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# Numerical Solutions of Fuzzy Two Coupled Nonlinear Differential Equations

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**Abstract**

In this manuscript, we will use the new modified version of the Runge-Kutta method suitable for solving fuzzy two coupled systems of Nonlinear Ordinary Differential Equations (ODE). With the aid of a numerical example, we will demonstrate the accuracy of the $RK-4$ coupled method for solving these two coupled differential equations. To find the analytical solutions we use Laplace Adomian Decomposition Method since it is a semianalytical method used well in many existing studies on dynamical systems. In order to tell the accuracy, we use the error analysis technique. With the help of numerical simulations, we are able to show at what point of $t$, both $x(t)$ and $y(t)$ will interact in order to support the theory.

**Keywords:** New theory of Numerical methods; Analytical Solution; Laplace Adomian Decomposition method; Runge-Kutta method; Two Coupled Differential Equations.

## 1 Introduction

The equivalence relations to a set of the non-crisp data set called fuzzy sets or fuzzy data set obtained by partisioning the existing relation that will not fail to satisfy the oprations satisfied by the crisp data set. The subsequent of differentiation as well as integration of fuzzy defined equations, and the ever existing theorems on existence and with it the uniqueness of FDE solutions in those space of quotients of fuzzy numbers are presented by various existing studies. The unique solution to the FDE's IVP will be well established if fuzzy normed $f$ satisfies Lipschitz condition.

Many recent studies also developed the fuzzy methods and they have been implemented in so many grounds, such as optimization of multi-objective problems with various decision criteria. The development of mathematics has reached a very high level and is still available today.

The need of RK-4 method was very first arisen at the time of Euler methods to solve ODE numerically. Since it was clearly found very first time by the mathematicians Runge-Kutta, that the convergence of Euler method is only about $O(h^2)$ and error existence affects the coincidense of approximate solutions obtained by Euler with that of Exact solutions. $O(h^2)$ is not a good approximation order. So RK-4 methood was developed and found with $O(h^4)$ which provides the confidence

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of least error and better approximation that coincdes to atleast four decimal places i.e., $O(h^4)$ for solving linear ode. It also helped the researchers to get the better approximate solutions to that of few non-linear problems like non-linear hybrid differential equations. But for two coupled system of differential equations there are still a research going on many fields like mathematical modelling in poplation dynamics, epidemiology etc.,

There are few noticable works are done on nonlinear epidemic modells and RK-4 methods have also been used but it is also to be mentioned that those modells are not completely three coupled differential equations. After this modell has been developed and if got published we hope strongely that it could be applied to get the solutions of three ccoupled or four coupled DE on the epidemic modells epidemic models. Also, The entire manuscript is brought up by the motivation of well established researches and some of the notable works are Allen, [1] gave his way of introduction mathematical biology. Abbasbandy extended a numerical method called Newtons method to deal with the nonlinear system of equations using modified Adomian Decomposition Method (ADM) in [2]. In [3] Bukley et al., researched on fuzzy differential equations (FDEs). Kermack et al., [4] mathematically analyzed theory of epidemics. Makinde et al., [5] applied ADM to a SIR epidemic model with uniform vaccination therapy. Farman [10] presented solution of SEIR epidemic model of meseales with non- integer time fractional derivatives by using LADM. Ongun [11], applied the LADM for solving a model for HIV infection of $CD4^{+}T$ cells. Palese [12] analysed Variation of Influenza A, B, and C. Saberiad [15] applied of Homotopy Perturbation Method for solving Hybrid Fuzzy Differential Equations. Pederson et al., [19] numerically solved hybrid fuzzy differential equation IVPs by a characterisation theorem. [20] Kandel et al., studied Fuzzy dynamical systems and nature of their solutions. In [21], [22], Lakshmikantham et al., Impulsive hybrid systems and stability theory, Theory of fuzzy differential equations and inclusions. In [23] Seikkala, On the fuzzy initial value problem. [24] Sepahvandzadeh et al., applied Variational Iteration method (VIM) for solving Hybrid Fuzzy Differential Equations. Also there are many researchers who are working on different types fuzzy differential equations in his research on hybrid systems, delay systems, epidemic models etc., in [13, 14], [16],[17, 18], [6, 7], [8, 9]. The manuscript consists of preliminaries in 2, fuzzy-two-coupled non-linear differential equations in 3, Analytical Solution, Semi Analytical Solution in 4, modified Fuzzy RK-4 Algorithm in 5, and finally conclusion in 6

## 2 Preliminaries

Let $E^1$ represents the set of functions $q : \mathscr{R} \to [0,1]$ such that

$$q(y) = \begin{cases} 4y-3, & \text{if } \ y \in (0.75, 1], \\ -2y+3, & \text{if } \ y \in (1, 1.5), \\ 0, & \text{if } \ y \notin (0.75, 1.5). \end{cases} \tag{2.1}$$

The $r$-level set of $q$ in (2.1) can be wriiten as

$$[q;r] = [0.75 + 0.25r,\ 1.5 - 0.5r]. \tag{2.2}$$

We define $\hat{0} \in E^1$ as $\hat{0}(y) = 1$ if $y = 0$ and $\hat{0}(y) = 0$ if $y \neq 0$ for future reference.

From [23] of $y : I \to E^1$ where $I \subset \mathscr{R}$ is an interval. If $\tilde{y}(t) = [\underline{y}(t;r), \overline{y}(t;r)]$ for all $t \in I$ and $r \in [0,1]$, then $\tilde{y}'(t) = [\underline{y}'(t;r), \overline{y}'(t;r)]$, if $y'(t;r) \in E^1$.
Following IVP,

$$y'(t) = g(t, y(t)),\ y(0) = y_0, \tag{2.3}$$

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where $g : [0, \infty) \times \mathscr{R} \rightarrow \mathscr{R}$ is continuous. We would like to interpret (2.3) using the Seikkala's derivative and $y_0 \in E^1$. Let $\tilde{y}_0 = [\underline{y}(0; r), \overline{y}(0; r)]$ and $\tilde{y}(t) = [\underline{y}(t; r), \overline{y}(t; r)]$.

## 2.1 Definitions and Basic Results

This secion consits of important results considered from [25, 23, 3, 26] "Let $G_k(\mathscr{R}^n)$ represents the house of complete nonempty, compacted, convex collection of subsets of $\mathscr{R}^n$ . Sum and product in $G_k(\mathscr{R}^n)$ are existing as usual. Let $y$ be a point in $\mathscr{R}^n$ and B be a non-empty sub set of $\mathscr{R}^n$ . The distance $D(y, B)$ from $y$ to $B$ is defined by

$$D(y, B) = \inf_{b \in B}\{\|y - b\|\}$$

Let $M$ and $N$ be two nonempty bounded subsets of $\mathscr{R}^n$ . The Housdorff separation of $M$ from $N$ is defined by

$$D_H^*(M, N) = \sup_{\mu \in M}\{d(\mu, \nu)\},$$

The Housdorff separation of N from M is defined by

$$D_H^*(N, M) = \sup_{\nu \in N}\{d(\nu, \mu)\},$$

The distance of separation between $M$ and $N$ as understood by the Housdorff sense

$$D_H(M, N) = \max\left\{\sup_{m \in M}\inf_{n \in N}\|m - n\|, \sup_{n \in M}\inf_{m \in M}\|m - n\|\right\},$$

where $\|\cdot\|$ is the traditional Euclidean norm $||.||$ in $\mathscr{R}^n$. Then it is clear that $(F_k(\mathscr{R}^n), D)$ becomes a complete metric space.

A fuzzy subset of $\mathscr{R}^n$ is explained in terms of a membership arguments which coins to each point $x \in \mathscr{R}^n$, a grade of membership in the fuzzy set. Such a membership function $q : \mathscr{R}^n \rightarrow I \in [0, 1]$ is used to denote the corresponding fuzzy set.

To every $r \in (0, 1]$, the $r$- level set $[q]^r$ of a fuzzy set $u$ is the subset of values $y \in \mathscr{R}^n$ with memberships $q(y)$ of $r$ powers, that is $[q]^r = \{y \in \mathscr{R}^n : q(y) \geq r\}$. The support $[q]^0$ of a fuzzy set is then defined as the closure of the union of all its level sets, that is, $[q]^0 = \overline{\bigcup_{r \in (0,1]} [q]^r}$. An inclusion result arrives spontaneously from the above definitions.

**Result 1**

To every $0 \leq r_1 \leq r_2 \leq 1$, $[q]^{r_2} \subseteq [q]^{r_1} \subseteq [q]^0$.

Universally, some level sets usually be null in a ordinary fuzzy set. Particularly, the triviality arise when $q(y) \equiv 0$ for all $y \in \mathscr{R}^n$, though the support is null: $q$ is null fuzzy set in this sense. Here we shall pay focus only to the normal fuzzy sets which satisfy.

In view of Result 1. we have

**Result 2**

$[q]^r$ is a compact subset of $\mathscr{R}^n$ for all $r \in I$.

**Result 3**

"If $u$ is fuzzy convex, then $[q]^r$ is convex for each $r \in I$.

Let $I = [0, 1] \subseteq R$ be as compact interval and let $E^n$ denote the set of all $q : \mathscr{R}^n \rightarrow I$ such that $q$ satisfies the following conditions.

(i) $q$ is normal, that is, there exist an $q_0 \in \mathscr{R}^n$ such that $q_0 = 1$,

(ii) $q$ is fuzzy convex,

(iii) $q$ is upper semicontinuous,

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(iv) $[q]^0 \equiv$ closure of $\{q \in \mathscr{R}^n : q(x) > 0\}$ is compact.
Then, from $(1) - (4)$, it follows that the $r$-level set $[q]^r \in P_k(\mathscr{R}^1)$ for all $0 \leq r \leq 1$. If $g : \mathscr{R}^n \times \mathscr{R}^n \to \mathscr{R}^n$ is a function, then using Zadeh's extension principle we can extend $g$ to $E^n \times E^n \to E^n$ by the equation"

$$\tilde{g}(q_1, q_2)(z) = \sup_{z=g(x,y)} \min\{q_1(x), q_2(y)\}. \tag{2.4}$$

It is well known that$[\tilde{g}(q_1, q_2)]^r = g([q_1]^r, [q_2]^r)$, for all $q_1, q_2 \in E^n$, $0 \leq r \leq 1$, and continuous function $g$. Further we have

$$[q_1^r + q_2^r] = ([q_1]^r + [q_2]^r), \tag{2.5}$$
$$[kq]^r = k[q]^r, \tag{2.6}$$

where $k \in \mathscr{R}$. The real numbers can be embedded in $E^n$ by the rule $c \to \hat{c}(t)$", where,

$$\hat{c}(t) = \begin{cases} 1 & for \;\; t = c, \\ 0 & elsewhere. \end{cases}$$

# 3 Fuzzy-Two-Coupled Non-linear Differential Equations

For preliminary definitins of fuzzy differential equations authors are encouraged to go through [25, 26], [18], etc., Two Coupled differential Equations have wide range of applications in any mathematicall modell of physical phenomena in epidemiology, ecology,etc., By the application of fuzzy it is used to eliminate the randomness and vagueness that arises in any dynamics of the system.

$$\begin{cases} x'(t) = c_1 x(t)y(t), \; t_0 \leq t \leq t_n \\ y'(t) = c_2 x(t)y(t), \; t_0 \leq t \leq t_n \\ x(t_0) = x_0, \\ y(t_0) = y_0 \end{cases} \tag{3.1}$$

where $c_1$, $c_2$ are numeric constants such that they are not equal to zero and also $c_1 \neq c_2$. By using the concept fuzzy, the equation (3.1) becomes,

$$\begin{cases} \tilde{x}'(t) = c_1 \tilde{x}(t)\tilde{y}(t), \; t_0 \leq t \leq t_n \\ \tilde{y}'(t) = c_2 \tilde{x}(t)\tilde{y}(t), \; t_0 \leq t \leq t_n \\ \tilde{x}(t_0) = x_0, \\ \tilde{y}(t_0) = y_0 \end{cases} \tag{3.2}$$

Such that $\tilde{x}(t) = [\underline{x}(t;r), \overline{x}(t;r)]$. In the same way for $\tilde{y}(t)\, \tilde{x}'(t)\, \tilde{y}'(t)$ and also for $\tilde{x}_0$, $\tilde{y}(0)$

# 4 Analytical Solution, Semi Analytical Solution

The analytical Solution of the system (3.2) is given by

$$\begin{aligned} \tilde{x}(t) &= \tilde{x}_0 e^{c_1 \int_0^t y(s)ds} \\ \tilde{y}(t) &= \tilde{y}_0 e^{c_2 \int_0^t x(s)ds} \end{aligned} \tag{4.1}$$

In order to obtain the semi analytical solution we are here by making use of well known Laplace Adomian Decomposition method (LADM). We prefer this method to compare the solutions of nonlinear coupled differential equations. The method is already defined and described in somany papers previously whereas the RK-4 algorith or method for nonlinear coupled differential equations is not defined clearly yet but found traces of the authors try over it in the literature. The method is taken since we are unable to process the analytical solutions even though its structure is expalined above.

### 4.1 Fuzzy Laplace Adomian Decomposition Method

$$\begin{aligned} X(k+1) &= L^{-1}(c_1/s \times L(A_k)) \\ Y(k+1) &= L^{-1}(c_2/s^{\alpha 2} \times L(A_k)) \end{aligned} \tag{4.2}$$

Where $(A_k)$ is an Adomian polynomial defined by $A_k = \frac{1}{k!}\frac{d^k}{\lambda^k}(\sum_{l=0}^{k}(\lambda^l.x_l\lambda^l.y_l)|_{\lambda=0}$ i.e.,
$A_0 = x_0y_0$
$A_1 = x_0y_1 + x_1y_0$
$A_2 = x_0y_2 + x_1y_1 + x_2y_0$ and so on.
$x(t) = \sum_{k=0}^{\infty}(x(k))$
$y(t) = \sum_{k=0}^{\infty}(y(k))$

## 5 Modified Fuzzy RK-4 Algorithm:

We are at present sharing the new algorithm for novel RK-4 method for solving nonlinear coupled differential equations. In this section we are using the fourth order Runge-Kutta method (RK-4). We are finding the values of $\tilde{x}(t)$,$\tilde{y}(t)$, at $h = 0.1$ for the best approximation. For $0 \leq r \leq 1$.
**To evaluate x(t), and y(t):**
Consider,

$$\begin{aligned} \tilde{x}(t+1) &= (\tilde{x}(t) + (1/6(A_1 + 2A_2 + 2A_3 + K_4))) \\ \tilde{y}(t+1) &= (\tilde{y}(t) + (1/6(L_1 + 2B_2 + 2B_3 + B_4))) \end{aligned} \tag{5.1}$$

To estimate (5.1), consider the following.

$$\begin{aligned} \tilde{A}_1 &= h \times c_1((\tilde{x}(t))(\tilde{y}(t)) \\ \tilde{B}_1 &= h \times c_2((\tilde{x}(t))(\tilde{y}(t)) \\ \tilde{A}_2 &= h \times c_1(\tilde{x}(t) + (\tilde{A}_1/2))(\tilde{y}(t) + (\tilde{B}_1/2)) \\ \tilde{B}_2 &= h \times c_2(\tilde{x}(t) + (\tilde{A}_1/2))(\tilde{y}(t) + (\tilde{B}_1/2)) \\ \tilde{A}_3 &= h \times c_1(\tilde{x}(t) + (\tilde{A}_2/2))(\tilde{y}(t) + (\tilde{B}_2/2)) \\ \tilde{B}_3 &= h \times c_2(\tilde{x}(t) + (\tilde{A}_2/2))(\tilde{y}(t) + (\tilde{B}_2/2)) \\ \tilde{A}_4 &= h \times c_1(\tilde{x}(t) + (\tilde{A}_3))(\tilde{y}(t) + (\tilde{B}_3)) \\ \tilde{B}_4 &= h \times c_2(\tilde{x}(t) + (\tilde{A}_3))(\tilde{y}(t) + (\tilde{B}_3)) \end{aligned} \tag{5.2}$$

For $1 \leq p \leq 4$ and $0 \leq r \leq 1$,
$\tilde{A}_p = \tilde{A}_p(t;r) = [\underline{A}_p(t;r), \overline{A}_p(t;r)]$,
$\tilde{B}_p = \tilde{B}_p(t;r) = [\underline{B}_p(t;r), \overline{B}_p(t;r)]$,
For $0 \leq t \leq n$, $n = 1, 2, 3, ...,$,
and for $\underline{q} = t$, $\overline{q} = t + 1$, $t = 0, 1, 2, 3, ...$
$\tilde{x}(q) = \tilde{x}(q)(t;r) = [\underline{x}_q(t;r), \overline{x}_q(t;r)]$,
$\tilde{y}(q) = \tilde{y}(q)(t;r) = [\underline{y}_q(t;r), \overline{y}_q(t;r)]$,
Where, $[\underline{f}(t;r), \overline{f}(t;r)] = [0.75 + 0.25r, 1.125 - 0.125r]f(t)$.

Table 1: Approximate solution by RK-4 for non-fuzzy case

| t | LADM-4 | | RK-4 | | Error | |
|---|---|---|---|---|---|---|
| | x(t) | y(t) | x(t) | y(t) | x(t) | y(t) |
| 0 | 5 | 3 | 5 | 3 | 0 | 0 |
| 0.1 | 4.991 | 3.0045 | 5.006 | 2.99325 | 0.015 | 0.01125 |
| 0.2 | 4.98201 | 3.009 | 4.99701 | 2.99775 | 0.015 | 0.01125 |
| 0.3 | 4.97301 | 3.01349 | 4.98802 | 3.00224 | 0.01501 | 0.01125 |
| 0.4 | 4.96402 | 3.01799 | 4.97904 | 3.00673 | 0.01502 | 0.01126 |
| 0.5 | 4.95503 | 3.02248 | 4.97006 | 3.01122 | 0.01503 | 0.01126 |
| 0.6 | 4.94605 | 3.02697 | 4.96108 | 3.01571 | 0.01503 | 0.01126 |
| 0.7 | 4.93707 | 3.03147 | 4.9521 | 3.0202 | 0.01503 | 0.01127 |
| 0.8 | 4.92809 | 3.03595 | 4.94313 | 3.02469 | 0.01504 | 0.01126 |
| 0.9 | 4.91912 | 3.04044 | 4.93416 | 3.02917 | 0.01504 | 0.01127 |
| 1.0 | 4.91014 | 3.04493 | 4.92519 | 3.03366 | 0.01505 | 0.01127 |

Table 2: Approximate solution by RK-4 for fuzzy case

| r | x(t;r) | | y(t;r) | |
|---|---|---|---|---|
| | min | max | min | max |
| 0 | 3.6939 | 5.625 | 2.27524 | 3.41286 |
| 0.1 | 3.81703 | 5.56917 | 2.35108 | 3.37494 |
| 0.2 | 3.94016 | 5.49671 | 2.42692 | 3.33702 |
| 0.3 | 4.06329 | 5.42447 | 2.50277 | 3.2991 |
| 0.4 | 4.18641 | 5.35246 | 2.57861 | 3.26118 |
| 0.5 | 4.30954 | 5.28068 | 2.65445 | 3.22326 |
| 0.6 | 4.43267 | 5.20913 | 2.73029 | 3.18534 |
| 0.7 | 4.5558 | 5.13781 | 2.80613 | 3.14742 |
| 0.8 | 4.67893 | 5.06671 | 2.88197 | 3.1095 |
| 0.9 | 4.80206 | 4.99584 | 2.95781 | 3.07158 |
| 1 | 4.92519 | 4.92519 | 3.03366 | 3.03366 |

## 5.1 An Example

Let us consider the following problem and compare the results of the method LADM RKM-4 in both non-fuzzy and as well as fuzzy. In Table 1, we are presenting the values of $x(t)$, $y(t)$, in non fuzzy by means of LADM-4 and RK-4, and also the error analysis between them. In table 1, we have presented only the values for $t \in [0, 1]$ but one can estimate the values for $t \in [0, 100]$. In that way we found that at $t = 15.5$, $x(t) = 3.66502$, $y(t) = 3.66749$, i.e $x(t) \approx y(t)$, $t = 20.$, there is a interaction between $x(t)$, and $y(t)$.

$$\begin{cases} \tilde{x}'(t) = -0.006\tilde{x}(t)\tilde{y}(t),\ 0 \le t \le 1 \\ \tilde{y}'(t) = 0.003\tilde{x}(t)\tilde{y}(t),\ 0 \le t \le 1 \\ \tilde{x}(0) = 5, \\ \tilde{y}(0) = 3. \end{cases} \tag{5.3}$$

We calculate error by means of $Error = |(LADM - 4) - (RK - 4)|$ Let us present below the table value of $x(t)$ and $y(t)$ in terms of fuzzy in Table 2. So that we have $x(t; r)$ and $y(t; r)$ for $t \in [0, 1]$ and $r \in [0, 1]$.

### 5.2 Numerical Simlations

For non-Fuzzy coupled case of above example:
For Fuzzy coupled case of above example:

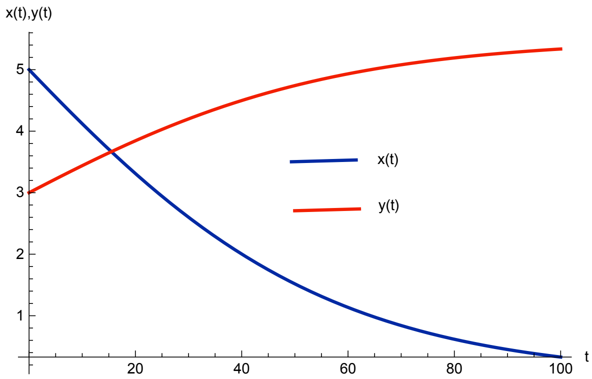


Figure 1: Non-Fuzzy Nonlinear Two Coupled Differential Systems

By the above figures, Figure1 and 2 we are able to understand the travel of

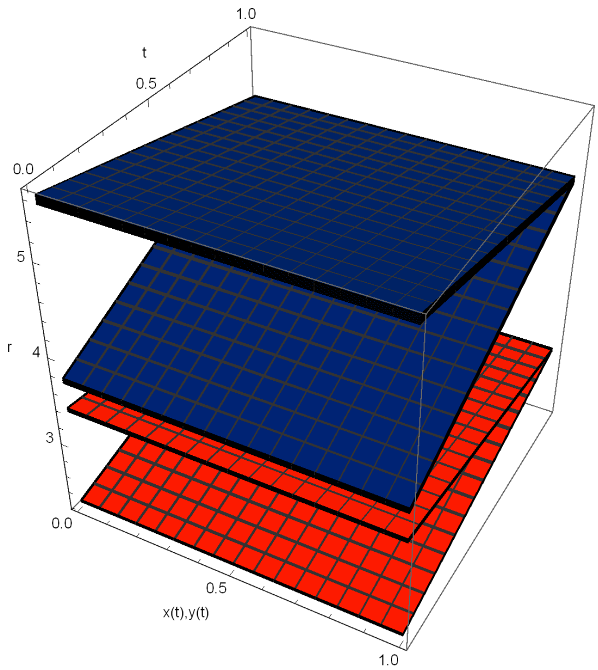


Figure 2: Fuzzy Nonlinear Two Coupled Differential Systems

solutions in $t \in 0, 100$ for non-fuzzy case and for $t \in [0, 1]$ and $r \in [0, 1]$ for fuzzy case respectively.

## 6 Conclusion

There are numerous numerical methods that one want to use other than Rung-Kutta method when it comes to the need to solve the function with non-linear

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ODE especially for coupled differential equaions we are having so many applications but the methods like Laplace Adomian Decomposition method etc., are used as presented in earlier section. But now we had presented the new coupled form of RK-4 algorithm for solving any kind of nonlinear two coupled nonlinear ODE. We recommend this RK-4 algorithm since its accuracy is of about $O(h^1)$ or one decimal place when it is compared with semianalytical method like LADM. The important aspect is that one can easily see the interaction between $x(t)$ and $y(t)$ in figure 1 which tells us at $t = 15.5$, $x(t) \approx y(t)$. As a future work, we will present this approach on completely coupled fuzzy disease modelling problems.

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# Parametrized hyperbolic tangent based Banach space valued multivariate multi layer neural network approximations

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**Abstract**

Here we examine the multivariate quantitative approximations of Banach space valued continuous multivariate functions on a box or $\mathbb{R}^N$, $N \in \mathbb{N}$, by the multivariate normalized, quasi-interpolation, Kantorovich type and quadrature type neural network operators. We research also the case of approximation by iterated operators of the last four types, that is multi hidden layer approximations. These approximations are achieved by establishing multidimensional Jackson type inequalities involving the multivariate modulus of continuity of the engaged function or its high order Fréchet derivatives. Our multivariate operators are defined by using a multidimensional density function induced by a parametrized hyperbolic tangent sigmoid function. The approximations are pointwise, uniform and $L_p$. The related feed-forward neural networks are with one or multi hidden layers.

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**Keywords and Phrases:** multi layer approximation, parametrized hyperbolic tangent sigmoid function, multivariate neural network approximation, quasi-interpolation operator, Kantorovich type operator, quadrature type operator, multivariate modulus of continuity, abstract approximation, iterated and

$L_p$ approximations.

# 1 Introduction

G.A. Anastassiou in [2] and [3], see chapters 2-5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support. Also in [3] he gives the $N$th order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see chapters 4-5 there.

Motivations for this work are the article [22] of Z. Chen and F. Cao, and [4]-[19], [23], [24].

Here we perform a parametrized hyperbolic tangent sigmoid function based neural network multivariate approximation to continuous functions over boxes or over the whole $\mathbb{R}^N$, $N \in \mathbb{N}$, and also iterated, multi layer and $L_p$ approximations. All convergences here are with rates expressed via the multivariate modulus of continuity of the involved function or its high order Fréchet derivative and given by very tight multidimensional Jackson type inequalities.

We come up with the "right" precisely defined multivariate normalized, quasi-interpolation neural network operators related to boxes or $\mathbb{R}^N$, as well as Kantorovich type and quadrature type related operators on $\mathbb{R}^N$. Our boxes are not necessarily symmetric to the origin. In preparation to prove our results we establish important properties of the basic multivariate density function induced by a parametrized hyperbolic tangent sigmoid function.

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle a_j \cdot x \rangle + b_j\right), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of $a_j$ and $x$, and $\sigma$ is the activation function of the network. In many fundamental neural network models, the activation function is based on the hyperbolic tangent sigmoid function. About neural networks read [25]-[27].

## 2 Background

We consider here the generalized hyperbolic tangent function $\tanh \lambda x$, $x \in \mathbb{R}$, $\lambda > 0$ :

$$\tanh \lambda x = \frac{e^{\lambda x} - e^{-\lambda x}}{e^{\lambda x} + e^{-\lambda x}}. \tag{1}$$

It is $\tanh \lambda 0 = 0, -1 < \tanh \lambda x < 1, \forall\ x \in \mathbb{R}$, and $\tanh \lambda (-x) = -\tanh \lambda x$. Furthermore we have $\tanh \lambda (\infty) = 1$ and $\tanh \lambda (-\infty) = -1$, and $\tanh \lambda x$ is strictly increasing on $\mathbb{R}$, with

$$\frac{d}{dx} \tanh \lambda x = \frac{\lambda}{\cos^2 \lambda x} > 0. \tag{2}$$

The induced activation function will be

$$\theta (x) := \frac{1}{4} (\tanh \lambda (x+1) - \tanh \lambda (x-1)) > 0, \forall x \in \mathbb{R}, \tag{3}$$

with $\theta (x) = \theta (-x)$.

Clearly $\theta (x)$ is differentiable and thus it is continuous.

**Proposition 1** *$\theta (x)$ is strictly decrasing on $(0, \infty)$ and strictly increasing on $(-\infty, 0]$. We have that $\theta (-\infty) = \theta (\infty) = 0$. So that $\theta$ has the bell shape with horizontal asymptote the x-axis. The maximum of $\theta$ is*

$$\theta (0) = \frac{\tanh \lambda}{2}. \tag{4}$$

We mention

**Theorem 2** *([20]) It holds*

$$\sum_{i=-\infty}^{\infty} \theta (x - i) = 1, \quad \forall\ x \in \mathbb{R}. \tag{5}$$

**Theorem 3** *([20]) We have that*

$$\int_{-\infty}^{\infty} \theta (x)\, dx = 1. \tag{6}$$

*So that $\theta$ is a density function on $\mathbb{R}$.*

**Theorem 4** *([20]) Let $0 < \alpha < 1$, $\lambda > 0$ and $n \in \mathbb{N}$. It holds*

$$\sum_{\begin{cases} k = -\infty \\ : |nx - k| \geq n^{1-\alpha} \end{cases}}^{\infty} \theta (nx - k) < e^{4\lambda} e^{-2\lambda n^{(1-\alpha)}}. \tag{7}$$

Denote by $\lfloor\cdot\rfloor$ the integral part of the number and by $\lceil\cdot\rceil$ the ceiling of the number.

**Theorem 5** *([20]) Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$, so that $\lceil na\rceil \leq \lfloor nb\rfloor$. Then*

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \theta(nx-k)} < \frac{4}{\tanh 2\lambda} = \frac{1}{\theta(1)}. \tag{8}$$

We make

**Remark 6** *([20])*

*(i) We have that*

$$\lim_{n\to\infty} \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \theta(nx-k) \neq 1, \tag{9}$$

*for at least some $x \in [a,b]$.*

*(ii) Let $[a,b] \subset \mathbb{R}$. For large $n$ we always have $\lceil na\rceil \leq \lfloor nb\rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na\rceil \leq k \leq \lfloor nb\rfloor$.*

*In general it holds*

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \theta(nx-k) \leq 1. \tag{10}$$

We introduce

$$Z(x_1,...,x_N) := Z(x) := \prod_{i=1}^{N} \theta(x_i), \quad x = (x_1,...,x_N) \in \mathbb{R}^N, \ N \in \mathbb{N}. \tag{11}$$

It has the properties:

(i) $Z(x) > 0, \ \forall\, x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} Z(x_1-k_1,...,x_N-k_N) = 1, \tag{12}$$

where $k := (k_1,...,k_n) \in \mathbb{Z}^N$, $\forall\, x \in \mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z(nx-k) = 1, \tag{13}$$

$\forall\, x \in \mathbb{R}^N$; $n \in \mathbb{N}$,

and

(iv)

$$\int_{\mathbb{R}^N} Z(x)\, dx = 1, \tag{14}$$

that is $Z$ is a multivariate density function.

Here denote $\|x\|_\infty := \max\{|x_1|, ..., |x_N|\}$, $x \in \mathbb{R}^N$, also set $\infty := (\infty, ..., \infty)$, $-\infty := (-\infty, ..., -\infty)$ upon the multivariate context, and

$$\begin{aligned} \lceil na \rceil &:= (\lceil na_1 \rceil, ..., \lceil na_N \rceil), \\ \lfloor nb \rfloor &:= (\lfloor nb_1 \rfloor, ..., \lfloor nb_N \rfloor), \end{aligned} \tag{15}$$

where $a := (a_1, ..., a_N)$, $b := (b_1, ..., b_N)$.

We obviously see that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left( \prod_{i=1}^N \theta(nx_i - k_i) \right) =$$

$$\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} ... \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} \left( \prod_{i=1}^N \theta(nx_i - k_i) \right) = \prod_{i=1}^N \left( \sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \theta(nx_i - k_i) \right). \tag{16}$$

For $0 < \beta < 1$ and $n \in \mathbb{N}$, a fixed $x \in \mathbb{R}^N$, we have that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) =$$

$$\sum_{\substack{k=\lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z(nx-k) + \sum_{\substack{k=\lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z(nx-k). \tag{17}$$

In the last two sums the counting is over disjoint vector sets of $k$'s, because the condition $\left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}$ implies that there exists at least one $\left| \frac{k_r}{n} - x_r \right| > \frac{1}{n^\beta}$, where $r \in \{1, ..., N\}$.

(v) As in, Theorem 4 we derive that

$$\sum_{\substack{k=\lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z(nx-k) \overset{(7)}{<} e^{4\lambda} e^{-2\lambda n^{(1-\beta)}}, \ \ 0 < \beta < 1, \ \lambda > 0. \tag{18}$$

with $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \prod_{i=1}^N [a_i, b_i]$.

(vi) By Theorem 5 we get that

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} < \left( \frac{4}{\tanh 2\lambda} \right)^N, \tag{19}$$

$\lambda > 0$, $\forall\, x \in \left(\prod_{i=1}^{N}[a_i, b_i]\right)$, $n \in \mathbb{N}$.

It is also clear that

(vii)

$$\sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\infty} Z(nx - k) < e^{4\lambda} e^{-2\lambda n^{(1-\beta)}}, \tag{20}$$

$\lambda > 0, 0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \mathbb{R}^N$.

Furthermore it holds

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) \neq 1, \tag{21}$$

for at least some $x \in \left(\prod_{i=1}^{N}[a_i, b_i]\right)$.

Here $\left(X, \|\cdot\|_{\gamma}\right)$ is a Banach space.

Let $f \in C\left(\prod_{i=1}^{N}[a_i, b_i], X\right)$, $x = (x_1, ..., x_N) \in \prod_{i=1}^{N}[a_i, b_i]$, $n \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

We introduce and define the following multivariate linear normalized neural network operator ($x := (x_1, ..., x_N) \in \left(\prod_{i=1}^{N}[a_i, b_i]\right)$):

$$A_n(f, x_1, ..., x_N) := A_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)} =$$

$$\frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} ... \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} \theta(nx_i - k_i)\right)}{\prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \theta(nx_i - k_i)\right)}. \tag{22}$$

For large enough $n \in \mathbb{N}$ we always obtain $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$. Also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

When $g \in C\left(\prod_{i=1}^{N}[a_i, b_i]\right)$ we define the companion operator

$$\widetilde{A}_n(g, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} g\left(\frac{k}{n}\right) Z(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)}. \tag{23}$$

Clearly $\widetilde{A}_n$ is a positive linear operator. We have that

$$\widetilde{A}_n(1, x) = 1, \ \forall\, x \in \left(\prod_{i=1}^{N}[a_i, b_i]\right).$$

Notice that $A_n(f) \in C\left(\prod_{i=1}^{N}[a_i, b_i], X\right)$ and $\widetilde{A}_n(g) \in C\left(\prod_{i=1}^{N}[a_i, b_i]\right)$.

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Furthermore it holds

$$\|A_n(f,x)\|_\gamma \leq \frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \left\|f\left(\frac{k}{n}\right)\right\|_\gamma Z(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)} = \widetilde{A}_n\left(\|f\|_\gamma, x\right), \tag{24}$$

$\forall\ x \in \prod_{i=1}^N [a_i, b_i]$.

Clearly $\|f\|_\gamma \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$.

So, we have that

$$\|A_n(f,x)\|_\gamma \leq \widetilde{A}_n\left(\|f\|_\gamma, x\right), \tag{25}$$

$\forall\ x \in \prod_{i=1}^N [a_i, b_i]$, $\forall\ n \in \mathbb{N}$, $\forall\ f \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$.

Let $c \in X$ and $g \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$, then $cg \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$.

Furthermore it holds

$$A_n(cg, x) = c\widetilde{A}_n(g, x),\ \ \forall\ x \in \prod_{i=1}^N [a_i, b_i]. \tag{26}$$

Since $\widetilde{A}_n(1) = 1$, we get that

$$A_n(c) = c,\ \ \forall\ c \in X. \tag{27}$$

We call $\widetilde{A}_n$ the companion operator of $A_n$.

For convenience we call

$$A_n^*(f,x) := \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right) Z(nx-k) =$$

$$\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor} \sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2\rfloor} \cdots \sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor} f\left(\frac{k_1}{n}, \ldots, \frac{k_N}{n}\right)\left(\prod_{i=1}^N \theta(nx_i - k_i)\right), \tag{28}$$

$\forall\ x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$.

That is

$$A_n(f,x) := \frac{A_n^*(f,x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)}, \tag{29}$$

$\forall\ x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$, $n \in \mathbb{N}$.

Hence

$$A_n(f,x) - f(x) = \frac{A_n^*(f,x) - f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)}. \tag{30}$$

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Consequently we derive

$$\|A_n(f,x)-f(x)\|_\gamma \overset{(19)}{\leq} \left(\frac{4}{\tanh 2\lambda}\right)^N \left\| A_n^*(f,x)-f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)\right\|_\gamma, \tag{31}$$

$\forall\ x\in\left(\prod_{i=1}^N [a_i,b_i]\right)$.

We will estimate the right hand side of (31).

For the last and others we need

**Definition 7** *([15], p. 274) Let $M$ be a convex and compact subset of $\left(\mathbb{R}^N,\|\cdot\|_p\right)$, $p\in[1,\infty]$, and $\left(X,\|\cdot\|_\gamma\right)$ be a Banach space. Let $f\in C(M,X)$. We define the first modulus of continuity of $f$ as*

$$\omega_1(f,\delta):=\sup_{\substack{x,y\in M:\\ \|x-y\|_p\leq\delta}} \|f(x)-f(y)\|_\gamma,\quad 0<\delta\leq diam(M). \tag{32}$$

*If $\delta>diam(M)$, then*

$$\omega_1(f,\delta)=\omega_1(f,diam(M)). \tag{33}$$

Notice $\omega_1(f,\delta)$ is increasing in $\delta>0$. For $f\in C_B(M,X)$ (continuous and bounded functions) $\omega_1(f,\delta)$ is defined similarly.

**Lemma 8** *([15], p. 274) We have $\omega_1(f,\delta)\to 0$ as $\delta\downarrow 0$, iff $f\in C(M,X)$, where $M$ is a convex compact subset of $\left(\mathbb{R}^N,\|\cdot\|_p\right)$, $p\in[1,\infty]$.*

Clearly we have also: $f\in C_U\left(\mathbb{R}^N,X\right)$ (uniformly continuous functions), iff $\omega_1(f,\delta)\to 0$ as $\delta\downarrow 0$, where $\omega_1$ is defined similarly to (32). The space $C_B\left(\mathbb{R}^N,X\right)$ denotes the continuous and bounded functions on $\mathbb{R}^N$.

When $f\in C_B\left(\mathbb{R}^N,X\right)$ we define,

$$B_n(f,x):=B_n(f,x_1,...,x_N):=\sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx-k):=$$

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n},\frac{k_2}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^N \theta(nx_i-k_i)\right), \tag{34}$$

$n\in\mathbb{N}$, $\forall\ x\in\mathbb{R}^N$, $N\in\mathbb{N}$, the multivariate quasi-interpolation neural network operator.

Also for $f\in C_B\left(\mathbb{R}^N,X\right)$ we define the multivariate Kantorovich type neural network operator

$$C_n(f,x):=C_n(f,x_1,...,x_N):=\sum_{k=-\infty}^{\infty}\left(n^N\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt\right) Z(nx-k)=$$

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty}\left(n^N\int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}}\int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}}...\int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}}f(t_1,...,t_N)\,dt_1...dt_N\right)$$
$$\cdot\left(\prod_{i=1}^{N}\theta(nx_i-k_i)\right), \tag{35}$$

$n\in\mathbb{N}$, $\forall\ x\in\mathbb{R}^N$.

Again for $f\in C_B(\mathbb{R}^N,X)$, $N\in\mathbb{N}$, we define the multivariate neural network operator of quadrature type $D_n(f,x)$, $n\in\mathbb{N}$, as follows.

Let $\theta=(\theta_1,...,\theta_N)\in\mathbb{N}^N$, $r=(r_1,...,r_N)\in\mathbb{Z}_+^N$, $w_r=w_{r_1,r_2,...r_N}\geq 0$, such that $\sum_{r=0}^{\theta}w_r=\sum_{r_1=0}^{\theta_1}\sum_{r_2=0}^{\theta_2}...\sum_{r_N=0}^{\theta_N}w_{r_1,r_2,...r_N}=1$; $k\in\mathbb{Z}^N$ and

$$\delta_{nk}(f):=\delta_{n,k_1,k_2,...,k_N}(f):=\sum_{r=0}^{\theta}w_rf\left(\frac{k}{n}+\frac{r}{n\theta}\right)=$$

$$\sum_{r_1=0}^{\theta_1}\sum_{r_2=0}^{\theta_2}...\sum_{r_N=0}^{\theta_N}w_{r_1,r_2,...r_N}f\left(\frac{k_1}{n}+\frac{r_1}{n\theta_1},\frac{k_2}{n}+\frac{r_2}{n\theta_2},...,\frac{k_N}{n}+\frac{r_N}{n\theta_N}\right), \tag{36}$$

where $\frac{r}{\theta}:=\left(\frac{r_1}{\theta_1},\frac{r_2}{\theta_2},...,\frac{r_N}{\theta_N}\right)$.

We set

$$D_n(f,x):=D_n(f,x_1,...,x_N):=\sum_{k=-\infty}^{\infty}\delta_{nk}(f)\,Z(nx-k)= \tag{37}$$

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty}\delta_{n,k_1,k_2,...,k_N}(f)\left(\prod_{i=1}^{N}\theta(nx_i-k_i)\right),$$

$\forall\ x\in\mathbb{R}^N$.

In this article we study the approximation properties of $A_n,B_n,C_n,\ D_n$ neural network operators and as well of their iterates. That is, the quantitative pointwise and uniform convergence of these operators to the unit operator $I$.

# 3 Multivariate Parametrized Hyperbolic Tangent Induced Banach Space Valued Network Approximations

Here we present several vectorial neural network approximations to Banach space valued functions given with rates.

We give

**Theorem 9** *Let* $f \in C\left(\prod_{i=1}^{N}[a_i,b_i], X\right)$, $0<\beta<1$, $\lambda>0$, $x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $N, n \in \mathbb{N}$ *with* $n^{1-\beta}>2$. *Then*

*1)*

$$\|A_n(f,x)-f(x)\|_\gamma \leq \left(\frac{4}{\tanh 2\lambda}\right)^N \left[\omega_1\left(f,\frac{1}{n^\beta}\right)+\frac{2e^{4\lambda}\left\|\|f\|_\gamma\right\|_\infty}{e^{2\lambda\left(n^{1-\beta}\right)}}\right] =: \Omega_1(n), \tag{38}$$

*and*

*2)*

$$\left\|\|A_n(f)-f\|_\gamma\right\|_\infty \leq \Omega_1(n). \tag{39}$$

*We notice that* $\lim_{n\to\infty} A_n(f) \overset{\|\cdot\|_\gamma}{=} f$, *pointwise and uniformly.*

*Above* $\omega_1$ *is with respect to* $p=\infty$ *and the speed of convergence is* $\max\left(\frac{1}{n^\beta},\frac{1}{e^{2\lambda n^{(1-\beta)}}}\right)=\frac{1}{n^\beta}$.

**Proof.** We observe that

$$\Delta(x) := A_n^*(f,x) - f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k) =$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right) Z(nx-k) - \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f(x) Z(nx-k) =$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(f\left(\frac{k}{n}\right)-f(x)\right) Z(nx-k). \tag{40}$$

Thus

$$\|\Delta(x)\|_\gamma \leq \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|f\left(\frac{k}{n}\right)-f(x)\right\|_\gamma Z(nx-k) =$$

$$\sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty \leq \frac{1}{n^\beta}}}^{\lfloor nb\rfloor}\left\|f\left(\frac{k}{n}\right)-f(x)\right\|_\gamma Z(nx-k) +$$

$$\sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb\rfloor}\left\|f\left(\frac{k}{n}\right)-f(x)\right\|_\gamma Z(nx-k) \overset{(13)}{\leq}$$

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$$\omega_1\left(f,\frac{1}{n^\beta}\right)+2\left\|\|f\|_\gamma\right\|_\infty \sum_{\left\{\begin{array}{c} k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}\end{array}\right.}^{\lfloor nb\rfloor} Z(nx-k) \overset{(18)}{\leq}$$

$$\omega_1\left(f,\frac{1}{n^\beta}\right)+2\left\|\|f\|_\gamma\right\|_\infty e^{4\lambda}e^{-2\lambda n^{(1-\beta)}},\ 0<\beta<1,\ \lambda>0. \tag{41}$$

So that

$$\|\Delta(x)\|_\gamma \leq \omega_1\left(f,\frac{1}{n^\beta}\right)+\frac{2e^{4\lambda}\left\|\|f\|_\gamma\right\|_\infty}{e^{2\lambda n^{(1-\beta)}}}. \tag{42}$$

Now using (31) we finish the proof. ■

We make

**Remark 10** *([15], pp. 263-266) Let* $\left(\mathbb{R}^N,\|\cdot\|_p\right)$, $N\in\mathbb{N}$*; where* $\|\cdot\|_p$ *is the* $L_p$*-norm,* $1\leq p\leq\infty$*.* $\mathbb{R}^N$ *is a Banach space, and* $\left(\mathbb{R}^N\right)^j$ *denotes the* $j$*-fold product space* $\mathbb{R}^N\times...\times\mathbb{R}^N$ *endowed with the max-norm* $\|x\|_{(\mathbb{R}^N)^j}:=\max_{1\leq\rho\leq j}\|x_\rho\|_p$*, where* $x:=(x_1,...,x_j)\in\left(\mathbb{R}^N\right)^j$*.*

*Let* $\left(X,\|\cdot\|_\gamma\right)$ *be a general Banach space. Then the space* $V_j:=V_j\left(\left(\mathbb{R}^N\right)^j;X\right)$ *of all* $j$*-multilinear continuous maps* $g:\left(\mathbb{R}^N\right)^j\to X$*,* $j=1,...,m$*, is a Banach space with norm*

$$\|g\|:=\|g\|_{V_j}:=\sup_{\left(\|x\|_{(\mathbb{R}^N)^j}=1\right)}\|g(x)\|_\gamma=\sup\frac{\|g(x)\|_\gamma}{\|x_1\|_p\cdots\|x_j\|_p}. \tag{43}$$

*Let* $M$ *be a non-empty convex and compact subset of* $\mathbb{R}^N$ *and* $x_0\in M$ *is fixed.*

*Let* $O$ *be an open subset of* $\mathbb{R}^N:M\subset O$*. Let* $f:O\to X$ *be a continuous function, whose Fréchet derivatives (see [28])* $f^{(j)}:O\to V_j=V_j\left(\left(\mathbb{R}^N\right)^j;X\right)$ *exist and are continuous for* $1\leq j\leq m$*,* $m\in\mathbb{N}$*.*

*Call* $(x-x_0)^j:=(x-x_0,...,x-x_0)\in\left(\mathbb{R}^N\right)^j$*,* $x\in M$*.*

*We will work with* $f|_M$*.*

*Then, by Taylor's formula ([21]), ([28], p. 124), we get*

$$f(x)=\sum_{j=0}^m\frac{f^{(j)}(x_0)(x-x_0)^j}{j!}+R_m(x,x_0),\quad \textit{all } x\in M, \tag{44}$$

*where the remainder is the Riemann integral*

$$R_m(x,x_0):=\int_0^1\frac{(1-u)^{m-1}}{(m-1)!}\left(f^{(m)}(x_0+u(x-x_0))-f^{(m)}(x_0)\right)(x-x_0)^m du, \tag{45}$$

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*here we set* $f^{(0)}(x_0)(x-x_0)^0 = f(x_0)$.

*We consider*

$$\omega := \omega_1\left(f^{(m)}, h\right) := \sup_{\substack{x,y\in M: \\ \|x-y\|_p \le h}} \left\|f^{(m)}(x) - f^{(m)}(y)\right\|, \tag{46}$$

$h > 0$.

*We obtain*

$$\left\|\left(f^{(m)}(x_0 + u(x-x_0)) - f^{(m)}(x_0)\right)(x-x_0)^m\right\|_\gamma \le$$

$$\left\|f^{(m)}(x_0 + u(x-x_0)) - f^{(m)}(x_0)\right\| \cdot \|x-x_0\|_p^m \le$$

$$\omega \|x-x_0\|_p^m \left\lceil \frac{u\|x-x_0\|_p}{h} \right\rceil, \tag{47}$$

*by Lemma 7.1.1, [1], p. 208, where* $\lceil\cdot\rceil$ *is the ceiling.*

*Therefore for all* $x \in M$ *(see [1], pp. 121-122):*

$$\|R_m(x,x_0)\|_\gamma \le \omega \|x-x_0\|_p^m \int_0^1 \left\lceil \frac{u\|x-x_0\|_p}{h} \right\rceil \frac{(1-u)^{m-1}}{(m-1)!} du$$

$$= \omega \Phi_m\left(\|x-x_0\|_p\right) \tag{48}$$

*by a change of variable, where*

$$\Phi_m(t) := \int_0^{|t|} \left\lceil \frac{s}{h} \right\rceil \frac{(|t|-s)^{m-1}}{(m-1)!} ds = \frac{1}{m!}\left(\sum_{j=0}^{\infty} (|t|-jh)_+^m\right), \quad \forall\, t \in \mathbb{R}, \tag{49}$$

*is a (polynomial) spline function, see [1], p. 210-211.*

*Also from there we get*

$$\Phi_m(t) \le \left(\frac{|t|^{m+1}}{(m+1)!h} + \frac{|t|^m}{2m!} + \frac{h|t|^{m-1}}{8(m-1)!}\right), \quad \forall\, t \in \mathbb{R}, \tag{50}$$

*with equality true only at* $t = 0$.

*Therefore it holds*

$$\|R_m(x,x_0)\|_\gamma \le \omega \left(\frac{\|x-x_0\|_p^{m+1}}{(m+1)!h} + \frac{\|x-x_0\|_p^m}{2m!} + \frac{h\|x-x_0\|_p^{m-1}}{8(m-1)!}\right), \quad \forall\, x \in M. \tag{51}$$

*We have found that*

$$\left\|f(x) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(x-x_0)^j}{j!}\right\|_\gamma \le$$

$$\omega_1\left(f^{(m)},h\right)\left(\frac{\|x-x_0\|_p^{m+1}}{(m+1)!h}+\frac{\|x-x_0\|_p^{m}}{2m!}+\frac{h\|x-x_0\|_p^{m-1}}{8(m-1)!}\right)<\infty, \tag{52}$$

$\forall\ x, x_0 \in M$.

*Here* $0<\omega_1\left(f^{(m)},h\right)<\infty$, *by* $M$ *being compact and* $f^{(m)}$ *being continuous on* $M$.

*One can rewrite (52) as follows:*

$$\left\|f(\cdot)-\sum_{j=0}^{m}\frac{f^{(j)}(x_0)(\cdot-x_0)^j}{j!}\right\|_\gamma \leq$$

$$\omega_1\left(f^{(m)},h\right)\left(\frac{\|\cdot-x_0\|_p^{m+1}}{(m+1)!h}+\frac{\|\cdot-x_0\|_p^{m}}{2m!}+\frac{h\|\cdot-x_0\|_p^{m-1}}{8(m-1)!}\right),\ \forall\ x_0\in M, \tag{53}$$

*a pointwise functional inequality on* $M$.

*Here* $(\cdot-x_0)^j$ *maps* $M$ *into* $\left(\mathbb{R}^N\right)^j$ *and it is continuous, also* $f^{(j)}(x_0)$ *maps* $\left(\mathbb{R}^N\right)^j$ *into* $X$ *and it is continuous. Hence their composition* $f^{(j)}(x_0)(\cdot-x_0)^j$ *is continuous from* $M$ *into* $X$.

*Clearly* $f(\cdot)-\sum_{j=0}^{m}\frac{f^{(j)}(x_0)(\cdot-x_0)^j}{j!}\in C(M,X)$, *hence* $\left\|f(\cdot)-\sum_{j=0}^{m}\frac{f^{(j)}(x_0)(\cdot-x_0)^j}{j!}\right\|_\gamma\in C(M)$.

*Let* $\left\{\widetilde{S}_N\right\}_{N\in\mathbb{N}}$ *be a sequence of positive linear operators mapping* $C(M)$ *into* $C(M)$.

*Therefore we obtain*

$$\left(\widetilde{S}_N\left(\left\|f(\cdot)-\sum_{j=0}^{m}\frac{f^{(j)}(x_0)(\cdot-x_0)^j}{j!}\right\|_\gamma\right)\right)(x_0)\leq$$

$$\omega_1\left(f^{(m)},h\right)\left[\frac{\left(\widetilde{S}_N\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)}{(m+1)!h}+\frac{\left(\widetilde{S}_N\left(\|\cdot-x_0\|_p^{m}\right)\right)(x_0)}{2m!}+\right.$$

$$\left.\frac{h\left(\widetilde{S}_N\left(\|\cdot-x_0\|_p^{m-1}\right)\right)(x_0)}{8(m-1)!}\right], \tag{54}$$

$\forall\ N\in\mathbb{N}$, $\forall\ x_0\in M$.

Clearly (54) is valid when $M=\prod_{i=1}^{N}[a_i,b_i]$ and $\widetilde{S}_n=\widetilde{A}_n$, see (23).

All the above is preparation for the following theorem, where we assume Fréchet differentiability of functions.

This will be a direct application of Theorem 10.2, [15], pp. 268-270. The operators $A_n$, $\widetilde{A}_n$ fulfill its assumptions, see (22), (23), (25), and (26).

We present the following high order approximation results.

**Theorem 11** *Let $O$ open subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1,\infty]$, such that $\prod_{i=1}^{N}[a_i,b_i] \subset O \subseteq \mathbb{R}^N$, and let $\left(X, \|\cdot\|_\gamma\right)$ be a general Banach space. Let $m \in \mathbb{N}$ and $f \in C^m(O,X)$, the space of $m$-times continuously Fréchet differentiable functions from $O$ into $X$. We study the approximation of $f|_{\prod_{i=1}^{N}[a_i,b_i]}$. Let $x_0 \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$ and $r>0$. Then*

*1)*

$$\left\|\left(A_n(f)\right)(x_0) - \sum_{j=0}^{m}\frac{1}{j!}\left(A_n\left(f^{(j)}(x_0)(\cdot - x_0)^j\right)\right)(x_0)\right\|_\gamma \leq$$

$$\frac{\omega_1\left(f^{(m)}, r\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)}$$
$$\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right], \tag{55}$$

*2) additionally if $f^{(j)}(x_0) = 0$, $j = 1,...,m$, we have*

$$\|(A_n(f))(x_0) - f(x_0)\|_\gamma \leq$$

$$\frac{\omega_1\left(f^{(m)}, r\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)} \tag{56}$$
$$\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right],$$

*3)*

$$\|(A_n(f))(x_0) - f(x_0)\|_\gamma \leq \sum_{j=1}^{m}\frac{1}{j!}\left\|\left(A_n\left(f^{(j)}(x_0)(\cdot - x_0)^j\right)\right)(x_0)\right\|_\gamma +$$

$$\frac{\omega_1\left(f^{(m)}, r\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)} \tag{57}$$
$$\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right],$$

*and*

*4)*

$$\left\|\|A_n(f) - f\|_\gamma\right\|_{\infty, \prod_{i=1}^{N}[a_i,b_i]} \leq$$

$$\sum_{j=1}^{m}\frac{1}{j!}\left\|\left\|\left(A_n\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\gamma\right\|_{\infty,x_0\in\prod_{i=1}^{N}[a_i,b_i]}+$$

$$\frac{\omega_1\left(f^{(m)},r\left\|\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right\|_{\infty,x_0\in\prod_{i=1}^{N}[a_i,b_i]}^{\frac{1}{m+1}}\right)}{rm!}$$

$$\left\|\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right\|_{\infty,x_0\in\prod_{i=1}^{N}[a_i,b_i]}^{\left(\frac{m}{m+1}\right)} \tag{58}$$

$$\left[\frac{1}{(m+1)}+\frac{r}{2}+\frac{mr^2}{8}\right].$$

We need

**Lemma 12** *The function* $\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^m\right)\right)(x_0)$ *is continuous in* $x_0\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $m\in\mathbb{N}$.

**Proof.** By Lemma 10.3, [15], p. 272.

**Remark 13** *By Remark 10.4 [15], p.273, we get that*

$$\left\|\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^k\right)\right)(x_0)\right\|_{\infty,x_0\in\prod_{i=1}^{N}[a_i,b_i]}\leq\left\|\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right\|_{\infty,x_0\in\prod_{i=1}^{N}[a_i,b_i]}^{\left(\frac{k}{m+1}\right)}, \tag{59}$$

*for all* $k=1,...,m$.

∎

We give

**Corollary 14** *(to Theorem 11, case of* $m=1$*) Then*

*1)*

$$\|(A_n(f))(x_0)-f(x_0)\|_\gamma\leq\left\|\left(A_n\left(f^{(1)}(x_0)(\cdot-x_0)\right)\right)(x_0)\right\|_\gamma+$$

$$\frac{1}{2r}\omega_1\left(f^{(1)},r\left(\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^2\right)\right)(x_0)\right)^{\frac{1}{2}}\right)\left(\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^2\right)\right)(x_0)\right)^{\frac{1}{2}} \tag{60}$$

$$\left[1+r+\frac{r^2}{4}\right],$$

*and*

*2)*

$$\left\|\|(A_n(f))-f\|_\gamma\right\|_{\infty,\prod_{i=1}^{N}[a_i,b_i]}\leq$$

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$$\left\|\left\|\left(A_n\left(f^{(1)}(x_0)(\cdot - x_0)\right)\right)(x_0)\right\|_\gamma\right\|_{\infty, x_0\in\prod_{i=1}^N[a_i,b_i]} +$$

$$\frac{1}{2r}\omega_1\left(f^{(1)}, r\left\|\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^2\right)\right)(x_0)\right\|_{\infty, x_0\in\prod_{i=1}^N[a_i,b_i]}^{\frac{1}{2}}\right)$$

$$\left\|\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^2\right)\right)(x_0)\right\|_{\infty, x_0\in\prod_{i=1}^N[a_i,b_i]}^{\frac{1}{2}}\left[1+r+\frac{r^2}{4}\right], \tag{61}$$

$r>0$.

We make

**Remark 15** *We estimate* $0<\alpha<1$, $\lambda>0$, $m,n\in\mathbb{N}: n^{1-\alpha}>2$,

$$\widetilde{A}_n\left(\|\cdot - x_0\|_\infty^{m+1}\right)(x_0) = \frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|\frac{k}{n}-x_0\right\|_\infty^{m+1}Z(nx_0-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx_0-k)} \overset{(19)}{<}$$

$$\left(\frac{4}{\tanh 2\lambda}\right)^N\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|\frac{k}{n}-x_0\right\|_\infty^{m+1}Z(nx_0-k) = \tag{62}$$

$$\left(\frac{4}{\tanh 2\lambda}\right)^N\left\{\sum_{\substack{k=\lceil na\rceil \\ :\left\|\frac{k}{n}-x_0\right\|_\infty\le\frac{1}{n^\alpha}}}^{\lfloor nb\rfloor}\left\|\frac{k}{n}-x_0\right\|_\infty^{m+1}Z(nx_0-k)+\right.$$

$$\left.\sum_{\substack{k=\lceil na\rceil \\ :\left\|\frac{k}{n}-x_0\right\|_\infty>\frac{1}{n^\alpha}}}^{\lfloor nb\rfloor}\left\|\frac{k}{n}-x_0\right\|_\infty^{m+1}Z(nx_0-k)\right\} \overset{(20)}{\le}$$

$$\left(\frac{4}{\tanh 2\lambda}\right)^N\left\{\frac{1}{n^{\alpha(m+1)}}+\frac{e^{4\lambda}\|b-a\|_\infty^{m+1}}{e^{2\lambda\left(n^{1-\beta}\right)}}\right\}, \tag{63}$$

*(where* $b-a=(b_1-a_1,...,b_N-a_N)$*).*

*We have proved that* ($\forall\ x_0\in\prod_{i=1}^N[a_i,b_i]$)

$$\widetilde{A}_n\left(\|\cdot - x_0\|_\infty^{m+1}\right)(x_0) < \left(\frac{4}{\tanh 2\lambda}\right)^N\left\{\frac{1}{n^{\alpha(m+1)}}+\frac{e^{4\lambda}\|b-a\|_\infty^{m+1}}{e^{2\lambda\left(n^{1-\beta}\right)}}\right\} =: \Lambda_1(n) \tag{64}$$

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*($0<\alpha<1,\ m,n\in\mathbb{N}:n^{1-\alpha}>2,\ \lambda>0$).*

*And, consequently it holds*

$$\left\|\widetilde{A}_n\left(\|\cdot-x_0\|_\infty^{m+1}\right)(x_0)\right\|_{\infty,x_0\in\prod_{i=1}^{N}[a_i,b_i]}<$$

$$\left(\frac{4}{\tanh 2\lambda}\right)^N\left\{\frac{1}{n^{\alpha(m+1)}}+\frac{e^{4\lambda}\|b-a\|_\infty^{m+1}}{e^{2\lambda\left(n^{1-\beta}\right)}}\right\}=\Lambda_1(n)\to 0,\quad as\ n\to+\infty. \tag{65}$$

*So, we have that $\Lambda_1(n)\to 0$, as $n\to+\infty$. Thus, when $p\in[1,\infty]$, from Theorem 11 we have the convergence to zero in the right hand sides of parts (1), (2).*

*Next we estimate $\left\|\left(\widetilde{A}_n\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\gamma$.*

*We have that*

$$\left(\widetilde{A}_n\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f^{(j)}(x_0)\left(\frac{k}{n}-x_0\right)^jZ(nx_0-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx_0-k)}. \tag{66}$$

*When $p=\infty$, $j=1,...,m$, we obtain*

$$\left\|f^{(j)}(x_0)\left(\frac{k}{n}-x_0\right)^j\right\|_\gamma\le\left\|f^{(j)}(x_0)\right\|\left\|\frac{k}{n}-x_0\right\|_\infty^j. \tag{67}$$

*We further have that*

$$\left\|\left(\widetilde{A}_n\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\gamma\overset{(19)}{<}$$

$$\left(\frac{4}{\tanh 2\lambda}\right)^N\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|f^{(j)}(x_0)\left(\frac{k}{n}-x_0\right)^j\right\|_\gamma Z(nx_0-k)\right)\le$$

$$\left(\frac{4}{\tanh 2\lambda}\right)^N\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|f^{(j)}(x_0)\right\|\left\|\frac{k}{n}-x_0\right\|_\infty^j Z(nx_0-k)\right)= \tag{68}$$

$$\left(\frac{4}{\tanh 2\lambda}\right)^N\left\|f^{(j)}(x_0)\right\|\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|\frac{k}{n}-x_0\right\|_\infty^j Z(nx_0-k)\right)=$$

$$\left(\frac{4}{\tanh 2\lambda}\right)^N\left\|f^{(j)}(x_0)\right\|\left\{\sum_{\substack{k=\lceil na\rceil\\ \left\{:\left\|\frac{k}{n}-x_0\right\|_\infty\le\frac{1}{n^\alpha}\right.}}^{\lfloor nb\rfloor}\left\|\frac{k}{n}-x_0\right\|_\infty^j Z(nx_0-k)\right.$$

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$$+ \sum_{\substack{k=\lceil na\rceil \\ \left\{ : \left\| \frac{k}{n} - x_0 \right\|_\infty > \frac{1}{n^\alpha} \right.}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^j Z(nx_0 - k) \Bigg\} \overset{(20)}{\leq} \tag{69}$$

$$\left(\frac{4}{\tanh 2\lambda}\right)^N \left\| f^{(j)}(x_0) \right\| \left\{ \frac{1}{n^{\alpha j}} + \frac{e^{4\lambda} \|b-a\|_\infty^j}{e^{2\lambda(n^{1-\beta})}} \right\} \to 0, \ as\ n \to \infty.$$

*That is*

$$\left\| \left( \widetilde{A}_n \left( f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma \to 0, \ as\ n \to \infty.$$

*Therefore when $p = \infty$, for $j = 1, ..., m$, we have proved:*

$$\left\| \left( \widetilde{A}_n \left( f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma <$$

$$\left(\frac{4}{\tanh 2\lambda}\right)^N \left\| f^{(j)}(x_0) \right\| \left\{ \frac{1}{n^{\alpha j}} + \frac{e^{4\lambda} \|b-a\|_\infty^j}{e^{2\lambda(n^{1-\beta})}} \right\} \leq$$

$$\left(\frac{4}{\tanh 2\lambda}\right)^N \left\| f^{(j)}(x_0) \right\|_\infty \left\{ \frac{1}{n^{\alpha j}} + \frac{e^{4\lambda} \|b-a\|_\infty^j}{e^{2\lambda(n^{1-\beta})}} \right\} =: \Lambda_{2j}(n) < \infty, \tag{70}$$

*and converges to zero, as $n \to \infty$.*

We conclude:

In Theorem 11, the right hand sides of (57) and (58) converge to zero as $n \to \infty$, for any $p \in [1, \infty]$.

Also in Corollary 14, the right hand sides of (60) and (61) converge to zero as $n \to \infty$, for any $p \in [1, \infty]$.

**Conclusion 16** *We have proved that the left hand sides of (55), (56), (57), (58) and (60), (61) converge to zero as $n \to \infty$, for $p \in [1, \infty]$. Consequently $A_n \to I$ (unit operator) pointwise and uniformly, as $n \to \infty$, where $p \in [1, \infty]$. In the presence of initial conditions we achieve a higher speed of convergence, see (56). Higher speed of convergence happens also to the left hand side of (55).*

We further give

**Corollary 17** *(to Theorem 11) Let $O$ open subset of $\left(\mathbb{R}^N, \|\cdot\|_\infty\right)$, such that $\prod_{i=1}^N [a_i, b_i] \subset O \subseteq \mathbb{R}^N$, and let $\left(X, \|\cdot\|_\gamma\right)$ be a general Banach space. Let $m \in \mathbb{N}$ and $f \in C^m(O, X)$, the space of $m$-times continuously Fréchet differentiable functions from $O$ into $X$. We study the approximation of $f|_{\prod_{i=1}^N [a_i, b_i]}$. Let $x_0 \in$*

$\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ *and* $r>0$. *Here* $\Lambda_1(n)$ *as in (65) and* $\Lambda_{2j}(n)$ *as in (70), where* $n\in\mathbb{N}:n^{1-\alpha}>2$, $0<\alpha<1$, $\lambda>0$, $j=1,...,m$. *Then*

*1)*

$$\left\|(A_n(f))(x_0)-\sum_{j=0}^{m}\frac{1}{j!}\left(A_n\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\gamma\leq$$

$$\frac{\omega_1\left(f^{(m)},r(\Lambda_1(n))^{\frac{1}{m+1}}\right)}{rm!}(\Lambda_1(n))^{\left(\frac{m}{m+1}\right)}\left[\frac{1}{(m+1)}+\frac{r}{2}+\frac{mr^2}{8}\right],\tag{71}$$

*2) additionally, if* $f^{(j)}(x_0)=0$, $j=1,...,m$, *we have*

$$\|(A_n(f))(x_0)-f(x_0)\|_\gamma\leq$$

$$\frac{\omega_1\left(f^{(m)},r(\Lambda_1(n))^{\frac{1}{m+1}}\right)}{rm!}(\Lambda_1(n))^{\left(\frac{m}{m+1}\right)}\left[\frac{1}{(m+1)}+\frac{r}{2}+\frac{mr^2}{8}\right],\tag{72}$$

*3)*

$$\left\|\|A_n(f)-f\|_\gamma\right\|_{\infty,\prod_{i=1}^{N}[a_i,b_i]}\leq\sum_{j=1}^{m}\frac{\Lambda_{2j}(n)}{j!}+$$

$$\frac{\omega_1\left(f^{(m)},r(\Lambda_1(n))^{\frac{1}{m+1}}\right)}{rm!}(\Lambda_1(n))^{\left(\frac{m}{m+1}\right)}$$

$$\left[\frac{1}{(m+1)}+\frac{r}{2}+\frac{mr^2}{8}\right]=:\Lambda_3(n)\to0,\text{ as }n\to\infty.\tag{73}$$

We continue with

**Theorem 18** *Let* $f\in C_B(\mathbb{R}^N,X)$, $0<\beta<1$, $\lambda>0$, $x\in\mathbb{R}^N,N,n\in\mathbb{N}$ *with* $n^{1-\beta}>2$, $\omega_1$ *is for* $p=\infty$. *Then*

*1)*

$$\|B_n(f,x)-f(x)\|_\gamma\leq\omega_1\left(f,\frac{1}{n^\beta}\right)+\frac{e^{4\lambda}\left\|\|f\|_\gamma\right\|_\infty}{e^{2\lambda(n^{1-\beta})}}=:\Omega_2(n),\tag{74}$$

*2)*

$$\left\|\|B_n(f)-f\|_\gamma\right\|_\infty\leq\Omega_2(n).\tag{75}$$

*Given that* $f\in\left(C_U(\mathbb{R}^N,X)\cap C_B(\mathbb{R}^N,X)\right)$, *we obtain* $\lim_{n\to\infty}B_n(f)=f$, *uniformly. The speed of convergence above is* $\max\left(\frac{1}{n^\beta},\frac{1}{e^{2\lambda n^{(1-\beta)}}}\right)=\frac{1}{n^\beta}$.

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**Proof.** We have that

$$B_n(f,x) - f(x) \overset{(13)}{=} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx-k) - f(x) \sum_{k=-\infty}^{\infty} Z(nx-k) = \tag{76}$$

$$\sum_{k=-\infty}^{\infty} \left( f\left(\frac{k}{n}\right) - f(x) \right) Z(nx-k).$$

Hence

$$\|B_n(f,x) - f(x)\|_\gamma \leq \sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z(nx-k) =$$

$$\sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty \leq \frac{1}{n^\beta} \end{cases}}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z(nx-k) +$$

$$\sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z(nx-k) \overset{(13)}{\leq}$$

$$\omega_1\left(f, \frac{1}{n^\beta}\right) + 2\left\| \|f\|_\gamma \right\|_\infty \sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} Z(nx-k) \overset{(20)}{\leq}$$

$$\omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{2e^{4\lambda}\left\| \|f\|_\gamma \right\|_\infty}{e^{2\lambda(n^{1-\beta})}}, \tag{77}$$

proving the claim. ■

We give

**Theorem 19** *Let $f \in C_B(\mathbb{R}^N, X)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $\lambda > 0, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, $\omega_1$ is for $p = \infty$. Then*

*1)*

$$\|C_n(f,x) - f(x)\|_\gamma \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) + \frac{2e^{4\lambda}\left\| \|f\|_\gamma \right\|_\infty}{e^{2\lambda(n^{1-\beta})}} =: \Omega_3(n), \tag{78}$$

*2)*

$$\left\| \|C_n(f) - f\|_\gamma \right\|_\infty \leq \Omega_3(n). \tag{79}$$

*Given that $f \in (C_U(\mathbb{R}^N, X) \cap C_B(\mathbb{R}^N, X))$, we obtain $\lim_{n\to\infty} C_n(f) = f$, uniformly.*

**Proof.** We notice that

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt = \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} \ldots \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1, t_2, \ldots, t_N)\,dt_1 dt_2 \ldots dt_N =$$

$$\int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} \ldots \int_0^{\frac{1}{n}} f\left(t_1 + \frac{k_1}{n}, t_2 + \frac{k_2}{n}, \ldots, t_N + \frac{k_N}{n}\right) dt_1 \ldots dt_N = \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt. \tag{80}$$

Thus it holds (by (35))

$$C_n(f,x) = \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt\right) Z(nx-k). \tag{81}$$

We observe that

$$\|C_n(f,x) - f(x)\|_\gamma =$$

$$\left\| \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt\right) Z(nx-k) - \sum_{k=-\infty}^{\infty} f(x) Z(nx-k) \right\|_\gamma =$$

$$\left\| \sum_{k=-\infty}^{\infty} \left( \left(n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt\right) - f(x) \right) Z(nx-k) \right\|_\gamma =$$

$$\left\| \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left( f\left(t + \frac{k}{n}\right) - f(x) \right) dt\right) Z(nx-k) \right\|_\gamma \leq \tag{82}$$

$$\sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left\| f\left(t + \frac{k}{n}\right) - f(x) \right\|_\gamma dt\right) Z(nx-k) =$$

$$\sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n} - x\right\|_\infty \leq \frac{1}{n^\beta} \end{cases}}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left\| f\left(t + \frac{k}{n}\right) - f(x) \right\|_\gamma dt\right) Z(nx-k) +$$

$$\sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n} - x\right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left\| f\left(t + \frac{k}{n}\right) - f(x) \right\|_\gamma dt\right) Z(nx-k) \leq$$

$$\sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n} - x\right\|_\infty \leq \frac{1}{n^\beta} \end{cases}}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \omega_1\left(f, \|t\|_\infty + \left\|\frac{k}{n} - x\right\|_\infty\right) dt\right) Z(nx-k) +$$

$$2\left\|\left\|f\right\|_{\gamma}\right\|_{\infty}\left(\sum_{\substack{k=-\infty \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\infty} Z\left(|nx-k|\right)\right) \leq$$

$$\omega_1\left(f,\frac{1}{n}+\frac{1}{n^{\beta}}\right)+\frac{2e^{4\lambda}\left\|\left\|f\right\|_{\gamma}\right\|_{\infty}}{e^{2\lambda\left(n^{1-\beta}\right)}}, \tag{83}$$

proving the claim. ■

We also present

**Theorem 20** *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $0<\beta<1$, $x \in \mathbb{R}^N$, $\lambda>0, N, n \in \mathbb{N}$ with $n^{1-\beta}>2$, $\omega_1$ is for $p=\infty$. Then*

*1)*

$$\left\|D_n\left(f,x\right)-f\left(x\right)\right\|_{\gamma} \leq \omega_1\left(f,\frac{1}{n}+\frac{1}{n^{\beta}}\right)+\frac{2e^{4\lambda}\left\|\left\|f\right\|_{\gamma}\right\|_{\infty}}{e^{2\lambda\left(n^{1-\beta}\right)}}=:\Omega_4\left(n\right), \tag{84}$$

*2)*

$$\left\|\left\|D_n\left(f\right)-f\right\|_{\gamma}\right\|_{\infty} \leq \Omega_4\left(n\right). \tag{85}$$

*Given that $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$, we obtain $\lim_{n\to\infty} D_n\left(f\right)=f$, uniformly.*

**Proof.** We have that (by (37))

$$\left\|D_n\left(f,x\right)-f\left(x\right)\right\|_{\gamma}=\left\|\sum_{k=-\infty}^{\infty}\delta_{nk}\left(f\right)Z\left(nx-k\right)-\sum_{k=-\infty}^{\infty}f\left(x\right)Z\left(nx-k\right)\right\|_{\gamma}=$$

$$\left\|\sum_{k=-\infty}^{\infty}\left(\delta_{nk}\left(f\right)-f\left(x\right)\right)Z\left(nx-k\right)\right\|_{\gamma}=\left\|\sum_{k=-\infty}^{\infty}\omega_r\left(f\left(\frac{k}{n}+\frac{r}{n\theta}\right)-f\left(x\right)\right)Z\left(nx-k\right)\right\|_{\gamma}\leq$$

$$\sum_{k=-\infty}^{\infty}\left(\sum_{r=0}^{\theta}\omega_r\left\|f\left(\frac{k}{n}+\frac{r}{n\theta}\right)-f\left(x\right)\right\|_{\gamma}\right)Z\left(nx-k\right)=$$

$$\sum_{\substack{k=-\infty \\ \left\|\frac{k}{n}-x\right\|_{\infty}\leq\frac{1}{n^{\beta}}}}^{\infty}\left(\sum_{r=0}^{\theta}\omega_r\left\|f\left(\frac{k}{n}+\frac{r}{n\theta}\right)-f\left(x\right)\right\|_{\gamma}\right)Z\left(nx-k\right)+$$

$$\sum_{\substack{k=-\infty \\ \left\| \frac{k}{n}-x \right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\infty} \left( \sum_{r=0}^{\theta} \omega_r \left\| f\left( \frac{k}{n} + \frac{r}{n\theta} \right) - f(x) \right\|_{\gamma} \right) Z(nx-k) \leq$$

$$\sum_{\substack{k=-\infty \\ \left\| \frac{k}{n}-x \right\|_{\infty} \leq \frac{1}{n^{\beta}}}}^{\infty} \left( \sum_{r=0}^{\theta} \omega_r \left\| f\left( \frac{k}{n} + \frac{r}{n\theta} \right) - f(x) \right\|_{\gamma} \right) Z(nx-k) +$$

$$2 \left\| \|f\|_{\gamma} \right\|_{\infty} \left( \sum_{\substack{k=-\infty \\ \left\| \frac{k}{n}-x \right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\infty} (Z(nx-k)) \right) \leq$$

$$\omega_1\left( f, \frac{1}{n} + \frac{1}{n^{\beta}} \right) + \frac{2e^{4\lambda} \left\| \|f\|_{\gamma} \right\|_{\infty}}{e^{2\lambda(n^{1-\beta})}} = \Omega_4(n),$$

proving the claim. ■

Next we perform multi layer neural network approximations.

We make

**Definition 21** *Let $f \in C_B(\mathbb{R}^N, X)$, $N \in \mathbb{N}$, where $\left(X, \|\cdot\|_{\gamma}\right)$ is a Banach space. We define the general neural network operator*

$$F_n(f,x) := \sum_{k=-\infty}^{\infty} l_{nk}(f) Z(nx-k) =$$

$$\begin{cases} B_n(f,x), & \text{if } l_{nk}(f) = f\left(\frac{k}{n}\right), \\ C_n(f,x), & \text{if } l_{nk}(f) = n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt, \\ D_n(f,x), & \text{if } l_{nk}(f) = \delta_{nk}(f). \end{cases} \tag{86}$$

Clearly $l_{nk}(f)$ is an $X$-valued bounded linear functional such that $\|l_{nk}(f)\|_{\gamma} \leq \left\| \|f\|_{\gamma} \right\|_{\infty}$.

Hence $F_n(f)$ is a bounded linear operator with $\left\| \|F_n(f)\|_{\gamma} \right\|_{\infty} \leq \left\| \|f\|_{\gamma} \right\|_{\infty}$.

We need

**Theorem 22** *Let $f \in C_B(\mathbb{R}^N, X)$, $N \geq 1$. Then $F_n(f) \in C_B(\mathbb{R}^N, X)$.*

**Proof.** Lengthy and similar to the proof of Theorem 11 of [18], as such is omitted. ■

**Remark 23** *By (22) it is obvious that* $\left\| \|A_n(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty < \infty$*, and* $A_n(f) \in C\left(\prod_{i=1}^{N}[a_i, b_i], X\right)$*, given that* $f \in C\left(\prod_{i=1}^{N}[a_i, b_i], X\right)$.

*Call* $K_n$ *any of the operators* $A_n, B_n, C_n, D_n$.

*Clearly then*

$$\left\| \|K_n^2(f)\|_\gamma \right\|_\infty = \left\| \|K_n(K_n(f))\|_\gamma \right\|_\infty \leq \left\| \|K_n(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty, \tag{87}$$

*etc.*

*Therefore we get*

$$\left\| \|K_n^k(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty, \ \ \forall\, k \in \mathbb{N}, \tag{88}$$

*the contraction property.*

*Also we see that*

$$\left\| \|K_n^k(f)\|_\gamma \right\|_\infty \leq \left\| \|K_n^{k-1}(f)\|_\gamma \right\|_\infty \leq \ldots \leq \left\| \|K_n(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty. \tag{89}$$

*Here* $K_n^k$ *are bounded linear operators.*

**Notation 24** *Here* $N \in \mathbb{N}$, $0 < \beta < 1$. *Denote by*

$$c_N := \begin{cases} \left(\frac{4}{\tanh 2\lambda}\right)^N, & \text{if } K_n = A_n, \\ 1, & \text{if } K_n = B_n, C_n, D_n, \end{cases} \tag{90}$$

$$\Lambda(n) := \begin{cases} \frac{1}{n^\beta}, & \text{if } K_n = A_n,\ B_n, \\ \frac{1}{n} + \frac{1}{n^\beta}, & \text{if } K_n = C_n, D_n, \end{cases} \tag{91}$$

$$\Gamma := \begin{cases} C\left(\prod_{i=1}^{N}[a_i, b_i], X\right), & \text{if } K_n = A_n, \\ C_B\left(\mathbb{R}^N, X\right), & \text{if } K_n = B_n, C_n, D_n, \end{cases} \tag{92}$$

*and*

$$Y := \begin{cases} \prod_{i=1}^{N}[a_i, b_i], & \text{if } K_n = A_n, \\ \mathbb{R}^N, & \text{if } K_n = B_n, C_n, D_n. \end{cases} \tag{93}$$

We give the condensed

**Theorem 25** *Let* $f \in \Gamma$, $0 < \beta < 1$, $x \in Y$; $n, \lambda > 0$*;* $N \in \mathbb{N}$ *with* $n^{1-\beta} > 2$. *Then*

*(i)*

$$\|K_n(f, x) - f(x)\|_\gamma \leq c_N \left[\omega_1(f, \Lambda(n)) + \frac{2e^{4\lambda}\left\| \|f\|_\gamma \right\|_\infty}{e^{2\lambda(n^{1-\beta})}}\right] =: \tau(n), \tag{94}$$

*where $\omega_1$ is for $p = \infty$,*

*and*

*(ii)*

$$\left\| \left\| K_n(f) - f \right\|_\gamma \right\|_\infty \leq \tau(n) \to 0, \text{ as } n \to \infty. \tag{95}$$

*For $f$ uniformly continuous and in $\Gamma$ we obtain*

$$\lim_{n\to\infty} K_n(f) = f,$$

*pointwise and uniformly.*

**Proof.** By Theorems 9, 18, 19, 20. ■

Next we do iterated, multi layer neural network approximation. (see also [10]).

We make

**Remark 26** *Let $r \in \mathbb{N}$ and $K_n$ as above. We observe that*

$$K_n^r f - f = \left(K_n^r f - K_n^{r-1} f\right) + \left(K_n^{r-1} f - K_n^{r-2} f\right) +$$

$$\left(K_n^{r-2} f - K_n^{r-3} f\right) + ... + \left(K_n^2 f - K_n f\right) + \left(K_n f - f\right).$$

*Then*

$$\left\| \left\| K_n^r f - f \right\|_\gamma \right\|_\infty \leq \left\| \left\| K_n^r f - K_n^{r-1} f \right\|_\gamma \right\|_\infty + \left\| \left\| K_n^{r-1} f - K_n^{r-2} f \right\|_\gamma \right\|_\infty +$$

$$\left\| \left\| K_n^{r-2} f - K_n^{r-3} f \right\|_\gamma \right\|_\infty + ... + \left\| \left\| K_n^2 f - K_n f \right\|_\gamma \right\|_\infty + \left\| \left\| K_n f - f \right\|_\gamma \right\|_\infty =$$

$$\left\| \left\| K_n^{r-1}(K_n f - f) \right\|_\gamma \right\|_\infty + \left\| \left\| K_n^{r-2}(K_n f - f) \right\|_\gamma \right\|_\infty + \left\| \left\| K_n^{r-3}(K_n f - f) \right\|_\gamma \right\|_\infty + ... +$$

$$\left\| \left\| K_n(K_n f - f) \right\|_\gamma \right\|_\infty + \left\| \left\| K_n f - f \right\|_\gamma \right\|_\infty \leq r \left\| \left\| K_n f - f \right\|_\gamma \right\|_\infty.$$

*That is*

$$\left\| \left\| K_n^r f - f \right\|_\gamma \right\|_\infty \leq r \left\| \left\| K_n f - f \right\|_\gamma \right\|_\infty. \tag{96}$$

*We give*

**Theorem 27** *All here as in Theorem 25 and $r \in \mathbb{N}$, $\tau(n)$ as in (94). Then*

$$\left\| \left\| K_n^r f - f \right\|_\gamma \right\|_\infty \leq r\tau(n). \tag{97}$$

*So that the speed of convergence to the unit operator of $K_n^r$ is not worse than of $K_n$.*

**Proof.** As similar to [18] is omitted. ■

**Remark 28** *Let* $m_1, m_2, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$, $0 < \beta < 1, \lambda > 0$, $f \in \Gamma$. *Then*

$$\Lambda(m_1) \geq \Lambda(m_2) \geq ... \geq \Lambda(m_r), \ \Lambda \text{ as in (91).}$$

*Therefore*

$$\omega_1(f, \Lambda(m_1)) \geq \omega_1(f, \Lambda(m_2)) \geq ... \geq \omega_1(f, \Lambda(m_r)).$$

*Assıme further that* $m_i^{(1-\beta)} > 2$, $i = 1, ..., r$. *Then*

$$\frac{e^{4\lambda}}{e^{2\lambda m_1^{(1-\beta)}}} \geq \frac{e^{4\lambda}}{e^{\lambda m_2^{(1-\beta)}}} \geq ... \geq \frac{e^{4\lambda}}{e^{\lambda m_r^{(1-\beta)}}}.$$

*Let* $K_{m_i}$ *as above,* $i = 1, ..., r$, *all of the same kind. We write*

$$K_{m_r}\left(K_{m_{r-1}}(...K_{m_2}(K_{m_1 f}))\right) - f =$$

$$K_{m_r}\left(K_{m_{r-1}}(...K_{m_2}(K_{m_1 f}))\right) - K_{m_r}\left(K_{m_{r-1}}(...K_{m_2} f)\right) +$$

$$K_{m_r}\left(K_{m_{r-1}}(...K_{m_2} f)\right) - K_{m_r}\left(K_{m_{r-1}}(...K_{m_3} f)\right) +$$

$$K_{m_r}\left(K_{m_{r-1}}(...K_{m3} f)\right) - K_{m_r}\left(K_{m_{r-1}}(...K_{m_4} f)\right) + ...+$$

$$K_{m_r}\left(K_{m_{r-1}} f\right) - K_{m_r} f + K_{m_r} f - f =$$

$$K_{m_r}\left(K_{m_{r-1}}(...K_{m_2})\right)(K_{m_1} f - f) + K_{m_r}\left(K_{m_{r-1}}(...K_{m_3})\right)(K_{m_2} f - f) +$$

$$K_{m_r}\left(K_{m_{r-1}}(...K_{m_4})\right)(K_{m3} f - f) + ... + K_{m_r}\left(K_{m_{r-1}} f - f\right) + K_{m_r} f - f.$$

*Hence by the triangle inequality of* $\left\| \|\cdot\|_\gamma \right\|_\infty$ *we get*

$$\left\| \left\| K_{m_r}\left(K_{m_{r-1}}(...K_{m_2}(K_{m_1} f))\right) - f \right\|_\gamma \right\|_\infty \leq$$

$$\left\| \left\| K_{m_r} K_{m_{r-1}} ... K_{m_2}(K_{m_1} f - f) \right\|_\gamma \right\|_\infty +$$

$$\left\| \left\| K_{m_r} K_{m_{r-1}} ... K_{m_2}(K_{m_1} f - f) \right\|_\gamma \right\|_\infty +$$

$$\left\| \left\| K_{m_r}\left(K_{m_{r-1}}(...K_{m_4})\right)(K_{m_3} f - f) \right\|_\gamma \right\|_\infty + ...+$$

$$\left\| \left\| K_{m_r}\left(K_{m_{r-1}} f - f\right) \right\|_\gamma \right\|_\infty + \left\| \left\| K_{m_r} f - f \right\|_\gamma \right\|_\infty \leq$$

*(repeatedly applying (87))*

$$\left\| \left\| K_{m_1} f - f \right\|_\gamma \right\|_\infty + \left\| \left\| K_{m_2} f - f \right\|_\gamma \right\|_\infty + \left\| \left\| K_{m_3} f - f \right\|_\gamma \right\|_\infty + ...+$$

$$\left\| \left\| K_{m_{r-1}} f - f \right\|_\gamma \right\|_\infty + \left\| \left\| K_{m_2} f - f \right\|_\gamma \right\|_\infty + \left\| \left\| K_{m_3} f - f \right\|_\gamma \right\|_\infty + ...+$$

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$$\left\| \left\| K_{m_{r-1}} f - f \right\|_\gamma \right\|_\infty + \left\| \left\| K_{m_r} f - f \right\|_\gamma \right\|_\infty =_{i=1}^{r} \left\| \left\| K_{m_i} f - f \right\|_\gamma \right\|_\infty .$$

*That is, we proved*

$$\left\| \left\| K_{m_r} \left( K_{m_{r-1}} \left( ... K_{m_2} \left( K_{m_1} f \right) \right) \right) - f \right\|_\gamma \right\|_\infty \leq_{i=1}^{r} \left\| \left\| K_{m_i} f - f \right\|_\gamma \right\|_\infty . \tag{98}$$

*We also present*

**Theorem 29** *Let* $f \in \Gamma$*;* $m, N, m_1, m_2, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$, $0 < \beta < 1, \lambda > 0$; $m_i^{(1-\beta)} > 2$, $i = 1, ..., r$, $x \in Y$, *and let* $(K_{m_1}, ..., K_{m_r})$ *as* $(A_{m_1}, ..., A_{m_r})$ *or* $(B_{m_1}, ..., B_{m_r})$ *or* $(C_{m_1}, ..., C_{m_r})$ *or* $(D_{m_1}, ..., D_{m_r})$, $p = \infty$. *Then*

$$\left\| K_{m_r} \left( K_{m_{r-1}} \left( ... K_{m_2} \left( K_{m_1} f \right) \right) \right) (x) - f(x) \right\|_\gamma \leq$$

$$\left\| \left\| K_{m_r} \left( K_{m_{r-1}} \left( ... K_{m_2} \left( K_{m_1} f \right) \right) \right) - f \right\|_\gamma \right\|_\infty \leq$$

$$\sum_{i=1}^{r} \left\| \left\| K_{m_i} f - f \right\|_\gamma \right\|_\infty \leq$$

$$c_N \sum_{i=1}^{r} \left[ \omega_1 \left( f, \Lambda(m_i) \right) + \frac{2e^{4\lambda} \left\| \left\| f \right\|_\gamma \right\|_\infty}{e^{2\lambda m_i^{(1-\beta)}}} \right] \leq$$

$$r c_N \left[ \omega_1 \left( f, \Lambda(m_1) \right) + \frac{2e^{4\lambda} \left\| \left\| f \right\|_\gamma \right\|_\infty}{e^{2\lambda m_1^{(1-\beta)}}} \right] . \tag{99}$$

*Clearly, we notice that the speed of convergence to the unit operator of the multiply iterated multi layer operator is not worse than the speed of* $K_{m_1}$.

**Proof.** As similar to [18] is omitted. ■

We continue with

**Theorem 30** *Let all as in Corollary 17, and* $r \in \mathbb{N}$. *Here* $\Lambda_3(n)$ *is as in (73). Then*

$$\left\| \left\| A_n^r f - f \right\|_\gamma \right\|_\infty \leq r \left\| \left\| A_n f - f \right\|_\gamma \right\|_\infty \leq r \Lambda_3(n) . \tag{100}$$

**Proof.** As similar to [18] is omitted. ■

Next we present some $L_{p_1}$, $p_1 \geq 1$, approximation related results.

**Theorem 31** *Let* $p_1 \geq 1$, $f \in C\left( \prod_{i=1}^{n} [a_i, b_i], X \right)$, $0 < \beta < 1, \lambda > 0$; $N, n \in \mathbb{N}$ *with* $n^{1-\beta} > 2$, *and* $\Omega_1(n)$ *as in (38),* $\omega_1$ *is for* $p = \infty$. *Then*

$$\left\| \left\| A_n f - f \right\|_\gamma \right\|_{p_1, \prod_{i=1}^{n} [a_i, b_i]} \leq \Omega_1(n) \left( \prod_{i=1}^{n} (b_i - a_i) \right)^{\frac{1}{p_1}} . \tag{101}$$

*We notice that* $\lim_{n \to \infty} \left\| \left\| A_n f - f \right\|_\gamma \right\|_{p_1, \prod_{i=1}^{n} [a_i, b_i]} = 0.$

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**Proof.** Obvious, by integrating (38), etc. ■

It follows

**Theorem 32** *Let* $p_1 \geq 1,\ f \in C_B\left(\mathbb{R}^N, X\right),\ 0 < \beta < 1, \lambda > 0;\ N, n \in \mathbb{N}$ *with* $n^{1-\beta} > 2$, *and* $\omega_1$ *is for* $p = \infty$; $\Omega_2(n)$ *as in (74) and* $P$ *a compact set of* $\mathbb{R}^N$. *Then*

$$\left\| \|B_n f - f\|_\gamma \right\|_{p_1,\ P} \leq \Omega_2(n) |P|^{\frac{1}{p_1}}, \tag{102}$$

*where* $|P| < \infty$, *is the Lebesgue measure of* $P$. *We notice that* $\lim_{n\to\infty} \left\| \|B_n f - f\|_\gamma \right\|_{p_1, P} = 0$ *for* $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$.

**Proof.** By integrating (74), etc. ■

Next come

**Theorem 33** *All as in Theorem 32, but we use* $\Omega_3(n)$ *of (78). Then*

$$\left\| \|C_n f - f\|_\gamma \right\|_{p_1,\ P} \leq \Omega_3(n) |P|^{\frac{1}{p_1}}. \tag{103}$$

*We have that* $\lim_{n\to\infty} \left\| \|C_n f - f\|_\gamma \right\|_{p_1,\ P} = 0$ *for* $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$.

**Proof.** By (78). ■

**Theorem 34** *All as in Theorem 32, but we use* $\Omega_4(n)$ *of (84). Then*

$$\left\| \|D_n f - f\|_\gamma \right\|_{p_1,\ P} \leq \Omega_4(n) |P|^{\frac{1}{p_1}}. \tag{104}$$

*We have that* $\lim_{n\to\infty} \left\| \|D_n f - f\|_\gamma \right\|_{p_1,\ P} = 0$ *for* $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$.

**Proof.** By (84). ■

**Application 35** *A typical application of all of our results is when* $\left(X, \|\cdot\|_\gamma\right) = (\mathbb{C}, |\cdot|)$, *where* $\mathbb{C}$ *is the set of the complex numbers.*

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# General sigmoid based Banach space valued neural network approximation

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**Abstract**

Here we study the univariate quantitative approximation of Banach space valued continuous functions on a compact interval or all the real line by quasi-interpolation Banach space valued neural network operators. We perform also the related Banach space valued fractional approximation. These approximations are derived by establishing Jackson type inequalities involving the modulus of continuity of the engaged function or its Banach space valued high order derivative or fractional derivaties. Our operators are defined by using a density function induced by a general sigmoid function. The approximations are pointwise and with respect to the uniform norm. The related Banach space valued feed-forward neural networks are with one hidden layer. We finish with a convergence analysis.

**2020 AMS Mathematics Subject Classification:** 26A33, 41A17, 41A25, 41A30, 46B25.

**Keywords and Phrases:** general sigmoid function, Banach space valued neural network approximation, Banach space valued quasi-interpolation operator, modulus of continuity, Banach space valued Caputo fractional derivative, Banach space valued fractional approximation.

## 1 Introduction

The author in [1] and [2], see Chapters 2-5, was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliagnet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact suport. Also in [2] he gives the $N$th order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see Chapters 4-5 there.

The author inspired by [14], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [3], [4], [5],[6], [7], by treating both the univariate and multivariate cases. He did also the corresponding fractional case [8].

In this article we are greatly inspired by the related works [15], [16].

The author here performs general sigmoid function based neural network approximations to continuous functions over compact intervals of the real line or over the whole $\mathbb{R}$ with values to an arbitrary Banach space $(X, \|\cdot\|)$. Finally he treats completely the related $X$-valued fractional approximation. All convergences here are with rates expressed via the modulus of continuity of the involved function or its $X$-valued high order derivative, or $X$-valued fractional derivatives and given by very tight Jackson type inequalities.

Our compact intervals are not necessarily symmetric to the origin. Some of our upper bounds to error quantity are very flexible and general. In preparation to prove our results we establish important properties of the basic density function defining our operators which is induced by a general sigmoid function.

Feed-forward $X$-valued neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma(\langle a_j \cdot x \rangle + b_j), \quad x \in \mathbb{R}^s,\ s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in X$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of $a_j$ and $x$, and $\sigma$ is the activation function of the network. In many fundamental neural network models, the activation function is derived from various specific sigmoid functions. Here we work for a general sigmoid function. About neural networks in general read [17], [18], [20]. See also [9] for a complete study of real valued approximation by neural network operators.

## 2 Basics

Let $h : \mathbb{R} \to [-1, 1]$ be a general sigmoid function, such that it is strictly increasing, $h(0) = 0$, $h(-x) = -h(x)$, $h(+\infty) = 1$, $h(-\infty) = -1$. Also $h$ is strictly convex over $(-\infty, 0]$ and striclty concave over $[0, +\infty)$, with $h^{(2)} \in C(\mathbb{R})$.

We consider the activation function

$$\psi(x) := \frac{1}{4}(h(x+1) - h(x-1)), \quad x \in \mathbb{R}, \tag{1}$$

As in [13], p. 285, we get that $\psi(-x) = \psi(x)$, thus $\psi$ is an even function. Since $x+1 > x-1$, then $h(x+1) > h(x-1)$, and $\psi(x) > 0$, all $x \in \mathbb{R}$.

We see that

$$\psi(0) = \frac{h(1)}{2}. \tag{2}$$

Let $x > 1$, we have that

$$\psi'(x) = \frac{1}{4}(h'(x+1) - h'(x-1)) < 0,$$

by $h'$ being strictly decreasing over $[0, +\infty)$.

Let now $0 < x < 1$, then $1 - x > 0$ and $0 < 1 - x < 1 + x$. It holds $h'(x-1) = h'(1-x) > h'(x+1)$, so that again $\psi'(x) < 0$. Consequently $\psi$ is stritly decreasing on $(0, +\infty)$.

Clearly, $\psi$ is strictly increasing on $(-\infty, 0)$, and $\psi'(0) = 0$.

See that

$$\lim_{x \to +\infty} \psi(x) = \frac{1}{4}(h(+\infty) - h(+\infty)) = 0, \tag{3}$$

and

$$\lim_{x \to -\infty} \psi(x) = \frac{1}{4}(h(-\infty) - h(-\infty)) = 0. \tag{4}$$

That is the $x$-axis is the horizontal asymptote on $\psi$.

Conclusion, $\psi$ is a bell symmetric function with maximum

$$\psi(0) = \frac{h(1)}{2}.$$

We need

**Theorem 1** *We have that*

$$\sum_{i=-\infty}^{\infty} \psi(x-i) = 1, \quad \forall\, x \in \mathbb{R}. \tag{5}$$

**Proof.** As exactly the same as in [13], p. 286 is omitted. ■

**Theorem 2** *It holds*

$$\int_{-\infty}^{\infty} \psi(x)\, dx = 1. \tag{6}$$

**Proof.** Similar to [13], p. 287. It is omitted. ■

Thus $\psi(x)$ is a density function on $\mathbb{R}$.

We give

**Theorem 3** *Let $0<\alpha<1$, and $n\in\mathbb{N}$ with $n^{1-\alpha}>2$. It holds*

$$\sum_{\begin{cases} k=-\infty \\ :|nx-k|\geq n^{1-\alpha} \end{cases}}^{\infty} \psi(nx-k) < \frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}. \tag{7}$$

*Notice that*

$$\lim_{n\to+\infty} \frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}=0.$$

**Proof.** Let $x\geq 1$. That is $0\leq x-1<x+1$. Applying the mean value theorem we get

$$\psi(x) \overset{(1)}{=} \frac{1}{4}\cdot 2\cdot h'(\xi) = \frac{h'(\xi)}{2}, \tag{8}$$

for some $x-1<\xi<x+1$.

Since $h'$ is strictly decreasing we obtain $h'(\xi)<h'(x-1)$ and

$$\psi(x) < \frac{h'(x-1)}{2}, \ \ \forall\, x\geq 1. \tag{9}$$

Therefore we have

$$\sum_{\begin{cases} k=-\infty \\ :|nx-k|\geq n^{1-\alpha} \end{cases}}^{\infty} \psi(nx-k) = \sum_{\begin{cases} k=-\infty \\ :|nx-k|\geq n^{1-\alpha} \end{cases}}^{\infty} \psi(|nx-k|) <$$

$$\frac{1}{2}\sum_{\begin{cases} k=-\infty \\ :|nx-k|\geq n^{1-\alpha} \end{cases}}^{\infty} h'(|nx-k|-1) \leq \frac{1}{2}\int_{(n^{1-\alpha}-1)}^{+\infty} h'(x-1)\,d(x-1) =$$

$$\frac{1}{2}\left(h(x-1)\,|_{(n^{1-\alpha}-1)}^{+\infty}\right) = \frac{1}{2}\left[h(+\infty)-h\left(n^{1-\alpha}-2\right)\right] = \frac{1}{2}\left(1-h\left(n^{1-\alpha}-2\right)\right). \tag{10}$$

The claim is proved. ■

Denote by $\lfloor\cdot\rfloor$ the integral part of the number and by $\lceil\cdot\rceil$ the ceiling of the number.

We further give

**Theorem 4** *Let $x\in[a,b]\subset\mathbb{R}$ and $n\in\mathbb{N}$ so that $\lceil na\rceil\leq\lfloor nb\rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)} < \frac{1}{\psi(1)}, \quad \forall\, x\in[a,b]. \tag{11}$$

**Proof.** As similar to [13], p. 289 is omitted. ■

**Remark 5** *We have that*

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi\left(nx-k\right)\neq 1, \tag{12}$$

*for at least some $x\in[a,b]$.*

*See [13], p. 290, same reasoning.*

**Note 6** *For large enough $n$ we always obtain $\lceil na\rceil\leq\lfloor nb\rfloor$. Also $a\leq\frac{k}{n}\leq b$, iff $\lceil na\rceil\leq k\leq\lfloor nb\rfloor$. In general it holds (by (5))*

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi\left(nx-k\right)\leq 1. \tag{13}$$

Let $(X,\|\cdot\|)$ be a Banach space.

**Definition 7** *Let $f\in C([a,b],X)$ and $n\in\mathbb{N}:\lceil na\rceil\leq\lfloor nb\rfloor$. We introduce and define the $X$-valued linear neural network operators*

$$A_n(f,x):=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\psi\left(nx-k\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi\left(nx-k\right)},\quad x\in[a,b]. \tag{14}$$

Clearly here $A_n(f,x)\in C([a,b],X)$. For convenience we use the same $A_n$ for real valued function when needed. We study here the pointwise and uniform convergence of $A_n(f,x)$ to $f(x)$ with rates.

For convenience also we call

$$A_n^*(f,x):=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\psi\left(nx-k\right), \tag{15}$$

(similarly $A_n^*$ can be defined for real valued function) that is

$$A_n(f,x)=\frac{A_n^*(f,x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi\left(nx-k\right)}. \tag{16}$$

So that

$$A_n(f,x)-f(x)=\frac{A_n^*(f,x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi\left(nx-k\right)}-f(x)$$

$$=\frac{A_n^*(f,x)-f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi\left(nx-k\right)\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi\left(nx-k\right)}. \tag{17}$$

Consequently we derive

$$\|A_n(f,x)-f(x)\|\leq\frac{1}{\psi(1)}\left\|A_n^*(f,x)-f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi\left(nx-k\right)\right)\right\|. \tag{18}$$

That is

$$\|A_n(f,x)-f(x)\| \leq \frac{1}{\psi(1)}\left\|\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(f\left(\frac{k}{n}\right)-f(x)\right)\psi(nx-k)\right\|. \tag{19}$$

We will estimate the right hand side of (19).

For that we need, for $f\in C([a,b],X)$ the first modulus of continuity

$$\omega_1(f,\delta)_{[a,b]} := \omega_1(f,\delta) := \sup_{\substack{x,y\in[a,b]\\ |x-y|\leq\delta}} \|f(x)-f(y)\|, \quad \delta>0. \tag{20}$$

Similarly, it is defined $\omega_1$ for $f\in C_{uB}(\mathbb{R},X)$ (uniformly continuous and bounded functions from $\mathbb{R}$ into $X$), for $f\in C_B(\mathbb{R},X)$ (continuous and bounded $X$-valued) and for $f\in C_u(\mathbb{R},X)$ (uniformly continuous).

The fact $f\in C([a,b],X)$ or $f\in C_u(\mathbb{R},X)$, is equivalent to $\lim_{\delta\to 0}\omega_1(f,\delta)=0$, see [11].

**Definition 8** *When $f\in C_{uB}(\mathbb{R},X)$, or $f\in C_B(\mathbb{R},X)$, we define*

$$\overline{A}_n(f,x) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right)\psi(nx-k), \quad n\in\mathbb{N},\ x\in\mathbb{R}, \tag{21}$$

*the $X$-valued quasi-interpolation neural network operator.*

**Remark 9** *We have that*

$$\left\|f\left(\frac{k}{n}\right)\right\| \leq \|f\|_{\infty,\mathbb{R}} < +\infty,$$

*and*

$$\left\|f\left(\frac{k}{n}\right)\right\|\psi(nx-k) \leq \|f\|_{\infty,\mathbb{R}}\,\psi(nx-k), \tag{22}$$

*and*

$$\sum_{k=-\lambda}^{\lambda}\left\|f\left(\frac{k}{n}\right)\right\|\psi(nx-k) \leq \|f\|_{\infty,\mathbb{R}}\left(\sum_{k=-\lambda}^{\lambda}\psi(nx-k)\right),$$

*and finally*

$$\sum_{k=-\infty}^{\infty}\left\|f\left(\frac{k}{n}\right)\right\|\psi(nx-k) \leq \|f\|_{\infty,\mathbb{R}}, \tag{23}$$

*a convergent in $\mathbb{R}$ series.*

*So the series $\sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right)\psi(nx-k)$ is absolutely convergent in $X$, hence it is convergent in $X$ and $\overline{A}_n(f,x)\in X$.*

We denote by $\|f\|_\infty := \sup_{x\in[a,b]}\|f(x)\|$, for $f\in C([a,b],X)$, similarly is defined for $f\in C_B(\mathbb{R},X)$.

6

## 3 Main Results

We present a series of $X$-valued neural network approximations to a function given with rates.

We first give

**Theorem 10** *Let $f \in C([a,b], X)$, $0<\alpha<1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in [a,b]$. Then*

*i)*

$$\|A_n(f,x) - f(x)\| \leq \frac{1}{\psi(1)} \left[ \omega_1 \left( f, \frac{1}{n^\alpha} \right) + \left(1 - h\left(n^{1-\alpha} - 2\right)\right) \|f\|_\infty \right] =: \rho, \tag{24}$$

*and*

*ii)*

$$\|A_n(f) - f\|_\infty \leq \rho. \tag{25}$$

*We notice $\lim_{n\to\infty} A_n(f) = f$, pointwise and uniformly.*

*The speed of convergence is* $\max\left(\frac{1}{n^\alpha}, \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\right)$.

**Proof.** As similar to [13], p. 293 is omitted. ■

Next we give

**Theorem 11** *Let $f \in C_B(\mathbb{R}, X)$, $0<\alpha<1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in \mathbb{R}$. Then*

*i)*

$$\left\|\overline{A}_n(f,x) - f(x)\right\| \leq \omega_1 \left( f, \frac{1}{n^\alpha} \right) + \left(1 - h\left(n^{1-\alpha} - 2\right)\right) \|f\|_\infty =: \mu, \tag{26}$$

*and*

*ii)*

$$\left\|\overline{A}_n(f) - f\right\|_\infty \leq \mu. \tag{27}$$

*For $f \in C_{uB}(\mathbb{R}, X)$ we get $\lim_{n\to\infty} \overline{A}_n(f) = f$, pointwise and uniformly.*

*The speed of convergence is* $\max\left(\frac{1}{n^\alpha}, \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\right)$.

**Proof.** As similar to [13], p. 294 is omitted. ■

In the next we discuss high order neural network $X$-valued approximation by using the smoothness of $f$.

**Theorem 12** *Let $f \in C^N([a,b], X)$, $n, N \in \mathbb{N}$, $0<\alpha<1$, $x \in [a,b]$ and $n^{1-\alpha} > 2$. Then*

*i)*

$$\|A_n(f,x) - f(x)\| \leq \frac{1}{\psi(1)} \left\{ \sum_{j=1}^{N} \frac{\left\|f^{(j)}(x)\right\|}{j!} \left[ \frac{1}{n^{\alpha j}} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2} (b-a)^j \right] + \right. \tag{28}$$

$$\left[\omega_1\left(f^{(N)},\frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N}N!}+\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)\left\|f^{(N)}\right\|_\infty (b-a)^N}{N!}\right]\right\}$$

*ii) assume further* $f^{(j)}(x_0)=0$, $j=1,...,N$, *for some* $x_0\in[a,b]$, *it holds*

$$\|A_n(f,x_0)-f(x_0)\|\leq\frac{1}{\psi(1)}$$

$$\left\{\omega_1\left(f^{(N)},\frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N}N!}+\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)\left\|f^{(N)}\right\|_\infty (b-a)^N}{N!}\right\}, \quad (29)$$

*and*

*iii)*

$$\|A_n(f)-f\|_\infty\leq\frac{1}{\psi(1)}\left\{\sum_{j=1}^N\frac{\left\|f^{(j)}\right\|_\infty}{j!}\left[\frac{1}{n^{\alpha j}}+\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}(b-a)^j\right]+\right.$$

$$\left.\left[\omega_1\left(f^{(N)},\frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N}N!}+\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)\left\|f^{(N)}\right\|_\infty (b-a)^N}{N!}\right]\right\}. \quad (30)$$

*Again we obtain* $\lim_{n\to\infty}A_n(f)=f$, *pointwise and uniformly.*

**Proof.** As similar to [13], pp. 296-301 is omitted. ■

All integrals from now on are of Bochner type [19].

We need

**Definition 13** *([12]) Let* $[a,b]\subset\mathbb{R}$, *X be a Banach space,* $\alpha>0$; $m=\lceil\alpha\rceil\in\mathbb{N}$, *(*$\lceil\cdot\rceil$ *is the ceiling of the number),* $f:[a,b]\to X$. *We assume that* $f^{(m)}\in L_1([a,b],X)$. *We call the Caputo-Bochner left fractional derivative of order* $\alpha$:

$$\left(D_{*a}^\alpha f\right)(x):=\frac{1}{\Gamma(m-\alpha)}\int_a^x(x-t)^{m-\alpha-1}f^{(m)}(t)\,dt,\quad\forall\ x\in[a,b]. \quad (31)$$

*If* $\alpha\in\mathbb{N}$, *we set* $D_{*a}^\alpha f:=f^{(m)}$ *the ordinary X-valued derivative (defined similar to numerical one, see [21], p. 83), and also set* $D_{*a}^0 f:=f$.

By [12], $\left(D_{*a}^\alpha f\right)(x)$ exists almost everywhere in $x\in[a,b]$ and $D_{*a}^\alpha f\in L_1([a,b],X)$.

If $\left\|f^{(m)}\right\|_{L_\infty([a,b],X)}<\infty$, then by [12], $D_{*a}^\alpha f\in C([a,b],X)$, hence $\left\|D_{*a}^\alpha f\right\|\in C([a,b])$.

We mention

**Lemma 14** *([11]) Let* $\alpha>0$, $\alpha\notin\mathbb{N}$, $m=\lceil\alpha\rceil$, $f\in C^{m-1}([a,b],X)$ *and* $f^{(m)}\in L_\infty([a,b],X)$. *Then* $D_{*a}^\alpha f(a)=0$.

We mention

**Definition 15** *([10]) Let $[a,b] \subset \mathbb{R}$, $X$ be a Banach space, $\alpha > 0$, $m := \lceil \alpha \rceil$. We assume that $f^{(m)} \in L_1([a,b], X)$, where $f : [a,b] \to X$. We call the Caputo-Bochner right fractional derivative of order $\alpha$:*

$$\left(D_{b-}^{\alpha} f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z)\, dz, \quad \forall\ x \in [a,b]. \tag{32}$$

*We observe that $\left(D_{b-}^{m} f\right)(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $\left(D_{b-}^{0} f\right)(x) = f(x)$.*

By [10], $\left(D_{b-}^{\alpha} f\right)(x)$ exists almost everywhere on $[a,b]$ and $\left(D_{b-}^{\alpha} f\right) \in L_1([a,b], X)$.

If $\left\|f^{(m)}\right\|_{L_\infty([a,b],X)} < \infty$, and $\alpha \notin \mathbb{N}$, by [10], $D_{b-}^{\alpha} f \in C([a,b], X)$, hence $\left\|D_{b-}^{\alpha} f\right\| \in C([a,b])$.

We need

**Lemma 16** *([11]) Let $f \in C^{m-1}([a,b], X)$, $f^{(m)} \in L_\infty([a,b], X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$. Then $D_{b-}^{\alpha} f(b) = 0$.*

**Convention 17** *We assume that*

$$D_{*x_0}^{\alpha} f(x) = 0, \text{ for } x < x_0, \tag{33}$$

*and*

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0, \tag{34}$$

*for all $x, x_0 \in [a,b]$.*

We mention

**Proposition 18** *([11]) Let $f \in C^n([a,b], X)$, $n = \lceil \nu \rceil$, $\nu > 0$. Then $D_{*a}^{\nu} f(x)$ is continuous in $x \in [a,b]$.*

**Proposition 19** *([11]) Let $f \in C^m([a,b], X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^{\nu} f(x)$ is continuous in $x \in [a,b]$.*

We also mention

**Proposition 20** *([11]) Let $f \in C^{m-1}([a,b], X)$, $f^{(m)} \in L_\infty([a,b], X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and*

$$D_{*x_0}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{35}$$

*for all $x, x_0 \in [a,b] : x \geq x_0$.*

*Then $D_{*x_0}^{\alpha} f(x)$ is continuous in $x_0$.*

**Proposition 21** *([11]) Let $f \in C^{m-1}([a,b],X)$, $f^{(m)} \in L_\infty([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and*

$$D_{x_0-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^{x_0} (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta)\, d\zeta, \tag{36}$$

*for all $x, x_0 \in [a,b] : x_0 \geq x$.*

*Then $D_{x_0-}^{\alpha} f(x)$ is continuous in $x_0$.*

**Corollary 22** *([11]) Let $f \in C^m([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x, x_0 \in [a,b]$. Then $D_{*x_0}^{\alpha} f(x)$, $D_{x_0-}^{\alpha} f(x)$ are jointly continuous functions in $(x, x_0)$ from $[a,b]^2$ into $X$, $X$ is a Banach space.*

We need

**Theorem 23** *([11]) Let $f : [a,b]^2 \to X$ be jointly continuous, $X$ is a Banach space. Consider*

$$G(x) = \omega_1(f(\cdot, x), \delta, [x,b]), \tag{37}$$

*$\delta > 0$, $x \in [a,b]$.*

*Then $G$ is continuous on $[a,b]$.*

**Theorem 24** *([11]) Let $f : [a,b]^2 \to X$ be jointly continuous, $X$ is a Banach space. Then*

$$H(x) = \omega_1(f(\cdot, x), \delta, [a,x]), \tag{38}$$

*$x \in [a,b]$, is continuous in $x \in [a,b]$, $\delta > 0$.*

We present the following $X$-valued fractional approximation result by neural networks.

**Theorem 25** *Let $\alpha > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in C^N([a,b],X)$, $0 < \beta < 1$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then*

*i)*

$$\left\| A_n(f,x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n\left((\cdot - x)^j\right)(x) - f(x) \right\| \leq$$

$$\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left(\frac{1 - h\left(n^{1-\beta} - 2\right)}{2}\right) \left(\left\|D_{x-}^{\alpha} f\right\|_{\infty,[a,x]} (x-a)^{\alpha} + \left\|D_{*x}^{\alpha} f\right\|_{\infty,[x,b]} (b-x)^{\alpha}\right) \right\}, \tag{39}$$

*ii) if $f^{(j)}(x)=0$, for $j=1,...,N-1$, we have*

$$\|A_n(f,x)-f(x)\| \leq \frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}, \tag{40}$$

*iii)*

$$\|A_n(f,x)-f(x)\| \leq (\psi(1))^{-1}$$

$$\left\{\sum_{j=1}^{N-1}\frac{\left\|f^{(j)}(x)\right\|}{j!}\left\{\frac{1}{n^{\beta j}}+(b-a)^j\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}\right\}, \tag{41}$$

$\forall\ x\in[a,b]$,

*and*

*iv)*

$$\|A_nf-f\|_{\infty} \leq (\psi(1))^{-1}$$

$$\left\{\sum_{j=1}^{N-1}\frac{\left\|f^{(j)}\right\|_{\infty}}{j!}\left\{\frac{1}{n^{\beta j}}+(b-a)^j\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(b-a)^{\alpha}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\right\}\right\}. \tag{42}$$

*Above, when $N=1$ the sum $\sum_{j=1}^{N-1}\cdot=0$.*

*As we see here we obtain $X$-valued fractionally type pointwise and uniform convergence with rates of $A_n\to I$ the unit operator, as $n\to\infty$.*

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**Proof.** It is very lengthy, as similar to [13], pp. 305-316, is omitted. ■

Next we apply Theorem 25 for $N = 1$.

**Theorem 26** *Let* $0 < \alpha, \beta < 1$, $f \in C^1([a,b], X)$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. *Then*

*i)*

$$\|A_n(f,x) - f(x)\| \leq$$

$$\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left(\frac{1 - h\left(n^{1-\beta} - 2\right)}{2}\right) \left(\left\|D_{x-}^{\alpha} f\right\|_{\infty,[a,x]} (x-a)^{\alpha} + \left\|D_{*x}^{\alpha} f\right\|_{\infty,[x,b]} (b-x)^{\alpha}\right) \right\}, \tag{43}$$

*and*

*ii)*

$$\|A_n f - f\|_{\infty} \leq \frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1\left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1\left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left(\frac{1 - h\left(n^{1-\beta} - 2\right)}{2}\right) (b-a)^{\alpha} \left(\sup_{x\in[a,b]} \left\|D_{x-}^{\alpha} f\right\|_{\infty,[a,x]} + \sup_{x\in[a,b]} \left\|D_{*x}^{\alpha} f\right\|_{\infty,[x,b]}\right) \right\}. \tag{44}$$

When $\alpha = \frac{1}{2}$ we derive

**Corollary 27** *Let* $0 < \beta < 1$, $f \in C^1([a,b], X)$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. *Then*

*i)*

$$\|A_n(f,x) - f(x)\| \leq$$

$$\frac{2(\psi(1))^{-1}}{\sqrt{\pi}} \left\{ \frac{\left(\omega_1\left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\left. \left(\frac{1 - h\left(n^{1-\beta} - 2\right)}{2}\right) \left(\left\|D_{x-}^{\frac{1}{2}} f\right\|_{\infty,[a,x]} \sqrt{(x-a)} + \left\|D_{*x}^{\frac{1}{2}} f\right\|_{\infty,[x,b]} \sqrt{(b-x)}\right) \right\}, \tag{45}$$

*and*

*ii)*

$$\|A_n f - f\|_\infty \leq \frac{2(\psi(1))^{-1}}{\sqrt{\pi}}$$

$$\left\{ \frac{\left( \sup_{x\in[a,b]} \omega_1\left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^\beta}\right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1\left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^\beta}\right)_{[x,b]} \right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\left. \left(\frac{1-h(n^{1-\beta}-2)}{2}\right)\sqrt{(b-a)}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\frac{1}{2}} f\right\|_{\infty,[a,x]} + \sup_{x\in[a,b]}\left\|D_{*x}^{\frac{1}{2}} f\right\|_{\infty,[x,b]}\right)\right\} < \infty. \tag{46}$$

We finish with

**Remark 28** *Some convergence analysis follows:*

*Let $0 < \beta < 1$, $f \in C^1([a,b], X)$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. We elaborate on (46). Assume that*

$$\omega_1\left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^\beta}\right)_{[a,x]} \leq \frac{K_1}{n^\beta}, \tag{47}$$

*and*

$$\omega_1\left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^\beta}\right)_{[x,b]} \leq \frac{K_2}{n^\beta}, \tag{48}$$

*$\forall\, x \in [a,b]$, $\forall\, n \in \mathbb{N}$, where $K_1, K_2 > 0$.*

*Then it holds*

$$\frac{\left[\sup_{x\in[a,b]} \omega_1\left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^\beta}\right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1\left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^\beta}\right)_{[x,b]}\right]}{n^{\frac{\beta}{2}}} \leq$$

$$\frac{\frac{(K_1+K_2)}{n^\beta}}{n^{\frac{\beta}{2}}} = \frac{(K_1+K_2)}{n^{\frac{3\beta}{2}}} = \frac{K}{n^{\frac{3\beta}{2}}}, \tag{49}$$

*where $K := K_1 + K_2 > 0$.*

*The other summand of the right hand side of (46), for large enough $n$, converges to zero at the speed $\left(\frac{1-h(n^{1-\beta}-2)}{2}\right)$.*

*Then, for large enough $n \in \mathbb{N}$, by (46) and (49) and the last comment, we obtain that*

$$\|A_n f - f\|_\infty \leq M \max\left(\frac{1}{n^{\frac{3\beta}{2}}}, \left(\frac{1-h(n^{1-\beta}-2)}{2}\right)\right), \tag{50}$$

*where* $M > 0$.

*If* $\frac{1}{n^{\frac{3\beta}{2}}} \geq \left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)$, *then* $\frac{1}{n^{\beta}} \geq \left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)$, *and consequently* $\|A_n f - f\|_{\infty}$ *in (50) converges to zero faster than in Theorem 10. This because the differentiability of* $f$.

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# Approximation of Time Separating Stochastic Processes by Neural Networks

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**Abstract**

Here we study the univariate quantitative approximation of time separating stochastic process over a compact interval or all the real line by quasi-interpolation neural network operators. We perform also the related fractional approximation. These approximations are derived by establishing Jackson type inequalities involving the modulus of continuity of the engaged stochastic function or its high order derivative or fractional derivatives. Our operators are defined by using a density function induced by a general sigmoid function. The approximations are pointwise and with respect to the uniform norm. The feed-forward neural networks are with one hidden layer. We finish with a lot of interesting applications.

**2020 AMS Subject Classification:** 26A33, 41A17, 41A25

**Keywords and Phrases:** general sigmoid function, time separating stochastic process, neural network approximation, quasi-interpolation operator, modulus of continuity, Caputo fractional derivative, fractional approximation.

## 1 Introduction

The first author in [1] and [2], see Chapters 2-5, was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact suport. Also in [2] he gives the $N$th order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see Chapters 4-5 there. The first author inspired by [15], continued his studies on neural networks approximation by introducing and using the proper quasi interpolation operators of sigmoidal and hyperbolic tangent type which resulted into

[3]-[7], by treating both the univariate and multivariate cases. He did also the corresponding fractional case [8]. In this article we are also inspired by the related works [16], [17]. The authors here use general sigmoid function based neural network quantitative approximations to continuous functions over compact intervals of the real line or over the whole $\mathbb{R}$ with values to $\mathbb{R}$. All convergences here are with rates expressed via the modulus of continuity of the involved function or its high order derivative, or fractional derivatives and given by very tight Jackson type inequalities. More precisely, here we perform quantitative approximations of time separating stochastic processes by neural networks. We give plenty of varied and interesting applications. Specific motivations came by:

1. Stationary Gaussian processes with an explicit representation such as

$$X_t = \cos(\alpha t)\,\xi_1 + \sin(\alpha t)\,\xi_2, \alpha \in \mathbb{R},$$

   where $\xi_1, \xi_2$ are independent random variables with the standard normal distribution, see [19],

2. by the "Fourier model" of a stationary process, see [20].

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle a_j \cdot x\rangle + b_j\right), \quad x \in \mathbb{R}^s, s \in \mathbb{N},$$

where for $0 \le j \le n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in X$ are the coefficients, $\langle a_j \cdot x\rangle$ is the inner product of $a_j$ and $x$, and $\sigma$ is the activation function of the network. In many fundamental neural network models, the activation function is derived from various specific sigmoid functions. Here we work for a general sigmoid function. About neural networks in general read [18], [21],[23]. See also [9] for a complete study of real valued approximation by neural network operators.

# 2 Background

Here we follow [14].

## 2.1 Basics on Neural Network

Let $h : \mathbb{R} \to [-1, 1]$ be a general sigmoid function, such that it is strictly increasing, $h(0) = 0, h(-x) = -h(x)$ for every $x \in \mathbb{R}$, $h(+\infty) = 1, h(-\infty) = -1$. Also $h$ is strictly convex over $(-\infty, 0]$ and strictly concave over $[0, +\infty)$, with $h^{(2)} \in C(\mathbb{R})$.
Some examples of related sigmoid functions follow: $\frac{1}{1+e^{-x}}$; $\tanh x$; $\frac{2}{\pi}\arctan\left(\frac{\pi}{2}x\right)$; $\frac{x}{\sqrt[2m]{1+x^{2m}}}, m \in \mathbb{N}$; $\frac{2}{\pi}gd(x)$; $\frac{x}{\left(1+|x|^{\lambda}\right)^{\frac{1}{\lambda}}}$, $\lambda$ is odd ; $erf\left(\frac{\sqrt{\pi}}{2}x\right)$ ; $\frac{1}{1+e^{-\mu x}}$; $\tanh(\mu x), \mu > 0$ for all $x \in \mathbb{R}$

We consider the activation function

$$\psi(x) := \frac{1}{4}\left(h(x+1) - h(x-1)\right), x \in \mathbb{R}, , \tag{1}$$

As in [13], p.285, we get that

$$\psi(-x) = \psi(x), \text{ for every } x \in \mathbb{R}.$$

Thus $\psi$ is an even function.
Since $x+1 > x-1$, then $h(x+1) > h(x-1)$, and $\psi(x) > 0$, for all $x \in \mathbb{R}$.
We see that

$$\psi(0) = \frac{h(1)}{2}. \tag{2}$$

Let $x > 1$, we have that

$$\psi'(x) = \frac{1}{4}\left(h'(x+1) - h'(x-1)\right) < 0, \tag{3}$$

by $h'$ be strictly decreasing on $[0,+\infty)$.
Let now $0 < x < 1$, then $1-x > 0$ and $0 < 1-x < 1+x$. It holds $h'(x-1) = h'(1-x) > h'(x+1)$, so that again $\psi'(x) < 0$. Consequently $\psi$ is strictly decreasing on $(0,+\infty)$. Clearly $\psi$ is strictly increasing on $(-\infty,0)$ and $\psi'(0) = 0$. See that

$$\lim_{x\to+\infty}\psi(x) = \frac{1}{4}\left(h(+\infty) - h(+\infty)\right) = 0,$$
and
$$\lim_{x\to-\infty}\psi(x) = \frac{1}{4}\left(h(-\infty) - h(-\infty)\right) = 0. \tag{4}$$

That is the $x$-axis is the horizontal asymptote on $\psi$ .
Conclusion, $\psi$ is a bell symmetric function with maximum $\psi(0) = \frac{h(1)}{2}$ .
We need

**Theorem 1.** *([14]) We have that*

$$\sum_{i=-\infty}^{+\infty}\psi(x-i) = 1, \text{ for every } x \in \mathbb{R}. \tag{5}$$

**Theorem 2.** *([14]) It holds*

$$\int_{-\infty}^{+\infty}\psi(x)dx = 1. \tag{6}$$

Thus, $\psi(x)$ is a density function on $\mathbb{R}$.
We give

**Theorem 3.** *([14]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty}\psi(nx-k) < \frac{\left(1 - h\left(n^{1-\alpha}-2\right)\right)}{2}. \tag{7}$$

*Notice that*

$$\lim_{n\to+\infty}\frac{\left(1 - h\left(n^{1-\alpha}-2\right)\right)}{2} = 0.$$

Denote by $\lfloor\cdot\rfloor$ the integral part of the number and by $\lceil\cdot\rceil$ the ceiling of the number.
We further give

**Theorem 4.** *([14])Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na\rceil \leq \lfloor nb\rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)} < \frac{1}{\psi(1)}, \quad \forall\, x \in [a,b]. \tag{8}$$

**Remark 5.** *We have that*

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)\neq 1, \tag{9}$$

*for at least some $x\in[a,b]$. See [13], p. 290, same reasoning.*

**Note 6.** *For large enough $n$ we always obtain $\lceil na\rceil \leq \lfloor nb\rfloor$. Also $a\leq \frac{k}{n}\leq b$, if and only if $\lceil na\rceil\leq k\leq\lfloor nb\rfloor$. In general it holds (by (5))*

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)\leq 1. \tag{10}$$

Let $(X,\|\cdot\|)$ be a Banach space.

**Definition 7.** *([14]) Let $f\in C([a,b],X)$ and $n\in\mathbb{N}:\lceil na\rceil\leq\lfloor nb\rfloor$. We introduce and define the $X$-valued linear neural network operators*

$$A_n(f,x):=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\psi(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)},\quad x\in[a,b]. \tag{11}$$

Clearly here $A_n(f,x)\in C([a,b],X)$. For convenience we use the same $A_n$ for real valued function when needed. We mention here the pointwise and uniform convergence of $A_n(f,x)$ to $f(x)$ with rates. For convenience also we call

$$A_n^*(f,x):=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\psi(nx-k), \tag{12}$$

(similarly $A_n^*$ can be defined for real valued function) that is

$$A_n(f,x)=\frac{A_n^*(f,x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)}. \tag{13}$$

So that

$$A_n(f,x)-f(x)=\frac{A_n^*(f,x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)}-f(x)$$

$$=\frac{A_n^*(f,x)-f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)}. \tag{14}$$

Consequently we derive

$$\|A_n(f,x)-f(x)\|\leq\frac{1}{\psi(1)}\left\|A_n^*(f,x)-f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi(nx-k)\right)\right\|. \tag{15}$$

That is

$$\|A_n(f,x)-f(x)\|\leq\frac{1}{\psi(1)}\left\|\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(f\left(\frac{k}{n}\right)-f(x)\right)\psi(nx-k)\right\|. \tag{16}$$

We will estimate the right hand side of (16).

For that we need, for $f\in C([a,b],X)$ the first modulus of continuity

$$\omega_1(f,\delta)_{[a,b]}:=\omega_1(f,\delta):=\sup_{\substack{x,y\in[a,b]\\|x-y|\leq\delta}}\|f(x)-f(y)\|,\quad\delta>0. \tag{17}$$

Similarly, it is defined $\omega_1$ for $f \in C_{uB}(\mathbb{R}, X)$ (uniformly continuous and bounded functions from $\mathbb{R}$ into $X$), for $f \in C_B(\mathbb{R}, X)$ (continuous and bounded $X$-valued) and for $f \in C_u(\mathbb{R}, X)$ (uniformly continuous). The fact $f \in C([a,b], X)$ or $f \in C_u(\mathbb{R}, X)$, is equivalent to $\lim_{\delta \to 0} \omega_1(f, \delta) = 0$, see [11].

**Definition 8.** *([14]) When $f \in C_{uB}(\mathbb{R}, X)$, or $f \in C_B(\mathbb{R}, X)$, we define*

$$\overline{A}_n(f, x) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \psi(nx - k), \quad n \in \mathbb{N}, \ x \in \mathbb{R}, \tag{18}$$

*the $X$-valued quasi-interpolation neural network operator.*

**Remark 9.** *([14]) We have that the series*

$$\sum_{k=-\infty}^{+\infty} f\left(\frac{k}{n}\right) \psi(nx - k)$$

*is absolutely convergent in $X$, hence it is convergent in $X$ and $\overline{A}_n(f, x) \in X$.*

We denote by $\|f\|_\infty := \sup_{x \in [a,b]} \|f(x)\|$, for $f \in C([a,b], X)$, similarly is defined for $f \in C_B(\mathbb{R}, X)$. We mention a series of $X$-valued neural network approximations to a function given with rates. We first give

**Theorem 10.** *([14]). Let $f \in C([a,b], X)$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in [a,b]$. Then*

*i)*

$$\|A_n(f, x) - f(x)\| \leq \frac{1}{\psi(1)} \left[\omega_1\left(f, \frac{1}{n^\alpha}\right) + \left(1 - h\left(n^{1-\alpha} - 2\right)\right) \|f\|_\infty\right] =: \rho, \tag{19}$$

*and*

*ii)*

$$\|A_n(f) - f\|_\infty \leq \rho. \tag{20}$$

*We notice $\lim_{n \to \infty} A_n(f) = f$, pointwise and uniformly.*
*The speed of convergence is $\max\left(\frac{1}{n^\alpha}, \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\right)$.*

Next we give

**Theorem 11.** *([14]). Let $f \in C_B(\mathbb{R}, X)$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in \mathbb{R}$. Then*

*i)*

$$\left\|\overline{A}_n(f, x) - f(x)\right\| \leq \omega_1\left(f, \frac{1}{n^\alpha}\right) + \left(1 - h\left(n^{1-\alpha} - 2\right)\right) \|f\|_\infty =: \mu, \tag{21}$$

*and*

*ii)*

$$\left\|\overline{A}_n(f) - f\right\|_\infty \leq \mu. \tag{22}$$

*For $f \in C_{uB}(\mathbb{R}, X)$ we get $\lim_{n \to \infty} \overline{A}_n(f) = f$, pointwise and uniformly. The speed of convergence is*

$$\max\left(\frac{1}{n^\alpha}, \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\right).$$

In the next we discuss high order neural network $X$-valued approximation by using the smoothness of $f$. The $X$-valued derivatives are as the numerical ones, see ([24]).

**Theorem 12.** *([14])Let $f \in C^N([a,b], X)$, $n, N \in \mathbb{N}$, $0 < \alpha < 1$, $x \in [a,b]$ and $n^{1-\alpha} > 2$. Then*

*i)*

$$\|A_n(f,x) - f(x)\| \leq \frac{1}{\psi(1)} \left\{ \sum_{j=1}^{N} \frac{\|f^{(j)}(x)\|}{j!} \left[ \frac{1}{n^{\alpha j}} + \frac{(1 - h(n^{1-\alpha} - 2))}{2} (b-a)^j \right] + \left[ \omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right) \frac{1}{n^{\alpha N} N!} + \frac{(1 - h(n^{1-\alpha} - 2)) \|f^{(N)}\|_\infty (b-a)^N}{N!} \right] \right\}. \tag{23}$$

*ii) Assume further $f^{(j)}(x_0) = 0$, $j = 1, ..., N$, for some $x_0 \in [a,b]$, it holds*

$$\|A_n(f,x_0) - f(x_0)\| \leq \frac{1}{\psi(1)} \left\{ \omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right) \frac{1}{n^{\alpha N} N!} + \frac{(1 - h(n^{1-\alpha} - 2)) \|f^{(N)}\|_\infty (b-a)^N}{N!} \right\}, \tag{24}$$

*and*

*iii)*

$$\|A_n(f) - f\|_\infty \leq \frac{1}{\psi(1)} \left\{ \sum_{j=1}^{N} \frac{\|f^{(j)}\|_\infty}{j!} \left[ \frac{1}{n^{\alpha j}} + \frac{(1 - h(n^{1-\alpha} - 2))}{2} (b-a)^j \right] + \left[ \omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right) \frac{1}{n^{\alpha N} N!} + \frac{(1 - h(n^{1-\alpha} - 2)) \|f^{(N)}\|_\infty (b-a)^N}{N!} \right] \right\}. \tag{25}$$

*Again we obtain $\lim_{n\to\infty} A_n(f) = f$, pointwise and uniformly.*

All integrals from now on are of Bochner type [22].
We need

**Definition 13.** *([12]) Let $[a,b] \subset \mathbb{R}$, $X$ be a Banach space, $\alpha > 0$; $m = \lceil \alpha \rceil \in \mathbb{N}$, ($\lceil \cdot \rceil$ is the ceiling of the number), $f : [a,b] \to X$. We assume that $f^{(m)} \in L_1([a,b], X)$. We call the Caputo-Bochner left fractional derivative of order $\alpha$:*

$$(D_{*a}^\alpha f)(x) := \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \quad \forall\, x \in [a,b]. \tag{26}$$

*If $\alpha \in \mathbb{N}$, we set $D_{*a}^\alpha f := f^{(m)}$ the ordinary $X$-valued derivative (defined similar to numerical one, see [24], p. 83), and also set $D_{*a}^0 f := f$.*

By [12], $(D_{*a}^\alpha f)(x)$ exists almost everywhere in $x \in [a,b]$ and $D_{*a}^\alpha f \in L_1([a,b], X)$.
If $\|f^{(m)}\|_{L_\infty([a,b],X)} < \infty$, then by [12], $D_{*a}^\alpha f \in C([a,b], X)$, hence $\|D_{*a}^\alpha f\| \in C([a,b])$.

**Definition 14.** *([10]) Let $[a,b] \subset \mathbb{R}$, $X$ be a Banach space, $\alpha > 0$, $m := \lceil \alpha \rceil$. We assume that $f^{(m)} \in L_1([a,b], X)$, where $f : [a,b] \to X$. We call the Caputo-Bochner right fractional derivative of order $\alpha$:*

$$\left(D_{b-}^\alpha f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z)\, dz, \quad \forall\, x \in [a,b]. \tag{27}$$

*We observe that $\left(D_{b-}^m f\right)(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $\left(D_{b-}^0 f\right)(x) = f(x)$.*

By [10], $\left(D_{b-}^{\alpha}f\right)(x)$ exists almost everywhere on $[a,b]$ and $\left(D_{b-}^{\alpha}f\right) \in L_1([a,b],X)$. If $\left\|f^{(m)}\right\|_{L_\infty([a,b],X)} < \infty$, and $\alpha \notin \mathbb{N}$, by [10], $D_{b-}^{\alpha}f \in C([a,b],X)$, hence $\left\|D_{b-}^{\alpha}f\right\| \in C([a,b])$.
In the next $\omega_1(f,\delta)_{[a,b]}$ refers to a modulus of continuity. $\omega_1$ defined over $[a,b]$.
We mention the following $X-$valued fractional approximation result by neural networks.

**Theorem 15.** *([14]). Let $\alpha > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in C^N([a,b],X)$, $0 < \beta < 1$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then*

*i)*

$$\left\|A_n(f,x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n\left((\cdot - x)^j\right)(x) - f(x)\right\| \leq$$

$$\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D_{x-}^{\alpha}f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha}f, \frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left(\frac{1 - h\left(n^{1-\beta} - 2\right)}{2}\right) \left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha} + \left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right) \right\}, \tag{28}$$

*ii) if $f^{(j)}(x) = 0$, for $j = 1, ..., N-1$, we have*

$$\|A_n(f,x) - f(x)\| \leq \frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\left(\omega_1\left(D_{x-}^{\alpha}f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha}f, \frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left(\frac{1 - h\left(n^{1-\beta} - 2\right)}{2}\right) \left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha} + \left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right) \right\}, \tag{29}$$

*iii)*

$$\|A_n(f,x) - f(x)\| \leq (\psi(1))^{-1}$$

$$\left\{ \sum_{j=1}^{N-1} \frac{\left\|f^{(j)}(x)\right\|}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j \left(\frac{1 - h\left(n^{1-\beta} - 2\right)}{2}\right) \right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D_{x-}^{\alpha}f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha}f, \frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left. \left(\frac{1 - h\left(n^{1-\beta} - 2\right)}{2}\right) \left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha} + \left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right) \right\} \right\}, \tag{30}$$

$\forall\, x \in [a,b]$,

*and*

*iv)*

$$\|A_n f - f\|_\infty \leq (\psi(1))^{-1}$$

$$\left\{ \sum_{j=1}^{N-1} \frac{\left\|f^{(j)}\right\|_\infty}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j \left(\frac{1 - h\left(n^{1-\beta} - 2\right)}{2}\right) \right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D^{\alpha}_{x-}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D^{\alpha}_{*x}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(b-a)^{\alpha}\left(\sup_{x\in[a,b]}\left\|D^{\alpha}_{x-}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D^{\alpha}_{*x}f\right\|_{\infty,[x,b]}\right)\right\}\Bigg\}. \tag{31}$$

*Above, when $N=1$ the sum $\sum_{j=1}^{N-1}\cdot=0$.*

*As we see here we obtain $X$-valued fractionally type pointwise and uniform convergence with rates of $A_n\to I$ the unit operator, as $n\to\infty$.*

Next we apply Theorem 15 for $N=1$.

**Corollary 16.** *([14]) Let $0<\alpha,\beta<1$, $f\in C^1([a,b],X)$, $x\in[a,b]$, $n\in\mathbb{N}:n^{1-\beta}>2$. Then*

*i)*

$$\|A_n(f,x)-f(x)\|\leq$$

$$\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D^{\alpha}_{x-}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D^{\alpha}_{*x}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D^{\alpha}_{x-}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D^{\alpha}_{*x}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}, \tag{32}$$

*and*

*ii)*

$$\|A_nf-f\|_{\infty}\leq\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D^{\alpha}_{x-}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D^{\alpha}_{*x}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(b-a)^{\alpha}\left(\sup_{x\in[a,b]}\left\|D^{\alpha}_{x-}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D^{\alpha}_{*x}f\right\|_{\infty,[x,b]}\right)\right\}. \tag{33}$$

When $\alpha=\frac{1}{2}$ we derive

**Corollary 17.** *([14]) Let $0<\beta<1$, $f\in C^1([a,b],X)$, $x\in[a,b]$, $n\in\mathbb{N}:n^{1-\beta}>2$. Then*

*i)*

$$\|A_n(f,x)-f(x)\|\leq$$

$$\frac{2(\psi(1))^{-1}}{\sqrt{\pi}}\left\{\frac{\left(\omega_1\left(D^{\frac{1}{2}}_{x-}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D^{\frac{1}{2}}_{*x}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\frac{\beta}{2}}}+\right.$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D^{\frac{1}{2}}_{x-}f\right\|_{\infty,[a,x]}\sqrt{(x-a)}+\left\|D^{\frac{1}{2}}_{*x}f\right\|_{\infty,[x,b]}\sqrt{(b-x)}\right)\right\}, \tag{34}$$

*and*

*ii)*

$$\|A_n f - f\|_\infty \leq \frac{2(\psi(1))^{-1}}{\sqrt{\pi}}$$

$$\left\{ \frac{\left( \sup_{x\in[a,b]} \omega_1 \left( D_{x-}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1 \left( D_{*x}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\left. \left( \frac{1 - h(n^{1-\beta} - 2)}{2} \right) \sqrt{(b-a)} \left( \sup_{x\in[a,b]} \left\| D_{x-}^{\frac{1}{2}} f \right\|_{\infty,[a,x]} + \sup_{x\in[a,b]} \left\| D_{*x}^{\frac{1}{2}} f \right\|_{\infty,[x,b]} \right) \right\} < \infty. \tag{35}$$

From now on we set $X = \mathbb{R}$.

## 2.2 Time Seperating Stochastic Processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\omega \in \Omega; Y_1, Y_2, \ldots, Y_m, m \in \mathbb{N}$, be real-valued random variables on $\Omega$ with finite expectations, and $h_1(t), h_2(t), \ldots h_m(t) : I \to \mathbb{R}$, where $I$ is an infinite subset of $\mathbb{R}$. Typically here $I$ is an infinite length interval of $\mathbb{R}$, usualy $I = \mathbb{R}$ or $I = \mathbb{R}_+$.
Clearly, then

$$Y(t, \omega) := \sum_{i=1}^{m} h_i(t) Y_i(\omega), t \in I, \tag{36}$$

is a quite common stochastic process separating time.
We can assume that $h_i \in C^r(I), i = 1, 2, ..., m; r \in \mathbb{N}$. Consequently, we have that the expectation

$$(EY)(t) = \sum_{i=1}^{m} h_i(t) EY_i \in C(I) \text{ or } C^r(I). \tag{37}$$

A classical example of a stochastic process separating time is

$$(\sin t) Y_1(\omega) + (\cos t) Y_2(\omega), t \in I.$$

Notice that $|\sin t| \leq 1$ and $|\cos t| \leq 1$.
Another typical example is

$$\sinh(t) Y_1(\omega) + \cosh(t) Y_2(\omega), t \in I. \tag{38}$$

In this article we will apply the main results of section 2.1, to $f(t) = (EY)(t)$. We will finish with several applications. See the related [19], [20].

# 3 Main Results

We present the following general approximation of the seperating stochastic processes by neural network operators.

**Theorem 18.** *Let $(EY)(t)$ as in (37), $t \in [t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2, h_i \in C([t_1, t_2])$ for every $i = 1, 2, ..., m, 0 < \alpha < 1, n \in \mathbb{N} : n^{1-\alpha} > 2$. Then*

*i)*
$$\left|A_n\left(\left(EY\right),t\right)-\left(EY\right)(t)\right|\leq\frac{1}{\psi(1)}\left[\omega_1\left(EY,\frac{1}{n^{\alpha}}\right)+\left(1-h\left(n^{1-\alpha}-2\right)\right)\|EY\|_{\infty}\right]=:\rho, \tag{39}$$

*and*

*ii)*
$$\left\|A_n\left(EY\right)-EY\right\|_{\infty}\leq\rho. \tag{40}$$

*We have that* $\lim_{n\to\infty}A_n\left(EY\right)=EY$, *pointwise and uniformly.*
*The speed of convergence is* $\max\left(\frac{1}{n^{\alpha}},\left(1-h\left(n^{1-\alpha}-2\right)\right)\right)$.

**Proof.** Notice that when $h_i\in C\left([t_1,t_2]\right)$ for every $i=1,2,...,m$, then $\left(EY\right)(t)\in C\left([t_1,t_2]\right)$. Thus, the conclusion comes from Theorem 10. $\square$
We continue with,

**Theorem 19.** *Let* $(EY)(t)$ *as in (37),* $h_i\in C_B\left(\mathbb{R}\right)$ *for every* $i=1,2,...,m,0<\alpha<1$, $n\in\mathbb{N}:n^{1-\alpha}>2$, $t\in\mathbb{R}$. *Then*

*i)*
$$\left|\overline{A}_n\left(EY,t\right)-\left(EY\right)(t)\right|\leq\omega_1\left(EY,\frac{1}{n^{\alpha}}\right)+\left(1-h\left(n^{1-\alpha}-2\right)\right)\|EY\|_{\infty}=:\mu, \tag{41}$$

*and*

*ii)*
$$\left\|\overline{A}_n\left(EY\right)-EY\right\|_{\infty}\leq\mu. \tag{42}$$

*For* $EY\in C_{uB}\left(\mathbb{R}\right)$ *we get* $\lim_{n\to\infty}\overline{A}_n\left(EY\right)=EY$, *pointwise and uniformly. The speed of convergence is* $\max\left(\frac{1}{n^{\alpha}},\left(1-h\left(n^{1-\alpha}-2\right)\right)\right)$.

**Proof.** Since that $h_i\in C_B\left(\mathbb{R}\right)$ for every $i=1,2,...,m$ and $t\in\mathbb{R}$, then $EX\in C_B\left(\mathbb{R}\right)$. Therefore the results come from Theorem 11. $\square$
Furthermore, we have

**Theorem 20.** *Let* $(EY)(t)$ *as in (37),* $t\in[t_1,t_2]$, *where* $t_1,t_2\in\mathbb{R}$, *with* $t_1<t_2,h_i\in C^N\left([t_1,t_2]\right)$ *for every* $i=1,2,...,m,n,N\in\mathbb{N}$, $0<\alpha<1$, *and* $n^{1-\alpha}>2$. *Then*

*i)*
$$\left|A_n\left(EY,t\right)-\left(EY\right)(t)\right|\leq\frac{1}{\psi(1)}\left\{\sum_{j=1}^{N}\frac{\left|\left(EY\right)^{(j)}(t)\right|}{j!}\left[\frac{1}{n^{\alpha j}}+\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\left(t_2-t_1\right)^j\right]+\right. \tag{43}$$
$$\left.\left[\omega_1\left(\left(EY\right)^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)\left\|\left(EY\right)^{(N)}\right\|_{\infty}\left(t_2-t_1\right)^N}{N!}\right]\right\}.$$

*ii) Assume further* $\left(EY\right)^{(j)}(t_0)=0$, $j=1,...,N$, *for some* $t_0\in[t_1,t_2]$, *it holds*

$$\left|A_n\left(EY,t_0\right)-\left(EY\right)(t_0)\right|\leq\frac{1}{\psi(1)}$$

$$\left\{\omega_1\left((EY)^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N}N!} + \frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)\left\|(EY)^{(N)}\right\|_\infty (t_2-t_1)^N}{N!}\right\}, \tag{44}$$

*and*

*iii)*

$$\|A_n(EY) - EY\|_\infty \leq \frac{1}{\psi(1)}\left\{\sum_{j=1}^{N}\frac{\left\|(EY)^{(j)}\right\|_\infty}{j!}\left[\frac{1}{n^{\alpha j}} + \frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}(t_2-t_1)^j\right] + \right.$$

$$\left.\left[\omega_1\left((EY)^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N}N!} + \frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)\left\|(EY)^{(N)}\right\|_\infty (t_2-t_1)^N}{N!}\right]\right\}. \tag{45}$$

*Again we obtain* $\lim_{n\to\infty} A_n(EY) = EY$, *pointwise and uniformly.*

**Proof.** By Theorem 12. □

We also present

**Theorem 21.** *Let* $\alpha > 0, N = \lceil\alpha\rceil, \alpha \notin \mathbb{N}, 0 < \beta < 1, t \in [t_1, t_2]$ *where* $t_1, t_2 \in \mathbb{R}$, *with* $t_1 < t_2$, $n \in \mathbb{N} : n^{1-\beta} > 2$. *Then*

*i)*

$$\left|A_n(EY,t) - \sum_{j=1}^{N-1}\frac{(EY)^{(j)}(t)}{j!}A_n\left((\cdot - t)^j\right)(t) - (EY)(t)\right| \leq$$

$$\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{t-}^\alpha(EY), \frac{1}{n^\beta}\right)_{[t_1,t]} + \omega_1\left(D_{*t}^\alpha(EY), \frac{1}{n^\beta}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{t-}^\alpha(EY)\right\|_{\infty,[t_1,t]}(t-t_1)^\alpha + \left\|D_{*t}^\alpha(EY)\right\|_{\infty,[t,t_2]}(t_2-t)^\alpha\right)\right\}, \tag{46}$$

*ii) if* $(EY)^{(j)}(t) = 0$, *for* $j = 1, ..., N-1$, *we have*

$$|A_n(EY,t) - (EY)(t)| \leq \frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\left(\omega_1\left(D_{t-}^\alpha(EY), \frac{1}{n^\beta}\right)_{[t_1,t]} + \omega_1\left(D_{*t}^\alpha(EY), \frac{1}{n^\beta}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{t-}^\alpha(EY)\right\|_{\infty,[t_1,t]}(t-t_1)^\alpha + \left\|D_{*t}^\alpha(EY)\right\|_{\infty,[t,t_2]}(t_2-t)^\alpha\right)\right\}, \tag{47}$$

*iii)*

$$\|A_n(EY,t) - (EY)(t)\| \leq (\psi(1))^{-1}$$

$$\left\{\sum_{j=1}^{N-1}\frac{\left|(EY)^{(j)}(t)\right|}{j!}\left\{\frac{1}{n^{\beta j}} + (t_2-t_1)^j\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{t-}^\alpha(EY), \frac{1}{n^\beta}\right)_{[t_1,t]} + \omega_1\left(D_{*t}^\alpha(EY), \frac{1}{n^\beta}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{t-}^{\alpha}\left(EY\right)\right\|_{\infty,[t_1,t]}\left(t-t_1\right)^{\alpha}+\left\|D_{*t}^{\alpha}\left(EY\right)\right\|_{\infty,[t,t_2]}\left(t_2-t\right)^{\alpha}\right)\Bigg\}\Bigg\}, \tag{48}$$

$\forall\ t\in[t_1,t_2]$,

*and*

*iv)*

$$\left\|A_n\left(EY\right)-EY\right\|_{\infty}\leq\left(\psi\left(1\right)\right)^{-1}$$

$$\left\{\sum_{j=1}^{N-1}\frac{\left\|\left(EY\right)^{(j)}\right\|_{\infty}}{j!}\left\{\frac{1}{n^{\beta j}}+\left(t_2-t_1\right)^{j}\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\right\}+\right.$$

$$\frac{1}{\Gamma\left(\alpha+1\right)}\left\{\frac{\left(\sup_{x\in[t_1,t_2]}\omega_1\left(D_{t-}^{\alpha}\left(EY\right),\frac{1}{n^{\beta}}\right)_{[t_1,t]}+\sup_{t\in[t_1,t_2]}\omega_1\left(D_{*t}^{\alpha}\left(EY\right),\frac{1}{n^{\beta}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(t_2-t_1\right)^{\alpha}\left(\sup_{t\in[t_1,t_2]}\left\|D_{t-}^{\alpha}\left(EY\right)\right\|_{\infty,[t_1,t]}+\sup_{t\in[t_1,t_2]}\left\|D_{*t}^{\alpha}\left(EY\right)\right\|_{\infty,[t,t_2]}\right)\right\}\right\}. \tag{49}$$

*Above, when $N=1$ the sum $\sum_{j=1}^{N-1}\cdot=0$.*

*As we see here we obtain t-valued fractionally type pointwise and uniform convergence with rates of $A_n\to I$ the unit operator, as $n\to\infty$.*

**Proof.** By Theorem 15. □

Next we apply Theorem 21 for $N=1$.

**Corollary 22.** *Let $(EY)(t)$ as in (37), $0<\alpha,\beta<1$, $t\in[t_1,t_2]$, where $t_1,t_2\in\mathbb{R}$, with $t_1<t_2$,$n\in\mathbb{N}$ : $n^{1-\beta}>2$. and $h_i\in C([t_1,t_2])$ for every $i=1,2,...,m$. Then*

*i)*

$$\left|A_n\left(EY,t\right)-\left(EY\right)\left(t\right)\right|\leq$$

$$\frac{\left(\psi\left(1\right)\right)^{-1}}{\Gamma\left(\alpha+1\right)}\left\{\frac{\left(\omega_1\left(D_{t-}^{\alpha}\left(EY\right),\frac{1}{n^{\beta}}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^{\alpha}\left(EY\right),\frac{1}{n^{\beta}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{t-}^{\alpha}\left(EY\right)\right\|_{\infty,[t_1,t]}\left(t-t_1\right)^{\alpha}+\left\|D_{*t}^{\alpha}\left(EY\right)\right\|_{\infty,[t,t_2]}\left(t_2-t\right)^{\alpha}\right)\right\}, \tag{50}$$

*and*

*ii)*

$$\left\|A_n\left(EY\right)-\left(EY\right)\right\|_{\infty}\leq\frac{\left(\psi\left(1\right)\right)^{-1}}{\Gamma\left(\alpha+1\right)}$$

$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D_{t-}^{\alpha}\left(EY\right),\frac{1}{n^{\beta}}\right)_{[t_1,t]}+\sup_{t\in[t_1,t_2]}\omega_1\left(D_{*t}^{\alpha}\left(EY\right),\frac{1}{n^{\beta}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(t_2-t_1\right)^{\alpha}\left(\sup_{t\in[t_1,t_2]}\left\|D_{t-}^{\alpha}\left(EY\right)\right\|_{\infty,[t_1,t]}+\sup_{x\in[t_1,t_2]}\left\|D_{*t}^{\alpha}\left(EY\right)\right\|_{\infty,[t,t_2]}\right)\right\}. \tag{51}$$

When $\alpha = \frac{1}{2}$ we derive

**Corollary 23.** *Assume again $(EY)(t)$ as in (37). Let $0 < \beta < 1$, $t \in [t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2$, $n \in \mathbb{N} : n^{1-\beta} > 2$ and $h_i \in C([t_1, t_2])$ for every $i = 1, 2, ..., m$. Then*

*i)*

$$|A_n(EY, t) - (EY)(t)| \leq$$

$$\frac{2(\psi(1))^{-1}}{\sqrt{\pi}} \left\{ \frac{\left( \omega_1 \left( D_{t-}^{\frac{1}{2}}(EY), \frac{1}{n^\beta} \right)_{[t_1,t]} + \omega_1 \left( D_{*t}^{\frac{1}{2}}(EY), \frac{1}{n^\beta} \right)_{[t,t_2]} \right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\left. \left( \frac{1 - h(n^{1-\beta} - 2)}{2} \right) \left( \left\| D_{t-}^{\frac{1}{2}}(EY) \right\|_{\infty,[t_1,t]} \sqrt{(t - t_1)} + \left\| D_{*t}^{\frac{1}{2}}(EY) \right\|_{\infty,[t,t_2]} \sqrt{(t_2 - t)} \right) \right\}, \tag{52}$$

*and*

*ii)*

$$\|A_n(EY) - (EY)\|_\infty \leq \frac{2(\psi(1))^{-1}}{\sqrt{\pi}}$$

$$\left\{ \frac{\left( \sup_{t\in[t_1,t_2]} \omega_1 \left( D_{t-}^{\frac{1}{2}}(EY), \frac{1}{n^\beta} \right)_{[t_1,t]} + \sup_{t\in[t_1,t_2]} \omega_1 \left( D_{*t}^{\frac{1}{2}}(EY), \frac{1}{n^\beta} \right)_{[t,t_2]} \right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\left. \left( \frac{1 - h(n^{1-\beta} - 2)}{2} \right) \sqrt{(t_2 - t_1)} \left( \sup_{t\in[t_1,t_2]} \left\| D_{t-}^{\frac{1}{2}}(EY) \right\|_{\infty,[t_1,t]} + \sup_{t\in[t_1,t_2]} \left\| D_{*t}^{\frac{1}{2}}(EY) \right\|_{\infty,[t,t_2]} \right) \right\} < \infty. \tag{53}$$

# 4 Applications

For the next applications we consider $(\Omega, F, P)$ be a probability space and $Y_0, Y_1, Y_2$ be real valued random variables on $\Omega$ with finite expectations. We consider the stochastic processes $Z_i(t, \omega)$ for $i = 1, 2, \ldots, 9$, where $t \in \mathbb{R}$ and $\omega \in \Omega$ as follows:

$$Z_1(t, \omega) = \left[ (t - t_0)^{\mu+1} + 1 \right] Y_0(\omega), \tag{54}$$

where $t_0 \in \mathbb{R}$ and $\mu \in \mathbb{N}$ are fixed;

$$Z_2(t, \omega) = \sin(\xi t) Y_1(\omega) + \cos(\xi t) Y_2(\omega), \tag{55}$$

where $\xi > 0$ is fixed;

$$Z_3(t, \omega) = \sinh(\mu t) Y_1(\omega) + \cosh(\mu t) Y_2(\omega), \tag{56}$$

where $\mu > 0$ is fixed;

$$Z_4(t, \omega) = sech(\mu t) Y_1(\omega) + \tanh(\mu t) Y_2(\omega), \tag{57}$$

where $\mu > 0$ is fixed.
Here $sech x := \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, x \in \mathbb{R}$.

$$Z_5(t, \omega) = e^{-\ell_1 t} Y_1(\omega) + e^{-\ell_2 t} Y_2(\omega), \tag{58}$$

where $\ell_1, \ell_2 > 0$ are fixed;

$$Z_6(t,\omega) = \frac{1}{1+e^{-\ell_1 t}} Y_1(\omega) + \frac{1}{1+e^{-\ell_2 t}} Y_2(\omega), \tag{59}$$

where $\ell_1, \ell_2 > 0$ are fixed;

$$Z_7(t,\omega) = e^{-e^{-\mu_1 t}} Y_1(\omega) + e^{-e^{-\mu_2 t}} Y_2(\omega), \tag{60}$$

where $\mu_1, \mu_2 > 0$ are fixed;

$$Z_8(t,\omega) = P_m(\ell_1 t) Y_1(\omega) + P_m(\ell_2 t) Y_2(\omega), \tag{61}$$

where $\ell_1, \ell_2 > 0$ and $m \in \mathbb{N}$ are fixed.
Here $P_m(x)$ is the Legendre Polynomial of degree $m \in \mathbb{N}$, i.e

$$P_m : [-1,1] \longrightarrow [-1,1]$$

such that,

$$P_m(x) = \frac{1}{2^m m!} \frac{d^m}{dx^m} \left(x^2-1\right)^m = \frac{1}{2^m} \sum_{k=0}^{m} \binom{m}{k}^2 (x-1)^{m-k} (x+1)^k, x \in [-1,1].$$

To define the stochastic process $Z_9(t,\omega)$, we consider the Cauchy function

$$\hat{f}(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Notice that, $\hat{f} \in C^{\infty}(\mathbb{R})$ and it has $\hat{f}^{(i)}(0) = 0$, for all $i = 1, 2, \ldots, \infty$.
We set,

$$Z_9(t,\omega) = \hat{f}(t) Y_0(\omega), \tag{62}$$

The expectations of $Z_i, i = 1, 2, ..., 9$ are

$$(EZ_1)(t) = \left[(t-t_0)^{\mu+1} + 1\right] E(Y_0), \tag{63}$$

$$(EZ_2)(t) = \sin(\xi t) E(Y_1) + \cos(\xi t) E(Y_2), \tag{64}$$

$$(EZ_3)(t) = \sinh(\mu t) E(Y_1) + \cosh(\mu t) E(Y_2), \tag{65}$$

$$(EZ_4)(t) = sech(\mu t) E(Y_1) + \tanh(\mu t) E(Y_2), \tag{66}$$

$$(EZ_5)(t) = e^{-\ell_1 t} E(Y_1) + e^{-\ell_2 t} E(Y_2), \tag{67}$$

$$(EZ_6)(t) = \frac{1}{1+e^{-\ell_1 t}} E(Y_1) + \frac{1}{1+e^{-\ell_2 t}} E(Y_2), \tag{68}$$

$$(EZ_7)(t) = e^{-e^{-\mu_1 t}} E(Y_1) + e^{-e^{-\mu_2 t}} E(Y_2), \tag{69}$$

$$(EZ_8)(t) = P_m(\ell_1 t) E(Y_1) + P_m(\ell_2 t) E(Y_2), \tag{70}$$

$$(EZ_9)(t) = \hat{f}(t) E(Y_0), \tag{71}$$

For the next $(EZ_i)(t), i = 1, 2, \ldots, 9$ are as defined in relations between (63) and (71) respectively.
We present the following result.

**Proposition 24.** *Let $t \in [t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2, 0 < \alpha < 1, n \in \mathbb{N} : n^{1-\alpha} > 2$. Then for $i = 1, 2, \dots, 9$*

*i)*

$$\left| A_n \left( \left( EZ_i \right), t \right) - \left( EZ_i \right)(t) \right| \leq \frac{1}{\psi(1)} \left[ \omega_1 \left( EZ_i, \frac{1}{n^\alpha} \right) + \left( 1 - h \left( n^{1-\alpha} - 2 \right) \right) \|EZ_i\|_\infty \right] =: \rho, \tag{72}$$

*and*

*ii)*

$$\left\| A_n \left( EZ_i \right) - EZ_i \right\|_\infty \leq \rho. \tag{73}$$

*We have that $\lim_{n\to\infty} A_n (EZ_i) = EZ_i$, pointwise and uniformly.*
*The speed of convergence is $\max\left(\frac{1}{n^\alpha}, \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\right)$.*

**Proof.** From Theorem 18. □

In the cases of stochastic processes $Z_i(t, \omega)$, for $i = 2, 4, 6, 7$ we have the next

**Proposition 25.** *Let $i \in \{2, 4, 6, 7\}, 0 < \alpha < 1, n \in \mathbb{N} : n^{1-\alpha} > 2, t \in \mathbb{R}$. Then*

*i)*

$$\left| \overline{A}_n \left( EZ_i, t \right) - \left( EZ_i \right)(t) \right| \leq \omega_1 \left( EZ_i, \frac{1}{n^\alpha} \right) + \left( 1 - h \left( n^{1-\alpha} - 2 \right) \right) \|EZ_i\|_\infty =: \mu, \tag{74}$$

*and*

*ii)*

$$\left\| \overline{A}_n \left( EZ_i \right) - EZ_i \right\|_\infty \leq \mu. \tag{75}$$

*For $EZ_i \in C_{uB}(\mathbb{R})$ we get $\lim_{n\to\infty} \overline{A}_n (EZ_i) = EZ_i$, pointwise and uniformly.*
*The speed of convergence is $\max\left(\frac{1}{n^\alpha}, \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\right)$.*

**Proof.** Notice that for every $t \in \mathbb{R}$ we have that:

for $Z_2(t, \omega)$, $|\sin(\xi t)| \leq 1$ and $|\cos(\xi t)| \leq 1$,

for $Z_4(t, \omega)$, $|sech(\mu t)| \leq 1$ and $|tanh(\mu t)| \leq 1$,

for $Z_6(t, \omega)$, $0 < \frac{1}{1+e^{-\ell_1 t}} < 1$ and $0 < \frac{1}{1+e^{-\ell_2 t}} < 1$,

for $Z_7(t, \omega)$, $0 < e^{-e^{-\mu_1 t}} < 1$ and $0 < e^{-e^{-\mu_2 t}} < 1$.

Thus, the results come from Theorem 19. □

Moreover, we present the next

**Proposition 26.** *Let $i = 1, 2, \dots, 9, t \in [t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2, 0 < \alpha < 1$, and $n^{1-\alpha} > 2$. Then*

*i)*

$$\left| A_n \left( EZ_i, t \right) - \left( EZ_i \right)(t) \right| \leq \frac{1}{\psi(1)} \left\{ \sum_{j=1}^{N} \frac{\left| \left( EZ_i \right)^{(j)}(t) \right|}{j!} \left[ \frac{1}{n^{\alpha j}} + \frac{\left( 1 - h \left( n^{1-\alpha} - 2 \right) \right)}{2} \left( t_2 - t_1 \right)^j \right] + \right. \tag{76}$$

$$\left. \left[ \omega_1 \left( \left( EZ_i \right)^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N} N!} + \frac{\left( 1 - h \left( n^{1-\alpha} - 2 \right) \right) \left\| \left( EZ_i \right)^{(N)} \right\|_\infty \left( t_2 - t_1 \right)^N}{N!} \right] \right\}.$$

*ii) Assume further $(EZ_i)^{(j)}(t_a) = 0$, $j = 1, ..., N$, for some $t_a \in [t_1, t_2]$, it holds*

$$|A_n(EZ_i, t_a) - (EZ_i)(t_a)| \leq \frac{1}{\psi(1)}$$

$$\left\{ \omega_1 \left( (EZ_i)^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N} N!} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right) \left\| (EZ_i)^{(N)} \right\|_\infty (t_2 - t_1)^N}{N!} \right\}, \tag{77}$$

*and*

*iii)*

$$\|A_n(EZ_i) - EZ_i\|_\infty \leq \frac{1}{\psi(1)} \left\{ \sum_{j=1}^{N} \frac{\left\| (EZ_i)^{(j)} \right\|_\infty}{j!} \left[ \frac{1}{n^{\alpha j}} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2} (t_2 - t_1)^j \right] + \right.$$

$$\left. \left[ \omega_1 \left( (EZ_i)^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N} N!} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right) \left\| (EZ_i)^{(N)} \right\|_\infty (t_2 - t_1)^N}{N!} \right] \right\}. \tag{78}$$

*Again we obtain $\lim_{n \to \infty} A_n(EZ_i) = EZ_i$, pointwise and uniformly.*

**Proof.** By Theorem 20. □

We also present

**Proposition 27.** *Let $i = 1, 2, \ldots, 9, \alpha > 0, N = \lceil \alpha \rceil, \alpha \notin \mathbb{N}, 0 < \beta < 1, t \in [t_1, t_2]$ where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then*

*i)*

$$\left| A_n(EZ_i, t) - \sum_{j=1}^{N-1} \frac{(EZ_i)^{(j)}(t)}{j!} A_n\left( (\cdot - t)^j \right)(t) - (EZ_i)(t) \right| \leq$$

$$\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)} \left\{ \frac{\left( \omega_1 \left( D_{t-}^\alpha (EZ_i), \frac{1}{n^\beta} \right)_{[t_1,t]} + \omega_1 \left( D_{*t}^\alpha (EZ_i), \frac{1}{n^\beta} \right)_{[t,t_2]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left( \frac{1 - h\left(n^{1-\beta} - 2\right)}{2} \right) \left( \left\| D_{t-}^\alpha (EZ_i) \right\|_{\infty,[t_1,t]} (t - t_1)^\alpha + \left\| D_{*t}^\alpha (EZ_i) \right\|_{\infty,[t,t_2]} (t_2 - t)^\alpha \right) \right\}, \tag{79}$$

*ii) if $(EZ_i)^{(j)}(t) = 0$, for $j = 1, ..., N-1$, we have*

$$|A_n(EZ_i, t) - (EZ_i)(t)| \leq \frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\left( \omega_1 \left( D_{t-}^\alpha (EZ_i), \frac{1}{n^\beta} \right)_{[t_1,t]} + \omega_1 \left( D_{*t}^\alpha (EZ_i), \frac{1}{n^\beta} \right)_{[t,t_2]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left( \frac{1 - h\left(n^{1-\beta} - 2\right)}{2} \right) \left( \left\| D_{t-}^\alpha (EZ_i) \right\|_{\infty,[t_1,t]} (t - t_1)^\alpha + \left\| D_{*t}^\alpha (EZ_i) \right\|_{\infty,[t,t_2]} (t_2 - t)^\alpha \right) \right\}, \tag{80}$$

*iii)*

$$\|A_n(EZ_i, t) - (EZ_i)(t)\| \leq (\psi(1))^{-1}$$

$$\left\{\sum_{j=1}^{N-1}\frac{\left|(EZ_i)^{(j)}(t)\right|}{j!}\left\{\frac{1}{n^{\beta j}}+(t_2-t_1)^j\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{t-}^{\alpha}(EZ_i),\frac{1}{n^{\beta}}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^{\alpha}(EZ_i),\frac{1}{n^{\beta}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{t-}^{\alpha}(EZ_i)\right\|_{\infty,[t_1,t]}(t-t_1)^{\alpha}+\left\|D_{*t}^{\alpha}(EZ_i)\right\|_{\infty,[t,t_2]}(t_2-t)^{\alpha}\right)\right\}\right\}, \tag{81}$$

$\forall\ t\in[t_1,t_2]$,

*and*

*iv)*

$$\|A_n(EZ_i)-EZ_i\|_{\infty}\leq(\psi(1))^{-1}$$

$$\left\{\sum_{j=1}^{N-1}\frac{\left\|(EZ_i)^{(j)}\right\|_{\infty}}{j!}\left\{\frac{1}{n^{\beta j}}+(t_2-t_1)^j\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\sup_{x\in[t_1,t_2]}\omega_1\left(D_{t-}^{\alpha}(EZ_i),\frac{1}{n^{\beta}}\right)_{[t_1,t]}+\sup_{t\in[t_1,t_2]}\omega_1\left(D_{*t}^{\alpha}(EZ_i),\frac{1}{n^{\beta}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(t_2-t_1)^{\alpha}\left(\sup_{t\in[t_1,t_2]}\left\|D_{t-}^{\alpha}(EZ_i)\right\|_{\infty,[t_1,t]}+\sup_{t\in[t_1,t_2]}\left\|D_{*t}^{\alpha}(EZ_i)\right\|_{\infty,[t,t_2]}\right)\right\}\right\}. \tag{82}$$

*Above, when $N=1$ the sum $\sum_{j=1}^{N-1}\cdot=0$.*

*As we see here we obtain $t$-valued fractionally type pointwise and uniform convergence with rates of $A_n\to I$ the unit operator, as $n\to\infty$.*

**Proof.** By Theorem 21. □

Next we apply Proposition 27 for $N=1$.

**Corollary 28.** *Let $i=1,2,\ldots,9$, $0<\alpha,\beta<1$, $t\in[t_1,t_2]$, where $t_1,t_2\in\mathbb{R}$, with $t_1<t_2$,$n\in\mathbb{N}:n^{1-\beta}>2$. Then*

*i)*

$$|A_n(EZ_i,t)-(EZ_i)(t)|\leq$$

$$\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{t-}^{\alpha}(EZ_i),\frac{1}{n^{\beta}}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^{\alpha}(EZ_i),\frac{1}{n^{\beta}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{t-}^{\alpha}(EZ_i)\right\|_{\infty,[t_1,t]}(t-t_1)^{\alpha}+\left\|D_{*t}^{\alpha}(EZ_i)\right\|_{\infty,[t,t_2]}(t_2-t)^{\alpha}\right)\right\}, \tag{83}$$

*and*

*ii)*

$$\|A_n(EZ_i)-(EZ_i)\|_{\infty}\leq\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\left( \sup_{t\in[t_1,t_2]} \omega_1 \left( D^{\alpha}_{t-} \left( EZ_i \right), \frac{1}{n^{\beta}} \right)_{[t_1,t]} + \sup_{t\in[t_1,t_2]} \omega_1 \left( D^{\alpha}_{*t} \left( EZ_i \right), \frac{1}{n^{\beta}} \right)_{[t,t_2]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left( \frac{1 - h\left(n^{1-\beta} - 2\right)}{2} \right) (t_2 - t_1)^{\alpha} \left( \sup_{t\in[t_1,t_2]} \left\| D^{\alpha}_{t-} \left( EZ_i \right) \right\|_{\infty,[t_1,t]} + \sup_{x\in[t_1,t_2]} \left\| D^{\alpha}_{*t} \left( EZ_i \right) \right\|_{\infty,[t,t_2]} \right) \right\}. \quad (84)$$

When $\alpha = \frac{1}{2}$ we derive

**Corollary 29.** *Assume $i = 1, 2, \ldots, 9$. Let $0 < \beta < 1$, $t \in [t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2$, and $n \in \mathbb{N} : n^{1-\beta} > 2$. Then*

*i)*

$$\left| A_n \left( EZ_i, t \right) - \left( EZ_i \right)(t) \right| \leq$$

$$\frac{2\left(\psi(1)\right)^{-1}}{\sqrt{\pi}} \left\{ \frac{\left( \omega_1 \left( D^{\frac{1}{2}}_{t-} \left( EZ_i \right), \frac{1}{n^{\beta}} \right)_{[t_1,t]} + \omega_1 \left( D^{\frac{1}{2}}_{*t} \left( EZ_i \right), \frac{1}{n^{\beta}} \right)_{[t,t_2]} \right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\left. \left( \frac{1 - h\left(n^{1-\beta} - 2\right)}{2} \right) \left( \left\| D^{\frac{1}{2}}_{t-} \left( EZ_i \right) \right\|_{\infty,[t_1,t]} \sqrt{(t - t_1)} + \left\| D^{\frac{1}{2}}_{*t} \left( EZ_i \right) \right\|_{\infty,[t,t_2]} \sqrt{(t_2 - t)} \right) \right\}, \quad (85)$$

*and*

*ii)*

$$\left\| A_n \left( EZ_i \right) - \left( EZ_i \right) \right\|_{\infty} \leq \frac{2\left(\psi(1)\right)^{-1}}{\sqrt{\pi}}$$

$$\left\{ \frac{\left( \sup_{t\in[t_1,t_2]} \omega_1 \left( D^{\frac{1}{2}}_{t-} \left( EZ_i \right), \frac{1}{n^{\beta}} \right)_{[t_1,t]} + \sup_{t\in[t_1,t_2]} \omega_1 \left( D^{\frac{1}{2}}_{*t} \left( EZ_i \right), \frac{1}{n^{\beta}} \right)_{[t,t_2]} \right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\left. \left( \frac{1 - h\left(n^{1-\beta} - 2\right)}{2} \right) \sqrt{(t_2 - t_1)} \left( \sup_{t\in[t_1,t_2]} \left\| D^{\frac{1}{2}}_{t-} \left( EZ_i \right) \right\|_{\infty,[t_1,t]} + \sup_{t\in[t_1,t_2]} \left\| D^{\frac{1}{2}}_{*t} \left( EZ_i \right) \right\|_{\infty,[t,t_2]} \right) \right\} < \infty. \quad (86)$$

# 5 Specific Applications

Let $(\Omega, \mathcal{F}, P)$, where $\Omega$ is the set of non-negative integers, be a probability space, $Y_{1,1}, Y_{2,1}$ be real-valued random variables on $\Omega$ following Poisson distributions with parameters $\lambda_1, \lambda_2 \in (0, \infty)$ respectively.
We consider the stochastic processes $Z_{i,1}(t, \omega)$ for $i = 1, 2, 3, 5$, where $t \in \mathbb{R}$ and $\omega \in \Omega$ as follows:

$$Z_{1,1}(t, \omega) = \left[ (t - t_0)^{\mu+1} + 1 \right] Y_{1,1}(\omega), \quad (87)$$

where $t_0 \in \mathbb{R}$ and $\mu \in \mathbb{N}$ are fixed;

$$Z_{2,1}(t, \omega) = \sin(\xi t)\, Y_{1,1}(\omega) + \cos(\xi t)\, Y_{2,1}(\omega), \quad (88)$$

where $\xi > 0$ is fixed;

$$Z_{3,1}(t, \omega) = \sinh(\mu t)\, Y_{1,1}(\omega) + \cosh(\mu t)\, Y_{2,1}(\omega), \quad (89)$$

where $\mu > 0$ is fixed;

$$Z_{5,1}(t,\omega) = e^{-\ell_1 t} Y_{1,1}(\omega) + e^{-\ell_2 t} Y_{2,1}(\omega), \tag{90}$$

where $\ell_1, \ell_2 > 0$ are fixed.
Since $E(Y_{1,1}) = \lambda_1$ and $E(Y_{2,1}) = \lambda_2$ , the expectations of $Z_{i,1}, i = 1,2,3,5$, are

$$(EZ_{1,1})(t) = \lambda_1 \left[ (t - t_0)^{\mu+1} + 1 \right], \tag{91}$$

$$(EZ_{2,1})(t) = \lambda_1 \sin(\xi t) + \lambda_2 \cos(\xi t), \tag{92}$$

$$(EZ_{3,1})(t) = \lambda_1 \sinh(\mu t) + \lambda_2 \cosh(\mu t), \tag{93}$$

$$(EZ_{5,1})(t) = \lambda_1 e^{-\ell_1 t} + \lambda_2 e^{-\ell_2 t}, \tag{94}$$

For the next we consider $(\Omega, \mathcal{F}, P)$, where $\Omega = \mathbb{R}$, be a probability space, $Y_{1,2}, Y_{2,2}$ be real-valued random variables on $\Omega$ following Gaussian distributions with expectations $\hat{\mu}_1, \hat{\mu}_2 \in \mathbb{R}$ respectively.
We consider the stochastic processes $Z_{i,2}(t,\omega)$ for $i = 1,2,3,5$, where $t \in \mathbb{R}$ and $\omega \in \Omega$ as follows:

$$Z_{1,2}(t,\omega) = \left[ (t - t_0)^{\mu+1} + 1 \right] Y_{1,2}(\omega), \tag{95}$$

where $t_0 \in \mathbb{R}$ and $\mu \in \mathbb{N}$ are fixed;

$$Z_{2,2}(t,\omega) = \sin(\xi t) Y_{1,2}(\omega) + \cos(\xi t) Y_{2,2}(\omega), \tag{96}$$

where $\xi > 0$ is fixed;

$$Z_{3,2}(t,\omega) = \sinh(\mu t) Y_{1,2}(\omega) + \cosh(\mu t) Y_{2,2}(\omega), \tag{97}$$

where $\mu > 0$ is fixed;

$$Z_{5,2}(t,\omega) = e^{-\ell_1 t} Y_{1,2}(\omega) + e^{-\ell_2 t} Y_{2,2}(\omega), \tag{98}$$

where $\ell_1, \ell_2 > 0$ are fixed.
Since $E(Y_{1,2}) = \hat{\mu}_1$ and $E(Y_{2,2}) = \hat{\mu}_2$ , The expectations of $Z_{i,2}, i = 1,2,3,5$ are

$$(EZ_{1,2})(t) = \hat{\mu}_1 \left[ (t - t_0)^{\mu+1} + 1 \right], \tag{99}$$

$$(EZ_{2,2})(t) = \hat{\mu}_1 \sin(\xi t) + \hat{\mu}_2 \cos(\xi t), \tag{100}$$

$$(EZ_{3,2})(t) = \hat{\mu}_1 \sinh(\mu t) + \hat{\mu}_2 \cosh(\mu t), \tag{101}$$

$$(EZ_{5,2})(t) = \hat{\mu}_1 e^{-\ell_1 t} + \hat{\mu}_2 e^{-\ell_2 t}. \tag{102}$$

Furthermore, we consider $(\Omega, \mathcal{F}, P)$, where $\Omega = [0, \infty)$, be a probability space, $Y_{1,3}, Y_{2,3}$ be real-valued random variables on $\Omega$ following Weibull distributions with scale parameters 1 and shape parameters $\gamma_1, \gamma_2 \in (0, \infty)$ respectively.
We consider the stochastic processes $Z_{i,3}(t,\omega)$ for $i = 1,2,3,5$, where $t \in \mathbb{R}$ and $\omega \in \Omega$ as follows:

$$Z_{1,3}(t,\omega) = \left[ (t - t_0)^{\mu+1} + 1 \right] Y_{1,3}(\omega), \tag{103}$$

where $t_0 \in \mathbb{R}$ and $\mu \in \mathbb{N}$ are fixed;

$$Z_{2,3}(t,\omega) = \sin(\xi t) Y_{1,3}(\omega) + \cos(\xi t) Y_{2,3}(\omega), \tag{104}$$

where $\xi > 0$ is fixed;

$$Z_{3,3}(t,\omega) = \sinh(\mu t) Y_{1,3}(\omega) + \cosh(\mu t) Y_{2,3}(\omega), \tag{105}$$

where $\mu > 0$ is fixed;

$$Z_{5,3}(t,\omega) = e^{-\ell_1 t} Y_{1,3}(\omega) + e^{-\ell_2 t} Y_{2,3}(\omega), \tag{106}$$

where $\ell_1, \ell_2 > 0$ are fixed.
Since $E(Y_{1,3}) = \Gamma\left(1 + \frac{1}{\gamma_1}\right)$ and $E(Y_{2,3}) = \Gamma\left(1 + \frac{1}{\gamma_2}\right)$ , where $\Gamma(\cdot)$ is the Gamma function, The expectations of $Z_{i,3}, i = 1,2,3,5,$ are

$$(EZ_{1,3})(t) = \Gamma\left(1 + \frac{1}{\gamma_1}\right)\left[(t - t_0)^{\mu+1} + 1\right], \tag{107}$$

$$(EZ_{2,3})(t) = \Gamma\left(1 + \frac{1}{\gamma_1}\right)\sin(\xi t) + \Gamma\left(1 + \frac{1}{\gamma_2}\right)\cos(\xi t), \tag{108}$$

$$(EZ_{3,3})(t) = \Gamma\left(1 + \frac{1}{\gamma_1}\right)\sinh(\mu t) + \Gamma\left(1 + \frac{1}{\gamma_2}\right)\cosh(\mu t), \tag{109}$$

$$(EZ_{5,3})(t) = \Gamma\left(1 + \frac{1}{\gamma_1}\right)e^{-\ell_1 t} + \Gamma\left(1 + \frac{1}{\gamma_2}\right)e^{-\ell_2 t}, \tag{110}$$

We present the following result.

**Proposition 30.** *Let $t \in [t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2, 0 < \alpha < 1, n \in \mathbb{N} : n^{1-\alpha} > 2$. Then for $i = 1,2,3,5$ and $k = 1,2,3$*
*i)*

$$\left|A_n\left(\left(EZ_{i,k}\right), t\right) - \left(EZ_{i,k}\right)(t)\right| \leq \frac{1}{\psi(1)}\left[\omega_1\left(EZ_{i,k}, \frac{1}{n^\alpha}\right) + \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\left\|EZ_{i,k}\right\|_\infty\right] =: \rho, \tag{111}$$

*and*

*ii)*

$$\left\|A_n\left(EZ_{i,k}\right) - EZ_{i,k}\right\|_\infty \leq \rho. \tag{112}$$

*We have that $\lim_{n\to\infty} A_n\left(EZ_{i,k}\right) = EZ_{i,k}$, pointwise and uniformly.*
*The speed of convergence is $\max\left(\frac{1}{n^\alpha}, \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\right)$.*

**Proof.** From Proposition 24. □
In the cases of stochastic processes $Z_{2,k}(t,\omega)$, for $k = 1,2,3$ we have the next

**Proposition 31.** *Let $k \in \{1,2,3\}, 0 < \alpha < 1, n \in \mathbb{N} : n^{1-\alpha} > 2, t \in \mathbb{R}$. Then*

*i)*

$$\left|\overline{A}_n\left(EZ_{2,k}, t\right) - \left(EZ_{2,k}\right)(t)\right| \leq \omega_1\left(EZ_{2,k}, \frac{1}{n^\alpha}\right) + \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\left\|EZ_{2,k}\right\|_\infty =: \mu, \tag{113}$$

*and*

*ii)*

$$\left\|\overline{A}_n\left(EZ_{2,k}\right) - EZ_{2,k}\right\|_\infty \leq \mu. \tag{114}$$

*For $EZ_{2,k} \in C_{uB}(\mathbb{R})$ we get $\lim_{n\to\infty} \overline{A}_n\left(EZ_{2,k}\right) = EZ_{2,k}$, pointwise and uniformly.*
*The speed of convergence is $\max\left(\frac{1}{n^\alpha}, \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\right)$.*

**Proof.** The results come from Proposition 25. □
Moreover, we present the next

**Corollary 32.** *Assume $i = 1, 2, 3, 5$ and $k = 1, 2, 3$. Let $0 < \beta < 1$, $t \in [t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2$, and $n \in \mathbb{N} : n^{1-\beta} > 2$. Then*

*i)*

$$|A_n (EZ_{i,k}, t) - (EZ_{i,k})(t)| \leq$$

$$\frac{2(\psi(1))^{-1}}{\sqrt{\pi}} \left\{ \frac{\left( \omega_1 \left( D_{t-}^{\frac{1}{2}} (EZ_{i,k}), \frac{1}{n^\beta} \right)_{[t_1,t]} + \omega_1 \left( D_{*t}^{\frac{1}{2}} (EZ_{i,k}), \frac{1}{n^\beta} \right)_{[t,t_2]} \right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\left. \left( \frac{1 - h(n^{1-\beta} - 2)}{2} \right) \left( \left\| D_{t-}^{\frac{1}{2}} (EZ_{i,k}) \right\|_{\infty,[t_1,t]} \sqrt{(t - t_1)} + \left\| D_{*t}^{\frac{1}{2}} (EZ_{i,k}) \right\|_{\infty,[t,t_2]} \sqrt{(t_2 - t)} \right) \right\}, \tag{115}$$

*and*

*ii)*

$$\|A_n (EZ_{i,k}) - (EZ_{i,k})\|_\infty \leq \frac{2(\psi(1))^{-1}}{\sqrt{\pi}}$$

$$\left\{ \frac{\left( \sup_{t\in[t_1,t_2]} \omega_1 \left( D_{t-}^{\frac{1}{2}} (EZ_{i,k}), \frac{1}{n^\beta} \right)_{[t_1,t]} + \sup_{t\in[t_1,t_2]} \omega_1 \left( D_{*t}^{\frac{1}{2}} (EZ_{i,k}), \frac{1}{n^\beta} \right)_{[t,t_2]} \right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\left. \left( \frac{1 - h(n^{1-\beta} - 2)}{2} \right) \sqrt{(t_2 - t_1)} \left( \sup_{t\in[t_1,t_2]} \left\| D_{t-}^{\frac{1}{2}} (EZ_{i,k}) \right\|_{\infty,[t_1,t]} + \sup_{t\in[t_1,t_2]} \left\| D_{*t}^{\frac{1}{2}} (EZ_{i,k}) \right\|_{\infty,[t,t_2]} \right) \right\} < \infty. \tag{116}$$

**Proof.** From Corollary 29. □

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# Multivariate Gudermannian function based neural network approximation

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**Abstract**

Here we present multivariate quantitative approximations of Banach space valued continuous multivariate functions on a box or $\mathbb{R}^N$, $N \in \mathbb{N}$, by the multivariate normalized, quasi-interpolation, Kantorovich type and quadrature type neural network operators. We examine also the case of approximation by iterated operators of the last four types. These approximations are achieved by establishing multidimensional Jackson type inequalities involving the multivariate modulus of continuity of the engaged function or its high order Fréchet derivatives. Our multivariate operators are defined by using a multidimensional density function induced by the Gudermannian sigmoid function. The approximations are pointwise and uniform. The related feed-forward neural network is with one hidden layer.

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**Keywords and Phrases:** Gudermannian sigmoid function, multivariate neural network approximation, quasi-interpolation operator, Kantorovich type operator, quadrature type operator, multivariate modulus of continuity, abstract approximation, iterated approximation.

## 1 Introduction

G.A. Anastassiou in [2] and [3], see chapters 2-5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities.

He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support. Also in [3] he gives the $N$th order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see chapters 4-5 there.

Motivations for this work are the article [17] of Z. Chen and F. Cao, and [4], [5], [6], [7], [8], [9], [10], [11], [12], [14], [15], [18], [19].

Here we perform multivariate Gudermannian sigmoid function based neural network approximations to continuous functions over boxes or over the whole $\mathbb{R}^N$, $N \in \mathbb{N}$, and also iterated approximations. All convergences here are with rates expressed via the multivariate modulus of continuity of the involved function or its high order Fréchet derivative and given by very tight multidimensional Jackson type inequalities.

We come up with the "right" precisely defined multivariate normalized, quasi-interpolation neural network operators related to boxes or $\mathbb{R}^N$, as well as Kantorovich type and quadrature type related operators on $\mathbb{R}^N$. Our boxes are not necessarily symmetric to the origin. In preparation to prove our results we establish important properties of the basic multivariate density function induced by Gudermannian sigmoid function and defining our operators.

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle a_j \cdot x \rangle + b_j\right), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of $a_j$ and $x$, and $\sigma$ is the activation function of the network. In many fundamental network models, the activation function is the Gudermannian sigmoid function. About neural networks see [20], [21], [22].

## 2 Background

See also [13], [24].

Here we consider $gd(x)$ the Gudermannian function [24], which is a sigmoid function, as a generator function:

$$\sigma(x) = 2 \arctan\left(\tanh\left(\frac{x}{2}\right)\right) = \int_0^x \frac{dt}{\cosh t} =: gd(x), \; x \in \mathbb{R}. \tag{1}$$

Let the normalized generator sigmoid function

$$f(x) := \frac{2}{\pi}\sigma(x) = \frac{2}{\pi}\int_0^x \frac{dt}{\cosh t} = \frac{4}{\pi}\int_0^x \frac{1}{e^t + e^{-t}} dt, \quad x \in \mathbb{R}. \tag{2}$$

Here

$$f'(x) = \frac{2}{\pi \cosh x} > 0, \quad \forall\, x \in \mathbb{R},$$

hence $f$is strictly increasing on $\mathbb{R}$.

Notice that $\tanh(-x) = -\tanh x$ and $\arctan(-x) = -\arctan x$, $x \in \mathbb{R}$.

So, here the neural network activation function will be:

$$W(x) = \frac{1}{4}\left[f(x+1) - f(x-1)\right], \; x \in \mathbb{R}. \tag{3}$$

By [3], we get that

$$W(x) = W(-x), \quad \forall\, x \in \mathbb{R}, \tag{4}$$

i.e. it is even and symmetric with respect to the $y$-axis. Here we have $f(+\infty) = 1$, $f(-\infty) = -1$ and $f(0) = 0$. Clearly it is

$$f(-x) = -f(x), \; \forall\, x \in \mathbb{R}, \tag{5}$$

an odd function, symmetric with respect to the origin. Since $x+1 > x-1$, and $f(x+1) > f(x-1)$, we obtain $W(x) > 0$, $\forall\, x \in \mathbb{R}$.

By [13], we have that

$$W(0) = \frac{1}{\pi} gd(1) \cong 0.2757. \tag{6}$$

By [13] $W$ is strictly decreasing on $(0, +\infty)$, and strictly increasing on $(-\infty, 0)$, and $W'(0) = 0$.

Also we have that

$$\lim_{x \to +\infty} W(x) = \lim_{x \to -\infty} W(x) = 0, \tag{7}$$

that is the $x$-axis is the horizontal asymptote for $W$.

Conclusion, $W$ is a bell shaped symmetric function with maximum $W(0) \cong 0.2757$.

We need

**Theorem 1** *([13]) It holds that*

$$\sum_{i=-\infty}^{\infty} W(x-i) = 1, \; \forall\, x \in \mathbb{R}. \tag{8}$$

**Theorem 2** *([13]) We have that*

$$\int_{-\infty}^{\infty} W(x)\, dx = 1. \tag{9}$$

*So $W(x)$ is a density function.*

**Theorem 3** *([13]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\begin{cases} k = -\infty \\ : |nx - k| \geq n^{1-\alpha} \end{cases}}^{\infty} W(nx - k) < \frac{2}{\pi e^{(n^{1-\alpha}-2)}} = \frac{2e^2}{\pi e^{n^{1-\alpha}}}. \tag{10}$$

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.

**Theorem 4** *([13]) Let $[a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$, so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx - k)} < \frac{2\pi}{gd(2)} \cong 4.824, \tag{11}$$

$\forall\ x \in [a, b]$.

We make

**Remark 5** *([13])*

*(i) We have that*

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx - k) \neq 1, \tag{12}$$

*for at least some $x \in [a, b]$.*

*(ii) Let $[a, b] \subset \mathbb{R}$. For large $n$ we always have $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.*

*In general it holds*

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} W(nx - k) \leq 1. \tag{13}$$

We introduce

$$Z(x_1, ..., x_N) := Z(x) := \prod_{i=1}^{N} W(x_i), \quad x = (x_1, ..., x_N) \in \mathbb{R}^N,\ N \in \mathbb{N}. \tag{14}$$

It has the properties:

(i) $Z(x) > 0,\ \forall\ x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z(x - k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} Z(x_1 - k_1, ..., x_N - k_N) = 1, \tag{15}$$

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where $k := (k_1, ..., k_n) \in \mathbb{Z}^N$, $\forall\, x \in \mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z(nx - k) = 1, \tag{16}$$

$\forall\, x \in \mathbb{R}^N$; $n \in \mathbb{N}$,

and

(iv)

$$\int_{\mathbb{R}^N} Z(x)\, dx = 1, \tag{17}$$

that is $Z$ is a multivariate density function.

Here denote $\|x\|_\infty := \max\{|x_1|, ..., |x_N|\}$, $x \in \mathbb{R}^N$, also set $\infty := (\infty, ..., \infty)$, $-\infty := (-\infty, ..., -\infty)$ upon the multivariate context, and

$$\begin{aligned} \lceil na \rceil &:= (\lceil na_1 \rceil, ..., \lceil na_N \rceil), \\ \lfloor nb \rfloor &:= (\lfloor nb_1 \rfloor, ..., \lfloor nb_N \rfloor), \end{aligned} \tag{18}$$

where $a := (a_1, ..., a_N)$, $b := (b_1, ..., b_N)$.

We obviously see that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left( \prod_{i=1}^{N} W(nx_i - k_i) \right) =$$

$$\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} \left( \prod_{i=1}^{N} W(nx_i - k_i) \right) = \prod_{i=1}^{N} \left( \sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} W(nx_i - k_i) \right). \tag{19}$$

For $0 < \beta < 1$ and $n \in \mathbb{N}$, a fixed $x \in \mathbb{R}^N$, we have that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) =$$

$$\sum_{\substack{k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty \le \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z(nx - k) + \sum_{\substack{k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z(nx - k). \tag{20}$$

In the last two sums the counting is over disjoint vector sets of $k$'s, because the condition $\left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}$ implies that there exists at least one $\left| \frac{k_r}{n} - x_r \right| > \frac{1}{n^\beta}$, where $r \in \{1, ..., N\}$.

(v) As in [10], pp. 379-380, we derive that

$$\sum_{\begin{cases} k=\lceil na\rceil \\ \left\| \frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}} \end{cases}}^{\lfloor nb\rfloor} Z(nx-k) \overset{(10)}{<} \frac{2e^2}{\pi e^{n^{1-\beta}}},\ \ 0<\beta<1,\ m\in\mathbb{N}, \tag{21}$$

with $n\in\mathbb{N}:n^{1-\beta}>2$, $x\in\prod_{i=1}^{N}[a_i,b_i]$.

(vi) By Theorem 4 we get that

$$0<\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)}<\left(\frac{2\pi}{gd(2)}\right)^{N}\cong(4.824)^{N}, \tag{22}$$

$\forall\ x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right),\ \ n\in\mathbb{N}.$

It is also clear that

(vii)

$$\sum_{\begin{cases} k=-\infty \\ \left\| \frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}} \end{cases}}^{\infty} Z(nx-k)<\frac{2e^2}{\pi e^{n^{1-\beta}}}, \tag{23}$$

$0<\beta<1$, $n\in\mathbb{N}:n^{1-\beta}>2$, $x\in\mathbb{R}^N$, $m\in\mathbb{N}$.

Furthermore it holds

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)\neq 1, \tag{24}$$

for at least some $x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

Here $\left(X,\|\cdot\|_{\gamma}\right)$ is a Banach space.

Let $f\in C\left(\prod_{i=1}^{N}[a_i,b_i],X\right)$, $x=(x_1,...,x_N)\in\prod_{i=1}^{N}[a_i,b_i]$, $n\in\mathbb{N}$ such that $\lceil na_i\rceil\leq\lfloor nb_i\rfloor$, $i=1,...,N$.

We introduce and define the following multivariate linear normalized neural network operator ($x:=(x_1,...,x_N)\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$):

$$A_n(f,x_1,...,x_N):=A_n(f,x):=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)Z(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)}=$$

$$\frac{\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor}\sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2\rfloor}...\sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor} f\left(\frac{k_1}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}W(nx_i-k_i)\right)}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor} W(nx_i-k_i)\right)}. \tag{25}$$

For large enough $n\in\mathbb{N}$ we always obtain $\lceil na_i\rceil\leq\lfloor nb_i\rfloor$, $i=1,...,N$. Also $a_i\leq\frac{k_i}{n}\leq b_i$, iff $\lceil na_i\rceil\leq k_i\leq\lfloor nb_i\rfloor$, $i=1,...,N$.

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When $g \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ we define the companion operator

$$\widetilde{A}_n(g,x) := \frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} g\left(\frac{k}{n}\right) Z(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)}. \tag{26}$$

Clearly $\widetilde{A}_n$ is a positive linear operator. We have that

$$\widetilde{A}_n(1,x) = 1, \ \forall\, x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right).$$

Notice that $A_n(f) \in C\left(\prod_{i=1}^{N}[a_i,b_i], X\right)$ and $\widetilde{A}_n(g) \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

Furthermore it holds

$$\|A_n(f,x)\|_\gamma \leq \frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \left\|f\left(\frac{k}{n}\right)\right\|_\gamma Z(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)} = \widetilde{A}_n\left(\|f\|_\gamma, x\right), \tag{27}$$

$\forall\, x \in \prod_{i=1}^{N}[a_i,b_i]$.

Clearly $\|f\|_\gamma \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

So, we have that

$$\|A_n(f,x)\|_\gamma \leq \widetilde{A}_n\left(\|f\|_\gamma, x\right), \tag{28}$$

$\forall\, x \in \prod_{i=1}^{N}[a_i,b_i]$, $\forall\, n \in \mathbb{N}$, $\forall\, f \in C\left(\prod_{i=1}^{N}[a_i,b_i], X\right)$.

Let $c \in X$ and $g \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, then $cg \in C\left(\prod_{i=1}^{N}[a_i,b_i], X\right)$.

Furthermore it holds

$$A_n(cg,x) = c\widetilde{A}_n(g,x), \ \forall\, x \in \prod_{i=1}^{N}[a_i,b_i]. \tag{29}$$

Since $\widetilde{A}_n(1) = 1$, we get that

$$A_n(c) = c, \ \forall\, c \in X. \tag{30}$$

We call $\widetilde{A}_n$ the companion operator of $A_n$.

For convinience we call

$$A_n^*(f,x) := \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right) Z(nx-k) =$$

$$\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor} \sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2\rfloor} \dots \sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor} f\left(\frac{k_1}{n}, \dots, \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} W(nx_i-k_i)\right), \tag{31}$$

$\forall\ x \in \left(\prod_{i=1}^{N}[a_i, b_i]\right)$.

That is

$$A_n(f, x) := \frac{A_n^*(f, x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)}, \tag{32}$$

$\forall\ x \in \left(\prod_{i=1}^{N}[a_i, b_i]\right)$, $n \in \mathbb{N}$.

Hence

$$A_n(f, x) - f(x) = \frac{A_n^*(f, x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)}. \tag{33}$$

Consequently we derive

$$\|A_n(f, x) - f(x)\|_\gamma \overset{(22)}{\leq} (4.824)^N \left\| A_n^*(f, x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) \right\|_\gamma, \tag{34}$$

$\forall\ x \in \left(\prod_{i=1}^{N}[a_i, b_i]\right)$.

We will estimate the right hand side of (34).

For the last and others we need

**Definition 6** *([11], p. 274) Let $M$ be a convex and compact subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1, \infty]$, and $\left(X, \|\cdot\|_\gamma\right)$ be a Banach space. Let $f \in C(M, X)$. We define the first modulus of continuity of $f$ as*

$$\omega_1(f, \delta) := \sup_{\substack{x, y \in M: \\ \|x - y\|_p \leq \delta}} \|f(x) - f(y)\|_\gamma, \quad 0 < \delta \leq diam(M). \tag{35}$$

*If $\delta > diam(M)$, then*

$$\omega_1(f, \delta) = \omega_1(f, diam(M)). \tag{36}$$

Notice $\omega_1(f, \delta)$ is increasing in $\delta > 0$. For $f \in C_B(M, X)$ (continuous and bounded functions) $\omega_1(f, \delta)$ is defined similarly.

**Lemma 7** *([11], p. 274) We have $\omega_1(f, \delta) \to 0$ as $\delta \downarrow 0$, iff $f \in C(M, X)$, where $M$ is a convex compact subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1, \infty]$.*

Clearly we have also: $f \in C_U\left(\mathbb{R}^N, X\right)$ (uniformly continuous functions), iff $\omega_1(f, \delta) \to 0$ as $\delta \downarrow 0$, where $\omega_1$ is defined similarly to (35). The space $C_B\left(\mathbb{R}^N, X\right)$ denotes the continuous and bounded functions on $\mathbb{R}^N$.

When $f \in C_B\left(\mathbb{R}^N, X\right)$ we define,

$$B_n(f,x) := B_n(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx-k) :=$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n}, \frac{k_2}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} W(nx_i - k_i)\right), \tag{37}$$

$n \in \mathbb{N}$, $\forall\, x \in \mathbb{R}^N$, $N \in \mathbb{N}$, the multivariate quasi-interpolation neural network operator.

Also for $f \in C_B\left(\mathbb{R}^N, X\right)$ we define the multivariate Kantorovich type neural network operator

$$C_n(f,x) := C_n(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty} \left(n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\, dt\right) Z(nx-k) =$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \left(n^N \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} ... \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1, ..., t_N)\, dt_1 ... dt_N\right)$$

$$\cdot \left(\prod_{i=1}^{N} W(nx_i - k_i)\right), \tag{38}$$

$n \in \mathbb{N}$, $\forall\, x \in \mathbb{R}^N$.

Again for $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \in \mathbb{N}$, we define the multivariate neural network operator of quadrature type $D_n(f,x)$, $n \in \mathbb{N}$, as follows.

Let $\theta = (\theta_1, ..., \theta_N) \in \mathbb{N}^N$, $r = (r_1, ..., r_N) \in \mathbb{Z}_+^N$, $w_r = w_{r_1, r_2, ... r_N} \geq 0$, such that $\sum\limits_{r=0}^{\theta} w_r = \sum\limits_{r_1=0}^{\theta_1} \sum\limits_{r_2=0}^{\theta_2} ... \sum\limits_{r_N=0}^{\theta_N} w_{r_1, r_2, ... r_N} = 1$; $k \in \mathbb{Z}^N$ and

$$\delta_{nk}(f) := \delta_{n, k_1, k_2, ..., k_N}(f) := \sum_{r=0}^{\theta} w_r f\left(\frac{k}{n} + \frac{r}{n\theta}\right) =$$

$$\sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} ... \sum_{r_N=0}^{\theta_N} w_{r_1, r_2, ... r_N} f\left(\frac{k_1}{n} + \frac{r_1}{n\theta_1}, \frac{k_2}{n} + \frac{r_2}{n\theta_2}, ..., \frac{k_N}{n} + \frac{r_N}{n\theta_N}\right), \tag{39}$$

where $\frac{r}{\theta} := \left(\frac{r_1}{\theta_1}, \frac{r_2}{\theta_2}, ..., \frac{r_N}{\theta_N}\right)$.

We set

$$D_n(f,x) := D_n(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty} \delta_{nk}(f)\, Z(nx-k) = \tag{40}$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \delta_{n, k_1, k_2, ..., k_N}(f) \left(\prod_{i=1}^{N} W(nx_i - k_i)\right),$$

$\forall\, x \in \mathbb{R}^N$.

In this article we study the approximation properties of $A_n, B_n, C_n,\ D_n$ neural network operators and as well of their iterates. That is, the quantitative pointwise and uniform convergence of these operators to the unit operator $I$.

# 3 Multivariate general Neural Network Approximations

Here we present several vectorial neural network approximations to Banach space valued functions given with rates.

We give

**Theorem 8** *Let* $f \in C\left(\prod_{i=1}^{N}[a_i,b_i], X\right)$, $0<\beta<1$, $x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $m, N, n \in \mathbb{N}$ *with* $n^{1-\beta}>2$. *Then*

*1)*

$$\|A_n(f,x)-f(x)\|_\gamma \leq (4.824)^N \left[\omega_1\left(f,\frac{1}{n^\beta}\right)+\frac{4e^2\left\|\|f\|_\gamma\right\|_\infty}{\pi e^{n^{1-\beta}}}\right] =: \lambda_1(n), \tag{41}$$

*and*

*2)*

$$\left\|\|A_n(f)-f\|_\gamma\right\|_\infty \leq \lambda_1(n). \tag{42}$$

*We notice that* $\lim_{n\to\infty} A_n(f) \overset{\|\cdot\|_\gamma}{=} f$, *pointwise and uniformly.*

*Above* $\omega_1$ *is with respect to* $p=\infty$.

**Proof.** We observe that

$$\Delta(x) := A_n^*(f,x) - f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k) =$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right) Z(nx-k) - \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f(x) Z(nx-k) =$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(f\left(\frac{k}{n}\right)-f(x)\right) Z(nx-k). \tag{43}$$

Thus

$$\|\Delta(x)\|_\gamma \leq \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|f\left(\frac{k}{n}\right)-f(x)\right\|_\gamma Z(nx-k) =$$

$$\sum_{\begin{cases} k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}\leq\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb\rfloor} \left\|f\left(\frac{k}{n}\right)-f(x)\right\|_{\gamma} Z(nx-k)+$$

$$\sum_{\begin{cases} k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb\rfloor} \left\|f\left(\frac{k}{n}\right)-f(x)\right\|_{\gamma} Z(nx-k) \overset{(16)}{\leq}$$

$$\omega_1\left(f,\frac{1}{n^{\beta}}\right)+2\left\|\|f\|_{\gamma}\right\|_{\infty} \sum_{\begin{cases} k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nb\rfloor} Z(nx-k) \overset{(21)}{\leq}$$

$$\omega_1\left(f,\frac{1}{n^{\beta}}\right)+\frac{4e^2\left\|\|f\|_{\gamma}\right\|_{\infty}}{\pi e^{n^{1-\beta}}}. \tag{44}$$

So that

$$\|\Delta(x)\|_{\gamma}\leq\omega_1\left(f,\frac{1}{n^{\beta}}\right)+\frac{4e^2\left\|\|f\|_{\gamma}\right\|_{\infty}}{\pi e^{n^{1-\beta}}}. \tag{45}$$

Now using (34) we finish the proof. ■

We make

**Remark 9** *([11], pp. 263-266) Let $\left(\mathbb{R}^N,\|\cdot\|_p\right)$, $N\in\mathbb{N}$; where $\|\cdot\|_p$ is the $L_p$-norm, $1\leq p\leq\infty$. $\mathbb{R}^N$ is a Banach space, and $\left(\mathbb{R}^N\right)^j$ denotes the $j$-fold product space $\mathbb{R}^N\times...\times\mathbb{R}^N$ endowed with the max-norm $\|x\|_{(\mathbb{R}^N)^j}:=\max_{1\leq\lambda\leq j}\|x_\lambda\|_p$, where $x:=(x_1,...,x_j)\in\left(\mathbb{R}^N\right)^j$.*

*Let $\left(X,\|\cdot\|_{\gamma}\right)$ be a general Banach space. Then the space $L_j:=L_j\left(\left(\mathbb{R}^N\right)^j;X\right)$ of all $j$-multilinear continuous maps $g:\left(\mathbb{R}^N\right)^j\to X$, $j=1,...,m$, is a Banach space with norm*

$$\|g\|:=\|g\|_{L_j}:=\sup_{\left(\|x\|_{(\mathbb{R}^N)^j}=1\right)}\|g(x)\|_{\gamma}=\sup\frac{\|g(x)\|_{\gamma}}{\|x_1\|_p...\|x_j\|_p}. \tag{46}$$

*Let $M$ be a non-empty convex and compact subset of $\mathbb{R}^N$ and $x_0\in M$ is fixed.*

*Let $O$ be an open subset of $\mathbb{R}^N: M\subset O$. Let $f:O\to X$ be a continuous function, whose Fréchet derivatives (see [23]) $f^{(j)}:O\to L_j=L_j\left(\left(\mathbb{R}^N\right)^j;X\right)$ exist and are continuous for $1\leq j\leq m$, $m\in\mathbb{N}$.*

*Call $(x-x_0)^j:=(x-x_0,...,x-x_0)\in\left(\mathbb{R}^N\right)^j$, $x\in M$.*

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*We will work with* $f|_M$.

*Then, by Taylor's formula ([16]), ([23], p. 124), we get*

$$f(x) = \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(x-x_0)^j}{j!} + R_m(x,x_0), \quad \text{all } x \in M, \tag{47}$$

*where the remainder is the Riemann integral*

$$R_m(x,x_0) := \int_0^1 \frac{(1-u)^{m-1}}{(m-1)!} \left( f^{(m)}(x_0 + u(x-x_0)) - f^{(m)}(x_0) \right) (x-x_0)^m du, \tag{48}$$

*here we set* $f^{(0)}(x_0)(x-x_0)^0 = f(x_0)$.

*We consider*

$$w := \omega_1\left(f^{(m)}, h\right) := \sup_{\substack{x,y\in M: \\ \|x-y\|_p \le h}} \left\| f^{(m)}(x) - f^{(m)}(y) \right\|, \tag{49}$$

$h > 0$.

*We obtain*

$$\left\| \left( f^{(m)}(x_0 + u(x-x_0)) - f^{(m)}(x_0) \right) (x-x_0)^m \right\|_\gamma \le$$

$$\left\| f^{(m)}(x_0 + u(x-x_0)) - f^{(m)}(x_0) \right\| \cdot \|x-x_0\|_p^m \le$$

$$w \|x-x_0\|_p^m \left\lceil \frac{u\|x-x_0\|_p}{h} \right\rceil, \tag{50}$$

*by Lemma 7.1.1, [1], p. 208, where* $\lceil\cdot\rceil$ *is the ceiling.*

*Therefore for all* $x \in M$ *(see [1], pp. 121-122):*

$$\|R_m(x,x_0)\|_\gamma \le w\|x-x_0\|_p^m \int_0^1 \left\lceil \frac{u\|x-x_0\|_p}{h} \right\rceil \frac{(1-u)^{m-1}}{(m-1)!} du$$

$$= w\Phi_m\left(\|x-x_0\|_p\right) \tag{51}$$

*by a change of variable, where*

$$\Phi_m(t) := \int_0^{|t|} \left\lceil \frac{s}{h} \right\rceil \frac{(|t|-s)^{m-1}}{(m-1)!} ds = \frac{1}{m!} \left( \sum_{j=0}^{\infty} (|t|-jh)_+^m \right), \quad \forall\, t \in \mathbb{R}, \tag{52}$$

*is a (polynomial) spline function, see [1], p. 210-211.*

*Also from there we get*

$$\Phi_m(t) \le \left( \frac{|t|^{m+1}}{(m+1)!h} + \frac{|t|^m}{2m!} + \frac{h|t|^{m-1}}{8(m-1)!} \right), \quad \forall\, t \in \mathbb{R}, \tag{53}$$

*with equality true only at* $t = 0$.

*Therefore it holds*

$$\left\| R_m(x, x_0) \right\|_\gamma \leq w \left( \frac{\|x - x_0\|_p^{m+1}}{(m+1)!h} + \frac{\|x - x_0\|_p^m}{2m!} + \frac{h \|x - x_0\|_p^{m-1}}{8(m-1)!} \right), \quad \forall\ x \in M. \tag{54}$$

*We have found that*

$$\left\| f(x) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(x - x_0)^j}{j!} \right\|_\gamma \leq$$

$$\omega_1 \left( f^{(m)}, h \right) \left( \frac{\|x - x_0\|_p^{m+1}}{(m+1)!h} + \frac{\|x - x_0\|_p^m}{2m!} + \frac{h \|x - x_0\|_p^{m-1}}{8(m-1)!} \right) < \infty, \tag{55}$$

$\forall\ x, x_0 \in M$.

*Here* $0 < \omega_1 \left( f^{(m)}, h \right) < \infty$, *by* $M$ *being compact and* $f^{(m)}$ *being continuous on* $M$.

*One can rewrite (55) as follows:*

$$\left\| f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \right\|_\gamma \leq$$

$$\omega_1 \left( f^{(m)}, h \right) \left( \frac{\|\cdot - x_0\|_p^{m+1}}{(m+1)!h} + \frac{\|\cdot - x_0\|_p^m}{2m!} + \frac{h \|\cdot - x_0\|_p^{m-1}}{8(m-1)!} \right), \ \forall\ x_0 \in M, \tag{56}$$

*a pointwise functional inequality on* $M$.

*Here* $(\cdot - x_0)^j$ *maps* $M$ *into* $\left(\mathbb{R}^N\right)^j$ *and it is continuous, also* $f^{(j)}(x_0)$ *maps* $\left(\mathbb{R}^N\right)^j$ *into* $X$ *and it is continuous. Hence their composition* $f^{(j)}(x_0)(\cdot - x_0)^j$ *is continuous from* $M$ *into* $X$.

*Clearly* $f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \in C(M, X)$, *hence* $\left\| f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \right\|_\gamma \in C(M)$.

*Let* $\left\{ \widetilde{L}_N \right\}_{N \in \mathbb{N}}$ *be a sequence of positive linear operators mapping* $C(M)$ *into* $C(M)$.

*Therefore we obtain*

$$\left( \widetilde{L}_N \left( \left\| f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \right\|_\gamma \right) \right)(x_0) \leq$$

$$\omega_1 \left( f^{(m)}, h \right) \left[ \frac{\left( \widetilde{L}_N \left( \|\cdot - x_0\|_p^{m+1} \right) \right)(x_0)}{(m+1)!h} + \frac{\left( \widetilde{L}_N \left( \|\cdot - x_0\|_p^m \right) \right)(x_0)}{2m!} \right.$$

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$$\left. + \frac{h\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{m-1}\right)\right)(x_0)}{8(m-1)!}\right], \tag{57}$$

$\forall\ N \in \mathbb{N}$, $\forall\ x_0 \in M$.

Clearly (57) is valid when $M = \prod_{i=1}^{N}[a_i, b_i]$ and $\widetilde{L}_n = \widetilde{A}_n$, see (26).

All the above is preparation for the following theorem, where we assume Fréchet differentiability of functions.

This will be a direct application of Theorem 10.2, [11], pp. 268-270. The operators $A_n$, $\widetilde{A}_n$ fulfill its assumptions, see (25), (26), (28), (29) and (30).

We present the following high order approximation results.

**Theorem 10** *Let $O$ open subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1, \infty]$, such that $\prod_{i=1}^{N}[a_i, b_i] \subset O \subseteq \mathbb{R}^N$, and let $\left(X, \|\cdot\|_\gamma\right)$ be a general Banach space. Let $m \in \mathbb{N}$ and $f \in C^m(O, X)$, the space of $\overline{m}$-times continuously Fréchet differentiable functions from $O$ into $X$. We study the approximation of $f|_{\prod_{i=1}^{N}[a_i,b_i]}$. Let $x_0 \in \left(\prod_{i=1}^{N}[a_i, b_i]\right)$ and $r > 0$. Then*

*1)*

$$\left\|(A_n(f))(x_0) - \sum_{j=0}^{m}\frac{1}{j!}\left(A_n\left(f^{(j)}(x_0)(\cdot - x_0)^j\right)\right)(x_0)\right\|_\gamma \leq$$

$$\frac{\omega_1\left(f^{(m)}, r\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)}$$

$$\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right], \tag{58}$$

*2) additionally if $f^{(j)}(x_0) = 0$, $j = 1, ..., \overline{m}$, we have*

$$\|(A_n(f))(x_0) - f(x_0)\|_\gamma \leq$$

$$\frac{\omega_1\left(f^{(m)}, r\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)} \tag{59}$$

$$\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right],$$

*3)*

$$\|(A_n(f))(x_0) - f(x_0)\|_\gamma \leq \sum_{j=1}^{m}\frac{1}{j!}\left\|\left(A_n\left(f^{(j)}(x_0)(\cdot - x_0)^j\right)\right)(x_0)\right\|_\gamma +$$

$$\frac{\omega_1\left(f^{(m)}, r\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)}$$

$$\left[\frac{1}{(m+1)}+\frac{r}{2}+\frac{mr^2}{8}\right], \tag{60}$$

*and*

*4)*

$$\left\|\|A_n(f)-f\|_\gamma\right\|_{\infty,\prod_{i=1}^{N}[a_i,b_i]} \leq$$

$$\sum_{j=1}^{m}\frac{1}{j!}\left\|\left\|\left(A_n\left(f^{(j)}(x_0)(\cdot - x_0)^j\right)\right)(x_0)\right\|_\gamma\right\|_{\infty,x_0\in\prod_{i=1}^{N}[a_i,b_i]}+$$

$$\frac{\omega_1\left(f^{(m)}, r\left\|\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right\|_{\infty,x_0\in\prod_{i=1}^{N}[a_i,b_i]}^{\frac{1}{m+1}}\right)}{rm!}$$

$$\left\|\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right\|_{\infty,x_0\in\prod_{i=1}^{N}[a_i,b_i]}^{\left(\frac{m}{m+1}\right)} \tag{61}$$

$$\left[\frac{1}{(m+1)}+\frac{r}{2}+\frac{mr^2}{8}\right].$$

We need

**Lemma 11** *The function* $\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^m\right)\right)(x_0)$ *is continuous in* $x_0 \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $m \in \mathbb{N}$.

**Proof.** By Lemma 10.3, [11], p. 272. ■

We give

**Corollary 12** *(to Theorem 10, case of* $m=1$*) Then*

*1)*

$$\|(A_n(f))(x_0)-f(x_0)\|_\gamma \leq \left\|\left(A_n\left(f^{(1)}(x_0)(\cdot - x_0)\right)\right)(x_0)\right\|_\gamma +$$

$$\frac{1}{2r}\omega_1\left(f^{(1)}, r\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^2\right)\right)(x_0)\right)^{\frac{1}{2}}\right)\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^2\right)\right)(x_0)\right)^{\frac{1}{2}} \tag{62}$$

$$\left[1+r+\frac{r^2}{4}\right],$$

*and*

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*2)*

$$\left\| \left\| \left( A_n(f) \right) - f \right\|_\gamma \right\|_{\infty, \prod_{i=1}^{N} [a_i, b_i]} \leq$$

$$\left\| \left\| \left( A_n \left( f^{(1)}(x_0)(\cdot - x_0) \right) \right)(x_0) \right\|_\gamma \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]} +$$

$$\frac{1}{2r} \omega_1 \left( f^{(1)}, r \left\| \left( \widetilde{A}_n \left( \| \cdot - x_0 \|_p^2 \right) \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]}^{\frac{1}{2}} \right)$$

$$\left\| \left( \widetilde{A}_n \left( \| \cdot - x_0 \|_p^2 \right) \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]}^{\frac{1}{2}} \left[ 1 + r + \frac{r^2}{4} \right], \tag{63}$$

$r > 0$.

We make

**Remark 13** *We estimate* $0 < \alpha < 1$, $m, n \in \mathbb{N} : n^{1-\alpha} > 2$,

$$\widetilde{A}_n \left( \| \cdot - x_0 \|_\infty^{m+1} \right)(x_0) = \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^{m+1} Z(nx_0 - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx_0 - k)} \overset{(22)}{<}$$

$$(4.824)^N \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^{m+1} Z(nx_0 - k) = \tag{64}$$

$$(4.824)^N \left\{ \sum_{\substack{k = \lceil na \rceil \\ : \left\| \frac{k}{n} - x_0 \right\|_\infty \leq \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^{m+1} Z(nx_0 - k) + \right.$$

$$\left. \sum_{\substack{k = \lceil na \rceil \\ : \left\| \frac{k}{n} - x_0 \right\|_\infty > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^{m+1} Z(nx_0 - k) \right\} \overset{(23)}{\leq}$$

$$(4.824)^N \left\{ \frac{1}{n^{\alpha(m+1)}} + \frac{2e^2 \|b - a\|_\infty^{m+1}}{\pi e^{n^{1-\alpha}}} \right\}, \tag{65}$$

*(where* $b - a = (b_1 - a_1, ..., b_N - a_N)$*).*

*We have proved that* ($\forall$ $x_0 \in \prod_{i=1}^{N} [a_i, b_i]$)

$$\widetilde{A}_n \left( \|\cdot - x_0\|_\infty^{m+1} \right)(x_0) < (4.824)^N \left\{ \frac{1}{n^{\alpha(m+1)}} + \frac{2e^2 \|b-a\|_\infty^{m+1}}{\pi e^{n^{1-\alpha}}} \right\} =: \varphi_1(n) \tag{66}$$

($0 < \alpha < 1$, $m, n \in \mathbb{N} : n^{1-\alpha} > 2$).

*And, consequently it holds*

$$\left\| \widetilde{A}_n \left( \|\cdot - x_0\|_\infty^{m+1} \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]} <$$

$$(4.824)^N \left\{ \frac{1}{n^{\alpha(m+1)}} + \frac{2e^2 \|b-a\|_\infty^{m+1}}{\pi e^{n^{1-\alpha}}} \right\} = \varphi_1(n) \to 0, \quad \text{as } n \to +\infty. \tag{67}$$

*So, we have that* $\varphi_1(n) \to 0$, *as* $n \to +\infty$. *Thus, when* $p \in [1, \infty]$, *from Theorem 10 we have the convergence to zero in the right hand sides of parts (1), (2).*

*Next we estimate* $\left\| \left( \widetilde{A}_n \left( f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma$.

*We have that*

$$\left( \widetilde{A}_n \left( f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) = \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f^{(j)}(x_0) \left( \frac{k}{n} - x_0 \right)^j Z(nx_0 - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx_0 - k)}. \tag{68}$$

*When* $p = \infty$, $j = 1, ..., m$, *we obtain*

$$\left\| f^{(j)}(x_0) \left( \frac{k}{n} - x_0 \right)^j \right\|_\gamma \leq \left\| f^{(j)}(x_0) \right\| \left\| \frac{k}{n} - x_0 \right\|_\infty^j. \tag{69}$$

*We further have that*

$$\left\| \left( \widetilde{A}_n \left( f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma \overset{(22)}{<}$$

$$(4.824)^N \left( \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f^{(j)}(x_0) \left( \frac{k}{n} - x_0 \right)^j \right\|_\gamma Z(nx_0 - k) \right) \leq$$

$$(4.824)^N \left( \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f^{(j)}(x_0) \right\| \left\| \frac{k}{n} - x_0 \right\|_\infty^j Z(nx_0 - k) \right) = \tag{70}$$

$$(4.824)^N \left\| f^{(j)}(x_0) \right\| \left( \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^j Z(nx_0 - k) \right) =$$

$$(4.824)^N \left\| f^{(j)}(x_0) \right\| \left\{ \sum_{\substack{k=\lceil na \rceil \\ : \left\| \frac{k}{n} - x_0 \right\|_\infty \leq \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^j Z(nx_0 - k) \right.$$

$$\left. + \sum_{\substack{k=\lceil na \rceil \\ : \left\| \frac{k}{n} - x_0 \right\|_\infty > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^j Z(nx_0 - k) \right\} \overset{(23)}{\leq} \tag{71}$$

$$(4.824)^N \left\| f^{(j)}(x_0) \right\| \left\{ \frac{1}{n^{\alpha j}} + \frac{2e^2 \|b-a\|_\infty^j}{\pi e^{n^{1-\alpha}}} \right\} \to 0, \text{ as } n \to \infty.$$

*That is*

$$\left\| \left( \widetilde{A}_n \left( f^{(j)}(x_0) (\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma \to 0, \text{ as } n \to \infty.$$

*Therefore when $p = \infty$, for $j = 1, ..., m$, we have proved:*

$$\left\| \left( \widetilde{A}_n \left( f^{(j)}(x_0) (\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma <$$

$$(4.824)^N \left\| f^{(j)}(x_0) \right\| \left\{ \frac{1}{n^{\alpha j}} + \frac{2e^2 \|b-a\|_\infty^j}{\pi e^{n^{1-\alpha}}} \right\} \leq \tag{72}$$

$$(4.824)^N \left\| f^{(j)} \right\|_\infty \left\{ \frac{1}{n^{\alpha j}} + \frac{2e^2 \|b-a\|_\infty^j}{\pi e^{n^{1-\alpha}}} \right\} =: \varphi_{2j}(n) < \infty,$$

*and converges to zero, as $n \to \infty$.*

We conclude:

In Theorem 10, the right hand sides of (60) and (61) converge to zero as $n \to \infty$, for any $p \in [1, \infty]$.

Also in Corollary 12, the right hand sides of (62) and (63) converge to zero as $n \to \infty$, for any $p \in [1, \infty]$.

**Conclusion 14** *We have proved that the left hand sides of (58), (59), (60), (61) and (62), (63) converge to zero as $n \to \infty$, for $p \in [1, \infty]$. Consequently $A_n \to I$ (unit operator) pointwise and uniformly, as $n \to \infty$, where $p \in [1, \infty]$. In the presence of initial conditions we achieve a higher speed of convergence, see (59). Higher speed of convergence happens also to the left hand side of (58).*

We further give

**Corollary 15** *(to Theorem 10) Let $O$ open subset of $\left(\mathbb{R}^N, \|\cdot\|_\infty\right)$, such that $\prod_{i=1}^{N}[a_i,b_i] \subset O \subseteq \mathbb{R}^N$, and let $\left(X, \|\cdot\|_\gamma\right)$ be a general Banach space. Let $m \in \mathbb{N}$ and $f \in C^m(O,X)$, the space of $\overline{m}$-times continuously Fréchet differentiable functions from $O$ into $X$. We study the approximation of $f|_{\prod_{i=1}^{N}[a_i,b_i]}$. Let $x_0 \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$ and $r > 0$. Here $\varphi_1(n)$ as in (67) and $\varphi_{2j}(n)$ as in (72), where $n \in \mathbb{N} : n^{1-\alpha} > 2$, $0 < \alpha < 1$, $j = 1, ..., m$. Then*

*1)*

$$\left\| (A_n(f))(x_0) - \sum_{j=0}^{m} \frac{1}{j!} \left( A_n\left( f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma \leq$$

$$\frac{\omega_1\left(f^{(m)}, r(\varphi_1(n))^{\frac{1}{m+1}}\right)}{rm!} (\varphi_1(n))^{\left(\frac{m}{m+1}\right)} \left[ \frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right], \tag{73}$$

*2) additionally, if $f^{(j)}(x_0) = 0$, $j = 1, ..., \overline{m}$, we have*

$$\|(A_n(f))(x_0) - f(x_0)\|_\gamma \leq$$

$$\frac{\omega_1\left(f^{(m)}, r(\varphi_1(n))^{\frac{1}{m+1}}\right)}{rm!} (\varphi_1(n))^{\left(\frac{m}{m+1}\right)} \left[ \frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right], \tag{74}$$

*3)*

$$\left\| \|A_n(f) - f\|_\gamma \right\|_{\infty, \prod_{i=1}^{N}[a_i,b_i]} \leq \sum_{j=1}^{m} \frac{\varphi_{2j}(n)}{j!} +$$

$$\frac{\omega_1\left(f^{(m)}, r(\varphi_1(n))^{\frac{1}{m+1}}\right)}{rm!} (\varphi_1(n))^{\left(\frac{m}{m+1}\right)} \tag{75}$$

$$\left[ \frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right] =: \varphi_3(n) \to 0, \text{ as } n \to \infty.$$

We continue with

**Theorem 16** *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $m, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, $\omega_1$ is for $p = \infty$. Then*

*1)*

$$\|B_n(f,x) - f(x)\|_\gamma \leq \omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{4e^2 \left\| \|f\|_\gamma \right\|_\infty}{\pi e^{n^{1-\beta}}} =: \lambda_2(n), \tag{76}$$

*2)*

$$\left\| \|B_n(f) - f\|_\gamma \right\|_\infty \leq \lambda_2(n). \tag{77}$$

*Given that* $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$, *we obtain* $\lim_{n\to\infty} B_n(f) = f$, *uniformly.*

**Proof.** We have that

$$B_n(f,x) - f(x) \overset{(16)}{=} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx-k) - f(x) \sum_{k=-\infty}^{\infty} Z(nx-k) = \tag{78}$$

$$\sum_{k=-\infty}^{\infty} \left(f\left(\frac{k}{n}\right) - f(x)\right) Z(nx-k).$$

Hence

$$\|B_n(f,x) - f(x)\|_\gamma \leq \sum_{k=-\infty}^{\infty} \left\|f\left(\frac{k}{n}\right) - f(x)\right\|_\gamma Z(nx-k) =$$

$$\sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty \leq \frac{1}{n^\beta} \end{cases}}^{\infty} \left\|f\left(\frac{k}{n}\right) - f(x)\right\|_\gamma Z(nx-k) +$$

$$\sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} \left\|f\left(\frac{k}{n}\right) - f(x)\right\|_\gamma Z(nx-k) \overset{(16)}{\leq}$$

$$\omega_1\left(f, \frac{1}{n^\beta}\right) + 2\left\|\|f\|_\gamma\right\|_\infty \sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} Z(nx-k) \overset{(23)}{\leq}$$

$$\omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{4e^2\left\|\|f\|_\gamma\right\|_\infty}{\pi e^{n^{1-\beta}}}, \tag{79}$$

proving the claim. ■

We give

**Theorem 17** *Let* $f \in C_B\left(\mathbb{R}^N, X\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $m, N, n \in \mathbb{N}$ *with* $n^{1-\beta} > 2$, $\omega_1$ *is for* $p = \infty$. *Then*

*1)*

$$\|C_n(f,x) - f(x)\|_\gamma \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) + \frac{4e^2\left\|\|f\|_\gamma\right\|_\infty}{\pi e^{n^{1-\beta}}} =: \lambda_3(n), \tag{80}$$

*2)*

$$\left\|\|C_n(f) - f\|_\gamma\right\|_\infty \leq \lambda_3(n). \tag{81}$$

*Given that* $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$, *we obtain* $\lim_{n\to\infty} C_n(f) = f$, *uniformly.*

**Proof.** We notice that

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt = \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} \dots \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1, t_2, \dots, t_N)\, dt_1 dt_2 \dots dt_N =$$

$$\int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} \dots \int_0^{\frac{1}{n}} f\left(t_1 + \frac{k_1}{n}, t_2 + \frac{k_2}{n}, \dots, t_N + \frac{k_N}{n}\right) dt_1 \dots dt_N = \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt. \tag{82}$$

Thus it holds (by (38))

$$C_n(f, x) = \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt\right) Z(nx - k). \tag{83}$$

We observe that

$$\|C_n(f, x) - f(x)\|_\gamma =$$

$$\left\| \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt\right) Z(nx - k) - \sum_{k=-\infty}^{\infty} f(x) Z(nx - k) \right\|_\gamma =$$

$$\left\| \sum_{k=-\infty}^{\infty} \left(\left(n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt\right) - f(x)\right) Z(nx - k) \right\|_\gamma =$$

$$\left\| \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left(f\left(t + \frac{k}{n}\right) - f(x)\right) dt\right) Z(nx - k) \right\|_\gamma \leq \tag{84}$$

$$\sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left\| f\left(t + \frac{k}{n}\right) - f(x) \right\|_\gamma dt\right) Z(nx - k) =$$

$$\sum_{\substack{k=-\infty \\ \left\|\frac{k}{n} - x\right\|_\infty \leq \frac{1}{n^\beta}}}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left\| f\left(t + \frac{k}{n}\right) - f(x) \right\|_\gamma dt\right) Z(nx - k) +$$

$$\sum_{\substack{k=-\infty \\ \left\|\frac{k}{n} - x\right\|_\infty > \frac{1}{n^\beta}}}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left\| f\left(t + \frac{k}{n}\right) - f(x) \right\|_\gamma dt\right) Z(nx - k) \leq$$

$$\sum_{\substack{k=-\infty \\ \left\|\frac{k}{n} - x\right\|_\infty \leq \frac{1}{n^\beta}}}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \omega_1\left(f, \|t\|_\infty + \left\|\frac{k}{n} - x\right\|_\infty\right) dt\right) Z(nx - k) +$$

$$2\left\|\|f\|_{\gamma}\right\|_{\infty}\left(\sum_{\substack{k=-\infty \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\infty} Z\left(|nx-k|\right)\right) \leq$$

$$\omega_1\left(f,\frac{1}{n}+\frac{1}{n^{\beta}}\right)+\frac{4e^2\left\|\|f\|_{\gamma}\right\|_{\infty}}{\pi e^{n^{1-\beta}}}, \tag{85}$$

proving the claim. ■

We also present

**Theorem 18** *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $0<\beta<1$, $x \in \mathbb{R}^N$, $m, N, n \in \mathbb{N}$ with $n^{1-\beta}>2$, $\omega_1$ is for $p=\infty$. Then*

*1)*

$$\|D_n(f,x)-f(x)\|_{\gamma} \leq \omega_1\left(f,\frac{1}{n}+\frac{1}{n^{\beta}}\right)+\frac{4e^2\left\|\|f\|_{\gamma}\right\|_{\infty}}{\pi e^{n^{1-\beta}}}=\lambda_4(n), \tag{86}$$

*2)*

$$\left\|\|D_n(f)-f\|_{\gamma}\right\|_{\infty} \leq \lambda_4(n). \tag{87}$$

*Given that $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$, we obtain $\lim_{n\to\infty} D_n(f)=f$, uniformly.*

**Proof.** Similar to the proof of Theorem 17, as such is omitted. ■

We make

**Definition 19** *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \in \mathbb{N}$, where $\left(X, \|\cdot\|_{\gamma}\right)$ is a Banach space. We define the general neural network operator*

$$F_n(f,x):=\sum_{k=-\infty}^{\infty} l_{nk}(f)\, Z(nx-k)=$$

$$\begin{cases} B_n(f,x), & \text{if } l_{nk}(f)=f\left(\frac{k}{n}\right), \\ C_n(f,x), & \text{if } l_{nk}(f)=n^N\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt, \\ D_n(f,x), & \text{if } l_{nk}(f)=\delta_{nk}(f). \end{cases} \tag{88}$$

Clearly $l_{nk}(f)$ is an $X$-valued bounded linear functional such that $\|l_{nk}(f)\|_{\gamma} \leq \left\|\|f\|_{\gamma}\right\|_{\infty}$.

Hence $F_n(f)$ is a bounded linear operator with $\left\|\|F_n(f)\|_{\gamma}\right\|_{\infty} \leq \left\|\|f\|_{\gamma}\right\|_{\infty}$.

We need

**Theorem 20** *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \geq 1$. Then $F_n(f) \in C_B\left(\mathbb{R}^N, X\right)$.*

**Proof.** Lengthy and similar to the proof of Theorem 21 of [14], as such is omitted. ∎

**Remark 21** *By (25) it is obvious that $\left\|\|A_n(f)\|_\gamma\right\|_\infty \leq \left\|\|f\|_\gamma\right\|_\infty < \infty$, and $A_n(f) \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$, given that $f \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$.*

*Call $L_n$ any of the operators $A_n, B_n, C_n, D_n$.*

*Clearly then*

$$\left\|\left\|L_n^2(f)\right\|_\gamma\right\|_\infty = \left\|\|L_n(L_n(f))\|_\gamma\right\|_\infty \leq \left\|\|L_n(f)\|_\gamma\right\|_\infty \leq \left\|\|f\|_\gamma\right\|_\infty, \tag{89}$$

*etc.*

*Therefore we get*

$$\left\|\left\|L_n^k(f)\right\|_\gamma\right\|_\infty \leq \left\|\|f\|_\gamma\right\|_\infty, \ \ \forall\ k \in \mathbb{N}, \tag{90}$$

*the contraction property.*

*Also we see that*

$$\left\|\left\|L_n^k(f)\right\|_\gamma\right\|_\infty \leq \left\|\left\|L_n^{k-1}(f)\right\|_\gamma\right\|_\infty \leq \ldots \leq \left\|\|L_n(f)\|_\gamma\right\|_\infty \leq \left\|\|f\|_\gamma\right\|_\infty. \tag{91}$$

*Here $L_n^k$ are bounded linear operators.*

**Notation 22** *Here $N \in \mathbb{N}$, $0 < \beta < 1$. Denote by*

$$c_N := \begin{cases} (4.824)^N, & \text{if } L_n = A_n, \\ 1, & \text{if } L_n = B_n, C_n, D_n, \end{cases} \tag{92}$$

$$\varphi(n) := \begin{cases} \frac{1}{n^\beta}, & \text{if } L_n = A_n,\ B_n, \\ \frac{1}{n} + \frac{1}{n^\beta}, & \text{if } L_n = C_n, D_n, \end{cases} \tag{93}$$

$$\Omega := \begin{cases} C\left(\prod_{i=1}^N [a_i, b_i], X\right), & \text{if } L_n = A_n, \\ C_B\left(\mathbb{R}^N, X\right), & \text{if } L_n = B_n, C_n, D_n, \end{cases} \tag{94}$$

*and*

$$Y := \begin{cases} \prod_{i=1}^N [a_i, b_i], & \text{if } L_n = A_n, \\ \mathbb{R}^N, & \text{if } L_n = B_n, C_n, D_n. \end{cases} \tag{95}$$

We give the condensed

**Theorem 23** *Let $f \in \Omega$, $0 < \beta < 1$, $x \in Y$; $n, m, N \in \mathbb{N}$ with $n^{1-\beta} > 2$. Then*
*(i)*

$$\left\| L_n(f,x) - f(x) \right\|_\gamma \leq c_N \left[ \omega_1(f, \varphi(n)) + \frac{4e^2 \left\| \|f\|_\gamma \right\|_\infty}{\pi e^{n^{1-\beta}}} \right] =: \tau(n), \tag{96}$$

*where $\omega_1$ is for $p = \infty$,*
*and*
*(ii)*

$$\left\| \left\| L_n(f) - f \right\|_\gamma \right\|_\infty \leq \tau(n) \to 0, \text{ as } n \to \infty. \tag{97}$$

*For $f$ uniformly continuous and in $\Omega$ we obtain*

$$\lim_{n \to \infty} L_n(f) = f,$$

*pointwise and uniformly.*

**Proof.** By Theorems 8, 16, 17, 18. ■

Next we talk about iterated neural network approximation (see also [9]).

We give

**Theorem 24** *All here as in Theorem 23 and $r \in \mathbb{N}$, $\tau(n)$ as in (96). Then*

$$\left\| \left\| L_n^r f - f \right\|_\gamma \right\|_\infty \leq r\tau(n). \tag{98}$$

*So that the speed of convergence to the unit operator of $L_n^r$ is not worse than of $L_n$.*

**Proof.** As similar to [14] is omitted. ■

We also present

**Theorem 25** *Let $f \in \Omega$; $m, N, m_1, m_2, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$, $0 < \beta < 1$; $m_i^{1-\beta} > 2$, $i = 1, ..., r$, $x \in Y$, and let $(L_{m_1}, ..., L_{m_r})$ as $(A_{m_1}, ..., A_{m_r})$ or $(B_{m_1}, ..., B_{m_r})$ or $(C_{m_1}, ..., C_{m_r})$ or $(D_{m_1}, ..., D_{m_r})$, $p = \infty$. Then*

$$\left\| L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right)(x) - f(x) \right\|_\gamma \leq$$

$$\left\| \left\| L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right) - f \right\|_\gamma \right\|_\infty \leq$$

$$\sum_{i=1}^{r} \left\| \left\| L_{m_i} f - f \right\|_\gamma \right\|_\infty \leq$$

$$c_N \sum_{i=1}^{r} \left[ \omega_1(f, \varphi(m_i)) + \frac{4e^2 \left\| \|f\|_\gamma \right\|_\infty}{\pi e^{m_i^{1-\beta}}} \right] \leq$$

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$$rc_N \left[ \omega_1 \left( f, \varphi \left( m_1 \right) \right) + \frac{4e^2 \left\| \left\| f \right\|_\gamma \right\|_\infty}{\pi e^{m_1^{1-\beta}}} \right]. \tag{99}$$

*Clearly, we notice that the speed of convergence to the unit operator of the multiply iterated operator is not worse than the speed of* $L_{m_1}$.

**Proof.** As similar to [14] is omitted. ■

We also give

**Theorem 26** *Let all as in Corollary 15, and* $r \in \mathbb{N}$. *Here* $\varphi_3(n)$ *is as in (75). Then*

$$\left\| \left\| A_n^r f - f \right\|_\gamma \right\|_\infty \leq r \left\| \left\| A_n f - f \right\|_\gamma \right\|_\infty \leq r \varphi_3(n). \tag{100}$$

**Proof.** As similar to [14] is omitted. ■

**Application 27** *A typical application of all of our results is when* $\left( X, \|\cdot\|_\gamma \right) = (\mathbb{C}, |\cdot|)$, *where* $\mathbb{C}$ *are the complex numbers.*

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# $p$-Schatten norm generalized Canavati fractional Ostrowski, Opial and Grüss type inequalities involving several functions

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**Abstract**

Using generalized Canavati fractional left and right vectorial Taylor formulae we establish generalized fractional Ostrowski, Opial and Grüss type inequalities for several functions that take values in the von Neumann-Schatten class $\mathcal{B}_p(H)$, $1 \leq p < \infty$. The estimates are with respect to all $p$-Schatten norms, $1 \leq p < \infty$. We finish with applications.

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**Keywords and Phrases:** $p$-Schatten norms, von Neumann-Schatten class, Ostrowski, Opial and Grüss inequalities, generalized Canavati fractional derivative, generalized Canavati fractional inequalities.

## 1 Introduction

The following results inspire our work.

**Theorem 1** *(1938, Ostrowski [16]) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)$ whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on $(a,b)$, i.e., $\|f'\|_\infty^{\sup} := \sup_{t \in (a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(x) \right| \leq \left[ \frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty^{\sup}, \tag{1}$$

*for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.*

Ostrowski type inequalities have great applications to integral approximations in Numerical Analysis.

We mention

**Theorem 2** *(1882, Čebyšev [8]) Let $f, g : [a, b] \to \mathbb{R}$ be absolutely continuous functions with $f', g' \in L_\infty([a, b])$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \left( \frac{1}{b-a} \int_a^b f(x) dx \right) \left( \frac{1}{b-a} \int_a^b g(x) dx \right) \right|$$

$$\leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty . \tag{2}$$

*The above integrals are assumed to exist.*

The related Grüss type inequalities have many applications to Probability Theory. We presented also ([3], Ch. 8,9) mixed fractional Ostrowski and Grüss-Cebysev type inequalities for several functions, acting to all possible directions. The estimates involve the left and right Caputo fractional derivatives. See also the monographs written by the author [1], Chapters 24-26 and [2], Chapters 2-6.

We are motivated also by S. Dragomir [11] recent work:

An operator $A \in \mathcal{B}(H)$ is said to belong to the von Neumann-Schatten class $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the $p$-Schatten norm is finite

$$\|A\|_p := [tr(|A|^p)]^{\frac{1}{p}} < \infty.$$

Assume that $A : [a, b] \to \mathcal{B}_p(H)$, $B : [a, b] \to \mathcal{B}_q(H)$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, are continuous and $B$ is strongly differentiable on $(a, b)$, then

$$\left\| \int_a^b A(t) B(t) dt - \left( \int_a^b A(s) ds \right) B(u) \right\|_1 \leq$$

$$\sup_{t\in[a,b]} \|B'(t)\|_q \times \begin{cases} \left[ \frac{1}{2}(b-a) + \left| u - \frac{a+b}{2} \right| \right] \int_a^b \|A(t)\|_p dt, \\ \left[ \frac{(u-a)^{\beta+1} + (b-u)^{\beta+1}}{\beta+1} \right]^{\frac{1}{\beta}} \left( \int_a^b \|A(t)\|_p^\alpha \right)^{\frac{1}{\alpha}}, \\ \text{for } \alpha, \beta > 1 \text{ with } \frac{1}{\alpha} + \frac{1}{\beta} = 1 \\ \left[ \frac{1}{4}(b-a)^2 + \left( u - \frac{a+b}{2} \right)^2 \right] \sup_{t\in[a,b]} \|A(t)\|_p , \end{cases} \tag{3}$$

for all $u \in [a, b]$, an Ostrowski type inequality.

Further inspiration comes from S. Dragomir [12] recent work on Grüss inequalities:

For two continuous functions $A, B : [a,b] \to \mathcal{B}(H)$ we define the noncommutative Cebysev fractional

$$D(A,B) := (b-a)\int_a^b A(t)B(t)\,dt - \int_a^b A(t)\,dt \int_a^b B(t)\,dt.$$

If $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, let $A : [a,b] \to \mathcal{B}_p(H)$, $B : [a,b] \to \mathcal{B}_q(H)$ be strongly differentiable functions on the interval $(a,b)$, then

$$\|D(A,B)\|_1 \leq D\left(\int_a^b \|A'(u)\|_p\,du, \int_a^b \|B'(u)\|_q\,du\right) \leq \tag{4}$$

$$\frac{1}{4}(b-a)^2 \int_a^b \|A'(u)\|_p\,du \int_a^b \|B'(u)\|_q\,du.$$

We are also inspired by Z. Opial [15], 1960, famous inequality.

**Theorem 3** *Let $x(t) \in C^1([0,h])$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then*

$$\int_0^h |x(t)\,x'(t)|\,dt \leq \frac{h}{4}\int_0^h (x'(t))^2\,dt. \tag{5}$$

*In (5), the constant $\frac{h}{4}$ is the best possible. Inequality (5) holds as equality for the optimal function*

$$x(t) = \begin{cases} ct, & 0 \leq t \leq \frac{h}{2}, \\ c(h-t), & \frac{h}{2} \leq t \leq h, \end{cases} \tag{6}$$

*where $c > 0$ is an arbitrary constant.*

Opial-type inequalities are used a lot in proving uniqueness of solutions to differential equations and also to give upper bounds to their solutions.

For an extensive study about fractional Opial inequalities see the author's monograph [1].

In this article we generalize [3], Ch. 8,9 for several Banach algebra $\mathcal{B}_p(H)$ valued functions, in the sense of developing fractional Ostrowski, Opial and Grüss type inequalities. Now our left and right generalized Canavati fractional derivatives are for Banach space valued functions and our integrals are of Bochner type [13]. Applications finish the article.

## 2 Background on Vectorial generalized Canavati fractional calculus

All in this section come from [5], pp. 109-115 and [4].

Let $g : [a,b] \to \mathbb{R}$ be a strictly increasing function. such that $g \in C^1([a,b])$, and $g^{-1} \in C^n([g(a),g(b)])$, $n \in \mathbb{N}$, $(X, \|\cdot\|)$ is a Banach space. Let $f \in C^n([a,b],X)$, and call $l := f \circ g^{-1} : [g(a), g(b)] \to X$. It is clear that $l, l', ..., l^{(n)}$ are continuous functions from $[g(a), g(b)]$ into $f([a,b]) \subseteq X$.

Let $\nu \geq 1$ such that $[\nu] = n$, $n \in \mathbb{N}$ as above, where $[\cdot]$ is the integral part of the number.

Clearly when $0 < \nu < 1$, $[\nu] = 0$.

I) Let $h \in C([g(a), g(b)], X)$, we define the left Riemann-Liouville Bochner fractional integral as

$$\left(J_\nu^{z_0} h\right)(z) := \frac{1}{\Gamma(\nu)} \int_{z_0}^{z} (z-t)^{\nu-1} h(t)\, dt, \tag{7}$$

for $g(a) \leq z_0 \leq z \leq g(b)$, where $\Gamma$ is the gamma function; $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt$. We set $J_0^{z_0} h = h$.

Let $\alpha := \nu - [\nu]$ $(0 < \alpha < 1)$. We define the subspace $C^\nu_{g(x_0)}([g(a), g(b)], X)$ of $C^{[\nu]}([g(a), g(b)], X)$, where $x_0 \in [a,b]$ as:

$$C^\nu_{g(x_0)}([g(a), g(b)], X) =$$

$$\left\{ h \in C^{[\nu]}([g(a), g(b)], X) : J_{1-\alpha}^{g(x_0)} h^{([\nu])} \in C^1([g(x_0), g(b)], X) \right\}. \tag{8}$$

So let $h \in C^\nu_{g(x_0)}([g(a), g(b)], X)$, we define the left $g$-generalized $X$-valued fractional derivative of $h$ of order $\nu$, of Canavati type, over $[g(x_0), g(b)]$ as

$$D^\nu_{g(x_0)} h := \left( J_{1-\alpha}^{g(x_0)} h^{([\nu])} \right)'. \tag{9}$$

Clearly, for $h \in C^\nu_{g(x_0)}([g(a), g(b)], X)$, there exists

$$\left( D^\nu_{g(x_0)} h \right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^{z} (z-t)^{-\alpha} h^{([\nu])}(t)\, dt, \tag{10}$$

for all $g(x_0) \leq z \leq g(b)$.

In particular, when $f \circ g^{-1} \in C^\nu_{g(x_0)}([g(a), g(b)], X)$, we have that

$$\left( D^\nu_{g(x_0)} \left( f \circ g^{-1} \right) \right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^{z} (z-t)^{-\alpha} \left( f \circ g^{-1} \right)^{([\nu])}(t)\, dt, \tag{11}$$

for all $g(x_0) \leq z \leq g(b)$. We have that $D^n_{g(x_0)} \left( f \circ g^{-1} \right) = \left( f \circ g^{-1} \right)^{(n)}$ and $D^0_{g(x_0)} \left( f \circ g^{-1} \right) = f \circ g^{-1}$, see [4].

By [4], we have for $f \circ g^{-1} \in C^\nu_{g(x_0)}([g(a), g(b)], X)$, where $x_0 \in [a,b]$ the following left generalized $g$-fractional, of Canavati type,$X$-valued Taylor's formula:

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**Theorem 4** *Let $f \circ g^{-1} \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, where $x_0 \in [a,b]$ is fixed.*

*(i) If $\nu \geq 1$, then*

$$f(x) - f(x_0) = \sum_{k=1}^{[\nu]-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(x_0))}{k!} (g(x) - g(x_0))^k +$$

$$\frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D^{\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{12}$$

*for all $x_0 \leq x \leq b$.*

*(ii) If $0 < \nu < 1$, we get*

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D^{\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{13}$$

*for all $x_0 \leq x \leq b$.*

II) Let $h \in C([g(a), g(b)], X)$, we define the right Riemann-Liouville Bochner fractional integral as

$$\left(J^{\nu}_{z_0-}h\right)(z) := \frac{1}{\Gamma(\nu)} \int_{z}^{z_0} (t - z)^{\nu-1} h(t)\, dt, \tag{14}$$

for $g(a) \leq z \leq z_0 \leq g(b)$. We set $J^0_{z_0-}h = h$.

Let $\alpha := \nu - [\nu]$ $(0 < \alpha < 1)$. We define the subspace $C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$ of $C^{[\nu]}([g(a), g(b)], X)$, where $x_0 \in [a,b]$ as:

$$C^{\nu}_{g(x_0)-}([g(a), g(b)], X) :=$$

$$\left\{h \in C^{[\nu]}([g(a), g(b)], X) : J^{1-\alpha}_{g(x_0)-}h^{([\nu])} \in C^1([g(a), g(x_0)], X)\right\}. \tag{15}$$

So let $h \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, we define the right $g$-generalized $X$-valued fractional derivative of $h$ of order $\nu$, of Canavati type, over $[g(a), g(x_0)]$ as

$$D^{\nu}_{g(x_0)-}h := (-1)^{n-1}\left(J^{1-\alpha}_{g(x_0)-}h^{([\nu])}\right)'. \tag{16}$$

Clearly, for $h \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, there exists

$$\left(D^{\nu}_{g(x_0)-}h\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{z}^{g(x_0)} (t - z)^{-\alpha} h^{([\nu])}(t)\, dt, \tag{17}$$

for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

In particular, when $f \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, we have that

$$\left(D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{z}^{g(x_0)} (t - z)^{-\alpha} \left(f \circ g^{-1}\right)^{([\nu])}(t)\, dt, \tag{18}$$

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for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

We get that

$$\left(D^n_{g(x_0)-}\left(f\circ g^{-1}\right)\right)(z) = (-1)^n \left(f\circ g^{-1}\right)^{(n)}(z) \tag{19}$$

and $\left(D^0_{g(x_0)-}\left(f\circ g^{-1}\right)\right)(z) = \left(f\circ g^{-1}\right)(z)$, all $z \in [g(a), g(b)]$, see [4].

By [4], we have for $f\circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, where $x_0 \in [a,b]$ is fixed, the following right generalized $g$-fractional, of Canavati type, $X$-valued Taylor's formula:

**Theorem 5** *Let $f\circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, where $x_0 \in [a,b]$ is fixed.*

*(i) If $\nu \geq 1$, then*

$$f(x) - f(x_0) = \sum_{k=1}^{[\nu]-1} \frac{\left(f\circ g^{-1}\right)^{(k)}(g(x_0))}{k!}(g(x) - g(x_0))^k +$$

$$\frac{1}{\Gamma(\nu)}\int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1}\left(D^{\nu}_{g(x_0)-}\left(f\circ g^{-1}\right)\right)(t)\,dt, \tag{20}$$

*for all $a \leq x \leq x_0$,*

*(ii) If $0 < \nu < 1$, we get*

$$f(x) = \frac{1}{\Gamma(\nu)}\int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1}\left(D^{\nu}_{g(x_0)-}\left(f\circ g^{-1}\right)\right)(t)\,dt, \tag{21}$$

*all $a \leq x \leq x_0$.*

III) Denote by

$$D^{m\nu}_{g(x_0)} = D^{\nu}_{g(x_0)} D^{\nu}_{g(x_0)} ... D^{\nu}_{g(x_0)} \quad (m\text{-times}),\ m \in \mathbb{N}. \tag{22}$$

We mention the following modified and generalized left $X$-valued fractional Taylor's formula of Canavati type:

**Theorem 6** *Let $f \in C^1([a,b], X)$, $g \in C^1([a,b])$, strictly increasing: $g^{-1} \in C^1([g(a), g(b)])$. Assume that $\left(D^{i\nu}_{g(x_0)}\left(f\circ g^{-1}\right)\right) \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, $0 < \nu < 1$, $x_0 \in [a,b]$, for $i = 0, 1, ..., m$. Then*

$$f(x) = \frac{1}{\Gamma((m+1)\nu)}\int_{g(x_0)}^{g(x)} (g(x) - z)^{(m+1)\nu-1}\left(D^{(m+1)\nu}_{g(x_0)}\left(f\circ g^{-1}\right)\right)(z)\,dz, \tag{23}$$

*all $x_0 \leq x \leq b$.*

IV) Denote by

$$D^{m\nu}_{g(x_0)-} = D^{\nu}_{g(x_0)-} D^{\nu}_{g(x_0)-} ... D^{\nu}_{g(x_0)-} \quad (m \text{ times}),\ m \in \mathbb{N}. \tag{24}$$

We mention the following modified and generalized right $X$-valued fractional Taylor's formula of Canavati type:

**Theorem 7** *Let $f \in C^1([a,b], X)$, $g \in C^1([a,b])$, strictly increasing: $g^{-1} \in C^1([g(a), g(b)])$. Assume that $\left(D^{i\nu}_{g(x_0)-}(f \circ g^{-1})\right) \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, $0 < \nu < 1$, $x_0 \in [a,b]$, for all $i = 0, 1, ..., m$. Then*

$$f(x) = \frac{1}{\Gamma((m+1)\nu)} \int_{g(x)}^{g(x_0)} (z - g(x))^{(m+1)\nu - 1} \left(D^{(m+1)\nu}_{g(x_0)-}(f \circ g^{-1})\right)(z)\, dz, \tag{25}$$

*all $a \le x \le x_0 \le b$.*

# 3 Basic Banach Algebras background

All here come from [17].

We need

**Definition 8** *([17], p. 245) A complex algebra is a vector space $A$ over the complex filed $\mathbb{C}$ in which a multiplication is defined that satisfies*

$$x(yz) = (xy)z, \tag{26}$$

$$(x+y)z = xz + yz, \quad x(y+z) = xy + xz, \tag{27}$$

*and*

$$\alpha(xy) = (\alpha x)y = x(\alpha y), \tag{28}$$

*for all $x, y$ and $z$ in $A$ and for all scalars $\alpha$.*

*Additionally if $A$ is a Banach space with respect to a norm that satisfies the multiplicative inequality*

$$\|xy\| \le \|x\| \|y\| \quad (x \in A,\ y \in A) \tag{29}$$

*and if $A$ contains a unit element $e$ such that*

$$xe = ex = x \quad (x \in A) \tag{30}$$

*and*

$$\|e\| = 1, \tag{31}$$

*then $A$ is called a Banach algebra.*

*$A$ is commutative iff $xy = yx$ for all $x, y \in A$.*

We make

**Remark 9** *Commutativity of $A$ is explicited stated when needed.*

*There exists at most one $e \in A$ that satisfies (30).*

*Inequality (29) makes multiplication to be continuous, more precisely left and right continuous, see [17], p. 246.*

*Multiplication in $A$ is not necessarily the numerical multiplication, it is something more general and it is defined abstractly, that is for $x, y \in A$ we have $xy \in A$, e.g. composition or convolution, etc.*

*For nice examples about Banach algebras see [17], p. 247-248, § 10.3.*

We also make

**Remark 10** *Next we mention about integration of $A$-valued functions, see [17], p. 259, § 10.22:*

*If $A$ is a Banach algebra and $f$ is a continuous $A$-valued function on some compact Hausdorff space $Q$ on which a complex Borel measure $\mu$ is defined, then $\int f d\mu$ exists and has all the properties that were discussed in Chapter 3 of [17], simply because $A$ is a Banach space. However, an additional property can be added to these, namely: If $x \in A$, then*

$$x \int_Q f \, d\mu = \int_Q x f(p) \, d\mu(p) \tag{32}$$

*and*

$$\left( \int_Q f \, d\mu \right) x = \int_Q f(p) \, x \, d\mu(p) . \tag{33}$$

*The Bochner integrals we will involve in our article follow (32) and (33). Also, let $f \in C([a,b], X)$, where $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ is a Banach space. By [5], p. 3, $f$ is Bochner integrable.*

## 4 $p$-Schatten norms background

In this advanced section all come from [11].

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on $H$. If $\{e_i\}_{i \in I}$ an orthonormal basis of $H$, we say that $A \in \mathcal{B}(H)$ is of trace class if

$$\|A\|_1 := \sum_{i \in I} \langle |A| \, e_i, e_i \rangle < \infty. \tag{34}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthornormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

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We define the trace of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$tr(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle, \tag{35}$$

where $\{e_i\}_{i \in I}$ an orthonormal basis of $H$. Note that this coincides with the usual definition of the trace if $H$ is finite-dimensional. We observe that the series (35) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

**Theorem 11** *We have:*

*(i) If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and*

$$tr(A^*) = \overline{tr(A)}; \tag{36}$$

*(ii) If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then $AT, TA \in \mathcal{B}_1(H)$ and*

$$tr(AT) = tr(TA) \quad \text{and} \quad |tr(AT)| \leq \|A\|_1 \|T\|; \tag{37}$$

*(iii) $tr(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|tr\| = 1$;*

*(iv) If $A, B \in \mathcal{B}_2(H)$ then $AB, BA \in \mathcal{B}_1(H)$ and $tr(AB) = tr(BA)$;*

*(v) $\mathcal{B}_{fin}(H)$ (finite rank operators) is a dense subspace of $\mathcal{B}_1(H)$.*

An operator $A \in \mathcal{B}(H)$ is said to belong to the von Neumann-Schatten class $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the $p$-Schatten norm is finite [19, p. 60-64]

$$\|A\|_p := [tr(|A|^p)]^{\frac{1}{p}} < \infty,$$

$|A|^p$ is an operator notation and not a power.

For $1 < p < q < \infty$ we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{38}$$

and

$$\|A\|_1 \geq \|A\|_p \geq \|A\|_q \geq \|A\|. \tag{39}$$

For $p \geq 1$ the functional $\|\cdot\|_p$ is a norm on the $*$-ideal $\mathcal{B}_p(H)$, which is a Banach algebra, and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [19, p. 60-64], for $p \geq 1$,

$$\|A\|_p = \|A^*\|_p, \ \ A \in \mathcal{B}_p(H) \tag{40}$$

$$\|AB\|_p \leq \|A\|_p \|B\|_p, \ \ A, B \in \mathcal{B}_p(H) \tag{41}$$

and

$$\|AB\|_p \leq \|A\|_p \|B\|, \ \ \|BA\|_p \leq \|B\| \|A\|_p, \ \ A \in \mathcal{B}_p(H), B \in \mathcal{B}(H). \tag{42}$$

This implies that

$$\|CAB\|_p \leq \|C\| \|A\|_p \|B\|, \quad A \in \mathcal{B}_p(H), \ B, C \in \mathcal{B}(H). \tag{43}$$

In terms of $p$-Schatten norm we have the Hölder inequality for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ :

$$(|tr(AB)| \leq) \|AB\|_1 \leq \|A\|_p \|B\|_q, \quad A \in \mathcal{B}_p(H), \ B \in \mathcal{B}_q(H). \tag{44}$$

For the theory of trace functionals and their applications the interested reader is referred to [18] and [19].

For some classical trace inequalities see [9], [10] and [14], which are continuations of the work of Bellman [7].

# 5 Main Results

We start with 1-Schatten norm weighted mixed generalized Canavati fractional Ostrowski type inequalities involving several functions taking values in the Banach algebra $\mathcal{B}_2(H) \subset \mathcal{B}(H)$:

**Theorem 12** *Let the $*$-ideal $\mathcal{B}_2(H)$, which $(\mathcal{B}_2(H), \|\cdot\|_2)$ is a Banach algebra; $x_0 \in [a,b] \subset \mathbb{R}$, $\nu \geq 1$, $n = [\nu]$; $f_i \in C^n([a,b], \mathcal{B}_2(H))$, $i = 1, ..., r \in \mathbb{N} - \{1\}$; $g \in C^1([a,b])$, strictly increasing such that $g^{-1} \in C^n([g(a), g(b)])$, with $(f_i \circ g^{-1})^{(k)}(g(x_0)) = 0$, $k = 1, ..., n-1$; $i = 1, ..., r$. Assume further that $f_i \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)], \mathcal{B}_2(H)) \cap C^{\nu}_{g(x_0)}([g(a), g(b)], \mathcal{B}_2(H))$, $i = 1, ..., r$.*

*Denote by*

$$K(f_1, ..., f_r)(x_0) :=$$
$$\sum_{i=1}^{r} \left[ \int_a^b \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x)\, dx - \left( \int_a^b \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) dx \right) f_i(x_0) \right]. \tag{45}$$

*Then*

$$\|K(f_1, ..., f_r)(x_0)\|_1 \leq \frac{1}{\Gamma(\nu+1)} \sum_{i=1}^{r} \Bigg[ \Bigg[ \left\| \left\| D^{\nu}_{g(x_0)-} (f_i \circ g^{-1}) \right\|_2 \right\|_{\infty, [g(a), g(x_0)]}$$
$$(g(x_0) - g(a))^{\nu} \left( \int_a^{x_0} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\|_2 \right) dx \right) \Bigg] + \tag{46}$$
$$\Bigg[ \left\| \left\| D^{\nu}_{g(x_0)} (f_i \circ g^{-1}) \right\|_2 \right\|_{\infty, [g(x_0), g(b)]} (g(b) - g(x_0))^{\nu} \left( \int_{x_0}^{b} \left( \prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\|_2 \right) dx \right) \Bigg] \Bigg].$$

**Proof.** Since $\left(f_i \circ g^{-1}\right)^{(k)} (g(x_0)) = 0$, $k = 1, ..., [\nu] - 1$; $i = 1, ..., r$; we have by Theorem 4 that

$$f_i(x) - f_i(x_0) = \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D_{g(x_0)}^{\nu} \left(f_i \circ g^{-1}\right)\right)(t)\, dt, \quad (47)$$

$\forall\, x \in [x_0, b]$,

and by Theorem 5 that

$$f_i(x) - f_i(x_0) = \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D_{g(x_0)-}^{\nu} \left(f_i \circ g^{-1}\right)\right)(t)\, dt, \quad (48)$$

$\forall\, x \in [a, x_0]$, for all $i = 1, ..., r$.

Left multiplying (47) and (48) with $\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)$ we get, respectively,

$$\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x) - \left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x_0) =$$

$$\frac{\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D_{g(x_0)}^{\nu} \left(f_i \circ g^{-1}\right)\right)(t)\, dt, \quad (49)$$

$\forall\, x \in [x_0, b]$,

and

$$\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x) - \left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x_0) = \quad (50)$$

$$\frac{\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D_{g(x_0)-}^{\nu} \left(f_i \circ g^{-1}\right)\right)(t)\, dt,$$

$\forall\, x \in [a, x_0]$, for all $i = 1, ..., r$.

Adding (49) and (50) as separate groups, we obtain

$$\sum_{i=1}^{r} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x) - \sum_{i=1}^{r} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x_0) =$$

$$\frac{1}{\Gamma(\nu)} \sum_{i=1}^{r} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D_{g(x_0)}^{\nu} \left(f_i \circ g^{-1}\right)\right)(t)\, dt, \quad (51)$$

$\forall\ x \in [x_0, b]$,

and

$$\sum_{i=1}^{r}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x) - \sum_{i=1}^{r}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x_0) =$$

$$\frac{1}{\Gamma(\nu)}\sum_{i=1}^{r}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\int_{g(x)}^{g(x_0)} (t-g(x))^{\nu-1}\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt, \quad (52)$$

$\forall\ x \in [a, x_0]$.

Next, we integrate (51) and (52) with respect to $x \in [a, b]$. We have

$$\sum_{i=1}^{r}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x)\,dx - \sum_{i=1}^{r}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)dx\right) f_i(x_0) =$$

$$\frac{1}{\Gamma(\nu)}\sum_{i=1}^{r}\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)} (g(x)-t)^{\nu-1}\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt\right)dx\right], \quad (53)$$

and

$$\sum_{i=1}^{r}\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x)\,dx - \sum_{i=1}^{r}\left(\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)dx\right) f_i(x_0) =$$

$$\frac{1}{\Gamma(\nu)}\sum_{i=1}^{r}\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x)}^{g(x_0)} (t-g(x))^{\nu-1}\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt\right)dx\right], \quad (54)$$

Finally, adding (53) and (54) we obtain the useful identity

$$K(f_1, ..., f_r)(x_0) :=$$

$$\sum_{i=1}^{r}\left[\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x)\,dx - \left(\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)dx\right) f_i(x_0)\right] =$$

$$\frac{1}{\Gamma(\nu)}\sum_{i=1}^{r}\left[\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x)}^{g(x_0)} (t-g(x))^{\nu-1}\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt\right)dx\right]\right.$$

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$$+\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-t)^{\nu-1}\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt\right)dx\right]\Bigg]. \tag{55}$$

Therefore, we get that

$$\|K(f_1,...,f_r)(x_0)\|_1 =$$

$$\left\|\sum_{i=1}^{r}\left[\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)f_i(x)\,dx-\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)dx\right)f_i(x_0)\right]\right\|_1 \leq \frac{1}{\Gamma(\nu)}$$

$$\sum_{i=1}^{r}\left[\left\|\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x)}^{g(x_0)}(t-g(x))^{\nu-1}\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt\right)dx\right]\right\|_1\right. \tag{56}$$

$$\left.+\left\|\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-t)^{\nu-1}\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt\right)dx\right]\right\|_1\right] \leq$$

$$\frac{1}{\Gamma(\nu)}\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left\|\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x)}^{g(x_0)}(t-g(x))^{\nu-1}\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt\right)\right\|_1 dx\right]\right. \tag{57}$$

$$\left.+\left[\int_{x_0}^{b}\left\|\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-t)^{\nu-1}\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt\right)\right\|_1 dx\right]\right] \leq$$

$$\frac{1}{\Gamma(\nu)}\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_2\right)\left(\int_{g(x)}^{g(x_0)}(t-g(x))^{\nu-1}\left\|\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\right\|_2 dt\right)dx\right]\right. \tag{58}$$

$$\left.+\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_2\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-t)^{\nu-1}\left\|\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\right\|_2 dt\right)dx\right]\right] =: (*).$$

Hence it holds

$$\|K(f_1,...,f_r)(x_0)\|_1 \leq (*). \tag{59}$$

We have that

$$(*) \leq \frac{1}{\Gamma(\nu+1)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)\right\|_2\right\|_{\infty,[g(a),g(x_0)]}\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)(g(x_0)-g(x))^{\nu}dx\right]\right.$$

$$\left.+\left[\left\|\left\|\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)\right\|_2\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)(g(x)-g(x_0))^{\nu}dx\right]\right]\leq \tag{60}$$

$$\frac{1}{\Gamma(\nu+1)}\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)\right\|_2\right\|_{\infty,[g(a),g(x_0)]}\right.\right.$$

$$\left.(g(x_0)-g(a))^{\nu}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)dx\right)\right]+ \tag{61}$$

$$\left.\left[\left\|\left\|\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)\right\|_2\right\|_{\infty,[g(x_0),g(b)]}(g(b)-g(x_0))^{\nu}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)dx\right)\right]\right],$$

proving (46). ■

Next comes an $L_1$ estimate.

**Theorem 13** *All as in Theorem 12. Then*

$$\||K(f_1,...,f_r)(x_0)\|_1\leq\frac{1}{\Gamma(\nu)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)\right\|_2\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)(g(x_0)-g(x))^{\nu-1}dx\right]\right. \tag{62}$$

$$\left.+\left[\left\|\left\|\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)\right\|_2\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)(g(x)-g(x_0))^{\nu-1}dx\right]\right].$$

**Proof.** We observe that (by (58), (59))

$$(*)\leq\frac{1}{\Gamma(\nu)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)\right\|_2\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)(g(x_0)-g(x))^{\nu-1}dx\right]\right. \tag{63}$$

$$+\left[\left\|\left\|\left(D_{g(x_0)}^{\nu}\left(f_i \circ g^{-1}\right)\right)\right\|_2\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|_2\right)(g(x)-g(x_0))^{\nu-1}dx\right]\Bigg],$$

proving (62). ■

An $L_p$ estimate follows.

**Theorem 14** *All as in Theorem 12. Let now $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\left\|K(f_1,...,f_r)(x_0)\right\|_1 \leq \frac{1}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{g(x_0)-}^{\nu}\left(f_i \circ g^{-1}\right)\right)\right\|_2\right\|_{q,[g(a),g(x_0)]}\left(\int_a^{x_0}(g(x_0)-g(x))^{\nu-\frac{1}{q}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|_2\right)dx\right)\right]\right.$$

$$\left.+\left[\left\|\left\|\left(D_{g(x_0)}^{\nu}\left(f_i \circ g^{-1}\right)\right)\right\|_2\right\|_{q,[g(x_0),g(b)]}\left(\int_{x_0}^{b}(g(x)-g(x_0))^{\nu-\frac{1}{q}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|_2\right)dx\right)\right]\right]. \tag{64}$$

**Proof.** We have that (by (58), (59))

$$(*) \leq \frac{1}{\Gamma(\nu)}\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|_2\right)\left(\int_{g(x)}^{g(x_0)}(t-g(x))^{p(\nu-1)}dt\right)^{\frac{1}{p}}\right.\right.$$

$$\left.\left(\int_{g(x)}^{g(x_0)}\left\|\left(D_{g(x_0)-}^{\nu}\left(f_i \circ g^{-1}\right)\right)(t)\right\|_2^q dt\right)^{\frac{1}{q}}dx\right]+$$

$$\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|_2\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-t)^{p(\nu-1)}dt\right)^{\frac{1}{p}}\right.$$

$$\left.\left.\left(\int_{g(x)}^{g(x_0)}\left\|\left(D_{g(x_0)}^{\nu}\left(f_i \circ g^{-1}\right)\right)(t)\right\|_2^q dt\right)^{\frac{1}{q}}dx\right]\right] \leq \tag{65}$$

$$\frac{1}{\Gamma(\nu)}\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|_2\right)\frac{(g(x_0)-g(x))^{\nu-1+\frac{1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}}\left\|\left\|\left(D_{g(x_0)-}^{\nu}\left(f_i \circ g^{-1}\right)\right)\right\|_2\right\|_{q,[g(a),g(x_0)]}dx\right]\right.$$

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$$+\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_2\right)\frac{(g(x)-g(x_0))^{\nu-1+\frac{1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}}\left\|\left\|\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)(z)\right\|_2\right\|_{q,[g(x_0),g(b)]}dx\right]\Bigg]$$

$$=\frac{1}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)}$$

$$\sum_{i=1}^{r}\left[\left\|\left\|\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)\right\|_2\right\|_{q,[g(a),g(x_0)]}\left(\int_a^{x_0}(g(x_0)-g(x))^{\nu-\frac{1}{q}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_2\right)dx\right)\right. \tag{66}$$

$$\left.+\left\|\left\|\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)\right\|_2\right\|_{q,[g(x_0),g(b)]}\left(\int_{x_0}^{b}(g(x)-g(x_0))^{\nu-\frac{1}{q}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_2\right)dx\right)\right],$$

proving (64). ■

We continue with $\gamma$-Schatten norm related Ostrowski fractional inequalities:

**Theorem 15** *Let $\gamma\geq 1$, the $*$-ideal $\mathcal{B}_\gamma(H)$, which $\left(\mathcal{B}_\gamma(H),\|\cdot\|_\gamma\right)$ is a Banach algebra; $x_0\in[a,b]\subset\mathbb{R}$, $\nu\geq 1$, $n=[\nu]$; $f_i\in C^n([a,b],\mathcal{B}_\gamma(H))$, $i=1,...,r\in\mathbb{N}-\{1\}$; $g\in C^1([a,b])$, strictly increasing such that $g^{-1}\in C^n([g(a),g(b)])$, with $\left(f_i\circ g^{-1}\right)^{(k)}(g(x_0))=0$, $k=1,...,n-1$; $i=1,...,r$. Assume further that $f_i\circ g^{-1}\in C_{g(x_0)-}^{\nu}([g(a),g(b)],\mathcal{B}_\gamma(H))\cap C_{g(x_0)}^{\nu}([g(a),g(b)],\mathcal{B}_\gamma(H))$, $i=1,...,r$.*

*Here $K(f_1,...,f_r)(x_0)$ is as in (45). Then*

$$\|K(f_1,...,f_r)(x_0)\|_\gamma\leq\frac{1}{\Gamma(\nu+1)}\sum_{i=1}^{r}\left[\left[\left\|\left\|D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right\|_\gamma\right\|_{\infty,[g(a),g(x_0)]}\right.\right.$$

$$\left.(g(x_0)-g(a))^{\nu}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_\gamma\right)dx\right)\right]+ \tag{67}$$

$$\left.\left[\left\|\left\|D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right\|_\gamma\right\|_{\infty,[g(x_0),g(b)]}(g(b)-g(x_0))^{\nu}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_\gamma\right)dx\right)\right]\right].$$

**Proof.** As similar to Theorem 12 is omitted. Use of (41). ■

An $L_1$ estimate follows:

**Theorem 16** *All as in Theorem 15. Then*

$$\left\| \left| K\left(f_{1},...,f_{r}\right)\left(x_{0}\right) \right\| \right\|_{\gamma} \leq \frac{1}{\Gamma\left(\nu\right)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{g(x_{0})-}^{\nu}\left(f_{i}\circ g^{-1}\right)\right)\right\|_{\gamma}\right\|_{L_{1}([g(a),g(x_{0})])}\int_{a}^{x_{0}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_{j}\left(x\right)\right\|_{\gamma}\right)\left(g\left(x_{0}\right)-g\left(x\right)\right)^{\nu-1}dx\right]\right.$$

$$\left.+\left[\left\|\left\|\left(D_{g(x_{0})}^{\nu}\left(f_{i}\circ g^{-1}\right)\right)\right\|_{\gamma}\right\|_{L_{1}([g(x_{0}),g(b)])}\int_{x_{0}}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_{j}\left(x\right)\right\|_{\gamma}\right)\left(g\left(x\right)-g\left(x_{0}\right)\right)^{\nu-1}dx\right]\right]. \tag{68}$$

**Proof.** As similar to Theorem 13 is omitted. ■

An $L_p$ estimate follows.

**Theorem 17** *All as in Theorem 15. Let now* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\left\|K\left(f_{1},...,f_{r}\right)\left(x_{0}\right)\right\|_{\gamma} \leq \frac{1}{\left(p\left(\nu-1\right)+1\right)^{\frac{1}{p}}\Gamma\left(\nu\right)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{g(x_{0})-}^{\nu}\left(f_{i}\circ g^{-1}\right)\right)\right\|_{\gamma}\right\|_{q,[g(a),g(x_{0})]}\left(\int_{a}^{x_{0}}\left(g\left(x_{0}\right)-g\left(x\right)\right)^{\nu-\frac{1}{q}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_{j}\left(x\right)\right\|_{\gamma}\right)dx\right)\right]\right.$$

$$\left.+\left[\left\|\left\|\left(D_{g(x_{0})}^{\nu}\left(f_{i}\circ g^{-1}\right)\right)\right\|_{\gamma}\right\|_{q,[g(x_{0}),g(b)]}\left(\int_{x_{0}}^{b}\left(g\left(x\right)-g\left(x_{0}\right)\right)^{\nu-\frac{1}{q}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_{j}\left(x\right)\right\|_{\gamma}\right)dx\right)\right]\right]. \tag{69}$$

**Proof.** As similar to Theorem 14 is omitted. ■

When $r=2$ we derive the following $p$-Schatten norm operator related Ostrowski type Canavati fractional inequalities.

**Theorem 18** *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, *and let the* $*$*-ideals* $\mathcal{B}_p(H)$, $\mathcal{B}_q(H)$, *for which* $\left(\mathcal{B}_p(H),\|\cdot\|_p\right)$, $\left(\mathcal{B}_q(H),\|\cdot\|_q\right)$ *are Banach algebras;* $x_0\in[a,b]\subset\mathbb{R}$, $\alpha\geq 1$, $n=[\alpha]$; $A_1\in C^n([a,b],\mathcal{B}_p(H))$, $A_2\in C^n([a,b],\mathcal{B}_q(H))$; $g\in C^1([a,b])$, *strictly increasing, such that* $g^{-1}\in C^n([g(a),g(b)])$, *with* $\left(A_i\circ g^{-1}\right)^{(k)}(g(x_0))=0$, $k=1,...,n-1$; $i=1,2$. *Assume further that* $A_1\circ g^{-1}\in C^{\alpha}_{g(x_0)-}([g(a),g(b)],\mathcal{B}_p(H))\cap C^{\alpha}_{g(x_0)}([g(a),g(b)],\mathcal{B}_p(H))$, *and* $A_2\circ g^{-1}\in C^{\alpha}_{g(x_0)-}([g(a),g(b)],\mathcal{B}_q(H))\cap C^{\alpha}_{g(x_0)}([g(a),g(b)],\mathcal{B}_q(H))$. *Then*

*1) it holds*

$$\Phi\left(A_1, A_2\right)\left(x_0\right) := \int_a^b A_2\left(x\right) A_1\left(x\right) + \int_a^b A_1\left(x\right) A_2\left(x\right) dx -$$

$$\left(\int_a^b A_2\left(x\right) dx\right) A_1\left(x_0\right) - \left(\int_a^b A_1\left(x\right) dx\right) A_2\left(x_0\right) =$$

$$\frac{1}{\Gamma\left(\alpha\right)}\left\{\left[\int_a^{x_0} A_2\left(x\right)\left(\int_{g(x)}^{g(x_0)}\left(z-g\left(x\right)\right)^{\alpha-1}\left(D_{g(x_0)-}^{\alpha}\left(A_1\circ g^{-1}\right)\right)\left(z\right) dz\right) dx\right] + \right.$$

$$\left[\int_{x_0}^b A_2\left(x\right)\left(\int_{g(x_0)}^{g(x)}\left(g\left(x\right)-z\right)^{\alpha-1}\left(D_{g(x_0)}^{\alpha}\left(A_1\circ g^{-1}\right)\right)\left(z\right) dz\right) dx\right] + \tag{70}$$

$$\left[\int_a^{x_0} A_1\left(x\right)\left(\int_{g(x)}^{g(x_0)}\left(z-g\left(x\right)\right)^{\alpha-1}\left(D_{g(x_0)-}^{\alpha}\left(A_2\circ g^{-1}\right)\right)\left(z\right) dz\right) dx\right] +$$

$$\left.\left[\int_{x_0}^b A_1\left(x\right)\left(\int_{g(x_0)}^{g(x)}\left(g\left(x\right)-z\right)^{\alpha-1}\left(D_{g(x_0)}^{\alpha}\left(A_2\circ g^{-1}\right)\right)\left(z\right) dz\right) dx\right]\right\},$$

*2) for $\gamma, \delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1$, we have that*

$$\left\|\Phi\left(A_1, A_2\right)\left(x_0\right)\right\|_1 \leq \frac{1}{\Gamma\left(\alpha\right)\left(\gamma\left(\alpha-1\right)+1\right)^{\frac{1}{\gamma}}}$$

$$\left\{\left[\left\|\left\|D_{g(x_0)-}^{\alpha}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{\delta,[g(a),g(x_0)]}\int_a^{x_0}\left\|A_2\left(x\right)\right\|_q\left(g\left(x_0\right)-g\left(x\right)\right)^{\alpha-\frac{1}{\delta}} dx\right] + \right.$$

$$\left[\left\|\left\|D_{g(x_0)}^{\alpha}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{\delta,[g(x_0),g(b)]}\int_{x_0}^b\left\|A_2\left(x\right)\right\|_q\left(g\left(x\right)-g\left(x_0\right)\right)^{\alpha-\frac{1}{\delta}} dx\right] + \tag{71}$$

$$\left[\left\|\left\|D_{g(x_0)-}^{\alpha}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{\delta,[g(a),g(x_0)]}\int_a^{x_0}\left\|A_1\left(x\right)\right\|_p\left(g\left(x_0\right)-g\left(x\right)\right)^{\alpha-\frac{1}{\delta}} dx\right] +$$

$$\left.\left[\left\|\left\|D_{g(x_0)}^{\alpha}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{\delta,[g(x_0),g(b)]}\int_{x_0}^b\left\|A_1\left(x\right)\right\|_p\left(g\left(x\right)-g\left(x_0\right)\right)^{\alpha-\frac{1}{\delta}} dx\right]\right\},$$

*3) we also obtain*

$$\left\|\Phi\left(A_1, A_2\right)\left(x_0\right)\right\|_1 \leq \frac{1}{\Gamma\left(\alpha\right)}$$

$$\left\{\left[\left\|\left\|D_{g(x_0)-}^{\alpha}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\left\|A_2\left(x\right)\right\|_q\left(g\left(x_0\right)-g\left(x\right)\right)^{\alpha-1} dx\right] + \right.$$

$$\left[\left\|\left\|D_{g(x_0)}^{\alpha}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\left\|A_2(x)\right\|_q\left(g(x)-g(x_0)\right)^{\alpha-1}dx\right]+$$
$$\left[\left\|\left\|D_{g(x_0)-}^{\alpha}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{L_1([g(a),g(x_0)])}\int_{a}^{x_0}\left\|A_1(x)\right\|_p\left(g(x_0)-g(x)\right)^{\alpha-1}dx\right]+$$
$$\left.\left[\left\|\left\|D_{g(x_0)}^{\alpha}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\left\|A_1(x)\right\|_p\left(g(x)-g(x_0)\right)^{\alpha-1}dx\right]\right\}, \tag{72}$$

*and*

*4)*

$$\left\|\Phi(A_1,A_2)(x_0)\right\|_1\leq\frac{1}{\Gamma(\alpha+1)}$$
$$\left\{\left[\left\|\left\|D_{g(x_0)-}^{\alpha}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{\infty,[g(a),g(x_0)]}\int_{a}^{x_0}\left\|A_2(x)\right\|_q\left(g(x_0)-g(x)\right)^{\alpha}dx\right]+\right.$$
$$\left[\left\|\left\|D_{g(x_0)}^{\alpha}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\left\|A_2(x)\right\|_q\left(g(x)-g(x_0)\right)^{\alpha}dx\right]+$$
$$\left[\left\|\left\|D_{g(x_0)-}^{\alpha}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{\infty,[g(a),g(x_0)]}\int_{a}^{x_0}\left\|A_1(x)\right\|_p\left(g(x_0)-g(x)\right)^{\alpha}dx\right]+$$
$$\left.\left[\left\|\left\|D_{g(x_0)}^{\alpha}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\left\|A_1(x)\right\|_p\left(g(x)-g(x_0)\right)^{\alpha}dx\right]\right\}. \tag{73}$$

**Proof.** Here we have that (acting as in the proof of Theorem 12 for $r=2$)

$$\Phi(A_1,A_2)(x_0):=\int_a^b A_2(x)A_1(x)+\int_a^b A_1(x)A_2(x)\,dx-$$
$$\left(\int_a^b A_2(x)\,dx\right)A_1(x_0)-\left(\int_a^b A_1(x)\,dx\right)A_2(x_0)\overset{(55)}{=}$$
$$\frac{1}{\Gamma(\alpha)}\left\{\left[\int_a^{x_0}A_2(x)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{\alpha-1}\left(D_{g(x_0)-}^{\alpha}\left(A_1\circ g^{-1}\right)\right)(z)\,dz\right)dx\right]+\right.$$
$$\left[\int_{x_0}^{b}A_2(x)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{\alpha-1}\left(D_{g(x_0)}^{\alpha}\left(A_1\circ g^{-1}\right)\right)(z)\,dz\right)dx\right]+$$
$$\left[\int_a^{x_0}A_1(x)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{\alpha-1}\left(D_{g(x_0)-}^{\alpha}\left(A_2\circ g^{-1}\right)\right)(z)\,dz\right)dx\right]+$$
$$\left.\left[\int_{x_0}^{b}A_1(x)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{\alpha-1}\left(D_{g(x_0)}^{\alpha}\left(A_2\circ g^{-1}\right)\right)(z)\,dz\right)dx\right]\right\}. \tag{74}$$

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Therefore it holds by taking the 1-Schatten norm that

$$\|\Phi(A_1, A_2)(x_0)\|_1 = \left\| \int_a^b A_2(x) A_1(x) + \int_a^b A_1(x) A_2(x) dx - \right.$$

$$\left. \left( \int_a^b A_2(x) dx \right) A_1(x_0) - \left( \int_a^b A_1(x) dx \right) A_2(x_0) \right\|_1 \leq$$

$$\frac{1}{\Gamma(\alpha)} \left\{ \left[ \left\| \int_a^{x_0} A_2(x) \left( \int_{g(x)}^{g(x_0)} (z - g(x))^{\alpha-1} \left( D_{g(x_0)-}^{\alpha} \left( A_1 \circ g^{-1} \right) \right)(z) dz \right) dx \right\|_1 \right] + \right.$$

$$\left[ \left\| \int_{x_0}^b A_2(x) \left( \int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha-1} \left( D_{g(x_0)}^{\alpha} \left( A_1 \circ g^{-1} \right) \right)(z) dz \right) dx \right\|_1 \right] +$$

$$\left[ \left\| \int_a^{x_0} A_1(x) \left( \int_{g(x)}^{g(x_0)} (z - g(x))^{\alpha-1} \left( D_{g(x_0)-}^{\alpha} \left( A_2 \circ g^{-1} \right) \right)(z) dz \right) dx \right\|_1 \right] +$$

$$\left. \left[ \left\| \int_{x_0}^b A_1(x) \left( \int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha-1} \left( D_{g(x_0)}^{\alpha} \left( A_2 \circ g^{-1} \right) \right)(z) dz \right) dx \right\|_1 \right] \right\} \leq \tag{75}$$

$$\frac{1}{\Gamma(\alpha)} \left\{ \left[ \int_a^{x_0} \left\| A_2(x) \left( \int_{g(x)}^{g(x_0)} (z - g(x))^{\alpha-1} \left( D_{g(x_0)-}^{\alpha} \left( A_1 \circ g^{-1} \right) \right)(z) dz \right) \right\|_1 dx \right] + \right.$$

$$\left[ \int_{x_0}^b \left\| A_2(x) \left( \int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha-1} \left( D_{g(x_0)}^{\alpha} \left( A_1 \circ g^{-1} \right) \right)(z) dz \right) \right\|_1 dx \right] +$$

$$\left[ \int_a^{x_0} \left\| A_1(x) \left( \int_{g(x)}^{g(x_0)} (z - g(x))^{\alpha-1} \left( D_{g(x_0)-}^{\alpha} \left( A_2 \circ g^{-1} \right) \right)(z) dz \right) \right\|_1 dx \right] +$$

$$\left. \left[ \int_{x_0}^b \left\| A_1(x) \left( \int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha-1} \left( D_{g(x_0)}^{\alpha} \left( A_2 \circ g^{-1} \right) \right)(z) dz \right) \right\|_1 dx \right] \right\} \leq \tag{76}$$

(by using the $p$-Schatten norm and Hölder's type inequality (44) for $p, q > 1$ : $\frac{1}{p} + \frac{1}{q} = 1$)

$$\frac{1}{\Gamma(\alpha)} \left\{ \left[ \int_a^{x_0} \|A_2(x)\|_q \left\| \left( \int_{g(x)}^{g(x_0)} (z - g(x))^{\alpha-1} \left( D_{g(x_0)-}^{\alpha} \left( A_1 \circ g^{-1} \right) \right)(z) dz \right) \right\|_p dx \right] + \right.$$

$$\left[ \int_{x_0}^b \|A_2(x)\|_q \left\| \left( \int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha-1} \left( D_{g(x_0)}^{\alpha} \left( A_1 \circ g^{-1} \right) \right)(z) dz \right) \right\|_p dx \right] +$$

$$\left[\int_a^{x_0} \|A_1(x)\|_p \left\| \left( \int_{g(x)}^{g(x_0)} (z - g(x))^{\alpha-1} \left( D^{\alpha}_{g(x_0)-} \left( A_2 \circ g^{-1} \right) \right)(z)\, dz \right) \right\|_q dx \right] +$$

$$\left[\int_{x_0}^{b} \|A_1(x)\|_p \left\| \left( \int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha-1} \left( D^{\alpha}_{g(x_0)} \left( A_2 \circ g^{-1} \right) \right)(z)\, dz \right) \right\|_q dx \right] \Bigg\} \leq \tag{77}$$

$$\frac{1}{\Gamma(\alpha)} \Bigg\{ \left[ \int_a^{x_0} \|A_2(x)\|_q \left( \int_{g(x)}^{g(x_0)} (z - g(x))^{\alpha-1} \left\| \left( D^{\alpha}_{g(x_0)-} \left( A_1 \circ g^{-1} \right) \right)(z) \right\|_p dz \right) dx \right] +$$

$$\left[ \int_{x_0}^{b} \|A_2(x)\|_q \left( \int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha-1} \left\| \left( D^{\alpha}_{g(x_0)} \left( A_1 \circ g^{-1} \right) \right)(z) \right\|_p dz \right) dx \right] +$$

$$\left[ \int_a^{x_0} \|A_1(x)\|_p \left( \int_{g(x)}^{g(x_0)} (z - g(x))^{\alpha-1} \left\| \left( D^{\alpha}_{g(x_0)-} \left( A_2 \circ g^{-1} \right) \right)(z) \right\|_q dz \right) dx \right] +$$

$$\left[ \int_{x_0}^{b} \|A_1(x)\|_p \left( \int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha-1} \left\| \left( D^{\alpha}_{g(x_0)} \left( A_2 \circ g^{-1} \right) \right)(z) \right\|_q dz \right) dx \right] \Bigg\}. \tag{78}$$

We have proved, so far, that

$$\|\Phi(A_1, A_2)(x_0)\|_1 \leq$$

$$\frac{1}{\Gamma(\alpha)} \Bigg\{ \left[ \int_a^{x_0} \|A_2(x)\|_q \left( \int_{g(x)}^{g(x_0)} (z - g(x))^{\alpha-1} \left\| \left( D^{\alpha}_{g(x_0)-} \left( A_1 \circ g^{-1} \right) \right)(z) \right\|_p dz \right) dx \right] +$$

$$\left[ \int_{x_0}^{b} \|A_2(x)\|_q \left( \int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha-1} \left\| \left( D^{\alpha}_{g(x_0)} \left( A_1 \circ g^{-1} \right) \right)(z) \right\|_p dz \right) dx \right] +$$

$$\left[ \int_a^{x_0} \|A_1(x)\|_p \left( \int_{g(x)}^{g(x_0)} (z - g(x))^{\alpha-1} \left\| \left( D^{\alpha}_{g(x_0)-} \left( A_2 \circ g^{-1} \right) \right)(z) \right\|_q dz \right) dx \right] +$$

$$\left[ \int_{x_0}^{b} \|A_1(x)\|_p \left( \int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha-1} \left\| \left( D^{\alpha}_{g(x_0)} \left( A_2 \circ g^{-1} \right) \right)(z) \right\|_q dz \right) dx \right] \Bigg\} =: (\lambda). \tag{79}$$

Let now $\gamma, \delta > 1$ such that $\frac{1}{\gamma} + \frac{1}{\delta} = 1$, and we apply the usual Hölder's inequality in (79). Then we have that

$$\|\Phi(A_1, A_2)(x_0)\|_1 \leq (\lambda) \leq \frac{1}{\Gamma(\alpha)(\gamma(\alpha-1)+1)^{\frac{1}{\gamma}}}$$

$$\Bigg\{ \left[ \int_a^{x_0} \|A_2(x)\|_q (g(x_0) - g(x))^{\frac{\gamma(\alpha-1)+1}{\gamma}} \left( \int_{g(x)}^{g(x_0)} \left\| \left( D^{\alpha}_{g(x_0)-} \left( A_1 \circ g^{-1} \right) \right)(z) \right\|_p^{\delta} dz \right)^{\frac{1}{\delta}} dx \right] +$$

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$$\left[\int_{x_0}^{b}\|A_2(x)\|_q (g(x)-g(x_0))^{\frac{\gamma(\alpha-1)+1}{\gamma}}\left(\int_{g(x_0)}^{g(x)}\left\|\left(D_{g(x_0)}^{\alpha}\left(A_1\circ g^{-1}\right)\right)(z)\right\|_p^{\delta}dz\right)^{\frac{1}{\delta}}dx\right]+$$

$$\left[\int_{a}^{x_0}\|A_1(x)\|_p (g(x_0)-g(x))^{\frac{\gamma(\alpha-1)+1}{\gamma}}\left(\int_{g(x)}^{g(x_0)}\left\|\left(D_{g(x_0)-}^{\alpha}\left(A_2\circ g^{-1}\right)\right)(z)\right\|_q^{\delta}dz\right)^{\frac{1}{\delta}}dx\right]+ \tag{80}$$

$$\left[\int_{x_0}^{b}\|A_1(x)\|_p (g(x)-g(x_0))^{\frac{\gamma(\alpha-1)+1}{\gamma}}\left(\int_{g(x_0)}^{g(x)}\left\|\left(D_{g(x_0)}^{\alpha}\left(A_2\circ g^{-1}\right)\right)(z)\right\|_q^{\delta}dz\right)^{\frac{1}{\delta}}dx\right]\Bigg\}$$

$$\leq \frac{1}{\Gamma(\alpha)(\gamma(\alpha-1)+1)^{\frac{1}{\gamma}}}$$

$$\Bigg\{\left[\left\|\left\|D_{g(x_0)-}^{\alpha}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{\delta,[g(a),g(x_0)]}\int_a^{x_0}\|A_2(x)\|_q (g(x_0)-g(x))^{\alpha-\frac{1}{\delta}}dx\right]+$$

$$\left[\left\|\left\|D_{g(x_0)}^{\alpha}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{\delta,[g(x_0),g(b)]}\int_{x_0}^{b}\|A_2(x)\|_q (g(x)-g(x_0))^{\alpha-\frac{1}{\delta}}dx\right]+ \tag{81}$$

$$\left[\left\|\left\|D_{g(x_0)-}^{\alpha}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{\delta,[g(a),g(x_0)]}\int_a^{x_0}\|A_1(x)\|_p (g(x_0)-g(x))^{\alpha-\frac{1}{\delta}}dx\right]+$$

$$\left[\left\|\left\|D_{g(x_0)}^{\alpha}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{\delta,[g(x_0),g(b)]}\int_{x_0}^{b}\|A_1(x)\|_p (g(x)-g(x_0))^{\alpha-\frac{1}{\delta}}dx\right]\Bigg\},$$

proving (71).

We also obtain

$$\|\Phi(A_1,A_2)(x_0)\|_1 \leq (\lambda) \leq \frac{1}{\Gamma(\alpha)}$$

$$\Bigg\{\left[\left\|\left\|D_{g(x_0)-}^{\alpha}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\|A_2(x)\|_q (g(x_0)-g(x))^{\alpha-1}dx\right]+$$

$$\left[\left\|\left\|D_{g(x_0)}^{\alpha}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\|A_2(x)\|_q (g(x)-g(x_0))^{\alpha-1}dx\right]+$$

$$\left[\left\|\left\|D_{g(x_0)-}^{\alpha}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\|A_1(x)\|_p (g(x_0)-g(x))^{\alpha-1}dx\right]+ \tag{82}$$

$$\left[\left\|\left\|D_{g(x_0)}^{\alpha}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\|A_1(x)\|_p (g(x)-g(x_0))^{\alpha-1}dx\right]\Bigg\},$$

proving (72).

At last we derive

$$\left\|\Phi\left(A_1,A_2\right)\left(x_0\right)\right\|_1 \leq (\lambda) \leq \frac{1}{\Gamma(\alpha+1)}$$

$$\left\{\left[\left\|\left\|D^{\alpha}_{g(x_0)-}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{\infty,[g(a),g(x_0)]}\int_a^{x_0}\left\|A_2(x)\right\|_q\left(g(x_0)-g(x)\right)^{\alpha}dx\right]+\right.$$
$$\left[\left\|\left\|D^{\alpha}_{g(x_0)}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\left\|A_2(x)\right\|_q\left(g(x)-g(x_0)\right)^{\alpha}dx\right]+$$
$$\left[\left\|\left\|D^{\alpha}_{g(x_0)-}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{\infty,[g(a),g(x_0)]}\int_a^{x_0}\left\|A_1(x)\right\|_p\left(g(x_0)-g(x)\right)^{\alpha}dx\right]+$$
$$\left.\left[\left\|\left\|D^{\alpha}_{g(x_0)}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\left\|A_1(x)\right\|_p\left(g(x)-g(x_0)\right)^{\alpha}dx\right]\right\}, \tag{83}$$

proving (73).

The theorem is proved. ■

Next we present $p$-Schatten left and right generalized Canavati fractional Opial type inequalities:

**Theorem 19** *Let the $*$-ideal $\mathcal{B}_2(H)$, which $\left(\mathcal{B}_2(H),\|\cdot\|_2\right)$ is a Banach algebra; $x_0\in[a,b]\subset\mathbb{R}$, $\nu\geq 1$, $n=[\nu]$; $f\in C^n\left([a,b],\mathcal{B}_2(H)\right)$, $g\in C^1([a,b])$, strictly increasing such that $g^{-1}\in C^n\left([g(a),g(b)]\right)$, with $\left(f\circ g^{-1}\right)^{(k)}(g(x_0))=0$, $k=0,1,...,n-1$. Assume further that $f\circ g^{-1}\in C^{\nu}_{g(x_0)}\left([g(a),g(b)],\mathcal{B}_2(H)\right)$.*

*Let also $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\int_{g(x_0)}^{z}\left\|\left(\left(f\circ g^{-1}\right)(w)\right)\left(\left(D^{\nu}_{g(x_0)}\left(f\circ g^{-1}\right)\right)(w)\right)\right\|_1 dw\leq \tag{84}$$

$$\frac{2^{-\frac{1}{q}}\left(z-g(x_0)\right)^{\nu+\frac{1}{p}-\frac{1}{q}}}{\Gamma(\nu)\left[\left(p(\nu-1)+1\right)\left(p(\nu-1)+2\right)\right]^{\frac{1}{p}}}\left(\int_{g(x_0)}^{z}\left\|\left(D^{\nu}_{g(x_0)}\left(f\circ g^{-1}\right)\right)(w)\right\|_2^q dw\right)^{\frac{2}{q}},$$

*for all $g(x_0)\leq z\leq g(b)$.*

**Proof.** Very similar to the proof of Theorem 13 of [6]. Use of (44) for $p=q=2$. ■

A similar result comex next:

**Theorem 20** *Let $\gamma\geq 1$, the $*$-ideal $\mathcal{B}_{\gamma}(H)$, which $\left(\mathcal{B}_{\gamma}(H),\|\cdot\|_{\gamma}\right)$ is a Banach algebra; $x_0\in[a,b]\subset\mathbb{R}$, $\nu\geq 1$, $n=[\nu]$; $f\in C^n\left([a,b],\mathcal{B}_{\gamma}(H)\right)$, $g\in C^1([a,b])$, strictly increasing such that $g^{-1}\in C^n\left([g(a),g(b)]\right)$, with $\left(f\circ g^{-1}\right)^{(k)}(g(x_0))=0$, $k=0,1,...,n-1$. Assume further that $f\circ g^{-1}\in C^{\nu}_{g(x_0)}\left([g(a),g(b)],\mathcal{B}_{\gamma}(H)\right)$.*

*Let also* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\int_{g(x_0)}^{z} \left\| \left( \left( f \circ g^{-1} \right) (w) \right) \left( \left( D_{g(x_0)}^{\nu} \left( f \circ g^{-1} \right) \right) (w) \right) \right\|_{\gamma} dw \leq \tag{85}$$

$$\frac{2^{-\frac{1}{q}} \left( z - g\left( x_0 \right) \right)^{\nu + \frac{1}{p} - \frac{1}{q}}}{\Gamma\left( \nu \right) \left[ \left( p\left( \nu - 1 \right) + 1 \right) \left( p\left( \nu - 1 \right) + 2 \right) \right]^{\frac{1}{p}}} \left( \int_{g(x_0)}^{z} \left\| \left( D_{g(x_0)}^{\nu} \left( f \circ g^{-1} \right) \right) (w) \right\|_{\gamma}^{q} dw \right)^{\frac{2}{q}},$$

*for all* $g\left( x_0 \right) \leq z \leq g\left( b \right)$.

**Proof.** Very similar to the proof of Theorem 13 of [6]. Use of (41) for $p = \gamma$.

■

It follows the corresponding right side fractional Opial type inequalities:

**Theorem 21** *All as in Theorem 19, however now it is* $f \circ g^{-1} \in C_{g(x_0)-}^{\nu} \left( \left[ g\left( a \right), g\left( b \right) \right], \mathcal{B}_2\left( H \right) \right)$. *Then*

$$\int_{z}^{g(x_0)} \left\| \left( \left( f \circ g^{-1} \right) (w) \right) \left( \left( D_{g(x_0)-}^{\nu} \left( f \circ g^{-1} \right) \right) (w) \right) \right\|_{1} dw \leq$$

$$\frac{2^{-\frac{1}{q}} \left( g\left( x_0 \right) - z \right)^{\nu + \frac{1}{p} - \frac{1}{q}}}{\Gamma\left( \nu \right) \left[ \left( p\left( \nu - 1 \right) + 1 \right) \left( p\left( \nu - 1 \right) + 2 \right) \right]^{\frac{1}{p}}} \left( \int_{z}^{g(x_0)} \left\| \left( D_{g(x_0)-}^{\nu} \left( f \circ g^{-1} \right) \right) (t) \right\|_{2}^{q} dt \right)^{\frac{2}{q}}, \tag{86}$$

*for all* $g\left( a \right) \leq z \leq g\left( x_0 \right)$.

**Proof.** Based on (20), and as similar to the proof of Theorem 19 is omitted.

■

Next comes another right ride fractional Opial type inequality:

**Theorem 22** *All as in Theorem 20, however now it is* $f \circ g^{-1} \in C_{g(x_0)-}^{\nu} \left( \left[ g\left( a \right), g\left( b \right) \right], \mathcal{B}_{\gamma}\left( H \right) \right)$. *Then*

$$\int_{z}^{g(x_0)} \left\| \left( \left( f \circ g^{-1} \right) (w) \right) \left( \left( D_{g(x_0)-}^{\nu} \left( f \circ g^{-1} \right) \right) (w) \right) \right\|_{\gamma} dw \leq$$

$$\frac{2^{-\frac{1}{q}} \left( g\left( x_0 \right) - z \right)^{\nu + \frac{1}{p} - \frac{1}{q}}}{\Gamma\left( \nu \right) \left[ \left( p\left( \nu - 1 \right) + 1 \right) \left( p\left( \nu - 1 \right) + 2 \right) \right]^{\frac{1}{p}}} \left( \int_{z}^{g(x_0)} \left\| \left( D_{g(x_0)-}^{\nu} \left( f \circ g^{-1} \right) \right) (t) \right\|_{\gamma}^{q} dt \right)^{\frac{2}{q}}, \tag{87}$$

*for all* $g\left( a \right) \leq z \leq g\left( x_0 \right)$.

**Proof.** Based on (20), and as similar to the proof of Theorem 19 is omitted.

■

It follows the modified generalized left $\mathcal{B}_2\left( H \right)$-valued fractional Opial inequality:

**Theorem 23** *All as in Theorem 6, where* $X = \mathcal{B}_2(H)$ *and let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Here we assume that* $\frac{1}{(m+1)q} < \nu < 1$. *Then*

$$\int_{g(x_0)}^{z} \left\| \left( \left( f \circ g^{-1} \right)(w) \right) \left( \left( D_{g(x_0)}^{(m+1)\nu} \left( f \circ g^{-1} \right) \right)(w) \right) \right\|_1 dw \leq \tag{88}$$

$$\frac{2^{-\frac{1}{q}} \left( z - g(x_0) \right)^{(m+1)\nu + \frac{1}{p} - \frac{1}{q}}}{\Gamma((m+1)\nu) \left[ \left( p((m+1)\nu - 1) + 1 \right) \left( p((m+1)\nu - 1) + 2 \right) \right]^{\frac{1}{p}}}$$

$$\left( \int_{g(x_0)}^{z} \left\| \left( D_{g(x_0)}^{(m+1)\nu} \left( f \circ g^{-1} \right) \right)(t) \right\|_2^q dt \right)^{\frac{2}{q}},$$

*for all* $g(x_0) \leq z \leq g(b)$.

**Proof.** As in Theorem 19. ■

Next comes another modified generalized left $\mathcal{B}_\gamma(H)$-valued fractional Opial inequality:

**Theorem 24** *All as in Theorem 6, where* $X = \mathcal{B}_\gamma(H)$ *and let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Here we assume that* $\frac{1}{(m+1)q} < \nu < 1$. *Then*

$$\int_{g(x_0)}^{z} \left\| \left( \left( f \circ g^{-1} \right)(w) \right) \left( \left( D_{g(x_0)}^{(m+1)\nu} \left( f \circ g^{-1} \right) \right)(w) \right) \right\|_\gamma dw \leq \tag{89}$$

$$\frac{2^{-\frac{1}{q}} \left( z - g(x_0) \right)^{(m+1)\nu + \frac{1}{p} - \frac{1}{q}}}{\Gamma((m+1)\nu) \left[ \left( p((m+1)\nu - 1) + 1 \right) \left( p((m+1)\nu - 1) + 2 \right) \right]^{\frac{1}{p}}}$$

$$\left( \int_{g(x_0)}^{z} \left\| \left( D_{g(x_0)}^{(m+1)\nu} \left( f \circ g^{-1} \right) \right)(t) \right\|_\gamma^q dt \right)^{\frac{2}{q}},$$

*for all* $g(x_0) \leq z \leq g(b)$.

**Proof.** As in Theorem 19. ■

The corresponding modified generalized right $\mathcal{B}_2(H)$-valued fractional Opial inequality comes next:

**Theorem 25** *All as in Theorem 7, where* $X = \mathcal{B}_2(H)$ *and let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Here we assume that* $\frac{1}{(m+1)q} < \nu < 1$. *Then*

$$\int_{z}^{g(x_0)} \left\| \left( \left( f \circ g^{-1} \right)(w) \right) \left( \left( D_{g(x_0)-}^{(m+1)\nu} \left( f \circ g^{-1} \right) \right)(w) \right) \right\|_1 dw \leq \tag{90}$$

$$\frac{2^{-\frac{1}{q}} \left( g(x_0) - z \right)^{(m+1)\nu + \frac{1}{p} - \frac{1}{q}}}{\Gamma((m+1)\nu) \left[ \left( p((m+1)\nu - 1) + 1 \right) \left( p((m+1)\nu - 1) + 2 \right) \right]^{\frac{1}{p}}}$$

$$\left(\int_{z}^{g(x_0)}\left\|\left(D_{g(x_0)-}^{(m+1)\nu}\left(f\circ g^{-1}\right)\right)(t)\right\|_2^q dt\right)^{\frac{2}{q}},$$

*for all* $g(a)\leq z\leq g(x_0)$.

**Proof.** As in Theorem 19. ∎

The corresponding modified generalized right $\mathcal{B}_\gamma(H)$-valued fractional Opial inequality comes next:

**Theorem 26** *All as in Theorem 7, where* $X=\mathcal{B}_\gamma(H)$ *and let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. *Here we assume that* $\frac{1}{(m+1)q}<\nu<1$. *Then*

$$\int_{z}^{g(x_0)}\left\|\left(\left(f\circ g^{-1}\right)(w)\right)\left(\left(D_{g(x_0)-}^{(m+1)\nu}\left(f\circ g^{-1}\right)\right)(w)\right)\right\|_\gamma dw\leq \tag{91}$$

$$\frac{2^{-\frac{1}{q}}\left(g(x_0)-z\right)^{(m+1)\nu+\frac{1}{p}-\frac{1}{q}}}{\Gamma((m+1)\nu)\left[\left(p\left((m+1)\nu-1\right)+1\right)\left(p\left((m+1)\nu-1\right)+2\right)\right]^{\frac{1}{p}}}$$

$$\left(\int_{z}^{g(x_0)}\left\|\left(D_{g(x_0)-}^{(m+1)\nu}\left(f\circ g^{-1}\right)\right)(t)\right\|_\gamma^q dt\right)^{\frac{2}{q}},$$

*for all* $g(a)\leq z\leq g(x_0)$.

**Proof.** As in Theorem 19. ∎

We make

**Remark 27** *(to Theorem 12)*

*Case of inequality (46):*

*Call and assume*

$$M_1(f_1,...,f_r):= \tag{92}$$

$$\max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|\left\|D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right\|_2\right\|_{\infty,[g(a),g(x_0)]},\right.$$

$$\left.\sup_{x_0\in[a,b]}\left\|\left\|D_{g(x_0)}^{\gamma}\left(f_i\circ g^{-1}\right)\right\|_2\right\|_{\infty,[g(x_0),g(b)]}\right\}<+\infty.$$

*Then*

$$\left\|K(f_1,...,f_r)(x_0)\right\|_1\leq \text{Right hand side (46)}\ \leq$$

$$\frac{M_1(f_1,...,f_r)\left(g(b)-g(a)\right)^\nu}{\Gamma(\nu+1)}\sum_{i=1}^{r}\left(\int_a^b\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|_2\right)dx\right). \tag{93}$$

We make

**Remark 28** *(to Theorem 13)*

*Case of inequality (62):*

*Call and assume*

$$M_2(f_1,...,f_r) := \tag{94}$$

$$\max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)-}\left(f_i\circ g^{-1}\right)\right\|_2\right\|_{L_1([g(a),g(x_0)])},\right.$$

$$\left.\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)}\left(f_i\circ g^{-1}\right)\right\|_2\right\|_{L_1([g(x_0),g(b)])}\right\}<+\infty.$$

*Then*

$$\|K(f_1,...,f_r)(x_0)\|_1 \leq \text{Right hand side (62)} \leq$$

$$\frac{M_2(f_1,...,f_r)(g(b)-g(a))^{\nu-1}}{\Gamma(\nu)}\sum_{i=1}^{r}\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_2\right)dx\right). \tag{95}$$

We make

**Remark 29** *(to Theorem 14)*

*Case of inequality (64):*

*Call and assume* $(p,q>1:\frac{1}{p}+\frac{1}{q}=1)$*:*

$$M_3(f_1,...,f_r) := \tag{96}$$

$$\max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)-}\left(f_i\circ g^{-1}\right)\right\|_2\right\|_{q,([g(a),g(x_0)])},\right.$$

$$\left.\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)}\left(f_i\circ g^{-1}\right)\right\|_2\right\|_{q,([g(x_0),g(b)])}\right\}<+\infty.$$

*Then*

$$\|K(f_1,...,f_r)(x_0)\|_1 \leq \text{Right hand side (64)} \leq$$

$$\frac{M_3(f_1,...,f_r)(g(b)-g(a))^{\nu-\frac{1}{q}}}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)}\sum_{i=1}^{r}\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_2\right)dx\right). \tag{97}$$

We make

**Remark 30** *(to Theorem 15)* $(\gamma\geq 1)$

*Case of inequality (67):*

*Call and assume*

$$M_1^{\gamma}(f_1,...,f_r) := \tag{98}$$

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$$\max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)-}\left(f_i\circ g^{-1}\right)\right\|_{\gamma}\right\|_{\infty,[g(a),g(x_0)]},\right.$$

$$\left.\sup_{x_0\in[a,b]}\left\|\left\|D^{\gamma}_{g(x_0)}\left(f_i\circ g^{-1}\right)\right\|_{\gamma}\right\|_{\infty,[g(x_0),g(b)]}\right\}<+\infty.$$

*Then*

$$\left\|K\left(f_1,...,f_r\right)(x_0)\right\|_{\gamma}\leq \textit{Right hand side (67)} \leq$$

$$\frac{M_1^{\gamma}\left(f_1,...,f_r\right)\left(g\left(b\right)-g\left(a\right)\right)^{\nu}}{\Gamma\left(\nu+1\right)}\sum_{i=1}^{r}\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j\left(x\right)\right\|_{\gamma}\right)dx\right). \tag{99}$$

We make

**Remark 31** *(to Theorem 16) ($\gamma\geq 1$)*

*Case of inequality (68):*

*Call and assume:*

$$M_2^{\gamma}\left(f_1,...,f_r\right):= \tag{100}$$

$$\max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)-}\left(f_i\circ g^{-1}\right)\right\|_{\gamma}\right\|_{L_1([g(a),g(x_0)])},\right.$$

$$\left.\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)}\left(f_i\circ g^{-1}\right)\right\|_{\gamma}\right\|_{L_1([g(x_0),g(b)])}\right\}<+\infty.$$

*Then*

$$\left\|K\left(f_1,...,f_r\right)(x_0)\right\|_{\gamma}\leq \textit{Right hand side (68)} \leq$$

$$\frac{M_2^{\gamma}\left(f_1,...,f_r\right)\left(g\left(b\right)-g\left(a\right)\right)^{\nu-1}}{\Gamma\left(\nu\right)}\sum_{i=1}^{r}\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j\left(x\right)\right\|_{\gamma}\right)dx\right). \tag{101}$$

We make

**Remark 32** *(to Theorem 17) ($\gamma\geq 1$)*

*Case of inequality (69):*

*Call and assume ($p,q>1:\frac{1}{p}+\frac{1}{q}=1$):*

$$M_3^{\gamma}\left(f_1,...,f_r\right):= \tag{102}$$

$$\max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)-}\left(f_i\circ g^{-1}\right)\right\|_{\gamma}\right\|_{q,([g(a),g(x_0)])},\right.$$

$$\left.\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)}\left(f_i\circ g^{-1}\right)\right\|_{\gamma}\right\|_{q,([g(x_0),g(b)])}\right\}<+\infty.$$

*Then*

$$\left\| K\left(f_1,...,f_r\right)(x_0)\right\|_\gamma \leq \textit{Right hand side (69)} \leq$$

$$\frac{M_3^\gamma\left(f_1,...,f_r\right)\left(g\left(b\right)-g\left(a\right)\right)^{\nu-\frac{1}{q}}}{\left(p\left(\nu-1\right)+1\right)^{\frac{1}{p}}\Gamma\left(\nu\right)}\sum_{i=1}^{r}\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j\left(x\right)\right\|_\gamma\right)dx\right). \quad (103)$$

**Remark 33** *(to Theorem 18)*

*i) for* $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$, *case of inequality (71):*

*Call and assume*

$$N_1\left(A_1,A_2\right):=$$

$$\max\left\{\sup_{x_0\in[a,b]}\left\|\left\|D^\nu_{g(x_0)-}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{\delta,[g(a),g(x_0)]},\sup_{x_0\in[a,b]}\left\|\left\|D^\nu_{g(x_0)}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{\delta,[g(x_0),g(b)]},\right.$$

$$\left.\sup_{x_0\in[a,b]}\left\|\left\|D^\nu_{g(x_0)-}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{\delta,[g(a),g(x_0)]},\sup_{x_0\in[a,b]}\left\|\left\|D^\nu_{g(x_0)}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{\delta,[g(x_0),g(b)]}\right\}<+\infty. \quad (104)$$

*Then*

$$\left\|\Phi\left(A_1,A_2\right)(x_0)\right\|_1 \leq \textit{right hand side (71)} \leq$$

$$\frac{N_1\left(A_1,A_2\right)\left(g\left(b\right)-g\left(a\right)\right)^{\alpha-\frac{1}{\delta}}}{\Gamma\left(\alpha\right)\left(\gamma\left(\alpha-1\right)+1\right)^{\frac{1}{\gamma}}}\left[\int_a^b\left\|A_1\left(x\right)\right\|_p dx+\int_a^b\left\|A_2\left(x\right)\right\|_q dx\right]. \quad (105)$$

*ii) case of inequality (72):*

*Call and assume*

$$N_2\left(A_1,A_2\right):= \quad (106)$$

$$\max\left\{\sup_{x_0\in[a,b]}\left\|\left\|D^\nu_{g(x_0)-}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{L_1([g(a),g(x_0)])},\sup_{x_0\in[a,b]}\left\|\left\|D^\nu_{g(x_0)}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{L_1([g(x_0),g(b)])},\right.$$

$$\left.\sup_{x_0\in[a,b]}\left\|\left\|D^\nu_{g(x_0)-}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{L_1([g(a),g(x_0)])},\sup_{x_0\in[a,b]}\left\|\left\|D^\nu_{g(x_0)}\left(A_2\circ g^{-1}\right)^{-1}\right\|_q\right\|_{L_1([g(x_0),g(b)])}\right\}<+\infty.$$

*Then*

$$\left\|\Phi\left(A_1,A_2\right)(x_0)\right\|_1 \leq \textit{right hand side (72)} \leq$$

$$\frac{N_2\left(A_1,A_2\right)\left(g\left(b\right)-g\left(a\right)\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}\left[\int_a^b\left\|A_1\left(x\right)\right\|_p dx+\int_a^b\left\|A_2\left(x\right)\right\|_q dx\right]. \quad (107)$$

*iii) case of inequality (73):*

*Call and assume*

$$N_3\left(A_1,A_2\right):= \quad (108)$$

$$\max\left\{\sup_{x_0\in[a,b]}\left\|\left\|D^\nu_{g(x_0)-}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{\infty,[g(a),g(x_0)]},\sup_{x_0\in[a,b]}\left\|\left\|D^\nu_{g(x_0)}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{\infty,[g(x_0),g(b)]},\right.$$

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$$\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)-}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{\infty,[g(a),g(x_0)]},\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)}\left(A_2\circ g^{-1}\right)^{-1}\right\|_q\right\|_{\infty,[g(x_0),g(b)]}\Bigg\}<+\infty.$$

*Then*

$$\left\|\Phi\left(A_1,A_2\right)(x_0)\right\|_1\leq \textit{right hand side (73)} \leq$$

$$\frac{N_3\left(A_1,A_2\right)\left(g\left(b\right)-g\left(a\right)\right)^{\alpha}}{\Gamma\left(\alpha+1\right)}\left[\int_a^b\left\|A_1\left(x\right)\right\|_p dx+\int_a^b\left\|A_2\left(x\right)\right\|_q dx\right]. \tag{109}$$

We need

**Remark 34** *(i) This is regarding Theorems 12-17. Here $K\left(f_1,...,f_r\right)(x_0)$, $x_0\in[a,b]$, is as in (45). Next we denote and have (case of $1\leq\nu<2$):*

$$\Delta\left(f_1,...,f_r\right):=\int_a^b K\left(f_1,...,f_r\right)(x_0)\,dx_0=$$

$$\sum_{i=1}^r\left[\left(b-a\right)\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^r f_j\left(x\right)\right)f_i\left(x\right)dx-\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^r f_j\left(x\right)\right)dx\right)\left(\int_a^b f_i\left(x\right)dx\right)\right], \tag{110}$$

*(ii) This is regarding Theorem 18. Here $\Phi\left(A_1,A_2\right)(x_0)$, $x_0\in[a,b]$, is as in (70). Next we denote and have (case of $1\leq\alpha<2$):*

$$\Delta\left(A_1,A_2\right):=\int_a^b\Phi\left(A_1,A_2\right)(x_0)\,dx_0=$$

$$\left(b-a\right)\left(\int_a^b A_2\left(x\right)A_1\left(x\right)dx+\int_a^b A_1\left(x\right)A_2\left(x\right)dx\right)- \tag{111}$$

$$\left(\int_a^b A_2\left(x\right)dx\right)\left(\int_a^b A_1\left(x\right)dx\right)-\left(\int_a^b A_1\left(x\right)dx\right)\left(\int_a^b A_2\left(x\right)dx\right).$$

*(iii) for $\gamma\geq1$, it holds*

$$\left\|\Delta\left(f_1,...,f_r\right)\right\|_{\gamma}\leq\int_a^b\left\|K\left(f_1,...,f_r\right)(x)\right\|_{\gamma}dx, \tag{112}$$

*and*

$$\left\|\Delta\left(A_1,A_2\right)\right\|_1\leq\int_a^b\left\|\Phi\left(A_1,A_2\right)(x)\right\|_1 dx. \tag{113}$$

We give the following set of $\gamma$-Schatten norm generalized Canavati type fractional Grüss type inequalities involving several functions over $\mathcal{B}_{\gamma}\left(H\right)$, $\gamma\geq1$.

**Theorem 35** *All as in Theorem 12, with $1 \leq \nu < 2$ (i.e. $n = 1$). Then*
*i)*

$$\|\Delta(f_1, ..., f_r)\|_1 \leq \frac{M_1(f_1, ..., f_r)(g(b) - g(a))^{\nu}(b-a)^2}{\Gamma(\nu+1)} \sum_{i=1}^{r}\left(\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right\|_{\infty,[a,b]}\right), \quad (114)$$

*where $M_1(f_1, ..., f_r)$ is as in (92),*
*ii)*

$$\|\Delta(f_1, ..., f_r)\|_1 \leq \frac{M_2(f_1, ..., f_r)(g(b) - g(a))^{\nu-1}(b-a)^2}{\Gamma(\nu)} \sum_{i=1}^{r}\left(\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right\|_{\infty,[a,b]}\right), \quad (115)$$

*where $M_2(f_1, ..., f_r)$ is as in (94),*
*iii) when $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\|\Delta(f_1, ..., f_r)\|_1 \leq \frac{M_3(f_1, ..., f_r)(g(b) - g(a))^{\nu-\frac{1}{q}}(b-a)^2}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)} \sum_{i=1}^{r}\left(\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right\|_{\infty,[a,b]}\right), \quad (116)$$

*where $M_3(f_1, ..., f_r)$ is as in (96).*

**Proof.** By Remarks 34, 27-29 and that

$$\int_a^b \left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right) dx \leq (b-a)\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right\|_{\infty,[a,b]}.$$

■

We continue with

**Theorem 36** *All as in Theorem 15, with $1 \leq \nu < 2$ (i.e. $n = 1$), $\gamma \geq 1$. Then*
*i)*

$$\|\Delta(f_1, ..., f_r)\|_{\gamma} \leq \frac{M_1^{\gamma}(f_1, ..., f_r)(g(b) - g(a))^{\nu}(b-a)^2}{\Gamma(\nu+1)}$$

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$$\sum_{i=1}^{r}\left(\left\|\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j(x)\right\|_{\gamma}\right\|_{\infty,[a,b]}\right), \tag{117}$$

*where* $M_1^{\gamma}(f_1,...,f_r)$ *is as in (98),*

*ii)*

$$\left\|\Delta(f_1,...,f_r)\right\|_{\gamma} \leq \frac{M_2^{\gamma}(f_1,...,f_r)(g(b)-g(a))^{\nu-1}(b-a)^2}{\Gamma(\nu)}$$

$$\sum_{i=1}^{r}\left(\left\|\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j(x)\right\|_{\gamma}\right\|_{\infty,[a,b]}\right), \tag{118}$$

*where* $M_2^{\gamma}(f_1,...,f_r)$ *is as in (100),*

*iii) when* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$*, we have*

$$\left\|\Delta(f_1,...,f_r)\right\|_{\gamma} \leq \frac{M_3^{\gamma}(f_1,...,f_r)(g(b)-g(a))^{\nu-\frac{1}{q}}(b-a)^2}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)}$$

$$\sum_{i=1}^{r}\left(\left\|\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j(x)\right\|_{\gamma}\right\|_{\infty,[a,b]}\right), \tag{119}$$

*where* $M_3^{\gamma}(f_1,...,f_r)$ *is as in (102).*

**Proof.** By Remarks 34, 30-32 and that

$$\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j(x)\right\|_{\gamma}\right)dx \leq (b-a)\left\|\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j(x)\right\|_{\gamma}\right\|_{\infty,[a,b]}.$$

■

Furthermore we have ($r=2$ case of $p$-Schatten norm Grüss inequalities)

**Theorem 37** *All as in Theorem 18, with* $1\leq\alpha<2$ *(i.e.* $[\alpha]=1$*). Then*

*i) for* $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$*, we have*

$$\left\|\Delta(A_1,A_2)\right\|_1 \leq \frac{N_1(A_1,A_2)(g(b)-g(a))^{\alpha-\frac{1}{\delta}}(b-a)}{\Gamma(\alpha)(\gamma(\alpha-1)+1)^{\frac{1}{\gamma}}}$$

$$\left[\int_a^b\left\|A_1(x)\right\|_p dx+\int_a^b\left\|A_2(x)\right\|_q dx\right], \tag{120}$$

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*where $N_1(A_1, A_2)$ is as in (104),*

*ii)*

$$\|\Delta(A_1, A_2)\|_1 \leq \frac{N_2(A_1, A_2)(g(b) - g(a))^{\alpha-1}(b-a)}{\Gamma(\alpha)}$$

$$\left[\int_a^b \|A_1(x)\|_p dx + \int_a^b \|A_2(x)\|_q dx\right], \tag{121}$$

*where $N_2(A_1, A_2)$ is as in (106),*

*and*

*iii)*

$$\|\Delta(A_1, A_2)\|_1 \leq \frac{N_3(A_1, A_2)(g(b) - g(a))^{\alpha}(b-a)}{\Gamma(\alpha+1)}$$

$$\left[\int_a^b \|A_1(x)\|_p dx + \int_a^b \|A_2(x)\|_q dx\right], \tag{122}$$

*where $N_3(A_1, A_2)$ is as in (108).*

**Proof.** By Remarks 34, 33. ■

# 6 Applications

We start with applications on Ostrowski type inequalities:

**Corollary 38** *(to Theorems 12-14) All as in Theorem 12 for $g(t) = t$. Then*

*i)*

$$\|K(f_1, ..., f_r)(x_0)\|_1 \leq \frac{1}{\Gamma(\nu+1)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{x_0-}^{\nu} f_i\right)\right\|_2\right\|_{\infty,[a,x_0]}(x_0-a)^{\nu}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_2\right)dx\right)\right]+\right. \tag{123}$$

$$\left.\left[\left\|\left\|\left(D_{x_0}^{\nu} f_i\right)\right\|_2\right\|_{\infty,[x_0,b]}(b-x_0)^{\nu}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_2\right)dx\right)\right]\right],$$

*ii)*

$$\|K(f_1, ..., f_r)(x_0)\|_1 \leq \frac{1}{\Gamma(\nu)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{x_0-}^{\nu} f_i\right)\right\|_2\right\|_{L_1([a,x_0])}\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_2\right)(x_0-x)^{\nu-1}dx\right]+\right.$$

$$\left[\left\|\left\|\left(D_{x_0}^{\nu}f_i\right)\right\|_2\right\|_{L_1([x_0,b])}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j(x)\right\|_2\right)(x-x_0)^{\nu-1}dx\right]\Bigg], \qquad (124)$$

*iii) when* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, *we have*

$$\left\|K(f_1,...,f_r)(x_0)\right\|_1 \leq \frac{1}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{x_0-}^{\nu}f_i\right)\right\|_2\right\|_{q,[a,x_0]}\left(\int_a^{x_0}(x_0-x)^{\nu-\frac{1}{q}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j(x)\right\|_2\right)dx\right)\right]+\right.$$

$$\left.\left[\left\|\left\|\left(D_{x_0}^{\nu}f_i\right)\right\|_2\right\|_{q,[x_0,b]}\left(\int_{x_0}^{b}(x-x_0)^{\nu-\frac{1}{q}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j(x)\right\|_2\right)dx\right)\right]\right]. \qquad (125)$$

It follows:

**Corollary 39** *(to Theorems 15-17) All as in Theorem 15 for* $g(t)=t$, $\gamma\geq 1$. *Then*

*i)*

$$\left\|K(f_1,...,f_r)(x_0)\right\|_\gamma \leq \frac{1}{\Gamma(\nu+1)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{x_0-}^{\nu}f_i\right)\right\|_\gamma\right\|_{\infty,[a,x_0]}(x_0-a)^{\nu}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j(x)\right\|_\gamma\right)dx\right)\right]+\right.$$

$$\left.\left[\left\|\left\|\left(D_{x_0}^{\nu}f_i\right)\right\|_\gamma\right\|_{\infty,[x_0,b]}(b-x_0)^{\nu}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j(x)\right\|_\gamma\right)dx\right)\right]\right], \qquad (126)$$

*ii)*

$$\left\|K(f_1,...,f_r)(x_0)\right\|_\gamma \leq \frac{1}{\Gamma(\nu)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{x_0-}^{\nu}f_i\right)\right\|_\gamma\right\|_{L_1([a,x_0])}\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j(x)\right\|_\gamma\right)(x_0-x)^{\nu-1}dx\right]+\right.$$

$$\left.\left[\left\|\left\|\left(D_{x_0}^{\nu}f_i\right)\right\|_\gamma\right\|_{L_1([x_0,b])}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j(x)\right\|_\gamma\right)(x-x_0)^{\nu-1}dx\right]\right], \qquad (127)$$

*iii) when* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, *we have*

$$\left\|K\left(f_1,...,f_r\right)(x_0)\right\|_\gamma \leq \frac{1}{\left(p(\nu-1)+1\right)^{\frac{1}{p}}\Gamma(\nu)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{x_0-}^{\nu}f_i\right)\right\|_\gamma\right\|_{q,[a,x_0]}\left(\int_a^{x_0}(x_0-x)^{\nu-\frac{1}{q}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j(x)\right\|_\gamma\right)dx\right)\right]+\right.$$

$$\left.\left[\left\|\left\|\left(D_{x_0}^{\nu}f_i\right)\right\|_\gamma\right\|_{q,[x_0,b]}\left(\int_{x_0}^{b}(x-x_0)^{\nu-\frac{1}{q}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j(x)\right\|_\gamma\right)dx\right)\right]\right]. \qquad (128)$$

We continue with

**Corollary 40** *(to Theorem 18) All as in Theorem 18, with* $g(t)=e^t$. *Then*
*i) for* $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$, *we have*

$$\left\|\Phi\left(A_1,A_2\right)(x_0)\right\|_1 \leq \frac{1}{\Gamma(\alpha)\left(\gamma(\alpha-1)+1\right)^{\frac{1}{\gamma}}}$$

$$\left\{\left[\left\|\left\|D_{e^{x_0}-}^{\alpha}\left(A_1\circ\log\right)\right\|_p\right\|_{\delta,[e^a,e^{x_0}]}\int_a^{x_0}\left\|A_2(x)\right\|_q\left(e^{x_0}-e^x\right)^{\alpha-\frac{1}{\delta}}dx\right]+\right.$$

$$\left[\left\|\left\|D_{e^{x_0}}^{\alpha}\left(A_1\circ\log\right)\right\|_p\right\|_{\delta,[e^{x_0},e^b]}\int_{x_0}^{b}\left\|A_2(x)\right\|_q\left(e^x-e^{x_0}\right)^{\alpha-\frac{1}{\delta}}dx\right]+$$

$$\left[\left\|\left\|D_{e^{x_0}-}^{\alpha}\left(A_2\circ\log\right)\right\|_q\right\|_{\delta,[e^a,e^{x_0}]}\int_a^{x_0}\left\|A_1(x)\right\|_p\left(e^{x_0}-e^x\right)^{\alpha-\frac{1}{\delta}}dx\right]+$$

$$\left.\left[\left\|\left\|D_{e^{x_0}}^{\alpha}\left(A_2\circ\log\right)\right\|_q\right\|_{\delta,[e^{x_0},e^b]}\int_{x_0}^{b}\left\|A_1(x)\right\|_p\left(e^x-e^{x_0}\right)^{\alpha-\frac{1}{\delta}}dx\right]\right\}, \qquad (129)$$

*ii) it holds*

$$\left\|\Phi\left(A_1,A_2\right)(x_0)\right\|_1 \leq \frac{1}{\Gamma(\alpha)}$$

$$\left\{\left[\left\|\left\|D_{e^{x_0}-}^{\alpha}\left(A_1\circ\log\right)\right\|_p\right\|_{L_1([e^a,e^{x_0}])}\int_a^{x_0}\left\|A_2(x)\right\|_q\left(e^{x_0}-e^x\right)^{\alpha-1}dx\right]+\right.$$

$$\left[\left\|\left\|D_{e^{x_0}}^{\alpha}\left(A_1\circ\log\right)\right\|_p\right\|_{L_1([e^{x_0},e^b])}\int_{x_0}^{b}\left\|A_2(x)\right\|_q\left(e^x-e^{x_0}\right)^{\alpha-1}dx\right]+$$

$$\left[\left\|\left\|D_{e^{x_0}-}^{\alpha}\left(A_2\circ\log\right)\right\|_q\right\|_{L_1([e^a,e^{x_0}])}\int_a^{x_0}\left\|A_1(x)\right\|_p\left(e^{x_0}-e^x\right)^{\alpha-1}dx\right]+ \qquad (130)$$

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$$\left[\left\|\left\|D_{e^{x_0}}^{\alpha}\left(A_2\circ\log\right)\right\|_q\right\|_{L_1\left(\left[e^{x_0},e^b\right]\right)}\int_{x_0}^{b}\left\|A_1\left(x\right)\right\|_p\left(e^x-e^{x_0}\right)^{\alpha-1}dx\right]\right\},$$

*and*

*iii)*

$$\left\|\Phi\left(A_1,A_2\right)\left(x_0\right)\right\|_1\leq\frac{1}{\Gamma\left(\alpha+1\right)}$$

$$\left\{\left[\left\|\left\|D_{e^{x_0}-}^{\alpha}\left(A_1\circ\log\right)\right\|_p\right\|_{\infty,\left[e^a,e^{x_0}\right]}\int_a^{x_0}\left\|A_2\left(x\right)\right\|_q\left(e^{x_0}-e^x\right)^{\alpha}dx\right]+\right.$$

$$\left[\left\|\left\|D_{e^{x_0}}^{\alpha}\left(A_1\circ\log\right)\right\|_p\right\|_{\infty,\left[e^{x_0},e^b\right]}\int_{x_0}^{b}\left\|A_2\left(x\right)\right\|_q\left(e^x-e^{x_0}\right)^{\alpha}dx\right]+$$

$$\left[\left\|\left\|D_{e^{x_0}-}^{\alpha}\left(A_2\circ\log\right)\right\|_q\right\|_{\infty,\left[e^a,e^{x_0}\right]}\int_a^{x_0}\left\|A_1\left(x\right)\right\|_p\left(e^{x_0}-e^x\right)^{\alpha}dx\right]+$$

$$\left.\left[\left\|\left\|D_{e^{x_0}}^{\alpha}\left(A_2\circ\log\right)\right\|_q\right\|_{\infty,\left[e^{x_0},e^b\right]}\int_{x_0}^{b}\left\|A_1\left(x\right)\right\|_p\left(e^x-e^{x_0}\right)^{\alpha}dx\right]\right\}. \tag{131}$$

We continue with applications on Opial inequalities

**Corollary 41** *(to Theorem 19) All as in Theorem 19 with $g\left(t\right)=t$. Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\int_{x_0}^{z}\left\|f\left(w\right)\left(D_{x_0}^{\nu}f\right)\left(w\right)\right\|_1dw\leq$$

$$\frac{2^{-\frac{1}{q}}\left(z-x_0\right)^{\nu+\frac{1}{p}-\frac{1}{q}}}{\Gamma\left(\nu\right)\left[\left(p\left(\nu-1\right)+1\right)\left(p\left(\nu-1\right)+2\right)\right]^{\frac{1}{p}}}\left(\int_{x_0}^{z}\left\|\left(D_{x_0}^{\nu}f\right)\left(w\right)\right\|_2^q dw\right)^{\frac{2}{q}}, \tag{132}$$

*for all $x_0\leq z\leq b$.*

It follows:

**Corollary 42** *(to Theorem 20) All as in Theorem 20, $\gamma\geq1$, with $g\left(t\right)=e^t$. Let also $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\int_{e^{x_0}}^{z}\left\|\left(\left(f\circ\log\right)\left(w\right)\right)\left(\left(D_{e^{x_0}}^{\nu}\left(f\circ\log\right)\right)\left(w\right)\right)\right\|_{\gamma}dw\leq$$

$$\frac{2^{-\frac{1}{q}}\left(z-e^{x_0}\right)^{\nu+\frac{1}{p}-\frac{1}{q}}}{\Gamma\left(\nu\right)\left[\left(p\left(\nu-1\right)+1\right)\left(p\left(\nu-1\right)+2\right)\right]^{\frac{1}{p}}}\left(\int_{e^{x_0}}^{z}\left\|\left(D_{e^{x_0}}^{\nu}\left(f\circ\log\right)\right)\left(w\right)\right\|_{\gamma}^q dw\right)^{\frac{2}{q}}, \tag{133}$$

*for all $e^{x_0}\leq z\leq e^b$.*

We finish with applications on Grüss inequalities:

**Corollary 43** *(to Theorem 35) All as in Theorem 35 with $g(t) = t$ $(1 \leq \nu < 2)$. Then*

*i)*

$$\|\Delta(f_1, ..., f_r)\|_1 \leq \frac{M_1(f_1, ..., f_r)(b-a)^{\nu+2}}{\Gamma(\nu+1)} \sum_{i=1}^{r} \left( \left\| \prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\|_2 \right\|_{\infty,[a,b]} \right), \tag{134}$$

*where $M_1(f_1, ..., f_r)$ is as in (92),*

*ii)*

$$\|\Delta(f_1, ..., f_r)\|_1 \leq \frac{M_2(f_1, ..., f_r)(b-a)^{\nu+1}}{\Gamma(\nu)} \sum_{i=1}^{r} \left( \left\| \prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\|_2 \right\|_{\infty,[a,b]} \right), \tag{135}$$

*where $M_2(f_1, ..., f_r)$ is as in (94),*

*iii) when $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\|\Delta(f_1, ..., f_r)\|_1 \leq \frac{M_3(f_1, ..., f_r)(b-a)^{\nu+1+\frac{1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}} \Gamma(\nu)} \sum_{i=1}^{r} \left( \left\| \prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\|_2 \right\|_{\infty,[a,b]} \right), \tag{136}$$

*where $M_3(f_1, ..., f_r)$ is as in (96).*

It follows ($r = 2$ case)

**Corollary 44** *(to Theorem 37) All as in Theorem 37, with $[a, b] \subset \mathbb{R}_+ - \{0\}$, and $g(t) = \log t$. Then*

*i) for $\gamma, \delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1$, we have*

$$\|\Delta(A_1, A_2)\|_1 \leq \frac{N_1(A_1, A_2)\left(\log \frac{b}{a}\right)^{\alpha - \frac{1}{\delta}} (b-a)}{\Gamma(\alpha)(\gamma(\alpha-1)+1)^{\frac{1}{\gamma}}}$$

$$\left[ \int_a^b \|A_1(x)\|_p \, dx + \int_a^b \|A_2(x)\|_q \, dx \right], \tag{137}$$

*where $N_1(A_1, A_2)$ is as in (104),*

*ii)*

$$\|\Delta(A_1, A_2)\|_1 \leq \frac{N_2(A_1, A_2)\left(\log \frac{b}{a}\right)^{\alpha-1} (b-a)}{\Gamma(\alpha)}$$

$$\left[\int_a^b \|A_1(x)\|_p \, dx + \int_a^b \|A_2(x)\|_q \, dx\right], \tag{138}$$

*where* $N_2(A_1, A_2)$ *is as in (106),*

*and*

*iii)*

$$\|\Delta(A_1, A_2)\|_1 \leq \frac{N_3(A_1, A_2)\left(\log \frac{b}{a}\right)^{\alpha}(b-a)}{\Gamma(\alpha+1)}$$
$$\left[\int_a^b \|A_1(x)\|_p \, dx + \int_a^b \|A_2(x)\|_q \, dx\right], \tag{139}$$

*where* $N_3(A_1, A_2)$ *is as in (108).*

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# Abstract multivariate algebraic function activated neural network approximations

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**Abstract**

Here we exhibit multivariate quantitative approximations of Banach space valued continuous multivariate functions on a box or $\mathbb{R}^N$, $N \in \mathbb{N}$, by the multivariate normalized, quasi-interpolation, Kantorovich type and quadrature type neural network operators. We study also the case of approximation by iterated operators of the last four types. These approximations are achieved by establishing multidimensional Jackson type inequalities involving the multivariate modulus of continuity of the engaged function or its high order Fréchet derivatives. Our multivariate operators are defined by using a multidimensional density function induced by the algebraic sigmoid function. The approximations are pointwise and uniform. The related feed-forward neural network is with one hidden layer.

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**Keywords and Phrases:** algebraic sigmoid function, multivariate neural network approximation, quasi-interpolation operator, Kantorovich type operator, quadrature type operator, multivariate modulus of continuity, abstract approximation, iterated approximation.

## 1 Introduction

G.A. Anastassiou in [2] and [3], see chapters 2-5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliagnet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support. Also in [3] he gives the $N$th order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see chapters 4-5 there.

Motivations for this work are the article [15] of Z. Chen and F. Cao, and [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [16], [17].

Here we perform multivariate algebraic sigmoid function based neural network approximations to continuous functions over boxes or over the whole $\mathbb{R}^N$, $N \in \mathbb{N}$, and also iterated approximations. All convergences here are with rates expressed via the multivariate modulus of continuity of the involved function or its high order Fréchet derivative and given by very tight multidimensional Jackson type inequalities.

We come up with the "right" precisely defined multivariate normalized, quasi-interpolation neural network operators related to boxes or $\mathbb{R}^N$, as well as Kantorovich type and quadrature type related operators on $\mathbb{R}^N$. Our boxes are not necessarily symmetric to the origin. In preparation to prove our results we establish important properties of the basic multivariate density function induced by algebraic sigmoid function and defining our operators.

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle a_j \cdot x\rangle + b_j\right), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x\rangle$ is the inner product of $a_j$ and $x$, and $\sigma$ is the activation function of the network. In many fundamental network models, the activation function is the algebrai sigmoid function. About neural networks see [18], [19], [20].

## 2 Basic

Here see also [12].

We consider the generator algebraic function

$$\varphi(x) = \frac{x}{\sqrt[2m]{1+x^{2m}}}, \quad m \in \mathbb{N}, x \in \mathbb{R}, \tag{1}$$

which is a sigmoid type of function and is a strictly increasing function.

We see that $\varphi(-x) = -\varphi(x)$ with $\varphi(0) = 0$. We get that

$$\varphi'(x) = \frac{1}{\left(1+x^{2m}\right)^{\frac{2m+1}{m}}} > 0, \ \forall\, x \in \mathbb{R}, \tag{2}$$

proving $\varphi$ as strictly increasing over $\mathbb{R}, \varphi'(x) = \varphi'(-x)$. We easily find that $\lim_{x \to +\infty} \varphi(x) = 1$, $\varphi(+\infty) = 1$, and $\lim_{x \to -\infty} \varphi(x) = -1$, $\varphi(-\infty) = -1$.

We consider the activation function

$$\Phi(x) = \frac{1}{4}\left[\varphi(x+1) - \varphi(x-1)\right]. \tag{3}$$

Clearly it is $\Phi(x) = \Phi(-x)$, $\forall\ x \in \mathbb{R}$, so that $\Phi$ is an even function and symmetric with respect to the $y$-axis. Clearly $\Phi(x) > 0$, $\forall\ x \in \mathbb{R}$.

Also it is

$$\Phi(0) = \frac{1}{2\sqrt[2m]{2}}. \tag{4}$$

By [12], we have that $\Phi'(x) < 0$ for $x > 0$. That is $\Phi$ is strictly decreasing over $(0, +\infty)$.

Clearly, $\Phi$ is strictly increasing over $(-\infty, 0)$ and $\Phi'(0) = 0$.

Furthermore we obtain that

$$\lim_{x \to +\infty} \Phi(x) = \frac{1}{4}\left[\varphi(+\infty) - \varphi(+\infty)\right] = 0, \tag{5}$$

and

$$\lim_{x \to -\infty} \Phi(x) = \frac{1}{4}\left[\varphi(-\infty) - \varphi(-\infty)\right] = 0. \tag{6}$$

That is the $x$-axis is the horizontal asymptote of $\Phi$.

Conclusion, $\Phi$ is a bell shape symmetric function with maximum

$$\Phi(0) = \frac{1}{2\sqrt[2m]{2}}, \quad m \in \mathbb{N}. \tag{7}$$

We need

**Theorem 1** *([12]) We have that*

$$\sum_{i=-\infty}^{\infty} \Phi(x - i) = 1, \quad \forall\ x \in \mathbb{R}. \tag{8}$$

**Theorem 2** *([12]) It holds*

$$\int_{-\infty}^{\infty} \Phi(x)\, dx = 1. \tag{9}$$

**Theorem 3** *([12]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} \Phi(nx - k) < \frac{1}{4m\left(n^{1-\alpha} - 2\right)^{2m}}, \quad m \in \mathbb{N}. \tag{10}$$

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.

We need

**Theorem 4** *([12]) Let $[a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)} < 2\left(\sqrt[2m]{1+4^m}\right), \tag{11}$$

$\forall\ x \in [a,b]$, $m \in \mathbb{N}$.

**Note 5** *1) By [12] we have that*

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k) \neq 1, \tag{12}$$

*for at least some $x \in [a,b]$.*

*2) Let $[a,b] \subset \mathbb{R}$. For large $n \in \mathbb{N}$ we always have $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.*

*In general it holds that*

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k) \leq 1. \tag{13}$$

We introduce

$$Z(x_1,...,x_N) := Z(x) := \prod_{i=1}^{N} \Phi(x_i), \quad x = (x_1,...,x_N) \in \mathbb{R}^N,\ N \in \mathbb{N}. \tag{14}$$

It has the properties:

(i) $Z(x) > 0$, $\forall\ x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} Z(x_1-k_1,...,x_N-k_N) = 1, \tag{15}$$

where $k := (k_1,...,k_n) \in \mathbb{Z}^N$, $\forall\ x \in \mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z(nx-k) = 1, \tag{16}$$

$\forall\ x \in \mathbb{R}^N$; $n \in \mathbb{N}$, and

(iv)

$$\int_{\mathbb{R}^N} Z(x)\,dx = 1, \tag{17}$$

that is $Z$ is a multivariate density function.

Here denote $\|x\|_\infty := \max\{|x_1|,...,|x_N|\}$, $x \in \mathbb{R}^N$, also set $\infty := (\infty,...,\infty)$, $-\infty := (-\infty,...,-\infty)$ upon the multivariate context, and

$$\begin{aligned} \lceil na \rceil &:= (\lceil na_1 \rceil,...,\lceil na_N \rceil), \\ \lfloor nb \rfloor &:= (\lfloor nb_1 \rfloor,...,\lfloor nb_N \rfloor), \end{aligned} \tag{18}$$

where $a := (a_1,...,a_N)$, $b := (b_1,...,b_N)$.

We obviously see that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left( \prod_{i=1}^{N} \Phi(nx_i - k_i) \right) =$$

$$\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} ... \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} \left( \prod_{i=1}^{N} \Phi(nx_i - k_i) \right) = \prod_{i=1}^{N} \left( \sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Phi(nx_i - k_i) \right). \tag{19}$$

For $0 < \beta < 1$ and $n \in \mathbb{N}$, a fixed $x \in \mathbb{R}^N$, we have that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) =$$

$$\sum_{\substack{k=\lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty \le \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z(nx-k) + \sum_{\substack{k=\lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z(nx-k). \tag{20}$$

In the last two sums the counting is over disjoint vector sets of $k$'s, because the condition $\left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}$ implies that there exists at least one $\left| \frac{k_r}{n} - x_r \right| > \frac{1}{n^\beta}$, where $r \in \{1,...,N\}$.

(v) As in [10], pp. 379-380, we derive that

$$\sum_{\substack{k=\lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z(nx-k) \overset{(10)}{<} \frac{1}{4m\left(n^{1-\beta}-2\right)^{2m}}, \quad 0<\beta<1, \ m \in \mathbb{N}, \tag{21}$$

with $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \prod_{i=1}^{N} [a_i, b_i]$.

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(vi) By Theorem 4 we get that

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} < \frac{1}{(\Phi(1))^N} \cong \left[2\left(\sqrt[2m]{1+4^m}\right)\right]^N, \tag{22}$$

$\forall\, x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$, $n \in \mathbb{N}$.

It is also clear that

(vii)

$$\sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} Z(nx-k) < \frac{1}{4m\left(n^{1-\beta}-2\right)^{2m}}, \tag{23}$$

$0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \mathbb{R}^N$, $m \in \mathbb{N}$.

Furthermore it holds

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) \neq 1, \tag{24}$$

for at least some $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$.

Here $\left(X, \|\cdot\|_\gamma\right)$ is a Banach space.

Let $f \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$, $x = (x_1, ..., x_N) \in \prod_{i=1}^N [a_i, b_i]$, $n \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

We introduce and define the following multivariate linear normalized neural network operator ($x := (x_1, ..., x_N) \in \left(\prod_{i=1}^N [a_i, b_i]\right)$):

$$A_n(f, x_1, ..., x_N) := A_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} =$$

$$\frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} ... \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^N \Phi(nx_i - k_i)\right)}{\prod_{i=1}^N \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \Phi(nx_i - k_i)\right)}. \tag{25}$$

For large enough $n \in \mathbb{N}$ we always obtain $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$. Also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

When $g \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$ we define the companion operator

$$\widetilde{A}_n(g, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} g\left(\frac{k}{n}\right) Z(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)}. \tag{26}$$

Clearly $\widetilde{A}_n$ is a positive linear operator. We have that

$$\widetilde{A}_n(1,x)=1,\ \ \forall\ x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right).$$

Notice that $A_n(f)\in C\left(\prod_{i=1}^{N}[a_i,b_i],X\right)$ and $\widetilde{A}_n(g)\in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

Furthermore it holds

$$\|A_n(f,x)\|_\gamma\leq\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|f\left(\frac{k}{n}\right)\right\|_\gamma Z(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)}=\widetilde{A}_n\left(\|f\|_\gamma,x\right),\tag{27}$$

$\forall\ x\in\prod_{i=1}^{N}[a_i,b_i]$.

Clearly $\|f\|_\gamma\in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

So, we have that

$$\|A_n(f,x)\|_\gamma\leq\widetilde{A}_n\left(\|f\|_\gamma,x\right),\tag{28}$$

$\forall\ x\in\prod_{i=1}^{N}[a_i,b_i]$, $\forall\ n\in\mathbb{N}$, $\forall\ f\in C\left(\prod_{i=1}^{N}[a_i,b_i],X\right)$.

Let $c\in X$ and $g\in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, then $cg\in C\left(\prod_{i=1}^{N}[a_i,b_i],X\right)$.

Furthermore it holds

$$A_n(cg,x)=c\widetilde{A}_n(g,x),\ \ \forall\ x\in\prod_{i=1}^{N}[a_i,b_i].\tag{29}$$

Since $\widetilde{A}_n(1)=1$, we get that

$$A_n(c)=c,\ \ \forall\ c\in X.\tag{30}$$

We call $\widetilde{A}_n$ the companion operator of $A_n$.

For convinience we call

$$A_n^*(f,x):=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)Z(nx-k)=$$

$$\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor}\sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2\rfloor}\cdots\sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor}f\left(\frac{k_1}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}\Phi(nx_i-k_i)\right),\tag{31}$$

$\forall\ x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

That is

$$A_n(f,x):=\frac{A_n^*(f,x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)},\tag{32}$$

$\forall\ x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$, $n \in \mathbb{N}$.

Hence

$$A_n(f,x) - f(x) = \frac{A_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)}. \tag{33}$$

Consequently we derive

$$\|A_n(f,x) - f(x)\|_\gamma \overset{(22)}{\leq} \left[2\left(\sqrt[2m]{1+4^m}\right)\right]^N \left\| A_n^*(f,x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) \right\|_\gamma, \tag{34}$$

$\forall\ x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$.

We will estimate the right hand side of (34).

For the last and others we need

**Definition 6** *([11], p. 274) Let $M$ be a convex and compact subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1, \infty]$, and $\left(X, \|\cdot\|_\gamma\right)$ be a Banach space. Let $f \in C(M, X)$. We define the first modulus of continuity of $f$ as*

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in M: \\ \|x-y\|_p \leq \delta}} \|f(x) - f(y)\|_\gamma, \quad 0 < \delta \leq diam(M). \tag{35}$$

*If $\delta > diam(M)$, then*

$$\omega_1(f,\delta) = \omega_1(f, diam(M)). \tag{36}$$

Notice $\omega_1(f,\delta)$ is increasing in $\delta > 0$. For $f \in C_B(M,X)$ (continuous and bounded functions) $\omega_1(f,\delta)$ is defined similarly.

**Lemma 7** *([11], p. 274) We have $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$, iff $f \in C(M,X)$, where $M$ is a convex compact subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1,\infty]$.*

Clearly we have also: $f \in C_U\left(\mathbb{R}^N, X\right)$ (uniformly continuous functions), iff $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$, where $\omega_1$ is defined similarly to (35). The space $C_B\left(\mathbb{R}^N, X\right)$ denotes the continuous and bounded functions on $\mathbb{R}^N$.

When $f \in C_B\left(\mathbb{R}^N, X\right)$ we define,

$$B_n(f,x) := B_n(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx-k) :=$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n}, \frac{k_2}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} \Phi(nx_i - k_i)\right), \tag{37}$$

$n \in \mathbb{N}$, $\forall\, x \in \mathbb{R}^N$, $N \in \mathbb{N}$, the multivariate quasi-interpolation neural network operator.

Also for $f \in C_B\left(\mathbb{R}^N, X\right)$ we define the multivariate Kantorovich type neural network operator

$$C_n\left(f, x\right) := C_n\left(f, x_1, ..., x_N\right) := \sum_{k=-\infty}^{\infty} \left(n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f\left(t\right) dt\right) Z\left(nx - k\right) =$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \left(n^N \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} ... \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f\left(t_1, ..., t_N\right) dt_1 ... dt_N\right)$$

$$\cdot \left(\prod_{i=1}^{N} \Phi\left(nx_i - k_i\right)\right), \tag{38}$$

$n \in \mathbb{N}$, $\forall\, x \in \mathbb{R}^N$.

Again for $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \in \mathbb{N}$, we define the multivariate neural network operator of quadrature type $D_n\left(f, x\right)$, $n \in \mathbb{N}$, as follows.

Let $\theta = \left(\theta_1, ..., \theta_N\right) \in \mathbb{N}^N$, $r = \left(r_1, ..., r_N\right) \in \mathbb{Z}_+^N$, $w_r = w_{r_1, r_2, ... r_N} \geq 0$, such that $\sum\limits_{r=0}^{\theta} w_r = \sum\limits_{r_1=0}^{\theta_1} \sum\limits_{r_2=0}^{\theta_2} ... \sum\limits_{r_N=0}^{\theta_N} w_{r_1, r_2, ... r_N} = 1$; $k \in \mathbb{Z}^N$ and

$$\delta_{nk}\left(f\right) := \delta_{n, k_1, k_2, ..., k_N}\left(f\right) := \sum_{r=0}^{\theta} w_r f\left(\frac{k}{n} + \frac{r}{n\theta}\right) =$$

$$\sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} ... \sum_{r_N=0}^{\theta_N} w_{r_1, r_2, ... r_N} f\left(\frac{k_1}{n} + \frac{r_1}{n\theta_1}, \frac{k_2}{n} + \frac{r_2}{n\theta_2}, ..., \frac{k_N}{n} + \frac{r_N}{n\theta_N}\right), \tag{39}$$

where $\frac{r}{\theta} := \left(\frac{r_1}{\theta_1}, \frac{r_2}{\theta_2}, ..., \frac{r_N}{\theta_N}\right)$.

We set

$$D_n\left(f, x\right) := D_n\left(f, x_1, ..., x_N\right) := \sum_{k=-\infty}^{\infty} \delta_{nk}\left(f\right) Z\left(nx - k\right) = \tag{40}$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \delta_{n, k_1, k_2, ..., k_N}\left(f\right) \left(\prod_{i=1}^{N} \Phi\left(nx_i - k_i\right)\right),$$

$\forall\, x \in \mathbb{R}^N$.

In this article we study the approximation properties of $A_n, B_n, C_n$, $D_n$ neural network operators and as well of their iterates. That is, the quantitative pointwise and uniform convergence of these operators to the unit operator $I$.

# 3 Multivariate general Neural Network Approximations

Here we present several vectorial neural network approximations to Banach space valued functions given with rates.

We give

**Theorem 8** *Let* $f \in C\left(\prod_{i=1}^{N}[a_i, b_i], X\right)$, $0 < \beta < 1$, $x \in \left(\prod_{i=1}^{N}[a_i, b_i]\right)$, $m, N, n \in \mathbb{N}$ *with* $n^{1-\beta} > 2$. *Then*

*1)*

$$\|A_n(f,x) - f(x)\|_\gamma \leq \left[2\left(\sqrt[2m]{1+4^m}\right)\right]^N \left[\omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{\left\|\|f\|_\gamma\right\|_\infty}{2m\left(n^{1-\beta}-2\right)^{2m}}\right] =: \lambda_1(n), \tag{41}$$

*and*

*2)*

$$\left\|\|A_n(f) - f\|_\gamma\right\|_\infty \leq \lambda_1(n). \tag{42}$$

*We notice that* $\lim_{n\to\infty} A_n(f) \overset{\|\cdot\|_\gamma}{=} f$, *pointwise and uniformly.*

*Above* $\omega_1$ *is with respect to* $p = \infty$.

**Proof.** We observe that

$$\Delta(x) := A_n^*(f,x) - f(x) \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k) =$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right) Z(nx-k) - \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f(x) Z(nx-k) =$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \left(f\left(\frac{k}{n}\right) - f(x)\right) Z(nx-k). \tag{43}$$

Thus

$$\|\Delta(x)\|_\gamma \leq \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \left\|f\left(\frac{k}{n}\right) - f(x)\right\|_\gamma Z(nx-k) =$$

$$\sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty \leq \frac{1}{n^\beta}}}^{\lfloor nb\rfloor} \left\|f\left(\frac{k}{n}\right) - f(x)\right\|_\gamma Z(nx-k) +$$

$$\sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor} \left\| f\left(\frac{k}{n}\right)-f(x)\right\|_{\gamma} Z(nx-k) \overset{(16)}{\leq}$$

$$\omega_1\left(f,\frac{1}{n^{\beta}}\right)+2\left\|\|f\|_{\gamma}\right\|_{\infty} \sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\lfloor nb\rfloor} Z(nx-k) \overset{(21)}{\leq}$$

$$\omega_1\left(f,\frac{1}{n^{\beta}}\right)+\frac{2\left\|\|f\|_{\gamma}\right\|_{\infty}}{4m\left(n^{1-\beta}-2\right)^{2m}}. \quad (44)$$

So that

$$\|\Delta(x)\|_{\gamma} \leq \omega_1\left(f,\frac{1}{n^{\beta}}\right)+\frac{\left\|\|f\|_{\gamma}\right\|_{\infty}}{2m\left(n^{1-\beta}-2\right)^{2m}}. \quad (45)$$

Now using (34) we finish the proof. ■

We make

**Remark 9** *([11], pp. 263-266) Let* $\left(\mathbb{R}^N,\|\cdot\|_p\right)$, $N\in\mathbb{N}$; *where* $\|\cdot\|_p$ *is the* $L_p$*-norm,* $1\leq p\leq\infty$. $\mathbb{R}^N$ *is a Banach space, and* $\left(\mathbb{R}^N\right)^j$ *denotes the* $j$*-fold product space* $\mathbb{R}^N\times...\times\mathbb{R}^N$ *endowed with the max-norm* $\|x\|_{(\mathbb{R}^N)^j}:=\max_{1\leq\lambda\leq j}\|x_\lambda\|_p$, *where* $x:=(x_1,...,x_j)\in\left(\mathbb{R}^N\right)^j$.

*Let* $\left(X,\|\cdot\|_{\gamma}\right)$ *be a general Banach space. Then the space* $L_j:=L_j\left(\left(\mathbb{R}^N\right)^j;X\right)$ *of all* $j$*-multilinear continuous maps* $g:\left(\mathbb{R}^N\right)^j\to X$, $j=1,...,m$, *is a Banach space with norm*

$$\|g\|:=\|g\|_{L_j}:=\sup_{\left(\|x\|_{(\mathbb{R}^N)^j}=1\right)}\|g(x)\|_{\gamma}=\sup\frac{\|g(x)\|_{\gamma}}{\|x_1\|_p...\|x_j\|_p}. \quad (46)$$

*Let* $M$ *be a non-empty convex and compact subset of* $\mathbb{R}^N$ *and* $x_0\in M$ *is fixed.*

*Let* $O$ *be an open subset of* $\mathbb{R}^N: M\subset O$. *Let* $f:O\to X$ *be a continuous function, whose Fréchet derivatives (see [21])* $f^{(j)}:O\to L_j=L_j\left(\left(\mathbb{R}^N\right)^j;X\right)$ *exist and are continuous for* $1\leq j\leq\overline{m}$, $\overline{m}\in\mathbb{N}$.

*Call* $(x-x_0)^j:=(x-x_0,...,x-x_0)\in\left(\mathbb{R}^N\right)^j$, $x\in M$.

*We will work with* $f|_M$.

*Then, by Taylor's formula ([14]), ([21], p. 124), we get*

$$f(x)=\sum_{j=0}^{\overline{m}}\frac{f^{(j)}(x_0)(x-x_0)^j}{j!}+R_{\overline{m}}(x,x_0), \quad \text{all } x\in M, \quad (47)$$

*where the remainder is the Riemann integral*

$$R_{\overline{m}}(x,x_0) := \int_0^1 \frac{(1-u)^{\overline{m}-1}}{(\overline{m}-1)!} \left( f^{(\overline{m})}(x_0 + u(x-x_0)) - f^{(\overline{m})}(x_0) \right) (x-x_0)^{\overline{m}} du, \tag{48}$$

*here we set* $f^{(0)}(x_0)(x-x_0)^0 = f(x_0)$.

*We consider*

$$w := \omega_1 \left( f^{(\overline{m})}, h \right) := \sup_{\substack{x,y \in M: \\ \|x-y\|_p \le h}} \left\| f^{(\overline{m})}(x) - f^{(\overline{m})}(y) \right\|, \tag{49}$$

$h > 0$.

*We obtain*

$$\left\| \left( f^{(\overline{m})}(x_0 + u(x-x_0)) - f^{(\overline{m})}(x_0) \right) (x-x_0)^{\overline{m}} \right\|_\gamma \le$$

$$\left\| f^{(\overline{m})}(x_0 + u(x-x_0)) - f^{(\overline{m})}(x_0) \right\| \cdot \|x-x_0\|_p^{\overline{m}} \le$$

$$w \|x-x_0\|_p^{\overline{m}} \left\lceil \frac{u\|x-x_0\|_p}{h} \right\rceil, \tag{50}$$

*by Lemma 7.1.1, [1], p. 208, where* $\lceil \cdot \rceil$ *is the ceiling.*

*Therefore for all* $x \in M$ *(see [1], pp. 121-122):*

$$\|R_{\overline{m}}(x,x_0)\|_\gamma \le w \|x-x_0\|_p^{\overline{m}} \int_0^1 \left\lceil \frac{u\|x-x_0\|_p}{h} \right\rceil \frac{(1-u)^{\overline{m}-1}}{(\overline{m}-1)!} du$$

$$= \overline{w} \Phi_{\overline{m}} \left( \|x-x_0\|_p \right) \tag{51}$$

*by a change of variable, where*

$$\Phi_{\overline{m}}(t) := \int_0^{|t|} \left\lceil \frac{s}{h} \right\rceil \frac{(|t|-s)^{\overline{m}-1}}{(\overline{m}-1)!} ds = \frac{1}{\overline{m}!} \left( \sum_{j=0}^{\infty} (|t|-jh)_+^{\overline{m}} \right), \quad \forall\, t \in \mathbb{R}, \tag{52}$$

*is a (polynomial) spline function, see [1], p. 210-211.*

*Also from there we get*

$$\Phi_{\overline{m}}(t) \le \left( \frac{|t|^{\overline{m}+1}}{(\overline{m}+1)!h} + \frac{|t|^{\overline{m}}}{2\overline{m}!} + \frac{h|t|^{\overline{m}-1}}{8(\overline{m}-1)!} \right), \quad \forall\, t \in \mathbb{R}, \tag{53}$$

*with equality true only at* $t = 0$.

*Therefore it holds*

$$\|R_{\overline{m}}(x,x_0)\|_\gamma \le w \left( \frac{\|x-x_0\|_p^{\overline{m}+1}}{(\overline{m}+1)!h} + \frac{\|x-x_0\|_p^{\overline{m}}}{2\overline{m}!} + \frac{h\|x-x_0\|_p^{\overline{m}-1}}{8(\overline{m}-1)!} \right), \quad \forall\, x \in M. \tag{54}$$

*We have found that*

$$\left\| f(x) - \sum_{j=0}^{\overline{m}} \frac{f^{(j)}(x_0)(x-x_0)^j}{j!} \right\|_\gamma \leq$$

$$\omega_1\left(f^{(\overline{m})}, h\right)\left(\frac{\|x-x_0\|_p^{\overline{m}+1}}{(\overline{m}+1)!h} + \frac{\|x-x_0\|_p^{\overline{m}}}{2\overline{m}!} + \frac{h\|x-x_0\|_p^{\overline{m}-1}}{8(\overline{m}-1)!}\right) < \infty, \quad (55)$$

$\forall\ x, x_0 \in M$.

*Here* $0 < \omega_1\left(f^{(\overline{m})}, h\right) < \infty$, *by* $M$ *being compact and* $f^{(\overline{m})}$ *being continuous on* $M$.

*One can rewrite (55) as follows:*

$$\left\| f(\cdot) - \sum_{j=0}^{\overline{m}} \frac{f^{(j)}(x_0)(\cdot-x_0)^j}{j!} \right\|_\gamma \leq$$

$$\omega_1\left(f^{(\overline{m})}, h\right)\left(\frac{\|\cdot-x_0\|_p^{\overline{m}+1}}{(\overline{m}+1)!h} + \frac{\|\cdot-x_0\|_p^{\overline{m}}}{2\overline{m}!} + \frac{h\|\cdot-x_0\|_p^{\overline{m}-1}}{8(\overline{m}-1)!}\right), \ \forall\ x_0 \in M, \quad (56)$$

*a pointwise functional inequality on* $M$.

*Here* $(\cdot-x_0)^j$ *maps* $M$ *into* $\left(\mathbb{R}^N\right)^j$ *and it is continuous, also* $f^{(j)}(x_0)$ *maps* $\left(\mathbb{R}^N\right)^j$ *into* $X$ *and it is continuous. Hence their composition* $f^{(j)}(x_0)(\cdot-x_0)^j$ *is continuous from* $M$ *into* $X$.

*Clearly* $f(\cdot) - \sum_{j=0}^{\overline{m}} \frac{f^{(j)}(x_0)(\cdot-x_0)^j}{j!} \in C(M,X)$, *hence* $\left\| f(\cdot) - \sum_{j=0}^{\overline{m}} \frac{f^{(j)}(x_0)(\cdot-x_0)^j}{j!} \right\|_\gamma \in C(M)$.

*Let* $\left\{\widetilde{L}_N\right\}_{N\in\mathbb{N}}$ *be a sequence of positive linear operators mapping* $C(M)$ *into* $C(M)$.

*Therefore we obtain*

$$\left(\widetilde{L}_N\left(\left\| f(\cdot) - \sum_{j=0}^{\overline{m}} \frac{f^{(j)}(x_0)(\cdot-x_0)^j}{j!} \right\|_\gamma\right)\right)(x_0) \leq$$

$$\omega_1\left(f^{(\overline{m})}, h\right)\left[\frac{\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^{\overline{m}+1}\right)\right)(x_0)}{(\overline{m}+1)!h} + \frac{\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^{\overline{m}}\right)\right)(x_0)}{2\overline{m}!}\right.$$

$$\left.+\frac{h\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^{\overline{m}-1}\right)\right)(x_0)}{8(\overline{m}-1)!}\right], \quad (57)$$

$\forall\ N \in \mathbb{N}$, $\forall\ x_0 \in M$.

Clearly (57) is valid when $M = \prod_{i=1}^{N} [a_i, b_i]$ and $\widetilde{L}_n = \widetilde{A}_n$, see (26).

All the above is preparation for the following theorem, where we assume Fréchet differentiability of functions.

This will be a direct application of Theorem 10.2, [11], pp. 268-270. The operators $A_n$, $\widetilde{A}_n$ fulfill its assumptions, see (25), (26), (28), (29) and (30).

We present the following high order approximation results.

**Theorem 10** *Let $O$ open subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1, \infty]$, such that $\prod_{i=1}^{N} [a_i, b_i] \subset O \subseteq \mathbb{R}^N$, and let $\left(X, \|\cdot\|_\gamma\right)$ be a general Banach space. Let $\overline{m} \in \mathbb{N}$ and $f \in C^{\overline{m}}(O, X)$, the space of $\overline{m}$-times continuously Fréchet differentiable functions from $O$ into $X$. We study the approximation of $f|_{\prod_{i=1}^{N} [a_i, b_i]}$. Let $x_0 \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$ and $r > 0$. Then*

*1)*

$$\left\| (A_n(f))(x_0) - \sum_{j=0}^{\overline{m}} \frac{1}{j!} \left( A_n \left( f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma \leq$$

$$\frac{\omega_1\left( f^{(\overline{m})}, r \left( \left( \widetilde{A}_n \left( \|\cdot - x_0\|_p^{\overline{m}+1} \right) \right)(x_0) \right)^{\frac{1}{\overline{m}+1}} \right)}{r\overline{m}!} \left( \left( \widetilde{A}_n \left( \|\cdot - x_0\|_p^{\overline{m}+1} \right) \right)(x_0) \right)^{\left(\frac{\overline{m}}{\overline{m}+1}\right)}$$

$$\left[ \frac{1}{(\overline{m}+1)} + \frac{r}{2} + \frac{\overline{m}r^2}{8} \right], \tag{58}$$

*2) additionally if $f^{(j)}(x_0) = 0$, $j = 1, ..., \overline{m}$, we have*

$$\|(A_n(f))(x_0) - f(x_0)\|_\gamma \leq$$

$$\frac{\omega_1\left( f^{(\overline{m})}, r \left( \left( \widetilde{A}_n \left( \|\cdot - x_0\|_p^{\overline{m}+1} \right) \right)(x_0) \right)^{\frac{1}{\overline{m}+1}} \right)}{r\overline{m}!} \left( \left( \widetilde{A}_n \left( \|\cdot - x_0\|_p^{\overline{m}+1} \right) \right)(x_0) \right)^{\left(\frac{\overline{m}}{\overline{m}+1}\right)} \tag{59}$$

$$\left[ \frac{1}{(\overline{m}+1)} + \frac{r}{2} + \frac{\overline{m}r^2}{8} \right],$$

*3)*

$$\|(A_n(f))(x_0) - f(x_0)\|_\gamma \leq \sum_{j=1}^{\overline{m}} \frac{1}{j!} \left\| \left( A_n \left( f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma +$$

$$\frac{\omega_1\left( f^{(\overline{m})}, r \left( \left( \widetilde{A}_n \left( \|\cdot - x_0\|_p^{\overline{m}+1} \right) \right)(x_0) \right)^{\frac{1}{\overline{m}+1}} \right)}{r\overline{m}!} \left( \left( \widetilde{A}_n \left( \|\cdot - x_0\|_p^{\overline{m}+1} \right) \right)(x_0) \right)^{\left(\frac{\overline{m}}{\overline{m}+1}\right)} \tag{60}$$

$$\left[\frac{1}{(\overline{m}+1)}+\frac{r}{2}+\frac{\overline{m}r^2}{8}\right],$$

*and*

*4)*

$$\left\| \left\| A_n(f) - f \right\|_\gamma \right\|_{\infty, \prod_{i=1}^{N}[a_i,b_i]} \leq$$

$$\sum_{j=1}^{\overline{m}} \frac{1}{j!} \left\| \left\| \left( A_n \left( f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma \right\|_{\infty, x_0 \in \prod_{i=1}^{N}[a_i,b_i]} +$$

$$\frac{\omega_1 \left( f^{(\overline{m})}, r \left\| \left( \widetilde{A}_n \left( \|\cdot - x_0\|_p^{\overline{m}+1} \right) \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N}[a_i,b_i]}^{\frac{1}{\overline{m}+1}} \right)}{r\overline{m}!}$$

$$\left\| \left( \widetilde{A}_n \left( \|\cdot - x_0\|_p^{\overline{m}+1} \right) \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N}[a_i,b_i]}^{\left(\frac{\overline{m}}{\overline{m}+1}\right)} \quad (61)$$

$$\left[\frac{1}{(\overline{m}+1)}+\frac{r}{2}+\frac{\overline{m}r^2}{8}\right].$$

We need

**Lemma 11** *The function* $\left( \widetilde{A}_n \left( \|\cdot - x_0\|_p^{\overline{m}} \right) \right)(x_0)$ *is continuous in* $x_0 \in \left( \prod_{i=1}^{N}[a_i,b_i] \right)$, $\overline{m} \in \mathbb{N}$.

**Proof.** By Lemma 10.3, [11], p. 272. ■

We give

**Corollary 12** *(to Theorem 10, case of* $\overline{m}=1$*) Then*

*1)*

$$\left\| (A_n(f))(x_0) - f(x_0) \right\|_\gamma \leq \left\| \left( A_n \left( f^{(1)}(x_0)(\cdot - x_0) \right) \right)(x_0) \right\|_\gamma +$$

$$\frac{1}{2r}\omega_1 \left( f^{(1)}, r \left( \left( \widetilde{A}_n \left( \|\cdot - x_0\|_p^2 \right) \right)(x_0) \right)^{\frac{1}{2}} \right) \left( \left( \widetilde{A}_n \left( \|\cdot - x_0\|_p^2 \right) \right)(x_0) \right)^{\frac{1}{2}} \quad (62)$$

$$\left[1 + r + \frac{r^2}{4}\right],$$

*and*

*2)*

$$\left\| \left\| (A_n(f)) - f \right\|_\gamma \right\|_{\infty, \prod_{i=1}^{N}[a_i,b_i]} \leq$$

$$\left\| \left\| \left( A_n \left( f^{(1)}(x_0)(\cdot - x_0) \right) \right)(x_0) \right\|_{\gamma} \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]} +$$

$$\frac{1}{2r} \omega_1 \left( f^{(1)}, r \left\| \left( \widetilde{A}_n \left( \| \cdot - x_0 \|_p^2 \right) \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]}^{\frac{1}{2}} \right)$$

$$\left\| \left( \widetilde{A}_n \left( \| \cdot - x_0 \|_p^2 \right) \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]}^{\frac{1}{2}} \left[ 1 + r + \frac{r^2}{4} \right], \tag{63}$$

$r > 0$.

We make

**Remark 13** *We estimate* $0 < \alpha < 1$, $m, \overline{m}, n \in \mathbb{N} : n^{1-\alpha} > 2$,

$$\widetilde{A}_n \left( \| \cdot - x_0 \|_\infty^{\overline{m}+1} \right)(x_0) = \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^{\overline{m}+1} Z(nx_0 - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx_0 - k)} \overset{(22)}{<}$$

$$\left[ 2 \left( \sqrt[2m]{1 + 4^m} \right) \right]^N \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^{\overline{m}+1} Z(nx_0 - k) = \tag{64}$$

$$\left[ 2 \left( \sqrt[2m]{1 + 4^m} \right) \right]^N \left\{ \sum_{\substack{k = \lceil na \rceil \\ : \left\| \frac{k}{n} - x_0 \right\|_\infty \leq \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^{\overline{m}+1} Z(nx_0 - k) + \right.$$

$$\left. \sum_{\substack{k = \lceil na \rceil \\ : \left\| \frac{k}{n} - x_0 \right\|_\infty > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^{\overline{m}+1} Z(nx_0 - k) \right\} \overset{(23)}{\leq}$$

$$\left[ 2 \left( \sqrt[2m]{1 + 4^m} \right) \right]^N \left\{ \frac{1}{n^{\alpha(\overline{m}+1)}} + \frac{\|b - a\|_\infty^{\overline{m}+1}}{4m \left( n^{1-\alpha} - 2 \right)^{2m}} \right\}, \tag{65}$$

*(where* $b - a = (b_1 - a_1, ..., b_N - a_N)$*).*

*We have proved that* ($\forall\ x_0 \in \prod_{i=1}^{N} [a_i, b_i]$)

$$\widetilde{A}_n \left( \| \cdot - x_0 \|_\infty^{\overline{m}+1} \right)(x_0) < \left[ 2 \left( \sqrt[2m]{1 + 4^m} \right) \right]^N \left\{ \frac{1}{n^{\alpha(\overline{m}+1)}} + \frac{\|b - a\|_\infty^{\overline{m}+1}}{4m \left( n^{1-\alpha} - 2 \right)^{2m}} \right\} =: \varphi_1(n) \tag{66}$$

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*($0<\alpha<1$, $m,\overline{m},n\in\mathbb{N}:n^{1-\alpha}>2$).*

*And, consequently it holds*

$$\left\|\widetilde{A}_n\left(\|\cdot-x_0\|_\infty^{\overline{m}+1}\right)(x_0)\right\|_{\infty,x_0\in\prod_{i=1}^{N}[a_i,b_i]}<$$

$$\left[2\left(\sqrt[2m]{1+4^m}\right)\right]^N\left\{\frac{1}{n^{\alpha(\overline{m}+1)}}+\frac{\|b-a\|_\infty^{\overline{m}+1}}{4m\left(n^{1-\alpha}-2\right)^{2m}}\right\}=\varphi_1(n)\to 0,\quad as\ n\to+\infty. \tag{67}$$

*So, we have that $\varphi_1(n)\to 0$, as $n\to+\infty$. Thus, when $p\in[1,\infty]$, from Theorem 10 we have the convergence to zero in the right hand sides of parts (1), (2).*

*Next we estimate* $\left\|\left(\widetilde{A}_n\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\gamma$.

*We have that*

$$\left(\widetilde{A}_n\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f^{(j)}(x_0)\left(\frac{k}{n}-x_0\right)^j Z(nx_0-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx_0-k)}. \tag{68}$$

*When $p=\infty$, $j=1,...,\overline{m}$, we obtain*

$$\left\|f^{(j)}(x_0)\left(\frac{k}{n}-x_0\right)^j\right\|_\gamma\le\left\|f^{(j)}(x_0)\right\|\left\|\frac{k}{n}-x_0\right\|_\infty^j. \tag{69}$$

*We further have that*

$$\left\|\left(\widetilde{A}_n\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\gamma\overset{(22)}{<}$$

$$\left[2\left(\sqrt[2m]{1+4^m}\right)\right]^N\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|f^{(j)}(x_0)\left(\frac{k}{n}-x_0\right)^j\right\|_\gamma Z(nx_0-k)\right)\le$$

$$\left[2\left(\sqrt[2m]{1+4^m}\right)\right]^N\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|f^{(j)}(x_0)\right\|\left\|\frac{k}{n}-x_0\right\|_\infty^j Z(nx_0-k)\right)= \tag{70}$$

$$\left[2\left(\sqrt[2m]{1+4^m}\right)\right]^N\left\|f^{(j)}(x_0)\right\|\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|\frac{k}{n}-x_0\right\|_\infty^j Z(nx_0-k)\right)=$$

$$\left[2\left(\sqrt[2m]{1+4^m}\right)\right]^N\left\|f^{(j)}(x_0)\right\|\left\{\sum_{\substack{k=\lceil na\rceil\\ \left\{:\left\|\frac{k}{n}-x_0\right\|_\infty\le\frac{1}{n^\alpha}\right.}}^{\lfloor nb\rfloor}\left\|\frac{k}{n}-x_0\right\|_\infty^j Z(nx_0-k)\right.$$

$$+ \sum_{\substack{k=\lceil na \rceil \\ : \left\| \frac{k}{n} - x_0 \right\|_\infty > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^j Z(nx_0 - k) \Bigg\} \overset{(21)}{\leq} \tag{71}$$

$$\left[2\left(\sqrt[2m]{1+4^m}\right)\right]^N \left\| f^{(j)}(x_0) \right\| \left\{ \frac{1}{n^{\alpha j}} + \frac{\|b-a\|_\infty^j}{4m\left(n^{1-\alpha}-2\right)^{2m}} \right\} \to 0, \text{ as } n \to \infty.$$

*That is*

$$\left\| \left( \widetilde{A}_n \left( f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma \to 0, \text{ as } n \to \infty.$$

*Therefore when* $p = \infty$, *for* $j = 1, ..., \overline{m}$, *we have proved:*

$$\left\| \left( \widetilde{A}_n \left( f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma <$$

$$\left[2\left(\sqrt[2m]{1+4^m}\right)\right]^N \left\| f^{(j)}(x_0) \right\| \left\{ \frac{1}{n^{\alpha j}} + \frac{\|b-a\|_\infty^j}{4m\left(n^{1-\alpha}-2\right)^{2m}} \right\} \leq \tag{72}$$

$$\left[2\left(\sqrt[2m]{1+4^m}\right)\right]^N \left\| f^{(j)} \right\|_\infty \left\{ \frac{1}{n^{\alpha j}} + \frac{\|b-a\|_\infty^j}{4m\left(n^{1-\alpha}-2\right)^{2m}} \right\} =: \varphi_{2j}(n) < \infty,$$

*and converges to zero, as* $n \to \infty$.

We conclude:

In Theorem 10, the right hand sides of (60) and (61) converge to zero as $n \to \infty$, for any $p \in [1, \infty]$.

Also in Corollary 12, the right hand sides of (62) and (63) converge to zero as $n \to \infty$, for any $p \in [1, \infty]$.

**Conclusion 14** *We have proved that the left hand sides of (58), (59), (60), (61) and (62), (63) converge to zero as* $n \to \infty$, *for* $p \in [1, \infty]$. *Consequently* $A_n \to I$ *(unit operator) pointwise and uniformly, as* $n \to \infty$, *where* $p \in [1, \infty]$. *In the presence of initial conditions we achieve a higher speed of convergence, see (59). Higher speed of convergence happens also to the left hand side of (58).*

We further give

**Corollary 15** *(to Theorem 10) Let* $O$ *open subset of* $\left(\mathbb{R}^N, \|\cdot\|_\infty\right)$, *such that* $\prod_{i=1}^N [a_i, b_i] \subset O \subseteq \mathbb{R}^N$, *and let* $\left(X, \|\cdot\|_\gamma\right)$ *be a general Banach space. Let* $\overline{m} \in \mathbb{N}$ *and* $f \in C^{\overline{m}}(O, X)$, *the space of* $\overline{m}$*-times continuously Fréchet differentiable functions from* $O$ *into* $X$. *We study the approximation of* $f|_{\prod_{i=1}^N [a_i,b_i]}$. *Let* $x_0 \in$

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$\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ *and* $r>0$. *Here* $\varphi_1(n)$ *as in (66) and* $\varphi_{2j}(n)$ *as in (72), where* $n\in\mathbb{N}: n^{1-\alpha}>2$, $0<\alpha<1$, $j=1,...,\overline{m}$. *Then*

*1)*

$$\left\|(A_n(f))(x_0)-\sum_{j=0}^{\overline{m}}\frac{1}{j!}\left(A_n\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\gamma\leq$$

$$\frac{\omega_1\left(f^{(\overline{m})},r(\varphi_1(n))^{\frac{1}{\overline{m}+1}}\right)}{r\overline{m}!}(\varphi_1(n))^{\left(\frac{\overline{m}}{\overline{m}+1}\right)}\left[\frac{1}{(\overline{m}+1)}+\frac{r}{2}+\frac{\overline{m}r^2}{8}\right], \tag{73}$$

*2) additionally, if* $f^{(j)}(x_0)=0$, $j=1,...,\overline{m}$, *we have*

$$\|(A_n(f))(x_0)-f(x_0)\|_\gamma\leq$$

$$\frac{\omega_1\left(f^{(\overline{m})},r(\varphi_1(n))^{\frac{1}{\overline{m}+1}}\right)}{r\overline{m}!}(\varphi_1(n))^{\left(\frac{\overline{m}}{\overline{m}+1}\right)}\left[\frac{1}{(\overline{m}+1)}+\frac{r}{2}+\frac{\overline{m}r^2}{8}\right], \tag{74}$$

*3)*

$$\left\|\|A_n(f)-f\|_\gamma\right\|_{\infty,\prod_{i=1}^{N}[a_i,b_i]}\leq\sum_{j=1}^{\overline{m}}\frac{\varphi_{2j}(n)}{j!}+$$

$$\frac{\omega_1\left(f^{(\overline{m})},r(\varphi_1(n))^{\frac{1}{\overline{m}+1}}\right)}{r\overline{m}!}(\varphi_1(n))^{\left(\frac{\overline{m}}{\overline{m}+1}\right)} \tag{75}$$

$$\left[\frac{1}{(\overline{m}+1)}+\frac{r}{2}+\frac{\overline{m}r^2}{8}\right]=:\varphi_3(n)\to 0,\ \text{as } n\to\infty.$$

We continue with

**Theorem 16** *Let* $f\in C_B\left(\mathbb{R}^N,X\right)$, $0<\beta<1$, $x\in\mathbb{R}^N$, $m,N,n\in\mathbb{N}$ *with* $n^{1-\beta}>2$, $\omega_1$ *is for* $p=\infty$. *Then*

*1)*

$$\|B_n(f,x)-f(x)\|_\gamma\leq\omega_1\left(f,\frac{1}{n^\beta}\right)+\frac{\left\|\|f\|_\gamma\right\|_\infty}{2m\left(n^{1-\beta}-2\right)^{2m}}=:\lambda_2(n), \tag{76}$$

*2)*

$$\left\|\|B_n(f)-f\|_\gamma\right\|_\infty\leq\lambda_2(n). \tag{77}$$

*Given that* $f\in\left(C_U\left(\mathbb{R}^N,X\right)\cap C_B\left(\mathbb{R}^N,X\right)\right)$, *we obtain* $\lim_{n\to\infty}B_n(f)=f$, *uniformly.*

**Proof.** We have that

$$B_n(f,x) - f(x) \overset{(16)}{=} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx-k) - f(x) \sum_{k=-\infty}^{\infty} Z(nx-k) = \tag{78}$$

$$\sum_{k=-\infty}^{\infty} \left( f\left(\frac{k}{n}\right) - f(x) \right) Z(nx-k).$$

Hence

$$\|B_n(f,x) - f(x)\|_\gamma \leq \sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z(nx-k) =$$

$$\sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta}}}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z(nx-k) +$$

$$\sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z(nx-k) \overset{(16)}{\leq}$$

$$\omega_1\left(f, \frac{1}{n^\beta}\right) + 2 \left\| \|f\|_\gamma \right\|_\infty \sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} Z(nx-k) \overset{(23)}{\leq}$$

$$\omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{\left\| \|f\|_\gamma \right\|_\infty}{2m\left(n^{1-\beta} - 2\right)^{2m}}, \tag{79}$$

proving the claim. ■

We give

**Theorem 17** *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $m, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, $\omega_1$ is for $p = \infty$. Then*

*1)*

$$\|C_n(f,x) - f(x)\|_\gamma \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) + \frac{\left\| \|f\|_\gamma \right\|_\infty}{2m\left(n^{1-\beta} - 2\right)^{2m}} =: \lambda_3(n), \tag{80}$$

*2)*

$$\left\| \|C_n(f) - f\|_\gamma \right\|_\infty \leq \lambda_3(n). \tag{81}$$

*Given that $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$, we obtain $\lim_{n\to\infty} C_n(f) = f$, uniformly.*

**Proof.** We notice that

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt = \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} \dots \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1, t_2, \dots, t_N)\, dt_1 dt_2 \dots dt_N =$$

$$\int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} \dots \int_0^{\frac{1}{n}} f\left(t_1 + \frac{k_1}{n}, t_2 + \frac{k_2}{n}, \dots, t_N + \frac{k_N}{n}\right) dt_1 \dots dt_N = \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt. \tag{82}$$

Thus it holds (by (38))

$$C_n(f, x) = \sum_{k=-\infty}^{\infty} \left( n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt \right) Z(nx - k). \tag{83}$$

We observe that

$$\|C_n(f, x) - f(x)\|_\gamma =$$

$$\left\| \sum_{k=-\infty}^{\infty} \left( n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt \right) Z(nx - k) - \sum_{k=-\infty}^{\infty} f(x) Z(nx - k) \right\|_\gamma =$$

$$\left\| \sum_{k=-\infty}^{\infty} \left( \left( n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt \right) - f(x) \right) Z(nx - k) \right\|_\gamma =$$

$$\left\| \sum_{k=-\infty}^{\infty} \left( n^N \int_0^{\frac{1}{n}} \left( f\left(t + \frac{k}{n}\right) - f(x) \right) dt \right) Z(nx - k) \right\|_\gamma \leq \tag{84}$$

$$\sum_{k=-\infty}^{\infty} \left( n^N \int_0^{\frac{1}{n}} \left\| f\left(t + \frac{k}{n}\right) - f(x) \right\|_\gamma dt \right) Z(nx - k) =$$

$$\sum_{\begin{cases} k = -\infty \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta} \end{cases}}^{\infty} \left( n^N \int_0^{\frac{1}{n}} \left\| f\left(t + \frac{k}{n}\right) - f(x) \right\|_\gamma dt \right) Z(nx - k) +$$

$$\sum_{\begin{cases} k = -\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} \left( n^N \int_0^{\frac{1}{n}} \left\| f\left(t + \frac{k}{n}\right) - f(x) \right\|_\gamma dt \right) Z(nx - k) \leq$$

$$\sum_{\begin{cases} k = -\infty \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta} \end{cases}}^{\infty} \left( n^N \int_0^{\frac{1}{n}} \omega_1\left( f, \|t\|_\infty + \left\| \frac{k}{n} - x \right\|_\infty \right) dt \right) Z(nx - k) +$$

$$2\left\|\|f\|_\gamma\right\|_\infty \left( \sum_{\substack{k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}}}^{\infty} Z(|nx-k|) \right) \leq$$

$$\omega_1\left(f, \frac{1}{n}+\frac{1}{n^\beta}\right) + \frac{\left\|\|f\|_\gamma\right\|_\infty}{2m\left(n^{1-\beta}-2\right)^{2m}}, \tag{85}$$

proving the claim. ■

We also present

**Theorem 18** *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $m, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, $\omega_1$ is for $p = \infty$. Then*

*1)*

$$\|D_n(f,x) - f(x)\|_\gamma \leq \omega_1\left(f, \frac{1}{n}+\frac{1}{n^\beta}\right) + \frac{\left\|\|f\|_\gamma\right\|_\infty}{2m\left(n^{1-\beta}-2\right)^{2m}} = \lambda_4(n), \tag{86}$$

*2)*

$$\left\|\|D_n(f) - f\|_\gamma\right\|_\infty \leq \lambda_4(n). \tag{87}$$

*Given that $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$, we obtain $\lim_{n\to\infty} D_n(f) = f$, uniformly.*

**Proof.** We have that (by (40))

$$\|D_n(f,x) - f(x)\|_\gamma = \left\| \sum_{k=-\infty}^{\infty} \delta_{nk}(f) Z(nx-k) - \sum_{k=-\infty}^{\infty} f(x) Z(nx-k) \right\|_\gamma = \tag{88}$$

$$\left\| \sum_{k=-\infty}^{\infty} \left(\delta_{nk}(f) - f(x)\right) Z(nx-k) \right\|_\gamma =$$

$$\left\| \sum_{k=-\infty}^{\infty} \left( \sum_{r=0}^{\theta} w_r \left( f\left(\frac{k}{n}+\frac{r}{n\theta}\right) - f(x) \right) \right) Z(nx-k) \right\|_\gamma \leq$$

$$\sum_{k=-\infty}^{\infty} \left( \sum_{r=0}^{\theta} w_r \left\| f\left(\frac{k}{n}+\frac{r}{n\theta}\right) - f(x) \right\|_\gamma \right) Z(nx-k) =$$

$$\sum_{\substack{k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty \leq \frac{1}{n^\beta}}}^{\infty} \left( \sum_{r=0}^{\theta} w_r \left\| f\left(\frac{k}{n}+\frac{r}{n\theta}\right) - f(x) \right\|_\gamma \right) Z(nx-k) +$$

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$$\sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} \left( \sum_{r=0}^{\theta} w_r \left\| f\left( \frac{k}{n} + \frac{r}{n\theta} \right) - f(x) \right\|_\gamma \right) Z(nx-k) \leq$$

$$\sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta}}}^{\infty} \left( \sum_{r=0}^{\theta} w_r \omega_1 \left( f, \left\| \frac{k}{n} - x \right\|_\infty + \left\| \frac{r}{n\theta} \right\|_\infty \right) \right) Z(nx-k) +$$

$$2 \left\| \|f\|_\gamma \right\|_\infty \left( \sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} Z(nx-k) \right) \leq$$

$$\omega_1 \left( f, \frac{1}{n} + \frac{1}{n^\beta} \right) + \frac{\left\| \|f\|_\gamma \right\|_\infty}{2m \left( n^{1-\beta} - 2 \right)^{2m}}, \tag{89}$$

proving the claim. ■

We make

**Definition 19** *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \in \mathbb{N}$, where $\left(X, \|\cdot\|_\gamma\right)$ is a Banach space. We define the general neural network operator*

$$F_n(f,x) := \sum_{k=-\infty}^{\infty} l_{nk}(f) Z(nx-k) =$$

$$\begin{cases} B_n(f,x), & \text{if } l_{nk}(f) = f\left(\frac{k}{n}\right), \\ C_n(f,x), & \text{if } l_{nk}(f) = n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt, \\ D_n(f,x), & \text{if } l_{nk}(f) = \delta_{nk}(f). \end{cases} \tag{90}$$

Clearly $l_{nk}(f)$ is an $X$-valued bounded linear functional such that $\|l_{nk}(f)\|_\gamma \leq \left\| \|f\|_\gamma \right\|_\infty$.

Hence $F_n(f)$ is a bounded linear operator with $\left\| \|F_n(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty$.

We need

**Theorem 20** *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \geq 1$. Then $F_n(f) \in C_B\left(\mathbb{R}^N, X\right)$.*

**Proof.** Clearly $F_n(f)$ is a bounded function.

Next we prove the continuity of $F_n(f)$. Notice for $N = 1$, $Z = \Phi$ by (14).

We will use the generalized Weierstrass $M$ test: If a sequence of positive constants $M_1, M_2, M_3, ...,$ can be found such that in some interval

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(a) $\|u_n(x)\|_\gamma \leq M_n$, $n = 1, 2, 3, ...$

(b) $\sum M_n$ converges,

then $\sum u_n(x)$ is uniformly and absolutely convergent in the interval.

Also we will use:

If $\{u_n(x)\}$, $n = 1, 2, 3, ...$ are continuous in $[a, b]$ and if $\sum u_n(x)$ converges uniformly to the sum $S(x)$ in $[a, b]$, then $S(x)$ is continuous in $[a, b]$. I.e. a uniformly convergent series of continuous functions is a continuous function. First we prove claim for $N = 1$.

We will prove that $\sum_{k=-\infty}^{\infty} l_{nk}(f) \Phi(nx - k)$ is continuous in $x \in \mathbb{R}$.

There always exists $\lambda \in \mathbb{N}$ such that $nx \in [-\lambda, \lambda]$.

Since $nx \leq \lambda$, then $-nx \geq -\lambda$ and $k - nx \geq k - \lambda \geq 0$, when $k \geq \lambda$. Therefore

$$\sum_{k=\lambda}^{\infty} \Phi(nx - k) = \sum_{k=\lambda}^{\infty} \Phi(k - nx) \leq \sum_{k=\lambda}^{\infty} \Phi(k - \lambda) = \sum_{k'=0}^{\infty} \Phi(k') \leq 1. \tag{91}$$

So for $k \geq \lambda$ we get

$$\|l_{nk}(f)\|_\gamma \Phi(nx - k) \leq \left\| \|f\|_\gamma \right\|_\infty \Phi(k - \lambda),$$

and

$$\left\| \|f\|_\gamma \right\|_\infty \sum_{k=\lambda}^{\infty} \Phi(k - \lambda) \leq \left\| \|f\|_\gamma \right\|_\infty.$$

Hence by the generalized Weierstrass $M$ test we obtain that $\sum_{k=\lambda}^{\infty} l_{nk}(f) \Phi(nx - k)$ is uniformly and absolutely convergent on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

Since $l_{nk}(f) \Phi(nx - k)$ is continuous in $x$, then $\sum_{k=\lambda}^{\infty} l_{nk}(f) \Phi(nx - k)$ is continuous on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

Because $nx \geq -\lambda$, then $-nx \leq \lambda$, and $k - nx \leq k + \lambda \leq 0$, when $k \leq -\lambda$. Therefore

$$\sum_{k=-\infty}^{-\lambda} \Phi(nx - k) = \sum_{k=-\infty}^{-\lambda} \Phi(k - nx) \leq \sum_{k=-\infty}^{-\lambda} \Phi(k + \lambda) = \sum_{k'=-\infty}^{0} \Phi(k') \leq 1.$$

So for $k \leq -\lambda$ we get

$$\|l_{nk}(f)\|_\gamma \Phi(nx - k) \leq \left\| \|f\|_\gamma \right\|_\infty \Phi(k + \lambda), \tag{92}$$

and

$$\left\| \|f\|_\gamma \right\|_\infty \sum_{k=-\infty}^{-\lambda} \Phi(k + \lambda) \leq \left\| \|f\|_\gamma \right\|_\infty.$$

Hence by Weierstrass $M$ test we obtain that $\sum_{k=-\infty}^{-\lambda} l_{nk}(f) \Phi(nx - k)$ is uniformly and absolutely convergent on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

Since $l_{nk}(f)\Phi(nx-k)$ is continuous in $x$, then $\sum_{k=-\infty}^{-\lambda} l_{nk}(f)\Phi(nx-k)$ is continuous on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

So we proved that $\sum_{k=\lambda}^{\infty} l_{nk}(f)\Phi(nx-k)$ and $\sum_{k=-\infty}^{-\lambda} l_{nk}(f)\Phi(nx-k)$ are continuous on $\mathbb{R}$. Since $\sum_{k=-\lambda+1}^{\lambda-1} l_{nk}(f)\Phi(nx-k)$ is a finite sum of continuous functions on $\mathbb{R}$, it is also a continuous function on $\mathbb{R}$.

Writing

$$\sum_{k=-\infty}^{\infty} l_{nk}(f)\Phi(nx-k) = \sum_{k=-\infty}^{-\lambda} l_{nk}(f)\Phi(nx-k) +$$

$$\sum_{k=-\lambda+1}^{\lambda-1} l_{nk}(f)\Phi(nx-k) + \sum_{k=\lambda}^{\infty} l_{nk}(f)\Phi(nx-k) \tag{93}$$

we have it as a continuous function on $\mathbb{R}$. Therefore $F_n(f)$, when $N=1$, is a continuous function on $\mathbb{R}$.

When $N=2$ we have

$$F_n(f,x_1,x_2) = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} l_{nk}(f)\Phi(nx_1-k_1)\Phi(nx_2-k_2) =$$

$$\sum_{k_1=-\infty}^{\infty}\Phi(nx_1-k_1)\left(\sum_{k_2=-\infty}^{\infty} l_{nk}(f)\Phi(nx_2-k_2)\right)$$

(there always exist $\lambda_1, \lambda_2 \in \mathbb{N}$ such that $nx_1 \in [-\lambda_1, \lambda_1]$ and $nx_2 \in [-\lambda_2, \lambda_2]$)

$$= \sum_{k_1=-\infty}^{\infty}\Phi(nx_1-k_1)\left[\sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f)\Phi(nx_2-k_2) +\right.$$

$$\left.\sum_{k_2=-\lambda_2+1}^{\lambda_2-1} l_{nk}(f)\Phi(nx_2-k_2) + \sum_{k_2=\lambda_2}^{\infty} l_{nk}(f)\Phi(nx_2-k_2)\right] =$$

$$= \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f)\Phi(nx_1-k_1)\Phi(nx_2-k_2) +$$

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\lambda_2+1}^{\lambda_2-1} l_{nk}(f)\Phi(nx_1-k_1)\Phi(nx_2-k_2) +$$

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=\lambda_2}^{\infty} l_{nk}(f)\Phi(nx_1-k_1)\Phi(nx_2-k_2) =: (*).$$

(For convenience call

$$F(k_1,k_2,x_1,x_2) := l_{nk}(f)\Phi(nx_1-k_1)\Phi(nx_2-k_2).\ )$$

Thus

$$(*) = \sum_{k_1=-\infty}^{-\lambda_1} \sum_{k_2=-\infty}^{-\lambda_2} F(k_1,k_2,x_1,x_2) + \sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=-\infty}^{-\lambda_2} F(k_1,k_2,x_1,x_2) +$$

$$\sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} F(k_1,k_2,x_1,x_2) + \sum_{k_1=-\infty}^{-\lambda_1} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1,k_2,x_1,x_2) +$$

$$\sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1,k_2,x_1,x_2) + \sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1,k_2,x_1,x_2) +$$

$$\sum_{k_1=-\infty}^{-\lambda_1} \sum_{k_2=\lambda_2}^{\infty} F(k_1,k_2,x_1,x_2) + \sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=\lambda_2}^{\infty} F(k_1,k_2,x_1,x_2) + \quad (94)$$

$$\sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=\lambda_2}^{\infty} F(k_1,k_2,x_1,x_2).$$

Notice that the finite sum of continuous functions $F(k_1,k_2,x_1,x_2)$, $\sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1,k_2,x_1,x_2)$ is a continuous function.

The rest of the summands of $F_n(f,x_1,x_2)$ are treated all the same way and similarly to the case of $N=1$. The method is demonstrated as follows.

We will prove that $\sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f)\,\Phi(nx_1-k_1)\,\Phi(nx_2-k_2)$ is continuous in $(x_1,x_2) \in \mathbb{R}^2$.

The continuous function

$$\|l_{nk}(f)\|_\gamma \,\Phi(nx_1-k_1)\,\Phi(nx_2-k_2) \le \left\| \|f\|_\gamma \right\|_\infty \Phi(k_1-\lambda_1)\,\Phi(k_2+\lambda_2),$$

and

$$\left\| \|f\|_\gamma \right\|_\infty \sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} \Phi(k_1-\lambda_1)\,\Phi(k_2+\lambda_2) =$$

$$\left\| \|f\|_\gamma \right\|_\infty \left( \sum_{k_1=\lambda_1}^{\infty} \Phi(k_1-\lambda_1) \right) \left( \sum_{k_2=-\infty}^{-\lambda_2} \Phi(k_2+\lambda_2) \right) \le$$

$$\left\| \|f\|_\gamma \right\|_\infty \left( \sum_{k_1'=0}^{\infty} \Phi(k_1') \right) \left( \sum_{k_2'=-\infty}^{0} \Phi(k_2') \right) \le \left\| \|f\|_\gamma \right\|_\infty .$$

So by the Weierstrass $M$ test we get that $\sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f)\,\Phi(nx_1-k_1)\,\Phi(nx_2-k_2)$ is uniformly and absolutely convergent. Therefore it is continuous on $\mathbb{R}^2$.

Next we prove continuity on $\mathbb{R}^2$ of $\sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f)\,\Phi(nx_1-k_1)\,\Phi(nx_2-k_2)$.

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Notice here that

$$\|l_{nk}(f)\|_{\gamma} \Phi(nx_1 - k_1) \Phi(nx_2 - k_2) \leq \left\| \|f\|_{\gamma} \right\|_{\infty} \Phi(nx_1 - k_1) \Phi(k_2 + \lambda_2)$$

$$\leq \left\| \|f\|_{\gamma} \right\|_{\infty} \Phi(0) \Phi(k_2 + \lambda_2) = \frac{1}{2\sqrt[2m]{2}} \cdot \left\| \|f\|_{\gamma} \right\|_{\infty} \Phi(k_2 + \lambda_2),$$

and

$$\frac{1}{2\sqrt[2m]{2}} \cdot \left\| \|f\|_{\gamma} \right\|_{\infty} \left( \sum_{k_1 = -\lambda_1 + 1}^{\lambda_1 - 1} 1 \right) \left( \sum_{k_2 = -\infty}^{-\lambda_2} \Phi(k_2 + \lambda_2) \right) =$$

$$\frac{1}{2\sqrt[2m]{2}} \cdot \left\| \|f\|_{\gamma} \right\|_{\infty} (2\lambda_1 - 1) \left( \sum_{k_2' = -\infty}^{0} \Phi(k_2') \right) \leq \frac{1}{2\sqrt[2m]{2}} \cdot (2\lambda_1 - 1) \left\| \|f\|_{\gamma} \right\|_{\infty}. \tag{95}$$

So the double series under consideration is uniformly convergent and continuous. Clearly $F_n(f, x_1, x_2)$ is proved to be continuous on $\mathbb{R}^2$.

Similarly reasoning one can prove easily now, but with more tedious work, that $F_n(f, x_1, ..., x_N)$ is continuous on $\mathbb{R}^N$, for any $N \geq 1$. We choose to omit this similar extra work. ■

**Remark 21** *By (25) it is obvious that* $\left\| \|A_n(f)\|_{\gamma} \right\|_{\infty} \leq \left\| \|f\|_{\gamma} \right\|_{\infty} < \infty$, *and* $A_n(f) \in C\left( \prod_{i=1}^{N} [a_i, b_i], X \right)$, *given that* $f \in C\left( \prod_{i=1}^{N} [a_i, b_i], X \right)$.

*Call* $L_n$ *any of the operators* $A_n, B_n, C_n, D_n$.

*Clearly then*

$$\left\| \|L_n^2(f)\|_{\gamma} \right\|_{\infty} = \left\| \|L_n(L_n(f))\|_{\gamma} \right\|_{\infty} \leq \left\| \|L_n(f)\|_{\gamma} \right\|_{\infty} \leq \left\| \|f\|_{\gamma} \right\|_{\infty}, \tag{96}$$

*etc.*

*Therefore we get*

$$\left\| \|L_n^k(f)\|_{\gamma} \right\|_{\infty} \leq \left\| \|f\|_{\gamma} \right\|_{\infty}, \quad \forall\, k \in \mathbb{N}, \tag{97}$$

*the contraction property.*

*Also we see that*

$$\left\| \|L_n^k(f)\|_{\gamma} \right\|_{\infty} \leq \left\| \|L_n^{k-1}(f)\|_{\gamma} \right\|_{\infty} \leq ... \leq \left\| \|L_n(f)\|_{\gamma} \right\|_{\infty} \leq \left\| \|f\|_{\gamma} \right\|_{\infty}. \tag{98}$$

*Here* $L_n^k$ *are bounded linear operators.*

**Notation 22** *Here* $N \in \mathbb{N}$, $0 < \beta < 1$. *Denote by*

$$c_N := \begin{cases} \left[ 2\left( \sqrt[2m]{1 + 4^m} \right) \right]^N, & \text{if } L_n = A_n, \\ 1, & \text{if } L_n = B_n, C_n, D_n, \end{cases} \tag{99}$$

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$$\varphi(n) := \begin{cases} \frac{1}{n^{\beta}}, & \text{if } L_n = A_n,\ B_n, \\ \frac{1}{n} + \frac{1}{n^{\beta}}, & \text{if } L_n = C_n, D_n, \end{cases} \tag{100}$$

$$\Omega := \begin{cases} C\left(\prod_{i=1}^{N} [a_i, b_i], X\right), & \text{if } L_n = A_n, \\ C_B\left(\mathbb{R}^N, X\right), & \text{if } L_n = B_n, C_n, D_n, \end{cases} \tag{101}$$

*and*

$$Y := \begin{cases} \prod_{i=1}^{N} [a_i, b_i], & \text{if } L_n = A_n, \\ \mathbb{R}^N, & \text{if } L_n = B_n, C_n, D_n. \end{cases} \tag{102}$$

We give the condensed

**Theorem 23** *Let $f \in \Omega$, $0 < \beta < 1$, $x \in Y$; $n$, $m$, $N \in \mathbb{N}$ with $n^{1-\beta} > 2$. Then*
*(i)*

$$\|L_n(f, x) - f(x)\|_{\gamma} \leq c_N \left[\omega_1(f, \varphi(n)) + \frac{\left\| \|f\|_{\gamma} \right\|_{\infty}}{2m\left(n^{1-\beta} - 2\right)^{2m}}\right] =: \tau(n), \tag{103}$$

*where $\omega_1$ is for $p = \infty$,*
*and*
*(ii)*

$$\left\| \|L_n(f) - f\|_{\gamma} \right\|_{\infty} \leq \tau(n) \to 0, \text{ as } n \to \infty. \tag{104}$$

*For $f$ uniformly continuous and in $\Omega$ we obtain*

$$\lim_{n\to\infty} L_n(f) = f,$$

*pointwise and uniformly.*

**Proof.** By Theorems 8, 16, 17, 18. ■

Next we do iterated neural network approximation (see also [9]).

We make

**Remark 24** *Let $r \in \mathbb{N}$ and $L_n$ as above. We observe that*

$$L_n^r f - f = \left(L_n^r f - L_n^{r-1} f\right) + \left(L_n^{r-1} f - L_n^{r-2} f\right) +$$

$$\left(L_n^{r-2} f - L_n^{r-3} f\right) + ... + \left(L_n^2 f - L_n f\right) + \left(L_n f - f\right).$$

*Then*

$$\left\| \|L_n^r f - f\|_{\gamma} \right\|_{\infty} \leq \left\| \|L_n^r f - L_n^{r-1} f\|_{\gamma} \right\|_{\infty} + \left\| \|L_n^{r-1} f - L_n^{r-2} f\|_{\gamma} \right\|_{\infty} +$$

$$\left\| \|L_n^{r-2} f - L_n^{r-3} f\|_{\gamma} \right\|_{\infty} + ... + \left\| \|L_n^2 f - L_n f\|_{\gamma} \right\|_{\infty} + \left\| \|L_n f - f\|_{\gamma} \right\|_{\infty} =$$

$$\left\| \left\| L_n^{r-1} \left( L_n f - f \right) \right\|_\gamma \right\|_\infty + \left\| \left\| L_n^{r-2} \left( L_n f - f \right) \right\|_\gamma \right\|_\infty + \left\| \left\| L_n^{r-3} \left( L_n f - f \right) \right\|_\gamma \right\|_\infty$$
$$+ ... + \left\| \left\| L_n \left( L_n f - f \right) \right\|_\gamma \right\|_\infty + \left\| \left\| L_n f - f \right\|_\gamma \right\|_\infty \leq r \left\| \left\| L_n f - f \right\|_\gamma \right\|_\infty . \quad (105)$$

*That is*

$$\left\| \left\| L_n^r f - f \right\|_\gamma \right\|_\infty \leq r \left\| \left\| L_n f - f \right\|_\gamma \right\|_\infty . \quad (106)$$

We give

**Theorem 25** *All here as in Theorem 23 and $r \in \mathbb{N}$, $\tau(n)$ as in (104). Then*

$$\left\| \left\| L_n^r f - f \right\|_\gamma \right\|_\infty \leq r\tau(n). \quad (107)$$

*So that the speed of convergence to the unit operator of $L_n^r$ is not worse than of $L_n$.*

**Proof.** By (106) and (104). ■

We make

**Remark 26** *Let $m, m_1, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$, $0 < \beta < 1$, $f \in \Omega$. Then $\varphi(m_1) \geq \varphi(m_2) \geq ... \geq \varphi(m_r)$, $\varphi$ as in (100).*

*Therefore*

$$\omega_1(f, \varphi(m_1)) \geq \omega_1(f, \varphi(m_2)) \geq ... \geq \omega_1(f, \varphi(m_r)). \quad (108)$$

*Assume further that $m_i^{1-\beta} > 2$, $i = 1, ..., r$. Then*

$$\frac{1}{4m\left(m_1^{1-\beta} - 2\right)^{2m}} \geq \frac{1}{4m\left(m_2^{1-\beta} - 2\right)^{2m}} \geq ... \geq \frac{1}{4m\left(m_r^{1-\beta} - 2\right)^{2m}}. \quad (109)$$

*Let $L_{m_i}$ as above, $i = 1, ..., r$, all of the same kind.*

*We write*

$$L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1} f\right)\right)\right) - f =$$
$$L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1} f\right)\right)\right) - L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2} f\right)\right) +$$
$$L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2} f\right)\right) - L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_3} f\right)\right) +$$
$$L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_3} f\right)\right) - L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_4} f\right)\right) + ...+ \quad (110)$$
$$L_{m_r}\left(L_{m_{r-1}} f\right) - L_{m_r} f + L_{m_r} f - f =$$
$$L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\right)\right)\left(L_{m_1} f - f\right) + L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_3}\right)\right)\left(L_{m_2} f - f\right) +$$
$$L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_4}\right)\right)\left(L_{m_3} f - f\right) + ... + L_{m_r}\left(L_{m_{r-1}} f - f\right) + L_{m_r} f - f.$$

*Hence by the triangle inequality property of $\left\| \left\| \cdot \right\|_\gamma \right\|_\infty$ we get*

$$\left\| \left\| L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1} f\right)\right)\right) - f \right\|_\gamma \right\|_\infty \leq$$

$$\left\| \left\| L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\right)\right)\left(L_{m_1}f-f\right)\right\|_\gamma \right\|_\infty +$$
$$\left\| \left\| L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_3}\right)\right)\left(L_{m_2}f-f\right)\right\|_\gamma \right\|_\infty +$$
$$\left\| \left\| L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_4}\right)\right)\left(L_{m_3}f-f\right)\right\|_\gamma \right\|_\infty + ...+$$
$$\left\| \left\| L_{m_r}\left(L_{m_{r-1}}f-f\right)\right\|_\gamma \right\|_\infty + \left\| \left\| L_{m_r}f-f\right\|_\gamma \right\|_\infty$$

*(repeatedly applying (96))*

$$\leq \left\| \left\| L_{m_1}f-f\right\|_\gamma \right\|_\infty + \left\| \left\| L_{m_2}f-f\right\|_\gamma \right\|_\infty + \left\| \left\| L_{m_3}f-f\right\|_\gamma \right\|_\infty + ...+$$
$$\left\| \left\| L_{m_{r-1}}f-f\right\|_\gamma \right\|_\infty + \left\| \left\| L_{m_r}f-f\right\|_\gamma \right\|_\infty = \sum_{i=1}^{r} \left\| \left\| L_{m_i}f-f\right\|_\gamma \right\|_\infty . \quad (111)$$

*That is, we proved*

$$\left\| \left\| L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right)-f\right\|_\gamma \right\|_\infty \leq \sum_{i=1}^{r} \left\| \left\| L_{m_i}f-f\right\|_\gamma \right\|_\infty . \quad (112)$$

We give

**Theorem 27** *Let $f \in \Omega$; $m, N, m_1, m_2, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$, $0 < \beta < 1$; $m_i^{1-\beta} > 2$, $i = 1, ..., r$, $x \in Y$, and let $(L_{m_1}, ..., L_{m_r})$ as $(A_{m_1}, ..., A_{m_r})$ or $(B_{m_1}, ..., B_{m_r})$ or $(C_{m_1}, ..., C_{m_r})$ or $(D_{m_1}, ..., D_{m_r})$, $p = \infty$. Then*

$$\left\| L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right)(x)-f(x)\right\|_\gamma \leq$$
$$\left\| \left\| L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right)-f\right\|_\gamma \right\|_\infty \leq$$
$$\sum_{i=1}^{r} \left\| \left\| L_{m_i}f-f\right\|_\gamma \right\|_\infty \leq$$
$$c_N \sum_{i=1}^{r} \left[ \omega_1\left(f, \varphi\left(m_i\right)\right) + \frac{\left\| \left\| f\right\|_\gamma \right\|_\infty}{2m\left(m_i^{1-\beta}-2\right)^{2m}} \right] \leq$$
$$r c_N \left[ \omega_1\left(f, \varphi\left(m_1\right)\right) + \frac{\left\| \left\| f\right\|_\gamma \right\|_\infty}{2m\left(m_1^{1-\beta}-2\right)^{2m}} \right] . \quad (113)$$

*Clearly, we notice that the speed of convergence to the unit operator of the multiply iterated operator is not worse than the speed of $L_{m_1}$.*

**Proof.** Using (112), (108), (109) and (103), (104). ■

We continue with

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**Theorem 28** *Let all as in Corollary 15, and $r \in \mathbb{N}$. Here $\varphi_3(n)$ is as in (75). Then*

$$\left\| \|A_n^r f - f\|_\gamma \right\|_\infty \leq r \left\| \|A_n f - f\|_\gamma \right\|_\infty \leq r\varphi_3(n). \tag{114}$$

**Proof.** By (106) and (75). ■

**Application 29** *A typical application of all of our results is when $\left(X, \|\cdot\|_\gamma\right) = (\mathbb{C}, |\cdot|)$, where $\mathbb{C}$ are the complex numbers.*

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