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Journal of Computational Analysis and Applications

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ON THE λ-DAEHEE POLYNOMIALS WITH q-PARAMETER

JIN-WOO PARK

ABSTRACT. In this paper, we consider the generalization of Daehee polynomials with q-parameter and investigate some properties of those polynomials.

1. INTRODUCTION

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completions of algebraic closure of $\mathbb{Q}_p$. The p-adic norm is defined $|p|_p = \frac{1}{p}$.

When one talks of q-extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. Throughout this paper, we use the notation :

$$[x]_q = \frac{1-q^x}{1-q}.$$

Note that $\lim_{q\to 1}[x]_q = x$ for each $x \in \mathbb{Z}_p$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the *p-adic invariant integral on* $\mathbb{Z}_p$ is defined by Kim as follows :

$$I(f) = \int_{\mathbb{Z}_p} f(x)\, d\mu_0(x) = \lim_{n\to\infty} \frac{1}{p^n} \sum_{x=0}^{p^n-1} f(x), \text{ (see [4, 5, 6]).} \tag{1.1}$$

Let f_1 be the translation of f with $f_1(x) = f(x+1)$. Then, by (1.1), we get

$$I(f_1) = I(f) + f'(0), \text{ where } f'(0) = \left.\frac{df(x)}{dx}\right|_{x=0}. \tag{1.2}$$

As it is well-known fact, the *Stirling number of the first kind* is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)\, x^l, \tag{1.3}$$

and the *Stirling number of the second kind* is given by the generating function to be

$$\left(e^t - 1\right)^m = m! \sum_{l=m}^{\infty} S_2(l,m) \frac{t^l}{l!}, \tag{1.4}$$

(see [1, 10]).

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Unsigned Stirling numbers of the first kind is given by

$$x^{\underline{n}} = x(x+1)\cdots(x+n-1) = \sum_{l=0}^{n} |S_1(n,l)|x^l. \tag{1.5}$$

Note that if we replace x to $-x$ in (1.3), then

$$\begin{aligned}(-x)_n =&(-1)^n x^{\underline{n}} = \sum_{l=0}^{n} S_1(n,l)(-1)^l x^l \\ =&(-1)^n \sum_{l=0}^{n} |S_1(n,l)|x^l.\end{aligned} \tag{1.6}$$

Hence $S_1(n,l) = |S_1(n,l)|(-1)^{n-l}$.

For $r \in \mathbb{N}$, the *Bernoulli polynomials of order* r are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}, \text{ (see [7, 8, 11])}. \tag{1.7}$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$ are called the *Bernoulli numbers of order* r, and in the special case, $r = 1$, $B_n^{(1)}(x) = B_n(x)$ are called the *ordinary Bernoulli polynomials.*

For $n \in \mathbb{N}$, let T_p be the p-adic locally constant space defined by

$$T_p = \bigcup_{n\geq 1} C_{p^n} = \lim_{n\to\infty} C_{p^n},$$

where $C_{p^n} = \left\{\omega|\omega^{p^n} = 1\right\}$ is the cyclic group of order p^n.

We assume that q is an indeterminate in $\mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$. Then we define the q-analogue of falling factorial sequence as follows :

$$(x)_{n,q} = x(x-q)(x-2q)\cdots(x-(n-1)q), \ (n \geq 1), \ (x)_{0,q} = 1.$$

Note that

$$\lim_{q\to 1}(x)_{n,q} = (x)_n = \sum_{l=0}^{n} S_1(n,l)x^l.$$

Recently, D. S. Kim and T. Kim introduced the *Daehee polynomials* as follows :

$$D_n(x) = \int_{\mathbb{Z}_p} (x+y)_n d\mu_0(y), (n \geq 0), \text{ (see [2, 5, 9])}. \tag{1.8}$$

When $x = 0$, $D_n = D_n(0)$ are called the *n's Daehee numbers.* From (1.8), we can derive the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}, \text{ (see [2])}. \tag{1.9}$$

In addition, D. S. Kim et. al. consider the *Daehee polynomials with q-parameter* which is defined by the generating function to be

$$\sum_{n=0}^{\infty} D_{n,q}\frac{t^n}{n!} = (1+qt)^{\frac{x}{q}} \frac{\log(1+qt)}{q\left((1+qt)^{\frac{1}{q}}-1\right)}, \text{ (see [3])}. \tag{1.10}$$

When $x = 0$, $D_{n,q} = D_{n,q}(0)$ are called the *Daehee numbers with q-parameter.*

In the viewpoint of generalization of the Daehee polynomials with q-parameter, we consider the λ-*Daehee polynomials with* q-*parameter* are defined to be

$$\sum_{n=0}^{\infty} D_{n,q}(\lambda|x)\frac{t^n}{n!} = (1+qt)^{\frac{x}{q}} \frac{\lambda \log(1+qt)}{q\left((1+qt)^{\frac{\lambda}{q}}-1\right)}. \tag{1.11}$$

When $x = 0$, $D_{n,q}(\lambda) = D_{n,q}(\lambda|0)$ are called the λ-*Daehee numbers with* q-*parameter*. In particular, the case $\lambda = 1$ is the Daehee polynomials with q-parameter.

In this paper, we give a p-adic integral representation of the λ- Daehee polynomials with q-parameter, which are called the Witt-type formula for the λ-Daehee polynomials with q-parameter. We can derive some interesting properties related to the λ-Daehee polynomials with q-parameter.

2. Witt-type formula for the n-th λ-Daehee polynomials with q-parameter

In this section, we assume that $t, q \in \mathbb{C}_p$ with $|t|_p < |q|_p p^{-\frac{1}{p-1}}$ and $\lambda \in \mathbb{Z}_p$. First, we consider the following integral representation associated with falling factorial sequences :

$$\int_{\mathbb{Z}_p} (x+\lambda y)_{n,q} d\mu_0(y), \text{ where } n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}. \tag{2.1}$$

By (2.1),

$$\begin{aligned}
\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x+\lambda y)_{n,q} d\mu_0(y)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} q^n \int_{\mathbb{Z}_p} \left(\frac{x+\lambda y}{q}\right)_n d\mu_0(y)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} q^n \int_{\mathbb{Z}_p} \binom{\frac{x+\lambda y}{q}}{n} d\mu_0(y) t^n \\
&= \int_{\mathbb{Z}_p} (1+qt)^{\frac{x+\lambda y}{q}} d\mu_0(y)
\end{aligned} \tag{2.2}$$

where $t \in \mathbb{C}_p$ with $|t|_p < |q|_p p^{-\frac{1}{p-1}}$. For $t \in \mathbb{C}_p$ with $|t|_p < |q|_p p^{-\frac{1}{p-1}}$, we get

$$\begin{aligned}
\int_{\mathbb{Z}_p} (1+qt)^{\frac{x+\lambda y}{q}} d\mu_0(y) &= (1+qt)^{\frac{x}{q}} \frac{\lambda \log(1+qt)}{q\left((1+qt)^{\frac{\lambda}{q}}-1\right)} \\
&= \sum_{n=0}^{\infty} D_{n,q}(\lambda|x)\frac{t^n}{n!}.
\end{aligned} \tag{2.3}$$

By (2.2) and (2.3), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$D_{n,q}(\lambda|x) = \int_{\mathbb{Z}_p} (x+\lambda y)_{n,q} d\mu_0(y).$$

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In (2.3), by replacing t by $\frac{1}{q}\left(e^t-1\right)$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{D_{n,q}(\lambda|x)}{q^n}\frac{\left(e^t-1\right)^n}{n!}&=e^{\frac{x}{q}t}\frac{\frac{\lambda}{q}t}{e^{\frac{\lambda}{q}t}-1}\\&=\sum_{n=0}^{\infty}B_n\left(\frac{x}{\lambda}\right)\frac{\lambda^n}{q^n}\frac{t^n}{n!},\end{aligned}\tag{2.4}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{D_{n,q}(\lambda|x)}{q^n}\frac{1}{n!}\left(e^t-1\right)^n&=\sum_{n=0}^{\infty}\frac{D_{n,q}(\lambda|x)}{q^n}\sum_{m=n}^{\infty}S_2(m,n)\frac{t^m}{m!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{m}\frac{D_{m,q}(\lambda|x)}{q^m}S_2(n,m)\right)\frac{t^n}{n!}.\end{aligned}\tag{2.5}$$

By (2.4) and (2.5), we obtain the following corollary.

Corollary 2.2. *For $n\geq 0$, we have*

$$B_n\left(\frac{x}{\lambda}\right)=\sum_{m=0}^{n}D_{m,q}(\lambda|x)q^{n-m}\lambda^{-n}S_2(n,m).$$

By the Theorem 2.1,

$$\begin{aligned}D_{n,q}(\lambda|x)&=\int_{\mathbb{Z}_p}(x+\lambda y)_{n,q}d\mu_0(y)\\&=q^n\int_{\mathbb{Z}_p}\left(\frac{x+\lambda y}{q}\right)_n d\mu_0(y)\\&=q^n\sum_{l=0}^{n}\frac{1}{q^l}S_1(n,l)\int_{\mathbb{Z}_p}(x+\lambda y)^l d\mu_0(y).\end{aligned}\tag{2.6}$$

By (1.2), we can derive easily that

$$\begin{aligned}\int_{\mathbb{Z}_p}e^{(x+\lambda y)t}d\mu_0(y)&=\frac{\lambda t}{e^{\lambda t}-1}e^{xt}=\sum_{n=0}^{\infty}B_n\left(\frac{x}{\lambda}\right)\frac{(\lambda t)^n}{n!}\\&=\sum_{l=0}^{\infty}\int_{\mathbb{Z}_p}(x+\lambda y)^l d\mu_0(y)\frac{t^l}{l!},\end{aligned}\tag{2.7}$$

and so

$$B_n\left(\frac{x}{\lambda}\right)=\int_{\mathbb{Z}_p}\left(\frac{x}{\lambda}+y\right)^n d\mu_0(y),\ (n\geq 0).\tag{2.8}$$

By (1.6), (2.7) and (2.8), we obtain the following corollary.

Corollary 2.3. *For $n\geq 0$, we have*

$$\begin{aligned}D_{n,q}(\lambda|x)&=\sum_{l=0}^{n}q^{n-l}S_1(n,l)\lambda^l B_l\left(\frac{x}{\lambda}\right)\\&=\sum_{l=0}^{n}|S_1(l,n)|(-q)^{n-l}\lambda^l B_l\left(\frac{x}{\lambda}\right).\end{aligned}$$

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From now on, we consider λ-*Daehee polynomials of order* $k(\in \mathbb{N})$ *with* q-*parameter*. λ-Daehee polynomials of order k with q-parameter are defined by the multivariant p-adic invariant integral on $\mathbb{Z}_p$:

$$D_{n,q}^{(k)}(\lambda|x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda(x_1 + \cdots + x_k) + x)_{n,q}\, d\mu_0(x_1) \cdots d\mu_0(x_k) \tag{2.9}$$

where n is an nonnegative integer and $k \in \mathbb{N}$. In the special case, $x = 0$, $D_{nq}^{(k)}(\lambda) = D_{n,q}^{(k)}(\lambda|0)$ are called the λ-*Daehee numbers of order* k *with* q-*parameter*.

From (2.9), we can derive the generating function of $D_{n,q}^{(k)}(x)$ as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty} D_{n,q}^{(k)}(\lambda|x)\frac{t^n}{n!} \\
&=\sum_{n=0}^{\infty} q^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{\frac{\lambda(x_1+\cdots+x_k)+x}{q}}{n} d\mu_0(x_1) \cdots d\mu_0(x_k) t^n \\
&=\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+qt)^{\frac{\lambda(x_1+\cdots+x_k)+x}{q}}\, d\mu_0(x_1) \cdots d\mu_0(x_k) \\
&=(1+qt)^{\frac{x}{q}} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+qt)^{\frac{\lambda(x_1+\cdots+x_k)}{q}}\, d\mu_0(x_1) \cdots d\mu_0(x_k) \\
&=(1+qt)^{\frac{x}{q}} \left(\frac{\lambda \log(1+qt)}{q\left((1+qt)^{\frac{\lambda}{q}} - 1\right)} \right)^k .
\end{aligned} \tag{2.10}$$

Note that, by (2.9),

$$\begin{aligned}
&D_{n,q}^{(k)}(\lambda|x) \\
=&q^n \sum_{m=0}^{n} \frac{S_1(n,m)}{q^m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda(x_1 + \cdots + x_k) + x)^m d\mu_0(x_1) \cdots d\mu_0(x_k).
\end{aligned} \tag{2.11}$$

Since

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(x_1+\cdots+x_k+x)t} d\mu_0(x_1) \cdots d\mu_0(x_k) \\
=&\left(\frac{t}{e^t - 1}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!},
\end{aligned}$$

we can derive easily

$$B_n^{(k)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_k + x)^n d\mu_0(x_1) \cdots d\mu_0(x_k). \tag{2.12}$$

Thus, by (2.11) and (2.12), we have

$$\begin{aligned}
D_{n,q}^{(k)}(\lambda|x) =&q^n \sum_{m=0}^{n} \frac{S_1(n,m)}{q^m} \lambda^m B_m^{(k)}\left(\frac{x}{\lambda}\right) \\
=&\sum_{m=0}^{n} q^{n-m} S_1(n,m) B_m^{(k)}\left(\frac{x}{\lambda}\right) \\
=&\sum_{m=0}^{n} |S_1(n,m)|(-q)^{n-m} B_m^{(k)}\left(\frac{x}{\lambda}\right).
\end{aligned} \tag{2.13}$$

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In (2.10), by replacing t by $\frac{1}{q}(e^t-1)$, we get

$$\begin{aligned}\sum_{n=0}^{\infty} D_{n,q}^{(k)}(\lambda|x)\frac{(e^t-1)^n}{q^n n!} &= e^{\frac{x}{q}t}\left(\frac{\frac{\lambda t}{q}}{e^{\frac{\lambda}{q}t}-1}\right)^k \\ &= \sum_{n=0}^{\infty}\lambda^n\frac{B_n^{(k)}\left(\frac{x}{\lambda}\right)}{q^n}\frac{t^n}{n!},\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{D_{n,q}^{(k)}(\lambda|x)}{q^n}\frac{1}{n!}(e^t-1)^n &= \sum_{n=0}^{\infty}\frac{D_{n,q}^{(k)}(\lambda|x)}{q^n}\sum_{l=n}^{\infty}S_2(l,n)\frac{t^l}{l!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\frac{D_{n,q}^{(k)}(\lambda|x)}{q^n}S_2(m,n)\right)\frac{t^m}{m!}.\end{aligned} \tag{2.15}$$

By (2.13), (2.14) and (2.15), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$ and $k \in \mathbb{N}$, we have*

$$\begin{aligned}D_{n,q}^{(k)}(\lambda|x) &= \sum_{m=0}^{n} q^{n-m}S_1(n,m)B_m^{(k)}\left(\frac{x}{\lambda}\right) \\ &= \sum_{m=0}^{n}|S_1(n,m)|(-q)^{n-m}B_m^{(k)}\left(\frac{x}{\lambda}\right),\end{aligned}$$

and

$$B_n^{(k)}\left(\frac{x}{\lambda}\right) = \lambda^{-n}\sum_{m=0}^{n}D_{m,q}^{(k)}(\lambda|x)q^{n-m}S_2(n,m).$$

Now, we consider the λ-*Daehee polynomials of the second kind with q-parameter* as follows :

$$\widehat{D}_{n,\xi,q}(\lambda|x) = \int_{\mathbb{Z}_p}(-\lambda y+x)_{n,q}\,d\mu_0(y),\ (n\geq 0). \tag{2.16}$$

In the special case, $x=0$, $\widehat{D}_{n,q}(\lambda) = \widehat{D}_{n,q}(\lambda|0)$ are called the λ-*Daehee numbers of the second kind with q-parameter.*

By (2.16), we have

$$\widehat{D}_{n,q}(\lambda|x) = q^n\int_{\mathbb{Z}_p}\left(\frac{-\lambda y+x}{q}\right)_n d\mu_0(y), \tag{2.17}$$

and so we can derive the generating function of $\widehat{D}_{n,q}(x)$ by (1.1) as follows :

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{D}_{n,q}(\lambda|x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}q^n\int_{\mathbb{Z}_p}\left(\frac{-\lambda y+x}{q}\right)_n d\mu_0(y)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}q^n\int_{\mathbb{Z}_p}\binom{\frac{-\lambda y+x}{q}}{n}d\mu_0(y)t^n \\ &= \int_{\mathbb{Z}_p}(1+qt)^{\frac{-\lambda y+x}{q}}\,d\mu_0(y) \\ &= (1+qt)^{\frac{x+\lambda}{q}}\frac{\lambda\log(1+qt)}{q\left((1+qt)^{\frac{\lambda}{q}}-1\right)}.\end{aligned} \tag{2.18}$$

From (1.3), (1.6) and (2.17), we get

$$\begin{aligned}\widehat{D}_{n,q}(\lambda|x) =&q^n \int_{\mathbb{Z}_p} \left(\frac{-\lambda y + x}{q}\right)_n d\mu_0(y)\\ =&q^n \int_{\mathbb{Z}_p} \sum_{l=0}^{n} \frac{S_1(n,l)}{q^l}(-\lambda y + x)^l d\mu_0(y)\\ =&\sum_{l=0}^{n} S_1(n,l)(-\lambda)^l \int_{\mathbb{Z}_p} \left(y - \frac{x}{\lambda}\right)^l d\mu_0(y) q^{n-l}\\ =&\sum_{l=0}^{n} S_1(n,l)(-\lambda)^l B_l\left(-\frac{x}{\lambda}\right) q^{n-l}\\ =&(-1)^n \sum_{l=0}^{n} |S_1(n,l)|\lambda^l B_l\left(-\frac{x}{\lambda}\right) q^{n-l}.\end{aligned} \tag{2.19}$$

By replacing qt to $e^t - 1$ in the equation (2.18), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \widehat{D}_{n,q}(\lambda|x)\frac{1}{n!}\left(e^t - 1\right)^n =&\frac{\frac{\lambda}{q}t}{q\left(e^{\frac{\lambda}{q}t} - 1\right)} e^{\frac{(x+\lambda)t}{q}}\\ =&\sum_{n=0}^{\infty} B_n\left(1 + \frac{x}{\lambda}\right)\lambda^n q^{-n}\frac{t^n}{n!},\end{aligned} \tag{2.20}$$

and, by (1.4),

$$\sum_{n=0}^{\infty} \widehat{D}_{n,q}(\lambda|x)\frac{1}{n!}\left(e^t - 1\right)^n = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} \widehat{D}_{m,q}(\lambda|x)S_2(n,m)\right)\frac{t^m}{m!}. \tag{2.21}$$

Note that , by (1.10), it is easy to show that $B_n(-x) = (-1)^n B_n(x+1)$. Thus, from (2.19), (2.20) and (2.21), we have the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$\begin{aligned}\widehat{D}_{n,q}(\lambda|x) =&\sum_{l=0}^{n} S_1(n,l)(-\lambda)^l B_l\left(-\frac{x}{\lambda}\right) q^{n-l}\\ =&(-1)^n \sum_{l=0}^{n} |S_1(n,l)|\lambda^l q^{n-l} B_l\left(-\frac{x}{\lambda}\right).\end{aligned}$$

and

$$\lambda^n B_n\left(1 + \frac{x}{\lambda}\right) = q^n \sum_{m=0}^{n} \widehat{D}_{m,q}(\lambda|x)S_2(n,m).$$

By Theorem 2.5, we obtain the following corollary.

Corollary 2.6. *For $n \geq 0$,*

$$\widehat{D}_{n,q}(\lambda|x) = q^n \sum_{l=0}^{n}\sum_{m=0}^{l} \widehat{D}_{m,q}(\lambda|x)S_1(n,l)S_2(l,m). \tag{2.22}$$

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Now, we observe that

$$\begin{aligned} q^{-n}(-1)^n \frac{D_{n,q}(\lambda|x)}{n!} =&(-1)^n \int_{\mathbb{Z}_p} \binom{\frac{x+\lambda y}{q}}{n} d\mu_0(y) \\ =& \int_{\mathbb{Z}_p} \binom{-\frac{x+\lambda y}{q}+n-1}{n} d\mu_0(y) \\ =& \sum_{m=1}^{n} \binom{n-1}{n-m} \int_{\mathbb{Z}_p} \binom{\frac{-x-\lambda y}{q}}{m} d\mu_0(y) \\ =& \sum_{m=1}^{n} \binom{n-1}{n-m} \frac{q^{-n}\widehat{D}_{m,q}(\lambda|-x)}{m!}, \end{aligned} \tag{2.23}$$

and, by the similar method to (2.23), we have

$$q^{-n}(-1)^n \frac{\widehat{D}_{n,q}(\lambda|x)}{n!} = \sum_{m=1}^{n} \binom{n-1}{n-m} \frac{D_{n,q}(\lambda|-x)}{m!} q^{-n}. \tag{2.24}$$

Hence, by (2.23) and (2.24), we obtain the following theorem.

Theorem 2.7. *For $n \geq 1$, we have*

$$q^{-n}(-1)^n \frac{D_{n,q}(\lambda|x)}{n!} = \sum_{m=1}^{n} \binom{n-1}{n-m} \frac{\widehat{D}_{m,q}(\lambda|-x)}{m!} q^{-n}$$

and

$$q^{-n}(-1)^n \frac{\widehat{D}_{n,q}(\lambda|x)}{n!} = \sum_{m=1}^{n} \binom{n-1}{n-m} \frac{D_{n,q}(\lambda|-x)}{m!} q^{-n}.$$

Now, we consider *higher-order λ-Daehee polynomials of second kind with q-parameter*. Higher-order λ-Daehee polynomials of second kind with q-parameter are defined by the multivariant p-adic invariant integral on $\mathbb{Z}_p$:

$$\widehat{D}^{(k)}_{n,\xi,q}(\lambda|x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (-\lambda(x_1+\cdots+x_k)+x)_{n,q} d\mu_0(x_1)\cdots d\mu_0(x_k) \tag{2.25}$$

where n is an nonnegative integer and $k \in \mathbb{N}$. In the special case, $x=0$, $\widehat{D}^{(k)}_{n,q}(\lambda) = \widehat{D}^{(k)}_{n,q}(\lambda|0)$ are called the *higher-order λ-Daehee numbers of second kind with q-parameter*.

From (2.25), we can derive the generating function of $\widehat{D}^{(k)}_{n,q}(\lambda|x)$ as follows:

$$\begin{aligned} &\sum_{n=0}^{\infty} \widehat{D}^{(k)}_{n,q}(\lambda|x) \frac{t^n}{n!} \\ =& \sum_{n=0}^{\infty} q^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{\frac{-\lambda(x_1+\cdots+x_k)+x}{q}}{n} d\mu_0(x_1)\cdots d\mu_0(x_k) t^n \\ =& \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+qt)^{\frac{-\lambda(x_1+\cdots+x_k)+x}{q}} d\mu_0(x_1)\cdots d\mu_0(x_k) \\ =& (1+qt)^{\frac{x+\lambda k}{q}} \left(\frac{\lambda \log(1+qt)}{q\left((1+qt)^{\frac{\lambda}{q}}-1\right)} \right)^k. \end{aligned} \tag{2.26}$$

By (2.25),

$$\begin{aligned}
&\widehat{D}_{n,q}^{(k)}(\lambda|x)\\
=&q^n\sum_{m=0}^{n}\frac{S_1(n,m)}{q^m}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-\lambda(x_1+\cdots+x_k)+x)^m d\mu_0(x_1)\cdots d\mu_0(x_k)\\
=&q^n\sum_{m=0}^{n}\frac{S_1(n,m)}{q^m}(-\lambda)^m\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(x_1+\cdots+x_k-\frac{x}{\lambda}\right)^m d\mu_0(x_1)\cdots d\mu_0(x_k)\\
=&q^n\sum_{m=0}^{n}\frac{S_1(n,m)}{q^m}(-\lambda)^m B_m^{(k)}\left(-\frac{x}{\lambda}\right)\\
=&(-1)^n\sum_{m=0}^{n}q^{n-m}\lambda^m|S_1(n,m)|B_m^{(k)}\left(-\frac{x}{\lambda}\right).
\end{aligned} \tag{2.27}$$

From (1.10), we know that $B_n^{(k)}(-x)=(-1)^nB_n^{(k)}(k+x)$. Hence, by (2.27), we obtain the following theorem.

Theorem 2.8. *For $n\geq 0$, we have*

$$\begin{aligned}
\widehat{D}_{n,q}^{(k)}(\lambda|x)=&\sum_{m=0}^{n}S_1(n,m)q^{n-m}(-\lambda)^mB_m^{(k)}\left(-\frac{x}{\lambda}\right)\\
=&(-1)^n\sum_{m=0}^{n}(-\lambda)^m q^{n-m}|S_1(n,m)|B_m^{(k)}\left(k+\frac{x}{\lambda}\right).
\end{aligned}$$

In (2.26), by replacing t by $\frac{1}{q}(e^t-1)$, we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\widehat{D}_{n,q}^{(k)}(\lambda|x)\frac{(e^t-1)^n}{q^n n!}=&e^{\frac{t}{q}(x+\lambda k)}\left(\frac{\frac{\lambda t}{q}}{e^{\frac{\lambda t}{q}}-1}\right)^k\\
=&\sum_{n=0}^{\infty}\frac{\lambda^nB_n^{(k)}\left(\frac{x}{\lambda}+k\right)}{q^n}\frac{t^n}{n!},
\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{\widehat{D}_{n,q}^{(k)}(\lambda|x)}{q^n}\frac{1}{n!}\left(e^t-1\right)^n=&\sum_{n=0}^{\infty}\frac{\widehat{D}_{n,q}^{(k)}(\lambda|x)}{q^n}\sum_{l=n}^{\infty}S_2(l,n)\frac{t^l}{l!}\\
=&\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\frac{\widehat{D}_{n,q}^{(k)}(\lambda|x)}{q^n}S_2(m,n)\right)\frac{t^m}{m!}.
\end{aligned} \tag{2.29}$$

By (2.28) and (2.29), we obtain the following theorem.

Theorem 2.9. *For $n\geq 0$ and $k\in\mathbb{N}$, we have*

$$B_n^{(k)}\left(\frac{x}{\lambda}+k\right)=\lambda^{-n}\sum_{m=0}^{n}\widehat{D}_{m,q}^{(k)}(\lambda|x)q^{n-m}S_2(n,m).$$

By Theorem 2.8 and Theorem 2.9, we obtain the following corollary.

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Corollary 2.10. *For $n \geq 0$, we have*

$$\widehat{D}_{n,q}^{(k)}(\lambda|x) = \sum_{m=0}^{n} \sum_{l=0}^{m} \widehat{D}_{l,q}^{(k)}(\lambda|x) q^{n-l} S_1(n,m) S_2(m,l).$$

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Stability of ternary quadratic derivation on ternary Banach algebras: revisited

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Abstract. In [6], Shagholi et al. defined ternary quadratic derivations on ternary Banach algebras and proved the Hyers-Ulam stability of ternary quadratic derivations on ternary Banach algebras. But the definition is not well-defined and so the proofs of the main results are wrong.

In this paper, we correct the definition of ternary quadratic derivation and the proofs of the main results.

1. Introduction

The study of stability problems for functional equations is related to a question of Ulam [7] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [3]. Subsequently, the result of Hyers was generalized by Aoki [1] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic mapping. The Hyers-Ulam stability problem for the quadratic functional equation (1.1) was proved by Skof for mappings $f: A \to B$, where A is a normed space and B is a Banach space (see [5]).

In [6], Shagholi et al. defined a ternary quadratic derivation D from a ternary Banach algebra A into a ternary Banach algebra B such that

$$D[x,y,z]=[D(x),y^2,z^2]+[x^2,D(y),z^2]+[x^2,y^2,D(z)]$$

for all $x,y,z\in A$. But x^2,y^2,z^2 are not defined and the brackets of the right side are not defined, since A is not an algebra and $D(x)\in B$ and $y^2,z^2\in A$. So we correct them as follows.

Definition 1.1. Let A be an algebra and ternary Banach algebra with norm $\|\cdot\|$. A mapping $D: A\to A$ is called a ternary quadratic derivation if

(1) D is a quadratic mapping,

(2) $D[x,y,z]=[D(x),y^2,z^2]+[x^2,D(y),z^2]+[x^2,y^2,D(z)]$ for all $x,y,z\in A$.

In this paper, the proofs of the main results given in [6] are corrected.

2. Stability of ternary quadratic derivations

Let A be an algebra and ternary Banach algebra with norm $\|\cdot\|$.

Theorem 2.1. *Let $f: A\to A$ be a mapping for which there exists a function $\phi: A\times A\times A\to[0,\infty)$ such that*

$$\tilde{\phi}(x,y,z):=\sum_{j=0}^{\infty}\frac{1}{4^j}\phi(2^jx,2^jy,2^jz)<\infty \tag{2.1}$$

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\|\le\phi(x,y,0), \tag{2.2}$$

$$\|f([x,y,z])-[f(x),y^2,z^2]-[x^2,f(y),z^2]-[x^2,y^2,f(z)])\|\le\phi(x,y,z) \tag{2.3}$$

for all $x,y,z\in A$. Then there exists a unique ternary quadratic derivation $D: A\to A$ such that

$$\|f(x)-D(x)\|\le\frac{1}{4}\tilde{\phi}(x,x,0), \tag{2.4}$$

for all $x\in A$.

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Proof. Putting $x = y = 0$ in (2.2), we get $f(0) = 0$. If we replace y in (2.2) by x and multiply both sides of (2.2) by $\frac{1}{4}$, we get

$$\|\frac{f(2x)}{4} - f(x)\| \le \frac{\phi(x,x,0)}{4} \tag{2.5}$$

for all $x \in A$. Now we use the Rassias' method on inequality (2.5) (see [2]). One can use induction on n to show that

$$\|\frac{f(2^n x)}{2^{2n}} - f(x)\| \le \frac{1}{4}\sum_{j=0}^{n-1} \frac{\phi(2^j x, 2^j x, 0)}{4^j} \tag{2.6}$$

for all $x \in A$ and all nonnegative integers n. Hence

$$\|\frac{f(2^{n+m} x)}{2^{2(n+m)}} - \frac{f(2^m x)}{2^{2m}}\| \le \frac{1}{4}\sum_{j=m}^{n+m-1} \frac{\phi(2^j x, 2^j x, 0)}{4^j}$$

for all nonnegative integers n and m with $n \ge m$ and all $x \in A$. It follows from (2.1) that the sequence $\{\frac{f(2^n x)}{2^{2n}}\}$ is Cauchy. Due to the completeness of A, this sequence is convergent. So one can define the mapping $D : A \to A$ by

$$D(x) := \lim_{n\to\infty} \frac{f(2^n x)}{2^{2n}} \tag{2.7}$$

for all $x \in A$. Replacing x, y by $2^n x, 2^n y$, respectively, in (2.2) and multiplying both sides of (2.2) by $\frac{1}{2^{2n}}$, we get

$$\begin{aligned}
&\|D(x+y) + D(x-y) - 2D(x) - 2D(y)\| \\
&= \lim_{n\to\infty} \frac{1}{2^{2n}} \|f(2^n(x+y)) + f(2^n(x-y)) - 2f(2^n x) - 2f(2^n y)\| \\
&\le \lim_{n\to\infty} \frac{\phi(2^n x, 2^n y, 0)}{2^{2n}} = 0
\end{aligned}$$

for all $x, y \in A$ and all nonnegative integers n. So

$$D(x+y) + D(x-y) = 2D(x) + 2D(y)$$

for all $x, y \in A$. Moreover, it follows from (2.6) and (2.7) that

$$\|f(x) - D(x)\| \le \frac{1}{4}\tilde{\phi}(x,x,0)$$

for all $x \in A$. It follows from (2.3) we get

$$\begin{aligned}
&\|D([x,y,z]) - [D(x), y^2, z^2] - [x^2, D(y), z^2] - [x^2, y^2, D(z)]\| \\
&\le \lim_{n\to\infty} \frac{1}{4^{3n}} \|f([2^n x, 2^n y, 2^n z]) - [f(2^n x), (2^n y)^2, (2^n z)^2] - [(2^n x)^2, f(2^n y), (2^n z)^2] - [(2^n x)^2, (2^n y)^2, f(2^n z)]\| \\
&\le \lim_{n\to\infty} \frac{\phi(2^n x, 2^n y, 2^n z)}{4^{3n}} \le \lim_{n\to\infty} \frac{\phi(2^n x, 2^n y, 2^n z)}{4^n} = 0
\end{aligned}$$

for all $x, y, z \in A$. So

$$D[x,y,z] = [D(x), y^2, z^2] + [x^2, D(y), z^2] + [x^2, y^2, D(z)]$$

for all $x, y, z \in A$.

Now, let $D' : A \to A$ be another ternary quadratic derivation satisfying (2.4). Then we have

$$\begin{aligned}
\|D(x) - D'(x)\| &= \frac{1}{2^{2n}} \|D(2^n x) - D'(2^n x)\| \\
&\le \frac{1}{2^{2n}} (\|D(2^n x) - f(2^n x)\|_B + \|f(2^n x) - D'(2^n x)\|) \\
&\le \frac{2}{2^{2n}} \phi(2^n x, 2^n x, 0)
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in A$. So we can conclude that $D(x) = D'(x)$ for all $x \in A$. This proves the uniqueness of D. Thus, the mapping $D : A \to A$ is a unique ternary quadratic derivation satisfying (2.4). □

Theorem 2.2. *Let $f : A \to A$ be a mapping for which there exists a function $\phi : A \times A \times A \to [0, \infty)$ satisfying* (2.2), (2.3) *and*

$$\sum_{j=0}^{\infty} 4^{3j} \phi(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}) < \infty \tag{2.8}$$

for all $x, y, z \in A$. Then there exists a unique ternary quadratic derivation $D : A \to A$ such that

$$\|f(x) - D(x)\| \le \tilde{\phi}(\frac{x}{2}, \frac{x}{2}, 0), \tag{2.9}$$

for all $x \in A$. Here,

$$\tilde{\phi}(x, y, z) := \sum_{j=0}^{\infty} 4^j \phi(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j})9$$

for all $x, y, z \in A$.

Proof. It follows from (2.5) that

$$\|f(x) - 4f(\frac{x}{2})\| \le \phi(\frac{x}{2}, \frac{x}{2}, 0)$$

for all $x \in A$.

By the same reasoning as in the proof od Theorem 2.1, one can define a quadartic unique mapping $D : A \to A$ by

$$D(x) := \lim_{n\to\infty} 2^{2n} f(\frac{x}{2^n}) \tag{2.10}$$

for all $x \in A$. It follows from (2.8) and (2.10) that

$$\begin{aligned}
&\|D([x, y, z]) - [D(x), y^2, z^2] - [x^2, D(y), z^2] - [x^2, y^2, D(z)]\| \\
&\le \lim_{n\to\infty} 4^{3n} \|f([\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}]) - [f(\frac{x}{2^n}), (\frac{y}{2^n})^2, (\frac{z}{2^n})^2] - [(\frac{x}{2^n})^2, f(\frac{y}{2^n}), (\frac{z}{2^n})^2] - [(\frac{x}{2^n})^2, (\frac{y}{2^n})^2, f(\frac{z}{2^n})]\| \\
&\le \lim_{n\to\infty} 4^{3n} \phi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}) = 0
\end{aligned}$$

for all $x, y, z \in A$. So

$$D[x, y, z] = [D(x), y^2, z^2] + [x^2, D(y), z^2] + [x^2, y^2, D(z)]$$

for all $x, y, z \in A$. Thus the mapping $D : A \to A$ is a unique ternary quadratic derivation satisfying (2.9). □

From Theorems 2.1 and 2.2, we obtain the following corollary concerning the Hyers-Ulam stability of the functional equation (1.1).

Corollary 2.3. *Let p and θ be nonnegative real numbers with $p \ne 2$, and let $f : A \to A$ be a mapping such that*

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \le \theta(\|x\|^p + \|y\|^p),$$

$$\|f([x, y, z]) - [f(x), y^2, z^2] - [x^2, f(y), z^2] - [x^2, y^2, f(z)])\| \le \theta(\|x\|^p + \|y\|^p + \|z\|^p),$$

for all $x, y, z \in A$. Then there exists a unique ternary quadratic derivation $D : A \to A$ such that

$$\|f(x) - D(x)\| \le \frac{2\theta}{4 - 2^p}\|x\|^p$$

holds for all $x \in X$, where $p < 2$, and the inequality

$$\|f(x) - D(x)\| \le \frac{2\theta}{2^p - 4}\|x\|^p$$

holds for all $x \in X$, where $p > 6$.

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SOME PROPERTIES OF MODULAR S-METRIC SPACES AND ITS FIXED POINT RESULTS

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Abstract. In this paper, we introduce modular S-metric spaces and deal with their some properties. We also prove some fixed point theorems on complete modular S-metric spaces.

1. Introduction

Fixed point theory in metric spaces begins with the Banach Contraction Principle which is published in 1922 [6]. Since it is simple and useful, it has become a very popular tool to solve existence problems in mathematical analysis. There are some authors introduced the generalization of metric spaces such as Gahler [16], which is called 2-metric space, and Dhage [14], which is called D-metric space. In 2013, Mustafa and Sims [24] found that the fundamental topology properties of the metric spaces are incorrect. They [25] introduced a generalization of metric spaces which is called G-metric spaces.

The concept of S-metric spaces was firstly introduced by Sedghi et al. [28] in 2012. Sedghi and Dung [29] proved a general fixed point theorem in S-metric spaces which is a generalization [[28], Theorem 3.1]. Gupta [17] introduced the concepts of cyclic contraction on S-metric space and proved some fixed point theorems on S-metric spaces. Chouhan [12] proved a common unique fixed point theorem for expansive mappings in S-metric space. Hieu et al. [18] gave a fixed point theorem for a class of maps depending on another map on S-metric spaces.

The notion of modular space was firstly introduced by Nakano [26] and developed by Koshi, Shimogaki, Yamamuro (see [22, 30]) and others. Recently, many researchers have been interested in fixed point of modular space. In 2008, Chistyakov [7] introduced the notion of modular metric space generated by F-modular and developed the theory of this space. He also defined the notion of a modular on an arbitrary set and the modular metric spaces in 2010 [8]. Abdou [1] studied and proved some new fixed points theorems for pointwise and asymptotic pointwise contraction mappings in modular metric spaces. Azadifer et. al. [3] introduced the notion of modular G-metric spaces and proved some fixed point theorems of contractive in this space. Many authors studied on modular metric spaces [4],[5],[10],[11],[19],[20],[21].

In this paper we introduce the concept of modular S-metric spaces and their properties. Then we give fixed point theorems for self mappings on complete modular S-metric spaces.

2. Preliminaries

Definition 2.1. [27]. A modular on a real linear space X is a functional $\rho : X \longrightarrow [0,\infty]$ satisfying the followings:

(A1) $\rho(0) = 0$;
(A2) If $x \in X$ and $\rho(\alpha x) = 0$ for all numbers $\alpha > 0$, then $x = 0$;
(A3) $\rho(-x) = \rho(x)$ for all $x \in X$;
(A4) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $x, y \in X$.

Let X be a non-empty set and $\lambda \in (0,\infty)$. We remark that the function $\omega : (0,\infty)\times X\times X \longrightarrow [0,\infty]$ is denoted by $\omega_\lambda(x, y) = \omega(\lambda, x, y)$ for all $\lambda > 0$ and $x, y \in X$.

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Definition 2.2. [8]. Let X be a non-empty set, a function $\omega : (0,\infty) \times X \times X \longrightarrow [0,\infty]$ is said to be a metric modular on X if satisfying, for all $x, y, z \in X$ the following conditions hold:

(i) $\omega_\lambda(x, y) = 0$ for all $\lambda > 0 \Leftrightarrow x = y$;
(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$;
(iii) $\omega_{\lambda+\mu}(x, y) \le \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0$.

Definition 2.3. [28] Let X be a non-empty set. An S-metric on X is a function $S : X^3 \to [0,\infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,

(i) $S(x, y, z) \ge 0$;
(ii) $S(x, y, z) = 0$ if and only if $x = y = z$;
(iii) $S(x, y, z) \le S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S-metric space.

3. Modular S-metric spaces

We define a new concept combining with S-metric and modular metric space.

Definition 3.1. Let X be a non-empty set. An modular S-metric on X is a function

$$s_\lambda : (0,\infty) \times X \times X \times X \to [0,\infty]$$

that satisfies the following conditions for all $x, y, z \in X$ and $\lambda > 0$:

$(S1)$ $s_\lambda(x, y, z) \ge 0$;
$(S2)$ $s_\lambda(x, y, z) = 0$ if and only if $x = y = z$;
$(S3)$ $s_{\lambda+\mu+\nu}(x, y, z) \le s_\lambda(x, x, a) + s_\mu(y, y, a) + s_\nu(z, z, a)$ for all $\lambda, \mu, \nu > 0$ and $a \in X$.

Example 3.2. **(1)** $s_\lambda(x, y, z) = 0$ if $x = y = z$ and $s_\lambda(x, y, z) = \infty$ if $x \ne y \ne z$.
(2) If S is an modular S-metric on X, we can get:

(a) $s_\lambda(x, y, z) = 0$ if $\lambda > S(x, y, z)$ and $s_\lambda(x, y, z) = \infty$ if $\lambda \le S(x, y, z)$.
(b) $s_\lambda(x, y, z) = 0$ if $\lambda \ge S(x, y, z)$ and $s_\lambda(x, y, z) = \infty$ if $\lambda < S(x, y, z)$.
(c) $s_\lambda(x, y, z) = \dfrac{S(x, y, z)}{\varphi(\lambda)}$; where $\varphi : (0,\infty) \to (0,\infty)$ is non-decreasing function.

Lemma 3.3. *If the function $0 < \lambda \to s_\lambda(x, y, z)$ is continuous on $(0,\infty)$ where $x, y, z \in X$, then we have $s_\lambda(x, x, y) = s_\lambda(y, y, x)$.*

Proof. There exists $\varepsilon > 0$ such that

$$s_\lambda(x, x, y) \le s_\varepsilon(x, x, x) + s_\varepsilon(x, x, x) + s_{\lambda-2\varepsilon}(y, y, x).$$

If we take limit as $\varepsilon \to 0$, we get $s_\lambda(x, x, y) \le s_\lambda(y, y, x)$. Similarly $s_\lambda(y, y, x) \le s_\lambda(x, x, y)$. So we get

$$s_\lambda(x, x, y) \le s_\lambda(y, y, x) \le s_\lambda(x, x, y)$$

and

$$s_\lambda(x, x, y) = s_\lambda(y, y, x).$$

□

Remark 3.4. The function $s_\lambda(x, y, z)$ for $\lambda > 0$ is non-increasing on $(0,\infty)$ where $x, y, z \in X$, if it is continuous on $(0,\infty)$. In fact if $0 < \nu < \mu < \lambda$, $(S3)$ implies

$$s_\lambda(x, x, y) \le s_{\lambda-\mu}(x, x, x) + s_{\mu-\nu}(x, x, x) + s_\nu(y, y, x)$$

and we have

$$s_\lambda(x, x, y) \le s_\nu(y, y, x)$$

from $(S2)$.

From Lemma 3.3, we conclude that $s_\lambda(x, x, y) \le s_\nu(x, x, y)$. From that inequality the function $s_\lambda(x, y, z)$ is non-increasing on $(0,\infty)$. It follows that at each point $\lambda > 0$ the right limit

$$s_{\lambda+0}(x, y, z) = \lim_{\mu\to\lambda+0} s_\mu(x, y, z)$$

and the left limit

$$s_{\lambda-0}(x,y,z)=\lim_{\varepsilon\to 0} s_{\lambda-\varepsilon}(x,y,z)$$

exist in $[0,\infty]$ and the following two inequalities hold:

$$s_{\lambda+0}(x,y,z)\le s_\lambda(x,y,z)\le s_{\lambda-0}(x,y,z).$$

Definition 3.5. Let s_λ be a modular S-metric on X. The binary relation $\overset{s}{\sim}$ on X defined for $x,y\in X$ by

$$x\overset{s}{\sim}y \Leftrightarrow \lim_{\lambda\to\infty} s_\lambda(x,x,y)=0 \tag{3.1}$$

is an equivalence relation. Indeed $x\overset{s}{\sim}x$ is clear by virtue of $(S2)$. From Lemma 3.3, we have

$$x\overset{s}{\sim}y \Leftrightarrow \lim_{\lambda\to\infty} s_\lambda(x,x,y)=0=\lim_{\lambda\to\infty} s_\lambda(y,y,x)\Leftrightarrow y\overset{s}{\sim}x.$$

If $x\overset{s}{\sim}y$ and $y\overset{s}{\sim}z$, we get $\lim_{\lambda\to\infty} s_\lambda(x,x,y)=0$ and $\lim_{\lambda\to\infty} s_\lambda(y,y,z)=0$. By $(S3)$ and Lemma 3.3, we conclude that

$$\begin{aligned}\lim_{\lambda\to\infty} s_{3\lambda}(x,x,z)\le&\lim_{\lambda\to\infty} s_\lambda(x,x,y)+\lim_{\lambda\to\infty} s_\lambda(x,x,y)+\lim_{\lambda\to\infty} s_\lambda(y,y,z)\\ =&0+0+0.\end{aligned}$$

It is clear that

$$\lim_{\lambda\to\infty} s_{3\lambda}(x,x,z)=0\Leftrightarrow x\overset{s}{\sim}z$$

by $(S1)$. The equivalence class of the element $x\in X$ in the quotient set $X/\overset{s}{\sim}$ is defined by

$$X_s\equiv X_s(x)=\{y\in X: y\overset{s}{\sim}x\}.$$

For $x_0\in X$, the set X_s^* is defined as follows:

$$X_s^*\equiv X_s^*(x_0)=\{x\in X:\exists\lambda=\lambda(x)>0 \text{ such that } s_\lambda(x,x,x_0)<\infty\}.$$

From Remark 3.4, the function

$$\widetilde{S}:(X/\overset{s}{\sim})\times(X/\overset{s}{\sim})\times(X/\overset{s}{\sim})\to[0,\infty]$$

given by

$$\widetilde{S}(X_s(x),X_s(x),X_s(y))=s_\lambda(x,x,y)$$

is well-defined and satisfies the axioms of S-metric.

Theorem 3.6. *If s_λ is a modular S-metric on X, then the modular set X_s is an modular S-metric space with S-metric given by*

$$S^\circ(x,x,y)=\inf\{\lambda>0: s_\lambda(x,x,y)\le\lambda\},$$

for all $x,y\in X_s$.

Proof. Since $x\overset{s}{\sim}y$, there exists $\lambda_0>0$ such that

$$s_\lambda(x,x,y)\le 1$$

for all $\lambda\ge\lambda_0$ by (3.1). Taking $\lambda_1=\max\{1,\lambda_0\}$, we get

$$s_{\lambda_1}(x,x,y)\le 1\le\lambda_1$$

which together with the definition of $S^\circ(x,x,y)$ gives

$$S^\circ(x,x,y)\le\lambda_1<\infty.$$

Given $x\in X_s$, $(S2)$ implies that $s_\lambda(x,x,x)=0$ for all $\lambda>0$, and so, $S^\circ(x,x,x)=0$. Let s_λ satisfy $(S2)$, $x,y\in X_s$ and $S^\circ(x,x,y)=0$. Then $s_\mu(x,x,y)$ does not exceed μ for all $\mu>0$. Hence for any $\lambda>0$ and $0<\mu<\lambda$, from Remark 3.4 we have $s_\lambda(x,x,y)\le s_\mu(x,x,y)\le\mu\to 0$ as $\mu\to+0$. It follows that $s_\lambda(x,x,y)=0$ for all $\lambda>0$. Thus axiom $(S2)$ implies $x=y$.

It is clear from $(S1)$ that $S^\circ(x,x,y) \geq 0$. Now we show the triangle inequality:

$$S^\circ(x,x,y) \leq 2S^\circ(x,x,z) + S^\circ(y,y,z)$$

for some $z \in X_s$. In fact by the definition of S° for any $\lambda > S^\circ(x,x,z)$ and $\mu > S^\circ(y,y,z)$, we find $s_\lambda(x,x,z) \leq \lambda$ and $s_\mu(y,y,z) \leq \mu$. As a result, we get

$$s_{2\lambda+\mu}(x,x,y) \leq 2s_\lambda(x,x,z) + s_\mu(y,y,z) \leq 2\lambda + \mu$$

by the axiom $(S3)$. It follows from the definition of S° that $S^\circ(x,x,y) \leq 2\lambda + \mu$ and it remains to pass limit as $\lambda \to S^\circ(x,x,z)$ and $\mu \to S^\circ(y,y,z)$. □

Theorem 3.7. *Let s_λ be a modular S-metric on a set X and*

$$S^1(x,x,y) = \inf\{\lambda + s_\lambda(x,x,y) : \lambda > 0\}$$

be defined for all $x, y \in X_s$. Then S^1 is an S-metric on X_s such that $S^\circ \leq S^1 \leq 2S^\circ$.

Proof. Since, for $x, y \in X_s$, the value $s_\lambda(x,x,y)$ is finite due to (3.1) for $\lambda > 0$ large enough, then the set $\{\lambda + s_\lambda(x,x,y) : \lambda > 0\} \subset \mathbb{R}^+$ is non-empty and bounded from below, therefore $S^1(x,x,y) \in \mathbb{R}^+$.

Since $s_\lambda(x,x,x) = 0$, then from the definition of S^1, $S^1(x,x,x) = \inf\{\lambda + \underbrace{s_\lambda(x,x,x)}_{0} : \lambda > 0\} = 0$.

Let s_λ satisfy $(S2)$, $x, y \in X_s$ and $S^1(x,x,y) = 0$. The equality $x = y$ will follow from $(S2)$ if we show that $s_\lambda(x,x,y) = 0$ for all $\lambda > 0$. On the contrary, suppose that $s_{\lambda_0}(x,x,y) > 0$ for some $\lambda_0 > 0$. Then for $\lambda \geq \lambda_0$ we find $\lambda + s_\lambda(x,x,y) \geq \lambda_0$, and if $0 < \lambda < \lambda_0$, then

$$0 < s_{\lambda_0}(x,x,y) \leq s_\lambda(x,x,y) \leq \lambda + s_\lambda(x,x,y)$$

from Remark 3.4. Thus, $\lambda + s_\lambda(x,x,y) \geq \lambda_1 = \min\{\lambda_0, s_{\lambda_0}(x,x,y)\}$ for all $\lambda > 0$. By the definition of S^1, $S^1(x,x,y) \geq \lambda_1 > 0$, which contradicts the assumption.

Now let us show that triangle inequality: $S^1(x,x,y) \leq 2S^1(x,x,z) + S^1(y,y,z)$. For any $\varepsilon > 0$ we find $\lambda = \lambda(\varepsilon) > 0$ and $\mu = \mu(\varepsilon) > 0$ such that

$$\lambda + s_\lambda(x,x,z) \leq S^1(x,x,z) + \varepsilon \quad \text{and} \quad \mu + s_\mu(y,y,z) \leq S^1(y,y,z) + \varepsilon$$

from the definition of S^1. Applying axiom $(S3)$,

$$\begin{aligned} S^1(x,x,y) \leq& (2\lambda + \mu) + s_{2\lambda+\mu}(x,x,y) \leq 2\lambda + \mu + 2s_\lambda(x,x,z) + s_\mu(y,y,z) \\ \leq& 2S^1(x,x,z) + 2\varepsilon + S^1(y,y,z) \end{aligned}$$

and it remains to take into account the arbitrariness of $\varepsilon > 0$.

Let us prove that metrics S° and S^1 are equivalent on X_s. In order to obtain the left-hand side inequality, suppose that $\lambda > 0$ is arbitrary. If $s_\lambda(x,x,y) \leq \lambda$, then the definition of S° implies $S^\circ(x,x,y) \leq \lambda$. If $s_\lambda(x,x,y) > \lambda$, then $S^\circ(x,x,y) \leq s_\lambda(x,x,y)$. Setting $\mu = s_\lambda(x,x,y)$ we find $\mu > \lambda$. Thus it follows from Remark 3.4 that

$$s_\mu(x,x,y) \leq s_\lambda(x,x,y) = \mu.$$

Hence

$$S^\circ(x,x,y) \leq \mu = s_\lambda(x,x,y).$$

Therefore for any $\lambda > 0$ we have

$$S^\circ(x,x,y) \leq \max\{\lambda, s_\lambda(x,x,y)\} \leq \lambda + s_\lambda(x,x,y).$$

Taking the infimum over all $\lambda > 0$, we get the inequality

$$S^\circ(x,x,y) \leq S^1(x,x,y)$$

To obtain the right-hand side inequality, we note that given $\lambda > 0$ such that $S^\circ(x,x,y) < \lambda$ by the definition of S°. We get $s_\lambda(x,x,y) \leq \lambda$. So $S^1(x,x,y) \leq \lambda + s_\lambda(x,x,y) \leq 2\lambda$. Passing to the limit as $\lambda \to S^\circ(x,x,y)$, we get

$$S^1(x,x,y) \leq 2S^\circ(x,x,y).$$

□

Theorem 3.8. *Let s_λ be a modular S-metric on a set X, $x, y \in X_s$ and $\lambda > 0$. We have*

(a) *If $S^\circ(x,x,y) < \lambda$, then $s_\lambda(x,x,y) \le S^\circ(x,x,y) < \lambda$.*
(b) *If $s_\lambda(x,x,y) = \lambda$, then $S^\circ(x,x,y) = \lambda$.*
(c) *If $\lambda = S^\circ(x,x,y) > 0$, then $s_{\lambda+0}(x,x,y) \le \lambda \le s_{\lambda-0}(x,x,y)$.*
(d) *If the function $\mu \to s_\mu(x,x,y)$ is continuous from the right on $(0,\infty)$, then along with $(a)-(c)$ we have:*

$$S^\circ(x,x,y) \le \lambda \Leftrightarrow s_\lambda(x,x,y) \le \lambda.$$

(e) *If the function $\mu \to s_\mu(x,x,y)$ is continuous from the left on $(0,\infty)$, then along with $(a)-(c)$ we have:*

$$S^\circ(x,x,y) < \lambda \Leftrightarrow s_\lambda(x,x,y) < \lambda.$$

(f) *If the function $\mu \to s_\mu(x,x,y)$ is continuous on $(0,\infty)$, then along with $(a)-(e)$ we have:*

$$S^\circ(x,x,y) = \lambda \Leftrightarrow s_\lambda(x,x,y) = \lambda.$$

Proof. (a) For any $\mu > 0$ such that $S^\circ(x,x,y) < \mu < \lambda$ by the definition of S° and Remark 3.4, we have $s_\mu(x,x,y) \le \mu$ and $s_\lambda(x,x,y) \le s_\mu(x,x,y)$. Hence $s_\lambda(x,x,y) \le \mu$ and it remains to pass to the limit as $\mu \to S^\circ(x,x,y)$.

(b) By the definition, $S^\circ(x,x,y) \le \lambda$ and item (a) implies $S^\circ(x,x,y) = \lambda$.

(c) For any $\mu > \lambda = S^\circ(x,x,y)$, the definition of S° implies $s_\mu(x,x,y) \le \mu$ and so

$$s_{\lambda+0}(x,x,y) = \lim_{\mu\to\lambda+0} s_\mu(x,x,y) \le \lim_{\mu\to\lambda+0} \mu = \lambda.$$

For any $0 < \mu < \lambda$ we find $s_\mu(x,x,y) > \mu$ and so

$$s_{\lambda-0}(x,x,y) = \lim_{\mu\to\lambda-0} s_\mu(x,x,y) \ge \lim_{\mu\to\lambda-0} \mu = \lambda.$$

(d) The sufficient condition follows from the definition of S°. Let us prove the reverse implication. If $S^\circ(x,x,y) < \lambda$, then by virtue of item (a), $s_\lambda(x,x,y) < \lambda$ and if $S^\circ(x,x,y) = \lambda$, then

$$s_\lambda(x,x,y) = s_{\lambda+0}(x,x,y) \le \lambda$$

which is a consequence of the continuity from the right of the function $\mu \to s_\mu(x,x,y)$ and item (c).

(e) By item (a), it suffices to prove the sufficient condition. The definition of S° gives $S^\circ(x,x,y) \le \lambda$ but if $S^\circ(x,x,y) = \lambda$, then by item (c) we would have

$$s_\lambda(x,x,y) = s_{\lambda-0}(x,x,y) \ge \lambda$$

which contradicts the assumption.

(f) Sufficient condition follows from (b). For the reverse asertion the two inequalities

$$s_\lambda(x,x,y) \le \lambda \le s_\lambda(x,x,y)$$

follows from (c). □

Definition 3.9. Let s_λ be a modular S-metric on a set X.

(1) A sequence $(x_n) \subset X_s^*$ converges to $x \in X_s^*$ if $s_\lambda(x_n,x_n,x) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we have $s_\lambda(x_n,x_n,x) < \varepsilon$. We write $x_n \to x$.
(2) A sequence $(x_n) \subset X_s^*$ is a s-Cauchy if $s_\lambda(x_n,x_n,x_m) \to 0$ as $m,n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we have $s_\lambda(x_n,x_n,x_m) < \varepsilon$.
(3) The modular S-metric space X_s^* is s-complete if every s-Cauchy is a s-convergent in X_s^*.

Lemma 3.10. *Let s_λ be a modular S-metric on a set X. If $x_n \to x$ and $y_n \to y$, then*

$$s_\lambda(x_n,x_n,y_n) \to s_\lambda(x,x,y).$$

Proof. Since $x_n \to x$ and $y_n \to y$, then for each $\varepsilon > 0$ there exist $n_1, n_2 \in \mathbb{N}$ such that

$$\begin{aligned}\forall n \geq & n_1,\ s_\lambda(x_n, x_n, x) < \varepsilon \\ \forall n \geq & n_2,\ s_\lambda(y_n, y_n, y) < \varepsilon.\end{aligned}$$

Without loss of generality we can assume

$$\begin{aligned}\forall n \geq & n_1, s_\delta(x_n, x_n, x) < \varepsilon(\delta) = \frac{\varepsilon}{4} \\ \forall n \geq & n_2, s_\delta(y_n, y_n, y) < \varepsilon(\delta) = \frac{\varepsilon}{4}.\end{aligned}$$

If we set $n_0 = \max\{n_1, n_2\}$, therefore for every $n \geq n_0$ we get

$$\begin{aligned}s_\lambda(x_n, x_n, y_n) \leq & 2s_\delta(x_n, x_n, x) + s_{\lambda-2\delta}(y_n, y_n, x) \\ \leq & 2s_\delta(x_n, x_n, x) + 2s_\delta(y_n, y_n, y) + s_{\lambda-4\delta}(x, x, y)\end{aligned}$$

for $\lambda > \delta > 0$ by triangle inequality. If we take $\delta \to 0$, we have

$$\begin{aligned}s_\lambda(x_n, x_n, y_n) \leq & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + s_\lambda(x, x, y) \\ s_\lambda(x_n, x_n, y_n) \leq & \varepsilon + s_\lambda(x, x, y)\end{aligned}$$

$$s_\lambda(x_n, x_n, y_n) - s_\lambda(x, x, y) \leq \varepsilon.$$

On the other hand we get

$$\begin{aligned}s_\lambda(x, x, y) \leq & 2s_\delta(x, x, x_n) + s_{\lambda-2\delta}(y, y, x_n) \\ \leq & 2s_\delta(x, x, x_n) + 2s_\delta(y, y, y_n) + s_{\lambda-4\delta}(x_n, x_n, y_n).\end{aligned}$$

From Lemma 3.3 and taking the limit as $\delta \to 0$ we have:

$$\begin{aligned}s_\lambda(x, x, y) \leq & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + s_\lambda(x_n, x_n, y_n) \\ \leq & \varepsilon + s_\lambda(x_n, x_n, y_n)\end{aligned}$$

$$s_\lambda(x, x, y) - s_\lambda(x_n, x_n, y_n) \leq \varepsilon.$$

So we get from that inequalities $|s_\lambda(x_n, x_n, y_n) - s_\lambda(x, x, y)| < \varepsilon$, that is, $s_\lambda(x_n, x_n, y_n) \to s_\lambda(x, x, y)$. □

4. Fixed Point Theorems

In this section we introduce some fixed point theorems on modular S-metric space.

Definition 4.1. Let s_λ be a modular S-metric on a set X. A map $T : X_s^* \to X_s^*$ is said to be a s-contraction if there exists a constant $0 \leq k < 1$ such that

$$s_\lambda(Tx, Tx, Ty) \leq k s_\lambda(x, x, y)$$

for all $x, y \in X$.

Corollary 4.2. *Let X_s^*, Y_s^* modular S-metric spaces and $f : X_s^* \to Y_s^*$ be a map. Then f is continuous at $x \in X_s^*$ if and only if $f(x_n) \to f(x)$ where $x_n \to x$.*

Theorem 4.3. *Let X_s^* be a s-complete and $T : X_s^* \to X_s^*$ be s-contraction. Then T has a unique fixed point $u \in X_s^*$.*

Proof. First, we show uniqueness. Suppose that there exist $x, y \in X_s^*$ with $x = Tx$ and $y = Ty$. Then

$$s_\lambda(x, x, y) = s_\lambda(Tx, Tx, Ty) \leq k s_\lambda(x, x, y).$$

Therefore $s_\lambda(x, x, y) = 0$.

To show the existence, we select $x \in X_s^*$ and show that $(T^n x)$ is a Cauchy sequence. For $n = 0, 1, 2, \cdots$, we get by induction

$$\begin{aligned} s_\lambda(T^n x, T^n x, T^{n+1} x) &\le k s_\lambda(T^{n-1} x, T^{n-1} x, T^n x) \\ &\vdots \\ &\le k^n s_\lambda(x, x, Tx). \end{aligned}$$

Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} s_\lambda(T^n x, T^n x, T^{n+1} x) = 0.$$

Thus there exists $\varepsilon > 0$ such that

$$s_\lambda(T^n x, T^n x, T^{n+1} x) \le \varepsilon.$$

Without loss of generality, we can assume that there exists $\frac{\varepsilon}{m-n}$ for $\frac{\lambda}{m-n}$ such that

$$\begin{aligned} s_\lambda(T^n x, T^n x, T^m x) &\le 2\sum_{i=n}^{m-2} s_{\frac{\lambda}{m-n}}(T^i x, T^i x, T^{i+1} x) + s_{\frac{\lambda}{m-n}}(T^{m-1} x, T^{m-1} x, T^m x) \\ &\le 2\sum_{i=n}^{m-2} k^i s_{\frac{\lambda}{m-n}}(x, x, Tx) + k^{m-1} s_{\frac{\lambda}{m-n}}(x, x, Tx) \\ &\le 2(\frac{\varepsilon}{m-n} + \cdots + \frac{\varepsilon}{m-n}) \\ &\le 2\varepsilon. \end{aligned}$$

That is for $m > n$,

$$s_\lambda(T^n x, T^n x, T^m x) \le 2\varepsilon.$$

This shows that $(T^n x)$ is a Cauchy sequence and since X_s^* is s-complete, there exists $u \in X_s^*$ with $\lim_{n\to\infty} T^n x = u$.

From the continuity of T, we get

$$u = \lim_{n\to\infty} T^{n+1} x = \lim_{n\to\infty} T(T^n x) = Tu.$$

Therefore u is a fixed point of T. □

Let $\mathcal{M}$ be the family of all continuous functions of five variables $M : \mathbb{R}_+^5 \to \mathbb{R}_+$. For some $k \in [0, 1)$, we consider the following conditions:

(C1) For all $x, y, z \in \mathbb{R}_+$, if $y \le M(x, x, 0, z, y)$ with $z \le 2x + y$, then $y \le kx$.
(C2) If $y \le M(y, 0, y, y, 0)$ for all $y \in \mathbb{R}_+$, then $y = 0$.

Theorem 4.4. *Let T be a self-map on s-complete X_s^* and*

$$s_\lambda(Tx, Tx, Ty) \le M(s_\lambda(x, x, y), s_\lambda(Tx, Tx, x), s_\lambda(Tx, Tx, y), s_{3\lambda}(Ty, Ty, x), s_\lambda(Ty, Ty, y)) \quad (4.1)$$

for all $x, y, z \in X_s^*$ and some $M \in \mathcal{M}$. Then we have

1. *If M satisfies the condition $(C1)$, then T has a fixed point.*
2. *If M satisfies the condition $(C2)$ and T has a fixed point x, then the fixed point is unique.*

Proof. **(1)** For each $x_0 \in X_s^*$ and $n \in \mathbb{N}$, we take $x_{n+1} = Tx_n$. It follows from (4.1) and Lemma 3.3 that

$$\begin{aligned} s_\lambda(x_{n+1}, x_{n+1}, x_{n+2}) =& s_\lambda(Tx_n, Tx_n, Tx_{n+1}) \\ \le& M(s_\lambda(x_n, x_n, x_{n+1}), s_\lambda(x_{n+1}, x_{n+1}, x_n), s_\lambda(x_{n+1}, x_{n+1}, x_{n+1}), \\ & s_{3\lambda}(x_{n+2}, x_{n+2}, x_n), s_\lambda(x_{n+2}, x_{n+2}, x_{n+1})) \\ =& M(s_\lambda(x_n, x_n, x_{n+1}), s_\lambda(x_n, x_n, x_{n+1}), 0, s_{3\lambda}(x_n, x_n, x_{n+2}), s_\lambda(x_{n+1}, x_{n+1}, x_{n+2})). \end{aligned}$$

By triangle inequality and Lemma 3.3, we have

$$s_{3\lambda}(x_n, x_n, x_{n+2}) \leq 2s_\lambda(x_n, x_n, x_{n+1}) + s_\lambda(x_{n+1}, x_{n+1}, x_{n+2}) \tag{4.2}$$

From (4.2), we see that $z \leq 2x + y$. Since M satisfies the condition $(C1)$, there exists $k \in [0, 1)$ such that

$$s_\lambda(x_{n+1}, x_{n+1}, x_{n+2}) \leq ks_\lambda(x_n, x_n, x_{n+1}) \leq \cdots \leq k^{n+1} s_\lambda(x_0, x_0, x_1). \tag{4.3}$$

Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} s_\lambda(x_n, x_n, x_{n+1}) = 0.$$

Hence there exists $\varepsilon > 0$ for $\lambda > 0$ such that

$$s_\lambda(x_n, x_n, x_{n+1}) \leq \varepsilon.$$

Without loss of generality, we can assume that there exists $\frac{\varepsilon}{m-n}$ for $\frac{\lambda}{m-n} > 0$ such that

$$s_{\frac{\lambda}{m-n}}(x_n, x_n, x_{n+1}) \leq \frac{\varepsilon}{m-n}.$$

Thus for all $n < m$ by using $(S3)$, Remark 3.4 and (4.3) we have

$$\begin{aligned} s_\lambda(x_n, x_n, x_m) \leq& 2s_{\frac{\lambda}{3}}(x_n, x_n, x_{n+1}) + s_{\frac{\lambda}{3}}(x_m, x_m, x_{n+1}) \\ \leq& 2s_{\frac{\lambda}{3}}(x_n, x_n, x_{n+1}) + s_{\frac{\lambda}{3}}(x_{n+1}, x_{n+1}, x_m) \\ &\vdots \\ \leq& 2\left(\frac{\varepsilon}{m-n} + \frac{\varepsilon}{m-n} + \cdots + \frac{\varepsilon}{m-n}\right) \\ \leq& 2\varepsilon. \end{aligned}$$

This proves that (x_n) is s-Cauchy in the s-complete X_s^*. Then (x_n) converges an $x \in X_s^*$.

Now we prove that x is a fixed point of T. By using (4.1), we get

$$\begin{aligned} s_\lambda(x_{n+1}, x_{n+1}, Tx) =& s_\lambda(Tx_n, Tx_n, Tx) \\ \leq& M(s_\lambda(x_n, x_n, x), s_\lambda(Tx_n, Tx_n, x_n), s_\lambda(Tx_n, Tx_n, x), s_{3\lambda}(Tx, Tx, x_n), s_\lambda(Tx, Tx, x)). \end{aligned}$$

Since $M \in \mathcal{M}$, then using Lemma 3.10 and taking the limit as $n \to \infty$, we obtain

$$s_\lambda(x, x, Tx) \leq M(0, 0, 0, s_{3\lambda}(Tx, Tx, x), s_\lambda(Tx, Tx, x)).$$

From Remark 3.4, we can rewrite

$$s_{3\lambda}(Tx, Tx, x) \leq s_\lambda(Tx, Tx, x).$$

Then the inequality can be written as follows:

$$s_\lambda(x, x, Tx) \leq M(0, 0, 0, s_\lambda(Tx, Tx, x), s_\lambda(Tx, Tx, x)).$$

Since M satisfies the condition $(C1)$, then $s_\lambda(x, x, Tx) \leq k.0 = 0$. This proves that $x = Tx$.

(2) Let x, y be fixed points of T. We prove that $x = y$. It follows from (4.1) that

$$\begin{aligned} s_\lambda(x, x, y) =& s_\lambda(Tx, Tx, Ty) \\ \leq& M(s_\lambda(x, x, y), s_\lambda(Tx, Tx, x), s_\lambda(Tx, Tx, y), s_{3\lambda}(Ty, Ty, x), s_\lambda(Ty, Ty, y)) \\ \leq& M(s_\lambda(x, x, y), 0, s_\lambda(x, x, y), s_{3\lambda}(y, y, x), 0). \end{aligned}$$

From Remark 3.4 and Lemma 3.3, we get

$$s_\lambda(x, x, y) \leq M(s_\lambda(x, x, y), 0, s_\lambda(x, x, y), s_\lambda(x, x, y), 0).$$

Since M satisfies the condition $(C2)$,

$$s_\lambda(x, x, y) = 0 \iff x = y.$$

□

Remark 4.5. Theorem 4.3 is a corollary of Theorem 4.4 when we take $M(x,y,z,s,t) = k.x$ for $k \in [0,1)$ and $x,y,z,s,t \in \mathbb{R}_+$.

Now we will give a new corollary of Theorem 4.4.

Corollary 4.6. *Let T be a self map on s-complete X_s^* and*

$$s_\lambda(Tx,Tx,Ty) \leq a(s_\lambda(Tx,Tx,x) + s_\lambda(Ty,Ty,y))$$

for some $a \in [0,\frac{1}{2})$ and all $x,y \in X_s^*$. Then T has a unique fixed point in X_s^*.

Proof. We must show that $M(x,y,z,s,t) = a(y+t)$ satisfies conditions $(C1)$ and $(C2)$. First, we have

$$M(x,x,0,z,y) = a(x+y).$$

So, if $y \leq M(x,x,0,z,y)$ with $z \leq 2x+y$, then

$$\begin{aligned} y \leq & M(x,x,0,z,y) = a(x+y) \\ y \leq & ax + ay \\ y \leq & \frac{a}{1-a}x \end{aligned}$$

with $\dfrac{a}{1-a} \in [0,1)$. Therefore, M satisfies condition $(C1)$.

If $y \leq M(y,0,y,y,0) = 0$, then $y = 0$. Therefore, M satisfies the condition $(C2)$.

Since

$$\begin{aligned} s_\lambda(Tx,Tx,Ty) \leq & M(s_\lambda(x,x,y), s_\lambda(Tx,Tx,x), s_\lambda(Tx,Tx,y), s_\lambda(Ty,Ty,x), s_\lambda(Ty,Ty,y)) \\ = & a(s_\lambda(Tx,Tx,x) + s_\lambda(Ty,Ty,y)), \end{aligned}$$

T has a unique fixed point in X_s^* by Theorem 4.4. □

Open problems How can obtain some similar results for the papers (see [2, 15]) in fuzzy metric spaces with the help of this technique?

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The strong converse inequality for de la Vallée Poussin means on the sphere *

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Abstract

This paper discusses the approximation by de la Vallée Poussin means $V_n f$ on the unit sphere. Especially, the lower bound of approximation is studied. As a main result, the strong converse inequality for the means is established. Namely, it is proved that there are constants C_1 and C_2 such that $C_1\omega\left(f, \frac{1}{\sqrt{n}}\right)_p \le \|V_n f - f\|_p \le C_2\omega\left(f, \frac{1}{\sqrt{n}}\right)_p$ for any p-th Lebesgue integrable or continuous function f defined on the sphere, where $\omega(f,t)_p$ is the modulus of smoothness of f.

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Keywords: sphere; de la Vallée Poussin means; approximation; modulus of smoothness; lower bound

1 Introduction

Motivated by geoscience, meteorology, and oceanography, sphere-oriented mathematics has gained increasing attention in recent decades. As main tools, spherical positive polynomial operators play prominent roles in the approximation and the interpolation on the sphere by means of orthonormal spherical harmonics. Several authors such as Ditzian [5], Dai and Ditzian [4], Bernes and Li [3], Wang and Li [16], Nikol'skiĭ and Lizorkin [10, 8] introduced and studied some spherical versions of some known one-dimensional polynomial operators, for example, spherical Jackson operators [8], spherical de la Vallée Poussin operators [3, 16], spherical delay mean operators [13] and best approximation operators [5, 4, 16] etc..

The main aim of the present paper is to study the approximation by the de la Vallée Poussin means on the unit sphere.

For to formulate our results, we first give some notations. Let $\mathbb{R}^d$, $d \ge 3$, be the Euclidean space of the points $x := (x_1, x_2, \ldots, x_d)$ endowed with the scalar product $x \cdot x' = \sum_{j=1}^{d} x_j x'_j (x, x' \in \mathbb{R}^d)$ and let $\sigma := \sigma^{d-1}$ be the unit sphere in $\mathbb{R}^d$ consisting of the points x satisfying $x^2 = x \cdot x = 1$.

We shall denote the points of σ by μ, and the elementary surface piece on σ by $d\sigma$. If it is necessary, we shall write $d\sigma :\equiv d\sigma(\mu)$ referring to the variable of integration. The surface area of σ^{d-1} is denoted by $|\sigma^{d-1}|$, and it is easy to deduce that $|\sigma^{d-1}| = \int_\sigma d\sigma = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$.

By $C(\sigma)$ and $L^p(\sigma)$, $1 \le p < +\infty$, we denote the space of continuous, real valued functions and the space of (the equivalence classes of) p-integrable functions defined on σ endowed with the respective norms $\|f\|_\infty := \max_{\mu\in\sigma} |f(\mu)|$, $\|f\|_p := \left(\int_\sigma |f(\mu)|^p d\sigma(\mu)\right)^{1/p}$ $(1 \le p < \infty)$. In the following, $L^p(\sigma)$ will always be one of the spaces $L^p(\sigma)$ for $1 \le p < \infty$, or $C(\sigma)$ for $p = \infty$.

Now we state some properties of spherical harmonics (see [16], [7], [9]). For integer $k \ge 0$, the restriction of a homogeneous harmonic polynomial of degree k on the unit sphere is called a spherical harmonic of degree k. The class of all spherical harmonics of degree k will be denoted by $\mathcal{H}_k$, and the class of all spherical harmonics of degree $k \le n$ will be denoted by Π_n^d. Of course,

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$\Pi_n^d = \bigoplus_{k=0}^n \mathcal{H}_k$, and it comprises the restriction to σ of all algebraic polynomials in d variables of total degree not exceeding n. The dimension of $\mathcal{H}_k$ is given by

$$d_k^d := \dim \mathcal{H}_k^d := \begin{cases} \frac{2k+d-2}{k+d-2}\binom{k+d-2}{k}, & k \geq 1; \\ 1, & k = 0, \end{cases}$$

and that of Π_n^d is $\sum_{k=0}^n d_k^d$.

The spherical harmonics have an intrinsic characterization. To describe this, we first introduce the Laplace-Beltrami operators (see [9]) to sufficiently smooth functions f defined on σ, which is the restriction of Laplace operator $\Delta := \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ on the sphere σ, and can be expressed as $Df(\mu) := \Delta f\left(\frac{\mu}{|\mu|}\right)\Big|_{\mu\in\sigma}$. Clearly, the operator D is an elliptic, (unbounded) selfadjoint operator on $L^2(\sigma)$, is invariant under arbitrary coordinate changes, and its spectrum comprises distinct eigenvalues $\lambda_k := -k(k+d-2),\ k = 0, 1, \ldots,$ each having finite multiplicity. The space $\mathcal{H}_k$ can be characterized intrinsically as the eigenspace corresponding to λ_k, i.e.

$$\mathcal{H}_k = \{\Psi \in C^\infty(\sigma) : D\Psi = -k(k+d-2)\Psi\}.$$

Since the λ_k's are distinct, and the operator is selfadjoint, the spaces $\mathcal{H}_k$ are mutually orthogonal; also, $L^2(\sigma) =$ closure $\{\bigoplus_k \mathcal{H}_k\}$. Hence, if we choose an orthogonal basis $\{Y_{k,l} : l = 1, \ldots, d_k^d\}$ for each $\mathcal{H}_k$, then the set $\{Y_{k,l} : k = 0, 1, \ldots, l = 1, \ldots, d_k^d\}$ is an orthogonal basis for $L^2(\sigma)$.

The orthogonal projection $Y_k : L^1(\sigma) \to \mathcal{H}_k$ is given by

$$Y_k(f;\mu) := \frac{\Gamma(\lambda)(k+\lambda)}{2\pi^{\lambda+1}} \int_\sigma P_k^\lambda(\mu\cdot\nu) f(\nu) d\sigma(\nu),$$

where $2\lambda = d-2$, and P_k^λ are the ultraspherical (or Gegenbauer) polynomials defined by the generating equation $(1-2r\cos\theta + r^2)^{-\lambda} = \sum_{k=0}^\infty r^k P_k^\lambda(\cos\theta)(0 \leq \theta \leq \pi)$. The further details for the ultraspherical polynomials can be found in [15].

For an arbitrary number θ, $0 < \theta < \pi$, we define the spherical translation operator of the function $f \in L^p(\sigma)$ with a step θ by the aid of the following equation (see [12], [2]):

$$S_\theta(f) := S_\theta(f;\mu) := \frac{1}{|\sigma^{d-2}|\sin^{d-2}\theta} \int_{\mu\cdot\nu=\cos\theta} f(\nu) d\sigma(\nu), \tag{1.1}$$

where $|\sigma^{d-2}|$ means the $(d-2)$-dimensional surface area of the unit sphere of $\mathbb{R}^{d-1}$. Here we integrate over the family of points $\nu \in \sigma$ whose spherical distance from the given point $\mu \in \sigma$ (i.e. the length of minor arc between μ and ν on the great circle passing through them) is equal to θ. Thus $S_\theta(f;\mu)$ can be interpreted as the mean value of the function f on the surface of $(d-2)$-dimensional sphere with radius $\sin\theta$.

The properties of spherical translation operator (1.1) are well known; see e.g., [2]. In particular, it can be expressed as the following series

$$S_\theta(f;\mu) = \sum_{k=0}^\infty \frac{P_k^\lambda(\cos\theta)}{P_k^\lambda(1)} Y_k(f;\mu) := \sum_{k=0}^\infty Q_k^\lambda(\cos\theta) Y_k(f;\mu)$$

where $Q_k^\lambda(\cos\theta) := \frac{P_k^\lambda(\cos\theta)}{P_k^\lambda(1)}$, and for any $f \in L^p(\sigma)$, $\|S_\theta(f)\|_p \leq \|f\|_p$, $\lim_{\theta\to 0}\|S_\theta f - f\|_p = 0$. We usually apply the translation operator to define spherical modulus of smoothness of a function $f \in L^p(\sigma)$, i.e. (see [16]) $\omega(f,t)_p := \sup_{0<\theta\leq t}\|f - S_\theta f\|_p$. Clearly, the modulus is meaningful to describe the approximation degree and the smoothness of functions on σ, which has been widely used in the study of approximation on sphere.

We also need a K-functional on sphere σ defined by (see [5], [16])

$$K(f;t)_p := \inf\left\{\|f-g\|_p + t^2\|Dg\|_p :\ g, Dg \in L^p(\sigma)\right\}. \quad 0 < t < t_0. \tag{1.2}$$

For the modulus of smoothness and K-functional, the following equivalent relationship has been proved (see [5])

$$\omega(f,t)_p \approx K(f,t)_p. \tag{1.3}$$

Here and in the following, $a \approx b$ means that there are positive constants C_1 and C_2 such that $C_1 \leq a \leq C_2 b$. We denote by $C_i (i = 1, 2, \ldots)$ the positive constants independent of f and n, and by $C(a)$ the positive constants depending only on a. Their value will be different at different occurrences, even within the same formula.

Define the kernel of de la Vallée Poussin as

$$v_n(t) = \frac{1}{I_{n,d}}\Big(\cos\frac{t}{2}\Big)^{2n}, \quad n \in \mathbb{N}, \tag{1.4}$$

where the constant $I_{n,d}$ satisfies $\int_\sigma v_n(\breve{\mu\nu}) d\sigma(\nu) = |\sigma^{d-2}|$, and $\breve{\mu\nu}$ is the spherical distance between the points μ and ν, i.e. the length of minor arc of great circle crossing μ and ν. Then the convolution resulted by the kernel is

$$V_n(f;\mu) = (f * v_n)(\mu) = \frac{1}{|\sigma^{d-2}|}\int_\sigma f(\nu) v_n(\breve{\mu\nu}) d\sigma(\nu), \quad f \in L^1(\sigma), \tag{1.5}$$

which is called de la Vallée Poussin means on the sphere.

The means were introduced by de la Vallée Poussin in 1908 for one dimensional Fourier series and were generalized to ultraspherical and Jacobi series by Kogbeliantz and Bavinck in 1925 and 1972, respectively (see also [16]). In 1993, Berens and Li [3] established the relation between the means and the best spherical polynomial approximation on the sphere, and discussed their approximation behavior by various of smoothness. Especially, they proved (see also [16]) the relation:

$$\max_{k \geq n} \|V_k f - f\|_p \approx \omega\Big(f, \frac{1}{\sqrt{n}}\Big)_p, \quad f \in L^p(\sigma). \tag{1.6}$$

Motivated by [1] and [6], we will improve the above result. Indeed, we will prove

$$\|V_n f - f\|_p \approx \omega\Big(f, \frac{1}{\sqrt{n}}\Big)_p$$

for any $f \in L^p(\sigma)$, $1 \leq p \leq +\infty$.

2 The kernel of de la Vallée Poussin

In the definition of de la Vallée-Poussin kernel v_n given by (1.4), the constants $I_{n,d}$ is requested to satisfy $\int_\sigma v_n(\breve{\mu\nu}) d\sigma(\nu) = |\sigma^{d-2}|$, which implies that $\int_0^\pi v_n(\theta) \sin^{2\lambda}\theta d\theta = 1 (2\lambda = d-2)$.

By computation, we have $I_{n,d} = 2^{2\lambda}\frac{\Gamma(\lambda+1/2)\Gamma(n+\lambda+1/2)}{\Gamma(n+2\lambda+1)}$, where $\Gamma(\lambda)$ is Gamma function. So,

$$v_n(t) = \frac{\Gamma(n+2\lambda+1)}{2^{2\lambda}\Gamma(\lambda+1/2)\Gamma(n+\lambda+1/2)}\left(\cos\frac{t}{2}\right)^{2n}.$$

Since $v_n(t)$ are even trigonometric polynomials with degree n, $V_n(f,\mu)$ are spherical polynomials with degree n. So we also call (1.5) spherical de la Vallée Poussin polynomial operators.

We can translate de la Vallée Poussin means given by (1.5) into the multiplier form:

$$V_n(f;\mu) = \sum_{k=0}^{\infty} \omega_{n,k}^{(\lambda)} Y_k(f;\mu) \tag{2.1}$$

where

$$\omega_{n,k}^{(\lambda)} := \begin{cases} \frac{n!(n+2\lambda)!}{(n-k)!(n+k+2\lambda)!}, & 0 \leq k \leq n; \\ 0, & k > n. \end{cases}$$

Since the means can be rewritten as

$$\begin{aligned} V_n(f;\mu) &= \int_0^\pi v_n(\theta) S_\theta(f;\mu) \sin^{2\lambda}\theta d\theta = \int_0^\pi v_n(\theta)\left(\sum_{k=0}^{\infty}\frac{P_k^\lambda(\cos\theta)}{P_k^\lambda(1)} Y_k(f;\mu)\right)\sin^{2\lambda}\theta d\theta \\ &= \sum_{k=0}^{\infty}\left(\int_0^\pi v_n(\theta)\frac{P_k^\lambda(\cos\theta)}{P_k^\lambda(1)}\sin^{2\lambda}\theta d\theta\right) Y_k(f;\mu), \end{aligned}$$

it is sufficient to prove $\int_0^\pi v_n(\theta)\frac{P_k^\lambda(\cos\theta)}{P_k^\lambda(1)}\sin^{2\lambda}\theta\, d\theta=\omega_{n,k}^{(\lambda)}(k\geq 0)$. Indeed, when $k=1$, one has

$$\begin{aligned}
&\int_0^\pi v_n(\theta)\frac{P_1^\lambda(\cos\theta)}{P_1^\lambda(1)}\sin^{2\lambda}\theta d\theta=\int_0^\pi v_n(\theta)\left(1-2\sin^2\frac{\theta}{2}\right)\sin^{2\lambda}\theta d\theta\\
=\;&\frac{2^{2\lambda}}{I_{n,d}}\left(\int_0^\pi\left(\cos\frac{\theta}{2}\right)^{2(n+\lambda)}\sin^{2\lambda}\frac{\theta}{2}d\theta-2\int_0^\pi\left(\cos\frac{\theta}{2}\right)^{2(n+\lambda)}\sin^{2(\lambda+1)}\frac{\theta}{2}d\theta\right)\\
=\;&\frac{2^{2\lambda+1}}{I_{n,d}}\left(\frac{1}{2}B\left(\lambda+\frac{1}{2},n+\lambda+\frac{1}{2}\right)-B\left(\lambda+1+\frac{1}{2},n+\lambda+\frac{1}{2}\right)\right)\\
=\;&\frac{n}{n+2\lambda+1}=\frac{n!(n+2\lambda)!}{(n-1)!(n+1+2\lambda)!}=\omega_{n,1}^{(\lambda)},
\end{aligned}$$

where $B(a,b)$ is Beta function.

Now, we suppose for $k\leq n$ that $\int_0^\pi v_n(\theta)\frac{P_k^\lambda(\cos\theta)}{P_k^\lambda(1)}\sin^{2\lambda}\theta d\theta=\omega_{n,k}^{(\lambda)}$. Then for $k+1$ we first recall the relation (see page 81 of [15])

$$(k+1)P_{k+1}^\lambda(x)-2(\lambda+k)xP_k^\lambda(x)+(2\lambda+k-1)P_{k-1}^\lambda(x)=0\quad(k\geq 1),$$

i.e.,

$$P_{k+1}^\lambda(\cos\theta)=\frac{1}{k+1}\left(2(\lambda+k)\cos\theta P_k^\lambda(\cos\theta)-(2\lambda+k-1)P_{k-1}^\lambda(\cos\theta)\right).$$

Then,

$$\begin{aligned}
&\int_0^\pi v_n(\theta)\frac{P_{k+1}^\lambda(\cos\theta)}{P_{k+1}^\lambda(1)}\sin^{2\lambda}\theta d\theta=\frac{1}{P_{k+1}^\lambda(1)(k+1)}\left(2(\lambda+k)\int_0^\pi v_n(\theta)\cos\theta P_k^\lambda(\cos\theta)\sin^{2\lambda}\theta d\theta\right.\\
&\left.-(2\lambda+k-1)\int_0^\pi v_n(\theta)P_{k-1}^\lambda(\cos\theta)\sin^{2\lambda}\theta d\theta\right):=\frac{1}{P_{k+1}^\lambda(1)(k+1)}\left(2(\lambda+k)J_2-J_1\right).
\end{aligned}$$

By the assumption, we obtain

$$J_1=(2\lambda+k-1)P_{k-1}^\lambda(1)\int_0^\pi v_n(\theta)\frac{P_{k-1}^\lambda(\cos\theta)}{P_{k-1}^\lambda(1)}\sin^{2\lambda}\theta d\theta=\frac{(2\lambda+k-1)!n!(n+2\lambda)!}{\Gamma(2\lambda)(k-1)!(n-k+1)!(n+k-1+2\lambda)!}.$$

For J_2 we have

$$\begin{aligned}
J_2\;&=\;\frac{1}{I_{n,d}}\int_0^\pi\left(\cos\frac{\theta}{2}\right)^{2n}\left(2\cos^2\frac{\theta}{2}-1\right)P_k^\lambda(\cos\theta)\sin^{2\lambda}\theta d\theta\\
&=\;\frac{2I_{n+1,d}}{I_{n,d}}P_k^\lambda(1)\int_0^\pi v_{n+1}(\theta)\frac{P_k^\lambda(\cos\theta)}{P_k^\lambda(1)}\sin^{2\lambda}\theta d\theta-P_k^\lambda(1)\int_0^\pi v_n(\theta)\frac{P_k^\lambda(\cos\theta)}{P_k^\lambda(1)}\sin^{2\lambda}\theta d\theta\\
&:=\;J_{21}-J_{22},
\end{aligned}$$

which implies from the assumption that $J_{22}=\frac{\Gamma(k+2\lambda)}{k!\Gamma(2\lambda)}\frac{n!(n+2\lambda)!}{(n-k)!(n+k+2\lambda)!}$, and

$$J_{21}=\frac{2\Gamma(n+1+\lambda+1/2)\Gamma(n+2\lambda+1)}{\Gamma(n+\lambda+1/2)\Gamma(n+1+2\lambda+1)}\frac{\Gamma(k+2\lambda)}{k!\Gamma(2\lambda)}\frac{(n+1)!(n+1+2\lambda)!}{(n+1-k)!(n+1+k+2\lambda)!}.$$

Therefore,

$$J_2=\frac{\Gamma(k+2\lambda)}{k!\Gamma(2\lambda)}\frac{n!(n+2\lambda)!}{(n+1-k)!(n+1+k+2\lambda)!}\left(n(n+1)+k(2\lambda+k)\right).$$

So,

$$\int_0^\pi v_n(\theta)\frac{P_{k+1}^\lambda(\cos\theta)}{P_{k+1}^\lambda(1)}\sin^{2\lambda}\theta d\theta=\frac{n!(n+2\lambda)!}{(n-k-1)!(n+k+1+2\lambda)!}=\omega_{n,k+1}^{(\lambda)}.$$

On the other hand, it is clear that for $k > n$, $\omega_{n,k}^{(\lambda)} = 0$. Hence, de la Vallée Poussin means $V_n(f;\mu)$ have the form of multiplier expression given in (2.1).

Now we give some properties for the de la Vallée Poussin kernel v_n.

Lemma 2.1. Let $v_n(t)$ be the kernel of de la Vallée Poussin defined by (1.4), $2\lambda = d-2$ and $d \geq 3$. Then there hold

$$\int_0^\pi \theta^{-\lambda} v_n(\theta) \sin^{2\lambda}\theta d\theta \leq C(d) n^{\frac{\lambda}{2}}, \tag{2.2}$$

and

$$\int_0^\pi \theta^{-\frac{2}{m}} v_n(\theta) \sin^{2\lambda}\theta d\theta \leq C(d) n^{\frac{1}{m}}, \quad m = 1, 2, \ldots. \tag{2.3}$$

Proof. We only prove (2.2). The proof of (2.3) is similar. First, a direct computation implies

$$I_{n,d} = C(d)\frac{(2n+d-3)!!}{(2n+2d-4)!!} \approx n^{-\frac{d-1}{2}}.$$

Then, $\int_0^\pi \theta^{-\lambda} v_n(\theta)\sin^{2\lambda}\theta d\theta = \frac{1}{I_{n,d}}\int_0^\pi \theta^{-\lambda}\Big(\cos\frac{\theta}{2}\Big)^{2n}\sin^{2\lambda} d\theta = \frac{J_{n,d}^{(-\lambda)}}{I_{n,d}}$, where

$$J_{n,d}^{(-\lambda)} = \int_0^{\frac{\pi}{2}} \theta^{-\frac{d-2}{2}} \sin^{d-2}\theta \cos^{2n}\frac{\theta}{2} d\theta.$$

So, we have

$$\begin{aligned} J_{n,d}^{(-\lambda)} &\leq 2^{\frac{d}{2}} \int_0^{\frac{\pi}{2}} \sin^{\frac{d-2}{2}} t \cos^{2n+d-2} t dt = 2^{\frac{d}{2}-1} B\Big(\frac{\frac{d-2}{2}+1}{2}, \frac{2n+d-2+1}{2}\Big) \\ &= 2^{\frac{d}{2}-1}\Gamma(\frac{d}{4})\frac{\Gamma(n+\frac{d-1}{2})}{\Gamma(n+\frac{3d-2}{4})} = C(d)\frac{\Gamma(n+\frac{d-1}{2})}{\Gamma(n+\frac{3d-2}{4})} \approx n^{-\frac{d}{4}}. \end{aligned}$$

Therefore

$$\int_0^\pi \theta^{-\lambda} v_n(\theta)\sin^{2\lambda}\theta d\theta = \frac{J_{n,d}^{(-2)}}{I_{n,d}} \leq C(d)\frac{n^{-\frac{d}{4}}}{n^{-\frac{d-1}{2}}} = C(d) n^{\frac{d-2}{4}}.$$

The proof of Lemma 2.1 is completed. □

Lemma 2.2. For the kernel of de la Vallée Poussin $v_n(t)$ defined by (1.4), we have

$$\int_0^\pi \theta^4 v_n(\theta)\sin^{2\lambda}\theta d\theta \leq C(d) n^{-2}.$$

Proof. Since

$$\begin{aligned} J_{n,d}^{(4)} &= \int_0^\pi \theta^4 \cos^{2n}\frac{\theta}{2}\sin^{2\lambda}\theta d\theta = 2^{d-1}\pi^4 \int_0^{\frac{\pi}{2}} \sin^{d+2}\theta\cos^{2n+d-2}\theta d\theta \\ &= \begin{cases} 2^{d-2}\pi^5\frac{(2n+d-3)!!(d+1)!!}{(2n+2d)!!}, & \text{if } d \text{ is even;} \\ 2^{d-1}\pi^4\frac{(2n+d-3)!!(d+1)!!}{(2n+2d)!!}, & \text{if } d \text{ is odd} \end{cases} \\ &= C(d)\frac{(2n+d-3)!!}{(2n+2d)!!}, \end{aligned}$$

we have

$$\int_0^\pi \theta^4 v_n(\theta)\sin^{2\lambda}\theta d\theta = \frac{J_{n,d}^{(4)}}{I_{n,d}} = C(d)\frac{(2n+2d-4)!!}{(2n+2d)!!} \leq C(d) n^{-2}.$$

This finishes the proof of Lemma 2.2. □

3 Lower bound of approximation for de la Vallée Poussin means

In this section we prove the main result of this paper, which can be stated as follows.

Theorem 3.1. Let $V_n(f;\mu)$ be de la Vallée Poussin means on the sphere given by (1.5). Then for $f \in L^p(\sigma), 1 \le p \le +\infty$, there exists a constant C which is independent of f and n, such that

$$\omega\left(f, \frac{1}{\sqrt{n}}\right)_p \le C\|V_n f - f\|_p.$$

In order to prove the result, we first prove the following lemma.

Lemma 3.1. For any $g, Dg, D^2 g \in L^p(\sigma)$, $1 \le p \le \infty$, there exist the constants A, B and C_2 which are are independent of n and g, such that $\|V_n g - g - \alpha(n) Dg\|_p \le C_2 n^{-2}\|D^2 g\|_p$, where $0 < \frac{A}{n} \le \alpha(n) \le \frac{B}{n}$.

Proof. Since (see (3.6) of [11]) $S_\theta(g;\mu) - g(\mu) = \int_0^\theta \sin^{-2\lambda} t\, dt \int_0^t \sin^{2\lambda} u\, S_u(Dg;\mu)du$, we have

$$S_u(Dg;\mu) - Dg(\mu) = \int_0^u \sin^{-2\lambda}\gamma d\gamma \int_0^\gamma \sin^{2\lambda}\nu\ S_\nu(D^2 g;\mu)d\nu.$$

Observing that

$$\begin{aligned} V_n(g;\mu) - g(\mu) &= \int_0^\pi v_n(\theta)\sin^{2\lambda}\theta d\theta \int_0^\theta \sin^{-2\lambda} t dt \int_0^t \sin^{2\lambda} u\ S_u(Dg;\mu)du \\ &= Dg(\mu)\int_0^\pi v_n(\theta)\sin^{2\lambda}\theta d\theta \int_0^\theta \sin^{-2\lambda} t dt \int_0^t \sin^{2\lambda} u du \\ &\quad + \int_0^\pi v_n(\theta)\sin^{2\lambda}\theta d\theta \int_0^\theta \sin^{-2\lambda} t dt \int_0^t \sin^{2\lambda} u\Big(S_u(Dg;\mu) - Dg(\mu)\Big)du \\ &:= Dg(\mu)\alpha(n) + \Psi(g;\mu), \end{aligned}$$

where $\alpha(n) = C(d)n^{-1}$ satisfies $0 < An^{-1} \le C(d)n^{-1} \le Bn^{-1}$, we obtain that from the Hölder-Minkowski's inequality and the contractility of translation operator

$$\begin{aligned} \|\Psi g\|_p &\le \|D^2 g\|_p \int_0^\pi v_n(\theta)\sin^{2\lambda}\theta d\theta \int_0^\theta \sin^{-2\lambda} t dt \int_0^t \sin^{2\lambda} u du \int_0^u \sin^{-2\lambda}\gamma d\gamma \int_0^\gamma \sin^{2\lambda}\nu d\nu \\ &\le C_3\|D^2 g\|_p \int_0^\pi v_n(\theta)\theta^4 \sin^{2\lambda}\theta d\theta. \end{aligned}$$

Thus, from Lemma 2.2 it follows that $\|\Psi g\|_p \le C_4 n^{-2}\|D^2 g\|_p$. The Lemma 3.1 has been proved. □

Now we turn to the proof of Theorem 3.1. We first introduce an operator V_n^m given by

$$V_n^m(f;\mu) = \sum_{k=0}^n \Big(\int_0^\pi v_n(\theta) Q_k^\lambda(\cos\theta)\sin^{2\lambda}\theta d\theta\Big)^m Y_k(f;\mu).$$

Then, form the orthogonality of projection operator Y_k, it follows that

$$\begin{aligned} V_n^{m+l} f &= \sum_{k=0}^n \Big(\int_0^\pi v_n(\theta) Q_k^\lambda(\cos\theta)\sin^{2\lambda}\theta d\theta\Big)^m Y_k\Big(\sum_{s=0}^n \Big(\int_0^\pi v_n(\theta) Q_s^\lambda(\cos\theta)\sin^{2\lambda}\theta d\theta\Big)^l Y_s f\Big) \\ &= V_n^m(V_n^l f). \end{aligned}$$

Thus, we take $g = V_n^m f$ and obtain that

$$\|f - g\|_p = \|f - V_n^m f\|_p \le \sum_{k=1}^m \|V_n^{k-1} f - V_n^k f\|_p \le m\|f - V_n f\|_p,$$

where $V_n^0 f = f$.

Next, we prove the estimate: $\|DV_n^m f\|_p \le \frac{A}{2C_2} C_1 n \|f\|_p$, where A and C_2 are the same as that in Lemma 3.1. In fact, we have

$$\|DV_n^m f\|_p \le \left\| \sum_{k=0}^{n} k(k+d-2)\Big(\int_0^\pi v_n(\theta)\Big|Q_k^\lambda(\cos\theta)\Big|\sin^{2\lambda}\theta d\theta\Big)^m Y_k(f)\right\|_p .$$

Since (see [1])

$$|Q_k^\lambda(\cos\theta)| \equiv \Big|\frac{P_k^\lambda(\cos\theta)}{P_k^\lambda(1)}\Big| \le C_5 \min\Big((k\theta)^{-\lambda}, 1\Big),$$

we use (2.2) and obtain for $k\theta \ge 1$ and $\theta \le \frac{\pi}{2}$, that

$$\begin{aligned}\|DV_n^m f\|_p &\le C_6 \left\| \sum_{k=0}^{n} k(k+d-2)k^{-\frac{d-2}{2}m}\Big(\int_0^\pi v_n(\theta)\theta^{-\lambda}\sin^{2\lambda}\theta d\theta\Big)^m Y_k(f)\right\|_p \\ &\le C_7 n^{\frac{d-2}{4}m}\|f\|_p \sum_{k=0}^{\infty} k^{2-\frac{d-2}{2}m}.\end{aligned}$$

For $2-\frac{d-2}{2}m < -1$, i.e. $m > \frac{6}{d-2}$, it is clear that the series $\sum_{k=0}^{\infty} k^{2-\frac{d-2}{2}m}$ is convergence. Thus

$$\|DV_n^m f\|_p \le C_8 n^{\frac{d-2}{4}m}\|f\|_p.$$

For $k\theta \le 1$, then (2.3) implies that

$$\|DV_n^m f\|_p \le \left\| \sum_{k=0}^{n} \Big(\int_0^\pi v_n(\theta)\theta^{-\frac{2}{m}}(\theta^2 k(k+d-2))^{\frac{1}{m}}\Big|Q_k^\lambda(\cos\theta)\Big|\sin^{2\lambda}\theta d\theta\Big)^m Y_k(f)\right\|_p$$

$$\le C_9 \left\| \sum_{k=0}^{n} \Big(\int_0^\pi v_n(\theta)\theta^{-\frac{2}{m}}\sin^{2\lambda}\theta d\theta\Big)^m Y_k(f)\right\|_p \le C_{10} n \left\|\sum_{k=0}^{\infty} Y_k(f)\right\|_p = \frac{A}{2C_2} C_1 n \|f\|_p,$$

where A and C_2 are the same as that in Lemma 3.1. Therefore, when $m > \frac{6}{d-2}$, we have

$$\|DV_n^m f\|_p \le \frac{A}{2C_2} C_1 n \|f\|_p.$$

In the next, without loss generality, we assume $m_1 > \frac{6}{d-2}$, and $m > \frac{6}{d-2} + m_1$. According to Lemma 3.1 we see that

$$\begin{aligned}&\alpha(n)\|DV_n^m f\|_p \le \|V_n^m f - f\|_p + C_2 n^{-2}\|D^2 V_n^m f\|_p \le m\|V_n f - f\|_p + \frac{AC_1}{2} n^{-1}\|DV_n^{m-m_1} f\|_p \\ &\le m\|V_n f - f\|_p + \frac{AC_1}{2} n^{-1}\|DV_n^m f\|_p + \frac{AC_1}{2} n^{-1}\|DV_n^{m-m_1}(V_n^{m_1} f - f)\|_p \\ &\le m\|V_n f - f\|_p + \frac{AC_1}{2n}\|DV_n^m f\|_p + \frac{AC_1C_{11}}{2}\|V_n^{m_1} f - f\|_p \\ &= C_{12}\|V_n f - f\|_p + \frac{AC_1}{2n}\|DV_n^m f\|_p.\end{aligned}$$

Taking $\alpha(n) = \frac{AC_1}{n}$, one has

$$\frac{1}{n}\|DV_n^m f\|_p \le \frac{2C_{12}}{AC_1}\|V_n f - f\|_p.$$

So from the definition of K-functional it follows

$$\begin{aligned}K\left(f, \frac{1}{\sqrt{n}}\right) &\le \|f - V_n^m f\|_p + \Big(\frac{1}{\sqrt{n}}\Big)^2 \|DV_n^m f\|_p \\ &\le m\|f - V_n f\|_p + \frac{2C_{12}}{AC_1}\|f - V_n f\|_p \le C_{14}\|f - V_n f\|_p,\end{aligned}$$

which together with (1.3) implies

$$\omega\left(f, \frac{1}{\sqrt{n}}\right)_p \leq C\|f - V_n f\|_p.$$

This finishes the proof of Theorem 3.1. □

From (1.6) and Theorem 3.1, the following Corollary 3.1 follows directly.

Corollary 3.1. For any $f \in L^p(\sigma), 1 \leq p \leq \infty$, there holds

$$\|V_n f - f\|_p \approx \omega\left(f, \frac{1}{\sqrt{n}}\right)_p.$$

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On the fixed point method for stability of a mixed type AQ-functional equation

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Abstract

In this article, we take into account the stability for the following functional equation of additive-quadratic type

$$f(x-y)-f(-x+y)-4f(x)+f(2x)-f(-y)+f(y)=0$$

with the fixed point method.

Keywords: Stability; Fixed point method; Additive-quadratic mapping.
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1 Introduction

Ulam [9] proposed the following question concerning the stability of homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon>0$, does there exist a $\delta>0$ such that if a function $h: G_1\to G_2$ satisfies the inequality $d(h(xy),h(x)h(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $H: G_1\to G_2$ with $d(h(x),H(x))<\varepsilon$ for all $x\in G_1$?

Hyers [5] answers the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately additive mappings was given by Aoki [1], and for approximately linear mappings was presented by Rassias [7] by considering an unbounded Cauchy difference. Thereafter, many interesting results of the stability of several functional equation have been extensively investigated.

On the contrary, Cădariu and Radu [2] observed that the existence of the solution for a functional equation and the estimation of the difference with the given mapping can be obtained from the fixed point alternative. This method is called a fixed point method. In particular, they [3, 4] applied this method to prove the stability theorems of the additive functional equation and the quadratic functional equation by using the fixed point method.

Now we consider the stability of the following mixed type additive-quadratic functional equation (briefly, AQ–functional equation)

$$f(x-y)-f(-x+y)-4f(x)+f(2x)-f(-y)+f(y)=0. \tag{1.1}$$

by using the fixed point method. In this case, every solution of the functional equation (1.1) is said to be an additive-quadratic mapping.

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2 Stability of Eq. (1.1) and its applications

Throughout this article, let V be a real or complex linear space and Y a Banach space. For a given mapping $f : V \to Y$, we use the following abbreviation

$$Df(x,y) := f(x-y) - f(-x+y) - 4f(x) + f(2x) - f(-y) + f(y)$$

for all $x, y \in V$. We first prove the following lemma.

Lemma 2.1 *Let $f : V \to Y$ be a mapping with $f(0) = 0$ such that $Df(x,y) = 0$ for all $x, y \in V\backslash\{0\}$. Then f is an additive-quadratic mapping.*

Proof. Since $f(0) = 0$, we get $Df(x,0) = Df(x,x) = 0$ for all $x \in V\backslash\{0\}$, and $Df(0,y) = 0$ for all $y \in V$. This completes the proof. □

For explicitly later use, we state the following theorem:

Theorem 2.2 (The alternative of fixed point) ([6] or [8]) *Suppose that a complete generalized metric space (X, d), which means that the metric d may assume infinite values, and a strictly contractive mapping $J : X \to X$ with the Lipschitz constant $0 < L < 1$ are given. Then, for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = +\infty, \ \forall n \in \mathbb{N} \cup \{0\},$$

or there exists a nonnegative integer k such that:

(1) *$d(J^n x, J^{n+1}x) < +\infty$ for all $n \geq k$;*

(2) *the sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in $Y := \{y \in X, d(J^k x, y) < +\infty\}$;*

(4) *$d(y, y^*) \leq (1/(1-L))d(y, Jy)$ for all $y \in Y$.*

Now, by the use of fixed point method, we obtain the main results as follow.

Theorem 2.3 *Let $\varphi : (V\backslash\{0\})^2 \to [0, \infty)$ be a function with $\varphi(x,y) = \varphi(-x,-y)$ for all $x, y \in V\backslash\{0\}$. Suppose that a mapping $f : V \to Y$ satisfies*

$$\|Df(x,y)\| \leq \varphi(x,y) \tag{2.1}$$

for all $x, y \in V\backslash\{0\}$ with $f(0) = 0$. If there exists a constant $0 < L < 1$ such that a function φ has the property

$$\varphi(2x, 2y) \leq 2L\varphi(x,y) \tag{2.2}$$

for all $x, y \in V\backslash\{0\}$, then there exists a unique additive-quadratic mapping $F : V \to Y$ such that

$$\|f(x) - F(x)\| \leq \frac{\varphi(x,x)}{2(1-L)} \tag{2.3}$$

for all $x \in V\backslash\{0\}$. In particular, F is represented by

$$F(x) = \lim_{n\to\infty} \left(\frac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n} + \frac{f(2^n x) - f(-2^n x)}{2^{n+1}} \right) \tag{2.4}$$

for all $x \in V$.

Proof. Consider the set

$$S := \{g : g : V \to Y, \ g(0) = 0\}$$

and introduce a generalized metric on S by

$$d(g,h) = \inf\{K \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq K\varphi(x,x) \ \text{ for all } \ x \in V\backslash\{0\}\}.$$

It is easy to see that (S, d) is a generalized complete metric space.

Now we define a mapping $J : S \to S$ by

$$Jg(x) := \frac{g(2x) - g(-2x)}{4} + \frac{g(2x) + g(-2x)}{8}$$

for all $x \in V$. Note that

$$J^n g(x) = \frac{g(2^n x) - g(-2^n x)}{2^{n+1}} + \frac{g(2^n x) + g(-2^n x)}{2 \cdot 4^n}$$

for all $n \in \mathbb{N}$ and all $x \in V$. Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \le K$. From the definition of d, we have

$$\begin{aligned}\|Jg(x) - Jh(x)\| =& \frac{3}{8}\|g(2x) - h(2x)\| + \frac{1}{8}\|g(-2x) - h(-2x)\| \\ \le& \frac{1}{2} K\varphi(2x, 2x) \\ \le& KL\varphi(x, x)\end{aligned}$$

for all $x \in V \backslash \{0\}$, which implies that

$$d(Jg, Jh) \le Ld(g, h)$$

for any $g, h \in S$, that is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Moreover, by (2.1), we see that

$$\|f(x) - Jf(x)\| = \frac{1}{8}\| - 3Df(x, x) + Df(-x, -x)\| \le \frac{\varphi(x, x)}{2}$$

for all $x \in V \backslash \{0\}$. It means that $d(f, Jf) \le \frac{1}{2} < \infty$ by the definition of d. Therefore, according to Theorem 2.2, the sequence $\{J^n f\}$ converges to the unique fixed point $F : V \to Y$ of J in the set $T = \{g \in S : d(f, g) < \infty\}$, which is represented by (2.4).

Note that

$$d(f, F) \le \frac{1}{1 - L} d(f, Jf) \le \frac{1}{2(1 - L)},$$

which implies (2.3).

By the definition of F, together with (2.1) and (2.4), we find that

$$\begin{aligned}\|DF(x, y)\| =& \lim_{n\to\infty} \Big\| \frac{Df(2^n x, 2^n y) - Df(-2^n x, -, 2^n y)}{2^{n+1}} \\ &+ \frac{Df(2^n x, 2^n y) + Df(-2^n x, -2^n y)}{2 \cdot 4^n} \Big\| \\ \le& \lim_{n\to\infty} \frac{2^n + 1}{2 \cdot 4^n} \left(\varphi(2^n x, 2^n y) + \varphi(-2^n x, -2^n y)\right) \\ =&0\end{aligned}$$

for all $x, y \in V \backslash \{0\}$. By Lemma 2.1, we have proved that $DF(x, y) = 0$ for all $x, y \in V$. This completes the proof. □

We continue our investigation with the following theorem.

Theorem 2.4 *Let $\varphi : (V \backslash \{0\})^2 \to [0, \infty)$ with $\varphi(x, y) = \varphi(-x, -y)$ for all $x, y \in V \backslash \{0\}$. Suppose that $f : V \to Y$ satisfies the inequality $\|Df(x, y)\| \le \varphi(x, y)$ for all $x, y \in V \backslash \{0\}$ with $f(0) = 0$. If there exists $0 < L < 1$ such that the mapping φ has the property*

$$L\varphi(2x, 2y) \ge 4\varphi(x, y) \tag{2.5}$$

for all $x, y \in V \backslash \{0\}$, then there exists a unique additive-quadratic mapping $F : V \to Y$ such that

$$\|f(x) - F(x)\| \le \frac{L\varphi(x, x)}{4(1 - L)} \tag{2.6}$$

for all $x \in V \backslash \{0\}$. In particular, F is given by

$$F(x) = \lim_{n\to\infty} \left(2^{n-1}\left(f\left(\frac{x}{2^n}\right) - f\left(-\frac{x}{2^n}\right)\right) + \frac{4^n}{2}\left(f\left(\frac{x}{2^n}\right) + f\left(-\frac{x}{2^n}\right)\right)\right) \tag{2.7}$$

for all $x \in V$.

Proof. Let (S, d) be the set as in the proof of Theorem 2.3, and we consider the mapping $J : S \to S$ defined by

$$Jg(x) := g\left(\frac{x}{2}\right) - g\left(-\frac{x}{2}\right) + 2\left(g\left(\frac{x}{2}\right) + g\left(-\frac{x}{2}\right)\right)$$

for all $g \in S$ and all $x \in V$. Observe that

$$J^n g(x) = 2^{n-1}\left(g\left(\frac{x}{2^n}\right) - g\left(-\frac{x}{2^n}\right)\right) + \frac{4^n}{2}\left(g\left(\frac{x}{2^n}\right) + g\left(-\frac{x}{2^n}\right)\right)$$

and $J^0 g(x) = g(x)$ for all $x \in V$. Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \le K$. The definition of d yields

$$\begin{aligned}\|Jg(x) - Jh(x)\| &= 3\left\|g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right)\right\| + \left\|g\left(-\frac{x}{2}\right) - h\left(-\frac{x}{2}\right)\right\| \\ &\le 4K\varphi\left(\frac{x}{2}, \frac{x}{2}\right) \\ &\le LK\varphi(x, x)\end{aligned}$$

for all $x \in V\backslash\{0\}$. So we get

$$d(Jg, Jh) \le Ld(g, h)$$

for any $g, h \in S$, that is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Also we see that

$$\|f(x) - Jf(x)\| = \left\|Df\left(\frac{x}{2}, \frac{x}{2}\right)\right\| \le \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \le \frac{L}{4}\varphi(x, x)$$

for all $x \in V\backslash\{0\}$, which implies that $d(f, Jf) \le \frac{L}{4} < \infty$.

Therefore, according to Theorem 2.2, the sequence $\{J^n f\}$ converges to the unique fixed point F of J in the set $T := \{g \in S : d(f, g) < \infty\}$, which is given by (2.7).

Since

$$d(f, F) \le \frac{1}{1-L}d(f, Jf) \le \frac{L}{4(1-L)}$$

the inequality (2.6) holds.

From the definition of F with (2.1) and (2.5), we have

$$\begin{aligned}\|DF(x, y)\| &= \lim_{n\to\infty}\left\|2^{n-1}\left(Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - Df\left(-\frac{x}{2^n}, -\frac{y}{2^n}\right)\right)\right. \\ &\quad \left. + \frac{4^n}{2}\left(Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + Df\left(-\frac{x}{2^n}, -\frac{y}{2^n}\right)\right)\right\| \\ &\le \lim_{n\to\infty}\frac{2^n + 4^n}{2}\left(\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + \varphi\left(-\frac{x}{2^n}, -\frac{y}{2^n}\right)\right) \\ &= 0\end{aligned}$$

for all $x, y \in V\backslash\{0\}$. So, by Lemma 2.1, F is an additive-quadratic mapping, which completes the proof. □

From now on, given a mapping $f : V \to Y$, we set

$$\begin{aligned}Af(x, y) &:= f(x + y) - f(x) - f(y), \\ Qf(x, y) &:= f(x + y) + f(x - y) - 2f(x) - 2f(y)\end{aligned}$$

for all $x, y \in V$. Using Theorem 2.3 and Theorem 2.4, we will prove the stability of the additive functional equation $Af \equiv 0$, and the quadratic functional equation $Qf \equiv 0$ in the following results.

Corollary 2.5 *Let $f_i : V \to Y$, $i = 1, 2$, be mappings for which there exist functions $\phi_i : (V\backslash\{0\})^2 \to [0, \infty)$, $i = 1, 2$, such that*

$$\|Af_i(x, y)\| \le \phi_i(x, y) \tag{2.8}$$

for all $x, y \in V\backslash\{0\}$. *If* $f_i(0) = 0$, $\phi_i(x,y) = \phi_i(-x,-y)$, $i = 1,2$, *for all* $x, y \in V\backslash\{0\}$, *and there exists* $0 < L < 1$ *such that*

$$\frac{1}{L}\phi_1(x,y) \le \phi_1(2x,2y) \le 2L\phi_1(x,y), \tag{2.9}$$

$$\phi_2(2x,2y) \le L\phi_2(2x,2y) \tag{2.10}$$

for all $x, y \in V\backslash\{0\}$, *then there exist unique additive mappings* $F_i : V \to Y$, $i = 1,2$, *such that*

$$\|f_1(x) - F_1(x)\| \le \frac{\phi_1(x,x) + 3\phi_1(x,-x)}{2(1-L)}, \tag{2.11}$$

$$\|f_2(x) - F_2(x)\| \le \frac{L(\phi_2(x,x) + 3\phi_2(x,-x))}{4(L-1)} \tag{2.12}$$

for all $x \in V\backslash\{0\}$. *In particular, the mappings* F_i, $i = 1,2$, *are represented by*

$$F_1(x) = \lim_{n\to\infty} \frac{f_1(2^n x)}{2^n}, \tag{2.13}$$

$$F_2(x) = \lim_{n\to\infty} 2^n f_2\left(\frac{x}{2^n}\right) \tag{2.14}$$

for all $x \in V$.

Proof. We first note that

$$Df_i(x,y) = Af_i(x,-y) - Af_i(-x,y) + Af_i(x,x) + Af_i(x,-x)$$

for all $x, y \in V$ and $i = 1,2$. Put

$$\varphi_i(x,y) := \phi_i(x,-y) + \phi_i(-x,y) + \phi_i(x,x) + \phi_i(x,-x)$$

for all $x, y \in V\backslash\{0\}$ and $i = 1,2$, then φ_1 satisfies (2.2) and φ_2 fulfills (2.5). Therefore $\|Df_i(x,y)\| \le \varphi_i(x,y)$ for all $x, y \in V\backslash\{0\}$ and $i = 1,2$. According to Theorem 2.3, there exists a unique mapping $F_1 : V \to Y$ satisfying (2.11), which is represented by (2.4).

Observe that, by virtue of (2.8) and (2.9),

$$\begin{aligned}\lim_{n\to\infty}\left\|\frac{f_1(2^n x) + f_1(-2^n x)}{2^{n+1}}\right\| &= \lim_{n\to\infty}\left\|\frac{f_1(2^n x) + f_1(-2^n x) - f_1(0)}{2^{n+1}}\right\| \\ &= \lim_{n\to\infty} \frac{1}{2^{n+1}}\|Af_1(2^n x, -2^n x)\| \\ &\le \lim_{n\to\infty} \frac{1}{2^{n+1}}\phi_1(2^n x, -2^n x) \\ &\le \lim_{n\to\infty} \frac{L^n}{2}\phi_1(x,-x) = 0\end{aligned}$$

and

$$\lim_{n\to\infty}\left\|\frac{f_1(2^n x) + f_1(-2^n x)}{2\cdot 4^n}\right\| \le \lim_{n\to\infty}\frac{2^n L^n}{2\cdot 4^n}\phi_1(x,-x) = 0$$

for all $x \in V\backslash\{0\}$. This inequality and (2.4) guarantees (2.13).

Moreover, we have

$$\left\|\frac{Af_1(2^n x, 2^n y)}{2^n}\right\| \le \frac{\phi_1(2^n x, 2^n y)}{2^n} \le L^n\phi_1(x,y)$$

for all $x, y \in V\backslash\{0\}$. Sending the limit as $n \to \infty$ in the above inequality, and using $F_1(0) = 0$, we get $AF_1(x,y) = 0$ for all $x, y \in V$.

On the other hand, according to Theorem 2.4, we see that there exists a unique mapping $F_2 : V \to Y$ satisfying (2.12), which is given by (2.7).

Notice that, by (2.8) and (2.11),

$$\begin{aligned}\lim_{n\to\infty} 2^{2n-1}\left\|f_2\left(\frac{x}{2^n}\right)+f_2\left(\frac{-x}{2^n}\right)\right\| &= \lim_{n\to\infty} 2^{2n-1}\left\|Af_2\left(\frac{x}{2^n},-\frac{x}{2^n}\right)\right\| \\ &\le \lim_{n\to\infty} 2^{2n-1}\phi_2\left(\frac{x}{2^n},-\frac{x}{2^n}\right) \\ &\le \lim_{n\to\infty}\frac{L^n}{2}\phi_2(x,-x)=0.\end{aligned}$$

as well as

$$\lim_{n\to\infty} 2^{n-1}\left\|f_2\left(\frac{x}{2^n}\right)+f_2\left(\frac{-x}{2^n}\right)\right\| \le \lim_{n\to\infty}\frac{L^n}{2^{n+1}}\phi_2(x,-x)=0$$

for all $x\in V\backslash\{0\}$. From these and (2.7), we obtain (2.14).

Moreover, we have

$$\left\|2^nAf_2\left(\frac{x}{2^n},\frac{y}{2^n}\right)\right\| \le 2^n\phi_2\left(\frac{x}{2^n},\frac{y}{2^n}\right)\le\frac{L^n}{2^n}\phi_2(x,y)$$

for all $x,y\in V\backslash\{0\}$. Taking the limit as $n\to\infty$ in the above inequality, and using $F_2(0)=0$, we fee that $AF_2(x,y)=0$ for all $x,y\in V$. The proof is ended. □

Corollary 2.6 *Let $f_i: V\to Y$, $i=1,2$, be mappings for which there exist functions $\phi_i:(V\backslash\{0\})^2\to[0,\infty)$, $i=1,2$, such that*

$$\|Qf_i(x,y)\|\le\phi_i(x,y)$$

for all $x,y\in V\backslash\{0\}$. If $f_i(0)=0$, $\phi_i(x,y)=\phi_i(-x,-y)$, $i=1,2$, for all $x,y\in V\backslash\{0\}$, and there exists $0<L<1$ such that the mapping ϕ_1 satisfies (2.9) and ϕ_2 satisfies (2.10) for all $x,y\in V\backslash\{0\}$, then there exist unique quadratic mappings $F_i: V\to Y$, $i=1,2$, such that

$$\|f_1(x)-F_1(x)\|\le\frac{3\phi_1(x,x)+5\phi_1(x,-x)}{4(1-L)}, \tag{2.15}$$

$$\|f_2(x)-F_2(x)\|\le\frac{L(3\phi_2(x,x)+5\phi_2(x,-x))}{8(1-L)} \tag{2.16}$$

for all $x\in V\backslash\{0\}$. In particular, the mappings $F_i, i=1,2$, are given by

$$F_1(x)=\lim_{n\to\infty}\frac{f_1(2^nx)}{4^n}, \tag{2.17}$$

$$F_2(x)=\lim_{n\to\infty}4^nf_2\left(\frac{x}{2^n}\right) \tag{2.18}$$

for all $x\in V$.

Proof. Note that

$$Df_i(x,y)=Qf_i(x,y)-Qf_i(y,-x)+f_i(x,-x)+\frac{1}{2}Qf_i(y,-y)-\frac{1}{2}Qf_i(y,y)$$

for all $x,y\in V$ and $i=1,2$. Put $\varphi_i(x,y):=\phi_i(x,y)+\phi_i(y,-x)+\phi_i(x,-x)+\frac{1}{2}\phi_i(y,y)+\frac{1}{2}\phi_i(y,-y)$ for all $x,y\in V\backslash\{0\}$ and $i=1,2$, then φ_1 (resp. φ_2) satisfies (2.2) (resp. (2.5)). Moreover,

$$\|Df_i(x,y)\|\le\varphi_i(x,y)$$

for all $x,y\in V\backslash\{0\}$ and $i=1,2$. It follows from Theorem 2.3 that there exists a unique mapping $F_1: V\to Y$ satisfying (2.15), which is represented by (2.4).

Observe that

$$\begin{aligned}\lim_{n\to\infty}\left\|\frac{f_1(2^nx)-f_1(-2^nx)}{2^{n+1}}\right\| &= \lim_{n\to\infty}\frac{1}{2^{n+1}}\left\|Qf_1(2^{n-1}x,-2^{n-1}x)-Qf_1(-2^{n-1}x,2^{n-1}x)\right\| \\ &\le \lim_{n\to\infty}\frac{1}{2^{n+1}}\left(\phi_1(2^{n-1}x,-2^{n-1}x)+\phi_1(-2^{n-1}x,2^{n-1}x)\right) \\ &\le \lim_{n\to\infty}\frac{L^n}{2}\left(\phi_1\left(\frac{x}{2},-\frac{x}{2}\right)+\phi_1\left(-\frac{x}{2},\frac{x}{2}\right)\right) \\ &=0\end{aligned}$$

and

$$\lim_{n\to\infty}\left\|\frac{f_1(2^n x)-f_1(-2^n x)}{2\cdot 4^n}\right\| \le \lim_{n\to\infty}\frac{L^n}{2^{n+1}}\left(\phi_1\left(\frac{x}{2},-\frac{x}{2}\right)+\phi_1\left(-\frac{x}{2},\frac{x}{2}\right)\right)=0$$

for all $x\in V\backslash\{0\}$. Due to this fact and (2.4), we get (2.17).

Moreover, we have

$$\left\|\frac{Qf_1(2^n x,2^n y)}{4^n}\right\| \le \frac{\phi_1(2^n x,2^n y)}{4^n} \le \frac{L^n}{2^n}\phi_1(x,y)$$

for all $x,y\in V\backslash\{0\}$. As $n\to\infty$ in the above inequality, we see that $QF_1(x,y)=0$ for all $x,y\in V\backslash\{0\}$. By using $F_1(0)=0$, then we have

$$QF_1(x,0)=0,\ QF_1(0,y)=-\,QF_1\left(\frac{y}{2},-\frac{y}{2}\right)+QF_1\left(-\frac{y}{2},\frac{y}{2}\right)=0$$

for all $x,y\in V\backslash\{0\}$. Therefore, $QF_1(x,y)=0$ for all $x,y\in V$.

On the other hand, Theorem 2.4 guarantees that there exists a unique mapping $F_2: V\to Y$ satisfying (2.16), which is represented by (2.7).

Observe that

$$\begin{aligned}4^n\left\|f_2\left(\frac{x}{2^n}\right)-f_2\left(-\frac{x}{2^n}\right)\right\| =&4^n\left\|Qf_2\left(\frac{x}{2^{n+1}},-\frac{x}{2^{n+1}}\right)-Qf_2\left(-\frac{x}{2^{n+1}},\frac{x}{2^{n+1}}\right)\right\|\\ \le&4^n\left(\phi_2\left(\frac{x}{2^{n+1}},-\frac{x}{2^{n+1}}\right)+\phi_2\left(-\frac{x}{2^{n+1}},\frac{x}{2^{n+1}}\right)\right)\\ \le&L^n\left(\phi_2\left(\frac{x}{2},-\frac{x}{2}\right)+\phi_2\left(-\frac{x}{2},\frac{x}{2}\right)\right)\end{aligned}$$

for all $x\in V\backslash\{0\}$. It leads us to get

$$\lim_{n\to\infty}4^n\left(f_2\left(\frac{x}{2^n}\right)-f_2\left(-\frac{x}{2^n}\right)\right)=0,\ \lim_{n\to\infty}2^n\left(f_2\left(\frac{x}{2^n}\right)-f_2\left(-\frac{x}{2^n}\right)\right)=0$$

for all $x\in V\backslash\{0\}$. Based on these facts and (2.7), we obtain (2.18).

Moreover, we have

$$\left\|4^nQf_2\left(\frac{x}{2^n},\frac{y}{2^n}\right)\right\| \le 4^n\phi_2\left(\frac{x}{2^n},\frac{y}{2^n}\right) \le L^n\phi_2(x,y)$$

for all $x,y\in V\backslash\{0\}$. Going the limit as $n\to\infty$ in the previous inequality, and using $F_2(0)=0$, we get $QF_2(x,y)=0$ for all $x,y\in V$, which complete the proof.

Now, we obtain the stability in the framework of normed spaces using Theorem 2.3 and Theorem 2.4.

Corollary 2.7 *Let X be a normed space and Y a Banach space. Suppose that the mapping $f: X\to Y$ satisfies the inequality*

$$\|Df(x,y)\|\le\theta(\|x\|^p+\|y\|^p)$$

for all $x,y\in X\backslash\{0\}$ with $f(0)=0$, where $\theta\ge 0$ and $p\in(-\infty,1)\cup(2,\infty)$. Then there exists a unique quadratic-additive mapping $F: X\to Y$ such that

$$\|f(x)-F(x)\|\le=\begin{cases}\frac{2\theta}{2^p-4}\|x\|^p & \text{if } p>2,\\ \frac{2\theta}{2-2^p}\|x\|^p & \text{if } p<1,\end{cases}$$

for all $x\in X\backslash\{0\}$.

Proof. This follows from Theorem 2.3 and Theorem 2.4 by putting

$$\varphi(x,y):=\theta(\|x\|^p+\|y\|^p)$$

for all $x,y\in X\backslash\{0\}$ with $L=2^{p-1}<1$ if $p<1$ and $L=2^{2-p}<1$ if $p>2$.

Corollary 2.8 *Let X be a normd space and Y a Banach space. Suppose that the mapping $f: X\to Y$ satisfies the inequality*

$$\|Df(x,y)\|\le\theta\|x\|^p\|y\|^q$$

for all $x, y \in X\backslash\{0\}$ *with* $f(0) = 0$*, where* $\theta \geq 0$ *and* $p + q \in (-\infty, 1) \cup (2, \infty)$*. Then there exists a unique quadratic-additive mapping* $F : X \to Y$ *such that*

$$\|f(x) - F(x)\| \leq= \begin{cases} \frac{\theta\|x\|^{p+q}}{2^{p+q}-4} & \text{if } p+q > 2, \\ \frac{\theta\|x\|^{p+q}}{2(2-2^{p+q})} & \text{if } p+q < 1 \end{cases}$$

for all $x \in X\backslash\{0\}$*.*

Proof. By considering

$$\varphi(x, y) := \theta\|x\|^p\|y\|^q$$

for all $x, y \in X\backslash\{0\}$ with $L = 2^{p+q-1} < 1$ if $p + q < 1$ and $L = 2^{2-p-q} < 1$ if $p + q > 2$, then by Theorem 2.3 and Theorem 2.4, we arrive at the conclusion of the corollary.

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DIFFERENCES OF COMPOSITION OPERATORS FROM LIPSCHITZ SPACE TO WEIGHTED BANACH SPACES IN POLYDISK

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Abstract. Let φ and ψ be holomorphic self-maps of the unit polydisk $\mathbb{D}_n$ in the n-dimensional complex space C^n, denote by C_φ and C_ψ the induced composition operators. In this paper, we estimate the essential norm of the differences of composition operators $C_\varphi - C_\psi$ from Lipschitz space to weighted Banach space in the unit polydisk.

1. Introduction

The algebra of all holomorphic functions on domain Ω will be denoted by $H(\Omega)$, where Ω is a bounded domain in C^n, where $n \geq 1$ is a fixed integer. Let $\mathbb{D}_n = \{z = (z_1, ..., z_n) \in C^n,\ |z_i| < 1,\ 1 \leq i \leq n\}$ be the open unit polydisk of the complex n-dimensional Euclidean space C^n and $H(\mathbb{D}_n)$ be the space of all holomorphic functions on $\mathbb{D}_n$. For $z = (z_1, ..., z_n)$ and $w = (w_1, ..., w_n)$ in C^n, the inner product of z and w is $\langle z, w\rangle = z_1\overline{w}_1 + ... + z_n\overline{w}_n$, where $\langle .,. \rangle$ denotes the inner product. Moreover, $|||z||| = \max_j\{|z_j|\}$ stands for the supremum norm on $\mathbb{D}_n$.

For $z, w \in \mathbb{D}$, the *pseudo-hyperbolic* distance between z and w is defined by

$$\rho(z, w) = |(z - w)/(1 - \overline{w}z)|.$$

It is well known that if $f \in H(\mathbb{D})$, then $\rho(f(z), f(w)) \leq \rho(z, w)$. The Bergman metric on the unit polydisk is given by

$$H_z(u, v) = \sum_{j=1}^{n} \frac{u_j\overline{v_j}}{(1 - |z_j|^2)^2}.$$

The *Kobayashi* distance $k_{\mathbb{D}_n}$ on $\mathbb{D}_n$ is defined by

$$k_{\mathbb{D}_n}(z, w) = \frac{1}{2}\log\frac{1 + |||\phi_z(w)|||}{1 - |||\phi_z(w)|||}, \tag{1.1}$$

where $\phi_z : \mathbb{D}_n \to \mathbb{D}_n$ is the automorphism of $\mathbb{D}_n$ given by

$$\phi_z(w) = \Big(\frac{w_1 - z_1}{1 - \overline{z_1}w_1}, ..., \frac{w_n - z_n}{1 - \overline{z_n}w_n}\Big).$$

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Let v be a strictly positive bounded continuous function (weight) on the open unit polydisk $\mathbb{D}_n$ in $C^n, n \geq 1$. We first introduce the weighted Banach spaces of analytic functions of the following form:

$$H_v^\infty := \Big\{ f \in H(\mathbb{D}_n);\ \|f\|_v = \sup_{z\in\mathbb{D}_n} v(z)|f(z)| < \infty \Big\}$$

endowed with the sup-norm $\|.\|_v$. Spaces of this type appear in the study of growth conditions of analytic functions and have been studied in various articles, see, e.g. [2, 8, 10].

For $0 \leq \alpha < 1$, an $f \in H(\mathbb{D}_n)$ belongs to the Lipschitz space $Lip_\alpha(\mathbb{D}_n)$, if

$$\|f\|_\alpha = |f(0)| + \sup_{z\in\mathbb{D}_n} \sum_{l=1}^{n} \Big| \frac{\partial f}{\partial z_l}(z) \Big| (1-|z_l|^2)^{1-\alpha} < \infty. \tag{1.2}$$

It is easy to show that $Lip_\alpha(\mathbb{D}_n)$ is a Banach space endowed with the norm $\|.\|_\alpha$ (see, e.g.[13, 14]).

Let $\varphi = (\varphi_1(z), ..., \varphi_n(z))$ and $\psi(z) = (\psi_1(z), ..., \psi_n(z))$ be holomorphic self-maps of $\mathbb{D}_n$. The composition operator C_φ induced by φ is defined by

$$(C_\varphi)f(z) = f(\varphi(z))$$

for $z \in \mathbb{D}_n$ and $f \in H(\mathbb{D}_n)$ (see, e.g.[3]). The essential norm of a continuous linear operator T is the distance from T to the set of all compact operators, that is, $\|T\|_e = \inf\{\|T-K\| :\ K$ is compact $\}$. Notice that $\|T\|_e = 0$ if and only if T is compact, so estimates on $\|T\|_e$ lead to conditions for T to be compact (see, e.g.[6, 14, **?**]). In the past few years, many authors have been interested in studying the mapping properties of the differences of two composition operators, that is, an operator of the form

$$T = C_\varphi - C_\psi.$$

The primary motivation for this has been the desire to understand the topological structure of the whole set of composition operators. Most papers in this area have focused on the classical reflexive spaces, but some classical nonreflexive spaces in the unit disc in the complex plane have also recently been discussed. We refer the readers to the recent papers [1, 4, 5, 6, 7, 9, 12] to learn more about the propertied about the differences.

Building on the above foundations we estimate the essential norm for the differences of composition operators induced by φ and ψ acting from Lipschitz space to weighted Banach space in the unit polydisk $\mathbb{D}_n$, where φ and ψ are two holomorphic self-maps of the unit polydisk in n-dimensional complex space C^n. The paper is organized as following: Some lemmas are given in section 2. Section 3 is devoted to the main results.

Throughout the remainder of this paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next.

2. Some Lemmas

Lemma 1. *Assume that $f \in Lip_\alpha(\mathbb{D}_n)$, then*

$$|f(z) - f(w)| \leq n\|f\|_\alpha k_{\mathbb{D}_n}(z, w)$$

for any $z, w \in \mathbb{D}_n$.

Proof. Empolying the definitions in (1.1) and (1.2) we have that

$$\begin{aligned} |f(z)-f(0)| &= \Big|\int_0^1 \frac{\Re f(tz)}{t}dt\Big| = \Big|\sum_{j=1}^n \int_0^1 z_j\frac{\partial f}{\partial \zeta_j}(tz)dt\Big| \\ &\le \sum_{j=1}^n \int_0^1 \frac{|z_j|}{(1-|tz_j|^2)^{1-\alpha}}\Big|\frac{\partial f}{\partial \zeta_j}(tz)\Big|(1-|tz_j|^2)^{1-\alpha}dt \\ &\le \|f\|_\alpha \sum_{j=1}^n \int_0^{|z_j|}\frac{1}{(1-t^2)^{1-\alpha}}dt \\ &\le \|f\|_\alpha \sum_{j=1}^n \int_0^{|z_j|}\frac{1}{1-t^2}dt \\ &= \frac{1}{2}\|f\|_\alpha \sum_{j=1}^n \log\frac{1+|z_j|}{1-|z_j|} \\ &\le n\|f\|_\alpha \frac{1}{2}\log\frac{1+|||z|||}{1-|||z|||}. \end{aligned} \tag{2.3}$$

The last inequality in (2.3) follows from the fact the map $t \to \log((1+t)/(1-t))$ is strictly increasing on $[0,1)$. Setting $z=\phi_w(z)$ and using (1.2), it's evident that

$$|f\circ\phi_w(z) - f\circ\phi_w(w)| \le n\|f\circ\phi_w\|_\alpha \frac{1}{2}\log\frac{1+|||\phi_w(z)|||}{1-|||\phi_w(z)|||}.$$

Replacing $f\circ\phi_w$ by $f\circ\phi_w\circ\phi_w^{-1}$,

$$|f(z)-f(w)| \le n\|f\|_\alpha \frac{1}{2}\log\frac{1+|||\phi_w(z)|||}{1-|||\phi_w(z)|||} \le n\|f\|_\alpha k_{\mathbb{D}_n}(z,w).$$

This completes the proof. □

Lemma 2. *For $f\in Lip_\alpha(\mathbb{D}_n)$ and a fixed $0<\delta<1$, define $G=\{z\in\mathbb{D}_n : |||z|||\le\delta\}$. Then*

$$\lim_{r\to 1}\sup_{\|f\|_\alpha\le 1}\sup_{z\in G}|f(z)-f(rz)| = 0.$$

Proof. Using the definition in (1.2) we obtain that

$$\begin{aligned} &\sup_{z\in G}|f(z)-f(rz)| \\ &= \sup_{z\in G}\Big|\sum_{j=1}^n\Big(f(rz_1,rz_2,...,rz_{j-1},z_j,...,z_n) - f(rz_1,rz_2,...,z_{j+1},...,z_n)\Big)\Big| \\ &\le \sup_{z\in G}\sum_{j=1}^n\Big|\int_r^1 z_j\frac{\partial f}{\partial z_j}(rz_1,...,rz_{j-1},tz_j,z_{j+1},...,z_n)dt\Big| \\ &\le (1-r)n\|f\|_\alpha\sup_{z\in G}\frac{1}{(1-|||z|||^2)^{1-\alpha}} \\ &\le \frac{(1-r)n\|f\|_\alpha}{(1-\delta^2)^{1-\alpha}} \to 0,\ r\to 1 \end{aligned}$$

This ends the proof. □

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3. Main result

In this section we estimate the essential norm of $C_\varphi - C_\psi : Lip_\alpha(\mathbb{D}_n) \to H_v^\infty(\mathbb{D}_n)$. We denote $F_\delta = \{z \in \mathbb{D}_n,\ \max\{|||\varphi(z)|||,\ |||\psi(z)|||\} \le 1-\delta\}$ and $E_\delta = \mathbb{D}_n - F_\delta$ for $0<\delta<1$. We consider the following two conditions

$$M_1 := \max_{1\le l\le n} \lim_{\delta\to 0} \sup_{z\in E_\delta} v(z)|(1-|\psi_l(z)|^2)^\alpha |\varphi_{\varphi_l(z)}(\psi_l(z))|$$

$$M_2 := \max_{1\le l\le n} \lim_{\delta\to 0} \sup_{z\in E_\delta} v(z)|(1-|\varphi_l(z)|^2)^\alpha |\varphi_{\psi_l(z)}(\varphi_l(z))|.$$

Theorem 1. *For any $0<\delta<1$, denote $F_\delta = \{z \in \mathbb{D}_n,\ \max\{|||\varphi(z)|||,\ |||\psi(z)|||\} \le 1-\delta\}$. Suppose $C_\varphi - C_\psi : Lip_\alpha(\mathbb{D}_n) \to H_v^\infty(\mathbb{D}_n)$ is bounded. Then*

$$\max\{M_1,\ M_2\} \le \|C_\varphi - C_\psi\|_e \le 2n \lim_{\delta\to 0} \sup_{z\in E_\delta} v(z) k_{\mathbb{D}_n}(\varphi(z),\psi(z)). \tag{3.4}$$

Proof. The upper estimate. For a fixed $0<r<1$, we have that both $C_{r\varphi}$ and $C_{r\psi}$ are compact operators. For any $0<\delta<1$,

$$\begin{aligned}
&\|C_\varphi - C_\psi\|_{e,Lip_\alpha\to H_v^\infty} \le \|C_\varphi - C_\psi - C_{r\varphi} + C_{r\psi}\|_{Lip_\alpha\to H_v^\infty}\\
&= \sup_{\|f\|_\alpha\le 1} \|(C_\varphi - C_\psi - C_{r\varphi} + C_{r\psi})f\|_{H_v^\infty}\\
&= \sup_{\|f\|_\alpha\le 1} \sup_{z\in\mathbb{D}_n} v(z)|f(\varphi(z)) - f(r\varphi(z)) + f(r\psi(z)) - f(\psi(z))|\\
&\le \sup_{\|f\|_\alpha\le 1} \sup_{z\in F_\delta} v(z)|f(\varphi(z)) - f(r\varphi(z)) + f(r\psi(z)) - f(\psi(z))|\\
&+ \sup_{\|f\|_\alpha\le 1} \sup_{z\in E_\delta} v(z)|f(\varphi(z)) - f(r\varphi(z)) + f(r\psi(z)) - f(\psi(z))|.
\end{aligned} \tag{3.5}$$

Since the weight $v(z)$ is a strictly positive bounded continuous function on the open unit polydisc $\mathbb{D}_n$ and using lemma 2 and we can choose r sufficiently close to 1 such that the first term in (3.5) is less than any given $\varepsilon>0$, and we denote the second term in (3.5) by I. Empolying lemma 1, it follows that

$$\begin{aligned}
I &\le \sup_{\|f\|_\alpha\le 1} \sup_{z\in E_\delta} v(z)(|f(\varphi(z)) - f(\psi(z))| + |f(r\varphi(z)) - f(r\psi(z))|)\\
&\le \sup_{\|f\|_\alpha\le 1} \sup_{z\in E_\delta} v(z) n\|f\|_\alpha (k_{\mathbb{D}_n}(\varphi(z),\psi(z)) + k_{\mathbb{D}_n}(r\varphi(z),r\psi(z)))\\
&\le 2n \sup_{z\in E_\delta} v(z) k_{\mathbb{D}_n}(\varphi(z),\psi(z)),
\end{aligned} \tag{3.6}$$

the last inequality is obtained from $k_{\mathbb{D}_n}(r\varphi(z),r\psi(z)) \le k_{\mathbb{D}_n}(\varphi(z),\psi(z))$. Firstly letting $r\to 1$ and then $\delta\to 0$, the upper estimate yeilds.

The lower estimate. For $l = 1,2,..,n$, set

$$E_\delta^l = \{z\in\mathbb{D}_n : \max(|\varphi_l(z)|, |\psi_1(z)|) > 1-\delta\}.$$

It is easy to see that $E_\delta = \bigcup_{l=1}^n E_\delta^l$. For a fixed l $(1\le l\le n)$, define

$$a_l = \lim_{\delta\to 0} \sup_{z\in E_\delta^l} v(z)(1-|\varphi_l(z)|^2)^\alpha |\varphi_{\psi_l(z)}(\varphi_l(z))|.$$

If we put $\delta_m = 1/m$, then $\delta_m\to 0$ as $m\to\infty$. For the case $\|\varphi_l\|_\infty = 1$ or $\|\psi_l\|_\infty = 1$, then for large enough m with $E_{\delta_m}^l \ne \emptyset$, there exists $z^m \in E_{\delta_m}^l$ such that

$$\lim_{m\to\infty} v(z^m)(1-|\varphi_l(z^m)|^2)^\alpha |\varphi_{\psi_l(z^m)}(\varphi_l(z^m))| = a_l. \tag{3.7}$$

Since $z^m \in E^l_{\delta_m}$ implies that $|\varphi_l(z^m)| > 1-\delta_m$ or $|\psi_l(z^m)| > 1-\delta_m$, without loss of generality we assume that $|\varphi_l(z^m)| \to 1$. Set

$$f_m(z) = \frac{1-|\varphi_l(z^m)|^2}{(1-\overline{\varphi_l(z^m)}z_l)^{1-\alpha}} \cdot \frac{\langle \varphi_{\psi_l(z^m)}(z), \varphi_{\psi_l(z^m)}(\varphi_l(z^m))\rangle}{|\varphi_{\psi_l(z^m)}(\varphi_l(z^m))|}.$$

We can easily obtain that $(f_m)_{m\in\mathbb{N}}$ converges to zero uniformly on compact subsets of $\mathbb{D}_n$ as $m\to\infty$ and $\sup_{k\in\mathbb{N}} \|f_m\|_\alpha \le C$. Thus for any compact operator $K : Lip_\alpha \to H^\infty_v$, we get $\|Kf_m\|_{H^\infty_v} \to 0,\ m\to\infty$. Moreover, it is obvious that

$$f_m(\varphi(z^m)) = (1-|\varphi_l(z^m)|^2)^\alpha |\varphi_{\psi_l(z^m)}(\varphi_l(z^m))|,\ f_m(\psi(z^m)) = 0. \tag{3.8}$$

Thus using the above results, (3.7) and (3.8), it is clear that

$$\begin{aligned}
&\|C_\varphi - C_\psi - K\|_{Lip_\alpha\to H^\infty_v} \ge C \limsup_{m\to\infty} \|(C_\varphi - C_\psi - K)f_m\|_{H^\infty_v} \\
&\ge C \limsup_{m\to\infty} (\|(C_\varphi - C_\psi)f_m\|_{H^\infty_v} - \|Kf_m\|_{H^\infty_v}) \\
&= C \limsup_{m\to\infty} \|(C_\varphi - C_\psi)f_m\|_{H^\infty_v} \\
&= C \limsup_{m\to\infty} \sup_{z\in\mathbb{D}_n} v(z)|f_m(\varphi(z)) - f_m(\psi(z))| \\
&\ge C \limsup_{m\to\infty} v(z^m)|f_m(\varphi(z^m)) - f_m(\psi(z^m))| \\
&= C \limsup_{m\to\infty} v(z^m)|(1-|\varphi_l(z^m)|^2)^\alpha |\varphi_{\psi_l(z^m)}(\varphi_l(z^m))| \\
&= Ca_l = C \lim_{\delta\to 0} \sup_{z\in E^l_\delta} v(z)(1-|\varphi_l(z)|^2)^\alpha |\varphi_{\psi_l(z)}(\varphi_l(z))|
\end{aligned}$$

From the above inequality we obtain that

$$\|C_\varphi - C_\psi\|_{e,Lip_\alpha\to H^\infty_v} \ge C \lim_{\delta\to 0} \sup_{z\in E^l_\delta} v(z)(1-|\varphi_l(z)|^2)^\alpha |\varphi_{\psi_l(z)}(\varphi_l(z))|. \tag{3.9}$$

If both $\|\varphi_l\|_\infty < 1$ and $\|\psi_l\|_\infty < 1$, in this condition, when δ is small enough, E^l_δ is empty, and without loss of generality we may assume that

$$\lim_{\delta\to 0} \sup_{z\in E^l_\delta} v(z)(1-|\varphi_l(z)|^2)^\alpha |\varphi_{\psi_l(z)}(\varphi_l(z))| = 0. \tag{3.10}$$

Since the above inequality (3.9) and (3.10) holds for every $1 \le l \le n$, thus we obtain that

$$\|C_\varphi - C_\psi\|_{e,Lip_\alpha\to H^\infty_v} \ge C \max_{1\le l\le n} \lim_{\delta\to 0} \sup_{z\in E^l_\delta} v(z)|(1-|\varphi_l(z)|^2)^\alpha |\varphi_{\psi_l(z)}(\varphi_l(z))|. \tag{3.11}$$

Now for each $l = 1, 2, ..., n$, we define

$$b_l = \lim_{\delta\to 0} \sup_{z\in E_\delta} v(z)(1-|\varphi_l(z)|^2)^\alpha |\varphi_{\psi_l(z)}(\varphi_l(z))|.$$

Then for any $\varepsilon > 0$, there exists a δ_0 with $0 < \delta_0 < 1$ such that

$$v(z)(1-|\varphi_l(z)|^2)^\alpha |\varphi_{\psi_l(z)}(\varphi_l(z))| > b_l - \varepsilon \tag{3.12}$$

whenever $z \in E_{\delta_0}$ and $l = 1, 2, ..., n$. From the above definition we know that $z \in E^l_{\delta_0}$ implies that $z \in E_{\delta_0}$, then by (3.11) and (3.12) we obtain that

$$\begin{aligned}
&\|C_\varphi - C_\psi\|_{e,Lip_\alpha\to H^\infty_v} \ge C \max_{1\le l\le n} (b_l - \varepsilon) \\
&= C \max_{1\le l\le n} \lim_{\delta\to 0} \sup_{z\in E_\delta} v(z)|(1-|\varphi_l(z)|^2)^\alpha |\varphi_{\psi_l(z)}(\varphi_l(z))| - C\varepsilon.
\end{aligned}$$

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Now letting $\varepsilon \to 0$ in the above inequality we obtain that

$$\|C_\varphi - C_\psi\|_{e,Lip_\alpha \to H_v^\infty} \geq C \max_{1\leq l\leq n} \lim_{\delta\to 0} \sup_{z\in E_\delta} v(z)(1-|\varphi_l(z)|^2)^\alpha |\varphi_{\psi_l(z)}(\varphi_l(z))|. \quad (3.13)$$

Using the similar proof of (3.13) we can get

$$\|C_\varphi - C_\psi\|_{e,Lip_\alpha \to H_v^\infty} \geq C \max_{1\leq l\leq n} \lim_{\delta\to 0} \sup_{z\in E_\delta} v(z)|(1-|\psi_l(z)|^2)^\alpha |\psi_{\varphi_l(z)}(\psi_l(z))|. \quad (3.14)$$

Combining (3.13) and (3.14), we get the lower estimate for the essential norm of the differences. □

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THE PATH COMPONENT OF THE SET OF GENERALIZED COMPOSITION OPERATORS ON THE BLOCH TYPE SPACES

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ABSTRACT. In this note, we give a characterization of the path component of the set of generalized composition operator on Bloch type spaces.

Keywords: Path component, composition operator, Bloch type spaces

1. INTRODUCTION

Let $\mathbb{D}$ be the unit disk of the complex plane $\mathbb{C}$, and $H(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$. $f \in H(\mathbb{D})$ belongs to the Bloch type space $\mathcal{B}^\alpha$, if

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f'(z)| < \infty,$$

where $0 < \alpha < \infty$. It is known that $\mathcal{B}^\alpha$ is a Banach space under the $\|\cdot\|_{\mathcal{B}^\alpha}$ norm. If $\alpha = 1$, $\mathcal{B}^\alpha$ is just the well-known Bloch space. More details about properties on Bloch type space are given in [4], [32] and [16].

We denote $S(\mathbb{D})$ be the set of analytic self-maps of $\mathbb{D}$. Every analytic self-map $\varphi \in S(\mathbb{D})$ induces a linear composition operator C_φ from $H(\mathbb{D})$ to itself. A general and concerning problem in the investigation of composition operator is to characterize operator theoretic properties of C_φ in terms of function theoretic properties of φ. To learn more conclusions about the composition operator, see [6].

For $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$, Li and Stevic [10] defined the generalized composition operator C^g_φ as follows:

$$C^g_\varphi(f)(z) = \int_0^z f'(\varphi(w))g(w)dw, \ f \in H(\mathbb{D}).$$

The boundedness and compactness of the generalized composition operator from Zygmund spaces to Bloch-type spaces were considered in [10]. Lindstrom and Sanatpour [15] gave the characterization of the generalized composition operator between Zygmund spaces. We can also refer to [11–14], [21–30] for the study of the operator C^g_φ and its generalizations. The composition operators between Bloch type spaces have been studied by several authors, for example [1–3, 5, 17, 19].

Recently, lots of researchers are interested in the difference of two composition operators, that is, an operator of the form $T = C_\varphi - C_\psi$, where $\varphi, \psi \in S(\mathbb{D})$. For example, Shapiro

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and Sundberg [20] studied the difference of composition operators on Hardy spaces. In [18], MacCluer, Ohno and Zhao considered it on H^∞. In [7] and [8], Hosokawa and Ohno investigated it on Bloch spaces. The purpose of studying the difference of composition operators is to investigate the topological structure of the set of composition operators acting on a given function space. Li [9] gave the sufficient and necessary conditions for the boundedness and compactness of the differences of generalized composition operator on the Bloch space. Yang, Luo, and Zhu [31] generalized Li's results between Bloch type spaces, which help us to study the topological structure of the set of generalized composition operators on the Bloch type spaces. In fact, we give a sufficient condition for the path component of the set of generalized composition operator on Bloch type spaces.

2. NOTATIONS AND AUXILIARY RESULTS

For $w, z \in \mathbb{D}$, the pseudo-hyperbolic distance between z and w is defined by

$$\rho(w,z) = |\frac{w-z}{1-\bar{w}z}|.$$

Let

$$u_s(z,w) = (1-s)z + sw, \phi_s(\varphi(z),\psi(z)) = (1-s)\varphi(z) + s\psi(z),$$

where $s \in [0,1], w \in \mathbb{D}$, $\varphi, \psi \in S(\mathbb{D})$ and simply denote $\phi_s(\varphi(z),\psi(z))$ by $\phi_s(z)$.

Let

$$\Gamma(\varphi) = \{\{z_n\} \in \mathbb{D} : |\varphi(z_n)| \to 1\},$$
$$\Gamma(\psi) = \{\{z_n\} \in \mathbb{D} : |\psi(z_n)| \to 1\}.$$

Obviously, $\Gamma(\phi_s) \subset \Gamma(\varphi) \cap \Gamma(\psi)$.

Define

$$D_\alpha^{\varphi,g}(z) = \frac{g(z)}{(1-|\varphi(z)|^2)^\alpha},\ D_{\alpha,\beta}^{\varphi,g}(z) = \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} g(z),$$

$$D_\alpha^{\psi,h}(z) = \frac{h(z)}{(1-|\psi(z)|^2)^\alpha},\ D_{\alpha,\beta}^{\psi,h}(z) = \frac{(1-|z|^2)^\beta}{(1-|\psi(z)|^2)^\alpha} h(z),$$

and

$$D_\alpha^{\phi_s}(z) = \frac{1-|z|^\alpha}{(1-|\phi(z)|^2)^\alpha}[(1-s)g(z) + sh(z)],$$

$$C_{\phi_s} f(z) = \int_0^z f'((1-s)\varphi(w) + s\psi(w))[(1-s)g(w) + sh(w)]dw, f \in \mathcal{B}^\alpha.$$

Let

$$I_1(z) = D_{\alpha,\beta}^{\varphi,g}\rho(\varphi(z),\psi(z)),$$
$$I_2(z) = D_{\alpha,\beta}^{\psi,h}\rho(\varphi(z),\psi(z)),$$

and

$$I_3(z) = D_{\alpha,\beta}^{\varphi,g}(z) - D_{\alpha,\beta}^{\psi,h}(z).$$

Lemma 2.1. *([7, Lemma 4.1]) Let $z, w \in \mathbb{D}$ and $\rho(z,w) = \lambda < 1$. Then the map $s \mapsto \rho(u_s, w)$ is continuous and decreasing on $[0,1]$.*

Lemma 2.2. *([31, Theorem 1.]) The following statements are equivalent:*
(1) $C_\varphi^g - C_\psi^h : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ *is bounded.*
(2) $\sup_{z\in\mathbb{D}} |I_1(z)| < \infty$ *and* $\sup_{z\in\mathbb{D}} |I_3(z)| < \infty$.
(3) $\sup_{z\in\mathbb{D}} |I_2(z)| < \infty$ *and* $\sup_{z\in\mathbb{D}} |I_3(z)| < \infty$.

Lemma 2.3. *([31, Theorem 4.]) Let $0 < \alpha, \beta < \infty$ and $\varphi, \psi \in S(\mathbb{D})$, $g, h \in H(\mathbb{D})$, if $C_\varphi^g - C_\psi^h : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded, and C_φ^g, C_ψ^h: $\mathcal{B}^\alpha \to \mathcal{B}^\beta$ are not compact, then $C_\varphi^g - C_\psi^h : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact if and only if the following two conditions hold.*
(1) $D(g,\varphi) = D(h,\psi) \neq \emptyset$, $D(g,\varphi) \subset \Gamma(\psi)$.
(2) *For arbitrary* $\{z_n\} \subset \Gamma(\varphi) \cap \Gamma(\psi)$,

$$\lim_{n\to\infty} |I_1(z_n)| = \lim_{n\to\infty} |I_2(z_n)| = \lim_{n\to\infty} |I_3(z_n)| = 0.$$

Lemma 2.4. *If $t < 0$ or $t > 1$, then $1 - x^t \leq t(1-x)$.*

Proof. Let $f(x) = 1 - x^t - t(1-x)$, then

$$f'(x) = -tx^(t-1) + t, f''(x) = -t(t-1)x(t-2).$$

Obviously, $f'(1) = 0$, $f''(1) \neq 0$, $f''(x) > 0$ for $t < 0$, $f''(x) < 0$ for $t > 1$. □

Lemma 2.5. *Let φ, ψ be analytic self maps of the unit disk $\mathbb{D}$, then*
(1) *For any $z \in \mathbb{D}$, when $\alpha < 1$, we have*

$$\left|D_\alpha^{\varphi,g}(z) - D_\alpha^{\phi_s}(z)\right| \leq \left|D_\alpha^{\varphi,g}(z) - D_\alpha^{\psi,h}(z)\right| + (2-\alpha)\left|D_\alpha^{\varphi,g}(z)\right|\rho^2(\varphi(z),\psi(z)).$$

(2) *For any $z \in \mathbb{D}$, when $\alpha \geq 1$, we have*

$$\left|D_\alpha^{\varphi,g}(z) - D_\alpha^{\phi_s}(z)\right| \leq \left|D_\alpha^{\varphi,g}(z) - D_\alpha^{\psi,h}(z)\right| + \alpha\left(\left|D_\alpha^{\varphi,g}(z)\right| + \left|D_\alpha^{\psi,h}(z)\right|\right)\rho^2(\varphi(z),\psi(z)).$$

Proof. (1) The lemma is trivially for $s = 0$ or 1. In the following, we assume $0 < s < 1$. For arbitrary $z \in \mathbb{D}$, denote $\zeta = \frac{1-|\varphi(z)|^2}{1-|\phi_s(z)|^2}$ and $\xi = \frac{1-|\psi(z)|^2}{1-|\phi_s(z)|^2}$. By the definition of $D_\alpha^{\varphi,g}(z)$, $D_\alpha^{\psi,h}(z)$ and $D_\alpha^{\phi_s}(z)$, it is easy to see

$$\begin{aligned}
D_\alpha^{\phi_s}(z) &= \frac{1-|z|^\alpha}{(1-|\phi(z)|^2)^\alpha}[(1-s)g(z) + sh(z)] \\
&= (1-s)\frac{(1-|\varphi(z)|^2)^\alpha}{(1-|\phi_s(z)|^2)^\alpha} D_\alpha^{\varphi,g}(z) + s\frac{(1-|\psi(z)|^2)^\alpha}{(1-|\phi_s(z)|^2)^\alpha} D_\alpha^{\psi,h}(z) \\
&= D_\alpha^{\varphi,g} - (1-s)\zeta^\alpha D_\alpha^{\varphi,g}(z) - s\xi^\alpha D_\alpha^{\psi,h}(z)
\end{aligned}$$

and

$$\begin{aligned}
\left|D_\alpha^{\varphi,g}(z) - D_\alpha^{\phi_s}(z)\right| &= \left|D_\alpha^{\varphi,g}(z) - (1-s)\frac{(1-|\varphi(z)|^2)^\alpha}{(1-|\phi_s(z)|^2)^\alpha} D_\alpha^{\varphi,g}(z) - s\frac{(1-|\psi(z)|^2)^\alpha}{(1-|\phi_s(z)|^2)^\alpha} D_\alpha^{\psi,h}(z)\right| \\
&= \left|D_\alpha^{\varphi,g}(z) - (1-s)\zeta^\alpha D_\alpha^{\varphi,g}(z) - s\xi^\alpha D_\alpha^{\psi,h}(z)\right| \\
&= \left|D_\alpha^{\varphi,g}(z)(1-(1-s)\zeta^\alpha) - D_\alpha^{\psi,h}(1-(1-s)\zeta^\alpha) + D_\alpha^{\psi,h}(1-(1-s)\zeta^\alpha) - s\xi^\alpha D_\alpha^{\psi,h}(z)\right| \\
&\leq \left|D_\alpha^{\varphi,g}(z) - D_\alpha^{\psi,h}\right|\left|(1-(1-s)\zeta^\alpha)\right| + \left|D_\alpha^{\psi,h}\right|\left|(1-(1-s)\zeta^\alpha) - s\xi^\alpha\right| \\
&\leq \left|D_\alpha^{\varphi,g}(z) - D_\alpha^{\psi,h}\right|\left|s\zeta^\alpha\right| + \left|D_\alpha^{\varphi,g}\right|\left|(1-(1-s)\zeta^\alpha) - s\xi^\alpha\right|.
\end{aligned} \tag{2.1}$$

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$$\left|D_\alpha^{\varphi,g}-D_\alpha^{\phi_s}(z)\right| \le \left|D_\alpha^{\varphi,g}-D_\alpha^{\psi,h}\right|\left|s\zeta^\alpha\right| + \left|D_\alpha^{\varphi,g}\right|\left|(1-(1-s)\zeta^\alpha)-s\xi^\alpha\right|. \tag{2.2}$$

By simply calculating and the proving process of Proposition 4.2 in [7], we get

$$0 \le \frac{s(1-s)|\varphi(z)-\psi(z)|^2}{1-|\phi_s(t)|^2} = 1-(1-s)\zeta - s\xi \le \rho^2(\varphi(z),\psi(z)). \tag{2.3}$$

Firstly, we consider the case $0<\alpha<1$.

Since $s\zeta = s\frac{1-|\varphi(z)|^2}{1-|\phi_s(z)|^2} \le 1$, then

$$s\zeta^\alpha \le s^{1-\alpha} \le 1. \tag{2.4}$$

Now, we estimate $(1-(1-s)\zeta^\alpha)-s\xi^\alpha$.

Choosing

$$f(\zeta) = 1-(1-s)\zeta^\alpha - s\xi^\alpha - (1-(1-s)\zeta - s\xi), \tag{2.5}$$

then

$$\begin{aligned} f(\zeta) &= (1-s)\zeta(1-\zeta^{\alpha-1}) + s\xi(1-\xi^{\alpha-1}) \\ &\le (\alpha-1)((1-s)\zeta(1-\zeta)+s\xi(1-\xi)) \\ &= (\alpha-1)((1-s)\zeta^2 - s\xi^2) - (\alpha-1)(1-(1-s)\zeta - s\xi). \end{aligned} \tag{2.6}$$

The last inequality above is obtained by Lemma 2.4. Uniting (2.5) and (2.6), we obtain

$$\begin{aligned} &1-(1-s)\zeta^\alpha - s\xi^\alpha - (1-(1-s)\zeta - s\xi) \\ \le&(1-(1-s\zeta)\alpha - s\xi) - (\alpha-1)(1-(1-s)\zeta^2 - s\xi^2) - (\alpha-1)(1-(1-s\zeta)-s\xi) \\ =&(2-\alpha)(1-(1-s)\zeta - s\xi) + (\alpha-1)(1-(1-s)\zeta^2 - s\xi^2). \end{aligned} \tag{2.7}$$

and

$$1-(1-s)\zeta^2 - s\xi^2 = \frac{s|\psi(z)|^2(1-|\psi(z)|^2)+(1-s)|\varphi(z)|^2(1-|\varphi(z)|^2)) + s(1-s)|\varphi(z)-\psi(z)|^2}{(1-|\phi_s(z)|^2)^2} > 0 \tag{2.8}$$

Hence,

$$1-(1-s)\zeta^\alpha - s\xi^\alpha \le (2-\alpha)(1-(1-s)\zeta - s\xi) \le (2-\alpha)\rho^2(\varphi(z),\psi(z)). \tag{2.9}$$

Combining (2.1), (2.4) and (2.9), we get

$$\left|D_\alpha^{\varphi,g}(z)-D_\alpha^{\phi_s}(z)\right| \le \left|D_\alpha^{\varphi,g}(z)-D_\alpha^{\psi,h}(z)\right| + (2-\alpha)\left|D_\alpha^{\varphi,g}(z)\right|\rho^2(\varphi(z),\psi(z)).$$

This complete the proof of (1).

Next, we are going to prove (2).

If $\alpha=1$, then by (2.3), we have

$$1-(1-s)\zeta^\alpha - s\xi^\alpha = 1-(1-s)\zeta - s\xi \le \rho^2(\varphi(z),\psi(z)) = \alpha\rho^2(\varphi(z),\psi(z). \tag{2.10}$$

If $\alpha > 1$, then

$$\begin{aligned} 1-(1-s)\zeta^\alpha - s\xi^\alpha &= 1-s-(1-s)\zeta^\alpha + s - s\xi^\alpha \\ &= (1-s)(1-\zeta^\alpha) + s(1-\xi^\alpha) \\ &\le \alpha(1-s)(1-\zeta) + s(1-\xi)) \\ &= \alpha(1-(1-s)\zeta - s\xi) \\ &\le \alpha\rho^2(\varphi(z),\psi(z)). \end{aligned} \tag{2.11}$$

The first inequality in (2.11) above is obtained by Lemma 2.4.

If $\xi \le 1$, using (2.1) and (2.11), we obtain

$$\big|D_\alpha^{\varphi,g}(z) - D_\alpha^{\phi_s}(z)\big| \le \big|D_\alpha^{\varphi,g}(z) - D_\alpha^{\psi,h}(z)\big| + \alpha\big|D_\alpha^{\varphi,g}(z)\big|\rho^2(\varphi(z),\psi(z)). \tag{2.12}$$

If $\xi \ge 1$, for any $s \in (0,1)$, we have $|\psi(z)| \le |\phi_s(z)| \le (1-s)|\varphi(z)| + s|\psi(z)|$ and $|\psi(z)| \le |\varphi(z)|$. Then $|\phi_s(z)| \le |\varphi(z)|$ and $\frac{1-|\varphi(z)|^2}{1-|\phi_s(z)|^2} = \zeta \le 1$. Combing (2.1), (2.10) with (2.11), it is obvious that

$$\big|D_\alpha^{\varphi,g}(z) - D_\alpha^{\phi_s}(z)\big| \le \big|D_\alpha^{\varphi,g}(z) - D_\alpha^{\psi,h}(z)\big| + \alpha\big|D_\alpha^{\psi,h}(z)\big|\big)\rho^2(\varphi(z),\psi(z)). \tag{2.13}$$

Due to (2.11), (2.12) and (2.13) above, we infer that

$$\big|D_\alpha^{\varphi,g}(z) - D_\alpha^{\phi_s}(z)\big| \le \big|D_\alpha^{\varphi,g}(z) - D_\alpha^{\psi,h}(z)\big| + \alpha\big(\big|D_\alpha^{\varphi,g}(z)\big| + \big|D_\alpha^{\psi,h}(z)\big|\big)\rho^2(\varphi(z),\psi(z)).$$

□

3. MAIN RESULTS

Proposition 3.1. *Let φ, ψ be analytic self maps of the unit disk $\mathbb{D}$, $g, h \in H(\mathbb{D})$. Suppose that C_φ^g and C_ψ^h are bounded but not compact on $\mathcal{B}^\alpha$. For any $s \in [0,1]$, when $C_\varphi^g - C_\psi^h$ is compact on $\mathcal{B}^\alpha$, then we have*

(1) $D_{\phi_s}^\alpha \subset \Gamma(\varphi) \cap \Gamma(\psi)$, where $D_{\phi_s}^\alpha = \{\{z_n\} \subset \mathbb{D} : |\varphi(z_n)| \to 1, |D_{\phi_s}^\alpha(z_n)| \not\to 1\}$.

(2) For any $\{z\}_n \subset \Gamma(\varphi) \cap \Gamma(\psi)$, we have

$$\lim_{n\to\infty}(D_\alpha^{\varphi,g}(z_n) - D_\alpha^{\phi_s}(z_n)) = \lim_{n\to\infty}(D_\alpha^{\varphi,g}(z_n)\rho(\varphi(z_n),\phi_s(z_n)) = 0.$$

Proof. (1) It is trivial.

(2) For any $\{z_n\} \subset \Gamma(\varphi) \cap \Gamma(\psi)$, it follows from Lemma 2.3 that

$$\begin{aligned} \lim_{n\to\infty}\big|D_\alpha^{\varphi,g}(z_n) - D_\alpha^{\phi_s}(z_n)\big| &= \lim_{n\to\infty}\big|(D_\alpha^{\varphi,g}(z_n)\big|\rho(\varphi(z_n),\phi_s(z_n)) \\ &= \lim_{n\to\infty}\big|D_\alpha^{\psi,h}(z_n)\big|\rho(\varphi(z_n),\phi_s(z_n)) \\ &= 0. \end{aligned}$$

Applying Lemma 2.5,

$$\lim_{n\to\infty}|D_\alpha^{\varphi,g}(z_n) - D_\alpha^{\phi_s}(z_n)| = 0,$$

then by Lemma 2.1,

$$|D_\alpha^{\varphi,g}(z_n)|\rho(\varphi(z_n),\phi_s(z_n)) \le |D_\alpha^{\varphi,g}|\rho(\varphi(z_n),\psi(z_n)) \to 0.$$

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Equivalently,

$$\lim_{n\to\infty}(D_\alpha^{\varphi,g}(z_n)-D_\alpha^{\phi_s}(z_n))=\lim_{n\to\infty}(D_\alpha^{\varphi,g}(z_n)\rho(\varphi(z_n),\phi_s(z_n))=0.$$

□

Theorem 3.2. *Let φ,ψ be analytic self maps of the unit disk $\mathbb{D}$, $g,h\in H(\mathbb{D})$. Suppose that C_φ^g and C_ψ^h are bounded but not compact on $\mathcal{B}^\alpha$. If $C_\varphi^g-C_\psi^h$ is compact on $\mathcal{B}^\alpha$, then the following two conclusions are equivalent:*

(1) *For any $\{z_n\}\subset\Gamma(\psi)\backslash\Gamma(\varphi)$, $D_\alpha^{\varphi,g}(z_n)\to 0$ as $n\to\infty$ and for any $\{z_n\}\subset\Gamma(\varphi)\backslash\Gamma(\psi)$, $D_\alpha^{\phi,h}(z_n)\to 0$ as $n\to\infty$.*

(2) *The map $s\mapsto C_{\phi_s}:[0,1]\to C_{\phi_s}(\mathcal{B}^\alpha)$ is continous.*

Proof. (1) $\Longrightarrow$ (2) We only need to prove the continuity at $s=0$.

Let

$$t(s)=\sup_{z\in\mathbb{D}}|D_\alpha^{\varphi,g}(z)-D_\alpha^{\phi_s}(z)|+\sup_{z\in\mathbb{D}}|D_\alpha^{\varphi,g}(z)|\rho(\varphi(z)),\psi(z)).$$

Then, it is easy to see that $\|C_\varphi^g-C_{\phi_s}\|_{\mathcal{B}^\alpha}\le t(s)$. By lemma 2.3 and the conditions of (1), we have

$$\begin{aligned}\lim_{n\to\infty}|D_\alpha^{\varphi,g}(z_n)-D_\alpha^{\psi,h}(z_n)|&=\lim_{n\to\infty}|D_\alpha^{\varphi,g}(z_n)|\rho(\varphi(z_n),\psi(z_n))\\&=\lim_{n\to\infty}|D_\alpha^{\psi,h}(z_n)|\rho(\varphi(z_n),\psi(z_n))\\&=0\end{aligned}$$

Hence, for any $\varepsilon>0$, there exists $r_1\in(0,1)$ such that for every $z\in\Gamma_{r_1}(\varphi)=\{z\in\mathbb{D}:|\varphi(z)|>r_1\}$,

$$|D_\alpha^{\varphi,g}(z)-D_\alpha^{\psi,h}(z)|<\frac{\varepsilon}{2},$$

and

$$|D_\alpha^{\varphi,g}(z)|\rho(\varphi(z),\psi(z)<\frac{\varepsilon}{2}.$$

Applying Lemma 2.5, we obtain that

$$|D_\alpha^{\varphi,g}(z)-D_\alpha^{\psi,h}(z)|<\frac{\varepsilon}{2}+\alpha\varepsilon=(\frac{1}{2}+\alpha)\varepsilon. \tag{3.1}$$

If $z\in\mathbb{D}\backslash\Gamma_{r_1}(\varphi)$, $D_\alpha^{\varphi,g}-D_\alpha^{\phi_s}$ is uniformly convergence to 0 when s approaches to 0, then there exists an s_1 very close to 0 such that for any $s<s_1$,

$$\sup_{z\in\mathbb{D}\backslash\Gamma_{r_1}}|D_\alpha^{\varphi,g}(z)-D_\alpha^{\phi_s}(z)|<\varepsilon. \tag{3.2}$$

For any $s<s_1$, uniting (3.1) and (3.2), we get

$$\sup_{z\in\mathbb{D}}|D_\alpha^{\varphi,g}(z)-D_\alpha^{\phi_s}(z)|<\varepsilon. \tag{3.3}$$

Hence,

$$\sup_{z\in\mathbb{D}}|D_\alpha^{\varphi,g}(z)-D_\alpha^{\phi_s}(z)|\to 0 \ as\ s\to 0. \tag{3.4}$$

Next, we are going to prove that

$$\sup_{z\in\mathbb{D}} |D_\alpha^{\varphi,g}(z)|\rho(\varphi(z)),\psi(z))\to 0\ as\ s\to 0.$$

For any $\{z_n\}\subset\Gamma(\varphi)$, applying Proposition 3.1 and Lemma 2.3, we have

$$\lim_{n\to\infty}(D_\alpha^{\varphi,g}\rho(\varphi(z_n),\phi_s(z_n))=0.$$

This implies that there exists an $r_2\in(0,1)$, such that for any $z\in\Gamma_{r_2}(\varphi)=\{z\in\mathbb{D}:|\varphi(z)|>r_2\}$,

$$|D_\alpha^{\varphi,g}(z)|\rho(\varphi(z),\psi(z))\le|D_\alpha^{\varphi,g}(z)|\rho(\varphi(z),\psi(z))<\varepsilon.$$

And because $\rho(\varphi(z),\psi(z))$ uniformly converges to 0 on $\mathbb{D}\backslash\Gamma_{r_2}(\varphi)$, we can find a sufficiently small positive number s_2, such that for any $s<s_2$,

$$\sup_{z\in\mathbb{D}\backslash\Gamma_{r_2}(\varphi)}|D_\alpha^{\varphi,g}(z)|\rho(\varphi(z),\phi_s(z))<\varepsilon.$$

Then,

$$\sup_{z\in\mathbb{D}}|D_\alpha^{\varphi,g}(z)|\rho(\varphi(z),\phi_s(z))\to 0\ as\ s\to 0. \tag{3.5}$$

Combing (3.4) with (3.5), we obtain that $t(s)$ converges to 0 as s approaches to 0, which finishes the proof of continuity.

(2) $\Longrightarrow$ (1) Assume there is a sequence $\{z_n\}\subset\Gamma(\psi)\backslash\Gamma(\varphi)$, such that $D_\alpha^{\varphi,g}(z_n)\to\delta\ne0$ as $n\to\infty$. Let $\lambda\in\mathbb{D}$ and $\lambda\ne0$, define the test function f_λ and g_λ respectively as follows:

$$f_\lambda(z)=\frac{1}{2^{\alpha+1}}\frac{1-|\lambda|^2}{\alpha\bar\lambda(1-\bar\lambda z)^\alpha},$$

$$g_\lambda(z)=\frac{1-|\lambda|^2}{(\alpha+1)2^{\alpha+1}}\Big(\frac{\lambda-z}{\bar\lambda(1-\bar\lambda z)^{\alpha+1}}+\frac{1}{\alpha\bar\lambda^2(1-\bar\lambda z)^{\alpha+1}}\Big).$$

Then $\|f_\lambda\|_{\mathcal{B}^\alpha}\le1$, $\|g_\lambda\|_{\mathcal{B}^\alpha}\le1$,

$$\begin{aligned}\|C_\varphi^g-C_{\phi_s}\|&\ge\|(C_\varphi^g-C_{\phi_s})g_{\varphi(z_n)}\|_{\mathcal{B}^\alpha}\\&\ge\frac{1}{2^{\alpha+1}}\Big|D_\alpha^{\phi_s}(z_n)\frac{(1-|\varphi(z_n)|^2)(1-|\varphi_s(z_n)|^2)^\alpha}{(1-\overline{\varphi(z_n)}\phi_s(z_n))^{\alpha+1}}\rho(\varphi(z_n),\phi_s(z_n))\Big|.\end{aligned}\tag{3.6}$$

Because $z_n\in\Gamma(\psi)\backslash\Gamma(\varphi)$, then $\phi_s(z_n)\not\to1$ and $\lim_{n\to\infty}\rho(\varphi(z_n),\phi_s(z_n))\ne0$. And $s\mapsto C_{\phi_s}$ is continous at 0, then by (3.6), we have

$$\Big|D_\alpha^{\phi_s}(z_n)\frac{(1-|\varphi(z_n)|^2)(1-|\varphi_s(z_n)|^2)^\alpha}{(1-\overline{\varphi(z_n)}\phi_s(z_n))^{\alpha+1}}\rho(\varphi(z_n),\phi_s(z_n)\Big|\to0,n\to\infty,s\to0.$$

By the compactness of $C_\varphi^g-C_\psi^h$, it is bounded. It follows from Lemma 2.1, Lemma 2.2 and lemma 2.5 that $C_\varphi^g-C_{\phi_s}$ is bounded. So

$$\begin{aligned}\|C_\varphi^g-C_{\phi_s}\|&\ge\|(C_\varphi^g-C_{\phi_s})g_{\varphi(z_n)}\|_{\mathcal{B}^\alpha}\\&\ge\frac{1}{2^{\alpha+1}}\Big(\Big|D_\alpha^{\varphi,g}(z_n)\Big|-\Big|D_\alpha^{\phi_s}(z_n)\Big|\Big|\frac{(1-|\varphi(z_n)|^2)(1-|\varphi_s(z_n)|^2)^\alpha}{(1-\overline{\varphi(z_n)}\phi_s(z_n))^{\alpha+1}}\Big|\Big).\end{aligned}\tag{3.7}$$

Letting $n\to\infty$ and $s\to0$, we have

$$\|C_\varphi^g-C_{\phi_s}\|\ge\frac{\delta}{2^{\alpha+1}}>0.\tag{3.8}$$

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For $\varphi(z_n) \equiv 0$, suppose $\lambda \in \mathbb{D}, \lambda \neq 0$ and

$$h_\lambda(z) = \frac{1}{2^{\alpha+1}} \frac{1}{\alpha\bar{\lambda}(1-\bar{\lambda}z)^\alpha}. \tag{3.9}$$

Then $h_\lambda \in \mathcal{B}^\alpha$ and $\|h_\lambda\|_{\mathcal{B}^\alpha} \leq 1$. If $s \neq 0$, then $\phi_s(z_n) \to s \neq 0$. Choosing $\lambda = \phi_s(z_n)$, we have

$$\begin{aligned} \|(C_\varphi^g - C_{\phi_s})h_{\phi_s(z_n)}\|_{\mathcal{B}^\alpha} &\geq \|(C_\varphi^g - C_{\phi_s})h_{\phi_s(z_n)}\|_{\mathcal{B}^\alpha} \\ &\geq \frac{1}{2^{\alpha+1}}\Big(|(1-|z_n|^2)^\alpha \varphi'(z_n)|D_\alpha^{\phi_s}(z_n) - \frac{D_\alpha^{\phi_s}(z_n)}{1-|\phi_s(z_n)|^\alpha}\Big). \end{aligned}$$

For $\Gamma(\psi)\backslash\Gamma(\varphi)$, Proposition 3.1 implies that $D_\alpha^{\phi_s}(z_n) \to 0$. Letting $n \to \infty$ and $s \to 0$, we get

$$\|C_\varphi^g - C_{\phi_s}\| \geq \delta > 0. \tag{3.10}$$

It follows from (3.8) and (3.10) that the map $s \mapsto C_{\phi_s}$ is not continuous at 0, which is a contradiction. So we complete the proof. □

Corollary 3.3. *Let φ, ψ be two analytic self maps of the unit disk $\mathbb{D}$, $g, h \in H(\mathbb{D})$. Suppose C_φ^g and C_ψ^h are bounded but not compact on $\mathcal{B}^\alpha$. If $C_\varphi^g - C_\psi^h$ is compact on $\mathcal{B}^\alpha$, then C_φ^g and C_ψ^h are in the same path component of $\mathcal{B}^\alpha$.*

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THE GENERALIZED HYERS-ULAM STABILITY OF QUADRATIC FUNCTIONAL EQUATIONS ON RESTRICTED DOMAINS

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Abstract. In this paper, we prove the generalized Hyers-Ulam stability for the functional equation

$$f(ax+by)+abf(x-y)=a(a+b)f(x)+b(a+b)f(y)$$

for some real numbers a, b with $2a+b=1$ on a restricted domain using the fixed point theorem.

Key words. Generalized Hyers-Ulam stability, Quadratic functional equation, Banach space, Restricted domains, Fixed point theorem

1. Introduction

In 1940, S. M. Ulam [15] proposed the following stability problem :

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In 1941, Hyers [7] answered this problem under the assumption that the groups are Banach spaces. Aoki [1] and Rassias [11] generalized the result of Hyers. Rassias [11] solved the generalized Hyers-Ulam stability of the functional inequality

$$\|f(x+y)-f(x)-f(y)\| \le \epsilon(\|x\|^p+\|y\|^p)$$

for some $\epsilon \ge 0$ and p with $0 < p < 1$ and all $x, y \in X$, where $f : X \longrightarrow Y$ is a function between Banach spaces. A generalization of the Rassias theorem was obtained by Găvruta [6] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassis approach.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called *a quadratic functional equation* and a solution of a quadratic functional equation is called *quadratic*. The generalized Hyers-Ulam stability problem for a quadratic functional equation was proved by Skof [13] for mappings $f : X \longrightarrow Y$, where X is a normed space and Y is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [3] proved the generalized Hyers-Ulam stability for a quadratic functional equation.

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Skof [14] was the first author to slove the Hyers-Ulam problem for additive mappings on a restricted domain and in 1998, Jung [8] investigated the Hyers-Ulam stability for additive and quadratic mappimgs on resticted domains. In 2002, Rassias [12] proved that if $f : X \longrightarrow Y$ satisfies the following inequality

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \le \delta, \tag{1.2}$$

then there exists a unique quadratic mapping which is approximately. Recently, Najati and Jung [9] showed that the functional equation

$$f(ax+by)+abf(x-y)=af(x)+bf(y) \tag{1.3}$$

is equivalent to (1.1) if a, b are non-zero real numbers with $a+b=1$ and proved that the Hyers-Ulam stability for the functional equation (1.3) on a resticted domain if f is even. Elhoucien and Youssef [5] showed the results in [9] by removing the Najati-Jung's assumption that f is even.

In this paper, we consider the functional equation

$$f(ax+by)+abf(x-y)=a(a+b)f(x)+b(a+b)f(y) \tag{1.4}$$

for fixed non-zero real numbers a, b with $2a+b=1$, $a \neq 1$ and we prove the generalized Hyers-Ulam stability of it on a restricted domain. Throughout this paper, we assume that X is a normed space and Y is a Banach space.

2. Solutions of (1.4)

Najati and Jung [9] showed that if an even mapping $f : X \longrightarrow Y$ satisfies (1.3), then f is quadratic and that if a, b are rational numbers, then f satisfies (1.3) if and only if f is quadratic. Elhoucien and Youssef [5] showed that if a mapping $f : X \longrightarrow Y$ satisfies (1.3), then f is additive-quadratic. In this section, we will show that if a mapping $f : X \longrightarrow Y$ satisfies (1.4), then f is quadratic.

Theorem 2.1. *Let $f : X \longrightarrow Y$ be a mapping satisfying (1.4). Then f is a quadratic mapping.*

Proof. Letting $x=y=0$ in (1.4), since $2a+b=1$, we have $(a^2+ab+b^2-1)f(0) = 3a(a-1)f(0)=0$. Since $a \neq 0, 1$, $f(0)=0$. Letting $y=0$ in (1.4), we have

$$f(ax)=a^2f(x) \tag{2.1}$$

for all $x \in X$. Letting $x=0$ in (1.4), we have

$$f(by)=b(a+b)f(y)-abf(-y) \tag{2.2}$$

for all $y \in X$. Let $f_o(x)=\frac{f(x)-f(-x)}{2}$. Then f_o satisfies (1.4), (2.1) and (2.2) and hence by (2.2), we have

$$f_o(bx)=bf_o(x) \tag{2.3}$$

for all $x \in X$. By (1.4), we have

$$f_o(ax+by)+f_o(ax-by)=2a(a+b)f_o(x)-ab[f_o(x+y)+f_o(x-y)] \tag{2.4}$$

for all $x, y \in X$. Letting $y=ay$ in (2.4), by (2.1), we have

$$a[f_o(x+by)+f_o(x-by)]=2(a+b)f_o(x)-b[f_o(x+ay)+f_o(x-ay)] \tag{2.5}$$

for all $x, y \in X$ and letting $x=bx$ in (2.5), by (2.3), we have

$$f_o(bx+ay)+f_o(bx-ay)=2(a+b)f_o(x)-a[f_o(x+y)+f_o(x-y)] \tag{2.6}$$

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for all $x, y \in X$. Interchanging x and y in (1.4), we have

$$f_o(bx + ay) + f_o(bx - ay) = 2b(a + b)f_o(x) + ab[f_o(x + y) + f_o(x - y)] \tag{2.7}$$

for all $x, y \in X$. By (2.6) and (2.7), since $a(a + b) \neq 0$, we have

$$f_o(x + y) + f_o(x - y) - 2f_o(x) = 0$$

for all $x, y \in X$ and hence f_o is additive. By (2.1), we have $a^2 f_o(x) = af_o(x)$ and since $a \neq 0, 1$, $f_o(x) = 0$ for all $x \in X$.

Let $f_e(x) = \frac{f(x)+f(-x)}{2}$. Then $f_e : X \longrightarrow Y$ is an even mapping satisfying (1.4) and so f_e satisfies (2.1) and (2.2). Replacing x and y by $2x$ and $x + y$ in (1.4), we have

$$f_e(x + by) + abf_e(x - y) - a(a + b)f_e(2x) - b(a + b)f_e(x + y) = 0 \tag{2.8}$$

for all $x, y \in X$. Since $a(a + b) \neq 0$ and f_e is even, by (2.8), we have

$$f_e(2x) = 4f_e(x), \ \ f_e(bx) = b^2 f(x) \tag{2.9}$$

for all $x \in X$. Letting $x = bx$ in (2.8), by (2.9), we have

$$bf_e(x + y) + af_e(bx - y) - 4ab(a + b)f_e(x) - (a + b)f_e(bx + y) = 0 \tag{2.10}$$

for all $x, y \in X$. Interchanging x and y in (2.10), we have

$$bf_e(x + y) + af_e(x - by) - 4ab(a + b)f_e(y) - (a + b)f_e(x + by) = 0 \tag{2.11}$$

for all $x, y \in X$. Letting $y = -y$ in (2.8), we have

$$f_e(x - by) + abf_e(x + y) - 4a(a + b)f_e(x) - b(a + b)f_e(x - y) = 0 \tag{2.12}$$

for all $x, y \in X$. Since $b(1 - 2a^2 - 2ab - b^2) = 2ab(a + b)$, by (2.8), (2.11), and (2.12), we have

$$f_e(x + y) + f_e(x - y) = 2f_e(x) + 2f_e(y)$$

for all $x, y \in X$ and so f_e is quadraric. Since $f = f_o + f_e = f_e$, f is quadratic. □

Corollary 2.2. *Let $f : X \longrightarrow Y$ be a mapping. If a, b are rational numbers, then f is quadratic if and only if f satisfies (1.4).*

3. Stability of (1.4)

In this section, we investigate the generalized Hyers-Ulam stability of (1.4) on a restricted domain. Jung [8] proved the Hyers-Ulam stability for additive and quadratic mappings on a resticted domain and Najati and Jung [9] proved the Hyers-Ulam syability of (1.3) on a resticted domain if f is an even mapping. Rahimi, Najati and Bae [10] investigated the generalized Hyers-Ulam syability of (1.1) with the bounded function $\delta + \epsilon(\|x\|^{2p} + \|y\|^{2p}) + \theta\|x\|^p\|y\|^p$ on a resticted domain.

Theorem 3.1. *Let $\phi : X^2 \longrightarrow [0, \infty)$ be a mapping and M a non-negative real number. Let $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$. Suppose that f satisfies the following inequality*

$$\|f(ax + by) + abf(x - y) - a(a + b)f(x) - b(a + b)f(y)\| \leq \delta + \phi(x, y) \tag{3.1}$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq M$ and for some non-negative real number δ. Then we have

$$\|f(2x) - 4f(x)\| \leq \Phi(x, y) \tag{3.2}$$

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for all $x, y \in X$ *with* $\|y\| \geq M$*, where*

$$\begin{aligned}\Phi(x,y) = \{&\phi(2x-2by, x+(1-b)y) + \phi(2x-2by, x-(1+b)y)\\ &+ \phi(2x+2by, x+(1+b)y) + \phi(2x+2by, x-(1-b)y)\\ &+ |b|[\phi(2x+2y, x+2y) + \phi(2x+2y, x) + \phi(2x-2y, x) + \phi(2x-2y, x-2y)]\\ &+ \phi(2x, x+2y) + \phi(2x, x-2y) + 4(|b|+2)\delta\} \times |2a(a+b)|^{-1}.\end{aligned}$$

Proof. Let $x, y \in X$ with $\|x\| + \|y\| \geq M$. Then $\|2x\| + \|x+y\| \geq \|x\| + \|y\| \geq M$. Hence by (3.1), we have

$$\begin{aligned}&\|f(x+by) + abf(x-y) - a(a+b)f(2x) - b(a+b)f(x+y)\|\\ &\leq \delta + \phi(2x, x+y)\end{aligned} \tag{3.3}$$

and letting $y = -y$ in (3.3), we have

$$\begin{aligned}&\|f(x-by) + abf(x+y) - a(a+b)f(2x) - b(a+b)f(x-y)\|\\ &\leq \delta + \phi(2x, x-y).\end{aligned} \tag{3.4}$$

By (3.3) and (3.4), we have

$$\begin{aligned}&\|f(x+by) - f(x-by) + bf(x-y) - bf(x+y)\|\\ &\leq 2\delta + \phi(2x, x+y) + \phi(2x, x-y).\end{aligned} \tag{3.5}$$

Let $x, y \in X$ with $\|y\| \geq M$. Since $\|x-by\| + \|y\| \geq M$ and $\|x+by\| + \|y\| \geq M$, by (3.5), we have

$$\begin{aligned}&\|f(x) - f(x-2by) + bf(x-(1+b)y) - bf(x+(1-b)y)\|\\ &\leq 2\delta + \phi(2x-2by, x+(1-b)y) + \phi(2x-2by, x-(1+b)y)\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}&\|f(x+2by) - f(x) + bf(x-(1-b)y) - bf(x+(1+b)y)\|\\ &\leq 2\delta + \phi(2x+2by, x+(1+b)y) + \phi(2x+2by, x-(1-b)y).\end{aligned} \tag{3.7}$$

Since $\|x+y\| + \|y\| \geq M$ and $\|x-y\| + \|-y\| \geq M$, by (3.5), we have

$$\begin{aligned}&\|f(x+(1+b)y) - f(x+(1-b)y) + bf(x) - bf(x+2y)\|\\ &\leq 2\delta + \phi(2x+2y, x+2y) + \phi(2x+2y, x)\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}&\|f(x-(1+b)y) - f(x-(1-b)y) + bf(x) - bf(x-2y)\|\\ &\leq 2\delta + \phi(2x-2y, x) + \phi(2x-2y, x-2y).\end{aligned} \tag{3.9}$$

Since $\|x\| + \|2y\| \geq M$, by (3.3) and (3.4), we have

$$\begin{aligned}&\|f(x+2by) + abf(x-2y) - a(a+b)f(2x) - b(a+b)f(x+2y)\|\\ &\leq \delta + \phi(2x, x+2y).\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}&\|f(x-2by) + abf(x+2y) - a(a+b)f(2x) - b(a+b)f(x-2y)\|\\ &\leq \delta + \phi(2x, x-2y).\end{aligned} \tag{3.11}$$

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Note that

$$\begin{aligned}
&2a(a+b)[f(2x)-4f(x)]\\
&=-[f(x)-f(x-2by)+bf(x-(1+b)y)-bf(x+(1-b)y)]\\
&+[f(x+2by)-f(x)+bf(x-(1-b)y)-bf(x+(1+b)y)]\\
&+b[f(x+(1+b)y)-f(x+(1-b)y)+bf(x)-bf(x+2y)]\\
&+b[f(x-(1+b)y)-f(x-(1-b)y)+bf(x)-bf(x-2y)]\\
&-[f(x+2by)+abf(x-2y)-a(a+b)f(2x)-b(a+b)f(x+2y)]\\
&-[f(x-2by)+abf(x+2y)-a(a+b)f(2x)-b(a+b)f(x-2y)]
\end{aligned} \tag{3.12}$$

for all $x, y \in X$ with $\|y\| \geq M$. By (3.6), (3.7), (3.8), (3.9), (3.10), (3.11), we have (3.2). □

We apply the fixed point method to investigate the generalized Hyers-Ulam stability for the functional equation (1.4).

Definition 3.2. Let X be a set. A function $d : X \times X \longrightarrow [0, \infty]$ is called *a generalized metric on* X if d satisfies

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$, and
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Now, we consider the following fixed point theorem :

Theorem 3.3. [4] *Let (X, d) be a complete generalized metric space and $J : X \longrightarrow X$ a strictly contractive mapping with a Lipschitz constant L with $0 < L < 1$. Then for each element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty \tag{3.13}$$

for all nonnegative integers n or there is a nonnegative integer k such that

(1) $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq k$,
(2) a sequence $\{J^n x\}$ converges to a fixed point y^* of J,
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^k x, y) < \infty\}$, and
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

Now, we will prove the stability of (1.4) on a restrcted domain.

Theorem 3.4. *Let $\phi : X^2 \longrightarrow [0, \infty)$ be a function such that*

$$\phi(2x, 2y) \leq L\phi(x, y) \tag{3.14}$$

for all $x, y \in X$ for some positive real number L with $L < 1$. Let $f : X \longrightarrow Y$ be a mapping with (3.1). Then there exists a unique quadratic mapping $Q : X \longrightarrow Y$ such that f satisfies (1.4) and

$$\|Q(x) - f(x)\| \leq \frac{1}{4(1-L)} \Phi(x, y) \tag{3.15}$$

for all $x \in X$ and $y \in X$ with $\|y\| \geq M$.

Proof. By Theorem 3.1, the following inequality

$$\|f(x) - 2^{-2} f(2x)\| \leq 2^{-2} \Phi(x, y) \tag{3.16}$$

holds for all $x, y \in X$ with $\|y\| \geq M$.

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Let $\Omega = \{g : X \longrightarrow Y \mid g(0) = 0\}$. Define a generalized metric d on Ω by $d(g, h) = inf\{C \in [0, \infty) \mid \|g(x) - h(x)\| \le C\Phi(x, y), \forall x, y \in X \ \ with \ \ \|y\| \ge M\}$. We claim that (Ω, d) is a complete metric space. Let $\{g_n\}$ be a Cauchy sequence in (Ω, d) and $\epsilon > 0$. Then there is a positive integer k such that $d(g_n, g_m) \le \epsilon$ for all $n, m \ge k$. Pick $y_0 \in X$ with $\|y_0\| \ge M$ and let $x \in X$. Since $\|g_n(x) - g_m(x)\| \le \epsilon\Phi(x, y_0)$ for all $n, m \ge k$, $\{g_n(x)\}$ is a Cauchy sequence in Y and hence we can define a mapping $g : X \longrightarrow Y$ by $g(x) = \lim_{n\longrightarrow\infty} g_n(x)$. Clearly, $g \in \Omega$ and $\lim_{n\longrightarrow\infty} g_n = g$. Thus (Ω, d) is a complete metric space.

Define a map $J : \Omega \longrightarrow \Omega$ by $Jh(x) = \frac{1}{4}h(2x)$ for all $x \in X$. Let $g, h \in \Omega$. Suppose that C is a positive real number such that

$$\|g(x) - h(x)\| \le C\Phi(x, y)$$

for all $x, y \in X$ with $\|y\| \ge M$. By (3.14), we have

$$\|Jg(x) - Jh(x)\| = \frac{1}{4}\|g(2x) - h(2x)\| \le \frac{1}{4}C\Phi(2x, 2y) \le \frac{1}{4}CL\Phi(x, y)$$

for all $x, y \in X$ with $\|y\| \ge M$ and hence we have

$$d(Jg, Jh) \le \frac{L}{4}d(g, h)$$

for all $g, h \in \Omega$. Since $0 < L < 4$, J is a strictly contractive mapping and by (3.16), we have

$$d(Jf, f) \le \frac{1}{4}.$$

By Theorem 3.3, $\{J^n f\}$ converges to the unique fixed element Q of J in $Y = \{h \in \Omega \mid d(f, h) < \infty\}$ and (3.15) holds. Further, we have

$$Q(x) = \lim_{n\longrightarrow\infty} J^n f(x) = \lim_{n\longrightarrow\infty} 2^{-2n} f(2^n x)$$

for all $x \in X$ and we have (3.15). Moreover, $Q(0) = 0$, because $f(0) = 0$.

Now, we claim that Q satisafies (1.4). First, suppose that $x \ne 0$ or $y \ne 0$. Replacing x and y by $2^n x$ and $2^n y$ in (3.1), respectively and deviding both sides of (3.1) by 2^{2n}, we have

$$\begin{aligned} &\|2^{-2n} f(2^n(ax + by)) + 2^{-2n} abf(2^n(x - y)) \\ &- a(a + b)2^{-2n} f(2^n x) - b(a + b)2^{-2n} f(2^n y)\| \le \frac{1}{4^n}[L^n\phi(x, y) + \delta] \end{aligned} \tag{3.17}$$

for all $x, y \in X$ and sufficiently large positive integer n. Letting $n \longrightarrow \infty$ in (3.17), Q satisfies (1.4). Clealy, if $x = 0$ and $y = 0$, then Q satisfies (1.4). By Theorem 2.1, Q is quadratic.

Assume that $Q_1 : X \longrightarrow Y$ is another quadratic mapping satisfying (1.4) and (3.15). Then we have

$$\|Q_1(x) - f(x)\| \le \frac{1}{4(1 - L)}\Phi(x, y)$$

for all $x \in X$ and $y \in X$ with $\|y\| \ge M$ and so

$$d(Q_1, f) \le \frac{1}{4(1 - L)} < \infty.$$

By (3) of Theorem 3.3, $Q = Q_1$. □

Skof [13](Jung [8], resp.) proved an asymptotic property of aditive (quadratic, resp.) mappings. We consider such property for (1.4).

Corollary 3.5. *A mapping $f : X \longrightarrow Y$ satisfies (1.4) if and only if the asymptotic condition*

$$\|f(ax+by)+abf(x-y)-a(a+b)f(x)-b(a+b)f(y)\| \longrightarrow 0 \ \ as \ \ \|x\|+\|y\| \longrightarrow \infty$$

holds.

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Hesitant fuzzy soft set and its lattice structures

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Abstract: Hesitant fuzzy set and soft set were introduced by Torra and Molodtsov, respectively. The two sets have been used successfully as effective mathematical tools for dealing with vagueness and uncertainties. By combining hesitant fuzzy set and soft set, in this paper, we propose a new model named hesitant fuzzy soft set, which can be regarded as an extension of many models, such as hesitant fuzzy set, soft set, fuzzy soft set, interval-valued fuzzy soft set and multi-fuzzy soft set. Some basic operations of hesitant fuzzy soft set are defined and some desirable properties of those operations are investigated. Furthermore, the lattice structures of hesitant fuzzy soft set are discussed.

Keywords: Hesitant fuzzy set; soft set; fuzzy soft set; hesitant fuzzy soft set; lattice

1 Introduction

Soft set was firstly proposed by Molodtsov [1], it is a new mathematical tool for modeling vagueness and uncertainty. Since its appearance, soft set theory has attracted more and more attention from many researchers and many important results on soft set have been achieved in theory and application. Maji and Biswas et al. [2] defined some basic operations. Ali et al. [3, 4] gave some new operations on soft sets and studied some algebraic structures of soft sets. Yang and Guo [5] introduced some kernels and closures of soft set relations. Many authors applied soft sets to some algebraic structures such as groups, rings, fields and modules [6–8]. The applications of soft set in decision making and other areas could be found in [9–12].

At the same time, in order to extend the application ranges of soft set, fuzzy extension of soft set theory has become a hot research topic. Maji et al. [13] introduced the notions of fuzzy soft set. Jiang et al. [14] and Majumdar and Samanta [15] further generalized fuzzy soft set to intuitionistic fuzzy soft set and generalised fuzzy soft set, respectively. Yang et al. [16] proposed the concept of interval-valued fuzzy soft set by combining the interval-valued fuzzy set and soft set. Some other generalized models of soft set could be seen in [17–19]

Recently, Torra [20] introduced hesitant fuzzy set which is a new extension of fuzzy set. It permits the membership degree of an element to a set to be represented as some possible values between 0 and 1. Presently, work on hesitant fuzzy set is making progress rapidly and lots of results on hesitant fuzzy set have been obtained [21–25]. The main goal of this paper is to combine the hesitant fuzzy set and soft set and obtain a new hybrid model named hesitant fuzzy soft set. It can be viewed as a hesitant fuzzy extension of the soft set or a generalization of the hesitant fuzzy set.

The rest of this paper is structured as follows. The following section briefly reviews some basic notions of soft set, fuzzy soft set and hesitant fuzzy set. Two new operations on hesitant fuzzy element are defined, and some of their properties are investigated. In Section 3, the concept of

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hesitant fuzzy soft set is first proposed by combining hesitant fuzzy set and soft set. Some operations on hesitant fuzzy soft set are given and some of their properties are studied. In Section 4, we discuss the lattice structures of hesitant fuzzy soft set. The conclusion is finally reached in Section 5.

2 Preliminary

Let U be an initial universe of objects and E the set of parameters in relation to objects in U. Parameters are often attributes, characteristics, or properties of objects. Let $P(U)$ denote the power set of U and $A \subseteq E$. Molodtsov [1] first gave the definition of soft set as follows.

Definition 2.1. [1] A pair (F, A) is called a soft set over U, where $A \subseteq E$ and F is a set valued mapping given by $F : A \to P(U)$.

Maji [13] introduced fuzzy soft set which is an fuzzy extension of soft set.

Definition 2.2. [13] Let $\mathcal{P}(U)$ be the set of all fuzzy subsets of U. A pair $(\mathcal{F}, A)$ is called a fuzzy soft set over U, where $\mathcal{F}$ is a set valued mapping given by $\mathcal{F} : A \to \mathcal{P}(U)$.

As a generalization form of fuzzy set, hesitant fuzzy set (HFS) was first introduced by Torra [20] as follows.

Definition 2.3. [20] Let X be a reference set, an HFS on X is in terms of a function that when applied to X returns a subset of $[0, 1]$, which can be represented as $H = \left\{\frac{h_H(x)}{x}|x \in X\right\}$, where $h_H(x)$ is a set of some values in $[0, 1]$, denoting the possible membership degrees of the element $x \in X$ to the set H.

For convenience, Xu and Xia [21,22] called $h_H(x)$ an hesitant fuzzy element (HFE) with respect to x of H. It is worth noting that the number of values of different $HFEs$ may be different, in this paper, let $l(h_H(x))$ denote the number of values of $h_H(x)$. We arrange the values of $h_H(x)$ in increasing order, and let $h_H^{\sigma(j)}(x)$ be the jth largest value of $h_H(x)$.

Definition 2.4. [20] Let $H = \left\{\frac{h_H(x)}{x}|x \in X\right\}$ be an HFS. Then
(1) H is said to be an empty hesitant set, denoted by Φ, if $h_H(x) = 0$ for all $x \in X$;
(2) H is said to be a full hesitant set, denoted by $\mathcal{I}$, if $h_H(x) = 1$ for all $x \in X$;
(3) H is said to be a complete ignorance set, denoted by $\mathcal{W}$, if $h_H(x) = [0, 1]$ for all $x \in X$.

Definition 2.5. [20] Let $\lambda > 0$, h, h_1 and h_2 be three $HFEs$, some operations on them are given as follows:
(1) $h_1 \cup h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{max(\gamma_1, \gamma_2)\}$;
(2) $h_1 \cap h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2}\{min(\gamma_1, \gamma_2)\}$;
(3) $h^c = \cup_{\gamma \in h}\{1 - \gamma\}$.

We further define the strict union and the strict intersection for $HFEs$ h_1 and h_2, which will be useful in the sequel.

Definition 2.6. *Let h_1 and h_2 be two $HFEs$, $h_i^- = min\{\gamma_i|\gamma_i \in h_i\}$ and $h_i^+ = max\{\gamma_i|\gamma_i \in h_i\}(i = 1, 2)$. The strict union and the strict intersection of h_1 and h_2 are defined as follows:*
(1) $h_1 \sqcup h_2 = \cup_{\gamma_i \in h_i, i=1,2}\{\gamma_i|\gamma_i > min(h_1^+, h_2^+)$ *or* $\gamma_1 = \gamma_2\}$;
(2) $h_1 \sqcap h_2 = \cup_{\gamma_i \in h_i, i=1,2}\{\gamma_i|\gamma_i < max(h_1^-, h_2^-)$ *or* $\gamma_1 = \gamma_2\}$;

For example, let $h_1 = \{0.2, 0.3, 0.6, 0.8\}$ and $h_2 = \{0.4, 0.5, 0.8, 0.9\}$, then $h_1 \sqcup h_2 = \{0.8, 0.9\} \neq \{0.4, 0.5, 0.6, 0.8, 0.9\} = h_1 \cup h_2$, $h_1 \sqcap h_2 = \{0.2, 0.3\} \neq \{0.2, 0.3, 0.4, 0.5, 0.6, 0.8\} = h_1 \cap h_2$.

In fact, all the above operations on $HFEs$ can be suitable for $HFSs$. Some relationships can be further established for these operations on $HFEs$.

Theorem 2.7. *For three HFEs h, h_1 and h_2, the following is valid:*
(1) $h_1^c \sqcup h_2^c = (h_1 \sqcap h_2)^c$;
(2) $h_1^c \sqcap h_2^c = (h_1 \sqcup h_2)^c$.

Proof. (1) Since $h_1 \sqcap h_2 = \cup_{\gamma_i \in h_i, i=1,2}\{\gamma_i | \gamma_i < max(h_1^-, h_2^-) \ or \ \gamma_1 = \gamma_2\}$, then

$$(h_1 \sqcap h_2)^c = \cup_{\gamma_i \in h_i, i=1,2}\{1-\gamma_i | \gamma_i < max(h_1^-, h_2^-) \ or \ \gamma_1 = \gamma_2\}$$

Since $h_1^c = \cup_{\gamma_1 \in h_1}\{1-\gamma_1\}$ and $h_2^c = \cup_{\gamma_2 \in h_2}\{1-\gamma_2\}$, then

$$\begin{aligned} h_1^c \sqcup h_2^c &= \{\cup_{\gamma_1 \in h_1}\{1-\gamma_1\}\} \sqcup \{\cup_{\gamma_2 \in h_2}\{1-\gamma_2\}\} \\ &= \cup_{\gamma_i \in h_i, i=1,2}\{1-\gamma_i | 1-\gamma_i < min(1-h_1^-, 1-h_2^-) \ or \ 1-\gamma_1 = 1-\gamma_2\} \\ &= \cup_{\gamma_i \in h_i, i=1,2}\{1-\gamma_i | \gamma_i < 1-min(1-h_1^-, 1-h_2^-) \ or \ \gamma_1 = \gamma_2\} \\ &= \cup_{\gamma_i \in h_i, i=1,2}\{1-\gamma_i | \gamma_i < max(h_1^-, h_2^-) \ or \ \gamma_1 = \gamma_2\} \end{aligned}$$

□

Theorem 2.8. *For three HFEs h_1, h_2 and h_3, the following is valid:*
(1) $(h_1 \cup h_2) \cup h_3 = h_1 \cup (h_2 \cup h_3)$;
(2) $(h_1 \cap h_2) \cap h_3 = h_1 \cap (h_2 \cap h_3)$;
(3) $h_1 \cup (h_2 \cap h_3) = (h_1 \cup h_2) \cap (h_1 \cup h_3)$;
(4) $h_1 \cap (h_2 \cup h_3) = (h_1 \cap h_2) \cup (h_1 \cap h_3)$.

Proof. (2) and (4) are similar to (1) and (3), respectively, so we only prove (1) and (3).
(1) Since $(h_1 \cup h_2) = \cup_{\gamma_i \in h_i, i=1,2}\{max(\gamma_1, \gamma_2)\}$, then

$$\begin{aligned} (h_1 \cup h_2) \cup h_3 &= \{\cup_{\gamma_i \in h_i, i=1,2}\{max(\gamma_1, \gamma_2)\}\} \cup h_3 \\ &= \cup_{\gamma_i \in h_i, i=1,2,3}\{max(max(\gamma_1, \gamma_2), \gamma_3)\} \\ &= \cup_{\gamma_i \in h_i, i=1,2,3}\{max(\gamma_1, \gamma_2, \gamma_3)\} \\ &= \cup_{\gamma_i \in h_i, i=1,2,3}\{max(\gamma_1, max(\gamma_2, \gamma_3))\} \\ &= h_1 \cup (h_2 \cup h_3). \end{aligned}$$

(3) Since $(h_2 \cap h_3) = \cup_{\gamma_i \in h_i, i=2,3}\{min(\gamma_2, \gamma_3)\}$, then

$$\begin{aligned} h_1 \cup (h_2 \cap h_3) &= h_1 \cup \{\cup_{\gamma_i \in h_i, i=2,3}\{min(\gamma_2, \gamma_3)\}\} \\ &= \cup_{\gamma_i \in h_i, i=1,2,3}\{max(\gamma_1, min(\gamma_2, \gamma_3))\} \\ &= \cup_{\gamma_i \in h_i, i=1,2,3}\{min(max(\gamma_1, \gamma_2), max(\gamma_2, \gamma_3))\} \\ &= \{\cup_{\gamma_i \in h_i, i=1,2}\{max(\gamma_1, \gamma_2)\}\} \cap \{\cup_{\gamma_i \in h_i, i=1,3}\{max(\gamma_1, \gamma_3)\}\} \\ &= (h_1 \cup h_2) \cap (h_1 \cup h_3) \end{aligned}$$

□

Theorem 2.9. *For three HFEs h_1, h_2 and h_3, the following is valid:*
(1) $(h_1 \sqcap h_2) \sqcap h_3 = h_1 \sqcap (h_2 \sqcap h_3)$;
(2) $(h_1 \sqcup h_2) \sqcup h_3 = h_1 \sqcup (h_2 \sqcup h_3)$;
(3) $(h_1 \sqcup h_2) \sqcap h_1 = h_1$;
(4) $(h_1 \sqcap h_2) \sqcup h_1 = h_1$.

Proof. (2) and (4) are similar to (1) and (3), respectively, so we only prove (1) and (3).
(1) Since $h_1 \sqcap h_2 = \cup_{\gamma_i \in h_i, i=1,2}\{\gamma_i | \gamma_i < max(h_1^-, h_2^-) \ or \ \gamma_1 = \gamma_2\}$, then

$$\begin{aligned} (h_1 \sqcap h_2) \sqcap h_3 &= \{\cup_{\gamma_i \in h_i, i=1,2}\{\gamma_i | \gamma_i < max(h_1^-, h_2^-) \ or \ \gamma_1 = \gamma_2\}\} \sqcap h_3 \\ &= \cup_{\gamma_i \in h_i, i=1,2,3}\{\gamma_i | \gamma_i < max(max(h_1^-, h_2^-), h_3^-) \ or \ \gamma_1 = \gamma_2 = \gamma_3\} \\ &= \cup_{\gamma_i \in h_i, i=1,2,3}\{max(h_1^-, h_2^-, h_3^-) \ or \ \gamma_1 = \gamma_2 = \gamma_3\} \\ &= \cup_{\gamma_i \in h_i, i=1,2,3}\{\gamma_i | \gamma_i < max(h_1^-, max(h_2^-, h_3^-)) \ or \ \gamma_1 = \gamma_2 = \gamma_3\} \\ &= h_1 \sqcap \{\cup_{\gamma_i \in h_i, i=2,3}\{\gamma_i | \gamma_i < max(h_2^-, h_3^-) \ or \ \gamma_2 = \gamma_3\}\} \\ &= h_1 \sqcap (h_2 \sqcap h_3). \end{aligned}$$

(3) Since $h_1 \sqcup h_2 = \cup_{\gamma_i \in h_i, i=1,2}\{\gamma_i | \gamma_i > min(h_1^+, h_2^+)\ or\ \gamma_1 = \gamma_2\}$,
i) If $h_1^+ \leq h_2^+$, then $min(h_1^+, h_2^+) = h_1^+$. It follows that $h_1 \sqcup h_2 = \cup_{\gamma_i \in h_i, i=1,2}\{\gamma_2 | \gamma_2 > h_1^+\ or\ \gamma_2 = \gamma_1\}$. By Definition 2.6, we have $(h_1 \sqcup h_2) \sqcap h_1 = h_1$.
ii) If $h_1^+ > h_2^+$, then $min(h_1^+, h_2^+) = h_2^+$. It follows that $h_1 \sqcup h_2 = \cup_{\gamma_i \in h_i, i=1,2}\{\gamma_1 | \gamma_1 > h_2^+\ or\ \gamma_1 = \gamma_2\}$. By Definition 2.6, we have $(h_1 \sqcup h_2) \sqcap h_1 = h_1$.

□

3 Hesitant fuzzy soft set

In this section, we present an extended soft set model which is called hesitant fuzzy soft set by combining the hesitant fuzzy set and soft set. Some operations and their properties on hesitant fuzzy soft set will also be discussed.

Definition 3.1. *Let $HF(U)$ be the class of all HFSs of the universe U, $A \subseteq E$. A pair $(\widetilde{F}, A)$ is called a hesitant fuzzy soft set ($HFSS$), where $\widetilde{F} : A \to HF(U)$ is a mapping.*

In other words, a hesitant fuzzy soft set over U is a parameterized family of hesitant fuzzy set of the universe U. To illustrate this idea, let us consider the following example.

Example 3.2. *Let $U = \{u_1, u_2, u_3\}$ be a set of mobile telephones and $A = \{e_1, e_2, e_3\} \subseteq E$ be a set of parameters. The $e_i (i = 1, 2, 3)$ stands for the parameters "expensive", "beautiful" and "multifunctional", respectively. Let $\widetilde{F} : A \to HF(U)$ be a function given as follows:*

$$\widetilde{F}(e_1) = \left\{ \frac{\{0.2, 0.7, 0.8\}}{u_1}, \frac{\{0.5, 0.8\}}{u_2}, \frac{\{0.4, 0.6, 0.8\}}{u_3} \right\},$$
$$\widetilde{F}(e_2) = \left\{ \frac{\{0.3, 0.5, 0.7\}}{u_1}, \frac{\{0.4, 0.6, 0.9\}}{u_2}, \frac{\{0.5, 0.7\}}{u_3} \right\},$$
$$\widetilde{F}(e_3) = \left\{ \frac{\{0.5, 0.8\}}{u_1}, \frac{\{0.3, 0.5, 0.8\}}{u_2}, \frac{\{0.5, 0.6, 0.9\}}{u_3} \right\}.$$

Then $(\widetilde{F}, A)$ is a hesitant fuzzy soft set.

Remark 3.3. (1) *If A has only an element, i.e. $A = \{e\}$, then hesitant fuzzy soft set becomes hesitant fuzzy set* [20];
(2) If $h_{\widetilde{F}(e)}(u)$ has only one value for all $e \in A$ and $u \in U$, then hesitant fuzzy soft set degenerates to traditional fuzzy soft set [13];
(3) If $h_{\widetilde{F}(e)}(u)$ is a subinterval of $[0, 1]$ for all $e \in A$ and $u \in U$, then hesitant fuzzy soft set reduces to interval-valued fuzzy soft set [17];
(4) For all $e \in A$, if $h_{\widetilde{F}(e)}(u)$ has the same number of values with respect to $u \in U$, then hesitant fuzzy soft set transforms to multi-fuzzy soft set [19].

Definition 3.4. *The complement of an HFSS $(\widetilde{F}, A)$ is denoted by $(\widetilde{F}, A)^c$ and is defined by $(\widetilde{F}, A)^c = (\widetilde{F}^c, A)$, where $\widetilde{F}^c : A \to HF(U)$ is a mapping given by $\widetilde{F}^c(e) = \left\{ \frac{h_{\widetilde{F}^c(e)}(u)}{u} | u \in U \right\}$, where $h_{\widetilde{F}^c(e)}(u) = \bigcup_{\gamma \in h_{\widetilde{F}(e)}(u)} \{1 - \gamma\}$.*

Example 3.5. (continued) The complement of $(\widetilde{F}, A)$ is following as:

$$\widetilde{F}^c(e_1) = \left\{ \frac{\{0.2, 0.3, 0.8\}}{u_1}, \frac{\{0.2, 0.5\}}{u_2}, \frac{\{0.2, 0.4, 0.6\}}{u_3} \right\},$$
$$\widetilde{F}^c(e_2) = \left\{ \frac{\{0.3, 0.5, 0.7\}}{u_1}, \frac{\{0.1, 0.4, 0.6\}}{u_2}, \frac{\{0.3, 0.5\}}{u_3} \right\},$$
$$\widetilde{F}^c(e_3) = \left\{ \frac{\{0.2, 0.5\}}{u_1}, \frac{\{0.2, 0.5, 0.7\}}{u_2}, \frac{\{0.1, 0.4, 0.5\}}{u_3} \right\}.$$

Definition 3.6. *Let* $(\widetilde{F}, A)$ *be an* $HFSS$ *over* U. *Then*
(1) $(\widetilde{F}, A)$ *is said to be an empty hesitant soft set, denoted by* $\widetilde{\Phi}_A$, *if* $h_{F(e)}(u) = 0$ *for all* $u \in U$ *and* $e \in A$;
(2) $(\widetilde{F}, A)$ *is said to be a full hesitant soft set, denoted by* $\widetilde{\mathcal{I}}_A$, *if* $h_{F(e)}(u) = 1$ *for all* $u \in U$ *and* $e \in A$;
(3) $(\widetilde{F}, A)$ *is said to be a complete hesitant soft set, denoted by* $\widetilde{\mathcal{W}}_A$, *if* $h_{F(e)}(u) = [0, 1]$ *for all* $u \in U$ *and* $e \in A$.

Proposition 3.7. *Let* $A \subseteq E$. *Then*
(1) $\widetilde{\Phi}^c_A = \widetilde{\mathcal{I}}_A$;
(2) $\widetilde{\mathcal{I}}^c_A = \widetilde{\Phi}_A$;
(3) $\widetilde{\mathcal{W}}^c_A = \widetilde{\mathcal{W}}_A$.

Definition 3.8. *Let* $(\widetilde{F}, A)$ *and* $(\widetilde{G}, B)$ *be two* $HFSSs$ *over* U *and* $A, B \subseteq E$. *We define a mapping* $\widetilde{H} : A \cup B \to HF(U)$ *such that for all* $e \in A \cup B \neq \emptyset$,

$$\widetilde{H}(e) = \begin{cases} \widetilde{F}(e), & \text{if } e \in A - B, \\ \widetilde{G}(e), & \text{if } e \in B - A, \\ \widetilde{H}(e), & \text{if } e \in A \cap B. \end{cases}$$

(1) *If* $\widetilde{H}(e) = \widetilde{F}(e) \cup \widetilde{G}(e)$, *then* $(\widetilde{H}, A \cup B)$ *is called the extended union of* $(\widetilde{F}, A)$ *and* $(\widetilde{G}, B)$, *denoted by* $(\widetilde{F}, A)\widetilde{\cup}(\widetilde{G}, B)$.
(2) *If* $\widetilde{H}(e) = \widetilde{F}(e) \cap \widetilde{G}(e)$, *then* $(\widetilde{H}, A \cup B)$ *is called the extended intersection of* $(\widetilde{F}, A)$ *and* $(\widetilde{G}, B)$, *denoted by* $(\widetilde{F}, A)\widetilde{\cap}(\widetilde{G}, B)$.
(3) *If* $\widetilde{H}(e) = \widetilde{F}(e) \sqcup \widetilde{G}(e)$, *then* $(\widetilde{H}, A \cup B)$ *is called the extended-strict union of* $(\widetilde{F}, A)$ *and* $(\widetilde{G}, B)$, *denoted by* $(\widetilde{F}, A)\widetilde{\sqcup}(\widetilde{G}, B)$.
(4) *If* $\widetilde{H}(e) = \widetilde{F}(e) \sqcap \widetilde{G}(e)$, *then* $(\widetilde{H}, A \cup B)$ *is called the extended-strict intersection of* $(\widetilde{F}, A)$ *and* $(\widetilde{G}, B)$, *denoted by* $(\widetilde{F}, A)\widetilde{\sqcap}(\widetilde{G}, B)$.

If $A \cup B = \emptyset$, *then* $(\widetilde{F}, A)\widetilde{\cup}(\widetilde{G}, B) = \widetilde{\Phi}_\emptyset$, $(\widetilde{F}, A)\widetilde{\cap}(\widetilde{G}, B) = \widetilde{\Phi}_\emptyset$, $(\widetilde{F}, A)\widetilde{\sqcup}(\widetilde{G}, B) = \widetilde{\Phi}_\emptyset$ *and* $(\widetilde{F}, A)\widetilde{\sqcap}(\widetilde{G}, B) = \widetilde{\Phi}_\emptyset$.

Definition 3.9. *Let* $(\widetilde{F}, A)$ *and* $(\widetilde{G}, B)$ *be two* $HFSSs$ *over* U *and* $A, B \subseteq E$. *We define a mapping* $\widetilde{H} : A \cap B \to HF(U)$ *such that for all* $e \in A \cap B \neq \emptyset$,
(1) *If* $\widetilde{H}(e) = \widetilde{F}(e) \cup \widetilde{G}(e)$, *then* $(\widetilde{H}, A \cap B)$ *is called the strict union of* $(\widetilde{F}, A)$ *and* $(\widetilde{G}, B)$, *denoted by* $(\widetilde{F}, A)\widetilde{\Cup}(\widetilde{G}, B)$.
(2) *If* $\widetilde{H}(e) = \widetilde{F}(e) \cap \widetilde{G}(e)$, *then* $(\widetilde{H}, A \cap B)$ *is called the strict intersection of* $(\widetilde{F}, A)$ *and* $(\widetilde{G}, B)$, *denoted by* $(\widetilde{F}, A)\widetilde{\Cap}(\widetilde{G}, B)$.
(3) *If* $\widetilde{H}(e) = \widetilde{F}(e) \sqcup \widetilde{G}(e)$, *then* $(\widetilde{H}, A \cap B)$ *is called the strict-strict union of* $(\widetilde{F}, A)$ *and* $(\widetilde{G}, B)$, *denoted by* $(\widetilde{F}, A)\widetilde{\uplus}(\widetilde{G}, B)$.
(4) *If* $\widetilde{H}(e) = \widetilde{F}(e) \sqcap \widetilde{G}(e)$, *then* $(\widetilde{H}, A \cap B)$ *is called the strict-strict intersection of* $(\widetilde{F}, A)$ *and* $(\widetilde{G}, B)$, *denoted by* $(\widetilde{F}, A)\widetilde{\pitchfork}(\widetilde{G}, B)$.

If $A \cap B = \emptyset$, *then* $(\widetilde{F}, A) \Cup (\widetilde{G}, B) = \widetilde{\Phi}_\emptyset$, $(\widetilde{F}, A) \Cap (\widetilde{G}, B) = \widetilde{\Phi}_\emptyset$, $(\widetilde{F}, A)\widetilde{\uplus}(\widetilde{G}, B) = \widetilde{\Phi}_\emptyset$ *and* $(\widetilde{F}, A)\widetilde{\pitchfork}(\widetilde{G}, B) = \widetilde{\Phi}_\emptyset$.

Proposition 3.10. *Let* $A \subseteq E$, $(\widetilde{F}, A)$ *be an* $HFSS$ *over* (U, E), $\theta_1 \in \{\widetilde{\Cap}, \widetilde{\pitchfork}\}$, $\theta_2 \in \{\widetilde{\cap}, \widetilde{\sqcap}\}$, $\theta_3 \in \{\widetilde{\Cup}, \widetilde{\uplus}\}$ *and* $\theta_4 \in \{\widetilde{\cup}, \widetilde{\sqcup}\}$. *Then*
(1) $(\widetilde{F}, A)\ \theta_1\ \widetilde{\mathcal{I}}_E = (\widetilde{F}, A)\ \theta_2\ \widetilde{\mathcal{I}}_A = (\widetilde{F}, A)$;
(2) $(\widetilde{F}, A)\ \theta_3\ \widetilde{\mathcal{I}}_E = (\widetilde{F}, A)\ \theta_4\ \widetilde{\mathcal{I}}_A = \widetilde{\mathcal{I}}_A$;
(3) $(\widetilde{F}, A)\ \theta_1\ \widetilde{\Phi}_E = (\widetilde{F}, A)\ \theta_2\ \widetilde{\Phi}_E = \widetilde{\Phi}_A$;
(4) $(\widetilde{F}, A)\ \theta_3\ \widetilde{\Phi}_E = (\widetilde{F}, A)\ \theta_4\ \widetilde{\Phi}_A = (\widetilde{F}, A)$;
(5) $(\widetilde{F}, A)\ \theta_1\ \widetilde{\Phi}_\phi = (\widetilde{F}, A)\ \theta_3\ \widetilde{\Phi}_\phi = \widetilde{\Phi}_\phi$;
(6) $(\widetilde{F}, A)\ \theta_2\ \widetilde{\Phi}_\phi = (\widetilde{F}, A)\ \theta_4\ \widetilde{\Phi}_\phi = (\widetilde{F}, A)$.

Theorem 3.11. *Let* $\alpha \in \{\widetilde{\uplus}, \widetilde{\sqcup}, \widetilde{\cap\!\!\!\!+}, \widetilde{\sqcap}, \widetilde{\cap}, \widetilde{\Cap}, \widetilde{\cup}, \widetilde{\Cup}\}$,$A, B, C \subseteq E$, $(\widetilde{F}, A)$, $(\widetilde{G}, B)$ *and* $(\widetilde{H}, C)$ *be* $HFSSs$ *over* (U, E). *Then the following holds:*
(1) $(\widetilde{F}, A)\ \alpha\ (\widetilde{F}, A) = (\widetilde{F}, A)$;
(2) $(\widetilde{F}, A)\ \alpha\ (\widetilde{G}, B) = (\widetilde{G}, B)\ \alpha\ (\widetilde{F}, A)$;
(3) $(\widetilde{F}, A)\ \alpha\ ((\widetilde{G}, B)\ \alpha\ (\widetilde{H}, C)) = ((\widetilde{F}, A)\ \alpha\ (\widetilde{G}, B))\ \alpha\ (\widetilde{H}, C)$.

Proof. (1) and (2) are trivial. We only prove (3). For example, let $\alpha = \widetilde{\uplus}$, the others can be proved analogously.

Suppose that $(\widetilde{F}, A)\widetilde{\uplus}((\widetilde{G}, B)\widetilde{\uplus}(\widetilde{H}, C)) = (\widetilde{J}, M)$ and $((\widetilde{F}, A)\widetilde{\uplus}(\widetilde{G}, B))\widetilde{\uplus}(\widetilde{H}, C) = (\widetilde{K}, N)$, thus $M = N = A \cap B \cap C$. If $M = \phi$, then $(\widetilde{F}, A)\ \widetilde{\uplus}((\widetilde{G}, B)\ \widetilde{\uplus}(\widetilde{H}, C)) = \Phi_\phi = ((\widetilde{F}, A)\ \widetilde{\uplus}(\widetilde{G}, B))\ \widetilde{\uplus}\ (\widetilde{H}, C)$. If $M \neq \phi$, then by (2) in Theorem 2.9, we have $h_{F(e)}(u) \sqcup (h_{G(e)}(u) \sqcup h_{H(e)}(u)) = (h_{F(e)}(u) \sqcup h_{G(e)}(u)) \sqcup h_{H(e)}(u)$ for all $e \in M$ and $u \in U$. It follows that $\widetilde{F}(e) \sqcup (\widetilde{G}(e) \sqcup \widetilde{H}(e)) = (\widetilde{F}(e) \sqcup \widetilde{G}(e)) \sqcup \widetilde{H}(e)$ for all $e \in M$. By the definition of the operation $\widetilde{\uplus}$, we have $(\widetilde{F}, A)\widetilde{\uplus}((\widetilde{G}, B)\widetilde{\uplus}(\widetilde{H}, C)) = ((\widetilde{F}, A)\widetilde{\uplus}(\widetilde{G}, B))\widetilde{\uplus}(\widetilde{H}, C)$. □

Remark 3.12. *Theorem 3.11 shows that the operations* $\widetilde{\cap}$, $\widetilde{\Cap}$, $\widetilde{\cup}$, $\widetilde{\Cup}$, $\widetilde{\uplus}$, $\widetilde{\sqcup}$, $\widetilde{\cap\!\!\!\!+}$ *and* $\widetilde{\sqcap}$ *are idempotent, commutative and associative, respectively.*

Theorem 3.13. *Let* $A, B \subseteq E$, $(\widetilde{F}, A)$ *and* $(\widetilde{G}, B)$ *be* $HFSSs$ *over* (U, E). *Then the following holds:*
(1) $((\widetilde{F}, A)\widetilde{\sqcap}(\widetilde{G}, B))^c = (\widetilde{F}, A)^c\widetilde{\sqcup}(\widetilde{G}, B)^c$;
(2) $((\widetilde{F}, A)\widetilde{\sqcup}(\widetilde{G}, B))^c = (\widetilde{F}, A)^c\widetilde{\sqcap}(\widetilde{G}, B)^c$;
(3) $((\widetilde{F}, A)\widetilde{\cap\!\!\!\!+}(\widetilde{G}, B))^c = (\widetilde{F}, A)^c\widetilde{\uplus}(\widetilde{G}, B)^c$;
(4) $((\widetilde{F}, A)\widetilde{\uplus}(\widetilde{G}, B))^c = (\widetilde{F}, A)^c\widetilde{\cap\!\!\!\!+}(\widetilde{G}, B)^c$;
(5) $((\widetilde{F}, A)\widetilde{\Cap}(\widetilde{G}, B))^c = (\widetilde{F}, A)^c\widetilde{\Cup}(\widetilde{G}, B)^c$;
(6) $((\widetilde{F}, A)\widetilde{\Cup}(\widetilde{G}, B))^c = (\widetilde{F}, A)^c\widetilde{\Cap}(\widetilde{G}, B)^c$;
(7) $((\widetilde{F}, A)\widetilde{\cap}(\widetilde{G}, B))^c = (\widetilde{F}, A)^c\widetilde{\cup}(\widetilde{G}, B)^c$;
(8) $((\widetilde{F}, A)\widetilde{\cup}(\widetilde{G}, B))^c = (\widetilde{F}, A)^c\widetilde{\cap}(\widetilde{G}, B)^c$,

Proof. We only prove (1). By using a similar technique, (2)-(8) can be proved, too.

Suppose that $(\widetilde{F}, A)\widetilde{\sqcap}(\widetilde{G}, B) = (\widetilde{H}, C)$. Then $C = A \cup B$,
(i) if $C = \phi$, then $A = \phi$ and $B = \phi$. Hence $((\widetilde{F}, A)\widetilde{\sqcap}(\widetilde{G}, B))^c = \widetilde{\Phi}_\phi$=$(\widetilde{F}, A)^c\widetilde{\sqcap}(\widetilde{G}, B)^c$.
(ii) if $C \neq \phi$, then for each $e \in C$ and $u \in U$, we have

$$h_{\widetilde{H}(e)}(u) = \begin{cases} h_{\widetilde{F}(e)}(u), & \text{if } e \in A - B, \\ h_{\widetilde{G}(e)}(u), & \text{if } e \in B - A, \\ h_{\widetilde{F}(e)}(u) \sqcap h_{\widetilde{G}(e)}(u), & \text{if } e \in A \cap B. \end{cases}$$

Then

$$h_{\widetilde{H}^c(e)}(u) = \begin{cases} h_{\widetilde{F}^c(e)}(u), & \text{if } e \in A - B, \\ h_{\widetilde{G}^c(e)}(u), & \text{if } e \in B - A, \\ (h_{\widetilde{F}(e)}(u) \sqcap h_{\widetilde{G}(e)}(u))^c, & \text{if } e \in A \cap B. \end{cases}$$

Again suppose that $(\widetilde{F}, A)^c\widetilde{\sqcup}(\widetilde{G}, B)^c = (\widetilde{J}, D)$. Then $D = A \cup B$ and for each $e \in D$ and $u \in U$, we have

$$h_{\widetilde{J}(e)}(u) = \begin{cases} h_{\widetilde{F}^c(e)}(u), & \text{if } e \in A - B, \\ h_{\widetilde{G}^c(e)}(u), & \text{if } e \in B - A, \\ h_{\widetilde{F}^c(e)}(u) \sqcup h_{\widetilde{G}^c(e)}(u), & \text{if } e \in A \cap B. \end{cases}$$

By Theorem 2.7, we have $h_{\widetilde{F}^c(e)}(u) \sqcup h_{\widetilde{G}^c(e)}(u) = (h_{\widetilde{F}(e)}(u) \sqcap h_{\widetilde{G}(e)}(u))^c$, i.e., $h_{\widetilde{J}(e)}(u) = h_{\widetilde{H}^c(e)}(u)$ for all $e \in A$ and $u \in U$.

Therefore, $(\widetilde{H}, C)$ and $(\widetilde{J}, D)$ are the same $HFSS$s. It follows that $((\widetilde{F}, A)\widetilde{\sqcap}(\widetilde{G}, B))^c = (\widetilde{F}, A)^c\widetilde{\sqcup}(\widetilde{G}, B)^c$. □

4 Lattice structures of hesitant fuzzy soft set

In this section, we first recall briefly the necessary definitions and notations. For convenience, we give the following axioms on an algebra $Q = (X, \vee, \wedge)$:
(1) $x \vee x = x, x \wedge x = x$;
(2) $x \vee y = y \vee x, x \wedge y = y \wedge x$;
(3) $(x \vee y) \vee z = x \vee (y \vee z), (x \wedge y) \wedge z = x \wedge (y \wedge z)$;
(4) $(x \vee y) \wedge x = x$, $(x \wedge y) \vee x = x$;
(5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$,
where $x, y, z \in X$.

The algebra Q is called a quasilattice, if it satisfies the axioms (1),(2) and (3). If a quasilattice further satisfies the axiom (4), then it is called a lattice. If a quasilattice (or lattice) further satisfies the axiom (5), then it is called a distributive quasilattice (or lattice).

For convenience, let $\widetilde{\mathfrak{S}}(U, E)$ denote the set of all $HFSS$s over U, i.e., $\widetilde{\mathfrak{S}}(U, E) = \{(\widetilde{F}, A) | A \subseteq E, \widetilde{F} : A \to HF(U)\}$. Then based on Theorem 3.11, we have the following property.

Proposition 4.1. *Let* $\alpha \in \{\widetilde{\cap}, \widetilde{\Cap}, \widetilde{⩀}, \widetilde{\sqcap}\}$ *and* $\beta \in \{\widetilde{\cup}, \widetilde{\Cup}, \widetilde{\uplus}, \widetilde{\sqcup}\}$, *then* $(\widetilde{\mathfrak{S}}(U, E), \alpha, \beta)$ *is a quasilattice.*

For the operations $\widetilde{\Cap}$ and $\widetilde{\cup}$, the distributive laws hold.

Theorem 4.2. *Let* $(\widetilde{F}, A)$, $(\widetilde{G}, B)$, $(\widetilde{H}, C) \in \widetilde{\mathfrak{S}}(U, E)$. *Then*
(1) $((\widetilde{F}, A)\widetilde{\Cap}(\widetilde{G}, B))\widetilde{\cup}(\widetilde{H}, C) = ((\widetilde{F}, A)\widetilde{\Cap}(\widetilde{G}, B))\widetilde{\cup}((\widetilde{F}, A)\widetilde{\Cap}(\widetilde{H}, C))$;
(2) $((\widetilde{F}, A)\widetilde{\cup}(\widetilde{G}, B))\widetilde{\Cap}(\widetilde{H}, C) = ((\widetilde{F}, A)\widetilde{\cup}(\widetilde{G}, B))\widetilde{\Cap}((\widetilde{F}, A)\widetilde{\cup}(\widetilde{H}, C))$.

Proof. we only prove (1). (2) can be proved by using a similar technique. Suppose that $(\widetilde{F}, A)\widetilde{\Cap}((\widetilde{G}, B)\widetilde{\cup}(\widetilde{H}, C)) = (\widetilde{J}, M)$ and $((\widetilde{F}, A)\widetilde{\Cap}(\widetilde{G}, B))\widetilde{\cup}((\widetilde{F}, A)\widetilde{\Cap}(\widetilde{H}, C)) = (\widetilde{K}, N)$. Then $M = A \cap (B \cup C) = (A \cap B) \cup (A \cap C) = N$. For each $e \in M$, it follows that $e \in A$ and $e \in B \cup C$.
(i) if $e \in A, e \notin B, e \in C$, then $\widetilde{J}(e) = \widetilde{F}(e) \cap \widetilde{H}(e) = \widetilde{K}(e)$.
(ii) if $e \in A, e \in B, e \notin C$, then $\widetilde{J}(e) = \widetilde{F}(e) \cap \widetilde{G}(e) = \widetilde{K}(e)$.
(iii) if $e \in A, e \in B, e \in C$, then by (4) in Theorem 2.8, we have $h_{\widetilde{F}(e)}(u) \cap (h_{\widetilde{G}(e)}(u) \cup h_{\widetilde{H}(e)}(u)) = (h_{\widetilde{F}(e)}(u) \cap h_{\widetilde{G}(e)}(u) \cup (\widetilde{F}(e) \cap h_{\widetilde{H}(e)}(u))$ for all $u \in U$. It follows that $\widetilde{J}(e) = \widetilde{F}(e) \cap (\widetilde{G}(e) \cup \widetilde{H}(e)) = (\widetilde{F}(e) \cap \widetilde{G}(e) \cup (\widetilde{F}(e) \cap (\widetilde{H}, C)) = \widetilde{K}(e)$.

Thus, $(\widetilde{J}, M)$ and $(\widetilde{K}, N)$ are the same $HFSS$, i.e., $(\widetilde{F}, A)\widetilde{\Cap}((\widetilde{G}, B)\widetilde{\cup}(\widetilde{H}, C)) = ((\widetilde{F}, A)\widetilde{\Cap}(\widetilde{G}, B))\widetilde{\cup}((\widetilde{F}, A)\widetilde{\Cap}(\widetilde{H}, C))$. □

Corollary 4.3. $(\widetilde{\mathfrak{S}}(U, E), \widetilde{\Cap}, \widetilde{\cup})$ *is a distributive quasilattice.*

The operations $\widetilde{\cap}$ and $\widetilde{\Cup}$ have the similar properties with the operations $\widetilde{\Cap}$ and $\widetilde{\cup}$.

Theorem 4.4. *Let* $(\widetilde{F}, A)$, $(\widetilde{G}, B)$, $(\widetilde{H}, C) \in \widetilde{\mathfrak{S}}(U, E)$. *Then*
(1) $((\widetilde{F}, A)\widetilde{\cap}(\widetilde{G}, B))\widetilde{\Cup}(\widetilde{H}, C) = ((\widetilde{F}, A)\widetilde{\cap}(\widetilde{G}, B))\widetilde{\Cup}((\widetilde{F}, A)\widetilde{\cap}(\widetilde{H}, C))$;
(2) $((\widetilde{F}, A)\widetilde{\Cup}(\widetilde{G}, B))\widetilde{\cap}(\widetilde{H}, C) = ((\widetilde{F}, A)\widetilde{\Cup}(\widetilde{G}, B))\widetilde{\cap}((\widetilde{F}, A)\widetilde{\Cup}(\widetilde{H}, C))$.

Corollary 4.5. $(\widetilde{\mathfrak{S}}(U, E), \widetilde{\Cap}, \widetilde{\cup})$ *is a distributive quasilattice.*

The following theorem shows that the absorption laws with respect to operations $\widetilde{\sqcap}$ and $\widetilde{\uplus}$ hold.

Theorem 4.6. *Let* $(\widetilde{F}, A)$, $(\widetilde{G}, B) \in \widetilde{\mathfrak{S}}(U, E)$. *Then*
(1) $((\widetilde{F}, A)\widetilde{\sqcap}(\widetilde{G}, B))\widetilde{\uplus}(\widetilde{F}, A) = (\widetilde{F}, A)$;
(2) $((\widetilde{F}, A)\widetilde{\uplus}(\widetilde{G}, B))\widetilde{\sqcap}(\widetilde{F}, A) = (\widetilde{F}, A)$.

Proof. We only prove (1) since (2) can be proved similarly. Suppose that $(\widetilde{F}, A)\widetilde{\sqcap}(\widetilde{G}, B) = (\widetilde{J}, M)$ and $((\widetilde{F}, A)\ \widetilde{\sqcap}\ (\widetilde{G}, B))\ \widetilde{⩀}\ (\widetilde{F}, A) = (\widetilde{K}, N)$. Then $M = A \cup B$, $N = (A \cup B) \cap A = A$, and for all $e \in A$ and $u \in U$,
(i) if $e \notin B$, then $h_{\widetilde{J}(e)}(u) = h_{\widetilde{F}(e)}(u)$ and $h_{\widetilde{K}(e)}(u) = h_{\widetilde{J}(e)}(u) \sqcap h_{\widetilde{F}(e)}(u) = h_{\widetilde{F}(e)}(u)$.

(ii) if $e \in B$, then $h_{\widetilde{J}(e)}(u) = h_{\widetilde{F}(e)}(u) \sqcup h_{\widetilde{G}(e)}(u)$ and $h_{\widetilde{K}(e)}(u) = h_{\widetilde{J}(e)}(u) \sqcap h_{\widetilde{F}(e)}(u) = (h_{\widetilde{F}(e)}(u) \sqcup h_{\widetilde{G}(e)}(u)) \sqcap h_{\widetilde{F}(e)}(u)$. By (3) in Theorem 2.9, we have $(h_{\widetilde{F}(e)}(u) \sqcup h_{\widetilde{G}(e)}(u)) \sqcap h_{\widetilde{F}(e)}(u) = h_{\widetilde{F}(e)}(u)$, i.e. $h_{\widetilde{K}(e)}(u) = h_{\widetilde{F}(e)}(u)$.

Thus $(\widetilde{K}, N) = (\widetilde{F}, A)$, i.e. $((\widetilde{F}, A)\widetilde{\sqcap}(\widetilde{G}, B))\widetilde{\uplus}(\widetilde{F}, A) = (\widetilde{F}, A)$. □

Theorem 4.7. $(\widetilde{\mathfrak{S}}(U,E), \widetilde{\sqcap}, \widetilde{\uplus})$ *is a bounded lattice.*

Proof. By Theorem 3.11 and Theorem 4.6, we get that $(\widetilde{\mathfrak{S}}(U,E), \widetilde{\sqcap}, \widetilde{\uplus})$ is a lattice. It is clear that $\widetilde{\mathcal{I}}_E$ and $\widetilde{\Phi}_\phi$ are the maximum element and the minimum element in $(\mathfrak{S}(U,E)$, respectively. □

Similar to $\widetilde{\sqcap}$ and $\widetilde{\uplus}$, the operations $\widetilde{\sqcup}$ and $\widetilde{\pitchfork}$ have also the following properties.

Theorem 4.8. *Let* $(\widetilde{F}, A)$, $(\widetilde{G}, B) \in \widetilde{\mathfrak{S}}(U,E)$. *Then*
(1) $((\widetilde{F}, A)\widetilde{\sqcup}(\widetilde{G}, B))\widetilde{\pitchfork}(\widetilde{F}, A) = (\widetilde{F}, A)$;
(2) $((\widetilde{F}, A)\widetilde{\pitchfork}(\widetilde{G}, B))\widetilde{\sqcup}(\widetilde{F}, A) = (\widetilde{F}, A)$.

Theorem 4.9. $(\widetilde{\mathfrak{S}}(U,E), \widetilde{\sqcup}, \widetilde{\pitchfork})$ *is a bounded lattice.*

Remark 4.10. *It is worth noting that* $(\widetilde{\mathfrak{S}}(U,E), \widetilde{\sqcup}, \widetilde{\sqcap})$, $(\widetilde{\mathfrak{S}}(U,E), \widetilde{\uplus}, \widetilde{\pitchfork})$ *and* $(\widetilde{\mathfrak{S}}(U,E), \alpha, \beta)$ *are not lattices, as the absorption laws do not hold necessarily, where* $\alpha \in \{\widetilde{\cap}, \widetilde{\Cap}\}$ *and* $\beta \in \{\widetilde{\cup}, \widetilde{\Cup}\}$. *To illustrate this idea, we give an example below.*

Example 4.11. *Let* $U = \{u_1, u_2, u_3\}$ *be the universe,* $E = \{e_1, e_2, e_3\}$ *the set of parameters,* $A = \{e_1, e_2\}$ *and* $B = \{e_2, e_3\}$. *The HFSSs* $(\widetilde{F}, A)$ *and* $(\widetilde{G}, B)$ *over U are given as:*

$$\begin{aligned}
\widetilde{F}(e_1) &= \left\{ \frac{\{0.2, 0.3, 0.7, 0.8\}}{u_1}, \frac{\{0.5, 0.8\}}{u_2}, \frac{\{0.4, 0.5, 0.6\}}{u_3} \right\}, \\
\widetilde{F}(e_2) &= \left\{ \frac{\{0.3, 0.4, 0.7\}}{u_1}, \frac{\{0.5, 0.7\}}{u_2}, \frac{\{0.1, 0.2, 0.4, 0.7\}}{u_3} \right\}, \\
\widetilde{G}(e_2) &= \left\{ \frac{\{0.5, 0.6\}}{u_1}, \frac{\{0.4, 0.8, 0.9\}}{u_2}, \frac{\{0.3, 0.5, 0.7, 0.8\}}{u_3} \right\}, \\
\widetilde{G}(e_3) &= \left\{ \frac{\{0.1, 0.3, 0.5\}}{u_1}, \frac{\{0.5, 0.6, 0.8\}}{u_2}, \frac{\{0.6, 0.9\}}{u_3} \right\}.
\end{aligned}$$

(1) *Let* $((\widetilde{F}, A)\widetilde{\sqcup}\,(\widetilde{G}, B))\widetilde{\sqcap}(\widetilde{F}, A) = (\widetilde{J}, M)$, *then* $M = A \cup B = \{e_1, e_2, e_3\} \neq A$. *So* $(\widetilde{J}, M) \neq (\widetilde{F}, A)$, *i.e.* $((\widetilde{F}, A)\widetilde{\sqcup}\,(\widetilde{G}, B))\widetilde{\sqcap}(\widetilde{F}, A) \neq (\widetilde{F}, A)$.

(2) *Let* $((\widetilde{F}, A)\widetilde{\uplus}\,(\widetilde{G}, B))\widetilde{\pitchfork}(\widetilde{F}, A) = (\widetilde{K}, N)$, *then* $N = A \cap B = \{e_2\} \neq A$, *Therefore,* $(\widetilde{K}, N) \neq (\widetilde{F}, A)$, *i.e.* $((\widetilde{F}, A)\widetilde{\uplus}\,(\widetilde{G}, B))\widetilde{\pitchfork}(\widetilde{F}, A) \neq (\widetilde{F}, A)$.

(3) *If* $e_2 \in A \cap B$, *then* $(h_{\widetilde{F}(e_2)}(u_1) \cap h_{\widetilde{G}(e_2)}(u_1)) \cup h_{\widetilde{F}(e_2)}(u_1) = (\{0.3, 0.4, 0.7\} \cap \{0.5, 0.6\}) \cup \{0.3, 0.4, 0.7\} = \{0.3, 0.4, 0.5, 0.6\} \cup \{0.3, 0.4, 0.7\} = \{0.3, 0.4, 0.5, 0.6, 0.7\} \neq \{0.3, 0.4, 0.7\} = h_{\widetilde{F}(e_2)}(u_1)$. *It follows that* $(\widetilde{F}(e_2) \cap \widetilde{G}(e_2)) \cup \widetilde{F}(e_2) \neq \widetilde{F}(e_2)$. *Consequently,* $((\widetilde{F}, A)\ \alpha\ (\widetilde{G}, B))\ \beta\ (\widetilde{F}, A) \neq (\widetilde{F}, A)$, *where* $\alpha \in \{\widetilde{\cap}, \widetilde{\Cap}\}$ *and* $\beta \in \{\widetilde{\cup}, \widetilde{\Cup}\}$.

5 Conclusion

Considering that soft set and its existing extension models cannot deal with the situations in which the evaluations of parameters have many possible values, in this paper, we have introduced the notion of $HFSS$ as an new extension to the HFS or the soft set model. We have also defined some basic operations on the $HFSS$ and discussed their properties. Finally, The lattice structures of $HFSS$ have been studied in detail based on the proposed operations.

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INCLUSION PROPERTIES FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH BESSEL FUNCTIONS

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Abstract: The purpose of the present paper is to investigate some characterization for generalized Bessel functions of first kind to be in the new subclasses $\mathcal{G}(\lambda, \alpha)$ and $\mathcal{K}(\lambda, \alpha)$ of analytic functions.

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1. Introduction

Let $\mathcal{A}$ be the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are normalized by $f(0) = 0 = f'(0) - 1$ and also univalent in $\mathbb{U}$. Denote by $\mathcal{T}$ [16] the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \quad n = 2, 3, \ldots. \tag{1.2}$$

Also, for functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad (z \in \mathbb{U}). \tag{1.3}$$

The class $\mathcal{S}^*(\alpha)$ of starlike functions of order α $(0 \leq \alpha < 1)$ may be defined as

$$\mathcal{S}^*(\alpha) = \left\{ f \in \mathcal{A} : \ \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \ z \in \mathbb{U} \right\}.$$

The class $\mathcal{S}^*(\alpha)$ and the class $\mathcal{K}(\alpha)$ of convex functions of order α $(0 \leq \alpha < 1)$

$$\begin{aligned} \mathcal{K}(\alpha) &= \left\{ f \in \mathcal{A} : \ \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \ z \in \mathbb{U} \right\} \\ &= \{ f \in \mathcal{A} : \ zf' \in \mathcal{S}^*(\alpha) \} \end{aligned}$$

were introduced by Robertson in [14]. We also write $\mathcal{S}^*(0) = \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ that $f(\mathbb{U})$ is starlike with respect to the origin. Further, $\mathcal{K}(0) = \mathcal{K}$ is the well-known standard class of convex functions. It is an established fact that $f \in \mathcal{K}(\alpha) \Longleftrightarrow zf' \in \mathcal{S}^*(\alpha)$.

A function $f \in \mathcal{A}$ is said to be in the class $\Re^\tau(A, B)$, $(\tau \in \mathbb{C}\backslash\{0\}, -1 \leq B < A \leq 1)$, if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(A - B)\tau - B[f'(z) - 1]} \right| < 1 \quad (z \in \mathbb{U}).$$

The class $\Re^\tau(A, B)$ was introduced earlier by Dixit and Pal [7]. If we put

$$\tau = 1, \ A = \alpha \text{ and } \ B = -\alpha \ (0 < \alpha \leq 1),$$

we obtain the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\left| \frac{f'(z) - 1}{f'(z) + 1} \right| < \alpha \quad (z \in \mathbb{U}; 0 < \alpha \leq 1)$$

which was studied by (among others) Padmanabhan [12] and Caplinger and Causey [5].

We recall here a generalized Bessel function $\omega_{p,b,c}(z) = \omega(z)$ defined in [1] and given by

$$\omega(z) = \omega_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \ c^n}{n! \ \Gamma\left(p + n + \frac{b+1}{2}\right)} \left(\frac{z}{2}\right)^{2n+p} \tag{1.4}$$

which is the particular solution of the second order linear homogeneous differential equation

$$z^2\omega''(z) + bz\omega'(z) + [cz^2 - p^2 + (1 - b)]\omega(z) = 0, \tag{1.5}$$

where $b, p, c \in \mathbb{C}$, which is a natural generalization of Bessel's equation.

The differential equation (1.5) permits the study of Bessel function, modified Bessel function, spherical Bessel function and modified spherical Bessel functions all together.

Solutions of (1.5) are referred to as the generalized Bessel function of order p. The particular solution given by (1.4) is called the generalized Bessel function of the first kind of order p. Although the series defined above is convergent everywhere, the function $\omega_{p,b,c}$ is generally not univalent in $\mathbb{U}$. It is of interest to note that when $b = c = 1$, we reobtain the Bessel function of the first kind $\omega_{p,1,1} = J_p$, and for $c = -1, b = 1$, the function $\omega_{p,1,-1}$ becomes the modified Bessel function $\mathcal{I}_p$. Now, we consider the function $u_{p,b,c}(z)$ defined by the transformation

$$u_{p,b,c}(z) = 2^p \Gamma\left(p + \frac{b+1}{2}\right) \, z^{\frac{-p}{2}} \, \omega_{p,b,c}\,(\sqrt{z}), \quad \sqrt{1} = 1.$$

By using well known Pochhammer symbol (or the shifted factorial) defined, in terms of the familiar Gamma function, by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & (n = 0), \\ a(a+1)(a+2)\cdots(a+n-1) & (n \in \mathbb{N} = \{1, 2, 3, \ldots\}), \end{cases}$$

we can express $u_{p,b,c}(z)$ as

$$u_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c/4)^n}{\left(p + \frac{b+1}{2}\right)_n} \left(\frac{z^n}{n!}\right), \tag{1.6}$$

where $p + \frac{b+1}{2} \neq 0, -1, -2, \cdots$. This function is analytic on $\mathbb{C}$ and satisfies the second-order linear differential equation

$$4z^2 u''(z) + 2(2p + b + 1) z u'(z) + c u(z) = 0.$$

Now, we considered the linear operator

$$\mathcal{I}(c, m) : \mathcal{A} \to \mathcal{A}$$

defined by

$$\mathcal{I}(c, m) f(z) = z u_{p,b,c}(z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1} \; (n-1)!} a_n \, z^n, \tag{1.7}$$

where $m = p + \frac{b+1}{2} \neq 0, -1, -2, \cdots$. For convenience throughout in the sequel, we use the following notations

$$u_{p,b,c} = u_p, \qquad m = p + \frac{b+1}{2}.$$

and if $c < 0$ and $m > 0$, then we let

$$z(2 - u_p(z)) = z - \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1} \; (n-1)!} z^n. \tag{1.8}$$

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For $0 \le \lambda < 1$ and $0 \le \alpha < 1$, we let $\mathcal{G}(\lambda, \alpha)$ the subclass of functions $f \in \mathcal{A}$ which satisfy the condition

$$\Re\left(\frac{zf'(z) + \lambda z^2 f''(z)}{f(z)}\right) > \alpha,\ (z \in \mathbb{U}). \tag{1.9}$$

and also let $\mathcal{K}(\lambda, \alpha)$ the subclass of functions $f \in \mathcal{A}$ which satisfy the condition

$$\Re\left(\frac{z[zf'(z) + \lambda z^2 f''(z)]'}{zf'(z)}\right) > \alpha,\ (z \in \mathbb{U}). \tag{1.10}$$

Also denote $\mathcal{G}^*(\lambda, \alpha) = \mathcal{G}(\lambda, \alpha) \cap \mathcal{T}$ and $\mathcal{K}^*(\lambda, \alpha) = \mathcal{K}(\lambda, \alpha) \cap \mathcal{T}$

The study of the generalized Bessel function is a recent interesting topic in geometric function theory (e.g. see the work of [1, 2, 3, 4] and [9]). In this paper, due to Ramesha et al. [13], Padmanabhan [12], and motivated by the works of Srivastava et al. [17], Murugusundaramoorthy and Magesh [11],(see [6, 8, 10, 15]) and by work of Baricz [1, 2, 3, 4], we obtain sufficient conditions for function $z(2 - u_p(z))$ in $\mathcal{G}(\lambda, \alpha)$ and $\mathcal{K}(\lambda, \alpha)$ and connections between $\mathcal{R}^\tau(A, B)$.

Remark 1. *It is of interest to note that for $\lambda = 0$, we have $\mathcal{G}(\lambda, \alpha) \equiv \mathcal{S}^*(\alpha)$ and $\mathcal{K}(\lambda, \alpha) \equiv \mathcal{K}(\alpha)$*

To prove the main results, we need the following Lemmas.

Lemma 1. [18] *A function $f \in \mathcal{A}$ belongs to the class $\mathcal{G}(\lambda, \alpha)$ if*

$$\sum_{n=2}^{\infty} (n + \lambda n(n-1) - \alpha)|a_n| \le 1 - \alpha.$$

Lemma 2. [18] *A function $f \in \mathcal{A}$ belongs to the class $\mathcal{K}(\lambda, \alpha)$ if*

$$\sum_{n=2}^{\infty} n(n + \lambda n(n-1) - \alpha)|a_n| \le 1 - \alpha.$$

Further we can easily prove that the conditions are also necessary if $f \in \mathcal{T}$.

Lemma 3. [18] *A function $f \in \mathcal{T}$ belongs to the class $\mathcal{G}^*(\lambda, \alpha)$ if and only if*

$$\sum_{n=2}^{\infty} (n + \lambda n(n-1) - \alpha)|a_n| \le 1 - \alpha.$$

Lemma 4. [18] *A function $f \in \mathcal{T}$ belongs to the class $\mathcal{K}^*(\lambda, \alpha)$ if and only if*

$$\sum_{n=2}^{\infty} n(n + \lambda n(n-1) - \alpha)|a_n| \le 1 - \alpha.$$

Lemma 5. [4] *If $b, p, c \in \mathbb{C}$* and *$m \ne 0, -1, -2, \ldots$ then the function u_p satisfies the recursive relation*

$$4mu_p'(z) = -cu_{p+1}(z)$$

for all $z \in \mathbb{C}$.

2. Main Results

Theorem 1. *If $c < 0$ and $m > 0$, then $z(2 - u_p(z))$ is in $\mathcal{G}(\lambda, \alpha)$ if*

$$\lambda u_p''(1) + [1 + 2\lambda]u_p'(1) + (1 - \alpha)u_p(1) \leq 2(1 - \alpha). \tag{2.1}$$

Proof. Since

$$z(2 - u_p(z)) = z - \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} z^n$$

and by virtue of Lemma 1, it suffices to show that

$$\mathcal{L}(c, m, \lambda, \alpha) = \sum_{n=2}^{\infty} (n + \lambda n(n-1) - \alpha) \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} \leq 1 - \alpha.$$

Writing $n^2 = (n-1)(n-2) + 3(n-1) + 1$ and $n = (n-1) + 1$, and by simple computation, we get

$$\begin{aligned}
\mathcal{L}(c, m, \lambda, \alpha) &= \sum_{n=2}^{\infty} (n^2\lambda + n(1-\lambda) - \alpha) \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} \\
&\leq \sum_{n=2}^{\infty} \lambda(n-1)(n-2) \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} + (1+2\lambda) \sum_{n=2}^{\infty} (n-1) \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} \\
&\quad + (1-\alpha) \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} \\
&= \lambda \sum_{n=3}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-3)!} + (1+2\lambda) \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-2)!} + (1-\alpha) \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} \\
&= \lambda \sum_{n=1}^{\infty} \frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n-1)!} + (1+2\lambda) \sum_{n=0}^{\infty} \frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n)!} + (1-\alpha) \sum_{n=0}^{\infty} \frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n+1)!} \\
&= \lambda \frac{(-c/4)^2}{m(m+1)} \sum_{n=0}^{\infty} \frac{(-c/4)^n}{(m+2)_n\ n!} + (1+2\lambda) \frac{(-c/4)}{m} \sum_{n=0}^{\infty} \frac{(-c/4)^n}{(m+1)_n\ n!} \\
&\quad + (1-\alpha) \sum_{n=0}^{\infty} \frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n+1)!} \\
&= \lambda \frac{(-c/4)^2}{m(m+1)} u_{p+2}(1) + (1+2\lambda) \frac{(-c/4)}{m} u_{p+1}(1) + (1-\alpha)[u_p(1) - 1] \\
&= \lambda u_p''(1) + (1+2\lambda)u_p'(1) + (1-\alpha)[u_p(1) - 1].
\end{aligned}$$

By a simplification, we see that the last expression is bounded above by $1 - \alpha$ if (2.1) is satisfied. □

By taking $\lambda = 0$, we state the following corollary.

Corollary 1. *If $c<0$ and $m>0$, then $z(2-u_p(z))$ is in $\mathcal{S}^*(\alpha)$ if*

$$u_p'(1)+(1-\alpha)u_p(1)\leq 2(1-\alpha). \tag{2.2}$$

Remark 2. *In particular, when $c=-1$ and $b=1$, the condition (2.1) becomes*

$$2^{p-2}\Gamma(p+1)\left[\lambda\mathcal{I}_{p+2}(1)+[1+2\lambda]\mathcal{I}_{p+1}(1)+2(1-\alpha)\mathcal{I}_p(1)\right]\leq 1-\alpha, \tag{2.3}$$

which is necessary and sufficient condition for $z(2-\zeta_p(z^{1/2})$ to be in $\mathcal{G}^*(\lambda,\alpha)$, where

$$u_p(z^{1/2})=2^p\Gamma(p+1)z^{-p/2}\mathcal{I}_p(z^{1/2}).$$

Theorem 2. *If $c<0$ and $m>0$, then $z(2-u_p(z))$ is in $\mathcal{K}(\lambda,\alpha)$ if*

$$\lambda u_p'''(1)+(1+5\lambda)u_p''(1)+(3+4\lambda-\alpha)u_p'(1)+(1-\alpha)u_p(1)\leq 2(1-\alpha). \tag{2.4}$$

Proof. Since

$$z(2-u_p(z))=z-\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!}z^n$$

and by virtue of Lemma 2, it suffices to show that

$$\mathcal{L}(c,m,\lambda,\alpha)=\sum_{n=2}^{\infty}(n^3\lambda+n^2(1-\lambda)-n\alpha)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!}\leq 1-\alpha.$$

Writing $n^3=(n-1)(n-2)(n-3)+6(n-1)(n-2)+7(n-1)+1$, $n^2=(n-1)(n-2)+3(n-1)+1$ and $n=(n-1)+1$, we can rewrite the above terms as

$$\begin{aligned}
\mathcal{L}(c,m,\lambda,\alpha) &\leq \lambda\sum_{n=2}^{\infty}(n-1)(n-2)(n-3)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!}\\
&\quad+(1+5\lambda)\sum_{n=2}^{\infty}(n-1)(n-2)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!}+(3+4\lambda-\alpha)\sum_{n=2}^{\infty}(n-1)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!}\\
&\quad+(1-\alpha)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!}\\
&=\lambda\sum_{n=4}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-4)!}+(1+5\lambda)\sum_{n=3}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-3)!}+(3+4\lambda-\alpha)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-2)!}\\
&\quad+(1-\alpha)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!}\\
&=\lambda\sum_{n=2}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n-2)!}+(1+5\lambda)\sum_{n=1}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n-1)!}\\
&\quad+(3+4\lambda-\alpha)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n)!}+(1-\alpha)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n+1)!}
\end{aligned}$$

$$= \lambda\frac{(-c/4)^3}{m(m+1)(m+2)}\sum_{n=2}^{\infty}\frac{(-c/4)^{n-2}}{(m+3)_{n-2}\ (n-2)!} + (1+5\lambda)\frac{(-c/4)^2}{m(m+1)}\sum_{n=1}^{\infty}\left(\frac{(-c/4)^{n-1}}{(m+2)_{n-1}\ (n-1)!}\right)$$

$$+ (3+4\lambda-\alpha)\frac{(-c/4)}{m}\sum_{n=0}^{\infty}\left(\frac{(-c/4)^{n}}{(m+1)_{n}\ (n)!}\right) + (1-\alpha)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n+1)!}$$

$$= \lambda\frac{(-c/4)^3}{m(m+1)(m+2)}u_{p+3}(1) + (1+5\lambda)\frac{(-c/4)^2}{m(m+1)}u_{p+2}(1)$$

$$+ (3+4\lambda-\alpha)\frac{(-c/4)}{m}u_{p+1}(1) + (1-\alpha)[u_p(1)-1]$$

$$= \lambda u_p'''(1) + (1+5\lambda)u_p''(1) + (3+4\lambda-\alpha)u_p'(1) + (1-\alpha)[u_p(1)-1].$$

By a simplification, we see that the last expression is bounded above by $1-\alpha$ if (2.4) is satisfied. □

By taking $\lambda = 0$, we state the following corollary.

Corollary 2. *If $c < 0$ and $m > 0$, then $z(2 - u_p(z))$ is in $\in \mathcal{K}(\alpha)$ if*

$$u_p''(1) + (3-\alpha)u_p'(1) + (1-\alpha)u_p(1) \leq 2(1-\alpha). \tag{2.5}$$

Remark 3. *We also note that the function $z(2 - u_p(z))$ is in $\mathcal{K}^*(\lambda, \alpha)$ if and only if the condition (2.4) is satisfied.*

3. Inclusion Properties

Making use of the following lemma, we will study the action of the Bessel function on the classes $\mathcal{K}(\lambda, \alpha)$.

Lemma 6. [7] *A function $f \in \Re^{\tau}(A, B)$ is of form (1.1), then*

$$|a_n| \leq (A-B)\frac{|\tau|}{n}, \quad n \in \mathbb{N}\backslash\{1\}. \tag{3.1}$$

The bound given in (3.1) is sharp.

Theorem 3. *Let $c < 0$ and $m > 0$. If $f \in \Re^{\tau}(A, B)$ and the inequality*

$$(A-B)|\tau|\left[\lambda u_p''(1) + (1+2\lambda)u_p'(1) + (1-\alpha)[u_p(1)-1]\right] \leq 1-\alpha \tag{3.2}$$

is satisfied, then $\mathcal{I}(c, m)f \in \mathcal{K}(\lambda, \alpha)$.

Proof. Let f be of the form (1.1) belong to the class $\Re^{\tau}(A, B)$ then by virtue of Lemma 2, it suffices to show that

$$\mathcal{P}(c, m, \lambda, \alpha) = \sum_{n=2}^{\infty} n(n^2\lambda + n(1-\lambda) - \alpha)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!}|a_n| \leq 1-\alpha. \tag{3.3}$$

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Writing $n^2 = (n-1)(n-2) + 3(n-1) + 1$ and $n = (n-1) + 1$, we get

$$\begin{aligned}
\mathcal{P}(c,m,\lambda,\alpha) &\leq \sum_{n=2}^{\infty}(n+\lambda n(n-1)-\alpha)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!}(A-B)|\tau| \\
&= (A-B)|\tau|\sum_{n=2}^{\infty}\lambda(n-1)(n-2)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} \\
&\quad + (A-B)|\tau|(1+2\lambda)\sum_{n=2}^{\infty}(n-1)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} \\
&\quad + (A-B)|\tau|(1-\alpha)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!}
\end{aligned}$$

$$\begin{aligned}
&= (A-B)|\tau|\left[\lambda\sum_{n=3}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-3)!} + (1+2\lambda)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-2)!} + (1-\alpha)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!}\right] \\
&= (A-B)|\tau|\left[\lambda\sum_{n=1}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n-1)!} + (1+2\lambda)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}\ n!} + (1-\alpha)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n+1)!}\right].
\end{aligned}$$

By using the similar method as in the proof of Theorem 1, we have

$$\begin{aligned}
&\mathcal{P}(c,m,\lambda,\alpha) = (A-B)|\tau|\left[\lambda\frac{(-c/4)^2}{m(m+1)}u_{p+2}(1) + (1+2\lambda)\frac{(-c/4)}{m}u_{p+1}(1) + (1-\alpha)[u_p(1)-1]\right] \\
&= (A-B)|\tau|\left[\lambda u_p''(1) + (1+2\lambda)u_p'(1) + (1-\alpha)[u_p(1)-1]\right],
\end{aligned}$$

the last expression is bounded above by $(1-\alpha)$ if and only if (3.2) is satisfied. Hence the proof is completed. □

Corollary 3. *Let $c < 0$ and $m > 0$. If $f \in \Re^{\tau}(A,B)$, and the inequality*

$$(A-B)|\tau|\left[u_p'(1) + (1-\alpha)\{u_p(1)-1\}\right] \leq 1-\alpha \tag{3.4}$$

is satisfied, then $\mathcal{I}(c,m)f \in \mathcal{K}(\alpha)$.

Theorem 4. *Let $c < 0$ and $m > 0$. Then*

$$\mathcal{L}(m,c,z) = \int_0^z (2-u_p(t))dt$$

is in $\mathcal{K}^*(\lambda,\alpha)$ if and only if the inequality

$$\lambda u_p''(1) + [1+2\lambda]u_p'(1) + (1-\alpha)[u_p(1)-1] \leq 1-\alpha. \tag{3.5}$$

Proof. Since

$$\mathcal{L}(m,c,z) = z - \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}} \, \frac{z^n}{(n)!},$$

by Lemma 4, we need only to show that

$$\mathcal{L}(c,m,\lambda,\alpha) = \sum_{n=2}^{\infty} n(n^2\lambda + n(1-\lambda) - \alpha)\frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n)!} \leq 1-\alpha.$$

Now, we have

$$\mathcal{L}(c,m,\lambda,\alpha) = \sum_{n=2}^{\infty} (n^2\lambda + n(1-\lambda) - \alpha)\frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-1)!}.$$

Writing $n^2 = (n-1)(n-2) + 3(n-1) + 1$ and $n = (n-1) + 1$, and proceeding as in Theorem 1, we get

$$\sum_{n=2}^{\infty} (n^2\lambda + n(1-\lambda) - \alpha)\frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-1)!} = \lambda u_p''(1) + [1+2\lambda]u_p'(1) + (1-\alpha)[u_p(1) - 1],$$

which is bounded above by $1-\alpha$ if and only if (3.5) holds. □

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Barnes-type Narumi of the second kind and poly-Cauchy of the second kind mixed-type polynomials

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Abstract

In this paper, by considering Barnes-type Narumi polynomials of the second kind as well as poly-Cauchy polynomials of the second kind, we define and investigate

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the mixed-type polynomials of these polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

1 Introduction

In this paper, we consider the polynomials $\widehat{N}_n^{(k)}(x|a_1,\dots,a_r)$ called the Barnes-type Narumi of the second kind and poly-Cauchy of the second kind mixed-type polynomials, whose generating function is given by

$$\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^x=\sum_{n=0}^{\infty}\widehat{N}_n^{(k)}(x|a_1,\dots,a_r)\frac{t^n}{n!}\,, \quad (1)$$

where $a_1,\dots,a_r\neq 0$. Here, $\mathrm{Lif}_k(x)$ $(k\in\mathbb{Z})$ is the polyfactorial function ([10]) defined by

$$\mathrm{Lif}_k(x)=\sum_{m=0}^{\infty}\frac{x^m}{m!(m+1)^k}\,.$$

When $x=0$, $\widehat{N}_n^{(k)}(a_1,\dots,a_r)=\widehat{N}_n^{(k)}(0|a_1,\dots,a_r)$ is called the the Barnes-type Narumi of the second kind and poly-Cauchy of the second kind mixed-type number.

Recall that the Barnes-type Narumi polynomials of the second kind, denoted by $\widehat{N}_n(x|a_1,\dots,a_r)$, are given by the generating function as

$$\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)(1+t)^x=\sum_{n=0}^{\infty}\widehat{N}_n(x|a_1,\dots,a_r)\frac{t^n}{n!}\,.$$

If $a_1=\dots=a_r=1$, then $\widehat{N}_n^{(r)}(x)=\widehat{N}_n(x|\underbrace{1,\dots,1}_{r})$ are the Narumi polynomials of the second kind of order r. Narumi polynomials were mentioned in [14, p.127] and have been investigated in e.g. [9, 12, 15].

The poly-Cauchy polynomials of the second kind, denoted by $\widehat{c}_n^{(k)}(x)$ ([8, 11]), are given by the generating function as

$$\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^x=\sum_{n=0}^{\infty}\widehat{c}_n^{(k)}(x)\frac{t^n}{n!}\,.$$

In this paper, by considering Barnes-type Narumi polynomials of the second kind as well as poly-Cauchy polynomials of the second kind, we define and investigate the mixed-type polynomials of these polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

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2 Umbral calculus

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \middle| a_k \in \mathbb{C} \right\}. \tag{2}$$

Let $\mathbb{P} = \mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ is the action of the linear functional L on the polynomial $p(x)$, and we recall that the vector space operations on $\mathbb{P}^*$ are defined by $\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$, $\langle cL|p(x)\rangle = c\langle L|p(x)\rangle$, where c is a complex constant in $\mathbb{C}$ (see [1-16]). For $f(t) \in \mathcal{F}$, let us define the linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n, \quad (n \geq 0). \tag{3}$$

In particular,

$$\left\langle t^k|x^n \right\rangle = n!\delta_{n,k} \quad (n, k \geq 0), \tag{4}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. That is, $L = f_L(t)$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the *umbral algebra* and the *umbral calculus* is the study of umbral algebra. The order $O\big(f(t)\big)$ of a power series $f(t)(\neq 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $O\big(f(t)\big) = 1$, then $f(t)$ is called a *delta series*; if $O\big(f(t)\big) = 0$, then $f(t)$ is called an *invertible series*. For $f(t), g(t) \in \mathcal{F}$ with $O\big(f(t)\big) = 1$ and $O\big(g(t)\big) = 0$, there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\left\langle g(t)f(t)^k|s_n(x)\right\rangle = n!\delta_{n,k}$, for $n, k \geq 0$. Such a sequence $s_n(x)$ is called the *Sheffer sequence* for $\big(g(t), f(t)\big)$ which is denoted by $s_n(x) \sim \big(g(t), f(t)\big)$.

For $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle \tag{5}$$

and

$$f(t) = \sum_{k=0}^{\infty} \left\langle f(t)|x^k \right\rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \left\langle t^k|p(x) \right\rangle \frac{x^k}{k!} \tag{6}$$

([14, Theorem 2.2.5]). Thus, by (6), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{and} \quad e^{yt}p(x) = p(x+y). \tag{7}$$

Sheffer sequences are characterized in the generating function ([14, Theorem 2.3.4]).

3

Lemma 1 *The sequence $s_n(x)$ is Sheffer for $\big(g(t), f(t)\big)$ if and only if*

$$\frac{1}{g\big(\bar{f}(t)\big)} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} t^k \quad (y \in \mathbb{C}),$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

For $s_n(x) \sim \big(g(t), f(t)\big)$, we have the following equations ([14, Theorem 2.3.7, Theorem 2.3.5, Theorem 2.3.9]):

$$f(t)s_n(x) = ns_{n-1}(x) \quad (n \geq 0), \tag{8}$$

$$s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \left\langle g\big(\bar{f}(t)\big)^{-1} \bar{f}(t)^j | x^n \right\rangle x^j, \tag{9}$$

$$s_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x) p_{n-j}(y), \tag{10}$$

where $p_n(x) = g(t)s_n(x)$.

Assume that $p_n(x) \sim \big(1, f(t)\big)$ and $q_n(x) \sim \big(1, g(t)\big)$. Then the transfer formula ([14, Corollary 3.8.2]) is given by

$$q_n(x) = x\left(\frac{f(t)}{g(t)}\right)^n x^{-1} p_n(x) \quad (n \geq 1).$$

For $s_n(x) \sim \big(g(t), f(t)\big)$ and $r_n(x) \sim \big(h(t), l(t)\big)$, assume that

$$s_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x) \quad (n \geq 0).$$

Then we have ([14, p.132])

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h\big(\bar{f}(t)\big)}{g\big(\bar{f}(t)\big)} l\big(\bar{f}(t)\big)^m \middle| x^n \right\rangle. \tag{11}$$

3 Main results

From the definition (1), $\widehat{N}_n^{(k)}(x|a_1, \ldots, a_r)$ is the Sheffer sequence for the pair

$$g(t) = \prod_{j=1}^{r} \left(\frac{te^{a_j t}}{e^{a_j t} - 1}\right) \frac{1}{\mathrm{Lif}_k(-t)} \quad \text{and} \quad f(t) = e^t - 1.$$

So,

$$\widehat{N}_n^{(k)}(x|a_1, \ldots, a_r) \sim \left(\prod_{j=1}^{r} \left(\frac{te^{a_j t}}{e^{a_j t} - 1}\right) \frac{1}{\mathrm{Lif}_k(-t)}, e^t - 1\right). \tag{12}$$

4

3.1 Explicit expressions

Let $(n)_j = n(n-1)\cdots(n-j+1)$ $(j \ge 1)$ with $(n)_0 = 1$. The (signed) Stirling numbers of the first kind $S_1(n,m)$ are defined by

$$(x)_n = \sum_{m=0}^{n} S_1(n,m)x^m\,.$$

Define the multinomial coefficient by

$$\binom{n}{l_1,\dots,l_r} = \frac{n!}{l_1!\cdots l_r!}$$

where $l_1+\cdots+l_r = n$.

Theorem 1

$$\begin{aligned}
\widehat{N}_n^{(k)}(x|a_1,\dots,a_r) &= \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=0}^{m-l}\sum_{l_1+\cdots+l_r=m-l-i}\frac{(-1)^{m-i}(m-l-i)!}{(m-l-i+r)!(l+1)^k}\\
&\quad\times\binom{m-l-i+r}{l_1+1,\dots,l_r+1}\binom{m}{l}\binom{m-l}{i}S_1(n,m)a_1^{l_1+1}\cdots a_r^{l_r+1}x^i \qquad (13)\\
&= \sum_{j=0}^{n}\sum_{l=j}^{n}\sum_{i=0}^{n-l}\binom{n}{l}\binom{n-l}{i}S_1(l,j)\widehat{c}_i^{(k)}\widehat{N}_{n-l-i}(a_1,\dots,a_r)x^j \qquad (14)\\
&= \sum_{l=0}^{n}\binom{n}{l}\widehat{N}_{n-l}(a_1,\dots,a_r)\widehat{c}_l^{(k)}(x)\,, \qquad (15)\\
&= \sum_{l=0}^{n}\binom{n}{l}\widehat{c}_{n-l}^{(k)}\widehat{N}_l(x|a_1,\dots,a_r)\,. \qquad (16)
\end{aligned}$$

Proof. Since

$$\prod_{j=1}^{r}\left(\frac{te^{a_j t}}{e^{a_j t}-1}\right)\frac{1}{\mathrm{Lif}_k(-t)}\widehat{N}_n^{(k)}(x|a_1,\dots,a_r) \sim (1, e^t-1) \qquad (17)$$

and

$$(x)_n \sim (1, e^t-1)\,, \qquad (18)$$

5

we have

$$\begin{aligned}
\widehat{N}_n^{(k)}(x|a_1,\ldots,a_r) &= \prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right)\mathrm{Lif}_k(-t)(x)_n \\
&= \sum_{m=0}^{n} S_1(n,m)\prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right)\mathrm{Lif}_k(-t)x^m \\
&= \sum_{m=0}^{n} S_1(n,m)\prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right)\sum_{l=0}^{m}\frac{(-1)^l t^l}{l!(l+1)^k}x^m \\
&= \sum_{m=0}^{n} S_1(n,m)\prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right)\sum_{l=0}^{m}\frac{(-1)^l (m)_l}{l!(l+1)^k}x^{m-l} \\
&= \sum_{m=0}^{n} S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^l\binom{m}{l}}{(l+1)^k}\prod_{j=1}^{r}\left(\frac{e^{-a_jt}-1}{-t}\right)x^{m-l} \\
&= \sum_{m=0}^{n} S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^l\binom{m}{l}}{(l+1)^k} \\
&\quad\times\sum_{i=0}^{\infty}\sum_{l_1+\cdots+l_r=i}\frac{a_1^{l_1+1}\cdots a_r^{l_r+1}}{(l_1+1)!\cdots(l_r+1)!}(-t)^i x^{m-l} \\
&= \sum_{m=0}^{n} S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^l\binom{m}{l}}{(l+1)^k} \\
&\quad\times\sum_{i=0}^{m-l}\sum_{l_1+\cdots+l_r=i}\frac{a_1^{l_1+1}\cdots a_r^{l_r+1}}{(l_1+1)!\cdots(l_r+1)!}(-1)^i(m-l)_i x^{m-l-i} \\
&= \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=0}^{m-l}\sum_{l_1+\cdots+l_r=i}\frac{(-1)^{l+i}i!}{(i+r)!(l+1)^k} \\
&\quad\times\binom{i+r}{l_1+1,\ldots,l_r+1}\binom{m}{l}\binom{m-l}{i}S_1(n,m)a_1^{l_1+1}\cdots a_r^{l_r+1}x^{m-l-i} \\
&= \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=0}^{m-l}\sum_{l_1+\cdots+l_r=m-l-i}\frac{(-1)^{m-i}(m-l-i)!}{(m-l-i+r)!(l+1)^k} \\
&\quad\times\binom{m-l-i+r}{l_1+1,\ldots,l_r+1}\binom{m}{l}\binom{m-l}{i}S_1(n,m)a_1^{l_1+1}\cdots a_r^{l_r+1}x^{i}.
\end{aligned}$$

So, we get (13).

By (9) with (12), we get

$$\begin{aligned}
&\left\langle g\big(\bar{f}(t)\big)^{-1}\bar{f}(t)^j|x^n\right\rangle \\
&=\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^j\Big|x^n\right\rangle \\
&=\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)\Big|j!\sum_{l=j}^{\infty}S_1(l,j)\frac{t^l}{l!}x^n\right\rangle \\
&=j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)\Big|x^{n-l}\right\rangle \\
&=j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\left\langle \sum_{i=0}^{\infty}\widehat{N}_i^{(k)}(a_1,\dots,a_r)\frac{t^i}{i!}\Big|x^{n-l}\right\rangle \\
&=j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\widehat{N}_{n-l}^{(k)}(a_1,\dots,a_r)\,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left\langle g\big(\bar{f}(t)\big)^{-1}\bar{f}(t)^j|x^n\right\rangle \\
&=j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\Big|\mathrm{Lif}_k\big(-\ln(1+t)\big)x^{n-l}\right\rangle \\
&=j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\sum_{i=0}^{n-l}\binom{n-l}{i}\widehat{c}_i^{(k)}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\Big|x^{n-l-i}\right\rangle \\
&=j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\sum_{i=0}^{n-l}\binom{n-l}{i}\widehat{c}_i^{(k)}\left\langle \sum_{m=0}^{\infty}\widehat{N}_m(a_1,\dots,a_r)\frac{t^m}{m!}\Big|x^{n-l-i}\right\rangle \\
&=j!\sum_{l=j}^{n}\sum_{i=0}^{n-l}\binom{n}{l}\binom{n-l}{i}S_1(l,j)\widehat{c}_i^{(k)}\widehat{N}_{n-l-i}(a_1,\dots,a_r)\,.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\widehat{N}_n^{(k)}(x|a_1,\dots,a_r)&=\sum_{j=0}^{n}\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\widehat{N}_{n-l}^{(k)}(a_1,\dots,a_r)x^j \\
&=\sum_{j=0}^{n}\sum_{l=j}^{n}\sum_{i=0}^{n-l}\binom{n}{l}\binom{n-l}{i}S_1(l,j)\widehat{c}_i^{(k)}\widehat{N}_{n-l-i}(a_1,\dots,a_r)x^j\,,
\end{aligned}$$

which is the identity (14).

Next,

$$\begin{aligned}
\widehat{N}_n^{(k)}(y|a_1,\ldots,a_r) &= \left\langle \sum_{i=0}^{\infty} \widehat{N}_i^{(k)}(y|a_1,\ldots,a_r)\frac{t^i}{i!}\Big|x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y\Big|x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\Big|\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\Big|\sum_{l=0}^{\infty}\widehat{c}_l^{(k)}(y)\frac{t^l}{l!}x^n \right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l}\widehat{c}_l^{(k)}(y)\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\Big|x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l}\widehat{c}_l^{(k)}(y)\left\langle \sum_{i=0}^{\infty}\widehat{N}_i(a_1,\ldots,a_r)\frac{t^i}{i!}\Big|x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l}\widehat{c}_l^{(k)}(y)\widehat{N}_{n-l}(a_1,\ldots,a_r)\,.
\end{aligned}$$

Thus, we obtain (15).
Finally, we obtain that

$$\begin{aligned}
\widehat{N}_n^{(k)}(y|a_1,\ldots,a_r) &= \left\langle \sum_{i=0}^{\infty} \widehat{N}_i^{(k)}(y|a_1,\ldots,a_r)\frac{t^i}{i!}\Big|x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y\Big|x^n \right\rangle \\
&= \left\langle \mathrm{Lif}_k\big(-\ln(1+t)\big)\Big|\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)(1+t)^y x^n \right\rangle \\
&= \left\langle \mathrm{Lif}_k\big(-\ln(1+t)\big)\Big|\sum_{l=0}^{\infty}\widehat{N}_l(y|a_1,\ldots,a_r)\frac{t^l}{l!}x^n \right\rangle \\
&= \sum_{l=0}^{n}\widehat{N}_l(y|a_1,\ldots,a_r)\binom{n}{l}\left\langle \mathrm{Lif}_k\big(-\ln(1+t)\big)\Big|x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n}\widehat{N}_l(y|a_1,\ldots,a_r)\binom{n}{l}\left\langle \sum_{i=0}^{\infty}\widehat{c}_i^{(k)}\frac{t^i}{i!}\Big|x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l}\widehat{N}_l(y|a_1,\ldots,a_r)\widehat{c}_{n-l}^{(k)}\,.
\end{aligned}$$

Thus, we get the identity (16). ∎

3.2 Sheffer identity

Theorem 2

$$\widehat{N}_n^{(k)}(x+y|a_1,\dots,a_r)=\sum_{j=0}^{n}\binom{n}{j}\widehat{N}_j^{(k)}(x|a_1,\dots,a_r)(y)_{n-j}\,. \tag{19}$$

Proof. By (12) with

$$\begin{aligned}p_n(x)&=\prod_{j=1}^{r}\left(\frac{te^{a_jt}}{e^{a_jt}-1}\right)\frac{1}{\mathrm{Lif}_k(-t)}\widehat{N}_n(x|a_1,\dots,a_r)\\&=(x)_n\sim(1,e^t-1)\,,\end{aligned}$$

using (10), we have (19). ∎

3.3 Difference relations

Theorem 3

$$\widehat{N}_n^{(k)}(x+1|a_1,\dots,a_r)-\widehat{N}_n^{(k)}(x|a_1,\dots,a_r)=n\widehat{N}_{n-1}^{(k)}(x|a_1,\dots,a_r)\,. \tag{20}$$

Proof. By (8) with (12), we get

$$(e^t-1)\widehat{N}_n^{(k)}(x|a_1,\dots,a_r)=n\widehat{N}_{n-1}^{(k)}(x|a_1,\dots,a_r)\,.$$

By (7), we have (20). ∎

3.4 Recurrence

Theorem 4

$$
\begin{aligned}
&\widehat{N}_{n+1}^{(k)}(x|a_1,\dots,a_r)\\
&= x\widehat{N}_n^{(k)}(x-1|a_1,\dots,a_r)\\
&\quad+\sum_{m=0}^{n}\sum_{j=1}^{r}\sum_{l=0}^{m}\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=l-i}\sum_{h=0}^{i}\frac{(-1)^{m+1-h}}{m+1}\frac{(l-i)!}{(l-i+r)!(i-h+1)^k}\\
&\quad\times\binom{m+1}{l}\binom{l-i+r}{l_1+1,\dots,l_r+1}\binom{l}{i}\binom{i}{h}S_1(n,m)B_{m+1-l}a_j^{m+1-l}a_1^{l_1+1}\cdots a_r^{l_r+1}(x-1)^h\\
&\quad-\sum_{j=1}^{r}a_j\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=l-i}\frac{(-1)^{m-i}}{(m-l+1)^k}\frac{(l-i)!}{(l-i+r)!}\\
&\quad\times\binom{m}{l}\binom{l-i+r}{l_1+1,\dots,l_r+1}\binom{l}{i}S_1(n,m)a_1^{l_1+1}\cdots a_r^{l_r+1}(x-1)^i\\
&\quad-\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=l-i}\frac{(-1)^{m-i+1}}{(m-l+2)^k}\frac{(l-i)!}{(l-i+r)!}\\
&\quad\times\binom{m}{l}\binom{l-i+r}{l_1+1,\dots,l_r+1}\binom{l}{i}S_1(n,m)a_1^{l_1+1}\cdots a_r^{l_r+1}(x-1)^i\,,
\end{aligned}
\tag{21}
$$

where B_n is the nth ordinary Bernoulli number.

Proof. By applying

$$s_{n+1}(x)=\left(x-\frac{g'(t)}{g(t)}\right)\frac{1}{f'(t)}s_n(x) \tag{22}$$

([14, Corollary 3.7.2]) with (12), we get

$$\widehat{N}_{n+1}^{(k)}(x|a_1,\dots,a_r)=x\widehat{N}_n^{(k)}(x-1|a_1,\dots,a_r)-e^{-t}\frac{g'(t)}{g(t)}\widehat{N}_n^{(k)}(x|a_1,\dots,a_r)\,.$$

Now,

$$
\begin{aligned}
\frac{g'(t)}{g(t)}&=\big(\ln g(t)\big)'\\
&=\left(r\ln t+\left(\sum_{j=1}^{r}a_j\right)t-\sum_{j=1}^{r}\ln(e^{a_jt}-1)-\ln\mathrm{Lif}_k(-t)\right)'\\
&=\frac{r}{t}+\sum_{j=1}^{r}a_j-\sum_{j=1}^{r}\frac{a_je^{a_jt}}{e^{a_jt}-1}+\frac{\mathrm{Lif}_k'(-t)}{\mathrm{Lif}_k(-t)}\\
&=\frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_it}-1)(e^{a_jt}-1-a_jte^{a_jt})}{t\prod_{j=1}^{r}(e^{a_jt}-1)}+\sum_{j=1}^{r}a_j+\frac{\mathrm{Lif}_k'(-t)}{\mathrm{Lif}_k(-t)}\,,
\end{aligned}
$$

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where

$$\begin{aligned}
&\frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_it}-1)(e^{a_jt}-1-a_jte^{a_jt})}{t\prod_{j=1}^{r}(e^{a_jt}-1)}\\
&=-\frac{\frac{1}{2}\left(\sum_{j=1}^{r}a_1\cdots a_{j-1}a_j^2a_{j+1}\cdots a_r\right)t^{r+1}+\cdots}{(a_1\cdots a_r)t^r+\cdots}\\
&=-\frac{1}{2}\left(\sum_{j=1}^{r}a_j\right)t+\cdots
\end{aligned}$$

is a series with order≥ 1. As seen in the proof of (13)

$$\widehat{N}_n^{(k)}(x|a_1,\ldots,a_r)=\sum_{m=0}^{n}S_1(n,m)\prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right)\mathrm{Lif}_k(-t)x^m\,,$$

so we have

$$\begin{aligned}
&\frac{g'(t)}{g(t)}\widehat{N}_n^{(k)}(x|a_1,\ldots,a_r)\\
&=\sum_{m=0}^{n}S_1(n,m)\frac{g'(t)}{g(t)}\left(\prod_{j=1}^{r}\frac{e^{a_jt}-1}{te^{a_jt}}\right)\mathrm{Lif}_k(-t)x^m\\
&=\sum_{m=0}^{n}S_1(n,m)\mathrm{Lif}_k(-t)\left(\prod_{j=1}^{r}\frac{e^{a_jt}-1}{te^{a_jt}}\right)\frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_it}-1)(e^{a_jt}-1-a_jte^{a_jt})}{t\prod_{j=1}^{r}(e^{a_jt}-1)}x^m\\
&\quad+\sum_{j=1}^{r}a_j\sum_{m=0}^{n}S_1(n,m)\left(\prod_{j=1}^{r}\frac{e^{a_jt}-1}{te^{a_jt}}\right)\mathrm{Lif}_k(-t)x^m\\
&\quad+\sum_{m=0}^{n}S_1(n,m)\left(\prod_{j=1}^{r}\frac{e^{a_jt}-1}{te^{a_jt}}\right)\mathrm{Lif}_k'(-t)x^m\,. \qquad (23)
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_i t}-1)(e^{a_j t}-1-a_j t e^{a_j t})}{t\prod_{j=1}^{r}(e^{a_j t}-1)}x^m \\
&=\frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_i t}-1)(e^{a_j t}-1-a_j t e^{a_j t})}{\prod_{j=1}^{r}(e^{a_j t}-1)}\frac{x^{m+1}}{m+1} \\
&=\frac{1}{m+1}\sum_{j=1}^{r}\left(1-\frac{a_j t e^{a_j t}}{e^{a_j t}-1}\right)x^{m+1} \\
&=\frac{1}{m+1}\sum_{j=1}^{r}\left(1-\sum_{l=0}^{\infty}\frac{(-a_j)^l B_l}{l!}t^l\right)x^{m+1} \\
&=\frac{1}{m+1}\sum_{j=1}^{r}\left(x^{m+1}-\sum_{l=0}^{m+1}\binom{m+1}{l}B_l(-a_j)^l x^{m+1-l}\right) \\
&=-\frac{1}{m+1}\sum_{j=1}^{r}\sum_{l=1}^{m+1}\binom{m+1}{l}B_l(-a_j)^l x^{m+1-l} \\
&=-\frac{1}{m+1}\sum_{j=1}^{r}\sum_{l=0}^{m}\binom{m+1}{l}B_{m+1-l}(-a_j)^{m+1-l}x^l\,,
\end{aligned}$$

the first term in (23) is

$$\begin{aligned}
&-\sum_{m=0}^{n}\sum_{j=1}^{r}\sum_{l=0}^{m}\frac{S_1(n,m)}{m+1}\binom{m+1}{l}B_{m+1-l}(-a_j)^{m+1-l}\mathrm{Lif}_k(-t)\left(\prod_{j=1}^{r}\frac{e^{-a_j t}-1}{-t}\right)x^l \\
&=-\sum_{m=0}^{n}\sum_{j=1}^{r}\sum_{l=0}^{m}\frac{S_1(n,m)}{m+1}\binom{m+1}{l}B_{m+1-l}(-a_j)^{m+1-l}\mathrm{Lif}_k(-t) \\
&\quad\times\sum_{i=0}^{\infty}\sum_{l_1+\cdots+l_r=i}\frac{a_1^{l_1+1}\cdots a_r^{l_r+1}}{(l_1+1)!\cdots(l_r+1)!}(-t)^i x^l \\
&=-\sum_{m=0}^{n}\sum_{j=1}^{r}\sum_{l=0}^{m}\frac{S_1(n,m)}{m+1}\binom{m+1}{l}B_{m+1-l}(-a_j)^{m+1-l}\mathrm{Lif}_k(-t) \\
&\quad\times\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=i}(-1)^i\frac{a_1^{l_1+1}\cdots a_r^{l_r+1}}{(l_1+1)!\cdots(l_r+1)!}(l)_i x^{l-i} \\
&=-\sum_{m=0}^{n}\sum_{j=1}^{r}\sum_{l=0}^{m}\frac{S_1(n,m)}{m+1}\binom{m+1}{l}B_{m+1-l}(-a_j)^{m+1-l} \\
&\quad\times\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=i}(-1)^i\frac{a_1^{l_1+1}\cdots a_r^{l_r+1}}{(l_1+1)!\cdots(l_r+1)!}(l)_i\sum_{h=0}^{l-i}\frac{(-1)^h}{h!(h+1)^k}t^h x^{l-i}
\end{aligned}$$

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$$
\begin{aligned}
&= -\sum_{m=0}^{n}\sum_{j=1}^{r}\sum_{l=0}^{m}\frac{S_1(n,m)}{m+1}\binom{m+1}{l}B_{m+1-l}(-a_j)^{m+1-l} \\
&\quad \times \sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=i}(-1)^i\frac{a_1^{l_1+1}\cdots a_r^{l_r+1}}{(l_1+1)!\cdots(l_r+1)!}(l)_i\sum_{h=0}^{l-i}\frac{(-1)^h}{(h+1)^k}\binom{l-i}{h}x^{l-i-h} \\
&= -\sum_{m=0}^{n}\sum_{j=1}^{r}\sum_{l=0}^{m}\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=i}\sum_{h=0}^{l-i}\frac{(-1)^{m+1-h}}{m+1}\frac{i!}{(i+r)!(l-i-h+1)^k} \\
&\quad \times \binom{m+1}{l}\binom{i+r}{l_1+1,\ldots,l_r+1}\binom{l}{i}\binom{l-i}{h}S_1(n,m)B_{m+1-l}a_j^{m+1-l}a_1^{l_1+1}\cdots a_r^{l_r+1}x^h \\
&= -\sum_{m=0}^{n}\sum_{j=1}^{r}\sum_{l=0}^{m}\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=l-i}\sum_{h=0}^{i}\frac{(-1)^{m+1-h}}{m+1}\frac{(l-i)!}{(l-i+r)!(i-h+1)^k} \\
&\quad \times \binom{m+1}{l}\binom{l-i+r}{l_1+1,\ldots,l_r+1}\binom{l}{i}\binom{i}{h}S_1(n,m)B_{m+1-l}a_j^{m+1-l}a_1^{l_1+1}\cdots a_r^{l_r+1}x^h\,.
\end{aligned}
$$

The second term in (23) is

$$
\begin{aligned}
&\sum_{j=1}^{r}a_j\sum_{m=0}^{n}S_1(n,m)\left(\prod_{j=1}^{r}\frac{e^{a_jt}-1}{te^{a_jt}}\right)\mathrm{Lif}_k(-t)x^m \\
&= \sum_{j=1}^{r}a_j\sum_{m=0}^{n}S_1(n,m)\left(\prod_{j=1}^{r}\frac{e^{a_jt}-1}{te^{a_jt}}\right)\sum_{l=0}^{m}\frac{(-1)^l t^l}{l!(l+1)^k}x^m \\
&= \sum_{j=1}^{r}a_j\sum_{m=0}^{n}S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^l}{(l+1)^k}\binom{m}{l}\left(\prod_{j=1}^{r}\frac{e^{-a_jt}-1}{-t}\right)x^{m-l} \\
&= \sum_{j=1}^{r}a_j\sum_{m=0}^{n}S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^l}{(l+1)^k}\binom{m}{l}\sum_{i=0}^{m-l}\sum_{l_1+\cdots+l_r=i}\frac{a_1^{l_1+1}\cdots a_r^{l_r+1}}{(l_1+1)!\cdots(l_r+1)!}(-1)^i t^i x^{m-l} \\
&= \left(\sum_{j=1}^{r}a_j\right)\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=0}^{m-l}\sum_{l_1+\cdots+l_r=i}\frac{(-1)^{i+l}}{(l+1)^k}\frac{i!}{(i+r)!} \\
&\quad \times \binom{m}{l}\binom{i+r}{l_1+1,\ldots,l_r+1}\binom{m-l}{i}S_1(n,m)a_1^{l_1+1}\cdots a_r^{l_r+1}x^{m-l-i} \\
&= \sum_{j=1}^{r}a_j\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=l-i}\frac{(-1)^{m-i}}{(m-l+1)^k}\frac{(l-i)!}{(l-i+r)!} \\
&\quad \times \binom{m}{l}\binom{l-i+r}{l_1+1,\ldots,l_r+1}\binom{l}{i}S_1(n,m)a_1^{l_1+1}\cdots a_r^{l_r+1}x^i\,.
\end{aligned}
$$

The third term in (23) is

$$\begin{aligned}
&\sum_{m=0}^{n} S_1(n,m)\left(\prod_{j=1}^{r}\frac{e^{a_jt}-1}{te^{a_jt}}\right)\mathrm{Lif}'_k(-t)x^m \\
&=\sum_{m=0}^{n} S_1(n,m)\left(\prod_{j=1}^{r}\frac{e^{a_jt}-1}{te^{a_jt}}\right)\frac{\mathrm{Lif}_{k-1}(-t)-\mathrm{Lif}_k(-t)}{t}x^m \\
&=\sum_{m=0}^{n} \frac{S_1(n,m)}{m+1}\left(\prod_{j=1}^{r}\frac{e^{a_jt}-1}{te^{a_jt}}\right)\big(\mathrm{Lif}_{k-1}(-t)-\mathrm{Lif}_k(-t)\big)x^{m+1} \\
&=\sum_{m=0}^{n} \frac{S_1(n,m)}{m+1}\left(\prod_{j=1}^{r}\frac{e^{a_jt}-1}{te^{a_jt}}\right)\sum_{l=0}^{m}\frac{(-1)^{l+1}t^{l+1}}{l!(l+2)^k}x^{m+1} \\
&=\sum_{m=0}^{n}\sum_{l=0}^{m}\frac{(-1)^{l+1}}{(l+2)^k}\binom{m}{l}S_1(n,m)\left(\prod_{j=1}^{r}\frac{e^{-a_jt}-1}{-t}\right)x^{m-l} \\
&=\sum_{m=0}^{n}\sum_{l=0}^{m}\frac{(-1)^{l+1}}{(l+2)^k}\binom{m}{l}S_1(n,m)\sum_{i=0}^{m-l}\sum_{l_1+\cdots+l_r=i}(-1)^i\frac{a_1^{l_1+1}\cdots a_r^{l_r+1}}{(l_1+1)!\cdots(l_r+1)!}t^ix^{m-l} \\
&=\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=0}^{m-l}\sum_{l_1+\cdots+l_r=i}\frac{(-1)^{i+l+1}}{(l+2)^k}\frac{i!}{(i+r)!} \\
&\quad\times\binom{m}{l}\binom{i+r}{l_1+1,\ldots,l_r+1}\binom{m-l}{i}S_1(n,m)a_1^{l_1+1}\cdots a_r^{l_r+1}x^{m-l-i} \\
&=\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=l-i}\frac{(-1)^{m-i+1}}{(m-l+2)^k}\frac{(l-i)!}{(l-i+r)!} \\
&\quad\times\binom{m}{l}\binom{l-i+r}{l_1+1,\ldots,l_r+1}\binom{l}{i}S_1(n,m)a_1^{l_1+1}\cdots a_r^{l_r+1}x^i\,.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\widehat{N}_{n+1}^{(k)}(x|a_1,\dots,a_r) \\
&= x\widehat{N}_n^{(k)}(x-1|a_1,\dots,a_r) \\
&\quad + \sum_{m=0}^{n}\sum_{j=1}^{r}\sum_{l=0}^{m}\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=l-i}\sum_{h=0}^{i}\frac{(-1)^{m+1-h}}{m+1}\frac{(l-i)!}{(l-i+r)!(i-h+1)^k} \\
&\quad \times \binom{m+1}{l}\binom{l-i+r}{l_1+1,\dots,l_r+1}\binom{l}{i}\binom{i}{h}S_1(n,m)B_{m+1-l}a_j^{m+1-l}a_1^{l_1+1}\cdots a_r^{l_r+1}(x-1)^h \\
&\quad - \sum_{j=1}^{r}a_j\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=l-i}\frac{(-1)^{m-i}}{(m-l+1)^k}\frac{(l-i)!}{(l-i+r)!} \\
&\quad \times \binom{m}{l}\binom{l-i+r}{l_1+1,\dots,l_r+1}\binom{l}{i}S_1(n,m)a_1^{l_1+1}\cdots a_r^{l_r+1}(x-1)^i \\
&\quad - \sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=0}^{l}\sum_{l_1+\cdots+l_r=l-i}\frac{(-1)^{m-i+1}}{(m-l+2)^k}\frac{(l-i)!}{(l-i+r)!} \\
&\quad \times \binom{m}{l}\binom{l-i+r}{l_1+1,\dots,l_r+1}\binom{l}{i}S_1(n,m)a_1^{l_1+1}\cdots a_r^{l_r+1}(x-1)^i\,,
\end{aligned}$$

which is the identity (21). ∎

3.5 Differentiation

Theorem 5

$$\frac{d}{dx}\widehat{N}_n^{(k)}(x|a_1,\dots,a_r) = n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}\widehat{N}_l^{(k)}(x|a_1,\dots,a_r)\,. \tag{24}$$

Proof. We shall use

$$\frac{d}{dx}s_n(x) = \sum_{l=0}^{n-1}\binom{n}{l}\left\langle \bar{f}(t)|x^{n-l}\right\rangle s_l(x)$$

(*Cf.* [14, Theorem 2.3.12]). Since

$$\begin{aligned}\left\langle \bar{f}(t)|x^{n-l}\right\rangle &= \left\langle \ln(1+t)|x^{n-l}\right\rangle \\ &= \left\langle \sum_{m=1}^{\infty}\frac{(-1)^{m-1}t^m}{m}\Big|x^{n-l}\right\rangle \\ &= \sum_{m=1}^{n-l}\frac{(-1)^{m-1}}{m}\left\langle t^m|x^{n-l}\right\rangle \\ &= \sum_{m=1}^{n-l}\frac{(-1)^{m-1}}{m}(n-l)!\delta_{m,n-l} \\ &= (-1)^{n-l-1}(n-l-1)!\,,\end{aligned}$$

with (12), we have

$$\begin{aligned}\frac{d}{dx}\widehat{N}_n^{(k)}(x|a_1,\dots,a_r) &= \sum_{l=0}^{n-1}\binom{n}{l}(-1)^{n-l-1}(n-l-1)!\widehat{N}_l^{(k)}(x|a_1,\dots,a_r) \\ &= n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}\widehat{N}_l^{(k)}(x|a_1,\dots,a_r)\,,\end{aligned}$$

which is the identity (24). ∎

3.6 A more relation

The classical Cauchy numbers c_n are defined by

$$\frac{t}{\ln(1+t)} = \sum_{n=0}^{\infty} c_n\frac{t^n}{n!}$$

(see e.g. [3, 10]).

Theorem 6

$$\begin{aligned}&\widehat{N}_n^{(k)}(x|a_1,\dots,a_r) \qquad (25)\\ &= \left(x-\sum_{i=1}^r a_i\right)\widehat{N}_{n-1}^{(k)}(x-1|a_1,\dots,a_r) \\ &\quad + \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l}c_l\big(\widehat{N}_{n-l}^{(k-1)}(x-1|a_1,\dots,a_r)-(r+1)\widehat{N}_{n-l}^{(k)}(x-1|a_1,\dots,a_r)\big) \\ &\quad + \frac{1}{n}\sum_{i=1}^{r}\sum_{l=0}^{n}\binom{n}{l}a_ic_l\widehat{N}_{n-l}^{(k)}(x-1|a_1,\dots,a_{i-1},a_{i+1},\dots,a_r)\,. \qquad (26)\end{aligned}$$

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Proof. For $n \geq 1$, we have

$$\begin{aligned}
\widehat{N}_n^{(k)}(y|a_1,\ldots,a_r) &= \left\langle \sum_{l=0}^{\infty} \widehat{N}_l^{(k)}(y|a_1,\ldots,a_r)\frac{t^l}{l!}\middle| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \middle| x^n \right\rangle \\
&= \left\langle \partial_t\left(\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y\right) \middle| x^{n-1} \right\rangle \\
&= \left\langle \left(\partial_t\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \middle| x^{n-1} \right\rangle \\
&\quad + \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\Big(\partial_t\mathrm{Lif}_k\big(-\ln(1+t)\big)\Big)(1+t)^y \middle| x^{n-1} \right\rangle \\
&\quad + \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)\left(\partial_t(1+t)^y\right) \middle| x^{n-1} \right\rangle .
\end{aligned}$$

The third term is

$$\begin{aligned}
& y\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{y-1} \middle| x^{n-1} \right\rangle \\
&= y\widehat{N}_{n-1}^{(k)}(y-1|a_1,\ldots,a_r)\,.
\end{aligned}$$

Since

$$\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big) - \mathrm{Lif}_k\big(-\ln(1+t)\big) = \left(\frac{1}{2^k}-\frac{1}{2^{k-1}}\right)t + \cdots ,$$

the second term is

$$\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)}{(1+t)\ln(1+t)}(1+t)^y \middle| x^{n-1}\right\rangle$$

$$=\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\right.$$

$$\left.\frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)}{t}(1+t)^{y-1} \middle| \frac{t}{\ln(1+t)}x^{n-1}\right\rangle$$

$$=\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\right.$$

$$\left.\frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)}{t}(1+t)^{y-1} \middle| \sum_{l=0}^{\infty}c_l\frac{t^l}{l!}x^{n-1}\right\rangle$$

$$=\sum_{l=0}^{n-1}\binom{n-1}{l}c_l\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\right.$$

$$\left.(1+t)^{y-1} \middle| \frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)}{t}x^{n-1-l}\right\rangle$$

$$=\sum_{l=0}^{n-1}\frac{1}{n-l}\binom{n-1}{l}c_l\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\right.$$

$$\left.(1+t)^{y-1} \middle| \Big(\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)\Big)\,x^{n-l}\right\rangle$$

$$=\sum_{l=0}^{n-1}\frac{1}{n-l}\binom{n}{l}c_l\left(\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)(1+t)^{y-1} \middle| x^{n-l}\right\rangle\right.$$

$$\left.-\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_{k}\big(-\ln(1+t)\big)(1+t)^{y-1} \middle| x^{n-l}\right\rangle\right)$$

$$=\frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}c_l\big(\widehat{N}_{n-l}^{(k-1)}(y-1|a_1,\ldots,a_r)-\widehat{N}_{n-l}^{(k)}(y-1|a_1,\ldots,a_r)\big)\,.$$

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Since

$$\begin{aligned}
&\partial_t \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\\
&=\frac{1}{1+t}\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\frac{\sum_{i=1}^{r}\left(\frac{a_i t(1+t)^{a_i}}{(1+t)^{a_i}-1}-\frac{t}{\ln(1+t)}\right)}{t}\\
&\quad-\frac{1}{1+t}\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\sum_{i=1}^{r}a_i\,,
\end{aligned}$$

the first term is

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{y-1}\middle|\frac{\sum_{i=1}^{r}\left(\frac{a_i t(1+t)^{a_i}}{(1+t)^{a_i}-1}-\frac{t}{\ln(1+t)}\right)}{t}x^{n-1}\right\rangle\\
&\quad-\sum_{i=1}^{r}a_i\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{y-1}\middle|x^{n-1}\right\rangle\\
&=\frac{1}{n}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{y-1}\right.\\
&\qquad\qquad\left.\Big|\sum_{i=1}^{r}\left(\frac{a_i t(1+t)^{a_i}}{(1+t)^{a_i}-1}-\frac{t}{\ln(1+t)}\right)x^{n}\right\rangle\\
&\quad-\sum_{i=1}^{r}a_i\widehat{N}_{n-1}^{(k)}(y-1|a_1,\dots,a_r)\\
&=\frac{1}{n}\sum_{i=1}^{r}a_i\left\langle\frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1}\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\right.\\
&\qquad\qquad\left.\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{y-1}\middle|\frac{t}{\ln(1+t)}x^{n}\right\rangle\\
&\quad-\frac{r}{n}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{y-1}\middle|\frac{t}{\ln(1+t)}x^{n}\right\rangle\\
&\quad-\sum_{i=1}^{r}a_i\widehat{N}_{n-1}^{(k)}(y-1|a_1,\dots,a_r)
\end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{n}\sum_{i=1}^{r} a_i \left\langle \frac{(1+t)^{a_i}\ln(1+t)}{(1+t)^{a_i}-1} \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\right. \\
&\qquad\qquad \left. \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{y-1} \Big| \sum_{l=0}^{\infty} c_l \frac{t^l}{l!} x^n \right\rangle \\
&\quad - \frac{r}{n}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right) \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{y-1} \Big| \sum_{l=0}^{\infty} c_l \frac{t^l}{l!} x^n \right\rangle \\
&\quad - \sum_{i=1}^{r} a_i \widehat{N}_{n-1}^{(k)}(y-1|a_1,\dots,a_r) \\
&= \frac{1}{n}\sum_{i=1}^{r} a_i \sum_{l=0}^{n}\binom{n}{l} c_l \widehat{N}_{n-l}^{(k)}(y-1|a_1,\dots,a_{i-1},a_{i+1},\dots,a_r) \\
&\quad - \frac{r}{n}\sum_{l=0}^{n}\binom{n}{l} c_l \widehat{N}_{n-l}^{(k)}(y-1|a_1,\dots,a_r) \\
&\quad - \sum_{i=1}^{r} a_i \widehat{N}_{n-1}^{(k)}(y-1|a_1,\dots,a_r)\,.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&\widehat{N}_n^{(k)}(x|a_1,\dots,a_r) \\
&= x\widehat{N}_{n-1}^{(k)}(x-1|a_1,\dots,a_r) \\
&\quad + \frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l} c_l \big(\widehat{N}_{n-l}^{(k-1)}(x-1|a_1,\dots,a_r) - \widehat{N}_{n-l}^{(k)}(x-1|a_1,\dots,a_r)\big) \\
&\quad + \frac{1}{n}\sum_{i=1}^{r} a_i \sum_{l=0}^{n}\binom{n}{l} c_l \widehat{N}_{n-l}^{(k)}(x-1|a_1,\dots,a_{i-1},a_{i+1},\dots,a_r) \\
&\quad - \frac{r}{n}\sum_{l=0}^{n}\binom{n}{l} c_l \widehat{N}_{n-l}^{(k)}(x-1|a_1,\dots,a_r) \\
&\quad - \sum_{i=1}^{r} a_i \widehat{N}_{n-1}^{(k)}(x-1|a_1,\dots,a_r) \\
&= \left(x - \sum_{i=1}^{r} a_i\right)\widehat{N}_{n-1}^{(k)}(x-1|a_1,\dots,a_r) \\
&\quad + \frac{1}{n}\sum_{l=0}^{n}\binom{n}{l} c_l \big(\widehat{N}_{n-l}^{(k-1)}(x-1|a_1,\dots,a_r) - (r+1)\widehat{N}_{n-l}^{(k)}(x-1|a_1,\dots,a_r)\big) \\
&\quad + \frac{1}{n}\sum_{i=1}^{r}\sum_{l=0}^{n}\binom{n}{l} a_i c_l \widehat{N}_{n-l}^{(k)}(x-1|a_1,\dots,a_{i-1},a_{i+1},\dots,a_r)\,,
\end{aligned}
$$

which is the identity (26). ∎

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3.7 A relation involving the Stirling numbers of the first kind

Theorem 7 *For $n-1 \geq m \geq 1$, we have*

$$\begin{aligned}
& m\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\widehat{N}_l^{(k)}(a_1,\dots,a_r) \\
&= \frac{m}{n}\sum_{i=0}^{n-m}\sum_{l=0}^{i}\binom{n}{i}\binom{i}{l}S_1(n-i,m)c_{i-l} \\
&\quad\times\left(\sum_{j=1}^{r}a_j\widehat{N}_l^{(k)}(-1|a_1,\dots,a_{j-1},a_{j+1},\dots,a_r)-r\widehat{N}_l^{(k)}(-1|a_1,\dots,a_r)\right) \\
&\quad - m\sum_{j=1}^{r}a_j\sum_{l=0}^{n-m-1}\binom{n-1}{l}S_1(n-1-l,m)\widehat{N}_l^{(k)}(-1|a_1,\dots,a_r) \\
&\quad + \sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-1-l,m-1) \\
&\quad\times\left(\widehat{N}_l^{(k-1)}(-1|a_1,\dots,a_r)+(m-1)\widehat{N}_l^{(k)}(-1|a_1,\dots,a_r)\right). \qquad (27)
\end{aligned}$$

Proof. We shall compute

$$\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m\Big|x^n\right\rangle$$

in two different ways. On the one hand, it is equal to

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)\Big|\bigl(\ln(1+t)\bigr)^m x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)\Big|m!\sum_{l=0}^{\infty}S_1(l,m)\frac{t^l}{l!}x^n\right\rangle \\
&= m!\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)\Big|x^{n-l}\right\rangle \\
&= m!\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\left\langle \sum_{i=0}^{\infty}\widehat{N}_i^{(k)}(a_1,\dots,a_r)\frac{t^i}{i!}\Big|x^{n-l}\right\rangle \\
&= m!\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\widehat{N}_{n-l}^{(k)}(a_1,\dots,a_r) \\
&= m!\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\widehat{N}_l^{(k)}(a_1,\dots,a_r).
\end{aligned}$$

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On the other hand, it is equal to

$$\begin{aligned}
&\left\langle \partial_t \left(\prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)} \right) \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m \right) \Big| x^{n-1} \right\rangle \\
&= \left\langle \left(\partial_t \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)} \right) \right) \mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m \Big| x^{n-1} \right\rangle \\
&\quad + \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)} \right) \big(\partial_t \mathrm{Lif}_k\big(-\ln(1+t)\big)\big) \big(\ln(1+t)\big)^m \Big| x^{n-1} \right\rangle \\
&\quad + \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)} \right) \mathrm{Lif}_k\big(-\ln(1+t)\big) \big(\partial_t \big(\ln(1+t)\big)^m\big) \Big| x^{n-1} \right\rangle . \qquad (28)
\end{aligned}$$

The third term of (28) is equal to

$$\begin{aligned}
&m \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)} \right) \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1} \Big| \big(\ln(1+t)\big)^{m-1} x^{n-1} \right\rangle \\
&= m \Bigg\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)} \right) \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1} \Big| \\
&\qquad (m-1)! \sum_{l=m-1}^{\infty} S_1(l,m-1)\frac{t^l}{l!} x^{n-1} \Bigg\rangle \\
&= m! \sum_{l=m-1}^{n-1} \binom{n-1}{l} S_1(l,m-1) \\
&\quad \times \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)} \right) \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1} \Big| x^{n-1-l} \right\rangle \\
&= m! \sum_{l=m-1}^{n-1} \binom{n-1}{l} S_1(l,m-1) \widehat{N}_{n-1-l}^{(k)}(-1|a_1,\dots,a_r) \\
&= m! \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1,m-1) \widehat{N}_{l}^{(k)}(-1|a_1,\dots,a_r) .
\end{aligned}$$

The second term of (28) is equal to

$$\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\left(\frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)}{(1+t)\ln(1+t)}\right)\big(\ln(1+t)\big)^m \Big| x^{n-1}\right\rangle$$
$$=\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)(1+t)^{-1}\Big|\big(\ln(1+t)\big)^{m-1}x^{n-1}\right\rangle$$
$$-\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_{k}\big(-\ln(1+t)\big)(1+t)^{-1}\Big|\big(\ln(1+t)\big)^{m-1}x^{n-1}\right\rangle$$
$$=(m-1)!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{N}_l^{(k-1)}(-1|a_1,\dots,a_r)$$
$$-(m-1)!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{N}_l^{(k)}(-1|a_1,\dots,a_r)\,.$$

The first term of (28) is equal to

$$\left\langle\left(\partial_t\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\Big|x^{n-1}\right\rangle$$
$$=\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\big(\ln(1+t)\big)^m\Big|\right.$$
$$\left.\frac{\sum_{j=1}^{r}\left(\frac{a_jt(1+t)^{a_j}}{(1+t)^{a_j}-1}-\frac{t}{\ln(1+t)}\right)}{t}x^{n-1}\right\rangle$$
$$-\sum_{j=1}^{r}a_j\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\big(\ln(1+t)\big)^m\Big|x^{n-1}\right\rangle$$
$$=\frac{1}{n}\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\Big|\right.$$
$$\left.\sum_{j=1}^{r}\left(\frac{a_jt(1+t)^{a_j}}{(1+t)^{a_j}-1}-\frac{t}{\ln(1+t)}\right)\big(\ln(1+t)\big)^m x^n\right\rangle$$
$$-\sum_{j=1}^{r}a_j\left\langle\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\Big|\big(\ln(1+t)\big)^m x^{n-1}\right\rangle$$

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$$
\begin{aligned}
&= \frac{1}{n}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\right| \\
&\qquad \sum_{j=1}^{r}\left(\frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1}-\frac{t}{\ln(1+t)}\right)m!\sum_{i=m}^{\infty}S_1(i,m)\frac{t^i}{i!}x^n\Bigg\rangle \\
&\quad -\sum_{j=1}^{r}a_j\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\right| m!\sum_{i=m}^{\infty}S_1(i,m)\frac{t^i}{i!}x^{n-1}\Bigg\rangle \\
&= \frac{m!}{n}\sum_{i=m}^{n}\binom{n}{i}S_1(i,m)\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\right. \\
&\qquad \left.\sum_{j=1}^{r}\left(\frac{a_j t(1+t)^{a_j}}{(1+t)^{a_j}-1}-\frac{t}{\ln(1+t)}\right)\right| x^{n-i}\Bigg\rangle \\
&\quad - m!\sum_{j=1}^{r}a_j\sum_{i=m}^{n-1}\binom{n-1}{i}S_1(i,m) \\
&\qquad \times\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\right| x^{n-1-i}\Bigg\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{m!}{n}\sum_{i=m}^{n}\binom{n}{i}S_1(i,m)\Bigg(\sum_{j=1}^{r}a_j\left\langle \frac{(1+t)^{a_j}\ln(1+t)}{(1+t)^{a_j}-1}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\right. \\
&\qquad \left.\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\right|\frac{t}{\ln(1+t)}x^{n-i}\Bigg\rangle \\
&\quad - r\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{(1+t)^{a_i}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\right|\frac{t}{\ln(1+t)}x^{n-i}\Bigg\rangle\Bigg) \\
&\quad - m!\sum_{j=1}^{r}a_j\sum_{i=m}^{n-1}\binom{n-1}{i}S_1(i,m)\widehat{N}_{n-1-i}^{(k)}(-1|a_1,\dots,a_r) \\
&= \frac{m!}{n}\sum_{i=m}^{n}\binom{n}{i}S_1(i,m)\sum_{l=0}^{n-i}\binom{n-i}{l}c_l \\
&\qquad \times\left(\sum_{j=1}^{r}a_j\widehat{N}_{n-i-l}^{(k)}(-1|a_1,\dots,a_{j-1},a_{j+1},\dots,a_r)-r\widehat{N}_{n-i-l}^{(k)}(-1|a_1,\dots,a_r)\right) \\
&\quad - m!\sum_{j=1}^{r}a_j\sum_{i=m}^{n-1}\binom{n-1}{i}S_1(i,m)\widehat{N}_{n-1-i}^{(k)}(-1|a_1,\dots,a_r)
\end{aligned}
$$

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$$
\begin{aligned}
&= \frac{m!}{n}\sum_{i=m}^{n}\sum_{l=0}^{n-i}\binom{n}{i}\binom{n-i}{l}S_1(i,m)c_{n-i-l} \\
&\quad\times\left(\sum_{j=1}^{r}a_j\widehat{N}_l^{(k)}(-1|a_1,\dots,a_{j-1},a_{j+1},\dots,a_r)-r\widehat{N}_l^{(k)}(-1|a_1,\dots,a_r)\right) \\
&\quad - m!\sum_{j=1}^{r}a_j\sum_{i=0}^{n-m-1}\binom{n-1}{i}S_1(n-1-i,m)\widehat{N}_i^{(k)}(-1|a_1,\dots,a_r) \\
&= \frac{m!}{n}\sum_{i=0}^{n-m}\sum_{l=0}^{i}\binom{n}{i}\binom{i}{l}S_1(n-i,m)c_{i-l} \\
&\quad\times\left(\sum_{j=1}^{r}a_j\widehat{N}_l^{(k)}(-1|a_1,\dots,a_{j-1},a_{j+1},\dots,a_r)-r\widehat{N}_l^{(k)}(-1|a_1,\dots,a_r)\right) \\
&\quad - m!\sum_{j=1}^{r}a_j\sum_{i=0}^{n-m-1}\binom{n-1}{i}S_1(n-1-i,m)\widehat{N}_i^{(k)}(-1|a_1,\dots,a_r)\,.
\end{aligned}
$$

Therefore, we get for $n-1\geq m\geq 1$

$$
\begin{aligned}
&m!\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\widehat{N}_l^{(k)}(a_1,\dots,a_r) \\
&= \frac{m!}{n}\sum_{i=0}^{n-m}\sum_{l=0}^{i}\binom{n}{i}\binom{i}{l}S_1(n-i,m)c_{i-l} \\
&\quad\times\left(\sum_{j=1}^{r}a_j\widehat{N}_l^{(k)}(-1|a_1,\dots,a_{j-1},a_{j+1},\dots,a_r)-r\widehat{N}_l^{(k)}(-1|a_1,\dots,a_r)\right) \\
&\quad - m!\sum_{j=1}^{r}a_j\sum_{i=0}^{n-m-1}\binom{n-1}{i}S_1(n-1-i,m)\widehat{N}_i^{(k)}(-1|a_1,\dots,a_r) \\
&\quad + (m-1)!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-1-l,m-1)\widehat{N}_l^{(k-1)}(-1|a_1,\dots,a_r) \\
&\quad - (m-1)!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-1-l,m-1)\widehat{N}_l^{(k)}(-1|a_1,\dots,a_r) \\
&\quad + m!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-1-l,m-1)\widehat{N}_l^{(k)}(-1|a_1,\dots,a_r)\,.
\end{aligned}
$$

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Dividing both sides by $(m-1)!$, we obtain, for $n-1 \geq m \geq 1$,

$$
\begin{aligned}
& m \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m) \widehat{N}_l^{(k)}(a_1,\dots,a_r) \\
& = \frac{m}{n} \sum_{i=0}^{n-m} \sum_{l=0}^{i} \binom{n}{i} \binom{i}{l} S_1(n-i,m) c_{i-l} \\
& \quad \times \left(\sum_{j=1}^{r} a_j \widehat{N}_l^{(k)}(-1|a_1,\dots,a_{j-1},a_{j+1},\dots,a_r) - r \widehat{N}_l^{(k)}(-1|a_1,\dots,a_r) \right) \\
& \quad - m \sum_{j=1}^{r} a_j \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-1-l,m) \widehat{N}_l^{(k)}(-1|a_1,\dots,a_r) \\
& \quad + \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-1-l,m-1) \\
& \quad \times \left(\widehat{N}_l^{(k-1)}(-1|a_1,\dots,a_r) + (m-1) \widehat{N}_l^{(k)}(-1|a_1,\dots,a_r) \right) .
\end{aligned}
$$

Thus, we get (27). ∎

3.8 A relation with the falling factorials

Theorem 8

$$
\widehat{N}_n^{(k)}(x|a_1,\dots,a_r) = \sum_{m=0}^{n} \binom{n}{m} \widehat{N}_{n-m}^{(k)}(a_1,\dots,a_r)(x)_m . \tag{29}
$$

Proof. For (12) and (18), assume that $\widehat{N}_n^{(k)}(x|a_1,\dots,a_r) = \sum_{m=0}^{n} C_{n,m}(x)_m$. By (11), we have

$$
\begin{aligned}
C_{n,m} & = \frac{1}{m!} \left\langle \frac{1}{\prod_{j=1}^{r} \left(\frac{\ln(1+t) e^{a_j \ln(1+t)}}{e^{a_j \ln(1+t)} - 1} \right)} \mathrm{Lif}_k\big(-\ln(1+t)\big) t^m \middle| x^n \right\rangle \\
& = \frac{1}{m!} \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j} - 1}{(1+t)^{a_j} \ln(1+t)} \right) \mathrm{Lif}_k\big(-\ln(1+t)\big) \middle| t^m x^n \right\rangle \\
& = \binom{n}{m} \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j} - 1}{(1+t)^{a_j} \ln(1+t)} \right) \mathrm{Lif}_k\big(-\ln(1+t)\big) \middle| x^{n-m} \right\rangle \\
& = \binom{n}{m} \widehat{N}_{n-m}^{(k)}(a_1,\dots,a_r) .
\end{aligned}
$$

Thus, we get the identity (29). ∎

3.9 A relation with higher-order Frobenius-Euler polynomials

For $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, the Frobenius-Euler polynomials of order r, $H_n^{(r)}(x|\lambda)$ are defined by the generating function

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda)\frac{t^n}{n!}$$

(see e.g. [6]).

Theorem 9

$$\widehat{N}_n^{(k)}(x|a_1,\dots,a_r) = \sum_{m=0}^{n}\left(\sum_{j=0}^{n-m}\sum_{l=0}^{n-m-j}\binom{s}{j}\binom{n-j}{l}(n)_j \times(1-\lambda)^{-j}S_1(n-j-l,m)\widehat{N}_l^{(k)}(a_1,\dots,a_r)\right)H_m^{(s)}(x|\lambda)\,. \tag{30}$$

Proof. For (12) and

$$H_n^{(s)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^s, t\right)\,, \tag{31}$$

assume that $\widehat{N}_n^{(k)}(x|a_1,\dots,a_r) = \sum_{m=0}^{n} C_{n,m}H_m^{(s)}(x|\lambda)$. By (11), similarly to the proof of (27), we have

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \frac{\left(\frac{e^{\ln(1+t)}-\lambda}{1-\lambda}\right)^s}{\prod_{j=1}^r\left(\frac{\ln(1+t)e^{a_j\ln(1+t)}}{e^{a_j\ln(1+t)}-1}\right)}\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\Big|x^n\right\rangle \\
&= \frac{1}{m!(1-\lambda)^s}\left\langle \prod_{j=1}^r\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m(1-\lambda+t)^s\Big|x^n\right\rangle \\
&= \frac{1}{m!(1-\lambda)^s} \\
&\quad\times\left\langle \prod_{j=1}^r\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\Big|\sum_{i=0}^{\min\{s,n\}}\binom{s}{i}(1-\lambda)^{s-i}t^i x^n\right\rangle \\
&= \frac{1}{m!(1-\lambda)^s}\sum_{i=0}^{n-m}\binom{s}{i}(1-\lambda)^{s-i}(n)_i \\
&\quad\times\left\langle \prod_{j=1}^r\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\Big|x^{n-i}\right\rangle \\
&= \frac{1}{m!(1-\lambda)^s}\sum_{i=0}^{n-m}\binom{s}{i}(1-\lambda)^{s-i}(n)_i\sum_{l=0}^{n-m-i}m!\binom{n-i}{l}S_1(n-i-l,m)\widehat{N}_l^{(k)}(a_1,\dots,a_r) \\
&= \sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{s}{i}\binom{n-i}{l}(n)_i(1-\lambda)^{-i}S_1(n-i-l,m)\widehat{N}_l^{(k)}(a_1,\dots,a_r)\,.
\end{aligned}$$

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Thus, we get the identity (30). ∎

3.10 A relation with higher-order Bernoulli polynomials

Bernoulli polynomials $\mathfrak{B}_n^{(r)}(x)$ of order r are defined by

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^\infty\frac{\mathfrak{B}_n^{(r)}(x)}{n!}t^n$$

(see e.g. [14, Section 2.2]). In addition, Cauchy numbers of the first kind $\mathfrak{C}_n^{(r)}$ of order r are defined by

$$\left(\frac{t}{\ln(1+t)}\right)^r=\sum_{n=0}^\infty\frac{\mathfrak{C}_n^{(r)}}{n!}t^n$$

(see e.g. [2, (2.1)], [13, (6)]).

Theorem 10

$$\begin{aligned}&\widehat{N}_n^{(k)}(x|a_1,\dots,a_r)\\&=\sum_{m=0}^n\left(\sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{n}{i}\binom{n-i}{l}\mathfrak{C}_i^{(s)}S_1(n-i-l,m)\widehat{N}_l^{(k)}(a_1,\dots,a_r)\right)\mathfrak{B}_m^{(s)}(x)\,.\end{aligned}\tag{32}$$

Proof. For (12) and

$$\mathfrak{B}_n^{(s)}(x)\sim\left(\left(\frac{e^t-1}{t}\right)^s,t\right),\tag{33}$$

assume that $\widehat{N}_n^{(k)}(x|a_1,\dots,a_r)=\sum_{m=0}^n C_{n,m}\mathfrak{B}_m^{(s)}(x)$. By (11), similarly to the proof of

(27), we have

$$
\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \frac{\left(\frac{e^{\ln(1+t)}-1}{\ln(1+t)}\right)^s}{\prod_{j=1}^r\left(\frac{\ln(1+t)e^{a_j\ln(1+t)}}{e^{a_j\ln(1+t)}-1}\right)}\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\middle| x^n\right\rangle \\
&= \frac{1}{m!}\left\langle \prod_{j=1}^r\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\middle| \left(\frac{t}{\ln(1+t)}\right)^s x^n\right\rangle \\
&= \frac{1}{m!}\left\langle \prod_{j=1}^r\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\middle| \sum_{i=0}^\infty \mathfrak{C}_i^{(s)}\frac{t^i}{i!}x^n\right\rangle \\
&= \frac{1}{m!}\sum_{i=0}^{n-m}\mathfrak{C}_i^{(s)}\binom{n}{i}\left\langle \prod_{j=1}^r\left(\frac{(1+t)^{a_j}-1}{(1+t)^{a_j}\ln(1+t)}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\middle| x^{n-i}\right\rangle \\
&= \frac{1}{m!}\sum_{i=0}^{n-m}\mathfrak{C}_i^{(s)}\binom{n}{i}\sum_{l=0}^{n-m-i}m!\binom{n-i}{l}S_1(n-i-l,m)\widehat{N}_l^{(k)}(a_1,\dots,a_r) \\
&= \sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{n}{i}\binom{n-i}{l}\mathfrak{C}_i^{(s)}S_1(n-i-l,m)\widehat{N}_l^{(k)}(a_1,\dots,a_r)\,.
\end{aligned}
$$

Thus, we get the identity (32). ∎

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SUPERSTABILITY AND STABILITY OF (r, s, t)-J^*-HOMOMORPHISMS: FIXED POINT AND DIRECT METHODS

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Abstract. In this paper, we introduce the following useful functional equations:

$$f(x+y)+f(x-2y)+f(y-x)=f(x), \tag{0.1}$$

$$f\left(\frac{\sum_{i=1}^{p} x_i}{p-1}\right)+\sum_{i=2}^{p} f\left(\frac{\sum_{\substack{j=1\\ j\neq i}}^{p} x_j - p x_i}{p-1}\right)+f\left(\frac{\sum_{i=2}^{p} x_i - x_1}{p-1}\right)=f(x_1) \tag{0.2}$$

and prove the superstability and the Hyers-Ulam stability of (r, s, t)-J^*-homomorphisms, associated with those, by using the fixed point method and the direct method.

1. Introduction and preliminaries

The stability of functional equations originated from a question of Ulam [54] in 1940. He proposed the following question "when and under what condition does an exact solution of a functional equation near an approximately solution of that exist?" A next year, this question was formulated and answered by Hyers [26] affirmatively, for Cauchy's additive equation on Banach spaces. In 1950, Aoki [1] was the second author to study this problem. In 1978, Rassias [49] obtained a generalization of the result of Hyers by considering the stability problem with unbounded Cauchy differences. For more epochal information and various aspects about the stability of functional equations theory, we refer the reader to monographs (cf. [2, 7, 9, 10, 12, 14, 21, 27, 29, 30, 31, 32, 33, 34, 35, 39, 41, 42, 46, 50, 51, 52, 53]), which also include many interesting results concerning the stability of different functional equations.

We say a functional equation (ξ) is *stable* if any function g satisfying the equation (ξ) *approximately* is near to true solution of (ξ). We say that a functional equation is *superstable* if every approximately solution is an exact solution of that [51].

Throughout this paper, $\mathcal{A}$ and $\mathcal{B}$ denote J^*-algebras and $\{r, s, t\}$ are positive integer constants. The notion of J^*-algebras has been posed by Harris [22] in 1974. By a J^*-algebra we mean a closed subspace $\mathcal{A}$ of a C^*-algebra such that $xx^*x \in \mathcal{A}$ whenever $x \in \mathcal{A}$ [22]. For more study about J^*-algebras, one can refer to (cf. [11, 22, 23, 24, 25]). Moreover, we introduce (r, s, t)-J^*-homomorphisms and (r, s, t)-J^*-derivations, which are an extension of J^*-derivations and J^*-homomorphisms (see [19, 44, 45]).

Definition 1.1. A linear mapping $h : \mathcal{A} \to \mathcal{B}$ is called an (r, s, t)-J^*-*homomorphism* if

$$h(x^r x^{*s} x^t) = h(x)^r h(x)^{*s} h(x)^t$$

for all $x \in \mathcal{A}$, and if $r = s = t = n$, then $h : \mathcal{A} \to \mathcal{B}$ is called an n-J^*-*homomorphism*.

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Definition 1.2. A linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called an (r, s, t)-J^*-*derivation* if

$$\delta(x^r x^{*s} x^t) = \delta(x)^r x^{*s} x^t + x^r \delta(x)^{*s} x^t + x^r x^{*s} \delta(x)^t$$

for all $x \in \mathcal{A}$, and if $r = s = t = n$, then $\delta : \mathcal{A} \to \mathcal{A}$ is called an n-J^*-*derivation.*

With $n = 1$, we have the definitions of J^*-homomorphisms and J^*-derivations.

We will use the following definition and fundamental result of fixed point theory:

Definition 1.3. ([3, 4, 5, 6]) Let $\mathcal{X}$ be a set. A function $d : \mathcal{X} \times \mathcal{X} \to [0, \infty]$ is called a *generalized metric* on $\mathcal{X}$ if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in \mathcal{X}$.

Theorem 1.4. ([3, 4, 5, 6]) *Let $(\mathcal{X}, d)$ be a complete generalized metric space and let $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ be a strictly contractive mapping with Lipschitz constant $\mathcal{L} < 1$. Then for each given element $x \in \mathcal{X}$, either*

$$d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence $\{\mathcal{J}^n x\}$ converges to a fixed point y^* of $\mathcal{J}$;*
(3) *y^* is the unique fixed point of $\mathcal{J}$ in the set $\mathcal{Y} = \{y \in \mathcal{X} \mid d(\mathcal{J}^{n_0} x, y) < \infty\}$;*
(4) $d(y, y^*) \leq \frac{1}{1-\mathcal{L}} d(y, \mathcal{J} y)$ *for all $y \in \mathcal{Y}$.*

This theorem was used by Cădariu and Radu (see [3, 4, 5, 47]) and then others to obtain the applications of fixed point theory in stability problems (cf. [8, 13, 15, 16, 17, 18, 19, 20, 28, 36, 38, 39, 40, 43, 48]).

Now consider the functional equation (0.2), which is a generalized version of the functional equation (0.1). In this paper, in order to investigate the functional equation (0.2), we will suppose that $p \geq 3$.

2. Superstability of (r, s, t)-J^*-homomorphisms

In this section, we prove the superstability of (r, s, t)-J^*-homomorphisms associated with the functional equation (0.2).

For the proof of our results, we first give some useful lemmas.

Lemma 2.1. ([37]) *Let $\mathcal{X}$ and $\mathcal{Y}$ be linear spaces and let $f : \mathcal{X} \to \mathcal{Y}$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x \in \mathcal{X}$. Then the mapping f is $\mathbb{C}$-linear.*

Lemma 2.2. *Let $n \geq 2$ be a fixed positive integer and $f : \mathcal{A} \to \mathcal{B}$ be a mapping such that*

$$\begin{aligned} &\left\| (n-1) f\left(\frac{x+y+z}{n}\right) + f\left(\frac{x+z-(n+1)y}{n}\right) + f\left(\frac{x+y-(n+1)z}{n}\right) \right\|_{\mathcal{B}} \\ &\leq \left\| f(x) - f\left(\frac{y+z-x}{n}\right) \right\|_{\mathcal{B}} \end{aligned} \tag{2.1}$$

for all $x, y, z \in \mathcal{A}$. Then f is additive.

Proof. From (2.1), it follows that $f(0) = 0$. Putting $x = 0,\ y = x,\ z = -x$ in (2.1), we have $f\left(-\frac{n+2}{n} x\right) + f\left(\frac{n+2}{n} x\right) = 0$ for all $x \in \mathcal{A}$. So $f(-x) = -f(x)$ for all $x \in \mathcal{A}$. Replacing x, y and z by $\frac{x+y}{n+1},\ x$ and y in (2.1), respectively, we get the equality

$$(n-1) f\left(\frac{n+2}{n(n+1)}(x+y)\right) = f\left(\frac{n+2}{n+1} x - \frac{n+2}{n(n+1)} y\right) + f\left(\frac{n+2}{n+1} y - \frac{n+2}{n(n+1)} x\right)$$

for all $x,y\in\mathcal{A}$. By putting $u=\frac{n+2}{n+1}x-\frac{n+2}{n(n+1)}y$ and $v=\frac{n+2}{n+1}y-\frac{n+2}{n(n+1)}x$, we conclude that

$$(n-1)f\left(\frac{1}{n-1}(u+v)\right)=f(u)+f(v)$$

for all $u,v\in\mathcal{A}$. Letting $v=0$, we see that $(n-1)f\left(\frac{1}{n-1}u\right)=f(u)$ and so $f(u+v)=f(u)+f(v)$ for all $u,v\in\mathcal{A}$. □

Lemma 2.3. *Let $p\geq 3$ be a fixed positive integer and $f:\mathcal{A}\to\mathcal{B}$ be a mapping such that*

$$\left\|f\left(\frac{\sum_{i=1}^{p}x_i}{p-1}\right)+\sum_{i=2}^{p}f\left(\frac{\sum_{j=1, j\neq i}^{p}x_j-px_i}{p-1}\right)\right\|_{\mathcal{B}}\leq\left\|f(x_1)-f\left(\frac{\sum_{i=2}^{p}x_i-x_1}{p-1}\right)\right\|_{\mathcal{B}}\tag{2.2}$$

for all $x_1,\cdots,x_p\in\mathcal{A}$. Then f is additive.

Proof. By (2.2), we have $f(0)=0$. Letting $x_1=x,\ x_2=y,\ x_3=z$ and $x_4=\cdots=x_p=0$ in (2.2), we obtain

$$\left\|(p-2)f\left(\frac{x+y+z}{p-1}\right)+f\left(\frac{x+z-py}{p-1}\right)+f\left(\frac{x+y-pz}{p-1}\right)\right\|_{\mathcal{B}}\leq\left\|f(x)-f\left(\frac{y+z-x}{p-1}\right)\right\|_{\mathcal{B}}$$

for all $x,y,z\in\mathcal{A}$, which is (2.1) for the case $n=p-1\geq 2$. Therefore f is additive, as desired. □

Theorem 2.4. *Let $p\geq 3$ be a fixed positive integer and $\varphi:\mathcal{A}^p\to[0,\infty)$ be a function such that*

$$\lim_{n\to\infty}b^{(r+s+t)n}\varphi(b^{-n}x,\cdots,b^{-n}x)=0$$

for all $x\in\mathcal{A}$, where $b\neq 1$ is a real number. Let $f:\mathcal{A}\to\mathcal{B}$ be a mapping satisfying

$$\left\|\mu f\left(\frac{\sum_{i=1}^{p}x_i}{p-1}\right)+\sum_{i=2}^{p}f\left(\mu\frac{\sum_{j=1, j\neq i}^{p}x_j-px_i}{p-1}\right)\right\|_{\mathcal{B}}\leq\left\|f(\mu x_1)-f\left(\mu\frac{\sum_{i=2}^{p}x_i-x_1}{p-1}\right)\right\|_{\mathcal{B}}\tag{2.3}$$

$$\|f(x^rx^{*s}x^t)-f(x)^rf(x)^{*s}f(x)^t\|_{\mathcal{B}}\leq\varphi(x,\cdots,x)\tag{2.4}$$

for all $\mu\in\mathbb{T}^1$ and all $x,x_1,\cdots,x_p\in\mathcal{A}$. Then the mapping $f:\mathcal{A}\to\mathcal{B}$ is an (r,s,t)-J^*-homomorphism.

Proof. Let $\mu=1$ in (2.3). By Lemma 2.3, the mapping $f:\mathcal{A}\to\mathcal{B}$ is additive. From (2.3), for $x_1=x_3=\cdots=x_p=x$ and $x_2=2x$, we have

$$\frac{p+1}{p-1}\|\mu f(x)-f(\mu x)\|_{\mathcal{B}}=\left\|\mu f\left(\frac{p+1}{p-1}x\right)+f\left(-\mu\frac{p+1}{p-1}\right)\right\|_{\mathcal{B}}\leq\|0\|_{\mathcal{B}}=0$$

for all $\mu\in\mathbb{T}^1$ and all $x\in\mathcal{A}$. Hence $f(\mu x)=\mu f(x)$ for all $\mu\in\mathbb{T}^1$ and all $x\in\mathcal{A}$. By Lemma 2.1, the mapping $f:\mathcal{A}\to\mathcal{B}$ is $\mathbb{C}$-linear. From (2.4) and the assumption on φ, it follows that

$$\begin{aligned}&\|f(x^rx^{*s}x^t)-f(x)^rf(x)^{*s}f(x)^t\|_{\mathcal{B}}\\&=\lim_{n\to\infty}b^{(r+s+t)n}\left\|f\left(\left(\frac{x}{b^n}\right)^r\left(\frac{x}{b^n}\right)^{*s}\left(\frac{x}{b^n}\right)^t\right)-f\left(\frac{x}{b^n}\right)^rf\left(\frac{x}{b^n}\right)^{*s}f\left(\frac{x}{b^n}\right)^t\right\|_{\mathcal{B}}\\&\leq\lim_{n\to\infty}b^{(r+s+t)n}\varphi\left(\frac{x}{b^n},\cdots,\frac{x}{b^n}\right)=0\end{aligned}$$

for all $\mu\in\mathbb{T}^1$ and all $x\in\mathcal{A}$. Hence $f(x^rx^{*s}x^t)=f(x)^rf(x)^{*s}f(x)^t$ for all $x\in\mathcal{A}$. □

Corollary 2.5. *Let θ be a nonnegative real number and $q_1, \cdots, q_p$ be positive real numbers such that $q_1, \cdots, q_p > r+s+t$ or $q_1, \cdots, q_p < r+s+t$. Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying* (2.3) *and*

$$\|f(x^r x^{*s} x^t) - f(x)^r f(x)^{*s} f(x)^t\|_{\mathcal{B}} \leq \theta \left(\|x\|_{\mathcal{A}}^{q_1} + \cdots + \|x\|_{\mathcal{A}}^{q_p}\right) \tag{2.5}$$

for all $x \in \mathcal{A}$. Then the mapping $f : \mathcal{A} \to \mathcal{B}$ is an (r,s,t)-J^*-homomorphism.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x_1, \cdots, x_p) := \theta\left(\|x_1\|_{\mathcal{A}}^{q_1} + \cdots + \|x_p\|_{\mathcal{A}}^{q_p}\right)$ with $b > 1$ for the case $q_1, \cdots, q_p > r+s+t$ and with $b < 1$ for the case $q_1, \cdots, q_p < r+s+t$. □

Corollary 2.6. *Let θ be a nonnegative real number and $q_1, \cdots, q_p$ be positive real numbers such that $q_1 + \cdots + q_p \neq r+s+t$. Let $f : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying* (2.3) *and*

$$\|f(x^r x^{*s} x^t) - f(x)^r f(x)^{*s} f(x)^t\|_{\mathcal{B}} \leq \theta \|x\|_{\mathcal{A}}^{q_1 + \cdots + q_p}$$

for all $x \in \mathcal{A}$. Then the mapping $f : \mathcal{A} \to \mathcal{B}$ is an (r,s,t)-J^*-homomorphism.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x_1, \cdots, x_p) := \theta\left(\|x_1\|_{\mathcal{A}}^{q_1} \cdots \|x_p\|_{\mathcal{A}}^{q_p}\right)$ with $b > 1$ for the case $q_1 + \cdots + q_p > r+s+t$ and with $b < 1$ for the case $q_1 + \cdots + q_p < r+s+t$. □

3. Hyers-Ulam stability of (r,s,t)-J^*-homomorphisms: fixed point method

In this section, by using the fixed point method, we prove the Hyers-Ulam stability of (r,s,t)-J^*-homomorphisms associated with the functional equation (0.2).

For a given mapping $f : \mathcal{A} \to \mathcal{B}$, we define

$$\varrho_\mu f(x_1, \cdots, x_p) := f\left(\mu \frac{\sum_{i=1}^p x_i}{p-1}\right) + \sum_{i=2}^p f\left(\mu \frac{\sum_{\substack{j=1 \\ j \neq i}}^p x_j - p x_i}{p-1}\right) + f\left(\mu \frac{\sum_{i=2}^p x_i - x_1}{p-1}\right) - \mu f(x_1)$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \cdots, x_p \in \mathcal{A}$.

Lemma 3.1. *The mapping $f : \mathcal{A} \to \mathcal{B}$ is a $\mathbb{C}$-linear mapping if and only if*

$$\varrho_\mu f(x_1, \cdots, x_p) = 0$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \cdots, x_p \in \mathcal{A}$.

Proof. The proof is easy and thus omitted. □

In the following theorems, we will except the case $p = 3$. This case will be considered individually.

Theorem 3.2. *Let $\varphi : \mathcal{A}^p \to [0, \infty)$ be a function with $\varphi(0, \cdots, 0) = 0$ and $p \neq 3$ such that there exists an $\mathcal{L} < 1$ with*

$$\varphi(x_1, \cdots, x_p) < \frac{\mathcal{L}}{k} \varphi(k x_1, \cdots, k x_p) \tag{3.1}$$

for all $x_1, \cdots, x_p \in \mathcal{A}$, where $k = \frac{2}{p-1}$. Suppose that $f : \mathcal{A} \to \mathcal{B}$ is an odd mapping satisfying (2.4) *and*

$$\|\varrho_\mu f(x_1, \cdots, x_p)\|_{\mathcal{B}} \leq \varphi(x_1, \cdots, x_p) \tag{3.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \cdots, x_p \in \mathcal{A}$. Then there exists a unique (r,s,t)-J^*-homomorphism $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - \mathcal{H}(x)\|_{\mathcal{B}} \leq \frac{\mathcal{L}}{2(1-\mathcal{L})} \varphi(0, x, \cdots, x) \tag{3.3}$$

for all $x \in \mathcal{A}$.

Proof. We first consider the set $\mathcal{S} := \{g : \mathcal{A} \to \mathcal{B}\}$ and introduce the generalized metric d as follows:

$$d(g,h) = \inf_{x\in\mathcal{A}} \left\{\mathcal{C} \in \mathbb{R}^+ : \quad \|g(x) - h(x)\|_{\mathcal{B}} \leq \mathcal{C}\varphi\left(0, x, \cdots, x\right)\right\}.$$

It is easy to show that $(\mathcal{S}, d)$ is complete (see the proof of [35, Lemma 2.1]). Now we define the linear mapping $\mathcal{J} : \mathcal{S} \to \mathcal{S}$ such that

$$\mathcal{J}\left(g(x)\right) := kg\left(\frac{x}{k}\right)$$

for all $x \in \mathcal{A}$. From (3.2), we can get $f(0) = 0$. By letting $\mu = 1,\ x_1 = 0$ and $x_2 = \cdots = x_p = x$ in (3.2) and the fact that $f(-x) = -f(x)$, (f is an odd mapping) and then by (3.1), we have

$$\begin{aligned} \left\|2f(x) + (p-1)f\left(\frac{-2}{p-1}x\right)\right\|_{\mathcal{B}} &\leq \varphi(0, x, \cdots, x), \\ \left\|kf\left(\frac{x}{k}\right) - f(x)\right\|_{\mathcal{B}} &\leq \frac{k}{2}\varphi\left(0, \frac{x}{k}, \cdots, \frac{x}{k}\right) \leq \frac{\mathcal{L}}{2}\varphi\left(0, x, \cdots, x\right) \end{aligned}$$

for all $x \in \mathcal{A}$. This means that

$$d(\mathcal{J}(f), f) \leq \frac{\mathcal{L}}{2} \tag{3.4}$$

Assume that $g, h \in \mathcal{S}$ are given with $d(g,h) = \varepsilon$. Then we have

$$\begin{aligned} \|\mathcal{J}(g(x)) - \mathcal{J}(h(x))\|_{\mathcal{B}} &= k\left\|g\left(\frac{x}{k}\right) - h\left(\frac{x}{k}\right)\right\|_{\mathcal{B}} \leq k\varepsilon\varphi\left(0, \frac{x}{k}, \cdots, \frac{x}{k}\right) \\ &< \mathcal{L}\varepsilon\varphi\left(0, x, \cdots, x\right) \end{aligned}$$

for all $x \in \mathcal{A}$. This implies that $d\left(\mathcal{J}(g), \mathcal{J}(h)\right) < \mathcal{L}\varepsilon = \mathcal{L}d(g,h)$, which means $\mathcal{J}$ is a strictly contractive mapping.

By Theorem 1.4, we have the following:

(1) $\mathcal{J}$ has a fixed point, i.e., there exists a mapping $\mathcal{H} : \mathcal{A} \to \mathcal{B}$, such that $\mathcal{J}(\mathcal{H}) = \mathcal{H}$. So

$$\mathcal{H}(x) = k\mathcal{H}\left(\frac{x}{k}\right) \tag{3.5}$$

for all $x \in \mathcal{A}$. The mapping $\mathcal{H}$ is also the unique fixed point of $\mathcal{J}$ in the set

$$\mathcal{M} = \{g \in \mathcal{S} : d(f,g) < \infty\}.$$

This signifies that $\mathcal{H}$ is a unique mapping satisfying (3.5), moreover there exists a $\mathcal{C} \in (0,\infty)$ such that

$$\|f(x) - \mathcal{H}(x)\|_{\mathcal{B}} \leq \mathcal{C}\varphi\left(0, x, \cdots, x\right)$$

for all $x \in \mathcal{A}$;

(2) The sequence $\{\mathcal{J}^n(g)\}$ converges to $\mathcal{H}$, for each given $g \in \mathcal{S}$. Thus $d\left(\mathcal{J}^n(f), \mathcal{H}\right) \to 0$, as $n \to \infty$. This implies the equality

$$\mathcal{H}(x) = \lim_{n\to\infty} k^n f\left(\frac{x}{k^n}\right)$$

for all $x \in \mathcal{A}$;

(3) $d(g,\mathcal{H}) \leq \frac{1}{1-\mathcal{L}}d\left(g, \mathcal{J}(g)\right)$, for all $g \in \mathcal{M}$. Therefore (3.4) shows us that

$$d(f,\mathcal{H}) \leq \frac{1}{1-\mathcal{L}}d\left(f, \mathcal{J}(f)\right) \leq \frac{\mathcal{L}}{2(1-\mathcal{L})}.$$

By this, we get the inequality (3.3).

It follows from (3.1) that

$$\begin{aligned}\|\varrho_\mu h(x_1,\cdots,x_p)\|_{\mathcal{B}} &= k^n \left\|\varrho_\mu f\left(\frac{x_1}{k^n},\cdots,\frac{x_p}{k^n}\right)\right\|_{\mathcal{B}} \le k^n\varphi\left(\frac{x_1}{k^n},\cdots,\frac{x_p}{k^n}\right)\\ &< \mathcal{L}^n\varphi\left(x_1,\cdots,x_p\right)\end{aligned}$$

for all $x_1,\cdots,x_p \in \mathcal{A}$, in which the right-hand side tends to zero as $n\to\infty$. Hence by Lemma 3.1, we deduce that $\mathcal{H}$ is $\mathbb{C}$-linear.

By (3.1) and (2.4), we obtain

$$\begin{aligned}&\left\|h(x^r x^{*s}x^t) - h(x)^r h(x)^{*s}h(x)^t\right\|_{\mathcal{B}}\\ &= k^{(r+s+t)n}\left\|f\left(\left(\frac{x}{k^n}\right)^r\left(\frac{x}{k^n}\right)^{*s}\left(\frac{x}{k^n}\right)^t\right) - f\left(\frac{x}{k^n}\right)^r f\left(\frac{x}{k^n}\right)^{*s} f\left(\frac{x}{k^n}\right)^t\right\|_{\mathcal{B}}\\ &\le k^{(r+s+t)n}\varphi\left(\frac{x}{k^n},\cdots,\frac{x}{k^n}\right) < \mathcal{L}^{(r+s+t)n}\varphi\left(x,\cdots,x\right)\end{aligned}$$

for all $x\in\mathcal{A}$. The right-hand side tends to zero as $n\to\infty$, and so the mapping $\mathcal{H}:\mathcal{A}\to\mathcal{B}$ is an (r,s,t)-J^*-homomorphism, as desired. □

Theorem 3.3. *Let $\varphi:\mathcal{A}^p\to[0,\infty)$ be a function with $\varphi(0,\cdots,0)=0$ and $p\neq 3$ such that there exists an $\mathcal{L}<1$ with*

$$\varphi(x_1,\cdots,x_p) < k\mathcal{L}\varphi\left(\frac{x_1}{k},\cdots,\frac{x_p}{k}\right) \tag{3.6}$$

for all $x_1,\cdots,x_p\in\mathcal{A}$, where $k=\frac{2}{p-1}$. Suppose that $f:\mathcal{A}\to\mathcal{B}$ is an odd mapping satisfying (3.2) *and* (2.4). *Then there exists a unique (r,s,t)-J^*-homomorphism $\mathcal{H}:\mathcal{A}\to\mathcal{B}$ such that*

$$\|f(x)-\mathcal{H}(x)\|_{\mathcal{B}} \le \frac{\mathcal{L}}{(1-\mathcal{L})(p-1)}\varphi\left(0,\frac{x}{k},\cdots,\frac{x}{k}\right) \tag{3.7}$$

for all $x\in\mathcal{A}$.

Proof. Let $\mathcal{S}$ be the defined set in the proof of Theorem 3.2. Consider the following generalized metric d:

$$d(g,h) = \inf_{x\in\mathcal{A}}\left\{\mathcal{C}\in\mathbb{R}^+ : \quad \|g(x)-h(x)\|_{\mathcal{B}} \le \mathcal{C}\varphi\left(0,\frac{x}{k},\cdots,\frac{x}{k}\right)\right\}.$$

It is easy to show that $(\mathcal{S},d)$ is complete (see the proof of [35, Lemma 2.1]). we define the linear mapping $\mathcal{J}:\mathcal{S}\to\mathcal{S}$ such that

$$\mathcal{J}\left(g(x)\right) := \frac{1}{k}g(kx)$$

for all $x\in\mathcal{A}$. By the same argument as in the proof of Theorem 3.2, we can obtain the mapping $\mathcal{H}:\mathcal{A}\to\mathcal{B}$, as the unique fixed point of $\mathcal{J}$ such that

$$\mathcal{H}(x) := \lim_{n\to\infty}\frac{1}{k^n}f(k^n x)$$

for all $x\in\mathcal{A}$. By (3.2) and (3.6), we have

$$\left\|f(x)-\frac{1}{k}f(kx)\right\|_{\mathcal{B}} \le \frac{1}{2}\varphi\left(0,x,\cdots,x\right) \le \frac{\mathcal{L}}{(p-1)}\varphi\left(0,\frac{x}{k},\cdots,\frac{x}{k}\right)$$

for all $x\in\mathcal{A}$. This means that $d(f,\mathcal{J}(f)) \le \frac{\mathcal{L}}{(p-1)}$. Hence

$$d(f,\mathcal{H}) \le \frac{1}{1-\mathcal{L}}d\left(f,\mathcal{J}(f)\right) \le \frac{\mathcal{L}}{(1-\mathcal{L})(p-1)}$$

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which implies that the inequality (3.7) holds.

The rest of the proof is similar to the proof of Theorem 3.2. □

Theorem 3.4. *Let $\varphi : \mathcal{A}^3 \to [0,\infty)$ be a function such that there exists an $\mathcal{L} < 1$ with*

$$\varphi(x,y,z) < \frac{\mathcal{L}}{2}\varphi(2x,2y,2z) \tag{3.8}$$

for all $x,y,z \in \mathcal{A}$. Suppose that $f : \mathcal{A} \to \mathcal{B}$ is an odd mapping satisfying

$$\left\| f\left(\mu\frac{x+y+z}{2}\right) + f\left(\mu\frac{x+z-3y}{2}\right) + f\left(\mu\frac{x+y-3z}{2}\right) \right.$$
$$\left. + f\left(\mu\frac{y+z-x}{2}\right) - \mu f(x) \right\|_{\mathcal{B}} \le \varphi(x,y,z), \tag{3.9}$$

$$\|f(x^r x^{*s} x^t) - f(x)^r f(x)^{*s} f(x)^t\|_{\mathcal{B}} \le \varphi(x,x,x) \tag{3.10}$$

for all $\mu \in \mathbb{T}^1$ and all $x,y,z \in \mathcal{A}$. Then there exists a unique (r,s,t)-J^*-homomorphism $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - \mathcal{H}(x)\|_{\mathcal{B}} \le \frac{\mathcal{L}}{2(1-\mathcal{L})}\varphi(2x,0,0) \tag{3.11}$$

for all $x \in \mathcal{A}$.

Proof. Consider the defined set $\mathcal{S}$ in the proof of Theorem 3.2 and the following generalized metric d:

$$d(g,h) = \inf_{x\in\mathcal{A}} \left\{\mathcal{C} \in \mathbb{R}^+ : \ \ \|g(x) - h(x)\|_{\mathcal{B}} \le \mathcal{C}\varphi(2x,0,0)\right\}.$$

Using the same method as in the proof of Theorem 3.2, we can get the mappings $\mathcal{J} : \mathcal{S} \to \mathcal{S}$ and $\mathcal{H} : \mathcal{A} \to \mathcal{B}$, with

$$\mathcal{J}(g(x)) := 2g\left(\frac{x}{2}\right), \qquad \mathcal{H}(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in \mathcal{A}$. By (3.8) and (3.9), we obtain

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\|_{\mathcal{B}} \le \varphi(x,0,0) < \frac{\mathcal{L}}{2}\varphi(2x,0,0)$$

for all $x \in \mathcal{A}$, which means $d(f,\mathcal{J}(f)) \le \frac{\mathcal{L}}{2}$. Hence $d(f,\mathcal{H}) \le \frac{\mathcal{L}}{2(1-\mathcal{L})}$. This implies that the inequality (3.11) holds. □

Theorem 3.5. *Let $\varphi : \mathcal{A}^3 \to [0,\infty)$ be a function such that there exists an $\mathcal{L} < 1$ with*

$$\varphi(x,y,z) < 2\mathcal{L}\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right)$$

for all $x,y,z \in \mathcal{A}$. Suppose that $f : \mathcal{A} \to \mathcal{B}$ is an odd mapping satisfying (3.9) and (3.10). Then there exists a unique (r,s,t)-J^*-homomorphism $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ such that

$$\|f(x) - \mathcal{H}(x)\|_{\mathcal{B}} \le \frac{\mathcal{L}}{(1-\mathcal{L})}\varphi(x,0,0)$$

for all $x \in \mathcal{A}$.

Proof. The proof is similar to the proof of Theorem 3.4. □

4. Hyers-Ulam stability of (r,s,t)-J^*-homomorphisms: direct method

In this section, by using the direct method, we prove the Hyers-Ulam stability of (r,s,t)-J^*-homomorphisms associated with the functional equation (0.2).

Theorem 4.1. *Let $\varphi : \mathcal{A}^p \to [0,\infty)$ be a function with $\varphi(0,\cdots,0)=0$ and $p \geq 4$. Denote by ϕ a function such that*

$$\phi(x_1,\cdots,x_p) := \sum_{n=0}^{\infty} k^{-(n+1)}\varphi\left(k^n x_1,\cdots,k^n x_p\right) < \infty, \tag{4.1}$$

$$\lim_{n\to\infty} k^{-(r+s+t)n}\varphi\left(k^n x,\cdots,k^n x\right) = 0 \tag{4.2}$$

for all $x, x_1,\cdots,x_p \in \mathcal{A}$, where $k = \frac{2}{p-1}$. Suppose that $f : \mathcal{A} \to \mathcal{B}$ is an odd mapping satisfying (3.2) *and* (2.4). *Then there exists a unique (r,s,t)-J^*-homomorphism $h : \mathcal{A} \to \mathcal{B}$ such that*

$$\|f(x) - h(x)\|_{\mathcal{B}} \leq \frac{1}{p-1}\phi(0,x,\cdots,x) \tag{4.3}$$

for all $x \in \mathcal{A}$.

Proof. It follows from (3.2) that

$$\left\|\frac{1}{k}f(kx) - f(x)\right\|_{\mathcal{B}} \leq \frac{1}{2}\varphi(0,x,\cdots,x)$$

for all $x \in \mathcal{A}$. Using the induction method, we obtain

$$\left\|\frac{1}{k^n}f(k^n x) - f(x)\right\|_{\mathcal{B}} \leq \frac{1}{p-1}\sum_{s=0}^{n-1} k^{-(s+1)}\varphi(0,k^s x,\cdots,k^s x) \tag{4.4}$$

for all $n \geq 1$ and all $x \in \mathcal{A}$. Assume that m, l are positive integers with $m > l$. By (4.4), for $m - l > 0$ and $k^l x$, we have

$$\begin{aligned}\left\|\frac{1}{k^m}f(k^m x) - \frac{1}{k^l}f(k^l x)\right\|_{\mathcal{B}} &= \frac{1}{k^l}\left\|\frac{1}{k^{m-l}}f(k^{m-l}k^l x) - f(k^l x)\right\|_{\mathcal{B}} \\ &\leq \frac{1}{p-1}\sum_{s=l}^{m-1} k^{-(s+1)}\varphi(0,k^s x,\cdots,k^s x) \\ &\leq \frac{1}{p-1}\sum_{s=l}^{\infty} k^{-(s+1)}\varphi(0,k^s x,\cdots,k^s x)\end{aligned}$$

for all $x \in \mathcal{A}$. By (4.1), the right-hand side tends to zero as $l \to \infty$. Therefore the sequence $\{\frac{1}{k^n}f(k^n x)\}$ is Cauchy. Since $\mathcal{A}$ is a complete space, the sequence $\{\frac{1}{k^n}f(k^n x)\}$ is convergent and we can define for all $x \in \mathcal{A}$, the mapping $h : \mathcal{A} \to \mathcal{B}$ by

$$h(x) := \lim_{n\to\infty} \frac{1}{k^n}f(k^n x).$$

Passing the limit $n \to \infty$ in (4.4) and then by (4.1), we obtain (4.3).

It follows from (4.1) and (3.2) that

$$\begin{aligned}\|\varrho_\mu h(x_1,\cdots,x_p)\|_{\mathcal{B}} &= \lim_{n\to\infty}\frac{1}{k^n}\left\|\varrho_\mu f\left(k^n x_1,\cdots,k^n x_p\right)\right\|_{\mathcal{B}} \\ &\leq \lim_{n\to\infty}\frac{1}{k^n}\varphi\left(k^n x_1\cdots,k^n x_p\right) = 0\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \cdots, x_p \in \mathcal{A}$. So by Lemma 3.1 we deduce that h is $\mathbb{C}$-linear.

By (4.2) and substituting x by $k^n x$ in (2.4), we obtain

$$\begin{aligned}
&\left\| h(x^r x^{*s} x^t) - h(x)^r h(x)^{*s} h(x)^t \right\|_{\mathcal{B}} \\
&= \lim_{n\to\infty} \frac{1}{k^{(r+s+t)n}} \left\| f\left((k^n x)^r (k^n x)^{*s} (k^n x)^t\right) - f\left(k^n x\right)^r f\left(k^n x\right)^{*s} f\left(k^n x\right)^t \right\|_{\mathcal{B}} \\
&\le \lim_{n\to\infty} \frac{1}{k^{(r+s+t)n}} \varphi\left(k^n x, \cdots, k^n x\right) = 0
\end{aligned}$$

for all $x \in \mathcal{A}$. Hence $h(x^r x^{*s} x^t) = h(x)^r h(x)^{*s} h(x)^t$ for all $x \in \mathcal{A}$.

Let $g : \mathcal{A} \to \mathcal{B}$ be another (r,s,t)-J^*-homomorphism satisfying (4.3). Then we have

$$\begin{aligned}
\|h(x) - g(x)\|_{\mathcal{B}} &\le \frac{1}{k^n} \|f(k^n x) - h(k^n x)\|_{\mathcal{B}} + \frac{1}{k^n} \|f(k^n x) - g(k^n x)\|_{\mathcal{B}} \\
&\le \frac{1}{k^n} \left(\frac{2}{p-1} \phi\left(0, k^n x, \cdots, k^n x\right) \right) \\
&= \frac{2}{p-1} \sum_{s=n}^{\infty} k^{-(s+1)} \varphi\left(0, k^s x, \cdots, k^s x\right)
\end{aligned}$$

for all $x \in \mathcal{A}$. By (4.1), the right-hand side tends to zero as $n \to \infty$, which means h is unique. □

Theorem 4.2. *Let $\varphi : \mathcal{A}^p \to [0, \infty)$ be a function with $\varphi(0, \cdots, 0) = 0$ and $p \ge 4$. Denote by ϕ a function such that*

$$\phi(x_1, \cdots, x_p) := \sum_{n=0}^{\infty} k^n \varphi\left(k^{-(n+1)} x_1, \cdots, k^{-(n+1)} x_p\right) < \infty \tag{4.5}$$

for all $x_1, \cdots, x_p \in \mathcal{A}$, where $k = \frac{2}{p-1}$. Suppose that $f : \mathcal{A} \to \mathcal{B}$ be an odd mapping satisfying (3.2) and (2.4). Then there exists a unique (r,s,t)-J^*-homomorphism $h : \mathcal{A} \to \mathcal{B}$ satisfying (4.3).

Proof. It follows from (3.2) that

$$\left\| k f\left(\frac{x}{k}\right) - f(x) \right\|_{\mathcal{B}} \le \frac{1}{p-1} \varphi\left(0, \frac{x}{k} \cdots, \frac{x}{k}\right)$$

for all $x \in \mathcal{A}$. By the same method which was done in the proof of Theorem 4.1, we can get the unique and $\mathbb{C}$-linear mapping $h(x) := \lim_{n\to\infty} k^n f\left(\frac{1}{k^n} x\right)$ satisfying (4.3). By (2.4), (4.5) and the fact that $k < 1$, we have

$$\begin{aligned}
&\left\| h(x^r x^{*s} x^t) - h(x)^r h(x)^{*s} h(x)^t \right\|_{\mathcal{B}} \\
&= \lim_{n\to\infty} k^{(r+s+t)n} \left\| f\left(\left(\frac{x}{k^n}\right)^r \left(\frac{x}{k^n}\right)^{*s} \left(\frac{x}{k^n}\right)^t \right) - f\left(\frac{x}{k^n}\right)^r f\left(\frac{x}{k^n}\right)^{*s} f\left(\frac{x}{k^n}\right)^t \right\|_{\mathcal{B}} \\
&\le \lim_{n\to\infty} k^{(r+s+t)n} \varphi\left(\frac{x}{k^n}, \cdots, \frac{x}{k^n}\right) \le \lim_{n\to\infty} k^n \varphi\left(\frac{x}{k^n}, \cdots, \frac{x}{k^n}\right) = 0
\end{aligned}$$

for all $x \in \mathcal{A}$. Hence $h(x^r x^{*s} x^t) = h(x)^r h(x)^{*s} h(x)^t$ for all $x \in \mathcal{A}$. □

Corollary 4.3. *Let θ be a nonnegative real number and $q_1, \cdots, q_p$ be positive real numbers such that $q_1, \cdots, q_p > r+s+t$ or $q_1, \cdots, q_p < 1$. Let $f : \mathcal{A} \to \mathcal{B}$ be an odd mapping satisfying (2.5) and*

$$\|\varrho_\mu f(x_1, \cdots, x_p)\|_{\mathcal{B}} \le \theta(\|x_1\|_{\mathcal{A}}^{q_1} + \cdots + \|x_p\|_{\mathcal{A}}^{q_p})$$

for all $\mu \in \mathbb{T}^1$ *and all* $x_1, \cdots, x_p \in \mathcal{A}$. *Then there exists a unique* (r, s, t)-J^*-*homomorphism* $h : \mathcal{A} \to \mathcal{B}$ *such that*

$$\|f(x) - h(x)\|_{\mathcal{B}} \leq \sum_{j=2}^{p} \frac{\theta \|x\|_{\mathcal{A}}^{q_j}}{2\,|1 - k^{q_j - 1}|}$$

for all $x \in \mathcal{A}$.

Proof. Defining $\varphi(x_1, \cdots, x_p) = \theta \left(\|x_1\|_{\mathcal{A}}^{q_1} + \cdots + \|x_p\|_{\mathcal{A}}^{q_p} \right)$ and applying Theorem 4.1 for the case $q_1, \cdots, q_p > r + s + t$, and Theorem 4.2 for the case $q_1, \cdots, q_p < 1$, we get the result. □

Theorem 4.4. *Let* $\varphi : \mathcal{A}^2 \to [0, \infty)$ *be a function with* $\varphi(0, 0) = 0$. *Denote by* ϕ *a function such that*

$$\phi(x, y) := \sum_{n=0}^{\infty} 2^{-(n+1)} \varphi\left(2^n x, 2^n y\right) < \infty$$

for all $x, y \in \mathcal{A}$. *Suppose that* $f : \mathcal{A} \to \mathcal{B}$ *is an odd mapping satisfying*

$$\|f(\mu x + \mu y) + f(\mu x - 2\mu y) + f(\mu y - \mu x) - \mu f(x)\|_{\mathcal{B}} \leq \varphi(x, y), \tag{4.6}$$

$$\|f(x^r x^{*s} x^t) - f(x)^r f(x)^{*s} f(x)^t\|_{\mathcal{B}} \leq \varphi(x, x) \tag{4.7}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, y \in \mathcal{A}$. *Then there exists a unique* (r, s, t)-J^*-*homomorphism* $h : \mathcal{A} \to \mathcal{B}$ *such that*

$$\|f(x) - h(x)\|_{\mathcal{B}} \leq \phi(0, x) \tag{4.8}$$

for all $x \in \mathcal{A}$.

Proof. From (4.6), it follows that

$$\left\| \frac{1}{2} f(2x) - f(x) \right\|_{\mathcal{B}} \leq \frac{1}{2} \varphi(0, x)$$

for all $x \in \mathcal{A}$. Using the same method as in the proof of Theorem 4.1, we conclude that the mapping $h(x) := \lim_{n \to \infty} \frac{1}{2^n} f\left(2^n x\right)$ is a unique (r, s, t)-J^*-homomorphism satisfying (4.8). □

Theorem 4.5. *Let* $\varphi : \mathcal{A}^2 \to [0, \infty)$ *be a function with* $\varphi(0, 0) = 0$. *Denote by* ϕ *a function such that*

$$\phi(x, y) := \sum_{n=0}^{\infty} 2^n \varphi\left(2^{-(n+1)} x, 2^{-(n+1)} y\right) < \infty,$$

$$\lim_{n \to \infty} 2^{(r+s+t)n} \varphi\left(2^{-n} x, 2^{-n} x\right) = 0$$

for all $x, y \in \mathcal{A}$. *Suppose that* $f : \mathcal{A} \to \mathcal{B}$ *is an odd mapping satisfying* (4.6) *and* (4.7). *Then there exists a unique* (r, s, t)-J^*-*homomorphism* $h : \mathcal{A} \to \mathcal{B}$ *satisfying* (4.8).

Proof. The proof is similar to the proof of Theorem 4.4. □

Corollary 4.6. *Let* θ *be a nonnegative real number and* q_1, q_2 *be positive real numbers such that* $q_1, q_2 < 1$ *or* $q_1, q_2 > r + s + t$. *Let* $f : \mathcal{A} \to \mathcal{B}$ *be an odd mapping satisfying*

$$\begin{aligned}
\|f(\mu x + \mu y) + f(\mu x - 2\mu y) + f(\mu y - \mu x) - \mu f(x)\|_{\mathcal{B}} &\leq \theta(\|x\|_{\mathcal{A}}^{q_1} + \|y\|_{\mathcal{A}}^{q_2}), \\
\|f(x^r x^{*s} x^t) - f(x)^r f(x)^{*s} f(x)^t\|_{\mathcal{B}} &\leq \theta(\|x\|_{\mathcal{A}}^{q_1} + \|x\|_{\mathcal{A}}^{q_2})
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, y \in \mathcal{A}$. *Then there exists a unique* (r, s, t)-J^*-*homomorphism* $h : \mathcal{A} \to \mathcal{B}$ *such that*

$$\|f(x) - h(x)\|_{\mathcal{B}} \leq \frac{\theta \|x\|_{\mathcal{A}}^{q_2}}{|2 - 2^{q_2}|}$$

for all $x \in \mathcal{A}$.

Proof. Defining $\varphi(x,y)=\theta\left(\|x\|_{\mathcal{A}}^{q_1}+\|y\|_{\mathcal{A}}^{q_2}\right)$ and applying Theorem 4.4 for the case $q_1,q_2<1$, and Theorem 4.5 for the case $q_1,q_2>r+s+t$, we get the result. □

Theorem 4.7. *Let $\varphi:\mathcal{A}^3\to[0,\infty)$ be a function. Denote by ϕ a function such that*

$$\phi(x,y,z):=\sum_{n=1}^{\infty}2^{-n}\varphi\left(2^nx,2^ny,2^nz\right)<\infty$$

for all $x,y,z\in\mathcal{A}$. Suppose that $f:\mathcal{A}\to\mathcal{B}$ is an odd mapping satisfying (3.9) *and* (3.10). *Then there exists a unique (r,s,t)-J^*-homomorphism $h:\mathcal{A}\to\mathcal{B}$ such that*

$$\|f(x)-h(x)\|_{\mathcal{B}}\le\phi(x,0,0)\tag{4.9}$$

for all $x\in\mathcal{A}$.

Proof. By (3.9), we get $\left\|\frac{1}{2}f(2x)-f(x)\right\|_{\mathcal{B}}\le\frac{1}{2}\varphi(2x,0,0)$ for all $x\in\mathcal{A}$. The same method as in the proof of Theorem 4.1, leads us to the unique (r,s,t)-J^*-homomorphism $h(x):=\lim_{n\to\infty}\frac{1}{2^n}f\left(2^nx\right)$satisfying (4.9). □

Theorem 4.8. *Let $\varphi:\mathcal{A}^3\to[0,\infty)$ be a function. Denote by ϕ a function such that*

$$\phi(x,y,z):=\sum_{n=0}^{\infty}2^{n}\varphi\left(2^{-n}x,2^{-n}y,2^{-n}z\right)<\infty,$$

$$\lim_{n\to\infty}2^{(r+s+t)n}\varphi\left(2^{-n}x,2^{-n}x,2^{-n}x\right)=0$$

for all $x,y,z\in\mathcal{A}$. Suppose that $f:\mathcal{A}\to\mathcal{B}$ is an odd mapping satisfying (3.9) *and* (3.10). *Then there exists a unique (r,s,t)-J^*-homomorphism $h:\mathcal{A}\to\mathcal{B}$ satisfying* (4.9).

Proof. The proof is similar to the proof of Theorem 4.7. □

Corollary 4.9. *Let θ be a nonnegative real number and q_1,q_2,q_3 be positive real numbers such that $q_1,q_2,q_3<1$ or $q_1,q_2,q_3>r+s+t$. Let $f:\mathcal{A}\to\mathcal{B}$ be an odd mapping satisfying*

$$\left\|f\left(\mu\frac{x+y+z}{2}\right)+f\left(\mu\frac{x+z-3y}{2}\right)+f\left(\mu\frac{x+y-3z}{2}\right)\right.$$
$$\left.+f\left(\mu\frac{y+z-x}{2}\right)-\mu f(x)\right\|_{\mathcal{B}}\le\theta\left(\|x\|_{\mathcal{A}}^{q_1}+\|y\|_{\mathcal{A}}^{q_2}+\|z\|_{\mathcal{A}}^{q_3}\right),$$
$$\|f(x^rx^{*s}x^t)-f(x)^rf(x)^{*s}f(x)^t\|_{\mathcal{B}}\le\theta\left(\|x\|_{\mathcal{A}}^{q_1}+\|x\|_{\mathcal{A}}^{q_2}+\|x\|_{\mathcal{A}}^{q_3}\right)$$

for all $\mu\in\mathbb{T}^1$ and all $x,y,z\in\mathcal{A}$. Then there exists a unique (r,s,t)-J^*-homomorphism $h:\mathcal{A}\to\mathcal{B}$ such that

$$\|f(x)-h(x)\|_{\mathcal{B}}\le\frac{2^{q_1}}{|2-2^{q_1}|}\theta\|x\|_{\mathcal{A}}^{q_1}$$

for all $x\in\mathcal{A}$.

Proof. Defining $\varphi(x,y,z)=\theta\left(\|x\|_{\mathcal{A}}^{q_1}+\|y\|_{\mathcal{A}}^{q_2}+\|z\|_{\mathcal{A}}^{q_3}\right)$ and applying Theorem 4.7 for the case $q_1,q_2,q_3<1$, and Theorem 4.8 for the case $q_1,q_2,q_3>r+s+t$, we get the result. □

Remark 4.10. The obtained results in this paper, could be more remarkable and interesting. In other words, as a consequence including simpler and better results, one can set $q_1=\cdots=q_p=q$, as well as $r=s=t=1$ (or a fixed $n\in\mathbb{N}$) in all the statements. Furthermore, all the obtained results do also hold for (r,s,t)-J^*-derivations similarly. The reader can directly verify this point just with a little difference in details.

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Differential subordinations obtained by using a generalization of Marx-Strohhäcker theorem

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Abstract

In [1] and [6] Marx and Strohhäcker have proved that if $f \in \mathcal{A}$ is a convex function, then it has the property of starlikeness of order $\frac{1}{2}$. In [5, Theorem 9.5.6], P. T. Mocanu extended this result to the class $\mathcal{A}_2$ for a convex function of order $-\frac{1}{2}$. In this paper we extend the results proven by Marx and Strohhäcker and by P. T. Mocanu and we'll prove that, if the function $f \in \mathcal{A}_n$, $n \geq 3$, is a close-to-convex function, then it is starlike of order $\frac{1}{2}$.

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1 Introduction and preliminaries

Let U be the unit disc of the complex plane $U = \{z \in \mathbb{C} : \ |z| < 1\}$. Let $\mathcal{H}(U)$ be the class of holomorphic functions in U. Also, let $\mathcal{A}_n = \{f \in \mathcal{H}(U) : \ f(z) = z + a_{n+1}z^{n+1} + \dots, \ z \in U\}$, with $\mathcal{A}_1 = \mathcal{A}$.

Let $\mathcal{S} = \{f \in \mathcal{A} : \ f \text{ univalent in } U\}$ be the class of holomorphic and univalent functions in the open unit disc U, with conditions $f(0) = 0$, $f'(0) = 1$, that is the holomorphic and univalent functions with the following power series development $f(z) = z + a_2z^2 + \dots$, $z \in U$.

Denote by $\mathcal{K} = \left\{f \in \mathcal{A} : \ \operatorname{Re}\frac{zf''(z)}{f'(z)} + 1 > 0, \ z \in U\right\}$ the class of normalized convex functions in U and by $\mathcal{C} = \left\{f \in \mathcal{A} : \ \exists\, \varphi \in \mathcal{K}, \ \operatorname{Re}\frac{f'(z)}{\varphi'(z)} > 0, \ z \in U\right\}$ the class of normalized close-to-convex functions in U.

An equivalent formulation would involve the existence of a starlike function h (not necessarily normalized) such that $\operatorname{Re}\frac{zf'(z)}{h(z)} > 0$, $z \in U$. We consider $\mathcal{K}\left(-\frac{1}{2\gamma}\right) = \left\{f \in \mathcal{A}_n : \ \operatorname{Re}\frac{zf''(z)}{f'(z)} + 1 > -\frac{1}{2\gamma}, \ z \in U, \ \gamma \geq 1\right\}$.

Let $\mathcal{S}^* = \left\{f \in \mathcal{A} : \ \operatorname{Re}\frac{zf'(z)}{f(z)} > 0, \ z \in U\right\}$ denote the class of starlike functions in U, and
$\mathcal{S}^*(\alpha) = \left\{f \in \mathcal{A} : \ \operatorname{Re}\frac{zf'(z)}{f(z)} > \alpha, \ z \in U\right\}$, denote the class of starlike functions of order α, with $0 \leq \alpha < 1$.

In order to prove our original results, we use the following lemmas:

Lemma 1.1 *[2], [3], [4, Theorem 2.3.i, p. 35] Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$, satisfy the condition $\operatorname{Re}\psi(is, t) \leq 0, \ z \in U$, for $s, t \in R$, $t \leq -\frac{n}{2}(1 + s^2)$. If $p(z) = 1 + p_nz^n + p_{n+1}z^{n+1} + \dots$ satisfies $\operatorname{Re}[p(z), zp'(z); z] > 0$, then $\operatorname{Re} p(z) > 0, \ z \in U$.*

More general forms of this lemma can be found in [6].

Lemma 1.2 *[5, Theorem 4.6.3, p. 84] The function $f \in \mathcal{A}$, with $f'(z) \neq 0$, $z \in U$, is close-to-convex if and only if $\int_{\theta_1}^{\theta_2} \operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] d\theta > -\pi$, $z = re^{i\theta}$, for all θ_1, θ_2, with $0 \leq \theta_1 < \theta_2 \leq 2\pi$ and $r \in (0, 1)$.*

Definition 1.1 *[4, Definition 2.2.b, p. 21] We denote by Q the set of functions q that are analytic and injective on $\overline{U} \setminus E(q)$, where $E(q) = \left\{\zeta \in \partial U : \ \lim\limits_{z\to\zeta} q(z) = \infty\right\}$ and are such that $q'(\zeta) \neq 0$, for $\zeta \in \partial U \setminus E(q)$. The set $E(q)$ is called exception set. The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.*

Definition 1.2 *[4, Definition 2.3.a, p. 27] Let Ω be a set in $\mathbb{C}$, $q \in Q$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C} \times U \to \mathbb{C}$ that satisfy the admissibility condition*

$$\text{(A)} \qquad \psi(r, s, t) \notin \Omega$$

where $r = q(\zeta)$, $s = m\zeta q'(\zeta)$, $R\left(\frac{t}{s}+1\right) \geq m\mathrm{Re}\left[\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right]$, $z \in U$, $\zeta \in \partial U \setminus E(q)$, $m \geq n$. We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

In the special case when Ω is a simply connected domain, $\Omega \neq \mathbb{C}$, and h is a conformal mapping of U onto Ω, we denote this class by $\Psi_n[h, q]$.

If $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$, then the admissibility condition (A) reduces to

$$\text{(A}'\text{)} \qquad \psi(q(\zeta), m\zeta q'(\zeta); z) \notin \Omega,$$

where $z \in U$, $\zeta \in \partial U \setminus E(q)$ and $m \geq n$.

If $\psi : \mathbb{C} \times U \to \mathbb{C}$, then the admissibility condition (A) reduces to

$$\text{(A}''\text{)} \qquad \psi(q(\zeta); z) \notin \Omega$$

where $z \in U$ and $\zeta \in \partial U \setminus E(q)$.

Definition 1.3 *[4, p. 36] Let f and F be members of $\mathcal{H}(U)$. The function f is said to be subordinate to F, written $f \prec F$ or $f(z) \prec F(z)$, if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, and $f(0) = F(0)$ and $f(U) \subset F(U)$.*

Definition 1.4 *[4, p. 16] Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second-order) differential subordination*

(i) $\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z)$,

then p is called a ***solution*** *of the differential subordination. The univalent function q is called a* ***dominant of the solutions of the differential subordination****, or more simply a* ***dominant****, if $p \prec q$ for all p satisfying (i).*

A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominant q of (i) is said to be the ***best dominant*** *of (i). (Note that the best dominant is unique up to a rotation of U).*

*If we require the more restrictive condition $p \in [a, n]$, then p will be called an (q, n)-****solution****, q an (a, n)-****dominant****, and $\widetilde{q}$ the best (a, n)-****dominant****,*

Lemma 1.3 *[4, Theorem 2.3.c, p. 30] Let $\psi \in \Psi_n[h, q]$ with $q(0) = a$. If $p \in \mathcal{H}[a, n]$, $\psi(p(z), zp'(z), z^2p''(z); z)$ is analytic in U, and*

(ii) $\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z)$,

then $p(z) \prec q(z)$, $z \in U$.

Theorem 1.1 *[1, 6, Marx-Strohhacker] If $f \in \mathcal{A}$ and satisfy the condition* $\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}+1\right) > 0$, *then*

(a) Re $\frac{zf'(z)}{f(z)} > \frac{1}{2}$ $\left[\text{ i.e., } f \in \mathcal{S}^*\left(\frac{1}{2}\right)\right]$ and

(b) Re $\frac{f(z)}{z} > \frac{1}{2}$, for $z \in U$.

In [5] has shown that the odd and convex functions of order $-\frac{1}{2}$ are starlike functions of order $\frac{1}{2}$.

Theorem 1.2 *[5, Marx-Strohhscker, Theorem 9.5.6, p. 218] If $f \in \mathcal{A}_2$ and satisfy the condition* $\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}+1\right) > -\frac{1}{2}$, *then* $\mathrm{Re}\,\frac{zf'(z)}{f(z)} > \frac{1}{2}$ $\left[\text{i.e., } f \in \mathcal{S}^*\left(\frac{1}{2}\right)\right]$, *for $z \in U$.*

2 Main results

We'll extend the theorem Marx-Strohhäcker for the functions $f \in \mathcal{A}_n$, $n \geq 3$, which are close-to-convex functions.

Theorem 2.1 *Let $n \geq 3$, $\gamma \geq 1$, $f \in \mathcal{A}_n$, satisfy the condition*

$$\operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > -\frac{1}{2\gamma}, \tag{2.1}$$

then $\operatorname{Re} \frac{zf'(z)}{f(z)} > \frac{1}{2}$.

Proof. According to Lemma 1.2 we obtain

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] d\theta \geq \int_{\theta_1}^{\theta_2} -\frac{1}{2\gamma} d\theta = -\frac{1}{2\gamma} \int_{\theta_1}^{\theta_2} d\theta = -\frac{1}{2\gamma}(\theta_2 - \theta_1) > -\frac{2\pi}{2\gamma} = -\frac{\pi}{\gamma} > -\pi, \quad \lambda \geq 1. \tag{2.2}$$

From (2.2) we have $f \in \mathbb{C}$, hence it is univalent.

Let $p(z) = 2 \cdot \frac{zf'(z)}{f(z)} - 1$. Since $f \in \mathrm{A}_n$ and f is close-to-convex function (univalent), the function p is analytic in U and $p(0) = 1$.

A simple computation leads to

$$\frac{p(z)+1}{2} = \frac{zf'(z)}{f(z)}. \tag{2.3}$$

By differentiating (2.3), we obtain

$$\frac{zp'(z)}{p(z)+1} = 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}. \tag{2.4}$$

Using (2.3) in (2.4), we have

$$\frac{p(z)+1}{2} + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)}. \tag{2.5}$$

Using (2.1) in (2.5), we obtain $\operatorname{Re}\left[\frac{p(z)+1}{2} + \frac{zp'(z)}{p(z)+1}\right] > -\frac{1}{2\gamma}$, which is equivalent to

$$\operatorname{Re}\left[\frac{p(z)+1}{2} + \frac{zp'(z)}{p(z)+1} + \frac{1}{2\gamma}\right] > 0. \tag{2.6}$$

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$, $\psi(p(z), zp'(z); z) = \frac{p(z)+1}{2} + \frac{zp'(z)}{p(z)+1} + \frac{1}{2\gamma}$, where $\psi(r,s) = \frac{r+1}{2} + \frac{s}{r+1} + \frac{1}{2\gamma}$. Then (2.6) is equivalent to $\operatorname{Re}\psi(p(z), zp'(z); z) > 0$, $z \in U$.

In order to prove Theorem 2.1, we use Lemma 1.1. For that we calculate $\operatorname{Re}\psi(is,t) = \operatorname{Re}\left(\frac{is+1}{2} + \frac{t}{1+is} + \frac{1}{2\gamma}\right)$ $= \operatorname{Re}\left(\frac{is+1}{2} + \frac{t(1-is)}{1+s^2} + \frac{1}{2\gamma}\right) = \frac{1}{2} + \frac{t}{1+s^2} + \frac{1}{2\gamma} \leq \frac{1}{2} - \frac{n(1+s^2)}{2(1+s^2)} + \frac{1}{2\gamma} = \frac{1-n}{2} + \frac{1}{2\gamma} = \frac{(1-n)\gamma+1}{2\gamma} \leq 0$. Since $n \geq 3$, $\gamma \geq 1$. Now, using Lemma 1.1, we get that $\operatorname{Re} p(z) > 0$, $z \in U$, i.e., $\operatorname{Re}\frac{zf'(z)}{f(z)} > \frac{1}{2}$, $z \in U$. ■

Remark 2.1 *Each of the four conditions in the Marx-Strohhäcker theorem can be rewritten in terms of subordination. This leads to the following equivalent form of the theorem.*

Theorem 2.2 *Let $n \geq 3$, $\gamma \geq 1$, $f \in \mathcal{A}_n$, satisfies the condition $\frac{zf''(z)}{f'(z)} + 1 \prec \frac{1-\left(\frac{1}{\gamma}+1\right)z}{1+z}$, then $\frac{zf'(z)}{f(z)} \prec \frac{1}{1+z}$.*

Theorem 2.3 *Let $n \geq 3$, $\gamma \geq 1$, $f \in \mathcal{A}_n$ satisfies the conditions*

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > -\frac{1}{2\gamma} \tag{2.7}$$

and

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \frac{1}{2} \tag{2.8}$$

then $\operatorname{Re}\frac{f(z)}{z} > \frac{1}{2}$, *for* $z \in U$.

Proof. In order to prove Theorem 2.1, we saw that, if $f \in \mathrm{A}_n$, $n \geq 3$ and satisfies the condition (2.1) or (2.7), then the function f is close-to-convex (univalent).

Let $p(z) = \frac{2f(z)}{z} - 1$. Since $f \in \mathcal{A}_n$, $n \geq 3$ and f is close-to-convex function (univalent) then the function p is analytic in U and $p(0) = 1$. A simple computation leads to

$$\frac{p(z)+1}{2} = \frac{f(z)}{z}. \tag{2.9}$$

By differentiating (2.9), we obtain

$$\frac{zp'(z)}{p(z)+1} = \frac{zf'(z)}{f(z)} - 1. \tag{2.10}$$

Using (2.8) in (2.10), we have

$$\mathrm{Re}\left(\frac{zp'(z)}{p(z)+1} + \frac{1}{2}\right) > 0, z \in U. \tag{2.11}$$

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$, $\psi(p(z), zp'(z); z) = \frac{zp'(z)}{1+p(z)} + \frac{1}{2}$, where $\psi(r,s) = \frac{1}{2} + \frac{s}{1+r}$. Then (2.11) is equivalent to $\mathrm{Re}\,\psi(p(z), zp'(z); z) > 0$, $z \in U$.

In order to prove Theorem 2.1, we use Lemma 1.1. For that we calculate $\mathrm{Re}\,\psi(is,t) = \mathrm{Re}\left[\frac{1}{2} + \frac{t}{1+is}\right] = \mathrm{Re}\left[\frac{1}{2} + \frac{t(1-is)}{1+s^2}\right] = \frac{1}{2} + \frac{t}{1+s^2} \leq \frac{1}{2} - \frac{n(1+s^2)}{2(1+s^2)} = \frac{1-n}{2} < 0$, since $n \geq 3$. Therefore, by applying Lemma 1.1 we conclude that p satisfies $\mathrm{Re}\,p(z) > 0$. This is equivalent to $\mathrm{Re}\,\frac{f(z)}{z} > \frac{1}{2}$, $z \in U$. ■

For $0 < \gamma < 1$, $n \geq 3$, Theorem 2.2 can be written as the following corollary.

Corollary 2.4 *Let $n \geq 3$, $0 < \gamma < 1$, $f \in \mathcal{A}_n$ satisfy the conditions* $\mathrm{Re}\left[\frac{zf''(z)}{f'(z)} + 1\right] > -\frac{\gamma}{2}$ *and* $\mathrm{Re}\,\frac{zf'(z)}{f(z)} > \frac{1}{2}$, *then* $\mathrm{Re}\,\frac{f(z)}{z} > \frac{1}{2}$, $z \in U$.

Theorem 2.5 *Let $n \geq 3$, $\gamma \geq 1$, $f \in \mathcal{A}_n$ satisfy differential subordination*

$$\frac{zf''(z)}{f'(z)} + 1 \prec \frac{1 - \left(\frac{1}{\gamma} + 1\right) z}{1+z}, \tag{2.12}$$

and

$$\frac{zf'(z)}{f(z)} \prec \frac{1}{1+z} \tag{2.13}$$

then $\frac{f(z)}{z} \prec \frac{1}{1+z}$, $z \in U$.

Proof. Consider

$$p(z) = \frac{2f(z)}{z} - 1. \tag{2.14}$$

Since $f \in \mathcal{A}_n$, and f is close-to-convex function (univalent) then the function p is analytic in U, and $p(0) = 1$.

By differentiating (2.14), we obtain

$$\frac{zp'(z)}{p(z)+1} + 1 = \frac{zf'(z)}{z}. \tag{2.15}$$

Using (2.13) in (2.15), we have

$$\frac{zp'(z)}{p(z)+1} + 1 \prec \frac{1}{1+z}. \tag{2.16}$$

Since $\mathrm{Re}\,\frac{1}{1+z} \geq \frac{1}{2}$, differential subordination (2.16) is equivalent to

$$\mathrm{Re}\left(\frac{zp'(z)}{p(z)+1} + \frac{1}{2}\right) > 0, \quad z \in U. \tag{2.17}$$

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$, $\psi(p(z), zp'(z); z) = \frac{zp'(z)}{p(z)+1} + \frac{1}{2}$, then (2.17) becomes $\mathrm{Re}\,\psi(p(z), zp'(z); z) > 0$, $z \in U$.

In order to prove Theorem 2.5, we use Lemma 1.3. For that we calculate $\mathrm{Re}\,\psi(is,t) = \mathrm{Re}\left(\frac{t}{1+is} + \frac{1}{2}\right) = \mathrm{Re}\left[\frac{t(1-is)}{1+s^2} + \frac{1}{2}\right] = \frac{t}{1+s^2} + \frac{1}{2} \leq \frac{-n(1+s^2)}{2(1+s^2)} + \frac{1}{2} = \frac{1-n}{2} < 0$. Using Definition 1.2, we have $\psi \in \Psi_n[h,q]$. Therefore by Lemma 1.3, we conclude that $p(z) \prec q(z)$, i.e., $\frac{f(z)}{z} \prec \frac{1}{1+z}$, for $z \in U$. ■

Theorem 2.6 *If $f \in \mathcal{A}_n$, $n \geq 3$, $\gamma \geq 1$ and satisfy the condition* $\operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > -\frac{1}{2\gamma}$, *then* $\operatorname{Re}\sqrt{f'(z)} > \frac{1}{2}$, *for $z \in U$.*

Proof. Consider $p(z) = 2\sqrt{f'(z)} - 1$, $z \in U$. Since $f \in \mathcal{A}_n$, $n \geq 3$, and f is close-to-convex function (univalent) then the function p is analytic in U and $p(0) = 1$. A simple computation leads to

$$\frac{p(z)+1}{2} = \sqrt{f'(z)}. \tag{2.18}$$

By differentiating (2.18), we have $\frac{2zp'(z)}{1+p(z)} + 1 = \frac{zf''(z)}{f'(z)} + 1$. Using (2.1), we have

$$\operatorname{Re}\left[\frac{2zp'(z)}{1+p(z)} + 1 + \frac{1}{2\gamma}\right] > 0. \tag{2.19}$$

If we let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$, $\psi(p(z), zp'(z)) = \frac{2zp'(z)}{1+p(z)} + \frac{1+2\gamma}{2\gamma}$, then (2.19) becomes $\operatorname{Re}\psi(p(z), zp'(z)) > 0$.

In order to prove Theorem 2.6, we use Lemma 1.1. For that, we calculate $\operatorname{Re}\psi(is, t) = \operatorname{Re}\left(\frac{2t}{1+is} + \frac{1+2\gamma}{2\gamma}\right)$ $= \operatorname{Re}\left(\frac{2t(1-is)}{1+s^2} + \frac{1+2\gamma}{2\gamma}\right) = \frac{2t}{1+s^2} + \frac{1+2\gamma}{2\gamma} \leq \frac{-n(1+s^2)}{1+s^2} + \frac{1+2\gamma}{2\gamma} = \frac{-2\gamma n+1+2\gamma}{2\gamma} = \frac{2\gamma(1-n)+1}{2\gamma} \leq 0$, since $n \geq 3$, $\gamma \geq 1$.

Using Lemma 1.1, we have $\operatorname{Re} p(z) > 0$, i.e., $\operatorname{Re}\sqrt{f'(z)} > \frac{1}{2}$. ■

For $0 < \gamma < 1$, $n \geq 3$, Theorem 2.6 can be written as the following corollary.

Corollary 2.7 *If $f \in \mathcal{A}_n$, $n \geq 3$, $0 < \gamma < 1$, satisfy the condition* $\operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > -\frac{\gamma}{2}$, *then* $\operatorname{Re}\sqrt{f'(z)} > \frac{1}{2}$, *for $z \in U$.*

In differential subordination language Theorem 2.6 can be written as

Theorem 2.8 *If $f \in \mathcal{A}_n$, $n \geq 3$, $\gamma \geq 1$, and satisfy the differential subordination*

$$\frac{zf''(z)}{f'(z)} + 1 \prec \frac{1 - \left(\frac{1}{\gamma} + 1\right)z}{1+z}, \tag{2.20}$$

then $\sqrt{f'(z)} \prec \frac{1}{1+z}$, *for $z \in U$.*

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A finite difference method for Burgers' equation in the unbounded domain using artificial boundary conditions *

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Abstract: This paper discusses the numerical solution of one-dimensional Burgers' equation in the infinite domain. The original problem is converted by Hopf-Cole transformation to the heat equation in the infinite domain, the latter is reduced to an equivalent problem in a finite computational domain with two artificial integral boundary conditions, a finite difference method is constructed for last problem by the method of reduction of order, and therefore the numerical solution of Burgers' equation is obtained. The method is proved and verified to be uniquely solvable, unconditionally stable and convergent with the order 2 in space and the order 3/2 in time for solving the heat equation as well as Burgers' equation in the computational domain.

Keywords: Burgers' equation; infinite domain; Hopf-Cole transformation; Artificial boundary condition; Finite difference method

1 Introduction

When an analytic solution is not available, or the analytic one is not suitable to be used, a numerical method is necessary for solving partial differential equations. Therefore, several kinds of exterior problems in the areas of heat transfer, fluid dynamics and other applications were solved numerically by using artificial boundary conditions [1-5].

The artificial boundary methods were established on bounded computational domains for various problems of heat equation on unbounded domains and the feasibility and effectiveness of the methods were shown by the numerical examples [6, 7]. Moreover, for the heat equation in

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a semi-unbounded domain $[-1,\infty)\times[0,\infty)$, by using an artificial integral boundary condition

$$u_x(0,t) = -\frac{1}{\sqrt{\pi}}\int_0^t \frac{u_\lambda(0,\lambda)}{\sqrt{t-\lambda}}d\lambda,$$

Sun and Wu [8] firstly proved that the finite difference scheme is uniquely solvable, unconditionally stable and convergent with the order 2 in space and the order 3/2 in time under an energy norm. Wu and Zhang [9] also obtained the high-order artificial boundary conditions for the heat equation in unbounded domains, but only proved that the reduced initial-boundary-value problems were stable.

Furthermore, Han, Wu and Xu [10] started to consider the nonlinear Burgers' equation in the unbounded domain as follows:

$$w_t + ww_x - \nu w_{xx} = F(x,t), \quad \forall (x,t)\in\mathbb{R}\times(0,T], \tag{1.1}$$

$$w(x,0) = f(x), \quad \forall x\in\mathbb{R}, \tag{1.2}$$

$$w(x,t)\to 0, \quad \text{when } |x|\to+\infty, \quad \forall t\in[0,T], \tag{1.3}$$

where $\nu = \frac{1}{Re}$, Re is the Reynolds number, and the given functions F and f are sufficiently smooth with compact supports $\text{supp}\{F(x,t)\}\subset[x_l,x_r]\times[0,T]$ and $\text{supp}\{f(x)\}\subset[x_l,x_r]$. They obtained nonlinear artificial boundary conditions, constructed a nonlinear difference method with no theoretical convergence analysis, and supported it by numerical examples. Recently, Sun and Wu [11] introduced a function transformation to reduce nonlinear Burgers' equation to a linear initial boundary value problem, deduced a linear finite difference scheme, and also proved that the finite difference scheme is uniquely solvable, unconditionally stable and convergent with the order 2 in space and 3/2 in time.

In this paper, we consider the problem (1.1)-(1.3) with $F\equiv 0$ and convert it into an initial value problem of heat equation by using Hopf-Cole transformation in the following. Let

$$\omega(x,t) = -\int_x^\infty w(y,t)dy, \quad \forall (x,t)\in\mathbb{R}\times(0,T],$$

we obtain

$$\omega_t + \frac{1}{2}\omega_x^2 - \nu\omega_{xx} = 0,$$
$$\omega(x,0) = -\int_x^\infty f(y)dy, \quad \forall x\in\mathbb{R},$$
$$\omega(x,t)\to 0, \quad \text{when } |x|\to+\infty, \quad \forall t\in[0,T].$$

Let $u = \exp(-\omega/2\nu) - 1$, then we have the initial value problem of heat equation:

$$u_t - \nu u_{xx} = 0, \quad \forall (x,t) \in \mathbb{R} \times (0,T], \tag{1.4}$$

$$u(x,0) = \phi(x) := \exp(\frac{1}{2\nu}\int_x^\infty f(y)dy) - 1, \tag{1.5}$$

$$u(x,t) \to 0, \quad \text{when } |x| \to +\infty, \quad \forall t \in [0,T], \tag{1.6}$$

where the sufficiently smooth given function $\phi(x)$ has compact support $\text{supp}\{\phi(x)\} \subset [x_l, x_r]$.

By using artificial linear integral boundary conditions similar to that in [8], we reduce the problem (1.4)-(1.6) to a problem in the bounded computational domain:

$$u_t - \nu u_{xx} = 0, \quad \forall (x,t) \in [x_l, x_r] \times [0,T], \tag{1.7}$$

$$u(x,0) = \phi(x), \quad \forall x \in [x_l, x_r], \tag{1.8}$$

$$u_x(x_l,t) = \frac{1}{\sqrt{\pi\nu}}\int_0^t \frac{u_\lambda(x_l,\lambda)}{\sqrt{t-\lambda}}d\lambda, \quad \forall t \in [0,T], \tag{1.9}$$

$$u_x(x_r,t) = -\frac{1}{\sqrt{\pi\nu}}\int_0^t \frac{u_\lambda(x_r,\lambda)}{\sqrt{t-\lambda}}d\lambda, \quad \forall t \in [0,T]. \tag{1.10}$$

In section 2, we construct a finite difference scheme for solving the problem (1.7)-(1.10). Then a new solution of Burgers' equation is obtained and the difficulty for solving the nonlinear problem is avoided. In section 3, we prove that the finite difference scheme is uniquely solvable, unconditionally stable and convergent with the order 2 in space and 3/2 in time. In section 4, a numerical example confirms the stability and convergence of the finite difference method.

2 The construction of the difference scheme

In order to construct the finite difference method, the bounded computational domain is divided into an $M \times N$ uniform mesh. Let $h = (x_r - x_l)/M$, $x_i = x_l + ih$ for $0 \le i \le M$, $\tau = T/N$, $t_n = n\tau$ for $0 \le n \le N$, $r = \frac{\nu\tau}{h^2}$, and u_i^n be the numerical solution of $u(x,t)$ at (x_i, t_n). Introduce the notations:

$$u_{i-\frac{1}{2}}^n = \frac{1}{2}(u_i^n + u_{i-1}^n), \quad \delta_x u_{i-\frac{1}{2}}^n = \frac{1}{h}(u_i^n - u_{i-1}^n), \quad u_i^{n-\frac{1}{2}} = \frac{1}{2}(u_i^n + u_i^{n-1}),$$

$$\delta_t u_i^{n-\frac{1}{2}} = \frac{1}{\tau}(u_i^n - u_i^{n-1}), \quad \delta_x^2 u_i^n = \frac{1}{h^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n),$$

$$\|u^n\|_A = \sqrt{h\sum_{i=1}^M (u_{i-\frac{1}{2}}^n)^2}, \quad \|\delta_x u^n\| = \sqrt{h\sum_{i=1}^M (\delta_x u_{i-\frac{1}{2}}^n)^2}.$$

Lemma 2.1 *Suppose $f(t) \in C^2[0, t_n]$, then*

$$|\int_0^{t_n} f'(t)\frac{dt}{\sqrt{t_n - t}} - \sum_{k=1}^{n} \frac{f(t_k) - f(t_{k-1})}{\tau} \int_{t_{k-1}}^{t_k} \frac{dt}{\sqrt{t_n - t}}| \leq \frac{1}{12}(20\sqrt{2} - 23) \max_{0\leq t\leq t_n} |f''(t)|\tau^{\frac{3}{2}}.$$

Proof Lemma 2.1 is proved by using $\sqrt{t_n - t} - (\frac{t_k - t}{\tau}\sqrt{t_n - t_{k-1}} + \frac{t - t_{k-1}}{\tau}\sqrt{t_n - t_k}) = \frac{1}{8}(t_n - \xi_k)^{-\frac{3}{2}}(t - t_{k-1})(t_k - t)$ to correct (2.2) and thereupon (2.1) in [8], as corrected in [12]. □

By introducing a new variable $v = \frac{\partial u}{\partial x}$ to reduce the order of heat equation, the problem (1.7)-(1.10) is equivalent to the problem of first-order differential equations:

$$\frac{\partial u}{\partial x} = \nu\frac{\partial v}{\partial x}, \quad \forall(x, t) \in [x_l, x_r] \times [0, T], \tag{2.1}$$

$$v - \frac{\partial u}{\partial x} = 0, \quad \forall(x, t) \in [x_l, x_r] \times [0, T], \tag{2.2}$$

$$u(x, 0) = \phi(x), \quad x_l \leq x \leq x_r, \tag{2.3}$$

$$v(x_l, t) = \frac{1}{\sqrt{\pi\nu}}\int_0^t \frac{\partial u(x_l, \lambda)}{\partial\lambda}\frac{1}{\sqrt{t - \lambda}}d\lambda, \tag{2.4}$$

$$v(x_r, t) = -\frac{1}{\sqrt{\pi\nu}}\int_0^t \frac{\partial u(x_r, \lambda)}{\partial\lambda}\frac{1}{\sqrt{t - \lambda}}d\lambda. \tag{2.5}$$

Define the grid functions:

$$U_i^n = u(x_i, t_n), \quad V_i^n = v(x_i, t_n), \quad 0 \leq i \leq M, \quad n \geq 0.$$

Using Lemma 2.1, it follows from (2.5) that

$$\begin{aligned} V_M^n &= -\frac{1}{\sqrt{\pi\nu}}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\frac{\partial u(x_r, \lambda)}{\partial\lambda}\frac{d\lambda}{\sqrt{t_n - \lambda}} \\ &= -\frac{1}{\sqrt{\pi\nu}}\sum_{k=1}^{n}\frac{U_M^k - U_M^{k-1}}{\tau}\int_{t_{k-1}}^{t_k}\frac{d\lambda}{\sqrt{t_n - \lambda}} + O(\tau^{\frac{3}{2}}) \\ &= -\frac{2}{\sqrt{\pi\nu}}\sum_{k=1}^{n}(U_M^k - U_M^{k-1})a_{n-k} + O(\tau^{\frac{3}{2}}) \\ &= -\frac{2}{\sqrt{\pi\nu}}[a_0U_M^n - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})U_M^k - a_{n-1}U_M^0] + O(\tau^{\frac{3}{2}}), \quad n = 1, 2, \cdots. \end{aligned}$$

Therefore, we have

$$V_M^{n-\frac{1}{2}} = \frac{1}{2}(V_M^{n-1} + V_M^n) = -\frac{2}{\sqrt{\pi\nu}}[a_0U_M^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})U_M^{k-\frac{1}{2}} - a_{n-1}U_M^0] + O(\tau^{\frac{3}{2}}),$$

and similarly,

$$V_0^{n-\frac{1}{2}} = \frac{1}{2}(V_0^{n-1} + V_0^n) = \frac{2}{\sqrt{\pi\nu}}[a_0U_0^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})U_0^{k-\frac{1}{2}} - a_{n-1}U_0^0] + O(\tau^{\frac{3}{2}}).$$

4

Using Taylor expansion, we have

$$\delta_t U_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \nu\delta_x V_{i-\frac{1}{2}}^{n-\frac{1}{2}} = p_{i-\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1 \le i \le M, \quad n \ge 1, \tag{2.6}$$

$$V_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \delta_x U_{i-\frac{1}{2}}^{n-\frac{1}{2}} = q_{i-\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1 \le i \le M, \quad n \ge 1, \tag{2.7}$$

$$U_i^0 = \phi(x_i), \quad 0 \le i \le M, \tag{2.8}$$

$$V_0^{n-\frac{1}{2}} = \frac{2}{\sqrt{\pi\nu}}[a_0 U_0^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})U_0^{k-\frac{1}{2}} - a_{n-1}U_0^0] + s^{n-\frac{1}{2}}, \quad n \ge 1, \tag{2.9}$$

$$V_M^{n-\frac{1}{2}} = -\frac{2}{\sqrt{\pi\nu}}[a_0 U_M^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})U_M^{k-\frac{1}{2}} - a_{n-1}U_M^0] + t^{n-\frac{1}{2}}, \quad n \ge 1, \tag{2.10}$$

where

$$|p_{i-\frac{1}{2}}^{n-\frac{1}{2}}| \le c(\tau^2 + h^2), \quad |q_{i-\frac{1}{2}}^{n-\frac{1}{2}}| \le c(\tau^2 + h^2), \quad 1 \le i \le M, \quad n \ge 1, \tag{2.11}$$

$$|t^{n-\frac{1}{2}}| \le c\tau^{\frac{3}{2}}, \quad |s^{n-\frac{1}{2}}| \le c\tau^{\frac{3}{2}}, \quad n \ge 1, \tag{2.12}$$

and c is a constant.

Thus, we construct a difference scheme for (2.1)-(2.5) in the following:

$$\delta_t u_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \nu\delta_x v_{i-\frac{1}{2}}^{n-\frac{1}{2}} = 0, \quad 1 \le i \le M, \quad n \ge 1, \tag{2.13}$$

$$v_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \delta_x u_{i-\frac{1}{2}}^{n-\frac{1}{2}} = 0, \quad 1 \le i \le M, \quad n \ge 1, \tag{2.14}$$

$$u_i^0 = \phi(x_i), \quad 0 \le i \le M, \tag{2.15}$$

$$v_0^{n-\frac{1}{2}} = \frac{2}{\sqrt{\pi\nu}}[a_0 u_0^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})u_0^{k-\frac{1}{2}} - a_{n-1}u_0^0], \quad n \ge 1, \tag{2.16}$$

$$v_M^{n-\frac{1}{2}} = -\frac{2}{\sqrt{\pi\nu}}[a_0 u_M^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})u_M^{k-\frac{1}{2}} - a_{n-1}u_M^0]. \quad n \ge 1, \tag{2.17}$$

Theorem 2.2 *The difference scheme (2.13)-(2.17) is equivalent to the following (2.18)-(2.22):*

$$u_i^0 = \phi(x_i), \quad 0 \le i \le M, \tag{2.18}$$

$$\frac{1}{2}(\delta_t u_{i-\frac{1}{2}}^{n-\frac{1}{2}} + \delta_t u_{i+\frac{1}{2}}^{n-\frac{1}{2}}) - \nu\delta_x^2 u_i^{n-\frac{1}{2}} = 0, \quad 1 \le i \le M-1, \quad n \ge 1, \tag{2.19}$$

$$\delta_t u_{\frac{1}{2}}^{n-\frac{1}{2}} + \frac{2\nu}{h}[\frac{2}{\sqrt{\pi\nu}}(a_0 u_0^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})u_0^{k-\frac{1}{2}} - a_{n-1}u_0^0) - \delta_x u_{\frac{1}{2}}^{n-\frac{1}{2}}], \quad n \ge 1, \tag{2.20}$$

$$\delta_t u_{M-\frac{1}{2}}^{n-\frac{1}{2}} + \frac{2\nu}{h}[\frac{2}{\sqrt{\pi\nu}}(a_0 u_M^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})u_0^{k-\frac{1}{2}} - a_{n-1}u_M^0) + \delta_x u_{M-\frac{1}{2}}^{n-\frac{1}{2}}], \quad n \ge 1, \tag{2.21}$$

where

$$a_m = \frac{1}{\sqrt{t_{m+1}} + \sqrt{t_m}} = \frac{1}{\sqrt{\tau}(\sqrt{m+1} + \sqrt{m})}, \quad m = 0, 1, 2, \cdots. \tag{2.22}$$

Proof Multiplying (2.13) by $\frac{1}{2}h$ and using (2.14) we obtain

$$v_i^{n-\frac{1}{2}} = \delta_x u_{i-\frac{1}{2}}^{n-\frac{1}{2}} + \frac{h}{2\nu}\delta_t u_{i-\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1 \le i \le M, \quad n \ge 1, \tag{2.23}$$

$$v_i^{n-\frac{1}{2}} = \delta_x u_{i+\frac{1}{2}}^{n-\frac{1}{2}} - \frac{h}{2\nu}\delta_t u_{i+\frac{1}{2}}^{n-\frac{1}{2}}, \quad 0 \le i \le M-1, \quad n \ge 1, \tag{2.24}$$

From (2.23) and (2.24) for i from 1 to $M-1$ we obtain

$$\delta_x u_{i-\frac{1}{2}}^{n-\frac{1}{2}} + \frac{h}{2\nu}\delta_t u_{i-\frac{1}{2}}^{n-\frac{1}{2}} = \delta_x u_{i+\frac{1}{2}}^{n-\frac{1}{2}} - \frac{h}{2\nu}\delta_t u_{i+\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1 \le i \le M-1, \quad n \ge 1,$$

or

$$\frac{1}{2}(\delta_t u_{i-\frac{1}{2}}^{n-\frac{1}{2}} + \delta_t u_{i+\frac{1}{2}}^{n-\frac{1}{2}}) - \nu\delta_x^2 u_i^{n-\frac{1}{2}} = 0, \quad 1 \le i \le M-1, \quad n \ge 1,$$

which is (2.19).

When $i = 0$, from (2.16) and (2.24), we know that

$$\frac{2\sqrt{\nu}}{\sqrt{\pi}}[a_0 u_0^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})u_0^{k-\frac{1}{2}} - a_{n-1}u_0^0] = \nu\delta_x u_{\frac{1}{2}}^{n-\frac{1}{2}} - \frac{h}{2}\delta_t u_{\frac{1}{2}}^{n-\frac{1}{2}}.$$

Dividing by $h/2$ on the both sides we obtain (2.20).

Similarly, when $i = M$, from (2.17) and (2.23), we know that

$$-\frac{2\sqrt{\nu}}{\sqrt{\pi}}[a_0 u_M^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})u_M^{k-\frac{1}{2}} - a_{n-1}u_M^0] = \nu\delta_x u_{M-\frac{1}{2}}^{n-\frac{1}{2}} + \frac{h}{2}\delta_t u_{M-\frac{1}{2}}^{n-\frac{1}{2}},$$

Dividing by $h/2$ on the both sides we obtain (2.21). □

The difference scheme (2.18)-(2.21) can be sorted as the following:

$$(\frac{1}{2} - r)u_{i+1}^n + (1+2r)u_i^n + (\frac{1}{2} - r)u_{i-1}^n = (\frac{1}{2} + r)u_{i+1}^{n-1} + (1-2r)u_i^{n-1} + (\frac{1}{2} + r)u_{i-1}^{n-1}, 1 \le i \le M-1, \tag{2.25}$$

$$(1 + 2r + \frac{4\sqrt{r}}{\sqrt{\pi}})u_0^n + (1-2r)u_1^n = (1 - 2r - \frac{4\sqrt{r}}{\sqrt{\pi}})u_0^{n-1} + (1+2r)u_1^{n-1}$$
$$+ \frac{4\sqrt{r\tau}}{\sqrt{\pi}}\sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})(u_0^k + u_0^{k-1}) + \frac{8\sqrt{r\tau}}{\sqrt{\pi}}a_{n-1}u_0^0, \tag{2.26}$$

$$(1 + 2r + \frac{4\sqrt{r}}{\sqrt{\pi}})u_M^n + (1-2r)u_{M-1}^n = (1 - 2r - \frac{4\sqrt{r}}{\sqrt{\pi}})u_M^{n-1} + (1+2r)u_{M-1}^{n-1}$$
$$+ \frac{4\sqrt{r\tau}}{\sqrt{\pi}}\sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})(u_M^k + u_M^{k-1}) + \frac{8\sqrt{r\tau}}{\sqrt{\pi}}a_{n-1}u_M^0. \tag{2.27}$$

3 The error estimate of the difference scheme

Lemma 3.1 *For any* $F = \{F_1, F_2, F_3, \cdots\}$*, we have*

$$\sum_{l=1}^{n}[a_0F_l - \sum_{k=1}^{l-1}(a_{l-k-1} - a_{l-k})F_k]F_l \geq \frac{1}{2\sqrt{t_n}}\sum_{l=1}^{n} F_l^2, \quad n = 1, 2, \cdots,$$

where a_m *is defined in (2.22).*

Proof Let $b_m = a_{m-1} - a_m = \frac{1}{\sqrt{\tau}}(\frac{1}{\sqrt{m}+\sqrt{m-1}} - \frac{1}{\sqrt{m+1}+\sqrt{m}})$, $m \geq 1$, then $b_m > 0$, and

$$\begin{aligned}
&\sum_{l=1}^{n}[a_0F_l - \sum_{k=1}^{l-1}(a_{l-k-1} - a_{l-k})F_k]F_l \\
= & \sum_{l=1}^{n} a_0F_l^2 - \sum_{l=1}^{n}\sum_{m=1}^{l-1}(a_{m-1} - a_m)F_{l-m}F_l \\
\geq & \sum_{l=1}^{n} a_0F_l^2 - \frac{1}{2}\sum_{l=1}^{n}\sum_{m=1}^{l-1} b_m(F_{l-m}^2 + F_l^2) \\
= & \sum_{l=1}^{n} a_0F_l^2 - \frac{1}{2}\sum_{l=1}^{n}\sum_{m=1}^{l-1} b_{l-m}F_m^2 - \frac{1}{2}\sum_{l=1}^{n}\sum_{m=1}^{l-1} b_mF_l^2 \\
= & \sum_{l=1}^{n} a_0F_l^2 - \frac{1}{2}\sum_{m=1}^{n}\sum_{l=m+1}^{n} b_{l-m}F_m^2 - \frac{1}{2}\sum_{l=1}^{n}\sum_{m=1}^{l-1} b_mF_l^2 \\
\geq & \sum_{l=1}^{n} a_0F_l^2 - (\sum_{m=1}^{n-1} b_m)\sum_{l=1}^{n} F_l^2 \\
= & [\frac{1}{\sqrt{\tau}} - \frac{1}{\sqrt{\tau}}(1 - \frac{1}{\sqrt{n} + \sqrt{n-1}})]\sum_{l=1}^{n} F_l^2 \\
\geq & \frac{1}{2\sqrt{t_n}}\sum_{l=1}^{n} F_l^2. \quad \square
\end{aligned}$$

Lemma 3.2 *Suppose* $\{u_i^n\}$ *be the solution of*

$$\delta_t u_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \nu\delta_x v_{i-\frac{1}{2}}^{n-\frac{1}{2}} = P_{i-\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1 \leq i \leq M, \quad n \geq 1, \tag{3.1}$$

$$v_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \delta_x u_{i-\frac{1}{2}}^{n-\frac{1}{2}} = Q_{i-\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1 \leq i \leq M, \quad n \geq 1, \tag{3.2}$$

$$u_i^0 = \phi(x_i), \quad 0 \leq i \leq M, \tag{3.3}$$

$$v_0^{n-\frac{1}{2}} = \frac{2}{\sqrt{\pi\nu}}[a_0u_0^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})u_0^{k-\frac{1}{2}} - a_{n-1}u_0^0] + S^{n-\frac{1}{2}}, \quad n \geq 1, \tag{3.4}$$

$$v_M^{n-\frac{1}{2}} = -\frac{2}{\sqrt{\pi\nu}}[a_0u_M^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})u_M^{k-\frac{1}{2}} - a_{n-1}u_M^0] + T^{n-\frac{1}{2}}. \quad n \geq 1, \tag{3.5}$$

where Supp$\{\phi(x)\} \subset [x_0, x_M]$*, then*

$$\begin{aligned} \|u^n\|_A^2 &\le \exp(\tfrac{2T}{4-\tau}) \cdot \tfrac{1}{1-\frac{\tau}{4}}\{\|u^0\|_A^2 + \tfrac{\sqrt{\pi\nu t_n}}{2}\tau\textstyle\sum_{l=1}^{n}[(T^{l-\frac{1}{2}})^2 + (S^{l-\frac{1}{2}})^2] \\ &\quad +2\tau\textstyle\sum_{l=1}^{n}(\|P^{l-\frac{1}{2}}\|_A^2 + \|Q^{l-\frac{1}{2}}\|_A^2)\}, \quad n = 1, 2, \cdots . \end{aligned} \tag{3.6}$$

Proof Multiplying (3.1) by $2u_{i-\frac{1}{2}}^{n-\frac{1}{2}}$ and multiplying (3.2) by $2v_{i-\frac{1}{2}}^{n-\frac{1}{2}}$, then adding the results, we have

$$\begin{aligned} &\tfrac{1}{\tau}[(u_{i-\frac{1}{2}}^{n})^2 - (u_{i-\frac{1}{2}}^{n-1})^2] + 2(v_{i-\frac{1}{2}}^{n-\frac{1}{2}})^2 = \tfrac{2}{h}(u_i^{n-\frac{1}{2}}v_i^{n-\frac{1}{2}} - u_{i-1}^{n-\frac{1}{2}}v_{i-1}^{n-\frac{1}{2}}) + 2u_{i-\frac{1}{2}}^{n-\frac{1}{2}}P_{i-\frac{1}{2}}^{n-\frac{1}{2}} + 2v_{i-\frac{1}{2}}^{n-\frac{1}{2}}Q_{i-\frac{1}{2}}^{n-\frac{1}{2}} \\ &\le \tfrac{2}{h}(u_i^{n-\frac{1}{2}}v_i^{n-\frac{1}{2}} - u_{i-1}^{n-\frac{1}{2}}v_{i-1}^{n-\frac{1}{2}}) + \tfrac{1}{2}(u_{i-\frac{1}{2}}^{n-\frac{1}{2}})^2 + 2(P_{i-\frac{1}{2}}^{n-\frac{1}{2}})^2 + \tfrac{1}{2}(v_{i-\frac{1}{2}}^{n-\frac{1}{2}})^2 + 2(Q_{i-\frac{1}{2}}^{n-\frac{1}{2}})^2, \\ &\qquad 1 \le i \le M, n \ge 1. \end{aligned} \tag{3.7}$$

Multiplying the above inequality by τh and summing up for i from 1 to M, we obtain

$$\begin{aligned} (\|u^n\|_A^2 - \|u^{n-1}\|_A^2) + 2\tau\|v^{n-\frac{1}{2}}\|_A^2 &\le 2\tau(u_M^{n-\frac{1}{2}}v_M^{n-\frac{1}{2}} - u_0^{n-\frac{1}{2}}v_0^{n-\frac{1}{2}}) + \frac{\tau}{2}\|u^{n-\frac{1}{2}}\|_A^2 + \frac{\tau}{2}\|v^{n-\frac{1}{2}}\|_A^2 \\ &\quad +2\tau\|P^{n-\frac{1}{2}}\|_A^2 + 2\tau\|Q^{n-\frac{1}{2}}\|_A^2, \quad n \ge 1. \end{aligned} \tag{3.8}$$

Noticing $\frac{\tau}{2}\|u^{n-\frac{1}{2}}\|_A^2 \le \frac{\tau}{4}(\|u^n\|_A^2 + \|u^{n-1}\|_A^2)$, thus

$$\begin{aligned} \|u^l\|_A^2 - \|u^{l-1}\|_A^2 &\le 2\tau(u_M^{l-\frac{1}{2}}v_M^{l-\frac{1}{2}} - u_0^{l-\frac{1}{2}}v_0^{l-\frac{1}{2}}) + \frac{\tau}{4}(\|u^l\|_A^2 + \|u^{l-1}\|_A^2) \\ &\quad +2\tau\|P^{l-\frac{1}{2}}\|_A^2 + 2\tau\|Q^{l-\frac{1}{2}}\|_A^2, \quad l = 1, 2, \ldots, n. \end{aligned}$$

Summing up for l from 1 to n, we have

$$\begin{aligned} \|u^n\|_A^2 &\le \|u^0\|_A^2 + 2\tau\sum_{l=1}^{n}(u_M^{l-\frac{1}{2}}v_M^{l-\frac{1}{2}} - u_0^{l-\frac{1}{2}}v_0^{l-\frac{1}{2}}) \\ &\quad +\frac{\tau}{4}\|u^n\|_A^2 + \frac{\tau}{2}\sum_{l=0}^{n-1}\|u^l\|_A^2 + 2\tau\sum_{l=1}^{n}(\|P^{l-\frac{1}{2}}\|_A^2 + \|Q^{l-\frac{1}{2}}\|_A^2). \end{aligned}$$

Substituting (3.4) and (3.5) into the above inequality, and using Lemma 3.1, we have

$$\begin{aligned} \|u^n\|_A^2 &\le \frac{1}{1-\frac{\tau}{4}}[\|u^0\|_A^2 + 2\tau\sum_{l=1}^{n}(u_M^{l-\frac{1}{2}}v_M^{l-\frac{1}{2}} - u_0^{l-\frac{1}{2}}v_0^{l-\frac{1}{2}}) + 2\tau\sum_{l=1}^{n}(\|P^{l-\frac{1}{2}}\|_A^2 + \|Q^{l-\frac{1}{2}}\|_A^2) \\ &\quad +\frac{\tau}{2}\sum_{l=0}^{n-1}\|u^l\|_A^2] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{1-\frac{\tau}{4}}\|u^0\|_A^2 + \frac{2\tau}{1-\frac{\tau}{4}}\cdot(-\frac{2}{\sqrt{\pi\nu}})\sum_{l=1}^{n}[a_0 u_M^{l-\frac{1}{2}} - \sum_{k=1}^{l-1}(a_{l-k-1}-a_{l-k})u_M^{k-\frac{1}{2}}]u_M^{l-\frac{1}{2}} \\
&\quad + \frac{2\tau}{1-\frac{\tau}{4}}\sum_{l=1}^{n} u_M^{l-\frac{1}{2}} T^{l-\frac{1}{2}} - \frac{2\tau}{1-\frac{\tau}{4}}\cdot(\frac{2}{\sqrt{\pi\nu}})\sum_{l=1}^{n}[a_0 u_0^{l-\frac{1}{2}} - \sum_{k=1}^{l-1}(a_{l-k-1}-a_{l-k})u_0^{k-\frac{1}{2}}]u_0^{l-\frac{1}{2}} \\
&\quad + \frac{2\tau}{1-\frac{\tau}{4}}\sum_{l=1}^{n} u_0^{l-\frac{1}{2}} S^{l-\frac{1}{2}} + \frac{2\tau}{1-\frac{\tau}{4}}\sum_{l=1}^{n}(\|P^{l-\frac{1}{2}}\|_A^2 + \|Q^{l-\frac{1}{2}}\|_A^2) + \frac{2\tau}{4-\tau}\sum_{l=0}^{n-1}\|u^l\|_A^2 \\
&\leq \frac{1}{1-\frac{\tau}{4}}\|u^0\|_A^2 - \frac{2\tau}{1-\frac{\tau}{4}}\cdot\frac{2}{\sqrt{\pi\nu}}\cdot\frac{1}{2\sqrt{t_n}}\sum_{l=1}^{n}(u_M^{l-\frac{1}{2}})^2 + \frac{\tau}{1-\frac{\tau}{4}}(\frac{2}{\sqrt{\pi\nu t_n}}\sum_{l=1}^{n}(u_M^{l-\frac{1}{2}})^2 \\
&\quad + \frac{\sqrt{\pi\nu t_n}}{2}\sum_{l=1}^{n}(T^{l-\frac{1}{2}})^2) - \frac{2\tau}{1-\frac{\tau}{4}}\cdot\frac{2}{\sqrt{\pi\nu}}\cdot\frac{1}{2\sqrt{t_n}}\sum_{l=1}^{n}(u_0^{l-\frac{1}{2}})^2 \\
&\quad + \frac{\tau}{1-\frac{\tau}{4}}(\frac{2}{\sqrt{\pi\nu t_n}}\sum_{l=1}^{n}(u_0^{l-\frac{1}{2}})^2 + \frac{\sqrt{\pi\nu t_n}}{2}\sum_{l=1}^{n}(S^{l-\frac{1}{2}})^2) \\
&\quad + \frac{2\tau}{1-\frac{\tau}{4}}\sum_{l=1}^{n}(\|P^{l-\frac{1}{2}}\|_A^2 + \|Q^{l-\frac{1}{2}}\|_A^2) + \frac{2\tau}{4-\tau}\sum_{l=0}^{n-1}\|u^l\|_A^2 \\
&\leq \frac{1}{1-\frac{\tau}{4}}\|u^0\|_A^2 + \frac{\tau}{1-\frac{\tau}{4}}\frac{\sqrt{\pi\nu t_n}}{2}\sum_{l=1}^{n}[(T^{l-\frac{1}{2}})^2 + (S^{l-\frac{1}{2}})^2] \\
&\quad + \frac{2\tau}{1-\frac{\tau}{4}}\sum_{l=1}^{n}(\|P^{l-\frac{1}{2}}\|_A^2 + \|Q^{l-\frac{1}{2}}\|_A^2) + \frac{2\tau}{4-\tau}\sum_{l=0}^{n-1}\|u^l\|_A^2, \quad n = 1, 2, \cdots.
\end{aligned}
$$

Using Gronwall's lemma, we can obtain (3.6). □

Theorem 3.3 *The difference scheme (2.18)-(2.22) is uniquely solvable.*

Proof By Theorem 2.2, it suffices to prove that the difference scheme (2.13)-(2.17) is solvable uniquely. When initial value is homogeneous, by Lemma 3.2, we have $\|u^n\|_A^2 = 0, n = 1, 2, \cdots$. □

Theorem 3.4 *Let $\{u_i^n | 0 \leq i \leq M, n \geq 1\}$ be the solution of (2.18)-(2.22), then*

$$\|u^n\|_A^2 \leq \frac{\exp(\frac{2T}{4-\tau})}{1-\frac{\tau}{4}}\|u^0\|_A^2, \quad n = 1, 2, \cdots. \tag{3.9}$$

Proof From Theorem 2.2, it suffices to prove that (3.9) hold for the difference scheme (2.13)-(2.17). Therefore, (3.9) follows directly from Lemma 3.2. □

Theorem 3.5 *Suppose (1.4)-(1.6) have solution $u(x,t) \in C_{x,t}^{4,3}(\mathbb{R}\times[0,T])$. Let $\{u_i^n\}$ be the solution of (2.18)-(2.22), and let $\tilde{u}_i^n = U_i^n - u_i^n$, then*

$$\|\tilde{u}^n\|_A^2 \leq \frac{CT}{4-\tau}(\sqrt{\pi\nu T}+4)\exp(\frac{2T}{4-\tau})(\tau^{\frac{3}{2}} + h^2)^2, \quad n = 1, 2, \cdots, [T/\tau], \tag{3.10}$$

where C is a constant independent of τ and h.

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Proof Subtracting (2.13)-(2.17) from (2.6)-(2.10), respectively, we obtain the error equations:

$$\delta_t\tilde{u}_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \nu\delta_x\tilde{v}_{i-\frac{1}{2}}^{n-\frac{1}{2}} = p_{i-\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1 \le i \le M, \quad n \ge 1, \tag{3.11}$$

$$\tilde{v}_{i-\frac{1}{2}}^{n-\frac{1}{2}} - \delta_x\tilde{u}_{i-\frac{1}{2}}^{n-\frac{1}{2}} = q_{i-\frac{1}{2}}^{n-\frac{1}{2}}, \quad 1 \le i \le M, \quad n \ge 1, \tag{3.12}$$

$$\tilde{u}_i^0 = 0, \quad 0 \le i \le M, \tag{3.13}$$

$$\tilde{v}_0^{n-\frac{1}{2}} = \frac{2}{\sqrt{\pi\nu}}[a_0\tilde{u}_0^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})\tilde{u}_0^{k-\frac{1}{2}} - a_{n-1}\tilde{u}_0^0] + s^{n-\frac{1}{2}}, \quad n \ge 1, \tag{3.14}$$

$$\tilde{v}_M^{n-\frac{1}{2}} = -\frac{2}{\sqrt{\pi\nu}}[a_0\tilde{u}_M^{n-\frac{1}{2}} - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})\tilde{u}_M^{k-\frac{1}{2}} - a_{n-1}\tilde{u}_M^0] + t^{n-\frac{1}{2}}, \quad n \ge 1. \tag{3.15}$$

Using Lemma 3.2 and applying (2.11), (2.12) and (3.13), we obtain

$$\begin{aligned}\|\tilde{u}^n\|_A^2 &= \exp(\frac{2T}{4-\tau}) \cdot \frac{1}{1-\frac{\tau}{4}} \cdot \{\|\tilde{u}^0\|_A^2 + \frac{\sqrt{\pi\nu t_n}}{2}\tau\sum_{l=1}^{n}[(t^{l-\frac{1}{2}})^2 + (s^{l-\frac{1}{2}})^2] \\ &\quad +2\tau\sum_{l=1}^{n}(\|p^{l-\frac{1}{2}}\|^2 + \|q^{l-\frac{1}{2}}\|^2)\} \\ &\le \frac{CT}{4-\tau}(\sqrt{\pi\nu T}+4)\exp(\frac{2T}{4-\tau})(\tau^{\frac{3}{2}} + h^2)^2, \quad n = 1, 2, \cdots, [T/\tau]. \quad \square\end{aligned}$$

Theorem 3.5 shows that the convergence order of (2.18)-(2.21) is 2 in space and 3/2 in time for the problem (1.7)-(1.10). Finally, the numerical solution of Burgers' equation is obtained by

$$w_i^n = -\frac{\nu}{h}\frac{u_{i+1}^n - u_{i-1}^n}{1 + u_i^n}, \tag{3.16}$$

which keeps the corresponding unique solvability, unconditional stability and convergence.

4 The numerical example

For the problem of Burgers' equation with an initial condition $f(x) = -\frac{8\nu x(x^2-9)}{(x^2-9)^2+1}$ in the support $[x_l, x_r] = [-3, 3]$, the exact solution is $w(x,t) = 2\nu\frac{\frac{1}{2\sqrt{\pi\nu t}}\int_{-3}^{3}\frac{x-\xi}{2\nu t}(\xi^2-9)^2\exp(-\frac{(x-\xi)^2}{4\nu t})d\xi}{1+\frac{1}{2\sqrt{\pi\nu t}}\int_{-3}^{3}(\xi^2-9)^2\exp(-\frac{(x-\xi)^2}{4\nu t})d\xi}$. The numerical solutions are obtained by the proposed scheme, then the convergence order w.r.t h is shown in Table 1, and the convergence order w.r.t τ is shown in Table 2.

Table 1. Convergence w.r.t. h of the problem for $T = 1$, $\nu = 0.1$, $\tau = 0.01$ and $\tau = h^{4/3}$ respectively.

M	N	L^∞-error	order	L^2-error	order	N	L^∞-error	order	L^2-error	order
50	100	2.2705e-3	—	2.0737e-3	—	9	3.1455e-3	—	2.5729e-3	—
100	100	6.0651e-4	1.9044	5.5643e-4	1.8979	22	7.6893e-4	2.0324	6.5174e-4	1.9810
200	100	1.6444e-4	1.8830	1.4962e-4	1.8949	54	1.8419e-4	2.0617	1.6620e-4	1.9714
400	100	5.0024e-5	1.7169	4.5653e-5	1.7125	137	4.5577e-5	2.0148	4.1607e-5	1.9980
800	100	3.0569e-5	0.7106	1.9714e-5	1.2115	345	1.1295e-5	2.0126	1.0393e-5	2.0012

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Table 2. Convergence w.r.t. τ of the problem for $T = 1$, $\nu = 0.1$, $h = 0.002$ and $h = \tau^{3/4}$ respectively.

N	M	L^∞-error	order	L^2-error	order	M	L^∞-error	order	L^2-error	order
20	3000	1.0398e-3	—	2.1342e-4	—	95	8.7265e-4	—	7.2610e-4	—
40	3000	3.6910e-4	1.4942	6.2138e-5	1.7801	159	2.9735e-4	1.5532	2.6197e-4	1.4708
80	3000	1.0386e-4	1.8294	1.8884e-5	1.7183	267	1.0258e-4	1.5354	9.3238e-5	1.4904
160	3000	2.6518e-5	1.9696	6.3713e-6	1.5675	450	3.5936e-5	1.5132	3.2868e-5	1.5042
320	3000	1.5322e-5	0.7914	2.6822e-6	1.2482	757	1.2623e-5	1.5094	1.1614e-5	1.5008

5 Conclusions

In this works, a new finite difference method for Burgers' equation in the unbounded domain is presented by (2.18), (2.25)-(2.27) and (3.16) succinctly. The inequality in Lemma 2.1 is slightly stronger than Lemma 1 in [8]. Lemma 3.2 is proved by using Gronwall's lemma, but for heat equation in the semi-infinite domain, similar Lemma 4 in [8], i.e. Lemma 3.2.4 in [12], was incorrectly proved by not using Gronwall's lemma, and the lemma can be modified and proved as Lemma 3.2. Finally, the proposed method is clearly proved and verified to be uniquely solvable, unconditionally stable and convergent with the order 2 in space and the order 3/2 in time to solve Burgers' equation in the unbounded domain.

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Barnes-type Peters polynomials associated with poly-Cauchy polynomials of the second kind

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Abstract

In this paper, by considering Barnes-type Peters polynomials of the second kind as well as poly-Cauchy polynomials of the second kind, we define and investigate the mixed-type polynomials of these polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

1 Introduction

In this paper, we consider the polynomials

$$\widehat{s}_n^{(k)}(x) = \widehat{s}_n^{(k)}(x|\lambda;\mu) = \widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)$$

called the Barnes-type Peters of the second kind and poly-Cauchy of the second kind mixed-type polynomials, whose generating function is given by

$$\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^x = \sum_{n=0}^{\infty}\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^n}{n!}\,, \tag{1}$$

where $\lambda_1,\dots,\lambda_r,\mu_1,\dots,\mu_r\in\mathbb{C}$ with $\lambda_1,\dots,\lambda_r\neq 0$. Here, $\mathrm{Lif}_k(x)$ $(k\in\mathbb{Z})$ is the polyfactorial function ([8]) defined by

$$\mathrm{Lif}_k(x) = \sum_{m=0}^{\infty}\frac{x^m}{m!(m+1)^k}\,.$$

When $x=0$, $\widehat{s}_n^{(k)} = \widehat{s}_n^{(k)}(0) = \widehat{s}_n^{(k)}(0|\lambda;\mu) = \widehat{s}_n^{(k)}(0;\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)$ are called the the Barnes-type Peters of the second kind and poly-Cauchy of the second kind mixed-type numbers.

Recall that the Barnes-type Peters polynomials of the second kind, denoted by $\widehat{s}_n(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)$, are given by the generating function as

$$\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}(1+t)^x = \sum_{n=0}^{\infty}\widehat{s}_n(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^n}{n!}\,.$$

If $r=1$, then $\widehat{s}_n(x|\lambda;\mu)$ are the Peters polynomials of the second kind. Peters polynomials were mentioned in [12, p.128] and have been investigated in e.g. [7].

The poly-Cauchy polynomials of the second kind, denoted by $\widehat{c}_n^{(k)}(x)$ ([6, 9]), are given by the generating function as

$$\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^x = \sum_{n=0}^{\infty}\widehat{c}_n^{(k)}(x)\frac{t^n}{n!}\,.$$

The generalized Barnes-type Euler polynomials $E_n(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)$ are defined by the generating function

$$\prod_{j=1}^{r}\left(\frac{2}{1+e^{\lambda_j t}}\right)^{\mu_j}e^{xt} = \sum_{n=0}^{\infty}E_n(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^n}{n!}\,.$$

If $\mu_1=\cdots=\mu_r=1$, then $E_n(x|\lambda_1,\dots,\lambda_r) = E_n(x|\lambda_1,\dots,\lambda_r;1,\dots,1)$ are called the Barnes-type Euler polynomials. If further $\lambda_1=\cdots=\lambda_r=1$, then $E_n^{(r)}(x) = E_n(x|1,\dots,1;1,\dots,1)$ are called the Euler polynomials of order r.

In this paper, by considering Barnes-type Peters polynomials of the second kind as well as poly-Cauchy polynomials of the second kind, we define and investigate the mixed-type polynomials of these polynomials. From the properties of Sheffer sequences of these polynomials arising from umbral calculus, we derive new and interesting identities.

2 Umbral calculus

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \middle| a_k \in \mathbb{C} \right\}. \tag{2}$$

Let $\mathbb{P} = \mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ is the action of the linear functional L on the polynomial $p(x)$, and we recall that the vector space operations on $\mathbb{P}^*$ are defined by $\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$, $\langle cL|p(x)\rangle = c\langle L|p(x)\rangle$, where c is a complex constant in $\mathbb{C}$. For $f(t) \in \mathcal{F}$, let us define the linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n, \quad (n \geq 0). \tag{3}$$

In particular,

$$\left\langle t^k|x^n\right\rangle = n!\delta_{n,k} \quad (n,k \geq 0), \tag{4}$$

where $\delta_{n,k}$ is the Kronecker's symbol.

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. That is, $L = f_L(t)$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the *umbral algebra* and the *umbral calculus* is the study of umbral algebra. The order $O(f(t))$ of a power series $f(t)(\neq 0)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $O(f(t)) = 1$, then $f(t)$ is called a *delta series*; if $O(f(t)) = 0$, then $f(t)$ is called an *invertible series*. For $f(t), g(t) \in \mathcal{F}$ with $O(f(t)) = 1$ and $O(g(t)) = 0$, there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\left\langle g(t)f(t)^k|s_n(x)\right\rangle = n!\delta_{n,k}$, for $n, k \geq 0$. Such a sequence $s_n(x)$ is called the *Sheffer sequence* for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$, (see [1, 4-12]).

For $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle \tag{5}$$

and

$$f(t) = \sum_{k=0}^{\infty} \left\langle f(t)|x^k\right\rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \left\langle t^k|p(x)\right\rangle \frac{x^k}{k!} \tag{6}$$

([12, Theorem 2.2.5]). Thus, by (6), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{and} \quad e^{yt}p(x) = p(x+y). \tag{7}$$

Sheffer sequences are characterized in the generating function ([12, Theorem 2.3.4]).

Lemma 1 *The sequence $s_n(x)$ is Sheffer for $(g(t), f(t))$ if and only if*

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} t^k \quad (y \in \mathbb{C}),$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

For $s_n(x) \sim \big(g(t), f(t)\big)$, we have the following equations ([12, Theorem 2.3.7, Theorem 2.3.5, Theorem 2.3.9]):

$$f(t)s_n(x) = ns_{n-1}(x) \quad (n \geq 1), \tag{8}$$

$$s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \left\langle g\big(\bar{f}(t)\big)^{-1} \bar{f}(t)^j | x^n \right\rangle x^j, \tag{9}$$

$$s_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x) p_{n-j}(y)\,, \tag{10}$$

where $p_n(x) = g(t)s_n(x)$.

Assume that $p_n(x) \sim \big(1, f(t)\big)$ and $q_n(x) \sim \big(1, g(t)\big)$. Then the transfer formula ([12, Corollary 3.8.2]) is given by

$$q_n(x) = x \left(\frac{f(t)}{g(t)} \right)^n x^{-1} p_n(x) \quad (n \geq 1).$$

For $s_n(x) \sim \big(g(t), f(t)\big)$ and $r_n(x) \sim \big(h(t), l(t)\big)$, assume that

$$s_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x) \quad (n \geq 0)\,.$$

Then we have ([12, p.132])

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{h\big(\bar{f}(t)\big)}{g\big(\bar{f}(t)\big)} l\big(\bar{f}(t)\big)^m \middle| x^n \right\rangle . \tag{11}$$

3 Main results

From the definition (1), $\widehat{s}_n^{(k)}(x|\lambda_1, \dots, \lambda_r; \mu_1, \dots, \mu_r)$ is the Sheffer sequence for the pair

$$g(t) = \prod_{j=1}^{r} \left(\frac{1 + e^{\lambda_j t}}{e^{\lambda_j t}} \right)^{\mu_j} \frac{1}{\mathrm{Lif}_k(-t)} \quad \text{and} \quad f(t) = e^t - 1.$$

So,

$$\widehat{s}_n^{(k)}(x|\lambda_1, \dots, \lambda_r; \mu_1, \dots, \mu_r) \sim \left(\prod_{j=1}^{r} \left(\frac{1 + e^{\lambda_j t}}{e^{\lambda_j t}} \right)^{\mu_j} \frac{1}{\mathrm{Lif}_k(-t)}, e^t - 1 \right) . \tag{12}$$

3.1 Explicit expressions

Let $(n)_j = n(n-1)\cdots(n-j+1)$ $(j \geq 1)$ with $(n)_0 = 1$. The (signed) Stirling numbers of the first kind $S_1(n,m)$ are defined by

$$(x)_n = \sum_{m=0}^{n} S_1(n,m)x^m \,.$$

Theorem 1 *Let $\lambda\mu = \sum_{j=1}^{r} \lambda_j \mu_j$. Then, we have*

$$\begin{aligned}
&\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \\
&= 2^{-\sum_{j=1}^{r}\mu_j} \sum_{m=0}^{n}\sum_{l=0}^{m} \frac{(-1)^l\binom{m}{l}}{(l+1)^k} S_1(n,m) E_{m-l}(x+\lambda\mu|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) &(13)\\
&= \sum_{j=0}^{n}\sum_{l=j}^{n}\binom{n}{l} S_1(l,j)\widehat{s}_{n-l}^{(k)} x^j &(14)\\
&= \sum_{j=0}^{n}\sum_{l=0}^{n-j}\sum_{i=0}^{l}\binom{n}{l}\binom{l}{i} S_1(n-l,j)\widehat{c}_i^{(k)}\widehat{s}_{l-i}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)x^j &(15)\\
&= \sum_{l=0}^{n}\binom{n}{l}\widehat{s}_{n-l}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\widehat{c}_l^{(k)}(x)\,, &(16)\\
&= \sum_{l=0}^{n}\binom{n}{l}\widehat{s}_l(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\widehat{c}_{n-l}^{(k)}\,. &(17)
\end{aligned}$$

Proof. Since

$$\prod_{j=1}^{r}\left(\frac{1+e^{\lambda_j t}}{e^{\lambda_j t}}\right)^{\mu_j} \frac{1}{\mathrm{Lif}_k(-t)}\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \sim (1, e^t-1) \tag{18}$$

and

$$(x)_n \sim (1, e^t-1)\,, \tag{19}$$

we have

$$\begin{aligned}
&\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\
&=\prod_{j=1}^{r}\left(\frac{e^{\lambda_j t}}{1+e^{\lambda_j t}}\right)^{\mu_j}\mathrm{Lif}_k(-t)(x)_n\\
&=\sum_{m=0}^{n}S_1(n,m)\prod_{j=1}^{r}\left(\frac{e^{\lambda_j t}}{1+e^{\lambda_j t}}\right)^{\mu_j}\mathrm{Lif}_k(-t)x^m\\
&=\sum_{m=0}^{n}S_1(n,m)\prod_{j=1}^{r}\left(\frac{e^{\lambda_j t}}{1+e^{\lambda_j t}}\right)^{\mu_j}\sum_{l=0}^{m}\frac{(-1)^l t^l}{l!(l+1)^k}x^m\\
&=\sum_{m=0}^{n}S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^l\binom{m}{l}}{(l+1)^k}\prod_{j=1}^{r}\left(\frac{e^{\lambda_j t}}{1+e^{\lambda_j t}}\right)^{\mu_j}x^{m-l}\\
&=2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^l\binom{m}{l}}{(l+1)^k}e^{\lambda\mu t}\prod_{j=1}^{r}\left(\frac{2}{1+e^{\lambda_j t}}\right)^{\mu_j}x^{m-l}\\
&=2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^l\binom{m}{l}}{(l+1)^k}E_{m-l}(x+\lambda\mu|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,.
\end{aligned}$$

So, we get (13).
By (9) with (12), we get

$$\begin{aligned}
&\left\langle g\big(\bar f(t)\big)^{-1}\bar f(t)^j|x^n\right\rangle\\
&=\left\langle\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^j\Big|x^n\right\rangle\\
&=\left\langle\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)\Big|j!\sum_{l=j}^{\infty}S_1(l,j)\frac{t^l}{l!}x^n\right\rangle\\
&=j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\left\langle\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)\Big|x^{n-l}\right\rangle\\
&=j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\left\langle\sum_{i=0}^{\infty}\widehat{s}_i^{(k)}\frac{t^i}{i!}\Big|x^{n-l}\right\rangle\\
&=j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\widehat{s}_{n-l}^{(k)}\,.
\end{aligned}$$

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On the other hand,

$$\begin{aligned}
&\left\langle g\big(\bar{f}(t)\big)^{-1}\bar{f}(t)^j|x^n\right\rangle \\
&= j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\left\langle\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\Big|\mathrm{Lif}_k\big(-\ln(1+t)\big)x^{n-l}\right\rangle \\
&= j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\sum_{i=0}^{n-l}\binom{n-l}{i}\widehat{c}_i^{(k)}\left\langle\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\Big|x^{n-l-i}\right\rangle \\
&= j!\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\sum_{i=0}^{n-l}\binom{n-l}{i}\widehat{c}_i^{(k)}\left\langle\sum_{m=0}^{\infty}\widehat{s}_m(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^m}{m!}\Big|x^{n-l-i}\right\rangle \\
&= j!\sum_{l=j}^{n}\sum_{i=0}^{n-l}\binom{n}{l}\binom{n-l}{i}S_1(l,j)\widehat{c}_i^{(k)}\widehat{s}_{n-l-i}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\widehat{s}_n^{(k)}(x) &= \sum_{j=0}^{n}\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\widehat{s}_{n-l}^{(k)}x^j \\
&= \sum_{j=0}^{n}\sum_{l=0}^{n-j}\sum_{i=0}^{l}\binom{n}{l}\binom{l}{i}S_1(n-l,j)\widehat{c}_i^{(k)}\widehat{s}_{l-i}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)x^j\,,
\end{aligned}$$

which are the identities (14) and (15).

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Next,

$$
\begin{aligned}
\widehat{s}_n^{(k)}(y|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r) &= \left\langle \sum_{i=0}^{\infty} \widehat{s}_i^{(k)}(y|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)\frac{t^i}{i!} \Big| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \Big| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \Big| \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \Big| \sum_{l=0}^{\infty} \widehat{c}_l^{(k)}(y)\frac{t^l}{l!} x^n \right\rangle \\
&= \sum_{l=0}^{n} \binom{n}{l} \widehat{c}_l^{(k)}(y) \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \Big| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n} \binom{n}{l} \widehat{c}_l^{(k)}(y) \left\langle \sum_{i=0}^{\infty} \widehat{s}_i(\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)\frac{t^i}{i!} \Big| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n} \binom{n}{l} \widehat{c}_l^{(k)}(y) \widehat{s}_{n-l}(\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)\,.
\end{aligned}
$$

Thus, we obtain (16).
Finally, we obtain that

$$
\begin{aligned}
\widehat{s}_n^{(k)}(y|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r) &= \left\langle \sum_{i=0}^{\infty} \widehat{s}_i^{(k)}(y|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)\frac{t^i}{i!} \Big| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \Big| x^n \right\rangle \\
&= \left\langle \mathrm{Lif}_k\big(-\ln(1+t)\big) \Big| \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}} \right)^{\mu_j} (1+t)^y x^n \right\rangle \\
&= \left\langle \mathrm{Lif}_k\big(-\ln(1+t)\big) \Big| \sum_{l=0}^{\infty} \widehat{s}_l(y|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r)\frac{t^l}{l!} x^n \right\rangle \\
&= \sum_{l=0}^{n} \widehat{s}_l(y|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r) \binom{n}{l} \left\langle \mathrm{Lif}_k\big(-\ln(1+t)\big) \Big| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n} \widehat{s}_l(y|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r) \binom{n}{l} \left\langle \sum_{i=0}^{\infty} \widehat{c}_i^{(k)} \frac{t^i}{i!} \Big| x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n} \binom{n}{l} \widehat{s}_l(y|\lambda_1,\ldots,\lambda_r;\mu_1,\ldots,\mu_r) \widehat{c}_{n-l}^{(k)}\,.
\end{aligned}
$$

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Thus, we get the identity (17). ∎

3.2 Sheffer identity

Theorem 2

$$\widehat{s}_n^{(k)}(x+y|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)=\sum_{j=0}^{n}\binom{n}{j}\widehat{s}_j^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)(y)_{n-j}\,. \tag{20}$$

Proof. By (12) with

$$\begin{aligned}p_n(x)&=\prod_{j=1}^{r}\left(\frac{1+e^{\lambda_j t}}{e^{\lambda_j t}}\right)^{\mu_j}\frac{1}{\mathrm{Lif}_k(-t)}\widehat{s}_n(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\&=(x)_n\sim(1,e^t-1)\,,\end{aligned}$$

using (10), we have (20). ∎

3.3 Difference relations

Theorem 3

$$\begin{aligned}\widehat{s}_n^{(k)}(x+1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)-\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\=n\widehat{s}_{n-1}^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,.\end{aligned} \tag{21}$$

Proof. By (8) with (12), we get

$$(e^t-1)\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)=n\widehat{s}_{n-1}^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,.$$

By (7), we have (21). ∎

3.4 Recurrence

Theorem 4

$$\begin{aligned}
&\widehat{s}_{n+1}^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\
&= (x+\lambda\mu)\widehat{s}_n^{(k)}(x-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\
&\quad - 2^{-1-\sum_{j=1}^r \mu_j}\sum_{m=0}^n\sum_{l=0}^m\sum_{i=1}^r S_1(n,m)\frac{(-1)^{m-l}\binom{m}{l}}{(m-l+1)^k}\mu_i\lambda_i E_l(x+\lambda(\mu+e_i)-1|\lambda;\mu+e_i)\\
&\quad - 2^{-\sum_{j=1}^r \mu_j}\sum_{m=0}^n\sum_{l=0}^m \frac{(-1)^{m-l}\binom{m}{l}}{(m-l+2)^k}S_1(n,m)E_l(x+\lambda\mu-1|\lambda;\mu) \qquad (22)\\
&= (x+\mu\lambda)\widehat{s}_n^{(k)}(x-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\
&\quad - \frac{1}{2}\sum_{j=0}^n\sum_{l=0}^{n-j}\sum_{i=1}^r \binom{n}{l}\mu_i\lambda_i^{j+1}S_1(n-l,j)\widehat{s}_l^{(k)}E_j\left(\frac{x+\lambda_i-1}{\lambda_i}\right)\\
&\quad - 2^{-\sum_{j=1}^r \mu_j}\sum_{m=0}^n\sum_{l=0}^m \frac{(-1)^{m-l}\binom{m}{l}}{(m-l+2)^k}S_1(n,m)E_l(x+\lambda\mu-1|\lambda;\mu)\,, \qquad (23)
\end{aligned}$$

$\lambda\mu = \sum_{j=1}^r \lambda_i\mu_i$.

Remark. Comparing (22) and (23),

$$\begin{aligned}
&2^{-\sum_{j=1}^r \mu_j}\sum_{m=0}^n\sum_{l=0}^m\sum_{i=1}^r \frac{(-1)^{m-l}\binom{m}{l}}{(m-l+1)^k}\mu_i\lambda_i S_1(n,m)E_l\big(x+\lambda(\mu+e_i)-1|\lambda;\mu+e_i\big)\\
&= \sum_{j=0}^n\sum_{l=0}^{n-j}\sum_{i=1}^r \binom{n}{l}\mu_i\lambda_i^{j+1}S_1(n-l,j)\widehat{s}_l^{(k)}E_j\left(\frac{x+\lambda_i-1}{\lambda_i}\right)\,.
\end{aligned}$$

Proof. By applying

$$s_{n+1}(x) = \left(x-\frac{g'(t)}{g(t)}\right)\frac{1}{f'(t)}s_n(x) \qquad (24)$$

([12, Corollary 3.7.2]) with (12), we get

$$\begin{aligned}
&\widehat{s}_{n+1}^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\
&\qquad = x\widehat{s}_n^{(k)}(x-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) - e^{-t}\frac{g'(t)}{g(t)}\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,.
\end{aligned}$$

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Since

$$\begin{aligned}\frac{g'(t)}{g(t)} &= \big(\ln g(t)\big)' \\ &= \left(\sum_{i=1}^{r}\mu_i \ln(1+e^{\lambda_i t}) - \left(\sum_{i=1}^{r}\mu_i\lambda_i\right)t - \ln \mathrm{Lif}_k(-t)\right)' \\ &= \sum_{i=1}^{r}\frac{\mu_i\lambda_i e^{\lambda_i t}}{1+e^{\lambda_i t}} - \sum_{i=1}^{r}\mu_i\lambda_i + \frac{\mathrm{Lif}_k'(-t)}{\mathrm{Lif}_k(-t)},\end{aligned}$$

by (13), we have

$$\begin{aligned}&\frac{g'(t)}{g(t)}\widehat{s}_n^{(k)}(x) \\ &= \left(\sum_{i=1}^{r}\frac{\mu_i\lambda_i e^{\lambda_i t}}{1+e^{\lambda_i t}} - \sum_{i=1}^{r}\mu_i\lambda_i + \frac{\mathrm{Lif}_k'(-t)}{\mathrm{Lif}_k(-t)}\right)\widehat{s}_n^{(k)}(x) \\ &= 2^{-1-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}S_1(n,m)\frac{(-1)^l\binom{m}{l}}{(l+1)^k}\sum_{i=1}^{r}\mu_i\lambda_i e^{(\lambda\mu+\lambda_i)t}\frac{2}{1+e^{\lambda_i t}}\prod_{j=1}^{r}\left(\frac{2}{1+e^{\lambda_j t}}\right)^{\mu_j} x^{m-l} \\ &\quad -\lambda\mu\widehat{s}_n^{(k)}(x) + \sum_{m=0}^{n}S_1(n,m)\prod_{j=1}^{r}\left(\frac{e^{\lambda_j t}}{1+e^{\lambda_j t}}\right)^{\mu_j}\mathrm{Lif}_k'(-t)x^m . \qquad (25)\end{aligned}$$

The first term in (25) is

$$2^{-1-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=1}^{r}S_1(n,m)\frac{(-1)^l\binom{m}{l}}{(l+1)^k}\mu_i\lambda_i E_{m-l}\big(x+\lambda(\mu+e_i)|\lambda;\mu+e_i\big),$$

where $\lambda=(\lambda_1,\dots,\lambda_r)$, $\mu=(\mu_1,\dots,\mu_r)$ and $e_i=(\underbrace{0,\dots,0}_{i-1},1,\underbrace{0,\dots,0}_{r-i})$ $(i=1,2,\dots,r)$.

Since

$$\mathrm{Lif}_{k-1}(-t)-\mathrm{Lif}_k(-t)=\left(\frac{1}{2^k}-\frac{1}{2^{k-1}}\right)t+\cdots,$$

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the third term in (25) is

$$
\begin{aligned}
&2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}S_1(n,m)e^{\lambda\mu t}\mathrm{Lif}'_k(-t)\prod_{j=1}^{r}\left(\frac{2}{1+e^{\lambda_j t}}\right)^{\mu_j}x^m\\
&=2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}S_1(n,m)e^{\lambda\mu t}\frac{\mathrm{Lif}_{k-1}(-t)-\mathrm{Lif}_k(-t)}{-t}E_m(x|\lambda;\mu)\\
&=-2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}S_1(n,m)e^{\lambda\mu t}\big(\mathrm{Lif}_{k-1}(-t)-\mathrm{Lif}_k(-t)\big)\frac{E_{m+1}(x|\lambda;\mu)}{m+1}\\
&=-2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}\frac{S_1(n,m)}{m+1}e^{\lambda\mu t}\\
&\quad\times\left(\sum_{l=0}^{m+1}\frac{(-1)^l t^l}{l!(l+1)^{k-1}}E_{m+1}(x|\lambda;\mu)-\sum_{l=0}^{m+1}\frac{(-1)^l t^l}{l!(l+1)^{k}}E_{m+1}(x|\lambda;\mu)\right)\\
&=-2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}\frac{S_1(n,m)}{m+1}e^{\lambda\mu t}\\
&\quad\times\left(\sum_{l=0}^{m+1}\frac{(-1)^l\binom{m+1}{l}}{(l+1)^{k-1}}E_{m+1-l}(x|\lambda;\mu)-\sum_{l=0}^{m+1}\frac{(-1)^l\binom{m+1}{l}}{(l+1)^{k}}E_{m+1-l}(x|\lambda;\mu)\right)\\
&=-2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}\frac{S_1(n,m)}{m+1}e^{\lambda\mu t}\sum_{l=1}^{m+1}\frac{(-1)^l\binom{m+1}{l}l}{(l+1)^{k}}E_{m+1-l}(x|\lambda;\mu)\\
&=2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}S_1(n,m)\sum_{l=0}^{m}\frac{(-1)^{m-l}\binom{m}{l}}{(m+2-l)^{k}}E_l(x+\lambda\mu|\lambda;\mu)\,.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&\widehat{s}_{n+1}^{(k)}(x)=(x+\lambda\mu)\widehat{s}_{n}^{(k)}(x-1)\\
&\quad-2^{-1-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}\sum_{i=1}^{r}S_1(n,m)\frac{(-1)^{m-l}\binom{m}{l}}{(m+1-l)^{k}}\mu_i\lambda_i E_l\big(x+\lambda(\mu+e_i)-1|\lambda;\mu+e_i\big)\\
&\quad-2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}\frac{(-1)^{m-l}\binom{m}{l}}{(m+2-l)^{k}}S_1(n,m)E_l(x+\lambda\mu-1|\lambda;\mu)\,,
\end{aligned}
$$

which is (22).

On the other hand, by (14) with (22), we have

$$
\begin{aligned}
&\frac{g'(t)}{g(t)}\widehat{s}_n^{(k)}(x)\\
&=\left(\sum_{i=1}^{r}\frac{\mu_i\lambda_i e^{\lambda_i t}}{1+e^{\lambda_i t}}-\sum_{i=1}^{r}\mu_i\lambda_i+\frac{\mathrm{Lif}'_k(-t)}{\mathrm{Lif}_k(-t)}\right)\widehat{s}_n^{(k)}(x)\\
&=\frac{1}{2}\sum_{i=1}^{r}\mu_i\lambda_i e^{\lambda_i t}\frac{2}{1+e^{\lambda_i t}}\sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{l}S_1(n-l,j)\widehat{s}_l^{(k)}x^j\\
&\quad-\mu\lambda\widehat{s}_n^{(k)}(x)+2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}\frac{(-1)^{m-l}\binom{m}{l}}{(m+2-l)^k}S_1(n,m)E_l(x+\lambda\mu|\lambda;\mu)\,. \qquad (26)
\end{aligned}
$$

The first term in (26) is

$$
\begin{aligned}
&\frac{1}{2}\sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{l}S_1(n-l,j)\widehat{s}_l^{(k)}\sum_{i=1}^{r}\mu_i\lambda_i e^{\lambda_i t}\frac{2}{1+e^{\lambda_i t}}x^j\\
&=\frac{1}{2}\sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{l}S_1(n-l,j)\widehat{s}_l^{(k)}\sum_{i=1}^{r}\mu_i\lambda_i e^{\lambda_i t}\lambda_i^j E_j\left(\frac{x}{\lambda_i}\right)\\
&=\frac{1}{2}\sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{l}S_1(n-l,j)\widehat{s}_l^{(k)}\sum_{i=1}^{r}\mu_i\lambda_i^{j+1}E_j\left(\frac{x+\lambda_i}{\lambda_i}\right)\,.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&\widehat{s}_{n+1}^{(k)}(x)=(x+\mu\lambda)\widehat{s}_n^{(k)}(x-1)\\
&-\frac{1}{2}\sum_{j=0}^{n}\sum_{l=0}^{n-j}\sum_{i=1}^{r}\binom{n}{l}\mu_i\lambda_i^{j+1}S_1(n-l,j)\widehat{s}_l^{(k)}E_j\left(\frac{x+\lambda_i-1}{\lambda_i}\right)\\
&-2^{-\sum_{j=1}^{r}\mu_j}\sum_{m=0}^{n}\sum_{l=0}^{m}\frac{(-1)^{m-l}\binom{m}{l}}{(m+2-l)^k}S_1(n,m)E_l(x+\lambda\mu-1|\lambda;\mu)\,.
\end{aligned}
$$

which is (23). ∎

3.5 Differentiation

Theorem 5

$$
\frac{d}{dx}\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)=n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}\widehat{s}_l^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,. \qquad (27)
$$

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Proof. We shall use

$$\frac{d}{dx}\widehat{s}_n(x) = \sum_{l=0}^{n-1} \binom{n}{l} \left\langle \bar{f}(t)|x^{n-l}\right\rangle \widehat{s}_l(x)$$

(*Cf.* [12, Theorem 2.3.12]). Since

$$\begin{aligned}\left\langle \bar{f}(t)|x^{n-l}\right\rangle &= \left\langle \ln(1+t)|x^{n-l}\right\rangle \\ &= \left\langle \sum_{m=1}^{\infty} \frac{(-1)^{m-1}t^m}{m}\Big|x^{n-l}\right\rangle \\ &= \sum_{m=1}^{n-l} \frac{(-1)^{m-1}}{m}\left\langle t^m|x^{n-l}\right\rangle \\ &= \sum_{m=1}^{n-l} \frac{(-1)^{m-1}}{m}(n-l)!\delta_{m,n-l} \\ &= (-1)^{n-l-1}(n-l-1)!\,,\end{aligned}$$

with (12), we have

$$\begin{aligned}&\frac{d}{dx}\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \\ &= \sum_{l=0}^{n-1} \binom{n}{l} (-1)^{n-l-1}(n-l-1)!\widehat{s}_l^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \\ &= n!\sum_{l=0}^{n-1} \frac{(-1)^{n-l-1}}{l!(n-l)}\widehat{s}_l^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,,\end{aligned}$$

which is the identity (27). ∎

3.6 A more relation

The classical Cauchy numbers c_n of the first kind are defined by

$$\frac{t}{\ln(1+t)} = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}$$

(see e.g. [3, 8]).

Theorem 6

$$\begin{aligned}&\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \\ &= x\widehat{s}_{n-1}^{(k)}(x-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) + \frac{1}{n}\sum_{l=1}^{n}\binom{n}{l}c_{n-l}\big(\widehat{s}_l^{(k-1)}(x-1) - \widehat{s}_l^{(k)}(x-1)\big) \\ &\quad + \sum_{i=1}^{r}\mu_i\lambda_i\widehat{s}_{n-1}^{(k)}(x-\lambda_i-1|\lambda;\mu+e_i)\,. \qquad (28)\end{aligned}$$

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Proof. For $n \geq 1$, we have

$$\begin{aligned}
&\widehat{s}_n^{(k)}(y|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \\
&= \left\langle \sum_{l=0}^{\infty} \widehat{s}_l^{(k)}(y|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^l}{l!} \middle| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \middle| x^n \right\rangle \\
&= \left\langle \partial_t \left(\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y\right) \middle| x^{n-1} \right\rangle \\
&= \left\langle \left(\partial_t \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\right) \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^y \middle| x^{n-1} \right\rangle \\
&\quad + \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} \Big(\partial_t \mathrm{Lif}_k\big(-\ln(1+t)\big)\Big)(1+t)^y \middle| x^{n-1} \right\rangle \\
&\quad + \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} \mathrm{Lif}_k\big(-\ln(1+t)\big)\left(\partial_t(1+t)^y\right) \middle| x^{n-1} \right\rangle .
\end{aligned}$$

The third term is

$$\begin{aligned}
&y\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} \mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{y-1} \middle| x^{n-1} \right\rangle \\
&= y\widehat{s}_{n-1}^{(k)}(y-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r).
\end{aligned}$$

Since

$$\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big) - \mathrm{Lif}_k\big(-\ln(1+t)\big) = \left(\frac{1}{2^k} - \frac{1}{2^{k-1}}\right)t + \cdots,$$

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the second term is

$$\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} \frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)}{(1+t)\ln(1+t)}(1+t)^y \middle| x^{n-1}\right\rangle$$

$$= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} \frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)}{t}(1+t)^{y-1} \middle| \frac{t}{\ln(1+t)}x^{n-1}\right\rangle$$

$$= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} \frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)}{t}(1+t)^{y-1} \middle| \sum_{l=0}^{\infty} c_l\frac{t^l}{l!}x^{n-1}\right\rangle$$

$$= \sum_{l=0}^{n-1}\binom{n-1}{l}c_l\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} (1+t)^{y-1} \middle| \frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)}{t}x^{n-1-l}\right\rangle$$

$$= \sum_{l=0}^{n-1}\frac{1}{n-l}\binom{n-1}{l}c_l\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} (1+t)^{y-1} \middle| \left(\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)\right)x^{n-l}\right\rangle$$

$$= \sum_{l=0}^{n-1}\frac{1}{n-l}\binom{n}{l}c_l\left(\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} \mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)(1+t)^{y-1} \middle| x^{n-l}\right\rangle - \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} \mathrm{Lif}_{k}\big(-\ln(1+t)\big)(1+t)^{y-1} \middle| x^{n-l}\right\rangle\right)$$

$$= \frac{1}{n}\sum_{l=0}^{n-1}\binom{n}{l}c_l\left(\widehat{s}_{n-l}^{(k-1)}(y-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)-\widehat{s}_{n-l}^{(k)}(y-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\right)$$

$$= \frac{1}{n}\sum_{l=1}^{n}\binom{n}{l}c_{n-l}\left(\widehat{s}_{l}^{(k-1)}(y-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)-\widehat{s}_{l}^{(k)}(y-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\right).$$

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Since

$$\begin{aligned}&\partial_t \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\\&=\sum_{i=1}^{r}\mu_i\lambda_i(1+t)^{-\lambda_i-1}\frac{(1+t)^{\lambda_i}}{(1+(1+t)\lambda_i}\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j},\end{aligned}$$

the first term is

$$\begin{aligned}&\sum_{i=1}^{r}\mu_i\lambda_i\left\langle\frac{(1+t)^{\lambda_i}}{(1+(1+t)\lambda_i}\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)(1+t)^{y-\lambda_i-1}|x^{n-1}\right\rangle\\&=\sum_{i=1}^{r}\mu_i\lambda_i\widehat{s}_{n-1}^{(k)}(y-\lambda_i-1|\lambda;\mu+e_i)\,.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\&=x\widehat{s}_{n-1}^{(k)}(x-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)+\frac{1}{n}\sum_{l=1}^{n}\binom{n}{l}c_{n-l}\bigl(\widehat{s}_l^{(k-1)}(x-1)-\widehat{s}_l^{(k)}(x-1)\bigr)\\&\quad+\sum_{i=1}^{r}\mu_i\lambda_i\widehat{s}_{n-1}^{(k)}(x-\lambda_i-1|\lambda;\mu+e_i)\,,\end{aligned}$$

which is the identity (28). ∎

3.7 A relation including the Stirling numbers of the first kind

Theorem 7 *For $n-1\ge m\ge 1$, we have*

$$\begin{aligned}&m\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\widehat{s}_l^{(k)}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\&=(m-1)\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{s}_l^{(k)}(-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\&\quad+\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{s}_l^{(k-1)}(-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\&\quad+m\sum_{l=0}^{n-m-1}\sum_{i=1}^{r}\binom{n-1}{l}S_1(n-l-1,m)\mu_i\lambda_i\widehat{s}_l^{(k)}(-\lambda_i-1|\lambda;\mu+e_i)\,.\end{aligned}\tag{29}$$

Proof. We shall compute

$$\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\Big|x^n\right\rangle$$

in two different ways. On the one hand, it is equal to

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)\Big|\big(\ln(1+t)\big)^m x^n\right\rangle\\
&=\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)\Big|m!\sum_{l=0}^{\infty}S_1(l,m)\frac{t^l}{l!}x^n\right\rangle\\
&=m!\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)\Big|x^{n-l}\right\rangle\\
&=m!\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\left\langle \sum_{i=0}^{\infty}\widehat{s}_i^{(k)}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\frac{t^i}{i!}\Big|x^{n-l}\right\rangle\\
&=m!\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\widehat{s}_{n-l}^{(k)}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\
&=m!\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\widehat{s}_{l}^{(k)}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,.
\end{aligned}$$

On the other hand, it is equal to

$$\begin{aligned}
&\left\langle \partial_t\left(\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\right)\Big|x^{n-1}\right\rangle\\
&=\left\langle \left(\partial_t\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\Big|x^{n-1}\right\rangle\\
&\quad+\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\left(\partial_t\mathrm{Lif}_k\big(-\ln(1+t)\big)\right)\big(\ln(1+t)\big)^m\Big|x^{n-1}\right\rangle\\
&\quad+\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)\left(\partial_t\big(\ln(1+t)\big)^m\right)\Big|x^{n-1}\right\rangle\,. \qquad (30)
\end{aligned}$$

The third term of (30) is equal to

$$
\begin{aligned}
& m\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\middle|\big(\ln(1+t)\big)^{m-1}x^{n-1}\right\rangle \\
&= m\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\right| \\
&\qquad \left.(m-1)!\sum_{l=m-1}^{\infty}S_1(l,m-1)\frac{t^l}{l!}x^{n-1}\right\rangle \\
&= m!\sum_{l=m-1}^{n-1}\binom{n-1}{l}S_1(l,m-1) \\
&\quad \times\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-1}\middle|x^{n-1-l}\right\rangle \\
&= m!\sum_{l=m-1}^{n-1}\binom{n-1}{l}S_1(l,m-1)\widehat{s}_{n-1-l}^{(k)}(-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \\
&= m!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{s}_{l}^{(k)}(-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,.
\end{aligned}
$$

The second term of (30) is equal to

$$
\begin{aligned}
& \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\left(\frac{\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)-\mathrm{Lif}_k\big(-\ln(1+t)\big)}{(1+t)\ln(1+t)}\right)\big(\ln(1+t)\big)^{m}\middle|x^{n-1}\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_{k-1}\big(-\ln(1+t)\big)(1+t)^{-1}\middle|\big(\ln(1+t)\big)^{m-1}x^{n-1}\right\rangle \\
&\quad -\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_{k}\big(-\ln(1+t)\big)(1+t)^{-1}\middle|\big(\ln(1+t)\big)^{m-1}x^{n-1}\right\rangle \\
&= (m-1)!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{s}_{l}^{(k-1)}(-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \\
&\quad -(m-1)!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{s}_{l}^{(k)}(-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\,.
\end{aligned}
$$

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The first term of (30) is equal to

$$\begin{aligned}
&\left\langle \left(\partial_t \prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\right)\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m \Big| x^{n-1}\right\rangle \\
&=\sum_{i=1}^{r}\mu_i\lambda_i\left\langle \frac{(1+t)^{\lambda_i}}{1+(1+t)^{\lambda_i}}\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\right.\\
&\qquad\qquad \left.\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-\lambda_i-1}\Big|\big(\ln(1+t)\big)^m x^{n-1}\right\rangle\\
&=\sum_{i=1}^{r}\mu_i\lambda_i\left\langle \frac{(1+t)^{\lambda_i}}{1+(1+t)^{\lambda_i}}\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\right.\\
&\qquad\qquad \left.\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-\lambda_i-1}\Big| m!\sum_{l=m}^{\infty}S_1(l,m)\frac{t^l}{l!}x^{n-1}\right\rangle\\
&=m!\sum_{i=1}^{r}\mu_i\lambda_i\sum_{l=m}^{n-1}\binom{n-1}{l}S_1(l,m)\\
&\quad\times\left\langle \frac{(1+t)^{\lambda_i}}{1+(1+t)^{\lambda_i}}\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)(1+t)^{-\lambda_i-1}\Big| x^{n-1-l}\right\rangle\\
&=m!\sum_{i=1}^{r}\mu_i\lambda_i\sum_{l=m}^{n-1}\binom{n-1}{l}S_1(l,m)\widehat{s}_{n-1-l}^{(k)}(-\lambda_i-1|\lambda;\mu+e_i)\\
&=m!\sum_{l=0}^{n-m-1}\sum_{i=1}^{r}\binom{n-1}{l}S_1(n-1-l,m)\mu_i\lambda_i\widehat{s}_l^{(k)}(-\lambda_i-1|\lambda;\mu+e_i)\,.
\end{aligned}$$

Therefore, we get, for $n-1\geq m\geq 1$,

$$\begin{aligned}
&m!\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)\widehat{s}_l^{(k)}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\
&=m!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{s}_l^{(k)}(-1)\\
&\quad+(m-1)!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{s}_l^{(k-1)}(-1)\\
&\quad-(m-1)!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-l-1,m-1)\widehat{s}_l^{(k)}(-1)\\
&\quad+m!\sum_{l=0}^{n-m-1}\sum_{i=1}^{r}\binom{n-1}{l}S_1(n-l-1,m)\mu_i\lambda_i\widehat{s}_l^{(k)}(-\lambda_i-1|\lambda;\mu+e_i)\,.
\end{aligned}$$

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Dividing both sides by $(m-1)!$, we obtain, for $n-1 \geq m \geq 1$,

$$
\begin{aligned}
& m \sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m) \widehat{s}_l^{(k)}(\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \\
& = (m-1) \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1,m-1) \widehat{s}_l^{(k)}(-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \\
& \quad + \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1,m-1) \widehat{s}_l^{(k-1)}(-1|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) \\
& \quad + m \sum_{l=0}^{n-m-1} \sum_{i=1}^{r} \binom{n-1}{l} S_1(n-l-1,m) \mu_i \lambda_i \widehat{s}_l^{(k)}(-\lambda_i-1|\lambda;\mu+e_i)\,.
\end{aligned}
$$

Thus, we get (29). ∎

3.8 A relation with the falling factorials

Theorem 8

$$
\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) = \sum_{m=0}^{n} \binom{n}{m} \widehat{s}_{n-m}^{(k)}(x)_m\,. \tag{31}
$$

Proof. For (12) and (19), assume that $\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r) = \sum_{m=0}^{n} C_{n,m}(x)_m$. By (11), we have

$$
\begin{aligned}
C_{n,m} & = \frac{1}{m!} \left\langle \frac{1}{\prod_{j=1}^{r}\left(\frac{1+e^{\lambda_j \ln(1+t)}}{e^{\lambda_j \ln(1+t)}}\right)^{\mu_j}} \mathrm{Lif}_k\big(-\ln(1+t)\big) t^m \Big| x^n \right\rangle \\
& = \frac{1}{m!} \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} \mathrm{Lif}_k\big(-\ln(1+t)\big) \Big| t^m x^n \right\rangle \\
& = \binom{n}{m} \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j} \mathrm{Lif}_k\big(-\ln(1+t)\big) \Big| x^{n-m} \right\rangle \\
& = \binom{n}{m} \widehat{s}_{n-m}^{(k)}\,.
\end{aligned}
$$

Thus, we get the identity (31). ∎

3.9 A relation with higher-order Frobenius-Euler polynomials

For $\alpha \in \mathbb{C}$ with $\alpha \neq 1$, the Frobenius-Euler polynomials of order r, $H_n^{(r)}(x|\alpha)$ are defined by the generating function

$$
\left(\frac{1-\alpha}{e^t-\alpha}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\alpha) \frac{t^n}{n!}
$$

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(see e.g. [10]).

Theorem 9

$$\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)=\sum_{m=0}^{n}\left(\sum_{j=0}^{n-m}\sum_{l=0}^{n-m-j}\binom{s}{j}\binom{n-j}{l}(n)_j \times(1-\alpha)^{-j}S_1(n-j-l,m)\widehat{s}_l^{(k)}\right)H_m^{(s)}(x|\alpha)\,. \tag{32}$$

Proof. For (12) and

$$H_n^{(s)}(x|\alpha)\sim\left(\left(\frac{e^t-\alpha}{1-\alpha}\right)^s,t\right)\,, \tag{33}$$

assume that $\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)=\sum_{m=0}^{n}C_{n,m}H_m^{(s)}(x|\alpha)$. By (11), similarly to the proof of (29), we have

$$\begin{aligned}
C_{n,m}&=\frac{1}{m!}\left\langle\frac{\left(\frac{e^{\ln(1+t)}-\alpha}{1-\alpha}\right)^s}{\prod_{j=1}^{r}\left(\frac{1+e^{\lambda_j\ln(1+t)}}{e^{\lambda_j\ln(1+t)}}\right)^{\mu_j}}\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m\middle|x^n\right\rangle\\
&=\frac{1}{m!(1-\alpha)^s}\left\langle\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m(1-\alpha+t)^s\middle|x^n\right\rangle\\
&=\frac{1}{m!(1-\alpha)^s}\\
&\quad\times\left\langle\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m\middle|\sum_{i=0}^{\min\{s,n\}}\binom{s}{i}(1-\alpha)^{s-i}t^ix^n\right\rangle\\
&=\frac{1}{m!(1-\alpha)^s}\sum_{i=0}^{n-m}\binom{s}{i}(1-\alpha)^{s-i}(n)_i\\
&\quad\times\left\langle\prod_{j=1}^{r}\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\bigl(-\ln(1+t)\bigr)\bigl(\ln(1+t)\bigr)^m\middle|x^{n-i}\right\rangle\\
&=\frac{1}{m!(1-\alpha)^s}\sum_{i=0}^{n-m}\binom{s}{i}(1-\alpha)^{s-i}(n)_i\sum_{l=0}^{n-m-i}m!\binom{n-i}{l}S_1(n-i-l,m)\widehat{s}_l^{(k)}\\
&=\sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{s}{i}\binom{n-i}{l}(n)_i(1-\alpha)^{-i}S_1(n-i-l,m)\widehat{s}_l^{(k)}\,.
\end{aligned}$$

Thus, we get the identity (32). ∎

3.10 A relation with higher-order Bernoulli polynomials

Bernoulli polynomials $\mathfrak{B}_n^{(r)}(x)$ of order r are defined by

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^\infty\frac{\mathfrak{B}_n^{(r)}(x)}{n!}t^n$$

(see e.g. [12, Section 2.2]). In addition, Cauchy numbers of the first kind $\mathfrak{C}_n^{(r)}$ of order r are defined by

$$\left(\frac{t}{\ln(1+t)}\right)^r=\sum_{n=0}^\infty\frac{\mathfrak{C}_n^{(r)}}{n!}t^n$$

(see e.g. [2, (2.1)], [11, (6)]).

Theorem 10

$$\begin{aligned}&\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)\\&=\sum_{m=0}^n\left(\sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{n}{i}\binom{n-i}{l}\mathfrak{C}_i^{(s)}S_1(n-i-l,m)\widehat{s}_l^{(k)}\right)\mathfrak{B}_m^{(s)}(x)\,.\end{aligned}\tag{34}$$

Proof. For (12) and

$$\mathfrak{B}_n^{(s)}(x)\sim\left(\left(\frac{e^t-1}{t}\right)^s,t\right),\tag{35}$$

assume that $\widehat{s}_n^{(k)}(x|\lambda_1,\dots,\lambda_r;\mu_1,\dots,\mu_r)=\sum_{m=0}^n C_{n,m}\mathfrak{B}_m^{(s)}(x)$. By (11), similarly to the proof of (29), we have

$$\begin{aligned}C_{n,m}&=\frac{1}{m!}\left\langle\frac{\left(\frac{e^{\ln(1+t)}-1}{\ln(1+t)}\right)^s}{\prod_{j=1}^r\left(\frac{1+e^{\lambda_j\ln(1+t)}}{e^{\lambda_j\ln(1+t)}}\right)^{\mu_j}}\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\Big|x^n\right\rangle\\
&=\frac{1}{m!}\left\langle\prod_{j=1}^r\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\Big|\left(\frac{t}{\ln(1+t)}\right)^s x^n\right\rangle\\
&=\frac{1}{m!}\left\langle\prod_{j=1}^r\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\Big|\sum_{i=0}^\infty\mathfrak{C}_i^{(s)}\frac{t^i}{i!}x^n\right\rangle\\
&=\frac{1}{m!}\sum_{i=0}^{n-m}\mathfrak{C}_i^{(s)}\binom{n}{i}\left\langle\prod_{j=1}^r\left(\frac{(1+t)^{\lambda_j}}{1+(1+t)^{\lambda_j}}\right)^{\mu_j}\mathrm{Lif}_k\big(-\ln(1+t)\big)\big(\ln(1+t)\big)^m\Big|x^{n-i}\right\rangle\\
&=\frac{1}{m!}\sum_{i=0}^{n-m}\mathfrak{C}_i^{(s)}\binom{n}{i}\sum_{l=0}^{n-m-i}m!\binom{n-i}{l}S_1(n-i-l,m)\widehat{s}_l^{(k)}\\
&=\sum_{i=0}^{n-m}\sum_{l=0}^{n-m-i}\binom{n}{i}\binom{n-i}{l}\mathfrak{C}_i^{(s)}S_1(n-i-l,m)\widehat{s}_l^{(k)}\,.\end{aligned}$$

Thus, we get the identity (34). ∎

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On the solution for a system of two rational difference equations

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Abstract: This paper is concerned with the dynamical behavior and the expression of the solution for a system of two rational difference equations

$$x_{n+1}=\frac{x_{n-3}}{A+x_{n-3}y_{n-1}},\quad y_{n+1}=\frac{y_{n-3}}{B+y_{n-3}x_{n-1}},\quad n=0,1,\cdots,$$

where the parameters A, B and the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0$ are positive real numbers.

Keywords: difference equations; expression of solutions; recursive sequences, equilibrium point; asymptotical stability.

1. Introduction

Rational difference equations that are one of the most important and practical classes of nonlinear difference equations have applications in various scientific branches such as biology, ecology, physiology, physics, engineering and economics, etc [1-4]. Although difference equations are very simple in form, it is extremely difficult to understand thoroughly the behaviors of their solutions. So recently there has been an increasing interest in the study of qualitative analysis of rational difference equation and systems of difference equations [5-7]. In particular, Papaschinopoulos and Schinas [8] studied the oscillatory behavior, the boundedness of the solutions, and the global asymptotic stability of the positive equilibrium of the system of two nonlinear difference equations

$$x_{n+1} = A + \frac{y_n}{x_{n-p}}, \quad y_{n+1} = A + \frac{x_n}{y_{n-q}}, \quad n = 0,1,\cdots, \tag{1.1}$$

where p, q are positive integers. Clark and Kulenovic [9, 10] investigated the global stability properties and asymptotic behavior of solutions of the recursive sequences

$$x_{n+1} = \frac{x_n}{a + cy_n}, \qquad y_{n+1} = \frac{y_n}{b + dx_n}, \qquad n = 0,1,\cdots. \tag{1.2}$$

where $a, b, c, d \in (0, \infty)$ and the initial conditions x_0 and y_0 are arbitrary nonnegative numbers. The periodicity of the positive solutions of the system of rational difference equations

$$x_{n+1} = \frac{1}{y_n}, \qquad y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}, \qquad n = 0,1,\cdots, \tag{1.3}$$

was studied by Cinar in [11]. Yalcinkaya [12] has obtained the sufficient conditions for the global asymptotic stability of the system of two nonlinear difference equations

$$x_{n+1} = \frac{x_n + y_{n-1}}{x_n y_{n-1} - 1}, \qquad y_{n+1} = \frac{y_n + x_{n-1}}{y_n x_{n-1} - 1}, \qquad n = 0,1,\cdots. \tag{1.4}$$

More recently, Din et al. [13] studied the equilibrium points, local asymptotic stability of an equilibrium point, instability of equilibrium points, periodicity behavior of positive solutions, and global character of an equilibrium point of the following fourth-order system of rational difference equations

$$x_{n+1} = \frac{\alpha x_{n-3}}{\beta + \gamma y_n y_{n-1} y_{n-2} y_{n-3}}, \, y_{n+1} = \frac{\alpha_1 y_{n-3}}{\beta_1 + \gamma_1 x_n x_{n-1} x_{n-2} x_{n-3}}, \, n = 0,1,\cdots. \tag{1.5}$$

In [14], Elsayed deals with the form of the solutions of the following rational difference system

$$x_{n+1} = \frac{x_{n-1}}{\pm 1 + x_{n-1} y_n}, \quad y_{n+1} = \frac{y_{n-1}}{\mp 1 + y_{n-1} x_n}, \quad n = 0, 1, \cdots, \tag{1.6}$$

with nonzero real number initial conditions. Other related results on the difference equation can be found in references [15-28] and references therein.

Based on the above results, we are mainly interested in study the asymptotic behavior and the expression of the solution for the following nonlinear rational difference equations

$$x_{n+1} = \frac{x_{n-3}}{A + x_{n-3} y_{n-1}}, \quad y_{n+1} = \frac{y_{n-3}}{B + y_{n-3} x_{n-1}}, \quad n = 0,1,\cdots, \tag{1.7}$$

where the parameters A, B and the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0$ are positive real numbers.

This paper proceeds as follows. In Section 2, we introduce some definitions and preliminary results. The main results and their proofs are given in Section 3.

2. Preliminaries and notations

In this section we prepare some materials used throughout this paper, namely notations, the basic definitions and preliminary results. We refer to the monographs of Kocic et al. [5, 29, 30].

Lemma 2.1 Let I_x, I_y be some intervals of real numbers and $f: I_x^4 \times I_y^4 \to I_x$, $g: I_x^4 \times I_y^4 \to I_y$ be continuously differentiable functions. Then for every initial conditions $(x_i, y_i) \in I_x \times I_y, (i = -3, -2, -1, 0)$, the system of difference equations

$$\begin{cases} x_{n+1} = f(x_n, x_{n-1}, x_{n-2}, x_{n-3}, y_n, y_{n-1}, y_{n-2}, y_{n-3}), \\ y_{n+1} = g(x_n, x_{n-1}, x_{n-2}, x_{n-3}, y_n, y_{n-1}, y_{n-2}, y_{n-3}), \end{cases} \quad n = 0, 1, 2, \cdots, \tag{2.1}$$

has a unique solution $\{(x_n, y_n)\}_{n=-3}^{\infty}$.

Definition 2.1 A point $(\bar{x}, \bar{y}) \in I_x \times I_y$ is called an equilibrium point of system (2.1) if

$$\bar{x} = f(\bar{x}, \bar{x}, \bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{y}, \bar{y}), \ \bar{y} = g(\bar{x}.\bar{x}, \bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{y}, \bar{y}).$$

That is, $(x_n, y_n) = (\bar{x}, \bar{y})$ for all $n \geq 1$ when the initial conditions

$$(x_0, x_{-1}, x_{-2}, x_{-3}, y_0, y_{-1}, y_{-2}, y_{-3}) = (\bar{x}, \bar{x}, \bar{x}, \bar{x}, \bar{y}, \bar{y}, \bar{y}, \bar{y}).$$

Definition 2.2 Let $(\bar{x}, \bar{y})$ be an equilibrium point of system (2.1). Then

(1) The equilibrium $(\bar{x}, \bar{y})$ of system (2.1) is said to be stable relative to $I_x \times I_y$ if for every $\varepsilon > 0$, there exits $\delta > 0$ such that for any initial conditions $(x_i, y_i) \in I_x \times I_y$ $(i = -3, -2, -1, 0)$, with $\sum_{i=-3}^{0} |x_i - \bar{x}| < \delta$, $\sum_{i=-3}^{0} |y_i - \bar{y}| < \delta$ implies $|x_n - \bar{x}| < \varepsilon$, $|y_n - \bar{y}| < \varepsilon$.

(2) The equilibrium $(\bar{x}, \bar{y})$ of system (2.1) is called an attractor relative to $I_x \times I_y$ if for all $(x_i, y_i) \in I_x \times I_y$ $(i = -3, -2, -1, 0)$, $\lim_{n\to\infty} x_n = \bar{x}$ and $\lim_{n\to\infty} y_n = \bar{y}$ hold.

(3) The equilibrium $(\bar{x}, \bar{y})$ of system (2.1) is called asymptotically stable relative to $I_x \times I_y$ if it is stable and an attractor.

(4) The equilibrium $(\bar{x}, \bar{y})$ of system (2.1) is called unstable if it is not stable.

Definition 2.3 Let $(\bar{x}, \bar{y})$ be an equilibrium point of the system (2.1), and f and g are continuously differentiable functions at $(\bar{x}, \bar{y})$. The linearized system of system (2.1) about the equilibrium point $(\overline{x}, \overline{y})$ is

$$X_{n+1} = F(X_n) = F_J X_n$$

where $X_n = (x_n, x_{n-1}, x_{n-2}, x_{n-3}, y_n, y_{n-1}, y_{n-2}, y_{n-3})^T$ and F_J is a Jacobian matrix of the system (2.1) about the equilibrium point $(\overline{x}, \overline{y})$.

Lemma 2.2 Assume that $X_{n+1} = F(X_n)$, $n = 0, 1, \cdots$, is a system of difference equations and $\overline{X}$ is the equilibrium point of this system i.e., $F(\overline{X}) = \overline{X}$. If all eigenvalues of the Jacobian matrix F_J about $\overline{X}$ lie inside the open unit disk $|\lambda| < 1$, then $\overline{X}$ is locally asymptotically stable. If one of them has a modulus greater than one, then $\overline{X}$ is unstable.

3. Main results and their proofs

It is obviously, if $A > 1, B \neq 1 \text{ or } B > 1, A \neq 1$, then (0,0) is the unique equilibrium point of the system (1.7).

Theorem 3.1 Let $\{x_n, y_n\}_{n=-3}^{\infty}$ be positive solutions of system (1.7), then for all $k \geq 0$,

$$(1)\quad 0 \leq x_n \leq \begin{cases} \dfrac{x_{-3}}{A^{k+1}}, & n = 4k+1, \\ \dfrac{x_{-2}}{A^{k+1}}, & n = 4k+2, \\ \dfrac{x_{-1}}{A^{k+1}}, & n = 4k+3, \\ \dfrac{x_0}{A^{k+1}}, & n = 4k+4. \end{cases} \qquad (2)\quad 0 \leq y_n \leq \begin{cases} \dfrac{y_{-3}}{B^{k+1}}, & n = 4k+1, \\ \dfrac{y_{-2}}{B^{k+1}}, & n = 4k+2, \\ \dfrac{y_{-1}}{B^{k+1}}, & n = 4k+3, \\ \dfrac{y_0}{B^{k+1}}, & n = 4k+4. \end{cases} \tag{3.1}$$

Proof. This assertion is true for $k = 0$, Assume that it is true for $k = m$, then for $k = m+1$, we have

$$x_n = \begin{cases} x_{4(m+1)+1} \leq \dfrac{x_{4(m+1)\text{-}3}}{A} = \dfrac{x_{4m+1}}{A} \leq \dfrac{1}{A}\dfrac{x_{\text{-}3}}{A^{m+1}} = \dfrac{x_{-3}}{A^{(m+1)+1}}, & n = 4(m+1)+1; \\ x_{4(m+1)+2} \leq \dfrac{x_{4(m+1)+1\text{-}3}}{A} = \dfrac{x_{4m+2}}{A} \leq \dfrac{1}{A}\dfrac{x_{\text{-}2}}{A^{m+1}} = \dfrac{x_{-2}}{A^{(m+1)+1}}, & n = 4(m+1)+2, \\ x_{4(m+1)+3} \leq \dfrac{x_{4(m+1)+2\text{-}3}}{A} = \dfrac{x_{4m+3}}{A} \leq \dfrac{1}{A}\dfrac{x_{\text{-}1}}{A^{m+1}} = \dfrac{x_{-1}}{A^{(m+1)+1}}, & n = 4(m+1)+3, \\ x_{4(m+1)+4} \leq \dfrac{x_{4(m+1)+3\text{-}3}}{A} = \dfrac{x_{4m+4}}{A} \leq \dfrac{1}{A}\dfrac{x_0}{A^{m+1}} = \dfrac{x_0}{A^{(m+1)+1}}, & n = 4(m+1)+4. \end{cases}$$

$$y_n=\begin{cases} y_{4(m+1)+1}\le \dfrac{y_{4(m+1)\text{-}3}}{B}=\dfrac{y_{4m+1}}{B}\le \dfrac{1}{B}\dfrac{y_{\text{-}3}}{B^{m+1}}=\dfrac{y_{-3}}{B^{(m+1)+1}}, & n=4(m+1)+1;\\ y_{4(m+1)+2}\le \dfrac{y_{4(m+1)+1\text{-}3}}{B}=\dfrac{y_{4m+2}}{B}\le \dfrac{1}{B}\dfrac{y_{\text{-}2}}{B^{m+1}}=\dfrac{y_{-2}}{B^{(m+1)+1}}, & n=4(m+1)+2,\\ y_{4(m+1)+3}\le \dfrac{y_{4(m+1)+2\text{-}3}}{B}=\dfrac{y_{4m+3}}{B}\le \dfrac{1}{B}\dfrac{y_{\text{-}1}}{B^{m+1}}=\dfrac{y_{-1}}{B^{(m+1)+1}}, & n=4(m+1)+3,\\ y_{4(m+1)+4}\le \dfrac{y_{4(m+1)+3\text{-}3}}{B}=\dfrac{y_{4m+4}}{B}\le \dfrac{1}{B}\dfrac{y_{0}}{B^{m+1}}=\dfrac{y_{0}}{B^{(m+1)+1}}, & n=4(m+1)+4.\end{cases}$$

This completes our inductive proof.

Corollary 3.1 If $A>1, B>1$, then by Theorem 3.1 $\{(x_n, y_n)\}_{n=-3}^{\infty}$ the solutions of the system (1.7) exponentially converges to the equilibrium point（0, 0）.

Theorem 3.2 For the equilibrium point（0, 0）of the system (1.7), the following results hold:

(1) If $A>1, B>1$, then the equilibrium point（0,0）of the system (1.7) is locally asymptotically stable.

(2) If $A<1$ or $B<1$, then the equilibrium point（0,0）of the system (1.7) is unstable.

Proof. We can easily obtain that the linearized system of (1.7) about the equilibrium point（0, 0）is

$$\varphi_{n+1}=D\varphi_n \tag{3.2}$$

where

$$\varphi_n=\begin{bmatrix} x_n\\ x_{n-1}\\ x_{n-2}\\ x_{n-3}\\ y_n\\ y_{n-1}\\ y_{n-2}\\ y_{n-3}\end{bmatrix}, \qquad D=\begin{bmatrix} 0&0&0&\frac{1}{A}&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&\frac{1}{B}\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\end{bmatrix}$$

the characteristic equation of (3.2) is

$$f(\lambda)=(\lambda^4-\frac{1}{A})(\lambda^4-\frac{1}{B})=0. \tag{3.3}$$

(1) If $A>1, B>1$, then we have $|\frac{1}{A}|<1, |\frac{1}{B}|<1$, this shows that all the roots of characteristic equation (3.3) lie inside unit disk. So the unique equilibrium（0, 0）is

locally asymptotically stable.

(2) It is easy to see that if $A<1$ or $B<1$, then there exists at least one root λ of the characteristic equation (3.3) such that $|\lambda|>1$. Thus, the equilibrium (0, 0) of the system (1.7) is unstable when $A<1$ or $B<1$.

By Corollary 3.1 and Theorem 3.2, we have the following result.

Corollary 3.2 If $A>1, B>1$, then the equilibrium point (0, 0) is globally asymptotically stable.

Theorem 3.3 If $A=B=1$, then every solution of the system (1.7) is bounded when the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}$ and y_0 are positive real numbers.

Proof. It follows from Eq. (1.7) that

$$x_{n+1}=\frac{x_{n\text{-}3}}{1+x_{n\text{-}3}y_{n\text{-}1}}\le x_{n\text{-}3},\ y_{n+1}=\frac{y_{n\text{-}3}}{1+y_{n\text{-}3}x_{n\text{-}1}}\le y_{n\text{-}3}.$$

Then the subsequences

$$\{x_{4n-3}\}_{n=0}^{\infty},\ \{x_{4n-2}\}_{n=0}^{\infty},\ \{x_{4n-1}\}_{n=0}^{\infty},\ \{x_{4n}\}_{n=0}^{\infty}$$

are decreasing and so are bounded from above by M =max$\{x_{-3}, x_{-2}, x_{-1}, x_0\}$, also, the subsequences

$$\{y_{4n-3}\}_{n=0}^{\infty},\ \{y_{4n-2}\}_{n=0}^{\infty},\ \{y_{4n-1}\}_{n=0}^{\infty},\ \{y_{4n}\}_{n=0}^{\infty}$$

are decreasing and so are bounded from above by m =max$\{y_{-3}, y_{-2}, y_{-1}, y_0\}$. Hence, every solution of the system (1.7) is bounded for any positive initial conditions.

In next section, we study the expressions of the solutions for the systems (1.7) with the parameters $A=B$.

Theorem 3.4 If $A=B$, suppose that $\{(x_n, y_n)\}_{n=-3}^{\infty}$ are solutions of the system (1.7). Also, assume that $x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}$ and y_0 are arbitrary positive numbers and let $x_{-3}=a, x_{-2}=b, x_{-1}=c, x_0=d, y_{-3}=e, y_{-2}=f, y_{-1}=g, y_0=h$. Then

$$\begin{aligned}
x_{4n-3}&=a\prod_{i=0}^{n-1}\frac{A^{2i}+A^{2i-1}ag+\cdots+Aag+ag}{A^{2i+1}+A^{2i}ag+A^{2i-1}ag+\cdots+Aag+ag},\\
x_{4n-2}&=b\prod_{i=0}^{n-1}\frac{A^{2i}+A^{2i-1}bh+\cdots+Abh+bh}{A^{2i+1}+A^{2i}bh+A^{2i-1}bh+\cdots+Abh+bh},\\
x_{4n-1}&=c\prod_{i=0}^{n-1}\frac{A^{2i+1}+A^{2i}ce+\cdots+Ace+ce}{A^{2i+2}+A^{2i+1}ce+A^{2i}ce+\cdots+Ace+ce},\\
x_{4n}&=d\prod_{i=0}^{n-1}\frac{A^{2i+1}+A^{2i}df+\cdots+Adf+df}{A^{2i+2}+A^{2i+1}df+A^{2i}df+\cdots+Adf+df},
\end{aligned}\tag{3.4}$$

$$\begin{aligned}
y_{4n-3} &= e\prod_{i=0}^{n-1}\frac{A^{2i}+A^{2i-1}ce+\cdots+Ace+ce}{A^{2i+1}+A^{2i}ce+A^{2i-1}ce+\cdots+Ace+ce},\\
y_{4n-2} &= f\prod_{i=0}^{n-1}\frac{A^{2i}+A^{2i-1}df+\cdots+Adf+df}{A^{2i+1}+A^{2i}df+A^{2i-1}df+\cdots+Adf+df},\\
y_{4n-1} &= g\prod_{i=0}^{n-1}\frac{A^{2i+1}+A^{2i}ag+\cdots+Aag+ag}{A^{2i+2}+A^{2i+1}ag+A^{2i}ag+\cdots+Aag+ag},\\
y_{4n} &= h\prod_{i=0}^{n-1}\frac{A^{2i+1}+A^{2i}bh+\cdots+Abh+bh}{A^{2i+2}+A^{2i+1}bh+A^{2i}bh+\cdots+Abh+bh},
\end{aligned} \tag{3.5}$$

where $n = 1, 2, \cdots$.

Proof. If $A = B$, then the system (1.7) is reduced to

$$x_{n+1} = \frac{x_{n-3}}{A + x_{n-3}y_{n-1}}, \quad y_{n+1} = \frac{y_{n-3}}{A + y_{n-3}x_{n-1}}. \tag{3.6}$$

It is easy to prove that Eqs. (3.4) and (3.5) hold for $n = 1$. Now suppose that $k \in N, k > 1$ and that Eqs. (3.4) and (3.5) hold for $n = k - 1$. That is,

$$\begin{aligned}
x_{4k-7} &= a\prod_{i=0}^{k-2}\frac{A^{2i}+A^{2i-1}ag+\cdots+Aag+ag}{A^{2i+1}+A^{2i}ag+A^{2i-1}ag+\cdots+Aag+ag},\\
x_{4k-6} &= b\prod_{i=0}^{k-2}\frac{A^{2i}+A^{2i-1}bh+\cdots+Abh+bh}{A^{2i+1}+A^{2i}bh+A^{2i-1}bh+\cdots+Abh+bh},\\
x_{4k-5} &= c\prod_{i=0}^{k-2}\frac{A^{2i+1}+A^{2i}ce+\cdots+Ace+ce}{A^{2i+2}+A^{2i+1}ce+A^{2i}ce+\cdots+Ace+ce},\\
x_{4k-4} &= d\prod_{i=0}^{k-2}\frac{A^{2i+1}+A^{2i}df+\cdots+Adf+df}{A^{2i+2}+A^{2i+1}df+A^{2i}df+\cdots+Adf+df},\\
y_{4k-7} &= e\prod_{i=0}^{k-2}\frac{A^{2i}+A^{2i-1}ce+\cdots+Ace+ce}{A^{2i+1}+A^{2i}ce+A^{2i-1}ce+\cdots+Ace+ce},\\
y_{4k-6} &= f\prod_{i=0}^{k-2}\frac{A^{2i}+A^{2i-1}df+\cdots+Adf+df}{A^{2i+1}+A^{2i}df+A^{2i-1}df+\cdots+Adf+df},\\
y_{4k-5} &= g\prod_{i=0}^{k-2}\frac{A^{2i+1}+A^{2i}ag+\cdots+Aag+ag}{A^{2i+2}+A^{2i+1}ag+A^{2i}ag+\cdots+Aag+ag},\\
y_{4k-4} &= h\prod_{i=0}^{k-2}\frac{A^{2i+1}+A^{2i}bh+\cdots+Abh+bh}{A^{2i+2}+A^{2i+1}bh+A^{2i}bh+\cdots+Abh+bh}.
\end{aligned}$$

Then, it follows from Eq. (3.6) and our assumptions that

$$x_{4k-3}=\frac{x_{4k-7}}{A+x_{4k-7}y_{4k-5}}$$

$$=\frac{a\prod_{i=0}^{k-2}\frac{A^{2i}+A^{2i-1}ag+\cdots+Aag+ag}{A^{2i+1}+A^{2i}ag+A^{2i-1}ag+\cdots+Aag+ag}}{A+a\prod_{i=0}^{k-2}\frac{A^{2i}+A^{2i-1}ag+\cdots+Aag+ag}{A^{2i+1}+A^{2i}ag+A^{2i-1}ag+\cdots+Aag+ag}\,g\prod_{i=0}^{k-2}\frac{A^{2i+1}+A^{2i}ag+\cdots+Aag+ag}{A^{2i+2}+A^{2i+1}ag+A^{2i}ag+\cdots+Aag+ag}}$$

$$=\frac{a\prod_{i=0}^{k-2}\frac{A^{2i}+A^{2i-1}ag+\cdots+Aag+ag}{A^{2i+1}+A^{2i}ag+A^{2i-1}ag+\cdots+Aag+ag}}{A+ag\frac{1}{A^{2k-2}+A^{2k-1}ag+\cdots+Aag+ag}}$$

$$=a\prod_{i=0}^{k-2}\frac{A^{2i}+A^{2i-1}ag+\cdots+Aag+ag}{A^{2i+1}+A^{2i}ag+A^{2i-1}ag+\cdots+Aag+ag}\left(\frac{A^{2k-2}+A^{2k-1}ag+\cdots+Aag+ag}{A^{2k-1}+A^{2k-2}ag+\cdots+Aag+ag}\right)$$

$$=a\prod_{i=0}^{k-1}\frac{A^{2i}+A^{2i-1}ag+\cdots+Aag+ag}{A^{2i+1}+A^{2i}ag+A^{2i-1}ag+\cdots+Aag+ag}.$$

That is

$$x_{4k-3}=a\prod_{i=0}^{k-1}\frac{A^{2i}+A^{2i-1}ag+\cdots+Aag+ag}{A^{2i+1}+A^{2i}ag+A^{2i-1}ag+\cdots+Aag+ag}.$$

In addition to, by Eq. (3.6) and our assumptions one has

$$y_{4k-3}=\frac{y_{4k-7}}{A+y_{4k-7}x_{4k-5}}$$

$$=\frac{e\prod_{i=0}^{k-2}\frac{A^{2i}+A^{2i-1}ce+\cdots+Ace+ce}{A^{2i+1}+A^{2i}ce+A^{2i-1}ce+\cdots+Ace+ce}}{A+e\prod_{i=0}^{k-2}\frac{A^{2i}+A^{2i-1}ce+\cdots+Ace+ce}{A^{2i+1}+A^{2i}ce+A^{2i-1}ce+\cdots+Ace+ce}\,c\prod_{i=0}^{k-2}\frac{A^{2i+1}+A^{2i}ce+\cdots+Ace+ce}{A^{2i+2}+A^{2i+1}ce+A^{2i}ce+\cdots+Ace+ce}}$$

$$=\frac{e\prod_{i=0}^{k-2}\frac{A^{2i}+A^{2i-1}ce+\cdots+Ace+ce}{A^{2i+1}+A^{2i}ce+A^{2i-1}ce+\cdots+Ace+ce}}{A+ce\frac{1}{A^{2n-2}+A^{2n-1}ce+\cdots+Ace+ce}}$$

$$=e\prod_{i=0}^{k-2}\frac{A^{2i}+A^{2i-1}ce+\cdots+Ace+ce}{A^{2i+1}+A^{2i}ce+A^{2i-1}ce+\cdots+Ace+ce}\left(\frac{A^{2k-2}+A^{2k-1}ce+\cdots+Ace+ce}{A^{2k-1}+A^{2k-2}ce+\cdots+Ace+ce}\right)$$

$$=e\prod_{i=0}^{k-1}\frac{A^{2i}+A^{2i-1}ce+\cdots+Ace+ce}{A^{2i+1}+A^{2i}ce+A^{2i-1}ce+\cdots+Ace+ce}.$$

That is,

$$y_{4k-3} = e\prod_{i=0}^{k-1}\frac{A^{2i}+A^{2i-1}ce+\cdots+Ace+ce}{A^{2i+1}+A^{2i}ce+A^{2i-1}ce+\cdots+Ace+ce}.$$

Similarly, one can prove

$$x_{4k-2} = b\prod_{i=0}^{k-1}\frac{A^{2i}+A^{2i-1}bh+\cdots+Abh+bh}{A^{2i+1}+A^{2i}bh+A^{2i-1}bh+\cdots+Abh+bh},$$

$$x_{4k-1} = c\prod_{i=0}^{k-1}\frac{A^{2i+1}+A^{2i}ce+\cdots+Ace+ce}{A^{2i+2}+A^{2i+1}ce+A^{2i}ce+\cdots+Ace+ce},$$

$$x_{4k} = d\prod_{i=0}^{k-1}\frac{A^{2i+1}+A^{2i}df+\cdots+Adf+df}{A^{2i+2}+A^{2i+1}df+A^{2i}df+\cdots+Adf+df},$$

$$y_{4k-2} = f\prod_{i=0}^{k-1}\frac{A^{2i}+A^{2i-1}df+\cdots+Adf+df}{A^{2i+1}+A^{2i}df+A^{2i-1}df+\cdots+Adf+df},$$

$$y_{4k-1} = g\prod_{i=0}^{k-1}\frac{A^{2i+1}+A^{2i}ag+\cdots+Aag+ag}{A^{2i+2}+A^{2i+1}ag+A^{2i}ag+\cdots+Aag+ag},$$

$$y_{4k} = h\prod_{i=0}^{k-1}\frac{A^{2i+1}+A^{2i}bh+\cdots+Abh+bh}{A^{2i+2}+A^{2i+1}bh+A^{2i}bh+\cdots+Abh+bh}.$$

Hence, Eqs. (3.4) and (3.5) hold for $n = k$. The proof is complete according to the mathematical induction.

Corollary 3.3 If $A = B = 1$, suppose that $\{(x_n, y_n)\}_{n=-3}^{\infty}$ are solutions of the system (1.7). Also, assume that $x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}$ and y_0 are arbitrary positive numbers and let $x_{-3} = a, x_{-2} = b, x_{-1} = c, x_0 = d, y_{-3} = e, y_{-2} = f, y_{-1} = g, y_0 = h$, then one has

$$x_{4n-3} = a\prod_{i=0}^{n-1}\frac{1+2iag}{1+(2i+1)ag}, \quad x_{4n-2} = b\prod_{i=0}^{n-1}\frac{1+2ibh}{1+(2i+1)bh},$$

$$x_{4n-1} = c\prod_{i=0}^{n-1}\frac{1+(2i+1)ce}{1+(2i+2)ce}, \quad x_{4n} = d\prod_{i=0}^{n-1}\frac{1+(2i+1)df}{1+(2i+2)df},$$

$$y_{4n-3} = e\prod_{i=0}^{n-1}\frac{1+2ice}{1+(2i+1)ce}, \quad y_{4n-2} = f\prod_{i=0}^{n-1}\frac{1+2idf}{1+(2i+1)df},$$

$$y_{4n-1} = g\prod_{i=0}^{n-1}\frac{1+(2i+1)ag}{1+(2i+2)ag}, \quad y_{4n} = h\prod_{i=0}^{n-1}\frac{1+(2i+1)bh}{1+(2i+2)bh},$$

where $n = 1, 2, \cdots$.

4. Conclusions

It is obvious that the system of two rational difference equations (1.7) is the extension of the models in [9, 10, 13, 14]. In this paper, we investigated the globally asymptotically stable of the equilibrium point $(0,0)$ for the difference equation (1.7) with the parameters $A > 1, B > 1$, and the unstable of the equilibrium point $(0,0)$ with the parameter $A < 1$ or $B < 1$ using linearization method. Moreover, the expressions of solutions of the system (1.7) with the parameters $A=B$ are obtained according to the mathematical induction. This paper presents the use of a variational iteration method and mathematical induction for systems of nonlinear difference equations. This technique is a powerful tool for solving various difference equations and can also be applied to other nonlinear differential equations in mathematical physics. In addition, the sufficient conditions that we obtained are very simple, which provide flexibility for the application and analysis of nonlinear difference equation. In addition, the system can be used to analyze and describe the pier buffering isolation system.

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Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

On Distributions of Discrete Order Statistics

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Abstract. In this study, the joint distributions of order statistics of *innid* discrete random variables are expressed in the form of an integral. Then, the results related to *pf* and *df* are given.

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1. Introduction

The joint probability density function(*pdf*) and marginal *pdf* of order statistics of independent but not necessarily identically distributed(*innid*) random variables was derived by Vaughan and Venables[22] by means of permanents. In addition, Balakrishnan[3], and Bapat and Beg[8] obtained the joint *pdf* and distribution function(*df*) of order statistics of *innid* random variables by means of permanents. In the first of two papers, Balasubramanian et al.[5] obtained the distribution of single order statistic in terms of distribution functions of the minimum and maximum order statistics of some subsets of $\{X_1, X_2, ..., X_n\}$ where X_i's are *innid* random variables. Later, Balasubramanian et al.[6] generalized their previous results[5] to the case of the joint distribution function of several order statistics. Recurrence relationships among the distribution functions of order statistics arising from *innid* random variables were obtained by Cao and West[10]. Using multinomial arguments, the *pdf* of $X_{r:n+1}$ $(1 \le r \le n+1)$ was obtained by Childs and Balakrishnan[11] by adding another independent random variable to the original *n* variables $X_1, X_2, ..., X_n$. Also, Balasubramanian et al.[7] established the identities satisfied by distributions of order statistics from non-independent non-identical variables through operator methods based on the difference and differential operators. In a paper published in 1991, Beg[9] obtained several recurrence relations and identities for product moments of order statistics of *innid* random variables using permanents. Recently, Cramer et al.[13] derived the expressions for the distribution and density functions by Ryser's method and the distributions of maxima and minima based on permanents.

A multivariate generalization of classical order statistics for random samples from a continuous multivariate distribution was defined by Corley[12]. Guilbaud[17] expressed the probability of the functions of *innid* random vectors as a linear combination of probabilities of the functions of independent and identically distributed(*iid*) random vectors and thus also for order statistics of random variables. Expressions for generalized joint densities of order statistics of *iid* random variables in terms of Radon-Nikodym derivatives with respect to product measures based on *df* were derived by Goldie and Maller[16]. Several identities and recurrence relations for *pdf* and *df* of order statistics of *iid* random variables were established by numerous authors including Arnold et al.[1], Balasubramanian and Beg[4], David[14], and Reiss[21]. Furthermore, Arnold et al.[1], David[14], Gan and Bain[15], and Khatri[18] obtained the probability function(*pf*) and *df* of order statistics of *iid* random variables from a discrete parent. Balakrishnan[2] showed that several relations and identities that have been derived for order statistics from continuous distributions also hold for the discrete case. In a paper published in 1986, Nagaraja[19] explored the behavior of higher order conditional probabilities of order statistics in a attempt to understand the structure of discrete order statistics. Later, Nagaraja[20] considered some results on order statistics of a random sample taken from a discrete population.

In general, the distribution theory for order statistics is complex when the parent distribution is discrete. In this study, the joint distributions of p order statistics of *innid* discrete random variables are obtained as an p fold integral.

As far as we know, these approaches have not been considered in the framework of order statistics from *innid* discrete random variables.

From now on, the subscripts and superscripts are defined in the first place in which they are used and these definitions will be valid unless they are redefined.

Let $X_1, X_2, ..., X_n$ be *innid* discrete random variables and $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$ be the order statistics obtained by arranging the n X_i's in increasing order of magnitude. Let F_i and f_i be *df* and *pf* of X_i (i =1, 2,…, n), respectively. For notational convenience we write $\sum\limits_{z_1, z_2, ..., z_p}$, $\sum\limits_{m_p, k_p ..., m_1, k_1}$, $\int$ and $\int\limits_V$ instead of $\sum\limits_{z_1=0}^{x_1} \sum\limits_{z_2=z_1}^{x_2} \sum\limits_{z_3=z_2}^{x_3} ... \sum\limits_{z_p=z_{p-1}}^{x_p}$,

$$\sum_{m_p=0}^{n-r_p} \sum_{k_p=0}^{r_p-1-r_{p-1}} ... \sum_{m_2=0}^{r_3-1-r_2} \sum_{k_2=0}^{r_2-1-r_1} \sum_{m_1=0}^{r_2-1-r_1} \sum_{k_1=0}^{r_1-1} , \int_{F_{i_{r_1}}(x_1-)}^{F_{i_{r_1}}(x_1)} \int_{F_{i_{r_2}}(x_2-)}^{F_{i_{r_2}}(x_2)} ... \int_{F_{i_{r_p}}(x_p-)}^{F_{i_{r_p}}(x_p)} \text{ and } \int_0^{F_{i_{r_1}}(x_1)} \int_{v_{i_{r_1}}^{(1)}}^{F_{i_{r_2}}(x_2)} ... \int_{v_{i_{r_{(p-1)}}}^{(p-1)}}^{F_{i_{r_p}}(x_p)} \text{ in}$$

the expressions below, respectively $(x_i = 0, 1, 2, \ldots)$ $(z_0 = 0)$.

2. Theorems for distribution and probability functions

In this section, the theorems related to *pf* and *df* of $X_{r_1:n}, X_{r_2:n}, ..., X_{r_p:n}$ ($1 \le r_1 < r_2 < ... < r_p \le n$, p=1, 2,…, n) will be given. We will now express the following theorem for the joint *pf* of order statistics of *innid* discrete random variables.

Theorem 2.1.

$$f_{r_1, r_2, ..., r_p:n}(x_1, x_2, ..., x_p) = \text{D} \sum_P \int \left(\prod_{w=1}^{p+1} \prod_{l=r_{w-1}+1}^{r_w-1} [v_{i_l}^{(w)} - v_{i_l}^{(w-1)}] \right) \prod_{w=1}^{p} dv_{i_{r_w}}^{(w)} , \tag{2.1}$$

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where $x_1 < x_2 < ... < x_p$, $\sum_P$ denotes the sum over all $n!$ permutations $(i_1, i_2, ..., i_n)$ of (1,2,…,n), $\mathrm{D} = \prod_{w=1}^{p+1} [(r_w - r_{w-1} - 1)!]^{-1}$, $r_0 = 0$, $r_{p+1} = n+1$, $v_{i_l}^{(0)} = 0$, $v_{i_l}^{(p+1)} = 1$ and $v_{i_l}^{(w)} = [v_{i_{r_w}}^{(w)} - F_{i_{r_w}}(x_w -)] \dfrac{f_{i_l}(x_w)}{f_{i_{r_w}}(x_w)} + F_{i_l}(x_w -)$.

Proof. Consider the event

$$\{X_{r_1:n} = x_1, X_{r_2:n} = x_2, ..., X_{r_p:n} = x_p\}.$$

The above event can be realized mutually exclusive as follows: $r_1 - 1 - k_1$ observations are less than x_1, $k_w + 1 + m_w$ (w=1, 2,…, p) observations are equal to x_w, $r_\xi - 1 - k_\xi - m_{\xi-1} - r_{\xi-1}$ ($\xi = 2, 3, ..., p$) observations are in interval $(x_{\xi-1}, x_\xi)$ and $n - m_p - r_p$ observation exceed x_p. The probability function of the above event can be written as

$$f_{r_1, r_2, ..., r_p:n}(x_1, x_2, ..., x_p) = P\{X_{r_1:n} = x_1, X_{r_2:n} = x_2, ..., X_{r_p:n} = x_p\}. \tag{2.2}$$

(2.2) can be expressed as

$$f_{r_1, r_2, ..., r_p:n}(x_1, x_2, ..., x_p)$$

$$= \sum_{m_p, k_p ..., m_1, k_1} \mathrm{C} \sum_P \left(\prod_{w=1}^{p+1} \prod_{l=r_{w-1}+m_{w-1}+1}^{r_w-1-k_w} [F_{i_l}(x_w -) - F_{i_l}(x_{w-1})] \right) \prod_{w=1}^{p} \prod_{j=r_w-k_w}^{r_w+m_w} f_{i_j}(x_w), \tag{2.3}$$

where $\mathrm{C} = \left(\prod_{w=1}^{p+1} [(r_w - 1 - k_w - m_{w-1} - r_{w-1})!]^{-1} \right) \prod_{w=1}^{p} [(k_w + 1 + m_w)!]^{-1}$, $m_0 = 0$, $k_{p+1} = 0$, $F_{i_l}(x_0) = 0$, $F_{i_l}(x_{p+1} -) = 1$, $F_{i_l}(x_w -) = P(X_{i_l} < x_w)$ and $m_{w-1} + k_w \le r_w - r_{w-1} - 1$ (w=1, 2,…, p+1).

(2.3) can be written as

$$f_{r_1, r_2, ..., r_p:n}(x_1, x_2, ..., x_p) = \sum_{m_p, k_p ..., m_1, k_1} \mathrm{C} \sum_P \left(\prod_{w=1}^{p+1} \prod_{l=r_{w-1}+m_{w-1}+1}^{r_w-1-k_w} [F_{i_l}(x_w -) - F_{i_l}(x_{w-1})] \right)$$

$$\cdot\prod_{w=1}^{p}\frac{(k_w+1+m_w)!}{k_w!\,m_w!}\left(\prod_{j=r_w-k_w}^{r_w-1} f_{i_j}(x_w)\right) f_{i_{r_w}}(x_w)\left(\prod_{j=r_w+1}^{r_w+m_w} f_{i_j}(x_w)\right)\int_0^1 y_w^{k_w}(1-y_w)^{m_w}\,dy_w\ . \qquad (2.4)$$

Also, (2.4) can be clearly written as

$$f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p)=\sum_{m_p,k_p\ldots,m_1,k_1}\sum_{P}\int_0^1\int_0^1\ldots\int_0^1$$

$$\cdot\frac{1}{(r_1-1-k_1)!k_1!}F_{i_1}(x_1-)F_{i_2}(x_1-)\ldots F_{i_{r_1-1-k_1}}(x_1-)y_1 f_{i_{r_1-k_1}}(x_1)y_1 f_{i_{r_1-k_1+1}}(x_1)\ldots y_1 f_{i_{r_1-1}}(x_1)dy_1 f_{i_{r_1}}(x_1)$$

$$\cdot\frac{1}{(r_2-r_1-m_1-k_2-1)!m_1!k_2!}(1-y_1)f_{i_{r_1+1}}(x_1)(1-y_1)f_{i_{r_1+2}}(x_1)\ldots(1-y_1)f_{i_{r_1+m_1}}(x_1)[F_{i_{r_1+m_1+1}}(x_2-)-F_{i_{r_1+m_1+1}}(x_1)]\ldots$$

$$\cdot[F_{i_{r_2-k_2-1}}(x_2-)-F_{i_{r_2-k_2-1}}(x_1)]\,y_2 f_{i_{r_2-k_2}}(x_2)\,y_2 f_{i_{r_2-k_2+1}}(x_2)\ldots y_2 f_{i_{r_2-1}}(x_2)\,dy_2 f_{i_{r_2}}(x_2)\ldots$$

$$\cdot\frac{1}{(r_p-r_{p-1}-m_{p-1}-k_p-1)!m_{p-1}!k_p!}(1-y_{p-1})f_{i_{r_{p-1}+1}}(x_{p-1})(1-y_{p-1})f_{i_{r_{p-1}+2}}(x_{p-1})\ldots(1-y_{p-1})f_{i_{r_{p-1}+m_{p-1}}}(x_{p-1})$$

$$\cdot[F_{i_{r_{p-1}+m_{p-1}+1}}(x_{r_p}-)-F_{i_{r_{p-1}+m_{p-1}+1}}(x_{r_{p-1}})]\ldots[F_{i_{r_p-k_p-1}}(x_{r_p}-)-F_{i_{r_p-k_p-1}}(x_{r_{p-1}})]$$

$$\cdot y_p f_{i_{r_p-k_p}}(x_p)y_p f_{i_{r_p-k_p+1}}(x_p)\ldots y_p f_{i_{r_p-1}}(x_p)dy_p f_{i_{r_p}}(x_p)$$

$$\cdot\frac{1}{(n-r_p-m_p)!m_p!}(1-y_p)f_{i_{r_p+1}}(x_p)(1-y_p)f_{i_{r_p+2}}(x_p)\ldots(1-y_p)f_{i_{r_p+m_p}}(x_p)[1-F_{i_{r_p+m_p+1}}(x_p)]\ldots[1-F_{i_n}(x_p)].$$

The following expression can be written from the last identity.

$$f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p)=\sum_{m_p,k_p\ldots,m_1,k_1}\sum_{P}\int_0^1\int_0^1\ldots\int_0^1\left\{\prod_{w=1}^{p+1}\frac{1}{(r_w-1-k_w-m_{w-1}-r_{w-1})!\;m_{w-1}!\,k_w!}\right.$$

$$\cdot\left(\prod_{\ell_1=r_{w-1}+1}^{r_{w-1}+m_{w-1}}(1-y_{w-1})f_{i_{\ell_1}}(x_{w-1})\right)\left(\prod_{\ell_2=r_{w-1}+m_{w-1}+1}^{r_w-1-k_w}[F_{i_{\ell_2}}(x_w-)-F_{i_{\ell_2}}(x_{w-1})]\right)$$

$$\left.\cdot\left(\prod_{\ell_3=r_w-k_w}^{r_w-1}y_w f_{i_{\ell_3}}(x_w)\right)\right\}\prod_{w=1}^{p}f_{i_{r_w}}(x_w)dy_w\ . \qquad (2.5)$$

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In (2.5), if $v_{i_j}^{(w)} = y_w f_{i_j}(x_w) + F_{i_j}(x_w -)$, the following identity is obtained.

$$f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p) = \sum_{m_p,k_p\ldots,m_1,k_1} \sum_P \int_{F_{i_{r_1}}(x_1-)}^{F_{i_{r_1}}(x_1)} \int_{F_{i_{r_2}}(x_2-)}^{F_{i_{r_2}}(x_2)} \cdots \int_{F_{i_{r_p}}(x_p-)}^{F_{i_{r_p}}(x_p)} \left\{ \prod_{w=1}^{p+1} \frac{1}{(r_w - 1 - k_w - m_{w-1} - r_{w-1})!\; m_{w-1}!\, k_w!} \right.$$

$$\cdot \left(\prod_{\ell_1 = r_{w-1}+1}^{r_{w-1}+m_{w-1}} [F_{i_{\ell_1}}(x_{w-1}) - v_{i_{\ell_1}}^{(w-1)}] \right) \left(\prod_{\ell_2 = r_{w-1}+m_{w-1}+1}^{r_w - 1 - k_w} [F_{i_{\ell_2}}(x_w -) - F_{i_{\ell_2}}(x_{w-1})] \right)$$

$$\left. \cdot \left(\prod_{\ell_3 = r_w - k_w}^{r_w - 1} [v_{i_{\ell_3}}^{(w)} - F_{i_{\ell_3}}(x_w -)] \right) \right\} \prod_{w=1}^{p} dv_{i_{r_w}}^{(w)}. \qquad (2.6)$$

By considering

$$\sum_{\tau=0}^{n} \sum_{\xi=0}^{n} \sum_P \frac{1}{\xi!(n-\tau-\xi)!\,\tau!} \left(\prod_{\ell_1=1}^{\xi} G_{i_{\ell_1}}^{(1)}(x) \right) \left(\prod_{\ell_2=\xi+1}^{n-\tau} G_{i_{\ell_2}}^{(2)}(x) \right) \prod_{\ell_3=n-\tau+1}^{n} G_{i_{\ell_3}}^{(3)}(x)$$

$$= \frac{1}{n!} \sum_P \prod_{l=1}^{n} [G_{i_l}^{(1)}(x) + G_{i_l}^{(2)}(x) + G_{i_l}^{(3)}(x)], \qquad (2.7)$$

where $\tau + \xi \le n$ and using (2.7) for each m_{w-1} and k_w in (2.6), we get

$$f_{r_1,r_2,\ldots,r_p:n}(x_1,x_2,\ldots,x_p)$$

$$= \left(\prod_{w=1}^{p+1} \frac{1}{(r_w - r_{w-1} - 1)!} \right) \sum_P \int \left(\prod_{w=1}^{p+1} \prod_{l=r_{w-1}+1}^{r_w - 1} [F_{i_l}(x_w -) - F_{i_l}(x_{w-1}) + v_{i_l}^{(w)} - F_{i_l}(x_w -) + F_{i_l}(x_{w-1}) - v_{i_l}^{(w-1)}] \right) \prod_{w=1}^{p} dv_{i_{r_w}}^{(w)}.$$

Thus, the proof is completed.

Specially, in Theorem 2.1, by taking $p=2$, $n=3$, $r_1=1$, $r_2=2$, $v_{i_3}^{(2)} = [v_{i_2}^{(2)} - F_{i_2}(x_2 -)] \dfrac{f_{i_3}(x_2)}{f_{i_2}(x_2)} + F_{i_3}(x_2 -)$ and for $x_1 < x_2$,

$$f_{1,2:3}(x_1,x_2) = \sum_P \int_{F_{i_1}(x_1-)}^{F_{i_1}(x_1)} \int_{F_{i_2}(x_2-)}^{F_{i_2}(x_2)} [1 - v_{i_3}^{(2)}]\, dv_{i_2}^{(2)} dv_{i_1}^{(1)}$$

$$=\sum_P f_{i_1}(x_1)\left\{f_{i_2}(x_2)-[\frac{1}{2}f_{i_2}(x_2)\left[F_{i_2}(x_2)+F_{i_2}(x_2-)\right]-f_{i_2}(x_2)F_{i_2}(x_2-)]\frac{f_{i_3}(x_2)}{f_{i_2}(x_2)}-f_{i_2}(x_2)F_{i_3}(x_2-)\right\}$$

$$=f_1(x_1)\left\{f_2(x_2)+\frac{1}{2}f_3(x_2)F_2(x_2-)-\frac{1}{2}f_3(x_2)F_2(x_2)-f_2(x_2)F_3(x_2-)\right\}$$

$$+f_1(x_1)\left\{f_3(x_2)+\frac{1}{2}f_2(x_2)F_3(x_2-)-\frac{1}{2}f_2(x_2)F_3(x_2)-f_3(x_2)F_2(x_2-)\right\}$$

$$+f_2(x_1)\left\{f_3(x_2)+\frac{1}{2}f_1(x_2)F_3(x_2-)-\frac{1}{2}f_1(x_2)F_3(x_2)-f_3(x_2)F_1(x_2-)\right\}$$

$$+f_2(x_1)\left\{f_1(x_2)+\frac{1}{2}f_3(x_2)F_1(x_2-)-\frac{1}{2}f_3(x_2)F_1(x_2)-f_1(x_2)F_3(x_2-)\right\}$$

$$+f_3(x_1)\left\{f_1(x_2)+\frac{1}{2}f_2(x_2)F_1(x_2-)-\frac{1}{2}f_2(x_2)F_1(x_2)-f_1(x_2)F_2(x_2-)\right\}$$

$$+f_3(x_1)\left\{f_2(x_2)+\frac{1}{2}f_1(x_2)F_2(x_2-)-\frac{1}{2}f_1(x_2)F_2(x_2)-f_2(x_2)F_1(x_2-)\right\}.$$

Morever, the above identity in the *iid* case can be expressed as

$$f_{1,2:3}(x_1,x_2)=6f(x_1)f(x_2)-6f(x_1)f(x_2)F(x_2)+3f(x_1)f^2(x_2)\ .$$

This result is obtained, if $i=1$, $j=2$ and $n=3$ in equation (6) in [18].

In case $x_1<x_2<...<x_p$, $v_{i_{r_1}}^{(1)}\le v_{i_{r_2}}^{(2)}\le...\le v_{i_{r_p}}^{(p)}$ is automatically satisfied because of

$$F_{i_{r_1}}\left(x_1-\right)\le v_{i_{r_1}}^{(1)}\le F_{i_{r_1}}\left(x_1\right),\ F_{i_{r_2}}\left(x_2-\right)\le v_{i_{r_2}}^{(2)}\le F_{i_{r_2}}\left(x_2\right),\ldots,F_{i_{r_p}}\left(x_p-\right)\le v_{i_{r_p}}^{(p)}\le F_{i_{r_p}}\left(x_p\right).$$

Also, in case $x_1=x_2=...=x_p=x$, the integration region is over

$$F_{i_{r_1}}\left(x-\right)\le v_{i_{r_1}}^{(1)}\le v_{i_{r_2}}^{(2)}\le...\le v_{i_{r_p}}^{(p)}\le F_{i_{r_p}}\left(x\right),\ F_{i_{r_1}}\left(x-\right)\le v_{i_{r_1}}^{(1)}\le F_{i_{r_1}}\left(x\right),$$

$$F_{i_{r_2}}\left(x-\right)\le v_{i_{r_2}}^{(2)}\le F_{i_{r_2}}\left(x\right),\ldots,F_{i_{r_p}}\left(x-\right)\le v_{i_{r_p}}^{(p)}\le F_{i_{r_p}}\left(x\right)\ .$$

So, if $x_1 \le x_2 \le ... \le x_p$, it should be written $\int\int ...\int$ instead of $\int_{F_{i_{r_1}}(x_1-)}^{F_{i_{r_1}}(x_1)} \int_{F_{i_{r_2}}(x_2-)}^{F_{i_{r_2}}(x_2)} ... \int_{F_{i_{r_p}}(x_p-)}^{F_{i_{r_p}}(x_p)}$

in (2.1), where $\int\int ...\int$ is to be carried out over the region: $v_{i_{r_1}}^{(1)} \le v_{i_{r_2}}^{(2)} \le ... \le v_{i_{r_p}}^{(p)}$,

$$F_{i_{r_1}}(x_1-) \le v_{i_{r_1}}^{(1)} \le F_{i_{r_1}}(x_1),\ F_{i_{r_2}}(x_2-) \le v_{i_{r_2}}^{(2)} \le F_{i_{r_2}}(x_2), \dots, F_{i_{r_p}}(x_p-) \le v_{i_{r_p}}^{(p)} \le F_{i_{r_p}}(x_p) .$$

The proof was given only in case $x_1 < x_2 < ... < x_p$, the proof for case $x_1 \le x_2 \le ... \le x_p$ is omitted.

Specially, in Theorem 2.1, by taking $p = 2$, $n = 3$, $r_1 = 1$, $r_2 = 2$, $v_{i_3}^{(2)} = [v_{i_2}^{(2)} - F_{i_2}(x_2-)]\dfrac{f_{i_3}(x_2)}{f_{i_2}(x_2)} + F_{i_3}(x_2-)$ and for $x_1 = x_2 = x$,

$$f_{1,2:3}(x,x) = \sum_P \int_{F_{i_1}(x-)}^{F_{i_1}(x)} \int_{v_{i_1}^{(1)}}^{F_{i_2}(x)} [1 - v_{i_3}^{(2)}]\, dv_{i_2}^{(2)} dv_{i_1}^{(1)}$$

$$= \sum_P \Bigg\{ F_{i_2}(x) f_{i_1}(x) - \frac{1}{2}\left[F_{i_1}(x) + F_{i_1}(x-)\right] f_{i_1}(x) - \frac{1}{2} F_{i_2}^2(x) f_{i_1}(x) \frac{f_{i_3}(x)}{f_{i_2}(x)} + \frac{1}{6}\left[F_{i_1}^3(x) - F_{i_1}^3(x-)\right]\frac{f_{i_3}(x)}{f_{i_2}(x)}$$
$$+ F_{i_2}(x) F_{i_2}(x-) \frac{f_{i_1}(x) f_{i_3}(x)}{f_{i_2}(x)} - \frac{1}{2}\left[F_{i_1}(x) + F_{i_1}(x-)\right] f_{i_1}(x) F_{i_2}(x-) \frac{f_{i_3}(x)}{f_{i_2}(x)} - F_{i_2}(x) F_{i_3}(x-) f_{i_1}(x)$$
$$+ \frac{1}{2}\left[F_{i_1}(x) + F_{i_1}(x-)\right] f_{i_1}(x) F_{i_3}(x-) \Bigg\}$$

$$= \Bigg\{ F_2(x) f_1(x) - \frac{1}{2}\left[F_1(x) + F_1(x-)\right] f_1(x) - \frac{1}{2} F_2^2(x) f_1(x) \frac{f_3(x)}{f_2(x)} + \frac{1}{6}\left[F_1^3(x) - F_1^3(x-)\right]\frac{f_3(x)}{f_2(x)}$$
$$+ F_2(x) F_2(x-) \frac{f_1(x) f_3(x)}{f_2(x)} - \frac{1}{2}\left[F_1(x) + F_1(x-)\right] f_1(x) F_2(x-) \frac{f_3(x)}{f_2(x)} - F_2(x) F_3(x-) f_1(x)$$
$$+ \frac{1}{2}\left[F_1(x) + F_1(x-)\right] f_1(x) F_3(x-) \Bigg\}$$

$$+\Bigg\{F_3(x)f_1(x)-\frac{1}{2}\big[F_1(x)+F_1(x-)\big]f_1(x)-\frac{1}{2}F_3^2(x)f_1(x)\frac{f_2(x)}{f_3(x)}+\frac{1}{6}\big[F_1^3(x)-F_1^3(x-)\big]\frac{f_2(x)}{f_3(x)}$$
$$+F_3(x)F_3(x-)\frac{f_1(x)f_2(x)}{f_3(x)}-\frac{1}{2}\big[F_1(x)+F_1(x-)\big]f_1(x)F_3(x-)\frac{f_2(x)}{f_3(x)}-F_3(x)F_2(x-)f_1(x)$$
$$+\frac{1}{2}\big[F_1(x)+F_1(x-)\big]f_1(x)F_2(x-)\Bigg\}$$

$$+\Bigg\{F_1(x)f_2(x)-\frac{1}{2}\big[F_2(x)+F_2(x-)\big]f_2(x)-\frac{1}{2}F_1^2(x)f_2(x)\frac{f_3(x)}{f_1(x)}+\frac{1}{6}\big[F_2^3(x)-F_2^3(x-)\big]\frac{f_3(x)}{f_1(x)}$$
$$+F_1(x)F_1(x-)\frac{f_2(x)f_3(x)}{f_1(x)}-\frac{1}{2}\big[F_2(x)+F_2(x-)\big]f_2(x)F_1(x-)\frac{f_3(x)}{f_1(x)}-F_1(x)F_3(x-)f_2(x)$$
$$+\frac{1}{2}\big[F_2(x)+F_2(x-)\big]f_2(x)F_3(x-)\Bigg\}$$

$$+\Bigg\{F_3(x)f_2(x)-\frac{1}{2}\big[F_2(x)+F_2(x-)\big]f_2(x)-\frac{1}{2}F_3^2(x)f_2(x)\frac{f_1(x)}{f_3(x)}+\frac{1}{6}\big[F_2^3(x)-F_2^3(x-)\big]\frac{f_1(x)}{f_3(x)}$$
$$+F_3(x)F_3(x-)\frac{f_2(x)f_1(x)}{f_3(x)}-\frac{1}{2}\big[F_2(x)+F_2(x-)\big]f_2(x)F_3(x-)\frac{f_1(x)}{f_3(x)}-F_3(x)F_1(x-)f_2(x)$$
$$+\frac{1}{2}\big[F_2(x)+F_2(x-)\big]f_2(x)F_1(x-)\Bigg\}$$

$$+\Bigg\{F_2(x)f_3(x)-\frac{1}{2}\big[F_3(x)+F_3(x-)\big]f_3(x)-\frac{1}{2}F_2^2(x)f_3(x)\frac{f_1(x)}{f_2(x)}+\frac{1}{6}\big[F_3^3(x)-F_3^3(x-)\big]\frac{f_1(x)}{f_2(x)}$$
$$+F_2(x)F_2(x-)\frac{f_3(x)f_1(x)}{f_2(x)}-\frac{1}{2}\big[F_3(x)+F_3(x-)\big]f_3(x)F_2(x-)\frac{f_1(x)}{f_2(x)}-F_2(x)F_1(x-)f_3(x)$$
$$+\frac{1}{2}\big[F_3(x)+F_3(x-)\big]f_3(x)F_1(x-)\Bigg\}$$

$$+\Bigg\{F_1(x)f_3(x)-\frac{1}{2}\big[F_3(x)+F_3(x-)\big]f_3(x)-\frac{1}{2}F_1^2(x)f_3(x)\frac{f_2(x)}{f_1(x)}+\frac{1}{6}\big[F_3^3(x)-F_3^3(x-)\big]\frac{f_2(x)}{f_1(x)}$$
$$+F_1(x)F_1(x-)\frac{f_3(x)f_2(x)}{f_1(x)}-\frac{1}{2}\big[F_3(x)+F_3(x-)\big]f_3(x)F_1(x-)\frac{f_2(x)}{f_1(x)}-F_1(x)F_2(x-)f_3(x)$$
$$+\frac{1}{2}\big[F_3(x)+F_3(x-)\big]f_3(x)F_2(x-)\Bigg\}.$$

Morever, the above identity in the *iid* case can be expressed as

$$\begin{aligned}
&= 6F(x)f(x) - 3[F(x)+F(x-)]f(x) - 3F^2(x)f(x) + [F^3(x) - F^3(x-)] + 6F(x)F(x-)f(x) \\
&-3[F(x)+F(x-)]F(x-)f(x) - 6F(x)F(x-)f(x) + 3[F(x)+F(x-)]f(x)F(x-) \\
&= 6F(x)f(x) - 3F(x)f(x) - 3F(x-)f(x) - 3F^2(x)f(x) + F^3(x) - F^3(x-) \\
&= 3f^2(x) - 3F^2(x)f(x) + f(x)[3F^2(x) - 3F(x)f(x) + f^2(x)] \\
&= f^3(x) + 3f^2(x)[1-F(x)].
\end{aligned}$$

This result is obtained, if $r=1$, $s=2$ and $n=3$ in equation (2.4.3) in [14].

We will now express the following theorem to obtain the joint *df* of order statistics of *innid* discrete random variables.

Theorem 2.2.

$$F_{r_1,r_2,\dots,r_p:n}(x_1,x_2,\dots,x_p) = \mathrm{D}\sum_P \int_V \left(\prod_{w=1}^{p+1} \prod_{l=r_{w-1}+1}^{r_w-1} [v_{i_l}^{(w)} - v_{i_l}^{(w-1)}] \right) \prod_{w=1}^{p} dv_{i_{r_w}}^{(w)} . \tag{2.8}$$

Proof. We have

$$F_{r_1,r_2,\dots,r_p:n}(x_1,x_2,\dots,x_p) = \sum_{z_1,z_2,\dots,z_p} f_{r_1,r_2,\dots,r_p:n}(z_1,z_2,\dots,z_p) . \tag{2.9}$$

Using (2.1) in (2.9), (2.8) is obtained.

3. Results for distribution and probability functions

In this section, the results related to *pf* and *df* of $X_{r_1:n}, X_{r_2:n}, \dots, X_{r_p:n}$ will be given. We will express the following result for *pf* of the *r*th order statistic of *innid* discrete random variables.

Result 3.1.

$$f_{r_1:n}(x_1) = \frac{1}{(r_1-1)!(n-r_1)!} \sum_P \int_{F_{i_{r_1}}(x_1-)}^{F_{i_{r_1}}(x_1)} \left(\prod_{l=1}^{r_1-1} v_{i_l}^{(1)} \right) \left(\prod_{l=r_1+1}^{n} [1-v_{i_l}^{(1)}] \right) dv_{i_{r_1}}^{(1)}. \tag{3.1}$$

Proof. In (2.1), if $p=1$, (3.1) is obtained.

In Result 3.2 and Result 3.3, the *pf* 's of minimum and maximum order statistics of *innid* discrete random variables are given, respectively.

Result 3.2.

$$f_{1:n}(x_1) = \frac{1}{(n-1)!} \sum_P \int_{F_{i_1}(x_1-)}^{F_{i_1}(x_1)} \left(\prod_{l=2}^{n} [1 - v_{i_l}^{(1)}] \right) dv_{i_1}^{(1)}. \tag{3.2}$$

Proof. Putting $r_1 = 1$ in (3.1), one will get (3.2).

Result 3.3.

$$f_{n:n}(x_1) = \frac{1}{(n-1)!} \sum_P \int_{F_{i_n}(x_1-)}^{F_{i_n}(x_1)} \left(\prod_{l=1}^{n-1} v_{i_l}^{(1)} \right) dv_{i_n}^{(1)}. \tag{3.3}$$

Proof. On taking $r_1 = n$ in (3.1), one will get (3.3).

In the following result, we will give the joint *pf* of $X_{1:n}, X_{2:n}, ..., X_{p:n}$.

Result 3.4. If $x_1 \le x_2 \le ... \le x_p$,

$$f_{1,2,...,p:n}(x_1, x_2, ..., x_p) = \frac{1}{(n-p)!} \sum_P \int\int...\int \left(\prod_{l=p+1}^{n} [1 - v_{i_l}^{(p)}] \right) \prod_{w=1}^{p} dv_{i_w}^{(w)}, \tag{3.4}$$

where $\int\int...\int$ is to be carried out over the region: $v_{i_1}^{(1)} \le v_{i_2}^{(2)} \le ... \le v_{i_p}^{(p)}$,

$F_{i_1}(x_1-) \le v_{i_1}^{(1)} \le F_{i_1}(x_1),\ F_{i_2}(x_2-) \le v_{i_2}^{(2)} \le F_{i_2}(x_2),\ \ldots, F_{i_p}(x_p-) \le v_{i_p}^{(p)} \le F_{i_p}(x_p)$.

Proof. On taking $r_w = w$ for $w = 1, 2, ..., p$ and $\int\int...\int$ instead of $\int$ in (2.1), one will get (3.4).

We will now give three results for the *df* of single order statistic of *innid* discrete random variables.

Result 3.5.

$$F_{r_1:n}(x_1)=\frac{1}{(r_1-1)!(n-r_1)!}\sum_P \int_0^{F_{i_{r_1}}(x_1)} \left(\prod_{l=1}^{r_1-1} v_{i_l}^{(1)}\right)\left(\prod_{l=r_1+1}^{n}[1-v_{i_l}^{(1)}]\right) dv_{i_{r_1}}^{(1)}. \tag{3.5}$$

Proof. On taking $p=1$ in (2.8), one will get (3.5).

Result 3.6.

$$F_{1:n}(x_1)=\frac{1}{(n-1)!}\sum_P \int_0^{F_{i_1}(x_1)} \left(\prod_{l=2}^{n}[1-v_{i_l}^{(1)}]\right) dv_{i_1}^{(1)}. \tag{3.6}$$

Proof. Putting $r_1=1$ in (3.5), one will get (3.6).

Result 3.7.

$$F_{n:n}(x_1)=\frac{1}{(n-1)!}\sum_P \int_0^{F_{i_n}(x_1)} \left(\prod_{l=1}^{n-1} v_{i_l}^{(1)}\right) dv_{i_n}^{(1)}. \tag{3.7}$$

Proof. On taking $r_1=n$ in (3.5), one will get (3.7).

Specially, in (3.7), by taking n=2 and $v_{i_1}^{(1)}=[v_{i_2}^{(1)}-F_{i_2}(x_1-)]\dfrac{f_{i_1}(x_1)}{f_{i_2}(x_1)}+F_{i_1}(x_1-)$, the following identity is obtained.

$$\begin{aligned}
F_{2:2}(x_1) &= \sum_P \int_0^{F_{i_2}(x_1)} v_{i_1}^{(1)} dv_{i_2}^{(1)} \\
&= \sum_P \left[\left(\frac{\left(v_{i_2}^{(1)}\right)^2}{2}-v_{i_2}^{(1)}F_{i_2}(x_1-)\right)\frac{f_{i_1}(x_1)}{f_{i_2}(x_1)}+v_{i_2}^{(1)}F_{i_1}(x_1-)\right]_0^{F_{i_2}(x_1)} \\
&= \sum_P \left\{\left(\frac{F_{i_2}^2(x_1)}{2}-F_{i_2}(x_1)F_{i_2}(x_1-)\right)\frac{f_{i_1}(x_1)}{f_{i_2}(x_1)}+F_{i_2}(x_1)F_{i_1}(x_1-)\right\} \\
&= \left[\frac{F_2^2(x_1)}{2}-F_2(x_1)F_2(x_1-)\right]\frac{f_1(x_1)}{f_2(x_1)}+F_2(x_1)F_1(x_1-) \\
&\quad +\left[\frac{F_1^2(x_1)}{2}-F_1(x_1)F_1(x_1-)\right]\frac{f_2(x_1)}{f_1(x_1)}+F_1(x_1)F_2(x_1-).
\end{aligned}$$

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Morever, the above identity for *iid* case can be expressed as

$F_{2:2}(x_1) = F^2(x_1).$

Also, the above identity for $x_1 = 1$ can be written as

$$F_{2:2}(1) = F^2(1)$$
$$= \left[f(0) + f(1)\right]^2.$$

In the following result, we will give the joint *df* of $X_{1:n}, X_{2:n}, ..., X_{p:n}$.

Result 3.8.

$$F_{1,2,...,p:n}(x_1, x_2, ..., x_p) = \frac{1}{(n-p)!} \sum_P \int_0^{F_{i_1}(x_1)} \int_{v_{i_1}^{(1)}}^{F_{i_2}(x_2)} \cdots \int_{v_{i_{p-1}}^{(p-1)}}^{F_{i_p}(x_p)} \left(\prod_{l=p+1}^{n} [1 - v_{i_l}^{(p)}] \right) \prod_{w=1}^{p} dv_{i_w}^{(w)} . \qquad (3.8)$$

Proof. On considering $r_w = w$ for $w = 1, 2, ..., p$ from (2.8), one will get (3.8).

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