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Some new Chebyshev type quantum integral inequalities on finite intervals

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Abstract: By using the two parameters of deformation q_1 and q_2, we establish some new Chebyshev type quantum integral inequalities on finite intervals. Furthermore, we also consider their relevance with other related known results.

Keywords: Chebyshev type inequalities; quantum integral inequalities; synchronous (asynchronous) functions

2010 Mathematics Subject Classification: 34A08; 26D10; 26D15

1 Introduction

Let us start by considering the following celebrated Chebyshev functional (see [1]):

$$T(f,g,p,q) = \left(\int_a^b q(x)dx\right)\left(\int_a^b p(x)f(x)g(x)dx\right) + \left(\int_a^b p(x)dx\right)\left(\int_a^b q(x)f(x)g(x)dx\right) - \left(\int_a^b q(x)f(x)dx\right)\left(\int_a^b p(x)g(x)dx\right) - \left(\int_a^b p(x)f(x)dx\right)\left(\int_a^b q(x)g(x)dx\right), \quad (1.1)$$

where $f,g:[a,b]\to\mathbb{R}$ are two integrable functions on $[a,b]$ and $p(x)$ and $q(x)$ are positive integrable functions on $[a,b]$. If f and g are *synchronous* on $[a,b]$, that is,

$$(f(x)-f(y))(g(x)-g(y)) \geq 0,$$

for any $x,y\in[a,b]$, then we have (see, e.g., [2, 3])

$$T(f,g,p,q) \geq 0, \quad (1.2)$$

The inequality in (1.2) is reversed if f and g are *asynchronous* on $[a,b]$, that is,

$$(f(x)-f(y))(g(x)-g(y)) \leq 0,$$

for any $x,y\in[a,b]$. If $p(x)=q(x)$ for any $x,y\in[a,b]$, we get the Chebyshev inequality, see [1].

Here we should point out that the Chebyshev functional (1.1) has attracted many researchers attention mainly due to its distinguished applications in numerical quadrature, probability and statistical problems and transform theory. At the same time, the Chebyshev functional (1.1) has also been employed to yield a number of integral inequalities, see [4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and the references therein.

The integral inequalities can be applied for the study of qualitative and quantitative properties of integrals, see [14, 15, 16, 17]. In order to generalize and spread the existing inequalities, we specify two ways to overcome the problems which ensue from the general definition of q-integral. In [18], Gauchman has introduced the restricted q-integral over $[a,b]$. In [19], Stanković, Rajković and Marinković have introduced the definition of the q-integral of the Riemann type. In [18], Gauchman gave the q-analogues of the well-known inequalities in the integral calculus, as Chebyshev, Grüss, Hermite-Hadamard for all the types of the q-integrals. In [19], Stanković, Rajković and Marinković obtained some new q-Chebyshev, q-Grüss, q-Hermite-Hadamard type inequalities. In [20, 21], by using the weighted q-integral Montgomery identity, Yang and Liu and Yang established the weighted q-Čebyšev-Grüss type inequalities for single and double integrals, respectively. Recently, Tariboon

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and Ntouyas [22] introduced the quantum calculus on finite intervals, they extended the Hölder, Hermite-Hadamard, trapezoid, Ostrowski, Cauchy-Bunyakovsky-Schwarz, Grüss, and Grüss-Čebyšev integral inequalities to quantum calculus on finite intervals in the paper [23].

Motivated by the results mentioned above, by using the two parameters of deformation q_1 and q_2, we establish some new Chebyshev type quantum integral inequalities on finite intervals. Furthermore, we also obtain their relevance with other related known results.

2 Preliminaries

Let $J := [a, b] \subset \mathbb{R}$, $K := [c, d] \subset \mathbb{R}$, $J_0 := (a, b)$ be interval and $0 < q, q_1, q_2 < 1$ be a constant. We give the definition q-derivative of a function $f : J \to \mathbb{R}$ at a point $x \in J$ on $[a, b]$ as follows.

Definition 2.1 ([22])**.** Assume $f : J \to \mathbb{R}$ is a continuous function and let $x \in J$. Then the expression

$$ {}_aD_qf(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a, \quad {}_aD_qf(a) = \lim_{x \to a} {}_aD_qf(x), \tag{2.1}$$

is called the q-derivative on J of function f at x.

We say that f is q-differentiable on J provided ${}_aD_qf(x)$ exists for all $x \in J$. Note that if $a = 0$ in (2.1), then ${}_0D_qf = D_qf$, where D_q is the well-known q-derivative of the function $f(x)$ defined by

$$D_qf(x) = \frac{f(x) - f(qx)}{(1-q)x}.$$

For more details, see [24].

Definition 2.2 ([22])**.** Assume $f : J \to \mathbb{R}$ is a continuous function. Then the q-integral on J is defined by

$$I_q^a f(x) = \int_a^x f(t)_a d_qt = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a), \tag{2.2}$$

for $x \in J$. Moreover, if $c \in (a, x)$ then the definite q-integral on J is defined by

$$\begin{aligned}\int_c^x f(t)_a d_qt &= \int_a^x f(t)_a d_qt - \int_a^c f(t)_a d_qt \\ &= (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) - (1-q)(c-a) \sum_{n=0}^{\infty} q^n f(q^n c + (1-q^n)a).\end{aligned}$$

Note that if $a = 0$, then (2.2) reduces to the classical q-integral of a function $f(x)$ defined by (see [24])

$$\int_0^x f(t)_0 d_qt = (1-q)x \sum_{n=0}^{\infty} q^n f(q^n x), \quad \forall x \in [0, \infty).$$

Lemma 2.3. *Assume $f, g : J \to \mathbb{R}$ are two continuous functions and $f(t) \leq g(t)$ for all $t \in J$. Then*

$$\int_a^x f(t)_a d_qt \leq \int_a^x g(t)_a d_qt. \tag{2.3}$$

Proof. For $x \in J$, then $q^n x + (1-q^n)a \in J$. Because $f, g : J \to \mathbb{R}$ are two continuous functions and $f(t) \leq g(t)$ for all $t \in J$. Then

$$f(q^n x + (1-q^n)a) \leq g(q^n x + (1-q^n)a). \tag{2.4}$$

Summing from 0 to ∞ with respect to n and multiplying both sides of (2.4) by $(1-q)(x-a) \geq 0$, then we get

$$\begin{aligned}\int_a^x f(t)_a d_qt &= (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) \\ &\leq (1-q)(x-a) \sum_{n=0}^{\infty} q^n g(q^n x + (1-q^n)a) = \int_a^x g(t)_a d_qt,\end{aligned}$$

which implies (2.3). The proof is completed. □

3 Chebyshev type quantum integral inequalities

In this section, we establish some new Chebyshev type quantum integral inequalities on finite intervals. For the sake of simplicity, we always assume that in this paper all of quantum integral exist and

$$I_q^a(uf)(b) = \int_a^b u(t)f(t)_a d_q t \quad \text{and} \quad I_q^a(ufg)(b) = \int_a^b u(t)f(t)g(t)_a d_q t.$$

Lemma 3.1. *Let f and g be two continuous and synchronous functions on J and let $u, v : J \to [0, \infty)$ be two continuous functions. Then the following inequality holds true*

$$I_q^a u(b) I_q^a(vfg)(b) + I_q^a v(b) I_q^a(ufg)(b) \geq I_q^a(uf)(b) I_q^a(vg)(b) + I_q^a(vf)(b) I_q^a(ug)(b). \tag{3.1}$$

Proof. Since f and g be two continuous and synchronous functions on J, then for all $\tau, \rho \in J$, we have

$$(f(\tau) - f(\rho))(g(\tau) - g(\rho)) \geq 0. \tag{3.2}$$

By (3.2), we write

$$f(\tau)g(\tau) + f(\rho)g(\rho) \geq f(\tau)g(\rho) + f(\rho)g(\tau). \tag{3.3}$$

Multiplying both sides of (3.3) by $v(\tau)$ and integrating the resulting identity with respect to τ from a to b, then we obtain

$$I_q^a(vfg)(b) + f(\rho)g(\rho) I_q^a v(b) \geq g(\rho) I_q^a(vf)(b) + f(\rho) I_q^a(vg)(b). \tag{3.4}$$

Multiplying both side of (3.4) by $u(\rho)$ and integrating the resulting identity with respect to ρ from a to b, then we get

$$I_q^a u(b) I_q^a(vfg)(b) + I_q^a v(b) I_q^a(ufg)(b) \geq I_q^a(uf)(b) I_q^a(vg)(b) + I_q^a(vf)(b) I_q^a(ug)(b),$$

which implies (3.1). □

Theorem 3.2. *Let f and g be two continuous and synchronous functions on J and let $\phi, \varphi, \psi : J \to [0, \infty)$ be three continuous functions. Then the following inequality holds true*

$$\begin{aligned} 2I_q^a\phi(b)\big(I_q^a\varphi(b)I_q^a(\psi fg)(b) + I_q^a\psi(b)I_q^a(\varphi fg)(b)\big) + 2I_q^a\varphi(b)I_q^a\psi(b)I_q^a(\phi fg)(b) \\ \geq I_q^a\phi(b)\big(I_q^a(\varphi f)(b)I_q^a(\psi g)(b) + I_q^a(\psi f)(b)I_q^a(\varphi g)(b)\big) + I_q^a\varphi(b)\big(I_q^a(\phi f)(b)I_q^a(\psi g)(b) \\ + I_q^a(\psi f)(b)I_q^a(\varphi g)(b)\big) + I_q^a\psi(b)\big(I_q^a(\phi f)(b)I_q^a(\varphi g)(b) + I_q^a(\varphi f)(b)I_q^a(\phi g)(b)\big). \end{aligned} \tag{3.5}$$

Proof. Putting $u = \varphi$, $v = \psi$ and using Lemma 3.1, we can write

$$I_q^a\varphi(b)I_q^a(\psi fg)(b) + I_q^a\psi(b)I_q^a(\varphi fg)(b) \geq I_q^a(\varphi f)(b)I_q^a(\psi g)(b) + I_q^a(\psi f)(b)I_q^a(\varphi g)(b). \tag{3.6}$$

Multiplying both sides of (3.6) by $I_q^a\phi(b)$, we obtain

$$I_q^a\phi(b)\left(I_q^a\varphi(b)I_q^a(\psi fg)(b) + I_q^a\psi(b)I_q^a(\varphi fg)(b)\right) \geq I_q^a\phi(b)\left(I_q^a(\varphi f)(b)I_q^a(\psi g)(b) + I_q^a(\psi f)(b)I_q^a(\varphi g)(b)\right). \tag{3.7}$$

Putting $u = \phi$, $v = \psi$ and using Lemma 3.1, we can write

$$I_q^a\phi(b)I_q^a(\psi fg)(b) + I_q^a\psi(b)I_q^a(\phi fg)(b) \geq I_q^a(\phi f)(b)I_q^a(\psi g)(b) + I_q^a(\psi f)(b)I_q^a(\phi g)(b). \tag{3.8}$$

Multiplying both sides of (3.8) by $I_q^a\varphi(b)$, we obtain

$$I_q^a\varphi(b)\left(I_q^a\phi(b)I_q^a(\psi fg)(b) + I_q^a\psi(b)I_q^a(\phi fg)(b)\right) \geq I_q^a\varphi(b)\left(I_q^a(\phi f)(b)I_q^a(\psi g)(b) + I_q^a(\psi f)(b)I_q^a(\phi g)(b)\right). \tag{3.9}$$

With the same arguments as before, we can get

$$I_q^a\psi(b)\left(I_q^a\phi(b)I_q^a(\varphi fg)(b) + I_q^a\varphi(b)I_q^a(\phi fg)(b)\right) \geq I_q^a\psi(b)\left(I_q^a(\phi f)(b)I_q^a(\varphi g)(b) + I_q^a(\varphi f)(b)I_q^a(\phi g)(b)\right). \tag{3.10}$$

The required inequality (3.5) follows on adding the inequalities (3.7), (3.9) and (3.10). □

Lemma 3.3. *Let f and g be two continuous and synchronous functions on $J \cup K$ and let $u, v : J \cup K \to [0, \infty)$ be two continuous functions. Then the following inequality holds true*

$$I_{q_1}^a u(b) I_{q_2}^c (vfg)(d) + I_{q_2}^c v(d) I_{q_1}^a (ufg)(b) \geq I_{q_1}^a (uf)(b) I_{q_2}^c (vg)(d) + I_{q_2}^c (vf)(d) I_{q_1}^a (ug)(b). \tag{3.11}$$

Proof. Multiplying both sides of (3.3) by $v(\rho)$ and q_2-integrating the resulting inequality obtained with respect to ρ from c to d, then we have

$$f(\tau)g(\tau) I_{q_2}^c v(d) + I_{q_2}^c (vfg)(d) \geq f(\tau) I_{q_2}^c (vg)(d) + g(\tau) I_{q_2}^c (vf)(d). \tag{3.12}$$

Multiplying both sides of (3.12) by $u(\tau)$ and q_1-integrating the resulting identity with respect to τ from a to b, then we obtain

$$I_{q_1}^a u(b) I_{q_2}^c (vfg)(d) + I_{q_2}^c v(d) I_{q_1}^a (ufg)(b) \geq I_{q_1}^a (uf)(b) I_{q_2}^c (vg)(d) + I_{q_2}^c (vf)(d) I_{q_1}^a (ug)(b),$$

which implies (3.11). □

Theorem 3.4. *Let f and g be two continuous and synchronous functions on $J \cup K$ and let $\phi, \varphi, \psi : J \cup K \to [0, \infty)$ be three continuous functions. Then the following inequality holds true*

$$\begin{aligned} &I_{q_1}^a \phi(b) \big(I_{q_1}^a \psi(b) I_{q_2}^c (\varphi fg)(d) + 2 I_{q_1}^a \varphi(b) I_{q_2}^c (\psi fg)(d) + I_{q_2}^c \psi(d) I_{q_1}^a (\varphi fg)(b) \big) \\ &\quad + \big(I_{q_1}^a \varphi(b) I_{q_2}^c \psi(d) + I_{q_2}^c \varphi(d) I_{q_1}^a \psi(b) \big) I_{q_1}^a (\phi fg)(b) \geq I_{q_1}^a \phi(b) \big(I_{q_1}^a (\varphi f)(b) I_{q_2}^c (\psi g)(d) + I_{q_2}^c (\varphi f)(d) I_{q_1}^a (\varphi g)(b) \big) \\ &+ I_{q_1}^a \varphi(b) \big(I_{q_1}^a (\phi f)(b) I_{q_2}^c (\psi g)(d) + I_{q_2}^c (\psi f)(d) I_{q_1}^a (\phi g)(b) \big) + I_{q_1}^a \psi(b) \big(I_{q_1}^a (\phi f)(b) I_{q_2}^c (\varphi g)(d) + I_{q_2}^c (\varphi f)(d) I_{q_1}^a (\phi g)(b) \big). \end{aligned} \tag{3.13}$$

Proof. Putting $u = \varphi$, $v = \psi$ and using Lemma 3.3, we can write

$$I_{q_1}^a \varphi(b) I_{q_2}^c (\psi fg)(d) + I_{q_2}^c \psi(d) I_{q_1}^a (\varphi fg)(b) \geq I_{q_1}^a (\varphi f)(b) I_{q_2}^c (\psi g)(d) + I_{q_2}^c (\psi f)(d) I_{q_1}^a (\varphi g)(b). \tag{3.14}$$

Multiplying both sides of (3.14) by $I_{q_1}^a \phi(b)$, we obtain

$$I_{q_1}^a \phi(b) \big(I_{q_1}^a \varphi(b) I_{q_2}^c (\psi fg)(d) + I_{q_2}^c \psi(d) I_{q_1}^a (\varphi fg)(b) \big) \geq I_{q_1}^a \phi(b) \big(I_{q_1}^a (\varphi f)(b) I_{q_2}^c (\psi g)(d) + I_{q_2}^c (\psi f)(d) I_{q_1}^a (\varphi g)(b) \big), \tag{3.15}$$

Putting $u = \phi$, $v = \psi$ and using Lemma 3.3, we can write

$$I_{q_1}^a \phi(b) I_{q_2}^c (\psi fg)(d) + I_{q_2}^c \psi(d) I_{q_1}^a (\phi fg)(b) \geq I_{q_1}^a (\phi f)(b) I_{q_2}^c (\psi g)(d) + I_{q_2}^c (\psi f)(d) I_{q_1}^a (\phi g)(b). \tag{3.16}$$

Multiplying both sides of (3.16) by $I_{q_1}^a \varphi(b)$, we obtain

$$I_{q_1}^a \varphi(b) \big(I_{q_1}^a \phi(b) I_{q_2}^c (\psi fg)(d) + I_{q_2}^c \psi(d) I_{q_1}^a (\phi fg)(b) \big) \geq I_{q_1}^a \varphi(b) \big(I_{q_1}^a (\phi f)(b) I_{q_2}^c (\psi g)(d) + I_{q_2}^c (\psi f)(d) I_{q_1}^a (\phi g)(b) \big), \tag{3.17}$$

With the same arguments as before, we can get

$$I_{q_1}^a \psi(b) \big(I_{q_1}^a \phi(b) I_{q_2}^c (\varphi fg)(d) + I_{q_2}^c \phi(d) I_{q_1}^a (\phi fg)(b) \big) \geq I_{q_1}^a \psi(b) \big(I_{q_1}^a (\phi f)(b) I_{q_2}^c (\varphi g)(d) + I_{q_2}^c (\varphi f)(d) I_{q_1}^a (\phi g)(b) \big), \tag{3.18}$$

The required inequality (3.14) follows on adding the inequalities (3.15), (3.17) and (3.18). □

Remark 3.5. The inequalities (3.5) and (3.13) are reversed in the following cases: (a) The functions f and g are synchronous on $J \cup K$. (b) The functions ϕ, φ and ψ are negative on $J \cup K$. (c) Two of he functions ϕ, φ and ψ are positive and the third one is negative on $J \cup K$.

Theorem 3.6. *Let f, g, h be three continuous and synchronous functions on $J \cup K$ and let $u : J \cup K \to [0, \infty)$ be a continuous function. Then the following inequality holds true*

$$\begin{aligned} &I_{q_1}^a u(b) I_{q_2}^c (ufgh)(d) + I_{q_1}^a (uh)(b) I_{q_2}^c (ufg)(d) + I_{q_1}^a (ufg)(b) I_{q_2}^c (uh)(d) + I_{q_1}^a (ufgh)(b) I_{q_2}^c u(d) \\ &\quad \geq I_{q_1}^a (uf)(b) I_{q_2}^c (ugh)(d) + I_{q_1}^a (ug)(b) I_{q_2}^c (ufh)(d) + I_{q_1}^a (ugh)(b) I_{q_2}^c (uf)(d) + I_{q_1}^a (ufh)(b) I_{q_2}^c (ug)(d). \end{aligned} \tag{3.19}$$

Proof. Let f, g, h be three continuous and synchronous functions on $J \cup K$, then for all $\tau, \rho \in J \cup K$, we have

$$(f(\tau) - f(\rho))(g(\tau) - g(\rho))(h(\tau) + h(\rho)) \geq 0,$$

which implies that

$$\begin{aligned} & f(\tau)g(\tau)h(\tau) + f(\rho)g(\rho)h(\rho) + f(\tau)g(\tau)h(\rho) + f(\rho)g(\rho)h(\tau) \\ & \qquad \geq f(\tau)g(\rho)h(\tau) + f(\tau)g(\rho)h(\rho) + f(\rho)g(\tau)h(\tau) + f(\rho)g(\tau)h(\rho). \end{aligned} \tag{3.20}$$

Multiplying both sides of (3.20) by $u(\tau)$ and q_2-integrating the resulting identity with respect to τ from c to d, then we obtain

$$\begin{aligned} & I_{q_2}^c(ufgh)(d) + f(\rho)g(\rho)h(\rho)I_{q_2}^c u(d) + h(\rho)I_{q_2}^c(ufg)(d) + f(\rho)g(\rho)I_{q_2}^c(uh)(d) \\ & \qquad \geq g(\rho)I_{q_2}^c(ufh)(d) + g(\rho)h(\rho)I_{q_2}^c(uf)(d) + f(\rho)I_{q_2}^c(ugh)(d) + f(\rho)h(\rho)I_{q_2}^c(ug)(d). \end{aligned} \tag{3.21}$$

Multiplying both sides of (3.21) by $u(\rho)$ and q_1-integrating the resulting inequality obtained with respect to ρ from c to d, then we have

$$\begin{aligned} & I_{q_1}^a u(b)I_{q_2}^c(ufgh)(d) + I_{q_1}^a(uh)(b)I_{q_2}^c(ufg)(d) + I_{q_1}^a(ufg)(b)I_{q_2}^c(uh)(d) + I_{q_1}^a(ufgh)(b)I_{q_2}^c u(d) \\ & \qquad \geq I_{q_1}^a(uf)(b)I_{q_2}^c(ugh)(d) + I_{q_1}^a(ug)(b)I_{q_2}^c(ufh)(d) + I_{q_1}^a(ugh)(b)I_{q_2}^c(uf)(d) + I_{q_1}^a(ufh)(b)I_{q_2}^c(ug)(d), \end{aligned}$$

which implies ((3.19). □

Theorem 3.7. *Let f, g, h be three continuous and synchronous functions on $J \cup K$ and let $u, v : J \cup K \to [0, \infty)$ be two continuous functions. Then the following inequality holds true*

$$\begin{aligned} & I_{q_1}^a u(b)I_{q_2}^c(vfgh)(d) + I_{q_1}^a(uh)(b)I_{q_2}^c(vfg)(d) + I_{q_1}^a(ufg)(b)I_{q_2}^c(vh)(d) + I_{q_1}^a(ufgh)(b)I_{q_2}^c v(d) \\ & \qquad \geq I_{q_1}^a(uf)(b)I_{q_2}^c(vgh)(d) + I_{q_1}^a(ug)(b)I_{q_2}^c(vfh)(d) + I_{q_1}^a(ugh)(b)I_{q_2}^c(vf)(d) + I_{q_1}^a(ufh)(b)I_{q_2}^c(vg)(d). \end{aligned} \tag{3.22}$$

Proof. Multiplying both sides of (3.20) by $v(\tau)$ and integrating the resulting identity with respect to τ from c to d, then we obtain

$$\begin{aligned} & I_{q_2}^c(vfgh)(d) + f(\rho)g(\rho)h(\rho)I_{q_2}^c u(d) + h(\rho)I_{q_2}^c(vfg)(d) + f(\rho)g(\rho)I_{q_2}^c(vh)(d) \\ & \qquad \geq g(\rho)I_{q_2}^c(vfh)(d) + g(\rho)h(\rho)I_{q_2}^c(vf)(d) + f(\rho)I_{q_2}^c(vgh)(d) + f(\rho)h(\rho)I_{q_2}^c(vg)(d). \end{aligned} \tag{3.23}$$

Multiplying both sides of (3.23) by $u(\rho)$ and integrating the resulting inequality obtained with respect to ρ from a to b, then we have

$$\begin{aligned} & I_{q_1}^a u(b)I_{q_2}^c(vfgh)(d) + I_{q_1}^a(uh)(b)I_{q_2}^c(vfg)(d) + I_{q_1}^a(ufg)(b)I_{q_2}^c(vh)(d) + I_{q_1}^a(ufgh)(b)I_{q_2}^c v(d) \\ & \qquad \geq I_{q_1}^a(uf)(b)I_{q_2}^c(vgh)(d) + I_{q_1}^a(ug)(b)I_{q_2}^c(vfh)(d) + I_{q_1}^a(ugh)(b)I_{q_2}^c(vf)(d) + I_{q_1}^a(ufh)(b)I_{q_2}^c(vg)(d), \end{aligned}$$

which implies (3.22). □

Remark 3.8. It may be noted that the inequalities in (3.19) and (3.22) are reversed if functions f, g and h are asynchronous. It is also easily seen that the special case $u = v$ of (3.22) in Theorem 3.7 reduces to that in Theorem 3.6.

4 Other quantum integral inequalities

The first class are the inequalities related to Cauchy's inequality.

Theorem 4.1. *Let ϕ, f and g be three continuous functions on J. Then the following inequality holds true*

$$[T(\phi, f, g)]^2 \leq T(\phi, f, f)T(\phi, g, g), \tag{4.1}$$

where $T(\phi, f, g) = I_q^a \phi(b) I_q^a(\phi fg)(b) - I_q^a(\phi f)(b) I_q^a(\phi g)(b)$.

Proof. By simple computation, we have the following fact that

$$T(\phi, f, g) = \frac{1}{2}\int_a^b \int_a^b \phi(\rho)\phi(\tau)[f(\rho) - f(\tau)][g(\rho) - g(\tau)]_a d_q\rho_a d_q\tau. \tag{4.2}$$

From (4.2) and weighted Cauchy's inequality, we easily obtain (4.1). □

Lemma 4.2. *Let f and h be two continuous functions on J and let $\phi : J \to [0, \infty)$ be a continuous function. Then the following inequality holds true*

$$m[g(\rho) - g(\tau)] \le f(\rho) - f(\tau) \le M[g(\rho) - g(\tau)], \quad \forall \rho, \tau \in J, \tag{4.3}$$

where m and M are given real numbers. Then for all $t > 0$ and $\nu > 0$, we have

$$T(\phi, f, f) + mMT(\phi, g, g) \le (m + M)T(\phi, f, g), \tag{4.4}$$

where $T(\phi, f, g)$ is defined as in Theorem 4.1.

Proof. If we use the condition (4.3), we get

$$(M[g(\rho) - g(\tau)] - [f(\rho) - f(\tau)])([f(\rho) - f(\tau)] - m[g(\rho) - g(\tau)]) \ge 0, \quad \forall \rho, \tau \in J. \tag{4.5}$$

From (4.5) and through simple computation, we have

$$[f(\rho) - f(\tau)]^2 + mM[g(\rho) - g(\tau)]^2 \le (m + M)[f(\rho) - f(\tau)][g(\rho) - g(\tau)]. \tag{4.6}$$

Multiplying both sides of (4.6) by $\phi(\rho)\phi(\tau)$ and integrating the resulting identity with respect to ρ and τ from a to b, we deduce the required inequality (4.4). □

Theorem 4.3. *Let f, g, ϕ be defined as in Lemma 4.2 and $0 < m \le M < \infty$. Then the following inequalities hold true*

$$T(\phi, f, f)T(\phi, g, g) \le \frac{(m + M)^2}{4mM}[T(\phi, f, g)]^2, \tag{4.7}$$

$$0 \le \sqrt{T(\phi, f, f)T(\phi, g, g)} - T(\phi, f, g) \le \frac{(\sqrt{M} - \sqrt{m})^2}{2\sqrt{mM}}T(\phi, f, g), \tag{4.8}$$

and

$$0 \le T(\phi, f, f)T(\phi, g, g) - [T(\phi, f, g)]^2 \le \frac{(M - m)^2}{4mM}[T(\phi, f, g)]^2, \tag{4.9}$$

where $T(\phi, f, g)$ is defined as in Theorem 4.1.

Proof. We use the following elementary inequality

$$2xy \le \alpha x^2 + \frac{1}{\alpha}y^2, \quad \forall x, y \ge 0, \ \alpha > 0,$$

to get, for the choices

$$\alpha = \sqrt{mM} > 0, \quad x = \sqrt{T(\phi, g, g)} \ge 0, \quad y = \sqrt{T(\phi, f, f)} \ge 0$$

the following inequality

$$2\sqrt{T(\phi, f, f)T(\phi, g, g)} \le \sqrt{mM}T(\phi, g, g) + \frac{1}{\sqrt{mM}}T(\phi, f, f). \tag{4.10}$$

Using (4.4) and (4.10), we deduce

$$2\sqrt{T(\phi, f, f)T(\phi, g, g)} \le \frac{m + M}{\sqrt{mM}}T(\phi, f, g).$$

which is clearly equivalent to (4.7). By a few transformations of (4.7), similarly, we obtain (4.8) and (4.9). □

The second class are the inequalities related to Hölder's inequality.

Theorem 4.4. *Let $\phi : J \to [0,\infty)$ be a continuous function on J and $f, g : J \to (0,\infty)$ be two continuous functions on J such that $0 < m \le f^\alpha(\tau)/g^\beta(\tau) \le M < \infty$ on J. If $1/\alpha + 1/\beta = 1$ with $\alpha > 1$, then the following inequality holds true*

$$\big(I_q^a(\phi f^\alpha)(b)\big)^{\frac{1}{\alpha}}\big(I_q^a(\phi g^\beta)(b)\big)^{\frac{1}{\beta}} \le \big(\frac{M}{m}\big)^{\frac{1}{\alpha\beta}} I_q^a(\phi f g)(b). \tag{4.11}$$

Proof. Since $f^\alpha(\tau)/g^\beta(\tau) \le M$, then $f^{\alpha/\beta} \le M^{1/\beta} g$. Multiplying by $\phi f > 0$, it follows that

$$\phi f^\alpha = \phi f^{1+\frac{\alpha}{\beta}} \le M^{\frac{1}{\beta}} \phi f g$$

and integrating the above inequality from a to b, then we have

$$\big(I_q^a(\phi f^\alpha)(b)\big)^{\frac{1}{\alpha}} \le M^{\frac{1}{\alpha\beta}} (I_q^a(\phi f g)(b))^{\frac{1}{\alpha}}. \tag{4.12}$$

On the other hand, since $m \le f^\alpha(\tau)/g^\beta(\tau)$, then $f \ge m^{1/\alpha} g^{\beta/\alpha}$. Multiplying by $\phi g > 0$, it follows that

$$\phi f g \ge m^{\frac{1}{\alpha}} \phi g^{1+\frac{\beta}{\alpha}} = m^{\frac{1}{\alpha}} \phi g^\beta.$$

Integrating the above inequality from a to b, we obtain that

$$\big(I_q^a(\phi f g)(b)\big)^{\frac{1}{\beta}} \ge m^{\frac{1}{\alpha\beta}} \big(I_q^a(\phi g^\alpha)(b)\big)^{\frac{1}{\beta}}. \tag{4.13}$$

Combining (4.12) and (4.13), we have the desired inequality (4.11). The proof is completed. □

Theorem 4.5. *Suppose that $1/\alpha + 1/\beta = 1/\gamma$ with $\alpha, \beta, \gamma > 0$. Let $\phi : J \to [0,\infty)$ be a continuous function on J and $f, g : J \to (0,\infty)$ be two continuous functions on J. If $0 < m \le f^\gamma(\tau)/g^{\beta\gamma/\alpha}(\tau) \le M < \infty$ for any $\tau \in J$, then the following inequalities hold true*

$$(M-m) I_q^a(\phi f^\alpha)(b) + \big(mM^{\frac{\alpha}{\gamma}} - Mm^{\frac{\alpha}{\gamma}}\big) I_q^a(\phi g^\beta)(b) \le \big(M^{\frac{\alpha}{\gamma}} - m^{\frac{\alpha}{\gamma}}\big) I_q^a(\phi f^\gamma g^\gamma)(b), \tag{4.14}$$

and

$$\alpha^{\frac{1}{\alpha}}\beta^{\frac{1}{\beta}}\gamma^{-\frac{1}{\gamma}} \frac{(M-m)^{\frac{1}{\alpha}}\big(mM^{\frac{\alpha}{\gamma}} - Mm^{\frac{\alpha}{\gamma}}\big)^{\frac{1}{\beta}}}{\big(M^{\frac{\alpha}{\gamma}} - m^{\frac{\alpha}{\gamma}}\big)^{\frac{1}{\gamma}}} \big(I_q^a(\phi f^\alpha)\big)^{\frac{1}{\alpha}} \big(I_q^a(\phi g^\beta)(b)\big)^{\frac{1}{\beta}} \le \big(I_q^a(\phi f^\gamma g^\gamma)(b)\big)^{\frac{1}{\gamma}}. \tag{4.15}$$

Proof. If $0 < m \le x^\gamma / y^{\beta\gamma/\alpha} \le M < \infty$, then the following inequality is valid (see [25]):

$$(M-m)x^\alpha + \big(mM^{\frac{\alpha}{\gamma}} - Mm^{\frac{\alpha}{\gamma}}\big) y^\beta \le \big(M^{\frac{\alpha}{\gamma}} - m^{\frac{\alpha}{\gamma}}\big) x^\gamma y^\gamma. \tag{4.16}$$

Substituting in the inequality (4.16) $x \to f(\tau)$ and $y \to g(\tau)$, and multiplying both sides of the obtained result by $\phi(\tau)$ and integrating the resulting identity with respect to τ from a to b, we obtain (4.14).

Now, rewrite (4.14) in the form

$$\left(\frac{\gamma}{\alpha}\right)\left(\left(\frac{\alpha}{\gamma}\right)(M-m) I_q^a(\phi f^\alpha)(b)\right) + \left(\frac{\gamma}{\beta}\right)\left(\left(\frac{\beta}{\gamma}\right)\big(mM^{\frac{\alpha}{\gamma}} - Mm^{\frac{\alpha}{\gamma}}\big) I_q^a(\phi g^\beta)(b)\right) \le \big(M^{\frac{\alpha}{\gamma}} - m^{\frac{\alpha}{\gamma}}\big) I_q^a(\phi f^\gamma g^\gamma)(b), \tag{4.17}$$

and applying arithmetic-geometric inequality on the left-hand side of (4.17) we get (4.15). □

The next class are the inequalities related to Minkowsky's inequality.

Theorem 4.6. *Let $p \ge 1$ and $\phi : J \to [0,\infty)$ be a continuous function on J and $f, g : J \to (0,\infty)$ be two continuous functions on J. If $0 < m \le f(\tau)/g(\tau) \le M < \infty$ for any $\tau \in J$, then we have*

$$\big(I_q^a(\phi f^p)(b)\big)^{\frac{1}{p}} + \big(I_q^a(\phi g^p)(b)\big)^{\frac{1}{p}} \le \frac{1 + M(m+2)}{(m+1)(M+1)} \big(I_q^a(\phi (f+g)^p)(b)\big)^{\frac{1}{p}}. \tag{4.18}$$

Proof. Using the condition $f(\tau)/g(\tau) \leq M$ for any $\tau \in J$, we can get

$$(M+1)^p f^p(\tau) \leq M^p (f+g)^p(\tau). \tag{4.19}$$

Multiplying both sides of (4.19) by $\phi(\tau)$ and integrating the resulting inequalities with respect to τ from a to b, we obtain

$$(M+1)^p I_q^a(f^p)(b) \leq M^p I_q^a((f+g)^p)(b).$$

Hence, we can write

$$\big(I_q^a(\phi f^p)(b)\big)^{\frac{1}{p}} \leq \frac{M}{M+1}\big(I_q^a(\phi(f+g)^p)(b)\big)^{\frac{1}{p}}. \tag{4.20}$$

On the other hand, using the condition $m \leq f(\tau)/g(\tau)$, we can get

$$(m+1)^p g^p(\tau) \leq (f+g)^p(\tau). \tag{4.21}$$

Multiplying both sides of (4.21) by $\phi(\tau)$ and integrating the resulting inequalities with respect to τ from a to b, we obtain

$$(m+1)^p I_q^a(\phi g^p)(b) \leq I_q^a(\phi(f+g)^p)(b).$$

Hence, we can write

$$\big(I_q^a(\phi g^p)(b)\big)^{\frac{1}{p}} \leq \frac{1}{m+1}\big(I_q^a(\phi(f+g)^p)(b)\big)^{\frac{1}{p}}. \tag{4.22}$$

Adding the inequalities (4.20) and (4.22), we obtain the inequality (4.19). □

Theorem 4.7. *Let $p \geq 1$ and $\phi : J \to [0,\infty)$ be a continuous function on J and $f,g : J \to (0,\infty)$ be two continuous functions on J. If $0 < m \leq f(\tau)/g(\tau) \leq M < \infty$ for any $\tau \in J$, then we have*

$$\big(I_q^a(\phi f^p)(b)\big)^{\frac{2}{p}} + \big(I_q^a(\phi g^p)(b)\big)^{\frac{2}{p}} \geq \left(\frac{(m+1)(M+1)}{M} - 2\right)\big(I_q^a(\phi f^p)(b)\big)^{\frac{1}{p}}\big(I_q^a(\phi g^p)(b)\big)^{\frac{1}{p}}. \tag{4.23}$$

Proof. Multiplying the inequalities (4.20) and (4.22), we obtain

$$\frac{(m+1)(M+1)}{M}\big(I_q^a(\phi f^p)(b)\big)^{\frac{1}{p}}\big(I_q^a(\phi g^p)(b)\big)^{\frac{1}{p}} \leq \Big(\big(I_q^a(\phi(f+g)^p)(b)\big)^{\frac{1}{p}}\Big)^2. \tag{4.24}$$

Applying Minkowski's inequality to the right hand side of (4.24), we get

$$\begin{aligned}\Big(\big(I_q^a(\phi(f+g)^p)(b)\big)^{\frac{1}{p}}\Big)^2 &\leq \Big(\big(I_q^a(\phi f^p)(b)\big)^{\frac{1}{p}} + [I_q^a(\phi g^p)(b)\big)^{\frac{1}{p}}\Big)^2 \\ &= \big(I_q^a(\phi f^p)(b)\big)^{\frac{2}{p}} + \big(I_q^a(\phi g^p)(b)\big)^{\frac{2}{p}} + 2\big(I_q^a(\phi f^p)(b)\big)^{\frac{1}{p}}\big(I_q^a(\phi g^p)(b)\big)^{\frac{1}{p}}. \end{aligned} \tag{4.25}$$

Combining (4.24) and (4.25), we obtain (4.23). □

Theorem 4.8. *Suppose that $1/\alpha + 1/\beta = 1$ with $\alpha > 1$. Let $\phi : J \to [0,\infty)$ be a continuous function on J and $f,g : J \to (0,\infty)$ be two continuous functions on J. If $0 < m \leq f(\tau)/g(\tau) \leq M < \infty$ for any $\tau \in J$, then the following inequality holds true*

$$I_q^a(\phi fg)(b) \leq \frac{2^\alpha}{\alpha}\left(\frac{M}{M+1}\right)^\alpha\left(\frac{I_q^a(\phi f^\alpha)(b) + I_q^a(\phi g^\alpha)(b)}{2}\right) + \frac{2^\beta}{\beta}\left(\frac{1}{m+1}\right)^\beta\left(\frac{I_q^a(\phi f^\beta)(b) + I_q^a(\phi g^\beta)(b)}{2}\right). \tag{4.26}$$

Proof. From $m \leq f(\tau)/g(\tau) \leq M$ for any $\tau \in J$, we have

$$f(\tau) \leq \frac{M}{M+1}(f(\tau)+g(\tau)), \qquad g(\tau) \leq \frac{1}{m+1}(f(\tau)+g(\tau)). \tag{4.27}$$

From (4.27) and the following Young-type inequality

$$xy \leq \frac{1}{\alpha}x^{\alpha} + \frac{1}{\beta}y^{\beta}, \quad \forall x, y \geq 0, \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

we obtain

$$\begin{aligned} I_q^a(\phi fg)(b) &\leq \frac{1}{\alpha} I_q^a(\phi f^{\alpha})(b) + \frac{1}{\beta} I_q^a(\phi g^{\beta})(b) \\ &\leq \frac{1}{\alpha}\left(\frac{M}{M+1}\right)^{\alpha} I_q^a(\phi(f+g)^{\alpha})(b) + \frac{1}{\beta}\left(\frac{1}{m+1}\right)^{\beta} I_q^a(\phi(f+g)^{\beta})(b). \quad (4.28) \end{aligned}$$

Using the elementary inequality $(c+d)^{\alpha} \leq 2^{\alpha-1}(c^{\alpha}+d^{\alpha})$, ($\alpha > 1$ and $c, d > 0$) in (4.28), we get

$$\begin{aligned} I_q^a(\phi fg)(b) &\leq \frac{1}{\alpha}\left(\frac{M}{M+1}\right)^{\alpha} 2^{\alpha-1} I_q^a(\phi(f^{\alpha}+g^{\alpha}))(b) + \frac{1}{\beta}\left(\frac{1}{m+1}\right)^{\beta} 2^{\beta-1} I_q^a(\phi(f^{\beta}+g^{\beta}))(b) \\ &= \frac{2^{\alpha}}{\alpha}\left(\frac{M}{M+1}\right)^{\alpha}\left(\frac{I_q^a(\phi f^{\alpha})(b) + I_q^a(\phi g^{\alpha})(b)}{2}\right) + \frac{2^{\beta}}{\beta}\left(\frac{1}{m+1}\right)^{\beta}\left(\frac{I_q^a(\phi f^{\beta})(b) + I_q^a(\phi g^{\beta})(b)}{2}\right). \end{aligned}$$

This completes the proof of the inequality in (4.26). □

Theorem 4.9. *Suppose that $1/\alpha + 1/\beta = 1$ with $\alpha, \beta > 0$. Let $\phi : J \to [0, \infty)$ be a continuous function on J and $f, g : J \to (0, \infty)$ be two continuous functions on J. If $0 < m \leq f(\tau)/(f(\tau)+g(\tau)) \leq M < \infty$ and $0 < m \leq g(\tau)/(f(\tau)+g(\tau)) \leq M < \infty$ for any $\tau \in J$, then we have*

$$\left(I_q^a(\phi(f+g)^{\alpha})(b)\right)^{\frac{1}{\alpha}} \geq \alpha^{\frac{1}{\alpha}}\beta^{\frac{1}{\beta}}\frac{(M-m)^{\frac{1}{\alpha}}\left(mM^{\alpha}-Mm^{\alpha}\right)^{\frac{1}{\beta}}}{M^{\alpha}-m^{\alpha}}\left(\left(I_q^a(\phi f^{\alpha})(b)\right)^{\frac{1}{\alpha}} + \left(I_q^a(\phi g^{\alpha})(b)\right)^{\frac{1}{\alpha}}\right). \quad (4.29)$$

Proof. Due to (4.15) with $\gamma = 1$ of Theorem 4.5, $m \leq f(\tau)/g^{\beta/\alpha}(\tau) \leq M$ for any $\tau \in J$, we have

$$\alpha^{\frac{1}{\alpha}}\beta^{\frac{1}{\beta}}\frac{(M-m)^{\frac{1}{\alpha}}(mM^{\alpha}-Mm^{\alpha})^{\frac{1}{\beta}}}{M^{\alpha}-m^{\alpha}}\left(I_q^a(\phi f^{\alpha})(b)\right)^{\frac{1}{\alpha}}\left(I_q^a(\phi g^{\beta})(b)\right)^{\frac{1}{\beta}} \leq I_q^a(\phi fg)(b). \quad (4.30)$$

By simple computation, we have

$$I_q^a(\phi(f+g)^{\alpha})(b) = I_q^a(\phi f(f+g)^{\alpha-1}) + I_q^a(\phi g(f+g)^{\alpha-1})(b). \quad (4.31)$$

From $m \leq f(\tau)/(f(\tau)+g(\tau)) \leq M$ and $m \leq g(\tau)/(f(\tau)+g(\tau)) \leq M$ for any $\tau \in J$, we have $m \leq f(\tau)/\left((f(\tau)+g(\tau))^{\alpha-1}\right)^{\beta/\alpha} \leq M$ and $m \leq g(\tau)/((f(\tau)+g(\tau))^{\alpha-1})^{\beta/\alpha} \leq M$ for any $\tau \in J$. Applying (4.30) on right hand of (4.31), we get

$$\begin{aligned} I_q^a(\phi(f(f+g)^{\alpha-1})(b) &\geq \alpha^{\frac{1}{\alpha}}\beta^{\frac{1}{\beta}}\frac{(M-m)^{\frac{1}{\alpha}}(mM^{\alpha}-Mm^{\alpha})^{\frac{1}{\beta}}}{M^{\alpha}-m^{\alpha}}[I_q^a(\phi f^{\alpha})(b)]^{\frac{1}{\alpha}}[I_q^a(\phi(f+g)^{(\alpha-1)\beta})(b)]^{\frac{1}{\beta}}, \\ I_q^a(\phi(g(f+g)^{\alpha-1})(b) &\geq \alpha^{\frac{1}{\alpha}}\beta^{\frac{1}{\beta}}\frac{(M-m)^{\frac{1}{\alpha}}(mM^{\alpha}-Mm^{\alpha})^{\frac{1}{\beta}}}{M^{\alpha}-m^{\alpha}}[I_q^a(\phi g^{\alpha})(b)]^{\frac{1}{\alpha}}[I_q^a(\phi(f+g)^{(\alpha-1)\beta})(b)]^{\frac{1}{\beta}}. \quad (4.32) \end{aligned}$$

Using (4.31) and adding two inequalities of (4.32), we obtain

$$I_q^a(\phi(f+g)^{\alpha})(b) \geq \alpha^{\frac{1}{\alpha}}\beta^{\frac{1}{\beta}}\frac{(M-m)^{\frac{1}{\alpha}}(mM^{\alpha}-Mm^{\alpha})^{\frac{1}{\beta}}}{M^{\alpha}-m^{\alpha}}\left(\left(I_q^a(\phi f^{\alpha})(b)\right)^{\frac{1}{\alpha}} + \left(I_q^a(\phi g^{\alpha})(b)\right)^{\frac{1}{\alpha}}\right)\left(I_q^a(\phi(f+g)^{\alpha})(b)\right)^{\frac{1}{\beta}}. \quad (4.33)$$

Dividing both sides of (4.33) by $\left(I_q^a(\phi(f+g)^{\alpha})(b)\right)^{\frac{1}{\beta}}$, we get (4.29). □

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A FIRST ORDER DIFFERENTIAL SUBORDINATION AND ITS APPLICATIONS

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Abstract. In this paper we use the differential subordinations techniques to obtain some properties of functions belonging to the class of analytic functions in the open unit disc $\mathbb{U}$. Also, some properties of the class of two fixed points in $\mathbb{U}$, are also discussed. Furthermore, some interesting results of Hurwitz Lerch Zeta function and Digamma function are obtained.

1. Introduction

Let A_k denote the class of functions f of the form

$$f(z) = z + \sum_{m=k+1}^{\infty} a_m z^m \qquad (k \in \mathbb{N} = \{1, 2, ...\}), \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$. Also, let $\mathcal{H}[a, k]$ denote the class of analytic functions in $\mathbb{U}$ in the form

$$r(z) = a + \sum_{m=k}^{\infty} a_m z^m \qquad (z \in \mathbb{U}), \tag{1.2}$$

for $a \in \mathbb{C}$ ($\mathbb{C}$ is the complex plane).

Usually the analytic functions with the normalization $f(0) = 0 = f'(0) - 1$ is studied. Moreover, we can obtain interesting results by using the Montel's normalization of f (*cf.* [16], [6]) as follows

$$f(z)|_{z=0} = 0 \qquad \text{and} \qquad \left.\frac{f(z)}{z}\right|_{z=\rho} = 1, \tag{1.3}$$

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where ρ is a fixed point in $\mathbb{U}$. We see that, when $\rho = 0$, we get the classical normalization in $\mathbb{U}$. We denote by $A_{k,\rho}$ the class of functions f in A_k with Montel's normalization. The class $A_{k,\rho}$ will be called the class of functions f with two fixed points.

A function f in the class A_k is said to be in the class $R_k(\alpha)$ if it satisfies

$$\operatorname{Re}\left(\frac{f(z)}{z}\right) > \alpha \qquad (z \in \mathbb{U}), \tag{1.4}$$

for some $\alpha\,(0 \le \alpha < 1)$. The classes $R_1(\alpha) = \mathcal{C}(\alpha)$, and $R_1(0) = \mathcal{C}(0)$, were earlier studied by Èzrohi [7] and MacGregor [19], respectively. Further; a function f in the class A_k is said to be in the class $P_k(\alpha)$ if it satisfies

$$\operatorname{Re}\left(f'(z)\right) > \alpha \qquad (z \in \mathbb{U}), \tag{1.5}$$

for some $\alpha\,(0 \le \alpha < 1)$. The class $P_1(0) = \mathcal{B}(0)$, was earlier studied by Yamaguchi [28]. For some $\alpha(0 \le \alpha < 1)$, $\lambda \neq 0$ with $\operatorname{Re}(\lambda) \ge 0$ and $z \in \mathbb{U}$ we write :

$$R_k^1(\alpha,\lambda) := \left\{ f(z) \in A_k : \operatorname{Re}\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right) > \alpha \right\} \tag{1.6}$$

and

$$R_k^2(\alpha,\lambda) := \left\{ f(z) \in A_k : \operatorname{Re}\left(f'(z) + \lambda z f''(z)\right) > \alpha \right\}. \tag{1.7}$$

We note that
(i) $R_k^1(\alpha, 1) = P_k(\alpha)$,
(ii) $f \in R_k^2(\alpha,\lambda)$ *if and only if* $z f' \in R_k^1(\alpha,\lambda)$.

Now, if $f \in A_k$, we define the function $G_k(\mu,\gamma;z)$ by

(1.8)
$$G_k(\mu,\gamma;z) := \frac{\left(f'(z)\right)^{\mu}\left(f(z)\right)^{1-\mu}}{\gamma\, z^{1-\mu}}\left((\gamma-1) + (1-\mu)\frac{zf'(z)}{f(z)} + \mu\left(1 + \frac{zf''(z)}{f'(z)}\right)\right),$$

with $\frac{z^{1-\mu}}{\left(f'(z)\right)^{\mu}\left(f(z)\right)^{1-\mu}} \neq 0$, for $\mu \in \mathbb{R}$, $\gamma \neq 0$ with $\operatorname{Re}(\gamma) \ge 0$ and $z \in \mathbb{U}$.

Let $H_k(\mu,\gamma,\alpha)$ denote the class of functions f satisfying the condition

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$$\operatorname{Re}\left(G_k(\mu,\gamma;z)\right) > \alpha \quad (z \in \mathbb{U}), \tag{1.9}$$

for some $\alpha(0 \leq \alpha < 1)$ and $G_k(\mu,\gamma;z)$ defined by (1.8).

Also, we note that

(i) For $\lambda \neq 0$ with $\operatorname{Re}(\lambda) \geq 0$

$$H_k\left(0, \frac{1}{\lambda}, \alpha\right) = R_k^1(\alpha,\lambda) \quad \text{and} \quad H_k\left(1, \frac{1}{\lambda}, \alpha\right) = R_k^2(\alpha,\lambda)\ . \tag{1.10}$$

(ii) One can define the $R_k^1(\alpha,\lambda)$ for $\lambda = 0$. Therefore we may use the following relations

$$R_k(\alpha) = R_k^1(\alpha, 0) = \lim_{\lambda \to 0} H_k\left(0, \frac{1}{\lambda}, \alpha\right)\ , \tag{1.11}$$

and

$$R_k^2(\alpha, 0) = \lim_{\lambda \to 0} H_k\left(1, \frac{1}{\lambda}, \alpha\right)\ . \tag{1.12}$$

A general Hurwitz- Lerch Zeta function (or Lerch transcendent) $\Phi(z,s,b)$ (cf., e.g., [24, Section 2.5, P. 121]) is the analytic continuation of the series

$$\Phi(z,s,b) = \sum_{k=0}^{\infty} \frac{z^k}{(k+b)^s}\ , \tag{1.13}$$

which converges for b ($b \in \mathbb{C}\setminus\mathbb{Z}_0^-$, $\mathbb{Z}_0^- = \{0,-1,-2,...\}$) if z and $s \in \mathbb{C}$ are any complex numbers with either $z \in \mathbb{U}$, or $|z| = 1$ and $\operatorname{Re}(s) > 1$. See also [2, Section 1.11].

Many authors obtained several properties of $\Phi(z,s,b)$, for example, Attiya and Hakami [1], Cho *et al.* [3], Choi and et al. [5], Ferreira and López [9], Guillera and Sondow [10, Section 2], Gupta *et al.* [11], Kutbi and Attiya ([13],[14]), Luo and Srivastava [15], Owa and Attiya [21], Prajapat and Bulboaca [22], Srivastava and Attiya [23], Srivastava *et al.* [25] and Wang *et al.* [27].

Moreover, the Digamma function (or Psi) (cf., e.g., [24, Section 1.2, P. 13]) is the logarithmic derivative of the classical gamma function,

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defined by

$$\Psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}=-C-\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{z+n}\right), \tag{1.14}$$

with the Euler constant $C = 0.57721566...$. See also [2, Section 1.7] and [18, Section 5.1]. Several properties of Ψ can be found in [17], [4], [8] and [26].

We shall also need the following definitions

Definition 1.1. Let f and F be analytic functions. The function f is said to be *subordinate* to F, written $f(z) \prec F(z)$, if there exists a function w analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| \leq 1$, and such that $f(z) = F(w(z))$. If F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$.

Definition 1.2. Let $\Psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If $q \in \mathcal{H}[a,k]$ satisfies the differential subordination

$$\Psi(p(z),\, z\,p'(z), z^2\,p''(z)\,;\, z) \prec h(z) \quad (z \in \mathbb{U})\,, \tag{1.15}$$

then q will be called $(a,k)-$*solution*. The univalent function s is called $(a,k)-$ *domainant*, if $q(z) \prec s(z)$ for all q satisfying (1.15) , $(a,k)-$ *domainant* $\bar{s}(z) \prec s(z)$ for all $(a,k)-$ *domainant* s of (1.15) is said to be *the best* $(a,k)-$ *domainant* of (1.15).

In this paper, using the technique of differential subordination, some properties of functions in the class $H_k(\mu,\gamma,\,\alpha)$ are obtained. Furthermore, some properties of the class of two fixed points in $\mathbb{U}$, are also introduced. Some applications to *Analytic Number Theory* are also discussed.

2. The class $H_k(\mu,\gamma,\,\alpha)$ with first order differential subordination

To prove our results, we need the following theorem due to Hallenbeck and Ruscheweyh [12] (see also Miller and Mocanu [20, P. 71]).

Theorem 2.1. Let h be convex univalent in $\mathbb{U}$, with $h(0) = a$, $\gamma \neq 0$ and $\mathrm{Re}(\gamma) \geq 0$. If $q \in \mathcal{H}[a,k]$ and

$$q(z)+\frac{z\,q'(z)}{\gamma} \prec h(z), \tag{2.1}$$

then

$$q(z) \prec S(z) \prec h(z), \tag{2.2}$$

where

$$S(z) = \frac{\gamma}{k\, z^{\frac{\gamma}{k}}} \int_0^z h(t)\, t^{\frac{\gamma}{k}-1}\, dt\ . \tag{2.3}$$

The function S is a convex univalent and is *the best* $(a,k)-$ *domainint.*

Now, we prove

Theorem 2.2. Let γ be a complex number satisfying $\gamma \neq 0$ with $\mathrm{Re}(\gamma) \geq 0$. If $q \in \mathcal{H}[1,k]$ and

$$\mathrm{Re}\left(q(z) \ + \ \frac{z\, q'(z)}{\gamma} \right) > \alpha\,, \tag{2.4}$$

then

(2.5)

$$\mathrm{Re}\,(q(z)) > (2\alpha - 1) + 2(1-\alpha)\,{}_2F_1\left(1, \frac{|\gamma|^2}{k\,\mathrm{Re}(\gamma)}; \frac{|\gamma|^2}{k\,\mathrm{Re}(\gamma)} + 1, -1\right) \quad (z \in \mathbb{U}),$$

for some $\alpha\,(0 \leq \alpha < 1)$ and $\lambda \neq 0$ with $\mathrm{Re}(\lambda) > 0$. The constant$(2\alpha - 1) + 2(1-\alpha)\,{}_2F_1\left(1, \frac{|\gamma|^2}{k\,\mathrm{Re}(\gamma)}; \frac{|\gamma|^2}{k\,\mathrm{Re}(\gamma)} + 1, -1\right)$ is the best possible.

Proof. If we take the convex univalent function h defined by

$$h(z) = \frac{1 + \ (2\,\alpha - 1)\, z}{1 + z} \qquad (\, 0 \leq \alpha < 1)\,, \tag{2.6}$$

then, we have

$$q(z) + \frac{z\, q'(z)}{\gamma} \ \prec h(z)\,, \tag{2.7}$$

where h is defined by (2.6) satisfying $h(0) = 1$.

Applying Theorem 2.1, then

$$q(z) \prec S(z)\,, \tag{2.8}$$

where the convex function S defined by

$$S(z) = \frac{\gamma}{k\, z^{\frac{\gamma}{k}}} \int_0^z \frac{1 + (2\,\alpha - 1)\, t}{1 + t} \quad t^{\frac{\gamma}{k}-1}\, dt,$$

$$= (2\alpha - 1) + 2(1-\alpha)\int_0^1 \frac{dt}{1+t^{\frac{k}{\gamma}}z}\,, \tag{2.9}$$

since $\operatorname{Re}(h(z)) > 0$ and $S(z) \prec h(z)$, we have $\operatorname{Re}(S(z)) > 0$.
Also, since

$$\inf_{z\in\mathbb{U}} \operatorname{Re}\left(\frac{1}{1+t^{\frac{k}{\gamma}}z}\right) = \frac{1}{1+t^{\frac{k\operatorname{Re}(\gamma)}{|\gamma|^2}}} \quad (0 \le t \le 1). \tag{2.10}$$

Hence

$$\inf_{z\in\mathbb{U}} \operatorname{Re}(S(z)) = (2\alpha - 1) + 2(1-\alpha)\int_0^1 \frac{dt}{1+t^{\frac{k\operatorname{Re}(\gamma)}{|\gamma|^2}}}$$

$$= (2\alpha-1) + 2(1-\alpha)\,{}_2F_1\left(1, \frac{|\gamma|^2}{k\operatorname{Re}(\gamma)}; \frac{|\gamma|^2}{k\operatorname{Re}(\gamma)}+1, -1\right). \tag{2.11}$$

Therefore, the constant $(2\alpha-1)+2(1-\alpha)\,{}_2F_1\left(1, \frac{|\gamma|^2}{k\operatorname{Re}(\gamma)}; \frac{|\gamma|^2}{k\operatorname{Re}(\gamma)}+1, -1\right)$ cannot be replaced by a larger one, which completes the proof of Theorem 2.2. □

Theorem 2.3. Let the function f defined by (1.1) be in the class $H_k(\mu, \gamma, \alpha)$, then

$$\operatorname{Re}\left(\frac{(f'(z))^{\mu}(f(z))^{1-\mu}}{z^{1-\mu}}\right) > (2\alpha-1)+2(1-\alpha)\,{}_2F_1\left(1, \frac{|\gamma|^2}{k\operatorname{Re}(\gamma)}; \frac{|\gamma|^2}{k\operatorname{Re}(\gamma)}+1, -1\right) \quad (z\in\mathbb{U}), \tag{2.12}$$

for some $\alpha\,(0 \le \alpha < 1)$, $\mu \in \mathbb{R}$ and $\gamma \ne 0$ with $\operatorname{Re}(\gamma) \ge 0$. The constant $(2\alpha-1) + 2(1-\alpha)\,{}_2F_1(1, \frac{|\gamma|^2}{k\operatorname{Re}(\gamma)}; \frac{|\gamma|^2}{k\operatorname{Re}(\gamma)}+1, -1)$ is the best possible.

Proof. Defining the function

$$q(z) = \frac{(f'(z))^{\mu}\,(f(z))^{1-\mu}}{z^{1-\mu}} \quad (z\in\mathbb{U}), \tag{2.13}$$

then, we have $q \in \mathcal{H}[1, k]$.

Taking the logarithmic differentiation in both sides of (2.13), we have

$$q(z) + \frac{z\,q'(z)}{\gamma} = G_k(\mu, \gamma; z), \tag{2.14}$$

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since $f \in H_k(\mu, \gamma, \alpha)$, then

$$\operatorname{Re}\left(q(z) + \frac{z\,q'(z)}{\gamma}\right) > \alpha. \tag{2.15}$$

Therefore, we have (2.12) by applying Theorem 2.2. □

Putting $\mu = 0$ and $\gamma = 1/\lambda$ $(\lambda \neq 0;\ \operatorname{Re}(\lambda) \geq 0)$, in Theorem 2.3, we have

Corollary 2.1. Let the function f defined by (1.1) be in the class $R_k^1(\alpha, \lambda)$, then

(2.16)
$$\operatorname{Re}\left(\frac{f(z)}{z}\right) > (2\alpha-1) + 2(1-\alpha)\,{}_2F_1\left(1, \frac{1}{k\operatorname{Re}(\lambda)}; \frac{1}{k\operatorname{Re}(\lambda)} + 1, -1\right) \quad (z \in \mathbb{U}),$$

for some $\alpha\,(0 \leq \alpha < 1)$ and $\lambda \neq 0$ with $\operatorname{Re}(\lambda) \geq 0$. The constant $(2\alpha-1) + 2(1-\alpha)\,{}_2F_1\left(1, \frac{1}{k\operatorname{Re}(\lambda)}; \frac{1}{k\operatorname{Re}(\lambda)} + 1, -1\right)$ is the best possible.

Assuming $\mu = 1$ and $\gamma = 1/\lambda$ $(\lambda \neq 0;\ \operatorname{Re}(\lambda) \geq 0)$, in Theorem 2.3 or putting zf' instead of f, in Corollary (2.1), we have

Corollary 2.2. Let the function f defined by (1.1) be in the class $R_k^2(\alpha, \lambda)$, then

(2.17)
$$\operatorname{Re}\left(f'(z)\right) > (2\alpha-1) + 2(1-\alpha)\,{}_2F_1\left(1, \frac{1}{k\operatorname{Re}(\lambda)}; \frac{1}{k\operatorname{Re}(\lambda)} + 1, -1\right) \qquad (z \in \mathbb{U}),$$

for some $\alpha\,(0 \leq \alpha < 1)$ and $\lambda \neq 0$ with $\operatorname{Re}(\lambda) \geq 0$. The constant $(2\alpha-1) + 2(1-\alpha)\,{}_2F_1\left(1, \frac{1}{k\operatorname{Re}(\lambda)}; \frac{1}{k\operatorname{Re}(\lambda)} + 1, -1\right)$ is the best possible.

Corollary 2.3. Let the function f defined by (1.1) be in the class $A_{k,\rho}$ of functions f with two fixed points. Also, let f be in the class $R_k^1(\alpha, \lambda)$, then

$$\operatorname{Re}\left(\sum_{m=k+2}^{\infty} a_m z^k \left(z^{m-k-1} - \rho^{m-k-1}\right)\right) > \tag{2.18}$$
$$2(1-\alpha)\left({}_2F_1\left(1, \frac{1}{k\operatorname{Re}(\lambda)}; \frac{1}{k\operatorname{Re}(\lambda)} + 1, -1\right) - 1\right) \quad (z \in \mathbb{U}),$$

for some $\alpha\,(0 \leq \alpha < 1)$, ρ is a fixed point in $\mathbb{U}$ defined in (1.3) and $\lambda \neq 0$ with $\operatorname{Re}(\lambda) \geq 0$. The constant $2(1-\alpha)\ \left({}_2F_1\left(1, \frac{1}{k\operatorname{Re}(\lambda)}; \frac{1}{k\operatorname{Re}(\lambda)} + 1, -1\right) - 1\right)$ is the best possible.

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Proof. Since $f \in A_{k,\rho}$, then we have

$$a_{k+1} = -\sum_{m=k+2}^{\infty} a_m \rho^{m-k-1}, \tag{2.19}$$

therefore, the function $f(z)/z$, takes the form

$$\frac{f(z)}{z} = 1 + \left(\sum_{m=k+2}^{\infty} a_m z^k (z^{m-k-2} - \rho^{m-k-1} \right), \tag{2.20}$$

Then, we have the Corollary by applying Corollary 2.1. □

By using the same technique in Corollary 2.3, we have

Corollary 2.4. Let the function f defined by (1.1) be in the class $A_{k,\rho}$ of functions f with two fixed points. Also, let f be in the class $R_k^2(\alpha,\, \lambda)$, then

$$\operatorname{Re}\left(\sum_{m=k+2}^{\infty} a_m z^k (m\ z^{m-k-1} - \rho^{m-k-1}) \right) > \tag{2.21}$$
$$2(1-\alpha)\ \left({}_2F_1\left(1, \frac{1}{k\operatorname{Re}(\lambda)}; \frac{1}{k\operatorname{Re}(\lambda)} + 1, -1\right) - 1 \right) \quad (z \in \mathbb{U}),$$

for some $\alpha\,(0 \le \alpha < 1)$, ρ is a fixed point in $\mathbb{U}$ defined in (1.3) and $\lambda \neq 0$ with $\operatorname{Re}(\lambda) \geq 0$. The constant ${}_2F_1\left(1, \frac{1}{k\operatorname{Re}(\lambda)}; \frac{1}{k\operatorname{Re}(\lambda)} + 1, -1\right)$ is the best possible.

3. Some applications in Analytic Number Theory

In this section we need the following lemma due to Guillera and Sondow [10].

Lemma 3.1. For $z \in \mathbb{C} - [1,\ \infty)$ and $\delta > 0$, we have

$$\int_0^1\int_0^1 \frac{-(x\ y)^{\delta-1}}{(1-xyz)\ \ln\ xy}\ dx\ dy\ =\ \Phi\left(z, 1, \delta\right), \tag{3.1}$$

and

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$$\int_0^1\int_0^1 \frac{-(x\ y)^{\delta-1}}{(1+xy)\ \ln\ xy}dx\ dy = \frac{1}{2}\left\{\Psi\left(\frac{\delta}{2}+\frac{1}{2}\right)-\Psi\left(\frac{\delta}{2}\right)\right\}, \tag{3.2}$$

where $\Phi(z,s,\delta)$ and $\Psi(\delta)$ defined by (1.13) and (1.14) respectively

Corollary 3.1. Let $\Phi(z,s,\ b)$ be the Lerch transcendental function defined by (1.13), then

$$\mathrm{Re}\left(\Phi(z,1,\delta)\right) > \frac{1}{2}\left(\Psi\left(\frac{\delta}{2}+\frac{1}{2}\right)-\Psi\left(\frac{\delta}{2}\right)\right) \quad (|z|<1;\ \delta>0), \tag{3.3}$$

and this result is the best possible.

Proof. We can show that the function

$$g(z) = z\left((2\alpha-1)+\frac{2(1-\alpha)}{\lambda}\int_0^1 \frac{t^{\frac{1-\lambda}{\lambda}}dt}{1-tz}\right) \quad (0\le\alpha<1;\ z\in\mathbb{U}), \tag{3.4}$$

is a member in the class $R_1^1(\alpha,\lambda)$. Using (3.4) and (1.13) we obtain

$$g(z) = z\left\{(2\alpha-1)+\frac{2(1-\alpha)}{\lambda}\Phi(z,1,\frac{1}{\lambda})\right\} \quad (0\le\alpha<1;\ z\in\mathbb{U}), \tag{3.5}$$

which a member in the class $R_1^1(\alpha,\lambda)$, where $\Phi(z,s,b)$ is the Lerch transcendental function defined by (1.13).

Using (2.16) and (3.5), we readily obtain the following property with $\lambda>0$; real

$$\mathrm{Re}\left(\Phi\left(z,1,\frac{1}{\lambda}\right)\right) > \lambda\int_0^1 \frac{dt}{1+t^\lambda} \quad (|z|<1), \tag{3.6}$$

which is equivalent to

$$\mathrm{Re}\left(\Phi(z,1,\delta)\right) > \Phi(-1,1,\delta), \quad (|z|<1;\ \delta>0), \tag{3.7}$$

the constant $\Phi(-1,1,\delta)$, cannot be replaced by a larger one .

Using (1.14) and (3.7), we have

$$\mathrm{Re}\left(\phi(z,1,\delta)\right) > \frac{1}{2}\left(\Psi\left(\frac{\delta}{2}+\frac{1}{2}\right)-\Psi\left(\frac{\delta}{2}\right)\right) \quad (|z|<1;\ \delta>0), \tag{3.8}$$

this result is the best possible in general, which completes the proof of Corollary 3.2 . □

Using (3.2) and (3.3), we have the following corollary

Corollary 3.2. For $\delta > 0$, we have

$$\int_0^1 \int_0^1 \frac{-(x\ y)^{\delta-1}}{(1+xy)\ \ln\ xy} dx\ dy < \mathrm{Re}\left(\phi\left(z,1,\delta\right)\right) \quad (|z|<1)\ . \tag{3.9}$$

This result is the best possible.

Using (3.1) and (3.3), we have the following corollary

Corollary 3.4. For $\delta > 0$, we have

$$\mathrm{Re}\left(\int_0^1 \int_0^1 \frac{-(x\ y)^{\delta-1}}{(1-xy\ z)\ \ln\ xy} dx\ dy\right) > \frac{1}{2}\left(\Psi\left(\frac{\delta}{2}+\frac{1}{2}\right)-\Psi\left(\frac{\delta}{2}\right)\right) \quad (|z|<1). \tag{3.10}$$

This result is the best possible.

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Approximation by a complex summation-integral type operators in compact disks

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Abstract. In this paper we introduce a kind of complex summation-integral type operators and study the approximation properties of these operators. We obtain a Voronovskaja-type result with quantitative estimate for these operators attached to analytic functions on compact disks. We also study the exact order of approximation. More important, our results show the overconvergence phenomenon for these complex operators.

Keywords: Complex summation-integral type operators; Voronovskaja-type result; Exact order of approximation; Simultaneous approximation; Overconvergence

Mathematical subject classification: 30E10, 41A25, 41A36

1. Introduction

In 1986, some approximation properties of complex Bernstein polynomials in compact disks were initially studied by Lorentz [15]. Very recently, the problem of the approximation of complex operators has been causing great concern, which is becoming a hot topic of research. A Voronovskaja-type result with quantitative estimate for complex Bernstein polynomials in compact disks was obtained by Gal [2]. Also, in [1, 3-14, 16-19] similar results for complex Bernstein-Kantorovich polynomials, Bernstein-Stancu polynomials, Kantorovich-Schurer polynomials, Kantorovich-Stancu polynomials, complex Favard-Szász-Mirakjan operators, complex Beta operators of first kind, complex Baskajov-Stancu operators, complex Bernstein-Durrmeyer polynomials, complex Bernstein-Durrmeyer operators based on Jacobi weights, complex genuine Durrmeyer Stancu polynomials and complex q-Durrmeyer type operators were obtained.

The aim of the present article is to obtain approximation results for a kind of complex summation-integral type operators (introduced and studied in the case of real variable by Ren [20]), which are defined for $f : [0, 1] \to \mathbf{C}$ continuous

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on $[0,1]$ by

$$M_n(f;z) := p_{n,0}(z)f(0) + \sum_{k=1}^{n-1} p_{n,k}(z)L_{n,k}(f) + p_{n,n}(z)f(1), \quad (1)$$

where $z \in \mathbf{C}$, $n = 1,2,\ldots,$ $p_{n,k}(z) = \binom{n}{k} z^k(1-z)^{n-k}$ is Bernstein basis function, and $L_{n,k}(f) = \frac{1}{B(n(n-k),nk)}\int_0^1 t^{nk-1}(1-t)^{n(n-k)-1}f(t)dt$, $B(x,y)$ is Beta function.

2. Auxiliary results

In the sequel, we shall need the following auxiliary results.

Lemma 1. Let $m,n \in \mathbf{N}$, $z \in \mathbf{C}$, we have $M_n(t^m;z)$ is a polynomial of degree less than or equal to min (m,n) and

$$M_n(t^m;z) = \frac{(n^2-1)!}{(n^2+m-1)!}\sum_{s=1}^{m} c_s(m)n^{2s}B_n(t^s;z),$$

where $c_s(m) \geq 0$ are constants depending on m and

$$B_n(f;z) = \sum_{k=0}^{n} p_{n,k}(z)f(\frac{k}{n}).$$

Proof. By the definition of Beta function, for all $m,n \in \mathbf{N}$, $z \in \mathbf{C}$, we have

$$M_n(t^m;z) = \frac{(n^2-1)!}{(n^2+m-1)!}\sum_{k=1}^{n-1} p_{n,k}(z)\frac{(nk+m-1)!}{(nk-1)!} + z^n.$$

Considering the definition of the $B_n(f;z)$, for any $m \in \mathbb{N}$, applying the principle of mathematical induction, we immediately obtain the desired conclusion.

Let $m = 0,1,2$, by Lemma 1, we have

$$M_n(1;z) = 1;$$
$$M_n(t;z) = z;$$
$$M_n(t^2;z) = \frac{n(n-1)}{n^2+1}z^2 + \frac{n+1}{n^2+1}z.$$

Lemma 2. For all $m,n \in \mathbb{N}$ we can get the equality

$$\frac{(n^2-1)!}{(n^2+m-1)!}\sum_{s=1}^{m} c_s(m)n^{2s} = 1.$$

Proof. For all $m,n \in \mathbb{N}$, by Lemma 1 we have

$$M_n(t^m;1) = \frac{(n^2-1)!}{(n^2+m-1)!}\sum_{s=1}^{m} c_s(m)n^{2s}.$$

On the other hand, we have $p_{n,k}(1) = 0, k = 0, 1, 2, \dots, n-1$, and $p_{n,n}(1) = 1$. So, by the formula (1) and using these above value, we have $M_n(t^m; 1) = 1$, which implies that we get desired conclusion.

Corollary 1. Let $e_m(t) = t^m$, $m \in \mathbf{N} \cup \{0\}$, $z \in \mathbf{C}$, $n \in \mathbf{N}$, for all $|z| \le r$, $r \ge 1$, we have $|M_n(e_m; z)| \le r^m$.

Proof. Since $M_n(e_0; z) = 1$, therefor this result is established for $m = 0$. When $m \in \mathbf{N}$, by using the methods Gal [5], p. 61, proof of Theorem 1.5.6, we have $|B_n(t^s; z)| \le r^s$. Thus, for all $m \in \mathbf{N}$ and $|z| \le r$, the proof follows directly by Lemma 1 and Lemma 2.

Lemma 3. Let $e_m(t) = t^m$, $m \in \mathbb{N} \cup \{0\}$ and $z \in \mathbb{C}$, we have

$$M_n(e_{m+1}; z) = \frac{nz(1-z)}{n^2+m} (M_n(e_m; z))' + \frac{m + n^2 z}{n^2+m} M_n(e_m; z). \tag{2}$$

Proof. By Lemma 1, we have $M_n(e_0; z) = 1$ and $M_n(e_1; z) = z$, therefore, this result is established for $m = 0$. Now let $m \in \mathbb{N}$, in view of

$$z(1-z)\,[p_{n,k}(z)]' = (k - nz)p_{n,k}(z),$$

it follows that

$$\begin{aligned}
&z(1-z)(M_n(e_m; z))' \\
&= \sum_{k=1}^{n-1} (k - nz) p_{n,k}(z) L_{n,k}(t^m) + nz^n(1-z) \\
&= \sum_{k=1}^{n-1} \left[\frac{(n^2+m)(nk+m)}{n(n^2+m)} - \frac{m}{n} \right] p_{n,k}(z) L_{n,k}(t^m) + nz^n - nzM_n(e_m; z) \\
&= \frac{n^2+m}{n} \sum_{k=1}^{n-1} p_{n,k}(z) L_{n,k}(t^{m+1}) - \frac{m}{n} \sum_{k=1}^{n-1} p_{n,k}(z) L_{n,k}(t^m) + nz^n - nzM_n(e_m; z) \\
&= \frac{n^2+m}{n} M_n(e_{m+1}; z) - \frac{m}{n} M_n(e_m; z) - nzM_n(e_m; z) \\
&= \frac{n^2+m}{n} M_n(e_{m+1}; z) - \frac{m + n^2 z}{n} M_n(e_m; z),
\end{aligned}$$

which implies the recurrence in the statement.

Lemma 4. Let m, $n \in \mathbf{N}$, $z \in \mathbf{C}$, $e_m(z) = z^m$, $S_{n,m}(z) := M_n(e_m; z) - z^m$, we have

$$\begin{aligned}
S_{n,m}(z) = {} & \frac{nz(1-z)}{n^2+m-1} (M_n(e_{m-1}; z))' + \frac{m-1+n^2 z}{n^2+m-1} S_{n,m-1}(z) \\
& + \frac{m-1+n^2 z}{n^2+m-1} z^{m-1} - z^m
\end{aligned} \tag{3}$$

Proof. Using the recurrence formula (2), by simple calculation, we can easily get the recurrence (3), the proof is omitted.

3. Main results

The first main result is expressed by the following upper estimates.

Theorem 1. Let $1 \leq r \leq R$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. Suppose that $f : D_R \to \mathbf{C}$ is analytic in D_R, i.e. $f(z) = \sum_{m=0}^{\infty} c_m z^m$ for all $z \in D_R$.

(i) for all $|z| \leq r$ and $n \in \mathbf{N}$, we have

$$|M_n(f;z) - f(z)| \leq \frac{K_r(f)}{n},$$

where $K_r(f) = (1+r) \sum_{m=1}^{\infty} |c_m| m(m-1) r^{m-1} < \infty$.

(ii) (Simultaneous approximation) If $1 \leq r < r_1 < R$ are arbitrary fired, then for all $|z| \leq r$ and $n, p \in \mathbf{N}$ we have

$$|(M_n(f;z))^{(p)} - f^{(p)}(z)| \leq \frac{K_{r_1}(f) p! r_1}{n(r_1 - r)^{p+1}},$$

where $K_{r_1}(f)$ is defined as at the above point (i).

Proof. Taking $e_m(z) = z^m$, by hypothesis that $f(z)$ is analytic in D_R, i.e. $f(z) = \sum_{m=0}^{\infty} c_m z^m$ for all $z \in D_R$, it is easy for us to obtain

$$M_n(f;z) = \sum_{m=0}^{\infty} c_m M_n(e_m;z),$$

therefore, we get

$$\begin{aligned} |M_n(f;z) - f(z)| &\leq \sum_{m=0}^{\infty} |c_m| \cdot |M_n(e_m;z) - e_m(z)| \\ &= \sum_{m=1}^{\infty} |c_m| \cdot |M_n(e_m;z) - e_m(z)|, \end{aligned}$$

as $M_n(e_0;z) = e_0(z) = 1$.

(i) For $m \in \mathbf{N}$, taking into account that $M_n(e_{m-1};z)$ is a polynomial degree $\leq min(m-1,n)$, by the well-known Bernstein inequality and Corollary 1, we get

$$|(M_n(e_{m-1};z))'| \leq \frac{m-1}{r} max\{|M_n(e_{m-1};z)| : |z| \leq r\} \leq (m-1) r^{m-2}.$$

On the one hand, when $m = 1$, for $|z| \leq r$, by Lemma 1, we have

$$|M_n(e_1;z) - e_1(z)| = (1+r) \frac{m(m-1)}{n} r^{m-1}.$$

On the other hand, when $m \geq 2$, for $n \in \mathbf{N}$, $|z| \leq r$, $r \geq 1$, using the

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recurrence formula (3) and the above inequality, we have

$$\begin{aligned}|M_n(e_m;z)-e_m(z)| &= |S_{n,m}(z)|\\ &\leq \frac{r(1+r)}{n}(m-1)r^{m-2}+r|S_{n,m-1}(z)|\\ &+\frac{m-1}{n}(1+r)r^{m-1}\\ &=\frac{2(m-1)}{n}(1+r)r^{m-1}+r|S_{n,m-1}(z)|.\end{aligned}$$

By writing the last inequality, for $m=2,\cdots$, we easily obtain step by step the following

$$\begin{aligned}|M_n(e_m;z)-e_m(z)| &\leq r\left(r|S_{n,m-2}(z)|+\frac{2(m-2)}{n}(1+r)r^{m-2}\right)\\ &+\frac{2(m-1)}{n}(1+r)r^{m-1}\\ &=r^2|S_{n,m-2}(z))|+\frac{2(m-2)+2(m-1)}{n}(1+r)r^{m-1}\\ &\leq\ldots\leq(1+r)\frac{m(m-1)}{n}r^{m-1}.\end{aligned}$$

In conclusion, for any $m,n\in\mathbf{N}$, $|z|\leq r$, $r\geq 1$, we have

$$|M_n(e_m;z)-e_m(z)|\leq(1+r)\frac{m(m-1)}{n}r^{m-1},$$

it follows that

$$|M_n(f;z)-f(z)|\leq\frac{1+r}{n}\sum_{m=1}^{\infty}|c_m|m(m-1)r^{m-1}.$$

By assuming that $f(z)$ is analytic in D_R, we have $f^{(2)}(z)=\sum\limits_{m=2}^{\infty}c_m m(m-1)z^{m-2}$ and the series is absolutely convergent in $|z|\leq r$, so we get $\sum\limits_{m=2}^{\infty}|c_m|m(m-1)r^{m-2}<\infty$, which implies $K_r(f)=(1+r)\sum\limits_{m=1}^{\infty}|c_m|m(m-1)r^{m-1}<\infty$.

(ii) For the simultaneous approximation, denoting by Γ the circle of radius $r_1>r$ and center 0, since for any $|z|\leq r$ and $\upsilon\in\Gamma$, we have $|\upsilon-z|\geq r_1-r$, by Cauchy's formulas it follows that for all $|z|\leq r$ and $n\in\mathbf{N}$, we have

$$\begin{aligned}|(M_n(f;z))^{(p)}-f^{(p)}(z)| &= \frac{p!}{2\pi}\left|\int_\Gamma\frac{M_n(f;\upsilon)-f(\upsilon)}{(\upsilon-z)^{p+1}}d\upsilon\right|\\ &\leq\frac{K_{r_1}(f)}{n}\frac{p!}{2\pi}\frac{2\pi r_1}{(r_1-r)^{p+1}}\\ &=\frac{K_{r_1}(f)}{n}\cdot\frac{p!r_1}{(r_1-r)^{p+1}},\end{aligned}$$

which proves the theorem.

Theorem 2. Let $R>1$, $D_R=\{z\in\mathbf{C}:|z|<R\}$. Suppose that $f:D_R\to\mathbf{C}$

is analytic in D_R, i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for all $z \in D_R$. For any fixed $r \in [1, R]$ and all $n \in \mathbf{N}$, $|z| \le r$, we have

$$\left| M_n(f; z) - f(z) - \frac{(n+1)z(1-z)f''(z)}{2(n^2+1)} \right| \le \frac{M_r(f)}{n^2}. \tag{4}$$

where $M_r(f) = \sum_{k=2}^{\infty} |c_k|(k-1)F_{k,r} r^k$ and $F_{k,r} = 10k^3 - 30k^2 + 39k - 16 + 4(k-2)(k-1)^2(1+r)$.

Proof. For all $z \in D_R$, we have

$$\left| M_n(f; z) - f(z) - \frac{(n+1)z(1-z)f''(z)}{2(n^2+1)} \right|$$
$$\le \sum_{k=2}^{\infty} |c_k| \left| M_n(e_k; z) - e_k(z) - \frac{(n+1)k(k-1)(1-z)z^{k-1}}{2(n^2+1)} \right|.$$

Denoting

$$E_{k,n}(z) = M_n(e_k; z) - e_k(z) - \frac{(n+1)k(k-1)(1-z)z^{k-1}}{2(n^2+1)},$$

it is obvious that $E_{k,n}(z)$ is a polynomial of degree less than or equal to k. For all $k \ge 2$, by simple computation and the use of Lemma 3, we can get

$$E_{k,n}(z) = \frac{nz(1-z)}{n^2+k-1}(E_{k-1,n}(z))' + \frac{k-1+n^2z}{n^2+k-1}E_{k-1,n}(z) + G_{k,n}(z), \tag{5}$$

where $G_{k,n}(z) = \frac{z^{k-2}}{n^2+k-1}(z^2A_{k,n} + zB_{k,n} + C_{k,n})$ and $A_{k,n} = -n(k-1) + \frac{n(n+1)(k-1)^2(k-2)}{2(n^2+1)} + n^2 - \frac{n^2(n+1)(k-1)(k-2)}{2(n^2+1)} - (n^2+k-1) + \frac{(n+1)k(k-1)(n^2+k-1)}{2(n^2+1)}$, $B_{k,n} = n(k-1) - \frac{n(n+1)(k-1)^2(k-2)}{2(n^2+1)} - \frac{n(n+1)(k-1)(k-2)^2}{2(n^2+1)} + k - 1 - \frac{(n+1)(k-1)^2(k-2)}{2(n^2+1)} + \frac{n^2(n+1)(k-1)(k-2)}{2(n^2+1)} - \frac{(n+1)k(k-1)(n^2+k-1)}{2(n^2+1)}$, $C_{k,n} = \frac{n(n+1)(k-1)(k-2)^2}{2(n^2+1)} + \frac{(n+1)(k-1)^2(k-2)}{2(n^2+1)}$.

For all $k \ge 2$, $n \in \mathbb{N}$ and $|z| \le r$, $r \ge 1$, we easily obtain

$$|C_{k,n}| \le (k-1)(k-2)(2k-3),$$

it follows that

$$\left|\frac{z^{k-2}C_{k,n}}{n^2+k-1}\right| \le \frac{(2k^3 - 9k^2 + 13k - 6)r^k}{n^2}.$$

By simple computation, we have $B_{k,n} = \frac{1}{2(n^2+1)}\{2n(k-1) - n(n+1)(k-1)^2(k-2) - n(n+1)(k-1)(k-2)^2 + 2(n^2+1)(k-1) - (n+1)(k-1)^2(k-2) + n^2(k-1)(k-2) - nk(k-1)^2 - n^2k(k-1) - k(k-1)^2\}$, so, we can get

$$\left|\frac{z^{k-1}B_{k,n}}{n^2+k-1}\right| \le \frac{(5k^3 - 15k^2 + 18k - 8)r^k}{n^2}.$$

By simple computation, we have $A_{k,n} = \frac{1}{2(n^2+1)}\{-2n(k-1) + n(n+1)(k-1)^2(k-2) + 2n^2 - n^2(k-1)(k-2) - 2(n^2k + k - 1) + nk(k-1)^2 + n^2k(k-1) + k(k-1)^2\}$, so, we can get

$$\left|\frac{z^k A_{k,n}}{n^2+k-1}\right| \le \frac{(3k^3 - 6k^2 + 8k - 2)r^k}{n^2}.$$

Thus, for all $k \geq 2$, $n \in \mathbb{N}$ and $|z| \leq r$, $r \geq 1$, we can obtain

$$|G_{k,n}(z)| \leq \frac{r^k}{n^2} D_k,$$

where$D_k = 10k^3 - 30k^2 + 39k - 16$.

By formula (5), for all $k \geq 2$, $n \in \mathbb{N}$ and $|z| \leq r$, $r \geq 1$, we have

$$|E_{k,n}(z)| \leq \frac{r(1+r)}{n} |(E_{k-1,n}(z))'| + r|E_{k-1,n}(z)| + |G_{k,n}(z)|.$$

Using the estimate in the proof of Theorem 1 (i), we get

$$|M_n(e_k; z) - e_k(z)| \leq \frac{1+r}{n} k(k-1)r^{k-1},$$

for all $k, n \in \mathbb{N}$, $|z| \leq r$, $r \geq 1$.

So, denote $\|f\|_r = max\{|f(z)|; |z| \leq r\}$, we have

$$|(E_{k-1.n}(z))'| \leq \frac{k-1}{r} \parallel E_{k-1,n} \parallel_r$$

$$\leq \frac{k-1}{r} \left[\|M_n(e_{k-1}; \cdot) - e_{k-1}\|_r + \| \frac{(n+1)(k-1)(k-2)(1-e_1)e_{k-2}}{2(n^2+1)} \|_r \right]$$

$$\leq \frac{k-1}{r} \left[\frac{(k-1)(k-2)(1+r)r^{k-2}}{n} + \frac{(k-1)(k-2)(1+r)r^{k-2}}{n} \right]$$

$$\leq \frac{4(k-2)(k-1)^2 r^{k-1}}{n},$$

for all $n \in \mathbb{N}$, $k \geq 2$ and $|z| \leq r$, $r \geq 1$.

It follows

$$|E_{k,n}(z)| \leq \frac{4(k-2)(k-1)^2(1+r)r^k}{n^2} + r|E_{k-1,n}(z)| + \frac{r^k}{n^2} D_k$$

$$:= \frac{r^k}{n^2} F_{k,r} + r|E_{k-1,n}(z)|,$$

where $F_{k,r}$ is a polynomial of degree 3 in k defined as $F_{k,r} = D_k + 4(k-2)(k-1)^2(1+r)$, D_k is expressed in the above.

Since $E_{0.n}(q; z) = E_{1.n}(q; z) = 0$ for any $z \in \mathbb{C}$, therefore, by writing the last inequality for $k = 2, 3, \ldots$, we easily obtain step by step the following

$$|E_{k,n}(z)| \leq \frac{r^k}{n^2} \sum_{j=2}^{k} F_{j,r} \leq \frac{(k-1)F_{k,r} r^k}{n^2}.$$

As a conclusion, we have

$$\left| M_n(f; z) - f(z) - \frac{(n+1)z(1-z)f''(z)}{2(n^2+1)} \right| \leq \sum_{k=2}^{\infty} |c_k||E_{k,n}(q; z)|$$

$$\leq \frac{1}{n^2} \sum_{k=2}^{\infty} |c_k|(k-1)F_{k,r} r^k.$$

As $f^{(4)}(z) = \sum_{k=4}^{\infty} c_k k(k-1)(k-2)(k-3)z^{k-4}$ and the series is absolutely convergent in $|z| \leq r$, it easily follows that $\sum_{k=4}^{\infty} |c_k| k(k-1)(k-2)(k-3)r^{k-4} < \infty$, which implies that $\sum_{k=2}^{\infty} |c_k|(k-1)F_{k,r}r^k < \infty$, this completes the proof of theorem.

In the following theorem, we will obtain the exact order in approximation.

Theorem 3. Let $R > 1$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. Suppose that $f : D_R \to \mathbf{C}$ is analytic in D_R. If f is not a polynomial of degree ≤ 1 , then for any $r \in [1, R)$ we have

$$\|M_n(f;\cdot) - f\|_r \geq \frac{C_r(f)}{n}, \ n \in \mathbf{N},$$

where $\|f\|_r = max\{|f(z)|; |z| \leq r\}$ and the constant $C_r(f) > 0$ depends on f, r but it is independent of n.

Proof. Denote $e_1(z) = z$ and

$$H_n(f; z) = M_n(f; z) - f(z) - \frac{(n+1)z(1-z)}{2(n^2+1)} f''(z).$$

For all $z \in D_R$ and $n \in \mathbf{N}$, we have

$$M_n(f; z) - f(z) = \frac{n+1}{2(n^2+1)} \left\{ z(1-z)f''(z) + \frac{2(n^2+1)}{n^2(n+1)} \left[n^2 H_n(f; z)\right] \right\}.$$

In view of the property: $\|F+G\|_r \geq |\|F\|_r - \|G\|_r| \geq \|F\|_r - \|G\|_r$, it follows

$$\|M_n(f;\cdot) - f\|_r \geq \frac{n+1}{2(n^2+1)} \left\{ \|e_1(1-e_1)f''\|_r - \frac{2(n^2+1)}{n^2(n+1)} \left[n^2 ||H_n(f;\cdot)||_r\right] \right\}.$$

Considering the hypothesis that f is not a polynomial of degree ≤ 1 in D_R, we have

$$\|e_1(1-e_1)f''\|_r > 0.$$

Indeed, supposing the contrary, it follows that

$$z(1-z)f''(z) = 0, \ \text{for all } z \in \overline{D_r}.$$

By hypothesis that $f(z)$ is analytic in D_R, we can denote $f(z) = \sum_{k=0}^{\infty} c_k z^k$, the identification of the coefficients method immediately leads to $c_k = 0, k = 2, 3, \cdots$. This implies that f is a polynomial of degree ≤ 1 on $\overline{D_r}$, a contradiction with the hypothesis.

Using the inequality (4), we get

$$n^2 \|H_n(f;\cdot)\|_r \leq M_r(f),$$

therefore, there exists an index n_0 depending only on f, r, such that for all $n \geq n_0$, we have

$$\|e_1(1-e_1)f''\|_r - \frac{2(n^2+1)}{n^2(n+1)} \left[n^2 ||H_n(f;\cdot)||_r\right] \geq \frac{1}{2} \|e_1(1-e_1)f''\|_r \, ,$$

which implies

$$\|M_n(f;\cdot)-f\|_r \geq \frac{n+1}{4(n^2+1)}\left\|e_1(1-e_1)f''\right\|_r \geq \frac{1}{4n}\left\|e_1(1-e_1)f''\right\|_r, \text{ for all } n \geq n_0.$$

For $n \in \{1, 2, \cdots, n_0 - 1\}$, we have

$$\|M_n(f;\cdot) - f\|_r \geq \frac{W_{r,n}(f)}{n},$$

where $W_{r,n}(f) = n\|M_n(f;\cdot) - f\|_r > 0$.

As a conclusion, we have

$$\|M_n(f;\cdot) - f\|_r \geq \frac{C_r(f)}{n}, \text{ for all } n \in \mathbf{N},$$

where

$$C_r(f) = min\left\{W_{r,1}(f), W_{r,2}(f), \ldots, W_{r,n_0-1}(f), \frac{1}{4}\|e_1(1-e_1)f''\|_r\right\},$$

this complete the proof.

Combining Theorem 3 with Theorem 1, we get the following result.

Corollary 2. Let $R > 1$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. Suppose that $f : D_R \to \mathbf{C}$ is analytic in D_R. If f is not a polynomial of degree ≤ 1 , then for any $r \in [1, R)$ we have

$$\|M_n(f;\cdot) - f\|_r \asymp \frac{1}{n}, \quad n \in \mathbf{N},$$

where $\|f\|_r = max\{|f(z)|; |z| \leq r\}$ and the constants in the equivalence depend on f, r but it is independent of n.

Theorem 4. Let $R > 1$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. Suppose that $f : D_R \to \mathbf{C}$ is analytic in D_R. Also, let $1 \leq r < r_1 < R$ and $p \in \mathbf{N}$ be fixed. If f is not a polynomial of degree $\leq max\{1, p-1\}$, then we have

$$\|(M_n(f;\cdot))^{(p)} - f^{(p)}\|_r \asymp \frac{1}{n}, \; n \in \mathbf{N},$$

where $\|f\|_r = max\{|f(z)|; |z| \leq r\}$ and the constants in the equivalence depend on f, r, r_1, p, but it is independent of n.

Proof. Taking into account the upper estimate in Theorem 1, it remains to prove the lower estimate only.

Denoting by Γ the circle of radius $r_1 > r$ and center 0 , by the Cauchy's formula it follows that for all $|z| \leq r$ and $n \in \mathbb{N}$ we have

$$M_n^{(p)}(f;z) - f^{(p)}(z) = \frac{p!}{2\pi i}\int_\Gamma \frac{M_n(f;v) - f(v)}{(v-z)^{p+1}}dv.$$

Keeping the notation there for $H_n(f;z)$, for all $n \in \mathbb{N}$, we have

$$M_n(f;z) - f(z) = \frac{n+1}{2(n^2+1)}\left\{z(1-z)f''(z) + \frac{2(n^2+1)}{n^2(n+1)}\left[n^2 H_n(f;z)\right]\right\}.$$

By using Cauchy's formula, for all $v \in \Gamma$, we get

$$M_n^{(p)}(f;z) - f^{(p)}(z) = \frac{n+1}{2(n^2+1)}\left\{[z(1-z)f''(z)]^p + \frac{2(n^2+1)}{n^2(n+1)}\frac{p!}{2\pi i}\int_\Gamma \frac{n^2 H_n(f;v)}{(v-z)^{p+1}}dv\right\},$$

passing now to $\|\cdot\|_r$ and denoting $e_1(z) = z$, it follows

$$\|M_n^{(p)}(f;\cdot) - f^{(p)}\|_r \geq \frac{n+1}{2(n^2+1)}\left\{\left\|[e_1(1-e_1)f'']^{(p)}\right\|_r - \frac{2(n^2+1)}{n^2(n+1)}\left\|\frac{p!}{2\pi i}\int_\Gamma \frac{n^2 H_n(f;v)}{(v-\cdot)^{p+1}}dv\right\|_r\right\},$$

Since for any $|z| \leq r$ and $v \in \Gamma$ we have $|v - z| \geq r_1 - r$, so, by using Theorem 2, we get

$$\left\|\frac{p!}{2\pi i}\int_\Gamma \frac{n^2 H_n(f;v)}{(v-\cdot)^{p+1}}dv\right\|_r \leq \frac{p!}{2\pi}\frac{2\pi r_1 n^2 \|H_n(f;\cdot)\|_{r_1}}{(r_1-r)^{p+1}} \leq \frac{M_{r_1}(f)p!r_1}{(r_1-r)^{p+1}}.$$

By hypothesis on f, we have

$$\|[e_1(1-e_1)f'']^{(p)}\|_r > 0.$$

Indeed, supposing the contrary, it follows that $\|[e_1(1-e_1)f'']^{(p)}\|_r = 0$, that is $z(1-z)f''(z)$ is a polynomial of degree $\leq p-1$. let $p = 1$ and $p = 2$, then the analyticity of f obviously implies that f is a polynomial of degree $\leq 1 = max(1, p-1)$, a contradiction.

Now let $p \geq 3$, then the analyticity of f obviously implies that f is a polynomial of degree $\leq p-1 = max(1, p-1)$, a contradiction with the hypothesis.

In conclusion, $\|[e_1(1-e_1)f'']^{(p)}\|_r > 0$ and in continuation, reasoning exactly as in the proof of Theorem 3, we can get the desired conclusion.

Remark 1. If we use King's approach to consider King type modification of the complex extension of the operators which was given by (1), we will obtain better approximation (cf. [21-23]).

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On the Convergence of Mann and Ishikawa Type Iterations in the Class of Quasi Contractive Operators

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Abstract

In this paper, we introduce two new iteration schemes, namely modified Mann and modified Ishikawa to approximate the fixed points of quasi contractive operators on a normed space. Various test problems are presented to reveal the validity and high efficiency of these iterative schemes.

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1 Introduction and preliminaries

In the last few decades, various researchers have explored the fixed points of contractive type operators in metric spaces, Hilbert spaces and different classes of Banach spaces, see [1] and references there in. To approximate unique fixed point of strict contractive type operators, Picard iterative scheme can be used effectively [1, 10, 15, 16]. But this scheme does not generally converge for the operators with slightly weaker contractive

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conditions. For such operators, Mann iteration [13] (cf. [6, 14]), Ishikawa iteration [7] and Krasnosel'okiĭ iteration [11] (cf. [3]) are much useful.

Let E be a normed space and C a nonempty convex subset of E. Let $T : C \to C$ be an operator and $\{\alpha_n\}$ and $\{\beta_n\}$ sequences of real numbers in $[0, 1]$.

The Mann iteration [13] is defined by the sequence $\{x_n\}_{n=0}^{\infty}$ as

$$x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n T x_n, \quad n \geq 0. \tag{1.1}$$

The sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)\, x_n + \alpha_n T y_n, \quad n \geq 0, \\ y_n &= (1 - \beta_n)\, x_n + \beta_n T x_n, \quad n \geq 0 \end{aligned} \tag{1.2}$$

is called Ishikawa iteration [7].

It is noticeable that for $\alpha_n = \lambda$ (constant), the iterative procedure (1.1) turn into Krasnosel'okiĭ iteration. Also for $\beta_n = 0$, Ishikawa iteration(1.2) reduces to Mann iteration (1.1).

Definition 1.1. Let (X, d) be a metric space and $a \in (0, 1)$. A mapping $T : X \to X$ satisfying

$$d\,(Tx, Ty) \leq a d\,(x, y) \quad \text{for all } x, y \in X \tag{1.3}$$

is called a contraction.

The following theorem is the classical Banach's contraction principle and of fundamental importance in the study of Fixed Point Theory.

Theorem 1.2. *Let (X, d) be a complete metric space and $T : X \to X$ be a contraction. Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by*

$$x_{n+1} = T x_n, \quad n \geq 0 \tag{1.4}$$

converges to p for any $x_0 \in X$.

The contraction in the above theorem forces T to be continuous. Despite this condition, Theorem 1.2 has many applications in solving the nonlinear equation $f\,(x) = 0$. Kannan [9] developed a fixed point theorem by relaxing the condition of continuity of T. He produced the following by taking b in $\left(0, \frac{1}{2}\right)$:

$$d\,(Tx, Ty) \leq b\,[d\,(x, Tx) + d\,(y, Ty)] \quad \text{for all } x, y \in X. \tag{1.5}$$

Chatterjea [4] obtained a similar result by considering $c \in \left(0, \frac{1}{2}\right)$ as follows:

$$d\,(Tx, Ty) \leq c\,[d\,(x, Ty) + d\,(y, Tx)] \quad \text{for all } x, y \in X. \tag{1.6}$$

In 1972, Zamferescu [17] proved the following very interesting and important fixed point theorem by taking into account (1.3), (1.5) and (1.6).

Theorem 1.3. *Let (X, d) be a metric space and $T : X \to X$ be a mapping for which there exist real numbers a, b and c satisfying $0 < a < 1$, $0 < b$ and $c < \frac{1}{2}$ such that for each $x, y \in X$, at least one of the following is true:*

(z_1) $d(Tx, Ty) \leq ad(x, y)$,

(z_2) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$,

(z_3) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \quad n \geq 0$$

converges to p for any $x_0 \in X$.

An operator $T : X \to X$ satisfying the contractive conditions (z_1), (z_2) and (z_3) is called Zamferescu operator.

In 1974, Ćirić [5] obtained a more general contraction to approximate unique fixed point with the help of Picard iteration: there exists $0 < h < 1$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq h \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \tag{1.7}$$

Definition 1.4. Let (X, d) be a metric space and $T : X \to X$ a mapping satisfying (1.7). Then T is called quasi contraction.

A new class of operators on an arbitrary Banach space E, satisfying

$$\|Tx - Ty\| \leq \delta \|x - y\| + 2\delta \|x - Tx\| \quad \text{for all } x, y \in E,\ 0 \leq \delta < 1, \tag{1.8}$$

was established by Berinde [2] in 2004. He approximated fixed points of this class of operators via Ishikawa iteration.

It is well known that a nonlinear equation $f(x) = 0$ can be expressed in terms of fixed point iteration method as follows:

$$x = Tx. \tag{1.9}$$

Taking up the technique of [8], if $T'x \neq 1$, $\theta \neq -1$, it can easily be seen by adding θx to both sides of (1.9) that

$$x = \frac{\theta x + Tx}{1 + \theta} = T_\theta x. \tag{1.10}$$

So as to make (1.10) to be efficient, we can choose $T'_\theta x = 0$, which gives

$$\theta = -T'x. \tag{1.11}$$

Now we are in a position to define modified Mann and modified Ishikawa iterative schemes.

Replacing Tx_n and Ty_n in (1.1) and (1.2) with $T_\theta x_n$ and $T_\theta y_n$, respectively, we get

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T_\theta x_n \tag{1.12}$$

and

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n T_\theta y_n, \\ y_n &= (1 - \beta_n) x_n + \beta_n T_\theta x_n. \end{aligned} \tag{1.13}$$

Using (1.10) in (1.12) and (1.13) and also the error term, we obtain

$$x_{n+1} = \left(1 - \frac{1}{1+\theta}\alpha_n\right) x_n + \frac{1}{1+\theta}\alpha_n T x_n + \mu_n \tag{1.14}$$

and

$$\begin{aligned} x_{n+1} &= \left[1 - \frac{\alpha_n}{1+\theta}\left(1 + \frac{\theta\beta_n}{1+\theta}\right)\right] x_n + \frac{\alpha_n}{1+\theta}\left[\frac{\theta\beta_n}{1+\theta} T x_n + T y_n\right] + \mu_n, \\ y_n &= \left(1 - \frac{1}{1+\theta}\beta_n\right) x_n + \frac{1}{1+\theta}\beta_n T x_n + \nu_n. \end{aligned} \tag{1.15}$$

We call the procedures defined in (1.14) and (1.15), the modified Mann and modified Ishikawa iterative procedures. It is obvious that (1.14) and (1.15) without error term reduce to (1.1) and (1.2), respectively for $\theta = 0$.

In this paper, we have proved the strong convergence of quasi contractive operator T satisfying (1.14) and (1.15) in the setting of normed space. We also present some test problems to compare the iterative procedures defined in (1.1), (1.2), (1.14) and (1.15). The numerical results obtained demonstrate the high performance and efficiency of modified Mann and modified Ishikawa iterative processes.

We use the following lemma in the sequal.

Lemma 1.5. ([12]) *Let $\{r_n\}$, $\{s_n\}$, $\{t_n\}$ and $\{k_n\}$ be the sequences of nonnegative numbers satisfying*

$$r_{n+1} \leq (1 - s_n) r_n + s_n t_n + k_n, \quad n \geq 0.$$

If $\sum_{n=0}^{\infty} s_n = \infty$ and $\lim_{n\to\infty} t_n = 0$ and $\sum_{n=0}^{\infty} k_n < \infty$ hold, then $\lim_{n\to\infty} r_n = 0$.

2 Main results

Assuming that the operator T has at least one fixed point, we prove the covergence theorems for iterative procedures (1.14) and (1.15).

Theorem 2.1. *Let C be a nonempty closed convex subset of a normed space E and $T : C \to C$ be an operator satisfying* (1.8). *For arbitrary $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by the iterative process* (1.14) *satisfying $\theta > -1$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\|\mu_n\| = 0(\alpha_n)$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T.*

Proof. Let p be the fixed point of the operator T. We consider

$$\begin{aligned} &\|x_{n+1} - p\| \\ &= \left\| \left(1 - \frac{1}{1+\theta}\alpha_n\right) x_n + \frac{1}{1+\theta}\alpha_n T x_n + \mu_n - p \right\| \\ &= \left\| \left(1 - \frac{1}{1+\theta}\alpha_n\right) x_n + \frac{1}{1+\theta}\alpha_n T x_n + \mu_n - \left(1 - \frac{1}{1+\theta}\alpha_n + \frac{1}{1+\theta}\alpha_n\right) p \right\| \\ &= \left\| \left(1 - \frac{1}{1+\theta}\alpha_n\right)(x_n - p) + \frac{1}{1+\theta}\alpha_n (T x_n - p) + \mu_n \right\| \\ &\leq \left(1 - \frac{1}{1+\theta}\alpha_n\right) \|x_n - p\| + \frac{1}{1+\theta}\alpha_n \|T x_n - p\| + \|\mu_n\|. \end{aligned} \tag{2.1}$$

Substituting $y = x_n$ and $x = p$ in (1.8), we get

$$\|Tx_n - p\| \le \delta \|x_n - p\|.$$

Thus (2.1) implies

$$\begin{aligned}\|x_{n+1} - p\| &\le \left(1 - \frac{1}{1+\theta}\alpha_n\right)\|x_n - p\| + \frac{\delta}{1+\theta}\alpha_n \|x_n - p\| + \|\mu_n\| \\ &= \left(1 - \frac{1}{1+\theta}\alpha_n + \frac{\delta}{1+\theta}\alpha_n\right)\|x_n - p\| + \|\mu_n\| \\ &= \left(1 - \frac{1-\delta}{1+\theta}\alpha_n\right)\|x_n - p\| + \|\mu_n\|.\end{aligned}$$

Using Lemma 1.5 and the fact that $0 \le \delta < 1$, $0 \le \alpha_n \le 1$, $\theta > -1$, $\|\mu_n\| = 0(\alpha_n)$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, we obtain

$$\lim_{n\to\infty} \|x_n - p\| = 0.$$

Hence $x_n \to p$. This completes the proof. □

Taking $\theta = 0$ in the setting of normed space and the contraction condition (1.8), we obtain the following corollary.

Corollary 2.2. *Let C be a nonempty closed convex subset of a normed space E and $T : C \to C$ be an operator satisfying* (1.8). *For arbitrary $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by the iterative process* (1.1) *satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T.*

Now we prove the convergence of modified Ishikawa iterative process in the form of the following theorem.

Theorem 2.3. *Let C a nonempty closed convex subset of a normed space E and $T : C \to C$ be an operator satisfying* (1.8). *For arbitrary $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by the iterative process* (1.15) *satisfying $\theta > -1$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\|\nu_n\| = 0(\alpha_n)$ and $\|\mu_n\| = 0(\alpha_n)$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T.*

Proof. Let p be the fixed point of the operator T. We consider

$$\begin{aligned}&\|x_{n+1} - p\| \\ &= \|(1-\alpha_n)x_n + \alpha_n T_\theta y_n + \mu_n - p\| \\ &= \left\|(1-\alpha_n)x_n + \alpha_n\left(\frac{\theta y_n + Ty_n}{1+\theta}\right) + \mu_n - p\right\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\left\|\frac{Ty_n + \theta y_n}{1+\theta} - p\right\| + \|\mu_n\| \\ &= (1-\alpha_n)\|x_n - p\| + \alpha_n\left\|\frac{(Ty_n - p) + \theta(y_n - p)}{1+\theta}\right\| + \|\mu_n\| \\ &\le (1-\alpha_n)\|x_n - p\| + \frac{\alpha_n}{1+\theta}\|Ty_n - p\| + \frac{\theta\alpha_n}{1+\theta}\|y_n - p\| + \|\mu_n\|. \qquad (2.2)\end{aligned}$$

Substituting $x = p$ and $y = y_n$ in (1.8), we get

$$\|Ty_n - p\| \leq \delta \|y_n - p\|. \tag{2.3}$$

Thus (2.2) implies

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \frac{\delta\alpha_n}{1+\theta}\|y_n - p\| + \frac{\theta\alpha_n}{1+\theta}\|y_n - p\| + \|\mu_n\|. \\
&= (1-\alpha_n)\|x_n - p\| + \frac{\delta+\theta}{1+\theta}\alpha_n\|y_n - p\| + \|\mu_n\|.
\end{aligned} \tag{2.4}$$

Consider

$$\begin{aligned}
\|y_n - p\| &= \left\|\left(1 - \frac{1}{1+\theta}\beta_n\right)x_n + \frac{1}{1+\theta}\beta_n Tx_n + \nu_n - p\right\| \\
&= \left\|\left(1 - \frac{1}{1+\theta}\beta_n\right)(x_n - p) + \frac{1}{1+\theta}\beta_n(Tx_n - p) + \nu_n\right\| \\
&\leq \left(1 - \frac{1}{1+\theta}\beta_n\right)\|x_n - p\| + \frac{1}{1+\theta}\beta_n\|Tx_n - p\| + \|\nu_n\|.
\end{aligned} \tag{2.5}$$

Substituting $x = p$ and $y = x_n$ in (1.8), we get

$$\|Tx_n - p\| \leq \delta\|x_n - p\|. \tag{2.6}$$

Thus (2.5) implies

$$\begin{aligned}
\|y_n - p\| &\leq \left(1 - \frac{1}{1+\theta}\beta_n\right)\|x_n - p\| + \frac{\delta}{1+\theta}\beta_n\|x_n - p\| + \|\nu_n\| \\
&= \left(1 - \frac{1-\delta}{1+\theta}\beta_n\right)\|x_n - p\| + \|\nu_n\|.
\end{aligned} \tag{2.7}$$

Using (2.7) in (2.4), we get

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \frac{\delta+\theta}{1+\theta}\alpha_n\left[\left(1 - \frac{1-\delta}{1+\theta}\beta_n\right)\|x_n - p\| + \|\nu_n\|\right] + \|\mu_n\| \\
&= \left[1 - \alpha_n + \frac{\delta+\theta}{1+\theta}\alpha_n\left(1 - \frac{1-\delta}{1+\theta}\beta_n\right)\right]\|x_n - p\| + \frac{\delta+\theta}{1+\theta}\alpha_n\|\nu_n\| + \|\mu_n\| \\
&= \left[1 - \alpha_n\left\{1 - \frac{\delta+\theta}{1+\theta}\left(1 - \frac{1-\delta}{1+\theta}\beta_n\right)\right\}\right]\|x_n - p\| \\
&\quad + \frac{\delta+\theta}{1+\theta}\alpha_n\|\nu_n\| + \|\mu_n\|.
\end{aligned} \tag{2.8}$$

Let

$$\begin{aligned}
A_n &= 1-\alpha_n\left[1-\frac{\delta+\theta}{1+\theta}\left(1-\frac{1-\delta}{1+\theta}\beta_n\right)\right]\\
&= 1-\alpha_n\left(1-\frac{\delta+\theta}{1+\theta}+\frac{(\delta+\theta)(1-\delta)}{(1+\theta)^2}\beta_n\right)\\
&= 1-\alpha_n\left(1-\frac{(1+\theta)(\delta+\theta)-(\delta+\theta)(1-\delta)\beta_n}{(1+\theta)^2}\right)\\
&= 1-\alpha_n\left(\frac{(1+\theta)^2-(1+\theta)(\delta+\theta)+(\delta+\theta)(1-\delta)\beta_n}{(1+\theta)^2}\right)\\
&= 1-\alpha_n\left(\frac{(1-\delta)(1+\theta)+(\delta+\theta)(1-\delta)\beta_n}{(1+\theta)^2}\right)\\
&= 1-\frac{(1-\delta)}{(1+\theta)}\alpha_n\left(\frac{(1+\theta)+(\delta+\theta)\beta_n}{(1+\theta)}\right)\\
&= 1-\frac{(1-\delta)}{(1+\theta)}\alpha_n\left(1+\frac{\delta+\theta}{1+\theta}\beta_n\right). \qquad (2.9)
\end{aligned}$$

Since $\beta_n \geq 0$, $0 \leq \delta < 1$ and $\theta > -1$, therefore $\frac{\delta+\theta}{1+\theta}\beta_n \geq 0$ and $1+\frac{\delta+\theta}{1+\theta}\beta_n \geq 1$.

Hence (2.9) gives

$$A_n = 1-\frac{1-\delta}{1+\theta}\alpha_n\left(1+\frac{\delta+\theta}{1+\theta}\beta_n\right) \leq 1-\frac{1-\delta}{1+\theta}\alpha_n.$$

Thus from (2.8), we get

$$\|x_{n+1}-p\| \leq \left(1-\frac{1-\delta}{1+\theta}\alpha_n\right)\|x_n-p\| + \frac{\delta+\theta}{1+\theta}\alpha_n\|\nu_n\| + \|\mu_n\|.$$

With the help of Lemma 1.5 and using the fact that $0 \leq \delta < 1$, $0 < \alpha_n < 1$, $\theta > -1$, $\|\nu_n\| = 0(\alpha_n)$, $\|\mu_n\| = 0(\alpha_n)$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$, we get

$$\lim_{n\to\infty}\|x_n-p\| = 0.$$

Consequently, $x_n \to p \in F$ and this completes the proof. □

Corollary 2.4. *Let C a nonempty closed convex subset of a normed space E and $T : C \to C$ be an operator satisfying* (1.8)*. For arbitrary $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by the iterative process* (1.2) *satisfying $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T.*

The above corollary in fact is the generalization of Theorem 2 of Berinde [2] in the context of a normed space and the contraction condition (1.8).

3 Applications

In this section, we consider various test problems to apply Mann (M), modified Mann (MM), Ishikawa (I) and modified Ishikawa (MI) iterative procedures for the estimation of fixed points. The data in the following table indicates the rapidness of convergence in each problem. We make use of Maple software and 10^{-3} tolerence for the purpose. Here we denote the number of iterations (NI).

Tx	θ	α_n	β_n	x_0	$Method$	NI	$x[k]$	Tx	$\lvert x[k]-Tx\rvert$
$3-x^2$	$2x$	$\frac{1}{\sqrt{n+1}}$	$\frac{1}{\sqrt{n+1}}$	1	M	9	1.3044	1.2985	0.0059
					MM	4	1.3047	1.2977	0.0070
					I	22	1.3009	1.3076	0.0067
					MI	2	1.3009	1.3077	0.0068
$3^{(1-x)}-\cos x$	$\ln 3(3^{(1-x)})-\sin x$	$\frac{1}{1+n}$	$\frac{1}{1+n}$	0.5	M	4	0.6657	0.6572	0.0085
					MM	4	0.6576	0.6652	0.0076
					I	11	0.6570	0.6658	0.0088
					MI	1	0.6588	0.6641	0.0053
$1-x-\cos x$	$1-\sin x$	$\frac{1}{\sqrt{n+1}}$	$\frac{1}{\sqrt{n+1}}$	0.1	M	4	0.0000	−0.0000	0.0000
					MM	1	−0.0026	0.0026	0.0052
					I	6	0.0037	−0.0036	0.0073
					MI	1	0.0001	−0.0001	0.0002
$\cos x-e^x+1$	$\sin x+e^x$	$\frac{1}{\sqrt{n+1}}$	$\frac{1}{\sqrt{n+1}}$	0.4	M	5	0.4120	0.4066	0.0054
					MM	1	0.4101	0.4100	0.0001
					I	12	0.4076	0.4150	0.0074
					MI	1	0.4101	0.4101	0.0000
$1-\frac{x}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{n+1}}$	$\frac{1}{\sqrt{n+1}}$	0.5	M	2	0.6616	0.6692	0.0076
					MM	1	0.6667	0.6667	0.0000
					I	3	0.6616	0.6692	0.0076
					MI	1	0.6667	0.6667	0.0000
$\frac{x}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{n+1}}$	$\frac{1}{\sqrt{n+1}}$	0.5	M	12	0.0181	0.0091	0.0091
					MM	1	0.0000	0.0000	0.0000
					I	6	0.0154	0.0077	0.0077
					MI	1	0.0000	0.0000	0.0000
$e^{(1-x)^2}-1$	$2(1-x)e^{(1-x)^2}$	$\frac{1}{1+n}$	$\frac{1}{1+n}$	0.5	M	4	0.4160	0.4065	0.0095
					MM	2	0.4089	0.4182	0.0093
					I	13	0.4159	0.4067	0.0092
					MI	1	0.4136	0.4104	0.0032

4 Conclusion

We have developed two new iterative schemes, namely modified Maan and modified Ishikawa. The convergence theorems for our proposed schemes have been proved. In Section 2, the table provides comparison between Mann, modified Mann, Ishikawa and modified Ishikawa iterative procedures. Our results clearly indicate that how rapidly our proposed methods converge to the fixed points. In some given test problems, due to large difference in number of iterations, it is obvious that modified Mann and modified Ishikawa iterative schemes require very little time to produce fixed point.

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MEAN ERGODIC THEOREMS FOR SEMIGROUPS OF LINEAR OPERATORS IN P-BANACH SPACES

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ABSTRACT. In this paper, by using the Rode's method, we extend Yosida's theorem to semigroups of linear operators in p-Banach spaces. Our paper is motivated from ideas in [7].

1. Introduction

In 1938, Yosida [14] Proved the following mean ergodic theorem for linear operators: Let E be a real Banach space and T be a linear operator of E into itself such that there exists a constant C with $\|T^n\| \le C$ for $n = 1, 2, 3, ...$, and T is weakly completely continuous,i.e., T maps the closed unite ball of E into a weakly compact subset of E. Then, the Cesaro mean

$$S_n x = \frac{1}{n}\sum_{k=1}^{n} T^k x$$

converges strongly as $n \to +\infty$ to a fixed point of T for each $x \in E$.
On the other hand, in 1975, Baillon [1] proved the following nonlinear ergodic theorem: Let X be a Banach space and C a closed convex subset of X. The mapping $T : C \to C$ is called nonexpansive on C if

$$\|Tx - Ty\| \le \|x - y\| \qquad \forall x, y \in C.$$

Let $F(T)$ be the set of fixed point of T. If X is stricly convex, $F(T)$ is closed and convex. In [1, 4], Baillon proved the first nonlinear ergodic theorem such that if X is a real Hilbert space and $F(T) \ne \emptyset$, then for each $x \in C$, the sequence $\{S_n x\}$ defined by

$$S_n x = (\frac{1}{n})(x + Tx + ... + T^{n-1}x)$$

converges weakly to a fiexd point of T. It was also shown by Pazy [8] thet if X a real Hilbert space and $S_n x$ converges weakly to $y \in C$, then $y \in F(T)$.

Recently, Rode [10] and Takahashi [13] tried to extend this nonlinear ergodic theorem to semigroup, generalizing the Cesaro means on $N = \{1, 2, ...\}$, such that the corresponding sequence of mappings converges to a projection onto the set of common fixed points. In this paper, by using the Rode's method, we extend Yosida's theorem to semigroups of linear operators in P-Banach spaces. The proofs employ the methods of Yosida[14], Greenleaf [5], Rode [10] and Takahashi [6, 12] . Our paper is motivated from ideas in [7]

2. p-Norm

Definition 2.1. ([3, 11]) Let X be a real linear space. A function $\|.\| : X \to \mathbb{R}$ is a quasi–norm (valuation) if it satisfies the following conditions :
(1) $\|x\| \ge 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;

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(2) $\|\lambda x\| = |\lambda|.\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$;
(3) There is a constant $M \geq 1$ such that $\|x+y\| \leq M(\|x\| + \|y\|)$ for all $x, y \in X$.
Then $(X, \|.\|)$ is called a quasi-normed space. The smallest possible M is called the modulus of concavity of $\|.\|$. A quasi-Banach space is a complete quas-normed space.

A quasi-norm $\|.\|$ is called a p-norm $0 < p < 1$ if

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a p-Banach space.
By the Aoki-Rolewicz [11], each quasi-norm is equivalent to some p-norm (see also [9]).
Since it is much easier to work with p-norm, henceforth we restrict our attention mainly to p-norms.

3. **Preliminaries and lemmas**

Let E a real p-Banach space and let E^* be the conjugate space of E, that is, the space of all continuous linear functionals on E. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $< x, x^* >$. We denote by co$D$ the convex hull of D, $\overline{co}D$ the closure of coD.
Let U be a linear continuous operator of E into itself. Then, we denote by U^* the conjugate operator of U.

Assumption (A). Let $(E, \|.\|_p)$ be a p-Banach space and $\{T_t : t \in G\}$, be a family of linear continuous operators of a real Banach space E into itself such that there exist a real number C with $\|T_t\|_p \leq C$ for all $t \in G$ and the weak closure of $\{T_t x : t \in G\}$ is weakly compact, for each $x \in E$. The index set G is a topological semigroup such that $T_{st} = T_s.T_t$ for all $s, t \in G$ and T is continuous with respect to the weak operator topology : $< T_s x, x^* > \to < T_t x, x^* >$ for all $x \in E$ and $x^* \in E^*$ if $s \to t$ in G.

We denote by $m(G)$ the p-Banach space of all bounded continuous real valued functions on the topological semigroup G with the p-norm. For each $s \in G$ and $f \in m(G)$, we define elements $l_s f$ and $r_s f$ in $m(G)$ given by $l_s f(t) = f(st)$ and $r_s f(t) = f(ts)$ for all $t \in G$. An element $\mu \in m(G)^*$ (the conjugate space of $m(G)$) is called a mean on G if $\|\mu\|_p = \mu(1) = 1$. A mean μ on G is called left (right) invariant if $\mu(l_s f) = \mu(f)$ $(\mu(r_s f) = \mu(f))$ for all $f \in m(G)$ and $s \in G$. An invariant mean is a left and right invariant mean. We know that $\mu \in m(G)^*$ is a mean on G if and only if

$$\inf\{f(t) : t \in G\} \leq \mu(f) \leq \sup\{f(t) : t \in G\}$$

for every $f \in m(G)$; see [4, 5, 9].

Let $\{T_t : t \in G\}$ be a family of linear continuous operators of E into itself satisfying the assumption (A) and μ be a mean on G. Fix $x \in E$. Then, for $x^* \in E^*$, the real valued function $t \to < T_t x, x^* >$ is in $m(G)$. Denote by $\mu_t < T_t x, x^* >$ the valued of μ at this function. By linearity of μ and of $< ., . >$, this is linear in x^*; moreover, since

$$|\mu_t < T_t x, x^* > | \quad \leq \quad \|\mu\|_p \cdot \sup_t | < T_t x, x^* > | \leq \sup_t \|T_t x\|_p \cdot \|x^*\|_p \leq C \cdot \|x\|_p \cdot \|x^*\|_p,$$

it is continuous in x^*. Hence $\mu_t < T_t x, . >$ is an element of E^{**}. So it follows from weak compactness of $\overline{co}\{T_t x : t \in G\}$ that $\mu_t < T_t x, x^* > = < T_\mu x, x^* >$ for every $x^* \in E^*$.
Put $K = \overline{co}\{T_t x : t \in G\}$ and suppose that the element $\mu_t < T_t x, . >$ is not contained in the $n(K)$, where n is the natural embedding of the p-Banach space E into its second conjugate space E^{**}. Since the convex set $n(K)$ is compact in the *weak** topology of E^{**}, there exists an element $y^* \in E^*$ such that

$$\mu_t < T_t x, y^* > < \inf\{< y^*, z^{**} >: z^{**} \in n(k)\}$$

Hence we have

$\mu_t < T_t x, y^* > \quad < \quad \inf\{< y^*, z^{**} >: z^{**} \in n(k)\} \leq \inf\{< T_t x, y^* >: t \in G\} \leq \mu_t < T_t x, y^* > .$

This is a contradiction. Thus, for a mean μ on G, we can define a linear continuous operator T_μ of E into itself such that $\|T_\mu\|_p \leq C$, $T\mu x \in \overline{co}\{T_t x : t \in G\}$ for all $x \in E$, and $\mu_t < T_t x, x^* >=< T_\mu x, x^* >$ for all $x \in E$ and $x^* \in E^*$. we denote by $F(G)$ the set all common fixed points of the mappings T_t, $t \in G$.

Lemma 3.1. *Assume that a left invariant mean μ exists on G. Then $T_\mu(E) \subset F(G)$. Especially, $F(G)$ is not empty.*

Proof. Let $x \in E$ and μ be a left invariant mean on G. Then since, for $s \in G$ and x^*,

$$\begin{aligned} < T_s T_\mu x, x^* > &= < T_\mu x, T_s^* x^* >= \mu_t < T_t x, T_s^* x^* >= \mu_t < T_s T_t x, x^* > \\ &= \mu_t < T_{st} x, x^* >= \mu_t < T_t x, x^* >=< T_\mu x, x^* >, \end{aligned}$$

we have $T_s T_\mu x = T_\mu x$. Hence $T_\mu(E) \subset F(G)$. □

Lemma 3.2. *Let λ be an invariant mean on G. Then $T_\lambda T_s = T_s T_\lambda = T_\lambda$ for each $s \in G$ and $T_\lambda T_\mu = T_\mu T_\lambda = T_\lambda$ for each mean μ on G. Especially, T_λ is a projection of E onto $F(G)$.*

Proof. Let $s \in G$. Since

$$< T_\lambda T_s x, x^* >= \lambda_t < T_t T_s x, x^* >= \lambda_t < T_{ts} x, x^* >= \lambda_t < T_t x, x^* >=< T_\lambda x, x^* >$$

for $x \in E$ and $x^* \in E^*$, we have $T_\lambda T_s = T_\lambda$. It follows from Lemma 3.1 that $T_s T\lambda = T\lambda$ for each $s \in G$. Let μ_j be a mean on G. Then, since

$$< T_\mu T_\lambda x, x^* >= \mu_t < T_t T_\lambda x, x^* >= \mu_t T_\lambda x, x^* >=< T_\lambda x, x^* >$$

and

$$\begin{aligned} < T_\lambda T_\mu x, x^* > &= < T_\mu x, T_\lambda^* x^* >= \mu_t < T_t x, T_\lambda^* x^* >= \mu_t < T_\lambda T_t x, x^* > \\ &= \mu_t < T_\lambda x, x^* >=< T_\lambda x, x^* > \end{aligned}$$

for $x \in E$ and $x^* \in E^*$, we have $T_\mu T_\lambda = T_\lambda T_\mu = T_\lambda$, Putting $\mu = \lambda$, we have $T_\lambda^2 = T_\lambda$ and hence T_λ is a projection of E onto $F(G)$. □

As a direct consequence of Lemma 3.2, we have the following.

Lemma 3.3. *Let μ and λ be invariant means on G. Then $T_\mu = T_\lambda$.*

Lemma 3.4. *Assume that an invariant mean exists on G. Then, for each $x \in E$, the set $\overline{co}\{T_t x : t \in G\} \cap F(G)$ consists of a single point.*

Proof. Let $x \in E$ and μ be an invariant mean on G. Then, we know that $T_\mu x \in F(G)$ and $T_\mu x \in \overline{co}\{T_t x : t \in G\}$. So, we show that $\overline{co}\{T_t x : t \in G\} \cap F(G) = \{T_\mu x\}$. Let $x_0 \in \overline{co}\{T_t x : t \in G\} \cap F(G)$ and $\epsilon > 0$. Then, for $x^* \in E^*$, there exists an element $\sum_{i=1}^n \alpha_i T_{t_i} x$ in the set $co\{T_t x : t \in G\}$ such that $\epsilon > C \cdot \|x^*\|_p . \|\sum_{i=1}^n \alpha_i T_{t_i} x - x_0\|_p$. Hence we have

$$\begin{aligned} \epsilon &> C \cdot \|x^*\|_p \cdot \|\sum_{i=1}^n \alpha_i T_{t_i} x - x_0\|_p \geq \sup_t \|T_t\|_p \cdot \|\sum_{i=1}^n \alpha_i T_{t_i} x - x_0\|_p \cdot \|x^*\|_p \\ &\geq \sup_t \|\sum_{i=1}^n \alpha_i T_{j,t} T_{j,t_i} x - x_0\|_j \cdot \|x^*\|_j \geq | < \sum_{i=1}^n \alpha_i T_t T_{t_i} x - x_0, x^* > | \\ &= |\sum_{i=1}^n \alpha_i \mu_t < T_{tt_i} x - x_0, x^* > | = |\mu_t < T_t x - x_0, x^* > | = | < T_\mu x - x_0, x^* > |. \end{aligned}$$

Since ϵ is arbitrary, we have $< T_\mu x, x^* >=< x_0, x^* >$ for every $x^* \in E^*$ and hence $T_\mu x = x_0$. □

4. **Ergodic Theorems**

Now we can prove mean ergodic theorems for semigroups of linear continuous operators in p-Banach space.

Theorem 4.1. *Let $\{T_t : t \in G\}$ be a family of linear continuous operators in a real p-Banach space E satisfying Assumption **(A)**. If a net $\{\mu^\alpha : \alpha \in I\}$ of means on G is asymptotically invariant, i.e.,*

$$\mu^\alpha - r_s^*\mu^\alpha \qquad \text{and} \qquad \mu^\alpha - l_s^*\mu^\alpha$$

converge to 0 in the weak* topology of $m(G)^*$ for each $s \in G$, then there exists a projection Q of E onto $F(G)$ such that $\|Q\|_p \leq C$, $T_{\mu^\alpha}x$ converges weakly to Qx for each $x \in E$, $QT_t = T_tQ = Q$ for each $t \in G$, and $Qx \in \overline{co}\{T_tx : t \in G\}$ for each $x \in E$. Furthermore, the projection Q onto $F(G)$ is the same for all asymptotically invariant nets.

Proof. Let μ be a cluster point of net $\{\mu^\alpha : \alpha \in I\}$ in the *weak** topology of $m(G)^*$. Then μ is an invariant mean on G. Hence, by Lemma 3.2, T_μ is a projection of E onto $F(G)$ such that $\|T_\mu\|_p \leq C$, $T_\mu T_t = T_tT_\mu = T_\mu$ for each $t \in G$ and $T_\mu x \in \overline{co}\{T_tx : t \in G\}$ for each $x \in E$. Setting $Q = T_\mu$, we show that $T_{\mu^\alpha}x$ converges weakly to Qx for each $x \in E$. Since $T_{\mu^\alpha}x \in \overline{co}\{T_tx : t \in G\}$ for all $\alpha \in I$ and $\overline{co}\{T_tx : t \in G\}$ is weakly compact, there exists a subnet $\{T_{\mu^\beta}x : \beta \in J\}$ of $\{T_{\mu^\alpha}x : \alpha \in I\}$ such that $T_{\mu^\beta}x$ converges weakly to an element $x_0 \in \overline{co}\{T_tx : t \in G\}$. To show that $T_{\mu^\alpha}x$ converges weakly to Qx, it is sufficient to show $x_0 = Qx$. Let $x^* \in E^*$ and $s \in G$. since $T_{\mu^\beta}x \to x_0$ weakly, we have $\mu_t^\beta < T_tx, x^* > \to < x_0, x^* >$ and $\mu_t^\beta < T_tx, T_s^*x^* > \to < x_0, T_s^*x^* > = < T_sx_0, x^* >$. On the other hand, since $\mu^\beta - l_s^*\mu^\beta \to 0$ in the *weak** topology, we have

$$\begin{aligned} \mu_t^\beta < T_tx, x^* > - l_s^*\mu_t^\beta < T_tx, x^* > &= \mu_t^\beta < T_tx, x^* > - \mu_t^\beta < T_{st}x, x^* > \\ &= \mu_t^\beta < T_tx, x^* > - \mu_t^\beta < T_tx, T_s^*x^* > \\ &\to 0. \end{aligned}$$

Hence, we have $< x_0, x^* > = < T_sx_0, x^* >$ and hence $x_0 \in F(G)$. So, we obtain $Qx = T\mu x = x_0$ by Lemma 3.4. That the projection Q is the same for all asymptotically invariant nets is obvious from Lemma 3.3. □

As a direct consequence of Theorem 4.1, we have the following.

Corollary 4.2. *Let $\{T_t : t \in G\}$ be as in Theorem 4.1 and assume that an invariant mean exists on G. Then, there exists a projection Q of E onto F such that $\|Q\|_p \leq C$, $QT_t = T_tQ = Q$ for each $t \in G$ and $Qx \in \overline{co}\{T_tx : t \in G\}$ for each $x \in E$*

Theorem 4.3. *Let $\{T_t : t \in G\}$ be as in Theorem 4.1. If a net $\{\mu^\alpha : \alpha \in I\}$ of means on G is asymptotically invariant and further $\mu^\alpha - r_s^*\mu^\alpha$ converges to 0 in the strong topology of $m(G)^*$, then exists a projection Q of E onto $F(G)$ such that $\|Q\|_p \leq C$, $T_{\mu^\alpha}x$ converges strongly to Qx for each $x \in E$, $QT_t = T_tQ = Q$ for each $t \in G$, and $Qx \in \overline{co}\{T_tx : t \in G\}$ for each $x \in E$.*

Proof. As in the proof of Theorem 4.1, let $Q = T_\mu$, where μ is a cluster point of the net $\{\mu^\alpha : \alpha \in I\}$ in the *weak** topology of $m(G)^*$. Then we show that $T_{\mu^\alpha}x$ converges strongly to Qx for each $x \in E$.

Let $E_0 = \overline{co}\{y - T_ty : y \in E, t \in G\}$. Then, for any $z \in E_0$, $T_{\mu^\alpha}z$ converges strongly to 0. In fact, if $z = y - T_sy$, then since, for any $y^* \in E^*$,

$$\begin{aligned} | < T_{\mu^\alpha}z, y^* > | &= |\mu_t^\alpha < T_t(y - T_sy), y^* > | = |\mu_t^\alpha < T_ty, y^* > - \mu_t^\alpha < T_{ts}y, y^* > | \\ &= |(\mu_t^\alpha - r_s^*\mu_t^\alpha) < T_ty, y^* > | \leq \|\mu^\alpha - r_s^*\mu^\alpha\|_p \cdot \sup_t | < T_ty, y^* > | \\ &\leq \|\mu^\alpha - r_s^*\mu^\alpha\|_p \cdot C \cdot \|y\|_p \cdot \|y^*\|_p, \end{aligned}$$

we have $\|T_{\mu^\alpha} z\|_p \leq C \cdot \|\mu^\alpha - r_s^* \mu^\alpha\|_p \cdot \|y\|_p$. Using this inequality, we show that $T_{\mu^\alpha} z$ converges strongly to 0 for any $z \in E_0$. Let z be any element of E_0 and ϵ be any positive number. By the definition of E_0, there exists an element $\sum_{i=1}^n a_i(y_i - T_{s_i} y_i)\epsilon$ in the set $co\{y - T_s y : y \in E, s \in G\}$ such that $\epsilon > 2C \cdot \|z - \sum_{i=1}^n a_i(y_i - T_{s_i} y_i)\|_p$. On the other hand, from $\|\mu^\alpha - r_s^* \mu^\alpha\|_p \to 0$ for all $s \in G$, there exists $a_0 \in I$ such that, for all $\alpha \geq \alpha_0$ and $i = 1, 2, ..., n$,

$$\epsilon > \|\mu^\alpha - r_{s_i}^* \mu^\alpha\|_p \cdot 2C\|y_i\|_p.$$

This implies

$$\begin{aligned}
\|T_{\mu^\alpha} z\|_p &\leq \|T_{\mu^\alpha} z - T_{\mu^\alpha}(\sum_{i=1}^n a_i(y_i - T_{s_i} y_i))\|_p + \|T_{\mu^\alpha}(\sum_{i=1}^n a_i(y_i - T_{s_i} y_i))\|_p \\
&\leq \|T_{\mu^\alpha}\|_p \cdot \|z - \sum_{i=1}^n a_i(y_i - T_{s_i} y_i)\|_p + |\sum_{i=1}^n a_i|^P \|T_{\mu^\alpha}(y_i - T_{s_i} y_i)\|_p \\
&\leq C \cdot \|z - \sum_{i=1}^n a_i(y_i - T_{s_i} y_i)\|_P + \sup_i \|\mu^\alpha - r_{s_i}^* \mu^\alpha\|_p \cdot C \cdot \|y_i\|_p \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Hence $T_{\mu^\alpha} z$ converges strongly to 0 for each $z \in E_0$.
Next, assume that $x - T_\mu x$ for some $x \in E$ is not contained in the set E_0. Then, by the Hahn–Banach theorem, there exists a linear continuous functional y^* such that $< x - T_\mu x, y^* >= 1$ and $< z, y^* >= 0$ for all $z \in E_0$. and so since $x - T_t x \in E_0$ for all $t \in G$, we have

$$< x - T_{\mu_j} x, y^* >= \mu_t < x - T_t x, y^* >= 0.$$

This is a contradiction. Hence $x - T_\mu$ for all $x \in E$ are contained in E_0. Therefore, we have $T_{\mu^\alpha} x - T_\mu x = T_{\mu^\alpha}(x - T_\mu)$ converges strongly to 0 for all $x \in E$. This completes the proof. □

By using Theorem 4.3, we can obtain the following corollary.

Corollary 4.4. *Let E be a real p-Banach space and T be a linear operator of E into itself such that exists a constant C with $\|T^n\|_p \leq C$ for $n = 1, 2, ...$ Assume that T is weakly completely continuous, i.e., T maps the closed unit ball of E into a weakly compact subset of E. Then there exists a projection Q of E onto the set $F(T)$ of all fixed points of T such that $\|Q\|_p \leq C$, the Cesaro means $S_n = \frac{1}{n}\sum_{k=1}^n T^k x$ converges strongly to Qx for each $x \in E$, and $TQ = QT = Q$.*

Proof. Let $x \in E$. Then, since $\{T^n x : n = 1, 2, ...\} = T(\{T^{n-1} x : n = 1, 2, ...\}) \subset T(B(0, \|x \cdot (c + 1)))$, where $B(x, r)$ means the closed ball with center x and radius r, the weak closure of $\{T^n x : n = 1, 2, ...\}$ is weakly compact. On the other hand, let $G = \{1, 2, 3, ...\}$ with the discrete topology and μ^n be a mean on G such that $\mu^n(f) = \sum_{i=1}^n (\frac{1}{n}) f(i)$ for each $f \in m(G)$. Then, it is obvious that $\|\mu^n - r_k^* \mu^n\|_p \leq \frac{2k}{n} \to 0$ for all $k \in G$. So, it follows from Theorem 4.3 that the corollary is true. □

If $G = [0, \infty)$ whit the natural topology, then we obtain the corresponding result.

Corollary 4.5. *Let E be a real p-Banach space and $\{T_t : t \in [0, \infty)\}$ be a family of linear operators of E into itself satisfying Assumption* **(A)**. *Then there exists a projection Q of E onto $F(G)$ such that $\|Q\|_p \leq C$, $\frac{1}{T}\int_0^T T \int_t x dt$ converges strongly to Qx for each $x \in E$, and $T_t Q = QT_t = Q$ for each $t \in [0, \infty)$.*

Remark. $\frac{1}{T}\int_0^T T \int_t x dt$ is a weak vector valued integral with respect to means on $G = [0, \infty)$. As in Section *IV* of Rode [10], we can also obtain the strong convergence of the sequences

$$(1-r)\sum_{k=1}^{\infty} r^k T^k x, \qquad r \to 1-$$

and

$$\lambda \int_0^{\infty} e^{-\lambda t} T_t x dt, \qquad \lambda \to 0+.$$

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Fixed Point Results for Ćirić type α-η-GF-Contractions

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Abstract: The aim of this paper is to establish some new fixed point results for Ćirić type α-η-GF-contraction in a complete metric space. We extend the concept of F-contraction and introduce the notion Ćirić type α-η-GF-contraction. An example is given to demonstrate the novelty of our work.

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1 Introduction

In metric fixed point theory the contractive conditions on underlying functions play an important role for finding solutions of fixed point problems. Banach contraction principle [4] is a fundamental result in metric fixed point theory. Due to its importance and simplicity, several authors have generalized/extended it in different directions. In 1973, Geraghty [9] studied a generalization of Banach contraction principle. Ćirić [5], introduced quasi contraction theorem, which generalized Banach contraction principle. Over the years, Banach contraction theorem has been generalized in different ways by several mathematicians (see [1-24]).

In 2012, Samet et al. [22], introduced a concept of $\alpha - \psi$- contractive type mappings and established various fixed point theorems for mappings in complete metric spaces. Afterwards Karapinar et al. [16], refined the notion and obtained various fixed point results. Hussain et al. [12], extended the concept of α-admissible mappings and obtained useful fixed point theorems. Subsequently, Abdeljawad [1] introduced pair of $\alpha-$admissible mappings satisfying new sufficient contractive conditions different from those in [12, 22], and proved fixed point and common fixed point theorems. Lately, Salimi et al. [21], modified the concept of $\alpha - \psi-$ contractive mapping and established fixed point results.

Definition 1 *([22]). Let $T : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. We say that T is α-admissible if $x, y \in X$, $\alpha(x, y) \geq 1$ implies that $\alpha(Tx, Ty) \geq 1$.*

Definition 2 *([21]). Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is α-admissible mapping with respect to η if $x, y \in X$, $\alpha(x, y) \geq \eta(x, y)$ implies that $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$.*

If $\eta(x, y) = 1$, then above definition reduces to definition 1. If $\alpha(x, y) = 1$, then T is called an η-subadmissible mapping.

Definition 3 *[11] Let (X, d) be a metric space. Let $T : X \to X$ and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is $\alpha-\eta$-continuous mapping on (X, d) if for given $x \in X$, and sequence $\{x_n\}$ with*

$$x_n \to x \text{ as } n \to \infty, \ \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N} \Rightarrow Tx_n \to Tx.$$

In 1962, Edelstein proved the following version of the Banach contraction principle.

Theorem 4 *[7]. Let (X, d) be a metric space and $T : X \to X$ be a self mapping. Assume that*

$$d(Tx, Ty) < d(x, y), \text{ holds for all } x, y \in X \text{ with } x \neq y.$$

Then T has a unique fixed point in X.

In 2012, Wardowski [24] introduce a new type of contractions called F-contraction and proved new fixed point theorems concerning F-contraction. He generalized the Banach contraction principle in a different way than as it was done by different investigators. Piri et al. [19] defined the F-contraction as follows.

Definition 5 *[19] Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F contraction if there exists $\tau > 0$ such that*

$$\forall x, y \in X,\ d(Tx, Ty) > 0 \Rightarrow \tau + F\left(d(Tx, Ty)\right) \leq F\left(d(x, y)\right), \tag{1.1}$$

where $F : \mathbb{R}_+ \to \mathbb{R}$ is a mapping satisfying the following conditions:
(F1) F is strictly increasing, i.e. for all $x, y \in \mathbb{R}_+$ such that $x < y$, $F(x) < F(y)$;
(F2) For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F3) There exists $k \in (0, 1)$ such that $\lim \alpha \to 0^+ \alpha^k F(\alpha) = 0$.

We denote by Δ_F, the set of all functions satisfying the conditions (F1)-(F3).

Example 6 *[24] Let $F : \mathbb{R}_+ \to \mathbb{R}$ be given by the formula $F(\alpha) = \ln \alpha$. It is clear that F satisfied (F1)-(F2)-(F3) for any $k \in (0, 1)$. Each mapping $T : X \to X$ satisfiying (1.1) is an F-contraction such that*

$$d(Tx, Ty) \leq e^{-\tau} d(x, y),\ \textit{for all } x, y \in X,\ Tx \neq Ty.$$

It is clear that for $x, y \in X$ such that $Tx = Ty$ the inequality $d(Tx, Ty) \leq e^{-\tau} d(x, y)$, also holds, i.e. T is a Banach contraction.

Example 7 *[24] If $F(\alpha) = \ln \alpha + \alpha$, $\alpha > 0$ then F satisfies (F1)-(F3) and the condition (1.1) is of the form*

$$\frac{d(Tx, Ty)}{d(x, y)} \leq e^{d(Tx,Ty)-d(x,y)} \leq e^{-\tau},\ \textit{for all } x, y \in X,\ Tx \neq Ty.$$

Remark 8 *From (F1) and (1.1) it is easy to conclude that every F-contraction is necessarily continuous.*

3

Wardowski [24] stated a modified version of the Banach contraction principle as follows.

Theorem 9 *[24] Let (X, d) be a complete metric space and let $T : X \to X$ be an F contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n \in \mathbb{N}}$ converges to x^*.*

Hussain et al. [11] introduced the following family of new functions.

Let Δ_G denotes the set of all functions $G : \mathbb{R}^{+4} \to \mathbb{R}^+$ satisfying:

(G) for all $t_1, t_2, t_3, t_4 \in \mathbb{R}^+$ with $t_1 t_2 t_3 t_4 = 0$ there exists $\tau > 0$ such that $G(t_1, t_2, t_3, t_4) = \tau$.

Definition 10 *[11] Let (X, d) be a metric space and T be a self mapping on X. Also suppose that $\alpha, \eta : X \times X \to [0, +\infty)$ be two function. We say that T is α-η-GF-contraction if for $x, y \in X$, with $\eta(x, Tx) \leq \alpha(x, y)$ and $d(Tx, Ty) > 0$ we have*

$$G(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) + F(d(Tx, Ty)) \leq F(d(x, y)),$$

where $G \in \Delta_G$ and $F \in \Delta_F$.

2 Main Result

In this section, we define a new contraction called Ćirić type α-η-GF-contraction and obtained some new fixed point theorems for such contraction in the setting of complete metric spaces. We define Ćirić type α-η-GF-contraction as follows:

Definition 11 *Let (X, d) be a metric space and T be a self mapping on X. Also suppose that $\alpha, \eta : X \times X \to [0, +\infty)$ two functions. We say that T is Ćirić type α-η-GF-contraction if for all $x, y \in X$, with $\eta(x, Tx) \leq \alpha(x, y)$ and $d(Tx, Ty) > 0$, we have*

$$G(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) + F(d(Tx, Ty)) \leq F(M(x, y)) \quad (2.1)$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$$

$G \in \Delta_G$ *and* $F \in \Delta_F$.

Now we state our main result.

Theorem 12 *Let (X,d) be a complete metric space. Let T be a Ćirić type α-η-GF-contraction satisfying the following assertions:*

(i) T is an α-admissible mapping with respect to η;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;

(iii) T is $\alpha - \eta$-continuous.

Then T has a fixed point in X. Moreover, T has a unique fixed point when $\alpha(x,y) \geq \eta(x,x)$ for all $x, y \in Fix(T)$.

Proof. Let x_0 in X such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. For $x_0 \in X$, we construct a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_1 = Tx_0$, $x_2 = Tx_1 = T^2 x_0$. Continuing this process, $x_{n+1} = Tx_n = T^{n+1}x_0$, for all $n \in \mathbb{N}$. Now since, T is an α-admissible mapping with respect to η then $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0) = \eta(x_0, x_1)$. By continuing in this process we have,

$$\eta(x_{n-1}, Tx_{n-1}) = \eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n), \text{ for all } n \in \mathbb{N}. \quad (2.2)$$

If there exists $n \in \mathbb{N}$ such that $d(x_n, Tx_n) = 0$, there is nothing to prove. So, we assume that $x_n \neq x_{n+1}$ with

$$d(Tx_{n-1}, Tx_n) = d(x_n, Tx_n) > 0, \forall n \in \mathbb{N}.$$

Since T is Ćirić type α-η-GF-contraction, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} & G(d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})) \\ +F\left(d(Tx_{n-1}, Tx_n)\right) \leq\ & F\left(M(x_{n-1}, x_n)\right) \end{aligned}$$

which implies

$$\begin{aligned} & G(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0) \\ +F\left(d(Tx_{n-1}, Tx_n)\right) \leq\ & F\left(M(x_{n-1}, x_n)\right) \quad (2.3) \end{aligned}$$

Now by definition of G, $d(x_{n-1},x_n).d(x_n,x_{n+1}).d(x_{n-1},x_{n+1}).0 = 0$, so there exists $\tau > 0$ such that,

$$G(d(x_{n-1},x_n),d(x_n,x_{n+1}),d(x_{n-1},x_{n+1}),0) = \tau.$$

Therefore

$$F\left(d(x_n,x_{n+1})\right) = F\left(d(Tx_{n-1},Tx_n)\right) \le F\left(M(x_{n-1},x_n)\right) - \tau. \tag{2.4}$$

Now

$$\begin{aligned} M(x_{n-1},x_n) &= \max\left\{\begin{array}{c} d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1}), \\ \frac{d(x_{n-1},x_{n+1})+d(x_n,x_n)}{2}; \end{array}\right\} \\ &= \max\left\{\begin{array}{c} d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1}), \\ \frac{d(x_{n-1},x_{n+1})}{2}; \end{array}\right\} \\ &\le \max\left\{d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\right\} \\ &= \max\left\{d(x_{n-1},x_n), d(x_n,x_{n+1})\right\}. \end{aligned}$$

So, we have

$$F\left(d(x_n,x_{n+1})\right) = F\left(d(Tx_{n-1},Tx_n)\right) \le F\left(\max\left\{d(x_{n-1},x_n), d(x_n,x_{n+1})\right\}\right) - \tau.$$

In this case $M(x_{n-1},x_n) = \max\left\{d(x_{n-1},x_n), d(x_n,x_{n+1})\right\} = d(x_n,x_{n+1})$ is impossible, because

$$F\left(d(x_n,x_{n+1})\right) = F\left(d(Tx_{n-1},Tx_n)\right) \le F\left(d(x_n,x_{n+1})\right) - \tau < F\left(d(x_n,x_{n+1})\right).$$

Which is a contradiction. So

$$M(x_{n-1},x_n) = \max\left\{d(x_{n-1},x_n), d(x_n,x_{n+1})\right\} = d(x_{n-1},x_n).$$

Thus from (2.4), we have

$$F\left(d(x_n,x_{n+1})\right) \le F\left(d(x_{n-1},x_n)\right) - \tau.$$

Continuing this process, we get

$$\begin{aligned} F\left(d(x_n, x_{n+1})\right) &\leq F\left(d(x_{n-1}, x_n)\right) - \tau \\ &= F\left(d(Tx_{n-2}, Tx_{n-1})\right) - \tau \\ &\leq F\left(d(x_{n-2}, x_{n-1})\right) - 2\tau \\ &= F\left(d(Tx_{n-3}, Tx_{n-2})\right) - 2\tau \\ &\leq F\left(d(x_{n-3}, x_{n-2})\right) - 3\tau \\ &\vdots \\ &\leq F\left(d(x_0, x_1)\right) - n\tau. \end{aligned}$$

This implies that

$$F\left(d(x_n, x_{n+1})\right) \leq F\left(d(x_0, x_1)\right) - n\tau. \tag{2.5}$$

From (2.5), we obtain $\lim_{n\to\infty} F\left(d(x_n, x_{n+1})\right) = -\infty$. Since $F \in \Delta_F$, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.6}$$

From $(F3)$, there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} \left(\left(d(x_n, x_{n+1})\right)^k F\left(d(x_n, x_{n+1})\right) \right) = 0. \tag{2.7}$$

From (2.5), for all $n \in \mathbb{N}$, we obtain

$$\left(d(x_n, x_{n+1})\right)^k \left(F\left(d(x_n, x_{n+1})\right) - F\left(d(x_0, x_1)\right)\right) \leq - \left(d(x_n, x_{n+1})\right)^k n\tau \leq 0. \tag{2.8}$$

By using (2.6), (2.7) and letting $n \to \infty$, in (2.8), we have

$$\lim_{n\to\infty} \left(n \left(d(x_n, x_{n+1})\right)^k \right) = 0. \tag{2.9}$$

We observe that from (2.9), then there exists $n_1 \in \mathbb{N}$, such that $n\left(d(x_n, x_{n+1})\right)^k \leq 1$ for all $n \geq n_1$, we get

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}} \text{ for all } n \geq n_1. \tag{2.10}$$

Now, $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Then, by the triangle inequality and from (2.10) we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + ... + d(x_{m-1}, x_m) \quad (2.11) \\ &= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

The series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is convergent. By taking limit as $n \to \infty$, in (2.11), we have $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. T is an α-η-continuous and $\eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n)$, for all $n \in \mathbb{N}$ then $x_{n+1} = Tx_n \to Tx^*$ as $n \to \infty$. That is, $x^* = Tx^*$. Hence x^* is a fixed point of T. To prove uniqueness, let $x \neq y$ be any two fixed point of T, then from (2.1), we have

$$G(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) + F(d(Tx, Ty)) \leq F(M(x, y))$$

we obtain

$$\tau + F(d(x, y)) \leq F(d(x, y)).$$

which is a contradiction. Hence, $x = y$. Therefore, T has a unique fixed point.

■

Theorem 13 *Let (X, d) be a complete metric space. Let T be a self mapping satisfying the following assertions:*

(i) T is an α-admissible mapping with respect to η;

(ii) T is Ćirić type α-η-GF-contraction;

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x^*$ as $n \to \infty$ then either

$$\alpha(Tx_n, x) \geq \eta(Tx_n, T^2x_n) \text{ or } \alpha(T^2x_n, x) \geq \eta(T^2x_n, T^3x_n)$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point in X. Moreover, T has a unique fixed point when $\alpha(x,y) \geq \eta(x,x)$ for all $x, y \in Fix(T)$.

Proof. As similar lines of the Theorem 12, we can conclude that

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \text{ and } x_n \to x^* \text{ as } n \to \infty.$$

Since, by (iv), either

$$\alpha(Tx_n, x^*) \geq \eta(Tx_n, T^2x_n) \text{ or } \alpha(T^2x_n, x^*) \geq \eta(T^2x_n, T^3x_n),$$

holds for all $n \in \mathbb{N}$. This implies

$$\alpha(x_{n+1}, x^*) \geq \eta(x_{n+1}, x_{n+2}) \text{ or } \alpha(x_{n+2}, x^*) \geq \eta(x_{n+2}, x_{n+3}), \text{ for all } n \in \mathbb{N}.$$

Then there exists a subsequence$\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\eta(x_{n_k}, Tx_{n_k}) = \eta(x_{n_k}, x_{n_k+1}) \leq \alpha(x_{n_k}, x^*).$$

From (2.1), we have

$$\begin{aligned}
& G(d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*), d(x_{n_k}, Tx^*), d(x^*, Tx_{n_k})) + F\left(d(Tx_{n_k}, Tx^*)\right) \\
\leq\ & F\left(M(x_{n_k}, x^*)\right) \\
=\ & F\left(\max\left\{\begin{array}{c} d(x_{n_k}, x^*), d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*), \\ \frac{d(x_{n_k}, Tx^*) + d(x^*, Tx_{n_k})}{2}; \end{array}\right\}\right) \\
=\ & F\left(\max\left\{\begin{array}{c} d(x_{n_k}, x^*), d(x_{n_k}, x_{n_k+1}), d(x^*, Tx^*), \\ \frac{d(x_{n_k}, Tx^*) + d(x^*, x_{n_k+1})}{2} \end{array}\right\}\right).
\end{aligned}$$

Using the continuity of F and the fact that

$$\lim_{k\to\infty} d(x_{n_k}, x^*) = 0 = \lim_{k\to\infty} d(x_{n_k+1}, x^*) \tag{2.12}$$

we obtain

$$\tau + F\left(d(x^*, Tx^*)\right) \leq F\left(d(x^*, Tx^*)\right). \tag{2.13}$$

Which is a contradiction. Therefore, $d(x^*, Tx^*) = 0$, implies x^* is a fixed point of T. Uniqueness follows similar lines as in Theorem 12. ■

In the following we extend the Wardowski type fixed point theorem.

Theorem 14 *Let T be a continuous self mapping on a complete metric space X. If for $x, y \in X$ with $d(x, Tx) \leq d(x, y)$ and $d(Tx, Ty) > 0$, we have*

$$G(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)) + F\left(d(Tx,Ty)\right) \leq F\left(M(x,y)\right),$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

$G \in \Delta_G$ and $F \in \Delta_F$. Then T has a fixed point in X.

Proof. Let us define $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(x,y) = d(x,y) \text{ and } \eta(x,y) = d(x,y) \text{ for all } x, y \in X.$$

Now, $d(x,y) \leq d(x,y)$ for all $x, y \in X$, so $\alpha(x,y) \geq \eta(x,y)$ for all $x, y \in X$. That is, conditions (i) and (iii) of Theorem 12 hold true. Since T is continuous, so T is α-η-continuous. Let $\eta(x,Tx) \leq \alpha(x,y)$ and $d(Tx,Ty) > 0$, we have $d(x,Tx) \leq d(x,y)$ with $d(Tx,Ty) > 0$, then

$$G(d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)) + F\left(d(Tx,Ty)\right) \leq F\left(M(x,y)\right).$$

That is, T is Ćirić type α-η-GF-contraction mapping. Hence, all conditions of Theorem 12 satisfied and T has a fixed point. ■

Corollary 15 *Let T be a continuous selfmapping on a complete metric space X. If for $x, y \in X$ with $d(x, Tx) \leq d(x, y)$ and $d(Tx, Ty) > 0$, we have*

$$\tau + F\left(d(Tx,Ty)\right) \leq F\left(M(x,y)\right),$$

where $\tau > 0$, and $F \in \Delta_F$. Then T has a fixed point in X.

Corollary 16 *Let T be a continuous selfmapping on a complete metric space X. If for $x, y \in X$ with $d(x, Tx) \leq d(x, y)$ and $d(Tx, Ty) > 0$, we have*

$$\tau + F\left(d(Tx,Ty)\right) \leq F\left(d(x,y)\right),$$

where $\tau > 0$, and $F \in \Delta_F$. Then T has a fixed point in X.

Corollary 17 *[11] Let (X,d) be a complete metric space. Let $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) T is an α-admissible mapping with respect to η;

(ii) T is an α-η-GF-contraction

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;

(iv) T is $\alpha - \eta$-continuous.

Then T has a fixed point in X. Moreover, T has a unique fixed point when $\alpha(x,y) \geq \eta(x,x)$ for all $x, y \in Fix(T)$.

Corollary 18 *[11] Let (X,d) be a complete metric space. Let $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) T is an α-admissible mapping with respect to η;

(ii) T is an α-η-GF-contraction

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;

(iv)if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x^*$ as $n \to \infty$ then either

$$\alpha(Tx_n, x) \geq \eta(Tx_n, T^2x_n) \text{ or } \alpha(T^2x_n, x) \geq \eta(T^2x_n, T^3x_n)$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point in X. Moreover, T has a unique fixed point when $\alpha(x,y) \geq \eta(x,x)$ for all $x, y \in Fix(T)$.

Example 19 *Consider the sequence,*

$S_1 = 1 \times 3$

$S_2 = 1 \times 3 + 2 \times 5$

$S_3 = 1 \times 3 + 2 \times 5 + 3 \times 7$

$S_n = 1 \times 3 + 2 \times 5 + 3 \times 7 \ldots + n \times (2n+1) = \frac{n(n+1)(4n+5)}{6}$.

Let X=$\{S_n : n \in \mathbb{N}\}$ and $d(x,y) = |x-y|$. Then (X,d) is a complete metric space. If $F(\alpha) = \alpha + \ln \alpha$, $\alpha > 0$ and $G(t_1, t_2, t_3, t_4) = \tau$ where $\tau = 1$. Define the mapping $T : X \to X$ by, $T(S_1) = S_1$ and $T(S_n) = S_{n-1}, n \geq 1$ and $\alpha(x, y) = 1$ if $x \in X, \eta(x, Tx) = \frac{1}{2}$ for all $x \in X$. we have

$$\lim_{n\to\infty} \frac{d(T(S_n), T(S_1))}{d(S_n, S_1)} = \lim_{n\to\infty} \frac{S_{n-1} - 3}{S_n - 3} = \frac{(n-1)\,n(4n+1) - 18}{n(n+1)(4n+5) - 18} = 1.$$

So we conclude the following two cases:

Case 1:

we observe that for every $m \in \mathbb{N}, m > 2, n = 1$ or $n = 1$ and $m > 1$ then $\alpha(S_m, S_n) \geq \eta(S_m, T(S_m))$, we have

$$\begin{aligned} \frac{d(T(S_m), T(S_1))}{M(S_m, S_1)} e^{d(T(S_m),T(S_1)) - M(S_m,S_1)} &= \frac{S_{m-1} - 3}{S_m - 3} e^{S_{m-1} - S_m} \\ &= \frac{(m-1)\,m\,(4m+1) - 18}{m\,(m+1)\,(4m+5) - 18} e^{-\frac{m(m+1)(4m+5)}{6}} \\ &< e^{-1}. \end{aligned}$$

Case 2:

for $m > n > 1$, then $\alpha(S_m, S_n) \geq \eta(S_m, T(S_m))$, we have

$$\begin{aligned} & \frac{d(T(S_m), T(S_n))}{M(S_m, S_n)} e^{d(T(S_m),T(S_n)) - M(S_m,S_n)} \\ = \; & \frac{S_{m-1} - S_{n-1}}{S_m - S_n} e^{S_n - S_{n-1} + S_{m-1} - S_m} \\ = \; & \frac{(m-1)\,m(4m+1) - (n-1)\,n(4n+1)}{m(m+1)(4m+5) - n(n+1)(4n+5)} e^{\frac{n(n+1)(4n+5)}{6} - \frac{m(m+1)(4m+5)}{6}} \leq e^{-1}. \end{aligned}$$

So all condition of theorems are satisfied, T has a fixed point in X.

Let $(X, d, \preceq)$ be a partially ordered metric space. Let $T : X \to X$ is such that for $x, y \in X$, with $x \preceq y$ implies $Tx \preceq Ty$, then the mapping T is said to be non-decreasing. We derive following important result in partially ordered metric spaces.

Theorem 20 *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Assume that the following assertions hold true:*

(i) T is nondecreasing and ordered GF-contraction;

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(ii) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(iii) either for a given $x \in X$ and sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$ and $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ we have $Tx_n \to Tx$

or if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ with $x_n \to x$ as $n \to \infty$ then either

$$Tx_n \preceq x \text{ or } T^2 x_n \preceq x$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point in X.

Define $\digamma = \{\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ : \phi$ is a Lebesgue integral mapping which is summable, nonnegative and satisfies $\int_0^{\epsilon} \phi(t)dt > 0$, for each $\epsilon > 0\}$.

We can easily deduce following result involving integral type inequalities.

Theorem 21 *Let T be a continuous selfmapping on a complete metric space X. If for $x, y \in X$ with*

$$\int_0^{d(x,Tx)} \phi(t)dt \leq \int_0^{d(x,y)} \phi(t)dt \text{ and } \int_0^{d(Tx,Ty)} \phi(t)dt > 0,$$

we have

$$G(\int_0^{d(x,Tx)} \phi(t)dt, \int_0^{d(y,Ty)} \phi(t)dt, \int_0^{d(x,Ty)} \phi(t)dt, \int_0^{d(y,Tx)} \phi(t)dt) + F\left(\int_0^{d(Tx,Ty)} \phi(t)dt\right)$$

$$\leq F\left(\int_0^{M(x,y)} \phi(t)dt\right),$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2}\right\}.$$

$\phi \in \digamma$, $G \in \Delta_G$ and $F \in \Delta_F$. Then T has a fixed point in X.

Conflict of Interests

The authors declare that they have no competing interests.

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FIXED POINT AND QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN NON-ARCHIMEDEAN BANACH SPACES

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ABSTRACT. In this paper, we solve the following quadratic ρ-functional inequalities

$$\begin{aligned} &\left\|f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)+f\left(\frac{z-x-y}{2}\right)\right. \\ &\qquad\qquad -f(x)-f(y)-f(z)\| \\ &\leq \|\rho(f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y) \qquad (0.1)\\ &\qquad\qquad -4f(x)-4f(y)-4f(z))\|, \end{aligned}$$

where ρ is a fixed non-Archimedean number with $|\rho| < \frac{1}{|4|}$, and

$$\begin{aligned} &\|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y) \\ &\qquad\qquad -4f(x)-4f(y)-4f(z)\| \qquad (0.2)\\ &\leq \left\|\rho\left(f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)+f\left(\frac{z-x-y}{2}\right)\right.\right. \\ &\qquad\qquad -f(x)-f(y)-f(z))\|, \end{aligned}$$

where ρ is a fixed non-Archimedean number with $|\rho| < |8|$.

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequalities (0.1) and (0.2) in non-Archimedean Banach spaces.

1. Introduction and preliminaries

A *valuation* is a function $|\cdot|$ from a field K into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s| \leq |r|+|s|, \qquad \forall r,s\in K.$$

A field K is called a *valued field* if K carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \leq \max\{|r|,|s|\}, \qquad \forall r,s\in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n\in\mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

Definition 1.1. ([19]) Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X\to[0,\infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

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(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|rx\| = |r|\|x\| \qquad (r \in K, x \in X)$;
(iii) the strong triangle inequality

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}, \qquad \forall x, y \in X$$

holds. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space.*

Definition 1.2. (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$ there is a positive integer N such that

$$\|x_n - x_m\| \leq \varepsilon$$

for all $n, m \geq N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$ there are a positive integer N and an $x \in X$ such that

$$\|x_n - x\| \leq \varepsilon$$

for all $n \geq N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space.*

The stability problem of functional equations originated from a question of Ulam [27] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation.* In particular, every solution of the Cauchy equation is said to be an *additive mapping.* Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [24] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [9] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.1}$$

is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping.* The stability of quadratic functional equation was proved by Skof [26] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. See [7, 15, 16] for more functional equations.

The functional equation $2f\left(\frac{x+y}{2}\right) + 2\left(\frac{x-y}{2}\right) = f(x) + f(y)$ is called a *Jensen type quadratic equation.*

In [10], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\| \tag{1.2}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

See also [25]. Gilányi [11] and Fechner [8] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [22] proved the Hyers-Ulam stability of additive functional inequalities.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.3. [3, 6] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that
(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*
(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [13] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 5, 14, 17, 20, 21, 23]).

In Section 2, we deal with quadratic functional equations. In Section 3, we solve the quadratic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.1) in non-Archimedean Banach spaces. In Section 4, we solve the quadratic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.2) in non-Archimedean Banach spaces.

Throughout this paper, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let $|2| \neq 1$.

2. Quadratic functional equations

Theorem 2.1. *Let X and Y be vector spaces. A mapping $f : X \to Y$ satisfies*

$$f\left(\frac{x+y+z}{2} + \frac{x-y-z}{2} + \frac{y-x-z}{2} + \frac{z-x-y}{2}\right) = f(x) + f(y) + f(z) \tag{2.1}$$

if and only if the mapping $f : X \to Y$ is a quadratic mapping.

Proof. Assume that $f : X \to Y$ satisfies (2.1)

Letting $x = y = z = 0$ in (2.1), we have $4f(0) = 3f(0)$. So $f(0) = 0$.

Letting $y = z = 0$ in (2.1), we get

$$2f\left(\frac{x}{2}\right) + 2f\left(-\frac{x}{2}\right) = f(x) \qquad \& \qquad 2f\left(-\frac{x}{2}\right) + 2f\left(\frac{x}{2}\right) = f(-x) \tag{2.2}$$

for all $x \in X$, which imply that $f(x) = f(-x)$ for all $x \in X$.

From this and (2.2), we obtain $4f\left(\frac{x}{2}\right) = f(x)$ or $f(2x) = 4f(x)$ for all $x \in X$.

Putting $z = 0$ in (2.1), we obtain $\frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) = f(x) + f(y)$ for all $x, y \in X$, which means that $f : X \to Y$ is a quadratic mapping.

The converse is obviously true. □

Corollary 2.2. *Let X and Y be vector spaces. An even mapping $f : X \to Y$ satisfies*

$$f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) = 4f(x) + 4f(y) + 4f(z) \tag{2.3}$$

for all $x, y, z \in X$. Then the mapping $f : X \to Y$ is a quadratic mapping.

Proof. Assume that $f : X \to Y$ satisfies (2.3).

Letting $x = y = z = 0$ in (2.3), we have $4f(0) = 12f(0)$. So $f(0) = 0$.

Letting $z = 0$ in (2.3), we get $2f(x+y) + 2f(x-y) = 4f(x) + 4f(y)$ and so $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$. □

3. Quadratic ρ-functional inequality (0.1)

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < \frac{1}{|4|}$.

In this section, we solve and investigate the quadratic ρ-functional inequality (0.1) in non-Archimedean normed spaces.

Lemma 3.1. *An even mapping $f : X \to Y$ satisfies*

$$\begin{aligned} &\left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \right. \\ &\qquad\qquad\qquad\qquad \left. - f(x) - f(y) - f(z) \right\| \\ &\leq \| \rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) \\ &\qquad\qquad\qquad\qquad -4f(x) - 4f(y) - 4f(z)) \| \end{aligned} \tag{3.1}$$

for all $x, y, z \in X$ if and only if $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $x = y = z = 0$ in (3.1), we get $\|f(0)\| \leq |\rho|\|8f(0)\|$. So $f(0) = 0$.

Letting $y = z = 0$ in (3.1), we get $\|4f\left(\frac{x}{2}\right) - f(x)\| \leq 0$ and so

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{3.2}$$

for all $x \in X$.

By (3.1) and (3.2), we have

$$f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) = f(x) + f(y) + f(z)$$

for all $x, y, z \in X$, since $|\rho| < \frac{1}{|4|}$.

The converse is obviously true. □

Now we prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (3.1) in non-Archimedean Banach spaces.

Theorem 3.2. *Let $\varphi : X^3 \to [0, \infty)$ be a function with $\varphi(0, 0, 0) = 0$ such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{|4|}\varphi(x, y, z) \tag{3.3}$$

for all $x, y, z \in X$. Let $f : X \to Y$ be an even mapping such that

$$\begin{aligned} &\left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \right. \\ &\qquad\qquad\qquad\qquad \left. - f(x) - f(y) - f(z) \right\| \\ &\leq \| \rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) \\ &\qquad\qquad\qquad\qquad -4f(x) - 4f(y) - 4f(z)) \| + \varphi(x, y, z) \end{aligned} \tag{3.4}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{1-L}\varphi(x, 0, 0) \tag{3.5}$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (3.4), we get $\|f(0)\| \leq |\rho|\|8f(0)\|$. So $f(0) = 0$.

Letting $y = z = 0$ in (3.4), we get

$$\left\| 4f\left(\frac{x}{2}\right) - f(x) \right\| \leq \varphi(x, 0, 0) \tag{3.6}$$

for all $x \in X$.

Consider the set $S := \{h : X \to Y, \quad h(0) = 0\}$ and introduce the generalized metric on S:

$$d(g,h) = \inf \{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \le \mu\varphi(x,0,0), \ \forall x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [18]).

Now we consider the linear mapping $J : S \to S$ such that $Jg(x) := 4g\left(\frac{x}{2}\right)$ for all $x \in X$. Let $g,h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \le \varepsilon\varphi(x,0,0)$$

for all $x \in X$. Hence

$$\begin{aligned}\|Jg(x) - Jh(x)\| &= \left\|4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right)\right\| \le |4|\varepsilon\varphi\left(\frac{x}{2},0,0\right) \\ &\le |4|\varepsilon\frac{L}{|4|}\varphi(x,0,0) \le L\varepsilon\varphi(x,0,0)\end{aligned}$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that $d(Jg, Jh) \le Ld(g,h)$ for all $g, h \in S$.

It follows from (3.6) that $d(f, Jf) \le 1$.

By Theorem 1.3, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q(x) = 4Q\left(\frac{x}{2}\right) \tag{3.7}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (3.7) such that there exists a $\mu \in (0,\infty)$ satisfying $\|f(x) - Q(x)\| \le \mu\varphi(x,0,0)$ for all $x \in X$;

(2) $d(J^l f, Q) \to 0$ as $l \to \infty$. This implies the equality $\lim_{l\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$ for all $x \in X$;

(3) $d(f,Q) \le \frac{1}{1-L}d(f,Jf)$, which implies the inequality $d(f,Q) \le \frac{1}{1-L}$. So $\|f(x) - Q(x)\| \le \frac{1}{1-L}\varphi(x,0,0)$ for all $x \in X$.

It follows from (3.3) and (3.4) that

$$\begin{aligned}&\left\|Q\left(\frac{x+y+z}{2}\right) + Q\left(\frac{x-y-z}{2}\right) + Q\left(\frac{y-x-z}{2}\right) + Q\left(\frac{z-x-y}{2}\right)\right. \\ &\qquad\qquad \left. -Q(x) - Q(y) - Q(z)\right\| \\ &= \lim_{n\to\infty} |4|^n \left\|f\left(\frac{x+y+z}{2^{n+1}}\right) + f\left(\frac{x-y-z}{2^{n+1}}\right) + f\left(\frac{y-x-z}{2^{n+1}}\right) + f\left(\frac{z-x-y}{2^{n+1}}\right)\right. \\ &\qquad\qquad \left. -f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\| \\ &\le \lim_{n\to\infty} |4|^n|\rho| \left\|f\left(\frac{x+y+z}{2^n}\right) + f\left(\frac{x-y-z}{2^n}\right) + f\left(\frac{y-x-z}{2^n}\right) + f\left(\frac{z-x-y}{2^n}\right)\right. \\ &\qquad \left. -4f\left(\frac{x}{2^n}\right) - 4f\left(\frac{y}{2^n}\right) - 4f\left(\frac{z}{2^n}\right)\right\| + \lim_{n\to\infty}\frac{1}{|4|^n}\varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right) \\ &= \|\rho(Q(x+y+z) + Q(x-y-z) + Q(y-x-z) + Q(z-x-y) \\ &\qquad -4Q(x) - 4Q(y) - 4Q(z))\|\end{aligned}$$

for all $x, y, z \in X$. So

$$\left\| Q\left(\frac{x+y+z}{2}\right) + Q\left(\frac{x-y-z}{2}\right) + Q\left(\frac{y-x-z}{2}\right) + Q\left(\frac{z-x-y}{2}\right) - Q(x) - Q(y) - Q(z) \right\|$$
$$\leq \|\rho(Q(x+y+z) + Q(x-y-z) + Q(y-x-z) + Q(z-x-y) - 4Q(x) - 4Q(y) - 4Q(z))\|$$

for all $x, y, z \in X$. By Lemma 3.1, the mapping $Q : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (3.5). Then we have

$$\|Q(x) - T(x)\| = \left\| 4^q Q\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right) \right\|$$
$$\leq \max\left\{ \left\| 4^q Q\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\|, \left\| 4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\| \right\} \leq \lim_{n\to\infty} \frac{1}{|4|^n} \varphi\left(\frac{x}{2^n}, 0, 0\right),$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q. Thus the mapping $Q : X \to Y$ is a unique quadratic mapping satisfying (3.5). □

Corollary 3.3. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping such that*

$$\left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z) \right\|$$
$$\leq \|\rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) - 4f(x) - 4f(y) - 4f(z))\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r) \tag{3.8}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{|2|^r \theta}{|2|^r - |2|^2} \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by takig $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Then we can choose $L = |2|^{2-r}$ and we get desired result. □

Theorem 3.4. *Let $\varphi : X^3 \to [0, \infty)$ be a function with $\varphi(0, 0, 0) = 0$ such that there exists an $L < 1$ with*

$$\varphi(x, y, z) \leq |4| L \varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$$

for all $x, y, z \in X$ Let $f : X \to Y$ be an even mapping satisfying (3.4)*. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{L}{1-L} \varphi(x, 0, 0)$$

for all $x \in X$.

Proof. It follows from (3.6) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{1}{|4|} \varphi(2x, 0, 0) \leq L\varphi(x, 0, 0) \tag{3.9}$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 3.2.

Now we consider the linear mapping $J : S \to S$ such that $Jg(x) := \frac{1}{4} g(2x)$ for all $x \in X$.

It follows from (3.9) that $d(f, Jf) \le L$. So $d(f, Q) \le \frac{L}{1-L}$. So $\|f(x) - Q(x)\| \le \frac{L}{1-L}\varphi(x, 0, 0)$ for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.2. □

Corollary 3.5. *Let $r > 2$ and θ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying* (3.8). *Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{|2|^r \theta}{|2|^2 - |2|^r}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by takig $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Then we can choose $L = |2|^{r-2}$ and we get desired result. □

4. Quadratic ρ-functional inequality (0.2)

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < |8|$.

In this section, we solve and investigate the quadratic ρ-functional inequality (0.2) in non-Archimedean normed spaces.

Lemma 4.1. *An even mapping $f : X \to Y$ satisfies*

$$\begin{aligned} &\|f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) - 4f(x) - 4f(y) - 4f(z)\| \\ &\le \left\|\rho\left(f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right)\right.\right. \\ &\qquad \left.\left. + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z)\right)\right\| \end{aligned} \tag{4.1}$$

for all $x, y, z \in X$ if and only if $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (4.1).

Letting $x = y = z = 0$ in (4.1), we get $\|8f(0)\| \le |\rho|\|f(0)\|$. So $f(0) = 0$.

Letting $x = y, z = 0$ in (4.1), we get

$$\|2f(2x) - 8f(x)\| \le 0 \tag{4.2}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$ for all $x \in X$.

By (4.1) and (4.2), we have

$$f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) = 4f(x) + 4f(y) + 4f(z)$$

for all $x, y, z \in X$, since $|\rho| < |8| \le |4|$.

The converse is obviously true. □

We prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (4.1) in non-Archimedean Banach spaces.

Theorem 4.2. *Let $\varphi : X^3 \to [0, \infty)$ be a function with $\varphi(0, 0, 0) = 0$ such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \le \frac{L}{|4|}\varphi(x, y, z) \tag{4.3}$$

for all $x, y, z \in X$. Let $f : X \to Y$ be an even mapping satisfying

$$\begin{aligned} &\|f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) - 4f(x) - 4f(y) - 4f(z)\| \\ &\le \left\|\rho\left(f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right)\right.\right. \\ &\qquad \left.\left. - f(x) - f(y) - f(z)\right)\right\| + \varphi(x, y, z) \end{aligned} \tag{4.4}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{L}{|4|(1-L)}\varphi(x, x, 0) \tag{4.5}$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (4.4), we get $\|8f(0)\| \leq |\rho|\|f(0)\|$. So $f(0) = 0$.

Letting $x = y, z = 0$ in (4.4), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right) \leq \frac{L}{|4|}\varphi(x, x, 0) \tag{4.6}$$

for all $x \in X$.

Consider the set $S := \{h : X \to Y, \quad h(0) = 0\}$ and introduce the generalized metric on S:

$$d(g, h) = \inf\left\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu\varphi(x, x, 0), \ \forall x \in X\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [18]).

Now we consider the linear mapping $J : S \to S$ such that $Jg(x) := 4g\left(\frac{x}{2}\right)$ for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then $\|g(x) - h(x)\| \leq \varepsilon\varphi(x, x, 0)$ for all $x \in X$. Hence

$$\|Jg(x) - Jh(x)\| = \left\|4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right)\right\| \leq L\varepsilon\varphi(x, x, 0)$$

for all $a \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that $d(Jg, Jh) \leq Ld(g, h)$ for all $g, h \in S$.

It follows from (4.6) that $d(f, Jf) \leq \frac{L}{|4|}$.

By Theorem 1.3, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q(x) = 4Q\left(\frac{x}{2}\right) \tag{4.7}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set $M = \{g \in S : d(f, g) < \infty\}$. This implies that Q is a unique mapping satisfying (4.7) such that there exists a $\mu \in (0, \infty)$ satisfying $\|f(x) - Q(x)\| \leq \mu\varphi(x, x, 0)$ for all $x \in X$;

(2) $d(J^l f, Q) \to 0$ as $l \to \infty$. This implies the equality $\lim_{l\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$ for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality $d(f, Q) \leq \frac{L}{|4|(1-L)}$. So

$$\|f(x) - Q(x)\| \leq \frac{L}{|4|(1-L)}\varphi(x, x, 0)$$

for all $x \in X$.

It follows from (4.3) and (4.4) that

$$\begin{aligned}
&\left\|Q\left(\frac{x+y+z}{2}\right)+Q\left(\frac{x-y-z}{2}\right)+Q\left(\frac{y-x-z}{2}\right)+Q\left(\frac{z-x-y}{2}\right)\right.\\
&\qquad\qquad\left.-Q(x)-Q(y)-Q(z)\right\|\\
&=\lim_{n\to\infty}|4|^n\left\|f\left(\frac{x+y+z}{2^{n+1}}\right)+f\left(\frac{x-y-z}{2^{n+1}}\right)+f\left(\frac{y-x-z}{2^{n+1}}\right)+f\left(\frac{z-x-y}{2^{n+1}}\right)\right.\\
&\qquad\qquad\left.-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)-f\left(\frac{z}{2^n}\right)\right\|\\
&\le\lim_{n\to\infty}|4|^n|\rho|\left\|f\left(\frac{x+y+z}{2^n}\right)+f\left(\frac{x-y-z}{2^n}\right)+f\left(\frac{y-x-z}{2^n}\right)+f\left(\frac{z-x-y}{2^n}\right)\right.\\
&\qquad\left.-4f\left(\frac{x}{2^n}\right)-4f\left(\frac{y}{2^n}\right)-4f\left(\frac{z}{2^n}\right)\right\|+\lim_{n\to\infty}|4|^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right)\\
&=\|\rho(Q(x+y+z)+Q(x-y-z)+Q(y-x-z)+Q(z-x-y)\\
&\qquad\qquad-4Q(x)-4Q(y)-4Q(z))\|
\end{aligned}$$

for all $x,y,z\in X$. So

$$\begin{aligned}
&\left\|Q\left(\frac{x+y+z}{2}\right)+Q\left(\frac{x-y-z}{2}\right)+Q\left(\frac{y-x-z}{2}\right)+Q\left(\frac{z-x-y}{2}\right)\right.\\
&\qquad\qquad\left.-Q(x)-Q(y)-Q(z)\right\|\\
&\le\|\rho(Q(x+y+z)+Q(x-y-z)+Q(y-x-z)+Q(z-x-y)\\
&\qquad\qquad-4Q(x)-4Q(y)-4Q(z))\|
\end{aligned}$$

for all $x,y,z\in X$. By Lemma 4.1, the mapping $Q:X\to Y$ is quadratic.

The rest of the proof is similar to the proof of Theorem 3.2. □

Corollary 4.3. *Let $r<2$ and θ be nonnegative real numbers, and let $f:X\to Y$ be an even mapping such that*

$$\begin{aligned}
&\|f(x+y+z)+f(x-y-z)+f(y-x-z)+f(z-x-y)-4f(x)-4f(y)-4f(z)\|\\
&\le\left\|\rho\left(f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)\right.\right.\\
&\qquad\left.\left.+f\left(\frac{z-x-y}{2}\right)-f(x)-f(y)-f(z)\right)\right\|+\theta(\|x\|^r+\|y\|^r+\|z\|^r)
\end{aligned}\tag{4.8}$$

for all $x,y,z\in X$. Then there exists a unique quadratic mapping $Q:X\to Y$ such that

$$\|f(x)-Q(x)\|\le\frac{2\theta}{|2|^r-|2|^2}\|x\|^r$$

for all $x\in X$.

Proof. The proof follows from Theorem 4.2 by takig $\varphi(x,y,z)=\theta(\|x\|^r+\|y\|^r+\|z\|^r)$ for all $x,y,z\in X$. Then we can choose $L=|2|^{2-r}$ and we get desired result. □

Theorem 4.4. *Let $\varphi:X^3\to[0,\infty)$ be a function with $\varphi(0,0,0)=0$ such that there exists an $L<1$ with*

$$\varphi(x,y,z)\le|4|L\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right)$$

for all $x, y, z \in X$ Let $f : X \to Y$ be an even mapping satisfying (4.4)*. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{1}{|4|(1-L)}\varphi(x, x, 0)$$

for all $x \in X$.

Proof. It follows from (4.6) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{1}{|4|}\varphi(x, x, 0) \tag{4.9}$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 4.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x)$$

for all $x \in X$.

It follows from (4.9) that $d(f, Jf) \leq \frac{1}{|4|}$. So $d(f, Q) \leq \frac{1}{|4|(1-L)}d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{1}{1-L}.$$

So

$$\|f(x) - Q(x)\| \leq \frac{1}{|4|(1-L)}\varphi(x, x, 0)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.2. □

Corollary 4.5. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying* (4.8)*. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{|2|^2 - |2|^r}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.4 by takig $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Then we can choose $L = |2|^{r-2}$ and we get desired result. □

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Dynamics and Global Stability of Higher Order Nonlinear Difference Equation

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ABSTRACT

In this paper, we study the behavior of the solutions of the following rational difference equation with big order

$$x_{n+1} = ax_{n-l} + bx_{n-k} + \frac{cx_{n-s} + dx_{n-t}}{ex_{n-s} + fx_{n-t}}.$$

where the parameters a, b, c, d, e and f are positive real numbers and the initial conditions $x_{-r},\ x_{-r+1}\ ...,\ x_{-1}$ and x_0 are positive real numbers where $r = max\{l, k, s, t\}$.

Keywords: recursive sequence, periodicity, boundedness, stability, difference equations.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

Difference equations appear as natural descriptions of observed evolution phenomena because most measurements of time evolving variables are discrete and as such these equations are in their own right important mathematical models. Difference equations related to differential equations as discrete mathematics related to continuous mathematics.

In recent years nonlinear difference equations have attracted the attention of many researchers, for example: Agarwal and Elsayed [1] studied the global stability, periodicity character and gave the solution form of some special cases of the recursive sequence

$$x_{n+1} = ax_n + \frac{bx_n x_{n-3}}{cx_{n-2} + dx_{n-3}}.$$

Cinar [5] obtained the solutions of the following difference equation

$$x_{n+1} = \frac{x_{n-1}}{-1 + ax_n x_{n-1}}.$$

El-Metwally et al.[10] dealed with the following difference equation

$$y_{n+1} = \frac{y_{n-(2k+1)} + p}{y_{n-(2k+1)} + qy_{n-2l}}.$$

Elsayed [12] studied the global stability, and periodicity character of the following recursive sequence

$$x_{n+1} = ax_{n-l} + \frac{bx_{n-l}}{cx_{n-l} - dx_{n-k}}.$$

Elsayed et al. [18] investigated the behavior of the following second order rational difference equation

$$x_{n+1} = ax_n + \frac{b + cx_{n-1}}{d + ex_{n-1}}.$$

Elsayed and El-Dessoky [16] investigated the global convergence, boundedness, and periodicity of solutions of the difference equation

$$x_{n+1} = ax_{n-s} + \frac{bx_{n-l} + cx_{n-k}}{dx_{n-l} + ex_{n-k}}.$$

Karatas et al. [21] got the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}x_{n-5}}.$$

Obaid et al. [24] studied the global attractivity and periodic character of the following fourth order difference equation

$$x_{n+1} = ax_n + \frac{bx_{n-1} + cx_{n-2} + dx_{n-3}}{\alpha x_{n-1} + \beta x_{n-2} + \gamma x_{n-3}}.$$

Yalcinkaya [29] dealed with the following difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

Zayed and El-Moeam [31], [32] studied the global asymptotic properties of the solutions of the following difference equations

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-k}}.$$
$$x_{n+1} = Ax_n + Bx_{n-k} + \frac{px_n + x_{n-k}}{q + x_{n-k}}.$$

For some related work see [1-33].

Our goal in this article is to investigate the global stability character and the periodicity of solutions of the recursive sequence

$$x_{n+1} = ax_{n-l} + bx_{n-k} + \frac{cx_{n-s} + dx_{n-t}}{ex_{n-s} + fx_{n-t}}. \tag{1}$$

where the parameters a, b, c, d, e and f are positive real numbers and the initial conditions $x_{-r},\ x_{-r+1}\ ...,\ x_{-1}$ and x_0 are positive real numbers where $r = max\{l, k, s, t\}$.

2. SOME BASIC PROPERTIES AND DEFINITIONS

Here, we recall some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let $F : I^{k+1} \to I$, be a continuously differentiable function. Then for every set of initial conditions $x_{-k},\ x_{-k+1},\ ...\ , x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), \qquad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1. (Equilibrium point) A point $\bar{x} \in I$ is called an equilibrium point of Equation (2) if

$$\bar{x} = F(\bar{x}, \bar{x}, ..., \bar{x}).$$

That is, $x_n = \bar{x}\ for\ n \geq 0$, is a solution of Equation (2), or equivalently, $\bar{x}$ is a fixed point of F.

Definition 2. (Periodicity)

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Definition 3. (Stability)

(i) The equilibrium point $\bar{x}$ of Equation (2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + ... + |x_0 - \bar{x}| < \delta,$$

we have

$$|x_n - \bar{x}| < \epsilon \ for\ all\ n \geq -k.$$

(ii) The equilibrium point $\bar{x}$ of Equation (2) is locally asymptotically stable if $\bar{x}$ is locally stable solution of Equation (2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + ... + |x_0 - \bar{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(iii) The equilibrium point $\bar{x}$ of Equation (2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(iv) The equilibrium point $\bar{x}$ of Equation (2) is globally asymptotically stable if $\bar{x}$ is locally stable, and $\bar{x}$ is also a global attractor of Equation (2).

(v) The equilibrium point $\bar{x}$ of Equation (2) is unstable if is not locally stable.

The linearized equation of Equation (2) about the equilibrium point $\bar{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\bar{x}, \bar{x}, ..., \bar{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Theorem A [22] *Assume that* $p_i \in R,\ i = 1, 2, ...$ *and* $k \in \{0, 1, 2, ...\}$. *Then*

$$\sum_{i=1}^{k} |p_i| < 1, \tag{4}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+k} + p_1 y_{n+k-1} + ... + p_k y_n = 0, \quad n = 0, 1, ... \tag{5}$$

Theorem B [23] Let $g : [a, b]^{k+1} \to [a, b]$, be a continuous function, where k is a positive integer, and where $[a, b]$ is an interval of real numbers. Consider the difference equation

$$x_{n+1} = g(x_n, x_{n-1}, ..., x_{n-k}), \qquad n = 0, 1, ... \tag{6}$$

Suppose that g satisfies the following conditions.

(1) For each integer i with $1 \leq i \leq k + 1$; the function $g(z_1, z_2, ..., z_{k+1})$ is weakly monotonic in z_i for fixed $z_1, z_2, ..., z_{i-1}, z_{i+1}, ..., z_{k+1}$.

(2) If m, M is a solution of the system

$$m = g(m_1, m_2, ..., m_{k+1}), \qquad M = g(M_1, M_2, ..., M_{k+1}),$$

then $m = M$, where for each $i = 1, 2, ..., k + 1$, we set

$$m_i = \begin{Bmatrix} m, & \text{if } g \text{ is non-decreasing in } z_i, \\ M, & \text{if } g \text{ is non-increasing in } z_i, \end{Bmatrix}, \quad M_i = \begin{Bmatrix} M, & \text{if } g \text{ is non-decreasing in } z_i, \\ m, & \text{if } g \text{ is non-increasing in } z_i. \end{Bmatrix}.$$

Then there exists exactly one equilibrium point $\bar{x}$ of Equation.(6), and every solution of Equation (6) converges to $\bar{x}$.

3. LOCAL STABILITY OF EQUATION (1)

In this section, we investigate the local stability character of the solutions of Equation (1). Equation (1) has a unique positive equilibrium point and is given by

$$\bar{x} = a\bar{x} + b\bar{x} + \frac{c\bar{x} + d\bar{x}}{e\bar{x} + f\bar{x}}.$$

If $(a+b) < 1$,then the unique positive equilibrium point is

$$\bar{x} = \frac{c+d}{[1-(a+b)](e+f)}.$$

Let $f : (0,\infty)^4 \longrightarrow (0,\infty)$ be a function defined by

$$f(u_0,u_1,u_2,u_3) = au_0 + bu_1 + \frac{cu_2 + du_3}{eu_2 + fu_3}.$$

Therefore it follows that

$$\begin{aligned} \frac{\partial f(u_0,u_1,u_2,u_3)}{\partial u_0} &= a, \qquad \frac{\partial f(u_0,u_1,u_2,u_3)}{\partial u_1} = b, \\ \frac{\partial f(u_0,u_1,u_2,u_3)}{\partial u_2} &= \frac{(cf-de)u_3}{(eu_2+fu_3)^2}, \quad \frac{\partial f(u_0,u_1,u_2,u_3)}{\partial u_3} = \frac{(de-cf)u_2}{(eu_2+fu_3)^2}. \end{aligned}$$

Then, we see that

$$\begin{aligned} \frac{\partial f(\bar{x},\bar{x},\bar{x},\bar{x})}{\partial u_0} &= a, \qquad \frac{\partial f(\bar{x},\bar{x},\bar{x},\bar{x})}{\partial u_1} = b, \\ \frac{\partial f(\bar{x},\bar{x},\bar{x},\bar{x})}{\partial u_2} &= \frac{(cf-de)[1-(a+b)]}{(e+f)(c+d)}, \quad \frac{\partial f(\bar{x},\bar{x},\bar{x},\bar{x})}{\partial u_3} = \frac{(de-cf)[1-(a+b)]}{(e+f)(c+d)}. \end{aligned}$$

Then, the linearized equation of Equation (1) about $\bar{x}$ is

$$y_{n+1} + ay_{n-l} + by_{n-k} + \frac{(cf-de)[1-(a+b)]}{(e+f)(c+d)}y_{n-s} + \frac{(de-cf)[1-(a+b)]}{(e+f)(c+d)}y_{n-p} = 0. \tag{7}$$

Theorem 1. Assume that

$$2\left|cf-de\right| < (e+f)(c+d).$$

Then the equilibrium point of Equation (1) is locally asymptotically stable.

Proof. It follows by Theorem A that Equation (7) is asymptotically stable if

$$|a| + |b| + \left|\frac{(cf-de)[1-(a+b)]}{(e+f)(c+d)}\right| + \left|\frac{(de-cf)[1-(a+b)]}{(e+f)(c+d)}\right| < 1,$$

or

$$2\left|\frac{(cf-de)[1-(a+b)]}{(e+f)(c+d)}\right| < [1-(a+b)],$$

and so

$$2\left|cf-de\right| < (e+f)(c+d).$$

This completes the proof.

4. GLOBAL ATTRACTIVITY OF THE EQUILIBRIUM POINT OF EQUATION (1)

In this section we deals the global attractivity character of solutions of Equation (1).

Theorem 2. The equilibrium point $\bar{x}$ is a global attractor of equation (1) if one of the following conditions holds:

$$\begin{aligned} (i)\ cf-de &\geq 0,\ d\geq c. \\ (ii)\ de-cf &\geq 0,\ c\geq d. \end{aligned}$$

Proof. Let r,s be nonnegative real numbers and assume that $f:[r,s]^4\rightarrow[r,s]$ be a function defined by

$$f(u_0,u_1,u_2,u_3)=au_0+bu_1+\frac{cu_2+du_3}{eu_2+fu_3}.$$

Then

$$\begin{aligned} \frac{\partial f(u_0,u_1,u_2,u_3)}{\partial u_0} &= a,\quad \frac{\partial f(u_0,u_1,u_2,u_3)}{\partial u_1}=b, \\ \frac{\partial f(u_0,u_1,u_2,u_3)}{\partial u_2} &= \frac{(cf-de)u_3}{(eu_2+fu_3)^2},\quad \frac{\partial f(u_0,u_1,u_2,u_3)}{\partial u_3}=\frac{(de-cf)u_2}{(eu_2+fu_3)^2}. \end{aligned}$$

We consider two cases:

Case1: Assume that $cf-de\geq 0$ is true, then we can easily see that the function $f(u_0,u_1,u_2,u_3)$ is increasing in u_0,u_1,u_2 and decreasing in u_3. Suppose that (m,M) is a solution of the system

$$M=f(M,M,M,m)\quad \textit{and}\quad m=f(m,m,m,M).$$

Then from Equation (1), we see that

$$\begin{aligned} M &= aM+bM+\frac{cM+dm}{eM+fm},\quad m=am+bm+\frac{cm+dM}{em+fM}, \\ M[1-(a+b)] &= \frac{cM+dm}{eM+fm},\ m[1-(a+b)]=\frac{cm+dM}{em+fM}, \end{aligned}$$

then

$$\begin{aligned} M^2e[1-(a+b)]+mMf[1-(a+b)] &= cM+dm, \\ m^2e[1-(a+b)]+mMf[1-(a+b)] &= cm+dM. \end{aligned}$$

Subtracting this two equations, we obtain

$$(M-m)\left\{e(M+m)[1-(a+b)]+(d-c)\right\}=0,$$

under the condition $(a+b)<1,\ d\geqslant c$, we see that $M=m$. It follows from Theorem B that $\bar{x}$ is a global attractor of Equation (1).

Case 2: Similar to Case 1.

5. BOUNDEDNESS OF SOLUTIONS OF EQUATION (1)

In this section we study the boundedness nature of the solutions of Equation (1).

Theorem 3. Every solution of Equation (1) is bounded if $a+b<1$.

Proof. Let $\{x_n\}_{n=-r}^{\infty}$ be a solution of Equation (1). It follows from Equation (1) that

$$x_{n+1}=ax_{n-l}+bx_{n-k}+\frac{cx_{n-s}+dx_{n-t}}{ex_{n-s}+fx_{n-t}}=ax_{n-l}+bx_{n-k}+\frac{cx_{n-s}}{ex_{n-s}+fx_{n-t}}+\frac{dx_{n-t}}{ex_{n-s}+fx_{n-t}}.$$

Then

$$x_{n+1} \leqslant ax_{n-l} + bx_{n-k} + \frac{cx_{n-s}}{ex_{n-s}} + \frac{dx_{n-t}}{fx_{n-t}} = ax_{n-l} + bx_{n-k} + \frac{c}{e} + \frac{d}{f} \ for \ all \ n \geq 1.$$

By using a comparison, we can right hand side as follows

$$z_{n+1} = az_{n-l} + bz_{n-k} + \frac{c}{e} + \frac{d}{f}.$$

and this equation is locally asymptotically stable if $a + b < 1$, and converges to the equilibrium point $\bar{z} = \frac{cf+de}{ef[1-(a+b)]}$. Therefore

$$\lim_{n\to\infty} \sup x_n \leqslant \tfrac{cf+de}{ef[1-(a+b)]}.$$

Thus the solution is bounded.

Theorem 4. Every solution of Equation (1) is unbounded if $a > 1$or $b > 1$.

Proof. Let $\{x_n\}_{n=-r}^{\infty}$ be a solution of Equation (1).Then from Equation (1) we see that

$$x_{n+1} = ax_{n-l} + bx_{n-k} + \frac{cx_{n-s} + dx_{n-p}}{ex_{n-s} + fx_{n-p}} > ax_{n-l} \ \ for \ all \ \ n \geq 1.$$

We see that the right hand side can be written as follows $z_{n+1} = az_{n-l}$. Then

$$z_{ln+i} = a^n z_{l+i} + const., \qquad i = 0, 1, ..., l,$$

and this equation is unstable because $a > 1$, and $\lim_{n\to\infty} z_n = \infty$.Then by using ratio test $\{x_n\}_{n=-r}^{\infty}$ is unbounded from above. When $b > 1$ is similar.

6. EXISTENCE OF PERIODIC SOLUTIONS

Here we study the existence of periodic solutions of Equation (1). The following theorem states the necessary and sufficient conditions that this equation has periodic solution of prime period two.

Theorem 5. Equation (1) has a prime period two solutions if and only if

$$\begin{array}{rcl}
(i)\ (d-c)(e-f)(a+b+1) & > & 4[cf + de(a+b)], \quad l,k,s - even \ and \ t - odd. \\
(ii)\ (c-d)(f-e)(a+b+1) & > & 4[cf(a+b) + de], \quad l,k,t - even \ and \ s - odd. \\
(iii)\ (c-d)(f-e)(1+a-b) & > & 4[de(1-b) + caf], \quad l,t - even \ and \ k,s - odd. \\
(iv)\ (d-c)(e-f)(1+a-b) & > & 4[cf(1-b) + dae], \ l,s - even \ and \ k,t - odd. \\
(v)\ (c-d)(f-e) & > & 4de, \ c > d, \ f > e, \ l,k,s - odd, \ and \ t - even. \\
(vi)\ (d-c)(e-f) & > & 4cf, \ d > c, \ e > f, \ \ l,k,t - odd \ and \ s - even. \\
(vii)\ (d-c)(e-f)(1-a+b) & > & 4[cf + dbe - daf], \quad l,t - odd \ and \ k,s - even. \\
(viii)\ (c-d)(f-e)(1-a+b) & > & 4[cbf - dae + de], \quad l,s - odd \ and \ k,t - even.
\end{array}$$

Proof. We prove first case when l, k and s are even, and t is odd (the other cases are similar and will be left to readers). First suppose that there exists a prime period two solution $...p, q, p, q, ...,$ of Equation (1).We will prove that Inequality (i) holds. We see from Equation (1) when l, k, s are even, and t is odd that

$$p = aq + bq + \frac{cq + dp}{eq + fp}, \qquad q = ap + bp + \frac{cp + dq}{ep + fq}.$$

Then

$$epq + fp^2 = (a+b)eq^2 + (a+b)fpq + cq + dp, \tag{8}$$
$$epq + fq^2 = (a+b)ep^2 + (a+b)fpq + cp + dq. \tag{9}$$

Subtracting (8) from (9) gives

$$\begin{aligned} f(p^2-q^2) &= -(a+b)e(p^2-q^2)-c(p-q)+d(p-q), \\ f(p-q)(p+q) &= -(a+b)e(p-q)(p+q)-c(p-q)+d(p-q), \end{aligned}$$

Since $p \neq q$, it follows that

$$f(p+q) = -(a+b)e(p+q)-c+d,$$

$$p+q = \frac{d-c}{f+(a+b)e}. \tag{10}$$

Again, adding (8) and (9) yields

$$\begin{aligned} 2epq+f(p^2+q^2) &= (a+b)e(p^2+q^2)+2(a+b)fpq+(c+d)(p+q), \\ (p^2+q^2)[f-(a+b)e] &= (c+d)(p+q)+2pq[(a+b)f-e], \end{aligned} \tag{11}$$

It follows by (10), (11) and the relation

$$p^2+q^2 = (p+q)^2-2pq \quad for\ all \quad p,q \in R,$$

that

$$\begin{aligned} [(p+q)^2-2pq][f-(a+b)e] &= (c+d)(p+q)+2pq[(a+b)f-e], \\ 2pq[(a+b)f-(a+b)e+f-e] &= (p+q)^2[f-(a+b)e]-(c+d)(p+q), \end{aligned}$$

$$pq = \frac{(d-c)[cf+de(a+b)]}{(a+b+1)(e-f)[f+(a+b)e]^2}. \tag{12}$$

Now it is clear from Equations (10) and (12) that p and q are the two distinct roots of the quadratic equation

$$\begin{aligned} r^2-\left(\frac{d-c}{f+(a+b)e}\right)r+\left(\frac{(d-c)[cf+de(a+b)]}{(a+b+1)(e-f)[f+(a+b)e]^2}\right) &= 0, \qquad (13) \\ (f+(a+b)e)r^2-(d-c)r+\left(\frac{(d-c)[cf+de(a+b)]}{(a+b+1)(e-f)[f+(a+b)e]}\right) &= 0, \end{aligned}$$

and so

$$\frac{(d-c)^2}{[f+(a+b)e]^2} > \frac{4(d-c)[cf+de(a+b)]}{(a+b+1)(e-f)[f+(a+b)e]^2}.$$

Thus

$$(d-c)(e-f)(a+b+1) > 4[cf+de(a+b)].$$

Therefore Inequality (i) holds.

Second suppose that Inequality (i) is true. We will show that Equation (1) has a prime period two solution. Assume that

$$p = \frac{d-c+\xi}{2(f+Ae)}, \qquad q = \frac{d-c-\xi}{2(f+Ae)},$$

where

$$\xi = \sqrt{(d-c)^2-\frac{4(d-c)(cf+Ade)}{(A+1)(e-f)}}, \quad and \quad A = (a+b).$$

We see from Inequality (1) that

$$(d-c)(e-f)(a+b+1) > 4[cf+de(a+b)].$$

which equivalents to

$$\frac{(d-c)^2}{[f+(a+b)e]^2} > 4\frac{(d-c)[cf+de(a+b)]}{(a+b+1)(e-f)[f+(a+b)e]^2},$$

Therefore p and q are distinct real numbers. Set $x_{-l} = p,\ x_{-k} = p,\ x_{-s} = p,\ x_{-t} = q, ..., x_{-2} = p,\ x_{-1} = q,\ x_0 = p$. We wish to show that

$$x_1 = x_{-1} = q \quad \text{and} \quad x_2 = x_0 = p.$$

It follows from Equation (1) that

$$x_1 = A\left[\frac{d-c+\xi}{2(f+Ae)}\right] + \frac{c[\frac{d-c+\xi}{2(f+Ae)}] + d[\frac{d-c-\xi}{2(f+Ae)}]}{e[\frac{d-c+\xi}{2(f+Ae)}] + f[\frac{d-c-\xi}{2(f+Ae)}]},$$

Dividing the denominator and numerator by $2(f+Ae)$ gives

$$x_1 = A\left[\frac{d-c+\xi}{2(f+Ae)}\right] + \frac{c[d-c+\xi] + d[d-c-\xi]}{e[d-c+\xi] + f[d-c-\xi]},$$

$$\begin{aligned}
&= A\left[\frac{d-c+\xi}{2(f+Ae)}\right] + \frac{d^2-c^2+\xi(c-d)}{e(d-c)+f(d-c)+\xi(e-f)}, \\
&= A\left[\frac{d-c+\xi}{2(f+Ae)}\right] + \frac{(d-c)(d+c-\xi)}{(d-c)(e+f)+\xi(e-f)},
\end{aligned}$$

Multiplying the denominator and numerator of the right side by $(d-c)(e+f) - \xi(e-f)$ gives

$$\begin{aligned}
x_1 &= A\left[\frac{d-c+\xi}{2(f+Ae)}\right] + \frac{(d-c)(d+c-\xi)[(d-c)(e+f)-\xi(e-f)]}{[(d-c)(e+f)+\xi(e-f)][(d-c)(e+f)-\xi(e-f)]}, \\
&= A\left[\frac{d-c+\xi}{2(f+Ae)}\right] + \frac{(d-c)(d+c-\xi)[(d-c)(e+f)-\xi(e-f)]}{(d-c)^2(e+f)^2-\xi^2(e-f)^2}, \\
&= A\left[\frac{d-c+\xi}{2(f+Ae)}\right] + \frac{(d-c)(d+c-\xi)[de+df-ce-cf-\xi e-\xi f]}{(d-c)^2(e+f)^2-(e-f)^2\left[(d-c)^2-\frac{4(d-c)(cf+Ade)}{(A+1)(e-f)}\right]}, \\
&= A\left[\frac{d-c+\xi}{2(f+Ae)}\right] + \frac{(d-c)[e(d^2-c^2)+f(d^2-c^2)+2\xi(cf-de)+\xi^2(e-f)]}{(d-c)^2[e^2+2ef+f^2-(e^2-2ef+f^2)]+\frac{4(d-c)(e-f)(cf+Ade)}{(A+1)}}, \\
&= A\left[\frac{d-c+\xi}{2(f+Ae)}\right] + \frac{(d-c)[(d^2-c^2)(e+f)+2\xi(cf-de)+\xi^2(e-f)]}{\frac{4ef(A+1)(d-c)^2+4(d-c)(e-f)(cf+Ade)}{(A+1)}}, \\
&= A\left[\frac{d-c+\xi}{2(f+Ae)}\right] + \frac{(d-c)\left[(d^2-c^2)(e+f)+2\xi(cf-de)+\frac{(d-c)(-3Ade-3cf+Afc-Afd-Aec+ed-ec-fd)}{(A+1)}\right]}{\frac{4(d-c)[ef(Ad-Ac+d-c)]+[ecf+Ae^2d-cf^2-Adef]}{(A+1)}},
\end{aligned}$$

$$\begin{aligned}
&= A\left[\frac{d-c+\xi}{2(f+Ae)}\right] + \frac{(d-c)\left[\frac{2(d-c)(A-1)(cf-de)}{A+1}+2\xi(cf-de)\right]}{\left[\frac{4(d-c)[efd-Acef+Ae^2d-cf^2]}{(A+1)}\right]}, \\
&= A\left[\frac{d-c+\xi}{2(f+Ae)}\right] + \frac{2(d-c)(cf-de)\left[\frac{(d-c)(A-1)}{A+1}+\xi\right]}{\left[\frac{4(d-c)[efd-Acef+Ae^2d-cf^2]}{(A+1)}\right]}, \\
&= A\left[\frac{d-c+\xi}{2(f+Ae)}\right] + \frac{(cf-de)\{(d-c)(A-1)+\xi(A+1)\}}{2[efd-Acef+Ae^2d-cf^2]}, \\
&= A\left[\frac{d-c+\xi}{2(f+Ae)}\right] + \frac{(cf-de)\{(d-c)(A-1)+\xi(A+1)\}}{2(f+Ae)(de-cf)}, \\
&= A\left[\frac{d-c+\xi}{2(f+Ae)}\right] + \frac{-\{(d-c)(A-1)+\xi(A+1)\}}{2(f+Ae)}, \\
&= \frac{Ad-Ac+A\xi-Ad+d+Ac-c-A\xi-\xi}{2(f+Ae)} = \frac{d-c-\xi}{2(f+Ae)} = q.
\end{aligned}$$

Similarly as before we can easily show that $x_2 = p$. Then it follows by induction that $x_{2n} = p$ and $x_{2n+1} = q$ for all $n \geq -1$. Thus Equation (1) has the prime period two solution $..., p, q, p, q, ...,$ where p and q are the distinct roots of the quadratic equation (13) and the proof is complete.

7. NUMERICAL EXAMPLES

For confirming the results of this article, we consider numerical examples which represent different types of solutions to Equation (1).

Example 1. We consider numerical example for the difference equation (1) when we take the constants and the initial conditions as follows: $l = 3,\ k = 2,\ s = 1\ , t = 3,\ x_{-3} = 5,\ x_{-2} = -12,\ x_{-1} = 6,\ x_0 = 8,\ a = 0.4,\ b = 0.3,\ c = 2,\ d = 4,\ e = 6,\ f = 8$. See Figure 1.

Example 2. See Figure (2) when we take Equation (1) with $l = 1,\ k = 3,\ s = 2\ , t = 3,\ x_{-3} = 13,\ x_{-2} = -9,\ x_{-1} = -7,\ x_0 = 5,\ \ a = 0.6,\ b = 0.4,\ c = 3,\ d = 2,\ e = 5,\ f = 8$.

Example 3. Figure (3) shows the behavior of the solution of the difference equation (1) when we put $l = 2,\ k = 1,\ s = 3\ , t = 3,\ x_{-3} = 15,\ x_{-2} = 11,\ x_{-1} = -9,\ x_0 = 5,\ \ a = 0.6,\ b = 1.4,\ c = 2,\ d = 4,\ e = 6,\ f = 9$.

Example 4. We assume $l = 2,\ k = 3,\ s = 1\ , t = 2,\ x_{-3} = 15,\ x_{-2} = 11,\ x_{-1} = -9,\ x_0 = 5,\ \ a = 1.5,\ b = 0.2,\ c = 2,\ d = 0,\ e = 6,\ f = 7$. See Figure 4.

Example 5. Figure (5) shows the period two solution of Equation (1) when $l = 0,\ k = 2,\ s = 2\ , t = 3,\ x_{-3} = p,\ x_{-2} = q,\ x_{-1} = p,\ x_0 = q,\ a = 0.06,\ b = 0.03,\ c = 1,\ d = 5,\ e = 7,\ f = 2$, since p and q as in the previous theorem.

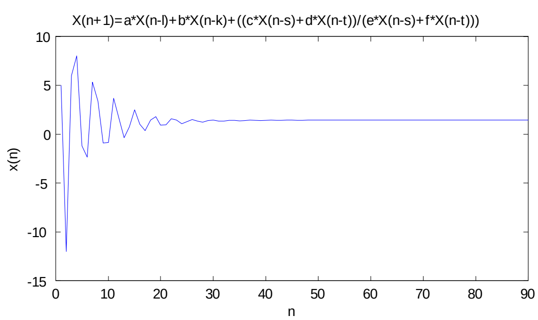

Figure 1.

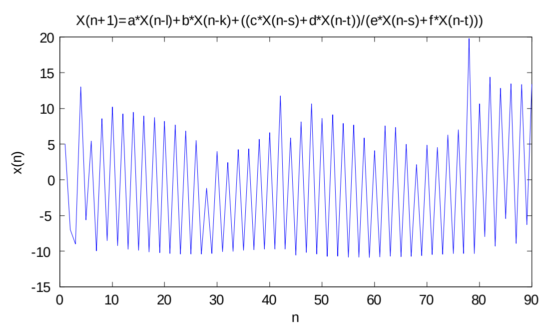

Figure 2.

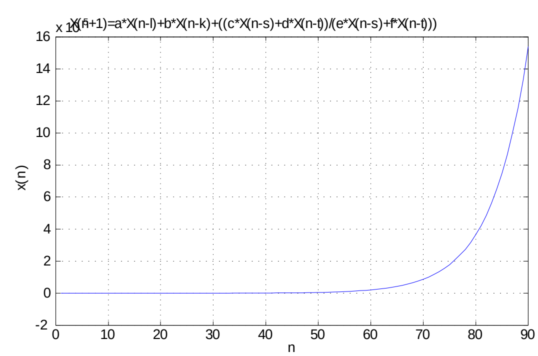

Figure 3.

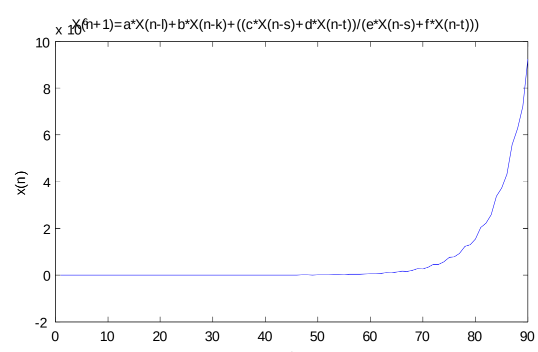

Figure 4.

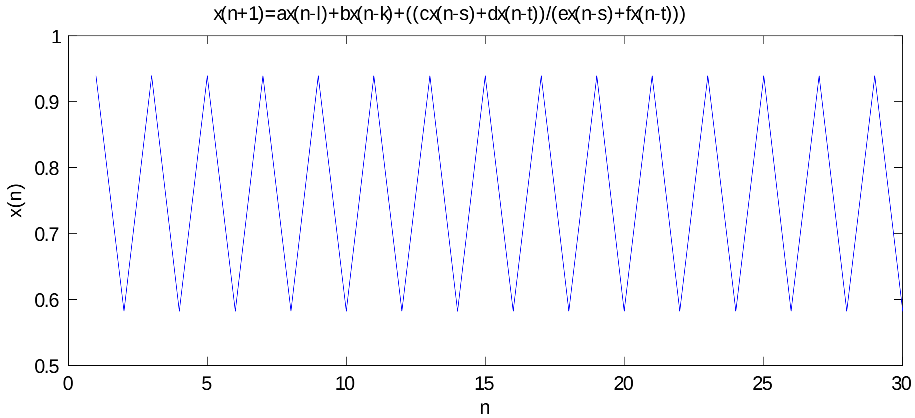

Figure 5.

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A fractional derivative inclusion problem via an integral boundary condition

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Abstract. We investigate the existence of solutions for the fractional differential inclusion ${}^cD^\alpha x(t) \in F(t, x(t))$ equipped with the boundary value problems $x(0) = 0$ and $x(1) = \int_0^\eta x(s)ds$, where $0 < \eta < 1$, $1 < \alpha \le 2$, ${}^cD^\alpha$ is the standard Caputo differentiation and $F : [0, 1] \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact valued multifunction. An illustrative example is also discussed.

Keywords: Fixed point, Fractional differential inclusion, Integral boundary value problem.

1 Introduction

During the last decade the fractional differential equations were investigated from theoretical and applied viewpoints (see for example, [1]-[6], [8]-[15], and [32]). A special attention was given to the real world applications where the power law effect is present and where the fractional models a give better results than the classical ones.

We recall that the Riemann-Liouville fractional integral of order $\alpha > 0$ of $f : (0, \infty) \to \mathbb{R}$ is given by $I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds$ provided the right side is pointwise defined on $(0, \infty)$ (see [26], [29], [31], [34] and [35]). Also, the Caputo fractional derivative of order α of f is defined by ${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds$, where $n = [\alpha] + 1$ (see [26], [29], [31], [34] and [35]).

We recall that the basic theory for fractional differential inclusions is represented by the fixed point theory of multivalued mappings which was intensively investigated during last years (the reader can find more details in [18]-[25], [30] and the related references). Thus, many papers about ordinary and fractional differential inclusions were written (e.g. [1]-[2],

[7], [16], [17] and [33]).

Let (X, d) be a metric space. Let us denote by $P(X)$ and 2^X the class of all subsets and the class of all nonempty subsets of X respectively. As a result, $P_{cl}(X)$, $P_{bd}(X)$, $P_{cv}(X)$ and $P_{cp}(X)$ denote the class of all closed, bounded, convex and compact subsets of X respectively. A mapping $Q : X \to 2^X$ is called a multifunction on X and $u \in X$ is called a fixed point of Q whenever $u \in Qu$ ([24]). Also, we say that Q is convex whenever Qx is convex for all $x \in X$ ([24]). A multifunction $G : [0,1] \to P_{cl}(\mathbb{R})$ is said to be measurable whenever the function $t \mapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$ is measurable for all $y \in \mathbb{R}$. Put $J = [0,1]$.

The aim of this manuscript is to investigate the existence of solutions for the fractional differential inclusion

$$^cD^\alpha x(t) \in F(t, x(t)) \qquad (*)$$

via the boundary value problems $x(0) = 0$ and $x(1) = \int_0^\eta x(s)ds$, where $^cD^\alpha$ is the standard Caputo differentiation, $0 < \eta < 1$, $1 < \alpha \leq 2$ and $F : J \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact valued multifunction. We say that $F : J \times \mathbb{R} \to 2^{\mathbb{R}}$ is a Caratheodory multifunction whenever $t \mapsto F(t, x)$ is measurable for all $x \in \mathbb{R}$ and $x \mapsto F(t, x)$ is upper semi-continuous for almost all $t \in J$. Also, a Caratheodory multifunction $F : J \times \mathbb{R} \to 2^{\mathbb{R}}$ is called L^1-Caratheodory whenever for each $\rho > 0$ there exists $\phi_\rho \in L^1(J, \mathbb{R}^+)$ such that $\| F(t, x) \| = \sup\{|v| : v \in F(t, x)\} \leq \phi_\rho(t)$ for all $\|x\|_\infty \leq \rho$ and for almost all $t \in J$. For each $x \in C(J, \mathbb{R})$, define the set of selections of F by

$$S_{F,x} := \{v \in L^1(J, \mathbb{R}) : \ v(t) \in F(t, x(t)) \text{ for almost all } t \in J\}.$$

Let E be a nonempty closed subset of a Banach space X and $G : E \to 2^X$ a multifunction with nonempty closed values. We say that the multifunction G is lower semi-continuous whenever the set $\{y \in E : G(y) \cap B \neq \emptyset\}$ is open for all open set B in X. It has been proved that each completely continuous multifunction is lower semi-continuous (see [24]). We shall use the following fixed point results.

Lemma 1.1. *([30]) Let X be a Banach space, $F : J \to P_{cp,cv}(X)$ an L^1-Caratheodory multifunction and Θ a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$. Then the operator $\Theta o S_F : C(J, X) \to P_{cp,cv}(C(J), X)$ defined by $(\Theta o S_F)(x) = \Theta(S_{F,x})$ is a closed graph operator in $C(J, X) \times C(J, X)$.*

It has been proved that if $dim X < \infty$, then $S_F(x) \neq \emptyset$ for all $x \in C(J, X)$ ([30]).

Lemma 1.2. *([24]) Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to P_{cp,cv}(C)$ is a upper semi-continuous compact map, where $P_{cp,cv}(C)$ denotes the family of nonempty, compact convex subsets of C. Then either F has a fixed point in $\overline{U}$ or there exist $u \in \partial U$ and $\lambda \in (0, 1)$ such that $u \in \lambda F(u)$.*

Let $(X, \|.\|)$ be a normed space. Define the Hausdorff metric $H_d : 2^X \times 2^X \to [0, \infty]\}$ by

$$H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\},$$

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where $d(A,b) = \inf_{a\in A} d(a;b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space ([27]). A multifunction $N : X \to P_{cl}(X)$ is called a contraction whenever there exists $\gamma > 0$ such that $H_d(N(x), N(y)) \leq \gamma d(x,y)$ for all $x, y \in X$.

Lemma 1.3. *([19]) Let (X,d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then N has a fixed point.*

Lemma 1.4. *[11] Let $0 < \eta < 1$. Then x is a solution for the differential equation ${}^cD^\alpha x(t) = v(t)$ $(t \in J$ and $1 < \alpha \leq 2)$ via the boundary value conditions $x(0) = 0$ and $x(1) = \int_0^\eta x(s)ds$ if and only if x is a solution of the integral equation*

$$x(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}v(s)ds$$
$$+\frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^\eta (\int_0^s (s-m)^{\alpha-1}v(m)dm)ds \quad (t \in J).$$

2 Main results

Here, we give our results about the existence of solutions for the inclusion problem $(*)$.

Theorem 2.1. *Suppose that $F : J\times\mathbb{R} \to 2^{\mathbb{R}}$ is a Caratheodory multifunction with compact and convex values and there exist a bounded continuous non-decreasing map $\psi : [0,\infty) \to (0,\infty)$ and a continuous function $p : J \to (0,\infty)$ such that*

$$\|F(t,x(t))\| = \sup\{|v| : \ v \in F(t,x(t))\} \leq p(t)\psi(\|x\|_\infty)$$

for all $t \in J$ and $x \in C(J,\mathbb{R})$. Then the problem $(*)$ has at least one solution.

Proof. By using Lemma 1.4, we know that the existence of solution for the problem $(*)$ is equivalent to the existence of solution for the integral equation

$$x(t) \in \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}v(s)ds$$
$$+\frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^\eta (\int_0^s (s-m)^{\alpha-1}v(m)dm)ds \quad (t \in J).$$

Put $E = C(J,\mathbb{R})$. Define the operator $N : E \to 2^E$ by

$$N(x) = \{h \in E : \ h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}v(s)ds$$
$$+\frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^\eta (\int_0^s (s-m)^{\alpha-1}v(m)dm)ds, \text{ for some } v \in S_{F,x}\}.$$

We show that the operator N satisfies the assumptions of Lemma 1.2. First, we show that $N(x)$ is convex for all $x \in C(J,\mathbb{R})$. Let $h_1, h_2 \in N(x)$. Choose $v_1, v_2 \in S_{F,x}$ such that

$$h_i(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} v_i(s)ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} v_i(s)ds$$
$$+\frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^\eta (\int_0^s (s-m)^{\alpha-1} v_i(m)dm)ds$$

for all $t \in J$ and $i = 1, 2$. Let $0 \le w \le 1$. Then, we have

$$[wh_1 + (1-w)h_2](t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[wv_1(s) + (1-w)v_2(s)]ds$$
$$-\frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}[wv_1(s) + (1-w)v_2(s)]ds$$
$$+\frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^\eta (\int_0^s (s-m)^{\alpha-1}[wv_1(m) + (1-w)v_2(m)]dm)ds$$

for all $t \in J$. Since $S_{F,x}$ is convex (because F has convex values), $wh_1 + (1-w)h_2 \in N(x)$. Now, we show that $N(x)$ maps bounded sets of $C(J,\mathbb{R})$ into bounded sets. Let $r > 0$ and $B_r = \{x \in C(J,\mathbb{R}) : \|x\|_\infty \le r\}$. For each $h \in N(x)$ and $x \in B_r$ choose $v \in S_{F,x}$ such that

$$h(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} v(s)ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} v(s)ds$$
$$+\frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^\eta (\int_0^s (s-m)^{\alpha-1} v(m)dm)ds$$

and

$$|h(t)| \le \sup_{t\in J} \mid \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} v(s)ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} v(s)ds$$
$$+\frac{2t}{(2-\eta^2)\Gamma(\alpha)}\int_0^\eta (\int_0^s (s-m)^{\alpha-1} v(m)dm)ds \mid$$
$$\le \sup_{t\in J} \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \mid v(s) \mid ds + \sup_{t\in[0,1]} \mid \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \mid \int_0^1 (1-s)^{\alpha-1} \mid v(s) \mid ds$$
$$+\sup_{t\in J} \mid \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \mid \int_0^\eta (\int_0^s (s-m)^{\alpha-1} \mid v(m) \mid dm)ds \le \|p\|_\infty \psi(\|x\|_\infty)A$$

for all $t \in J$, where $\|p\|_\infty = \sup_{t\in J} p(t)$ and $A = \frac{(\alpha+1)(2-\eta^2)+2(\alpha+1)+2\eta^{\alpha+1}}{(2-\eta^2)\Gamma(\alpha+2)}$. Thus,

$$\|h(t)\|_\infty = \sup_{t\in J}|h(t)| \le A\|p\|_\infty \psi(\|x\|_\infty).$$

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Now, we show that N maps bounded sets into equi-continuous sets of $C(J,\mathbb{R})$. Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $x \in B_r$. Then,

$$|h(t_2) - h(t_1)| = | \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} v(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} v(s) ds$$

$$- \frac{2t_2}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s) ds + \frac{2t_1}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s) ds$$

$$+ \frac{2t_2}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta (\int_0^s (s-m)^{\alpha-1} v(m) dm) ds$$

$$- \frac{2t_1}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta (\int_0^s (s-m)^{\alpha-1} v(m) dm) ds \,|$$

$$\leq \|p\|_\infty \psi(\|x\|_\infty) [\frac{(2-\eta^2)(t_2^\alpha - t_1^\alpha) + 2(t_1 - t_2)}{(2-\eta^2)\Gamma(\alpha+1)} + \frac{2(t_2 - t_1)\eta^{\alpha+1}}{(2-\eta^2)\Gamma(\alpha+2)}]$$

For all $h \in N(x)$. Thus, $\lim_{t_2 \to t_1} |h(t_2) - h(t_1)| = 0$ for all $x \in B_r$. Hence by using the Arzela-Ascoli theorem, N is completely continuous. Here, we show that N has a closed graph. Let $x_n \to x_0$, $h_n \in N(x_n)$ for all n and $h_n \to h_0$. We have to show that $h_0 \in N(x_0)$. For each n choose $v_n \in S_{F,x_n}$ such that

$$h_n(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_n(s) ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v_n(s) ds$$

$$+ \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta (\int_0^s (s-m)^{\alpha-1} v_n(m) dm) ds$$

for all $t \in J$. Define the continuous linear operator $\theta : L^1(J,\mathbb{R}) \to C(J,\mathbb{R})$ by

$$\theta(v) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s) ds$$

$$+ \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta (\int_0^s (s-m)^{\alpha-1} v(m) dm) ds.$$

Note that,

$$\|h_n(t) - h_0(t)\| = \| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (v_n(s) - v_0(s)) ds$$

$$- \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} (v_n(s) - v_0(s)) ds$$

$$+ \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta (\int_0^s (s-m)^{\alpha-1} (v_n(m) - v_0(m)) dm) ds \|$$

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for all n and so $\lim n \to \infty \|h_n(t) - h_0(t)\| = 0$. By using Lemma 1.1, $\theta o S_F$ is a closed graph operator. Since $h_n(t) \in \theta(S_{F,x_n})$ for all n and $x_n \to x_0$, there exists $v_0 \in S_{F,x_0}$ such that

$$h_0(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_0(s)ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v_0(s)ds$$
$$+\frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta (\int_0^s (s-m)^{\alpha-1} v_0(m)dm)ds.$$

Thus, N has a closed graph. If there exists $\lambda \in (0,1)$ such that $x \in \lambda N(x)$, then there exists $v \in S_{F,x}$ such that

$$x(t) = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)ds - \frac{2\lambda t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s)ds$$
$$+\frac{2\lambda t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta (\int_0^s (s-m)^{\alpha-1} v(m)dm)ds$$

for all $t \in J$. Now, choose $M > 0$ such that $\frac{\|p\|_\infty \psi(\|x\|_\infty)([\alpha+1)(2-\eta^2)+2(\alpha+1)+2\eta^{\alpha+1}]}{(2-\eta^2)\Gamma(\alpha+2)} < M$ for all $x \in E$. This is possible because ψ is bounded. Thus,

$$\|x\|_\infty \leq \frac{\|p\|_\infty \psi(\|x\|_\infty)([\alpha+1)(2-\eta^2)+2(\alpha+1)+2\eta^{\alpha+1}]}{(2-\eta^2)\Gamma(\alpha+2)} < M.$$

Now, put $U = \{x \in C(J,\mathbb{R}) : \|x\|_\infty < M+1\}$. Thus, there are not $x \in \partial U$ and $\lambda \in (0,1)$ such that $x \in \lambda N(x)$. Note that, the operator $N : \overline{U} \to P_{cp,cv}(\overline{U})$ is upper semi-continuous because it is completely continuous. Now by using Lemma 1.2, N has a fixed point in $\overline{U}$ which is a solution of the problem $(*)$. This completes the proof. □

Now, we present our next result about the existence of solutions for the problem $(*)$ with non-convex valued assumption.

Theorem 2.2. *Let $m \in C(J,\mathbb{R}^+)$ be such that $\|m\|_\infty(\frac{4-\eta^2}{(2-\eta^2)\Gamma(\alpha+1)} + \frac{2\eta^{\alpha+1}}{(2-\eta^2)\Gamma(\alpha+2)}) < 1$. Suppose that $F : J \times \mathbb{R} \to P_{cp}(\mathbb{R})$ is a multifunction such that $H_d(F(t,x), F(t,y)) \leq m(t)|x-y|$ and $d(x, F(t,x)) \leq m(t)$ for almost all $t \in J$ and $x, y \in \mathbb{R}$. Then the boundary value inclusion problem $(*)$ has a solution.*

Proof. Note that, $S_{F,x}$ is nonempty for all $x \in C(J,\mathbb{R})$. By using Theorem III.6 in [18], we get F has a measurable selection. Now, similar to the proof of Theorem 2.1, consider the operator $N : E \to 2^E$ by

$$N(x) = \{h \in E : \ h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s)ds$$
$$+\frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta (\int_0^s (s-m)^{\alpha-1} v(m)dm)ds, \text{ for some } v \in S_{F,x}\},$$

where $E = C(J, \mathbb{R})$. First, we show that $N(x)$ is a closed subset of E for all $x \in E$. Let $x \in E$ and $\{u_n\}_{n\geq 1}$ be a sequence in $N(x)$ with $u_n \to u$. For each n, choose $v_n \in S_{F,x}$ such that

$$u_n(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_n(s)ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v_n(s)ds$$
$$+ \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta (\int_0^s (s-m)^{\alpha-1} v_n(m)dm)ds$$

for all $t \in J$. Since F has compact values, $\{v_n\}_{n\geq 1}$ has a subsequence which converges to some $v \in L^1(J, \mathbb{R})$. We denote this subsequence again by $\{v_n\}_{n\geq 1}$. It is easy to check that $v \in S_{F,x}$ and

$$u_n(t) \to u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s)ds$$
$$+ \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta (\int_0^s (s-m)^{\alpha-1} v(m)dm)ds$$

for all $t \in J$. This implies that $u \in N(x)$ and so the multifunction N has closed values. Now, we show that N is a contractive multifunction with constant

$$\gamma = \|m\|_\infty (\frac{4-\eta^2}{(2-\eta^2)\Gamma(\alpha+1)} + \frac{2\eta^{\alpha+1}}{(2-\eta^2)\Gamma(\alpha+2)}) < 1.$$

Let $x, y \in E$ and $h_1 \in N(x)$. Choose $v_1 \in S_{F,x}$ such that

$$h_1(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_1(s)ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v_1(s)ds$$
$$+ \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta (\int_0^s (s-m)^{\alpha-1} v_1(m)dm)ds$$

for all $t \in J$. Since $H_d(F(t,x), F(t,y)) \leq m(t)|x(t) - y(t)|$ for almost all $t \in J$, there exists $w_0 \in F(t, y(t))$ such that $|v_1 - w_0| \leq m(t)|x(t) - y(t)|$ for almost all $t \in [0,1]$. Define the multifunction $U : J \to 2^{\mathbb{R}}$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq m(t)|x(t) - y(t)|\} \text{ for almost all } t \in J\}.$$

By using Proposition III.4 in [18], we get the multifunction $U(t) \cap F(t, y(t))$ is measurable. It is easy to see that there exists $v_2 \in S_{F,y}$ such that $|v_1(t) - v_2(t)| \leq m(t)|x(t) - y(t)|$ For all $t \in J$. Now, define

$$h_2(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_2(s)ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v_2(s)ds$$
$$+ \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta (\int_0^s (s-m)^{\alpha-1} v_2(m)dm)ds$$

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for all $t \in J$. Thus,

$$| h_1(t) - h_2(t) | \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} | v_1(s) - v_2(s) | ds$$

$$+ | \frac{2t}{(2-\eta^2)\Gamma(\alpha)} | \int_0^1 (1-s)^{\alpha-1} | v_1(s) - v_2(s) | ds$$

$$+ | \frac{2t}{(2-\eta^2)\Gamma(\alpha)} | \int_0^\eta (\int_0^s (s-m)^{\alpha-1} | v_1(m) - v_2(m) | dm) ds$$

$$\leq \|m\|_\infty (\frac{4-\eta^2}{(2-\eta^2)\Gamma(\alpha+1)} + \frac{2\eta^{\alpha+1}}{(2-\eta^2)\Gamma(\alpha+2)}) \|x-y\|_\infty = \gamma \|x-y\|_\infty.$$

Therefore, the multifunction N is a contraction with closed values. By using Lemma 1.3, N has a fixed point which is a solution of the inclusion problem $(*)$. □

3 Application

Consider the problem

$$^cD^{3/2}x(t) \in F(t, x(t)) \quad (t \in [0,1])$$

via the boundary value conditions $x(0) = 0$ and $x(1) = \int_0^{3/4} x(s) ds$, where $F : [0,1] \times \mathbb{R} \to 2^{\mathbb{R}}$ is the multifunction defined by

$$F(t, x) = [\frac{x^5}{4(x^5+3)} + \frac{t+1}{8}, \frac{1}{4} \sin x + \frac{1}{4}(t+1)].$$

Since $\max[\frac{x^5}{4(x^5+3)} + \frac{t+1}{8}, \frac{1}{4} \sin x + \frac{1}{4}(t+1)] \leq \frac{3}{4}$, it is easy to check that

$$\sup\{|\gamma| : \gamma \in F(t, x)\} \leq p(t)\psi(\|x\|_\infty)$$

for all $x \in C([0,1], \mathbb{R})$, where $p(t) = 1$ and $\psi(t) = \frac{3}{4}$ for all $t \in [0,1]$. Thus by using Theorem 2.1, this inclusion problem has at least one solution.

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Stability and hyperstability of generalized orthogonally quadratic ternary homomorphisms in non-Archimedean ternary Banach algebras: a fixed point approach

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Abstract: Using the fixed point method, we prove the stability and the hyperstability of generalized orthogonally quadratic ternary homomorphisms in non-Archimedean ternary Banach algebras.

1. Introduction and preliminaries

The stability problem of functional equations had been first raised by Ulam [29]. This problem solved by Hyers [16] in the framework of Banach spaces. For more details about the result concerning such problems, we refer the reader to ([1, 3, 11, 17, 22, 25, 26, 27, 28, 31, 32]). The stability of homomorphisms and derivations in Banach algebras, Banach ternary algebras, C^*-algebras, Lie C^*-algebras, C^*-ternary algebras has been studied by many authors (see [9, 25, 26, 27, 28]).

Let $\mathcal{A}, \mathcal{B}$ be two ternary algebras. A mapping $f : \mathcal{A} \to \mathcal{B}$ is called a quadratic ternary homomorphism if f is a quadratic mapping (i.e. $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ for all $x, y \in \mathcal{A}$) and satisfies

$$f([a, b, c]) = [f(a), f(b), f(c)]$$

for all $a, b, c \in \mathcal{A}$.

A mapping $g : \mathcal{A} \to \mathcal{B}$ is called a generalized quadratic ternary homomorphism if there exists a quadratic ternary homomorphism $f : \mathcal{A} \to \mathcal{B}$ such that

$$g([a, b, c]) = [g(a), f(b), f(c)]$$

for all $a, b, c \in \mathcal{A}$.

In 2003, Cădariu and Radu applied the fixed point methods to the investigation of Jensen functional equations [4] (see also [5, 6, 12, 21, 24]).

Arriola and Beyer [2] initiated the stability of functional equations in non-Archimedean spaces. In fact they established the stability of the Cauchy functional equation over p-adic fields. After their results some papers (see, for instance, ([7, 8, 9, 10]) on the stability of other equations in such spaces have been published.

In 1897, Hensel [15] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. During the last three decades p-adic numbers have gained the interest in of physicists for their research, in particular, in the problems coming from quantum physics, p-adic strings and hyperstrings [18, 19]. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: For any $x, y > 0$, there exists $n \in \mathbb{N}$ such that $x < ny$ (see [13, 30]).

Let $\mathbb{K}$ denote a field and function (valuation absolute) $|.|$ from $\mathbb{K}$ into $[0, \infty)$. A non-Archimedean valuation is a function $|.|$ that satisfies the strong triangle inequality; namely $|x + y| \le max\{|x|, |y|\} \le |x| + |y|$ for all $x, y \in \mathbb{K}$. The associated field $\mathbb{K}$ is referred to as a non-Archimedean filed. Clearly, $|1| = |-1| = 1$ and $|n| \le 1$ for all $n \ge 1$. A trivial example of a non-Archimedean valuation is the function $|.|$ taking everything except 0 into 1 and $|0| = 0$. We always assume in addition that $|.|$ is non trivial, i.e., there is a $z \in \mathbb{K}$ such that $|z| \ne 0, 1$.

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Let X be a linear space over a field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|.|$. A function $\|.\| : X \to [0, \infty)$ is said to be a non-Archimedean norm if it is a norm over $\mathbb{K}$ with the strong triangle inequality (ultrametric); namely, $\|x + y\| \leq max\{\|x\|, \|y\|\}$ for all $x, y \in \mathbb{K}$. Then $(X, \|.\|)$ is called a non-Archimedean space. In any such a space a sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if and only if $\{x_{n+1}, x_n\}_{n\in\mathbb{N}}$ converges to zero. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent. A non-Archimedean ternary Banach algebra is a complete non-Archimedean space $\mathcal{A}$ equipped with a ternary product $(x, y, z) \to [x, y, z]$ of $\mathcal{A}^3$ into $\mathcal{A}$ which is $\mathcal{K}$-linear in each variables and associative in the sense that

$$[x, y, [z, w, v]] = [x, [w, z, y], v] = [[x, y, z], w, v]$$

and satisfies the following:

$$\|[x, y, z]\| \leq \|x\| \cdot \|y\| \cdot \|z\|$$

(see [14]).

Let X be a nonempty set and let $d : X \times X \to [0, \infty]$ satisfy: $d(x, y) = 0$ if and only if $x = y$, $d(x, y) = d(y, x)$ and $d(x, z) \leq max\{d(x, y), d(y, z)\}$ (strong triangle inequality), for all $x, y, z \in X$. Then (X, d) is called a non-Archimedean generalized metric space. (X, d) is called complete if every d-Cauchy sequence in X is d-convergent.

Suppose that X is a real vector space (or an algebra) with $\dim X \geq 2$ and $\perp$ is a binary relation on X with the following properties:

(O_1) totality of $\perp$ for zero: $x \perp 0$, $0 \perp x$ for all $x \in X$;
(O_2) independence: if $x, y \in X - \{0\}$, $x \perp y$, then x, y are linearly independent;
(O_3) homogeneity: if $x, y \in X$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;
(O_4) the Thalesian property: if P is a $2-$dimensional subspace (subalgebra) of X, $x \in P$ and $\lambda \in R_+$, then there exists $u_x \in P$ such that $x \perp u_x$ and $x + u_x \perp \lambda x - u_x$.

The pair $(X, \perp)$ is called an orthogonality space (algebra). By an orthogonality normed space (normed algebra) we mean an orthogonality space (algebra) having a normed structure (see [23]).

2. **Main results**

Using the strong triangle inequality in the proof of the main result of [20], we get to the following result:

Theorem 2.1. *(Non-Archimedean Alternative Contraction Principle) Suppose that (Ω, d) is a non-Archimedean generalized complete metric space and $T : \Omega \to \Omega$ is a strictly contractive mapping with the Lipschitz constant L. Let $x \in \Omega$. If either*
(i) $d(T^m(x), T^{m+1}(x)) = \infty$ for all $m \geq 0$, or
(ii) there exists some m_0 such that $d(T^m(x), T^{m+1}(x)) < \infty$ for all $m \geq m_0$, then the sequence $\{T^m(x)\}$ is convergent to a fixed point x^* of T; x^* is the unique fixed point of T in the set

$$\Lambda = \{y \in \Omega : d(T^{m_0}(x), y) < \infty\};$$

and $d(y, x^*) \leq d(y, T(y))$ for all y in this set.

In this section, we suppose that $\mathcal{A}$ is a non-Archimedean ternary Banach algebra with $\perp := \acute{\perp} \bigcup \{(x, \alpha x) : x \in \mathcal{A}, \alpha \in \mathbb{R}\}$, where $\perp \bigcup$ is an orthogonality on $\mathcal{A}$, and $\mathcal{B}$ is a non-Archimedean ternary Banach algebra and $l \in \{1, -1\}$ is fixed. Also, let $|4| < 1$ and we assume that $4 \neq 0$ in $\mathbb{K}$ (i.e., the characteristic of $\mathbb{K}$ is not 4).

Theorem 2.2. *Let $g, f : \mathcal{A} \to \mathcal{B}$ be two mappings with $g(0) = f(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^8 \to [0, \infty]$ such that*

$$\begin{aligned} &\|\eta(ax + by) + \eta(ax - by) - 2a^2\eta(x) - 2b^2\eta(y)\| + \|f([u, v, w]) - [f(u), f(v), f(w)]\| \\ &\quad + \|g([r, s, t]) - [g(r), f(s), f(t)]\| \leq \varphi(x, y, u, v, w, r, s, t) \end{aligned} \tag{2.1}$$

for all $\eta \in \{f, g\}$, $x, y \in \mathcal{A}$ with $x \perp y$ and for all $u, v, w, r, s, t, \in \mathcal{A}$, that are mutually orthogonal and nonzero fixed integers a, b. Suppose that there exists $L < 1$ such that

$$\varphi(x, y, u, v, w, r, s, t) \leq |4|^{l(l+2)} L\varphi(\frac{x}{2^l}, \frac{y}{2^l}, \frac{u}{2^l}, \frac{v}{2^l}, \frac{w}{2^l}, \frac{r}{2^l}, \frac{s}{2^l}, \frac{t}{2^l}) \tag{2.2}$$

for all $x, y \in \mathcal{A}$ with $x \perp y$ and for all $u, v, w, r, s, t, \in \mathcal{A}$, that are mutually orthogonal. Then there exist a unique orthogonally quadratic ternary homomorphism $h : \mathcal{A} \to \mathcal{B}$ and a unique generalized orthogonally quadratic ternary homomorphism $H : \mathcal{A} \to \mathcal{B}$ (respect to h) such that

$$max\{\|g(x) - H(x)\|, \|f(x) - h(x)\|\} \leq \frac{L^{\frac{1-l}{2}}}{|4|}\psi(x) \tag{2.3}$$

for all $x \in \mathcal{A}$, where

$$\psi(x) := max\{\varphi(\frac{x}{a}, \frac{x}{b}, 0, 0, 0, 0, 0, 0), \varphi(\frac{x}{a}, 0, 0, 0, 0, 0, 0, 0), \frac{1}{|2b^2|}\varphi(x, x, 0, 0, 0, 0, 0, 0),$$
$$\frac{1}{|2b^2|}\varphi(x, -x, 0, 0, 0, 0, 0, 0), \varphi(0, \frac{x}{b}, 0, 0, 0, 0, 0, 0)\}.$$

for all $x \in \mathcal{A}$.

Proof. By (2.2), one can show that

$$\lim_{n\to\infty} \frac{1}{|4|^{l(l+2)n}}\varphi(2^{ln}x, 2^{ln}y, 2^{ln}u, 2^{ln}v, 2^{ln}w, 2^{ln}r, 2^{ln}s, 2^{ln}t) = 0 \tag{2.4}$$

for all $x, y \in \mathcal{A}$ with $x \perp y$ and for all $u, v, w, r, s, t, \in \mathcal{A}$, that are mutually orthogonal. Putting $\eta = g$ in (2.1) and $u = v = w = r = s = t = 0$ in (2.1), we get

$$\|g(ax + by) + g(ax - by) - 2a^2g(x) - 2b^2g(y)\| \leq \varphi(x, y, 0, 0, 0, 0, 0, 0) \tag{2.5}$$

for all $x, y \in \mathcal{A}$ with $x \perp y$. Putting $y = 0$ in (2.5). Since $x \perp 0$, we get

$$\|2g(ax) - 2a^2g(x)\| \leq \varphi(x, 0, 0, 0, 0, 0, 0, 0) \tag{2.6}$$

for all $x \in \mathcal{A}$. Setting $y = -y$ in (2.5), by the definition of $\perp$, we get

$$\|g(ax - by) + g(ax + by) - 2a^2g(x) - 2b^2g(-y)\| \leq \varphi(x, -y, 0, 0, 0, 0, 0, 0) \tag{2.7}$$

for all $x, y \in \mathcal{A}$ with $x \perp y$. It follows from (2.5) and (2.7) that

$$\|2b^2g(y) - 2b^2g(-y)\| \leq max\{\varphi(x, y, 0, 0, 0, 0, 0, 0), \varphi(x, -y, 0, 0, 0, 0, 0, 0)\} \tag{2.8}$$

for all $x, y \in \mathcal{A}$ with $x \perp y$. Putting $y = by$ in (2.8), by the definition of $\perp$, we get

$$\|g(by) - g(-by)\| \leq max\{\frac{1}{|2b^2|}\varphi(x, by, 0, 0, 0, 0, 0, 0), \frac{1}{|2b^2|}\varphi(x, -by, 0, 0, 0, 0, 0, 0)\} \tag{2.9}$$

for all $x, y \in \mathcal{A}$ with $x \perp y$. Let $x = 0$ in (2.5). Since $0 \perp x$, we get

$$\|g(by) + g(-by) - 2b^2g(y)\| \leq \varphi(0, y, 0, 0, 0, 0, 0, 0) \tag{2.10}$$

for all $y \in \mathcal{A}$. It follows from (2.9) and (2.10) that

$$\|2g(by) - 2b^2g(y)\| \leq max\{\frac{1}{|2b^2|}\varphi(x, by, 0, 0, 0, 0, 0, 0), \frac{1}{|2b^2|}\varphi(x, -by, 0, 0, 0, 0, 0, 0), \varphi(0, y, 0, 0, 0, 0, 0, 0)\} \tag{2.11}$$

for all $x, y \in \mathcal{A}$ with $x \perp y$. Replacing x and y by $\frac{x}{a}$ and $\frac{y}{b}$ in (2.5), respectively, and by the definition of $\perp$, we get

$$\|g(2x) - 2a^2g(\frac{x}{a}) - 2b^2g(\frac{x}{b})\| \leq \varphi(\frac{x}{a}, \frac{x}{b}, 0, 0, 0, 0, 0, 0) \tag{2.12}$$

for all $x \in \mathcal{A}$. Setting $x = \frac{x}{a}$ in (2.6), by the definition of $\perp$, we get

$$\|2a^2g(\frac{x}{a}) - 2g(x)\| \leq \varphi(\frac{x}{a}, 0, 0, 0, 0, 0, 0, 0) \tag{2.13}$$

for all $x \in \mathcal{A}$. Putting $y = \frac{x}{b}$ in (2.9), by the definition of $\perp$, we get

$$\|2b^2g(\frac{x}{b}) - 2g(x)\| \leq max\{\frac{1}{|2b^2|}\varphi(x, x, 0, 0, 0, 0, 0, 0), \frac{1}{|2b^2|}\varphi(x, -x, 0, 0, 0, 0, 0, 0), \varphi(0, \frac{x}{b}, 0, 0, 0, 0, 0, 0)\} \tag{2.14}$$

for all $x \in \mathcal{A}$. It follows from (2.12), (2.13) and (2.14) that

$$\|g(2x) - 4g(x)\| \leq \psi(x) \tag{2.15}$$

for all $x \in \mathcal{A}$. Consider the set

$$X := \{\acute{g} : \quad \acute{g} : \mathcal{A} \to \mathcal{B} \quad \acute{g}(0) = 0\}.$$

For every $\acute{g}, \acute{h} \in X$, define

$$d(\acute{g}, \acute{h}) := \inf\{K \in (0, \infty) : \quad \|\acute{g}(x) - \acute{h}(x)\| \leq K\psi(x), \quad \forall x \in \mathcal{A}\}.$$

It is easy to show that (X, d) is a complete generalized non-Archimedean metric space. Now, we consider the $\mathcal{J} : X \to X$ such that

$$\mathcal{J}(\acute{g})(x) := \frac{1}{4^l}\acute{g}(2^l x)$$

for all $x \in \mathcal{A}$. For any $\acute{g}, \acute{h} \in X$, it follows that for all $x \in \mathcal{A}$

$$\begin{aligned} d(\acute{g}, \acute{h}) < K \;&\Rightarrow\; \|\acute{g}(x) - \acute{h}(x)\| \leq K\psi(x) \\ &\Rightarrow\; \|\frac{\acute{g}(2^l x)}{4^l} - \frac{\acute{h}(2^l x)}{4^l}\| \leq K\frac{\psi(2^l x)}{|4|^l} \\ &\Rightarrow\; \|\mathcal{J}\acute{g}(x) - \mathcal{J}\acute{h}(x)\| \leq LK\psi(x). \end{aligned}$$

Hence we have

$$d(\mathcal{J}(\acute{g}), \mathcal{J}(\acute{h})) \leq Ld(\acute{g}, \acute{h}).$$

By applying the inequality (2.15), we see that $d(J(f), f) \leq \frac{L^{\frac{1-l}{2}}}{|4|}$. It follows from Theorem 2.1 that $\mathcal{J}$ has a unique fixed point $H : \mathcal{A} \to \mathcal{B}$ in the set $\Lambda : \{\acute{g} \in X : d(\acute{g}, g) < \infty\}$, where H is defined by

$$H(x) = \lim_{n\to\infty} \mathcal{J}^n g(x) = \lim_{n\to\infty} \frac{1}{4^{ln}} g(2^{ln} x) \tag{2.16}$$

for all $x \in \mathcal{A}$. It follows from (2.4), (2.5) and (2.16) that

$$\begin{aligned} &\|H(ax + by) + H(ax - by) - 2a^2 H(x) - 2b^2 H(y)\| \\ &= \lim_{n\to\infty} \frac{1}{|4|^{ln}} \|g(2^{ln} ax + 2^{ln} by) + g(2^{ln} ax - 2^{ln} by) - 2a^2 g(2^{ln} x) - 2b^2 g(2^{ln} y)\| \\ &\leq \lim_{n\to\infty} \frac{1}{|4|^{ln}} \varphi(2^{ln} x, 2^{ln} y, 0, 0, 0, 0, 0, 0) \\ &\leq \lim_{n\to\infty} \frac{1}{|4|^{ln(l+2)}} \varphi(2^{ln} x, 2^{ln} y, 0, 0, 0, 0, 0, 0) = 0 \end{aligned}$$

for all $x, y \in \mathcal{A}$ with $x \perp y$. This shows that H is an orthogonally quadratic.

Putting $\eta = f$, $u = v = w = r = s = t = 0$ in (2.1), we get

$$\|f(ax + by) + f(ax - by) - 2a^2 f(x) - 2b^2 f(y)\| \leq \varphi(x, y, 0, 0, 0, 0, 0, 0)$$

for all $x, y \in \mathcal{A}$ with $x \perp y$. By the same reasoning as above, we can show that the limit

$$h(x) =: \lim_{n\to\infty} \frac{1}{4^{ln}} f(2^{ln} x)$$

exists for all $x \in \mathcal{A}$. Moreover, we can show that h is an orthogonally quadratic mapping on $\mathcal{A}$ satisfying (2.3). On the other hand, we have

$$\begin{aligned} \|h([u, v, w]) - [h(u), h(v), h(w)]\| &= \lim_{n\to\infty} \frac{1}{|4|^{2ln}} \|f(4^{ln}[u, v, w]) - [f(2^{ln} u), f(2^{ln} v), f(2^{ln} w)]\| \\ &\leq \lim_{n\to\infty} \frac{1}{|4|^{2ln}} \varphi(0, 0, 2^{ln} u, 2^{ln} v, 2^{ln} w, 0, 0, 0) \\ &\leq \lim_{n\to\infty} \frac{1}{|4|^{l(l+2)n}} \varphi(0, 0, 2^{ln} u, 2^{ln} v, 2^{ln} w, 0, 0, 0) = 0 \end{aligned}$$

for all $u, v, w \in \mathcal{A}$, that are mutually orthogonal. Therefore, h is an orthogonally quadratic ternary homomorphism on $\mathcal{A}$. Also, we have

$$\begin{aligned} \|H([r, s, t]) - [H(r), h(s), h(t)]\| &= \lim_{n\to\infty} \frac{1}{|4|^{2ln}} \|g(4^{ln}[r, s, t]) - [g(2^{ln} r), f(2^{ln} s), f(2^{ln} t)]\| \\ &\leq \lim_{n\to\infty} \frac{1}{|4|^{2ln}} \varphi(0, 0, 0, 0, 0, 2^{ln} r, 2^{ln} s, 2^{ln} t) \\ &\leq \lim_{n\to\infty} \frac{1}{|4|^{l(l+2)n}} \varphi(0, 0, 0, 0, 0, 2^{ln} r, 2^{ln} s, 2^{ln} t) = 0 \end{aligned}$$

for all $r, s, t \in \mathcal{A}$, that are mutually orthogonal. It follows that H is a generalized orthogonally quadratic ternary homomorphism (respect to h) on $\mathcal{A}$. This completes the proof. $\square$

From now on, we use the following abbreviation for any mappings $g, f : \mathcal{A} \to \mathcal{B}$:

$$\begin{aligned}\Delta(g,f)(z_1,...,z_8) :=& \|f(az_1+bz_2)+f(az_1-bz_2)-2a^2f(z_1)-2b^2f(z_2)\|\\ &+\|g(az_1+bz_2)+g(az_1-bz_2)-2a^2g(z_1)-2b^2g(z_2)\|\\ &+\|f([z_3,z_4,z_5])-[f(z_3),f(z_4),f(z_5)]\|\\ &+\|g([z_6,z_7,z_8])-[g(z_6),f(z_7),f(z_8)]\|.\end{aligned}$$

Corollary 2.3. *Let $\mathbb{K} = \mathbb{Q}_2$ be the 2-adic number field. Let $\mathcal{A}$ be a non-Archimedean ternary Banach algebra on $\mathbb{K}$ with $\perp = \acute{\perp} \bigcup \{(x, \alpha x) : x \in X, \alpha \in \mathbb{R}\}$ and $\mathcal{B}$ be a non-Archimedean ternary Banach algebra on $\mathbb{K}$. Let ϵ be a nonnegative real number and let p be a real number such that $p > 6$ if $l = 1$ and $0 < p < 2$ if $l = -1$. Suppose that mappings $g, f : \mathcal{A} \to \mathcal{B}$ satisfy $f(0) = g(0) = 0$ and*

$$\Delta(g,f)(z_1,...,z_8) \le \epsilon \; max\{\|z_i\|^p : 1 \le i \le 8\}$$

for all $z_1, z_2 \in \mathcal{A}$ with $z_1 \perp z_2$ and for all $z_3, ..., z_8 \in \mathcal{A}$, that are mutually orthogonal. Then there exist a unique orthogonally quadratic ternary homomorphism $h : \mathcal{A} \to \mathcal{B}$ and a unique generalized orthogonally quadratic ternary homomorphism $H : \mathcal{A} \to \mathcal{B}$ (respect to h) such that

$$\begin{aligned}&max\{\|g(z)-H(z)\|,\|f(z)-h(z)\|\}\\ &\le |2|^{\frac{l(4-p)+p}{2}}\epsilon\|z\|^p \begin{cases} 2, & \gcd(a,2)=\gcd(b,2)=1;\\ max\{2^{ip},2\}, & a=k2^i, \gcd(b,2)=1;\\ max\{2^{jp},2^{2j+1}\}, & \gcd(a,2)=1, b=m2^j \vee a=k2^i, b=m2^j (j\ge i);\\ max\{2^{jp},2^{2j+1}\}, & a=k2^i, b=m2^j (i\ge j)\end{cases}\end{aligned}$$

for all $x \in \mathcal{A}$, where $i, j, k, m \ge 1$ are integers and $\gcd(k, 2) = \gcd(m, 2) = 1$.

Now, we have the following result on hyperstability of generalized orthogonally quadratic ternary homomorphisms in non-Archimedean ternary Banach algebras.

Corollary 2.4. *Let $p > 0$ be a nonnegative real number such that $|2|^{(2l+4)p} \ge 1$ and let $j \in \{3, 4, ..., 8\}$ be fixed. Suppose that mappings $g, f : \mathcal{A} \to \mathcal{B}$ satisfy $f(0) = g(0) = 0$ and*

$$\Delta(g,f)(z_1,...,z_8) \le (\sum_{i=1}^{8}\|z_i\|^p)\|z_j\|^p$$

for all $z_1, z_2 \in \mathcal{A}$ with $z_1 \perp z_2$ and for all $z_3, ..., z_8 \in \mathcal{A}$, that are mutually orthogonal, where a, b are positive fixed integers. Then f is an orthogonally quadratic ternary homomorphism and g is a generalized orthogonally quadratic ternary homomorphism related to f.

Proof. It follows from Theorem 2.2 by taking

$$\varphi(z_1,...,z_8) = (\sum_{i=1}^{8}\|z_i\|^p)\|z_j\|^p$$

for all $z_1, z_2 \in \mathcal{A}$ with $z_1 \perp z_2$ and for all $z_3, ..., z_8 \in \mathcal{A}$, that are mutually orthogonal and putting $L = |2|^{-(2l+4)p}$. $\square$

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SYMMETRY IDENTITIES OF HIGHER-ORDER q-EULER POLYNOMIALS UNDER THE SYMMETRIC GROUP OF DEGREE FOUR

DAE SAN KIM AND TAEKYUN KIM

Abstract. In this paper, we give some new identities of symmetry for the higher-order q-Euler polynomials under the symmetric group of degree four which are derived from the fermionic p-adic q-integrals on $\mathbb{Z}_p$.

1. Introduction

Let p be a fiixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$. Let us assume that q is an indeterminate in $\mathbb{C}_p$ such that $|1-q|_p < p^{-\frac{1}{p-1}}$. The q-number of x is defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$. Let $C(\mathbb{Z}_p)$ be the space of all $\mathbb{C}_p$-valued continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as

$$
\begin{aligned}
(1.1)\qquad & I_{-q}(f) \\
&= \int_{\mathbb{Z}_p} f(x)\, d\mu_{-q}(x) \\
&= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \quad (\text{see [9, 10, 12, 13]}).
\end{aligned}
$$

Thus, by (1.1), we get

$$
qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \quad (\text{see [9]}), \tag{1.2}
$$

where $f_1(x) = f(x+1)$. The Carlitz-type q-Euler numbers are defined by

$$
q(E_q+1)^n + E_{n,q} = [2]_q \delta_{0,n}, \quad E_{0,q} = 1, \quad (\text{see [9, 10]}), \tag{1.3}
$$

with the usual convention about replacing E_q^n by $E_{n,q}$.

The q-Euler polynomials are given by

$$
E_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} E_{l,q}, \quad (\text{see [9]}). \tag{1.4}
$$

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From (1.1) and (1.4), we have

$$\int_{\mathbb{Z}_p} [x+y]_q^n \, d\mu_{-q}(y) = E_{n,q}(x), \quad (n \geq 0), \quad (\text{see [9, 10, 12]}). \tag{1.5}$$

For $r \in \mathbb{N}$, we consider the higher-order q-Euler polynomials as follows:

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{t[x_1+\cdots+x_r+x]_q} d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) = \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x) \frac{t^n}{n!}. \tag{1.6}$$

Thus, by (1.3), we get

$$\begin{aligned} &E_{n,q}^{(r)}(x) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1+\cdots+x_r+x]_q^n \, d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r), \quad (\text{see [9]}). \end{aligned} \tag{1.7}$$

When $x=0$, $E_{n,q}^{(r)} = E_{n,q}^{(r)}(0)$ are called the higher-order q-Euler numbers.

In this paper, we give some new identities of symmetry for the higher-order q-Euler polynomials under the symmetric group S_4 of degree four.

Recently, several authors have studied q-extensions of Euler numbers and polynomials in the several different areas (see [1–23]).

2. Symmetry identities of $E_{n,q}^{(r)}(x)$ under S_4

Let $w_1, w_2, w_3, w_4 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, $w_3 \equiv 1 \pmod 2$, $w_4 \equiv 1 \pmod 2$. Then we have

(2.1)

$$\begin{aligned} &\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{\left[w_1w_2w_3\sum_{l=1}^r x_l + w_1w_2w_3w_4x + w_4w_2w_3\sum_{l=1}^r i_l + w_4w_1w_3\sum_{l=1}^r j_l + w_4w_1w_2\sum_{l=1}^r k_l\right]_q t} \\ &\times d\mu_{-q^{w_1w_2w_3}}(x_1)\cdots d\mu_{-q^{w_1w_2w_3}}(x_r) \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_{-q^{w_1w_2w_3}}^r} \sum_{x_1,\ldots,x_r=0}^{p^N-1} \left(-q^{w_1w_2w_3}\right)^{\sum_{l=1}^r x_l} \\ &\quad \times e^{\left[w_1w_2w_3\sum_{l=1}^r x_l + w_1w_2w_3w_4x + w_4w_2w_3\sum_{l=1}^r i_l + w_4w_1w_3\sum_{l=1}^r j_l + w_4w_1w_2\sum_{l=1}^r k_l\right]_q t} \\ &= \lim_{N\to\infty} \frac{1}{[w_4p^N]_{-q^{w_1w_2w_3}}^r} \sum_{l_1,\ldots,l_r=0}^{w_4-1} \sum_{x_1,\ldots,x_r=0}^{p^N-1} (-1)^{\sum_{i=1}^r l_i} q^{w_1w_2w_3\sum_{i=1}^r (l_i+w_4x_i)} (-1)^{x_1+\cdots+x_r} \\ &\quad \times e^{\left[w_1w_2w_3\sum_{i=1}^r (l_i+w_4x_i) + w_1w_2w_3w_4x + w_4w_2w_3\sum_{l=1}^r i_l + w_4w_1w_3\sum_{l=1}^r j_l + w_4w_1w_2\sum_{l=1}^r k_l\right]_q t}. \end{aligned}$$

Now, we observe that

(2.2)

$$\begin{aligned} &\frac{1}{[2]_{q^{w_1w_2w_3}}^r} \sum_{i_1,\ldots,i_r=0}^{w_1-1} \sum_{j_1,\ldots,j_r=0}^{w_2-1} \sum_{k_1,\ldots,k_r=0}^{w_3-1} (-1)^{\sum_{l=1}^r (i_l+j_l+k_l)} \\ &\times q^{w_4w_2w_3\sum_{l=1}^r i_l + w_4w_1w_3\sum_{l=1}^r j_l + w_4w_1w_2\sum_{l=1}^r k_l} \\ &\times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{\left[w_1w_2w_3\sum_{l=1}^r x_l + w_1w_2w_3w_4x + w_4w_2w_3\sum_{l=1}^r i_l + w_4w_1w_3\sum_{l=1}^r j_l + w_4w_1w_2\sum_{l=1}^r k_l\right]_q t} \\ &\times d\mu_{-q^{w_1w_2w_3}}(x_1)\cdots d\mu_{-q^{w_1w_2w_3}}(x_r) \end{aligned}$$

$$=\lim_{N\to\infty}\left(\frac{1}{1+q^{w_1w_2w_3w_4p^N}}\right)^r\sum_{i_1,\ldots,i_r=0}^{w_1-1}\sum_{j_1,\ldots,j_r=0}^{w_2-1}\sum_{k_1,\ldots,k_r=0}^{w_3-1}\sum_{l_1,\ldots,l_r=0}^{w_4-1}(-1)^{\sum_{n=1}^r(l_n+j_n+i_n+k_n)}$$
$$\times q^{w_4w_2w_3\sum_{l=1}^r i_l+w_4w_1w_3\sum_{j=1}^r j_l+w_4w_1w_2\sum_{l=1}^r k_l+w_1w_2w_3\sum_{i=1}^r l_i}$$
$$\times\sum_{x_1\ldots,x_r=0}^{p^N-1}q^{w_1w_2w_3\sum_{i=1}^r x_i}(-1)^{x_1+\cdots+x_r}$$
$$\times e^{\left[w_1w_2w_3\sum_{i=1}^r(l_i+x_iw_4)+w_1w_2w_3w_4x+w_4w_2w_3\sum_{l=1}^r i_l+w_4w_1w_3\sum_{l=1}^r j_l+w_4w_1w_2\sum_{l=1}^r k_l\right]_q t}.$$

As this expression is invariant under S_4, we have the following theorem.

Theorem 2.1. *For $w_1,w_2,w_3,w_4\in\mathbb{N}$ with $w_1\equiv 1 \pmod 2$, $w_2\equiv 1\pmod 2$, $w_3\equiv 1\pmod 2$, $w_4\equiv 1\pmod 2$, the following expression*

$$\frac{1}{[2]^r_{q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}}}}\sum_{i_1,\ldots,i_r=0}^{w_{\sigma(1)}-1}\sum_{j_1,\ldots,j_r=0}^{w_{\sigma(2)}-1}\sum_{k_1,\ldots,k_r=0}^{w_{\sigma(3)}-1}(-1)^{\sum_{l=1}^r(i_l+j_l+k_l)}$$
$$\times q^{w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}\sum_{l=1}^r i_l+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}\sum_{l=1}^r j_l+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}\sum_{l=1}^r k_l}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{[A]_qt}$$
$$\times d\mu_{-q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}}}(x_1)\cdots d\mu_{-q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}}}(x_r)$$

are the same for any $\sigma\in S_4$,
where $A=w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}\sum_{l=1}^r i_l+w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}w_{\sigma(4)}x+w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}\sum_{l=1}^r i_l$ $+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}\sum_{l=1}^r j_l+w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}\sum_{l=1}^r k_l$.

From (1.7), we have

(2.3)
$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}e^{\left[w_1w_2w_3\sum_{l=1}^r x_l+w_1w_2w_3w_4x+w_4w_2w_3\sum_{l=1}^r i_l+w_4w_1w_3\sum_{l=1}^r j_l+w_4w_1w_2\sum_{l=1}^r k_l\right]_q t}$$
$$\times d\mu_{-q^{w_1w_2w_3}}(x_1)\cdots d\mu_{-q^{w_1w_2w_3}}(x_r)$$
$$=\sum_{n=0}^\infty[w_1w_2w_3]_q^n\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[\sum_{l=1}^r x_l+w_4x+\frac{w_4}{w_1}\sum_{l=1}^r i_l+\frac{w_4}{w_2}\sum_{l=1}^r j_l+\frac{w_4}{w_3}\sum_{l=1}^r k_l\right]^n_{q^{w_1w_2w_3}}$$
$$\times d\mu_{-q^{w_1w_2w_3}}(x_1)\cdots d\mu_{-q^{w_1w_2w_3}}(x_r)\frac{t^n}{n!}$$
$$=\sum_{n=0}^\infty[w_1w_2w_3]_q^n E^{(r)}_{n,q^{w_1w_2w_3}}\left(w_4x+\frac{w_4}{w_1}\sum_{l=1}^r i_l+\frac{w_4}{w_2}\sum_{l=1}^r j_l+\frac{w_4}{w_3}k_l\right)\frac{t^n}{n!}.$$

Thus, by (2.3), we get

(2.4)
$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[w_1w_2w_3\sum_{l=1}^r x_l+w_4w_2w_3\sum_{l=1}^r i_l+w_4w_1w_3\sum_{l=1}^r j_l+w_4w_1w_2\sum_{l=1}^r k_l\right]^n_q$$
$$\times d\mu_{-q^{w_1w_2w_3}}(x_1)\cdots d\mu_{-q^{w_1w_2w_3}}(x_r)$$
$$=[w_1w_2w_3]_q^n E^{(r)}_{n,q^{w_1w_2w_3}}\left(w_4x+\frac{w_4}{w_1}\sum_{l=1}^r i_l+\frac{w_4}{w_2}\sum_{l=1}^r j_l+\frac{w_4}{w_3}\sum_{l=1}^r k_l\right).$$

Therefore, by (2.4) and Theorem 2.1, we obtain the following theorem.

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Theorem 2.2. *For $n \geq 0$, $w_1, w_2, w_3, w_4 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, $w_3 \equiv 1 \pmod 2$, $w_4 \equiv 1 \pmod 2$, the following expression*

$$\frac{\left[w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}\right]_q^n}{[2]^r_{q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}}}} \sum_{i_1,\ldots,i_r=0}^{w_{\sigma(1)}-1} \sum_{j_1,\ldots,j_r=0}^{w_{\sigma(2)}-1} \sum_{k_1,\ldots,k_r=0}^{w_{\sigma(3)}-1} (-1)^{\sum_{l=1}^r (i_l+j_l+k_l)}$$
$$\times q^{w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)}\sum_{l=1}^r i_l + w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)}\sum_{l=1}^r j_l + w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)}\sum_{l=1}^r k_l}$$
$$\times E^{(r)}_{n,q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}}}\left(w_{\sigma(4)}x + \frac{w_{\sigma(4)}}{w_{\sigma(1)}}\sum_{l=1}^r i_l + \frac{w_{\sigma(4)}}{w_{\sigma(2)}}\sum_{l=1}^r j_l + \frac{w_{\sigma(4)}}{w_{\sigma(3)}}\sum_{l=1}^r k_l\right)$$

are the same for any $\sigma \in S_4$.

Now, we observe that

$$\left[\sum_{l=1}^r x_l + w_4 x + \frac{w_4}{w_1}\sum_{l=1}^r i_l + \frac{w_4}{w_2}\sum_{l=1}^r j_l + \frac{w_4}{w_3}\sum_{l=1}^r k_l\right]_{q^{w_1w_2w_3}} \tag{2.5}$$
$$= \frac{[w_4]_q}{[w_1w_2w_3]_q}\left[w_2w_3\sum_{l=1}^r i_l + w_1w_3\sum_{l=1}^r j_l + w_1w_2\sum_{l=1}^r k_l\right]_{q^{w_4}}$$
$$+ q^{w_2w_3w_4\sum_{l=1}^r i_l + w_1w_3w_4\sum_{l=1}^r j_l + w_1w_2w_4\sum_{l=1}^r k_l}.$$

By (2.5), we get

$$\left[\sum_{l=1}^r x_l + w_4 x + \frac{w_4}{w_1}\sum_{l=1}^r i_l + \frac{w_4}{w_2}\sum_{l=1}^r j_l + \frac{w_4}{w_3}\sum_{l=1}^r k_l\right]^n_{q^{w_1w_2w_3}} \tag{2.6}$$
$$= \sum_{m=0}^n \binom{n}{m}\left(\frac{[w_4]_q}{[w_1w_2w_3]_q}\right)^{n-m}\left[w_2w_3\sum_{l=1}^r i_l + w_1w_3\sum_{l=1}^r j_l + w_1w_2\sum_{l=1}^r k_l\right]^{n-m}_{q^{w_4}}$$
$$\times q^{m\left(w_2w_3w_4\sum_{l=1}^r i_l + w_1w_3w_4\sum_{l=1}^r j_l + w_1w_2w_4\sum_{l=1}^r k_l\right)}\left[\sum_{l=1}^r x_l + w_4x\right]^m_{q^{w_1w_2w_3}}.$$

From (2.6), we can derive the following equation:

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[\sum_{l=1}^r x_l + w_4 x + \frac{w_4}{w_1}\sum_{l=1}^r i_l + \frac{w_4}{w_2}\sum_{l=1}^r j_l + \frac{w_4}{w_3}\sum_{l=1}^r k_l\right]^n_{q^{w_1w_2}} \tag{2.7}$$
$$\times d\mu_{-q^{w_1w_2w_3}}(x_1)\cdots d\mu_{-q^{w_1w_2w_3}}(x_r)$$
$$= \sum_{m=0}^n \binom{n}{m}\left(\frac{[w_4]_q}{[w_1w_2w_3]_q}\right)^{n-m}\left[w_2w_3\sum_{l=1}^r i_l + w_1w_3\sum_{l=1}^r j_l + w_1w_2\sum_{l=1}^r k_l\right]^{n-m}_{q^{w_4}}$$
$$\times E^{(r)}_{m,q^{w_1w_2w_3}}(w_4x)$$
$$\times q^{m\left(w_2w_3w_4\sum_{l=1}^r i_l + w_1w_3w_4\sum_{l=1}^r j_l + w_1w_2w_4\sum_{l=1}^r k_l\right)}.$$

Thus, by (2.7), we get

(2.8)

$$\frac{[w_1w_2w_3]_q^n}{[2]^r_{q^{w_1w_2w_3}}}\sum_{i_1,\ldots,i_r=0}^{w_1-1}\sum_{j_1,\ldots,j_r=0}^{w_2-1}\sum_{k_1,\ldots,k_r=0}^{w_3-1}(-1)^{\sum_{l=1}^r(i_l+j_l+k_l)}\,q^{w_2w_3w_4\sum_{l=1}^r i_l + w_4w_1w_3\sum_{l=1}^r j_l}$$

$$\times q^{w_1w_2w_4\sum_{l=1}^{r}k_l}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[\sum_{l=1}^{r}x_l+w_4x+\frac{w_4}{w_1}\sum_{l=1}^{r}i_l+\frac{w_4}{w_2}\sum_{l=1}^{r}j_l+\frac{w_4}{w_3}\sum_{l=1}^{r}k_l\right]_{q^{w_1w_2w_3}}^{n}$$
$$\times d\mu_{-q^{w_1w_2w_3}}(x_1)\cdots d\mu_{-q^{w_1w_2w_3}}(x_r)$$
$$=\sum_{m=0}^{n}\binom{n}{m}\frac{[w_1w_2w_3]_q^m}{[2]_{q^{w_1w_2w_3}}^r}[w_4]_q^{n-m}E_{m,q^{w_1w_2w_3}}^{(r)}(w_4x)\sum_{i_1,\ldots,i_r=0}^{w_1-1}\sum_{j_1,\ldots,j_r=0}^{w_2-1}\sum_{k_1,\ldots,k_r=0}^{w_3-1}$$
$$\times(-1)^{\sum_{l=1}^{r}(i_l+j_l+k_l)}q^{(m+1)\left(w_2w_3w_4\sum_{l=1}^{r}i_l+w_4w_1w_3\sum_{l=1}^{r}j_l+w_4w_1w_2\sum_{l=1}^{r}k_l\right)}$$
$$\times\left[w_2w_3\sum_{l=1}^{r}i_l+w_1w_3\sum_{l=1}^{r}j_l+w_1w_2\sum_{l=1}^{r}k_l\right]_{q^{w_4}}^{n-m}$$
$$=\sum_{m=0}^{n}\binom{n}{m}\frac{[w_1w_2w_3]_q^m}{[2]_{q^{w_1w_2w_3}}^r}[w_4]_q^{n-m}E_{m,q^{w_1w_2w_3}}^{(r)}(w_4x)\,T_{n,q^{w_4}}^{(r)}(w_1,w_2,w_3\mid m),$$

where

$$T_{n,q}^{(r)}(w_1,w_2,w_3\mid m)=\sum_{i_1,\ldots,i_r=0}^{w_1-1}\sum_{j_1,\ldots,j_r=0}^{w_2-1}\sum_{k_1,\ldots,k_r=0}^{w_3-1}(-1)^{\sum_{l=1}^{r}(i_l+j_l+k_l)}\times q^{(m+1)\left(w_2w_3\sum_{l=1}^{r}j_l+w_1w_3\sum_{l=1}^{r}j_l+w_1w_2\sum_{l=1}^{r}k_l\right)}\times\left[w_2w_3\sum_{l=1}^{r}i_l+w_1w_3\sum_{l=1}^{r}j_l+w_1w_2\sum_{l=1}^{r}k_l\right]_q^{n-m}. \tag{2.9}$$

As this expression is invariant under S_4, we have the following theorem.

Theorem 2.3. *For $n\geq 0$, $w_1,w_2,w_3,w_4\in\mathbb{N}$ with $w_1\equiv 1 \pmod 2$, $w_2\equiv 1 \pmod 2$, $w_3\equiv 1 \pmod 2$, $w_4\equiv 1 \pmod 2$, the following expression*

$$\sum_{m=0}^{n}\binom{n}{m}\frac{\left[w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}\right]_q^m}{[2]_{q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}}}^r}\left[w_{\sigma(4)}\right]_q^{n-m}$$
$$\times E_{m,q^{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}}}^{(r)}\left(w_{\sigma(4)}x\right)T_{n,q^{w_{\sigma(4)}}}^{(r)}\left(w_{\sigma(1)},w_{\sigma(2)},w_{\sigma(3)}\mid m\right)$$

are the same for any $\sigma\in S_4$.

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Soft saturated and dried values with applications in BCK/BCI-algebras

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Abstract

The notions of soft saturated values and soft dried values are introduced, and their applications in BCK/BCI-algebras are discussed. Using these notions, properties of energetic subsets are investigated. Using the concepts of intersectional (union) ideals, properties of right vanished (stable) subsets are explored.

Keywords:

Energetic subset, Right vanished subset, Right stable subset, Saturated value, Dried value.

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1 Introduction

The real world is inherently uncertain, imprecise and vague. Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [32]. In response to this situation Zadeh [33] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a

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more general framework, the approach to uncertainty is outlined by Zadeh [34]. To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [29]. Maji et al. [26] and Molodtsov [29] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [29] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Worldwide, there has been a rapid growth in interest in soft set theory and its applications in recent years. Evidence of this can be found in the increasing number of high-quality articles on soft sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences in recent years. Maji et al. [26] described the application of soft set theory to a decision making problem. Maji et al. [25] also studied several operations on the theory of soft sets. Aktaş and Çağman [2] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. They also discussed the notion of soft groups.

BCK and BCI-algebras are two classes of logical algebras which are introduced by Imai and Iséki (see [9, 10]). This notion originated from two different ways:

(1) set theory, and

(2) classical and non-classical propositional calculi.

In set theory, we have the following simple relations: $(A - B) - (A - C) \subseteq C - B$ and $A-(A-B) \subseteq B$. Several properties on BCK/BCI-algebras are investigated in the papers [11, 12, 13, 14] and [27]. There is a deep relation between BCK/BCI-algebras and posets. Today BCK/BCI-algebras have been studied by many authors and they have been applied to many branches of mathematics, such as group, functional analysis, probability theory, topology, fuzzy set theory, and so on. Jun and Park [24] studied applications of soft sets in ideal theory of BCK/BCI-algebras. Jun et al. [20, 22] introduced the notion of intersectional soft sets, and considered its applications to BCK/BCI-algebras. Also, Jun [16] discussed the union soft sets with applications in BCK/BCI-algebras. We refer the reader to the papers [1, 3, 5, 6, 7, 15, 18, 19, 21, 23, 30, 31, 35] for further information regarding algebraic structures/properties of soft set theory.

In this paper, we introduce the notions of soft saturated values and soft dried values, and discuss their applications in BCK/BCI-algebras. Using these notions, we investigate several properties of energetic subsets. Using the concepts of intersectional (union) ideals, we explore some properties of right vanished (stable) subsets.

2 Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra.* Any BCK/BCI-algebra X satisfies the following axioms:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$

$$(\forall x, y, z \in X)\,(x \le y \ \Rightarrow\ x * z \le y * z,\ z * y \le z * x)\,, \tag{2.2}$$

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \le x * y) \tag{2.4}$$

where $x \le y$ if and only if $x * y = 0$. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra

X is called an *ideal* of X if it satisfies:

$$0 \in I, \tag{2.5}$$

$$(\forall x \in X)\,(\forall y \in I)\,(x * y \in I \;\Rightarrow\; x \in I)\,. \tag{2.6}$$

We refer the reader to the books [8, 28] for further information regarding BCK/BCI-algebras.

A soft set theory is introduced by Molodtsov [29], and Çağman et al. [4] provided new definitions and various results on soft set theory.

In what follows, let U be an initial universe set and E be a set of parameters. We say that the pair (U, E) is a *soft universe.* Let $\mathscr{P}(U)$ denote the power set of U and $A, B, C, \cdots \subseteq E$.

Definition 2.1 ([4, 29])**.** A *soft set* (f, A) over U is defined to be the set of ordered pairs

$$(f, A) := \{(x, f(x)) : x \in E,\ f(x) \in \mathscr{P}(U)\}\,,$$

where $f : E \to \mathscr{P}(U)$ such that $f(x) = \emptyset$ if $x \notin A$.

The function f is called an approximate function of the soft set (f, A). The subscript A in the notation f indicates that f is the approximate function of (f, A).

Definition 2.2 ([16])**.** Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra. A soft set (f, X) over U is called a *union soft subalgebra* over U if the following condition holds:

$$(\forall x, y \in X)\,(f(x * y) \subseteq f(x) \cup f(y))\,. \tag{2.7}$$

Definition 2.3 ([16])**.** Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra. A soft set (f, X) over U is called a *union soft ideal* over U if it satisfies:

$$(\forall x, y \in X)\,(f(0) \subseteq f(x) \subseteq f(x * y) \cup f(y))\,. \tag{2.8}$$

Proposition 2.4 ([16])**.** *Let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra. Every union soft ideal (f, X) over U satisfies the following condition:*

$$(\forall x, y \in X)\,(x \leq y \;\Rightarrow\; f(x) \subseteq f(y))\,. \tag{2.9}$$

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3 Energetic subsets and soft saturated (dried) values

In what follows, let $\left\{ \begin{array}{c} \mathscr{Q}(U) \\ \mathscr{R}(U) \end{array} \right\}$ be the class of all subsets of U such that

$$(\forall A, B, C \in \mathscr{P}(U)) \left\{ \begin{array}{l} A \cap B \subseteq C \;\Rightarrow\; A \subseteq C \text{ or } B \subseteq C \\ A \subseteq B \cup C \;\Rightarrow\; A \subseteq B \text{ or } A \subseteq C \end{array} \right\},$$

and let $(U, E) = (U, X)$ where X is a BCK/BCI-algebra unless otherwise specified.

Definition 3.1 ([17]). A non-empty subset G of X is said to be *S-energetic* if it satisfies:

$$(\forall a, b \in X)\,(a * b \in G \;\Rightarrow\; \{a, b\} \cap G \neq \emptyset)\,. \tag{3.1}$$

Example 3.2 ([17]). Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	0	a
c	c	b	a	0	b
d	d	a	a	a	0

The set $G := \{a, b, c\}$ is an S-energetic subset of X, but $H := \{a, b\}$ is not an S-energetic subset of X since $d * c = a \in H$ but $\{d, c\} \cap H = \emptyset$.

Definition 3.3 ([22]). A soft set (f, X) over U is called an *int-soft subalgebra* over U if it satisfies:

$$(\forall x, y \in X)\,(f(x * y) \supseteq f(x) \cap f(y))\,. \tag{3.2}$$

Definition 3.4 ([22]). A soft set (f, X) over U is called an *int-soft ideal* over U if it satisfies:

$$(\forall x \in X)\,(f(x) \subseteq f(0))\,, \tag{3.3}$$

$$(\forall x, y \in X)\,(f(x * y) \cap f(y) \subseteq f(x))\,. \tag{3.4}$$

Lemma 3.5 ([22]). *Every int-soft ideal (f, X) over U satisfies the following conditions:*

(1) $(\forall x, y \in X)\,(x \leq y \;\Rightarrow\; f(y) \subseteq f(x))\,.$

(2) $(\forall x, y, z \in X)\,(x * y \leq z \;\Rightarrow\; f(y) \cap f(z) \subseteq f(x))$.

Given a soft set (f, X) over U and $\alpha \in \mathscr{P}(U)$, we define useful subsets of X.

$$f_\alpha^{\subseteq} := \{x \in X \mid f(x) \subseteq \alpha\},\; f_\alpha^{\subset} := \{x \in X \mid f(x) \subset \alpha\},$$
$$f_\alpha^{\supseteq} := \{x \in X \mid f(x) \supseteq \alpha\},\; f_\alpha^{\supset} := \{x \in X \mid f(x) \supset \alpha\}.$$

Proposition 3.6. *If (f, X) is an int-soft subalgebra over U with $f : X \to \mathscr{Q}(U)$, then*

$$(\forall \alpha \in \mathscr{Q}(U))\,\big(f_\alpha^{\subseteq} \neq \emptyset \;\Rightarrow\; f_\alpha^{\subseteq} \textit{ is an S-energetic subset of } X\big).$$

Proof. Let $x, y \in X$ be such that $x * y \in f_\alpha^{\subseteq}$. Then

$$f(x) \cap f(y) \subseteq f(x * y) \subseteq \alpha,$$

and so $f(x) \subseteq \alpha$ or $f(y) \subseteq \alpha$, that is, $x \in f_\alpha^{\subseteq}$ or $y \in f_\alpha^{\subseteq}$. Hence $\{x, y\} \cap f_\alpha^{\subseteq} \neq \emptyset$. Therefore $f_\alpha^{\subseteq}$ is an S-energetic subset of X. □

Corollary 3.7. *If (f, X) is an int-soft subalgebra over U with $f : X \to \mathscr{Q}(U)$, then*

$$(\forall \alpha \in \mathscr{Q}(U))\,(f_\alpha^{\subset} \neq \emptyset \;\Rightarrow\; f_\alpha^{\subset} \textit{ is an S-energetic subset of } X).$$

Proof. Straightforward. □

The following example shows that the converse of Proposition 3.6 is not true.

Example 3.8. Let $(U, E) = (U, X)$ where $X = \{0, a, b, c, d\}$ is a BCK-algebra as in Example 3.2. Let (f, X) be a soft set over U in which f is given as follows:

$$f : X \to \mathscr{Q}(U), \quad x \mapsto \begin{cases} \gamma_2 & \text{if } x = 0, \\ \gamma_3 & \text{if } x = d, \\ \gamma_1 & \text{if } x \in \{a, b, c\}, \end{cases}$$

where γ_1, γ_2, $\gamma_3 \in \mathscr{Q}(U)$ with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3$. For any $\alpha \in \mathscr{Q}(U)$, if $\gamma_1 \subseteq \alpha \subsetneq \gamma_2$ then $f_\alpha^{\subseteq} = \{a, b, c\}$ is an S-energetic subset of X. But (f, X) is not an int-soft subalgebra over U since

$$f(d * d) = f(0) = \gamma_2 \not\supseteq \gamma_3 = f(d) \cap f(d).$$

Let $(U, E) = (U, X)$ where X is a BCK-algebra. Then every int-soft ideal over U is an int-soft subalgebra over U (see [22]). Hence we have the following corollary.

Corollary 3.9. *Let $(U, E) = (U, X)$ where X is a BCK-algebra. If (f, X) is an int-soft ideal over U with $f : X \to \mathscr{Q}(U)$, then*

$$(\forall \alpha \in \mathscr{Q}(U)) \left(f_{\alpha}^{\subseteq} \neq \emptyset \ \Rightarrow \ f_{\alpha}^{\subseteq} \text{ is an S-energetic subset of } X \right).$$

The following example shows that the converse of Corollary 3.9 is not true.

Example 3.10. Consider the soft set (f, X) over U as in Example 3.8. For any $\alpha \in \mathscr{Q}(U)$, if $\gamma_1 \subseteq \alpha \subsetneq \gamma_2$ then $f_{\alpha}^{\subseteq} = \{a, b, c\}$ is an S-energetic subset of X. But (f, X) is not an int-soft ideal over U since $f(d) = \gamma_3 \nsubseteq \gamma_2 = f(0)$.

Definition 3.11. Let (f, X) be a soft set over U and $\alpha \in \mathscr{P}(U)$ with $f_{\alpha}^{\supseteq} \neq \emptyset$. Then α is called a *soft saturated S-value* for (f, X) if the following assertion is valid:

$$(\forall a, b \in X) \left(f(a * b) \supseteq \alpha \ \Rightarrow f(a) \cup f(b) \supseteq \alpha \right). \tag{3.5}$$

Example 3.12. Let $(U, E) = (U, X)$ where $X = \{0, 1, 2, 3\}$ is a BCK-algebra with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	2
3	3	1	3	0

Consider a soft set (f, X) over U in which f is given as follows:

$$f : X \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \gamma_1 & \text{if } x = 0, \\ \gamma_2 & \text{if } x = 1, \\ \gamma_3 & \text{if } x \in \{2, 3\}, \end{cases}$$

where γ_1, γ_2 and γ_3 are subsets of U with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3$. Take $\alpha \in \mathscr{P}(U)$ with $\gamma_2 \subsetneq \alpha \subseteq \gamma_3$. Then $f_{\alpha}^{\supseteq} = \{2, 3\}$, and it is easy to check that α is a soft saturated S-value for (f, X).

Example 3.13. Let $(U, E) = (\mathbb{N}, X)$ where $\mathbb{N}$ is the set of all natural numbers and $X = \{0, 1, 2, a, b\}$ is a BCI-algebra with the following Cayley table:

$*$	0	1	2	a	b
0	0	0	0	b	a
1	1	0	1	b	a
2	2	2	0	b	a
a	a	a	a	0	b
b	b	b	b	a	0

Consider a soft set (f, X) over U in which f is given as follows:

$$f : X \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \mathbb{N} & \text{if } x = 0, \\ 2\mathbb{N} & \text{if } x \in \{1, a\}, \\ 2\mathbb{N} - \{2, 4, 6\} & \text{if } x = 2, \\ 2\mathbb{N} - \{4, 6, 8\} & \text{if } x = b. \end{cases}$$

If $\alpha = 2\mathbb{N} - \{4\}$, then $f_{\alpha}^{\supseteq} = \{0, 1, a\} \neq \emptyset$, $f(2 * b) = f(a) = 2\mathbb{N} \supseteq \alpha$, and $f(2) \cup f(b) = 2\mathbb{N} - \{4, 6\} \not\supseteq \alpha$. Hence α is not a soft saturated S-value for (f, X).

Proposition 3.14. *Let (f, X) be an int-soft subalgebra over U with $f : X \to \mathscr{R}(U)$. If $\alpha \in \mathscr{R}(U)$ is a soft saturated S-value for (f, X), then*

$$f_{\alpha}^{\supseteq} \neq \emptyset \ \Rightarrow \ f_{\alpha}^{\supseteq} \text{ is an S-energetic subset of } X.$$

Proof. Let $a, b \in X$ be such that $a * b \in f_{\alpha}^{\supseteq}$. Then $f(a * b) \supseteq \alpha$, which implies from (3.5) that $f(a) \cup f(b) \supseteq \alpha$. Thus $f(a) \supseteq \alpha$ or $f(b) \supseteq \alpha$, that is, $a \in f_{\alpha}^{\supseteq}$ or $b \in f_{\alpha}^{\supseteq}$. Hence $\{a, b\} \cap f_{\alpha}^{\supseteq} \neq \emptyset$. Therefore $f_{\alpha}^{\supseteq}$ is an S-energetic subset of X. □

Theorem 3.15. *Let (f, X) be a soft set over U and $\alpha \in \mathscr{P}(U)$ be such that $f_{\alpha}^{\supseteq} \neq \emptyset$. If (f, X) is a union soft subalgebra over U, then α is a soft saturated S-value for (f, X).*

Proof. Let $x, y \in X$ be such that $f(x * y) \supseteq \alpha$. Then

$$\alpha \subseteq f(x * y) \subseteq f(x) \cup f(y),$$

and so α is a soft saturated S-value for (f, X). □

Corollary 3.16. *Let $(U, E) = (U, X)$ where X is a BCK-algebra. Let (f, X) be a soft set over U and let $\alpha \in \mathscr{P}(U)$ be such that $f_{\alpha}^{\supseteq} \neq \emptyset$. If (f, X) is a union soft ideal over U, then α is a soft saturated S-value for (f, X).*

Definition 3.17. Let (f, X) be a soft set over U and $\alpha \in \mathscr{P}(U)$ with $f_{\alpha}^{\subseteq} \neq \emptyset$. Then α is called a *soft dried S-value* for (f, X) if the following assertion is valid:

$$(\forall a, b \in X) \left(f(a * b) \subseteq \alpha \ \Rightarrow f(a) \cap f(b) \subseteq \alpha \right). \tag{3.6}$$

Example 3.18. Let $(U, E) = (U, X)$ where $X = \{0, 1, 2, 3\}$ is a BCK-algebra as in Example 3.12. Consider a soft set (f, X) over U in which f is given as follows:

$$f : X \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \gamma_2 & \text{if } x = 0, \\ \gamma_1 & \text{if } x = 1, \\ \gamma_3 & \text{if } x \in \{2, 3\}, \end{cases}$$

where γ_1, γ_2 and γ_3 are subsets of U with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3$. For any $\alpha \in \mathscr{P}(U)$ with $\gamma_1 \subseteq \alpha \subsetneq \gamma_2$, $f_\alpha^\subseteq = \{1\}$ and α is a soft dried S-value for (f, X).

Theorem 3.19. *Let (f, X) be a union soft subalgebra over U with $f : X \to \mathscr{Q}(U)$. For any soft dried S-value $\alpha \in \mathscr{Q}(U)$ for (f, X), we have*

$$f_\alpha^\subseteq \neq \emptyset \;\Rightarrow\; f_\alpha^\subseteq \text{ is an S-energetic subset of } X.$$

Proof. Let $a, b \in X$ be such that $a * b \in f_\alpha^\subseteq$. Then $f(a * b) \subseteq \alpha$, and so $f(a) \cap f(b) \subseteq \alpha$ by (3.6). Thus $f(a) \subseteq \alpha$ or $f(b) \subseteq \alpha$, i.e., $a \in f_\alpha^\subseteq$ or $b \in f_\alpha^\subseteq$. Hence $\{a, b\} \cap f_\alpha^\subseteq \neq \emptyset$. Therefore $f_\alpha^\subseteq$ is an S-energetic subset of X. □

Corollary 3.20. *Let $(U, E) = (U, X)$ where X is a BCK-algebra. Let (f, X) be a union soft ideal over U with $f : X \to \mathscr{Q}(U)$. For any soft dried S-value $\alpha \in \mathscr{Q}(U)$ for (f, X), we have*

$$f_\alpha^\subseteq \neq \emptyset \;\Rightarrow\; f_\alpha^\subseteq \text{ is an S-energetic subset of } X.$$

Theorem 3.21. *Let (f, X) be an int-soft subalgebra over U and let $\alpha \in \mathscr{P}(U)$ be such that $f_\alpha^\subseteq \neq \emptyset$. Then α is a soft dried S-value for (f, X).*

Proof. Let $a, b \in X$ be such that $f(a * b) \subseteq \alpha$. Then $\alpha \supseteq f(a * b) \supseteq f(a) \cap f(b)$, which shows that α is a soft dried S-value for (f, X). □

Corollary 3.22. *Let $(U, E) = (U, X)$ where X is a BCK-algebra. Let (f, X) be an int-soft ideal over U and let $\alpha \in \mathscr{P}(U)$ be such that $f_\alpha^\subseteq \neq \emptyset$. Then α is a soft dried S-value for (f, X).*

Definition 3.23 ([17])**.** Let X be a BCK/BCI-algebra. A non-empty subset G of X is said to be *I-energetic* if it satisfies:

$$(\forall x, y \in X)\,(y \in G \;\Rightarrow\; \{x, y * x\} \cap G \neq \emptyset)\,. \tag{3.7}$$

Example 3.24 ([17])**.** Let $X = \{0, 1, 2, a, b\}$ be a BCI-algebra with the following Cayley table:

$*$	0	1	2	a	b
0	0	0	0	b	a
1	1	0	1	b	a
2	2	2	0	b	a
a	a	a	a	0	b
b	b	b	b	a	0

It is routine to verify that $G := \{a, b\}$ is an I-energetic subset of X.

Example 3.25 ([17]). Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	1
2	2	1	0	1	2
3	3	1	1	0	3
4	4	4	4	4	0

It is routine to verify that $G := \{0, 1, 4\}$ is an I-energetic subset of X.

The notion of I-energetic subsets is independent to the notion of S-energetic subsets. In fact, the S-energetic subset $G := \{a, b, c\}$ in Example 3.2 is not an I-energetic subset of X since $\{d, a * d\} \cap G = \emptyset$. Also, in Example 3.25, the I-energetic subset $G := \{0, 1, 4\}$ is not an S-energetic subset of X since $3 * 2 = 1 \in G$ and $\{3, 2\} \cap G = \emptyset$ (see [17]).

Definition 3.26. Let (f, X) be a soft set over U and $\alpha \in \mathscr{P}(U)$ with $f_{\alpha}^{\supseteq} \neq \emptyset$. Then α is called a *soft saturated I-value* for (f, X) if the following assertion is valid:

$$(\forall x, y \in X)\,(f(y) \supseteq \alpha \;\Rightarrow f(y * x) \cup f(x) \supseteq \alpha)\,. \tag{3.8}$$

Example 3.27. Let $(U, E) = (U, X)$ where $X = \{0, 1, 2, 3\}$ is a BCK-algebra as in Example 3.12. Consider a soft set (f, X) over U in which f is given as follows:

$$f : X \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \gamma_3 & \text{if } x = 0, \\ \gamma_2 & \text{if } x \in \{1, 3\}, \\ \gamma_1 & \text{if } x = 2, \end{cases}$$

where γ_1, γ_2 and γ_3 are subsets of U with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3$. Put $\alpha \in \mathscr{P}(U)$ with $\gamma_1 \subsetneq \alpha \subseteq \gamma_2$. Then $f_{\alpha}^{\supseteq} = \{0, 1, 3\}$. It is easy to check that α is a soft saturated I-value for (f, X).

Theorem 3.28. *Let $(U, E) = (U, X)$ where X is a BCK-algebra. If (f, X) is a union soft subalgebra over U, then every soft saturated I-value for (f, X) is a soft saturated S-value for (f, X).*

Proof. Since (f, X) is a union soft subalgebra over U, $f(0) \subseteq f(x)$ for all $x \in X$. Let $\alpha \in \mathscr{P}(U)$ be a soft saturated I-value for (f, X). Assume that $f(a * b) \supseteq \alpha$ for all $a, b \in X$. Using (3.8), (2.3), (III) and (V), we have

$$\begin{aligned} &\alpha \subseteq f((a * b) * a) \cup f(a) = f((a * a) * b) \cup f(a) \\ &= f(0 * b) \cup f(a) = f(0) \cup f(a) = f(a). \end{aligned}$$

Thus $f(a) \cup f(b) \supseteq f(a) \supseteq \alpha$ and therefore α is a soft saturated S-value for (f, X). □

Corollary 3.29. *Let $(U, E) = (U, X)$ where X is a BCK-algebra. If (f, X) is a union soft ideal over U, then every soft saturated I-value for (f, X) is a soft saturated S-value for (f, X).*

Proof. Straightforward. □

The converse of Theorem 3.28 is not true as seen in the following example.

Example 3.30. Let $(U, E) = (U, X)$ where $X = \{0, a, b, c\}$ is a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Consider a soft set (f, X) over U in which f is given as follows:

$$f : X \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \gamma_1 & \text{if } x = 0, \\ \gamma_2 & \text{if } x = a, \\ \gamma_3 & \text{if } x \in \{b, c\}, \end{cases}$$

where γ_1, γ_2 and γ_3 are subsets of U with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3$. Take $\alpha \in \mathscr{P}(U)$ with $\gamma_2 \subsetneq \alpha \subseteq \gamma_3$. Then $f_\alpha^\supseteq = \{b, c\}$. It is easy to check that α is a soft saturated S-value for (f, X), but not a soft saturated I-value for (f, X) since $f(b) \supseteq \gamma_3$ and $f(b * a) \cup f(a) = f(a) = \gamma_2 \not\supseteq \gamma_3$.

Theorem 3.31. *Let (f, X) be an int-soft ideal over U with $f : X \to \mathscr{Q}(U)$. Then*

$$(\forall \alpha \in \mathscr{Q}(U)) \left(f_\alpha^\subseteq \neq \emptyset \;\Rightarrow\; f_\alpha^\subseteq \text{ is an I-energetic subset of } X \right).$$

Proof. Let $x, y \in X$ be such that $y \in f_\alpha^\subseteq$. Then $f(y) \subseteq \alpha$. It follows from (3.4) that

$$\alpha \supseteq f(y) \supseteq f(y * x) \cap f(x).$$

Thus $f(y * x) \subseteq \alpha$ or $f(x) \subseteq \alpha$, i.e., $y * x \in f_\alpha^\subseteq$ or $x \in f_\alpha^\subseteq$. Hence $\{x, y * x\} \cap f_\alpha^\subseteq \neq \emptyset$, and so $f_\alpha^\subseteq$ is an I-energetic subset of X. □

Theorem 3.32. *Let (f, X) be an int-soft ideal over U with $f : X \to \mathscr{R}(U)$. If $\alpha \in \mathscr{R}(U)$ is a soft saturated I-value for (f, X), then*

$$f_\alpha^\supseteq \neq \emptyset \;\Rightarrow\; f_\alpha^\supseteq \text{ is an I-energetic subset of } X.$$

Proof. Let $x, y \in X$ be such that $y \in f_\alpha^\supseteq$. Then $f(y) \supseteq \alpha$, which implies from (3.8) that $f(y * x) \cup f(x) \supseteq \alpha$. Hence $f(y * x) \supseteq \alpha$ or $f(x) \supseteq \alpha$, that is, $y * x \in f_\alpha^\supseteq$ or $x \in f_\alpha^\supseteq$. Thus $\{x, y * x\} \cap f_\alpha^\supseteq \neq \emptyset$, and therefore $f_\alpha^\supseteq$ is an I-energetic subset of X. □

Theorem 3.33. *Let $\alpha \in \mathscr{P}(U)$ be such that $f_\alpha^\supseteq \neq \emptyset$. If (f, X) is a union soft ideal over U, then α is a soft saturated I-value for (f, X).*

Proof. Let $x, y \in X$ be such that $f(y) \supseteq \alpha$. Then $\alpha \subseteq f(y) \subseteq f(y * x) \cup f(x)$ by (2.8). Hence α is a soft saturated I-value for (f, X). □

Theorem 3.34. *If (f, X) is a union soft ideal over U with $f : X \to \mathscr{R}(U)$, then*

$$(\forall \alpha \in \mathscr{R}(U)) \left(f_\alpha^\supseteq \neq \emptyset \;\Rightarrow\; f_\alpha^\supseteq \text{ is an I-energetic subset of } X \right).$$

Proof. Let $x, y \in X$ be such that $y \in f_\alpha^\supseteq$. Then $f(y) \supseteq \alpha$, and so

$$\alpha \subseteq f(y) \subseteq f(y * x) \cup f(x)$$

by (2.8). Thus $f(y*x) \supseteq \alpha$ or $f(x) \supseteq \alpha$, i.e., $y*x \in f_\alpha^\supseteq$ or $x \in f_\alpha^\supseteq$. Hence $\{x, y*x\} \cap f_\alpha^\supseteq \neq \emptyset$, and so $f_\alpha^\supseteq$ is an I-energetic subset of X. □

Definition 3.35. Let (f, X) be a soft set over U and $\alpha \in \mathscr{P}(U)$ with $f_\alpha^\subseteq \neq \emptyset$. Then α is called a *soft dried I-value* for (f, X) if the following assertion is valid:

$$(\forall x, y \in X) \left(f(y) \subseteq \alpha \;\Rightarrow f(y * x) \cap f(x) \subseteq \alpha \right). \tag{3.9}$$

Example 3.36. Let $(U, E) = (U, X)$ where $X = \{0, a, b, c\}$ is a BCK-algebra as in Example 3.30. Consider the soft set (f, X) over U in Example 3.30. Take $\alpha \in \mathscr{P}(U)$ with $\gamma_2 \subseteq \alpha \subsetneq \gamma_3$. Then $f_\alpha^\subseteq = \{0, a\}$. It is easy to check that α is a soft dried I-value for (f, X).

Theorem 3.37. *Let $(U, E) = (U, X)$ where X is a BCK-algebra. If (f, X) is an int-soft subalgebra over U, then every soft dried I-value for (f, X) is a soft dried S-value for (f, X).*

Proof. Since (f, X) is an int-soft subalgebra over U, $f(0) \supseteq f(x)$ for all $x \in X$. Let $\alpha \in \mathscr{P}(U)$ be a soft dried I-value for (f, X). Assume that $f(a * b) \subseteq \alpha$ for all $a, b \in X$. Using (3.9), (2.3), (III) and (V), we have

$$\begin{aligned} \alpha &\supseteq f((a * b) * a) \cap f(a) = f((a * a) * b) \cap f(a) \\ &= f(0 * b) \cap f(a) = f(0) \cap f(a) = f(a). \end{aligned}$$

Thus $f(a) \cap f(b) \subseteq f(a) \subseteq \alpha$ and therefore α is a soft dried S-value for (f, X). □

Theorem 3.38. *If (f, X) is a union soft ideal over U with $f : X \to \mathscr{R}(U)$, then*

$$(\forall \alpha \in \mathscr{R}(U)) \left(f_{\alpha}^{\supsetneq} \neq \emptyset \;\Rightarrow\; f_{\alpha}^{\supsetneq} \text{ is an I-energetic subset of } X \right).$$

Proof. Let $x, y \in X$ be such that $y \in f_{\alpha}^{\supsetneq}$. Then $f(y) \supset \alpha$. It follows from (2.8) that

$$\alpha \subseteq f(y) \subseteq f(y * x) \cup f(x).$$

Thus $f(y * x) \supseteq \alpha$ or $f(x) \supseteq \alpha$, i.e., $y * x \in f_{\alpha}^{\supsetneq}$ or $x \in f_{\alpha}^{\supsetneq}$. Hence $\{x, y * x\} \cap f_{\alpha}^{\supsetneq} \neq \emptyset$, and so $f_{\alpha}^{\supsetneq}$ is an I-energetic subset of X. □

Theorem 3.39. *Let (f, X) be a union soft ideal over U with $f : X \to \mathscr{Q}(U)$. If $\alpha \in \mathscr{Q}(U)$ is a soft dried I-value for (f, X), then*

$$f_{\alpha}^{\subsetneq} \neq \emptyset \;\Rightarrow\; f_{\alpha}^{\subsetneq} \text{ is an I-energetic subset of } X.$$

Proof. Let $x, y \in X$ be such that $y \in f_{\alpha}^{\subsetneq}$. Then $f(y) \subseteq \alpha$, which implies from (3.9) that $f(y * x) \cap f(x) \subseteq \alpha$. Hence $f(y * x) \subseteq \alpha$ or $f(x) \subseteq \alpha$, that is, $y * x \in f_{\alpha}^{\subsetneq}$ or $x \in f_{\alpha}^{\subsetneq}$. Thus $\{x, y * x\} \cap f_{\alpha}^{\subsetneq} \neq \emptyset$, and therefore $f_{\alpha}^{\subsetneq}$ is an I-energetic subset of X. □

Definition 3.40 ([17])**.** Let Q be a non-empty subset of a BCK/BCI-algebra X. Then Q is said to be *right vanished* if it satisfies:

$$(\forall a, b \in X) \, (a * b \in Q \;\Rightarrow\; a \in Q) \, . \tag{3.10}$$

Q is said to be *right stable* if $Q * X := \{a * x \mid a \in Q, \ x \in X\} \subseteq Q$.

Theorem 3.41. *Let $(U, E) = (U, X)$ where X is a BCK-algebra and let (f, X) be an int-soft ideal over U. Then $f_{\alpha}^{\supsetneq}$ and $f_{\alpha}^{\supset}$ are right stable subsets of X for any $\alpha \in \mathscr{P}(U)$ with $f_{\alpha}^{\supsetneq} \neq \emptyset \neq f_{\alpha}^{\supset}$.*

Proof. Let $x \in X$ and $a \in f_{\alpha}^{\supsetneq}$. Then $f(a) \supseteq \alpha$. Since $a * x \leq a$ and (f, X) is an int-soft ideal over U, it follows from Lemma 3.5(1) that $f(a * x) \supseteq f(a) \supseteq \alpha$, i.e., $a * x \in f_{\alpha}^{\supsetneq}$. Hence $f_{\alpha}^{\supsetneq}$ is a right stable subset of X. Similarly, $f_{\alpha}^{\supset}$ is a right stable subset of X. □

Theorem 3.42. *Let $(U, E) = (U, X)$ where X is a BCK-algebra. If (f, X) is a union soft ideal over U, then $f_{\alpha}^{\subsetneq}$ and $f_{\alpha}^{\subset}$ are right stable subsets of X for any $\alpha \in \mathscr{P}(U)$ with $f_{\alpha}^{\subsetneq} \neq \emptyset \neq f_{\alpha}^{\subset}$.*

Proof. Let $\alpha \in \mathscr{P}(U)$ and $x, a \in X$ be such that $a \in f_\alpha^{\subseteq}$. Then $f(a) \subseteq \alpha$. Note that $a * x \leq a$, i.e., $(a * x) * a = 0$. Since (f, X) is a union soft ideal of X, it follows that

$$f(a * x) \subseteq f((a * x) * a) \cup f(a) = f(0) \cup f(a) = f(a) \subseteq \alpha.$$

Hence $a * x \in f_\alpha^{\subseteq}$, and so $f_\alpha^{\subseteq}$ is a right stable subset of X. Similarly, $f_\alpha^{\subset}$ is a right stable subset of X. □

Theorem 3.43. *Let $(U, E) = (U, X)$ where X is a BCK-algebra. If (f, X) is a union soft ideal over U, then $f_\alpha^{\supseteq}$ and $f_\alpha^{\supset}$ are right vanished subsets of X for any $\alpha \in \mathscr{P}(U)$ with $f_\alpha^{\supseteq} \neq \emptyset \neq f_\alpha^{\supset}$.*

Proof. Let $\alpha \in \mathscr{P}(U)$ and $a, b \in X$ be such that $a * b \in f_\alpha^{\supseteq}$. Then $f(a * b) \supseteq \alpha$. Note that $a * b \leq a$, i.e., $(a * b) * a = 0$. Since (f, X) is a union soft ideal of X, it follows from (2.8), (2.3), (III) and (V) that

$$\begin{aligned} \alpha &\subseteq f(a * b) \subseteq f((a * b) * a) \cup f(a) \\ &= f((a * a) * b) \cup f(a) = f(0 * b) \cup f(a) \\ &= f(0) \cup f(a) = f(a), \end{aligned}$$

and so $a \in f_\alpha^{\supseteq}$. Therefore $f_\alpha^{\supseteq}$ is a right vanished subset of X. Similarly, $f_\alpha^{\supset}$ is a right vanished subset of X. □

4 Conclusions

We have introduced the notions of soft saturated values and soft dried values, and discussed their applications in BCK/BCI-algebras. Using these notions, we have investigated several properties of energetic subsets. Using the concepts of int-soft ideals (union ideals), we have explored some properties of right vanished (stable) subsets. Work is on going. Some important issues for further work are:

1. To develop strategies for obtaining more valuable results,
2. To apply these notions and results for studying related notions in other (soft) algebraic structures such as soft (semi-, near-, Γ-) rings, soft lattices, soft BL-algebras, soft R_0-algebras, soft MV-algebras and soft MTL-algebras, etc.,
3. To study (fuzzy) rough set theoretical aspects based on this article.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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Boundedness from Below of Composition Followed by Differentiation on Bloch-type Spaces

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Abstract. Let φ be an analytic self-map of the unit disk $\mathbb{D}$. The composition followed by differentiation operator, denoted by DC_φ, is defined by

$$DC_\varphi f(z) = (f \circ \varphi)' = f'(\varphi)\varphi', \quad f \in H(\mathbb{D}).$$

In this paper, under some assumption conditions, we give a necessary and sufficient condition for the operator $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ to be bounded below.

MSC 2000: 47B33, 30H30.

Keywords: Bloch-type space, composition operator, differentiation operator, bounded below.

1 Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $\partial\mathbb{D}$ be its boundary. Let $H(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$. For $0 < \alpha < \infty$, an $f \in H(\mathbb{D})$ is said to belong to Bloch-type space(or α-Bloch space), denoted by $\mathcal{B}^\alpha = \mathcal{B}^\alpha(\mathbb{D})$, if

$$\|f\|_\alpha = \sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

It is easy to check that $\mathcal{B}^\alpha$ is a Banach space with the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \|f\|_\alpha$. When $\alpha = 1$, $\mathcal{B}^1 = \mathcal{B}$ is the well-known Bloch space.

Throughout the paper, $S(\mathbb{D})$ denotes the set of all analytic self-maps of $\mathbb{D}$. Associated with $\varphi \in S(\mathbb{D})$ is the composition operator C_φ defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{D})$. The main subject in the study of composition operators is to describe operator theoretic properties of C_φ in terms of function theoretic properties of φ. See [4] and the references therein for the study of the composition operator. See [7, 8, 9, 10, 11, 12, 13, 14, 15] for the study of composition operators on Bloch-type spaces.

Let D be the differentiation operator and $\varphi \in S(\mathbb{D})$. The composition followed by differentiation operator, denoted by DC_φ, is defined as follows.

$$DC_\varphi f(z) = (f \circ \varphi)' = f'(\varphi)\varphi', \quad f \in H(\mathbb{D}).$$

In [7], the authors studied the boundedness and compactness of DC_φ between Bloch-type spaces. For example, they obtained the following results:

Theorem A. [7] *Let $\alpha, \beta > 0$ and $\varphi \in S(\mathbb{D})$. Then DC_φ: $\mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded if and only if*

$$M_1 := \sup_{z\in\mathbb{D}} \frac{|\varphi'(z)|^2(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}} < \infty, \qquad M_2 := \sup_{z\in\mathbb{D}} \frac{|\varphi''(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha}} < \infty.$$

Theorem B. [7] *Let $\alpha, \beta > 0$, $\varphi \in S(\mathbb{D})$ such that $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded. Then $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact if and only if*

$$\lim_{|\varphi(z)|\to 1} \frac{|\varphi'(z)|^2(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}} = 0 \tag{1}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{|\varphi''(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha}} = 0. \tag{2}$$

Recall that the operator $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is said to be bounded, if there exists a $C > 0$, such that $\|DC_\varphi f\|_{\mathcal{B}^\beta} \le C\|f\|_{\mathcal{B}^\alpha}$ for all $f \in \mathcal{B}^\alpha$. A bounded operator $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is said to be bounded below, if there exists a $\delta > 0$, such that

$$\|DC_\varphi f\|_{\mathcal{B}^\beta} \ge \delta\|f\|_{\mathcal{B}^\alpha}$$

for all $f \in \mathcal{B}^\alpha$. We notice that $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded below if and only if DC_φ has closed range. The boundedness from below of composition operator C_φ on $\mathcal{B}$ was studied by Gathage, Zheng and Zorboska in terms of sampling sets, see [6]. More precisely, they proved that C_φ is bounded below on $\mathcal{B}$ if and only if there exists $\varepsilon > 0$, such that $G_\varepsilon = \varphi(\Omega_\varepsilon)$ is a sampling set for $\mathcal{B}$, where

$$\Omega_\varepsilon = \{z \in \mathbb{D} : \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2} \ge \varepsilon\}.$$

See [1, 2, 5, 6] for other characterizations of the boundedness from below of composition operator on $\mathcal{B}$. The boundedness from below of multiplication operator on Bloch-type spaces was studied in [3].

In this paper, we give a necessary and sufficient condition for the boundedness from below of the operator $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$, i.e., we obtain the following results.

Theorem 1. *Let $0 < \alpha, \beta < \infty$. Let $\varphi \in S(\mathbb{D})$ such that $\varphi'(z) \not\equiv 0$ and (2) holds. Suppose that DC_φ is bounded from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$ and $\lim_{\varphi(z)\to\partial\mathbb{D}} \frac{|\varphi'(z)|^2(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}}$ exists. Then DC_φ: $\mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded below if and only if*

$$\lim_{\varphi(z)\to\partial\mathbb{D}} \frac{|\varphi'(z)|^2(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}} > 0. \tag{3}$$

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Theorem 2. *Let $0 < \alpha, \beta < \infty$. Let $\varphi \in S(\mathbb{D})$ such that $\varphi'(z) \not\equiv 0$ and (1) holds. Suppose that DC_φ is bounded from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$ and $\lim_{\varphi(z)\to\partial\mathbb{D}} \frac{|\varphi''(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}$ exists. Then DC_φ: $\mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded below if and only if*

$$\lim_{\varphi(z)\to\partial\mathbb{D}} \frac{|\varphi''(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} > 0. \tag{4}$$

Throughout the paper, we denote by C a positive constant which may differ from one occurrence to the next. We say that $P \preceq Q$ if there exists a constant C such that $P \leq CQ$. The symbol $P \approx Q$ means that $P \preceq Q \preceq P$.

2 Proof of main results

In this section, we prove the main results in this paper. For this purpose, we need the following lemma.

Lemma 1. *Let $\varphi \in S(\mathbb{D})$ such that $\varphi'(z) \not\equiv 0$. Suppose that $\beta > 0$ and $f_n \in H(\mathbb{D})$ for $n = 1, 2, \cdots$. If $\|DC_\varphi f_n\|_{\mathcal{B}^\beta} \to 0$ as $n \to \infty$, then $f_n' \circ \varphi \to 0$ as $n \to \infty$, locally uniformly in $\mathbb{D}$.*

Proof. The proof is similar to the proof of Lemma 2.9 in [3]. For the convenience of the readers, we give the detail of the proof. Since $\varphi'(z) \not\equiv 0$, then for any $r_0 \in (0, 1)$, there exists an r' such that $r_0 < r' < 1$ and $\varphi'(z) \neq 0$ for $|z| = r'$. By Lemma 2.2 of [3],

$$|\varphi'(z) f_n'(\varphi(z))| \leq C_{\beta,r'} \|\varphi' f_n' \circ \varphi\|_{\mathcal{B}^\beta}$$

for $n = 1, 2, \cdots$, and $|z| = r'$. Let $\delta = \min_{|z|=r'} |\varphi'(z)| > 0$. Then we have $|f_n'(\varphi(z))| \leq (C_{\beta,r'}/\delta)\|\varphi' f_n' \circ \varphi\|_{\mathcal{B}^\beta}$, for $n = 1, 2, \cdots$, and $|z| = r'$. By Maximum principle and the assumption that $\|\varphi' f_n' \circ \varphi\|_{\mathcal{B}^\beta} \to 0$ as $n \to \infty$, we have $f_n' \circ \varphi \to 0$ as $n \to \infty$, uniformly for $|z| \leq r'$. The proof of the lemma is finished.

Lemma 2. [16] *Let m be a positive integer and $\alpha > 0$. Then $f \in \mathcal{B}^\alpha$ if and only if*

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^{m+\alpha-1} |f^{(m)}(z)| < \infty.$$

Moreover,

$$\|f\|_{\mathcal{B}^\alpha} \approx \sum_{j=0}^{m-1} |f^{(j)}(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^{m+\alpha-1} |f^{(m)}(z)|.$$

Proof of Theorem 1. Necessity. By the assumption that DC_φ is bounded from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$, from Theorem A, we have

$$M_1 := \sup_{z\in\mathbb{D}} \frac{|\varphi'(z)|^2(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}} < \infty.$$

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Hence,

$$M_3 := \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\varphi'(z)|^2 < \sup_{z\in\mathbb{D}} \frac{|\varphi'(z)|^2(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}} < \infty.$$

Assume that (3) does not hold, i.e., for any $\eta_1 > 0$, there exists a $\delta_1 > 0$ such that

$$\frac{|\varphi'(z)|^2(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}} < \eta_1 \tag{5}$$

for $|\varphi(z)| > \delta_1$. Let $a_n \in \mathbb{D}$ such that $\varphi(a_n) \to \partial\mathbb{D}$ as $n \to \infty (n = 1, 2, \cdots)$. Set

$$f_n(z) = \frac{1}{\alpha\overline{\varphi(a_n)}} \frac{1-|\varphi(a_n)|^2}{(1-\overline{\varphi(a_n)}z)^\alpha}.$$

It is easy to check that $1 \le \|f_n\|_\alpha \le 2^{\alpha+1}$. Since DC_φ from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$ is bounded below, then, there exists a $\delta > 0$ such that

$$\|DC_\varphi f_n\|_{\mathcal{B}^\beta} \ge \delta\|f_n\|_{\mathcal{B}^\alpha} \ge \delta\|f_n\|_\alpha \ge \delta. \tag{6}$$

On the other hand, we obtain

$$\begin{aligned}
&\|DC_\varphi f_n\|_{\mathcal{B}^\beta} = |\varphi'(0)f_n'(\varphi(0))| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\big(\varphi'\cdot f_n'\circ\varphi\big)'(z)| \\
= \ & |\varphi'(0)f_n'(\varphi(0))| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\varphi'^2(z)f_n''(\varphi(z)) + \varphi''(z)f_n'(\varphi(z))| \\
\le \ & |\varphi'(0)f_n'(\varphi(0))| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\varphi'^2(z)f_n''(\varphi(z))| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\varphi''(z)f_n'(\varphi(z))| \\
= \ & |\varphi'(0)f_n'(\varphi(0))| + E_1 + E_2,
\end{aligned}$$

where

$$E_1 = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\varphi'^2(z)f_n''(\varphi(z))| \ and \ E_2 = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\varphi''(z)f_n'(\varphi(z))|.$$

First we estimate E_1. For any $z \in \mathbb{D}$ such that $|\varphi(z)| > \delta_1$, by (5), we have

$$\begin{aligned}
& (1-|z|^2)^\beta|\varphi'^2(z)f_n''(\varphi(z))| \\
= \ & (\alpha+1)(1-|z|^2)^\beta|\varphi'(z)|^2|\varphi(a_n)|\frac{1-|\varphi(a_n)|^2}{|1-\overline{\varphi(a_n)}\varphi(z)|^{\alpha+2}} \\
\le \ & (\alpha+1)\frac{|\varphi'(z)|^2(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}}\frac{(1-|\varphi(a_n)|^2)(1-|\varphi(z)|^2)^{\alpha+1}}{|1-\overline{\varphi(a_n)}\varphi(z)|^{\alpha+2}} \\
\le \ & (\alpha+1)\frac{|\varphi'(z)|^2(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}}(1+|\varphi(a_n)|)(1+|\varphi(z)|)^{\alpha+1} \\
\le \ & (\alpha+1)2^{\alpha+2}\eta_1.
\end{aligned} \tag{7}$$

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For any $\eta_2 > 0$, there exists a positive integer N, $1-|\varphi(a_n)|^2 < \eta_2$ holds for all $n > N$. For any $z \in \mathbb{D}$ such that $|\varphi(z)| \le \delta_1$ and $n > N$, we deduce

$$\begin{aligned} & (1-|z|^2)^\beta |\varphi'^2(z) f_n''(\varphi(z))| \\ = \;& (\alpha+1)(1-|z|^2)^\beta |\varphi'(z)|^2 |\varphi(a_n)| \frac{1-|\varphi(a_n)|^2}{|1-\overline{\varphi(a_n)}\varphi(z)|^{\alpha+2}} \\ \le \;& (\alpha+1)(1-|z|^2)^\beta |\varphi'(z)|^2 \frac{1-|\varphi(a_n)|^2}{(1-|\varphi(z)|)^{\alpha+2}} \\ \le \;& (\alpha+1)(1-|z|^2)^\beta |\varphi'(z)|^2 \frac{\eta_2}{(1-\delta_1)^{\alpha+2}} \\ \le \;& (\alpha+1)\frac{M_3}{(1-\delta_1)^{\alpha+2}}\eta_2. \end{aligned} \tag{8}$$

From (7) and (8), we have

$$\begin{aligned} E_1 \le \;& \sup_{|\varphi(z)|>\delta_1} (1-|z|^2)|\varphi'(z)|^2 |f_n''(\varphi(z))| \\ & + \sup_{|\varphi(z)|\le\delta_1} (1-|z|^2)|\varphi'(z)|^2 |f_n''(\varphi(z))| \\ < \;& (\alpha+1)2^{\alpha+2}\eta_1 + (\alpha+1)\frac{M_3}{(1-\delta_1)^{\alpha+2}}\eta_2, \qquad \text{as} \quad n > N. \end{aligned}$$

By the arbitrary of η_1 and η_2, we see that $E_1 \to 0$ as $n \to \infty$.

Next we estimate E_2. From (2), for any $\eta_3 > 0$, there exists a $\delta_2 > 0$ such that

$$\frac{|\varphi''(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} < \eta_3, \quad \text{when} \quad |\varphi(z)| > \delta_2.$$

For any $z \in \mathbb{D}$ such that $|\varphi(z)| > \delta_2$,

$$\begin{aligned} (1-|z|^2)^\beta |\varphi''(z) f_n'(\varphi(z))| &= (1-|z|^2)^\beta |\varphi''(z)| \frac{1-|\varphi(a_n)|^2}{|1-\overline{\varphi(a_n)}\varphi(z)|^{\alpha+1}} \\ &\le 2^\alpha (1+|\varphi(a_n)|) \frac{|\varphi''(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} \\ &< 2^{\alpha+1}\eta_3. \end{aligned} \tag{9}$$

For any $z \in \mathbb{D}$ such that $|\varphi(z)| \le \delta_2$ and $n > N$, we have

$$\begin{aligned} (1-|z|^2)^\beta |\varphi''(z) f_n'(\varphi(z))| &= (1-|z|^2)^\beta |\varphi''(z)| \frac{1-|\varphi(a_n)|^2}{|1-\overline{\varphi(a_n)}\varphi(z)|^{\alpha+1}} \\ &\le 2^\alpha \frac{|\varphi''(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} \frac{1-|\varphi(a_n)|^2}{1-|\varphi(z)|} \\ &\le 2^\alpha \frac{M_2\eta_2}{1-\delta_2}. \end{aligned} \tag{10}$$

Then,

$$\begin{aligned} E_2 &\leq \sup_{|\varphi(z)|>\delta_2}(1-|z|^2)|\varphi''(z)f_n'(\varphi(z))| + \sup_{|\varphi(z)|\leq\delta_2}(1-|z|^2)|\varphi''(z)f_n'(\varphi(z))| \\ &\leq 2^{\alpha+1}\eta_3 + 2^{\alpha}\frac{M_2}{1-\delta_2}\eta_2, \qquad \text{as} \quad n > N. \end{aligned}$$

Since η_2 and η_3 are arbitrary, then $E_2 \to 0$ as $n \to \infty$. In addition, $|\varphi'(0)f_n'(\varphi(0))| \to 0$ as $n \to \infty$. Therefore, $\|DC_\varphi f_n\|_{\mathcal{B}^\beta} \to 0$ as $n \to \infty$, which contradicts (6). Therefore (3) holds.

Sufficiency. Now assume that (3) holds. Denoted

$$\epsilon = \lim_{\varphi(z)\to\partial\mathbb{D}} \frac{|\varphi'(z)|^2(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}} > 0. \tag{11}$$

Suppose on the contrary that DC_φ is not bounded below from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$. Then, there exists a sequence $\{f_n\} \subset \mathcal{B}^\alpha$ such that $\|f_n\|_{\mathcal{B}^\alpha} = 1$ for $n = 1, 2, \cdots$, and

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|(f_n'(\varphi)\varphi')'(z)| \leq \|DC_\varphi f_n\|_{\mathcal{B}^\beta} \to 0$$

as $n \to \infty$. By Lemma 1, $f_n' \circ \varphi \to 0$ and hence $f_n' \to 0$ as $n \to \infty$, locally uniformly in $\mathbb{D}$. By Cauchy's estimate we see that $f_n'' \to 0$ as $n \to \infty$, locally uniformly in $\mathbb{D}$. Let $z_n \in \mathbb{D}$ be a sequence such that

$$(1-|\varphi(z_n)|^2)^{\alpha+1}|f_n''(\varphi(z_n))| \geq \frac{1}{2}. \tag{12}$$

Since for every $n = 1, 2, \cdots$, $\|f_n\|_{\mathcal{B}^\alpha} = 1$, we see that the above $\{z_n\}$ exist by Lemma 2. Then $\varphi(z_n) \to \partial\mathbb{D}$ as $n \to \infty$. Hence by (2) and (11), we get

$$\frac{|\varphi''(z_n)|(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^\alpha} \to 0 \tag{13}$$

and

$$\frac{|\varphi'(z_n)|^2(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^{\alpha+1}} \geq \epsilon/2 \tag{14}$$

for sufficiently large n, respectively. Therefore, by (12), (13), (14) and Lemma 2, we obtain

$$\begin{aligned} \|DC_\varphi f_n\|_{\mathcal{B}^\beta} &\geq (1-|z|^2)^\beta|(\varphi' \cdot f_n' \circ \varphi)'(z)| \\ &\geq \frac{|\varphi'(z_n)|^2(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^{\alpha+1}}(1-|\varphi(z_n)|^2)^{\alpha+1}|f''(\varphi(z_n))| \\ &\quad -\frac{|\varphi''(z_n)|(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^\alpha}(1-|\varphi(z_n)|^2)^\alpha|f'(\varphi(z_n))| \\ &\geq \frac{\epsilon}{4}, \ \ as \ \ n \to \infty. \end{aligned}$$

We arrive at a contradiction. Therefore DC_φ is bounded below from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$. This completes the proof of this theorem.

Proof of Theorem 2. The proof of Theorem 2 is similar to the proof of Theorem 1. Hence we omit the details.

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Uni-soft filters of BE-algebras

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Abstract

Further properties of uni-soft filters in a BE-algebra are investigated. The problem of classifying uni-soft filters by their τ-exclusive filter is solved. New uni-soft filter from old one is established.

Keywords: (Self distributive) BE-algebra, Filter, Uni-soft filter,

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1 Introduction

To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [8]. Maji et al. [7] and Molodtsov [8] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [8] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from

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the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [7] described the application of soft set theory to a decision making problem. Maji et al. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. Çağman et al. [3] introduced fuzzy parameterized (FP) soft sets and their related properties. They proposed a decision making method based on FP-soft set theory, and provided an example which shows that the method can be successfully applied to the problems that contain uncertainties. Feng [4] considered the application of soft rough approximations in multicriteria group decision making problems. Aktaş and Çağman [2] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. They also discussed the notion of soft groups. As a generalization of a BCK-algebra, Kim and Kim [6] introduced the notion of a BE-algebra, and investigated several properties. In [1], Ahn and So introduced the notion of ideals in BE-algebras. They gave several descriptions of ideals in BE-algebras. Jun et al. [5] introduced the notion of uni-soft filter of a BE-algebra, and investigated their properties. They considered characterizations of a uni-soft filter, and provided conditions for a soft set to be a uni-soft filter.

In this paper, we investigate further properties of a uni-soft filter. We solve the problem of classifying uni-soft filters by their τ-exclusive filters. We make a new uni-soft filter from old one.

2 Preliminaries

Let $K(\tau)$ be the class of all algebras of type $\tau = (2, 0)$. By a *BE-algebra* (see [6]) we mean a system $(X; *, 1) \in K(\tau)$ in which the following axioms hold:

$$(\forall x \in X)\,(x * x = 1), \tag{2.1}$$

$$(\forall x \in X)\,(x * 1 = 1), \tag{2.2}$$

$$(\forall x \in X)\,(1 * x = x), \tag{2.3}$$

$$(\forall x, y, z \in X)\,(x * (y * z) = y * (x * z)). \quad \text{(exchange)} \tag{2.4}$$

A relation "$\leq$" on a BE-algebra X is defined by

$$(\forall x, y \in X)\,(x \leq y \iff x * y = 1). \tag{2.5}$$

A BE-algebra $(X; *, 1)$ is said to be *transitive* (see [1]) if it satisfies:

$$(\forall x, y, z \in X)\,(y * z \leq (x * y) * (x * z)). \tag{2.6}$$

A BE-algebra $(X;*,1)$ is said to be *self distributive* (see [6]) if it satisfies:

$$(\forall x,y,z\in X)\,(x*(y*z)=(x*y)*(x*z)). \tag{2.7}$$

Every self distributive BE-algebra $(X;*,1)$ satisfies the following properties:

$$(\forall x,y,z\in X)\,(x\le y\ \Rightarrow\ z*x\le z*y,\ y*z\le x*z)\,. \tag{2.8}$$

$$(\forall x,y\in X)\,(x*(x*y)=x*y)\,, \tag{2.9}$$

$$(\forall x,y,z\in X)\,(x*y\le (z*x)*(z*y))\,, \tag{2.10}$$

$$(\forall x,y,z\in X)\,((x*y)*(x*z)\le x*(y*z))\,. \tag{2.11}$$

Note that every self distributive BE-algebra is transitive, but the converse is not true in general (see [1]).

Let $(X;*,1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is a *filter* of X (see [6]) if

(F1) $1\in F$;

(F2) $(\forall x,y\in X)(x*y,x\in F\Rightarrow y\in F)$.

A soft set theory is introduced by Molodtsov [8]. In what follows, let U be an initial universe set and X be a set of parameters. Let $\mathcal{P}(U)$ denotes the power set of U and $A,B,C,\cdots\subseteq X$.

A *soft set* $\left(\tilde{\mathscr{F}},A\right)$ of X over U is defined to be the set of ordered pairs

$$\left(\tilde{\mathscr{F}},A\right):=\left\{\left(x,\tilde{\mathscr{F}}(x)\right):x\in X,\ \tilde{\mathscr{F}}(x)\in\mathcal{P}(U)\right\},$$

where $\tilde{\mathscr{F}}:X\to\mathcal{P}(U)$ such that $\tilde{\mathscr{F}}(x)=\emptyset$ if $x\notin A$.

For a soft set $\left(\tilde{\mathscr{F}},A\right)$ of X and a subset τ of U, the *τ-exclusive set* of $\left(\tilde{\mathscr{F}},A\right)$, denoted by $e_A\left(\tilde{\mathscr{F}};\tau\right)$, is defined to be the set

$$e_A\left(\tilde{\mathscr{F}};\tau\right):=\left\{x\in A\mid \tau\supseteq\tilde{\mathscr{F}}(x)\right\}.$$

For any soft sets $\left(\tilde{\mathscr{F}},X\right)$ and $\left(\tilde{\mathscr{G}},X\right)$ of X, we call $\left(\tilde{\mathscr{F}},X\right)$ a *soft subset* of $\left(\tilde{\mathscr{G}},X\right)$, denoted by $\left(\tilde{\mathscr{F}},X\right)\tilde{\subseteq}\left(\tilde{\mathscr{G}},X\right)$, if $\tilde{\mathscr{F}}(x)\subseteq\tilde{\mathscr{G}}(x)$ for all $x\in X$. The *soft union* of $\left(\tilde{\mathscr{F}},X\right)$ and $\left(\tilde{\mathscr{G}},X\right)$, denoted by $\left(\tilde{\mathscr{F}},X\right)\tilde{\cup}\left(\tilde{\mathscr{G}},X\right)$, is defined to be the soft set $\left(\tilde{\mathscr{F}}\tilde{\cup}\tilde{\mathscr{G}},X\right)$ of X over U in which $\tilde{\mathscr{F}}\tilde{\cup}\tilde{\mathscr{G}}$ is defined by

$$\left(\tilde{\mathscr{F}}\tilde{\cup}\tilde{\mathscr{G}}\right)(x)=\tilde{\mathscr{F}}(x)\cup\tilde{\mathscr{G}}(x)\text{ for all }x\in M.$$

The *soft intersection* of $\left(\tilde{\mathscr{F}}, X\right)$ and $\left(\tilde{\mathscr{G}}, X\right)$, denoted by $\left(\tilde{\mathscr{F}}, X\right) \tilde{\cap} \left(\tilde{\mathscr{G}}, X\right)$, is defined to be the soft set $\left(\tilde{\mathscr{F}} \tilde{\cap} \tilde{\mathscr{G}}, M\right)$ of X over U in which $\tilde{\mathscr{F}} \tilde{\cap} \tilde{\mathscr{G}}$ is defined by

$$\left(\tilde{\mathscr{F}} \tilde{\cap} \tilde{\mathscr{G}}\right)(x) = \tilde{\mathscr{F}}(x) \cap \tilde{\mathscr{G}}(x) \text{ for all } x \in S.$$

3 Uni-soft filters

In what follows, we take a BE-algebra X, as a set of parameters unless otherwise specified.

Definition 3.1 ([5]). A soft set $\left(\tilde{\mathscr{F}}, X\right)$ of X over U is called a *uni-soft filter* of X if it satisfies:

$$(\forall x \in X)\left(\tilde{\mathscr{F}}(1) \subseteq \tilde{\mathscr{F}}(x)\right), \tag{3.1}$$

$$(\forall x, y \in X)\left(\tilde{\mathscr{F}}(y) \subseteq \tilde{\mathscr{F}}(x * y) \cup \tilde{\mathscr{F}}(x)\right). \tag{3.2}$$

We make a new uni-soft filter from old one.

Lemma 3.2 ([5]). *For a soft set $\left(\tilde{\mathscr{F}}, X\right)$ over U, the following are equivalent.*

(i) $\left(\tilde{\mathscr{F}}, X\right)$ *is a uni-soft filter of X over U.*

(ii) *The τ-exclusive set $e_X\left(\tilde{\mathscr{F}}; \tau\right)$ is a filter of X for all $\tau \in \mathcal{P}(U)$ with $e_X\left(\tilde{\mathscr{F}}; \tau\right) \neq \emptyset$.*

Theorem 3.3. *For a soft set $\left(\tilde{\mathscr{F}}, X\right)$ over U, define a soft set $(\tilde{\mathscr{F}}^*, X)$ over U by*

$$\tilde{\mathscr{F}}^* : X \to \mathcal{P}(U),\ x \mapsto \begin{cases} \tilde{\mathscr{F}}(x) & \text{if } x \in e_X(\tilde{\mathscr{F}}; \tau), \\ U & \text{otherwise} \end{cases}$$

where τ is a nonempty subset of U. If $\left(\tilde{\mathscr{F}}, X\right)$ is a uni-soft filter of X over U, then so is $(\tilde{\mathscr{F}}^*, X)$.

Proof. Assume that $\left(\tilde{\mathscr{F}}, X\right)$ is a uni-soft filter of X over U. Then $e_X(\tilde{\mathscr{F}}; \tau)(\neq \emptyset)$ is a filter of X over U for all $\tau \subseteq U$ by Lemma 3.2. Hence $1 \in e_X(\tilde{\mathscr{F}}; \tau)$, and so $\tilde{\mathscr{F}}^*(1) = \tilde{\mathscr{F}}(1) \subseteq \tilde{\mathscr{F}}(x) \subseteq \tilde{\mathscr{F}}^*(x)$ for all $x \in X$. Let $x, y \in X$. If $x * y \in e_X(\tilde{\mathscr{F}}; \tau)$ and $x \in e_X(\tilde{\mathscr{F}}; \tau)$, then $y \in e_X(\tilde{\mathscr{F}}; \tau)$. Hence

$$\tilde{\mathscr{F}}^*(y) = \tilde{\mathscr{F}}(y) \subseteq \tilde{\mathscr{F}}(x * y) \cup \tilde{\mathscr{F}}(x) = \tilde{\mathscr{F}}^*(x * y) \cup \tilde{\mathscr{F}}^*(x).$$

If $x * y \notin e_X(\tilde{\mathscr{F}};\tau)$ or $x \notin e_X(\tilde{\mathscr{F}};\tau)$, then $\tilde{\mathscr{F}}^*(x*y) = U$ or $\tilde{\mathscr{F}}^*(x) = U$. Thus

$$\tilde{\mathscr{F}}^*(y) \subseteq U = \tilde{\mathscr{F}}^*(x*y) \cup \tilde{\mathscr{F}}^*(x).$$

Therefore $(\tilde{\mathscr{F}}^*, X)$ is a uni-soft filter of X over U. □

Theorem 3.4. *If $\left(\tilde{\mathscr{F}}, X\right)$ and $\left(\tilde{\mathscr{G}}, X\right)$ are uni-soft filters of X over U, then the soft union $\left(\tilde{\mathscr{F}}, X\right) \tilde{\cup} \left(\tilde{\mathscr{G}}, X\right)$ of $\left(\tilde{\mathscr{F}}, X\right)$ and $\left(\tilde{\mathscr{G}}, X\right)$ is a uni-soft filter of X over U.*

Proof. For any $x \in X$, we have

$$\left(\tilde{\mathscr{F}} \tilde{\cup} \tilde{\mathscr{G}}\right)(1) = \tilde{\mathscr{F}}(1) \cup \tilde{\mathscr{G}}(1) \subseteq \tilde{\mathscr{F}}(x) \cup \tilde{\mathscr{G}}(x) = (\tilde{\mathscr{F}} \tilde{\cup} \tilde{\mathscr{G}})(x).$$

Let $x, y \in X$. Then

$$\begin{aligned}\left(\tilde{\mathscr{F}} \tilde{\cup} \tilde{\mathscr{G}}\right)(y) &= \tilde{\mathscr{F}}(y) \cup \tilde{\mathscr{G}}(y) \\ &\subseteq \left(\tilde{\mathscr{F}}(x*y) \cup \tilde{\mathscr{F}}(x)\right) \cup \left(\tilde{\mathscr{G}}(x*y) \cup \tilde{\mathscr{G}}(x)\right) \\ &= \left(\tilde{\mathscr{F}}(x*y) \cup \tilde{\mathscr{G}}(x*y)\right) \cup \left(\tilde{\mathscr{F}}(x) \cup \tilde{\mathscr{G}}(x)\right) \\ &= \left(\tilde{\mathscr{F}} \tilde{\cup} \tilde{\mathscr{G}}\right)(x*y) \cup \left(\tilde{\mathscr{F}} \tilde{\cup} \tilde{\mathscr{G}}\right)(x).\end{aligned}$$

Hence $\left(\tilde{\mathscr{F}}, X\right) \tilde{\cup} \left(\tilde{\mathscr{G}}, X\right)$ is a uni-soft filter of X over U. □

The following example shows that the soft intersection of uni-soft filters of X over U may not be a uni-soft filter of X over U

Example 3.5. Consider a BE-algebra $X = \{1, a, b, c, d, 0\}$ with the Cayley table which is given in Table 1 (see [1]).
Let $E = X$ be the set of parameters and $U = \mathbb{Z}$ be the initial universe set. Define two soft sets $\left(\tilde{\mathscr{F}}, X\right)$ and $\left(\tilde{\mathscr{G}}, X\right)$ over U as follows:

$$\tilde{\mathscr{F}} : X \to \mathcal{P}(U), \quad x \mapsto \begin{cases} 4\mathbb{N} & \text{if } x \in \{1, c\} \\ 2\mathbb{Z} & \text{if } x \in \{a, b, d, 0\} \end{cases}$$

and

$$\tilde{\mathscr{G}} : X \to \mathcal{P}(U), \quad x \mapsto \begin{cases} 8\mathbb{N} & \text{if } x \in \{1, a, b\} \\ 4\mathbb{Z} & \text{if } x \in \{c, d, 0\} \end{cases}$$

Table 1: Cayley table for the "$*$"-operation

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

respectively. It is easy to check that $\left(\tilde{\mathscr{F}}, X\right)$ and $\left(\tilde{\mathscr{G}}, X\right)$ are uni-soft filters of X over U. But $\left(\tilde{\mathscr{F}}, X\right) \tilde{\cap} \left(\tilde{\mathscr{G}}, X\right) = (\tilde{\mathscr{F}} \tilde{\cap} \tilde{\mathscr{G}}, X)$ is not a uni-soft filter of X over U, since

$$\begin{aligned}(\tilde{\mathscr{F}} \tilde{\cap} \tilde{\mathscr{G}})(d) &= \tilde{\mathscr{F}}(d) \cap \tilde{\mathscr{G}}(d) = 2\mathbb{Z} \cap 4\mathbb{Z} \\ &= 4\mathbb{Z} \nsubseteq 4\mathbb{N} = 8\mathbb{N} \cup 4\mathbb{N} \\ &= (\tilde{\mathscr{F}}(a) \cap \tilde{\mathscr{G}}(a)) \cup (\tilde{\mathscr{F}}(c) \cap \tilde{\mathscr{G}}(c)) \\ &= (\tilde{\mathscr{F}} \tilde{\cap} \tilde{\mathscr{G}})(a) \cup (\tilde{\mathscr{F}} \tilde{\cap} \tilde{\mathscr{G}})(c) \\ &= (\tilde{\mathscr{F}} \tilde{\cap} \tilde{\mathscr{G}})(c * d) \cup (\tilde{\mathscr{F}} \tilde{\cap} \tilde{\mathscr{G}})(c).\end{aligned}$$

Theorem 3.6. *Let $\left(\tilde{\mathscr{F}}, X\right)$ be a uni-soft filter of X. Let τ_1 and τ_2 be subsets of U such that $\tau_1 \supsetneq \tau_2$. If the τ_1-exclusive set of $\left(\tilde{\mathscr{F}}, X\right)$ is equal to the τ_2-exclusive set of $\left(\tilde{\mathscr{F}}, X\right)$, then there is no $x \in X$ such that $\tau_1 \supsetneq \tilde{\mathscr{F}}(x) \supsetneq \tau_2$.*

Proof. Straightforward. □

The converse of Theorem 3.6 is not true in general as seen in the following example.

Example 3.7. Consider a BE-algebra $X = \{1, a, b, c\}$ with the Cayley table which is given in Table 2.
Given $U = X$, consider a soft set $\left(\tilde{\mathscr{F}}, X\right)$ of X over U which is given by

$$\tilde{\mathscr{F}} : X \to \mathcal{P}(U), \quad x \mapsto \begin{cases} \emptyset & \text{if } x = 1, \\ \{1, a, c\} & \text{if } x \in \{a, b\}, \\ \{1, a\} & \text{if } x = c. \end{cases}$$

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Table 2: Cayley table for the "$*$"-operation

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	a	a	1

Then $\left(\tilde{\mathscr{F}}, X\right)$ is a uni-soft filter of X. The τ-exclusive sets of $\left(\tilde{\mathscr{F}}, X\right)$ are described as follows:

$$e_X(\tilde{\mathscr{F}};\tau) = \begin{cases} X & \text{if } \tau \in \{X, \{1,a,c\}\} \\ \{1,c\} & \text{if } \{1,a\} \subseteq \tau \subsetneq \{1,a,c\}, \\ \{1\} & \text{otherwise.} \end{cases}$$

If we take $\tau_1 = X$ and $\tau_2 = \{1, b\}$, then $\tau_1 \supsetneq \tau_2$ and there is no $x \in X$ such that $\tau_1 \supsetneq \tilde{\mathscr{F}}(x) \supsetneq \tau_2$. But $e_X(\tilde{\mathscr{F}};\tau_1) = X \neq \{1\} = e_X(\tilde{\mathscr{F}};\tau_2)$.

Theorem 3.8. *Let $\left(\tilde{\mathscr{F}}, X\right)$ be a uni-soft filter of X. Let τ_1 and τ_2 be subsets of U such that $\tau_1 \supsetneq \tau_2$ and $\{\tau_1, \tau_2, \tilde{\mathscr{F}}(x)\}$ is totally ordered by set inclusion for all $x \in X$. If there is no $x \in X$ such that $\tau_1 \supseteq \tilde{\mathscr{F}}(x) \supsetneq \tau_2$, then the τ_1-exclusive set of $\left(\tilde{\mathscr{F}}, X\right)$ is equal to the τ_2-exclusive set of $\left(\tilde{\mathscr{F}}, X\right)$.*

Proof. Since $\tau_1 \supsetneq \tau_2$, we have $e_X(\tilde{\mathscr{F}};\tau_2) \subseteq e_X(\tilde{\mathscr{F}};\tau_1)$. If $x \in e_X(\tilde{\mathscr{F}};\tau_1)$, then $\tau_1 \supseteq \tilde{\mathscr{F}}(x)$. Since $\{\tau_1, \tau_2, \tilde{\mathscr{F}}(x) \mid x \in X\}$ is totally ordered by set inclusion and there is no $x \in X$ such that $\tau_1 \supseteq \tilde{\mathscr{F}}(x) \supsetneq \tau_2$, it follows that $\tau_2 \supseteq \tilde{\mathscr{F}}(x)$, that is, $x \in e_X(\tilde{\mathscr{F}};\tau_2)$. Therefore the τ_1-exclusive set of $\left(\tilde{\mathscr{F}}, X\right)$ is equal to the τ_2-exclusive set of $\left(\tilde{\mathscr{F}}, X\right)$. □

We have the following question.

Question. *Given a uni-soft filter $\left(\tilde{\mathscr{F}}, X\right)$ of X, does any filter can be represented as a τ-exclusive set of $\left(\tilde{\mathscr{F}}, X\right)$?*

The following example shows that the answer to the question above is false.

Example 3.9. Let $X = \{1, a, b, c\}$ be the BE-algebra as in Example 3.7. Given $U = X$, consider a soft set $\left(\tilde{\mathscr{F}}, X\right)$ of X over U which is given by

$$\tilde{\mathscr{F}} : X \to \mathcal{P}(U), \quad x \mapsto \begin{cases} \{c\} & \text{if } x = 1, \\ \{1, c\} & \text{if } x \in \{a, b, c\}. \end{cases}$$

Then $\left(\tilde{\mathscr{F}}, X\right)$ is a uni-soft filter of X. The τ-exclusive sets of $\left(\tilde{\mathscr{F}}, X\right)$ are described as follows:

$$e_X(\tilde{\mathscr{F}}; \tau) = \begin{cases} X & \text{if } \tau \supseteq \{1, c\}, \\ \{1\} & \text{if } \{c\} \subseteq \tau \subsetneq \{1, c\}, \\ \emptyset & \text{otherwise.} \end{cases}$$

The filter $\{1, b\}$ cannot be a τ-exclusive set $e_X(\tilde{\mathscr{F}}; \tau)$, since there is no $\tau \subseteq U$ such that $e_X(\tilde{\mathscr{F}}; \tau) = \{1, b\}$.

However, we have the following theorem.

Theorem 3.10. *Every filter of a BE-algebra can be represented as a τ-exclusive set of a uni-soft filter, that is, given a filter F of a BE-algebra X, there exists a uni-soft filter $\left(\tilde{\mathscr{F}}, X\right)$ of X over U such that F is the τ-exclusive set of $\left(\tilde{\mathscr{F}}, X\right)$ for a nonempty subset τ of U.*

Proof. Let F be a filter of a BE-algebra X. For a nonempty subset τ of U, define a soft set $\left(\tilde{\mathscr{F}}, X\right)$ over U by

$$\tilde{\mathscr{F}} : X \to \mathcal{P}(U), \; x \mapsto \begin{cases} \tau & \text{if } x \in F, \\ U & \text{if } x \notin F. \end{cases}$$

Obviously, $F = e_X(\tilde{\mathscr{F}}; \tau)$. We now prove that $\left(\tilde{\mathscr{F}}, X\right)$ is a uni-soft filter of X. Since $1 \in F = e_X(\tilde{\mathscr{F}}; \tau)$, we have $\tilde{\mathscr{F}}(1) \subseteq \tau \subseteq \tilde{\mathscr{F}}(x)$ for all $x \in X$. Let $x, y \in X$. If $x * y \in F$ and $x \in F$, then $y \in F$ because F is a filter of X. Hence $\tilde{\mathscr{F}}(x * y) = \tilde{\mathscr{F}}(x) = \tilde{\mathscr{F}}(y) = \tau$, and so $\tilde{\mathscr{F}}(y) \subseteq \tilde{\mathscr{F}}(x * y) \cup \tilde{\mathscr{F}}(x)$. If $x * y \notin F$ or $x \notin F$, then $\tilde{\mathscr{F}}(x * y) = U$ or $\tilde{\mathscr{F}}(x) = U$. Hence $\tilde{\mathscr{F}}(y) \subseteq U = \tilde{\mathscr{F}}(x * y) \cup \tilde{\mathscr{F}}(x)$. Therefore $\left(\tilde{\mathscr{F}}, X\right)$ is a uni-soft filter of X. □

Note that if $E = X$ is a finite BE-algebra, then the number of filters of X is finite whereas the number of τ-exclusive sets of a uni-soft filter of X over $U = \mathbb{Z}$ appears to be infinite. But, since every τ-exclusive set is indeed a filter of X, not all these τ-exclusive sets are distinct. The next theorem characterizes this aspect.

Theorem 3.11. *Let* $\left(\tilde{\mathscr{F}}, X\right)$ *be a uni-soft filter of* X *over* $U = \mathbb{Z}$ *and let* $\tau_1 \subsetneq \tau_2 \subseteq U$ *be such that* $\{\tau_1, \tau_2, \tilde{\mathscr{F}}(x)\}$ *is a chain for all* $x \in X$. *Two* τ*-exclusive sets* $e_X(\tilde{\mathscr{F}}; \tau_1)$ *and* $e_X(\tilde{\mathscr{F}}; \tau_2)$ *are equal if and only if there is no* $x \in X$ *such that* $\tau_1 \subsetneq \tilde{\mathscr{F}}(x) \subsetneq \tau_2$.

Proof. Let τ_1 and τ_2 be subsets of U such that $e_X(\tilde{\mathscr{F}}; \tau_1) = e_X(\tilde{\mathscr{F}}; \tau_2)$. Assume that there exists $x \in X$ such that $\tau_1 \subsetneq \tilde{\mathscr{F}}(x) \subsetneq \tau_2$. Then $e_X(\tilde{\mathscr{F}}; \tau_2)$ is a proper superset of $e_X(\tilde{\mathscr{F}}; \tau_1)$, which contradicts the hypothesis.

Conversely, suppose that there is no $x \in X$ such that $\tau_1 \subsetneq \tilde{\mathscr{F}}(x) \subsetneq \tau_2$. Obviously, $e_X(\tilde{\mathscr{F}}; \tau_2) \supseteq e_X(\tilde{\mathscr{F}}; \tau_1)$. If $x \in e_X(\tilde{\mathscr{F}}; \tau_2)$, then $\tau_2 \supseteq \tilde{\mathscr{F}}(x)$. It follows from the hypothesis that $\tau_1 \supseteq \tilde{\mathscr{F}}(x)$, i.e., $x \in e_X(\tilde{\mathscr{F}}; \tau_1)$. Therefore $e_X(\tilde{\mathscr{F}}; \tau_1) = e_X(\tilde{\mathscr{F}}; \tau_2)$. □

Let $\left(\tilde{\mathscr{F}}, X\right)$ be a soft set of X over U. For any $a, b \in X$ and $k \in \mathbb{N}$, consider the set

$$\tilde{\mathscr{F}}[a^k; b] := \left\{x \in X \mid \tilde{\mathscr{F}}\left(a^k * (b * x)\right) = \tilde{\mathscr{F}}(1)\right\}$$

where $\tilde{\mathscr{F}}(a^k * x) = \tilde{\mathscr{F}}(a * (a * (\cdots * (a * (a * x)) \cdots)))$ in which a appears k-times. Note that $a, b, 1 \in \tilde{\mathscr{F}}[a^k; b]$ for all $a, b \in X$ and $k \in \mathbb{N}$.

Proposition 3.12. *Let* $\left(\tilde{\mathscr{F}}, X\right)$ *be a soft set of* X *over* U *satisfying the condition* (3.1) *and* $\tilde{\mathscr{F}}(x*y) = \tilde{\mathscr{F}}(x) \cap \tilde{\mathscr{F}}(y)$ *for all* $x, y \in X$. *For any* $a, b \in X$ *and* $k \in \mathbb{N}$, *if* $x \in \tilde{\mathscr{F}}[a^k; b]$, *then* $y * x \in \tilde{\mathscr{F}}[a^k; b]$ *for all* $y \in X$.

Proof. Assume that $x \in \tilde{\mathscr{F}}[a^k; b]$. Then $\tilde{\mathscr{F}}(a^k * (b * x)) = \tilde{\mathscr{F}}(1)$, and so

$$\begin{aligned}
\tilde{\mathscr{F}}(a^k * (b * (y * x))) &= \tilde{\mathscr{F}}(a^k * (y * (b * x))) \\
&= \tilde{\mathscr{F}}(y * (a^k * (b * x))) \\
&= \tilde{\mathscr{F}}(y) \cap \tilde{\mathscr{F}}(a^k * (b * x)) \\
&= \tilde{\mathscr{F}}(y) \cap \tilde{\mathscr{F}}(1) = \tilde{\mathscr{F}}(1)
\end{aligned}$$

for all $y \in X$ by the exchange property of the operation $*$ and (3.1). Hence $y * x \in \tilde{\mathscr{F}}[a^k; b]$ for all $y \in X$. □

Proposition 3.13. *For any soft set* $\left(\tilde{\mathscr{F}}, X\right)$ *of* X *over* U, *if an element* $a \in X$ *satisfies* $a * x = 1$ *for all* $x \in X$ *then* $\tilde{\mathscr{F}}[a^k; b] = X = \tilde{\mathscr{F}}[b^k; a]$ *for all* $b \in X$ *and* $k \in \mathbb{N}$.

Proof. For any $x \in X$, we have

$$\begin{aligned}
\tilde{\mathscr{F}}(a^k * (b * x)) &= \tilde{\mathscr{F}}(a^{k-1} * (a * (b * x))) \\
&= \tilde{\mathscr{F}}(a^{k-1} * (b * (a * x))) \\
&= \tilde{\mathscr{F}}(a^{k-1} * (b * 1)) \\
&= \tilde{\mathscr{F}}(1),
\end{aligned}$$

and so $x \in \tilde{\mathscr{F}}[a^k; b]$. Similarly, $x \in \tilde{\mathscr{F}}[b^k; a]$. □

Proposition 3.14. *Let X be a self distributive BE-algebra and let $\left(\tilde{\mathscr{F}}, X\right)$ be an order-reversing soft set of X over U with the property (3.1). If $b \leq c$ in X, then $\tilde{\mathscr{F}}[a^k; c] \subseteq \tilde{\mathscr{F}}[a^k; b]$ for all $a \in X$ and $k \in \mathbb{N}$.*

Proof. Let $a, b, c, \in X$ be such that $b \leq c$. For any $k \in \mathbb{N}$, if $x \in \tilde{\mathscr{F}}[a^k; c]$, then

$$\begin{aligned}\tilde{\mathscr{F}}(1) &= \tilde{\mathscr{F}}(a^k * (c * x)) = \tilde{\mathscr{F}}(c * (a^k * x)) \\ &\supseteq \tilde{\mathscr{F}}(b * (a^k * x)) = \tilde{\mathscr{F}}(a^k * (b * x))\end{aligned}$$

by (2.4) and (2.8), and so $\tilde{\mathscr{F}}(a^k * (b * x)) = \tilde{\mathscr{F}}(1)$. Thus $x \in \tilde{\mathscr{F}}[a^k; b]$, which completes the proof. □

The following example shows that there exists a soft set $\left(\tilde{\mathscr{F}}, X\right)$ of X over U, $a, b \in X$ and $k \in \mathbb{N}$ such that $\tilde{\mathscr{F}}[a^k; b]$ is not a filter of X.

Example 3.15. Consider the BE-algebra $X = \{1, a, b, c\}$ in Example 3.7. Let $\left(\tilde{\mathscr{F}}, X\right)$ be a soft set of X over $U = \mathbb{N}$ which is given by

$$\tilde{\mathscr{F}} : X \to \mathcal{P}(U), \quad x \mapsto \begin{cases} 6\mathbb{N} & \text{if } x = 1, \\ 3\mathbb{N} & \text{if } x \in \{a, b, c\}. \end{cases}$$

Then it is a soft set of X over U. But $\tilde{\mathscr{F}}[c; b] = \{x \in X | \tilde{\mathscr{F}}(c*(b*x)) = \tilde{\mathscr{F}}(1)\} = \{1, a, b\}$ is not a filter, since $a * c = a \in \tilde{\mathscr{F}}[c; b]$ and $c \notin \tilde{\mathscr{F}}[c; b]$.

We provide conditions for a set $\tilde{\mathscr{F}}[a^k; b]$ to be a filter.

Theorem 3.16. *Let $\left(\tilde{\mathscr{F}}, X\right)$ be a soft set over X. If X is a self distributive BE-algebra and $\tilde{\mathscr{F}}$ is injective, then $\tilde{\mathscr{F}}[a^k; b]$ is a filter of X for all $a, b \in X$ and $k \in \mathbb{N}$.*

Proof. Assume that X is a self distributive BE-algebra and $\tilde{\mathscr{F}}$ is injective. Obviously, $1 \in \tilde{\mathscr{F}}[a^k; b]$. Let $a, b, x, y \in X$ and $k \in \mathbb{N}$ be such that $x * y \in \tilde{\mathscr{F}}[a^k; b]$ and $x \in \tilde{\mathscr{F}}[a^k; b]$. Then $\tilde{\mathscr{F}}(a^k * (b * x)) = \tilde{\mathscr{F}}(1)$ which implies that $a^k * (b * x) = 1$ since $\tilde{\mathscr{F}}$ is injective.

Using (2.7), we have

$$\begin{aligned}\tilde{\mathscr{F}}(1) &= \tilde{\mathscr{F}}(a^k * (b * (x * y)))\\ &= \tilde{\mathscr{F}}(a^{k-1} * (a * (b * (x * y))))\\ &= \tilde{\mathscr{F}}(a^{k-1} * (a * ((b * x) * (b * y))))\\ &= \cdots\\ &= \tilde{\mathscr{F}}((a^k * (b * x)) * (a^k * (b * y)))\\ &= \tilde{\mathscr{F}}(1 * (a^k * (b * y)))\\ &= \tilde{\mathscr{F}}(a^k * (b * y)),\end{aligned}$$

which implies that $y \in \tilde{\mathscr{F}}[a^k; b]$. Therefore $\tilde{\mathscr{F}}[a^k; b]$ is a filter of X for all $a, b \in X$ and $k \in \mathbb{N}$. □

Theorem 3.17. *Let X be a self distributive BE-algebra. Let $\left(\tilde{\mathscr{F}}, X\right)$ be a soft set of X over U satisfying the condition (3.1) and*

$$(\forall x, y \in X)\left(\tilde{\mathscr{F}}(x * y) = \tilde{\mathscr{F}}(x) \cup \tilde{\mathscr{F}}(y)\right). \tag{3.3}$$

Then $\tilde{\mathscr{F}}[a^k; b]$ is a filter of X for all $a, b \in X$ and $k \in \mathbb{N}$.

Proof. Let $a, b \in X$ and $k \in \mathbb{N}$. Obviously, $1 \in \tilde{\mathscr{F}}[a^k; b]$. Let $x, y \in X$ be such that $x * y \in \tilde{\mathscr{F}}[a^k; b]$ and $x \in \tilde{\mathscr{F}}[a^k; b]$. Then $\tilde{\mathscr{F}}\left(a^k * (b * x)\right) = \tilde{\mathscr{F}}(1)$, which implies from (3.3) and (3.1) that

$$\begin{aligned}\tilde{\mathscr{F}}(1) &= \tilde{\mathscr{F}}(a^k * (b * (x * y)))\\ &= \tilde{\mathscr{F}}(a^{k-1} * (a * (b * (x * y))))\\ &= \tilde{\mathscr{F}}(a^{k-1} * (a * ((b * x) * (b * y))))\\ &= \cdots\\ &= \tilde{\mathscr{F}}((a^k * (b * x)) * (a^k * (b * y)))\\ &= \tilde{\mathscr{F}}(a^k * (b * x)) \cup \tilde{\mathscr{F}}(a^k * (b * y))\\ &= \tilde{\mathscr{F}}(1) \cup \tilde{\mathscr{F}}(a^k * (b * y))\\ &= \tilde{\mathscr{F}}(a^k * (b * y)).\end{aligned}$$

Hence $y \in \tilde{\mathscr{F}}[a^k; b]$ and therefore $\tilde{\mathscr{F}}[a^k; b]$ is a filter of X for all $a, b \in X$ and $k \in \mathbb{N}$. □

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On q-analogue of Stancu-Schurer-Kantorovich operators based on q-Riemann integral

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Abstract

In the present paper we introduce the Kantorovich type generalization of Stancu-Schurer operators based on q-Riemann integral. A convergence theorem using the well known Bohman-Korovkin criterion is proven and the rate of convergence involving the modulus of continuity is established. Also, we obtain a Voronovskaja type theorem for these operators.

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1 Introduction

In recent years, the applications of q-calculus have played an important role in the area of approximation theory, generalizations of many well-known linear and positive operators based on the q-integers were studied by numbers of authors ([2, 5, 7, 10–12, 14, 16, 18–20]). In 1987, Lupaş [9] introduced and studied q-analogue of Bernstein operators and in 1996 another generalization of these operators were introduced by Philips [17]. More results on q-Bernstein polynomials were obtained by Ostrowska in [15]. In [1], Agratini introduced a new class of q-Bernstein type operators which fix certain polynomials and studied their approximation properties. Very recently, Muraru [14] proposed the q-Bernstein-Schurer

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operators. Agrawal et al. [3] introduced the Stancu type generalization of Bernstein-Schurer operators. Aral et al. [4] also presented many results on convergence of different q-operators recently in their book.

First, we present some basic definitions and notations from q-calculus. Let $q > 0$. For each nonnegative integer k, the q-integer $[k]_q$ and q-factorial $[k]_q!$ are respectively defined by

$$[k]_q := \begin{cases} \frac{1-q^k}{1-q}, & q \neq 1, \\ k, & q = 1, \end{cases} \qquad [k]_q! := \begin{cases} [k]_q[k-1]_q \cdots [1]_q, & k \geq 1, \\ 1, & k = 0. \end{cases}$$

For the integers n, k satisfying $n \geq k \geq 0$, the q-binomial coefficients are defined by

$$\left[\begin{array}{c} n \\ k \end{array} \right]_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

We denote $(a+b)_q^k = \prod_{j=0}^{k-1}(a + bq^j)$.

The q-Jackson integral on the interval $[0, b]$ is defined as

$$\int_0^b f(t) d_q t = (1-q)b \sum_{j=0}^{\infty} f(q^j b) q^j, \quad 0 < q < 1,$$

provided that sums converge absolutely. Suppose $0 < a < b$. The q-Jackson integral on the interval $[a, b]$ is defined as

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t, \quad 0 < q < 1.$$

Dalmanoğlu [5], Mahmudov and Sabancigil [12] defined some q-type generalizations of Bernstein-Kantorovich operators using q-Jackson integral. In [18], Ren and Zeng were introduced two kinds of Kantorovich-type q-Bernstein-Stancu operators. The first version is defined using q-Jackson integral as follows

$$S_{n,q}^{(\alpha,\beta)}(f, x) = ([n+1]_q + \beta) \sum_{k=0}^{n} q^{-k} p_{n,k}(q; x) \int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} f(t) d_q t, \tag{1.1}$$

where $0 \leq \alpha \leq \beta$, $f \in C[0, 1]$ and $p_{n,k}(q; x) = \left[\begin{array}{c} n \\ k \end{array} \right]_q x^k (1-x)_q^{n-k}$.

To guarantee the positivity of q-Bernstein-Stancu-Kantorovich operators, in [18] is considered the Riemann-type q-integral (see [13]) defined by

$$\int_a^b f(t) d_q^R t = (1-q)(b-a) \sum_{j=0}^{\infty} f\left(a + (b-a)q^j\right) q^j. \tag{1.2}$$

Ren and Zeng [18] redefine $S_{n,q}^{(\alpha,\beta)}$ by putting the Riemann-type q-integral into the operators instead of the q-Jackson integral as

$$\tilde{S}_{n,q}^{(\alpha,\beta)}(f, x) = ([n+1]_q + \beta) \sum_{k=0}^{n} q^{-k} p_{n,k}(q; x) \int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} f(t) d_q^R t. \tag{1.3}$$

Very recently, the q-Kantorovich extension of the Bernstein-Schurer operators have been considered by Kumar et al. [8] as follows:

$$K_{n,p}(f,q;x) = [n+1]_q \sum_{k=0}^{n+p} b_{n,k}(q;x) q^{-k} \int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q} f(t) d_q^R t, \tag{1.4}$$

where $x \in [0,1]$, $f \in C[0,1+p]$, $p \in \mathbb{N}^0 = \mathbb{N} \cup \{0\}$ and $b_{n,k}(q;x) = \begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k (1-x)_q^{n+p-k}$.

In the present paper, inspired by the new Kantorovich type generalization of the q-Bernstein-Schurer operators we introduce the Kantorovich type of Stancu-Schurer operators involving the Riemann-type q-integral.

2 Construction of the operators

In this section we construct the Stancu-Schurer-Kantorovich operators based on q-integers. Let $\alpha, \beta \in R$ be such that $0 \le \alpha \le \beta$ and $p \in \mathbb{N}^0 = \mathbb{N} \cup \{0\}$, then for any $f \in C[0,1+p]$, $q \in (0,1)$, the Stancu-Schurer-Kantorovich operators are defined using q-Riemann integral as follows

$$K_{n,p}^{(\alpha,\beta)}(f,q;x) = ([n+1]_q + \beta) \sum_{k=0}^{n+p} q^{-k} b_{n,k}(q;x) \int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} f(t) d_q^R t. \tag{2.1}$$

Lemma 2.1. *Let $K_{n,p}^{(\alpha,\beta)}(f,q;x)$ be given by (2.1). Then the following equalities hold:*

(i) $K_{n,p}^{(\alpha,\beta)}(1,q;x) = 1$;

(ii) $K_{n,p}^{(\alpha,\beta)}(t,q;x) = \frac{1}{[n+1]_q+\beta}\left\{\frac{1}{[2]_q} + \alpha + \frac{2q}{[2]_q}[n+p]_q x\right\}$;

(iii) $K_{n,p}^{(\alpha,\beta)}(t^2,q;x) = \frac{1}{([n+1]_q+\beta)^2}\left\{q\left(1+\frac{2(q-1)}{[2]_q}+\frac{(q-1)^2}{[3]_q}\right)[n+p]_q[n+p-1]_q x^2 + \left(\frac{3q+1}{[2]_q} + \frac{4\alpha q}{[2]_q} + \frac{q^2-1}{[3]_q}\right)[n+p]_q x + \frac{1}{[3]_q} + \frac{2\alpha}{[2]_q} + \alpha^2\right\}$.

Proof. By definition of q-Riemann integral (1.2), we have

$$\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} d_q^R t = \frac{q^k}{[n+1]_q+\beta}; \tag{2.2}$$

$$\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} t d_q^R t = \frac{1}{([n+1]_q+\beta)^2}\left\{q^k([k]_q+\alpha) + \frac{q^{2k}}{[2]_q}\right\}; \tag{2.3}$$

$$\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} t^2 d_q^R t = \frac{1}{([n+1]_q+\beta)^3}\left\{q^k([k]_q+\alpha)^2 + \frac{2q^{2k}}{[2]}([k]_q+\alpha) + \frac{q^{3k}}{[3]_q}\right\}. \tag{2.4}$$

The following identities hold

$$\sum_{k=0}^{n+p} b_{n,k}(q;x) q^k = 1 - (1-q)[n+p]_q x; \tag{2.5}$$

$$\sum_{k=0}^{n+p} b_{n,k}(q;x) q^{2k} = 1 - (1-q^2)[n+p]_q x + q(1-q)^2 [n+p]_q [n+p-1]_q x^2. \tag{2.6}$$

Hence, by using equality $\sum_{k=0}^{n+p} b_{n,k}(q;x) = 1$ and equation (2.2), we get

$$K_{n,p}^{(\alpha,\beta)}(1,q;x) = 1.$$

By using relations (2.3) and (2.5) we have

$$\begin{aligned} K_{n,p}^{(\alpha,\beta)}(t,q;x) &= \frac{1}{[n+1]_q+\beta}\sum_{k=0}^{n+p} q^{-k} b_{n,k}(q;x)\left\{q^k([k]_q+\alpha)+\frac{q^{2k}}{[2]_q}\right\} \\ &= \frac{1}{[n+1]_q+\beta}\left\{\frac{1}{[2]_q}+\alpha+\frac{2q}{[2]_q}[n+p]_q x\right\}. \end{aligned} \tag{2.7}$$

Now, from the equations (2.4) and (2.6), we get

$$\begin{aligned} &K_{n,p}^{(\alpha,\beta)}(t^2,q;x) \\ &= \frac{1}{([n+1]_q+\beta)^2}\sum_{k=0}^{n+p} q^{-k} b_{n,k}(q;x)\left\{q^k([k]_q+\alpha)^2+\frac{2q^{2k}}{[2]}([k]_q+\alpha)+\frac{q^{3k}}{[3]_q}\right\} \\ &= \frac{1}{([n+1]_q+\beta)^2}\sum_{k=0}^{n+p} b_{n,k}(q;x)\left\{\left(\frac{1}{1-q}+\alpha\right)^2\right. \\ &\quad + 2q^k\left[\frac{1}{[2]_q}\left(\frac{1}{1-q}+\alpha\right)-\left(\frac{1}{1-q}+\alpha\right)\frac{1}{1-q}\right]+q^{2k}\left.\left(\frac{1}{(1-q)^2}-\frac{2}{(1-q)[2]_q}+\frac{1}{[3]_q}\right)\right\} \\ &= \frac{1}{([n+1]_q+\beta)^2}\left\{q\left(1+\frac{2(q-1)}{[2]_q}+\frac{(q-1)^2}{[3]_q}\right)[n+p]_q[n+p-1]_q x^2\right. \\ &\quad + \left.\left(\frac{3q+1}{[2]_q}+\frac{4\alpha q}{[2]_q}+\frac{q^2-1}{[3]_q}\right)[n+p]_q x+\frac{1}{[3]_q}+\frac{2\alpha}{[2]_q}+\alpha^2\right\}. \end{aligned}$$

□

Remark 2.2. From Lemma 2.1, we get

$$K_{n,p}^{(\alpha,\beta)}(t-x,q;x) = \frac{1+[2]_q\alpha}{[2]_q([n+1]_q+\beta)}+\left(\frac{2q}{[2]_q}\frac{[n+p]_q}{[n+1]_q+\beta}-1\right)x;$$

$$\begin{aligned} &K_{n,p}^{(\alpha,\beta)}\left((t-x)^2,q;x\right) \\ &= K_{n,p}^{(\alpha,\beta)}(t^2;x)-2xK_{n,p}^{(\alpha,\beta)}(t;x)+x^2K_{n,p}^{(\alpha,\beta)}(1;x) \\ &= \frac{1}{([n+1]_q+\beta)^2}\left\{\frac{q^2(4q^2+q+1)}{[2]_q[3]_q}[n+p]_q[n+p-1]_q x^2\right. \\ &\quad + \left.\frac{q(4q^2+5q+3)+4\alpha q(q^2+q+1)}{[2]_q[3]_q}[n+p]_q x+\frac{1}{[3]_q}+\frac{2\alpha}{[2]_q}+\alpha^2\right\} \\ &\quad - \frac{2x}{[n+1]_q+\beta}\left\{\frac{1}{[2]_q}+\alpha+\frac{2q}{[2]_q}[n+p]_q x\right\}+x^2 \end{aligned}$$

$$= \left[\frac{q^2(4q^2+q+1)}{[2]_q[3]_q}\frac{[n+p]_q[n+p-1]_q}{([n+1]_q+\beta)^2} - \frac{4q}{[2]_q}\frac{[n+p]_q}{[n+1]_q+\beta} + 1\right] x^2$$
$$+ \left[\frac{q(4q^2+5q+3)+4\alpha q(q^2+q+1)}{[2]_q[3]_q}\frac{[n+p]_q}{([n+1]_q+\beta)^2} - \frac{2(1+[2]_q\alpha)}{[2]_q([n+1]_q+\beta)}\right] x$$
$$+ \frac{1}{([n+1]_q+\beta)^2}\left(\frac{1}{[3]_q} + \frac{2\alpha}{[2]_q} + \alpha^2\right).$$

Lemma 2.3. *For $f \in C[0, p+1]$, we have*

$$\|K_{n,p}^{(\alpha,\beta)}(f, q; \cdot)\|_{C[0,1]} \leq \|f\|_{C[0,p+1]},$$

where $\|\cdot\|_{C[0,p+1]}$ is the sup-norm on $[0, p+1]$.

Proof. We have

$$\left|K_{n,p}^{(\alpha,\beta)}(f, q; \cdot)\right| \leq ([n+1]_q+\beta)\sum_{k=0}^{n+p} q^{-k} b_{n,k}(q; x)\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} |f(t)| d_q^R t$$
$$\leq \|f\|_{C[0,p+1]} K_{n,p}^{(\alpha,\beta)}(1, q; x) = \|f\|_{C[0,p+1]}.$$

□

Lemma 2.4. *For each $x \in [0, 1]$ and $0 < q < 1$, we have*

$$K_{n,p}^{(\alpha,\beta)}\left((t-x)^2, q; x\right) \leq 4C\frac{\varphi^2(x)}{[n+p]_q} + \frac{8(\alpha^2+3\beta^2+3[p]_q^2+4)}{([n+1]_q+\beta)^2}, \tag{2.8}$$

$$K_{n,p}^{(\alpha,\beta)}\left((t-x)^4, q; x\right) \leq 64\tilde{C}\frac{\varphi^2(x)}{[n+p]_q^2} + \frac{8^3(\alpha^4+27[p]_q^4+27\beta^4+28)}{([n+1]_q+\beta)^4}, \tag{2.9}$$

where $\varphi^2(x) = x(1-x)$ and $C, \tilde{C}$ are some constants.

Proof. We have

$$K_{n,p}^{(\alpha,\beta)}\left((t-x)^2, q; x\right)$$
$$= ([n+1]_q+\beta)\sum_{k=0}^{n+p} q^{-k} b_{n,k}(q; x)\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} (t-x)^2 d_q^R t$$
$$= (1-q)\sum_{k=0}^{n+p} b_{n,k}(q; x)\sum_{j=0}^{\infty}\left(\frac{[k]_q+\alpha}{[n+1]_q+\beta} + \frac{q^k}{[n+1]_q+\beta}q^j - x\right)^2 q^j$$
$$\leq 2(1-q)\sum_{k=0}^{n+p} b_{n,k}(q; x)\left[\sum_{j=0}^{\infty}\left(\frac{[k]_q+\alpha}{[n+1]_q+\beta} - x\right)^2 q^j + \sum_{j=0}^{\infty}\frac{q^{2k}q^{3j}}{([n+1]_q+\beta)^2}\right]$$
$$\leq 2\sum_{k=0}^{n+p} b_{n,k}(q; x)\left[\frac{[k]_q+\alpha}{[n+1]_q+\beta} - \frac{[k]_q}{[n+p]_q} - \left(x - \frac{[k]_q}{[n+p]_q}\right)\right]^2$$
$$+ \frac{2}{[3]_q}\sum_{k=0}^{n+p} b_{n,k}(q; x)\frac{q^{2k}}{([n+1]_q+\beta)^2}$$

$$\le 4\sum_{k=0}^{n+p} b_{n,k}(q;x)\left(\frac{[k]_q+\alpha}{[n+1]_q+\beta}-\frac{[k]_q}{[n+p]_q}\right)^2+4\sum_{k=0}^{n+p} b_{n,k}(q;x)\left(x-\frac{[k]_q}{[n+p]_q}\right)^2$$
$$+\frac{2}{[3]_q([n+1]_q+\beta)^2}$$
$$\le 8\sum_{k=0}^{n+p} b_{n,k}(q;x)\left(\frac{[k]_q+\alpha}{[n+1]_q+\beta}-\frac{[k]_q}{[n+1]_q+\beta}\right)^2$$
$$+8\sum_{k=0}^{n+p} b_{n,k}(q;x)\left(\frac{[k]_q}{[n+1]_q+\beta}-\frac{[k]_q}{[n+p]_q}\right)^2$$
$$+4B_{n+p}\left((t-x)^2,q;x\right)+\frac{2}{[3]_q}\frac{1}{([n+1]_q+\beta)^2},$$

where $B_n(f,q;x)=\sum_{k=0}^{n}\binom{n}{k}_q x^k(1-x)_q^{n-k}f\left([k]_q/[n]_q\right)$ is the q-Bernstein operators. On the other hand by [10], we have

$$|B_n\left((t-x)^m,q;x\right)|\le K_m\frac{x(1-x)}{[n]_q^{\lfloor (m+1)/2\rfloor}},$$

for some constant $K_m>0$, where $\lfloor a\rfloor$ is the integer part of $a\ge 0$. We find that

$$K_{n,p}^{(\alpha,\beta)}\left((t-x)^2,q;x\right)$$
$$\le\frac{8\alpha^2}{([n+1]_q+\beta)^2}+8\sum_{k=0}^{n+p} b_{n,k}(q;x)[k]_q^2\frac{(q^{n+1}[p]_q-q^{n+p}-\beta)^2}{[n+p]_q^2([n+1]_q+\beta)^2}$$
$$+4C\frac{\varphi^2(x)}{[n+p]_q}+\frac{2}{[3]_q([n+1]_q+\beta)^2}$$
$$\le\frac{8\alpha^2}{([n+1]_q+\beta)^2}+\frac{24([p]^2+1+\beta^2)}{([n+1]_q+\beta)^2}+4C\frac{\varphi^2(x)}{[n+p]_q}+\frac{2}{[3]_q([n+1]_q+\beta)^2}$$
$$\le 4C\frac{\varphi^2(x)}{[n+p]_q}+\frac{8(\alpha^2+3\beta^2+3[p]_q^2+4)}{([n+1]_q+\beta)^2}.$$

Also, we obtain

$$K_{n,p}^{(\alpha,\beta)}\left((t-x)^4,q;x\right)$$
$$=([n+1]_q+\beta)\sum_{k=0}^{n+p} q^{-k}b_{n,k}(q;x)\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}}(t-x)^4 d_q^R t$$
$$=(1-q)\sum_{k=0}^{n+p} b_{n,k}(q;x)\sum_{j=0}^{\infty}\left(\frac{[k]_q+\alpha}{[n+1]_q+\beta}+\frac{q^k}{[n+1]_q+\beta}q^j-x\right)^4 q^j$$
$$\le 8(1-q)\sum_{k=0}^{n+p} b_{n,k}(q;x)\sum_{j=0}^{\infty}\left(\frac{[k]_q+\alpha}{[n+1]_q+\beta}-x\right)^4 q^j$$
$$+8(1-q)\sum_{k=0}^{n+p} b_{n,k}(q;x)\sum_{j=0}^{\infty}\left(\frac{q^k}{[n+1]_q+\beta}\right)^4 q^{5j}$$

6

$$
\begin{aligned}
&= 8\sum_{k=0}^{n+p} b_{n,k}(q;x)\left(\frac{[k]_q+\alpha}{[n+1]_q+\beta}-x\right)^4 + \frac{8}{[5]_q}\sum_{k=0}^{n+p} b_{n,k}(q;x)\left(\frac{q^k}{[n+1]_q+\beta}\right)^4 \\
&= 8\sum_{k=0}^{n+p} b_{n,k}(q;x)\left[\frac{[k]_q+\alpha}{[n+1]_q+\beta}-\frac{[k]_q}{[n+p]_q}-\left(x-\frac{[k]_q}{[n+p]_q}\right)\right]^4 \\
&\quad + \frac{8}{[5]_q}\sum_{k=0}^{n+p} b_{n,k}(q;k)\left(\frac{q^k}{[n+1]_q+\beta}\right)^4 \\
&\le 64B_{n+p}\left((t-x)^4,q;x\right) + 64\sum_{k=0}^{n+p} b_{n,k}(q;x)\left[\frac{[k]_q+\alpha}{[n+1]_q+\beta}-\frac{[k]_q}{[n+p]_q}\right]^4 \\
&\quad + \frac{8}{[5]_q([n+1]_q+\beta)^4} \\
&\le 64B_{n+p}\left((t-x)^4,q;x\right) + 8^3\sum_{k=0}^{n+p} b_{n,k}(q;x)\left(\frac{[k]_q+\alpha}{[n+1]_q+\beta}-\frac{[k]_q}{[n+1]_q+\beta}\right)^4 \\
&\quad + 8^3\sum_{k=0}^{n+p} b_{n,k}(q;x)\left(\frac{[k]_q}{[n+1]_q+\beta}-\frac{[k]_q}{[n+p]_q}\right)^4 + \frac{8}{[5]_q([n+1]_q+\beta)^4} \\
&\le 64\tilde{C}\frac{\varphi^2(x)}{[n+p]_q^2} + \frac{8^3\alpha^4}{([n+1]_q+\beta)^4} + 8^3\sum_{k=0}^{n+p} b_{n,k}(q;x)[k]_q^4\frac{(q^{n+1}[p]_q-q^{n+p}-\beta)^4}{[n+p]_q^4([n+1]_q+\beta)^4} \\
&\quad + \frac{8}{[5]_q([n+1]_q+\beta)^4} \\
&\le 64\tilde{C}\frac{\varphi^2(x)}{[n+p]_q^2} + \frac{8^3\alpha^4}{([n+1]_q+\beta)^4} + 24^3\frac{[p]_q^4+1+\beta^4}{([n+1]_q+\beta)^4} + \frac{8}{[5]_q([n+1]_q+\beta)^4} \\
&\le 64\tilde{C}\frac{\varphi^2(x)}{[n+p]_q^2} + \frac{8^3(\alpha^4+27[p]_q^4+27\beta^4+28)}{([n+1]_q+\beta)^4}.
\end{aligned}
$$

□

3 Direct theorems

In this section we propose to study some approximation properties of the Stancu-Schurer-Kantorovich operators defined in (2.1). First, we prove the basic convergence theorem of $K_{n,p}^{(\alpha,\beta)}$ and then obtain the rate convergence of these operators in term of the modulus of continuity. Further, we study local direct results for the q-analogue of Stancu-Schurer-Kantorovich operators.

Theorem 3.1. *Let $(q_n)_n$, $0 < q_n < 1$ be a sequence satisfying the following conditions*

$$\lim_{n\to\infty} q_n = 1, \quad \lim_{n\to\infty} q_n^n = a, \quad a \in [0,1). \tag{3.1}$$

Then for any $f \in C[0,p+1]$, the sequence $K_{n,p}^{(\alpha,\beta)}(f,q_n;x)$ converges to f uniformly on $[0,1]$.

Proof. From (3.1) we obtain $[n+1]_{q_n} \to \infty$ as $n \to \infty$. Further $\frac{[n+p]_{q_n}}{[n+1]_{q_n}} \to 1$, hence $K_{n,p}^{(\alpha,\beta)}(1, q_n; x) \to 1$, $K_{n,p}^{(\alpha,\beta)}(t, q_n; x) \to x$ and $K_{n,p}^{(\alpha,\beta)}(t^2, q_n; x) \to x^2$ uniformly on $[0,1]$ as $n \to \infty$. Therefore, using the Bohman-Korovkin theorem implies that $K_{n,p}(f, q_n; \cdot)$ converges to f uniformly on $[0,1]$. □

Let us consider the following K-functional

$$K_2(f,\delta) := \inf\left\{\|f-g\| + \delta\|g''\|, g \in C^2[0,p+1]\right\}, \quad \text{where } \delta \geq 0. \tag{3.2}$$

It is known (see [6]) there exist an absolute constant $C > 0$ such that

$$K_2(f,\delta) \leq C\omega_2(f,\sqrt{\delta}), \tag{3.3}$$

where the second order modulus of smoothness for $f \in C[0,p+1]$ is defined as

$$\omega_2(f,\delta) := \sup_{0<h<\delta;\, x,x+2h\in[0,p+1]} |f(x+2h) - 2f(x+h) + f(x)|, \quad \text{where } \delta > 0.$$

The usual modulus of continuity for $f \in C[0,p+1]$ is defined as

$$\omega(f,\delta) := \sup_{0<h<\delta;\, x,x+h\in[0,p+1]} |f(x+h) - f(x)|.$$

Theorem 3.2. *Let $(q_n)_n$ be a sequence satisfying conditions (3.1) and $f \in C[0,1+p]$. Then*

$$\left|K_{n,p}^{(\alpha,\beta)}(f, q_n; x) - f(x)\right| \leq 2\omega(f,\delta_n),$$

where

$$\delta_n = \sqrt{\frac{C}{[n+p]_{q_n}} + \frac{8(\alpha^2 + 3\beta^2 + 3[p]_{q_n}^2 + 4)}{([n+1]_{q_n} + \beta)^2}},$$

and C is a constant.

Proof. For any $t, x \in [a,b]$, it is known that

$$|f(t) - f(x)| \leq \omega(f,\delta)\left(\frac{(t-x)^2}{\delta^2} + 1\right).$$

Therefore, we obtain

$$\left|K_{n,p}^{(\alpha,\beta)}(f, q_n; x) - f(x))\right| \leq \omega(f,\delta)\left(1 + \frac{1}{\delta^2} K_{n,p}^{(\alpha,\beta)}\left((t-x)^2, q_n; x\right)\right).$$

By using the relation (2.8), we obtain the required result. □

In what follows, we give estimate of the rate of convergence by means of the Lipschitz function for the operators defined in (2.1). Let

$$Lip_M(\gamma) = \{f \in C[0,p+1], |f(t) - f(x)| \leq M|t-x|^\gamma\}, 0 < \gamma \leq 1,$$

be the Lipschitz class.

Theorem 3.3. *Let $(q_n)_n$ be a sequence satisfying conditions* (3.1) *and $f \in Lip_M(\gamma)$. Then*

$$|K_{n,p}^{(\alpha,\beta)}(f, q_n; x) - f(x)| \leq M(\delta_n(x))^{\gamma/2},$$

where $\delta_n(x) = K_{n,p}^{(\alpha,\beta)}\left((t-x)^2, q_n; x\right)$.

Proof. Since $K_{n,p}^{(\alpha,\beta)}(e_0, q_n; \cdot) = e_0$ and $f \in Lip_M(\gamma)$, we have

$$\left|K_{n,p}^{(\alpha,\beta)}(f, q_n; x) - f(x)\right| \leq K_{n,p}^{(\alpha,\beta)}\left(|f(t) - f(x)|, q_n; x\right) \leq MK_{n,p}^{(\alpha,\beta)}\left(|t-x|^{\gamma}, q_n; x\right).$$

Applying the Hölder's inequality with $p = \frac{2}{\gamma}$ and $q = \frac{2}{2-\gamma}$, we get

$$\left|K_{n,p}^{(\alpha,\beta)}(f, q_n; x) - f(x)\right| \leq MK_{n,p}^{(\alpha,\beta)}\left(|t-x|^2, q_n; x\right)^{\frac{\gamma}{2}} = M\left(\delta_n(x)\right)^{\frac{\gamma}{2}}.$$

□

Theorem 3.4. *Let $(q_n)_n$ be a sequence satisfying conditions* (3.1) *and $f \in C[0, p+1]$. Then, for every $x \in [0,1]$ we have*

$$\left|K_{n,p}^{(\alpha,\beta)}(f, q_n; x) - f(x)\right| \leq 4K_2\left(f, \psi_{n,p}(q_n; x)\right) + \omega\left(f, \gamma_{n,p}\right),$$

where C is a constant and

$$\psi_{n,p}(q_n; x) = \frac{C\varphi^2(x)}{[n+p]_{q_n}} + \frac{12[p]_{q_n}^2 + 7\beta^2 + 3(\alpha+1)^2 + 10}{([n+1]_{q_n} + \beta)^2},$$

$$\gamma_{n,p} = \frac{\alpha+\beta+2+2[p]_{q_n}}{[n+1]_{q_n}+\beta}, \quad \varphi^2(x) = x(1-x).$$

Proof. We define the auxiliary operators

$$K_{n,p}^{*(\alpha,\beta)}(f, q_n; x) = K_{n,p}^{(\alpha,\beta)}(f, q_n; x) + f(x) - f(a_n x + b_n), \tag{3.4}$$

where

$$a_n = \frac{2q_n}{[2]_{q_n}} \frac{[n+p]_{q_n}}{[n+1]_{q_n} + \beta},\ b_n = \frac{1}{[n+1]_{q_n} + \beta}\left(\frac{1}{[2]_{q_n}} + \alpha\right).$$

From Lemma 2.1 we obtain

$$K_{n,p}^{*(\alpha,\beta)}(1, q_n; x) = 1 \text{ and } K_{n,p}^{*(\alpha,\beta)}(t, q_n; x) = x.$$

Using Taylor's formula,

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-s)g''(s)ds,\ g \in C^2[0, p+1],$$

we get

$$\begin{aligned}
&K_{n,p}^{*(\alpha,\beta)}(g, q_n; x) - g(x)\\
&= g'(x)K_{n,p}^{*(\alpha,\beta)}(t-x, q_n; x) + K_{n,p}^{*(\alpha,\beta)}\left(\int_x^t (t-s)g''(s)ds, q_n; x\right)\\
&= K_{n,p}^{*(\alpha,\beta)}\left(\int_x^t (t-s)g''(s)ds, q_n; x\right)\\
&= K_{n,p}^{(\alpha,\beta)}\left(\int_x^t (t-s)g''(s)ds, q_n; x\right) - \int_x^{a_n x + b_n} (a_n x + b_n - s)g''(s)ds.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\left|K_{n,p}^{*(\alpha,\beta)}(g,q_n;x)-g(x)\right| \\
&\leq K_{n,p}^{(\alpha,\beta)}\left(\left|\int_x^t (t-s)g''(s)ds\right|,q_n;x\right)+\left|\int_x^{a_nx+\beta}(a_nx+b_n-s)g''(s)ds\right| \\
&\leq K_{n,p}^{(\alpha,\beta)}\left((t-x)^2,q_n;x\right)\|g''\|_{C[0,p+1]}+(a_nx+b_n-x)^2\|g''\|_{C[0,p+1]}. \qquad (3.5)
\end{aligned}$$

Using the relation (3.4) we obtain

$$\begin{aligned}
\left|K_{n,p}^{(\alpha,\beta)}(f,q_n;x)-f(x)\right| \leq{}& \left|K_{n,p}^{*(\alpha,\beta)}(f-g,q_n;x)\right|+\left|K_{n,p}^{*(\alpha,\beta)}(g,q_n;x)-g(x)\right| \\
&+|f(x)-g(x)|+|f(a_nx+b_n)-f(x)|.
\end{aligned}$$

Since $\|K_{n,p}^{*(\alpha,\beta)}\|_{C[0,1]}\leq 3\|f\|_{C[0,p+1]}$, and using (3.5) we have

$$\begin{aligned}
&\left|K_{n,p}^{(\alpha,\beta)}(f,q_n;x)-f(x)\right| \\
&\leq 4\|f-g\|_{C[0,p+1]}+\left[K_{n,p}^{(\alpha,\beta)}\left((t-x)^2,q_n;x\right)+(a_nx+b_n-x)^2\right]\|g''\|_{C[0,p+1]} \\
&\quad +\omega\left(f,|(a_n-1)x+b_n|\right).
\end{aligned}$$

Since

$$\begin{aligned}
(a_nx+b_n-x)^2 &= \left[\frac{2q_n}{[2]_{q_n}}\frac{[n+p]_{q_n}}{[n+1]_{q_n}+\beta}x+\frac{1}{[n+1]_{q_n}+\beta}\left(\alpha+\frac{1}{[2]_{q_n}}\right)-x\right]^2 \\
&= \frac{1}{([n+1]_{q_n}+\beta)^2}\left\{\left(\frac{2q_n}{[2]_{q_n}}[n+p]_{q_n}-[n+1]_{q_n}\right)x-\beta x+\alpha+\frac{1}{[2]_{q_n}}\right\}^2 \\
&\leq \frac{3}{([n+1]_{q_n}+\beta)^2}\left\{\left(\frac{-1+2q_n^{n+1}[p]_{q_n}-q_n^{n+1}}{1+q_n}\right)^2+\beta^2+\left(\alpha+\frac{1}{[2]_{q_n}}\right)^2\right\} \\
&\leq \frac{6+24[p]_{q_n}^2+3\beta^2+3(\alpha+1)^2}{([n+1]_{q_n}+\beta)^2},
\end{aligned}$$

we have

$$\begin{aligned}
K_{n,p}^{(\alpha,\beta)}\left((t-x)^2,q_n;x\right)+(a_nx+b_n-x)^2 &\leq \frac{4C\varphi^2(x)}{[n+p]_{q_n}^2}+\frac{4(12[p]_{q_n}^2+7\beta^2+3(\alpha+1)^2)+10)}{([n+1]_{q_n}+\beta)^2} \\
&\leq 4\psi_{n,p}(q_n;x).
\end{aligned}$$

Also $|(a_n-1)x+b_n|\leq\gamma_{n,p}$. Therefore

$$\left|K_{n,p}^{(\alpha,\beta)}(f,q_n;x)-f(x)\right|\leq 4\|f-g\|_{C[0,p+1]}+4\psi_{n,p}(q_n;x)\|g''\|_{C[0,p+1]}+\omega(f,\gamma_{n,p}).$$

Taking the infimum over all $g\in C^2[0,p+1]$ and using (3.2), the proof of the theorem is completed. □

4 A Voronovskaya theorem for q-Stancu-Schurer-Kantorovich operators

In this section we shall establish a Voronovskaja type theorem for q-Stancu-Schurer-Kantorovich operators. First, we need the auxiliary results contained in the following lemmas.

Lemma 4.1. *Let $(q_n)_n$ be a sequence satisfying conditions* (3.1). *Then we have*

$$\lim_{n\to\infty}[n]_{q_n}K_{n,p}^{(\alpha,\beta)}(t-x,q_n;x)=-\frac{1+a-2ap+2\beta}{2}x+\alpha+\frac{1}{2},$$
$$\lim_{n\to\infty}[n]_{q_n}K_{n,q_n}^{(\alpha,\beta)}((t-x)^2,q_n;x)=-x^2+x.$$

Proof. Using the formulas for $K_{n,p}^{(\alpha,\beta)}(t^i,q_n;x), i=0,1,2$ given in Lemma 2.1, we get

$$\begin{aligned}
&\lim_{n\to\infty}[n]_{q_n}K_{n,p}^{(\alpha,\beta)}(t-x,q_n;x)\\
&=\lim_{n\to\infty}[n]_{q_n}\left\{\left(\frac{2q_n}{[2]_{q_n}}\frac{[n+p]_{q_n}}{[n+1]_{q_n}+\beta}-1\right)x+\frac{1}{[n+1]_{q_n}+\beta}\left(\frac{1}{[2]_{q_n}}+\alpha\right)\right\}\\
&=\lim_{n\to\infty}\left\{\frac{[n]_{q_n}}{[2]_{q_n}([n+1]_{q_n}+\beta)}(-1-q_n^{n+p+1}+q_n^{n+1}(1+q_n)[p]_{q_n}-[2]_{q_n}\beta)x\right.\\
&\quad\left.+\frac{\alpha[n]_{q_n}}{[n+1]_{q_n}+\beta}+\frac{[n]_{q_n}}{[2]_{q_n}([n+1]_{q_n}+\beta)}\right\}\\
&=-\frac{1+a-2ap+2\beta}{2}x+\alpha+\frac{1}{2},
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\to\infty}[n]_{q_n}K_{n,p}^{(\alpha,\beta)}((t-x)^2,q_n;x)\\
&=\lim_{n\to\infty}[n]_{q_n}\left\{K_{n,p}^{(\alpha,\beta)}(t^2,q_n;x)-x^2-2xK_{n,p}^{(\alpha,\beta)}(t-x,q_n;x)\right\}\\
&=\lim_{n\to\infty}[n]_{q_n}\left(\frac{4q_n^4+q_n^3+q_n^2}{[2]_{q_n}[3]_{q_n}}\cdot\frac{[n+p-1]_{q_n}[n+p]_{q_n}}{([n+1]_{q_n}+\beta)^2}-1\right)x^2\\
&\quad+\lim_{n\to\infty}\frac{q_n(4q_n^2+5q_n+3)+4\alpha q_n(q_n^2+q_n+1))}{[2]_{q_n}[3]_{q_n}([n+1]_{q_n}+\beta)^2}[n]_{q_n}[n+p]_{q_n}x\\
&\quad+\lim_{n\to\infty}\frac{[n]_{q_n}}{([n+1]_{q_n}+\beta)^2}\left(\alpha^2+\frac{2\alpha}{[2]_{q_n}}+\frac{1}{[3]_{q_n}}\right)-\lim_{n\to\infty}2x[n]_{q_n}K_{n,p}^{(\alpha,\beta)}(t-x,q_n;x)\\
&=-x^2+x.
\end{aligned}$$

□

Lemma 4.2. *Let $(q_n)_n$ be a sequence satisfying conditions* (3.1). *Then for each $x\in[0,1]$ we have*

$$K_{n,p}^{(\alpha,\beta)}\left((t-x)^2,q_n;x\right)=O\left(\frac{1}{[n]_{q_n}}\right);\quad K_{n,p}^{(\alpha,\beta)}\left((t-x)^4,q_n;x\right)=O\left(\frac{1}{[n]_{q_n}^2}\right).$$

Proof. This result follows from Lemma 2.4. □

The main result of this section is the following Voronovskaja type theorem:

Theorem 4.3. *Let* $(q_n)_n$ *be a sequence satisfying conditions* (3.1) *and* $f'' \in C[0, p+1]$*. Then we have*

$$\lim_{n\to\infty}[n]_{q_n}(K_{n,p}^{(\alpha,\beta)}(f,q;x)-f(x)) = \left(-\frac{1+a-2ap+2\beta}{2}x+\alpha+\frac{1}{2}\right)f'(x)+\frac{1}{2}\left(-x^2+x\right)f''(x).$$

Proof. From the Taylor's theorem, we have

$$f(t) = f(x)+(t-x)f'(x)+\frac{1}{2}(t-x)^2f''(x)+\xi(t,x)(t-x)^2, \tag{4.1}$$

where the function $\xi(\cdot,x)$ is the Peano form of remainder, $\xi(\cdot,x)\in C[0,p+1]$ and $\xi(t,x)\to 0$ as $t\to x$.

Applying $K_{n,p}^{(\alpha,\beta)}$ on both side of (4.1), we obtain

$$\begin{aligned}&[n]_{q_n}\left(K_{n,p}^{(\alpha,\beta)}(f,q_n;x)-f(x)\right)\\&=[n]_{q_n}f'(x)K_{n,p}^{(\alpha,\beta)}(t-x,q_n;x)+\frac{1}{2}[n]_{q_n}f''(x)K_{n,p}^{(\alpha,\beta)}\left((t-x)^2,q_n;x\right)\\&\quad+[n]_{q_n}K_{n,p}^{(\alpha,\beta)}\left(\xi(t,x)(t-x)^2,q_n;x\right).\end{aligned} \tag{4.2}$$

Now, we shall show that

$$\lim_{n\to\infty}[n]_{q_n}K_{n,p}^{(\alpha,\beta)}\left(\xi(t,x)(t-x)^2,q_n;x\right)=0. \tag{4.3}$$

From the Cauchy-Schwarz inequality, we have

$$K_{n,p}^{(\alpha,\beta)}\left(\xi(t,x)(t-x)^2,q_n;x\right)\le\sqrt{K_{n,p}^{(\alpha,\beta)}\left(\xi^2(t,x),q_n;x\right)}\sqrt{K_{n,p}^{(\alpha,\beta)}\left((t-x)^4,q_n;x\right)}.$$

From Theorem 3.1, we have

$$\lim_{n\to\infty}K_{n,p}^{(\alpha,\beta)}\left(\xi^2(t,x),q_n;x\right)=\xi^2(x,x)=0$$

uniformly with respect to $x\in[0,1]$. Since $K_{n,p}^{(\alpha,\beta)}\left((t-x)^4,q_n;x\right)=O\left(\frac{1}{[n]_{q_n}^2}\right)$ (see Lemma 2.4), it follows (4.3). In view of Lemma 4.1 the theorem is proved. □

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Mathematical analysis of a cell mediated immunity in a virus dynamics model with nonlinear infection rate and removal

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Abstract

In this paper, we investigate the dynamical behavior of a nonlinear model for viral infection with Cytotoxic T Lymphocyte (CTL) immune response. The model is a generalization of several models presented in the literature by considering more general functions for the: (i) intrinsic growth rate of uninfected cells; (ii) incidence rate of infection; (iii) natural death rate of infected cells; (iv) rate at which the infected cells are killed by CTL cells; (v) production and removal rates of viruses; (vi) activation and natural death rates of CTLs. We derive two threshold parameters R_0 (the basic infection reproduction number) and R_1 (the CTL immune response activation number) and establish a set of conditions on the general functions which are sufficient to determine the global dynamics of the model. By using suitable Lyapunov functions and LaSalle's invariance principle, we prove the global asymptotic stability of all equilibria of the model.

Keywords: Viral infection; global stability; CTL immune response; Lyapunov functional.

1 Introduction

During the last decades, several mathematical models have been proposed to describe the interaction of the virus with target cells (see e.g. [1]-[15]. The immune response is universal and necessary to eliminate or control the disease after viral infection. The Cytotoxic T Lymphocyte (CTL) cells are responsible to attack and kill the infected cells. Several viral infection models have been introduced in the literature to model the CTL immune response to several diseases [16]-[20]. The basic virus dynamics model with CTL immune response has four state variables: x, the population of uninfected target cells; y, the population of infected cells; v, the population of free virus particles in the blood; and z, the population of CTL cells. The model equations are as follows [1]:

$$\dot{x} = s - dx - \beta xv, \tag{1}$$

$$\dot{y} = \beta xv - ay - qyz, \tag{2}$$

$$\dot{v} = ky - cv, \tag{3}$$

$$\dot{z} = ryz - \mu z. \tag{4}$$

Parameters s, k and r represent, respectively, the rate at which new healthy cells are generated from the source within the body, the generation rate constant of free viruses produced from the infected cells and the rate at which the CTL cells are produced. Parameters d, a, c and μ are the natural death rate constants of the uninfected target cells, infected cells, free virus particles and CTL cells, respectively. Parameter β is the infection rate constant and q is the removal rate constant of the infected cells due to the CTLs. All the parameters given in model (1)-(4) are positive. Our aim in this paper is to propose and analyze a nonlinear viral infection model which generalizes model (1)-(4) and several models presented in the literature. We consider the following nonlinear viral infection model with CTL immune response.

$$\dot{x} = n(x) - \psi(x, y, v), \tag{5}$$

$$\dot{y} = \psi(x, y, v) - a\varphi_1(y) - q\varphi_3(z)\varphi_1(y), \tag{6}$$

$$\dot{v} = k\varphi_1(y) - c\varphi_2(v), \tag{7}$$

$$\dot{z} = r\varphi_3(z)\varphi_1(y) - \mu\varphi_3(z), \tag{8}$$

where $n(x)$ represents the intrinsic growth rate of uninfected cells accounting for both production and natural mortality; $\psi(x, y, v)$ denotes the incidence rate of infection; $a\varphi_1(y)$ refers to the natural death rate of infected cells; $q\varphi_3(z)\varphi_1(y)$ represents the rate at which the infected cells are killed by the CTL cells; $k\varphi_1(y)$ denotes the generation rate of free virus particles; $c\varphi_2(v)$ accounts for the removal rate of free virus particles; $r\varphi_3(z)\varphi_1(y)$ and $\mu\varphi_3(z)$ refer to the activation and natural death rates of CTLs, respectively. Functions n, ψ, φ_i, $i = 1, 2, 3$ are continuously differentiable and satisfy the following assumptions:

Assumption A1. (i) there exists x_0 such that $n(x_0) = 0$, $n(x) > 0$ for $x \in [0, x_0)$,

(ii) $n'(x) < 0$ for all $x > 0$,

(iii) there are s, $\bar{s} > 0$ such that $n(x) \leq s - \bar{s}x$ for $x \geq 0$.

Assumption A2. (i) $\psi(x, y, v) > 0$ and $\psi(0, y, v) = \psi(x, y, 0) = 0$ for all $x > 0$, $y \geq 0$, $v > 0$,

(ii) $\dfrac{\partial\psi(x, y, v)}{\partial x} > 0$, $\dfrac{\partial\psi(x, y, v)}{\partial y} < 0$, $\dfrac{\partial\psi(x, y, v)}{\partial v} > 0$ and $\dfrac{\partial\psi(x, 0, 0)}{\partial v} > 0$ for all $x > 0$, $y \geq 0$, $v > 0$,

(iii) $\dfrac{d}{dx}\left(\dfrac{\partial\psi(x, 0, 0)}{\partial v}\right) > 0$ for all $x > 0$.

Assumption A3. (i) $\varphi_j(u) > 0$ for all $u > 0$, $\varphi_j(0) = 0$, $j = 1, 2, 3$,

(ii) $\varphi_j'(u) > 0$, for all $u > 0$, $j = 1, 3$, $\varphi_2'(u) > 0$, for all $u \geq 0$,

(iii) there are $\alpha_j \geq 0$, $j = 1, 2, 3$ such that $\varphi_j(u) \geq \alpha_j u$, for all $u \geq 0$.

Assumption A4. $\dfrac{\psi(x, y, v)}{\varphi_2(v)}$ is decreasing with respect to v for all $v > 0$.

1.1 Properties of solutions

In this subsection, we study some properties of the solution of the model such as the non-negativity and boundedness of solutions.

Proposition. Assume that Assumptions A1-A3 are satisfied. Then there exist positive numbers L_i, $i = 1, 2, 3$, such that the compact set

$$\Gamma = \left\{(x, y, v, z) \in \mathbb{R}^4_{\geq 0} : 0 \leq x, y \leq L_1, 0 \leq v \leq L_2, 0 \leq z \leq L_3\right\}$$

is positively invariant.

Proof. We have

$$\dot{x} \mid_{x=0} = n(0) > 0, \qquad \dot{y} \mid_{y=0} = \psi(x, 0, v) \geq 0 \text{ for all } x \geq 0, v \geq 0,$$

$$\dot{v} \mid_{v=0} = k\varphi_1(y) \geq 0 \text{ for all } y \geq 0, \qquad \dot{z} \mid_{z=0} = 0.$$

Hence, the orthant $\mathbb{R}^4_{\geq 0}$ is positively invariant for system (5)-(8). Next, we show that the solutions of the system are bounded. Let $T(t) = x(t) + y(t) + \frac{a}{2k}v(t) + \frac{q}{r}z(t)$, then

$$\begin{aligned}\dot{T}(t) &= n(x) - \frac{a}{2}\varphi_1(y) - \frac{ac}{2k}\varphi_2(v) - \frac{q\mu}{r}\varphi_3(z) \leq s - \bar{s}x - \frac{a}{2}\alpha_1 y - \frac{ac}{2k}\alpha_2 v - \frac{q\mu}{r}\alpha_3 z \\ &\leq s - \sigma\left(x + y + \frac{a}{2k}v + \frac{q}{r}z\right) = s - \sigma T(t),\end{aligned}$$

where $\sigma = \min\{\bar{s}, \frac{a}{2}\alpha_1, c\alpha_2, \mu\alpha_3\}$. Then,

$$T(t) \leq T(0)e^{-\sigma t} + \frac{s}{\sigma}\left(1 - e^{-\sigma t}\right) = e^{-\sigma t}\left(T(0) - \frac{s}{\sigma}\right) + \frac{s}{\sigma}.$$

Hence, $0 \leq T(t) \leq L_1$ if $T(0) \leq L_1$ for $t \geq 0$ where $L_1 = \frac{s}{\sigma}$. It follows that, $0 \leq x(t), y(t) \leq L_1$, $0 \leq v(t) \leq L_2$ and $0 \leq z(t) \leq L_3$ for all $t \geq 0$ if $x(0) + y(0) + \frac{a}{2k}v(0) + \frac{q}{r}z(0) \leq L_1$, where $L_2 = \dfrac{2kL_1}{a}$ and $L_3 = \dfrac{rL_1}{q}$. Therefore, $x(t), y(t), v(t)$ and $z(t)$ are all bounded. □

1.2 The equilibria and threshold parameters

In this subsection we calculate the equilibria of the model and derive two threshold parameters.

Lemma. Assume that Assumptions A1-A4 are satisfied, then there exist two threshold parameters $R_0 > 0$ and $R_1 > 0$ with $R_1 < R_0$ such that

(i) if $R_0 \leq 1$, then there exists only one positive equilibrium $E_0 \in \Gamma$,

(ii) if $R_1 \leq 1 < R_0$, then there exist only two positive equilibria $E_0 \in \Gamma$ and $E_1 \in \Gamma$, and

(iii) if $R_1 > 1$, then there exist three positive equilibria $E_0 \in \Gamma$, $E_1 \in \Gamma$ and $E_2 \in \overset{\circ}{\Gamma}$.

Proof. At any equilibrium we have

$$n(x) - \psi(x, y, v) = 0, \tag{9}$$

$$\psi(x, y, v) - a\varphi_1(y) - q\varphi_1(y)\varphi_3(z) = 0, \tag{10}$$

$$k\varphi_1(y) - c\varphi_2(v) = 0, \tag{11}$$

$$(r\varphi_1(y) - \mu)\,\varphi_3(z) = 0. \tag{12}$$

From Eq. (12), either $\varphi_3(z) = 0$ or $\varphi_3(z) \neq 0$. If $\varphi_3(z) = 0$, then from Assumption A3 we get, $z = 0$ and from Eqs. (9)-(11) we have

$$n(x) = \psi(x, y, v) = a\varphi_1(y) = \frac{ac\varphi_2(v)}{k}. \tag{13}$$

From Eq. (13), we have $\varphi_1(y) = \frac{n(x)}{a}$, $\varphi_2(v) = \frac{kn(x)}{ac}$. Since φ_1, φ_2 are continuous and strictly increasing functions with $\varphi_1(0) = \varphi_2(0) = 0$, then φ_1^{-1}, φ_2^{-1} exist and they are also continuous and strictly increasing [21]. Let $\varkappa_1(x) = \varphi_1^{-1}\left(\frac{n(x)}{a}\right)$ and $\varkappa_2(x) = \varphi_2^{-1}\left(\frac{kn(x)}{ac}\right)$, then

$$y = \varkappa_1(x), \;\; v = \varkappa_2(x). \tag{14}$$

Obviously from Assumption A1, $\varkappa_1(x), \varkappa_2(x) > 0$ for $x \in [0, x_0)$ and $\varkappa_1(x_0) = \varkappa_2(x_0) = 0$. Substituting from Eq. (14) into Eq. (13) we get

$$\psi\left(x, \varkappa_1(x), \varkappa_2(x)\right) - \frac{ac}{k}\varphi_2(\varkappa_2(x)) = 0. \tag{15}$$

We note that, $x = x_0$ is a solution of Eq. (15). Then, from Eq. (14) we have $y = v = 0$, and this leads to the infection-free equilibrium $E_0 = (x_0, 0, 0, 0)$. Let

$$\Phi_1\left(x\right) = \psi\left(x, \varkappa_1(x), \varkappa_2(x)\right) - \frac{ac}{k}\varphi_2(\varkappa_2(x)) = 0.$$

Then from Assumptions A1-A3, we have

$$\Phi_1(0) = -\frac{ac}{k}\varphi_2(\varkappa_2(0)) < 0,$$
$$\Phi_1(x_0) = \psi\left(x_0, 0, 0\right) - \frac{ac}{k}\varphi_2(0) = 0.$$

Moreover,

$$\Phi_1'\left(x_0\right) = \frac{\partial\psi(x_0, 0, 0)}{\partial x} + \varkappa_1'(x_0)\frac{\partial\psi(x_0, 0, 0)}{\partial y} + \varkappa_2'(x_0)\frac{\partial\psi(x_0, 0, 0)}{\partial v} - \frac{ac}{k}\varphi_2'(0)\varkappa_2'(x_0).$$

Assumption A2 implies that $\frac{\partial\psi(x_0,0,0)}{\partial x} = \frac{\partial\psi(x_0,0,0)}{\partial y} = 0$. Also, from Assumption A3, we have $\varphi_2'(0) > 0$, then

$$\Phi_1'\left(x_0\right) = \frac{ac}{k}\varkappa_2'(x_0)\varphi_2'(0)\left(\frac{k}{ac\varphi_2'(0)}\frac{\partial\psi(x_0, 0, 0)}{\partial v} - 1\right).$$

From Eq. (14), we get

$$\Phi_1'\left(x_0\right) = n'(x_0)\left(\frac{k}{ac\varphi_2'(0)}\frac{\partial\psi(x_0, 0, 0)}{\partial v} - 1\right).$$

From Assumption A1, we have $n'(x_0) < 0$. Therefore, if $\dfrac{k}{ac\varphi_2'(0)}\dfrac{\partial\psi(x_0, 0, 0)}{\partial v} > 1$, then $\Phi_1'\left(x_0\right) < 0$ and there exists a $x_1 \in (0, x_0)$ such that $\Phi_1(x_1) = 0$. From Eq. (14), we have $y_1 = \varkappa_1(x_1) > 0$ and $v_1 = \varkappa_2(x_1) > 0$.

It follows that, a chronic-infection equilibrium without CTL immune response $E_1 = (x_1, y_1, v_1, 0)$ exists when $\frac{k}{ac\varphi_2'(0)}\frac{\partial\psi(x_0,0,0)}{\partial v} > 1$. Let us define

$$R_0 = \frac{k}{ac\varphi_2'(0)}\frac{\partial\psi(x_0,0,0)}{\partial v},$$

which describes the average number of newly infected cells generated from one infected cell at the beginning of the infectious process. The other possibility of Eq. (12) is $\varphi_3(z) \neq 0$ which leads to $y_2 = \varphi_1^{-1}\left(\frac{\mu}{r}\right) > 0$ and $v_2 = \varphi_2^{-1}\left(\frac{k\mu}{cr}\right) > 0$. Substituting $y = y_2$ and $v = v_2$ in Eq. (9), we letting

$$\Phi_2(x) = n(x) - \psi(x, y_2, v_2) = 0.$$

Clearly,

$$\Phi_2(0) = n(0) > 0 \text{ and } \Phi_2(x_0) = -\psi(x_0, y_2, v_2) < 0.$$

According to Assumptions A1 and A2, $\Phi_2(x)$ is a strictly decreasing function of x. Thus, there exists a unique $x_2 \in (0, x_0)$ such that $\Phi_2(x_2) = 0$. Now from Eq. (10) we have

$$z_2 = \varphi_3^{-1}\left(\frac{a}{q}\left(\frac{k\psi(x_2,y_2,v_2)}{ac\varphi_2(v_2)} - 1\right)\right).$$

From Assumption A3, we have if $\frac{k\psi(x_2,y_2,v_2)}{ac\varphi_2(v_2)} > 1$, then $z_2 > 0$. Now we define

$$R_1 = \frac{k\psi(x_2,y_2,v_2)}{ac\varphi_2(v_2)},$$

which represents the immune response reproduction ratio which expresses the CTL load during the lifespan of a CTL cell. Hence, z_2 can be rewritten as $z_2 = \varphi_3^{-1}\left(\frac{a}{q}(R_1 - 1)\right)$. It follows that, there is a chronic-infection equilibrium with CTL immune response $E_2 = (x_2, y_2, v_2, z_2)$ if $R_1 > 1$.

Now we show that $E_0, E_1 \in \Gamma$ and $E_2 \in \mathring{\Gamma}$. Clearly, $E_0 \in \Gamma$. We have $x_1 < x_0$, then from Assumption A1

$$0 = n(x_0) < n(x_1) \leq s - \bar{s}x_1.$$

It follows that

$$0 < x_1 < \frac{s}{\bar{s}} \leq \frac{s}{\sigma} = L_1.$$

From Eqs. (9)-(10), we get

$$a\alpha_1 y_1 \leq a\varphi_1(y_1) = n(x_1) < n(0) \leq s \Rightarrow 0 < y_1 < \frac{s}{a\alpha_1} < \frac{s}{\frac{a}{2}\alpha_1} \leq L_1.$$

Eq. (13) implies that,

$$c\alpha_2 v_1 \leq c\varphi_2(v_1) = k\varphi_1(y_1) = \frac{k}{a}n(x_1) < \frac{k}{a}n(0) \leq \frac{ks}{a} \Rightarrow 0 < v_1 < \frac{ks}{ac\alpha_2} < \frac{2ks}{ac\alpha_2} \leq L_2.$$

Moreover, $z_1 = 0$ and then, $E_1 \in \Gamma$. Let $R_1 > 1$, then one can show that $0 < x_2 < L_1$ and $0 < v_2 < L_2$. From Eq. (10), we have

$$a\varphi_1(y_2) + q\varphi_1(y_2)\varphi_3(z_2) = n(x_2).$$

Then

$$a\alpha_1 y_2 \leq a\varphi_1(y_2) \leq n(x_2) \Rightarrow a\alpha_1 y_2 \leq n(x_2) < n(0) \leq s \Rightarrow 0 < y_2 \leq \frac{s}{a\alpha_1} \leq L_1.$$

and

$$\frac{q\mu\alpha_3}{r}z_2 \leq q\varphi_1(y_2)\varphi_3(z_2) \leq n(x_2) < n(0) \leq s \Rightarrow 0 < z_2 \leq \frac{sr}{q\mu\alpha_3} \leq L_3.$$

Then, $E_2 \in \mathring{\Gamma}$. Clearly from Assumptions A2 and A4, we obtain

$$\begin{aligned} R_1 &= \frac{k\psi(x_2,y_2,v_2)}{ac\varphi_2(v_2)} < \frac{k\psi(x_2,0,v_2)}{ac\varphi_2(v_2)} \leq \frac{k}{ac}\lim_{v\to 0^+}\frac{\psi(x_2,0,v)}{\varphi_2(v)} \\ &= \frac{k}{ac\varphi_2'(0)}\frac{\partial\psi(x_2,0,0)}{\partial v} < \frac{k}{ac\varphi_2'(0)}\frac{\partial\psi(x_0,0,0)}{\partial v} = R_0. \ \square \end{aligned}$$

2 Global stability analysis

In this section, we investigate the global asymptotic stability of the infection-free, chronic-infection without CTL immune response and chronic-infection with CTL immune response equilibria of model (5)-(8) by using direct Lyapunov method and applying LaSalle's invariance principle. Throughout the paper, we define the function $H:(0,\infty)\to[0,\infty)$ as: $H(w)=w-1-\ln w$, where $H(w)\geq 0$ for any $w>0$ and H has the global minimum $H(1)=0$.

Theorem 1. Let Assumptions A1-A4 hold true and $R_0\leq 1$, then the infection-free equilibrium E_0 is globally asymptotically stable (GAS) in Γ.

Proof. We construct a Lyapunov functional as:

$$U_0(x,y,v,z)=x-x_0-\int_{x_0}^{x}\lim_{v\to 0^+}\frac{\psi(x_0,0,v)}{\psi(\eta,0,v)}d\eta+y+\frac{a}{k}v+\frac{aq}{rk}z. \tag{16}$$

It is seen that, $U_0(x,y,v,z)>0$ for all $x,y,v,z>0$ while $U_0(x,y,v,z)$ reaches its global minimum at E_0. We calculate $\frac{dU_0}{dt}$ along the solutions of model (5)-(8) as:

$$\begin{aligned}\frac{dU_0}{dt}&=\left(1-\lim_{v\to 0^+}\frac{\psi(x_0,0,v)}{\psi(x,0,v)}\right)(n(x)-\psi(x,y,v))+\psi(x,y,v)-a\varphi_1(y)\\&\quad+\frac{a}{k}\left(k\varphi_1(y)-c\varphi_2(v)-q\varphi_2(v)\varphi_3(z)\right)+\frac{aq}{rk}(r\varphi_3(z)\varphi_2(v)-\mu\varphi_3(z))\\&=n(x)\left(1-\lim_{v\to 0^+}\frac{\psi(x_0,0,v)}{\psi(x,0,v)}\right)+\psi(x,y,v)\lim_{v\to 0^+}\frac{\psi(x_0,0,v)}{\psi(x,0,v)}-\frac{ac}{k}\varphi_2(v)-\frac{aq\mu}{rk}\varphi_3(z).\end{aligned} \tag{17}$$

Since $n(x_0)=0$, then we get

$$\frac{dU_0}{dt}=(n(x)-n(x_0))\left(1-\lim_{v\to 0^+}\frac{\psi(x_0,0,v)}{\psi(x,0,v)}\right)+\frac{ac}{k}\left(\frac{k}{ac}\frac{\psi(x,y,v)}{\varphi_2(v)}\lim_{v\to 0^+}\frac{\psi(x_0,0,v)}{\psi(x,0,v)}-1\right)\varphi_2(v)-\frac{aq\mu}{rk}\varphi_3(z). \tag{18}$$

From Assumptions A2 and A4 we have

$$\frac{\psi(x,y,v)}{\varphi_2(v)}<\frac{\psi(x,0,v)}{\varphi_2(v)}\leq\lim_{v\to 0^+}\frac{\psi(x,0,v)}{\varphi_2(v)}=\frac{1}{\varphi_2'(0)}\frac{\partial\psi(x,0,0)}{\partial v}.$$

Then,

$$\begin{aligned}\frac{dU_0}{dt}&\leq(n(x)-n(x_0))\left(1-\frac{(\partial\psi(x_0,0,0)/\partial v)}{(\partial\psi(x,0,0)/\partial v)}\right)+\frac{ac}{k}\left(\frac{k}{ac\varphi_2'(0)}\frac{\partial\psi(x_0,0,0)}{\partial v}-1\right)\varphi_2(v)-\frac{aq\mu}{rk}\varphi_3(z)\\&=(n(x)-n(x_0))\left(1-\frac{(\partial\psi(x_0,0,0)/\partial v)}{(\partial\psi(x,0,0)/\partial v)}\right)+\frac{ac}{k}\left(R_0-1\right)\varphi_2(v)-\frac{aq\mu}{rk}\varphi_3(z).\end{aligned} \tag{19}$$

From Assumptions A1 and A2, we have

$$(n(x)-n(x_0))\left(1-\frac{(\partial\psi(x_0,0,0)/\partial v)}{(\partial\psi(x,0,0)/\partial v)}\right)\leq 0.$$

Therefore, if $R_0\leq 1$, then $\frac{dU_0}{dt}\leq 0$ for all $x,v,z>0$. We note that solutions of system (5)-(8) limited to Υ, the largest invariant subset of $\left\{\frac{dU_0}{dt}=0\right\}$[22]. We see that, $\frac{dU_0}{dt}=0$ if and only if $x(t)=x_0$, $v(t)=0$ and $z(t)=0$ for all t. By the above discussion, each element of Υ satisfies $v(t)=0$ and $z(t)=0$. Then from Eq. (7) we get

$$\dot{v}(t)=0=k\varphi_1(y(t)).$$

It follows from Assumption A3 that, $y(t)=0$ for all t. Using LaSalle's invariance principle, we derive that E_0 is GAS. □

To prove the global stability of the two equilibria E_1 and E_2, we need the following condition on the incidence rate function.

Assumption A5.

$$\left(\frac{\psi(x,y,v)}{\psi(x,y_i,v_i)}-\frac{\varphi_2(v)}{\varphi_2(v_i)}\right)\left(1-\frac{\psi(x,y_i,v_i)}{\psi(x,y,v)}\right)\leq 0, \quad x,y,v>0,\ i=1,2$$

Theorem 2. Assume that Assumptions A1-A5 are satisfied and $R_1 \leq 1 < R_0$, then the chronic-infection equilibrium without CTL immune response E_1 is GAS in Γ.

Proof. We define the following Lyapunov functional

$$U_1(x,y,v,z) = x - x_1 - \int_{x_1}^{x}\frac{\psi(x_1,y_1,v_1)}{\psi(\eta,y_1,v_1)}d\eta + y - y_1 - \int_{y_1}^{y}\frac{\varphi_1(y_1)}{\varphi_1(\eta)}d\eta + \frac{a}{k}\left(v - v_1 - \int_{v_1}^{v}\frac{\varphi_2(v_1)}{\varphi_2(\eta)}d\eta\right) + \frac{q}{r}z. \tag{20}$$

It is seen that, $U_1(x,y,v,z) > 0$ for all $x,y,v,z>0$ while $U_1(x,y,v,z)$ reaches its global minimum at E_1. The time derivative of U_1 along the trajectories of (5)-(8) is given by

$$\begin{aligned}\frac{dU_1}{dt} &= \left(1-\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right)(n(x)-\psi(x,y,v)) + \left(1-\frac{\varphi_1(y_1)}{\varphi_1(y)}\right)(\psi(x,y,v)-a\varphi_1(y)-q\varphi_1(y)\varphi_3(z))\\ &+\frac{a}{k}\left(1-\frac{\varphi_2(v_1)}{\varphi_2(v)}\right)(k\varphi_1(y)-c\varphi_2(v)) + \frac{q}{r}(r\varphi_1(y)\varphi_3(z)-\mu\varphi_3(z))\\ &= \left(1-\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right)n(x) + \psi(x_1,y_1,v_1)\frac{\psi(x,y,v)}{\psi(x,y_1,v_1)} - \frac{\varphi_1(y_1)}{\varphi_1(y)}\psi(x,y,v)\\ &+ a\varphi_1(y_1) + q\varphi_1(y_1)\varphi_3(z) - \frac{ac}{k}\varphi_2(v) - a\varphi_1(y)\frac{\varphi_2(v_1)}{\varphi_2(v)} + \frac{ac}{k}\varphi_2(v_1) - \frac{q\mu}{r}\varphi_3(z).\end{aligned} \tag{21}$$

Using the equilibrium conditions for E_1:

$$n(x_1) = \psi(x_1,y_1,v_1) = a\varphi_1(y_1) = \frac{ac}{k}\varphi_2(v_1),$$

we obtain

$$\begin{aligned}\frac{dU_1}{dt} &= (n(x)-n(x_1))\left(1-\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right) + 3a\varphi_1(y_1) - a\varphi_1(y_1)\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)} + a\varphi_1(y_1)\frac{\psi(x,y,v)}{\psi(x,y_1,v_1)}\\ &- a\varphi_1(y_1)\frac{\varphi_1(y_1)\psi(x,y,v)}{\varphi_1(y)\psi(x_1,y_1,v_1)} - a\varphi_1(y_1)\frac{\varphi_2(v)}{\varphi_2(v_1)} - a\varphi_1(y_1)\frac{\varphi_2(v_1)\varphi_1(y)}{\varphi_2(v)\varphi_1(y_1)} + q\left(\varphi_1(y_1)-\frac{\mu}{r}\right)\varphi_3(z).\end{aligned} \tag{22}$$

Collecting terms to get

$$\begin{aligned}\frac{dU_1}{dt} &= (n(x)-n(x_1))\left(1-\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right) + a\varphi_1(y_1)\left(\frac{\psi(x,y,v)}{\psi(x,y_1,v_1)} - \frac{\varphi_2(v)}{\varphi_2(v_1)} - 1 + \frac{\varphi_2(v)\psi(x,y_1,v_1)}{\varphi_2(v_1)\psi(x,y,v)}\right)\\ &+ a\varphi_1(y_1)\left[4 - \frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)} - \frac{\varphi_1(y_1)\psi(x,y,v)}{\varphi_1(y)\psi(x_1,y_1,v_1)} - \frac{\varphi_2(v_1)\varphi_1(y)}{\varphi_2(v)\varphi_1(y_1)} - \frac{\varphi_2(v)\psi(x,y_1,v_1)}{\varphi_2(v_1)\psi(x,y,v)}\right]\\ &+ q\left(\varphi_1(y_1)-\varphi_1(y_2)\right)\varphi_3(z).\end{aligned} \tag{23}$$

Eq. (23) can be rewritten as:

$$\begin{aligned}\frac{dU_1}{dt} &= (n(x)-n(x_1))\left(1-\frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)}\right) + a\varphi_1(y_1)\left(\frac{\psi(x,y,v)}{\psi(x,y_1,v_1)} - \frac{\varphi_2(v)}{\varphi_2(v_1)}\right)\left(1-\frac{\psi(x,y_1,v_1)}{\psi(x,y,v)}\right)\\ &+ a\varphi_1(y_1)\left[4 - \frac{\psi(x_1,y_1,v_1)}{\psi(x,y_1,v_1)} - \frac{\varphi_1(y_1)\psi(x,y,v)}{\varphi_1(y)\psi(x_1,y_1,v_1)} - \frac{\varphi_2(v_1)\varphi_1(y)}{\varphi_2(v)\varphi_1(y_1)} - \frac{\varphi_2(v)\psi(x,y_1,v_1)}{\varphi_2(v_1)\psi(x,y,v)}\right]\\ &+ q\left(\varphi_1(y_1)-\varphi_1(y_2)\right)\varphi_3(z).\end{aligned} \tag{24}$$

From Assumptions A1-A5, we get that the first and second terms of Eq. (24) are less than or equal zero. Since the geometrical mean is less than or equal to the arithmetical mean, the third term of Eq. (24) is also less than or equal zero. Now, if $R_1 \leq 1$, then E_2 does not exist since $z_2 = \frac{a}{q}(R_1 - 1) \leq 0$. It follows that, $\dot{z}(t) = (r\varphi_1(y) - \mu)\varphi_3(z) \leq 0$, i.e. $\varphi_1(y_1) \leq \varphi_1(y_2)$. It follows from above that, $\frac{dU_1}{dt} \leq 0$ for all $x, y, v, z > 0$. The solutions of system (5)-(8) limited to Υ, the largest invariant subset of $\left\{(x, y, v, z) : \frac{dU_1}{dt} = 0\right\}$[22]. We have $\frac{dU_1}{dt} = 0$ if and only if $x(t) = x_1$, $y(t) = y_1$, $v(t) = v_1$ and $z(t) = 0$. So, Υ contains a unique point, that is E_1. Thus, the global asymptotic stability of the chronic-infection equilibrium without CTL immune response E_1 follows from LaSalle's invariance principle. □

Theorem 3. Let Assumptions A1-A5 hold true and $R_1 > 1$, then the chronic-infection equilibrium with CTL immune response E_2 is GAS in $\overset{\circ}{\Gamma}$.

Proof. We construct a Lyapunov functional as follows:

$$U_2(x, y, v, z) = x - x_2 - \int_{x_2}^{x} \frac{\psi(x_2, y_2, v_2)}{\psi(\eta, y_2, v_2)} d\eta + y - y_2 - \int_{y_2}^{y} \frac{\varphi_1(y_2)}{\varphi_1(\eta)} d\eta$$
$$+ \left(\frac{a + q\varphi_3(z_2)}{k}\right)\left(v - v_2 - \int_{v_2}^{v} \frac{\varphi_2(v_2)}{\varphi_2(\eta)} d\eta\right) + \frac{q}{r}\left(z - z_2 - \int_{z_2}^{z} \frac{\varphi_3(z_2)}{\varphi_3(\eta)} d\eta\right). \quad (25)$$

We have $U_2(x, y, v, z) > 0$ for all $x, y, v, z > 0$ and $U_2(x_2, y_2, v_2, z_2) = 0$. Calculating the derivative of U_2 along positive solutions of (5)-(8) gives us the following

$$\frac{dU_2}{dt} = \left(1 - \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)}\right)(n(x) - \psi(x, y, v)) + \left(1 - \frac{\varphi_1(y_2)}{\varphi_1(y)}\right)(\psi(x, y, v) - a\varphi_1(y) - q\varphi_1(y)\varphi_3(z))$$
$$+ \left(\frac{a + q\varphi_3(z_2)}{k}\right)\left(1 - \frac{\varphi_2(v_2)}{\varphi_2(v)}\right)(k\varphi_1(y) - c\varphi_2(v)) + \frac{q}{r}\left(1 - \frac{\varphi_3(z_2)}{\varphi_3(z)}\right)(r\varphi_1(y)\varphi_3(z) - \mu\varphi_3(z)). \quad (26)$$

Collecting terms of Eq. (26) and using $n(x_2) = \psi(x_2, y_2, v_2)$ we obtain

$$\frac{dU_2}{dt} = (n(x) - n(x_2))\left(1 - \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)}\right) + \psi(x_2, y_2, v_2) - \psi(x_2, y_2, v_2)\frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)}$$
$$+ \psi(x, y, v)\frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)} - \frac{\varphi_1(y_2)\psi(x, y, v)}{\varphi_1(y)} + a\varphi_1(y_2) + q\varphi_1(y_2)\varphi_3(z) - \frac{ac}{k}\varphi_2(v) - a\varphi_1(y)\frac{\varphi_2(v_2)}{\varphi_2(v)}$$
$$+ \frac{ac}{k}\varphi_2(v_2) - \frac{qc}{k}\varphi_3(z_2)\varphi_2(v) - q\varphi_3(z_2)\varphi_1(y)\frac{\varphi_2(v_2)}{\varphi_2(v)} + \frac{qc}{k}\varphi_3(z_2)\varphi_2(v_2) - \frac{q\mu}{r}\varphi_3(z) + \frac{q\mu}{r}\varphi_3(z_2). \quad (27)$$

By using the equilibrium conditions of E_2

$$\psi(x_2, y_2, v_2) = a\varphi_1(y_2) + q\varphi_1(y_2)\varphi_3(z_2) = \frac{ac}{k}\varphi_2(v_2) + \frac{qc}{k}\varphi_3(z_2)\varphi_2(v_2), \quad \varphi_1(y_2) = \frac{\mu}{r},$$

we obtain

$$\frac{dU_2}{dt} = (n(x) - n(x_2))\left(1 - \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)}\right) + 3\psi(x_2, y_2, v_2) - \psi(x_2, y_2, v_2)\frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)}$$
$$+ \psi(x_2, y_2, v_2)\frac{\psi(x, y, v)}{\psi(x, y_2, v_2)} - \psi(x_2, y_2, v_2)\frac{\varphi_1(y_2)\psi(x, y, v)}{\varphi_1(y)\psi(x_2, y_2, v_2)}$$
$$- \psi(x_2, y_2, v_2)\frac{\varphi_2(v)}{\varphi_2(v_2)} - a\varphi_1(y_2)\frac{\varphi_2(v_2)\varphi_1(y)}{\varphi_2(v)\varphi_1(y_2)} - q\varphi_1(y_2)\varphi_3(z_2)\frac{\varphi_2(v_2)\varphi_1(y)}{\varphi_2(v)\varphi_1(y_2)}. \quad (28)$$

Collecting terms of Eq. (28), we get

$$\frac{dU_2}{dt} = (n(x) - n(x_2))\left(1 - \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)}\right) + \psi(x_2, y_2, v_2)\left(\frac{\psi(x, y, v)}{\psi(x, y_2, v_2)} - \frac{\varphi_2(v)}{\varphi_2(v_2)} - 1 + \frac{\varphi_2(v)\psi(x, y_2, v_2)}{\varphi_2(v_2)\psi(x, y, v)}\right)$$
$$+ \psi(x_2, y_2, v_2)\left[4 - \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)} - \frac{\varphi_1(y_2)\psi(x, y, v)}{\varphi_1(y)\psi(x_2, y_2, v_2)} - \frac{\varphi_2(v_2)\varphi_1(y)}{\varphi_2(v)\varphi_1(y_2)} - \frac{\varphi_2(v)\psi(x, y_2, v_2)}{\varphi_2(v_2)\psi(x, y, v)}\right]. \quad (29)$$

We can rewrite (29) as

$$\frac{dU_2}{dt} = (n(x) - n(x_2))\left(1 - \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)}\right) + \psi(x_2, y_2, v_2)\left(\frac{\psi(x, y, v)}{\psi(x, y_2, v_2)} - \frac{\varphi_2(v)}{\varphi_2(v_2)}\right)\left(1 - \frac{\psi(x, y_2, v_2)}{\psi(x, y, v)}\right)$$
$$+ \psi(x_2, y_2, v_2)\left[4 - \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)} - \frac{\varphi_1(y_2)\psi(x, y, v)}{\varphi_1(y)\psi(x_2, y_2, v_2)} - \frac{\varphi_2(v_2)\varphi_1(y)}{\varphi_2(v)\varphi_1(y_2)} - \frac{\varphi_2(v)\psi(x, y_2, v_2)}{\varphi_2(v_2)\psi(x, y, v)}\right]. \quad (30)$$

We note that from Assumptions A1-A5 and the relationship between the arithmetical and geometrical means, we have $\frac{dU_2}{dt} \leq 0$ for all $x, y, v, z > 0$. The solutions of model (5)-(8) limited to Υ, the largest invariant subset of $\left\{(x, y, v, z) : \frac{dU_2}{dt} = 0\right\}$[22]. We have $\frac{dU_2}{dt} = 0$ if and only if $x(t) = x_2$, $y(t) = y_2$ and $v(t) = v_2$ for all t. Therefore, if $v(t) = v_2$ and $y(t) = y_2$, then from Eq. (6), $\psi(x_2, y_2, v_2) - a\varphi_1(y_2) - q\varphi_1(y_2)\varphi_3(z(t)) = 0$, which gives $z(t) = z_2$ for all t. Thus, $\frac{dU_2}{dt} = 0$ occurs at E_2. The global asymptotic stability of the chronic-infection equilibrium with CTL immune response E_2 follows from LaSalle's invariance principle. □

3 Conclusion

In this paper, we have proposed and analyzed a nonlinear viral infection model with CTL immune response. We have considered more general nonlinear functions for the: (i) intrinsic growth rate of uninfected cells; (ii) incidence rate of infection; (iii) natural death rate of infected cells; (iv) rate at which the infected cells are killed by CTL cells; (v) production and removal rates of viruses; (vi) activation and natural death rates of CTLs. We have derived a set of conditions on these general functions and have determined two threshold parameters to prove the existence and the global stability of the model's equilibria. The global asymptotic stability of the three equilibria, infection-free, chronic-infection without CTL immune response and chronic-infection with CTL immune response has been proven by using direct Lyapunov method and LaSalle's invariance principle.

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ON THE STABILITY OF CUBIC LIE ∗-DERIVATIONS

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Abstract. We will show the general solution of the functional equation $f(sx+y)+f(x-sy)-s^2f(x+y)-sf(x-y)=(s-1)(s^2-1)f(x)-(s+1)(s^2-1)f(y)$ and investigate the stability of cubic Lie ∗-derivations associated with the given functional equation.

1. Introduction

The concept of stability problem of a functional equation was first posed by Ulam [14] concerning the stability of group homomorphisms as follows: *Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?* In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism then there exists a true homomorphism near it. By the problem raised by Ulam, several stability problems of a large variety of functional equations have been extensively studied and generalized by a number of authors and many interesting results have been obtained for the last nearly fifty years. For further information about the topic, we refer the reader to [9], [5], [1] and [2].

Recall that a Banach ∗-algebra is a Banach algebra (complete normed algebra) which has an isometric involution. Jang and Park [6] investigated the stability of ∗-derivations and of quadratic ∗-derivations with Cauchy functional equation and the Jensen functional equation on Banach ∗-algebra. The stability of ∗-derivations on Banach ∗-algebra by using fixed point alternative was proved by Park and Bodaghi and also Yang et al.; see [12] and [15], respectively. Also, the stability of cubic Lie derivations was introduced by Fošner and Fošner; see [4].

Jun and Kim [8] introduced the following cubic functional equation:

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x)$$

and established a general solution. Najati [11] investigated the following generalized cubic functional equation:

$$f(sx+y)+f(sx-y)=sf(x+y)+sf(x-y)+2(s^3-s)f(x) \tag{1.1}$$

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for a positive integer $s \geq 2$. Also, Jun and Kim [7] proved the Hyers-Ulam-Rassias stability of a Euler-Lagrange type cubic mapping as follows:

$$f(sx+y)+f(x+sy) \\ =(s+1)(s-1)^2[f(x)+f(y)]+s(s+1)f(x+y)\,, \tag{1.2}$$

where $s \in \mathbb{Z}\,(s \neq 0\,, \pm 1)$.

In this paper, we deal with the following the functional equation:

$$f(sx+y)+f(x-sy)-s^2f(x+y)-sf(x-y) \\ =(s-1)(s^2-1)f(x)-(s+1)(s^2-1)f(y) \tag{1.3}$$

for all $s \in \mathbb{Z}\,(s \neq 0\,, \pm 1)$. We will show the general solution of the functional equation (1.3) and investigate the stability of cubic Lie $*$-derivations associated with the given functional equation on normed algebras.

2. Cubic Functional Equations

In this section let X and Y be vector spaces and we investigate the general solution of the functional equation (1.3).

Theorem 2.1. *A function $f : X \to Y$ satisfies the functional equation (1.3) if and only if it satisfies the functional equation (1.1).*

Proof. Suppose that f satisfies the equation (1.3). It is easy to show that $f(0)=0\,,\ f(sx)=s^3f(x)$ for all $x \in X$ and all $s \in \mathbb{Z}\,(s \neq 0\,, \pm 1)$. Letting $x=-x$ in the equation (1.3), we have

$$-f(sx-y)-f(x+sy)+(s+1)(s^2-1)f(y) \\ =-s^2f(x-y)-sf(x+y)-(s-1)(s^2-1)f(x) \tag{2.1}$$

for all $x\,, y \in X$. Replacing x and y in the equation (2.1), we get

$$f(x-sy)-f(sx+y)+(s+1)(s^2-1)f(x) \\ =s^2f(x-y)-sf(x+y)-(s-1)(s^2-1)f(y) \tag{2.2}$$

for all $x\,, y \in X$. Subtracting the equation (2.2) from the equation (1.3), we obtain

$$2f(sx+y)+2(s^2-1)f(y) \\ =(s^2+s)f(x+y)+(s-s^2)f(x-y)+2s(s^2-1)f(x) \tag{2.3}$$

for all $x\,, y \in X$. Now, letting $y=-y$ in the equation (2.3)

$$2f(sx-y)-2(s^2-1)f(y) \\ =(s^2+s)f(x-y)+(s-s^2)f(x+y)+2s(s^2-1)f(x) \tag{2.4}$$

for all $x\,, y \in X$. Adding two equations (2.3) and (2.4), we have

$$2f(sx+y)+2f(sx-y)=2sf(x+y)+2sf(x-y)+4s(s^2-1)f(x) \tag{2.5}$$

for all $x\,, y \in X$. Thus we have the equation (1.1). Conversely, suppose that f satisfies the equation (1.1). It is easy to see that $f(0)=0\,,\ f(sx)=s^3f(x)$

for all $x \in X$ and all $s \in \mathbb{Z}(s \neq 0)$. Letting $y = sy$ in the equation (1.1), we have

$$f(x+sy)+f(x-sy)=s^2f(x+y)+s^2f(x-y)-2(s^2-1)f(x) \tag{2.6}$$

for all $x, y \in X$. Replacing x and y in the equation (2.6), we get

$$f(sx+y)-f(sx-y)=s^2f(x+y)-s^2f(x-y)-2(s^2-1)f(y) \tag{2.7}$$

for all $x, y \in X$. By adding two equations (1.1) and (2.7), we obtain

$$2f(sx+y)=(s^2+s)f(x+y)+(s-s^2)f(x-y)+2s(s^2-1)f(x)-2(s^2-1)f(y) \tag{2.8}$$

for all $x, y \in X$. Now, letting $y = sy$ in the equation (2.7), we have

$$f(x+sy)-f(x-sy)=sf(x+y)-sf(x-y)+2s(s^2-1)f(y) \tag{2.9}$$

for all $x, y \in X$. Subtracting the equation (2.9) from the equation (2.6), we know that

$$2f(x-sy)=(s^2-s)f(x+y)+(s^2+s)f(x-y)-2(s^2-1)f(x)-2s(s^2-1)f(y) \tag{2.10}$$

for all $x, y \in X$. By adding two equations (2.8) and (2.10), we have the desired equation (1.3). □

3. Cubic Lie $*$-Derivations

Throughout this section, we assume that A is a complex normed $*$-algebra and M is a Banach A-bimodule. We will use the same symbol $||\cdot||$ as norms on a normed algebra A and a normed A-bimodule M. A mapping $f : A \to M$ is a *cubic homogeneous mapping* if $f(\mu a) = \mu^3 f(a)$, for all $a \in A$ and $\mu \in \mathbb{C}$. A cubic homogeneous mapping $f : A \to M$ is called a *cubic derivation* if

$$f(xy) = f(x)y^3 + x^3 f(y)$$

holds for all $x, y \in A$. For all $x, y \in A$, the symbol $[x, y]$ will denote the commutator $xy-yx$. We say that a cubic homogeneous mapping $f : A \to M$ is a cubic Lie derivation if

$$f([x, y]) = [f(x), y^3] + [x^3, f(y)]$$

for all $x, y \in A$. In addition, if f satisfies in condition $f(x^*) = f(x)^*$ for all $x \in A$, then it is called the *cubic Lie $*$-derivation.*

Example 3.1. Let $A = \mathbb{C}$ be a complex field endowed with the map $z \mapsto z^* = \bar{z}$ (where $\bar{z}$ is the complex conjugate of z). We define $f : A \to A$ by $f(a) = a^3$ for all $a \in A$. Then f is cubic and

$$f([a, b]) = [f(a), b^3] + [a^3, f(b)] = 0$$

for all $a \in A$. Also,

$$f(a^*) = f(\bar{a}) = \bar{a}^3 = f(\bar{a}) = f(a)^*$$

for all $a \in A$. Thus f is a cubic Lie $*$-derivation.

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In the following, $\mathbb{T}^1$ will stand for the set of all complex units, that is,

$$\mathbb{T}^1 = \{\mu \in \mathbb{C} \,|\, |\mu| = 1\}\,.$$

For the given mapping $f : A \to M$, we consider

$$\begin{aligned}(3.1)\quad \Delta_\mu f(x,y) &:= f(s\mu x+\mu y)+f(\mu x-s\mu y)-s^2\mu^3 f(x+y)-s\mu^3 f(x-y)\\ &\quad -\mu^3(s-1)(s^2-1)f(x)+\mu^3(s+1)(s^2-1)f(y)\,,\\ \Delta f(x,y) &:= f([x,\, y]) - [f(x),\, y^3] - [x^3,\, f(y)]\end{aligned}$$

for all $x,\, y \in A$, $\mu \in \mathbb{C}$ and $s \in \mathbb{Z}\,(s \neq 0\,, \pm 1)$.

Theorem 3.2. *Suppose that $f : A \to M$ is a mapping with $f(0) = 0$ for which there exists a function $\phi : A^5 \to [0,\, \infty)$ such that*

$$(3.2)\qquad \widetilde{\phi}(a,b,x,y,z) := \sum_{j=0}^{\infty} \frac{1}{|s|^{3j}}\phi(s^j a, s^j b, s^j x, s^j y, s^j z) < \infty$$

$$(3.3)\qquad ||\Delta_\mu f(a,b)|| \le \phi(a,b,0,0,0)$$

$$(3.4)\qquad ||\Delta f(x,y) + f(z^*) - f(z)^*|| \le \phi(0,0,x,y,z)$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}} = \{e^{i\theta} \,|\, 0 \le \theta \le \frac{2\pi}{n_0}\}$ and all $a,b,x,y,z \in A$ in which $n_0 \in \mathbb{N}$. Also, if for each fixed $a \in A$ the mapping $r \mapsto f(ra)$ from $\mathbb{R}$ to M is continuous, then there exists a unique cubic Lie $*$-derivation $L : A \to M$ satisfying

$$(3.5)\qquad ||f(a) - L(a)|| \le \frac{1}{|s|^3}\widetilde{\phi}(a,0,0,0,0)\,.$$

Proof. Let $b = 0$ and $\mu = 1$ in the inequality (3.3), we have

$$(3.6)\qquad ||f(a) - \frac{1}{s^3}f(sa)|| \le \frac{1}{|s|^3}\phi(a,0,0,0,0)$$

for all $a \in A$. Using the induction, it is easy to show that

$$(3.7)\qquad ||\frac{1}{s^{3t}}f(s^t a) - \frac{1}{s^{3k}}f(s^k a)|| \le \frac{1}{|s|^3}\sum_{j=k}^{t-1}\frac{\phi(s^j a,0,0,0,0)}{|s|^{3j}}$$

for $t > k \ge 0$ and $a \in A$. The inequalities (3.2) and (3.7) imply that the sequence $\{\frac{1}{s^{3n}}f(s^n a)\}_{n=0}^{\infty}$ is a Cauchy sequence. Since M is complete, the sequence is convergent. Hence we can define a mapping $L : A \to M$ as

$$(3.8)\qquad L(a) = \lim_{n\to\infty}\frac{1}{s^{3n}}f(s^n a)$$

for $a \in A$. By letting $t = n$ and $k = 0$ in the inequality (3.7), we have

$$(3.9)\qquad ||\frac{1}{s^{3n}}f(s^n a) - f(a)|| \le \frac{1}{|s|^3}\sum_{j=0}^{n-1}\frac{\phi(s^j a,0,0,0,0)}{|s|^{3j}}$$

for $n > 0$ and $a \in A$. By taking $n \to \infty$ in the inequality (3.9), the inequalities (3.2) implies that the inequality (3.5) holds.

Now, we will show that the mapping L is a unique cubic Lie $*$-derivation such that the inequality (3.5) holds for all $a \in A$. We note that

$$||\Delta_\mu L(a,b)|| = \lim_{n\to\infty} \frac{1}{|s|^{3n}} ||\Delta_\mu f(s^n a, s^n b)|| \leq \lim_{n\to\infty} \frac{\phi(s^n a, s^n b, 0, 0, 0)}{|s|^{3n}} = 0\,, \tag{3.10}$$

for all $a, b \in A$ and $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$. By taking $\mu = 1$ in the inequality (3.10), it follows that the mapping L is a Euler-Lagrange cubic mapping. Also, the inequality (3.10) implies that $\Delta_\mu L(a, 0) = 0$. Hence

$$L(\mu a) = \mu^3 L(a)$$

for all $a \in A$ and $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$. Let $\mu \in \mathbb{T}^1 = \{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\}$. Then $\mu = e^{i\theta}$, where $0 \leq \theta \leq 2\pi$. Let $\mu_1 = \mu^{\frac{1}{n_0}} = e^{\frac{i\theta}{n_0}}$. Hence we have $\mu_1 \in \mathbb{T}^1_{\frac{1}{n_0}}$. Then

$$L(\mu a) = L(\mu_1^{n_0} a) = \mu_1^{3n_0} L(a) = \mu^3 L(a)$$

for all $\mu \in \mathbb{T}^1$ and $a \in A$. Suppose that ρ is any continuous linear functional on A and a is a fixed element in A. Then we can define a function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(r) = \rho(L(ra))$$

for all $r \in \mathbb{R}$. It is easy to check that g is cubic. Let

$$g_k(r) = \rho\Big(\frac{f(s^k ra)}{s^{3k}}\Big)$$

for all $k \in \mathbb{N}$ and $r \in \mathbb{R}$.

Note that g as the pointwise limit of the sequence of measurable functions g_k is measurable. Hence g as a measurable cubic function is continuous (see [3]) and

$$g(r) = r^3 g(1)$$

for all $r \in \mathbb{R}$. Thus

$$\rho(L(ra)) = g(r) = r^3 g(1) = r^3 \rho(L(a)) = \rho(r^3 L(a))$$

for all $r \in \mathbb{R}$. Since ρ was an arbitrary continuous linear functional on A we may conclude that

$$L(ra) = r^3 L(a)$$

for all $r \in \mathbb{R}$. Let $\mu \in \mathbb{C}\,(\mu \neq 0)$. Then $\frac{\mu}{|\mu|} \in \mathbb{T}^1$. Hence

$$L(\mu a) = L\Big(\frac{\mu}{|\mu|}|\mu| a\Big) = \Big(\frac{\mu}{|\mu|}\Big)^3 L(|\mu| a) = \Big(\frac{\mu}{|\mu|}\Big)^3 |\mu|^3 L(a) = \mu^3 L(a)$$

for all $a \in A$ and $\mu \in \mathbb{C}\,(\mu \neq 0)$. Since a was an arbitrary element in A, we may conclude that L is cubic homogeneous.

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Next, replacing x, y by $s^k x, s^k y$, respectively, and $z = 0$ in the inequality (3.4), we have

$$\begin{aligned} ||\Delta L(x,y)|| &= \lim_{n\to\infty} ||\frac{\Delta f(s^n x, s^n y)}{s^{3n}}|| \\ &\leq \lim_{n\to\infty} \frac{1}{|s|^{3n}} \phi(0,0,s^n x, s^n y, 0) = 0 \end{aligned}$$

for all $x, y \in A$. Hence we have $\Delta L(x,y) = 0$ for all $x, y \in A$. That is, L is a cubic Lie derivation. Letting $x = y = 0$ and replacing z by $s^k z$ in the inequality (3.4), we get

$$\Big|\Big|\frac{f(s^n z^*)}{s^{3n}} - \frac{f(s^n z)^*}{s^{3n}}\Big|\Big| \leq \frac{\phi(0,0,0,0,s^n z)}{|s|^{3n}} \tag{3.11}$$

for all $z \in A$. As $n \to \infty$ in the inequality (3.11), we have

$$L(z^*) = L(z)^*$$

for all $z \in A$. This means that L is a cubic Lie $*$-derivation. Now, assume $L' : A \to A$ is another cubic $*$-derivation satisfying the inequality (3.5). Then

$$\begin{aligned} ||L(a) - L'(a)|| &= \frac{1}{|s|^{3n}} ||L(s^n a) - L'(s^n a)|| \\ &\leq \frac{1}{|s|^{3n}} \Big(||L(s^n a) - f(s^n a)|| + ||f(s^n a) - L'(s^n a)|| \Big) \\ &\leq \frac{1}{|s|^{3n+1}} \sum_{j=0}^{\infty} \frac{1}{|s|^{3j}} \phi(s^{j+n} a, 0,0,0,0) \\ &= \frac{1}{|s|^3} \sum_{j=n}^{\infty} \frac{1}{|s|^{3j}} \phi(s^j a, 0,0,0,0) , \end{aligned}$$

which tends to zero as $k \to \infty$, for all $a \in A$. Thus $L(a) = L'(a)$ for all $a \in A$. This proves the uniqueness of L . ☐

Corollary 3.3. *Let θ , r be positive real numbers with $r < 3$ and let $f : A \to M$ be a mapping with $f(0) = 0$ such that*

$$||\Delta_\mu f(a,b)|| \leq \theta(||a||^r + ||b||^r)$$

$$||\Delta f(x,y) + f(z^*) - f(z)^*|| \leq \theta(||x||^r + ||y||^r + ||z||^r)$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and $a, b, x, y, z \in A$. Then there exists a unique cubic Lie $*$-derivation $L : A \to M$ satisfying

$$||f(a) - L(a)|| \leq \frac{\theta ||a||^r}{|s|^3 - |s|^r}$$

for all $a \in A$.

Proof. The proof follows from Theorem 3.2 by taking $\phi(a,b,x,y,z) = \theta(||a||^r + ||b||^r + ||x||^r + ||y||^r + ||z||^r)$ for all $a, b, x, y, z \in A$. ☐

Now, we will investigate the stability of the given functional equation (3.1) using the alternative fixed point method. Before proceeding the proof, we will state the theorem, the alternative of fixed point; see [10] and [13].

Definition 3.4. Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies
(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 3.5 (The alternative of fixed point [10], [13]). *Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant l . Then for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1}x) = \infty \text{ for all } n \ge 0 ,$$

or there exists a natural number n_0 such that
(1) *$d(T^n x, T^{n+1}x) < \infty$ for all $n \ge n_0$;*
(2) *The sequence $(T^n x)$ is convergent to a fixed point y^* of T ;*
(3) *y^* is the unique fixed point of T in the set*

$$\triangle = \{y \in \Omega | d(T^{n_0} x, y) < \infty\} ;$$

(4) *$d(y, y^*) \le \frac{1}{1-l}\, d(y, Ty)$ for all $y \in \triangle$.*

Theorem 3.6. *Let $f : A \to M$ be a continuous mapping with $f(0) = 0$ and let $\phi : A^5 \to [0, \infty)$ be a continuous mapping such that*

$$||\Delta_\mu f(a, b)|| \le \phi(a, b, 0, 0, 0) \tag{3.12}$$

$$||\Delta f(x, y) + f(z^*) - f(z)^*|| \le \phi(0, 0, x, y, z) \tag{3.13}$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and $a, b, x, y, z \in A$. If there exists a constant $l \in (0, 1)$ such that

$$\phi(sa, sb, sx, sy, sz) \le |s|^3 l \phi(a, b, x, y, z) \tag{3.14}$$

for all $a, b, x, y, z \in A$, then there exists a cubic Lie $*$-derivation $L : A \to M$ satisfying

$$||f(a) - L(a)|| \le \frac{1}{|s|^3(1 - l)} \phi(a, 0, 0, 0, 0) \tag{3.15}$$

for all $a \in A$.

Proof. Consider the set

$$\Omega = \{g \,|\, g : A \to A \,, g(0) = 0\}$$

and introduce the generalized metric on Ω ,

$$d(g, h) = \inf\{c \in (0, \infty) \mid \| g(a) - h(a) \| \le c\phi(a, 0, 0, 0, 0) \,, \text{for all } a \in A\} .$$

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It is easy to show that (Ω, d) is complete. Now we define a function $T : \Omega \to \Omega$ by

$$T(g)(a) = \frac{1}{s^3} g(sa) \tag{3.16}$$

for all $a \in A$. Note that for all $g, h \in \Omega$, let $c \in (0, \infty)$ be an arbitrary constant with $d(g, h) \leq c$. Then

$$||g(a) - h(a)|| \leq c\phi(a, 0, 0, 0, 0) \tag{3.17}$$

for all $a \in A$. Letting $a = sa$ in the inequality (3.17) and using (3.14) and (3.16), we have

$$\begin{aligned} ||T(g)(a) - T(h)(a)|| &= \frac{1}{|s|^3} ||g(sa) - h(sa)|| \\ &\leq \frac{1}{|s|^3}\, c\, \phi(sa, 0, 0, 0, 0) \leq c\, l\, \phi(a, 0, 0, 0, 0)\,, \end{aligned}$$

that is,

$$d(Tg, Th) \leq c\, l\,.$$

Hence we have that

$$d(Tg, Th) \leq l\, d(g, h)\,,$$

for all $g, h \in \Omega$, that is, T is a strictly self-mapping of Ω with the Lipschitz constant l. Letting $\mu = 1$, $b = 0$ in the inequality (3.12), we get

$$||\frac{1}{s^3} f(sa) - f(a)|| \leq \frac{1}{|s|^3} \phi(a, 0, 0, 0, 0)$$

for all $a \in A$. This means that

$$d(Tf, f) \leq \frac{1}{|s|^3}\,.$$

We can apply the alternative of fixed point and since $\lim_{n\to\infty} d(T^n f, L) = 0$, there exists a fixed point L of T in Ω such that

$$L(a) = \lim_{n\to\infty} \frac{f(s^n a)}{s^{3n}}\,, \tag{3.18}$$

for all $a \in A$. Hence

$$d(f, L) \leq \frac{1}{1 - l} d(Tf, f) \leq \frac{1}{|s|^3} \frac{1}{1 - l}\,.$$

This implies that the inequality (3.15) holds for all $a \in A$. Since $l \in (0, 1)$, the inequality (3.14) shows that

$$\lim_{n\to\infty} \frac{\phi(s^n a, s^n b, s^n x, s^n y, s^n z)}{|s|^{3n}} = 0\,. \tag{3.19}$$

Replacing a, b by $s^n a$, $s^n b$, respectively, in the inequality (3.12), we have

$$\frac{1}{|s|^{3n}} ||\Delta_\mu f(s^n a, s^n b)|| \leq \frac{\phi(s^n a, s^n b, 0, 0, 0)}{|s|^{3n}}\,.$$

Taking the limit as k tend to infinity, we have $\Delta_\mu f(a,b) = 0$ for all $a\,, b \in A$ and all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$. The remains are similar to the proof of Theorem 3.2. □

Corollary 3.7. *Let $\theta\,, r$ be positive real numbers with $r < 3$ and let $f : A \to M$ be a mapping with $f(0) = 0$ such that*

$$||\Delta_\mu f(a,b)|| \leq \theta(||a||^r + ||b||^r)$$
$$||\Delta f(x,y) + f(z^*) - f(z)^*|| \leq \theta(||x||^r + ||y||^r + ||z||^r)$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and $a, b, x, y, z \in A$. Then there exists a unique cubic Lie $*$-derivation $L : A \to M$ satisfying

$$||f(a) - L(a)|| \leq \frac{\theta||a||^r}{|s|^3(1-l)}$$

for all $a \in A$.

Proof. The proof follows from Theorem 3.6 by taking $\phi(a,b,x,y,z) = \theta(||a||^r + ||b||^r + ||x||^r + ||y||^r + ||z||^r)$ for all $a, b, x, y, z \in A$. □

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Fuzzy share functions for cooperative fuzzy games[†]

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Abstract In this paper, the concept of fuzzy share functions of cooperative fuzzy games with fuzzy characteristic functions is proposed. Players in the proposed cooperative fuzzy game do not need to know precise information about the payoff value. We generalize the axiom of additivity by introducing a positive fuzzy value function $\tilde{\mu}$ on the class of cooperative fuzzy games in fuzzy characteristic function form. The so-called axiom of $\tilde{\mu}$-additivity generalizes the classical axiom of additivity by putting the weight $\tilde{\mu}(\tilde{v})$ on the value of the game $\tilde{v}$. We show that any additive function $\tilde{\mu}$ determines a unique fuzzy share function satisfying the axioms of efficient shares, null player property, symmetry and $\tilde{\mu}$-additivity on the subclass of games on which $\tilde{\mu}$ is positive and which contains all positively scaled unanimity games. Finally, we introduce the fuzzy Shapley share functions and fuzzy Banzhaf share functions for the cooperative fuzzy games with fuzzy characteristic functions.
Keywords: Cooperative fuzzy game; Fuzzy share functions; Characteristic functions; Fuzzy numbers.

1. Introduction

A cooperative game with transferable utility, or simply a TU-game, is a finite set of players N and for any subset (coalition) of players a worth representing the total payoff that the coalition can obtain by cooperating. A value function for TU-games is a function that assigns to every TU-game with n players an n-dimensional vector representing a distribution of payoffs among the players. A value function is efficient if for every game it distributes exactly the worth of the 'grand coalition', N, over all players. The most famous efficient value function is the Shapley value[16]. An example of a value function that is not efficient is the Banzhaf value[3, 8, 14]. Since the Banzhaf value is not efficient, it is not adequate in allocating the worth $v(N)$. In order to allocate $v(N)$ and according to the Banzhaf value, Van der Laan et al. in [18] characterize the normalized Banzhaf value, which distributes the worth $v(N)$ proportional to the Banzhaf values of the players.

A different approach to efficiently allocate the worth $v(N)$ is described in [19], who introduce share functions as an alternative type of solution for TU-games. A share vector for an n-player game is an n-dimensional real vector whose components add up to one. The ith component is player i's share in the total payoff that is to be distributed among the players. A share function assigns such a share vector to every game. The share function corresponding to the Shapley value is the Shapley share function, which is obtained by dividing the Shapley value of each player by $v(N)$, i.e., by the sum of the Shapley values of all players. Similarly, the Banzhaf share function is obtained by dividing the Banzhaf-value or normalized Banzhaf-value by the corresponding sum of payoffs over all players. One advantage of share functions over value functions is that share functions avoid the "efficiency issue", i.e., they avoid the question of what is the final worth to be distributed over the players. This yields some major simplifications. For example, although the Banzhaf and normalized Banzhaf value are very different value functions (e.g. the Banzhaf value satisfies linearity and the dummy player property which are not satisfied by the normalized Banzhaf value), they correspond to the same Banzhaf share function. Another main advantage of share functions has been discovered by [15], who shows that on a ratio scale meaningful statements can be made for a certain class of share functions, whereas all statements with respect to value functions are meaningless. Besides the advantages of share functions for general TU-games, in [2, 20] they study share

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functions for so-called games in coalition structure, for which an extra advantage is that they provide a natural method to define solutions for such games. Share functions, when multiplied by the worth of the grand coalition N, yield a distribution of the worth of the grand coalition reflecting the individual bargaining position of the players.

Mares and Vlach [12, 13] were concerned about the uncertainty in the value of the characteristic function associated with a game. In their models, the domain of the characteristic function of a game remains to be the class of crisp (deterministic) coalitions but the values assigned to them are fuzzy quantities. However, the implicit assumption that all players and coalitions know the expected payoffs even before the negotiation process, is evidently unrealistic. In fact, during the process of negotiation and coalition forming, the players can have only vague idea about the real outcome of the situation, and this vague expectation can be modeled by mathematical tools (see [12]).

In this paper, we consider the fuzzy share functions of a cooperative fuzzy game with fuzzy characteristic function. The paper will be organized as follows. In Section 2, we introduce the concepts of fuzzy numbers and the Hukuhara difference on fuzzy numbers. Then, the model of cooperative fuzzy games is introduced. Moreover, some basic concepts of crisp games will be discussed. In Section 3, the fuzzy share functions of cooperative fuzzy games with fuzzy characteristic function is proposed, we generalize the axiom of additivity by introducing a positive fuzzy valued function $\tilde{\mu}$ on the class of cooperative fuzzy games in fuzzy characteristic function form. The so-called axiom of $\tilde{\mu}$-additivity generalizes the classical axiom of additivity by putting the weight $\tilde{\mu}(\tilde{v})$ on the value of the game $\tilde{v}$. We show that any additive function $\tilde{\mu}$ determines a unique fuzzy share function satisfying the axioms of efficient shares, null player property, symmetry and $\tilde{\mu}$-additivity on the subclass of games on which $\tilde{\mu}$ is positive and which contains all positively scaled unanimity games. In Section 4, we introduce fuzzy Shapley share functions and fuzzy Banzhaf share functions, furthermore, an applicable example is given. Finally, some conclusions will be discussed in Section 5.

2. Preliminaries

In this section, we first recall the concept of fuzzy number, and then introduce some basic concepts and notations in cooperative games with fuzzy characteristic functions.

2.1 A review of fuzzy numbers

Let us start by recalling the most general definition of a fuzzy number. Let $\mathbb{R}$ be $(-\infty, +\infty)$, i.e., the set of all real numbers.

Definition 2.1. A fuzzy number, denoted by $\tilde{a}$, is a fuzzy subset of $\mathbb{R}$ with membership function $u_{\tilde{a}} : \mathbb{R} \to [0, 1]$ satisfying the following conditions:

(1) there exists at least one number $a_0 \in \mathbb{R}$ such that $u_{\tilde{a}}(a_0) = 1$;

(2) $u_{\tilde{a}}(x)$ is nondecreasing on $(-\infty, a_0)$ and nonincreasing on $(a_0, +\infty)$;

(3) $u_{\tilde{a}}(x)$ is upper semi-continuous, i.e., $lim_{x\to x_0^+} u_{\tilde{a}}(x) = u_{\tilde{a}}(x_0)$ if $x_0 < a_0$; and $lim_{x\to x_0^-} u_{\tilde{a}}(x) = u_{\tilde{a}}(x_0)$ if $x_0 > a_0$;

(4) Supp$(u_{\tilde{a}})$, the support set of $\tilde{a}$, is compact, where Supp$(u_{\tilde{a}}) = cl\{x \in (R) | u_{\tilde{a}}(x) > 0\}$.

We denote the set of all fuzzy numbers by $\Re$. An important type of fuzzy numbers in common use is the triangular fuzzy number [9], whose membership function has the form

$$u_{\tilde{a}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & x \in [a_1, a_2], \\ \dfrac{a_3 - x}{a_3 - a_2}, & x \in [a_2, a_3], \\ 0, & otherwise. \end{cases}$$

where $a_1, a_2, a_3 \in \mathbb{R}$ with $a_1 \le a_2 \le a_3$.

For a fuzzy number $\tilde{a} \in \Re$, the level set is defined as $\tilde{a}_\lambda = \{x \in \mathbb{R} | u_{\tilde{a}}(x) \ge \lambda\}$, $u_{\tilde{a}}(x) \in [0, 1]$. It follows from the properties of the membership function of a fuzzy number $\tilde{a}$ that each of its λ-cuts $\tilde{a}_\lambda$

is an interval number, denoted by $\tilde{a}_\lambda = [\tilde{a}^L_\lambda, \tilde{a}^R_\lambda]$, $\lambda \in (0,1]$, where $\tilde{a}^L_\lambda$ and $\tilde{a}^R_\lambda$ mean the lower and upper bounds of $\tilde{a}_\lambda$.

Let $\tilde{a}, \tilde{b} \in \Re$, and let $*$ be a binary operation on $\mathbb{R}$. The $*$ operation can be extended to fuzzy numbers by means of Zadeh's extension principle [22] in the following way:

$u_{\tilde{a}*\tilde{b}}(z) = \sup_{x*y} min\{u_{\tilde{a}}(x), u_{\tilde{b}}(x)\}, z \in \mathbb{R},$ (2.1)

where $\tilde{a} * \tilde{b}$ is a fuzzy number with the membership function $u_{\tilde{a}*\tilde{b}}$.

It is not easy to apply Eq.(2.1) in calculation directly. However, calculating λ-cuts of the fuzzy number $\tilde{a} * \tilde{b}$ is an easy task in each case because

$(\tilde{a} + \tilde{b})_\lambda = \tilde{a}_\lambda + \tilde{b}_\lambda = [\tilde{a}^L_\lambda + \tilde{b}^L_\lambda, \tilde{a}^R_\lambda + \tilde{b}^R_\lambda],$

$(\tilde{a} - \tilde{b})_\lambda = \tilde{a}_\lambda - \tilde{b}_\lambda = [\tilde{a}^L_\lambda - \tilde{b}^R_\lambda, \tilde{a}^R_\lambda - \tilde{b}^L_\lambda],$

$(m\tilde{a})_\lambda = m\tilde{a}_\lambda = [m\tilde{a}^L_\lambda, m\tilde{a}^R_\lambda], \forall m \in \mathbb{R}, m > 0,$

$(\tilde{a}\tilde{b})_\lambda = [min\{\tilde{a}^L_\lambda\tilde{b}^L_\lambda, \tilde{a}^L_\lambda\tilde{b}^R_\lambda, \tilde{a}^R_\lambda\tilde{b}^L_\lambda, \tilde{a}^R_\lambda\tilde{b}^R_\lambda\}, max\{\tilde{a}^L_\lambda\tilde{b}^L_\lambda, \tilde{a}^L_\lambda\tilde{b}^R_\lambda, \tilde{a}^R_\lambda\tilde{b}^L_\lambda, \tilde{a}^R_\lambda\tilde{b}^R_\lambda\}].$

Definition 2.2. For any two fuzzy numbers $\tilde{a}, \tilde{b} \in \Re$, we write

(1) $\tilde{a} \geq \tilde{b}$ if and only if $\tilde{a}^L_\lambda \geq \tilde{b}^L_\lambda$ and $\tilde{a}^R_\lambda \geq \tilde{b}^R_\lambda$, $\forall \lambda \in (0,1]$;

(2) $\tilde{a} = \tilde{b}$ if and only if $\tilde{a} \geq \tilde{b}$ and $\tilde{b} \geq \tilde{a}$;

(3) $\tilde{a} \subseteq \tilde{b}$ if and only if $\tilde{a}^L_\lambda \geq \tilde{b}^L_\lambda$ and $\tilde{a}^R_\lambda \leq \tilde{b}^R_\lambda$, $\forall \lambda \in (0,1]$.

Remark 2.1. The ordering "$\geq$" between fuzzy numbers in Definition 2.2 has been defined in [9], which is the extension of the max operator to fuzzy numbers with Zadeh's extension principle, i.e.,

$\tilde{a} \geq \tilde{b}$ if and only $max\{\tilde{a}, \tilde{b}\} = \tilde{a},\ \forall \tilde{a}, \tilde{b} \in \Re$.

In this paper, we will use the Hukuhara difference between fuzzy numbers [4,10] as follows.

Definition 2.3. Let $\tilde{a}, \tilde{b} \in \Re$. If there exists $\tilde{c} \in \Re$ such that $\tilde{a} = \tilde{b} + \tilde{c}$, then $\tilde{c}$ is called the Hukuhara difference, and denoted by $\tilde{c} = \tilde{a} -_H \tilde{b}$.

Remark 2.2. The Hukuhara difference is defined as an inverse calculation of the "+" operator defined based on Zadeh's extension principle. But the Hukuhara difference between two fuzzy numbers does not always exists. Regarding the existence of the Hukuhara difference, there is an extensive literature described in [9].

Theorem 2.1. Let $\tilde{a}, \tilde{b} \in \Re$. The Hukuhara difference $\tilde{c} = \tilde{a} -_H \tilde{b}$ exists if and only if
$\tilde{a}^L_\lambda - \tilde{b}^L_\lambda \leq \tilde{a}^L_\beta - \tilde{b}^L_\beta \leq \tilde{a}^R_\beta - \tilde{b}^R_\beta \leq \tilde{a}^R_\lambda - \tilde{b}^R_\lambda,\ \forall \lambda, \beta \in (0,1],\ \beta > \lambda.$

Lemma 2.1. Let $\tilde{a}, \tilde{b} \in \Re$. If $\tilde{a} -_H \tilde{b}$ exists, then for any $\lambda \in (0,1]$,
$(\tilde{a} -_H \tilde{b})_\lambda = \tilde{a}_\lambda -_H \tilde{b}_\lambda = [\tilde{a}^L_\lambda - \tilde{b}^L_\lambda, \tilde{a}^R_\lambda - \tilde{b}^R_\lambda].$

Lemma 2.2. Let $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \Re$. If $\tilde{a} -_H \tilde{b}$ and $\tilde{c} -_H \tilde{d}$ exists, then
$(\tilde{a} + \tilde{c}) -_H (\tilde{b} + \tilde{d}) = (\tilde{a} -_H \tilde{b}) + (\tilde{c} -_H \tilde{d}).$

2.2 Cooperative games with fuzzy characteristic functions

We consider cooperative games with the set of players $N = \{1, 2, \ldots, n\}$. A cooperative crisp game is defined by (N, v), in which N is the set of players and the characteristic function $v : 2^N \to \mathbb{R}_+ = \{r \in \mathbb{R} | r \geq 0\}$ satisfies the condition that $v(\emptyset) = 0$.

In a cooperative crisp game, a characteristic function v describes a cooperative game and associates a crisp coalition S with the worth $v(S)$, which is interpreted as the payoff that the coalition S can acquire only through the action of S. The cooperative crisp game is based on the assumption that all players and coalitions know the payoff value v before the cooperation begins. As Borkotokey [5] says, this assumption is not realistic because there are many uncertain factors during negotiation and coalition formation. In many situations, the players can have only vague ideas about the real payoff value. Taking into account the imprecision of information in decision making problems, we incorporate a fuzzy characteristic function, which is represented by fuzzy numbers $\tilde{v}(S)$. Therefore, the characteristic function of such a game associates a crisp coalition $S \in \mathcal{P}(N)$ with a fuzzy number $\tilde{v}(S)$. Assessing such fuzzy numbers for any crisp coalition $S \in \mathcal{P}(N)$, we define a cooperative game with fuzzy characteristic values by a pair $(N, \tilde{v})$, where the fuzzy characteristic function $\tilde{v} : \mathcal{P}(N) \to \Re_+$ is such that $\tilde{v}(\emptyset) = 0$. Obviously, games with fuzzy characteristic functions are a kind of cooperative fuzzy games. Hereinafter, a cooperative game with fuzzy characteristic function will be called a "cooperative fuzzy game" for short.

Along the paper we use the $|\cdot|$ operator to denote the cardinality of a finite set, i.e., $|S|$ is the number of players in S, for any $S \subseteq N$. Alternatively, sometimes we use lowercase letters to denote cardinalities, and thus $s = |S|$ for any $S \subseteq N$. A fuzzy game $(N, \tilde{v})$ is called *monotone* if for every $S, T \subseteq N$ with $T \subseteq S$, it holds that $\tilde{v}^L_\lambda(T) \leq \tilde{v}^L_\lambda(S)$ and $\tilde{v}^R_\lambda(T) \leq \tilde{v}^R_\lambda(S)$. That is, monotone fuzzy games are those in which the cooperation among players is never pernicious. Since the whole paper deals with monotone games, henceforth we will simply say game instead of monotone game.

For each $S \subseteq N$ and $i \in N$, we will write $S \cup i$ instead of $S \cup \{i\}$ and $S \setminus i$ instead of $S \setminus \{i\}$. For a pair of fuzzy games $(N, \tilde{w}), (N, \tilde{v}) \in \mathcal{FG}$, the game $(N, \tilde{z})$ is defined by $\tilde{z}(S) = \tilde{w}(S) + \tilde{v}(S)$ for all $S \subseteq N$. Further, given $S \in \mathcal{P}(N)$, the unanimity game with carrier S, $(N, u_T(S))$, is defined by $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise. Notice that $(N, u_T(S)) \in \mathcal{FG}$ for every $S \in \mathcal{P}(N)$.

Given $(N, \tilde{v}) \in \mathcal{FG}$, a player $i \in N$ is a *dummy* if $\tilde{v}(S \cup i) = \tilde{v}(S) + \tilde{v}(i)$ for all $S \subseteq N \setminus i$, that is, if all her marginal contributions are equal to $\tilde{v}(i)$. A player $i \in N$ is called a *null player* if she is a dummy and $\tilde{v}(i) = 0$. Two players $i, j \in N$ are *symmetric* if $\tilde{v}(S \cup i) = \tilde{v}(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$, that is, if their marginal contributions to each coalition coincide.

3. Fuzzy Share Functions

In this section we extend the share function introduced by Van der Laan et al. in [18] to a fuzzy environment. We consider a class of fuzzy share functions for n-person, the basic concept of fuzzy share functions is that it assigns to each player his fuzzy share in the payoff $\tilde{v}(N)$ of the grand coalition N, i.e., a fuzzy share function on a class $\mathcal{FC}$ of games is a function $\tilde{\rho} : \mathcal{FC} \to \Re^n$ giving player i the share $\tilde{\rho}_i(\tilde{v})$ in the value $\tilde{v}(N)$, where $\mathcal{FC}$ is the subset of $\mathcal{FG}$ that is $\mathcal{FC} \subset \mathcal{FG}$. So, for any game $\tilde{v}$, a fuzzy share function $\tilde{\rho}$ gives a fuzzy payoff $\tilde{\rho}_i(\tilde{v})\tilde{v}(N)$ to player $i, i = 1, 2, \ldots, n$. Observe that we do not require a priori that the share is nonnegative, although for monotone games this seems to be reasonable. We return to this point at the end of this section. Of course the total payoff equals $\tilde{v}(N)$ if and only if $\sum_{i=1}^n \tilde{\rho}_i(\tilde{v}) = 1$. Therefore, for a share function $\tilde{\rho}$ on $\mathcal{FC}$, we redefine the axiom of efficiency as follows.

AXIOM 3.1. For any $\tilde{v} \in \mathcal{FC}$, $\sum_{i=1}^n \tilde{\rho}_i(\tilde{v}) = 1$.

Now, let $\tilde{\mu} : \mathcal{FC} \to \Re$ be a fuzzy valued function on the class $\mathcal{FC}$ of games. Then we have the following definition.

Definition 3.1.

(1) A fuzzy valued function $\tilde{\mu} : \mathcal{FC} \to \Re$ is called *additive* on the class $\mathcal{FC}$ of fuzzy games if for any pair $\tilde{w}, \tilde{v}$ on the class $\mathcal{FC}$ such that $\tilde{w} + \tilde{v} \in \mathcal{FC}$, it holds that $\tilde{\mu}^L_\lambda(\tilde{w} + \tilde{v}) = \tilde{\mu}^L_\lambda(\tilde{w}) + \tilde{\mu}^L_\lambda(\tilde{v})$, $\tilde{\mu}^R_\lambda(\tilde{w} + \tilde{v}) = \tilde{\mu}^R_\lambda(\tilde{w}) + \tilde{\mu}^R_\lambda(\tilde{v})$.

(2) A fuzzy valued function $\tilde{\mu} : \mathcal{FC} \to \Re$ is called *linear* on the class $\mathcal{FC}$ of fuzzy games if it is additive and if for any $\tilde{v}$ on $\mathcal{FC}$ it holds that $\tilde{\mu}(\alpha\tilde{v}) = \alpha\tilde{\mu}(\tilde{v})$.

(3) A fuzzy valued function $\tilde{\mu} : \mathcal{FC} \to \Re$ is called *positive* on the class $\mathcal{FC}$ of fuzzy games if $\tilde{\mu}(\tilde{v}) > 0$ for all $\tilde{v} \in \mathcal{FC}$.

For instance the function $\tilde{\mu}$ defined by $\tilde{\mu}(\tilde{v}) = \sum_{T \subset N} \tilde{v}(T)$ is additive function. For a given function $\tilde{\mu}$, we generalize the axioms of additivity and linearity to the concepts of $\tilde{\mu}$-additivity and $\tilde{\mu}$-linearity of a share function $\tilde{\rho}$ on a class $\mathcal{FC}$.

AXIOM 3.2.($\tilde{\mu}$-additivity) Let $\tilde{\mu} : \mathcal{FC} \to \Re$ be given. Then for any pair $\tilde{w}$ and $\tilde{v}$ of games in $\mathcal{FC}$ such that $\tilde{w} + \tilde{v} \in \mathcal{FC}$ it holds that $\tilde{\mu}_\lambda(\tilde{w} + \tilde{v})\tilde{\rho}_\lambda(\tilde{w} + \tilde{v}) = [\tilde{\mu}(\tilde{w})\tilde{\rho}(\tilde{w})]_\lambda + [\tilde{\mu}(\tilde{v})\tilde{\rho}(\tilde{v})]_\lambda$.

AXIOM 3.3.($\tilde{\mu}$-linearity) Let $\tilde{\mu} : \mathcal{FC} \to \Re$ be given. Then for any pair $\tilde{w}$ and $\tilde{v}$ of games in $\mathcal{FC}$ such that $\tilde{\mu}_\lambda(a\tilde{w} + b\tilde{v})\tilde{\rho}_\lambda(a\tilde{w} + b\tilde{v}) = a[\tilde{\mu}(\tilde{w})\tilde{\rho}(\tilde{w})]_\lambda + b[\tilde{\mu}(\tilde{v})\tilde{\rho}(\tilde{v})]_\lambda$ for any pair of real numbers a and b such that $a\tilde{w} + b\tilde{v} \in \mathcal{FC}$.

Definition 3.2[7]. Let $I : \Re \to \mathbb{R}$, this function is defined by

$$I(m) = \frac{\int_0^1 xm(x)dx}{\int_0^1 m(x)dx}.$$

$I(m)$ denotes the center of gravity of m .

If $m = (a_1, a_2, a_3)$ be a triangular fuzzy number, the center of gravity of m is defined by

$$I_t(m) = \frac{\int_0^1 xm(x)dx}{\int_0^1 m(x)dx} = \frac{a_1 + a_2 + a_3}{3}.$$

Since the center of gravity could approximately denote the value of fuzzy number, so we use the center of gravity $I(\tilde{v})$ to denote the value $\tilde{v}(N)$.

We are now able to define a class of fuzzy share functions. Therefore, let $\mathcal{FG}$ be the collection of all subclasses of games such that for any subclass $\mathcal{FC} \in \mathcal{FG}$ holds that $\alpha u_T \in \mathcal{FC}$ for any $T \subset N$ and any real number $\alpha > 0$, i.e., $\mathcal{FG}$ is the collection of all subclasses containing all positively scaled unanimity games. As is known, see [10, 11, 21], every $\tilde{v} \in \mathcal{FG}$ can be expressed as $\tilde{v}(S) = \sum_{T \subset P(I):T \neq \emptyset} u_T(S)c_T(\tilde{v})$ with the so-called *dividends* $c_T(\tilde{v})$. So, any game $\tilde{v} \in \mathcal{FG}$ can be written as the sum of scaled unanimity games with the dividend $c_T(\tilde{v})$ as the scale of u_T, $T \subset N$. In [21] we know that

$$\tilde{v}(S) = \sum_{T \in P(I):T \neq \emptyset} u_T(S)\tilde{c}_T(\tilde{v}) = \sum_{\substack{T \in P(I):T \neq \emptyset \\ \tilde{c}_T(\tilde{v}) \geq 0}} u_T(S)\tilde{c}_T(\tilde{v}) - \sum_{\substack{T \in P(I):T \neq \emptyset \\ \tilde{c}_T(\tilde{v}) < 0}} u_T(S)(-\tilde{c}_T(\tilde{v}))$$

where $\tilde{c}_T(\tilde{v}) = Sup\{\lambda \in [0,1] | x \in \bar{c}_T^{\lambda}(\tilde{v})\}$, $\bar{c}_T^{\lambda}(\tilde{v}) = [c_T(\tilde{v}_\lambda^L), c_T(\tilde{v}_\lambda^R)]$ and

$$c_T(\tilde{v}_\lambda^L) = \sum_{T \subset N} (-1)^{(|T|-|S|)} \tilde{v}_\lambda^L(S), \quad c_T(\tilde{v}_\lambda^R) = \sum_{T \subset N} (-1)^{(|T|-|S|)} \tilde{v}_\lambda^R(S).$$

Especially, for any $S \in \mathcal{P}(N)$, we let $[\tilde{v}_0^L(S), \tilde{v}_0^R(S)] = cl\{x \in R | \tilde{v}(S)(x) > 0\}$, where cl denotes the closure of sets, and let

$$c_T(\tilde{v}_0^L) = \sum_{T \subset N} (-1)^{(|T|-|S|)} \tilde{v}_0^L(S), \quad c_T(\tilde{v}_0^R) = \sum_{T \subset N} (-1)^{(|T|-|S|)} \tilde{v}_0^R(S),$$

for $T \in \mathcal{P}(N) \setminus \emptyset$.

Proposition 3.1[21]. Let $v \in G_H(I)$ satisfy the following three conditions:

(i) $c_T(v_\lambda^R) \geq c_T(v_\lambda^L)$, $\forall \lambda \in (0, 1]$, $\forall T \in P(I)$;

(ii) $c_T(v_0^R) \leq 0$ or $c_T(v_0^L) \geq 0$, $\forall T \in P(I)$;

(iii) $[c_T(v_\beta^L), c_T(v_\beta^R)] \subseteq [c_T(v_\lambda^L), c_T(v_\lambda^R)]$, $\forall T \in P(I)$, $\forall \lambda, \beta \in (0, 1]$, $\lambda < \beta$.

Then the Hukuhara-Shapley function is the unique Shapley value for game v.

Theorem 3.1. For some subclass of games $\mathcal{FC} \in \mathcal{FG}$, let $\tilde{\mu} : \mathcal{FC} \to \Re$ be a positive fuzzy value function on $\mathcal{FC}$. Then on the subclass $\mathcal{FC}$ there exists a unique fuzzy share function $\tilde{\rho} : \mathcal{FC} \to \Re^n$ satisfying the axioms of efficient shares, null player property, symmetry and $\tilde{\mu}$-additivity if and only if $\tilde{\mu}$ is additive on $\mathcal{FC}$.

Proof. Firstly, we suppose $\tilde{\rho}$ satisfies efficiency and $\tilde{\mu}$-additivity. From the $\tilde{\mu}$-additivity it follows that

$$\begin{aligned}
& \tilde{\mu}_\lambda(\tilde{w} + \tilde{v}) \sum_{i=1}^n \rho_i(\tilde{w} + \tilde{v}) \\
= \;& \tilde{\mu}_\lambda(\tilde{w} + \tilde{v})[(\rho_1(\tilde{w} + \tilde{v})_\lambda + \ldots + (\rho_n(\tilde{w} + \tilde{v})_\lambda] \\
= \;& \tilde{\mu}_\lambda(\tilde{w} + \tilde{v})[\rho_1(\tilde{w} + \tilde{v})]_\lambda + \ldots + \tilde{\mu}_\lambda(\tilde{w} + \tilde{v})[\rho_n(\tilde{w} + \tilde{v})]_\lambda \\
= \;& [\tilde{\mu}_\lambda(\tilde{w})(\rho_1(\tilde{w}))_\lambda + \tilde{\mu}_\lambda(\tilde{v})(\rho_1(\tilde{v}))_\lambda] + \ldots + [\tilde{\mu}_\lambda(\tilde{w})(\rho_n(\tilde{w}))_\lambda + \tilde{\mu}_\lambda(\tilde{v})(\rho_n(\tilde{v}))_\lambda] \\
= \;& \tilde{\mu}_\lambda(\tilde{w})[(\rho_1(\tilde{w}))_\lambda + \ldots + (\rho_n(\tilde{w}))_\lambda] + \tilde{\mu}_\lambda(\tilde{v})[(\rho_1(\tilde{v}))_\lambda + \ldots + (\rho_n(\tilde{v}))_\lambda] \\
= \;& \tilde{\mu}_\lambda(\tilde{w}) \sum_{i=1}^n \rho_i(\tilde{w}) + \tilde{\mu}_\lambda(\tilde{v}) \sum_{i=1}^n \rho_i(\tilde{v})
\end{aligned}$$

and for any $\tilde{w}, \tilde{v} \in \mathcal{FC}$ such that $\tilde{w}+\tilde{v} \in \mathcal{FC}$. Efficiency then implies that $\tilde{\mu}(\tilde{w}+\tilde{v}) = \tilde{\mu}(\tilde{w})+\tilde{\mu}(\tilde{v})$. Hence $\tilde{\mu}$ must be additive.

Secondly, we assume that $\tilde{\mu}$ is additive. We shall show that there can be at most one share function $\tilde{\rho} : \mathcal{FC} \to \Re^n$ satisfying the four axioms. Therefore, let $\tilde{\rho} : \mathcal{FC} \to \Re^n$ be a function satisfying the axioms. Recall that any positively scaled unanimity game belongs to the subclass $\mathcal{FC}$. For a unanimity game u_T, we have that two players i and j are symmetric if they are both in T, whereas a player not in T is a null player. Hence from the symmetry, null player property and efficient shares axioms it follows that for any positively scaled unanimity game αu_T, $\alpha > 0$, it holds that

$$\tilde{\rho}_i(\alpha u_T) = \frac{1}{|T|}, \quad \text{when } i \in T, \qquad (3.1)$$

$$\tilde{\rho}_i(\alpha u_T) = 0, \quad \text{when } i \notin T, \qquad (3.2)$$

Now, with $\tilde{c}_T(\tilde{v})$ the dividends of the game $\tilde{v}$, we can rewrite $\tilde{c}_T(\tilde{v})$ as the difference of two sums of positively scaled unanimity games by

$$\tilde{v}(S) = \sum_{T\in P(I):T\neq\emptyset} u_T(S)\tilde{c}_T(\tilde{v}) = \sum_{\substack{T\in P(I):T\neq\emptyset\\ \tilde{c}_T(\tilde{v})\geq 0}} u_T(S)\tilde{c}_T(\tilde{v}) - \sum_{\substack{T\in P(I):T\neq\emptyset\\ \tilde{c}_T(\tilde{v})<0}} u_T(S)(-\tilde{c}_T(\tilde{v})).$$

Since $\tilde{v}$ is a positive function on $\mathcal{FC}$ and $\mathcal{FC}$ contains $\tilde{v}$ and all positively scaled unanimity games, it follows by applying the axiom of $\tilde{u}$-additivity repeatedly that $\tilde{\rho}(\tilde{v})$ is uniquely defined by

$$\tilde{\mu}(\tilde{v})\tilde{\rho}(\tilde{v}) = \textstyle\sum_{\tilde{c}_T(\tilde{v})\geq 0} \tilde{\mu}(\tilde{c}_T(\tilde{v})u_T)\tilde{\rho}(\tilde{c}_T(\tilde{v})u_T) - \sum_{\tilde{c}_T(\tilde{v})<0} \tilde{\mu}(-\tilde{c}_T(\tilde{v})u_T)\tilde{\rho}(-\tilde{c}_T(\tilde{v})u_T). \quad (3.3)$$

It only need to prove that $\tilde{\rho}$ indeed satisfies the axioms.

First, because of the additivity of $\tilde{\mu}$ it holds that

$$\tilde{\mu}(\tilde{v}) = \textstyle\sum_{\tilde{c}_T(\tilde{v})\geq 0} \tilde{\mu}(\tilde{c}_T(\tilde{v})u_T) - \sum_{\tilde{c}_T(\tilde{v})<0} \tilde{\mu}(-\tilde{c}_T(\tilde{v})u_T). \qquad (3.4)$$

Hence it follows from equation (3.1), (3.2), (3.3) and (3.4) that $\sum_{j=1}^{n} \tilde{\rho}_j(\tilde{v}) = 1$ and therefore the axiom of efficient shares is satisfied. Second, observe that a null player in $\tilde{v}$ is a null player in any u_T with nonzero dividend $\tilde{c}_T(\tilde{v})$. Hence, by Equations (3.1) and (3.3) and the positiveness of $\tilde{v}$ it follows that $\tilde{\rho}$ satisfies the null player property. Third, if i and j are two symmetric players in $\tilde{v}$, then $\tilde{c}_T(\tilde{v})_i = \tilde{c}_T(\tilde{v})_j$, whereas for each other $T \subset N$ with nonzero weight $\tilde{c}_T(\tilde{v})$, i and j are either both in T or both not in T. Hence by Equations (3.1), (3.2) and (3.3) and the positiveness of $\tilde{\mu}$ it follows that $\tilde{\rho}$ satisfies the symmetry property. Finally, for any two games $\tilde{v}, \tilde{w} \in \mathcal{FC}$ we have that $\tilde{v}+\tilde{w} = \sum_{T\subset N}(\tilde{c}_T(\tilde{v})+\tilde{c}_T(\tilde{w}))u_T$. Together with Equation (3.3) and the additivity of $\tilde{u}$ this implies that $\tilde{\mu}(\tilde{v}+\tilde{w})\tilde{\rho}(\tilde{v}+\tilde{w}) = \tilde{\mu}(\tilde{v})\tilde{\rho}(\tilde{v}) + \tilde{\mu}(\tilde{w})\tilde{\rho}(\tilde{w})$ and hence $\tilde{\rho}$ is $\tilde{\mu}$-additive.

Theorem 3.2. For given positive numbers ω_t, $t = 1, \ldots, n$, let the function $\tilde{\mu}^\omega$ be defined by

$$\tilde{\mu}^\omega(\tilde{v}) = \textstyle\sum_{i\in N}\sum_{\{T|i\in T\}} \omega_t m_T^i = \sum_{i\in N}\sum_{\{T|i\in T\}} \omega_t[\tilde{v}(T\cup i) -_H \tilde{v}(T)] = I(\tilde{v}),$$

where $t = |T|$. Then the share function $\tilde{\rho}^\omega$ defined by

$$\tilde{\rho}_i^\omega(\tilde{v}) = \frac{\sum_{\{T|i\in T\}} \omega_t m_T^i}{I(\tilde{v})} = \frac{\sum_{\{T|i\in T\}} \omega_t[\tilde{v}(T\cup i) -_H \tilde{v}(T)]}{I(\tilde{v})}, \; i \in N, \quad (3.5)$$

is the unique share function satisfying the axioms of efficient shares, null player property, symmetry and $\tilde{\mu}^\omega$-additive on the subclass $\mathcal{FC}$ of $\mathcal{FG}$ on which $\tilde{\mu}^\omega$ is positive.

Proof. By definition of $\tilde{\mu}^\omega$, all positively scaled unanimity games αu_T are $\tilde{\mu}^\omega$ positively and hence $\mathcal{FC} \in \mathcal{FG}$. Moreover, $\tilde{\mu}^\omega$ is additive. Hence, it follows from Theorem 3.1 that there exists a unique share function that satisfies the four axioms with respect to $\tilde{\mu}^\omega$ on the class $\mathcal{FC}$ of $\tilde{\mu}^\omega$-positive games.

It remains to show that $\tilde{\rho}^\omega$ indeed satisfies the four axioms. First, by definition we have that $\tilde{\rho}^\omega$ satisfies the efficient shares axiom. Second, since $m_T^i(\tilde{v}) = 0$ for all $T \subset N$ if i is a null player, the null player property is satisfied. Third, if i and j are symmetric we have that $m_T^i(\tilde{v}) = m_T^j(\tilde{v})$ for all $T \subset N$ containing both i and j, $m_{T\cup\{i\}}^i(\tilde{v}) = m_{T\cup\{j\}}^j(\tilde{v})$ for all $T \subset N$ such that both $i, j \notin T$ and $m_T^i(\tilde{v}) = m_{T\cup\{j\}\setminus\{i\}}^j(\tilde{v})$ for all $T \subset N$ such that $i \in T$ and $j \notin T$. Since the weights ω_t only depend on t this implies that the symmetry axiom holds. Finally, observe that

$\tilde{\mu}^{\omega}(\tilde{v})\rho_i^{\omega}(\tilde{v}) = \sum_{\{T|i\in T\}} \omega_t m_T^i(\tilde{v}),\ i = 1, \ldots, n.$

Since for all i and T it holds that $m_T^i(a\tilde{v} + b\tilde{w}) = am_T^i(\tilde{v}) + bm_T^i(\tilde{w})$, it follows that $\tilde{\rho}^{\omega}$ is $\tilde{\mu}^{\omega}$-linear and hence also $\tilde{\mu}^{\omega}$-additive.

Using the same method in [21], we could get the following Lemma 3.1.

Lemma 3.1[21]. Let $S \in P(N)$ and $i \in S$. Then we have

$[\tilde{\rho}(\tilde{v})(S)]_{\lambda} = \tilde{\rho}(\tilde{v})_{\lambda}(S), \forall\lambda \in (0, 1],$

where the function is defined by Eq.(3.5).

4. Fuzzy Share Functions: Examples

In this section, we will introduce fuzzy Shapley share functions and fuzzy Banzhaf share functions, furthermore, an applicable example is given.

Definition 4.1[21]. The Shapley value, assigns to any game $(N, v) \in \mathcal{G}$ a vector in $\Re^n$ defined as

$$\phi_i^S(\tilde{v}) = \sum_{S\subseteq N\setminus i} \frac{s!(n-s-1)!}{n!}[\tilde{v}(S \cup i) -_H \tilde{v}(S)],\ i \in N. \quad (4.1)$$

Definition 4.2[21]. The Banzhaf value, assigns to any game $(N, v) \in G$ a vector in $\Re^n$ defined as

$$\phi_i^B(\tilde{v}) = \sum_{S\subseteq N\setminus i} \frac{1}{2^{n-1}}[\tilde{v}(S \cup i) -_H \tilde{v}(S)],\ i \in N. \quad (4.2)$$

Definition 4.3[2].

(i) Given a game $(N, v) \in G$, the Shapley share function, ρ^S, assigns to any game $(N, v) \in \mathcal{G}$ a vector in $\mathbb{R}^n$ defined as $\rho_i^S(N, v) = \dfrac{\phi_i^S(v)}{v(N)}$, $i \in N$, if $v \neq v_0$, and $\rho_i^S(N, v) = \frac{1}{|N|}$, $i \in N$.

(ii) Given a game $(N, v) \in G$, the Banzhaf share function, ρ^B, assigns to any game $(N, v) \in \mathcal{G}$ a vector in $\mathbb{R}^n$ defined as $\rho_i^B(N, v) = \dfrac{\phi_i^B(v)}{\sum_{j\in N}\phi_j^B(v)}$, $i \in N$, if $v \neq v_0$, and $\rho_i^B(N, v) = \frac{1}{|N|}$, $i \in N$.

Definition 4.4 Given a game $(N, v) \in \mathcal{FG}$, the fuzzy Shapley share function, $\tilde{\rho}^S$, assigns to any game $(N, v) \in \mathcal{FG}$ a vector in $\Re^n$ defined as $\tilde{\rho}_i^S(N, v) = \frac{\tilde{\phi}_i^S(N,v)}{I(\tilde{v})}, i \in N$, if $v \neq v_0$, and $\tilde{\rho}_i^S(N, v) = \frac{1}{|N|}$, $i \in N$.

Definition 4.5 Given a game $(N, v) \in \mathcal{FG}$, the fuzzy Banzhaf share function, $\tilde{\rho}^B$, assigns to any game $(N, v) \in \mathcal{FG}$ a vector in $\Re^n$ defined as $\tilde{\rho}_i^B(N, v) = \frac{\tilde{\phi}_i^B(N,v)}{I(\tilde{v})}, i \in N$, if $v \neq v_0$, and $\tilde{\rho}_i^B(N, v) = \frac{1}{|N|}$, $i \in N$.

In Theorem 3.1 the class $\mathcal{FC}$ is restricted by the condition that the function $\tilde{\mu}$ must satisfy $\tilde{\mu}(\tilde{v}) > 0$ for all $\tilde{v} \in \mathcal{FC}$. So, the restrictions on the class $\mathcal{FC}$ depend on the way in which $\tilde{\mu}$ is specified. For instance, if $\tilde{\mu}(\tilde{v}) = \tilde{v}(N)$ we have to exclude games with $\tilde{v}(N) \leq 0$. In the remaining of this paper, let $\mathcal{FG}_{\tilde{\mu}}$ denote the class of $\tilde{\mu}$-positive games, i.e.

$\mathcal{FG}_{\tilde{\mu}} = \{\tilde{v} \in \mathcal{FG} | \tilde{\mu}(\tilde{v}) > 0\}.$

For a positive constant $\alpha > 0$ and a function $\tilde{\mu}$ it holds that $\mathcal{FG}_{\tilde{\mu}} = \mathcal{FG}_{\alpha\tilde{\mu}}$. Moreover, for an additive function $\tilde{\mu}$ we have that the class $\mathcal{FG}_{\tilde{\mu}}$ is additive, i.e., the game $\tilde{v} + \tilde{w}$ is $\tilde{\mu}$-additive if both $\tilde{v}, \tilde{w}$ are $\tilde{\mu}$-positive.

As shown in Theorem 4.1, $\tilde{\rho}^S$ is defined on the class $\mathcal{FG}_{\tilde{\mu}}$ with $\tilde{\mu}(\tilde{v}) = \tilde{v}(N) > 0$. Clearly on this class we have the Shapley value $\tilde{\phi}^S(\tilde{v})$ of player i is equal to his Shapley share $\tilde{\rho}_i^S(\tilde{v})$ times the value $\tilde{v}(N)$ of grand coalition.

Theorem4.1. Let the function $\tilde{\mu}^S$ be defined by $\tilde{\mu}^S = \tilde{v}(N) = I(\tilde{v})$ and let $\mathcal{FC} \subset \mathcal{FG}_{\tilde{\mu}^S}$ be a subclass of games in $\mathcal{FG}$. Then the fuzzy Shapley share function $\tilde{\rho}^S$ is the unique fuzzy share function satisfying the axioms of efficient shares, null player property, symmetry and $\tilde{u}^S$-linearity on the class $\mathcal{FC}$.

Proof. For $T \subset N$ with $T = t$, take $\omega_t = \frac{t!(n-t-1)!}{n!}$. Then, we have that $\tilde{\mu}^{\omega}$ as defined in Theorem 3.2 is given by

$\tilde{\mu}^{\omega} = \sum_{i\in N}\sum_{\{T|i\in T\}} \omega_t m_T^i(\tilde{v}) = \tilde{v}(N) = \tilde{\mu}^S(\tilde{v}) = I(\tilde{v}).$

Further, the share function $\tilde{\rho}^{\omega}$ is given by

$$\rho_i^{\omega}(\tilde{v}) = \frac{\sum_{\{T|i\in T\}}\omega_t m_T^i(\tilde{v})}{I(\tilde{v})} = \frac{\sum_{\{T|i\in T\}}\frac{t!(n-t-1)!}{n!}m_T^i(\tilde{v})}{I(\tilde{v})} = \frac{\phi_i^S(\tilde{v})}{I(\tilde{v})} = \rho_i^S(\tilde{v}), i\in N.$$

Since all positively scaled unanimity games belong to $\mathcal{FC}$ and $\tilde{\mu}^S$ is linear, it follows from Theorem 3.1 that $\tilde{\rho}^{\omega} = \tilde{\rho}^S$ is the unique share function on $\mathcal{FC}$ that satisfies the axioms. □

Since $\mathcal{FG}_{\tilde{\mu}^S} \subset \mathcal{FG}$, Theorem 4.1 holds on the class $\mathcal{FG}_{\tilde{\mu}^S}$ and the restriction to the class of $\tilde{\mu}^S$-positive games only requires that the value of the grand coalition is positive. Therefore the class of essential zero normalized games is a subset of $\tilde{\mu}^S$-positive games, so that Theorem 4.1 also holds on this class of games.

Theorem4.2. Let the function $\tilde{\mu}^B$ be defined by $\tilde{\mu}^B(\tilde{v}) = I(\tilde{v})$ and let $\mathcal{FC} \subset \mathcal{FG}_{\tilde{\mu}^B}$ be a subclass of games in $\mathcal{FG}$. Then the fuzzy Banzhaf share function $\tilde{\rho}^B$ is the unique share function satisfying the axioms of efficient shares, null player property, symmetry and $\tilde{\mu}^B$-linearity on the class $\mathcal{FC}$.
Proof. The function $\tilde{\mu}^{\omega}$ as defined in Theorem 3.2 is given by

$$\tilde{\mu}^{\omega}(\tilde{v}) = \sum_{i\in N}\sum_{\{T|i\in T\}}\omega_t m_T^i(\tilde{v}) = \tilde{\mu}^B(\tilde{v}) = I(\tilde{v}).$$

Further, the share function $\tilde{\rho}^{\omega}$ as defined in Theorem 3.2 is given by

$$\tilde{\rho}_i^{\omega}(\tilde{v}) = \frac{\sum_{\{T|i\in T\}}\omega_t m_T^i(\tilde{v})}{I(\tilde{v})} = \frac{\sum_{\{T|i\in T\}}\frac{1}{2^{n-1}}m_T^i(\tilde{v})}{I(\tilde{v})} = \tilde{\rho}_i^B(\tilde{v}),\ i\in N.$$

Since all positively scaled unanimity games belong to $\mathcal{FC}$ and $\tilde{\mu}^B$ is linear, it follows from Theorem 3.1 that $\tilde{\rho}^{\omega} = \tilde{\rho}^B$ is the unique share function on $\mathcal{FC}$ that satisfies the axioms. □

Example 4.1. Consider a joint production model in which three decision makers pool three resources to make seven finished products. Three decision makers, named $1, 2$ *and* 3, possess three different initial resources. Decision maker i has 10 tons of resource R_i and can produce n_i tons of Product P_{ii} , $i = 1, 2, 3$. Now, decision makers decide to undertake a joint project: if decision makers i and j cooperate, they will produce n_{ij} tons of product P_{ij}, and if all three cooperate, n_{123} tons of product P_{123} can be produced. The effective output of each finished product is shown in Table 1.

It is natural for the three decision makers to try to evaluate the revenue of the joint project in the early period of the project in order to decide whether the project can be realized or not. However, the average profit per ton of each product is dependent on a number of factors such as product market price, product cost, consumer demand, the relation of commodity supply and demand, etc. Hence, the average profit of each product is an approximate evaluation, which is represented by triangular fuzzy numbers as shown in Table 1.

Table 1. The effective output and the average profit of each finished product

Product	Output of product(tons)	Average Profit(thousands of dollars)
P_{11}	8.0	(1.8,2.0,2.2)
P_{12}	18.0	(2.9,3.1,3.3)
P_{13}	17.5	(2.0,2.3,2.6)
P_{22}	9.0	(2.9,3.0,3.1)
P_{23}	18.0	(3.0,3.2,3.4)
P_{33}	10.0	(0.9,1.0,1.2)
P_{123}	28.0	(3.2,3.5,3.8)

Now,we can make an imprecise assessment of the worth of each crisp coalition (i.e.,the fuzzy worth of each crisp coalition) as follows:

$$\tilde{v}(\{1\}) = 8.0\cdot(1.8, 2.0, 2.2) = (14.4, 16.0, 17.6),$$

$$\tilde{v}(\{2\}) = 9.0\cdot(2.9, 3.0, 3.1) = (26.1, 27.0, 27.9),$$

$$\tilde{v}(\{3\}) = 10.0\cdot(0.9, 1.0, 1.2) = (9.0, 10.0, 12.0),$$

$$\tilde{v}(\{1, 2\}) = 18.0\cdot(2.9, 3.1, 3.3) = (52.2, 55.8, 59.4),$$

$$\tilde{v}(\{1, 3\}) = 17.5\cdot(2.0, 2.3, 2.6) = (35.0, 40.25, 45.5),$$

$\tilde{v}(\{2,3\}) = 18.0 \cdot (3.0, 3.2, 3.4) = (54.0, 57.6, 61.2),$

$\tilde{v}(\{1,2,3\}) = 28.0 \cdot (3.2, 3.5, 3.8) = (89.6, 98.0, 106.4),$

Fuzzy share Shapley function:

We can employ the proposed Hukuhara - Shapley function in Eq.(4.1) to estimate each decision maker's share in crisp coalition $T \subseteq \{1,2,3\}$.

For example,decision maker 1 in the grand coalition $\{1,2,3\}$ has the profit share $\rho_1(\tilde{v})(\{1,2,3\})$,

$$\begin{aligned}\phi_1^S(\tilde{v})(\{1,2,3\}) &= \tfrac{1}{3}\tilde{v}(\{1\}) + \tfrac{1}{6}[\tilde{v}(\{1,2\}) -_H \tilde{v}(\{2\})] + \tfrac{1}{6}[\tilde{v}(\{1,3\}) -_H \tilde{v}(\{3\})] \\ &\quad + \tfrac{1}{3}[\tilde{v}(\{1,2,3\}) -_H \tilde{v}(\{2,3\})] \\ &= \tfrac{1}{3}(14.4, 16.0, 17.6) + \tfrac{1}{6}(26.1, 28.8, 31.5) + \tfrac{1}{6}(26.0, 30.25, 33.5) \\ &\quad + \tfrac{1}{3}(35.6, 40.4, 45.2) \\ &= (25.35, 28.64, 31.77).\end{aligned}$$

$$\tilde{v}(N) = I(\tilde{v}) = \frac{89.6 + 98.0 + 106.4}{3} = 98$$

$$\tilde{\rho}_1^S(\tilde{v})(\{1,2,3\}) = \frac{\phi_1^S(\tilde{v})(\{1,2,3\})}{I(\tilde{v})} = \frac{(25.35, 28.64, 31.77)}{98} = (0.2587, 0.2922, 0.3242)$$

Using a similar method, the fuzzy share Shapley value for this game can be obtained as shown in Table 2.

Table 2. The fuzzy share Shapley values of game with fuzzy characteristic function

Coalition	Decision maker 1	Decision maker 2	Decision maker 3
$\{1\}$	(0.1469,0.1633,0.1796)	0	0
$\{2\}$	0	(0.2663,0.2755,0.2847)	0
$\{3\}$	0	0	(0.0918,0.1020,0.1224)
$\{1,2\}$	(0.2066,0.2286,0.2505)	(0.3260,0.3408,0.3556)	0
$\{1,3\}$	(0.2061,0.2360,0.2607)	0	(0.1510,0.1747,0.2036)
$\{2,3\}$	0	(0.3628,0.3806,0.3934)	(0.1882,0.2071,0.2311)
$\{1,2,3\}$	(0.2587,0.2922,0.3242)	(0.4153,0.4369,0.4568)	(0.2403,0.2708,0.3047)

By judging the allocations in Table 2, decision makers can conclude whether the joint project can be realized or not. To do so, decision makers can investigate the problem by varying parameter λ, which is the degree of all the membership functions of the fuzzy numbers involved in the game, from 0.0 to 1.0. For example, consider the case of $\lambda = 0.7$. The expected worth of all the resources is the interval $\tilde{v}_{0.7}(\{1,2,3\}) = [0.9743, 1.0257]$, which is allocated among three decision makers. By Eq.(3.4), we estimate the interval Shapley function for each decision maker, i.e.,

$\tilde{\rho}_i(\tilde{v}_{0.7})(\{1,2,3\}) = \tilde{\rho}_i(\tilde{v})(\{1,2,3\})_{0.7},\ i = 1,2,3.$

Therefore, $\tilde{\rho}_1(\tilde{v}_{0.7})(\{1,2,3\}) = [0.2822, 0.3018]$,

$\tilde{\rho}_2(\tilde{v}_{0.7})(\{1,2,3\}) = [0.4304, 0.4429]$,

$\tilde{\rho}_3(\tilde{v}_{0.7})(\{1,2,3\}) = [0.2617, 0.2810]$.

In other words, the expected worth is interval $[0.9743, 1.0257]$, which is allocated among three decision makers, i.e., $[0.2822, 0.3018]$ for decision makers 1, $[0.4304, 0.4429]$ for decision makers 2, and $[0.2617, 0.2810]$ for decision makers 3.

Fuzzy share Banzhaf function:

From definition 4.2, we could know that

$$\begin{aligned}\phi_1^B(\tilde{v})(\{1,2,3\}) &= \tfrac{1}{4}\tilde{v}(\{1\}) + \tfrac{1}{4}[\tilde{v}(\{1,2\}) -_H \tilde{v}(\{2\})] + \tfrac{1}{4}[\tilde{v}(\{1,3\}) -_H \tilde{v}(\{3\})] \\ &\quad + \tfrac{1}{4}[\tilde{v}(\{1,2,3\}) -_H \tilde{v}(\{2,3\})] \\ &= \tfrac{1}{4}(14.4, 16.0, 17.6) + \tfrac{1}{4}(26.1, 28.8, 31.5) + \tfrac{1}{4}(26.0, 30.25, 33.5) \\ &\quad + \tfrac{1}{4}(35.6, 40.4, 45.2) \\ &= (25.525, 28.8625, 31.95).\end{aligned}$$

Using the same way, we could get

$\phi_2^B(\tilde{v})(\{1,2,3\}) = (40.875, 43.0375, 44.95)$, $\phi_3^B(\tilde{v})(\{1,2,3\}) = (23.725, 26.7625, 30.05)$.

so $\tilde{\mu}^B(\tilde{v}) = (90.125, 98.6625, 106.95)$,

$$\tilde{\mu}^B(\tilde{v}) = I(\tilde{v}) = \frac{90.125 + 98.6625 + 106.95}{3} = 98.5792,$$

$$\tilde{\rho}_1^B(\tilde{v})(\{1,2,3\}) = \frac{\phi_1^B(\tilde{v})(\{1,2,3\})}{I(\tilde{v})} = \frac{(25.525, 28.8625, 31.95)}{98.5792} = (0.2589, 0.2928, 0.3241).$$

Using a similar method, the fuzzy share Shapley value for this game can be obtained as shown in Table 3.

Table 3. The fuzzy share Banzhaf values of game with cooperative fuzzy game

Coalition	Decision maker 1	Decision maker 2	Decision maker 3
$\{1\}$	(0.1461,0.1623,0.1785)	0	0
$\{2\}$	0	(0.2648,0.2739,0.2830)	0
$\{3\}$	0	0	(0.0913,0.1014,0.1217)
$\{1,2\}$	(0.2054,0.2272,0.2490)	(0.3241,0.3388,0.3535)	0
$\{1,3\}$	(0.2049,0.2346,0.2592)	0	(0.1501,0.1737,0.2024)
$\{2,3\}$	0	(0.3606,0.3784,0.3911)	(0.1872,0.2059,0.2298)
$\{1,2,3\}$	(0.2589,0.2928,0.3241)	(0.4146,0.4366,0.4560)	(0.2407,0.2715,0.3048)

We also consider the case of $\lambda = 0.7$. The expected worth of all the resources is the interval $\tilde{v}_{0.7}(\{1,2,3\}) = [0.9749, 1.0261]$, which is allocated among three decision makers. By Eq.(4.2), we estimate the interval Shapley function for each decision maker, i.e.,

$$\tilde{\rho}_i(\tilde{v}_{0.7})(\{1,2,3\}) = \tilde{\rho}_i(\tilde{v})(\{1,2,3\})_{0.7},\ i = 1,2,3.$$

Therefore, $\tilde{\rho}_1(\tilde{v}_{0.7})(\{1,2,3\}) = [0.2826, 0.3022]$,

$\tilde{\rho}_2(\tilde{v}_{0.7})(\{1,2,3\}) = [0.4300, 0.4424]$,

$\tilde{\rho}_3(\tilde{v}_{0.7})(\{1,2,3\}) = [0.2622, 0.2815]$.

In other words, the expected worth is interval$[0.9749, 1.0261]$, which is allocated among three decision makers, i.e., $[0.2826, 0.3022]$ for decision makers 1, $[0.4300, 0.4424]$ for decision makers 2, and $[0.2622, 0.2815]$ for decision makers 3, and it satisfies efficiency.

5. Conclusion

Game theoretic approaches to cooperative situations in fuzzy environments have given rise to several kinds of fuzzy games. We mention here only the games with fuzzy characteristic functions. In this paper, we have extended the share function introduced by Van der Laan et al. in [18] to a fuzzy environment, we generalize the axiom of additivity by introducing a positive fuzzy valued function $\tilde{\mu}$ on the class of cooperative fuzzy games in fuzzy characteristic function form. The so-called axiom of $\tilde{\mu}$-additivity generalizes the classical axiom of additivity by putting the weight $\tilde{\mu}(\tilde{v})$ on the value of the game $\tilde{v}$. We show that any additive function $\tilde{\mu}$ determines a unique fuzzy share function satisfying the axioms of efficient shares, null player property, symmetry and $\tilde{\mu}$-additivity on the subclass of games on which $\tilde{\mu}$ is positive and which contains all positively scaled unanimity games. Then we introduce fuzzy Shapley share functions and fuzzy Banzhaf share functions, and at last, we give an applicable example for the cooperative fuzzy games with fuzzy characteristic functions.

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