Volume 30, Number 1 January 2022
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS, LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(twice annually)

Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, St.Martin Univ.,Olympia,WA,USA.

Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com
http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $300. For any other part of the world add $140 more(handling and postages) to the above prices for Print. No credit card payments.

JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI, Elsevier-Scopus.

Editorial Board

Associate Editors of Journal of Computational Analysis and Applications

George Cybenko
Thayer School of Engineering
Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences

Probability

Jagdev Singh
JECRC University, Jaipur, India
jagdevsinghrathore@gmail.com
Fractional Calculus, Mathematical
Modelling, Special Functions,
Numerical Methods

Instructions to Contributors
Journal of Computational Analysis and Applications
An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .
Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).
If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.
References should include (in the following order):
initials of first and middle name, last name of author(s)
title of article,

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

<u>Journal Article</u>

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

<u>Book</u>

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

<u>Contribution to a Book</u>

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus <u>homepage</u>.

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

Riesz Basis in de Branges Spaces of Entire Functions

Sa'ud Al-Sa'di[1*] and Hamed Obiedat[2]

[1,2] Department of Mathematics, Faculty of Science, The Hashemite University, P.O Box 330127, Zarqa 13133, Jordan

Abstract

In this paper we consider the problem of Riesz basis in de Branges spaces of entire functions $\mathcal{H}(E)$ with the condition that $\varphi'(x) \geq \alpha > 0$, where φ is the corresponding phase function. We are concerned with the sets of real numbers $\{\lambda_n\}$ such that the normalized reproducing kernels $k(\lambda_n, .)/\|k(\lambda_n, .)\|$ satisfies the restricted isometry property, which in turn constitute a Riesz basis in $\mathcal{H}(E)$. Then we give a criterion on stability of reproducing kernels corresponding to real points which form a Riesz basis in $\mathcal{H}(E)$ with respect to small perturbations, which generalize some well-known Riesz basis perturbation results in the Paley-Wiener space.

2010 Mathematics Subject Classification: 46E22; 41A99; 30B99; 30D10

Key words and phrases: de Branges Spaces; Reproducing kernels; phase function; Restricted isometry property; Riesz basis.

1 Introduction

Compressive sensing provides an alternative method for efficiently acquiring and reconstructing a signal to the Shannon sampling theorem when the signal under acquisition is known to be sparse or compressible. Recently, Candès and Tao [4] introduced very intense activity related to compressed sensing, known as the restricted isometry property, which is also known as the uniform uncertainty principle. The restricted isometry property generalizes the notion of coherence, and allow recovering and extending many known compressive sampling results.

In this paper we work in the context of a reproducing kernel Hilbert spaces. In these spaces the restricted isometry property is a very convenient tool which allows one to reconstruct a signal from its sampling values. It is known that a frame which satisfies a restricted isometry property with isometry constant $\delta < 1$ act as an orthogonal basis. For this reason, one of the main interests of the present paper is to understand what properties of a sequence $\{\lambda_n\}$ of real numbers guarantee that the corresponding normalized reproducing kernels

* Corresponding author: saud@hu.edu.jo

satisfies a restricted isometry property in de Branges spaces $\mathcal{H}(E)$ of entire functions as a special class of reproducing kernel Hilbert spaces. Theory of de Branges spaces is an important branch of modern analysis having numerous interesting applications in mathematical physics, harmonic analysis and even number theory.

The problem of description of Riesz bases of normalized reproducing kernels is one of intriguing open problems in the area, results in this direction would be of interests for specialists in de Branges theory and its applications. In spite of many deep and important results, there is still no explicit description of bases in general de Branges spaces. The present paper studies stability of Riesz bases of reproducing kernels in the class of de Branges spaces with the condition that $\varphi'(x) \geq \alpha > 0$ on $\mathbb{R}$, where φ is an important characteristic of a de Branges space known as a phase function. Specifically, we are concerned with the sets of real numbers $\Lambda = \{\lambda_n\}$ such that the normalized reproducing kernels $k(\lambda_n, .)/\|k(\lambda_n, .)\|$ constitute a Riesz basis. We also prove new results on stability of reproducing kernels corresponding to real points which form a Riesz basis in $\mathcal{H}(E)$ with respect to small perturbations, which generalize some well-known Riesz basis perturbation results in the Paley-Wiener space.

In order to properly state our results, we need to review the main concepts and terminology of the theory of de Branges spaces of entire functions introduced by L. de Branges [13] in connection with inverse spectral problems for differential operators. These spaces generalize the classical Paley-Wiener space which consists of the entire functions of exponential type and square integrable on the real line. More information about these spaces can be found in [8–11].

2 Theory of de Branges spaces

In this section, we present a brief review and some relevant results on de Branges spaces theory. Assume f is an analytic function on the upper half-plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$, then f is said to be of *bounded type* in $\mathbb{C}^+$ if it can be written as a quotient of two bounded analytic functions in $\mathbb{C}^+$. The *mean type* of f in $\mathbb{C}^+$ is defined by

$$\mathrm{mt}_+(f) := \limsup_{y \to +\infty} \frac{\log |f(iy)|}{y}.$$

For an entire function f, we define the function f^* as $f^*(z) := \overline{f(\bar{z})}$. The *Hermite-Biehler* class, denoted by $\mathcal{HB}$, consists of all entire functions $E(z)$ that has no zeros in the upper half-plane and satisfies the condition

$$|E(\bar{z})| < |E(z)|, \quad \text{whenever } \Im z > 0. \tag{1}$$

Given a function $E \in \mathcal{HB}$, the associated de Branges space $\mathcal{H}(E)$ consists of all entire functions $f(z)$ such that

$$||f||_E^2 := \int_{\mathbb{R}} \left| \frac{f(t)}{E(t)} \right|^2 dt < \infty, \tag{2}$$

and $f(z)/E(z)$ and $f^*(z)/E(z)$ are of bounded type and nonpositive mean type in the upper half-plane. This is a Hilbert space with respect to the inner product

$$\langle f, g\rangle_E = \int_{\mathbb{R}} \frac{f(t)\overline{g(t)}}{|E(t)|^2}\, dt.$$

The Hilbert space $\mathcal{H}(E)$ has the special property that, for every nonreal number w, the linear functional defined on the space by $f \mapsto f(w)$ is continuous. Therefore, for every nonreal $w \in \mathbb{C}$ there exists a function $k(w, z)$ in $\mathcal{H}(E)$ such that

$$f(w) = \langle f(t), k(w, t)\rangle_E, \tag{3}$$

for every $f \in \mathcal{H}(E)$. Property (3) is known as the *reproducing kernel property*. The function $k(w, z)$ is called the *reproducing kernel* of $\mathcal{H}(E)$, which is given by (see [13, Theorem 19])

$$k(w, z) = \frac{\bar{E}(w)E(z) - E(\bar{w})E^*(z)}{2\pi i(\bar{w} - z)}. \tag{4}$$

An important feature of the de Branges space $\mathcal{H}(E)$ is the phase function corresponding to the generating function E, that is, for any entire function $E \in \mathcal{HB}$, there exists a continuous and strictly increasing function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $E(x)e^{i\varphi(x)} \in \mathbb{R}$ for all $x \in \mathbb{R}$, essentially, $\varphi = -\arg(E)$ on $\mathbb{R}$, and $E(x)$ can be written as

$$E(x) = |E(x)|e^{-i\varphi(x)}, \;\; x \in \mathbb{R}. \tag{5}$$

If a function φ has these properties then it is referred to as a *phase function* of E. It follows that a phase function of E is defined uniquely up to an additive constant, a multiple of 2π. If $\varphi(x)$ is any such function, and $E(x) \neq 0$, then using (4) and (5), an easy computation gives

$$\|k(x, .)\|^2 = k(x, x) = \frac{1}{\pi}\varphi'(x)|E(x)|^2. \tag{6}$$

The leading example of de Branges spaces is the Paley-Wiener space

$$\mathcal{H}(e^{-i\pi z}) = \mathcal{P}\mathrm{W}_\pi,$$

consists of square-integrable functions on the real line whose Fourier transforms are supported on $[-\pi, \pi]$. The reproducing kernel for $\mathcal{P}\mathrm{W}_\pi$ is $k(w, z) = \frac{\sin \pi(z-\bar{w})}{\pi(z-\bar{w})}$, $w, z \in \mathbb{C}$, $z \neq \bar{w}$, and the corresponding phase function $\varphi(x) = \pi x$.

A key feature of a de Branges space is that it always has a basis consisting of reproducing kernels corresponding to real points, [2].

Theorem 2.1. *Let $\mathcal{H}(E)$ be a de Branges space and $\varphi(x)$ be a phase function associated with E. If $\alpha \in \mathbb{R}$, and $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ is a sequence of real numbers, such that $\varphi(\lambda_n) = \alpha + \pi n$, $n \in \mathbb{Z}$, then The functions $\{k(\lambda_n, z)\}_{n\in\mathbb{Z}}$ form an orthogonal set in $\mathcal{H}(E)$.*

If $e^{i\alpha}E(z)-e^{-i\alpha}E^*(z)\notin\mathcal{H}(E)$*, then* $\left\{\frac{k(\lambda_n,z)}{\|k(\lambda_n,.)\|}\right\}_{n\in\mathbb{Z}}$ *is an orthonormal basis for* $\mathcal{H}(E)$*. Moreover, for every* $f(z)\in\mathcal{H}(E)$*,*

$$f(z)=\sum_{n\in\mathbb{Z}} f(\lambda_n)\frac{k(\lambda_n,z)}{\|k(\lambda_n,.)\|^2}\,, \tag{7}$$

and

$$\|f\|^2=\sum_{n\in\mathbb{Z}}\left|\frac{f(\lambda_n)}{E(\lambda_n)}\right|^2\frac{\pi}{\varphi'(\lambda_n)}. \tag{8}$$

A central tool in our proofs is the following Bernstein inequality in de Branges spaces introduced by A. Baranov, whose proof can be found in [2]:

Lemma 2.2. *Let* $E\in\mathcal{HB}$ *be such that* $E'/E\in\mathbb{H}^\infty(\mathbb{C}^+)$*, then*

$$\|f'/E\|_2\le C_{_{Ber}}\|f\|_E$$

for all $f\in\mathcal{H}(E)$*, where* $C_{_{Ber}}=(4+\sqrt{6})\|E'/E\|_\infty$*.*

3 Basis Theory

In this section we recall some basic concept of frames and Riesz bases for Hilbert spaces (see for example, Daubechies [7]; Duffin and Schaeffer [14]).

A family of elements $\{f_n\}_{n=1}^\infty$ in a separable Hilbert space $\mathcal{H}$ forms a frame if there exist $0<A\le B<\infty$ such that

$$A\|f\|^2\le\sum_{n=1}^\infty|\langle f,f_n\rangle|^2\le B\|f\|^2,\quad\text{for all}\ \ f\in\mathcal{H}. \tag{9}$$

The constants A,B in (9) are called the *frame bounds* for $\{f_n\}_{n=1}^\infty$. If the two frame bounds are equal we call a frame $\{f_n\}_{n=1}^\infty$ a *tight frame*. For each $f\in\mathcal{H}$ we have the *frame expansions*

$$f=\sum_{n=1}^\infty\langle f,f_n\rangle\tilde f_n=\sum_{n=1}^\infty\langle f,\tilde f_n\rangle f_n, \tag{10}$$

with unconditional convergence of these series, where $\{\tilde f_n\}$ is the dual frame of $\{f_n\}$. If, in addition to (9), $\{f_n\}_{n=1}^\infty$ is a linearly independent set, we call it a *Riesz basis* for $\mathcal{H}$. An equivalent characterization for a sequence $\{f_n\}_{n=1}^\infty$ to be a Riesz basis is that $\{f_n\}_{n=1}^\infty$ be a complete sequence in $\mathcal{H}$ and there exist positive constants A and B such that

$$A\sum_n|c_n|^2\le\left\|\sum_n c_nf_n\right\|_{\mathcal{H}}^2\le B\sum_n|c_n|^2\,, \tag{11}$$

for all finite sequences of scalars $\{c_n\}$, see [20].

If the Reisz basis is an orthogonal basis, then $A = B = 1$. Hence, a Riesz basis is automatically a frame, moreover, inequality in (9) holds with the same constants A and B as the inequality in (11). A Riesz basis $\{f_n\}_{n=1}^{\infty}$ is equivalent to an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for $\mathcal{H}$, namely, if there is a bounded invertible operator $U : \mathcal{H} \to \mathcal{H}$ such that $Uf_n = e_n$. Consequently, any Riesz basis of $\mathcal{H}$ is an unconditional basis of $\mathcal{H}$ but not conversely in general. Because of this parallelism, the Riesz bases is the appropriate framework from which to obtain nonorthogonal sampling formulas. It follows that every $f \in \mathcal{H}$ has a unique expression

$$f = \sum_n \langle f, \tilde{f}_n \rangle f_n$$

where $\tilde{f}_n = U^*Uf_n$ are the elements of the dual basis of $\{f_n\}$.

If $\mathcal{H}$ is a reproducing kernel Hilbert space, a sequence $\Lambda = \{\lambda_n\}$ is *interpolating* for $\mathcal{H}$ if there exists an $f \in \mathcal{H}$ satisfying $f(\lambda_n) = a_n$ for any choice of interpolation data $\{a_n / \|k(\lambda_n, .)\|\} \in \ell^2(\mathbb{C})$. It is *complete interpolating* if in addition f is unique. From an equivalent point of view, it is well known that a sequence Λ is an *interpolating sequence* in $\mathcal{H}$ if and only if $\{k(\lambda_n, .)/\|k(\lambda_n, .)\|\}$ is a Riesz sequence, and Λ is a *complete interpolating sequence* if and only if $\{k(\lambda_n, .)/\|k(\lambda_n, .)\|\}$ is a Riesz basis in $\mathcal{H}$, see [17] for more details and discussions.

Definition 3.1. *A sequence $\{f_n\}_{n=1}^{\infty}$ is said to have the* restricted isometry property *if there exists $\delta \in (0, 1)$ such that*

$$(1 - \delta) \sum_{n=1}^{\infty} |c_n|^2 \le \left\| \sum_{n=1}^{\infty} c_n f_n \right\|^2 \le (1 + \delta) \sum_{n=1}^{\infty} |c_n|^2, \tag{12}$$

for any sequence of scalars $\{c_n\}$, where δ is known as the isometry constant.

Although the restricted isometry property is difficult to verify, small restricted isometry constants are desired; the closed δ to zero, the closer to orthogonal basis. On the other hand, this definition in particular means that $\{f_n\}$ is a Riesz basis for its linear span. Conversely, if $\{f_n\}$ is a Riesz basis satisfying (11) then the scaled sequence $\{\sqrt{\frac{2}{B+A}} f_n\}$ satisfies (12) with $\delta = \frac{B-A}{B+A}$. In this work, we approach the problem of stability of Riesz basis of a Hilbert space $\mathcal{H}$. Specifically, given a family $\{g_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$ which is close, in some sense, to the Riesz basis (or a frame) $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{H}$, we find conditions to ensure that $\{g_n\}_{n=1}^{\infty}$ is also a Riesz basis (or a frame). This problem is important in practice, and has been studied widely by many authors in the context of bases of exponentials in L^2 on some interval. The first result due to Paley and N. Wiener [18] states that if $\{\lambda_n\}_{n\in\mathbb{Z}} \subseteq \mathbb{R}$ and $\sup_{n\in\mathbb{Z}} |\lambda_n - n| \le \delta < \frac{1}{\pi^2}$, then the set $\{e^{i\lambda_n x}\}_{n\in\mathbb{Z}}$ is a Riesz basis for the Paley-Wiener space $\mathcal{P}\mathrm{W}_\pi$ (in this cae $f_n = e^{inx}$ and $g_n = e^{i\lambda_n x}$). In [19] M. Kadec proved that the result is true for $\delta < \frac{1}{4}$, whereas the conclusion may fail if $\sup_{n\in\mathbb{Z}} |\lambda_n - n| = \frac{1}{4}$ (see [5]). Recently, some results obtained in [3] on the stability of bases and frames of reproducing kernels based on the estimates of derivatives in terms of Carleson measure in model spaces

$K^2_\Theta = \mathbb{H}^2 \ominus \Theta\mathbb{H}^2$ of the Hardy class $\mathbb{H}^2$ in the upper half plane $\mathbb{C}^+$, where Θ is an inner function in $\mathbb{C}^+$.

In the present paper we are particularly interested in the reproducing kernel Hilbert space $\mathcal{H}(E)$, we shall take for the f_n's the normalized reproducing kernel functions $\frac{k(\lambda_n,.)}{\|k(\lambda_n,.)\|}$, where $\Lambda = \{\lambda_n\}$ is a sequence of real numbers. To be exact, we are interested in stability of the basis $\frac{k(\lambda_n,.)}{\|k(\lambda_n,.)\|}$: given a Riesz basis $\frac{k(\lambda_n,.)}{\|k(\lambda_n,.)\|}$ for $\mathcal{H}(E)$ and a set of points μ_n which, in some sense, close to λ_n, whether the system $\frac{k(\mu_n,.)}{\|k(\mu_n,.)\|}$ is also a Riesz basis for $\mathcal{H}(E)$, which, as a result, leads to a Riesz basis expansion.

We will need below the following lemma which will play the key role in our proofs, see Corollary 15.1.5 in [6].

Lemma 3.1. *Let $\{f_n\}_{n=1}^\infty$ be a frame for a Hilbert space $\mathcal{H}$ with bounds A, B, and let $\{g_n\}_{n=1}^\infty$ be a sequence in $\mathcal{H}$. If there exists a constant $R < A$ such that*

$$\sum_{n=1}^{\infty} |\langle f, f_n - g_n\rangle_{\mathcal{H}}|^2 \leqslant R\,\|f\|^2_{\mathcal{H}}, \quad \forall f \in \mathcal{H},$$

then $\{g_n\}_{n=1}^\infty$ is a frame for $\mathcal{H}$ with bounds

$$A(1-\sqrt{R/A}\,)^2, B(1+\sqrt{R/B}\,)^2.$$

If $\{f_n\}_{n=1}^\infty$ is a Riesz basis, then $\{g_n\}_{n=1}^\infty$ is a Riesz basis.

4 Riesz Basis in de Branges Spaces

Given a de Branges space $\mathcal{H}(E)$ with reproducing kernel $k(w,z)$, we can assume, without loss of generality, that E has no real zeros (see [16]), hence $k(x,x) > 0$ for all $x \in \mathbb{R}$ by (6). Let $\Lambda = \{\lambda_n\}_{n=1}^\infty$ be a sequence of real numbers, from now on, we set

$$f_n(z) := \frac{k(\lambda_n, z)}{\|k(\lambda_n, .)\|}, n \in \mathbb{N}, z \in \mathbb{C}. \tag{13}$$

Definition 4.1. *Let $\Lambda = \{\lambda_n\}_{n=1}^\infty$ be a sequence of distinct points. We say that Λ is* sequentially separated *if $|\lambda_{n+1} - \lambda_n| \geq \sigma_n$, for all $n \geq 1$, and $\sigma_n \leq \sigma_{n+1}$ for all $n \geq 1$.*

Next we derive an estimate of the isometry constant δ. This estimate leads to a sufficient condition for a sequence $\{f_n\}$ to have the Restricted Isometry Property.

Lemma 4.1. *Given a de Branges space $\mathcal{H}(E)$, and $\varphi(x)$ a phase function associated with E such that $\varphi'(x) \geq \alpha > 0$ on $\mathbb{R}$. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequentially* *separated sequence of real numbers such that* $\sigma_n \geq 1$. *If* $\sum_{n=1}^{\infty} \frac{1}{\sigma_n^2} < \frac{3\alpha^2}{\pi^2}$, *then*

$$\delta := \left(\sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} |\langle f_n, f_m \rangle|^2 \right)^{\frac{1}{2}} < 1 \tag{14}$$

Proof. For any real number x, $E(x) = e^{-i\varphi(x)}|E(x)|$, which implies that $\frac{E(x)}{\overline{E}(x)} = e^{-2i\varphi(x)}$. Let $a, b \in \mathbb{R}$, then using (4) and the fact that $k(a,b) = \langle k(a,.), k(b,.) \rangle$ we get,

$$\begin{aligned} \frac{k(a,b)}{\overline{E}(a)} &= \frac{1}{\overline{E}(a)} \frac{\overline{E}(a)E(b) - E(a)\overline{E}(b)}{2\pi i(a-b)} \\ &= \frac{E(b) - e^{-2i\varphi(a)}\overline{E}(b)}{2\pi i(a-b)}. \end{aligned}$$

Simple calculations then shows that

$$\begin{aligned} \langle \frac{k(a,.)}{\overline{E}(a)}, \frac{k(b,.)}{\overline{E}(b)} \rangle &= \frac{1}{E(b)} \frac{k(a,b)}{\overline{E}(a)} \\ &= \frac{1 - e^{2i(\varphi(b) - \varphi(a))}}{2\pi i(a-b)} \end{aligned}$$

and,

$$\frac{k^2(a,b)}{|E(a)|^2|E(b)|^2} = \frac{\sin^2(\varphi(a) - \varphi(b))}{\pi^2(a-b)^2}.$$

Consequently, since $k(x,x) = \frac{1}{\pi}\varphi'(x)|E(x)|^2$ for all $x \in \mathbb{R}$, we have

$$\begin{aligned} \frac{k^2(a,b)}{k(a,a)k(b,b)} &= \pi^2 \frac{k^2(a,b)}{\varphi'(a)\varphi'(b)|E(a)|^2|E(b)|^2} \\ &= \frac{1}{\varphi'(a)\varphi'(b)} \frac{\sin^2(\varphi(a) - \varphi(b))}{(a-b)^2} \end{aligned}$$

In particular, for f_n defined in (13) we have

$$\begin{aligned} |\langle f_n, f_m \rangle|^2 &= \left| \langle \frac{k(\lambda_n,.)}{\|k(\lambda_n,.)\|}, \frac{k(\lambda_m,.)}{\|k(\lambda_m,.)\|} \rangle \right|^2 \\ &= \frac{1}{\varphi'(\lambda_m)\varphi'(\lambda_n)} \frac{\sin^2(\varphi(\lambda_m) - \varphi(\lambda_n))}{(\lambda_m - \lambda_n)^2} \\ &\leq \frac{1}{\alpha^2} \frac{1}{(\lambda_m - \lambda_n)^2} \end{aligned}$$

because $\varphi'(x) \geq \alpha$ on $\mathbb{R}$ by the hypothesis. Since $\{\lambda_n\}$ is sequentially separated and $\sigma_n \geq 1$ then for $m > n$, $m = n + k$, for some $k \geq 1$, and

$$(\lambda_m - \lambda_n) \geq (m-n)\sigma_n = k\sigma_n$$

Therefore, for any $n \geq 1$,

$$\begin{aligned}\sum_{m=n+1}^{\infty} |\langle f_n, f_m\rangle|^2 &\leq \frac{1}{\alpha^2}\sum_{m=n+1}^{\infty}\frac{1}{(\lambda_m-\lambda_n)^2}\\ &\leq \frac{1}{\alpha^2}\sum_{m=n+1}^{\infty}\frac{1}{(m-n)^2\sigma_n^2}\\ &\leq \frac{1}{\alpha^2\sigma_n^2}\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}\frac{1}{\alpha^2\sigma_n^2}.\end{aligned}$$

Consequently,

$$\sum_{\substack{m,n=1\\ m\neq n}}^{\infty} |\langle f_n, f_m\rangle|^2 = 2\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}|\langle f_n, f_m\rangle|^2 \leq \frac{\pi^2}{3\alpha^2}\sum_{n=1}^{\infty}\frac{1}{\sigma_n^2}.$$

From this the conclusion follows with $\delta < 1$. □

Next we apply the estimate obtained in Lemma 4.1 to give conditions for the sequence $\{f_n\}$ to have the Restricted Isometry Property.

Theorem 4.2. *Given a de Branges space $\mathcal{H}(E)$, and $\varphi(x)$ a phase function associated with E such that $\varphi'(x) \geq \alpha > 0$ on $\mathbb{R}$. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequentially separated sequence of real numbers such that $\sigma_n \geq 1$, $\forall n \geq 1$. If $\sum_{n=1}^{\infty}\frac{1}{\sigma_n^2} < \frac{3\alpha^2}{\pi^2}$, then the sequence $\{f_n\}_{n=1}^{\infty}$ satisfies the Restricted Isometry Property.*

Proof. From the definition of f_n, $\|f_n\| = 1$, for $n \geq 1$, then for any finite sequence of complex numbers $\{c_n\}_{n\geq 1}$ we have

$$\begin{aligned}
\Big\|\sum_{n=1}^{\infty} c_n f_n\Big\|^2 &= \sum_{m,n=1}^{\infty} c_n \bar{c}_m \langle f_n, f_m\rangle \\
&= \sum_{n=1}^{\infty} |c_n|^2 \big\|f_n\big\|^2 + \sum_{\substack{m,n=1\\ m\neq n}}^{\infty} c_n \bar{c}_m \langle f_n, f_m\rangle \\
&\leq \sum_{n=1}^{\infty} |c_n|^2 + \sum_{\substack{m,n=1\\ m\neq n}}^{\infty} |c_n \bar{c}_m \langle f_n, f_m\rangle| \\
&\leq \sum_{n=1}^{\infty} |c_n|^2 + \Bigg(\sum_{\substack{m,n=1\\ m\neq n}}^{\infty} |c_n|^2|c_m|^2\Bigg)^{\frac{1}{2}} \Bigg(\sum_{\substack{m,n=1\\ m\neq n}}^{\infty} |\langle f_n, f_m\rangle|^2\Bigg)^{\frac{1}{2}} \\
&\leq \sum_{n=1}^{\infty} |c_n|^2 + \Bigg(\sum_{n=1}^{\infty} |c_n|^2\Bigg)^{\frac{1}{2}} \Bigg(\sum_{m=1}^{\infty} |c_m|^2\Bigg)^{\frac{1}{2}} \Bigg(\sum_{\substack{m,n=1\\ m\neq n}}^{\infty} |\langle f_n, f_m\rangle|^2\Bigg)^{\frac{1}{2}} \\
&= \sum_{n=1}^{\infty} |c_n|^2 + \sum_{n=1}^{\infty} |c_n|^2 \Bigg(\sum_{\substack{m,n=1\\ m\neq n}}^{\infty} |\langle f_n, f_m\rangle|^2\Bigg)^{\frac{1}{2}} \\
&= \Bigg(1 + \Bigg(\sum_{\substack{m,n=1\\ m\neq n}}^{\infty} |\langle f_n, f_m\rangle|^2\Bigg)^{\frac{1}{2}}\Bigg) \sum_{n=1}^{\infty} |c_n|^2 \\
&= (1+\delta) \sum_{n=1}^{\infty} |c_n|^2
\end{aligned}$$

where $\Bigg(\sum\limits_{\substack{m,n=1\\ m\neq n}}^{\infty} |\langle f_n, f_m\rangle|^2\Bigg)^{\frac{1}{2}} = \delta$, by Lemma 4.1.

Similarly, we prove the first part of the inequality. We use the claim in equation (14) above, we have

$$\begin{aligned}
\Big\|\sum_{n=1}^{\infty} c_n f_n\Big\|^2 &\geq \Bigg(1 - \Bigg(\sum_{\substack{m,n=1\\ m\neq n}}^{\infty} |\langle f_n, f_m\rangle|^2\Bigg)^{\frac{1}{2}}\Bigg) \sum_{n=1}^{\infty} |c_n|^2 \\
&= (1-\delta) \sum_{n=1}^{\infty} |c_n|^2.
\end{aligned}$$

Therefore, the sequence $\{f_n\}$ satisfies the Restricted Isometry Property for some $\delta \in (0,1)$, completing the proof. □

If $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ is a given sequence, then for $\epsilon > 0$, we define a perturbation

9

sequence

$$\mathcal{M}_\epsilon := \left\{ \mu_n \in \mathbb{R} : \mu_n = \lambda_n + \epsilon_n,\, 0 < \epsilon_n \leq \epsilon\, \frac{k(\lambda_n, \lambda_n)}{\tau_n},\, n \geq 1 \right\}, \tag{15}$$

where $\tau_n = \max\limits_{t \in [\lambda_n, \lambda_{n+1}]} k(t,t)$. In what follows, the constant A_f is the lower frame bound of the sequence $\{f_n\}$ in (9) and (11), and C_{Ber} is the Berntein constant from Lemma 2.2.

Theorem 4.3. *Given a de Branges space $\mathcal{H}(E)$, such that $E'/E \in \mathbb{H}^\infty(\mathbb{C}^+)$, and $\varphi(x)$ a phase function associated with E such that $\varphi'(x) \geq \alpha > 0$ on $\mathbb{R}$. If $\{f_n\}$ is a Riesz basis in $\mathcal{H}(E)$, then the sequence $\{\frac{k(\mu_n, z)}{\|k(\lambda_n, .)\|} : \mu_n \in \mathcal{M}_\epsilon\}$ is also a Riesz basis in $\mathcal{H}(E)$ whenever $\epsilon < \frac{\alpha A_f}{\pi C^2_{Ber}}$.*

Proof. Since the function $k(t,t)$ is continuous for all $t \in \mathbb{R}$, the Mean Value Theorem implies that there exists $t_n \in (\lambda_n, \mu_n)$ such that

$$\int_{\lambda_n}^{\mu_n} \frac{k(t,t)}{k(\lambda_n, \lambda_n)}\, dt = \epsilon_n \frac{k(t_n, t_n)}{k(\lambda_n, \lambda_n)}, \text{ for all } n \geq 1.$$

Moreover, since $\mu_n \in \mathcal{M}_\epsilon$, then

$$\epsilon_n \frac{k(t_n, t_n)}{k(\lambda_n, \lambda_n)} \leq \epsilon\, \frac{k(\lambda_n, \lambda_n)}{\tau_n} \frac{k(t_n, t_n)}{k(\lambda_n, \lambda_n)} \leq \epsilon, \text{for all } n \geq 1.$$

Let $f \in \mathcal{H}(E)$, and $h_n(z) := \frac{k(\mu_n, z)}{\|k(\lambda_n, .)\|}$, for $\mu_n \in \mathcal{M}_\epsilon$. Then

$$\begin{aligned}
|\langle f, f_n - h_n \rangle|^2 &= \frac{1}{k(\lambda_n, \lambda_n)}\, |f(\lambda_n) - f(\mu_n)|^2 \\
&= \frac{1}{k(\lambda_n, \lambda_n)} \left| \int_{\lambda_n}^{\mu_n} (f(t))'\, dt \right|^2 \\
&\leq \frac{1}{k(\lambda_n, \lambda_n)} \int_{\lambda_n}^{\mu_n} \left| \frac{f'(t)}{E(t)} \right|^2\, dt \int_{\lambda_n}^{\mu_n} |E(t)|^2\, dt \\
&= \int_{\lambda_n}^{\mu_n} \left| \frac{f'(t)}{E(t)} \right|^2\, dt \int_{\lambda_n}^{\mu_n} \pi\, \frac{k(t,t)}{k(\lambda_n, \lambda_n)} \frac{1}{\varphi'(t)}\, dt \\
&\leq \frac{\pi}{\alpha} \int_{\lambda_n}^{\mu_n} \left| \frac{f'(t)}{E(t)} \right|^2\, dt \int_{\lambda_n}^{\mu_n} \frac{k(t,t)}{k(\lambda_n, \lambda_n)}\, dt \\
&\leq \frac{\pi\, \epsilon}{\alpha} \int_{\lambda_n}^{\mu_n} \left| \frac{f'(t)}{E(t)} \right|^2\, dt.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} |\langle f, f_n - h_n \rangle|^2 &\leq \frac{\pi\, \epsilon}{\alpha} \int_{\mathbb{R}} \left| \frac{f'(t)}{E(t)} \right|^2\, dt \\
&= \frac{\pi\, \epsilon}{\alpha} \|f'/E\|^2 \\
&\leq \frac{\pi\, \epsilon}{\alpha} C^2_{\text{Ber}}\, \|f\|^2,
\end{aligned}$$

10

where the last inequality follows from Lemma 2.2. Consequently, $\{h_n\}$ is a Riesz basis by Lemma 3.1 with $R = \frac{\pi \epsilon}{\alpha} C^2_{\text{Ber}} < A_f$ by the hypothesis.

□

Theorem 4.4. *Let $\mathcal{H}(E)$ be a de Branges space, with reproducing kernel function $k(w,z)$. Let $\{\lambda_n\}, \{\mu_n\}$ be two sequences of real numbers, and $\{h_n(z) := \frac{k(\mu_n,z)}{\|k(\lambda_n,.)\|}\}$ be a Riesz basis in $\mathcal{H}(E)$ with frame bounds A_h and B_h. If there exits positive constants C_1, C_2 such that*

$$C_1 k(\lambda_n, \lambda_n) \leq k(\mu_n, \mu_n) \leq C_2\, k(\lambda_n, \lambda_n), \tag{16}$$

for all $n \geq 1$, then the sequence $\{\frac{k(\mu_n,z)}{\|k(\mu_n,.)\|}\}$ is also a Riesz basis in $\mathcal{H}(E)$, whenever $CB_h < A_h$, where $C = (1 + \frac{1}{C_1} - \frac{2}{\sqrt{C_2}})$.

Proof. Since the sequence $\{h_n\}$ is a Riesz basis, then for all $f \in \mathcal{H}(E)$,

$$A_h \|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, h_n\rangle|^2 \leq B_h \|f\|^2.$$

Let $f \in \mathcal{H}(E)$, and $g_n(z) := \frac{k(\mu_n,z)}{\|k(\mu_n,.)\|}$. Then

$$\begin{aligned}
|\langle f, h_n - g_n\rangle|^2 &= \left| \frac{f(\mu_n)}{\sqrt{k(\lambda_n,\lambda_n)}} - \frac{f(\mu_n)}{\sqrt{k(\mu_n,\mu_n)}} \right|^2 \\
&= |f(\mu_n)|^2 \left| \frac{1}{\sqrt{k(\lambda_n,\lambda_n)}} - \frac{1}{\sqrt{k(\mu_n,\mu_n)}} \right|^2 \\
&= |f(\mu_n)|^2 \left| \frac{1}{k(\lambda_n,\lambda_n)} + \frac{1}{k(\mu_n,\mu_n)} - \frac{2}{\sqrt{k(\lambda_n,\lambda_n)k(\mu_n,\mu_n)}} \right| \\
&\leq R \frac{|f(\mu_n)|^2}{k(\lambda_n,\lambda_n)}
\end{aligned}$$

where $R = 1 + \frac{1}{C_1} - \frac{2}{\sqrt{C_2}}$. Thus, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} |\langle f, h_n - g_n\rangle|^2 &\leq R \sum_{n=1}^{\infty} \frac{|f(\mu_n)|^2}{k(\lambda_n,\lambda_n)} \\
&= R \sum_{n=1}^{\infty} |\langle f, h_n\rangle|^2 \\
&\leq R B_h \|f\|^2.
\end{aligned}$$

Consequently, $\{g_n\}$ is a Riesz basis by Lemma 3.1 as $RB_h < A_h$. □

Now we state the main result on stability of Riesz basis in de Branges spaces, the proof is an immediate consequence of Theorem 4.3 and Theorem 4.4.

Theorem 4.5. *Given a de Branges space $\mathcal{H}(E)$, such that $E'/E \in \mathbb{H}^\infty(\mathbb{C}^+)$, and $\varphi(x)$ a phase function associated with E such that $\varphi'(x) \geq \alpha > 0$ on $\mathbb{R}$. Let $\{f_n\}$ be a Riesz basis in $\mathcal{H}(E)$ with bounds A_f, B_f. Let $\mathcal{M}_\epsilon$ be the sequence defined in (15), and assume that there exits positive constants C_1, C_2 such that*

$$C_1 k(\lambda_n, \lambda_n) \leq k(\mu_n, \mu_n) \leq C_2\, k(\lambda_n, \lambda_n), \text{ for all } n \geq 1. \tag{17}$$

Then the sequence $\{\frac{k(\mu_n,z)}{\|k(\mu_n,.)\|} : \mu_n \in \mathcal{M}_\epsilon\}$ is also a Riesz basis in $\mathcal{H}(E)$ whenever

$$\epsilon < \frac{\alpha A_f}{\pi C^2_{_{Ber}}} \quad \text{and} \quad C\, B_f(1+\sqrt{R/B_f}\,)^2 < A_f(1-\sqrt{R/A_f}\,)^2$$

where $R = \frac{\pi\,\epsilon}{\alpha} C^2_{_{Ber}}$ and $C = (1 + \frac{1}{C_1} - \frac{2}{\sqrt{C_2}})$.

Remark 4.1. *de Branges spaces $\mathcal{H}(E)$ that satisfy the conditions of the previous theorems in general do not have simple analytic characterizations. We would like to emphasize that the best way to construct the corresponding generating functions $E \in \mathcal{HB}$ is via their Weierstrass factorization formula. A special class of Hermite-Biehler functions is the Pólya class where any function can be characterized by its Hadamard factorization formula. For the sake of completeness, we include some examples of such functions, see [1] and [13]:*

(1) Let E have the form

$$E(z) = \gamma\, e^{bz} e^{-iaz} \prod_{n\in\mathbb{Z}} \left(1 - \frac{z}{z_n}\right) e^{zRe(\frac{1}{z_n})}, \tag{18}$$

and let the zeros z_n satisfy the following conditions:

(a). $z_n = \beta n + w_n$, for all $n \in \mathbb{Z}$, where $\beta > 0$, and the sequence $\{w_n\}_{n\in\mathbb{Z}}$ is bounded,

(b). $Im(w_n) \geq \alpha > 0$.

Then $\frac{E'}{E} \in \mathbb{H}^\infty(\mathbb{C}^+)$. If, in addition, $w_n = u_n + iv_n$ where $u_n \in [\alpha_1, \alpha_2]$ and $v_n \in [a_1, a_2]$, $a_1 > 0$ for all $n \in \mathbb{Z}$, then $E'/E \in \mathbb{H}^\infty(\mathbb{C}^+)$. and $\varphi'(x)$ is bounded away from zero.

(2) Let

$$E(z) = \gamma e^{-iaz} S(z) \prod_{n=1}^{\infty} \left(1 - \frac{z}{\bar{z}_n}\right) e^{h_n z},$$

for all $z \in \mathbb{C}$, where the sequence $\{z_n\}_{n=1}^\infty \subset \mathbb{C}^+$ has no condensation points in $\mathbb{C}$ and satisfies the Blaschke condition

$$\sum_{n=1}^{\infty} y_n / \left(x_n^2 + y_n^2\right) < +\infty,$$

which guarantee the convergence of the previous product, and

$$h_n = x_n / \left(x_n^2 + y_n^2\right), \; n \in \mathbb{N},$$

$a > 0$, S is an entire function taking the real values on the real line and having only real zeros, and γ is a complex number with modulus 1. If the sequence $\{z_n\}_{n=1}^{\infty}$ is contained in the set $\Gamma_\tau = \{z \in \mathbb{C}^+ : \tau < \arg z < \pi - \tau\}$, $\tau > 0$, then $\frac{E'}{E} \in \mathbb{H}^\infty(\mathbb{C}^+)$ and $\varphi'(x)$ is bounded away from zero.

Furthermore, a wide class of de Branges spaces for which the previous theorems may be applied is the homogeneous *de Branges spaces. Such spaces are related to the classical Bessel functions and more general confluent hypergeometric functions, and were characterized by L. de Branges [12, 13]. We present a brief review of the construction of these spaces. Let $\nu > -1$. A space $\mathcal{H}(E)$ is said to be* homogeneous *of order ν if, for all $0 < a < 1$ and all $F \in \mathcal{H}(E)$, the function $z \mapsto a^{\nu+1}F(az)$ belongs to $\mathcal{H}(E)$ and has the same norm as F. For $\nu > -1$ consider the real entire functions $A_\nu(z) : \mathbb{C} \to \mathbb{C}$ and $B_\nu(z) : \mathbb{C} \to \mathbb{C}$ given by*

$$A_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}z\right)^{2n}}{n!(\nu+1)(\nu+2)\ldots(\nu+n)} = \Gamma(\nu+1)\left(\frac{1}{2}z\right)^{-\nu} J_\nu(z)$$

and

$$B_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}z\right)^{2n+1}}{n!(\nu+1)(\nu+2)\ldots(\nu+n+1)} = \Gamma(\nu+1)\left(\frac{1}{2}z\right)^{-\nu+1} J_\nu(z)$$

where

$$J_\nu(z) = \sum_{n \geq 0} \frac{(-1)^n \left(\frac{1}{2}z\right)^{2n+\nu}}{n!\Gamma(\nu+n+1)}$$

is the classical Bessel function of the first kind. These special functions have only real, simple zeros and have no common zeros. Furthermore, they satisfy the following differential equations

$$A'_\nu(z) = -B_\nu(z) \quad \text{and} \quad B'_\nu(z) = A_\nu(z) - (2\nu+1)B_\nu(z)/z. \tag{19}$$

If we define

$$E_\nu(z) := A_\nu(z) - iB_\nu(z),$$

then the function $E_\nu(z)$ is a Hermite-Biehler function with no real zeros, of bounded type in the upper-half, and is of exponential type 1 in $\mathbb{C}$. Also we have that

$$c_\nu|x|^{2\nu+1} \leq |E_\nu(x)|^{-2} \leq C_\nu|x|^{2\nu+1},$$

for all real $|x| \geq 1$ and for some $c_\nu, C_\nu > 0$, see [15]. Moreover, it is known that $A_\nu, B_\nu \notin \mathcal{H}(E_\nu)$. Note that if $\nu = -1/2$ we have $A_{-1/2}(z) = \cos z$ and

$B_{-1/2}(z) = \sin z$, hence, $E_{-1/2}(z) = e^{-iz}$ and the space $\mathcal{H}(E_{-1/2})$ coincides with the Paley-Wiener space $\mathcal{P}W_1$. By (19) we have

$$i\frac{E'_\nu(z)}{E_\nu(z)} = 1 - (2\nu+1)\frac{B_\nu(z)}{zE_\nu(z)},$$

for all $z \in \mathbb{C}^+$. Hence $E'_\nu(z)/E_\nu(z) \in H^\infty(\mathbb{C}^+)$. This also implies that the phase function $\varphi_\nu(z)$ associated with $E_\nu(z)$ satisfies

$$\varphi'_\nu(x) = 1 - \frac{(2\nu+1)A_\nu(x)B_\nu(x)}{x\,|E_\nu(x)|^2}.$$

Hence, $\varphi'_\nu(x) \simeq 1$ for all real x.

Acknowledgements

The authors would like to thank Prof. Eric S. Weber, Dept. of Mathematics, Iowa State University, for introducing the topic and for his suggestions on preparing the manuscript.

Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] A. D. Baranov, Differentiation in the Branges spaces and embedding theorems, *Journal of Mathematical Sciences*, 101(2), 2881–2913 (2000).

[2] A. D. Baranov, The Bernstein inequality in the de Branges spaces and embedding theorems, *Translations of the American Mathematical Society Series*, 2 (209), 21–50 (2003).

[3] A. D. Baranov, Stability of bases and frames of reproducing kernels in model spaces, *Ann. Inst. Fourier*, 55(7), 2399–2422 (2005).

[4] E. J. Candes, J. K. Romberg, and T. Tao, Stable signal recovery from incomplete and inaccurate measurements, *Communications on Pure and Applied Mathematics*: A Journal Issued by the Courant Institute of Mathematical Sciences, 59(8), 1207–1223 (2006).

[5] O. Christensen, Frames, Riesz bases, and discrete Gabor wavelet expansions, *Bulletin of the American Mathematical Society*, 38(3), 273–291 (2001).

[6] O. Christensen, *An introduction to frames and Riesz bases*, Applied and Numerical Harmonic Analysis, Birkhuser Boston. Inc., Boston, MA 4, 2003.

[7] I. Daubechies, *Ten lectures on wavelets*, CBMS-NSF Series in Applied Mathematics, SIAM, Philadelphia, 1992.

[8] L. de Branges, Some Hilbert spaces of entire functions, *Transactions of the American Mathematical Society*, 10, 840–846 (1959).

[9] L. de Branges, Some Hilbert spaces of entire functions, *Transactions of the American Mathematical Society*, 96(2), 259–295 (1960).

[10] L. de Branges, Some Hilbert spaces of entire functions II, *Transactions of the American Mathematical Society*, 99(1), 118–152 (1961).

[11] L. de Branges, Some Hilbert spaces of entire functions III, *Transactions of the American Mathematical Society*, 100(1), 73–115 (1961).

[12] L. de Branges, Homogeneous and periodic spaces of entire functions, *Duke Mathematical Journal*, 29(2), 203–224 (1962).

[13] L. de Branges, *Hilbert spaces of entire functions*, Prentice- Hall,Englewood Cliffs, NJ, 1968.

[14] R. J. Duffin and A. C. Schaeffer, A class of nonharmonic Fourier series, *Trans. Amer. Math. Soc.*, 72, 341–366 (1952).

[15] J. Holt, and J. D. Vaaler, The Beurling-Selberg extremal functions for a ball in Euclidean space, *Duke Math. J.*, 83(1), 203–247 (1996).

[16] M. Kaltenba and H. Woracek, Pontryagin spaces of entire functions I, *Integral Equations and Operator Theory*, 33(1), 34–97 (1999).

[17] K. N. Nikolai, *Skiı, Treatise on the shift operator*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 273, 1986.

[18] R.E.A.C. Paley, and N. Wiener, *Fourier transforms in the complex domain*, vol. 19. American Mathematical Soc., New York, 1934.

[19] M. R. Young, *An introduction to nonharmonic Fourier series*, Academic press, New York, 1980.

[20] R. Y. Young, An introduction to nonharmonic analysis, *Academic Press*, New York 21, 1139–1142 (1980).

Solving the linear moment problems for nonhomogeneous linear recursive sequences

Mohammed Mouniane[1], Mustapha Rachidi[2] and Bouazza El Wahbi[1]

[1]Laboratory of Analysis, Geometry and Applications (LAGA).
Department of Mathematics, Faculty of Sciences.
Ibn Tofail University, B.P. 133, Kenitra, Morocco
mohammed.mouniane@uit.ac.ma, bouazza.elwahbi@uit.ac.ma

[2]Institute of Mathematics - INMA,
Federal University of Mato Grosso do Sul - UFMS,
Campo Grande, MS, 79070-900, Brazil
mu.rachidi@gmail.com, mustapha.rachidi@ufms.br

August 10, 2021

Abstract

The present paper aimed to explore the linear moment problem for the real sequences defined by the nonhomogeneous linear recursive relation. Various properties are provided, especially, those related to the Hankel matrices. Some considerations in connection with K-moment problem, for the nonhomogeneous recursive are discussed.

Keywords: Linear moment problem, $K-$moment problem, Hankel matrix, nonhomogeneous linear recursive sequences.

1 Introduction

In view of its fundamental role in various fields of mathematics and applied science, the linear moment problem has been extensively studied in the literature (see [4,5,9,11–13]). Especially, it has been shown that this problem is useful for some topics in physics, such that the quantum dynamical systems, the resolvent $R_\varphi(\lambda)$ of a given Hamiltonian A, which can be written as an infinite series in terms of $1/\lambda$, whose coefficients are the moment $\mu_n = \langle\varphi|A^n|\varphi\rangle$ of order n of the operator A, where φ is a state vector of the given system (see [4,12] for example). Furthermore, the linear moment problem is also related to the Lanczos numerical method, which is an important technique for finding the positions of n particles such that the first $2n-1$ moments own given values (see [5,13] for example).

1

Recently, the linear moment problem has been investigated in the literature, by various methods (see, for example, [4, 9, 11, 12]).

The linear moment problem is simple to formulate. Indeed, let $\mathcal{H}$ be a real separable Hilbert space, $\mathcal{L}(\mathcal{H})$ be the space of linear operators on $\mathcal{H}$ and $\mathcal{S}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ the subspace of self adjoint operators on $\mathcal{H}$. For a given operator $A \in \mathcal{L}(\mathcal{H})$ and non-vanishing $x \in \mathcal{H}$, the sequence $\Gamma = \{\alpha_n\}_{n\geq 0}$ defined by $\alpha_n = \langle A^n x | x\rangle$ for $n \geq 0$, is called the moment sequence of A on x, and α_n is the moment of order n of the operator A on x. The linear moment problem is the reciprocal of the previous situation. More precisely, let $\Gamma = \{\alpha_n\}_{0\leq n\leq p}$ ($p \leq +\infty$) be a sequence of real numbers, the linear moment problem associated with Γ consists to find a self-adjoint operator $A \in \mathcal{S}(\mathcal{H})$ and a non-vanishing vector $x \in \mathcal{H}$ such that,

$$\alpha_n = \langle A^n x | x\rangle, \quad \text{for } \ 0 \leq n \leq p. \tag{1}$$

The problem (1) is called the *full linear moment problem* when $p = +\infty$ and the *truncated linear moment problem* for $p < +\infty$ (see [7–9, 12], for example).

On the other hand, the linear moment problem (1) for the sequence Γ, is also related to the classical power K-moment problem (K is a closed set of $\mathbb{R}$), whose aim is to find a positive Borelean measure μ with $\operatorname{supp}(\mu) \subset K$ such that

$$\alpha_n = \int_K t^n d\mu(t), \quad \text{for } \ 0 \leq n \leq p, \tag{2}$$

where $p \leq +\infty$. The moment problem (2) is important in operator theory, particularly, it is related to the study of the shift of subnormal operators and subnormal extension (see [1, 3, 6–8]). Recently, the two preceding moment problems (1) and (2) have been studied in [3, 9–11], for some sequences defined by linear recursive relations. Moreover, it was established the closed connection between the full and the truncated moment problem for recursive sequences in [9, 11]. More precisely, let $\{u_n\}_{n\geq 0}$ be the sequence satisfying the following linear recursive relation of order r,

$$u_{n+1} = a_0 u_n + a_1 u_{n-1} + \cdots + a_{r-1} u_{n-r+1} \text{ for } n \geq r-1, \tag{3}$$

where $u_0, u_1, \ldots, u_{r-1}$ are the initial data, it was shown in [9–11] that, for the linear moment problems (1), the full one ($p = +\infty$) and the truncated one ($p < +\infty$) are closely related. Especially, it was shown in [9] that in the finite dimensional case ($\dim_{\mathbb{R}} \mathcal{H} < +\infty$), the two preceding linear moment problems (the full and the truncated) are identical. On the other side, it was shown in [9] that the full and truncated moment problem (2), for the recursive sequence (3), are equivalent.

The purpose of this paper is to study the linear moment problem (1), for a real non-homogeneous recursive sequence $\{v_n\}_{n\geq 0}$ of order r, defined by the following recursive relation,

$$v_{n+1} = a_0 v_n + a_1 v_{n-1} + \cdots + a_{r-1} v_{n-r+1} + c_{n+1} \text{ for } n \geq r-1, \tag{4}$$

where the coefficients $a_0, \ldots, a_{r-1}$ ($r \geq 2$, $a_{r-1} \neq 0$) are real numbers, $v_0 = \alpha_0, \ldots, v_{r-1} = \alpha_{r-1}$ are the initial values, and $\mathcal{C} = \{c_n\}_{n \geq r}$ is a (non trivial) real sequence. It seems to us that properties of the linear moment problem (1) for nonhomogeneous sequences (4), can be useful for the study of certain related perturbed physical systems. For the K-moment problem (2), it can be also, for studying the perturbed moment, of the shift of operators.

In this study, we characterize the solution of the linear moment problem (1) for sequences (4) in the general setting, especially, when the operator $A \in \mathcal{S}(\mathcal{H})$, namely, A is self-adjoint. When the real separable Hilbert space $\mathcal{H}$ is of finite dimension and the non-homogeneous sequence $\{v_n\}_{n \geq 0}$ is a moment sequence of an operator A, on a non-vanishing $x \in \mathcal{H}$, we establish that the sequence $\{c_n\}_{n \geq r}$ is a linear recursive sequence of type (3). And when the real separable Hilbert space $\mathcal{H}$ is of infinite dimension and the non-homogeneous sequence $\{v_n\}_{n \geq 0}$ is a moment sequence of an operator A, on a non-vanishing $x \in \mathcal{H}$, then the general term of the sequence $\{c_n\}_{n \geq r}$, is expressed as a limit of $c_n = \lim_{s \to +\infty} c_n^{(s)}$, where $c_n^{(s)}$ is a linear recursive sequence of type (3). We establish the solution of the linear moment problem (1), using the properties of the Hankel matrices. The special case when $\{c_n\}_{n \geq r}$ is a linear recursive sequence of type (3), is discussed. Moreover, the K-moment problem (2) for nonhomogeneous recursive sequences (4) is provided, using the spectral measures of self-adjoint operators. By the way, some other consequences are derived, especially, the Stieltjes and Hamburger moment problems (2), for the nonhomogeneous recursive sequences (4), are discussed through the spectral measures of self-adjoint operators. It should be noted that the study of these two problems for the sequences (4), is not common in the literature.

2 Linear moment problem and sequences (4)

Let improve the connections between solutions of (4) considered as a difference equation and the linear moment problem (1). Let $\{Q_n\}_{n \geq r}$ be the family of polynomials defined by $Q_n(z) = z^{n-r}P(z)$, where $P(z) = z^r - a_0 z^{r-1} - a_1 z^{r-2} - \cdots - a_{r-1}$, is the so-called characteristic polynomial of the homogeneous part of the sequence (4). Let $x \neq 0$ be an element of $\mathcal{H}$ and $A \in \mathcal{S}(\mathcal{H})$. Suppose that $v_n = \langle A^n x | x \rangle$, for every $n \geq 0$. Then, we have, $\langle A^{n+1} x | x \rangle = a_0 \langle A^n x | x \rangle + \cdots + a_{r-1} \langle A^{n-r+1} x | x \rangle + c_{n+1}$, for every $n \geq r-1$. Therefore, we derive $c_{n+1} = \langle Q_{n+1}(A) x | x \rangle$, for every $n \geq r-1$. Consequently, we can state the following proposition.

Proposition 2.1. *Let $\mathcal{T} = \{v_n\}_{n \geq 0}$ be a sequence (4), of characteristic polynomial $P(z) = z^r - a_0 z^{r-1} - a_1 z^{r-2} - \cdots - a_{r-1}$. Suppose that $\mathcal{T} = \{v_n\}_{n \geq 0}$, is a moment sequence of an operator $A \in \mathcal{S}(\mathcal{H})$, namely, $v_n = \langle A^n x | x \rangle$, for every $n \geq 0$, where $x \neq 0$. Then, the sequence $\{c_n\}_{n \geq r}$ is given by $c_{n+1} = \langle Q_{n+1}(A) x | x \rangle$, for every $n \geq r-1$, where $Q_n(z) = z^{n-r}P(z)$.*

Therefore, the question of studying the converse of the preceding affirmation of Proposition 2.1 arises.

Theorem 2.2. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence (4), of characteristic polynomial $P(z) = z^r - a_0 z^{r-1} - a_1 z^{r-2} - \cdots - a_{r-1}$. Let $A \in \mathcal{S}(\mathcal{H})$ and $x \neq 0 \in \mathcal{H}$. Then, we have $v_n = \langle A^n x|x\rangle$, for every $n \geq 0$, if and only if, $v_n = \langle A^n x|x\rangle$ for $n = 0, 1, \ldots, r-1$ and $c_n = \langle A^{n-r} P(A)x|x\rangle$, for $n \geq r$.*

Proof. Suppose $v_n = \langle A^n x|x\rangle$ $(n \geq 0)$, for some $x \neq 0$ in $\mathcal{H}$ and $A \in \mathcal{S}(\mathcal{H})$. Then, we have $c_k = v_k - \sum_{j=0}^{r-1} a_j v_{k-j-1} = \left\langle (A^k - \sum_{j=0}^{r-1} a_j A^{k-j-1})x|x \right\rangle = \langle A^{k-r} P(A)x|x\rangle$, for every $k \geq r$. Conversely, suppose that $v_n = \langle A^n x|x\rangle$, for $n = 0, 1, \ldots, r-1$ and $c_n = \langle A^{n-r} P(A)x|x\rangle$ for every $n \geq r$. Therefore, we have

$$v_r = \sum_{j=0}^{r-1} a_j \langle A^{r-j-1} x|x\rangle + \langle P(A)x|x\rangle = \langle A^r x|x\rangle.$$

And, by induction, we derive that $v_n = \langle A^n x|x\rangle$, for every $n \geq 0$. □

As a consequence of Theorem 2.2, we obtain the following corollary.

Corollary 2.3. *Let $A \in \mathcal{S}(\mathcal{H})$ and $x \in \mathcal{H}$, then under the data of Theorem 2.2, the following statements are equivalent,*

(i) $v_n = \langle A^n x|x\rangle$, *for every* $n \geq 0$.

(ii) $v_n = \langle A^n x|x\rangle$, *for* $n = 0, 1, \ldots, 2r-1$, *and* $c_n = \sum_{j=0}^{r-1} a_j c_{n-j-1} + \langle A^{n-2r} z|z\rangle$ *for every* $n \geq 2r$, *where* $z = P(A)x$.

Proof. It suffices to establish the equivalence between (ii) and the second statement of Theorem 2.2. Let A be a self-adjoint operator, suppose that $v_n = \langle A^n x|x\rangle$ for $n = 0, 1, \ldots, r-1$ and $c_n = \langle A^{n-r} P(A)x|x\rangle$, for every $n \geq r$. Then, for $z = P(A)x$, we have, $\langle A^{n-2r} z|z\rangle = \langle A^{n-r} x|P(A)x\rangle - \sum_{j=0}^{r-1} a_j \langle A^{n-r-j-1} P(A)x|x\rangle = c_n - \sum_{j=0}^{r-1} a_j c_{n-j-1}$, for any $n \geq 2r$. Conversely, suppose that (ii) holds. A direct computation shows that $c_n = \langle A^{n-r} P(A)x|x\rangle$, for $n = r, r+1, \ldots, 2r-1$. On the other hand, by induction we prove that $c_n = \langle A^{n-r} P(A)x|x\rangle$, for every $n \geq 2r$. It follows that (i) and (ii) are equivalent. □

We conclude this section by the following observation. Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence (4), whose characteristic polynomial is $P(z) = z^r - a_0 z^{r-1} - a_1 z^{r-2} - \cdots - a_{r-1}$. Suppose that there exist $A \in \mathcal{S}(\mathcal{H})$ and $x \in \mathcal{H}$ such that $v_n = \langle A^n x|x\rangle$. Then, we have, $c_{2k} - \sum_{j=0}^{r-1} a_j c_{2k-j-1} = \|A^{k-r} P(A)x\|^2$ for every $k \geq r$. Therefore, when $c_{2k} \neq 0$, for some $k \in \mathbb{N}$, we have $c_{2k} > \sum_{j=0}^{r-1} a_j c_{2k-j-1}$, for any $k \geq r$. This later inequality is a necessary condition for the existence of the solution of the linear moment problem (1), for the sequence $\mathcal{T} = \{v_n\}_{n\geq 0}$ defined by (4).

4

3 The linear moment problem (1) for sequences (4)

Let $\mathcal{H}$ be a finite dimensional Hilbert space over $\mathbb{R}$ ($m = \dim_{\mathbb{R}} \mathcal{H}$) and $\mathcal{T} = \{v_n\}_{n\geq 0}$ a sequence (4). A straightforward computation and by using Theorem 2.2, allows us to see that $\mathcal{T} = \{v_n\}_{n\geq 0}$ is a moment sequences of a self-adjoint operator A on a non-vanishing vector x of $\mathcal{H}$ if and only if $v_n = \sum_{j=1}^{s} {\lambda_j}^n \|x_j\|^2$ for $n = 0, 1, \ldots, r-1$ and

$$c_n = \sum_{j=1}^{s} \frac{P(\lambda_j)}{\lambda_j^r} \|x_j\|^2 {\lambda_j}^n, \tag{5}$$

where $x_j = \Pi_j x \in \mathcal{H}_j$ ($0 \leq j \leq s$), the subspace of the eigenvectors of A, corresponding to the eigenvalues λ_j ($0 \leq j \leq s$). Expression (5) is nothing else but the analytic formula of the sequence $\{c_n\}_{n\geq r}$, viewed as a linear recursive sequence of type (3) of order s. More precisely, (5) implies that $\{c_n\}_{n\geq r}$ is a linear recursive sequence of type (3), of characteristic polynomial $K(z) = \prod_{j=1}^{s}(z - \lambda_j)$. Thus, we can state the following proposition.

Proposition 3.1. *Let $\mathcal{T}$ be a sequence* (4). *Suppose that $\mathcal{T}$ is a moment sequences of a self-adjoint operator A on the finite dimensional Hilbert space $\mathcal{H}$. Then, the nonhomogeneous part $\mathcal{C}$ is a linear recursive sequence of type* (3) *of order s (with $s \leq \dim \mathcal{H}$). More precisely, the characteristic polynomial of $\mathcal{C}$ is $K(z) = \prod_{j=1}^{s}(z - \lambda_j)$, where the λ_j ($0 \leq j \leq s$) are the eigenvalues of A.*

Suppose that $\mathcal{H}$ is a separable real Hilbert space (over $\mathbb{C}$) of infinite dimension. The simplest spectral theorem (after the algebraic case) concerns a compact self-adjoint and a compact normal operator A on $\mathcal{H}$, and asserts that $\mathcal{H}$ coincide with the closure of the orthogonal sum of the eigenspaces $\mathcal{H}_n$, corresponding to all possible eigenvalues $\{\lambda_n\}_{n\geq 0}$. With a view to generalization it is convenient to express it under the spectral resolution form $Ax = \sum_{n=0}^{+\infty} \lambda_n \Pi_n x$, where Π_n is an orthoprojection onto $\mathcal{H}_n$, the eigenspace corresponding to the eigenvalue λ_j, and $x = \sum_{n=0}^{+\infty} \Pi_n x$. We consider the class of operators satisfying the Spectral Theorem, which are called spectral operators or S-operators for short.

Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence (4), with characteristic polynomial P. Suppose that $\mathcal{T}$ is a sequence of moments of an S-operator A of $\mathcal{L}(\mathcal{H})$, on a non-vanishing vector $x \in \mathcal{H}$, namely, $v_n = \langle A^n x | x\rangle$, for every $n \geq 0$, where A is an S-operator and $x = \sum_{n=0}^{+\infty} \Pi_n x \in \mathcal{H}$.

Let $s \geq 1$ and consider the sequence $\{v_n^{(s)}\}_{n\geq 0}$ defined as follows: $v_j^{(s)} = v_j$

5

for $i = 0, 1, \dots, r-1$, and

$$v_{n+1}^{(s)} = a_0 v_n^{(s)} + a_1 v_{n-1}^{(s)} + \cdots + a_{r-1} v_{n-r+1}^{(s)} + c_{n+1}^{(s)}, \tag{6}$$

for $n \geq r-1$, where $c_n^{(s)} = \sum_{p=0}^{s} \frac{P(\lambda_p)}{\lambda_p^r} \|x_p\|^2 {\lambda_p}^n$. It is easy to see that $c_n = \lim_{s \to +\infty} c_n^{(s)}$. For $n = r$, expression (6) shows that we have $v_r = \lim_{s \to +\infty} v_r^{(s)}$. By induction on n, we have $v_n = \lim_{s \to +\infty} v_n^{(s)}$, for every $n \geq r$. In conclusion, we have the following result.

Theorem 3.2. *Let $\mathcal{T} = \{v_n\}_{n \geq 0}$ be a sequence* (4)*, with characteristic polynomial P. Suppose the Hilbert space $\mathcal{H}$ is of infinite dimension and that $\mathcal{T}$ is a moment sequences of an S-operator A on $\mathcal{H}$, on a non-vanishing vector $x = \sum_{n=0}^{+\infty} \Pi_n x$. Then, we have $v_n = \lim_{s \to +\infty} v_n^{(s)}$, for every $n \geq r$, where $\{v_n^{(s)}\}_{n \geq 0}$ is a sequence* (4)*, whose associate nonhomogeneous term is*

$$c_n^{(s)} = \sum_{p=0}^{s} \frac{P(\lambda_p)}{\lambda_p^r} \|x_p\|^2 {\lambda_p}^n, \tag{7}$$

where $P(z) = z^r - a_0 z^{r-1} - a_1 z^{r-2} - \cdots - a_{r-1}$ $(a_{r-1} \neq 0)$ is the characteristic polynomial of $\mathcal{T}$ and $x_p = \Pi_p x \in \mathcal{H}$. Moreover, expression (7) *stands for the analytic formula of the sequence $\{c_n^{(s)}\}_{n \geq 0}$, viewed as a linear recursive sequence of type* (3)*.*

From Theorem 3.2, we derive that

$$c_n = \sum_{p=0}^{+\infty} \frac{P(\lambda_p)}{\lambda_p^r} \|x_p\|^2 {\lambda_p}^n. \tag{8}$$

Remark 3.3. If there exists $s \geq 1$ such that $\lambda_p = 0$, for every $p \geq s+1$, we show that expressions (5) and (8) are identical. Suppose that for every $N > 0$ there exists $k \geq N$ such that $\lambda_k \neq 0$. Therefore, expression (8) doesn't represent a recursive sequence of finite order. Meanwhile, we can approximate this situation by a family of sequences (4), whose associated c_n is given by expression (7).

4 Hankel matrices and solution of the linear moment problem (1)

In this section, we present algebraic treatment of the Hankel matrix related to the sequences defined by (4), and its use for characterizing the existence of solutions for the linear moment problem (1).

Let H_k be the Hankel matrix of size $k+1$, whose entries are defined from the elements of the sequence $\mathcal{T} = \{v_i\}_{i \geq 0}$, in the sense that $H_k := (v_{i+j})_{0 \leq i,j \leq k}$.

The j^{th} column of H_k will be denoted by $\mathbf{V}_j := (v_{j+\ell})_{\ell=0}^{k}, 0 \leq j \leq k$, so that H_k can be briefly written as $H_k = (\mathbf{V}_0 \ \ \mathbf{V}_1 \ \ \cdots \ \ \mathbf{V}_k)$. Observe that we can verify that

$$\mathbf{V}_{r+k} = a_0 \mathbf{V}_{r+k-1} + a_1 \mathbf{V}_{r+k-2} + \cdots + a_{r-1} \mathbf{V}_k + \widehat{\mathbf{C}}_{r+k}, \tag{9}$$

where $\widehat{\mathbf{C}}_{r+k} := (c_{r+\ell})_{\ell=0}^{r+k-1}$.
With a vectorial representation, we can write the matrix H_{r+n} as follows

$$H_{r+n} = \begin{pmatrix} \mathbf{V}_0 & \mathbf{V}_1 & \cdots & \mathbf{V}_{r-1} & | & \mathbf{V}_r & \cdots & \mathbf{V}_{r+k} & \cdots & \mathbf{V}_{r+n-1} \end{pmatrix}.$$

Using expression (9) and some computational techniques emanated from determinant properties, we get,

$$\det H_{r+n} = \det \begin{pmatrix} \mathbf{V}_0 & \mathbf{V}_1 & \cdots & \mathbf{V}_{r-1} & | & \widehat{\mathbf{C}}_r & \cdots & \widehat{\mathbf{C}}_{r+k} & \cdots & \widehat{\mathbf{C}}_{r+n-1} \end{pmatrix}.$$

Repeating the same treatment on the matrix $S_k := (v_{i+j+1})_{0 \leq i,j \leq k}$, one gets out of it by the following result.

Proposition 4.1. *Let $\mathcal{T} = \{v_n\}_{n \geq 0}$ be a sequence (4),*

$$H_{r+n} = (v_{i+j})_{0 \leq i,j \leq r+n-1} \ \textit{and} \ S_{r+n} = (v_{i+j+1})_{0 \leq i,j \leq r+n-1}$$

be the Hankel matrices associated with $\mathcal{T}$. Then, we have

$$\det H_{r+n} = \begin{vmatrix} v_0 & \cdots & v_{r-1} & c_r & \cdots & c_{r+n-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_{r-1} & \cdots & v_{2r-2} & c_{2r-1} & \cdots & c_{2r+n-2} \\ v_r & \cdots & v_{2r-1} & c_{2r} & \cdots & c_{2r+n-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_{r+n-1} & \cdots & v_{2r+n-2} & c_{2r+n-1} & \cdots & c_{2r+2n-2} \end{vmatrix} \tag{10}$$

and

$$\det S_{r+n} = \begin{vmatrix} v_1 & \cdots & v_r & c_{r+1} & \cdots & c_{r+n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_r & \cdots & v_{2r-1} & c_{2r} & \cdots & c_{2r+n-1} \\ v_{r+1} & \cdots & v_{2r} & c_{2r+1} & \cdots & c_{2r+n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ v_{r+n} & \cdots & v_{2r+n-1} & c_{2r+n} & \cdots & c_{2r+2n-1} \end{vmatrix}. \tag{11}$$

Expression (10) shows that, for $n \geq 0$, it appears only the columns which depend on the entries of the sequence $\{c_n\}_{n \geq r}$ after the r-th column, in the determinant of the Hankel matrix H_{r+n}. A similar situation is observed for the matrix $S_k = (v_{i+j+1})_{0 \leq i,j \leq k}$.

If the sequence $\mathcal{C} = \{c_n\}_{n \geq r}$ is also of type (3) of order s, then the $r+s-th$ column of the matrix H_{r+n} is a linear combination of the columns $r, r+1, \ldots, r+s-1$, and the $r+s+1-th$ column of the matrix S_{r+n} is a linear combination of the columns $r+1, r+2, \ldots, r+s$. Therefore, by Proposition 4.1, we get the following property.

Proposition 4.2. *If the sequence $\{c_n\}_{n\geq r}$ is also a linear recursive sequence of type* (3) *of order s, then we have,*

1. $\det H_{r+n} = 0$*, for $n \geq s$, if and only if, the $r+s+1$-column of the matrix H_{r+n} is a linear combination of the previous s columns, namely, the $r, r+1, \ldots, r+s-1$ columns of the matrix H_{r+n}.*

2. $\det S_{r+n} = 0$*, for $n \geq s+1$, if and only if, the $s+1$-column of the matrix S_{r+n} is a linear combination of the previous s columns, namely, the $r+1, r+2, \ldots, r+s$ columns of the matrix H_{r+n}.*

The two Hankel matrices $H_{r+n} = (v_{i+j})_{0\leq i,j\leq r+n-1}$ and $S_{r+n} = (v_{i+j+1})_{0\leq i,j\leq r+n-1}$ and their determinants (10)-(11), play a central role for solving the two moment problems (1)-(2) and their applications.
We recall that it was established in [12, Lemma 1.1] that a $N \times N$ Hermitean matrix A is strictly positive definite if and only if each sub-matrix $A_k = (a_{ij})_{1\leq i,j\leq k}$ has $\det(A_k) > 0$, for $k = 1, 2, \ldots, N$. For a given Hankel matrix $H = (m_{i+j})_{i,j\geq 0}$, we consider the family of sub-matrices $H_n = (m_{i+j})_{0\leq i,j\leq n}$. Then, [12, Proposition 1.2] shows that for a Hankel matrix the family of sesquilinear form $\mathcal{F} = \{\mathtt{H}_n\}_{n\geq 0}$, defined by $\mathtt{H}_n(\alpha, \beta) = \sum_{j,k=0}^{n} m_{j+k}\alpha_j\bar{\beta}_k$, is (strictly) positive definite if and only if $\det(H_n) > 0$, where $H_n = (m_{i+j})_{0\leq i,j\leq n}$. Equivalently, we say that the Hankel matrix $H_n = (m_{i+j})_{0\leq i,j\leq n}$ is positive definite if and only if $\det(H_n) > 0$, where $H_n = (m_{i+j})_{0\leq i,j\leq n}$.

In order to establish the existence of solution of the linear moment problem (1), we will present a result of the closed relation between Hankel positive matrix, self-adjoint operator and measure. More precisely, we recall that from [6] the following theorem.

Theorem 4.3. *If $\{v_n\}_{n\geq 0}$ is a sequence of real numbers, the following statements are equivalent.*

(a) There is a self-adjoint operator A and a vector e such that $e \in \operatorname{dom} A^n$ for all n and $v_n = \langle A^n e, e\rangle$, for all $n \geq 0$.

(b) If $\alpha = (\alpha_0, \ldots, \alpha_n)$, where $\alpha_j \in \mathbb{C}$, then we have $\displaystyle\sum_{j,k=0}^{n} m_{j+k}\alpha_j\bar{\alpha}_k \geq 0$, for every $n \geq 0$.

(c) There is a positive regular Borelean measure μ on $\mathbb{R}$ such that $\displaystyle\int |t|^n d\mu(t) < \infty$ for all $n \geq 0$ and $v_n = \displaystyle\int t^n d\mu(t)$.

Therefore, for the Hankel matrix $H = (m_{i+j})_{i,j\geq 0}$, the second assertion of Theorem 4.3, implies that the sesquilinear form defined by $\mathtt{H}_n(\alpha, \beta) = \sum_{j,k=0}^{n} m_{j+k}\alpha_j\bar{\beta}_k$, is a (strictly) positive definite form if and only if the matrix $H_n = (m_{i+j})_{0\leq i,j\leq n}$ is (strictly) positive definite, for every $n \geq 0$. Equivalently, the second assertion

8

of Theorem 4.3, shows that the Hankel matrix $H = (m_{i+j})_{i,j\geq 0}$ is positive, or in an equivalent way, $\det H_n \geq 0$, for every $n \geq 0$, where $H_n = (m_{i+j})_{0\leq i,j\leq n}$.

Combining Proposition 4.1 and Theorem 4.3, we can formulate the following result.

Theorem 4.4. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence* (4). *Then, the following assertions are equivalent,*

1. *The linear moment problem* (1) *for sequence* (4) *owns a solution.*
2. *The Hankel matrix $H = (v_{i+j})_{i,j\geq 0}$ is positive.*
3. $\det H_n \geq 0$, *for every $0 \leq n \leq r-1$ and* $\det H_{n+r} \geq 0$, *for every $n \geq 0$, where* $\det H_{n+r}$ *is given by* (10).

Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence (4) and suppose that the associated nonhomogeneous part $\mathcal{C} = \{c_n\}_{n\geq r}$ is a sequence of type (3) of order s, whose characteristic polynomial is $Q(z) = z^s - b_0 z^{s-1} - b_1 z^{s-2} - \cdots - b_{s-1}$. Let $R(z) = z^r - a_0 z^{r-1} - a_1 z^{r-2} - \cdots - a_{r-1}$ be the characteristic polynomial of the homogeneous part of (4). The linearization process of [2, Theorem 2.1 (Linearization Process)] applied to the sequence (4), allows us to show that $\mathcal{T} = \{v_n\}_{n\geq 0}$ is a sequence of type (3) of order $r+s$, with initial data $v_0, v_1, \ldots, v_{r+s-1}$ and whose coefficients $c_0, c_1, \ldots, c_{r+s}$ are obtained from its characteristic polynomial given by $P(z) = Q(z)R(z)$. Therefore, following Proposition 4.2, we get the following property.

Proposition 4.5. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence* (4) *and $H_{r+n} = (v_{i+j})_{0\leq i,j\leq r+n-1}$ its associated Hankel matrices of order $r+n$. Suppose that $\mathcal{C}$ is a sequence of type* (3) *of order s. Then, we have* $\det H_{r+n} = 0$, *for every $n \geq s$.*

On the other hand, let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$ be a solution of the linear moment problem (1) on a vector on a non-vanishing $x \in \mathcal{H}$, associated with the sequence $\mathcal{T} = \{v_n\}_{n\geq 0}$ defined by (4). By the linear recursive relation (3), related to the linearized expression of (4), we have $\langle A^n P(A)x|x\rangle = \langle P(A)x|A^n x\rangle = 0$, for every $n \geq 0$, where $P(z) = Q(z)R(z)$ is the characteristic polynomial of the linearized sequence of (4). Therefore, we have $\langle A^n P(A)x|A^m P(A)x\rangle = 0$, for every $n \geq 0$, $m \geq 0$, especially $\|A^n P(A)x\| = 0$, for every $n \geq 0$. This implies that $A^n x$ is a linear combination of x, Ax, $\ldots$, $A^{r+s-1}x$. Therefore, when the nonhomogeneous part $\mathcal{C}$ is an $s-GFS$, if the linear moment problem owns a solution A, a self-adjoint operator on a Hilbert space $\mathcal{H}$, then it has a solution A on some $r+s$-dimensional Hilbert space (for more details see [11, Proposition 2.2]). This allows us to suppose that the Hilbert space $\mathcal{H}$ is of finite dimension $(r+s)$. Therefore, we have the following result.

Proposition 4.6. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence* (4), *with positive definite associated Hankel matrix H_r, and let $P(z)$ the characteristic polynomial of its homogeneous part. Suppose that $\mathcal{C}$ is a linear recursive sequence of type* (3) *of order s, whose characteristic polynomial is $Q(z)$. Then, there exists a* $(\deg(P)+\deg(Q))$*-dimensional Hilbert space $\mathcal{H}_{(\mathcal{T})}$ and a self-adjoint operator A on $\mathcal{H}_{(\mathcal{T})}$, solution of the moment problem* (1).

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 30, NO.1, 2022, COPYRIGHT 2022 EUDOXUS PRESS, LLC

Proposition 4.6 shows the main role of the recursiveness of the sequence $\{c_n\}_{n\geq 0}$, in reducing the study of the linear moment problem (1) to the finite dimensional Hilbert space $\mathcal{H}$.

5 Some considerations on the K-moment problems (2) for sequences (4)

The aim here is to apply results of the preceding sections for solving the K-moment problem (2) for nonhomogeneous recursive sequences (4), using results of the linear moments problems in Hilbert spaces $\mathcal{H}$. More precisely, the solution of K-moment problem (2) is obtained in terms of representing measure of the self-adjoint operator A and the vector $x \in \mathcal{H}$ solution of the linear moment problem (1), for the nonhomogeneous recursive sequences (4). The Stieltjes and Hamburger moment problems for the nonhomogeneous recursive sequences (4) are discussed.

5.1 $K-$moment problems associated with sequences (4)

Recall that the purpose of the $K-$moment problem associated with a given sequence $\mathcal{T} = \{v_n\}_{0\leq n\leq p}$, where K is a closed subset of $\mathbb{R}$, is to find a positive Borel measure μ such that Expression (2) is verified, namely,

$$v_n = \int_K t^n d\mu(t) \qquad \text{and} \qquad \text{supp}(\mu) \subset K.$$

As mentioned above, the problem (2) has been studied in the literature, by various methods and techniques. It is called the *full moment problem* when $p = +\infty$ and the *truncated moment problem*, for $p < +\infty$ (see [7–9]). Using the spectral representation of the self-adjoint operators, we can show that the linear moment problem (1) and the moment problem (5.1) are equivalent (see for example [6]). Moreover, using Theorem 4.3 and Theorem 4.4, we get,

Theorem 5.1. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence (4). Suppose that the Hankel matrix $H = (v_{i+j})_{i,j\geq 0}$ is positive. Then, there exists a positive Borel measure μ such that*

$$v_n = \int_K t^n d\mu(t),$$

where $K = \text{supp}(\mu)$. Namely, the there exists a positive Borel measure μ solution of the K-moment problem (2).

Now consider the moment problem (2) for a sequence $\mathcal{T} = \{v_n\}_{n\geq 0}$ given by (4). Let μ be a positive Borel measure of support K. Then, following the proof of Theorem 2.2, we have $v_n = \displaystyle\int_K t^n d\mu(t)$ for every $n \geq 0$, if and only if, $v_n = \displaystyle\int_K t^n d\mu(t)$ for any $n = 0, \ldots, r-1$ and $c_n = \displaystyle\int_K t^{n-r} P(t) d\mu(t)$ for $n \geq r$, where

$K = \text{supp}(\mu)$. Moreover, a direct computation allows us to get the following result.

Proposition 5.2. *Under the preceding data, the following assertions are equivalent.*

(i) $v_n = \int_K t^n d\mu(t)$, *for every* $n \geq 0$, *where* $K = \text{supp}(\mu)$.

(ii) $v_n = \int_K t^n d\mu(t)$ *for* $n = 0, \ldots, 2r-1$ *and* $c_n - \sum_{j=0}^{r-1} a_j c_{n-j-1} = \int_K t^{n-2r} P(t)^2 d\mu(t)$, *for every* $n \geq 2r$, *where* $K = \text{supp}(\mu)$.

It is easy to show that the second assertion of the Proposition 5.2 implies that $c_{2k} - \sum_{j=0}^{r-1} a_j c_{2k-j-1} = \int [t^{k-r}P(t)]^2 d\mu(t)$, for any $k \geq r$, and if there exists $k_0 \geq r$ such that $c_{2k_0} - \sum_{j=0}^{r-1} a_j c_{2k_0-j-1} = 0$, then $\text{supp}(\mu) \subset \mathcal{Z}(P) \cup \{0\}$ or equivalently the sequence $\mathcal{T}$ is an $r - GFS$, in which case the sequence $\mathcal{C}$ vanish. This allows us to give a necessary condition for a sequence (4) to be a moment sequences of some positive Borel measure. Thus, we recover Lemma 2.2 of [10], considered for the special case of the Hausdorff moment problem. Since the sequence $\mathcal{C}$ is a nontrivial, if a sequence (4) is a moment sequence of a positive Borel measure μ, we have $c_{2k} > \sum_{j=0}^{r-1} a_j c_{2k-j-1}$, for $k \geq r$. Hence, we can obtain the following.

Proposition 5.3. *Let* $\mathcal{T} = \{v_n\}_{n\geq 0}$ *be a sequence* (4). *If* $\mathcal{T}$ *is a moment sequences of a positive Borel measure* μ, *then* $c_{2k} > \sum_{j=0}^{r-1} a_j c_{2k-j-1}$ *for any* $k \geq r$.

Using Proposition 4.5, we can easily establish the following.

Proposition 5.4. *Let* $\mathcal{T} = \{v_n\}_{n\geq 0}$ *be a sequence* (4), μ *a positive Borel measure and* ρ *a measure given by* $t^r d\rho(t) = P(t)d\mu(t)$. *Then* μ *is a solution of the full moment problem* (2) *associated with* $\mathcal{T}$ *if and only if* μ *is a solution of the truncated moment problem* (2) *associated with* $\mathcal{T}_r = \{v_n\}_{0\leq n\leq r-1}$ *and* $\{c_{n+r}\}_{n\geq 0}$ *is a moment sequences of* ρ.

Particularly, when $\mathcal{T} = \{v_n\}_{n\geq 0}$ is a sequence of type (3) of order r (i.e $c_n = 0$, for every $n \geq 0$), then the second assertion of the preceding proposition is equivalent to the fact that μ is a solution of the truncated moment problem (2) associated with $\mathcal{T}_r = \{v_n\}_{0\leq n\leq r-1}$ and $\int_K t^n d\rho(t) = \int_K t^{n-r} P(t) d\mu(t)$, for every $n \geq r$. The last statement is equivalent to $\text{supp}(\mu) \subset \mathcal{Z}(P)$, and we obtain Lemma 2.2 of [10] in the particular case of the Hausdorff moment problem.

5.2 Moment problems (2) associated with sequences (4), with c_n satisfying (3)

Let consider the linear moment problem (1) for sequence sequences (4), where the sequence $\mathcal{C} = \{c_n\}_{n\geq r}$ satisfies the linear recursive relation (3). Then, by

Proposition 4.2, Theorem 4.4, Proposition 4.6 and Theorem 5.1, we get the following result concerning the Hamburger moment problem for sequences (4).

Theorem 5.5. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence* (4). *Suppose that $\mathcal{C} = \{c_n\}_{n\geq 0}$ is a sequence of type* (3) *of order s. Then, a necessary and sufficient condition that there exists a measure μ solution of the truncated Hamburger moment problem associated with a sequence $\mathcal{T} = \{v_n\}_{n\geq 0}$ is that the Hankel matrix H_{r+s} is positive definite or equivalently* $\det H_n > 0$ *for* $n = 0, 1, \ldots, r+s$.

Similarly, we get the following result concerning the Stieltjes moment problem for sequences (4).

Theorem 5.6. *Let $\mathcal{T} = \{v_n\}_{n\geq 0}$ be a sequence* (4). *Suppose that $\mathcal{C} = \{c_n\}_{n\geq 0}$ is a sequence of type* (3) *of order s. Then, a necessary and sufficient condition that there exists a measure μ solution of the truncated Stieltjes moment problem associated with a sequence $\mathcal{T} = \{v_n\}_{n\geq 0}$ is that the two matrices H_{r+s} and S_{r+s} are positive definite or equivalently* $\det H_n > 0$ *and* $\det S_n > 0$ *for* $n = 0, 1, \ldots, r+s$.

Note that a similar result can be established for the Hausdorff moment problem.

Acknowledgements.

The second author expresses his sincere thanks to the INMA and Universidade Federal de Mato Grosso do Sul – UFMS – Brazil for their valuable support. Special thanks to Prof. Elen Pereira.

References

[1] N. I. Akhiezer, *The classical moment problem and some related questions in analysis*, 2 Eds., New York: Hafner Publ. Co. 1965.

[2] R. Ben Taher, M. Mouline and M. Rachidi, Solving some general nonhomogeneous recurrence relations of order r by a linearization method and an application to polynomial and factorial polynomial cases, *Fibonacci Quart.* 40, No. 1, 79-84, 2002.

[3] R. Ben Taher, M. Rachidi, E. H. Zerouali, Recursive subnormal completion and the truncated moment problem, *Bull. Lond. Math. Soc.* 33 , No. 4, 425-432, 2001.

[4] D. Bessis, M. Vallini, Perturbative-variational approximations to the spectral properties of semibounded Hilbert space operator, based on the moment problem with finite or divergin moments. Application to quantum mechanical systems, *J. Math. Phys.* 16, No. 3, 462-474, 1975 .

[5] C. Brezinski, The methods of Vorobyev and Lanczos, *Linear Algebra Appl.* 234, 21-41, 1997.

[6] J. B. Conway, *A course in functional analysis*, 2 Eds., Springer-Verlag, 1990.

[7] R. Curto, L. Fialkow, Flat extensions of positive moment matrices : Recursively generated relations, *Mem. Amer. Math. Soc.* 136, No. 648, 1-54, 1998.

[8] R. Curto, L. Fialkow, Solution of the truncated complex moment problem for flat data, *Mem. Amer. Math. Soc.* 119, No. 568, 1-50, 1996.

[9] B. El Wahbi, M. Rachidi, r-Generalized Fibonacci sequences and the linear moment problem, *Fibonacci Quart.* 38, No. 5, 368-394, 2000.

[10] B. El Wahbi, M. Rachidi, On r-Generalized Fibonacci sequences and Hausdorff moment problem, *Fibonacci Quart.* 39, No. 1, 5-11, 2001.

[11] B. El Wahbi, M. Rachidi, E. H. Zerouali, Recursive relations, Jacobi matrices, moment problems and continued fractions, *Pacific J. Math.* 216, 39-50, 2004.

[12] B. Simon, The classical moment problem as a self-adjoint finite difference operator, *Adv. Math. (N. Y),* 137, No. 1, 82-203, 1998.

[13] R. R. Whitehead, Moment Methods and Lanczos Methods. In B.J. Dalton, S.M. Grimes, J.P. Vary, S.A. Williams (eds), Theory and Applications of Moment Methods in Many-Fermion Systems, Springer US, Boston, MA, pp. 235-255, 1980.

The Atomic Solution for Fractional Wave Type Equation

Iman Aldarawi*

Department of mathematics, The University of Jordan, Amman, Jordan

September 23, 2021

ABSTRACT

Sometimes, it is not possible to find a general solution for some differential equations using some classical methods, like separation of variables. In such a case, one can try to use theory of tensor product of Banach spaces to find certain solutions, called atomic solution. The aim of this paper is to find atomic solution for conformable non-linear wave equation.

Key Words: fractional wave type equation; conformable derivative; atomic solution.

1 Introduction

In [Khalil et al., 2014], a new definition called α-conformable fractional derivative was introduced as follows:
Letting $\alpha \in (0,1)$, and $f : E \subseteq (0,\infty)$. Then for $x \in E$

$$D^{\alpha} f(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}. \tag{1}$$

If the limit exists then it is called the α-conformable fractional derivative of f at x.
For x=0, if f is α-differentiable on $(0, r)$ for some $r > 0$, and $\lim_{x\to 0} D^{\alpha} f(0)$ exists then we define $D^{\alpha} f(0) = \lim_{x\to 0} D^{\alpha} f(0)$. The new definition satisfies:

1. $T_{\alpha}(af + bg) = aT_{\alpha}(f) + bT_{\alpha}(g)$, for all $a, b \in R$.

*i.aldarawi@ju.edu.jo

2. $T_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$.

Further, for $\alpha \in (0,1]$ and f,g are α-differentiable at a point t, with $g(t) \neq 0$. Then

1. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
2. $T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$, $g(t) \neq 0$.

We list here the fractional derivatives of certain functions,

1. $T_\alpha(t^p) = pt^{p-\alpha}$.
2. $T_\alpha(\sin \frac{1}{\alpha}t^\alpha) = \cos \frac{1}{\alpha}t^\alpha$.
3. $T_\alpha(\cos \frac{1}{\alpha}t^\alpha) = -\sin \frac{1}{\alpha}t^\alpha$.
4. $T_\alpha e^{\frac{1}{\alpha}t^\alpha} = e^{\frac{1}{\alpha}t^\alpha}$.

On letting $\alpha = 1$ in these derivatives, we get the corresponding classical rules for the ordinary derivatives.
One should notice that a function could be α-conformable differentiable at a point but not differentiable, for example, take $f(t) = 2\sqrt{t}$. Then $T_{\frac{1}{2}}(f)(0) = 1$.
This is not the case for the known classical fractional derivatives, since $T_1(f)(0)$ does not exist.
A vast number of researcher dedicated so much of their work to study conformable derivatives and its applications. Among them, [Abdeljawad, 2015], [Abu Hammad and Khalil, 2014], [Aldarawi, 2018], [Alhabees and Aldarawi, 2020], [ALHabees, 2021], [ALHorani and Khalil, 2018], [Anderson et al., 2018], [Atangana et al., 2015], [Chung, 2015], [Hammad and Khalil, 2014], [Khalil et al., 2016], [Kilbas,], [Mhailan et al., 2020].

2 Atomic Solution

Let X and Y be two Banach spaces and X^* be the dual of X. Assume $x \in X$ and $y \in Y$. The operator $T : X^* \to Y$, defined by

$$T(x^*) = x^*(x)y \tag{2}$$

is bounded one rank linear operator. We write $x \otimes y$ for T. Such operators are called atoms. Atoms are among the main ingredient in the theory of tensor products.

Atoms are used in theory of best approximation in Banach spaces, see [Al Horani et al., 2016]. According to [Khalil, 1985], one of the known results that we need in our paper is: if the sum of two atoms is an atom, then either the first components are dependent or the second are dependent.
For more on tensor product of Banach spaces we refer to [Deeb and Khalil, 1988] and [Khalil, 1985].
Our main object in this paper is to find an atomic solution of the equation

$$D_t^\alpha D_t^\alpha u = c^2 D_x^\beta D_x^\beta u + D_t^\alpha D_x^\beta u. \tag{3}$$

This is called the conformable non-linear wave equation, where c is constant. Let $c = 1$ for simplicity to get

$$D_t^\alpha D_t^\alpha u = D_x^\beta D_x^\beta u + D_t^\alpha D_x^\beta u. \tag{4}$$

If one tries to solve this equation via separation of variables, then it is not possible since the variables can not be separated.

3 Procedure

Let $u(x,t) = X(x)T(t)$. substitute in equation (4) to get:

$$X(x)T^{2\alpha}(t) = X^{2\beta}(x)T(t) + X^\beta(x)T^\alpha(t). \tag{5}$$

This can be written in tensor product form as:

$$X(x) \otimes T^{2\alpha}(t) = X^{2\beta}(x) \otimes T(t) + X^\beta(x) \otimes T^\alpha(t). \tag{6}$$

Let us consider the following conditions: $X(0) = 1$, $X^\beta(0) = 1$.
In equation (6), we have the situation: the sum of two atoms is an atom. Hence, we have two cases:

3.1 case I: $X^{2\beta}(x) = X^\beta(x)$

The situation of case I: $X^{2\beta}(x) = X^\beta(x)$, using the result in [Al-Horani et al., 2020], we get

$$X(x) = e^{\frac{x^\beta}{\beta}}. \tag{7}$$

Now, we substitute in (6) to get

3

$$\begin{aligned} e^{\frac{x^\beta}{\beta}} \otimes T^{2\alpha}(t) &= e^{\frac{x^\beta}{\beta}} \otimes T(t) + e^{\frac{x^\beta}{\beta}} \otimes T^\alpha(t). \\ e^{\frac{x^\beta}{\beta}} \otimes T^{2\alpha}(t) &= e^{\frac{x^\beta}{\beta}} \otimes [T(t) + T^\alpha(t)]. \\ T^{2\alpha}(t) &= T(t) + T^\alpha(t). \end{aligned} \tag{8}$$

Hence, $T^{2\alpha}(t) = T(t)+T^\alpha(t)$. Again, using the result in [Al-Horani et al., 2020],

$$T(t) = c_1 e^{(\frac{1+\sqrt{5}}{2})\frac{t^\alpha}{\alpha}} + c_2 e^{(\frac{1-\sqrt{5}}{2})\frac{t^\alpha}{\alpha}}. \tag{9}$$

Using the conditions $T(0) = T^\alpha(0) = 1$, we get

$$T(t) = \frac{\sqrt{5}+1}{2\sqrt{5}} e^{(\frac{1+\sqrt{5}}{2})\frac{t^\alpha}{\alpha}} + \frac{\sqrt{5}-1}{2\sqrt{5}} e^{(\frac{\sqrt{5}-1}{2})\frac{t^\alpha}{\alpha}}. \tag{10}$$

From (7) and (10), we obtain the atomic solution of (4) as follows:

$$u(x,t) = e^{\frac{x^\beta}{\beta}} \left(\frac{\sqrt{5}+1}{2\sqrt{5}} e^{(\frac{1+\sqrt{5}}{2})\frac{t^\alpha}{\alpha}} + \frac{\sqrt{5}-1}{2\sqrt{5}} e^{(\frac{1-\sqrt{5}}{2})\frac{t^\alpha}{\alpha}} \right). \tag{11}$$

3.2 case II: $T(t) = T^\alpha(t)$

This is conformable linear differential equation. Hence, we can use the result in [Khalil, 1985], or use the fact that

$$T^\alpha(t) = t^{1-\alpha} T\prime(t). \tag{12}$$

To get

$$\begin{aligned} T(t) &= t^{1-\alpha} T\prime(t) \\ \frac{dT(t)}{T(t)} &= t^{\alpha-1} dt \\ LnT(t) &= \frac{t^\alpha}{\alpha} + k. \end{aligned} \tag{13}$$

Where k is constant. Hence,

$$T(t) = K e^{\frac{t^\alpha}{\alpha}}, K = e^k. \tag{14}$$

Again, by using the conditions $T(0) = T^\alpha(0) = 1$, we get

$$T(t) = e^{\frac{t^\alpha}{\alpha}}. \tag{15}$$

Substitute in equation (4) to get

$$\begin{aligned} X(x) \otimes e^{\frac{t^\alpha}{\alpha}} &= (X^{2\beta}(x) + X^{\beta}(x)) \otimes e^{\frac{t^\alpha}{\alpha}} \\ X(x) &= X^{2\beta}(x) + X^{\beta}(x). \end{aligned} \tag{16}$$

Again, by using the result in [Khalil, 1985], and the conditions $X(0) = X^{\beta}(0) = 1$, we get

$$X(x) = \left(\frac{3+\sqrt{5}}{2\sqrt{5}}\right) e^{\frac{-1+\sqrt{5}}{2}\frac{x^\beta}{\beta}} + \left(\frac{-3+\sqrt{5}}{2\sqrt{5}}\right) e^{\frac{-1-\sqrt{5}}{2}\frac{x^\beta}{\beta}} \tag{17}$$

From (15) and (17), we obtain the atomic solution of (4) as follows:

$$u(x,t) = \left(\left(\frac{3+\sqrt{5}}{2\sqrt{5}}\right) e^{\frac{-1+\sqrt{5}}{2}\frac{x^\beta}{\beta}} + \left(\frac{-3+\sqrt{5}}{2\sqrt{5}}\right) e^{\frac{-1-\sqrt{5}}{2}\frac{x^\beta}{\beta}}\right) e^{\frac{t^\alpha}{\alpha}} \tag{18}$$

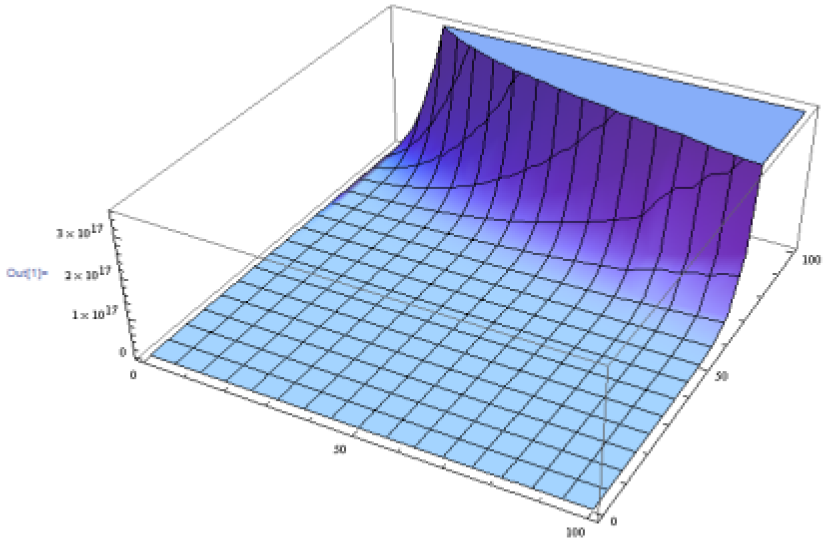
3.3 Example

Considering the following fractional wave equation

$$D_t^{0.5} D_t^{0.5} u = D_x^{0.2} D_x^{0.2} u + D_t^{0.5} D_x^{0.2} u. \tag{19}$$

The solution of (19) is

$$u(x,t) = e^{\frac{x^{0.2}}{0.2}} \left(\frac{\sqrt{5}+1}{2\sqrt{5}} e^{(\frac{1+\sqrt{5}}{2})\frac{t^{0.5}}{0.5}} + \frac{\sqrt{5}-1}{2\sqrt{5}} e^{(\frac{1-\sqrt{5}}{2})\frac{t^{0.5}}{0.5}} \right). \tag{20}$$



References

[Abdeljawad, 2015] Abdeljawad, T. (2015). On conformable fractional calculus. *Journal of computational and Applied Mathematics*, 279:57–66.

[Abu Hammad and Khalil, 2014] Abu Hammad, I. and Khalil, R. (2014). Fractional fourier series with applications. *Am. J. Comput. Appl. Math*, 4(6):187–191.

[Al Horani et al., 2016] Al Horani, M., Hammad, M. A., and Khalil, R. (2016). Variation of parameters for local fractional nonhomogenous lineardifferential equations. *J. Math. Computer Sci*, 16:147–153.

[Al-Horani et al., 2020] Al-Horani, M., Khalil, R., and Aldarawi, I. (2020). Fractional cauchy euler di¤ erential equation. *Journal of Computational Analysis & Applications*, 28(2).

[Aldarawi, 2018] Aldarawi, I. (2018). Conformable fractional hyper geometric equation. *J. Semigroup Theory Appl.*, 2018:Article–ID.

6

[ALHabees, 2021] ALHabees, A. (2021). Exact solutions of conformable fractional harry dym equation. *Journal of Computational Analysis & Applications*, 29(4).

[Alhabees and Aldarawi, 2020] Alhabees, A. and Aldarawi, I. (2020). Integrating factors for non-exact conformable differential equation. *J. Math. Comput. Sci.*, 10(4):964–979.

[ALHorani and Khalil, 2018] ALHorani, M. and Khalil, R. (2018). Total fractional differentials with applications to exact fractional differential equations. *International Journal of Computer Mathematics*, 95(6-7):1444–1452.

[Anderson et al., 2018] Anderson, D. R., Camrud, E., and Ulness, D. J. (2018). On the nature of the conformable derivative and its applications to physics. *arXiv preprint arXiv:1810.02005*.

[Atangana et al., 2015] Atangana, A., Baleanu, D., and Alsaedi, A. (2015). New properties of conformable derivative. *Open Mathematics*, 13(1).

[Chung, 2015] Chung, W. S. (2015). Fractional newton mechanics with conformable fractional derivative. *Journal of computational and applied mathematics*, 290:150–158.

[Deeb and Khalil, 1988] Deeb, W. and Khalil, R. (1988). Best approximation in l (x, y). In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 104, pages 527–531. Cambridge University Press.

[Hammad and Khalil, 2014] Hammad, M. A. and Khalil, R. (2014). Conformable fractional heat differential equation. *Int. J. Pure Appl. Math*, 94(2):215–221.

[Khalil, 1985] Khalil, R. (1985). Isometries of lp* lp. *Tam. J. Math.*, 16:77–85.

[Khalil et al., 2016] Khalil, R., Al Horani, M., and Anderson, D. (2016). Undetermined coefficients for local fractional differential equations. *J. Math. Comput. Sci*, 16:140–146.

[Khalil et al., 2014] Khalil, R., Al Horani, M., Yousef, A., and Sababheh, M. (2014). A new definition of fractional derivative. *Journal of computational and applied mathematics*, 264:65–70.

[Kilbas,] Kilbas, A. *Theory and applications of fractional differential equations.*

[Mhailan et al., 2020] Mhailan, M., Hammad, M. A., Horani, M. A., and Khalil, R. (2020). On fractional vector analysis. *J. Math. Comput. Sci.*, 10(6):2320–2326.

Asymptotic behavior of solutions of a class of time-varying systems with periodic perturbation

M. A. Hammami[a] H. Meghnafi[a] M. Meghnafi[b]

[a] University of Sfax
Faculty of Sciences of Sfax
Department of Mathematics
BP 1171, Sfax, 3000 Tunisia
{Email: MohamedAli.Hammami@fss.rnu.tn}
[b] Department of Mathematics and Computer Science
University of Bechar

Abstract This paper deals with stability of nonlinear differential equations with parameter with periodic perturbation. We determine values of the parameter under which the solutions of the perturbed systems could be uniformly exponentially stable. Sufficient conditions for global uniform asymptotic stability and/or practical stability in terms of Lyapunov-like functions are obtained in the sense that the trajectories converge to a small ball centered at the origin. Moreover, to illustrate the applicability of our result, we study the stabilization problem for a class of control system.

Keywords: Differential equations, parametric systems, perturbation, asymptotic behavior of solutions.

Mathematics Subject Classification (2000): 34D20, 37B25, 37B55.

1 Introduction

The investigation of stability analysis of nonlinear uncertain systems is an important topic in systems theory. The problem of stability analysis of nonlinear time-varying systems has attracted the attention of several researchers and has produced a vast body of important results (see [2]-[26], [29], [32], [33], [34] and the references therein). There have been a number of interesting developments in searching the stability criteria for nonlinear differential systems, but most have been restricted to finding the asymptotic stability conditions for some classes of certain systems. In particular, parametric stability for nonlinear systems is an interesting area of research, and it naturally arises in diverse fields such as population biology, economics, neural networks, and chemical processes.

Basically, parametric stability for nonlinear systems addresses the stability of equilibria for nonlinear systems with real parametric uncertainty, especially the feasibility of equilibria and the stability nature of the equilibria with respect to small variations of the real parametric uncertainty (see [25]). Dynamic systems governed by ordinary differential equations with periodically varying coefficients have been studied since one and a half centuries ago (see [12], [14], [19] and the references therein).
Mathieu [31] introduced a differential equation with periodic coefficient and Hill [24] presented the first ever solution technique of linear periodic equations. Lyapunov [30] demonstrated the Lyapunov-Floquet transformation for autonomous systems which is a linear periodic system into a dynamically equivalent time-invariant form. Unlike the differential systems without parameters, studying stability of differential parametric systems with periodic coefficients may not be easily verified ([16]-[17]).
It is well known that for linear parametric systems of the form: $\dot{x} = A(\alpha)x$, α is a real parameter which can be constant or depending on time. For technical reasons, it is important to distinguish between constant and time-varying parameters. Constant parameters have a fixed value that is known only approximately. In this case, the underlying dynamical linear system is time invariant. Time-varying parameter $\alpha(t)$ is a certain function which varies in some range and the resulting system is then time-varying. Kharitonov's theorem (see [27]) gives a simple necessary and sufficient condition for parametric system where a quadratic Lyapunov function is used to solve the problem of stability. Barmish in [3] introduced the notion of parameter dependent Lyapunov functions for continuous-time linear systems whose dynamic matrices are affected by bounded uncertain time-varying parameters. Floquet [20] developed the complete study for stability of linear time-periodic differential equations. Based on Floquet theory the stability of the linear system with time-periodic coefficients can be determined from the eigenvalues of a certain matrix. These eigenvalues are often called Floquet multipliers. He proved that, if all Floquet multipliers have magnitude less than one, the linear system with time-periodic coefficient is asymptotically stable.
In general to solve the problem of stability the usual techniques are related to some linear matrices inequalities that finding an adequate Lyapunov matrix to solve a system of Lyapunov inequalities which is a convex program. Perturbation theory is a pertinent discipline for the applications of time parametric dynamics which is a compilation of methods systematically used to evaluate the global behavior of solutions to differential equations. This motivates us to study the problem of uniform exponential stability of perturbed systems by assuming that the nominal associated system is globally uniformly asymptotically stable by imposing some restrictions on the size of perturbations in particular that are periodic in time.
The goal is to obtain estimates for the solutions of perturbed differential equations and to get uniform boundedness and uniform convergence to a small neighborhood of the origin. The notion of practical stability, (see [6]), is introduced in a special case. We determine values of parameters under which the systems are uniformly practically exponentially stable where some estimates on the decay rate of solutions at infinity are obtained. Finally, we give an application for the stabilization a class of control parametric system.

2 General definitions

Consider the non-autonomous system

$$\frac{dx}{dt} = f(t,x) \tag{1}$$

where $f : [0,\infty)\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous in t and locally Lipschitz in x on $[0,\infty)\times\mathbb{R}^n$. The origin is an equilibrium point for (1), if $f(t,0) = 0, \quad \forall t \geq 0$.

Definition 1. ***(Exponential stability)*** *The zero solution of system (1) is exponentially stable if there exist positive constants c, μ, and λ such that*

$$\|x(t)\| \leq \mu\|x(t_0)\|e^{-\lambda(t-t_0)}, \quad \forall\ \|x(t_0)\| < c \tag{2}$$

and globally exponentially stable if (2) *is satisfied for any initial state $x(t_0) \in \mathbb{R}^n$.*

The exponential stability is more important than stability, also the desired system may be unstable and yet the system may oscillate sufficiently near this state that its performance is acceptable, in particular when $f(t,0) \neq 0$, thus the notion of *practical stability* is more suitable in several situations than Lyapunov stability, it means that the trajectories converge to a small neighborhood of the origin, in the sense of uniform stability and uniform attractivity of system (1) with respect a certain ball $B_r = \{x \in \mathbb{R}^n / \|x\| \leq r\}$.

Definition 2. ***(Uniform stability of** B_r**)*** *B_r is uniformly stable if for all $\varepsilon > r$, there exists $\delta = \delta(\varepsilon) > 0$, such that*

$$\|x(t_0)\| < \delta \Longrightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq t_0. \tag{3}$$

Definition 3. ***(Uniform attractivity of** B_r**)*** *B_r is uniformly attractive, if for $\varepsilon > r$, $t_0 > 0$ and $x(t_0) \in D$, there exists $T(\varepsilon, x(t_0)) > 0$, such that*

$$\|x(t)\| < \varepsilon, \quad \forall t \geq t_0 + T(\varepsilon, x(t_0)). \tag{4}$$

B_r is globally uniformly attractive if (4) *is satisfied for all $x(t_0) \in \mathbb{R}^n$.*

Definition 4. ***(Practical stability)*** *System (1) is said uniformly practically asymptotically stable, if there exists $B_r \subset \mathbb{R}^n$, such that B_r is uniformly stable and uniformly attractive. It is globally uniformly practically asymptotically stable if $x(t_0) \in \mathbb{R}^n$.*

Definition 5. *System (1) is said uniformly exponentially convergent to B_r, if there exist $\gamma > 0$ and $k \geq 0$, such that*

$$\|x(t)\| \leq k\|x(t_0)\| \exp(-\gamma(t-t_0)) + r, \quad \forall t \geq t_0, \ \ \forall x(t_0) \in \mathbb{R}^n. \tag{5}$$

If $x(t_0) \in \mathbb{R}^n$, the system is globally uniformly exponentially convergent to B_r.
We say that the system is globally uniformly practically exponentially stable if for $r > 0$, it is globally uniformly exponentially convergent to B_r.

Here, we study the asymptotic behavior of a small ball centered at the origin for $0 \leq \| x(t) \| - r$, so that if $r = 0$ we find the classical definition of the uniform asymptotic or exponential stability of the origin viewed as an equilibrium point.

3

3 Problem formulation

We consider the following system of differential equations

$$\frac{dx}{dt} = \mu(A(\alpha(t)) + B(t))x + \nu\varphi(t,x), \quad t \geq 0, \tag{6}$$

where $A(\alpha(t)) \in \mathbb{R}^{n\times n}$ is a matrix given by $A(\alpha(t)) = \alpha_1(t)A_1 + \alpha_2(t)A_2$, with $\alpha_1(t)+\alpha_2(t) = 1, \alpha_i(t) \in \mathbb{R}^+,\ \forall t \geq 0,\ B(t) \in \mathbb{R}^{n\times n}$ is T-periodic matrix, $\mu, \nu \in \mathbb{R}$ are parameters and $\varphi(t,x)$ is a smooth vector function such that, for all $t \geq 0$ and $x \in \mathbb{R}^n$

$$\varphi(t+T,x) = \varphi(t,x)$$

and

$$\|\varphi(t,x)\| \leq k\|x\|^{1+\delta} + r, \quad \delta \geq 0,\ k > 0,\ r > 0. \tag{7}$$

Suppose that the spectrum of matrices A_1 and A_2 belong to the left half-plane $\{\lambda \in \mathbb{C}, \mathcal{R}(\lambda) < 0\}$ and

$$\int_0^T B(t)d(t) = 0. \tag{8}$$

Throughout this paper, we indicate the following domains:

$$I_1 = \{\mu \in \mathbb{R}, 0 < \mu < \mu_0\}, \qquad I_2 = \{\nu \in \mathbb{R}, |\nu| < \nu_0\},$$

such that the system (6) is practically uniformly exponentially stable for $\mu \in I_1$, $\nu \in I_2$. Moreover, we obtain estimates on the solutions of (6) that guarantee exponential decay when $t \longrightarrow +\infty$ to a certain ball $B(0, r_i)$ with a radius $r_i, i = 1, 2$.

Remark For $\mu = \nu = 1$, the system (6) can be seen as a perturbed system (see [8], [9]).

Notations: The following notations will be used throughout this paper. For a matrix X, the notation X^* denotes the transpose of matrix X. $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ denote the minimum and the maximum eigenvalues of X respectively.

Since

$$spect(A_i)_{i=1,2} \subset \{\lambda \in \mathbb{C}, Re(\lambda) < 0\},$$

then, there exist symmetric and positive definite matrices H_1 and H_2 solutions of the matrices Lyapunov equations (see [26] for the existence and uniqueness of the matrices $H_i,\ i = 1, 2$),

$$H_1A_1 + A_1^*H_1 = -I \tag{9}$$

and

$$H_2A_2 + A_2^*H_2 = -I. \tag{10}$$

The matrices $H_i,\ i = 1, 2$ satisfy:

$$H_i = \int_0^\infty e^{sA_i^*}e^{sA_i}ds.$$

In many cases, it is hard to find a common positive-definite matrix $H = H_1 = H_2$. In fact, the existence of a common positive-definite matrix depends on the difference of the two matrices $A_i, i = 1, 2$. In order to solve these problems, many scholars have made many further investigations. For example, in [28], the authors showed that, if the matrices A_1 and A_2 are real Hurwitz matrices, and that their difference is rank one, then A_1 and A_2 have a common quadratic Lyapunov function if and only if the product A_1A_2 has no real negative eigenvalue. We can solve this problem, in the special case when $A_1 + A_1^* = A_2 + A_2^*$, we get

$$H = \int_0^{\infty} e^{sA_1^*} e^{sA_1} ds = \int_0^{\infty} e^{sA_2^*} e^{sA_2} ds.$$

To facilitate our task, we will suppose that, (9) and (10) have a unique solution $H = H^* > 0$.
We have

$$\gamma_1 \|x\|^2 \leq \langle Hx, x\rangle \leq \|H\|\|x\|^2,$$

where $\gamma_1 = \lambda_{\min}(H)$.

Now, In order to study the asymptotic behavior of solutions, we shall impose some conditions on the parameters under which the system (6) can be practically uniformly exponentially stable.

Theorem 1. *Let*

$$\beta_1 = \max_{\tau\in[t_0,t_0+T]} \|H \int_{t_0}^{\tau} B(s)ds + \int_{t_0}^{\tau} B^*(s)dsH\|,$$

$$\beta_2 = \max_{\tau\in[t_0,t_0+T]} \|(H \int_{t_0}^{\tau} B(s)ds + \int_{t_0}^{\tau} B^*(s)dsH)(A_1 + B(\tau))\|,$$

$$\beta_3 = \max_{\tau\in[t_0,t_0+T]} \|(H \int_{t_0}^{\tau} B(s)ds + \int_{t_0}^{\tau} B^*(s)dsH)(A_2 + B(\tau))\|,$$

and

$$\mu_0 = \min\{\frac{\gamma_1}{\beta_1}, \frac{1}{2\beta}\} \quad \text{where } \beta = \max\{\beta_2, \beta_3\}.$$

Let H be a solution to the matrices Lyapunov equations (9) and (10) and $\delta = 0$. Then, for parameters μ and ν such that

$$0 < \mu < \mu_0 \qquad \text{and} \quad 2\mu\beta + 2|\nu|\ k\left(\frac{\|H\|}{\mu} + \beta_1\right) < 1,$$

and for any initial data $x(t_0) \in \mathbb{R}^n$, the solutions of system (6) converge exponentially towards the ball $B(0, r_1)$ whose radius is given by

$$r_1 = 2|\nu|r \frac{(\frac{\|H\|}{\mu} + \beta_1)^2}{(\frac{\gamma_1}{\mu} - \beta_1)\left(1 - 2\mu\beta - 2|\nu|k(\frac{\|H\|}{\mu} + \beta_1)\right)}.$$

5

Remark Note that, if $\nu = \nu(t)$ with $|\nu(t)| \longrightarrow 0$ as $t \longrightarrow +\infty$, then the solution of system (6) tend to zero when t tends to infinity.

Proof Define the following matrix

$$H(t,\mu) = \frac{1}{\mu}H - H\int_{t_0}^{t} B(s)ds - \int_{t_0}^{t} B^*(s)ds\ H. \tag{11}$$

Since $H = H^*$, it follows that

$$H(t,\mu) = H^*(t,\mu)$$

and by (8), the matrix $H(t,\mu)$ is T-periodic, i.e.

$$H(t+T,\mu) = H(t,\mu).$$

Let $x(t)$ be a solution to (6), then the function

$$h(t,\mu,\nu) = \langle H(t,\mu)x(t), x(t)\rangle$$

is continuously differentiable on t. It follows that, the derivative of $h(t,\mu,\nu)$ is given by

$$\frac{d}{dt}h(t,\mu,\nu) = \langle \frac{d}{dt}H(t,\mu)x(t), x(t)\rangle + \langle H(t,\mu)\frac{d}{dt}x(t), x(t)\rangle + \langle H(t,\mu)x(t), \frac{d}{dt}x(t)\rangle.$$

Since

$$\frac{d}{dt}H(t,\mu) = -HB(t) - B^*(t)H,$$

then

$$\begin{aligned}\frac{d}{dt}h(t,\mu,\nu) &= -\langle (HB(t) + B^*(t)H)x(t), x(t)\rangle \\ &\quad +\langle \mu H(t,\mu)(A(\alpha(t)) + B(t))x(t), x(t)\rangle \\ &\quad +\langle \mu(A(\alpha(t))^* + B^*(t))H(t,\mu)x(t), x(t)\rangle \\ &\quad +\nu\langle H(t,\mu)\varphi(t,x), x(t)\rangle + \nu\langle H(t,\mu)x(t), \varphi(t,x)\rangle.\end{aligned}$$

Using the definition of matrix $H(t,\mu)$, we obtain

$$\begin{aligned}\frac{d}{dt}h(t,\mu,\nu) &= \langle(-HB(t) - B^*(t)H)x(t), x(t)\rangle + \langle H(A(\alpha(t)) + B(t))x(t), x(t)\rangle \\ &\quad -\mu\langle (H\int_{t_0}^{t} B(s)ds + \int_{t_0}^{t} B^*(s)ds\ H)(A(\alpha(t)) + B(t))x(t), x(t)\rangle \\ &\quad +\langle (A(\alpha(t))^* + B^*(t))Hx(t), x(t)\rangle \\ &\quad -\mu\langle (A(\alpha(t))^* + B^*(t))(H\int_{t_0}^{t} B(s)ds + \int_{t_0}^{t} B^*(s)ds\ H)x(t), x(t)\rangle \\ &\quad +2\left(\nu\langle H(t,\mu)\varphi(t,x), x(t)\rangle\right).\end{aligned}$$

6

Replacing $A(\alpha(t))$ by its value and multiplying $B(t)$ by $(\alpha_1(t)+\alpha_2(t))$, we get

$$\begin{aligned}
\frac{d}{dt}h(t,\mu,\nu) &= \langle \alpha_1(t)(HA_1+A_1^*H)+\alpha_2(t)(HA_2+A_2^*H)x(t),x(t)\rangle \\
&\quad -\alpha_1(t)\mu\Big(\langle (H\int_{t_0}^t B(s)ds+\int_{t_0}^t B^*(s)ds\ H)(A_1+B(t))x(t),x(t)\rangle \\
&\quad +\langle (A_1+B(t))^*(H\int_{t_0}^t B(s)ds+\int_{t_0}^t B^*(s)ds\ H)x(t),x(t)\rangle\Big) \\
&\quad -\alpha_2(t)\mu\Big(\langle (H\int_{t_0}^t B(s)ds+\int_{t_0}^t B^*(s)ds\ H)(A_2+B(t))x(t),x(t)\rangle \\
&\quad +\langle (A_2+B(t))^*(H\int_{t_0}^t B(s)ds+\int_{t_0}^t B^*(s)ds\ H)x(t),x(t)\rangle\Big) \\
&\quad +2\left(\nu\langle H(t,\mu)\varphi(t,x),x(t)\rangle\right).
\end{aligned} \tag{12}$$

Taking into account (9) and (10) and using the fact that $0<\mu<\mu_0$, we obtain the following estimate

$$\begin{aligned}
\frac{d}{dt}h(t,\mu,\nu) &\le -\|x(t)\|^2 \\
&\quad +2\mu\alpha_1(t)\max_{\tau\in[t_0,t_0+T]}\|(H\int_{t_0}^\tau B(s)ds+\int_{t_0}^\tau B^*(s)dsH)(A_1+B(\tau))\|\|x(t)\|^2 \\
&\quad +2\mu\alpha_2(t)\max_{\tau\in[t_0,t_0+T]}\|(H\int_{t_0}^\tau B(s)ds+\int_{t_0}^\tau B^*(s)dsH)(A_2+B(\tau))\|\|x(t)\|^2 \\
&\quad +2|\nu|k\left(\frac{\|H\|}{\mu}+\beta_1\right)\|x(t)\|^2+2|\nu|r\left(\frac{\|H\|}{\mu}+\beta_1\right)\|x(t)\| \\
&\le -\left(1-2\mu\beta-2|\nu|\ k\left(\frac{\|H\|}{\mu}+\beta_1\right)\right)\|x(t)\|^2 \\
&\quad +2|\nu|r\left(\frac{\|H\|}{\mu}+\beta_1\right)\|x(t)\|.
\end{aligned}$$

Since the matrix $H(t,\mu)$ is positive definite for $0<\mu<\mu_0$, it follows that

$$0<(\frac{1}{\mu}\gamma_1-\beta_1)I\le H(t,\mu)\le(\frac{1}{\mu}\|H\|+\beta_1)I.$$

Thus,

$$\begin{aligned}
\frac{d}{dt}h(t,\mu,\nu) &\le -\frac{1-2\mu\beta-2|\nu|\ k\left(\frac{\|H\|}{\mu}+\beta_1\right)}{\frac{1}{\mu}\|H\|+\beta_1}h(t,\mu,\nu) \\
&\quad +\ 2|\nu|r\frac{(\frac{\|H\|}{\mu}+\beta_1)}{\sqrt{\frac{\gamma_1}{\mu}-\beta_1}}\sqrt{h(t,\mu,\nu)}.
\end{aligned}$$

7

Let $\mathcal{H}(t,\mu,\nu)=\sqrt{h(t,\mu,\nu)}$, it follows that,

$$\begin{aligned}\frac{d}{dt}\mathcal{H}(t,\mu,\nu) &\le -\frac{1-2\mu\beta-2|\nu|\ k\left(\frac{\|H\|}{\mu}+\beta_1\right)}{2(\frac{\|H\|}{\mu}+\beta_1)}\mathcal{H}(t,\mu,\nu)\\ &+ |\nu|r\frac{\frac{\|H\|}{\mu}+\beta_1}{\sqrt{\frac{\gamma_1}{\mu}-\beta_1}}\end{aligned}$$

which implies that

$$\begin{aligned}\mathcal{H}(t,\mu,\nu) &\le \mathcal{H}(t_0,\mu,\nu)\exp\left(-\frac{1-2\mu\beta-2|\nu|\ k\left(\frac{\|H\|}{\mu}+\beta_1\right)}{2(\|H\|+\mu\beta_1)}\mu(t-t_0)\right)\\ &+2|\nu|r\frac{(\frac{\|H\|}{\mu}+\beta_1)^2}{\sqrt{\frac{\gamma_1}{\mu}-\beta_1}\left(1-2\mu\beta-2|\nu|k(\frac{\|H\|}{\mu}+\beta_1)\right)}\\ &\le \sqrt{\frac{\|H\|}{\mu}}\|x(t_0)\|\exp\left(-\frac{1-2\mu\beta-2|\nu|\ k\left(\frac{\|H\|}{\mu}+\beta_1\right)}{2(\|H\|+\mu\beta_1)}\mu(t-t_0)\right)\\ &+2|\nu|r\frac{(\frac{\|H\|}{\mu}+\beta_1)^2}{\sqrt{\frac{\gamma_1}{\mu}-\beta_1}\left(1-2\mu\beta-2|\nu|k(\frac{\|H\|}{\mu}+\beta_1)\right)}\end{aligned}$$

and consequently

$$\begin{aligned}\|x(t)\| &\le \sqrt{\frac{\|H\|}{\gamma_1-\mu\beta_1}}\exp\left(-\frac{1-2\mu\beta-2|\nu|\ k\left(\frac{\|H\|}{\mu}+\beta_1\right)}{2(\|H\|+\mu\beta_1)}\ \mu(t-t_0)\right)\|x(t_0)\|\\ &+ 2|\nu|r\frac{(\frac{\|H\|}{\mu}+\beta_1)^2}{(\frac{\gamma_1}{\mu}-\beta_1)\left(1-2\mu\beta-2|\nu|k(\frac{\|H\|}{\mu}+\beta_1)\right)}.\end{aligned}$$

Thus, we obtain an estimation as in Definition 5. Hence, the solutions of system (6) converge exponentially towards the ball $B(0,r_1)$ whose radius is given by

$$r_1=2|\nu|r\frac{(\frac{\|H\|}{\mu}+\beta_1)^2}{(\frac{\gamma_1}{\mu}-\beta_1)\left(1-2\mu\beta-2|\nu|k(\frac{\|H\|}{\mu}+\beta_1)\right)}.$$

Remark A simple verification shows that $r_1>0$.

In the next part of this paper, a new class of functions appears: functions that depend on a set of constant parameters, that is, $f=f(t,x,\varepsilon)$, where $\varepsilon\in\mathbb{R}^p$. The constant parameters could represent physical parameters of the system and the study of perturbation of these parameters accounts for modeling errors or changes in the parameter values due to aging. Let begin by introducing the following lemma.

Lemma (see [26]) Let $f(t, x, \varepsilon)$ be continuous in (t, x, ε) and locally Lipschitz in x (uniformly in t and ε) on $[t_0, +\infty[\times \mathbb{R}^n \times \{\|\varepsilon - \varepsilon_0\| \leq c\}$. Let $y(t, \varepsilon_0)$ be a solution of $\dot{x} = f(t, x, \varepsilon_0)$ with $y(t_0, \varepsilon_0) = y_0 \in \mathbb{R}^n$. Suppose $y(t, \varepsilon_0)$ is defined and belongs to $\mathbb{R}^n$ for all $t \geq t_0$. Then, given $\lambda > 0$, there is $\gamma > 0$ such that, if

$$\|z_0 - y_0\| < \gamma \textit{ and } \|\varepsilon - \varepsilon_0\| < \gamma$$

then there is a unique solution $z(t, \varepsilon)$ of $\dot{x} = f(t, x, \varepsilon)$ defined for $t \geq t_0$, with $z(t_0, \varepsilon) = z_0$, and $z(t, \varepsilon)$ satisfies

$$\|z(t, \varepsilon) - y(t, \varepsilon_0)\| < \lambda, \quad \forall t \geq t_0.$$

Quite often when we study the state equation $\dot{x} = f(t, x, \varepsilon)$, where $\varepsilon \in \mathbb{R}^p$, we need to compute bounds on the solution $x(t)$ without computing the solution itself. That is why, in order to make our tache more easy, we will solve the differential equation $\dot{x} = f(t, x, \varepsilon_0)$ where ε_0 is a parameter sufficiently close to ε, i.e., $\|\varepsilon - \varepsilon_0\|$ sufficiently small and after that we will approximate the solution of $\dot{x} = f(t, x, \varepsilon)$.

Theorem 2. *Let H be a solution to the matrices Lyapunov equations (9) and (10). Let β_1, β_2, β_3, β and μ_0 be defined in the Theorem 1, let $\delta > 0$, $\rho > 0$ and*

$$\nu_0 = \frac{\mu^{1-\delta/2}\ (\gamma_1 - \mu\beta_1)^{1+\delta/2}\ (1 - 2\mu\beta)}{2\ k(\|H\| + \mu\beta_1)^2\ (\sqrt{\frac{\|H\|}{\mu}}\rho + \gamma)^\delta}$$

with γ is some constant. Then, for $0 < \mu < \mu_0$, $|\nu| < \nu_0$ and for any initial data

$$x(t_0) \in \mathbb{R}^n, \quad \|x(t_0)\| \leq \rho,$$

the system (6) is practically uniformly exponentially stable.

Proof Let $x(t)$ be a solution to system (6) and $H(t, \mu)$ be defined by (11). From the proof of Theorem 1, the function $h(t, \mu, \nu)$ satisfy the inequality (12). By the definition of matrix $H(t, \mu)$ and taking into account that $\|\varphi(t, x)\| \leq k\|x\|^{1+\delta} + r$, we obtain the following estimate

$$\begin{aligned} \frac{d}{dt}h(t, \mu, \nu) \quad &\leq \quad -(1 - 2\mu\beta)\|x(t)\|^2 + 2|\nu|k\left(\frac{\|H\|}{\mu} + \beta_1\right)\|x(t)\|^{2+\delta} \\ &\quad +2|\nu|r\left(\frac{\|H\|}{\mu} + \beta_1\right)\|x(t)\|. \end{aligned}$$

Since

$$\|x(t)\|^2 \leq \frac{h(t, \mu, \nu)}{(\frac{1}{\mu}\gamma_1 - \beta_1)} \quad \textit{and} \quad \|x(t)\|^\delta \leq \frac{h(t, \mu, \nu)^{\delta/2}}{(\frac{1}{\mu}\gamma_1 - \beta_1)^{\delta/2}},$$

then,

$$\|x(t)\|^{2+\delta} \leq \frac{h(t, \mu, \nu)^{1+\delta/2}}{(\frac{1}{\mu}\gamma_1 - \beta_1)^{1+\delta/2}}.$$

9

It follows that

$$\begin{aligned}\frac{d}{dt}h(t,\mu,\nu) \;\leq\; & -\frac{1-2\mu\beta}{\frac{1}{\mu}\|H\|+\beta_1}h(t,\mu,\nu) \\ & +\frac{2|\nu|k\left(\frac{1}{\mu}\|H\|+\beta_1\right)}{(\frac{1}{\mu}\gamma_1-\beta_1)^{1+\delta/2}}h(t,\mu,\nu)^{1+\delta/2} \\ & +2|\nu|r\frac{\left(\frac{\|H\|}{\mu}+\beta_1\right)}{\sqrt{\frac{1}{\mu}\gamma_1-\beta_1}}\sqrt{h(t,\mu,\nu)}.\end{aligned}$$

Introduce the following notation

$$\epsilon_1=\frac{1-2\mu\beta}{\frac{1}{\mu}\|H\|+\beta_1},\;\; \epsilon_2=\frac{2|\nu|k\left(\frac{1}{\mu}\|H\|+\beta_1\right)}{(\frac{1}{\mu}\gamma_1-\beta_1)^{1+\delta/2}} \;\; and \;\; \epsilon_3=2|\nu|r\frac{\left(\frac{\|H\|}{\mu}+\beta_1\right)}{\sqrt{\frac{1}{\mu}\gamma_1-\beta_1}},$$

hence

$$\frac{d}{dt}h(t,\mu,\nu)\leq -\epsilon_1 h(t,\mu,\nu)+\epsilon_2 h(t,\mu,\nu)^{1+\delta/2}+\epsilon_3\sqrt{h(t,\mu,\nu)}.$$

Let

$$z(t)=\sqrt{h(t,\mu,\nu)},$$

we have

$$\frac{d}{dt}z(t)\leq -\frac{\epsilon_1}{2}z(t)+\frac{\epsilon_2}{2}z(t)^{1+\delta}+\frac{\epsilon_3}{2}. \tag{13}$$

Let $z(t,\varepsilon)$ the solution of (13) where $\varepsilon=(\epsilon_1,\epsilon_2,\epsilon_3)\in\mathbb{R}^3_+$ and $y_1(t,\varepsilon_0)$ the solution of

$$\frac{d}{dt}z(t)\leq -\frac{\epsilon_1}{2}z(t)+\frac{\epsilon_2}{2}z(t)^{1+\delta} \tag{14}$$

where $\varepsilon_0=(\epsilon_1,\epsilon_2,0)\in\mathbb{R}^3_+$.

In order to solve (14), we can take $\eta=1+\delta$ and $w(t)=y_1(t,\varepsilon_0)^{1-\eta}=y_1(t,\varepsilon_0)^{-\delta}$. Thus,

$$\frac{d}{dt}w(t)=\frac{\epsilon_1\delta}{2}w-\frac{\epsilon_2\delta}{2}.$$

Solving the homogenous equation

$$\frac{d}{dt}w(t)=\frac{\epsilon_1\delta}{2}w,$$

we get

$$w(t)=L\;e^{\frac{\epsilon_1\delta}{2}t}.$$

Now, suppose that L is a function that depends on t, i.e. we have

$$w(t)=L(t)\;e^{\frac{\epsilon_1\delta}{2}t}.$$

A simple computation shows that

$$L(t) = \frac{\epsilon_2}{\epsilon_1}\, e^{-\frac{\epsilon_1 \delta}{2} t} + \theta, \quad \forall \theta \geq 0,$$

and consequently

$$w(t) = \frac{\epsilon_2}{\epsilon_1} + \theta e^{\frac{\epsilon_1 \delta}{2} t}$$

where

$$\theta = \left(w(t_0) - \frac{\epsilon_2}{\epsilon_1} \right)\, e^{-\frac{\epsilon_1 \delta}{2} t_0}.$$

It follows that,

$$w(t) = \frac{\epsilon_2}{\epsilon_1} + \left(w(t_0) - \frac{\epsilon_2}{\epsilon_1} \right)\, e^{\frac{\epsilon_1 \delta}{2}(t - t_0)}.$$

Since $y_1(t, \varepsilon_0) = w(t)^{-1/\delta}$ and $w(t_0) = y_1(t_0, \varepsilon_0)^{-\delta}$, we obtain

$$y_1(t, \varepsilon_0) = \left(y_1(t_0, \varepsilon_0)^{-\delta}\, e^{\frac{\epsilon_1 \delta}{2}(t - t_0)} + \frac{\epsilon_2}{\epsilon_1} - \frac{\epsilon_2}{\epsilon_1}\, e^{\frac{\epsilon_1 \delta}{2}(t - t_0)} \right)^{-1/\delta}.$$

If

$$\epsilon_2 y_1^{\delta}(t_0, \varepsilon_0) < \epsilon_1, \tag{15}$$

which will be verified later on, and using the fact that for all $a \geq 0$ and $b \geq 0$, we have

$$(a + b)^p \leq a^p (1 + \frac{b}{a})^p, \quad \forall p \in \mathbb{R},$$

Thus,

$$y_1(t, \varepsilon_0) \leq y_1(t_0, \varepsilon_0) e^{-\frac{\epsilon_1}{2}(t - t_0)} \quad \times \quad \left(1 - y_1^{\delta}(t_0, \varepsilon_0) \frac{\epsilon_2}{\epsilon_1} + y_1^{\delta}(t_0, \varepsilon_0) \frac{\epsilon_2}{\epsilon_1} e^{-\frac{\epsilon_1}{2}\delta(t - t_0)} \right)^{-1/\delta}$$

yields,

$$y_1(t, \varepsilon_0) \leq y_1(t_0, \varepsilon_0) e^{-\frac{\epsilon_1}{2}(t - t_0)} \left(1 - y_1^{\delta}(t_0, \varepsilon_0) \frac{\epsilon_2}{\epsilon_1} \right)^{-1/\delta}.$$

Then, by the Lemma, for $\|\epsilon_3\|_2 < \gamma$ and $\lambda > 0$, we get

$$\|z(t, \varepsilon) - y_1(t, \varepsilon_0)\| < \lambda,$$

which implies that

$$\begin{aligned} \|z(t, \varepsilon)\| &< \lambda + \left\| y_1(t_0, \varepsilon_0) e^{-\frac{\epsilon_1}{2}(t - t_0)} \left(1 - y_1^{\delta}(t_0, \varepsilon_0) \frac{\epsilon_2}{\epsilon_1} \right)^{-1/\delta} \right\| \\ &< \lambda + (\|z(t_0, \varepsilon)\| + \gamma) e^{-\frac{\epsilon_1}{2}(t - t_0)} \left\| 1 - y_1^{\delta}(t_0, \varepsilon_0) \frac{\epsilon_2}{\epsilon_1} \right\|^{-1/\delta} \\ &< \lambda + (\sqrt{\frac{\|H\|}{\mu}} \|x(t_0)\| + \gamma) e^{-\frac{\epsilon_1}{2}(t - t_0)} \left\| 1 - y_1^{\delta}(t_0, \varepsilon_0) \frac{\epsilon_2}{\epsilon_1} \right\|^{-1/\delta}. \end{aligned}$$

11

Since

$$\sqrt{\frac{\gamma_1}{\mu}-\beta_1}\|x(t)\| \leq z(t,\varepsilon) \leq \sqrt{\frac{\|H\|}{\mu}+\beta_1}\|x(t)\|,$$

then,

$$\begin{aligned}\|x(t)\| &\leq \sqrt{\frac{\|H\|}{\gamma_1-\mu\beta_1}}\left\|1-y_1^\delta(t_0,\varepsilon_0)\frac{\epsilon_2}{\epsilon_1}\right\|^{-1/\delta}\|x(t_0)\|e^{-\frac{\epsilon_1}{2}(t-t_0)}\\ &\quad+\frac{\lambda}{\sqrt{\frac{\gamma_1}{\mu}-\beta_1}}+\frac{\gamma}{\sqrt{\frac{\gamma_1}{\mu}-\beta_1}}\left\|1-y_1^\delta(t_0,\varepsilon_0)\frac{\epsilon_2}{\epsilon_1}\right\|^{-1/\delta}.\end{aligned}$$

The last inequality implies that the solutions of system (6) converge exponentially toward the ball $B(0,r_2)$ whose radius is given by

$$r_2=\frac{\lambda}{\sqrt{\frac{\gamma_1}{\mu}-\beta_1}}+\frac{\gamma}{\sqrt{\frac{\gamma_1}{\mu}-\beta_1}}\left\|1-y_1^\delta(t_0,\varepsilon_0)\frac{\epsilon_2}{\epsilon_1}\right\|^{-1/\delta}$$

which is clearly positive.

Finally, let verify the condition (15). Since $|\nu|<\nu_0$, $0<\mu<\mu_0$ and $\|x(t_0)\|\leq\rho$, then

$$\begin{aligned}\frac{\epsilon_2}{\epsilon_1}y_1^\delta(t_0,\varepsilon_0) &\leq \frac{2|\nu|k\left(\frac{\|H\|}{\mu}+\beta_1\right)^2}{(\frac{1}{\mu}\gamma_1-\beta_1)^{1+\delta/2}\,(1-2\mu\beta)}(\|z(t_0,\varepsilon)\|+\gamma))^\delta\\ &\leq \frac{2\nu_0k}{\mu^{1-\delta/2}}\,\frac{(\|H\|+\mu\beta_1)^2}{(\gamma_1-\mu\beta_1)^{1+\delta/2}\,(1-2\mu\beta)}\left(\sqrt{\frac{\|H\|}{\mu}}\rho+\gamma\right)^\delta.\end{aligned}$$

Hence, according to the definition of ν_0, we have

$$\frac{\epsilon_2}{\epsilon_1}y_1^\delta(t_0,\varepsilon_0)<1.$$

4 Application to control

In this section we study the stabilization problem of a control system modeled by the same dynamic as (6).

Definition 6. *A function $\alpha:[0,a[\to[0,+\infty[$ is said to be of class $\mathcal{K}$, if it is continuous, strictly increasing and $\alpha(0)=0$. It is of class $\mathcal{K}_\infty$ if, in addition, $a=+\infty$ and $\alpha(r)\to+\infty$ as $r\to+\infty$.*

Let as recall the following result (see [6]).

Theorem 3. *Let consider system* (1) *and suppose that there exist a continuously differentiable real function* $h(\cdot,\cdot)$ *on* $\mathbb{R}_+ \times \mathbb{R}^n$, $\mathcal{K}_\infty$ *functions* $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, *a* $\mathcal{K}$ *function* $\alpha_3(\cdot)$ *and a small positive real number* ϱ *such that the following inequalities hold for all* $t \in \mathbb{R}_+$ *and* $x \in \mathbb{R}^n$

$$\alpha_1(\|x\|) \leq h(t,x) \leq \alpha_2(\|x\|)$$

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} f(t,x) \leq -\alpha_3(\|x\|) + \varrho.$$

Then the system is globally uniformly practically stable with $r = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(\varrho)$.

When the function satisfying $f(t,0) \neq 0$ for certain $t \in \mathbb{R}_+$, we shall study the asymptotic stability of the system at a neighborhood of the origin viewed as a small ball centered at the origin. The state approaches the origin (or some sufficiently small neighborhood of it) in a sufficiently fast manner. The following result gives sufficient conditions for practical global exponential stability.

Theorem 4. *Consider system* (1). *Let* $h : [0,+\infty[\times\mathbb{R}^n \to \mathbb{R}$ *be a continuously differentiable Lyapunov function such that*

$$c_1\|x\|^2 \leq h(t,x) \leq c_2\|x\|^2$$

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} f(t,x) \leq -c_3 h(t,x) + \varrho$$

for all $t \geq 0$ *and* $x \in \mathbb{R}^n$, *where* c_1, c_2 *and* c_3 *are positive constants. Then* B_r *is globally uniformly exponentially stable, with* $r = \sqrt{\varrho/c_1c_2}$.

Now we state the stabilizability problem associated with the following nonlinear time-varying control system:

$$\frac{dx}{dt} = f(t,x(t),u(t)), \quad t \geq 0, \tag{16}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $f(t,x,u) : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$.

Definition 7. *The feedback controller* $u(t) = u(t,x(t))$, *where* $u(t,x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^m$ *stabilizes globally uniformly asymptotically or exponentially the control system* (16) *if the closed-loop system*

$$\frac{dx}{dt} = f(t,x(t),u(t,x(t))) \tag{17}$$

is globally uniformly asymptotic or exponential stable.

In the case where $f(t,0,0) \neq 0$ for a certain $t \geq 0$. We can formulate the above definition as:

Definition 8. *The feedback controller* $u(t) = u(t,x(t))$ *stabilizes globally uniformly asymptotically or exponentially the control system* (16) *with respect* B_r, *if the associated closed-loop system* (17) *is globally practically uniformly asymptotically or exponentially stable.*

From Theorem 3, one has the following result which concern the asymptotic stabilizability problem of system (16).

Theorem 5. *Suppose that there exist a stabilizing feedback controller $u(t) = u(t, x(t))$ for control system (16) and a continuously differentiable function $h(\cdot,\cdot) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, $\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, a $\mathcal{K}$ function $\alpha_3(\cdot)$ and a small positive real number ϱ such that the following inequalities hold for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$*

$$\alpha_1(\|x\|) \leq h(t, x) \leq \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial h}{\partial x}\, f(t, x, u(t, x(t))) \leq -\alpha_3(\|x\|) + \varrho.$$

Then system (16) in closed-loop with the feedback controller $u = u(t, x(t))$ is globally uniformly practically asymptotically stable with $r = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(\varrho)$.

Also, we can say that the control system (16) is globally uniformly exponentially stabilizable by the feedback control $u(t) = u(t, x(t))$, where $u(t, x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^m$, if the closed-loop system (17) is globally uniformly exponentially stable.

Definition 9. *B_r is globally uniformly exponentially stabilizable by the feedback control $u(t) = u(t, x(t))$ if there exist $\gamma > 0$ and $k > 0$ such that for all $t \geq t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$, the solution $x(t)$ of the closed-loop system (17) satisfies:*

$$\|x(t)\| \leq k\|x_0\| exp(-\gamma(t - t_0)) + r.$$

In this case, system (16) is globally practically uniformly exponentially stabilizable by the feedback control $u(t) = u(t, x(t))$.

One has the following result which concern the exponential stabilizability problem of system (16).

Theorem 6. *Let $u = u(t, x(t))$ an exponential stabilizing feedback law and*

$$h : [0, +\infty[\times \mathbb{R}^n \to \mathbb{R}$$

be a continuously differentiable Lyapunov function such that

$$c_1\|x\|^2 \leq h(t, x) \leq c_2\|x\|^2$$

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}\, f(t, x, u(t, x(t))) \leq -c_3 h(t, x) + \varrho$$

for all $t \geq 0$ and $x \in \mathbb{R}^n$, where c_1, c_2 and c_3 are positive constants. Then B_r is globally uniformly exponentially stable with $r = \sqrt{\varrho/c_1c_2}$, with respect the closed-loop system (17).

Now, we will study the practical exponential stability problem a class of nonlinear systems of the form (6). It is worth to notice that the origin is not required to be an equilibrium point for the system (6). This may be in many situations meaningful from

14

a practical point of view specially, when stability for control systems is investigated.

Consider the class of systems that can be modeled by:

$$\frac{dx}{dt} = \mu(A(\alpha(t)) + B(t))x + \nu\varphi(t, x, u), \quad t \geq 0, \tag{18}$$

where $A(\alpha(t)) \in \mathbb{R}^{n\times n}$ is a matrix given by $A(\alpha(t)) = \alpha_1(t)A_1 + \alpha_2(t)A_2$, with $\alpha_1(t) + \alpha_2(t) = 1, \alpha_i(t) \in \mathbb{R}^+,\ \forall t \geq 0,\ B(t) \in \mathbb{R}^{n\times n}$ is T-periodic matrix, $\mu \in \mathbb{R}$, $\nu \in \mathbb{R}$ are parameters and $\varphi(t, x, u)$ is a smooth vector function. u denotes the control of the system. We suppose that there exists a stabilizing feedback control $u(t) = u(t, x(t))$, where the function u is a suitable feedback controller such that the condition (7) is replaced as follows: $\varphi(t, x, u)$ is a smooth vector function such that, for all $t \geq 0$ and $x \in \mathbb{R}^n$,

$$\varphi(t + T, x, u(t, x(t))) = \varphi(t, x, u(t, x(t)))$$

and

$$\|\varphi(t, x, u(t, x(t)))\| \leq k\|x\|^{1+\delta} + r, \quad \delta \geq 0,\ k > 0,\ r > 0.$$

The practical uniform exponential stability can therefore be established as in Theorem 2, an d an estimation as in Definition 9 can be obtained which gives that the system (18) in closed-loop with $u(t) = u(t, x(t))$ is practically globally uniformly exponentially stable.

5 Conclusion

Asymptotic stability of a class of parametric differential equations has been studied. New sufficient conditions for practical uniform asymptotic exponential stability of solutions for parametric systems with periodic coefficients are obtained. An application to control system is given.

References

[1] O. Aeyels and P. Penteman, A new asymptotic stability criterion for nonlinear time-varying differntial equations. *IEEE Trans. Aut. Contr.*, 43(1998), 968-971.

[2] S. Akhalaia, M. Ashordia and N Kekelia, On the necessary and sufficient conditions for the stability of linear generalized ordinary differential, linear impulsive and linear difference systems, *Georgian Mathematical Journal*, 16 (4), 597-616.

[3] B.R. Barmish, Necessary and sufficient conditions for quadratic stabilizabilityof an uncertain system, *Journal of Optimization Theory and Applications*, 46 (4) 399-408 (1985).

[4] B. Benaser, K. Boukerrioua, M. Defoort, M. Djemai, M.A. Hammami, T.M.L. Kirati, *Sufficient conditions for uniform exponential stability and h-stability of some classes of dynamic equations on arbitrary time scales.* Nonlinear Anal.: Hybrid Systems **32** (2019), 54-64.

[5] A.Benabdallah and M.A.Hammami, On the output stability for nonlinear uncertain control systems. *International Journal of Control*, vol 74 N6 (2001), pp. 547-551.

[6] A.Ben Abdallah, I.Ellouze and M.A.Hammami, Practical stability of nonlinear time-varying cascade systems, *Journal of Dynamical and Control Systems* **15** No. 1, (2009), 45-62.

[7] A. Benabdallah, I. Ellouze and M.A. Hammami, Practical exponential stability of perturbed triangular systems and a separation principle, *Asian journal of control*, 13 (3), 445-448.

[8] A. Ben Abdallah, M. Dlala and M.A. Hammami, A new Lyapunov function for stability of perturbed nonlinear systems. *Systems and Control Letters*, vol 56 N3 pp.179-187 (2007).

[9] A. Ben Abdallah, M. Dlala and M.A. Hammami, Exponential stability of perturbed nonlinear systems. *Nonlinear Dynamics and Systems Theory*, 5(4) pp. 357-367 (2005).

[10] B. Bellman, Stability Theory of Differntial Equations. *Mac Grow-Hill*, 1959.

[11] B. Ben Hamed, I. Ellouze, M.A. HammamiPractical uniform stability of nonlinear differential delay equations, *Mediterranean Journal of Mathematics*, 8 (4), 603-616.

[12] V.V. Bolotin, The dynamic stability of elastic systems. *Holden-Day, San Francisco,* 1964.

[13] T. Caraballo, M.A. Hammami and L. Mchiri, Practical exponential stability of impulsive stochastic functional differential equations *Systems and Control Letters*, 109, 43-48.

[14] M.P. Cartmell, Introduction to linear, parametric and nonlinear vibrations. *Chapman and Hall*, London, 1990.

[15] H. Damak, M.A. Hammami and B.Kalitine, *On the global uniform asymptotic stability of time-varying systems*. Differ. Equ. Dyn. Syst. **22** (2014), 113-124.

[16] G. Demidenko and I. Matveeva, On stability of solutions to linear systems with perioic coefficients, *Siberian Math. J.*, 42(2001), No. 2, 282-296.

[17] G. Demidenko and I. Matveeva, On asymptotic stability of solutions to nonlinear systems of differential equations with periodic coefficients, *Sel. Journal of Applied Mathematics*, Vol. 3, No. 2,pp.37-48,(2002).

[18] I Ellouze, MA Hammami, A separation principle of time ^{a}varying dynamical systems: A practical stability approach, *Mathematical Modelling and analysis*, 12 (3), 297-308.

[19] M. Farkas. Periodic motions, volume 104. *Springer Science and Business Media*, 2013.

[20] G. Floquet, Sur les équations différentielles linéaires à coefficients périodiques. *Annales scientifiques de l'Ecole normale supérieure*, volume 12, pages 4788, 1883.

[21] B. Ghanmi, N. Hadj Taieb, M.A. Hammami, Growth conditions for exponential stability of time-varying perturbed systems, *International Journal of Control*, 86 (6), 1086-1097.

[22] M.A.Hammami, On the stability of nonlinear control systems with uncertainty. *Journal of Dynamical Control systems*, vol 7 N2 (2001), pp. 171-179.

[23] A. Hamza and K. Oraby, *Stability of abstract dynamic equations on time scales by Lyapunov's second method.* Turkish J. Math. **42** (2018), 841-861.

[24] Hill G.W. On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and moon. *Acta mathematica*, 8(1):136, (1886).

[25] M. Ikeda, Y. Ohta, D.D. Siljak, Parametric stability, *G. Conte, A.M. Perdon, B. Wyman (Eds.), New Trends in Systems Theory*, Birkhauser, Boston, 1991, pp. 120.

[26] Hassan. K. Khalil. Nonlinear Systems. *Macmillan Publishing Company*, Singapore, (1992).

[27] V. L. Kharitonov, Asymptotic stability of an equilibrium position of a family of systems of differential equations, *Differentsialnye uravneniya,* 14 (1978), 2086-2088. (in Russian).

[28] C. King, M. Nathanson, On the existence of a common quadratic Lyapunov function for a rank one difference, *Linear Algebra and its Applications* 419 (2006) 400416.

[29] V. Lakshmikantham, S. Leela and A. Martynyuk, Stability Analysis of Non-linear Systems. Marcel Dekker, New York, (1989).

[30] Lyapunov A.M. Sur une série relative a la théorie des équations différentielles linéaires avec coeficient périodiques. *Comptes Rendus de l' Academie des Sciences, Paris*, 123:12481252, (1896).

[31] E. Mathieu, Mémoire sur le mouvement vibratoire dune membrane de forme elliptique. *Journal de mathematiques pures et appliquees*, 13: 137203, (1868).

[32] M. Meghnafi, M. A. Hammami and T. Blouhi, Existence results on impulsive stochastic semilinear differential inclusions, *Int. J. Dynamical Systems and Differential Equations*, Vol. 11, No. 2, 2021 131.

[33] V. Slynko, C. Tunc, S. Erdur, On the interval stability of impulsive systems with time delay, *Journal of Computer and Systems Sciences International*, 59 (1),(2020) 8-18.

[34] C. Tunc, A remark on the qualitative conditions of nonlinear IDEs, *Int. J. Math. Comput. Sci.* 15 (3), (2020) 905-922.

Generalized Canavati Fractional Hilbert-Pachpatte type inequalities for Banach algebra valued functions

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Using generalized Canavati fractional left and right vectorial Taylor formulae we prove corresponding left and right fractional Hilbert-Pachpatte type inequalities for Banach algebra valued functions. We cover also the sequential fractional case. We finish with applications.

2020 Mathematics Subject Classification : 26A33, 26D10, 26D15.
Keywords and Phrases: generalized Canavati fractional derivative, generalized Canavati fractional inequalities, Hilbert-Pachpatte inequalities, Banach algebra.

1 Introduction

Motivation follows:

We need

Definition 1 *(see [5]) A definition of the Hausdorff measure h_α goes as follows: if (T,d) is a metric space, $A \subseteq T$ and $\delta > 0$, let $\Lambda(A,\delta)$ be the set of all arbitrary collections $(C)_i$ of subsets of T, such that $A \subseteq \cup_i C_i$ and $diam(C_i) \leq \delta$ (diam =diameter) for every i. Now, for every $\alpha > 0$ define*

$$h_\alpha^\delta(A) := \inf\left\{\sum (diamC_i)^\alpha \mid (C_i)_i \in \Lambda(A,\delta)\right\}. \quad (1)$$

Then there exists $\lim_{\delta\to 0} h_\alpha^\delta(A) = \sup_{\delta>0} h_\alpha^\delta(A)$, and $h_\alpha(A) := \lim_{\delta\to 0} h_\alpha^\delta(A)$ gives an outer measure on the power set $\mathcal{P}(T)$, which is countably additive on the σ-field *of all Borel subsets of T. If $T = \mathbb{R}^n$, then the Hausdorff measure h_n, restricted to the σ-field of the Borel subsets of $\mathbb{R}^n$, equals the Lebesgue measure on $\mathbb{R}^n$ up to a constant multiple. In particular, $h_1(C) = \mu(C)$ for every Borel set $C \subseteq \mathbb{R}$, where μ is the Lebesgue measure.*

We also need

Definition 2 *([2], Ch. 1) Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\nu > 0$; $n := \lceil \nu \rceil \in \mathbb{N}$, $\lceil \cdot \rceil$ is the ceiling of the number, $f : [a,b] \to X$. We assume that $f^{(n)} \in L_1([a,b], X)$. We call the Caputo-Bochner left fractional derivative of order ν:*

$$\left(D_{*a}^{\nu} f\right)(x) := \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt, \quad \forall\ x \in [a,b]. \tag{2}$$

If $\nu \in \mathbb{N}$, we set $D_{*a}^{\nu} f := f^{(\nu)}$ the ordinary X-valued derivative, and also set $D_{*a}^{0} f := f$. Here Γ is the gamma function and integrals are of Bochner type [3].

By [2], Ch. 1, $\left(D_{*a}^{\nu} f\right)(x)$ exists almost everywhere in $x \in [a,b]$ and $D_{*a}^{\nu} f \in L_1([a,b], X)$.

If $\left\|f^{(n)}\right\|_{L_\infty([a,b],X)} < \infty$, then by [2], Ch. 1, $D_{*a}^{\nu} f \in C([a,b], X)$.

We are motivated by a Hilbert-Pachpatte left fractional inequality:

Theorem 3 *([2], Ch. 1) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\nu_1 > \frac{1}{q}$, $\nu_2 > \frac{1}{p}$, $n_i := \lceil \nu_i \rceil$, $i = 1,2$. Here $[a_i, b_i] \subset \mathbb{R}$, $i = 1,2$; X is a Banach space. Let $f_i \in C^{n_i - 1}([a_i, b_i], X)$, $i = 1,2$. Set*

$$F_{x_i}(t_i) := \sum_{j_i=0}^{n_i-1} \frac{(x_i - t_i)^{j_i}}{j_i!} f_i^{(j_i)}(t_i), \tag{3}$$

$\forall\ t_i \in [a_i, x_i]$, where $x_i \in [a_i, b_i]$; $i = 1,2$. Assume that $f_i^{(n_i)}$ exists outside a μ-null Borel set $B_{x_i} \subseteq [a_i, x_i]$, such that

$$h_1\left(F_{x_i}\left(B_{x_i}\right)\right) = 0, \ \ \forall\ x_i \in [a_i, b_i];\ i = 1,2. \tag{4}$$

We also assume that $f_i^{(n_i)} \in L_1([a_i, b_i], X)$, and

$$f_i^{(k_i)}(a_i) = 0, \ \ k_i = 0,1,...,n_i - 1;\ i = 1,2, \tag{5}$$

and

$$\left(D_{*a_1}^{\nu_1} f_1\right) \in L_q([a_1, b_1], X), \quad \left(D_{*a_2}^{\nu_2} f_2\right) \in L_p([a_2, b_2], X). \tag{6}$$

Then

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\|f_1(x_1)\| \|f_2(x_2)\|\, dx_1 dx_2}{\left(\frac{(x_1-a_1)^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)} + \frac{(x_2-a_2)^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)} \leq$$

$$\frac{(b_1-a_1)(b_2-a_2)}{\Gamma(\nu_1)\Gamma(\nu_2)} \left\|D_{*a_1}^{\nu_1} f_1\right\|_{L_q([a_1,b_1],X)} \left\|D_{*a_2}^{\nu_2} f_2\right\|_{L_p([a_2,b_2],X)}. \tag{7}$$

We need

Definition 4 *([2], Ch. 2) Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m := \lceil \alpha \rceil$. We assume that $f^{(m)} \in L_1([a,b], X)$, where $f : [a,b] \to X$. We call the Caputo-Bochner right fractional derivative of order α:*

$$\left(D_{b-}^{\alpha} f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (J-x)^{m-\alpha-1} f^{(m)}(J)\, dJ, \quad \forall\, x \in [a,b]. \tag{8}$$

We observe that $D_{b-}^{m} f(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $D_{b-}^{0} f(x) = f(x)$.

By [2], Ch. 2, $\left(D_{b-}^{\alpha} f\right)(x)$ exists almost everywhere on $[a,b]$ and $\left(D_{b-}^{\alpha} f\right) \in L_1([a,b], X)$.

If $\left\| f^{(m)} \right\|_{L_\infty([a,b],X)} < \infty$, and $\alpha \notin \mathbb{N}$, then by [2], Ch. 2, $D_{b-}^{\alpha} f \in C([a,b], X)$, hence $\left\| D_{b-}^{\alpha} f \right\| \in C([a,b])$.

We are motivated also by the following Hilbert-Pachpatte right fractional inequality:

Theorem 5 *([2], Ch. 2) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\alpha_1 > \frac{1}{q}$, $\alpha_2 > \frac{1}{p}$, $m_i := \lceil \alpha_i \rceil$, $i = 1, 2$. Here $[a_i, b_i] \subset \mathbb{R}$, $i = 1, 2$; X is a Banach space. Let $f_i \in C^{m_i - 1}([a_i, b_i], X)$, $i = 1, 2$. Set*

$$F_{x_i}(t_i) := \sum_{j_i=0}^{m_i-1} \frac{(x_i - t_i)^{j_i}}{j_i!} f_i^{(j_i)}(t_i), \tag{9}$$

$\forall\, t_i \in [x_i, b_i]$, where $x_i \in [a_i, b_i]$; $i = 1, 2$. Assume that $f_i^{(m_i)}$ exists outside a μ-null Borel set $B_{x_i} \subseteq [x_i, b_i]$, such that

$$h_1\left(F_{x_i}\left(B_{x_i}\right)\right) = 0, \quad \forall\, x_i \in [a_i, b_i];\ i = 1, 2. \tag{10}$$

We also assume that $f_i^{(m_i)} \in L_1([a_i, b_i], X)$, and

$$f_i^{(k_i)}(b_i) = 0, \quad k_i = 0, 1, ..., , m_i - 1;\ i = 1, 2, \tag{11}$$

and

$$\left(D_{b_1-}^{a_1} f_1\right) \in L_q([a_1, b_1], X), \quad \left(D_{b_2-}^{\alpha_2} f_2\right) \in L_p([a_2, b_2], X). \tag{12}$$

Then

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\|f_1(x_1)\| \, \|f_2(x_2)\| \, dx_1 dx_2}{\left(\frac{(b_1-x_1)^{p(\alpha_1-1)+1}}{p(p(\alpha_1-1)+1)} + \frac{(b_2-x_2)^{q(\alpha_2-1)+1}}{q(q(\alpha_2-1)+1)}\right)} \leq$$

$$\frac{(b_1-a_1)(b_2-a_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \left\| D_{b_1-}^{\alpha_1} f_1 \right\|_{L_q([a_1,b_1],X)} \left\| D_{b_2-}^{\alpha_2} f_2 \right\|_{L_p([a_2,b_2],X)}. \tag{13}$$

In this work we derive Hilbert-Pachpatte inequalities for Banach algebra valued functions with respect to their Canavati type generalized left and right fractional derivatives. We cover also the sequential fractional case. We finish with applications.

3

2 Background on Vectorial generalized Canavati fractional calculus

All in this section come from [2], pp. 109-115 and [1].

Let $g : [a, b] \to \mathbb{R}$ be a strictly increasing function. such that $g \in C^1([a, b])$, and $g^{-1} \in C^n([g(a), g(b)])$, $n \in \mathbb{N}$, $(X, \|\cdot\|)$ is a Banach space. Let $f \in C^n([a, b], X)$, and call $l := f \circ g^{-1} : [g(a), g(b)] \to X$. It is clear that $l, l', ..., l^{(n)}$ are continuous functions from $[g(a), g(b)]$ into $f([a, b]) \subseteq X$.

Let $\nu \geq 1$ such that $[\nu] = n$, $n \in \mathbb{N}$ as above, where $[\cdot]$ is the integral part of the number.

Clearly when $0 < \nu < 1$, $[\nu] = 0$.

I) Let $h \in C([g(a), g(b)], X)$, we define the left Riemann-Liouville Bochner fractional integral as

$$\left(J_{\nu}^{z_0} h\right)(z) := \frac{1}{\Gamma(\nu)} \int_{z_0}^{z} (z - t)^{\nu - 1} h(t)\, dt, \tag{14}$$

for $g(a) \leq z_0 \leq z \leq g(b)$, where Γ is the gamma function; $\Gamma(\nu) = \int_0^{\infty} e^{-t} t^{\nu-1} dt$. We set $J_0^{z_0} h = h$.

Let $\alpha := \nu - [\nu]$ $(0 < \alpha < 1)$. We define the subspace $C_{g(x_0)}^{\nu}([g(a), g(b)], X)$ of $C^{[\nu]}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ as:

$$C_{g(x_0)}^{\nu}([g(a), g(b)], X) = \left\{ h \in C^{[\nu]}([g(a), g(b)], X) : J_{1-\alpha}^{g(x_0)} h^{([\nu])} \in C^1([g(x_0), g(b)], X) \right\}. \tag{15}$$

So let $h \in C_{g(x_0)}^{\nu}([g(a), g(b)], X)$, we define the left g-generalized X-valued fractional derivative of h of order ν, of Canavati type, over $[g(x_0), g(b)]$ as

$$D_{g(x_0)}^{\nu} h := \left(J_{1-\alpha}^{g(x_0)} h^{([\nu])} \right)'. \tag{16}$$

Clearly, for $h \in C_{g(x_0)}^{\nu}([g(a), g(b)], X)$, there exists

$$\left(D_{g(x_0)}^{\nu} h \right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^{z} (z - t)^{-\alpha} h^{([\nu])}(t)\, dt, \tag{17}$$

for all $g(x_0) \leq z \leq g(b)$.

In particular, when $f \circ g^{-1} \in C_{g(x_0)}^{\nu}([g(a), g(b)], X)$, we have that

$$\left(D_{g(x_0)}^{\nu} \left(f \circ g^{-1} \right) \right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^{z} (z - t)^{-\alpha} \left(f \circ g^{-1} \right)^{([\nu])}(t)\, dt, \tag{18}$$

for all $g(x_0) \leq z \leq g(b)$. We have that $D_{g(x_0)}^{n}\left(f \circ g^{-1}\right) = \left(f \circ g^{-1}\right)^{(n)}$ and $D_{g(x_0)}^{0}\left(f \circ g^{-1}\right) = f \circ g^{-1}$, see [1].

4

By [1], we have for $f \circ g^{-1} \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, where $x_0 \in [a,b]$ the following left generalized g-fractional, of Canavati type,X-valued Taylor's formula:

Theorem 6 *Let $f \circ g^{-1} \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, where $x_0 \in [a,b]$ is fixed.*
(i) If $\nu \geq 1$, then

$$f(x) - f(x_0) = \sum_{k=1}^{[\nu]-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(x_0))}{k!} (g(x) - g(x_0))^k +$$

$$\frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D^{\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{19}$$

for all $x_0 \leq x \leq b$.
(ii) If $0 < \nu < 1$, we get

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D^{\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{20}$$

for all $x_0 \leq x \leq b$.

II) Let $h \in C([g(a), g(b)], X)$, we define the right Riemann-Liouville Bochner fractional integral as

$$\left(J^{\nu}_{z_0-}h\right)(z) := \frac{1}{\Gamma(\nu)} \int_{z}^{z_0} (t - z)^{\nu-1} h(t)\, dt, \tag{21}$$

for $g(a) \leq z \leq z_0 \leq g(b)$. We set $J^{0}_{z_0-}h = h$.

Let $\alpha := \nu - [\nu]$ $(0 < \alpha < 1)$. We define the subspace $C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$ of $C^{[\nu]}([g(a), g(b)], X)$, where $x_0 \in [a,b]$ as:

$$C^{\nu}_{g(x_0)-}([g(a), g(b)], X) :=$$

$$\left\{h \in C^{[\nu]}([g(a), g(b)], X) : J^{1-\alpha}_{g(x_0)-}h^{([\nu])} \in C^1([g(a), g(x_0)], X)\right\}. \tag{22}$$

So let $h \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, we define the right g-generalized X-valued fractional derivative of h of order ν, of Canavati type, over $[g(a), g(x_0)]$ as

$$D^{\nu}_{g(x_0)-}h := (-1)^{n-1} \left(J^{1-\alpha}_{g(x_0)-}h^{([\nu])}\right)'. \tag{23}$$

Clearly, for $h \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, there exists

$$\left(D^{\nu}_{g(x_0)-}h\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{z}^{g(x_0)} (t - z)^{-\alpha} h^{([\nu])}(t)\, dt, \tag{24}$$

for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

In particular, when $f \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, we have that

$$\left(D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_z^{g(x_0)} (t-z)^{-\alpha} \left(f \circ g^{-1}\right)^{([\nu])}(t)\, dt, \tag{25}$$

for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

We get that

$$\left(D^{n}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(z) = (-1)^n \left(f \circ g^{-1}\right)^{(n)}(z) \tag{26}$$

and $\left(D^{0}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(z) = \left(f \circ g^{-1}\right)(z)$, all $z \in [g(a), g(b)]$, see [1].

By [1], we have for $f \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ is fixed, the following right generalized g-fractional, of Canavati type, X-valued Taylor's formula:

Theorem 7 *Let $f \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ is fixed.*

(i) If $\nu \geq 1$, then

$$f(x) - f(x_0) = \sum_{k=1}^{[\nu]-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(x_0))}{k!} (g(x) - g(x_0))^k +$$

$$\frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{27}$$

for all $a \leq x \leq x_0$,

(ii) If $0 < \nu < 1$, we get

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{28}$$

all $a \leq x \leq x_0$.

III) Denote by

$$D^{m\nu}_{g(x_0)} = D^{\nu}_{g(x_0)} D^{\nu}_{g(x_0)} \ldots D^{\nu}_{g(x_0)} \quad (m\text{-times}),\ m \in \mathbb{N}. \tag{29}$$

We mention the following modified and generalized left X-valued fractional Taylor's formula of Canavati type:

Theorem 8 *Let $f \in C^1([a, b], X)$, $g \in C^1([a, b])$, strictly increasing: $g^{-1} \in C^1([g(a), g(b)])$. Assume that $\left(D^{i\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right) \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, $0 < \nu < 1$, $x_0 \in [a, b]$, for $i = 0, 1, \ldots, m$. Then*

$$f(x) = \frac{1}{\Gamma((m+1)\nu)} \int_{g(x_0)}^{g(x)} (g(x) - z)^{(m+1)\nu-1} \left(D^{(m+1)\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right)(z)\, dz, \tag{30}$$

all $x_0 \leq x \leq b$.

6

IV) Denote by

$$D_{g(x_0)-}^{m\nu} = D_{g(x_0)-}^{\nu} D_{g(x_0)-}^{\nu} ... D_{g(x_0)-}^{\nu} \quad (m \text{ times}),\ m \in \mathbb{N}. \tag{31}$$

We mention the following modified and generalized right X-valued fractional Taylor's formula of Canavati type:

Theorem 9 *Let $f \in C^1([a,b], X)$, $g \in C^1([a,b])$, strictly increasing: $g^{-1} \in C^1([g(a), g(b)])$. Assume that $\left(D_{g(x_0)-}^{i\nu}(f \circ g^{-1})\right) \in C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$, $0 < \nu < 1$, $x_0 \in [a,b]$, for all $i = 0, 1, ..., m$. Then*

$$f(x) = \frac{1}{\Gamma((m+1)\nu)} \int_{g(x)}^{g(x_0)} (z - g(x))^{(m+1)\nu - 1} \left(D_{g(x_0)-}^{(m+1)\nu}(f \circ g^{-1})\right)(z)\, dz, \tag{32}$$

all $a \leq x \leq x_0 \leq b$.

3 Banach Algebras background

All here come from [4].

We need

Definition 10 *([4], p. 245) A complex algebra is a vector space A over the complex filed $\mathbb{C}$ in which a multiplication is defined that satisfies*

$$x(yz) = (xy)z, \tag{33}$$

$$(x+y)z = xz + yz, \ \ x(y+z) = xy + xz, \tag{34}$$

and

$$\alpha(xy) = (\alpha x)y = x(\alpha y), \tag{35}$$

for all x, y and z in A and for all scalars α.

Additionally if A is a Banach space with respect to a norm that satisfies the multiplicative inequality

$$\|xy\| \leq \|x\| \|y\| \quad (x \in A,\ y \in A) \tag{36}$$

and if A contains a unit element e such that

$$xe = ex = x \quad (x \in A) \tag{37}$$

and

$$\|e\| = 1, \tag{38}$$

then A is called a Banach algebra.

A is commutative iff $xy = yx$ for all $x, y \in A$.

We make

Remark 11 *Commutativity of A will be explicited stated when needed.*

There exists at most one $e \in A$ that satisfies (37).

Inequality (36) makes multiplication to be continuous, more precisely left and right continuous, see [4], p. 246.

Multiplication in A is not necessarily the numerical multiplication, it is something more general and it is defined abstractly, that is for $x, y \in A$ we have $xy \in A$, e.g. composition or convolution, etc.

For nice examples about Banach algebras see [4], p. 247-248, § 10.3.

We also make

Remark 12 *Next we mention about integration of A-valued functions, see [4], p. 259, § 10.22:*

If A is a Banach algebra and f is a continuous A-valued function on some compact Hausdorff space Q on which a complex Borel measure μ is defined, then $\int f d\mu$ exists and has all the properties that were discussed in Chapter 3 of [4], simply because A is a Banach space. However, an additional property can be added to these, namely: If $x \in A$, then

$$x \int_Q f \; d\mu = \int_Q x f(p) \; d\mu(p) \tag{39}$$

and

$$\left(\int_Q f \; d\mu \right) x = \int_Q f(p) \, x \; d\mu(p) \, . \tag{40}$$

The Bochner integrals we will involve in our article follow (39) and (40). Also, let $f \in C([a,b], X)$, where $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ is a Banach space. By [2], p. 3, f is Bochner integrable.

4 Main Results

We start with a left generalized Canavati fractional Hilbert-Pachpatte type inequality over a Banach algebra.

Theorem 13 *Let $p, q > 1$, such that $\frac{1}{p} + \frac{1}{q} = 1$, and $(A, \|\cdot\|)$ is a Banach algebra; and $i = 1, 2$. Let also $x_{0i} \in [a_i, b_i] \subset \mathbb{R}$, $\nu_i \geq 1$, $n_i = [\nu_i]$, $f_i \in C^{n_i}([a_i, b_i], A)$; $g_i \in C^1([a_i, b_i])$, strictly increasing, such that $g_i^{-1} \in C^{n_i}([g_i(a_i), g_i(b_i)])$, with $\left(f_i \circ g_i^{-1}\right)^{(k_i)}(g_i(x_{0i})) = 0$, $k_i = 0, 1, ..., n_i - 1$. Assume further that $f_i \circ g_i^{-1} \in C^{\nu_i}_{g_i(x_{0i})}([g_i(a_i), g_i(b_i)], A)$. Then*

$$\int_{g_1(x_{01})}^{g_1(b_1)} \int_{g_2(x_{02})}^{g_2(b_2)} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\| dz_1 dz_2}{\left(\frac{(z_1 - g_1(x_{01}))^{p(\nu_1 - 1) + 1}}{p(p(\nu_1 - 1) + 1)} + \frac{(z_2 - g_2(x_{02}))^{q(\nu_2 - 1) + 1}}{q(q(\nu_2 - 1) + 1)} \right)} \leq$$

8

$$\frac{\left(g_1\left(b_1\right)-g_1\left(x_{01}\right)\right)\left(g_2\left(b_2\right)-g_2\left(x_{02}\right)\right)}{\Gamma\left(\nu_1\right)\Gamma\left(\nu_2\right)} \tag{41}$$

$$\left\|\left\|D_{g_1(x_{01})}^{\nu_1}\left(f_1\circ g_1^{-1}\right)\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)],A)}\left\|\left\|D_{g_2(x_{02})}^{\nu_2}\left(f_2\circ g_2^{-1}\right)\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)],A)}.$$

Proof. By (19) and assumptions we get that

$$\left(f_i\circ g_i^{-1}\right)\left(z_i\right)=\frac{1}{\Gamma\left(\nu_i\right)}\int_{g_i(x_{0i})}^{z_i}\left(z_i-t_i\right)^{\nu_i-1}\left(D_{g_i(x_{0i})}^{\nu_i}\left(f_i\circ g_i^{-1}\right)\right)\left(t_i\right)dt_i, \tag{42}$$

for all $g_i\left(x_{0i}\right)\leq z_i\leq g_i\left(b_i\right)$; $i=1,2$.

By Hölder's inequality we obtain

$$\left\|\left(f_1\circ g_1^{-1}\right)\left(z_1\right)\right\|\leq\frac{1}{\Gamma\left(\nu_1\right)}\int_{g_1(x_{01})}^{z_1}\left(z_1-t_1\right)^{\nu_1-1}\left\|\left(D_{g_1(x_{01})}^{\nu_1}\left(f_1\circ g_1^{-1}\right)\right)\left(t_1\right)\right\|dt_1\leq$$

$$\frac{1}{\Gamma\left(\nu_1\right)}\left(\int_{g_1(x_{01})}^{z_1}\left(z_1-t_1\right)^{p(\nu_1-1)}dt_1\right)^{\frac{1}{p}}\left(\int_{g_1(x_{01})}^{z_1}\left\|\left(D_{g_1(x_{01})}^{\nu_1}\left(f_1\circ g_1^{-1}\right)\right)\left(t_1\right)\right\|^q dt_1\right)^{\frac{1}{q}}=$$

$$\frac{1}{\Gamma\left(\nu_1\right)}\frac{\left(z_1-g_1\left(x_{01}\right)\right)^{\frac{p(\nu_1-1)+1}{p}}}{\left(p\left(\nu_1-1\right)+1\right)^{\frac{1}{p}}}\left(\int_{g_1(x_{01})}^{z_1}\left\|\left(D_{g_1(x_{01})}^{\nu_1}\left(f_1\circ g_1^{-1}\right)\right)\left(t_1\right)\right\|^q dt_1\right)^{\frac{1}{q}}. \tag{43}$$

That is

$$\left\|\left(f_1\circ g_1^{-1}\right)\left(z_1\right)\right\|\leq\frac{1}{\Gamma\left(\nu_1\right)}\frac{\left(z_1-g_1\left(x_{01}\right)\right)^{\frac{p(\nu_1-1)+1}{p}}}{\left(p\left(\nu_1-1\right)+1\right)^{\frac{1}{p}}}$$

$$\left(\int_{g_1(x_{01})}^{z_1}\left\|\left(D_{g_1(x_{01})}^{\nu_1}\left(f_1\circ g_1^{-1}\right)\right)\left(t_1\right)\right\|^q dt_1\right)^{\frac{1}{q}}, \tag{44}$$

for all $g_1\left(x_{01}\right)\leq z_1\leq g_1\left(b_1\right)$.

Similarly, we prove that

$$\left\|\left(f_2\circ g_2^{-1}\right)\left(z_2\right)\right\|\leq\frac{1}{\Gamma\left(\nu_2\right)}\frac{\left(z_2-g_2\left(x_{02}\right)\right)^{\frac{q(\nu_2-1)+1}{q}}}{\left(q\left(\nu_2-1\right)+1\right)^{\frac{1}{q}}}$$

$$\left(\int_{g_2(x_{02})}^{z_2}\left\|\left(D_{g_2(x_{02})}^{\nu_2}\left(f_2\circ g_2^{-1}\right)\right)\left(t_2\right)\right\|^p dt_2\right)^{\frac{1}{p}}, \tag{45}$$

for all $g_2\left(x_{02}\right)\leq z_2\leq g_2\left(b_2\right)$.

Therefore we have

$$\left\|\left(f_1\circ g_1^{-1}\right)\left(z_1\right)\right\|\leq\frac{1}{\Gamma\left(\nu_1\right)}\frac{\left(z_1-g_1\left(x_{01}\right)\right)^{\frac{p(\nu_1-1)+1}{p}}}{\left(p\left(\nu_1-1\right)+1\right)^{\frac{1}{p}}}$$

$$\left\|\left\|\left(D^{\nu_1}_{g_1(x_{01})}\left(f_1\circ g_1^{-1}\right)\right)\right\|\right\|_{q,[g_1(x_{01}),g_1(b_1)]}, \tag{46}$$

for all $g_1(x_{01})\leq z_1\leq g_1(b_1)$;

and

$$\left\|\left(f_2\circ g_2^{-1}\right)(z_2)\right\|\leq\frac{1}{\Gamma(\nu_2)}\frac{(z_2-g_2(x_{02}))^{\frac{q(\nu_2-1)+1}{q}}}{(q(\nu_2-1)+1)^{\frac{1}{q}}}$$
$$\left\|\left\|\left(D^{\nu_2}_{g_2(x_{02})}\left(f_2\circ g_2^{-1}\right)\right)\right\|\right\|_{p,[g_2(x_{02}),g_2(b_2)]}, \tag{47}$$

for all $g_2(x_{02})\leq z_2\leq g_2(b_2)$.

Hence we get that

$$\left\|\left(f_1\circ g_1^{-1}\right)(z_1)\right\|\left\|\left(f_2\circ g_2^{-1}\right)(z_2)\right\|\leq\frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)(p(\nu_1-1)+1)^{\frac{1}{p}}(q(\nu_2-1)+1)^{\frac{1}{q}}}$$
$$(z_1-g_1(x_{01}))^{\frac{p(\nu_1-1)+1}{p}}(z_2-g_2(x_{02}))^{\frac{q(\nu_2-1)+1}{q}} \tag{48}$$
$$\left\|\left\|\left(D^{\nu_1}_{g_1(x_{01})}\left(f_1\circ g_1^{-1}\right)\right)\right\|\right\|_{q,[g_1(x_{01}),g_1(b_1)]}\left\|\left\|\left(D^{\nu_2}_{g_2(x_{02})}\left(f_2\circ g_2^{-1}\right)\right)\right\|\right\|_{p,[g_2(x_{02}),g_2(b_2)]}\leq$$

(using Young's inequality for $a,b\geq 0$, $a^{\frac{1}{p}}b^{\frac{1}{q}}\leq\frac{a}{p}+\frac{b}{q}$)

$$\frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)}\left(\frac{(z_1-g_1(x_{01}))^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(z_2-g_2(x_{02}))^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)$$
$$\left\|\left\|\left(D^{\nu_1}_{g_1(x_{01})}\left(f_1\circ g_1^{-1}\right)\right)\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)],A)}\left\|\left\|\left(D^{\nu_2}_{g_2(x_{02})}\left(f_2\circ g_2^{-1}\right)\right)\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)],A)},$$

$\forall\,(z_1,z_2)\in[g_1(x_{01}),g_1(b_1)]\times[g_2(x_{02}),g_2(b_2)]$.

So far we have

$$\frac{\left\|\left(f_1\circ g_1^{-1}\right)(z_1)\left(f_2\circ g_2^{-1}\right)(z_2)\right\|}{\left(\frac{(z_1-g_1(x_{01}))^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(z_2-g_2(x_{02}))^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)}\leq \tag{50}$$

$$\frac{\left\|\left(f_1\circ g_1^{-1}\right)(z_1)\right\|\left\|\left(f_2\circ g_2^{-1}\right)(z_2)\right\|}{\left(\frac{(z_1-g_1(x_{01}))^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(z_2-g_2(x_{02}))^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)}\leq \tag{51}$$

$$\frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)}\left\|\left\|\left(D^{\nu_1}_{g_1(x_{01})}\left(f_1\circ g_1^{-1}\right)\right)\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)],A)}$$
$$\left\|\left\|\left(D^{\nu_2}_{g_2(x_{02})}\left(f_2\circ g_2^{-1}\right)\right)\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)],A)},$$

$\forall\,(z_1,z_2)\in[g_1(x_{01}),g_1(b_1)]\times[g_2(x_{02}),g_2(b_2)]$.

The denominators in (50), (51) can be zero only when both $z_1=g_1(x_{01})$ and $z_2=g_2(x_{02})$.

Therefore we obtain (41), by integrating (50), (51) over $[g_1(x_{01}),g_1(b_1)]\times[g_2(x_{02}),g_2(b_2)]$. ■

We continue with a right generalized Canavati fractional Hilbert-Pachpatte type inequality over a Banach algebra.

Theorem 14 *All as in Theorem 13, however now it is* $f_i \circ g_i^{-1} \in C^{\nu_i}_{g_i(x_{0i})-}([g_i(a_i), g_i(b_i)], A)$, *for* $i = 1, 2$. *Then*

$$\int_{g_1(a_1)}^{g_1(x_{01})} \int_{g_2(a_2)}^{g_2(x_{02})} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\| dz_1 dz_2}{\left(\frac{(g_1(x_{01}) - z_1)^{p(\nu_1 - 1) + 1}}{p(p(\nu_1 - 1) + 1)} + \frac{(g_2(x_{02}) - z_2)^{q(\nu_2 - 1) + 1}}{q(q(\nu_2 - 1) + 1)} \right)} \leq$$

$$\frac{(g_1(x_{01}) - g_1(a_1))(g_2(x_{02}) - g_2(a_2))}{\Gamma(\nu_1)\Gamma(\nu_2)} \tag{52}$$

$$\left\| \left\| D^{\nu_1}_{g_1(x_{01})-} \left(f_1 \circ g_1^{-1}\right) \right\| \right\|_{L_q([g_1(a_1), g_1(x_{01})], A)} \left\| \left\| D^{\nu_2}_{g_2(x_{02})-} \left(f_2 \circ g_2^{-1}\right) \right\| \right\|_{L_p([g_2(a_2), g_2(x_{02})], A)}.$$

Proof. Similar to Theorem 13, by using now (27). ■

Next comes a sequential left generalized Canavati fractional Hilbert-Pachpatte type inequality over a Banach algebra.

Theorem 15 *Let* $p, q > 1$, *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *and* $(A, \|\cdot\|)$ *is a Banach algebra; and* $i = 1, 2$. *Let also* $f_i \in C^1([a_i, b_i], A)$; $g_i \in C^1([a_i, b_i])$, *strictly increasing, such that* $g_i^{-1} \in C^1([g_i(a_i), g_i(b_i)])$. *Assume that* $\frac{1}{(m_i+1)q} < \nu_i < 1$, $x_{0i} \in [a_i, b_i]$, *and* $D^{j_i \nu_i}_{g_i(x_{0i})} \left(f_i \circ g_i^{-1}\right) \in C^{\nu_i}_{g_i(x_{0i})}([g_i(a_i), g_i(b_i)], A)$, *for* $j_i = 0, 1, ..., m_i \in \mathbb{N}$. *Then*

$$\int_{g_1(x_{01})}^{g_1(b_1)} \int_{g_2(x_{02})}^{g_2(b_2)} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\| dz_1 dz_2}{\left(\frac{(z_1 - g_1(x_{01}))^{p((m_1+1)\nu_1 - 1) + 1}}{p(p((m_1+1)\nu_1 - 1) + 1)} + \frac{(z_2 - g_2(x_{02}))^{q((m_2+1)\nu_2 - 1) + 1}}{q(q((m_2+1)\nu_2 - 1) + 1)} \right)} \leq$$

$$\frac{(g_1(b_1) - g_1(x_{01}))(g_2(b_2) - g_2(x_{02}))}{\Gamma((m_1+1)\nu_1)\Gamma((m_2+1)\nu_2)} \tag{53}$$

$$\left\| \left\| D^{(m_1+1)\nu_1}_{g_1(x_{01})} \left(f_1 \circ g_1^{-1}\right) \right\| \right\|_{L_q([g_1(x_{01}), g_1(b_1)], A)} \left\| \left\| D^{(m_2+1)\nu_2}_{g_2(x_{02})} \left(f_2 \circ g_2^{-1}\right) \right\| \right\|_{L_p([g_2(x_{02}), g_2(b_2)], A)}.$$

Proof. Using (30), as similar to Theorem 13 the proof is omitted. ■

The right side analog of Theorem 15 follows:

Theorem 16 *Let* $p, q > 1$, *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *and* $(A, \|\cdot\|)$ *is a Banach algebra; and* $i = 1, 2$. *Let also* $f_i \in C^1([a_i, b_i], A)$; $g_i \in C^1([a_i, b_i])$, *strictly increasing, such that* $g_i^{-1} \in C^1([g_i(a_i), g_i(b_i)])$. *Assume that* $\frac{1}{(m_i+1)q} < \nu_i < 1$, $x_{0i} \in [a_i, b_i]$, *and* $D^{j_i \nu_i}_{g_i(x_{0i})-} \left(f_i \circ g_i^{-1}\right) \in C^{\nu_i}_{g_i(x_{0i})-}([g_i(a_i), g_i(b_i)], A)$, *for* $j_i = 0, 1, ..., m_i \in \mathbb{N}$. *Then*

$$\int_{g_1(a_1)}^{g_1(x_{01})} \int_{g_2(a_2)}^{g_2(x_{02})} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\| dz_1 dz_2}{\left(\frac{(g_1(x_{01}) - z_1)^{p((m_1+1)\nu_1 - 1) + 1}}{p(p((m_1+1)\nu_1 - 1) + 1)} + \frac{(g_2(x_{02}) - z_2)^{q((m_2+1)\nu_2 - 1) + 1}}{q(q((m_2+1)\nu_2 - 1) + 1)} \right)} \leq$$

$$\frac{(g_1(x_{01}) - g_1(a_1))(g_2(x_{02}) - g_2(a_2))}{\Gamma((m_1+1)\nu_1)\Gamma((m_2+1)\nu_2)} \tag{54}$$

$$\left\| \left\| D^{(m_1+1)\nu_1}_{g_1(x_{01})-} \left(f_1 \circ g_1^{-1}\right) \right\| \right\|_{L_q([g_1(a_1), g_1(x_{01})], A)} \left\| \left\| D^{(m_2+1)\nu_2}_{g_2(x_{02})-} \left(f_2 \circ g_2^{-1}\right) \right\| \right\|_{L_p([g_2(a_2), g_2(x_{02})], A)}.$$

Proof. Using (32), as similar to Theorem 13 is omitted. ■

5 Applications

We give

Corollary 17 *(to Theorem 13) All as in Theorem 13 for* $g_i(t) = e^t$, $i = 1, 2$. *Then*

$$\int_{e^{x_{01}}}^{e^{b_1}} \int_{e^{x_{02}}}^{e^{b_2}} \frac{\|(f_1 \circ \log)(z_1)(f_2 \circ \log)(z_2)\| \, dz_1 dz_2}{\left(\frac{(z_1 - e^{x_{01}})^{p(\nu_1 - 1)+1}}{p(p(\nu_1-1)+1)} + \frac{(z_2 - e^{x_{02}})^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)} \right)} \leq$$

$$\frac{\left(e^{b_1} - e^{x_{01}}\right)\left(e^{b_2} - e^{x_{02}}\right)}{\Gamma(\nu_1)\Gamma(\nu_2)} \tag{55}$$

$$\left\| \left\| D_{e^{x_{01}}}^{\nu_1} (f_1 \circ \log) \right\| \right\|_{L_q\left(\left[e^{x_{01}}, e^{b_1}\right], A\right)} \left\| \left\| D_{e^{x_{02}}}^{\nu_2} (f_2 \circ \log) \right\| \right\|_{L_p\left(\left[e^{x_{02}}, e^{b_2}\right], A\right)}.$$

We finish with

Corollary 18 *(to Theorem 15) All as in Theorem 15 for* $[a_1, b_1] \subset \mathbb{R}$, $[a_2, b_2] \subset (0, \infty)$, *and* $g_1(t) = e^t$ *and* $g_2(t) = \log t$. *Then*

$$\int_{e^{x_{01}}}^{e^{b_1}} \int_{\log(x_{02})}^{\log(b_2)} \frac{\|(f_1 \circ \log)(z_1)(f_2 \circ e^t)(z_2)\| \, dz_1 dz_2}{\left(\frac{(z_1 - e^{x_{01}})^{p((m_1+1)\nu_1 - 1)+1}}{p(p((m_1+1)\nu_1-1)+1)} + \frac{(z_2 - \log(x_{02}))^{q((m_2+1)\nu_2-1)+1}}{q(q((m_2+1)\nu_2-1)+1)} \right)} \leq$$

$$\frac{\left(e^{b_1} - e^{x_{01}}\right)\log(b_2/x_{02})}{\Gamma((m_1+1)\nu_1)\Gamma((m_2+1)\nu_2)} \tag{56}$$

$$\left\| \left\| D_{e^{x_{01}}}^{(m_1+1)\nu_1} (f_1 \circ \log) \right\| \right\|_{L_q\left(\left[e^{x_{01}}, e^{b_1}\right], A\right)} \left\| \left\| D_{\log(x_{02})}^{(m_2+1)\nu_2} \left(f_2 \circ e^t\right) \right\| \right\|_{L_p([\log(x_{02}), \log(b_2)], A)}.$$

References

[1] G.A. Anastassiou, *Strong mixed and generalized fractional calculus for Banach space valued functions*, Mat. Vesnik, 69(3) (2017), 176-191.

[2] G.A. Anastassiou, *Intelligent Computations: Abstract Fractional Calculus, Inequalities, Approximations*, Springer, Heidelberg, New York, 2018.

[3] J. Mikusinski, *The Bochner integral*, Academic Press, New York, 1978.

[4] W. Rudin, *Functional Analysis*, Second Edition, McGraw-Hill, Inc., New York, 1991.

[5] C. Volintiru, *A proof of the fundamental theorem of Calculus using Hausdorff measures*, Real Analysis Exchange, 26 (1), 2000/2001, 381-390.

Generalized Ostrowski, Opial and Hilbert-Pachpatte type inequalities for Banach algebra valued functions involving integer vectorial derivatives

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Using a generalized vectorial Taylor formula involving ordinary vector derivatives we establish mixed Ostrowski, Opial and Hilbert-Pachpatte type inequalities for several Banach algebra valued functions. The estimates are with respect to all norms $\|\cdot\|_p$, $1 \leq p \leq \infty$. We finish with applications.

2020 Mathematics Subject Classification : 26D10, 26D15.
Keywords and Phrases: vector valued derivative, generalized integral inequalities, Ostrowski-Opial-Hilbert-Pachpatte inequalities, Banach algebra.

1 Introduction

The following result motivates our work.

Theorem 1 *(1938, Ostrowski [6]) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty^{\sup} := \sup_{t \in (a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty^{\sup}, \qquad (1)$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

1

Ostrowski type inequalities have great applications to integral approximations in Numerical Analysis.

We present ([1], Ch. 8,9) mixed fractional Ostrowski inequalities for several functions for various norms.

In this article we generalize [1], Ch. 8,9 for several Banach algebra valued functions by using ordinary vector valued derivatives and our integrals here are of Bochner type [4]. Motivation comes also from [3].

We are also inspired by Z. Opial [5], 1960, famous inequality.

Theorem 2 *Let $x(t) \in C^1([0,h])$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then*

$$\int_0^h |x(t)\, x'(t)|\, dt \leq \frac{h}{4} \int_0^h (x'(t))^2\, dt. \tag{2}$$

In (2), the constant $\frac{h}{4}$ is the best possible. Inequality (2) holds as equality for the optimal function

$$x(t) = \begin{cases} ct, & 0 \leq t \leq \frac{h}{2}, \\ c(h-t), & \frac{h}{2} \leq t \leq h, \end{cases} \tag{3}$$

where $c > 0$ is an arbitrary constant.

Opial-type inequalities are used a lot in proving uniqueness of solutions to differential equations and also to give upper bounds to their solutions.

In this work we also derive Opial type inequalities for Banach algebra valued functions with respect to ordinary vector valued derivatives.

Additionally we include in this article related Hilbert-Pachpatte type inequalities, [7]. We finish with selective applications to Ostrowski, Opial and Hilbert-Pachpatte inequalities.

2 About Banach Algebras

All here come from [8].

We need

Definition 3 *([8], p. 245) A complex algebra is a vector space A over the complex field $\mathbb{C}$ in which a multiplication is defined that satisfies*

$$x(yz) = (xy)z, \tag{4}$$

$$(x+y)z = xz + yz, \quad x(y+z) = xy + xz, \tag{5}$$

and

$$\alpha(xy) = (\alpha x)y = x(\alpha y), \tag{6}$$

for all x, y and z in A and for all scalars α.

Additionally if A is a Banach space with respect to a norm that satisfies the multiplicative inequality

$$\|xy\| \leq \|x\| \, \|y\| \quad (x \in A, \, y \in A) \tag{7}$$

and if A contains a unit element e such that

$$xe = ex = x \quad (x \in A) \tag{8}$$

and

$$\|e\| = 1, \tag{9}$$

then A is called a Banach algebra.

A is commutative iff $xy = yx$ for all $x, y \in A$.

We make

Remark 4 *Commutativity of A will be explicited stated when needed.*

There exists at most one $e \in A$ that satisfies (8).

Inequality (7) makes multiplication to be continuous, more precisely left and right continuous, see [8], p. 246.

Multiplication in A is not necessarily the numerical multiplication, it is something more general and it is defined abstractly, that is for $x, y \in A$ we have $xy \in A$, e.g. composition or convolution, etc.

For nice examples about Banach algebras see [8], p. 247-248, § 10.3.

We also make

Remark 5 *Next we mention about integration of A-valued functions, see [8], p. 259, § 10.22:*

If A is a Banach algebra and f is a continuous A-valued function on some compact Hausdorff space Q on which a complex Borel measure μ is defined, then $\int f d\mu$ exists and has all the properties that were discussed in Chapter 3 of [8], simply because A is a Banach space. However, an additional property can be added to these, namely: If $x \in A$, then

$$x \int_Q f \, d\mu = \int_Q x f(p) \, d\mu(p) \tag{10}$$

and

$$\left(\int_Q f \, d\mu \right) x = \int_Q f(p) \, x \, d\mu(p) . \tag{11}$$

The Bochner integrals we will involve in our article follow (10) and (11).

3

3 Background

We use the following generalized vector Taylor's formula:

Theorem 6 *([2], p. 97) Let $n \in \mathbb{N}$ and $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. Let any $x, y \in [a,b]$. Then*

$$f(x) = f(y) + \sum_{i=1}^{n-1} \frac{(g(x) - g(y))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(y)) \tag{12}$$

$$+ \frac{1}{(n-1)!} \int_{g(y)}^{g(x)} (g(x) - z)^{n-1} \left(f \circ g^{-1}\right)^{(n)} (z) \, dz.$$

The derivatives here are defined similarly to the numerical ones, see [9], pp. 83-86.

The above integral is of Bochner type [4], and so are the integrals in this work. By [2], p. 3, if $f \in C([a,b], X)$ then f is Bochner integrable.

4 Main Results

We start with mixed generalized Ostrowski type inequalities for several functions that are Banach algebra valued. A uniform estimate follows.

Theorem 7 *Let $n \in \mathbb{N}$ and $f_i \in C^n([a,b], A)$, $i = 1, ..., r \in \mathbb{N} - \{1\}$; where $[a,b] \subset \mathbb{R}$ and $(A, \|\cdot\|)$ is a Banach algebra. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. We assume that $\left(f_i \circ g^{-1}\right)^{(j)} (g(x_0)) = 0$, $j = 1, ..., n-1$; $i = 1, ..., r$; where $x_0 \in [a,b]$ be fixed. Denote by*

$$E(f_1, ..., f_r)(x_0) :=$$

$$\sum_{i=1}^{r} \left[\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x) \, dx - \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) dx \right) f_i(x_0) \right]. \tag{13}$$

Then

1)

$$E(f_1, ..., f_r)(x_0) = \frac{1}{(n-1)!}$$

$$\sum_{i=1}^{r} \left[(-1)^n \left[\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) \left(\int_{g(x)}^{g(x_0)} (z - g(x))^{n-1} \left(f_i \circ g^{-1}\right)^{(n)} (z) \, dz \right) dx \right] + \right. \tag{14}$$

4

$$\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]\Bigg],$$

and

2)

$$\|E(f_1,...,f_r)(x_0)\|\leq\frac{1}{n!}$$

$$\left\{\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{\infty,[g(a),g(x_0)]}(g(x_0)-g(a))^n\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]+\right.\right.$$

$$\left.\left.\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{\infty,[g(x_0),g(b)]}(g(b)-g(x_0))^n\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]\right]\right\}. \tag{15}$$

Proof. Let $x_0\in[a,b]$ such that $\left(f_i\circ g^{-1}\right)^{(j)}(g(x_0))=0$, $j=1,...,n-1$; $i=1,...,r$. Let $x\in[a,x_0]$, then by Theorem 6 we have

$$f_i(x)-f_i(x_0)=\frac{1}{(n-1)!}\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz \tag{16}$$

$$=\frac{(-1)^n}{(n-1)!}\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz,$$

for $i=1,...,r$.

And for $x\in[x_0,b]$, then again by Theorem 6 we get

$$f_i(x)-f_i(x_0)=\frac{1}{(n-1)!}\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz, \tag{17}$$

for $i=1,...,r$.

We multiply (16) by $\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)$ to get:

$$\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)f_i(x)-\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)f_i(x_0)=$$

$$\frac{\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)(-1)^n}{(n-1)!}\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz, \tag{18}$$

5

$\forall\ x \in [a, x_0]$; for $i = 1, ..., r$.

Similarly, we get (by (17))

$$\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x) - \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x_0) =$$

$$\frac{\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)}{(n-1)!} \int_{g(x_0)}^{g(x)} (g(x) - z)^{n-1} \left(f_i \circ g^{-1}\right)^{(n)}(z)\, dz, \tag{19}$$

$\forall\ x \in [x_0, b]$; for $i = 1, ..., r$.

Adding (18) and (19) as separate groups, we obtain

$$\sum_{i=1}^{r} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x) - \sum_{i=1}^{r} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x_0) =$$

$$\frac{(-1)^n}{(n-1)!} \sum_{i=1}^{r} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) \int_{g(x)}^{g(x_0)} (z - g(x))^{n-1} \left(f_i \circ g^{-1}\right)^{(n)}(z)\, dz, \tag{20}$$

$\forall\ x \in [a, x_0]$,

and

$$\sum_{i=1}^{r} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x) - \sum_{i=1}^{r} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x_0) =$$

$$\frac{1}{(n-1)!} \sum_{i=1}^{r} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) \int_{g(x_0)}^{g(x)} (g(x) - z)^{n-1} \left(f_i \circ g^{-1}\right)^{(n)}(z)\, dz, \tag{21}$$

$\forall\ x \in [x_0, b]$.

Next, we integrate (20) and (21) with respect to $x \in [a, b]$. We have

$$\sum_{i=1}^{r} \int_a^{x_0} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x)\, dx - \sum_{i=1}^{r} \left(\int_a^{x_0} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) dx\right) f_i(x_0) = \tag{22}$$

$$\frac{(-1)^n}{(n-1)!} \sum_{i=1}^{r} \left[\int_a^{x_0} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) \left(\int_{g(x)}^{g(x_0)} (z - g(x))^{n-1} \left(f_i \circ g^{-1}\right)^{(n)}(z)\, dz\right) dx\right],$$

and

$$\sum_{i=1}^{r}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x)\,dx-\sum_{i=1}^{r}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) dx\right) f_i(x_0)= \tag{23}$$

$$\frac{1}{(n-1)!}\sum_{i=1}^{r}\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right].$$

Finally, adding (22) and (23) we obtain the useful identity

$$E(f_1,...,f_r)(x_0):=$$

$$\sum_{i=1}^{r}\left[\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x)\,dx-\left(\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) dx\right) f_i(x_0)\right]=\frac{1}{(n-1)!}$$

$$\sum_{i=1}^{r}\left[(-1)^n\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]+\right.$$

$$\left.\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]\right], \tag{24}$$

proving (14).

Therefore, we get that

$$\|E(f_1,...,f_r)(x_0)\|=$$

$$\left\|\sum_{i=1}^{r}\left[\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x)\,dx-\left(\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) dx\right) f_i(x_0)\right]\right\|\leq\frac{1}{(n-1)!}$$

$$\left\{\sum_{i=1}^{r}\left[\left\|\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]\right\|\right.\right.$$

$$\left.\left.+\left\|\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]\right\|\right]\right\}\leq$$

$$\frac{1}{(n-1)!}\left\{\sum_{i=1}^{r}\left[\left[\int_{a}^{x_0}\left\|\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)\right\|dx\right]\right.\right.$$

$$\left.\left.+\left[\int_{x_0}^{b}\left\|\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)\right\|dx\right]\right]\right\}\leq \tag{25}$$

$$\frac{1}{(n-1)!}\left\{\sum_{i=1}^{r}\left[\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left\|\left(f_i\circ g^{-1}\right)^{(n)}(z)\right\|dz\right)dx\right]\right.\right.$$

$$\left.\left.+\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left\|\left(f_i\circ g^{-1}\right)^{(n)}(z)\right\|dz\right)dx\right]\right]\right\}=:(\xi). \tag{26}$$

Hence it holds

$$\|E(f_1,...,f_r)(x_0)\|\leq(\xi). \tag{27}$$

We have that

$$(\xi)\leq\frac{1}{n!}\left\{\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{\infty,[g(a),g(x_0)]}\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)(g(x_0)-g(x))^n\,dx\right]\right.\right.$$

$$\left.\left.+\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)(g(x)-g(x_0))^n\,dx\right]\right]\right\}\leq \tag{28}$$

$$\frac{1}{n!}\left\{\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{\infty,[g(a),g(x_0)]}(g(x_0)-g(a))^n\left(\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]\right.\right.$$

$$\left.\left.+\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{\infty,[g(x_0),g(b)]}(g(b)-g(x_0))^n\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]\right]\right\}, \tag{29}$$

proving (15). ■

Next comes an L_1 estimate.

Theorem 8 *All as in Theorem 7. Then*

$$\|E(f_1,...,f_r)(x_0)\| \leq \frac{1}{(n-1)!}$$

$$\left\{\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)(g(x_0)-g(x))^{n-1}dx\right]\right.\right.$$

$$\left.\left.+\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)(g(x)-g(x_0))^{n-1}dx\right]\right]\right\}. \tag{30}$$

Proof. By (26), (27), we get that

$$\|E(f_1,...,f_r)(x_0)\| \leq (\xi) \leq \frac{1}{(n-1)!}$$

$$\left\{\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)(g(x_0)-g(x))^{n-1}dx\right]\right.\right.$$

$$\left.\left.+\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)(g(x)-g(x_0))^{n-1}dx\right]\right]\right\}, \tag{31}$$

proving (30). ■

An L_p estimate follows.

Theorem 9 *All as in Theorem 7, and let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\|E(f_1,...,f_r)(x_0)\| \leq \frac{1}{(n-1)!\,(p(n-1)+1)^{\frac{1}{p}}}$$

$$\sum_{i=1}^{r}\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{L_q([g(a),g(x_0)])}\left(\int_a^{x_0}(g(x_0)-g(x))^{n-\frac{1}{q}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right.$$

$$\left.+\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{L_q([g(x_0),g(b)])}\left(\int_{x_0}^{b}(g(x)-g(x_0))^{n-\frac{1}{q}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]. \tag{32}$$

9

Proof. By (26), (27), we get that

$$
\begin{aligned}
&\left\|E\left(f_1,...,f_r\right)\left(x_0\right)\right\| \leq (\xi) \leq \frac{1}{(n-1)!} \\
&\left\{\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|\right)\left(\int_{g(x)}^{g(x_0)}\left(z-g(x)\right)^{p(n-1)}dz\right)^{\frac{1}{p}}\right.\right.\right. \\
&\left.\left(\int_{g(x)}^{g(x_0)}\left\|\left(f_i\circ g^{-1}\right)^{(n)}(z)\right\|^{q}dz\right)^{\frac{1}{q}}dx\right]+ \\
&\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|\right)\left(\int_{g(x_0)}^{g(x)}\left(g(x)-z\right)^{p(n-1)}dz\right)^{\frac{1}{p}}\right. \\
&\left.\left.\left.\left(\int_{g(x_0)}^{g(x)}\left\|\left(f_i\circ g^{-1}\right)^{(n)}(z)\right\|^{q}dz\right)^{\frac{1}{q}}dx\right]\right]\right\} = \frac{1}{(n-1)!}
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
&\left\{\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|\right)\frac{\left(g(x_0)-g(x)\right)^{\frac{p(n-1)|+1}{p}}}{\left(p(n-1)+1\right)^{\frac{1}{p}}}\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{L_q([g(a),g(x_0)])}dx\right]\right.\right. \\
&\left.\left.+\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|\right)\frac{\left(g(x)-g(x_0)\right)^{\frac{p(n-1)|+1}{p}}}{\left(p(n-1)+1\right)^{\frac{1}{p}}}\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{L_q([g(x_0),g(b)])}dx\right]\right]\right\} \\
&= \frac{1}{(n-1)!\left(p(n-1)+1\right)^{\frac{1}{p}}} \\
&\left\{\sum_{i=1}^{r}\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{L_q([g(a),g(x_0)])}\left(\int_a^{x_0}\left(g(x_0)-g(x)\right)^{n-\frac{1}{q}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|\right)dx\right)\right.\right. \\
&\left.\left.+\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{L_q([g(x_0),g(b)])}\left(\int_{x_0}^{b}\left(g(x)-g(x_0)\right)^{n-\frac{1}{q}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|\right)dx\right)\right]\right\},
\end{aligned}
\tag{34}
$$

proving (32). ■

Next we present a left generalized Opial type inequality for ordinary derivatives:

Theorem 10 *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $n \in \mathbb{N}$, $f \in C^n([a,b], A)$; where $[a,b] \subset \mathbb{R}$ and $(A, \|\cdot\|)$ is a Banach algebra. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. We assume that $\left(f \circ g^{-1}\right)^{(j)}(g(x_0)) = 0$, $j = 0, 1, ..., n-1$; where $x_0 \in [a,b]$ be fixed. Then*

$$\int_{g(x_0)}^{g(x)} \left\| \left(f \circ g^{-1}\right)(z) \left(f \circ g^{-1}\right)^{(n)}(z) \right\| dz \leq$$

$$\frac{(g(x) - g(x_0))^{n + \frac{1}{p} - \frac{1}{q}}}{2^{\frac{1}{q}} (n-1)! \left[(p(n-1)+1)(p(n-1)+2)\right]^{\frac{1}{p}}} \left(\int_{g(x_0)}^{g(x)} \left\| \left(f \circ g^{-1}\right)^{(n)}(z) \right\|^q dz \right)^{\frac{2}{q}}, \tag{35}$$

for all $x_0 \leq x \leq b$.

Proof. Let $x_0 \in [a,b]$ such that $\left(f \circ g^{-1}\right)^{(j)}(g(x_0)) = 0$, $j = 0, 1, ..., n-1$. For $x \in [x_0, b]$ by Theorem 6 we have

$$\left(f \circ g^{-1}\right)(g(z)) = \frac{1}{(n-1)!} \int_{g(x_0)}^{g(x)} (g(x) - z)^{n-1} \left(f \circ g^{-1}\right)^{(n)}(z)\, dz. \tag{36}$$

By Hölder's inequality we obtain

$$\left\| \left(f \circ g^{-1}\right)(g(x)) \right\| \leq \frac{1}{(n-1)!} \int_{g(x_0)}^{g(x)} (g(x) - z)^{n-1} \left\| \left(f \circ g^{-1}\right)^{(n)}(z) \right\| dz \leq \tag{37}$$

$$\frac{1}{(n-1)!} \left(\int_{g(x_0)}^{g(x)} (g(x) - z)^{p(n-1)}\, dt \right)^{\frac{1}{p}} \left(\int_{g(x_0)}^{g(x)} \left\| \left(f \circ g^{-1}\right)^{(n)}(z) \right\|^q dz \right)^{\frac{1}{q}} =$$

$$\frac{1}{(n-1)!} \frac{(g(x) - g(x_0))^{\frac{p(n-1)+1}{p}}}{(p(n-1)+1)^{\frac{1}{p}}} \left(\int_{g(x_0)}^{g(x)} \left\| \left(f \circ g^{-1}\right)^{(n)}(z) \right\|^q dz \right)^{\frac{1}{q}}.$$

Call

$$\varphi(g(x)) := \int_{g(x_0)}^{g(x)} \left\| \left(f \circ g^{-1}\right)^{(n)}(z) \right\|^q dz, \tag{38}$$

$\varphi(g(x_0)) = 0$.

Thus

$$\frac{d\varphi(g(x))}{dg(x)} = \left\| \left(f \circ g^{-1}\right)^{(n)}(g(x)) \right\|^q \geq 0, \tag{39}$$

and

$$\left(\frac{d\varphi(g(x))}{dg(x)} \right)^{\frac{1}{q}} = \left\| \left(f \circ g^{-1}\right)^{(n)}(g(x)) \right\| \geq 0, \tag{40}$$

$\forall\ g(x) \in [g(x_0), g(b)]$.

11

Consequently, we get

$$\left\| \left(f \circ g^{-1} \right) (g(w)) \right\| \left\| \left(f \circ g^{-1} \right)^{(n)} (g(w)) \right\| \leq$$

$$\frac{(g(w) - g(x_0))^{\frac{p(n-1)+1}{p}}}{(n-1)! \left(p(n-1)+1 \right)^{\frac{1}{p}}} \left(\varphi(g(w)) \frac{d\varphi(g(w))}{dg(w)} \right)^{\frac{1}{q}}, \tag{41}$$

$\forall\ g(w) \in [g(x_0), g(b)]$.

Then we observe that

$$\int_{g(x_0)}^{g(x)} \left\| \left(f \circ g^{-1} \right) (g(w)) \left(f \circ g^{-1} \right)^{(n)} (g(w)) \right\| dg(w) \overset{(7)}{\leq}$$

$$\int_{g(x_0)}^{g(x)} \left\| \left(f \circ g^{-1} \right) (g(w)) \right\| \left\| \left(f \circ g^{-1} \right)^{(n)} (g(w)) \right\| dg(w) \leq$$

$$\frac{1}{(n-1)! \left(p(n-1)+1 \right)^{\frac{1}{p}}}$$

$$\int_{g(x_0)}^{g(x)} (g(w) - g(x_0))^{\frac{p(n-1)+1}{p}} \left(\varphi(g(w)) \frac{d\varphi(g(w))}{dg(w)} \right)^{\frac{1}{q}} dg(w) \leq \tag{42}$$

$$\frac{1}{(n-1)! \left(p(n-1)+1 \right)^{\frac{1}{p}}}$$

$$\left(\int_{g(x_0)}^{g(x)} (g(w) - g(x_0))^{p(n-1)+1} dg(w) \right)^{\frac{1}{p}} \left(\int_{g(x_0)}^{g(x)} \varphi(g(w)) \frac{d\varphi(g(w))}{dg(w)} dg(w) \right)^{\frac{1}{q}} =$$

$$\frac{1}{(n-1)! \left(p(n-1)+1 \right)^{\frac{1}{p}} \left(p(n-1)+2 \right)^{\frac{1}{p}}}$$

$$(g(x) - g(x_0))^{\frac{p(n-1)+2}{p}} \left(\int_{g(x_0)}^{g(x)} \varphi(g(w))\ d\varphi(g(w)) \right)^{\frac{1}{q}} = \tag{43}$$

$$\frac{(g(x) - g(x_0))^{n+\frac{1}{p}-\frac{1}{q}}}{(n-1)! \left(p(n-1)+1 \right)^{\frac{1}{p}} \left(p(n-1)+2 \right)^{\frac{1}{p}}} \left(\frac{\varphi^2(g(x))}{2} \right)^{\frac{1}{q}} =$$

$$\frac{(g(x) - g(x_0))^{n+\frac{1}{p}-\frac{1}{q}}}{2^{\frac{1}{q}} (n-1)! \left(\left(p(n-1)+1 \right) \left(p(n-1)+2 \right) \right)^{\frac{1}{p}}} \left(\int_{g(x_0)}^{g(x)} \left\| \left(f \circ g^{-1} \right)^{(n)} (z) \right\|^q dz \right)^{\frac{2}{q}}, \tag{44}$$

for all $g(x_0) \leq g(x) \leq g(b)$, proving (35). ∎

The corresponding right generalized Opial type inequality follows:

Theorem 11 *All as in Theorem 10. Then*

$$\int_{g(x)}^{g(x_0)} \left\| \left(f \circ g^{-1}\right)(z) \left(f \circ g^{-1}\right)^{(n)}(z) \right\| dz \leq$$

$$\frac{\left(g\left(x_0\right) - g\left(x\right)\right)^{n+\frac{1}{p}-\frac{1}{q}}}{2^{\frac{1}{q}}\left(n-1\right)!\left(\left(p\left(n-1\right)+1\right)\left(p\left(n-1\right)+2\right)\right)^{\frac{1}{p}}} \left(\int_{g(x)}^{g(x_0)} \left\| \left(f \circ g^{-1}\right)^{(n)}(z) \right\|^q dz \right)^{\frac{2}{q}}, \tag{45}$$

for all $a \leq x \leq x_0$.

Proof. As similar to Theorem 10 is omitted. ■

Next we present a left generalized Hilbert-Pachpatte inequality for ordinary derivatives.

Theorem 12 *Let* $i = 1, 2$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, *and* $n_i \in \mathbb{N}$, $f_i \in C^{n_i}\left(\left[a_i, b_i\right], A\right)$; *where* $\left[a_i, b_i\right] \subset \mathbb{R}$ *and* $(A, \|\cdot\|)$ *is a Banach algebra. Let* $g_i \in C^1\left(\left[a_i, b_i\right]\right)$, *strictly increasing, such that* $g_i^{-1} \in C^{n_i}\left(\left[g_i\left(a_i\right), g_i\left(b_i\right)\right]\right)$. *We assume that* $\left(f_i \circ g_i^{-1}\right)^{(j_i)}\left(g_i\left(x_{0i}\right)\right) = 0$, $j_i = 0, 1, ..., n_i - 1$; *where* $x_{0i} \in \left[a_i, b_i\right]$ *be fixed. Then*

$$\int_{g_1(x_{01})}^{g_1(b_1)} \int_{g_2(x_{02})}^{g_2(b_2)} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\| dz_1 dz_2}{\left(\frac{(z_1 - g_1(x_{01}))^{p(n_1-1)+1}}{p(p(n_1-1)+1)} + \frac{(z_2 - g_2(x_{02}))^{q(n_2-1)+1}}{q(q(n_2-1)+1)} \right)} \leq$$

$$\frac{\left(g_1\left(b_1\right) - g_1\left(x_{01}\right)\right)\left(g_2\left(b_2\right) - g_2\left(x_{02}\right)\right)}{\left(n_1 - 1\right)!\left(n_2 - 1\right)!} \tag{46}$$

$$\left\| \left\| \left(f_1 \circ g_1^{-1}\right)^{(n_1)} \right\| \right\|_{L_q([g_1(x_{01}), g_1(b_1)], A)} \left\| \left\| \left(f_2 \circ g_2^{-1}\right)^{(n_2)} \right\| \right\|_{L_p([g_2(x_{02}), g_2(b_2)], A)}.$$

Proof. Let $i = 1, 2$; $x_0 \in \left[a_i, b_i\right]$, such that $\left(f_i \circ g_i^{-1}\right)^{(j_i)}\left(g_i\left(x_{0i}\right)\right) = 0$, $j_i = 0, 1, ..., n_i - 1$.

For $x_i \in \left[x_{0i}, b_i\right]$ by Theorem 6 we have

$$\left(f_i \circ g_i^{-1}\right)\left(g_i\left(x_i\right)\right) = \frac{1}{\left(n_i - 1\right)!} \int_{g_i(x_{0i})}^{g_i(x_i)} \left(g_i\left(x_i\right) - z_i\right)^{n_i - 1} \left(f_i \circ g_i^{-1}\right)^{(n_i)}(z_i)\, dz_i. \tag{47}$$

As in (37) we have

$$\left\| \left(f_1 \circ g_1^{-1}\right)\left(g_1\left(x_1\right)\right) \right\| \leq \frac{1}{\left(n_1 - 1\right)!} \frac{\left(g_1\left(x_1\right) - g_1\left(x_{01}\right)\right)^{\frac{p(n_1-1)+1}{p}}}{\left(p\left(n_1 - 1\right) + 1\right)^{\frac{1}{p}}}$$

$$\left(\int_{g_1(x_{01})}^{g_1(x_1)} \left\| \left(f_1 \circ g_1^{-1}\right)^{(n_1)}(z) \right\|^q dz \right)^{\frac{1}{q}} \leq$$

13

$$\frac{1}{(n_1-1)!}\frac{(g_1(x_1)-g_1(x_{01}))^{\frac{p(n_1-1)+1}{p}}}{(p(n_1-1)+1)^{\frac{1}{p}}}\left\|\left\|\left(f_1\circ g_1^{-1}\right)^{(n_1)}\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)])}, \tag{48}$$

for all $x_1 \in [x_{01}, b_1]$.

Similarly, we obtain that

$$\left\|\left(f_2\circ g_2^{-1}\right)(g_2(x_2))\right\| \leq \frac{1}{(n_2-1)!}\frac{(g_2(x_2)-g_2(x_{02}))^{\frac{q(n_2-1)+1}{q}}}{(q(n_2-1)+1)^{\frac{1}{q}}}$$

$$\left\|\left\|\left(f_2\circ g_2^{-1}\right)^{(n_2)}\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)])}, \tag{49}$$

for all $x_2 \in [x_{02}, b_2]$.

By (48) and (49) we get

$$\left\|\left(f_1\circ g_1^{-1}\right)(g_1(x_1))\left(f_2\circ g_2^{-1}\right)(g_2(x_2))\right\| \leq$$

$$\left\|\left(f_1\circ g_1^{-1}\right)(g_1(x_1))\right\|\left\|\left(f_2\circ g_2^{-1}\right)(g_2(x_2))\right\| \leq \frac{1}{(n_1-1)!(n_2-1)!}$$

$$\frac{(g_1(x_1)-g_1(x_{01}))^{\frac{p(n_1-1)+1}{p}}}{(p(n_1-1)+1)^{\frac{1}{p}}}\frac{(g_2(x_2)-g_2(x_{02}))^{\frac{q(n_2-1)+1}{q}}}{(q(n_2-1)+1)^{\frac{1}{q}}} \tag{50}$$

$$\left\|\left\|\left(f_1\circ g_1^{-1}\right)^{(n_1)}\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)])}\left\|\left\|\left(f_2\circ g_2^{-1}\right)^{(n_2)}\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)])} \leq$$

(using Young's inequality for $a, b \geq 0$, $a^{\frac{1}{p}}b^{\frac{1}{q}} \leq \frac{a}{p}+\frac{b}{q}$)

$$\frac{1}{(n_1-1)!(n_2-1)!}\left(\frac{(g_1(x_1)-g_1(x_{01}))^{p(n_1-1)+1}}{p(p(n_1-1)+1)}+\frac{(g_2(x_2)-g_2(x_{02}))^{q(n_2-1)+1}}{q(q(n_2-1)+1)}\right) \tag{51}$$

$$\left\|\left\|\left(f_1\circ g_1^{-1}\right)^{(n_1)}\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)])}\left\|\left\|\left(f_2\circ g_2^{-1}\right)^{(n_2)}\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)])},$$

$\forall (x_1, x_2) \in [x_{01}, b_1] \times [x_{02}, b_2]$.

So far we have

$$\frac{\left\|\left(f_1\circ g_1^{-1}\right)(g_1(x_1))\left(f_2\circ g_2^{-1}\right)(g_2(x_2))\right\|}{\left(\frac{(g_1(x_1)-g_1(x_{01}))^{p(n_1-1)+1}}{p(p(n_1-1)+1)}+\frac{(g_2(x_2)-g_2(x_{02}))^{q(n_2-1)+1}}{q(q(n_2-1)+1)}\right)} \leq \tag{52}$$

$$\frac{1}{(n_1-1)!(n_2-1)!}\left\|\left\|\left(f_1\circ g_1^{-1}\right)^{(n_1)}\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)],A)}$$

$$\left\|\left\|\left(f_2\circ g_2^{-1}\right)^{(n_2)}\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)],A)},$$

$\forall (x_1, x_2) \in [x_{01}, b_1] \times [x_{02}, b_2]$.

14

The denominator in (52) can be zero, only when both $g_1(x_1) = g_1(x_{01})$ and $g_2(x_2) = g_2(x_{02})$.

Therefore we obtain (46), by integrating (52) over $[g_1(x_{01}), g_1(b_1)] \times [g_2(x_{02}), g_2(b_2)]$. ■

It follows the right generalized Hilbert-Pachpate inequality for ordinary derivatives.

Theorem 13 *All as in Theorem 12. Then*

$$\int_{g_1(a_1)}^{g_1(x_{01})} \int_{g_2(a_2)}^{g_2(x_{02})} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\| dz_1 dz_2}{\left(\frac{(g_1(x_{01}) - z_1)^{p(n_1-1)+1}}{p(p(n_1-1)+1)} + \frac{(g_2(x_{02}) - z_2)^{q(n_2-1)+1}}{q(q(n_2-1)+1)} \right)} \leq$$

$$\frac{(g_1(x_{01}) - g_1(a_1))(g_2(x_{02}) - g_2(a_2))}{(n_1-1)!(n_2-1)!} \tag{53}$$

$$\left\| \left\| \left(f_1 \circ g_1^{-1}\right)^{(n_1)} \right\| \right\|_{L_q([g_1(a_1), g_1(x_{01})], A)} \left\| \left\| \left(f_2 \circ g_2^{-1}\right)^{(n_2)} \right\| \right\|_{L_p([g_2(a_2), g_2(x_{02})], A)}.$$

Proof. As similar to theorem 12 is omitted. ■

5 Applications

We make

Remark 14 *Assume next that $(A, \|\cdot\|)$ is a commutative Banach algebra. Then, we get that*

$$E(f_1, ..., f_r)(x_0) \overset{(13)}{=} r \int_a^b \left(\prod_{j=1}^r f_j(x) \right) dx - \sum_{i=1}^r \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r f_j(x) \right) dx \right) f_i(x_0), \tag{54}$$

$x_0 \in [a, b]$.

When $r = 2$, we have that

$$E(f_1, f_2)(x_0) = 2 \int_a^b f_1(x) f_2(x) dx - f_1(x_0) \int_a^b f_2(x) dx - f_2(x_0) \int_a^b f_1(x) dx, \tag{55}$$

$x_0 \in [a, b]$.

We give

Corollary 15 *(to Theorem 7) All as in Theorem 7, $(A, \|\cdot\|)$ is a commutative Banach algebra, $r = 2$. Then*

$$\|E(f_1, f_2)(x_0)\| \leq \frac{1}{n!} \sum_{i=1}^2 \left[\left[\left\| \left\| \left(f_i \circ g^{-1}\right)^{(n)} \right\| \right\|_{\infty, [g(a), g(x_0)]} \right. \right.$$

15

$$(g(x_0) - g(a))^n \left(\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{2} \|f_j(x)\| \right) dx \right) \Bigg] +$$

$$\left[\left\| \left\| \left(f_i \circ g^{-1} \right)^{(n)} \right\| \right\|_{\infty, [g(x_0), g(b)]} (g(b) - g(x_0))^n \left(\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{2} \|f_j(x)\| \right) dx \right) \right] \Bigg]. \tag{56}$$

It follows

Corollary 16 *(to Corollary 15) All as in Corollary 15, with* $g(t) = e^t$*. Then*

$$\|E(f_1, f_2)(x_0)\| \leq \frac{1}{n!} \sum_{i=1}^{2} \Bigg[\Bigg[\left\| \left\| (f_i \circ \log)^{(n)} \right\| \right\|_{\infty, [e^a, e^{x_0}]}$$

$$(e^{x_0} - e^a)^n \left(\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{2} \|f_j(x)\| \right) dx \right) \Bigg] +$$

$$\left[\left\| \left\| (f_i \circ \log)^{(n)} \right\| \right\|_{\infty, [e^{x_0}, e^b]} \left(e^b - e^{x_0} \right)^n \left(\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{2} \|f_j(x)\| \right) dx \right) \right] \Bigg]. \tag{57}$$

We continue with

Corollary 17 *(to Theorem 10) All as in Theorem 10 for* $g(t) = e^t$*. Then*

$$\int_{e^{x_0}}^{e^x} \left\| ((f \circ \log)(z)) (f \circ \log)^{(n)}(z) \right\| dz \leq$$

$$\frac{(e^x - e^{x_0})^{n + \frac{1}{p} - \frac{1}{q}}}{2^{\frac{1}{q}} (n-1)! \left[(p(n-1)+1)(p(n-1)+2) \right]^{\frac{1}{p}}} \left(\int_{e^{x_0}}^{z} \left\| (f \circ \log)^{(n)}(z) \right\|^q dz \right)^{\frac{2}{q}}, \tag{58}$$

for all $x_0 \leq x \leq b$*.*

We finish with

Corollary 18 *(to Theorem 12) All as in Theorem 12 for* $g_i(t) = e^t$*,* $i = 1, 2$*. Then*

$$\int_{e^{x_{01}}}^{e^{b_1}} \int_{e^{x_{02}}}^{e^{b_2}} \frac{\|(f_1 \circ \log)(z_1)(f_2 \circ \log)(z_2)\| \, dz_1 dz_2}{\left(\frac{(z_1 - e^{x_{01}})^{p(n_1 - 1) + 1}}{p(p(n_1 - 1) + 1)} + \frac{(z_2 - e^{x_{02}})^{q(n_2 - 1) + 1}}{q(q(n_2 - 1) + 1)} \right)} \leq$$

16

$$\frac{\left(e^{b_1} - e^{x_{01}}\right)\left(e^{b_2} - e^{x_{02}}\right)}{(n_1 - 1)!\,(n_2 - 1)!} \tag{59}$$
$$\left\|\left\|\left\|(f_1 \circ \log)^{(n_1)}\right\|\right\|\right\|_{L_q\left(\left[e^{x_{01}}, e^{b_1}\right], A\right)} \left\|\left\|\left\|(f_2 \circ \log)^{(n_2)}\right\|\right\|\right\|_{L_p\left(\left[e^{x_{02}}, e^{b_2}\right], A\right)}.$$

The simplest applications derive when $g(t) = t$ and $A = \mathbb{R}$, leading to basic known results.

References

[1] G.A. Anastassiou, *Intelligent Comparisons: Analytic Inequalities*, Springer, Heidelberg, New York, 2016.

[2] G.A. Anastassiou, *Intelligent Computations: Abstract Fractional Calculus, Inequalities, Approximations*, Springer, Heidelberg, New York, 2018.

[3] S.S. Dragomir, *Noncommutative Ostrowski type inequalities for functions in Banach algebras*, RGMIA Res. Rep. Coll. 24 (2021), Art. 10, 24 pp.

[4] J. Mikusinski, *The Bochner integral*, Academic Press, New York, 1978.

[5] Z. Opial, *Sur une inegalite*, Ann. Polon. Math. 8(1960), 29-32.

[6] A. Ostrowski, *Über die Absolutabweichung einer differentiabaren Funcktion von ihrem Integralmittelwert*, Comment. Math. Helv., 10 (1938), 226-227.

[7] B.G. Pachpatte, *Inequalities similar to the integral analogue of Hilbert's inequalities*, Tamkang J. Math., 30 (1) (1999), 139-146.

[8] W. Rudin, *Functional Analysis*, Second Edition, McGraw-Hill, Inc., New York, 1991.

[9] G.E. Shilov, *Elementary Functional Analysis*, Dover Publications Inc., New York, 1996.

Multivariate Ostrowski type inequalities for several Banach algebra valued functions

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we are dealing with several smooth functions from a compact convex set of $\mathbb{R}^k$, $k \geq 2$ to a Banach algebra. For these we prove general multivariate Ostrowski type inequalities with estimates in norms $\|\cdot\|_p$, for all $1 \leq p \leq \infty$. We provide also interesting applications.

2020 Mathematics Subject Classification : 26D10, 26D15.
Keywords and Phrases: Multivariate integral inequality, Ostrowski inequality, mixed partial derivative, Banach algebra valued functions.

1 Introduction

In 1938, A Ostrowski [5] proved the following famous inequality:

Theorem 1 *(1938, Ostrowski [6]) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty^{\sup} := \sup_{t \in (a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty^{\sup}, \quad (1)$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

Since then there has been a lot of activity around these inequalities with important applications to Numerical Analysis and Probability.

This article is also greatly motivated by the following result:

Theorem 2 *(see [1]) Let* $f \in C^1\left(\prod_{i=1}^{k}[a_i,b_i]\right)$*, where* $a_i < b_i$*;* $a_i, b_i \in \mathbb{R}$*,* $i = 1, ..., k$*, ans let* $\overrightarrow{x_0} := (x_{01}, ..., x_{0k}) \in \prod_{i=1}^{k}[a_i,b_i]$ *be fixed. Then*

$$\left| \frac{1}{\prod_{i=1}^{k}(b_i - a_i)} \int_{a_1}^{b_1} ... \int_{a_i}^{b_i} ... \int_{a_k}^{b_k} f(z_1, ..., z_k)\, dz_1 ... dz_k - f(\overrightarrow{x_0}) \right| \leq \tag{2}$$

$$\sum_{i=1}^{k} \left(\frac{(x_{0i} - a_i)^2 + (b_i - x_{0i})^2}{2(b_i - a_i)} \right) \left\| \frac{\partial f}{\partial z_i} \right\|_{\infty}.$$

Inequality (2) is sharp, here the optimal function is

$$f^*(z_1, ..., z_k) := \sum_{i=1}^{k} |z_i - x_{0i}|^{\alpha_i}, \quad \alpha_i > 1.$$

Clearly inequality (2) generalizes inequality (1) to multidimension.

We are inspired also by [2].

In this article we establish multivariate Ostrowski type inequalities for several smooth functions from a compact convex subset of $\mathbb{R}^k$, $k \geq 2$, to a Banach algebra. These involve the norms $\|\cdot\|_p$, $1 \leq p \leq \infty$.

2 About Banach Algebras

All here come from [6].

We need

Definition 3 *([6], p. 245) A complex algebra is a vector space* A *over the complex field* $\mathbb{C}$ *in which a multiplication is defined that satisfies*

$$x(yz) = (xy)z, \tag{3}$$

$$(x + y)z = xz + yz, \quad x(y + z) = xy + xz, \tag{4}$$

and

$$\alpha(xy) = (\alpha x)y = x(\alpha y), \tag{5}$$

for all x, y *and* z *in* A *and for all scalars* α*.*

Additionally if A *is a Banach space with respect to a norm that satisfies the multiplicative inequality*

$$\|xy\| \leq \|x\| \|y\| \quad (x \in A,\ y \in A) \tag{6}$$

and if A contains a unit element e such that

$$xe = ex = x \quad (x \in A) \tag{7}$$

and

$$\|e\| = 1, \tag{8}$$

then A is called a Banach algebra.

A is commutative iff $xy = yx$ for all $x, y \in A$.

We make

Remark 4 *Commutativity of A will be explicited stated when needed.*

There exists at most one $e \in A$ that satisfies (7).

Inequality (6) makes multiplication to be continuous, more precisely left and right continuous, see [6], p. 246.

Multiplication in A is not necessarily the numerical multiplication, it is something more general and it is defined abstractly, that is for $x, y \in A$ we have $xy \in A$, e.g. composition or convolution, etc.

For nice examples about Banach algebras see [6], p. 247-248, § 10.3.

We also make

Remark 5 *Next we mention about integration of A-valued functions, see [6], p. 259, § 10.22:*

If A is a Banach algebra and f is a continuous A-valued function on some compact Hausdorff space Q on which a complex Borel measure μ is defined, then $\int f d\mu$ exists and has all the properties that were discussed in Chapter 3 of [6], simply because A is a Banach space. However, an additional property can be added to these, namely: If $x \in A$, then

$$x \int_Q f \, d\mu = \int_Q x f(p) \, d\mu(p) \tag{9}$$

and

$$\left(\int_Q f \, d\mu \right) x = \int_Q f(p) \, x \, d\mu(p). \tag{10}$$

The vector integrals we will involve in our article follow (9) and (10).

3 Vector Analysis Background

(see [8], pp. 83-94)

3

Let $f(t)$ be a function defined on $[a,b] \subseteq \mathbb{R}$ taking values in a real or complex normed linear space $(X, \|\cdot\|)$, Then $f(t)$ is said to be differentiable at a point $t_0 \in [a,b]$ if the limit

$$f'(t_0) = \lim_{h \to 0} \frac{f(t_0+h) - f(t_0)}{h} \tag{11}$$

exists in X, the convergence is in $\|\cdot\|$. This is called the derivative of $f(t)$ at $t = t_0$.

We call $f(t)$ differentiable on $[a,b]$, iff there exists $f'(t) \in X$ for all $t \in [a,b]$.

Similarly and inductively are defined higher order derivatives of f, denoted $f'', f^{(3)}, ..., f^{(k)}$, $k \in \mathbb{N}$, just as for numerical functions.

For all the properties of derivatives see [8], pp. 83-86.

Let now $(X, \|\cdot\|)$ be a Banach space, and $f : [a,b] \to X$.

We define the vector valued Riemann integral $\int_a^b f(t)\,dt \in X$ as the limit of the vector valued Riemann sums in X, convergence is in $\|\cdot\|$. The definition is as for the numerical valued functions.

If $\int_a^b f(t)\,dt \in X$ we call f integrable on $[a,b]$. If $f \in C([a,b], X)$, then f is integrable, [8], p. 87.

For all the properties of vector valued Riemann integrals see [8], pp. 86-91.

We define the space $C^n([a,b], X)$, $n \in \mathbb{N}$, of n-times continuousky differentiable functions from $[a,b]$ into X; here continuity is with respect to $\|\cdot\|$ and defined in the usual way as for numerical functions·.

Let $(X, \|\cdot\|)$ be a Banach space and $f \in C^n([a,b], X)$, then we have the vector valued Taylor's formula, see [8], pp. 93-94, and also [7], (IV, 9; 47).

It holds

$$f(y) - f(x) - f'(x)(y-x) - \frac{1}{2} f''(x)(y-x)^2 - ... - \frac{1}{(n-1)!} f^{(n-1)}(x)(y-x)^{n-1}$$
$$= \frac{1}{(n-1)!} \int_x^y (y-t)^{n-1} f^{(n)}(t)\,dt, \quad \forall\, x, y \in [a,b]. \tag{12}$$

In particular (12) is true when $X = \mathbb{R}^m, \mathbb{C}^m$, $m \in \mathbb{N}$, etc.

A function $f(t)$ with values in a normed linear space X is said to be piecewise continuous (see [8], p. 85) on the interval $a \leq t \leq b$ if there exists a partition $a = t_0 < t_1 < t_2 < ... < t_n = b$ such that $f(t)$ is continuous on every open interval $t_k < t < t_{k+1}$ and has finite limits $f(t_0+0)$, $f(t_1-0)$, $f(t_1+0)$, $f(t_2-0)$, $f(t_2+0)$, ..., $f(t_n-0)$.

Here $f(t_k - 0) = \lim_{t \uparrow t_k} f(t)$, $f(t_k+0) = \lim_{t \downarrow t_k} f(t)$.

The values of $f(t)$ at the points t_k can be arbitrary or even undefined.

A function $f(t)$ with values in normed linear space X is said to be piecewise smooth on $[a,b]$, if it is continuous on $[a,b]$ and has a derivative $f'(t)$ at all but a finite number of points of $[a,b]$, and if $f'(t)$ is piecewise continuous on $[a,b]$ (see [8], p. 85).

4

Let $u(t)$ and $v(t)$ be two piecewise smooth functions on $[a,b]$, one a numerical function and the other a vector function with values in Banach space X. Then we have the following integration by parts formula

$$\int_a^b u(t)\,dv(t) = u(t)\,v(t)\,|_a^b - \int_a^b v(t)\,du(t)\,, \tag{13}$$

see [8], p. 93.

We mention also the mean value theorem for Banach space valued functions.

Theorem 6 *(see [4], p. 3) Let $f \in C([a,b], X)$, where X is a Banach space. Assume f' exists on $[a,b]$ and $\|f'(t)\| \leq K$, $a < t < b$, then*

$$\|f(b) - f(a)\| \leq K(b-a)\,. \tag{14}$$

Here the multiple Riemann integral of a function from a real box or a real compact and convex subset to a Banach space is defined similarly to numerical one however convergence is with respect to $\|\cdot\|$. Similarly are defined the vector valued partial derivatives as in the numerical case.

We mention the equality of vector valued mixed partiasl derivatives.

Proposition 7 *(see Proposition 4.11 of [3], p. 90) Let $Q = (a,b) \times (c,d) \subseteq \mathbb{R}^2$ and $f \in C(Q,X)$, where $(X, \|\cdot\|)$ is a Banach space. Assume that $\frac{\partial}{\partial t} f(s,t)$, $\frac{\partial}{\partial s} f(s,t)$ and $\frac{\partial^2}{\partial t \partial s} f(s,t)$ exist and are continuous for $(s,t) \in Q$, then $\frac{\partial^2}{\partial s \partial t} f(s,t)$ exists for $(s,t) \in Q$ and*

$$\frac{\partial^2}{\partial s \partial t} f(s,t) = \frac{\partial^2}{\partial t \partial s} f(s,t)\,, \ \text{ for } (s,t) \in Q. \tag{15}$$

4 Main Results

We present general Ostrowski type inequalities results regarding several Banach algebra valued functions.

Theorem 8 *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$; $(A, \|\cdot\|)$ a Banach algebra and $f_i \in C^{n+1}(Q, A)$, $i = 1, ..., r$; $r \in \mathbb{N}$, $n \in \mathbb{Z}_+$, and fixed $\overrightarrow{x_0} \in Q \subset \mathbb{R}^k$, $k \geq 2$, where Q is a compact and convex subset. Here all vector partial derivatives $f_{i\alpha} := \frac{\partial^\alpha f_i}{\partial z^\alpha}$, where $\alpha = (\alpha_1, ..., \alpha_k)$, $\alpha_\lambda \in \mathbb{Z}^+$, $\lambda = 1, ..., k$, $|\alpha| = \sum_{\lambda=1}^{k} \alpha_\lambda = j$, $j = 1, ..., n$, fulfill $f_{i\alpha}(\overrightarrow{x_0}) = 0$, $i = 1, ..., r$.*

Denote

$$D_{n+1}(f_i) := \max_{\alpha : |\alpha| = n+1} \|\|f_{i\alpha}\|\|_{\infty, Q}\,, \tag{16}$$

$i = 1, ..., r$, and

$$\|\overrightarrow{z} - \overrightarrow{x_0}\|_{l_1} := \sum_{\lambda=1}^{k} |z_\lambda - x_{0\lambda}|\,. \tag{17}$$

Then

$$\left\| \sum_{i=1}^{r} \int_Q \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} f_\rho\left(\overrightarrow{z}\right) \right) f_i\left(\overrightarrow{z}\right) d\overrightarrow{z} - \sum_{i=1}^{r} \left(\int_Q \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} f_\rho\left(\overrightarrow{z}\right) \right) d\overrightarrow{z} \right) f_i\left(\overrightarrow{x_0}\right) \right\| \leq \tag{18}$$

$$\frac{\max_{i\in\{1,...,r\}} D_{n+1}(f_i)}{(n+1)!} \sum_{i=1}^{r} \left(\int_Q \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \left\| f_\rho\left(\overrightarrow{z}\right) \right\| \right) \left\| \overrightarrow{z} - \overrightarrow{x_0} \right\|_{l_1}^{n+1} d\overrightarrow{z} \right) \leq$$

$$\frac{\max_{i\in\{1,...,r\}} D_{n+1}(f_i)}{(n+1)!} \min \left\{ \left(\int_Q \left\| \overrightarrow{z} - \overrightarrow{x_0} \right\|_{l_1}^{n+1} d\overrightarrow{z} \right) \left[\sum_{i=1}^{r} \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \left\| \left\| f_\rho \right\| \right\|_{\infty,Q} \right) \right], \right.$$

$$\left\| \left\| \cdot - \overrightarrow{x_0} \right\|_{l_1}^{n+1} \right\|_{\infty,Q} \left[\sum_{i=1}^{r} \left\| \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \left\| f_\rho \right\| \right) \right\|_{L_1(Q,A)} \right],$$

$$\left. \left\| \left\| \cdot - \overrightarrow{x_0} \right\|_{l_1}^{n+1} \right\|_{L_p(Q,A)} \left[\sum_{i=1}^{r} \left[\left\| \left(\prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \left\| f_\rho \right\| \right) \right\|_{L_q(Q,A)} \right] \right] \right\}. \tag{19}$$

Proof. Take $g_{i\overrightarrow{z}}(t) := f_i\left(\overrightarrow{x_0} + t\left(\overrightarrow{z} - \overrightarrow{x_0}\right)\right)$, $0 \leq t \leq 1$; $i = 1, ..., r$. Notice that $g_{i\overrightarrow{z}}(0) = f_i\left(\overrightarrow{x_0}\right)$ and $g_{i\overrightarrow{z}}(1) = f_i\left(\overrightarrow{z}\right)$. The jth derivative of $g_{i\overrightarrow{z}}(t)$, based on Proposition 7, is given by

$$g_{i\overrightarrow{z}}^{(j)}(t) = \left[\left(\sum_{\lambda=1}^{k} (z_\lambda - x_{0\lambda}) \frac{\partial}{\partial z_\lambda} \right)^j f_i \right] (x_{01} + t(z_1 - x_{01}), ..., x_{0k} + t(z_k - x_{0k})) \tag{20}$$

and

$$g_{i\overrightarrow{z}}^{(j)}(0) = \left[\left(\sum_{\lambda=1}^{k} (z_\lambda - x_{0\lambda}) \frac{\partial}{\partial z_\lambda} \right)^j f_i \right] \left(\overrightarrow{x_0}\right), \tag{21}$$

for $j = 1, ..., n+1$; $i = 1, ..., r$.

Let $f_{i\alpha}$ be a partial derivative of $f_i \in C^{n+1}(Q, A)$. Because by assumption of the theorem we have $f_{i\alpha}\left(\overrightarrow{x_0}\right) = 0$ for all $\alpha : |\alpha| = j$, $j = 1, ..., n$, we find that

$$g_{i\overrightarrow{z}}^{(j)}(0) = 0, \quad j = 1, ..., n; \ i = 1, ..., r.$$

Hence by vector Taylor's theorem (12) we see that

$$f_i\left(\overrightarrow{z}\right) - f_i\left(\overrightarrow{x_0}\right) = \sum_{j=1}^{n} \frac{g_{i\overrightarrow{z}}^{(j)}(0)}{j!} + R_{in}\left(\overrightarrow{z}, 0\right) = R_{in}\left(\overrightarrow{z}, 0\right), \tag{22}$$

where

$$R_{in}\left(\overrightarrow{z},0\right):=\int_{0}^{1}\left(\int_{0}^{t_{1}}...\left(\int_{0}^{t_{n-1}}\left(g_{i\overrightarrow{z}}^{(n)}\left(t_{n}\right)-g_{i\overrightarrow{z}}^{(n)}\left(0\right)\right)dt_{n}\right)...\right)dt_{1}, \tag{23}$$

$i=1,...,r$.

Therefore,

$$\left\|R_{in}\left(\overrightarrow{z},0\right)\right\|\leq\int_{0}^{1}\left(\int_{0}^{t_{1}}...\left(\int_{0}^{t_{n-1}}\left\|\left\|g_{i\overrightarrow{z}}^{(n+1)}\left(\xi\left(t_{n}\right)\right)\right\|\right\|_{\infty}t_{n}dt_{n}\right)...\right)dt_{1}, \tag{24}$$

by the vector mean value Theorem 6 applied on $g_{i\overrightarrow{z}}^{(n)}$ over $(0,t_n)$. Moreover, we get

$$\begin{aligned}\left\|R_{in}\left(\overrightarrow{z},0\right)\right\| &\leq\left\|\left\|g_{i\overrightarrow{z}}^{(n+1)}\right\|\right\|_{\infty,[0,1]}\int_{0}^{1}\int_{0}^{t_{1}}...\left(\int_{0}^{t_{n-1}}t_{n}dt_{n}\right)...dt_{1}\\ &=\frac{\left\|\left\|g_{i\overrightarrow{z}}^{(n+1)}\right\|\right\|_{\infty,[0,1]}}{(n+1)!}. \end{aligned}\tag{25}$$

However, there exists a $t_{i0}\in[0,1]$ such that $\left\|\left\|g_{i\overrightarrow{z}}^{(n+1)}\right\|\right\|_{\infty,[0,1]}=\left\|g_{i\overrightarrow{z}}^{(n+1)}\left(t_{i0}\right)\right\|$.

That is

$$\begin{aligned}\left\|\left\|g_{i\overrightarrow{z}}^{(n+1)}\right\|\right\|_{\infty,[0,1]} &=\left\|\left[\left(\sum_{\lambda=1}^{k}\left(z_{\lambda}-x_{0\lambda}\right)\frac{\partial}{\partial z_{\lambda}}\right)^{n+1}f_{i}\right]\left(\overrightarrow{x_{0}}+t_{i0}\left(\overrightarrow{z}-\overrightarrow{z_{0i}}\right)\right)\right\|\\ &\leq\left[\left(\sum_{\lambda=1}^{k}\left|z_{\lambda}-x_{0\lambda}\right|\left\|\frac{\partial}{\partial z_{\lambda}}\right\|\right)^{n+1}f_{i}\right]\left(\overrightarrow{x_{0}}+t_{i0}\left(\overrightarrow{z}-\overrightarrow{z_{0i}}\right)\right).\end{aligned}$$

I.e.,

$$\left\|\left\|g_{i\overrightarrow{z}}^{(n+1)}\right\|\right\|_{\infty,[0,1]}\leq\left[\left(\sum_{\lambda=1}^{k}\left|z_{\lambda}-x_{0\lambda}\right|\left\|\left\|\frac{\partial}{\partial z_{\lambda}}\right\|\right\|_{\infty}\right)^{n+1}f_{i}\right], \tag{26}$$

$i=1,...,r$.

Hence by (26) we get

$$\left\|R_{in}\left(\overrightarrow{z},0\right)\right\|\leq\frac{\left[\left(\sum_{\lambda=1}^{k}\left|z_{\lambda}-x_{0\lambda}\right|\left\|\left\|\frac{\partial}{\partial z_{\lambda}}\right\|\right\|_{\infty}\right)^{n+1}f_{i}\right]}{(n+1)!}\leq$$

$$\frac{D_{n+1}\left(f_{i}\right)}{(n+1)!}\left(\sum_{\lambda=1}^{k}\left|z_{\lambda}-x_{0\lambda}\right|\right)^{n+1}=\frac{D_{n+1}\left(f_{i}\right)}{(n+1)!}\left\|\overrightarrow{z}-\overrightarrow{x_{0}}\right\|_{l_{1}}^{n+1}, \tag{27}$$

$i=1,...,r$.

Therefore it holds

$$\left\| R_{in}\left(\overrightarrow{z},0\right)\right\| \leq \frac{\max\limits_{i\in\{1,...,r\}} D_{n+1}\left(f_i\right)}{(n+1)!}\left\|\overrightarrow{z}-\overrightarrow{x_0}\right\|_{l_1}^{n+1}, \tag{28}$$

for $i=1,...,r$.

By (22) we get that

$$\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) f_i\left(\overrightarrow{z}\right)-\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) f_i\left(\overrightarrow{x_0}\right)=\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) R_{in}\left(\overrightarrow{z},0\right), \tag{29}$$

for all $i=1,...,r$.

Hence

$$\sum_{i=1}^{r}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) f_i\left(\overrightarrow{z}\right)-\sum_{i=1}^{r}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) f_i\left(\overrightarrow{x_0}\right)$$

$$=\sum_{i=1}^{r}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) R_{in}\left(\overrightarrow{z},0\right). \tag{30}$$

Therefore we find

$$E\left(f_1,...,f_r\right)\left(x_0\right):=$$

$$\sum_{i=1}^{r}\int_Q\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) f_i\left(\overrightarrow{z}\right)d\overrightarrow{z}-\sum_{i=1}^{r}\left(\int_Q\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right)d\overrightarrow{z}\right) f_i\left(\overrightarrow{x_0}\right)=$$

$$\sum_{i=1}^{r}\int_Q\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) R_{in}\left(\overrightarrow{z},0\right)d\overrightarrow{z}. \tag{31}$$

Consequently, we have that

$$\left\|E\left(f_1,...,f_r\right)\left(x_0\right)\right\|=$$

$$\left\|\sum_{i=1}^{r}\int_Q\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) f_i\left(\overrightarrow{z}\right)d\overrightarrow{z}-\sum_{i=1}^{r}\left(\int_Q\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right)d\overrightarrow{z}\right) f_i\left(\overrightarrow{x_0}\right)\right\|=$$

$$\left\| \sum_{i=1}^{r} \int_{Q} \left(\prod_{\substack{\rho=1 \\ \rho\neq i}}^{r} f_{\rho}\left(\overrightarrow{z}\right) \right) R_{in}\left(\overrightarrow{z},0\right) d\overrightarrow{z} \right\| \leq \tag{32}$$

$$\sum_{i=1}^{r} \left\| \int_{Q} \left(\prod_{\substack{\rho=1 \\ \rho\neq i}}^{r} f_{\rho}\left(\overrightarrow{z}\right) \right) R_{in}\left(\overrightarrow{z},0\right) d\overrightarrow{z} \right\| \leq$$

$$\sum_{i=1}^{r} \left(\int_{Q} \left\| \left(\prod_{\substack{\rho=1 \\ \rho\neq i}}^{r} f_{\rho}\left(\overrightarrow{z}\right) \right) R_{in}\left(\overrightarrow{z},0\right) \right\| d\overrightarrow{z} \right) \overset{(6)}{\leq}$$

$$\sum_{i=1}^{r} \left(\int_{Q} \left(\prod_{\substack{\rho=1 \\ \rho\neq i}}^{r} \left\| f_{\rho}\left(\overrightarrow{z}\right) \right\| \right) \left\| R_{in}\left(\overrightarrow{z},0\right) \right\| d\overrightarrow{z} \right) \overset{(28)}{\leq} \tag{33}$$

$$\frac{\max\limits_{i\in\{1,\dots,r\}} D_{n+1}\left(f_i\right)}{(n+1)!} \sum_{i=1}^{r} \left(\int_{Q} \left(\prod_{\substack{\rho=1 \\ \rho\neq i}}^{r} \left\| f_{\rho}\left(\overrightarrow{z}\right) \right\| \right) \left\| \overrightarrow{z} - \overrightarrow{x_0} \right\|_{l_1}^{n+1} d\overrightarrow{z} \right).$$

So far we have proved

$$\left\| E\left(f_1,\dots,f_r\right)\left(x_0\right) \right\| \leq$$

$$\frac{\max\limits_{i\in\{1,\dots,r\}} D_{n+1}\left(f_i\right)}{(n+1)!} \sum_{i=1}^{r} \left(\int_{Q} \left(\prod_{\substack{\rho=1 \\ \rho\neq i}}^{r} \left\| f_{\rho}\left(\overrightarrow{z}\right) \right\| \right) \left\| \overrightarrow{z} - \overrightarrow{x_0} \right\|_{l_1}^{n+1} d\overrightarrow{z} \right) =: (\xi). \tag{34}$$

Furthermore it holds

$$(\xi) \leq \frac{\max\limits_{i\in\{1,\dots,r\}} D_{n+1}\left(f_i\right)}{(n+1)!} \left(\int_{Q} \left\| \overrightarrow{z} - \overrightarrow{x_0} \right\|_{l_1}^{n+1} d\overrightarrow{z} \right) \left[\sum_{i=1}^{r} \left(\prod_{\substack{\rho=1 \\ \rho\neq i}}^{r} \left\| \left\| f_{\rho} \right\| \right\|_{\infty,Q} \right) \right], \tag{35}$$

and

$$(\xi) \leq \frac{\max\limits_{i\in\{1,\dots,r\}} D_{n+1}\left(f_i\right)}{(n+1)!} \left\| \left\| \cdot - \overrightarrow{x_0} \right\|_{l_1}^{n+1} \right\|_{\infty,Q} \left[\sum_{i=1}^{r} \left\| \left(\prod_{\substack{\rho=1 \\ \rho\neq i}}^{r} \left\| f_{\rho} \right\| \right) \right\|_{L_1(Q,A)} \right], \tag{36}$$

9

and finally

$$(\xi) \leq \frac{\max\limits_{i\in\{1,...,r\}} D_{n+1}(f_i)}{(n+1)!} \left[\sum_{i=1}^{r} \left[\left\| \left(\prod_{\substack{\rho=1 \\ \rho\neq i}}^{r} \|f_\rho\| \right) \right\|_{L_q(Q,A)} \right] \right] \left\| \|\cdot - \overrightarrow{x_0}\|_{l_1}^{n+1} \right\|_{L_p(Q,A)}, \tag{37}$$

proving (18), (19). ■

We give

Corollary 9 *(to Theorem 8) All as in Theorem 8, with $f_1 = ... = f_r = f$, $r \in \mathbb{N}$. Then*

$$\left\| \int_Q f^r(\overrightarrow{z})\, d\overrightarrow{z} - \left(\int_Q f^{r-1}(\overrightarrow{z})\, d\overrightarrow{z} \right) f(\overrightarrow{x_0}) \right\| \leq$$

$$\frac{D_{n+1}(f)}{(n+1)!} \left(\int_Q \|f(\overrightarrow{z})\|^{r-1} \|\overrightarrow{z} - \overrightarrow{x_0}\|_{l_1}^{n+1} d\overrightarrow{z} \right) \leq \tag{38}$$

$$\frac{D_{n+1}(f)}{(n+1)!} \min \left\{ \left(\int_Q \|\overrightarrow{z} - \overrightarrow{x_0}\|_{l_1}^{n+1} d\overrightarrow{z} \right) \left(\|\|f\|\|_{\infty,Q} \right)^{r-1}, \right.$$

$$\left. \left\| \|\cdot - \overrightarrow{x_0}\|_{l_1}^{n+1} \right\|_{\infty,Q} \left\| \|f\|^{r-1} \right\|_{L_1(Q,A)}, \left\| \|\cdot - \overrightarrow{x_0}\|_{l_1}^{n+1} \right\|_{L_p(Q,A)} \left\| \|f\|^{r-1} \right\|_{L_q(Q,A)} \right\}. \tag{39}$$

We also give

Corollary 10 *(to Theorem 8) All as in Theorem 8, with $(A, \|\cdot\|)$ being a commutative Banach algebra. Then*

$$\left\| r \int_Q \left(\prod_{\rho=1}^{r} f_\rho(\overrightarrow{z}) \right) d\overrightarrow{z} - \sum_{i=1}^{r} \left(\int_Q \left(\prod_{\substack{\rho=1 \\ \rho\neq i}}^{r} f_\rho(\overrightarrow{z}) \right) d\overrightarrow{z} \right) f_i(\overrightarrow{x_0}) \right\| \leq$$

$$\text{Right hand side of (18)} \leq \text{Right hand side of (19).} \tag{40}$$

We make

Remark 11 *Of great interest are applications of Theorem 8 when $Q = \prod\limits_{\lambda=1}^{k} [a_\lambda, b_\lambda]$, where $[a_\lambda, b_\lambda] \subset \mathbb{R}$, $\lambda = 1, ..., k$.*

We observe that by the multinomial theorem we get:

$$\int_{\prod\limits_{\lambda=1}^{k} [a_\lambda, b_\lambda]} \left(\sum_{\lambda=1}^{k} |z_\lambda - x_{0\lambda}| \right)^{n+1} dz_1 ... dz_k = \sum_{\rho_1+\rho_2+...\rho_k=n+1} \frac{(n+1)!}{\rho_1!\rho_2!...\rho_k!}$$

$$\int_{\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]} |z_1-x_{01}|^{\rho_1}|z_2-x_{02}|^{\rho_2}...|z_k-x_{0k}|^{\rho_k}dz_1...dz_k = \tag{41}$$

$$\sum_{\rho_1+\rho_2+...\rho_k=n+1}\frac{(n+1)!}{\rho_1!\rho_2!...\rho_k!}\prod_{\lambda=1}^{k}\left(\int_{a_\lambda}^{b_\lambda}|z_\lambda-x_{0\lambda}|^{\rho_\lambda}dz_\lambda\right)=$$

$$\sum_{\sum_{\lambda=1}^{k}\rho_\lambda=n+1}\frac{(n+1)!}{\prod_{\lambda=1}^{k}\rho_\lambda!}\prod_{\lambda=1}^{k}\left(\int_{a_\lambda}^{x_{0\lambda}}(x_{0\lambda}-z_\lambda)^{\rho_\lambda}dz_\lambda+\int_{x_{0\lambda}}^{b_\lambda}(z_\lambda-x_{0\lambda})^{\rho_\lambda}dz_\lambda\right)=$$

$$\sum_{\sum_{\lambda=1}^{k}\rho_\lambda=n+1}\frac{(n+1)!}{\prod_{\lambda=1}^{k}\rho_\lambda!}\prod_{\lambda=1}^{k}\left(\frac{(x_{0\lambda}-a_\lambda)^{\rho_\lambda+1}+(b_\lambda-x_{0\lambda})^{\rho_\lambda+1}}{\rho_\lambda+1}\right). \tag{42}$$

We have found that

$$\int_{\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]}\left\|\overrightarrow{z}-\overrightarrow{x_0}\right\|_{l_1}^{n+1}d\overrightarrow{z}= \tag{43}$$

$$\sum_{\sum_{\lambda=1}^{k}\rho_\lambda=n+1}\frac{(n+1)!}{\prod_{\lambda=1}^{k}\rho_\lambda!}\prod_{\lambda=1}^{k}\left(\frac{(b_\lambda-x_{0\lambda})^{\rho_\lambda+1}+(x_{0\lambda}-a_\lambda)^{\rho_\lambda+1}}{\rho_\lambda+1}\right).$$

Based on (18), (19) and (43) we conclude:

Theorem 12 *Let $(A,\|\cdot\|)$ a Banach algebra and $f_i\in C^{n+1}\left(\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda],A\right)$, $i=1,...,r$; $r\in\mathbb{N}$, $n\in\mathbb{Z}_+$, and fixed $\overrightarrow{x_0}\in\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]\subset\mathbb{R}^k$, $k\geq 2$. Here all vector partial derivatives $f_{i\alpha}:=\frac{\partial^\alpha f_i}{\partial z^\alpha}$, where $\alpha=(\alpha_1,...,\alpha_k)$, $\alpha_\lambda\in\mathbb{Z}^+$, $\lambda=1,...,k$, $|\alpha|=\sum_{\lambda=1}^{k}\alpha_\lambda=j$, $j=1,...,n$, fulfill $f_{i\alpha}(\overrightarrow{x_0})=0$, $i=1,...,r$.*

Denote

$$D_{n+1}(f_i):=\max_{\alpha:|\alpha|=n+1}\left\|\|f_{i\alpha}\|\right\|_{\infty,\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]}, \tag{44}$$

$i=1,...,r$.

Then

$$\left\|\sum_{i=1}^{r}\int_{\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r}f_\rho(\overrightarrow{z})\right)f_i(\overrightarrow{z})d\overrightarrow{z}-\right.$$

$$\left.\sum_{i=1}^{r}\left(\int_{\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r}f_\rho(\overrightarrow{z})\right)d\overrightarrow{z}\right)f_i(\overrightarrow{x_0})\right\|\leq \tag{45}$$

$$\left(\max_{i\in\{1,\dots,r\}} D_{n+1}\left(f_i\right)\right)\left[\sum_{i=1}^{r}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r}\left\|\|f_\rho\|\right\|_{\infty,\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]}\right)\right]$$

$$\left[\sum_{\sum_{\lambda=1}^{k}\rho_\lambda=n+1}\frac{1}{\prod_{\lambda=1}^{k}\rho_\lambda!\prod_{\lambda=1}^{k}\left(\rho_\lambda+1\right)}\prod_{\lambda=1}^{k}\left(\left(b_\lambda-x_{0\lambda}\right)^{\rho_\lambda+1}+\left(x_{0\lambda}-a_\lambda\right)^{\rho_\lambda+1}\right)\right].$$

References

[1] G.A. Anastassiou, *Multivariate Ostrowski type inequalities*, Acta Math. Hungarica 76 (4) (1997), 267-278.

[2] S.S. Dragomir, *Noncommutative Ostrowski type inequalities for functions in Banach algebras*, RGMIA Res. Rep. Coll. 24 (2021), Art. 10, 24 pp.

[3] B. Driver, *Analysis Tools with Applications*, Springer, N.Y., Heidelberg, 2003.

[4] G. Ladas, V. Laksmikantham, *Differential Equations in Abstract Spaces*, Academic Press, New York, London, 1972.

[5] A. Ostrowski, *Über die Absolutabweichung einer differentiabaren Funcktion von ihrem Integralmittelwert*, Comment. Math. Helv., 10 (1938), 226-227.

[6] W. Rudin, *Functional Analysis*, Second Edition, McGraw-Hill, Inc., New York, 1991.

[7] L. Schwartz, *Analyse Mathematique*, Hermann, Paris, 1967.

[8] G.E. Shilov, *Elementary Functional Analysis*, The MIT Press Cambridge, Massachusetts, 1974.

Gap Formula for the Mexican hat wavelet transform

Abhishek Singh[1], Aparna Rawat and Nikhila Raghuthaman

December 14, 2021

Abstract

In this paper, we study the Mexican hat wavelet formulated from the Gaussian function. The Mexican hat wavelet transform (MHWT) is defined using this basic wavelet. A standard method is introduced to obtain the gap formula for the MHWT. Further, an example for the gap formula is also presented.

Key words: Fourier transform; Wavelet transform; Schwartz distributions; Tempered Boehmians

Mathematics Subject Classification(2010): 44A15; 46F12; 54B15; 46F99

1

1 Introduction

By utilizing the theory of distributional as well as classical Fourier and Hilbert transforms, the theory of wavelet transform in L^p-spaces $(1 \leq p \leq \infty)$ is formulated. The wavelet transform has been rising as a major mathematical tool for the past two decades and its contribution to signal analysis is significant. The major reason for this is the representation of functions in a time-frequency plane is possible with wavelet transform. Hence, the wavelet transform can be treated as an operator which localizes time and frequency. Moreover, one can regulate wavelets within a fixed time period to acquire varied frequency components that are useful in enhancing the study of signals having localized impulses and oscillations. Based on the idea of wavelets as a family of functions, the mother wavelet $\psi_{b,a}(t)$ is defined by dilating and translating the function $\psi \in L^2(\mathbb{R})$ and is given by

$$\psi_{b,a}(u) = (\sqrt{a})^{-1}\psi\left(\frac{u-b}{a}\right), \quad b, u \in \mathbb{R}, a \in \mathbb{R}_+ = (0, \infty), \tag{1.1}$$

[1] Corresponding author: Department of Mathematics and Statistics
Banasthali Vidhyapith, Banasthali, India

where a is the dilation, which calculates the level of compression, and b is called shifting parameter, which works out the wavelet's time location. If $|a|< 1$, then (1.1) is the compressed version of the mother wavelet and represents higher frequencies.
For a square integrable function f, the wavelet transform with respect to $\psi_{b,a}$ is defined by [5],

$$W(b,a) = \int_{-\infty}^{\infty} f(u)\overline{\psi_{b,a}}(u)du \text{ for } a \in T_{+} \text{ and } u, b \in \mathbb{R}. \tag{1.2}$$

The inversion formula for (1.2) is given as follows:

$$f(x) = \frac{2}{C_\psi} \int_0^\infty \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{a}} W(b,a) \psi\left(\frac{x-b}{a}\right) db \right] \frac{da}{a^2}, \quad x \in \mathbb{R} \tag{1.3}$$

where

$$\frac{1}{2}C_\psi = \int_0^\infty \frac{|\hat{\psi}(u)|^2}{|u|} du = \int_0^\infty \frac{|\hat{\psi}(-u)|^2}{|u|} du < \infty \qquad [1, \text{p. } 64].$$

Recently among very many authors, the researches carried out by R. S. Pathak *et al.* [4-10] have investigated the theory of wavelet transform to distributions and ultradistribution spaces. Singh *et al.* have extended the theory for distributional wavelet and mexican hat wavelet transform [11-14]. Further, inversion formulae for the same are established in the sense of distributions and ultradistributions.

Mexican hat wavelet that is formulated by taking the second derivative of Gaussian function is defined by

$$\psi(u) = exp\left(\frac{-u^2}{2}\right) (1-u^2) = -\frac{d^2}{du^2}\, exp\left(\frac{-u^2}{2}\right). \tag{1.4}$$

Therefore,

$$\psi_{b,a}(u) = -a^{3/2} D_u^2 exp\left(-\frac{(b-u)^2}{2a^2}\right), \qquad \left(D_u = \frac{d}{du}\right). \tag{1.5}$$

Thus from (1.2), we have

$$W(b,a) = -a^{3/2} \int_{-\infty}^{\infty} f(t)\, D_t^2\, exp\left(-\frac{(b-t)^2}{2a^2}\right) dt, \qquad a > 0. \tag{1.6}$$

Then, under certain conditions on f, we have

$$W(b,a) = -a^{3/2} \int_{-\infty}^{\infty} f^{(2)}(t)\, exp\left(-\frac{(b-t)^2}{2a^2}\right) dt, \qquad a > 0. \tag{1.7}$$

From the above two equations we can consider the MHWT as the Weierstrass transform of $\left(\frac{d}{du}\right)^2 f(u)$. This relation can further be utilized to explore various

2

properties of $W(b,a)$. Also, as Weierstrass transform is defined for complex values of b, therefore, the definition of the MHWT can be extended for b being complex, whenever required.

Now for $a \in (0,\infty)$ and $b \in \mathbb{C}$, we define

$$k(b,a) = \frac{1}{\sqrt{2\pi a}} exp\left(\frac{-b^2}{2a}\right). \tag{1.8}$$

Clearly,

$$D_u^2 k(b-u,a^2) = \frac{1}{\sqrt{2\pi a}} D_u^2\left(exp\left(\frac{-(b-u)^2}{2a^2}\right)\right). \tag{1.9}$$

Hence the Mexican hat wavelet transform of a function $f(t)$ is given by [7]

$$W(b,a) = a^{3/2}\int_{-\infty}^{\infty} f^{(2)}(u) exp\left(\frac{-(b-u)^2}{2a^2}\right) du. \tag{1.10}$$

2 Gap formula for Mexican hat wavelet transform

The gap formula which is also known as the jump operator provides a unified approach to obtain a relation between the determining function at a given point in terms of the transform. Here, it acts as an operator which gives $f^{(2)}(b+) - f^{(2)}(b-)$ in terms of $W(b,a)$ where $W(b,a)$ and $f^{(2)}(b)$ are related by (1.10). Such representations have been obtained for various integral transform like Laplace transform, Stieltjes transform, Weierstrass transform, and many more [2, 15, 16]. In the next theorem, we present Gap formula for the Mexican hat wavelet transform.

Theorem 2.1. *Let $f^{(2)}(y) \in L_1(m,n)$ for any finite interval such that the integral (1.10) relating $W(b,a)$ to $f^{(2)}(y)$ converges for $m < b < n$. Also, there exists numbers $f^{(2)}(b \pm 0)$ satisfying*

$$\int_0^h [f^{(2)}(b \pm u) - f^{(2)}(b \pm 0)]du = o(h), \quad h \to 0.$$

Then for d satisfying $m < d < n$ we have for $-\infty < b < \infty$,

$$\lim_{a^2 \to 1-} -i(1-a^2)^{3/2} a \int_{d-i\infty}^{d+i\infty} (s-b) \exp\left(\frac{(s-b)^2}{2a^2}\right) W(s,1) ds = f^{(2)}(b+0) - f^{(2)}(b-0).$$

Proof. Let $\alpha(u) = \int_0^u f^{(2)}(v)dv,\ \forall d \in (m,n)$. Also, let $\alpha(u)$ be locally bounded variation, such that

$$|\alpha(u)| = \begin{cases} M\exp\left(\frac{(u-\eta)^2}{2}\right), & u > x, \\ M\exp\left(\frac{(u-\xi)^2}{2}\right), & u < x. \end{cases} \tag{2.1}$$

3

Then the MHWT of $f(v)$ is defined by

$$W(b,1) = \int_{-\infty}^{\infty} k(b-u,1) f^{(2)}(v) dv. \tag{2.2}$$

Now, using integration by parts on (2.2), we get

$$W(b,1) = \int_{-\infty}^{\infty} k_1(b-u,1)\alpha(u) du, \tag{2.3}$$

where

$$k_1(b-u,1) = \frac{\partial}{\partial b} k(b-u,1).$$

Consider

$$\begin{aligned} I &= -i(1-a^2)^{13/2} \int_{d-i\infty}^{d+i\infty} (s-b) \exp\left(\frac{(s-b)^2}{2a^2}\right) W(s,1) ds \\ &= -i(1-a^2)^{3/2} \int_{d-i\infty}^{d+i\infty} (s-b) \exp\left(\frac{(s-b)^2}{2a^2}\right) \int_{-\infty}^{\infty} k_1(s-u,1)\alpha(u) du \\ &= -i(1-a^2)^{3/2}\sqrt{2\pi}a \int_{-\infty}^{\infty} \alpha(u) du \int_{d-i\infty}^{d+i\infty} \frac{(s-b)}{\sqrt{2\pi}a} \exp\left(\frac{(s-b)^2}{2a^2}\right) k_1(s-u,1) ds. \end{aligned}$$

Let us consider

$$\begin{aligned} J &= \frac{-i}{\sqrt{2\pi}a} \int_{d-i\infty}^{d+i\infty} (s-b) \exp\left(\frac{(s-b)^2}{2a^2}\right) k_1(s-u,1) ds \\ &= \frac{1}{\sqrt{2\pi}a} \int_{-\infty}^{\infty} (d+iy-b) \exp\left(\frac{(d+iy-b)^2}{2a^2}\right) k_1(d+iy-u,1) dy, \quad (s=d+iy) \\ &= \frac{1}{\sqrt{2\pi}a} \int_{-\infty}^{\infty} i(y-i(d-b)) \exp\left(\frac{-(y-i(d-b))^2}{2a^2}\right) k_1(iy+d-u,1) dy \\ &= \int_{-\infty}^{\infty} k(d+iy-b,a^2) k_2(d+iy-u,1) dy, \end{aligned}$$

where

$$k_2(s-u,1) = \frac{\partial^2 k(s-u,1)}{\partial s^2} = (s-u)k_1(s-u,1).$$

By [7, Theorem 2.1], we have

$$\begin{aligned} J &= \int_{-\infty}^{\infty} k(d+iy-b,a^2) k_2(d+iy-u,1) dy \\ &= k_2(d+iy-u-d-iy+b, 1-a^2) \\ &= k_2(b-u, 1-a^2). \end{aligned} \tag{2.4}$$

Hence, we obtain $J = k_2(b-u, 1-a^2)$, by combining (2.4) with Corollary 2.2 of [3], where $f^{(2)}(b) = k_2(b-u, 1-a^2)$. Further, breaking the integral I into

4

4 parts, corresponding to the intervals $(-\infty, b-\delta)$, $(b-\delta, b)$, $(b, b+\delta)$ and $(b+\delta, \infty)$, we have

$$\begin{aligned} I &= (1-a^2)^{3/2}(2\pi)^{1/2}a\left\{\int_{-\infty}^{b-\delta} + \int_{b-\delta}^{b} + \int_{b}^{b+\delta} + \int_{b+\delta}^{\infty}\right\}\alpha(u)(u)k_2(b-u, 1-a^2)du \\ &= I_1(a) + I_2(a) + I_3(a) + I_4(a). \end{aligned}$$

For $I_2(a)$, we can choose a $\delta > 0$ so that $|f^{(2)}(u) - f^{(2)}(b-)| < \epsilon$ for $b - \delta < u < b$ and therefore,

$$\begin{aligned} |I_2(a) + f^{(2)}(b-)| &= \left|\int_{b-\delta}^{b} k_1(b-u, 1-a^2)[f^{(2)}(u) - f^{(2)}(b-)]du\right| + o(1) \\ &= \left|\int_{b-\delta}^{b} k_2(b-u, 1-a^2)\beta(u)du\right| + o(1) \\ &\leq \epsilon\int_{b-\delta}^{b} k_2(b-u, 1-a^2)|s-u|du + o(1) \\ &\leq \epsilon M + o(1) \quad as\ a^2 \to 1-. \end{aligned}$$

Similarly $|I_3(a) - f^{(2)}(b+)| \leq \epsilon M + o(1)$.

For ϵ being arbitrary, we have $I_2(a) \approx -f^{(2)}(b-)$ and $I_3(a) \approx f^{(2)}(b+)$.

For $I_1(a)$ and $I_4(a)$ by Lemma 2.1c of [3], for some ξ and η such that $m < \xi < \eta < n$, at $a = 1$

$$\begin{aligned} f^{(2)}(u) &= o\left[exp\left(\frac{(u-\eta)^2}{2}\right)\right], \quad u \to \infty, \\ f^{(2)}(u) &= o\left[exp\left(\frac{(u-\xi)^2}{2}\right)\right], \quad u \to \infty. \end{aligned}$$

Therefore,

$$\begin{aligned} |I_1(a)| &= \lim_{a^2\to 1-}\left|(2\pi)^{1/2}(1-a^2)^{3/2}\int_{-\infty}^{b-\delta} k_1(b-u, 1-a^2)f^{(2)}(u)du\right| \\ &\leq \lim_{a^2\to 1-}(1-a^2)^{-3/2}\int_{-\infty}^{b-\delta}\exp\left(\frac{-(b-u)^2}{2(1-a^2)}\right)|f^{(2)}(u)|du \\ &\leq \lim_{a^2\to 1-} M(1-a^2)^{-3/2}\int_{-\infty}^{b-\delta}\exp\left(\frac{-(b-u)^2}{2(1-a^2)}\right)\exp\left(\frac{-(u-\xi)^2}{2}\right)du \\ &= \mathrm{o}(1). \end{aligned}$$

Hence, $I_1(a) = \mathrm{o}(1)$ and similarly $I_4(a) = \mathrm{o}(1)$ as $a^2 \to 1-$, which concludes the proof of the theorem.

□

5

Example 2.2. As a simple example take the MHWT at $a = 1$,

$$\begin{aligned} W(s,1) &= \int_{-\infty}^{\infty} k_1(s-u,1)\alpha(u)du \\ &= \exp\left(\frac{-s^2}{2}\right), \end{aligned} \tag{2.5}$$

where

$$\alpha(u) = \int_0^u f^{(2)}(v)dv = \begin{cases} 0 & u < 0 \\ 1 & u > 0. \end{cases}$$

Since the integral (1.10) converges always, therefore by Theorem 2.1, we have

$$\begin{aligned} &= \lim_{a^2 \to 1-} -i(1-a^2)^{3/2} \int_{-\infty}^{\infty} (s-b)\exp\left(\frac{(s-b)^2}{2a^2}\right) W(s,1)ds \\ &= \lim_{a^2 \to 1-} -i(1-a^2)^{3/2} \int_{-\infty}^{\infty} (s-b)\exp\left(\frac{(s-b)^2}{2a^2}\right) \exp\left(\frac{-s^2}{2}\right) ds \\ &= \lim_{a^2 \to 1-} \frac{i(1-a^2)^{3/2}\sqrt{2\pi}a^4}{(a^2-1)^{3/2}} \exp\left(\frac{-b^2}{2(1-a^2)}\right) \\ &= \begin{cases} 1 & b = 0, \\ 0 & otherwise. \end{cases} \end{aligned} \tag{2.6}$$

Conclusions

In this article, we studied the conditions needed to obtain a relation between the determining function at a point of discontinuity with its MHWT. As the Gaussian function derives the Mexican hat wavelet, therefore it satisfies the Gaussian decays in both frequency and space. Further, as the MHWT has localization in both space and frequency, it has a strong appeal to applications in space-frequency analysis, mixed boundary value problems, approximation theory, mathematical modeling, other digital modulation.

Acknowledgement

This work is supported by Major Research Project by SERB-DST, Government of India, through sanction No. ECR/2017/000394.

References

[1] Chui CK (1992) An Introduction to Wavelets. San Diego: Academic press

[2] Ditzian, Z (1968) Gap Formulae for the Weierstrass Transforms. Canadian Mathematical Bulletin, 11(2):249-254

[3] Hirschman II, Widder DV (1955) The convolution transform. Princeton Univ. Press, New Jersey.

[4] Pathak RS (1997) Integral transforms of generalized functions and their applications. Amsterdam:Gordon and Breach Science Publishers

[5] Pathak RS (2009) The Wavelet Transform. Amsterdam, Paris: Atlantis Press, World Scientific

[6] Pathak RS, Singh A (2019) PaleyWienerSchwartz type theorem for the wavelet transform. Applicable Analysis, 98(7):1324-1332

[7] Pathak RS, Singh A (2016) Mexican hat wavelet transform of distributions. Integral Transforms and Special Functions, 27(6):468-483

[8] Pathak RS, Singh A (2016) Wavelet transform of generalized functions in $K'\{M_p\}$ spaces. Proceedings Mathematical Sciences 126(2):213-226

[9] Pathak RS, Singh A (2017) Wavelet transform of Beurling-Bjrck type ultradistributions. Rendiconti del Seminario Matematico della Universita di Padova, 137(1):211-222

[10] Pathak RS, Singh A (2016) Distributional wavelet transform. Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, 86(2), 273-277.

[11] Singh A (2021) Some characterizations of wavelet transform, Natl. Acad. Sci. Lett. 44, 143145.

[12] Singh A (2020) Distributional Mexican hat wavelet transforms, The Journal of Analysis, 28 (2), 533-544.

[13] Singh A, Raghuthaman N and Rawat A (2021) Paley-Wiener-Schwartz type theorem for ultradistributional wavelet transform, Complex Analysis and Operator Theory 15, 75, doi.org/10.1007/s11785-021-01124-4.

[14] Singh A and Rawat A (2021) Mexican hat wavelet transforms on generalized functions function, Proc. Math. Sci., 131:31. https://doi.org/10.1007/s12044-021-00627-6.

[15] Widder DV (1946) The Laplace transformation. Princeton University Press

[16] Zemanian AH (1996) Generalized Integral Transformations. Interscience Publishers, New York

Modelling the fear effect in prey predator ecosystem incorporating prey patches

Anal Chatterjee

Department of Mathematics,
Barrackpore Rastraguru Surendranath College,
Barrackpore, Kolkata-700120, India.
E-mail address: chatterjeeanal172@gmail.com

Abstract

In an ecosystem, the balance of prey-predator system is greatly influenced by the availability of prey and the fear imposed on it's population. In this paper, it is proposed that a prey-predator model in which prey is assumed to be able to detect the presence of predator and to counteract it by forming patches and incorporating the cost of fear into prey reproduction. **Equilibrium points are calculated and analysis of the local and global asymptotic behaviors of the system are done. Hopf-bifurcation is seen in case of adequate availability of prey. The system stabilizes in presence of high levels of fear.** Availability of prey act as a crucial role to change the dynamics of the system. **Numerical simulations showcases the relationship between prey patches and other related parameters like level of fear, conversion rate of predator and availability of prey. These simulations reveal the impact of fear on the prey-predator system and also justify the theoretical findings.** In the end, the bifurcation scenarios are derived when two different parameters switch together at a same time. Numerical simulations are justified the theoretical findings.

Keywords: Fear; Patches; Hunting Stability; Bifurcation.

1 Introduction

The survey of prey-predator dynamics is one of the blooming topics of ecosystem in last few decades. Predation process perform an indispensable part to maintain ecological balance. In real field application, the predator do not capture all the prey population due to refuge property of prey [1, 2]. In biomathematics, the research of prey refuge is one of the hot spot area. As a result, many researchers focus in this aspect [3, 4, 5]. Some experimental finding confirm that fear effect on predator may alter the behavior of prey [6, 7, 8]. Some theoretical studies have revealed that growth rate of prey need to improve through implementation of fear effects [9, 10, 11]. Recently, the authors in [12] studied the hunting cooperation and the fear factor among prey in a Leslie-Gower model. This study revealed that fear factor is more effective than hunting cooperation to stabilize the system. Also, the scientists in [13] proposed a Beddington-DeAngelis functional response of predator-prey model and investigated the impact of anti-predator activity on whole system. They noted that the system may exhibits multiple Hopf-bifurcation. The researchers in [14] investigated that chaotic system turned into stable system in presence of cost of fear in three species model. But very few numbers of researchers explored the combine effects of hunting cooperation and anti-predator activity in predator-prey system. In recent past, the authors in [15] studied the combine effects of hunting cooperation and fear factor in prey-predator system and observed that strong demographic Allee phenomenon. **Recently, the authors in [16] studies the influence of harvesting and allee effects in disease induced prey-predator system and reveals that allee effect and harvesting can be a handy technique for controlling the spread of disease. Fractional order mathematical models are a new research field in non-linear dynamics [17, 18]. The authors in [19] apply the homotopy analysis transform technique in prey-predator model to evaluate approximate solution which converges to the exact solution of time-fractional nonlinear subject to initial conditions.**

Anti-grazing strategy is a vital part in prey-predator system to protect prey from predator. In marine system, size of phytoplankton are very small compare to the predatory enemies but they can survive from consumes by using anti-grazing strategies like morphology [20] formation of colonies [21] which resist the grazing pressure by higher trophic organisms. Toxin ejected by phytoplankton is one of another anti-grazing strategies to protect from zooplankton [22]. The author in [23] studied the formulation of patches for defense mechanism and discussed the ability of releasing toxin chemicals. Thus, paired mechanism over with patching and poison release outcomes will act a crucial role for the coexistence species. Some experimental researches noted that the patch size depend on organism density and also proportional with it [24]. In real field, phytoplankton are allowed to form spherical patches or colonies and release toxin chemicals [25].

Motivated by the above theoretical and experimental literatures, the dynamics of such system in which hunting by predator and fear of prey is studied. The aim of the present study is to investigate the impact of hunting, fear effect and toxin effect due to formulation of patches. As per my knowledge, the combine effect of three above factors has not to explore yet. **The main target in present manuscript is to investigate the subsequent biological topics:**

- How does availability of prey density influence on the dynamics of prey-predator system.
- Can fear factor among prey influence to stabilize the prey-predator system.

- How does patches influence the prey-predator dynamics.

It is considered that, birth rate of prey population is reduced due to fear of hunting by predator. In the next section, proposed model is developed with incorporate prey patches. Section 2 represents the construction of mathematical model based on some assumptions. Basic properties such as boundedness is discussed in Section 3. Analytical results based on the model and **global stability** are discussed in Section 4. Section 5 represents the local bifurcation such as Hopf and transcritical-bifurcation analysis. Numerical simulations and discussion are illustrated in Section 6 & 7. Finally, the paper summarize with a brief conclusion.

2 Basic assumptions and model formulation

Let us consider the assumption to construct the following mathematical model: Let $x(t)$ and $y(t)$ be the density of prey and predator population at time $t > 0$ respectively. Here r and r_1 be the intrinsic growth rate and the intra-species competition rate of prey. c **and** e **represent the predation rate and conversion rate of predator.** Here $(1-k_1)$ terms represents the amount of availability of prey for predation by the predator where, $k_1 \in (0, 1]$. It is assumed that predation term is the Holling-II functional form. According to literature review, a fraction part k_1 of prey aggregate to form N patches. Therefore, each patches represent as $\frac{1}{N}k_1x$. It is assume that the three dimensional patch is roughly spherical in ocean. Therefore, the radius of patch is proportional to $[\frac{1}{N}k_1x]^{1/3}$. As a result the surface of patch is proportional to $[\frac{1}{N}k_1x]^{2/3} = \rho x^{2/3}$, where $\rho = [\frac{1}{N}k_1]^{2/3}$. **The effect of fear has a direct impact on prey reproduction [26, 27, 28]**. In presence of predator, intrinsic growth of prey becomes a function of the predator density like $F(y;K) = \frac{r}{1+Ky}$ in which K is defined as level of fear of the prey according to anti-predator response. **This above function follows some conditions**:
(i) $F(y; 0) = r$**:** in the absence of fear effect, the prey reproduction rate remain unaltered.
(ii) $F(0; K) = r$**:** in the absence of predator, **the prey reproduction rate remain unaltered**.
(iii) $\lim_{K\to\infty} F(y; K) = 0$**: extremely fearful prey fails to reproduce.**
(iv) $\lim_{y\to\infty} F(y; K) = 0$**: at a extremely higher predator density, prey fails to reproduce.**
(v) $\frac{\partial F(y;K)}{\partial K} < 0$**:** the prey reproduction rate low with high amount of fear effect.
(vi) $\frac{\partial F(y;K)}{\partial y} < 0$**:** the prey reproduction rate low with high amount of predator density.

$$\begin{aligned} \frac{dx}{dt} &= \frac{rx}{1+Ky} - r_1x^2 - \frac{c(1-k_1)xy}{1+a(1-k_1)x} \equiv G_1(x,y) \\ \frac{dy}{dt} &= \frac{e(1-k_1)xy}{1+a(1-k_1)x} - dy - e\rho x^{2/3}y \equiv G_2(x,y). \end{aligned} \tag{1}$$

The system (1) will be analyzed with the following initial conditions,

$$x(0) \geq 0, \; y(0) \geq 0. \tag{2}$$

3 Mathematical preliminaries

Theorem 1. *All non negative solutions* $(x(t), y(t)$ *of the system (1) initiate in* $R^2_+ - \{0,0\}$ *are uniformly bounded.*

Proof. Let us choose a function $\Theta = x + y$.
Therefore,

$$\frac{d\Theta}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = \frac{rx}{1+Ky} - r_1x^2 - \frac{c(1-k_1)xy}{1+a(1-k_1)x} + \frac{e(1-k_1)xy}{1+a(1-k_1)x} - dy - e\rho x^{2/3}y.$$

Let us consider a positive constant ζ such that $\zeta \leq d$. Therefore,

$$\frac{d\Theta}{dt} + \zeta\Theta \leq r_0x - r_1x^2 + \zeta x - \frac{(1-k_1)(c-e)}{1+a(1-k_1)x} - y(d-\zeta) - e\rho x^{2/3}y$$

$$\leq (r_0+\zeta)x - r_1x^2 \leq \frac{(r_0+\zeta)^2}{4r_1}.$$

By choosing $\Gamma = \frac{(r_0+\zeta)^2}{4r_1}$, we obtain

$$0 \leq \Theta(x(t), y(t)) \leq \frac{\Gamma}{\zeta}(1-e^{-\zeta t}) + \Theta(x(0), y(0))e^{-\zeta t},$$

which indicates that $0 \leq \Theta(x(t), y(t)) \leq \frac{\Gamma}{\zeta}$ as $t \to \infty$. Therefore, all non negatives solutions of the system (1) are originated from $R^2_+ - \{0,0\}$ will be restricted in the region $\nabla = \{(x,y) \in R^2_+ : x(t) + y(t) \leq \frac{\Gamma}{\zeta} + \varepsilon\}$.
In ecology, it means that the system act in a specified manner. Boundedness of the system implies that none of the two interacting species grow unexpectedly or exponentially for a long period of time. Clearly, as a result of limited resource, numbers of each species is surely bounded. □

From the ecological point of view, let us first consider the following region $R^2_+ = \{(x,y) : x \geq 0, y \geq 0\}$. Here, the function $G_1(x,y) = xf(x,y)$ and $G_2 = yg(x,y)$ of the system (1) are continuously differentiable and locally Lipschitz in $R^2_+ = \{(x,y) : x \geq 0, y \geq 0\}$. Therefore, Theorem A.4, page 423 in H. R. Thieme's book [29] implies that the solutions of the initial value problem with non-negative initial conditions exist on the interval $[0, S)$ and unique, where S is a sufficiently large number.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 30, NO.1, 2022, COPYRIGHT 2022 EUDOXUS PRESS, LLC

4 Equilibria: Existence and stability

All possible equilibria are catalogued below:
(**i**) The predator free equilibrium $E_1 = (\frac{r}{r_1}, 0)$.
(**ii**) The positive coexistence equilibrium $E^* = (x^*, y^*)$,
while x^* is ensured by solving $\{a(1-k_1)\}^3 e^3\rho^3 x^{*5} + 3\{a(1-k_1)\}^2 e^3\rho^3 x^{*4} + [3e^3\rho^3 a(1-k_1) - \{(1-k_1)(e-da)\}^3]x^{*3} + [e^3\rho^3 + 3\{(1-k_1)(e-da)\}^2 d]x^{*2} - 3\{(1-k_1)(e-da)\}d^2 x^* + d^3 = 0$.
Also, y^* is ensured by solving $cK(1-k_1)y^2 + [c(1-k_1) + r_1 x^*(1 + a(1-k_1)x^*)K]y^* - (1 + a(1-k_1)x^*)(r - r_1 x^*) = 0$.

Thus the condition for the existence of the interior equilibrium point $E^*(x^*,\ y^*)$ is given by, $x^* > 0,\ y^* > 0$.

Explicitly, general form of the Jacobian matrix at $\overline{E} = (\overline{x}, \overline{y})$ is defined as

$$\overline{J} = \begin{bmatrix} \frac{r}{(1+K\overline{y})} - 2r_1\overline{x} - \frac{c(1-k_1)\overline{y}}{(1+a(1-k_1)\overline{x})^2} & -\frac{rK\overline{x}}{(1+K\overline{y})^2} - \frac{c(1-k_1)\overline{x}}{1+a(1-k_1)\overline{x}} \\ \frac{e(1-k_1)\overline{y}}{(1+a(1-k_1)\overline{x})^2} - \frac{2}{3}e\rho\overline{y}\frac{1}{\overline{x}^{1/3}} & \frac{e(1-k_1)\overline{x}}{1+a(1-k_1)\overline{x}} - d - e\rho\overline{x}^{2/3} \end{bmatrix}. \quad (3)$$

There exists a feasible predator free steady state E_1 of the system (1) which is unstable if $\frac{d}{e} + \rho\frac{r}{r_1}^{2/3} < \frac{(1-k_1)r}{a(1-k_1)r+r_1}$.
The Jacobian matrix at E^* can be written as
$$J^* = \begin{bmatrix} \frac{r}{(1+Ky^*)} - 2r_1x^* - \frac{c(1-k_1)y^*}{(1+a(1-k_1)x^*)^2} & -\frac{rKx^*}{(1+Ky^*)^2} - \frac{c(1-k_1)x^*}{1+a(1-k_1)x^*} \\ \frac{e(1-k_1)y^*}{(1+a(1-k_1)x^*)^2} - \frac{2}{3}e\rho\frac{y^*}{x^{*1/3}} & 0 \end{bmatrix}.$$
Thus the eigenvalues in this case are obtained as roots of the quadratic
$\lambda^2 - tr(J^*) + det(J^*) = 0$,
$tr(J^*) = \frac{r}{(1+Ky^*)} - 2r_1x^* - \frac{c(1-k_1)y^*}{(1+a(1-k_1)x^*)^2}$,
$det(J^*) = [\frac{rK}{(1+Ky^*)^2} + \frac{c(1-k_1)}{1+a(1-k_1)x^*}][\frac{e(1-k_1)}{(1+a(1-k_1)x^*)^2} - \frac{2}{3}e\rho\frac{1}{x^{*1/3}}]x^*y^*$.
Now $tr(J^*) < 0$ if $\frac{r}{(1+Ky^*)} < 2r_1x^* + \frac{c(1-k_1)y^*}{(1+a(1-k_1)x^*)^2}$ as well as $det(J^*) > 0$ if $\rho < \frac{27}{8}\frac{(1-k_1)^3x^*}{(1+a(1-k_1)x^*)^6}$.
Therefore, according Routh–Hurwitz criterion we can admit that E^* is locally asymptotically stable providing the above two conditions are fulfilled.

Theorem 2. ***If the non negative equilibrium*** E^* ***exists, then*** (x^*, y^*) ***is globally asymptotically stable in the*** $x-y$ ***plane if*** $r_1 > \frac{c(1-k_1)^2a}{1+a(1-k_1)x^*}$.

Proof. Let us consider a Lyapunov function about E^*
$V = x - x^* - x^*ln\frac{x}{x^*} + \frac{c}{e}(1 + a(1-k_1)x^*)(y - y^* - y^*ln\frac{y}{y^*})$.
Differentiating V with respect to t of the system (1), we get
$\frac{dV}{dt} = (x-x^*)(\frac{r}{1+Ky} - r_1x - \frac{c(1-k_1)y}{1+a(1-k_1)x}) + \frac{c}{e}(1+a(1-k_1)x^*)(y-y^*)(\frac{e(1-k_1)xy}{1+a(1-k_1)x} - dy - e\rho x^{2/3}y)$
$=(x-x^*)\left(\frac{rK(y-y^*)}{(1+Ky)(1+Ky^*)} - r_1(x-x^*) + \frac{c(1-k_1)(y-y^*)}{1+a(1-k_1)x} + \frac{c(1-k_1)^2a(x-x^*)}{[1+a(1-k_1)x][1+a(1-k_1)x^*]}\right)+$
$\frac{c}{e}(1 + a(1-k_1)x^*)(y-y^*)\left[\frac{e(1-k_1)(x-x^*)}{(1+a(1-k_1)x)(1+a(1-k_1)x^*)} - e\rho(x^{\frac{2}{3}} - x^{*\frac{2}{3}})\right]$.

After some calculation and simplification we get
$\leq -\left[r_1 - \frac{c(1-k_1)^2 a}{1+a(1-k_1)x^*}\right](x-x^*)^2 - \frac{rK}{(1+Ky)}(x-x^*)(y-y^*)$.

Clearly, $\dot{V}$ is negative definite if $r_1 > \frac{c(1-k_1)^2 a}{1+a(1-k_1)x^*}$. Therefore by LaSalle's theorem [30] E^* is globally asymptotically stable in $x-y$ plane. □

5 Local bifurcation

5.1 Hopf-Bifurcation

Theorem 3. *The necessary and sufficient conditions for Hopf bifurcation of the system (1) around E^* at $k_1 = k_1^c$ are $[tr(J^*)]_{k_1=k_1^c} = 0$, $[det(J^*)]_{k_1=k_1^c} > 0$ and $\frac{d}{dk_1}[tr(J^*)]_{k_1=k_1^c} \neq 0$.*

Proof. The condition $[tr(J^*)]_{k_1=k_1^c} = 0$ gives $\frac{r}{(1+Ky^*)} - 2r_1x^* - \frac{c(1-k_1)y^*}{(1+a(1-k_1)x^*)^2} = 0$, in which $[tr(J^*)]_{k_1=k_1^c} = 0$.
Now $[det(J^*)]_{k_1=k_1^c} > 0$ which is equivalent to the characteristic equation $\lambda^2 + [det(J^*)]_{k_1=k_1^c} = 0$ whose roots are purely imaginary,
For $k_1 = k_1^c$, the characteristic can be written as

$$\chi^2 + \omega = 0, \tag{4}$$

where $\omega = [det(J^*)]_{k_1=k_1^c} > 0$. Therefore, the above equation has two roots of the form $\chi_1 = +i\sqrt{\omega}$ and $\chi_2 = -i\sqrt{\omega}$. Let at any neighbouring point k_1 of k_1^c, we can express the above roots in general form like $\chi_{1,2} = \theta_1(k_1) + \pm i\theta_2(k_1)$, where $\theta_1(k_1) = \frac{tr(J^*)}{2}$ and $\theta_2(k_1) = \sqrt{det(J^*) - \frac{tr(J^*)}{4}}$. Now it is to be verified the transversality condition $\frac{d}{dk_1}(Re(\chi_j(k_1)))_{k_1=k_1^c} \neq 0$ for $j = 1, 2$.
Substituting $\chi_1 = \theta_1(k_1) + i\theta_2(k_1)$ in (4) and calculate the derivative, we have

$$\begin{aligned} 2\theta_1(k_1)\theta_1'(k_1) - 2\theta_2(k_1)\theta_2'(k_1) + \omega' = 0, \\ 2\theta_2(k_1)\theta_1'(k_1) + 2\theta_1(k_1)\theta_2'(k_1) = 0. \end{aligned} \tag{5}$$

Solving (5), we get
$\frac{d}{dk_1}(Re(\chi_j(k_1)))_{k_1=k_1^c} = \frac{-2\theta_1\omega'}{2(\theta_1^2+\theta_2^2)} \neq 0$, i.e., $\frac{d}{dk_1}[tr(J^*)]_{k_1=k_1^c} \neq 0$, which satisfy the transversality condition. This implies that the system undergoes a Hopf-bifurcation at $k_1 = k_1^c$. □

5.2 Transcritical-bifurcation

Theorem 4. *System (1) undergoes a transcritical bifurcation when the system parameters satisfy the restriction $k_1 = k_1^{TC}$. Here, k_1 is seen as the bifurcation parameter.*

Proof. For $k_1 = k_1^{TC}$, the Jacobian matrix J_1 of the system (1) around E_1 has one zero eigenvalue. Let U_1 and V_1 be the eigenvectors of the matrix J_1 and $(J_1)^T$ corresponding to zero eigenvalue respectively. Therefore, we obtain $U_1 = \left(-(\frac{r}{r_1} + \frac{c(1-k_1)}{r_1+a(1-k_1)r}) \;\; 1\right)^T$ and $V_1 = (0 \;\; 1)^T$. We have $F_{k_1}(x, y) = \begin{pmatrix} 0 & -y \end{pmatrix}^T$, $F_{k_1}\left(E_1; k_1 = k_1^{TC}\right) = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$ and $(V_1)^T F_{k_1}\left(E_1; k_1 = k_1^{TC}\right) = 0$.
Also, $DF_{k_1}\left(E_1; k_1 = k_1^{TC}\right) U_1 = (0 \;\; -1)^T$.
Therefore, we obtain $(V_1)^T \left[DF_{k_1}\left(E_1; k_1 = k_1^{TC}\right)(U_1)\right] = -1$.
Further, $(V_1)^T D^2F\left(E_1; k_1 = k_1^{TC}\right)(U_1, U_1)$
$= -2e\left[\frac{r_1^2(1-k_1)}{(r_1+a(1-k_1)r)^2} - \frac{2e\rho}{3}(\frac{r_1}{r})^{1/3}\right]\left[\frac{r_1}{r} + \frac{e(1-k_1)}{r_1+a(1-k_1)r}\right] < 0.$
By applying Sotomayor's theorem [31] we can conclude that the system experiences a transcritical bifurcation at E_1 when k_1 crosses k_1^{TC}.

□

6 Numerical simulations

In order to visualize the analytical finding, we perform the numerical simulation over the set of parametric values **[32, 33, 34]**

$$r = 1.2,\; r_1 = 0.05, K = 0.1,\; k_1 = 0.7,$$
$$c = 0.45,\; e = 0.25,\; a = 0.3,\; d = 0.1,\; \rho = 0.15. \qquad (6)$$

It is noted that the system (1) shows stable dynamics around at $E^*(3.06, 5.74)$ (cf. Fig. 1(a)).

6.1 Effect of k_1

It is observed that when availability of prey species is high for predation, i.e., the low value of k_1, the dynamical system switches to unstable behavior (viz. $k_1 = 0.66$). But high level of fear can stabilize the system (1) (viz. $K = 0.2$). It is illustrated in Fig. 1(b). **Thus, the fear effect can prevent the occurrence of limit cycle oscillation and increase the stability of the system.** Fig. 2(a-b) depicts various steady state behavior of prey and predator for the parameter k_1. Here, it is noted that a Hopf point are situated (H) at $k_1 = 0.673026$ with eigenvalue $\pm 0.284862i$ and one Limit point (LP) and a Branch point (BP) coincide at $k_1 = 0.864180$ with eigenvalue $(0. - 1.2)$. **Branch point (BP) indicates that at that particular point, predator goes to extension and the transcritical bifurcation occurs. The Limit point (LP) is a collision and disappearance of two equilibria in the dynamical system. The system switches from stable to unstable or unstable to stable behavior after crossing the Hopf point(H).** It is observed that the first Lyapunov coefficient being $-2.654148e^{-03}$ at Hopf point (H) which confirm that a family of stable limit cycle generate from H (viz. Fig. 3(a)). **It is clearly indicates that increasing the amount of prey refuge**

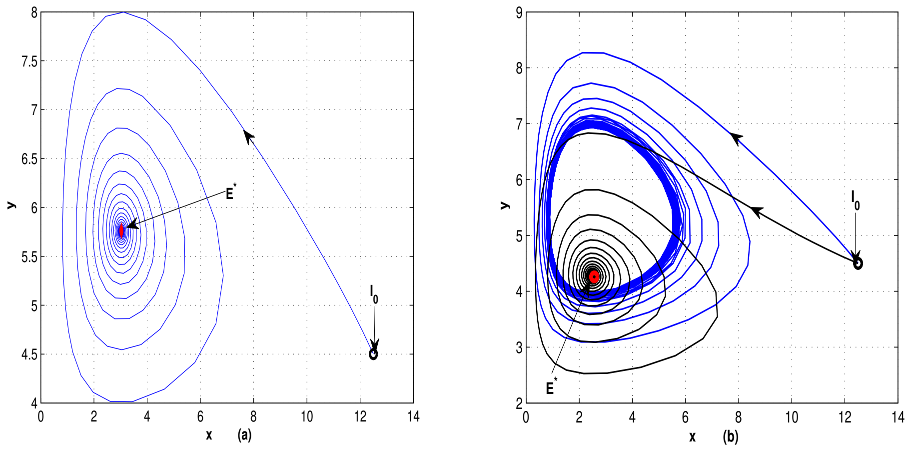


Figure 1: **(a)** The equilibrium point E^* is stable for the set of parametric values. **(b)** The figure depicts oscillatory behavior around at E^* of system (1) for $k_1 = 0.66$ and $K = 0.1$(blue line), stable behaviour at E^* for $k_1 = 0.66$ and $K = 0.2$(black line).

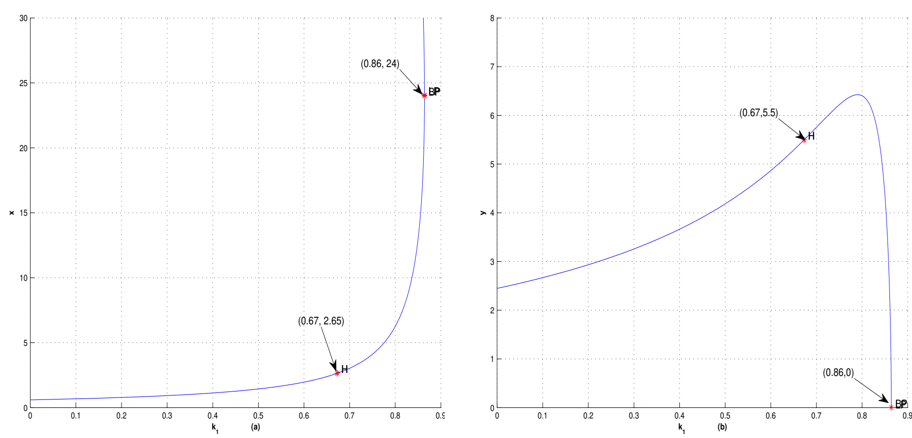


Figure 2: **(a-b)** The trajectory represents the different dynamical behaviors of prey and predator respectively for k_1.

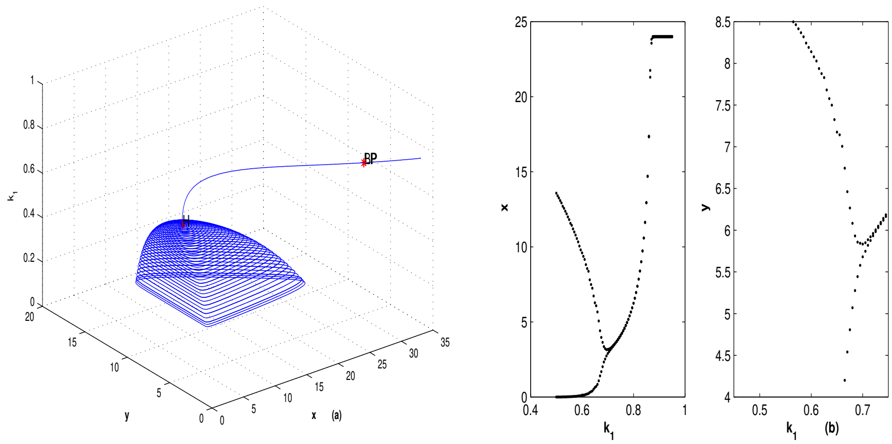


Figure 3: **(a)** The trajectory represents a family of stable limit cycles generate from Hopf (H) point for k_1 in $x - y - k_1$ plane. **(b)** Bifurcation diagram for k_1.

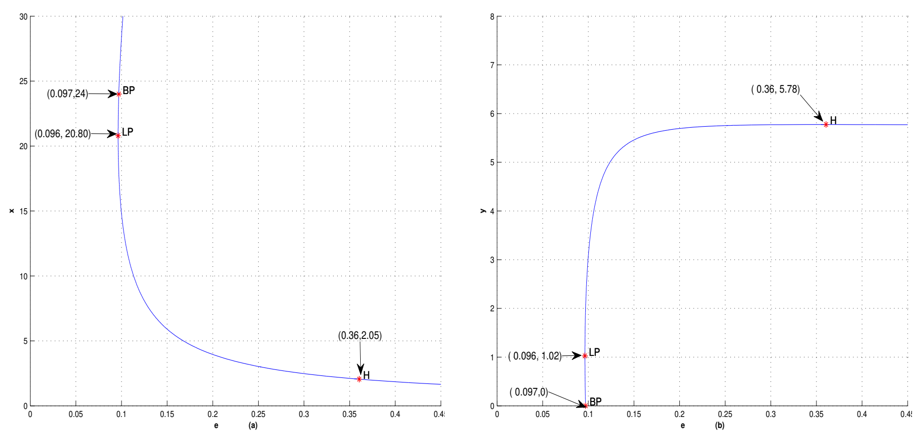


Figure 4: **(a-b)** The trajectory represents the different dynamical behaviors of prey and predator respectively for e.

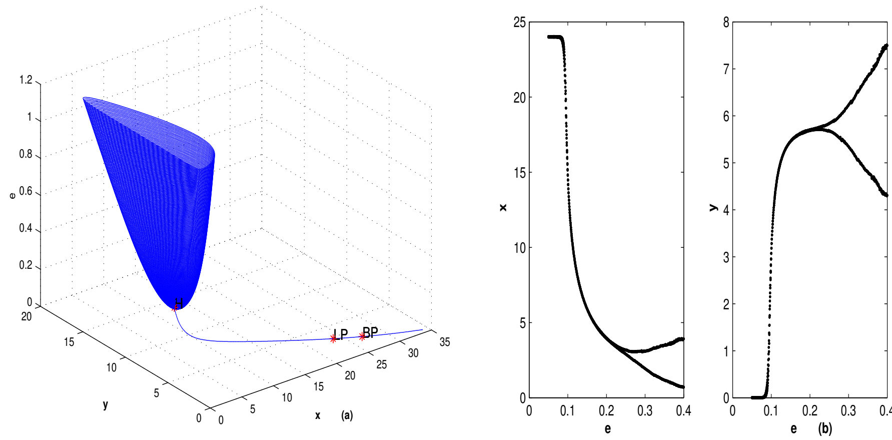


Figure 5: **(a)** The trajectory represents a family of stable limit cycles generate from Hopf (H) point for e in $x-y-e$ plane. **(b)** Bifurcation diagram for e.

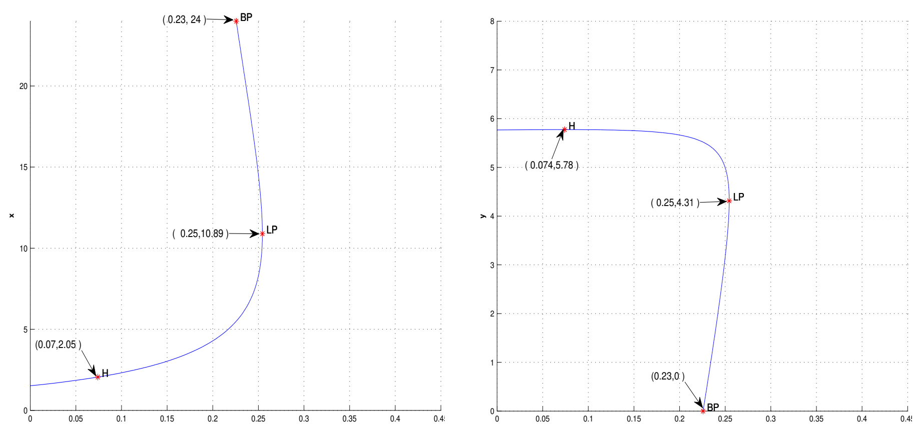


Figure 6: **(a-b)** The trajectory represents the different dynamical behaviors of prey and predator respectively for ρ.

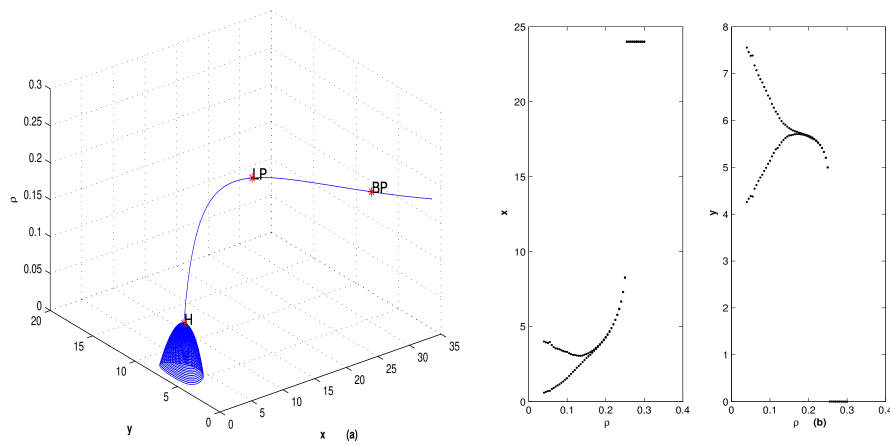


Figure 7: **(a)** The trajectory represents a family of stable limit cycles generate from Hopf (H) point for e in $x - y - \rho$ plane.. **(b)** Bifurcation diagram for ρ.

can increase both densities of prey and predator. On the other hand, when k_1 reaches a high risk threshold of the prey refuge the predator goes to extinct and the equilibrium E_1 is globally asymptotically stable.

6.2 Effect of e

Fig. 4(a-b) indicates that predator's conversion rate (e) play a crucial role to switch the prey and predator natures. Here, we have one Hopf point ($e = 0.360577$), Branch point ($e = 0.097047$) and a Limit point ($e = 0.096319$). Further, the system experiences a family of stable limit cycle generate from Hopf point (viz. Fig. 5(a)).

6.3 Effect of ρ

It is observed that the prey patches play a big impact in the system (1). From Fig. 6(a-b) & Fig. 7(a) it follow several stability behaviour and family of stable limit cycle for the free parameter ρ respectively. At $\rho = 1.416971$, the system experiences a super critical bifurcation with first Lyapunov coefficient $-2.031921e^{-03}$ and predator becomes extinct at $\rho = 0.225770$ i.e., at BP point. Also, a Limit point (LP) is obtained at $\rho = 0.254407$.

6.4 Bifurcation

The bifurcation diagrams (cf. Fig. 3(b), Fig. 5(b) and Fig. 7(b)) illustrate the complete dynamic pictures of the system (1) for the effect of parameter k_1, e

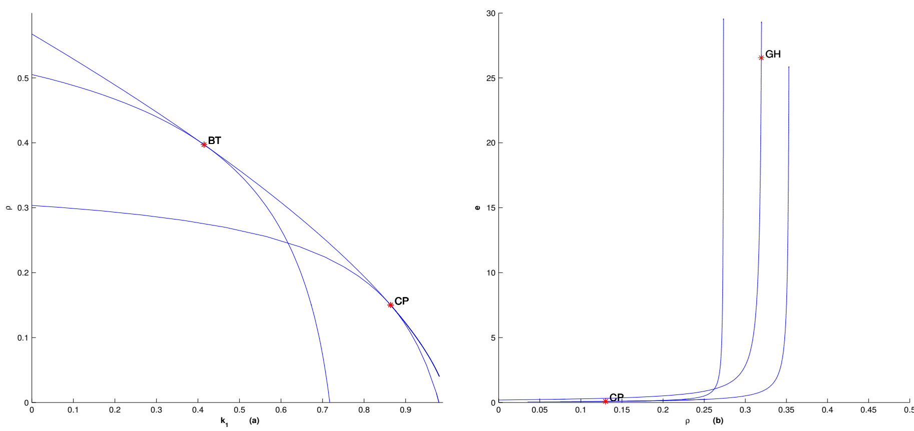


Figure 8: **(a)** Two parameters bifurcation diagram for $k_1 - \rho$. **(b)** Two parameters bifurcation diagram for $\rho - e$.

and ρ respectively. Fig. 5(a-b) display the two parameters bifurcation diagram for $k_1 - \rho$ and $\rho - e$ respectively. In this case, we see a Bogdanov-Takens (BT), Cusp bifurcation (CP) and Generalized Hopf (GH). **Generalized Hopf separates branches of sub-and supercritical Andronov-Hopf bifurcations in the two parameter plain**. The It is clearly indicates that a saddle-node bifurcation curve meet at transcritical curve at Cusp point(CP), i.e., **SN-TC** point and **saddle-node and Hopf bifurcation curve touch at** BT **point**. Also, the bifurcation curve exhibits a Generalized Hopf point (GH) where the 1^{st} Lyapunov coefficient turn out to be zero. All the numerical finding are summarized in Table 1.

7 Discussion

In this present article, a prey-predator model is designed by incorporating patches, prey refuge and fear effect to discover the dynamics of prey-predator systems. **It is assumed that prey population grows logistically and predators consume prey population under Holling II functional response.** Firstly, some basic properties are analyzed and verified which are ecologically well behaved such as boundedness and properties of existence of equilibria. **The local stability behavior of the system is carried out around each equilibrium. In order to explore the dynamics of proposed system, it is identified that, the system (1) has two equilibrium point such as axial** (E_1) **and coexistence equilibrium** (E^*)**. We also perform the global stability of coexistence equilibrium by choosing a suitable Lyapunov function.** Throughout the analysis, availability of prey, i.e., the parameter k_1 play crucial role to exhibit Hopf bifurcation and stability

Table 1: Natures of equilibrium points.

Parameters	Values	Eigenvalues	Equilibrium points
k_1	0.673026	$(\pm 0.284862i)$	Hopf (H)
	0.864180	$(0, -1.2029)$	Limit Point (LP)
	0.864180	$(0, -1.2029)$	Branch Point (BP)
e	0.360577	$(\pm 0.305907i)$	Hopf (H)
	0.096319	$(0, -1.00886)$	Limit Point(LP)
	0.097047	$(0, -1.2)$	Branch Point(BP)
ρ	0.074021	$(\pm 0.289879i)$	Hopf (H)
	0.225770	$(0, -1.2)$	Branch Point(BP)
	0.254407	$(0, -0.398958)$	Limit Point(BP)
(k_1, ρ)	$(0.4146440.397248)$	$(\approx \pm 0.00)$	Bogdanov-Takens (BT)
	$(0.8639100.150199)$	$(0, -1.20027)$	Cusp bifurcation (CP)
(ρ, e)	$(0.0835480.129990)$	$(0, -1.2)$	Cusp bifurcation (CP)
	$(0.319445, 26.549989)$	$(\pm 1.53468i)$	Generalized Hopf (GH)

switching behavior. Numerically, we observe that when $k_1 < k_1^c = 0.673026$, the system exhibits oscillatory behavior and each population shows stable coexistence between $0.673026 < k_1 < 0.864180$. When processed further, coexistence equilibrium looses stability via transcritical bifurcation i.e., branch point and the predator population will die out. Similar characteristic nature of prey and predator have been seen for the effect of conversion rate of predator and toxicity level due to patches. Further, to study the impact of fear effect, prey shows anti-predator behaviours. **Several two parameter bifurcations are drawn that show different stability nature of dynamics.** It is observed that high value of fear level can stabilize the whole system in presence of high availability of prey species for predation. **So, availability of prey species, conversion rate of predator, prey patches and fear level acts an crucial roles in in determining the long-term population dynamics. We hope that this study will contribute in understanding the impact of fear, effect of conversion rate of predator and toxicity level due to patches. The system (1) can also be modified further for two prey and one or two predator which may be more significant to the biological diversity.**

8 Conclusion

In this article, we consider fear effect prey-predator model and a prey refuge with forming patches. By examining the characteristic equation of the corresponding linearized system we obtain the threshold conditions for the stability of system. It is observed that level of fear, availability of prey due to refuge mechanism, conversion rate of predator and toxicity level due to patches play major role to stabilize the system. We find that combined effects of more than one parameters results in complex dynamical behaviour.

References

[1] A.Sih,Prey refuges and predator-prey stability,*Theoretical Population Biology*,31(1),1-12(1987).

[2] Y.Huang, F.Chen, & L.Zhong,Stability analysis of a prey–predator model with Holling type III response function incorporating a prey refuge,*Applied Mathematics and Computation*, 182(1),672-683(2006).

[3] D.Mukherjee,The effect of prey refuges on a three species food chain model,*Differential Equations and Dynamical Systems*,22(4), 413-426(2014).

[4] S.Sarwardi,P.K.Mandal,& S.Ray,Analysis of a competitive prey–predator system with a prey refuge, *Biosystems*,110(3),133-148(2012).

[5] T.K.Kar,Stability analysis of a prey–predator model incorporating a prey refuge,*Communications in Nonlinear Science and Numerical Simulation*,10(6),681-691(2005).

[6] W.Cresswell,Predation in bird populations,*Journal of Ornithology*, 152(1),251-263(2011) .

[7] S.D.Peacor,B.L.Peckarsky,G.C.Trussell,&J.R.Vonesh,Costs of predator-induced phenotypic plasticity: a graphical model for predicting the contribution of nonconsumptive and consumptive effects of predators on prey, *Oecologia*, 171(1), 1-10(2013).

[8] E.L.Preisser,&D.I.Bolnick,The many faces of fear: comparing the pathways and impacts of nonconsumptive predator effects on prey populations,*PloS one*, 3(6), e2465(2008).

[9] F.Hua,K.E.Sieving,R.J.Fletcher Jr,&C.A.Wright,Increased perception of predation risk to adults and offspring alters avian reproductive strategy and performance, *Behavioral Ecology*, 25(3), 509-519(2014).

[10] X.Wang,L.Zanette,& X.Zou,Modelling the fear effect in predator–prey interactions,*Journal of mathematical biology*,73(5), 1179-1204(2016).

[11] X.Wang,&X.Zou,Modeling the fear effect in predator–prey interactions with adaptive avoidance of predators,*Bulletin of mathematical biology*,79(6),1325-1359(2017).

[12] S.Pal,N.Pal,S.Samanta,&J.Chattopadhyay,Fear effect in prey and hunting cooperation among predators in a Leslie-Gower model,*Math. Biosci. Eng*,16(5),5146-5179(2019).

[13] S.Pal,S.Majhi,S.Mandal,&N.Pal,Role of fear in a predator–prey model with Beddington–DeAngelis functional response,*Zeitschrift für Naturforschung A*,74(7),581-595(2019).

[14] P.Panday,N.Pal,S.Samanta,&J.Chattopadhyay,Stability and bifurcation analysis of a three-species food chain model with fear,*International Journal of Bifurcation and Chaos*,28(01),1850009(2018).

[15] S.Pal,N.Pal,S.Samanta,&J.Chattopadhyay,Effect of hunting cooperation and fear in a predator-prey model, *Ecological Complexity*, 39, 100770(2019).

[16] J.P.Tripathi, K.P.Das, S.Bugalia, H.Choudhary, D. Kumar, & J.Singh,Role of harvesting and allee in a predator-prey model with disease in the both populations, *Nonlinear Studies*, 28(4), 939-968(2021).

[17] J.Singh, A.Gupta, & D.Baleanu, On the analysis of an analytical approach for fractional Caudrey-Dodd-Gibbon equations. *Alexandria Engineering Journal*,https://doi.org/10.1016/j.aej.2021.09.053.

[18] J.Singh, D.Kumar, S.D.Purohit, A.M.Mishra, & M.Bohra,An efficient numerical approach for fractional multidimensional diffusion equations with exponential memory, *Numerical Methods for Partial Differential Equations*, 37(2), 1631-1651 (2021).

[19] J.Singh, A.Kilicman, D.Kumar, & R.Swroop,Numerical study for fractional model of nonlinear predator-prey biological population dynamical system, *Thermal Science*, 23(6) S2017-S2025, (2019).

[20] D.O.Hessen & E.Van Donk,Morphological changes in Scenedesmus induced by substances released from Daphnia,*Archiv fur Hydrobiologie*, 127,129-129(1993).

[21] W.Lampert,Laboratory studies on zooplankton-cyanobacteria interactions,*New Zealand journal of marine and freshwater research*,21(3), 483-490(1987).

[22] M.F.Watanabe,H.D.Park,&M.Watanabe,Compositions of Microcystis species and heptapeptide toxins,*Internationale Vereinigung für theoretische und angewandte Limnologie: Verhandlungen*, 25(4),2226-2229(1994).

[23] T.Smayda,&Y.Shimizu,Toxic phytoplankton blooms in the sea,*Develop. Mar. Biol*, 976(1993).

[24] J.Bowman,N.Cappuccino,&L.Fahrig, Patch size and population density: the effect of immigration behavior,*Conservation ecology*, 6(1)(2002).

[25] U.Riebesell,Aggregation of Phaeocystis during phytoplankton spring blooms in the southern North Sea,*Marine Ecology Progress Series*,96, 281-289(1993).

[26] R.Ma, Y.Bai, & F.Wang,Dynamical behavior analysis of a two-dimensional discrete predator-prey model with prey refuge and fear factor, *Journal of Applied Analysis & Computation*, 10(4), 1683-1697 (2020).

[27] Z.Zhu, R.Wu, L.Lai, & X.Yu,The influence of fear effect to the Lotka–Volterra predator–prey system with predator has other food resource, *Advances in Difference Equations*, 2020(1), 1-13 (2020).

[28] D.Barman, J.Roy, H.Alrabaiah,P.Panja, S.P.Mondal, & S.Alam,Impact of predator incited fear and prey refuge in a fractional order prey predator model, *Chaos, Solitons & Fractals*, 142, 110420 (2021).

[29] Horst R Thieme, *Mathematics in population biology*, Princeton University Press, 2018.

[30] H.K. Khalil, *Nonlinear Systems*. Vol. 8 of Second Prentice-Hall International Editions. New York: Macmillan Publishing Company, 11, 589-597(1992).

[31] L.Perko,*Differential Equations and Dynamical Systems*,vol.7,Springer,Berlin,(2013).

[32] J.Chattopadhyay,S.Chatterjee,& E.Venturino,Patchy agglomeration as a transition from monospecies to recurrent plankton blooms, *Journal of theoretical biology*,253(2),289-295(2008).

[33] H.Zhang,Y.Cai,S.Fu,&W.Wang,Impact of the fear effect in a prey-predator model incorporating a prey refuge,*Applied Mathematics and Computation*, 356, 328-337(2019).

[34] A.Kumar & B.Dubey, Modeling the effect of fear in a prey–predator system with prey refuge and gestation delay, *International Journal of Bifurcation and Chaos*, 29(14), 1950195(2019).

Non-polynomial fractal quintic spline method for nonlinear boundary-value problems

Arshad Khan[a], Zainav Khatoon[a,*], Talat Sultana[b]

[a]*Department of Mathematics, Jamia Millia Islamia, New Delhi-110025, India.*

[b]*Department of Mathematics, Lakshmibai College, University of Delhi, New Delhi-110052, India.*

Abstract

In this study, we have proposed second, fourth and sixth order convergent numerical techniques for approximating linear and non-linear boundary value problems of second order with the help of fractal non-polynomial spline function. We have discussed the convergence analysis and error bound for sixth order method to prove the theoretical aspects of the presented method. Numerical problems are experimented to validate the theoretical results. Comparison with fractal polynomial and few other existing methods leads us to the conclusion that the proposed technique is more efficient.

Keywords: Difference equations, fractal non-polynomial spline, quasilinearisation, convergence analysis, truncation error.

Mathematics Subject Classification: 28A80, 65D07, 34B15

1. Introduction

With the help of fractal non-polynomial spline, we have developed numerical techniques to find the approximate solution of boundary value problems(BVPs) of the type:

$$\begin{cases} w_{tt}(t) + \mathtt{p}(t)w(t) = \mathtt{f}(t), & t \in (0,\ 1), \\ w(0) = \sigma_0, & w(1) = \sigma_1, \end{cases} \tag{1.1}$$

and

$$\begin{cases} w_{tt}(t) + \mathtt{F}(t, w(t)) = 0, & t \in (0,\ 1), \\ w(0) = \sigma_0, & w(1) = \sigma_1, \end{cases} \tag{1.2}$$

where σ_0 and σ_1 are constants. In (1.1), $\mathtt{p}(t)$ and $\mathtt{f}(t)$ are continuous functions in closed interval $I = [0,\ 1]$. For random choices of $\mathtt{p}$ and $\mathtt{f}$, exact solution of these BVPs cannot be find.

Email addresses: akhan1234in@rediffmail.com (Arshad Khan[a]), zainavk@gmail.com (Zainav Khatoon[a,*]), talat17m@gmail.com (Talat Sultana[b])

Therefore we approach numerical methods to get approximate solution of (1.1). In (1.2), presume that for $(t,w(t)) \in \mathtt{D} = \{0 \le t \le 1,\ -\infty < w(t) < \infty\}$, $\mathtt{F}$ and $\frac{\partial \mathtt{F}}{\partial w}$ are continuous. We know that (1.2) admits unique solution, if $\sup_{(t,w(t))\in D} \frac{\partial \mathtt{F}}{\partial w} < \pi^2$,[22]. Here we assume that $\frac{\partial \mathtt{F}}{\partial w} \le 0$ on D and $\frac{\partial \mathtt{F}}{\partial w} < 0$ on $D^* = \{0 < t < 1,\ -\infty < w(t) < \infty\}$. The notation w_{tt} symbolizes second derivative of w with respect to t.

Various authors have used different techniques to find numerical solution of linear as well as non-linear BVPs. Authors in [11] used cubic spline functions to find the approximate solution of nonlinear BVPs. Few numerical techniques derived by various authors for solving non-linear BVPs are given in [1, 2, 8, 14, 23, 27, 28, 32] and fractional differential equations are given in [13, 15, 16, 17, 18, 19, 29, 30].

With the help of quasilinearisation technique[6, 21, 26], the non-linear BVP (1.2) is converted into a system of linear BVPs, which in turn are solved by derived numerical scheme using fractal non-polynomial quintic spline function. A parameter λ called scaling factor is used in fractal spline which is suitably restricted to obtain the approximate solution of the linearized BVPs. Fractal interpolation function was introduced by Barnsley[4] using Iterated function system. Although fractals are difficult to constrain but they are best suitable for generation of various irregular shapes found in nature. It provides the possibility of simulating and describing landscapes precisely with the help of mathematical models. To find the numerical solution of (1.2), Balasubramani et. al.[3] have worked upon fractal quintic polynomial spline functions. In this paper we have worked upon finding the approximate solution using fractal non-polynomial spline functions and observed that the proposed scheme provides better results.

The description of paper is as follows:

At the beginning ,we have given a brief description of the presented method which uses fractal non-polynomial quintic spline to get a relation between $w(t)$ and $M(t)$ using continuity conditions. In section 3, we have discussed the truncation error. Thereafter, possible classes of method are discussed in section 4. Then we have discussed the convergence analysis of sixth order method in section 5. Error bounds are carried out. Thereafter, we have given a briefing about finite-difference method and Numerov's method, and experimented four numerical problems to testify the efficacy of proposed method in section 6. Concluding remarks are provided in section 7.

2. Fractal Nonpolynomial spline

Let $0 = t_0 < t_1 < t_2 < \ldots < t_n = 1$ be the partition of the interval $I = [0,1]$ given in (1.1) and (1.2). Let $w(t)$ and $W_{\mathtt{j}}$ denote the analytical and approximate solutions respectively. For $t_{\mathtt{j}} = \mathtt{j}h,\ h = 1/n,\ \mathtt{j} = 0,1,\ldots,n$. Let $M_{\mathtt{j}}$ and $S_{\mathtt{j}}$ denote the approximation corresponding to $w_{tt}(t_{\mathtt{j}})$ and $w_{tttt}(t_{\mathtt{j}})$ respectively.

Concept of Iterated functions system (IFS) is used to develop fractal interpolation functions(FIF). Basic details related to fractal interpolation are provided in [5, 9, 10].

Define $\mathtt{H_j} : I \to I_{\mathtt{j}}$, where $I_{\mathtt{j}} = [t_{\mathtt{j}-1}, t_{\mathtt{j}}]$ such that

$$\mathtt{H_j}(t) = ht + t_{\mathtt{j}-1},\ t \in I.$$

Clearly, $\mathtt{H_j}(t_0) = t_{\mathtt{j}-1}$ and $\mathtt{H_j}(t_n) = t_{\mathtt{j}}$,
and define $\mathbb{F}_{\mathtt{j}} : I \times \mathbb{R} \to \mathbb{R}$ such that

$$\mathbb{F}_{\mathtt{j}}(t,w) = \lambda w + r_{\mathtt{j}}(t),\ (t,w)\ \in\ I \times \mathbb{R},$$

where λ is scaling factor such that $|\lambda| < h^4$ and

$$r_{\mathtt{j}}(t) = \mathtt{A_j} cos\xi(t-t_0) + \mathtt{B_j} sin\xi(t-t_0) + \mathtt{C_j}(t-t_0)^3 + \mathtt{D_j}(t-t_0)^2 + \mathtt{E_j}(t-t_0) + \mathtt{F_j}.$$

Constructing the IFS as follows

$$I \times \mathbb{R}; X_{\mathtt{j}}(t,w) = (\mathtt{H_j}(t), (\mathbb{F}_{\mathtt{j}}(t,w))) : \mathtt{j} = 1,2,\ldots,n,$$

which satisfies the following conditions:

$$\begin{cases} \mathbb{F}_{\mathtt{j}}(t_0, W_0) = W_{\mathtt{j}-1},\ \ \mathbb{F}_{\mathtt{j}}(t_n, W_n) = W_{\mathtt{j}}, \\ \mathbb{F}_{\mathtt{j},1}(t_n, W_{n,1}) = \mathbb{F}_{\mathtt{j}+1,1}(t_0, W_{0,1}), \\ \mathbb{F}_{\mathtt{j},2}(t_0, M_0) = M_{\mathtt{j}-1},\ \ \mathbb{F}_{\mathtt{j},2}(t_n, M_n) = M_{\mathtt{j}}, \\ \mathbb{F}_{\mathtt{j},3}(t_n, W_{n,3}) = \mathbb{F}_{\mathtt{j}+1,3}(t_0, W_{0,3}), \\ \mathbb{F}_{\mathtt{j},4}(t_0, S_0) = S_{\mathtt{j}-1},\ \ \mathbb{F}_{\mathtt{j},4}(t_n, S_n) = S_{\mathtt{j}}, \end{cases}$$

where $\mathtt{j} = 1,2,....,n-1$, and $\mathbb{F}_{\mathtt{j},\mathtt{k}}(t,w) = \frac{\lambda w + r_{\mathtt{j}}^{\mathtt{k}}(t)}{h^{\mathtt{k}}}$, $\mathtt{k} = 1,2,3,4$ and

$$W_{0,1} = \frac{r_1^{(1)}(t_0)}{h-\lambda},\ \ W_{n,1} = \frac{r_n^{(1)}(t_n)}{h-\lambda},\ W_{0,3} = \frac{r_1^{(3)}(t_0)}{h^3-\lambda},\ \ W_{n,3} = \frac{r_n^{(3)}(t_n)}{h^3-\lambda}\ .$$

Clearly, IFS is satisfying C^4-differentiability conditions on FIFs[5, 9, 10].

Let $\mathcal{F} = \big\{\Phi\ \in\ \mathtt{C}^4(I,\mathbb{R}) \mid \Phi(t_0) = W_0,\ \Phi(t_n) = W_n,\ \Phi^{(2)}(t_0) = M_0,$
$\Phi^{(2)}(t_n) = M_n,\ \Phi^{(4)}(t_0) = S_0,\ \Phi^{(4)}(t_n) = S_n\big\}.$
Then $(\mathcal{F}, d)$ is a complete metric space and d is a metric induced on $\mathcal{F}$ by $\mathtt{C}^4-$norm. Let us define the Read-Bajraktarevic operator $\mathbb{T}$ on $(\mathcal{F}, d)$ as

$$\mathbb{T}(\Phi(\mathtt{H_j}(t))) = \lambda\Phi(t) + \mathtt{A_j} cos\xi(t-t_0) + \mathtt{B_j} sin\xi(t-t_0) + \mathtt{C_j}(t-t_0)^3 + \mathtt{D_j}(t-t_0)^2 \\ + \mathtt{E_j}(t-t_0) + \mathtt{F_j}\ , \qquad t \in [t_0, t_n], \qquad \mathtt{j} = 1,2,....,n.$$

As operator $\mathbb{T}$ is contraction map, it must have a unique fixed point φ (say) which will satisfy the following conditions:

$$\varphi(\mathtt{H_j}(t)) = \lambda\varphi(t) + \mathtt{A_j} cos\xi(t-t_0) + \mathtt{B_j} sin\xi(t-t_0) + \mathtt{C_j}(t-t_0)^3 + \mathtt{D_j}(t-t_0)^2$$

$$+\mathtt{E_j}(t-t_0)+\mathtt{F_j}, \quad t\in[t_0,t_n], \;\; \mathtt{j}=1,2,....,n. \tag{2.1}$$

From [10], it can be seen that

$$\begin{cases}\mathbb{F}_{\mathtt{j}}(t_0,W_0)=W_{\mathtt{j}-1}, \;\; \mathbb{F}_{\mathtt{j}}(t_n,W_n)=W_{\mathtt{j}}, \;\; \mathbb{F}_{\mathtt{j},2}(t_0,M_0)=M_{\mathtt{j}-1},\\ \mathbb{F}_{\mathtt{j},2}(t_n,M_n)=M_{\mathtt{j}}, \;\; \mathbb{F}_{\mathtt{j},4}(t_0,S_0)=S_{\mathtt{j}-1}, \;\; \mathbb{F}_{\mathtt{j},2}(t_n,S_n)=S_{\mathtt{j}},\end{cases}$$

are equivalent to

$$\begin{cases}\varphi(t_{\mathtt{j}-1})=W_{\mathtt{j}-1}, \; \varphi(t_{\mathtt{j}})=W_{\mathtt{j}}, \; \varphi^{(2)}(t_{\mathtt{j}-1})=M_{\mathtt{j}-1},\\ \varphi^{(2)}(t_{\mathtt{j}})=M_{\mathtt{j}}, \; \varphi^{(4)}(t_{\mathtt{j}-1})=S_{\mathtt{j}-1}, \; \varphi^{(4)}(t_{\mathtt{j}})=S_{\mathtt{j}}.\end{cases} \tag{2.2}$$

The conditions $\mathbb{F}_{\mathtt{j},1}(t_n,W_{n,1})=\mathbb{F}_{\mathtt{j}+1,1}(t_0,W_{0,1})$, and $\mathbb{F}_{\mathtt{j},3}(t_n,W_{n,3})=\mathbb{F}_{\mathtt{j}+1,3}(t_0,W_{0,3})$, can be reevaluated as $\varphi^{(1)}(\mathtt{H_j}(t_n))=\varphi^{(1)}(\mathtt{H_{j+1}}(t_0))$ and $\varphi^{(3)}(\mathtt{H_j}(t_n))=\varphi^{(3)}(\mathtt{H_{j+1}}(t_0))$ respectively. The coefficients $\mathtt{A}_j$, $\mathtt{B}_j$, $\mathtt{C}_j$, $\mathtt{D}_j$, $\mathtt{E}_j$ and $\mathtt{F}_j$ used in (2.1) are evaluated using (2.2). We get

$$\begin{aligned}
\mathtt{A_j} &= \tfrac{h^4}{\xi^4}\Big(S_{\mathtt{j}-1}-\tfrac{\lambda}{h^4}S_0\Big),\\
\mathtt{B_j} &= \tfrac{h^4}{\xi^4\sin\xi}\Big(S_{\mathtt{j}}-\tfrac{\lambda}{h^4}S_n\Big)-\tfrac{h^4\cos\xi}{\xi^4\sin\xi}\Big(S_{\mathtt{j}-1}-\tfrac{\lambda}{h^4}S_0\Big),\\
\mathtt{C_j} &= \tfrac{h^2}{6}\Big(M_{\mathtt{j}}-\tfrac{\lambda}{h^2}M_n\Big)-\tfrac{h^2}{6}\Big(M_{\mathtt{j}-1}-\tfrac{\lambda}{h^2}M_0\Big)+\tfrac{h^4}{6\xi^2}\Big(S_{\mathtt{j}}-\tfrac{\lambda}{h^4}S_n\Big)-\tfrac{h^4}{6\xi^2}\Big(S_{\mathtt{j}-1}-\tfrac{\lambda}{h^4}S_0\Big),\\
\mathtt{D_j} &= \tfrac{h^2}{2}\Big(M_{\mathtt{j}-1}-\tfrac{\lambda}{h^2}M_0\Big)+\tfrac{h^4}{2\xi^2}\Big(S_{\mathtt{j}-1}-\tfrac{\lambda}{h^4}S_0\Big),\\
\mathtt{E_j} &= \Big(W_{\mathtt{j}}-\lambda W_n\Big)-\Big(W_{\mathtt{j}-1}-\lambda W_0\Big)-\tfrac{h^4}{6\xi^4}(6+\xi^2)\Big(S_{\mathtt{j}}-\tfrac{\lambda}{h^4}S_n\Big)+\tfrac{h^4}{6\xi^4}(6-2\xi^2)\Big(S_{\mathtt{j}-1}-\tfrac{\lambda}{h^4}S_0\Big)\\
&\quad -\tfrac{h^2}{6}\Big(M_{\mathtt{j}}-\tfrac{\lambda}{h^2}M_n\Big)-\tfrac{2h^2}{6}\Big(M_{\mathtt{j}-1}-\tfrac{\lambda}{h^2}M_0\Big),\\
\mathtt{F_j} &= \Big(W_{\mathtt{j}-1}-\lambda W_0\Big)+\tfrac{h^4}{\xi^4}\Big(S_{\mathtt{j}-1}-\tfrac{\lambda}{h^4}S_0\Big).
\end{aligned}$$

For continuity of $\varphi^{(1)}$, we have used $\varphi^{(1)}(t_{\mathtt{j}}^-)=\varphi^{(1)}(t_{\mathtt{j}}^+)$ i.e., $\varphi^{(1)}(\mathtt{H_j}(t_n))=\varphi^{(1)}(\mathtt{H_{j+1}}(t_0))$ and eventually get the following condition:

$$\lambda\varphi^{(1)}(t_n)-\mathtt{A_j}\xi\sin\xi+\mathtt{B_j}\xi\cos\xi+3\mathtt{C_j}+2\mathtt{D_j}+\mathtt{E_j}=\lambda\varphi^{(1)}(t_0)+\xi\mathtt{B_{j+1}}+\mathtt{E_{j+1}}. \tag{2.3}$$

Similarly for continuity of $\varphi^{(3)}$ we have used $\varphi^{(3)}(t_{\mathtt{j}}^-)=\varphi^{(3)}(t_{\mathtt{j}}^+)$ i.e., $\varphi^{(3)}(\mathtt{H_j}(t_n))=\varphi^{(3)}(\mathtt{H_{j+1}}(t_0))$ and get

$$\lambda\varphi^{(3)}(t_n)+\mathtt{A_j}\xi^3\sin\xi-\mathtt{B_j}\xi^3\cos\xi+6\mathtt{C_j}=\lambda\varphi^{(3)}(t_0)+\xi^3\mathtt{B_{j+1}}+6\mathtt{C_{j+1}}. \tag{2.4}$$

After substituting the values of $\mathtt{A_j}$, $\mathtt{B_j}$, $\mathtt{C_j}$, $\mathtt{D_j}$, $\mathtt{E_j}$, $\mathtt{B_{j+1}}$, $\mathtt{C_{j+1}}$ and $\mathtt{E_{j+1}}$ in (2.3) and (2.4), we obtain

$$\Big(S_0+S_n\Big)\Big(\frac{\lambda}{2\xi^2}+\frac{\lambda}{\xi^3}\frac{\cos\xi}{\sin\xi}-\frac{\lambda}{\xi^3}\sin\xi\Big)+\Big(S_{\mathtt{j}-1}+S_{\mathtt{j}+1}\Big)\Big(\frac{h^4}{\xi^3\sin\xi}-\frac{h^4}{6\xi^4}(6+\xi^2)\Big)$$

$$+S_j\Big(\frac{h^4}{6\xi^4}(12-4\xi^2)-\frac{2h^4}{\xi^3}\frac{\cos\xi}{\sin\xi}\Big)=\lambda\varphi^{(1)}(t_n)-\lambda\varphi^{(1)}(t_0)-(W_{j+1}-2W_j+W_{j-1})$$
$$-\frac{\lambda}{2}(M_0+M_n)+\frac{h^2}{6}(M_{j+1}+4M_j+M_{j-1}), \tag{2.5}$$

and

$$(S_0+S_n)\Big(\frac{\lambda}{\xi}\frac{\cos\xi}{\sin\xi}-\frac{\lambda}{\xi\sin\xi}\Big)+\Big(\frac{h^4}{\xi\sin\xi}-\frac{h^4}{\xi^2}\Big)(S_{j-1}+S_{j+1})+S_j\Big(\frac{2h^4}{\xi^2}-\frac{2h^4}{\xi}\frac{\cos\xi}{\sin\xi}\Big)$$
$$=\lambda(\varphi^{(3)}(t_0)-\varphi^{(3)}(t_n))+h^2(M_{j-1}-2M_j+M_{j+1}), \tag{2.6}$$

respectively. From equation (2.5), we have

$$\Big(\alpha_2S_{j-1}+2\beta_2S_j+\alpha_2S_{j+1}\Big)=-\frac{1}{6h^2}(M_{j+1}+4M_j+M_{j-1})+\frac{1}{h^4}k_2\Big(S_0+S_n\Big)-\frac{\lambda}{h^4}(\varphi^{(1)}(t_n)$$
$$-\varphi^{(1)}(t_0))+\frac{\lambda}{2h^4}(M_0+M_n)+\frac{1}{h^4}(W_{j+1}-2W_j+W_{j+1}), \tag{2.7}$$

and from equation (2.6), we have

$$(\alpha_1S_{j-1}+2\beta_1S_j+\alpha_1S_{j+1})=\frac{1}{h^2}(M_{j+1}-2M_j+M_{j-1})-\frac{1}{h^4}k_1(S_0+S_n)$$
$$-\frac{\lambda}{h^4}\big(\varphi^{(3)}(t_n)-\varphi^{(3)}(t_0)\big), \tag{2.8}$$

where
$\alpha_1=\frac{1}{\xi^2}\Big(\xi cosec(\xi)-1\Big)$, $\beta_1=\frac{1}{\xi^2}\Big(1-\xi cot(\xi)\Big)$,
$\alpha_2=\frac{1}{\xi^2}\Big(\frac{1}{6}-\alpha_1\Big)$, $\beta_2=\frac{1}{\xi^2}\Big(\frac{1}{3}-\beta_1\Big)$,
$k_1=\frac{cot\xi}{\xi}-\frac{cosec\xi}{\xi}$, $k_2=\frac{1}{\xi^2}\Big(\frac{1}{2}+k_1\Big)$.
Solving (2.7) and (2.8), we get

$$S_j=\frac{(S_0+S_n)}{2h^4}\frac{(\alpha_1k_2+\alpha_2k_1)}{(\alpha_1\beta_2-\alpha_2\beta_1)}-\frac{\alpha_1\lambda}{2h^4}\frac{\big(\varphi^{(1)}(t_n)-\varphi^{(1)}(t_0)\big)}{(\alpha_1\beta_2-\alpha_2\beta_1)}+\frac{\alpha_2\lambda}{2h^4}\frac{\big(\varphi^{(3)}(t_n)-\varphi^{(3)}(t_0)\big)}{(\alpha_1\beta_2-\alpha_2\beta_1)}$$
$$+\frac{\alpha_1\lambda}{4h^4}\frac{(M_0+M_n)}{(\alpha_1\beta_2-\alpha_2\beta_1)}+\frac{\alpha_1}{2h^4}\frac{(W_{j+1}-2W_j+W_{j-1})}{(\alpha_1\beta_2-\alpha_2\beta_1)}-\frac{\alpha_1}{12h^2}\frac{(M_{j+1}+4M_j+M_{j-1})}{(\alpha_1\beta_2-\alpha_2\beta_1)}$$
$$-\frac{\alpha_2}{2h^2}\frac{(M_{j+1}-2M_j+M_{j-1})}{(\alpha_1\beta_2-\alpha_2\beta_1)}. \tag{2.9}$$

Using equation (2.9) in equation (2.8), we have

$$\alpha_1(W_{j+2}+W_{j-2})+2(\beta_1-\alpha_1)(W_{j+1}+W_{j-1})+(2\alpha_1-4\beta_1)W_j$$
$$=-2(\alpha_1+\beta_1)\lambda(\varphi^{(1)}(t_0)-\varphi^{(1)}(t_n))+2(\alpha_2+\beta_2)\lambda(\varphi^{(3)}(t_0)-\varphi^{(3)}(t_n))$$
$$-(\alpha_1+\beta_1)\lambda(M_0+M_n)+h^2(pM_{j+2}+qM_{j+1}+rM_j+qM_{j-1}+pM_{j-2}), \tag{2.10}$$

where
$p = \alpha_2 + \frac{\alpha_1}{6}$,
$q = 2\left[\frac{1}{6}(2\alpha_1 + \beta_1) - (\alpha_2 - \beta_2)\right]$,
$r = 2\left[\frac{1}{6}(\alpha_1 + 4\beta_1) + (\alpha_2 - 2\beta_2)\right]$.
Remark 1: When $\left(\alpha_1, \beta_1, \alpha_2, \beta_2\right) = \left(\frac{1}{6}, \frac{2}{6}, \frac{-7}{360}, \frac{-8}{360}\right)$ equation (2.10) reduces to (2.5) of Balasubramani et al.[3].
Remark 2: When $\lambda = 0$, equation (2.10) reduces to quintic non-polynomial spline method by P. Srivastav et al.[31].

2.1. Spline Solution for Linear BVPs

Equation (1.1) is discretized at $t = t_j$, since $M_j + p_j W_j = f_j$, where $p_j = p(t_j)$, $f_j = f(t_j)$. The boundary equations are discretized as $W_0 = \sigma_0$, $W_n = \sigma_1$.
Substitute

$$\varphi^{(3)}(t_0) = \frac{-W_0+3W_1-3W_2+W_3}{h^3}, \qquad \varphi^{(3)}(t_n) = \frac{W_n-3W_{n-1}+3W_{n-2}-W_{n-3}}{h^3},$$
$$\varphi^{(1)}(t_0) = \frac{W_1-W_0}{h}, \qquad \varphi^{(1)}(t_n) = \frac{W_n-W_{n-1}}{h},$$
$$M_j = f_j - p_j W_j,$$

in (2.10), and after some calculations we get,

$$\begin{cases} -\left[\frac{2(\alpha_1+\beta_1)\lambda}{h} + \frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_1 + \left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_2 - \left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_3 - \left[\alpha_1 + ph^2 p_{j-2}\right]W_{j-2} \\ -\left[2(\beta_1-\alpha_1) + qh^2 p_{j-1}\right]W_{j-1} - \left[(2\alpha_1 - 4\beta_1) + rh^2 p_j\right]W_j - \left[2(\beta_1-\alpha_1)\right. \\ \left.+qh^2 p_{j+1}\right]W_{j+1} - \left[\alpha_1 + ph^2 p_{j+2}\right]W_{j+2} - \left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-3} + \left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-2} \\ -\left[\frac{2(\alpha_1+\beta_1)\lambda}{h} + \frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-1} = -h^2\left[p(f_{j+2}+f_{j-2}) + q(f_{j+1}+f_{j-1}) + rf_j\right] \\ +\lambda(\alpha_1+\beta_1)[(f_0+f_n) - (p_0\sigma_0 + p_n\sigma_n)] - \left[\frac{2(\alpha_1+\beta_1)\lambda}{h} + \frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]\sigma_0 \\ -\left[\frac{2(\alpha_1+\beta_1)\lambda}{h} + \frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]\sigma_1, \qquad j = 2,3,\ldots,(n-2). \end{cases} \tag{2.11}$$

In (2.11) we have $(n-1)$ unknowns $W_1, W_2, \ldots W_{n-1}$ and $(n-3)$ equations. Therefore two more equations are required to find unique solution. Hence we derive two boundary equations as follows:

Boundary equations

Let the equation at $j = 1$ and $j = n-1$ be

$$\begin{cases} \left[\frac{2(\alpha_1+\beta_1)\lambda}{h} + \frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_0 - \left[\frac{2(\alpha_1+\beta_1)\lambda}{h} + \frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_1 + \left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_2 \\ -\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_3 - \left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-3} + \left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-2} - \left[\frac{2(\alpha_1+\beta_1)\lambda}{h}\right. \\ \left.+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-1} + \left[\frac{2(\alpha_1+\beta_1)\lambda}{h} + \frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_n = \lambda(\alpha_1+\beta_1)[(f_0+f_n) \\ -(q_0\sigma_0 + q_n\sigma_n)] + \sum_{k=0}^{k=5}\left(l_k w(t_k) + m_k h^2 w_{tt}(t_k)\right), \end{cases} \tag{2.12}$$

and

$$\begin{cases} \left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_0-\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_1+\left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_2 \\ -\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_3-\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-3}+\left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-2}-\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}\right. \\ \left.+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-1}+\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_n=\lambda(\alpha_1+\beta_1)[(\mathtt{f}_0+\mathtt{f}_n) \\ -(q_0\sigma_0+q_n\sigma_n)]+\sum_{\mathtt{k}=n-5}^{\mathtt{k}=n}\left(l_{\mathtt{k}}w(t_{\mathtt{k}})+\mathtt{m}_{\mathtt{k}}h^2w_{tt}(t_{\mathtt{k}})\right), \end{cases} \tag{2.13}$$

respectively. The system (2.11), (2.12) and (2.13) provides the numerical solution $W_{\mathtt{j}},\ \mathtt{j}=1,2,\ldots,n-1$ for linear BVPs.

2.2. Spline Solution for nonlinear BVPs

2.2.1. Quasilinearisation technique

We use quasilinearisation technique to convert the non-linear BVP given in (1.2) into a system of linear BVPs. Here $w^{(0)}(t)$ denotes the initial approximation and the function $\mathtt{F}(t,w(t))$ is expanded around the $w^{(0)}(t)$ to obtain

$$\mathtt{F}(t,w^{(1)}(t))=\mathtt{F}(t,w^{(0)}(t))+(w^{(1)}-w^{(0)})\Big(\frac{\partial\mathtt{F}}{\partial w}\Big)_{(t,w^{(0)}(t))}+\cdots.$$

In general,

$$\mathtt{F}(t,w^{(\mathtt{r}+1)}(t))=\mathtt{F}(t,w^{(\mathtt{r})}(t))+(w^{(\mathtt{r}+1)}-w^{(\mathtt{r})})\Big(\frac{\partial\mathtt{F}}{\partial w}\Big)_{(t,w^{(\mathtt{r})}(t))}+\cdots,$$

where $\mathtt{r}$ is the iteration index such that $\mathtt{r}=0,1,2,...$
The nonlinear BVP (1.2) can be written as

$$\begin{cases} w_{tt}^{(\mathtt{r}+1)}(t)+\mathtt{F}(t,w^{(\mathtt{r}+1)}(t))=0, \quad t\in(0,1), \\ w^{(\mathtt{r}+1)}(0)=\sigma_0, \quad w^{(\mathtt{r}+1)}(1)=\sigma_1. \end{cases} \tag{2.14}$$

By substituting

$$\mathtt{F}(t,w^{(\mathtt{r}+1)}(t))=\mathtt{F}(t,w^{(\mathtt{r})}(t))+(w^{(\mathtt{r}+1)}-w^{(\mathtt{r})})\Big(\frac{\partial\mathtt{F}}{\partial w}\Big)_{(t,w^{(\mathtt{r})}(t))}$$

in (2.14), we get

$$\begin{cases} w_{tt}^{(\mathtt{r}+1)}(t)+q^{(\mathtt{r})}(t)w^{(\mathtt{r}+1)}(t)=f^{(\mathtt{r})}(t), \quad t\in(0,1), \quad \mathtt{r}=0,1,..., \\ w^{(\mathtt{r}+1)}(0)=\sigma_0, \quad w^{(\mathtt{r}+1)}(1)=\sigma_1, \end{cases} \tag{2.15}$$

where

$$q^{(\mathtt{r})}(t)=\Big(\frac{\partial\mathtt{F}}{\partial w}\Big)_{(t,w^{(\mathtt{r})}(t))}, \quad f^{(\mathtt{r})}(t)=w^{(\mathtt{r})}(t)\Big(\frac{\partial\mathtt{F}}{\partial w}\Big)_{(t,w^{(\mathtt{r})}(t))}-\mathtt{F}(t,w^{(\mathtt{r})}(t)).$$

Hence the non-linear BVP (1.2) is converted into a system of linear BVPs. Now we will proceed to solve this system numerically.

2.2.2. **Numerical scheme**

Let $W_{\mathtt{j}}^{(\mathtt{r})}$ is the approximate value of $w^{(\mathtt{r})}(t_{\mathtt{j}})$ and $M_{\mathtt{j}}^{(\mathtt{r})}$ is the approximate value of $w_{tt}^{(\mathtt{r})}(t_{\mathtt{j}})$. Now, at $t=t_{\mathtt{j}}$, the differential equation (2.15) can be discretized as

$$M_{\mathtt{j}}^{(\mathtt{r}+1)}+q_{\mathtt{j}}^{(\mathtt{r})}W_{\mathtt{j}}^{(\mathtt{r}+1)}=f_{\mathtt{j}}^{(\mathtt{r})},$$

where

$$q_{\mathtt{j}}^{(\mathtt{r})}=\Big(\frac{\partial F}{\partial w}\Big)_{(t_{\mathtt{j}},w_{\mathtt{j}}^{(\mathtt{r})})},\quad f_{\mathtt{j}}^{(\mathtt{r})}=w_{\mathtt{j}}^{(\mathtt{r})}\Big(\frac{\partial F}{\partial w}\Big)_{(t_{\mathtt{j}},w_{\mathtt{j}}^{(\mathtt{r})})}-\mathtt{F}(t_{\mathtt{j}},w_{\mathtt{j}}^{(\mathtt{r})}).$$

Also, the boundary conditions can be discretised as $W_0^{(\mathtt{r}+1)}=\sigma_0,\quad W_n^{(\mathtt{r}+1)}=\sigma_1$.

Substitute

$$\varphi^{(3)}(t_0)=\frac{-W_0^{(\mathtt{r}+1)}+3W_1^{(\mathtt{r}+1)}-3W_2^{(\mathtt{r}+1)}+W_3^{(\mathtt{r}+1)}}{h^3},$$
$$\varphi^{(3)}(t_n)=\frac{W_n^{(\mathtt{r}+1)}-3W_{n-1}^{(\mathtt{r}+1)}+3W_{n-2}^{(\mathtt{r}+1)}-W_{n-3}^{(\mathtt{r}+1)}}{h^3},$$
$$\varphi^{(1)}(t_0)=\frac{W_1^{(\mathtt{r}+1)}-W_0^{(\mathtt{r}+1)}}{h},\qquad \varphi^{(1)}(t_n)=\frac{W_n^{(\mathtt{r}+1)}-W_{n-1}^{(\mathtt{r}+1)}}{h},$$
$$M_{\mathtt{j}}^{(\mathtt{r}+1)}=f_{\mathtt{j}}^{(\mathtt{r})}-q_{\mathtt{j}}^{(\mathtt{r})}W_{\mathtt{j}}^{(\mathtt{r}+1)},$$

in equation (2.10) we have

$$\begin{cases}-\Big[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_1^{(\mathtt{r}+1)}+\Big[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_2^{(\mathtt{r}+1)}-\Big[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_3^{(\mathtt{r}+1)}-\Big[\alpha_1\\ +ph^2q_{\mathtt{j}-2}^{(\mathtt{r})}\Big]W_{\mathtt{j}-2}^{(\mathtt{r}+1)}-\Big[2(\beta_1-\alpha_1)+qh^2q_{\mathtt{j}-1}^{(\mathtt{r})}\Big]W_{\mathtt{j}-1}^{(\mathtt{r}+1)}-\Big[(2\alpha_1-4\beta_1)+rh^2q_{\mathtt{j}}^{(\mathtt{r})}\Big]W_{\mathtt{j}}^{(\mathtt{r}+1)}\\ -\Big[2(\beta_1-\alpha_1)+qh^2q_{\mathtt{j}+1}^{(\mathtt{r})}\Big]W_{\mathtt{j}+1}^{(\mathtt{r}+1)}-\Big[\alpha_1+ph^2q_{\mathtt{j}+2}^{(\mathtt{r})}\Big]W_{\mathtt{j}+2}^{(\mathtt{r}+1)}-\Big[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_{n-3}^{(\mathtt{r}+1)}\\ +\Big[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_{n-2}^{(\mathtt{r}+1)}-\Big[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\Big]W_{n-1}^{(\mathtt{r}+1)}=-h^2\Big[p(f_{\mathtt{j}+2}^{(\mathtt{r})}+f_{\mathtt{j}-2}^{(\mathtt{r})})\\ +q(f_{\mathtt{j}+1}^{(\mathtt{r})}+f_{\mathtt{j}-1}^{(\mathtt{r})})+rf_{\mathtt{j}}^{(\mathtt{r})}\Big]+\lambda(\alpha_1+\beta_1)[(f_0^{(\mathtt{r})}+f_n^{(\mathtt{r})})-(q_0^{(\mathtt{r})}\sigma_0+q_n^{(\mathtt{r})}\sigma_n)]\\ -\Big[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big]\sigma_0-\Big[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\Big]\sigma_1,\quad \mathtt{j}=2,3,\ldots,(n-2).\end{cases} \tag{2.16}$$

In (2.16) we have $(n-1)$ unknowns $W_1^{(\mathtt{r}+1)},\ W_2^{(\mathtt{r}+1)},\ldots W_{n-1}^{(\mathtt{r}+1)}$ and $(n-3)$ equations. Therefore two more equations are required to find unique solution. Hence we derive two boundary equations as follows:

Boundary equations

Let the equation at $\mathtt{j}=1$ and $\mathtt{j}=n-1$ be

$$\begin{cases}\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_0^{(\mathtt{r}+1)}-\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_1^{(\mathtt{r}+1)}+\left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_2^{(\mathtt{r}+1)}\\ -\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_3^{(\mathtt{r}+1)}-\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-3}^{(\mathtt{r}+1)}+\left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-2}^{(\mathtt{r}+1)}-\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}\right.\\ \left.+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-1}^{(\mathtt{r}+1)}+\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_n^{(\mathtt{r}+1)}=\lambda(\alpha_1+\beta_1)[(f_0^{(\mathtt{r})}+f_n^{(\mathtt{r})})\\ -(q_0^{(\mathtt{r})}\sigma_0+q_n^{(\mathtt{r})}\sigma_n)]+\sum_{\mathtt{k}=0}^{\mathtt{k}=5}\left(l_{\mathtt{k}}w^{(\mathtt{r}+1)}(t_{\mathtt{k}})+\mathtt{m}_{\mathtt{k}}h^2w_{tt}^{(\mathtt{r}+1)}(t_{\mathtt{k}})\right),\end{cases}\tag{2.17}$$

and

$$\begin{cases}\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_0^{(\mathtt{r}+1)}-\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_1^{(\mathtt{r}+1)}+\left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_2^{(\mathtt{r}+1)}\\ -\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_3^{(\mathtt{r}+1)}-\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-3}^{(\mathtt{r}+1)}+\left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-2}^{(\mathtt{r}+1)}-\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}\right.\\ \left.+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W_{n-1}^{(\mathtt{r}+1)}+\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W_n^{(\mathtt{r}+1)}=\lambda(\alpha_1+\beta_1)[(f_0^{(\mathtt{r})}+f_n^{(\mathtt{r})})\\ -(q_0^{(\mathtt{r})}\sigma_0+q_n^{(\mathtt{r})}\sigma_n)]+\sum_{\mathtt{k}=n-5}^{\mathtt{k}=n}\left(l_{\mathtt{k}}w^{(\mathtt{r}+1)}(t_{\mathtt{k}})+\mathtt{m}_{\mathtt{k}}h^2w_{tt}^{(\mathtt{r}+1)}(t_{\mathtt{k}})\right),\end{cases}\tag{2.18}$$

respectively. For non-linear BVPs, system (2.16), (2.17) and (2.18) gives the approximate solution $W_{\mathtt{j}}^{(\mathtt{r}+1)},\ \mathtt{j}=1,2,\ldots,n-1$.

3. Truncation error

From (2.16), we have

$$\begin{cases}T_{\mathtt{j}}^{(\mathtt{r})}(h)=\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W^{(\mathtt{r}+1)}(t_0)-\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W^{(\mathtt{r}+1)}(t_1)\\ \qquad+\left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W^{(\mathtt{r}+1)}(t_2)-\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W^{(\mathtt{r}+1)}(t_3)-\Big[\alpha_1\\ \qquad+ph^2q^{(\mathtt{r})}(t_{\mathtt{j}-2})\Big]W^{(\mathtt{r}+1)}(t_{\mathtt{j}-2})-\Big[2(\beta_1-\alpha_1)+qh^2q^{(\mathtt{r})}(t_{\mathtt{j}-1})\Big]W^{(\mathtt{r}+1)}(t_{\mathtt{j}-1})\\ \qquad-\Big[(2\alpha_1-4\beta_1)+rh^2q^{(\mathtt{r})}(t_{\mathtt{j}})\Big]W^{(\mathtt{r}+1)}(t_{\mathtt{j}})-\Big[2(\beta_1-\alpha_1)\\ \qquad+qh^2q^{(\mathtt{r})}(t_{\mathtt{j}+1})\Big]W^{(\mathtt{r}+1)}(t_{\mathtt{j}+1})-\Big[\alpha_1+ph^2q^{(\mathtt{r})}(t_{\mathtt{j}+2})\Big]W^{(\mathtt{r}+1)}(t_{\mathtt{j}+2})\\ \qquad-\left[\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W^{(\mathtt{r}+1)}(t_{n-3})+\left[\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W^{(\mathtt{r}+1)}(t_{n-2})\\ \qquad-\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right]W^{(\mathtt{r}+1)}(t_{n-1})+\left[\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right]W^{(\mathtt{r}+1)}(t_n)\\ \qquad+h^2\Big[\mathtt{p}(f^{(\mathtt{r})}(t_{\mathtt{j}+2})+f^{(\mathtt{r})}(t_{\mathtt{j}-2}))+q(f^{(\mathtt{r})}(t_{\mathtt{j}+1})+f^{(\mathtt{r})}(t_{\mathtt{j}-1}))+rf^{(\mathtt{r})}(t_{\mathtt{j}})\Big]\\ \qquad-\lambda(\alpha_1+\beta_1)[(f^{(\mathtt{r})}(t_0)+f^{(\mathtt{r})}(t_n))-(q^{(\mathtt{r})}(t_0)W^{(\mathtt{r}+1)}(t_0)\\ \qquad+q^{(\mathtt{r})}W^{(\mathtt{r}+1)}(t_n))],\quad \mathtt{j}=2,3,\ldots,(n-2).\end{cases}\tag{3.1}$$

Substituting $f^{(\mathtt{r})}(t_{\mathtt{j}})=w_{tt}^{(\mathtt{r}+1)}(t_{\mathtt{j}})+q^{(\mathtt{r})}(t_{\mathtt{j}})w^{(\mathtt{r}+1)}(t_{\mathtt{j}})$ in (3.1), we get

$$\begin{cases} T_{\mathtt{j}}^{(\mathtt{r})}(h) = -2(\alpha_2+\beta_2)\lambda\left[\frac{-W^{(\mathtt{r}+1)}(t_0)+3W^{(\mathtt{r}+1)}(t_1)-3W^{(\mathtt{r}+1)}(t_2)+W^{(\mathtt{r}+1)}(t_3)}{h^3}\right] \\ \quad +2(\alpha_2+\beta_2)\lambda\left[\frac{W^{(\mathtt{r}+1)}(t_n)-W^{(\mathtt{r}+1)}(t_{n-1})+3W^{(\mathtt{r}+1)}(t_{n-2})-W^{(\mathtt{r}+1)}(t_{n-3})}{h^3}\right] \\ \quad -2(\alpha_1+\beta_1)\lambda\left[\frac{W_1^{(\mathtt{r}+1)}-W_0^{(\mathtt{r}+1)}}{h}\right]+2(\alpha_1+\beta_1)\lambda\left[\frac{W_n^{(\mathtt{r}+1)}-W_{n-1}^{(\mathtt{r}+1)}}{h}\right] \\ \quad -(\alpha_1+\beta_1)\lambda W_{tt}^{(\mathtt{r}+1)}(t_0)-(\alpha_1+\beta_1)\lambda W_{tt}^{(\mathtt{r}+1)}(t_n) \\ \quad -\alpha_1(w^{(\mathtt{r}+1)}(t_{\mathtt{j}+2})+w^{(\mathtt{r}+1)}(t_{\mathtt{j}-2}))-2(\beta_1-\alpha_1)(w^{(\mathtt{r}+1)}(t_{\mathtt{j}+1})+w^{(\mathtt{r}+1)}(t_{\mathtt{j}-1})) \\ \quad -(2\alpha_1-4\beta_1)w^{(\mathtt{r}+1)}(t_{\mathtt{j}})+ph^2w_{tt}^{(\mathtt{r}+1)}(t_{\mathtt{j}+2})+qh^2w_{tt}^{(\mathtt{r}+1)}(t_{\mathtt{j}+1})+rh^2w_{tt}^{(\mathtt{r}+1)}(t_{\mathtt{j}}) \\ \quad +qh^2w_{tt}^{(\mathtt{r}+1)}(t_{\mathtt{j}+1})+ph^2w_{tt}^{(\mathtt{r}+1)}(t_{\mathtt{j}+2}). \end{cases} \tag{3.2}$$

After further simplification we obtain,

$$\begin{cases} T_{\mathtt{j}}^{(\mathtt{r})}(h) = -2(\alpha_2+\beta_2)\lambda\left[W_{ttt}^{(\mathtt{r}+1)}(t_0)+O(h)\right]+2(\alpha_2+\beta_2)\lambda\left[W_{ttt}^{(\mathtt{r}+1)}(t_n)+O(h)\right] \\ \quad -2(\alpha_1+\beta_1)\lambda\left[W_{t}^{(\mathtt{r}+1)}(t_0)+O(h)\right]+2(\alpha_1+\beta_1)\lambda\left[W_{t}^{(\mathtt{r}+1)}(t_n)+O(h)\right] \\ \quad -(\alpha_1+\beta_1)\lambda W_{tt}^{(\mathtt{r}+1)}(t_0)-(\alpha_1+\beta_1)\lambda W_{tt}^{(\mathtt{r}+1)}(t_n) \\ \quad +\left[\frac{1}{6}(7\alpha_1+\beta_1)-(4p+q)\right]h^4w_{tttt}^{(\mathtt{r}+1)}(t_{\mathtt{j}})+\left[\frac{1}{180}(31\alpha_1+\beta_1)\right. \\ \quad \left.-\frac{1}{12}(16p+q)\right]h^6w_{tttttt}^{(\mathtt{r}+1)}(t_{\mathtt{j}})+\left[\frac{1}{131040}(1611\alpha_1+31\beta_1)\right. \\ \quad \left.-\frac{1}{360}(4p+q)\right]h^8w_{tttttttt}^{(\mathtt{r}+1)}(t_{\mathtt{j}})+O(h^9). \end{cases} \tag{3.3}$$

We write

$$T_{\mathtt{j}}^{(\mathtt{r})}(h) = T_{\lambda}^{(\mathtt{r})}(h)+T_{*}^{(\mathtt{r})}(h),$$

where

$$\begin{aligned} T_{\lambda}^{(\mathtt{r})}(h) &= -2(\alpha_2+\beta_2)\lambda\left[W_{ttt}^{(\mathtt{r}+1)}(t_0)+O(h)\right]+2(\alpha_2+\beta_2)\lambda\left[W_{ttt}^{(\mathtt{r}+1)}(t_n)+O(h)\right] \\ &\quad -2(\alpha_1+\beta_1)\lambda\left[W_{t}^{(\mathtt{r}+1)}(t_0)+O(h)\right]+2(\alpha_1+\beta_1)\lambda\left[W_{t}^{(\mathtt{r}+1)}(t_n)+O(h)\right] \\ &\quad -(\alpha_1+\beta_1)\lambda W_{tt}^{(\mathtt{r}+1)}(t_0)-(\alpha_1+\beta_1)\lambda W_{tt}^{(\mathtt{r}+1)}(t_n), \end{aligned}$$

and

$$\begin{aligned} T_{*}^{(\mathtt{r})}(h) &= \left[\frac{1}{6}(7\alpha_1+\beta_1)-(4p+q)\right]h^4w_{tttt}^{(\mathtt{r}+1)}(t_{\mathtt{j}})+\left[\frac{1}{180}(31\alpha_1+\beta_1)-\frac{1}{12}(16p+q)\right]h^6w_{tttttt}^{(\mathtt{r}+1)}(t_{\mathtt{j}}) \\ &\quad +\left[\frac{1}{131040}(1611\alpha_1+31\beta_1)-\frac{1}{360}(4p+q)\right]h^8w_{tttttttt}^{(\mathtt{r}+1)}(t_{\mathtt{j}})+O(h^9). \end{aligned}$$

4. Class of methods

4.1. Second order method

Choose λ such that $|\lambda| < h^4$. For getting method of second order, unknown coefficients must satisfy conditions:

$(\alpha_1+\beta_1) = \frac{1}{2}$.

$\left[\frac{1}{6}(7\alpha_1+\beta_1)-(4p+q)\right] \neq 0$.

One such set of values are:

$(\alpha_1,\beta_1)=\left(\frac{1}{4},\frac{1}{4}\right)$ and
$p=1/4$, $q=0$, $r=1/2$.
Also

at $\mathtt{j}=1$, $\quad \left(l_0,l_1,l_2,l_3,l_4,l_5\right)=\left(0,-1,2,-1,0,0\right),$
$\left(\mathtt{m}_0,\mathtt{m}_1,\mathtt{m}_2,\mathtt{m}_3,\mathtt{m}_4,\mathtt{m}_5\right)=\left(0,\frac{1}{6},\frac{4}{6},\frac{1}{6},0,0\right),$

and

at $\mathtt{j}=n-1$, $\quad \left(l_n,l_{n-1},l_{n-2},l_{n-3},l_{n-4},l_{n-5}\right)=\left(0,-1,2,-1,0,0\right),$
$\left(\mathtt{m}_n,\mathtt{m}_{n-1},\mathtt{m}_{n-2},\mathtt{m}_{n-3},\mathtt{m}_{n-4},\mathtt{m}_{n-5}\right)=\left(0,\frac{1}{6},\frac{4}{6},\frac{1}{6},0,0\right).$

Since $|\lambda|<h^4$, we have $T_{\lambda}^{(\mathtt{r})}(h)=O(h^4)$ and $T_{*}^{(\mathtt{r})}(h)=\frac{-2}{3}h^4w_{tttt}^{(\mathtt{r}+1)}(t_{\mathtt{j}})+O(h^5)$. Therefore

$$T_{\mathtt{j}}^{(\mathtt{r})}(h)=O(h^4). \tag{4.1}$$

4.2. Fourth order method

Choose λ such that $|\lambda|<h^6$. For getting method of order four, values of unknown coefficients must satisfy conditions:
$(\alpha_1+\beta_1)=\frac{1}{2}$,
$\left[\frac{1}{6}\left(7\alpha_1+\beta_1\right)-(4p+q)\right]=0,$
$\left[\frac{1}{180}\left(31\alpha_1+\beta_1\right)-\frac{1}{12}(16p+q)\right]\neq 0.$
One such set of values are $(\alpha_1,\beta_1)=\left(\frac{1}{6},\frac{1}{3}\right)$ and
$p=\frac{1}{120}, q=\frac{26}{120}, r=\frac{66}{120}$.
Also

at $\mathtt{j}=1$, $\quad \left(l_0,l_1,l_2,l_3,l_4,l_5\right)=\left(0,-1,2,-1,0,0\right),$
$\left(\mathtt{m}_0,\mathtt{m}_1,\mathtt{m}_2,\mathtt{m}_3,\mathtt{m}_4,\mathtt{m}_5\right)=\left(0,\frac{1}{12},\frac{10}{12},\frac{1}{12},0,0\right),$

and

at $\mathtt{j}=n-1$, $\quad \left(l_n,l_{n-1},l_{n-2},l_{n-3},l_{n-4},l_{n-5}\right)=\left(0,-1,2,-1,0,0\right),$
$\left(\mathtt{m}_n,\mathtt{m}_{n-1},\mathtt{m}_{n-2},\mathtt{m}_{n-3},\mathtt{m}_{n-4},\mathtt{m}_{n-5}\right)=\left(0,\frac{1}{12},\frac{10}{12},\frac{1}{12},0,0\right).$

Since $|\lambda|<h^6$, we have $T_{\lambda}^{(\mathtt{r})}(h)=O(h^6)$ and $T_{*}^{(\mathtt{r})}(h)=\frac{7}{5000}h^6w_{tttt}^{(\mathtt{r}+1)}(t_{\mathtt{j}})+O(h^7)$. Therefore

$$T_{\mathtt{j}}^{(\mathtt{r})}(h)=O(h^6). \tag{4.2}$$

4.3. Sixth order method

Choose λ such that $|\lambda|<h^8$. For getting method of order six, values of unknown coefficients must satisfy conditions:

$(\alpha_1+\beta_1)=\frac{1}{2}$,
$\frac{1}{6}(7\alpha_1+\beta_1)-(4p+q)=0$,
$\frac{1}{180}(31\alpha_1+\beta_1)-\frac{1}{12}(16p+q)=0$,
$\left[\frac{1}{131040}(1611\alpha_1+31\beta_1)-\frac{1}{360}(4p+q)\right]\neq 0$.
The only set of such values are $(\alpha_1,\beta_1)=\left(\frac{1}{12},\frac{5}{12}\right)$ and $p=\frac{1}{360}, q=\frac{56}{360}, r=\frac{246}{360}$.
Also

at $\mathtt{j}=1$, $\quad \big(l_0,l_1,l_2,l_3,l_4,l_5\big)=\big(-4,7,-2,-1,0,0\big)$,
$\big(\mathtt{m}_0,\mathtt{m}_1,\mathtt{m}_2,\mathtt{m}_3,\mathtt{m}_4,\mathtt{m}_5\big)=\big(\frac{71}{240},\frac{43}{12},\frac{7}{8},\frac{1}{3},\frac{-5}{48},\frac{1}{60}\big)$,

and

at $\mathtt{j}=n-1$, $\quad \big(l_n,l_{n-1},l_{n-2},l_{n-3},l_{n-4},l_{n-5}\big)=\big(-4,7,-2,-1,0,0\big)$,
$\big(\mathtt{m}_n,\mathtt{m}_{n-1},\mathtt{m}_{n-2},\mathtt{m}_{n-3},\mathtt{m}_{n-4},\mathtt{m}_{n-5}\big)=\big(\frac{71}{240},\frac{43}{12},\frac{7}{8},\frac{1}{3},\frac{-5}{48},\frac{1}{60}\big)$.

Since $|\lambda|<h^8$, we have $T_{\lambda}^{(\mathtt{r})}(h)=O(h^8)$ and $T_{*}^{(\mathtt{r})}(h)=\frac{7}{5000}h^8 w_{tttttttt}^{(\mathtt{r}+1)}(t_{\mathtt{j}})+O(h^9)$.
Therefore

$$T_{\mathtt{j}}^{(\mathtt{r})}(h)=O(h^8). \tag{4.3}$$

Remark 3: Since $\alpha_2=\frac{1}{\xi^2}\left(\frac{1}{6}-\alpha_1\right)$ and $\beta_2=\frac{1}{\xi^2}\left(\frac{1}{3}-\beta_1\right)$,
i.e.$(\alpha_2+\beta_2)=\frac{1}{\xi^2}\left(\frac{1}{2}-(\alpha_1+\beta_1)\right)$,
therefore $(\alpha_1+\beta_1)=\frac{1}{2}$ implies $(\alpha_2+\beta_2)=0$.

5. Convergence analysis

The system given in (2.16), (2.17) and (2.18) can be written as

$$M^{(\mathtt{r})}W^{(\mathtt{r}+1)}=d^{(\mathtt{r})}, \tag{5.1}$$

where

$$M^{(\mathtt{r})}=\begin{bmatrix}
M_{1,1}^{(\mathtt{r})} & M_{1,2}^{(\mathtt{r})} & M_{1,3}^{(\mathtt{r})} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & M_{1,n-3}^{(\mathtt{r})} & M_{1,n-2}^{(\mathtt{r})} & M_{1,n-1}^{(\mathtt{r})}\\
M_{2,1}^{(\mathtt{r})} & M_{2,2}^{(\mathtt{r})} & M_{2,3}^{(\mathtt{r})} & M_{2,4}^{(\mathtt{r})} & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & M_{2,n-3}^{(\mathtt{r})} & M_{2,n-2}^{(\mathtt{r})} & M_{2,n-1}^{(\mathtt{r})}\\
M_{3,1}^{(\mathtt{r})} & M_{3,2}^{(\mathtt{r})} & M_{3,3}^{(\mathtt{r})} & M_{3,4}^{(\mathtt{r})} & M_{3,5}^{(\mathtt{r})} & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & M_{3,n-3}^{(\mathtt{r})} & M_{3,n-2}^{(\mathtt{r})} & M_{3,n-1}^{(\mathtt{r})}\\
M_{4,1}^{(\mathtt{r})} & M_{4,2}^{(\mathtt{r})} & M_{4,3}^{(\mathtt{r})} & M_{4,4}^{(\mathtt{r})} & M_{4,5}^{(\mathtt{r})} & M_{4,6}^{(\mathtt{r})} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & M_{4,n-3}^{(\mathtt{r})} & M_{4,n-2}^{(\mathtt{r})} & M_{4,n-1}^{(\mathtt{r})}\\
M_{5,1}^{(\mathtt{r})} & M_{5,2}^{(\mathtt{r})} & M_{5,3}^{(\mathtt{r})} & M_{5,4}^{(\mathtt{r})} & M_{5,5}^{(\mathtt{r})} & M_{5,6}^{(\mathtt{r})} & M_{5,7}^{(\mathtt{r})} & 0 & \dots & 0 & 0 & 0 & 0 & 0 & M_{5,n-3}^{(\mathtt{r})} & M_{5,n-2}^{(\mathtt{r})} & M_{5,n-1}^{(\mathtt{r})}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
M_{n-5,1}^{(\mathtt{r})} & M_{n-5,2}^{(\mathtt{r})} & M_{n-5,3}^{(\mathtt{r})} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & M_{n-5,n-7}^{(\mathtt{r})} & M_{n-5,n-6}^{(\mathtt{r})} & M_{n-5,n-5}^{(\mathtt{r})} & M_{n-5,n-4}^{(\mathtt{r})} & M_{n-5,n-3}^{(\mathtt{r})} & M_{n-5,n-2}^{(\mathtt{r})} & M_{n-5,n-1}^{(\mathtt{r})}\\
M_{n-4,1}^{(\mathtt{r})} & M_{n-4,2}^{(\mathtt{r})} & M_{n-4,3}^{(\mathtt{r})} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & M_{n-4,n-6}^{(\mathtt{r})} & M_{n-4,n-5}^{(\mathtt{r})} & M_{n-4,n-4}^{(\mathtt{r})} & M_{n-4,n-3}^{(\mathtt{r})} & M_{n-4,n-2}^{(\mathtt{r})} & M_{n-4,n-1}^{(\mathtt{r})}\\
M_{n-3,1}^{(\mathtt{r})} & M_{n-3,2}^{(\mathtt{r})} & M_{n-3,3}^{(\mathtt{r})} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & M_{n-3,n-5}^{(\mathtt{r})} & M_{n-3,n-4}^{(\mathtt{r})} & M_{n-3,n-3}^{(\mathtt{r})} & M_{n-3,n-2}^{(\mathtt{r})} & M_{n-3,n-1}^{(\mathtt{r})}\\
M_{n-2,1}^{(\mathtt{r})} & M_{n-2,2}^{(\mathtt{r})} & M_{n-2,3}^{(\mathtt{r})} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & M_{n-2,n-4}^{(\mathtt{r})} & M_{n-2,n-3}^{(\mathtt{r})} & M_{n-2,n-2}^{(\mathtt{r})} & M_{n-2,n-1}^{(\mathtt{r})}\\
M_{n-1,1}^{(\mathtt{r})} & M_{n-1,2}^{(\mathtt{r})} & M_{n-1,3}^{(\mathtt{r})} & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & M_{n-1,n-3}^{(\mathtt{r})} & M_{n-1,n-2}^{(\mathtt{r})} & M_{n-1,n-1}^{(\mathtt{r})}
\end{bmatrix},$$

where $W^{(\mathtt{r}+1)} = (W_1^{(\mathtt{r}+1)}, W_2^{(\mathtt{r}+1)}, \ldots, W_{n-1}^{(\mathtt{r}+1)})^T$, $M^{(\mathtt{r})}$ is coefficient matrix of $W^{(\mathtt{r}+1)}$and $d^{(\mathtt{r})} = (d_1^{(\mathtt{r})}, d_2^{(\mathtt{r})}, \ldots, d_{n-1}^{(\mathtt{r})})^T$.
Let $N^{(\mathtt{r})}(\mathtt{r})$ be the matrix when $\lambda = 0$. Note that,

$$\|M^{(\mathtt{r})} - N^{(\mathtt{r})}\|_\infty = \max_i \sum_{i=1}^{n-1} \|M_{i,\mathtt{j}}^{(\mathtt{r})} - N_{i,\mathtt{j}}^{(\mathtt{r})}\|.$$

Thus we get

$$\|M^{(\mathtt{r})} - N^{(\mathtt{r})}\|_\infty = 2\left|\frac{2(\alpha_1+\beta_1)\lambda}{h} + \frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right| + 2\left|\frac{-6(\alpha_2+\beta_2)\lambda}{h^3}\right| + 2\left|\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right|.$$

Theorem 5.1. *[7] : Let Q_1andQ_2 be any two matrices having matrix norm as $\|.\|$. If the eigen values of Q_1 are given as $\theta_1, \theta_2, \ldots, \theta_n$ and eigenvalues of Q_2 be given as $\mu_1, \mu_2, \ldots, \mu_{\mathtt{n}}$. Then*

$$\max_{\mathtt{j}} |\theta_{\mathtt{j}} - \mu_{\mathtt{j}}| \leq 2^{\frac{2N-1}{N}} N^{\frac{1}{N}} (2P)^{\frac{N-1}{N}} \|Q_1 - Q_2\|^{\frac{1}{N}}, \tag{5.2}$$

where $P = max(\|Q_1\|,\ \|Q_2\|)$.

In our case, we take the matrices $M^{(\mathtt{r})} = Q_1$, $N^{(\mathtt{r})} = Q_2$, $N = n-1$. Using $\|.\|_\infty$ in theorem 5.1, we get

$$\max_{\mathtt{j}} |\theta_{\mathtt{j}} - \mu_{\mathtt{j}}| \leq 2^{\left(\frac{2n-3}{n-1}\right)} (n-1)^{\left(\frac{1}{n-1}\right)} (2P)^{\left(\frac{n-2}{n-1}\right)} \|M^{(\mathtt{r})} - N^{(\mathtt{r})}\|_\infty^{\left(\frac{1}{n-1}\right)}, \tag{5.3}$$

where $P = max(\|M^{(\mathtt{r})}\|_\infty,\ \|N^{(\mathtt{r})}\|_\infty)$ and $M^{(\mathtt{r})}$ and $N^{(\mathtt{r})}$ have eigenvalues $\theta_{\mathtt{j}}$ *and* $\mu_{\mathtt{j}}, \mathtt{j} = 1, 2, \ldots, n-1$ respectively.

For sufficiently small values of h, $N^{(\mathtt{r})}(\mathtt{r})$ becomes irreducible, $N_{i,i}^{(\mathtt{r})} > 0$, $N_{i,\mathtt{j}}^{(\mathtt{r})} \leq 0$, $i \neq \mathtt{j}$ and the row sums give

$R_1^{(\mathtt{r})} = 4 - \frac{43}{12}h^2 q_1^{(\mathtt{r})} - \frac{7}{8}h^2 q_2^{(\mathtt{r})} - \frac{1}{3}h^2 q_3^{(\mathtt{r})} \quad > 0,$

$R_2^{(\mathtt{r})} = \frac{1}{12} - \frac{56}{360}h^2 q_1^{(\mathtt{r})} - \frac{246}{360}h^2 q_2^{(\mathtt{r})} - \frac{56}{360}h^2 q_3^{(\mathtt{r})} - \frac{1}{360}h^2 q_4^{(\mathtt{r})} \quad > 0,$

$R_{\mathtt{j}}^{(\mathtt{r})} = -\frac{1}{360}h^2 q_{i-2}^{(\mathtt{r})} - \frac{56}{360}h^2 q_{i-1}^{(\mathtt{r})} - \frac{246}{360}h^2 q_i^{(\mathtt{r})} - \frac{56}{360}h^2 q_{i+1}^{(\mathtt{r})} - \frac{1}{360}h^2 q_{i+2}^{(\mathtt{r})} \quad > 0,$
where $\mathtt{j} = 3, 4, \ldots n-3$,

$R_{n-2}^{(\mathtt{r})} = \frac{1}{12} - \frac{56}{360}h^2 q_{n-1}^{(\mathtt{r})} - \frac{246}{360}h^2 q_{n-2}^{(\mathtt{r})} - \frac{56}{360}h^2 q_{n-3}^{(\mathtt{r})} - \frac{1}{360}h^2 q_{n-4}^{(\mathtt{r})} \quad > 0,$

$R_{n-1}^{(\mathtt{r})} = 4 - \frac{43}{12}h^2 q_{n-1}^{(\mathtt{r})} - \frac{7}{8}h^2 q_{n-2}^{(\mathtt{r})} - \frac{1}{3}h^2 q_{n-3}^{(\mathtt{r})} \quad > 0.$

Here $N^{(\mathtt{r})}$ is a monotone matrix [20]. Therefore for adequately small values of h, $(N^{(\mathtt{r})})^{-1}$

exist and we get non-zero eigenvalues $\mu_{\mathtt{j}},\ \mathtt{j}=1,2,\ldots n-1$. Thus for these values of h (corresponding to which $N^{(\mathtt{r})}$ is a monotone matrix), λ lies in the region $(-h^8,\ h^8)$. We select λ in such a manner that it must satisfy the following two conditions :

(i) $M^{(\mathtt{r})}$ is invertible matrix, since $\|M^{(\mathtt{r})}-N^{(\mathtt{r})}\|_\infty = 2\left|\frac{2(\alpha_1+\beta_1)\lambda}{h}+\frac{6(\alpha_2+\beta_2)\lambda}{h^3}\right|+2\left|\frac{-6(\alpha_2+\beta_2)\lambda}{h^3}\right|+2\left|\frac{2(\alpha_2+\beta_2)\lambda}{h^3}\right|$, and from (5.3) we find that eigenvalues of $M^{(\mathtt{r})}$ are non-zero, whenever λ is sufficiently small.

(ii) Since $N_{\mathtt{j}}^{(\mathtt{r})}>0,\ \mathtt{j}=1,2,\ldots,n-1,$, the row sum corresponding to $M^{(\mathtt{r})}$ is

$$S_{\mathtt{j}}^{(\mathtt{r})}=R_{\mathtt{j}}-\frac{4(\alpha_1+\beta_1)\lambda}{h}-\frac{4(\alpha_2+\beta_2)\lambda}{h^3},\quad \mathtt{j}=1,2,\ldots,n-1, \tag{5.4}$$

when λ is sufficiently small.

When $N^{(\mathtt{r})}$ is monotone (i.e. when h is adequately small) and $M^{(\mathtt{r})}$ invertible and row sum of $M^{(\mathtt{r})}$ is positive (i.e. for sufficiently small $\lambda\in(-h^8,h^8)$).We derive the error bound as follows:

5.1. Error Bound for Sixth order method

The system (2.16), (2.17), and (2.18) with analytic solutions can be written as

$$M^{(\mathtt{r})}\bar{w}^{(\mathtt{r}+1)}=d^{(\mathtt{r})}+\mathtt{T}^{(\mathtt{r})}(h), \tag{5.5}$$

where

$$\bar{w}^{(\mathtt{r}+1)}=(\bar{w}^{(\mathtt{r}+1)}(t_1),\bar{w}^{(\mathtt{r}+1)}(t_2),\ldots,\bar{w}^{(\mathtt{r}+1)}(t_{n-1}))^T,$$

and

$$\mathtt{T}^{(\mathtt{r})}(h)=(\mathtt{T}_1^{(\mathtt{r})}(h),\mathtt{T}_2^{(\mathtt{r})}(h),\ldots,\mathtt{T}_{n-1}^{(\mathtt{r})}(h))^T.$$

Since from (5.1) we have

$$M^{(\mathtt{r})}W^{(\mathtt{r}+1)}=d^{(\mathtt{r})}. \tag{5.6}$$

Using (5.5) and (5.6) we get

$$M^{(\mathtt{r})}(\bar{w}^{(\mathtt{r}+1)}-W^{(\mathtt{r}+1)})=\mathtt{T}^{(\mathtt{r})}(h),$$

that is,

$$M^{(\mathtt{r})}\mathtt{E}^{(\mathtt{r}+1)}=\mathtt{T}^{(\mathtt{r})}(h), \tag{5.7}$$

where $\mathtt{E}^{(\mathtt{r}+1)} = (\mathtt{E}_1^{(\mathtt{r}+1)}, \mathtt{E}_2^{(\mathtt{r}+1)}, \ldots, \mathtt{E}_{n-1}^{(\mathtt{r}+1)})^T$, $\mathtt{E}_\mathtt{j}^{(\mathtt{r}+1)} = w^{(\mathtt{r}+1)}(t_\mathtt{j}) - W_\mathtt{j}^{(\mathtt{r}+1)}$.
Consequently, using (5.7) we obtain

$$\mathtt{E}^{(\mathtt{r}+1)} = (M^{(\mathtt{r})})^{-1}\mathtt{T}^{(\mathtt{r})}(h). \tag{5.8}$$

Using the definition of product of inverse of matrix with the matrix itself, we get

$$\sum_{\mathtt{j}=1}^{n-1} M_{i,\mathtt{j}}^{(\mathtt{r})^{-1}} \; S_\mathtt{j}^{(\mathtt{r})} = 1, \;\; i = 1,2,\ldots,n-1.$$

Hence by (5.4) we get

$$\sum_{\mathtt{j}=1}^{n-1} M_{i,\mathtt{j}}^{(\mathtt{r})^{-1}} \leq \frac{1}{\underset{1\leq \mathtt{j}\leq n-1}{S_\mathtt{j}^{(\mathtt{r})}}} = \frac{1}{C_i^{(\mathtt{r})} h^2}, \tag{5.9}$$

such that $C^{(\mathtt{r})}$ is constant. Using (5.8) and (5.9) we get

$$\mathtt{E}_i^{(\mathtt{r}+1)} = \sum_{\mathtt{j}=1}^{n-1} M_{i,\mathtt{j}}^{(\mathtt{r})^{-1}} \mathtt{T}_\mathtt{j}^{(\mathtt{r})}(h), \quad i = 1,2,\ldots,n-1. \tag{5.10}$$

Substituting (4.3) and (5.9) in (5.10), we get

$$|\mathtt{E}_i^{(\mathtt{r}+1)}| \leq \frac{qh^8}{C_i^{(\mathtt{r})} h^2},$$

where q is a constant.
Hence we obtain

$$\|E\|_\infty = O(h^6),$$

which proves that the proposed scheme is sixth-order convergent. Similar procedure can be used to derive the convergence of second as well as fourth order methods.

6. Numerical experiments

We take adequate number of iterations till the maximum error between the two succeeding iterations satisfy the following tolerance bound:

$$\max_\mathtt{j} |W_\mathtt{j}^{(\mathtt{r}+1)} - W_\mathtt{j}^{(\mathtt{r})}| < TOL,$$

where TOL is convergence tolerance. When the condition is met, we believe $W^{(\mathtt{r}+1)}$ is the approximate value W of the given problem. Here we have considered $TOL = 10^{-15}$.
For each n, $\mathtt{E}_N$ denotes the maximum point-wise error which is determined by

$$\max_\mathtt{j} |w(t_\mathtt{j}) - W_\mathtt{j}|,$$

where $w(t_{\mathtt{j}})$ and $W_{\mathtt{j}}$ are the analytic and approximate solutions respectively at $t = t_{\mathtt{j}}$. Order of convergence of the proposed method is determined as

$$\mathrm{p}^n = log_2\left(\frac{\mathrm{E}^n}{\mathrm{E}^{2n}}\right).$$

6.1. Numerical Schemes for comparison

As we compare the presented method with Numerov's method and second order finite difference method, here we give a brief particulars about these two methods.

6.1.1. Finite-difference method

Consider BVP given in (1.1) and (1.2), let $W^{(\mathtt{r}+1)}$ be the approximate value of $w^{(\mathtt{r}+1)}(t)$. Putting

$$W_{tt}^{(\mathtt{r}+1)(t)} \approx \frac{1}{h^2}\Big[W_{\mathtt{j}-1}^{(\mathtt{r}+1)} - 2W_{\mathtt{j}}^{(\mathtt{r}+1)} + W_{\mathtt{j}+1}^{(\mathtt{r}+1)}\Big], \tag{6.1}$$

in (1.2) and after simplifying, we get

$$W_{\mathtt{j}-1}^{(\mathtt{r}+1)} + \big[-2 + h^2 q_{\mathtt{j}}^{(\mathtt{r}+1)}\big] W_{\mathtt{j}}^{(\mathtt{r}+1)} + W_{\mathtt{j}+1}^{(\mathtt{r}+1)} = h^2 \mathtt{f}_{\mathtt{j}}^{(\mathtt{r})}, \tag{6.2}$$

for $\mathtt{j} = 1, 2, \ldots n$. Here $W_0 = \sigma_0$ and $W_1 = \sigma_1$.

6.1.2. Numerov's method

For BVP given in (1.1) and (1.2), Numerov's method can be written as

$$W_{\mathtt{j}-1} - 2W_{\mathtt{j}} + W_{\mathtt{j}+1} = \frac{h^2}{12}\big[\mathtt{f}_{\mathtt{j}-1} + 10\mathtt{f}_{\mathtt{j}} + \mathtt{f}_{\mathtt{j}+1}\big], \tag{6.3}$$

where $\mathtt{f}_{\mathtt{j}} = \mathtt{f}(t_{\mathtt{j}}, W_{\mathtt{j}}),\ \mathtt{j} = 0, 1 \ldots n,\ W_0 = \sigma_0$ and $W_1 = \sigma_1$. To get more details about this method, one can refer [12].

Problem 1: Consider the following linear BVP[25, 31]

$$\begin{cases} w_{tt}(t) + w(t) = -1, \quad 0 < t < 1, \\ w(0) = 0, \quad w(1) = 0, \end{cases} \tag{6.4}$$

with exact solution $w(t) = cos(t) + \frac{1-cos(1)}{sin(1)} sin(t) - 1$. Approximate results are shown in Table 1 along with results given by Srivastava et al.[31] and Ramadan et al.[25]. λ varies according to the order of method.

Problem 2: Consider the following nonlinear BVP[3]

$$\begin{cases} w_{tt}(t) + exp(-2w(t)) = 0, \quad 0 < t < 1, \\ w(0) = 0, \quad w(1) = log(2), \end{cases} \tag{6.5}$$

Table 1: **M.A.E. for problem 1.**

h	**1/8**	**1/16**	**1/32**	**1/64**
Second Order Method $p = 0.04063483994113$, $q = 0.25412730690212$, $r = 0.41047570631347$	1.5516×10^{-03}	2.0410×10^{-04}	3.0770×10^{-05}	5.2534×10^{-06}
p^N	2.9263	2.7296	2.5502	
$(p,q,r) = \left(\frac{1}{4}, 0, \frac{1}{2}\right)$	3.4324×10^{-03}	6.0707×10^{-04}	1.2491×10^{-04}	2.8070×10^{-05}
p^N	2.4992	2.2809	2.1538	
Fourth Order Method $(p,q,r) = \left(\frac{1}{120}, \frac{26}{120}, \frac{66}{120}\right)$	1.9214×10^{-05}	5.8656×10^{-07}	1.7739×10^{-08}	5.2095×10^{-10}
p^N	5.0337	5.0472	5.0896	
$(p,q,r) = \left(\frac{1}{720}, \frac{11}{45}, \frac{183}{360}\right)$	1.9558×10^{-05}	6.0424×10^{-07}	1.8788×10^{-08}	5.8564×10^{-10}
p^N	5.0164	5.0072	5.0036	
Sixth Order Method $(p,q,r) = \left(\frac{1}{360}, \frac{56}{360}, \frac{246}{360}\right)$	2.6594×10^{-07}	2.2124×10^{-09}	1.6972×10^{-11}	1.2678×10^{-13}
p^N	6.9093	7.0262	7.0646	
Srivastava et al.[31]	7.1329×10^{-08}	5.2213×10^{-09}	3.6359×10^{-10}	3.1275×10^{-11}
p^N	3.7720	3.8440	3.5392	
Ramadan et al.[25]	1.7538×10^{-04}	2.1600×10^{-05}	2.6770×10^{-06}	3.3310×10^{-07}
p^N	3.0213	3.0123	3.0065	

with exact solution $w(t) = log(1+t)$. Approximate results are shown in Table 2 along with results given by Balasubramani et al.[3], finite difference method and Mohanty et al.[24].

Table 2: **M.A.E for problem 2.**

h	**1/8**	**1/16**	**1/32**	**1/64**
Second Order Method				
$(p,q,r) = \left(\frac{1}{4}, 0, \frac{1}{2}\right)$	1.9977×10^{-03}	4.5767×10^{-04}	1.1324×10^{-04}	2.8198×10^{-05}
p^N	2.1259	2.0148	2.0058	
$(p,q,r) = \left(\frac{1}{4}, \frac{1}{4}, 0\right)$	2.7119×10^{-03}	6.2781×10^{-04}	1.5566×10^{-04}	3.8770×10^{-05}
p^N	2.1109	2.0119	2.0053	
Fourth Order Method				
$(p,q,r) = \left(\frac{1}{720}, \frac{11}{45}, \frac{183}{360}\right)$	2.6377×10^{-05}	9.0287×10^{-07}	3.0209×10^{-08}	1.0626×10^{-09}
p^N	4.8686	4.9014	4.8292	
Balasubramani et al.[3]				
$(p,q,r) = \left(\frac{1}{120}, \frac{26}{120}, \frac{66}{120}\right)$	3.7039×10^{-06}	1.3093×10^{-07}	4.6024×10^{-09}	1.6823×10^{-10}
p^N	4.8222	4.8303	4.7739	
Sixth Order Method				
$(p,q,r) = \left(\frac{1}{360}, \frac{56}{360}, \frac{246}{360}\right)$	2.4456×10^{-07}	2.1358×10^{-09}	1.6419×10^{-11}	1.1984×10^{-13}
p^N	6.8392	7.0233	7.0980	
Finite difference method	2.2281×10^{-04}	5.6130×10^{-05}	1.4060×10^{-05}	3.5166×10^{-06}
p^N	1.9890	1.9972	1.9993	
Mohanty et al.[24]	1.6424×10^{-05}	1.0481×10^{-06}	6.5976×10^{-08}	3.8966×10^{-09}
p^N	3.9699	3.9896	4.0816	

Problem 3: Consider the following nonlinear BVP[3]

$$\begin{cases} w_{tt}(t) - \frac{(2-t)exp(2w(t))+(1/(t+1))}{3} = 0, & 0 < t < 1, \\ w(0) = 0, \qquad w(1) = log(1/2), \end{cases} \tag{6.6}$$

with exact solution $w(t) = log(1/1+t)$. Approximate results are shown in Table 3 along with results given by Balasubramani et al.[3], finite difference method and Numerov's method.

Table 3: **M.A.E for problem 3.**

h	**1/8**	**1/16**	**1/32**	**1/64**
Second Order Method $(p,q,r)=\left(\frac{1}{4},0,\frac{1}{2}\right)$	1.3688×10^{-03}	4.1286×10^{-04}	1.1600×10^{-04}	3.0846×10^{-05}
p^N	1.7292	1.8314	1.9110	
$(p,q,r)=\left(\frac{1}{4},\frac{1}{4},0\right)$	2.3839×10^{-03}	6.2248×10^{-04}	1.6526×10^{-04}	4.2528×10^{-05}
p^N	1.9372	1.9132	1.9582	
Fourth Order Method $(p,q,r)=\left(\frac{1}{720},\frac{11}{45},\frac{183}{360}\right)$	2.7594×10^{-05}	9.4434×10^{-07}	3.1573×10^{-08}	1.1062×10^{-09}
p^N	4.8689	4.9025	4.8349	
Balasubramani et al.[3] $(p,q,r)=\left(\frac{1}{120},\frac{26}{120},\frac{66}{120}\right)$	3.8662×10^{-06}	1.3680×10^{-07}	4.8082×10^{-09}	1.7524×10^{-10}
p^N	4.8207	4.8304	4.7781	
Sixth Order Method $(p,q,r)=\left(\frac{1}{360},\frac{56}{360},\frac{246}{360}\right)$	1.3851×10^{-07}	1.2157×10^{-09}	6.9262×10^{-12}	1.2062×10^{-13}
p^N	6.8320	7.4555	5.8434	
Finite difference method	2.3261×10^{-04}	5.8573×10^{-05}	1.4670×10^{-05}	3.6702×10^{-06}
p^N	1.9890	1.9974	1.9989	
Numerov′s Method	2.1034×10^{-06}	1.3382×10^{-07}	8.4017×10^{-09}	5.2577×10^{-10}
p^N	3.9743	3.9935	3.9982	

Problem 4: Consider the following nonlinear BVP[3]

$$\begin{cases} w_{tt}(t) - \frac{25t^8 exp(w(t)) - 20t^3}{4+t^5} = 0, \quad 0 < t < 1, \\ w(0) = -log(4), \qquad w(1) = -log(5), \end{cases} \tag{6.7}$$

with exact solution $w(t) = -log(4+t^5)$. Approximate results are shown in Table 4 along with results given by Balasubramani et al.[3], finite difference method and Numerov's method.

Table 4: **M.A.E for problem 4.**

h	**1/8**	**1/16**	**1/32**	**1/64**
Second Order Method $(p,q,r) = \left(\frac{1}{4},0,\frac{1}{2}\right)$	5.5212×10^{-03}	1.2773×10^{-03}	3.3060×10^{-04}	8.4161×10^{-05}
p^N	2.1118	1.9500	1.9738	
$(p,q,r) = \left(\frac{1}{4},\frac{1}{4},0\right)$	9.5912×10^{-03}	2.0448×10^{-03}	5.2840×10^{-04}	1.3576×10^{-04}
p^N	2.2296	1.9523	1.9605	
Fourth Order Method $(p,q,r) = \left(\frac{1}{720},\frac{11}{45},\frac{183}{360}\right)$	6.2487×10^{-05}	1.0123×10^{-06}	3.8928×10^{-08}	2.7550×10^{-09}
p^N	5.9477	4.7007	3.8206	
Balasubramani el al.[3] $(p,q,r) = \left(\frac{1}{120},\frac{26}{120},\frac{66}{120}\right)$	3.9439×10^{-06}	3.3929×10^{-07}	1.4653×10^{-08}	6.6424×10^{-10}
p^N	3.5391	4.5332	4.4633	
Sixth Order Method $(p,q,r) = \left(\frac{1}{360},\frac{56}{360},\frac{246}{360}\right)$	5.1118×10^{-06}	1.2322×10^{-08}	2.1551×10^{-10}	3.1186×10^{-12}
p^N	8.6963	5.8374	6.1107	
Finite difference Method	1.1795×10^{-03}	2.9324×10^{-04}	7.3024×10^{-05}	1.8265×10^{-05}
p^N	2.0080	2.0056	1.9992	
Numerov's Method	3.0070×10^{-05}	1.8480×10^{-06}	1.1585×10^{-07}	7.2337×10^{-09}
p^N	4.0242	3.9956	4.0014	

7. Conclusion

This study deals with developing second, fourth and sixth order convergent numerical schemes by using fractal non-polynomial spline function. With the help of quasilinearisation technique, the non-linear BVPs is converted into a system of linear BVPs, which in turn are solved by using the proposed schemes. These schemes are used to find approximate solution

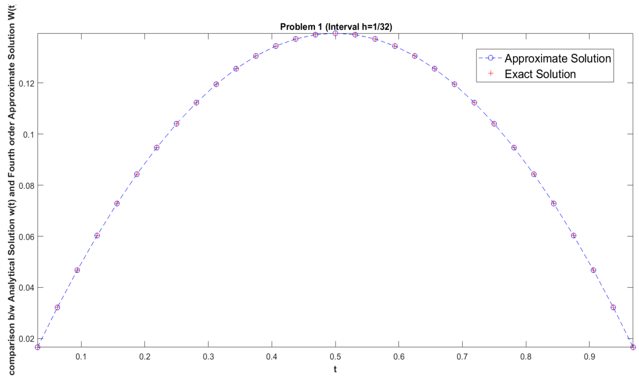


Figure 1: Relationship between analytical and approximate solution for problem 1.

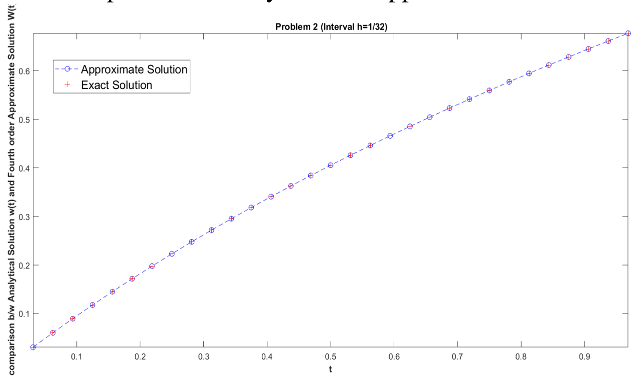


Figure 2: Relationship between analytical and approximate solution for problem 2.

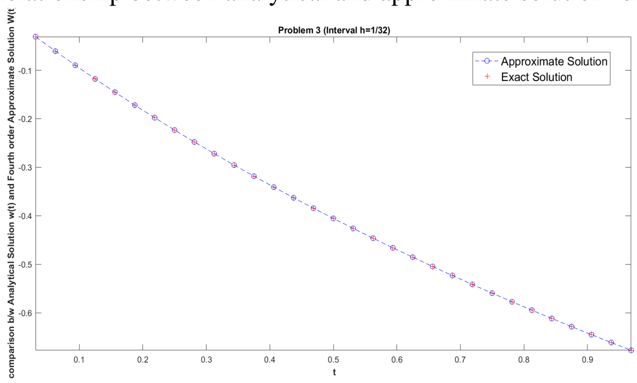


Figure 3: Relationship between analytical and approximate solution for problem 3.

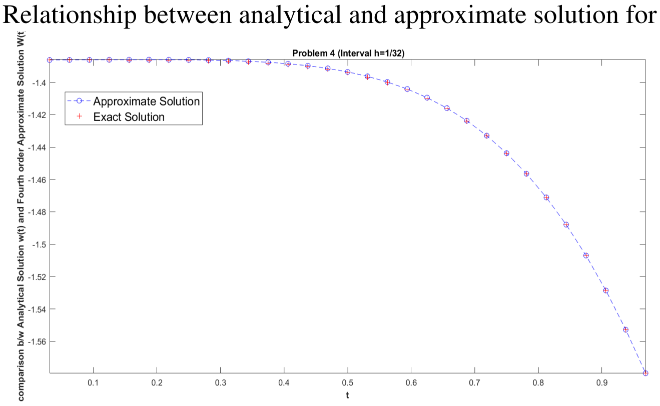


Figure 4: Relationship between analytical and approximate solution for problem 4.

of second order linear as well as nonlinear BVPs. Comparison with polynomial fractal quintic spline and few other methods leads us to the conclusion that the presented methods are more efficient.

References

[1] Baccouch M. (2018) A superconvergent local discontinuous Galerkin method for nonlinear two-point boundary-value problems, *Numer. Algor.* 79 (3), 697–718.

[2] Baccouch M. (2012) An adaptive local discontinuous Galerkin method for nonlinear two-point boundary-value problems, *Numer. Algor.*, 84,1121-1153.

[3] Balasubramani N., Guru Prem Prasad M. and Natesan S.(2020) Fractal quintic spline method for nonlinear boundary-value problems, *Hacettepe Journal of Mathematics and Statistics* 49(6),1885-1903.

[4] Barnsley M.F. (1986) Fractal functions and interpolation. *Constr. Approx.* 2(1),303–329.

[5] Barnsley M.F. and Harrington A.N. (1989) The calculus of fractal interpolation functions. *J.Approx. Theory*;57(1),14–34 .

[6] Bellman R.E. and Kalaba R.E.(1965) Quasilinearization and Nonlinear Boundary Value Problems, *American Elsevier,* New York.

[7] Bhatia R., Elsner L., and Krause G. (1990) Bounds for the variation of the roots of a polynomial and the eigenvalues of a matrix, *Linear Algebra Appl.* 142, 195–209.

[8] Bhatta S.K. and Sastri K.S. (1993) A sixth order spline procedure for a class of nonlinear boundary-value problems, *Int. J. Comput. Math.* 49 (3–4), 255–271.

[9] Chand A.K.B. and Kapoor G.P. (2006); Generalized cubic spline fractal interpolation functions. *SIAM J. Numer. Anal.* 44(2), 655–676.

[10] Chand A.K.B. and Viswanathan P. (2013) A constructive approach to cubic hermite fractal interpolation function and its constrained aspects. *BIT* 53(4), 841–865.

[11] Chawla M.M. and Subramanian R. (1987) A new fourth-order cubic spline method for nonlinear two-point boundary-value problems,*Int. J. Comput. Math.* 22 (3-4), 321–341.

[12] Chawla M.M. and Shivakumar P.N. (1985) Numerov's method for nonlinear two-point boundary-value problems, *Int. J. Comput. Math.* 17 (2), 167–176.

[13] Dwivedi K.D. and Singh J. (2021) Numerical solution of two-dimensional fractional-order reaction advection sub-diffusion equation with finite-difference Fibonacci collocation method. *Mathematics and Computers in Simulation.* 181, 38-50.

[14] Erdogan U. and Ozis T. (2011) A smart non-standard finite difference scheme for second order nonlinear boundary-value problems, *J. Comput. Phys.* 230 (17), 6464–6474.

[15] Goswami A., Sushila, Singh J. and Kumar D. (2021) Analytical study of fractional nonlinear Schrödinger equation with harmonic oscillator. *Discrete & Continuous Dynamical Systems S.* 14(10),3589-3610,.

[16] Goswami A., Sushila , Singh J. and Kumar D. (2020) Numerical computation of fractional Kersten-Krasil'shchik coupled KdV-mKdV system occurring in multi-component plasmas. *AIMS Mathematics.* 5(3), 2346-2368,.

[17] Goswami A., Singh J., Kumar D. and Sushila (2019); An efficient analytical approach for fractional equal width equations describing hydro-magnetic waves in cold plasma. *Physica A: Statistical Mechanics and its Applications.* 524, 563-575.

[18] Goswami A., Singh J. , Kumar D. , Gupta S. and Sushila (2019) An efficient analytical technique for fractional partial differential equations occurring in ion acoustic waves in plasma. *Journal of Ocean Engineering and Science.* 4(2), 85-99, 2019.

[19] Goswami A., Singh J. and Kumar D. (2018) Numerical simulation of fifth order KdV equations occurring in magneto-acoustic waves. *Ain Shams Engineering Journal.* 9(4), 2265-2273.

[20] Henrici P. (1962) Discrete variable methods in ordinary differential equations, *John Wiley and Sons*, New York.

[21] Kadalbajoo M.K. and Patidar K.C.(2002) Spline techniques for solving singularly-perturbed nonlinear problems on nonuniform grids, *J. Optim. Theory Appl.* 114 (3), 573–591.

[22] Lees M. (1966) Discrete method for nonlinear two-point boundary-value problems, in:*Numerical Solution of Partial Differential Equations*, ed. J.H. Bramble, Academic Press,New York.

[23] Liu L.B., Liu H.W., and Chen Y. (2011) Polynomial spline approach for solving second-order boundary-value problems with Neumann conditions,*Appl. Math. Comput.* 217 (16), 6872–6882.

[24] Mohanty R.K., Manchanda G., Khurana G. and Khan A. (2020) A third order exponentially fitted discretization for the solution of non-linear two point boundary value problems on a graded mesh. *J. Appl. Anal. Comput.* 10(5), 1741–1770.

[25] Ramadan M.A., Lashien I.F. and Zahra W.K. (2007) Polynomial and nonpolynomial spline approaches to the numerical solution of second order boundary value problems,*Applied Mathematics and Computation,* 184(2), 476-484.

[26] Rashidinia J., Mohammadi R. and Jalilian R. (2008) Spline solution of nonlinear singular boundary-value problems. *Int. J. Comput. Math.* 85(1), 39–52.

[27] Ravi Kanth A.S.V. and Bhattacharya V. (2006) Cubic spline for a class of nonlinear singular boundary-value problems arising in physiology. *Appl. Math. Comput.* 174(1), 768–774.

[28] Sahlan M.N., Hashemizadeh E.(2015) Wavelet Galerkin method for solving nonlinear singular boundary-value problems arising in physiology. *Appl. Math. Comput.* 250, 260–269.

[29] Singh J., Ganbari B., Kumar D. and Baleanu D. (2021) Analysis of fractional model of guava for biological pest control with memory effect. *Journal of Advanced Research.* 32, 99-108.

[30] Singh J., Kumar D., Purohit S.D., Mishra A.M. and Bohra M. (2021) An efficient numerical approach for fractional multi-dimensional diffusion equations with exponential memory. *Numerical Methods for Partial Differential Equations* 37(2), 1631-1651.

[31] Srivastava P.K., Kumar M. and Mohapatra R.N. (2011) Quintic nonpolynomial spline method for the solution of a second-order boundary value problem with engineering applications,*Computers and Mathematics with Applications*; 62, 1707–1714.

[32] Tirmizi I.A. and Twizell E.H. (2002) Higher-order finite-difference methods for nonlinear second-order two-point boundary-value problems, *Appl. Math. Lett.* 15 (7), 897–902.

Numerical analysis of Non-Linear Waves Propagation and interactions in Plasma

Chiman Lal[a], Ram Dayal Pankaj[1], Arun Kumar[b]
[a]Research Scholar JNVU Jodhpur, Rajasthan (India)
[b]Govt. College Kota,Rajasthan (India)

Abstract

Solitary wave propagation and interaction in plasma using numerical tools like Galerkin Finite Element scheme are discussed in this paper. A one-dimensional nonlinear Schrodinger-Korteweg De-Vries (Sch-KdV) equation is taken as model equation for Non-linear waves propagation in the said media. The derived system, with the help of cubic B-spline source functions are engaged as element and weight functions, after finite element formulation is solved with Runge Kutta Fourth Order method (RK^4). Previously the finite element methods with some numerical simulations do not exhibit the complex nature of wave interaction, especially solitary wave interaction. A combination of Galerkin Finite Element scheme with RK^4 is a very prominent instrument to study the nature of Non-linear evolution equations in ionic medias, which is the novelty of the paper.

Key words: Schrödinger - Korteweg - De Vries (Sch-KdV)equations, Galerkin Finite Element Scheme,Cubic B-spline source functions, Solitary Wave

Mathematics Subject Classification(2010): 35M10, 65Z05.

1

1 Introduction

Several physical phenomena are described either by nonlinear coupled partial differential equations or by nonlinear evolution equation. This Non-linear wave propagation phenomenon appears in one or other ways can be well explained by travelling and solitary wave solution of the said equations. Most of these equations do not have an analytical solution, or it is extremely difficult and expensive to compute their analytical solutions. Hence numerical study of these nonlinear partial differential equations is important in practice. The Non-linear

[1]Corresponding author: drrdpankaj@yahoo.com
Department of Mathematics and Statistics
Jai Narain Vyas University Jodhpur, Rajasthan (India)

waves propagation in plasma can also be explained by these solutions. In the past study, many methods for finding the Solitary and periodic solutions [1]-[8] and numerical method [8]-[12],[21]-[24] are used for Non-linear evolution equations (NLEEs).
In this paper, we study a Galerkin finite element Scheme for the 1D nonlinear Schrödinger -Korteweg-De -Vries (Sch-KdV) equation by using linear finite elements in space and extrapolation to remove the nonlinear term. We discuss the properties of this method and compare its accuracy with previous studies. The interaction of two solitons is also studied. Moreover, the propagation of the Maxwellian initial condition is simulated.
The outline of this paper is as follows, In the next section the model equation is discretized to form a numerical scheme. In section 3 a numerical scheme is developed and results are explained graphically. Finally, we give a brief conclusion in Section 4

2 Model Equation and Discretization

Non-linear waves propagation and interactions in plasma for this purpose we consider the 1D nonlinear Schrödinger -Korteweg-De -Vries (Sch-KdV) equation [13]-[15] as model equation as -

$$i\theta_t = \theta_{xx} + \theta\upsilon \tag{2.1}$$

$$\upsilon_t = -6\theta\upsilon_x - \upsilon_{xxx} + (|\theta|^2)_x \tag{2.2}$$

Here $\theta(x,t)$is complex function and υ(x,t)is real-valued function. This system appeared as model equation for describing various types of wave propagation such as Langmuir wave, dust-acoustic wave and electromagnetic waves in plasma physics. with initial conditions

$$\theta(x,0) = f(x) = 9\sqrt{2}\, e^{i\alpha x}\, k^2 \text{sech}^2(kx) \tag{2.3}$$

$$\upsilon(x,0) = g(x) = \frac{\alpha + 16k^2}{3} - 6k^2 \tanh^2(kx) \tag{2.4}$$

and boundary conditions

$$\theta(t,a) = 0,\ \upsilon(t,b) = 0,\ \ x \in [a,b]\, and\ t \in [0,T] \tag{2.5}$$

Here $\theta = \theta(x,t)$ and $\upsilon = \upsilon$(x,t) are going to be considered as sufficiently differentiable functions.
We multiplied weight function to the equations (2.1)-(2.2) and integrated over the x domain for finite element method [16]-[20], so we get

$$\int_a^b (i\omega\theta_t - \omega\theta_{xx} - \omega\theta\upsilon)dx = 0 \tag{2.6}$$

$$\int_a^b (\omega\upsilon_t + 6\omega\theta\upsilon_x + \omega\upsilon_{xxx} - \omega(|\theta|^2)_x)dx = 0 \tag{2.7}$$

The domain [a, b] of x is separated into N finite subdivision as

2

$$a = x_0 < x_1 < x_2 < ... < x_{N-1} < x_N = b$$

Here nodal point is $\{x_m\}_{m=0}^{N}$ i.e. m = 0,1,2,...,N and length of subdivision will be $h = x_{m+1}$ - x_m . We construct the approximate solutions for the system with cubic B-spline base functions

$$\theta_N(x,t) = \sum_{j=-1}^{N+1} \psi_j(x)u_j(t) \tag{2.8}$$

$$\upsilon_N(x,t) = \sum_{j=-1}^{N+1} \psi_j(x)v_j(t) \tag{2.9}$$

where u_j(t) and v_j(t) are function of time t and ψ_j(x) are function of x, called element size functions. A local coordinate $\xi = x - x_m$ for 0$\leq \xi \leq$ h introduced for cubic B-spline base functions with typical element $[x_m, x_{m+1}]$, which has the form;

$$\begin{aligned} \psi_{m-1} &= \frac{(h-\xi)^3}{h^3} \\ \psi_m &= \frac{(h^3 + 3h^2(h-\xi) + 3h(h-\xi)^2 - 3(h-\xi)^3)}{h^3} \\ \psi_{m+1} &= \frac{(h^3 + 3h^2\xi + 3h\xi^2 - 3\xi^3)}{h^3} \\ \psi_{m+2} &= \frac{\xi^3}{h^3} \end{aligned} \tag{2.10}$$

The approximate solutions of Eqs.((2.8)-(2.9)) with element size function eq.(2.10) may be define as with typical element $[x_m, x_{m+1}]$;

$$\theta_N(\xi,t) = \sum_{j=m-1}^{m+2} u_j^e(t)\psi_j^e(\xi) \tag{2.11}$$

$$\upsilon_N(\xi,t) = \sum_{j=m-1}^{m+2} v_j^e(t)\psi_j^e(\xi) \tag{2.12}$$

The point-wise values of θ_N and υ_N in terms u and v will be

$$\theta_N(x_m,t) = u_{m-1} + 4u_m + u_{m+1} \tag{2.13}$$

$$\upsilon_N(x_m,t) = v_{m-1} + 4v_m + v_{m+1} \tag{2.14}$$

So Eqs. ((2.6)-(2.7)) with $[x_m, x_{m+1}]$ will be

$$\int_{x_m}^{x_{m+1}} (i\omega\theta_t - \omega\theta_{xx} - \omega\theta\upsilon)dx \tag{2.15}$$

$$\int_{x_m}^{x_{m+1}} (\omega\upsilon_t + 6\omega\theta\upsilon_x + \omega_{xx}\upsilon_x - 2\omega\theta\theta_x)dx + [\omega\upsilon_{xx} - \omega_x\upsilon_x] \tag{2.16}$$

3

here weight function ω_i with size functions ψ_j are takenfor the Galerkin finite element method, Substituting Eqs. ((2.11)-(2.12)) into Eqs. ((2.15)-(2.16)),we get

$$\sum_{j=m-1}^{m+2}\{\int_0^h[(i\psi_i\psi_j)\dot{u}_j-(\psi_i\psi_j^{''})u_j-\sum_{k=m-1}^{m+2}((\psi_i\psi_j\psi_k)u_j)u_k]dx\}=0 \qquad (2.17)$$

$$\sum_{j=m-1}^{m+2}\{\int_0^h[(\psi_i\psi_j)\dot{v}_j+(\psi_j^{"}\psi_k')v_j+\sum_{k=m-1}^{m+2}((6(\psi_i\psi_j\psi_k')u_j)v_k-2((\psi_i\psi_j \psi_k')u_j)u_k)]dx+[((\psi_i\psi_j^{"})-(\psi_i'\psi_j'))v_j]_0^h\}=0 \qquad (2.18)$$

where i, j, k = m-1, m, m+1, m+2, $u^e=(u_{m-1},\ u_m,\ u_{m+1},\ u_{m+2})$ and $v^e=(v_{m-1},\ v_m,v_{m+1},\ u_{m+2})$ are element parameters where

$$A_{ij}=\int_0^h(i\psi_i\psi_j)d\xi,\ \ B_{ij}=\int_0^h(\psi_i\psi_j^{''})d\xi,\ \ C_{jk}=\int_0^h(\psi_j^{"}\psi_k')d\xi$$

$$D_{ij}=\int_0^h(\psi_i\psi_j)d\xi,\ F_{ijk}=\int_0^h 6(\psi_i\psi_j\psi_k')d\xi,\ G_{ijk}=\int_0^h(\psi_i\psi_j\psi_k)d\xi$$

$$H_{ijk}=\int_0^h 2(\psi_i\psi_j\psi_k')d\xi,\qquad I_{ij}=[(\psi_i\psi_j^{"})]_0^h,\quad J_{ij}=[(\psi_i'\psi_j^{'})]_0^h$$

The element matrices in ((2.17)-(2.18)) are computed as follows:

$$A_{ij}=\frac{ih}{140}\begin{bmatrix}20&129&60&1\\129&1188&933&60\\60&933&1188&129\\1&60&129&20\end{bmatrix}\qquad B_{ij}=\frac{3}{10h}\begin{bmatrix}4&-7&2&1\\33&-44&-11&22\\22&-11&-44&33\\1&2&-7&4\end{bmatrix}$$

$$C_{ij}=\frac{3}{2h^2}\begin{bmatrix}-3&-5&7&1\\5&3&-9&1\\-1&9&-3&-5\\-1&-7&5&3\end{bmatrix}\qquad D_{ij}=\frac{h}{140}\begin{bmatrix}20&129&60&1\\129&1188&933&60\\60&933&1188&129\\1&60&129&20\end{bmatrix}$$

$$I_{ij}=\frac{6}{h^2}\begin{bmatrix}-1&2&-1&0\\-4&9&-6&1\\-1&6&-9&4\\0&1&-2&1\end{bmatrix}\qquad J_{ij}=\frac{9}{h^2}\begin{bmatrix}-1&0&1&0\\0&1&0&-1\\1&0&-1&0\\0&-1&0&1\end{bmatrix}$$

$$G_{ij}(u)=\frac{h}{840}\begin{bmatrix}G_{11}(u)&G_{12}(u)&G_{13}(u)&G_{14}(u)\\G_{21}(u)&G_{22}(u)&G_{23}(u)&G_{24}(u)\\G_{31}(u)&G_{32}(u)&G_{33}(u)&G_{34}(u)\\G_{41}(u)&G_{42}(u)&G_{43}(u)&G_{44}(u)\end{bmatrix}$$

$$F_{ij}(v) = \frac{6h}{840}\begin{bmatrix} F_{11}(v) & F_{12}(v) & F_{13}(v) & F_{14}(v) \\ F_{21}(v) & F_{22}(v) & F_{23}(v) & F_{24}(v) \\ F_{31}(v) & F_{32}(v) & F_{33}(v) & F_{34}(v) \\ F_{41}(v) & F_{42}(v) & F_{43}(v) & F_{44}(v) \end{bmatrix};$$

$$H_{ij}(u) = \frac{2h}{840}\begin{bmatrix} H_{11}(u) & H_{12}(u) & H_{13}(u) & H_{14}(u) \\ H_{21}(u) & H_{22}(u) & H_{23}(u) & H_{24}(u) \\ H_{31}(u) & H_{32}(u) & H_{33}(u) & H_{34}(u) \\ H_{41}(u) & H_{42}(u) & H_{43}(u) & H_{44}(u) \end{bmatrix}$$

where

$G_{11}(u)$=(84,463,172,1)(u), $G_{12}(u)$=(463,2889,1275,17)(u),
$G_{13}(u)$=(172,1275,696,17)(u), $G_{14}(u)$=(1,17,17,1)(u)
$G_{21}(u)$=(463,2889,1275,17)(u), $G_{22}(u)$=(2889,23664,15519,696)(u)
$G_{23}(u)$=(1275,15519,15519,1275)(u), $G_{24}(u)$=(17,696,1275,172)(u),
$G_{31}(u)$=(172,1275,696,17)(u), $G_{32}(u)$=(1275,15519,15519,1275)(u),
$G_{33}(u)$=(696,15519,23664,2889)(u), $G_{34}(u)$=(17,1275,2889,463)(u),
$G_{41}(u)$=(1,17,17,1)(u), $G_{42}(u)$=(17,696,1275,172)(u),
$G_{43}(u)$=(17,1275,2889,463)(u), $G_{44}(u)$=(1,172,463,84)(u)

$F_{11}(v)$=(-280,-150,420,10)(v) $F_{12}(v)$=(-1605,-1305,2781,129)(v)
$F_{13}(v)$=(-630,-792,1314,108)(v) $F_{14}(v)$=(-5,-21,21,5)(v)
$F_{21}(v)$=(-1605,-1305,2781,129)(v) $F_{22}(v)$=(-10830,-17640,25002,3468)(v)
$F_{23}(v)$=(-5349,-17541,17541,5349)(v) $F_{24}(v)$=(-108,-1314,792,630)(v)
$F_{31}(v)$=(-630,-792,1314,108)(v) $F_{32}(v)$=(-5349,-17541,17541,5349)(v)
$F_{33}(v)$=(-3468,-25002,17640,10830)(v) $F_{34}(v)$=(-129,-2781,1305,1605)(v)
$F_{41}(v)$=(-5,-21,21,5)(v) $F_{42}(v)$=(-108,-1314,792,630)(v)
$F_{43}(v)$=(-129,-2781,1305,1605)(v) $F_{44}(v)$=(-10,-420,150,280)(v)

$H_{11}(u)$=(-280,-150,420,10)(u) $H_{12}(u)$=(-1605,-1305,2781,129)(u)
$H_{13}(u)$=(-630,-792,1314,108)(u) $H_{14}(u)$=(-5,-21,21,5)(u)
$H_{21}(u)$=(-1605,-1305,2781,129)(u) $H_{22}(u)$=(-10830,-17640,25002,3468)(u)
$H_{23}(u)$=(-5349,-17541,17541,5349)(u) $H_{24}(u)$=(-108,-1314,792,630)(u)
$H_{31}(u)$=(-630,-792,1314,108)(u) $H_{32}(u)$=(-5349,-17541,17541,5349)(u)
$H_{33}(u)$=(-3468,-25002,17640,10830)(u) $H_{34}(u)$=(-129,-2781,1305,1605)(u)
$H_{41}(u)$=(-5,-21,21,5)(u) $H_{42}(u)$=(-108,-1314,792,630)(u)
$H_{43}(u)$=(129,2781,1305,1605)(u) $H_{44}(u)$=(-10,-420,150,280)(u)

Here A_{ij}, B_{ij}, C_{jk}, D_{ij}, F_{ijk}, G_{ijk}, H_{ijk}, I_{ij} and J_{ij} are element matrices. so, the new obtained system in matrix form

$$\dot{u} = A^{-1}[\{B - G(u)\}u] \tag{2.19}$$

$$\dot{v} = D^{-1}[H(u)u + (J - I)v - Cv - F(u)v] \tag{2.20}$$

5

Here $u = (u_{-1}, u_0, u_1, ..., u_N, u_{N+1})$ and $v = (v_{-1}, v_0, v_1, ..., v_N, v_{N+1})$ are time dependent constraints, The generalized rows of the combined matrices are:

A = $\frac{ih}{140}$(1,120,1191,2416,1191,120,1)
B = $\frac{3}{10h}$(1,24,15,-80,15,24,1)
C = $\frac{3}{2h^2}$(-1,-8,19,0,-19,8,1)
D = $\frac{h}{140}$(1,120,1191,2416,1191,120,1)
I = $\frac{6}{h^2}$(0,0,0,0,0,0,0)
J = $\frac{9}{h^2}$ (0,0,0,0,0,0,0)
G(u) = $\frac{h}{840}${(1,17,17,1,0,0,0)u,(17,868,2550,868,17,0,0)u,(17,2550,18871,18871,2550,17,0)u, (1,868,18871,47496,18871,868,1)u,(0,17,2550,18871,18871,2550,17)u,(0,0,17,868,2550,868, 17)u,(0,0,0,1,17,17,1)u}

F(v) = $\frac{6h}{840}$ {(-5,-21,21,5,0,0,0)v,(-108,-1944,0,1944,108,0,0)v,(-129,-8130,-17841,17841,8130, 129,0)v, (-10,-3888,-35682,0,35682,3888,10)v,(0,-129,-8130,-17841,17841,8130,129)v,(0,0,-108,-1944,0,1944,108)v,
(0,0,0,-5,-21,21,5)v}

H(u) = $\frac{2h}{840}$ {(-5,-21,21,5,0,0,0)u,(-108,-1944,0,1944,108,0,0)u,(-129,-8130,-17841,17841,8130, 129,0)u, (-10,-3888,-35682,0,35682,3888,10)u,(0,-129,-8130,-17841,17841,8130,129)u,(0,0,-108,-1944,0,1944,108)u, (0,0,0,-5,-21,21,5)u }

The system equations (2.19) and (2.20) has $(N+3) \times (N+1)$ ordered unknown equations. if we use time dependent boundary condition in Eqs.(2.13) and (2.14) with $m = 0$, then so parameters can be written as other parameters;

$$u_{-1},\ v_{-1} \rightarrow u_0,\ u_1 \text{ and } v_0, v_1 \text{ ; when we take } m = 0$$

Similarly

$$u_{N+1}\ ,\ v_{N+1} \rightarrow u_{N-1}, u_N \text{ and } v_{N-1}, v_N \text{ we take } m = N$$

Then, the system of Eqs. (2.19) and(2.20) will be two matrix systems of $(N+1) \times (N+1)$ orders.These equations of systems will be solved by RK^4 (Runge-Kutta fourth order method) to known initial condition u_j^0 and v_j^0 with nodal points x_m for m=0(1)N as follows:

$$u(x_m, 0) = \theta_N(x_m, 0)$$

$$v(x_m, 0) = v_N(x_m, 0)$$

If we write the system explicitly as

$$\theta_N(x_0, 0) = u_{-1} + 4u_0 + u_1 = u(x_0, 0),$$

$$\theta_N(x_1, 0) = u_0 + 4u_1 + u_2 = u(x_1, 0),$$

$$\theta_N(x_2, 0) = u_1 + 4u_2 + u_3 = u(x_2, 0),$$

$$\vdots$$

$$\theta_N(x_N,0) = u_{N-1} + 4u_N + u_{N+1} = u(x_N,0),$$

and

$$\upsilon_N(x_0,0) = v_{-1} + 4v_0 + v_1 = v(x_0,0),$$
$$\upsilon_N(x_1,0) = v_0 + 4v_1 + v_2 = v(x_1,0),$$
$$\upsilon_N(x_2,0) = v_1 + 4v_2 + v_3 = v(x_2,0),$$
$$\vdots$$
$$\upsilon_N(x_N,0) = v_{N-11} + 4v_N + v_{N+1} = v(x_N,0),$$

if we write u_{-1} , $u_{N+1} \to u_0$, u_N, and v_{-1} , $v_{N+1}\to v_0$ and v_N respectively. then we get a new system $(N+1)\times(N+1)$ order in matrix form as :

$$\begin{bmatrix} 4 & 2 & & & & & & \\ 1 & 4 & 1 & & & & & \\ & 1 & 4 & 1 & & & & \\ & & & . & & & & \\ & & & & . & & & \\ & & & & & . & & \\ & & & & & 1 & 4 & 1 \\ & & & & & & 2 & 4 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ . \\ . \\ . \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} u(x_0,0) \\ u(x_1,0) \\ u(x_2,0) \\ . \\ . \\ . \\ u(x_{N-1},0) \\ u(x_N,0) \end{bmatrix} \tag{2.21}$$

and

$$\begin{bmatrix} 4 & 2 & & & & & & \\ 1 & 4 & 1 & & & & & \\ & 1 & 4 & 1 & & & & \\ & & & . & & & & \\ & & & & . & & & \\ & & & & & . & & \\ & & & & & 1 & 4 & 1 \\ & & & & & & 2 & 4 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ . \\ . \\ . \\ v_{N-1} \\ v_N \end{bmatrix} = \begin{bmatrix} v(x_0,0) \\ v(x_1,0) \\ v(x_2,0) \\ v(x_{N-1},0) \\ v(x_N,0) \end{bmatrix} \tag{2.22}$$

By Matleb solving the algebraic Equations (2.21) and (2.22) with initial parameters u_j^0 and v_j^0 are gained for j=0(1)N.

3 Numerical Scheme

Non-Linear waves propagations and interaction are investigated to the system of equations (2.1)-(2.2) numerically for numerous values of x and t. L_2 , L_∞ and L'_2 , L'_∞ are error norms and used to investigate consistency with numerical solutions(Soliton) for $\theta(x,t)$ and $\upsilon(x,t)$ respectively for initial conditions for the Sch-KdV equation.

$$\theta(x,0) = f(x) = 9\sqrt{2}\, e^{i\alpha x}\, k^2 \text{sech}^2(kx), \tag{3.1}$$

7

$$v(x,0) = g(x) = \frac{\alpha + 16k^2}{3} - 6k^2 \tanh^2(kx) \tag{3.2}$$

$$L_2 = \|\theta - \theta_N\|_2 = \sqrt{h \sum_{j=-1}^{N+1} \left|\theta_j - (\theta_N)_j\right|^2} \tag{3.3}$$

$$L_\infty = \|\theta - \theta_N\|_\infty = \underset{0 \le j \ge N}{Max} \left|\theta_j - (\theta_N)_j\right| \tag{3.4}$$

And

$$L'_2 = \|v - v_N\|_2 = \sqrt{h \sum_{j=-1}^{N+1} \left|v_j - (v_N)_j\right|^2} \tag{3.5}$$

$$L'_\infty = \|v - v_N\|_\infty = \underset{0 \le j \ge N}{Max} \left|v_j - (v_N)_j\right| \tag{3.6}$$

Numerical error L_2 *and* L_∞ For $\theta(x,t)$ with $k = \sqrt{2},\ \alpha = 1/20$

h	$\Delta t = 0.001$	$\Delta t = 0.002$	$\Delta t = 0.003$	$\Delta t = 0.01$
	L_2 ; L_∞	L_2 ; L_∞	L_2 ; L_∞	L_2 ; L_∞
0.2	15.82×10^{-7} ; 51.24×10^{-7}	30.71×10^{-7} ; 95.17×10^{-7}	41.39×10^{-7} ; 97.41×10^{-7}	55.56×10^{-7} ; 99.56×10^{-7}
0.4	54.21×10^{-7} ; 55.88×10^{-7}	63.56×10^{-7} ; 68.67×10^{-7}	71.39×10^{-7} ; 70.70×10^{-7}	86.66×10^{-7} ; 85.01×10^{-7}
0.625	57.24×10^{-7} ; 59.82×10^{-7}	68.21×10^{-7} ; 61.23×10^{-7}	78.29×10^{-7} ; 68.21×10^{-7}	87.21×10^{-7} ; 82.01×10^{-7}
0.8	68.19×10^{-7} ; 64.21×10^{-7}	72.21×10^{-7} ; 59.52×10^{-7}	80.19×10^{-7} ; 63.11×10^{-7}	89.21×10^{-7} ; 78.21×10^{-7}
0.1	75.21×10^{-7} ; 72.24×10^{-7}	75.11×10^{-7} ; 52.11×10^{-7}	83.12×10^{-7} ; 59.29×10^{-7}	91.11×10^{-7} ; 72.31×10^{-7}

Numerical error L_2 *and* L_∞ For $v(x,t)$

h	$\Delta t = 0.001$	$\Delta t = 0.002$	$\Delta t = 0.003$	$\Delta t = 0.01$
	L'_2 ; L'_∞	L'_2 ; L'_∞	L'_2 ; L'_∞	L'_2 ; L'_∞
0.25	07.85×10^{-8} ; 05.68×10^{-8}	08.75×10^{-7} ; 07.05×10^{-7}	15.16×10^{-7} ; 06.65×10^{-7}	19.56×10^{-7} ; 08.75×10^{-7}
0.5	09.95×10^{-8} ; 06.95×10^{-8}	10.72×10^{-7} ; 08.75×10^{-7}	20.61×10^{-7} ; 08.85×10^{-7}	25.11×10^{-7} ; 09.85×10^{-7}
0.625	10.01×10^{-8} ; 07.02×10^{-8}	13.56×10^{-7} ; 09.11×10^{-7}	22.56×10^{-7} ; 09.21×10^{-7}	26.92×10^{-7} ; 10.96×10^{-7}
0.8	12.21×10^{-8} ; 08.11×10^{-8}	15.11×10^{-7} ; 10.21×10^{-7}	24.11×10^{-7} ; 10.09×10^{-7}	28.11×10^{-7} ; 12.11×10^{-7}
0.1	14.02×10^{-8} ; 09.75×10^{-8}	19.21×10^{-7} ; 11.25×10^{-7}	27.72×10^{-7} ; 12.21×10^{-7}	31.27×10^{-7} ; 14.25×10^{-7}

In figure 1 and 2 nonlinear wave propagation and its travelling wave solution is presented. The coupled equations (2.1) and (2.2) are plotted for some fix values of k, α,h and t $(-5 < t < 5)$. the space step is taken as 0.001. It is shown in the figure that the solution of said equation exhibit a soliton for the small values of x $(0 \le x \le 0.1)$. If we extend the range of x $(-15 \le x \le 15)$ the solution converted from soliton to a wave natured system. A solitary wave interaction is presented in the figure 3 for the same values of k, α, h and step lengths with

Solitary wave propagation for model equations

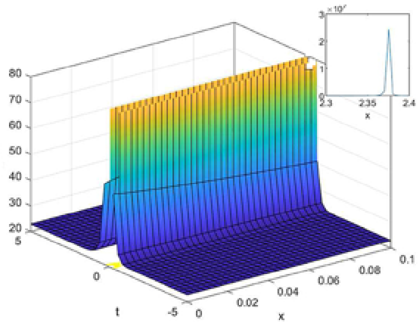


Figure 1. Modulus in 3D, 2D plot the *solitary wave propagation of* θ *when* $k=\sqrt{2},\ \alpha=1/20, h=0.4$
$\Delta t=0.001,\ \Delta x=0.001, -5\le t\le 5,\ 0\le x\le 0.1,$

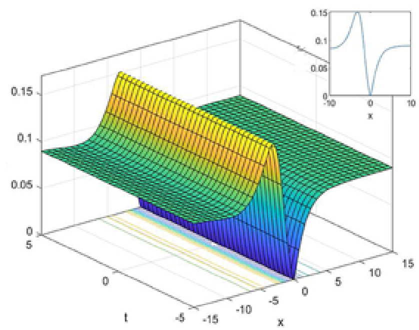


Figure 2. Modulus in 3D, 2D plot the *solitary wave propagation of* v *when* $k=\sqrt{2},\ \alpha=1/20, h=0.4$
$\Delta t=0.001,\ \Delta x=0.001, -5\le t\le 5,\ -15\le x\le 15,$

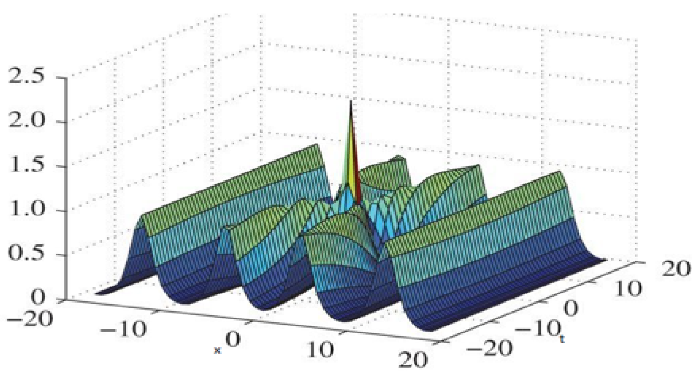


Fig.-3 Solitary wave interaction
$k=\sqrt{2},\ \alpha=1/20,\ \Delta t=0.02,\ \Delta x=0.02,\ -20\le(x,t)\le 20,$

large values of x and t $(-20 \leq (x,t) \leq 20)$. It clearly exhibit that solitons are developed when the values of x and t coincides. For different values of x and t the system represent the travelling wave solution.

4 Conclusions

In the present paper, we have investigated numerically a physical model for wave propagation in a nonlinear, dispersive medium i.e a relativistic plasma. A Galerkin finite element Scheme is exhibited to locate Solitary wave(Solitons) propagation and interactions in plasma for Schrödinger - KortewegDe Vries (Sch-KdV) equations. The new obtained systems (finite element formulation) solved

9

by RK^4 (Runge-Kutta fourth order method). The different values of x, t and error norms L_2 , L_∞ are used for numerical solutions of Sch-KdV equations.The numerical results obtained by this method are in good agreement with the exact solutions available in the literature. The errors obtained by the proposed method are less when compared with those of available in the literature. The solitary wave solution in fig.-1, 2 and its interaction in fig.-3 of this system are presented which are new. here, we learn that this method will emulates development of many exact travelling wave solutions with new solitons.This scheme is a significant instrument for Non-linear evolution equations (NLEEs).
The advantages of the present scheme for oscillatory problems are discussed in detail. It can be expected that the main ideas will also be useful for other physical problems being highly oscillatory in nature, e.g., the nonlinearized model.

Acknowledgement

The authors are thankful to the CSIR- New Delhi for financial protection $(NET(SRF) - 09\backslash 098(0138)\backslash 2019 - EMR - I)$.

References

[1] Malomed, B. A. (2006) Soliton Management in Periodic Systems. Springer: New York.

[2] Yang Y (2001) Solitons in Field Theory and Nonlinear Analysis. Springer: New York,

[3] Dauxois, T., M. Peyrard, (2006.) Physics of Solitons. Cambridge University Press: Cambridge,

[4] Kumar A., Pankaj R. D. (2012), Some Solitary and Periodic Solutions of the non-linear wave Equation by Variational Approach, Journals of Rajasthan Academy of Physical Sciences, 11(2), 133-139, ISSN:0972-6306

[5] Pankaj R. D. and Sindhi C. (2016). Traveling Wave Solutions for Wave-Wave Interaction Model in Ionic Media. International Educational Scientific Research Journal 2(6) 70-72.

[6] Pankaj, R. D. (2013) Laplace - Modified Decomposition Method to Study Solitary Wave Solutions of Coupled Nonlinear Klein-Gordon Schrdinger Equation, International Journal of Statistika and Mathematika, V. 5(1) 01-05, ISSN: 2277- 2790

[7] Kumar, A. and Pankaj R. D. (2013) Solitary Wave Solutions of Schrdinger Equation by LaplaceAdomian Decomposition Method, Physical Review & Research International 3(4): 702-712

[8] Pankaj, R. D. and. Kumar, A. (2013) Solutions of the Coupled Klein-Gordon Equation by Modified Exp-Function Method, Indian Journal of Applied Research 3(3) 276-278

[9] Kaya D., El-Sayed S.M. (2004). A numerical simulation and explicit solutions of the generalized BurgerFisher equation. Appl. Math. Comput., 152, 403413

[10] He J.H., Guo-Cheng Wu, Austin F. (2010). The variational iteration method which should be followed. Nonlinear Sci. Lett. A–Math. Phys. Mech., 1 (1), 130

[11] Kumar, A. and Pankaj R. D. (2014) Finite Difference Scheme for the Zakharov Equation as a Model for Nonlinear Wave-Wave Interaction in Ionic Media, International Journal of Scientific & Engineering Research, 5(2) 759-762. ISSN 2229-5518

[12] Kumar, A., Pankaj R. D. and Manish Gaur (2011) Finite Difference Scheme of the Model for Nonlinear Wave-Wave Interaction in Ionic Media. Computational Mathematics and Modeling. 22(3) 255265, ISSN: 1046283X

[13] Kumar, A., Pankaj R. D. and Gupta C.P. (2011) A Description of a wave-wave interaction model by Variational and Decomposition methods. Mathematica Aeterna, 1(1), 55 63, ISSN:1314-3344

[14] Kumar, A. and Pankaj R. D. (2012) Laplace-Decomposition Method to Study Solitary Wave Solutions of Coupled Non-Linear Partial Differential Equation, International Scholarly Research Network (ISRN) Computational Mathematic Volume 2012, Article ID 423469, 1-5 ISSN: 2090-7842

[15] Pankaj, R. D. and. Kumar, A and Sindhi Chandrawati (2017) A Description of the Coupled Schrdinger-KDV Equation of Dusty Plasma. International Journal of Mathematics Trends and Technology (IJMTT) 52 (8) 537-544 ISSN: 2231-5373

[16] Ucar Y, Alaattin E.and Karaagac B. (2020) Numerical solutions of Boussinesq equation using Galerkin finite element method. Numer Methods Partial Differential Eq. Wiley;119. DOI: 10.1002/num.22600

[17] Pani A. K. and Saranga, H. (1997) Finite element Galerkin method for the good Boussinesq equation, Nonlinear Analysis. Theory. Methods& Applications, 29(8) 937-956,

[18] Daripa P. and Hua W. (1999), A numerical study of an ill-posed Boussinesq equation arising in water waves and nonlinear lattices: Filtering and regularization techniques, Appl. Math. Comput. 101, 159207.

[19] Wazwaz A. M. (2001) Construction of soliton solutions and periodic solutions of the Boussinesq equation by the modified decomposition method, Chaos Solitons Frac. 12, 15491556.

[20] Mohebbi A., Asgari, Z. (2011) Efficient numerical algorithms for the solution of good Boussinesq equation in water wave propagation, Comp. Phys. Commun. 182 24642470.

[21] Goswami A, Sushila, Singh J, and Kumar D, (2020) Numerical computation of fractional Kersten-Krasilshchik coupled KdV-mKdV system arising in multi-component plasmas, AIMS Mathematics, 5 (3),2346-2368.

[22] Goswami A, Sushila, Singh J, and Kumar D, (2018) Numerical simulation of fifth order KdV equations occurring in magneto-acoustic waves, Ain Shams Engineering Journal, Volume 9(4) 2265-2273, ISSN 2090-4479,

[23] Karapinar E. et al. (2020). Identifying the space source term problem for time-space-fractional diffusion equation. Adv Differ Equ 2020, 557 https://doi.org/10.1186/s13662-020-02998-y

[24] Pankaj, R.D. and Lal C (2021) Numerical Elucidation of Klein-Gordon-Zakharov System. Jñãnãbha 51(1), 207-212

MULTIPLE SUMMATION FORMULAE FOR THE MODIFIED MULTIVARIABLE I-FUNCTION

D.K.PAVAN KUMAR[1*], FREDRIC AYANT[2], Y. PRAGATHI KUMAR[3], N.SRIMANNARAYANA[4], AND B.SATYANARAYANA[5]

ABSTRACT. The importance of I-function, H-function and many more special functions has a wide range of applications in applied mathematics and applied physics. Some of the multiple summations for the modified multivariable I-function(MMIF) has been discussed in the present article. Some of the summation formulae are concluded at the end of the paper as special cases of our primary results. Also these summation formulae leads to develop the solution of a boundary value problem.

1. INTRODUCTION

Recent advancements of special functions and their applications in mathematical modelling attracting researchers. The motivation of this work is by the applications of special functions like G, H and I-functions by several authors([1], [2], [3]). The generalization of H-function, namely I-function has great importance in Physics and Applied Mathematics. Prasad [15] generalized the I-function and studied many results. In the literature of the special functions like H, G, Meijer etc., many authors established integral results and solved boundary value problems also([7], [11], [5]). Recently, I-function has found its applications in wireless communication.

Srivastava and Panda [8, 9] studied multivariable H-function. The extension of the same as two functions H and I studied by Prasad and Singh [14, 15]. Here we establish four different summation formulae for the MMIF defined by Prasad [15] and a number of summation formulae derived as particular cases.

---

[1]*D.K.Pavan Kumar*

2010 *Mathematics Subject Classification.* 33C99, 33C60, 44A20.

Key words and phrases. Modified multivariable I and H-functions, Multiple Integral Contours, Multiple Summation Formulae.

Assume $\mathbb{C}, \mathbb{R}$ and $\mathbb{N}$ as set of complex, real and positive integers respectively and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We define MMIF as :

(1.1)
$$I(Z_1, ..., Z_r) =$$

$$I^{0,n_2;0,n_3;....;0,n_r:\left|R^1:m^1,n^1;....;m^{(r)},n^{(r)}\right.}_{p_2,q_2;p_3,q_3;....;p_r,n_r:\left|R:p^1,q^1;....;p^{(r)},q^{(r)}\right.} \left[\begin{matrix} Z_1 \\ \vdots \\ Z_r \end{matrix} \left| \begin{matrix} \left(a_{2j};\alpha^1_{2j},\alpha^{11}_{2j}\right)_{1,p_2};\left(\alpha_{3j};\alpha^1_{3j},\alpha^{11}_{3j},\alpha^{111}_{3j}\right)_{1,p_3}; \\ \\ \left(b_{2j};\beta^1_{2j},\beta^{11}_{2j}\right)_{1,q_2};\left(\beta_{3j};\beta^1_{3j},\beta^{11}_{3j},\beta^{111}_{3j}\right)_{1,q_3}; \end{matrix} \right. \right.$$

$$\begin{matrix} ...;(a_{rj};\alpha'_{rj},...,\alpha^{(r)}_{rj})_{1,p_r} : (e_j;u'_jg'_j,...,u^{(r)}_j g^{(r)}_j)_{1,R'} : \\ . \\ . \\ ...;(b_{rj};\beta'_{rj},...,\beta^{(r)}_{rj})_{1,q_r} : (l_j;U'_jf'_j,...,U^{(r)}_j f^{(r)}_j)_{1,R} : \end{matrix}$$

$$\left. \begin{matrix} (a'_j;\alpha'_j)_{1,p^{(1)}},(a^{(r)}_j;\alpha^{(r)}_j)_{1,p^{(r)}} \\ . \\ . \\ (b'_j;\beta'_j)_{1,q^{(1)}},(b^{(r)}_j;\beta^{(r)}_j)_{1,q^{(r)}} \end{matrix} \right)$$

$$= \frac{1}{(2\pi\, w)^r} \int_{L_1} .... \int_{L_r} \xi(s_1,....,s_r) \prod_{i=1}^{r} \phi(s_i)\, {z_i}^{s_i} ds_1 ....ds_r$$

where $\xi(s_1,....,s_r)$ and $\phi(s_i)$ clearly mentioned in [6]. The MMIF is analytic if

(1.2)
$$\sum_{k=1}^{p_2} \alpha^{(i)}_{2k} + \sum_{k=1}^{p_3} \alpha^{(i)}_{3k} + ... + \sum_{k=1}^{p_s} \alpha^{(i)}_{sk} - \sum_{k=1}^{q_2} \beta^{(i)}_{2k} - \sum_{k=1}^{q_3} \beta^{(i)}_{3k} - ... - \sum_{k=1}^{q_s} \beta^{(i)}_{sk} - \sum_{j=1}^{R} f^{(i)}_j \leq 0$$

The contour integral in (1.1) converges absolutely if $|\arg z_i| < \frac{1}{2}\Omega_i\pi$, where

(1.3)
$$\Omega_i = \sum_{k=1}^{n^{(i)}} \alpha^{(i)}_k - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha^{(i)}_k + \sum_{k=1}^{m^{(i)}} \beta^{(i)}_k - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta^{(i)}_k + \sum_{k=1}^{n_2} \alpha^{(i)}_{2k} - \sum_{k=n_2+1}^{p_2} \alpha^{(i)}_{2k}$$
$$+ \sum_{k=1}^{n_3} \alpha^{(i)}_{3k} - \sum_{k=n_3+1}^{p_3} \alpha^{(i)}_{3k} + ... + \sum_{k=1}^{n_r} \alpha^{(i)}_{rk} - \sum_{k=n_r+1}^{p_r} \alpha^{(i)}_{rk}$$
$$- \sum_{k=1}^{q_2} \beta^{(i)}_{2k} - \sum_{k=1}^{q_3} \beta^{(i)}_{3k} .... - \sum_{k=1}^{q_r} \beta^{(i)}_{rk} + \sum_{j=1}^{R'} g^{(i)}_j - \sum_{j=1}^{R} f^{(i)}_j > 0 \text{ (i=1,...,r).}$$

We note

$$\text{A}=(a_{2j};\alpha'_{2j},\alpha''_{2j})_{1,p_2};....;\left(a_{(r-1)j};\alpha'_{(r-1)j},....,\alpha^{r-1}_{(r-1)j}\right)_{1,p_{r-1}} \tag{1.4}$$

$$\text{B}=\left(b_{2j};\beta'_{2j},\beta''_{2j}\right)_{1,q_2};....;\left(b_{(r-1)j};\beta'_{(r-1)j},....,\beta^{r-1}_{(r-1)j}\right)_{1,q_{r-1}} \tag{1.5}$$

$$\mathbf{A}=\left(a_{rj};\alpha'_{rj},....,\alpha^{(r)}_{rj}\right)_{1,p_r};\Im=(a'_j,\alpha'_j)_{1,p'};....;\left(a^{(r)}_j,\alpha^{(r)}_j\right)_{1,p^{(r)}} \tag{1.6}$$

$$\mathbf{B}=\left(b_{rj};\beta'_{rj},....,\beta^{(r)}_{rj}\right)_{1,q_r};\Re=\left(b'_j,\beta'_j\right)_{1,q'};....;\left(b^{(r)}_j,\beta^{(r)}_j\right)_{1,q^{(r)}} \tag{1.7}$$

$$\mathbf{E}=\left(e_j;u'_jg'_j,....,u^{(r)}_j g^{(r)}_j\right)_{1,R'};L=\left(l_j;U'_jf'_j,....,U^{(r)}_j f^{(r)}_j\right)_{1,R}$$

$$U=p_2,q_2;p_3,q_3;...;p_{r-1},q_{r-1}\,;\;V=0,n_2;0,n_3;...;0,n_{r-1} \tag{1.8}$$

$$Y=(p',q')\,;....\left(p^{(r)},q^{(r)}\right);X=(m',n')\,:....;\left(m^{(r)},n^{(r)}\right) \tag{1.9}$$

2. Main results

In this section, we establish the summation formulae for the MMIF as follows:

Theorem 2.1.

(2.1)

$$\sum_{u_1,...u_m=0}^{\infty}\prod_{J=1}^{m}\frac{((w_j))_{u_j}}{u_j!}\;I^{V;0,n_r+1:|R':X}_{U;p_r+1,q_r+1:|R:Y}\left(\begin{array}{c|c} z_1 & A; \\ . & . \\ . & . \\ z_r & B;\text{B}, \end{array}\right.$$

$$\left.\begin{array}{c}(1-g-\sum_{j=1}^m t_j;a_1,...,a_r),\text{A}:E:\Im \\ . \\ . \\ (1-h-\sum_{j=1}^m t_j;b_1,...,b_r):L:\Re\end{array}\right)$$

$$=I^{V;0,n_r+2:|R':X}_{U;p_r+2,q_r+2:|R:Y}\left(\begin{array}{c|c} z_1 & A;(1-g;a_1,...a_r), \\ . & . \\ . & . \\ z_r & B;\text{B}, \end{array}\right.$$

$$\left. \begin{array}{c} (1+g-h+\sum_{j=1}^{m} w_j; b_1-a_1,...,b_r-a_r), \mathrm{A}: E: \Im \\ . \\ . \\ (1-h+\sum_{j=1}^{m} w_j; b_1,...,b_r), (1+g-h; b_1-a_1,...,b_r-a_r): L: \Re \end{array} \right)$$

Following the lines of Braaksma([4], p.278), we may establish the asymptotic expansion in the following convenient way :

$$a_i, b_i, b_i-a_i>0 (i=1,...,r),\ \mathrm{Re}(h-g-\sum_{j=1}^{m} w_j)>0 \textit{ and } |arg(z_i)|<\tfrac{1}{2}(\Omega_i-2b_i)\pi$$

Proof. To establish the Theorem (2.1), expressing the MMIF by Prasad [15] in the Mellin-Barnes multiple integrals contour using (1.1) and interchanging the order of summation and integration, we obtain

$$I=\frac{1}{(2\pi w)^r}\int_{L_1}...\int_{L_r}\phi(s_1,...,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\frac{\Gamma(g+\sum_{j=1}^{m}a_is_i)}{\Gamma(h+\sum_{j=1}^{m}b_is_i)}$$

$$\times\sum_{u_1,...u_m=0}^{\infty}\prod_{J=1}^{m}\frac{((w_j))_{u_j}}{u_j!}\frac{\left(g+\sum_{j=1}^{m}a_is_i\right)_{\sum_{j=1}^{m}t_j}}{\left(h+\sum_{j=1}^{m}b_is_i\right)_{\sum_{j=1}^{m}t_j}}ds_1...ds_2$$

Now applying result of Panda([12], p.108, Eq.2) and Gauss's theorem ([10], p.28, Eq.1.7.6) in the above equation and interpreting the resulting expression with the help of (1.1), we arrive at Theorem (2.1).

□

Theorem 2.2.

(2.2)

$$\sum_{u_1,...u_m=0}^{\infty}\prod_{j=1}^{m}\frac{((w_j))_{u_j}}{u_j!}\ I_{U;p_r+2,q_r+2:|R:Y}^{V;0,n_r+2:|R':X}\left(\begin{array}{c|c} z_1 & A;(1-g-\sum_{j=1}^{m}t_j;a_1,...,a_r), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B;\mathrm{B}, \end{array}\right.$$

$$\left.\begin{array}{c} (1-g'-\sum_{j=1}^{m}t_j;a_1',...,a_r'),\mathrm{A}:E:\Im \\ . \\ . \\ (g'-g-\sum_{j=1}^{m}t_j;a_1-a_1',...,a_r-a_r'),(\sum_{j=1}^{m}w_j-g-\sum_{j=1}^{m}t_j;a_1,...,a_r):L:\Re \end{array}\right)$$

$$= I^{V;0,n_r+3:|R':X}_{U;p_r+3,q_r+3:|R:Y}\left(\begin{array}{c|c} z_1 & A;(1-\frac{g}{2};\frac{a_1}{2},...,\frac{a_r}{2}),(1-g';a'_1,...,a'_r), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B;\mathrm{B},(g'-\frac{g}{2};\frac{a_1}{2}-a'_1,...,\frac{a_r}{2}-a'_r), \end{array}\right.$$

$$\left.\begin{array}{c} (g'+\sum_{j=1}^m w_j-\frac{a}{2};\frac{a_1}{2}-a'_1,...,\frac{a_r}{2}-a'_r),\mathrm{A}:E:\Im \\ \cdot \\ \cdot \\ (\sum_{j=1}^m w_j+g'-g;a_1-a'_1,...,a_r-a'_r):L:\Re \end{array}\right)$$

provided

$a_i, a'_i, a_i - 2a_i > 0 (i = 1,...,r)$, $\mathrm{Re}(g' - \frac{g}{2} - \sum_{j=1}^{m} w_j) > 0$ *and* $|arg(z_i)| < \frac{1}{2}(\Omega_i - 2a_i)\pi$

Theorem 2.3.

(2.3)

$$\sum_{u_1,...u_m=0}^{\infty}\prod_{j=1}^{m}\frac{((w_j))_{u_j}}{u_j!}\ I^{V;0,n_r+4:|R':X}_{U;p_r+4,q_r+4:|R:Y}\left(\begin{array}{c|c} z_1 & A;(1-g-\sum_{j=1}^m t_j;a_1,...,a_r), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B;\mathrm{B}, \end{array}\right.$$

$$\begin{array}{c} (1-g'-\sum_{j=1}^m t_j;a'_1,...,a'_r),(1-g''-\sum_{j=1}^m t_j;a''_1,...,a''_r), \\ \cdot \\ \cdot \\ (g'-g-\sum_{j=1}^m t_j;a_1-a'_1,...,a_r-a'_r),(\sum_{j=1}^m w_j-g-\sum_{j=1}^m t_j;a_1,...,a_r) \end{array}$$

$$\left.\begin{array}{c} (-\frac{g}{2}-\sum_{j=1}^m t_j;\frac{a_1}{2},...,\frac{a_r}{2}),\mathrm{A}:E:\Im \\ \cdot \\ \cdot \\ (1-\frac{g}{2}-\sum_{j=1}^m t_j;\frac{a_1}{2},...,\frac{a_r}{2}),(g''-g-\sum_{j=1}^m t_j;a_1-a''_1,...,a_r-a''_r):L:\Re \end{array}\right)$$

$$= I^{V;0,n_r+3:|R':X}_{U;p_r+3,q_r+3:|R:Y}\left(\begin{array}{c|c} z_1 & A;(1-g';a'_1,...,a'_r) \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B;\mathrm{B},(g'-g+\sum_{j=1}^m w_j;a_1-a'_1,...,a_r-a'_r), \end{array}\right.$$

$$\begin{array}{c} (1-g'';a''_1,...,a''_r),(g'+g''-g+\sum_{j=1}^m w_j;a_1-a''_1,...,a_r-a''_r), \\ \cdot \\ \cdot \\ (g''-g+\sum_{j=1}^m w_j;a_1-a''_1,...,a_r-a''_r), \end{array}$$

$$\left. \begin{array}{c} \mathrm{A}:E:\Im \\ \cdot \\ \cdot \\ (g'+g''-g+\sum_{j=1}^{m}w_j;a_1-a_1'-a_1'',...,a_r-a_r'-a_r''):L:\Re \end{array} \right)$$

provided $a_i, a_i', a_i'', a_i - a_i' - a_i'' > 0 (i = 1,...,r)$, $\mathrm{Re}(g' + g'' - g - \sum_{j=1}^{m} w_j) < 1$

and $|arg(z_i)| < \frac{1}{2}(\Omega_i - \frac{7}{2}a_i)\pi$.

Theorem 2.4.

(2.4)

$$\sum_{u_1,...u_m=0}^{\infty}\prod_{j=1}^{m}\frac{((w_j))_{u_j}}{u_j!}\, I_{U;p_r+3,q_r+3:|R:Y}^{V;0,n_r+3:|R':X}\left(\begin{array}{c|c} z_1 & A;(-\frac{g}{2}-\sum_{j=1}^{m}t_j;\frac{a_1}{2},...,\frac{a_r}{2}), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B;\mathrm{B},(1-\frac{g}{2}-\sum_{j=1}^{m}t_j;\frac{a_1}{2},...,\frac{a_r}{2}), \end{array} \right.$$

$$\begin{array}{c} (-\frac{g}{2}-\sum_{j=1}^{m}t_j;\frac{a_1}{2},...,\frac{a_r}{2}),(1-g-\sum_{j=1}^{m}t_j;a_1,...,a_r), \\ \cdot \\ \cdot \\ (1-\frac{g}{2}-\sum_{j=1}^{m}t_j;\frac{a_1}{2},...,\frac{a_r}{2}),(-g-\sum_{j=1}^{m}t_j+\sum_{j=1}^{m}w_j;a_1,...,a_r), \end{array}$$

$$\left. \begin{array}{c} (g''-\sum_{j=1}^{m}t_j;a_1'',...,a_r''),\mathrm{A}:E:\Im \\ \cdot \\ \cdot \\ (g'-g-\sum_{j=1}^{m}t_j;a_1-a_1'',...,a_r-a_r''):L:\Re \end{array} \right)$$

$$= I_{U;p_r+3,q_r+3:|R:Y}^{V;0,n_r+3:|R':X}\left(\begin{array}{c|c} z_1 & A; \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B;\mathrm{B}, \end{array} \right.$$

$$\begin{array}{c} (\frac{1-g}{2};\frac{a_1}{2},...,\frac{a_r}{2}),(1-g'';a_1'',...,a_r''), \\ \cdot \\ \cdot \\ (\frac{1-g}{2}+g';\frac{a_1}{2}-a_1',...,\frac{a_r}{2}-a_r'),(\frac{1-g}{2}+\sum_{j=1}^{m}w_j;\frac{a_1}{2},...,a_r), \end{array}$$

$$\left. \begin{array}{c} (\frac{1-g}{2}+\sum_{j=1}^{m}w_j;\frac{a_1}{2},...,\frac{a_r}{2},\mathrm{A}:E:\Im \\ \cdot \\ \cdot \\ (g'-g+\sum_{j=1}^{m}w_j;a_1-a_1',...,a_r-a_r'):L:\Re \end{array} \right)$$

provided
$a_i, a'_i, a_i - a'_i - 2a''_i > 0 (i = 1, ..., r)$, $\mathrm{Re}(g' - \frac{g}{2} - \sum_{j=1}^{m} w_j) < \frac{1}{2}$ *and* $|arg(z_i)| < \frac{1}{2}(\Omega_i - \frac{5}{2}a_i)\pi$

To prove Theorems (2.2, 2.3 and 2.4), we follow the similar lines with the help of ([10], p.52, Eq.(2.3.3.5)), ([10], p.56, Eq.(2.3.4.5)) and ([10], p.245, Eq.(III.22)) respectively, instead of Gauss's theorem.

3. Particular Cases

In this section, we observe several particular cases. If we take $a'_i = 0 (i = 1, ..., r)$and assume $g' \to \infty$ in Theorem (2.2) and Theorem (2.4), also using the following properties of confluence,

$$\lim_{\lambda \to \infty} \left[(\lambda)_m \left(\frac{z}{\lambda} \right)^m \right] = z^m \tag{3.1}$$

and

$$\lim_{\rho \to \infty} \left[\frac{(\rho w)^m}{(\rho)_m} \right] = w^m, \; m = 0, 1, ... \tag{3.2}$$

After algebraic simplification, we obtain the following corollaries :

Corollary 3.1.

(3.3)

$$\sum_{u_1, ... u_m = 0}^{\infty} \prod_{j=1}^{m} \frac{(-1)^{t_j} ((w_j))_{u_j}}{u_j!} \; I_{U;p_r+1,q_r+1:|R:Y}^{V;0,n_r+1:|R':X} \left(\begin{array}{c|c} z_1 & A; \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B; \mathrm{B}, \end{array} \begin{array}{c} (1 - g - \sum_{j=1}^{m} t_j; a_1, ..., a_r), \mathrm{A} : E : \Im \\ \cdot \\ \cdot \\ (\sum_{j=1}^{m} w_j - g - \sum_{j=1}^{m} t_j; a_1, ..., a_r) : L : \Re \end{array} \right)$$

$$= I_{U;p_r+1,q_r+1:|R:Y}^{V;0,n_r+1:|R':X} \left(\begin{array}{c|c} z_1 & A; (1 - \frac{g}{2}; \frac{a_1}{2}, ..., \frac{a_r}{2}), \mathrm{A} : E : \Im \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B; \mathrm{B}, (\sum_{j=1}^{m} w_j - \frac{g}{2}; \frac{a_1}{2}, ..., \frac{a_r}{2}) : L : \Re \end{array} \right)$$

provided $a_i > 0 (i = 1, ..., r)$, $\mathrm{Re}(\sum_{j=1}^{m} w_j) > 0$ *and* $|arg(z_i)| < \frac{1}{2}(\Omega_i - a_i)\pi$.

Corollary 3.2.

(3.4)

$$\sum_{u_1,\dots u_m=0}^{\infty}\prod_{j=1}^{m}\frac{(-1)^{t_j}(w_j)_{u_j}}{u_j!}\; I^{V;0,n_r+2:|R':X}_{U;p_r+2,q_r+2:|R:Y}\left(\begin{array}{c|c} z_1 & A; \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B;\mathrm{B}, \end{array}\right.$$

$$\left.\begin{array}{c} (-\frac{g}{2}-\sum_{j=1}^{m}t_j;\frac{a_1}{2},\dots,\frac{a_r}{2}),(1-g-\sum_{j=1}^{m}t_j;a_1,\dots,a_r),\mathrm{A}:E:\Im \\ \cdot \\ \cdot \\ (1-\frac{g}{2}-\sum_{j=1}^{m}t_j;\frac{a_1}{2},\dots,\frac{a_r}{2}),(-g-\sum_{j=1}^{m}t_j+\sum_{j=1}^{m}w_j;a_1,\dots,a_r):L:\Re \end{array}\right)$$

$$= I^{V;0,n_r+1:|R':X}_{U;p_r+1,q_r+1:|R:Y}\left(\begin{array}{c|c} z_1 & A;(\frac{1-g}{2};\frac{a_1}{2},\dots,\frac{a_r}{2}),\mathrm{A}:E:\Im \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B;\mathrm{B},(\frac{1-g}{2}+\sum_{j=1}^{m}w_j;\frac{a_1}{2},\dots,\frac{a_r}{2}):L:\Re \end{array}\right)$$

provided $a_i>0(i=1,\dots,r)$, $\mathrm{Re}(\sum_{j=1}^{m}w_j)<\frac{1}{2}$ *and* $|arg(z_i)|<\frac{1}{2}(\Omega_i-\frac{3}{2}a_i)\pi$.

Taking $a_i=0(i=1,\dots,r)$ and assume $g''\to\infty$ in Theorem (2.3). Also using the equations (2.4),(3.1) and after algebraic manipulations, we obtain the following corollary.

Corollary 3.3.

(3.5)

$$\sum_{u_1,\dots u_m=0}^{\infty}\prod_{j=1}^{m}\frac{(-1)^{t_j}(w_j)_{u_j}}{u_j!}\; I^{V;0,n_r+3:|R':X}_{U;p_r+3,q_r+3:|R:Y}\left(\begin{array}{c|c} z_1 & A; \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B;\mathrm{B}, \end{array}\right.$$

$$\begin{array}{c} (1-g-\sum_{j=1}^{m}t_j;a_1,\dots,a_r),(1-g'-\sum_{j=1}^{m}t_j;a'_1,\dots,a'_r), \\ \cdot \\ \cdot \\ (g'-g-\sum_{j=1}^{m}t_j;a_1-a'_1,\dots,a_r-a'_r),(\sum_{j=1}^{m}w_j-g-\sum_{j=1}^{m}t_j;a_1,\dots,a_r) \end{array}$$

$$\left.\begin{array}{c} (-\frac{g}{2}-\sum_{j=1}^{m}t_j;\frac{a_1}{2},\dots,\frac{a_r}{2}),\mathrm{A}:E:\Im \\ \cdot \\ \cdot \\ (1-\frac{g}{2}-\sum_{j=1}^{m}t_j;\frac{a_1}{2},\dots,\frac{a_r}{2}):L:\Re \end{array}\right)$$

$$= I^{V;0,n_r+1:|R':X}_{U;p_r+1,q_r+1:|R:Y} \left(\begin{array}{c|c} z_1 & A; (1-g'; a'_1, ..., a'_r), \mathrm{A} : E : \Im \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & B; \mathrm{B}, (g'-g+\sum_{j=1}^{m} w_j; a_1 - a'_1, ..., a_r - a'_r) : L : \Re \end{array} \right)$$

provided $a_i > 0 (i = 1, ..., r)$, $\mathrm{Re}(\sum\limits_{j=1}^{m} w_j) < \frac{1}{2}$
and $|arg(z_i)| < \frac{1}{2}(\Omega_i - \frac{3}{2}a_i)\pi$.

We can give a number of corollaries by specializing the parameters. The multiple summation formulae involved in this article are general in nature in their manifold.

4. Concluding Remarks

If I-function defined by Prasad [15] reduces in generalized form of H-function defined by Prasad and Singh [14], we obtain the similar relations using analogue techniques. Also by modifying the functions defined by Srivastava and Panda([8], [9]) and Goyal and Garg [13], we can obtain similar type of relations.

The importance of all these results are common in nature. We can obtain single, double or multiple summation formulae by making use of general multiple summation formulae used here. By specializing various parameters and variables in the MMIF, we get several useful product of such functions like E, F, G, H and I of one and several variables. These formulae are useful in many interesting cases of Applied Mathematics and Mathematical Physics. In the next extension of this work, we are going to apply these summation formulae to obtain the solutions of Boundary value problems.

Acknowledgment

Corresponding author is thankful to all co-authors for their valuable contribution towards the preparation of this article.

References

[1] Manish Kumar Bansal, Shiv Lal, Devendra Kumar, Sunil Kumar, Jagdev Singh. Fractional Differential Equation Pertaining to an Integral operator Involving Incomplete H-Function in the Kernel, *Math Meth Appl Sci.*, 1–12(2020).

[2] Sanjay Bhatter, Amit Mathur, Devendra Kumar, Jagdev Singh. New Extension of Fractional-Calculus Results Associated with Product of Certain Special Functions, *Int. J. Appl. Comput. Math*, 7:97(2021).

[3] Jagdev Singh, Devendra Kumar, Manish Kumar, Bansal. Solution of nonlinear differential equation and special functions, *Math Meth Appl Sci.*, 43, 2106-2116(2020).

[4] B. L. J. Braaksma. Asymptotic expansions and analytic continuations for a class of Barnes-Integrals, *Compositio Mathematica*, 15, 239-341(1962-1964).

[5] B. Satyanarayana, Y. Pragathi Kumar, N. Srimannarayana, B. V. Purnima. Solution of a Boundary Value Problem Involving I-Function and Struve's Function, *Int. j. recent technol*,8,411-415(2019).

[6] Fredric Ayant, Pragathi Kumar Y, Srimannarayana N, Satyanarayana B. Transformation formulae for modified multivariable I-function of Prasad , *Adv. Math.: Sci. J.*,9,3663-3674(2020).

[7] Fredric Ayant, Pragathi Kumar Y, Srimannarayana N, Satyanarayana B. Certain integrals and series expansions involving modified generalized I-function of Prasad, *Adv. Math.: Sci. J.*, 9, 5835-5847(2020).

[8] H. M. Srivastava, R. Panda. Some expansion theorems and generating relations for the H-function of several complex variables, *Comment. Math. Univ. St. Paul*,24,119-137(1975).

[9] H. M. Srivastava, R. Panda. Some expansion theorems and generating relations for the H-function of several complex variables II , *Comment. Math. Univ. St. Paul*,25,167-197(1976).

[10] L. J. Slater. *Generalized Hypergeometric functions*, Cambridge Univ. Press,1976.

[11] Pragathi Kumar Y, Alem Mebrahtu, Satyanarayana B, Srimannarayana N. Boundary value problem solution with general class of polynomial, M-series and I-function, *Int. J. Adv. Sci.*,29, 915-923(2020).

[12] R. Panda. Some Multiple Series Transformations , *Jnanabha Sect. A*,4,165-168(1976).

[13] S. P. Goyal and R. S. Garg. On Multiple Summation formulas for the multivariable H-function, *Indian J. Math*,26,135-144(1984).

[14] Y. N. Prasad and A. K. Singh. Basic properties of the transform involving H-function of r-variables as Kernal, *Indian Acad. Math.*,2,109-115(1982).

[15] Y. N. Prasad. Multivariable I-function, *Vijnana Parishad Anusandhan Patrika*,29,231-237(1986).

(D.K.PAVAN KUMAR) DEPARTMENT OF MATHEMATICS, GUDLAVALLERU ENGINEERING COLLEGE, GUDLAVALLERU, A.P., INDIA.
Email address: krish.pav@gmail.com

(FREDRIC AYANT) PROFESSOR(RETD.,), SIX-FOURS LES PLAGES 83140, VAR, FRANCE.
Email address: fredericayant@gmail.com

(Y. PRAGATHI KUMAR) DEPARTMENT OF MATHEMATICS, CHALAPATHI INSTITUTE OF ENGINEERING AND TECHNOLOGY, LAM, GUNTUR, A.P, INDIA.
Email address: pragathi.ys@gmail.com

(N.SRIMANNARAYANA) DEPARTMENT OF MATHEMATICS, KONERU LAKSHMAIAH EDUCATION FOUNDATION, VADDESWARAM, A.P., INDIA.
Email address: sriman72@gmail.com

(B.SATYANARAYANA) DEPARTMENT OF MATHEMATICS, ACHARYA NAGARJUNA UNIVERSITY, NAGARJUNA NAGAR, A.P, INDIA.
Email address: drbsn63@yahoo.co.in

Analysis of unsteady MHD Williamson nanofluid flow past a stretching sheet with nonlinear mixed convection, thermal radiation and velocity slips

M. Das[1], B. Kumbhakar[1,*] and J. Singh[2]

[1]Department of Mathematics,
NIT Meghalaya, Shillong-793003, Meghalaya, India
[2]Department of Mathematics,
JECRC University, Jaipur-303905, Rajasthan, India

Abstract

This article examines the transient MHD convective flow with heat and mass transport of Williamson nanofluid over a stretching sheet in the presence of a chemical reaction. Velocity slips, convective heating and vanishing mass flux conditions at the surface are imposed. As a novelty, the effects of nonlinear thermal radiation, mixed convection, velocity slips and activation energy are incorporated. Such problems find significant applications in aircraft, missiles, gas turbines, etc. Similarity transformations are employed to convert controlling PDEs into a system of ODEs and the resulting nonlinear BVP is solved numerically using *bvp4c*. The effects of various parameters on velocity, temperature and concentration distributions are demonstrated and depicted graphically. However, the numerical values of local skin friction coefficients, Nusselt and Sherwood numbers are tabulated and discussed. The graphs show that the nonlinear convection parameters, for both temperature and concentration, boost the primary flow. Higher values of the velocity slip parameters result in diminishing the flow. The fluid temperature rises as a result of both radiation and convective heating. The activation energy improves the concentration profile within the boundary layer. The current findings would appeal to a broad audience in mechanical engineering, medical sciences, industrial engineering, etc.

Keywords Williamson nanofluid · Thermal radiation · Velocity slip · Convective heating · Activation energy · Chemical reaction

PACS 44.20.+b · 44.25.+f · 44.40.+a · 47.70.Fw · 52.30Cv

Mathematics Subject Classification (2010) 76A05 · 76W05 · 80A20 · 80A32

*Corresponding author. *E-mail address:* bkmath@nitm.ac.in

Nomenclature

u, v, w	Velocity components	L	Dimensionless quantity
x, y, z	Space coordinate	N	Buoyancy ratio parameter
T_∞	Ambient temperature	Le	Lewis number
C_∞	Ambient concentration	D_T	Thermal diffusion coefficient
S	Unsteadiness parameter	Bi	Biot number
n	Fitted rate constant		
kr	Chemical reaction coefficient	**Greek Symbols**	
$u_{\mathrm{w}}, v_{\mathrm{w}}$	Stretching velocities along x and y directions	ν	Kinetic viscosity
		μ	Dynamic viscosity
B_0	Constant magnetic field	Γ_1	Chemical reaction parameter
k	Thermal conductivity	σ	Electrical conductivity
D_B	Mass diffusivity	κ	Boltzmann constant
Rd	Thermal radiation parameter	σ^*	Stefan-Boltzmann constant
Pr	Prandtl number	θ	Dimensionless temperature
g	Gravitational acceleration	Γ	Material parameter
c_p	Specific heat	α	Thermal diffusivity
T	Fluid Temperature	ϕ	Dimensionless concentration
C	Species concentration	α_1, α_2	Velocity slip parameters
Nt	Thermophoresis parameter	λ_1, λ_2	Nonlinear thermal and solutal convection parameters
T_{w}	Wall temperature		
C_{w}	Wall concentration	ρ	Fluid density
Nb	Brownian motion parameter	θ_{w}	Temperature ratio parameter
d_1^*, d_2^*	Velocity slip coefficients	β_1	Velocity ratio parameter
We_1, We_2	Weissenberg numbers along x and y directions	τ	Heat capacity ratio
		β_C, β_C^*	Linear and non-linear solutal expansion coefficients
E_a	Activation energy		
M	Magnetic parameter	β_T, β_T^*	Linear and non-linear thermal expansion coefficients
k^*	Mean absorption coefficient		
E	Dimensionless activation energy	λ	Mixed convection parameter

1 Introduction

For the past few decades, nanotechnology-based techniques have been used to create nanoscale particles with a size of less than 100 nm. Stable suspensions can be made using nanoparticles to increase the thermal characteristics of the base fluid. It has been demonstrated that adding tiny quantities of metal or metal oxide nanoparticles to liquid improves thermal conductivity. Nanofluids, like current working fluids, have high heat absorption and heat transmission characteristics. Recent years have seen a significant increase in interest in nanofluids research owing to its numerous usages in communication, electronics and computer systems, as well as optical devices. Hayat et al. [1] discussed the movement of a non-Newtonian fluid across a wedge as a mixed convection flow.

Nourazar et al. [2] used the HPM to solve an MHD nanofluid flow on a horizontal flat plate with a changing magnetic field and viscous dissipation. A study on the effect of natural convection in viscoelastic fluid past a cone taking viscous dissipation was done by Makanda et al. [3]. In a rotating device, Sheikholeslami et al. [4] examined the nanofluid flow and heat transmission properties between two parallel horizontal plates. Using a fixed wedge with changing wall temperature and concentration, Srinivasacharya et al. [5] investigated the influence of a varied magnetic field on nanofluid flow.

Understanding the boundary layer flow with heat transfer along a stretched sheet has become more significant because of several engineering activities. Extrusion of polymers, paper manufacturing, and other similar processes are examples of chemical engineering and metallurgy applications. The rate of heat transfer between the fluid and stretching surface considering heating and/or cooling has a significant impact on the quality of the final product. As a result, the choice of heating or cooling fluid is critical to the heat transfer rate. In light of the physical relevance of heat transmission across moving surfaces, several researchers have been obliged to publish their discoveries in this area. Crane [6] examined the flow past a stretched plate that is subject to the relation between the velocity and the distance from a slit. This yielded an accurate result. Following Crane's work, MHD viscous flow across a stretched sheet was given by Azimi et al. [7], who discussed the analysis of momentum features in the flow. Dessie and Kishan [8] investigated the effect of viscous dissipation and heat source/sink over a stretching sheet. Mishra et al. [9] studied numerically MHD power-law fluid flow over a stretching sheet taking a non-uniform heat source.

Regarding the MHD heat transfer fluxes, thermal radiation is a crucial factor in controlling heat transfer rate. It may impact many industrial processes such as glass manufacture, gas turbine production, furnace design, and re-entry vehicle engine design. As a result, this generated extensive studies on the influence of heat radiation in hydromagnetic fluxes. Daniel and Daniel [10] explored the impact of thermal radiation and buoyancy force on MHD flow through a stretchable sheet with the help of the homotopy analysis method. Kumbhakar and Rao [11] discussed MHD stagnation point flow of an electrically conducting fluid over a nonlinearly stretching surface considering thermal radiation and viscous dissipation. Kho et al. [12] studied thermal radiation effect in the flow of Williamson nanofluid passing through a stretching sheet. With heat and mass transfer through an unstable stretched surface in a uniform magnetic field, Ishaq et al. [13] explored entropy production and thermal radiation. Alharbi et al. [14] conducted experiments on MHD Eyring-Powell flow in an unstable oscillatory stretching sheet to evaluate the influence of thermal radiation and a heat source/sink. Kumar et al. [15] examined the transient natural/free convective nanofluid flow past a vertical plate with effects of radiation and magnetic field.

According to current trends in chemical reaction analysis, it is essential to create a mathematical model of a system to forecast its performance. Especially in the chemical and hydro-metallurgical sectors, heat and mass transport research during chemical reactions is of great significance. Some examples of combined heat and mass transfer applications with chemical reaction effects are chemical processing equipment design, fog formation and dispersion, temperature and moisture distribution over agricultural fields and fruit tree groves, crop damage due to freezing, cooling towers, and food processing. An excellent example of a first-order homogeneous chemical reaction is the production of smog. Das [16] examined the effects of thermal radiation and chemical reaction on MHD micropolar fluid flow near an inclined porous plate. Sheikh and Abbas [17] studied chemical reaction impact on MHD viscous fluid flow over an oscillating stretching sheet under the influence of heat generation/absorption. Tarakaramu and Narayan [18] explored the effect of chemical reactions on unsteady MHD nanofluid flow towards a stretchable sheet. Kumar et al. [19] investigated the influence of binary chemical reaction with Arrhenius activation energy on the MHD Carreau fluid flow over a stretched surface. They found that the chemical reaction has a significant impact on the flow. Khan et al. [20] studied the aspects of activation energy and thermal radiation on MHD flow containing Ti_6Al_4V nanoparticle past a stretching sheet. Chu et al. [21] discussed the action of a chemical reaction and activation energy on MHD third grade nanofluid flow past a stretching sheet.

The assumption that the flow field obeys the standard no-slip condition at the sheet is quite common in the preceding research and all relevant references. However, the no-slip criterion is inadequate when the fluid is made up of particle emulsions and polymers. Furthermore, boundary-slipping fluids have crucial technological uses, such as cleaning prosthetic heart valves and interior cavities. In such circumstances, the partial slip is an appropriate boundary condition. Additionally, when micro-scale dimensions are included in the flow field, such a slip is necessary. Slip at the wall boundary significantly alters the fluid's flow behavior and shear stress than no-slip circumstances. Using a low-magnetic Reynolds number assumption, Zheng et al. [22] investigated the slip consequences of Oldroyd-B fluid flow across a plate. Hayat et al. [23] explored velocity slip condition on MHD nanofluid flow past a rotating disk. Amanulla et al. [24] discussed the slip effects on MHD Prandtl flow past an isothermal sphere in a non-Darcy porous medium. Ellahi et al. [25] analyzed the combined impact of slip and entropy generation on MHD flow through a moving plate. Khan et al. [26] explored the significance of slip conditions for a magnetite Jeffrey nanofluid flow over a porous stretching sheet in the existence of thermal radiation and the Soret effect. Das et al. [27] studied mutiple slip effects on tangent hyperbolic fluid flow along a stretching sheet considering Soret and Dufour effects, thermal radiation and heat source.

In processes in which high temperatures are involved, convective heat transfer is essential. Consider the following examples: gas turbines, nuclear power plants, thermal energy storage, and so forth. It is more feasible to use convective boundary conditions in industrial and technical processes, such as material drying and transpiration cooling operations [28]. Because of the practical significance of convective boundary conditions in viscous fluids, Many scholars have investigated and presented their findings on this issue. Ramzan et al. [29] investigated the impact of convective heating conditions and Cattaneo-Christov heat flux with heat production/absorption on MHD 3D Maxwell fluid flow across a bidirectional stretching surface. Nayak et al. [30] studied MHD nanofluid flow over a linearly stretching sheet considering the convective heating boundary constraint along with viscous dissipation, velocity slip, nonlinear thermal radiation and Joule heating. Shah et al. [31] observed simultaneous effects of convective boundary condition and thermal radiation on MHD Carbon nanotubes nanofluid flow across a stretching sheet. Aspects of convective boundary condition, Joule heating, thermal radiation, and a changing heat source/sink were studied in detail by Kumar et al. [32] concerning the flow and heat transfer properties of an electrically conducting Casson fluid due to an exponentially expanding curved surface. Loganathan et al. [33] examined the impact of convective heating, Cattaneo-Christov double diffusion and thermal radiation on MHD Maxwell fluid flow along an extended surface. Recently, Jamshed and Nisar [34] studied convective heating, thermal radiation and heat source effects on Williamson nanofluid flow over a stretching sheet.

Based on the above literature survey, the authors have found that no attempt has been made yet to study the impacts of nonlinear thermal radiation and Arrhenius activation energy on unsteady mixed convective flow of Williamson nanofluid over a stretching surface. Therefore, this research aims to fill such gap by exploring the novel circumstances of nonlinear thermal radiation and activation energy on unsteady MHD convective flow with heat and mass transport of Williamson nanofluid over a stretching sheet in the presence of a chemical reaction. The outcomes of this study may have significant bearings on several practical applications such as in aircraft, missiles, gas turbines, food processing, etc. Numerical solutions are obtained for the velocity, temperature and concentration distributions with the help of *bvp4c* routine of MATLAB software. The impacts of significant flow parameters on velocity, temperature and concentration profiles are illustrated and presented graphically. However, the variations in surface drag-coefficients, Nusselt and Sherwood numbers are discussed using numerical data. Moreover, for a limiting case of the present study, a data comparison is made just to ensure that the obtained results are correct and reliable.

2 Mathematical formulation

Consider a three-dimensional, unsteady and incompressible MHD Williamson nanofluid flow along a stretching surface with velocity slip. Further, the influences of nonlinear thermal radiation and chemical reaction with activation energy are also considered. A physical configuration of the flow problem is demonstrated in Fig. 1. The figure shows that the sheet is positioned in the Cartesian coordinate system (x, y, z) such that the x-axis is along the surface in the direction of flow, y-axis is along the width of the surface, and z-axis is normal to xy plane. A constant magnetic field of magnitude B_0 is applied along the z-direction. The surface is stretched along x and y-directions with velocities $u_{\mathrm{w}} = \frac{ax}{1-\beta t}$ and $v_{\mathrm{w}} = \frac{by}{1-\beta t}$ $(a, b$ being positive constants and β is a parameter having dimension time^{-1}) respectively. The nanofluid temperature and species concentration at the surface are kept at constant values of T_{w} and C_{w} respectively. In contrast, the ambient fluid temperature and species concentration are maintained at constant values of T_∞ and C_∞, respectively.

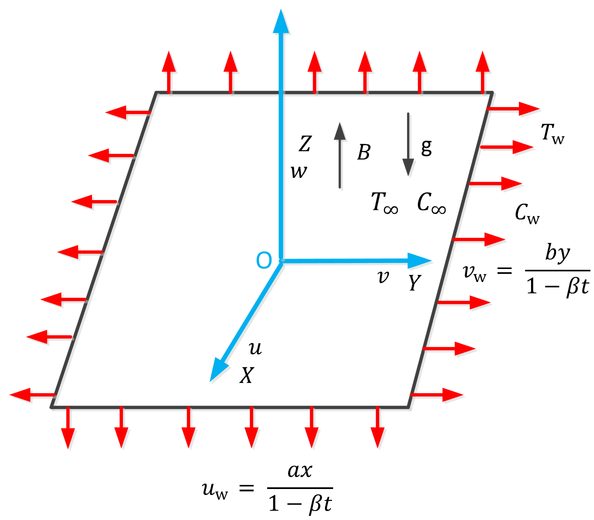


Figure 1: Physical configuration of the problem

Based on the aforementioned assumptions, the governing equations of the current fluid flow (continuity, momentum, energy and species concentration) may be modeled as ([35], [36]):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1}$$

$$\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= \nu\frac{\partial^2 u}{\partial z^2} + \sqrt{2}\nu\Gamma\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} + \frac{\nu\Gamma}{\sqrt{2}}\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2} - \frac{\sigma B^2(t)}{\rho_f}u \\ &+ \mathrm{g}\left[\beta_T(T-T_\infty) + \beta_T^*(T-T_\infty)^2 + \beta_C(C-C_\infty) + \beta_C^*(C-C_\infty)^2\right],\end{aligned} \tag{2}$$

$$\begin{aligned}\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} =& \nu\frac{\partial^2 v}{\partial z^2} + \sqrt{2}\nu\Gamma\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2} + \frac{\nu\Gamma}{\sqrt{2}}\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} \\ &- \frac{\sigma B^2(t)}{\rho_f}v,\end{aligned} \tag{3}$$

$$\begin{aligned}\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=&\alpha\frac{\partial^2 T}{\partial z^2}+\tau\left\{D_B\frac{\partial T}{\partial z}\frac{\partial C}{\partial z}+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial z}\right)^2\right\}\\&-\frac{1}{(\rho c_p)_f}\frac{\partial q_r}{\partial z},\end{aligned}\tag{4}$$

$$\begin{aligned}\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=&D_B\frac{\partial^2 C}{\partial z^2}+\frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial z^2}\\&-kr\left(C-C_\infty\right)\left(\frac{T}{T_\infty}\right)^n e^{-\frac{E_a}{\kappa T}}.\end{aligned}\tag{5}$$

The physical boundary conditions for the current problem are given as follows:

$$\left.\begin{aligned}&u=u_{\mathrm{w}}+d_1^*\frac{\partial u}{\partial z},\ v=v_{\mathrm{w}}+d_2^*\frac{\partial v}{\partial z},\ w=0,\ \ -k\frac{\partial T}{\partial z}=h_f\left(T_{\mathrm{w}}-T\right),\\&\quad D_B\frac{\partial C}{\partial z}+\frac{D_T}{T_\infty}\frac{\partial T}{\partial z}=0,\ \text{at } z=0,\\&u\to 0,\quad v\to 0,\quad T\to T_\infty,\quad C\to C_\infty\quad \text{as}\quad z\to\infty.\end{aligned}\right\}\tag{6}$$

In order to approximate the radiative heat flux q_r, the following Rosseland's approximation for an optically thick fluid is employed (Fatunmbi and Adeniyan [37]):

$$q_r=-\frac{16\sigma^* T^3}{3k^*}\frac{\partial T}{\partial z}.\tag{7}$$

The energy equation has the form after applying expression (7) to equation (4)

$$\begin{aligned}\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=&\alpha\frac{\partial^2 T}{\partial z^2}+\tau\left\{D_B\frac{\partial T}{\partial z}\frac{\partial C}{\partial z}+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial z}\right)^2\right\}\\&+\frac{16\sigma^* T^2}{3\left(\rho c_p\right)_f k^*}\left\{T\frac{\partial^2 T}{\partial Z^2}+3\left(\frac{\partial T}{\partial Z}\right)^2\right\}.\end{aligned}\tag{8}$$

The variable aspects of wall temperature, wall concentration and magnetic field are given by the following form [38]

$$T_{\mathrm{w}}(x,t)=\frac{T_0 x u_{\mathrm{w}}}{\nu(1-\beta t)^{\frac{1}{2}}}+T_\infty,\quad C_{\mathrm{w}}(x,t)=\frac{C_0 x u_{\mathrm{w}}}{\nu(1-\beta t)^{\frac{1}{2}}}+C_\infty,\quad B(t)=\frac{B_0}{(1-\beta t)^{\frac{1}{2}}}.$$

To obtain similar solutions of equations (2), (3), (8) and (5) subject to the boundary conditions (6), the following similarity variables are introduced:

$$\left.\begin{aligned}&u=\frac{ax}{1-\beta t}f'(\eta),\quad v=\frac{ay}{1-\beta t}g'(\eta),\quad w=-\sqrt{\frac{a\nu}{1-\beta t}}\left\{f(\eta)+g(\eta)\right\},\\&\theta(\eta)=\frac{T-T_\infty}{T_{\mathrm{w}}-T_\infty},\quad \phi(\eta)=\frac{C-C_\infty}{C_{\mathrm{w}}-C_\infty},\quad \eta=z\sqrt{\frac{a}{\nu(1-\beta t)}}.\end{aligned}\right\}\tag{9}$$

7

Substitution of the above similarity variables in equations (2), (3), (8) and (5) yields the following ordinary differential equations:

$$f''' \left[1 + We_1 f''\right] + \frac{We_1}{2} L^2 g'' g''' - f'^2 + (f+g) f'' - S\left(f' + \frac{1}{2}\eta f''\right) \\ - Mf' + \lambda\left(1+\lambda_1\theta\right)\theta + \lambda N\left(1+\lambda_2\phi\right)\phi = 0, \tag{10}$$

$$g''' \left[1 + We_2 g''\right] + \frac{We_2}{2L^2} f'' f''' - g'^2 + (f+g) g'' - S\left(g' + \frac{1}{2}\eta g''\right) \\ - Mg' = 0, \tag{11}$$

$$\theta'' + Pr\left(f+g\right)\theta' - Pr\frac{S}{2}(3\theta + \eta\theta') - 2Pr\theta f' + +PrNb\theta'\phi' + PrNt\theta'^2 \\ + Rd\left\{1+\theta\left(\theta_{\mathrm{w}}-1\right)\right\}^2\left[3\theta'^2\left(\theta_{\mathrm{w}}-1\right) + \left\{1+\theta\left(\theta_{\mathrm{w}}-1\right)\right\}\theta''\right] = 0, \tag{12}$$

$$\phi'' + PrLe\left(f+g\right)\phi' - PrLe\frac{S}{2}(3\phi + \eta\phi') - 2PrLe\phi f' + \frac{Nt}{Nb}\theta'' \\ - PrLe\Gamma_1\left\{1+(\theta_{\mathrm{w}}-1)\theta\right\}^n e^{\left(-\frac{E}{1+(\theta_{\mathrm{w}}-1)\theta}\right)}\phi = 0. \tag{13}$$

The dimensionless boundary conditions are stated as

$$\left.\begin{aligned} &f'(0) = 1 + \alpha_1 f''(0), \quad g'(0) = \beta_1 + \alpha_2 g''(0), \quad f(0) = 0, \quad g(0) = 0, \\ &\theta'(0) = -Bi\left(1-\theta(0)\right), \quad \phi'(0) = -\frac{Nt}{Nb}\theta'(0), \\ &f'(\infty) \to 0, \quad g'(\infty) \to 0, \quad \theta(\infty) \to 0, \quad \phi(\infty) \to 0. \end{aligned}\right\} \tag{14}$$

where

$We_1 = \sqrt{\frac{2\Gamma^2 a u_{\mathrm{w}}^2}{\nu(1-\beta t)}}$, $We_2 = \sqrt{\frac{2\Gamma^2 a v_{\mathrm{w}}^2}{\beta_1^2\nu(1-\beta t)}}$, $N = \frac{\beta_C(C_{\mathrm{w}} - C_\infty)}{\beta_T(T_{\mathrm{w}} - T_\infty)}$, $S = \frac{\beta}{a}$,
$\lambda = \frac{\beta_T \mathrm{g}(1-\beta t)(T_{\mathrm{w}} - T_\infty)}{a u_{\mathrm{w}}}$, $M = \frac{\sigma B_0^2}{a\rho_f}$, $L = \frac{y}{x}$, $Pr = \frac{\nu}{\alpha}$, $Le = \frac{\alpha}{D_B}$,
$Nb = \frac{\tau D_B\left(C_{\mathrm{w}} - C_\infty\right)}{\nu}$, $Nt = \frac{\tau D_T\left(T_{\mathrm{w}} - T_\infty\right)}{\nu T_\infty}$, $\theta_{\mathrm{w}} = \frac{T_{\mathrm{w}}}{T_\infty}$, $E = \frac{E_a}{\kappa T_\infty}$,
$\Gamma_1 = \frac{kr(1-\beta t)}{a}$, $\alpha_1 = d_1^*\sqrt{\frac{a}{\nu(1-\beta t)}}$, $\alpha_2 = d_2^*\sqrt{\frac{a}{\nu(1-\beta t)}}$, $\beta_1 = \frac{b}{a}$,
$Bi = \frac{h_f}{k}\sqrt{\frac{\nu(1-\beta t)}{a}}$, $\lambda_1 = \frac{\beta_T^*\left(T_{\mathrm{w}} - T_\infty\right)}{\beta_T}$, $\lambda_2 = \frac{\beta_C^*\left(C_{\mathrm{w}} - C_\infty\right)}{\beta_C}$
$\delta = \frac{Q_1(1-\beta t)}{a(\rho c_p)_f}$, $Rd = \frac{16\sigma^* T_\infty^3}{3(\rho c_p)_f \alpha k^*}$.

3 Skin-friction coefficients, Nusselt number and Sherwood number

The physical quantities of engineering interest for the present fluid flow problem are the local skin-friction coefficients, Nusselt number and Sherwood number. The skin-friction coefficient measures the shear stress, whereas the Nusselt number and Sherwood number describe the rate of heat and mass transfer at the surface. A low Nusselt number signifies that conductive heat transport is more than the convective heat transfer, whereas a high Nusselt number indicates that convective heat transfer dominates the conductive heat transfer. Thermal engineering devices may be designed more effectively with this in mind. Convective mass transfer is divided by diffusive mass transport, and this ratio is known as the Sherwood number. It is used to conduct mass transfer analyses on systems such as liquid-liquid extraction. Mathematically, the local skin-friction coefficients (C_{fx}, C_{fy}), Nusselt number (Nu_x) and Sherwood number (Sh_x) are expressed as

$$C_{fx} = \frac{\nu}{u_{\mathrm{w}}^2}\left[\frac{\partial u}{\partial z}\left\{1+\frac{\Gamma}{\sqrt{2}}\sqrt{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2}\right\}\right]_{z=0}, \tag{15}$$

$$C_{fy} = \frac{\nu}{v_{\mathrm{w}}^2}\left[\frac{\partial v}{\partial z}\left\{1+\frac{\Gamma}{\sqrt{2}}\sqrt{\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2}\right\}\right]_{z=0}, \tag{16}$$

$$Nu_x = -\frac{x}{k\left(T_{\mathrm{w}}-T_\infty\right)}\left[\left(k+\frac{16\sigma^* T^3}{3k^*}\right)\frac{\partial T}{\partial z}\right]_{z=0}, \tag{17}$$

$$Sh_x = -\frac{x D_B}{D_B\left(C_{\mathrm{w}}-C_\infty\right)}\left(\frac{\partial C}{\partial z}\right)_{z=0}. \tag{18}$$

The aforementioned physical values can be expressed in non-dimensional form using the dimensionless variables specified in (9)

$$C_{fx}\sqrt{Re_x} = f''(0)\left[1+\frac{We_1}{2}\sqrt{f''^2(0)+L^2 g''^2(0)}\right], \tag{19}$$

$$C_{fy}\sqrt{Re_y} = g''(0)\left[1+\frac{We_2}{2}\sqrt{\frac{1}{L^2}f''^2(0)+g''^2(0)}\right]\sqrt{\beta_1^{-3}}, \tag{20}$$

$$\frac{Nu_x}{\sqrt{Re_x}} = -\left[1+Rd\left\{1+\left(\theta_w-1\right)\theta(0)\right\}^3\right]\theta'(0), \tag{21}$$

$$\frac{Sh_x}{\sqrt{Re_x}} = -\phi'(0), \tag{22}$$

where $Re_x = \dfrac{u_{\mathrm{w}} x}{\nu}$ and $Re_y = \dfrac{v_{\mathrm{w}} y}{\nu}$ are the local Reynolds numbers.

4 Numerical solution

4.1 Methodology

The coupled and highly nonlinear ordinary differential equations (10)-(13) subject to the boundary conditions (14) are solved numerically by employing the *bvp4c* solver in MATLAB. The higher-order equations (10)-(13) are converted into a set of first-order equations. Furthermore, while implementing the numerical technique, the boundary value problem is metamorphosed into an initial value problem by assuming some suitable guess values to those missing initial conditions.

Table 1: Comparison of values of $-f''(0)$ for altered values of M when $\beta_1 = 0.5$

	$-f''(0)$		
M	Present	Oyelakin et al. [39]	Nadeem et al. [40]
0	1.093096	1.09310	1.0932
10	3.342030	3.34204	3.3420
100	10.058166	10.05818	10.058

Table 2: Comparison of values of $-g''(0)$ for altered values of M when $\beta_1 = 0.5$

	$-g''(0)$		
M	Present	Oyelakin et al. [39]	Nadeem et al. [40]
0	0.465206	0.46520	0.4653
10	1.645891	1.64590	1.6459
100	5.020785	5.02080	5.0208

4.2 Validation

The numerical values of $-f''(0)$ and $-g''(0)$ displayed in Tables 1 and 2 have been computed for different values of magnetic parameter M for a specific situation of the current problem, i.e., when $We_1 = We_2 = \lambda = \lambda_1 = \lambda_2 = \alpha_1 = \alpha_2 = N = 0$, $\beta_1 = 0.5$ and $n = 1$ to test the correctness of the obtained results and the reliability of the employed numerical approach. From the tables, it is clearly observed that our results have a firm agreement with the results reported by Oyelakin et al. [39] and Nadeem et al. [40].

5 Results and discussion

This section presents the analysis of the obtained results for the current heat and mass transport phenomenon. The behavior of the flow profiles as well as the physical quantities of practical importance, is investigated in depth with respect to the changes of the emergent parameters. For the computational purpose, we have assumed the parameters' values as $We_1 = We_2 = S = 0.2$,

$N = n = Nt = 0.5$, $Pr = \theta_w = 1.2$, $L = Nb = \lambda = \alpha_1 = \alpha_2 = 0.4$, $Rd = 0.1$, $Le = M = 1.0$, $\beta_1 = 0.7$, $Bi = \lambda_1 = \lambda_2 = K_1 = 0.3$, $E = 0.6$. Throughout the study, the same values for parameters are adopted, while the altered values of the parameters are shown separately in the respective figures.

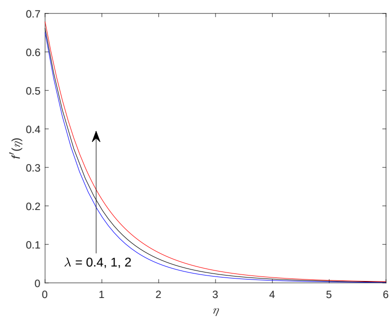


Figure 2: Changes in $f'(\eta)$ vs λ

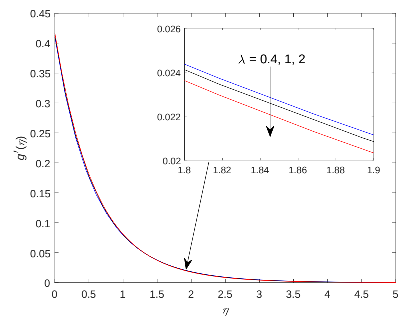


Figure 3: Changes in $g'(\eta)$ vs λ

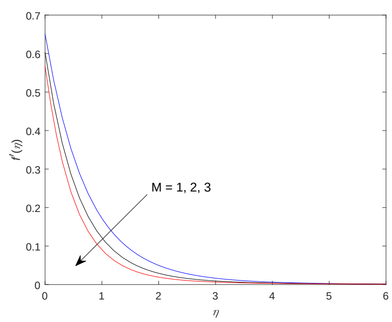


Figure 4: Changes in $f'(\eta)$ vs M

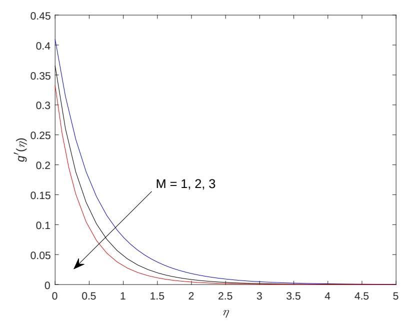


Figure 5: Changes in $g'(\eta)$ vs M

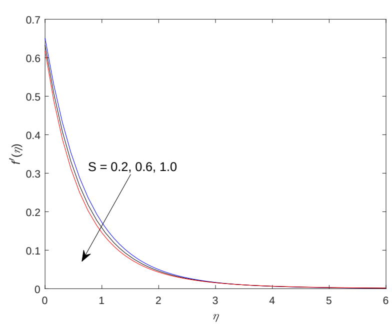


Figure 6: Changes in $f'(\eta)$ vs S

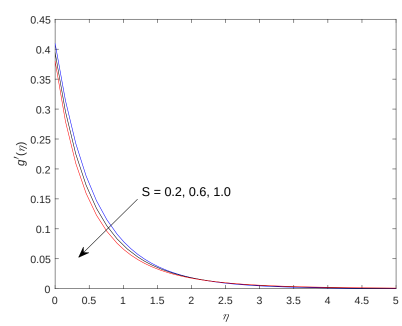


Figure 7: Changes in $g'(\eta)$ vs S

11

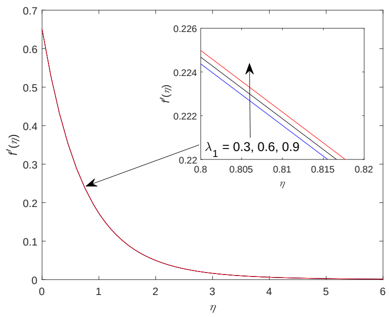


Figure 8: Changes in $f'(\eta)$ vs λ_1

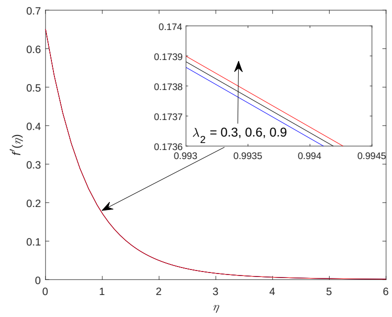


Figure 9: Changes in $f'(\eta)$ vs λ_2

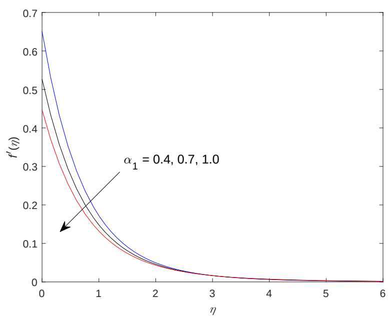


Figure 10: Changes in $f'(\eta)$ vs α_1

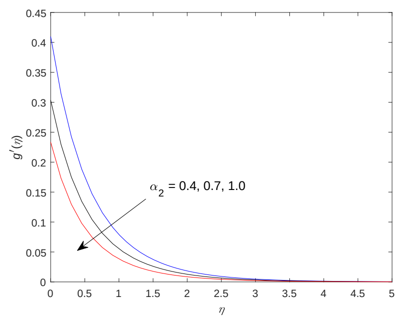


Figure 11: Changes in $g'(\eta)$ vs α_2

Figures 2-11 illustrate the influence of λ, M, S, λ_1, λ_2, α_1 and α_2 on the velocity field. Growth in $f'(\eta)$ and reduction in $g'(\eta)$ are observed in Figures 2 and 3. The higher mixed convection parameter contributes to a larger buoyancy force. This powerful force accelerates the primary flow by suppressing the flow in the secondary direction. A significant increase in the magnetic parameter has caused a significant drop in the nanofluid velocity profile. Increased M leads to a corresponding rise in the resistive Lorentz force, which causes the fluid flow to decrease as depicted in Figures 4 and 5. Decreasing nature of $f'(\eta)$ and $g'(\eta)$ for improvement in S is noted in Figures 6 and 7. In Figures 8 and 9, it is noticed that larger values of λ_1 and λ_2 indicate an upsurge in $f'(\eta)$. Temperature and concentration differences arise from nonlinear convection parameters λ_1 and λ_2 that are greater than the equivalent linear convection values. Velocity is therefore emphasized. Figures 10 and 11 express diminishing character of $f'(\eta)$ and $g'(\eta)$ w.r.t. α_1 and α_2. An increase in velocity slip parameters lead to increase the slip between the fluid and surface of the sheet. So a partial slip velocity moved to the flow field that has the tendency to decelerate the flow.

Figure 12 shows that $\theta(\eta)$ heightens on rising values of M. When the magnetic parameter increases, a stronger Lorentz force is generated. This force

12

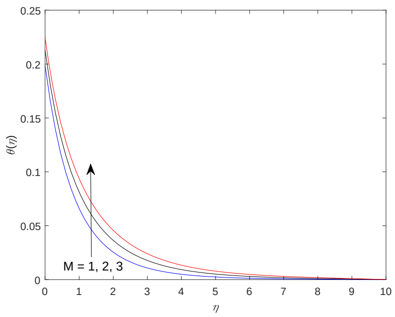


Figure 12: Changes in $\theta(\eta)$ vs M

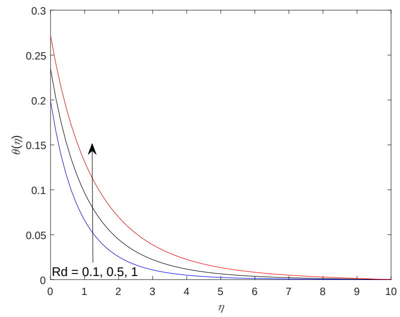


Figure 13: Changes in $\theta(\eta)$ vs Rd

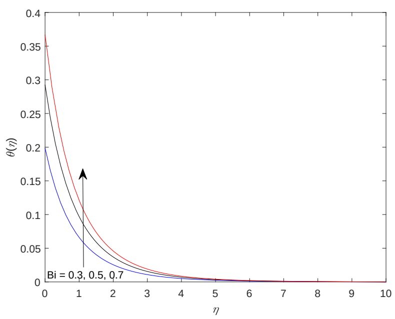


Figure 14: Changes in $\theta(\eta)$ vs Bi

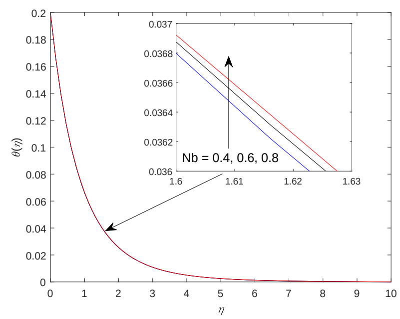


Figure 15: Changes in $\theta(\eta)$ vs Nb

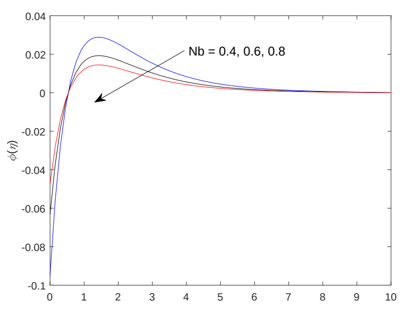


Figure 16: Changes in $\phi(\eta)$ vs Nb

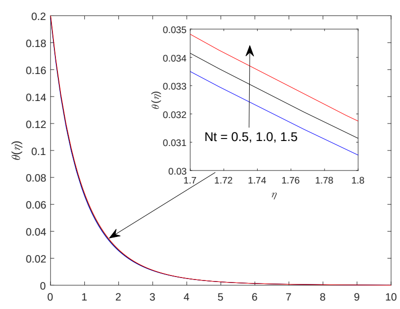


Figure 17: Changes in $\theta(\eta)$ vs Nt

provides resistance against the flow and thereby, the fluid temperature is intensified. Figure 13 elucidates a rising trend for $\theta(\eta)$ on enhanced values of Rd. Improved radiation parameter reduces the mean heat absorption coefficient. As a result, the fluid temperature gets hiked. From Figure 14, an increase in $\theta(\eta)$ is noticed for enlarged values of Bi. An increase in the Biot number leads to

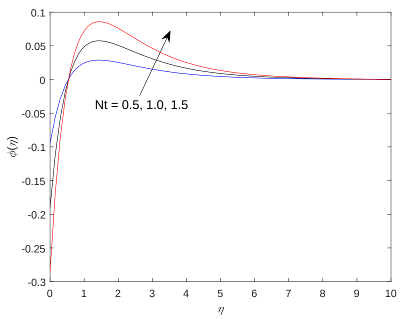


Figure 18: Changes in $\phi(\eta)$ vs Nt

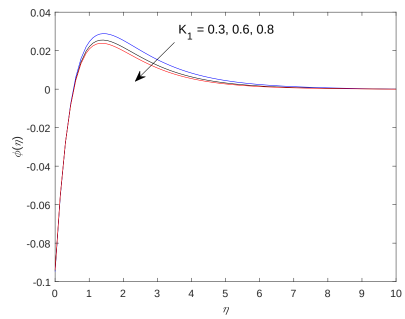


Figure 19: Changes in $\phi(\eta)$ vs K_1

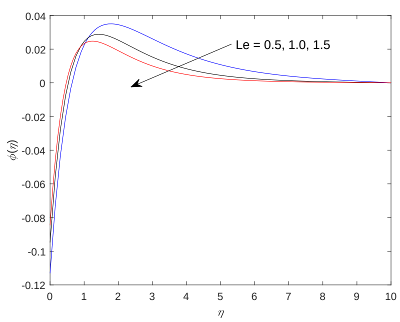


Figure 20: Changes in $\phi(\eta)$ vs Le

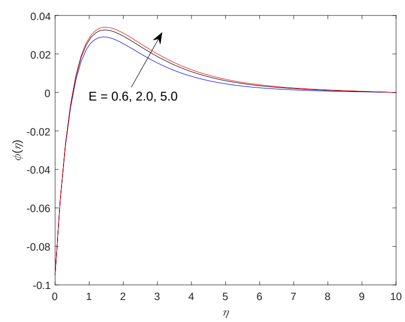


Figure 21: Changes in $\phi(\eta)$ vs E

enhance the heat transfer due to convective heating with hot fluid. So, temperature of the fluid is augmented. Figure 15 shows that when Nb increases, $\theta(\eta)$ decreases near the sheet and takes on an inverse nature far away from it. In reality, a larger Nb causes more Brownian diffusion with lesser viscous forces, and therefore, a hike in the temperature profile is observed. $\phi(\eta)$ is enhanced near the sheet for uplifting Nb values, while a reverse influence is seen away from the sheet, as shown in Figure 16. According to Figure 17, with upsurging values of Nt, $\theta(\eta)$ is increased. Physically, an increase in Nt causes a stronger thermophoretic force, which enriches the fluid's temperature. Figure 18 shows that $\phi(\eta)$ decreases towards the sheet, while the opposite trend is seen further away from the sheet in terms of Nt.

Figure 19 shows that an improvement in K_1 leads to a significant fall in $\phi(\eta)$. A devastating chemical reaction corresponds to a positive K_1. As a result, an improvement in K_1 causes a decrease in species concentration. In Figure 20, it is seen that for growing values of Le, $\phi(\eta)$ is reduced. Lewis number is basically the relation between thermal diffusivity to mass diffusivity. So, higher Lewis number implies less mass diffusion in the fluid flow. Hence, species concentration is lessened. Figure 21 reveals that there is an upward

Table 3: Numerical values of the skin friction coefficients when $\zeta_C = \zeta_T = 0.4$, $L = \lambda = N = n = 0.5$, $We_1 = We_2 = 1.6$ and $\beta_1 = 0.6$

λ	M	S	λ_1	λ_2	α_1	α_2	$-\sqrt{Re_x}C_{fx}$	$-\sqrt{Re_y}C_{fy}$
0.4	1	0.2	0.3	0.3	0.4	0.4	0.953465	1.833773
1							0.921040	1.766040
2							0.871657	1.663539
0.4	2						1.095809	2.137023
	3						1.204741	2.373294
	1	0.6					1.006186	1.945268
		1.0					1.052698	2.044400
		0.2	0.6				0.952571	–
			0.9				0.951679	–
			0.3	0.6			0.953415	–
				0.9			0.953364	–
				0.3	0.7		0.725064	–
					1.0		0.586788	–
					0.4	0.7	–	1.812118
						1.0	–	1.799073

trend in $\phi(\eta)$ with the progress of the parameter E. Boosted E values aid in the speeding up of chemical reactions and hence the species concentration is escalated.

The numerical values of the local skin-friction coefficients for various values of the controlling parameters λ, M, S, λ_1, λ_2, α_1 and α_2 are set forth in Table 3. For higher values of M and S, both $\sqrt{Re_x}C_{fx}$ and $\sqrt{Re_y}C_{fy}$ are increased whereas reverse trend is detected w.r.t. λ, λ_1, λ_2, α_1 and α_2. Local Nusselt and Sherwood numbers calculated for flow parameters M, Rd, Bi, Nb, Nt, K_1, Le and E are described in Table 4. Increasing trend of $\frac{Nu_x}{\sqrt{Re_x}}$ is found for Rd and Bi but opposite nature is noticed for M, Nb and Nt. Growing values of Nt and E imply increasing tendency of $\frac{Sh_x}{\sqrt{Re_x}}$ whereas converse behavior is found w.r.t. Nb, Le and K_1.

6 Conclusions

The present analysis explores the aspects of nonlinear thermal radiation and activation energy on unsteady convective heat and mass transport phenomena of Williamson nanofluid over a stretching sheet in the existence of Lorentz force and chemical reaction. Moreover, Navier's velocity slip and convective heating conditions are imposed at the surface boundary. The following are some of the significant outcomes from the simulation of the problem:

- A diminishing nature is observed for the velocity profiles with the improvement in unsteadiness and the intensity of the Lorentz force.

Table 4: Numerical values of the local Nusselt and Sherwood numbers when $Pr = 1.4$, $\theta_w = 1.1$ and $Le = 1.5$

M	Rd	Bi	Nb	Nt	K_1	Le	E	$\frac{Nu_x}{\sqrt{Re_x}}$	$-\frac{Sh_x}{\sqrt{Re_x}}$
1	0.1	0.3	0.4	0.5	0.3	1.0	0.6	0.267377	0.300448
2								0.262851	–
3								0.259108	–
1	0.5							0.361351	–
	1.0							0.474500	–
	0.1	0.5						0.395575	–
		0.7						0.497861	–
		0.3	0.6					0.267364	0.200288
			0.8					0.267357	0.150212
			0.4	1.0				0.267028	0.600074
				1.5				0.266675	0.898864
				0.5	0.6			–	0.300408
					0.8			–	0.300385
					0.3	0.5		–	0.300598
						1.5		–	0.300337
						1.0	2.0	–	0.300487
							5.0	–	0.300502

- The temperature distribution is enhanced as the thermal radiation and the convective heating at the bottom of the surface is boosted.
- The thermophoretic force and the activation energy are found to have strong influence on rising the species concentration far away from the sheet. However, the impact is getting reversed near the sheet.
- The skin friction coefficients are uplifted with the increase of unsteadiness and the magnetic impact.
- There is an enhancement in heat transfer rate at the surface for growing value of Biot number and thermal radiation parameter.
- Rate of mass transfer at the wall is improved as the values of thermophoresis and the activation energy parameters increase.

Acknowledgments

The first two authors are grateful to National Institute of Technology Meghalaya for providing financial assistance and research facilities.

16

References

[1] Hayat T, Hussain M, Nadeem S and Mesloub S (2011) Falkner-Skan wedge flow of a power-law fluid with mixed convection and porous medium, Comput. Fluids, 49(1):22-28

[2] Nourazar SS, Matin MH and Simiari M (2011) The HPM applied to MHD nanofluid flow over a horizontal stretching plate, J. Appl. Math., 2011:876437

[3] Makanda G, Makinde OD and Sibanda P (2013) Natural convection of viscoelastic fluid from a cone embedded in a porous medium with viscous dissipation, Math. Probl. Eng., 2013:934712

[4] Sheikholeslami M, Abelman S and Ganji DD (2014) Numerical simulation of MHD nanofluid flow and heat transfer considering viscous dissipation, Int. J. Heat Mass Transf., 79:212-222

[5] Srinivasacharya D, Mendu U and Venumadhav K (2015) MHD boundary layer flow of a nanofluid past a wedge, Procedia Eng., 127:1064-1070

[6] Crane LJ (1970) Flow past a stretching plate, J. Appl. Math. Phys., 21:645-647

[7] Azimi M, Ganji DD and Abbassi F (2012) Study on MHD viscous flow over a stretching sheet using DTM-Pade technique, Mod. Mech. Eng., 2(4):126-129

[8] Dessie H and Kishan N (2014) MHD effects on heat transfer over stretching sheet embedded in porous medium with variable viscosity, viscous dissipation and heat source/sink, Ain Shams Eng. J., 5(3):967-977

[9] Mishra SR, Baag S, Dash GC and Acharya MR (2020) Numerical approach to MHD flow of power-law fluid on a stretching sheet with non-uniform heat source, Nonlinear Eng., 9(1):81-93

[10] Daniel YS and Daniel SK (2015) Effects of buoyancy and thermal radiation on MHD flow over a stretching porous sheet using homotopy analysis method, Alexandria Eng. J., 54(3):705-712

[11] Kumbhakar B and Rao PS (2015) Dissipative Boundary Layer Flow over a Nonlinearly Stretching Sheet in the Presence of Magnetic Field and Thermal Radiation, Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci., 85(1):117-125

[12] Kho YB, Hussanan A, Mohamed MKA, Sarif NM, Ismail Z and Salleh MZ (2017) Thermal radiation effect on MHD flow and heat transfer analysis of Williamson nanofluid past over a stretching sheet with constant wall temperature, J. Phys. Conf. Ser., 890:012034

[13] Ishaq M, Ali G, Shah Z, Islam S and Muhammad S (2018) Entropy generation on nanofluid thin film flow of Eyring-Powell fluid with thermal radiation and MHD effect on an unsteady porous stretching sheet, Entropy, 20(6):412

[14] Alharbi SO, Dawar A, Shah Z, Khan W, Idrees M, Islam S and Khan I (2018) Entropy generation in MHD Eyring-Powell fluid flow over an unsteady oscillatory porous stretching surface under the impact of thermal radiation and heat source/sink, Appl. Sci., 8(12):2588

[15] Kumar MA, Reddy YD, Rao VS and Goud BS (2021) Thermal radiation impact on MHD heat transfer natural convective nanofluid flow over an impulsively started vertical plate, Case Stud. Therm. Eng., 24:100826

[16] Das K (2012) Influence of thermophoresis and chemical reaction on MHD micropolar fluid flow with variable fluid properties, Int. J. Heat Mass Transf., 55(23-24):7166-7174

[17] Sheikh M and Abbas Z (2015) Effects of thermophoresis and heat generation/absorption on MHD flow due to an oscillatory stretching sheet with chemically reactive species, J. Magn. Magn. Mater., 396:204-213

[18] Tarakaramu N and Narayan PVS (2017) Unsteady MHD nanofluid flow over a stretching sheet with chemical reaction, IOP Conf. Ser.: Mater. Sci. Eng., 263:062030

[19] Kumar RVMSSK, Kumar GV, Raju CSK, Shehzad SA and Varma SVK (2018) Analysis of Arrhenius activation energy in magnetohydrodynamic Carreau fluid flow through improved theory of heat diffusion and binary chemical reaction, J. Phys. Commun., 2:035004

[20] Khan U, Zaib A, Khan I and Nisar KS (2019) Activation energy on MHD flow of titanium alloy (Ti_6Al_4V) nanoparticle along with a cross flow and streamwise direction with binary chemical reaction and non-linear radiation: Dual Solutions, J Mater Res Technol., https://doi.org/10.1016/j.jmrt.2019.10.044

[21] Chu YM, Khan MI, Khan NB, Kadrye S, Khan SU, Tlili I and Nayak MK (2020) Significance of activation energy, bio-convection and magnetohydrodynamic in flow of third grade fluid (non-Newtonian) towards stretched surface: A Buongiorno model analysis, Int. Commun. Heat Mass Transf., 118:104893

[22] Zheng L, Liu Y and Zhang X (2012) Slip effects on MHD flow of a generalized Oldroyd-B fluid with fractional derivative, Nonlinear Anal.: Real World Appl., 13(2):513-523

[23] Hayat T, Muhammad T, Shehzad SA and Alsaedi A (2017) On magnetohydrodynamic flow of nanofluid due to a rotating disk with slip effect: A numerical study, Comput. Methods Appl. Mech. Eng., 315:467-477

[24] Amanulla CH, Saleem S, Wakif A and AlQarni MM (2019) MHD Prandtl fluid flow past an isothermal permeable sphere with slip effects, Case Stud. Therm. Eng., 14:100447

[25] Ellahi R, Alamri SZ, Basit A and Majeed A (2018) Effects of MHD and slip on heat transfer boundary layer flow over a moving plate based on specific entropy generation, J. Taibah Univ. Sci., 12(4):476-482

[26] Khan SA, Nie Y and Ali B (2020) Multiple slip effects on MHD unsteady viscoelastic nanofluid flow over a permeable stretching sheet with radiation using the finite element method, SN Appl. Sci., 2:66

[27] Das M, Nandi S, Kumbhakar B and Seth GS (2021) Soret and Dufour effects on MHD nonlinear convective flow of tangent hyperbolic nanofluid over a bidirectional stretching sheet with multiple slips, J. Nanofluids, 10(2):200-213

[28] Nandi S and Kumbhakar B (2021) Viscous dissipation and chemical reaction effects on tangent hyperbolic nanofluid flow past a stretching wedge with convective heating and Naviers slip conditions, Iran. J. Sci. Technol. - Trans. Mech. Eng., https://doi.org/10.1007/s40997-021-00437-1

[29] Ramzan M, Bilal M and Chung JD (2017) Influence of homogeneous-heterogeneous reactions on MHD 3D Maxwell fluid flow with Cattaneo-Christov heat flux and convective boundary condition, J. Mol. Liq., 230:415-422

[30] Nayak MK, Shaw S, Pandey VS and Chamkha AJ (2018) Combined effects of slip and convective boundary condition on MHD 3D stretched flow of nanofluid through porous media inspired by non-linear thermal radiation, Indian J. Phys., 92:1017-1028

[31] Shah Z, Bonyah E, Islam S and Gul T (2019) Impact of thermal radiation on electrical MHD rotating flow of Carbon nanotubes over a stretching sheet, AIP Adv., 9(4):015115

[32] Kumar KA, Sugunamma V and Sandeep N (2020) Effect of thermal radiation on MHD Casson fluid flow over an exponentially stretching curved sheet, J. Therm. Anal. Calorim., 140:2377-2385

[33] Loganathan K, Alessa N, Namgyel N and Karthik TS (2021) MHD flow of thermally radiative Maxwell fluid past a heated stretching sheet with Cattaneo-Christov dual diffusion, J. Math., 2021:5562667

[34] Jamshed W and Nisar KS (2021) Computational single-phase comparative study of a Williamson nanofluid in a parabolic trough solar collector via the Keller box method, Int J Energy Res., https://doi.org/10.1002/er.6554

[35] Hayat T, Kiyani MZ, Alsaedi A, Khan MI and Ahmad I (2018) Mixed convective three-dimensional flow of Williamson nanofluid subject to chemical reaction, Int. J. Heat Mass Transf., 127(Part A):422-429

[36] Nandi S, Kumbhakar B, Seth GS and Chamkha AJ (2021) Features of 3D magneto-convective nonlinear radiative Williamson nanofluid flow with activation energy, multiple slips and Hall effect, Phys. Scr., 96:065206

[37] Fatunmbi EO and Adeniyan A (2020) Nonlinear thermal radiation and entropy generation on steady flow of magneto-micropolar fluid passing a stretchable sheet with variable properties, Results Eng., 6:100142

[38] Irfan M (2021) Study of Brownian motion and thermophoretic diffusion on non-linear mixed convection flow of Carreau nanofluid subject to variable properties, Surf. Interfaces, 23:100926

[39] Oyelakin IS, Mondal S and Sibanda P (2017) Unsteady MHD three-dimensional Casson nanofluid flow over a porous linear stretching sheet with slip condition, Front. Heat Mass Transf., 8, https://doi.org/10.5098/hmt.8.37

[40] Nadeem S, Haq RU, Akbar NS and Khan ZH (2013) MHD three-dimensional Casson fluid flow past a porous linearly stretching sheet, Alexandria Eng. J., 52:577-582

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 30, NO. 1, 2022