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ON THE STABILITY OF THE GENERALIZED QUADRATIC SET-VALUED FUNCTIONAL EQUATION

HAHNG-YUN CHU† AND SEUNG KI YOO*

Abstract. In this article, we focus on the n-dimensional quadratic set-valued functional equation $(4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j) = 4\sum_{i=1}^{n} f(x_i)$, where $n \geq 2$ is an integer. We prove the Hyers-Ulam stability for the set-valued functional equation.

1. Introduction

The stability problem of functional equation concerning group homomorphisms had been first raised by S. M. Ulam [18] in 1940.

Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x * y), h(x) \diamond h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The first partial solution to Ulam's question was provided by D. H. Hyers [8] for Banach spaces. Hyers' theorem was generalized by T. Aoki [1] for additive mapping. Th. M. Rassias [15] generalized the result of Hyers as follows:

Let $f : X \to Y$ be a mapping between Banach spaces and let $0 \leq p < 1$ be fixed. If f satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{1.1}$$

for some $\theta \geq 0$ and for all $x, y \in X$, then there exists a unique additive mapping $A : X \to Y$ such that $\|A(x) - f(x)\| \leq \frac{2\theta}{2-2^p}\|x\|^p$ for all $x \in X$. If $f(tx)$ is continuous in t for each fixed $x \in X$, then A is linear.

Thereafter, P. Găvruta [7] provided a generalization of Th. M. Rassias' theorem, more precisely speaking, in which he replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ in (1.1) by control functions $\phi(x, y)$ with more general types for the existence of a unique linear mapping. The functional equation $f(x+y)+f(x-y) =$

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Key words and Phrases. Hyers-Ulam stability, generalized quadratic set-valued functional equation.

$2f(x)+2f(y)$ is called the *quadratic functional equation* and every solution of the quadratic functional equation is called a *quadratic function.*

The Hyers-Ulam stsbility of quadratic functional equation was proved by F. Skof [17] for a function $f : E_1 \to E_2$ where E_1 is a normed space and E_2 is a Banach space. P. W. Cholewa [3] considered Skof's theorem to a version of abelian groups. Skof's result was generalized by S. Czerwik [6] who proved the generalized Hyers-Ulam stability of quadratic functional equation in the spirit of Rassias approach. Kang and Chu [10] extended the quadratic functional equation to the generalized form $(4-n)f(\sum_{i=1}^n x_i)+\sum_{i=1}^n f(\sum_{j=1}^n \theta(i,j)x_j) = 4\sum_{i=1}^n f(x_i)$ where $n \geq 2$ is an integer and the function θ is defined by

$$\theta(i,j) = \begin{cases} 1 \text{ if } i \neq j \\ -1 \text{ if } i = j \end{cases}$$

and also investigated the Hyers-Ulam stability for the generalized quadratic functional equation. In [12], Lu and Park defined the additive set-valued functional equations $f(\alpha x + \beta y) = rf(x) + sf(y)$ and $f(x + y + z) = 2f(\frac{x+y}{2}) + f(z)$ and proved the Hyers-Ulam stability of the set-valued functional equations. In [14], Park et al. investigated stability problems of the Jensen additive, quadratic, cubic and quartic set-valued functional equation. Kenary et al. [11] proved the stability for various types of the set-valued functional equation using the fixed point alternative. In recent years, Chu and Yoo [5] studied the Hyers-Ulam stability of the n-dimensional additive set-valued functional equation. In [4], they also investigated the Hyers-Ulam stability of the n-dimensional cubic set-valued functional equation.

Let $CB(Y)$ be the set of all closed bounded subsets of Y and $CC(Y)$ the set of all closed convex subsets of Y. Let $CBC(Y)$ be the set of all closed bounded convex subsets of Y. For any elements A, B of $CC(Y)$, we denote $A \oplus B = \overline{A+B}$. If A is convex, then we obtain that $(\alpha+\beta)A = \alpha A + \beta A$ for all $\alpha, \beta \in \mathbb{R}^+$. Let $f : X \to CBC(Y)$ be a mapping. The *quadratic set-valued functional equation* is defined by $f(x + y) \oplus f(x - y) = 2f(x) \oplus 2f(y)$ for all $x, y \in X$. Every solution of the quadratic set-valued functional equation is said to be a *quadratic set-valued mapping.*

In this paper, we introduce the generalized n-dimensional quadratic set-valued functional equation

$$(4-n)f(\sum_{i=1}^n x_i) \oplus \sum_{i=1}^n f(\sum_{j=1}^n \theta(i,j)x_j) = 4\sum_{i=1}^n f(x_i) \tag{1.2}$$

where $n \geq 2$ is an integer and the function θ is defined by

$$\theta(i,j) = \begin{cases} 1 \text{ if } i \neq j \\ -1 \text{ if } i = j \end{cases}$$

and investigate the Hyers-Ulam stability of the functional equation.

In the set-valued dynamics, every solution of the generalized n-dimensional quadratic set-valued functional equation is called a *n-dimensional quadratic set-valued mapping.*

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For a subset $A \subset Y$, the distance function $d(\cdot, A)$ is defined by $d(x, A) := inf\{\| x - y \| : y \in A\}$ for $x \in Y$. For $A, B \in CB(Y)$, the *Hausdorff distance* $h(A, B)$ is defined by

$$h(A, B) := inf\{\alpha \geq 0 |\ A \subseteq B + \alpha B_Y, B \subseteq A + \alpha B_Y\},$$

where B_Y is the closed unit ball in Y. In [2], it was proved that $(CBC(Y), \oplus, h)$ is a complete metric semigroup. Rådström [16] proved that $(CBC(Y), \oplus, h)$ is isometrically embedded in a Banach space. The following remark is easily proved by using the notion of the Hausdorff distance.

Remark 1.1. Let $A, A', B, B', C \in CBC(Y)$ and $\alpha > 0$. Then we have that

(1) $h(A \oplus A', B \oplus B') \leq h(A, B) + h(A', B')$;
(2) $h(\alpha A, \alpha B) = \alpha h(A, B)$;
(3) $h(A, B) = h(A \oplus C, B \oplus C)$.

This paper is organized as follows. In section 2, we prove that the generalized n-dimensional set-valued mapping is actually general type of the quadratic set-valued mapping. We also investigate Hyers-Ulam stability for the generalized n-dimensional set-valued funational equation.

As applications of the stability, we take to change the control function and obtain the different approaches to unique generalized n-dimensional functional equation. In section 3, we also get the Hyers-Ulam staility for the generalized n-dimensional set-valued functional equation by using the fixed point method which is developed by Margolis and Diaz.

2. Stability of the set-valued functional equation

In this section, we mainly deal with the Hyers-Ulam stability for the generalized n-dimensional quadratic set-valued functional equation. We first study for properties of the n-dimensional quadratic set-valued mapping. Next we prove the Hyers-Ulam stabilities for the generalized n-dimensional quadratic set-valued equation. Especially when n is an even numbers, we find the precise control function depending upon the original function and n-dimensional quadratic set-valued mapping. Similarly we also obtain the precise control function in the odd case for the generalized n-dimensional quadratic set-valued functional equation.

Proposition 2.1. *Suppose that a mapping $f : X \to CBC(Y)$ defined by*

$$(4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j) = 4\sum_{i=1}^{n} f(x_i) \tag{2.1}$$

for all $x_1, \ldots, x_n \in X$. Then f has the following properties:

(1) $f(0) = \{0\}$
(2) $f(x) = f(-x)$ *for all* $x \in X$

(3) *f is a quadratic set-valued mapping.*

Proof. (1) Putting $x_i = 0$ $(i = 1, \dots, n)$ in (2.1), we have $f(0) = \{0\}$.
(2) Putting $x_1 = x$ and $x_i = 0$ $(i = 2, \dots, n)$ in (2.1), we get $(4-n)f(x)\oplus f(-x)\oplus(n-1)f(x) = 4f(x)$. Thus $f(x) = f(-x)$ for all $x \in X$.
(3) Replacing $x_1 = x$, $x_2 = y$ and $x_i = 0$ $(i = 3, \dots, n)$, we have $(4-n)f(x+y)\oplus f(-x+y)\oplus f(x-y)\oplus (n-2)f(x+y) = 4f(x)\oplus 4f(y)\oplus(n-2)f(0)$. So we conclude that $f(x+y)\oplus f(x-y) = 2f(x)\oplus 2f(y)$. This completes the proof. □

Next, we prove the stability of the generalized n-dimensional quadratic set-valued functional equation. To extended precisely to the stability theory for the set-valued functional equation, we state the stability according to dimensions of the equation.

Theorem 2.2. *Let $n \geq 2$ be an integer and let $\phi : X^n \to [0,\infty)$ be a function such that*

$$\tilde{\phi}(x_1, \dots, x_n) := \sum_{i=0}^{\infty} \frac{1}{4^i}\phi(2^i x_1, \dots, 2^i x_n) < \infty \tag{2.2}$$

for all $x_1, \dots, x_n \in X$. Suppose that $f : X \longrightarrow (CBC(Y), h)$ is a mapping with $f(0) = \{0\}$ and

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n}\theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \phi(x_1, \dots, x_n) \tag{2.3}$$

for all $x_1, \dots, x_n \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ such that

$$h(f(x), T(x)) \leq \frac{1}{8}\tilde{\phi}(x, x, 0, \dots, 0) \tag{2.4}$$

for all $x \in X$.

Proof. Putting $x_1 = x_2 = x$ and $x_3 = \cdots = x_n = 0$ in (2.3), we have

$$h(\frac{f(2x)}{4}, f(x)) \leq \frac{1}{8}\phi(x, x, 0, \dots, 0) \tag{2.5}$$

for all $x \in X$. Replacing x by $2x$ and dividing by 4 in (2.5)

$$h(\frac{f(4x)}{4^2}, f(2x)) \leq \frac{1}{32}\phi(2x, 2x, 0, \dots, 0) \tag{2.6}$$

for all $x \in X$. By (2.5) and (2.6), we get

$$h(\frac{f(4x)}{4^2}, f(x)) \leq \frac{1}{8}\phi(x, x, 0, \dots, 0) + \frac{1}{4\cdot 8}\phi(2x, 2x, 0, \dots, 0) \tag{2.7}$$

for all $x \in X$. Using the induction on i, we have that

$$h(\frac{f(2^r x)}{4^r}, f(x)) \leq \frac{1}{8}\sum_{i=0}^{r-1}\frac{1}{4^i}\phi(2^i x, 2^i x, 0, \dots, 0) \tag{2.8}$$

for any positive integer r and for all $x \in X$.

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Now, we show that the sequence $\{\frac{f(2^r x)}{4^r}\}$ converges for all $x \in X$. For any positive integer r and s, we divide inequality (2.8) by 4^s and replace x by $2^s x$. Then we obtain that the following inequality

$$h(\frac{f(2^{r+s}x)}{4^{r+s}}, \frac{f(2^s x)}{4^s}) \leq \frac{1}{4^s}\frac{1}{8}\sum_{i=0}^{r-1}\frac{1}{4^i} \leq \phi(2^{s+i}x, 2^{s+i}x, 0, \dots, 0) \tag{2.9}$$

for all $x \in X$. Since the right-hand side of the inequality (2.9) tends to zero as s tends to infinity, the sequence $\{\frac{f(2^r x)}{4^r}\}$ is a Cauchy sequence in (CBC(Y), h). Therefore, we can define a mapping $T : X \to (CBC(Y), h)$ as $T(x) := \lim_{r\to\infty} \frac{f(2^r x)}{4^r}$ for all $x \in X$. It follows from the definition of T and (2.2) that

$$h\Big((4-n)T(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} T(\sum_{j=1}^{n}\theta(i,j)x_j),\ 4\sum_{i=1}^{n} T(x_i)\Big) \leq \lim_{r\to\infty}\frac{1}{4^r}\phi(2^r x_1, \dots, 2^r x_n) = 0 \tag{2.10}$$

for all $x_1, \dots, x_n \in X$. Hence, we claim that T is an n-dimensional quadratic set-valued mapping. By letting $r \to \infty$ in (2.8), we have the desired inequality (2.4). Now we prove the uniqueness of T. Let $T' : X \to (CBC(Y), h)$ be another n-dimensional quadratic set-valued mapping satisfying (2.4). Therefore, we get the following inequality

$$h(T(x), T'(x)) = \frac{1}{4^r}h(T(2^r x), T'(2^r x)) \leq \frac{1}{4^r}\frac{1}{8}\tilde{\phi}(2^r x, 2^r x, 0, \dots, 0)$$

for all $x \in X$. Hence, letting $r \to \infty$, the right-hand side of above inequality goes to zero, and it follows that $T(x) = T'(x)$ for all $x \in X$. □

Corollary 2.3. *Let $n \geq 2$ be an integer, $0 < p < 2$ and $\theta \geq 0$ be real numbers and let X be a real normed space. Suppose that $f : X \to (CBC(Y), h)$ is a mapping satisfying*

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n}\theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \theta\sum_{i=1}^{n}\|x_i\|^p$$

for all $x_1, \dots, x_n \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ that satisfies

$$h(f(x), T(x)) \leq \frac{\theta}{2^2 - 2^p}\|x\|^p$$

for all $x \in X$.

Proof. The result follows Theorem 2.2 by setting $\phi(x_1, \dots, x_n) = \theta\sum_{i=1}^{n}\|x_i\|^p$ for all $x_1, \dots, x_n \in X$. □

Theorem 2.4. *Let $n \geq 2$ be an integer and let $\phi : X^n \to [0, \infty)$ be a function such that*

$$\tilde{\phi}(x_1, \dots, x_n) := \sum_{i=1}^{\infty} 4^i \phi(\frac{x_1}{2^i}, \dots, \frac{x_n}{2^i}) < \infty \tag{2.11}$$

for all $x_1, \dots, x_n \in X$. *Suppose that* $f : X \longrightarrow (CBC(Y), h)$ *is a mapping and*

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \le \phi(x_1, \dots, x_n) \tag{2.12}$$

for all $x_1, \dots, x_n \in X$. *Then there exists a unique n-dimensional quadratic set-valued mapping* $T : X \to (CBC(Y), h)$ *such that*

$$h(f(x), T(x)) \le \frac{1}{8}\tilde{\phi}(x, x, 0, \dots, 0) \tag{2.13}$$

for all $x \in X$.

Proof. By (2.11) and (2.12), we get $f(0) = \{0\}$. Replacing x by $\frac{x}{2}$ and multiplying by 4 in (2.5), we have the following inequality

$$h(f(x), 4f(\frac{x}{2})) \le \frac{1}{2}\phi(\frac{x}{2}, \frac{x}{2}, 0, \dots, 0)$$

for all $x \in X$. The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 2.5. *Let* $n \ge 2$ *be an integer,* $p > 2$ *and* $\theta \ge 0$ *be real numbers and let* X *be a real normed space. Suppose that* $f : X \to (CBC(Y), h)$ *is a mapping satisfying*

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \le \theta\sum_{i=1}^{n} \|x_i\|^p$$

for all $x_1, \dots, x_n \in X$. *Then there exists a unique n-dimensional quadratic set-valued mapping* $T : X \to (CBC(Y), h)$ *that satisfies*

$$h(f(x), T(x)) \le \frac{\theta}{2^p - 2^2}\|x\|^p$$

for all $x \in X$.

Proof. The result follows Theorem 2.4 by setting $\phi(x_1, \dots, x_n) = \theta\sum_{i=1}^{n} \|x_i\|^p$ for all $x_1, \dots, x_n \in X$. □

Let n be an even positive integer. In this case, we can obtain the control function for the Hausdorff distence between the original mapping and n-dimensional quadratic set-valued mapping.

Theorem 2.6. *Let* $n \ge 2$ *be even and let* $\phi : X^n \to [0, \infty)$ *be a function such that*

$$\tilde{\phi}(x_1, \dots, x_n) := \sum_{i=0}^{\infty} \frac{1}{4^i}\phi(2^i x_1, \dots, 2^i x_n) < \infty \tag{2.14}$$

for all $x_1, \dots, x_n \in X$. *Suppose that* $f : X \longrightarrow (CBC(Y), h)$ *is an even mapping with* $f(0) = \{0\}$ *and*

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \le \phi(x_1, \dots, x_n) \tag{2.15}$$

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for all $x_1, \dots, x_n \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ such that

$$h(f(x), T(x)) \leq \frac{1}{4n}\tilde{\phi}(x, -x, x, -x, \dots, x, -x) \tag{2.16}$$

for all $x \in X$.

Proof. Put $x_k = (-1)^{k-1}x$ $(k = 1, \dots, n)$ in (2.15). Since f is even and the range of f is convex, we have that

$$h(\frac{f(2x)}{4}, f(x)) \leq \frac{1}{4n}\phi(x, -x, x, -x, \dots, x, -x) \tag{2.17}$$

for all $x \in X$. The rest of the proof is similar to proof of Theorem 2.2. □

Corollary 2.7. *Let $n \geq 2$ be even, $0 < p < 2$ and $\theta \geq 0$ be real numbers and let X be a real normed space. Suppose that $f : X \to (CBC(Y), h)$ is an even mapping satisfying*

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \theta \sum_{i=1}^{n} \|x_i\|^p$$

for all $x_1, \dots, x_n \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ that satisfies

$$h(f(x), T(x)) \leq \frac{\theta}{2^2 - 2^p}\|x\|^p$$

for all $x \in X$.

Proof. The result follows Theorem 2.6 by setting $\phi(x_1, \dots, x_n) = \theta \sum_{i=1}^{n} \|x_i\|^p$ for all $x_1, \dots, x_n \in X$. □

Theorem 2.8. *Let $n \geq 2$ be even and let $\phi : X^n \to [0, \infty)$ be a function such that*

$$\tilde{\phi}(x_1, \dots, x_n) := \sum_{i=0}^{\infty} 4^i \phi(\frac{x_1}{2^{i+1}}, \dots, \frac{x_n}{2^{i+1}}) < \infty \tag{2.18}$$

for all $x_1, \dots, x_n \in X$. Suppose that $f : X \longrightarrow (CBC(Y), h)$ is an even mapping with $f(0) = \{0\}$ and

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \phi(x_1, \dots, x_n) \tag{2.19}$$

for all $x_1, \dots, x_n \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ such that

$$h(f(x), T(x)) \leq \frac{1}{n}\tilde{\phi}(x, -x, x, -x, \dots, x, -x) \tag{2.20}$$

for all $x \in X$.

Proof. Replacing x by $\frac{x}{2}$ and multiplying by 4 in (2.17), we have the following inequality

$$h(f(x),4f(\frac{x}{2}))\le\frac{1}{n}\phi(\frac{x}{2},-\frac{x}{2},\frac{x}{2},-\frac{x}{2},\dots,\frac{x}{2},-\frac{x}{2})$$

for all $x\in X$. The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 2.9. *Let $n\ge 2$ be even, $p>2$ and $\theta\ge 0$ be real numbers and let X be a real normed space. Suppose that $f:X\to(CBC(Y),h)$ is an even mapping satisfying*

$$h\Big((4-n)f(\sum_{i=1}^{n}x_i)\oplus\sum_{i=1}^{n}f(\sum_{j=1}^{n}\theta(i,j)x_j),\ 4\sum_{i=1}^{n}f(x_i)\Big)\le\theta\sum_{i=1}^{n}\|x_i\|^p$$

for all $x_1,\dots,x_n\in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $T:X\to(CBC(Y),h)$ that satisfies

$$h(f(x),T(x))\le\frac{\theta}{2^p-2^2}\|x\|^p$$

for all $x\in X$.

Proof. The result follows Theorem 2.8 by setting $\phi(x_1,\dots,x_n)=\theta\sum_{i=1}^{n}\|x_i\|^p$ for all $x_1,\dots,x_n\in X$. □

As applications for the theorem, we get the Hyers-Ulam stability for the generalized n-dimensional set-valued functional equation and especially we deal with the odd case for n.

Theorem 2.10. *Let $n\ge 2$ be odd and let $\phi:X^n\to[0,\infty)$ be a function such that*

$$\tilde{\phi}(x_1,\dots,x_n):=\sum_{i=0}^{\infty}\frac{1}{9^i}\phi(3^ix_1,\dots,3^ix_n)<\infty \tag{2.21}$$

for all $x_1,\dots,x_n\in X$. Suppose that $f:X\longrightarrow(CBC(Y),h)$ is an even mapping with $f(0)=\{0\}$ and

$$h\Big((4-n)f(\sum_{i=1}^{n}x_i)\oplus\sum_{i=1}^{n}f(\sum_{j=1}^{n}\theta(i,j)x_j),\ 4\sum_{i=1}^{n}f(x_i)\Big)\le\phi(x_1,\dots,x_n) \tag{2.22}$$

for all $x_1,\dots,x_n\in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $T:X\to(CBC(Y),h)$ such that

$$h(f(x),T(x))\le\frac{2}{9(n-1)}\tilde{\phi}(x,-x,x,-x,\dots,-x,x) \tag{2.23}$$

for all $x\in X$.

Proof. Put $x_k=(-1)^{k-1}x$ $(k=1,\dots,n)$ in (2.22). Since f is even and the range of f is convex, we have that

$$h(\frac{f(3x)}{9},f(x))\le\frac{2}{9(n-1)}\phi(x,-x,x,-x,\dots,-x,x)$$

for all $x\in X$. The rest of the proof is similar to proof of Theorem 2.2. □

Corollary 2.11. *Let $n > 2$ be odd, $0 < p < 2$ and $\theta \geq 0$ be real numbers and let X be a real normed space. Suppose that $f : X \to (CBC(Y), h)$ is an even mapping satisfying*

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \theta \sum_{i=1}^{n} \|x_i\|^p$$

for all $x_1, \dots, x_n \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ that satisfies

$$h(f(x), T(x)) \leq \frac{2n\theta}{(n-1)(3^2-3^p)}\|x\|^p$$

for all $x \in X$.

Proof. The result follows Theorem 2.10 by setting $\phi(x_1, \dots, x_n) = \theta \sum_{i=1}^{n} \|x_i\|^p$ for all $x_1, \dots, x_n \in X$. □

Theorem 2.12. *Let $n > 2$ be odd and let $\phi : X^n \to [0, \infty)$ be a function such that*

$$\tilde{\phi}(x_1, \dots, x_n) := \sum_{i=0}^{\infty} 9^i \phi(\frac{x_1}{3^{i+1}}, \dots, \frac{x_n}{3^{i+1}}) < \infty \tag{2.24}$$

for all $x_1, \dots, x_n \in X$. Suppose that $f : X \longrightarrow (CBC(Y), h)$ is an even mapping with $f(0) = \{0\}$ and

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \phi(x_1, \dots, x_n) \tag{2.25}$$

for all $x_1, \dots, x_n \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ such that

$$h(f(x), T(x)) \leq \frac{2}{n-1}\tilde{\phi}(x, -x, x, -x, \dots, -x, x) \tag{2.26}$$

for all $x \in X$.

Proof. Put $x_k = (-1)^{k-1}x$ $(k = 1, \dots, n)$ in (2.25). Since f is even and the range of f is convex, we have that

$$h(9f(\frac{x}{3}), f(x)) \leq \frac{2}{n-1}\phi(\frac{x}{3}, -\frac{x}{3}, \frac{x}{3}, -\frac{x}{3}, \dots, -\frac{x}{3}, \frac{x}{3})$$

for all $x \in X$. The rest of the proof is similar to proof of Theorem 2.2. □

Corollary 2.13. *Let $n > 2$ be odd, $p > 2$ and $\theta \geq 0$ be real numbers and let X be a real normed space. Suppose that $f : X \to (CBC(Y), h)$ is an even mapping satisfying*

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \theta \sum_{i=1}^{n} \|x_i\|^p$$

for all $x_1, \dots, x_n \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ that satisfies

$$h(f(x), T(x)) \leq \frac{2n\theta}{(n-1)(3^p - 3^2)} \|x\|^p$$

for all $x \in X$.

Proof. The result follows Theorem 2.12 by setting $\phi(x_1, \dots, x_n) = \theta \sum_{i=1}^{n} \|x_i\|^p$ for all $x_1, \dots, x_n \in X$. □

3. Stability of the set-valued functional equation by fixed point method

As using the fixed point method, we get plenty of the results related to the generalized n-dimensional quadratic set-valued functional equation. We first introduce the generalized metric on the given phase space and recall fundamental results for the fixed point theory. Let X be a set. A function $d : X \times X \to [0, \infty)$ is the *generalized metric* on X if d satisfies the following properties:

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The following theorem is very useful for proving Hyers-Ulam stability which is due to Margolis and Diaz [13].

Theorem 3.1. *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) *$d(J^n x, J^{n+1} x) < \infty, \ \forall n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

Using the alternative fixed point theorem, we investigate the stability of the even dimensional quadratic set-valued functional equation.

Theorem 3.2. *Let $n \geq 2$ be even. Suppose that an even mapping $f : X \longrightarrow (CBC(Y), h)$ with $f(0) = \{0\}$ satisfies the inequality*

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \phi(x_1, \ldots, x_n) \tag{3.1}$$

for all $x_1, \ldots, x_n \in X$, and there exists a constant L with $0 < L < 1$ for which the function $\phi : X^n \to [0, \infty)$ satisfies

$$\phi(2x, -2x, 2x, -2x, \ldots, 2x, -2x) \leq 4L\phi(x, -x, x, -x, \ldots, x, -x) \tag{3.2}$$

for all $x \in X$. Then there exists a n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ given by $T(x) = \lim_{k\to\infty} \frac{f(2^k x)}{4^k}$ such that

$$h(f(x), T(x)) \leq \frac{1}{4n(1-L)}\phi(x, -x, x, -x, \ldots, x, -x) \tag{3.3}$$

for all $x \in X$.

Proof. Put $x_k = (-1)^{k-1}x$ $(k = 1, \ldots, n)$ in (3.1). Since f is even and the range of f is convex, we have that

$$h(\frac{f(2x)}{4}, f(x)) \leq \frac{1}{4n}\phi(x, -x, x, -x, \ldots, x, -x) \tag{3.4}$$

for all $x \in X$.

Let $S := \{g \mid g : X \to CBC(Y),\ g(0) = \{0\}\}$. We define a generalized metric on S defined by

$$d(g_1, g_2) := \inf\{\mu \in (0, \infty) \mid h(g_1(x), g_2(x)) \leq \mu\phi(x, -x, x, -x, \ldots, x, -x), x \in X\}.$$

It is easy to show that (S, d) is complete (see [9]). Now, we define the mapping $J : S \to S$ given by $Jg(x) = \frac{1}{4}g(2x)$ for all $x \in X$. For $g_1, g_2 \in S$, let $d(g_1, g_2) = \mu$. Then

$$h(\frac{1}{4}g_1(2x), \frac{1}{4}g_2(2x)) \leq \frac{1}{4}\mu\phi(2x, -2x, 2x, -2x \ldots, 2x, -2x)$$

for all $x \in X$. Then by (3.2), we have $h(Jg_1(x), Jg_2(x)) \leq \mu L\phi(x, -x, x, -x, \ldots, x, -x)$ for all $x \in X$. Therefore, we get $d(Jg_1, Jg_2) \leq Ld(g_1, g_2)$ for any $g_1, g_2 \in S$. Hence J is a strictly contractive mapping with Lipschitz constant L. It follows from (3.4) that $d(Jf, f) \leq \frac{1}{4n}$. By Theorem 3.1, the sequence $\{J^k f\}$ converges to a fixed point $T : X \to (CBC(Y), h)$ of J in the set $\{g \in S \mid d(f, g) < \infty\}$ such that $\{J^k f\} \to 0$ as $k \to \infty$. This implies $T(x) = \lim_{k\to\infty} \frac{f(2^k x)}{4^k}$ for all $x \in X$. And we also have $d(f, T) \leq \frac{1}{1-L}d(Jf, f) \leq \frac{1}{4n(1-L)}$. This means that the inequality (3.3) holds. By (3.1),

$$h\Big((4-n)T(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} T(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} T(x_i)\Big) \leq \lim_{k\to\infty} \frac{1}{4^k}\phi(x, -x, x, -x, \ldots, x, -x) = 0$$

Therefore, T is a unique n-dimensional quadratic set-valued mapping as desired. □

Corollary 3.3. *Let $n \geq 2$ be even, $0 < p < 2$ and $\theta \geq 0$ be real numbers and let $n \geq 2$ be even. Suppose that $f : X \longrightarrow (CBC(Y), h)$ is an even mapping satisfying*

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \theta \sum_{i=1}^{n} \|x_i\|^p$$

for all $x_1, \dots, x_n \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ that satisfies

$$h(f(x), T(x)) \leq \frac{\theta}{2^2 - 2^p}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by setting $\phi(x_2, \dots, x_n) = \theta \sum_{i=1}^{n} \|x_i\|^p$ for every $x_1, \dots, x_n \in X$. Then we can choose $L = 2^{p-2}$ and we get the desired result. □

Theorem 3.4. *Let $n \geq 2$ be even. Suppose that an even mapping $f : X \longrightarrow (CBC(Y), h)$ with $f(0) = \{0\}$ satisfies the inequality*

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \phi(x_1, \dots, x_n) \tag{3.5}$$

for all $x_1, \dots, x_n \in X$, and there exists a constant L with $0 < L < 1$ for which the function $\phi : X^n \to [0, \infty)$ satisfies

$$\phi(\frac{x}{2}, -\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}, \dots, \frac{x}{2}, -\frac{x}{2}) \leq \frac{L}{4}\phi(x, -x, x, -x, \dots, x, -x) \tag{3.6}$$

for all $x \in X$. Then there exists a n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ given by $T(x) = \lim_{k\to\infty} 4^k f(\frac{x}{2^k})$ such that

$$h(f(x), T(x)) \leq \frac{L}{4n(1-L)}\phi(x, -x, x, -x, \dots, x, -x) \tag{3.7}$$

for all $x \in X$.

Proof. Replacing x by $\frac{x}{2}$ and multiplying 4 in (3.4), we have

$$h(f(x), 4f(\frac{x}{2})) \leq \frac{1}{n}\phi(\frac{x}{2}, -\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}, \dots, \frac{x}{2}, -\frac{x}{2}) \leq \frac{L}{4n}\phi(x, -x, x, -x, \dots, x, -x)$$

for all $x \in X$. The rest of the proof is similar to proof of Theorem 2.2. □

Corollary 3.5. *Let $n \geq 2$ be even, $p > 2$ and $\theta \geq 0$ be real numbers and let $n \geq 2$ be even. Suppose that $f : X \longrightarrow (CBC(Y), h)$ is an even mapping satisfying*

$$h\Big((4-n)f(\sum_{i=1}^{n} x_i) \oplus \sum_{i=1}^{n} f(\sum_{j=1}^{n} \theta(i,j)x_j),\ 4\sum_{i=1}^{n} f(x_i)\Big) \leq \theta \sum_{i=1}^{n} \|x_i\|^p$$

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for all $x_1, \dots, x_n \in X$. Then there exists a unique n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ that satisfies

$$h(f(x), T(x)) \le \frac{\theta}{n(2^p - 2^2)} \|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.9 by setting $\phi(x_2, \dots, x_n) = \theta \sum_{i=1}^n \|x_i\|^p$ for every $x_1, \dots, x_n \in X$. Then we can choose $L = 2^{2-p}$ and we get the desired result. □

Finally, we deal with the Hyers-Ulam stability for the odd dimensional quadratic set-valued functional equation.

Theorem 3.6. *Let $n > 2$ be odd. Suppose that an even mapping $f : X \longrightarrow (CBC(Y), h)$ with $f(0) = \{0\}$ satisfies the inequality*

$$h\Big((4-n)f(\sum_{i=1}^n x_i) \oplus \sum_{i=1}^n f(\sum_{j=1}^n \theta(i,j)x_j),\ 4\sum_{i=1}^n f(x_i)\Big) \le \phi(x_1, \dots, x_n) \tag{3.8}$$

for all $x_1, \dots, x_n \in X$, and there exists a constant L with $0 < L < 1$ for which the function $\phi : X^n \to [0, \infty)$ satisfies

$$\phi(3x, -3x, 3x, -3x, \dots, -3x, 3x) \le 9L\phi(x, -x, x, -x, \dots, -x, x) \tag{3.9}$$

for all $x \in X$. Then there exists a n-dimensional quadratic set-valued mapping $T : X \to (CBC(Y), h)$ given by $T(x) = \lim_{k\to\infty} \frac{f(3^k x)}{9^k}$ such that

$$h(f(x), T(x)) \le \frac{2}{9(n-1)(1-L)} \phi(x, -x, x, -x, \dots, -x, x) \tag{3.10}$$

for all $x \in X$.

Proof. Put $x_k = (-1)^{k-1}x$ $(k = 1, \dots, n)$ in (3.8). Since f is even and the range of f is convex, we have that

$$h(9f(\frac{x}{3}), f(x)) \le \frac{2}{n-1}\phi(\frac{x}{3}, -\frac{x}{3}, \frac{x}{3}, -\frac{x}{3}, \dots, -\frac{x}{3}, \frac{x}{3})$$

for all $x \in X$. The rest of the proof is similar to proof of Theorem 2.2.

□

Corollary 3.7. *Let $n > 2$ be odd, $0 < p < 2$ and $\theta \ge 0$ be real numbers and let $n \ge 2$ be odd. Suppose that $f : X \longrightarrow (CBC(Y), h)$ is an even mapping satisfying*

$$h\Big((4-n)f(\sum_{i=1}^n x_i) \oplus \sum_{i=1}^n f(\sum_{j=1}^n \theta(i,j)x_j),\ 4\sum_{i=1}^n f(x_i)\Big) \le \theta \sum_{i=1}^n \|x_i\|^p$$

for all $x_1, \dots, x_n \in X$. *Then there exists a unique* n*-dimensional quadratic set-valued mapping* $T : X \to (CBC(Y), h)$ *that satisfies*

$$h(f(x), T(x)) \leq \frac{2\theta}{(n-1)(3^2 - 3^p)} \|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by setting $\phi(x_2, \dots, x_n) = \theta \sum_{i=1}^{n} \|x_i\|^p$ for every $x_1, \dots, x_n \in X$. Then we can choose $L = 3^{p-2}$ and we get the desired result. □

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COMMON BEST PROXIMITY POINTS FOR PROXIMALLY COMMUTING MAPPINGS IN NON-ARCHIMEDEAN PM-SPACES

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Abstract. In this paper, we prove new common best proximity point theorems for proximally commuting mappings in complete non-Archimedean PM-spaces. Our results generalized the recent results of S. Basha [Common best proximity points: global minimization of multi-objective functions, J. Global Optim. 49(2011), 15–21] and C. Mongkolkeha, P. Kumam [Some common best proximity points for proximity commuting mappings, Optim. Lett. 7 (2013), 1825–1836].

1. Introduction

Best proximity point theorems provide sufficient conditions that ensure the existence of approximate solutions which are optimal as well. In fact, if there is no solution to the fixed point equation $Tx = x$ for a non-self mapping $T : A \to B$, then it is desirable to determine an approximate solution x such that the error $F_{x,Tx}(t)$ is maximum.

A classical best approximation theorem was introduced by Fan [13], that is, if A is a nonempty compact convex subset of a Hausdorff locally convex topological vector space B and $T : A \to B$ is a continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$. Afterward, several authors, including Prolla [22], Reich [23], Sehgal and Singh [32, 33] and others, have derived some extensions of Fan's theorem in many directions. Other works of the existence of a best proximity point for contractions can be seen in [2, 5, 12, 15].

In 2005, Anthony Eldred, Kirk and Veeramani [6] have obtained best proximity point theorems for relatively nonexpansive mappings. Since then, best proximity point theorems for several types of contractions have been established in [3, 4, 8, 12, 16, 17, 19, 20, 26, 27, 28, 29, 30, 36, 37, 38, 39, 40].

2. Preliminaries

Throughout this paper, the space of all probability distribution functions (briefly, d.f.'s) is denoted by $\Delta^+ = \{F : \mathbb{R} \cup \{-\infty, +\infty\} \longrightarrow [0, 1] : F$ is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1\}$ and the subset $D^+ \subseteq \Delta^+$ is the set $D^+ = \{F \in \Delta^+ : l^- F(+\infty) = 1\}$. Here $l^- f(x)$ denotes the left limit of the function f at the point x and $l^- f(x) = \lim_{t\to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \le G$ if and only if $F(t) \le G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the d.f. given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \le 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 2.1. ([31]) A mapping $* : [0, 1] \times [0, 1] \to [0, 1]$ is a *continuous t–norm* if $*$ satisfies the following conditions:

(a) $*$ is commutative and associative;
(b) $*$ is continuous;
(c) $a * 1 = a$ for all $a \in [0, 1]$;

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(d) $a * b \leq c * d$ whenever $a \leq c$ and $c \leq d$, and $a, b, c, d \in [0, 1]$.

Two typical examples of continuous t–norm are $a * b = ab$ and $a * b = \min(a, b)$.

A t–norm $*$ is said to be *positive* ([31]) if $a * b > 0$ whenever $a, b \in (0, 1]$. The notation $* < *'$ means that $a * b < a *' b$ for all $a, b \in (0, 1)$.

Definition 2.2. (1) A *Probabilistic Metric space* (briefly, PM-space) is a triple $(X, F, *)$, where X is a nonempty set, T is a continuous t–norm and F is a mapping from $X \times X$ into D^+ such that, if $F_{x,y}$ denotes the value of F at the pair (x, y), the following conditions hold:

(PM1) $F_{x,y}(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = y$;
(PM2) $F_{x,y}(t) = F_{y,x}(t)$;
(PM3) $F_{x,z}(t + s) \geq F_{x,y}(t) * F_{y,z}(s)$ for all $x, y, z \in X$ and $t, s \geq 0$.

(2) If, in the above definition, the triangular inequality (PM3) is replaced by

(PM4) $F_{x,z}(\max\{t, s\}) \geq F_{x,y}(t) * F_{y,z}(s)$ for all $x, y, z \in X$ and $t, s \geq 0$,

then the triple $(X, F, *)$ is called a *non-Archimedean PM-space* (briefly, NA-PM-space).

It is easy to check that the triangular inequality (PM4) implies (PM3), that is, every NA-PM-space is itself a PM-space. It is easy to show that (PM4) is equivalent to the following condition:

(PM5) $F_{x,z}(t) \geq F_{x,y}(t) * F_{y,z}(t)$ for all $x, y, z \in X$ and $t \geq 0$.

Example 2.3. Let (X, d) be an ordinary metric space and let θ be a nondecreasing and continuous function from $(0, \infty)$ into $(0, 1)$ such that $\lim_{t\to\infty} \theta(t) = 1$. Some examples of these functions are as follows:

$$\theta(t) = \frac{t}{t+1}, \quad \theta(t) = 1 - e^{-t}, \quad \theta(t) = e^{-1/t}.$$

Let $a * b \leq ab$ for each $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define

$$F_{x,y}(t) = [\theta(t)]^{d(x,y)}$$

for all $x, y \in X$. Then $(X, F, *)$ is a NA-PM-space ([1]).

For more details and examples of these spaces see also [7], [9], [10], [11], [14], [18], [21], [24], [25], [34], [35], [41] and [42].

Definition 2.4. Let (X, F, T) be a NA-PM-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ if, for any $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that

$$F_{x_n,x}(\epsilon) > 1 - \lambda$$

whenever $n \geq N$;

(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for any $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that

$$F_{x_{n+p},x_n}(\epsilon) > 1 - \lambda$$

whenever $n \geq N$ and $p \in \mathcal{N}$;

(3) A PM-space $(X, F, *)$ is said to be *complete* if every Cauchy sequence in X is convergent to a point in X.

Definition 2.5. Let $(X, F, *)$ be a PM-space. For each p in X and $\lambda > 0$, the strong λ-neighborhood of p is the set

$$N_p(\lambda) = \{q \in X : F_{p,q}(\lambda) > 1 - \lambda\}$$

and the strong neighborhood system for X is the union $\bigcup_{p \in V} \mathcal{N}_p$, where

$$\mathcal{N}_p = \{N_p(\lambda) : \lambda > 0\}.$$

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The strong neighborhood system for X determines a Hausdorff topology for X.

Theorem 2.6. ([31]) *If $(X, F, *)$ is a PM-space and $\{p_n\}$ and $\{q_n\}$ are sequences such that $p_n \to p$ and $q_n \to q$ as $n \to \infty$, then $\lim_{n\to\infty} F_{p_n,q_n}(t) = F_{p,q}(t)$ for all continuity point t of $F_{p,q}$.*

Let A and B be two nonempty subsets of a PM-space and $t > 0$. The following notions and notations are used in the sequel.

$$F_{A,B}(t) := \sup\{F_{x,y}(t) : x \in A, y \in B\},$$
$$A_0 := \{x \in A : F_{x,y}(t) = F_{A,B}(t) \text{ for some } y \in B\},$$
$$B_0 := \{y \in B : F_{x,y}(t) = F_{A,B}(t) \text{ for some } x \in A\}.$$

Definition 2.7. A mapping $T : X \to X$ is said to be a *contraction* if there exists a constant $k \in [0, 1)$ such that

$$F_{Tx,Ty}(kt) \geq F_{x,y}(t) \tag{2.1}$$

for all $x, y \in X$ and $t > 0$.

Definition 2.8. A mapping $T : X \to X$ is said to be a *weak contraction* if

$$F_{Tx,Ty}(\phi(t)) * F_{x,y}(t) \geq F_{x,y}(\phi(t)) \tag{2.2}$$

for all $x, y \in X$ and $t > 0$, where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that, for any $t > 0$, $0 < \varphi(t) < t$.

Definition 2.9. A point $x \in A$ is said to be a *best proximity point* of a mapping $S : A \to B$ if it satisfies the following condition:

$$F_{x,Sx}(t)) = F_{A,B}(t)$$

for all $x, y \in X$ and $t > 0$.

It can be observed that a best proximity reduces to a fixed point if the underlying mapping is a self-mapping.

Definition 2.10. Let $S : A \to B$ and $T : A \to B$ be two mappings. An element $x^* \in A$ is said to be a *common best proximity point* if it satisfies the following condition:

$$F_{x^*,Sx^*}(t) = F_{x^*,Tx^*}(t) = F_{A,B}(t)$$

for each $t > 0$.

Observe that a common best proximity point is an element at which the multi-objective functions $x \to F_{x,Sx}(t)$ and $x \to F_{x,Tx}(t)$ attain a common global maximum since $F_{x,Sx}(t) \leq F_{A,B}(t)$ and $F_{x,Tx}(t) \leq F_{A,B}(t)$ for all x and $t > 0$.

Definition 2.11. A mapping $S : A \to B$ and $T : A \to B$ is said to be a *proximally commuting* if they satisfy the following condition:

$$[F_{u,Sx}(t) = F_{v,Tx}(t) = F_{A,B}(t)] \implies Sv = Tu$$

for all $u, v, x, \in A$ and $t > 0$.

It is easy to see that the proximal commutativity of self-mappings become commutativity of the mappings.

Definition 2.12. Two mappings $S : A \to B$ and $T : A \to B$ are said to be a *proximally swapped* if they satisfy the following condition:

$$[F_{y,u}(t) = F_{y,v}(t) = F_{A,B}(t), \quad Su = Tv] \implies Sv = Tu$$

for all $u, v, \in A$, $y \in B$ and $t > 0$.

Definition 2.13. A set A is said to be *approximatively compact* with respect to a set B if every sequence $\{x_n\}$ in A satisfies the condition that $F_{y,x_n}(t) \to F_{y,A}(t)$ for some $y \in B$ and for each $t > 0$ has a convergent subsequence.

Observe that every set is approximatively compact with respect to itself. Also, every compact set is approximatively compact with respect to any set. Moreover, A_0 and B_0 are nonempty set if A is compact and B is approximatively compact with respect to A.

3. Main result

Now, we give our main results in this paper.

Theorem 3.1. *Let A and B be nonempty closed subsets of a complete NA-PM-space $(X, F, *)$ in which the t–norm $*$ is positive and $* < \min$ such that A is approximatively compact with respect to B. Also, assume that A_0 and B_0 are nonempty. Let $S : A \to B$, $T : A \to B$ be the non-self mappings satisfying the following conditions:*

(a) *for each x and y are elements in A and $t > 0$,*

$$F_{Sx,Sy}(\phi(t)) * F_{Tx,Ty}(t) \geq F_{Tx,Ty}(\phi(t)),$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that, for any $t > 0$, $0 < \varphi(t) < t$;

(b) *T is continuous;*

(c) *S and T commute proximally;*

(d) *S and T can be swapped proximally;*

(e) *$S(A_0) \subseteq B_0$ and $S(A_0) \subseteq T(A_0)$.*

Then there exists an element $x \in A$ such that $F_{x,Tx}(t) = F_{A,B}(t)$ and $F_{x,Sx}(t) = F_{A,B}(t)$.

Moreover, if x^* is another common best proximity point of the mappings S and T, then it is necessary that $F_{x,x^*}(t) \geq F_{A,B}(t) * F_{A,B}(t)$ for all $t > 0$.

Proof. Let x_0 be a fixed element in A_0. In view of the fact that $S(A_0) \subseteq T(A_0)$, there exists an element $x_1 \in A_0$ such that $Sx_0 = Tx_1$. Again, since $S(A_0) \subseteq T(A_0)$, there exists an element $x_2 \in A_0$ such that $Sx_1 = Tx_2$. By the similar fashion, we can find a sequence $\{x_n\}$ in A_0 such that

$$Sx_{n-1} = Tx_n \tag{3.1}$$

for all $n \in \mathbb{N}$. It follows that

$$F_{Sx_n,Sx_{n+1}}(\phi(t)) * F_{Tx_n,Tx_{n+1}}(t) \geq F_{Tx_n,Tx_{n+1}}(\phi(t)) \tag{3.2}$$

and

$$F_{Sx_n,Sx_{n+1}}(\phi(t)) * F_{Sx_{n-1},Sx_n}(t) \geq F_{Sx_{n-1},Sx_n}(\phi(t)) \tag{3.3}$$

for all $t > 0$. Thus we have

$$F_{Sx_n,Sx_{n+1}}(t) \geq F_{Sx_{n-1},Sx_n}(t) \tag{3.4}$$

for all $t > 0$, which means that the sequence $\{F_{Sx_{n-1},Sx_n}(t)\}$ is non-decreasing and bounded above. Hence there exists $r \leq 1$ such that, for any $t > 0$,

$$\lim_{n\to\infty} F_{Sx_{n-1},Sx_n}(t) = r. \tag{3.5}$$

If $r < 1$, then we have

$$F_{Sx_n,Sx_{n+1}}(\phi(t)) * F_{Sx_{n-1},Sx_n}(t) \geq F_{Sx_{n-1},Sx_n}(\phi(t)) \tag{3.6}$$

for all $t > 0$. Taking $n \to \infty$ in the inequality (3.6), by the continuity of φ, we get $a * r \geq a$, where $a = \lim_{n\to\infty} F_{Sx_{n-1},Sx_n}(\phi(t))$, which is a contradiction unless $r = 1$. Therefore, it follows that

$$\lim_{n\to\infty} F_{Sx_{n-1},Sx_n}(t) = 1. \tag{3.7}$$

By the property of F, we conclude that $F_{Sx_{n-1},Sx_n}(t)$ tend to 1 for all $t > 0$.

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Next, we prove that $\{Sx_n\}$ is a Cauchy sequence. We consider two cases.

Case I. Suppose that there exits $n \in \mathbb{N}$ such that $Sx_n = Sx_{n+1}$. Then we observe that

$$F_{Sx_{n+1},Sx_{n+2}}(\phi(t)) * F_{Tx_{n+1},Tx_{n+2}}(t) \geq F_{Tx_{n+1},Tx_{n+2}}(\phi(t))$$

and

$$F_{Sx_{n+1},Sx_{n+2}}(\phi(t)) * F_{Sx_n,Sx_{n+1}}(t) \geq F_{Sx_n,Sx_{n+1}}(\phi(t))$$

for all $t > 0$. Then we have

$$F_{Sx_{n+1},Sx_{n+2}}(t) = 1$$

for all $t > 0$, which implies that $Sx_{n+1} = Sx_{n+2}$ and so, for each $m > n$, we conclude that $Sx_m = Sx_n$. Hence $\{Sx_n\}$ is a Cauchy sequence in B.

Case II. The successive terms of $\{Sx_n\}$ are different. Suppose that $\{Sx_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$, $t > 0$ and the subsequences $\{Sx_{m_k}\}$, $\{Sx_{n_k}\}$ of $\{Sx_n\}$ with $n_k > m_k \geq k$ such that

$$F_{Sx_{m_k},Sx_{n_k}}(t) \leq 1 - \varepsilon, \quad F_{Sx_{m_k},Sx_{n_k-1}}(t) > 1 - \varepsilon. \tag{3.8}$$

By using (3.8) and the triangular inequality, we have

$$\begin{aligned} 1 - \varepsilon &\geq F_{Sx_{m_k},Sx_{n_k}}(t) \\ &\geq F_{Sx_{m_k},Sx_{n_k-1}}(t) * F_{Sx_{n_k-1},Sx_{n_k}}(t) \\ &\geq (1-\varepsilon) * F_{Sx_{n_k-1},Sx_{n_k}}(t). \end{aligned} \tag{3.9}$$

Thus, using (3.9) and (3.7), we have

$$F_{Sx_{m_k},Sx_{n_k}}(t) \to 1 - \varepsilon \tag{3.10}$$

as $k \to \infty$. Again, by the triangular inequality, we have

$$F_{Sx_{m_k},Sx_{n_k}}(t) \geq F_{Sx_{m_k},Sx_{m_k+1}}(t) * F_{Sx_{m_k+1},Sx_{n_k+1}}(t) * F_{Sx_{n_k+1},Sx_{n_k}}(t) \tag{3.11}$$

and

$$F_{Sx_{m_k+1},Sx_{n_k+1}}(t) \geq F_{Sx_{m_k+1},Sx_{m_k}}(t) * F_{Sx_{m_k},Sx_{n_k}}(t) * F_{Sx_{n_k},Sx_{n_k+1}}(t). \tag{3.12}$$

From (3.7), (3.10), (3.11) and (3.12), it follows that

$$F_{Sx_{m_k+1},Sx_{n_k+1}}(t) \to 1 - \varepsilon \tag{3.13}$$

as $k \to \infty$. In view of the fact that

$$F_{Sx_{m_k+1},Sx_{n_k+1}}(\phi(t)) * F_{Tx_{m_k+1},Tx_{n_k+1}}(t) \geq F_{Tx_{m_k+1},Tx_{n_k+1}}(\phi(t)), \tag{3.14}$$

we have

$$F_{Sx_{m_k+1},Sx_{n_k+1}}(\phi(t)) * F_{Sx_{m_k},Sx_{n_k}}(t) \geq F_{Sx_{m_k},Sx_{n_k}}(\phi(t)). \tag{3.15}$$

Letting $k \to \infty$ in the inequality (3.15), we obtain

$$a * (1 - \varepsilon) \geq a,$$

where $a = F_{Sx_{m_k+1},Sx_{n_k+1}}(\phi(t))$, which is a contradiction by the property of φ. Then we deduce that $\{Sx_n\}$ is a Cauchy sequence in B. Since B is a closed subset a complete NA-PM-space X, there exists $y \in B$ such that $Sx_n \to y$ as $n \to \infty$. Consequently, it follows that the sequence $\{Tx_n\}$ also converges to y. From $S(A_0) \subseteq B_0$, there exists an element $u_n \in A$ such that

$$F_{Sx_n,u_n}(t) = F_{A,B}(t) \tag{3.16}$$

for all $n \in \mathbb{N}$ and $t > 0$. Thus it follows from (3.1) and (3.16) that

$$F_{Tx_n,u_{n-1}}(t) = F_{Sx_{n-1},u_{n-1}}(t) = F_{A,B}(t) \tag{3.17}$$

for all $n \in \mathbb{N}$ and $t > 0$. By (3.16), (3.17) and the fact that the mappings S and T are proximally commuting, we obtain

$$Tu_n = Su_{n-1} \tag{3.18}$$

for all $n \in \mathbb{N}$. Moreover, we have

$$\begin{aligned} F_{y,A}(t) &\geq F_{y,u_n}(t) \\ &\geq F_{y,Sx_n}(t) * F_{Sx_n,u_n}(t) \\ &= F_{y,Sx_n}(t) * F_{A,B}(t) \\ &\geq F_{y,Sx_n}(t) * F_{y,A}(t), \end{aligned} \tag{3.19}$$

for all $t > 0$. Therefore, it follows that, for all $t > 0$, $F_{y,u_n}(t) \to F_{y,A}(t)$ as $n \to \infty$. Since A is approximatively compact with respect to B, there exists a subsequence $\{u_{n_k}\}$ of the sequence $\{u_n\}$ such that $\{u_{n_k}\}$ converges to some element $u \in A$. Further, since $F_{y,u_{n_k-1}}(t) \to F_{y,A}(t)$ for all $t > 0$ and A is approximatively compact with respect to B, there exists a subsequence $\{u_{n_{k_j}-1}\}$ of the sequence $\{u_{n_k-1}\}$ such that $\{u_{n_{k_j}-1}\}$ converges to some element $v \in A$. By the continuity of the mappings S and T, we have

$$Tu = \lim_{j\to\infty} Tu_{n_{k_j}} = \lim_{k\to\infty} Su_{n_{k_j}-1} = Sv \tag{3.20}$$

and

$$\begin{aligned} F_{y,u}(t) &= \lim_{k\to\infty} F_{Sx_{n_k},u_{n_k}}(t) = F_{A,B}(t), \\ F_{y,v}(t) &= \lim_{j\to\infty} F_{Tx_{n_{k_j}},u_{n_{k_j}-1}}(t) = F_{A,B}(t). \end{aligned} \tag{3.21}$$

Since S and T can be swapped proximally, we have

$$Tv = Su. \tag{3.22}$$

Next, we prove that $Su = Sv$. Suppose that $Su \neq Sv$. Then, by (3.20), (3.21), (3.22) and the property of φ, we have

$$F_{Su,Sv}(\phi(t)) * F_{Tu,Tv}(t) \geq F_{Tu,Tv}(\phi(t))$$

and so

$$F_{Su,Sv}(\phi(t)) * F_{Su,Sv}(t) \geq F_{Su,Sv}(\phi(t))$$

for all $t > 0$, which is a contradiction. Thus $Su = Sv$ and also $Tu = Su$. Since $S(A_0)$ is contained in B_0, there exists an element $x \in A$ such that $F_{x,Tu}(t) = F_{A,B}(t)$ and $F_{x,Su}(t) = F_{A,B}(t)$. Since S and T are proximally commuting, we have $Sx = Tx$. Consequently, we have

$$F_{Su,Sx}(\phi(t)) * F_{Tu,Tx}(t) \geq F_{Tu,Tx}(\phi(t)) \tag{3.23}$$

and so

$$F_{Su,Sx}(\phi(t)) * F_{Su,Sx}(t) \geq F_{Su,Sx}(\phi(t)) \tag{3.24}$$

for all $t > 0$, which is impossible if $Su \neq Sx$. Thus we have $Su = Sx$ and hence $Tu = Tx$. It follows that

$$F_{x,Tx}(t) = F_{x,Tu}(t) = F_{A,B}(t)$$

and

$$F_{x,Sx}(t) = F_{x,Su}(t) = F_{A,B}(t)$$

for all $t > 0$. Therefore, x is a common best proximity point of S and T.

To prove the uniqueness of the point x, suppose that x^* is another common best proximity point of the mappings S and T. Then we have

$$F_{x^*,Tx^*}(t) = F_{A,B}(t), \quad F_{x^*,Sx^*}(t) = F_{A,B}(t)$$

for all $t > 0$. Since S and T are proximally commuting, we get $Sx = Tx$ and $Sx^* = Tx^*$. Consequently, we have

$$F_{Sx^*,Sx}(\phi(t)) * F_{Tx^*,Tx}(t) \geq F_{Tx^*,Tx}(\phi(t)) \tag{3.25}$$

and so

$$F_{Sx^*,Sx}(\phi(t)) * F_{Sx^*,Sx}(t) \geq F_{Sx^*,Sx}(\phi(t)) \tag{3.26}$$

for all $t > 0$, which is impossible if $Sx^* \neq Sx$. Thus we have $Sx = Sx^*$. Moreover, it can be concluded that

$$\begin{aligned} F_{x,x^*}(t) &\geq F_{x,Sx}(t) * F_{Sx,Sx^*}(t) * F_{Sx^*,x^*}(t) \\ &\geq F_{A,B}(t) * F_{A,B}(t) \end{aligned}$$

for all $t > 0$. This completes the proof. □

Corollary 3.2. *Let A be a nonempty closed subset of a complete NA-PM-space $(X, F, *)$ in which the t–norm $*$ is positive and $* <$ min such that A is compact. Let $S : A \to A$, $T : A \to A$ be the self mappings satisfying the following conditions:*

(a) *for each x and y are elements in A and $t > 0$,*

$$F_{Sx,Sy}(\phi(t)) * F_{Tx,Ty}(t) \geq F_{Tx,Ty}(\phi(t)),$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that, for any $t > 0$, $0 < \varphi(t) < t$;

(b) *T is continuous;*

(c) *S and T commutative;*

(e) *$S(A) \subseteq A$ and $S(A) \subseteq T(A)$.*

Then S and T have common fixed point.

4. An example

Now, we give an example to illustrate Theorem 3.1.

Example 4.1. Consider the complete metric space $\mathbb{R}^2$ with Euclidean metric. Define

$$F_{(x_1,x_2),(y_1,y_2)}(t) = \frac{t}{t + |x_1 - y_1| + |x_2 - y_2|}$$

for all $t > 0$ and

$$F_{(x_1,x_2),(y_1,y_2)}(t) = 0$$

for all $t \leq 0$. It is easy to show that $(X, F, \cdot)$ is a NA-PM-space. Let

$$A = \{(x, 1) : 0 \leq x \leq 1\}, \quad B = \{(x, -1) : 0 \leq x \leq 1\}.$$

Define two mappings $S : A \to B$, $T : A \to B$ as follows:

$$S(x, 1) = \big(0, -1\big), \quad T(x, 1) = \big(x, -1\big),$$

respectively. It is easy to see that $F_{A,B}(t) = \frac{t}{t+2}$, $A_0 = A$ and $B_0 = B$. Further, S and T are continuous and A is approximatively compact with respect to B.

First, we show that S and T are satisfy the condition (a) of Theorem 3.1 with $\varphi : [0, \infty) \to [0, \infty)$ defined by $\varphi(t) = \dfrac{t}{2}$ for all $t \in [0, \infty)$. Let $(x, 1), (y, 1) \in A$. Without loss of generality, we can take $x > y$. Then we have

$$\begin{aligned} F_{S(x,1),S(y,1)}(\phi(t)) * F_{T(x,1),T(y,1)}(t) &= 1 \cdot \tfrac{t}{t+|x-y|} \\ &\geq \tfrac{\frac{t}{2}}{\frac{t}{2}+|x-y|} \\ &= F_{T(x,1),T(y,1)}(\phi(t)) \end{aligned}$$

for all $t > 0$.

Next, we show that S and T are proximally commuting. Let $(u, 1), (v, 1), (x, 1) \in A$ be such that

$$F_{(u,1),S(x,1)}(t) = F_{A,B}(t) = \frac{t}{t+2}, \quad F_{(v,1),T(x,1)}(t) = F_{A,B}(t) = \frac{t}{t+2}$$

for all $t > 0$. It follows that $u = 0$ and $v = x$ and hence

$$S(v, 1) = \big(0, -1\big) = (u, -1) = T(u, 1).$$

Finally, we show that S and T are proximally swapped. If it is true that

$$F_{(u,1),(y,-1)}(t) = F_{(v,1),(y,-1)}(t) = F_{A,B}(t) = \frac{t}{t+2}, \quad S(u,1) = T(v,1)$$

for some $(u,1),(v,1) \in A$ and $(y,-1) \in B$, then we get $u = v = 0$ and so

$$S(v,1) = T(u,1).$$

Therefore, all the hypothesis of Theorem 3.1 are satisfied. Furthermore, $(0,1) \in A$ is a common best proximity point of the mappings S and T since

$$F_{(0,1),S(0,1)}(t) = F_{(0,1),(0,-1)}(t) = F_{(0,1),T((0,1)}(t) = F_{A,B}(t)$$

for all $t > 0$.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors carried out the proof. All authors conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

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The Value Distribution of Some Difference Polynomials of Meromorphic Functions *

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Abstract

The purpose of this paper is to investigate the value distribution of some difference polynomials $G_1(z) = \prod_{j=1}^{m} f(z + c_j) - af(z)^n$, $G_2(z) = f(z)^n \prod_{j=1}^{m} f(z + c_j)$ and $G_3(z) = f(z)^n \prod_{j=1}^{m}(f(z + c_j) - f(z))$, where $f(z)$ is a meromorphic function and $a \in \mathbb{C} \setminus \{0\}$ and $c_j, j = 1, 2, \ldots, m$ are complex constants.

Key words: meromorphic function; difference polynomial; zeros.
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1 Introduction and main results

This purpose of this paper is to study some properties of value distribution of some complex difference polynomials of meromorphic functions. The fundamental theorems and the standard notations of the Nevanlinna value distribution theory of meromorphic functions will be used(see [7, 15]). In addition, for meromorphic function f, we will use $S(r, f)$ to denote any quantity satisfying $S(r, f) = o(T(r, f))$ for all r outside a possible exceptional set E of finite logarithmic measure $\lim_{r\to\infty} \int_{[1,r)\cap E} \frac{dt}{t} < \infty$. We use $\rho(f), \lambda(f)$ and $\lambda(\frac{1}{f})$ to denote the order, the exponent of convergence of zeros and the exponent of convergence of poles of $f(z)$ respectively.

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Many people were interested in the value distribution of different expressions of meromorphic function and obtained lots of valuable theorems. In 1959, Hayman [8] studied value distribution of meromorphic function and its derivatives, and obtained the following famous theorems.

Theorem 1.1 *[8]. Let $f(z)$ be a transcendental entire function. Then*

(i) for $n \geq 3$ and $a \neq 0, \Psi(z) = f'(z) - af(z)^n$ assumes all finite values infinitely often.

(ii) for $n \geq 2$, $\Phi(z) = f'(z)f(z)^n$ assumes all finite values except possibly zero infinitely often.

However, Mues [12] proved that the conclusion of Theorem 1.1 is not true for $n = 3$ by providing a counter example and proved that $f'(z) - af(z)^4$ has infinitely many zeros.

Recently, the topic of difference product in the complex plane $\mathbb{C}$ has attracted many researchers, a number of papers have focused on value distribution of differences and differences operator analogues of Nevanlinna theory (including [2, 4, 5, 6, 11]).

In 2007, Laine and Yang [9] proved the following result, which is regarded as a difference counterpart of Theorem 1.1.

Theorem 1.2 *[9]. Let $f(z)$ be a transcendental entire function of finite order, and c be a non-zero complex constant. Then for $n \geq 2$, $\Phi_1(z) = f(z+c)f(z)^n$ assumes every non-zero value $a \in \mathbb{C}$ infinitely often.*

It is well known that $\Delta f(z) = f(z+c) - f(z)$, where $c \in \mathbb{C} \setminus \{0\}$ is a constant satisfying $f(z+c) - f(z) \not\equiv 0$, which can be considered as the difference counterpart of $f'(z)$. Similarly, $\Delta f(z) - af(z)^n$ can be considered as the difference counterpart $f'(z) - af(z)^n$, where $a \in \mathbb{C} \setminus \{0\}$.

In 2011, Chen [1] considered the difference counterpart of Theorem 1.1 and obtained the following theorems.

Theorem 1.3 *[1]. Let $f(z)$ be a transcendental entire function of finite order, and let $a, c \in \mathbb{C} \setminus \{0\}$ be constants, with c such that $f(z+c) \not\equiv f(z)$. Set $\Psi_n(z) = \Delta f(z) - af(z)^n$ and $n \geq 3$ is an integer. Then $\Psi_n(z)$ assumes all finite values infinitely often, and for every $b \in \mathbb{C}$ one has $\lambda(\Psi_n(z) - b) = \rho(f)$.*

Theorem 1.4 *[1]. Let $f(z)$ be a transcendental entire function of finite order with a Borel exceptional value 0, and let $a, c \in \mathbb{C} \setminus \{0\}$ be constants, with c such that $f(z+c) \not\equiv f(z)$. Then $\Psi_2(z)$ assumes all finite values infinitely often, and for every $b \in \mathbb{C}$ one has $\lambda(\Psi_2(z) - b) = \rho(f)$.*

Theorem 1.5 *[1]. Let $f(z)$ be a transcendental entire function of finite order with a finite nonzero Borel exceptional value d, and let $a, c \in \mathbb{C} \setminus \{0\}$ be constants, with c such that $f(z+c) \not\equiv f(z)$. Then for every $b \in \mathbb{C}$ with $b \neq -ad^2$, $\Psi_2(z)$ assumes the value b infinitely often, and $\lambda(\Psi_2(z) - b) = \rho(f)$.*

In 2013, Zheng and Chen [16] further investigated the value distribution of some difference polynomial of entire function and obtained the following theorem.

Theorem 1.6 *[16]. Let $f(z)$ be a transcendental entire function of finite order with a finite nonzero Borel exceptional value d, and let $a \in \mathbb{C} \setminus \{0\}$, $c_1, c_2, \ldots, c_m$ be complex constants satisfying that at least one of them is non-zero. Then for $1 \leq m < n$ and every $b(\neq d^m - ad^n) \in \mathbb{C}$, $G_1(z) = \prod_{j=1}^{m} f(z+c_j) - af(z)^n$ assumes the value b infinitely often and $\lambda(G_1(z) - b) = \rho(f)$.*

Thus, it is natural to ask: *On What condition can Theorem 1.6 still hold when $f(z)$ is a transcendental meromorphic function?*

The main purpose of this article is to study the above questions and obtain the following theorem.

Theorem 1.7 *Let $f(z)$ be a transcendental meromorphic function of finite order with two Borel exceptional values d, ∞, and let $a \in \mathbb{C} \setminus \{0\}$, $c_1, c_2, \ldots, c_m$ be complex constants satisfying that at least one of them is non-zero. Then for $1 \leq m < n$ and every $b(\neq d^m - ad^n) \in \mathbb{C}$, $G_1(z) = \prod_{j=1}^{m} f(z+c_j) - af(z)^n$ assumes the value b infinitely often and $\lambda(G_1(z) - b) = \rho(f)$.*

In addition, we further study the value distribution of some difference polynomials of meromorphic function of more general form

$$G_2(z) = f(z)^n \prod_{j=1}^{m} f(z+c_j), \quad G_3(z) = f(z)^n \prod_{j=1}^{m} [f(z+c_j) - f(z)]$$

and obtain the following results:

Theorem 1.8 *Let $f(z)$ be a transcendental meromorphic function of finite order with two Borel exceptional values d, ∞, and let $c_1, c_2, \ldots, c_m$ be nonzero complex constants. Then for $n \geq 1$, $G_2(z)$ assumes every value $b(\neq d^{n+m}) \in \mathbb{C}$ infinitely often and $\lambda(G_2(z) - b) = \rho(f)$.*

Corollary 1.1 *Let $f(z)$ be a transcendental meromorphic function of finite order with two Borel exceptional values $0, \infty$, and let $c_1, c_2, \ldots, c_m$ be nonzero complex constants. Then for $n \geq 1$, $G_2(z)$ assumes every nonzero value $b \in \mathbb{C}$ infinitely often and $\lambda(G_2(z) - b) = \rho(f)$.*

Example 1.1 *Let $f(z) = \frac{e^z - 2}{e^z + 2}$, it is easy to see that $0, \infty$ are not Borel exceptional values. Let $n = 1, m = 2, c_1 = \pi i, c_2 = -\pi i$ and $b = 1$, then we have $G_2(z) = f(z)f(z + c_1)f(z + c_2) - 1 = \frac{4}{e^z - 2}$ has no zeros. Let $n = 2, m = 2, c_1 = \pi i, c_2 = -\pi i$ and $b = 1$, then we have $G_2(z) = f(z)^3 f(z+c_1)f(z+c_2) - 1 = \frac{-4}{e^z+1}$ has no zeros. Hence, this shows that the condition in Corollary 1.1 is sharp in a sense.*

Theorem 1.9 *Let $f(z)$ be a transcendental meromorphic function of finite order with two Borel exceptional values d, ∞, and let $c_1, c_2, \ldots, c_m$ be nonzero complex constants and $f(z + c_j) \not\equiv f(z)$ for $j = 1, 2, \ldots, m$. Then for $n, m \geq 1$ are two integers, $G_3(z)$ assumes every value $b \in \mathbb{C} \setminus \{0\}$ infinitely often and $\lambda(G_3(z) - b) = \rho(f)$.*

Remark 1.1 *When $b = 0$, the conclusion may not hold. For example, let $f(z) = e^z$, $G_3(z) = f(z)^n[f(z + \pi i) - f(z)]$. Then $G_3(z) = -2e^{(n+1)z}$ has no zeros.*

Corollary 1.2 *Let $f(z)$ be a transcendental entire function of finite order with a Borel exceptional values d, and let $c_1, c_2, \ldots, c_m$ be nonzero complex constants and $f(z+c_j) \not\equiv f(z)$ for $j = 1, 2, \ldots, m$. Then for $n, m \geq 1$ are two integers, $G_3(z)$ assumes every value $b \in \mathbb{C}$ infinitely often and $\lambda(G_3(z) - b) = \rho(f)$.*

Remark 1.2 *It is easily to see that Theorem 1.9 is an improvement of the result in [10, Theorem 1.4], where they consider the case of $m = 1$ and the value b can replaced by a small function $\alpha(z)$. In fact, our results also can allow the value b to be a polynomial, even be a meromorphic function $\alpha(z) \not\equiv 0$ satisfying $\rho(\alpha) < \rho(f)$.*

Example 1.2 *Let $f(z) = e^z + 2z, c = 2\pi i, \alpha(z) = 4cz, n = 1$ and $m = 1$. Then we know that $f(z)$ has no Borel exceptional value, and we have $G_3(z) = f(z)\Delta f(z) - 4cz = 2ce^z$, which has no zeros. Hence, the condition on $f(z)$ having a Borel exceptional value is necessary in Corollary 1.2.*

The following result of this paper is the value distribution of differential and difference polynomial of entire function.

Theorem 1.10 *Let $f(z)$ be a transcendental entire function of finite order, and $a, c_1, \ldots, c_m$ be nonzero complex constants. Then for any positive integers $n \geq 2m + 3$, $\Psi(z) = f^{(k)}(z) \prod_{j=1}^{m} f(z + c_j) - af(z)^n$ assumes all finite values $b \in \mathbb{C}$ infinitely often.*

Regarding Theorem 1.2, we pose the following question.

Question 1.1 *What can be said if the condition $n \geq 2m+3$ in Theorem 1.10 is replaced with $1 \leq n \leq 2m + 2$?*

2 Some Lemmas

The following lemma is important in the fields of factorization and uniqueness theory of meromorphic functions which is given by Gross [3]. In 2010, Xu and Yi [13] made a small changed form as follows.

Lemma 2.1 *[13]. Suppose that $f_j(z)(j = 1, 2, \ldots, n+1)$ are meromorphic functions and $g_j(j = 1, 2 \ldots, n)$ are entire functions satisfying the following conditions.*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv f_{n+1}$.

(ii) If $1 \leq j \leq n+1, 1 \leq k \leq n$, the order of f_j is less than the order of $e^{g_k(z)}$. If $n \geq 2$, $1 \leq j \leq n+1, 1 \leq h < k \leq n$, and the order of $f_j(z)$ is less than the order of $e^{g_h - g_k}$.

Then $f_j(z) \equiv 0(j = 1, 2, \ldots, n+1)$.

Lemma 2.2 *[15]. Let f be a nonconstant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \cdots + a_n f^n$, where $a_0, a_1, a_2, \cdots, a_n$ are constants and $a_n \neq 0$. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 2.3 *[2, Theorem 2.1]. Let $f(z)$ be a meromorphic function of finite order ρ and let c be a fixed nonzero complex number, then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = O(r^{\rho-1+\varepsilon}) = S(r,f).$$

Lemma 2.4 *[2, Corollary 2.5]. Let $f(z)$ be a meromorphic function with order $\rho = \rho(f), \rho < +\infty$, and let η be a fixed nonzero complex number, then for each $\varepsilon > 0$, we have*

$$T(r, f(z+\eta)) = T(r,f) + O(r^{\rho-1+\varepsilon}) + O(\log r).$$

Lemma 2.5 *[2, Theorem 2.2]. Let f be a meromorphic function with exponent of convergence of poles $\lambda(\frac{1}{f}) = \lambda < +\infty$, $c \neq 0$ be fixed, then for each $\varepsilon > 0$,*

$$N(r, f(z+\eta)) = N(r,f) + O(r^{\lambda-1+\varepsilon}) + O(\log r).$$

Lemma 2.6 *[14]. If $f(z)$ is a transcendental meromorphic function with exponent of convergence of poles $\lambda(\frac{1}{f}) = \lambda < +\infty$, $c \neq 0$. Then, for each $\varepsilon > 0$, one has*

$$\lambda\left(\frac{1}{f(z+c)}\right) = \lambda\left(\frac{1}{f(z)}\right) = \lambda, \quad \lambda\left(\frac{1}{\Delta f}\right) \leq \lambda.$$

Lemma 2.7 *Let $f(z)$ be a transcendental meromorphic function with exponent of convergence of poles $\lambda(\frac{1}{f}) = \lambda < \rho(f) = \rho < +\infty$, and let $c_1, c_2, \ldots, c_m$ be nonzero complex constants, and $n, m \geq 1$ be integers. Then $\rho(G_2) = \rho(f)$.*

Proof: We firstly prove that $\rho(G_2) \leq \rho(f)$. We can rewrite $G_2(z)$ as the form

$$G_2(z) = f(z)^{n+m} \prod_{j=1}^{m} \frac{f(z+c_j)}{f(z)}. \tag{1}$$

For each $\varepsilon (0 < \varepsilon < \rho - \lambda)$, it follows by Lemma 2.3 and (1) that

$$\begin{aligned} m(r, G_2) &\leq (n+m)m(r,f) + \sum_{j=1}^{m} m(r, \frac{f(z+c_j)}{f(z)}) + O(1) \\ &= (n+m)m(r,f) + O(r^{\rho-1+\varepsilon}). \end{aligned} \tag{2}$$

By Lemma 2.5, we have

$$\begin{aligned} N(r, G_2) &\leq nN(r,f) + \sum_{j=1}^{m} N(r, f(z+c_j)) \\ &\leq (n+m)N(r,f) + O(r^{\lambda-1+\varepsilon}) + O(\log r). \end{aligned} \tag{3}$$

Since $\lambda < \rho$, it follows from (2) and (3) that

$$T(r, G_2) \leq (n+m)T(r,f) + O(r^{\rho-1+\varepsilon}) + O(\log r). \tag{4}$$

So, we can get that $\rho(G_2) \leq \rho(f)$ easily.

Next, we prove that $\rho(G_2) \geq \rho(f)$. From Lemma 2.3 and (1), we have

$$\begin{aligned}(n+m)m(r,f) = m(r,f^{n+m}) &\leq m(r,G_2) + \sum_{j=1}^{m} m(r, \frac{f(z)}{f(z+c_j)}) + O(1) \\ &= m(r,G_2) + O(r^{\rho-1+\varepsilon}).\end{aligned} \tag{5}$$

Since $\lambda(1/f) = \lambda < \rho$, for any given $\varepsilon > 0$ there exists $r_0 > 0$ such that for all $r > r_0$ we have

$$N(r,f) \leq r^{\lambda+\varepsilon}. \tag{6}$$

Thus, it follows from (5) and (6) that

$$T(r,f) \leq \frac{1}{n+m} m(r,G_2) + O(r^{\rho-1+\varepsilon}) + O(r^{\lambda+\varepsilon}), \quad r > r_0. \tag{7}$$

Since $\lambda < \rho$ and $0 < \varepsilon < \rho - \lambda$, it follows from (7) that $\rho(G_2) \geq \rho(f)$.

Hence, the proof of Lemma 2.7 is proved. □

By using the same argument as in Lemma 2.7, we can prove the following lemma easily.

Lemma 2.8 *Let $f(z)$ be a transcendental meromorphic function with exponent of convergence of poles* $\max\{\lambda(f), \lambda(\frac{1}{f})\} = \lambda < \rho(f) = \rho < +\infty$, *and let $c_1, c_2, \ldots, c_m$ be nonzero complex constants such that $f(z+c_j) \not\equiv f(z) (j = 1, 2, \ldots, m)$, and $n, m \geq 1$ be integers. Then $\rho(G_3) = \rho(f)$.*

Lemma 2.9 *[15, page 37]. Let $f(z)$ be a nonconstant meromorphic function in the complex plane and l be a positive integer. Then*

$$T(r, f^{(l)}(z)) \leq (l+1)T(r,f) + S(r,f), \quad N(r, f^{(l)}(z)) = N(r,f) + l\overline{N}(r,f).$$

3 Proofs of Theorems 1.7, 1.8 and 1.9

3.1 The Proof of Theorem 1.7

We first prove $\rho(G_1) = \rho(f)$. By Lemma 2.2 and Lemma 2.4, we have $\rho(G_1) \leq \rho(f)$. On the other hand, it follows from Lemma 2.4 that

$$\begin{aligned} nT(r,f) = T(r,af^n) + O(1) &= T\left(r, \prod_{j=1}^{m} f(z+c_j) - G_1(z)\right) + O(1) \\ &\leq \sum_{j=1}^{m} T(r, f(z+c_j)) + T(r, G_1(z)) + O(1) \\ &= mT(r,f) + T(r,G_1(z)) + O(r^{\rho-1+\varepsilon}) + O(\log r), \end{aligned}$$

that is,

$$(n-m)T(r,f) \leq T(r,G_1(z)) + O(r^{\rho-1+\varepsilon}) + O(\log r). \tag{8}$$

Since $1 \leq m < n$, it follows from (8) that $\rho(f) \leq \rho(G_1)$. Hence, we can prove that $\rho(G_1) = \rho(f)$.

Since $f(z)$ has two Borel exceptional values d, ∞, then $f(z), f(z+c_j)$ can be written as the form

$$f(z) = d + \frac{g(z)}{p(z)} \exp\{\alpha z^k\}, \quad f(z+c_j) = d + \frac{g(z+c_j)}{p(z+c_j)} h_j(z) \exp\{\alpha z^k\}, \tag{9}$$

where $\alpha \neq 0$ is a constant, $k(\geq 1)$ is an integer satisfying $\rho(f) = k$, and $g(z), h_j(z) = e^{\alpha k c_j z^{k-1} + \cdots + \alpha c_j^k}$ are entire functions such that $g(z)h_j(z) \not\equiv 0, \rho(g) < k, \rho(h_j) \leq k-1$, $j = 1, 2 \ldots, m$, and $p(z)$ is the canonical product formed with the poles of $f(z)$ satisfying $\rho(p) = \lambda(p) = \lambda(\frac{1}{f}) < \rho(f)$. Set $H(z) = \frac{g(z)}{p(z)} \not\equiv 0$, then we can see that $\rho(H) < \rho(f)$.

Now we prove that $\lambda(G_1 - b) = \rho(f)$. Suppose that $\lambda(G_1 - b) < \rho(f)$. Since $\rho(G_1) = \rho(f) = \rho(G_1 - b)$, then $\lambda(G_1 - b) < \rho(G_1 - b) = \rho(f) = k$ and $G_1(z) - b$ can be rewritten as the form

$$G_1(z) - b = \frac{g_1^*(z)}{p_1^*(z)} \exp\{\beta z^k\} = H_1^*(z) \exp\{\beta z^k\}, \tag{10}$$

where $\beta(\neq 0)$ is a constant, $g_1^*(z)(\not\equiv 0)$ is an entire function satisfying $\rho(g_1^*) < k$. Thus, by Lemma 2.6, we have $\rho(p_1^*) = \lambda(p_1^*) \leq \max\{\lambda(\frac{1}{f(z)}), \lambda(\frac{1}{f(z+c_j)}), j = 1, 2, \ldots, m\} = \lambda(\frac{1}{f}) < \rho(f) = k$. So, we have $\rho(H_1^*) < \rho(f) = k$.

Thus, from (9), (10) and the definition of $G_1(z)$, we have

$$\begin{aligned}
&\prod_{j=1}^{m} H(z+c_j)h_j(z)e^{m\alpha z^k} + \cdots + d^{m-2}\left(\sum_{1\leq i<j\leq m} H(z+c_i)H(z+c_j)h_i(z)h_j(z)\right)e^{2\alpha z^k} \\
&\quad + d^{m-1}\left(\sum_{j=1}^{m} H(z+c_j)h_j(z)\right)e^{\alpha z^k} + d^m \\
&\quad - a\left(d^n + nd^{n-1}H(z)e^{\alpha z^k} + \cdots + H(z)^n e^{n\alpha z^k}\right) = b + H_1^*(z)e^{\beta z^k}.
\end{aligned}$$

Since $1 \leq m < n, aH(z)H_1^*(z) \not\equiv 0$, by comparing growths of both sides of the above equality, we have $\beta = n\alpha$. Thus, we can rewrite the above equality as the form

$$f_n(z)e^{n\alpha z^k} + f_{n-1}(z)e^{(n-1)\alpha z^k} + \cdots + f_1(z)e^{\alpha z^k} = f_{n+1}(z), \tag{11}$$

where $f_{n+1}(z) = b - d^m + ad^n$, and $f_1, \ldots, f_n$ are algebraic expressions in the terms $a, d, n, m, H(z), H_1^*(z), H(z+c_j), h_j(z), j = 1, 2, \ldots, m$, such as addition, subtraction and multiplication. Since $\rho(H) < \rho(f) = k, \rho(h_j) \leq k-1$ and $\rho(H_1^*) < \rho(f) = k$, then we have $\rho(f_t) < k = \rho(e^{t\alpha z^k})$ for $t = 1, 2, \ldots, n$. Thus, by Lemma 2.1 and (11), we have $f_t(z) \equiv 0$ for $t = 1, 2, \ldots, n+1$, that is, $b - d^m + ad^n \equiv 0$, which is a contradiction with the assumption $b \neq d^m - ad^n$. Hence, we have that $\lambda(G_1 - b) = \rho(f)$.

This completes the proof of Theorem 1.7.

3.2 The proof of Theorem 1.8

Similar to the proof of Theorem 1.7, we can obtain (9).

Now we prove that $\lambda(G_2 - b) = \rho(f)$. Suppose that $\lambda(G_2 - b) < \rho(f)$.

By Lemma 2.7, we have $\rho(G_2) = \rho(f) = \rho(G_2 - b)$, then $\lambda(G_2 - b) < \rho(G_2 - b) = \rho(f) = k$ and $G_2(z) - b$ can be rewritten as the form

$$G_2(z) - b = \frac{g_2^*(z)}{p_2^*(z)} \exp\{\beta z^k\} = H_2^*(z) \exp\{\beta z^k\}, \tag{12}$$

where $\beta(\neq 0)$ is a constant, $g_2^*(z)(\not\equiv 0)$ is an entire function satisfying $\rho(g_2^*) < k$. Thus, by Lemma 2.6, we have $\rho(p_2^*) = \lambda(p_2^*) \leq \max\{\lambda(\frac{1}{f(z)}), \lambda(\frac{1}{f(z+c_j)}), j = 1, 2, \ldots, m\} = \lambda(\frac{1}{f}) < \rho(f) = k$. So, we have $\rho(H_2^*) < \rho(f) = k$.

From (9), (12) and the definition of $G_2(z)$, we have

$$(d + H(z)e^{\alpha z^k})^n \prod_{j=1}^{m} \left[d + H(z + c_j)h_j(z)e^{\alpha z^k}\right] = b + H_2^*(z) \exp\{\beta z^k\}.$$

By simple calculation, we can rewrite the above equation as the form

$$f_{n+m}(z)e^{(n+m)\alpha z^k} + \cdots + f_1(z)e^{\alpha z^k} + d^{n+m} - b = H_2^*(z) \exp\{\beta z^k\}, \tag{13}$$

where $f_{n+m}(z) = H(z)^n \prod_{j=1}^m H(z + c_j)h_j(z) \not\equiv 0$ and $f_1, f_2, \ldots, f_{n+m-1}$ are algebraic expressions in the terms $d, n, m, H(z), H(z + c_j), h_j(z), j = 1, 2, \ldots, m$, such as addition, subtraction and multiplication. Since $\rho(H) < \rho(f) = k, \rho(h_j) \leq k - 1$ and $\rho(H_2^*) < \rho(f) = k$, then we have $\rho(f_t) < k = \rho(e^{t\alpha z^k})$ for $t = 1, 2, \ldots, n + m$. By comparing growths of both sides of the above equality, we have $\beta = (n+m)\alpha$. Thus, it follows from (13) that

$$[f_{n+m}(z) - H_2^*(z)]e^{(n+m)\alpha z^k} + \cdots + f_1(z)e^{\alpha z^k} = b - d^{n+m}, \tag{14}$$

By Lemma 2.1, we have $b = d^{n+m}$, a contradiction. Hence, we have $\lambda(G_2 - b) = \rho(f)$.

This completes the proof of Theorem 1.8.

3.3 The proof of Theorem 1.9

Similar to the proof of Theorem 1.7, we can obtain (9).

Now we prove that $\lambda(G_3 - b) = \rho(f)$. Suppose that $\lambda(G_3 - b) < \rho(f)$.

By Lemma 2.8, we have $\rho(G_3) = \rho(f) = \rho(G_3 - b)$, then $\lambda(G_3 - b) < \rho(G_3 - b) = \rho(f) = k$ and $G_3(z) - b$ can be rewritten as the form

$$G_3(z) - b = \frac{g_3^*(z)}{p_3^*(z)} \exp\{\beta z^k\} = H_3^*(z) \exp\{\beta z^k\}, \tag{15}$$

where $\beta(\neq 0)$ is a constant, $g_3^*(z)(\not\equiv 0)$ is an entire function satisfying $\rho(g_3^*) < k$. Thus, by Lemma 2.6, we have $\rho(p_3^*) = \lambda(p_3^*) \leq \max\{\lambda(\frac{1}{f(z)}), \lambda(\frac{1}{\Delta f(z)})\} = \lambda(\frac{1}{f}) < \rho(f) = k$. So, we have $\rho(H_3^*) < \rho(f) = k$.

From (9), (13) and the definition of $G_3(z)$, we have

$$\left(d + H(z)e^{\alpha z^k}\right)^n \prod_{j=1}^{m} \left\{[H(z+c_j)h_j(z) - H(z)]e^{\alpha z^k}\right\} = b + H_3^*(z)\exp\{\beta z^k\}.$$

By simple calculation, we can rewrite the above equation as the form

$$f_{n+1}(z)e^{(n+m)\alpha z^k} + \cdots + f_1(z)e^{\alpha z^k} - b = H_3^*(z)\exp\{\beta z^k\}, \tag{16}$$

where $f_{n+1}(z) = H(z)^n \prod_{j=1}^{m}[H(z+c_j)h_j(z) - H(z)] \not\equiv 0$ and

$$f_{i+1}(z) = C_n^i d^{n-i} H(z)^i \prod_{j=1}^{m}[H(z+c_j)h_j(z) - H(z)], \quad i = 0, 1, \ldots, n.$$

Since $\rho(H) < \rho(f) = k, \rho(h_j) \leq k-1$ and $\rho(H_3^*) < \rho(f) = k$, then we have $\rho(f_i) < k = \rho(e^{t\alpha z^k})$ for $i = 1, 2, \ldots, n+1$. By comparing growths of both sides of (16), we have $\beta = (n+m)\alpha$. Thus, it follows from (13) that

$$[f_{n+1}(z) - H_3^*(z)]e^{(n+m)\alpha z^k} + \cdots + f_1(z)e^{\alpha z^k} = b, \tag{17}$$

By Lemma 2.1, we have $f_1(z) \equiv 0$, that is,

$$d^n \prod_{j=1}^{m}[H(z+c_j)h_j(z) - H(z)] \equiv 0,$$

which is a contradiction with the assumptions $f(z+c_j) \not\equiv f(z)$ for $j = 1, 2, \ldots, m$. Hence, we have $\lambda(G_3 - b) = \rho(f)$.

This completes the proof of Theorem 1.9.

4 The proof of Theorem 1.10

We will take two case as follows into consideration by using the idea of Theorem 1 in [16].

Case 1. Suppose that $0 < \sigma(f) < \infty$. We assume that there exists $b \in \mathbb{C}$ such that $\Psi(z) - b$ has finitely many zeros only. Set

$$F(z) = \frac{f^{(k)}(z) \prod_{j=1}^{m} f(z+c_j) - b}{af(z)^n}. \tag{18}$$

It follows from (18) that $T(r, F) \leq (n+m+1)T(r, f) + S(r, f)$ and $F(z)$ has only finite 1-points, *i.e.*,

$$\overline{N}(r, \frac{1}{F-1}) = O(\log r). \tag{19}$$

Since $f(z)$ is entire, from (18), we have that the poles of $F(z)$ occur only at zeros of $f(z)$, and those poles which are not zeros of $f^{(k)}(z) \prod_{j=1}^{m} f(z+c_j) - b$ having multiplicities $\geq n$

at the same time. Moreover, the zeros of $F(z)$ can only occur at zeros of $f^{(k)}(z)\prod_{j=1}^{m} f(z+c_j)-b$ which are not poles of $F(z)$. Thus, it follows by Lemma 2.3 and Lemma 2.9 that

$$\begin{aligned}\overline{N}(r,F)+\overline{N}(r,\frac{1}{F}) &\le \frac{1}{n}N(r,F)+\overline{N}(r,\frac{1}{f^{(k)}(z)\prod_{j=1}^{m}f(z+c_j)-b}) \qquad (20)\\ &\le \frac{1}{n}T(r,F)+T(r,f^{(k)}(z)\prod_{j=1}^{m}f(z+c_j))+O(1)\\ &\le \frac{1}{n}T(r,F)+(m+1)T(r,f)+S(r,f).\end{aligned}$$

By the second fundamental theorem, it follows from (19) and (20) that

$$\begin{aligned}T(r,F) &\le \overline{N}(r,F)+\overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{F-1})+S(r,F)\\ &\le \frac{1}{n}T(r,F)+(m+1)T(r,f)+S(r,f),\end{aligned}$$

i.e.,

$$(1-\frac{1}{n})T(r,F)\le (m+1)T(r,f)+S(r,f). \qquad (21)$$

On the other hand, we have

$$\begin{aligned}nT(r,f)=T(r,f^n) &= T(r,\frac{f^{(k)}(z)\prod_{j=1}^{m}f(z+c_j)-b}{aF(z)})\\ &\le T(r,f^{(k)}(z)\prod_{j=1}^{m}f(z+c_j))+T(r,F)+O(1)\\ &\le (m+1)T(r,f)+T(r,F)+S(r,f),\end{aligned}$$

i.e.,

$$(n-m-1)T(r,f)\le T(r,F)+S(r,f). \qquad (22)$$

Thus, it follows from (21) and (22) that

$$(1-\frac{1}{n}-\frac{m+1}{n-m-1})T(r,F)\le S(r,f). \qquad (23)$$

Since $f(z)$ is a transcendental entire function and $n\ge 2m+3$, from (23) we can deduce a contradiction. Hence, for any $b\in\mathbb{C}$, $\Psi(z)-b$ has infinitely many zeros.

Case 2. Suppose that $\sigma(f)=0$, then $\Psi(z)$ is also of zero order. We assume that $\Psi(z)$ is a polynomial, then

$$T(r,\Psi(z))=O(\log r). \qquad (24)$$

Thus, it follows from (24) and Lemma 2.4 that

$$T(r,f^{(k)}(z)\prod_{j=0}^{m}f(z+c_j)-b)=T(r,\Psi(z)+af(z)^n)=nT(r,f)+S(r,f). \qquad (25)$$

On the other hand, we have

$$\begin{aligned} T(r, f^{(k)}(z)\prod_{j=1}^{m} f(z+c_j)) &= m(r, f^{(k)}(z)\prod_{j=1}^{m} f(z+c_j)) \qquad (26)\\ &= m\left(r, f^{m+1}(z)\frac{f^{(k)}(z)\prod_{j=1}^{m} f(z+c_j)}{f^{m+1}(z)}\right)\\ &\leq (m+1)m(r,f) + m\left(r, \frac{f^{(k)}(z)}{f(z)}\right) + \sum_{j=1}^{m} m\left(r, \frac{f(z+c_j)}{f(z)}\right)\\ &\quad + O(1) + S(r,f)\\ &= (m+1)T(r,f) + S(r,f). \end{aligned}$$

From (25), (26) and $n \geq m+2$, we can deduce a contradiction with the assumption that f is transcendental.

Thus, it is easy to see that for any $b \in \mathbb{C}$, $\Psi(z) - b$ is a transcendental entire function with zero order and has infinitely many zeros.

Therefore, this completes the proof of Theorem 1.10.

Competing interests

The authors declare that they have no competing interests.

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On properties of decomposable measures and pseudo-integrals

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Abstract

In this paper, we mainly discuss two classes of σ-$\oplus$-decomposable measures and the corresponding pseudo-integrals: one is based on the generated pseudo-addition (g-case) and the other is based on the idempotent pseudo-operation (sup and inf). In particular, we obtained the correlation between the measure zero sets with respect to a σ-$\oplus$-decomposable measure and the corresponding pseudo-integrals on them. As an application of the main results, we generalized the classical Radon-Nikodym theorem to the decomposable measure theory based on pseudo-integrals.

Keywords: Pseudo-addition; Pseudo-multiplication; Pseudo-integral; Radon-Nikodym theorem

1 Introduction

Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers a semiring is taken on a real interval $[a, b] \subset [-\infty, +\infty]$ endowed with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$ (see [6, 22, 24, 25, 36]). Based on this structure there were developed the concepts of $\oplus$-measure (pseudo-additive measure), pseudo-integral, pseudo-convolution, pseudo-Laplace transform, etc. The advantage of the pseudo-analysis is that there are covered with one theory, and so with unified methods, problems (usually nonlinear and under uncertainty) from many different fields (system theory, optimization, decision making, control theory, differential equations, difference equations, etc.). Pseudo-analysis uses many mathematical tools from different field as functional equations, variational calculus, measure theory, functional analysis, optimization theory, semiring theory, etc.

Similar ideas were developed independently by Maslov and his collaborators in the framework of idempotent analysis and idempotent mathematics, with important applications [14, 15]. In particular, idempotent analysis is fundamental for the theory of weak solutions to Hamilton-Jacobi equations with non-smooth Hamiltonian, see [14, 15] and also [26, 27] (in the framework of pseudo-analysis). In some cases, this theory enables one to obtain exact solutions in the similar form as for the linear equations. Some further developments relate more general pseudo-operations with applications to nonlinear partial differential equations, see [29]. Recently, these applications have become important also in the field of image processing [27].

The classical measure theory is one of the most important theories in mathematics and based on countable additive measures [11, 40]. Although the additive measures are widely used, they do not allow modeling many phenomena involving interaction between criteria. For this reason, the fuzzy measure proposed by Sugeno as an extension of classical measure in which the additivity is replaced by a weaker condition, i.e., monotonicity [39]. So far, there have been many different fuzzy measures, such as the decomposable measure, the λ-additive measure, the belief measure,the possibility measure and the plausibility measure, etc. Among the fuzzy measure mentioned before, the decomposable measure was independently introduced by Dubois and Prade [8] and Weber [42], because of the close relation with the classical measure theory. Further developments of decomposable measures and related integrals have been extensive studied [6, 23, 31, 32, 33, 35]. Decomposable measures include several well-known fuzzy measures such as the λ-additive measure and probability and possibility measures, and they are a natural setting for relaxing probabilistic assumptions regarding the modeling of uncertainty [9, 34]. Decomposable measures and the corresponding integrals are very useful in decision theory and the theory of nonlinear differential and integral equations [26, 28, 30, 38].

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Based on the above these, the notions of σ-$\oplus$-decomposable measure (pseudo-additive measure) and corresponding integral (pseudo-integral) based on this measure were introduced [22, 24, 25, 27, 36]. Since integrals bases on non-additive measure have wide application, there were obtained generalizations of the classical integral inequalities for integrals with respect to non-additive measures, such as the ineqalities for Choqut and Sugeno integral were given in [2, 13, 18, 19, 41, 44] and the inequalities for the pseudo-integrals with respect to σ-$\oplus$-decomposable measure were considered in [1, 3, 4, 17, 20, 21]. In [37], Sugeno generalize the classical Radon-Nikodym derivatives for functions with respect to fuzzy measures.

In this paper, we will discuss two classes of σ-$\oplus$-decomposable measures and the corresponding pseudo-integrals: one is based on the generated pseudo-addition (g-case, see [16, 22]) and the other is based on the idempotent pseudo-operation (sup and inf, see [23, 39]). In Section 2, we recall the notions of pseudo-addition $\oplus$ and pseudo-multiplication $\odot$ forming a real semiring on the interval $[a, b] \subset [-\infty, +\infty]$. Then we will give the notion of σ-$\oplus$-decomposable measure and corresponding pseudo-integral based on this measure. In Section 3, we will discuss several important properties as monotonicity, continuous from above and continuous from below for σ-$\oplus$-decomposable measures, and will show the relationship between the measure zero sets with respect to a σ-$\oplus$-decomposable measure and the corresponding pseudo-integrals on them. Finally, we will generalize the classical Radon-Nikodym theorem to decomposable measure theory based on pseudo-integrals.

2 Preliminaries

Let $[a, b]$ be a closed subinterval of $[-\infty, \infty]$ (in some cases we will also take semiclosed subintervals). The total order on $[a, b]$ will be denoted by $\preceq$. This can be the usual order of the real line, but it can also be another order.

Definition 2.1 *[17] A binary operation $\oplus : [a, b] \times [a, b] \to [a, b]$ is called a pseudo-addition, if it satisfies the following conditions, for all $x, y, z, w \in [a, b]$:*

(1) $x \oplus y = y \oplus x$; *(commutativity)*
(2) $x \oplus z \preceq y \oplus w$ whenever $x \preceq y$ and $z \preceq w$; *(monotonicity)*
(3) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$; *(associativity)*
(4) $\mathbf{0} \oplus x = x$, where $\mathbf{0}$ is a zero element (usually $\mathbf{0}$ is either a or b). *(boundary condition)*

Let $[a, b]_+ = \{x \mid x \in [a, b], \mathbf{0} \preceq x\}$.

Definition 2.2 *[17] A binary operation $\odot : [a, b] \times [a, b] \to [a, b]$ is called a pseudo-multiplication, if it satisfies the following conditions, for all $x, y, z \in [a, b]$:*

(1) $x \odot y = y \odot x$; *(commutativity)*
(2) $x \odot z \preceq y \odot z$ whenever $x \preceq y$ and $z \in [a, b]_+$; *(positively monotonicity)*
(3) $(x \odot y) \odot z = x \odot (y \odot z)$; *(associativity)*
(4) $\mathbf{1} \odot x = x$, where $\mathbf{1} \in [a, b]$ is an unit element. *(boundary condition)*

We assume also $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e.,

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z).$$

The structure $([a, b], \oplus, \odot)$ is called a real semiring [1, 20]. In this paper we will consider semirings with the following continuous operations:

Case 1: The pseudo-addition is an idempotent operation and the pseudo-multiplication is not.

(a) $x \oplus y = \sup(x, y)$, $\odot$ is an arbitrary non-idempotent pseudo-multiplication on the interval $[a, b]$.

We have $\mathbf{0} = a$ and the idempotent operation sup induces a total order in the following way: $x \preceq y$ if and only if $\sup(x, y) = y$. In order to keep the semiring structure, $\odot$ has to be pseudo-multiplication of the first type, i.e., $a \odot b = a$ and then $a \neq \mathbf{1}$. Special important case is when this pseudo-multiplication can be represented by a strictly increasing and continuous generator surjective function $g : [a, b] \to [0, \infty]$, i.e., $\odot$ is given with

$$x \odot y = g^{-1}(g(x) \cdot g(y)),$$

such that $g(\mathbf{0}) = g(a) = 0$, with the usual convention $0 \cdot \infty = 0$.

(b) $x \oplus y = \inf(x, y)$, $\odot$ is an arbitrary non-idempotent pseudo-multiplication on the interval $[a, b]$.

We have $\mathbf{0} = b$ and the idempotent operation inf induces a total order in the following way: $x \preceq y$ if and only if $\inf(x, y) = y$. In order to keep the semiring structure, $\odot$ has to be pseudo-multiplication of the second type, i.e., $a \odot b = b$ and then $b \neq \mathbf{1}$. Special important case is when this pseudo-multiplication can be represented by a strictly decreasing and continuous generator surjective function $g : [a, b] \to [0, \infty]$, i.e., $\odot$ is given with

$$x \odot y = g^{-1}(g(x) \cdot g(y)),$$

such that $g(\mathbf{0}) = g(b) = 0$.

Case 2: The pseudo-operations are defined by a strictly monotone and continuous generator surjective function $g : [a, b] \to [0, \infty]$, i.e., pseudo-operations are given with

$$x \oplus y = g^{-1}(g(x) + g(y)) \text{ and } x \odot y = g^{-1}(g(x) \cdot g(y)),$$

such that $g(\mathbf{0}) = 0$.

If the zero element for the pseudo-addition is a, we will consider increasing generators. Then $g(a) = 0$ and $g(b) = \infty$. If the zero element for the pseudo-addition is b, we will consider decreasing generators. Then $g(b) = 0$ and $g(a) = \infty$. If the generator g is increasing (decreasing), then the operation $\oplus$ induces the usual order (opposite to the usual order) on the interval $[a, b]$ in the following way: $x \preceq y$ if and only if $g(x) \leq g(y)$.

Case 3: Both operations are idempotent. We have

(a) $x \oplus y = \sup(x, y)$, $x \odot y = \inf(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = a$ and $\mathbf{1} = b$. The idempotent operation sup induces the usual order ($x \preceq y$ if and only if $\sup(x, y) = y$).

(b) $x \oplus y = \inf(x, y)$, $x \odot y = \sup(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = b$ and $\mathbf{1} = a$. The idempotent operation inf induces the usual order ($x \preceq y$ if and only if $\inf(x, y) = y$).

Let X be a non-empty set, we shall denote by $\mathscr{A}$ and $\mathcal{B}$ are σ-algebra and Borel σ-algebra of subsets of a set X, respectively.

Definition 2.3 *[3] A set function $m : \mathscr{A} \to [a, b]_+$ (or semiclosed interval) is called a σ-$\oplus$-decomposable measure if it satisfies the following conditions:*

(1) $m(\emptyset) = \mathbf{0}$;

(2) $m(\bigcup_{i=1}^{\infty} E_i) = \bigoplus_{i=1}^{\infty} m(E_i)$ for any sequence $\{E_i\}_{i\in\mathbb{N}}$ of pairwise disjoint sets from $\mathscr{A}$, where $\bigoplus_{i=1}^{\infty} x_i = \lim_{n\to\infty} \bigoplus_{i=1}^{n} x_i$ for all $\{x_i\} \subset [a, b]$.

A σ-$\oplus$-decomposable measure m also is called σ-sup-decomposable measure if $x \oplus y = \sup(x, y)$ on the interval $[a, b]$. A set E is called σ-$\oplus$-decomposable measure zero set if $m(E) = \mathbf{0}$. It is obvious that $\emptyset$ is σ-$\oplus$-decomposable measure zero set.

We notice that if m is a σ-$\oplus$-decomposable measure, where $\oplus$ has a generator g, then $\mu = g \circ m$ is a σ-additive measure, and if a set E is σ-$\oplus$-decomposable measure zero set, then E is a measure zero set with respect to μ. In fact, we have that

(1) $\mu(\emptyset) = g(m(\emptyset)) = g(\mathbf{0}) = 0$;

(2) $\mu(\bigcup_{i=1}^{\infty} E_i) = g\left(m(\bigcup_{i=1}^{\infty} E_i)\right) = g(\bigoplus_{i=1}^{\infty} m(E_i)) = \sum_{i=1}^{\infty} g(m(E_i)) = \sum_{i=1}^{\infty} \mu(E_i)$ for any sequence $\{E_i\}_{i\in\mathbb{N}}$ of pairwise disjoint sets from $\mathscr{A}$;

(3) If E is a σ-$\oplus$-decomposable measure zero set, then $\mu(E) = g(m(E)) = g(\mathbf{0}) = 0$.

We call that m is g-finite, σ-g-finite, totally g-finite, totally σ-g-finite and g-complete, if $\mu = g \circ m$ is finite, σ-finite, totally finite, totally σ-finite and complete (see [11]), respectively.

Definition 2.4 *If $(X, \mathscr{A})$ is a measurable space and m, ν are two σ-$\oplus$-decomposable measure on $\mathscr{A}$. ν is called absolutely $\oplus$-continuous with respect to m, if $\nu(E) = \mathbf{0}$ for every measurable set E for which $m(E) = \mathbf{0}$.*

It should be noted that if ν is absolutely $\oplus$-continuous with respect to m, where $\oplus$ has a generator g, then $g \circ \nu$ is absolutely continuous with respect to $g \circ m$ (see [11]).

Let f and h be two functions defined on X and with values in $[a,b]$. Then, for any $x \in X$ and $\lambda \in [a,b]$ we define

$$(f \oplus h)(x) = f(x) \oplus h(x),$$
$$(f \odot h)(x) = f(x) \odot h(x),$$
$$(\lambda \odot f)(x) = \lambda \odot f(x).$$

Definition 2.5 *[4] The pseudo-characteristic function of a set E is defined with:*

$$\chi_E(x) = \begin{cases} \mathbf{0}, & x \notin E, \\ \mathbf{1}, & x \in E, \end{cases}$$

where $\mathbf{0}$ is zero element for $\oplus$ and $\mathbf{1}$ is unit element for $\odot$.

Definition 2.6 *A set function $m : \mathscr{A} \to [a,b]$ (or semiclosed interval) is monotone if*

$$m(E) \preceq m(F)$$

whenever $E, F \in \mathscr{A}$ and $E \subset F$.

Denote by μ the usual Lebesgue measure on $\mathbb{R}$. We have

$$m(E) = ess\sup_{\mu}\{x | x \in E\} = \sup\{a | \mu(\{x | x \in E, x > a\}) > 0\}.$$

Further, m is a σ-sup-decomposable measure [17]. More, Mesiar and Pap(see [17]) have showed that any σ-sup-decomposable measure generated as essential supremum of a continuous density can be obtained as a limit of pseudo-additive measures with respect to generted pseudo-addition.

In this paper we will consider the semiring $([a,b], \oplus, \odot)$ for three (with completely different behavior) cases, namely Case 1(a), 2 and 3(a). Observe that the Case 1(b) and 3(b) are linked to Case 1(a) and 3(a) by duality [21].

First, if the pseudo-operations are defined by a monotone and continuous surjective function $g : [a,b] \to [0,\infty]$ (i.e., Case 2), then the pseudo-integral for a measurable function $f : X \to [a,b]$ is given by

$$\int_X^{\oplus} f \odot dm = g^{-1}\left(\int_X (g \circ f)\, d\,(g \circ m)\right),$$

where the integral applied on the right side is the standard Lebesgue integral. In a special case, when $X = [c,d]$, $\mathscr{A} = \mathcal{B}(X)$ and $m = g^{-1} \circ \mu$, then the pseudo-integral reduces on the g-integral

$$\int_{[c,d]}^{\oplus} f dx = g^{-1}\left(\int_c^d g(f(x))dx\right).$$

Second, if the semiring is of the form $([a,b], \sup, \odot)$ (i.e., Case 1(a) and Case 3(a)), then we shall consider complete sup-measure (shortly sup-measure) m only and $\mathscr{A} = 2^X$, i.e., for any family $\{E_i\}_{i \in I}$ of measurable sets,

$$m(\bigcup_{i \in I} E_i) = \sup_{i \in I} m(E_i).$$

If X is countable (especially, if X is finite) then any σ-sup-decomposable measure m is complete and, moreover, $m(E) = \sup_{x \in E} \psi(x)$, where $\psi : X \to [a,b]$ is a density function given by $\psi(x) = m(\{x\})$. Then the pseudo-integral for a function $f : X \to [a,b]$ is given by

$$\int_X^{\oplus} f \odot dm = \sup_{x \in X}(f(x) \odot \psi(x)),$$

where function ψ defines σ-sup-decomposable measure m.

3 Main results

Theorem 3.1 *A σ-$\oplus$-decomposable measure $m : \mathscr{A} \to [a,b]$ is monotone, if $\oplus$ satisfies one of the following conditions:*
(1) $x \oplus y = \sup(x,y)$ on the interval $[a,b]$;
(2) $\oplus$ has a strictly monotone and continuous surjective generator g.

Proof. If $E, F \in \mathscr{A}$ and $E \subset F$, then $F - E \in \mathscr{A}$. Since $F = E \cup (F - E)$, we get

$$m(F) = m(F - E) \oplus m(E).$$

If $\oplus$ satisfies Condition (1), then we have

$$m(F) = \sup\{m(F - E), m(E)\} \geq m(E),$$

i.e., $\sup\{m(E), m(F)\} = m(F)$. Hence, by $x \preceq y$ if and only if $\sup(x,y) = y$ for all $x, y \in [a,b]$, we have

$$m(E) \preceq m(F).$$

If $\oplus$ satisfies Condition (2), then we have

$$g(m(F)) = g(m(F - E)) + g(m(E)).$$

Since $g(x) \geq 0$ for all $x \in [a,b]$, we have $g(m(F)) \geq g(m(E))$. Hence, by $x \preceq y$ if and only if $g(x) \leq g(y)$ for all $x, y \in [a,b]$, we have

$$m(E) \preceq m(F).$$

□

It is easy to see that if F is a σ-$\oplus$-decomposable measure zero set with respect to m, where m is a σ-$\oplus$-decomposable measure, then E is a σ-$\oplus$-decomposable measure zero set with respect to m, for all $E \subset F$.

Theorem 3.2 *Let $(X, \mathscr{A})$ be a measurable space. If $m : \mathscr{A} \to [a,b]$ is a σ-$\oplus$-decomposable measure and $\{E_n\} \subset \mathscr{A}(X)$ is an increasing sequence for which $\lim\limits_{n\to\infty} E_n \in \mathscr{A}(X)$, then*

$$m\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} m(E_n).$$

Proof. We might write $E_0 = \emptyset$. Since $\{E_n\}$ is an increasing sequence, $\{E_i - E_{i-1}\}$ is a sequence of pairwise disjoint sets from $\mathscr{A}$, then

$$E_n = \bigcup_{i=1}^{n} (E_i - E_{i-1}) \text{ and } \lim_{n\to\infty} E_n = \bigcup_{i=1}^{\infty} (E_i - E_{i-1}).$$

Hence, we have

$$m(E_n) = m\left(\bigcup_{i=1}^{n} (E_i - E_{i-1})\right) = \mathop{\oplus}_{i=1}^{n} m(E_i - E_{i-1})$$

and

$$m\left(\lim_{n\to\infty} E_n\right) = m\left(\bigcup_{i=1}^{\infty} (E_i - E_{i-1})\right) = \mathop{\oplus}_{i=1}^{\infty} m(E_i - E_{i-1}).$$

By $\mathop{\oplus}\limits_{i=1}^{\infty} x_i = \lim\limits_{n\to\infty} \mathop{\oplus}\limits_{i=1}^{n} x_i$ for all $\{x_i\} \subset [a,b]$, we have

$$\mathop{\oplus}_{i=1}^{\infty} m(E_i - E_{i-1}) = \lim_{n\to\infty} \mathop{\oplus}_{i=1}^{n} m(E_i - E_{i-1}).$$

Consequently, we get $m\left(\lim\limits_{n\to\infty} E_n\right) = \lim\limits_{n\to\infty} m(E_n)$. □

Theorem 3.3 *Let $(X, \mathscr{A})$ be a measurable space and $m : \mathscr{A} \to [a,b]$ be a σ-$\oplus$-decomposable measure, where $\oplus$ has a strictly increasing (or decreasing) and continuous surjective generator g. If $\{E_n\} \subset \mathscr{A}(X)$ is a decreasing sequence, and there exists at least one $l \in \mathbb{N}$ such that $m(E_l) \prec b$ (or $m(E_l) \prec a$). Then*

$$m\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} m(E_n).$$

Proof. Suppose $m(E_l) \prec b$ for some $l \in \mathbb{N}$. For the case $m(E_l) \prec a$, we can prove it by a similar proof. By Theorem 3.1 and $\{E_n\}$ is a decreasing sequence, we have

$$m(E_n) \preceq m(E_l) \prec b \text{ for all } n \geq l,$$

and therefore

$$m\left(\lim_{n\to\infty} E_n\right) \prec b.$$

By the monotonicity of g, we get

$$g(m(E_l)) < +\infty \text{ and } g\left(m\left(\lim_{n\to\infty} E_n\right)\right) < +\infty.$$

Since $\{E_n\}$ ia a decreasing sequence, $\{E_l - E_n\}$ is an increasing sequence. By Theorem 3.2, we obtain

$$m\left(E_l - \lim_{n\to\infty} E_n\right) = m\left(\lim_{n\to\infty}(E_l - E_n)\right) = \lim_{n\to\infty} m(E_l - E_n),$$

i.e.,

$$g\left(m\left(E_l - \lim_{n\to\infty} E_n\right)\right) = g\left(\lim_{n\to\infty} m(E_l - E_n)\right).$$

By the continuity of g and $g \circ m$ is a σ-additive measure, we have

$$\begin{aligned} g(m(E_l)) - g\left(m\left(\lim_{n\to\infty} E_n\right)\right) &= g\left(m\left(E_l - \lim_{n\to\infty} E_n\right)\right) \\ &= g\left(\lim_{n\to\infty} m(E_l - E_n)\right) \\ &= \lim_{n\to\infty} g(m(E_l - E_n)) \\ &= g(m(E_l)) - \lim_{n\to\infty} g(m(E_n)) \\ &= g(m(E_l)) - g\left(\lim_{n\to\infty} m(E_n)\right). \end{aligned}$$

Since $g(m(E_l)) < +\infty$, we have

$$g\left(m\left(\lim_{n\to\infty} E_n\right)\right) = g\left(\lim_{n\to\infty} m(E_n)\right).$$

By the strictly monotonicity of g, we get $m\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} m(E_n)$. □

Lemma 3.1 *[17] Let m be a σ-sup-decomposable measure which is defined by*

$$m(E) = ess\sup_{\mu}\{\psi(x)|x \in E\},$$

on $([0,\infty], \mathcal{B})$, where $\psi : [0,\infty] \to [0,\infty]$ is a continuous density. Then for any generator g there exist a family $\{m_\lambda\}$ of σ-$\oplus_\lambda$-decomposable measures on $([0,\infty], \mathcal{B})$, where $\oplus_\lambda$ is generated by g^λ (the function g on the power λ), $\lambda \in (0,\infty)$, such that $\lim_{\lambda\to\infty} m_\lambda = m$.

Theorem 3.4 *Let $([0,\infty], \sup, \odot)$ be a semiring with $\odot$ generated by a strictly increasing and continuous surjective generator g. Let m be the same as in Lemma 3.1. If $\{E_n\} \subset \mathcal{B}([0,\infty])$ is a decreasing sequence, $m(E_n) \prec b$ for at least one $n \in \mathbb{N}$, then*

$$m\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} m(E_n).$$

Proof. Since m is the same one in Lemma 3.1, for the generator g there exist a family $\{m_\lambda\}$ of σ-$\oplus_\lambda$-decomposable measures on $([0,\infty], \mathcal{B})$, where $\oplus_\lambda$ is generated by g^λ, $\lambda \in (0,\infty)$, such that

$$\lim_{\lambda\to\infty} m_\lambda = m.$$

Let $l \in \mathbb{N}$ such that $m(E_l) \prec b$. Thus we have

$$m_\lambda(E_l) \prec b,\ \lambda \in (0, \infty).$$

Since $f(x) = x^\mu\ (\mu \neq 0)$ is strictly increasing, whenever $\mu > 0$, g^λ and g are comonotone functions. Hence, by Theorem 3.3, we have

$$m_\lambda\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} m_\lambda(E_n),\ \lambda \in (0, \infty).$$

Consequently, we obtain that

$$\lim_{\lambda\to\infty} m_\lambda\left(\lim_{n\to\infty} E_n\right) = \lim_{\lambda\to\infty}\lim_{n\to\infty} m_\lambda(E_n) = \lim_{n\to\infty}\lim_{\lambda\to\infty} m_\lambda(E_n),$$

i.e., $m\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} m(E_n)$. □

Theorem 3.5 *Let $(X, \oplus, \odot)$ be a semiring with generated pseudo-operations by a strictly monotone and continuous surjective generator g, and let $m : \mathscr{A}(X) \to [a, b]$ be a σ-$\oplus$-decomposable measure. If $f : X \to [a, b]$ is a measurable function with respect to m and E is a σ-$\oplus$-decomposable measure zero set on $\mathscr{A}(X)$, then*

$$\int_E^{\oplus} f \odot dm = \boldsymbol{0}.$$

Proof. If the pseudo-operations are generated by a strictly monotone and continuous surjective generator g, i.e., $x \oplus y = g^{-1}(g(x)+g(y))$ and $x \odot y = g^{-1}(g(x)\cdot g(y))$ for every $x, y \in [a, b]$. Then the pseudo-integral for a measurable function $f : X \to [a, b]$ is given by

$$\int_E^{\oplus} f \odot dm = g^{-1}\left(\int_E g \circ f d\mu\right),$$

for every measurable set E, where $\mu = g \circ m$ is a σ-additive measure. Since E is a set of σ-$\oplus$-decomposable measure zero on $\mathscr{A}(X)$ and $g(\mathbf{0}) = 0$, we have that E is a measure zero set with respect to μ. Hence, by Theorem **C** of § 25 of [11], we have $\int_E g \circ f d\mu = 0$. Consequently, by the strictly monotonicity of g, we obtain that $\int_E^{\oplus} f \odot dm = g^{-1}(0) = \mathbf{0}$. □

Example 3.1 *Let $X = \mathbb{R}$ and let the pseudo-addition be represented by a strictly increasing and continuous generator surjective function $g : [0, +\infty] \to [0, +\infty]$, which is defined by $g(x) = x$ for all $x \in [0, +\infty]$. It is easy to see that $x \oplus y = x + y$ and $x \odot y = x \cdot y$ for all $x, y \in [0, +\infty]$. Hence, we have $\boldsymbol{0} = 0, \boldsymbol{1} = 1$. We define a set function m on $\mathcal{B}(X)$ by*

$$m(E) = \mu(E)$$

for all $E \in \mathcal{B}(X)$, where μ is a Lebesgue measure. It is obvious that m satisfies (1) and (2) of Definition 2.3. Consequently, the set function m is a σ-$\oplus$-decomposable measure. Let E be the set of rational numbers. Since $\mu(E) = 0$, thus $m(E) = \boldsymbol{0}$, i.e., E is a set of σ-$\oplus$-decomposable measure zero on $\mathcal{B}(X)$. Hence, for any measurable function f we have

$$\int_E^{\oplus} f \odot dm = \boldsymbol{0}.$$

Theorem 3.6 *Let $(X, \sup, \odot)$ be a semiring with $\odot$ generated by a strictly increasing and continuous surjective generator g, and let $m : \mathscr{A}(X) \to [a, b]$ be a σ-sup-decomposable measure which is defined by $m(E) = \sup_{x\in E} \psi(x)$, where $\psi : X \to [a, b]$ is a density function given by $\psi(x) = m(\{x\})$. If $f : X \to [a, b]$ is a measurable function with respect to m and E is a set of σ-sup-decomposable measure zero on $\mathscr{A}(X)$, then*

$$\int_E^{\sup} f \odot dm = \boldsymbol{0}.$$

Proof. Let $m : \mathscr{A} \to [a,b]$ be a σ-sup-decomposable measure which is defined by $m(E) = \sup_{x\in E} \psi(x)$, where $\psi : X \to [a,b]$ is a density function given by $\psi(x) = m(\{x\})$. Then the pseudo-integral for a measurable function $f : X \to [a,b]$ is given by

$$\int_E^{\sup} f \odot dm = \sup_{x\in E}(f(x) \odot \psi(x)),$$

for every measurable set E, where function ψ defines σ-sup-decomposable measure m. Since $f(x) \preceq \sup_{x\in E} f(x)$ and $\psi(x) \preceq \sup_{x\in E} \psi(x) = m(E)$ for all $x \in E$, by the monotonicity of g, we have

$$g\left(f\left(x\right)\right) \leq g\left(\sup_{x\in E} f\left(x\right)\right) \text{ and } g\left(\psi\left(x\right)\right) \leq g\left(m\left(E\right)\right),$$

for all $x \in E$. Hence, by $g(x) \geq 0$ for all $x \in [a,b]$, we have

$$g\left(f\left(x\right)\right) \cdot g\left(\psi\left(x\right)\right) \leq g\left(\sup_{x\in E} f\left(x\right)\right) \cdot g\left(m\left(E\right)\right),$$

which implies that

$$g\left(g^{-1}\left(g\left(f\left(x\right)\right) \cdot g\left(\psi\left(x\right)\right)\right)\right) \leq g\left(g^{-1}\left(g\left(\sup_{x\in E} f\left(x\right)\right) \cdot g\left(m\left(E\right)\right)\right)\right),$$

for all $x \in E$. Since $\odot$ is generated by a generator g, i.e., $y \odot z = g^{-1}(g(y)g(z))$ for all $y, z \in [a,b]$, we get that

$$g\left(f(x) \odot \psi(x)\right) \leq g\left(\sup_{x\in E} f(x) \odot m(E)\right),$$

for all $x \in E$. Hence, we obtain that

$$f(x) \odot \psi(x) \preceq \sup_{x\in E} f(x) \odot m(E)$$

for all $x \in E$, which implies that

$$\sup_{x\in E}(f(x) \odot \psi(x)) \preceq \sup_{x\in E} f(x) \odot m(E).$$

Since E is a set of σ-sup-decomposable measure zero on $\mathscr{A}(X)$, we have $\sup_{x\in E} f(x) \odot m(E) = \mathbf{0}$. Consequently, we have $\sup_{x\in E}(f(x) \odot \psi(x)) \preceq \mathbf{0}$. By the monotonicity of g and $g(y) \geq 0$ for $y \in [a,b]$, we have $0 \leq g\left(\sup_{x\in E}(f(x) \odot \psi(x))\right) \leq g(\mathbf{0}) = 0$, i.e., $g\left(\sup_{x\in E}(f(x) \odot \psi(x))\right) = 0$, which implies that $\int_E^{\sup} f \odot dm = \mathbf{0}$. □

Theorem 3.7 *Let $(X, \sup, \inf)$ be a semiring, and let m be the same as in Theorem 3.6. If $f : X \to [a,b]$ is a measurable function with respect to m and E is a set of σ-sup-decomposable measure zero on $\mathscr{A}(X)$, then*

$$\int_E^{\sup} f \odot dm = \mathbf{0}.$$

Proof. Let m be the same as in Theorem 3.6. Then the pseudo-integral for a measurable function $f : X \to [a,b]$ is given by

$$\int_E^{\sup} f \odot dm = \sup_{x\in E}(f(x) \odot \psi(x)),$$

for every measurable set E, where function ψ defines σ-sup-decomposable measure m. Since $f(x) \preceq \sup_{x\in E} f(x)$ and $\psi(x) \preceq \sup_{x\in E} \psi(x) = m(E)$ for all $x \in E$, and $y \odot z = \inf(y,z)$ for all $y, z \in [a,b]$, we have

$$f(x) \odot \psi(x) = \inf\{f(x), \psi(x)\} \preceq \inf\left\{\sup_{x\in E} f(x), m(E)\right\} = \sup_{x\in E} f(x) \odot m(E),$$

for all $x \in E$, which implies that

$$\sup_{x\in E}(f(x) \odot \psi(x)) \preceq \sup_{x\in E} f(x) \odot m(E).$$

Since E is a set of σ-sup-decomposable measure zero on $\mathscr{A}(X)$, we have $\sup_{x\in E} f(x) \odot m(E) = \mathbf{0}$. Hence, we have $\sup_{x\in E}(f(x) \odot \psi(x)) \preceq \mathbf{0}$. By $y \preceq z$ if and only if $\sup(y, z) = z$ and $y \oplus z = \sup(y, z)$ for all $y, z \in [a, b]$, we obtain that

$$\sup_{x\in E}(f(x) \odot \psi(x)) \oplus \mathbf{0} = \mathbf{0}.$$

By (4) of Definition 2.1, we have $\sup_{x\in E}(f(x) \odot \psi(x)) = \mathbf{0}$, which implies that $\int_E^{\sup} f \odot dm = \mathbf{0}$. □

Theorem 3.8 *Let $(X, \oplus, \odot)$ be a semiring with generated pseudo-operations by a strictly monotone and continuous surjective generator g, and let $m : \mathscr{A}(X) \to [a, b]$ be a σ-$\oplus$-decomposable measure. If $f : X \to [a, b]$ is a measurable function and $\mathbf{0} \prec f$ a.e. on a measurable set E with respect to m, and if*

$$\int_E^{\oplus} f \odot dm = \mathbf{0},$$

then $m(E) = \mathbf{0}$.

Proof. If the pseudo-operations are generated by a strictly monotone and continuous surjective generator g, i.e., $y \oplus z = g^{-1}(g(y)+g(z))$ and $y \odot z = g^{-1}(g(y) \cdot g(z))$ for every $y, z \in [a, b]$. Then the pseudo-integral for a measurable function $f : X \to [a, b]$ is given by

$$\int_E^{\oplus} f \odot dm = g^{-1}\left(\int_E g \circ f d\mu\right),$$

for every measurable set E, where $\mu = g \circ m$ is a σ-additive measure.

Let $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$, where $E_2 = \{x \in E \,|\, \mathbf{0} \prec f(x)\}$. By $y \preceq z$ if and only if $g(y) \leq g(z)$ for all $y, z \in [a, b]$, the strictly monotonicity of g and $g(\mathbf{0}) = 0$, we obtain that $g(f(x)) > 0$ for all $x \in E_2$. Since $\mathbf{0} \prec f$ a.e. on a measurable set E with respect to m, we have $m(E_1) = \mathbf{0}$. Hence, by Theorem 3.5, we have $\int_{E_1}^{\oplus} f \odot dm = \mathbf{0}$, which implies that

$$\int_E^{\oplus} f \odot dm = \int_{E_1}^{\oplus} f \odot dm \oplus \int_{E_2}^{\oplus} f \odot dm = \int_{E_2}^{\oplus} f \odot dm.$$

If $\int_E^{\oplus} f \odot dm = \mathbf{0}$, then we get that

$$\int_{E_2}^{\oplus} f \odot dm = g^{-1}\left(\int_{E_2} g \circ f d\mu\right) = \mathbf{0},$$

i.e., $\int_{E_2} g \circ f d\mu = g(\mathbf{0}) = 0$. By Theorem **D** of § 25 of [11], we have $\mu(E_2) = 0$. Hence, we obtain that $m(E_2) = g^{-1}(\mu(E)) = \mathbf{0}$. Consequently, $m(E) = m(E_1) \oplus m(E_2) = \mathbf{0}$. □

Theorem 3.9 *Let $(X, \sup, \odot)$ be a semiring with $\odot$ generated by a strictly increasing and continuous surjective generator g, and let m be the same as in Theorem 3.6. If $f : X \to [a, b]$ is a measurable function and $\mathbf{0} \prec f$ a.e. on a measurable set E with respect to m, and if*

$$\int_E^{\sup} f \odot dm = \mathbf{0},$$

then $m(E) = \mathbf{0}$.

Proof. Let m be the same as in Theorem 3.6. Then the pseudo-integral for a measurable function $f : X \to [a, b]$ is given by

$$\int_E^{\sup} f \odot dm = \sup_{x\in E}(f(x) \odot \psi(x)),$$

for every measurable set E, where function ψ defines σ-sup-decomposable measure m. Let $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$, where $E_2 = \{x \in E \,|\mathbf{0} \prec f(x)\}$. If $\mathbf{0} \prec f$ a.e. on a measurable set E with respect to m, then $m(E_1) = \mathbf{0}$, which implies that

$$m(E) = \sup\{m(E_1), m(E_2)\} = m(E_2).$$

If $\int_E^{\sup} f \odot dm = \mathbf{0}$, then by $g(\mathbf{0}) = 0$, we have $g\left(\int_E^{\sup} f \odot dm\right) = g\left(\sup_{x\in E}(f(x) \odot \psi(x))\right) = 0$. By the strictly monotonicity of g, we have

$$g(f(x) \odot \psi(x)) \le g\left(\sup_{x\in E}(f(x) \odot \psi(x))\right) = 0,$$

for all $x \in E$. Since $g(y) \ge 0$ for all $y \in [a, b]$, we have $g(f(x) \odot \psi(x)) = 0$, for all $x \in E$. If $\odot$ is generated by a generator g, i.e., $y \odot z = g^{-1}(g(y)g(z))$ for all $y, z \in [a, b]$, then we get that

$$g(f(x)) \cdot g(\psi(x)) = g(f(x) \odot \psi(x)) = 0.$$

Since $\mathbf{0} \prec f(x)$ for all $x \in E_2$, we get $g(f(x)) > 0$ for all $x \in E_2$. Thus, we have $g(\psi(x)) = 0$, i.e., $\psi(x) = \mathbf{0}$ for all $x \in E_2 \subset E$. Consequently, we obtain taht $m(E_2) = \sup_{x\in E_2} \psi(x) = \mathbf{0}$, which implies that $m(E) = \mathbf{0}$. □

Theorem 3.10 *Let $(X, \sup, \inf)$ be a semiring, and let m be the same as in Theorem 3.6. If $f : X \to [a, b]$ is a measurable function and $\mathbf{0} \prec f$ a.e. on a measurable set E with respect to m, and if*

$$\int_E^{\sup} f \odot dm = \mathbf{0},$$

then $m(E) = \mathbf{0}$.

Proof. Let m be the same as in Theorem 3.6. Then the pseudo-integral for a measurable function $f : X \to [a, b]$ is given by

$$\int_E^{\sup} f \odot dm = \sup_{x\in E}(f(x) \odot \psi(x)),$$

for every measurable set E, where function ψ defines σ-sup-decomposable measure m.

Let $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$, where $E_2 = \{x \in E \,|\mathbf{0} \prec f(x)\}$. If $\mathbf{0} \prec f$ a.e. on a measurable set E with respect to m, then $m(E_1) = \mathbf{0}$. Hence, by Theorem 3.7, we have $\int_{E_1}^{\sup} f \odot dm = \mathbf{0}$, which implies that

$$\int_E^{\sup} f \odot dm = \int_{E_1}^{\sup} f \odot dm \oplus \int_{E_2}^{\sup} f \odot dm = \int_{E_2}^{\sup} f \odot dm.$$

If $\int_E^{\sup} f \odot dm = \mathbf{0}$, then we get that $\int_{E_2}^{\sup} f \odot dm = \sup_{x\in E_2} (f(x) \odot \psi(x)) = \mathbf{0}$. Hence, we obtain that

$$f(x) \odot \psi(x) \preceq \sup_{x\in E_2} (f(x) \odot \psi(x)) = \mathbf{0} = f(x) \odot \mathbf{0},$$

for all $x \in E_2$. By $\mathbf{0} \prec f(x)$ for all $x \in E_2$ and (2) of Definition 2.2, we have $\psi(x) \preceq \mathbf{0}$ for all $x \in E_2$, which implies that $m(E_2) = \sup_{x\in E_2} \psi(x) \preceq \mathbf{0}$. Since $f(x) \preceq \sup_{x\in E_2} f(x)$ and $\psi(x) \preceq \sup_{x\in E_2} \psi(x) = m(E_2)$ for all $x \in E_2$, and $y \odot z = \inf(y, z)$ for all $y, z \in [a, b]$, we have

$$f(x) \odot \psi(x) = \inf\{f(x), \psi(x)\} \preceq \inf\left\{\sup_{x\in E_2} f(x), m(E_2)\right\} = \sup_{x\in E_2} f(x) \odot m(E_2),$$

for all $x \in E_2$, which implies that

$$\sup_{x\in E_2} f(x) \odot \mathbf{0} = \mathbf{0} = \sup_{x\in E_2} (f(x) \odot \psi(x)) \preceq \sup_{x\in E_2} f(x) \odot m(E_2).$$

By $\mathbf{0} \prec f(x)$ for all $x \in E_2$ and (2) of Definition 2.2, we have $\mathbf{0} \preceq m(E_2)$. Consequently, we obtain that $m(E_2) = \mathbf{0}$, which implies that $m(E) = m(E_1) \oplus m(E_2) = \mathbf{0}$. □

Theorem 3.11 *Let $(X, \oplus, \odot)$ be a semiring with generated pseudo-operations by a strictly monotone and continuous surjective generator g, and let m be a σ-$\oplus$-decomposable measure on $\mathscr{A}(X)$ and f be a measurable function with respect to m on X. Then $\int_E^{\oplus} f \odot dm = \mathbf{0}$ if and only if $f = \mathbf{0}$ a.e. for every measurable set E.*

Proof. If the pseudo-operations are generated by a strictly monotone and continuous surjective generator g, i.e., we have

$$y \oplus z = g^{-1}(g(y) + g(z)) \text{ and } x \odot y = g^{-1}(g(y) \cdot g(z))$$

for every $y, z \in [a, b]$. Then the pseudo-integral for a measurable function $f : X \to [a, b]$ is given by

$$\int_E^{\oplus} f \odot dm = g^{-1}\left(\int_E g \circ f d\mu\right)$$

for every measurable set E, where $\mu = g \circ m$ is a σ-additive measure.

Suppose $\int_E^{\oplus} f \odot dm = \mathbf{0}$ for every measurable set E. Since $g(\mathbf{0}) = 0$, we have $\int_E g \circ f d\mu = g(\mathbf{0}) = 0$ for every measurable set E. By Theorem **E** of § 25 of [11], we have $g \circ f = 0$ a.e. for every measurable set E, which implies that $f = \mathbf{0}$ a.e. for every measurable set E.

Suppose $f = \mathbf{0}$ a.e. for every measurable set E. Let $E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \emptyset$, where $E_1 = \{x \in E | f(x) = \mathbf{0}\}$. Then, we have $m(E_2) = \mathbf{0}$. By Theorem 3.5, we have $\int_{E_2}^{\oplus} f \odot dm = \mathbf{0}$. Hence, we get that

$$\int_E^{\oplus} f \odot dm = \int_{E_1}^{\oplus} f \odot dm \oplus \int_{E_2}^{\oplus} f \odot dm = \int_{E_1}^{\oplus} f \odot dm.$$

Since $f(x) = \mathbf{0}$ for all $x \in E_1$, we have $g(f(x)) = 0$ for all $x \in E_1$. Hence, we obtain that $\int_{E_1} g \circ f d\mu = 0$, i.e., $\int_{E_1}^{\oplus} f \odot dm = g^{-1}(0) = \mathbf{0}$. Consequently, $\int_E^{\oplus} f \odot dm = g^{-1}(0) = \mathbf{0}$. □

Theorem 3.12 *Let $(X, \sup, \odot)$ be a semiring with $\odot$ generated by a strictly increasing and continuous surjective generator g, and let m be the same as in Theorem 3.6 and f be a measurable function with respect to m on X. Then $\int_E^{\sup} f \odot dm = \mathbf{0}$ if and only if $f = \mathbf{0}$ a.e. for every measurable set E.*

Proof. Let m be the same as in Theorem 3.6. Then the pseudo-integral for a measurable function $f : X \to [a, b]$ is given by

$$\int_E^{\sup} f \odot dm = \sup_{x\in E}(f(x) \odot \psi(x)),$$

for every measurable set E, where function ψ defines σ-sup-decomposable measure m.

Suppose $\int_E^{\sup} f \odot dm = \mathbf{0}$ for every measurable set E. By $g(\mathbf{0}) = 0$, we have

$$g\left(\int_E^{\sup} f \odot dm\right) = g\left(\sup_{x\in E}(f(x) \odot \psi(x))\right) = 0.$$

By the strictly monotonicity of g, we have

$$g(f(x) \odot \psi(x)) \leq g\left(\sup_{x \in E}(f(x) \odot \psi(x))\right) = 0,$$

for all $x \in E$. Since $g(y) \geq 0$ for all $y \in [a,b]$, we have $g(f(x) \odot \psi(x)) = 0$, for all $x \in E$. Since $\odot$ is generated by a generator g, i.e., $y \odot z = g^{-1}(g(y)g(z))$ for all $y, z \in [a,b]$, we get that

$$g(f(x)) \cdot g(\psi(x)) = g(f(x) \odot \psi(x)) = 0,$$

for all $x \in E$. Let $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$, where $E_1 = \{x \in E | f(x) = \mathbf{0}\}$. Then, we have $f(x) \neq \mathbf{0}$ for all $x \in E_2$, which implies that $g(f(x)) \neq 0$ for all $x \in E_2$. Hence, we have $g(\psi(x)) = 0$, i.e., $\psi(x) = \mathbf{0}$ for all $x \in E_2 \subset E$, which implies that $m(E_2) = \sup_{x \in E_2} \psi(x) = \mathbf{0}$. Hence, $f = \mathbf{0}$ a.e. for every measurable set E.

Suppose $f = \mathbf{0}$ a.e. for every measurable set E. Let $E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \emptyset$, where $E_1 = \{x \in E | f(x) = \mathbf{0}\}$, then $m(E_2) = \mathbf{0}$. By Theorem 3.6, we have $\int_{E_2}^{\sup} f \odot dm = \mathbf{0}$. Hence, we get that

$$\int_{E}^{\sup} f \odot dm = \int_{E_1}^{\sup} f \odot dm \oplus \int_{E_2}^{\sup} f \odot dm = \int_{E_1}^{\sup} f \odot dm.$$

Since $f(x) = \mathbf{0}$ for all $x \in E_1$, we have $f(x) \odot \psi(x) = \mathbf{0}$ for all $x \in E_1$. Hence, we obtain that $\int_{E_1}^{\sup} f \odot dm = \sup_{x \in E_1} (f(x) \odot \psi(x)) = \mathbf{0}$, which implies that $\int_{E}^{\sup} f \odot dm = \mathbf{0}$. □

Theorem 3.13 *Let $(X, \sup, \inf)$ be a semiring, and let m be the same as in Theorem 3.6 and f be a measurable function with respect to m and $\mathbf{0} \preceq f$. Then $\int_{E}^{\sup} f \odot dm = \mathbf{0}$ if and only if $f = \mathbf{0}$ a.e. for every measurable set E.*

Proof. Let m be the same as in Theorem 3.6. Then the pseudo-integral for a measurable function $f : X \to [a,b]$ is given by

$$\int_{E}^{\sup} f \odot dm = \sup_{x \in E}(f(x) \odot \psi(x)),$$

for every measurable set E, where function ψ defines σ-sup-decomposable measure m. For every measurable set E, let $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$, where $E_1 = \{x \in E | f(x) = \mathbf{0}\}$.

Suppose $\int_{E}^{\sup} f \odot dm = \mathbf{0}$ for every measurable set E. By the proof of Theorem 3.10, we have $m(E_2) = \mathbf{0}$. Hence, $f = \mathbf{0}$ a.e. for every measurable set E.

Suppose $f = \mathbf{0}$ a.e. for every measurable set E, then $m(E_2) = \mathbf{0}$. By Theorem 3.7, we have $\int_{E_2}^{\sup} f \odot dm = \mathbf{0}$. Hence, we get that

$$\int_{E}^{\sup} f \odot dm = \int_{E_1}^{\sup} f \odot dm \oplus \int_{E_2}^{\sup} f \odot dm = \int_{E_1}^{\sup} f \odot dm.$$

Since $f(x) = \mathbf{0}$ for all $x \in E_1$, we have $f(x) \odot \psi(x) = \mathbf{0}$ for all $x \in E_1$. Hence, we obtain that $\int_{E_1}^{\sup} f \odot dm = \sup_{x \in E_1} (f(x) \odot \psi(x)) = \mathbf{0}$, which implies that $\int_{E}^{\sup} f \odot dm = \mathbf{0}$. □

Theorem 3.14 *Let $(X, \oplus, \odot)$ be a semiring with generated pseudo-operations by a strictly monotone and continuous surjective generator $g : [a,b] \to [0, +\infty]$, and let $m : \mathscr{A}(X) \to [a,b]$ be a σ-$\oplus$-decomposable measure. For a measurable function $f : X \to [a,b]$ with respect to m, define the set function $\nu : X \to [a,b]$ by*

$$\nu(E) = \int_{E}^{\oplus} f \odot dm,$$

for any measurable set E. Then ν is a σ-$\oplus$-decomposable measure and absolutely $\oplus$-continuous with respect to m.

Proof. If the pseudo-operations are generated by a strictly monotone and continuous surjective generator $g : [a, b] \to [0, +\infty]$, i.e., $y \oplus z = g^{-1}(g(y) + g(z))$ and $y \odot z = g^{-1}(g(y) \cdot g(z))$ for every $y, z \in [a, b]$. Then the pseudo-integral for a measurable function $f : X \to [a, b]$ is given by

$$\nu(E) = \int_E^{\oplus} f \odot dm = g^{-1}\left(\int_E g \circ f d\mu\right),$$

for every measurable set E, where $\mu = g \circ m$ is a σ-additive measure. Hence, ν is a set function of $\mathscr{A}$ to [a,b].

(1) By $\emptyset$ is σ-$\oplus$-decomposable measure zero set and Theorem 3.5, we have

$$\nu(\emptyset) = \int_\emptyset^{\oplus} f \odot dm = \mathbf{0};$$

(2) For any sequence $\{E_i\}_{i\in\mathbb{N}}$ of pairwise disjoint sets from $\mathscr{A}$, we have

$$\begin{aligned}
\nu\left(\bigcup_{i=1}^{\infty} E_i\right) &= \int_{\bigcup_{i=1}^{\infty} E_i}^{\oplus} f \odot dm = g^{-1}\left(\int_{\bigcup_{i=1}^{\infty} E_i} g \circ f d\mu\right) \\
&= g^{-1}\left(\sum_{i=1}^{\infty} \int_{E_i} g \circ f d\mu\right) = g^{-1}\left(\sum_{i=1}^{\infty} g\left(\int_{E_i}^{\oplus} f \odot dm\right)\right) \\
&= \bigoplus_{i=1}^{\infty} \int_{E_i}^{\oplus} f \odot dm = \bigoplus_{i=1}^{\infty} \nu(E_i).
\end{aligned}$$

Consequently, ν is a σ-$\oplus$-decomposable measure. By Theorem 3.5, we obtain that ν is absolutely $\oplus$-continuous with respect to m. □

Theorem 3.15 *Let $(X, \oplus, \odot)$ be a semiring with generated pseudo-operations by a strictly monotone and continuous surjective generator g, and let $m : \mathscr{A}(X) \to [a, b]$ be a totally σ-g-finite-decomposable measure. If the σ-g-finite-decomposable measure ν is absolutely $\oplus$-continuous with respect to m, then there exists a measurable function f on X and $g(f(x)) < +\infty$ for all $x \in X$, such that*

$$\nu(E) = \int_E^{\oplus} f \odot dm,$$

for every measurable set E. The function f is unique in the sense that if also $\nu(E) = \int_E^{\oplus} h \odot dm$, then $f = h$ a.e. with respect to m.

Proof. If the pseudo-operations are generated by a strictly monotone and continuous surjective generator g, i.e.,

$$y \oplus z = g^{-1}(g(y) + g(z)) \text{ and } x \odot y = g^{-1}(g(y) \cdot g(z))$$

for every $y, z \in [a, b]$. Then the pseudo-integral for a measurable function $f : X \to [a, b]$ is given by

$$\int_E^{\oplus} f \odot dm = g^{-1}\left(\int_E g \circ f d\mu\right)$$

for every measurable set E, where $\mu = g \circ m$ is a σ-additive measure.

Since m is a totally σ-g-finite-decomposable measure, we have μ is a totally σ-finite measure. If the σ-g-finite-decomposable measure ν is absolutely continuous with respect to m, then $g \circ \nu$ is σ-finite measure and absolutely $\oplus$-continuous with respect to μ. Hence, by Theorem **B** of § 31 of [11], there exists a finite valued measurable function p with respect to μ on X such that

$$g(\nu(E)) = \int_E p d\mu,$$

for every measurable set E. By the strictly monotonicity of g, we have $p(x) = (g(g^{-1}(p(x))))$ for all $x \in X$. Let $f = g^{-1} \circ p$, then

$$g(\nu(E)) = \int_E p d\mu = \int_E g \circ f d\mu,$$

for all measurable set E, which implies that

$$\nu(E) = g^{-1}\left(\int_E g \circ f d\mu\right) = \int_E^{\oplus} f \odot dm,$$

for every measurable set E. If $\nu(E) = \int_E^{\oplus} h \odot dm$, then $g(\nu(E)) = \int_E g \circ h d\mu$. Hence $g \circ f = g \circ h$ a.e. with respect to μ, i.e., $f = h$ a.e. with respect to m. □

4 Conclusions

In this paper, we mainly discussed two classes of σ-$\oplus$-decomposable measures and the corresponding pseudo-integrals: one is based on the generated pseudo-addition (g-case, see [16, 22]) and the other is based on the idempotent pseudo-operation (sup and inf, see [23, 39]). We got several properties as monotonicity, continuous from above and continuous from below for σ-$\oplus$-decomposable measures. In particular, we obtained the correlation between the measure zero sets with respect to a σ-$\oplus$-decomposable measure and the corresponding pseudo-integrals on them. As an application of the main results, we generalized the classical Radon-Nikodym theorem, which has been extensively studied and discussed [5, 7, 10, 12, 43], to the decomposable measure theory based on pseudo-integrals. We also hope that our results in this paper may lead to significant, new and innovative results in other related fields.

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COMPOSITION OPERATOR ON ZYGMUND-ORLICZ SPACE

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Abstract. In this paper, we use Young's function to define Zygmund-Orlicz space. We study boundedness and compactness of composition operator on Zygmund-Orlicz space.

1. Introduction

Let $\mathbb{D}$ be the unit disk of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of all holomorphic function on $\mathbb{D}$. Let μ be a bounded, continuous and positive function denfined on $\mathbb{D}$. A function $f \in H(\mathbb{D})$ belongs to $\mu-$Zygmund space , denoted as $f \in Z^{\mu}$, if

$$\|f\|_{\mu} := \sup_{z\in\mathbb{D}} \mu(z)|f''(z)| < \infty.$$

Clearly, if $\mu(z) = 1 - |z|^2$, the space Z^{μ} is just the Zygmund space, which is denoted by Z, while when $\mu(z) = (1 - |z|^2)^{\alpha}$ with $\alpha > 0$, the space Z^{μ} becomes the α-Zygmund space which is denoted by Z^{α}. It is readily seen that Z^{μ} is a Banach space with the norm

$$\|f\|_{Z^{\mu}} = |f(0)| + |f'(0)| + \|f\|_{\mu}.$$

For some more information of μ-Zygmund space on the unit disk see [3], while for composition and integral-type operators between them on the unit disk see for example [4, 6, 7, 8, 9].

Let A_1, A_2 be two linear subspaces of $H(\mathbb{D})$. If ϕ is a holomorphic self-map of $\mathbb{D}$, such that $f \circ \phi$ belongs to A_2 for all $f \in A_1$, then ϕ induces a linear operator $C_{\phi} : A_1 \to A_2$ defined as

$$C_{\phi}f = f \circ \phi,$$

called the composition operator with symbol ϕ. This type of operator appears in studies on isometries of various function spaces. Composition operator has been studied by numerous authors on many subspaces of $H(\mathbb{D})$ and in paticular on Zygmund spaces and μ-Zygmund spaces. In [5], Julio C. Ramos Fernández characterized boundedness and compactness of composition operators on Bloch-Orlicz spaces denoted by $\mathcal{B}^{\varphi}$, where φ is Young's function. More precisely, let $\varphi : [0, \infty) \to [0, \infty)$ be a strictly increasing convex function such that $\varphi(0) = 0$ and $\lim_{t\to\infty} \frac{t}{\varphi(t)} = \lim_{t\to 0+} \frac{\varphi(t)}{t} = 0$,

$$\mathcal{B}^{\varphi} = \{f \in H(D) : \sup_{z\in D}(1 - |z|^2)\varphi(\lambda|f'(z)|) < \infty\}$$

for some $\lambda > 0$ depending on f.

It is easy to see that $\mathcal{B}^{\varphi}$ is a Banach space with the norm

$$\|f\|_{\mathcal{B}^{\varphi}} = |f(0)| + \|f\|_{\varphi},$$

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where $\|f\|_\varphi = \inf\{k > 0 : S_\varphi(\frac{f'}{k}) \le 1\}$ is a Minkowski's functional and $S_\varphi(f) = \sup_{z\in\mathbb{D}}(1 - |z|^2)\varphi(|f(z)|)$.

This paper is organized as follows: In section 2, we use Young's function to define the Zygmund-Orlicz space, as a generalization of Zygmund space. The spaces is defined in a similar way as Korenblem-Orlicz space and Bloch-Orlicz space in [1, 2, 5]. We study some of its properties and show that the Zygmund-Orlicz space is isometrically equal to certain μ-Zygmund space for a very special weight μ. In section 3, we characterize boundedness and compactness of composition operator on Zygmund-Orlicz space.

Throughout the rest of this paper, C will denote a finite positive constant, and it may differ from one occurrence to the other.

2. Zygmund-Orlicz spaces

In this section, we define the Zygmund-Orlicz space Z^φ using Young's function. More precisely, Z^φ is the class of all analytic functions f in $\mathbb{D}$ such that

$$\sup_{z\in D}(1 - |z|^2)\varphi(\lambda|f''(z)|) < \infty$$

for some $\lambda > 0$ depending on f. The set Z^φ is an F-space which we call Zygmund-Orlicz space associated to the function φ. We can observe that when $\varphi(t) = t$ with $t \ge 0$, we get back the Zygmund spaces Z.

It is not hard to see that

$$\|f\|_\varphi = \inf\{k > 0 : S_\varphi(\frac{f''}{k}) \le 1\}$$

define a seminorm for Z^φ, where $S_\varphi(f) = \sup_{z\in\mathbb{D}}(1 - |z|^2)\varphi(|f(z)|)$.

In fact, it can be show that Z^φ is a Banach space with the norm

$$\|f\|_{Z^\varphi} = |f(0)| + |f'(0)| + \|f\|_\varphi. \tag{2.1}$$

Also, we can observe that for any $f \in Z^\varphi \setminus \{0\}$, the following relation

$$S_\varphi(\frac{f''}{\|f\|_{Z^\varphi}}) \le S_\varphi(\frac{f''}{\|f\|_\varphi}) \le 1 \tag{2.2}$$

holds. The inequality above allow us to obtain that

$$|f''(z)| \le \varphi^{-1}(\frac{1}{1 - |z|^2})\|f\|_\varphi$$

for all $f \in Z^\varphi$ and for all $z \in \mathbb{D}$. Furthermore, we have

$$|f'(z)| \le |f'(0)| + |\int_0^{|z|} |f''(\zeta)||d\zeta| \le (1 + \int_0^1 \varphi^{-1}(\frac{1}{1 - |z|^2 t^2})dt)\|f\|_{Z^\varphi}. \tag{2.3}$$

Lemma 1. *The Zygmund-Orlicz space is isometrically equal to μ-Zygmund space, where*

$$\mu(z) = \frac{1}{\varphi^{-1}(\frac{1}{1-|z|^2})}$$

with $z \in \mathbb{D}$. Thus, for any $f \in Z^\varphi$, we have $\|f\|_\varphi = \|f\|_\mu = \sup_{z\in\mathbb{D}} \mu(z)|f''(z)|$.

Proof. From (2.2), for any $f \in Z^\varphi \setminus \{0\}$ and any $z \in \mathbb{D}$, we have

$$(1 - |z|^2)\varphi(\frac{|f''(z)|}{\|f\|_\varphi}) \le 1$$

which implies that $\mu(z)|f''(z)| \le \|f\|_\varphi$ for all $z \in \mathbb{D}$. Thus $Z^\varphi \subset Z^\mu$ and $\|f\|_\mu \le \|f\|_\varphi$.

Conversely, if $f \in Z^\mu$, then $\mu(z)|f''(z)| \le \|f\|_\mu$, for all $z \in \mathbb{D}$. From here, we have $S_\varphi(\frac{f''}{\|f\|_\varphi}) \le 1$. Thus, $f \in Z^\varphi$ and $\|f\|_\varphi \le \|f\|_\mu$. □

The following result will be very useful in the next section and it is a version of Lemma 6 in [5]. for completeness, we include an outline of its proof.

Lemma 2. *Let $a \in \mathbb{D}$ fixed. There exists a holomorphic function $f_a \in H(\mathbb{D})$, such that*

$$\varphi(|f'_a(z)|) = \frac{1-|a|^2}{|1-\bar{a}z|^2}$$

for all $z \in \mathbb{D}$.

Proof. For $z \in \mathbb{D}$, we set $u(z) = \varphi^{-1}(\frac{1-|a|^2}{|1-\bar{a}z|^2})$, then u is a real and continuously differentiable function. Therefore, its partial derivatives exist and are continuous throughout $\mathbb{D}$. It is clear that function u satisfies $u(z) \geq \varphi^{-1}(\frac{1}{|1-|a|^2}) > 0$ for all $z \in \mathbb{D}$. Now we set $f'_a(z) = u(z)e^{iv(z)}$ where v is a real function defined on $\mathbb{D}$. Then, in order for f'_a to be an analytic function on $\mathbb{D}$, its real parts $U(z) = u(z)\cos v(z)$ and its imaginary parts $V(z) = u(z)\sin v(z)$ must satisfy the Cauchy-Riemann equations. We get the relation $uv_x = -u_y$ and $uv_y = u_x$. We can choose a C^1 real function v defined on $\mathbb{D}$ such that f'_a is an analytic function on $\mathbb{D}$ satisfying $\varphi(|f'_a(z)|) = \frac{1-|a|^2}{|1-\bar{a}z|^2}$. Of course, $f_a(z) = \int_0^z f'_a(\zeta)d\zeta + f_a(0)$ is an analytic function on $\mathbb{D}$, too. □

Remark. It is clear that for any $a \in \mathbb{D}$, the function $g_a(z) = \int_0^z f_a(s)ds$ with $z \in \mathbb{D}$ and f_a is the function found in Lemma 2.2, belongs to the space Z^φ.

$$S(g''_a) = \sup_{z\in\mathbb{D}}(1-|z|^2)\frac{1-|a|^2}{|1-\bar{a}z|^2} = \sup_{z\in\mathbb{D}}(1-|\sigma_a(z)|^2) = 1 \tag{2.4}$$

where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ denote the automorphism of the disk $\mathbb{D}$. From (2.4), we get $\|g_a\|_\varphi = 1$ for all $a \in \mathbb{D}$.

Lemma 3. *The composition opreator C_ϕ is compact on Z^φ if and only if given a bounded sequence $\{f_n\}$ in Z^φ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $\|C_\phi f_n\|_\varphi \to 0$ as $n \to \infty$.*

3. Main results

Theorem 1. *Let ϕ be a holomorphic self-map of $\mathbb{D}$. Then $C_\phi : Z^\varphi \to Z^\varphi$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)}{\mu(\phi(z))}|\phi'(z)|^2 < \infty \quad \textit{and}$$

$$\sup_{z\in\mathbb{D}} \mu(z)|\phi''(z)| \left(1 + \int_0^1 \varphi^{-1}(\frac{1}{1-|\phi(z)t|^2})dt\right) < \infty.$$

Proof. Suppose that

$$L_1 = \sup_{z\in\mathbb{D}} \frac{\mu(z)}{\mu(\phi(z))}|\phi'(z)|^2 < \infty,$$

$$L_2 = \sup_{z\in\mathbb{D}} \mu(z)|\phi''(z)| \left(1 + \int_0^1 \varphi^{-1}(\frac{1}{1-|\phi(z)t|^2})\right) dt < \infty. \tag{3.5}$$

Then for all $f \in Z^\varphi \setminus \{0\}$, We have the following estimate

$$\begin{aligned}
S_\varphi\Big(\frac{(f\circ\phi)''(z)}{(L_1+L_2)\|f\|_{Z^\varphi}}\Big) &= \sup_{z\in\mathbb{D}}(1-|z|^2)\varphi\Big(\frac{|f''(\phi(z))\phi'^2(z) + f'(\phi(z))\phi''(z)|}{(L_1+L_2)\|f\|_{Z^\varphi}}\Big) \\
&\leq \sup_{z\in\mathbb{D}}(1-|z|^2)\varphi\Big(\frac{|f''(\phi(z))||\phi'(z)|^2 + |f'(\phi(z))||\phi''(z)|}{(L_1+L_2)\|f\|_{Z^\varphi}}\Big) \\
&\leq \sup_{z\in\mathbb{D}}(1-|z|^2)\varphi\Big(\frac{\frac{|\phi'(z)|^2}{\mu(\phi(z))}\|f\|_\varphi + (1+\int_0^1\varphi^{-1}(\frac{1}{1-|\phi(z)t|^2})dt)|\phi''(z)|\|f\|_{Z^\varphi}}{(L_1+L_2)\|f\|_{Z^\varphi}}\Big) \\
&\leq \sup_{z\in\mathbb{D}}(1-|z|^2)\varphi\Big(\frac{1}{\mu(z)}\frac{(L_1+L_2)\|f\|_{Z^\varphi}}{(L_1+L_2)\|f\|_{Z^\varphi}}\Big) \\
&= \sup_{z\in\mathbb{D}}(1-|z|^2)\varphi\Big(\frac{1}{\mu(z)}\Big) = 1
\end{aligned}$$

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where we have used the relations (2.1) (2.2) (2.3) and Lemma 2.1. From here, we can conclude that $\|C_\phi f\|_\varphi \leq (L_1 + L_2)\|f\|_{Z^\varphi}$. Moreover from (3.5), take $z = 0$, we have

$$\varphi^{-1}(1)|\phi''(0)|\Big(1 + \int_0^1 \varphi^{-1}(\frac{1}{1-|\phi(z)t|^2})dt\Big) < \infty,$$

and therefore

$$1 + \int_0^1 \varphi^{-1}(\frac{1}{1-|\phi(0)t|^2})dt < \infty. \tag{3.6}$$

Hence with (2.1) (2.3)

$$\begin{aligned} |f(\phi(0))| + |f'(\phi(0))\phi'(0)| &\leq |f(0)| + \int_0^{|\phi(0)|} |f'(\xi)||d\xi| + |f'(0)| + \int_0^{|\phi(0)|} |f''(\zeta)||d\zeta| \\ &\leq \|f\|_{Z^\varphi} + 2(1 + \int_0^{|\phi(0)|} |f''(\zeta)||d\zeta|)\|f\|_{Z^\varphi} \\ &= \|f\|_{Z^\varphi} + 2(1 + \int_0^1 \varphi^{-1}(\frac{1}{1-|\phi(0)t|^2})dt)\|f\|_{Z^\varphi}. \end{aligned} \tag{3.7}$$

Combined with (3.6) (3.7), we have $\|C_\phi f\|_{Z^\varphi} \leq C\|f\|_{Z^\varphi}$ for all $f \in Z^\varphi$, and $C_\phi : Z^\varphi \to Z^\varphi$ is bounded.

Conversely, suppose that there exists a constant $C > 0$ such that

$$\|f \circ \phi\|_\varphi \leq C\|f\|_\varphi$$

for all $f \in Z^\varphi$. By taking the function $f(z) = z \in Z^\varphi$, we obtain

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\varphi(\frac{|(C_\phi f)''(z)|}{k}) = \sup_{z\in\mathbb{D}}(1-|z|^2)\varphi(\frac{|\phi''(z)|}{k}) < \infty.$$

That is

$$\sup_{z\in\mathbb{D}} \mu(z)|\phi''(z)| < \infty. \tag{3.8}$$

For $a \in \mathbb{D}$, set $g_a(z) = \int_0^z f_a(s)ds$ and from the remark , we see that $g_a \in Z^\varphi$ and $\|g_a\|_\varphi = 1$. With (2.2), we have that

$$1 \geq S_\varphi(\frac{(g_a \circ \phi)''(z)}{C\|g_a\|_\varphi}) = \sup_{z\in\mathbb{D}}(1-|z|^2)\varphi(\frac{|f_a'(\phi(z))\phi'^2(z) + f_a(\phi(z))\phi''(z)|}{C}).$$

Hence

$$\begin{aligned} &\mu(z)|f_a'(\phi(z))||\phi'(z)|^2 - \mu(z)|f_a(\phi(z))||\phi''(z)| \\ &\leq \mu(z)|f_a'(\phi(z))\phi'^2(z) + f_a(\phi(z))\phi''(z)| \leq C, \end{aligned}$$

or

$$\mu(z)|f_a'(\phi(z))||\phi'(z)|^2 \leq C + \mu(z)|f_a(\phi(z))||\phi''(z)|.$$

For $g_a \in Z^\varphi$, we have $g_a' \in \mathcal{B}^\varphi$. That is to say $f_a \in \mathcal{B}^\varphi$ and from Lemma 2.1 we get $f_a \in \mathcal{B}_\mu$, where $\mu(z) = \frac{1}{\varphi^{-1}(\frac{1}{1-|z|^2})}$. Thus, from [11], we have

$$|f_a(z)| \leq C(1 + \int_0^{|z|} \frac{1}{\mu(t)}dt)\|f_a\|_{\mathcal{B}_\mu}. \tag{3.9}$$

Hence

$$\sup_{z\in\mathbb{D}} \mu(z)|f_a'(\phi(z))||\phi'(z)|^2 \leq C + C\mu(z)(1 + \int_0^{|\phi(z)|} \frac{1}{\mu(t)}dt)|\phi''(z)|\|f_a\|_{\mathcal{B}_\mu}. \tag{3.10}$$

It is obvious that $\sup_{z\in\mathbb{D}} \mu(z)|f_a'(\phi(z))||\phi'(z)|^2 < \infty$ when $|\phi(z)| \leq \frac{1}{\sqrt{2}}$.

Now for $\frac{1}{\sqrt{2}} < |\phi(z)| < 1$, fix $a \in \mathbb{D}$, we set

$$h_a(z) = (1-|a|^2)\int_0^z (\int_0^{\bar{a}\zeta} h(t)dt - \frac{1}{2}\frac{(\int_0^{\bar{a}\zeta} h(t)dt)^2}{\int_0^{|a|^2} h(t)dt})d\zeta,$$

where $h(t) = \frac{1}{\mu(t)}$. Then

$$h'_a(z) = (1-|a|^2)\Big(\int_0^{\bar{a}z} h(t)dt - \frac{1}{2}\frac{(\int_0^{\bar{a}z} h(t)dt)^2}{\int_0^{|a|^2} h(t)dt}\Big),$$

$$h''_a(z) = (1-|a|^2)\Big(\bar{a}h(\bar{a}z) - \frac{\bar{a}h(\bar{a}z)\int_0^{\bar{a}z} h(t)dt}{\int_0^{|a|^2} h(t)dt}\Big).$$

It is easy to see that $h'_a(a) = \frac{1-|a|^2}{2}\int_0^{|a|^2} h(t)dt$ and $h''_a(a) = 0$. From above, we can see that $h_a \in Z^\varphi$, and $\|h_a\|_\varphi \le C$. Therefore, $\|C_\phi h_a\|_\varphi \le C$. Hence, we have

$$\begin{aligned} &\sup_{z\in\mathbb{D}}(1-|z|^2)\varphi\Big(\frac{|(C_\phi h_a)''(z)|}{C}\Big) \\ = \ &\sup_{z\in\mathbb{D}}(1-|z|^2)\varphi\Big(\frac{|h''_a(\phi(z))||\phi'(z)|^2 + |h'_a(\phi(z))||\phi''(z)|}{C}\Big) < \infty. \end{aligned}$$

From above, put $a = \phi(z)$, we obtain

$$\sup_{z\in\mathbb{D}} \mu(z)|\phi''(z)|\int_0^{|\phi(z)|^2}\frac{1}{\mu(t)}dt \le C.$$

Combined with the boundedness of ϕ, (3.8) and the following inequality of [12]

$$\int_0^{|\phi(z)|^2}\frac{1}{\mu(t)}dt \le \int_0^{|\phi(z)|}\frac{1}{\mu(t)}dt \le C + C\int_0^{|\phi(z)|^2}\frac{1}{\mu(t)}dt, \tag{3.11}$$

we have

$$\begin{aligned} &\sup_{z\in\mathbb{D}} \mu(z)|\phi''(z)||\phi(z)|\int_0^1 \varphi^{-1}\Big(\frac{1}{1-|\phi(z)t|^2}\Big)dt \\ \le \ &\sup_{z\in\mathbb{D}} \mu(z)|\phi''(z)|\int_0^{|\phi(z)|}\frac{1}{\mu(t)}dt \\ \le \ &\sup_{z\in\mathbb{D}} \mu(z)|\phi''(z)|\Big(C + C\int_0^{|\phi(z)|^2}\frac{1}{\mu(t)}dt\Big) < \infty \end{aligned} \tag{3.12}$$

Hence with (3.10), we have

$$\sup_{z\in\mathbb{D}} \mu(z)|f'_a(\phi(z))||\phi'(z)|^2 = \sup_{z\in D}\frac{\mu(z)}{\mu(\phi(z))}|\phi'(z)|^2 \le C,$$

$$\sup_{z\in\mathbb{D}} \mu(z)|\phi''(z)|\Big(1 + \int_0^1 \varphi^{-1}\Big(\frac{1}{1-|\phi(z)t|^2}\Big)dt\Big) \le C.$$

This completes the proof. □

Theorem 2. *Let ϕ be a holomorphic self-map of $\mathbb{D}$. Then $C_\phi : Z^\varphi \to Z^\varphi$ is compact if and only if C_ϕ is bounded and*

$$\lim_{|\phi(z)|\to 1}\frac{\mu(z)}{\mu(\phi(z))}|\phi'(z)|^2 = 0, \tag{3.13}$$

$$\lim_{|\phi(z)|\to 1}\mu(z)|\phi''(z)|\Big(1 + \int_0^1 \varphi^{-1}\Big(\frac{1}{1-|\phi(z)t|^2}\Big)dt\Big) = 0. \tag{3.14}$$

Proof. Suppose first that C_ϕ is bounded and (3.5) holds. Let $\{f_n\}$ be a bounded sequence in Z^φ converging to 0 uniformly on compact subsets of $\mathbb{D}$. Then, by Lemma 2.3, it is sufficient to show that$\|C_\phi f_n\|_\varphi \to 0$ as $n \to \infty$. From (3.13) and (3.14), for $\varepsilon_1, \varepsilon_2 > 0$, we can find an $r \in (0, 1)$ such that

$$\frac{\mu(z)}{\mu(\phi(z))}|\phi'(z)|^2 < \varepsilon_1 \quad \textit{and} \quad \mu(z)|\phi''(z)|\Big(1 + \int_0^1 \varphi^{-1}\Big(\frac{1}{1-|\phi(z)t|^2}\Big) < \varepsilon_2,$$

whenever $r < |\phi(z)| < 1$. From here, we have that

$$\begin{aligned}\mu(z)|(C_\phi f_n)''(z)| &\leq \mu(z)|f_n''(\phi(z)||\phi'(z)|^2 + |f_n'(\phi(z))||\phi''(z)|)\\ &\leq \frac{\mu(z)}{\mu(\phi(z))}|\phi'(z)|^2\|f_n\|_\varphi\\ &+ \mu(z)|\phi''(z)|(1+\int_0^1 \varphi^{-1}(\frac{1}{1-|\phi(z)t|^2})dt\|f_n\|_{Z^\varphi}\\ &< \|f_n\|_\varphi\varepsilon_1 + \|f_n\|_{Z^\varphi}\varepsilon_2. \end{aligned} \tag{3.15}$$

On the other hand, since $\{f_n\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$, $\sup\limits_{|\phi(r)|\leq r}|f_n''(\phi(z))| \to 0$, and $\sup\limits_{|\phi(r)|\leq r}|f_n'(\phi(z))| \to 0$ as $n\to\infty$. From the boundedness of C_ϕ, set $f(z) = z \in Z^\varphi$ and $f(z) = z^2 \in Z^\varphi$, we have

$$M_1 = \sup_{z\in D}\mu(z)|\phi'(z)|^2 < \infty, \quad M_2 = \sup_{z\in D}\mu(z)|\phi''(z)| < \infty. \tag{3.16}$$

Hence, we have

$$\begin{aligned} &\sup_{|\phi(r)|\leq r}\mu(z)|(C_\phi f_n)''(z)|\\ \leq &\sup_{|\phi(r)|\leq r}\mu(z)|\phi'(z)|^2|f_n''(\phi(z))| + \sup_{|\phi(r)|\leq r}\mu(z)|\phi''(z)||f_n'(\phi(z))|\\ \leq &M_1\sup_{|\phi(r)|\leq r}|f_n''(\phi(z))| + M_2\sup_{|\phi(r)|\leq r}|f_n'(\phi(z))| \to 0\end{aligned}$$

With (3.15), we obtain that C_ϕ is a compact operator on Z^φ.

Suppose that $C_\phi : Z^\varphi \to Z^\varphi$ is compact. It is clear that $C_\phi : Z^\varphi \to Z^\varphi$ is bounded. Let $\{z_n\}$ be a sequence in $\mathbb{D}$ such that $|w_n| = |\phi(z_n)| \to 1$ as $n\to\infty$. Set

$$g_n(z) = \int_0^z (f_{w_n}(s) - f_{w_n}(0))ds$$

then $g_n \in Z^\varphi$ and $\|g_n\|_\varphi = 1$. Furthermore, $g_n \to 0$ uniformly on compact subsets of D as $n\to\infty$. Hence

$$\begin{aligned} 0 &\leftarrow \|C_\phi g_n\|_\mu \geq \mu(z_n)|g_n''(\phi(z_n))\phi'^2(z_n) + g_n'(\phi(z_n))\phi''(z_n)|\\ &\geq \mu(z_n)|f'_{\phi(z_n)}(\phi(z_n))||\phi'(z_n)|^2\\ &- \mu(z_n)|f_{\phi(z_n)}(\phi(z_n)) - f_{\phi(z_n)}(0)||\phi''(z_n)|. \end{aligned} \tag{3.17}$$

Since C_ϕ is compact operator on Z^φ, set

$$s_{w_n}(z) = \frac{1}{\ln(1-|w_n|^2)}\int_0^z \Big(\frac{(\ln(1-\bar{w_n}\zeta))^2}{2\ln(1-|w_n|^2)} - \ln(1-\bar{w_n}\zeta)\Big)d\zeta.$$

It is easy to see that $s'_{w_n}(w_n) = -\frac{1}{2}$ and $s''_{w_n}(w_n) = 0$. So $s_{w_n} \in Z^\varphi$ and $s_{w_n} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n\to\infty$. For $\varepsilon_3 > 0$, we have

$$\varepsilon_3 > \|C_\phi s_{w_n}\|_\mu = \frac{1}{2}\sup_{z\in D}\mu(z_n)|\phi''(z_n)|,$$

which means

$$\lim_{n\to\infty}\mu(z_n)|\phi''(z_n)| = 0. \tag{3.18}$$

Set

$$h_n(z) = (1-|w_n|^2)\int_0^z(\int_0^{\bar{w_n}\zeta}h(t)dt - \frac{1}{2}\frac{(\int_0^{\bar{w_n}\zeta}h(t)dt)^2}{\int_0^{|w_n|^2}h(t)dt})d\zeta,$$

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then $\{h_n\} \subset Z^{\varphi}$ and $\{h_n\}$ is a sequence converging to 0 uniformly on compact subsets of $\mathbb{D}$. Furthermore, for $\varepsilon_4 > 0$, we have

$$\begin{aligned} & \sup_{z_n\in\mathbb{D}}(1-|z_n|^2)\varphi\Big(\frac{|(C_\phi h_n)''(z_n)|}{k}\Big) \\ = & \sup_{z_n\in\mathbb{D}}(1-|z_n|^2)\varphi\Big(\frac{|h_n''(\phi(z_n))\phi'^2(z_n)+h_n'(\phi(z_n))\phi''(z_n)|}{k}\Big) \\ = & \sup_{z_n\in\mathbb{D}}(1-|z_n|^2)\varphi\Big(\frac{\frac{1-|\phi(z_n)|^2}{2}|\phi''(z_n)|\int_0^{|\phi(z_n)|^2}\frac{1}{\mu(t)}dt}{k}\Big) < \varepsilon_4. \end{aligned}$$

That is to say, for $\varepsilon_4 > 0, \exists N$, such that $\frac{1}{\sqrt{2}} < |\phi(z_n)| < 1$ whenever $n > N$, we have

$$\frac{1-|\phi(z_n)|^2}{2}\mu(z_n)|\phi''(z_n)|\int_0^{|\phi(z_n)|^2}\frac{1}{\mu(t)}dt < \varepsilon_4. \tag{3.19}$$

Hence, with (3.18) (3.19), we obtain that

$$\begin{aligned} \mu(z_n)|\phi''(z_n)|\int_0^{|\phi(z_n)|}\frac{1}{\mu(t)}dt & \le \mu(z_n)|\phi''(z_n)|(C_1+C_2\int_0^{|\phi(z_n)|^2}\frac{1}{\mu(t)}dt) \\ & \le C\varepsilon_3 + C\frac{1-|\phi(z_n)|^2}{2}\mu(z_n)|\phi''(z_n)|\int_0^{|\phi(z_n)|^2}\frac{1}{\mu(t)}dt \\ & \le C\varepsilon_3 + C\varepsilon_4. \end{aligned} \tag{3.20}$$

Therefore, with the boundedness of ϕ and (3.18) (3.20), we have

$$\lim_{|\phi(z)|\to 1}\mu(z)|\phi''(z)|(1+\int_0^1\varphi^{-1}(\frac{1}{1-|\phi(z)t|^2})dt) = 0$$

and (3.14) hold. Moreover, with (3.17) we have that

$$\begin{aligned} \mu(z_n)|f_{w_n}(w_n)-f_{w_n}(0)||\phi''(z_n)| & \le \mu(z_n)|\phi''(z_n)|(1+\int_0^{|w_n|}\frac{1}{\mu(t)}dt)\|f_{w_n}-f_{w_n}(0)\|_{\mathcal{B}_\mu} \\ & \le (C\varepsilon_3+C\varepsilon_4)\|f_{w_n}-f_{w_n}(0)\|_{\mathcal{B}_\mu}. \end{aligned} \tag{3.21}$$

From (3.17) (3.21), we obtain that

$$\begin{aligned} \frac{\mu(z_n)}{\mu(\phi(z_n))}|\phi'(z_n)|^2 & = \mu(z_n)|f_{w_n}'(w_n)||\phi'(z_n)|^2 \\ & \le \|C_\phi g_n\|_\mu + \mu(z_n)|f_{w_n}(w_n)-f_{w_n}(0)||\phi''(z_n)| \\ & \le \|C_\phi g_n\|_\mu + (C\varepsilon_3+C\varepsilon_4)\|f_{w_n}-f_{w_n}(0)\|_{\mathcal{B}_\mu}. \end{aligned}$$

This implies that

$$\lim_{|\phi(z)|\to 1}\frac{\mu(z)}{\mu(\phi(z))}|\phi'(z)|^2 = 0,$$

and (3.13) holds. This completes the proof. □

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Multiple positive solutions for m-point boundary value problems with one-dimensional p-Laplacian systems and sign changing nonlinearity *

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Abstract: In this paper, we consider the multipoint boundary value problem for the one-dimensional p-Laplacian system

$$\begin{cases} (\phi_{p_1}(u'))' + q_1(t)f(t,u,v) = 0, \ t \in (0,1), \\ (\phi_{p_2}(v'))' + q_2(t)g(t,u,v) = 0, \ t \in (0,1), \end{cases}$$

$$\begin{cases} u(0) = \sum\limits_{i=1}^{m-2} a_i u(\xi_i), \ u'(1) = \beta u'(0), \\ v(0) = \sum\limits_{i=1}^{m-2} a_i v(\xi_i), \ v'(1) = \beta v'(0), \end{cases}$$

where $\phi_{p_i}(s) = |s|^{p_i-2}s$, $p_i > 1$, $i = 1,2$, $\xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ and $a_i \in [0,1)$, $\sum\limits_{i=1}^{m-2} a_i < 1$, $\beta \in (0,1)$. By using the fixed point index theorem on cones, we study the existence of positive solutions for the m-point boundary value problem with sign changing nonlinear term. Some sufficient conditions for the existence of multiple positive solutions are obtained. Finally, an example is also included to illustrate the importance of the main result obtained.

Keywords: Multipoint boundary value problem, Fixed point theorem, Cone, Positive solution, One-dimensional p-Laplacian.

2010 MR Subject Classification: 34B10, 34B15, 34B18

1 Introduction

In this paper, we study the existence of multiple positive solutions to the boundary value problem (BVP for short) for the one-dimensional p-Laplacian system

$$\begin{cases} (\phi_{p_1}(u'))' + q_1(t)f(t,u,v) = 0, \ t \in (0,1), \\ (\phi_{p_2}(v'))' + q_2(t)g(t,u,v) = 0, \ t \in (0,1), \end{cases} \tag{1.1}$$

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$$\begin{cases} u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i),\ u'(1) = \beta u'(0), \\ v(0) = \sum_{i=1}^{m-2} a_i v(\xi_i),\ v'(1) = \beta v'(0), \end{cases} \tag{1.2}$$

where $\phi_{p_i}(s) = |s|^{p_i-2}s,\ p_i > 1,\ i = 1, 2,\ \xi_i \in (0, 1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$.

Multipoint boundary value problems of ordinary differential equations arise in a variety of areas of applied mathematics and physics. For example, the vibrations of a guy wire of a uniform cross-section and composed of N parts of different densities can be set up as a multipoint boundary value problem (see [10]). The study of multipoint boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [2]. Since then there has been much current attention focused on the study of nonlinear multipoint boundary value problems, see ([1, 3, 4, 5, 6, 8, 9, 11, 12, 13, 14, 15]).

Karakostas [4] proved the existence of positive solutions for the two-point boundary value problem

$$x''(t) - \text{sign}(1-\alpha)q(t)f(x, x')x' = 0,\ t \in (0, 1),$$

with one of the following sets of boundary conditions:

$$x(0) = 0,\ x'(1) = \alpha x'(0),$$

or

$$x(1) = 0,\ x'(1) = \alpha x'(0),$$

where $\alpha > 0,\ \alpha \neq 1$. By using indices of convergence of the nonlinearities at 0 and at $+\infty$, they provided a priori upper and lower bounds for the slope of the solutions.

Ma [11] proved the existence of positive solutions for the multipoint boundary value problem

$$x''(t) - q(t)f(x, x')x' = 0,\ t \in (0, 1),$$

$$x(0) = \sum_{i=1}^{n-2} b_i x(\xi_i),\ x'(1) = \alpha x'(0),$$

where $\xi_i \in (0, 1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1,\ b_i \in [0, 1),\ \alpha > 1$. They provided sufficient conditions for the existence of multiple positive solutions to the above BVP by applying the fixed point theorem in cones.

Recently, Ji [3] investigated the following m-point boundary value problem

$$(\phi_p(u'))' + q(t)f(t, u) = 0,\ t \in (0, 1),$$

$$u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i),\ u(1) = \sum_{i=1}^{m-2} \beta_i u(\xi_i).$$

They obtained sufficient conditions that guarantee the existence of positive solutions by using fixed point theorems on cones.

Motivated by these results, our purpose of this paper is to show the existence of multiple positive solutions to multipoint BVP (1.1), (1.2). To date no paper has appeared in the literature which discusses the multipoint boundary value problem for one-dimensional p-Laplacian systems when nonlinearity in the differential equation may change sign. This paper attempts to fill this gap in the literature. The interesting point of this paper is the nonlinear terms f and g may change sign.

For convenience, we list the following assumptions:

(H_1) $a_i \in [0,1)$ satisfies $\sum_{i=1}^{m-2} a_i < 1,\ \beta \in (0,1)$;
(H_2) $f,\ g \in C([0,1] \times [0,+\infty) \times [0,+\infty),\ (-\infty,+\infty))$;
(H_3) $q_1,\ q_2 \in L^1[0,1]$ are nonnegative on $(0,1)$ and $q_1,\ q_2$ are not identically zero on any subinterval of $(0,1)$. Furthermore, $q_1,\ q_2$ satisfy $0 < \int_0^1 q_1(t)dt < +\infty,\ 0 < \int_0^1 q_2(t)dt < +\infty$.

2 Preliminaries

For the convenience of readers, we provide some background material from the theory of cones in Banach spaces. We also state in this section the fixed point index theorem on cones.
Definition 2.1. Let E be a real Banach space over R. A nonempty closed set $K \subset E$ is said to be a cone provide that
(i) $au + bv \in K$ for all $u, v \in K$ and all $a \geq 0, b \geq 0$, and
(ii) $u, -u \in K$ implies $u = 0$.
Every cone $K \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in K$.

Definition 2.2. The map α is said to be a nonnegative continuous concave functional on a cone K of a real Banach space E provided that $\alpha : K \to [0,\infty)$ is continuous and
$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$
for all $x,\ y \in K$ and $0 \leq t \leq 1$. Similarly, we say the map γ is a nonnegative continuous convex functional on a cone K of a real Banach space E provided that $\gamma : P \to [0,\infty)$ is continuous and
$\gamma(tx + (1-t)y) \leq t\gamma(x) + (1-t)\gamma(y)$
for all $x,\ y \in K$ and $0 \leq t \leq 1$.

To prove our results, the following fixed point theorem in cones is fundamental.

Theorem 2.1. ([7]) Let K be a cone in a real Banach space E. Let D be an open bounded subset of E with $D_k = D \cap K \neq \emptyset$. Assume that $A : \overline{D}_k \to K$ is completely continuous such that $A \neq Ax$ for $x \in D_k K$. Then the following results hold:
(1) If $\|Ax\| \leq \|x\|,\ x \in \partial D_k$, then $i_k(A, D_k) = 1$.
(2) If there exists $e \in K\backslash\{0\}$ such that $x \neq Ax + \lambda e$ for all $x \in \partial D_k$ and all $\lambda > 0$, then $i_k(A, D_k) = 0$.
(3) Let U be open in X such that $\overline{U} \subset D_k$. If $i_k(A, D_k) = 1$ and $i_k(A, U_k) = 0$, then A has a fixed point in $D_k \backslash \overline{U}_k$. The same result holds if $i_k(A, D_k) = 0$ and $i_k(A, U_k) = 1$.

3 Related lemmas

In this paper, we denote $C^+[0,1] = \{x \in C[0,1] : x(t) \geq 0,\ t \in [0,1]\}$. $\phi_{p_1}^{-1},\ \phi_{p_2}^{-1}$ are, respectively, the inverse function to $\phi_{p_1},\ \phi_{p_2}$.

Let $E = C[0,1] \times C[0,1]$, define norm $\|(u,v)\| = \|u\| + \|v\|$, where $\|u\| = \max_{t\in[0,1]} |u(t)|$, $\|v\| = \max_{t\in[0,1]} |v(t)|$, then E is a Banach space.

Define the cone $K \subset E$ by
$K = \{(u,v) \in E \mid u(t),\ v(t) \geq 0,\ u \text{ and } v \text{ are concave and nondecreasing on } [0,1]\}$.
Lemma 3.1. Assume that $(H_1) - (H_3)$ hold. Then, For any $x,\ y \in C^+[0,1]$, the problem

$$\begin{cases} (\phi_{p_1}(u'))' + q_1(t)f(t,x,y) = 0,\ t \in (0,1), \\ (\phi_{p_2}(v'))' + q_2(t)g(t,x,y) = 0,\ t \in (0,1), \end{cases} \tag{3.1}$$

$$\begin{cases} u(0)=\sum\limits_{i=1}^{m-2} a_i u(\xi_i),\ u'(1)=\beta u'(0), \\ v(0)=\sum\limits_{i=1}^{m-2} a_i v(\xi_i),\ v'(1)=\beta v'(0), \end{cases} \tag{3.2}$$

has the unique solution (u, v) as

$$\begin{aligned} u(t) = & \int_0^t \phi_{p_1}^{-1}\left(\int_s^1 q_1(\tau)f(\tau,x(\tau),y(\tau))d\tau + \frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)f(\tau,x(\tau),y(\tau))d\tau\right)ds \\ & + \frac{1}{1-\sum\limits_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_{p_1}^{-1}\left(\int_s^1 q_1(\tau)f(\tau,x(\tau),y(\tau))d\tau\right. \\ & \left. + \frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)f(\tau,x(\tau),y(\tau))d\tau\right)ds, \end{aligned} \tag{3.3}$$

$$\begin{aligned} v(t) = & \int_0^t \phi_{p_2}^{-1}\left(\int_s^1 q_2(\tau)g(\tau,x(\tau),y(\tau))d\tau + \frac{\phi_{p_2}(\beta)}{1-\phi_{p_2}(\beta)}\int_0^1 q_2(\tau)g(\tau,x(\tau),y(\tau))d\tau\right)ds \\ & + \frac{1}{1-\sum\limits_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_{p_2}^{-1}\left(\int_s^1 q_2(\tau)g(\tau,x(\tau),y(\tau))d\tau\right. \\ & \left. + \frac{\phi_{p_2}(\beta)}{1-\phi_{p_2}(\beta)}\int_0^1 q_2(\tau)g(\tau,x(\tau),y(\tau))d\tau\right)ds. \end{aligned} \tag{3.4}$$

Proof. For any $x, y \in C^+[0,1]$, suppose (u, v) is a solution of BVP (3.1), (3.2). By integration of (3.1), it follows that

$$\begin{aligned} u'(t) &= \phi_{p_1}^{-1}\left(\phi_{p_1}(u'(0)) - \int_0^t q_1(\tau)f(\tau,x(\tau),y(\tau))d\tau\right), \\ u(t) &= u(0) + \int_0^t \phi_{p_1}^{-1}\left(\phi_{p_1}(u'(0)) - \int_0^s q_1(\tau)f(\tau,x(\tau),y(\tau))d\tau\right)ds. \end{aligned}$$

Using the boundary condition (3.2), we can easily have

$$\begin{aligned} u(t) = & \int_0^t \phi_{p_1}^{-1}\left(\int_s^1 q_1(\tau)f(\tau,x(\tau),y(\tau))d\tau + \frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)f(\tau,x(\tau),y(\tau))d\tau\right)ds \\ & + \frac{1}{1-\sum\limits_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_{p_1}^{-1}\left(\int_s^1 q_1(\tau)f(\tau,x(\tau),y(\tau))d\tau\right. \\ & \left. + \frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)f(\tau,x(\tau),y(\tau))d\tau\right)ds. \end{aligned}$$

In a similar way, we can prove

$$\begin{aligned} v(t) = & \int_0^t \phi_{p_2}^{-1}\left(\int_s^1 q_2(\tau)g(\tau,x(\tau),y(\tau))d\tau + \frac{\phi_{p_2}(\beta)}{1-\phi_{p_2}(\beta)}\int_0^1 q_2(\tau)g(\tau,x(\tau),y(\tau))d\tau\right)ds \\ & + \frac{1}{1-\sum\limits_{i=1}^{m-2} a_i}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi_{p_2}^{-1}\left(\int_s^1 q_2(\tau)g(\tau,x(\tau),y(\tau))d\tau\right. \\ & \left. + \frac{\phi_{p_2}(\beta)}{1-\phi_{p_2}(\beta)}\int_0^1 q_2(\tau)g(\tau,x(\tau),y(\tau))d\tau\right)ds. \end{aligned}$$

Lemma 3.2. Assume that $(H_1)-(H_3)$ hold. If $f(t,x,y),\ g(t,x,y)>0$, for $x,\ y\in C^+[0,1]$, $t\in[0,1]$, for the unique solution (u,v) of BVP (3.1), (3.2), then $u(t)$ and $v(t)$ are concave, and $u(t),\ v(t)\geq 0,\ u'(t),\ v'(t)\geq 0,\ t\in[0,1]$.
Proof. From the fact that $(\phi_p(u'))'(t)=-q(t)f(t,x(t),y(t))\leq 0$, we have $\phi_p(u'(t))$ is nonincreasing. It follows that $u'(t)$ is also nonincreasing. Thus, we know that the graph of $u(t)$ is concave down on $(0,1)$. Then the concavity of u together with boundary $u'(1)=\beta u'(0)$ implies that $u'(t)\geq 0$ for $t\in[0,1]$. Similarly, we can prove the graph of $v(t)$ is concave down on (0,1) and $v'(t)\geq 0$ for $t\in[0,1]$.

From $u'(t)\geq 0$, we know that

$$u(\xi_i)\geq u(0),\ \text{for } i=1,2,\dots,m-2.$$

This implies

$$u(0)=\sum_{i=1}^{m-2}a_iu(\xi_i)\geq\sum_{i=1}^{m-2}a_iu(0).$$

By $1-\sum\limits_{i=1}^{m-2}a_i>0$, it is obvious that $u(0)\geq 0$. Hence $u(1)\geq u(0)\geq 0$. So from the concavity of u, we know that $u(t)\geq 0$, $t\in[0,1]$. In a similar way, we can know $v(t)\geq 0$, $t\in[0,1]$.
Lemma 3.3. If $(u,v)\in K,\ \eta\in(0,1)$, then $u(t)\geq\eta\|u\|,\ v(t)\geq\eta\|v\|,\ t\in[\eta,1]$.
Proof. For $u\in K$, we know $u(t)$ and $v(t)$ are nonnegative, nondecreasing and concave on $[0,1]$, then

$$u(t)\geq tu(1)\geq\eta u(1)=\eta\|u\|,\ v(t)\geq tv(1)\geq\eta v(1)=\eta\|v\|,\ t\in[\eta,1].$$

We define

$$\varphi(t)=\theta t,\ \theta\in(0,1),$$

$$L=1+\frac{\sum\limits_{i=1}^{m-2}a_i\xi_i}{1-\sum\limits_{i=1}^{m-2}a_i},$$

$$\gamma_1=\min\left\{\frac{\eta\int_\eta^1\phi_{p_1}^{-1}\left(\frac{1}{1-\phi_{p_1}(\beta)}\int_s^1 q_1(\tau)d\tau\right)ds}{L\phi_{p_1}^{-1}\left(\frac{1}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)d\tau\right)},\ \frac{\eta\int_\eta^1\phi_{p_2}^{-1}\left(\frac{1}{1-\phi_{p_2}(\beta)}\int_s^1 q_2(\tau)d\tau\right)ds}{L\phi_{p_2}^{-1}\left(\frac{1}{1-\phi_{p_2}(\beta)}\int_0^1 q_2(\tau)d\tau\right)}\right\},$$

$$\gamma=\eta\gamma_1,$$
$$K_\rho=\{(u,v)\in K:\|(u,v)\|<\rho\},$$
$$K_\rho^*=\{(u,v)\in K:\rho\varphi(t)<u(t)+v(t)<\rho\},$$
$$\Omega_\rho=\{(u,v)\in K:\min_{\eta\leq t\leq 1}(u(t)+v(t))<\gamma\rho\}$$
$$=\{(u,v)\in K:\gamma\|(u,v)\|\leq\min_{\eta\leq t\leq 1}(u(t)+v(t))<\gamma\rho\}.$$

Lemma 3.4. ([7]) Ω_ρ has the following properties:
(a) Ω_ρ is open relative to K.
(b) $K_{\gamma\rho}\subset\Omega_\rho\subset K_\rho$.
(c) $u\in\partial\Omega_\rho$ if and only if $\min\limits_{\eta\leq t\leq 1}(u(t)+v(t))=\gamma\rho$.
(d) If $u\in\partial\Omega_\rho$, then $\gamma\rho\leq u(t)\leq\rho$, for $t\in[\eta,1]$.

Now for convenience we introduce the following notations. Let

$$f^{\rho}_{\gamma\rho}=\left\{\min_{t\in[\eta,1]}\frac{f(t,u,v)}{\phi_{p_1}(\rho)}:u+v\in[\gamma\rho,\rho]\right\},\ g^{\rho}_{\gamma\rho}=\left\{\min_{t\in[\eta,1]}\frac{g(t,u,v)}{\phi_{p_2}(\rho)}:u+v\in[\gamma\rho,\rho]\right\},$$
$$f^{\rho}_{\rho\varphi(t)}=\left\{\max_{t\in[0,1]}\frac{f(t,u,v)}{\phi_{p_1}(\rho)}:u+v\in[\rho\varphi(t),\rho]\right\},\ g^{\rho}_{\rho\varphi(t)}=\left\{\max_{t\in[0,1]}\frac{g(t,u,v)}{\phi_{p_2}(\rho)}:u+v\in[\rho\varphi(t),\rho]\right\},$$
$$f_{\infty}=\lim_{(u,v)\to\infty}inf\min_{t\in[\eta,1]}\frac{f(t,u,v)}{\phi_{p_1}(u+v)},\ g_{\infty}=\lim_{(u,v)\to\infty}inf\min_{t\in[\eta,1]}\frac{g(t,u,v)}{\phi_{p_2}(u+v)},$$
$$f^{\infty}=\lim_{(u,v)\to\infty}sup\max_{t\in[0,1]}\frac{f(t,u,v)}{\phi_{p_1}(u+v)},\ g^{\infty}=\lim_{(u,v)\to\infty}sup\max_{t\in[0,1]}\frac{g(t,u,v)}{\phi_{p_2}(u+v)},$$
$$\frac{1}{m_1}=2L\phi^{-1}_{p_1}\left(\frac{1}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)d\tau\right),\ \frac{1}{m_2}=2L\phi^{-1}_{p_2}\left(\frac{1}{1-\phi_{p_2}(\beta)}\int_0^1 q_2(\tau)d\tau\right),$$
$$\frac{1}{M_1}=2\eta\int_\eta^1\phi^{-1}_{p_1}\left(\frac{1}{1-\phi_{p_1}(\beta)}\int_s^1 q_1(\tau)d\tau\right)ds,\ \frac{1}{M_2}=2\eta\int_\eta^1\phi^{-1}_{p_2}\left(\frac{1}{1-\phi_{p_2}(\beta)}\int_s^1 q_2(\tau)d\tau\right)ds,$$

where $(u,v)\to\infty\Leftrightarrow\|u\|+\|v\|\to\infty$.

Remark 2.1. By (H_3), it is easy to see that $0<m_1,m_2,M_1,M_2<\infty$, and
$M_1\gamma=M_1\eta\gamma_1\leq\eta m_1<m_1,\ M_2\gamma=M_2\eta\gamma_1\leq\eta m_2<m_2$.

4 Existence of positive solutions

We now give our results on the existence of multiple positive solutions of BVP (1.1), (1.2).

Theorem 4.1. Assume $(H_1)-(H_3)$ hold. In addition, the following condition (H_4) holds:
(H_4) There exist $\rho_1,\rho_2,\rho_3\in(0,\infty)$, with $\rho_1<\gamma\rho_2<\rho_2<\rho_3$ such that
(1) $f(t,u,v),\ g(t,u,v)>0,\ t\in[0,1],\ u+v\in[\rho_1\varphi(t),\infty)$;
(2) $f^{\rho_1}_{\rho_1\varphi(t)}<\phi_{p_1}(m_1),\ g^{\rho_1}_{\rho_1\varphi(t)}<\phi_{p_2}(m_2),\ f^{\rho_2}_{\gamma\rho_2}\geq\phi_{p_1}(M_1\gamma),\ g^{\rho_2}_{\gamma\rho_2}\geq\phi_{p_2}(M_2\gamma)$,
$f^{\rho_3}_{\rho_3\varphi(t)}\leq\phi_{p_1}(m_1),\ g^{\rho_3}_{\rho_3\varphi(t)}\leq\phi_{p_2}(m_2)$.

Then BVP (1.1), (1.2) has at least three positive solutions in K.

Proof. We assume that (H_4) holds. Denote

$$f^*(t,u,v)=\begin{cases}f(t,u,v), & u+v\geq\rho_1\varphi(t),\\ f(t,u,\rho_1\varphi(t)-u), & 0\leq u+v<\rho_1\varphi(t).\end{cases}$$

$$g^*(t,u,v)=\begin{cases}g(t,u,v), & u+v\geq\rho_1\varphi(t),\\ g(t,u,\rho_1\varphi(t)-u), & 0\leq u+v<\rho_1\varphi(t).\end{cases}$$

We can see that $f^*(t,u,v),\ g^*(t,u,v)\in C([0,1]\times[0,+\infty)\times[0,+\infty),\ (0,+\infty))$.

Define the following integral equation systems:

$$\begin{aligned}A(u,v)(t)=&\int_0^t\phi^{-1}_{p_1}\left(\int_s^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))d\tau\right.\\&\left.+\frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))d\tau\right)ds\\&+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi^{-1}_{p_1}\left(\int_s^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))\,d\tau\right.\\&\left.+\frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))d\tau\right)ds,\end{aligned}\quad(4.1)$$

$$
\begin{aligned}
B(u,v)(t) = &\int_0^t \phi_{p_2}^{-1}\left(\int_s^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))d\tau\right.\\
&\left.+\frac{\phi_{p_2}(\beta)}{1-\phi_{p_2}(\beta)}\int_0^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))d\tau\right)ds\\
&+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_{p_2}^{-1}\left(\int_s^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))\,d\tau\right.\\
&\left.+\frac{\phi_{p_2}(\beta)}{1-\phi_{p_2}(\beta)}\int_0^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))d\tau\right)ds. \qquad (4.2)
\end{aligned}
$$

Define operator

$F(u,v)(t) = (A(u,v)(t), B(u,v)(t))$.

According to the definition of F and Lemma 3.2, it is easy to show that $F(K) \subset K$. By similar arguments in [5, 12], $F : K \to K$ is completely continuous.

Now we consider the following modified problem of (1.1) and (1.2):

$$
\begin{cases}
(\phi_{p_1}(u'))' + q_1(t)f^*(t,u,v) = 0,\ t\in(0,1),\\
(\phi_{p_2}(v'))' + q_2(t)g^*(t,u,v) = 0,\ t\in(0,1),
\end{cases} \qquad (4.3)
$$

$$
\begin{cases}
u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i),\ u'(1) = \beta u'(0),\\
v(0) = \sum_{i=1}^{m-2} a_i v(\xi_i),\ v'(1) = \beta v'(0),
\end{cases} \qquad (4.4)
$$

From the condition (H_4), we have

$f^{*\rho_1}_{\rho_1\varphi(t)} < \phi_{p_1}(m_1),\ g^{*\rho_1}_{\rho_1\varphi(t)} < \phi_{p_2}(m_2),\ f^{*\rho_2}_{\gamma\rho_2} \ge \phi_{p_1}(M_1\gamma),\ g^{*\rho_2}_{\gamma\rho_2} \ge \phi_{p_2}(M_2\gamma)$,

$f^{*\rho_3}_{\rho_3\varphi(t)} \le \phi_{p_1}(m_1),\ g^{*\rho_3}_{\rho_3\varphi(t)} \le \phi_{p_2}(m_2)$.

Firstly, we show that $i_k(F, K^*_{\rho_1}) = 1$.

In fact, by (4.1), (4.2), $f^{*\rho_1}_{\rho_1\varphi(t)} < \phi_{p_1}(m_1)$ and $g^{*\rho_1}_{\rho_1\varphi(t)} < \phi_{p_2}(m_2)$, for $(u,v) \in \partial K^*_{\rho_1}$, we have

$$
\begin{aligned}
\|A(u,v)(t)\| &= \max_{0\le t\le 1}|A(u,v)(t)| = A(u,v)(1)\\
&= \int_0^1 \phi_{p_1}^{-1}\left(\int_s^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))d\tau\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))d\tau\Bigg)ds\\
&+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_{p_1}^{-1}\Bigg(\int_s^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))d\tau\\
&+\frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))d\tau\Bigg)ds\\
\le&\int_0^1\phi_{p_1}^{-1}\Bigg(\int_0^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))d\tau\\
&+\frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))d\tau\Bigg)ds\\
&+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_{p_1}^{-1}\Bigg(\int_0^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))d\tau\\
&+\frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))d\tau\Bigg)ds\\
=&L\phi_{p_1}^{-1}\left(\frac{1}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)f^*(\tau,u(\tau),v(\tau))d\tau\right)\\
<&L\phi_{p_1}^{-1}\left(\frac{\phi_{p_1}(\rho_1)\phi_{p_1}(m_1)}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)d\tau\right)=\frac{\rho_1}{2m_1}m_1=\frac{\rho_1}{2}=\frac{\|(u,v)\|}{2},
\end{aligned}
$$

$$
\begin{aligned}
\|B(u,v)(t)\|=&\max_{0\le t\le 1}|B(u,v)(t)|=B(u,v)(1)\\
=&\int_0^1\phi_{p_2}^{-1}\Bigg(\int_s^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))d\tau\\
&+\frac{\phi_{p_2}(\beta)}{1-\phi_{p_2}(\beta)}\int_0^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))d\tau\Bigg)ds\\
&+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_{p_2}^{-1}\Bigg(\int_s^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))d\tau\\
&+\frac{\phi_{p_2}(\beta)}{1-\phi_{p_2}(\beta)}\int_0^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))d\tau\Bigg)ds\\
\le&\int_0^1\phi_{p_2}^{-1}\Bigg(\int_0^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))d\tau\\
&+\frac{\phi_{p_2}(\beta)}{1-\phi_{p_2}(\beta)}\int_0^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))d\tau\Bigg)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{1-\sum\limits_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_{p_2}^{-1}\left(\int_0^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))d\tau\right.\\
&\left.+\frac{\phi_{p_2}(\beta)}{1-\phi_{p_2}(\beta)}\int_0^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))d\tau\right)ds\\
=&L\phi_{p_2}^{-1}\left(\frac{1}{1-\phi_{p_2}(\beta)}\int_0^1 q_2(\tau)g^*(\tau,u(\tau),v(\tau))d\tau\right)\\
<&L\phi_{p_2}^{-1}\left(\frac{\phi_{p_2}(\rho_1)\phi_{p_2}(m_1)}{1-\phi_{p_2}(\beta)}\int_0^1 q_2(\tau)d\tau\right)=\frac{\rho_1}{2m_2}m_2=\frac{\rho_1}{2}=\frac{\|(u,v)\|}{2}.
\end{aligned}
$$

Therefore, $\|F(u,v)(t)\|=\|(A(u,v)(t),B(u,v)(t))\|=\|A(u,v)(t)\|+\|B(u,v)(t)\|<\|(u,v)\|$ for $(u,v)\in\partial K^*_{\rho_1}$. By Theorem 2.1, we have $i_k(F,K^*_{\rho_1})=1$.

Secondly, we show that $i_k(F,\Omega_{\rho_2})=0$.

Let $(e_1(t),e_2(t))\equiv(\frac{1}{2},\frac{1}{2})$ for $t\in[0,1]$, then $(e_1(t),e_2(t))\in\partial K_1$. We claim that

$(u(t),v(t))\neq F(u,v)(t)+\lambda(e_1(t),e_2(t)),\ (u,v)\in\partial\Omega_{\rho_2},\ \lambda\geq 0.$

In fact, if not, there exist $(u_0,v_0)\in\partial\Omega_{\rho_2}$ and $\lambda_0\geq 0$ such that

$(u_0(t),v_0(t))=F(u_0,v_0)(t)+\lambda_0(e_1(t),e_2(t)).$

Hence, from Lemma 3.3 and $f^{*\rho_2}_{\gamma\rho_2}\geq\phi_{p_1}(M_1\gamma)$, we have that for $t\in[\eta,1]$,

$$
\begin{aligned}
u_0(t)=&A(u_0,v_0)(t)+\lambda_0e_1(t)\geq\eta\|A(u_0,v_0)(t)\|+\frac{\lambda_0}{2}=\eta A(u_0,v_0)(1)+\frac{\lambda_0}{2}\\
=&\eta\int_0^1\phi_{p_1}^{-1}\left(\int_s^1 q_1(\tau)f^*(\tau,u_0(\tau),v_0(\tau))d\tau\right.\\
&\left.+\frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)f^*(\tau,u_0(\tau),v_0(\tau))d\tau\right)ds\\
&+\frac{\eta}{1-\sum\limits_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\phi_{p_1}^{-1}\left(\int_s^1 q_1(\tau)f^*(\tau,u_0(\tau),v_0(\tau))d\tau\right.\\
&\left.+\frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)f^*(\tau,u_0(\tau),v_0(\tau))d\tau\right)ds+\frac{\lambda_0}{2}\\
\geq&\eta\int_0^1\phi_{p_1}^{-1}\left(\int_s^1 q_1(\tau)f^*(\tau,u_0(\tau),v_0(\tau))d\tau\right.\\
&\left.+\frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)}\int_0^1 q_1(\tau)f^*(\tau,u_0(\tau),v_0(\tau))d\tau\right)ds+\frac{\lambda_0}{2}\\
>&\eta\int_\eta^1\phi_{p_1}^{-1}\left(\int_s^1 q_1(\tau)f^*(\tau,u_0(\tau),v_0(\tau))d\tau\right.\\
&\left.+\frac{\phi_{p_1}(\beta)}{1-\phi_{p_1}(\beta)}\int_s^1 q_1(\tau)f^*(\tau,u_0(\tau),v_0(\tau))d\tau\right)ds+\frac{\lambda_0}{2}\\
=&\eta\int_\eta^1\phi_{p_1}^{-1}\left(\frac{1}{1-\phi_{p_1}(\beta)}\int_s^1 q_1(\tau)f^*(\tau,u_0(\tau),v_0(\tau))d\tau\right)ds+\frac{\lambda_0}{2}\\
\geq&\eta\int_\eta^1\phi_{p_1}^{-1}\left(\frac{\phi_{p_1}(\rho_2)\phi_{p_1}(M_1\gamma)}{1-\phi_{p_1}(\beta)}\int_s^1 q_1(\tau)d\tau\right)ds+\frac{\lambda_0}{2}\\
=&M_1\gamma\rho_2\frac{1}{2M_1}+\frac{\lambda_0}{2}=\frac{\gamma\rho_2}{2}+\frac{\lambda_0}{2}.
\end{aligned}
$$

Similarly, from Lemma 3.3 and $g^{*\rho_2}_{\gamma\rho_2} \geq \phi_{p_2}(M_2\gamma)$, we have that for $t \in [\eta, 1]$, We can prove $v_0(t) > \frac{\gamma\rho_2}{2} + \frac{\lambda_0}{2}$. This implies that $\min\limits_{\eta\leq t\leq 1}(u(t) + v(t)) = \gamma\rho_2 > \gamma\rho_2 + \lambda_0$, which is a contradiction. Hence, by Theorem 2.1, it follows that $i_k(F, \Omega_{\rho_2}) = 0$.

Finally, similar to the proof of $i_k(F, K^*_{\rho_1}) = 1$, we can show that $i_k(F, K^*_{\rho_3}) = 1$. We can get the BVP (4.3), (4.4) has at least three positive solutions (u_1, v_1), (u_2, v_2) and (u_3, v_3) such that

$(u_1, v_1) \in K^*_{\rho_1}$, $(u_1, v_1) \in \Omega_{\rho_2}\backslash\overline{K^*_{\rho_1}}$, $(u_3, v_3) \in K^*_{\rho_3}\backslash\overline{\Omega_{\rho_2}}$.

As a result, the BVP (4.3), (4.4) has at least three positive solutions (u_1, v_1), (u_2, v_2) and (u_3, v_3) such that $u_1 + v_1, u_2 + v_2, u_3 + v_3 \in [\rho_1\varphi(t), \infty)$, and

$f^*(t, u, v) = f(t, u, v)$, $g^*(t, u, v) = g(t, u, v)$, $u + v \geq \rho_1\varphi(t)$,

which mean (u_1, v_1), (u_2, v_2) and (u_3, v_3) are also solutions of BVP (1.1), (1.2).

Similarly, we can obtain the following conclusions.

Theorem 4.2. Assume $(H_1) - (H_3)$ hold. In addition, the following condition (H_5) holds:
(H_5) There exist $\rho_1, \rho_2, \rho_3 \in (0, \infty)$, with $\rho_1 < \rho_2 < \gamma\rho_3$ such that

(1) $f(t, u, v)$, $g(t, u, v) > 0$, $t \in [0, 1]$, $u + v \in [\min\{\gamma\rho_1, \rho_2\varphi(t)\}, \infty)$;

(2) $f^{\rho_1}_{\gamma\rho_1} \geq \phi_{p_1}(M_1\gamma)$, $g^{\rho_1}_{\gamma\rho_1} \geq \phi_{p_2}(M_2\gamma)$, $f^{\rho_2}_{\rho_2\varphi(t)} < \phi_{p_1}(m_1)$, $g^{\rho_2}_{\rho_2\varphi(t)} < \phi_{p_2}(m_2)$, $f^{\rho_3}_{\gamma\rho_3} \geq \phi_{p_1}(M_1\gamma)$, $g^{\rho_3}_{\gamma\rho_3} \geq \phi_{p_2}(M_2\gamma)$.

Then BVP (1.1) and (1.2) has at least two positive solutions in K.

Theorem 4.3. Assume $(H_1) - (H_3)$ hold. In addition, the following condition (H_6) holds:
(H_6) There exist $\rho_1, \rho_2 \in (0, \infty)$, with $\rho_1 < \gamma\rho_2$ such that

(1) $f(t, u, v)$, $g(t, u, v) > 0$, $t \in [0, 1]$, $u + v \in [\rho_1\varphi(t), \infty)$;

(2) $f^{\rho_1}_{\rho_1\varphi(t)} < \phi_{p_1}(m_1)$, $g^{\rho_1}_{\rho_1\varphi(t)} < \phi_{p_2}(m_2)$, $f^{\rho_2}_{\gamma\rho_2} \geq \phi_{p_1}(M_1\gamma)$, $g^{\rho_2}_{\gamma\rho_2} \geq \phi_{p_2}(M_2\gamma)$, $0 \leq f^\infty < \phi_{p_1}(m_1)$, $0 \leq g^\infty < \phi_{p_2}(m_2)$.

Then BVP (1.1) and (1.2) has at least three positive solutions in K.

Proof. We show that (H_6) implies (H_4). Let $k \in (f^\infty, \phi_{p_1}(m_1))$. Then there exists $r > \rho_2$, such that $\max\limits_{t\in[0,1]} f(t, u, v) \leq k\phi_{p_1}(u + v)$ for $u + v \in [r, \infty)$ since $0 \leq f^\infty < \phi_{p_1}(m_1)$. Let

$$\alpha = \left\{\max_{t\in[0,1]} f(t, u, v) : \rho_1\varphi(t) \leq u + v \leq r\right\} \text{ and } \rho_3 > \max\left\{\phi^{-1}_{p_1}\left(\frac{\alpha}{\phi_{p_1(m)} - k}\right), \rho_2\right\}.$$

Then we have

$$\max_{t\in[0,1]} f(t, u, v) \leq k\phi_{p_1}(u+v)+\alpha \leq k\phi_{p_1}(\rho_3)+\alpha < \phi_{p_1}(m_1)\phi_{p_1}(\rho_3), \text{ for } u+v \in [\rho_1\varphi(t), \rho_3).$$

This implies that $f^{\rho_3}_{\rho_3\varphi(t)} < \phi_{p_1}(m_1)$. Similarly, $0 \leq g^\infty < \phi_{p_2}(m_2)$ implies that $g^{\rho_3}_{\rho_3\varphi(t)} < \phi_{p_2}(m_2)$. Hence, (H_4) holds, by Theorem 4.1, BVP (1.1), (1.2) has at least three positive solutions in K.

Theorem 4.4. Assume $(H_1) - (H_3)$ hold. In addition, the following condition (H_7) holds:
(H_7) There exist $\rho_1, \rho_2 \in (0, \infty)$, with $\rho_1 < \rho_2$ such that

(1) $f(t, u, v)$, $g(t, u, v) > 0$, $t \in [0, 1]$, $u + v \in [\min\{\gamma\rho_1, \rho_2\varphi(t)\}, \infty)$;

(2) $f^{\rho_1}_{\gamma\rho_1} \geq \phi_{p_1}(M_1\gamma)$, $g^{\rho_1}_{\gamma\rho_1} \geq \phi_{p_2}(M_2\gamma)$, $f^{\rho_2}_{\rho_2\varphi(t)} < \phi_{p_1}(m_1)$, $g^{\rho_2}_{\rho_2\varphi(t)} < \phi_{p_2}(m_2)$, $\phi_{p_1}(M_1) < f_\infty \leq \infty$, $\phi_{p_2}(M_2) < g_\infty \leq \infty$.

Then BVP (1.1) and (1.2) has at least two positive solutions in K.

Proof. We show that (H_7) implies (H_5). Since $\phi_{p_1}(M_1) < f_\infty \leq \infty$, then there exists $\rho_3 > \frac{\rho_2}{\gamma}$, such that

$$\min_{t\in[\eta,1]} f(t, u, v) \geq \phi_{p_1}(u+v)\phi_{p_1}(M_1) \geq \phi_{p_1}(\gamma\rho_3)\phi_{p_1}(M_1) = \phi_{p_1}(\rho_3)\phi_{p_1}(M_1\gamma), u+v \in [\gamma\rho_3, \rho_3).$$

This implies that $f^{\rho_3}_{\gamma\rho_3} \geq \phi_{p_1}(M_1\gamma)$. Similarly, $\phi_{p_2}(M_2) < g_\infty \leq \infty$ implies that $g^{\rho_3}_{\gamma\rho_3} \geq \phi_{p_2}(M_2\gamma)$. Hence, (H_5) holds, by Theorem 4.2, BVP (1.1), (1.2) has at least two positive solutions in K.

By the arguments similar to that of Theorem 3.1 and Theorem 3.2, we obtain the following results.

Theorem 4.5. Assume $(H_1)-(H_3)$ hold. In addition, the following condition (H_8) holds:
(H_8) There exist $\rho_1, \rho_2 \in (0,\infty)$, with $\rho_1 < \gamma\rho_2$ such that

(1) $f(t,u,v),\ g(t,u,v) > 0,\ t \in [0,1],\ u+v \in [\rho_1\varphi(t),\infty)$;

(2) $f^{\rho_1}_{\rho_1\varphi(t)} \le \phi_{p_1}(m_1),\ g^{\rho_1}_{\rho_1\varphi(t)} \le \phi_{p_2}(m_2),\ f^{\rho_2}_{\gamma\rho_2} \ge \phi_{p_1}(M_1\gamma),\ g^{\rho_2}_{\gamma\rho_2} \ge \phi_{p_2}(M_2\gamma)$.

Then BVP (1.1), (1.2) has at least one positive solutions in K.

Theorem 4.6. Assume $(H_1)-(H_3)$ hold. In addition, the following condition (H_9) holds:
(H_9) There exist $\rho_1, \rho_2 \in (0,\infty)$, with $\rho_1 < \rho_2$ such that

(1) $f(t,u,v),\ g(t,u,v) > 0,\ t \in [0,1],\ u+v \in [\min\{\gamma\rho_1, \rho_2\varphi(t)\},\infty)$;

(2) $f^{\rho_1}_{\gamma\rho_1} \ge \phi_{p_1}(M_1\gamma),\ g^{\rho_1}_{\gamma\rho_1} \ge \phi_{p_2}(M_2\gamma),\ f^{\rho_2}_{\rho_2\varphi(t)} \le \phi_{p_1}(m_1),\ g^{\rho_2}_{\rho_2\varphi(t)} \le \phi_{p_2}(m_2)$.

Then BVP (1.1) and (1.2) has at least one positive solutions in K.

5 Example

Now we present an example to illustrate the main result.

Example 5.1. Consider the following BVP

$$\begin{cases} (|u'(t)|u'(t))' + q_1(t)f(t,u,v) = 0,\ t \in (0,1), \\ (|v'(t)|v'(t))' + q_2(t)g(t,u,v) = 0,\ t \in (0,1), \end{cases} \tag{5.1}$$

$$\begin{cases} u(0) = \frac{1}{2}u(\frac{1}{3}) + \frac{1}{4}u(\frac{2}{3}),\ u'(1) = \frac{1}{2}u'(0), \\ v(0) = \frac{1}{2}v(\frac{1}{3}) + \frac{1}{4}v(\frac{2}{3}),\ v'(1) = \frac{1}{2}v'(0), \end{cases} \tag{5.2}$$

where

$$f(t,u,v) = \begin{cases} \frac{1}{80}(1+t)^{\frac{1}{2}}(u+v-\frac{t}{4})^{21} + \frac{1}{10^{35}}, & 0 \le u+v \le 2, \\ \frac{1}{80}(1+t)^{\frac{1}{2}}(2-\frac{t}{4})^{21} + \frac{1}{10^{35}}, & u+v > 2, \end{cases}$$

$$g(t,u,v) = \begin{cases} \frac{1}{40}(1+t)^{\frac{1}{4}}(u+v-\frac{t}{4})^{19} + \frac{1}{10^{40}}, & 0 \le u+v \le 2, \\ \frac{1}{40}(1+t)^{\frac{1}{4}}(2-\frac{t}{4})^{19} + \frac{1}{10^{40}}, & u+v > 2, \end{cases}$$

$q_1(t) = q_2(t) = 1$.

Obviously, $p_1 = p_2 = 3,\ \beta = 1,\ \xi_1 = \frac{1}{3},\ \xi_2 = \frac{2}{3},\ a_1 = \frac{1}{2},\ a_2 = \frac{1}{4}$. Choose $\rho_1 = 1,\ \rho_2 = \frac{448\sqrt{3}}{9},\ \rho_3 = 1200,\ \eta = \frac{1}{4},\ \theta = \frac{1}{2}$, then $\varphi(t) = \frac{t}{2}$. We note $\gamma = \frac{3\sqrt{3}}{224},\ m_1 = m_2 = \frac{3\sqrt{3}}{28},\ M_1 = M_2 = 4$. Consequently, $f(t,u,v)$ satisfies

(1) $f(t,u,v),\ g(t,u,v) > 0,\ t \in [0,1],\ u+v \in [\frac{t}{2},\infty)$;

(2) $f^{\rho_1}_{\rho_1\varphi(t)} \le 0.018 < \phi_{p_1}(m_1) \approx 0.034,\ g^{\rho_1}_{\rho_1\varphi(t)} \le 0.03 < \phi_{p_2}(m_2) \approx 0.034$,
$f^{\rho_2}_{\gamma\rho_2} \ge 0.239 > \phi_{p_1}(M_1\gamma) \approx 0.009,\ g^{\rho_2}_{\gamma\rho_2} \ge 0.147 > \phi_{p_2}(M_2\gamma) \approx 0.009$,
$f^{\rho_3}_{\rho_3\varphi(t)} \le 0.026 < \phi_{p_1}(m_1) \approx 0.034,\ g^{\rho_3}_{\rho_3\varphi(t)} \le 0.011 < \phi_{p_2}(m_2) \approx 0.034$.

Thus with Theorem (4.1), BVP (5.1), (5.2) has at least three positive solutions in K.

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An S-partially contractive mapping with a control function ϕ

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Abstract. In this article, we introduce a ϕ-contraction principle in a partial S-metric space, we show the existence of a fixed point for a self mapping in a partial S-metric space. Also, we show that we have uniqueness only under some specific conditions.

Keywords. Partial S-metric space, Banach contraction principle, Fixed point.

1 Introduction and Preliminaries

Finding a fixed point for a self mapping on different types of metric spaces has been one the main topics of research in pure mathematics. it starts with the Banach contraction principle which was introduced by Banach in the early nineties.
Since the Banach contraction was introduced, many results were found in fixed point theory field in different type of metric spaces, such as [13], [14], [15],[16],[22], [23], [24],[25], [4], [5], [6], [8],[9], [10], [11], [12], [19], [20], [21],[26],[27],[28].

An S- metric space was introduced in [2].

Definition 1. [2] Let X be a nonempty set. An *S-metric space* on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions, for all $x, y, z, a \in X$:

- $S(x, y, z) \geq 0$,
- $S(x, y, z) = 0$ if and only if $x = y = z$,
- $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an *S-metric space.*

In this article, we are interested in partial S-metric space which was introduced in [1]. We recall some definitions of partial metric spaces and state some of their

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properties.

Definition 2. [1] Let X be a nonempty set. A *partial S-metric space* on X is a function $S_p : X^3 \to [0, \infty)$ that satisfies the following conditions, for all $x, y, z, t \in X$:

(P1) $x = y$ if and only if $S_p(x, x, x) = S_p(y, y, y) = S_p(x, x, y)$

(P2) $S_p(x, y, z) \leq S_p(x, x, t) + S_p(y, y, t) + S_p(z, z, t) - S_p(t, t, t)$

(P3) $S_p(x, x, x) \leq S_p(x, y, z)$

(P4) $S_p(x, x, y) = S_p(y, y, x)$.

The pair (X, S_p) is called a partial S-metric space.

We recall some definitions of partial S-metric spaces and state some of their properties.

Definition 3. A sequence $\{x_n\}_{n=0}^{\infty}$ of elements in X is called *Cauchy* if the limit $\lim_{n,m\to\infty} S_p(x_n, x_n, x_m)$ exists and finite. The partial S-metric space (X, S_p) is called *complete* if for each Cauchy sequence $\{x_n\}_{n=0}^{\infty}$ there exists $z \in X$ such that

$$S_p(z, z, z) = \lim_n S_p(z, z, x_n) = \lim_{n,m} S_p(x_n, x_n, x_m).$$

Also, (X, S_p) is a complete partial S-metric space if and only if (X, S_p^s) is a complete S-metric space. A sequence $\{x_n\}_n$ in a partial S-metric space (X, S_p) is called 0-*Cauchy* if $\lim_{n,m\to\infty} S_p(x_n, x_n, x_m) = 0$. We say that (X, S_p) is 0-*complete* if every 0-Cauchy in X converges to a point $x \in X$ such that $S_p(x, x, x) = 0$.

Example 1. *(see [1]) Let $X = \mathbb{Q} \cap [0, \infty)$ with the partial metric $p(x, y, z) = max\{x, y, z\}$. Then (X, S_p) is a 0-complete partial metric space which is not complete.*

Definition 4. Let (X, S_p) be a complete partial S-metric space. Set $\rho_p = \inf\{S_p(x, y, z) : x, y, z \in X\}$ and define the set $X_p = \{x \in X : S_p(x, x, x) = \rho_p\}$.

The following Lemma summarizes the relation between certain comparison functions that usually act as control functions in the studied contractive typed mappings in fixed point theory. For such a summary and fixed point theory for $\phi-$ contractive mappings, see [18].

Lemma 1. *Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function and relative to the function ϕ consider the following conditions:*

- *(i) ϕ is monotone increasing.*
- *(ii) $\phi(t) < t$ for all $t > 0$.*
- *(iii) $\phi(0) = 0$.*
- *(iv) ϕ is right uppersemicontinuous.*
- *(v) ϕ is right continuous.*
- *(vi) $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \geq 0$.*

Then the following are valid:

- *(1) The conditions (i) and (ii) imply (iii).*
- *(2) The conditions (ii) and (v) imply (iii).*
- *(3) The conditions (i) and (vi) imply (ii).*
- *(4) The conditions (i) and (iv) imply imply (vi).*
- *(5) If ϕ satisfies (i) then (iv) $\Leftrightarrow$ (v).*

2 Main Results

Now, we prove our main result.

Theorem 1. *Let (X, S_p) be a complete partial S-metric space. Suppose $T : X \to X$ is a given self mapping satisfying:*

$$S_p(Tx, Tx, Ty) \leq \max\{\phi(S_p(x, x, y)), S_p(x, x, x), S_p(y, y, y)\}, \tag{1}$$

where ϕ is defined as in Lemma 1. Then:

(1) the set X_p is nonempty;

(2) there is a unique $u \in X_p$ such that $Tu = u$;

Proof. For any $x \in X$, we have $S_p(Tx,Tx,Tx) \leq S_p(x,x,x)$ and hence the sequence $\{S_p(T^n x, T^n x, T^n x)\}_{n\geq 0}$ is a nonincreasing sequence. Now Define

$$M_x := 2[f^{-1}(S_p(x,x,Tx)) + S_p(x,x,x)],$$

where $f(t) = t - \phi(t)$. Notice that $f(0) = 0$ (and hence $f^{-1}(0) = 0$) and $f(t) < t$ for $t > 0$ and hence $f^{-1}(t) > t$ for $t > 0$. Now we prove by induction that

$$S_p(T^n x, T^n x, x) \leq M_x, \quad \forall n \geq 0. \tag{2}$$

Notice that the inequality (2) is true for $n = 0, 1$ since: $S_p(x,x,x) \leq M_x$ and $S_p(Tx,Tx,x) \leq f^{-1}(S_p(Tx,Tx,x)) \leq M_x$.

Suppose that (2) is true for each $n \leq n_0 - 1$ for some positive integer $n_0 \geq 2$. Then we have

$$\begin{aligned} S_p(T^{n_0}x, T^{n_0}x, x) &\leq 2S_p(T^{n_0}x, T^{n_0}x, Tx) + S_p(Tx,Tx,x) \\ &\leq 2\max\{\phi(S_p(T^{n_0-1}x, T^{n_0-1}x, x)), S_p(T^{n_0-1}x, T^{n_0-1}x, T^{n_0-1}x), \\ &\quad S_p(x,x,x)\} + S_p(Tx,Tx,x) \\ &\leq 2\max\{\phi(S_p(T^{n_0-1}x, T^{n_0-1}x, x))), S_p(x,x,x)\} + S_p(Tx,Tx,x) \end{aligned}$$

Therefore, we have two cases.

Case 1:

$$\begin{aligned} S_p(T^{n_0}x, T^{n_0}x, x) &\leq \phi(S_p(T^{n_0-1}x, T^{n_0-1}x, Tx)) + S_p(Tx,Tx,x) \\ &\leq 2[\phi(f^{-1}(S_p(Tx,Tx,x)) + S_p(x,x,x))] + S_p(Tx,Tx,x) \\ &= 2[f^{-1}(S_p(Tx,Tx,x)) + S_p(x,x,x) - f(f^{-1}(S_p(Tx,Tx,x)) \\ &\quad + S_p(x,x,x))] + S_p(Tx,Tx,x) \\ &\leq M_x - 2f(f^{-1}(S_p(Tx,Tx,x)) + S_p(x,x,x)) + S_p(Tx,Tx,x) \\ &= M_x - S_p(Tx,Tx,x) - S_p(x,x,x) \leq M_x. \end{aligned}$$

Case 2:

$$\begin{aligned} S_p(T^{n_0}x, T^{n_0}x, x) &\leq S_p(x,x,x) + S_p(Tx,Tx,x) \\ &\leq S_p(x,x,x) + f^{-1}(S_p(Tx,Tx,x)) = M_x. \end{aligned}$$

Hence, we obtain (2). Next we prove that the sequence $\{S_p(T^n x, T^n x, T^n x)\}_{n\geq 0}$ is Cauchy. Equivalently, we show that

$$\lim_{n,m\to\infty} S_p(T^n x, T^n x, T^m x) = r_x, \tag{3}$$

where $r_x := \inf_n S_p(T^n x, T^n x, T^n x)$. Its clear that $r_x \leq S_p(T^n x, T^n x, T^n x) \leq S_p(T^n x, T^n x, T^m x)$ for all n, m. Also, given any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S_p(T^{n_0} x, T^{n_0} x, T^{n_0} x) < r_x + \epsilon$ and $\phi^{n_0}(2M_x) < r_x + \epsilon$. Therefore, for any $m, n > 2n_0$ we have

$$\begin{aligned}
r_x &\leq S_p(T^n x, T^n x, T^m x) \\
&\leq \max\{\phi(S_p(T^{n-1}x, T^{n-1}x, T^{m-1}x)), S_p(T^{n-1}x, T^{n-1}x, T^{n-1}x), \\
&\quad S_p(T^{m-1}x, T^{m-1}x, T^{m-1}x)\} \\
&\leq \max\{\phi^2(S_p(T^{n-2}x, T^{n-2}x, T^{m-2}x)), S_p(T^{n-2}x, T^{n-2}x, T^{n-2}x), \\
&\quad S_p(T^{m-2}x, T^{m-2}x, T^{m-2}x)\} \\
&\leq \max\{\phi^{n_0}(S_p(T^{n-n_0}x, T^{n-n_0}x, T^{m-n_0}x)), \\
&\quad S_p(T^{n-n_0}x, T^{n-n_0}x, T^{n-n_0}x), S_p(T^{m-n_0}x, T^{m-n_0}x, T^{m-n_0}x)\} \\
&\leq \max\{\phi^{n_0}(S_p(T^{n-n_0}x, T^{n-n_0}x, x) + S_p(T^{m-n_0}x, T^{m-n_0}x, x)), \\
&\quad S_p(T^{n-n_0}x, T^{n-n_0}x, T^{n-n_0}x), S_p(T^{m-n_0}x, T^{m-n_0}x, T^{m-n_0}x)\} \\
&< \max\{\phi^{n_0}(2M_x), r_x + \epsilon, r_x + \epsilon\} \\
&< r_x + \epsilon.
\end{aligned}$$

Hence, we obtain (3). Since (X, S_p) is a complete partial S-metric space, there exists $z \in X$ such that $S_p(z, z, z) = r_x$. Next, we show that $S_p(z, z, z) = p(Tz, Tz, z)$.

Next, we show that $S_p(z, z, z) = S_p(z, z, Tz) = S_p(Tz, Tz, z)$. For each natural number n we have

$$S_p(z, z, Tz) \leq 2S_p(z, z, z_n) - S_p(z_n, z_n, z_n) + S_p(Tz, Tz, z_n).$$

From the contraction condition of our theorem, we deduce that there exists a subsequence of natural numbers $\{n_l\}$ such that $S_p(Tz, Tz, z_{n_l}) \leq \phi(S_p(z, z, z_{n_l-1}))$, for $l \geq 1$, or $S_p(Tz, Tz, z_{n_l}) \leq S_p(z, z, z)$ for $l \geq 1$, or $S_p(Tz, z, z_{n_l}) \leq S_p(z_{n_l-1}, z_{n_l-1}, z_{n_l-1})$, for $l \geq 1$, in all of these three cases, if we take the limit as l goes toward ∞ we get $S_p(z, z, Tz) \leq S_p(z, z, z)$. But, we know by the property (iv) of the partial S-metric space that $S_p(z, z, z) \leq S_p(z, z, Tz)$. Therefore,

$$S_p(z, z, z) = S_p(z, z, Tz). \tag{4}$$

Now we show that X_p (see Definition 4) is nonempty. For each $k \in \mathbb{N}$ choose $x_k \in X$ with $S_p(x_k, x_k, x_k) < \rho_p + 1/k$, where $x_k = T^k x$. First, we prove that

$$\lim_{m,n\to\infty} S_p(z_n, z_n, z_m) = \rho_p. \tag{5}$$

Given $\epsilon > 0$, take $n_0 := [f^{-1}(3/\epsilon)] + 1$. If $k > n_0$, then

$$\begin{aligned}
\rho_p &\leq S_p(Tz_k, Tz_k, Tz_k) \leq S_p(z_k, z_k, z_k) = r_{x_k} \leq S_p(x_k, x_k, x_k) < \rho_p + 1/k \\
&< \rho_p + 1/n_0 < \rho_p + 1/f^{-1}(3/\epsilon).
\end{aligned}$$

Set $U_k := S_p(z_k, z_k, z_k) - S_p(Tz_k, Tz_k, Tz_k)$. Then $U_k < 1/f^{-1}(3/\epsilon)$ for $k > n_0$. Thus, if $m, n > n_0$ then by (4) and the fact that f (and hence f^{-1}) is increasing, we have

$$\begin{aligned} S_p(z_n, z_n, z_m) &\leq S_p(z_n, z_n, Tz_n) + S_p(Tz_n, Tz_n, Tz_m) + S_p(Tz_m, Tz_m, z_m) \\ &\quad - S_p(Tz_n, Tz_n, Tz_n) - S_p(Tz_m, Tz_m, Tz_m) \\ &= U_n + U_m + S_p(Tz_n, Tz_n, Tz_m) \\ &< 2/f^{-1}(3/\epsilon) + \max\{\phi(S_p(z_n, z_n, z_m)), S_p(z_n, z_n, z_n), S_p(z_m, z_m, z_m)\} \\ &\leq \max\{f^{-1}\left(2/f^{-1}(3/\epsilon)\right), 3/f^{-1}(3/\epsilon) + \rho_p\} \\ &\leq \max\{f^{-1}(2\epsilon/3)), \rho_p + \epsilon\} \\ &\leq \rho_p + \epsilon + f^{-1}(2\epsilon/3). \end{aligned}$$

Therefore, if we let $\epsilon \to 0$ we get (5). Since (X, S_p) is a complete partial metric space, there exists $u \in X$ such that $S_p(u, u, u) = \lim_{m,n\to\infty} S_p(z_n, z_n, z_m) = \rho_p$. Consequently, $u \in X_p$ and hence X_p is nonempty.

Now choose an arbitrary $x \in X_p$. Then

$$\rho_p \leq S_p(Tz, Tz, Tz) \leq S_p(Tz, Tz, z) = S_p(z, z, z) = r_x = \rho_p,$$

which, using P2, implies that $Tz = z$. To prove uniqueness of the fixed point we suppose that $u, v \in X_p$ are both fixed points of T. Then

$$\begin{aligned} \rho_p \leq S_p(u, u, v) = S_p(Tu, Tu, Tv) &\leq \max\{\phi(S_p(u, u, v)), S_p(u, u, u), S_p(v, v, v)\} \\ &\leq \max\{\phi(p(u, v)), \rho_p\}. \end{aligned}$$

Case 1: $\rho_p \leq S_p(u, u, v) \leq \rho_p \Rightarrow S_p(u, u, v) = \rho_p = S_p(u, u, u) = S_p(v, v, v) \Rightarrow u = v$.

Case 2:

$$\begin{aligned} & S_p(u, u, v) \leq \phi(S_p(u, u, v)) \\ \Rightarrow \quad & S_p(u, u, v) - \phi(S_p(u, u, v)) \leq 0 \\ \Rightarrow \quad & f(S_p(u, u, v)) \leq 0 \\ \Rightarrow \quad & f(S_p(u, u, v)) = 0 \\ \Rightarrow \quad & S_p(u, u, v) = 0 \\ \Rightarrow \quad & u = v. \end{aligned}$$

Thus, the fixed point is unique. □

Note that the above theorem does not guarantee uniqueness of the fixed point in X. However, if (1) is replaced by the condition below, we can show uniqueness in X.

In the next result, we change our contraction condition so that we obtain uniqueness of the fixed point in the whole space X.

Theorem 2. *Let (X, S_p) be a complete partial S-metric space. Suppose $T : X \to X$ is a given self mapping satisfying:*

$$S_p(Tx, Tx, Ty) \leq \max\left\{\phi(S_p(x,x,y)), \frac{S_p(x,x,x) + S_p(y,y,y)}{2}\right\}, \tag{6}$$

where $\phi : [0,\infty) \to [0,\infty)$ is as in Theorem 1. Then there is a unique point $z \in X$ such that $Tz = z$. Furthermore, $z \in X_p$.

Proof. Using Theorem 1 we only need to prove uniqueness. Suppose there exists $u, v \in X$ such that $Tu = u$ and $Tv = v$. Now

$$S_p(u,u,v) = S_p(Tu,Tu,Tv) \leq \max\left\{\phi(S_p(u,u,v)), \frac{S_p(u,u,u) + S_p(v,v,v)}{2}\right\}.$$

Case 1:

$$\begin{aligned} & S_p(u,u,v) \leq \phi(S_p(u,u,v)) \\ \Rightarrow \quad & S_p(u,u,v) - \phi(S_p(u,u,v)) \leq 0 \\ \Rightarrow \quad & f(S_p(u,u,v)) \leq 0 \\ \Rightarrow \quad & f(S_p(u,u,v)) = 0 \\ \Rightarrow \quad & S_p(u,u,v) = 0 \\ \Rightarrow \quad & u = v. \end{aligned}$$

Case 2:

$$\begin{aligned} & S_p(u,u,v) \leq \frac{S_p(u,u,u) + S_p(v,v,v)}{2} \\ \Rightarrow \quad & 2S_p(u,u,v) - S_p(u,u,u) - S_p(v,v,v) \leq 0 \\ \Rightarrow \quad & 2S_p(u,u,v) - S_p(u,u,u) - S_p(v,v,v) = 0 \\ \Rightarrow \quad & u = v. \end{aligned}$$

□

As a consequence of Theorem2, we obtain the following Corollary.

Corollary 1. *Let (X, S_p) be a 0-complete partial S-metric space. Suppose $T : X \to X$ is a given self mapping satisfying:*

$$S_p(Tx, Tx, Ty) \leq \phi(S_p(x,x,y)), \tag{7}$$

where $\phi : [0,\infty) \to [0,\infty)$ is an increasing function such that $f(t) = t - \phi(t)$ is increasing with f^{-1} is right continuous at 0. Also assume $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \geq 0$ (and hence $\phi(0) = 0, \phi(t) < t$ for $t > 0$). Then there is a unique $z \in X$ such that $Tz = z$. Also $S_p(z,z,z) = 0$.

Example 2. *Let* $X = [0,1] \cup [3,4]$*. Define* $S_p : X^3 \to [0,\infty)$, $T : X \to X$ *and* $\phi : [0,\infty) \to [0,\infty)$ *as follows:*

$$S_p(x,y,z) = \max\{x,y,z\}$$
$$T(x) = \begin{cases} \frac{x}{2} & , \quad x \in [0,1] \\ \frac{7}{5} & , \quad x \in [3,4] \end{cases}$$
$$\phi(t) = \frac{t}{1+t}$$

The above definitions satisfy the hypothesis of Theorem 2. In particular, we make the following observations:

- (X,p) *is a complete partial metric space.*
- *We can easily prove by induction that* $\phi^n(t) = \frac{t}{1+nt}$ *which implies that* $\lim_{n\to\infty} \phi^n(t) = 0$.
- T *satisfies condition (6):*

  1) If $\{x,y,z\} \cap [3,4] \neq \emptyset$ *then*

$$\begin{aligned} S_p(Tx,Ty,Tz) &= \max\{Tx,Ty,Tz\} = \frac{7}{5} \\ &\leq \max\left\{\phi(S_p(x,y,z)), \frac{S_p(x,x,x)+S_p(y,y,y)}{2}\right\} \end{aligned}$$

  2) If $\{x,y,z\} \subset [0,1]$ *then*

$$\begin{aligned} S_p(Tx,Ty,Tz) &= \max\{Tx,Ty,Tz\} = \max\left\{\frac{x}{2},\frac{y}{2},\frac{z}{2}\right\} \\ &\leq \max\left\{\phi(S_p(x,y)), \frac{S_p(x,x,x)+S_p(y,y,y)}{2}\right\}. \end{aligned}$$

By Theorem 2, there is a unique fixed point which is $z = 0$.

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Approximation by complex q-Gamma operators in compact disks

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Abstract. In this paper, the order of simultaneous approximation and Voronovskaya type theorems with quantitative estimate for complex q-Gamma operators attached to analytic functions in compact disks are obtained.

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1 Introduction

In recent years, an intensive research has been conducted on polynomials and operators in compact disks, such as [1], [3]-[8].

For a real function of real variable f: $[0,\infty) \to \mathbb{R}$, it is well known that the Gamma operators are given by $G_n(f;x) = \frac{1}{x^n\Gamma(n)}\int_0^\infty f(t/n)t^{n-1}e^{-t/x}dt$, $x \in [0,\infty)$. In 2005, Zeng [9] obtained the approximation properties of G_n defined above, supposed f satisfies exponential growth condition. he studied the approximation properties to the locally bounded functions and the absolutely continuous functions and obtained some good properties in real disks.

In this paper, we introduce complex q-Gamma operators as follows

$$G_{n,q}(f;z) = \frac{1}{z^n\Gamma_q(n)}\int_0^{\infty/A} f\left(\frac{t}{[n]_q}\right)t^{n-1}E_q\left(-\frac{qt}{z}\right)d_qt. \tag{1}$$

We give a suitable exponential growth condition in a parabolic domain for $f(z)$. Let $\mathbb{D}_R$=$\{z \in \mathbb{C}; |z| < R\}$ be with $1 < R < \infty$ and suppose that $f : [R,+\infty) \cup \overline{\mathbb{D}_R} \to \mathbb{C}$ is continuous in $[R,+\infty) \cup \overline{\mathbb{D}_R}$, analytic in $\mathbb{D}_R$, *i.e.* $f(z) = \sum_{k=0}^\infty c_k z^k$, for all $z \in \mathbb{D}_R$, and

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that there exist $M, C, B > 0$ and $A \in \left(\frac{1}{R}, 1\right)$, with the property $|c_k| \le M q^{k(k-1)/2} \frac{A^k}{[k]_q!}$, for all $k = 0, 1, ...$, which implies $|f(z)| \le M E_q(A|z|)$ for all $z \in \mathbb{D}_R$ and $|f(x)| \le C e^{Bx}$, for all $x \in [R, +\infty)$.

We recall some concepts of q-calculus. All of the results can be found in [7]. For any fixed real number $0 < q \le 1$ and each nonnegative integer k, we denote q-integers by $[k]_q$, where

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q}, & q \neq 1; \\ k, & q = 1. \end{cases}$$

Also q-factorial and q-binomial coefficients are defined as follows:

$$[k]_q! = \begin{cases} [k]_q[k-1]_q...[1]_q, & k = 1, 2, ...; \\ 1, & k = 0, \end{cases}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad (n \ge k \ge 0).$$

The q-improper integrals are defined as

$$\int_0^{\infty/A} f(x) d_q x = (1-q) \sum_{-\infty}^{\infty} f\left(\frac{q^n}{A}\right) \frac{q^n}{A}, \quad A > 0,$$

provided the sums converge absolutely.

The $q-$analogs $e_q(x)$ and $E_q(x)$ of the exponential function are given as

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!} = \frac{1}{(1-(1-q)x)_q^{\infty}}, \quad |x| < \frac{1}{1-q}, \quad |q| < 1,$$

$$E_q(x) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{x^k}{[k]_q!} = (1+(1-q)x)_q^{\infty}, \quad |q| < 1,$$

where $(1-x)_q^{\infty} = \prod_{j=0}^{\infty}(1-q^j x)$. It is easily observed that $e_q(x)E_q(-x) = e_q(-x)E_q(x) = 1$. The q-Gamma integral is defined as

$$\Gamma_q(t) = \int_0^{\infty/A} x^{t-1} E_q(-qx) d_q x, \quad t > 0, \tag{2}$$

which satisfies the following functional equations: $\Gamma_q(t+1) = [t]_q \Gamma_q(t), \quad \Gamma_q(1) = 1$.

2 Auxiliary Results

In the sequel, we suppose that $e_k(t) = t^k$, $k = 0, 1, 2, ...$ In order to obtain the main results, we need the following lemmas:

Lemma 2.1. *For $n \in \mathbb{N}$ and $z \in \mathbb{C}$, we have the following identities:*

$$G_{n,q}(e_k; z) = \frac{[n+k-1]_q!}{[n-1]_q![n]_q^k} e_k(z), \tag{3}$$

$$G_{n,q}(e_k; z) = \frac{[n+k-1]_q z}{[n]_q} G_n(e_{k-1}; z). \tag{4}$$

Proof. From (1) and (2), we have

$$\begin{aligned} G_{n,q}(e_k; z) &= \frac{1}{z^n \Gamma_q(n)} \int_0^{\infty/A} \left(\frac{t}{[n]_q}\right)^k t^{n-1} E_q\left(-\frac{qt}{z}\right) d_q t \\ &= \frac{z^k}{[n]_q^k \Gamma_q(n)} \int_0^{\infty/A} \left(\frac{t}{z}\right)^{n+k-1} E_q\left(-\frac{qt}{z}\right) d_q\left(\frac{t}{z}\right) \\ &= \frac{\Gamma_q(n+k) z^k}{[n]_q^k [n-1]_q!} = \frac{[n+k-1]_q!}{[n-1]_q![n]_q^k} e_k(z), \end{aligned}$$

so we proved (3), and (4) is easily obtained according to (3). □

Lemma 2.2. *If f is analytic in D_R, $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in D_R$, then for all $n \in \mathbb{N}$ and $1 \le r \le R$, we have*

$$G_{n,q}(f; z) = \sum_{k=0}^{\infty} c_k \cdot G_{n,q}(e_k; z). \tag{5}$$

Proof. By Lemma 2.1, we obtain that $G_{n,q}(e_k; z)$ is a polynomial of degree $\le k$, $k = 0, 1, 2, ...$ for all $z \in \mathbb{C}$. From the hypothesis on f in section 1, it follows that $G_{n,q}(f; z)$ is analytic in D_R (see [2], pp. 1171-1172 and p. 1178). Therefore, it is easy to obtain Lemma 2.2. □

3 Main Results

We start with the following quantitative estimates of the convergence for complex q-Gamma operators attached to an analytic function in a disk of radius $R > 1$ and center 0.

Theorem 3.1. *Let $\mathbb{D}_R = \{z \in \mathbb{C}; |z| < R\}$ be with $1 < R < \infty$ and suppose that $f : [R, +\infty) \bigcup \overline{\mathbb{D}_R} \to \mathbb{C}$ is continuous in $[R, +\infty) \cup \overline{\mathbb{D}_R}$, analytic in $\mathbb{D}_R$, i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$, and $f(z)$ satisfies exponential-type growth condition in the statement of section 1.*
(i) Let $1 \le r < \frac{1}{A}$ be arbitrary fixed. For all $|z| \le r$, $n \ge 2$ ($n \in \mathbb{N}$), we have

$$|G_{n,q}(f; z) - f(z)| \le \frac{L_{q,r,A}}{[n]_q},$$

where $L_{q,r,A} = M r^2 A^2 C_{q,r,A}$, $C_{q,r,A}$ is a constant depends only on q, r, A.
(ii) For the simultaneous approximation by complex q-Gamma operators, we have: if $1 \le$

$r \leq r_1 < \frac{1}{A}$ *are arbitrary fixed, then for all* $|z| \leq r$ *and* $n, p \in \mathbb{N}$, $n \geq 2$, *we have*

$$|G_{n,q}^{(p)}(f;z) - f^{(p)}(z)| \leq \frac{L_{q,r_1,A}}{[n]_q} \frac{p! r_1}{(r_1 - r)^{p+1}},$$

where $L_{q,r_1,A}$ *is defined at the above point (i).*

Proof. (i) Suppose that $|z| \leq r$, by Lemma 2.2, we have $G_{n,q}(f;z) = \sum_{k=0}^{\infty} c_k G_{n,q}(e_k;z)$, so we get

$$|G_{n,q}(f;z) - f(z)| \leq \sum_{k=0}^{\infty} |c_k| \cdot |G_{n,q}(e_k;z) - e_k(z)| = \sum_{k=2}^{\infty} |c_k| \cdot |G_{n,q}(e_k;z) - e_k(z)|,$$

since $G_{n,q}(e_0;z) = e_0(z) = 1$ and $G_{n.q}(e_1;z) = e_1(z) = z$.

By Lemma 2.1, for all $|z| \leq r$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
& |G_{n,q}(e_k;z) - e_k(z)| \\
= & \left| \frac{[n+k-1]_q z}{[n]_q} G_{n,q}(e_{k-1};z) - \frac{[n+k-1]_q z}{[n]_q} e_{k-1}(z) + \frac{[n+k-1]_q z}{[n]_q} e_{k-1}(z) - e_k(z) \right| \\
\leq & \left| \frac{[n+k-1]_q z}{[n]_q} \right| \cdot |G_{n,q}(e_{k-1};z) - e_{k-1}(z)| + |e_k(z)| \cdot \left| \frac{[n+k-1]_q}{[n]_q} - 1 \right| \\
\leq & \frac{[n+k-1]_q r}{[n]_q} |G_{n,q}(e_{k-1};z) - e_{k-1}(z)| + \frac{[k-1]_q r^k}{[n]_q},
\end{aligned}$$

step by step, we get by the above recurrence that

$$\begin{aligned}
& |G_{n,q}(e_k;z) - e_k(z)| \\
\leq & \frac{[n+k-1]_q}{[n]_q} \frac{[n+k-2]_q}{[n]_q} ... \frac{[n+2]_q}{[n]_q} \frac{q^n r^k}{[n]_q} + \frac{[n+k-1]_q}{[n]_q} \frac{[n+k-2]_q}{[n]_q} ... \frac{[n+3]_q}{[n]_q} \frac{q^n [2]_q r^k}{[n]_q} \\
& + \frac{[n+k-1]_q}{[n]_q} \frac{[n+k-2]_q}{[n]_q} ... \frac{[n+4]_q}{[n]_q} \frac{q^n [3]_q r^k}{[n]_q} + ... + \frac{[n+k-1]_q}{[n]_q} \frac{[n+k-2]_q}{[n]_q} \frac{q^n [k-3]_q r^k}{[n]_q} \\
& + \frac{[n+k-1]_q}{[n]_q} \frac{q^n [k-2]_q r^k}{[n]_q} + \frac{q^n [k-1]_q r^k}{[n]_q} \\
\leq & \frac{[n+k-1]_q}{[n]_q} \frac{[n+k-2]_q}{[n]_q} ... \frac{[n+2]_q}{[n]_q} r^k q^n \left(\frac{1}{[n]_q} + \frac{[2]_q}{[n]_q} + ... + \frac{[k-1]_q}{[n]_q} \right) \\
= & \frac{[n+k-1]_q!}{[n+1]_q! [n]_q^{k-2}} \frac{(1 + [2]_q + ... + [k-1]_q) q^n r^k}{[n]_q} \leq \frac{[n+k-1]_q!}{[n+1]_q! [n]_q^{k-2}} \frac{[k]_q [k-1]_q r^k}{[n]_q},
\end{aligned}$$

for all $|z| \leq r$ and $n \in \mathbb{N}$.

From Lemma 2.2 and the hypothesis on c_k, immediately implies for all $n \geq 2$ and

$|z| \le r$

$$\begin{aligned} |G_{n,q}(f;z)-f(z)| &\le \sum_{k=2}^{\infty}|c_k|\cdot|G_{n,q}(e_k;z)-e_k(z)| \\ &\le M\sum_{k=2}^{\infty}\frac{[n+k-1]_q!}{[n+1]_q![n]_q^{k-2}}\frac{[k]_q[k-1]_q r^k}{[n]_q}\frac{q^{k(k-1)/2}A^k}{[k]_q!} \\ &\le \frac{Mr^2A^2}{[n]_q}\sum_{k=2}^{\infty}\begin{bmatrix} n+k-1 \\ k-2 \end{bmatrix}_q\left(\frac{rA}{[n]_q}\right)^{k-2}, \end{aligned}$$

by Heine's binomial formula (see [7]), we have

$$\sum_{k=0}^{\infty}\begin{bmatrix} n+k+1 \\ k \end{bmatrix}_q\left(\frac{rA}{[n]_q}\right)^{k} = \frac{1}{\left(1-\frac{rA}{[n]_q}\right)_q^{n+2}}$$

$$\le \begin{cases} \dfrac{1}{\left(1-\frac{rA}{[n_0]_q}\right)_q^{n_0+2}}, & n=n_0, \\ \dfrac{1}{\left(1-\frac{rA}{[n]_q}\right)_q^{\infty}} = \displaystyle\sum_{j=0}^{\infty}\frac{\left(\frac{rA}{(1-q)[n]_q}\right)^j}{[j]_q!} = e_q\left(\frac{rA}{[n]_q}\right), & n=\infty, \end{cases} \le C_{q,r,A},$$

where, n_0 is a finite number and $C_{q,r,A}$ is a constant depends only on $q,\ r,\ A$. Thus

$$|G_{n,q}(f;z)-f(z)| \le \frac{Mr^2A^2C_{q,r,A}}{[n]_q},$$

therefore we have

$$|G_{n,q}(f;z)-f(z)| \le \frac{L_{q,r,A}}{[n]_q},$$

where $L_{q,r,A} = Mr^2A^2C_{q,r,A}$, for all $1 \le r < \frac{1}{A}$.

(ii) Denoting by γ the circle of radius $r_1 > r$ and center 0, since for any $|z| \le r$ and $v \in \gamma$, we have $|v-z| \ge r_1 - r$, by Cauchy's formulas it follows that for all $|z| \le r$ and $n \in \mathbb{N}$, $n \ge 2$, we have

$$\begin{aligned} |G_{n,q}^{(p)}(f;z)-f^{(p)}(z)| &= \frac{p!}{2\pi}\left|\int_\gamma \frac{G_{n,q}(f;v)-f(v)}{(v-z)^{p+1}}dv\right| \\ &\le \frac{L_{q,r_1,A}}{[n]_q}\frac{p!}{2\pi}\frac{2\pi r_1}{(r_1-r)^{p+1}} = \frac{L_{q,r_1,A}}{[n]_q}\frac{p!r_1}{(r_1-r)^{p+1}}, \end{aligned}$$

which proves (ii) and Theorem 3.1. □

Next, we will give Voronovskaya type result in compact disks, for complex q-Gamma operators attached to an analytic function in D_R, $R > 1$ and center 0.

Theorem 3.2. *Suppose that $f : \overline{\mathbb{D}_R} \cup [R,\infty) \to \mathbb{C}$ is continuous and bounded in $\overline{\mathbb{D}_R}\cup[R,\infty)$ and analytic in $\mathbb{D}_R$. Let $1 \le r < \frac{1}{A}$ be arbitrary fixed, $n \ge 2$ $(n \in \mathbb{N})$, then we have the following Voronovskaya type result*

$$\left|G_{n,q}(f;z) - f(z) - \frac{z^2 f''(z)}{[2]_q[n]_q}\right| \le \frac{J_{q,r,A}}{[n]_q^2},$$

where $J_{q,r,A} = \frac{1}{[2]_q^2}\sum_{k=3}^{\infty}[k-2]_q^2[k-1]_q(rA)^k < \infty$.

Proof. Denoting $E_{q,k,n}(z) = G_{n,q}(e_k;z) - e_k(z) - \frac{q^n[k]_q[k-1]_q e_k(z)}{[2]_q[n]_q}$, since $E_{q,0,n}(z) = E_{q,1,n}(z) = E_{q,2,n}(z) = 0$, then we have

$$\left|G_{n,q}(f;z) - f(z) - \frac{z^2 f''(z)}{[2]_q[n]_q}\right| \le \sum_{k=3}^{\infty}|c_k|\cdot|E_{q,k,n}(z)|,$$

so, it remains to estimate $E_{q,k,n}(z)$ for $k \ge 3$.

By Lemma 2.1 and simple calculation, we have

$$E_{q,k,n}(z) = \frac{[n+k-1]_q z}{[n]_q}E_{q,k-1,n}(z) + \frac{q^n[k-1]_q[k-2]_q\left([n+k-1]_q - q^2[n]_q\right)}{[2]_q[n]_q^2}z^k,$$

this implies, for all $|z| \le r$, $k \ge 3$, $n \in \mathbb{N}$,

$$|E_{q,k,n}(z)| \le \frac{[n+k-1]_q r}{[n]_q}|E_{q,k-1,n}(z)| + \frac{q^n[k-1]_q^2[k-2]_q}{[2]_q[n]_q^2}r^k,$$

taking in the last inequality, $k = 3,4,...$, and reasoning by recurrence, we obtain

$$\begin{aligned}
|E_{q,k,n}(z)| &\le \frac{[n+k-1]_q}{[n]_q}\frac{[n+k-2]_q}{[n]_q}\cdots\frac{[n+3]_q}{[n]_q}\frac{q^n\cdot[2]_q^2}{[2]_q[n]_q^2}r^k \\
&\quad + \frac{[n+k-1]_q}{[n]_q}\frac{[n+k-2]_q}{[n]_q}\cdots\frac{[n+4]_q}{[n]_q}\frac{q^n[2]_q\cdot[3]_q^2}{[2]_q[n]_q^2}r^k \\
&\quad + \ldots + \frac{[n+k-1]_q}{[n]_q}\frac{q^n[k-3]_q[k-2]_q^2}{[2]_q[n]_q^2}r^k + \frac{q^n[k-2]_q[k-1]_q^2}{[2]_q[n]_q^2}r^k \\
&\le \frac{[n+k-1]_q}{[n]_q}\frac{[n+k-2]_q}{[n]_q}\cdots\frac{[n+3]_q}{[n]_q}\frac{q^n r^k}{[2]_q[n]_q^2}\sum_{j=3}^{k}[j-2]_q[j-1]_q^2 \\
&\le \frac{[n+k-1]_q}{[n]_q}\frac{[n+k-2]_q}{[n]_q}\cdots\frac{[n+3]_q}{[n]_q}\frac{[k-2]_q^2[k-1]_q^2 q^n r^k}{[2]_q[n]_q^2},
\end{aligned}$$

by the hypothesis on c_k, we have

$$\begin{aligned}
&\left|G_{n,q}(f;z) - f(z) - \frac{z^2 f''(z)}{[2]_q[n]_q}\right| \le \sum_{k=3}^{\infty}|c_k|\cdot|E_{q,k,n}(z)| \\
&\le \frac{1}{[2]_q[n]_q^2}\sum_{k=3}^{\infty}\frac{[n+k-1]_q}{[n]_q[k-1]_q}\frac{[n+k-2]_q}{[n]_q[k-2]_q}\cdots\frac{[n+3]_q}{[3]_q[n]_q}\frac{[k-2]_q^2[k-1]_q^2(rA)^k}{[2]_q[k]_q} \\
&\le \frac{1}{[2]_q^2[n]_q^2}\sum_{k=3}^{\infty}[k-2]_q^2[k-1]_q(rA)^k,
\end{aligned}$$

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for all $|z| \leq r$ and $n \geq 2$ $(n \in \mathbb{N})$. Since the series $\sum_{k=3}^{\infty} u^{k+1}$ and its q-derivative $\sum_{k=3}^{\infty}[k+1]_q u^k$ are uniformly and absolutely convergent in any compact disk included in the open unit disk, therefore, for $1 \leq r < \frac{1}{A}$, we have $\frac{1}{[2]_q^2[n]_q^2}\sum_{k=3}^{\infty}[k-2]_q^2[k-1]_q(rA)^k < \infty$. □

Denoting with $||P_k||_r = max\{|P_k(z)| : |z| \leq r\}$, where $P_k(z)$ is a complex polynomial of degree ≤ k. Now we will give the exact order of approximation by complex q-Gamma operators.

Theorem 3.3. *In the hypothesis of Theorem 3.1, if f is not a polynomial of degree ≤ 1 in the case (i), we have*

$$||G_{n,q}(f) - f||_r \geq \frac{1}{[n]_q} U_r(f), \quad n \in \mathbb{N},$$

where the constant $U_r(f)$ depends only on f and r.

Proof. Applying the norm $||\cdot||_r$ to the identity

$$G_{n,q}(f;z) - f(z) = \frac{1}{[n]_q}\left\{\frac{z^2}{[2]_q}f''(z) + \frac{1}{[n]_q}\left[[n]_q^2\left(G_{n,q}(f;z) - f(z) - \frac{z^2}{[2]_q[n]_q}f''(z)\right)\right]\right\},$$

we get

$$||G_{n,q}(f) - f||_r \geq \frac{1}{[n]_q}\left\{\left|\left|\frac{e_2}{[2]_q}f''\right|\right|_r - \frac{1}{[n]_q}\left[[n]_q^2\left|\left|G_{n,q}(f) - f - \frac{e_2}{[2]_q[n]_q}f''\right|\right|_r\right]\right\}.$$

Since f is not a polynomial of degree ≤ 1 in $\mathbb{D}_R$, it follows that $||\frac{e_2}{[2]_q}f''||_r > 0$. Indeed, supposing the contrary it follows that $z^2 f''(z) = 0$ for all $z \in \overline{\mathbb{D}_R}$, therefore we get $f''(z) = 0$ for all $z \in \overline{\mathbb{D}_R}$, by the uniqueness of analytic functions we get $f''(z) = 0$ for all $z \in \overline{\mathbb{D}_R}$, that is f is a linear function in $\mathbb{D}_R$, which is in contradiction with the hypothesis.

Now, by Theorem 3.2, we have

$$\left|G_{n,q}(f;z) - f(z) - \frac{z^2 f''(z)}{[2]_q[n]_q}\right| \leq \frac{1}{[2]^2[n]_q^2}\sum_{k=3}^{\infty}[k-2]_q^2[k-1]_q(rA)^k.$$

Therefore, there exists an index n_0 (depending only on f and r) such that for all $n \geq n_0$, we have

$$\left|\left|\frac{e_2}{[2]_q}f''\right|\right|_r - \frac{1}{[n]_q}\left[[n]_q^2\left|\left|G_{n,q}(f) - f - \frac{e_2}{[2]_q[n]_q}f''\right|\right|_r\right] \geq \frac{1}{[2]_q^2}||e_2 f''||_r,$$

which implies

$$||G_{n,q}(f) - f||_r \geq \frac{1}{[2]_q^2}||e_2 f''||_r,$$

for all $n \geq n_0$.

For $1 \leq n \leq n_0 - 1$, we have

$$||G_{n,q}(f) - f||_r \geq \frac{1}{[n]_q}([n]_q||G_{n,q}(f) - f||_r) = \frac{1}{[n]_q}V_{r,n}(f) > 0,$$

Therefore, finally we obtain

$$||G_{n,q}(f)-f||_r \geq \frac{1}{[n]_q}U_r(f),$$

for all n, with $U_r(f)=min\left\{V_{r,1}(f),V_{r,2}(f),...,V_{r,n_0}(f),\frac{1}{[2]_q^2}||e_2f''||_r\right\}$. □

Combining Theorem 3.1 with Theorem 3.3, we immediately get the following result:

Corollary 3.4. *In the hypothesis of Theorem 3.1 and Theorem 3.3, we have*

$$||G_{n,q}(f)-f||_r \sim \frac{1}{[n]_q}, \quad n\in\mathbb{N}.$$

Theorem 3.5. *In the hypothesis of Theorem 3.1, if $1\leq r\leq r_1<\frac{1}{A}$ are arbitrary fixed and f is not a polynomial of degree $\leq p-1$, then for all $|z|\leq r$ and $n,p\in\mathbb{N}$ $(n\geq 2)$, we have*

$$||G_{n,q}^{(p)}(f)-f^{(p)}||_r \sim \frac{1}{[n]_q}.$$

Proof. Taking into account the upper estimate in case (ii) of Theorem 3.1, it remains to prove the lower estimate only.

Denoting by Γ the circle of radius $r_1>r$ and center 0, since for any $|z|\leq r$ and $v\in\Gamma$, we have $|v-z|\geq r_1-r$, by Cauchy's formulas it follows that for all $|z|\leq r$ and $n\in\mathbb{N}$, we get

$$G_{n,q}^{(p)}(f;z)-f^{(p)}(z)=\frac{p!}{2\pi i}\int_\Gamma \frac{G_{n,q}(f;v)-f(v)}{(v-z)^{p+1}}dv, \tag{6}$$

as we have the identity

$$G_{n,q}(f;z)-f(z)=\frac{1}{[n]_q}\left\{\frac{z^2}{[2]_q}f''(z)+\frac{1}{[n]_q}\left[[n]_q^2\left(G_{n,q}(f;z)-f(z)-\frac{z^2}{[2]_q[n]_q}f''(z)\right)\right]\right\}, \tag{7}$$

applying (7) to (6), we have

$$\begin{aligned}
&G_{n,q}^{(p)}(f;z)-f^{(p)}(z)\\
=\ &\frac{1}{[n]_q}\left\{\frac{p!}{2\pi i}\int_\Gamma\frac{v^2f''(v)}{[2]_q(v-z)^{p+1}}dv+\frac{1}{[n]_q}\frac{p!}{2\pi i}\int_\Gamma\frac{[n]_q^2\left[G_{n,q}(f;v)-f(v)-\frac{v^2}{[2]_q[n]_q}f''(v)\right]}{(v-z)^{p+1}}dv\right\}\\
=\ &\frac{1}{[n]_q}\left\{\left|\left|\left(\frac{e_2f''}{[2]_q}\right)^{(p)}\right|\right|_r+\frac{1}{[n]_q}\frac{p!}{2\pi i}\int_\Gamma\frac{[n]_q^2\left[G_{n,q}(f;v)-f(v)-\frac{v^2}{[2]_q[n]_q}f''(v)\right]}{(v-z)^{p+1}}dv\right\},
\end{aligned}$$

applying the norm $||\cdot||_r$ to the above identity, we have

$$\begin{aligned}
&||G_{n,q}^{(p)}(f)-f^{(p)}||_r\\
\geq\ &\frac{1}{[n]_q}\left\{\left|\left|\left(\frac{e_2f''}{[2]_q}\right)^{(p)}\right|\right|_r-\frac{1}{[n]_q}\left|\left|\frac{p!}{2\pi}\int_\Gamma\frac{[n]_q^2\left[G_{n,q}(f;v)-f(v)-\frac{v^2}{[2]_q[n]_q}f''(v)\right]}{(v-z)^{p+1}}dv\right|\right|_r\right\},
\end{aligned}$$

by using Theorem 3.2, we have

$$\left\|\frac{p!}{2\pi}\int_\Gamma \frac{[n]_q^2\left[G_{n,q}(f;v)-f(v)-\frac{v^2}{[2]_q[n]_q}f''(v)\right]}{(v-z)^{p+1}}dv\right\|_r \leq \frac{p!}{2\pi}\frac{2\pi r_1}{(r_1-r)^{p+1}}J_{q,r_1,A}=\frac{J_{q,r_1,A}p!r_1}{(r_1-r)^{p+1}},$$

by the hypothesis on f, we have $\left\|\left(\frac{e_2f''}{[2]_q}\right)^{(p)}\right\|_r > 0$, reasoning exactly as in the proof of Theorem 3.3, we immediately get the desired result. □

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Value sharing of meromorphic functions of differential polynomials of finite order

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Abstract

In this paper, we shall study the uniqueness problems on meromorphic functions of differential polynomials of finite order sharing a value. Our results improve or generalize many previous results on value sharing of meromorphic functions, such as Fang and Hua, Yang and Hua, Lin and Yi, Zhang, Xu, et al.

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1 Introduction and main results

Let $\mathbb{C}$ denote the complex plane and $f(z)$ be a non-constant meromorphic function on $\mathbb{C}$. We assume the reader is familiar with the standard notion used in the Nevanlinna value distribution theory such as $T(r,f)$, $m(r,f)$, $N(r,f)$ (see [7, 13, 14]), and $S(r,f)$ denotes any quantity that satisfies the condition $S(r,f)=o(T(r,f))$ as $r\to\infty$ outside of a possible exceptional set of finite linear measure. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$, provided that $T(r,a)=S(r,f)$.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $a\in\mathbb{C}\bigcup\{\infty\}$, we say that $f(z)$, $g(z)$ share a CM (counting multiplicities) if $f(z)-a$, $g(z)-a$ have the same zeros with the same multiplicities and we say that $f(z)$, $g(z)$ share a IM (ignoring multiplicities) if we do not consider the multiplicities. $N_k(r,f)$ denotes the truncated counting function bounded by k.

Define the order of f as

$$\sigma(f)=\limsup_{r\longrightarrow\infty}\frac{\log^+ T(r,f)}{\log r},$$

The following well known theorem in value distribution theory was posed by Hayman and settled by several authors almost at the same time [2, 4].

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Theorem A: *Let $f(z)$ be a transcendental meromorphic function, $n \geq 1$ a positive integer. Then $f^n f' = 1$ has infinitely many solutions.*

Fang and Hua [5], Yang and Hua [12] got a unicity theorem respectively corresponding to Theorem A.

Theorem B: *Let f and g be two non-constant entire (meromorphic) functions, $n \geq 6 (n \geq 11)$ be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.*

Note that $f^n(z)f'(z) = \frac{1}{n+1}(f^{n+1}(z))'$, Fang [6] considered the case of kth derivative and proved

Theorem C: *Let f and g be two non-constant entire functions, and let n, k be two positive integers with $n > 2k + 4$. If $(f^n(z))^{(k)}$ and $(g^n(z))^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$.*

Theorem D: *Let f and g be two non-constant entire functions, and let n, k be two positive integers with $n > 2k + 8$. If $(f^n(z)(f(z) - 1))^{(k)}$ and $(g^n(z)(g(z) - 1))^{(k)}$ share 1 CM, then $f(z) \equiv g(z)$.*
For more results on this field, see [8, 9, 17]. Corresponding to Theorems C and D, It is natural to ask the following question.

Question 1.1. *Does Theorem C or D holds if f and g are meromorphic functions?*

Remark 1.1. Question 1.1 seems to have been solved by Bhoosnurmath and Dyavanal [3], but their proofs contain some gaps that were pointed out by Zhang [15, Annex remarks], Xu et al [10, Remark 2], respectively. The gaps have not been filled as far as we know. Here we give a partial answer to Problem 1.1.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions with $\sigma(f) < +\infty$. Let n, k be two positive integers with $n > \max\{3k + 8, 2(\sigma(f) - 1)k\}$. If $[f^n(z)]^{(k)}$ and $[g^n(z)]^{(k)}$ share 1 CM, then one of the following two conclusions holds:*
(1) $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$;
(2) $f = c_3 e^{dz}$, $g = c_4 e^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k(c_3c_4)^n(nd)^{2k} = 1$.

Remark 1.2. Theorem 1.1 affirmatively answered Problems 1.1. Namely, Theorem C holds for the case of meromorphic functions of finite order, provided that n is sufficiently large. But unfortunately, Theorems D fails if $f(z)$ and $g(z)$ are meromorphic functions without the condition $\Theta(\infty, f) > 2/n$, even if f and g share ∞ CM. We give the following counterexample.

Example 1.1. *Let*

$$f(z) = \frac{h(z)(1 - h^n(z))}{1 - h^{n+1}(z)}, \quad g(z) = \frac{1 - h^n(z)}{1 - h^{n+1}(z)}, \tag{1.1}$$

where n is a positive integer and $h(z)$ is a non-constant meromorphic function.

We deduce from (1.1) that $f^n(f-1) = g^n(g-1)$, thus f and g satisfy the conditions of Theorem D, but $f \not\equiv g$.
Note that

$$T(r, f) = T(r, gh) = nT(r, h) + S(r, f).$$

By the second fundamental theorem, we deduce

$$\overline{N}(r, f) = \sum_{j=1}^{n} \overline{N}(\frac{1}{h - a_j}) \geq (n-2)T(r, h) + S(r, f),$$

where $a_j (\neq 1)$ $(j = 1, 2, \cdots, n)$ are distinct roots of the algebraic equation $h^{n+1} = 1$.
Therefore,

$$\Theta(\infty, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, f)}{T(r, f)} \leq 2/n.$$

When $n > 3k+8$, then $\frac{n}{2k} + 1 > \frac{5}{2}$, so from Theorem 1.1 we have

Corollary 1.1. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions with $\sigma(f) < 3$. Let n, k be two positive integers with $n > 3k+8$. If $[f^n(z)]^{(k)}$ and $[g^n(z)]^{(k)}$ share 1 CM, then one of the following two conclusions holds:*
(1) $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$;
(2) $f = c_3 e^{dz}$, $g = c_4 e^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k (c_3c_4)^n (nd)^{2k} = 1$.

Consider IM sharing value and we have

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions with $\sigma(f) < +\infty$. Let n, k be two positive integers with $n > \max\{9k+14, 2(\sigma(f)-1)k\}$. If $[f^n(z)]^{(k)}$ and $[g^n(z)]^{(k)}$ share 1 IM, then one of the following two conclusions holds:*
(1) $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$;
(2) $f = c_3 e^{dz}$, $g = c_4 e^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k (c_3c_4)^n (nd)^{2k} = 1$.

Corollary 1.2. *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions with $\sigma(f) < 6$. Let n, k be two positive integers with $n > 9k+14$. If $[f^n(z)]^{(k)}$ and $[g^n(z)]^{(k)}$ share 1 IM, then one of the following two conclusions holds:*
(1) $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$;
(2) $f = c_3 e^{dz}$, $g = c_4 e^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k (c_3c_4)^n (nd)^{2k} = 1$.

2 Preliminary lemmas and a main proposition

Lemma 2.1. *[11] Let $f(z)$ be a non-constant meromorphic function and let $a_0(z), a_1(z), \cdots, a_n(z) (\not\equiv 0)$ be small functions of f. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.2. *[16] Let $f(z)$ be a non-constant meromorphic function, s, k be two positive integers. Then*

$$N_s(r, \frac{1}{f^{(k)}}) \leq T(r, f^{(k)}) - T(r, f) + N_{s+k}(r, \frac{1}{f}) + S(r, f),$$

$$N_s(r, \frac{1}{f^{(k)}}) \leq k\overline{N}(r, f) + N_{s+k}(r, \frac{1}{f}) + S(r, f).$$

Lemma 2.3. *Let $f(z)$ be a non-constant meromorphic function of finite order, and let k be a positive integer. Suppose that $f^{(k)} \not\equiv 0$, then*

$$N(r, \frac{1}{f^{(k)}}) \leq N(r, \frac{1}{f}) + k\overline{N}(r, f) + O(\log r).$$

Proof. Note that f is of finite order, by Lemma $1.4'$ in [14, P. 21], we have

$$m(r, \frac{f'}{f}) = O(\log r).$$

Now we prove $m(r, \frac{f^{(k)}}{f}) = O(\log r)$ by mathematical induction. Suppose that the conclusion is true for the case of $k = m$, if $k = m + 1$, we have

$$\frac{f^{(m+1)}}{f} = (\frac{f^{(m)}}{f})' + \frac{f^{(m)}}{f}\frac{f'}{f}.$$

Then we get

$$\begin{aligned} m(r, \frac{f^{(m+1)}}{f}) &\leq m(r, (\frac{f^{(m)}}{f})') + m(r, \frac{f^{(m)}}{f}) + m(r, \frac{f'}{f}) + O(1) \\ &= m(r, \frac{(\frac{f^{(m)}}{f})'}{\frac{f^{(m)}}{f}}\frac{f^{(m)}}{f}) + O(\log r) \\ &\leq m(r, \frac{(\frac{f^{(m)}}{f})'}{\frac{f^{(m)}}{f}}) + m(r, \frac{f^{(m)}}{f}) + O(\log r) \\ &= O(\log r). \end{aligned}$$

Moreover, we have

$$m(r, \frac{1}{f}) \leq m(r, \frac{1}{f^{(k)}}) + m(r, \frac{f^{(k)}}{f}) = m(r, \frac{1}{f^{(k)}}) + O(\log r).$$

Hence

$$T(r, f) - N(r, \frac{1}{f}) \leq T(r, f^{(k)}) - N(r, \frac{1}{f^{(k)}}) + O(\log r).$$

That is

$$\begin{aligned} N(r,\frac{1}{f^{(k)}}) &\leq T(r,f^{(k)})-T(r,f)+N(r,\frac{1}{f})+O(\log r)\\ &= m(r,f^{(k)})+N(r,f^{(k)})-T(r,f)+N(r,\frac{1}{f})+O(\log r)\\ &\leq m(r,f)+m(r,\frac{f^{(k)}}{f})+N(r,f)+k\overline{N}(r,f)-T(r,f)+N(r,\frac{1}{f})+O(\log r)\\ &= N(r,\frac{1}{f})+k\overline{N}(r,f)+O(\log r). \end{aligned}$$

This completes the proof of Lemma 2.3.

Lemma 2.4. *[12] Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and n, k be two positive integers, a be a finite nonzero constant. If f and g share a CM, then one of the following cases holds:*
(i) $T(r,f) \leq N_2(r,1/f)+N_2(r,1/g)+N_2(r,f)+N_2(r,g)+S(r,f)+S(r,g)$, the same inequality holding for $T(r,g)$;
(ii) $fg\equiv a^2$; (iii) $f\equiv g$.

Lemma 2.5. *Let $f(z)$ and $g(z)$ be non-constant meromorphic functions, n, k be two positive integers with $n > k+2$, a be a finite nonzero constant. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share a IM. Then $T(r,f)=O(T(r,g))$, $T(r,g)=O(T(r,f))$ and $\sigma(f)=\sigma(g)$.*

Proof. Let $F=f^n$. By the second fundamental theorem for small functions, we have

$$T(r,F^{(k)}) \leq \overline{N}(r,f)+\overline{N}(r,\frac{1}{F^{(k)}})+\overline{N}(r,\frac{1}{F^{(k)}-a})+S(r,F). \tag{2.1}$$

By (2.1) and Lemma 2.1 and Lemma 2.2 with $s=1$ applied to F, we have

$$\begin{aligned} nT(r,f) &\leq N_{k+1}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{F^{(k)}-a})+\overline{N}(r,f)+S(r,F)\\ &\leq (k+1)\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{[f^n]^{(k)}-a})+\overline{N}(r,f)+S(r.f)\\ &\leq (k+2)T(r,f)+\overline{N}(r,\frac{1}{[g^n]^{(k)}-a})+S(r,f). \end{aligned}$$

Namely,

$$\begin{aligned} (n-k-2)T(r,f) &\leq \overline{N}(r,\frac{1}{[g^n]^{(k)}-a})+S(r,f)\\ &\leq n(k+1)T(r,g)+S(r,f). \end{aligned}$$

Since $n > k+2$, we have $T(r,f)=O(T(r,g))$. Similarly we have $T(r,g)=O(T(r,f))$. Thus $\sigma(f)=\sigma(g)$.
This completes the proof of Lemma 2.5.

By the arguments similar to the proof of Lemma 2.5, we get the following proposition.

Proposition 2.1. *Let f be a transcendental meromorphic function, n, k be two positive integers with $n > k+2$, $a(z)(\not\equiv 0, \infty)$ be a small function of f. Then $[f^n]^{(k)} - a(z)$ has infinitely many zeros.*

Lemma 2.6. *[10] Let f and g be two non-constant meromorphic functions, $k, n > 2k+1$ be two positive integers. If $[f^n]^{(k)} = [g^n]^{(k)}$, then $f = tg$ for a constant t such that $t^n = 1$.*

Lemma 2.7. *Let f, g be two nonconstant meromorphic functions with $\sigma(f) < +\infty$, n, k be two positive integers with $n > \max\{3k+8, 2(\sigma(f)-1)k\}$. If $[f^n]^{(k)}[g^n]^{(k)} = 1$, then $f = c_3e^{dz}$, $g = c_4e^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k(c_3c_4)^n(nd)^{2k} = 1$.*

Proof. Note that $n > k+2$, $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 1 IM. Then by Lemma 2.5 we get $\sigma(f) = \sigma(g) < +\infty$.

First, we prove

$$f \neq 0, \quad g \neq 0. \tag{2.2}$$

Suppose that z_0 is a zero of f with multiplicity s, then z_0 is a pole of g, say multiplicity t, and z_0 is a zero of $[f^n]^{(k)}$ with multiplicity $ns-k$, a pole of $[g^n]^{(k)}$ with multiplicity $nt+k$, thus we have

$$ns - k = nt + k,$$

namley

$$n(s-t) = 2k. \tag{2.3}$$

Note that $n > 3k+8$ and we get a contradiction from (2.3). Thus f has no zero. Similarly, g has no zero. Thus (2.2) holds.
Next we prove

$$N(r, f) = O(\log r), \quad N(r, g) = O(\log r). \tag{2.4}$$

Rewrite $[f^n]^{(k)}[g^n]^{(k)} = 1$ as

$$[f^n]^{(k)} = \frac{1}{[g^n]^{(k)}}. \tag{2.5}$$

We deduce from (2.5) that

$$N(r, [f^n]^{(k)}) = N(r, \frac{1}{[g^n]^{(k)}}). \tag{2.6}$$

As $N(r, [f^n]^{(k)}) = nN(r, f) + k\overline{N}(r, f)$, this together with (2.2), (2.6) and Lemma 2.3 implies that

$$nN(r, f) + k\overline{N}(r, f) \leq k\overline{N}(r, g) + O(\log r). \tag{2.7}$$

Similarly we get

$$nN(r, g) + k\overline{N}(r, g) \leq k\overline{N}(r, f) + O(\log r). \tag{2.8}$$

Combining (2.7) and (2.8) yields

$$N(r,f)+N(r,g)=O(\log r). \tag{2.9}$$

Thus we obtain (2.4), which means that both f and g have at most finitely many poles. Let

$$f=\frac{e^{h(z)}}{p(z)},\quad g=\frac{e^{h_1(z)}}{q(z)}, \tag{2.10}$$

where $p(z)$ and $q(z)$ are polynomials with $\deg(p(z))=p$, $\deg(q(z))=q$, $h(z)$ and $h_1(z)$ are nonconstant entire functions. By Corollary 1 in [14, P. 65], $h(z)$ and $h_1(z)$ are polynomials with $\deg(h(z))=\deg(h_1(z))=h=\sigma(f)$. Then

$$f^n=\frac{e^{nh(z)}}{p^n(z)},\quad g^n=\frac{e^{nh_1(z)}}{q^n(z)}. \tag{2.11}$$

Let $H(z)=nh(z)$, $P(z)=p^n(z)$, $H_1(z)=nh_1(z)$, $Q(z)=q^n(z)$. By mathematical induction we get that

$$[f^n]^{(k)}=\frac{e^{H(z)}P_k(z)}{P^{k+1}(z)},\quad [g^n]^{(k)}=\frac{e^{H_1(z)}Q_k(z)}{Q^{k+1}(z)}, \tag{2.12}$$

where $P_k(z)$ and $Q_k(z)$ are two polynomials with $\deg(P_k(z))=k(h-1+np)$ and $\deg(Q_k(z))=k(h-1+nq)$. By $[f^n]^{(k)}[g^n]^{(k)}=1$, we have

$$h(z)+h_1(z)\equiv C, \tag{2.13}$$

where C is a constant. Furthermore, we get

$$\deg(P_k(z))+\deg(Q_k(z))=\deg(P^{k+1}(z)Q^{k+1}(z)), \tag{2.14}$$

which implies that

$$2k(h-1)=n(p+q). \tag{2.15}$$

By (2.4), if

$$N(r,f)+N(r,g)\neq 0, \tag{2.16}$$

then $p+q\geq 1$, we deduce from (2.15) that

$$n\leq 2k(h-1)=2k(\sigma(f)-1), \tag{2.17}$$

which contradicts the assumption. Therefore

$$N(r,f)+N(r,g)=0, \tag{2.18}$$

namely both f and g are entire functions and $p=q=0$. From (2.15) we obtain that $h=1$. Thus $h(z)=dz+l_3$, $h_1(z)=-dz+l_4$.
Rewrite f and g as

$$f=c_3e^{dz},\qquad g=c_4e^{-dz},$$

where c_3,c_4 and d are nonzero constants. We deduce that $(-1)^k(c_3c_4)^n(nd)^{2k}=1$.

This completes the proof of Lemma 2.7.

Lemma 2.8. *[1] Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and n, k be two positive integers, a be a finite nonzero constant. If f and g share a IM, then one of the following cases holds:*
(i) $T(r,f) \leq N_2(r,1/f)+N_2(r,1/g)+N_2(r,f)+N_2(r,g)+2\overline{N}(r,1/f)+\overline{N}(r,1/g)+2\overline{N}(r,f)+\overline{N}(r,g)+S(r,f)+S(r,g)$*, the similar inequality holding for $T(r,g)$;*
(ii) $fg \equiv a^2$; *(iii)* $f \equiv g$.

3 Proof of Theorem 1.1

Let $F = [f^n]^{(k)}$, $G = [g^n]^{(k)}$, $F^* = f^n$, $G^* = g^n$, then F and G share 1 CM.
Thus by Lemma 2.5, one of the following cases holds:
(i) $T(r,F) \leq N_2(r,1/F) + N_2(r,1/G) + N_2(r,F) + N_2(r,G) + S(r,F) + S(r,G)$, the same inequality holding for $T(r,G)$;
(ii) $FG \equiv 1$; *(iii)* $F \equiv G$.

Case *(i)***.** By Lemma 2.1 and Lemma 2.2 with $s = 2$, we obtain

$$\begin{aligned} T(r,F^*) &\leq N_{k+2}(r,1/F^*) + N_{k+2}(r,1/G^*) + (k+2)\overline{N}(r,g) + 2\overline{N}(r,f) \\ &\quad +S(r,f) + S(r,g) \\ &\leq (k+2)\overline{N}(r,1/f) + (k+2)\overline{N}(r,1/g) + (k+2)\overline{N}(r,g) + 2\overline{N}(r,f) \\ &\quad +S(r,f) + S(r,g) \\ &\leq (2k+4)T(r,g) + (k+4)T(r,f) + S(r,f) + S(r,g), \end{aligned}$$

namely

$$nT(r,f) \leq (2k+4)T(r,g) + (k+4)T(r,f) + S(r,f) + S(r,g). \tag{3.1}$$

Similarly we have

$$nT(r,g) \leq (2k+4)T(r,f) + (k+4)T(r,g) + S(r,f) + S(r,g). \tag{3.2}$$

From (3.1) and (3.2) we deduce that

$$(n-3k-8)(T(r,f) + T(r,g) \leq S(r,f) + S(r,g), \tag{3.3}$$

which is a contradiction since $n > 3k+8$.

Case *(ii)***.** We have $[f^n]^{(k)}[g^n]^{(k)} = 1$. By Lemma 2.7 we get the conclusion (2) of Theorem 1.1.

Case *(iii)***.** We have $[f^n]^{(k)} \equiv [g^n]^{(k)}$. By Lemma 2.6 we get the conclusion (1) of Theorem 1.1.

This completes the proof of Theorem 1.1.

4 Proof of Theorem 1.2

Let $F = [f^n]^{(k)}$, $G = [g^n]^{(k)}$, $F^* = f^n$, $G^* = g^n$, then F and G share 1 IM. Thus by Lemma 2.8, one of the following cases holds:
(i) $T(r, F) \le N_2(r, 1/F) + N_2(r, 1/G) + N_2(r, F) + N_2(r, G) + 2\overline{N}(r, 1/F) + \overline{N}(r, 1/G) + 2\overline{N}(r, F) + \overline{N}(r, G) + S(r, F) + S(r, G)$, the same inequality holding for $T(r, G)$;
(ii) $FG \equiv 1$; (iii) $F \equiv G$.

Case (i)**.** By Lemma 2.1 and Lemma 2.2 with $s = 1, 2$, we obtain

$$\begin{aligned} T(r, F^*) &\le N_{k+2}(r, 1/F^*) + N_{k+2}(r, 1/G^*) + (k+2)\overline{N}(r, g) + 2\overline{N}(r, f) \\ &\quad 2(N_{k+1}(r, 1/F^*) + k\overline{N}(r, f)) + N_{k+1}(r, 1/G^*) + k\overline{N}(r, g) \\ &\quad +2\overline{N}(r, f) + \overline{N}(r, g) + S(r, f) + S(r, g) \\ &\le (3k+4)\overline{N}(r, 1/f) + (2k+3)\overline{N}(r, 1/g) + (2k+4)\overline{N}(r, f) \\ &\quad +(2k+3)\overline{N}(r, g) + S(r, f) + S(r, g) \\ &\le (5k+8)T(r, f) + (4k+6)T(r, g) + S(r, f) + S(r, g), \end{aligned}$$

namely

$$nT(r, f) \le (5k+8)T(r, f) + (4k+6)T(r, g) + S(r, f) + S(r, g). \tag{4.1}$$

Similarly we have

$$nT(r, g) \le (5k+8)T(r, g) + (4k+6)T(r, f) + S(r, f) + S(r, g). \tag{4.2}$$

From (4.1) and (4.2) we deduce that

$$(n - 9k - 14)(T(r, f) + T(r, g) \le S(r, f) + S(r, g), \tag{4.3}$$

which is a contradiction since $n > 9k + 14$.

Case (ii)**.** We have $[f^n]^{(k)}[g^n]^{(k)} = 1$. By Lemma 2.7 we get the conclusion (2) of Theorem 1.2.

Case (iii)**.** We have $[f^n]^{(k)} \equiv [g^n]^{(k)}$. By Lemma 2.6 we get the conclusion (1) of Theorem 1.2.

This completes the proof of Theorem 1.2.

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Regularized optimization method for determining the space-dependent source in a parabolic equation without iteration☆

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Abstract

In this paper, we consider an inverse problem of identifying a space-dependent source in the parabolic equation which is a classical ill-posed problem. The inverse source problem is formulated into a regularized optimization problem. Then, a non-iterative algorithm based on a sequence well-posed direct problems solved by the finite element method is proposed for solving the optimization problem. In order to obtain a reasonable regularization solution, we utilize the damped Morozov discrepancy principle together with the linear model function method for choosing regularization parameters. Numerical results for one- and two-dimensional examples show that the proposed method is efficient and robust with respect to data noise, especially for reconstructing the discontinuous source functions. Furthermore, the proposed method is successfully used to solve a real example of identifying the magnitude of groundwater pollution source.

Keywords: inverse source problem, parabolic equation, optimization, finite element method, discrepancy principle.

1. Introduction

Inverse source identification problems arise in many branches of applied science and engineering science, which aim to determine the unknown source from some measurable information related to the source. For example, Identification of a pollution source intensity from some given measurements of the pollutant concentrations is crucial to environmental safeguard in watersheds [1]. In this paper, we consider the inverse problem for determining the unknown space-dependent source in a parabolic equation from a final measurement. As we all know, this inverse source problem is ill-posed since small errors inherently presented in the practical measurement can induce enormous and highly oscillatory errors in

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reconstructing the unknown heat source.

The inverse problem of determining an unknown space-dependent source in the parabolic equation has been considered in a few theoretical papers concerned with existence and uniqueness of the solution[2, 3]. Recently, many authors are interested in numerical reconstruction of the space-dependent source in parabolic equations [4, 5, 7, 6, 8, 9, 10, 11, 12]. In [4], the authors transferred the inverse heat source problems to the problems of numerical differentiation for obtaining stable solutions. An effective meshless numerical method and a finite difference approximate method were proposed in [7] and [6], respectively. In [9], a regularization method based on the quasi-reversibility method together with the error estimate was proposed for identifying an unknown space-dependent source in one dimensional standard heat equation. In [10] and [11], two iterative methods were proposed for finding the spacewise dependent source: one is an iterative algorithm based on a sequence of well-posed direct problems; the other is a variational conjugate gradient-type iterative algorithm which also need to solve a sequence of well-posed direct problems at each iteration. The paper [12] is devoted to identify an unknown heat source depending simultaneously on both space and time variables that is transformed into an optimization problem. The aim of this paper is to construct a regularized optimization method, which is a non-iterative method. We firstly formulate the inverse problem of determining the spacewise dependent source into a regularized optimization problem. Then, the optimization problem is reduced to a system of linear algebraic equations based on a sequence well-posed direct problems solved by the finite element method.

This paper is organized as follows. In section 2, the source identification problem is formulated and some properties of the solution of direct problem are given. In section 3, a regularized optimization method is proposed for solve the source identification problem. In section 4, implementations of the regularized optimization method are presented. In section 5, numerical results for one- and two-dimensional examples are given to illustrate the efficiency and stability of the proposed method with respect to data noise. Finally, some conclusions are drawn.

2. Mathematical formulation of the source identification problem

Let Ω be a bounded domain possessing piecewise-smooth boundaries in the Euclidean space $\mathbf{R}^n$, $n \geq 1$. $x = (x_1, x_2, \cdots, x_n)$ denotes an arbitrary point in Ω, and $\partial\Omega$ is used for the boundary of the domain Ω. Let us denote by Q_T a cylinder $\Omega \times (0,T)$ consisting of all points $(x,t) \in \mathbf{R}^{n+1}$ with $x \in \Omega$ and $t \in (0,T)$.

2.1. functional spaces

The space $L_2(\Omega)$ is a Banach space consisting of all square integrable functions on the domain Ω with the norm

$$\|u\|_{2,\Omega} = \left(\int_\Omega |u(x)|^2 dx \right)^{1/2}$$

and the scalar product

$$(u,v) = \int_\Omega u(x)v(x)dx.$$

The Sobolev spaces $W_2^l(\Omega)$, where l is a positive integer, consists of all functions from $L_2(\Omega)$ having all generalized derivatives of the first l orders that are square integrable over Ω. The norm of the space $W_2^l(\Omega)$ is defined by

$$\|u\|_{2,\Omega}^{(l)} = \left(\sum_{k=0}^{l} \sum_{|\alpha|=k} \|D_x^\alpha u\|_{2,\Omega}^2 \right)^{1/2},$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is a multi-index, and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$,

$$D_x^\alpha u \equiv \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

The space $\overset{0}{W}{}_2^l(\Omega)$ is a subspace of $W_2^l(\Omega)$ in which the set all functions in Ω that are infinite differentiable and have compact support is dense.

The Sobolev space $W_2^{l_1,l_2}(Q_T)$ with positive integers $l_i \geq 0, i = 1, 2$ is defined as a Banach space of all functions u belonging to the space $L_2(Q_T)$ along with their weak x-derivatives of the first l_1 orders and t-derivatives of the first l_2 orders. The norm of the space $W_2^{l_1,l_2}(Q_T)$ is defined by

$$\|u\|_{2,Q_T}^{(l_1,l_2)} = \left(\int_{Q_T} \left(\sum_{k=0}^{l_1} \sum_{|\alpha|=k} |D_x^\alpha u|^2 + \sum_{k=1}^{l_2} |D_t^k u|^2 \right) dxdt \right)^{1/2}.$$

The space $W_{2,0}^{l_1,l_2}(Q_T)$ is a subspace of $W_2^{l_1,l_2}(Q_T)$ in which the set of all smooth functions in Q_T that vanish on the lateral $\partial\Omega \times [0, T]$ is dense.

2.2. The source identification problem

The source identification problem considered in this paper is stated as follows: find the function $u(x, t)$ and the unknown source function $f(x)$ which satisfy the following parabolic equation and boundary conditions

$$\begin{cases} u_t(x,t) = (Lu)(x,t) + f(x), \ (x,t) \in \Omega \times (0,T), \\ u(x,0) = 0, \ x \in \Omega, \\ u(x,t) = 0, \ (x,t) \in \partial\Omega \times [0,T], \end{cases} \tag{2.1}$$

and the final over-specified measurement

$$u(x, T) = g(x), \ x \in \Omega, \tag{2.2}$$

where

$$Lu \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u.$$

Moreover, the operator L is supposed to be uniformly elliptic, which means that $a_{ij}(x) = a_{ji}(x)$ and

$$0 < \nu \sum_{i=1}^{n} \zeta_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \zeta_i \zeta_j \leq \mu \sum_{i=1}^{n} \zeta_i^2 \tag{2.3}$$

with positive constants ν and μ, and arbitrary point $\zeta = (\zeta_1, \cdots, \zeta_n) \in \mathbf{R}^n$. Considering the practical engineering applications, we confine the coefficients of the operator L to satisfying that

$$a_{ij}(x), b_i(x), c(x) \in C(\bar{\Omega}), \ \frac{\partial a_{ij}(x)}{\partial x_k} \in C(\bar{\Omega}), k = 1, 2, \cdots, n. \tag{2.4}$$

Remark 1. *As mentioned in the introduction, there are many numerical methods [4, 5, 7, 6, 8, 9, 10, 11, 12, 22] for identifying the space-dependent source $f(x)$ in parabolic equations. However, these methods are mainly concerned and implemented in the one-dimensional or standard parabolic equations. In other words, some of these methods maybe not adapted to the generally parabolic equation in (2.1). Limited to our knowledge, we think the methods proposed in [4, 6, 9, 10, 11, 22] should be adapted and extended to the general n-dimensional inverse problem (2.1), but some new difficulties maybe occur and should be overcome. For example, numerical differential problems of both the second order and the first order, which are both ill-posed, should be computed in [4]. The finite difference approximation applied to reconstructing the source term in [6, 9] should be improved to deal with any n-dimensional domain. We pay more attention to the two iteration methods constructed for the generally problem (2.1) in [10, 11]. In the two iteration methods, the boundary element method with fundamental solutions of parabolic equations is used to solve a sequence of well-posed direct problems. Generally, the fundamental solutions of linear parabolic equations with variable coefficients are very complex, and their existence is also no general results [23]. Compared to these known methods, the regularized optimization method proposed in this paper is very simple for solving the general n-dimensional inverse source problem (2.1), and more suitable for parallel computing which greatly enhance the efficiency of the regularized method.*

2.3. Properties of the direct problem

The direct problem is finding a solution $u(x,t)$ satisfying the problem (2.1) when the coefficients of the operator L and the source $f(x)$ are known. From results of Chapter 1 in [13], we have the following lemma for the direct problem.

Lemma 1. *Let the operator L be uniformly elliptic and its coefficients satisfy (2.4), and let $f(x) \in L_2(\Omega)$. Then the direct problem (2.1) has a solution $u \in W^{2,1}_{2,0}(Q_T)$, this solution is unique and the following estimate is valid:*

$$\|u\|^{(2,1)}_{2,Q_T} \leq C_1 \sqrt{T} \, \|f\|_{2,\Omega} , \tag{2.5}$$

where the constant C_1 does not depend on u.

Therefore, given $f(x) \in L_2(\Omega)$, $u(x,T)$ is well defined since $u(x,t) \in W^{2,1}_{2,0}(Q_T)$. Moreover, it is reasonable to assume that the over-specified measurement $g(x)$ satisfies

$$g(x) \in W^1_2(\Omega). \tag{2.6}$$

3. Regularized Optimization method

3.1. Regularized optimization functional

We now consider the inverse source problem (2.1)-(2.2) as the following constrained optimization problem: finding a source function $f(x)$ such that

$$\min_{f\in\Phi} J(f) = \int_\Omega |u(x,T;f) - g(x)|^2 dx + \alpha \int_\Omega |f(x)|^2 dx, \tag{3.1}$$

where α is the regularization parameter and the constrained set is

$$\Phi = \{f(x) \mid |f(x)| \le M,\ f(x) \in L_2(\Omega)\}, \tag{3.2}$$

M is a constant. The solution $u(x,t;f)$ in (3.1) with respect to the source term $f(x)$ is a weak solution of (2.1) which satisfies

$$u(x,0;f) = 0 \tag{3.3}$$

and the variational formulation

$$\int_\Omega u_t \psi dx + \int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x) u_{x_i} \psi_{x_j} - \sum_{i=1}^n b_i(x) u_{x_i} \psi - c(x) u \psi \right) dx = \int_\Omega f(x)\psi dx \tag{3.4}$$

for any $\psi(x) \in \overset{0}{W}{}_2^1$ and a.e. $t \in (0,T)$.

Theorem 1. *There exists a minimizer $\tilde{f}(x) \in \Phi$ such that*

$$J(\tilde{f}) = \min_{f\in\Phi} J(f).$$

Proof. From the non-negativeness of the functional $J(f)$, it follows that $J(f)$ has the greatest lower bound $\inf_{f\in\Phi} J(f)$, which means that there exists a minimizing sequence $\{f_m\}$ in Φ such that

$$\inf_{f\in\Phi} J(f) \le J(f_m) \le \inf_{f\in\Phi} J(f) + \frac{1}{m}$$

with the associated weak solution $u_m := u(x,t;f_m)$. Obviously, there exists a constant C_2 such that

$$\|f_m\|_{2,\Omega} \le C_2,$$

where C_2 is independent of m. Thus, we can extract a subsequence, again denoted by $\{f_m\}$, such that f_m converges weakly to $\tilde{f}$ in Φ due to the closure of Φ.

From Lemma 1, we know that the sequence $\{u_m\}$ is bounded in $W_2^{2,1}(Q_T)$. Hence, we can also extract a subsequence, still denoted by $\{u_m\}$, such that u_m converges weakly to u^*. Therefore, the rest we need to prove that $u^* = u(x,t;\tilde{f})$. In order to do this, multiplying both side of the equation

$$\int_\Omega u_{m,t} \psi dx + \int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x) u_{m,x_i} \psi_{x_j} - \sum_{i=1}^n b_i(x) u_{m,x_i} \psi - c(x) u_m \psi \right) dx = \int_\Omega f_m(x)\psi dx \tag{3.5}$$

by any function $\gamma(t) \in C^1[0,T]$ with $\gamma(T)=0$, then integrating with respect to t on $[0,T]$, we derive that

$$-\int_\Omega u(x,0)\gamma(0)\psi dx - \int_0^T \gamma \int_\Omega u_m \psi dxdt + \int_0^T \gamma \int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x) u_{m,x_i} \psi_{x_j} - \sum_{i=1}^n b_i(x) u_{m,x_i} \psi - c(x) u_m \psi \right) dxdt = \int_0^T \gamma \int_\Omega \tilde{f} \psi dxdt + \int_0^T \gamma \int_\Omega (f_m - \tilde{f}) \psi dxdt.$$

The last term of the above equality converges to zero since f_m converges weakly to $\tilde{f}$. Noting that u_m converges weakly to u^* in $W_2^{2,1}(Q_T)$ and Letting $m \to \infty$, we have

$$-\int_\Omega u(x,0)\gamma(0)\psi dx - \int_0^T \gamma \int_\Omega u^* \psi dxdt + \int_0^T \gamma \int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x) u^*_{x_i} \psi_{x_j} - \sum_{i=1}^n b_i(x) u^*_{x_i} \psi - c(x) u^* \psi \right) dxdt = \int_0^T \gamma \int_\Omega \tilde{f} \psi dxdt. \tag{3.6}$$

Obviously, (3.6) is also true for any $\gamma(t) \in C_0^\infty(0,T)$, this implies that

$$\int_\Omega u_t^* \psi dx + \int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x) u^*_{x_i} \psi_{x_j} - \sum_{i=1}^n b_i(x) u^*_{x_i} \psi - c(x) u^* \psi \right) dx = \int_\Omega \tilde{f}(x) \psi dx$$

for any $\psi(x) \in \overset{0}{W}{}_2^1$ and $u^*(x,0)=0$. Hence, it follows that $u^* = u(x,t;\tilde{f})$ by the definition of $u(x,t;\tilde{f})$. Then, the weakly lower semi-continuity of $J(f)$ ensures that $\tilde{f}$ is a minimizer of $J(f)$.

3.2. Approximation by the finite element method

In this subsection, we introduce the finite element method for solving the continuous minimization problems (3.1), (3.2) and (3.4). Similarly to that done in [14, 15], we first triangulate the domain Ω with a regular triangulation T^h of simplicial elements, and define S_h to be the continuous piecewise linear finite element space defined over T^h. The space $\overset{0}{S}_h$, in which all functions vanish on the boundary $\partial\Omega$, is a subspace of S_h. Let $\{P_i\}_{i=1}^{M_h}$ be the set of interior nodes, i.e., those that do not lie on the boundary $\partial\Omega$. So, a function in the space $\overset{0}{S}_h$ is uniquely determined by its value at the point P_i, and the set of pyramid functions $\{\phi_j\}_{j=1}^{M_h} \subset \overset{0}{S}_h$, defined by

$$\phi_j(P_i) = \begin{cases} 1, & i=j, \\ 0, & i \neq j, \end{cases} \tag{3.7}$$

forms a basis of $\overset{0}{S}_h$. Obviously, a function $v(x)$ in $\overset{0}{S}_h$ can be extract that $v(x) = \sum_{j=1}^{M_h} v_j \phi_j(x)$, where $v_j = v(P_j)$ is the value of $v(x)$ at P_j. The time interval $[0,T]$ is partitioned into N equal subintervals by using nodal points

$$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$$

with $t_m = m\Delta t$ and $\Delta t = \frac{T}{N}$. Let $u^m = u(x, t_m)$ for $0 \leq m \leq N$. Then, we define the difference quotient

$$D_t u^m = \frac{u^m - u^{m-1}}{\Delta t}$$

for a given sequence $\{u^m\}_{m=1}^N \subset L^2(\Omega)$.

Let $f(x)$ be extended to the boundary $\partial\Omega$. Then, we define $f_h = \sum_{j=1}^{K_h} f_j \phi_j(x)$ that approximate $f(x) \in L_2(\Omega)$ and project it into the space S_h, where K_h is the numbers of all nodes of T^h, and f_j is the value of $f(x)$ at the j-th node. And now we can formulate the continuous optimal problem (3.1) as the following finite element approximation

$$\min_{f \in S_h \bigcap \Phi} J(f_h) = \int_\Omega |u_h^N(f_h) - g(x)|^2 dx + \alpha \int_\Omega |f_h|^2 dx, \tag{3.8}$$

where $u_h^m(f_h) = \sum_{j=1}^{M_h} u_j^m \phi_j(x)$ for $m = 0, 1, \cdots, N$ satisfies that

$$u_h^0(f_h) = 0 \tag{3.9}$$

and

$$\int_\Omega \psi_h D_t u_h^m(f_h) dx + \int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x)(u_h^m(f_h))_{x_i}(\psi_h)_{x_j} - \sum_{i=1}^n b_i(x)(u_h^m(f_h))_{x_i}\psi_h - c(x)u_h^m(f_h)\psi_h \right) dx = \int_\Omega f_h(x)\psi_h dx \tag{3.10}$$

for any $\psi_h \in \overset{0}{S}_h$.

Theorem 2. *There exists at least a minimizer to the discrete minimization problem (3.8)-(3.10).*

The proof of Theorem 2 follows the same lines as the proof of Theorem 3.1 in [15]. So, we omit it. On the other hand, from results of [14, 15] we can also obtain that the minimizer sequence of the discrete minimization problem corresponding to h and Δt has a subsequence that converges to a minimizer of the continuous problem (3.1)-(3.4) as $h \to 0, \Delta t \to 0$.

4. Implementations of the regularized optimization method

4.1. Regularized least square method

Due to the linearity of the governing equation and the homogenous boundary and initial conditions, we easily see that the problem (2.1) satisfies the principle of superposition in terms of the source function, which is noted in [16, 17] and used to construct inverse methods for recovering the initial function. Here, we also use this principle of superposition to formulate the finite element approximation (3.8) into a linear algebraic system. Noting that

$$f_h = \sum_{j=1}^{K_h} f_j \phi_j(x), \tag{4.1}$$

we have

$$u_h^N(f_h) = \sum_{j=1}^{K_h} f_j u_h^N(\phi_j(x)), \tag{4.2}$$

where $u_h^N(\phi_j(x))$ is computed by the finite element method proposed in [18] when the spatial domain is one-dimensional; otherwise, for the two-dimensional domain, $u_h^N(\phi_j(x))$ is computed by the functions of PDE Toolbox in Matlab. Therefore, we rewrite the approximation functional $J(f_h)$ as the form of $\tilde{f} = (f_1, f_2, \cdots, f_{K_h})^T$ in the form

$$J(\tilde{f}) = \int_\Omega \left| \sum_{j=1}^{K_h} f_j u_h^N(\phi_j(x)) - g(x) \right|^2 dx + \alpha \int_\Omega \left| \sum_{j=1}^{K_h} f_j \phi_j(x) \right|^2 dx, \tag{4.3}$$

From the necessary condition for minimizing the approximation function $J(\tilde{f})$

$$\frac{\partial J(\tilde{f})}{\partial f_i} = 0, \ i = 1, 2, \cdots, K_h, \tag{4.4}$$

we obtain the following linear algebraic system

$$(A + \alpha G)\tilde{f} = b, \tag{4.5}$$

where $A = (a_{ij})_{K_h \times K_h}$, $G = (g_{ij})_{K_h \times K_h}$, $b = (b_1, b_2, \cdots, b_{K_h})^T$, and

$$a_{ij} = \int_\Omega u_h^N(\phi_i(x)) u_h^N(\phi_j(x)) dx, \ g_{ij} = \int_\Omega \phi_i(x)\phi_j(x) dx, \ b_i = \int_\Omega u_h^N(\phi_i(x)) g(x) dx. \tag{4.6}$$

For a given regularization parameter α, the solution $\tilde{f}^*$ of equation (4.5) is a discrete reconstruction of the unknown source $f(x)$; and $f_h^* = \sum_{j=1}^{K_h} f_j^* \phi_j(x)$ is an approximation of $f(x)$ in the space S_h. Hence, the algorithm for reconstructing the unknown source $f(x)$ from the final over-specified measurement $g(x) = u(x, T)$ is summarized as the following algorithm.

Algorithm 1 : Algorithm with non-iteration for reconstructing the unknown source.

Given the final measurement $g(x) = u(x, T)$ and a regularization parameter α.

Step 1. Solve the direct problem (2.1) for each basis source term $f(x) = \phi_i(x)$ via the finite element method, then obtain $u_h^N(\phi_i(x))$.

Step 2. Compute the Matrices A and G and the vector b by using (4.6).

Step 3. Solve the regularized linear algebraic system (4.5).

Step 4. Reconstruct the source $f(x)$ by using the formulation (4.1).

Remark 2. *The cost of Algorithm 1 is mainly taken in the first step if we run it by serial computing. Fortunately, we note that the first step of Algorithm 1 can be computed in parallel. Therefore, the efficiency of Algorithm 1 will be very high if we run it on a parallel computer system. In addition, we can choose other bases instead of the continuous piecewise linear finite element basis, such as the polynomial basis, and the trigonometric function basis, which can greatly reduce the amount of computation if the number of the basis functions is relatively small.*

4.2. Strategy for choosing regularization parameters

Due to the ill-posedness of the inverse source problem, the regularization parameter α play an important role for reconstructing a reasonable solution. The measurement noises and the round-off errors may be highly amplified due to the choice of an unreasonable regularization parameter and therefore making the inverse solution completely useless. In this study, we employ the damped Morozov discrepancy principle [19, 20] to choose regularization parameters, i.e., choosing regularization parameters α such that

$$\int_\Omega \left| \sum_{j=1}^{K_h} f_j u_h^N(\phi_j(x)) - g^\delta(x) \right|^2 dx + \alpha^\gamma \int_\Omega \left| \sum_{j=1}^{K_h} f_j \phi_j(x) \right|^2 dx = C\delta^2, \tag{4.7}$$

where γ is the damped coefficient, C is a constant, δ is the noise level which meet that $\|g - g^\delta\| \le \delta$. Here, g is the exact data and g^δ is the measurement data. In order to obtain regularization parameters in a stable and quick way, we adopt the linear model function method proposed in [19, 20] to solve the discrepancy equation (4.7) with $\gamma = 1.5$ and $C = 1.5$.

5. Numerical examples

The solution of the governing equation in (2.1) with nonhomogeneous boundary condition $B(x,t)$ and initial condition $u_0(x)$ is not a linear mapping for the source term $f(x)$. Therefore, we first divide the problem with nonhomogeneous boundary conditions into the following two problems, i.e.,

$$u(x,t;f) = u_1(x,t;f) + u_2(x,t), \tag{5.1}$$

where $u_1(x,t;f)$ satisfies the problem

$$\begin{cases} (u_1)_t(x,t) = (Lu_1)(x,t) + f(x), \ (x,t) \in \Omega \times (0,T), \\ u_1(x,0) = 0, \ x \in \Omega, \\ u_1(x,t) = 0, \ (x,t) \in \partial\Omega \times [0,T], \end{cases} \tag{5.2}$$

and $u_2(x,t)$ is the solution of the following homogeneous equation with nonhomogeneous initial and boundary conditions

$$\begin{cases} (u_2)_t(x,t) = (Lu_2)(x,t), \ (x,t) \in \Omega \times (0,T), \\ u_2(x,0) = u_0(x), \ x \in \Omega, \\ u_2(x,t) = B(x,t), \ (x,t) \in \partial\Omega \times [0,T]. \end{cases} \tag{5.3}$$

Then, By using the data $g(x)-u_2(x,T)$, Algorithm 1 can be implemented for reconstructing the source function $f(x)$.

In all one-dimensional examples of this section, we divide $[0,1]$ into 100 equal subintervals which means that there are 100 elements and 101 nodes; while in all two-dimensional examples, we divide $[0,1]\times[0,1]$ into 50×50 equal sub-rectangles which indicates that the mesh grid has a total of 5000 triangle elements and 2601 nodes. In the computational process, we obtain actually the final data vector $g=\{g(P_i)\}$ at the points of the mesh grid in our simulations, and add a random distributed perturbation to the data vector g with relative noise level $\hat{\delta}$, i.e., $g^\delta = g+\hat{\delta}(2*\text{rand}(\text{size}(g))-1)*g$. The function rand(size(g)) in Matlab generates a random vector whose elements are the standard uniform distribution on the interval (0,1).

In the numerical results listed in Table 1, we report the relative noise levels $\hat{\delta}$, the regularization parameters, the relative error of the inverse solution computed by the formula

$$\text{RelError} = \frac{\|f_h^* - f\|_2}{\|f\|_2}.$$

The comparisons between the exact solutions and the inverse solutions are showed in Figure 1 to Figure 8, respectively.

Table 1. Some numerical results for examples 1-5.

Examples	$\hat{\delta}$	α	RelError
Example 1	0.001	4.2806e-006	3.8244e-003
	0.01	4.1359e-005	1.1608e-002
Example 2	0.001	2.7407e-007	3.6889e-002
	0.01	3.5558e-006	9.3541e-002
Example 3	0.001	1.0017e-007	1.6677e-001
	0.01	3.7607e-006	2.5831e-001
Example 4	0.001	4.7644e-008	1.9297e-002
	0.01	4.9258e-007	5.7354e-002
Example 5	0.001	3.5247e-008	1.7401e-001
	0.01	1.1364e-006	2.6286e-001

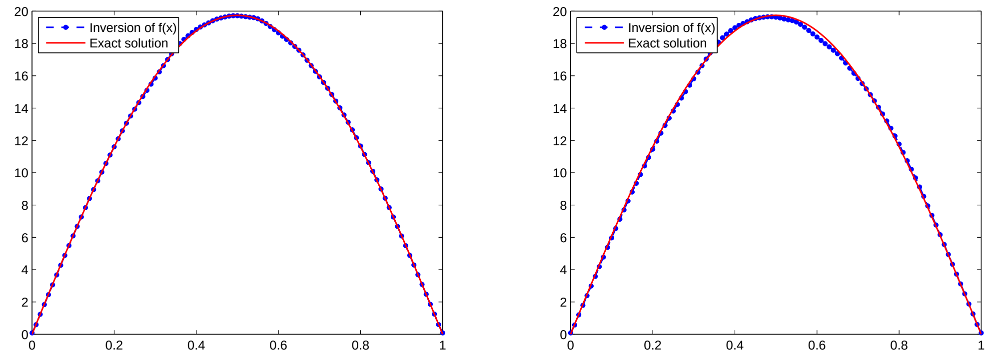
Example 1. We take $\Omega=(0,1), T=1$, and $Lu=\Delta u=\frac{\partial^2 u}{\partial x^2}$. Let

$$u(x,t) = \left(2-\exp\left(-\pi^2 t\right)\right)\sin(\pi x),\ (x,t)\in[0,1]\times[0,1].$$

In this case, $f(x)=2\pi^2\sin(\pi x)$, $u_0(x)=\sin(\pi x)$, $u(0,t)=u(1,t)=0$, and the final measurement is given by

$$g(x)=u(x,1)=\left(2-\exp\left(-\pi^2\right)\right)\sin(\pi x),\ x\in[0,1].$$

The inverse solutions for different noise levels are showed in Figure 1.



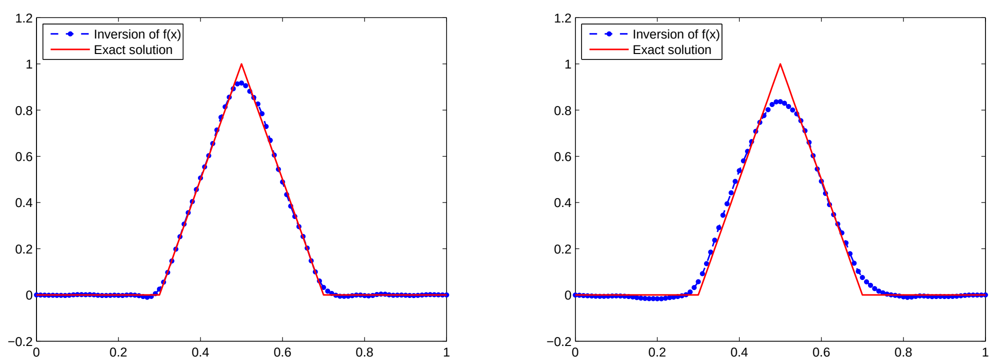
(a) $\hat{\delta} = 0.001$ (b) $\hat{\delta} = 0.01$

Figure 1: The comparison between the exact solution and its inverse solution.

Example 2. Consider a piecewise smooth heat source:

$$f(x) = \begin{cases} 0, & x \in [0, 0.3], \\ 5(x - 0.3), & x \in (0.3, 0.5], \\ -5(x - 0.7), & x \in (0.5, 0.7], \\ 0, & x \in (0.7, 1]. \end{cases}$$

We take $\Omega = (0, 1), T = 1$, $Lu = \Delta u = \frac{\partial^2 u}{\partial x^2}$, $u_0(x) = 0$, $u(0, t) = u(1, t) = 0$. The final over-specified measurement $u(x, T)$ is computed by the finite element method proposed in [18]. Numerical results for different noise levels are showed in Figure 2.



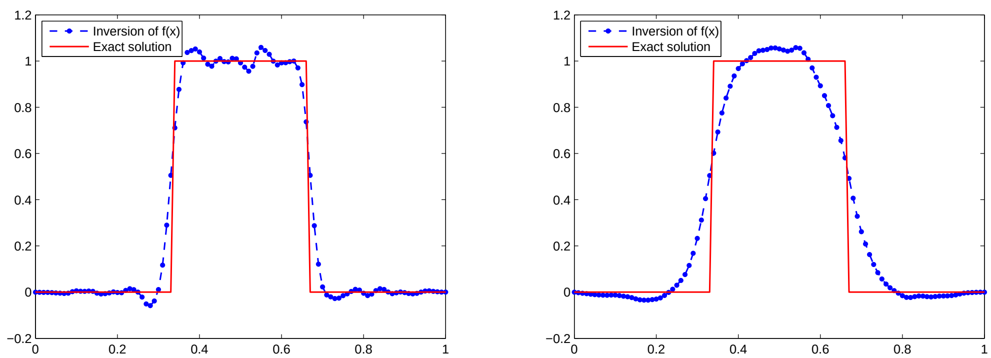
(a) $\hat{\delta} = 0.001$ (b) $\hat{\delta} = 0.01$

Figure 2: The comparison between the exact solution and its inverse solution.

Example 3. Consider a discontinuous source

$$f(x) = \begin{cases} 0, & x \in [0, 1/3), \\ 1, & x \in [1/3, 2/3], \\ 0, & x \in (2/3, 1]. \end{cases}$$

In this example, we also take $\Omega = (0,1), T = 1,\ Lu = \Delta u = \frac{\partial^2 u}{\partial x^2}$, and the homogenous boundary and initial conditions. Because the source $f(x)$ is a discontinuous function, the direct problem has no analytic solution. So, we obtain the final over-specified measurement $u(x,T)$ by solving the direct problem with the finite element method [18]. The reconstructed solutions for different noise levels are showed in Figure 3.



(a) $\hat{\delta} = 0.001$ (b) $\hat{\delta} = 0.01$

Figure 3: The comparison between the exact solution and its inverse solution ($n = 100$).

Example 4.[6] For this two-dimensional example, we take $\Omega = (0,1) \times (0,1), T = 1$, $Lu = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$, the initial value $u_0(x,y) = \sin(\pi x)\sin(\pi y), (x,y) \in \Omega$, and the homogenous boundary conditions. The exact source function is defined by

$$f(x,y) = \exp\left(-\sigma\left[(x-\mu_1)^2 + (y-\mu_2)^2\right]\right).$$

Note that when σ is large enough the above source mimics a Dirac delta distribution $\delta(x-\mu_1, y-\mu_2)$. Here, we take $\sigma = 80$, $\mu_1 = \frac{3}{4}$ and $\mu_2 = \frac{1}{2}$. The final measurement $u(x,y,T)$ is obtained by the functions of PDE Toolbox in Matlab. The exact solution is showed in Figure 4. The reconstructed solutions and their errors are showed in Figure 5 and Figure 6 for $\hat{\delta} = 0.001$ and $\hat{\delta} = 0.01$, respectively.

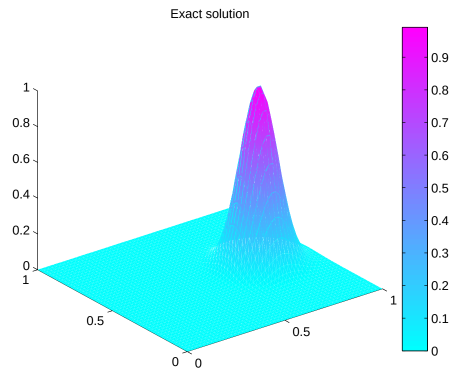
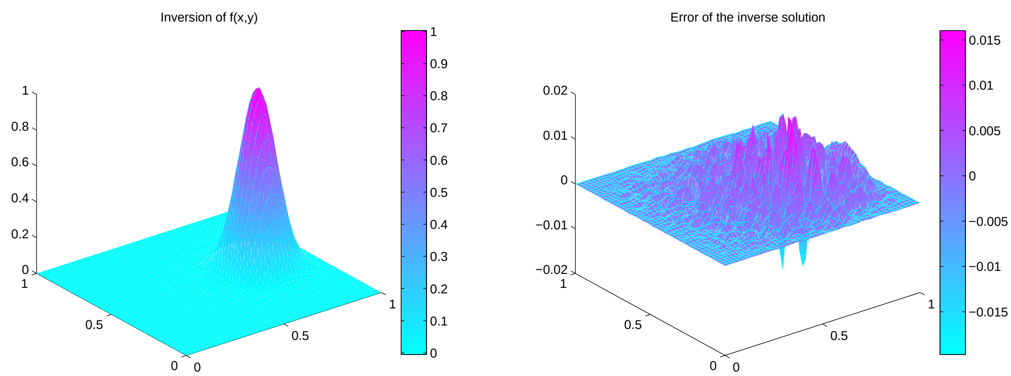


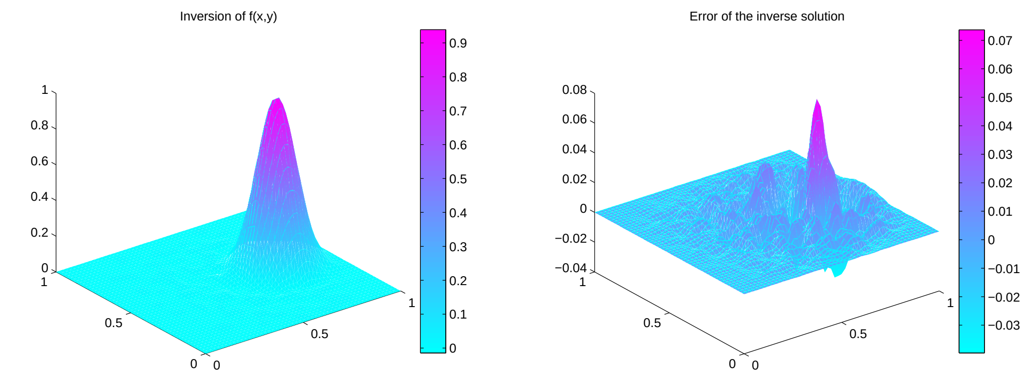
Figure 4: The exact solution of example 4.



(a) Inverse solution

(b) Error of the inverse solution

Figure 5: The inverse solution and its error for $\hat{\delta} = 0.001$.



(a) Inverse solution (b) Error of the inverse solution

Figure 6: The inverse solution and its error for $\hat{\delta} = 0.01$.

Example 5. In this two-dimensional example, we consider a discontinuous function

$$f(x,y) = \begin{cases} 0, & (x,y) \in \left\{(x,y) \middle| 0 < x, y < 1, \sqrt{(x-0.5)^2 + (y-0.5)^2} \geq 0.25 \right\} \\ 1, & (x,y) \in \left\{(x,y) \middle| \sqrt{(x-0.5)^2 + (y-0.5)^2} < 0.25 \right\}. \end{cases}$$

We also take $\Omega = (0,1) \times (0,1), T = 1$, $Lu = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$, and the homogenous boundary and initial conditions. The final measurement $u(x,y,T)$ is also obtained by the functions of PDE Toolbox in Matlab. The exact solution is showed in Figure 7. The reconstructed solutions and their errors are showed in Figure 8 and Figure 9 for $\hat{\delta} = 0.001$ and $\hat{\delta} = 0.01$, respectively.

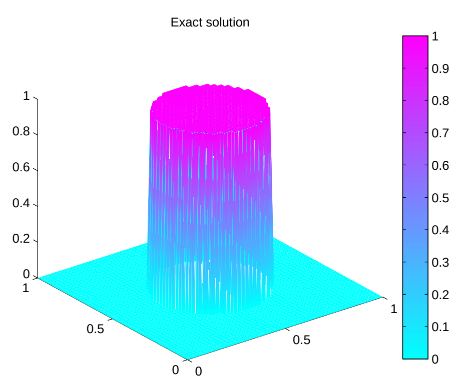
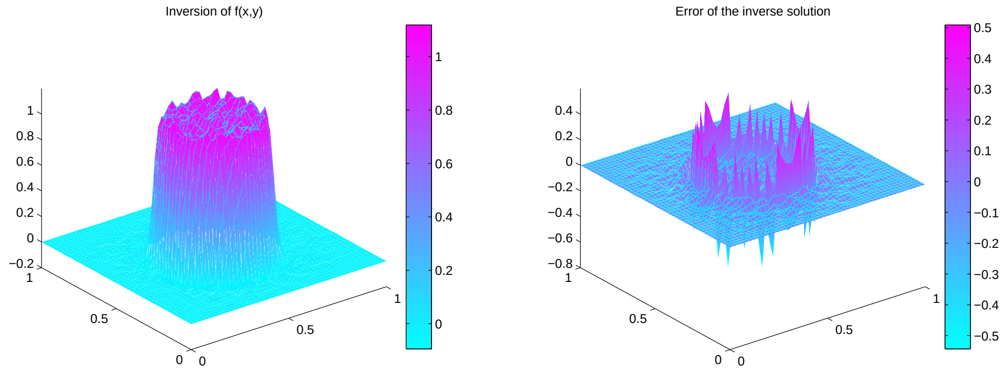
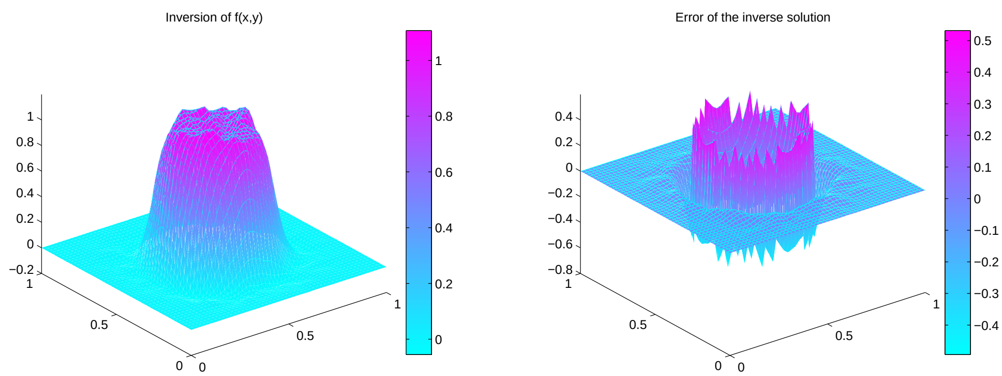


Figure 7: The exact solution of example 5.



(a) Inverse solution (b) Error of the inverse solution

Figure 8: The inverse solution and its error for $\hat{\delta} = 0.001$.



(a) Inverse solution (b) Error of the inverse solution

Figure 9: The inverse solution and its error for $\hat{\delta} = 0.01$.

6. Application to a real source determination[21, 22]

This real example is taken from references [21] and [22]. Consider acid contaminant in the groundwater in Fengshui, Zibo of Shandong Province, China. The studied region is a relatively integrated unit of hydrogeology whose area is about 45 km^2. In this region, the groundwater flow accumulated by atmosphere precipitation is gradually pressed when it seeps from the southeast to the north-west until it encountered the coal-seam, and so a strip containing rich groundwater is formed. For this reason, Yuedian and Zhanghua wellsprings were established in 1980s. However, with the excess exploitation of mines, e.g. the exploitation of coal-wells, groundwater pollution has become more and more serious in this region. In particular, acid contaminant of SO_4^{2-} in Zhanghua wellspring becomes higher and higher year after year. Based on the measured concentration data in this region from 1988 to 1999 along the groundwater flow direction, we are try to determine the average magnitude of acid contaminant seeping into the aquifer every year.

Under some suitable assumptions on the aquifer, choosing the direction of groundwater flow which is from Sijiaofang to Zhanghau as the direction of x axis, and Sijiaofang as zero point, the year of 1988 as initial moment, then this real problem of acid contaminant in the groundwater system can be characterized by the following system:

$$\begin{cases} \frac{\partial u}{\partial t} = a_L v \frac{\partial^2 u}{\partial x^2} - v\frac{\partial u}{\partial x} - \lambda u + \frac{f(x)}{n_e}, \ 0 < x < L, 0 < t < T, \\ u(0,t) = 7.96t + 45.6, 0 \le t \le T, \\ u(L,t) = 1.75t^2 + 331.6, 0 \le t \le T, \\ u(x,0) = 0.0715x + 45.6, 0 \le x \le L, \end{cases} \tag{6.1}$$

where $u = u(x,t)$[mg/l] is the solute concentration at time t and space point x, and the model parameters: $v = 365$[m/y] is the average pore water velocity, $a_L = 1$[m] is the longitude dispersivity, $\lambda = 0.05[\mathrm{y}^{-l}]$ is the attenuation coefficient, $n_e = 0.25$[dimensionless] is the effective porosity and $f(x)$ represents an average magnitude of the pollutants seeping into the aquifer every year. In addition, $L = 4000$[m] denotes the distance from Sijiaofang to Zhanghau, and $T = 11$[y]. The boundary conditions and the initial condition in system (6.1) are obtained by applying data fitting skills from the actually measured data, see [22]. The additional data at the final year of $T = 11$ is also obtained by the similar technique as follows:

$$u(x,T) = 0.1026x + 133.2, 0 \le x \le L. \tag{6.2}$$

The inverse problem considered here is to determine the source magnitude function $f(x)$ in (6.1) from the measured data $u(x,T)$ by the regularized optimization method. In the numerical implementation, we firstly transform this inverse problem into a dimensionless form [21] by setting $U = \frac{u}{45.6}, y = \frac{x}{L}, \tau = \frac{vt}{L}$, then apply Algorithm 1 to solve it by dividing $[0,1]$ with 200 equal subintervals. Firstly, assuming that all of the initial boundary data in the model (6.1) and the additional final data (6.2) are accurate, we reconstruct the source with regularization parameter $\alpha = 0.5 \times 10^{-4}$. Secondly, in the case of the additional final data having random noises, we carry out similar computations with the linear model function method for choosing regularization parameters.

Case 1. Find a solution $f_h^* = \sum_{j=1}^{K_h} f_j^* \phi_j(x)$ in the space S_h. See Figure 10 and Table 2.

Case 2. Find a solution as the form $f_h^* = \sum_{j=0}^{N_p} f_j^* x^j$ in the polynomial function space $\mathbb{P}_{N_p}[x]$. Based on analyses of [22], we only take $N_p = 1$ and $N_p = 2$ to reconstruct the source, respectively. See Table 2.

To show accurateness and reasonableness of the above solutions, we substitute these solutions into the model (6.1) and reconstruct the additional data denoted by $u(x,T;f_h^*)$. Then the residuals $\|u(x,T;f_h^*) - u(x,T)\|_2$ are computed at the 201 nodes and listed in Table 2 as compared with the actually additional final data (6.2).

From Figure 10 and Table 2, we see that the source magnitude function $f(x)$ in the model (6.1) can be determined numerically from the additional final data by the proposed regularized optimization method. We also find from the last column of Table 2 that the method used here is better than that in paper [22, 21] in the sense of smaller residuals. In addition, Algorithm 1 is very fast in the above second case since only two of three final data for the corresponding basis functions need to be computed.

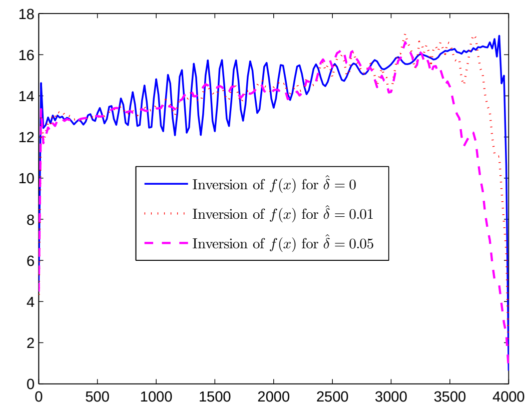


Figure 10: Inverse solutions, where $\hat{\delta}$ is the relative noise level.

Table 2. Numerical results for Example 6.

Cases	Relative noise level	α	$f_h^*(x)$	$\|u(x,T;f_h^*) -u(x,T)\|_2$
Solutions in the space S_h	0.00	5.0000e-005	See Figure 10(-)	7.0182e-001
	0.01	5.1865e-004	See Figure 10(..)	9.5460e-001
	0.05	3.8324e-003	See Figure 10(- -)	2.6560e+000
Solutions in the space $\mathbb{P}_1[x]$	0.00	5.0000e-005	$12.997+0.0010584x$	2.0369e-001
	0.01	1.0177e-001	$13.052+0.0010055x$	2.7136e-001
	0.05	3.4462e-001	$13.129+0.00090556x$	9.7427e-001
Solutions in the space $\mathbb{P}_2[x]$	0.00	5.0000e-005	$13.001+0.0010514x$ $+0.0000000019274x^2$	2.0386e-001
	0.01	9.4045e-002	$13.343+0.00048213x$ $+0.00000015466x^2$	5.6136e-001
	0.05	3.5021e-001	$13.341+0.00039093x$ $+0.00000017290x^2$	9.1224e-001
Results of [21]	0.00	—	$14.507+0.000016411x$	3.2952e+000
	0.01	4.357e-3	$14.507+0.000015817x$	3.2972e+000
	0.05	1.274e-2	$14.515+0.000011783x$	3.3075e+000

7. Conclusions

In this paper, we mainly study the inverse problem of determining a space-dependent source in the parabolic equation. As we all know, the inverse source problem is a classical ill-posed problem. Basing on a sequence well-posed direct problems solved by the finite element method, we propose a regularized optimization method for solving the inverse source problem, and use the linear model function method to choose regularization parameters for obtaining a stable solution. The proposed method is a non-iterative method, and can be extended to the parabolic equation with other boundary conditions, even mixed boundary conditions. In addition, we find that the regularization parameter plays an important role in numerically solving the regularized optimization problem. Numerical results for one- and two-dimensional examples show that the proposed method together with regularization parameter chosen strategy is efficient and robust with respect to data noise, especially for reconstructing the discontinuous source functions. Furthermore, the proposed method is successfully used to solve a real example of identifying the magnitude of groundwater pollution source.

In Algorithm 1, Matlab is used to compute the final dada for each basis function, it introduces some errors due to the finite element approximation. Therefore it is desirable to keep this computational errors less than the noise level. Here, we thank the reviewers very much for pointing out this fact. Obviously, the mesh grid is denser the error of the finite element approximation is smaller. Consequently, the computation amount will be increase if Algorithm 1 is run by serial computing. As mentioned above, we can improve it by parallel computing. On the other hand, instead of using the linear finite element basis we approximate the source function $f(x)$ by applying the polynomial function basis such as in the real life example, or the trigonometric function basis. In this case, the number of computing the final data for each basis is independent on the mesh. Thereby, we can improve greatly the efficiency of Algorithm 1 by selecting the number of the basis under some *a priori* information about the source function. Results of our numerical examples show that the proposed regularized optimization method is robust to the error of the finite element approximation. And we will study the error estimation of the proposed method in our future work.

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Knowledge reduction in knowledge bases and its algorithm *

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December 1, 2014

Abstract: One of rough set theory's strengths is the fact that an unknown target concept can be approximately characterized by existing knowledge structures in a knowledge base. In this paper, we investigate Knowledge reduction in knowledge bases and give its algorithm.

Keywords: Knowledge base; Knowledge reduction; Necessary relation; Algorithm.

1 Introduction

Rough set theory was proposed by Pawlak [10] as a mathematical tool for data reasoning. It may be seen as an extension of classical set theory, has been proved to be an effective approach to deal with intelligent systems characterized by insufficient and incomplete information and has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems and many other fields [6, 7, 8, 9].

Basic opinion in rough set theory is that knowledge (human intelligence) is the ability to classify elements [1, 4]. That is to say, knowledge is a family of classifications (or partitions) on the universe. Rough set theory mainly consider equivalence relations on the universe, which determine partitions on the universe. One deals with not only a single classification (or partition) on the universe, but also a family of classifications (or partitions) on the universe. This leads to the definition of a knowledge base, which is a important concept in rough set theory.

For a given knowledge base, one of the tasks in data mining and knowledge discovery is to generate new knowledge through known knowledge.

The purpose of this paper is to investigate knowledge reduction in knowledge bases.

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2 Preliminaries

In this section, we recall basic concepts about rough sets and knowledge bases.

Throughout this paper, U denotes a non-empty finite set called the universe, 2^U denotes the set of all subsets of U, $\mathcal{R}^*$ denote the set of all equivalence relations on U. All mappings are assumed to be surjective.

For $\mathcal{R} \subseteq \mathcal{R}^*$, denote $ind(\mathcal{R}) = \bigcap_{R\in\mathcal{R}} R$. Obviously, $ind(\mathcal{R}) \in \mathcal{R}^*$. For $R \in \mathcal{R}^*$, denote $[x]_R = \{y \in U : xRy\}$.

Let $R \in \mathcal{R}^*$. The pair (U, R) is called a Pawlak approximation space. Based on (U, R), one can define the following two rough approximations:

$$\underline{R}(X) = \{x \in U : [x]_R \subseteq X\}, \quad \overline{R}(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}.$$

$\underline{R}(X)$ and $\overline{R}(X)$ are called the Pawlak lower approximation and the Pawlak upper approximation of X, respectively.

The boundary region of X, defined by the difference between these rough approximations, that is $Bnd_R(X) = \overline{R}(X) - \underline{R}(X)$.

A set is rough if its boundary region is not empty; Otherwise, it is crisp. Thus, X is rough if $\underline{R}(X) \neq \overline{R}(X)$.

Definition 2.1 ([13])**.** *A pair $(U, \mathcal{R})$ is called a knowledge base, if $\mathcal{R} \subseteq \mathcal{R}^*$.*

It is well-know that elements in a knowledge base are not of the same importance, some even are redundant. So we often consider knowledge reductions in a knowledge base by deleting unrelated or unimportant elements with the requirement of keeping the ability of classification. This is the meaning of knowledge reduction in knowledge bases.

Definition 2.2 ([13])**.** *Let $(U, \mathcal{R})$ be a knowledge base and $\mathcal{P} \subseteq \mathcal{R}$.*

(1) *$\mathcal{P}$ is called a coordinate subfamily of $\mathcal{R}$, if $ind(\mathcal{P}) = ind(\mathcal{R})$.*

(2) *$R \in \mathcal{P}$ is called independent in $\mathcal{P}$, if $ind(\mathcal{P} - \{R\}) \neq ind(\mathcal{P})$; $\mathcal{P}$ is called a independent subfamily of $\mathcal{R}$, if $\forall\ R \in \mathcal{P}$, R is independent in $\mathcal{P}$.*

(3) *$\mathcal{P}$ is called a knowledge reduction (for short, reduction) of $\mathcal{R}$, if $\mathcal{P}$ is both coordinate and independent.*

In this paper, the set of all coordinate subfamilies (resp., all knowledge reductions) of $\mathcal{R}$ is denoted by $co(\mathcal{R})$ (resp., $red(\mathcal{R})$).

Obviously,

$$\mathcal{P} \in red(\mathcal{R}) \iff \mathcal{P} \in co(\mathcal{R}) \text{ and } \forall\ \mathcal{Q} \subset \mathcal{P}, \mathcal{Q} \notin co(\mathcal{R}).$$

3 Knowledge reduction in knowledge bases

Proposition 3.1. *Let $(U, \mathcal{R})$ be a knowledge base. Then there always exist a knowledge reduction of $\mathcal{R}$.*

Proof. Suppose $\forall\ R \in \mathcal{R}$, $\mathcal{R} - \{R\} \notin co(\mathcal{R})$. Then $\mathcal{R} \in red(\mathcal{R})$.

Suppose $\exists\ R_1 \in \mathcal{R}$, $\mathcal{R} - \{R_1\} \in co(\mathcal{R})$. Then, we consider $\mathcal{R} - \{R_1\}$. Again suppose $\forall\ R \in \mathcal{R} - \{R_1\}$, $(\mathcal{R} - \{R_1\}) - \{R\} \notin co(\mathcal{R})$. Then $\mathcal{R} - \{R_1\} \in red(\mathcal{R})$. Again suppose $\exists\ R_2 \in \mathcal{R} - \{R_1\}$, $(\mathcal{R} - \{R_1\}) - \{R_2\} \in co(\mathcal{R})$. Then, we consider $\mathcal{R} - \{R_1, R_2\}$. Repeat this process. Since $\mathcal{R}$ is finite, we can find a knowledge reduction of $\mathcal{R}$.

Thus, there always exist a knowledge reduction of $\mathcal{R}$. □

Definition 3.2. *Let $(U, \mathcal{R})$ be a knowledge base. Put*

$$\mathcal{D}(x, y) = \{R \in \mathcal{R} | (x, y) \notin R\}\ (x, y \in U).$$

Then

(1) *$\mathcal{D}(x, y)$ is called the discernibility subfamily of $\mathcal{R}$ on x and y.*

(2) *$\mathfrak{D}(\mathcal{R}) = (d_{ij})_{n\times n}$ is called the discernibility matrix of $\mathcal{R}$ where $U = \{x_1, x_2, \cdots, x_n\}$ and $d_{ij} = \mathcal{D}(x_i, x_j)$ $(1 \le i, j \le n)$.*

Example 3.3. *Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. We consider the knowledge base $(U, \mathcal{R})$ where $\mathcal{R} = \{R_1, R_2, R_3, R_4\}$ and*

$U/R_1 = \{\{x_1, x_2, x_5\}, \{x_3, x_4, x_6\}\}$,
$U/R_2 = \{\{x_1, x_6\}, \{x_2, x_3, x_4, x_5\}\}$,
$U/R_3 = \{\{x_1, x_2, x_5, x_6\}, \{x_3, x_4\}\}$,
$U/R_4 = \{\{x_1, x_2, x_5\}, \{x_3, x_4, x_6\}\}$.

We can obtain the discernibility matrix $\mathfrak{D}(\mathcal{R})$ as follows:

$$\begin{pmatrix} \emptyset & \{R_2\} & \mathcal{R} & \mathcal{R} & \{R_2\} & \{R_1, R_4\} \\ \{R_2\} & \emptyset & \{R_1, R_3, R_4\} & \{R_1, R_3, R_4\} & \emptyset & \{R_1, R_2, R_4\} \\ \mathcal{R} & \{R_1, R_3, R_4\} & \emptyset & \emptyset & \{R_1, R_3, R_4\} & \{R_2, R_3\} \\ \mathcal{R} & \{R_1, R_3, R_4\} & \emptyset & \emptyset & \{R_1, R_3, R_4\} & \{R_2, R_3\} \\ \{R_2\} & \emptyset & \{R_1, R_3, R_4\} & \{R_1, R_3, R_4\} & \emptyset & \{R_1, R_2, R_4\} \\ \{R_1, R_4\} & \{R_1, R_2, R_4\} & \{R_2, R_3\} & \{R_2, R_3\} & \{R_1, R_2, R_4\} & \emptyset \end{pmatrix}$$

Proposition 3.4. *Let $(U, \mathcal{R})$ be a knowledge base. The $\forall\ x, y, z \in U$,*

(1) $\mathcal{D}(x, x) = \emptyset$.

(2) $\mathcal{D}(x, y) = \mathcal{D}(y, z)$.

(3) $\mathcal{D}(x, y) \subseteq \mathcal{D}(x, z) \cup \mathcal{D}(z, y)$.

Proof. (1) and (2) are obvious.

(3) Suppose $\mathcal{D}(x, y) \not\subseteq \mathcal{D}(x, z) \cup \mathcal{D}(z, y)$. Then $\mathcal{D}(x, y) - \mathcal{D}(x, z) \cup \mathcal{D}(z, y) \ne \emptyset$. Pick

$$R \in \mathcal{D}(x, y) - \mathcal{D}(x, z) \cup \mathcal{D}(z, y).$$

$R \in \mathcal{D}(x, y)$ implies $(x, y) \notin R$.

Since $R \notin \mathcal{D}(x, z) \cup \mathcal{D}(z, y)$, we have $R \notin \mathcal{D}(x, z)$ and $R \notin \mathcal{D}(z, y)$. Then $(x, z) \in R$ and $(z, y) \in R$. So $(x, y) \in R$. This is a contradiction.

Thus $\mathcal{D}(x, y) \subseteq \mathcal{D}(x, z) \cup \mathcal{D}(z, y)$. □

Corollary 3.5. *Let* $(U, \mathcal{R})$ *be a knowledge base. Then* d *is a distance function on* U *where*

$$d(x, y) = |\mathcal{D}(x, y)| \ (x, y \in U).$$

Proposition 3.6. *Let* $(U, \mathcal{R})$ *be a knowledge base. Then*

$$\mathcal{P} \in co(\mathcal{R}) \iff If \ (x, y) \notin ind(\mathcal{R}), \ then \ \mathcal{P} \cap \mathcal{D}(x, y) \neq \emptyset.$$

Proof. (1) "$\Longrightarrow$". Since $\mathcal{P} \in co(\mathcal{R})$, we have $ind(\mathcal{P}) = ind(\mathcal{R})$. Then $(x, y) \notin ind(\mathcal{P})$. So $(x, y) \notin P \ for \ some \ P \in \mathcal{P}$.

$(x, y) \notin P$ implies $P \in \mathcal{D}(x, y)$. Then $P \in \mathcal{P} \cap \mathcal{D}(x, y)$.

Thus $\mathcal{P} \cap \mathcal{D}(x, y) \neq \emptyset$.

"$\Longleftarrow$". Suppose $\mathcal{P} \notin co(\mathcal{R})$. Then $ind(\mathcal{P}) \neq ind(\mathcal{R})$. This implies $ind(\mathcal{P}) - ind(\mathcal{R}) \neq \emptyset$. Pick

$$(x, y) \in ind(\mathcal{P}) - ind(\mathcal{R}).$$

Since $(x, y) \notin ind(\mathcal{R})$, we have $\mathcal{P} \cap \mathcal{D}(x, y) \neq \emptyset$.

Note that $(x, y) \in ind(\mathcal{P})$. Then $\forall \ P \in \mathcal{P}$, $(x, y) \in P$. So $P \notin \mathcal{D}(x, y)$. Thus $\mathcal{P} \cap \mathcal{D}(x, y) = \emptyset$. This is a contradiction.

Thus $\mathcal{P} \in co(\mathcal{R})$. □

The discernibility family can easily determine knowledge reductions.

Theorem 3.7. *Let* $(U, \mathcal{R})$ *be a knowledge base. Then* $\mathcal{P} \in red(\mathcal{R}) \iff$

(1) *If* $(x, y) \notin ind(\mathcal{R})$, *then* $\mathcal{P} \cap \mathcal{D}(x, y) \neq \emptyset$;
(2) $\forall \ R \in \mathcal{P}$, $\exists \ (x_R, y_R) \in ind(\mathcal{R})$, $(\mathcal{P} - \{R\}) \cap \mathcal{D}(x_R, y_R) = \emptyset$.

Proof. This holds by Proposition 3.6. □

Definition 3.8. *Let* $(U, \mathcal{R})$ *be a knowledge base. Put*

$$core(\mathcal{R}) = \bigcap_{\mathcal{P} \in red(\mathcal{R})} \mathcal{P}.$$

Then $core(\mathcal{R})$ *is called the core of* $\mathcal{R}$. *Moreover,*

(1) $R \in \mathcal{R}$ *is called a necessary relation, if* $R \in core(\mathcal{R})$.

(2) $R \in \mathcal{R}$ *is called a relatively necessary relation, if* $R \in \bigcup_{\mathcal{P} \in red(\mathcal{R})} \mathcal{P} - core(\mathcal{R})$.

(3) $R \in \mathcal{R}$ *is called a absolutely dispensable relation, if* $R \in \mathcal{R} - \bigcup_{\mathcal{P} \in red(\mathcal{R})} \mathcal{P}$.

(4) $R \in \mathcal{R}$ *is called a dispensable relation, if* $R \in \mathcal{R} - core(\mathcal{R})$.

Obviously, R is dispensable $\iff$ R is relatively necessary or absolutely dispensable.

Proposition 3.9. *Let* $(U, \mathcal{R})$ *be a knowledge base. Then*

$$|red(\mathcal{R})| = 1 \iff core(\mathcal{R}) \in red(\mathcal{R}).$$

Proof. Necessity. This is obvious.

Sufficiency. Denote $red(\mathcal{R}) = \{\mathcal{P}_k : 1 \leq k \leq n\}$. We only need to prove $n = 1$.

Suppose $n \geq 2$. Since $core(\mathcal{R}) \in red(\mathcal{R})$, we have $core(\mathcal{R}) = \mathcal{P}_i$ for some i. Pick $j \neq i$. Then $\mathcal{P}_i = \bigcap\limits_{k=1}^{n} \mathcal{P}_k \subseteq \mathcal{P}_j$. But $\mathcal{P}_i \neq \mathcal{P}_j$. Thus $\mathcal{P}_i \subset \mathcal{P}_j$. Since $\mathcal{P}_j \in red(\mathcal{R})$, we have $\mathcal{P}_i \notin co(\mathcal{R})$. Then $\mathcal{P}_i \notin red(\mathcal{R})$. This is a contradiction.

Thus $n = 1$. □

Discernibility family can easily determine the core.

Proposition 3.10. *Let $(U, \mathcal{R})$ be a knowledge base. The following are equivalent:*

(1) *R is a necessary relation;*

(2) *R is independent in $\mathcal{R}$;*

(3) *$\exists\, x, y \in U$, $\mathcal{D}(x, y) = \{R\}$.*

Proof. (1) $\Longrightarrow$ (2). Suppose that R is not independent in $\mathcal{R}$. Then

$$ind(\mathcal{R} - \{R\}) = ind(\mathcal{R}).$$

This implies $\mathcal{R} - \{R\} \in co(\mathcal{R})$. Consider $\mathcal{R} - \{R\}$. By Proposition 3.1, $\exists\, \mathcal{P} \subseteq \mathcal{R} - \{R\}$, $\mathcal{P} \in red(\mathcal{R})$.

$\mathcal{P} \subseteq \mathcal{R} - \{R\}$ implies $R \notin \mathcal{P}$. Then R is not a necessary relation. This is a contradiction.

(2) $\Longrightarrow$ (1). Suppose that R is not a necessary relation. Then $\exists\, \mathcal{P} \in red(\mathcal{R})$, $R \notin \mathcal{P}$. So $\mathcal{P} \subseteq \mathcal{R} - \{R\} \subseteq \mathcal{R}$. This implies

$$ind(\mathcal{P}) \supseteq ind(\mathcal{R} - \{R\}) \supseteq ind(\mathcal{R}).$$

By $\mathcal{P} \in red(\mathcal{R})$, $ind(\mathcal{P}) = ind(\mathcal{R})$. Then $ind(\mathcal{R} - \{R\}) = ind(\mathcal{R})$. So R is not independent in $\mathcal{R}$. This is a contradiction.

(2) $\Longrightarrow$ (3). Since R is independent in $\mathcal{R}$, we have $ind(\mathcal{R} - \{R\}) \neq ind(\mathcal{R})$. Then $ind(\mathcal{R} - \{R\}) - ind(\mathcal{R}) \neq \emptyset$. Pick

$$(x, y) \in ind(\mathcal{R} - \{R\}) - ind(\mathcal{R}).$$

Denote $\mathcal{R} = \{R_1, R_2, \ldots, R_n\}$. Then $R = R_j$ for some $j \leq n$. So

$$(x, y) \in \bigcap_{1 \leq i \leq n, i \neq j} R_i - \bigcap_{1 \leq i \leq n} R_i.$$

This implies $(x, y) \notin R_j$ and $(x, y) \in R_i$ $(i \neq j)$.

Thus $\mathcal{D}(x, y) = \{R\}$.

(3) $\Longrightarrow$ (2). Since $\exists\, x, y \in U$, $\mathcal{D}(x, y) = \{R\}$, we have

$$(x, y) \notin R, \ (x, y) \in R^{'} \ (R^{'} \neq R).$$

Then $(x, y) \in ind(\mathcal{R} - \{R\})$. But $(x, y) \notin ind(\mathcal{R})$.

Thus $ind(\mathcal{R} - \{R\}) \neq ind(\mathcal{R})$. Hence R is independent in $\mathcal{R}$. □

Proposition 3.11. *Let $(U, \mathcal{R})$ be a knowledge base. Denote*

$$R^\star = \bigcup_{\mathcal{P} \in co(\mathcal{R})} ind(\mathcal{P} - \{R\}).$$

Then the following are equivalent:

(1) *R is a absolutely dispensable relation;*
(2) $\forall\ \mathcal{P} \in co(\mathcal{R})$, $ind(\mathcal{P} - \{R\}) = ind(\mathcal{R})$;
(3) *$R^\star = ind(\mathcal{R})$;*
(4) *$R^\star \subseteq R$.*

Proof. (1) $\Longrightarrow$ (2). By Proposition 3.1, $\exists\ \mathcal{Q} \subseteq \mathcal{P}$, $\mathcal{Q} \in red(\mathcal{R})$. Since R is not a necessary relation, we have $R \notin \mathcal{Q}$, which implies $\mathcal{Q} \subseteq \mathcal{R} - \{R\}$. Then

$$\mathcal{Q} \subseteq \mathcal{P} \cap (\mathcal{R} - \{R\}) = \mathcal{P} - \{R\} \subseteq \mathcal{P}.$$

We have

$$ind(\mathcal{Q}) \supseteq ind(\mathcal{P} - \{R\})) \supseteq ind(\mathcal{P}).$$

Note that $\mathcal{P} \in co(\mathcal{R})$ and $\mathcal{Q} \in red(\mathcal{R})$. Then $ind(\mathcal{P}) = ind(\mathcal{R}) = ind(\mathcal{Q})$. Thus

$$ind(\mathcal{P} - \{R\}) = ind(\mathcal{R}).$$

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are obvious.

(4) $\Longrightarrow$ (1). Suppose $\exists\ \mathcal{P} \in red(\mathcal{R})$, $R \in \mathcal{P}$. Then $\mathcal{P} - \{R\} \subset \mathcal{P}$. Since $\mathcal{P} \in red(\mathcal{R})$, we have $\mathcal{P} - \{R\} \notin co(\mathcal{R})$. Then $ind(\mathcal{P} - \{R\}) - ind(\mathcal{R}) \neq \emptyset$. $\mathcal{P} \in red(\mathcal{R})$ implies $ind(\mathcal{P}) = ind(\mathcal{R})$. Then

$$ind(\mathcal{P} - \{R\}) - ind(\mathcal{P}) \neq \emptyset.$$

Pick $(x, y) \in ind(\mathcal{P} - \{R\}) - ind(\mathcal{P})$. Note that $ind(\mathcal{P}) = ind(\mathcal{P} - \{R\}) \cap R$. Then $(x, y) \notin R$.

Since $\mathcal{P} \in co(\mathcal{R})$ and $R^\star \subseteq R$, we have $ind(\mathcal{P} - \{R\}) \subseteq R$. Then $(x, y) \in R$. This is a contradiction. □

Theorem 3.12. *Let $(U, \mathcal{R})$ be a knowledge base. Then*

(1) *R is necessary $\Leftrightarrow \mathcal{R} - \{R\} \notin co(\mathcal{R})$.*
(2) *R is relatively necessary $\Leftrightarrow \mathcal{R} - \{R\} \in co(\mathcal{R})$ and $R^\star \nsubseteq R$.*
(3) *R is absolutely dispensable $\Leftrightarrow R^\star \subseteq R$.*
(4) *R is dispensable $\Leftrightarrow \mathcal{R} - \{R\} \in co(\mathcal{R})$.*

Proof. This holds by Proposition 3.10 and Proposition 3.11. □

Example 3.13. *In Example 3.3, we have*

(1) *R_2 is necessary.*
(2) *R_1 and R_4 are relatively necessary.*
(3) *R_3 is absolutely dispensable.*
(4) *R_1, R_3 and R_4 are dispensable.*

4 A algorithm on knowledge reduction

It is more convenient to calculate knowledge reductions and the core in knowledge bases by using the following discernibility function when there are many equivalence relations in knowledge bases.

Below, we give a algorithm on the knowledge reductions with the help of mathematical logic.

"$\bigvee$"(disjunction), "$\bigwedge$"(conjunction), "$\longrightarrow$"(implication), "$\longleftrightarrow$"(biimplication) are propositional connectives in mathematical logic. They are read as "or", "and", "if-then", "if and only if", respectively.

Let $(U, \mathcal{R})$ be a knowledge base. $\forall\ R \in \mathcal{R}$, we specify a Boolean variable "R". If $\mathcal{D}(x, y) = \{R_1, R_2, \cdots, R_k\}$ with $x, y \in U$, then we specify a Boolean function $R_1 \vee R_2 \vee \cdots \vee R_k$.

Denote

$$\bigvee\{R_1, R_2, \cdots, R_k\}\ or\ \bigvee_{i=1}^{k} R_i = R_1 \vee R_2 \vee \cdots \vee R_k,$$

$$\bigwedge\{R_1, R_2, \cdots, R_k\}\ or\ \bigwedge_{i=1}^{k} R_i = R_1 \wedge R_2 \wedge \cdots \wedge R_k.$$

We stipulate that $\vee\ \emptyset = 1$ and $\wedge\ \emptyset = 0$ where 0 and 1 are two Boolean constants.

Definition 4.1. *Let $(U, \mathcal{R})$ be a knowledge base where $U = \{x_1, x_2, \cdots, x_n\}$ and $\mathfrak{D}(\mathcal{R}) = (d_{ij})_{n\times n}$ the discernibility matrix of $\mathcal{R}$. We define the discernibility function $\Delta(\mathcal{R})$ of $\mathcal{R}$ as follows:*

$$\Delta(\mathcal{R}) = \bigwedge(\bigvee d_{ij}).$$

Example 4.2. *In Example 3.3, we have*

$$\triangle(\mathcal{R}) = R_2 \wedge (R_1 \vee R_2 \vee R_3 \vee R_4) \wedge (R_1 \vee R_4) \wedge (R_1 \vee R_3 \vee R_4) \wedge (R_1 \vee R_2 \vee R_4) \wedge (R_2 \vee R_3).$$

Denote

$$L(\mathcal{R}) = \{\bigvee d_{ij} : 1 \leq i, j \leq n\}.$$

We define a binary relation "$\leq$" on $L(\mathcal{R})$ as follows:

$$\bigvee d_{ij} \leq \bigvee d_{kl} \iff d_{ij} \subseteq d_{kl}\ for\ any\ \bigvee d_{ij}, \bigvee d_{kl} \in L(\mathcal{R}).$$

For any $\bigvee d_{ij}, \bigvee d_{kl} \in L(\mathcal{R})$, we denote

$$(\bigvee d_{ij}) \bigsqcup (\bigvee d_{kl}) = \bigvee (d_{ij} \cup d_{kl}),\quad (\bigvee d_{ij}) \bigsqcap (\bigvee d_{kl}) = \bigvee (d_{ij} \cap d_{kl}).$$

Proposition 4.3. *$(L(\mathcal{R}),\ \leq)$ is a poset.*

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Proof. (1) $\bigvee d_{ij} \le \bigvee d_{ij}$ for any $\bigvee d_{ij} \in L(\mathcal{R})$.

(2) Let $\bigvee d_{ij}, \bigvee d_{kl} \in L(\mathcal{R})$. Suppose $\bigvee d_{ij} \le \bigvee d_{kl}$ and $\bigvee d_{kl} \le \bigvee d_{ij}$. Then $d_{ij} \subseteq d_{kl}$ and $d_{kl} \subseteq d_{ij}$. This implies $d_{ij} = d_{kl}$. So $\bigvee d_{ij} = \bigvee d_{kl}$.

(3) Let $\bigvee d_{ij}, \bigvee d_{kl}, \bigvee d_{hv} \in L(\mathcal{R})$. Suppose $\bigvee d_{ij} \le \bigvee d_{kl}$ and $\bigvee d_{kl} \le \bigvee d_{hv}$. Then $d_{ij} \subseteq d_{kl}$ and $d_{kl} \subseteq d_{hv}$. This implies $d_{ij} \subseteq d_{hv}$. So $\bigvee d_{ij} \le \bigvee d_{hv}$.

Thus $(L(\mathcal{R}), \le)$ is a poset. □

Proposition 4.4. *Let $(U, \mathcal{R})$ be a knowledge base where $U = \{x_1, x_2, \cdots, x_n\}$. If $\{d_{ij} : 1 \le i, j \le n\}$ is a topology on $\mathcal{R}$, then $(L(\mathcal{R}), \le, \sqcup, \sqcap)$ is a lattice with top element and bottom element.*

Proof. Denote $\tau = \{d_{ij} : 1 \le i, j \le n\}$. By Proposition 4.3, $(L(\mathcal{R}), \le)$ is a poset.

For $\bigvee d_{ij}, \bigvee d_{kl} \in L(\mathcal{R})$, since τ is a topology on $\mathcal{R}$, we have $d_{ij} \cup d_{kl} \in \tau, d_{ij} \cap d_{kl} \in \tau$. This implies

$$(\bigvee d_{ij}) \bigsqcup (\bigvee d_{kl}) = \bigvee (d_{ij} \cup d_{kl}) \in L(\mathcal{R}),$$

$$(\bigvee d_{ij}) \bigsqcap (\bigvee d_{kl}) = \bigvee (d_{ij} \cap d_{kl}) \in L(\mathcal{R}).$$

Obviously, $1_{L(\mathcal{R})} = \vee \mathcal{R}$, $0_{L(\mathcal{R})} = \vee \emptyset$.
Thus $(L(\mathcal{R}), \le, \bigsqcup, \bigsqcap)$ is a lattice with top element and bottom element. □

Example 4.5. *In Example 3.3, $(\bigvee d_{23}) \bigsqcap (\bigvee d_{63}) = R_3 \notin L(\mathcal{R})$. Then $(L(\mathcal{R}), \le, \sqcup, \sqcap)$ is not a lattice.*

Definition 4.6. *Let $(U, \mathcal{R})$ be a knowledge base. If $\Delta(\mathcal{R}) = \bigvee\limits_{k=1}^{q} (\bigwedge\limits_{l=1}^{p_k} R_{kl})$, where every $\mathcal{P}_k = \{R_{kl} : l \le p_k\} \subseteq \mathcal{R}$ has not repetitive elements, then $\bigvee\limits_{k=1}^{q} (\bigwedge\limits_{l=1}^{p_k} R_{kl})$ is called the standard minimum formula of $\Delta(\mathcal{R})$. We denote it by $\Delta^*(\mathcal{R})$. That is,*

$$\Delta^*(\mathcal{R}) = \bigvee_{k=1}^{q} (\bigwedge_{l=1}^{p_k} R_{kl}).$$

Example 4.7. *In Example 3.3, we have*

$$R_2 \le (R_1 \vee R_2 \vee R_3 \vee R_4), \ \ R_2 \le (R_1 \vee R_2 \vee R_4), R_2 \le (R_2 \vee R_3), \ \ (R_1 \vee R_4) \le (R_1 \vee R_3 \vee R_4).$$

Obviously,

$$R_2 \wedge (R_1 \vee R_2 \vee R_3 \vee R_4) = R_2, \quad R_2 \wedge (R_1 \vee R_2 \vee R_4) = R_2,$$

$$R_2 \wedge (R_2 \vee R_3) = R_2, \quad (R_1 \vee R_4) \wedge (R_1 \vee R_3 \vee R_4) = R_1 \vee R_4.$$

Then $\Delta(\mathcal{R}) = R_2 \wedge (R_1 \vee R_2 \vee R_3 \vee R_4) \wedge (R_1 \vee R_4) \wedge (R_1 \vee R_3 \vee R_4) \wedge (R_1 \vee R_2 \vee R_4) \wedge (R_2 \vee R_3)$
$= R_2 \wedge (R_1 \vee R_4)$
$= (R_1 \wedge R_2) \vee (R_2 \wedge R_4)$.

Thus $\Delta^*(\mathcal{R}) = (R_1 \wedge R_2) \vee (R_2 \wedge R_4)$.

Theorem 4.8. *Let* $(U, \mathcal{R})$ *be a knowledge base. If* $\Delta^*(\mathcal{R}) = \bigvee_{k=1}^{q} (\bigwedge_{l=1}^{p_k} R_{kl})$ *is the standard minimum formula of* $\Delta(\mathcal{R})$*. Then* $red(\mathcal{R}) = \{\mathcal{P}_k : k \leq q\}$ *where* $\mathcal{P}_k = \{R_{kl} : l \leq p_k\}$.

Proof. (1) Let $\mathcal{P}_{k_0} \in \{\mathcal{P}_k : k \leq q\}$.

(i) Obviously, $\Delta^*(\mathcal{R}) = \bigvee_{k=1}^{q} (\bigwedge_{l=1}^{p_k} R_{kl}) = \bigvee_{k=1}^{q} (\bigwedge \mathcal{P}_k)$. Then $\bigwedge \mathcal{P}_{k_0} \longrightarrow \Delta^*(\mathcal{R})$.

Since $\Delta^*(\mathcal{R}) = \Delta(\mathcal{R}) = \bigwedge(\bigvee d_{ij})$, we have

$$\Delta^*(\mathcal{R}) \Longleftrightarrow \bigvee d_{ij} \ for \ any \ 1 \leq i, j \leq n.$$

Then $\forall\ x, y \in U$, $\bigwedge \mathcal{P}_{k_0} \longrightarrow \bigvee \mathcal{D}(x, y)$.

So $\forall\ (x, y) \notin ind(\mathcal{R})$, $\bigwedge \mathcal{P}_{k_0} \longrightarrow \bigvee \mathcal{D}(x, y)$.

Now $\bigwedge \mathcal{P}_{k_0} \Longleftrightarrow R_{k_0 l}$ for any $l \leq p_{k_0}$ and $\bigvee \mathcal{D}(x, y) \longleftrightarrow R$ for some $R \in \mathcal{D}(x, y)$. Then $\forall\ (x, y) \notin ind(\mathcal{R})$, $R_{k_0 l}$ for any $l \leq p_{k_0} \longrightarrow R$ for some $R \in \mathcal{D}(x, y)$. So $\forall\ (x, y) \notin ind(\mathcal{R})$, there exists $l_0 \leq p_{k_0}$ such that $R = R_{k_0 l_0}$, i.e., $R \in \mathcal{P}_{k_0} \cap \mathcal{D}(x, y)$. Thus $\forall\ (x, y) \notin ind(\mathcal{R})$, $\mathcal{P}_{k_0} \cap \mathcal{D}(x, y) \neq \emptyset$.

By Proposition 3.6, $\mathcal{P}_{k_0} \in co(\mathcal{R})$.

(ii) To prove $\mathcal{P}_{k_0} \in red(\mathcal{R})$, by Theorem 3.7, we only need to show that

$$\forall\ R \in \mathcal{P}_{k_0},\ \exists\ (x_R, y_R) \in ind(\mathcal{R}),\ (\mathcal{P}_{k_0} - \{R\}) \cap \mathcal{D}(x_R, y_R) = \emptyset.$$

Suppose that $\exists\ R_0 \in \mathcal{P}_{k_0}$ such that $(\mathcal{P}_{k_0} - \{R_0\}) \cap \mathcal{D}(x, y) \neq \emptyset$ for any $(x, y) \notin ind(\mathcal{R})$. Pick $R_{xy} \in (\mathcal{P}_{k_0} - \{R_0\}) \cap \mathcal{D}(x, y)$. Then $\bigwedge(\mathcal{P}_{k_0} - \{R_0\}) \longrightarrow R_{xy}$ and $R_{xy} \longrightarrow \bigvee \mathcal{D}(x, y)$. Thus $\forall\ (x, y) \notin ind(\mathcal{R})$, $\bigwedge(\mathcal{P}_{k_0} - \{R_0\}) \longrightarrow \bigvee \mathcal{D}(x, y)$.

$\forall\ (x, y) \in ind(\mathcal{R})$, we have $\mathcal{D}(x, y) = \emptyset$. Then $\bigwedge(\mathcal{P}_{k_0} - \{R_0\}) \longrightarrow \bigvee \mathcal{D}(x, y)$.

It follows that $\forall\ x, y \in U$,

$$\bigwedge(\mathcal{P}_{k_0} - \{R_0\}) \longrightarrow \bigvee \mathcal{D}(x, y).$$

Since $\Delta^*(\mathcal{R})$ contains all true explanations of $\Delta(\mathcal{R})$, we have $\mathcal{P}_{k_0} - \{R_0\} \in \{\mathcal{P}_k : k \leq q\}$. Then

$(\bigwedge \mathcal{P}_{k_0}) \bigvee (\bigwedge(\mathcal{P}_{k_0} - \{R_0\}))$
$= ((\bigwedge(\mathcal{P}_{k_0} - \{R_0\})) \bigwedge \{R_0\}) \bigvee ((\bigwedge(\mathcal{P}_{k_0} - \{R_0\})) \bigwedge 1)$
$= (\bigwedge(\mathcal{P}_{k_0} - \{R_0\})) \bigwedge (\{R_0\} \bigvee 1)$
$= (\bigwedge(\mathcal{P}_{k_0} - \{R_0\})) \bigwedge 1$
$= \bigwedge(\mathcal{P}_{k_0} - \{R_0\})$.

This implies $\mathcal{P}_{k_0} \notin \{\mathcal{P}_k : k \leq q\}$. This is a contradiction.

Thus $\mathcal{P}_{k_0} \in red(\mathcal{R})$. This show $red(\mathcal{R}) \supseteq \{\mathcal{P}_k : k \leq q\}$.

(2) Let $\mathcal{P} \in red(\mathcal{R})$. Then $\mathcal{P} \in co(\mathcal{R})$. By Proposition 3.6, $\mathcal{P} \cap \mathcal{D}(x,y) \neq \emptyset$ for any $(x,y) \notin ind(\mathcal{R})$. Similar to the proof of (1) (ii), we have $\mathcal{P} \in \{\mathcal{P}_k : k \leq q\}$.

Thus $red(\mathcal{R}) \subseteq \{\mathcal{P}_k : k \leq q\}$. Hence $red(\mathcal{R}) = \{\mathcal{P}_k : k \leq q\}$.

□

Algorithms 4.9. *Let $(U, \mathcal{R})$ be a knowledge base. The algorithm of knowledge reductions of $\mathcal{R}$ is shown as follows:*

Input: the knowledge base $(U, \mathcal{R})$;
Output: $red(\mathcal{R})$ and $core(\mathcal{R})$.

Step 1. *Input the knowledge base $(U, \mathcal{R})$;*
Step 2. *Calculate the discernibility matrix $\mathfrak{D}(\mathcal{R})$ of $\mathcal{R}$;*
Step 3. *Give discernibility function $\Delta(\mathcal{R})$ of $\mathcal{R}$;*
Step 4. *Calculate standard minimum formula $\Delta^*(\mathcal{R})$ of $\Delta(\mathcal{R})$;*
Step 5. *Output all knowledge reductions of $\mathcal{R}$ and the core of $\mathcal{R}$.*

Example 4.10. *We consider Example 3.3.*

In Step 1, *we input the knowledge base $(U, \mathcal{R})$.*
In Step 2, *we obtain the discernibility matrix $\mathfrak{D}(\mathcal{R})$.*
In Step 3, *we obtain*

$$\Delta(\mathcal{R}) = R_2 \wedge (R_1 \vee R_2 \vee R_3 \vee R_4) \wedge (R_1 \vee R_4) \wedge (R_1 \vee R_3 \vee R_4) \wedge (R_1 \vee R_2 \vee R_4) \wedge (R_2 \vee R_3).$$

In Step 4, *we obtain $\Delta^*(\mathcal{R}) = (R_1 \wedge R_2) \vee (R_2 \wedge R_4)$.*
In Step 5, *we obtain all knowledge reductions of $\mathcal{R}$: $\{R_1, R_2\}$, $\{R_2, R_4\}$ and $core(\mathcal{R}) = \{R_2\}$.*

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Belief reduction in IVF decision information systems and its algorithm *

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Abstract: Attribute reduction is one of the main problems in the study of information systems. This paper investigates belief reduction in IVF decision information systems by using the generalized D-S theory of evidence and rough set theory.

Keywords: IVF; Decision information system; Similarity relation; Belief reduction.

1 Introduction

Imprecision and uncertainty are two important aspects of incompleteness of information. One theory for the study of insufficient and incomplete information in intelligent systems is rough set theory [3]. Another important method used to deal with uncertainty in information systems is D-S theory of evidence [4]. There are strong connections between these two theory. It has been demonstrated that various belief structures are associated with various approximation spaces such that the different dual pairs of lower and upper approximation operators induced by approximation spaces may be used to interpret the corresponding dual pairs of belief functions induced by belief structures [5, 8, 10]. Based on this observation, D-S theory of evidence may be used to analyze attribute reduction and knowledge acquisition in information systems [7, 9, 12]. In the traditional rough set approach, the values of attributes are assumed to be nominal data, i.e. symbols. In many applications, however, the decision attribute-values can be linguistic terms (i.e. interval value fuzzy sets). The traditional rough set approach would treat these values as symbols, thereby some important information included in these values such as the partial ordering and membership degrees is ignored, which means that the traditional rough set approach cannot effectively deal with interval value fuzzy initial data (e.g. linguistic terms). Thus a new rough set model is needed to deal with such data.

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The purpose of this paper is to investigate belief reduction in IVF decision information systems by using the generalized D-S theory of evidence and rough set theory.

2 Preliminaries

Throughout this paper, "interval-valued fuzzy" denote briefly by " IVF ". U denotes a finite and nonempty set called the universe. 2^U denotes the family of all subsets of U. $F(U)$ denotes the set of all fuzzy sets in U. I denotes $[0,1]$ and $[I]$ denotes $\{[a,b] : a,b \in I \ and \ a \leq b\}$.

2.1 IVF sets

Definition 2.1 ([1]). *$\forall\, a,b \in [I]$, define*

(1) $a = b \Longleftrightarrow a^- = b^-,\ a^+ = b^+$.
(2) $a \leq b \Longleftrightarrow a^- \leq b^-,\ a^+ \leq b^+$; $\quad a < b \Longleftrightarrow a \leq b,\ a \neq b$.
(3) $a^c = [1,1] - a = [1 - a^+, 1 - a^-]$.

Definition 2.2 ([1]). *$\forall\, \{a_i : i \in J\} \subseteq [I]$, define*

(1) $\bigvee\limits_{i\in J} a_i = [\bigvee\limits_{i\in J} a_i^-,\ \bigvee\limits_{i\in J} a_i^+]$.
(2) $\bigwedge\limits_{i\in J} a_i = [\bigwedge\limits_{i\in J} a_i^-,\ \bigwedge\limits_{i\in J} a_i^+]$.

Definition 2.3 ([1]). *A mapping $A : U \to [I]$ is called an IVF set on U. Denote*

$$A(x) = [A^-(x), A^+(x)] \quad (x \in U).$$

Then $A^-(x)$ (resp. $A^+(x)$) is called the lower (resp. upper) degree to which x belongs to A. A^- (resp. A^+) is called the lower (resp. upper) IVF set of A.

The set of all IVF sets in U is denoted by $F^{(i)}(U)$.

Similar to fuzzy sets, the operators $\subseteq, \cap, \cup$ and the complement of IVF sets can be defined.

2.2 IVF decision information systems

Definition 2.4 ([6]). *$(U, A \cup D)$ is called an IVF decision information system, where $U = \{x_0, x_1, \cdots, x_{n-1}\}$ is the universe, A is a condition attribute set and $D = \{d_k \in F^{(i)}(U) : k = 1, 2, \cdots, r\}$ is a decision attribute set.*

Example 2.5 ([6]). *Table 1 gives an IVF decision information system $(U, A \cup D)$ where $U = \{x_0, x_1, \cdots, x_9\}$, $A = \{a_1, a_2, a_3\}$, $D = \{d_1, d_2, d_3\}$.*

Definition 2.6. *Let $(U, A \cup D)$ be an IVF decision information system. Then $B \subseteq A$ determines an equivalence relation as follows:*

$$R_B = \{(x,y) \in U \times U : a(x) = a(y)\ (a \in B)\}.$$

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Table 1: An IVF decision information system $(U, A \cup D)$

	a_1	a_2	a_3	d_1	d_2	d_3
x_0	2	1	3	[0.7,0.9]	[0.15,0.2]	[0.4,0.5]
x_1	3	2	1	[0.3,0.5]	[0.5,0.7]	[0.35,0.4]
x_2	2	1	3	[0.7,0.8]	[0.3,0.4]	[0.1,0.2]
x_3	2	2	3	[0.15,0.2]	[0.5,0.8]	[0.2,0.3]
x_4	1	1	4	[0.05,0.1]	[0.2,0.3]	[0.65,0.9]
x_5	1	1	2	[0.1,0.2]	[0.35,0.5]	[1.0,1.0]
x_6	3	2	1	[0.25,0.4]	[1.0,1.0]	[0.3,0.4]
x_7	1	1	4	[0.1,0.2]	[0.25,0.4]	[0.5,0.6]
x_8	2	1	3	[0.45,0.6]	[0.25,0.3]	[0.2,0.3]
x_9	3	2	1	[0.05,0.1]	[0.8,0.9]	[0.05,0.2]

R_B forms a partition $U/R_B = \{[x]_B : x \in U\}$ of U where $[x]_B = \{y \in U : (x, y) \in R_B\}$.

The lower and upper approximations of $X \in F^{(i)}(U)$ with regard to (U, R_B) as follows:

$$\underline{R_B}(X)(x) = \bigwedge_{y \in [x]_B} X(y), \quad \overline{R_B}(X)(x) = \bigvee_{y \in [x]_B} X(y) \ (x \in U).$$

Remark 2.7. *If $y \in [x]_B$, then $[y]_B = [x]_B$. So $\underline{R_B}(X)(y) = \underline{R_B}(X)(x)$ and $\overline{R_B}(X)(y) = \overline{R_B}(X)(x)$.*

Denote $\underline{R_B}(X)([x]_B) = \underline{R_B}(X)(x)$, $\overline{R_B}(X)([x]_B) = \overline{R_B}(X)(x)$ $\quad (*)$

Proposition 2.8. *Let $(U, A \cup D)$ be an IVF decision information system and let $C \subseteq B \subseteq A$. Then $\forall\ X \in F^{(i)}(U)$,*

(1) $\underline{R_B}(\widetilde{U} - X) = \widetilde{U} - \overline{R_B}(X)$.

(2) $\underline{R_C}(X) \subseteq \underline{R_B}(X) \subseteq X \subseteq \overline{R_B}(X) \subseteq \overline{R_C}(X)$.

Proof. (1) $\forall\ x \in U$,

$$\begin{aligned}\underline{R_B}(\widetilde{U} - X)(x) &= \bigwedge_{y \in [x]_B} (\widetilde{U} - X)(y) = \bigwedge_{y \in [x]_B} (\widetilde{U}(y) - X(y)) \\ &= \bigwedge_{y \in [x]_B} \widetilde{U}(y) - \bigvee_{y \in [x]_B} X(y) \\ &= \widetilde{U}(x) - \bigvee_{y \in [x]_B} X(y) = (\widetilde{U} - \overline{R_B}(X))(x).\end{aligned}$$

Then $\underline{R_B}(\widetilde{U} - X) = \widetilde{U} - \overline{R_B}(X)$.

(2) Since $C \subseteq B$, $\forall\ x \in U$, $[x]_C \supseteq [x]_B$. Then

$$\underline{R_C}(X)(x) = \bigwedge_{y \in [x]_C} X(y) \leq \bigwedge_{y \in [x]_B} X(y) = \underline{R_B}(X)(x).$$

So $\underline{R_C}(X) \subseteq \underline{R_B}(X)$.

Similarly, $\overline{R_B}(X) \subseteq \overline{R_C}(X)$.

$\forall\, x \in U,\ x \in [x]_B$. Then $\underline{R_B}(X)(x) = \bigwedge\limits_{y\in[x]_B} X(y) \leq X(x)$. So $\underline{R_B}(X) \subseteq X$.

Similarly, $X \subseteq \overline{R_B}(X)$.

Hence

$$\underline{R_C}(X) \subseteq \underline{R_B}(X) \subseteq X \subseteq \overline{R_B}(X) \subseteq \overline{R_C}(X).$$

□

3 The generalized D-S theory of evidence

3.1 Necessity IVF measures and possibility IVF measures

Zadeh's theory of possibility [11] is based on the idea that the possibility of an event is determined by its most favorable case only.

$N^{(i)} : F^{(i)}(U) \to [I]$ is called a necessity IVF measure if it satisfies

$$N^{(i)}(\widetilde{\emptyset}) = \overline{0},\ N^{(i)}(\widetilde{U}) = [1,1],\ N^{(i)}(X \cap Y) = N^{(i)}(X) \wedge N^{(i)}(Y).$$

$\Pi^{(i)} : F^{(i)}(U) \to [I]$ is called a possibility IVF measure if it satisfies

$$\Pi^{(i)}(\widetilde{\emptyset}) = \overline{0},\ \Pi^{(i)}(\widetilde{U}) = [1,1],\ \Pi^{(i)}(X \cup Y) = \Pi^{(i)}(X) \vee \Pi^{(i)}Y).$$

It can easily be checked that $N^{(i)} : F^{(i)}(U) \to [I]$ is a necessity IVF measure iff the function Π^i defined by

$$\Pi^{(i)}(X) = [1,1] - N^{(i)}(X)\ (\forall X \in F^{(i)}(U))$$

is a possibility IVF measure.

Proposition 3.1. *Let $A \in 2^U$. For $X \in F^{(i)}(U)$, denote*

$$N_A^{(i)}(X) = \bigwedge_{y\in A} X(y) = [\bigwedge_{y\in A} X^-(y), \bigwedge_{y\in A} X^+(y)],$$

$$\Pi_A^{(i)}(X) = \bigvee_{y\in A} X(y) = [\bigvee_{y\in A} X^-(y), \bigvee_{y\in A} X^+(y)].$$

Then $N_A^{(i)}$ (resp. $\Pi_A^{(i)}$) is a necessity (resp. possibility) IVF measure.

Proof. The proof is obvious. □

3.2 IVF belief functions

Definition 3.2. *Let $(\mathcal{M}, m)$ be a belief structure on U. $\mathrm{Bel}^{(i)} : F^{(i)}(U) \to [I]$ is called a IVF belief function induced by $(\mathcal{M}, m)$ on U, if $\mathrm{Bel}^{(i)}(X) = \sum\limits_{\{Y:Y\in\mathcal{M}\}} m(Y)N_A^{(i)}(X)$.*

It can be prove that $\mathrm{Bel}^{(i)}$ is a IVF belief function iff (i) $\mathrm{Bel}^{(i)}(\widetilde{\emptyset}) = \overline{0}$, (ii) $\mathrm{Bel}^{(i)}(\widetilde{U}) = [1,1]$, (iii) $\mathrm{Bel}^{(i)}(\bigcup\limits_{i=1}^{k} X_i) \geq \sum\limits_{\emptyset\neq J\subseteq\{1,2,\cdots,k\}} (-1)^{|J|+1}\mathrm{Bel}^{(i)}(\bigcap\limits_{i\in J} X_i)$.

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4 Belief reduction in IVF decision information systems

4.1 The similarity relation R_D

Definition 4.1. *Let $S = (U, A \cup D)$ be a IVF decision information system where $U = \{x_0, x_1, \cdots, x_n\}$, A is a condition attribute set, $D = \{d_1, d_2, \cdots, d_r\}$ is a decision attribute set.*

Denote

$$d_k(x_i) = D_{ik}\ (i = 0, 1, \cdots, n-1,\ k = 1, 2, \cdots, r),$$

For $i, j \in \{0, 1, \cdots, n-1\}$, define

$$R_D(x_i, x_j) = \bigwedge\{[1,1] - D_{ik} \wedge D_{jk}| : k = 1, 2, \cdots, r\}.$$

Obviously, $R_D(x_i, x_i) = [1,1]$, $R_D(x_i, x_j) = R_D(x_j, x_i)$. Then R_D is a similarity relation on U. We can obtain the similar decision class $S_D(x)$:

$$S_D(x)(y) = R_D(x, y)\ (y \in U).$$

Denote

$$S_D(x_i) = \frac{x_0}{S_D(x_i)(x_0)} + \frac{x_1}{S_D(x_i)(x_1)} + \frac{x_2}{S_D(x_i)(x_2)} + \frac{x_3}{S_D(x_i)(x_3)} + \frac{x_4}{S_D(x_i)(x_4)} + \frac{x_5}{S_D(x_i)(x_5)} + \frac{x_6}{S_D(x_i)(x_6)}$$

$$+ \frac{x_7}{S_D(x_i)(x_7)} + \frac{x_8}{S_D(x_i)(x_8)} + \frac{x_9}{S_D(x_i)(x_9)}\quad (i = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9),$$

$$U/R_D = \{S_D(x) : x \in U\}.$$

Example 4.2. *Consider the IVF decision information system $S = (U, A \cup D)$ in Example* 2.5.

$$\begin{aligned} S_D(x_0)(x_1) &= R_D(x_0, x_1) \\ &= ([1,1] - D_{01} \wedge D_{11}) \bigwedge([1,1] - D_{02} \wedge D_{12}) \bigwedge([1,1] - D_{03} \wedge D_{13}) \\ &= ([1,1] - [0.7, 0.9] \wedge [0.3, 0.5]) \bigwedge([1,1] - [0.15, 0.2] \wedge [0.5, 0.7]) \\ &\quad \bigwedge([1,1] - [0.4, 0.5] \wedge [0.35, 0.4]) \\ &= ([1,1] - [0.3, 0.5]) \bigwedge([1,1] - [0.15, 0.2]) \bigwedge([1,1] - [0.35, 0.4]) \\ &= [0.5, 0.7] \bigwedge[0.8, 0.85]) \bigwedge[0.6, 0.65] \\ &= [0.5, 0.65]. \end{aligned}$$

Similarly,

$$S_D(x_0)(x_0) = [1,1],\quad S_D(x_0)(x_2) = [0.2, 0.3],\quad S_D(x_0)(x_3) = [0.7, 0.8],$$

$$S_D(x_0)(x_4) = [0.5, 0.6],\quad S_D(x_0)(x_5) = [0.5, 0.6],\quad S_D(x_0)(x_6) = [0.6, 0.7],$$

$S_D(x_0)(x_7) = [0.5, 0.6], \quad S_D(x_0)(x_8) = [0.4, 0.55], \quad S_D(x_0)(x_9) = [0.8, 0.85].$

Thus

$$S_D(x_0) = \frac{x_0}{[1,1]} + \frac{x_1}{[0.5, 0.65]} + \frac{x_2}{[0.2, 0.3]} + \frac{x_3}{[0.7, 0.8]} + \frac{x_4}{[0.5, 0.6]} + \frac{x_5}{[0.5, 0.6]} + \frac{x_6}{[0.6, 0.7]} + \frac{x_7}{[0.5, 0.6]} + \frac{x_8}{[0.4, 0.55]} + \frac{x_9}{[0.8, 0.85]}.$$

We can also calculate $S_D(x_i)$ $(i = 1, 2, 3, 4, 5, 6, 7, 8, 9)$. *They record in Table 2.*

Table 2: $S_D(x_i)(x_j)$

	x_0	x_1	x_2	x_3	x_4
x_0	[1,1]	[0.5,0.65]	[0.2,0.3]	[0.7,0.8]	[0.5,0.6]
x_1	[0.5,0.65]	[1,1]	[0.5,0.7]	[0.3,0.5]	[0.6,0.65]
x_2	[0.2,0.3]	[0.5,0.7]	[1,1]	[0.6,0.7]	[0.7,0.8]
x_3	[0.7,0.8]	[0.3,0.5]	[0.6,0.7]	[1,1]	[0.7,0.8]
x_4	[0.5,0.6]	[0.6,0.65]	[0.7,0.8]	[0.7,0.8]	[1,1]
x_5	[0.5,0.6]	[0.5,0.65]]	[0.6,0.7]	[0.5,0.65]	[0.1,0.35]
x_6	[0.6,0.7]	[0.3,0.5]	[0.6,0.7]	[0.2,0.5]	[0.6,0.7]
x_7	[0.5,0.6]	[0.6,0.65]	[0.4,0.75]	[0.6,0.75]	[0.4,0.5]
x_8	[0.4,0.55]	[0.5,0.7]	[0.4,0.55]	[0.7,0.75]	[0.7,0.8]
x_9	[0.8,0.85]	[0.3,0.5]	[0.6,0.7]	[0.2,0.5]	[0.7,0.8]
−−	−−	−−	−−	−−	−−
	x_5	x_6	x_7	x_8	x_9
x_0	[0.5,0.6]	[0.6,0.7]	[0.5,0.6]	[0.4,0.55]	[0.8,0.85]
x_1	[0.5,0.65]	[0.3,0.5]	[0.6,0.65]	[0.5,0.7]	[0.3,0.5]
x_2	[0.6,0.7]	[0.6,0.7]	[0.4,0.75]	[0.4,0.55]	[0.6,0.7]
x_3	[0.5,0.65]	[0.2,0.5]	[0.6,0.75]	[0.7,0.75]	[0.2,0.5]
x_4	[0.1,0.35]	[0.6,0.7]	[0.4,0.5]	[0.7,0.8]	[0.7,0.8]
x_5	[1,1]	[0.5,0.65]	[0.4,0.5]	[0.7,0.75]	[0.5,0.65]
x_6	[0.5,0.65]	[1,1]	[0.6,0.7]	[0.6,0.75]	[0.1,0.2]
x_7	[0.4,0.5]	[0.6,0.7]	[1,1]	[0.7,0.75]	[0.6,0.75]
x_8	[0.7,0.75]	[0.6,0.75]	[0.7,0.75]	[1,1]	[0.7,0.75]
x_9	[0.5,0.65]	[0.6,0.75]	[0.6,0.75]	[0.7,0.75]	[1,1]

4.2 Belief reduction

Denote the probability of $X \in F(U)$ by $\mathrm{P}(X)$. In [4], define $\mathrm{P}(X) = \sum_{x \in U} X(x)\mathrm{P}(\{x\})$.

Now we define the probability $\mathrm{P}^{(i)}(X)$ of $X \in F^{(i)}(U)$ by

$$\mathrm{P}^{(i)}(X) = \sum_{x \in U} X(x)\mathrm{P}^{(i)}(\{x\}) = [\sum_{x \in U} X^{-}(x)\mathrm{P}^{(i)}(\{x\}), \sum_{x \in U} X^{+}(x)\mathrm{P}^{(i)}(\{x\})].$$

Proposition 4.3. *Let $(U, A \cup D)$ be an IVF decision information system and $B \subseteq A$. For $X \in F^{(i)}(U)$, denote*

$$\mathrm{Bel}_B^{(i)}(X) = \mathrm{P}^{(i)}(\underline{R_B}(X)) = \sum_{x \in U} \underline{R_B}(X)(x)\mathrm{P}(\{x\}).$$

Pick $\mathcal{M} = U/R_B = \{Y_x : x \in U\}$. Define the basic probability assignment m_B by

$$m_B(Y) = \begin{cases} \frac{|Y|}{|U|}, & if\ Y \in \mathcal{M}, \\ 0, & otherwise. \end{cases}$$

Then $\mathrm{Bel}_B^{(i)} : F^{(i)}(U) \to [I]$ is an IVF belief function induced by $(\mathcal{M}, m)$ on U.

Proof.

$$\begin{aligned} \mathrm{Bel}_B^{(i)}(X) &= \mathrm{P}^{(i)}(\underline{R_B}(X)) = \sum_{x \in U} [\underline{R_B}(X)(x)\mathrm{P}(\{x\})] \\ &= \sum_{x \in U} [\mathrm{P}(\{x\})(\bigwedge_{y \in Y_x} X(y))] = \sum_{Y_x \in \mathcal{M}} [\sum_{x \in Y_x} \mathrm{P}(\{x\})(\bigwedge_{y \in Y_x} X(y))] \\ &= \sum_{Y_x \in \mathcal{M}} [\mathrm{P}(Y_x)(\bigwedge_{y \in Y_x} X(y))] = \sum_{Y_x \in \mathcal{M}} m(Y_x)(\bigwedge_{y \in Y_x} X(y)) \\ &= \sum_{\{Y_x : Y_x \in \mathcal{M}\}} m(Y_x) N_{Y_x}^{(i)}(X). \qquad (4.1) \end{aligned}$$

Thus $\mathrm{Bel}_B^{(i)}$ is an IVF belief function induced by $(\mathcal{M}, m)$ on U. □

Proposition 4.4. *Let $(U, A \cup D)$ be an IVF decision information system. If $C \subseteq B \subseteq A$ and $X \in F^{(i)}(U)$, then*

$$\mathrm{Bel}_C^{(i)}(X) \le \mathrm{Bel}_B^{(i)}(X) \le X.$$

Proof. This holds by Proposition 4.3. □

Definition 4.5. *Let $S = (U, A \cup D)$ be an IVF decision information system.*

(1) *$B \subseteq A$ is called a belief consistent set in S, if*

$$\mathrm{Bel}_B^{(i)}(S_D(x)) = \mathrm{Bel}_A^{(i)}(S_D(x)) \ (x \in U).$$

(2) *If $B \subseteq A$ is a belief consistent set in S and $\forall\ C \subsetneqq B$,*

$$\mathrm{Bel}_C^{(i)}(S_D(x)) \ne \mathrm{Bel}_A^{(i)}(S_D(x)) \ (x \in U),$$

then B is called a belief reduction in S.

Lemma 4.6. *Let $S = (U, A \cup D)$ be an IVF decision information system. Then $B \subseteq A$ is a belief consistent set in $S \iff$*

$$\sum_{x \in U} \mathrm{Bel}_B^{(i)}(S_D(x)) = \sum_{x \in U} \mathrm{Bel}_A^{(i)}(S_D(x)).$$

Proof. " $\Longrightarrow$ " Suppose that B is a belief consistent set in S. Then

$$\forall\ x \in U,\quad \mathrm{Bel}_B^{(i)}(S_D(x)) = \mathrm{Bel}_A^{(i)}(S_D(x)).$$

Thus

$$\sum_{x\in U} \mathrm{Bel}_B^{(i)}(S_D(x)) = \sum_{x\in U} \mathrm{Bel}_A^{(i)}(S_D(x)).$$

" $\Longleftarrow$ " Suppose that $\sum_{x\in U} \mathrm{Bel}_B^{(i)}(S_D(x)) = \sum_{x\in U} \mathrm{Bel}_A^{(i)}(S_D(x))$.
Then by Proposition 4.4,

$$\forall\ x \in U,\quad \mathrm{Bel}_B^{(i)}(S_D(x)) \leq \mathrm{Bel}_A^{(i)}(S_D(x)).$$

This implies that $\forall\ x \in U, \mathrm{Bel}_B^{(i)}(S_D(x)) = \mathrm{Bel}_A^{(i)}(S_D(x))$.
Thus B is a belief consistent set in S. □

Theorem 4.7. *Let $S = (U, A \cup D)$ be an IVF decision information system. Then*

$$B \subseteq A \text{ is a belief reduction in } S \Longleftrightarrow \sum_{x\in U} \mathrm{Bel}_B^{(i)}(S_D(x)) = \sum_{x\in U} \mathrm{Bel}_A^{(i)}(S_D(x))$$

$$\text{and}\ \ \forall\ C \subsetneq B,\ \sum_{x\in U} \mathrm{Bel}_C^{(i)}(S_D(x)) < \sum_{x\in U} \mathrm{Bel}_A^{(i)}(S_D(x)).$$

Proof. " $\Longrightarrow$ " Suppose that B is of belief reduction in S. Then B is a belief consistent set in S. By Lemma 4.6,

$$\sum_{x\in U} \mathrm{Bel}_B^{(i)}(S_D(x)) = \sum_{x\in U} \mathrm{Bel}_A^{(i)}(S_D(x)).$$

By Definition 4.5, $\forall\ C \subsetneq B,\ x \in U,\ \mathrm{Bel}_C^{(i)}(S_D(x)) \neq \mathrm{Bel}_A^{(i)}(S_D(x))$.
By Proposition 4.4, $\forall\ C \subsetneq B,\ x \in U,\ \mathrm{Bel}_C^{(i)}(S_D(x)) \leq \mathrm{Bel}_A^{(i)}(S_D(x))$.
Thus

$$\sum_{x\in U} \mathrm{Bel}_C^{(i)}(S_D(x)) < \sum_{x\in U} \mathrm{Bel}_A^{(i)}(S_D(x)).$$

" $\Longleftarrow$ " Suppose that $\sum_{x\in U} \mathrm{Bel}_B^{(i)}(S_D(x)) = \sum_{x\in U} \mathrm{Bel}_A^{(i)}(S_D(x))$

$$\text{and}\ \ \forall\ C \subsetneq B,\ \sum_{x\in U} \mathrm{Bel}_C^{(i)}(S_D(x)) < \sum_{x\in U} \mathrm{Bel}_A^{(i)}(S_D(x)).$$

By Lemma 4.6, B is a belief consistent set in S.
By Proposition 4.4, $\forall\ x \in U, \mathrm{Bel}_C^{(i)}(S_D(x)) \leq \mathrm{Bel}_A^{(i)}(S_D(x))$.
Then

$$\mathrm{Bel}_C^{(i)}(S_D(x)) \neq \mathrm{Bel}_A^{(i)}(S_D(x)).$$

Thus B is a belief reduction in S.

□

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Algorithms 4.8. *Let $S = (U, A \cup D)$ be an IVF decision information system. The algorithm of belief reduction in S is shown as follows:*

Input: the IVF decision information system S.
Output: All belief reductions in S.

Step 1. *Input the IVF decision information system S;*
Step 2. *Pick $B \subseteq A$;*
Step 3. *Calculate the similar decision class $S_D(x_j)$;*
Step 4. *Calculate $\mathrm{Bel}_A^{(i)}(S_D(x_j))$ and $\mathrm{Bel}_B^{(i)}(S_D(x_j))$;*
Step 5. *Compare $\mathrm{Bel}_A^{(i)}(S_D(x_j))$ and $\mathrm{Bel}_B^{(i)}(S_D(x_j))$;*
Step 6. *By Theorem 4.7, B is a belief reduction in S.*

Example 4.9. *Consider the IVF decision information system $S = (U, A \cup D)$ in Example* 4.2.

By (4.1) *and Proposition* 3.1,

$$\begin{aligned}\mathrm{Bel}_A^{(i)}(S_D(x_1)) &= \sum_{i=1}^{5} m(X_i) \bigwedge_{y \in X_i} S_D(x_1)(y) \\ &= \frac{|X_0|}{|U|} \times \bigwedge_{y \in X_1} S_D(x_0)(y) + \cdots + \frac{|X_5|}{|U|} \times \bigwedge_{y \in X_5} S_D(x_0)(y) \\ &= \frac{3}{10} \times [0.2, 0.3] + \cdots + \frac{1}{10} \times [0.5, 0.6] \\ &= [0.430, 0.525]\end{aligned}$$

Similarly, we can calculate that

$$\mathrm{Bel}_A^{(i)}(S_D(x_1)) = [0.440, 0.590], \quad \mathrm{Bel}_A^{(i)}(S_D(x_2)) = [0.410, 0.590],$$
$$\mathrm{Bel}_A^{(i)}(S_D(x_3)) = [0.510, 0.675], \quad \mathrm{Bel}_A^{(i)}(S_D(x_4)) = [0.490, 0.590],$$
$$\mathrm{Bel}_A^{(i)}(S_D(x_5)) = [0.470, 0.610], \quad \mathrm{Bel}_A^{(i)}(S_D(x_6)) = [0.400, 0.525],$$
$$\mathrm{Bel}_A^{(i)}(S_D(x_7)) = [0.480, 0.600], \quad \mathrm{Bel}_A^{(i)}(S_D(x_8)) = [0.550, 0.675],$$
$$\mathrm{Bel}_A^{(i)}(S_D(x_9)) = [0.460, 0.625] \quad \textit{and}$$
$$\mathrm{Bel}_B^{(i)}(S_D(x_0)) = [0.430, 0.515], \quad \mathrm{Bel}_B^{(i)}(S_D(x_1)) = [0.420, 0.590],$$
$$\mathrm{Bel}_B^{(i)}(S_D(x_2)) = [0.390, 0.580], \quad \mathrm{Bel}_B^{(i)}(S_D(x_3)) = [0.490, 0.655],$$
$$\mathrm{Bel}_B^{(i)}(S_D(x_4)) = [0.430, 0.560], \quad \mathrm{Bel}_B^{(i)}(S_D(x_5)) = [0.380, 0.545],$$
$$\mathrm{Bel}_B^{(i)}(S_D(x_6)) = [0.380, 0.515], \quad \mathrm{Bel}_B^{(i)}(S_D(x_7)) = [0.480, 0.600],$$
$$\mathrm{Bel}_B^{(i)}(S_D(x_8)) = [0.550, 0.675], \quad \mathrm{Bel}_B^{(i)}(S_D(x_9)) = [0.440, 0.605].$$

By Lemma 4.6, $B = \{a_1, a_2\}$ is not a belief consistent set in S.
Thus $B = \{a_1, a_2\}$ is not a belief reduction in S.

5 Conclusions

In this paper, we have researched belief reduction in IVF decision information systems. In future work, we will investigate knowledge acquisition in IVF decision information systems.

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EXISTENCE AND BIFURCATION OF POSITIVE GLOBAL SOLUTIONS FOR PARAMETERIZED NONHOMOGENEOUS ELLIPTIC EQUATIONS INVOLVING CRITICAL EXPONENTS

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Abstract

We establish extence and bifurcation of positive global solutions for parametrized nonhomogeneous elliptic equations involving critical Sobolev exponents.

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Key words: elliptic problem; critical exponent; multiplicity; bifurcation; parametrized nonhomogeneous problem; positive global solution

1. Introduction

Let $N \geq 3$ and $2^* := 2N/(N-2)$. Let consider a Hilbert space

$$H^1(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$$

with the inner product

$$(u,v) := \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + uv)dx$$

and the corresponding norm

$$||u|| := \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)dx \right)^{1/2}.$$

By $H^{-1}(\mathbb{R}^N)$, we denote its dual with the dual norm $||\cdot||_*$ and, by $\langle , \rangle$, the pairing of $H^1(\mathbb{R}^N)$ with its dual. We denote by $||\cdot||_p$ the usual norm of $L^p(\mathbb{R}^N)$ for $p \in [1,\infty]$. Let $D^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$ be a Hilbert space with the inner product $\int_{\mathbb{R}^N} \nabla u \cdot \nabla v$ and the corresponding norm $||\nabla u||_2$.

In this paper, we are concerned with the multiple existence and bifurcation of positive solutions of the following problem:

$$(P_\mu) \qquad \begin{cases} -\Delta u + u = u^{2^*-1} + \mu f \text{ in } \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N, \ N \geq 3, \end{cases}$$

where $\mu \in \mathbb{R}^+$, $f \in H^{-1}(\mathbb{R}^N)$, $f \geq 0$ and $f \not\equiv 0$ in $\mathbb{R}^N$.

A well-known result for the homoneneous case is that all positive regular solution of

$$-\Delta u = u^{2^*-1}$$

in $\mathbb{R}^N$ are given by

$$\omega_\epsilon := \left(\frac{\epsilon\sqrt{N(N-2)}}{\epsilon^2 + |x|^2} \right)^{(N-2)/2}$$

with $\epsilon > 0$. Every ω_ϵ is a minimizer for the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$. Namely, the Sobolev constant

$$S := \inf_{0 \neq u \in D^{1,2}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{2/2^*}}$$

is achived by ω_ϵ and

$$(1,1) \qquad ||\nabla \omega_\epsilon||_2^2 = ||\omega_\epsilon||_{2^*}^{2^*} = S^{N/2} (cf.[2,6]).$$

For convenience, we omit "$\mathbb{R}^N$" and "dx" in integration and, throughout this paper, we will use the letter $C > 0$ to denote the natural various contents independent of u.

Our attempt to show multiplicity of positive solutions for problem (P_μ) relies on the Ekeland's variational principle in [13] and the Mountain Pass Theorem in [5]. Since our problem (P_μ) posesses the critical nonlinearity and the embedding

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$H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is not compact, in taking the opportunity of variational structure of problem, the (PS) condition is no longer valid and so the Mountain Pass Theorem in [1] could not be applied directly. However, we can use the Mountain Pass Theorem *without* the (PS) condition in [5] to get some $(PS)_c$ sequence of the variational functional for the second solution with $c > 0$.

In the last decade, the existence and properties of solutions of the problem:

$$(P_0) \qquad \begin{cases} -\Delta u + u = g(x,u), u > 0 \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \ N \geq 2 \end{cases}$$

has been stuied by Struss[24], Lions[22, 23], Ding and Ni[12], Cao[7], Zhu[25] and other authors for the case where $g(x,0) = 0$ on $\mathbb{R}^N$ and $g(x,t)$ has a subcritical superlinear growth. On the other hand, the nonhomogeneous problem with $1 < p < 2^* - 1$:

$$(P) \qquad \begin{cases} -\Delta u + u = |u|^{p-2}u + \mu f, u > 0 \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \ N \geq 2, \end{cases}$$

where $\mu \in \mathbb{R}^+$, $f \geq 0$, $f \in L^2(\mathbb{R}^N)$ was studied in [26,11,14,15].

In the critical case $p = 2^*$, the problem is much more difficult than the subcritical case. As we mentioned, the Palais-Smale condition does not hold at some critical levels and the effect of the nonhomogeneous term f to the multiple existence of solutions is delicate. The multiplicity of the solutions of (P_μ) not only depends on the norm of f but also the decay rate of f. In [10], it has shown that if $2 < N < 6$ and $|x|^{N-2}f$ is bounded, then there exists $\mu^* > 0$ such that problem (P_μ) possesses at least two positive solutions with $\mu \in]0, \mu^*[$. In case that $N \geq 6$, there exist $\mu^{**}, \mu_* > 0$ with $\mu_* < \mu^{**}$ such that for each $\mu \in]\mu^{**}, \mu^*[$, problem (P_μ) possesses two positive solutions and for $\mu \in]0, \mu_*[$ problem (P_μ) has a unique positive solution. In [11], the authors also gave similar multiplicity results for subcritical caseas. For critical case, In [18], Hirano and Kim consider the multiplicity of solutions of (P_μ) with $-\Delta + I$ replaced by $-\Delta + \alpha I$, $\alpha > 0$. They assume that $p = 2^*$, $3 \leq N \leq 5$, $f \in L^{2^*/(2^*-1)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $f \geq 0$ and $f \not\equiv 0$, and $|x|^{N-2}f$ is bounded. It was shown that there exist μ_* and a function $\alpha : (0, \mu_*) \to \mathbb{R}^+$ such that for each $\alpha \in (0, \alpha(\mu))$, problem (P_μ) posesses at least three solutions; the third solution is sign-changing if we assume that there exist exactly two positive solutions. we also refer [21] for critical case. In [19], the effect of the shape of the multiplicity of (P) was investigated when $-\Delta + I$ replaced by $-\epsilon\Delta$, $\epsilon > 0$.

In this paper, we do not assume the decay rate on f but assume only uniform boundedness of f which is independent of solution u and $x \in \mathbb{R}^N$. We study also bifurcation phenomenon and get a bifurcation point of (P_μ). There seems to have some progress on existence result in elliptic equations. We also refer a multiplicity result on parabolic equations for subcritical case in [20, 16] and elliptic with Neumann boundary condition in [17].

We now state our main results:

Proposition 2.3. *Assume $f \in H^{-1}(\mathbb{R}^N)$, $f \geq 0$, $f \not\equiv 0$ in $\mathbb{R}^N$ and $||\mu f||_* \leq C_N^*$, then problem (P_μ) has at least one positive solution u_μ such that*

$$(2.1) \qquad I_\mu(u_\mu) := c_1 = \inf\{I_\mu : u \in \bar{B}_{R_0}\},$$

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where $\bar{B}_{R_0} = \{u \in H^1(R^N) : ||u|| \leq R_0\}$ *and* $C_N^* = \frac{1}{2}\left(\frac{4}{N+2}\right)\left(\frac{N}{N+2}\right)^{(N-2)/4} S^{N/4}$.

THEOREM 3.6. *Assume* $f \in H^{-1}(\mathbb{R}^N)$, $f \geq 0$ $f \not\equiv 0$ *in* $\mathbb{R}^N$ *and satisfies* $||\mu f||_* \leq C_N^*$. *Then there exists a positive constant* $\mu^* > 0$ *such that* (P_μ) *possesses at least two positive solutions for* $0 < \mu < \mu^*$, *a unique solution for* $\mu = \mu^*$ *and no positive solution if* $\mu > \mu^*$.

By U_μ, we denote the second solution of (P_μ).

THEOREM 4.5. *(i) The set* $\{U_\mu\}$ *is bounded uniformly in* $H^1(\mathbb{R}^N)$,
(ii) (μ^*, u_{μ^*}) *is a bifurcation point.*

2. Existence of minimal positive solutions

LEMMA 2.1. *The operator* $-\Delta + I$ *has the maximum principle in* $H^1(\mathbb{R}^N)$.

Proof. Let $h \geq 0$ and $-\Delta u + u = h$. Suppose that $u_- \not\equiv 0$, where $u_+(x) = \max\{u(x), 0\}$ and $u_-(x) = \min\{u(x), 0\}$. then $0 < \int |\nabla u_-|^2 + |u_-|^2) = \int hu_- dx$ which leads a contradiction. This completes the proof. ■

In order to get the existence of positive solutions of (P_μ), we consider the energy functional I_μ of the problem (P_μ) defined by

$$I_\mu(u) := \frac{1}{2}\int(|\nabla u|^2 + |u|^2) - \frac{1}{2^*}\int(u^+)^{2^*} - \mu\int fu, \quad \text{for} \quad u \in H^1(\mathbb{R}^N).$$

First, we study the existence of a local mininum for energy functional I_μ and its properities. We denote

$$(2,1) \qquad C_N^* := \frac{1}{2}\left(\frac{4}{N+2}\right)\left(\frac{N}{N+2}\right)^{(N-2)/4} S^{N/4}.$$

LEMMA 2.2. *Assume* $f \in H^{-1}(\mathbb{R}^N)$, $f(x) \geq 0$, $f(x) \not\equiv 0$ *and* $||\mu f||_* \leq C_N^*$, *then there exits a positive constant* $R_0 > 0$ *such that* $I_\mu(u) \geq 0$ *for any* $u \in \partial B_{R_0} = \{u \in H^1(\mathbb{R}^N) : ||u|| = R_0\}$.

Proof. We consider the function $h(t) : [0, +\infty) \to \mathbb{R}^N$ defined by

$$h(t) = \frac{1}{2}t - \frac{1}{2^*}S^{-2^*/2}t^{2^*-1}.$$

Note that $h(0) = 0$, $2^* - 1 > 1$ and $h(t) \to -\infty$ as t $\to \infty$. We can show easly there a unique $t_0 > 0$ achieving the maxinum of $h(t)$ at t_0. Since

$$h'(t_0) = \frac{1}{2} - \frac{2^* - 1}{2^*}S^{-2^*/2}t_0^{2^*-2} = 0,$$

we have

$$t_0 = \left(\frac{2^*}{2(2^* - 1)}\right)^{1/(2^*-2)} S^{2^*/2(2^*-2)}.$$

Hence, we have

$$h(t_0) = \frac{1}{2}\left(\frac{4}{N+2}\right)\left(\frac{N}{N+2}\right)^{(N-2)/4} S^{N/4}. \tag{2,2}$$

Taking $R_0 = t_0$, for all $u \in \partial B_{R_0}$,

$$\begin{aligned} I_\mu(u) &= \frac{1}{2}\int (|\nabla u|^2 + |u|^2) - \frac{1}{2^*}\int (u^+)^{2^*} - \mu \int fu \\ &\geq \frac{1}{2}||u||^2 - \frac{1}{2^*}S^{-2^*/2}||u||^{2^*} - ||\mu f||_*||u|| \\ &= t_0\left[h(t_0) - ||\mu f||_*\right] \end{aligned} \tag{2,3}$$

From $(2,2)$ and $(2,3)$, we have $I_\mu(u)|_{\partial B_{R_0}} \geq 0$. This completes the proof. ■

Proposition 2.3. *Assume $f \in H^{-1}(\mathbb{R}^N)$, $f(x) \geq 0$, $f(x) \not\equiv 0$ in $\mathbb{R}^N$ and $||\mu f||_* \leq C_N^*$, then problem (P_μ) has at least one positive solution u_μ such that*

$$I_\mu(u_\mu) := c_1 = \inf\{I_\mu : u \in \bar{B}_{R_0}\}, \tag{2.4}$$

where $\bar{B}_{R_0} = \{u \in H^1(R^N) : ||u|| \leq R_0\}$.

Proof. By Sobolev inequality, the generalized Hölder and Young's inequality with $\epsilon > 0$, there exists $C_\epsilon > 0$, we have

$$\begin{aligned} I_\mu(u) &= \frac{1}{2}\int (|\nabla u|^2 + |u|^2) - \frac{1}{2^*}\int (u^+)^{2^*} - \mu \int fu \\ &\geq \frac{1}{2}||u||^2 - \frac{1}{2^*}S^{-2^*/2}||u||_{2^*}^{2^*} - ||\mu f||_*||u|| \\ &\geq \left(\frac{1}{2} - \epsilon\right)||u||^2 - \frac{1}{2^*}S^{-2^*/2}||u||^{2^*} - C_\epsilon||\mu f||_*^2. \end{aligned}$$

Taking $\epsilon < \frac{1}{2}$, then, for $R_0 = t_0$ as in Lemma 2,2, we can find a $C_{R_0} > 0$ small enough such that

$$I_\mu(u)|_{\partial B_{R_0}} \geq C_{R_0} \text{ for } ||\mu f||_* \leq C_N^*. \tag{2.5}$$

Since there exists a $\tilde{C}_{R_0} > 0$ such that $|I_\mu(u)| \leq \tilde{C}_{R_0}$ for all $u \in \bar{B}_{R_0}$ and $\bar{B}_{R_0}$ is a complete metric space with respect to the metric $d(u,v) = ||u - v||, u, v \in \bar{B}_{R_0}$, by using the Ekeland's variational principle, from (2.5), we can prove that there exists a sequence $\{u_n\} \subset \bar{B}_{R_0}$ and $u_\mu \in \bar{B}_{R_0}$ such that

$$I_\mu(u_n) \to c_1, \tag{2.6}$$

$$I'_\mu(u_n) \to 0, \tag{2.7}$$

$$\begin{aligned} u_n \to u_\mu &\ \text{ weakly in } H^1(\mathbb{R}^N), \\ u_n \to u_\mu &\ \text{ a.e. in } \mathbb{R}^N, \\ \nabla u_n \to \nabla u_\mu &\ \text{ a.e. in } \mathbb{R}^N \end{aligned} \tag{2.8}$$

and

$$u_n{}^{2^*-1} \to u_\mu{}^{2^*-1} \quad \text{weakly} \quad \text{in} \quad \left(L^{2^*}(\mathbb{R}^N)\right)^* \quad \text{as } n \to \infty.$$

Therefore, u_μ is a weak solution of (P_μ). Hence,

$$\left\langle I'_\mu(u_\mu), \varphi \right\rangle = 0 \quad \forall \varphi \in H^1(\mathbb{R}^N). \tag{2.9}$$

Moreover, by Lemma 2.1, u_μ is positive on $\mathbb{R}^N$, where I'_μ is the Frėchlet derivative of I_μ.

Next, we are going to prove (2.4). In fact, by the definition of c_1, we know that $I_\mu(u_\mu) \geq c_1$ since $u_\mu \in \bar{B}_{R_0}$, that is,

$$I_\mu(u_\mu) = \frac{1}{2}\int (|\nabla u_\mu|^2 + |u_\mu|^2) - \frac{1}{2^*}\int |u_\mu|^{2^*} - \mu \int f u_\mu \geq c_1 \tag{2.10}$$

By (2.9) and (2.10), we have

$$\left(\frac{1}{2} - \frac{1}{2^*}\right)\int (|\nabla u_\mu|^2 + |u_\mu|^2) - \left(1 - \frac{1}{2^*}\right)\mu \int f u_\mu \geq c_1 \tag{2.11}$$

On the other hand, by (2.6) - (2.8) and Fatou's lemma, we get

$$\begin{aligned} c_1 &= \liminf_n \left(\frac{1}{2} - \frac{1}{2^*}\right)\int (|\nabla u_n|^2 + |u_n|^2) - \limsup_n (1 - \frac{1}{2^*})\mu \int f u_n \\ &\geq \left(\frac{1}{2} - \frac{1}{2^*}\right)\int (|\nabla u_\mu|^2 + |u_\mu|^2) - \left(1 - \frac{1}{2^*}\right)\mu \int f u_\mu. \end{aligned} \tag{2.12}$$

Thus, (2.10) and (2.12) imply (2.4) holds. This completes the proof. ∎

Remark. *(i)* $c_1 < 0$, *(ii)* c_1 *is bounded below, (iii)* $||u_\mu|| = o(1)$ *as* $\mu \to 0^+$.

Indeed: (i) For $t > 0$ and $\varphi > 0$, we have

$$I_\mu(t\varphi) = \frac{t^2}{2}\int (|\nabla \varphi|^2 + |\varphi|^2) - \frac{t^{2^*}}{2^*}\int |\varphi|^{2^*} - t\mu \int f\varphi \leq \frac{t^2}{2}||\varphi||^2 - t\mu \int f\varphi.$$

By taking $t > 0$ sufficiently small, we can see $c_1 < 0$.

(ii) By (2.9) with $\varphi = u_\mu$, and $c_1 = I_\mu(u_\mu)$, we have

$$\begin{aligned} c_1 &= \left(\frac{1}{2} - \frac{1}{2^*}\right)\int (|\nabla u_\mu|^2 + |u_\mu|^2) - \left(1 - \frac{1}{2^*}\right)\mu \int f u_\mu \\ &\geq \left(\frac{1}{2} - \frac{1}{2^*}\right)||u_\mu||^2 - \left(1 - \frac{1}{2^*}\right)||\mu f||_*||u_\mu|| \\ &\geq -\frac{1}{22^*}\left(\frac{(2^*-1)^2}{2^*-2}\right)||\mu f||_*^2 \end{aligned} \tag{2.13}$$

by Young's inequality.

(iii) Since $c_1 < 0$, from (2.13), we see that $||u_\mu|| \to 0$ as $\mu \to 0^+$. Hence, $||u_\mu|| = o(1)$ as $\mu \to 0^+$. We also have that $||u_\mu||$ is uniformly bounded with respect to μ. We will restate results relating to this remark in Proposition 3.4 more precisely.

Proposition 2.4. *Problem* (P_μ) *possesses at least one minimal positive solution of* (P_μ).

Proof. Let $\mathcal{N}$ be the Nehari manifold

$$\mathcal{N} := \left\{ u \in H^1(\mathbb{R}^N) : \int |\nabla u|^2 + |u|^2 = \int |u|^{2^*} + \int \mu f u \right\} \setminus \{0\} .$$

Note that $||\mu f||_* \ll 1$ for μ small enough and for each $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, there exists a unique $t_u > 0$ such that

$$t_u^2 \int |\nabla u|^2 + |u|^2 - t_u^{2^*} \int |u|^{2^*} - t_u \int \mu f u = 0$$

and $I_\mu(t_u u) > 0$. Then

$$\mathcal{N} = \left\{ t_u u \ : \ u \in H^1(\mathbb{R}^N) \setminus \{0\} \right\}$$

and

$$\mathcal{N} \cong S^{N-1} = \left\{ u \in H^1(\mathbb{R}^N) \ : \ ||u|| = 1 \right\} .$$

Hence,

$$H^1(\mathbb{R}^N) = H_1 \cup H_2 \cup \mathcal{N}, \quad H_1 \cap H_2 = \phi \text{ and } 0 \in H_1,$$

where

$$\begin{aligned} H_1 &= \left\{ tu \ : \ u \in H^1(\mathbb{R}^N) \setminus \{0\} , \ t \in [0, t_u[\right\} \\ H_2 &= \left\{ tu \ : \ u \in H^1(\mathbb{R}^N) \setminus \{0\} , \ t > t_u \right\} . \end{aligned}$$

This implies that for $t > 0$ with $t < t_u, \ tu \in H_1$.

Here, we need to switch our view point, by associating with v a mapping

$$v : [0, \infty[\to H^1(\mathbb{R}^N)$$

defined by

$$(v(t))x = v(x,t), \quad x \in \mathbb{R}^N, t \in [0, \infty[.$$

In other words, we consider v not as a function of x and t together, but rather as a mapping v of t into the space $H^1(\mathbb{R}^N)$ of a function of x.

We have, for any $v_0 \in H_1$, the solution v of the initial value problem:

$$\begin{cases} \dfrac{dv}{dt} - \Delta v + v = \quad v^{2^*-1} + \mu f(x), \\ v(0) = v_0, \end{cases}$$

converges to u_μ as $t \to \infty$,

Indeed, in the proof of Proposition 2.3, we know that $I_\mu(v(t))$ is decreasing and $\lim_{t\to\infty} I_\mu(v(t)) = I_\mu(u_\mu)$, where $I_\mu(u_\mu)$ is the local minimum.
Since

$$\begin{aligned} I_\mu(v(t)) - I_\mu(v(s)) &= \int_s^t \frac{d}{dt} I_\mu(v(t)) dt \\ &= \int_s^t \left\langle \frac{d}{dt} v, \nabla I_\mu(v(t)) \right\rangle dt \\ &= -\int_t^s \left\| \frac{d}{dt} v \right\|^2 dt, \end{aligned}$$

we have, $\lim_{s,t\to\infty} \left\| \frac{d}{dt} v \right\|^2 = 0$. Thus, $v' \to 0$ a.e. in $\mathbb{R}^N$ as $t \to \infty$ and hence, $\left\langle I'_\mu(v), \varphi \right\rangle \to 0$, $\forall \varphi \in C^\infty(\mathbb{R}^N)$. Therefore, we have $v \to u_\mu$ as $t \to \infty$, since $I_\mu(v(t))$ is decreasing and converges to the local minimum $I_\mu(u_\mu)$.
Now, let $v_0 = tu$, where $t \in]0, 1[$ and u is a positive solution. Then $u \in \mathcal{N}$ and $v_0 \in H_1$. Since $v_0 \leq u$ and the solution v converges u_μ as $t \to \infty$, by the order preserving principle, $u_\mu \leq u$. This completes the proof. ∎

Proposition 2.5. *Suppose that $f \in H^{-1}(\mathbb{R}^N)$, $f \geq 0$, $f \not\equiv 0$ in $\mathbb{R}^N$ and $||\mu f||_* \leq C_N^*$. Then there exist $\tilde{\mu} \geq \bar{\mu} > 0$ such that (P_μ) possesses a positive solution for $0 < \mu \leq \bar{\mu}$ and no positive solution for $\mu > \bar{\mu}$.*

Proof. By Proposition 2.3, (P_μ) has a positive solution if $\mu \leq C_N^*/||f||_*$. Suppose (P_μ) has a positive solution for some $\mu = \bar{\mu}$. We show that (P_μ) has a positive solution for any $0 < \mu \leq \bar{\mu}$. For fixed $0 < \mu < \bar{\mu}$, using the Lax-Milgram Theorem, we construct a positive sequence $\{u_n\}$ as following;

Let

$$-\Delta u_1 + u_1 = \mu f$$

and

$$-\Delta u_n + u_n = u_{n-1}^{2^*-1} + \mu f \quad \text{for} \quad n \geq 2. \tag{2.14}$$

Then, by the maximum principle, we have $0 < u_n < u_{n+1} < \cdots < \bar{u}$ for $n \geq 1$. And $||u_1|| \leq \mu ||f||_*$ and $||u_1||_{2^*} \leq S^{-1/2}||u_1|| \leq S^{-1/2}\mu||f||_*$. Multiplying (2.14) by u_n, we have $||u_n|| \leq S^{-2^*/2}||\bar{u}||^{2^*-1} + \mu||f||_*$. Therefore, there exists u in $H^1(\mathbb{R}^N)$ such that

$$u_n \to u \text{ weakly in } H^1(\mathbb{R}^N) \text{ as } n \to \infty,$$

$$u_n \to u \text{ a.e. in } \mathbb{R}^N \text{ as } n \to \infty,$$
$$\nabla u_n \to \nabla u \text{ a.e. in } \mathbb{R}^{\mathrm{N}},$$
$$u_n^{2^*-1} \to u^{2^*-1} \text{ weakly in } (L^{2^*}(\mathbb{R}^N))^* \text{ as } n \to \infty.$$

Thus, u is a positive solution of (P_μ).

Next, let u be a positive solution of (P_μ). Then, for any $\epsilon > 0$, multiplying (P_μ) by $\omega_\epsilon^{2^*}$, we have

$$-\Delta u \omega_\epsilon^{2^*} + u\omega_\epsilon^{2^*} = u^{2^*-1}\omega_\epsilon^{2^*} + \mu f(x)\omega_\epsilon^{2^*}. \tag{2.15}$$

Since $2^* > 2$, for any $M > 0$, there exists a constant $C > 0$ such that

$$u^{2^*-1} \geq Mu - C \quad \forall u > 0.$$

Hence, we have, from (2.15),

$$-\int \Delta u \omega_\epsilon^{2^*} + \int u\omega_\epsilon^{2^*} \geq \int \left((Mu - C)\omega_\epsilon^{2^*} + \mu f(x)\omega_\epsilon^{2^*}) \right).$$

By Green's formular, we have

$$\int \Delta u \omega_\epsilon^{2^*} = \int u \Delta \omega_\epsilon^{2^*}.$$

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Thus,

$$\mu \int f(x)\omega_\epsilon^{2^*} \leq C \int \omega_\epsilon^{2^*} + \int \left(1 - M - \frac{\Delta \omega_\epsilon^{2^*}}{w_\epsilon^{2^*}}\right) \omega_\epsilon^{2^*} u. \tag{2.16}$$

Since

$$\begin{aligned}\frac{\Delta w_\epsilon^{2^*}}{\omega_\epsilon^{2^*}} = \frac{\Delta(\epsilon + |x|^2)^{-N}}{(\epsilon + |x|^2)^{-N}} &= 2N(N+1)(\epsilon + |x|^2)^{-2} \left(\frac{N+2}{N+1}|x|^2 - \frac{N}{N+1}\epsilon\right) \\ &= 2N(N+1)(\epsilon + 0^2)^{-2} \left(\frac{N+2}{N+1}0^2 - \frac{N}{N+1}\epsilon\right) \\ &= -2N^2\epsilon^{-1},\end{aligned}$$

we get, from (2.16),

$$\mu \int f(x)\omega_\epsilon^{2^*} \leq C \int \omega_\epsilon^{2^*} + \left(2N^2\epsilon^{-1} + 1 - M\right) \int \omega_\epsilon^{2^*} u.$$

If we choose $M = 2N^2\epsilon^{-1} + 1$, then, by (1.1), we have

$$\mu \leq \frac{C \int \omega_\epsilon^{2^*}}{\int f(x)\omega_\epsilon^{2^*}} = \frac{CS^{N/2}}{\int f(x)\omega_\epsilon^{2^*}}.$$

Hence, there exists $\bar{\mu} > 0$ such that

$$\bar{\mu} \leq \tilde{\mu} \doteqdot \inf_{\epsilon > 0} \frac{C \int w_\epsilon^{2^*}}{\int f(x)\omega_\epsilon^{2^*}} = \inf_{\epsilon > 0} \frac{CS^{N/2}}{\int f(x)\omega_\epsilon^{2^*}}. \tag{2.17}$$

Therefore, if $\mu > \bar{\mu}$, then (P_μ) has no solution and this completes the proof. ■

3. Multiplicity of positive solutions

From now on, we assume that $f \in H^{-1}(\mathbb{R}^N)$, $f \geq 0$, $f \not\equiv 0$ in $\mathbb{R}^N$and f satisfies $||\mu f||_* \leq C_N^*$.

We set

$$\mu^* := \sup\{\mu \in \mathbb{R}^+ : (P_\mu) \text{ has at least one positive solution in } H^1(\mathbb{R}^N)\}.$$

Then, by Proposition 2.5, we have $0 < \bar{\mu} \leq \mu^* < \infty$.

Remark. The minimal solution u_μ of (P_μ) is increasing with respect to μ. Indeed, suppose $\mu^* > \nu > \mu$. Since

$$-\Delta u_\nu + u_\nu - u_\nu^{2^*-1} - \mu f(x) = (\nu - \mu) f \geq 0,$$

$u_\nu > 0$ is a supersolution of (P_μ). Since $f(x) \geq 0$ and $f(x) \not\equiv 0$, $u \equiv 0$ is a subsolution of (P_μ) for any $\mu > 0$. By the standard barrier method, we can obtain a solution u_μ of (P_μ) such that $0 \leq u_\mu \leq u_\nu$ on $\mathbb{R}^N$. We note that 0 is not a solution of (P_μ), $\nu > \mu$ and u_μ is a minimal solution of (P_μ) because u_μ also can be derived by an iteration scheme with initial value $u_{(0)} = 0$. Therefore, by the maximal principle, $0 < u_\mu < u_\nu$ on $\mathbb{R}^N$ which completes the proof. ■

Now, consider the corresponding eigenvalue problem:

$$(3.1)_\mu \qquad \begin{cases} -\Delta\varphi + \varphi = \lambda(\mu)(2^*-1)u_\mu^{2^*-2}\varphi, \\ \quad \varphi \text{ in } H^1(\mathbb{R}^N). \end{cases}$$

Let λ_1 be the first eigenvalue of $(3.1)_\mu$;i.e.,

$$\lambda_1 = \lambda_1(\mu) := \inf\{\int \left(|\nabla\varphi|^2 + |\varphi|^2\right) : \varphi \in H^1(\mathbb{R}^N), (2^*-1)\int u_\mu^{2^*-2}\varphi^2 dx = 1\}.$$

Then, $0 < \lambda_1 < \infty$ and we can achieve the minimum by some function $\varphi_1 = \varphi_1(\mu) \in H^1(\mathbb{R}^N)$ and $\varphi_1 > 0$ in $\mathbb{R}^N$ if $\mu \in]0, \mu^*[$ $(cf.$ [27]).

We summarize basic properties for $\lambda_1(\mu)$:

Lemma 3.1. *(i) For* $\mu \in]0, \mu^*[,\ \lambda_1(\mu) > 1$,
(ii) If $0 < \mu < \nu \le \mu^*$, *then* $\lambda_1(\nu) < \lambda_1(\mu)$,
(iii) $\lambda_1(\mu) \to +\infty$ *as* $\mu \to 0^+$.

Proof. (i) For given $0 < \mu < \nu \le \mu^*$, every solution u_ν of (P_μ) with $\nu \in (\mu, \mu^*)$ is a supersolution of (P_μ). By Taylor expansion, we have

$$\begin{aligned} -\Delta(u_\nu - u_\mu) + u(u_\nu - u_\mu) &= u_\nu^{2^*-1} - u_\mu^{2^*-1} + (\nu-\mu)f \\ &> (2^*-1)u_\mu^{2^*-2}(u_\nu - u_\mu) \end{aligned}$$

and moreover, we get

$$\begin{aligned} \int \nabla(u_\nu - u_\mu)\nabla\varphi_1 + \int (u_\nu - u_\mu)\varphi_1 &= \int \left(u_\nu^{2^*-1} - u_\mu^{2^*-1}\right)\varphi_1 + \int(\nu-\mu)f\varphi_1 \\ &> (2^*-1)\int u_\mu^{2^*-2}(u_\nu - u_\mu)\varphi_1. \end{aligned}$$

Therefore, from $(3.1)_\mu$, we have

$$\int \nabla(u_\nu - u_\mu)\nabla\varphi_1 + \int (u_\nu - u_\mu)\varphi_1 = \lambda_1(\mu)(2^*-1)\int u_\mu^{2^*-2}(u_\nu - u_\mu)\varphi_1,$$

which implies $\lambda_1(\mu) > 1$.

(ii) Since, for $0 < \mu < \nu \le \mu^*$, $u_\mu < u_\nu$ and

$$\begin{aligned} \lambda_1(\mu)(2^*-1)\int u_\mu^{2^*-2}\varphi_1(\mu)\varphi_1(\nu) &= \int \nabla\varphi_1(\mu)\nabla\varphi_1(\nu) + \int \varphi_1(\mu)\varphi_1(\nu) \\ &= \lambda_1(\nu)(2^*-1)\int u_\nu^{2^*-2}\varphi_1(\nu)\varphi_1(\mu), \end{aligned}$$

we have $\lambda_1(\mu) > \lambda_1(\nu)$.

(iii) First, we show that $||u_\mu|| \to 0$ as $\mu \to 0^+$. Multiplying (P_μ) by u_μ, we have,

$$\int \left(|\nabla u_\mu|^2 + |u_\mu|^2\right) = \int u_\mu^{2^*} + \int \mu f u_\mu$$

and hence, for $\epsilon > 0$, we have, by Young's inequality with ϵ,

$$\left(1 - \frac{1}{\lambda_1(2^*-1)} - \frac{\epsilon}{2}\right)||u_\mu||^2 \le \frac{\mu^2}{2\epsilon}||f||_*^2 \ \text{ for } \epsilon > 0.$$

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Thus, for $\epsilon > 0$ small, we have $||u_\mu||^2 \le C_\epsilon \mu^2$ for some constant $C_\epsilon > 0$, and hence, $||u_\mu|| = o(1)$ as $\mu \to 0^+$. Next, Multiplying $(3.1)_\mu$ by $\varphi_1(\mu)$, we have, by Hölder's inequality, that

$$\begin{aligned}\int \left(|\nabla\varphi_1|^2 + |\varphi_1|^2\right) &= \lambda_1 \cdot (2^* - 1)\int u_\mu^{2^*-2}\varphi_1^2 \\ &\le \lambda_1 \cdot (2^* - 1)\left(\int u_\mu^{2^*}\right)^{(2^*-2)/2^*}\left(\int \varphi_1^{2^*}\right)^{2/2^*} \\ &\le \lambda_1 \cdot (2^* - 1)\left(\int u_\mu^{2^*}\right)^{(2^*-2)/2^*}\left(\int |\nabla\varphi_1|^2 + |\varphi_1|^2\right) \\ &\le \lambda_1 \cdot (2^* - 1) S^{-(2^*-2)/2}||u_\mu||^{2^*-2}||\varphi_1||^2\end{aligned}$$

and thus, $S^{(2^*-2)/2} \le \lambda_1 \cdot (2^*-1)||u_\mu||^{2^*-2}$. Therefore, we have the desired result. This completes the proof. ■

Lemma 3.2. *Let u_μ be a positive solution of $(1.3)_\mu$ for which $\lambda_1(\mu) > 1$. Then, for any $g \in H^1(\mathbb{R}^N)$, the problem:*

$$-\Delta w + w = (2^* - 1)u_\mu^{2^*-2}w + g(x), \quad w \in H^1(\mathbb{R}^N) \tag{3.2}$$

has a solution.

Proof. Consider the functional defined by

$$J(w) = \frac{1}{2}\int \left(|\nabla w|^2 + |w|^2\right) - \frac{1}{2}(2^* - 1)\int u_\mu^{2^*-2}w^2 - \int gw, \quad w \in H^1(\mathbb{R}^N).$$

From Hölder's inequality and Young's inequality, we have, for any $\epsilon > 0$,

$$\begin{aligned}J(w) &\ge \left(\frac{1}{2} - \frac{1}{2\lambda_1(\mu)}\right)||w||^2 - \frac{\epsilon}{2}||w||^2 - \frac{1}{2\epsilon}||g||_*^2 \\ &= \left(\frac{1}{2} - \frac{1}{2\lambda_1(\mu)} - \frac{\epsilon}{2}\right)||w||^2 - \frac{1}{2\epsilon}||g||_*^2\end{aligned}$$

and hence, for small $\epsilon > 0$, there exist $C_{1,\epsilon} > 0$ and $C_{2,\epsilon} > 0$ such that

$$J(w) \ge C_{1,\epsilon}||w||^2 - C_{2,\epsilon}||g||_*^2. \tag{3.3}$$

Let $\{w_n\} \subset H^1(\mathbb{R}^N)$ be the minimizing sequence of J. From (3.3), we have $\{w_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Hence, passing subsequence, we may have that there exists $w \in H^1(\mathbb{R}^N)$ such that

$$w_n \to w \text{ weakly in } H^1(\mathbb{R}^N) \text{ as } n \to \infty,$$
$$w_n \to w \text{ a.e. in } \mathbb{R}^N \text{ as } n \to \infty$$

Here, we also note that

$$\nabla w_n \to \nabla w \text{ a.e. in } \mathbb{R}^N \text{ as } n \to \infty.$$

And

$$u_n^{2^*-1} \to \tilde{u}^{2^*-1} \text{ weakly in } (L^{2^*}(\mathbb{R}^N))^* \text{ as } n \to \infty.$$

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By Fatou's Lemma

$$||w||^2 \leq \liminf_{n\to\infty} ||w_n||^2.$$

The weak convergence and the fact that $\int u_\mu^{2^*-2} w_n^2 < \infty$ for $n \geq 1$ imply

$$\lim_{n\to\infty} \int g w_n = \int g w, \quad \lim_{n\to\infty} \int u_\mu^{2^*-2} w_n = \int u_\mu^{2^*-2} w$$

and hence,

$$J(w) \leq \lim_{n\to\infty} J(w_n) = d.$$

Then, $J(w) = d$ and w is a minimizer of J. Therefore, w is a critical point of J and w is a solution of (3.2). This completes the proof. ∎

Proposition 3.3. *For $\mu = \mu^*$, the problem (P_μ) has a positive solution u_{μ^*} and $\lambda_1(\mu^*) = 1$. Moreover, the solution u_{μ^*} is unique in $H^1(\mathbb{R}^N)$.*

Proof. For $\mu \in]0, \mu^*[$, multiplying (P_μ) by u_μ, we have, by $(3.1)_\mu$,

$$\begin{aligned}\int \left(|\nabla u_\mu|^2 + |u_\mu|^2\right) &= \int u_\mu^{2^*} + \mu \int f u_\mu \\ &\leq \frac{1}{\lambda_1(\mu)(2^*-1)} \int (|\nabla u_\mu|^2 + |u_\mu|^2) + \mu^* ||f||_* ||u_\mu|| \\ &= \left(\frac{1}{\lambda_1(\mu)(2^*-1)} + \frac{\epsilon \mu^*}{2}\right) ||u_\mu||^2 + \frac{\mu^*}{2\epsilon} ||f||_*^2.\end{aligned}$$

By taking $\epsilon > 0$ small enough, there exists an constant $C_\epsilon > 0$ such that $||u_\mu|| \leq C_\epsilon$ for all $\mu \in]0, \mu^*[$. Then, there exists u_{μ^*} in $H^1(\mathbb{R}^N)$ such that u_μ monotonically increasing to u_{μ^*} as $\mu \to \mu^*$ and $u_\mu \to u_{\mu^*}$ weakly in $H^1(\mathbb{R}^N)$ as $\mu \to \mu^*$. Hence, u_{μ^*} is a positive solution of (P_μ) with $\mu = \mu^*$. We note that $\lambda_1(\mu)$ is a continuous function of $\mu \in]0, \mu^*]$.

Define $F : \mathbb{R}^1 \times H^1(\mathbb{R}^N) \to H^{-1}(\mathbb{R}^N)$ by

$$F(\mu, u) := \Delta u - u + (u^+)^{2^*-1} + \mu f(x).$$

Since $u_\mu \to u_{\mu*}$ weakly as $\mu \to \mu^*$, from Lemma 3.1, $\lambda(\mu^*) \geq 1$. If $\lambda_1(\mu^*) > 1$, then $F_u(\mu^*, u_{\mu^*})\varphi = \Delta\varphi - \varphi + (2^*-1)u_{\mu^*}^{2^*-2}\varphi = 0$ has no nontrivial solution. From Lemma 3.2, $F(\mu^*, u_{\mu^*})$ is an isomorphism of $\mathbb{R}^1 \times H^1(\mathbb{R}^N)$ onto $H^{-1}(\mathbb{R}^N)$, and by the implicitly function theorem to F, we find a neighborhood $]\mu^* - \delta,\ \mu^* + \delta[$ of μ^* such that (P_μ) possesses a positive solution if $\mu \in]\mu^* - \delta,\ \mu^* + \delta[$, which contradicts the definition of μ^*. Therefore, $\lambda_1(\mu^*) = 1$.

Suppose v_{μ^*} is a positive solution of (P_{μ^*}). Then $v_{\mu*} \geq u_{\mu^*}$ since u_{μ^*} is minimal. Let $w = v_{\mu^*} - u_{\mu^*}$. Then, since $\lambda_1(\mu^*) = 1$, we have

$$-\Delta w + w \geq (2^* - 1) u_{\mu^*}^{2^*-2} w.$$

Since $\varphi_1 = \varphi_1(\mu^*)$ is the eigenfunction of the problem $(3,1)_{\mu^*}$, we have,

$$(2^*-1) \int u_{\mu^*}^{2^*-2} \varphi_1 w = \int \nabla w \nabla \varphi_1 + \int w \varphi_1 \geq (2^*-1) \int u_{\mu^*}^{2^*-1} w \varphi_1$$

and hence, $w \equiv 0$. This completes the proof. ∎

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PROPOSITION 3.4. *The minimal solution u_μ of (P_μ) increasing continuously to u_{μ^*} as $\mu \to \mu^*$ and uniformly bounded in $H^1(\mathbb{R}^N)$ for all $\mu \in]0, \mu^*]$. Moreover, $||u_\mu|| \le O(\mu^2)$ as $\mu \to 0^+$.*

Proof. It suffices to find the uniform bound of u_μ. Multiplying (P_μ) by u_μ, we have

$$\int \left(|\nabla u_\mu|^2 + |u_\mu|^2\right) = \int u_\mu^{2^*} + \int \mu f u_\mu$$

and hence, for $\epsilon > 0$, we have

$$\left(1 - \frac{1}{\lambda_1(2^* - 1)} - \frac{\epsilon}{2}\right) ||u_\mu||^2 \le \frac{\mu^2}{2\epsilon}||f||_*^2 \ \text{ for } \epsilon > 0.$$

Therefore, for $\epsilon > 0$ small, we have $||u_\mu|| \le C_\epsilon \mu$ for some constant $C_\epsilon > 0$. Next, fix $\tau \in]0, \mu^*]$. If μ increases to τ, then u_μ is increasing up to u_τ and $u_\mu \to u_\tau$ in $H^1(\mathbb{R}^N)$. If it is not the case, then, by multiplying u_μ on (P_μ) again, we have

$$||u_\mu||^2 \le \left\langle u_\tau^{2^*-1}, u_\mu \right\rangle + \tau \left\langle f, u_\mu \right\rangle$$

and so

$$||u_\mu|| \le CS^{-(2^*-1)/2}||u_\tau||^{2^*-1} + \tau||f||_*$$

for some $C > 0$. Hence, there exists a sequence $\{u_{\mu_j}\}$ in $H^1(\mathbb{R}^N)$ conversing weakly to a solution $\tilde{u}$ of (P_τ) but $\tilde{u} \ne u_\tau$. Since $\{u_{\mu_j}\}$ coverge to $\tilde{u}$ strongly in local L^1 sense, by the maximum principle, we have $u_{\mu_j} \le \tilde{u} < u_\tau$ which leads a contradiction to the minimality of u_τ. This completes the proof. ∎

REMARK. *From Proposition 3.4 , we have that $\lambda(\mu)$ is a continuously decreasing function from $[0, \mu^*]$ onto $[1, \infty[$ and $||u_\mu|| = o(1)$ as $\mu \to 0^+$.*

Next, we are going to find the second solution. In order to get another positive solution of (P_μ), we consider the following problem:

$$(3.4)_\mu \qquad \begin{cases} -\Delta v + v = (v + u_\mu)^{2^*-1} - u_\mu^{2^*-1} \ \text{ in } \ \mathbb{R}^N, \\ \qquad v \in H^1(\mathbb{R}^N), \ v > 0 \text{ in } \mathbb{R}^N \end{cases}$$

and the corresponding variational functional:

$$J_\mu(v) := \frac{1}{2}\int \left(|\nabla v|^2 + |v|^2\right) - \frac{1}{2^*}\int \left((v^+ + u_\mu)^{2^*} - u_\mu^{2^*} - 2^* u_\mu^{2^*-1} v^+\right)$$

for $v \in H^1(\mathbb{R}^N)$.

Clearly, we can have another positive solution $U_\mu = u_\mu + v_\mu$ if we show the problem $(3.4)_\mu$ possesses a positive solution for $\mu \in]0, \mu^*[$. We look for a critical point of J_μ which is a weak solution of $(3.4)_\mu$ by employing standard argument of the Mountain Pass method without the (PS) condition.

In the proof of the existance second solution, we make use of some arguments in [9, 10, 11].

THEOREM 3.5. *The problem (P_μ) possesses at least two positive solutions for all $\mu \in]0, \mu^*[$.*

Proof.
(i) Let $v \in H^1(\mathbb{R}^N) \setminus \{0\}$, Then, for $\epsilon > 0$, by Young's inequality,

$$\begin{aligned} J_\mu(v) &= \frac{1}{2}\int \left(|\nabla v|^2 + |v|^2\right) dx - \int\int_0^{v^+} \left((u_\mu + s)^{2^*-1} - u_\mu^{2^*-1}\right) ds dx \\ &\geq \frac{1}{2}\left(1 - \frac{1}{\lambda_1}\right)\int \left(|\nabla v|^2 + |v|^2\right) dx \\ &\quad - \int\int_0^{v^+} \left((u_\mu + s)^{2^*-1} - u_\mu^{2^*-1} - (2^*-1)u_\mu^{2^*-2}s\right) ds dx \\ &\geq \frac{1}{2}\left(1 - \frac{1}{\lambda_1}\right)\int \left(|\nabla v|^2 + |v|^2\right) dx - \int\int_0^{v^+} \left(\epsilon u_\mu^{2^*-2}s + C_\epsilon s^{2^*-1}\right) ds dx \\ &\geq \frac{1}{2}\left(1 - \frac{1}{\lambda_1}\right)||v||^2 - \frac{\epsilon}{2}\int u_\mu^{2^*-2}\left(v^+\right)^2 dx - \frac{C_\epsilon}{2^*}\int \left(v^+\right)^{2^*} dx \\ &\geq \frac{1}{2}\left(1 - \frac{1}{\lambda_1} - \frac{\epsilon}{2(2^*-1)\lambda_1}\right)||v||^2 - \frac{C_\epsilon}{2^*}S^{-2^*/2}||v||^{2^*} \end{aligned}$$

for some constant $C_\epsilon > 0$. Hence, for sufficiently small $\epsilon > 0$, there exist $\rho > 0, \delta > 0$ such that

$$J_\mu(v)|_{\partial \tilde{B}_\rho} \geq \delta > 0,$$

where $\tilde{B}_\rho = \{u \in H^1(\mathbb{R}^N) |\ ||u|| \leq \rho\}$.
(ii) Let $v \in H^1(\mathbb{R}^N), v \geq 0$ and $v \not\equiv 0$, then, for $t > 0$, we have

$$\begin{aligned} J_\mu(tv) &= \frac{t^2}{2}\int \left(|\nabla v|^2 + |v|^2\right) dx - \frac{1}{2^*}\int \left((u_\mu + tv)^{2^*} - u_\mu^{2^*} - 2^* u_\mu^{2^*-1}tv\right) dx \\ &\leq \frac{t^2}{2}\int \left(|\nabla v|^2 + |v|^2\right) dx - \frac{t^{2^*}}{2^*}\int |v|^{2^*} dx \\ &\leq \frac{t^2}{2}||v||^2 - \frac{t^{2^*}}{2^*}||v||_{2^*}^{2^*} \end{aligned}$$

Hence, we deduce

$$J_\mu(tv) \to -\infty$$

as $t \to \infty$. Therefore, for each $0 \not\equiv v \in H^1(\mathbb{R}^N)$ with $v \geq 0$, there exists a constant $t_v > 0$ such that $J_\mu(t_v v) \leq 0$ for $t \geq t_v$.
Let $K_1(v) := \frac{1}{2}\int (|\nabla v|^2 + v^2) - \frac{1}{2^*}\int (v^+)^{2^*} - \mu \int fv$.
Because u_μ is the critical point of $K_1(u)$, we can prove that, for $v \in H^1(\mathbb{R}^N)$,

$$J_\mu(v) = K_\mu(v) - K_\mu(0) = K_\mu(v) - K_1(u_\mu). \tag{3,5}$$

where

$$K_\mu(v) := \frac{1}{2}\int \left(|\nabla(v + u_\mu)|^2 + (v + u_\mu)^2 - \frac{1}{2^*}\int (v^+ + u_\mu)^{2^*} - \mu \int f(x)(v + u_\mu).\right.$$

(iii) From (ii), there exist small $t_1 > 0$ such that, for $0 < t < t_1$, $J_\mu(t\omega_\epsilon) < \frac{1}{N}S^{N/2}$.

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Choose $t_2 > t_1$ such that $J_\mu(t\omega_\epsilon) \leq 0$ for all $t \geq t_2$. For $t_1 \leq t \leq t_2$,

$$\begin{aligned} J_\mu(t\omega_\epsilon) &= \frac{t^2}{2}\int \left(|\nabla\omega|^2 + |\omega_\epsilon|^2\right) dx - \frac{1}{2^*}\int \left((u_\mu + t\omega_\epsilon)^{2^*} - u_\mu^{2^*} - 2^* u_\mu^{2^*-1} t\omega_\epsilon\right) dx \\ &< \frac{t^2}{2}||\omega_\epsilon||^2 - \frac{t^{2^*}}{2^*}||\omega_\epsilon||_{2^*}^{2^*} \\ &= \left(\frac{t^2}{2} - \frac{t^{2^*}}{2^*}\right) S^{N/2} \leq \frac{1}{N} S^{N/2}. \end{aligned}$$

(iv) Let

$$\Gamma := \{\gamma \in \mathcal{C}([0,1], H^1); \gamma(0) = 0,\ \gamma(1) = t_2\omega_\epsilon\}$$

and

$$c_\mu = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} J_\mu(\gamma(s)).$$

Then, we have

$$0 < \alpha \leq c_\mu \leq \sup_{t\geq 0} J_\mu(t\omega_\epsilon) < \frac{1}{N} S^{N/2}. \tag{3.6}$$

We now applying the Mountain Pass Theorem without Palais-Smale condition in [5] to get a sequence $\{v_n\} \subset H^1(\mathbb{R}^N)$ such that

$$J_\mu(v_n) \to c_\mu, \quad J'_\mu(v_n) \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^N). \tag{3.7}$$

Since

$$\begin{aligned} 1 + c_\mu + ||v_n|| + ||u_\mu|| &\geq 1 + c_\mu + ||v_n + u_\mu|| \\ &\geq J_\mu(v_n) - \frac{1}{2^*} J'_\mu(v_n)(v_n^+ + u_\mu) \\ &\geq \left(\frac{1}{2} - \frac{1}{2^*}\right) ||v_n||^2 - \frac{2}{2^*}||v_n||||u_\mu|| - \left(1 - \frac{1}{2^*}\right) ||u_\mu||_{2^*}^{2^*}, \end{aligned}$$

we see that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Hence, there exists a subsequence, say again, $\{v_n\}$ such that

$$\begin{gathered} v_n \to v_\mu \text{ weakly in } H^1(\mathbb{R}^N), \\ v_n \to v_\mu \text{ a.e. in } \mathbb{R}^N, \\ \nabla v_n \to \nabla v_\mu \text{ a.e. in } \mathbb{R}^N, \end{gathered}$$

and

$$(v_n + u_\mu)^{2^*-1} - u_\mu^{2^*-1} \to (v^+ + u_\mu)^{2^*-1} - u_\mu^{2^*-1} \text{ weakly in } (L^{2^*}(\mathbb{R}^N))^*.$$

Hence, v_μ is a weak solution of $-\Delta v + v = (v^+ + u_\mu)^{2^*-1} - u_\mu^{2^*-1}$.

Using the maximal principle, we get $v_\mu \geq 0$ in $\mathbb{R}^N$. Set $u_n := v_n + u_\mu$, $u := v_\mu + u_\mu$. Then

$$\begin{gathered} u_n \to u \text{ weakly in } H^1(\mathbb{R}^N), \\ u_n \to u \text{ a.e. in } \mathbb{R}^N, \\ \nabla u_n \to \nabla u \text{ a.e. in } \mathbb{R}^N. \end{gathered}$$

From (3.5),

$$J_\mu(v_n) = K_\mu(v_n) - K_\mu(0) = K_1(u_n) - K_1(u_\mu) \to c_\mu \text{ as } n \to \infty \tag{3.8}$$

and u is a solution of

$$-\Delta u + u = u^{2^*} + \mu f(x). \tag{3.9}$$

Now, we are going to show that $u \not\equiv u_\mu$. In fact, if $u \equiv u_\mu$, i.e., $v_\mu \equiv 0$, then $u_n \not\to u$ strongly in $H^1(\mathbb{R}^N)$ since $J_\mu(0) = 0 < u_\mu$. Let $c_2 := c_\mu + K_1(u_\mu)$. By the Brezis-Lieb Lemma($cf.$ [4]) we have

$$\begin{cases} ||u_n||^2 = ||u_\mu||^2 + ||v_n||^2 + o(1), \\ |u_n^+|^{2^*} = |u_\mu|^{2^*} + |v_n^+|^{2^*} + o(1), \\ \displaystyle\int fu_n = \int fu_\mu + o(1) \text{ as } n \to \infty. \end{cases} \tag{3.10}$$

By (3.8), (3.9), we have

$$\int \left(|\nabla u_n|^2 + u_n^2\right) = \int (u_n^+)^{2^*} + \mu \int f(x)u_n + o(1),$$

$$\int \left(|\nabla u_\mu|^2 + u_\mu^2\right) = \int (u_\mu^+)^{2^*} + \mu \int f(x)u_\mu.$$

Hence,

$$\int \left(|\nabla v_n|^2 + v_n^2\right) = \int (v_n^+)^{2^*} + o(1), \tag{3.11}$$

by substracting the two identities above and by (3.10).
Using (3.8), (3.9), (3.10) and (3.11), we have that, as $n \to \infty$

$$\begin{aligned} c_2 &= c_\mu + K_1(u_\mu) \\ &= J_\mu(v_n) + K_1(u_\mu) + o(1) \\ &= K_1(u_n) + o(1) \\ &= K_1(u_\mu) + \frac{1}{2}\int |\nabla v_n|^2 + v_n^2 - \frac{1}{2^*}\int v_n^{2^*} + o(1) \\ &= K_1(u_\mu) + \left(\frac{1}{2} - \frac{1}{2^*}\right)\int (v_n)^{2^*} + o(1) \\ &= K_1(u_\mu) + \frac{1}{N}\int (v_n)^{2^*} + o(1). \end{aligned}$$

By Sobolev inequality:

$$S||v_n||_{2^*}^2 \le ||v_n||^2 = ||v_n||_{2^*}^{2^*} + o(1),$$

we have $||v_n||_{2^*}^{2^*} \ge S^{N/2}$. Thus,

$$c_2 = c_\mu + K_1(u_\mu) \ge K_1(u_\mu) + \frac{1}{N}S^{N/2}.$$

This leads a contradiction to (3.6). Therefore, we have $v_\mu > 0$. This completes the proof. ∎

Consequently, we have:

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THEOREM 3.6. *Assume $f \in H^{-1}(\mathbb{R}^N)$, $f \geq 0$, $f \not\equiv 0$ in $\mathbb{R}^N$ and $||\mu f||_* \leq C_N^*$. Then there exists a positive constant $\mu^* > 0$ such that (P_μ) possesses at least two positive solutions for $0 < \mu < \mu^*$, a unique solution for $\mu = \mu^*$ and no positive solution if $\mu > \mu^*$.*

4. Bifurcation

In order to study bifurcation phenomenon, we consider following eigenvalue problem:

$$(4.1)_\mu \qquad \begin{cases} -\Delta\phi + \phi = \eta(\mu)(2^* - 1)U_\mu^{2^*-2}\phi, \\ \qquad \phi \text{ in } H^1(\mathbb{R}^N). \end{cases}$$

Let η_1 be the first eigenvalue of $(4.1)_\mu$;i.e.,

$$\eta_1 = \eta_1(\mu)\inf\{\int |\nabla\phi|^2 + |\phi|^2; \phi \in H^1(\mathbb{R}^N), \int (2^* - 1)U_\mu^{2^*-2}\phi^2 = 1\}$$

and $\phi_1 > 0$ be the corresponding eigenfunction.

In the proof of the following lemma, we make use of arguments in [3].

LEMMA 4.1. *Let U_μ be a second positive solution of (P_μ) obtained in Theorem 3.5. Then $\eta_1(\mu) < 1$ for $0 < \mu < \mu^*$.*

Proof. Suppose contrary that $\eta_1(\mu) \geq 1$, Let $\psi = U_\mu - u_\mu > 0$. Then ϕ_1 and ψ satisfies

$$(4.2) \qquad \Delta\phi_1 - \phi_1 + (2^* - 1)U_\mu^{2^*-2}\phi_1 \leq 0 \text{ and } \Delta\psi - \psi + (2^* - 1)_\mu^{2^*-2}\psi \geq 0,$$

respectively. Set $\sigma = \psi/\phi_1$;i.e., $\psi = \sigma\phi_1$. Then, by (4.2),

$$(4.3) \qquad \sigma\nabla(\phi_1^2\nabla\sigma) = \psi\nabla\psi - \frac{\psi}{\phi_1}\nabla\phi_1 \geq 0.$$

Let ζ be a C^∞ function on $\mathbb{R}^+$ with $0 \leq \zeta(t) \leq 1$,

$$\zeta(t) := \begin{cases} 1 \text{ for } 0 \leq t \leq 1, \\ 0 \text{ for } t \geq 2. \end{cases}$$

For $R > 0$, set $\zeta_R(t) = \zeta\left(\frac{|x|}{R}\right)$ in $\mathbb{R}^N$. Multiplying (4.3) by ζ_R^2 and intergrating over $\mathbb{R}^N$, we have by Green' theorem,

$$(4.4) \qquad \begin{aligned} \int \zeta_R^2\phi_1^2|\nabla\sigma|^2 &\leq 2\left|\int \phi_1^2\zeta_R\sigma\nabla\sigma\cdot\nabla\zeta_R\right| \\ &\leq 2\left[\int_{R<|x|<2R} \zeta_R^2\phi_1^2|\nabla\sigma|^2\right]^{1/2}\left[\int \phi_1^2\sigma^2|\nabla\zeta_R|^2\right]^{1/2} \\ &\leq C_1\left[\int_{R<|x|<2R} \zeta_R^2\phi_1^2|\nabla\sigma|^2\right]^{1/2}\left[\int_{R<|x|<2R} \psi^{2^*}\right]^{1/2} \\ &\leq C_2\left[\int_{R<|x|<2R} \zeta_R^2\phi_1^2|\nabla\sigma|^2\right]^{1/2} \end{aligned}$$

for some constants C_1 and C_2, which implies

$$\int \zeta_R^2 \phi_1^2 |\nabla \sigma|^2 \leq C_3$$

for some constant $C_3 > 0$.
Letting $R \to \infty$, we see that

$$\int \phi_1^2 |\nabla \sigma|^2 \leq C_3.$$

But then it follows that the last term in (4.4) tends to 0 as $R \to \infty$, so that

$$\int \phi_1^2 |\nabla \sigma|^2 = 0.$$

Therefore, σ is a positive constant and by (4.2), $\phi_1 \equiv \psi = U_\mu - u_\mu$, and thus $U_\mu \equiv u_\mu$, which leads a contradiction. This completes the proof. ∎

LEMMA 4.2. *For $\mu \in]0, \mu^*[$, U_μ decreases contonusely to u_{μ^*} as $\mu \to \mu^*$ in $H^1(\mathbb{R}^N)$. Moreover,*
(i) $U_\mu \to u_{\mu^*}$ as $\mu \to \mu^*$ by the uniqueness of u_{μ^*},
(ii) $\lim_{\mu \to 0^+} ||U_\mu|| = S^{N/4}$.

Proof. First, we note that

$$\begin{aligned}
\left(\frac{1}{2} - \frac{1}{2^*}\right) ||U_\mu||^2 &= \frac{1}{2}||U_\mu||^2 - \frac{1}{2^*} \int \left(U_\mu^{2^*} + \mu \int f U_\mu \right) \\
&= \mu \left(1 - \frac{1}{2^*}\right) \int f U_\mu - \mu \int f u_\mu - \mu \int f v_\mu \\
&+ \frac{1}{2}||u_\mu||^2 + \frac{1}{2}||v_\mu||^2 + \int \nabla u_\mu \nabla v_\mu + \int u_\mu v_\mu - \frac{1}{2^*} \int U_\mu^{2^*} \\
&\geq \mu \left(1 - \frac{1}{2^*}\right) \int f U_\mu + J_\mu(v_\mu) + H(u_\mu),
\end{aligned}$$

where $H(u) = \frac{1}{2}||u||^2 - \frac{1}{2^*} \int u^{2^*} - \mu \int fu$.

From Hölder's and Young's inequality, for $\epsilon > 0$, we have

$$\left(\frac{2^* - 2}{22^*} - \frac{\epsilon(2^* - 1)}{22^*}\right) ||U_\mu||^2 \leq \frac{2^* - 1}{\epsilon 22^*} \mu^2 ||f||_*^2 + \frac{1}{N} S^{N/2} + H(u_\mu).$$

Since

$$\begin{aligned}
H(u_\mu) &= \left(\frac{1}{2} - \frac{1}{2^*}\right) ||u_\mu||^2 - \mu \left(1 - \frac{1}{2^*}\right) \int f u_\mu \\
&\leq \left(\frac{1}{2} - \frac{1}{2^*}\right) ||u_{\mu^*}||^2,
\end{aligned}$$

$H(u_\mu)$ is uniformly bounded for $\mu \in (0, \mu^*]$. Moreover, by the remark of Proposition 3.4, $H(u_\mu) = o(1)$ as $\mu \to 0^+$. Taking $\epsilon > 0$ small enought, we have $||U_\mu|| \leq$ C for some C > 0. Since $0 < u_\mu \leq U_\mu$, (i) follows from Proposition 3.3 and Proposition 3.4.

For (ii). By (ii) of Lemma 3.1, and (i) and (iii) in the proof of Theorem 3.5, there exists $d > 0$ such that

$$0 < d < J_\mu(v_\mu) = H(U_\mu) - H(u_\mu) < \frac{1}{N} S^{N/2}$$

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and thus, since $J'_\mu(U_\mu)U_\mu = 0$,

$$d + H(u_\mu) \le \frac{1}{N}||U_\mu||^2 - \frac{2^* - 1}{2^*}\mu \int fU_\mu \le H(u_\mu) + \frac{1}{N}S^{N/2}.$$

Since U_μ is uniformly bounded,

$$d + o(1) \le \frac{1}{N}||U_\mu||^2 \le \frac{1}{N}S^{N/2} + o(1). \tag{4.5}$$

By Sobolev's inequality, $S||U_\mu||^2_{2^*} \le ||U_\mu||^2 = ||U_\mu||^{2^*}_{2^*} + o(1)$. Then $||U_\mu||^{2^*}_{2^*} \ge S^{N/2} + o(1)$ and so $||U_\mu||^2 \ge S^{N/2} + o(1)$. Therefore by (4.5), we have

$$\lim_{\mu\to 0^+} ||U_\mu|| = S^{N/2}.$$

Now, fix $\rho \in]0, \mu^*]$. Suppose μ increase to ρ, then U_μ is decreasing to U_ρ in $H^1(\mathbb{R}^N)$ and we have

$$||U_\mu|| \le S^{-2^*/2}||U_\rho||^{2^*-1} + \rho||f||_*$$

and so, there exists a sequence U_{μ_j} conving weakly to a solution $\tilde{U}$ of (P_μ) in $H^1(\mathbb{R}^N)$ with $\rho = \mu$ but $\tilde{U} \ne U_\rho$. By the maximum principle, we have $U_\rho < \tilde{U} \le U_{\mu^*}$ which contradicts the uniqueness of solutions bigger than u_μ. Therefore, U_μ is decreasing continuously to U_ρ and $U_\mu \to U_\rho$ in $H^1(\mathbb{R}^N)$. This completes the proof. ■

Lemma 4.3. *Let V be a positive supersolution of (P_μ) bigger than u_μ, then $V \le U_\mu$.*

Proof. Suppose $V > U_\mu$ in $\mathbb{R}^N$, then $W = V - U_\mu$ satisfies

$$(2^* - 1)\int U_\mu^{2^*-2}W\phi_1 \le \int \nabla W \cdot \nabla\phi_1 = \eta_1(2^* - 1)\int U_\mu^{2^*-2}W\phi_1$$

and thus, $\eta_1(\mu) \ge 1$, which leads a conrradiction. This completes the proof. ■

Remark. *From Lemma 4.1 and Lemma 4.3, we can see the uniqueness of second solutions which are bigger than the minimal solutions u_μ.*

Now, we state basic properties of the eigenvalue problem $(4.1)_\mu$:

Lemma 4.4. *(i) $1/(2^* - 1) < \eta_1(\mu) < 1$ for $0 < \mu < \mu^*$,*
(ii) $\eta_1(\mu) \to 1/(2^* - 1) \to 1/(2^* - 1)$ as $\mu \to 0^+$,
(iii) $\eta_1(\mu) \to 1$ as $\mu \to \mu^*$.

Proof. (i) Since $\phi_1 > 0$ is an eigenvector corresponding to the the first eigenvalue $\eta_1(\mu)$, we know

$$\eta_1(\mu)(2^* - 1)\int U_\mu^{2^*-1}\phi_1 = \int \nabla U_\mu \cdot \nabla\phi_1 = \int U_\mu^{2^*-1}\phi_1 + \mu\int f\phi_1.$$

and so,

$$\eta_1(\mu)\left((2^* - 1) - 1\right)\int U_\mu^{2^*-1}\phi_1 = \mu\int f\phi_1.$$

Therefore, by Lemma 4.1, $1 > \eta_1(\mu) > \frac{1}{2^*-1}$.

(ii) As $\mu \to 0^+$,

$$\frac{1}{2^*-1} < \eta_1(\mu) \leq \frac{||U_\mu||^2}{(2^*-1)\,||U_\mu||_{2^*}^{2^*}} \leq \frac{S^{N/2}+o(1)}{(2^*-1)\,(S^{N/2}+o(1))} \to \frac{1}{2^*-1}.$$

Thus, $\eta_1(\mu) \to 1/(2^*-1)$ as $\mu \to 0^+$.

(iii) follows from (i) of Lemma 3.1, Proposition 3.3, Lemma 4.1 and (i) of Lemma 4.2. This completes the proof. ∎

In order to show the existence of a bifurcation point, we make use of Theorem 3.2 in [8].

Now, we have:

THEOREM 4.5. *(i) The set $\{U_\mu\}$ is bounded uniformly in $H^1(\mathbb{R}^N)$,*
(ii) (μ^*, u_{μ^*}) is a bifurcation point.

Proof. (i) It follows immediately from the proof of Lemma 4.2.
(ii) For this, define $F : R \times H^1(\mathbb{R}^N) \to H^{-1}(\mathbb{R}^N)$ by

$$F(\mu, u) := \Delta u - u + (u^+)^{2^*-1} + \mu f(x).$$

It is easy to see that $F(\mu, u)$ is differentiable at solution point (μ, u) for $]0, \mu^*[$ and

$$F_u(\mu, u_\mu)w = \Delta w - w + (2^*-1)u_\mu^{2^*-2}w$$

is an ismorphism of $R \times H(\mathbb{R}^N)$ onto $H^{-1}(\mathbb{R}^N)$. Then, by the Implicit Function Theorem, the solution of $F(\mu, u)$ near (μ, u_μ) are given by a single continuous cuver and $u_\mu \to 0$ in $H^1(\mathbb{R}^N)$ as $\mu \to 0$.

We now are going to prove that (μ^*, u_{μ^*}) is a bifurcation point of F. Since $F_u(\mu^*, u_{\mu^*})\phi = 0, \phi \in H^1(\mathbb{R}^N)$ has a solution $\phi_1 > 0$ in $\mathbb{R}^N$, $\mathcal{N}\left(F_u\left(\mu^*, u_{\mu^*}\right)\right) = \text{span}\{\phi_1\}$ is one dimensional and $\text{codim}\mathcal{R}\left(F_u\left(\mu^*, u_{\mu^*}\right)\right) = 1$ by the Fredholm alternative. Suppose there exists $v \in H^1(\mathbb{R}^N)$ satisfying

$$\Delta v - v + (2^*-1)\,u_{\mu^*}^{2^*-2}v = -f(x).$$

Then

$$0 = \int \left(\nabla v \cdot \nabla \phi_1 + v\phi_1 - (2^*-1)\,u_{\mu^*}^{2^*-2}v\phi_1\right) = \int f\phi_1,$$

which is impossible because $0 \not\equiv f \geq 0$. Hence, $F_u\left(\mu^*, u_{\mu^*}\right) \notin \mathcal{R}\left(F_u\left(\mu^*, u_{\mu^*}\right)\right)$. Thus, by Theorem 3.2 in [8], (μ^*, u_{μ^*}) is the bifurcation point near which, the solution of (p_μ) form a curve $(\mu^* + \tau(s), u_{\mu^*} + s\phi_1 + z(s))$ with s near $s = 0$ and $\tau(0) = \tau'(0) = 0, z(0) = z'(0) = 0$. Finally, we will show that $\tau''(0) < 0$ which implies that the bifurcation curve only turns to the left in the $\mu u-$plane. For this, differentiate (P_μ) in s, we have

$$\Delta u_s - u_s + (2^*-1)\,u^{2^*-2}u_s + \tau'(s)f(x) = 0, \tag{4.6}$$

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where $u_s = \phi_1 + z'(s)$. Multiplying $F_u(\mu^*, u_{\mu^*})\,\phi_1 = 0$ by u_s and (4,6) by ϕ_1, integrating and substracting, we have

$$\begin{aligned}
&\tau'(s)\int f\phi_1 \\
&= (2^*-1)\int \left(u_{\mu^*}^{2^*-2} - (u_{\mu^*} + s\phi_1 + z(s))^{2^*-2}\right)(\phi_1 + z'(s))\phi_1 \\
&= -s(2^*-1)(2^*-2)\int (u_{\mu^*} + \theta(s\phi_1 + z(s)))^{2^*-3}\left(\phi_1 + \frac{z(s)}{s}\right)(\phi_1 + z'(s))\,\phi_1
\end{aligned}$$

for some $\theta(s) \in (0,1)$. Therefore,

$$\tau''(0)\int f\phi_1 = \left(lim_{s\to 0}\frac{\tau'(s)}{s}\right)\int f\phi_1 = -\left(2^*-1\right)\left(2^*-2\right)\int \left(u_{\mu^*}\right)^{2^*-3}\phi_1^3$$

and $\tau''(0) < 0$. This completes proof. ∎

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Some inequalities on meromorphic function and its derivative on its Borel direction *

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Abstract

In view of Nevanlinna theory in the angular domain, we establish some inequalities of meromorphic function concerning its derivation in its Borel direction. By applying these inequalities, we also investigate exceptional values of meromorphic functions with infinite order in the Borel direction.

Key words: Infinite order; Borel direction; Exceptional value.
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1 Introduction and main results

It is assumed that the reader is familiar with the basic results and the standard notations of the Nevanlinna theory of meromorphic functions (see [7, 17, 20]). We denote by $\mathbb{C}$ the open complex plane, by $\widehat{\mathbb{C}}(=\mathbb{C}\bigcup\{\infty\})$ the extended complex plane, and by $\Omega(\subset\mathbb{C})$ an angular domain. In addition, the order of meromorphic function f is defined by

$$\rho(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r},$$

and the exponent of convergence of distinct a-points of f is defined by

$$\overline{\rho}(a,f)=\limsup_{r\to\infty}\frac{\log^{+}\overline{N}(r,a,f)}{\log r}.$$

Let f be a meromorophic function of order $\rho(0<\rho<\infty)$, then we say that a is an exceptional value in the sense of Borel (evB for short) for f for the distinct zeros if $\overline{\rho}(a,f)<\rho$.

It is well known that the singular direction of meromorphic function is an interesting topic in the field of complex analysis, such as, Julia direction, Borel direction, T direction, Hayman direction, and so on (see [1, 3, 4, 8, 10, 12, 13, 14]). Moreover, we know that every one singular direction is always responding to exceptional value, such as, the Julia's direction relating with Picard exceptional value and the Borel's direction relating with Borel exceptional value, and so

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on. 2011, Peng and Sun [11] gave some examples on T direction which is a singular direction relating with T exceptional value. In the discussion of the topic of singular direction, we find that the characteristics of meromorphic functions in the angular domain played an important role(see [6, 18, 19, 24, 25]). So, we firstly introduce the characteristics of meromorphic functions in the angular domain as follows [5, 25].

For a meromorphic function f on the angular domain $\Omega(\alpha,\beta) = \{z : \alpha \leq \arg z \leq \beta\}$ and $0 < \beta - \alpha \leq 2\pi$. Define

$$A_{\alpha,\beta}(r,f) = \frac{\omega}{\pi}\int_1^r (\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}})\{\log^+|f(te^{i\alpha})| + \log^+|f(te^{i\beta})|\}\frac{dt}{t},$$

$$B_{\alpha,\beta}(r,f) = \frac{2\omega}{\pi r^\omega}\int_\alpha^\beta \log^+|f(re^{i\theta})|\sin\omega(\theta-\alpha)d\theta,$$

$$C_{\alpha,\beta}(r,f) = 2\sum_{1<|b_\mu|<r}(\frac{1}{|b_\mu|^\omega} - \frac{|b_\mu|^\omega}{r^{2\omega}})\sin\omega(\theta_\mu - \alpha),$$

$$S_{\alpha,\beta}(r,f) = D_{\alpha,\beta}(r,f) + C_{\alpha,\beta}(r,f),$$

where $D_{\alpha,\beta}(r,f) = A_{\alpha,\beta}(r,f) + B_{\alpha,\beta}(r,f)$, $\omega = \frac{\pi}{\beta-\alpha}$ and $b_\mu = |b_\mu|e^{i\theta_\mu}(\mu = 1,2,\cdots)$ are the poles of f on $\Omega(\alpha,\beta)$ counted according to their multiplicities. $S_{\alpha,\beta}(r,f)$ is called the Nevanlinna's angular characteristic, and $C_{\alpha,\beta}(r,f)$ is called the angular counting function of the poles of f on $\Omega(\alpha,\beta)$, and $\overline{C}_{\alpha,\beta}(r,f)$ is the reduced function of $C_{\alpha,\beta}(r,f)$. Similarly, the order of meromorphic function f on $\Omega(\alpha,\beta)$ is defined by

$$\rho_{\alpha,\beta}(f) = \limsup_{r\to\infty}\frac{\log S_{\alpha,\beta}(r,f)}{\log r},$$

and the exponent of convergence of distinct a-points of f on $\Omega(\alpha,\beta)$ is defined by

$$\overline{\rho}_{\alpha,\beta}(a,f) = \limsup_{r\to\infty}\frac{\log^+\overline{C}_{\alpha,\beta}(r,a,f)}{\log r}.$$

Suppose that f is a meromorphic function of order $\rho_{\alpha,\beta}(f)(0 < \rho_{\alpha,\beta}(f) < \infty)$, then we say that a is an exceptional value on the angular domain in the sense of Borel (evaB for short) for f for the distinct zeros if $\overline{\rho}_{\alpha,\beta}(a,f) < \rho_{\alpha,\beta}(f)$.

Remark 1.1 *By the second fundamental theorem in the whole complex plane, we know that a meromorophic function f of order $\rho(0 < \rho < \infty)$ at most has two evB for the distinct zeros. However, the corresponding conclusion can not hold for meromorphic function f with order $\rho_{\alpha,\beta}(0 < \rho_{\alpha,\beta} < \infty)$ on $\Omega(\alpha,\beta)$ since $Q_{\alpha,\beta}(r,f) = O\{\log(rS_{\alpha,\beta}(r,f))\}$ is not valid, as $r\to\infty(r \notin E)$ and E is the set with finite linear measure.*

Thus,*it is an interesting topic to research the exceptional value of meromorphic functions on the angular domain.*

Before stating the our results, we will introduce the definition as follows.

Definition 1.1 *[2]. Let f be a meromorphic function of infinite order, $\rho(r)$ be a real function satisfying the following conditions:*

(i) $\rho(r)$ is continuous, non-decreasing for $r \geq r_0$ and $\rho(r)\to\infty$ as $r\to\infty$;

(ii)

$$\lim_{r\to\infty}\frac{\log U(R)}{\log U(r)} = 1, \quad R = r + \frac{r}{\log U(r)},$$

where $U(r) = r^{\rho(r)}$ $(r \geq r_0)$;

(iii)

$$\limsup_{r\to\infty}\frac{\log T(r,f)}{\log U(r)}=1.$$

Then $\rho(r)$ is called infinite order of meromorphic function f. This definition was given by Xiong Qinglai[2].

We will give the definition of Borel direction of meromorphic functions f of infinite order $\rho(r)$ as follows.

Definition 1.2 *[2]. Let f be a meromorphic function of infinite order $\rho(r)$. If for any $\varepsilon(0<\varepsilon<\pi)$, the equality*

$$\limsup_{r\to\infty}\frac{\log n(\Omega(\theta-\varepsilon,\theta+\varepsilon,r),f=a)}{\rho(r)\log r}=1,$$

holds for any complex number $a\in\widehat{\mathbb{C}}$, at most except two exception, where $n(\Omega(\theta-\varepsilon,\theta+\varepsilon,r),f=a)$ is the counting function of zero of the function $f-a$ in the angular domain $\Omega(\theta-\varepsilon,\theta+\varepsilon)$, counting multiplicities. Then the ray $\arg z=\theta$ is called a Borel direction of $\rho(r)$ order of meromorphic function f.

Remark 1.2 *Chuang [2] proved that every meromorphic function f with infinite order $\rho(r)$ has as least one Borel direction of infinite order $\rho(r)$.*

In 2012, Long and Wu [9] studied the uniqueness of meromorphic functions with infinite order sharing some values in the Borel direction. Later, Zhang, Xu and Yi [21] further investigated the uniqueness of meromorphic functions sharing some values in the Borel direction, and improved the results of Long and Wu. In 2013, Zhang [23] also studied the problems of Borel directions of meromorphic functions concerning shared values and obtained that if two meromorphic functions with infinite order share three distinct values, their Borel direction are same. In the same year, Xu, Wu and Tu [15] investigated the relations between exceptional values and Borel direction, and obtained a series of results. In this paper, we mainly further investigate the exceptional values of meromrophic function and its derivation in its Borel direction. Now, we give the main theorem of this paper as follows.

Theorem 1.1 *Let f be a transcendental meromorphic function of infinite order $\rho(r)$ on the whole complex plane, $\arg z=\theta(0\le\theta<2\pi)$ be one Borel direction of $\rho(r)$ order of function f and $\Omega:=\Omega(\theta-\varepsilon,\theta+\varepsilon)$ for any $\varepsilon(0<\varepsilon<\pi)$. Let $a,b(\neq 0)$ be distinct points and k be a positive integer. Then*

$$\begin{aligned}S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\le&\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f)+(k+1)\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a,f)\\&+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}\left(r,b,f^{(k)}\right)+Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f),\end{aligned}\tag{1}$$

where $Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f)$ is defined as in Lemma 2.2 of Section 2.

In order to prove Theorem 1.1, we will prove the more general form of the inequality of meromorphic function and its derivation in the Borel direction as follows.

Theorem 1.2 *Let f be a transcendental meromorphic function of infinite order $\rho(r)$ on the whole complex plane, $\arg z=\theta(0\le\theta<2\pi)$ be one Borel direction of $\rho(r)$ order of function f and $\Omega:=\Omega(\theta-\varepsilon,\theta+\varepsilon)$ for any $\varepsilon(0<\varepsilon<\pi)$. Let $a_j,b_l(j=1,2,\ldots,p;l=1,2,\ldots,q)$ be distinct* *complex numbers satisfying* $b_l \neq 0$, *and* m_j, n_l, s *be any positive integers. Then*

$$\begin{aligned}\Bigg\{pq - \Bigg[\sum_{j=1}^{p}\frac{kq+1}{m_j+1} &+ \sum_{l=1}^{q}\frac{1}{n_l+1} \\ &+\frac{1}{s+1}\left(1+k\sum_{l=1}^{q}\frac{1}{n_l+1}\right)\Bigg]\Bigg\} S_{\theta-\varepsilon,\theta+\varepsilon}(r,f) \\ &\leq \frac{s}{s+1}\left(1+k\sum_{l=1}^{q}\frac{1}{n_l+1}\right)\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f|\leq s) \\ &+(kq+1)\sum_{j=1}^{p}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq m_j) \\ &+\sum_{l=1}^{q}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}\left(r,b_l,f^{(k)}|\leq n_l\right)+Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f), \qquad (2)\end{aligned}$$

where $\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a,f|\leq k)$ *is the counting function of distinct a-points of* f *on* Ω *whose multiplicities do not exceed* k.

Let $p=q=1$ and $s\to\infty, m_j\to\infty$ and $n_l\to\infty$ in Theorem 1.2, we can get Theorem 1.1 easily.

We also investigate the problem on exceptional value of meromorphic function and its derivation in its Borel direction, by applying the conclusions of Theorems 1.1 and 1.2. To state the theorem, we will introduce the definitions as follows.

Definition 1.3 *Let* $\arg z=\theta(0\leq\theta<2\pi)$ *be one Borel direction of* $\rho(r)$ *order of function* f *and* k *be a positive integer, we call that* a *is*

(i) an exceptional value in the sense of Borel for f *in the Borel direction (evBB for short) for distinct zeros of multiplicity* $\leq k$, *if* $\overline{\rho}_\theta^k(a,f)<1$;

(ii) an exceptional value in the sense of Borel for f *in the Borel direction (evBB for short) for distinct zeros , if* $\overline{\rho}_\theta(a,f)<1$; *where*

$$\overline{\rho}_\theta^k(a,f)=\limsup_{r\to\infty}\frac{\log^+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a,f|\leq k)}{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)},\quad \overline{\rho}_\theta(a,f)=\limsup_{r\to\infty}\frac{\log^+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a,f)}{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}.$$

In particular, we say that a *is an evBB for* f *for simple zeros if* $k=1$, a *is an evBB for* f *for simple and double zeros if* $k=2$.

Definition 1.4 *Let* $\arg z=\theta(0\leq\theta<2\pi)$ *be one Borel direction of* $\rho(r)$ *order of function* f *and* k,l *be two positive integers, then we call* a *an evBB for* f^l *for distinct zeros of order* $\leq k$, *if* $\overline{\rho}_\theta^k(a,f^{(l)})<1$, *where*

$$\overline{\rho}_\theta^k(a,f^{(l)})=\limsup_{r\to\infty}\frac{\log^+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a,f^{(l)}|\leq k)}{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}.$$

Theorem 1.3 *Let* f *be a transcendental meromorphic function of infinite order* $\rho(r)$ *on the whole complex plane,* $\arg z=\theta(0\leq\theta<2\pi)$ *be one Borel direction of* $\rho(r)$ *order of function* f *and* $\Omega:=\Omega(\theta-\varepsilon,\theta+\varepsilon)$ *for any* $\varepsilon(0<\varepsilon<\pi)$. *If* ∞ *is an evBB for* f *for distinct poles of order* $\leq s$, *and* $a_j(j=1,2,\ldots,p)$ *are evBB for* f *for distinct zeros of order* $\leq m_j$, *and* $b_l(\neq 0)(l=1,2,\ldots,q)$ *are evBB for* $f^{(k)}$ *for distinct zeros of order* $\leq n_l$, *where* k,p,q,s *and all of* m_j,n_l *are positive integers. Then*

$$\sum_{j=1}^{p}\frac{kq+1}{m_j+1}+\sum_{l=1}^{q}\frac{1}{n_l+1}+\frac{1}{s+1}\left(1+k\sum_{l=1}^{q}\frac{1}{n_l+1}\right)\geq pq.$$

Let $p = q = 1$ in Theorem 1.3, we can obtain the corollary as follows.

Corollary 1.1 *Let f be a transcendental meromorphic function of infinite order $\rho(r)$ on the whole complex plane, $\arg z = \theta(0 \le \theta < 2\pi)$ be one Borel direction of $\rho(r)$ order of function f and $\Omega := \Omega(\theta - \varepsilon, \theta + \varepsilon)$ for any $\varepsilon(0 < \varepsilon < \pi)$. If ∞ is an evBB for f for distinct poles of order $\le s$, and a is an evBB for f for distinct zeros of order $\le m$, $b(\neq 0)$ is an evBB for $f^{(k)}$ for distinct zeros of order $\le n$, and s, m, n, k are positive integers. Then*

$$\frac{k+1}{m+1} + \frac{1}{n+1} + \frac{n+1+k}{(n+1)(s+1)} \ge 1. \tag{3}$$

Let $s \to \infty$ and $m \to \infty$ in (3), that is, ∞, a are evBB for f for distinct zeros. From Corollary 1.1, we have $\frac{1}{n+1} \ge 1$, which implies $n = 0$. Thus, we can obtain the following corollary.

Corollary 1.2 *Let f be a transcendental meromorphic function of infinite order $\rho(r)$ on the whole complex plane, $\arg z = \theta(0 \le \theta < 2\pi)$ be one Borel direction of $\rho(r)$ order of function f and $\Omega := \Omega(\theta - \varepsilon, \theta + \varepsilon)$ for any $\varepsilon(0 < \varepsilon < \pi)$. If ∞, a are evBB for f for distinct zeros. Then, for all positive integers k and n, we have $\overline{\rho}^n_\theta(b, f^{(k)}) = 1$ for all $b \neq 0, \infty$.*

2 Some Lemmas

To prove our results, we need the following Lemmas.

Lemma 2.1 *(see [6, 16]). Let f be a nonconstant meromorphic function on $\Omega(\alpha, \beta)$. Then for arbitrary complex number a, we have*

$$S_{\alpha,\beta}\left(r, \frac{1}{f-a}\right) = S_{\alpha,\beta}(r, f) + \varepsilon(r, a),$$

where $\varepsilon(r, a) = O(1)$ as $r \to \infty$.

Lemma 2.2 *(see [5, 6, 25]). Suppose that f is a non-constant meromorphic function in one angular domain $\Omega(\alpha, \beta)$ with $0 < \beta - \alpha \le 2\pi$, then for arbitrary q distinct $a_j \in \widehat{\mathbb{C}}(1 \le j \le q)$, we have*

$$\sum_{i=}^{q} D_{\alpha,\beta}(r, \frac{1}{f-a_j}) \le 2S_{\alpha,\beta}(r, f) - C_1(r) + Q_{\alpha,\beta}(r, f),$$

where $C_1(r) = 2C_{\alpha,\beta}(r, f) - C_{\alpha,\beta}(r, f') + C_{\alpha,\beta}(r, \frac{1}{f'})$ and

$$(q-2)S_{\alpha,\beta}(r, f) \le \sum_{j=1}^{q} \overline{C}_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right) + Q_{\alpha,\beta}(r, f),$$

where the term $\overline{C}_{\alpha,\beta}(r, \frac{1}{f-a_j})$ will be replaced by $\overline{C}_{\alpha,\beta}(r, f)$ when some $a_j = \infty$ and

$$\begin{aligned} Q_{\alpha,\beta}(r, f) =& A_{\alpha,\beta}\left(r, \frac{f'}{f}\right) + B_{\alpha,\beta}\left(r, \frac{f'}{f}\right) \\ &+ \sum_{j=1}^{q}\left\{A_{\alpha,\beta}\left(r, \frac{f'}{f-a_j}\right) + B_{\alpha,\beta}\left(r, \frac{f'}{f-a_j}\right)\right\} + O(1). \end{aligned} \tag{4}$$

Lemma 2.3 *(see [6, P138].) Let f be a nonconstant meromorphic function in the whole complex plane $\mathbb{C}$. Given one angular domain on $\Omega(\alpha, \beta)$. Then for any $1 \le r < R$, we have*

$$A_{\alpha,\beta}\left(r, \frac{f'}{f}\right) \le K\left\{\left(\frac{R}{r}\right)^{\omega} \int_1^R \frac{\log^+ T(r, f)}{t^{1+\omega}} dt + \log^+ \frac{r}{R-r} + \log\frac{R}{r} + 1\right\},$$

and

$$B_{\alpha,\beta}\left(r,\frac{f'}{f}\right)\leq\frac{4\omega}{r^{\omega}}m\left(r,\frac{f'}{f}\right),$$

where $\omega=\frac{\pi}{\beta-\alpha}$ *and* K *is a positive constant not depending on* r *and* R.

Remark 2.1 *Nevanlinna conjectured that*

$$D_{\alpha,\beta}\left(r,\frac{f'}{f}\right)=A_{\alpha,\beta}\left(r,\frac{f'}{f}\right)+B_{\alpha,\beta}\left(r,\frac{f'}{f}\right)=o\left(S_{\alpha,\beta}\left(r,f\right)\right)\tag{5}$$

when r *tends to* $+\infty$ *outside an exceptional set of finite linear measure, and he proved that* $A_{\alpha,\beta}\left(r,\frac{f'}{f}\right)+B_{\alpha,\beta}\left(r,\frac{f'}{f}\right)=O(1)$ *when the function* f *is meromorphic in* $\mathbb{C}$ *and has finite order. In 1974, Gol'dberg[5] constructed a counter-example to show that (3) is not valid.*

Lemma 2.4 *(see [22, Lemma 4]). Let* f *be a meromorphic function in* $\mathbb{C}$, $\Omega(\alpha,\beta)$ $(0<\beta-\alpha\leq 2\pi)$ *be a closed angular domain, then*

$$Q_{\alpha,\beta}\left(r,f\right)=\begin{cases}O(1), & f\ is\ of\ finite\ order,\\ O(\log U(r)), & f\ is\ of\ infinite\ order,\end{cases}$$

where $Q_{\alpha,\beta}\left(r,f\right)$ *is stated as in (4),* $U(r)=r^{\rho(r)}$, $\rho(r)$ *is the precise order of* $T(r,f)$ *when* f *is of infinite order,* E *is a set of finite linear measure.*

Lemma 2.5 *(see [22, Lemma 5]). Let* f *be a meromorphic function on a closed angular domain* $\Omega(\alpha,\beta)$ *and* $\omega=\frac{\pi}{\beta-\alpha}$, *then for any* $a\in\widehat{\mathbb{C}}$ *and for any* $\varepsilon\in(0,\frac{\beta-\alpha}{2})$,

$$\begin{aligned}
&C_{\alpha,\beta}(r,a,f)\geq 2\omega\sin(\omega\varepsilon)\int_1^r\frac{n(t,\Omega_\varepsilon,f=a)}{t^{\omega+1}}dt+O(1),\\
&C_{\alpha,\beta}(r,a,f)\geq\frac{4\omega\sin(\omega\varepsilon)}{r^{\omega}}N(r,\Omega_\varepsilon,f=a)+o(1),\\
&C_{\alpha,\beta}(r,a,f)\leq 4\omega\int_1^r\frac{n(t,\Omega,f=a)}{t^{\omega+1}}dt,\\
&C_{\alpha,\beta}(r,a,f)\leq 2n(r,\Omega,f=a),
\end{aligned}$$

where $\Omega_\varepsilon=(\alpha+\varepsilon,\beta-\varepsilon)$.

Remark 2.2 *For the reduced case that each multiple zero of* $f-a$ *in* $\Omega(\alpha,\beta)$ *is counted only once (ignoring multiplicities), Lemma 2.5 still holds, and its proof is similar to the case counting multiplicities.*

Lemma 2.6 *(see [3]). Let* f *be a meromorphic function of infinite order* $\rho(r)$. *Then the ray* $\arg z=\theta$ *is one Borel direction of* $\rho(r)$ *order of meromorphic function* f *if and only if* f *satisfies the equality*

$$\limsup_{r\to\infty}\frac{\log S_{\theta-\epsilon,\theta+\epsilon}(r,f)}{\rho(r)\log r}=1,$$

for any $\epsilon(0<\epsilon<\frac{\pi}{2})$.

3 Proof of Theorem 1.2

Proof: Since f is a meromorphic function of infinite order $\rho(r)$ and $\arg z=\theta(0\leq\theta<2\pi)$ is one Borel direction of $\rho(r)$ order of meromorphic function f, by Lemma 2.6, we can get for any $\varepsilon(0<\varepsilon<\pi)$

$$\limsup_{r\to\infty}\frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}{\rho(r)\log r}=1.\tag{6}$$

By Lemmas 2.2-2.4, we have

$$D_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{f'}{f}) = O(\log U(r)).$$

By Ref. [6, 25], we can get $Q_{\theta-\varepsilon,\theta+\varepsilon}(r,^{(k)}) = Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f)$. Thus, we have

$$D_{\theta-\varepsilon,\theta+\varepsilon}(r,\frac{f^{(k)}}{f}) = O(\log U(r)) = Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f). \tag{7}$$

Hence, for any positive integer k, we have

$$\sum_{j=1}^{p} D_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f) \le D_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k)}) + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f). \tag{8}$$

By Lemma 2.1, we have

$$S_{\theta-\varepsilon,\theta+\varepsilon}(r,f^{(k)}) = D_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k)}) + C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k)}) + O(1). \tag{9}$$

Then, it follows from (8) and (9) that

$$\sum_{j=1}^{p} D_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f) \le S_{\theta-\varepsilon,\theta+\varepsilon}(r,f^{(k)}) - C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k)}) + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f). \tag{10}$$

From (10), we have

$$\begin{aligned} pS_{\theta-\varepsilon,\theta+\varepsilon}(r,f) \le & S_{\theta-\varepsilon,\theta+\varepsilon}(r,f^{(k)}) + \sum_{j=1}^{p} C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f) - C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k)}) \\ & + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f). \end{aligned} \tag{11}$$

By applying Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned} qS_{\theta-\varepsilon,\theta+\varepsilon}(r,f^{(k)}) \le & \sum_{j=1}^{q} C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f^{(k)}) + C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k)}) \\ & + C_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f^{(k)}) - C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k+1)}) \\ & - 2C_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f^{(k)}) + C_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f^{(k+1)}) \\ & + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f) \\ \le & \sum_{j=1}^{q} C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f^{(k)}) + C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k)}) \\ & + C_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f^{(k+1)}) - C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k+1)}) \\ & - C_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f^{(k)}) + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f) \\ \le & \sum_{j=1}^{q} C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f^{(k)}) + C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k)}) \\ & + \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f) - C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k+1)}) + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f). \end{aligned} \tag{12}$$

It follows from (11) and (12) that

$$\begin{aligned} pqS_{\theta-\varepsilon,\theta+\varepsilon}(r,f) \leq & C_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f) + (q-1)\sum_{j=1}^{p} C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f) \\ & - (q-1)C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k)}) + \sum_{j=1}^{p} C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f) \\ & + \sum_{l=1}^{q} C_{\theta-\varepsilon,\theta+\varepsilon}(r,b_l,f^{(k)}) - C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k+1)}) \\ & + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f). \end{aligned} \tag{13}$$

If z_0 is a zero of $f-a$ of order $j>k$ in $\Omega(\theta-\varepsilon,\theta+\varepsilon)$, then z_0 is a zero of $f^{(k+1)}$ of order $j-(k+1)$ in $\Omega(\theta-\varepsilon,\theta+\varepsilon)$, and if z_0 is a zero of $f^{(k)}-b$ of order m in $\Omega(\theta-\varepsilon,\theta+\varepsilon)$, then z_0 is a zero of $f^{(k+1)}$ of order $m-1$ in $\Omega(\theta-\varepsilon,\theta+\varepsilon)$. Moreover, If z_0 is the zero of $f-a$ of order $>k$ and also zero of $f^{(k)}$ in $\Omega(\theta-\varepsilon,\theta+\varepsilon)$, then z_0 is not zero of $f^{(k)}-b$ in $\Omega(\theta-\varepsilon,\theta+\varepsilon)$ as $b\neq 0$. Thus, we have

$$\begin{aligned} & \sum_{j=1}^{p} C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f) + \sum_{l=1}^{q} C_{\theta-\varepsilon,\theta+\varepsilon}(r,b_l,f^{(k)}) - C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k+1)}) \\ & \qquad \leq \sum_{j=1}^{p} C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f| \leq k+1) + \sum_{l=1}^{q} \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,b_l,f^{(k)}), \end{aligned} \tag{14}$$

$$\sum_{j=1}^{p} C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f) - C_{\theta-\varepsilon,\theta+\varepsilon}(r,0,f^{(k)}) \leq \sum_{j=1}^{p} C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f| \leq k).$$

Substituting (14) to (13), we get

$$\begin{aligned} pqS_{\theta-\varepsilon,\theta+\varepsilon}(r,f) \leq & \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f) + (q-1)\sum_{j=1}^{p} C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f| \leq k) \\ & + \sum_{j=1}^{p} C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f| \leq k+1) + \sum_{l=1}^{q} \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,b_l,f^{(k)}) \\ & + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f). \end{aligned} \tag{15}$$

For any positive integer k, we have

$$\begin{aligned} C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f| \leq k) & \leq k\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f) \\ & \leq \frac{k}{m_j+1}\left[m_j\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f| \leq m_j) + C_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f)\right] \\ & \leq \frac{k}{m_j+1}\left[m_j\overline{C}(r,a_j,f| \leq m_j) + S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\right] + O(1), \end{aligned} \tag{16}$$

and

$$\begin{aligned} \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,b_l,f^{(k)}) & \leq \frac{1}{n_l+1}\left[n_l\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,b_j,f^{(k)}| \leq n_l) + S_{\theta-\varepsilon,\theta+\varepsilon}(r,f^{(k)})\right] + O(1) \\ \overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f) & \leq \frac{1}{s+1}\left[s\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f| \leq s) + S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\right], \end{aligned} \tag{17}$$

and since $S_{\theta-\varepsilon,\theta+\varepsilon}(r,f^{(k)}) \leq S_{\theta-\varepsilon,\theta+\varepsilon}(r,f) + k\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f) + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f)$, then it follows from (15)-(17) that

$$\begin{aligned}
pqS_{\theta-\varepsilon,\theta+\varepsilon}(r,f) \leq\ & (q-1)\sum_{j=1}^{p}\frac{k}{m_j+1}\left[m_j\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq m_j) + S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\right]\\
&+\sum_{j=1}^{p}\frac{k+1}{m_j+1}\left[m_j\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq m_j) + S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\right]\\
&+\sum_{l=1}^{q}\frac{1}{n_l+1}\left[n_l\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,b_l,f^{(k)}|\leq n_l) + S_{\theta-\varepsilon,\theta+\varepsilon}(r,f^{(k)})\right]\\
&+\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f) + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\\
\leq\ & (q-1)\sum_{j=1}^{p}\frac{km_j}{m_j+1}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq m_j)\\
&+\sum_{j=1}^{p}\frac{m_j(k+1)}{m_j+1}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq m_j)\\
&+\sum_{l=1}^{q}\frac{n_l}{n_l+1}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,b_l,f^{(k)}|\leq n_l)\\
&+(1+\sum_{l=1}^{q}\frac{k}{n_l+1})\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f)\\
&+\left(\sum_{j=1}^{p}\frac{kq+1}{m_j+1}+\sum_{j=1}^{q}\frac{1}{n_l+1}\right)S_{\theta-\varepsilon,\theta+\varepsilon}(r,f) + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\\
\leq\ & (kq+1)\sum_{j=1}^{p}\frac{km_j}{m_j+1}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq m_j)\\
&+\sum_{l=1}^{q}\frac{n_l}{n_l+1}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,b_l,f^{(k)}|\leq n_l)\\
&+\left(1+\sum_{l=1}^{q}\frac{k}{n_l+1}\right)\frac{s}{s+1}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f|\leq s)
\end{aligned}$$

$$\begin{aligned}
&+\left(\sum_{j=1}^{p}\frac{kq+1}{m_j+1}+\sum_{j=1}^{q}\frac{1}{n_l+1}+(1+\sum_{l=1}^{q}\frac{k}{n_l+1})\frac{1}{s+1}\right)\\
&\times S_{\theta-\varepsilon,\theta+\varepsilon}(r,f) + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f).
\end{aligned}$$

Since m_j, n_l, k, p, q and s are positive integers, it follows from the above inequality that

$$\begin{aligned} pqS_{\theta-\varepsilon,\theta+\varepsilon}(r,f) \leq & \left(\sum_{j=1}^{p}\frac{kq+1}{m_j+1}+\sum_{j=1}^{q}\frac{1}{n_l+1}+(1+\sum_{l=1}^{q}\frac{k}{n_l+1})\frac{1}{s+1}\right)S_{\theta-\varepsilon,\theta+\varepsilon}(r,f) \\ & + (kq+1)\sum_{j=1}^{p}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq m_j) \\ & + \sum_{l=1}^{q}\frac{n_l}{n_l+1}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,b_l,f^{(k)}|\leq n_l) \\ & + \left(1+\sum_{l=1}^{q}\frac{k}{n_l+1}\right)\frac{s}{s+1}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f|\leq s) \\ & + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f). \end{aligned} \tag{18}$$

Thus, from (18), we can prove (2) easily.

Therefore, this completes the proof of Theorem 1.2.

□

4 The proof of Theorem 1.3

Proof: Since f is a meromorphic function of infinite order $\rho(r)$ and $\arg z=\theta(0\leq\theta<2\pi)$ is one Borel direction of $\rho(r)$ order of meromorphic function f, by Lemma 2.6, we can get for any $\varepsilon(0<\varepsilon<\pi)$

$$\limsup_{r\to\infty}\frac{\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)}{\rho(r)\log r}=1. \tag{19}$$

Since ∞ is an evBB for f for distinct poles of order $\leq s$, and $a_j(j=1,2,\ldots,p)$ are evBB for f for distinct zeros of order $\leq m_j$, and $b_l(\neq 0)(l=1,2,\ldots,q)$ are evBB for $f^{(k)}$ for distinct zeros of order $\leq n_l$, from Definition 1.3 and (19), we have that there exists a number $\eta(0<\eta<1)$ such that for sufficiently large r,

$$\overline{C}(r,\infty,f|\leq s)\leq (U(r))^{\eta}, \tag{20}$$

$$\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq m_j)<(U(r))^{\eta}, j=1,2,\ldots,p, \tag{21}$$

$$\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,b_l,f^{(k)}|\leq n_l)<(U(r))^{\eta}, l=1,2,\ldots,q. \tag{22}$$

Set

$$\Lambda:=\sum_{j=1}^{p}\frac{kq+1}{m_j+1}+\sum_{j=1}^{q}\frac{1}{n_l+1}+(1+\sum_{l=1}^{q}\frac{k}{n_l+1})\frac{1}{s+1}.$$

From Theorem 1.2, we have

$$\begin{aligned} (pq-\Lambda)S_{\theta-\varepsilon,\theta+\varepsilon}(r,f) \leq & (kq+1)\sum_{j=1}^{p}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq m_j) \\ & + \sum_{l=1}^{q}\frac{n_l}{n_l+1}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,b_l,f^{(k)}|\leq n_l) \\ & + \left(1+\sum_{l=1}^{q}\frac{k}{n_l+1}\right)\frac{s}{s+1}\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,\infty,f|\leq s) \\ & + Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f). \end{aligned} \tag{23}$$

From (20)-(23), for sufficiently large r, it follows that

$$(pq-\Lambda)S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\leq O((U(r))^{\eta})+Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f). \tag{24}$$

Since $\eta<1$, from (19) and (24) for sufficiently large r, we can get $pq-\Lambda\leq 0$, that is,

$$\sum_{j=1}^{p}\frac{kq+1}{m_j+1}+\sum_{l=1}^{q}\frac{1}{n_l+1}+\frac{1}{s+1}\left(1+k\sum_{l=1}^{q}\frac{1}{n_l+1}\right)\geq pq.$$

Thus, this completes the proof of Theorem 1.3.

□

5 Remarks

From the procedure of proofs of Theorems 1.1 and 1.2, we find that the conclusions of Theorems 1.1 and 1.2 can still hold for transcendental meromorphic function f with finite order $\rho(0<\rho<\infty)$ on the whole complex plane.

Thus, it is a natural question to ask: Does the conclusion of Theorem 1.3 still holds when f is a transcendental meromorphic function with finite order $\rho(0<\rho<\infty)$ on the whole complex plane?

In fact, we can not give a positive answer to the above question. Now, we will give a simple procedure to prove this assertion as follows.

Firstly, similar to Definitions 1.3 and 1.4, we can get some definitions of exception values of meromorphic function with finite order in the Borel direction, if $\overline{\rho}_{\theta}^{k}(a,f)<\rho,\overline{\rho}_{\theta}(a,f)<\rho,\overline{\rho}_{\theta}^{k}(a,f^{(l)})<\rho$, when $\log S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)$ is replaced by $\log r$. Thus, from the definition of Borel direction, (20)-(23) can be replaced by

$$\overline{C}(r,\infty,f|\leq s)\leq r^{\eta'}, \tag{25}$$

$$\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,a_j,f|\leq m_j)<r^{\eta'}, j=1,2,\ldots,p, \tag{26}$$

$$\overline{C}_{\theta-\varepsilon,\theta+\varepsilon}(r,b_l,f^{(k)}|\leq n_l)<r^{\eta'}, l=1,2,\ldots,q. \tag{27}$$

where $\eta'<\rho$ and r is sufficiently large, and (24) can be replaced by

$$(pq-\Lambda)\,S_{\theta-\varepsilon,\theta+\varepsilon}(r,f)\leq O\left(r^{\eta'}\right)+Q_{\theta-\varepsilon,\theta+\varepsilon}(r,f). \tag{28}$$

However, by Lemmas 2.1-2.5, we can not be sure to derive a contradiction from (28). Therefore, Theorem 1.3 may not be true when f is of finite order.

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