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Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
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715 Stadium Dr.
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Ordinary Differential Equations,
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Christodoulos A. Floudas
Department of Chemical Engineering
Princeton University
Princeton,NJ 08544-5263
609-258-4595(x4619 assistant)
e-mail: floudas@titan.princeton.edu
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J .A. Goldstein
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The University of Memphis
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901-678-3130
jgoldste@memphis.edu
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H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer
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John R. Graef
Department of Mathematics
University of Tennessee at
Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional
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Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element
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Tian-Xiao He
Department of Mathematics and
Computer Science
P.O. Box 2900, Illinois Wesleyan
University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet,
Integration Theory, Numerical
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Margareta Heilmann
Faculty of Mathematics and Natural
Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de

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Journal of Computational Analysis and Applications

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Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
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Existence and uniqueness of fuzzy solutions for the nonlinear second-order fuzzy Volterra integrodifferential equations

Shaher Momani[1,2], Omar Abu Arqub[3,*], Saleh Al-Mezel[2], Marwan Kutbi[2]

[1]*Department of Mathematics, Faculty of Science, The University of Jordan, Amman 11942, Jordan*
[2]*Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University (KAU), Jeddah 21589, Saudi Arabia*
[3]*Department of Mathematics, Faculty of Science, Al Balqa Applied University, Salt 19117, Jordan*

Abstract. Formulation of uncertainty Volterra integrodifferential equations (VIDEs) is very important issue in applied sciences and engineering; whilst the natural way to model such dynamical systems is to use the fuzzy approach. In this work, we present and prove the existence and uniqueness of four solutions of fuzzy VIDEs based on the Hausdorff distance under the assumption of strongly generalized differentiability for the fuzzy-valued mappings of a real variable whose values are normal, convex, upper semicontinuous, and compactly supported fuzzy sets in $\mathbb{R}$. In addition to that, we utilize and prove the characterization theorem for solutions of fuzzy VIDEs which allow us to translate a fuzzy VIDE into a system of crisp equations. The proof methodology is based on the assumption of the generalized Lipchitz property for each nonlinear term appears in the fuzzy equation subject to the specific metric used, while the main tools employed in the analysis are founded on the applications of the Banach fixed point theorem and a certain integral inequality with explicit estimate. An efficient computational algorithm is provided to guarantee the procedure and to confirm the performance of the proposed approach.

Keywords: Fuzzy VIDE; Banach fixed point theorem; Existence and uniqueness

AMS Subject Classification: 26E50; 46S40; 34A07

1. Introduction

There is an inexhaustible supply of applications of VIDEs, especially, in characterizing many social, physical, biological, and engineering problems. On the other aspect as well, since many real-world problems are too complex to be defined in precise terms, uncertainty is often involved in any real-world design process. Fuzzy sets provide a widely appreciated tool to introduce uncertain parameters into mathematical applications. In many applications, at least some of the parameters of the model should be represented by fuzzy rather than crisp numbers. Thus, it is immensely important to develop appropriate and applicable definitions and theorems to accomplish the mathematical construction that would appropriately treat fuzzy VIDEs and solve them.

In this work we are interested in the following main questions; firstly, under what conditions can we be sure that solutions of fuzzy VIDE exist; secondly, under what conditions can we be sure that there are four unique solutions; one solution for each lateral derivative; to fuzzy VIDE, thirdly under what conditions can we be sure that fuzzy VIDE is equivalent into system of crisp VIDEs. Anyhow, in this paper we will answered the aforementioned questions and present an efficient computational algorithm to guarantee the procedure and to confirm the performance of the proposed approach. More precisely, we consider the following second-order fuzzy VIDE under the assumption of strongly generalized differentiability of the general form:

$$x''(t) = f\big(t, x(t), x'(t)\big) + \int_0^t g\big(t, \tau, x(\tau), x'(\tau)\big) d\tau, 0 \le \tau < t \le 1, \tag{1.1}$$

subject to the fuzzy initial conditions

$$x(0) = \alpha, x'(0) = \beta, \tag{1.2}$$

where $f: [0,1] \times \mathbb{R}_{\mathcal{F}}^2 \to \mathbb{R}_{\mathcal{F}}$ and $g: [0,1]^2 \times \mathbb{R}_{\mathcal{F}}^2 \to \mathbb{R}_{\mathcal{F}}$ are continuous fuzzy-valued functions that satisfy a generalized Lipchitz condition and $\alpha, \beta \in \mathbb{R}_{\mathcal{F}}$.

The topics of fuzzy VIDEs which is growing interest for some time, in particular in relation to fuzzy control, fuzzy population growth model, fuzzy oscillating magnetic fields, have been rapidly developed in recent years. Anyhow, in

***Correspondence author: Omar Abu arqub, e-mail: o.abuarqub@bau.edu.jo**

this work, we are focusing our attention on second-order fuzzy VIDEs subject to given fuzzy initial conditions. At the beginning, approaches to fuzzy IDEs and other fuzzy equations can be of three types. The first approach assumes that even if only the initial values are fuzzy, the solution is a fuzzy function, and consequently the derivatives in the IDE must be considered as fuzzy derivatives [1,2]. These can be done by the use of the Hukuhara derivative for fuzzy-valued functions. Generally, this approach has a drawback; the solution becomes fuzzier as time goes, hence, the fuzzy solution behaves quite differently from the crisp solution. In the second approach, the fuzzy IDE is transformed to a crisp one by interpreted it as a family of differential inclusions [3,4]. The main shortcoming of using differential inclusions is that we do not have a derivative of a fuzzy-valued function. The third approach based on the Zadeh's extension principle, where the associated crisp problem is solved and in the solution the initial fuzzy values are substituted instead of the real constants, and in the final solution, arithmetic operations are considered to be operations on fuzzy numbers [5,6]. The weakness of this approach is the need to rewrite the solution in the fuzzy setting which in turn makes the methods of solution are not user-friendly and more restricted with more computation steps. As a conclusion, to overcome the above-mentioned shortcoming, the concept of a strongly generalized differentiability was developed and investigated in [7-14]. Anyhow, using the strongly generalized differentiability, the fuzzy IDE has locally four solutions. Indeed, with this approach, we can find solutions for a larger class of fuzzy IDEs than using other types of differentiability.

The solvability analysis of fuzzy VIDEs has been studied by several researchers by using the strongly generalized differentiability, the Hukuhara derivative, or the Zadeh's extension principle for the fuzzy-valued mappings of a real variable whose values are normal, convex, upper semicontinuous, and compactly supported fuzzy sets in $\mathbb{R}$. The reader is asked to refer to [15-22] in order to know more details about these analyzes, including their kinds and history, their modifications and conditions for use, their scientific applications, their importance and characteristics, and their relationship including the differences. But on the other aspect as well, more details about characterization theorem can be found in [23,24].

The organization of the paper is as follows. In the next section, we present some necessary definitions and preliminary results from the fuzzy calculus theory. The procedure of solving fuzzy VIDEs is presented in section 3. In section 4, existence and uniqueness of four solutions are introduced. In section 5, we utilize the characterization theorem for the solution of fuzzy VIDEs. This article ends in section 6 with some concluding remarks.

2. Excerpts of fuzzy calculus theory

Fuzzy calculus is the study of theory and applications of integrals and derivatives of uncertain functions. This branch of mathematical analysis, extensively investigated in the recent years, has emerged as an effective and powerful tool for the mathematical modeling of several engineering and scientific phenomena. In this section, we present some necessary definitions from fuzzy calculus theory and preliminary results. For the concept of fuzzy derivative, we will adopt strongly generalized differentiability, which is a modification of the Hukuhara differentiability and has the advantage of dealing properly with fuzzy VIDEs.

Let X be a nonempty set. A fuzzy set u in X is characterized by its membership function $u: X \to [0,1]$. Thus, $u(s)$ is interpreted as the degree of membership of an element s in the fuzzy set u for each $s \in X$. A fuzzy set u on $\mathbb{R}$ is called convex if for each $s, t \in \mathbb{R}$ and $\lambda \in [0,1]$, $u(\lambda s + (1-\lambda)t) \geq \min\{u(s), u(t)\}$, is called upper semicontinuous if $\{s \in \mathbb{R}: u(s) > r\}$ is closed for each $r \in [0,1]$, and is called normal if there is $s \in \mathbb{R}$ such that $u(s) = 1$. The support of a fuzzy set u is defined as $\{s \in \mathbb{R}: u(s) > 0\}$.

Definition 2.1. [25] A fuzzy number u is a fuzzy subset of the real line with a normal, convex, and upper semicontinuous membership function of bounded support.

For each $r \in (0,1]$, set $[u]^r = \{s \in \mathbb{R}: u(s) \geq r\}$ and $[u]^0 = \overline{\{s \in \mathbb{R}: u(s) > 0\}}$, where $\overline{\{\cdot\}}$ denote the closure of $\{\cdot\}$. Then, it easily to establish that u is a fuzzy number if and only if $[u]^r$ is compact convex subset of $\mathbb{R}$ for each $r = [0,1]$ and $[u]^1 \neq \phi$ [26]. Thus, if u is a fuzzy number, then $[u]^r = [\underline{u}(r), \overline{u}(r)]$, where $\underline{u}(r) = \min\{s: s \in [u]^r\}$ and $\overline{u}(r) = \max\{s: s \in [u]^r\}$ for each $r \in [0,1]$. The symbol $[u]^r$ is called the r-cut representation or parametric form of a fuzzy number u. We will let $\mathbb{R}_{\mathcal{F}}$ denote the set of fuzzy numbers on $\mathbb{R}$.

The question arises here is, if we have an interval-valued function $[\underline{z}(r), \overline{z}(r)]$ defined on $[0,1]$, then is there a fuzzy number u such that $[u(r)]^r = [\underline{z}(r), \overline{z}(r)]$. The next theorem characterizes fuzzy numbers through their r-cut representations.

Theorem 2.1. [26] Suppose that $\underline{u}:[0,1] \to \mathbb{R}$ and $\overline{u}:[0,1] \to \mathbb{R}$ satisfy the following conditions; first, $\underline{u}$ is a bounded increasing function and $\overline{u}$ is a bounded decreasing function with $\underline{u}(1) \leq \overline{u}(1)$; second, for each $k \in (0,1]$ $\underline{u}$ and $\overline{u}$ are left-hand continuous functions at $r = k$; third, $\underline{u}$ and $\overline{u}$ are right-hand continuous functions at $r = 0$. Then $u:\mathbb{R} \to [0,1]$ defined by

$$u(s) = \sup\{r: \underline{u}(r) \leq s \leq \overline{u}(r)\}, \tag{2.1}$$

is a fuzzy number with parameterization $[\underline{u}(r), \overline{u}(r)]$. Furthermore, if $u:\mathbb{R} \to [0,1]$ is a fuzzy number with parameterization $[\underline{u}(r), \overline{u}(r)]$, then the functions $\underline{u}$ and $\overline{u}$ satisfy the aforementioned conditions.

In general, we can represent an arbitrary fuzzy number u by an order pair of functions $(\underline{u}, \overline{u})$ which satisfy the requirements of Theorem 2.1 Frequently, we will write simply $\underline{u}_r$ and $\overline{u}_r$ instead of $\underline{u}(r)$ and $\overline{u}(r)$, respectively.

The metric structure on $\mathbb{R}_{\mathcal{F}}$ is given by $d_\infty: \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}^+ \cup \{0\}$ such that $d_\infty(u,v) = \sup_{r\in[0,1]} d_H([u]^r,[v]^r)$ for arbitrary fuzzy numbers u and v, where d_H is the Hausdorff metric between $[u]^r$ and $[v]^r$. This metric is defined as $d_H([u]^r,[v]^r) = \inf\{\varepsilon: [u]^r \subset N([u]^r,\varepsilon), [v]^r \subset N([v]^r,\varepsilon)\} = \max\{|\underline{u}_r - \underline{v}_r|, |\overline{u}_r - \overline{v}_r|\}$, where the two set $N([u]^r,\varepsilon)$ and $N([v]^r,\varepsilon)$ are the ε-neighborhoods of $[u]^r$ and $[v]^r$, respectively. It is shown in [27] that $(\mathbb{R}_{\mathcal{F}}, d_\infty)$ is a complete metric space.

Lemma 2.1. [27] For each $A,B,C,D \in \mathbb{R}_{\mathcal{F}}$ with $\lambda \in \mathbb{R}$ the metric function d_∞ satisfies the following properties:

i. $d_\infty(A+C, B+C) = d_\infty(A,B)$,
ii. $d_\infty(A,B) \leq d_\infty(A,C) + d_\infty(C,B)$,
iii. $d_\infty(A+C, B+D) \leq d_\infty(A,B) + d_\infty(C,D)$,
iv. $d_\infty(\lambda A, \lambda B) = |\lambda| d_\infty(A,B)$.

For arithmetic operations on fuzzy numbers, the following results are well-known and follow from the theory of interval analysis. If u and v are two fuzzy number, then for each $r = [0,1]$, we have; firstly, $[u+v]^r = [u]^r + [v]^r = [\underline{u}_r + \underline{v}_r, \overline{u}_r + \overline{v}_r]$; secondly, $[\lambda u]^r = \lambda[u]^r = [\min\{\lambda\underline{u}_r, \lambda\overline{u}_r\}, \max\{\lambda\underline{u}_r, \lambda\overline{u}_r\}]$; thirdly, $[uv]^r = [u]^r[v]^r = [\min\{\underline{u}_r\underline{v}_r, \underline{u}_r\overline{v}_r, \overline{u}_r\underline{v}_r, \overline{u}_r\overline{v}_r\}, \max\{\underline{u}_r\underline{v}_r, \underline{u}_r\overline{v}_r, \overline{u}_r\underline{v}_r, \overline{u}_r\overline{v}_r\}]$; fourthly, $u = v$ if and only if $[u]^r = [v]^r$ if and only if $\underline{u}_r = \underline{v}_r$ and $\overline{u}_r = \overline{v}_r$. In fact, the collection of all fuzzy number with aforementioned addition and scalar multiplication is a convex cone [28].

Let $u, v \in \mathbb{R}_{\mathcal{F}}$. If there exists a $w \in \mathbb{R}_{\mathcal{F}}$ such that $u = v + w$, then w is called the H-difference of u and v, denoted by $u \ominus v$. Here, the sign "$\ominus$" stands always for H-difference and let us remark that $u \ominus v \neq u + (-1)v$. Usually we denote $u + (-1)v$ by $u - v$, while $u \ominus v$ stands for the H-difference. It follows that Hukuhara differentiable function has increasing length of support [25]. To avoid this difficulty, we consider the following definition.

Definition 2.2. [8] Let $x:[0,1] \to \mathbb{R}_{\mathcal{F}}$ and $t^* \in [0,1]$. We say that x is strongly generalized differentiable at t^*, if there exists an element $x'(t^*) \in \mathbb{R}_{\mathcal{F}}$ such that either

i. for all $h > 0$ sufficiently close to 0, the H-differences $x(t^*+h) \ominus x(t^*)$, $x(t^*) \ominus x(t^*-h)$ exist and $\lim_{h\to0^+} \frac{x(t^*+h)\ominus x(t^*)}{h} = \lim_{h\to0^+} \frac{x(t^*)\ominus x(t^*-h)}{h} = x'(t^*)$,
ii. for all $h > 0$ sufficiently close to 0, the H-differences $x(t^*) \ominus x(t^*+h)$, $x(t^*-h) \ominus x(t^*)$ exist and $\lim_{h\to0^+} \frac{x(t^*)\ominus x(t^*+h)}{-h} = \lim_{h\to0^+} \frac{x(t^*-h)\ominus x(t^*)}{-h} = x'(t^*)$.

Here, the limit is taken in the metric space $(\mathbb{R}_{\mathcal{F}}, d_\infty)$ and at the endpoints of $[0,1]$, we consider only one-sided derivatives. For customizing, in Definition 2.2, the first case corresponds to the H-derivative introduced in [28], so this differentiability concept is a generalization of the Hukuhara derivative.

Definition 2.3. [10] Let $x:[0,1] \to \mathbb{R}_{\mathcal{F}}$. We say that x is (1)-differentiable on $[0,1]$ if x is differentiable in the sense (i) of Definition 2.2 and its derivative is denoted D_1^1x. Similarly, we say that x is (2)-differentiable on $[0,1]$ if x is differentiable in the sense (ii) of Definition 2.2 and its derivative is denoted D_2^1x.

The subsequent theorems show us a way to translate a fuzzy VIDE into a system of crisp VIDEs without the need to consider the fuzzy setting approach. Anyhow, these theorems have many uses in the applied mathematics and the numerical analysis fields.

Theorem 2.2. [10] Let $x:[0,1] \to \mathbb{R}_{\mathcal{F}}$ and put $[x(t)]^r = [\underline{x}_r(t), \overline{x}_r(t)]$ for each $r \in [0,1]$.

i. if x is (1)-differentiable, then $\underline{x}_r$ and $\overline{x}_r$ are differentiable functions on $[0,1]$ and $[D_1^1 x(t)]^r = [\underline{x}'_r(t), \overline{x}'_r(t)]$,
ii. if x is (2)-differentiable, then $\underline{x}_r$ and $\overline{x}_r$ are differentiable functions on $[0,1]$ and $[D_2^1 x(t)]^r = [\overline{x}'_r(t), \underline{x}'_r(t)]$.

Next, we introduce the definitions for second fuzzy derivatives based on the selection of derivative type in each step of differentiation. For a given fuzzy-valued function x, we have two possibilities according to Definition 2.3 in order to obtain the derivative of x as follows: $D_1^1 x(t)$ and $D_2^1 x(t)$. Anyhow, for each of these two derivative, we have again two possibilities of derivatives: $D_1^1\left(D_1^1 x(t)\right), D_2^1\left(D_1^1 x(t)\right)$ and $D_1^1\left(D_2^1 x(t)\right), D_2^1\left(D_2^1 x(t)\right)$, respectively.

Definition 2.4. [29] Let $x:[0,1] \to \mathbb{R}_{\mathcal{F}}$ and $n, m \in \{1,2\}$, we say that x is (n,m)-differentiable on $[0,1]$ if $D_2^1 x$ exist and its (m)-differentiable. The second derivatives of x are denoted by $D_{n,m}^2 x$.

Theorem 2.3. [29] Let $D_1^1 x:[0,1] \to \mathbb{R}_{\mathcal{F}}$ or $D_2^1 x:[0,1] \to \mathbb{R}_{\mathcal{F}}$, where $[x(t)]^r = [\underline{x}_r(t), \overline{x}_r(t)]$ for each $r \in [0,1]$:
i. if $D_1^1 x$ is (1)-differentiable, then $\underline{x}'_r$ and $\overline{x}'_r$ are differentiable functions on $[0,1]$ and $\left[D_{1,1}^2 x(t)\right]^r = [\underline{x}''_r(t), \overline{x}''_r(t)]$,
ii. if $D_1^1 x$ is (2)-differentiable, then $\underline{x}'_r$ and $\overline{x}'_r$ are differentiable functions on $[0,1]$ and $\left[D_{1,2}^2 x(t)\right]^r = [\overline{x}''_r(t), \underline{x}''_r(t)]$,
iii. if $D_2^1 x$ is (1)-differentiable, then $\underline{x}'_r$ and $\overline{x}'_r$ are differentiable functions on $[0,1]$ and $\left[D_{2,1}^2 x(t)\right]^r = [\overline{x}''_r(t), \underline{x}''_r(t)]$,
iv. if $D_2^1 x$ is (2)-differentiable, then $\underline{x}'_r$ and $\overline{x}'_r$ are differentiable functions on $[0,1]$ and $\left[D_{2,2}^2 x(t)\right]^r = [\underline{x}''_r(t), \overline{x}''_r(t)]$.

A fuzzy-valued function $x:[0,1] \to \mathbb{R}_{\mathcal{F}}$ is called continuous at a point $t^* \in [0,1]$ provided for arbitrary fixed $\varepsilon > 0$, there exists an $\delta > 0$ such that $d_\infty\left(x(t), x(t^*)\right) < \varepsilon$ whenever $|t^* - t| < \delta$ for each $t \in [0,1]$. We say that x is continuous on $[0,1]$ if x is continuous at each $t^* \in [0,1]$ such that the continuity is one-sided at endpoints 0 and 1.

In order to complete the expert results about the fuzzy calculus theory we finalize the present section by some preliminary information about the fuzzy integral. Following [26], we define the integral of a fuzzy-valued function using the Riemann integral concept.

Definition 2.5. [26] Suppose that $x:[0,1] \to \mathbb{R}_{\mathcal{F}}$, for each partition $\wp = \{t_0^*, t_1^*, \ldots, t_n^*\}$ of $[0,1]$ and for arbitrary points $\xi_i \in [t_{i-1}^*, t_i^*]$, $1 \le i \le n$, let $\Re_\wp = \sum_{i=1}^n x(\xi_i)(t_i^* - t_{i-1}^*)$ and $\Delta = \max_{1 \le i \le n}|t_i^* - t_{i-1}^*|$. Then the definite integral of $x(t)$ over $[t_0, t_0 + a]$ is defined by $\int_0^1 x(t)dt = \lim_{\Delta \to 0} \Re_\wp$ provided the limit exists in the metric space $(\mathbb{R}_{\mathcal{F}}, d_\infty)$.

Theorem 2.4. [26] Let $x:[0,1] \to \mathbb{R}_{\mathcal{F}}$ be continuous fuzzy-valued function and put $[x(t)]^r = [\underline{x}_r(t), \overline{x}_r(t)]$ for each $r \in [0,1]$. Then $\int_0^1 x(t)dt$ exist, belong to $\mathbb{R}_{\mathcal{F}}$, $\underline{x}_r$ and $\overline{x}_r$ are integrable functions on [0,1], and $\left[\int_0^1 x(t)dt\right]^r = \left[\int_0^1 \underline{x}_r(t)dt, \int_0^1 \overline{x}_r(t)dt\right]$.

Lemma 2.2. [30] Let $x, y:[0,1] \to \mathbb{R}_{\mathcal{F}}$ be integrable fuzzy-valued functions and $\lambda \in \mathbb{R}$. Then the following are hold:
i. $d_\infty\left(x(t), y(t)\right)$ is integrable,
ii. $d_\infty\left(\int_0^1 x(t)dt, \int_0^1 y(t)dt\right) \le \int_0^1 d_\infty\left(x(t), y(t)\right)dt$,
iii. $\int_0^1 \lambda x(t)dt = \lambda \int_0^1 x(t)dt$,
iv. $\int_0^1\left(x(t) + y(t)\right)dt = \int_0^1 x(t)dt + \int_0^1 y(t)dt$.

It should be noted that the fuzzy integral can be also defined using the Lebesgue-type approach [25] or the Henstock-type approach [31]. However, if x is continuous function, then all approaches yield the same value and results. Moreover, the representation of the fuzzy integral using Defintion 2.5 is more convenient for numerical calculations and computational mathematics. The reader is kindly requested to go through [25,26,30-32] in order to know more details about the fuzzy integral, including its history and kinds, its properties and modification for use, its applications and characteristics, its justification and conditions for use, and its mathematical and geometric properties.

3. Algorithm of solving fuzzy VIDEs

The topic of fuzzy VIDEs are one of the most important modern mathematical fields that result from modeling of uncertain physical, engineering, and economical problems. In this section, we study fuzzy VIDEs using the concept of strongly generalized differentiability in which fuzzy equation is converted into equivalent system of crisp equations for each type of differentiability. Furthermore, we present an algorithm to solve the new system which consists of four crisp VIDEs.

Problem formulation is normally the most important part of the process. It is the determination of r-cut representation form of nonlinear terms f, g, the selection of the differentiability type, and the separation of fuzzy initial conditions. Next, fuzzy VIDE (1.1) and (1.2) is first formulated as an crisp set of VIDEs subject to crisp set of initial conditions, after that, a new discretized form of fuzzy VIDE (1.1) and (1.2) is presented. Anyhow, by considering the parametric form for both sides of fuzzy VIDE (1.1) and (1.2), one can write

$$[D_{n,m}^2 x(t)]^r = [f(t, x(t), D_n^1(t))]^r + \int_0^t [g(t,\tau,x(\tau),D_n^1(\tau))]^r d\tau, \tag{3.1}$$

subject to the crisp initial conditions

$$[x(0)]^r = [\alpha]^r, [D_n^1(0)]^r = [\beta]^r, \tag{3.2}$$

in which the endpoints functions of $[f(t, x(t), D_n^1(t))]^r$ and $[g(t,\tau,x(\tau),D_n^1(\tau))]^r$ are given, respectively, as follows:

$$\begin{aligned}[f(t,x(t),D_n^1(t))]^r &= f(t,[x(t)]^r,[D_n^1(t)]^r) = \left[\underline{f}_r(t,[x(t)]^r,[D_n^1(t)]^r), \overline{f}_r(t,[x(t)]^r,[D_n^1(t)]^r)\right] \\ &= \left[f_{1,r}\left(t,\underline{x}_r(t),\overline{x}_r(t),\underline{x}_r'(t),\overline{x}_r'(t)\right), f_{2,r}\left(t,\underline{x}_r(t),\overline{x}_r(t),\underline{x}_r'(t),\overline{x}_r'(t)\right)\right],\end{aligned} \tag{3.3}$$

$$\begin{aligned}[g(t,\tau,x(\tau),D_n^1(\tau))]^r &= g(t,\tau,[x(\tau)]^r,[D_n^1(\tau)]^r) = \left[\underline{g}_r(t,\tau,[x(\tau)]^r,[D_n^1(\tau)]^r), \overline{g}_r(t,\tau,[x(\tau)]^r,[D_n^1(\tau)]^r)\right] \\ &= \left[g_{1,r}\left(t,\tau,\underline{x}_r(\tau),\overline{x}_r(\tau),\underline{x}_r'(\tau),\overline{x}_r'(\tau)\right), g_{2,r}\left(t,\tau,\underline{x}_r(\tau),\overline{x}_r(\tau),\underline{x}_r'(\tau),\overline{x}_r'(\tau)\right)\right].\end{aligned} \tag{3.4}$$

Definition 3.1. Let $x: [0,1] \to \mathbb{R}_{\mathcal{F}}$ and $(n,m) \in \{1,2\}$, we say that x is a (n,m)-solution for fuzzy VIDE (1.1) and (1.2) on [0,1], if $D_n^1 x$ and $D_{n,m}^2 x$ exist on [0,1] and $D_{n,m}^2 x(t) = f(t, x(t), D_n^1 x(t)) + \int_0^t g(t,\tau,x(\tau),D_n^1 x(\tau))d\tau$ with $x(0) = \alpha, x'(0) = \beta$.

The object of the next algorithm is to implement a procedure to solve fuzzy VIDE in parametric form in term of its r-cut representation. To do so, let x be a (n,m)-solution, utilizing Theorems 2.2 and 2.3, and considering fuzzy VIDE (1.1) and (1.2), we can thus translate it into system of crisp VIDEs, hereafter, called corresponding (n,m)-system. Anyhow, four IDEs systems are possible as given in the follow algorithm.

Algorithm 3.1: To find (n,m)-solution of fuzzy VIDE (1.1) and (1.2), we discuss the following four cases:

Input: The independent interval [0,1], the unit truth interval [0,1], and the fuzzy numbers α, β.

Output: The (n,m)-differentiable solution of VIDE (1.1) and (1.2) on [0,1].

Step 1: Set $[f(t,x(t),D_n^1(t))]^r = \left[\underline{f}_r(t,[x(t)]^r,[D_n^1(t)]^r), \overline{f}_r(t,[x(t)]^r,[D_n^1(t)]^r)\right]$,

Set $[g(t,\tau,x(\tau),D_n^1(\tau))]^r = \left[\underline{g}_r(t,\tau,[x(\tau)]^r,[D_n^1(\tau)]^r), \overline{g}_r(t,\tau,[x(\tau)]^r,[D_n^1(\tau)]^r)\right]$,

Set $[\alpha]^r = [\underline{\sigma}_r, \overline{\sigma}_r]$ and $[\beta]^r = [\underline{\beta}_r, \overline{\beta}_r]$.

Case I. If $x(t)$ is (1,1)-differentiable, then use $[D_1^1 x(t)]^r$ and $[D_{1,1}^2 x(t)]^r$, and solving fuzzy VIDE (1.1) and (1.2) translates into the following (1,1)-system:

$$\begin{aligned}\underline{x}_r''(t) &= \underline{f}_r(t,[x(t)]^r,[D_1^1(t)]^r) + \int_0^t \underline{g}_r(t,\tau,[x(\tau)]^r,[D_1^1(\tau)]^r)d\tau, \\ \overline{x}_r''(t) &= \overline{f}_r(t,[x(t)]^r,[D_1^1(t)]^r) + \int_0^t \overline{g}_r(t,\tau,[x(\tau)]^r,[D_1^1(\tau)]^r)d\tau,\end{aligned} \tag{3.5}$$

subject to the crisp initial conditions

$$\underline{x}_r(0) = \underline{\sigma}_r, \overline{x}_r(0) = \overline{\sigma}_r, \underline{x}_r'(0) = \underline{\beta}_r, \overline{x}_r'(0) = \overline{\beta}_r. \tag{3.6}$$

Case II. If $x(t)$ is (1,2)-differentiable, then use $[D_1^1 x(t)]^r$ and $[D_{1,2}^2 x(t)]^r$, and solving fuzzy VIDE (1.1) and (1.2) translates into the following (1,2)-system:

$$\begin{aligned}\underline{x}_r''(t) &= \overline{f}_r(t,[x(t)]^r,[D_1^1(t)]^r) + \int_0^t \overline{g}_r(t,\tau,[x(\tau)]^r,[D_1^1(\tau)]^r)d\tau, \\ \overline{x}_r''(t) &= \underline{f}_r(t,[x(t)]^r,[D_1^1(t)]^r) + \int_0^t \underline{g}_r(t,\tau,[x(\tau)]^r,[D_1^1(\tau)]^r)d\tau,\end{aligned} \tag{3.7}$$

subject to the crisp initial conditions

$$\underline{x}_r(0) = \underline{\sigma}_r, \overline{x}_r(0) = \overline{\sigma}_r, \underline{x}'_r(0) = \underline{\beta}_r, \overline{x}'_r(0) = \overline{\beta}_r. \tag{3.8}$$

Case III. If $x(t)$ is (2,1)-differentiable, then use $[D_2^1 x(t)]^r$ and $\left[D_{2,1}^2 x(t)\right]^r$, and solving fuzzy VIDE (1.1) and (1.2) translates into the following $(2,1)$-system:

$$\begin{aligned} \underline{x}''_r(t) &= \overline{f}_r(t, [x(t)]^r, [D_2^1(t)]^r) + \int_0^t \overline{g}_r(t, \tau, [x(\tau)]^r, [D_2^1(\tau)]^r) d\tau, \\ \overline{x}''_r(t) &= \underline{f}_r(t, [x(t)]^r, [D_2^1(t)]^r) + \int_0^t \underline{g}_r(t, \tau, [x(\tau)]^r, [D_2^1(\tau)]^r) d\tau, \end{aligned} \tag{3.9}$$

subject to the crisp initial conditions

$$\underline{x}_r(0) = \underline{\sigma}_r, \overline{x}_r(0) = \overline{\sigma}_r, \underline{x}'_r(0) = \overline{\beta}_r, \overline{x}'_r(0) = \underline{\beta}_r. \tag{3.10}$$

Case IV. If $x(t)$ is $(2,2)$-differentiable, then use $[D_2^1 x(t)]^r$ and $\left[D_{2,2}^2 x(t)\right]^r$, and solving fuzzy VIDE (1.1) and (1.2) translates into the following $(2,2)$-system:

$$\begin{aligned} \underline{x}''_r(t) &= \underline{f}_r(t, [x(t)]^r, [D_2^1(t)]^r) + \int_0^t \underline{g}_r(t, \tau, [x(\tau)]^r, [D_2^1(\tau)]^r) d\tau, \\ \overline{x}''_r(t) &= \overline{f}_r(t, [x(t)]^r, [D_2^1(t)]^r) + \int_0^t \overline{g}_r(t, \tau, [x(\tau)]^r, [D_2^1(\tau)]^r) d\tau, \end{aligned} \tag{3.11}$$

subject to the crisp initial conditions

$$\underline{x}_r(0) = \underline{\sigma}_r, \overline{x}_r(0) = \overline{\sigma}_r, \underline{x}'_r(0) = \overline{\beta}_r, \overline{x}'_r(0) = \underline{\beta}_r. \tag{3.12}$$

Step 2: Solve the obtained (n, m)-system of crisp VIDEs for $\underline{x}_r(t)$ and $\overline{x}_r(t)$.
Step 3: Ensure that $x(t)$ is (n, m)-solution on the interval $[0,1]$.
Step 4: Construct a (n, m)-differentiable solution such that $x(t) = \left[\underline{x}_r(t), \overline{x}_r(t)\right]$.
Step 5: Stop.

Sometimes, we can't decompose the membership function of the fuzzy solution $x(t)$ as a function defined on $\mathbb{R}$ for each $t \in [0,1]$. Then, using identity (2.1) we can leave a (n, m)-solution in term of its r-cut representation form. To summarize the evolution process; our strategy for solving fuzzy VIDE (1.1) and (1.2) is based on the selection of derivatives type in the given fuzzy VIDE. The first step is to choose the type of solution and translate fuzzy VIDE into the corresponding system of equations with coupled crisp VIDE for each type of differentiability. The second step is to solve the obtained VIDEs system, while aim of the third step is to use the representation Theorem 2.1 in order to construct the fuzzy solution.

Next, we construct a procedure based on Algorithm 3.1 to obtain the solutions of fuzzy VIDE (1.1) and (1.2). Here, we discussing and considering the $(1,1)$-differentiability in Case I of Algorithm 3.1 only; since the same procedure can be applied directly for the remaining cases. Anyhow, without the loss of generality and for simplicity, we assume that the function g takes the form $g\big(t, \tau, x(\tau), x'(\tau)\big) = k(t, \tau) G\big(x(\tau), x'(\tau)\big)$. So, based on this, fuzzy VIDE (1.1) can be written in a new discretized form as $x''(t) = f\big(t, x(t), x'(t)\big) + \int_0^t k(t, \tau) G\big(x(\tau), x'(\tau)\big) d\tau$, in which the r-cut representation form of $G\big(x(\tau), x'(\tau)\big)$ should be of the form

$$\left[G\big(x(\tau), x'(\tau)\big)\right]^r = \left[\underline{G}_r\big([x(\tau)]^r, \left[D_1^1(\tau)\right]^r\big), \overline{G}_r\big([x(\tau)]^r, \left[D_1^1(\tau)\right]^r\big)\right]. \tag{3.13}$$

In order to design a scheme for solving fuzzy VIDE (1.1) and (1.2), we first replace it by the following equivalent crisp system of VIDEs:

$$\begin{aligned} \underline{x}''_r(t) &= \underline{f}_r(t, [x(t)]^r, [D_1^1(t)]^r) + \int_0^t K_1\big(t, \tau, [x(\tau)]^r, \left[D_1^1(\tau)\right]^r\big) d\tau, \\ \overline{x}''_r(t) &= \overline{f}_r(t, [x(t)]^r, [D_1^1(t)]^r) + \int_0^t K_2\big(t, \tau, [x(\tau)]^r, \left[D_1^1(\tau)\right]^r\big) d\tau, \end{aligned} \tag{3.14}$$

subject to the crisp initial conditions

$$\underline{x}_r(0) = \underline{\sigma}_r, \overline{x}_r(0) = \overline{\sigma}_r, \underline{x}'_r(0) = \underline{\beta}_r, \overline{x}'_r(0) = \overline{\beta}_r, \tag{3.15}$$

where the new functions K_1, K_2 are given, respectively, as

$$K_1\big(t,\tau,[x(\tau)]^r,[D_1^1(\tau)]^r\big)=\begin{cases}k(t,\tau)\underline{G}_r\big([x(\tau)]^r,[D_1^1(\tau)]^r\big), & k(t,\tau)\geq 0,\\ k(t,\tau)\overline{G}_r\big([x(\tau)]^r,[D_1^1(\tau)]^r\big), & k(t,\tau)<0,\end{cases}$$
$$K_2\big(t,\tau,[x(\tau)]^r,[D_1^1(\tau)]^r\big)=\begin{cases}k(t,\tau)\overline{G}_r\big([x(\tau)]^r,[D_1^1(\tau)]^r\big), & k(t,\tau)\geq 0,\\ k(t,\tau)\underline{G}_r\big([x(\tau)]^r,[D_1^1(\tau)]^r\big), & k(t,\tau)<0.\end{cases} \tag{3.16}$$

Prior to applying the analytic or the numerical methods for solving system of crisp VIDEs (3.14) and (3.15), we suppose that the kernel function $k(t,\tau)$ is nonnegative for $0\leq\tau\leq c$ and nonpositive for $c\leq\tau\leq t$. Therefore, system of crisp VIDEs (3.14) can be translated again into the following form:

$$\underline{x}'_r(t)=\underline{f}_r(t,[x(t)]^r,[D_1^1(t)]^r)+\int_0^c k(t,\tau)\underline{G}_r\big([x(\tau)]^r,[D_1^1(\tau)]^r\big)d\tau+\int_c^t k(t,\tau)\overline{G}_r\big([x(\tau)]^r,[D_1^1(\tau)]^r\big)d\tau,$$
$$\overline{x}'_r(t)=\overline{f}_r(t,[x(t)]^r,[D_1^1(t)]^r)+\int_0^c k(t,\tau)\overline{G}_r\big([x(\tau)]^r,[D_1^1(\tau)]^r\big)d\tau+\int_c^t k(t,\tau)\underline{G}_r\big([x(\tau)]^r,[D_1^1(\tau)]^r\big)d\tau. \tag{3.17}$$

4. Existence and uniqueness of four fuzzy solutions

It is worth stating that in many cases, since fuzzy VIDEs are often derived from problems in physical world, existence and uniqueness are often obvious for physical reasons. Notwithstanding this, a mathematical statement about existence and uniqueness is worthwhile. Uniqueness would be of importance if, for instance, we wished to approximate the solutions. If two solutions passed through a point, then successive approximations could very well jump from one solution to the other with misleading consequences.

Denote by $C([0,1],\mathbb{R}_{\mathcal{F}})$ the set of all continuous mapping from [0,1] to $\mathbb{R}_{\mathcal{F}}$. The supremum metric on $C([0,1],\mathbb{R}_{\mathcal{F}})$ is defined by $d: C([0,1],\mathbb{R}_{\mathcal{F}})\times C([0,1],\mathbb{R}_{\mathcal{F}})\to\mathbb{R}^+\cup\{0\}$ such that $d(x,y)=\sup_{t\in[0,1]}\big(d_\infty\big(x(t),y(t)\big)e^{-\eta t}\big)$ for each $x,y\in C([0,1],\mathbb{R}_{\mathcal{F}})$, where $\eta\in\mathbb{R}$ is fixed. It is shown in [33] that $\big(C([0,1],\mathbb{R}_{\mathcal{F}}),d\big)$ is a complete metric space. On the other aspect as well, by $C^1([0,1],\mathbb{R}_{\mathcal{F}})$, we denote the set of all continuous mapping from $[0,1]$ to $\mathbb{R}_{\mathcal{F}}$ such that $x':[0,1]\to\mathbb{R}_{\mathcal{F}}$ exists as a continuous function. Anyhow, for $C^1([0,1],\mathbb{R}_{\mathcal{F}})$, we define the distance function $D: C^1([0,1],\mathbb{R}_{\mathcal{F}})\times C^1([0,1],\mathbb{R}_{\mathcal{F}})\to\mathbb{R}^+\cup\{0\}$ such that $D(x,y)=d(x,y)+d(x',y')$. Indeed, it is shown in [33] that $\big(C^1([0,1],\mathbb{R}_{\mathcal{F}}),d\big)$ is also a complete metric space.

The following lemma transforms a fuzzy VIDE into four fuzzy Volterra integral equations. Here the equivalence between equations means that any solution of an equation is a solution too for the other one with respect to the differentiability type used.

Lemma 4.1. The fuzzy VIDE (1.1) and (1.2), where $f:[0,1]\times\mathbb{R}_{\mathcal{F}}^2\to\mathbb{R}_{\mathcal{F}}$ and $g:[0,1]^2\times\mathbb{R}_{\mathcal{F}}^2\to\mathbb{R}_{\mathcal{F}}$ are supposed to be continuous is equivalent to one of the following fuzzy Volterra integral equations:

i. $x(t)=\alpha+\beta t+\int_0^t\big(\int_0^z f\big(s,x(s),x'(s)\big)ds\big)dz+\int_0^t\big(\int_0^z\big(\int_0^s g\big(s,\tau,x(\tau),x'(\tau)\big)d\tau\big)ds\big)dz$, when x is (1,1)-differentiable,
ii. $x(t)=\alpha+\beta t\ominus(-1)\int_0^t\big(\int_0^z f\big(s,x(s),x'(s)\big)ds\big)dz\ominus(-1)\int_0^t\big(\int_0^z\big(\int_0^s g\big(s,\tau,x(\tau),x'(\tau)\big)d\tau\big)ds\big)dz$, when x is (1,2)- differentiable,
iii. $x(t)=\alpha\ominus(-1)\Big(\beta t+\int_0^t\big(\int_0^z f\big(s,x(s),x'(s)\big)ds\big)dz+\int_0^t\big(\int_0^z\big(\int_0^s g\big(s,\tau,x(\tau),x'(\tau)\big)d\tau\big)ds\big)dz\Big)$, when x is (2,1)-differentiable,
iv. $x(t)=\alpha\ominus(-1)\Big(\beta t\ominus(-1)\int_0^t\big(\int_0^z f\big(s,x(s),x'(s)\big)ds\big)dz\ominus(-1)\int_0^t\big(\int_0^z\big(\int_0^s g\big(s,\tau,x(\tau),x'(\tau)\big)d\tau\big)ds\big)dz\Big)$, when x is (2,2)- differentiable.

Proof. Since f and g are continuous functions; so they are integrable. Now, we determine the equivalent integral forms of fuzzy VIDE (1.1) and (1.2) under each type of strongly generalized differentiability as follows. Firstly, let us consider x is (1,1)-differentiable, then the equivalent integral form of fuzzy VIDE (1.1) and (1.2) can be written by implementation of fuzzy integration on both sides of the original equation two times as follows:

$$x'(z)=x'(0)+\int_0^z f\big(s,x(s),x'(s)\big)ds+\int_0^z\left(\int_0^s g\big(s,\tau,\xi(\tau),x'(\tau)\big)d\tau\right)ds, \tag{4.1}$$

for $z\in[0,1]$ and again for $t\in[0,1]$, one can write

$$x(t)=x(0)+x'(0)t+\int_0^t\left(\int_0^z f\big(s,x(s),x'(s)\big)ds\right)dz+\int_0^t\left(\int_0^z\left(\int_0^s g\big(s,\tau,\xi(\tau),x'(\tau)\big)d\tau\right)ds\right)dz. \tag{4.2}$$

Secondly, let us consider x is $(1,2)$-differentiable, then the equivalent integral form of fuzzy VIDE (1.1) and (1.2) can be written by implementation of fuzzy integration on both sides of the original equation two times as

$$x'(z) = x'(0) \ominus (-1)\int_0^z f\big(s, x(s), x'(s)\big)ds \ominus (-1)\int_0^z \left(\int_0^s g\big(s,\tau,\xi(\tau),x'(\tau)\big)d\tau\right)ds, \tag{4.3}$$

for $z \in [0,1]$, again for $t \in [0,1]$, we must have

$$\begin{aligned} x(t) = x(0) + \Bigg(x'(0)t \ominus (-1)\int_0^t \left(\int_0^z f\big(s,x(s),x'(s)\big)ds\right)dz \\ \ominus (-1)\int_0^t \left(\int_0^z \left(\int_0^s g\big(s,\tau,\xi(\tau),x'(\tau)\big)d\tau\right)ds\right)dz\Bigg). \end{aligned} \tag{4.4}$$

Thirdly, if x is $(2,1)$-differentiable, then the equivalent form of fuzzy VIDE (1.1) and (1.2) can be written as

$$x'(z) = x'(0) + \int_0^z f\big(s,x(s),x'(s)\big)ds + \int_0^z \left(\int_0^s g\big(s,\tau,\xi(\tau),x'(\tau)\big)d\tau\right)ds, \tag{4.5}$$

for $z \in [0,1]$, which is equivalent for $t \in [0,1]$ to the integral equation of the form

$$\begin{aligned} x(t) = x(0) \ominus (-1)\Bigg(x'(0)t + \int_0^t \left(\int_0^z f\big(s,x(s),x'(s)\big)ds\right)dz \\ + \int_0^t \left(\int_0^z \left(\int_0^s g\big(s,\tau,\xi(\tau),x'(\tau)\big)d\tau\right)ds\right)dz\Bigg). \end{aligned} \tag{4.6}$$

Fourthly, since x is $(2,2)$-differentiable, then one can write

$$x'(z) = x'(0) \ominus (-1)\int_0^z f\big(s,x(s),x'(s)\big)ds \ominus (-1)\int_0^z \left(\int_0^s g\big(s,\tau,\xi(\tau),x'(\tau)\big)d\tau\right)ds, \tag{4.7}$$

for $z \in [0,1]$ and for $t \in [0,1]$, we can also write

$$\begin{aligned} x(t) = x(0) \ominus (-1)\Bigg(x'(0)t \ominus (-1)\int_0^t \left(\int_0^z f\big(s,x(s),x'(s)\big)ds\right)dz \\ \ominus (-1)\int_0^t \left(\int_0^z \left(\int_0^s g\big(s,\tau,\xi(\tau),x'(\tau)\big)d\tau\right)ds\right)dz\Bigg), \end{aligned} \tag{4.8}$$

which is equivalent to the form of part (iv).

In mathematics, the Banach fixed-point theorem; also known as the contraction mapping theorem; is an important tool in the theory of metric spaces; it guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces, and provides a constructive method to find those fixed points. The following results (Definition 4.1 and Theorem 4.1) were collected from [34].

Definition 4.1. Let (X, d_X) be a metric space. A mapping $G: X \to X$ is said to be a contraction mapping, if there exist a positive real number ρ with $\rho < 1$ such that $d_X\big(G(x), G(y)\big) \leq \rho d_X(x,y)$ for each $x, y \in X$.

We observe that, applying G to each of the two points of the space contracts the distance between them; obviously G is continuous. Anyhow, a point $x \in X$ is called a fixed point of the mapping $G: X \to X$ if $G(x) = x$. Next, we present the Banach fixed-point theorem.

Theorem 4.1. Any contraction mapping G of a nonempty complete metric space (X, d_X) into itself has a unique fixed point.

Lemma 4.2. The real-valued functions $\nu, \omega, \mu: [0,1] \to \mathbb{R}$ with $\eta \in \mathbb{R}$ represented by

$$\begin{aligned} \nu(t) &= \frac{1}{\eta}(1 - e^{-\eta t}), \\ \omega(t) &= \frac{1}{\eta^2}(1 - e^{-\eta t} - \eta t e^{-\eta t}), \\ \mu(t) &= \frac{1}{\eta^3}\left(1 - e^{-\eta t} - \eta t e^{-\eta t} - \frac{\eta^2}{2}t^2 e^{-\eta t}\right), \end{aligned} \tag{4.9}$$

are continuous nondecreasing functions on $[0,1]$. Furthermore, $\nu(1) = \sup_{t\in[0,1]} \nu(t)$, $\omega(1) = \sup_{t\in[0,1]} \omega(t)$, $\mu(1) = \sup_{t\in[0,1]} \mu(t)$, and $\lim_{\eta\to+\infty}\big(\nu(1) + \omega(1) + \mu(1)\big) = 0$.

Proof. Clearly ν, ω, μ are continuous functions on $[0,1]$ for each $\eta \in \mathbb{R}$. Since $\nu'(t) = e^{-\eta t} > 0$, $\omega'(t) = te^{-\eta t} > 0$, and $\mu'(t) = \frac{1}{2}t^2 e^{-\eta t} > 0$ for each $t \in [0,1]$ and $\eta \in \mathbb{R}$; thus, ν, ω, μ are nondecreasing functions. As a result one can conclude that $\nu(1) = \sup_{t\in[0,1]} \nu(t)$, $\omega(1) = \sup_{t\in[0,1]} \omega(t)$, and $\mu(1) = \sup_{t\in[0,1]} \mu(t)$. On the other aspect as well, using the limit functions techniques it yields that

$$\begin{aligned} &\lim_{\eta\to+\infty}\big(\nu(1)+\omega(1)+\mu(1)\big) \\ &\quad = \lim_{\eta\to+\infty}\left(\frac{1}{\eta}(1-e^{-\eta t}) + \frac{1}{\eta^2}(1-e^{-\eta t}-\eta te^{-\eta t}) + \frac{1}{\eta^3}\left(1-e^{-\eta t}-\eta te^{-\eta t}-\frac{\eta^2}{2}t^2e^{-\eta t}\right)\right) \\ &\quad = 0. \end{aligned} \tag{4.10}$$

It should be mention here that Lemma 4.2 guarantees the existence of a unique fixed point for the next theorem. In other word, an existence of a unique solution for fuzzy VIDE (1.1) and (1.2) for each type of differentiability.

Throughout this paper, we will try to give the results of the all theorems; however, in some cases we will switch between the results obtained for the four type of differentiability in order not to increase the length of the paper without the loss of generality for the remaining results. Actually, in the same manner, we can employ the same technique to construct the proof for the omitted cases.

Theorem 4.2. Let $f: [0,1] \times \mathbb{R}_{\mathcal{F}}^2 \to \mathbb{R}_{\mathcal{F}}$ and $g: [0,1]^2 \times \mathbb{R}_{\mathcal{F}}^2 \to \mathbb{R}_{\mathcal{F}}$ are continuous fuzzy-valued functions. If there exists $K_1, K_2, L_1, L_2 > 0$ such that

$$\begin{aligned} &d_\infty\Big(f\big(t,\xi_1(t),\xi_2(t)\big), f\big(t,\zeta_1(t),\zeta_2(t)\big)\Big) \le K_1 d_\infty\big(\xi_1(t),\zeta_1(t)\big) + K_2 d_\infty\big(\xi_2(t),\zeta_2(t)\big), \\ &d_\infty\Big(g\big(t,\tau,\xi_1(\tau),\xi_2(\tau)\big), g\big(s,\tau,\zeta_1(\tau),\zeta_2(\tau)\big)\Big) \le L_1 d_\infty\big(\xi_1(\tau),\zeta_1(\tau)\big) + L_2 d_\infty\big(\xi_2(\tau),\zeta_2(\tau)\big), \end{aligned} \tag{4.11}$$

for each $t, \tau \in [0,1]$ and $\xi_1(\tau), \xi_2(\tau), \zeta_1(t), \zeta_2(t) \in \mathbb{R}_{\mathcal{F}}$. Then, the fuzzy VIDE (1.1) and (1.2) has four unique solutions on $[0,1]$ for each type of differentiability.

Proof. Without the loss of generality, we consider the $(1,1)$-differentiability only; actually, in the same manner, we can employ the same technique for the remaining types. For each $\xi(t) \in \mathbb{R}_{\mathcal{F}}$ and $t \in [0,1]$ define the operator $G\xi$ and $(G\xi)'$, respectively, as follows:

$$\begin{aligned} (G\xi)(t) &= \alpha + \beta t + \int_0^t\left(\int_0^z f\big(s,\xi(s),\xi'(s)\big)ds\right)dz + \int_0^t\left(\int_0^z\left(\int_0^s g\big(s,\tau,\xi(\tau),\xi'(\tau)\big)d\tau\right)ds\right)dz, \\ (G\xi)'(t) &= \beta + \int_0^t f\big(s,\xi(s),\xi'(s)\big)ds + \int_0^t\left(\int_0^s g\big(s,\tau,\xi(\tau),\xi'(\tau)\big)d\tau\right)ds. \end{aligned} \tag{4.12}$$

Thus, $G\xi: [0,1] \to \mathbb{R}_{\mathcal{F}}$ is continuous and $G: \mathrm{C}^1([t_0, t_0 + a], \mathbb{R}_{\mathcal{F}}) \to \mathrm{C}^1([t_0, t_0 + a], \mathbb{R}_{\mathcal{F}})$. Now, we are going to show that the operator $G\xi$ satisfies the hypothesis of the Banach-fixed point theorem. For each $\xi, \zeta \in \mathrm{C}^1([t_0, t_0 + a], \mathbb{R}_{\mathcal{F}})$, we have

$$\begin{aligned} &D(G\xi, G\zeta) = d(G\xi, G\zeta) + d((G\xi)', (G\zeta)') \\ &= \sup_{t\in[0,1]}\big(d_\infty\big((G\xi)(t),(G\zeta)(t)\big)e^{-\eta t}\big) + \sup_{t\in[0,1]}\big(d_\infty\big((G\xi)'(t),(G\zeta)'(t)\big)e^{-\eta t}\big) \\ &= \sup_{t\in[0,1]}\Bigg\{d_\infty\Bigg(\alpha+\beta t+\int_0^t\left(\int_0^z f\big(s,\xi(s),\xi'(s)\big)ds\right)dz + \int_0^t\left(\int_0^z\left(\int_0^s g\big(s,\tau,\xi(\tau),\xi'(\tau)\big)d\tau\right)ds\right)dz, \alpha \\ &\qquad + \beta t + \int_0^t\left(\int_0^z f\big(s,\zeta(s),\zeta'(s)\big)ds\right)dz \\ &\qquad + \int_0^t\left(\int_0^z\left(\int_0^s g\big(s,\tau,\zeta(\tau),\zeta'(\tau)\big)d\tau\right)ds\right)dz\Bigg)e^{-\eta t}\Bigg\} \\ &\quad + \sup_{t\in[0,1]}\Bigg\{d_\infty\Bigg(\beta + \int_0^t f\big(s,\xi(s),\xi'(s)\big)ds + \int_0^t\left(\int_0^s g\big(s,\tau,\xi(\tau),\xi'(\tau)\big)d\tau\right)ds, \beta \\ &\qquad + \int_0^t f\big(s,\zeta(s),\zeta'(s)\big)ds + \int_0^t\left(\int_0^s g\big(s,\tau,\zeta(\tau),\zeta'(\tau)\big)d\tau\right)ds\Bigg)e^{-\eta t}\Bigg\} \end{aligned} \tag{4.13}$$

$$= \sup_{t\in[0,1]}\left\{d_\infty\left(\int_0^t\left(\int_0^z f\big(s,\xi(s),\xi'(s)\big)ds\right)dz + \int_0^t\left(\int_0^z\left(\int_0^s g\big(s,\tau,\xi(\tau),\xi'(\tau)\big)d\tau\right)ds\right)dz\,, \int_0^t\left(\int_0^z f\big(s,\zeta(s),\zeta'(s)\big)ds\right)dz + \int_0^t\left(\int_0^z\left(\int_0^s g\big(s,\tau,\zeta(\tau),\zeta'(\tau)\big)d\tau\right)ds\right)dz\right)e^{-\eta t}\right\}$$

$$+ \sup_{t\in[0,1]}\left\{d_\infty\left(\int_0^t f\big(s,\xi(s),\xi'(s)\big)ds + \int_0^t\left(\int_0^s g\big(s,\tau,\xi(\tau),\xi'(\tau)\big)d\tau\right)ds\,, \int_0^t f\big(s,\zeta(s),\zeta'(s)\big)ds + \int_0^t\left(\int_0^s g\big(s,\tau,\zeta(\tau),\zeta'(\tau)\big)d\tau\right)ds\right)e^{-\eta t}\right\}$$

$$\le \sup_{t\in[0,1]}\left\{d_\infty\left(\int_0^t\left(\int_0^z f\big(s,\xi(s),\xi'(s)\big)ds\right)dz\,, \int_0^t\left(\int_0^z f\big(s,\zeta(s),\zeta'(s)\big)ds\right)dz\right)e^{-\eta t} + d_\infty\left(\int_0^t\left(\int_0^z\left(\int_0^s g\big(s,\tau,\xi(\tau),\xi'(\tau)\big)d\tau\right)ds\right)dz\,, \int_0^t\left(\int_0^z\left(\int_0^s g\big(s,\tau,\zeta(\tau),\zeta'(\tau)\big)d\tau\right)ds\right)dz\right)e^{-\eta t}\right\}$$

$$+ \sup_{t\in[0,1]}\left\{d_\infty\left(\int_0^t f\big(s,\xi(s),\xi'(s)\big)ds\,, \int_0^t f\big(s,\zeta(s),\zeta'(s)\big)ds\right)e^{-\eta t} + d_\infty\left(\int_0^t\left(\int_0^s g\big(s,\tau,\xi(\tau),\xi'(\tau)\big)d\tau\right)ds\,, \int_0^t\left(\int_0^s g\big(s,\tau,\zeta(\tau),\zeta'(\tau)\big)d\tau\right)ds\right)e^{-\eta t}\right\}$$

$$\le \sup_{t\in[0,1]}\left\{\int_0^t\int_0^z d_\infty\left(f\big(s,\xi(s),\xi'(s)\big), f\big(s,\zeta(s),\zeta'(s)\big)\right)ds\,dz e^{-\eta t} + \int_0^t\int_0^z\int_0^s d_\infty\left(g\big(s,\tau,\xi(\tau),\xi'(\tau)\big), g\big(s,\tau,\zeta(\tau),\zeta'(\tau)\big)\right)d\tau ds\,dz\,e^{-\eta t}\right\}$$

$$+ \sup_{t\in[0,1]}\left\{\int_0^t d_\infty\left(f\big(s,\xi(s),\xi'(s)\big), f\big(s,\zeta(s),\zeta'(s)\big)\right)ds\,e^{-\eta t} + \int_0^t\int_0^s d_\infty\left(g\big(s,\tau,\xi(\tau),\xi'(\tau)\big), g\big(s,\tau,\zeta(\tau),\zeta'(\tau)\big)\right)d\tau\,ds e^{-\eta t}\right\}$$

$$\le \sup_{t\in[0,1]}\left\{\int_0^t\int_0^z \left(K_1 d_\infty\big(\xi(s),\zeta(s)\big) + K_2 d_\infty\big(\xi'(s),\zeta'(s)\big)\right)ds\,dz e^{-\eta t} + \int_0^t\int_0^z\int_0^s \left(L_1 d_\infty\big(\xi(\tau),\zeta(\tau)\big) + L_2 d_\infty\big(\xi'(\tau),\zeta'(\tau)\big)\right)d\tau ds\,dz\,e^{-\eta t}\right\}$$

$$+ \sup_{t\in[0,1]}\left\{\int_0^t \left(K_1 d_\infty\big(\xi(s),\zeta(s)\big) + K_2 d_\infty\big(\xi'(s),\zeta'(s)\big)\right)ds\,e^{-\eta t} + \int_0^t\int_0^s \left(L_1 d_\infty\big(\xi(\tau),\zeta(\tau)\big) + L_2 d_\infty\big(\xi'(\tau),\zeta'(\tau)\big)\right)d\tau\,ds e^{-\eta t}\right\}$$

$$\le \sup_{t\in[0,1]}\left\{\int_0^t\int_0^z \big(K_1 d(\xi,\zeta) + K_2 d(\xi',\zeta')\big)e^{\eta s}ds\,dz e^{-\eta t} + \int_0^t\int_0^z\int_0^s \big(L_1 d(\xi,\zeta) + L_2 d(\xi',\zeta')\big)\,e^{\eta\tau}d\tau ds\,dz\,e^{-\eta t}\right\}$$

$$+ \sup_{t\in[0,1]}\left\{\int_0^t \big(K_1 d(\xi,\zeta) + K_2 d(\xi',\zeta')\big)e^{\eta s}ds\,e^{-\eta t} + \int_0^t\int_0^s \big(L_1 d(\xi,\zeta) + L_2 d(\xi',\zeta')\big)e^{\eta\tau}d\tau\,ds e^{-\eta t}\right\}$$

$$\le \big(K_1 d(\xi,\zeta) + K_2 d(\xi',\zeta')\big)\left(\sup_{t\in[0,1]}\int_0^t\int_0^z e^{\eta s}ds\,dz e^{-\eta t} + \sup_{t\in[0,1]}\int_0^t e^{\eta s}ds\,e^{-\eta t}\right)$$

$$+\big(L_1 d(\xi,\zeta) + L_2 d(\xi',\zeta')\big)\left(\sup_{t\in[0,1]}\int_0^t\int_0^z\int_0^s e^{\eta\tau}d\tau ds\,dz\,e^{-\eta t} + \sup_{t\in[0,1]}\int_0^t\int_0^s e^{\eta\tau}d\tau\,ds e^{-\eta t}\right)$$

$$= \big(K_1 d(\xi,\zeta) + K_2 d(\xi',\zeta')\big)\left(\sup_{t\in[0,1]} e^{-\eta t}\int_0^t\int_0^z e^{\eta s}ds\,dz + \sup_{t\in[0,1]} e^{-\eta t}\int_0^t e^{\eta s}ds\right)$$

$$+\big(L_1 d(\xi,\zeta) + L_2 d(\xi',\zeta')\big)\left(\sup_{t\in[0,1]} e^{-\eta t}\int_0^t\int_0^z\int_0^s e^{\eta\tau}d\tau ds\,dz + \sup_{t\in[0,1]} e^{-\eta t}\int_0^t\int_0^s e^{\eta\tau}d\tau\,ds\right)$$

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$$\leq \max\{K_1,K_2\}\left(d(\xi,\zeta)+d(\xi',\zeta')\right)\left(\sup_{t\in[0,1]} e^{-\eta t}\int_0^t\int_0^z e^{\eta s}ds\,dz+\sup_{t\in[0,1]} e^{-\eta t}\int_0^t e^{\eta s}ds\right)$$
$$+\max\{L_1,L_2\}\left(d(\xi,\zeta)+d(\xi',\zeta')\right)\left(\sup_{t\in[0,1]} e^{-\eta t}\int_0^t\int_0^z\int_0^s e^{\eta\tau}d\tau ds\,dz+\sup_{t\in[0,1]} e^{-\eta t}\int_0^t\int_0^s e^{\eta\tau}d\tau\,ds\right)$$
$$\leq \max\{K_1,K_2\}\,D(\xi,\zeta)\left(\sup_{t\in[0,1]} e^{-\eta t}\left(\frac{1}{\eta^2}(e^{\eta t}-1-\eta t)\right)+\sup_{t\in[0,1]} e^{-\eta t}\left(\frac{1}{\eta}(e^{\eta t}-1)\right)\right)$$
$$+\max\{L_1,L_2\}\,D(\xi,\zeta)\left(\sup_{t\in[0,1]} e^{-\eta t}\left(\frac{1}{\eta^3}\left(e^{\eta t}-1-\eta t-\frac{\eta^2}{2}t^2\right)\right)+\sup_{t\in[0,1]} e^{-\eta t}\left(\frac{1}{\eta^2}(e^{\eta t}-1-\eta t)\right)\right)$$
$$\leq \max\{K_1,K_2\}\,D(\xi,\zeta)\left(e^{-\eta}\left(\frac{1}{\eta^2}(e^{\eta}-1-\eta)\right)+e^{-\eta}\left(\frac{1}{\eta}(e^{\eta}-1)\right)\right)$$
$$+\max\{L_1,L_2\}\,D(\xi,\zeta)\left(e^{-\eta}\left(\frac{1}{\eta^3}\left(e^{\eta}-1-\eta-\frac{\eta^2}{2}\right)\right)+e^{-\eta}\left(\frac{1}{\eta^2}(e^{\eta}-1-\eta)\right)\right)$$
$$\leq \max\{K_1,K_2,L_1,L_2\}\,\psi(\eta)D(\xi,\zeta),$$

where $\psi(\eta)=e^{-\eta}\left(\frac{1}{\eta^3}\left(e^{\eta}-1-\eta-\frac{\eta^2}{2}\right)+\frac{2}{\eta^2}(e^{\eta}-1-\eta)+\frac{1}{\eta}(e^{\eta}-1)\right)$. But since $\lim_{\eta\to+\infty}\psi(\eta)=0$ from Lemma 4.2, So, we can choose $\eta>0$ such that

$$\max\{K_1,K_2,L_1,L_2\}\,\psi(\eta)<1. \tag{4.14}$$

Anyhow, G is a contractive mapping; whilst the unique fixed point of G is in the space $C^1([0,1],\mathbb{R}_{\mathcal{F}})$. Using that $G\xi$ is the integral of a continuous function, we conclude that it is actually in the space $C^2([0,1],\mathbb{R}_{\mathcal{F}})$. Hence, by the Banach fixed-point theorem, fuzzy VIDE (1.1) and (1.2) has a unique fixed point $x\in C^1([0,1],\mathbb{R}_{\mathcal{F}})$. That is, a contiunuous function x on $[0,1]$ satisfying $Gx=x$. As a result, writing $(Gx)(t)=x(t)$ out, we have by Eq. (4.12)

$$x(t)=\alpha+\beta t+\int_0^t\left(\int_0^z f\big(s,x(s),x'(s)\big)ds\right)dz+\int_0^t\left(\int_0^z\left(\int_0^s g\big(s,\tau,x(\tau),x'(\tau)\big)d\tau\right)ds\right)dz. \tag{4.15}$$

On the other aspect as well, differentiate both sides Eq. (4.15) and substitute $t=0$ to obtain fuzzy VIDE (1.1) and (1.2). Hence, every solution of fuzzy VIDE (1.1) and (1.2) must satisfy Eq. (4.15), and conversely. So, the proof of the theorem is complete.

Remark 4.1: The continuous nonlinear terms $f:[0,1]\times\mathbb{R}_{\mathcal{F}}^2\to\mathbb{R}_{\mathcal{F}}$ and $g:[0,1]^2\times\mathbb{R}_{\mathcal{F}}^2\to\mathbb{R}_{\mathcal{F}}$ are said to satisfy a generalized Lipchitz condition relative to their last argument in fuzzy sense with respect to the metric space $(\mathbb{R}_{\mathcal{F}},d_\infty)$ if the conditions of Eq. (4.11) of Theorem 4.2 are hold.

5. Generalized characterization theorem

The characterization theorem shows us the following general hint on how to deal with the analytical or the numerical solutions of fuzzy VIDEs. We can translate the original fuzzy VIDE equivalently into a system of crisp VIDEs. The solutions techniques of the system of crisp VIDEs are extremely well studied in the literature, so any method we can consider for the system of crisp VIDEs, since the solution will be as well solution of the fuzzy VIDE under study. As a conclusion one does not need to rewrite the methods of solution for system of crisp VIDEs in fuzzy setting, but instead, we can use the methods directly on the obtained crisp system.

A function $f:[0,1]\times\mathbb{R}^4\to\mathbb{R}$ is said to be equicontinuous if for any $\epsilon>0$ and any $(t,x,y,z,w)\in[0,1]\times\mathbb{R}^4$, we have $|f(t,x,y,z,w)|-\left|f\big((t,x_1,y_1,z_1,w_1)\big)\right|<\epsilon$, whenever $\|(t,x_1,y_1,z_1,w_1)-(t,x,y,z,w)\|<\delta$, and uniformly bounded on any bounded set. Similarly, for a function defined on $[0,1]^2\times\mathbb{R}^4$ with the need for attention to change the metric used on $[0,1]^2\times\mathbb{R}^4$.

Theorem 5.1. Consider the fuzzy VIDE (1.1) and (1.2), where $f:[0,1]\times\mathbb{R}_{\mathcal{F}}^2\to\mathbb{R}_{\mathcal{F}}$ and $g:[0,1]^2\times\mathbb{R}_{\mathcal{F}}^2\to\mathbb{R}_{\mathcal{F}}$ are such that

i. $\left[f\big(t,x(t),D_n^1(t)\big)\right]^r=\left[f_{1,r}\left(t,\underline{x}_r(t),\overline{x}_r(t),\underline{x}_r'(t),\overline{x}_r'(t)\right),f_{2,r}\left(t,\underline{x}_r(t),\overline{x}_r(t),\underline{x}_r'(t),\overline{x}_r'(t)\right)\right]$,
$\left[g\big(t,\tau,x(\tau),D_n^1(\tau)\big)\right]^r=\left[g_{1,r}\left(t,\tau,\underline{x}_r(\tau),\overline{x}_r(\tau),\underline{x}_r'(\tau),\overline{x}_r'(\tau)\right),g_{2,r}\left(t,\tau,\underline{x}_r(\tau),\overline{x}_r(\tau),\underline{x}_r'(\tau),\overline{x}_r'(\tau)\right)\right]$,

ii. $f_{1,r}, f_{2,r}$ and $g_{1,r}, g_{2,r}$ are equicontinuous functions and uniformly bounded on any bounded set,

iii. there exists real-finite constants $L, K, M, N > 0$ such that

$$\left|f_{1,2,r}(t, x_1, y_1, z_1, w_1) - f_{1,2,r}(t, x_2, y_2, z_2, w_2)\right| \leq L \max\{|x_1 - x_2|, |y_1 - y_2|\} + K \max\{|z_1 - z_2|, |w_1 - w_2|\},$$

$$\left|g_{1,2,r}(t, \tau, x_1, y_1, z_1, w_1) - g_{1,2,r}(t, \tau, x_2, y_2, z_2, w_2)\right| \leq M \max\{|x_1 - x_2|, |y_1 - y_2|\} + N \max\{|z_1 - z_2|, |w_1 - w_2|\},$$

for each $t, \tau \in [t_0, t_0 + a]$, $r \in [0,1]$, and $x_{1,2}, y_{1,2}, z_{1,2}, w_{1,2} \in \mathbb{R}$. Then, for (n, m)-differentiability, the fuzzy VIDE (1.1) and (1.2) and the corresponding (n, m)-system are equivalent.

Proof. Since the proof procedure is similar for each type of differentiability with respect to the corresponding (n, m)-system. Anyhow, we assume that x is $(1,1)$-differentiable (Case I of Algorithm 3.1) without the loss of generality. The equicontinuity of $f_{1,r}, f_{2,r}$ and $g_{1,r}, g_{2,r}$ implies the continuity of f and g, respectively. Furthermore, the Lipchitz property of condition (iii) ensures that f and g are satisfies a Lipchitz property in the metric space $(\mathbb{R}_{\mathcal{F}}, d_\infty)$ as follows:

$$\begin{aligned}
&d_\infty\Big(f\big(t, x(t), x'(t)\big), f\big(t, y(t), y'(t)\big)\Big) = \sup_{r\in[0,1]} d_H\big(\big[f\big(t, x(t), x'(t)\big)\big]^r, \big[f\big(t, y(t), y'(t)\big)\big]^r\big) \\
&= \sup_{r\in[0,1]} \max\Big\{\Big|\underline{f}_r\big(t, x(t), x'(t)\big) - \underline{f}_r\big(t, y(t), y'(t)\big)\Big|, \Big|\overline{f}_r\big(t, x(t), x'(t)\big) - \overline{f}_r\big(t, y(t), y'(t)\big)\Big|\Big\} \\
&= \sup_{r\in[0,1]} \max\Big\{\Big|f_{1,r}\Big(t, \underline{x}_r(t), \overline{x}_r(t), \underline{x}'_r(t), \overline{x}'_r(t)\Big) \\
&\qquad - f_{1,r}\Big(t, \underline{y}_r(t), \overline{y}_r(t), \underline{y}'_r(t), \overline{y}'_r(t)\Big)\Big|, \Big|f_{2,r}\Big(t, \underline{x}_r(t), \overline{x}_r(t), \underline{x}'_r(t), \overline{x}'_r(t)\Big) \\
&\qquad - f_{2,r}\Big(t, \underline{y}_r(t), \overline{y}_r(t), \underline{y}'_r(t), \overline{y}'_r(t)\Big)\Big|\Big\} \\
&\leq L \sup_{r\in[0,1]} \max\Big\{\Big|\underline{x}_r(t) - \underline{y}_r(t)\Big|, \big|\overline{x}_r(t) - \overline{y}_r(t)\big|\Big\} + K \sup_{r\in[0,1]} \max\Big\{\Big|\underline{x}'_r(t) - \underline{y}'_r(t)\Big|, \Big|\overline{x}'_r(t) - \overline{y}'_r(t)\Big|\Big\} \\
&= L \sup_{r\in[0,1]} d_H([x(t)]^r, [y(t)]^r) + K \sup_{r\in[0,1]} d_H\Big(\big[x'(t)\big]^r, \big[y'(t)\big]^r\Big) \\
&= L d_\infty\big(x(t), y(t)\big) + K d_\infty\big(x'(t), y'(t)\big).
\end{aligned} \tag{5.1}$$

Whilst on the other aspect as well, by similar fashion, it is easy to conclude that

$$d_\infty\Big(g\big(t, \tau, x(\tau), x'(\tau)\big), g\big(t, \tau, y(\tau), y'(\tau)\big)\Big) \leq M d_\infty\big(x(\tau), y(\tau)\big) + N d_\infty\big(x'(\tau), y'(\tau)\big). \tag{5.2}$$

By the continuity of f and g, from this last Lipchitz conditions of Eqs. (5.1) and (5.2), and the boundedness property of condition (ii), it follows that fuzzy VIDE (1.1) and (1.2) has a unique solution on $[0,1]$. Whilst, the solution of fuzzy VIDE (1.1) and (1.2) is (1,1)-differentiable and so, by Theorems 2.2 and 2.3, the functions $\underline{x}_r, \overline{x}_r$ and $\underline{x}'_r, \overline{x}'_r$ are differentiable on $[0,1]$. As a conclusion one can obtained that $\big(\underline{x}_r(t), \overline{x}_r(t)\big)$ is a solution of crisp VIDEs (3.5) and (3.6).

Conversely, suppose that we have a solution $\big(\underline{x}_r(t), \overline{x}_r(t)\big)$ with $r \in [0,1]$ is fixed, of fuzzy VIDE (1.1) and (1.2) (note that this solution exists by property of condition (iii)). Whilst, the Lipchitz conditions of Eqs. (5.1) and (5.2) implies the existence and uniqueness of fuzzy solution $\tilde{x}(t)$. Indeed, since $\tilde{x}$ is (1,1)-differentiable, then $\underline{\tilde{x}}_r(t)$ and $\overline{\tilde{x}}_r(t)$ the endpoints of $[\tilde{x}(t)]^r$ are a solution of crisp VIDEs (3.5) and (3.6) (note that $[\tilde{x}]^r$ and $[D_1^1\tilde{x}]^r$ are obviously valid level sets of fuzzy-valued functions). But since the solution of crisp VIDEs (3.5) and (3.6) is unique, we have $[\tilde{x}(t)]^r = \big[\underline{\tilde{x}}_r(t), \overline{\tilde{x}}_r(t)\big]^r = \big[\underline{x}_r(t), \overline{x}_r(t)\big]^r = [x(t)]^r$. That is the fuzzy VIDE (1.1) and (1.2) and the system of crisp VIDEs (3.5) and (3.6) are equivalent. This completes the proof of the theorem.

The purpose of the next corollary is not to make an essential improvement of Theorem 5.1, but rather to give alternate conditions under which fuzzy VIDE (1.1) and (1.2) and the corresponding system of crisp VIDEs are equivalent.

Corollary 5.1. Suppose that $f: [0,1] \times \mathbb{R}_{\mathcal{F}}^2 \to \mathbb{R}_{\mathcal{F}}$ and $g: [0,1]^2 \times \mathbb{R}_{\mathcal{F}}^2 \to \mathbb{R}_{\mathcal{F}}$ are such that the condition (i) of Theorem 5.1 hold. If there exists real-finite constants $L, K, M, N > 0$ such that

$$\begin{aligned}
&\left|f_{1,2,r}(t_1, x_1, y_1, z_1, w_1) - f_{1,2,r}(t_2, x_2, y_2, z_2, w_2)\right| \\
&\qquad \leq L \max\{|t_1 - t_2|, |x_1 - x_2|, |y_1 - y_2|\} + K \max\{|t_1 - t_2|, |z_1 - z_2|, |w_1 - w_2|\}, \\
&\left|g_{1,2,r}(t_1, \tau_1, x_1, y_1, z_1, w_1) - g_{1,2,r}(t_2, \tau_2, x_2, y_2, z_2, w_2)\right| \\
&\qquad \leq M \max\{|t_1 - t_2|, |\tau_1 - \tau_2|, |x_1 - x_2|, |y_1 - y_2|\} + N \max\{|t_1 - t_2|, |\tau_1 - \tau_2|, |z_1 - z_2|, |w_1 - w_2|\},
\end{aligned} \tag{5.3}$$

for each $t_{1,2}, \tau_{1,2} \in [0,1]$, $r \in [0,1]$, and $x_{1,2}, y_{1,2}, z_{1,2}, w_{1,2} \in \mathbb{R}$. Then, for (n, m)-differentiability, the fuzzy VIDE (1.1) and (1.2) and the corresponding (n, m)-system are equivalent.

Proof. Here, we consider the $(1,1)$-differentiability only; actually, in the same manner, we can employ the same technique for the remaining types of (n,m)-differentiability. To this end, assume the hypothesis of Corollary 5.1, then the conditions (i) and (iii) of Theorem 5.1 are clearly hold. To establish condition (ii), apply the following: fix $\epsilon > 0$, choose $\delta_1 = \epsilon/(2L)$ and $\delta_2 = \epsilon/(2K)$, and suppose $\|(t,x,y)-(t_1,x_1,y_1)\| < \delta_1$ and $\|(t,z,w)-(t_1,z_1,w_1)\| < \delta_2$. Then, for each $r \in [0,1]$, one can write

$$\begin{aligned} &\left|f_{1,2,r}(t,x,y,z,w) - f_{1,2,r}(t_1,x_1,y_1,z_1,w_1)\right| \\ &\qquad \le L\max\{|t-t_1|,|x-x_1|,|y-y_1|\} + K\max\{|t-t_1|,|z-z_1|,|w-w_1|\} \\ &\qquad \le L\|(t,x,y)-(t_1,x_1,y_1)\| + K\|(t,z,w)-(t_1,z_1,w_1)\| \\ &\qquad \le L\delta_1 + K\delta_2 = \epsilon. \end{aligned} \tag{5.4}$$

Next, we want to show that $f_{1,r}, f_{2,r}$ are uniformly bounded on any bounded set. To do so, let S be any bounded subset of $[0,1]\times\mathbb{R}^4$. Then there exist constants $x_1,y_1,z_1,w_1,x_2,y_2,z_2,w_2 \in \mathbb{R}$ such that if $w=(t,x,y,y,z) \in S$, then $t \in [0,1]$, $x \in [x_1,x_2]$, $y \in [y_1,y_2]$, $z \in [z_1,z_2]$, and $w \in [w_1,w_2]$. For the conduct of proceedings in the proof, fix $r^* \in [0,1]$ and $w^* \in S$, further, let $L^* = \max\{1,|x_1-x_2|,|y_1-y_2|\}$, $K^* = \max\{1,|z_1-z_2|,|w_1-w_2|\}$, and $C = LL^* + KK^* + \operatorname{supp} f(w^*)$, where $\operatorname{supp} f(w^*)$ is the support of $f(w^*)$. Suppose that $r \in [0,1]$ and $w \in S$. Then one can write

$$\left|f_{1,r}(w) - f_{1,r}(w^*)\right| \le L\max\{1,|x_1-x_2|,|y_1-y_2|\} + K\max\{1,|z_1-z_2|,|w_1-w_2|\} = LL^* + KK^*, \tag{5.5}$$

while on the other aspect as well, the triangle inequality will gives

$$\begin{aligned} \left|f_{1,r}(w) - f_{1,r^*}(w^*)\right| &= \left|f_{1,r}(w) - f_{1,r}(w^*) + f_{1,r}(w^*) - f_{1,r^*}(w^*)\right| \\ &\le \left|f_{1,r}(w) - f_{1,r}(w^*)\right| + \left|f_{1,r}(w^*) - f_{1,r^*}(w^*)\right| \\ &= LL^* + KK^* + \operatorname{supp} f(w^*) = C. \end{aligned} \tag{5.6}$$

But since $\left|f_{1,r}(w)\right| - \left|f_{1,r^*}(w^*)\right| \le \left|f_{1,r}(w) - f_{1,r^*}(w^*)\right| \le C$ or $\left|f_{1,r}(w)\right| \le C + \left|f_{1,r^*}(w^*)\right|$, therefore $f_{1,r}$ is uniformly bounded on S. Similarly, $f_{2,r}$ is uniformly bounded on any bounded set. The same procedure can be applied directly for $g_{1,r}, g_{2,r}$. Hence, fuzzy VIDE (1.1) and (1.2) and the corresponding $(1,1)$-system are equivalent by Theorem 5.1.

Remark 5.1. The following requirement conditions on f and g:

$$\begin{aligned} \left[f\left(t,x(t),x'(t)\right)\right]^r &= \left[f_{1,r}\left(t,\underline{x}_r(t),\overline{x}_r(t),\underline{x}'_r(t),\overline{x}'_r(t)\right), f_{2,r}\left(t,\underline{x}_r(t),\overline{x}_r(t),\underline{x}'_r(t),\overline{x}'_r(t)\right)\right], \\ \left[g\left(t,\tau,x(\tau),x'(\tau)\right)\right]^r &= \left[g_{1,r}\left(t,\tau,\underline{x}_r(\tau),\overline{x}_r(\tau),\underline{x}'_r(\tau),\overline{x}'_r(\tau)\right), g_{2,r}\left(t,\tau,\underline{x}_r(\tau),\overline{x}_r(\tau),\underline{x}'_r(\tau),\overline{x}'_r(\tau)\right)\right], \end{aligned} \tag{5.7}$$

are fulfilled by any fuzzy-valued functions obtained from continuous real-valued functions by Zadeh's extension principle and Nguyen theorem [35-37]. So these conditions are not too restrictive.

6. Conclusion

Existence and uniqueness theorem is the tool which makes it possible for us to conclude that there exists only one solution to a given problem which satisfies a constraint condition. How does it work? Why is it the case? We believe it but it would be interesting to see the main ideas behind. To this end, in this paper we investigated and proved the existence, uniqueness, and other properties of solutions of a certain nonlinear second-order fuzzy VIDE under strongly generalized differentiability by considered four cases of differentiability. We make use of the standard tools of the fixed point theorem and a certain integral inequality with explicit estimate to establish the main results. In addition to that, some results for characterizing solution by an equivalent system of crisp VIDEs are presented and proved.

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$\alpha\beta$-statistical convergence and strong $\alpha\beta$-convergence of order γ for a sequence of fuzzy numbers[†]

Zeng-Tai Gong[a,*], Xue Feng[a,b]

[a] *College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China*

[b] *School of Mathematics and Statistics, Qinghai University for Nationalities, Xining 810007, China*

Abstract The purpose of this paper is to introduce the concepts of $\alpha\beta$-statistical convergence of order γ and strong $\alpha\beta$-convergence of order γ for a sequence of fuzzy numbers. At the same time, some connections between $\alpha\beta$-statistical convergence of order γ and strong $\alpha\beta$-convergence of order γ for a sequence of fuzzy numbers are established. It also shows that if a sequence of fuzzy numbers is strongly $\alpha\beta$-convergent of order γ then it is $\alpha\beta$-statistically convergent of order γ.

Keywords: Fuzzy numbers; sequence of fuzzy numbers; statistical convergence.

1. Introduction

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [1] and subsequently several authors have discussed various aspects of theory and applications of fuzzy sets. Recently Matloka [2] introduced bounded and convergent sequences of fuzzy numbers, studied some of their properties, and showed that every convergent sequence of fuzzy numbers is bounded. In addition, sequences of fuzzy numbers have been discussed by Aytar and Pehlivan [3], Basarir and Mursaleen [4,5] and many others. The notion of statistical convergence was introduced by Fast [6] which is a very useful functional tool for studying the convergence problems of numerical sequences. Some applications of statistical convergence in number theory and mathematical analysis can be found in [7, 8]. The idea is based on the notion of natural density of subsets of N, and the natural density of s subset A of N is denoted by $\delta(A)$ and defined by

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} |\{k < n : k \in A\}|.$$

In 2014, Hüseyin Aktuğlu [9] introduced the concepts of $\alpha\beta$-statistically convergence and $\alpha\beta$-statistically convergence of order γ for a sequence, which shows that $\alpha\beta$-statistically convergence is a non-trivial extension of ordinary and statistical convergences.

In this paper, we define the sequence spaces of $\alpha\beta$-statistical convergence of order γ and strong $\alpha\beta$-convergence of order γ, and testify some properties of these spaces. At the same time, some connections between $\alpha\beta$-statistical convergence of order γ and strong $\alpha\beta$-convergence for a sequence of order γ of fuzzy numbers are established. In Section 2 we will give a brief overview about fuzzy numbers, statistical convergence, and present $\delta^{\alpha,\beta}(k,\gamma)$. In Section 3 we show that $\alpha\beta$-statistical convergence for a sequence of fuzzy numbers can reduce to statistical convergence, λ-statistical convergence, and lacunary statistical convergence. Meanwhile, strong $\alpha\beta$-convergence for a sequence of fuzzy numbers can reduce to strong convergence, strong $\lambda-$convergence and strongly lacunary convergence.

2. Definitions and preliminaries

Let $\tilde{A} \in \tilde{F}(R)$ be a fuzzy subset on R. If $\tilde{A}$ is convex, normal, upper semi-continuous and has compact support, we say that $\tilde{A}$ is a fuzzy number. Let $\tilde{R}^c$ denote the set of all fuzzy numbers [10,11,12].

For $\tilde{A} \in \tilde{R}^c$, we write the level set of $\tilde{A}$ as $A_\lambda = \{x : A(x) \geq \lambda\}$ and $A_\lambda = [A_\lambda^-, A_\lambda^+]$. Let $\tilde{A}, \tilde{B} \in \tilde{R}^c$, we define $\tilde{A} + \tilde{B} = \tilde{C}$ iff $A_\lambda + B_\lambda = C_\lambda,\ \lambda \in [0,1]$ iff $A_\lambda^- + B_\lambda^- = C_\lambda^-$ and $A_\lambda^+ + B_\lambda^+ = C_\lambda^+$ for any $\lambda \in [0,1]$.

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[*]Corresponding Author:Zeng-Tai Gong. Tel.: +869317971430. E-mail addresses: zt-gong@163.com

Define

$$D(\tilde{A},\tilde{B}) = \sup_{\lambda\in[0,1]} d(A_\lambda, B_\lambda) = \sup_{\lambda\in[0,1]} \max\{|A_\lambda^- - B_\lambda^-|, |A_\lambda^+ - B_\lambda^+|\},$$

where d is the Hausdorff metric. $D(\tilde{A},\tilde{B})$ is called the distance between $\tilde{A}$ and $\tilde{B}$ [11,13,14].

Using the results of [10,11], we see that

(1) $(\tilde{R}^c, D)$ is a complete metric space,
(2) $D(u+w, v+w) = D(u, v)$,
(3) $D(ku, kv) = |k|D(u, v),\ k \in R$,
(4) $D(u+v, w+e) \le D(u, w) + D(v, e)$,
(5) $D(u+v, \bar{0}) \le D(u, \bar{0}) + D(v, \bar{0})$,
(6) $D(u+v, w) \le D(u, w) + D(v + \bar{0})$,

where $u, v, w, e \in \tilde{R}^c,\ \tilde{0}(t) = \begin{cases} 1, & t = (0,0,0...,0), \\ 0, & \text{otherwise.} \end{cases}$

Definitions 2.1.[15] A sequence $\{x_n\}$ of fuzzy numbers is said to be statistically convergent to a fuzzy number x_0 if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in N : D(x_n, x_0) \ge \varepsilon\}$ has natural density zero. The fuzzy number x_0 is called the statistical limit of the sequence $\{x_n\}$ and we write $st\text{-}\lim_{n\to\infty} x_n = x_0$.

Now let $\alpha(n)$ and $\beta(n)$ be two sequences of positive numbers satisfying the following conditions:
(1) $\alpha(n)$ and $\beta(n)$ are both non-decreasing,
(2) $\beta(n) \ge \alpha(n)$,
(3) $\beta(n) - \alpha(n) \to \infty$, as $n \to \infty$,
and let Λ denote the set of pairs (α, β) satisfying (1), (2) and (3).

For each pair $(\alpha,\beta) \in \Lambda,\ 0 < \gamma \le 1$ and $K \subset N$, we define $\delta^{\alpha,\beta}(K,\gamma)$ in the following way:

$$\delta^{\alpha,\beta}(K,\gamma) = \lim_n \frac{|K \cap P_n^{\alpha,\beta}|}{(\beta(n) - \alpha(n) + 1)^\gamma}$$

where $P_n^{\alpha,\beta}$ is the closed interval $[\alpha(n), \beta(n)]$ and $|S|$ represents the cardinality of S.

Lemma 2.1. Let K and M be two subsets of N and $0 < \gamma \le \delta \le 1$. Then for all $(\alpha,\beta) \in \Lambda$, we have
(1) $\delta^{\alpha,\beta}(\varnothing,\gamma)$=0,
(2) $\delta^{\alpha,\beta}(N,1)$=1,
(3) if K is a finite set, the $\delta^{\alpha,\beta}(K,\gamma) = 0$,
(4) $K \subset M \Rightarrow \delta^{\alpha,\beta}(K,\gamma) \le \delta^{\alpha,\beta}(M,\gamma)$,
(5) $\delta^{\alpha,\beta}(K,\delta) \le \delta^{\alpha,\beta}(K,\gamma)$.

3. Main results

Definition 3.1. A sequence of fuzzy numbers is said to be $\alpha\beta$-statistically convergent of order γ to x_0, if for every $\varepsilon > 0$,

$$\delta^{\alpha,\beta}(\{k : D(x_k, x_0) \ge \varepsilon\},\gamma) = \lim_n \frac{|\{k \in P_n^{\alpha,\beta} : D(x_k, x_0) \ge \varepsilon\}|}{(\beta(n) - \alpha(n) + 1)^\gamma} = 0.$$

In this case, we write $\tilde{S}_\gamma^{\alpha,\beta} - \lim x_k = x_0$. The set of all $\alpha\beta$-statistically convergent of order γ will be denoted simply by $\tilde{S}_\gamma^{\alpha,\beta}$.

For $\gamma = 1$, we say that x is $\alpha\beta$-statistically convergent to x_0 and this is denoted by $\tilde{S}^{\alpha,\beta} - \lim x_k = x_0$.

The following example shows that Definition 3.1 is non-trivial generalization of both ordinary and statistical convergence.

Example 3.1. Taking $\alpha(n) = 1$ and $\beta(n) = n^{\frac{1}{\gamma}}$, where $0 < \gamma < 1$ is fixed, then

$$\delta^{\alpha,\beta}(\{k : D(x_k, x_0) \ge \varepsilon\},\gamma) = \lim_n \frac{|\{k \in [1, n^{\frac{1}{\gamma}}] : D(x_k, x_0) \ge \varepsilon\}|}{n}$$

and, in particular, for $\gamma = \frac{1}{2}$ we have

$$\delta^{\alpha,\beta}(\{k : D(x_k, x_0) \ge \varepsilon\}, \frac{1}{2}) = \lim_n \frac{|\{k \in [1, n^2] : D(x_k, x_0) \ge \varepsilon\}|}{n}.$$

Consider the sequence of fuzzy numbers

$$x_k(t)=\begin{cases} t+1, & -1\le t\le 0, k\neq n^2,\\ -t+1, & 0<t\le 1, k\neq n^2,\\ t, & 0\le t\le 1, k=n^2,\\ 2-t, & 1<t\le 2, k=n^2,\\ 0, & \text{others};\end{cases}\qquad x_0(t)=\begin{cases} t+1, & -1\le t\le 0,\\ -t+1, & 0<t\le 1,\\ 0, & \text{others};\end{cases}$$

Obviously $st-\lim_n x_k=x_0$, however

$$\delta^{\alpha,\beta}(\{k: D(x_k,x_0)\ge\varepsilon\},\frac{1}{2})=\lim_n\frac{|\{k\in[1,n^2]: D(x_k,x_0)\ge\varepsilon\}|}{n}\neq 0.$$

for all $\varepsilon>0, \tilde{S}_\gamma^{\alpha,\beta}-\lim x_k\neq x_0$.

Definition 3.2. Based on strongly $\alpha\beta$-convergence of order γ, for every $\varepsilon>0$, we define the following sets

$$\tilde{W}_\gamma^{\alpha,\beta}=\big\{x=\{x_k\}:\lim_n\frac{1}{(\beta(n)-\alpha(n)+1)^\gamma}\sum_{k\in P_n^{\alpha,\beta}}D(x_k,x_0)=0\big\},$$

$$\tilde{W}_{\gamma 0}^{\alpha,\beta}=\big\{x=\{x_k\}:\lim_n\frac{1}{(\beta(n)-\alpha(n)+1)^\gamma}\sum_{k\in P_n^{\alpha,\beta}}D(x_k,\bar{0})=0\big\},$$

$$\tilde{W}_{\gamma\infty}^{\alpha,\beta}=\big\{x=\{x_k\}:\sup_n\frac{1}{(\beta(n)-\alpha(n)+1)^\gamma}\sum_{k\in P_n^{\alpha,\beta}}D(x_k,\bar{0})<\infty\big\},$$

where

$$\tilde{0}(t)=\begin{cases} 1, & t=(0,0,0...,0),\\ 0, & \text{otherwise}.\end{cases}$$

If $x\in\tilde{W}_\gamma^{\alpha,\beta}$, we say that x is strongly $\alpha\beta$-convergent of order γ to x_0 and we write $\tilde{W}_\gamma^{\alpha,\beta}-\lim x_k=x_0$. For $\gamma=1$, we say that x is strongly $\alpha\beta$-convergent to x_0 and this is denoted by $\tilde{W}^{\alpha,\beta}-\lim x_k=x_0$.

Remark 3.1. Take $\alpha(n)=1,\ \beta(n)=n$ and $\gamma=1$, then $P_n^{\alpha,\beta}=[1,n]$ and

$$\delta^{\alpha,\beta}(\{k: D(x_k,x_0)\ge\varepsilon\},\gamma)=\lim_n\frac{|\{k\le n: D(x_k,x_0)\ge\varepsilon\}|}{n}=0.$$

This shows that in this case, $\alpha\beta$-statistical convergence of order γ reduces to statistical convergence which we denoted by $\tilde{S}$. Meanwhile, the sequences space $\tilde{W}_\gamma^{\alpha,\beta}$ reduces to $\tilde{W}$, $\tilde{W}_{\gamma 0}^{\alpha,\beta}$ reduces to $\tilde{W}_0$ and $\tilde{W}_{\gamma\infty}^{\alpha,\beta}$ reduces to $\tilde{W}_\infty$. Where $\tilde{W}$, $\tilde{W}_0$ and $\tilde{W}_\infty$ are defined by Mursaleen and Basarir [16].

$$\tilde{W}=\big\{x=\{x_k\}:\lim_n\frac{1}{n}\sum_{k=1}^n D(x_k,x_0)=0\big\},$$

$$\tilde{W}_0=\big\{x=\{x_k\}:\lim_n\frac{1}{n}\sum_{k=1}^n D(x_k,\bar{0})=0\big\},$$

$$\tilde{W}_\infty=\big\{x=\{x_k\}:\sup_n\frac{1}{n}\sum_{k=1}^n D(x_k,\bar{0})<\infty\big\}.$$

Remark 3.2. Let λ_n be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1}\le\lambda_n+1,\ \lambda_1=1$ and $I_n=[n-\lambda_n+1,n]$. We choose $\alpha(n)=n-\lambda_n+1,\ \beta(n)=n$ and $\gamma=1$, then $P_n^{\alpha,\beta}=[n-\lambda_n+1,n]$. Moreover,

$$\delta^{\alpha,\beta}(\{k: D(x_k,x_0)\ge\varepsilon\},\gamma)=\lim_n\frac{|\{k\in I_n: D(x_k,x_0)\ge\varepsilon\}|}{\lambda_n}=0.$$

This shows that in this case, $\alpha\beta$-statistical convergence of order γ reduces to λ-statistical convergence which we denoted by $\tilde{S}(\lambda)$. Meanwhile, the sequences space $\tilde{W}_{\gamma}^{\alpha,\beta}$ reduces to $\tilde{W}(\lambda)$, $\tilde{W}_{\gamma 0}^{\alpha,\beta}$ reduces to $\tilde{W}_0(\lambda)$ and $\tilde{W}_{\gamma\infty}^{\alpha,\beta}$ reduces to $\tilde{W}_\infty(\lambda)$. Where $\tilde{W}(\lambda)$, $\tilde{W}_0(\lambda)$ and $\tilde{W}_\infty(\lambda)$ are defined by Savas [17].

$$\tilde{W}(\lambda) = \big\{x = \{x_k\} : \lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} D(x_k, x_0) = 0\big\},$$

$$\tilde{W}_0(\lambda) = \big\{x = \{x_k\} : \lim_n \frac{1}{\lambda_n} \sum_{k\in I_n} D(x_k, \bar{0}) = 0\big\},$$

$$\tilde{W}_\infty(\lambda) = \big\{x = \{x_k\} : \sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} D(x_k, \bar{0}) < \infty\big\}.$$

Remark 3.3. A lacunary sequence $\theta = \{k_r\}$ is an increasing sequence such that $k_0 = 0,\ h_r = k_r - k_{r-1} \to \infty,\ r \to \infty$ and $I_r = (k_{r-1}, k_r]$. Take $\alpha(r) = k_{r-1} + 1,\ \beta(r) = k_r$ and $\gamma = 1$, then $P_r^{\alpha,\beta} = [k_{r-1} + 1, k_r]$. However $(k_{r-1}, k_r] \cap N = [k_{r-1} + 1, k_r] \cap N$, we have

$$\delta^{\alpha,\beta}(\{k : D(x_k, x_0) \geq \varepsilon\}, \gamma) = \lim_r \frac{|\{k \in I_r : D(x_k, x_0) \geq \varepsilon\}|}{h_r} = 0.$$

This shows that in this case, $\alpha\beta$-statistical convergence of order γ coincides with lacunary statistical convergence which we denoted by $\tilde{S}(\theta)$. Meanwhile, the sequences space $\tilde{W}_{\gamma}^{\alpha,\beta}$ reduces to $\tilde{W}(\theta)$, $\tilde{W}_{\gamma 0}^{\alpha,\beta}$ reduces to $\tilde{W}_0(\theta)$ and $\tilde{W}_{\gamma\infty}^{\alpha,\beta}$ reduces to $\tilde{W}_\infty(\theta)$.
Where

$$\tilde{W}(\theta) = \big\{x = \{x_k\} : \lim_r \frac{1}{h_r} \sum_{k\in I_r} D(x_k, x_0) = 0\big\},$$

$$\tilde{W}_0(\theta) = \big\{x = \{x_k\} : \lim_r \frac{1}{h_r} \sum_{k\in I_r} D(x_k, \bar{0}) = 0\big\},$$

$$\tilde{W}_\infty(\theta) = \big\{x = \{x_k\} : \sup_r \frac{1}{h_r} \sum_{k\in I_r} D(x_k, \bar{0}) < \infty\big\}.$$

Theorem 3.1. Let $x = \{x_k\},\ y = \{y_k\}$ be two sequences of fuzzy numbers. We have
(1) If $\tilde{S}_{\gamma}^{\alpha,\beta} - \lim x_k = x_0$ and $c \in R$, then $\tilde{S}_{\gamma}^{\alpha,\beta} - \lim cx_k = cx_0$;
(2) If $\tilde{S}_{\gamma}^{\alpha,\beta} - \lim x_k = x_0,\ \tilde{S}_{\gamma}^{\alpha,\beta} - \lim y_k = y_0$, then $\tilde{S}_{\gamma}^{\alpha,\beta} - \lim(x_k + y_k) = x_0 + y_0$.

Proof. (1) The proof is obvious when c=0. Suppose that $c \neq 0$, then the proof of (1) follows from the following inequality,

$$\frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(cx_k, cx_0) \geq \varepsilon\}| \leq \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_0) \geq \frac{\varepsilon}{|c|}\}|.$$

(2)Suppose that $\tilde{S}_{\gamma}^{\alpha,\beta} - \lim x_k = x_0,\ \tilde{S}_{\gamma}^{\alpha,\beta} - \lim y_k = y_0$, then

$$\lim_n \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_0) \geq \frac{\varepsilon}{2}\}| = 0,$$

$$\lim_n \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(y_k, y_0) \geq \frac{\varepsilon}{2}\}| = 0.$$

Since

$$D(x_k + y_k, x_0 + y_0) \leq D(x_k + y_k, x_0 + y_k) + D(x_0 + y_k, x_0 + y_0) = D(x_k, x_0) + D(y_k, y_0).$$

For $\varepsilon > 0$, we have

$$\begin{aligned} & \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_k + y_k, x_0 + y_0) \geq \varepsilon\}| \\ \leq\ & \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_0) \geq \frac{\varepsilon}{2}\}| + \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(y_k, y_0) \geq \frac{\varepsilon}{2}\}| \end{aligned}$$

$\to 0,\ n \to \infty$. Hence $\tilde{S}_{\gamma}^{\alpha,\beta} - \lim(x_k + y_k) = x_0 + y_0$.

Definition 3.3. The sequence of fuzzy numbers $x = \{x_k\}$ is a $\alpha\beta$-statistically Cauchy sequence of order γ, if for every $\varepsilon > 0$ there exists a number $N(= N(\varepsilon))$ such that

$$\lim_{n} \frac{1}{(\beta(n) - \alpha(n) + 1)^{\gamma}} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_N) \geq \varepsilon\}| = 0.$$

Theorem 3.2. Let $x = \{x_k\}$ be a sequence of fuzzy numbers. It is a $\alpha\beta$-statistically convergent sequence of order γ if and only if x is a $\alpha\beta$-statistical Cauchy sequence of order γ.

Proof. Suppose that $\tilde{S}_{\gamma}^{\alpha,\beta} - \lim x_k = x_0$ and let $\varepsilon > 0$, then

$$\lim_{n} \frac{1}{(\beta(n) - \alpha(n) + 1)^{\gamma}} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_0) \geq \varepsilon\}| = 0,$$

and N is choosen such that $\lim_{n} \frac{1}{(\beta(n)-\alpha(n)+1)^{\gamma}} |\{k \in P_n^{\alpha,\beta} : D(x_N, x_0) \geq \varepsilon\}| = 0$,
then we have

$$\begin{aligned} & \frac{1}{(\beta(n) - \alpha(n) + 1)^{\gamma}} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_N) \geq \varepsilon\}| \\ \leq\ & \frac{1}{(\beta(n) - \alpha(n) + 1)^{\gamma}} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_0) \geq \varepsilon\}| + \frac{1}{(\beta(n) - \alpha(n) + 1)^{\gamma}} |\{k \in P_n^{\alpha,\beta} : D(x_N, x_0) \geq \varepsilon\}|. \end{aligned}$$

Hence $x = \{x_k\}$ is a $\alpha\beta$-statistically Cauchy sequence of order γ.

Next, assume that $x = \{x_k\}$ be $\alpha\beta$-statistical Cauchy sequence of order γ, then there exists a strictly increasing sequence N_p of positive integers such that $\lim_{n} \frac{1}{(\beta(n)-\alpha(n)+1)^{\gamma}} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_{N_p}) \geq \varepsilon_p\}| = 0$, where $\varepsilon_p : p = 1, 2, 3, \cdots$ is a strictly decreasing sequence of numbers converging to zero for each p and q pair $(p \neq q)$ of positive integers, we can select K_{pq} such $D(x_{K_{pq}}, x_{N_p}) < \varepsilon_p$ and $D(x_{K_{pq}}, x_{N_q}) < \varepsilon_q$.
It follows that

$$D(x_{N_p}, x_{N_q}) \leq D(x_{K_{pq}}, x_{N_p}) + D(x_{K_{pq}}, x_{N_q}) < \varepsilon_p + \varepsilon_q \to 0,\ p, q \to \infty.$$

Hence, $\{x_{N_p}\} : p = 1, 2, \cdots$ is a Cauchy sequence and statisfies the Cauchy convergence criterion. Let $\{x_{N_p}\}$ converge to x_0. Since $\varepsilon_p : p = 1, 2, \cdots \to 0$, so for $\varepsilon > 0$, there exists p_0 such that $\varepsilon_{p_0} < \frac{\varepsilon}{2}$ and $D(x_{N_p}, x_0) < \frac{\varepsilon}{2},\ p \geq p_0$, then

$$D(x_k, x_0) \leq D(x_k, x_{N_{p_0}}) + D(x_{N_{p_0}}, x_0) \leq D(x_k, x_{N_{p_0}}) + \frac{\varepsilon}{2},$$

we have

$$\begin{aligned} & \frac{1}{(\beta(n) - \alpha(n) + 1)^{\gamma}} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_0) \geq \varepsilon\}| \leq \frac{1}{(\beta(n) - \alpha(n) + 1)^{\gamma}} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_{N_{P_0}}) \geq \frac{\varepsilon}{2}\}| \\ \leq\ & \frac{1}{(\beta(n) - \alpha(n) + 1)^{\gamma}} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_{N_{P_0}}) \geq \varepsilon_{p_0}\}| \to 0,\ n \to \infty. \end{aligned}$$

This shows that $x = \{x_k\}$ is $\alpha\beta$-statistically convergent of order γ.

Theorem 3.3. Let $x = \{x_k\}$ is a sequence of fuzzy numbers. There exsit a $\alpha\beta$-statistically convergent of order γ sequence $y = \{y_k\}$ such that $x_k = y_k$ for almost all k according to γ, then $x = \{x_k\}$ is a $\alpha\beta$-statistically convergent sequence of order γ.

Proof. Let $x_k = y_k$ for almost all k according to γ and $\tilde{S}_{\gamma}^{\alpha,\beta} - \lim y_k = x_0$. Suppose $\varepsilon > 0$. Then for each n,

$$\{k \in P_n^{\alpha,\beta} : D(x_k, x_0) \geq \varepsilon\} \subseteq \{k \in P_n^{\alpha,\beta} : D(y_k, x_0) \geq \varepsilon\} \cup \{k \in P_n^{\alpha,\beta} : x_k \neq y_k\}.$$

Since $x_k = y_k$ for almost all k according to γ, the latter set contains a fixed number of integers, say $S = S(\varepsilon)$. Then

$$\begin{aligned} & \lim_{n} \frac{1}{(\beta(n) - \alpha(n) + 1)^{\gamma}} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_0) \geq \varepsilon\}| \\ \leq\ & \lim_{n} \frac{1}{(\beta(n) - \alpha(n) + 1)^{\gamma}} |\{k \in P_n^{\alpha,\beta} : D(y_k, x_0) \geq \varepsilon\}| + \lim_{n} \frac{S}{(\beta(n) - \alpha(n) + 1)^{\gamma}}, \end{aligned}$$

Hence $\tilde{S}_{\gamma}^{\alpha,\beta}-\lim x_k=x_0$, i.e. $x=\{x_k\}$ is a $\alpha\beta$-statistically convergent sequence of order γ.
Theorem 3.4. Let $0<\gamma_1\le\gamma_2\le 1$, then $\tilde{S}_{\gamma_1}^{\alpha,\beta}\subseteq\tilde{S}_{\gamma_2}^{\alpha,\beta}$.
Proof. Let $0<\gamma_1\le\gamma_2\le 1$. Then we have

$$\frac{1}{(\beta(n)-\alpha(n)+1)^{\gamma_2}}|\{k\in P_n^{\alpha,\beta}:D(x_k,x_0)\ge\varepsilon\}|\le\frac{1}{(\beta(n)-\alpha(n)+1)^{\gamma_1}}|\{k\in P_n^{\alpha,\beta}:D(x_k,x_0)\ge\varepsilon\}|,$$

for every $\varepsilon>0$ and so we get $\tilde{S}_{\gamma_1}^{\alpha,\beta}\subseteq\tilde{S}_{\gamma_2}^{\alpha,\beta}$.
Corollary 3.1. If a sequence $x=\{x_k\}$ of fuzzy numbers is $\alpha\beta$-statistically convergent of order γ, then it is $\alpha\beta$-statistically convergent, for each $\gamma\in(0,1]$, i.e. $\tilde{S}_{\gamma}^{\alpha,\beta}\subseteq\tilde{S}^{\alpha,\beta}$.
Theorem 3.5. The sequence spaces of fuzzy numbers $\tilde{W}_{\gamma 0}^{\alpha,\beta}$, $\tilde{W}_{\gamma}^{\alpha,\beta}$ and $\tilde{W}_{\gamma\infty}^{\alpha,\beta}$ satisfy the relationship: $\tilde{W}_{\gamma 0}^{\alpha,\beta}\subset\tilde{W}_{\gamma}^{\alpha,\beta}\subset\tilde{W}_{\gamma\infty}^{\alpha,\beta}$.
Proof. Let $x=\{x_k\}\in\tilde{W}_{\gamma}^{\alpha,\beta}$. Note that

$$\begin{aligned}&\frac{1}{(\beta(n)-\alpha(n)+1)^{\gamma}}\sum_{k\in P_n^{\alpha,\beta}}D(x_k,\bar{0})\\ \le\ &\frac{1}{(\beta(n)-\alpha(n)+1)^{\gamma}}\sum_{k\in P_n^{\alpha,\beta}}D(x_k,x_0)+\frac{1}{(\beta(n)-\alpha(n)+1)^{\gamma}}\sum_{k\in P_n^{\alpha,\beta}}D(x_0,\bar{0})\\ \le\ &\frac{1}{(\beta(n)-\alpha(n)+1)^{\gamma}}\sum_{k\in P_n^{\alpha,\beta}}D(x_k,x_0)+\frac{1}{(\beta(n)-\alpha(n)+1)^{\gamma}}D(x_0,\bar{0}),\end{aligned}$$

according to the above inequality, we have $\sup_n\frac{1}{(\beta(n)-\alpha(n)+1)^{\gamma}}\sum_{k\in P_n^{\alpha,\beta}}D(x_k,\bar{0})<\infty$, thus we get $x\in\tilde{W}_{\gamma\infty}^{\alpha,\beta}$.
The proof of $\tilde{W}_{\gamma 0}^{\alpha,\beta}\subset\tilde{W}_{\gamma}^{\alpha,\beta}$ is obvious.
Theorem 3.6. The sequence spaces of fuzzy numbers $\tilde{W}_{\gamma 0}^{\alpha,\beta}$, $\tilde{W}_{\gamma}^{\alpha,\beta}$ and $\tilde{W}_{\gamma\infty}^{\alpha,\beta}$ are linear spaces over the set of real numbers.
Proof. Let $x=\{x_k\},\ y=\{y_k\}\in\tilde{W}_{\gamma 0}^{\alpha,\beta},\ \alpha,\beta\in R$. In order to get result we need to prove the following

$$\lim_n\frac{1}{(\beta(n)-\alpha(n)+1)^{\gamma}}\sum_{k\in P_n^{\alpha,\beta}}D(\alpha x_k+\beta y_k,\bar{0})=0.$$

Since $x=\{x_k\},\ y=\{y_k\}\in\tilde{W}_{\gamma 0}^{\alpha,\beta}$, we have

$$\lim_n\frac{1}{(\beta(n)-\alpha(n)+1)^{\gamma}}\sum_{k\in P_n^{\alpha,\beta}}D(x_k,\bar{0})=0,$$

$$\lim_n\frac{1}{(\beta(n)-\alpha(n)+1)^{\gamma}}\sum_{k\in P_n^{\alpha,\beta}}D(y_k,\bar{0})=0.$$

And $D(\alpha x_k+\beta y_k,\bar{0})\le D(\alpha x_k,\bar{0})+D(\beta y_k,\bar{0})=|\alpha|D(x_k,\bar{0})+|\beta|D(y_k,\bar{0})$,
we get

$$\begin{aligned}&\frac{1}{(\beta(n)-\alpha(n)+1)^{\gamma}}\sum_{k\in P_n^{\alpha,\beta}}D(\alpha x_k+\beta y_k,\bar{0})\le\frac{1}{(\beta(n)-\alpha(n)+1)^{\gamma}}\sum_{k\in P_n^{\alpha,\beta}}\left[D(\alpha x_k,\bar{0})+D(\beta y_k,\bar{0})\right]\\ \le\ &\frac{|\alpha|}{(\beta(n)-\alpha(n)+1)^{\gamma}}\sum_{k\in P_n^{\alpha,\beta}}D(x_k,x_0)+\frac{|\beta|}{(\beta(n)-\alpha(n)+1)^{\gamma}}\sum_{k\in P_n^{\alpha,\beta}}D(y_k,x_0)\to 0,\ n\to\infty.\end{aligned}$$

Thus $\alpha x+\beta y\in\tilde{W}_{\gamma 0}^{\alpha,\beta}$. Similarly it can be shown that the other spaces are also linear spaces.
Theorem 3.7. Let $0<\gamma\le 1$. If a sequence $x=\{x_k\}$ of fuzzy number is strongly $\alpha\beta$-convergent of order γ, then it is $\alpha\beta$-statistically convergent of order γ, i.e. $\tilde{W}_{\gamma}^{\alpha,\beta}\subset\tilde{S}_{\gamma}^{\alpha,\beta}$.

Proof. Given $\varepsilon > 0$ and any sequence $x = \{x_k\}$ of fuzzy numbers, we write

$$\begin{aligned}\sum_{k\in P_n^{\alpha,\beta}} D(x_k,x_0) &= \sum_{k\in P_n^{\alpha,\beta},D(x_k,x_0)<\varepsilon} D(x_k,x_0) + \sum_{k\in P_n^{\alpha,\beta},D(x_k,x_0)\geq\varepsilon} D(x_k,x_0)\\ &\geq \sum_{k\in P_n^{\alpha,\beta},D(x_k,x_0)\geq\varepsilon} D(x_k,x_0) \geq |\{k\in P_n^{\alpha,\beta} : D(x_k,x_0)\geq\varepsilon\}|\cdot\varepsilon\end{aligned}$$

and hence

$$\frac{1}{(\beta(n)-\alpha(n)+1)^\gamma}\sum_{k\in P_n^{\alpha,\beta}} D(x_k,x_0) \geq \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma}|\{k\in P_n^{\alpha,\beta} : D(x_k,x_0)\geq\varepsilon\}|\cdot\varepsilon.$$

Here, it can be easily to see that if a sequence $x = \{x_k\}$ of fuzzy number is strongly $\alpha\beta$-convergent of order γ, then it is $\alpha\beta$-statistically convergent of order γ.

Corollary 3.2. Let $0 < \gamma \leq \eta \leq 1$. If a sequence $x = \{x_k\}$ of fuzzy number is strongly $\alpha\beta$-convergent of order γ, then it is $\alpha\beta$-statistically convergent of order η, i.e. $\tilde{W}_\gamma^{\alpha,\beta} \subset \tilde{S}_\eta^{\alpha,\beta}$.

Definition 3.4. Let $p = \{p_k\}$ be any sequence of strictly positive real numbers. A sequence $x = \{x_k\}$ of fuzzy numbers is said to be strongly $\alpha\beta(p)$-convergent of order γ, if for $\gamma \in (0,1]$, there is a fuzzy number x_0 such that

$$\lim_n \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma}\sum_{k\in P_n^{\alpha,\beta}} \big[D(x_k,x_0)\big]^{p_k} = 0,$$

we denote the set of all strongly $\alpha\beta(p)$-convergent of order γ for fuzzy sequences by $\tilde{W}_\gamma^{\alpha,\beta}(p)$. Where

$$\tilde{W}_\gamma^{\alpha,\beta}(p) = \big\{x=\{x_k\} : \lim_n \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma}\sum_{k\in P_n^{\alpha,\beta}} \big[D(x_k,x_0)\big]^{p_k} = 0\big\},$$

$$\tilde{W}_{\gamma 0}^{\alpha,\beta}(p) = \big\{x=\{x_k\} : \lim_n \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma}\sum_{k\in P_n^{\alpha,\beta}} \big[D(x_k,\bar{0})\big]^{p_k} = 0\big\},$$

$$\tilde{W}_{\gamma\infty}^{\alpha,\beta}(p) = \big\{x=\{x_k\} : \sup_n \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma}\sum_{k\in P_n^{\alpha,\beta}} \big[D(x_k,\bar{0})\big]^{p_k} < \infty\big\}.$$

It similar to the proofs of Theorem 3.5, 3.6, for strongly $\alpha\beta(p)$-convergent of order γ we have the following results.

Theorem 3.8. The sequence spaces of fuzzy numbers $\tilde{W}_{\gamma 0}^{\alpha,\beta}(p)$, $\tilde{W}_\gamma^{\alpha,\beta}(p)$ and $\tilde{W}_{\gamma\infty}^{\alpha,\beta}(p)$ satisfy the relationship: $\tilde{W}_{\gamma 0}^{\alpha,\beta}(p) \subset \tilde{W}_\gamma^{\alpha,\beta}(p) \subset \tilde{W}_{\gamma\infty}^{\alpha,\beta}(p)$.

Theorem 3.9. The sequence spaces of fuzzy numbers $\tilde{W}_{\gamma 0}^{\alpha,\beta}(p)$, $\tilde{W}_\gamma^{\alpha,\beta}(p)$ and $\tilde{W}_{\gamma\infty}^{\alpha,\beta}(p)$ are linear spaces over the set of real numbers.

Theorem 3.10. Let $0 < p_k \leq q_k$, and $\{\frac{q_k}{p_k}\}$ be bounded. Then $\tilde{W}_\gamma^{\alpha,\beta}(q) \subset \tilde{W}_\gamma^{\alpha,\beta}(p)$.

Proof. Let $x = \{x_k\} \in \tilde{W}_\gamma^{\alpha,\beta}(q)$, and $t_k = \big[D(x_k,x_0)\big]^{q_k}$, $\lambda_k = \frac{p_k}{q_k}$, $0 < \lambda_k \leq 1$. Let $0 < \lambda < \lambda_k$, and define $u_k = \begin{cases} t_k, & t_k \geq 1,\\ 0, & t_k < 1,\end{cases}$ $v_k = \begin{cases} 0, & t_k \geq 1,\\ t_k, & t_k < 1,\end{cases}$ then $t_k = u_k + v_k$, $t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$, and $u_k^{\lambda_k} \leq u_k \leq t_k$, $v_k^{\lambda_k} \leq v_k^{\lambda}$. We have

$$\begin{aligned}\frac{1}{(\beta(n)-\alpha(n)+1)^\gamma}\sum_{k\in P_n^{\alpha,\beta}} \big[D(x_k,x_0)\big]^{p_k} &= \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma}\sum_{k\in P_n^{\alpha,\beta}} t_k^{\lambda_k}\\ = \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma}\sum_{k\in P_n^{\alpha,\beta}} (u_k^{\lambda_k}+v_k^{\lambda_k}) &\leq \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma}\sum_{k\in P_n^{\alpha,\beta}} t_k\\ + \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma}\sum_{k\in P_n^{\alpha,\beta}} v_k^{\lambda} &\to 0,\ n\to\infty.\end{aligned}$$

Since $x = \{x_k\} \in \tilde{W}_\gamma^{\alpha,\beta}(q)$, we have $\lim_n \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} \sum_{k\in P_n^{\alpha,\beta}} t_k = 0$. And since $v_k < 1$, $\lambda < 1$, we get $\lim_n \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} \sum_{k\in P_n^{\alpha,\beta}} v_k^\lambda = 0$. Hence, $\tilde{W}_\gamma^{\alpha,\beta}(q) \subset \tilde{W}_\gamma^{\alpha,\beta}(p)$.

In the following theorem, we shall discuss the relationship between the space $\tilde{W}_\gamma^{\alpha,\beta}(p)$ and $\tilde{S}_\gamma^{\alpha,\beta}$.

Theorem 3.11. Let $0 < h = \inf_k p_k \le \sup_k p_k = H < \infty$, l_∞ be a set of all bounded sequence of fuzzy numbers. Then

(1) $\tilde{W}_\gamma^{\alpha,\beta}(p) \subset \tilde{S}_\gamma^{\alpha,\beta}$;

(2) If $x = \{x_k\} \in l_\infty \cap \tilde{S}_\gamma^{\alpha,\beta}$, then $x = \{x_k\} \in \tilde{W}_\gamma^{\alpha,\beta}(p)$;

(3) $l_\infty \cap \tilde{S}_\gamma^{\alpha,\beta} = l_\infty \cap \tilde{W}_\gamma^{\alpha,\beta}(p)$.

Proof. (1) Let $x = \{x_k\} \in \tilde{W}_\gamma^{\alpha,\beta}(p)$, Note that

$$\begin{aligned} & \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} \sum_{k\in P_n^{\alpha,\beta}} \big[D(x_k,x_0)\big]^{p_k} \ge \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} \sum_{k\in P_n^{\alpha,\beta}, D(x_k,x_0)\ge\varepsilon} \big[D(x_k,x_0)\big]^{p_k} \\ \ge\ & \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} \sum_{k\in P_n^{\alpha,\beta}, D(x_k,x_0)\ge\varepsilon} \min\{\varepsilon^h, \varepsilon^H\} \\ =\ & \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_k,x_0) \ge \varepsilon\}| \cdot \min\{\varepsilon^h, \varepsilon^H\}, \end{aligned}$$

follow from the above inequality, we have $\lim_n \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_k,x_0) \ge \varepsilon\}| = 0$. Thus we get $x = \{x_k\} \in \tilde{S}_\gamma^{\alpha,\beta}$.

(2) Let $x = \{x_k\} \in l_\infty \cap \tilde{S}_\gamma^{\alpha,\beta}$, then there is a constant $T > 0$, such that $D(x_k,x_0) \le T$. Therefore

$$\begin{aligned} & \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} \sum_{k\in P_n^{\alpha,\beta}} \big[D(x_k,x_0)\big]^{p_k} \\ =\ & \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} \sum_{k\in P_n^{\alpha,\beta}, D(x_k,x_0)\ge\varepsilon} \big[D(x_k,x_0)\big]^{p_k} + \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} \sum_{k\in P_n^{\alpha,\beta}, D(x_k,x_0)<\varepsilon} \big[D(x_k,x_0)\big]^{p_k} \\ \le\ & \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} \sum_{k\in P_n^{\alpha,\beta}, D(x_k,x_0)\ge\varepsilon} \max\{T^h, T^H\} + \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} \sum_{k\in P_n^{\alpha,\beta}, D(x_k,x_0)<\varepsilon} \varepsilon^{p_k} \\ \le\ & \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_k,x_0) \ge \varepsilon\}| \cdot \max\{T^h, T^H\} \\ & + \max\{\varepsilon^h, \varepsilon^H\}, \end{aligned}$$

follow from the above inequality, we have $\lim_n \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} \sum_{k\in P_n^{\alpha,\beta}} \big[D(x_k,x_0)\big]^{p_k} = 0$. Thus we get $x = \{x_k\} \in \tilde{W}_\gamma^{\alpha,\beta}(p)$.

(3) From (1) and (2), (3) is obvious.

4. Conclusion

In this article, we introduced some classes of sequences of fuzzy numbers defined by $\alpha\beta$-statistically convergence of order γ, strong $\alpha\beta$-convergence of order γ, and strong $\alpha\beta(p)$-convergence of order γ. We have proved some properties and relationships of these spaces. At the same time, it also shows that if a sequence of fuzzy numbers is strongly $\alpha\beta$-convergent of order γ then it is $\alpha\beta$-statistically convergent of order γ.

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IF rough approximations based on lattices*

Gangqiang Zhang† Yu Han‡ Zhaowen Li§

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Abstract: An IF rough set, which is the result of approximation of an IF set with respect to an IF approximation space, is an extension of fuzzy rough sets. This paper studies rough set theory within the context of lattices. First, we introduce the concepts of IF rough sets and IF rough approximation operators based on lattices. Then, we give some properties on IF rough approximations of IF sublattices such as IF ideals and IF filters.

Keywords: Lattice; IF set; Full congruence relation; IF approximate space; IF rough set; IF sublattice; IF rough approximation.

1 Introduction

Rough set theory was originally proposed by Pawlak [11, 12] as a mathematical approach to handle imprecision and uncertainty in data analysis. Usefulness and versatility of this theory have amply been demonstrated by successful applications in a variety of problems [15, 16].

The basic structure of rough set theory is an approximation space. Based on it, lower and upper approximations can be induced. Using these approximations, knowledge hidden in information systems may be revealed and expressed in the form of decision rules [11].

Intuitionistic fuzzy (IF, for short) sets were originated by Atanassov [1, 2]. It is an intuitively straightforward extension of Zadeh's fuzzy sets [19]. IF sets have played an useful role in the research of uncertainty theories. Unlike a fuzzy set, which gives a degree of which element belongs to a set, an IF set gives both

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†Corresponding Author, College of Information Science and Engineering, Guangxi University for Nationalities, Nanning, Guangxi 530006, P.R.China. zhanggangqiang100@126.com

‡College of Science, Guangxi University for Nationalities, Nanning, Guangxi 530006, P.R.China.

§Guangxi Key Laboratory of Universities Optimization Control and Engineering Calculation, and College of Sciences, Guangxi University for Nationalities, Nanning, Guangxi 530006, P.R.China.

a membership degree and a nonmembership degree. Thus, an IF set is more objective than a fuzzy set to describe the vagueness of data or information.

Recently, rough set approximation was introduced into IF sets [14, 20, 21, 22]. For example, Zhou et al. [20, 21, 22] proposed a general framework for the study of IF rough sets, Zhang et al. [24] gave a general frame for IF rough sets on two universes.

The purpose of this paper is to investigate IF rough approximations based on lattices.

2 Preliminaries

Throughout this paper, " Intuitionistic fuzzy " is briefly written " IF ", U denotes a universe, I denotes $[0,1]$, L denotes a lattice with the least element 0_L and the greatest element 1_L. $J=\{(a,b)\in I\times I: a+b\leq 1\}$.

In this section, we recall some basic notions and properties.

2.1 IF sets

Definition 2.1 ([8]). *Let $(a,b),(c,d)\in I\times I$. Define*

(1) $(a,b)=(c,d) \iff a=c,\ b=d$.

(2) $(a,b)\sqcup(c,d)=(a\vee c, b\wedge d)$, $(a,b)\sqcap(c,d)=(a\wedge c, b\vee d)$.

(3) $(a,b)^c=(b,a)$.

Moreover, for $\{(a_\alpha,b_\alpha):\alpha\in\Gamma\}\subseteq I\times I$,

$\bigsqcup\limits_{\alpha\in\Gamma}(a_\alpha,b_\alpha)=(\bigvee\limits_{\alpha\in\Gamma}a_\alpha,\ \bigwedge\limits_{\alpha\in\Gamma}b_\alpha)$, $\bigsqcap\limits_{\alpha\in\Gamma}(a_\alpha,b_\alpha)=(\bigwedge\limits_{\alpha\in\Gamma}a_\alpha,\ \bigvee\limits_{\alpha\in\Gamma}b_\alpha)$.

Definition 2.2 ([8]). *Let $(a,b),(c,d)\in J$ and let $S\subseteq J\times J$. $(a,b)S(c,d)$, if $a\leq c$ and $b\geq d$. We denote S by $\leq$.*

Remark 2.3. (1) *Let $(J,\leq)$ be a poset with $0_J=(0,1)$ and $1_J=(1,0)$.*

(2) $(a,b)^{cc}=(a,b)$.

(3) $((a,b)\sqcup(c,d))\sqcup(e,f)=(a,b)\sqcup((c,d)\sqcup(e,f))$,
$((a,b)\sqcap(c,d))\sqcap(e,f)=(a,b)\sqcap((c,d)\sqcap(e,f))$.

(4) $(a,b)\sqcup(c,d)=(c,d)\sqcup(a,b)$, $(a,b)\sqcap(c,d)=(c,d)\sqcap(a,b)$.

(5) $((a,b)\sqcup(c,d))\sqcap(e,f)=((a,b)\sqcap(e,f))\sqcup((c,d)\sqcap(e,f))$.
$((a,b)\sqcap(c,d))\sqcup(e,f)=((a,b)\sqcup(e,f))\sqcap((c,d)\sqcup(e,f))$.

(6) $(\bigsqcup\limits_{\alpha\in\Gamma}(a_\alpha,b_\alpha))^c=\bigsqcap\limits_{\alpha\in\Gamma}(a_\alpha,b_\alpha)^c$, $(\bigsqcap\limits_{\alpha\in\Gamma}(a_\alpha,b_\alpha))^c=\bigsqcup\limits_{\alpha\in\Gamma}(a_\alpha,b_\alpha)^c$.

Definition 2.4 ([1]). *An IF set A in U is an object having the form*

$$A=\{<x,\mu_A(x),\nu_A(x)>: x\in U\},$$

where $\mu_A,\nu_A\in F(U)$ satisfying $0\leq\mu_A(x)+\nu_A(x)\leq 1$ for each $x\in U$, and $\mu_A(x),\nu_A(x)$ are used to define the degree of membership and the degree of non-membership of the element x to A, respectively.

$IF(U)$) denotes the family of all IF sets in U.

For the sake of simplicity, we give the following definition.

Definition 2.5. *A is called an IF set in U, if $A = (A^*, A_*) \in F(U) \times F(U)$ and for each $x \in U$, $A(x) = (A^*(x), A_*(x)) \in J$, where $A^*(x), A_*(x)$ are used to define the degree of membership and the degree of non-membership of the element x to A, respectively.*

For each $\mathcal{A} \subseteq IF(U)$, we denote

$$\mathcal{A}^c = \{A^c : A \in \mathcal{A}\},$$

$$\mathcal{A}^* = \{A^* : A \in \mathcal{A}\} \text{ and } \mathcal{A}_* = \{A_* : A \in \mathcal{A}\}.$$

For each $\lambda \in J$, $\widehat{\lambda}$ represents a constant IF set which satisfies $\widehat{\lambda}(x) = \lambda$ for each $x \in U$.

$A \in IF(U)$ is called proper if $A \neq \widehat{\lambda}$ for any $\lambda \in J$.

In this paper, if we concern IF sets in U without special statements, we always refer to the proper IF subset.

Some IF relations and IF operations are defined as follows ([19]): for any $A, B \in IF(U)$ and $\{A_\alpha : \alpha \in \Gamma\} \subseteq IF(U)$,

(1) $A = B \Longleftrightarrow A(x) = B(x)$ for each $x \in U$.

(2) $A \subseteq B \Longleftrightarrow A(x) \leq B(x)$ for each $x \in U$.

(3) $(\bigcup\limits_{\alpha\in\Gamma} A_\alpha)(x) = \bigsqcup\limits_{\alpha\in\Gamma} A_\alpha(x)$ for each $x \in U$.

(4) $(\bigcap\limits_{\alpha\in\Gamma} A_\alpha)(x) = \bigsqcap\limits_{\alpha\in\Gamma} A_\alpha(x)$ for each $x \in U$.

(5) $A^c(x) = A(x)^c$ for each $x \in U$.

(6) $(\lambda A)(x) = \lambda \sqcap (A^*(x), A_*(x))$ for any $x \in U$ and $\lambda \in J$.

Obviously, $A = B \Longleftrightarrow A^* = B^*$ and $A_* = B_* \Longleftrightarrow A \subseteq B$ and $B \subseteq A$.

We define a special IF sets $1_y = ((1_y)^*, (1_y)_*)$ for some $y \in U$ as follows:

$$(1_y)^*(x) = \begin{cases} 1, & x = y, \\ 0, & x \neq y. \end{cases} \qquad (1_y)_*(x) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

Remark 2.6. *For each $A \in IF(U)$,*

$$A = \bigcup_{y\in U} (A(y)1_y).$$

Let $\mu \in IF(U)$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$, the (α, β)-level cut set of μ,denoted by μ_α^β,is defined as follows:

$$\mu_\alpha^\beta = \{x \in U : \mu^*(x) \geq \alpha, \mu_*(x) \leq \beta\}.$$

We respectively call the sets

$$\mu_\alpha = \{x \in U : \mu^*(x) \geq \alpha\}, \quad \mu^\beta = \{x \in U : \mu_*(x) \leq \beta\}$$

the α-level cut set, the β-level set of membership generated by A.

For $x \in U$ and $(a,b) \in J - \{(0,1)\}$, $x^{(a,b)} \in IF(U)$ is called an IF point if

$$x^{(a,b)}(y) = \begin{cases} (0,1), & y \neq x, \\ (a,b), & y = x. \end{cases}$$

It is said that the IF point $x^{(a,b)}$ belongs to $\mu \in IF(U)$, which is written $x^{(a,b)} \in \mu$. Obviously,

$$x^{(a,b)} \in \mu \iff \mu(x) \geq (a,b).$$

$IFP(U)$ denotes the set of all IF point of U.

For $\mu, \lambda \in IF(U)$,

$$\mu \subseteq \lambda \iff \forall\, x^{(a,b)} \in IFP(U),\ x^{(a,b)} \in \mu \text{ implies } x^{(a,b)} \in \lambda.$$

2.2 Lattices

Definition 2.7. *Let L be a set and let $\leq$ be a binary relation on L. Then $\leq$ is called a partial order on L, if*

(i) $a \leq a$ for any $a \in L$, (ii) $a \leq b$ and $b \leq a$ imply $a = b$ for any $a, b \in L$, (iii) $a \leq b$ and $b \leq c$ imply $a \leq c$ for any $a, b, c \in L$.

Moreover, the pair $(L, \leq)$ is called a partial order set (briefly, a poset).

Definition 2.8. *Let $(L, \leq)$ be a poset and $a, b \in L$.*

(1) a is called a top (or maximal) element of L, if $x \leq a$ for any $x \in L$.

(2) b is called a bottom (or minimal) element of L,, if $b \leq x$ for any $x \in L$.

If a poset L has top elements a_1, a_2 (resp. bottom elements b_1, b_2), then $a_1 = a_2$ (resp. $b_1 = b_2$). We denote this sole top element (resp. this sole bottom element) by 1_L (resp. 0_L).

Definition 2.9. *Let $(L, \leq)$ be a poset, $S \subseteq L$ and $a, b \in L$.*

(1) a is called a above boundary in S, if $x \leq a$ for any $x \in S$.

(2) b is called a under boundary in S, if $b \leq x$ for any $x \in S$.

(3) $a = sup\ S$ or $\vee S$, if a is a minimal above boundary in S.

(4) $b = inf\ S$ or $\wedge S$, if b is a maximal under boundary in S.

Let $(L, \leq)$ be a poset and $S \subseteq L$. If $S = \{a, b\}$, then we denote $\vee S = a \vee b$ and $\wedge S = a \wedge b$.

Obviously, if $(L, \leq)$ is a poset and $a, b \in L$, then

$$a = a \wedge b \Leftrightarrow a \leq b \Leftrightarrow b = a \vee b.$$

A poset L is called a lattice, if for any $a, b \in L$, $a \vee b \in L$ and $a \wedge b \in L$.

Let L be a lattice. For $X \subseteq L$, we denote

(1) $\downarrow X = \{y \in L : y \leq x \text{ for some } x \in X\}$,

(2) $\uparrow X = \{y \in L : y \geq x \text{ for some } x \in X\}$.

Especially, $\downarrow x = \downarrow \{x\}$, $\uparrow x = \uparrow \{x\}$.

$F(L)$(resp. IF(L)) denotes the family of all fuzzy (resp. IF) sets in L.

$\mu \in F(L)$ is called a fuzzy sublattice of L, if $\mu(x\wedge y)\wedge\mu(x\vee y) \geq \mu(x)\wedge\mu(y)$ for any $x, y \in L$.

Let μ be a fuzzy sublattice of L.

(1) μ is a fuzzy ideal of L, if $\mu(x \vee y) = \mu(x) \wedge \mu(y)$ for any $x, y \in L$.

(2) μ is a fuzzy filter of L, if $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$ for any $x, y \in L$.

2.3 Fuzzy rough approximation operators based on lattices

Definition 2.10 ([3]). *Let θ be an equivalence relation on L. The pair (L, θ) is called Pawlak approximation space. For each $\mu \in F(L)$, the fuzzy lower and the fuzzy upper approximation of μ with respect to (L, θ), denoted by $\underline{\theta}(\mu)$ and $\overline{\theta}(\mu)$, are defined as follows: for each $x \in L$,*

$$\underline{\theta}(\mu)(x) = \bigwedge_{a\in[x]_\theta} A(a), \quad \overline{\theta}(\mu)(x) = \bigvee_{a\in[x]_\theta} A(a).$$

The pair $(\underline{\theta}(\mu), \overline{\theta}(\mu))$ is called the fuzzy rough set of μ with respect to (L, θ). $\underline{\theta} : F(L) \to F(L)$ and $\overline{\theta} : F(L) \to F(L)$ are called the fuzzy lower approximation operator and the fuzzy upper approximation operator, respectively. In general, we refer to $\underline{\theta}$ and $\overline{\theta}$ as the fuzzy rough approximation operators.

Proposition 2.11 ([3]). *Let θ be an equivalence relation on L. Then for $\mu, \lambda \in F(L)$,*

(1) $\underline{\theta}(\mu) \subseteq \mu \subseteq \overline{\theta}(\mu)$.

(2) *If* $\mu \subseteq \lambda$, *then* $\overline{\theta}(\mu) \subseteq \overline{\theta}(\lambda)$ *and* $\underline{\theta}(\mu) \subseteq \underline{\theta}(\lambda)$.

(3) $\underline{\theta}(\mu^c) = (\overline{\theta}(\mu))^c$ *and* $\overline{\theta}(\mu^c) = (\underline{\theta}(\mu))^c$.

(4) $\overline{\theta}\overline{\theta}(\mu) = \overline{\theta}(\mu)$ *and* $\underline{\theta}\underline{\theta}(\mu) = \underline{\theta}(\mu)$.

(5) $\underline{\theta}(\mu)(x) = \underline{\theta}(\mu)(a)$ *and* $\overline{\theta}(\mu)(x) = \overline{\theta}(\mu)(a)$ *for any* $x \in L$ *and* $a \in [x]_\theta$.

(6) $\underline{\theta}\overline{\theta}(\mu) = \overline{\theta}(\mu)$ *and* $\overline{\theta}\underline{\theta}(\mu) = \underline{\theta}(\mu)$.

Definition 2.12 ([3]). *Let θ be an equivalence relation on L. Then θ is called a full congruence relation, if $(a, b) \in \theta$ implies that $(a \vee x, b \vee x) \in \theta$ and $(a \wedge x, b \wedge x) \in \theta$ for any $x \in L$.*

For $a \in L$, denote

$$[a]_\theta = \{x \in L : (a, x) \in \theta\}, \quad L/\theta = \{[a]_\theta : a \in L\}.$$

Lemma 2.13 ([3]). *Let θ be a full congruence relation on L. Then for any $a, b, c, d \in L$,*

(1) *If* $(a, b), (c, d) \in \theta$, *then* $(a \vee c, b \vee d), (a \wedge c, b \wedge d) \in \theta$.

(2) *If* $x \in [a]_\theta, y \in [b]_\theta$, *then* $x \vee y \in [a \vee b]_\theta$.

(3) *If* $x \in [a]_\theta, y \in [b]_\theta$, *then* $x \wedge y \in [a \wedge b]_\theta$.

Proposition 2.14 ([3]). *Let θ be a full congruence relation on L.*

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(1) *If μ is a fuzzy ideal, then for $x, y \in L$,*

$$\underline{\theta}(\mu)(x \wedge y) = \bigwedge_{a\in[x]_\theta, b\in[y]_\theta} \mu(a \wedge b), \quad \overline{\theta}(\mu)(x \vee y) = \bigvee_{a\in[x]_\theta, b\in[y]_\theta} \mu(a \vee b).$$

(2) *If μ is a fuzzy filter, then for $x, y \in L$,*

$$\underline{\theta}(\mu)(x \vee y) = \bigwedge_{a\in[x]_\theta, b\in[y]_\theta} \mu(a \vee b), \quad \overline{\theta}(\mu)(x \wedge y) = \bigvee_{a\in[x]_\theta, b\in[y]_\theta} \mu(a \wedge b).$$

3 IF rough sets and IF rough approximation operators based on lattices

Definition 3.1. *Let θ be an equivalence relation on L. The pair (L, θ) is called Pawlak approximation space. For each $\mu \in IF(L)$, the IF lower and the IF upper approximation of μ with respect to (L, θ), denoted by $\underline{\theta}(\mu)$ and $\overline{\theta}(\mu)$, are defined as follows:*

$$\underline{\theta}(\mu) = ((\underline{\theta}(\mu))^*, (\underline{\theta}(\mu))_*),$$
$$\overline{\theta}(\mu) = ((\overline{\theta}(\mu))^*, (\overline{\theta}(\mu))_*),$$

where for each $x \in L$,

$$(\underline{\theta}(\mu))^*(x) = \bigwedge_{a\in[x]_\theta} \mu^*(a), \quad (\underline{\theta}(\mu))_*(x) = \bigvee_{a\in[x]_\theta} \mu_*(a),$$

$$(\overline{\theta}(\mu))^*(x) = \bigvee_{a\in[x]_\theta} \mu^*(a), \quad (\overline{\theta}(\mu))_*(x) = \bigwedge_{a\in[x]_\theta} \mu_*(a).$$

The pair $(\underline{\theta}(\mu), \overline{\theta}(\mu))$ is called the IF rough set of μ with respect to (L, θ).
$\underline{\theta} : IF(L) \to IF(L)$ and $\overline{\theta} : IF(L) \to IF(L)$ are called the IF lower approximation operator and the IF upper approximation operator, respectively. In general, we refer to $\underline{\theta}$ and $\overline{\theta}$ as the IF rough approximation operators.

Remark 3.2. (1) $(\underline{\theta}(\mu))^* = \underline{\theta}(\mu^*)$ $\quad (\underline{\theta}(\mu))_* = \overline{\theta}(\mu_*)$
(2) $(\overline{\theta}(\mu))^* = \overline{\theta}(\mu^*)$ $\quad (\overline{\theta}(\mu))_* = \underline{\theta}(\mu_*)$

Proposition 3.3. *For any $x \in L$,*

$$\underline{\theta}(\mu)(x) = \bigsqcap_{a\in[x]_\theta} \mu(a), \quad \overline{\theta}(\mu)(x) = \bigsqcup_{a\in[x]_\theta} \mu(a).$$

Proof.

$$\begin{aligned} \underline{\theta}(\mu)(x) &= \Big(\bigwedge_{a\in[x]_\theta} \mu^*(a), \bigvee_{a\in[x]_\theta} \mu_*(a)\Big) \\ &= \bigsqcap_{a\in[x]_\theta} (\mu^*(a), \mu_*(a)) \\ &= \bigsqcap_{a\in[x]_\theta} \mu(a). \end{aligned}$$

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$$\begin{aligned}\overline{\theta}(\mu)(x) &= (\bigvee_{a\in[x]_\theta}\mu^*(a), \bigwedge_{a\in[x]_\theta}\mu_*(a))\\ &= \bigsqcup_{a\in[x]_\theta}(\mu^*(a),\mu_*(a))\\ &= \bigsqcup_{a\in[x]_\theta}\mu(a).\end{aligned}$$

□

Proposition 3.4. *Let θ be an equivalence relation on L. Then for any $\mu,\lambda \in IF(L)$,*
(1) $\underline{\theta}(\mu) \subseteq \mu \subseteq \overline{\theta}(\mu)$.
(2) *If* $\mu \subseteq \lambda$, *then* $\overline{\theta}(\mu) \subseteq \overline{\theta}(\lambda)$ *and* $\underline{\theta}(\mu) \subseteq \underline{\theta}(\lambda)$.
(3) $\overline{\theta}\overline{\theta}(\mu) = \overline{\theta}(\mu)$ *and* $\underline{\theta}\underline{\theta}(\mu) = \underline{\theta}(\mu)$.
(4) $\underline{\theta}(\mu)(x) = \underline{\theta}(\mu)(a)$ *and* $\overline{\theta}(\mu)(x) = \overline{\theta}(\mu)(a)$ *for any* $x \in L$ *and* $a \in [x]_\theta$
(5) $\underline{\theta}\overline{\theta}(\mu) = \overline{\theta}(\mu)$ *and* $\overline{\theta}\underline{\theta}(\mu) = \underline{\theta}(\mu)$.

Proof. It is straightforward. □

Proposition 3.5. *Let θ be an equivalence relation on L. Then for any $\{\mu_i : i \in I\} \subseteq IF(L)$,*
(1) $\underline{\theta}(\bigsqcup_{i\in I}\mu_i) \supseteq \bigsqcup_{i\in I}\underline{\theta}(\mu_i)$, $\underline{\theta}(\bigsqcap_{i\in I}\mu_i) = \bigsqcap_{i\in I}\underline{\theta}(\mu_i)$.
(2) $\overline{\theta}(\bigsqcup_{i\in I}\mu_i) = \bigsqcup_{i\in I}\overline{\theta}(\mu_i)$, $\overline{\theta}(\bigsqcap_{i\in I}\mu_i) \subseteq \bigsqcap_{i\in I}\overline{\theta}(\mu_i)$.

Proof. (1) For any $x \in L$,
$\underline{\theta}(\bigsqcup_{i\in I}\mu_i)(x) = \bigsqcap_{a\in[x]_\theta}\bigsqcup_{i\in I}\mu_i(a) \supseteq \bigsqcup_{i\in I}\bigsqcap_{a\in[x]_\theta}\mu_i(a) = \bigsqcup_{i\in I}\underline{\theta}(\mu_i)(x)$,
$\underline{\theta}(\bigsqcap_{i\in I}\mu_i)(x) = \bigsqcap_{a\in[x]_\theta}\bigsqcap_{i\in I}\mu_i(a) = \bigsqcap_{i\in I}\bigsqcap_{a\in[x]_\theta}\mu_i(a) = \bigsqcap_{i\in I}\underline{\theta}(\mu_i)(x)$.
Thus, $\underline{\theta}(\bigsqcup_{i\in I}\mu_i) \supseteq \bigsqcup_{i\in I}\underline{\theta}(\mu_i)$, $\underline{\theta}(\bigsqcap_{i\in I}\mu_i) = \bigsqcap_{i\in I}\underline{\theta}(\mu_i)$.
(2) The proof is similar to (1). □

4 IF sublattices and IF rough approximations based on lattices

4.1 IF sublattices

Definition 4.1. *$\mu \in IF(L)$ is called an IF sublattice of L, if*
$\mu(x \wedge y) \sqcap \mu(x \vee y) \geq \mu(x) \sqcap \mu(y)$ *for any* $x, y \in L$.

Definition 4.2. *Let μ be an IF sublattice of L. Then*
(1) μ *is an IF ideal of* L, *if* $\mu(x \vee y) = \mu(x) \sqcap \mu(y)$ *for any* $x, y \in L$.
(2) μ *is an IF filter of* L, *if* $\mu(x \wedge y) = \mu(x) \sqcap \mu(y)$ *for any* $x, y \in L$.

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Denote the set of all IF ideals of L by $IFI(L)$.

Proposition 4.3. *Let μ be an IF sublattice of L. Then*
(1) *μ is an IF ideal of $L \iff x \leq y$ implies that $\mu(x) \geq \mu(y)$ for any $x, y \in L$.*
(2) *μ is an IF filterof $L \iff x \leq y$ implies that $\mu(x) \leq \mu(y)$ for any $x, y \in L$.*

Proof. It is straightforward. □

Let μ be a proper IF ideal of L. Then
(1) μ is called an IF prime ideal of L, if $\mu(x \wedge y) \leq \mu(x) \sqcup \mu(y)$ for any $x, y \in L$.
(2) μ is called an IF prime filter of L, if $\mu(x \vee y) \leq \mu(x) \sqcup \mu(y)$ for any $x, y \in L$.

4.2 IF rough approximations of some IF sublattices

Lemma 4.4. *Let θ be a full congruence relation on L.*
(1) *If μ is an IF ideal of L, then for any $x, y \in L$,*

$$\underline{\theta}(\mu)(x \wedge y) = \bigsqcap_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \wedge b), \quad \overline{\theta}(\mu)(x \vee y) = \bigsqcup_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \vee b).$$

(2) *If μ is an IF filter of L, then for any $x, y \in L$,*

$$\underline{\theta}(\mu)(x \vee y) = \bigsqcap_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \vee b), \quad \overline{\theta}(\mu)(x \wedge y) = \bigsqcup_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \wedge b).$$

Proof. (1) By Lemma 2.12,
$\bigwedge_{z \in [x \wedge y]_\theta} \mu^*(z) \leq \bigwedge_{a \in [x]_\theta, b \in [y]_\theta} \mu^*(a \wedge b), \quad \bigvee_{z \in [x \wedge y]_\theta} \mu_*(z) \geq \bigvee_{a \in [x]_\theta, b \in [y]_\theta} \mu_*(a \wedge b).$
Then

$$\underline{\theta}(\mu)(x \wedge y) = \bigsqcap_{z \in [x \wedge y]_\theta} \mu(z) \leq \bigsqcap_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \wedge b).$$

Now assume that $z \in [x \wedge y]_\theta$. Then $z \vee x \in [x]_\theta, \ z \vee y \in [y]_\theta$.
Since $z \leq (z \vee x) \wedge (z \vee y)$, by Proposition 2.14, we have

$$\mu(z) \geq \mu(z \vee x) \wedge (z \vee y).$$

Then

$$\mu^*(z) \geq \mu^*((z \vee x) \wedge (z \vee y)), \quad \mu_*(z) \leq \mu_*((z \vee x) \wedge (z \vee y)).$$

Note that

$$\bigwedge_{z \in [x \wedge y]_\theta} \mu^*(z) \geq \bigwedge_{a \in [x]_\theta, b \in [y]_\theta} \mu^*(a \wedge b), \quad \bigvee_{z \in [x \wedge y]_\theta} \mu_*(z) \leq \bigvee_{a \in [x]_\theta, b \in [y]_\theta} \mu_*(a \wedge b).$$

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Then $\prod_{z\in[x\wedge y]_\theta} \mu(z) \geq \prod_{a\in[x]_\theta, b\in[y]_\theta} \mu(a\wedge b)$.
Thus

$$\underline{\theta}(\mu)(x\wedge y) = \prod_{a\in[x]_\theta, b\in[y]_\theta} \mu(a\wedge b).$$

Note that

$$\bigvee_{z\in[x\wedge y]_\theta} \mu^*(z) \geq \bigvee_{a\in[x]_\theta, b\in[y]_\theta} \mu^*(a\vee b), \quad \bigwedge_{z\in[x\vee y]_\theta} \mu_*(z) \geq \bigwedge_{a\in[x]_\theta, b\in[y]_\theta} \mu_*(a\wedge b).$$

Then

$$\overline{\theta}(\mu)(x\vee y) = \bigsqcup_{z\in[x\vee y]_\theta} \mu(z) \geq \bigsqcup_{a\in[x]_\theta, b\in[y]_\theta} \mu(a\vee b).$$

Now assume that $z\in[x\vee y]_\theta$. Then

$$z\wedge x\in[x]_\theta, \quad z\wedge y\in[y]_\theta.$$

Since $z\geq(z\wedge x)\vee(z\wedge y)$, by Proposition 2.14, we have

$$\mu(z)\leq\mu(z\wedge x)\vee(z\wedge y).$$

Then $\mu^*(z)\leq\mu^*((z\wedge x)\vee(z\wedge y))$, $\mu_*(z)\geq\mu_*((z\wedge x)\vee(z\wedge y))$.
Since

$$\bigvee_{z\in[x\vee y]_\theta} \mu^*(z) \leq \bigvee_{a\in[x]_\theta, b\in[y]_\theta} \mu^*(a\vee b), \quad \bigwedge_{z\in[x\vee y]_\theta} \mu_*(z) \geq \bigwedge_{a\in[x]_\theta, b\in[y]_\theta} \mu_*(a\vee b).$$

we have

$$\bigsqcup_{z\in[x\vee y]_\theta} \mu(z) \leq \bigsqcup_{a\in[x]_\theta, b\in[y]_\theta} \mu(a\vee b).$$

Thus

$$\overline{\theta}(\mu)(x\vee y) = \bigsqcup_{a\in[x]_\theta, b\in[y]_\theta} \mu(a\vee b).$$

(2) The proof is similar to (1). □

Proposition 4.5. *Let θ be a full congruence relation on L. Let $\mu\in IF(L)$ and let $\underline{\theta}(\mu)$ be an IF sublattice of L. Then*
(1) If μ is an IF ideal of L, then $\underline{\theta}(\mu)$ is an IF ideal of L.
(2) If μ is an IF filter of L, then $\underline{\theta}(\mu)$ is an IF filter of L.

Proof. (1) Since $\underline{\theta}(u)$ is an IF sublattice of L, we conclude that for any $x, y\in L$
,

$$\underline{\theta}(\mu)(x\vee y)\geq\underline{\theta}(\mu)(x\wedge y)\sqcap\underline{\theta}(\mu)(x\vee y)\geq\underline{\theta}(\mu)(x)\sqcap\underline{\theta}(\mu)(y).$$

By Lemma 4.4, for any $x, y \in L$,

$$\begin{aligned}
\underline{\theta}(\mu)(x \vee y) &= \prod_{z \in [x \vee y]_\theta} \mu(z) \\
&\le \prod_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \vee b) \\
&= \prod_{a \in [x]_\theta, b \in [y]_\theta} (\mu(a) \sqcap \mu(b)) \\
&= \prod_{a \in [x]_\theta, b \in [y]_\theta} (\mu^*(a) \wedge \mu^*(b), \mu_*(a) \vee \mu_*(b)) \\
&= (\bigwedge_{a \in [x]_\theta, b \in [y]_\theta} (\mu^*(a) \wedge \mu^*(b)), \bigvee_{a \in [x]_\theta, b \in [y]_\theta} (\mu_*(a) \vee \mu_*(b))) \\
&= ((\bigwedge_{a \in [x]_\theta} \mu(a)) \wedge (\bigwedge_{b \in [y]_\theta} \mu(b)), (\bigvee_{a \in [x]_\theta} \mu(a)) \vee (\bigvee_{b \in [y]_\theta} \mu(b))) \\
&= (\bigwedge_{a \in [x]_\theta} \mu(a), \bigvee_{a \in [x]_\theta} \mu(a)) \sqcap (\bigwedge_{b \in [y]_\theta} \mu(b), \bigvee_{b \in [y]_\theta} \mu(b)) \\
&= (\prod_{a \in [x]_\theta} \mu(a)) \sqcap (\prod_{b \in [y]_\theta} \mu(b)) \\
&= \underline{\theta}(\mu)(x) \sqcap \underline{\theta}(\mu)(y).
\end{aligned}$$

(2) The proof is similar to (1). □

Proposition 4.6. *Let θ be a full congruence relation on L. Then for $\mu \in IF(L)$,*
(1) If μ is an IF sublattice of L, then $\overline{\theta}(\mu)$ is an IF sublattice of L.
(2) If μ is an IF ideal of L, then $\overline{\theta}(\mu)$ is an IF ideal of L.
(3) If μ is an IF filter of L, then $\overline{\theta}(\mu)$ is an IF filter of L.

Proof. (1) Suppose that μ is an IF sublattice of L. Then for any $x, y \in L$,

$$\begin{aligned}
&\overline{\theta}(\mu)(x \wedge y) \prod \overline{\theta}(\mu)(x \vee y) \\
= &(\bigvee_{a \in [x \wedge y]_\theta} \mu^*(a), \bigwedge_{a \in [x \wedge y]_\theta} \mu_*(a)) \prod (\bigvee_{b \in [x \vee y]_\theta} \mu^*(b), \bigwedge_{b \in [x \vee y]_\theta} \mu_*(b)) \\
&= (\bigvee_{a \in [x \wedge y]_\theta} \mu^*(a) \wedge \bigvee_{b \in [x \vee y]_\theta} \mu^*(b), \bigwedge_{a \in [x \wedge y]_\theta} \mu^*(a) \vee \bigwedge_{b \in [x \vee y]_\theta} \mu^*(b)) \\
&\ge (\bigvee_{a \in [x]_\theta, c \in [y]_\theta} \mu^*(a \wedge c) \wedge \bigvee_{b \in [x]_\theta, d \in [y]_\theta} \mu^*(b \vee d), \\
&\quad \bigwedge_{a \in [x]_\theta, c \in [y]_\theta} \mu_*(a \wedge c) \vee \bigwedge_{b \in [x]_\theta, d \in [y]_\theta} \mu_*(b \vee d)) \\
&\ge (\bigvee_{a, b \in [x]_\theta, c, d \in [y]_\theta} \mu^*(a \wedge c) \wedge \mu^*(b \vee d), \bigwedge_{a, b \in [x]_\theta, c, d \in [y]_\theta} \mu_*(a \wedge c) \vee \mu_*(b \vee d)) \\
&\ge (\bigvee_{a \in [x]_\theta, c \in [y]_\theta} \mu^*(a \wedge c) \wedge \mu^*(a \vee c), \bigwedge_{a \in [x]_\theta, c \in [y]_\theta} \mu_*(a \wedge c) \vee \mu_*(a \vee c)) \\
&\ge (\bigvee_{a \in [x]_\theta, c \in [y]_\theta} \mu^*(a) \wedge \mu^*(c), \bigwedge_{a \in [x]_\theta, c \in [y]_\theta} \mu_*(a) \vee \mu_*(c))
\end{aligned}$$

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$= (\bigvee_{a\in[x]_\theta} \mu^*(a) \wedge \bigvee_{c\in[y]_\theta} \mu^*(c), \bigwedge_{a\in[x]_\theta,} \mu_*(a) \vee \bigwedge_{c\in[y]_\theta} \mu_*(c))$
$= (\bigvee_{a\in[x]_\theta} \mu^*(a), \bigwedge_{a\in[x]_\theta} \mu_*(a)) \bigsqcap (\bigvee_{b\in[y]_\theta} \mu^*(b), \bigwedge_{b\in[y]_\theta} \mu_*(b))$
$= \overline{\theta}(\mu)(x) \bigsqcap \overline{\theta}(\mu)(y)$
Thus $\overline{\theta}(\mu)$ is an IF sublattice of L.
(2) Suppose that μ is an IF ideal of L. Then μ is an IF sublattice of L.
By (1), $\overline{\theta}(\mu)$ is an IF sublattice of L.
For any $x, y \in L$,

$$\begin{aligned}
\overline{\theta}(\mu)(x \vee y) &= \bigsqcup_{a\in[x]_\theta, b\in[y]_\theta} \mu(a \vee b) \\
&= \bigsqcup_{a\in[x]_\theta, b\in[y]_\theta} (\mu(a) \sqcap \mu(b)) \\
&= \bigsqcup_{a\in[x]_\theta, b\in[y]_\theta} (\mu^*(a) \wedge \mu^*(b), \mu_*(a) \vee \mu_*(b)) \\
&= (\bigvee_{a\in[x]_\theta, b\in[y]_\theta} (\mu^*(a) \wedge \mu^*(b)), \bigwedge_{a\in[x]_\theta, b\in[y]_\theta} (\mu_*(a) \vee \mu_*(b))) \\
&= ((\bigvee_{a\in[x]_\theta} \mu^*(a)) \wedge (\bigvee_{b\in[y]_\theta} \mu^*(b)), (\bigwedge_{a\in[x]_\theta} \mu_*(a)) \vee (\bigwedge_{b\in[y]_\theta} \mu_*(b))) \\
&= (\bigvee_{a\in[x]_\theta} \mu^*(a), \bigwedge_{a\in[x]_\theta} \mu_*(a)) \sqcap (\bigvee_{b\in[y]_\theta} \mu^*(b), \bigwedge_{b\in[y]_\theta} \mu_*(b)) \\
&= \bigsqcup_{a\in[x]_\theta} \mu(a) \sqcap \bigsqcup_{b\in[x]_\theta} \mu(b) \\
&= \overline{\theta}(\mu)(x) \bigsqcap \overline{\theta}(\mu)(y)
\end{aligned}$$

Thus $\overline{\theta}(\mu)$ is an IF ideal of L.
(3) The proof is similar to (2). □

Proposition 4.7. *Let θ be a full congruence relation on L and let $\underline{\theta}(\mu)$ is a proper IF sublattice of L.*
(1) *If $\mu \in IF(L)$ is an IF prime ideal of L, then $\underline{\theta}(\mu)$ is an IF prime ideal of L.*
(2) *If $\mu \in IF(L)$ is an IF prime filter of L, then $\underline{\theta}(\mu)$ is an IF prime filter of L.*

Proof. (1) Suppose that μ is an IF prime ideal of L.
By Proposition 4.5, $\underline{\theta}(\mu)$ is an IF ideal of L.

By Proposition 4.3 and Lemma 4.4, for any $x, y \in L$.

$$\begin{aligned}
\underline{\theta}(\mu)(x \wedge y) &= \bigsqcap_{a\in[x]_\theta, b\in[y]_\theta} \mu(a \wedge b) \\
&= \bigsqcap_{a\in[x]_\theta, b\in[y]_\theta} (\mu^*(a \wedge b), \mu_*(a \wedge b)) \\
&= (\bigwedge_{a\in[x]_\theta, b\in[y]_\theta} \mu^*(a \wedge b), \bigvee_{a\in[x]_\theta, b\in[y]_\theta} \mu_*(a \wedge b)) \\
&\leq (\bigwedge_{a\in[x]_\theta, b\in[y]_\theta} (\mu^*(a) \wedge \mu^*(b)), \bigvee_{a\in[x]_\theta, b\in[y]_\theta} (\mu_*(a) \wedge \mu_*(b))) \\
&= (\bigwedge_{a\in[x]_\theta} \mu^*(a) \vee \bigwedge_{b\in[y]_\theta} \mu^*(b), \bigvee_{a\in[x]_\theta} \mu_*(a) \wedge \bigvee_{b\in[y]_\theta} \mu_*(b)) \\
&= (\bigwedge_{a\in[x]_\theta} \mu^*(a), \bigvee_{a\in[x]_\theta} \mu_*(a)) \sqcup (\bigwedge_{b\in[y]_\theta} \mu^*(b), \bigvee_{b\in[y]_\theta} \mu_*(b)) \\
&= \bigsqcap_{a\in[x]_\theta} \mu(a) \sqcup \bigsqcap_{b\in[y]_\theta} \mu(b) \\
&= \underline{\theta}(\mu)(x) \sqcup \underline{\theta}(\mu)(y).
\end{aligned}$$

Thus $\underline{\theta}(\mu)$ is an IF prime ideal of L.

(2) The proof is similar to (1). □

Definition 4.8. *Let θ be a full congruence relation on L. Then*

(1) θ is called $\vee$-complete, if $\{x \vee y : x \in [a]_\theta, y \in [b]_\theta\} = [a \vee b]_\theta$ for any $a, b \in L$.

(2) θ is called $\wedge$-complete, if $\{x \wedge y : x \in [a]_\theta, y \in [b]_\theta\} = [a \wedge b]_\theta$ for any $a, b \in L$.

(3) θ is called complete, if θ is both $\vee$-complete and $\wedge$-complete.

Proposition 4.9. *Let θ be a full congruence relation on L.*

(1) Let μ be an IF prime ideal of L and let θ be $\wedge$-complete. If $\overline{\theta}(\mu)$ is proper, then $\overline{\theta}(\mu)$ is an IF prime ideal of L.

(2) Let μ be an IF prime filter of L and let θ be $\vee$-complete. If $\overline{\theta}(\mu)$ is proper, then $\overline{\theta}(\mu)$ is an IF filter ideal.

Proof. (1) By Proposition 4.6, $\overline{\theta}(\mu)$ is an IF ideal of L.

Since θ is $\wedge$-complete, for any $x, y \in L$, we have

$$\begin{aligned}
\overline{\theta}(\mu)(x \wedge y) &= \bigsqcup_{a\in[x]_\theta, b\in[y]_\theta} \mu(a \wedge b) \\
&\leq \bigsqcup_{a\in[x]_\theta, b\in[y]_\theta} \mu(a) \sqcup \mu(b) \\
&= \bigsqcup_{a\in[x]_\theta, b\in[y]_\theta} (\mu^*(a) \vee \mu^*(b), \mu_*(a) \wedge \mu_*(b)) \\
&= (\bigvee_{a\in[x]_\theta, b\in[y]_\theta} (\mu^*(a) \vee \mu^*(b)), \bigwedge_{a\in[x]_\theta, b\in[y]_\theta} (\mu_*(a) \wedge \mu_*(b))) \\
&= ((\bigvee_{a\in[x]_\theta} \mu^*(a)) \vee (\bigvee_{b\in[y]_\theta} \mu^*(b)), (\bigwedge_{a\in[x]_\theta} \mu_*(a)) \wedge (\bigwedge_{b\in[y]_\theta} \mu_*(b))) \\
&= (\bigvee_{a\in[x]_\theta} \mu^*(a), \bigwedge_{a\in[x]_\theta} \mu_*(a)) \sqcup (\bigvee_{b\in[y]_\theta} \mu^*(b), \bigwedge_{b\in[y]_\theta} \mu_*(b)) \\
&= \bigsqcup_{a\in[x]_\theta} \mu(a) \sqcup \bigsqcup_{b\in[y]_\theta} \mu(b) \\
&= \overline{\theta}(\mu)(x) \sqcup \overline{\theta}(\mu)(y)
\end{aligned}$$

Thus $\overline{\theta}(\mu)$ is an IF prime ideal of L.
(2) The proof is similar to (1) □

Definition 4.10. *Let $\mu \in IF(L)$. The least IF ideal of L containing μ is called an IF ideal of L induced by μ. We denoted it by $< \mu >$.*

For any $\mu \in IF(L)$, we denote

$$\mu^\diamond(x) = \bigsqcup\{(\alpha, \beta) \in J : x \in I(\mu_\alpha^\beta)\} \ (x \in L).$$

Proposition 4.11. *Let $\mu \in IF(L)$. Then*
(1) $\mu \subseteq \mu^\diamond$.
(2) $\mu^\diamond = \bigcap\{\nu \in IFI(L) : \mu \subseteq \nu \ , \nu(0_L) = 1_J\}$.

Proof. (1) Consider that $\mu_\alpha^\beta = \{x \in U : \mu(x) \geq (\alpha, \beta)\}$. Then

$$\mu(x) = \sqcup\{(\alpha, \beta) : x \in \mu_\alpha^\beta\} \leq \sqcup\{(\alpha, \beta) : x \in I(\mu_\alpha^\beta)\} = \mu^\diamond.$$

(2) Firstly, we can prove that $\mu^\diamond \in IFI(L)$.
For any $x, y \in L$,

$$\mu^\diamond(x) = \bigsqcup\{(\alpha, \beta) \in J : x \in I(\mu_\alpha^\beta)\},$$
$$\mu^\diamond(y) = \bigsqcup\{(\alpha, \beta) \in J : y \in I(\mu_\alpha^\beta)\}.$$

Put

$$A = \{(\alpha, \beta) \in J : x \in I(\mu_\alpha^\beta)\}, \quad B = \{(\alpha, \beta) \in J : y \in I(\mu_\alpha^\beta)\}.$$

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Suppose $x \leq y$. Then $A \subseteq B$. So $\sqcup A \leq \sqcup B$. This implies that

$$\mu^{\diamond}(y) \leq \mu^{\diamond}(x).$$

Secondly, since $0_L \in I(\mu_1^0)$, we have $1_J \leq \mu^{\diamond}(0_L)$. Then $\mu^{\diamond}(0_L) = 1_J$. Combined with (1), we have

$$\mu^{\diamond} \in \{\nu \in IFI(L) : \mu \subseteq \nu \ , \nu(0_L) = 1_J\}.$$

Then

$$\mu^{\diamond} \supseteq \cap\{\nu \in IFI(L) : \mu \subseteq \nu \ , \nu(0_L) = 1_J\}.$$

Now, we need to prove that

$$\mu^{\diamond} \subseteq \cap\{\nu \in IFI(L) : \mu \subseteq \nu, \nu(0_L) = 1_J\}.$$

For any $\nu \in \{\nu \in IFI(L) : \mu \subseteq \nu \ , \nu(0_L) = 1_J\}$, we have $\mu \subseteq \nu$. This implies

$$\mu_\alpha^\beta \subseteq \nu_\alpha^\beta.$$

Then $I(\mu_\alpha^\beta) \subseteq I(\nu_\alpha^\beta)$.
Denote

$$C = \{(\alpha, \beta) \in J : x \in I(\nu_\alpha^\beta)\}.$$

Then $A \subseteq C$ and so $\mu^{\diamond}(x) \leq \nu^{\diamond}(x)$.
Note that $\nu \in IFI(L)$. Then $\nu_\alpha^\beta \in IFI(L)$. So $I(\nu_\alpha^\beta) = \nu_\alpha^\beta$.
This implies that

$$\nu^{\diamond}(x) = \sqcup C = \sqcup\{(\alpha, \beta) \in J : x \in \nu_\alpha^\beta\} = \nu(x).$$

Thus $\mu^{\diamond}(x) \leq \nu(x)$.
Hence

$$\mu^{\diamond} \subseteq \cap\{\nu \in IFI(L) : \mu \subseteq \nu, \nu(0_L) = 1_J\}.$$

□

Proposition 4.12. *Let θ be a full congruence relation on L. Then for any $\mu \in IF(L)$,*
(1) $\overline{\theta}(<\mu>) = \overline{\theta}(<\overline{\theta}(\mu)>)$.
(2) $\overline{\theta}(\mu^{\diamond}) = \overline{\theta}((\overline{\theta}(\mu))^{\diamond})$.

Proof. (1) Since $\mu \subseteq <\mu>$, we conclude from Proposition 3.4 that

$$\overline{\theta}(\mu) \subseteq \overline{\theta}(<\mu>).$$

By Proposition 4.6 and Proposition 4.11,

$$<\overline{\theta}(\mu)> \subseteq \overline{\theta}(<\mu>).$$

By Proposition 3.4,

$$\overline{\theta}(<\overline{\theta}(\mu)>) \subseteq \overline{\theta}(<\mu>).$$

Note that $\mu \subseteq \overline{\theta}(\mu)$. Then $<\mu>\subseteq<\overline{\theta}(\mu)>$.
By Proposition 3.4,

$$\overline{\theta}(<\mu>) \subseteq \overline{\theta}(<\overline{\theta}(\mu)>).$$

Thus

$$\overline{\theta}(<\mu>) = \overline{\theta}(<\overline{\theta}(\mu)>).$$

(2) Since $<\mu>\subseteq \mu^{\diamond}$, by Proposition 3.4, we have $\overline{\theta}(<\mu>) \subseteq \overline{\theta}(\mu^{\diamond})$.
It is clear that

$$\overline{\theta}(\mu^{\diamond})(0_L) = 1_J, \quad <\overline{\theta}(\mu)> \subseteq <\overline{\theta}(\mu^{\diamond})> = \overline{\theta}(\mu^{\diamond}).$$

Then $(\overline{\theta}(\mu))^{\diamond} \subseteq \overline{\theta}(\mu^{\diamond})$
By Proposition 3.4,

$$\overline{\theta}((\overline{\theta}(\mu))^{\diamond}) \subseteq \overline{\theta}(\mu^{\diamond}).$$

Since $\mu^{\diamond} \subseteq (\overline{\theta}(\mu))^{\diamond}$ we conclude $\overline{\theta}(\mu^{\diamond}) \subseteq \overline{\theta}((\overline{\theta}(\mu))^{\diamond})$.
Thus

$$\overline{\theta}(\mu^{\diamond}) = \overline{\theta}((\overline{\theta}(\mu))^{\diamond}).$$

□

Proposition 4.13. *Let $a^{(r,s)}, b^{(p,q)} \in IFP(L)$ and $\mu \in IF(L)$. Then*
(1) $\overline{\theta}(a^{(r,s)}) = \chi^{(r,s)}_{[a]_\theta}$.

(2) $<a^{(r,s)}>(x) = \chi^{(r,s)}_{\downarrow a}$ *and* $(a^{(r,s)})^{\diamond}(x) = \begin{cases} (1,0) & x = 0_L, \\ (r,s) & 0_L \neq x, \\ (0,1) & otherwise. \end{cases}$

(3) $\overline{\theta}(<a^{(r,s)}>)(x) = \begin{cases} (r,s) & \downarrow a \cap [x]_\theta \neq \emptyset, \\ (0,1) & otherwise. \end{cases}$

$$\overline{\theta}((a^{(r,s)})^{\diamond})(x) = \begin{cases} (1,0) & 0_L \in [x]_\theta \\ (r,s) & \downarrow a \cap [x]_\theta \neq \emptyset, \\ (0,1) & otherwise. \end{cases}$$

(4) $<a^{(r,s)}> \wedge <b^{(p,q)}>=<(a \wedge b)^{(r,s)\wedge(p,q)}>$

Proof. It is straightforward. □

Let θ be an equivalence relation on L. $\mu \in F(L)$ is called a fixed-point of θ-upper (resp. θ-lower) rough approximation, if $\overline{\theta}(\mu) = \mu$ (resp. $\underline{\theta}(\mu) = \mu$).
Denote

$$Fix(\overline{\theta}) = \{\mu \in F(L) \mid \overline{\theta}(\mu) = \mu\}, \quad Fix(\underline{\theta}) = \{\mu \in F(L) \mid \underline{\theta}(\mu) = \mu\}.$$

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Proposition 4.14. *Let θ_1 and θ_2 be two equivalence relations on L. Then the following are equivalent:*

(1) *For each $\mu \in F(L)$, $\overline{\theta}_1(\mu) \leq \overline{\theta}_2(\mu)$;*
(2) *For each $\mu \in F(L)$, $\underline{\theta}_1(\mu) \geq \underline{\theta}_2(\mu)$;*
(3) *$Fix(\overline{\theta}_2) \subseteq Fix(\overline{\theta}_1)$;*
(4) *$Fix(\underline{\theta}_2) \subseteq Fix(\underline{\theta}_1)$.*

Proof. (1) $\Longrightarrow$ (2). This holds by Proposition 2.10.
(2) $\Longrightarrow$ (3) Let $\mu \in F(L)$ and $\underline{\theta}_1(\mu^c) \geq \underline{\theta}_2(\mu^c)$.
By Proposition 2.10, $(\overline{\theta}_1(\mu))^c \geq \overline{\theta}_2(\mu))^c$.
Thus $\overline{\theta}_1(\mu) \leq \overline{\theta}_2(\mu)$.
Note that $\overline{\theta}_2(\mu) = \mu$. Then $\mu \leq \overline{\theta}_1(\mu) \leq \overline{\theta}_2(\mu) = \mu$.
It follows that $\overline{\theta}_1(\mu) = \mu$.
(3) $\Longrightarrow$ (1) Let $\mu \in F(L)$. Since $\overline{\theta}_2(\mu) \in Fix(\overline{\theta}_2)$, we have $\overline{\theta}_2(\mu) \in Fix(\overline{\theta}_1)$.
Thus $\overline{\theta}_1(\mu) \leq \overline{\theta}_1(\overline{\theta}_2(\mu)) = \overline{\theta}_2(\mu)$.
(2) $\Longrightarrow$ (4) Let $\mu \in F(L)$ and $\underline{\theta}_2(\mu) = \mu$. Then $\mu = \underline{\theta}_2(\mu) \leq \underline{\theta}_1(\mu) = \mu$.
It follows that $\underline{\theta}_1(\mu) = \mu$.
(4) $\Longrightarrow$ (2) Let $\mu \in F(L)$. By Proposition 3.4, $\underline{\theta}_2(\mu) \in Fix(\theta_2)$.
Then $\underline{\theta}_2(\mu) \in Fix(\theta_1)$. Thus

$$\underline{\theta}_2(\mu) = \underline{\theta}_1(\underline{\theta}_2(\mu)) \leq \underline{\theta}_1(\mu).$$

□

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Some results on approximating spaces *

Neiping Chen†

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Abstract: Topology and rough set theory are widely used in research field of computer science. In this paper, we study properties of topologies induced by binary relations, investigate a particular type of topological spaces which associate with some equivalence relation (i.e., approximating spaces) and obtain some characteristic conditions of approximating spaces.

Keywords: Binary relation; Rough set; Topology; Approximating space

1 Introduction

Rough set theory, proposed by Pawlak [8], is a new mathematical tool for data reasoning. It may be seen as an extension of classical set theory and has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems and many other fields [9, 10, 11, 12].

The basic structure of rough set theory is an approximation space. Based on it, lower and upper approximations can be induced. Using these approximations, knowledge hidden in information systems may be revealed and expressed in the form of decision rules. A key notion in Pawlak rough set model is equivalence relations. The equivalence classes are the building blocks for the construction of these approximations. In the real world, the equivalence relation is, however, too restrictive for many practical applications. To address this issue, many interesting and meaningful extensions of Pawlak rough sets have been presented. Equivalence relations can be replaced by tolerance relations [15], binary relations [20] and so on.

Topological structure is an important base for knowledge extraction and processing. Then, an interesting research topic in rough set theory is to study relationships between rough sets and topologies. Many authors studied topological properties of rough sets [3, 4, 7, 18, 22]. It is known that the pair of lower and upper approximation operators induced by a reflexive and transitive relation is exactly the pair of interior and closure operators of a topology [21].

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†Corresponding Author, S School of Mathematics and Statistics, Hunan University of Commerce, Changsha 410205, China, neipingchen100@126.com

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The purpose of this paper is to investigate further approximating spaces.

2 Preliminaries

Throughout this paper, I denotes $[0,1]$, N is the set of natural number. U denotes a non-empty set, 2^U denotes the set of all subsets of U, $|X|$ denotes the cardinality of X.

2.1 Binary relations

Recall that R is called a binary relation on U if $R \in 2^{U\times U}$.

Let R be a binary relation on U. R is called preorder if R is reflexive and transitive. R is called tolerance if R is both reflexive and symmetric. R is called equivalence if R is reflexive, symmetric and transitive.

Let R be a binary relation on U. For $u, v, w \in U$, we define

$$R^{uvw} = R \cup S^{uvw} \quad and \quad R^{uv} = \bigcup_{w\in U} R^{uvw},$$

$$where \quad S^{uvw} = \begin{cases} \{(u,v)\}, & (u,w) \in R \;\; and \;\; (w,v) \in R \\ \emptyset, & (u,w) \notin R \quad or \quad (w,v) \notin R \end{cases}.$$

If $S^{uvw} \neq \emptyset$, then

$$S^{uvw}(x) = \begin{cases} \{v\}, & x = u \\ \emptyset, & x \neq u \end{cases}.$$

Definition 2.1 ([4]). *Let R and R_s be two binary relations on U. If for all $x, y \in U$, xR_sy if and only if xRy or there exists $\{v_1, v_2, \ldots, v_n\} \subseteq U$ such that $x\theta v_1$, v_1Rv_2, $\ldots$, v_nRy, then R_s is called the transmitting expression of R.*

Theorem 2.2 ([4]). *Let R be a binary relation on U and R_s the transmitting expression of R. Then R_s is a transitive relation on U. Moreover,*

(1) *If R is reflexive, then R_s is also reflexive;*
(2) *If R is transitive, then $R_s = R$;*
(3) *If R is symmetric, then R_s is also symmetric.*

2.2 Rough sets

Let R be an equivalence relation on U. Then the pair (U, R) is called a Pawlak approximation space. Based on (U, R), one can define the following two rough approximations:

$$R_*(X) = \{x \in U : [x]_R \subseteq X\},$$

$$R^*(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}.$$

$R_*(X)$ and $R^*(X)$ are called the Pawlak lower approximation and the Pawlak upper approximation of X, respectively.

Definition 2.3 ([19]). *Let R be a binary relation on U. $\forall\ x \in U$, denote*

$$R(x) = \{y \in U : (x, y) \in R\}.$$

Then $R(x)$ is called the successor neighborhood of x, the pair (U, R) is called an approximation space. The lower and upper approximations of $X \in 2^U$ with regard to (U, R), denoted by $\underline{R}(X)$ and $\overline{R}(X)$ are respectively, defined as follows:

$$\underline{R}(X) = \{x \in U : R(x) \subseteq X\} \text{ and } \overline{R}(X) = \{x \in U : R(x) \cap X \neq \emptyset\}.$$

Proposition 2.4. *Let $\{R_\alpha : \alpha \in \Gamma\}$ be a family of binary relations on U. Then $\forall\ X \in 2^U$,*

$$\bigcap_{\alpha\in\Gamma} \underline{R_\alpha}(X) = \underline{\bigcup_{\alpha\in\Gamma} R_\alpha}(X).$$

Proof. Put $R = \bigcup\limits_{\alpha\in\Gamma} R_\alpha$. By $R_\beta \subseteq R$ for each $\beta \in \Gamma$, $\underline{R_\beta}(X) \supseteq \underline{R}(X)$. Then $\bigcap\limits_{\alpha\in\Gamma} \underline{R_\alpha}(X) \supseteq \underline{R}(X)$.

Let $x \in \bigcap\limits_{\alpha\in\Gamma} \underline{R_\alpha}(X)$. Then $x \in \underline{R_\alpha}(X)$ and so $R_\alpha(x) \subseteq X$ for each $\alpha \in \Gamma$. Thus $(\bigcup\limits_{\alpha\in\Gamma} R_\alpha)(x) = \bigcup\limits_{\alpha\in\Gamma} (R_\alpha(x)) \subseteq X$. So $x \in \underline{\bigcup\limits_{\alpha\in\Gamma} R_\alpha}(X)$. Hence $\underline{R_\beta}(X) \subseteq \underline{\bigcup\limits_{\alpha\in\Gamma} R_\alpha}(X)$.

Therefore, $\bigcap\limits_{\alpha\in\Gamma} \underline{R_\alpha}(X) = \underline{\bigcup\limits_{\alpha\in\Gamma} R_\alpha}(X)$. □

Proposition 2.5. *Let R be a binary relation on U. Then $\forall\ u, v.w \in U$,*

$$\underline{R^{uvw}}(X) - \{u\} = \underline{R}(X) - \{u\}.$$

Proof. (1) If $R^{uvw} = R$, then $\underline{R^{uvw}}(X) - \{u\} = \underline{R}(X) - \{u\}$.

(2) If $R^{uvw} \neq R$, then $(u, w), (w, v) \in R$ and $(u, v) \notin R$.

Obviously, $\underline{R^{uvw}}(X) - \{u\} \subseteq \underline{R}(X) - \{u\}$.

For $x \in \underline{R}(X) - \{u\}$, note that $S^{uvw}(x) = \emptyset$ $(x \in U - \{u\})$, then

$$R^{uvw}(x) = (R \cup S^{uvw})(x) = R(x) \cup S^{uvw}(x) = R(x) \subseteq X \ (x \in U - \{u\}).$$

So $x \in \underline{R^{uvw}}(X) - \{u\}$. It follows $\underline{R^{uvw}}(X) - \{u\} \supseteq \underline{R}(X) - \{u\}$.

Hence

$$\underline{R^{uvw}}(X) - \{u\} = \underline{R}(X) - \{u\}.$$

□

Theorem 2.6. *Let R be a binary relation on U and τ a topology on U. If one of the following conditions is satisfied, then R is preorder.*

(1) *$\overline{R}$ is the closure operator of τ.*

(2) *$\underline{R}$ is the interior operator of τ.*

Proof. (1) Let $x, y, z \in U$. Denote $cl_\tau(z_1)(y) = \lambda$.

Note that $\underline{R}$ is the interior operator of τ and $x \in cl_\tau(\{x\}) = \overline{R}(\{x\})$. Then $(x, x) \in R$. So R is reflexive.

Let $(x, y), (y, z) \in R$. Then $x \in \overline{R}(\{y\}), y \in \overline{R}(\{z\})$.

Note that $\overline{R}$ is the closure operator of τ. Then $x \in cl(\{y\}), y \in cl(\{z\})$. So

$$x \in cl(\{x\}) \subseteq cl(cl(\{y\})) = cl(\{y\}) \subseteq cl(cl(\{z\})) = cl(\{z\}) = \overline{R}(\{z\}).$$

This implies $(x, z) \in R$. So R is transitive.

Hence R is preorder.

(2) This proof is similar to (1). □

3 Topologies induced by binary relations

3.1 Topologies induced by reflexive relations

Let R be a reflexive relation on U. Denote

$$\tau_R = \{X \in 2^U : \underline{R}(X) = X\},$$

$$\sigma_R = \{\underline{R}(X) : X \in 2^U\}.$$

Kondo [2] proved that if R is a reflexive relation on X, then τ_R is a topology on X, which may be called the topology induced by R on X.

Remark 3.1. (1) *If R is preorder, then* $\tau_R = \sigma_R$.

(2) *If R is equivalence, then* $\tau_R = \{ \bigcup_{x \in X} [x]_R : X \in 2^U\}$.

Theorem 3.2 ([7])**.** *Let R be a preorder relation on U. Then*

(1) *σ_R is a topology on U.*

(2) *$\underline{R}$ is an interior operator of σ_R.*

(3) *$\overline{R}$ is a closure operator of σ_R.*

Proposition 3.3. *Let ρ and R be two reflexive relations on U. Then*

(1) $\rho \subseteq R \Longrightarrow \tau_\rho \supseteq \tau_R$.

(2) *If ρ and R are preorder, then* $\tau_\rho = \tau_R \Longleftrightarrow \rho = R$.

Proof. (1) $\forall\ X \in \tau_R$, $\underline{R}(X) = X$. By $\rho \subseteq R$ and the reflexivity of ρ,

$$X = \underline{R}(X) \subseteq \underline{\rho}(X) \subseteq X.$$

Then $\underline{\rho}(X) = X$ and so $X \in \tau_\rho$. Thus $\tau_\rho \supseteq \tau_R$.

(2) Necessity. Suppose $\tau_\rho = \tau_R$. Note that ρ and R are preorder. Then $\tau_\rho = \sigma_\rho = \sigma_R = \tau_R$.

By Theorem 3.2(3),

$$\begin{aligned}(x, y) \in \rho \quad &\Longleftrightarrow \quad x \in \overline{\rho}(\{y\}) \Longleftrightarrow x \in cl_{\sigma_\rho}(\{y\}) \\ &\Longleftrightarrow \quad x \in cl_{\sigma_R}(\{y\}) \Longleftrightarrow x \in \overline{R}(\{y\}) \Longleftrightarrow (x, y) \in R.\end{aligned}$$

Then $\rho = R$.

Sufficiency. Obviously. □

Proposition 3.4. *Let $\{R_\alpha : \alpha \in \Gamma\}$ be a family of reflexive relations on U. Then*

$$\tau_{\bigcup\limits_{\alpha\in\Gamma} R_\alpha} = \bigcap_{\alpha\in\Gamma} \tau_{R_\alpha}.$$

Proof. By Proposition 3.3(1), $\tau_{\bigcup\limits_{\alpha\in\Gamma} R_\alpha} \subseteq \bigcap\limits_{\alpha\in\Gamma} \tau_{R_\alpha}$.

Let $X \in \bigcap\limits_{\alpha\in\Gamma} \sigma_{R_\alpha}$. Then $\forall\ \alpha \in \Gamma$, $\underline{R_\alpha}(X) = X$. By Proposition 2.4,

$$X = \bigcap_{\alpha\in\Gamma} \underline{R_\alpha}(X) = \underline{\bigcup_{\alpha\in\Gamma} R_\alpha}(X).$$

So $X \in \tau_{\bigcup\limits_{\alpha\in\Gamma} R_\alpha}$. This implies $\tau_{\bigcup\limits_{\alpha\in\Gamma} R_\alpha} \supseteq \bigcap\limits_{\alpha\in\Gamma} \tau_{R_\alpha}$.

Hence $\tau_{\bigcup\limits_{\alpha\in\Gamma} R_\alpha} = \bigcap\limits_{\alpha\in\Gamma} \tau_{R_\alpha}$. □

3.2 The topologies induced by some binary relations

Theorem 3.5. *Let ρ, λ, R be three reflexive relations on U. If $\tau_\rho = \tau_R = \tau_\lambda$ and $\rho \subseteq \delta \subseteq \lambda$, then $\tau_\delta = \tau_R$.*

Proof. By $\rho \subseteq \delta \subseteq \lambda$ and Proposition 3.3(1),

$$\tau_R = \tau_\lambda \subseteq \tau_\delta \subseteq \tau_\rho = \tau_R.$$

Then $\tau_\delta = \tau_R$. □

Theorem 3.6. *Let R be a reflexive relation on U. Then $\forall\ u, v.w \in U$, $\tau_{R^{uvw}} = \tau_R = \tau_{R^{uv}}$.*

Proof. Obviously, R^{uvw} and R^{uv} both are reflexive.

(1) 1) If $u = v$, then $R^{uvw} = R$ and so $\tau_{R^{uvw}} = \tau_R$.

2) If $u \neq v$, $R^{uvw} = R$, we have $\tau_{R^{uvw}} = \tau_R$.

3) If $u \neq v$, $R^{uvw} \neq R$, we have $(u, w) \in R$, $(w, v) \in R, (u, v) \notin R$ and $S^{uvw} = \{(u, v)\}$.

Let $X \in \sigma_R$. Then $X \subseteq \underline{R}(X)$. By Proposition 3.3(1), $\sigma_R \supseteq \sigma_{R^{uvw}}$. By Proposition 2.5, $X - \{u\} \subseteq \underline{R}(X) - \{u\} = \underline{R^{uvw}}(X) - \{u\}$.

i) If $u \notin X$, then $X \subseteq \underline{R^{uvw}}(X)$.

ii) If $u \in X$, then $u \in \underline{R}(X)$ and so

$$w \in R(u) \subseteq X \subseteq \underline{R}(X).$$

We can obtain $R(w) \subseteq \underline{R}(X)$. Note that $v \in R(w)$. Then $v \in \underline{R}(X)$. We have

$$R^{uvw}(u) = (R \cup S^{uvw})(u) = R(u) \cup S^{uvw}(u) = R(u) \cup \{v\} \subseteq X.$$

Then $u \in \underline{R^{uvw}}(X)$. Thus $X \subseteq \underline{R^{uvw}}(X)$. By the reflexivity of ρ, $X \supseteq \underline{R^{uvw}}(X)$. Then $\underline{R^{uvw}}(X) = X$ and So $X \in \sigma_{R^{uvw}}$.

By *i*) and *ii*), $\tau_R \subseteq \tau_{R^{uvw}}$.

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Thus

$$\tau_{R^{uvw}} = \tau_R \ (w \in U).$$

(2) By (1) and Proposition 3.4,

$$\tau_{R^{uv}} = \tau_{\bigcup\limits_{w\in U} R^{uvw}} = \bigcap_{w\in U} \tau_{R^{uvw}} = \tau_R.$$

□

Denote $R_0 = R$. R_n $(n \in \omega)$ are defined as follows:

$$R_{n+1} = \bigcup_{u,v\in X} (R_n)^{uv}.$$

Put

$$R^* = \lim_{n\to\infty} R_n.$$

Obviously, $R^* = \bigcup\limits_{n=0}^{\infty} R_n$.

Corollary 3.7. *Let R be a reflexive relation on U. Then $\tau_{R_n} = \tau_R = \tau_{R^*}$.*

Proof. This holds by Proposition 3.4 and Theorem 3.6. □

Theorem 3.8. *Let R be a binary relation on U. Then*

$$R \textit{ is transitive} \iff R = R_1.$$

Proof. Necessity. Obviously.

Sufficiency. Suppose that R is not translative. Then there exist x, y, z such that $(x,z),(z,y) \in R$, $(x,y) \notin R$. So $(x,y) \in R^{xy}$. This implies

$$(x,y) \in R_1 = \bigcup_{u,v\in X} R^{uv}.$$

We have $R_1 \neq R$. This is a contradiction.

Thus R is translative. □

Corollary 3.9. *If R is a preorder relation on U, then $\forall\ n \in \omega$, $R_n = R$.*

Proof. This holds by Theorem 4.6. □

Denote $R_0 = R$. R_n $(n \in \omega)$ are defined as follows: $R_{n+1} = \bigcup\limits_{u,v\in X} (R_n)^{uv}$

Denote

$$R^* = \lim_{n\to\infty} R_n.$$

Theorem 3.10. *If R is a reflexive relation on U, then R^* is translative.*

Proof. Let $(u,w),(w,v) \in R^*$. Then there exist $n_1, n_2 \in N$ such that $(u,w) \in R_{n_1}, (w,v) \in R_{n_2}$. Pick $n_0 = n_1 + n_2$. Then $(u,w) \in R_{n_0}, (w,v) \in R_{n_0}$. So

$$(u,v) \in (R_{n_0})^{uvw} \subseteq (R_{n_0})^{uv} \subseteq R_{n_0+1} \subseteq R^*.$$

So R^* is translative. □

Theorem 3.11. *Let R be a reflexive relation on U. Then $R_s = R^*$.*

Proof. Note that

$$\begin{array}{lll} & (x,y) \in R^*. & (R^* = \bigcup_{n=0}^{\infty} R_n) \\ \iff & \exists n \in N, (x,y) \in R_n. & (R_n = \bigcup_{u,v\in X} (R_{n-1})^{uv}) \\ \iff & (x,y) \in (R_{n-1})^{xy}. & ((R_{n-1})^{xy} = \bigcup_{w\in U} (R_{n-1})^{xyw}) \\ \iff & \exists w_{2^n} \in U, (x,y) \in (R_{n-1})^{xyw_{2^n}}. & \\ \iff & \exists w_{2^n} \in U, (x,w_{2^n}),(w_{2^n},y) \in R_{n-1}. & \\ \iff & \exists w_{2^n-1}, w_{2^n}, w_{2^n-2} \in U, & \\ & (x,w_{2^n-1}),(w_{2^n-1},w_{2^n}),(w_{2^n},w_{2^n-2}),(w_{2^n-2},y) \in R_{n-2}. & \\ \cdots & \cdots\cdots & \\ \iff & \exists w_2, w_3, \cdots, w_{2^n} \in U, & \\ & (x,w_3),\cdots,(w_{2^n-1},w_{2^n}),(w_{2^n},w_{2^n-2}),\cdots,(w_2,y) \in R_0 = R. & \\ \iff & (x,y) \in R_s. & \end{array}$$

Then $R_s = R^*$. □

Corollary 3.12. *Let R be a tolerance relation on U. Then*
(1) *R_s is equivalence.*
(2) *$\tau_{R_s} = \tau_R$.*
(3) *$\underline{\underline{R_s}}$ is an interior operator of τ_R.*
(4) *$\overline{\overline{R_s}}$ is a closure operator of τ_R.*

Proof. (1) This holds by Theorem 3.11.
(2) This holds by Corollary 3.7 and Theorem 3.11
(3) This holds by (2) and Theorem 3.2.
(4) This holds by (2) and Theorem 3.2. □

4 Some characteristic conditions of approximating spaces

Definition 4.1 ([4])**.** *Let (U,μ) be a topological space. If there exists an equivalence relation R on U such that $\tau_R = \mu$, then (U,τ) is called a approximating space.*

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Definition 4.2. *Let μ be a topology on U. Define a binary relation R_μ on U by*

$$(x, y) \in R_\mu \iff x \in cl_\mu(\{y\}).$$

Then R_μ is called the binary relation induced by μ on U.

Theorem 4.3. *Let (U, μ) be a topological space. Then the following are equivalent:*

(1) *(U, μ) is an approximating space;*

(2) *There exists a tolerance relation R on U such that $\tau_R = \mu$;*

(3) *There exists a tolerance relation R on U such that $\underline{R}$ is an interior operator of μ;*

(4) *There exists a tolerance relation R on U such that $\overline{R}$ is a closure operator of μ;*

(5) *There exists an equivalence relation R on U such that*

$$\mu = \{\bigcup_{x\in X} [x]_R : X \in 2^U\}.$$

Proof. (1) $\Longrightarrow$ (2) is obvious.

(1) $\Longrightarrow$ (3) and (1)$\Longrightarrow$ (4) hold by Theorem 3.2.

(1) $\Longrightarrow$ (5) holds by Remark 3.2.

(2) $\Longrightarrow$ (1) Suppose that there exists a tolerance relation R on U such that $\tau_R = \mu$.

By Theorem 2.2 and Corollary 3.12, R_s is equivalence and $\tau_{R_s} = \tau_R$.

Then $\tau_{R_s} = \mu$.

Thus (U, μ) is an approximating space.

(3) $\Longrightarrow$ (1) Suppose that there exists a tolerance relation R on U such that $\underline{R}$ is an interior operator of μ. Then

$$X \in \tau_R \Longleftrightarrow \underline{R}(X) = X \Longleftrightarrow int_\mu(X) = X \Longleftrightarrow X \in \mu.$$

Then $\tau_R = \mu$.

By Theorem 2.6(2), R is preorder. So R is equivalence.

Thus (U, μ) is an approximating space.

(4) $\Longrightarrow$ (1) The proof is similar to (3) $\Longrightarrow$ (1).

(5) $\Longrightarrow$ (1) holds by Remark 3.2. □

Corollary 4.4. *If (U, μ) is an approximating space, then R_μ is an equivalence relation.*

Proof. Obviously, R_μ is reflexive.

By Theorem 4.3, there exists an equivalence relation R on U such that

$$\mu = \{\bigcup_{x\in X} [x]_R : X \in 2^U\}.$$

By Remark 2.4, we have

$$(x, y) \in R_\mu \Rightarrow x \in cl_\mu(\{y\}) = [y]_R \Rightarrow y \in [y]_R = [x]_R = cl_\mu(\{x\}) \Rightarrow (y, x) \in R_\mu.$$

$$\begin{aligned}(x,y),(y,z)\in R_\mu &\Longrightarrow x\in cl_\mu(\{y\})=[y]_R, y\in cl_\mu(\{z\})=[z]_R\\ &\Longrightarrow x\in [y]_R=[z]_R=cl_\mu(\{z\})\Longrightarrow (x,z)\in R_\mu.\end{aligned}$$

Thus R_μ is equivalence. □

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Divisible and strong fuzzy filters of residuated lattices

Young Bae Jun[1], Xiaohong Zhang[2] and Sun Shin Ahn[3,*]

[1] *Department of Mathematics Education, Gyeongsang National University, Jinju 660-701, Korea*
[2] *Department of Mathematics, College of Arts and Sciences, Shanghai Maritime University, Shanghai 201306, P.R. China*
[3] *Department of Mathematics Education, Dongguk University, Seoul 100-715, Korea*

Abstract. In a residuated lattice, divisible fuzzy filters and strong fuzzy filters are introduced, and their properties are investigated. Characterizations of a divisible and strong fuzzy filter are discussed. Conditions for a fuzzy filter to be divisible are established. Relations between a divisible fuzzy filter and a strong fuzzy filter are considered.

1. Introduction

In order to deal with fuzzy and uncertain informations, non-classical logic has become a formal and useful tool. As the semantical systems of non-classical logic systems, various logical algebras have been proposed. Residuated lattices are important algebraic structures which are basic of MTL-algebras, BL-algebras, MV-algebras, Gödel algebras, R_0-algebras, lattice implication algebras, etc. The filter theory plays an important role in studying logical systems and the related algebraic structures, and various filters have been proposed in the literature. Zhang et al. [8] introduced the notions of IMTL-filters (NM-filters, MV-filters) of residuated lattices, and presented their characterizations. Ma and Hu [4] introduced divisible filters, strong filters and n-contractive filters in residuated lattices.

In this paper, we consider the fuzzification of divisible filters and strong filters in residuated lattices. We define divisible fuzzy filters and strong fuzzy filters, and investigate related properties. We discussed characterizations of a divisible and strong fuzzy filter, and provided conditions for a fuzzy filter to be divisible. We establish relations between a divisible fuzzy filter and a strong fuzzy filter.

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* The corresponding author.
[0]**E-mail**: `skywine@gmail.com` (Y. B. Jun); `zxhonghz@263.net` (X. Zhang); `sunshine@dongguk.edu` (S. S. Ahn)

2. Preliminaries

Definition 2.1 ([1, 2, 3]). A *residuated lattice* is an algebra $\mathcal{L} := (L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that

(1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice.
(2) $(L, \odot, 1)$ is a commutative monoid.
(3) $\odot$ and $\rightarrow$ form an adjoint pair, that is,

$$(\forall x, y, z \in L)\,(x \leq y \rightarrow z \;\Leftrightarrow\; x \odot y \leq z)\,.$$

In a residuated lattice $\mathcal{L}$, the ordering $\leq$ and negation $\neg$ are defined as follows:

$$(\forall x, y \in L)\,(x \leq y \;\Leftrightarrow\; x \wedge y = x \;\Leftrightarrow\; x \vee y = y \;\Leftrightarrow\; x \rightarrow y = 1)$$

and $\neg x = x \rightarrow 0$ for all $x \in L$.

Proposition 2.2 ([1, 2, 3, 4, 6, 7]). *In a residuated lattice $\mathcal{L}$, the following properties are valid.*

$$1 \rightarrow x = x,\; x \rightarrow 1 = 1,\; x \rightarrow x = 1,\; 0 \rightarrow x = 1,\; x \rightarrow (y \rightarrow x) = 1. \tag{2.1}$$

$$x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z). \tag{2.2}$$

$$x \leq y \;\Rightarrow\; z \rightarrow x \leq z \rightarrow y,\; y \rightarrow z \leq x \rightarrow z. \tag{2.3}$$

$$z \rightarrow y \leq (x \rightarrow z) \rightarrow (x \rightarrow y),\; z \rightarrow y \leq (y \rightarrow x) \rightarrow (z \rightarrow x). \tag{2.4}$$

$$(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z. \tag{2.5}$$

$$\neg x = \neg\neg\neg x,\; x \leq \neg\neg x,\; \neg 1 = 0,\; \neg 0 = 1. \tag{2.6}$$

$$x \odot y \leq x \odot (x \rightarrow y) \leq x \wedge y \leq x \wedge (x \rightarrow y) \leq x. \tag{2.7}$$

$$x \leq y \;\Rightarrow\; x \odot z \leq y \odot z. \tag{2.8}$$

$$x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z),\; (x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z). \tag{2.9}$$

$$x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z). \tag{2.10}$$

$$\neg\neg(x \rightarrow y) \leq \neg\neg x \rightarrow \neg\neg y. \tag{2.11}$$

$$x \rightarrow (x \wedge y) = x \rightarrow y. \tag{2.12}$$

Definition 2.3 ([5]). A nonempty subset F of a residuated lattice $\mathcal{L}$ is called a *filter* of $\mathcal{L}$ if it satisfies the conditions:

$$(\forall x, y \in L)\,(x, y \in F \;\Rightarrow\; x \odot y \in F)\,. \tag{2.13}$$

$$(\forall x, y \in L)\,(x \in F,\; x \leq y \;\Rightarrow\; y \in F)\,. \tag{2.14}$$

Proposition 2.4 ([5]). *A nonempty subset F of a residuated lattice $\mathcal{L}$ is a filter of $\mathcal{L}$ if and only if it satisfies:*

$$1 \in F. \tag{2.15}$$

$$(\forall x \in F)\,(\forall y \in L)\,(x \to y \in F \;\Rightarrow\; y \in F)\,. \tag{2.16}$$

Definition 2.5 ([9]). A fuzzy set μ in a residuated lattice $\mathcal{L}$ is called a *fuzzy filter* of $\mathcal{L}$ if it satisfies:

$$(\forall x, y \in L)\,(\mu(x \odot y) \geq \min\{\mu(x), \mu(y)\})\,. \tag{2.17}$$

$$(\forall x, y \in L)\,(x \leq y \;\Rightarrow\; \mu(x) \leq \mu(y))\,. \tag{2.18}$$

Theorem 2.6 ([9]). *A fuzzy set μ in a residuated lattice $\mathcal{L}$ is a fuzzy filter of $\mathcal{L}$ if and only if the following assertions are valid:*

$$(\forall x \in L)\,(\mu(1) \geq \mu(x))\,. \tag{2.19}$$

$$(\forall x, y \in L)\,(\mu(y) \geq \min\{\mu(x \to y), \mu(x)\})\,. \tag{2.20}$$

3. **Divisible and strong fuzzy filters**

In what follows let $\mathcal{L}$ denote a residuated lattice unless otherwise specified.

Definition 3.1 ([4]). A filter F of $\mathcal{L}$ is said to be *divisible* if it satisfies:

$$(\forall x, y \in L)\,((x \wedge y) \to [x \odot (x \to y)] \in F)\,. \tag{3.1}$$

Definition 3.2. A fuzzy filter μ of $\mathcal{L}$ is said to be *divisible* if it satisfies:

$$(\forall x, y \in L)\,\big(\mu\big((x \wedge y) \to [x \odot (x \to y)]\big) = \mu(1)\big)\,. \tag{3.2}$$

Example 3.3. Let $L = \{0, a, b, 1\}$ be a chain with Cayley tables which are given in Tables 1 and 2.

Table 1. Cayley table for the "$\odot$"-operation

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Then $\mathcal{L} := (L, \vee, \wedge, \odot, \to, 0, 1)$ is a residuated lattice. Define a fuzzy set μ in $\mathcal{L}$ by $\mu(1) = 0.7$ and $\mu(x) = 0.2$ for all $x(\neq 1) \in L$. It is routine to verify that μ is a divisible fuzzy filter of $\mathcal{L}$.

TABLE 2. Cayley table for the "→"-operation

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Example 3.4. Consider a residuated lattice $L = [0, 1]$ in which two operations "$\odot$" and "$\to$" are defined as follows:

$$x \odot y = \begin{cases} 0 & \text{if } x + y \leq \frac{1}{2}, \\ x \wedge y & \text{otherwise.} \end{cases}$$

$$x \to y = \begin{cases} 1 & \text{if } x \leq y, \\ \left(\frac{1}{2} - x\right) \vee y & \text{otherwise.} \end{cases}$$

The fuzzy set μ of $\mathcal{L}$ given by $\mu(1) = 0.9$ and $\mu(x) = 0.2$ for all $x(\neq 1) \in L$ is a fuzzy filter of $\mathcal{L}$. But it is not divisible since

$$\mu((0.3 \wedge 0.2) \to (0.3 \odot (0.3 \to 0.2)) = \mu(0.3) \neq \mu(1).$$

Proposition 3.5. *Every divisible fuzzy filter μ of $\mathcal{L}$ satisfies the following identity.*

$$(\forall x, y, z \in L)\left(\mu(((x \odot y) \wedge (x \odot z)) \to (x \odot (y \wedge z))) = \mu(1)\right). \tag{3.3}$$

Proof. Let $x, y, z \in L$. If we let $x := x \odot y$ and $y := x \odot z$ in (3.2), then

$$\mu(((x \odot y) \wedge (x \odot z)) \to ((x \odot y) \odot ((x \odot y) \to (x \odot z)))) = \mu(1). \tag{3.4}$$

Using (2.2) and (2.7), we have

$$\begin{aligned}(x \odot y) \odot ((x \odot y) \to (x \odot z)) &= x \odot y \odot (y \to (x \to (x \odot z))) \\ &\leq x \odot (y \wedge (x \to (x \odot z))),\end{aligned}$$

and so

$$\begin{aligned}&((x \odot y) \wedge (x \odot z)) \to ((x \odot y) \odot ((x \odot y) \to (x \odot z))) \\ &\leq ((x \odot y) \wedge (x \odot z)) \to (x \odot (y \wedge (x \to (x \odot z))))\end{aligned}$$

by (2.3). It follows from (3.4) and (2.18) that

$$\begin{aligned}\mu(1) &= \mu(((x \odot y) \wedge (x \odot z)) \to ((x \odot y) \odot ((x \odot y) \to (x \odot z)))) \\ &\leq \mu(((x \odot y) \wedge (x \odot z)) \to (x \odot (y \wedge (x \to (x \odot z)))))\end{aligned}$$

and so that

$$\mu(((x \odot y) \wedge (x \odot z)) \to (x \odot (y \wedge (x \to (x \odot z))))) = \mu(1) \tag{3.5}$$

since $\mu(1) \geq \mu(x)$ for all $x \in L$. On the other hand, if we take $x := x \to (x \odot z)$ in (3.2) then

$$\begin{aligned}\mu(1) &= \mu((y \wedge (x \to (x \odot z))) \to ((x \to (x \odot z)) \odot ((x \to (x \odot z)) \to y))) \\ &\leq \mu((x \odot (y \wedge (x \to (x \odot z)))) \to (x \odot ((x \to (x \odot z)) \odot ((x \to (x \odot z)) \to y)))) \\ &= \mu((x \odot (y \wedge (x \to (x \odot z)))) \to (x \odot (x \to (x \odot z)) \odot ((x \to (x \odot z)) \to y)))\end{aligned}$$

by using (2.10), (2.18) and the commutativity and associativity of $\odot$. Hence

$$\mu((x \odot (y \wedge (x \to (x \odot z)))) \to (x \odot (x \to (x \odot z)) \odot ((x \to (x \odot z)) \to y))) = \mu(1). \tag{3.6}$$

Using (2.5), we get

$$\begin{aligned}&(((x \odot y) \wedge (x \odot z)) \to (x \odot (y \wedge (x \to (x \odot z)))))\odot \\ &((x \odot (y \wedge (x \to (x \odot z)))) \to (x \odot (x \to (x \odot z)) \odot ((x \to (x \odot z)) \to y))) \\ &\leq ((x \odot y) \wedge (x \odot z)) \to (x \odot (x \to (x \odot z)) \odot ((x \to (x \odot z)) \to y)).\end{aligned}$$

It follows from (2.18), (2.17), (3.5) and (3.6) that

$$\begin{aligned}&\mu(((x \odot y) \wedge (x \odot z)) \to (x \odot (x \to (x \odot z)) \odot ((x \to (x \odot z)) \to y))) \\ &\geq \mu((((x \odot y) \wedge (x \odot z)) \to (x \odot (y \wedge (x \to (x \odot z)))))\odot \\ &((x \odot (y \wedge (x \to (x \odot z)))) \to (x \odot (x \to (x \odot z)) \odot ((x \to (x \odot z)) \to y)))) \\ &\geq \min\{\mu((((x \odot y) \wedge (x \odot z)) \to (x \odot (y \wedge (x \to (x \odot z)))))), \\ &\mu(((x \odot (y \wedge (x \to (x \odot z)))) \to (x \odot (x \to (x \odot z)) \odot ((x \to (x \odot z)) \to y))))\} \\ &= \mu(1)\end{aligned}$$

Thus

$$\mu(((x \odot y) \wedge (x \odot z)) \to (x \odot (x \to (x \odot z)) \odot ((x \to (x \odot z)) \to y))) = \mu(1). \tag{3.7}$$

Since $x \odot (x \to (x \odot z)) \odot ((x \to (x \odot z)) \to y)) \leq x \odot z \odot (z \to y) \leq x \odot (y \wedge z)$, we obtain

$$\begin{aligned}&((x \odot y) \wedge (x \odot z)) \to (x \odot (x \to (x \odot z)) \odot ((x \to (x \odot z)) \to y))) \\ &\leq ((x \odot y) \wedge (x \odot z)) \to (x \odot (y \wedge z)).\end{aligned}$$

It follows that

$$\begin{aligned}&\mu(((x \odot y) \wedge (x \odot z)) \to (x \odot (y \wedge z))) \\ &\geq \mu(((x \odot y) \wedge (x \odot z)) \to (x \odot (x \to (x \odot z)) \odot ((x \to (x \odot z)) \to y)))) \\ &= \mu(1)\end{aligned}$$

and that $\mu(((x \odot y) \wedge (x \odot z)) \to (x \odot (y \wedge z))) = \mu(1)$. □

We consider characterizations of a divisible fuzzy filter.

Theorem 3.6. *A fuzzy filter μ of $\mathcal{L}$ is divisible if and only if the following assertion is valid:*

$$(\forall x, y, z \in L)\left(\mu\big([x \to (y \wedge z)] \to [(x \to y) \odot ((x \wedge y) \to z)]\big) = \mu(1)\right). \tag{3.8}$$

Proof. Assume that μ is a divisible fuzzy filter of $\mathcal{L}$. If we take $x := x \to y$ and $y := x \to z$ in (3.2) and use (2.9) and (2.2), then

$$\begin{aligned}\mu(1) &= \mu\left([(x \to y) \wedge (x \to z)] \to [(x \to y) \odot ((x \to y) \to (x \to z))]\right)\\ &= \mu\left([x \to (y \wedge z)] \to [(x \to y) \odot ((x \odot (x \to y)) \to z)]\right).\end{aligned}$$

Using (2.4) and (2.10), we have

$$\begin{aligned}&(x \wedge y) \to [x \odot (x \to y)] \leq [(x \odot (x \to y)) \to z] \to [(x \wedge y) \to z]\\ &\leq [(x \to y) \odot ((x \odot (x \to y)) \to z)] \to [(x \to y) \odot ((x \wedge y) \to z)]\end{aligned}$$

for all $x, y, z \in L$. Since μ is a divisible fuzzy filter of $\mathcal{L}$, it follows from (3.2) and (2.18) that

$$\begin{aligned}\mu(1) &= \mu((x \wedge y) \to [x \odot (x \to y)])\\ &\leq \mu([(x \to y) \odot ((x \odot (x \to y)) \to z)] \to [(x \to y) \odot ((x \wedge y) \to z)])\end{aligned}$$

and so from (2.19) that

$$\mu([(x \to y) \odot ((x \odot (x \to y)) \to z)] \to [(x \to y) \odot ((x \wedge y) \to z)]) = \mu(1)$$

for all $x, y, z \in L$. Using (2.5), we get

$$\begin{aligned}&\Big([x \to (y \wedge z)] \to [(x \to y) \odot ((x \odot (x \to y)) \to z)]\Big) \odot\\ &\qquad\Big([(x \to y) \odot ((x \odot (x \to y)) \to z)] \to [(x \to y) \odot ((x \wedge y) \to z)]\Big)\\ &\leq [x \to (y \wedge z)] \to [(x \to y) \odot ((x \wedge y) \to z)],\end{aligned}$$

and so

$$\begin{aligned}&\mu\Big([x \to (y \wedge z)] \to [(x \to y) \odot ((x \wedge y) \to z)]\Big)\\ &\geq \mu\Big(\big([x \to (y \wedge z)] \to [(x \to y) \odot ((x \odot (x \to y)) \to z)]\big) \odot\\ &\qquad\big([(x \to y) \odot ((x \odot (x \to y)) \to z)] \to [(x \to y) \odot ((x \wedge y) \to z)]\big)\Big)\\ &\geq \min\Big\{\mu\big([x \to (y \wedge z)] \to [(x \to y) \odot ((x \odot (x \to y)) \to z)]\big),\\ &\qquad\mu\big([(x \to y) \odot ((x \odot (x \to y)) \to z)] \to [(x \to y) \odot ((x \wedge y) \to z)]\big)\Big\}\\ &= \mu(1).\end{aligned}$$

Therefore $\mu\Big([x \to (y \wedge z)] \to [(x \to y) \odot ((x \wedge y) \to z)]\Big) = \mu(1)$ for all $x, y, z \in L$.

Conversely, let μ be a fuzzy filter that satisfies the condition (3.8). if we take $x := 1$ in (3.8) and use (2.1), then we obtain (3.2). □

Theorem 3.7. *A fuzzy filter μ of $\mathcal{L}$ is divisible if and only if it satisfies:*

$$(\forall x, y \in L)\left(\mu\left([y \odot (y \to x)] \to [x \odot (x \to y)]\right) = \mu(1)\right). \tag{3.9}$$

Proof. Suppose that μ is a divisible fuzzy filter of $\mathcal{L}$. Note that

$$(x \wedge y) \to [x \odot (x \to y)] \leq [y \odot (y \to x)] \to [x \odot (x \to y)]$$

for all $x, y \in L$. It follows from (3.2) and (2.18) that

$$\begin{aligned}\mu(1) &= \mu\left((x \wedge y) \to [x \odot (x \to y)]\right)\\ &\leq \mu\left([y \odot (y \to x)] \to [x \odot (x \to y)]\right)\end{aligned}$$

and that $\mu\left([y \odot (y \to x)] \to [x \odot (x \to y)]\right) = \mu(1)$.

Conversely, let μ be a fuzzy filter of $\mathcal{L}$ that satisfies the condition (3.9). Since

$$y \to x = y \to (y \wedge x) \text{ for all } x, y \in L,$$

the condition (3.9) implies that

$$\mu\left([y \odot (y \to (x \wedge y))] \to [x \odot (x \to (x \wedge y))]\right) = \mu(1). \tag{3.10}$$

If we take $y := x \wedge z$ in (3.10), then

$$\begin{aligned}\mu(1) &= \mu\left([(x \wedge z) \odot ((x \wedge z) \to (x \wedge (x \wedge z)))] \to [x \odot (x \to (x \wedge (x \wedge z)))]\right)\\ &= \mu\left((x \wedge z) \to [x \odot (x \to z)]\right).\end{aligned}$$

Therefore μ is a divisible fuzzy filter of $\mathcal{L}$. □

We discuss conditions for a fuzzy filter to be divisible.

Theorem 3.8. *If a fuzzy filter μ of $\mathcal{L}$ satisfies the following assertion:*

$$(\forall x, y \in L)\left(\mu((x \wedge y) \to (x \odot y)) = \mu(1)\right), \tag{3.11}$$

then μ is divisible.

Proof. Note that $x \odot y \leq x \odot (x \to y)$ for all $x, y \in L$. It follows from (2.3) that

$$(x \wedge y) \to (x \odot y) \leq (x \wedge y) \to (x \odot (x \to y)).$$

Hence, by (3.11) and (2.18), we have

$$\mu(1) = \mu((x \wedge y) \to (x \odot y)) \leq \mu((x \wedge y) \to (x \odot (x \to y))),$$

and so $\mu((x \wedge y) \to (x \odot (x \to y))) = \mu(1)$ for all $x, y \in L$. Therefore μ is a divisible fuzzy filter of $\mathcal{L}$. □

Theorem 3.9. *If a fuzzy filter μ of $\mathcal{L}$ satisfies the following assertion:*

$$(\forall x, y \in L)\,(\mu((x \wedge (x \to y)) \to y) = \mu(1))\,, \tag{3.12}$$

then μ is divisible.

Proof. Taking $y := x \odot y$ in (3.12) implies that

$$\begin{aligned}\mu(1) &= \mu((x \wedge (x \to (x \odot y))) \to (x \odot y)) \\ &\leq \mu((x \wedge y) \to (x \odot y))\end{aligned}$$

and so $\mu((x \wedge y) \to (x \odot y)) = \mu(1)$ for all $x, y \in L$. It follows from Theorem 3.8 that μ is a divisible fuzzy filter of $\mathcal{L}$. □

Theorem 3.10. *If a fuzzy filter μ of $\mathcal{L}$ satisfies the following assertion:*

$$(\forall x, y, z \in L)\,(\mu(x \to z) \geq \min\{\mu((x \odot y) \to z), \mu(x \to y)\})\,, \tag{3.13}$$

then μ is divisible.

Proof. If we take $x := x \wedge (x \to y)$, $y := x$ and $z := y$ in (3.13), then

$$\begin{aligned}\mu((x \wedge (x \to y)) \to y) &\geq \min\{\mu(((x \wedge (x \to y)) \odot x) \to y), \mu((x \wedge (x \to y)) \to x)\} \\ &= \mu(1)\end{aligned}$$

Thus $\mu((x \wedge (x \to y)) \to y) = \mu(1)$ for all $x, y \in L$, and so μ is a divisible fuzzy filter of $\mathcal{L}$ by Theorem 3.9. □

Theorem 3.11. *If a fuzzy filter μ of $\mathcal{L}$ satisfies the following assertion:*

$$(\forall x \in L)\,(\mu(x \to (x \odot x)) = \mu(1))\,, \tag{3.14}$$

then μ is divisible.

Proof. Let μ be a fuzzy filter of $\mathcal{L}$ that satisfies the condition (3.14). Using (2.10) and the commutativity of $\odot$, we have $x \to y \leq (x \odot x) \to (x \odot y)$, and so

$$(x \to (x \odot x)) \odot (x \to y) \leq (x \to (x \odot x)) \odot ((x \odot x) \to (x \odot y))$$

for all $x, y \in L$ by (2.8) and the commutativity of $\odot$. It follows from (2.5), (2.8) and the commutativity of $\odot$ that

$$\begin{aligned}&((x \to (x \odot x)) \odot (x \to y)) \odot ((x \odot y) \to z) \\ &\leq ((x \to (x \odot x)) \odot ((x \odot x) \to (x \odot y))) \odot ((x \odot y) \to z) \\ &\leq (x \to (x \odot y)) \odot ((x \odot y) \to z) \\ &\leq x \to z\end{aligned}$$

and so from (2.17), (2.18), (2.19) and (3.14) that

$$\begin{aligned}\mu(x \to z) &\geq \mu(((x \to (x \odot x)) \odot (x \to y)) \odot ((x \odot y) \to z)) \\ &\geq \min\{\mu((x \to (x \odot x)) \odot (x \to y)), \mu((x \odot y) \to z)\} \\ &\geq \min\{\mu(x \to (x \odot x)), \mu(x \to y), \mu((x \odot y) \to z)\} \\ &= \min\{\mu(1), \mu(x \to y), \mu((x \odot y) \to z)\} \\ &= \min\{\mu((x \odot y) \to z), \mu(x \to y)\}\end{aligned}$$

for all $x, y, z \in L$. Therefore μ is a divisible fuzzy filter of $\mathcal{L}$ by Theorem 3.10. □

Definition 3.12 ([4])**.** A filter F of $\mathcal{L}$ is said to be *strong* if it satisfies:

$$(\forall x \in L)\,(\neg\neg(\neg\neg x \to x) \in F)\,. \tag{3.15}$$

Definition 3.13. A fuzzy filter μ of $\mathcal{L}$ is said to be *strong* if it satisfies:

$$(\forall x \in L)\,\big(\mu\big(\neg\neg(\neg\neg x \to x)\big) = \mu(1)\big)\,. \tag{3.16}$$

Example 3.14. Consider a residuated lattice $L := \{0, a, b, c, d, 1\}$ with the following Hasse diagram (Figure 3.1) and Cayley tables (see Table 3 and Table 4).

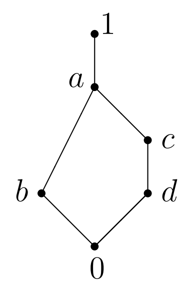

Figure 3.1

TABLE 3. Cayley table for the "$\odot$"-operation

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	b	d	d	a
b	c	b	b	0	0	b
c	b	d	0	d	d	c
d	b	d	0	d	d	d
1	0	a	b	c	d	1

Define a fuzzy set μ in $\mathcal{L}$ by $\mu(1) = 0.6$ and $\mu(x) = 0.5$ for all $x(\neq 1) \in L$. It is routine to check that μ is a strong fuzzy filter of $\mathcal{L}$.

TABLE 4. Cayley table for the "→"-operation

→	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	c	c	1
b	c	a	1	c	c	1
c	b	a	b	1	a	1
d	b	a	b	a	1	1
1	0	a	b	c	d	1

We provide characterizations of a strong fuzzy filter.

Theorem 3.15. *Given a fuzzy set μ of $\mathcal{L}$, the following assertions are equivalent.*

(1) *μ is a strong fuzzy filter of $\mathcal{L}$.*

(2) *μ is a fuzzy filter of $\mathcal{L}$ that satisfies*

$$(\forall x, y \in L)\,(\mu((y \to \neg\neg x) \to \neg\neg(y \to x) = \mu(1))\,. \tag{3.17}$$

(3) *μ is a fuzzy filter of $\mathcal{L}$ that satisfies*

$$(\forall x, y \in L)\,(\mu((\neg x \to y) \to \neg\neg(\neg y \to x)) = \mu(1))\,. \tag{3.18}$$

Proof. Assume that μ is a strong fuzzy filter of $\mathcal{L}$. Then μ is a fuzzy filter of $\mathcal{L}$. Note that

$$\begin{aligned}\neg\neg(\neg\neg x \to x) &\le \neg\neg((y \to \neg\neg x) \to (y \to x))\\ &\le \neg\neg((y \to \neg\neg x) \to \neg\neg(y \to x))\\ &= (y \to \neg\neg x) \to \neg\neg(y \to x)\end{aligned}$$

and

$$\begin{aligned}\neg\neg(\neg\neg x \to x) &\le \neg\neg(((\neg x \to y) \odot \neg y) \to x)\\ &= \neg\neg((\neg x \to y) \to (\neg y \to x))\\ &\le \neg\neg((\neg x \to y) \to \neg\neg(\neg y \to x))\\ &= (\neg x \to y) \to \neg\neg(\neg y \to x)\end{aligned}$$

for all $x, y \in L$. If follows from (3.16) and (2.18) that

$$\mu(1) = \mu(\neg\neg(\neg\neg x \to x)) \le \mu((y \to \neg\neg x) \to \neg\neg(y \to x)) \tag{3.19}$$

and

$$\mu(1) = \mu(\neg\neg(\neg\neg x \to x)) \le \mu((\neg x \to y) \to \neg\neg(\neg y \to x)). \tag{3.20}$$

Combining (2.19), (3.19) and (3.20), we have $\mu((y \to \neg\neg x) \to \neg\neg(y \to x)) = \mu(1)$ and $\mu((\neg x \to y) \to \neg\neg(\neg y \to x)) = \mu(1)$ for all $x, y \in L$. Therefore (2) and (3) are valid. Let μ be a fuzzy filter of $\mathcal{L}$ that satisfies the condition (3.17). If we take $y := \neg\neg x$ in (3.17) and use (2.1), then we can induce the condition (3.16) and so μ is a strong fuzzy filter of $\mathcal{L}$. Let μ be a fuzzy filter of $\mathcal{L}$ that satisfies the condition (3.18). Taking $y := \neg x$ in (3.18) and using (2.1) induces the condition (3.16). Hence μ is a strong fuzzy filter of $\mathcal{L}$. □

We investigate relationship between a divisible fuzzy filter and a strong fuzzy filter.

Theorem 3.16. *Every divisible fuzzy filter is a strong fuzzy filter.*

Proof. Let μ be a divisible fuzzy filter of $\mathcal{L}$. If we put $x := \neg\neg x$ and $y := x$ in (3.2), then we have

$$\mu((\neg\neg x \wedge x) \to (\neg\neg x \odot (\neg\neg x \to x))) = \mu(1). \tag{3.21}$$

Using (2.4) and (2.8), we get

$$\begin{aligned}(\neg\neg x \wedge x) \to (\neg\neg x \odot (\neg\neg x \to x)) &\leq \neg(\neg\neg x \odot (\neg\neg x \to x)) \to \neg(\neg\neg x \wedge x)\\ &\leq (\neg\neg x \odot \neg(\neg\neg x \odot (\neg\neg x \to x))) \to (\neg\neg x \odot \neg(\neg\neg x \wedge x))\\ &\leq \neg(\neg\neg x \odot \neg(\neg\neg x \wedge x)) \to \neg(\neg\neg x \odot \neg(\neg\neg x \odot (\neg\neg x \to x)))\end{aligned}$$

for all $x \in L$. It follows from (3.21) and (2.18) that

$$\begin{aligned}\mu(1) &= \mu((\neg\neg x \wedge x) \to (\neg\neg \odot (\neg\neg x \to x)))\\ &\leq \mu(\neg(\neg\neg x \odot \neg(\neg\neg x \wedge x)) \to \neg(\neg\neg x \odot \neg(\neg\neg x \odot (\neg\neg x \to x)))).\end{aligned} \tag{3.22}$$

Combining (3.22) with (2.19), we have

$$\mu(\neg(\neg\neg x \odot \neg(\neg\neg x \wedge x)) \to \neg(\neg\neg x \odot \neg(\neg\neg x \odot (\neg\neg x \to x)))) = \mu(1) \tag{3.23}$$

for all $x \in L$. Using (2.2), (2.11), (2.12) and (2.6), we get

$$\begin{aligned}\neg(\neg\neg x \odot \neg(\neg\neg x \wedge x)) &= \neg\neg x \to \neg\neg(\neg\neg x \wedge x)\\ &\geq \neg\neg(x \to (\neg\neg x \wedge x))\\ &= \neg\neg(x \to (x \wedge \neg\neg x))\\ &= \neg\neg(x \to \neg\neg x) = \neg\neg 1 = 1\end{aligned}$$

and so $\neg(\neg\neg x \odot \neg(\neg\neg x \wedge x)) = 1$ for all $x \in L$. It follows from (3.23) and (2.20) that

$$\begin{aligned}&\mu(\neg(\neg\neg x \odot \neg(\neg\neg x \odot (\neg\neg x \to x))))\\ &\geq \min\{\mu(\neg(\neg\neg x \odot \neg(\neg\neg x \wedge x)) \to \neg(\neg\neg x \odot \neg(\neg\neg x \odot (\neg\neg x \to x)))),\\ &\quad \mu(\neg(\neg\neg x \odot \neg(\neg\neg x \wedge x)))\}\\ &= \mu(1)\end{aligned}$$

and so that

$$\begin{aligned}\mu(1) &= \mu(\neg(\neg\neg x \odot \neg(\neg\neg x \odot (\neg\neg x \to x)))) \\ &= \mu(\neg(\neg\neg x \odot (\neg\neg x \to \neg(\neg\neg x \to x)))).\end{aligned} \tag{3.24}$$

Taking $x := \neg\neg x$ and $y := \neg(\neg\neg x \to x)$ in (3.2) induces

$$\begin{aligned}\mu(1) &= \mu((\neg\neg x \wedge \neg(\neg\neg x \to x)) \to (\neg\neg x \odot (\neg\neg x \to \neg(\neg\neg x \to x)))) \\ &\leq \mu(\neg(\neg\neg x \odot (\neg\neg x \to \neg(\neg\neg x \to x))) \to \neg(\neg\neg x \wedge \neg(\neg\neg x \to x)))\end{aligned}$$

by using (2.3) and (2.18). Thus

$$\mu(\neg(\neg\neg x \odot (\neg\neg x \to \neg(\neg\neg x \to x))) \to \neg(\neg\neg x \wedge \neg(\neg\neg x \to x))) = \mu(1). \tag{3.25}$$

Since $\neg(\neg\neg x \to x) \leq \neg\neg x$ for all $x \in L$, it follows from (2.19), (2.20), (3.24) and (3.25) that

$$\mu(1) = \mu(\neg(\neg\neg x \wedge \neg(\neg\neg x \to x))) = \mu(\neg\neg(\neg\neg x \to x))$$

for all $x \in L$. Therefore μ is a strong fuzzy filter of $\mathcal{L}$. □

Corollary 3.17. *If a fuzzy filter μ of $\mathcal{L}$ satisfies one of conditions* (3.8)*,* (3.9)*,* (3.11)*,* (3.12)*,* (3.13) *and* (3.14)*, then μ is a strong fuzzy filter of $\mathcal{L}$.*

The following example shows that the converse of Theorem 3.16 may not be true in general.

Example 3.18. The strong fuzzy filter μ of $\mathcal{L}$ which is given in Example 3.14 is not a divisible fuzzy filter of $\mathcal{L}$ since $\mu((a \wedge c) \to (a \odot (a \to c))) = \mu(a) \neq \mu(1)$.

4. Conclusions

The filter theory plays an important role in studying logical systems and the related algebraic structures, and various filters have been proposed in the literature. Zhang et al. [8] introduced the notions of IMTL-filters (NM-filters, MV-filters) of residuated lattices, and presented their characterizations. Ma and Hu [4] introduced divisible filters, strong filters and n-contractive filters in residuated lattices.

In this paper, we have considered the fuzzification of divisible filters and strong filters in residuated lattices. We have defined divisible fuzzy filters and strong fuzzy filters, and have investigated related properties. We have discussed characterizations of a divisible and strong fuzzy filter, and have provided conditions for a fuzzy filter to be divisible. We have establish relations between a divisible fuzzy filter and a strong fuzzy filter. In a forthcoming paper, we will study the fuzzification of n-contractive filters in residuated lattices, and apply the results to other algebraic structures.

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FREQUENT HYPERCYCLICITY OF WEIGHTED COMPOSITION OPERATORS ON CLASSICAL BANACH SPACES

SHI-AN HAN AND LIANG ZHANG*

Abstract. In this paper we characterize the frequent hypercyclicity of weighted composition operators on some classical Banach spaces, such as the weighted Dirichlet space S_v. Besides, we also discuss the frequent hypercyclicity of the weighted composition operators on the weighted Bergman space A^p_α.

1. Introduction and terminology

Let $H(\mathbb{D})$ be the space of all holomorphic functions on $\mathbb{D}$, where $\mathbb{D}$ is the open unit disk of the complex plane $\mathbb{C}$. The collection of all holomorphic self-maps of $\mathbb{D}$ will be denoted by $S(\mathbb{D})$, and let $Aut(\mathbb{D})$ denote the set of all automorphisms on $\mathbb{D}$. The disk algebra, denoted by $A(\mathbb{D})$, consists of all functions in $H(\mathbb{D})$ that are continuous up to the boundary $\partial\mathbb{D}$ of the unit disk $\mathbb{D}$. Let dA denote the normalized Lebesegue measure on $\mathbb{D}$. The space of bounded analytic functions on $\mathbb{D}$ will be denoted by H^∞, with the sup norm $\|\cdot\|_\infty$.

For $\alpha > -1$ and $1 < p < \infty$, the weighted Bergman space A^p_α consists of analytic functions f such that

$$\|f\|^p = \int_{\mathbb{D}} |f(z)|^p \, d\nu_\alpha(z) < \infty,$$

where $d\nu_\alpha$ on $\mathbb{D}$ is defined by

$$d\nu_\alpha = (\alpha+1)\left(1-|z|^2\right)^\alpha d\nu(z)$$

and $\nu_\alpha(\mathbb{D}) = 1$. Under the norm $\|.\|$, A^p_α is a separable infinite dimensional Banach space, since the set of polynomials is dense in A^p_α.

For each real number v, the weighted Dirichlet space S_v is the space of holomorphic functions $f(z) = \sum_{n=0}^\infty a_n z^n, z \in \mathbb{D}$ such that the following norm

$$\|f\|_v^2 = \sum_{n=0}^\infty |a_n|^2 (n+1)^{2v}$$

is finite. Observe that the space S_v is Hilbert space, where the inner product is defined by

$$\langle f, g\rangle = \sum_{n=0}^\infty a_n \overline{b_n} (n+1)^{2v},$$

where $f(z) = \sum_{n=0}^\infty a_n z^n$ and $g(z) = \sum_{n=0}^\infty b_n z^n$. For instance, if $v = 0, -1/2, 1/2$, then S_v is, respectively, the classical Hardy space H^2, the Bergman space A^2, and the Dirichlet space $\mathcal{D}$.

By Lemma 1.2 in [5], we know the following expression

$$\|f\|^2 = \sum_{i=0}^{l} |f^{(i)}(0)| + \int_{\mathbb{D}} |f^{(l+1)}(z)|^2 (1-|z|^2)^{2l+1-2v} dA(z)$$

defines an equivalent norm on S_v, where $l \geq -1$ is an integer such that $v < l+1$, and when $l = -1$, the first term in the right hand side above does not appear.

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*Corresponding author.

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A bounded linear operator T acting on a separable Banach space X is said to be hypercyclic if there is an $f \in X$ such that orbit $\{T^n f\}_{n\geq 0}$ is dense in X. One bounded operator T is called similar to another bounded operator S on X if there exists a bounded and invertible operator V on H such that $TV = VS$. And the similarity preserve hypercyclicity. A continuous linear operator T acting on a separable Banach space X is said to be mixing, if for any pair U, V of nonempty open subsets of X, there exists some $N \geq 0$ such that

$$T^n(U) \cap (V) \neq \emptyset, \quad for \;\; all \; n \geq N.$$

The lower density of a subset A of $\mathbb{N}$ is defined as

$$\underline{dens}(A) = \liminf_{N\to\infty} \frac{card\{0 \leq n \leq N; n \in A\}}{N+1}.$$

A vector $x \in X$ is called frequently hypercyclic for T, if for every non-empty open subset U of X,

$$\underline{dens}\{n \in \mathbb{N}, T^n x \in U\} > 0.$$

The operator T is called frequently hypercyclic if it possesses a frequently hypercyclic vector. It is obvious that if the operator T is frequently hypercyclic, then T is hypercyclic. More related details can be founded in chapter 9 in the book [6].

Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$, the weighted composition operator uC_φ is defined as

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \;\; f \in H(\mathbb{D}), \; z \in \mathbb{D}.$$

And when $u \equiv 1$, we just have the composition operator C_φ and when $\varphi(z) = z$, we get the multiplication operator M_u.

For $\varphi \in LFT(\mathbb{D})$, we define φ as following:

$$\varphi(z) = \frac{az+b}{cz+d},$$

where $ad - bc \neq 0$.

Note that the linear fractional self-maps of $\mathbb{D}$ fall into distinct classes determined by their fixed point properties (see [1]). There are:

(a) Maps with interior fixed point. By the Schwarz Lemma the interior fixed point is either attractive, or the map is an elliptic automorphism.

(b) Parabolic maps. Its fixed point is on $\partial\mathbb{D}$, and the derivative $= 1$ at the fixed point.

(c) Hyperbolic maps with attractive fixed point on $\partial\mathbb{D}$ and their repulsive fixed point outside of $\mathbb{D}$. Both fixed points are on $\partial\mathbb{D}$ if and only if the map is the automorphism of $\mathbb{D}$. In this case, the derivative < 1 at the attractive fixed point.

According to a result by P.R. Hurst [8], the composition operator $C_\varphi : S_v \to S_v$ is bounded for any $v \in \mathbb{R}$ and any $\varphi \in LFT(\mathbb{D})$. In [4], the authors partially characterized the frequent hypercyclicity of scalar multiples of composition operators, whose symbols are linear fractional maps, acting on the weighted Dirichlet space S_v. E. Gallado and A. Montes [5] have furnished a complete characterization of the hypercyclicity of λC_φ on S_v in terms of λ, v, φ. Readers interested in related topics can refer to [3, 7, 9, 12, 13].

In this note, we will discuss the conditions of the frequent hypercyclicity of weighted composition operators on some classical Banach spaces, such as the weighted Dirichlet space S_v and the weighted Bergman space A^p_α.

2. Frequent hypercyclicity of uC_φ on S_v

In this section, we begin to discuss the frequent hypecyclicity of the weighted composition operator uC_φ on S_v.

Theorem 2.1. *If uC_φ is frequently hypercyclic on S_v, then φ is univalent and has no fixed point in $\mathbb{D}$, and $u(z) \neq 0$ for every $z \in \mathbb{D}$.*

Proof. It is well known that uC_φ is hypercyclic on S_v, so by Theorem 1 in [11], we obtain it. □

The following result can be found in [11, Theorem 2].

Theorem 2.2. *Let $v > 1/2$. Then*

(a) No weighted composition operator on S_v is hypercyclic.

(b) If φ has two fixed points α, β in $\overline{\mathbb{D}}$, and $u(\alpha) = u(\beta)$, then uC_φ is not cyclic on S_v.

Combining with the comparison principle, to discuss frequent hypecyclicity of the weighted composition operator uC_φ on S_v, we may assume without loss of generality that $0 \le v \le \frac{1}{2}$.

2.1. **The case for** $v = 0$**.** In general, composition operators are bounded on H^2 (see [2, Charpter 3]). M_u is also a bounded operator on S_v if $u \in H^\infty$. So when $v = 0$, $\varphi \in S(\mathbb{D})$ and $u \in H^\infty$, $uC_\varphi = M_u C_\varphi$.

According to the definition of [9], for any $w \in \partial\mathbb{D}$ and any positive number α, $Lip_\alpha(w)$ corresponds to the class of holomorphic functions φ such that there is some neighborhood G of w in $\partial\mathbb{D}$ and a positive constant M with

$$|\varphi(z) - \varphi(w)| \le M\,|z - w|^\alpha\,, \quad \text{for } z \in G.$$

For example, if an analytic function φ on $\mathbb{D}$ is also analytic at $w \in \partial\mathbb{D}$, then $\varphi \in Lip_\alpha(w)$ whenever $0 \le \alpha \le 1$. Moreover, if $\varphi'(\omega) = 0$, then $\varphi \in Lip_\alpha(w)$ whenever $0 \le \alpha \le 2$.

We have the following proposition.

Proposition 2.3. *Let $\varphi \in LFT(\mathbb{D})$, $w \in \partial\mathbb{D}$ be the Denjoy-Wolff point of φ, $u \in Lip_\alpha(w) \cap A(\mathbb{D})$, $\|u\|_\infty = |u(w)| \ne 0$. Then $u(w)$ is an eigenvalue for uC_φ, whenever φ is hyperbolic and $\alpha > 0$ or φ is parabolic automorphism and $\alpha > 1$. Moreover, if u never vanishes on $\overline{\mathbb{D}}$, then the eigenfunction also never vanishes.*

Proof. According to the proof of Propositioin 2.4 of [9], we have that the function $g(z) = \prod_{n=0}^{\infty} \frac{u(\varphi_n(z))}{u(w)}$ is a nonzero holomorphic function on $\mathbb{D}$. Since $\|u\|_\infty = |u(w)| \ne 0$, then for every $j \ge 0$ and $z \in \mathbb{D}$, $\left|\frac{u(\varphi_j(z))}{u(w)}\right| \le 1$. And note that for fixed $z \in \mathbb{D}$, $\prod_{j=0}^{n}\left|\frac{u(\varphi_j(z))}{u(w)}\right|$ is decreasing with respect to n. Therefore, $\|g\|_\infty = \sup_{z\in\mathbb{D}}\left|\prod_{n=0}^{\infty}\frac{u(\varphi_n(z))}{u(w)}\right| \le \sup_{z\in\mathbb{D}}\left|\frac{u(\varphi(z))}{u(w)}\right| \le 1$. That is, $g \in H^\infty \subset S_v$ and $u(z)\, g(\varphi(z)) = u(w)\, g(z)$. Thus $u(w)$ is an eigenvalue for uC_φ. Since $u(z) \ne 0$ for every $z \in \overline{\mathbb{D}}$, $g(z) \ne 0$ for $z \in \overline{\mathbb{D}}$. □

Next, we obtain the following result.

Theorem 2.4. *Let $\varphi \in LFT(\mathbb{D})$, $w \in \partial\mathbb{D}$ be the Denjoy-Wolff point of φ, $u \in Lip_\alpha(w) \cap A(\mathbb{D})$, $\|u\|_\infty = |u(w)| \ne 0$ and $u(z) \ne 0$ for every $z \in \overline{\mathbb{D}}$, then*

(a) If φ is hyperbolic automorphism, $\alpha > 0$ and $\varphi'(w)^{1/2} < |u(w)| < \varphi'(w)^{-1/2}$, then uC_φ is frequently hypecyclic on $H^2(\mathbb{D})$.

(b) If φ is parabolic automorphism, $\alpha > 1$ and $|u(w)| = 1$, then uC_φ is frequently hypecyclic on $H^2(\mathbb{D})$.

(c) If φ is hyperbolic non-automorphism, $\alpha > 0$ and $|u(w)| > \varphi'(w)^{1/2}$, then uC_φ is frequently hypecyclic on $H^2(\mathbb{D})$.

Proof. By the proof of Proposition 2.4, $g(z) = \prod_{n=0}^{\infty} \frac{u(\varphi_n(z))}{u(w)} \ne 0$ for $z \in \overline{\mathbb{D}}$, it is easy to see that M_g is a bounded operator on $H^2(\mathbb{D})$ and $uC_\varphi M_g = u(w)\, M_g C_\varphi$. Combining with the comparison principle, we obtain this theorem. □

2.2. **The case for** $0 < v < 1/2$**.** For $v \in (0, 1/2)$, using the equivalent norm in S_v, we define the Banach space Q_c as follows:

$$Q_c = \{f \in S_v : \|f\|_{Q_c} = |f(0)| + \sup_{w\in\mathbb{D}} \|f \circ \varphi_w - f\| < \infty\},$$

where $c = 1 - 2v$ and $\varphi_w(z) = (w - z)/(1 - \bar{w}z)$. For different $p \in (0, 1)$, $Q_{p_1} \subset Q_{p_2}$ when $0 < p_1 < p_2 \le 1$. In particular, $Q_1 = BMOA$, the bounded mean oscillation space of analytic functions and when $p > 1$, $Q_p = \mathcal{B}$, the Bloch space on $\mathbb{D}$.

Let $g \in Q_{1-2v}$, by Corollary 2 in [10], we know that if

$$\sup_{\zeta\in\partial\mathbb{D}} \int_{D(\zeta,r)} |g(z)|^2(1-|z|)^{1-2v} dA(z) = O(r^{3-2v}), \tag{2.1}$$

then M_g is bounded on S_v.

Thus we get the following theorems.

Theorem 2.5. *Let $0 < v < 1/2$ and $\alpha > 0$. And let $\varphi \in LFM(\mathbb{D})$ and φ be a hyperbolic automorphism of the unit disc with Denjoy-Wolff point $w \in \partial\mathbb{D}$, $u \in Lip_\alpha(w)$ and $u(w) \neq 0$, the function $g = \prod_{i=0}^{\infty} \frac{1}{u(w)} u(\varphi_i(w)) \in Q_{1-2v}$, $\|I - M_g\|_{S_v\to S_v} < 1$ and (2.1) holds, then the following are equivalent:*

(a) uC_φ is frequently hypercyclic.

(b) uC_φ is hypercyclic.

(c) $\varphi'(w)^{(1-2v)/2} < |u(w)| < \varphi'(w)^{(2v-1)/2}$.

Proof. The implication $(a) \Rightarrow (b)$ is trivial. If $\varphi \in LFM(\mathbb{D})$ with Denjoy-Wolff point $w \in \partial\mathbb{D}$ and $u \in Lip_\alpha(w)$, $u(w) \neq 0$, as we saw in the proof of Proposition 2.4 in [9], the map $g(z) = \prod_{i=0}^{\infty} \frac{1}{u(w)} u(\varphi_i(w))$ is a nonzero holomorphic function satisfying $uC_\varphi g = u(w)g$.

Since $g \in Q_{1-2v}$ and (2.1) holds, we have M_g is bounded operator on S_v, so $g \in S_v$, thus the function g is an eigenfunction of uC_φ corresponding to $u(w)$ on S_v, and $uC_\varphi M_g = u(w)M_g C_\varphi$.

Note that $\|I - M_g\|_{S_v\to S_v} \leq 1 + \|M_g\|_{S_v\to S_v}$. So $I - M_g$ is also a bounded operator on S_v. Because $\|I - M_g\|_{S_v\to S_v} < 1$, then M_g is a invertible operator. It is obvious that $(b) \Leftrightarrow (c)$. Besides, suppose that the condition (c) holds, by the proof of Theorem 2.6 in [4], $u(w)C_\varphi$ satisfies the Frequent Hypercyclicity Criterion. The implication $(c) \Rightarrow (a)$ is obvious. □

2.3. **The case for $v = 1/2$.** If so, we know that S_v is the Dirichlet space $\mathcal{D}$.

Theorem 2.6. *Let $\varphi \in LFT(\mathbb{D})$, $\alpha > 1$, $w \in \partial\mathbb{D}$ be the Denjoy-Wolff point of φ, $u \in Lip_\alpha(w) \cap A(\mathbb{D})$, $\|u\|_\infty = |u(w)| > 1$ and $u(z) \neq 0$ for every $z \in \overline{\mathbb{D}}$. If φ is hyperbolic non-automorphism, then uC_φ is frequently hypecyclic on the Dirichlet space $\mathcal{D}$.*

Proof. By the proof of Proposition 2.4, $g(z) = \prod\limits_{n=0}^{\infty} \frac{u(\varphi_n(z))}{u(w)} \neq 0$ for $z \in \overline{\mathbb{D}}$, Since $\|u\|_\infty = |u(w)| > 1$, so $g \in H^\infty \subset \mathcal{D}$ and $u(z)\, g(\varphi(z)) = u(w)\, g(z)$. It is easy to see that M_g is a bounded operator on the Dirichlet space $\mathcal{D}$ and $uC_\varphi M_g = u(w) M_g C_\varphi$. By Theorem 1.8 in [5] and the comparison principle, we complete the proof. □

3. Frequent hypercyclicity of uC_φ on A^p_α

In this section, we study in detail frequent hypercyclicity of uC_φ on the weighted Bergman space A^p_α and we suppose that the weighted composition operator uC_φ is bounded on A^p_α.

Proposition 3.1. *Let $\alpha > -1$, $1 < p < \infty$ and $\varphi \in LFT(\mathbb{D})$. If uC_φ is frequently hypercyclic on A^p_α, then*

(i) φ has no fixed point in $\mathbb{D}$ and φ is univalent.

(ii) $u(z) \neq 0$ for every $z \in \mathbb{D}$.

Proof. The proof is obvious, so we omit it. □

Next, we obtain the following results.

Theorem 3.2. *Let $\alpha > -1$, $\beta > 0$, $1 < p < \infty$, $\varphi \in LFT(\mathbb{D})$ and φ be a hyperbolic automorphism and $w \in \partial\mathbb{D}$ be the Denjoy-Wolff point of φ, $u \in Lip_\beta(w) \cap A(\mathbb{D})$, $\|u\|_\infty = |u(w)| \neq 0$ and $u(z) \neq 0$ for every $z \in \overline{\mathbb{D}}$. If $\varphi'(w)^{(2+\alpha)/p} < |u(w)| < \varphi'(w)^{(-2-\alpha)/p}$, then uC_φ is frequently hypecyclic on A^p_α.*

Proof. First, since this space under consideration is unitarily invariant, we may assume that 1 and -1 are fixed points of φ and 1 is the attractive fixed point. The change of variables

$$\sigma(z) = \frac{i(1-z)}{1+z}$$

takes the unit disk onto the upper half plane, 1 and -1 to 0 and ∞. We obtain that φ is conjugate to the translation map

$$\psi(z) = \rho z,$$

where $0 < \rho < 1$. By using the equation $\sigma \circ \varphi = \psi \circ \sigma$, we can get

$$\varphi(z) = \frac{(1+\rho)z + 1 - \rho}{(1-\rho)z + 1 + \rho},$$

where $\varphi'(1) = \rho$.

Let X_0 denote the subspace of polynomials vanishing m at 1, where $m > \frac{2(\alpha+2)}{p}$. It is obvious that X_0 is dense on A^p_α. Fix $f \in X_0$. It is similarly proved as in Theorem 3.5 in [5] that

$$\left\|\lambda^n C^n_\varphi f\right\|^p \leq C\left|\lambda\right|^{np} \rho^{(\alpha+2)n}, n \in \mathbb{N},$$

where C is a constant independent of n. If $\varphi'(1)^{(2+\alpha)/p} < |\lambda| < \varphi'(1)^{(-2-\alpha)/p}$, we obtain that

$$\sum_{n=1}^{\infty} \left\|(\lambda C_\varphi)^n f\right\| < \infty, \ for\ all\ \ f \in X_0. \tag{3.1}$$

Similarly, let Y_0 denote the subspace of polynomials vanishing m at -1 and Y_0 is dense on A^p_α. We take $S = (\lambda C_\varphi)^{-1}$. Observe that -1 is the attractive fixed point of φ^{-1} with $\left(\varphi^{-1}\right)'(-1) = \frac{1}{\varphi'(-1)} = \rho$ and $\varphi'(1)^{(2+\alpha)/p} < |\lambda| < \varphi'(1)^{(-2-\alpha)/p}$. Therefore, a similar argument leads to

$$\sum_{n=1}^{\infty} \left\|S^n f\right\| < \infty, \ for\ all\ \ f \in Y_0. \tag{3.2}$$

If we set $X := X_0 \cap Y_0$, then we obtain that X is dense in A^p_α. Clearly (3.1) and (3.2) hold for all $f \in X$. It is obvious that $\lambda C_\varphi S$ is the identity on X. Consequently, λC_φ satisfies the Frequent Hypercyclicity Criterion. By Proposition 2.4, then uC_φ is frequently hypecyclic on A^p_α. □

Theorem 3.3. *Let $\alpha > -1$, $\beta > 0$, $1 < p < \infty$, $\varphi \in LFT(\mathbb{D})$ and φ is a hyperbolic non-automorphism, $w \in \partial\mathbb{D}$ be the Denjoy-Wolff point of φ, $u \in Lip_\beta(w) \cap A(\mathbb{D})$, $\|u\|_\infty = |u(w)| \neq 0$ and $u(z) \neq 0$ for every $z \in \overline{\mathbb{D}}$. If $|u(w)| > \varphi'(\zeta)^{(2+\alpha)/p}$, then uC_φ is frequently hypecyclic on A^p_α.*

Proof. First, we prove that if $|\lambda| > \varphi'(w)^{(2+\alpha)/p}$, then λC_φ is frequently hypecyclic on A^p_α.

Now, we assume that $w = 1$ is the boundary fixed point and β is a exterior fixed point. Upon conjugating with an appropriate map, φ is conjugate to

$$\rho z + 1 - \rho,$$

where $0 < \rho < 1$. Hence we may assume that $\varphi(z) = \rho z + 1 - \rho$, where $\varphi'(1) = \rho$. For any $n \in \mathbb{N}$, we have

$$\varphi_n(z) = \rho^n z + 1 - \rho^n. \tag{3.3}$$

Let X_0 denote the subspace of polynomials vanishing m at 1, where m is to be determined later on. Obviously, X_0 is dense on A^p_α. Fix $f \in X_0$. It is similarly proved as in Theorem 2.11 in [5] that

$$\left\|\lambda^n C^n_\varphi f\right\|^p \leq C\left|\lambda\right|^{np} \rho^{mnp}, n \in \mathbb{N},$$

where C is a constant independent of n. Since $0 < \rho < 1$, we can choose m large enough to have $|\lambda\rho^m| < 1$. By the assumption, we obtain that

$$\sum_{n=1}^{\infty} \left\|(\lambda C_\varphi)^n f\right\| < \infty, \ for\ all\ \ f \in X_0. \tag{3.4}$$

Define $T = \lambda C_\varphi$ and the inverse $S = \lambda^{-1} C_{\varphi_{-1}}$. Let Y be the set of all polynomials that vanish m times at β where m will be suitable number. The set Y_0 will be

$$Y_0 = \bigcup_{n=0}^{\infty} \lambda^{-n} C^n_{\varphi_{-1}}(Y) = \bigcup_{n=0}^{\infty} \lambda^{-n} C_{\varphi_{-n}}(Y).$$

Similarly, we obtain that for n large enough

$$\left\|\lambda^{-n} C_{\varphi_{-n}} f\right\|^p \leq C\,|\lambda|^{-np}\,\rho^{n(\alpha+2)},$$

where C is a constant independent of n. By the assumption, we have

$$\sum_{n=1}^{\infty} \|S^n f\| < \infty, \ for\ all\ \ f \in Y_0. \tag{3.5}$$

If we set $X := \cup_{n=0}^{\infty} S^n (X \cap Y)$, then we obtain that X is dense in A^p_α. Clearly (3.4) and (3.5) hold for all $f \in X$. It is obvious that $\lambda C_\varphi S$ is the identity on X. Consequently, λC_φ satisfies the Frequent Hypercyclicity Criterion. By Proposition 2.4, then uC_φ is frequently hypecyclic on A^p_α. □

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Shi-An Han
Department of Mathematics
Tianjin University
Tianjin, 300072
P.R. China.
E-mail address: hsatju@163.com, 844709515@qq.com

Liang Zhang
School of Marine Science and Technology
Tianjin University
Tianjin, 300072
P.R. China.
E-mail address: 168zhangliang2011@163.com

ON THE SPECIAL TWISTED q-POLYNOMIALS

JIN-WOO PARK

Abstract. In this paper, we found some interesting identities of q-extension of special twisted polynomials which are derive from the bosonic q-integral and fermionic q-integral on $\mathbb{Z}_p$.

1. Introduction

Let p be a given odd prime number. Throughout this paper, we assume that $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the rings of p-adic integers, the fields of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized by $|p|_p = \frac{1}{p}$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the *bosonic p-adic q-integral on $\mathbb{Z}_p$* is defined by T. Kim to be

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \text{ (see [9, 10])}, \tag{1.1}$$

and the *fermionic p-adic q-integral* on $\mathbb{Z}_p$ is also defined by Kim to be

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{ (see [9, 11])}. \tag{1.2}$$

Let $f_1(x) = f(x+1)$. Then, by (1.1) and (1.2), we get

$$qI_q(f_1) - I_q(f) = (q-1)f(0) + \frac{q-1}{\log q} f'(0), \tag{1.3}$$

and

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \tag{1.4}$$

where $f'(0) = \frac{d}{dx} f(x)\big|_{x=0}$ (see [9, 10, 11]).

It is well known that the *q-Bernoulli polynomials* are defined by the generating function to be

$$\frac{q - 1 + \frac{(q-1)t}{\log q}}{qe^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!}, \tag{1.5}$$

and the *q-Euler polynomials* are given by

$$\frac{[2]_q}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \tag{1.6}$$

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When $x = 0$, $B_{n,q} = B_{n,q}(0)(E_{n,q} = E_{n,q}(0))$ are called the *nth q-Bernoulli numbers*(*nth q-Euler numbers*, respectively)(see [7, 8, 14, 16]).

The *Stirling numbers of the first kind* are defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l,\ (n \geq 0),$$

and the *Stirling numbers of the second kind* are defined by

$$(e^t-1)^n = n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!}, \text{(see [1, 12])}.$$

The *Daehee polynomials of the first kind* are defined by the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!} \text{ (see [4, 5])}.$$

Recently, the q-Daehee polynomials are defined by the generating function to be

$$\left(\frac{1-q+\frac{1-q}{\log q}}{1-q-qt}\right)(1+t)^x = \sum_{n=0}^{\infty} D_{n,q}(x)\frac{t^n}{n!},\ \text{(see [2, 13])}, \tag{1.7}$$

and the *q-Changhee polynomials* are defined by the generating function to be

$$\frac{[2]_q}{[2]_q+qt}(1+t)^x = \sum_{n=0}^{\infty} Ch_{n,q}(x)\frac{t^n}{n!},\ \text{(see [3])} \tag{1.8}$$

where $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$. When $x = 0$, $D_{n,q} = D_{n,q}(0)(Ch_{n,q} = Ch_{n,q}(0))$ are called the *nth q-Daehee numbers*(*nth q-Changhee numbers*, respectively).

The Daehee polynomials and Changhee polynomials are introduced by T. Kim et. al. in [4, 6], and found interesting identities in [2, 4, 5, 6, 13, 15, 16]. In this paper, we found some interesting identities of q-extension of special twisted polynomials which are derive from the bosonic q-integral and fermionic q-integral on $\mathbb{Z}_p$.

2. Twisted q-Daehee numbers and polynomials of higher-order

For $n \in \mathbb{N}$, let T_p be the p-adic locally constant space defined by

$$T_p = \bigcup_{n\geq 1} C_{p^n} = \lim_{n\to\infty} C_{p^n},$$

where $C_{p^n} = \{\omega | \omega^{p^n} = 1\}$ is the cyclic group of order p^n.

In this section, we assume that $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$. We define the *higher order q-Beroulli polynomials* as follows:

$$\left(\frac{q-1+\frac{q-1}{\log q}t}{qe^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(r)}(x)\frac{t^n}{n!}. \tag{2.1}$$

When $x = 0$, $B_{n,q}^{(r)}(0) = B_{n,q}^{(r)}$ are called the *higher order q-Bernoulli numbers.*

For $\varepsilon \in T_p$, we consider the *twisted q-Daehee polynomials of order* r as follows:

$$\left(\frac{q-1+\frac{q-1}{\log q}\log(1+\epsilon t)}{q\epsilon t+q-1}\right)^r (1+\epsilon t)^x = \sum_{n=0}^{\infty} D_{n,\epsilon,q}^{(r)}(x)\frac{t^n}{n!}. \tag{2.2}$$

When $x = 0$, $D_{n,\epsilon,q}^{(r)}(0) = D_{n,\epsilon,q}^{(r)}$ are called *twisted q-Daehee numbers of order* r.

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From (1.1), we can obtain the equation:

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\epsilon t)^{x_1+\cdots+x_r+x}d\mu_q(x_1)\cdots d\mu_q(x_r)\\&=\left(\frac{q-1+\frac{q-1}{\log q}\log(1+\epsilon t)}{q\epsilon t+q-1}\right)^r(1+\epsilon t)^x\\&=\sum_{n=0}^{\infty}D_{n,\epsilon,q}^{(r)}(x)\frac{t^n}{n!}.\end{aligned}\tag{2.3}$$

By (2.3), we get

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\epsilon^n\binom{x_1+\cdots+x_r+x}{n}d\mu_q(x_1)\cdots d\mu_q(x_r)=\frac{D_{n,\epsilon,q}^{(r)}(x)}{n!}\ (n\geq 0).\tag{2.4}$$

By replacing t by $\frac{1}{\epsilon}(e^t-1)$ in (2.3), we have

$$\sum_{n=0}^{\infty}D_{n,\epsilon,q}^{(r)}(x)\frac{\left(\frac{1}{\epsilon}(e^t-1)\right)^n}{n!}=\left(\frac{q-1+\frac{q-1}{\log q}t}{qe^t-1}\right)^r e^{xt}=\sum_{n=0}^{\infty}B_{n,q}^{(r)}(x)\frac{t^n}{n!}\tag{2.5}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}D_{n,\epsilon,q}^{(r)}(x)\frac{1}{\epsilon^n n!}\left(e^t-1\right)^n&=\sum_{n=0}^{\infty}D_{n,\epsilon,q}^{(r)}(x)\frac{1}{\epsilon^n n!}n!\sum_{m=n}^{\infty}S_2(m,n)\frac{t^m}{m!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{D_{m,\epsilon,q}^{(r)}(x)S_2(n,m)}{\epsilon^n}\right)\frac{t^n}{n!}.\end{aligned}\tag{2.6}$$

Thus, by (2.5) and (2.6), we have

$$B_{n,q}^{(r)}(x)=\sum_{m=0}^{n}\frac{D_{m,\epsilon,q}^{(r)}(x)S_2(n,m)}{\epsilon^n}.\tag{2.7}$$

Therefore, by (2.4) and (2.7), we obtain the following theorem.

Theorem 2.1. *For $n\geq 0$, we have*

$$B_{n,q}^{(r)}(x)=\sum_{m=0}^{n}\frac{D_{m,\epsilon,q}^{(r)}(x)S_2(n,m)}{\epsilon^n}$$

and

$$\frac{D_{n,\epsilon,q}^{(r)}(x)}{n!}=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\epsilon^n\binom{x_1+\cdots+x_r+x}{n}d\mu_q(x_1)\cdots d\mu_q(x_r)$$

where $S_2(m,n)$ is the Stirling number of the second kind.

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From (2.1), by replacing t by $\log(1+\epsilon t)$, we have

$$\begin{aligned}&\left(\frac{q-1+\frac{q-1}{\log q}\log(1+\epsilon t)}{q\epsilon t+q-1}\right)^r(1+\epsilon t)^x\\&=\sum_{n=0}^{\infty}B_{n,q}^{(r)}(x)\frac{1}{n!}\left(\log(1+\epsilon t)\right)\\&=\sum_{n=0}^{\infty}B_{n,q}^{(r)}(x)\frac{1}{n!}n!\sum_{m=n}^{\infty}S_1(m,n)\frac{(\epsilon t)^m}{m!}\\&=\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\epsilon^m S_1(m,n)B_{n,q}^{(r)}(x)\right)\frac{t^m}{m!},\end{aligned}\tag{2.8}$$

where $S_1(m,n)$ is the Stirling number of the first kind. Thus, by (2.2) and (2.8), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$D_{n,\epsilon,q}^{(r)}(x)=\sum_{n=0}^{m}\epsilon^m S_1(m,n)B_{n,q}^{(r)}(x).$$

Now, we consider the *q-Changhee polynomials of order r* which are defined by the generating function as follows:

$$\frac{[2]_q}{q\varepsilon t+[2]_q}(1+\varepsilon t)^x=\sum_{n=0}^{\infty}Ch_{n,\varepsilon,q}^{(r)}(x)\frac{t^n}{n!}.\tag{2.9}$$

In the special case $x=0$, $Ch_{n,\epsilon,q}^{(r)}(0)=Ch_{n,\epsilon,q}^{(r)}$ are called the *q-Changhee numbers of order r*.

From (1.2), we note that

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\epsilon t)^{x_1+\cdots+x_r+x}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\\&=\left(\frac{[2]_q}{q\varepsilon t+[2]_q}\right)^r(1+\varepsilon t)^x.\end{aligned}\tag{2.10}$$

By (2.10), we have

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\epsilon^n\binom{x_1+\cdots+x_r+x}{n}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)=\frac{Ch_{n,\epsilon,q}^{(r)}(x)}{n!}.\tag{2.11}$$

In view of (1.6), we define the *higher order q-Euler polynomials* by generating function to be

$$\left(\frac{[2]_q}{qe^t+1}\right)^r e^{xt}=\sum_{n=0}^{\infty}E_{n,q}^{(r)}(x)\frac{t^n}{n!}.\tag{2.12}$$

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From (2.10), we note that

$$\begin{aligned}
&\int_{\mathbb{Z}_p}(1+\epsilon t)^{x_1+\cdots+x_r+x}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\\
&=\left(\frac{[2]_q}{qe^{\log(1+\epsilon t)}+1}\right)^r e^{x\log(1+\varepsilon t)}\\
&=\sum_{n=0}^{\infty}E_{n,q}^{(r)}\frac{1}{n!}\left(\log(1+\epsilon t)\right)^n\\
&=\sum_{n=0}^{\infty}E_{n,q}^{(r)}(x)\frac{1}{n!}n!\sum_{m=n}^{\infty}S_1(m,n)\frac{(\epsilon t)^m}{m!}\\
&=\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\epsilon^m E_{n,q}^{(r)}(x)S_1(m,n)\right)\frac{t^m}{m!}.
\end{aligned}\tag{2.13}$$

Hence, by (2.11) and (2.13), we obtain the following theorem.

Theorem 2.3. *For $n\geq 0$, we have*

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\epsilon^n\binom{x_1+\cdots+x_r+x}{n}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r)\\
=&\frac{Ch_{n,\epsilon,q}^{(r)}(x)}{n!}=\frac{1}{n!}\sum_{m=0}^{n}\epsilon^n E_{m,q}^{(r)}(x)S_1(n,m).
\end{aligned}$$

By replacing t by $\frac{1}{\epsilon}(e^t-1)$ in (2.9), we have

$$\sum_{n=0}^{\infty}Ch_{n,\epsilon,q}^{(r)}(x)\frac{(e^t-1)^n}{\epsilon^n n!}=\left(\frac{[2]_q}{qe^t+1}\right)^r e^{xt}\tag{2.14}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty}\epsilon^{-n}Ch_{n,\epsilon,q}^{(r)}(x)\frac{1}{n!}\left(e^t-1\right)^n&=\sum_{n=0}^{\infty}\epsilon^{-n}Ch_{n,\epsilon,q}^{(r)}(x)\sum_{m=n}^{\infty}S_2(m,n)\frac{t^m}{m!}\\
&=\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty}\epsilon^{-n}Ch_{n,\epsilon,q}^{(r)}(x)S_2(m,n)\right)\frac{t^m}{m!}
\end{aligned}\tag{2.15}$$

By (2.12), (2.14) and (2.15), we obtain the following theorem.

Theorem 2.4. *For $n\geq 0$, we have*

$$E_{n,q}^{(r)}(x)=\sum_{m=0}^{n}\epsilon^{-m}Ch_{m,\epsilon,q}^{(r)}(x)S_2(n,m).$$

From now on, we consider the q-analogue of the *twisted Cauchy polynomials of order* r, which are defined by the generating function to be

$$\left(\frac{q(1+\epsilon t)-1}{(q-1)+\frac{q-1}{\log q}\log(1+\epsilon t)}\right)^r(1+\epsilon t)^x=\sum_{n=0}^{\infty}C_{n,\epsilon,q}^{(r)}(x)\frac{t^n}{n!}.\tag{2.16}$$

In the special case $x=0$, $C_{n,\epsilon,q}^{(r)}(0)=C_{n,\epsilon,q}^{(r)}$ are called the *twisted Cauchy numbers of order* r. Note that

$$\begin{aligned}&\lim_{q\to1}\left(\frac{q(1+\epsilon t)-1}{(q-1)+\frac{q-1}{\log q}\log(1+\epsilon t)}\right)^r(1+r)^x\\&=\left(\frac{\epsilon t}{\log(1+\epsilon t)}\right)^r(1+t)^x=\sum_{n=0}^{\infty}C_{n,\epsilon}^{(r)}(x)\frac{t^n}{n!},\end{aligned}\tag{2.17}$$

where $C_{n,\epsilon}^{(r)}$ are called the *Cauchy polynomials of order* r.

By (2.2), we can derive the followings:

$$\begin{aligned}(1+\epsilon t)^x&=\left(\frac{q(1+\epsilon t)-1}{(q-1)+\frac{q-1}{\log q}\log(1+\epsilon t)}\right)^r(1+\epsilon t)^x\left(\frac{(q-1)+\frac{q-1}{\log q}\log(1+\epsilon t)}{q(1+\epsilon t)-1}\right)^r\\&=\left(\sum_{k=0}^{\infty}C_{n,\epsilon,q}^{(r)}\right)\left(\sum_{m=0}^{\infty}D_{m,\epsilon,q}^{(r)}\frac{t^m}{m!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}C_{l,\epsilon,q}^{(r)}(x)D_{n-l,\epsilon,q}^{(r)}\right)\frac{t^n}{n!}\end{aligned}\tag{2.18}$$

and

$$(1+\epsilon t)^x=\sum_{n=0}^{\infty}\epsilon^n(x)_n\frac{t^n}{n!}.\tag{2.19}$$

By (2.18) and (2.19), we obtain the following theorem.

Theorem 2.5. *For* $n\geq0$*, we have*

$$\binom{x}{n}=\frac{1}{\epsilon^n n!}\sum_{l=0}^{n}\binom{n}{l}C_{l,\epsilon,q}^{(r)}(x)D_{n-l,\epsilon,q}^{(r)}.$$

Let n be a given nonnegative integer. In [2], authors defined *q-analogue of the Bernoulli-Euler mixed-type polynomials of order* (r,s) $BE_{n,q}^{(r,s)}(x)$, and derived the following equation.

$$\sum_{n=0}^{\infty}BE_{n,q}^{(r,s)}(x)\frac{t^n}{n!}=\left(\frac{[2]_q}{qe^t+1}\right)^s\left(\frac{q-1+\frac{q-1}{\log q}t}{qe^t-1}\right)^r e^{xt}.\tag{2.20}$$

By replacing t by $\log(1+\epsilon t)$, we get

$$\begin{aligned}&\sum_{n=0}^{\infty}BE_{n,q}^{(r,s)}(x)\frac{(\log(1+\epsilon t))^n}{n!}\\&=\left(\frac{[2]_q}{q(1+\epsilon t)+1}\right)^s\left(\frac{q-1+\frac{q-1}{\log q}\log(1+\epsilon t)}{q(1+\epsilon t)-1}\right)^r(1+\epsilon t)^x\\&=\left(\sum_{m=0}^{\infty}Ch_{m,\epsilon,q}^{(s)}(x)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}D_{l,\epsilon,q}^{(r)}\frac{t^l}{l!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}Ch_{l,\epsilon,q}^{(s)}(x)D_{n-l,\epsilon,q}^{(r)}\right)\frac{t^n}{n!},\end{aligned}\tag{2.21}$$

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and

$$\sum_{n=0}^{\infty} BE_{n,q}^{(r,s)}(x)\frac{(\log(1+\epsilon t))^n}{n!} = \sum_{n=0}^{\infty}\left(\epsilon^n \sum_{m=0}^{n} BE_{m,q}^{(r,s)}(x)S_1(n,m)\right)\frac{t^m}{m!}. \tag{2.22}$$

Thus, by (2.21) and (2.22), we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, we have*

$$\sum_{l=0}^{n}\binom{n}{l} Ch_{l,\epsilon,q}^{(s)}(x)D_{n-l,\epsilon,q}^{(r)} = \epsilon^n \sum_{m=0}^{n} BE_{m,q}^{(r,s)}(x)S_1(n,m).$$

From now on, we consider the q-analogue of the *twisted Daehee-Changhee mixed-type polynomials of order* (r,s) as follows:

$$DC_{n,\epsilon,q}^{(r,s)}(x) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} D_{n,\epsilon,q}^{(r)}(x+y_1+\cdots+y_s)d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_s) \tag{2.23}$$

where n is a given nonnegative integer.

By (2.23), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} DC_{n,\epsilon,q}^{(r,s)}(x)\frac{t^n}{n!}\\
&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} D_{n,\epsilon,q}^{(r)}(x+y_1+\cdots+y_s)\frac{t^n}{n!}d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_s)\\
&=\left(\frac{q-1+\frac{q-1}{\log q}\log(1+\epsilon t)}{q\epsilon t+q-1}\right)^r \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\epsilon t)^{x+y_1+\cdots+y_s}d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_s)\\
&=\left(\frac{q-1+\frac{q-1}{\log q}\log(1+\epsilon t)}{q\epsilon t+q-1}\right)^r\left(\frac{[2]_q}{q\varepsilon t+[2]_q}\right)^s(1+\varepsilon t)^x\\
&=\left(\sum_{n=0}^{\infty} D_{n,\epsilon,q}^{(r)}(x)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} Ch_{m,\epsilon,q}^{(s)}\frac{t^m}{m!}\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\binom{n}{m}D_{m,\epsilon,q}^{(r)}(x)Ch_{m-n,\epsilon,q}^{(s)}\right)\frac{t^n}{n!}
\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} DC_{n,\epsilon,q}^{(r,s)}(x)\frac{\left(\frac{1}{\epsilon}(e^t-1)\right)^n}{n!} &= \left(\frac{q-1+\frac{q-1}{\log q}t}{qe^t-1}\right)^r\left(\frac{[2]_q}{qe^t+1}\right)^s e^{xt}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}B_{m,q}^{(r)}(x)E_{n-m,q}^{(s)}\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.25}$$

Note that

$$\begin{aligned}
\sum_{n=0}^{\infty} DC_{m,\epsilon,q}^{(r,s)}(x)\frac{\left(\frac{1}{\epsilon}(e^t-1)\right)^n}{n!} &= \sum_{n=0}^{\infty} DC_{n,\epsilon,q}^{(r,s)}(x)\frac{1}{\epsilon^n n!}n!\sum_{m=n}^{\infty}S_2(m,n)\frac{t^m}{m!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\epsilon^{-m}DC_{m,\epsilon,q}^{(r,s)}(x)S_2(n,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.26}$$

Hence, by (2.24), (2.25) and (2.26), we obtain the following theorem.

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Theorem 2.7. *For $n \geq 0$, we get*

$$DC_{n,\epsilon,q}^{(r,s)}(x) = \sum_{m=0}^{\infty} \binom{n}{m} D_{m,\epsilon,q}^{(r)}(x) Ch_{m-n,\epsilon,q}^{(s)}$$

and

$$\sum_{m=0}^{n} \binom{n}{m} B_{m,q}^{(r)}(x) E_{n-m,q}^{(s)} = \sum_{m=0}^{n} \epsilon^{-m} DC_{m,\epsilon,q}^{(r,s)}(x) S_2(n,m).$$

Now, we consider the q-analogue of the *twisted Cauchy-Changhee mixed-type polynomials of order* (r,s) as follows:

$$CC_{n,\epsilon,q}^{(r,s)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} C_{n,\epsilon,q}^{(r)}(x+y_1+\cdots+y_s) d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_s) \tag{2.27}$$

where n is a given nonnegative integer.

By (2.27), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} CC_{n,\epsilon,q}^{(r,s)}(x) \frac{t^n}{n!} \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} C_{n,\epsilon,q}^{(r)}(x+y_1+\cdots+y_s) \frac{t^n}{n!} d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_s) \\
&= \left(\frac{q(1+\epsilon t)-1}{(q-1)+\frac{q-1}{\log q}\log(1+\epsilon t)} \right)^r \left(\frac{[2]_q}{q\varepsilon t+[2]_q} \right)^s (1+\varepsilon t)^x \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \binom{n}{m} C_{m,\epsilon,q}^{(r)}(x) Ch_{m-n,\epsilon,q}^{(s)} \right) \frac{t^n}{n!}
\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} CC_{n,\epsilon,q}^{(r,s)}(x) \frac{\left(\frac{1}{\epsilon}(e^t-1)\right)^n}{n!} &= \left(\frac{qe^t-1}{q-1+\frac{q-1}{\log q}t} \right)^r \left(\frac{[2]_q}{qe^t+1} \right)^s e^{xt} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} B_{m,q}^{(-r)}(x) E_{n-m,q}^{(s)} \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.29}$$

Now, we observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} CC_{m,\epsilon,q}^{(r,s)}(x) \frac{\left(\frac{1}{\epsilon}(e^t-1)\right)^n}{n!} &= \sum_{n=0}^{\infty} CC_{n,\epsilon,q}^{(r,s)}(x) \frac{1}{\epsilon^n n!} n! \sum_{m=n}^{\infty} S_2(m,n) \frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \epsilon^{-m} CC_{m,\epsilon,q}^{(r,s)}(x) S_2(n,m) \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.30}$$

Therefore, by (2.28), (2.29) and (2.30), we obtain the following theorem.

Theorem 2.8. *For $n \geq 0$, we have*

$$CC_{n,\epsilon,q}^{(r,s)}(x) = \sum_{m=0}^{\infty} \binom{n}{m} C_{m,\epsilon,q}^{(r)}(x) Ch_{m-n,\epsilon,q}^{(s)}$$

and

$$\sum_{m=0}^{n} \binom{n}{m} B_{m,q}^{(-r)}(x) E_{n-m,q}^{(s)} = \sum_{m=0}^{n} \epsilon^{-m} CC_{m,\epsilon,q}^{(r,s)}(x) S_2(n,m).$$

From now on, we consider the q-analogue of *twisted Cauchy-Daehee mixed-type polynomials of order* (r,s) as follows:

$$CD_{n,\epsilon,q}^{(r,s)}(x)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}C_{n,\epsilon,q}^{(r)}(x+y_1+\cdots+y_s)d\mu_q(y_1)\cdots d\mu_q(y_s) \tag{2.31}$$

where n is a given nonnegative integer.

By (2.31), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}CD_{n,\epsilon,q}^{(r,s)}(x)\frac{t^n}{n!}\\
&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{n=0}^{\infty}C_{n,\epsilon,q}^{(r)}(x+y_1+\cdots+y_s)\frac{t^n}{n!}d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_s)\\
&=\left(\frac{q(1+\epsilon t)-1}{(q-1)+\frac{q-1}{\log q}\log(1+\epsilon t)}\right)^r\left(\frac{q-1+\frac{q-1}{\log q}\log(1+\epsilon t)}{q\epsilon t+q-1}\right)^s(1+\varepsilon t)^x \qquad (2.32)\\
&=\begin{cases}\sum_{n=0}^{\infty}C_{n,\epsilon,q}^{(r-s)}(x)\frac{t^n}{n!} & \text{if } r>s,\\ \sum_{n=0}^{\infty}D_{n,\epsilon,q}^{(s-r)}(x)\frac{t^n}{n!} & \text{if } r<s,\\ \sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!} & \text{if } r=s.\end{cases}
\end{aligned}$$

Thus, by (2.32), we obtain the following theorem.

Theorem 2.9. *For $n\geq 0$, we have*

$$CD_{n,\epsilon,q}^{(r,s)}=\begin{cases}C_{n,\epsilon,q}^{(r-s)}(x) & \textit{if } r>s,\\ D_{n,\epsilon,q}^{(s-r)}(x) & \textit{if } r<s,\\ (x)_n & \textit{if } r=s.\end{cases}$$

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Department of Mathematics Education, Daegu University, Gyeongsan-si, Gyeongsangbuk-do, 712-714, Republic of Korea.

E-mail address: a04170001@knu.ac.kr

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Equicontinuity of Maps on $[0, 1)$

Kesong Yan, Fanping Zeng and Bin Qin*

School of Information and Statistics
Guangxi University of Finance and Economics
Nanning, Guangxi, 530003, P.R China

Abstract: We mainly study the equicontinuity of maps on [0,1). Let $f : X \to X$ be a continuous map on $X = [0, 1)$. We show that if f is an equicontinuous map with $F(f)$ nonempty, then one of the following two conditions holds: (1) $F(f)$ consists of a single point and $F(f^2) = \bigcap_{n=1}^{\infty} f^n(X)$;(2) $F(f) = \bigcap_{n=1}^{\infty} f^n(X)$. Last we construct two examples to show that the converse result doesn't hold.

Keywords: *Interval , Equicontinous, Periodic point.*

Mathematics Subject Classification (2000): Primary: 54B20, 54E40

1 Introduction

Let (X, d) be a metric space with the metric d (not necessary compact) and $f : X \to X$ be a continuous map. For every nonnegative integer n define f^n inductively by $f^n = f \circ f^{n-1}$, where f^0 is the identity map on X. A point x of X is said to be a *periodic point* of f if there is a positive integer n such that $f^n(x) = x$. The least such n is called the *period* of x. A point of period one is called a *fixed point*. Let $F(f)$ denote the fixed point set of f and $P(f)$ the set of periodic points of f.

If $x \in X$ then the *trajectory* (or *orbit*) of x is the sequence $orb(x, f) = \{f^n(x) : n \geqslant 0\}$ and the $\omega-$*limit set* of x is

$$\omega(x, f) = \bigcap_{m \geqslant 0} \overline{\bigcup_{n \geqslant m} f^n(x)}.$$

Equivalently, $y \in \omega(x, f)$ if and only if $y \in X$ is a limit point of the trajectory $orb(x, f)$, i.e., $f^{n_k}(x) \to y$ for some sequence of integers $n_k \to \infty$.

The map f is said to be *equicontinuous* (in some terminology also *Lyapunov stable*) if given $\epsilon > 0$ there exists a $\delta > 0$ such that $d(f^i(x), f^i(y)) < \epsilon$ whenever $d(x, y) < \delta$ for all $x, y \in X$ and all $i \geqslant 1$.

In 1982, J. Cano [4] proved the following theorem on equicontinuous map for the closed interval I.

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* Corresponding author.

Email address: ksyan@mail.ustc.edu.cn (K. Yan), fpzeng@gxu.edu.cn (F. Zeng), binqin8846@126.com(B. Qin)

Theorem 1.1 *Let $f : I \to I$ be an equicontinuous map. Then $F(f)$ is connected and if it is non-degenerate then $F(f) = P(f)$.*

The next theorem was due to Bruckner and Hu [3]. This result was also proved by Blokh in [2].

Theorem 1.2 *Let $f : I \to I$ be a continuous map. Then f is equicontinuous if and only if $\bigcap_{i=1}^{\infty} f^n(I) = F(f^2)$.*

In [9], Valaristos described the characters of equicontinuous circle maps: A continuous map f of the unit circle S^1 to itself is equicontinuous if and only if one of the following four statements holds: (1) f is topologically conjugate to a rotation; (2) $F(f)$ contains exactly two points and $F(f^2) = S^1$; (3) $F(f)$ contains exactly one point and $F(f^2) = \bigcap_{n=1}^{\infty} f^n(S^1)$; (4) $F(f) = \bigcap_{n=1}^{\infty} f^n(S^1)$. In 2000, Sun [8] obtained some necessary and sufficient conditions of equicontinuous σ-maps. Later, Mai [6] studied the structure of equicontinuous maps of general metric spaces, and given some still simpler necessary and sufficient conditions of equicontinuous graph maps.

In [5], Gu showed that a map on Warsaw circle W is equicontinuous if and only if $F(f)$ consists of a single point and $F(f^2) = \bigcap_{n=1}^{\infty} f^n(X)$ or $F(f) = \bigcap_{n=1}^{\infty} f^n(X)$.

Warsaw circle W is simple connected but not locally connected, and it often appears as an example of circle-like and non arc-like in the theory of continuum (see [7]). In addition, Warsaw circle is not a continuous image of the closed interval. So it is not a Peano continuum. However, it is easily to see that there is a continuous bijective map $\phi : [0, 1) \to X$. Moreover, if f is a continuous self-map of Warsaw circle W, then there is unique continuous map $\tilde{f} : [0, 1) \to [0, 1)$ such that $\phi \circ \tilde{f} = f \circ \phi$ (see [10]). Note that ϕ is not a homeomorphism since $[0, 1)$ is not compact but Warsaw circle W is compact. It follows that f and $\tilde{f}$ are not topologically conjugate. So, it may be that there are some different dynamical properties between maps on $[0, 1)$ and on Warsaw circle.

In this paper we shall deal with the problem of equicontinuity of maps on $[0, 1)$. Our main results are the following theorems.

Theorem 1.3 *Let $X = [0, 1)$ and $f : X \to X$ be an equicontinuous map. If $F(f) \neq \emptyset$, then every periodic point of f has periodic 1 or 2, both $F(f^2)$ and $F(f)$ are connected. Furthermore, if $F(f)$ is non-degenerate then $F(f) = P(f)$.*

Theorem 1.4 *Let $X = [0, 1)$ and $f : X \to X$ be a continuous map with $F(f) \neq \emptyset$. If f is equicontinuous, then one of the following two conditions holds:*

(1) $F(f)$ consists of a single point and $F(f^2) = \bigcap_{n=1}^{\infty} f^n(X)$;

(2) $F(f) = \bigcap_{n=1}^{\infty} f^n(X)$.

Moreover, it is equivalent whenever f is uniformly continuous.

In Section 3, we will construct two examples to show that the converse result of Theorem 1.4 doesn't hold.

2 Proof of Theorem 1.3 and 1.4

In this section, we mainly prove Theorem 1.3 and 1.4.

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2.1 Some lemmas

In this section, we give some lemmas which are needed in proof of Theorem 1.3 and 1.4.

Lemma 2.1 *Let $f : X \to X$ be an continuous map of $X = [0, 1)$. If $F(f^2) = X$, then f is the identity map on X.*

Proof It is not hard to see that $f(X) = X$. Assume there exist $x \in X$ such that $f(x) \neq x$. Then we can choose $0 \leqslant p_1 < p_2 < 1$ such that $f(p_1) = p_2$ and $f(p_2) = p_1$. Let $m = \max_{x\in[0,p_2]} f(x)$. It is clear that $p_2 \leqslant m < 1$. For each $x \in (p_2, 1)$, we have $f(x) < p_2$ or there exists $q \in (p_2, 1)$ such that $f(q) = p_2$ by the continuity of f, which contradicts to $q \in F(f^2)$. Hence $f(X) = [0, m]$. This also contradicts to $f(X) = X$. Therefore, f is the identity map on X.

Lemma 2.2 *Let $X = [0, 1)$ and $f : X \to X$ be an equicontinuous map with a fixed point p. Suppose J is a component of $F(f^4)$ containing p. Then $\omega(x, f) \subset J$ for every $x \in X$.*

Proof Without loss of generality, we may assume that J is a proper subset of X (note that it is clearly hold whenever $J = X$). Firstly, we prove that there is a connected open subset $K \supset J$ such that $\omega(x, f) \subset J$ for every $x \in K$.

Case 1 $J = \{p\}$. Let $\epsilon = (1/2)\min(p, 1 - p)$. By the equicontinuity of f, there is an open interval K of p such that $|f^n(x) - p| < \epsilon$ for every x in K and every positive integer n. Let $L = \overline{\bigcup_{j\geqslant 0} f^j(K)}$. Then L is a closed proper invariant interval of X. It follows from Theorem 1.1 and 1.2 that the fixed point set of $f|_L$ and $f^2|_L$ is connected, and therefore it is $\{p\}$. Moreover, all periodic points of $f|_L$ have period 1 or 2. But the fixed point p is the only periodic point of f in L. Therefore $P(f|_L) = F(f|_L)$ and by Proposition 15 in [1, p. 78] the ω-limit points coincide with the fixed points. Hence p is the only ω-limit point of f in L. Thus $\omega(x, f) = \{p\} = J$ for every $x \in L$. Since $K \subset L$, we have $\omega(x, f) = J$ for every $x \in K$.

Case 2 $J = [q_1, q_2]$ is a closed interval of X. For every $i = 1, 2$ we consider the orbit $\{q_i, f(q_i), f^2(q_i), f^3(q_i)\}$ of q_i. Let $\epsilon = (1/2)\min(f^j(q_i), 1 - f^j(q_i))$. By the equicontinuity of f, there is an open interval K_{ij} containing $f^j(q_i)$ such that $|f^{4n}(x) - f^j(q_i)| < \epsilon$ for every $x \in K_{ij}$ and every positive integer n. Let $K_i = \bigcap_{j=0}^{3} f^{-j}(K_{ij})$, define $L = \overline{\bigcup_{j=0}^{\infty} f^j(K_1 \cup J \cup K_2)}$. Then L is a closed proper invariant interval of X. We know from Theorem 1.1 and 1.2 that fixed point set of $f|_L$ and $f^2|_L$ is connected and therefore, it is contained in J. Moreover, all periodic points of $f|_L$ have period 1 and 2. Since $P(f|_L)$ is closed, by Proposition 15 in [1, p. 78], it coincides with the set of ω-limits points. Therefore, $\omega(x, f) \subset J$ for each $x \in L$. Let $K = K_1 \cup J \cup K_2$. Then $K \supset J$ and $\omega(x, f) \subset J$ for each $x \in K$.

Case 3 $J = [q, 1)$, where $0 < q < 1$. Obviously, $f(J) \subset J$. By Lemma 2.1, we have $J \subset F(f)$. Then $\lim_{x\to 1} f(x) = 1$. Let $g : [0, 1] \to [0, 1]$ such that $g|_X = f$ and $g(1) = 1$. So g is a equicontinuous map on $[0, 1]$. It follows from Theorem 1.1 and 1.2 that the fixed point set of g^2 is connected and $P(g) = F(g^2)$. Hence $P(g) = [q, 1]$. By Proposition 15 in [1, p. 78], $\omega(x, g) \subset [q, 1]$ for each $x \in [0, 1]$. Let $K = X$, then $\omega(x, f) \subset J$ for each $x \in K$.

Secondly, we show that $\omega(x, f) \subset J$ for each $x \in X$. Let

$$S = \{x \in X : \omega(x, f) \subset J\}.$$

Note that S is a nonempty set since $K \subset S$. Let $y \in S$. Then there is a positive integer m such that $f^m(y) \in K$. By the continuity of f^m, there exists an open subset U containing y such that $f^m(U) \subset K$. Hence $U \subset S$ and S is an open set. Let T be the component of S containing J and therefore K as well. Then T is open and connected. It is sufficient to show that $T = X$. Suppose that $T \neq X$. Let $\epsilon = (1/2)\min\{|x - y| : x \in J, y \in X - T\}$. Then $\epsilon > 0$. Assume that z is an endpoint of $X - T$. Then we have $f^n(z) \notin T$ for each positive integer n. On the other hand, for any $\delta > 0$ we can choose $x \in T$ such that $|x - z| < \delta$. Since $\omega(x, f) \subset J$, there is a positive integer m such that $f^m(x) \in B(J, \epsilon/2)$. Hence $|f^m(x) - f^m(z)| > \epsilon/2$. This is a contradiction. Therefore, $T = X$ and the proof is completed.

The following two lemmas are obviously facts on any compact metric space.

Lemma 2.3 *Let $f : X \to X$ be a continuous map, where X is a compact metric space. Let k be a positive integer and $g = f^k$. Then f is equicontinuous if and only if g is equicontinuous.*

Lemma 2.4 *Let $f : X \to X$ be a continuous map, where X is a compact metric space. If $f|_{f(X)}$ is equicontinuous then f is equicontinuous.*

2.2 Proof of Theorem 1.3

Let $X = [0, 1)$ and $f : X \to X$ be an equicontinuous map. If p is a fixed point of f and J is a component of $F(f^4)$ containing p, then we consider the following three case.

Case 1 $J = \{p\}$. By Lemma 2.2, $\omega(x, f) \subset \{p\}$ for each $x \in X$. This shows that p is a unique periodic point of f. Hence $F(f) = F(f^2) = \{p\}$ is connected.

Case 2 $J = [q_1, q_2]$. By Lemma 2.2, $\omega(x, f) \subset J$ for each $x \in X$. This shows that $P(f) \subset J$. Hence $P(f) = F(f^4) = J$ and $F(f^4)$ is connected. Applying Theorem 1.1 to $f|_J$, we know that all periodic points of f have period 1 or 2, both $F(f)$ and $F(f^2)$ are connected. Furthermore, if $F(f)$ is non-degenerate then $F(f) = P(f)$.

Case 3 $J = [q, 1)$. By Lemma 2.2, $\omega(x, f) \subset J$ for each $x \in X$. This shows that $P(f) \subset J$. Hence $P(f) = F(f^4) = J$ and $F(f^4)$ is connected. Applying Lemma 2.1 to $f|_J$, we have $P(f) = F(f) = J$ is connected.

This complete the proof of Theorem 1.3.

2.3 Proof of Theorem 1.4

Let $X = [0, 1)$ and $f : X \to X$ be a continuous map. We suppose that f is equicontinuous. By Theorem 1.3, both $F(f^2)$ and $F(f)$ are connected.

(1) If $F(f)$ consists a single point p then $F(f^2) = P(f)$. Moreover by Lemma 2.2, we have $\omega(x, f) \subset F(f^2)$ for every $x \in X$.

Case 1 If $F(f^2) = [q_1, q_2]$, where $0 \leqslant q_1 \leqslant q_2 < 1$. Similar the proof of Lemma 2.2, there exists an open, connected subset K containing $F(f^2)$ such that $L = \overline{\bigcup_{j=0}^{\infty} f^j(K)} \subset X$ is a closed and invariant interval. Fixed $\epsilon > 0$, there is $\delta > 0$ such that $|x-y| < \delta$ implies $|f^n(x) - f^n(y)| < \epsilon$ for all $n \geqslant 0$. For $x \in X$, since $\omega(x, f) \subset F(f^2) \subset K \subset L$, there exists a positive integer N_x such that $f^{N_x}(x) \in K$. Then $f^m(x) \in L$ for each $m > N_x$. By the continuity of f^{N_x}, there is an open neighborhood V_x of x such that $f^{N_x}(V_x) \subset K$ and hence $f^m(V_x) \subset L$ for every $m \geqslant N_x$. Note that the collection $\{V_x\}_{x \in I_\delta}$ forms an open cover of $I_\delta = [0, 1-\delta]$. By the compactness of I_δ, there is a finite subcover $\{V_{x_1}, \ldots, V_{x_s}\}$. Set $N = \max\{N_{x_1}, \ldots, N_{x_s}\}$. Then $f^m(V_{x_i}) \subset L$ for every $m \geqslant N$ and any $1 \leqslant i \leqslant s$. Thus, $f^m(I_\delta) \subset L$ for every $m \geqslant N$, and hence $f^m(X) \subset B(L, \epsilon)$ for all $m \geqslant N$, where $B(L, \epsilon) = \{y \in X : d(x, y) < \epsilon \text{ for some } x \in L\}$. By the arbitrary of ϵ, we can get $\bigcap_{n=1}^{\infty} f^n(X) \subset L$. Using Theorem 1.2, we have $\bigcap_{n=1}^{\infty} f^n(L) = F(f^2)$. It follows that $F(f^2) = \bigcap_{n=1}^{\infty} f^n(L) = \bigcap_{n=1}^{\infty} f^n(X)$, i.e., (1) holds.

Case 2 If $F(f^2) = [q, 1)$. Applying Lemma 2.1 to $f|_{[q,1)}$, we have $f(x) = x$ for all $x \in [q, 1)$, i.e., $[q, 1) \subset F(f)$. This contradicts to $F(f)$ consists a single point.

(2) If $F(f)$ is non-degenerate, then $F(f) = P(f)$ by Theorem 1.3. Similar to the above argument we can get $F(f) = \bigcap_{n=1}^{\infty} f^n(X)$ whenever $F(f) = [q_1, q_2]$. Now we assume $F(f) = [q, 1)$ for some $0 \leqslant q < 1$. Define $g : [0, 1] \to [0, 1]$ as $g|_X = f$ and $g(1) = 1$. So g is a equicontinuous map on $[0, 1]$. It follows from Theorem 1.1 and 1.2 that $\bigcap_{n=1}^{\infty} g^n([0, 1]) = F(g^2) = F(g)$. Thus, $\bigcap_{n=1}^{\infty} f^n(X) = F(f)$.

3 Examples

In this section, we will construct two examples to show that the converse result of Theorem 1.4 doesn't hold.

Example 3.1 *Let $I = [0, 1)$ and let $a_n = 1 - 1/2^n$ for every $n = 1, 2, \cdots$. Now we define a piecewise linear continuous map $f : I \to I$ as follows (See Figure 1):*

(1) $f(x) = 1 - x$ for each $x \in [0, 1/2]$;

(2) $f(a_{2n}) = 1/2$ and $f(a_{2n-1}) = 0$ for all $n = 1, 2, \cdots$.

It is easily to see that $F(f)$ consists of a single point and $F(f^2) = \bigcap_{n=1}^{\infty} f^n(I) = [0, 1/2]$. However, f is not equicontinuous since $|a_{n+1} - a_n| = \frac{1}{2^{n+1}} \to 0$ but $|f(a_{n+1}) - f(a_n)| = 1/2$ for all $n \geqslant 1$.

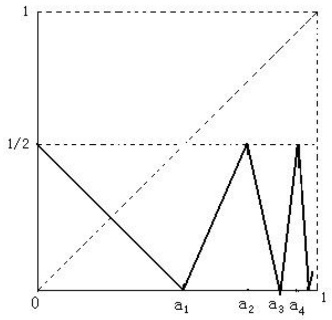

Figure 1

Example 3.2 *Let* $I = [0, 1)$ *and let* $a_n = 1 - 1/2^n$ *for every* $n = 1, 2, \cdots$. *Now we define a piecewise linear continuous map* $f : I \to I$ *as follows (See Figure 2):*

(1) $f(x) = x$ *for each* $x \in [0, 1/2]$*;*

(2) $f(a_{2n}) = 0$ *and* $f(a_{2n-1}) = 1/2$ *for all* $n = 1, 2, \cdots$.

It is easily to see that $F(f)$ *is non-degenerate and* $F(f) = \bigcap_{n=1}^{\infty} f^n(I) = [0, 1/2]$. *However,* f *is not equicontinuous since* $|a_{n+1} - a_n| = \frac{1}{2^{n+1}} \to 0$ *but* $|f(a_{n+1}) - f(a_n)| = 1/2$ *for all* $n \geqslant 1$.

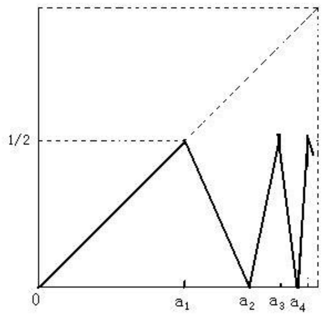

Figure 2

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On mixed type Riemann-Liouville and Hadamard fractional integral inequalities

Weerawat Sudsutad, [1] S.K. Ntouyas [2,3] and Jessada Tariboon [1]

[1]Nonlinear Dynamic Analysis Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, 10800 Thailand
e-mail: wrw.sst@gmail.com (W. Sudsutad), jessadat@kmutnb.ac.th (J. Tariboon)

[2]Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece
e-mail: sntouyas@uoi.gr
[3]Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

Abstract

In this paper, some new mixed type Riemann-Liouville and Hadamard fractional integral inequalities are established, in the case where the functions are bounded by integrable functions. Moreover, mixed type Riemann-Liouville and Hadamard fractional integral inequalities of Chebyshev type are presented.

Key words and phrases: Fractional integral; fractional integral inequalities; Riemann-Liouville fractional integral; Hadamard fractional integral; Chebyshev inequalities.
AMS (MOS) Subject Classifications: 26D10; 26A33.

1 Introduction

The study of mathematical inequalities play very important role in classical differential and integral equations which has applications in many fields. Fractional inequalities are important in studying the existence, uniqueness and other properties of fractional differential equations. Recently many authors have studied integral inequalities on fractional calculus using Riemann-Liouville and Caputo derivative, see [1], [2], [3], [4], [5], [6] and the references therein.

Another kind of fractional derivative that appears in the literature is the fractional derivative due to Hadamard introduced in 1892 [7], which differs from the Riemann-Liouville and Caputo derivatives in the sense that the kernel of the integral contains logarithmic function of arbitrary exponent. Details and properties of Hadamard fractional derivative and integral can be found in [8, 9, 10, 11, 12, 13]. Recently in the literature, were appeared some results on fractional integral inequalities using Hadamard fractional integral; see [14, 15, 16].

Recently, we have been established some new Riemann-Liouville fractional integral inequalities in [17], and some fractional integral inequalities via Hadamard's fractional integral in [18]. In the present paper we combine the results of [17] and [18] and obtain some new mixed type Riemann-Liouville and Hadamard fractional integral inequalities. In Section 3, we consider the case where the functions are bounded by integrable functions and are not necessary increasing or decreasing as are the synchronous functions. In Section 4, we establish mixed type Riemann-Liouville and Hadamard fractional integral inequalities of Chebyshev type, concerning the integral of the product of two functions and the product of two integrals. As applications, in Section 5, we present a way for constructing the four bounding functions, and use them to give some estimates of Chebyshev type inequalities of Riemann-Liouville and Hadamard fractional integrals for two unknown functions.

2 Preliminaries

In this section, we give some preliminaries and basic properties used in our subsequent discussion. The necessary background details are given in the book by Kilbas et al. [8].

Definition 2.1 *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a continuous function $f : (a, \infty) \to \mathbb{R}$ is defined by*

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds,$$

provided the right-hand side is point-wise defined on (a, ∞), where Γ is the gamma function.

Definition 2.2 *The Hadamard fractional integral of order $\alpha \in \mathbb{R}^+$ of a function $f(t)$, for all $0 < a < t < \infty$, is defined as*

$$J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s)\frac{ds}{s},$$

provided the integral exists.

From Definitions (2.1) and (2.2), we derive the following properties:

$$\begin{aligned} I_a^\alpha I_a^\beta f(t) &= I_a^{\alpha+\beta} f(t) = I_a^\beta I_a^\alpha f(t), \\ J_a^\alpha J_a^\beta f(t) &= J_a^{\alpha+\beta} f(t) = J_a^\beta J_a^\alpha f(t), \end{aligned}$$

for $\alpha, \beta > 0$ and

$$\begin{aligned} I_a^\alpha (t^\gamma) &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}(t-a)^{\gamma+\alpha}, \\ J_a^\alpha (\log t)^\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}\left(\log \frac{t}{a}\right)^{\gamma+\alpha}, \end{aligned}$$

for $\alpha > 0$, $\gamma > -1$, $t > a > 0$.

3 Inequalities Involving Mixed Type of Riemann-Liouville and Hadamard Fractional Integral for Bounded Functions

In this section we obtain some new inequalities of mixed type for Riemann-Liouville and Hadamard fractional integral in the case where the functions are bounded by integrable functions and are not necessary increasing or decreasing as are the synchronous functions.

Theorem 3.1 *Let f be an integrable function on $[a, \infty)$, $a > 0$. Assume that:*

(H_1) *There exist two integrable functions φ_1, φ_2 on $[a, \infty)$ such that*

$$\varphi_1(t) \le f(t) \le \varphi_2(t), \quad \text{for all } t \in [a, \infty),\ a > 0. \tag{1}$$

Then for $0 < a < t < \infty$ and $\alpha, \beta > 0$, the following two inequalities hold:

(A_1) $J_a^\alpha \varphi_2(t) I_a^\beta f(t) + J_a^\alpha f(t) I_a^\beta \varphi_1(t) \ge J_a^\alpha \varphi_2(t) I_a^\beta \varphi_1(t) + J_a^\alpha f(t) I_a^\beta f(t)$,

(B_1) $I_a^\alpha \varphi_2(t) J_a^\beta f(t) + I_a^\alpha f(t) J_a^\beta \varphi_1(t) \ge I_a^\alpha \varphi_2(t) J_a^\beta \varphi_1(t) + I_a^\alpha f(t) J_a^\beta f(t)$.

Proof. From condition $(H1)$, for all $\tau, \rho > a$, we have

$$(\varphi_2(\tau) - f(\tau))(f(\rho) - \varphi_1(\rho)) \geq 0,$$

which implies

$$\varphi_2(\tau)f(\rho) + \varphi_1(\rho)f(\tau) \geq \varphi_1(\rho)\varphi_2(\tau) + f(\tau)f(\rho). \tag{2}$$

Multiplying both sides of (2) by $(\log(t/\tau))^{\alpha-1}/\tau\Gamma(\alpha)$, $\tau \in (a,t)$, we get

$$\begin{aligned} f(\rho)\,\frac{(\log(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}\,\varphi_2(\tau) + \varphi_1(\rho)\,\frac{(\log(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}\,f(\tau) \\ \geq \varphi_1(\rho)\,\frac{(\log(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}\,\varphi_2(\tau) + f(\rho)\,\frac{(\log(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}\,f(\tau). \end{aligned} \tag{3}$$

Integrating both sides of (3) with respect to τ on (a,t), we obtain

$$\begin{aligned} f(\rho)\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \varphi_2(\tau)\frac{d\tau}{\tau} + \varphi_1(\rho)\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} f(\tau)\frac{d\tau}{\tau} \\ \geq \varphi_1(\rho)\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \varphi_2(\tau)\frac{d\tau}{\tau} + f(\rho)\frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} f(\tau)\frac{d\tau}{\tau}, \end{aligned}$$

which yields

$$f(\rho)\,J_a^\alpha\varphi_2(t) + \varphi_1(\rho)\,J_a^\alpha f(t) \geq \varphi_1(\rho)\,J_a^\alpha\varphi_2(t) + f(\rho)\,J_a^\alpha f(t). \tag{4}$$

Multiplying both sides of (4) by $(t-\rho)^{\beta-1}/\Gamma(\beta)$, $\rho \in (a,t)$, we have

$$\begin{aligned} J_a^\alpha\varphi_2(t)\,\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}\,f(\rho) + J_a^\alpha f(t)\,\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}\,\varphi_1(\rho) \\ \geq J_a^\alpha\varphi_2(t)\,\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}\,\varphi_1(\rho) + J_a^\alpha f(t)\,\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}\,f(\rho). \end{aligned} \tag{5}$$

Integrating both sides of (5) with respect to ρ on (a,t), we get

$$\begin{aligned} J_a^\alpha\varphi_2(t)\frac{1}{\Gamma(\beta)}\int_a^t (t-\rho)^{\beta-1} f(\rho)d\rho + J_a^\alpha f(t)\frac{1}{\Gamma(\beta)}\int_a^t (t-\rho)^{\beta-1}\varphi_1(\rho)d\rho \\ \geq J_a^\alpha\varphi_2(t)\frac{1}{\Gamma(\beta)}\int_a^t (t-\rho)^{\beta-1}\,\varphi_1(\rho)d\rho + J_a^\alpha f(t)\frac{1}{\Gamma(\beta)}\int_a^t (t-\rho)^{\beta-1}\,f(\rho)d\rho. \end{aligned} \tag{6}$$

Hence, we get the desired inequality in (A_1). The inequality (B_1), is proved by similar arguments. □

Corollary 3.2 *Let f be an integrable function on $[a,\infty)$, $a > 0$ satisfying $m \leq f(t) \leq M$, for all $t \in [a,\infty)$ and $m, M \in \mathbb{R}$. Then for $0 < a < t < \infty$ and $\alpha, \beta > 0$, the following two inequalities hold:*

(A_2) $$M\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\beta f(t) + m\frac{(t-a)^\beta}{\Gamma(\beta+1)}J_a^\alpha f(t) \geq mM\frac{(\log\frac{t}{a})^\alpha(t-a)^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} + J_a^\alpha f(t)I_a^\beta f(t),$$

(B_2) $$M\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J_a^\beta f(t) + m\frac{(\log\frac{t}{a})^\beta}{\Gamma(\beta+1)}I_a^\alpha f(t) \geq mM\frac{(t-a)^\alpha(\log\frac{t}{a})^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} + I_a^\alpha f(t)J_a^\beta f(t).$$

Theorem 3.3 *Let f be an integrable function on $[a,\infty)$, $a > 0$ and $\theta_1, \theta_2 > 0$ satisfying $1/\theta_1 + 1/\theta_2 = 1$. In addition, suppose that the condition (H_1) holds. Then for $0 < a < t < \infty$ and α, $\beta > 0$, the following two inequalities hold:*

(A_3) $J_a^\alpha\varphi_2(t)I_a^\beta\varphi_1(t)+J_a^\alpha f(t)I_a^\beta f(t)+\frac{1}{\theta_1}\frac{(t-a)^\beta}{\Gamma(\beta+1)}J_a^\alpha(\varphi_2-f)^{\theta_1}(t)+\frac{1}{\theta_2}\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\beta(f-\varphi_1)^{\theta_2}(t)$

$\geq J_a^\alpha\varphi_2(t)I_a^\beta f(t)+J_a^\alpha f(t)I_a^\beta\varphi_1(t),$

(B_3) $I_a^\alpha\varphi_2(t)J_a^\beta\varphi_1(t),+I_a^\alpha f(t)J_a^\beta f(t)+\frac{1}{\theta_1}\frac{(\log\frac{t}{a})^\beta}{\Gamma(\beta+1)}I_a^\alpha(\varphi_2-f)^{\theta_1}(t)+\frac{1}{\theta_2}\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J_a^\beta(f-\varphi_1)^{\theta_2}(t)$

$\geq I_a^\alpha\varphi_2(t)J_a^\beta f(t)+I_a^\alpha f(t)J_a^\beta\varphi_1(t).$

Proof. Firstly, we recall the well-known Young's inequality as

$$\frac{1}{\theta_1}x^{\theta_1}+\frac{1}{\theta_2}y^{\theta_2}\geq xy,\quad \forall x,y\geq 0,\quad \theta_1,\theta_2>0,$$

where $1/\theta_1+1/\theta_2=1$. By setting $x=\varphi_2(\tau)-f(\tau)$ and $y=f(\rho)-\varphi_1(\rho)$, $\tau,\rho>a$, we have

$$\frac{1}{\theta_1}(\varphi_2(\tau)-f(\tau))^{\theta_1}+\frac{1}{\theta_2}(f(\rho)-\varphi_1(\rho))^{\theta_2}\geq(\varphi_2(\tau)-f(\tau))(f(\rho)-\varphi_1(\rho)). \tag{7}$$

Multiplying both sides of (7) by $(\log(t/\tau))^{\alpha-1}(t-\rho)^{\beta-1}/\tau\Gamma(\alpha)\Gamma(\beta)$, $\tau,\rho\in(a,t)$, we get

$$\frac{1}{\theta_1}\frac{(\log t/\tau)^{\alpha-1}(t-\rho)^{\beta-1}}{\tau\Gamma(\alpha)\Gamma(\beta)}(\varphi_2(\tau)-f(\tau))^{\theta_1}+\frac{1}{\theta_2}\frac{(\log t/\tau)^{\alpha-1}(t-\rho)^{\beta-1}}{\tau\Gamma(\alpha)\Gamma(\beta)}(f(\rho)-\varphi_1(\rho))^{\theta_2}$$
$$\geq\frac{(\log t/\tau)^{\alpha-1}}{\tau\Gamma(\alpha)}(\varphi_2(\tau)-f(\tau))\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}(f(\rho)-\varphi_1(\rho)).$$

Double integrating the above inequality with respect to τ and ρ from a to t, we have

$$\frac{1}{\theta_1}J_a^\alpha(\varphi_2-f)^{\theta_1}(t)I_a^\beta(1)(t)+\frac{1}{\theta_2}J_a^\alpha(1)(t)I_a^\beta(f-\varphi_1)^{\theta_2}(t)\geq J_a^\alpha(\varphi_2-f)(t)I_a^\beta(f-\varphi_1)(t),$$

which implies the result in (A_3). By using the similar method, we obtain the inequality in (B_3). □

Corollary 3.4 *Let f be an integrable function on $[a,\infty)$, $a>0$ satisfying $m\leq f(t)\leq M,\theta_1=\theta_2=2$ for all $t\in[a,\infty)$ and $m,M\in\mathbb{R}$. Then for $0<a<t<\infty$ and $\alpha,\beta>0$, the following two inequalities hold:*

(A_4) $(m+M)^2\frac{(\log\frac{t}{a})^\alpha(t-a)^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{(t-a)^\beta}{\Gamma(\beta+1)}J_a^\alpha f^2(t)+\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\beta f^2(t)+2J_a^\alpha f(t)I_a^\beta f(t)$

$\geq 2(m+M)\left(\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\beta f(t)+\frac{(t-a)^\beta}{\Gamma(\beta+1)}J_a^\alpha f(t)\right),$

(B_4) $(m+M)^2\frac{(t-a)^\alpha(\log\frac{t}{a})^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J_a^\beta f^2(t)+\frac{(\log\frac{t}{a})^\beta}{\Gamma(\beta+1)}I_a^\alpha f^2(t)+2J_a^\beta f(t)I_a^\alpha f(t)$

$\geq 2(m+M)\left(\frac{(\log\frac{t}{a})^\beta}{\Gamma(\beta+1)}I_a^\alpha f(t)+\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J_a^\beta f(t)\right).$

Theorem 3.5 *Let f be an integrable function on $[a,\infty)$, $a>0$ and $\theta_1,\theta_2>0$ satisfying $\theta_1+\theta_2=1$. In addition, suppose that the condition (H_1) holds. Then for $0<a<t<\infty$, and α, $\beta>0$, the following two inequalities hold:*

(A_5) $\theta_1\frac{(t-a)^\beta}{\Gamma(\beta+1)}J_a^\alpha\varphi_2(t)+\theta_2\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\beta f(t)$

$\geq J_a^\alpha(\varphi_2-f)^{\theta_1}(t)I_a^\beta(f-\varphi_1)^{\theta_2}(t)+\theta_1\frac{(t-a)^\beta}{\Gamma(\beta+1)}J_a^\alpha f(t)+\theta_2\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\beta\varphi_1(t),$

$$(B_5)\ \theta_1\frac{(\log\frac{t}{a})^{\beta}}{\Gamma(\beta+1)}I_a^{\alpha}\varphi_2(t)+\theta_2\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J_a^{\beta}f(t)$$
$$\geq I_a^{\alpha}(\varphi_2-f)^{\theta_1}(t)J_a^{\beta}(f-\varphi_1)^{\theta_2}(t)+\theta_1\frac{(\log\frac{t}{a})^{\beta}}{\Gamma(\beta+1)}I_a^{\alpha}f(t)+\theta_2\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J_a^{\beta}\varphi_1(t).$$

Proof. From the well-known weighted AM-GM inequality

$$\theta_1x+\theta_2y\geq x^{\theta_1}y^{\theta_2},\quad \forall x,y\geq 0,\quad \theta_1,\theta_2>0,$$

where $\theta_1+\theta_2=1$, and setting $x=\varphi_2(\tau)-f(\tau)$ and $y=f(\rho)-\varphi_1(\rho),\tau,\rho>a$, we have

$$\theta_1(\varphi_2(\tau)-f(\tau))+\theta_2(f(\rho)-\varphi_1(\rho))\geq(\varphi_2(\tau)-f(\tau))^{\theta_1}(f(\rho)-\varphi_1(\rho))^{\theta_2}. \tag{8}$$

Multiplying both sides of (8) by $(\log(t/\tau))^{\alpha-1}(t-\rho)^{\beta-1}/\tau\Gamma(\alpha)\Gamma(\beta)$, $\tau,\rho\in(a,t)$, we obtain

$$\theta_1\frac{(\log(t/\tau))^{\alpha-1}(t-\rho)^{\beta-1}}{\tau\Gamma(\alpha)\Gamma(\beta)}(\varphi_2(\tau)-f(\tau))+\theta_2\frac{(\log(t/\tau))^{\alpha-1}(t-\rho)^{\beta-1}}{\tau\Gamma(\alpha)\Gamma(\beta)}(f(\rho)-\varphi_1(\rho))$$
$$\geq\frac{(\log t/\tau)^{\alpha-1}}{\tau\Gamma(\alpha)}(\varphi_2(\tau)-f(\tau))^{\theta_1}\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}(f(\rho)-\varphi_1(\rho))^{\theta_2}.$$

Double integration the above inequality with respect to τ and ρ from a to t, we have

$$\theta_1J_a^{\alpha}(\varphi_2-f)(t)I_a^{\beta}(1)(t)+\theta_2J_a^{\alpha}(1)(t)I_a^{\beta}(f-\varphi_1)(t)\geq J_a^{\alpha}(\varphi_2-f)^{\theta_1}(t)I_a^{\beta}(f-\varphi_1)^{\theta_2}(t).$$

Therefore, we deduce the inequality in (A_5). By using the similar method, we obtain the desired bound in (B_5). □

Corollary 3.6 *Let f be an integrable function on $[a,\infty)$, $a>0$ satisfying $m\leq f(t)\leq M$, $\theta_1=\theta_2=1/2$ for all $0<a<t<\infty$ and $m,M\in\mathbb{R}$. Then for $0<a<t<\infty$ and $\alpha,\beta>0$, the following two inequalities hold:*

$$(A_6)\ M\frac{(\log\frac{t}{a})^{\alpha}(t-a)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}I_a^{\beta}f(t)$$
$$\geq m\frac{(\log\frac{t}{a})^{\alpha}(t-a)^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J_a^{\alpha}f(t)+2J_a^{\alpha}(M-f)^{1/2}(t)I_a^{\beta}(f-m)^{1/2}(t),$$

$$(B_6)\ M\frac{(t-a)^{\alpha}(\log\frac{t}{a})^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}J_a^{\beta}f(t)$$
$$\geq m\frac{(t-a)^{\alpha}(\log\frac{t}{a})^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+\frac{(\log\frac{t}{a})^{\beta}}{\Gamma(\beta+1)}I_a^{\alpha}f(t)+2I_a^{\alpha}(M-f)^{1/2}(t)J_a^{\beta}(f-m)^{1/2}(t).$$

Lemma 3.7 *[19] Assume that $a\geq 0$, $p\geq q\geq 0$, and $p\neq 0$. Then, we have*

$$a^{q/p}\leq\left(\frac{q}{p}k^{(q-p)/p}a+\frac{p-q}{p}k^{q/p}\right).$$

Theorem 3.8 *Let f be an integrable function on $[a,\infty)$, $a>0$ and constants $p\geq q\geq 0$, $p\neq 0$. In addition, assume that the condition (H_1) holds. Then for any $k>0$, $0<a<t<\infty$, $\alpha>0$, the following two inequalities hold:*

$$(A_7)\ J_a^{\alpha}(\varphi_2-f)^{q/p}(t)I_a^{\alpha}(f-\varphi_1)^{q/p}(t)+\frac{q}{p}k^{(q-p)/p}\left(J_a^{\alpha}\varphi_2(t)I_a^{\alpha}\varphi_1(t)+J_a^{\alpha}f(t)I_a^{\alpha}f(t)\right)$$
$$\leq\frac{q}{p}k^{(q-p)/p}\left(J_a^{\alpha}\varphi_2(t)I_a^{\alpha}f(t)+J_a^{\alpha}f(t)I_a^{\alpha}\varphi_1(t)\right)+\frac{p-q}{p}k^{q/p}\frac{(t-a)^{\alpha}(\log\frac{t}{a})^{\alpha}}{\Gamma^2(\alpha+1)},$$

(B_7) $I_a^\alpha(\varphi_2 - f)^{q/p}(t)J_a^\alpha(f-\varphi_1)^{q/p}(t) + \frac{q}{p}k^{(q-p)/p}\left(I_a^\alpha\varphi_2(t)J_a^\alpha\varphi_1(t) + I_a^\alpha f(t)J_a^\alpha f(t)\right)$

$\leq \frac{q}{p}k^{(q-p)/p}\left(I_a^\alpha\varphi_2(t)J_a^\alpha f(t) + I_a^\alpha f(t)J_a^\alpha\varphi_1(t)\right) + \frac{p-q}{p}k^{q/p}\frac{(t-a)^\alpha(\log\frac{t}{a})^\alpha}{\Gamma^2(\alpha+1)}.$

Proof. From condition (H_1) and Lemma 3.7, for $p \geq q \geq 0$, $p \neq 0$, it follows that

$$((\varphi_2(\tau) - f(\tau))(f(\rho) - \varphi_1(\rho)))^{q/p} \leq \frac{q}{p}k^{(q-p)/p}(\varphi_2(\tau) - f(\tau))(f(\rho) - \varphi_1(\rho)) + \frac{p-q}{p}k^{q/p}, \qquad (9)$$

for any $k > 0$. Multiplying both sides of (9) by $(\log(t/\tau))^{\alpha-1}/\tau\Gamma(\alpha)$, $\tau \in (a,t)$, and integrating the resulting identity with respect to τ from a to t, one has

$$(f(\rho) - \varphi_1(\rho))^{q/p}J_a^\alpha(\varphi_2 - f)^{q/p}(t)$$
$$\leq \frac{q}{p}k^{(q-p)/p}(f(\rho) - \varphi_1(\rho))J_a^\alpha(\varphi_2 - f)(t) + \frac{p-q}{p}k^{q/p}\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}. \qquad (10)$$

Multiplying both sides of (10) by $(t-\rho)^{\alpha-1}/\Gamma(\alpha)$, $\rho \in (a,t)$, and integrating the resulting identity with respect to ρ from a to t, we obtain

$$J_a^\alpha(\varphi_2 - f)^{q/p}(t)I_a^\alpha(f - \varphi_1)(t)^{q/p}$$
$$\leq \frac{q}{p}k^{(q-p)/p}J_a^\alpha(\varphi_2 - f)(t)I_a^\alpha(f-\varphi_1)(t) + \frac{p-q}{p}k^{q/p}\frac{(t-a)^\alpha(\log\frac{t}{a})^\alpha}{\Gamma^2(\alpha+1)},$$

which leads to inequality in (A_6). Using the similar arguments, we get the required inequality in (B_6). □

Corollary 3.9 *Let f be an integrable function on $[a,\infty)$, $a > 0$ satisfying $m \leq f(t) \leq M$ for all $t \in [a,\infty)$, constants $q = 1$, $p = 2$, $k = 1$ and $m, M \in \mathbb{R}$. Then for $0 < a < t < \infty$ and $\alpha > 0$, the following two inequalities hold:*

(A_8) $2J_a^\alpha(M-f)^{1/2}(t)I_a^\alpha(f-m)^{1/2}(t) + J_a^\alpha f(t)I_a^\alpha f(t)$

$\leq \frac{M(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha f(t) + \frac{m(t-a)^\alpha}{\Gamma(\alpha+1)}J_a^\alpha f(t) + (1-mM)\frac{(t-a)^\alpha(\log\frac{t}{a})^\alpha}{\Gamma^2(\alpha+1)},$

(B_8) $2I_a^\alpha(M-f)^{1/2}(t)J_a^\alpha(f-m)^{1/2}(t) + I_a^\alpha f(t)J_a^\alpha f(t)$

$\leq \frac{M(t-a)^\alpha}{\Gamma(\alpha+1)}J_a^\alpha f(t) + \frac{m(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha f(t) + (1-mM)\frac{(t-a)^\alpha(\log\frac{t}{a})^\alpha}{\Gamma^2(\alpha+1)}.$

4 Chebyshev Type Inequalities for Riemann-Liouville and Hadamard Fractional Integrals

In this section, we establish our main fractional integral inequalities of Chebyshev type, concerning the integral of the product of two functions and the product of two integrals, with the help of the following lemma.

Lemma 4.1 *Let f be an integrable function on $[a,\infty)$, $a > 0$ and φ_1, φ_2 are two integrable functions on $[a,\infty)$. Assume that the condition (H_1) holds. Then for $0 < a < t < \infty$, and $\alpha, \beta > 0$, the following two equalities hold:*

(A_9) $\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J_a^\alpha f^2(t) + \frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha f^2(t) - 2J_a^\alpha f(t)I_a^\alpha f(t)$

$= J_a^\alpha(f-\varphi_1)(t)I_a^\alpha(\varphi_2 - f)(t) + J_a^\alpha(\varphi_2 - f)(t)I_a^\alpha(f-\varphi_1)(t)$

$$
\begin{aligned}
&+\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}\left(J_a^{\alpha}(\varphi_1 f+\varphi_2 f-\varphi_1\varphi_2)(t)-J_a^{\alpha}((\varphi_2-f)(f-\varphi_1))(t)\right)\\
&+\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}\left(I_a^{\alpha}(\varphi_1 f+\varphi_2 f-\varphi_1\varphi_2)(t)-I_a^{\alpha}((\varphi_2-f)(f-\varphi_1))(t)\right)\\
&+J_a^{\alpha}\varphi_1(t)I_a^{\alpha}(\varphi_2-f)(t)+J_a^{\alpha}\varphi_2(t)I_a^{\alpha}(\varphi_1-f)(t)-J_a^{\alpha}f(t)I_a^{\alpha}(\varphi_1+\varphi_2)(t),
\end{aligned}
$$

(B_9)
$$
\begin{aligned}
&\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J_a^{\alpha}f^2(t)+\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}I_a^{\beta}f^2(t)-2J_a^{\alpha}f(t)I_a^{\beta}f(t)\\
&=J_a^{\alpha}(f-\varphi_1)(t)I_a^{\beta}(\varphi_2-f)(t)+J_a^{\alpha}(\varphi_2-f)(t)I_a^{\beta}(f-\varphi_1)(t)\\
&+\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}\left(J_a^{\alpha}(\varphi_1 f+\varphi_2 f-\varphi_1\varphi_2)(t)-J_a^{\alpha}((\varphi_2-f)(f-\varphi_1))(t)\right)\\
&+\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}\left(I_a^{\beta}(\varphi_1 f+\varphi_2 f-\varphi_1\varphi_2)(t)-I_a^{\beta}((\varphi_2-f)(f-\varphi_1))(t)\right)\\
&+J_a^{\alpha}\varphi_1(t)I_a^{\beta}(\varphi_2-f)(t)+J_a^{\alpha}\varphi_2(t)I_a^{\beta}(\varphi_1-f)(t)-J_a^{\alpha}f(t)I_a^{\beta}(\varphi_1+\varphi_2)(t).
\end{aligned}
$$

Proof. For any $0<a<\tau,\rho<t<\infty$, we have

$$
\begin{aligned}
&(\varphi_2(\rho)-f(\rho))(f(\tau)-\varphi_1(\tau))+(\varphi_2(\tau)-f(\tau))(f(\rho)-\varphi_1(\rho))\\
&-(\varphi_2(\tau)-f(\tau))(f(\tau)-\varphi_1(\tau))-(\varphi_2(\rho)-f(\rho))(f(\rho)-\varphi_1(\rho))\\
&=f^2(\tau)+f^2(\rho)-2f(\tau)f(\rho)+\varphi_2(\rho)f(\tau)+\varphi_1(\tau)f(\rho)-\varphi_1(\tau)\varphi_2(\rho)\\
&+\varphi_2(\tau)f(\rho)+\varphi_1(\rho)f(\tau)-\varphi_1(\rho)\varphi_2(\tau)-\varphi_2(\tau)f(\tau)+\varphi_1(\tau)\varphi_2(\tau)\\
&-\varphi_1(\tau)f(\tau)-\varphi_2(\rho)f(\rho)+\varphi_1(\rho)\varphi_2(\rho)-\varphi_1(\rho)f(\rho).
\end{aligned}
\tag{11}
$$

Multiplying (11) by $(\log(t/\tau))^{\alpha-1}/\tau\Gamma(\alpha)$, $\tau\in(a,t)$, $0<a<t<\infty$, and integrating the resulting identity with respect to τ from a to t, we get

$$
\begin{aligned}
&(\varphi_2(\rho)-f(\rho))(J_a^{\alpha}f(t)-J_a^{\alpha}\varphi_1(t))+(J_a^{\alpha}\varphi_2(t)-J_a^{\alpha}f(t))(f(\rho)-\varphi_1(\rho))\\
&-J_a^{\alpha}((\varphi_2-f)(f-\varphi_1))(t)-(\varphi_2(\rho)-f(\rho))(f(\rho)-\varphi_1(\rho))\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}\\
&=J_a^{\alpha}f^2(t)+f^2(\rho)\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}-2f(\rho)_HI_a^{\alpha}f(t)+\varphi_2(\rho)_HI_a^{\alpha}f(t)+f(\rho)J_a^{\alpha}\varphi_1(t)\\
&-\varphi_2(\rho)J_a^{\alpha}\varphi_1(t)+f(\rho)J_a^{\alpha}\varphi_2(t)+\varphi_1(\rho)J_a^{\alpha}f(t)-\varphi_1(\rho)J_a^{\alpha}\varphi_2(t)\\
&-J_a^{\alpha}\varphi_2f(t)+J_a^{\alpha}\varphi_1\varphi_2(t)-J_a^{\alpha}\varphi_1f(t)-\varphi_2(\rho)f(\rho)\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}\\
&+\varphi_1(\rho)\varphi_2(\rho)\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}-\varphi_1(\rho)f(\rho)\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}
\tag{12}
$$

Multiplying (12) by $(t-\rho)^{\alpha-1}/\Gamma(\alpha)$, $\rho\in(a,t)$, $0<a<t<\infty$, and integrating the resulting identity with respect to ρ from a to t, we have

$$
\begin{aligned}
&(J_a^{\alpha}f(t)-J_a^{\alpha}\varphi_1(t))(I_a^{\alpha}\varphi_2(t)-I_a^{\alpha}f(t))+(J_a^{\alpha}\varphi_2(t)-J_a^{\alpha}f(t))(I_a^{\alpha}f(t)-I_a^{\alpha}\varphi_1(t))\\
&-J_a^{\alpha}((\varphi_2-f)(f-\varphi_1))(t)\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}-I_a^{\alpha}((\varphi_2-f)(f-\varphi_1))(t)\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}\\
&=J_a^{\alpha}f^2(t)\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}+I_a^{\alpha}f^2(t)\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}-2J_a^{\alpha}f(t)I_a^{\alpha}f(t)\\
&+J_a^{\alpha}f(t)I_a^{\alpha}\varphi_2(t)+J_a^{\alpha}\varphi_1(t)I_a^{\alpha}f(t)-J_a^{\alpha}\varphi_1(t)I_a^{\alpha}\varphi_2(t)\\
&+J_a^{\alpha}\varphi_2(t)I_a^{\alpha}f(t)+J_a^{\alpha}f(t)I_a^{\alpha}\varphi_1(t)-J_a^{\alpha}\varphi_2(t)I_a^{\alpha}\varphi_1(t)\\
&-J_a^{\alpha}\varphi_2f(t)\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}+J_a^{\alpha}\varphi_1\varphi_2(t)\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}-J_a^{\alpha}\varphi_1f(t)\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)}
\end{aligned}
$$

$$- I_a^\alpha \varphi_2 f(t)\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)} + I_a^\alpha \varphi_1\varphi_2(t)\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)} - I_a^\alpha \varphi_1 f(t)\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}.$$

Therefore, the desired equality (A_9) is proved. The equality (B_9) is derived by using the similar arguments. □

Let now g be an integrable function on $[a,\infty)$, $a>0$ satisfying the assumption:

(H_2) There exist ψ_1 and ψ_2 integrable functions on $[a,\infty)$ such that

$$\psi_1(t) \le g(t) \le \psi_2(t) \quad \text{for} \quad 0 < a < t < \infty.$$

Theorem 4.2 *Let f and g be two integrable functions on $[a,\infty)$, $a>0$ and φ_1, φ_2, ψ_1 and ψ_2 are four integrable functions on $[a,\infty)$ satisfying the conditions (H_1) and (H_2) on $[a,\infty)$. Then for all $0<a<t<\infty$ and $\alpha>0$, the following inequality holds:*

$$\begin{aligned} &\left|\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J_a^\alpha fg(t) + \frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha fg(t) - J_a^\alpha f(t) I_a^\alpha g(t) - I_a^\alpha f(t) J_a^\alpha g(t)\right| \\ &\quad \le |K(f,\varphi_1,\varphi_2)|^{1/2}\,|K(g,\psi_1,\psi_2)|^{1/2}. \end{aligned} \tag{13}$$

where $K(u,v,w)$ is defined by

$$K(u,v,w) = \frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J_a^\alpha(uw+uv-vw)(t) + \frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha(uw+uv-vw)(t) - 2J_a^\alpha u(t) I_a^\alpha u(t).$$

Proof. Let f and g be two integrable functions defined on $[a,\infty)$ satisfying (H_1) and (H_2), respectively. We define a function H for $0<a<t<\infty$ as follows

$$H(\tau,\rho) := (f(\tau)-f(\rho))(g(\tau)-g(\rho)), \qquad \tau,\rho\in(a,t). \tag{14}$$

Multiplying both sides of (14) by $(\log(t/\tau))^{\alpha-1}(t-\rho)^{\alpha-1}/\tau\Gamma^2(\alpha)$, $\tau,\rho\in(a,t)$, and double integrating the resulting identity with respect to τ and ρ from a to t, we have

$$\begin{aligned} &\frac{1}{\Gamma^2(\alpha)}\int_a^t\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}(t-\rho)^{\alpha-1}H(\tau,\rho)d\rho\frac{d\tau}{\tau} \\ &= \frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J_a^\alpha fg(t) + \frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha fg(t) - J_a^\alpha f(t) I_a^\alpha g(t) - I_a^\alpha f(t) J_a^\alpha g(t). \end{aligned} \tag{15}$$

Applying the Cauchy-Schwarz inequality to (15), we have

$$\begin{aligned} &\left(\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J_a^\alpha fg(t) + \frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha fg(t) - J_a^\alpha f(t) I_a^\alpha g(t) - I_a^\alpha f(t) J_a^\alpha g(t)\right)^2 \\ &\le\left(\frac{1}{\Gamma^2(\alpha)}\int_a^t\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}(t-\rho)^{\alpha-1}(f(\tau)-f(\rho))^2d\rho\frac{d\tau}{\tau}\right) \\ &\quad\times\left(\frac{1}{\Gamma^2(\alpha)}\int_a^t\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}(t-\rho)^{\alpha-1}(g(\tau)-g(\rho))^2d\rho\frac{d\tau}{\tau}\right) \\ &=\left(\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J_a^\alpha f^2(t) + \frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha f^2(t) - 2J_a^\alpha f(t) I_a^\alpha f(t)\right) \\ &\quad\times\left(\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J_a^\alpha g^2(t) + \frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha g^2(t) - 2J_a^\alpha g(t) I_a^\alpha g(t)\right). \end{aligned} \tag{16}$$

Since $(\varphi_2(t)-f(t))(f(t)-\varphi_1(t))\geq 0$ and $(\psi_2(t)-f(t))(f(t)-\psi_1(t))\geq 0$ for $t\in[a,\infty)$, we get

$$\begin{aligned}
\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J_a^\alpha((\varphi_2-f)(f-\varphi_1))(t) &\geq 0,\\
\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha((\varphi_2-f)(f-\varphi_1))(t) &\geq 0,\\
\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J_a^\alpha((\psi_2-g)(g-\psi_1))(t) &\geq 0,\\
\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha((\psi_2-g)(g-\psi_1))(t) &\geq 0.
\end{aligned}$$

Thus, from Lemma 4.1, we obtain

$$\begin{aligned}
&\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J_a^\alpha f^2(t)+\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha f^2(t)-2J_a^\alpha f(t)I_a^\alpha f(t)\\
&\leq J_a^\alpha(f-\varphi_1)(t)I_a^\alpha(\varphi_2-f)(t)+J_a^\alpha(\varphi_2-f)(t)I_a^\alpha(f-\varphi_1)(t)\\
&\quad+\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J_a^\alpha(\varphi_2 f+\varphi_1 f-\varphi_1\varphi_2)(t)+\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha(\varphi_2 f+\varphi_1 f-\varphi_1\varphi_2)(t)\\
&\quad+J_a^\alpha\varphi_1(t)I_a^\alpha(\varphi_2-f)(t)+J_a^\alpha\varphi_2(t)I_a^\alpha(\varphi_1-f)(t)-J_a^\alpha f(t)I_a^\alpha(\varphi_1+\varphi_2)(t)\\
&=\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J_a^\alpha(\varphi_2 f+\varphi_1 f-\varphi_1\varphi_2)(t)+\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha(\varphi_2 f+\varphi_1 f-\varphi_1\varphi_2)(t)-2J_a^\alpha f(t)I_a^\alpha f(t)\\
&=K(f,\varphi_1,\varphi_2),
\end{aligned}\tag{17}$$

and

$$\begin{aligned}
&\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J_a^\alpha g^2(t)+\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha g^2(t)-2J_a^\alpha g(t)I_a^\alpha g(t)\\
&\leq J_a^\alpha(g-\psi_1)(t)I_a^\alpha(\psi_2-g)(t)+J_a^\alpha(\psi_2-g)(t)I_a^\alpha(g-\psi_1)(t)\\
&\quad+\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J_a^\alpha(\psi_2 g+\psi_1 g-\psi_1\psi_2)(t)+\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha(\psi_2 g+\psi_1 g-\psi_1\psi_2)(t)\\
&\quad+J_a^\alpha\psi_1(t)I_a^\alpha(\psi_2-g)(t)+J_a^\alpha\psi_2(t)I_a^\alpha(\psi_1-g)(t)-J_a^\alpha g(t)I_a^\alpha(\psi_1+\psi_2)(t),\\
&=\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J_a^\alpha(\varphi_2 g+\varphi_1 g-\varphi_1\varphi_2)(t)+\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha(\varphi_2 g+\varphi_1 g-\varphi_1\varphi_2)(t)-2J_a^\alpha g(t)I_a^\alpha g(t)\\
&=K(g,\psi_1,\psi_2).
\end{aligned}\tag{18}$$

From (16), (17) and (18), the required inequality in (13) is proved. □

Corollary 4.3 *If $K(f,\varphi_1,\varphi_2)=K(f,m,M)$ and $K(g,\psi_1,\psi_2)=K(g,p,P)$, $m,M,p,P\in\mathbb{R}$, then inequality* (13) *reduces to the following fractional integral inequality:*

$$\begin{aligned}
&\left|\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J_a^\alpha fg(t)+\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha fg(t)-J_a^\alpha f(t)I_a^\alpha g(t)-I_a^\alpha f(t)J_a^\alpha g(t)\right|\\
&\leq\frac{1}{4}\Bigg\{\Bigg[\left(J_a^\alpha f(t)-I_a^\alpha f(t)+M\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}-m\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}\right)^2\\
&\quad+\left(I_a^\alpha f(t)-J_a^\alpha f(t)+M\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}-m\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}\right)^2\Bigg]^{1/2}\\
&\quad\times\Bigg[\left(J_a^\alpha g(t)-I_a^\alpha g(t)+P\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}-p\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}\right)^2
\end{aligned}$$

$$+\left(J_a^\alpha g(t)-I_a^\alpha g(t)+p\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}-P\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}\right)^2\Bigg]^{1/2}\Bigg\}.$$

Theorem 4.4 *Let f and g be two integrable function on $[a,\infty)$, $a>0$. Assume that there exist four integrable functions $\varphi_1,\varphi_2,\psi_1$ and ψ_2 satisfying the conditions (H_1) and (H_2) on $[a,\infty)$. Then for all $0<a<t<\infty$ and $\alpha,\beta>0$, the following inequality holds:*

$$\begin{aligned}&\left|\frac{(t-a)^\beta}{\Gamma(\beta+1)}J_a^\alpha fg(t)+\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\beta fg(t)-J_a^\alpha f(t)I_a^\beta g(t)-I_a^\beta f(t)J_a^\alpha g(t)\right|\\&\quad\le|K_1(f,\varphi_1,\varphi_2)|^{1/2}\,|K_1(g,\psi_1,\psi_2)|^{1/2},\end{aligned}\tag{19}$$

where $K_1(u,v,w)$ is defined by

$$K_1(u,v,w)=\frac{(t-a)^\beta}{\Gamma(\beta+1)}J_a^\alpha(uw+uv-vw)(t)+\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\beta(uw+uv-vw)(t)-2J_a^\alpha u(t)I_a^\beta u(t).$$

Proof. Multiplying both sides of (14) by $(\log(t/\tau))^{\alpha-1}(t-\rho)^{\beta-1}/\tau\Gamma(\alpha)\Gamma(\beta)$, $\tau,\rho\in(a,t)$, and double integrating with respect to τ and ρ from a to t we get

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}(t-\rho)^{\beta-1}H(\tau,\rho)d\rho\frac{d\tau}{\tau}\\&=\frac{(t-a)^\beta}{\Gamma(\alpha+1)}J_a^\alpha fg(t)+\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\beta fg(t)-J_a^\alpha f(t)I_a^\beta g(t)-I_a^\beta f(t)J_a^\alpha g(t).\end{aligned}\tag{20}$$

By using the Cauchy-Schwarz inequality for double integrals, we have

$$\begin{aligned}&\left|\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J_a^\alpha fg(t)+\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha fg(t)-J_a^\alpha f(t)I_a^\alpha g(t)-I_a^\alpha f(t)J_a^\alpha g(t)\right|\\&\le\Bigg[\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}(t-\rho)^{\beta-1}f^2(\tau)d\rho\frac{d\tau}{\tau}\\&\quad+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}(t-\rho)^{\beta-1}f^2(\rho)d\rho\frac{d\tau}{\tau}\\&\quad-\frac{2}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}(t-\rho)^{\beta-1}f(\tau)f(\rho)d\rho\frac{d\tau}{\tau}\Bigg]^{1/2}\\&\quad\times\Bigg[\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}(t-\rho)^{\beta-1}g^2(\tau)d\rho\frac{d\tau}{\tau}\\&\quad+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}(t-\rho)^{\beta-1}g^2(\rho)d\rho\frac{d\tau}{\tau}\\&\quad-\frac{2}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^t\left(\log\frac{t}{\tau}\right)^{\alpha-1}(t-\rho)^{\beta-1}g(\tau)g(\rho)d\rho\frac{d\tau}{\tau}\Bigg]^{1/2}.\end{aligned}$$

Therefore, we get

$$\begin{aligned}&\left|\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}J_a^\alpha fg(t)+\frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)}I_a^\alpha fg(t)-J_a^\alpha f(t)I_a^\alpha g(t)-I_a^\alpha f(t)J_a^\alpha g(t)\right|\\&\le\left[\frac{(t-a)^\beta}{\Gamma(\beta+1)}J_a^\alpha f^2(t)+\frac{\left(\log\frac{t}{\tau}\right)^\alpha}{\Gamma(\alpha+1)}I_a^\beta f^2(t)-2J_a^\alpha f(t)I_a^\beta f(t)\right]^{1/2}\\&\quad\times\left[\frac{(t-a)^\beta}{\Gamma(\beta+1)}J_a^\alpha g^2(t)+\frac{\left(\log\frac{t}{\tau}\right)^\alpha}{\Gamma(\alpha+1)}I_a^\beta g^2(t)-2J_a^\alpha g(t)I_a^\beta g(t)\right]^{1/2}.\end{aligned}$$

Thus, from Lemma 4.1, we get

$$\begin{aligned}
&\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J_a^{\alpha}f^2(t)+\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}I_a^{\beta}f^2(t)-2J_a^{\alpha}f(t)I_a^{\beta}f(t)\\
\le{}& J_a^{\alpha}(f-\varphi_1)(t)I_a^{\beta}(\varphi_2-f)(t)+J_a^{\alpha}(\varphi_2-f)(t)I_a^{\beta}(f-\varphi_1)(t)\\
&+\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J_a^{\alpha}(\varphi_2 f+\varphi_1 f-\varphi_1\varphi_2)(t)+\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}I_a^{\beta}(\varphi_2 f+\varphi_1 f-\varphi_1\varphi_2)(t)\\
&+J_a^{\alpha}\varphi_1(t)I_a^{\beta}(\varphi_2-f)(t)+J_a^{\alpha}\varphi_2(t)I_a^{\beta}(\varphi_1-f)(t)-J_a^{\alpha}f(t)I_a^{\beta}(\varphi_1+\varphi_2)(t)\\
={}&\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J_a^{\alpha}(\varphi_2 f+\varphi_1 f-\varphi_1\varphi_2)(t)+\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}I_a^{\beta}(\varphi_2 f+\varphi_1 f-\varphi_1\varphi_2)(t)-2J_a^{\alpha}f(t)I_a^{\beta}f(t)\\
={}&K_1(f,\varphi_1,\varphi_2),
\end{aligned}\tag{21}$$

and

$$\begin{aligned}
&\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J_a^{\alpha}g^2(t)+\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}I_a^{\beta}g^2(t)-2J_a^{\alpha}g(t)I_a^{\beta}g(t)\\
\le{}& J_a^{\alpha}(g-\psi_1)(t)I_a^{\beta}(\psi_2-g)(t)+J_a^{\alpha}(\psi_2-g)(t)I_a^{\beta}(g-\psi_1)(t)\\
&+\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J_a^{\alpha}(\psi_2 g+\psi_1 g-\psi_1\psi_2)(t)+\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}I_a^{\beta}(\psi_2 g+\psi_1 g-\psi_1\psi_2)(t)\\
&+J_a^{\alpha}\psi_1(t)I_a^{\beta}(\psi_2-g)(t)+J_a^{\alpha}\psi_2(t)I_a^{\beta}(\psi_1-g)(t)-J_a^{\alpha}g(t)I_a^{\beta}(\psi_1+\psi_2)(t),\\
={}&\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J_a^{\alpha}(\varphi_2 g+\varphi_1 g-\varphi_1\varphi_2)(t)+\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}I_a^{\beta}(\varphi_2 g+\varphi_1 g-\varphi_1\varphi_2)(t)-2J_a^{\alpha}g(t)I_a^{\beta}g(t)\\
={}&K_1(g,\psi_1,\psi_2).
\end{aligned}\tag{22}$$

From (15), (21) and (22), we obtain the desired bound in (19). □

Corollary 4.5 *If $K(f,\varphi_1,\varphi_2)=K(f,m,M)$ and $K(g,\psi_1,\psi_2)=K(g,p,P)$, $m,M,p,P\in\mathbb{R}$, then inequality* (13) *reduces to the following fractional integral inequality:*

$$\begin{aligned}
&\left|\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}J_a^{\alpha}fg(t)+\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}I_a^{\beta}fg(t)-J_a^{\alpha}f(t)I_a^{\beta}g(t)-I_a^{\beta}f(t)J_a^{\alpha}g(t)\right|\\
\le{}&\frac{1}{4}\Bigg\{\Bigg[\left(J_a^{\alpha}f(t)-I_a^{\beta}f(t)+M\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}-m\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}\right)^2\\
&+\left(I_a^{\beta}f(t)-J_a^{\alpha}f(t)+M\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}-m\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}\right)^2\Bigg]^{1/2}\\
&\times\Bigg[\left(J_a^{\alpha}g(t)-I_a^{\alpha}g(t)+P\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}-p\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}\right)^2\\
&+\left(J_a^{\alpha}g(t)-I_a^{\beta}g(t)+p\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}-P\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}\right)^2\Bigg]^{1/2}\Bigg\}.
\end{aligned}$$

5 Applications

In this section we present a way for constructing the four bounding functions, and use them to give some estimates of Chebyshev type inequalities of Riemann-Liouville and Hadamard fractional integrals for two unknown functions.

From the Definitions 2.1 and 2.2, for $0<a=t_0<t_1<t_2<\cdots<t_p<t_{p+1}=T$, we define two

notations of sub-integrals for Riemann-Liouville and Hadamard fractional integrals as

$$I^{\alpha}_{t_j,t_{j+1}}f(T)=\frac{1}{\Gamma(\alpha)}\int_{t_j}^{t_{j+1}}(T-\tau)^{\alpha-1}f(\tau)d\tau,\quad j=0,1,\dots,p. \tag{23}$$

and

$$J^{\alpha}_{t_j,t_{j+1}}f(T)=\frac{1}{\Gamma(\alpha)}\int_{t_j}^{t_{j+1}}\left(\log\frac{T}{\tau}\right)^{\alpha-1}f(\tau)\frac{d\tau}{\tau},\quad j=0,1,\dots,p. \tag{24}$$

Note that

$$\begin{aligned} I^{\alpha}_{a}f(T) &= \sum_{j=0}^{p} I^{\alpha}_{t_j,t_{j+1}}f(T) \\ &= \frac{1}{\Gamma(\alpha)}\int_a^{t_1}(T-\tau)^{\alpha-1}f(\tau)d\tau+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(T-\tau)^{\alpha-1}f(\tau)d\tau \\ &\quad +\cdots+\frac{1}{\Gamma(\alpha)}\int_{t_p}^{T}(T-\tau)^{\alpha-1}f(\tau)d\tau, \end{aligned}$$

and

$$\begin{aligned} J^{\alpha}_{a}f(T) &= \sum_{j=0}^{p} J^{\alpha}_{t_j,t_{j+1}}f(T) \\ &= \frac{1}{\Gamma(\alpha)}\int_a^{t_1}\left(\log\frac{T}{\tau}\right)^{\alpha-1}f(\tau)d\tau+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\frac{T}{\tau}\right)^{\alpha-1}f(\tau)d\tau \\ &\quad +\cdots+\frac{1}{\Gamma(\alpha)}\int_{t_p}^{T}\left(\log\frac{T}{\tau}\right)^{\alpha-1}f(\tau)d\tau. \end{aligned}$$

Let u be a unit step function defined by

$$u(t)=\begin{cases}1, & t>0,\\ 0, & t\le 0,\end{cases} \tag{25}$$

and let $u_a(t)$ be the Heaviside unit step function defined by

$$u_a(t)=u(t-a)=\begin{cases}1, & t>a,\\ 0, & t\le a.\end{cases} \tag{26}$$

Let φ_1 be a piecewise continuous functions on $[0,T]$ defined by

$$\begin{aligned} \varphi_1(t) &= m_1(u_0(t)-u_{t_1}(t))+m_2(u_{t_1}(t)-u_{t_2}(t))+m_3(u_{t_2}(t)-u_{t_3}(t))+\ldots+m_{p+1}u_{t_p}(t) \\ &= m_1u_0(t)+(m_2-m_1)u_{t_1}(t)+(m_3-m_2)u_{t_2}(t)+\ldots+(m_{p+1}-m_p)u_{t_p}(t) \\ &= \sum_{j=0}^{p}(m_{j+1}-m_j)u_{t_j}(t), \end{aligned} \tag{27}$$

where $m_0=0$ and $0<a=t_0<t_1<t_2<\cdots<t_p<t_{p+1}=T$.

Analogously, we define the functions φ_2,ψ_1 and ψ_2 as

$$\varphi_2(t)=\sum_{j=0}^{p}(M_{j+1}-M_j)u_{t_j}(t), \tag{28}$$

$$\psi_1(t)=\sum_{j=0}^{p}(n_{j+1}-n_j)u_{t_j}(t), \tag{29}$$

$$\psi_2(t) = \sum_{j=0}^{p}(N_{j+1} - N_j)u_{t_j}(t), \tag{30}$$

where the constants $n_0 = N_0 = M_0 = 0$. If there is an integrable function f on $[a, T]$ satisfying condition (H_1) then we get $m_{j+1} \leq f(t) \leq M_{j+1}$ for each $t \in (t_j, t_{j+1}]$, $j = 0, 1, 2, \ldots, p$. In particular, $p = 4$, the time history of f can be shown as in figure 1.

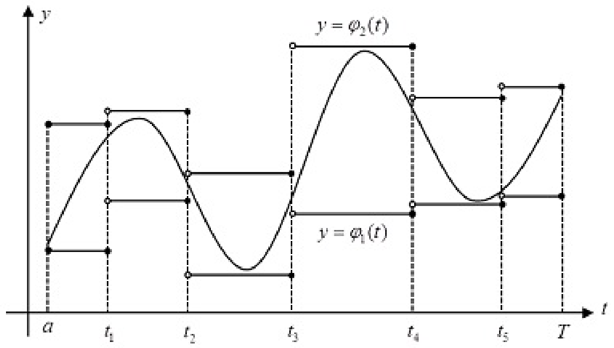

Figure 1: Functions f, φ_1 and φ_2.

Proposition 5.1 *Let f and g be two integrable functions on $[a, T]$, $a > 0$. Assume that the functions $\varphi_1, \varphi_2, \psi_1$ and ψ_2 are defined by (27), (28), (29) and (30), respectively, satisfying (H_1)-(H_2). Then for $\alpha > 0$, the following inequality holds:*

$$\begin{aligned} &\left|\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha fg(t) + \frac{(\log\frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha fg(t) - J_a^\alpha f(t) I_a^\alpha g(t) - I_a^\alpha f(t) J_a^\alpha g(t)\right| \\ &\quad \leq |K^*(f,\varphi_1,\varphi_2)|^{1/2} \, |K^*(g,\psi_1,\psi_2)|^{1/2}, \end{aligned} \tag{31}$$

where

$$\begin{aligned} &K^*(u,v,w)(T) \\ &\leq \frac{(T-a)^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{p} \left\{ wJ^\alpha_{t_i,t_{j+1}}u(T) + vJ^\alpha_{t_i,t_{j+1}}u(T) - vw\left[\left(\log\frac{T}{t_j}\right)^\alpha - \left(\log\frac{T}{t_{j+1}}\right)^\alpha\right]\right\} \\ &+ \frac{(\log\frac{T}{a})^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^{p} \left\{ wI^\alpha_{t_i,t_{j+1}}u(T) + vI^\alpha_{t_i,t_{j+1}}u(T) - vw\left[(T-t_j)^\alpha - (T-t_{j+1})^\alpha\right]\right\} \\ &- 2\left(\sum_{j=0}^{p} J^\alpha_{t_i,t_{j+1}}u(T)\right)\left(\sum_{j=0}^{p} I^\alpha_{t_i,t_{j+1}}u(T)\right). \end{aligned}$$

Proof. Since

$$\begin{aligned} I^\alpha_{t_j,t_{j+1}}(1)(T) &= \frac{1}{\Gamma(\alpha)}\int_{t_j}^{t_{j+1}} (T-\tau)^{\alpha-1}\, d\tau \\ &= \frac{1}{\Gamma(\alpha+1)}\left[(T-t_j)^\alpha - (T-t_{j+1})^\alpha\right], \\ J^\alpha_{t_j,t_{j+1}}(1)(T) &= \frac{1}{\Gamma(\alpha)}\int_{t_j}^{t_{j+1}} \left(\log\frac{T}{\tau}\right)^{\alpha-1}\frac{d\tau}{\tau} \\ &= \frac{1}{\Gamma(\alpha+1)}\left[\left(\log\frac{T}{t_j}\right)^\alpha - \left(\log\frac{T}{t_{j+1}}\right)^\alpha\right], \end{aligned}$$

we have

$$I_a^{\alpha}(\varphi_1\varphi_2)(T) = \sum_{j=0}^{p} \frac{m_{j+1}M_{j+1}}{\Gamma(\alpha+1)} \left[(T-t_j)^{\alpha} - (T-t_{j+1})^{\alpha}\right],$$

$$J_{t_j,t_{j+1}}^{\alpha}(\psi_1\psi_2)(T) = \sum_{j=0}^{p} \frac{n_{j+1}N_{j+1}}{\Gamma(\alpha+1)} \left[\left(\log\frac{T}{t_j}\right)^{\alpha} - \left(\log\frac{T}{t_{j+1}}\right)^{\alpha}\right].$$

Therefore, two functional $K^*(f,\varphi_1,\varphi_2)(T)$ and $K^*(g,\psi_1,\psi_2)(T)$ can be expressed by

$$\begin{aligned} K^*(f,\varphi_1,\varphi_2)(T) \le{} & \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=0}^{p} \Bigg\{ M_{j+1}J_{t_i,t_{j+1}}^{\alpha} f(T) + m_{j+1}J_{t_i,t_{j+1}}^{\alpha} f(T) \\ & - m_{j+1}M_{j+1}\left[\left(\log\frac{T}{t_j}\right)^{\alpha} - \left(\log\frac{T}{t_{j+1}}\right)^{\alpha}\right] \Bigg\} \\ & + \frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=0}^{p} \Bigg\{ M_{j+1}I_{t_i,t_{j+1}}^{\alpha} f(T) + m_{j+1}I_{t_i,t_{j+1}}^{\alpha} f(T) \\ & - m_{j+1}M_{j+1}\left[(T-t_j)^{\alpha} - (T-t_{j+1})^{\alpha}\right] \Bigg\} \\ & - 2\left(\sum_{j=0}^{p} J_{t_i,t_{j+1}}^{\alpha} f(T)\right)\left(\sum_{j=0}^{p} I_{t_i,t_{j+1}}^{\alpha} f(T)\right), \end{aligned}$$

and

$$\begin{aligned} K^*(g,\psi_1,\psi_2)(T) \le{} & \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=0}^{p} \Bigg\{ N_{j+1}J_{t_i,t_{j+1}}^{\alpha} g(T) + n_{j+1}J_{t_i,t_{j+1}}^{\alpha} g(T) \\ & - n_{j+1}M_{j+1}\left[\left(\log\frac{T}{t_j}\right)^{\alpha} - \left(\log\frac{T}{t_{j+1}}\right)^{\alpha}\right] \Bigg\} \\ & + \frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=0}^{p} \Bigg\{ N_{j+1}I_{t_i,t_{j+1}}^{\alpha} g(T) + n_{j+1}I_{t_i,t_{j+1}}^{\alpha} g(T) \\ & - n_{j+1}N_{j+1}\left[(T-t_j)^{\alpha} - (T-t_{j+1})^{\alpha}\right] \Bigg\} \\ & - 2\left(\sum_{j=0}^{p} J_{t_i,t_{j+1}}^{\alpha} g(T)\right)\left(\sum_{j=0}^{p} I_{t_i,t_{j+1}}^{\alpha} g(T)\right). \end{aligned}$$

By applying Theorem (4.2), the required inequality (31) is established. □

Proposition 5.2 *Let f and g be two integrable functions on $[a,T]$, $a>0$. Assume that the functions $\varphi_1,\varphi_2,\psi_1$ and ψ_2 are defined by* (27), (28), (29) *and* (30), *respectively, satisfying* (H_1)-(H_2). *Then for $\alpha,\beta>0$, the following inequality holds:*

$$\begin{aligned} & \left| \frac{(t-a)^{\beta}}{\Gamma(\beta+1)} J_a^{\alpha} fg(t) + \frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)} I_a^{\beta} fg(t) - J_a^{\alpha} f(t) I_a^{\beta} g(t) - I_a^{\beta} f(t) J_a^{\alpha} g(t) \right| \\ & \quad \le |K_1^*(f,\varphi_1,\varphi_2)|^{1/2} \, |K_1^*(g,\psi_1,\psi_2)|^{1/2}, \end{aligned} \tag{32}$$

where

$$K_1^*(u,v,w)(T) \le \frac{(T-a)^{\beta}}{\Gamma(\beta+1)} \sum_{j=0}^{p} \left\{ wJ_{t_i,t_{j+1}}^{\alpha} u(T) + vJ_{t_i,t_{j+1}}^{\alpha} u(T) - vw\left[\left(\log\frac{T}{t_j}\right)^{\alpha} - \left(\log\frac{T}{t_{j+1}}\right)^{\alpha}\right] \right\}$$

$$
\begin{aligned}
&+\frac{(\log\frac{T}{a})^{\alpha}}{\Gamma(\alpha+1)}\sum_{j=0}^{p}\left\{wI^{\beta}_{t_i,t_{j+1}}u(T)+vI^{\beta}_{t_i,t_{j+1}}u(T)-vw\left[(T-t_j)^{\beta}-(T-t_{j+1})^{\beta}\right]\right\}\\
&-2\left(\sum_{j=0}^{p}J^{\alpha}_{t_i,t_{j+1}}u(T)\right)\left(\sum_{j=0}^{p}I^{\beta}_{t_i,t_{j+1}}u(T)\right).
\end{aligned}
$$

Proof. By direct computations, we have

$$
\begin{aligned}
K_1^*(f,\varphi_1,\varphi_2)(T)\leq\ &\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}\sum_{j=0}^{p}\left\{M_{j+1}J^{\alpha}_{t_i,t_{j+1}}f(T)+m_{j+1}J^{\alpha}_{t_i,t_{j+1}}f(T)\right.\\
&\left.-m_{j+1}M_{j+1}\left[\left(\log\frac{T}{t_j}\right)^{\alpha}-\left(\log\frac{T}{t_{j+1}}\right)^{\alpha}\right]\right\}\\
&+\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}\sum_{j=0}^{p}\left\{M_{j+1}I^{\alpha}_{t_i,t_{j+1}}f(T)+m_{j+1}I^{\beta}_{t_i,t_{j+1}}f(T)\right.\\
&\left.-m_{j+1}M_{j+1}\left[(T-t_j)^{\beta}-(T-t_{j+1})^{\beta}\right]\right\}\\
&-2\left(\sum_{j=0}^{p}J^{\alpha}_{t_i,t_{j+1}}f(T)\right)\left(\sum_{j=0}^{p}I^{\beta}_{t_i,t_{j+1}}f(T)\right),
\end{aligned}
$$

and

$$
\begin{aligned}
K_1^*(g,\psi_1,\psi_2)(T)\leq\ &\frac{(t-a)^{\beta}}{\Gamma(\beta+1)}\sum_{j=0}^{p}\left\{N_{j+1}J^{\alpha}_{t_i,t_{j+1}}g(T)+n_{j+1}J^{\alpha}_{t_i,t_{j+1}}g(T)\right.\\
&\left.-n_{j+1}M_{j+1}\left[\left(\log\frac{T}{t_j}\right)^{\alpha}-\left(\log\frac{T}{t_{j+1}}\right)^{\alpha}\right]\right\}\\
&+\frac{(\log\frac{t}{a})^{\alpha}}{\Gamma(\alpha+1)}\sum_{j=0}^{p}\left\{N_{j+1}I^{\beta}_{t_i,t_{j+1}}g(T)+n_{j+1}I^{\beta}_{t_i,t_{j+1}}g(T)\right.\\
&\left.-n_{j+1}N_{j+1}\left[(T-t_j)^{\beta}-(T-t_{j+1})^{\beta}\right]\right\}\\
&-2\left(\sum_{j=0}^{p}J^{\alpha}_{t_i,t_{j+1}}g(T)\right)\left(\sum_{j=0}^{p}I^{\beta}_{t_i,t_{j+1}}g(T)\right),
\end{aligned}
$$

By applying Theorem (4.4), the required inequality (32) is established. □

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Weighted composition operators from $F(p, q, s)$ spaces to nth weighted-Orlicz spaces

Haiying Li and Zhitao Guo
Henan Engineering Laboratory for Big Data Statistical Analysis and Optimal Control,
School of Mathematics and Information Science,
Henan Normal University, Xinxiang 453007.
Email address: haiyingli2012@yahoo.com(Haiying Li)

Abstract

The boundedness and compactness of the weighted composition operator from $F(p, q, s)$ spaces to nth weighted-Orlicz spaces are characterized in this paper.

Keywords: weighted composition operator, $F(p, q, s)$ spaces, nth weighted-Orlicz spaces.

1 Introduction

Let $\mathcal{H}(\mathbb{D})$ be the space of all holomorphic functions on the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$, $\mathbb{N}_0$ the set of all nonnegative integers, $\mathbb{N}$ the set of all positive integers, and dA the Lebesgue measure on $\mathbb{D}$ normalized so that $A(\mathbb{D}) = 1$. Let $u \in \mathcal{H}(\mathbb{D})$, the weighted composition operator uC_ϕ is defined by $(uC_\phi f)(z) = u(z)f(\phi(z))$, $f \in \mathcal{H}(\mathbb{D})$, for more details, see, [1, 3, 16, 18].

For $0 < p, s < \infty$, $-2 < q < \infty$, a function $f \in \mathcal{H}(\mathbb{D})$ is said to belong to the general function space $F(p, q, s)$ if

$$\|f\|_{F(p,q,s)} = |f(0)|^p + \sup_{z\in\mathbb{D}} \int_D |f'(z)|^p (1 - |z|^2)^q (1 - |\psi_a(z)|^2)^s dA(z) < \infty,$$

where $\psi_a(z) = (a - z)/(1 - \overline{a}z)$, $a \in \mathbb{D}$. The space $F(p, q, s)$ was introduced by Zhao in [14]. Since for $q + s \le -1$, $F(p, q, s)$ is the space of constant functions, we assume that $q + s > -1$. For some results on $F(p, q, s)$ space see, for example, [4, 5, 7, 8, 10, 11, 15, 16, 17, 18].

Let μ be a positive continuous function on [0,1). We say that μ is normal if there exist two positive numbers a and b with $0 < a < b$, and $\delta \in [0, 1)$ such that (see [6])

$$\frac{\mu(r)}{(1-r)^a} \text{ is decreasing on } [\delta, 1), \lim_{r\to1} \frac{\mu(r)}{(1-r)^a} = 0; \frac{\mu(r)}{(1-r)^b} \text{ is increasing on } [\delta, 1), \lim_{r\to1} \frac{\mu(r)}{(1-r)^b} = \infty.$$

Let $\mu(z) = \mu(|z|)$ be a normal function on $\mathbb{D}$. The nth weighted-type space on $\mathbb{D}$, denoted by $\mathcal{W}_\mu^{(n)} = \mathcal{W}_\mu^{(n)}(\mathbb{D})$ which was introduced by Stević in [9], consists of all $f \in \mathcal{H}(\mathbb{D})$ such that

$$b_{\mathcal{W}_\mu^{(n)}}(f) = \sup_{z\in\mathbb{D}} \mu(z)|f^{(n)}(z)| < \infty.$$

For $n = 0$ the space becomes the weighted-type space $H_\mu^\infty(\mathbb{D})$, for $n = 1$ the Bloch-type space $\mathcal{B}_\mu(\mathbb{D})$ and for $n = 2$ the Zygmund-type space $\mathcal{Z}_\mu(\mathbb{D})$. From now on, we will assume that $n \in \mathbb{N}$. Set

$$\|f\|_{\mathcal{W}_\mu^{(n)}} = \sum_{j=0}^{n-1} |f^{(j)}(0)| + b_{\mathcal{W}_\mu^{(n)}}(f).$$

With this norm the nth weighted-type space becomes a Banach space.

Recently, Fernándz in [2] uses Young's functions to define the Bloch-Orlicz space. More precisely, let $\varphi : [0, \infty) \to [0, \infty)$ be a strictly increasing convex function such that $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.The Bloch-Orlicz space associated with the function φ, denoted by $\mathcal{B}^\varphi$, is the class of all analytic functions f in $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}} (1 - |z|^2)\varphi(\lambda|f'(z)|) < \infty$$

for some $\lambda > 0$ depending on f. Also, since φ is convex, it is not hard to see that the Minkowski's functional

$$\|f\|_{b^\varphi} = \inf \left\{ k > 0 : S_\varphi\left(\frac{f'}{k}\right) \le 1 \right\}$$

defines a seminorm for $\mathcal{B}^\varphi$, which, in this case, is known as Luxemburgs seminorm, where

$$S_\varphi(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)\varphi(|f(z)|).$$

In fact, it can be shown that $\mathcal{B}^\varphi$ is a Banach space with the norm $\|f\|_{\mathcal{B}^\varphi} = |f(0)| + \|f\|_{b^\varphi}$. For more details, see [2]. We also have that the Bloch-Orlicz space is isometrically equal to the μ-Bloch space, where $\mu(z) = \frac{1}{\varphi^{-1}(\frac{1}{1-|z|^2})}$, $z \in \mathbb{D}$.

Inspired by this, now we define the nth weighted-Orlicz space, which is denoted by $\mathcal{W}_\varphi^{(n)}$, as the class of all analytic function f in $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\varphi(\lambda|f^{(n)}(z)|) < \infty$$

for some $\lambda > 0$ depending on f. Same as the Bloch-Orlicz space, it is not difficult to see that the Minkowski's functional

$$\|f\|_{w^\varphi} = \inf\left\{k>0 : S_\varphi\left(\frac{f^{(n)}}{k}\right) \le 1\right\}$$

defines a seminorm for $\mathcal{W}_\varphi^{(n)}$. Furthermore, it can be shown that $\mathcal{W}_\varphi^{(n)}$ is a Banach space with the norm

$$\|f\|_{\mathcal{W}_\varphi^{(n)}} = \sum_{j=0}^{n-1}|f^{(j)}(0)| + \|f\|_{w^\varphi}.$$

In the same way as in the case B^φ, for any $f \in \mathcal{W}_\varphi^{(n)}\backslash\{0\}$, the relation

$$S_\varphi\left(\frac{f^{(n)}}{\|f\|_{\mathcal{W}_\varphi^{(n)}}}\right) \le 1$$

holds. Also, as a direct consequence of this, we have that the nth weighted-Orlicz space is isometrically equal to the nth weighted-type space, where $\mu(z) = \frac{1}{\varphi^{-1}(\frac{1}{1-|z|^2})}$, $z \in \mathbb{D}$. Thus, for any $f \in \mathcal{W}_\varphi^{(n)}$, we have

$$\|f\|_{\mathcal{W}_\varphi^{(n)}} = \sum_{j=0}^{n-1}|f^{(j)}(0)| + \sup_{z\in\mathbb{D}}\frac{|f^{(n)}(z)|}{\varphi^{-1}(\frac{1}{1-|z|^2})}.$$

Clearly, for $n = 1$, the nth weighted-Orlicz space $\mathcal{W}_\varphi^{(n)}$ becomes the Bloch-Orlicz space, and for $n = 2$ the Zygmund-Orlicz space. In this paper, we are devoted to investigating the boundedness and compactness of the weighted composition operator uC_ϕ from $F(p,q,s)$ spaces to nth weighted-Orlicz spaces. In what follows, we use the letter C to denote a positive constant whose value may change its value at each occurrence.

2 Auxiliary Results

In this section we formulate some auxiliary results which will be used in the proof of the main results. Lemma 1 and Lemma 2 can be found in [5].

Lemma 1. *Assume that $f \in F(p,q,s)$, $0 < p, s < \infty$, $-2 < q < \infty$, $q+s > -1$. Then, for each $n \in \mathbb{N}$, there is a positive constant C, independent of f such that $\|f\|_{\mathcal{B}^{\frac{2+q}{p}}} \le C\|f\|_{F(p,q,s)}$ and*

$$|f^{(n)}(z)| \le \frac{C\|f\|_{F(p,q,s)}}{(1-|z|^2)^{\frac{2+q-p}{p}+n}}, \quad z \in \mathbb{D}.$$

Lemma 2. *Let $\alpha > 0$ and $f \in \mathcal{B}^\alpha$. Then,*

$$|f(z)| \le \begin{cases} C\|f\|_{B^\alpha}, & 0<\alpha<1, \\ C\log\frac{2}{1-|z|^2}\|f\|_{B^\alpha}, & \alpha = 1, \\ \frac{C}{(1-|z|^2)^{\alpha-1}}\|f\|_{B^\alpha}, & \alpha>1. \end{cases}$$

Lemma 3 and Lemma 4 can be found in [12].

Lemma 3. *Assume* $a > 0$ *and*

$$D_{n+1} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & a+1 & \cdots & a+n \\ \vdots & \vdots & & \vdots \\ \prod_{j=0}^{n-1}(a+j) & \prod_{j=0}^{n-1}(a+j+1) & \cdots & \prod_{j=0}^{n-1}(a+j+n) \end{vmatrix}.$$

Then, $D_{n+1} = \prod_{j=1}^{n} j!$.

Lemma 4. *Assume* $n \in \mathbb{N}$, $u, f \in \mathcal{H}(\mathbb{D})$ *and* ϕ *is an analytic self-map of* $\mathbb{D}$. *Then,*

$$\big(u(z)f(\phi(z))\big)^{(n)} = \sum_{k=0}^{n} f^{(k)}(\phi(z)) \sum_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}\big(\phi'(z), \phi''(z), \cdots, \phi^{(l-k+1)}(z)\big),$$

where

$$B_{l,k}\big(\phi'(z), \phi''(z), \cdots, \phi^{(l-k+1)}(z)\big) = \sum_{k_1,\cdots,k_l} \frac{l!}{k_1!, \cdots, k_l!} \prod_{j=1}^{l} \left(\frac{\phi^{(j)}(z)}{j!}\right)^{k_j}, \tag{1}$$

and the sum in (1) *is overall non-negative integer* $k_1, \cdots, k_l$ *satisfying* $k_1+k_2+\cdots+k_l = k$ *and* $k_1+2k_2+\cdots+lk_l = l$.

The next characterization of compactness is proved in a standard way (see, e.g., the proofs of [1], Prop 3.11]). Hence we omit it. The following Lemma 6 can be found in [13].

Lemma 5. *Suppose that* $u \in \mathcal{H}(\mathbb{D})$, $n \in \mathbb{N}$, ϕ *is an analytic self-map of* $\mathbb{D}$. *Then,* $uC_\phi : F(p,q,s) \to \mathcal{W}_{\varphi}^{(n)}$ *is compact if and only if* $uC_\phi : F(p,q,s) \to \mathcal{W}_{\varphi}^{(n)}$ *is bounded and for any bounded sequence* $\{f_k\}_{k\in\mathbb{N}}$ *in* $F(p,q,s)$ *which converges to zero uniformly on compact subsets of* $\mathbb{D}$ *as* $k \to \infty$, *we have* $\|uC_\phi f\|_{\mathcal{W}_{\varphi}^{(n)}} \to 0$ *as* $k \to \infty$.

Lemma 6. *Fix* $0 < \alpha < 1$ *and let* $\{f_k\}_{k\in\mathbb{N}}$ *be a bounded sequence in* $\mathcal{B}^\alpha$ *which converges to zero uniformly on compact subsets of* $\mathbb{D}$ *as* $k \to \infty$. *Then we have*

$$\lim_{k\to\infty} \sup_{z\in\mathbb{D}} |f_k(z)| = 0. \tag{2}$$

3 The Boundedness of $uC_\phi : F(p,q,s) \to \mathcal{W}_{\varphi}^{(n)}$

Theorem 7. *Let* $u \in \mathcal{H}(\mathbb{D})$, $0 < p, s < \infty$, $-2 < q < \infty$, $q+s > -1$, $n \in \mathbb{N}$ *and* ϕ *be an analytic self-map of* $\mathbb{D}$.

(a) If $2+q < p$, *then* $uC_\phi : F(p,q,s) \to \mathcal{W}_{\varphi}^{(n)}$ *is bounded if and only if*

$$M_0 = \sup_{z\in\mathbb{D}} \frac{|u^{(n)}(z)|}{\varphi^{-1}(\frac{1}{1-|z|^2})} < \infty, \tag{3}$$

and

$$M_k = \sup_{z\in\mathbb{D}} \frac{\big|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}\big(\phi'(z), \phi''(z), \cdots, \phi^{(l-k+1)}(z)\big)\big|}{\varphi^{-1}(\frac{1}{1-|z|^2})(1-|\phi(z)|^2)^{\frac{2+q-p}{p}+k}} < \infty, \tag{4}$$

where $k = 1, 2, \cdots, n$.

(b) If $2+q = p$, *then* $uC_\phi : F(p,q,s) \to \mathcal{W}_{\varphi}^{(n)}$ *is bounded if and only if* (4) *holds and*

$$M_0' = \sup_{z\in\mathbb{D}} \frac{|u^{(n)}(z)| \log \frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}(\frac{1}{1-|z|^2})} < \infty. \tag{5}$$

(c) If $2+q > p$, *then* $uC_\phi : F(p,q,s) \to \mathcal{W}_{\varphi}^{(n)}$ *is bounded if and only if* (4) *holds and*

$$M_0'' = \sup_{z\in\mathbb{D}} \frac{|u^{(n)}(z)|}{\varphi^{-1}(\frac{1}{1-|z|^2})(1-|\phi(z)|^2)^{\frac{2+q-p}{p}}} < \infty. \tag{6}$$

Proof. If $2+q<p$. Assume that (3) and (4) hold, then for each $f\in\mathcal{W}_\varphi^{(n)}\backslash\{0\}$, by Lemma 1, Lemma 2 and Lemma 4, we have

$$\begin{aligned}
&S_\varphi\Big(\frac{(uC_\phi f)^{(n)}(z)}{C\|f\|_{F(p,q,s)}}\Big)\\
\leq\ &\sup_{z\in\mathbb{D}}(1-|z|^2)\cdot\\
&\varphi\Big(\frac{|u^{(n)}(z)||f(\phi(z))|+\big|\sum_{k=1}^n f^{(k)}(\phi(z))\sum_{l=k}^n C_n^l u^{(n-l)}(z)B_{l,k}\big(\phi'(z),\phi''(z),\cdots,\phi^{(l-k+1)}(z)\big)\big|}{C\|f\|_{F(p,q,s)}}\Big)\\
\leq\ &\sup_{z\in\mathbb{D}}(1-|z|^2)\cdot\\
&\varphi\Big(\frac{\varphi^{-1}(\frac{1}{1-|z|^2})M_0|f(\phi(z))|+\varphi^{-1}(\frac{1}{1-|z|^2})\sum_{k=1}^n M_k(1-|\phi(z)|^2)^{\frac{2+q-p}{p}+k}\big|f^{(k)}(\phi(z))\big|}{C\|f\|_{F(p,q,s)}}\Big)\\
\leq\ &\sup_{z\in\mathbb{D}}(1-|z|^2)\varphi\Big(\frac{CM_0\|f\|_{\mathcal{B}^{\frac{2+q}{p}}}+\sum_{k=1}^n C_kM_k\|f\|_{F(p,q,s)}}{C\|f\|_{F(p,q,s)}}\varphi^{-1}\Big(\frac{1}{1-|z|^2}\Big)\Big)\\
\leq\ &\sup_{z\in\mathbb{D}}(1-|z|^2)\varphi\Big(\frac{\sum_{j=0}^n C_jM_j}{C}\varphi^{-1}\Big(\frac{1}{1-|z|^2}\Big)\Big)\leq 1.
\end{aligned}$$

Here $C_j(j=0,1,\cdots,n)$ are all constants, and $C\geq\sum_{j=0}^n C_jM_j$. Now, we can conclude that there exists a constant C such that $\|uC_\phi f\|_{\mathcal{W}_\varphi^{(n)}}\leq C\|f\|_{F(p,q,s)}$ and the weighted composition operator $uC_\phi: F(p,q,s)\to\mathcal{W}_\varphi^{(n)}$ is bounded. If $2+q=p$, or $2+q>p$, from (4) (5), or (4) (6), we can get $uC_\phi: F(p,q,s)\to\mathcal{W}_\varphi^{(n)}$ is bounded similarly.

Conversely, suppose that $uC_\phi: F(p,q,s)\to\mathcal{W}_\varphi^{(n)}$ is bounded, that is, for all $f\in F(p,q,s)$, there exists a constant C such that $\|uC_\phi f\|_{\mathcal{W}_\varphi^{(n)}}\leq C$. For $\omega\in\mathbb{D}$, and constants $c_0,c_1,\cdots,c_n$, set

$$f_\omega(z)=\sum_{j=0}^n c_j\frac{(1-|\omega|^2)^{j+1}}{(1-\overline{\omega}z)^{\alpha+j}},\tag{7}$$

where $\alpha=\frac{2+q}{p}$. It is well known that $f_\omega\in F(p,q,s)$, and

$$f_\omega(\omega)=\frac{1}{(1-|\omega|^2)^{\alpha-1}}\sum_{j=0}^n c_j,\tag{8}$$

$$f_\omega^{(l)}(\omega)=\frac{\overline{\omega}^l}{(1-|\omega|^2)^{\alpha-1+l}}\sum_{j=0}^n c_j\prod_{r=0}^{l-1}(\alpha+j+r),\quad l=1,2,\cdots,n.\tag{9}$$

We claim that for each $k\in\{1,2,\cdots,n\}$, there are constants $c_0,c_1,\cdots,c_n$ such that $\sum_{j=0}^n c_j\neq 0$ and

$$f_\omega^{(k)}(\omega)=\frac{\overline{\omega}^k}{(1-|\omega|^2)^{\alpha-1+k}},\quad f_\omega^{(t)}(\omega)=0,\quad t\in\{0,1,2,\cdots,n\}\backslash\{k\}.\tag{10}$$

In fact, by (8) and (9), (10) is equivalent to the following system of liner equations

$$\begin{cases}
c_0+c_1+\cdots+c_n=0,\\
c_0\alpha+c_1(\alpha+1)+\cdots+c_n(\alpha+n)=0,\\
c_0\alpha(\alpha+1)+c_1(\alpha+1)(\alpha+2)+\cdots+c_n(\alpha+n)(\alpha+n+1)=0,\\
\cdots\cdots\\
c_0\prod_{r=0}^{k-1}(\alpha+r)+c_1\prod_{r=0}^{k-1}(\alpha+1+r)+\cdots+c_n\prod_{r=0}^{k-1}(\alpha+n+r)=1,\\
\cdots\cdots\\
c_0\prod_{r=0}^{n-1}(\alpha+r)+c_1\prod_{r=0}^{n-1}(\alpha+1+r)+\cdots+c_n\prod_{r=0}^{n-1}(\alpha+n+r)=0.
\end{cases}\tag{11}$$

By using Lemma 3, we obtain that the determinant of system of linear Eq.(11) is different from zero, from which the claim follows. For each $k \in \{1, 2, \cdots, n\}$, we choose the corresponding family of functions that satisfy (10) and denote it by $f_{\omega,k}$. Then, from Lemma 4 and the boundedness of $uC_\phi : F(p,q,s) \to \mathcal{W}_\varphi^{(n)}$, for $\omega \in \mathbb{D}$ such that $|\phi(\omega)| > \frac{1}{2}$,

$$\begin{aligned} 1 &\geq S_\varphi\bigg(\frac{(uC_\phi f_{\phi(\omega),k})^{(n)}(z)}{C}\bigg) \geq \sup_{|\phi(\omega)|>\frac{1}{2}} (1-|\omega|^2)\varphi\bigg(\frac{|(uC_\phi f_{\phi(\omega),k})^{(n)}(\omega)|}{C}\bigg) \\ &= \sup_{|\phi(\omega)|>\frac{1}{2}} (1-|\omega|^2)\varphi\bigg(\frac{|\phi(\omega)|^k\big|\sum_{l=k}^n C_n^l u^{(n-l)}(z)B_{l,k}\big(\phi'(z),\phi''(z),\cdots,\phi^{(l-k+1)}(z)\big)\big|}{C(1-|\phi(\omega)|^2)^{\frac{2+q-p}{p}+k}}\bigg) \end{aligned}$$

It follows that

$$\sup_{|\phi(\omega)|>\frac{1}{2}} \frac{\big|\sum_{l=k}^n C_n^l u^{(n-l)}(z)B_{l,k}\big(\phi'(z),\phi''(z),\cdots,\phi^{(l-k+1)}(z)\big)\big|}{\varphi^{-1}(\frac{1}{1-|z|^2})(1-|\phi(z)|^2)^{\frac{2+q-p}{p}+k}} < \infty. \tag{12}$$

By the test functions $f_k(z) = z^k (k = 1, 2, \cdots, n)$, use the mathematical induction as in [12], we can get that

$$\sup_{z\in\mathbb{D}} \frac{\big|\sum_{l=k}^n C_n^l u^{(n-l)}(z)B_{l,k}\big(\phi'(z),\phi''(z),\cdots,\phi^{(l-k+1)}(z)\big)\big|}{\varphi^{-1}(\frac{1}{1-|z|^2})} < \infty.$$

Then, for each $k \in \{1, 2, \cdots, n\}$,

$$\sup_{|\phi(\omega)|\leq\frac{1}{2}} \frac{\big|\sum_{l=k}^n C_n^l u^{(n-l)}(z)B_{l,k}\big(\phi'(z),\phi''(z),\cdots,\phi^{(l-k+1)}(z)\big)\big|}{\varphi^{-1}(\frac{1}{1-|z|^2})(1-|\phi(z)|^2)^{\frac{2+q-p}{p}+k}} < \infty. \tag{13}$$

Combining (12) with (13), we obtain that (4) is necessary for all cases.

If $2+q<p$, taking $f(z)=1$, then $(uC_\phi f)(z) = u(z)$, by the boundedness of $uC_\phi : F(p,q,s) \to \mathcal{W}_\varphi^{(n)}$, we have

$$S_\varphi\bigg(\frac{(uC_\phi f)^{(n)}(z)}{C}\bigg) = \sup_{z\in\mathbb{D}}(1-|z|^2)\varphi\bigg(\frac{|u^{(n)}(z)|}{C}\bigg) \leq 1.$$

It follows that (3) holds.

If $2+q=p$, for a fixed $\omega \in \mathbb{D}$, set

$$g_\omega(z) = \log\frac{2}{1-\overline{\omega}z}.$$

Then it is easy to see that $g_\omega \in F(p,q,s)$ and we have

$$g_\omega(\omega) = \log\frac{2}{1-|\omega|^2}, \quad g_\omega^{(k)}(\omega) = (k-1)!\frac{\overline{\omega}^k}{(1-|\omega|^2)^k}, \quad k = 1, 2, \cdots, n.$$

From Lemma 4 and the boundedness of $uC_\phi : F(p,q,s) \to \mathcal{W}_\varphi^{(n)}$, we have

$$\begin{aligned} 1 &\geq S_\varphi\bigg(\frac{(uC_\phi g_{\phi(\omega)})^{(n)}(z)}{C}\bigg) \geq \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)\varphi\bigg(\frac{|(uC_\phi f_{\phi(\omega)})^{(n)}(\omega)|}{C}\bigg) \\ &\geq \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)\cdot \\ &\quad \varphi\bigg(\frac{\big|u^{(n)}(\omega)\log\frac{2}{1-|\phi(\omega)|^2}\big|}{C} - \sum_{k=1}^n \frac{|\phi(\omega)|^k\big|\sum_{l=k}^n C_n^l u^{(n-l)}(z)B_{l,k}\big(\phi'(z),\cdots,\phi^{(l-k+1)}(z)\big)\big|}{C(1-|\phi(\omega)|^2)^k}\bigg). \end{aligned}$$

By $M_k < \infty$ and the boundedness of $\phi(\omega)$, it follows that

$$\sup_{\omega\in\mathbb{D}} \frac{|u^{(n)}(\omega)|\log\frac{2}{1-|\phi(\omega)|^2}}{\varphi^{-1}(\frac{1}{1-|\omega|^2})} \leq C + \sum_{k=1}^n \frac{\big|\sum_{l=k}^n C_n^l u^{(n-l)}(z)B_{l,k}\big(\phi'(z),\phi''(z),\cdots,\phi^{(l-k+1)}(z)\big)\big|}{\varphi^{-1}(\frac{1}{1-|z|^2})(1-|\phi(z)|^2)^k} < \infty.$$

If $2+q>p$, using the function in (7), and in the system of linear Eq.(11), we can also find $c_0, c_1, \cdots, c_n$ and denote the corresponding function $h_\omega(z)$ such that

$$h_\omega(\omega)=\frac{1}{(1-|\omega|^2)^{\frac{2+q-p}{p}}},\quad h_\omega^{(k)}(\omega)=0,\quad k=1,2,\cdots,n.$$

Then from Lemma 4 and the boundedness of $uC_\phi: F(p,q,s)\to\mathcal{W}_\varphi^{(n)}$, we have

$$\begin{aligned} 1 &\geq S_\varphi\bigg(\frac{(uC_\phi h_{\phi(\omega)})^{(n)}(z)}{C}\bigg)\geq \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)\varphi\bigg(\frac{|(uC_\phi h_{\phi(\omega)})^{(n)}(\omega)|}{C}\bigg)\\ &= \sup_{\omega\in\mathbb{D}}(1-|\omega|^2)\varphi\bigg(\frac{|u^{(n)}(\omega)|}{C(1-|\phi(\omega)|^2)^{\frac{2+q-p}{p}}}\bigg),\end{aligned}$$

from which we can see that (6) holds. □

4 The Compactness of $uC_\phi: F(p,q,s)\to\mathcal{W}_\varphi^{(n)}$

Theorem 8. *Let $u\in\mathcal{H}(\mathbb{D})$, $0<p,s<\infty$, $-2<q<\infty$, $q+s>-1$, $n\in\mathbb{N}$ and ϕ be an analytic self-map of $\mathbb{D}$.*

(a) If $2+q<p$, then $uC_\phi: F(p,q,s)\to\mathcal{W}_\varphi^{(n)}$ is compact if and only if $uC_\phi: F(p,q,s)\to\mathcal{W}_\varphi^{(n)}$ is bounded and

$$\lim_{|\phi(z)|\to1}\frac{\big|\sum_{l=k}^n C_n^l u^{(n-l)}(z)B_{l,k}\big(\phi'(z),\phi''(z),\cdots,\phi^{(l-k+1)}(z)\big)\big|}{\varphi^{-1}(\frac{1}{1-|z|^2})(1-|\phi(z)|^2)^{\frac{2+q-p}{p}+k}}=0, \tag{14}$$

where $k=1,2,\cdots,n$.

(b) If $2+q=p$, then $uC_\phi: F(p,q,s)\to\mathcal{W}_\varphi^{(n)}$ is compact if and only if $uC_\phi: F(p,q,s)\to\mathcal{W}_\varphi^{(n)}$ is bounded, (14) *holds and*

$$\lim_{|\phi(z)|\to1}\frac{|u^{(n)}(z)|\log\frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}(\frac{1}{1-|z|^2})}=0. \tag{15}$$

(c) If $2+q>p$, then $uC_\phi: F(p,q,s)\to\mathcal{W}_\varphi^{(n)}$ is compact if and only if $uC_\phi: F(p,q,s)\to\mathcal{W}_\varphi^{(n)}$ is bounded, (14) *holds and*

$$\lim_{|\phi(z)|\to1}\frac{|u^{(n)}(z)|}{\varphi^{-1}(\frac{1}{1-|z|^2})(1-|\phi(z)|^2)^{\frac{2+q-p}{p}}}=0. \tag{16}$$

Proof. Let $\{f_i\}_{i\in\mathbb{N}}$ be a sequence in $F(p,q,s)$ with $\|f_i\|_{F(p,q,s)}\leq L$, and f_i converges to zero uniformly on compact subsets of $\mathbb{D}$ as $i\to\infty$. To prove that $uC_\phi: F(p,q,s)\to\mathcal{W}_\varphi^{(n)}$ is compact, by Lemma 5, we only need to show $\lim_{i\to\infty}\|uC_\phi f_i\|_{\mathcal{W}_\varphi^{(n)}}=0$.

If $2+q<p$, suppose that $uC_\phi: F(p,q,s)\to\mathcal{W}_\varphi^{(n)}$ is bounded and (14) holds, then for given $\epsilon>0$, there exists a $\delta\in(0,1)$, when $\delta<|\phi(z)|<1$, we have

$$\frac{\big|\sum_{l=k}^n C_n^l u^{(n-l)}(z)B_{l,k}\big(\phi'(z),\phi''(z),\cdots,\phi^{(l-k+1)}(z)\big)\big|}{\varphi^{-1}(\frac{1}{1-|z|^2})(1-|\phi(z)|^2)^{\frac{2+q-p}{p}+k}}<\epsilon,\quad k=1,2,\cdots,n. \tag{17}$$

By the proof of the boundedness, we know that $M_0<\infty$ and

$$\sup_{z\in\mathbb{D}}\frac{\big|\sum_{l=k}^n C_n^l u^{(n-l)}(z)B_{l,k}\big(\phi'(z),\phi''(z),\cdots,\phi^{(l-k+1)}(z)\big)\big|}{\varphi^{-1}(\frac{1}{1-|z|^2})}\leq C,\quad k=1,2,\cdots,n.$$

Let $K = \{z \in \mathbb{D}, |\phi(z)| \le \delta\}$, then by Lemma 1 and (17), we have

$$\begin{aligned}
& \sup_{z\in\mathbb{D}} \frac{\left|(uC_\phi f_i)^{(n)}(z)\right|}{\varphi^{-1}(\frac{1}{1-|z|^2})} \\
\le\ & \sup_{z\in\mathbb{D}} \frac{|u^{(n)}(z)|}{\varphi^{-1}(\frac{1}{1-|z|^2})}|f_i(\phi(z))| \\
& + \sum_{k=1}^{n} \sup_{z\in K} \frac{\left|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}\big(\phi'(z),\phi''(z),\cdots,\phi^{(l-k+1)}(z)\big)\right|}{\varphi^{-1}(\frac{1}{1-|z|^2})}|f_i^{(k)}(\phi(z))| \\
& + \sum_{k=1}^{n} \sup_{z\in\mathbb{D}\setminus K} \frac{\left|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}\big(\phi'(z),\phi''(z),\cdots,\phi^{(l-k+1)}(z)\big)\right|}{\varphi^{-1}(\frac{1}{1-|z|^2})}|f_i^{(k)}(\phi(z))| \\
\le\ & M_0|f_i(\phi(z))| + C\sum_{k=1}^{n}\sup_{z\in K}|f_i^{(k)}(\phi(z))| \\
& + \sum_{k=1}^{n} \sup_{z\in\mathbb{D}\setminus K} \frac{\left|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z) B_{l,k}\big(\phi'(z),\phi''(z),\cdots,\phi^{(l-k+1)}(z)\big)\right|\|f_k\|_{F(p,q,s)}}{\varphi^{-1}(\frac{1}{1-|z|^2})(1-|\phi(z)|^2)^{\frac{2+q-p}{p}+k}} \\
\le\ & M_0|f_i(\phi(z))| + nC\sup_{|\omega|\le\delta}|f_i^{(k)}(\omega)| + nL\epsilon.
\end{aligned}$$

Since $f_k \in F(p,q,s) \subset \mathcal{B}^{\frac{2+q}{p}}$, by Lemma 6, we have $\lim_{i\to\infty}\sup_{z\in\mathbb{D}}|f_i(\phi(z))| = 0$. By Cauchy's estimate, we know $\sup_{|\omega|\le\delta}|f_i^{(k)}(\omega)| \to 0$, as $i\to\infty$. On the other hand, since $\{\phi(0)\}$ is also compact subset of $\mathbb{D}$, we have $\sum_{j=0}^{n-1}|f_i^{(j)}(0)| \to 0$, as $i\to\infty$. So $\|uC_\phi f_i\|_{\mathcal{W}_{\varphi}^{(n)}} \to 0$, as $i\to\infty$. Hence $uC_\phi : F(p,q,s)\to\mathcal{W}_{\varphi}^{(n)}$ is compact.

If $2+q=p$ or $2+q>p$, assume that $uC_\phi : F(p,q,s)\to\mathcal{W}_{\varphi}^{(n)}$ is bounded, (14), (15) or (14), (16) hold respectively. Then given $\epsilon>0$, there exists a $\delta\in(0,1)$, when $\delta<|\phi(z)|<1$, we have

$$\frac{|u^{(n)}(z)|\log\frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}(\frac{1}{1-|z|^2})} < \epsilon \quad\text{or}\quad \frac{|u^{(n)}(z)|}{\varphi^{-1}(\frac{1}{1-|z|^2})(1-|\phi(z)|^2)^{\frac{2+q-p}{p}}} < \epsilon.$$

Then by Lemma 1, Lemma 2 and Lemma 5 and similar to the above, we can easily get that $uC_\phi : F(p,q,s)\to\mathcal{W}_{\varphi}^{(n)}$ is compact.

Conversely, assume that $uC_\phi : F(p,q,s)\to\mathcal{W}_{\varphi}^{(n)}$ is compact, then it is clear that $uC_\phi : F(p,q,s)\to\mathcal{W}_{\varphi}^{(n)}$ is bounded. Let $\{z_i\}_{i\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\phi(z_i)|\to 1$, as $i\to\infty$. (If such a sequence does not exist, then the condition in (14), (15), (16) automatically hold.) Let $f_{\omega,k}(z)(k=1,2,\cdots,n)$ be as defined in the proof of Theorem 7. Then the sequence $\{f_{\phi(z_i),k}\}$ are bounded in $F(p,q,s)$ and converge to zero uniformly on compact subsets of $\mathbb{D}$ as $i\to\infty$. By Lemma 5 and the compactness of $uC_\phi : F(p,q,s)\to\mathcal{W}_{\varphi}^{(n)}$, we have

$$\lim_{i\to\infty}\|uC_\phi f_{\phi(z_i),k}\|_{\mathcal{W}_{\varphi}^{(n)}} = 0. \tag{18}$$

Then

$$\begin{aligned}
1 \ \ge\ & S_\varphi\bigg(\frac{(uC_\phi f_{\phi(z_i),k})^{(n)}(z_i)}{\|uC_\phi f_{\phi(z_i),k}\|_{\mathcal{W}_{\varphi}^{(n)}}}\bigg) \\
\ge\ & (1-|z_i|^2)\varphi\bigg(\frac{|\phi(z_i)|^k\left|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z_i) B_{l,k}\big(\phi'(z_i),\phi''(z_i),\cdots,\phi^{(l-k+1)}(z_i)\big)\right|}{\|uC_\phi f_{\phi(z_i),k}\|_{\mathcal{W}_{\varphi}^{(n)}}(1-|\phi(z_i)|^2)^{\frac{2+q-p}{p}+k}}\bigg)
\end{aligned}$$

It follows that

$$\frac{|\phi(z_i)|^k\left|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z_i) B_{l,k}\big(\phi'(z_i),\phi''(z_i),\cdots,\phi^{(l-k+1)}(z_i)\big)\right|}{\varphi^{-1}(\frac{1}{1-|z_i|^2})(1-|\phi(z_i)|^2)^{\frac{2+q-p}{p}+k}} \le \|uC_\phi f_{\phi(z_i),k}\|_{\mathcal{W}_{\varphi}^{(n)}}.$$

$$\begin{aligned}
&\lim_{|\phi(z_i)|\to 1} \frac{\big|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z_i) B_{l,k}\big(\phi'(z_i), \phi''(z_i), \cdots, \phi^{(l-k+1)}(z_i)\big)\big|}{\varphi^{-1}(\frac{1}{1-|z_i|^2})(1-|\phi(z_i)|^2)^{\frac{2+q-p}{p}+k}} \\
= &\lim_{i\to\infty} \frac{|\phi(z_i)|^k\big|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z_i) B_{l,k}\big(\phi'(z_i), \phi''(z_i), \cdots, \phi^{(l-k+1)}(z_i)\big)\big|}{\varphi^{-1}(\frac{1}{1-|z_i|^2})(1-|\phi(z_i)|^2)^{\frac{2+q-p}{p}+k}} = 0,
\end{aligned}$$

which implies that (14) is necessary for all cases. If $2+q=p$, set

$$g_i(z) = \left(\log \frac{2}{1-\overline{\phi(z_i)}z}\right)^2 \left(\log \frac{2}{1-|\phi(z_i)|^2}\right)^{-1}.$$

Then $\{g_i(z)\}$ is a bounded sequence in $F(p,q,s)$ and converges to zero uniformly on compact subsets of $\mathbb{D}$, and we have

$$g_i(\phi(z_i)) = \log \frac{2}{1-|\phi(z_i)|^2},\ g_i^{(k)}(\phi(z_i)) = \frac{2(k-1)!\overline{\phi(z_i)}^k}{(1-|\phi(z_i)|^2)^k} + C_k \frac{\overline{\phi(z_i)}^k}{(1-|\phi(z_i)|^2)^k}\left(\log \frac{2}{1-|\phi(z_i)|^2}\right)^{-1},$$

where $C_k (k=1,2,\cdots,n)$ is constants about k. By Lemma 5 and the compactness of $uC_\phi : F(p,q,s) \to \mathcal{W}_\varphi^{(n)}$, we have

$$\lim_{i\to\infty} \|uC_\phi g_i\|_{\mathcal{W}_\varphi^{(n)}} = 0. \tag{19}$$

Then

$$\begin{aligned}
1 \ &\geq \ S_\varphi\left(\frac{(uC_\phi g_i)^{(n)}(z_i)}{\|uC_\phi g_i\|_{\mathcal{W}_\varphi^{(n)}}}\right) \\
&\geq \ (1-|z_i|^2)\cdot \\
&\quad \varphi\left(\frac{|u^{(n)}(z_i)| \log \frac{2}{1-|\phi(z_i)|^2}}{\|uC_\phi g_i\|_{\mathcal{W}_\varphi^{(n)}}} - \sum_{k=1}^{n} \frac{C|\phi(z_i)|^k\big|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z_i) B_{l,k}\big(\phi'(z_i), \phi''(z_i), \cdots, \phi^{(l-k+1)}(z_i)\big)\big|}{\|uC_\phi g_i\|_{\mathcal{W}_\varphi^{(n)}}(1-|\phi(z_i)|^2)^{\frac{2+q-p}{p}+k}}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{|u^{(n)}(z_i)| \log \frac{2}{1-|\phi(z_i)|^2}}{\varphi^{-1}(\frac{1}{1-|z_i|^2})} \\
\leq \ &\|uC_\phi g_i\|_{\mathcal{W}_\varphi^{(n)}} + \sum_{k=1}^{n} \frac{C\big|\sum_{l=k}^{n} C_n^l u^{(n-l)}(z_i) B_{l,k}\big(\phi'(z_i), \phi''(z_i), \cdots, \phi^{(l-k+1)}(z_i)\big)\big|}{\varphi^{-1}(\frac{1}{1-|z_i|^2})(1-|\phi(z_i)|^2)^{\frac{2+q-p}{p}+k}}.
\end{aligned}$$

Then by (14) and (19), we can get (15) holds.
If $2+q>p$, let $h_\omega(z)$ be as defined in the proof of Theorem 7. Then the sequence $\{h_{\phi(z_i)}\}$ is bounded in $F(p,q,s)$ and converges to zero uniformly on compact subsets of $\mathbb{D}$ as $i\to\infty$. By Lemma 5 and the compactness of $uC_\phi : F(p,q,s) \to \mathcal{W}_\varphi^{(n)}$, we have

$$\lim_{i\to\infty} \|uC_\phi h_{\phi(z_i)}\|_{\mathcal{W}_\varphi^{(n)}} = 0. \tag{20}$$

Then

$$1 \geq S_\varphi\left(\frac{(uC_\phi h_{\phi(z_i)})^{(n)}(z_i)}{\|uC_\phi h_{\phi(z_i)}\|_{\mathcal{W}_\varphi^{(n)}}}\right) \geq (1-|z_i|^2)\varphi\left(\frac{|u^{(n)}(z_i)|}{\|uC_\phi h_{\phi(z_i)}\|_{\mathcal{W}_\varphi^{(n)}}(1-|\phi(z_i)|^2)^{\frac{2+q-p}{p}}}\right).$$

It follows that

$$\frac{|u^{(n)}(z_i)|}{\varphi^{-1}(\frac{1}{1-|z_i|^2})(1-|\phi(z_i)|^2)^{\frac{2+q-p}{p}}} \leq \|uC_\phi h_{\phi(z_i)}\|_{\mathcal{W}_\varphi^{(n)}}$$

from which we can get (16) holds by (20). □

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MODIFIED q-DAEHEE NUMBERS AND POLYNOMIALS

DONGKYU LIM

ABSTRACT. The p-adic q-integral was defined by T. Kim to be

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} \frac{q^x}{[p^N]_q} f(x) \quad \text{(see [9, 10])}.$$

From p-adic q-integrals' equations, we can derive various q-extension of Bernoulli polynomials and numbers (see [1-20]). In [4], T. Kim have studied Daehee polynomials and numbers and their applications. Recently, many properties and valuable identities related to Daehee polynomials and numbers are introduced by several authors (see [1-20]). In [11], T. Kim *et al.* introduced the q-analogue of Daehee numbers and polynomials which are called q-Daehee numbers and polynomials. In this paper, we consider the modified q-Daehee numbers and polynomials which are different the q-Daehee numbers and polynomials of T. Kim *et al.* and give some useful properties and identities of those polynomials which are derived the new p-adic q-integral equations.

MSC: 11B68, 11S40, 11S80

KEYWORDS AND PHRASES. Modified q-Daehee number; Modified q-Daehee polynomial; Modified q-Bernoulli number; p-adic q-integral

1. Introduction

The q-Daehee polynomials $D_{n,q}(x)$ are defined and studied by T. Kim *et al.*, the generating function to be

$$\frac{1-q+\frac{1-q}{\log q}\log(1+t)}{1-q-qt}(1+t)^x = \sum_{n=0}^{\infty} D_{n,q}(x)\frac{t^n}{n!} \quad \text{(see [11])}. \tag{1}$$

This generating function for $D_{n,q}(x)$ is related with p-adic q-integral on $\mathbb{Z}_p$ defined by T. Kim(see [9, 10]).

In this paper, we consider modified p-adic q-integration on $\mathbb{Z}_p$ which are used by many authors(see [1-20]). We define modified q-Daehee polynomials $D_n(x|q)$ from modified p-adic $q-$integrals, and relate $D_n(x|q)$ with modified q-Bernoulli polynomials $B_n(x|q)$.

Throughout this paper, we denote the ring of p-adic integers, the field of p-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$ by $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$, respectively. The p-adic norm $|\cdot|_p$ is normalized by $|p|_p = \frac{1}{p}$. We denote the space of uniformly differentiable function on $\mathbb{Z}_p$ by $UD[\mathbb{Z}_p]$. The q-Haar measure is defined as (see [9, 10]) $\mu_q(a+p^m\mathbb{Z}_p) = \frac{q^a}{[P^m]_q}$, where $[x]_q = \frac{1-q^x}{1-q}$. For a function f in $UD[\mathbb{Z}_p]$, the modified p-adic q-integral on $\mathbb{Z}_p$ is given by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} \frac{q^x}{[p^N]_q} f(x) \quad \text{(see [9-20])}. \tag{2}$$

The bosonic integral on $\mathbb{Z}_p$ is given by $I_1(f) = \lim_{q\to 1} I_q(f)$.

From (2), we have the following integral identity.

$$qI_q(f_1) - I_q(f) = \frac{q-1}{\log q} f'(0) + (q-1)f(0), \tag{3}$$

where $f_1(x) = f(x+1)$ and $f'(x) = \frac{d}{dx}f(x)$.

In special case, we apply $f(x) = q^{-x}e^{tx}$ on (3), we have

$$(e^t - 1)\int_{\mathbb{Z}_p} q^{-x}e^{xt}d\mu_q(x) = \frac{q-1}{\log q}t.$$

Thus

$$\int_{\mathbb{Z}_p} q^{-x}e^{xt}d\mu_q(x) = \frac{q-1}{\log q}\frac{t}{e^t-1}. \tag{4}$$

The q-analogue Bernoulli numbers $B_n(q)$ are known as follows:

$$\sum_{n=0}^{\infty} B_n(q)\frac{t^n}{n!} = \frac{q-1}{\log q}\frac{t}{e^t-1} \quad \text{(see [3, 5, 9])}. \tag{5}$$

Indeed if $q \to 1$, we have $\lim_{q\to 1} B_n(q) = B_n$. So we call $B_n(x|q)$ as the nth modified q-Bernoulli polynomials and the generating function to be

$$\frac{q-1}{\log q}\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x|q)\frac{t^n}{n!}. \tag{6}$$

When $x = 0$, $B_n(0|q) = B_n(q)$ are the nth modified q-Bernoulli numbers.

From (3) and (6), we have

$$B_n(x|q) = \int_{\mathbb{Z}_p} q^{-y}(x+y)^n d\mu_q(y).$$

From (6), we note that

$$B_n(x|q) = \sum_{l=0}^{n} \binom{n}{l} B_l(q)x^{n-l}. \tag{7}$$

For the case $|t|_p \le p^{-\frac{1}{p-1}}$, the Daehee polynomials are defined as follows:

$$\sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!} = \frac{\log(1+t)}{t}(1+t)^x \quad \text{(see [11])}. \tag{8}$$

From p-adic q-integrals' equations, we can derive various q-extension of Bernoulli polynomials and numbers(see [1-20]). In [4], T. Kim have studied Daehee polynomials and numbers and their applications. Recently, many properties and valuable identities related to Daehee polynomials and numbers are introduced by several authors(see [1-20]). In [11], T. Kim *et al.* introduced the q-analogue of Daehee numbers and polynomials which are called q-Daehee numbers and polynomials. In this paper, we consider the modified q-Daehee numbers and polynomials which are different the q-Daehee numbers and polynomials of T. Kim *et al.* and give some useful properties and identities of those polynomials which are derived the new p-adic q-integral equations.

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2. Modified q-Daehee numbers and polynomials

Let us now consider the p-adic q-integral representation as follows:

$$\int_{\mathbb{Z}_p} q^{-y}(x+y)_n d\mu_q(y) \quad (n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}), \tag{9}$$

where $(x)_n$ is known as the *Pochhammer symbol*(or *decreasing fractorial*) defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} S_1(n,k)x^k \tag{10}$$

and here $S_1(n,k)$ is the Stirling number of the first kind (see [4, 11]).

From (9), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p} q^{-y}(x+y)_n d\mu_q(y)\right)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p} q^{-y}\left(\sum_{n=0}^{\infty}\binom{x+y}{n}t^n\right)d\mu_q(y)\\ &= \int_{\mathbb{Z}_p} q^{-y}(1+t)^{x+y}d\mu_q(y),\end{aligned} \tag{11}$$

where $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$.

For $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$, from (3), we have

$$\int_{\mathbb{Z}_p} q^{-y}(1+t)^{x+y}d\mu_q(y) = \frac{q-1}{\log q}\frac{\log(1+t)}{t}(1+t)^x. \tag{12}$$

Let

$$F_q(x,t) = \frac{q-1}{\log q}\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x|q)\frac{t^n}{n!}. \tag{13}$$

In here the polynomial $D_n(x|q)$ is called modified nth q-Daehee polynomials of the first kind. Moreover, we have

$$D_n(x|q) = \int_{\mathbb{Z}_p} q^{-y}(x+y)_n d\mu_q(y). \tag{14}$$

When $x = 0$, $D_n(0|q) = D_n(q)$ is called modified the n-th q-Daehee numbers.

Notice that $F_q(x,t)$ seems to be a new q-extension of the generating function for Daehee polynomials of the first kind. Therefore, from (8) and the following fact,

$$\lim_{q\to 1} F_q(x,t) = \frac{\log(1+t)}{t}(1+t)^x.$$

On the other hand, we can derive

$$\sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p} q^{-x}(x)_n d\mu_q(x)\right)\frac{t^n}{n!} = \frac{q-1}{\log q}\frac{\log(1+t)}{t} = \sum_{n=0}^{\infty} D_n(q)\frac{t^n}{n!}. \tag{15}$$

From (10) and (15), we have

$$\frac{q-1}{\log q}D_n(x) = D_n(x|q). \tag{16}$$

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From (10) and (11), we have

$$D_n(x|q) = \int_{\mathbb{Z}_p} q^{-y}(x+y)_n d\mu_q(y) = \sum_{k=0}^{n} S_1(n,k)B_k(x|q). \tag{17}$$

$B_k(x|q)$ are the modified q-Bernoulli polynomials introduced in (6).

Thus we have the following theorem, which relates modified q-Bernoulli polynomials and modified q-Daehee polynomials.

Theorem 1. *For $n, m \in \mathbb{Z}_+$, we have the following equalities.*

$$D_n(x|q) = \sum_{k=0}^{n} S_1(n,k)B_k(x|q)$$

and

$$D_n(q) = \sum_{k=0}^{n} S_1(n,k)B_k(q).$$

From the generating function of modified q-Daehee polynomials in $D_n(x|q)$ in (13), by replacing t to $e^t - 1$, we have

$$\sum_{n=0}^{\infty} D_n(x|q)\frac{(e^t-1)^n}{n!} = \sum_{n=0}^{\infty} B_n(x|q)\frac{t^n}{n!} = \sum_{m=0}^{\infty} D_m(x|q)\sum_{n=0}^{\infty} S_2(n,m)\frac{t^n}{n!}. \tag{18}$$

Thus by comparing the coefficients of t^n, we have

$$B_n(x|q) = \sum_{m=0}^{n} D_m(x|q)S_2(n,m).$$

In here, $S_2(n,m)$ is the Stirling number of the second kind defined by the following generating series:

$$\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!} \quad cf.[4, 11]. \tag{19}$$

Therefore, we obtan the following theorem.

Theorem 2. *For $n, m \in \mathbb{Z}_+$, we have the following identity.*

$$B_n(x|q) = \sum_{m=0}^{n} D_m(x|q)S_2(n,m).$$

The increasing factorial sequence is known as

$$x^{(n)} = x(x+1)(x+2)\cdots(x+n-1) \quad (n \in \mathbb{Z}_+).$$

Let us define the modified q-Daehee numbers of the second kind as follows:

$$\widehat{D}_n(q) = \int_{\mathbb{Z}_p} q^{-y}(-y)_n d\mu_q(y) \quad (n \in \mathbb{Z}_+). \tag{20}$$

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It is easy to observe that

$$x^{(n)} = (-1)^n(-x)_n = \sum_{k=0}^{n} S_1(n,k)(-1)^{n-k}x^k. \tag{21}$$

From (20) and (21), we have

$$\begin{aligned}\widehat{D}_n(q) &= \int_{\mathbb{Z}_p} q^{-y}(-y)_n d\mu_q(y)\\ &= \int_{\mathbb{Z}_p} q^{-y}y^{(n)}(-1)^n d\mu_q(y)\\ &= \sum_{k=0}^{n} S_1(n,k)(-1)^k B_k(q).\end{aligned} \tag{22}$$

Thus, we state the following theorem.

Theorem 3. *The following holds true:*

$$\widehat{D}_n(q) = \sum_{k=0}^{n} S_1(n,k)(-1)^k B_k(q).$$

Let us now consider the generating function of the modified q-Daehee numbers of the second kind as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} \widehat{D}_n(q)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p} q^{-y}(-y)_n d\mu_q(y)\right)\frac{t^n}{n!}\\ &= \int_{\mathbb{Z}_p} q^{-y}\left(\sum_{n=0}^{\infty}\binom{-y}{n}t^n\right)d\mu_q(y)\\ &= \int_{\mathbb{Z}_p} q^{-y}(1+t)^{-y}d\mu_q(y).\end{aligned} \tag{23}$$

From (23), we denote the generating function for the modified q-Daehee numbers of the second as follows:

$$\widehat{F}_q(t) = \frac{q-1}{\log q}\frac{\log(1+t)}{t}(1+t). \tag{24}$$

Let us consider the modified q-Daehee polynomials of the second kind as follows:

$$\frac{q-1}{\log q}\frac{\log(1+t)}{t}(1+t)^{x+1} = \sum_{n=0}^{\infty} \widehat{D}_n(x|q)\frac{t^n}{n!}. \tag{25}$$

It follows from (25) that

$$\int_{\mathbb{Z}_p} q^{-y}(1+t)^{x-y}d\mu_q(y) = \sum_{n=0}^{\infty} \widehat{D}_n(x|q)\frac{t^n}{n!}. \tag{26}$$

From (26) gives

$$\begin{aligned}\widehat{D}_n(x|q) &= \int_{\mathbb{Z}_p} q^{-y}(x-y)_n d\mu_q(y)\\ &= q^{-1}\sum_{k=0}^{n} |S_1(n,k)|B_k(x+1|q^{-1}),\end{aligned} \tag{27}$$

where $n \geq 0$ and $|S_1(n,k)|$ is the unsigned stirling numbers of the first kind.

Then, by (27), we have the following theorem.

Theorem 4. *For $n \geq 0$, the following are true.*

$$\widehat{D}_n(x|q) = q^{-1} \sum_{k=0}^{n} |S_1(n,k)| B_k(x+1|q^{-1}).$$

From the modified q-Bernoulli polynomials in (6),

$$q \sum_{n=0}^{\infty} B_n(x|q^{-1}) \frac{t^n}{n!} = \frac{q-1}{\log q} \frac{t}{e^t - 1} e^{(1-x)t} = \sum_{n=0}^{\infty} B_n(1-x|q) \frac{t^n}{n!}. \tag{28}$$

Thus, we have

$$q(-1)^n B_n(x|q^{-1}) = B_n(1-x|q). \tag{29}$$

From (29), the value at $x = 1$, we have

$$q(-1)^n B_n(1|q^{-1}) = B_n(q).$$

On the other hand, we can check easily the following

$$(x+y)_n = (-1)^n(-x-y+n-1)_n \tag{30}$$

and

$$\frac{(x+y)_n}{n!} = (-1)^n \binom{-x+y+n-1}{n}. \tag{31}$$

From (13), (27), (30) and (31), we have

$$\begin{aligned} (-1)^n \frac{D_n(x|q)}{n!} &= \int_{\mathbb{Z}_p} q^{-y} \binom{-x-y+n-1}{n} d\mu_q(y) \\ &= \sum_{m=0}^{n} \binom{n-1}{n-m} \int_{\mathbb{Z}_p} q^{-y} \binom{-x-y}{m} d\mu_q(y) \\ &= \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{\widehat{D}_m(-x|q)}{m!} \end{aligned} \tag{32}$$

and

$$\begin{aligned} (-1)^n \frac{\widehat{D}_n(x|q)}{n!} &= (-1)^n \int_{\mathbb{Z}_p} q^{-y} \binom{-x+y}{n} d\mu_q(y) \\ &= \int_{\mathbb{Z}_p} q^{-y} \binom{-x+y+n-1}{n} d\mu_q(y) \\ &= \sum_{m=0}^{n} \binom{n-1}{n-m} \int_{\mathbb{Z}_p} q^{-y} \binom{-x+y}{m} d\mu_q(y) \\ &= \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{D_m(-x|q)}{m!}. \end{aligned} \tag{33}$$

Therefore, we get the following theorem, which relates modified q-Daehee polynomials of the first and the second kind.

Theorem 5. *For $n \in \mathbb{N}$, the following equlity hold true.*

$$(-1)^n \frac{D_n(x|q)}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{\widehat{D}_m(-x|q)}{m!}$$

and

$$(-1)^n \frac{\widehat{D}_n(x|q)}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{D_m(-x|q)}{m!}.$$

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School of Mathematical Sciences, Nankai University, Tianjin Ciy, 300071, China
E-mail address: dgrim84@gmail.com

A class of BVPS for second-order impulsive integro-differential equations of mixed type in Banach space*

Jitai Liang[1] Liping Wang[2] Xuhuan Wang[2,3]

[1]Yunnan Normal University Business School,Kunming, Yunnan, 650106, P.R. China

[2]Department of Mathematics, Pingxiang University, Pingxiang, Jiangxi, 337000, P.R. China

[3]Department of Mathematics, Baoshan University, Baoshan, Yunnan, 678000, P.R. China

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This paper is concerned with a class of boundary value problems for the nonlinear mixed impulsive integro-differential equations with the derivative u' and deviating arguments in Banach space by using the cone theory and upper and lower solutions method together with monotone iterative technique. Sufficient conditions are established for the existence of extremal solutions of the given problem.

Keywords Integro-differential equations; cone; upper and lower solutions; monotone iterative technique; Impulsive

Mathematics Subject Classifications (2000) 34B15, 34B37.

1 Introduction

Impulsive differential equations have become more important in recent years in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics etc. and there have appeared many papers (see [1-28]) and the references therein). There has been a significant development in impulse theory. Especially, there is an increasing interest in the study of nonlinear mixed integro-differential

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equations with deviating arguments and multipiont BVPS[7-14] for impulsive differential equations.

In this article, we are concerned with the following BVPS for the nonlinear mixed impulsive integro-differential equations with the derivative u' and deviating arguments in Banach space E:

$$\begin{cases} u''(t) = f(t, u(t), u(\alpha(t)), u'(t), Tu, Su) \quad t \neq t_k, \quad t \in J = [0,1] \\ \Delta u(t_k) = Q_k u'(t_k) \qquad k = 1, 2, \cdots, m \\ \Delta u'(t_k) = I_k(u'(t_k), u(t_k)) \qquad k = 1, 2, \cdots, m \\ u(0) = \lambda_1 u(1) + k_1 \qquad u'(0) = \lambda_2 u'(1) + \lambda_3 \int_0^1 w(s, u(s))ds + \mu u'(\eta) + k_2 \end{cases} \tag{1.1}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = 1$, $f \in C(J \times E^5, E)$, $I_k \in C(E \times E, E)$, $Q_k \geq 0$, $(Tu)(t) = \displaystyle\int_0^{\beta(t)} k(t,s)u(\gamma(s))ds$, $(Su)(t) = \displaystyle\int_0^1 h(t,s)u(\delta(s))ds$, $D = \{(t,s) \in J^2 \mid 0 \leq s \leq \beta(t)\}$, $k \in C(D, R^+)$, and $h \in C(J^2, R^+)$, $w \in (J \times E, E)$, $\alpha, \beta, \gamma, \delta \in C(J, J)$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$,$\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$, $0 \leq \eta \leq 1$, $0 \leq \mu$, $0 < \lambda_1, \lambda_2 < 1$, $0 \leq \lambda_3$, $k_1, k_2 \in E$.

The article is organized as follow. In section 2, we establish comparison principles and lemmas. In Section 3, we prove the existence of the result of minimal and maximal solutions for the first order impulsive differential equations, which nonlinearly involve the operator A by using upper and lower solutions, i.e. Theorem 3.1. In Section 4, we obtain the main results (Theorem4.1) by applying Theorem 3.1, that is the existence of the theorem of minimal and maximal solutions of (1.1).

2 Preliminaries and lemmas

Let $PC(J, E) = \{x : J \to E; x(t)$ is continuous everywhere expect for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k) = x(t_k^-),\ k = 1, 2, \cdots, m\}$;$PC^1(J, E) = \{x \in PC(J, E) : x'(t)$ is continuous everywhere expect for some t_k at which $x'(t_k^+)$ and $x'(t_k^-)$ exist and $x'(t_k) = x'(t_k^-),\ k = 1, 2, \cdots, m\}$. Evidently,$PC(J, E)$ and $PC^1(J, E)$ are Banach spaces with the norms $\| x \|_{PC} = \sup\{|x(t)| : t \in J\}$ and $\| x \|_{PC^1} = \max\{\| x \|_{PC}, \| x' \|_{PC}\}$. Let $J^- = J \backslash \{t_k, k = 1, 2, \cdots, m\}$, $\Omega = PC^1(J, E) \cap C^2(J^-, E)$.

If P is a normal cone in E, then $P_c = \{x \in PC(J, E) \mid x(t) \geq \theta, \forall t \in J\}$ is a normal cone in $PC(J, E)$, $P^* = \{f \in E^* \mid f(x) \geq 0, \forall x \in P\}$ denotes the dual cone of P.

A function $x \in \Omega$ is called a solution of BVPS (1.1) if it satisfies $Eq.$(1.1). In this paper, we always assume that E is a real Banach space and P is a regular cone in E,and denote $K_0 = \max\{k(t,s), (t,s) \in D\}$ and $H_0 = \max\{h(t,s), (t,s) \in J^2\}$.

We consider the following first order impulsive differential equation in Banach space E:

$$\begin{cases} x'(t) = f(t, Ax(t), Ax(\alpha(t)), x(t), TAx, SAx) & t \neq t_k, \quad t \in J = [0,1] \\ \Delta x(t_k) = I_k(Ax(t_k), x(t_k)) & k = 1, 2, \cdots, m \\ x(0) = \lambda_2 x(1) + \lambda_3 \int_0^1 w(s, Ax(s))ds + \mu x(\eta) + k_2 \end{cases} \tag{2.1}$$

where $f, I_k, T, S, w, Q_k, t_k, \lambda_2, \lambda_3, \mu, k_2$ are difined as (1.1) and

$$Ax(t) = \frac{k_1}{1-\lambda_1} + \int_0^1 G(t,s)x(s)ds + \sum_{k=1}^{m} G(t,t_k)Q_k x(t_k)$$

with

$$G(t,s) = \begin{cases} \dfrac{1}{1-\lambda_1}, & 0 \leq s \leq t \leq 1 \\ \dfrac{\lambda_1}{1-\lambda_1}, & 0 \leq t \leq s \leq 1 \end{cases}$$

Lemma 2.1 Suppose $x \in PC(J, E) \cap C^1(J^-, E)$ satisfies

$$\begin{cases} x'(t) + Mx(t) + M_1 Bx + M_2 Bx(\alpha(t)) + M_3 TBx + M_4 SBx \leq 0 & t \neq t_k, \quad t \in J = [0,1] \\ \Delta x(t_k) \leq -L_k Bx(t_k) & k = 1, 2, \cdots, m \\ x(0) \leq \lambda_2 x(1) \end{cases} \tag{2.2}$$

where

$$Bx(t) = \int_0^1 G(t,s)x(s)ds + \sum_{k=1}^{m} G(t,t_k)Q_k x(t_k)$$

$0 < \lambda_1, \lambda_2 < 1,\ L_k \geq 0$ and constants $M, M_i (i = 1, 2, 3, 4)$ satisfy

$$M > 0,\ M_i \geq 0,\ M + (M_1 + M_2 + M_3 K_0 + M_4 H_0 + \sum_{k=1}^{m} L_k)(\frac{1}{1-\lambda_1} + \sum_{k=1}^{m} \frac{Q_k}{1-\lambda_1}) \leq \lambda_2. \tag{2.3}$$

then $x(t) \leq \theta$ for $t \in J$.(θ denotes the zero elment of E)

Proof. For any given $g \in P^*$, let $y(t) = g(x(t))$, then $y \in PC(J, R) \cap C^1(J^-, R)$ and $y'(t) = g(x'(t))$.
In view of (2.2), we get

$$\begin{cases} y'(t) + My(t) + M_1 By + M_2 By(\alpha(t)) + M_3 TBy + M_4 SBy \leq 0 & t \neq t_k, \quad t \in J = [0,1] \\ \Delta y(t_k) \leq -L_k By(t_k) & k = 1, 2, \cdots, m \\ y(0) \leq \lambda_2 y(1) \end{cases} \tag{2.4}$$

We will show that $y(t) \leq 0, t \in J$.

(i)Suppose to contrary that $y(t) \geq 0,\ y(t) \not\equiv 0$ for $t \in J$,
In view of the first inequality of (2.4),we get $y'(t) \leq 0$. And by the second one in (2.4),we obtain that $y(t)$ is decreasing in J. Then $0 \leq y(1) \leq y(t) \leq y(0)$. By the third inequality of(2.4), we have $y(1) > 0$ and $\lambda_2 \geq 1$, which is contradiction.

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(ii) Suppose there are $\bar{t}, \underline{t} \in J$ such that $y(\bar{t}) > 0$ *and* $y(\underline{t}) < 0$.
Let $y(t_*) = \min_{t\in J} y(t) = -\lambda$, then $\lambda > 0$.By (2.4),we get

$$\begin{aligned} y'(t) \quad &\leq \quad \{M + (M_1 + M_2)(\textstyle\int_0^1 G(t,s)ds + \sum_{k=1}^m G(t,t_k)Q_k) \\ &\quad\quad +M_3(\textstyle\int_0^{\beta(t)} K(t,s)[\int_0^1 G(s,r)dr + \sum_{k=1}^m G(s,t_k)Q_k]ds) \\ &\quad\quad +M_4(\textstyle\int_0^1 H(t,s)[\int_0^1 G(s,r)dr + \sum_{k=1}^m G(s,t_k)Q_k]ds)\}\lambda \\ &\leq \quad \{M + (M_1 + M_2)(\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1}) \\ &\quad\quad +M_3K_0(\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1}) + M_4H_0(\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1})\}\lambda \\ &\leq \quad [M + (M_1 + M_2 + M_3K_0 + M_4H_0)(\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1})]\lambda \qquad t \neq t_k \end{aligned}$$

$$\Delta y(t_k) \quad \leq \quad -L_kBy(t_k)) \leq \lambda L_k(\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1}) \qquad k = 1, 2, \cdots, m$$

Case 1 If $t_* \in [0, \bar{t})$, integrating from t_* to $\bar{t}$,we get

$$\begin{aligned} 0 < y(\bar{t}) \quad &= \quad y(t_*) + \int_{t_*}^{\bar{t}} y'(s)ds + \sum_{t_* \leq t_k < \bar{t}} \Delta y(t_k) \\ &\leq \quad -\lambda + [M + (M_1 + M_2 + M_3K_0 + M_4H_0) \\ &\quad\quad (\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1})]\lambda - \sum_{t_* \leq t_k < \bar{t}} L_kBy(t_k) \\ &\leq \quad -\lambda + [M + (M_1 + M_2 + M_3K_0 + M_4H_0) \\ &\quad\quad (\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1})]\lambda + \lambda \sum_{k=1}^m L_k(\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1}) \end{aligned}$$

Hence

$$1 < [M + (M_1 + M_2 + M_3K_0 + M_4H_0)(\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1})] + \sum_{k=1}^m L_k(\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1})$$

It is contradiction to (2.3).
Case 2 If $t_* \in [\bar{t}, 1]$, we have

$$\begin{aligned} 0 < y(\bar{t}) \quad &= \quad y(0) + \int_0^{\bar{t}} y'(s)ds + \sum_{0 < t_k < \bar{t}} \Delta y(t_k) \\ &\leq \quad y(0) + \int_0^{\bar{t}} [M + (M_1 + M_2 + M_3K_0 + M_4H_0) \\ &\quad\quad (\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1})]\lambda ds + \lambda \sum_{0 < t_k < \bar{t}} L_k(\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1}) \end{aligned}$$

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$$\begin{aligned}
y(1) &= u(t^*) + \int_{t_*}^{1} y'(s)ds + \sum_{t_* \le t_k < 1} \Delta y(t_k) \\
&\le -\lambda + \int_{t_*}^{1} [M + (M_1 + M_2 + M_3K_0 + M_4H_0) \\
&\quad (\frac{1}{1-\lambda_1} + \sum_{k=1}^{m} \frac{Q_k}{1-\lambda_1})]\lambda ds + \lambda \sum_{t_* \le t_k < 1} L_k(\frac{1}{1-\lambda_1} + \sum_{k=1}^{m} \frac{Q_k}{1-\lambda_1})
\end{aligned}$$

Hence

$$\begin{aligned}
& -\lambda + \frac{1}{\lambda_2} \int_{t_*}^{1} [M + (M_1 + M_2 + M_3K_0 + M_4H_0) \\
& (\frac{1}{1-\lambda_1} + \sum_{k=1}^{m} \frac{Q_k}{1-\lambda_1})]\lambda ds + \frac{1}{\lambda_2}\lambda \sum_{t_* \le t_k < 1} L_k(\frac{1}{1-\lambda_1} + \sum_{k=1}^{m} \frac{Q_k}{1-\lambda_1}) \\
> \; & -\lambda + \int_{t_*}^{1} [M + (M_1 + M_2 + M_3K_0 + M_4H_0) \\
& (\frac{1}{1-\lambda_1} + \sum_{k=1}^{m} \frac{Q_k}{1-\lambda_1})]\lambda ds + \lambda \sum_{t_* \le t_k < 1} L_k(\frac{1}{1-\lambda_1} + \sum_{k=1}^{m} \frac{Q_k}{1-\lambda_1}) \\
\ge \; & y(1) \ge \frac{1}{\lambda_2} y(0) \\
> \; & -\frac{1}{\lambda_2} \int_0^{\bar{t}} y'(s)ds - \frac{1}{\lambda_2} \sum_{0 < t_k < \bar{t}} \Delta y(t_k) \\
\ge \; & -\frac{1}{\lambda_2} \int_0^{\bar{t}} [M + (M_1 + M_2 + M_3K_0 + M_4H_0) \\
& (\frac{1}{1-\lambda_1} + \sum_{k=1}^{m} \frac{Q_k}{1-\lambda_1})]\lambda ds - \frac{1}{\lambda_2}\lambda \sum_{0 < t_k < \bar{t}} L_k(\frac{1}{1-\lambda_1} + \sum_{k=1}^{m} \frac{Q_k}{1-\lambda_1}) \\
\ge \; & -\frac{1}{\lambda_2}\lambda \int_0^{t_*} ([M + (M_1 + M_2 + M_3K_0 + M_4H_0) \\
& (\frac{1}{1-\lambda_1} + \sum_{k=1}^{m} \frac{Q_k}{1-\lambda_1})]ds - \frac{1}{\lambda_2}\lambda \sum_{0 < t_k < t_*} L_k(\frac{1}{1-\lambda_1} + \sum_{k=1}^{m} \frac{Q_k}{1-\lambda_1}).
\end{aligned}$$

We obtain that $M+(M_1+M_2+M_3K_0+M_4H_0+\sum_{k=1}^{m} L_k)(\frac{1}{1-\lambda_1}+\sum_{k=1}^{m} \frac{Q_k}{1-\lambda_1}) > \lambda_2$ which is contradiction.
Since $g \in P^*$ is arbitrary, we have $x(t) \le \theta$, $\forall t \in J$.
We complete the proof.

Lemma 2.2 Assume that (2.3) is satisfied.Let $e_k, a \in E, \sigma \in PC(J,E)$. Then the linear problem

$$\begin{cases} x'(t) = -Mx(t) - M_1Ax - M_2Ax(\alpha(t)) - M_3TAx - M_4SAx + \sigma(t) & t \ne t_k, \quad t \in J = [0,1] \\ \Delta x(t_k) = -L_kAx(t_k) + e_k & k = 1, 2, \cdots, m \\ x(0) = \lambda_2 x(1) + a \end{cases} \tag{2.5}$$

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has a unique solution $x \in PC^1(J,E)$ if and only if $x \in PC(J,E)$ is a solution of the integral equation:

$$\begin{aligned} x(t) &= aDe^{-Mt} + \textstyle\int_0^1 H(t,s)(\sigma(s) - M_1 Ax(s) - M_2 Ax(\alpha(s)) \\ &\quad -M_3 TAx(s) - M_4 SAx(s))ds + \sum_{k=1}^{m} H(t,t_k)(-L_k Ax(t_k) + e_k), \end{aligned} \tag{2.6}$$

where $D = (1 - \lambda_2 e^{-M})^{-1}$,

$$H(t,s) = \begin{cases} De^{-M(t-s)}, & 0 \le s \le t \le 1, \\ D\lambda_2 e^{-M(1+t-s)}, & 0 \le t \le s \le 1. \end{cases} \tag{2.7}$$

Proof. First, differentiating (2.6), we have

$$\begin{aligned} x'(t) &= (aDe^{-Mt} + \textstyle\int_0^1 H(t,s)(\sigma(s) - M_1 Ax(s) - M_2 Ax(\alpha(s)) \\ &\quad -M_3 TAx(s) - M_4 SAx(s))ds + \sum_{k=1}^{m} H(t,t_k)(-L_k Ax(t_k) + e_k))' \\ &= -M(t)[aDe^{-Mt} + \textstyle\int_0^1 H(t,s)(\sigma(s) - M_1 Ax(s) \\ &\quad -M_2 Ax(\alpha(s)) - M_3 TAx(s) - M_4 SAx(s))ds \\ &\quad + \sum_{k=1}^{m} H(t,t_k)(-L_k Ax(t_k) + e_k)] - M_1 Ax \\ &\quad -M_2 Ax(\alpha(t)) - M_3 TAx - M_4 SAx + \sigma(t) \\ &= -M(t)x(t) - M_1 Ax(t) - M_2 Ax(\alpha(t)) - M_3 TAx(t) - M_4 SAx(t) + \sigma(t) \end{aligned}$$

$$\begin{aligned} \Delta x(t_k) &= x(t_k^+) - x(t_k^-) \\ &= \sum_{0<t_j<t_k} \Delta x(t_j) - \sum_{0<t_j<t_k^-} \Delta x(t_j) \\ &= \sum_{j=1}^{k} (-L_j Ax(t_j) + e_j) - \sum_{j=1}^{k-1} (-L_j Ax(t_j) + e_j) \\ &= -L_k Ax(t_k) + e_k. \end{aligned}$$

Also

$$\begin{aligned} x(0) &= \lambda_2 D \textstyle\int_0^1 e^{-M(1-s)}(\sigma(s) - M_1 Ax(s) - M_2 Ax(\alpha(s)) \\ &\quad -M_3 TAx(s) - M_4 SAx(s))ds + \lambda_2 D \textstyle\sum_{k=1}^{m} e^{-M(1-t_k)} \Delta x(t_k) + aD \end{aligned}$$

$$\begin{aligned} x(1) &= D \textstyle\int_0^1 e^{-M(1-s)}(\sigma(s) - M_1 Ax(s) - M_2 Ax(\alpha(s)) \\ &\quad -M_3 TAx(s) - M_4 SAx(s))ds + e^{-M} aD \textstyle\sum_{k=1}^{m} e^{-M(1-t_k)} \Delta x(t_k) + aD. \end{aligned}$$

It is easy to check that $x(0) = \lambda_2 x(1) + a$.
Hence, we know that (2.6) is a solution of (2.5).
Next we show that the solution of (2.5) is unique. Let x_1, x_2 are the solutions of (2.5) and set $p = x_1 - x_2$, we get

$$\begin{aligned} p' &= x_1' - x_2' \\ &= -Mx_1(t) - M_1 Ax_1 - M_2 Ax_1(\alpha(t)) - M_3 TAx_1 - M_4 SAx_1 + \sigma(t) \\ &\quad -(-Mx_2(t) - M_1 Ax_2 - M_2 Ax_2(\alpha(t)) - M_3 TAx_2 - M_4 SAx_2 + \sigma(t)) \\ &= -Mp - M_1 Ap - M_2 Ap(\alpha(t)) - M_3 TAp - M_4 SAp, \end{aligned}$$

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$$\begin{aligned}
\Delta p(t_k) &= \Delta x_1 - \Delta x_2 \\
&= -L_k A x_1(t_k) + e_k - (-L_k A x_2(t_k) + e_k) \\
&= -L_k A p(t_k), \\
p(0) &= x_1(0) - x_2(0) \\
&= \lambda_2 x_1(T) + a - (\lambda_2 x_2(1) + a) \\
&= \lambda_2 p(1).
\end{aligned}$$

In view of Lemma 2.1, we get $p \leq \theta$ which implies $x_1 \leq x_2$. Similarly, we have $x_1 \geq x_2$. Hence $x_1 = x_2$. The proof is complete.

3 Results for first order impulsive differential equation

For convenience, let us list the following conditions:
(H_1) There exit $x_0, y_0 \in PC^1(J, E)$ satisfying

$$\begin{cases}
x_0'(t) \leq f(t, Ax_0(t), Ax_0(\alpha(t)), x_0(t), TAx_0, SAx_0) \quad t \neq t_k, \quad t \in J = [0,1] \\
\Delta x_0(t_k) \leq I_k(Ax_0(t_k), x_0(t_k)) \qquad k = 1, 2, \cdots, m \\
x_0(0) \leq \lambda_2 x_0(1) + \lambda_3 \int_0^1 w(s, Ax_0(s))ds + \mu x_0(\eta) + k_2 \\
y_0'(t) \geq f(t, Ay_0(t), Ay_0(\alpha(t)), y_0(t), TAy_0, SAy_0) \quad t \neq t_k, \quad t \in J = [0,1] \\
\Delta y_0(t_k) \geq I_k(Ay_0(t_k), y_0(t_k)) \qquad k = 1, 2, \cdots, m \\
y_0(0) \geq \lambda_2 y_0(T) + \lambda_3 \int_0^1 w(s, Ay_0(s))ds + \mu y_0(\eta) + k_2
\end{cases} \tag{3.1}$$

(H_2)

$$\begin{aligned}
&f(t, \overline{x}, \overline{y}, \overline{z}, \overline{u}, \overline{v}) - f(t, x, y, z, u, v) \\
&\geq -M_1(\overline{x} - x) - M_2(\overline{y} - y) - M(\overline{z} - z) - M_3(\overline{u} - u) - M_4(\overline{v} - v)
\end{aligned} \tag{3.2}$$

$$I_k(\overline{x}, \overline{z}) - I_k(x, z) \geq -L_k(\overline{x} - x) \tag{3.3}$$

Where $Ax_0 \leq x \leq \overline{x} \leq Ay_0$, $Ax_0(\alpha(t)) \leq y \leq \overline{y} \leq Ay_0(\alpha(t))$, $x_0 \leq z \leq \overline{z} \leq y_0$, $TAx_0 \leq u \leq \overline{u} \leq TAy_0$, $SAx_0 \leq v \leq \overline{v} \leq SAy_0$, $\forall t \in J$.
(H_3) Constants $L_k, M, M_i, i = 1, 2, 3, 4$ satisfy (2.3).
(H_4) Assume that $a(t)$ is non-negative integral function, such that

$$w(t, A\overline{u}) - w(t, Au) \geq a(t)(A\overline{u} - Au) \tag{3.4}$$

Where $x_0 \leq u \leq \overline{u} \leq y_0$.
If $x_0, y_0 \in PC^1(J, E)$ and $x_0 \leq y_0, t \in J$,then the interval $[x_0, y_0]$ denotes the set

$$\{x \in PC^1(J, E) : x_0(t) \leq x(t) \leq y_0(t), t \in J\}$$

Theorem 3.1 Assume the hypotheses $(H_1) - (H_4)$ hold. Then Eq.(2.1) has the extremal solutions $x^*(t), y^*(t) \in [x_0, y_0]$. Moreover there exist two iterative sequences $\{x_n\}$ and $\{y_n\}$ satisfying

$$x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots \leq y_n \leq \cdots \leq y_1 \leq y_0 \tag{3.5}$$

such that $\{x_n\}$,$\{y_n\}$ uniformly converge in $PC(J,E)\bigcap C^1(J^-,E)$ to x^*, y^*, respectively.

Proof. For $z \in [x_0, y_0]$, considering (2.5) with
$\sigma(t) = f(t, Az(t), Az(\alpha(t)), z, TAz, SAz) + M(t)z(t) + M_1Az(t) + M_2Az(\alpha(t)) + M_3TAz + M_4SAz$,
$e_k = I_k(Az(t_k), z(t_k)) + L_kAz(t_k)$,
$a = \lambda_3 \int_0^T w(s, Az(s)) + \mu z(\eta)ds + k_2$.
By Lemma 2.2, the BVPS has a unique solution $z \in [x_0, y_0]$.
We define an operator φ by $x = \varphi z$, then φ is an operator from $[x_0, y_0]$ to $PC(J, E)$.
We claim that
(a) $x_0 \le \varphi x_0$, $\quad \varphi y_0 \le y_0$,
(b) φ is nondecreasing on $[x_0, y_0]$.
We prove (a), let $x_1 = \varphi x_0$, $p(t) = x_0(t) - x_1(t)$

$$\begin{aligned}
p' &= x_0' - x_1' \\
&\le f(t, Ax_0(t), Ax_0(\alpha(t)), x_0(t), TAx_0, SAx_0) - [f(t, Ax_0(t), Ax_0(\alpha(t)), x_0(t), TAx_0, SAx_0) \\
&\quad +M(x_0(t) - x_1(t)) + M_1(Ax_0(t) - Ax_1(t)) + M_2(Ax_0(\alpha(t)) - Ax_1(\alpha(t))) \\
&\quad +M_3(TAx_0 - TAx_1) + M_4(SAx_0 - SAx_1)] \\
&= -Mp(t) - M_1Ap - M_2Ap(\alpha(t)) - M_3TAp - M_4SAp,
\end{aligned}$$

$$\begin{aligned}
\Delta p(t_k) &= \Delta x_0(t_k) - \Delta x_1(t_k) \\
&\le I_k(Ax_0(t_k), x_0(t_k)) - [I_k(Ax_0(t_k), x_0(t_k)) - L_k(Ax_1 - Ax_0)] \\
&= -L_kAp(t_k),
\end{aligned}$$

$$\begin{aligned}
p(0) &= x_0(0) - x_1(0) \\
&\le \lambda_2 x_0(1) + \mu u_0(\eta) + \lambda_3 \int_0^1 w(s, Ax_0(s))ds + k_2 \\
&\quad -(\lambda_2 u_1(1) + \mu u_0(\eta) + \lambda_3 \int_0^1 w(s, Ax_0(s))ds + k_2) \\
&= \lambda_1 p(1).
\end{aligned}$$

By Lemma 2.1, we have $p \le \theta$. That is $x_0 \le \varphi x_0$. Similarly, we can prove $\varphi y_0 \le y_0$.
To prove (b), let $x_1 = \varphi x_0$, $\quad y_1 = \varphi y_0$, $\quad p = x_1 - y_1$, then

$$\begin{aligned}
p'(t) &= x_1' - y_1' \\
&= f(t, Ax_0(t), Ax_0(\alpha(t)), x_0(t), TAx_0, SAx_0) + M(x_0(t) - x_1(t)) \\
&\quad +M_1(Ax_0(t) - Ax_1(t)) + M_2(Ax_0(\alpha(t)) - Ax_1(\alpha(t))) \\
&\quad +M_3(TAx_0 - TAx_1) + M_4(SAx_0 - SAx_1) \\
&\quad -[f(t, Ay_0(t), Ay_0(\alpha(t)), y_0(t), TAy_0, SAy_0) \\
&\quad +M(y_0(t) - y_1(t)) + M_1(Ay_0(t) - Ay_1(t)) \\
&\quad +M_2(Ay_0(\alpha(t)) - Ay_1(\alpha(t))) + M_3(TAy_0 - TAy_1) + M_4(SAy_0 - SAy_1)] \\
&\le -Mp(t) - M_1Ap - M_2Ap(\alpha(t)) - M_3TAp - M_4SAp,
\end{aligned}$$

$$\begin{aligned}
\Delta p(t_k) &= \Delta x_1(t_k) - \Delta y_1(t_k) \\
&= -L_kAx_1 + I_k(Ax_0(t_k), x_0(t_k)) + L_kAx_0 \\
&\quad -(-L_kAy_1 + I_k(Ay_0(t_k), y_0(t_k)) + L_kAy_0) \\
&\le -L_k(Ax_0 - Ay_0) + L_kAx_0 - L_kAy_0 - L_kAp \\
&\le -L_kAp(t_k),
\end{aligned}$$

$$\begin{aligned} p(0) &= x_1(0) - y_1(0) \\ &\le \lambda_2 x_1(1) + \mu x_0(\eta) + \lambda_3 \int_0^1 w(s, Ax_0(s))ds + k_2 \\ &\quad -(\lambda_2 y_1(1) + \mu y_0(\eta) + \lambda_3 \int_0^1 w(s, Ay_0(s))ds + k_2) \\ &= \lambda_2 p(1) + \mu(x_0(\eta) - y_0(\eta)) + \lambda_3 \int_0^1 a(s)(Ax_0(s) - Ay_0(s))ds \\ &\le \lambda_1 p(1). \end{aligned}$$

In view of Lemma 2.1 , we know $\varphi x_0 \le \varphi y_0$. Hence (b) holds.
We define two sequences $\{x_n\}$, $\{y_n\}$

$$x_{n+1} = \varphi x_n, \quad y_{n+1} = \varphi y_n, \quad (n = 0, 1, 2, \cdots)$$

By (a) and (b), we know that (3.5) holds.
And each x_n, y_n satisfies

$$\begin{cases} x_n'(t) = f(t, Ax_{n-1}(t), Ax_{n-1}(\alpha(t)), x_{n-1}(t), TAx_{n-1}, SAx_{n-1}) \\ +M(x_{n-1}(t) - x_1(t)) + M_1(Ax_{n-1}(t) - Ax_n(t)) + M_2(Ax_{n-1}(\alpha(t)) - Ax_n(\alpha(t))) \\ +M_3(TAx_{n-1} - TAx_n) + M_4(SAx_{n-1} - SAx_n) \quad t \ne t_k, \quad t \in J = [0,1] \\ \Delta x_n(t_k) = -L_k Ax_n(t_k) + I_k(Ax_{n-1}(t_k), x_{n-1}(t_k)) + L_k Ax_{n-1}(t_k) \qquad k = 1, 2, \cdots, m \\ x_n(0) = \lambda_2 x_n(1) + \lambda_3 \int_0^1 w(s, Ax_{n-1}(s))ds + \mu x_{n-1}(\eta) + k_2 \end{cases} \tag{3.6}$$

$$\begin{cases} y_n'(t) = f(t, Ay_{n-1}(t), Ay_{n-1}(\alpha(t)), y_{n-1}(t), TAy_{n-1}, SAy_{n-1}) \\ +M(y_{n-1}(t) - y_1(t)) + M_1(Ay_{n-1}(t) - Ay_n(t)) + M_2(Ay_{n-1}(\alpha(t)) - Ay_n(\alpha(t))) \\ +M_3(TAy_{n-1} - TAy_n) + M_4(SAy_{n-1} - SAy_n) \quad t \ne t_k, \quad t \in J = [0,1] \\ \Delta y_n(t_k) = -L_k Ay_n(t_k) + I_k(Ay_{n-1}(t_k), y_{n-1}(t_k)) + L_k Ay_{n-1}(t_k) \qquad k = 1, 2, \cdots, m \\ y_n(0) = \lambda_2 y_n(1) + \lambda_3 \int_0^1 w(s, Ay_{n-1}(s))ds + \mu y_{n-1}(\eta) + k_2. \end{cases} \tag{3.7}$$

By virtue of the regularity of the cone P, we obtain that there exist $x^*, y^* \in [x_0, y_0]$ such that

$$\lim_{n \longrightarrow \infty} x_n(t) = x^*(t) \qquad \lim_{n \longrightarrow \infty} y_n(t) = y^*(t) \tag{3.8}$$

and $\{x_n | n = 1, 2, \cdots\}$ is a bounded subset in $PC(J, E)$.
Let $X = \{x_n | n = 1, 2, \cdots\}$, $X(t) = \{x_n(t) | n = 1, 2, \cdots\}$ $t \in J$, in view of (3.8) we get

$$\alpha(X(t)) = 0 \qquad t \in J$$

which implies that $X(t)$ is relatively compact for $t \in J$.
For any $z \in [x_0, y_0]$, by (H_1) (H_2) we have

$$\begin{aligned} & x_0'(t) + Mx_0(t) + M_1 Ax_0(t) + M_2 Ax_0(\alpha(t)) + M_3 TAx_0 + M_4 SAx_0 \\ \le\ & f(t, Ax_0(t), Ax_0(\alpha(t)), x_0(t), TAx_0, SAx_0) + Mx_0(t) \\ & +M_1 Ax_0(t) + M_2 Ax_0(\alpha(t)) + M_3 TAx_0 + M_4 SAx_0 \\ \le\ & f(t, Az_0(t), Az_0(\alpha(t)), z_0(t), TAz_0, SAz_0) + Mz_0(t) \\ & +M_1 Az_0(t) + M_2 Az_0(\alpha(t)) + M_3 TAz_0 + M_4 SAz_0 \\ \le\ & f(t, Ay_0(t), Ay_0(\alpha(t)), y_0(t), TAy_0, SAy_0) + My_0(t) \\ & +M_1 Ay_0(t) + M_2 Ay_0(\alpha(t)) + M_3 TAy_0 + M_4 SAy_0 \\ \le\ & y_0'(t) + My_0(t) + M_1 Ay_0(t) + M_2 Ay_0(\alpha(t)) + M_3 TAy_0 + M_4 SAy_0. \end{aligned}$$

In view of the normality of the cone P_c, we get that there exists a constant $C > 0$, such that

$$\begin{aligned}&\parallel f(t, Az_0(t), Az_0(\alpha(t)), z_0(t), TAz_0, SAz_0) + Mz_0(t)\\&+M_1Az_0(t) + M_2Az_0(\alpha(t)) + M_3TAz_0 + M_4SAz_0 \parallel \leq C,\end{aligned}$$

$\forall z \in [x_0, y_0], t \in J$. From (3.5) (3.6), it is obviously to show that $\{x'_n | n = 1, 2, \cdots\}$ is a bounded subset in $PC(J, E)$. It follows in view of the mean value theorem that X is equicontinuous on $J_k,\ k = 0, 1, 2, \cdots, m$. So we obtain by virtue of Ascoli-Arzela's theorem and $\alpha(X(t)) = 0$ that $\alpha(X) = \sup_{t\in J} \alpha(X(t)) = 0$ which implies X is relatively compact in $PC(J, E)$ and so there exists a sequence of $\{x_n(t)\}$ which converges uniformly on J to $x^*(t)$. Since $\{x_n | n = 1, 2, \cdots\}$ is nondecreasing and the cone P_c is normal, we get that $\{x_n | n = 1, 2, \cdots\}$ itself converges uniformly on J to $x^*(t)$, which implies $x^* \in PC(J, E)$. By the lemma 2.2 and (3.6), we see that x^* satisfies (2.1).

Similarly, we also can prove that y_n converges uniformly on J to $y^*(t)$, and y^* satisfies (2.1).

Finally, we assert that if $z \in [x_0, y_0]$ is any solution of $Eq.(2.1)$,then $x^*(t) \leq z(t) \leq y^*(t)$ on J. We will prove that if $x_n \leq z \leq y_n$, for $n = 0, 1, 2, \cdots$, then $x_{n+1}(t) \leq z(t) \leq y_{n+1}(t)$.

Letting $p(t) = x_{n+1}(t) - z(t)$, then

$$\begin{cases} p'(t) \leq -Mp(t) - M_1Ap - M_2Ap(\alpha(t)) - M_3TAp - M_4SAp \leq 0 & t \neq t_k,\quad t \in J = [0, 1] \\ \Delta p(t_k) \leq -L_kAp(t_k) \qquad k = 1, 2, \cdots, m \\ p(0) \leq \lambda_2 p(1) \end{cases}$$

By Lemma 2.1 , we have $p(t) \leq \theta$ for all $t \in J$, that is $x_{n+1}(t) \leq z(t)$. Similarly , we can prove $z(t) \leq y_{n+1}(t)$. for all $t \in J$. Thus $x_{n+1}(t) \leq z(t) \leq y_{n+1}(t)$ for all $t \in J$, which implies $x^*(t) \leq z(t) \leq y^*(t)$.

The proof is complete.

Remark In (2.1), if $w(s, Ax(s)) = a(s)Ax(s)$, where $a(t)$ is non-negative integral function ,then (H_4) is not required in Theorem 3.1, and we have the following theorem.

Theorem 3.2 Suppose that conditions $(H_1) - (H_3)$are satisfied. In additional that $x_0, y_0 \in PC^1(J, E)$ be such that $x_0 \leq y_0$. Then the conclusion of Theorem 3.1 holds.

The proof is almost similar to theorem 3.1, so we omit it.

4 Results for second order impulsive differential equation

In this section, we prove the existence theorem of maximal and minimal solutions of (1.1) by applying Theorem 3.1 in Section 3.

Let us list other conditions for convenience.

(G_1) There exists $u_0, v_0 \in \Omega$, satisfying $u_0(t) \le v_0(t)$, $u_0'(t) \le v_0'(t)$,

$$\begin{cases} u_0''(t) \le f(t, u_0(t), u_0(\alpha(t)), u_0(t), Tu_0, Su_0) \quad t \ne t_k, \quad t \in J = [0,1] \\ \Delta u_0(t_k) = Q_k u_0'(t_k) \\ \Delta u_0'(t_k) \le I_k(u_0(t_k), u_0'(t_k)) \qquad\qquad k = 1, 2, \cdots, m \\ u_0(0) = \lambda_1 u_0(1) + k_1 \\ u_0'(0) \le \lambda_2 u_0'(1) + \lambda_3 \int_0^1 w(s, u_0(s))ds + \mu u_0'(\eta) + k_2 \end{cases} \tag{4.1}$$

and v_0 satisfies inverse inequalities of (4.1)
(G_2)

$$\begin{aligned} & f(t, \overline{x}, \overline{y}, \overline{z}, \overline{u}, SA\overline{v}) - f(t, x, y, z, u, v) \\ & \ge -M_1(\overline{x} - x) - M_2(\overline{y} - y) - M(\overline{z} - z) - M_3(\overline{u} - u) - M_4(\overline{v} - v) \end{aligned} \tag{4.2}$$

$$I_k(\overline{x}, \overline{z}) - I_k(x, z) \ge -L_k(\overline{x} - x) \tag{4.3}$$

Where $u_0 \le x \le \overline{x} \le v_0$, $u_0(\alpha(t)) \le y \le \overline{y} \le v_0(\alpha(t))$, $u_0' \le z \le \overline{z} \le v_0'$, $Tu_0 \le u \le \overline{u} \le Tv_0$, $Su_0 \le v \le \overline{v} \le Sv_0$, $\forall t \in J$.
(G_3) Constants $L_k, M, M_i, i = 1, 2, 3, 4$ satisfy (2.3).
(G_4) Assume that $a(t)$ is non-negative integral function, such that

$$w(t, \overline{u}) - w(t, u) \ge a(t)(\overline{u} - u) \tag{4.4}$$

Where $u_0 \le u \le \overline{u} \le v_0$.
Let $\Lambda = \{z \in [x_0, y_0] \bigcap PC^1(J, E) \mid u_0'(t) \le z'(t) \le v_0'(t)\}$.
Theorem 4.1 Assume the conditions $(G_1) - (G_4)$ hold.Then Eq.(1.1) has minimal and maximal solutions $u^*, v^* \in \Omega$ in Λ.
Proof. In Eq.(1.1),let $u'(t) = x(t)$. Then (1.1) is equivalent to the following system:

$$\begin{cases} u'(t) = x(t) \\ x'(t) = f(t, u, u(\alpha), x, Tu(t), Su(t)) \\ \Delta u(t_k) = Q_k x(t_k) \\ \Delta x(t_k) = I_k(x(t_k), u(t_k)) \\ u(0) = \lambda_1 u(1) + k_1 \\ x(0) = \lambda_2 x(1) + \lambda_3 \int_0^1 w(s, u(s))ds + \mu x(\eta) + k_2 \end{cases} \tag{4.5}$$

For $x \in PC(J, E)$, the system

$$\begin{cases} u'(t) = x(t) \\ \Delta u(t_k) = Q_k x(t_k) \\ u(0) = \lambda_1 u(1) + k_1 \end{cases} \tag{4.6}$$

has a unique solution $x \in PC(J, E) \bigcap C^1(J^-, E)$, which satisfies

$$u(t) = \frac{k_1}{1 - \lambda_1} + \int_0^1 G(t, s)x(s)ds + \sum_{k=1}^{m} G(t, t_k)Q_k x(t_k) \tag{4.7}$$

It is easy to prove, so we omit it .
Define an operator A by $u = Ax(t)$, $t \in J$. It is easy to show that
$A : PC(J,E)\bigcap C^1(J^-,E) \longrightarrow \Omega$ is continuous and nondecreasing .
Hence, from (4.5)-(4.7), Eq.(1.1) is transformed into first order boundary value problem (2.1).

Let $x_0 = u_0', y_0 = v_0'$, by (G_1) we have $x_0 \leq y_0$ and

$$u_0(t) = \frac{k_1}{1-\lambda_1} + \int_0^1 G(t,s)x_0(s)ds + \sum_{k=1}^{m} G(t,t_k)Q_k x_0(t_k) \tag{4.8}$$

$$v_0(t) = \frac{k_1}{1-\lambda_1} + \int_0^1 G(t,s)y_0(s)ds + \sum_{k=1}^{m} G(t,t_k)Q_k y_0(t_k) \tag{4.9}$$

which imply that $u_0 = Ax_0$, $v_0 = Ay_0$, and x_0, y_0 satisfies (H_1).
By the condition (G_2) (G_4) it is easy to see that (H_2) (H_4) hold .
Therefore, it follows from Theorem 3.1 that (2.1) has minimal and maximal solutions $x^*, y^* \in PC(J,E)\bigcap C^1(J^-,E)$ in $[x_0, y_0]$.
Let $u^* = Ax^*$, $v^* = Ay^*$, then $u^*, v^* \in \Omega$ and

$$u^*(t) = \frac{k_1}{1-\lambda_1} + \int_0^1 G(t,s)x^*(s)ds + \sum_{k=1}^{m} G(t,t_k)Q_k x^*(t_k) \tag{4.10}$$

In view of (4.10), we have

$$\begin{cases} u^{*'}(t) = x^*(t) \\ \Delta u^*(t_k) = Q_k x^*(t_k) \\ u^*(0) = \lambda_1 u^*(1) + k_1 \end{cases} \tag{4.11}$$

The fact that x^* satisfies (2.1) and u^* satisfies (4.11) implies u^* is a solution of (1.1). Similarly, we can prove v^* is a solution of (1.1).
It is easy to show that $u^*, v^* \in \Omega$ are minimal and maximal solutions for (1.1) in Λ. We complete the proof.
Remark In (1.1), if $w(s, x(s)) = a(s)x(s)$, where $a(t)$ is non-negative integral function, then (H_4) is not required in Theorem 4.1, and we have the following theorem.
Theorem 4.2 Suppose that conditions $(G_1) - (G_3)$are satisfied. Then the conclusion of Theorem 4.1 holds.
The proof is almost similar to theorem 4.1, so we omit it.

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DISTRIBUTION AND SURVIVAL FUNCTIONS WITH APPLICATIONS IN INTUITIONISTIC RANDOM LIE C^*-ALGEBRAS

AFRAH A. N. ABDOU, YEOL JE CHO*, AND REZA SAADATI

ABSTRACT. In this paper, first, we consider the distribution and survival functions and we define intuitionistic random Lie C^*-algebras. As an application, using the fixed point method, we approximate the derivations on intuitionistic random Lie C^*-algebras for the the following additive functional equation

$$\sum_{i=1}^{m} f\Big(mx_i + \sum_{j=1,\, j\neq i}^{m} x_j\Big) + f\Big(\sum_{i=1}^{m} x_i\Big) = 2f\Big(\sum_{i=1}^{m} mx_i\Big)$$

for all $m \in \mathbb{N}$ with $m \geq 2$.

1. Introduction

Distribution and survival functions are important in probability theory. In this paper, we use these functions to define intuitionistic random Lie C^*-algebras and find an approximation of an m-variable functional equation.

2. Preliminaries

Now, we give some definitions and lemmas for our main results in this paper.

Definition 2.1. A function $\mu : \mathbb{R} \to [0, 1]$ is called a *distribution function* if it is left continuous on $\mathbb{R}$, non-decreasing and

$$\inf_{t\in\mathbb{R}} \mu(t) = 0, \quad \sup_{t\in\mathbb{R}} \mu(t) = 1.$$

We denote by D the family of all measure distribution functions and by H a special element of D defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Forward, $\mu(x)$ is denoted by μ_x.

Definition 2.2. A function $\nu : \mathbb{R} \to [0, 1]$ is called a *survival function* if it is right continuous on $\mathbb{R}$, non-increasing and

$$\inf_{t\in\mathbb{R}} \nu(t) = 0, \quad \sup_{t\in\mathbb{R}} \nu(t) = 1.$$

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*The corresponding author.

We denote by B the family of all survival functions and by G a special element of B defined by

$$G(t)=\begin{cases}1, & \text{if } t\le 0,\\ 0, & \text{if } t>0.\end{cases}$$

Forward, $\nu(x)$ is denoted by ν_x.

Lemma 2.3. ([1]) *Consider the set L^* and the operation $\le_{L^*}$ defined by:*

$$L^*=\{(x_1,x_2):(x_1,x_2)\in[0,1]^2 \text{ and } x_1+x_2\le 1\},$$

$$(x_1,x_2)\le_{L^*}(y_1,y_2)\Longleftrightarrow x_1\le y_1,\ x_2\ge y_2$$

for all $(x_1,x_2),(y_1,y_2)\in L^*$. Then $(L^*,\le_{L^*})$ is a complete lattice.

We denote the bottom and the top elements of lattices by $0_{L^*}=(0,1)$ and $1_{L^*}=(1,0)$. Classically, the *triangular norm* $*=T$ on $[0,1]$ is defined as an increasing, commutative and associative mapping $T:[0,1]^2\longrightarrow[0,1]$ satisfying

$$T(1,x)=1*x=x$$

for all $x\in[0,1]$. The *triangular conorm* $S=\diamond$ is defined as an increasing, commutative, associative mapping $S:[0,1]^2\longrightarrow[0,1]$ satisfying $S(0,x)=0\diamond x=x$ for all $x\in[0,1]$.

Using the lattice $(L^*,\le_{L^*})$, these definitions can be straightforwardly extended.

Definition 2.4. ([1]) A *triangular norm* (*t–norm*) on L^* is a mapping $\mathcal{T}:(L^*)^2\longrightarrow L^*$ satisfying the following conditions:
(1) for all $x\in L^*$, $\mathcal{T}(x,1_{L^*})=x$ (: boundary condition);
(2) for all $(x,y)\in(L^*)^2$, $\mathcal{T}(x,y)=\mathcal{T}(y,x)$ (: commutativity);
(3) for all $(x,y,z)\in(L^*)^3$, $\mathcal{T}(x,\mathcal{T}(y,z))=\mathcal{T}(\mathcal{T}(x,y),z)$ (: associativity);
(4) for all $(x,x',y,y')\in(L^*)^4$, $x\le_{L^*}x'$ and $y\le_{L^*}y'\Longrightarrow\mathcal{T}(x,y)\le_{L^*}\mathcal{T}(x',y')$ (: monotonicity).

In this paper, $(L^*,\le_{L^*},\mathcal{T})$ has an Abelian topological monoid with the top element 1_{L^*} and so $\mathcal{T}$ is a *continuous t–norm.*

Definition 2.5. A continuous t–norm $\mathcal{T}$ on L^* is said to be *continuous representable t-norm* if there exist a continuous t–norm $*$ and a continuous t–conorm $\diamond$ on $[0,1]$ such that, for all $x=(x_1,x_2),y=(y_1,y_2)\in L^*$,

$$\mathcal{T}(x,y)=(x_1*y_1,x_2\diamond y_2).$$

For example,

$$\mathcal{T}(a,b)=(a_1b_1,\min\{a_2+b_2,1\})$$

and

$$\mathbf{M}(a,b)=(\min\{a_1,b_1\},\max\{a_2,b_2\})$$

for all $a=(a_1,a_2)$, $b=(b_1,b_2)\in L^*$ are the continuous representable t-norm.

Definition 2.6. (1) A *negator* on L^* is any decreasing mapping $\mathcal{N}:L^*\longrightarrow L^*$ satisfying $\mathcal{N}(0_{L^*})=1_{L^*}$ and $\mathcal{N}(1_{L^*})=0_{L^*}$.
(2) If $\mathcal{N}(\mathcal{N}(x))=x$ for all $x\in L^*$, then $\mathcal{N}$ is called an *involutive negator*.

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(3) A *negator* on $[0,1]$ is a decreasing mapping $N : [0,1] \longrightarrow [0,1]$ satisfying $N(0) = 1$ and $N(1) = 0$, where N_s denotes the *standard negator* on $[0,1]$ defined by

$$N_s(x) = 1 - x$$

for all $x \in [0,1]$.

Definition 2.7. Let μ and ν be a distribution function and a survival function from $X \times (0, +\infty)$ to $[0,1]$ such that $\mu_x(t) + \nu_x(t) \leq 1$ for all $x \in X$ and $t > 0$. The 3-tuple $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be an *intuitionistic random normed space* (briefly, IRN-space) if X is a vector space, $\mathcal{T}$ is a continuous representable t-norm and $\mathcal{P}_{\mu,\nu}$ is a mapping $X \times (0, +\infty) \to L^*$ satisfying the following conditions: for all $x, y \in X$ and $t, s > 0$,

(1) $\mathcal{P}_{\mu,\nu}(x, 0) = 0_{L^*}$;

(2) $\mathcal{P}_{\mu,\nu}(x, t) = 1_{L^*}$ if and only if $x = 0$;

(3) $\mathcal{P}_{\mu,\nu}(\alpha x, t) = \mathcal{P}_{\mu,\nu}(x, \frac{t}{\alpha})$ for all $\alpha \neq 0$;

(4) $\mathcal{P}_{\mu,\nu}(x + y, t + s) \geq_{L^*} \mathcal{T}(\mathcal{P}_{\mu,\nu}(x, t), \mathcal{P}_{\mu,\nu}(y, s))$.

In this case, $\mathcal{P}_{\mu,\nu}$ is called an *intuitionistic random norm*, where

$$\mathcal{P}_{\mu,\nu}(x, t) = (\mu_x(t), \nu_x(t)).$$

Note that, if $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is an IRN-space and define $\mathcal{P}_{\mu,\nu}(x - y, t) = \mathcal{M}_{\mu,\nu}(x, y, t)$, then

$$(X, \mathcal{M}_{\mu,\nu}, \mathcal{T})$$

is an *intuitionistic Menger spaces*.

Example 2.8. Let $(X, \|\cdot\|)$ be a normed space. Let $\mathcal{T}(a, b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$ for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in L^*$ and μ, ν be a distribution function and a survival function defined by

$$\mathcal{P}_{\mu,\nu}(x, t) = (\mu_x(t), \nu_x(t)) = \Big(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|}\Big)$$

for all $t \in \mathbb{R}^+$. Then $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is an IRN-space.

Definition 2.9. (1) A sequence $\{x_n\}$ in an IRN-space $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is called a *Cauchy sequence* if, for any $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\mathcal{P}_{\mu,\nu}(x_n - x_m, t) >_{L^*} (N_s(\varepsilon), \varepsilon)$$

for all $n, m \geq n_0$, where N_s is the standard negator.

(2) A sequence $\{x_n\}$ in an IRN-space $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be *convergent* to a point $x \in X$ (denoted by $x_n \stackrel{\mathcal{P}_{\mu,\nu}}{\longrightarrow} x$) if $\mathcal{P}_{\mu,\nu}(x_n - x, t) \longrightarrow 1_{L^*}$ as $n \longrightarrow \infty$ for all $t > 0$.

(3) An IRN-space $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be *complete* if every Cauchy sequence in X is convergent to a point $x \in X$.

Definition 2.10. A *intuitionistic random normed algebra* $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T}, \mathcal{T}')$ is a IRN-space $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ with algebraic structure such that

(4) $\mathcal{P}_{\mu,\nu}(xy, ts) \geq \mathcal{T}'(\mathcal{P}_{\mu,\nu}(x, t), \mathcal{P}_{\mu,\nu}(y, s))$ for all $x, y \in X$ and $t, s > 0$, in which $\mathcal{T}'$ is a continuous representable t-norm.

Every normed algebra $(X, \|\cdot\|)$ defines a random normed algebra (X, μ, T_M, T_P), where

$$\mathcal{P}_{\mu,\nu}(x,t) = \Big(\frac{t}{t+\|x\|}, \frac{\|x\|}{t+\|x\|}\Big)$$

for all $t > 0$ if and only if

$$\|xy\| \leq \|x\|\|y\| + s\|y\| + t\|x\|$$

for all $x, y \in X$ and $t, s > 0$. This space is called the *induced random normed algebra* (see [6]). For more properties and example of theory of random normed spaces, we refer to [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24].

Definition 2.11. Let $(\mathcal{U}, \mathcal{P}_{\mu,\nu}, \mathcal{T}, \mathcal{T}')$ be an intuitionistic random Banach algebra. An *involution* on $\mathcal{U}$ is a mapping $u \to u^*$ from $\mathcal{U}$ into $\mathcal{U}$ satisfying the following conditions:

(1) $u^{**} = u$ for all $u \in \mathcal{U}$;
(2) $(\alpha u + \beta v)^* = \overline{\alpha} u^* + \overline{\beta} v^*$ for all $u, v \in \mathcal{U}$ and $\alpha, \beta \in \mathbb{C}$;
(3) $(uv)^* = v^* u^*$ for all $u, v \in \mathcal{U}$.

If, in addition, $\nu_{u^*u}(ts) = T'(\nu_u(t), \nu_u(s))$ for all $u \in \mathcal{U}$ and $t, s > 0$, then $\mathcal{U}$ is an *intuitionistic random C^*–algebra*.

Now, we recall a fundamental result in fixed point theory.

Let Ω be a set. A function $d : \Omega \times \Omega \to [0, \infty]$ is called a *generalized metric* on Ω if d satisfies the following conditions:

(1) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in \Omega$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in \Omega$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in \Omega$.

Theorem 2.12. ([2]) *Let (Ω, d) be a complete generalized metric space and let $J : \Omega \to \Omega$ be a contractive mapping with Lipschitz constant $L < 1$. Then, for each given element $x \in \Omega$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*

(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $\Gamma = \{y \in \Omega \mid d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in \Gamma$.*

In this paper, using the fixed point method, we approximate the derivations on intuitionistic random Lie C^*-algebras for the the following additive functional equation

$$\sum_{i=1}^{m} f\Big(mx_i + \sum_{j=1,\, j\neq i}^{m} x_j\Big) + f\Big(\sum_{i=1}^{m} x_i\Big) = 2f\Big(\sum_{i=1}^{m} mx_i\Big) \tag{2.1}$$

for all $m \in \mathbb{N}$ with $m \geq 2$.

3. Approximation of derivations in intuitionistic random Lie C^*-algebras

In this section, we approximate the derivations on intuitionistic random Lie C^*-algebras (see also [32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 44]).

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For any mapping $f : A \to A$, we define

$$D_\omega f(x_1, \cdots, x_m) := \sum_{i=1}^{m} \mu f\Big(mx_i + \sum_{j=1,\, j\neq i}^{m} x_j\Big) + f\Big(\mu \sum_{i=1}^{m} x_i\Big) - 2f\Big(\mu \sum_{i=1}^{m} mx_i\Big)$$

for all $\omega \in \mathbb{T}^1 := \{\xi \in \mathbb{C} : |\xi| = 1\}$ and $x_1, \cdots, x_m \in A$.

Note that a $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *derivation* on intuitionistic random C^*-algebras if δ satisfies $\delta(xy) = y\delta(x) + x\delta(y)$ and $\delta(x^*) = \delta(x)^*$ for all $x, y \in A$.

Now, we approximate the derivations on intuitionistic random Lie C^*-algebras for the functional equation $D_\omega f(x_1, \cdots, x_m) = 0$.

Theorem 3.1. *Let $f : A \to A$ be a mapping for which there are functions $\varphi : A^m \to L^*$, $\psi : A^2 \to L^*$ and $\eta : A \to L^*$ such that*

$$\mathcal{P}_{\mu,\nu}(D_\omega f(x_1, \cdots, x_m), t) \geq_L \varphi(x_1, \cdots, x_m, t), \tag{3.1}$$

$$\lim_{j\to\infty} \varphi(m^j x_1, \cdots, m^j x_m, m^j t) = 1_{\mathcal{L}}, \tag{3.2}$$

$$\mathcal{P}_{\mu,\nu}(f(xy) - xf(y) - xf(y), t) \geq_L \psi(x, y, t), \tag{3.3}$$

$$\lim_{j\to\infty} \psi(m^j x, m^j y, m^{2j} t) = 1_{\mathcal{L}}, \tag{3.4}$$

$$\mathcal{P}_{\mu,\nu}(f(x^*) - f(x)^*, t) \geq_L \eta(x, t), \tag{3.5}$$

$$\lim_{j\to\infty} \eta(m^j x, m^j t) = 1_{\mathcal{L}} \tag{3.6}$$

for all $\omega \in \mathbb{T}^1$, $x_1, \cdots, x_m, x, y \in A$ and $t > 0$. If there exists $R < 1$ such that

$$\varphi(mx, 0, \cdots, 0, mRt) \geq_L \varphi(x, 0, \cdots, 0, t) \tag{3.7}$$

for all $x \in A$ and $t > 0$, then there exists a unique derivation $\delta : A \to A$ such that

$$\mathcal{P}_{\mu,\nu}(f(x) - \delta(x), t) \geq_L \varphi(x, 0, \cdots, 0, (m - mR)t) \tag{3.8}$$

for all $x \in A$ and $t > 0$.

Proof. Consider the set $X := \{g : A \to A\}$ and introduce the *generalized metric* on X defined by

$$d(g, h) = \inf\{C \in \mathbb{R}_+ : \mathcal{P}_{\mu,\nu}(g(x) - h(x), Ct) \geq_L \varphi(x, 0, \cdots, 0, t),\ \forall x \in A,\ t > 0\}.$$

It is easy to show that (X, d) is complete.

Now, we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{1}{m} g(mx)$$

for all $x \in A$. By Theorem 3.1 of [46],

$$d(Jg, Jh) \leq Rd(g, h)$$

for all $g, h \in X$. Letting $\omega = 1$, $x = x_1$ and $x_2 = \cdots = x_m = 0$ in (3.1), we have

$$\mathcal{P}_{\mu,\nu}(f(mx) - mf(x), t) \geq_L \varphi(x, 0, \cdots, 0, t) \tag{3.9}$$

for all $x \in A$ and $t > 0$ and so

$$\mathcal{P}_{\mu,\nu}(f(x) - \frac{1}{m} f(mx), t) \geq_L \varphi(x, 0, \cdots, 0, mt)$$

for all $x \in A$ and $t > 0$. Hence $d(f, Jf) \leq \frac{1}{m}$. By Theorem 2.12, there exists a mapping $\delta : A \to A$ such that

(1) δ is a fixed point of J, i.e.,

$$\delta(mx) = m\delta(x) \tag{3.10}$$

for all $x \in A$. The mapping δ is a unique fixed point of J in the set

$$Y = \{g \in X : d(f, g) < \infty\}.$$

This implies that δ is a unique mapping satisfying (3.10) such that there exists $C \in (0, \infty)$ satisfying

$$\mathcal{P}_{\mu,\nu}(\delta(x) - f(x), Ct) \geq_L \varphi(x, 0, \cdots, 0, t)$$

for all $x \in A$ and $t > 0$.

(2) $d(J^n f, \delta) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(m^n x)}{m^n} = \delta(x) \tag{3.11}$$

for all $x \in A$.

(3) $d(f, \delta) \leq \frac{1}{1-R} d(f, Jf)$, which implies the inequality $d(f, \delta) \leq \frac{1}{m-mR}$. This implies that the inequality (3.8) holds.

Thus it follows from (3.1), (3.2) and (3.11) that

$$\begin{aligned}
&\mathcal{P}_{\mu,\nu}\Big(\sum_{i=1}^{m} \delta\Big(mx_i + \sum_{j=1,\, j\neq i}^{m} x_j\Big) + \delta\Big(\sum_{i=1}^{m} x_i\Big) - 2\delta\Big(\sum_{i=1}^{m} mx_i\Big), t\Big) \\
= &\lim_{n\to\infty} \mathcal{P}_{\mu,\nu}\Big(\sum_{i=1}^{m} f\Big(m^{n+1}x_i + \sum_{j=1,\, j\neq i}^{m} m^n x_j\Big) + f\Big(\sum_{i=1}^{m} m^n x_i\Big) - 2f\Big(\sum_{i=1}^{m} m^{n+1} x_i\Big), m^n t\Big) \\
\leq_L &\lim_{n\to\infty} \varphi(m^n x_1, \cdots, m^n x_m, m^n t) \\
= &1_{\mathcal{L}}
\end{aligned}$$

for all $x_1, \cdots, x_m \in A$ and $t > 0$ and so

$$\sum_{i=1}^{m} \delta\Big(mx_i + \sum_{j=1,\, j\neq i}^{m} x_j\Big) + \delta\Big(\sum_{i=1}^{m} x_i\Big) = 2\delta\Big(\sum_{i=1}^{m} mx_i\Big)$$

for all $x_1, \cdots, x_m \in A$.

By a similar method to above, we get

$$\omega\delta(mx) = \delta(m\omega x)$$

for all $\omega \in \mathbb{T}^1$ and $x \in A$. Thus one can show that the mapping $H : A \to A$ is $\mathbb{C}$-linear.

Also, it follows from (3.3), (3.4) and (3.11) that

$$\begin{aligned}
&\mathcal{P}_{\mu,\nu}(\delta(xy) - y\delta(x) - x\delta(y), t) \\
= &\lim_{n\to\infty} \mathcal{P}_{\mu,\nu}(f(m^n xy) - m^n y f(m^n x) - m^n x f(m^n y), m^n t) \\
\leq &\lim_{n\to\infty} \psi(m^n x, m^n y, m^{2n} t) \\
= &1_{\mathcal{L}}
\end{aligned}$$

for all $x, y \in A$ and so

$$\delta(xy) = y\delta(x) + x\delta(y)$$

for all $x, y \in A$. Thus $\delta : A \to A$ is a derivation satisfying (3.7), as desired.

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Also, Similarly, by (3.5), (3.6) and (3.11), we have $\delta(x^*) = \delta(x)^*$. This completes the proof. □

4. **Approximation of derivations on intuitionistic random Lie C^*-algebras**

An intuitionistic random C^*-algebra $\mathcal{C}$, endowed with the Lie product

$$[x, y] := \frac{xy - yx}{2}$$

in $\mathcal{C}$, is called a *intuitionistic random Lie C^*-algebra.*

Definition 4.1. Let A and B be intuitionistic random Lie C^*-algebras. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called an *intuitionistic random Lie C^*-algebra derivation* if

$$\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$$

for all $x, y \in \mathcal{A}$.

Throughout this Section, assume that A is an intuitionistic random Lie C^*-algebra with norm $\mathcal{P}_{\mu,\nu}$.

Now, we approximate the derivations on intuitionistic random Lie C^*-algebras for the functional equation

$$D_\omega f(x_1, \cdots, x_m) = 0.$$

Theorem 4.2. *Let $f : A \to A$ be a mapping for which there are functions $\varphi : A^m \to L^*$ and $\psi : A^2 \to L^*$ such that*

$$\lim_{j\to\infty} \varphi(m^j x_1, \cdots, m^j x_m, m^j t) = 1_{\mathcal{L}}, \tag{4.1}$$

$$\mathcal{P}_{\mu,\nu}(D_\omega f(x_1, \cdots, x_m), t) \geq_L \varphi(x_1, \cdots, x_m, t), \tag{4.2}$$

$$\mathcal{P}_{\mu,\nu}(f([x, y]) - [f(x), y] - [x, f(y)], t) \geq_L \psi(x, y, t), \tag{4.3}$$

$$\lim_{j\to\infty} \psi(m^j x, m^j y, m^{2j} t) = 1_{\mathcal{L}} \tag{4.4}$$

for all $\omega \in \mathbb{T}^1$, $x_1, \cdots, x_m, x, y \in A$ and $t > 0$. If there exists $R < 1$ such that

$$\varphi(mx, 0, \cdots, 0, mx) \geq_L \varphi(x, 0, \cdots, 0, t)$$

for all $x \in A$ and $t > 0$, then there exists a unique homomorphism $\delta : A \to A$ such that

$$\mathcal{P}_{\mu,\nu} f(x) - \delta(x), t) \geq_L \varphi(x, 0, \cdots, 0, (m - mR)t) \tag{4.5}$$

for all $x \in A$ and $t > 0$.

Proof. By the same reasoning as the proof of Theorem 3.1, we can find the mapping $\delta : A \to A$ given by

$$\delta(x) = \lim_{n\to\infty} \frac{f(m^n x)}{m^n}$$

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for all $x \in A$. It follows from (4.3) that

$$\begin{aligned} & \mathcal{P}_{\mu,\nu}(\delta([x,y]) - [\delta(x),y] - [x,\delta(y)],t) \\ = \quad & \lim_{n\to\infty} \mathcal{P}_{\mu,\nu}(f(m^{2n}[x,y]) - [f(m^n x),\cdot m^n y] - [m^n x, f(m^n y)], m^{2n}t) \\ \geq_L \quad & \lim_{n\to\infty} \psi(m^n x, m^n y, m^{2n}t) = 1_{\mathcal{L}} \end{aligned}$$

for all $x, y \in A$ and $t > 0$ and so

$$\delta([x,y]) = [\delta(x),y] + [x,\delta(y)]$$

for all $x, y \in A$. Thus $\delta : A \to B$ is an intuitionistic random Lie C^*-algebra derivation satisfying (4.5). This completes the proof. □

Corollary 4.3. *Let $0 < r < 1$ and θ be nonnegative real numbers and $f : A \to A$ be a mapping such that*

$$\begin{aligned} & \mathcal{P}_{\mu,\nu}(D_\omega f(x_1,\cdots,x_m),t) \\ \geq_L \quad & \Big(\frac{t}{t+\theta(\|x_1\|_A^r + \|x_2\|_A^r + \cdots + \|x_m\|_A^r)}, \frac{\theta(\|x_1\|_A^r + \|x_2\|_A^r + \cdots + \|x_m\|_A^r)}{t+\theta(\|x_1\|_A^r + \|x_2\|_A^r + \cdots + \|x_m\|_A^r)}\Big), \end{aligned}$$

$$\begin{aligned} & \mathcal{P}_{\mu,\nu}(f([x,y]) - [f(x),y] - [x,f(y)],t) \\ \geq_L \quad & \Big(\frac{t}{t+\theta\cdot\|x\|_A^r\cdot\|y\|_A^r}, \frac{\theta\cdot\|x\|_A^r\cdot\|y\|_A^r}{t+\theta\cdot\|x\|_A^r\cdot\|y\|_A^r}\Big) \end{aligned}$$

for all $\omega \in \mathbb{T}^1$, $x_1,\cdots,x_m,x,y \in A$ and $t > 0$. Then there exists a unique derivation $\delta : A \to A$ such that

$$\mathcal{P}_{\mu,\nu}(f(x) - \delta(x),t) \leq_L \Big(\frac{t}{t+\frac{\theta}{m-m^r}\|x\|_A^r}, \frac{\frac{\theta}{m-m^r}\|x\|_A^r}{t+\frac{\theta}{m-m^r}\|x\|_A^r}\Big)$$

for all $x \in A$ and $t > 0$.

Proof. The proof follows from Theorem 4.2 by taking

$$\begin{aligned} & \varphi(x_1,\cdots,x_m,t) \\ = \quad & \Big(\frac{t}{t+\theta(\|x_1\|_A^r + \|x_2\|_A^r + \cdots + \|x_m\|_A^r)}, \frac{\theta(\|x_1\|_A^r + \|x_2\|_A^r + \cdots + \|x_m\|_A^r)}{t+\theta(\|x_1\|_A^r + \|x_2\|_A^r + \cdots + \|x_m\|_A^r)}\Big), \end{aligned}$$

$$\psi(x,y,t) := \Big(\frac{t}{t+\theta\cdot\|x\|_A^r\cdot\|y\|_A^r}, \frac{\theta\cdot\|x\|_A^r\cdot\|y\|_A^r}{t+\theta\cdot\|x\|_A^r\cdot\|y\|_A^r}\Big)$$

and

$$R = m^{r-1}$$

for all $x_1,\cdots,x_m,x,y \in A$ and $t > 0$. This completes the proof. □

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(Afrah A. N. Abdou) Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia
E-mail address: aabdou@kau.edu.sa

(Yeol Je Cho) Department of Mathematics Education and the RINS, Gyeongsang National University, Chinju 660-701, Korea, and Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia
E-mail address: yjcho@gnu.ac.kr

(Reza Saadati) Department of Mathematics, Iran University of Science and Technology, Tehran, Iran
E-mail address: rsaadati@iust.ac.ir

CUBIC ρ-FUNCTIONAL INEQUALITY AND QUARTIC ρ-FUNCTIONAL INEQUALITY

CHOONKIL PARK, JUNG RYE LEE*, AND DONG YUN SHIN

ABSTRACT. In this paper, we solve the following cubic ρ-functional inequality

$$\begin{aligned}&\|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)\|\\ &\leq \left\|\rho\left(4f\left(x+\frac{y}{2}\right)+4f\left(x-\frac{y}{2}\right)-f(x+y)-f(x-y)-6f(x)\right)\right\|, \qquad (0.1)\end{aligned}$$

where ρ is a fixed complex number with $|\rho|<2$, and the quartic ρ-functional inequality

$$\begin{aligned}&\|f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y)\| \qquad (0.2)\\ &\leq \left\|\rho\left(8f\left(x+\frac{y}{2}\right)+8f\left(x-\frac{y}{2}\right)-2f(x+y)-2f(x-y)-12f(x)+3f(y)\right)\right\|,\end{aligned}$$

where ρ is a fixed complex number with $|\rho|<2$.

Using the direct method, we prove the Hyers-Ulam stability of the cubic ρ-functional inequality (0.1) and the quartic ρ-functional inequality (0.2) in complex Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [17] concerning the stability of group homomorphisms. Hyers [8] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [12] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

In [9], Jun and Kim considered the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \qquad (1.1)$$

It is easy to show that the function $f(x)=x^3$ satisfies the functional equation (1.1), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*. We can define the following Jensen type cubic functional equation

$$4f\left(x+\frac{y}{2}\right)+4f\left(x-\frac{y}{2}\right)=f(x+y)+f(x-y)+6f(x).$$

Note that if $f(2x)=8f(x)$ then the Jensen type cubic functional equation is equivalent to the cubic functional equation (1.1).

In [10], Lee et al. considered the following quartic functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \qquad (1.2)$$

It is easy to show that the function $f(x)=x^4$ satisfies the functional equation (1.2), which is called a *quartic functional equation* and every solution of the quartic functional equation is said

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*Corresponding author (AMAT 2015).

to be a *quartic mapping*. We can define the following Jensen type quartic functional equation

$$8f\left(x+\frac{y}{2}\right)+8f\left(x-\frac{y}{2}\right)=2f(x+y)+2f(x-y)+12f(x)-3f(y).$$

Note that if $f(2x)=16f(x)$ then the Jensen type quartic functional equation is equivalent to the quartic functional equation (1.2).

Recently, considerable attention has been increasing to the problem of the Hyers-Ulam stability of functional equations. Several Hyers-Ulam stability results concerning Cauchy, Jensen, quadratic, cubic and quartic functional equations have been investigated in [1, 3, 13, 14, 15, 16, 18].

In [6], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x)+2f(y)-f(xy^{-1})\|\le\|f(xy)\| \tag{1.3}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x)+2f(y)=f(xy)+f(xy^{-1}).$$

Gilányi [7] and Fechner [4] proved the Hyers-Ulam stability of the functional inequality (1.3). Park, Cho and Han [11] proved the Hyers-Ulam stability of additive functional inequalities.

In Section 3, we solve the cubic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the cubic ρ-functional inequality (0.1) in complex Banach spaces.

In Section 4, we solve the quartic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the quartic ρ-functional inequality (0.2) in complex Banach spaces.

Throughout this paper, assume that X is a complex normed space and that Y is a complex Banach space.

2. Cubic ρ-functional inequality (0.1)

Throughout this section, assume that ρ is a fixed complex number with $|\rho|<2$.

In this section, we solve and investigate the cubic ρ-functional inequality (0.1) in complex normed spaces.

Lemma 2.1. *Let X and Y be vector spaces. A mapping $f: X\to Y$ satisfies $f(2x)=8f(x)$ and*

$$4f\left(x+\frac{y}{2}\right)+4f\left(x-\frac{y}{2}\right)=f(x+y)+f(x-y)+6f(x)$$

if and only if the mapping $f: X\to Y$ is a cubic mapping.

Proof. One can easily prove it. We omit the proof. □

Lemma 2.2. *If a mapping $f: X\to Y$ satisfies*

$$\begin{aligned}&\|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)\|\\&\le\left\|\rho\left(4f\left(x+\frac{y}{2}\right)+4f\left(x-\frac{y}{2}\right)-f(x+y)-f(x-y)-6f(x)\right)\right\|\end{aligned} \tag{2.1}$$

for all $x,y\in X$, then $f: X\to Y$ is cubic.

Proof. Assume that $f: X\to Y$ satisfies (2.1).

Letting $x=y=0$ in (2.1), we get $\|-14f(0)\|\le|\rho|\|0\|=0$. So $f(0)=0$.

Letting $y=0$ in (2.1), we get $\|2f(2x)-16f(x)\|\le 0$ and so $f(2x)=8f(x)$ for all $x\in X$. Thus

$$f\left(\frac{x}{2}\right)=\frac{1}{8}f(x) \tag{2.2}$$

for all $x\in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned}
&\|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)\| \\
&\le \left\|\rho\left(4f\left(x+\frac{y}{2}\right)+4f\left(x-\frac{y}{2}\right)-f(x+y)-f(x-y)-6f(x)\right)\right\| \\
&= \frac{|\rho|}{2}\|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)\|
\end{aligned}$$

and so

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x)$$

for all $x, y \in X$, since $|\rho| < 2$. So $f : X \to Y$ is cubic. □

We prove the Hyers-Ulam stability of the cubic ρ-functional inequality (2.1) in complex Banach spaces.

Theorem 2.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping such that*

$$\Psi(x,y) := \sum_{j=1}^{\infty} 8^j \varphi(\frac{x}{2^j}, \frac{y}{2^j}) < \infty, \tag{2.3}$$

$$\begin{aligned}
&\|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)\| \qquad (2.4)\\
&\le \left\|\rho\left(4f\left(x+\frac{y}{2}\right)+4f\left(x-\frac{y}{2}\right)-f(x+y)-f(x-y)-6f(x)\right)\right\|+\varphi(x,y)
\end{aligned}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\|f(x) - C(x)\| \le \frac{1}{16}\Psi(x,0) \tag{2.5}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (2.4), we get

$$\|2f(2x) - 16f(x)\| \le \varphi(x,0) \tag{2.6}$$

and so $\|f(x) - 8f\left(\frac{x}{2}\right)\| \le \frac{1}{2}\varphi\left(\frac{x}{2}, 0\right)$ for all $x \in X$. So

$$\begin{aligned}
\left\|8^l f\left(\frac{x}{2^l}\right) - 8^m f\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l+1}^{m}\left\|8^j f\left(\frac{x}{2^j}\right) - 8^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\
&\le \frac{1}{16}\sum_{j=l+1}^{m} 8^j \varphi\left(\frac{x}{2^j}, 0\right) \qquad (2.7)
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.7) that the sequence $\{8^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{8^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $C : X \to Y$ by

$$C(x) := \lim_{n\to\infty} 8^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.7), we get (2.5).

It follows from (2.3) and (2.4) that

$$\begin{aligned}
&\|C(2x+y)+C(2x-y)-2C(x+y)-2C(x-y)-12C(x)\| \\
&= \lim_{n\to\infty} 8^n \left\| f\left(\frac{2x+y}{2^n}\right) + f\left(\frac{2x-y}{2^n}\right) - 2f\left(\frac{x+y}{2^n}\right) - 2f\left(\frac{x-y}{2^n}\right) - 12f\left(\frac{x}{2^n}\right)\right\| \\
&\le \lim_{n\to\infty} 8^n|\rho| \left\| 4f\left(\frac{2x+y}{2^{n+1}}\right) + 4f\left(\frac{2x-y}{2^{n+1}}\right) - f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x-y}{2^n}\right) - 6f\left(\frac{x}{2^n}\right)\right\| \\
&\qquad + \lim_{n\to\infty} 8^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \\
&= \left\| \rho\left(4C\left(x+\frac{y}{2}\right) + 4C\left(x-\frac{y}{2}\right) - C(x+y) - C(x-y) - 6C(x)\right)\right\|
\end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned}
&\|C(2x+y)+C(2x-y)-2C(x+y)-2C(x-y)-12C(x)\| \\
&\le \left\| \rho\left(4C\left(x+\frac{y}{2}\right) + 4C\left(x-\frac{y}{2}\right) - C(x+y) - C(x-y) - 6C(x)\right)\right\|
\end{aligned}$$

for all $x, y, z \in X$. By Lemma 2.2, the mapping $C : X \to Y$ is cubic.

Now, let $T : X \to Y$ be another cubic mapping satisfying (2.5). Then we have

$$\begin{aligned}
\|C(x) - T(x)\| &= \left\| 8^q C\left(\frac{x}{2^q}\right) - 8^q T\left(\frac{x}{2^q}\right)\right\| \\
&\le \left\| 8^q C\left(\frac{x}{2^q}\right) - 8^q f\left(\frac{x}{2^q}\right)\right\| + \left\| 8^q T\left(\frac{x}{2^q}\right) - 8^q f\left(\frac{x}{2^q}\right)\right\| \\
&\le \frac{2}{16} \cdot 8^q \Psi\left(\frac{x}{2^q}, 0\right),
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $C(x) = T(x)$ for all $x \in X$. This proves the uniqueness of C. Thus the mapping $C : X \to Y$ is a unique cubic mapping satisfying (2.5). □

Corollary 2.4. *Let $r > 3$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned}
&\|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)\| \qquad (2.8)\\
&\le \left\| \rho\left(4f\left(x+\frac{y}{2}\right) + 4f\left(x-\frac{y}{2}\right) - f(x+y) - f(x-y) - 6f(x)\right)\right\| + \theta(\|x\|^r + \|y\|^r)
\end{aligned}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\|f(x) - C(x)\| \le \frac{\theta}{2^{r+1} - 16}\|x\|^r$$

for all $x \in X$.

Theorem 2.5. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying (2.4) and*

$$\Psi(x, y) := \sum_{j=0}^{\infty} \frac{1}{8^j} \varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\|f(x) - C(x)\| \le \frac{1}{16}\Psi(x, 0) \qquad (2.9)$$

for all $x \in X$.

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Proof. It follows from (2.6) that

$$\left\|f(x) - \frac{1}{8}f(2x)\right\| \leq \frac{1}{16}\varphi(x,0)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{8^l}f(2^l x) - \frac{1}{8^m}f(2^m x)\right\| &\leq \sum_{j=l}^{m-1}\left\|\frac{1}{8^j}f\left(2^j x\right) - \frac{1}{8^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\leq \frac{1}{16}\sum_{j=l}^{m-1}\frac{1}{8^j}\varphi(2^j x, 0) \qquad (2.10)\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.10) that the sequence $\{\frac{1}{8^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{8^n}f(2^n x)\}$ converges. So one can define the mapping $C : X \to Y$ by

$$C(x) := \lim_{n\to\infty}\frac{1}{8^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.10), we get (2.9).

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 2.6. *Let $r < 3$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (2.8). Then there exists a unique cubic mapping $C : X \to Y$ such that*

$$\|f(x) - C(x)\| \leq \frac{\theta}{16 - 2^{r+1}}\|x\|^r \qquad (2.11)$$

for all $x \in X$.

Remark 2.7. If ρ is a real number such that $-2 < \rho < 2$ and Y is a real Banach space, then all the assertions in this section remain valid.

3. Quartic ρ-functional inequality (0.2)

Throughout this section, assume that ρ is a fixed complex number with $|\rho| < 2$.

In this section, we solve and investigate the quartic ρ-functional inequality (0.2) in complex normed spaces.

Lemma 3.1. *Let X and Y be vector spaces. An even mapping $f : X \to Y$ satisfies*

$$8f\left(x + \frac{y}{2}\right) + 8f\left(x - \frac{y}{2}\right) = 2f(x+y) + 2f(x-y) + 12f(x) - 3f(y) \qquad (3.1)$$

if and only if the mapping $f : X \to Y$ is a quartic mapping.

Proof. Sufficiency. Assume that $f : X \to Y$ satisfies (3.1)

Letting $x = y = 0$ in (3.1), we have $16f(0) = 13f(0)$. So $f(0) = 0$.

Letting $x = 0$ in (3.1), we get $16f\left(\frac{y}{2}\right) = f(y)$ for all $y \in X$. So $f : X \to Y$ satisfies the quartic functional equation.

Necessity. Assume that $f : X \to Y$ is a quartic mapping. Then $f(2x) = 16f(x)$ for all $x \in X$. So $f : X \to Y$ satisfies (3.1). □

Lemma 3.2. *If a mapping $f : X \to Y$ satisfies*

$$\begin{aligned}&\|f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y)\| \qquad (3.2)\\ &\leq \left\|\rho\left(8f\left(x+\frac{y}{2}\right) + 8f\left(x - \frac{y}{2}\right) - 2f(x+y) - 2f(x-y) - 12f(x) + 3f(y)\right)\right\|\end{aligned}$$

for all $x, y \in X$, then the mapping $f : X \to Y$ is quartic.

Proof. Assume that $f : X \to Y$ satisfies (3.2).

Letting $x = y = 0$ in (3.2), we get $\|24f(0)\| \le |\rho|\|3f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.2), we get

$$\|2f(2x) - 32f(x)\| \le 0 \tag{3.3}$$

and so

$$f\left(\frac{x}{2}\right) = \frac{1}{16}f(x) \tag{3.4}$$

for all $x \in X$.

It follows from (3.2) and (3.4) that

$$\begin{aligned}
&\|f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y)\| \\
&\le \left\|\rho\left(8f\left(x+\frac{y}{2}\right) + 8f\left(x-\frac{y}{2}\right) - 2f(x+y) - 2f(x-y) - 12f(x) + 3f(y)\right)\right\| \\
&= \frac{|\rho|}{2}\|f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y)\|
\end{aligned}$$

and so

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y)$$

for all $x, y \in X$, since $|\rho| < 2$. So $f : X \to Y$ is quartic. □

We prove the Hyers-Ulam stability of the quartic ρ-functional inequality (3.2) in complex Banach spaces.

Theorem 3.3. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$,*

$$\Psi(x,y) := \sum_{j=1}^{\infty} 16^j \varphi(\frac{x}{2^j}, \frac{y}{2^j}) < \infty,$$

$$\begin{aligned}
&\|f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y)\| \qquad (3.5)\\
&\le \left\|\rho\left(8f\left(x+\frac{y}{2}\right) + 8f\left(x-\frac{y}{2}\right) - 2f(x+y) - 2f(x-y) - 12f(x) + 3f(y)\right)\right\| + \varphi(x,y)
\end{aligned}$$

for all $x, y \in X$. Then there exists a unique quartic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{32}\Psi(x,0) \tag{3.6}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.5), we get

$$\|2f(2x) - 32f(x)\| \le \varphi(x,0) \tag{3.7}$$

and so $\left\|f(x) - 16f\left(\frac{x}{2}\right)\right\| \le \frac{1}{2}\varphi\left(\frac{x}{2}, 0\right)$ for all $x \in X$. So

$$\begin{aligned}
\left\|16^l f\left(\frac{x}{2^l}\right) - 16^m f\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l+1}^{m}\left\|16^j f\left(\frac{x}{2^j}\right) - 16^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\
&\le \frac{1}{32}\sum_{j=l+1}^{m} 16^j \varphi\left(\frac{x}{2^j}, 0\right) \qquad (3.8)
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.8) that the sequence $\{16^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{16^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 16^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get (3.6).

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 3.4. *Let $r > 4$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} &\|f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y)\| \quad (3.9)\\ &\le \left\|\rho\left(8f\left(x+\frac{y}{2}\right)+8f\left(x-\frac{y}{2}\right)-2f(x+y)-2f(x-y)-12f(x)+3f(y)\right)\right\| \\ &\quad +\theta(\|x\|^r+\|y\|^r)\end{aligned}$$

for all $x, y \in X$. Then there exists a unique quartic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{\theta}{2^{r+1} - 32}\|x\|^r$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (3.9), we get $\|24f(0)\| \le |\rho|\|3f(0)\|$, So $f(0) = 0$. Letting $\varphi(x, y) := \theta(\|x\|^r + \|y\|^r)$ in Theorem 3.3, we obtain the desired result. □

Theorem 3.5. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (3.5) and*

$$\Psi(x, y) := \sum_{j=0}^{\infty} \frac{1}{16^j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique quartic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{32}\Psi(x, 0) \quad (3.10)$$

for all $x \in X$.

Proof. It follows from (3.7) that

$$\left\|f(x) - \frac{1}{16}f(2x)\right\| \le \frac{1}{32}\varphi(x, 0)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{16^l}f(2^l x) - \frac{1}{16^m}f(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{16^j}f\left(2^j x\right) - \frac{1}{16^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\le \frac{1}{32}\sum_{j=l}^{m-1}\frac{1}{16^j}\varphi(2^j x, 0) \quad (3.11)\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.11) that the sequence $\{\frac{1}{16^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{16^n}f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}\frac{1}{16^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.11), we get (3.10).

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 3.6. *Let $r < 4$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (3.9)*. Then there exists a unique quartic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{\theta}{32 - 2^{r+1}} \|x\|^r$$

for all $x \in X$.

Remark 3.7. If ρ is a real number such that $-2 < \rho < 2$ and Y is a real Banach space, then all the assertions in this section remain valid.

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Choonkil Park
Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Korea
E-mail address: baak@hanyang.ac.kr

Jung Rye Lee
Department of Mathematics, Daejin University, Kyeonggi 487-711, Korea
E-mail address: jrlee@daejin.ac.kr

Dong Yun Shin
Department of Mathematics, University of Seoul, Seoul 130-743, Korea
E-mail address: dyshin@uos.ac.kr

Complex Valued G_b-Metric Spaces

Ozgur EGE
Celal Bayar University
Department of Mathematics
45140 Manisa, Turkey
E-mail: ozgur.ege@cbu.edu.tr

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Abstract

In this paper, we introduce the concept of complex valued G_b-metric spaces. We also prove Banach contraction principle and Kannan's fixed point theorem in this space. Our result generalizes some well-known results in the fixed point theory.

Keywords: Complex valued G_b-metric space, fixed point, Banach contraction principle, Kannan's fixed point theorem.
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1 Introduction

The concept of a metric space was introduced by Frechet [11]. Then many mathematicians study of fixed points of contractive mappings. After the introduction of Banach contraction principle, the study of existence and uniqueness of fixed points and common fixed points have been a major area of interest. In a number of generalized metric spaces, many researchers proved the Banach fixed point theorem.

Bakhtin [6] presented b-metric spaces as a generalization of metric spaces. He also proved generalized Banach contraction principle in b-metric spaces. After that, many papers related to variational principle for single-valued and multi-valued operators have studied in b-metric spaces (see [7, 8, 9, 10, 18]). Azam et al. [4] defined the notion of complex valued metric spaces and gave common fixed point result for mappings. Rao et al. [21] introduced the complex valued b-metric spaces. Mustafa and Sims [13] presented the notion of G-metric spaces. Many researchers [1, 2, 3, 12, 14, 15, 19, 20, 22, 23, 25] obtained common fixed point results for G-metric spaces. The concept of G_b-metric space was given in [5]. Mustafa et al. [16] prove some coupled coincidence fixed point theorems for nonlinear (ψ, φ)-weakly contractive mappings in partially ordered G_b-metric spaces. Other important studies on G_b-metric spaces, see [17, 24].

In this work, our aim is to prove Banach contraction principle and Kannan's fixed point theorem in complex valued G_b-metric spaces. For this purpose, we give new definitions and additional theorems with proofs.

2 Preliminaries

In this section, we recall some properties of G_b-metric spaces.

Definition 2.1. [5]. Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a mapping $G : X \times X \times X \to \mathbb{R}^+$ satisfies:

(G_b1) $G(x, y, z) = 0$ if $x = y = z$;

(G_b2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;

(G_b3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;

(G_b4) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z;

(G_b5) $G(x, y, z) \leq s(G(x, a, a) + G(a, y, z))$ for all $x, y, z, a \in X$ (rectangle inequality).

Then, G is called a generalized b-metric and (X, G) is called a generalized b-metric or a G_b-metric space.

Note that each G-metric space is a G_b-metric space with $s = 1$.

Proposition 2.2. *[5]. Let X be a G_b-metric space. Then for each $x, y, z, a \in X$ it follows that:*

(i) *if* $G(x, y, z) = 0$ *then* $x = y = z$,

(ii) $G(x, y, z) \leq s(G(x, x, y) + G(x, x, z))$,

(iii) $G(x, y, y) \leq 2sG(y, x, x)$,

(iv) $G(x, y, z) \leq s(G(x, a, z) + G(a, y, z))$.

Definition 2.3. [5]. Let X be a G_b-metric space. A sequence $\{x_n\}$ in X is said to be:
• G_b-Cauchy if for each $\epsilon > 0$, there exists a positive integer n_0 such that for all $m, n, l \geq n_0$, $G(x_n, x_m, x_l) < \epsilon$,
• G_b-convergent to a point $x \in X$ if for each $\epsilon > 0$, there exists a positive integer n_0 such that for all $m, n \geq n_0$, $G(x_n, x_m, x) < \epsilon$.

Proposition 2.4. *[5]. Let X be a G_b-metric space.*

(1) The sequence $\{x_n\}$ is G_b-Cauchy.

(2) For any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $m, n \geq n_0$.

Proposition 2.5. *[5]. Let X be a G_b-metric space. The following are equivalent:*

(1) $\{x_n\}$ is G_b-convergent to x.

(2) $G(x_n, x_n, x) \to 0$ as $n \to \infty$.

(3) $G(x_n, x, x) \to 0$ as $n \to \infty$.

Definition 2.6. [5]. A G_b-metric space X is called complete if every G_b-Cauchy sequence is G_b-convergent in X.

The complex metric space was initiated by Azam et al. [4]. Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(C_1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(C_2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,

(C_3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,

(C_4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

Particularly, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of $(C_2), (C_3)$ and (C_4) is satisfied and we write $z_1 \prec z_2$ if only (C_4) is satisfied. The following statements hold:

(1) If $a, b \in \mathbb{R}$ with $a \leq b$, then $az \prec bz$ for all $z \in \mathbb{C}$.

(2) If $0 \precsim z_1 \precnsim z_2$, then $|z_1| < |z_2|$.

(3) If $z_1 \precsim z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

3 Complex Valued G_b-Metric Spaces

In this section, we define the complex valued G_b-metric space.

Definition 3.1. Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a mapping $G : X \times X \times X \to \mathbb{C}$ satisfies:

(CG_b1) $G(x, y, z) = 0$ if $x = y = z$;

(CG_b2) $0 \prec G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;

(CG_b3) $G(x, x, y) \precsim G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;

(CG_b4) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z;

(CG_b5) $G(x, y, z) \precsim s(G(x, a, a) + G(a, y, z))$ for all $x, y, z, a \in X$ (rectangle inequality).

Then, G is called a complex valued G_b-metric and (X, G) is called a complex valued G_b-metric space.

From (CG_b5), we have the following proposition.

Proposition 3.2. *Let (X, G) be a complex valued G_b-metric space. Then for any $x, y, z \in X$,*
• $G(x, y, z) \precsim s(G(x, x, y) + G(x, x, z))$,
• $G(x, y, y) \precsim 2sG(y, x, y)$.

Definition 3.3. Let (X, G) be a complex valued G_b-metric space, let $\{x_n\}$ be a sequence in X.
(i) $\{x_n\}$ is complex valued G_b-convergent to x if for every $a \in \mathbb{C}$ with $0 \prec a$, there exists $k \in \mathbb{N}$ such that $G(x, x_n, x_m) \prec a$ for all $n, m \geq k$.
(ii) A sequence $\{x_n\}$ is called complex valued G_b-Cauchy if for every $a \in \mathbb{C}$ with $0 \prec a$, there exists $k \in \mathbb{N}$ such that $G(x_n, x_m, x_l) \prec a$ for all $n, m, l \geq k$.
(iii) If every complex valued G_b-Cauchy sequence is complex valued G_b-convergent in (X, G), then (X, G) is said to be complex valued G_b-complete.

Proposition 3.4. *Let (X, G) be a complex valued G_b-metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is complex valued G_b-convergent to x if and only if $|G(x, x_n, x_m)| \to 0$ as $n, m \to \infty$.*

Proof. $(\Rightarrow)$ Assume that $\{x_n\}$ is complex valued G_b-convergent to x and let

$$a = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}$$

for a real number $\epsilon > 0$. Then we have $0 \prec a \in \mathbb{C}$ and there is a natural number k such that $G(x, x_n, x_m) \prec a$ for all $n, m \geq k$. Thus, $|G(x, x_n, x_m)| < |a| = \epsilon$ for all $n, m \geq k$ and so $|G(x, x_n, x_m)| \to 0$ as $n, m \to \infty$.
$(\Leftarrow)$ Suppose that $|G(x, x_n, x_m)| \to 0$ as $n, m \to \infty$. For a given $a \in \mathbb{C}$ with $0 \prec a$, there exists a real number $\delta > 0$ such that for $z \in \mathbb{C}$

$$|z| < \delta \quad \Rightarrow \quad z \prec a.$$

Considering δ, we have a natural number k such that $|G(x, x_n, x_m)| < \delta$ for all $n, m \geq k$. This means that $G(x, x_n, x_m) \prec a$ for all $n, m \geq k$, i.e., $\{x_n\}$ is complex valued G_b-convergent to x. □

From Propositions 3.2 and 3.4, we can prove the following theorem.

Theorem 3.5. *Let (X, G) be a complex valued G_b-metric space, then for a sequence $\{x_n\}$ in X and point $x \in X$, the following are equivalent:*

(1) $\{x_n\}$ *is complex valued G_b-convergent to x.*
(2) $|G(x_n, x_n, x)| \to 0$ *as* $n \to \infty$.
(3) $|G(x_n, x, x)| \to 0$ *as* $n \to \infty$.
(4) $|G(x_m, x_n, x)| \to 0$ *as* $m, n \to \infty$.

Proof. (1) $\Rightarrow$ (2) It is clear from Proposition 3.4.
(2) $\Rightarrow$ (3) By Proposition 3.2, we have

$$G(x_n, x, x) \precsim s(G(x_n, x_n, x) + G(x_n, x_n, x))$$

and using (2), we get

$$|G(x_n, x, x)| \to 0$$

as $n \to \infty$.
(3) $\Rightarrow$ (4) If we use (CG_b4) and Proposition 3.2, then

$$\begin{aligned} G(x_m, x_n, x) = G(x, x_m, x_n) &\precsim s(G(x, x, x_m) + G(x, x, x_n)) \\ &= s(G(x_m, x, x) + G(x_n, x, x)) \end{aligned}$$

and $|G(x_m, x_n, x)| \to 0$ as $m, n \to \infty$.
(4) $\Rightarrow$ (1) We will use the equivalence in Proposition 3.4, (CG_b3) and (CG_b4). Since

$$\begin{aligned} G(x, x_n, x_m) = G(x_m, x, x_n) &\precsim s(G(x_m, x_m, x) + G(x_m, x_m, x_n)) \\ &\precsim s(G(x_m, x_n, x)) \end{aligned}$$

and $|G(x_m, x_n, x)| \to 0$ as $m, n \to \infty$, we obtain $|G(x, x_n, x_m)| \to 0$ and this completes the proof. □

Theorem 3.6. *Let (X, G) be a complex valued G_b-metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is complex valued G_b-Cauchy sequence if and only if $|G(x_n, x_m, x_l)| \to 0$ as $n, m, l \to \infty$.*

Proof. $(\Rightarrow)$ Let $\{x_n\}$ be complex valued G_b-Cauchy sequence and

$$b = \frac{\varepsilon}{\sqrt{2}} + i\frac{\varepsilon}{\sqrt{2}}$$

where $\varepsilon > 0$ is a real number. Then $0 \prec b \in \mathbb{C}$ and there is a natural number k such that $G(x_n, x_m, x_l) \prec b$ for all $n, m, l \geq k$. Therefore, we get $|G(x_n, x_m, x_l)| < |b| = \varepsilon$ for all $n, m, l \geq k$ and the required result.
$(\Leftarrow)$ Assume that $|G(x_n, x_m, x_l)| \to 0$ as $n, m, l \to \infty$. Then there exists a real number $\gamma > 0$ such that for $z \in \mathbb{C}$

$$|z| < \gamma \ \text{ implies } \ z \prec b$$

where $b \in \mathbb{C}$ with $0 \prec b$. For this γ, there is a natural number k such that $|G(x_n, x_m, x_l)| < \gamma$ for all $n, m, l \geq k$. This means that $G(x_n, x_m, x_l) \prec b$ for all $n, m, l \geq k$. Hence $\{x_n\}$ is complex valued G_b-Cauchy sequence. □

We prove the contraction principle in complex valued G_b-metric spaces as follows:

Theorem 3.7. *Let (X, G) be a complete complex valued G_b-metric space with coefficient $s > 1$ and $T : X \to X$ be a mapping satisfying:*

$$G(Tx, Ty, Tz) \precsim kG(x, y, z) \tag{3.1}$$

for all $x, y, z \in X$, where $k \in [0, \frac{1}{s})$. Then T has a unique fixed point.

Proof. Let T satisfy (3.1), $x_0 \in X$ be an arbitrary point and define the sequence $\{x_n\}$ by $x_n = T^n x_0$. From (3.1), we obtain

$$G(x_n, x_{n+1}, x_{n+1}) \precsim kG(x_{n-1}, x_n, x_n). \tag{3.2}$$

Using again (3.1), we have

$$G(x_{n-1}, x_n, x_n) \precsim kG(x_{n-2}, x_{n-1}, x_{n-1})$$

and by (3.2), we get

$$G(x_n, x_{n+1}, x_{n+1}) \precsim k^2 G(x_{n-2}, x_{n-1}, x_{n-1}).$$

If we continue in this way, we find

$$G(x_n, x_{n+1}, x_{n+1}) \precsim k^n G(x_0, x_1, x_1). \tag{3.3}$$

Using (CG_b5) and (3.3) for all $n, m \in \mathbb{N}$ with $n < m$,

$$\begin{aligned} G(x_n, x_m, x_m) &\precsim s[G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_m, x_m)] \\ &\precsim s[G(x_n, x_{n+1}, x_{n+1})] + s^2[G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G(x_{n+2}, x_m, x_m)] \\ &\precsim s[G(x_n, x_{n+1}, x_{n+1})] + s^2[G(x_{n+1}, x_{n+2}, x_{n+2})] + \\ &\quad s^3[G(x_{n+2}, x_{n+3}, x_{n+3})] + \ldots + s^{m-n} G(x_{m-1}, x_m, x_m)] \\ &\precsim (sk^n + s^2 k^{n+1} + s^3 k^{n+2} + \ldots + s^{m-n} k^{m-1}) G(x_0, x_1, x_1) \\ &\precsim sk^n [1 + sk + (sk)^2 + (sk)^3 + \ldots + (sk)^{m-n-1}] G(x_0, x_1, x_1) \\ &\precsim \frac{sk^n}{1 - sk} G(x_0, x_1, x_1). \end{aligned}$$

Thus, we obtain

$$|G(x_n, x_m, x_m)| \leq \frac{sk^n}{1 - sk} |G(x_0, x_1, x_1)|.$$

Since $k \in [0, \frac{1}{s})$ where $s > 1$, taking limits as $n \to \infty$, then

$$\frac{sk^n}{1 - sk} |G(x_0, x_1, x_1)| \to 0.$$

This means that

$$|G(x_n, x_m, x_m)| \to 0.$$

By Proposition 3.2, we get

$$G(x_n, x_m, x_l) \precsim G(x_n, x_m, x_m) + G(x_l, x_m, x_m)$$

for $n, m, l \in \mathbb{N}$. Thus,

$$|G(x_n, x_m, x_l)| \leq |G(x_n, x_m, x_m)| + |G(x_l, x_m, x_m)|.$$

If we take limit as $n, m, l \to \infty$, we obtain $|G(x_n, x_m, x_l)| \to 0$. So $\{x_n\}$ is complex valued G_b-Cauchy sequence by Theorem 3.6. Completeness of (X, G) gives us that there is an element $u \in X$ such that $\{x_n\}$ is complex valued G_b-convergent to u.

To prove $Tu = u$, we will assume the contrary. From (3.1), we obtain

$$G(x_{n+1}, Tu, Tu) \precsim kG(x_n, u, u)$$

and

$$|G(x_{n+1}, Tu, Tu)| \leq k|G(x_n, u, u)|.$$

If we take the limit as $n \geq \infty$, we get

$$|G(u, Tu, Tu)| \leq k|G(u, u, u)|,$$

which is a contradiction since $k \in [0, \frac{1}{s})$. As a result, $Tu = u$.

Lastly, we prove the uniqueness. Let $w \neq u$ be another fixed point of T. Using (3.1),

$$G(z, w, w) = G(Tz, Tw, Tw) \precsim kG(z, w, w).$$

and

$$|G(z, w, w)| \leq k|G(z, w, w)|.$$

Since $k \in [0, \frac{1}{s})$, we have $|G(z, w, w)| \leq 0$. Thus, $u = w$ and so u is a unique fixed point of T. □

Example 3.8. Let $X = [-1, 1]$ and $G : X \times X \times X \to \mathbb{C}$ be defined as follows:

$$G(x, y, z) = |x - y| + |y - z| + |z - x|$$

for all $x, y, z \in X$. (X, G) is complex valued G-metric space [12]. Define

$$G_*(x, y, z) = G(x, y, z)^2.$$

G_* is a complex valued G_b-metric with $s = 2$ (see [5]). If we define $T : X \to X$ as $Tx = \frac{x}{3}$, then T satisfies the following condition for all $x, y, z \in X$:

$$G(Tx, Ty, Tz) = G(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}) = \frac{1}{3} G(x, y, z) \precsim kG(x, y, z)$$

where $k \in [\frac{1}{3}, \frac{1}{s})$, $s > 1$. Thus $x = 0$ is the unique fixed point of T in X.

We will prove Kannan's fixed point theorem for complex valued G_b-metric spaces.

Theorem 3.9. *Let (X, G) be a complete complex valued G_b-metric space and the mapping $T : X \to X$ satisfies for every $x, y \in X$*

$$G(Tx, Ty, Ty) \precsim \alpha[G(x, Tx, Tx) + G(y, Ty, Ty)] \tag{3.4}$$

where $\alpha \in [0, \frac{1}{2})$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. We shall show that $\{x_n\}$ is G_b-Cauchy sequence. If $x_n = x_{n+1}$, then x_n is the fixed point of T. Thus, suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Setting $G(x_n, x_{n+1}, x_{n+1}) = G_n$, it follows from (3.4) that

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(Tx_{n-1}, Tx_n, Tx_n) \\ &\precsim \alpha[G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + G(x_n, Tx_n, Tx_n)] \\ &\precsim \alpha[G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})] \\ &\precsim \alpha[G_{n-1} + G_n] \\ G_n &\precsim \frac{\alpha}{1-\alpha} G_{n-1} = \beta G_{n-1}, \end{aligned}$$

where $\beta = \frac{\alpha}{1-\alpha} < 1$ as $\alpha \in [0, \frac{1}{2})$. If we repeat this process, then we get

$$G_n \precsim \beta^n G_0. \tag{3.5}$$

We can also suppose that x_0 is not a periodic point. If $x_n = x_0$, then we have

$$G_0 \precsim \beta^n G_0.$$

Since $\beta < 1$, then $1 - \beta^n < 1$ and

$$(1 - \beta^n)|G_0| \leq 0 \quad \Rightarrow \quad |G_0| = 0.$$

It follows that x_0 is a fixed point of T. Therefore in the sequel of proof we can assume $T^n x_0 \neq x_0$ for $n = 1, 2, 3, \ldots$ From inequality (3.4), we obtain

$$\begin{aligned} G(T^n x_0, T^{n+m} x_0, T^{n+m} x_0) &\precsim \alpha[G(T^{n-1} x_0, T^{n+m} x_0, T^{n+m} x_0) \\ &\quad + G(T^{n+m-1} x_0, T^{n+m} x_0, T^{n+m} x_0)] \\ &\precsim \alpha[\beta^{n-1} G(x_0, Tx_0, Tx_0) + \beta^{n+m-1} G(x_0, Tx_0, Tx_0)]. \end{aligned}$$

So, $|G(x_n, x_{n+m}, x_{n+m})| \to 0$ as $n \to \infty$. It implies that $\{x_n\}$ is a G_b-Cauchy in X. By the completeness of X, there exists $u \in X$ such that $x_n \to u$. From (CG_b5), we get

$$\begin{aligned} G(Tu, u, u) &\precsim s[G(Tu, T^{n+1} x_0, T^{n+1} x_0) + G(T^{n+1} x_0, u, u)] \\ &\precsim s(\alpha[G(u, Tu, Tu) + G(T^n x_0, T^{n+1} x_0, T^{n+1} x_0)]) + sG(T^{n+1} x_0, u, u) \\ &\precsim s\alpha[G(u, Tu, Tu) + s\alpha G(T^n x_0, T^{n+1} x_0, T^{n+1} x_0)]) + sG(T^{n+1} x_0, u, u). \end{aligned}$$

Letting $n \to \infty$, since $s\alpha < 1$ and $x_n \to u$, we have $|G(Tu, u, u)| \to 0$, i.e., $u = Tu$.

Now we show that T has a unique fixed point. For this, assume that there exists another point v in X such that $v = Tv$. Now,

$$\begin{aligned} G(v, u, u) &\precsim G(Tv, Tu, Tu) \\ &\precsim \alpha[G(v, Tv, Tv) + G(u, Tu, Tu)] \\ &\precsim \alpha[G(v, v, v) + G(u, u, u)] \\ &\precsim 0. \end{aligned}$$

Hence, we conclude that $u = v$. ☐

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Finite Difference approximations for the Two-side Space-time Fractional Advection-diffusion Equations*

Yabin Shao[1,2] †, Weiyuan Ma [2]

1. Department of Applied Mathematics

Chongqing University of Posts and Telecommunications, Chongqing, 400065, China

2. College of Mathematics and Computer Science

Northwest University for Nationalities, Lanzhou, 730030, China

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Abstract

Fractional order advection-diffusion equation is viewed as generalizations of classical diffusion equations, treating super-diffusive flow processes. In this paper, we present a new weighted finite difference approximation for the equation with initial and boundary conditions in a finite domain. Using mathematical induction, we prove that the weighted finite difference approximation is conditionally stable and convergent. Numerical computations are presented which demonstrate the effectiveness of the method and confirm the theoretical claims.

Keywords: Fractional order advection-diffusion equation; Weighted finite difference approximation; Stability; Convergence.

1 INTRODUCTION

In recent years, fractional differential equations have attracted much attention. Many important phenomena in physics [1, 2, 3], finance [4, 5], hydrology [6], engineering [7], mathematics [8] and material science are well described by differential equations of fractional order. These fractional order models tend to be more appropriate than the traditional integer-order models. So, the fractional derivatives are considered to be a very powerful and useful tool.

The fractional advection-diffusion equation provides a useful description of transport dynamics in complex systems which are governed by anomalous diffusion and non-exponential relaxation [9]. In this paper, we consider a special case

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†Corresponding author. E-mail: yb-shao@163.com(Y. Shao)

of anomalous diffusion, the two-sided space-time fractional advection-diffusion equation can be written in the following way

$$\begin{aligned} \tfrac{\partial^\beta u(x,t)}{\partial t^\beta} &= -v(x)\tfrac{\partial u(x,t)}{\partial x} + d_+(x)\tfrac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha} \\ + \quad & d_-(x)\tfrac{\partial^\alpha u(x,t)}{\partial_- x^\alpha} + f(x,t), \quad x \in [L,R], t \in (0,T], \end{aligned} \tag{1}$$

$$u(L,t) = 0, u(R,t) = \varphi(t), \quad t \in [0,T], \tag{2}$$

$$u(x,0) = u_0(x), \quad x \in (L,R], \tag{3}$$

where α and β are parameters describing the order of the space- and time-fractional derivatives, respectively, physical considerations restrict $0 < \beta < 1, 1 < \alpha < 2$. The functions $v(x,t), d_+(x,t)$ and $d_-(x,t)$ are all non-negative, bounded and $d_+(x,t), d_-(x,t) \geq v(x,t)$.

The left-sided (+) and the right-sided (−) Riemann-Liouville fractional derivatives of order α of a function $u(x,t)$ are defined as follows

$$\frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha} = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_L^x \frac{u(\xi,t)}{(x-\xi)^{\alpha+1-n}}d\xi \tag{4}$$

and

$$\frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha} = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_x^R \frac{u(\xi,t)}{(x-\xi)^{\alpha+1-n}}d\xi, \tag{5}$$

where n is an integer such that $n-1 < \alpha \leq n$. The time derivative $\frac{\partial^\beta u(x,t)}{\partial t^\beta}$ is given by a Caputo fractional derivative

$$\frac{\partial^\beta u(x,t)}{\partial t^\beta} = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-\eta)^{-\beta}\frac{\partial u(x,\eta)}{\partial \eta}d\eta, \tag{6}$$

where $\Gamma(\cdot)$ is the gamma function.

As is well known, the fractional order differential operator is a nonlocal operator, which requires more involved computational schemes for its handling. Finite difference schemes for fractional partial differential equations are more complex than partial differential equations [1, 2, 4, 10, 11, 12, 13, 14]. It should note the following work for fractional advection-diffusion equation. Su et al. [13] presented a Crank-Nicolson type finite difference scheme for two-sided space fractional advection-diffusion equation. Liu et al. [14] considered a space-time fractional advection-diffusion with Caputo time fractional derivative and Riemann-Liouville space fractional derivatives. In this paper, we present a new weighted finite difference approximation for the equation.

The rest of the paper is as follows. In Section 2, we derive the new weighted finite difference approximation (NWFDM) for the fractional advection-diffusion equation. The convergence and stability of the finite difference scheme is given in Section 3, where we apply discrete energy method. In Section 4, numerical results are shown which confirm that the numerical method is effective.

2 NEW WEIGHTED FINITE DIFFERENCE SCHEME

To present the numerical approximation scheme, we give some notations: τ is the time step, u_j^n be the numerical solution at (x_i, t_n) for $x_j = L + ih, t_n = n\tau, j = 0, 1, \cdots, J, n = 0, 1, \cdots, N$.

The shifted Grünwald formula is applied to discretize the left-handed fractional derivative and right-handed fractional derivative [15],

$$\frac{\partial^\alpha u(x_i, t_n)}{\partial_+ x^\alpha} = \frac{1}{h^\alpha} \sum_{j=0}^{i+1} g_j u(x_i - (j-1)h, t_n) + o(h), \tag{7}$$

$$\frac{\partial^\alpha u(x_i, t_n)}{\partial_- x^\alpha} = \frac{1}{h^\alpha} \sum_{j=0}^{N-i+1} g_j u(x_i + (j-1)h, t_n) + o(h), \tag{8}$$

where the Grünwald coefficients are defined by

$$g_0 = 1, g_j = (1 - \frac{\alpha+1}{j}) g_{j-1}, \quad j = 1, 2, 3, \cdots.$$

Adopting the discrete scheme in [15], we discretize the Caputo time fractional derivative as,

$$\frac{\partial^\beta u(x_i, t_n)}{\partial t^\beta} = \frac{\tau^{1-\beta}}{\Gamma(2-\beta)} \sum_{j=0}^{n} \frac{u(x_i, t_{n+1-j}) - u(x_i, t_{n-j})}{\tau} \sigma_j + o(\tau),$$

where $\sigma_j = (j+1)^{1-\beta} - j^{1-\beta}$.

Now we replace (1) with the following weighted finite difference approximation:

$$\begin{aligned} \frac{\tau^{1-\beta}}{\Gamma(2-\beta)} \sum_{j=0}^{n} \frac{u_i^{n+1-j} - u_i^{n-j}}{\tau} \sigma_j &= -v_i[\theta \frac{u_{i+1}^n - u_{i-1}^n}{2h} \\ +(1-\theta) \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2h}] &+ \frac{d_{+i}}{h^\alpha}[\theta \sum_{k=0}^{i+1} g_k u_{i-k+1}^n \\ +(1-\theta) \sum_{k=0}^{i+1} g_k u_{i-k+1}^{n+1}] &+ \frac{d_{-i}}{h^\alpha}[\theta \sum_{k=0}^{N-i+1} g_k u_{i+k-1}^n \\ +(1-\theta) \sum_{k=0}^{N-i+1} g_k u_{i+k-1}^{n+1}] &+ \theta f_i^n + (1-\theta) f_i^{n+1}, \end{aligned} \tag{9}$$

for $i = 1, 2, \cdots, J-1, n = 0, 1, \cdots, N-1$, where θ is the weighting parameter subjected to $0 \leq \theta \leq 1$. When $\theta = 0, 1, \frac{1}{2}$, we get the space-time fractional implicit, explicit, Crank-Nicolson type difference scheme, respectively.

The above equation (9) can be simplified, for $n=0$,

$$\begin{aligned}
&-(1-\theta)(\xi_i+\eta_i g_2+\zeta_i)u_{i-1}^1-(1-\theta)\eta_i\sum_{k=3}^{i+1}g_k u_{i-k+1}^1\\
&-(1-\theta)\zeta_i\sum_{k=3}^{J-i+1}g_k u_{i+k-1}^1+(1-\theta)(\xi_i-\eta_i-\zeta_i g_2)u_{i+1}^1\\
&+[1-(1-\theta)(\eta_i g_1+\zeta_i g_1)]u_i^1=\theta(\xi_i+\eta_i g_2+\zeta_i)u_{i-1}^0\\
&+[1+\theta(\eta_i g_1+\zeta_i g_1)]u_i^0+\theta(-\xi_i+\eta_i+\zeta_i g_2)u_{i+1}^0\\
&+\theta\eta_i\sum_{k=3}^{i+1}g_k u_{i-k+1}^0+\theta\zeta_i\sum_{k=3}^{J-i+1}g_k u_{i+k-1}^0\\
&+\Gamma(1-\beta)\tau^\beta(\theta f_i^0+(1-\theta)f_i^1), \qquad (10)
\end{aligned}$$

and for $n>0$,

$$\begin{aligned}
&-(1-\theta)(\xi_i+\eta_i g_2+\zeta_i)u_{i-1}^{n+1}-(1-\theta)\eta_i\sum_{k=3}^{i+1}g_k u_{i-k+1}^{n+1}\\
&-(1-\theta)\zeta_i\sum_{k=3}^{J-i+1}g_k u_{i+k-1}^{n+1}+(1-\theta)(\xi_i-\eta_i-\zeta_i g_2)u_{i+1}^{n+1}\\
&+[1-(1-\theta)(\eta_i g_1+\zeta_i g_1)]u_i^{n+1}=\theta(\xi_i+\eta_i g_2+\zeta_i)u_{i-1}^n\\
&+[2-2^{1-\beta}+\theta(\eta_i g_1+\zeta_i g_1)]u_i^n+\theta(-\xi_i+\eta_i+\zeta_i g_2)u_{i+1}^n\\
&+\theta\eta_i\sum_{k=3}^{i+1}g_k u_{i-k+1}^n+\theta\zeta_i\sum_{k=3}^{J-i+1}g_k u_{i+k-1}^n+\sum_{j=1}^{n-1}d_j u_i^{n-j}\\
&+u_i^0\sigma_n+\Gamma(1-\beta)\tau^\beta(\theta f_i^n+(1-\theta)f_i^{n+1}), \qquad (11)
\end{aligned}$$

and Dirichlet boundary conditions

$$u_0^n=0, u_J^n=\varphi(t_n), \quad n=1,2,\cdots,N-1,$$

and initial conditions

$$u_i^0=u_0(x_i), \quad i=0,1,\cdots,J,$$

where $\xi_i=\frac{v_i\tau^\beta\Gamma(2-\beta)}{2h}, \eta_i=\frac{d_{+i}\tau^\beta\Gamma(2-\beta)}{h^\alpha}, \zeta_i=\frac{d_{-i}\tau^\beta\Gamma(2-\beta)}{h^\alpha}$ and $d_j=\sigma_{j+1}-\sigma_j, j=1,2,\cdots,n-1$.

The numerical method (10) and (11) can be written in the matrix form:

$$AU^1=B_0U^0+Q^0,$$
$$AU^{n+1}=BU^n+d_1U^{n-1}+\cdots+d_{n-1}U^1+\sigma_nU^0+Q^n,$$

where

$$U^n = (u_1^n,u_2^n,\cdots,u_{J-1}^n)^T,$$

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$$
\begin{aligned}
U^0 &= [u_0(x_1), u_0(x_2), \cdots, u_0(x_{J-1})]^T, \\
b &= (\eta_{J-1} + \zeta_{J-1} g_2)[(1-\theta)u_J^{n+1} + \theta u_J^n], \\
F^n &= (f_1^n, f_2^n, \cdots, f_{J-1}^n + b)^T, \\
E &= (\zeta_1 g_J, \zeta_2 g_{J-1}, \cdots, \zeta_{J-1} g_2)^T, \\
Q^n &= \Gamma(2-\beta)\tau^\beta(\theta F^n + (1-\theta)F^{n+1}) \\
&\quad +(1-\theta)U_J^{n+1}E + \theta U_J^n E,
\end{aligned}
$$

and matrix $A = (A_{ij})_{(J-1)\times(J-1)}$ is defined as follows:

$$
A_{ij} = \begin{cases}
-(1-\theta)(\xi_i + \eta_i g_2 + \zeta_i), & j = i-1, \\
1-(1-\theta)(\eta_i g_1 + \zeta_i g_1), & j = i, \\
(1-\theta)(\xi - \eta_i - \zeta_i g_2), & j = i+1, \\
-(1-\theta)\eta_i g_{i+1-j}, & j = 1, 2, \cdots, i-2, \\
-(1-\theta)\zeta_i g_{j+1-i}, & j = i+2, i+3, \cdots, J-1.
\end{cases}
$$

It is obvious that matrix A is strictly dominant, the system defined by (10) and (11) has unique solution.

3 STABILITY AND CONVERGENCE

In this section, we investigate the stability and convergence of the numerical scheme (9).

Theorem 1 *For*

$$
\frac{\theta\alpha\Gamma(2-\beta)\tau^\beta}{h^\alpha} \max_{x\in[L,R]} (d_+(x) + d_-(x)) \le 2 - 2^{1-\beta}, \tag{12}
$$

the weighted finite difference scheme (9) for solving equation (1)-(3) is stable.

Proof. Let $u_i^n, \tilde{u}_i^n (i = 1, 2, \cdots, J, n = 0, 1, 2, \cdots, N-1)$ be the numerical solutions of (9) corresponding to the initial data u_i^0 and $\tilde{u}_i^0$, respectively. Let $\varepsilon_i^n = \tilde{u}_i^n - u_i^n$, the stability condition is equivalent to

$$
\|E^n\|_\infty \le \|E^0\|_\infty, \quad n = 0, 1, \cdots, N-1, \tag{13}
$$

where $E^n = (\varepsilon_1^k, \varepsilon_2^k, \cdots, \varepsilon_{J-1}^k)$. We will use mathematical induction to get the above result.

For $n = 0$, we have

$$
\begin{aligned}
&-(1-\theta)[(\xi_i + \eta_i g_2 + \zeta_i)\varepsilon_{i-1}^1 + \eta_i \sum_{k=3}^{i+1} g_k \varepsilon_{i-k+1}^1 \\
&+\zeta_i \sum_{k=3}^{J-i+1} g_k \varepsilon_{i+k-1}^1 - (\xi_i - \eta_i - \zeta_i g_2)\varepsilon_{i+1}^1]
\end{aligned}
$$

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$$
\begin{aligned}
&+[1-(1-\theta)(\eta_i g_1+\zeta_i g_1)]\varepsilon_i^1=\theta[(\xi_i+\eta_i g_2+\zeta_i)\varepsilon_{i-1}^0\\
&+\zeta_i\sum_{k=3}^{J-i+1} g_k\varepsilon_{i+k-1}^0+(-\xi_i+\eta_i+\zeta_i g_2)\varepsilon_{i+1}^0\\
&+\eta_i\sum_{k=3}^{i+1} g_k\varepsilon_{i-k+1}^0]+[1+\theta(\eta_i g_1+\zeta_i g_1)]\varepsilon_i^0, \qquad (14)
\end{aligned}
$$

for $n>0$,

$$
\begin{aligned}
&-(1-\theta)[(\xi_i+\eta_i g_2+\zeta_i)\varepsilon_{i-1}^{n+1}+\eta_i\sum_{k=3}^{i+1} g_k\varepsilon_{i-k+1}^{n+1}\\
&+\zeta_i\sum_{k=3}^{J-i+1} g_k\varepsilon_{i+k-1}^{n+1}-(\xi_i-\eta_i-\zeta_i g_2)\varepsilon_{i+1}^{n+1}]\\
&+[1-(1-\theta)(\eta_i g_1+\zeta_i g_1)]\varepsilon_i^{n+1}=\sum_{j=1}^{n-1} d_j\varepsilon_i^{n-j}\\
&+\sigma_n\varepsilon_i^0+\theta[(-\xi_i+\eta_i+\zeta_i g_2)\varepsilon_{i+1}^n+\eta_i\sum_{k=3}^{i+1} g_k\varepsilon_{i-k+1}^n\\
&+\zeta_i\sum_{k=3}^{J-i+1} g_k\varepsilon_{i+k-1}^n+(\xi_i+\eta_i g_2+\zeta_i)\varepsilon_{i-1}^n]\\
&+[2-2^{1-\beta}+\theta(\eta_i g_1+\zeta_i g_1)]\varepsilon_i^n. \qquad (15)
\end{aligned}
$$

Note that $d_+(x,t), d_-(x,t)\geq v(x,t)$, we have

$$\xi_i-\eta_i-\zeta_i g_2\leq 0. \qquad (16)$$

In fact, if $n=0$, suppose $|\varepsilon_l^1|=\max_{1\leq i\leq J-1}|\varepsilon_i^1|$, note that $\xi_i,\eta_i,\zeta_i>0$ and for any integer number m, $\sum_{j=0}^m g_j<0$, from (12), (16), we derive

$$
\begin{aligned}
\|E^1\|_\infty=|\varepsilon_l^1| &\leq -(1-\theta)\eta_l\sum_{k=0}^{l+1} g_k|\varepsilon_l^1|+|\varepsilon_l^1|-(1-\theta)\zeta_l\sum_{k=0}^{J-l+1}|\varepsilon_l^1|\\
&\leq \;|-(1-\theta)[(\xi_l+\eta_l g_2+\zeta_l)\varepsilon_{l-1}^1+\zeta_l\sum_{k=3}^{J-l+1} g_l\varepsilon_{l+k-1}^1\\
&\quad+(\eta_l+\zeta_l g_2-\xi_l)\varepsilon_{l+1}^1+\eta_l\sum_{k=3}^{l+1} g_k\varepsilon_{l-k+1}^1]\\
&\quad+[1-(1-\theta)(\eta_l g_1+\zeta_l g_1)]\varepsilon_l^1|\\
&\leq \;\theta[(\xi_l+\eta_l g_2+\zeta_l)|\varepsilon_{l-1}^0|+\zeta_l\sum_{k=3}^{J-l+1} g_k|\varepsilon_{l+k-1}^0|
\end{aligned}
$$

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$$
\begin{aligned}
&+(\eta_l+\zeta_l g_2)|\varepsilon^0_{l+1}|+\eta_l\sum_{k=3}^{l+1}g_k|\varepsilon^0_{l-k+1}|]\\
&+[1-\theta(\xi_l-\eta_l g_1-\zeta_l g_1)]|\varepsilon^0_l|\le\|E^0\|_\infty,
\end{aligned}
$$

Suppose that $\|E^n\|_\infty\le\|E^0\|_\infty, n=1,2,\cdots,s$, then when $n=s+1$, let $|\varepsilon^{s+1}_l|=\max_{1\le i\le J-1}|\varepsilon^{s+1}_i|$. Similar to former estimate, we obtain

$$
\begin{aligned}
&\|E^{s+1}\|_\infty\le|-(1-\theta)[(\xi_l+\eta_l g_2+\zeta_l)\varepsilon^{n+1}_{l-1}+\eta_l\sum_{k=3}^{l+1}g_k\varepsilon^{n+1}_{l-k+1}\\
&+\zeta_l\sum_{k=3}^{J-l+1}g_k\varepsilon^{n+1}_{l+k-1}-(\xi_l-\eta_l-\zeta_l g_2)\varepsilon^{n+1}_{l+1}]\\
&+[1-(1-\theta)(\eta_l g_1+\zeta_l g_1)]\varepsilon^{n+1}_l|\\
\le\;&\theta(\xi_l+\eta_l g_2+\zeta_l)|\varepsilon^s_{l-1}|+\theta(-\xi_l+\eta_l+\zeta_l g_2)|\varepsilon^s_{l+1}|\\
&+[2-2^{1-\beta}+\theta(\eta_l g_1+\zeta_l g_1)]|\varepsilon^s_l|+\theta\eta_l\sum_{k=3}^{l+1}g_k|\varepsilon^s_{l-k+1}|\\
&+\theta\zeta_l\sum_{k=3}^{J-l+1}g_k|\varepsilon^s_{l+k-1}|+\sum_{j=1}^{s-1}d_j|\varepsilon^{s-j}_l|+\sigma_s|\varepsilon^0_l|\\
\le\;&\|E^0\|_\infty.
\end{aligned}
$$

Hence, $\|E^{s+1}\|_\infty\le\|E^0\|_\infty$. The proof is completed.

Theorem 2 *Suppose that $u(x,t)$ is the sufficiently smooth solution of (1)-(3) and u^k_i is the difference solution of difference scheme (9). If the condition (12) is satisfied, then*

$$\|u(x_i,t_n)-u^n_i\|_\infty\le M\sigma^{-1}_{n-1}(\tau^{1+\beta}+\tau^\beta h),$$

where M is a positive constant.

Proof. Define $e^n_i=u(x_i,t_n)-u^n_i$ and $e^n=(e^n_1,e^n_2,\cdots,e^n_{J-1})$. Notice that $e^0_j=0$, we have: when $n=0$,

$$
\begin{aligned}
&-(1-\theta)[(\xi_i+\eta_i g_2+\zeta_i)e^1_{i-1}+\eta_i\sum_{k=3}^{i+1}g_k e^1_{i-k+1}\\
&+\zeta_i\sum_{k=3}^{J-i+1}g_k e^1_{i+k-1}-(\xi_i-\eta_i-\zeta_i g_2)e^1_{i+1}]\\
&+[1-(1-\theta)(\eta_i g_1+\zeta_i g_1)]e^1_i=R^1_i, \qquad (17)
\end{aligned}
$$

when $n>0$,

$$-(1-\theta)[(\xi_i+\eta_i g_2+\zeta_i)e^{n+1}_{i-1}+\eta_i\sum_{k=3}^{i+1}g_k e^{n+1}_{i-k+1}$$

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$$
\begin{aligned}
&+\zeta_i \sum_{k=3}^{J-i+1} g_k e_{i+k-1}^{n+1} - (\xi_i - \eta_i - \zeta_i g_2) e_{i+1}^{n+1}] \\
&+[1-(1-\theta)(\eta_i g_1 + \zeta_i g_1)] e_i^{n+1} - \theta\eta_i \sum_{k=3}^{i+1} g_k e_{i-k+1}^{n} \\
&-[2-2^{1-\beta} + \theta(\eta_i g_1 + \zeta_i g_1)] e_i^n - \theta\zeta_i \sum_{k=3}^{J-i+1} g_k e_{i+k-1}^{n} \\
&-\theta(\xi_i + \eta_i g_2 + \zeta_i) e_{i-1}^n - \sum_{j=1}^{n-1} d_j e_i^{n-j} \\
&-\theta(-\xi_i + \eta_i + \zeta_i g_2) e_{i+1}^n = R_i^{n+1}, \qquad (18)
\end{aligned}
$$

where R_i^{n+1} is the truncation error of difference scheme (9). Furthermore, there exists a positive constant M independent of step sizes such that $|R_i^{n+1}| \leq M(\tau^{1+\beta} + \tau^\beta h)$.

We will prove by inductive method. Let $|e_l^1| = \max_{1\leq i\leq J-1} |e_i^1|$. If $k = 1$, subject to the condition (12), based on (17), we have

$$
\begin{aligned}
&\|e^1\|_\infty \leq |-(1-\theta)[(\xi_i + \eta_i g_2 + \zeta_i) e_{i-1}^1 + \eta_i \sum_{k=3}^{i+1} g_k e_{i-k+1}^1 \\
&+\zeta_i \sum_{k=3}^{J-i+1} g_k e_{i+k-1}^1 - (\xi_i - \eta_i - \zeta_i g_2) e_{i+1}^1] \\
&+[1-(1-\theta)(\eta_i g_1 + \zeta_i g_1)] e_i^1| \\
&\leq M(\tau^{1+\beta} + \tau^\beta h) = \sigma_0^{-1} M(\tau^{1+\beta} + \tau^\beta h).
\end{aligned}
$$

Assume that $\|e^n\|_\infty \leq M\sigma_{n-1}^{-1}(\tau^{1+\beta} + \tau^\beta h), n = 1, 2, \cdots, s$, then when $n = s+1$, let $|e_l^{s+1}| = max_{1\leq i\leq J-1}|e_i^{s+1}|$, notice that $\sigma_j^{-1} < \sigma_k^{-1}, j = 0, 1, \cdots, k-1$. Similarly, we obtain

$$
\begin{aligned}
\|e^{s+1}\|_\infty &\leq d_1\|e^s\|_\infty + \sum_{j=1}^{n-1} d_j \|e^{s-j}\|_\infty + M(\tau^{1+\beta} + \tau^\beta h) \\
&\leq (d_1\sigma_{s-1}^{-1} + d_2\sigma_{s-1}^{-1} + \cdots + d_s\sigma_0^{-1} + 1) M(\tau^{1+\beta} + \tau^\beta h) \\
&\leq \sigma_s^{-1} M(\tau^{1+\beta} + \tau^\beta h).
\end{aligned}
$$

Thus, the proof is completed.

In additional, since

$$
\lim_{n\to\infty} \frac{\sigma_n^{-1}}{n^\beta} = \lim_{n\to\infty} \frac{n^{-1}}{(1-\beta)n^{-1}} = \frac{1}{1-\beta}.
$$

there is a constant C_1 for which

$$
\|e^n\|_\infty \leq C_1 n^\beta (\tau^{1+\beta} + \tau^\beta h).
$$

and $n\tau \leq T$ is finite, we obtain the following result.

Theorem 3 *Under the conditions of Theorem 2, then numerical solution converges to exact solution as h and τ tend to zero. Furthermore there exists positive constant $C > 0$, such that*

$$\|u(x_i, t_n) - u_i^n\| \leq C(\tau + h),$$

where $i = 1, 2, \cdots, J-1; n = 1, 2, \cdots, N$.

4 NUMERICAL RESULTS

In this section, the following two-sided space-time space-time fractional advection-diffusion equation in a bounded domain is considered in [15]:

$$\begin{aligned}
&\frac{\partial^{0.6} u(x,t)}{\partial t^{0.6}} = -\frac{\partial u(x,t)}{\partial x} + d_+(x,t)\frac{\partial^{1.6} u(x,t)}{\partial_+ x^{1.6}} \\
&+d_-(x,t)\frac{\partial^{1.6} u(x,t)}{\partial_- x^{1.6}} + f(x,t), \quad (x,t) \in [0,1] \times [0,1] \\
&u(0,t) = 0, \qquad u(1,t) = 1 + 4t^2, \quad t \in [0,1], \\
&u(x,0) = x^2, \quad x \in [0,1],
\end{aligned}$$

where $d_+(x,t) = \frac{2}{5}\Gamma(0.4)x^{0.6}$, $d_-(x,t) = 5\Gamma(0.4)(1-x)^{1.6}$, and $f(x,t) = \frac{100}{7\Gamma(0.4)}x^2t^{1.4} + (1+4t^2)(-25x^2 + 40x - 12)$. The exact solution is $u(x,t) = (1+4t^2)x^2$.

Table 1: The error $\max|u_i^k - u(x_i, t^k)|$ for the IWFDMs with $\theta = 1$

N	J	State	The error
10	10	Divergence	1.1305e+019
100	10	Divergence	2.3237e+163
10000	10	Divergence	Infinity
30000	10	Convergence	1.3230

Table 1 shows the maximum absolute numerical error between the exact solution and the numerical solution obtained by NWFDM with $\theta = 1$. From Table 1, it can see that our scheme is conditionally stable.

Table 2 and Table 3 show the maximum absolute error, at time $t = 1.0$, between the exact analytical solution and the numerical solution obtained by NWFDM with $\theta = 1/2$ and $\theta = 0$, respectively.

Table 4 and Table 5 show the comparison of maximum absolute numerical error of the weighted finite difference scheme in [12] (WFDM) and new weighted finite difference (NWFDM). We can see that the NNWDM is more accurate than WFDM at $\theta = 0$, but at $\theta = 0.4$ is opposite. From the above five tables, it can seen that the numerical tests are in excellent agreement with theoretical analysis.

Table 2: The error and convergence rate for the scheme with $\theta = 1/2$

N	J	Maximum error	Convergence rate
200	200	0.0809	-
400	400	0.0486	1.6646
800	800	0.0298	1.6309
1600	1600	0.0055	1.6022

Table 3: The error and convergence rate for the scheme with $\theta = 0$

N	J	Maximum error	Convergence rate
200	200	0.0415	-
400	400	0.0209	1.9378
800	800	0.0107	1.9533
1600	1600	0.0054	1.9815

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Table 4: The comparison of two schemes with $\theta = 0$

N	J	NWFDM	WFDM
50	50	0.1514	0.1522
100	100	0.0783	0.0797
150	150	0.0533	0.0545
200	200	0.0405	0.0415

Table 5: The comparison of two schemes with $\theta = 0.4$

N	J	NWFDM	WFDM
50	50	0.2198	0.1498
100	100	0.1242	0.0785
150	150	0.0899	0.0537
200	200	0.0717	0.0409

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A modified Newton-Shamanskii method for a nonsymmetric algebraic Riccati equation*

Jian-Lei Li[a]†, Li-Tao Zhang[b], Xu-Dong Li[c], Qing-Bin Li[b]

[a] *College of Mathematics and Information Science, North China University of, Water Resources and Electric Power, Zhengzhou, Henan, 450011, PR China.*

[b] *Department of Mathematics and Physics, Zhengzhou Institute of, Aeronautical Industry Management, Zhengzhou, Henan, 450015, PR China.*

[c] *Investment Projects Construction Agent Center of Luohe City People's, Government, Luohe, Henan, 462001, PR China.*

Abstract

The non-symmetric algebraic Riccati equation arising in transport theory can be rewritten as a vector equation and the minimal positive solution of the non-symmetric algebraic Riccati equation can be obtained by solving the vector equation. In this paper, based on the Newton-Shamanskii method, we propose a new iterative method called modified Newton-Shamanskii method for solving the vector equation. Some convergence results are presented. The convergence analysis shows that sequence of vectors generated by the modified Newton-Shamanskii method is monotonically increasing and converges to the minimal positive solution of the vector equation. Finally, numerical experiments are presented to illustrate the performance of the modified Newton-Shamanskii method.

Key words: non-symmetric algebraic Riccati equation; M-matrix; transport theory; minimal positive solution; modified Newton-Shamanskii method.

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1 Introduction

For convenience, firstly, we give some definitions and notations. For any matrices $A = [a_{i,j}]$ and $B = [b_{i,j}] \in R^{m\times n}$, we write $A \geq B(A > B)$ if $a_{i,j} \geq b_{i,j}(a_{i,j} > b_{i,j})$

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†Corresponding author. E-mail:hnmaths@163.com.

holds for all i, j. The Hadamard product of A and B is defined by $A \circ B = [a_{i,j} \cdot b_{i,j}]$. I denotes the identity matrix with appropriate dimension. The superscript T denotes the transpose of a vector or a matrix. We denote the norm by $\|\cdot\|$ for a vector or a matrix.

In this paper we are interested in iteratively solving the following nonsymmetric algebraic Riccati equation (NARE) arising in transport theory (see [3–5, 21] and the references cited therein):

$$XCX - XE - AX + B = 0, \tag{1.1}$$

where $A, B, C, E \in R^{n\times n}$ have the following special form:

$$A = \Delta - eq^T,\ B = ee^T,\ C = qq^T,\ E = D - qe^T. \tag{1.2}$$

Here and in the following, $e = (1, 1, ..., 1)^T,\ q = (q_1, q_2, ..., q_n)^T$ with $q_i = c_i/2\omega_i$,

$$\begin{cases} \Delta = \operatorname{diag}(\delta_1, \delta_2, ..., \delta_n) \quad \text{with} \quad \delta_i = \dfrac{1}{c\omega_i(1+\alpha)}, \\ D = \operatorname{diag}(d_1, d_2, ..., d_n) \quad \text{with} \quad d_i = \dfrac{1}{c\omega_i(1-\alpha)}, \end{cases} \tag{1.3}$$

and

$$0 < c \le 1,\ 0 \le \alpha < 1, \quad 0 < \omega_n < ... < \omega_2 < \omega_1 < 1 \tag{1.4}$$

$\sum_{i=1}^{n} c_i = 1,\ c_i > 0,\ i = 1, 2, ..., n.$

The form of the Riccati equation (1.1) arises in Markov models [22] and in nuclear physics [3, 24], and it has many positive solutions in the componentwise sense. There have been a lot of studies about algebraic properties [11, 21] and iterative methods for the nonnegative solution of the nonsymmetric algebraic Riccati equations (1.1), including the basic fixed-point iterations [5–8,19], the doubling algorithm [9], the Schur method [23,28], the Matrix Sign Function method [13,25] and the alternately linearized implicit iteration method [15], and so on; see related references therein. The existence of positive solutions of (1.1) has been shown in [3] and [4], but only the minimal positive solution is physically meaningful. So it is important to develop some effective and efficient procedures to compute the minimal positive solution of Equation (1.1).

Recently, Lu [10] has shown that the matrix equation (1.1) is equivalent to a vector equation and has developed a simple and efficient iterative procedure to compute the minimal positive solution of (1.1). The fixed-point iteration methods were further studied in [14, 16] for solving the vector equation. In [14] Bai, Gao and Lu proposed two nonlinear splitting iteration methods: the nonlinear block Jacobi and the nonlinear block Gauss-Seidel iteration methods. In [16] Bao, Lin and Wei proposed a modified simple iteration method for solving the vector equation. Furthermore, the convergence rates of various fixed-point iterations [10,14,16] were determined and compared in [20].

The Newton method has been presented and analyzed by Lu for solving the vector equation in [12]. It has been shown that the Newton method for the vector equation is more simple and efficient than using the corresponding Newton method directly for the original Riccati equation (1.1). Li, Huang and Zhang present a relaxed Newton-like method [17] for solving the vector equation. Especially, in [18] Lin and Bao applied the Newton-Shamanskii method [2, 26] to solve the vector equation.

Based on the Newton-Shamanskii method [18], in this paper, we propose a modified Newton-Shamanskii method to solve the vector equation. The convergence analysis shows that the sequence of vectors generated by the new iterative method is monotonically increasing and converges to the minimal positive solution of the vector equation, which can be used to obtain the minimal positive solution of the original Riccati equation. Our method extends the recent work done by Lu [12] and Lin and Bao [18].

Now, we give the definition of Z-matrix and M-matrix, and also give the following two Lemmas which will be used later.

Definition 1 [1] *A real square matrix A is called a Z-matrix if all its off-diagonal elements are non-positive. Any Z-matrix A can be written as $A = sI - B$ with $B \geq 0$, $s > 0$.*

Definition 2 [1] *Any matrix A of the form $A = sI - B$ for which $s > \rho(B)$, the spectral radius of B, is called an M-matrix.*

Lemma 1.1 [1] *For a Z-matrix A, the following statements are equivalent:*
(1) A is a nonsingular M-matrix;
(2) A is nonsingular and $A^{-1} \geq 0$;
(3) $Av > 0$ for some vector $v \geq 0$.

Lemma 1.2 [1] *Let $A \in R^{n\times n}$ be a nonsingular M-matrix. If $B \in R^{n\times n}$ is a Z-matrix and satisfies the relation $B \geq A$, then $B \in R^{n\times n}$ is also a nonsingular M-matrix.*

The rest of the paper is organized as follows. In Section 2, we review the Newton-Shamanskii method and some useful results, and present the modified Newton-Shamanskii method. Some convergence results are given in Section 3. Section 4 and 5 give numerical experiments and conclusions, respectively.

2 The modified Newton-Shamanskii method

It has been shown in [10, 12] that the solution of (1.1) must have the following form:

$$X = T \circ (uv^T) = (uv^T) \circ T,$$

where $T = [t_{i,j}] = [1/(\delta_i + d_j)]$ and u, v are two vectors, which satisfy the vector equations:

$$\begin{cases} u = u \circ (Pv) + e, \\ v = v \circ (\tilde{P}u) + e, \end{cases} \tag{2.1}$$

where $P = [p_{i,j}] = [q_j/(\delta_i + d_j)]$, $\tilde{P} = [\tilde{p}_{i,j}] = [q_j/(\delta_j + d_i)]$. Define $w = [u^T, v^T]^T$. The equation (2.1) can be rewritten equivalently as

$$f(w) = w - w \circ \mathcal{P}w - e = 0, \tag{2.2}$$

where

$$\mathcal{P} = \begin{bmatrix} 0 & P \\ \tilde{P} & 0 \end{bmatrix}.$$

The minimal positive solution of (1.1) can be obtained via computing the minimal positive solution of the vector equation (2.2).

The Newton method presented by Lu in [12] for the vector equation (2.2) is the following:

$$w_{k+1} = w_k - f'(w_k)^{-1} f(w_k),\ k = 0, 1, 2...$$

where for any $w \in R^{2n}$, the Jacobian matrix $f'(w)$ of $f(w)$ is given by

$$f'(w) = I_{2n} - G(w),\ \text{with } G(w) = \begin{bmatrix} G_1(v) & H_1(u) \\ H_2(v) & G_2(u) \end{bmatrix} \tag{2.3}$$

where $G_1(v) = \text{diag}(Pv)$, $G_2(u) = \text{diag}(\tilde{P}u)$, $H_1(u) = [u \circ p_1, u \circ p_2, ..., u \circ p_n]$ and $H_2(v) = [v \circ \tilde{p}_1, v \circ \tilde{p}_2, ..., v \circ \tilde{p}_n]$. For $i = 1, 2, ..., n$, p_i and $\tilde{p}_i$ are the ith column of P and $\tilde{P}$, respectively. Obviously, when $w > 0$, $G(w) \geq 0$ and $f'(w)$ is a Z-matrix.

The Newton-Shamanskii method for solving the vector equation (2.2) is given in [18] as follows:

Algorithm 2.1 (Newton-Shamanskii method) *For a given $m \geq 1$ and $k = 0, 1, 2, ...$,*

$$\begin{cases} \tilde{w}_{k,1} = w_k - f'(w_k)^{-1} f(w_k), \\ \tilde{w}_{k,p+1} = \tilde{w}_{k,p} - f'(w_k)^{-1} f(\tilde{w}_{k,p}), \quad 1 \leq p \leq m-1, \\ w_{k+1} = \tilde{w}_{k,m}. \end{cases} \tag{2.4}$$

It has been shown in [18] that the Newton-Shamanskii method has a better convergence than the Newton method [12]. However, if the inversion of the Jacobian matrix $f'(w)$ is difficult to compute, the Newton-Shamanskii method may converge slowly. Hence, based on the Newton-Shamanskii method, we propose the following modified Newton-Shamanskii method:

Algorithm 2.2 (Modified Newton-Shamanskii method) *For a given* $m \geq 1$ *and* $k = 0, 1, 2, ...$, *the Modified Newton-Shamanskii method is defined as follows:*

$$\begin{cases} \tilde{w}_{k,1} = w_k - T_k^{-1} f(w_k), \\ \tilde{w}_{k,p+1} = \tilde{w}_{k,p} - T_k^{-1} f(\tilde{w}_{k,p}), \quad 1 \leq p \leq m-1, \\ w_{k+1} = \tilde{w}_{k,m}. \end{cases} \tag{2.5}$$

where T_k *is a* Z*-matrix and* $T_k \geq f'(w_k)$.

Remark 2.1 When $T_k = f'(w_k)$, the modified Newton-Shamanskii method becomes the Newton-Shamanskii method [18]. When $m = 1$ and $T_k = f'(w_k)$, the modified Newton-Shamanskii method becomes the Newton method [12].

Before we give the convergence analysis of the Modified Newton-Shamanskii method, let us now state some results which are indispensable for our subsequent discussions.

Lemma 2.1 [18] *For any vectors* $w_1, w_2 \in R^{2n}$, $f'(w_1) - f'(w_2) = G(w_2 - w_1)$. *Furthermore, if* $w_2 > w_1$, *we have* $f'(w_1) - f'(w_2) = G(w_2 - w_1) \geq 0$.

Here and in the subsequent section, for convenience, $[f''(w)y]y$ is define as $f''(w)y^2$. Let

$$f''(w)y = [L_1 y, L_2 y, ..., L_{2n} y]^T \in R^{2n \times 2n},$$

where $L_i \in R^{2n \times 2n}$, $y \in R^{2n}$ and for $k = 1, 2, ..., n$,

$$L_k = \begin{bmatrix} 0 & (-e_k P_k^T) \\ (-e_k P_k^T)^T & 0 \end{bmatrix}, \quad L_{n+k} = \begin{bmatrix} 0 & (-\tilde{P}_k e_k^T) \\ (-\tilde{P}_k e_k^T)^T & 0 \end{bmatrix}$$

with $e_k^T = (0, ..., 0, 1, 0, ...)$, P_k^T and $\tilde{P}_k^T$ are the kth rows of the matrices P and $\tilde{P}$, respectively.

Lemma 2.2 [12] *For any vectors* $w_+, w \in R^{2n}$, *we have*

$$f(w_+) = f(w) + f'(w)(w_+ - w) + \frac{1}{2} f''(w)(w_+ - w, w_+ - w). \tag{2.6}$$

In particular, if $w_+ = w_*$, *the minimal positive solution of* (2.2), *then*

$$0 = f(w) + f'(w)(w_* - w) + \frac{1}{2} f''(w)(w_* - w, w_* - w). \tag{2.7}$$

Furthermore, for any $y > 0$ *or* $y < 0$,

$$f''(w)y^2 < 0 \tag{2.8}$$

and $f''(w)y^2$ is independent of w.

Because of the independence, in the following, we denote the operator $f''(w)$ by $\mathscr{L}$, i.e., $\mathscr{L}(y, y) = f''(w)(y, y)$ for any $y \in R^{2n}$. By (2.7), we have

$$f(w) = f'(w)(w - w_*) - \frac{1}{2}\mathscr{L}(w - w_*, w - w_*), \tag{2.9}$$

$$f'(w)(w - w_*) = f(w) + \frac{1}{2}\mathscr{L}(w - w_*, w - w_*). \tag{2.10}$$

Lemma 2.3 [12] *If $0 \le w < w_*$ and $f(w) < 0$, then $f'(w)$ is a nonsingular M-matrix.*

3 Convergence analysis of the Modified Newton-Shamanskii method

Now, we analyse convergence of the modified Newton-Shamanskii method (2.5).

Theorem 3.1 *Given a vector $w_k \in R^{2n}$. $\tilde{w}_{k,1}, \tilde{w}_{k,2}, ..., \tilde{w}_{k,m}, w_{k+1}$ are obtained by the modified Newton-Shamanskii method (2.5). If $w_k < w_*$ and $f(w_k) < 0$, then, $f'(w_k)$ is a nonsingular M-matrix, moreover,*

(1) $w_k < \tilde{w}_{k,1} < \tilde{w}_{k,2} < ... < \tilde{w}_{k,m} = w_{k+1} < w_*$;

(2) $f(\tilde{w}_{k,p}) < 0$ for $p = 1, 2, ..., m$;

(3) $f'(\tilde{w}_{k,p})$ is a nonsingular M-matrix for $p = 1, 2, ..., m$.

Therefore, $w_{k+1} < w_*$, $f(w_{k+1}) < 0$ and $f'(w_{k+1})$ is a nonsingular M-matrix.

Proof. Since $w_k < w_*$ and $f(w_k) < 0$, by Lemma 2.3, we can easily obtain that $f'(w_k)$ is a nonsingular M-matrix. By Lemma 1.2, we can conclude that T_k is also a nonsingular M-matrix. Now, we prove the theorem by mathematical induction. Define the error vectors $\tilde{e}_{k,i} = \tilde{w}_{k,i} - w_*$ and $e_k = w_k - w_*$, then $e_k < 0$. For $p = 1$, we have $\tilde{w}_{k,1} = w_k - T_k^{-1} f(w_k)$. Since $f(w_k) < 0$ and T_k is also a nonsingular M-matrix, then $\tilde{w}_{k,1} > w_k$ by Lemma 1.1.

By Eqs. (2.5) and (2.9), we obtain

$$\begin{aligned}\tilde{e}_{k,1} &= e_k - T_k^{-1} f(w_k) \\ &= e_k - T_k^{-1}[f'(w_k)e_k - \frac{1}{2}\mathscr{L}(e_k, e_k)] \\ &= T_k^{-1}[T_k - f'(w_k)]e_k + \frac{1}{2}T_k^{-1}\mathscr{L}(e_k, e_k) < 0. \end{aligned} \tag{3.1}$$

Thus, $\tilde{w}_{k,1} < w_*$.

By Eq. (2.6) and Lemma 1.1, we have

$$\begin{aligned} f(\tilde{w}_{k,1}) &= f(w_k - T_k^{-1}f(w_k)) \\ &= f(w_k) - f'(w_k)T_k^{-1}f(w_k) + \frac{1}{2}\mathscr{L}(T_k^{-1}f(w_k), T_k^{-1}f(w_k)) \\ &= [T_k - f'(w_k)]T_k^{-1}f(w_k) + \frac{1}{2}\mathscr{L}(T_k^{-1}f(w_k), T_k^{-1}f(w_k)) < 0. \qquad (3.2) \end{aligned}$$

By Lemma 2.3, it can be concluded that $f'(\tilde{w}_{k,1})$ is a nonsingular M-matrix. Therefore, the results hold for $p = 1$.

Assume the results are true for $1 \leq p \leq t$. Then, for $p = t + 1$, we have $\tilde{w}_{k,t+1} = \tilde{w}_{k,t} - T_k^{-1}f(\tilde{w}_{k,t})$. Since $f(\tilde{w}_{k,t}) < 0$ and T_k is a nonsingular M-matrix, then $\tilde{w}_{k,t+1} > \tilde{w}_{k,t}$.

Since $w_k < \tilde{w}_{k,1} < \tilde{w}_{k,2} < ... < \tilde{w}_{k,t}$, by Lemma 2.1, we have $f'(w_k) > f'(\tilde{w}_{k,1}) > f'(\tilde{w}_{k,2}) > ... > f'(\tilde{w}_{k,t})$. Therefore,

$$T_k - f'(\tilde{w}_{k,t}) > ... > T_k - f'(\tilde{w}_{k,1}) > T_k - f'(w_k) \geq 0.$$

By Eqs. (2.5) and (2.9), we have the following error vectors equation

$$\begin{aligned} \tilde{e}_{k,t+1} &= \tilde{e}_{k,t} - T_k^{-1}f(\tilde{w}_{k,t}) \\ &= \tilde{e}_{k,t} - T_k^{-1}[f'(\tilde{w}_{k,t})\tilde{e}_{k,t} - \frac{1}{2}\mathscr{L}(\tilde{e}_{k,t}, \tilde{e}_{k,t})] \\ &= T_k^{-1}[T_k - f'(\tilde{w}_{k,t})]\tilde{e}_{k,t} + \frac{1}{2}T_k^{-1}\mathscr{L}(\tilde{e}_{k,t}, \tilde{e}_{k,t}) < 0. \qquad (3.3) \end{aligned}$$

Therefore, $\tilde{w}_{k,t+1} < w_*$.

Similarly, by Eq. (2.6) and Lemma 1.1, we have

$$\begin{aligned} f(\tilde{w}_{k,t+1}) &= f(\tilde{w}_{k,t} - T_k^{-1}f(\tilde{w}_{k,t})) \\ &= f(\tilde{w}_{k,t}) - f'(\tilde{w}_{k,t})T_k^{-1}f(\tilde{w}_{k,t}) + \frac{1}{2}\mathscr{L}(T_k^{-1}f(\tilde{w}_{k,t}), T_k^{-1}f(\tilde{w}_{k,t})) \\ &= [T_k - f'(\tilde{w}_{k,t})]T_k^{-1}f(\tilde{w}_{k,t}) + \frac{1}{2}\mathscr{L}(T_k^{-1}f(\tilde{w}_{k,t}), T_k^{-1}f(\tilde{w}_{k,t})) < 0. \qquad (3.4) \end{aligned}$$

By Lemma 2.3, we have that $f'(\tilde{w}_{k,t+1})$ is a nonsingular M-matrix. Therefore, the results hold for $p = t+1$. Hence, by the principle of mathematical induction, the proof of the theorem is completed. □

In practical computation, we should choose T_k such that the iteration step (2.5) is less expensive to implement. For any $w_k \in R^{2n}$, according to the structure of the Jacobian $f'(w_k)$, T_k may be chosen as

$$T_k = I_{2n} - \begin{bmatrix} G_1(v_k) & 0 \\ 0 & G_2(u_k) \end{bmatrix} \qquad (3.5)$$

or

$$T_k = I_{2n} - \begin{bmatrix} G_1(v_k) & H_1(u_k) \\ 0 & G_2(u_k) \end{bmatrix}. \tag{3.6}$$

Another choice for T_k is

$$T_k = I_{2n} - \begin{bmatrix} G_1(v_k) & 0 \\ H_2(v_k) & G_2(u_k) \end{bmatrix}.$$

Numerical experiments show that the performance for this choice is almost the same as that for T_k given by (3.6).

The following theorem provides some results concerning the convergence of the modified Newton-Shamanskii method for the vector equation (2.2).

Theorem 3.2 *Let w_* be the minimal positive solution of the vector equation* (2.2). *The sequence of the vector sets $\{w_k, \tilde{w}_{k,1}, \tilde{w}_{k,2}, ..., \tilde{w}_{k,m}\}$ obtained by the modified Newton-Shamanskii method* (2.5) *with the initial vector $w_0 = 0$ is well defined. For all $k \geq 0$ and $1 \leq p \leq m$, we have*

(1) $f(w_k) < 0$ and $f(\tilde{w}_{k,p}) < 0$;

(2) $f'(w_k)$ and $f'(\tilde{w}_{k,p})$ are nonsingular M-matrices;

(3) $w_0 < \tilde{w}_{0,1} < \tilde{w}_{0,2} < ... < \tilde{w}_{0,m} = w_1 < \tilde{w}_{1,1} < \tilde{w}_{1,2} < ... < \tilde{w}_{1,m} = w_2 < ... < \tilde{w}_{k-1,m} = w_k < \tilde{w}_{k,1} < ... < \tilde{w}_{k,m} = w_{k+1} < ... < w_*$.

Furthermore, we have

$$\lim_{k\to\infty} w_k = w_*.$$

Proof. This theorem can also be proved by mathematical induction. The proof is similar to that of the Theorem 1 in [18]. Therefore, it is omitted. □

4 Numerical experiments

In this section, we give numerical experiments to illustrate the performance of the modified Newton-Shamanskii method presented in Section 3 with two different choices of the matrix T_k. Let NS denote the Newton-Shamanskii iterative method [18], MNS1 and MNS2 denote the modified Newton-Shamanskii iterative method (2.5) with T_k given by(3.5) and (3.6), respectively. In order to show numerically the performance of the modified Newton-Shamanskii iterative method, we list the number of iteration steps (denoted as IT), the CPU time in seconds (denoted as CPU), and relative residual error (denoted as ERR). The residual error is defined by

$$\text{ERR} = \max\left\{\frac{\|u_{k+1} - u_k\|_2}{\|u_{k+1}\|_2}, \frac{\|v_{k+1} - v_k\|_2}{\|v_{k+1}\|_2}\right\},$$

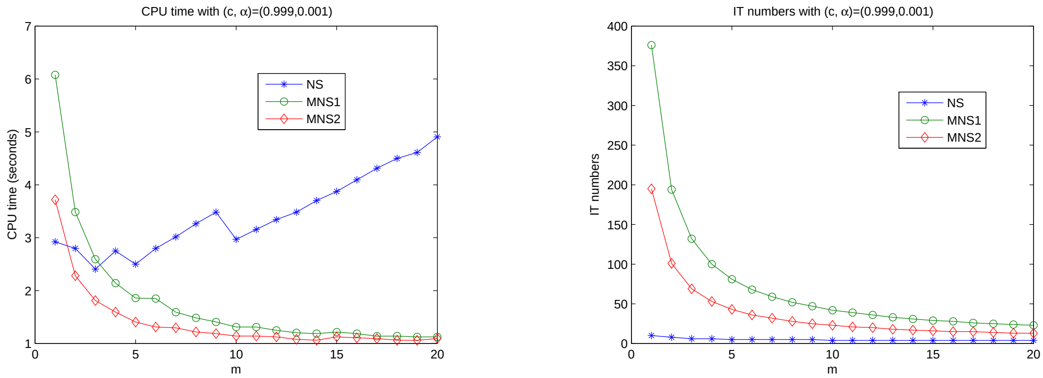

Figure 1: *CPU time and IT numbers for* $(c, \alpha) = (0.999, 0.001)$ *and* $n = 512$ *with different* m*. Left: CPU time; right: IT numbers*

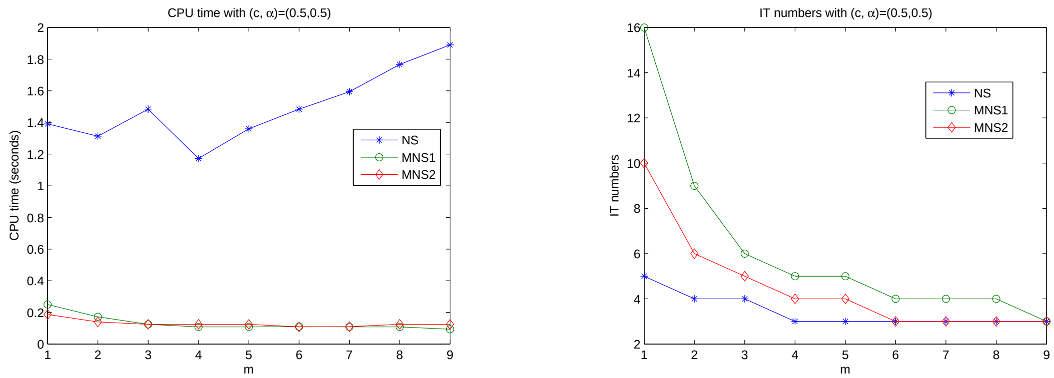

Figure 2: *CPU time and IT numbers for* $(c, \alpha) = (0.5, 0.5)$ *and* $n = 512$ *with different* m*. Left: CPU time; right: IT numbers*

where $\|\cdot\|_2$ is the 2-norm for a vector. For comparison, every experiment is repeated 5 times, and the average of the 5 CPU times is shown here. All the experiments are run in MATLAB 7.0 on a personal computer with Intel(R) Pentium(R) D 3.00GHz CPU and 0.99 GB memory, and all iterations are terminated once the current iterate satisfies ERR $\leq n \cdot \text{eps}$, where eps $= 1 \times 10^{-16}$.

In the test example, the constants c_i and w_i, $i = 1, 2, ...n$, are given by the numerical quadrature formula on the interval $[0, 1]$, which are obtained by dividing $[0, 1]$ into $\frac{n}{4}$ subintervals of equal length and applying a Gauss-Legendre quadrature [27] with 4 nodes to each subinterval; see the Example 5.2 in [6]

We test several different values (c, α). In Table 1, for $n = 512$ with different m and pairs of (c, α), and in Table 2, for the fixed $(c, \alpha) = (0.99, 0.01)$ with different n, we list ITs, CPUs and ERRs for the NS method and MNS methods, respectively. Figure 1 and

Table 1: Numerical results for $n = 512$ and different pairs of (c, α)

m	method		(c,α)			
			$(0.999, 0.001)$	$(0.99, 0.01)$	$(0.9, 0.1)$	$(0.5, 0.5)$
1	NS	IT	10	9	7	5
		CPU	2.9380	2.6100	2.2810	1.6090
		ERR	2.1776e-15	1.5433e-15	1.4280e-15	1.5773e-14
	MNS1	IT	376	130	43	16
		CPU	5.9370	2.0630	0.7820	0.2810
		ERR	4.7938e-14	4.3618e-14	2.7158e-14	7.0829e-15
	MNS2	IT	195	69	24	10
		CPU	3.7500	1.3430	0.5150	0.2190
		ERR	4.7717e-014	3.3654e-14	1.6087e-14	1.7311e-15
3	NS	IT	6	5	5	4
		CPU	2.5310	2.0630	2.0780	1.7190
		ERR	5.3953e-15	4.6570e-14	1.2318e-15	1.0553e-15
	MNS1	IT	132	46	16	6
		CPU	2.5780	0.9370	0.3130	0.1410
		ERR	4.3397e-14	3.7357e-14	1.0270e-14	2.6302e-14
	MNS2	IT	69	25	10	5
		CPU	1.8440	0.6720	0.2810	0.1410
		ERR	4.4170e-14	2.9843e-14	8.7831e-16	1.5640e-16
6	NS	IT	5	4	4	3
		CPU	2.9220	2.2340	2.2810	1.7350
		ERR	1.9497e-15	1.7274e-15	1.3919e-15	1.0832e-15
	MNS1	IT	68	24	9	4
		CPU	1.9060	0.6100	0.2340	0.1100
		ERR	4.7883e-14	4.0900e-14	3.3139e-15	2.9604e-16
	MNS2	IT	36	14	6	3
		CPU	1.3280	0.5160	0.2340	0.1250
		ERR	4.7025e-14	8.5873e-15	5.5104e-16	2.1332e-15
12	NS	IT	4	4	3	3
		CPU	3.4530	3.5160	2.5630	2.6410
		ERR	1.9512e-15	1.6885e-15	1.3243e-15	1.1225e-15
	MNS1	IT	36	13	5	3
		CPU	1.2660	0.4680	0.1880	0.1250
		ERR	2.9584e-14	2.6812e-14	1.9980e-14	1.64101e-16
	MNS2	IT	20	8	4	3
		CPU	1.2190	0.4530	0.2180	0.2030
		ERR	1.1402e-14	4.8204e-15	5.5981e-16	1.6410e-16

Table 2: Numerical results for $(c, \alpha) = (0.99, 0.01)$ and different n, m

m	method		n				
			64	128	256	512	1024
1	NS	IT	9	9	9	9	9
		CPU	0.0310	0.0620	0.4220	2.6100	16.9060
		ERR	6.8597e-16	9.5495e-16	1.1845e-15	1.5433e-15	2.7143e-15
	MNS1	IT	140	136	133	130	126
		CPU	0.0630	0.0940	0.4850	2.0630	7.4530
		ERR	5.3918e-15	1.2247e-14	2.3157e-14	4.3618e-14	1.0212e-14
	MNS2	IT	73	72	70	69	67
		CPU	0.0160	0.0630	0.2970	1.3430	4.7180
		ERR	6.1723e-15	9.44380e-15	2.1990e-14	3.3654e-14	7.8688e-14
5	NS	IT	5	5	5	5	5
		CPU	0.0160	0.0630	0.4220	2.3750	14.6560
		ERR	8.1022e-16	8.6594e-16	1.2226e-15	1.6581e-15	2.1565e-15
	MNS1	IT	31	30	29	29	28
		CPU	0.0150	0.0310	0.1410	0.6410	2.3590
		ERR	2.7024e-15	7.6059e-15	2.2028e-14	2.1877e-14	6.3491e-14
	MNS2	IT	17	17	16	16	16
		CPU	0.0160	0.0310	0.0930	0.5160	1.9530
		ERR	2.4804e-15	2.4814e-15	2.0690e-14	2.0656e-14	2.0698e-14
10	NS	IT	4	4	4	4	4
		CPU	0.0160	0.0780	0.5000	2.7500	16.6090
		ERR	7.2604e-16	8.1207e-16	1.1571e-15	1.5460e-15	2.2290e-15
	MNS1	IT	16	16	16	15	15
		CPU	0.0150	0.0150	0.0780	0.4680	1.7350
		ERR	6.0508e-15	5.8374e-15	6.0658e-15	4.9045e-14	4.8935e-14
	MNS2	IT	10	10	9	9	9
		CPU	0.0160	0.0320	0.0630	0.4380	1.6400
		ERR	5.6442e-16	4.2340e-16	1.4346e-14	1.3941e-14	1.3698e-14

Figure 2 describe the CPU time and IT numbers of those methods when $n = 512$ for $(c, \alpha) = (0.999, 0.001)$ and $(c, \alpha) = (0.5, 0.5)$. From these Tables and Figures, we can see that the optimal choice of m for the modified Newton-Shamanskii method is larger when $(c, \alpha) = (0.999, 0.001)$, compared with $(c, \alpha) = (0.5, 0.5)$. Obviously, compared with the Newton-Shamanskii iterative method, though the iterations number of the modified Newton-Shamanskii iterative method is more, according to the CPU time, we can find that the modified Newton-Shamanskii iterative method outperforms the Newton-Shamanskii iterative method. Among these methods, the MNS2 method is the best one.

5 Conclusion

In this paper, based on the Newton-Shamanskii method, we have proposed a modified Newton-Shamanskii method for solving the minimal positive solution of the nonsymmetric algebraic Riccati equation arising in transport theory and have given the convergence analysis. The convergence analysis shows that the iteration sequence generated by the modified Newton-Shamanskii method is monotonically increasing and converges to the minimal positive solution of the vector equation. Numerical experiments show that the modified Newton-Shamanskii method has a better performance than the Newton-Shamanskii method for the nonsymmetric algebraic Riccati equation. We find that when T_k is chosen as the block triangular of the Jacobian matrix, the modified Newton-Shamanskii method has a better convergence rate. The choice of the matrix T_k impacts the convergence rate of the modified Newton-Shamanskii method, hence, the determination of the optimum matrix T_k such that the modified Newton-Shamanskii method has a better convergence rate needs further to be studied.

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Hesitant fuzzy filters and hesitant fuzzy G-filters in residuated lattices

G. Muhiuddin

Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia

Abstract.

Characterizations of a hesitant fuzzy filter in a residuated lattice are considered. Given a hesitant fuzzy set, a new hesitant fuzzy filter of a residuated lattice is constructed. The notion of a hesitant fuzzy G-filter of a residuated lattice is introduced, and its characterizations are discussed. Conditions for a hesitant fuzzy filter to be a hesitant fuzzy G-filter are provided. Finally, the extension property of a hesitant fuzzy G-filter is established.

1. Introduction

The notions of Atanassov's intuitionistic fuzzy sets, type 2 fuzzy sets and fuzzy multisets etc. are a generalization of fuzzy sets. As another generalization of fuzzy sets, Torra and Narukawa [5] and Torra [6] introduced the notion of hesitant fuzzy sets and discussed the relationship between hesitant fuzzy sets and intuitionistic fuzzy sets. Xia and Xu [11] studied hesitant fuzzy information aggregation techniques and their application in decision making. They developed some hesitant fuzzy operational rules based on the interconnection between the hesitant fuzzy set and the intuitionsitic fuzzy set. Xu and Xia [12] proposed a variety of distance measures for hesitant fuzzy sets, and investigated the connections of the aforementioned distance measures and further developed a number of hesitant ordered weighted distance measures and hesitant ordered weighted similarity measures. Xu and Xia [13] defined the distance and correlation measures for hesitant fuzzy information and then considered their properties in detail. Wei [9] investigated the hesitant fuzzy multiple attribute decision making problems in which the attributes are in different priority level.

Residuated lattices are a non-classical logic system which is a formal and useful tool for computer science to deal with uncertain and fuzzy information. Filter theory, which is an important notion, in residuated lattices is studied by Shen and Zhang [4] and Zhu and Xu [15]. Wei [10] introduced the notion of hesitant fuzzy (implicative, regular and Boolean) filters in residuated lattice, and discussed its properties.

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E-mail: chishtygm@gmail.com

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In this paper, we deal with further properties of a hesitant fuzzy filter in a residuated lattice. We consider characterizations of a hesitant fuzzy filter in a residuated lattice. Given a hesitant fuzzy set, we construct a new hesitant fuzzy filter of a residuated lattice. We introduce the notion of a hesitant fuzzy G-filter of a residuated lattice, and discuss its characterizations. We provide conditions for a hesitant fuzzy filter to be a hesitant fuzzy G-filter. Finally, we establish the extension property of a hesitant fuzzy G-filter.

2. Preliminaries

Definition 2.1 ([1, 2, 3])**.** A *residuated lattice* is an algebra $(L, \vee, \wedge, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that

(1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice.
(2) $(L, \odot, 1)$ is a commutative monoid.
(3) $\odot$ and $\to$ form an adjoint pair, that is,

$$(\forall x, y, z \in L)\,(x \leq y \to z \;\Leftrightarrow\; x \odot y \leq z)\,.$$

In a residuated lattice L, the ordering $\leq$ and negation $\neg$ are defined as follows:

$$(\forall x, y \in L)\,(x \leq y \;\Leftrightarrow\; x \wedge y = x \;\Leftrightarrow\; x \vee y = y \;\Leftrightarrow\; x \to y = 1)$$

and $\neg x = x \to 0$ for all $x \in L$.

Proposition 2.2 ([1, 2, 3, 7, 8])**.** *In a residuated lattice L, the following properties are valid.*

(2.1) $1 \to x = x,\; x \to 1 = 1,\; x \to x = 1,\; 0 \to x = 1,\; x \to (y \to x) = 1.$

(2.2) $y \leq (y \to x) \to x.$

(2.3) $x \leq y \to z \;\Leftrightarrow\; y \leq x \to z.$

(2.4) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z).$

(2.5) $x \leq y \;\Rightarrow\; z \to x \leq z \to y,\; y \to z \leq x \to z.$

(2.6) $z \to y \leq (x \to z) \to (x \to y),\; z \to y \leq (y \to x) \to (z \to x).$

(2.7) $(x \to y) \odot (y \to z) \leq x \to z.$

(2.8) $x \odot y \leq x \wedge y.$

(2.9) $x \leq y \;\Rightarrow\; x \odot z \leq y \odot z.$

(2.10) $y \to z \leq x \vee y \to x \vee z.$

(2.11) $(x \vee y) \to z = (x \to z) \wedge (y \to z).$

Definition 2.3 ([4]). A nonempty subset F of a residuated lattice L is called a *filter* of L if it satisfies the conditions:

(2.12) $$(\forall x, y \in L)\,(x, y \in F \;\Rightarrow\; x \odot y \in F)\,.$$

(2.13) $$(\forall x, y \in L)\,(x \in F,\; x \leq y \;\Rightarrow\; y \in F)\,.$$

Proposition 2.4 ([4]). *A nonempty subset F of a residuated lattice L is a filter of L if and only if it satisfies:*

(2.14) $$1 \in F.$$

(2.15) $$(\forall x \in F)\,(\forall y \in L)\,(x \to y \in F \;\Rightarrow\; y \in F)\,.$$

3. Hesitant fuzzy filters

Let E be a reference set. A *hesitant fuzzy set* on E (see [6]) is defined in terms of a function h that when applied to E returns a subset of $[0, 1]$, that is, $h : E \to \mathscr{P}([0, 1])$.

In what follows, we take a residuated lattice L as a reference set.

Definition 3.1 ([10]). A hesitant fuzzy set h on L is called a *hesitant fuzzy filter* of L if it satisfies:

(3.1) $$(\forall x, y \in L)\,(x \leq y \;\Rightarrow\; h(x) \subseteq h(y))\,,$$

(3.2) $$(\forall x, y \in L)\,(h(x) \cap h(y) \subseteq h(x \odot y))\,.$$

Example 3.2. Let $L = [0, 1]$ be a subset of $\mathbb{R}$. For any $a, b \in L$, define

$$a \vee b = \max\{a, b\},\; a \wedge b = \min\{a, b\},$$

$$a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ (1 - a) \vee b & \text{otherwise,} \end{cases}$$

and

$$a \odot b = \begin{cases} 0 & \text{if } a + b \leq 1, \\ a \wedge b & \text{otherwise.} \end{cases}$$

Then $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a residuated lattice (see [15]). We define a hesitant fuzzy set

$$h : L \to \mathscr{P}([0, 1]),\; x \mapsto \begin{cases} (0.2, 0.7) & \text{if } x \in (c, 1] \text{ where } 0.5 \leq c \leq 1, \\ (0.3, 0.6] & \text{otherwise.} \end{cases}$$

It is routine to verify that h is a hesitant fuzzy filter of L.

Example 3.3. Let $L = \{0, a, b, c, d, 1\}$ be a set with the lattice diagram appears in Figure 1.

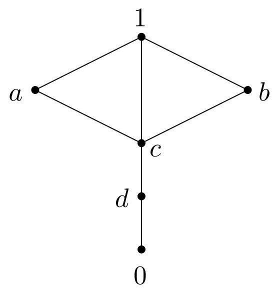

Figure 1

Consider two operation '$\odot$' and '$\to$' shown in Table 1 and Table 2, respectively.

TABLE 1. Cayley table for the binary operation '$\odot$'

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	c	c	0	a
b	0	c	b	c	d	b
c	0	c	c	c	0	c
d	0	0	d	0	0	d
1	0	a	b	c	d	1

TABLE 2. Cayley table for the binary operation '$\to$'

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	b	b	d	1
b	0	a	1	a	d	1
c	d	1	1	1	d	1
d	a	1	1	1	1	1
1	0	a	b	c	d	1

Then $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a residuated lattice. We define a hesitant fuzzy set

$$h : L \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} [0.2, 0.9) & \text{if } x \in \{1, a\}, \\ (0.3, 0.8] & \text{otherwise.} \end{cases}$$

It is routine to verify that h is a hesitant fuzzy filter of L.

Wei [10] provided a characterization of a hesitant fuzzy filter as follows.

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Lemma 3.4 ([10]). *A hesitant fuzzy set h on L is a hesitant fuzzy filter of L if and only if it satisfies*

$$(\forall x \in L)\,(h(x) \subseteq h(1))\,. \tag{3.3}$$

$$(\forall x, y \in L)\,(h(x) \cap h(x \to y) \subseteq h(y))\,. \tag{3.4}$$

We provide other characterizations of a hesitant fuzzy filter.

Theorem 3.5. *A hesitant fuzzy set h on L is a hesitant fuzzy filter of L if and only if it satisfies:*

$$(\forall x, y, z \in L)\,(x \leq y \to z \;\Rightarrow\; h(x) \cap h(y) \subseteq h(z))\,. \tag{3.5}$$

Proof. Assume that h is a hesitant fuzzy filter of L. Let $x, y, z \in L$ be such that $x \leq y \to z$. Then $h(x) \subseteq h(y \to z)$ by (3.1), and so

$$h(z) \supseteq h(y) \cap h(y \to z) \supseteq h(x) \cap h(y)$$

by (3.4).

Conversely let h be a hesitant fuzzy set on L satisfying (3.5). Since $x \leq x \to 1$ for all $x \in L$, it follows from (3.5) that

$$h(1) \supseteq h(x) \cap h(x) = h(x)$$

for all $x \in L$. Since $x \to y \leq x \to y$ for all $x, y \in L$, we have

$$h(y) \supseteq h(x) \cap h(x \to y)$$

for all $x, y \in L$. Hence h is a hesitant fuzzy filter of L. □

Theorem 3.6. *A hesitant fuzzy set h on L is a hesitant fuzzy filter of L if and only if h satisfies the condition (3.3) and*

$$(\forall x, y, z \in L)\,(h(x \to (y \to z)) \cap h(y) \subseteq h(x \to z))\,. \tag{3.6}$$

Proof. Assume that h is a hesitant fuzzy filter of L. Then the condition (3.3) is valid. Using (2.4) and (3.4), we have

$$\begin{aligned} h(x \to z) &\supseteq h(y) \cap h(y \to (x \to z)) \\ &= h(y) \cap h(x \to (y \to z)) \end{aligned}$$

for all $x, y, z \in L$.

Conversely, let h be a hesitant fuzzy set on L satisfying (3.3) and (3.6). Taking $x := 1$ in (3.6) and using (2.1), we get

$$\begin{aligned} h(z) = h(1 \to z) &\supseteq h(1 \to (y \to z)) \cap h(y) \\ &= h(y \to z) \cap h(y) \end{aligned}$$

for all $y, z \in L$. Thus h is a hesitant fuzzy filter of L by Lemma 3.4. □

Lemma 3.7. *Every hesitant fuzzy filter h on L satisfies the following condition:*

$$(\forall a, x \in L)\,(h(a) \subseteq h((a \to x) \to x))\,. \tag{3.7}$$

Proof. If we take $y = (a \to x) \to x$ and $x = a$ in (3.4), then

$$\begin{aligned} h((a \to x) \to x) &\supseteq h(a) \cap h(a \to ((a \to x) \to x)) \\ &= h(a) \cap h((a \to x) \to (a \to x)) \\ &= h(a) \cap h(1) = h(a). \end{aligned}$$

This completes the proof. □

Theorem 3.8. *A hesitant fuzzy set h on L is a hesitant fuzzy filter of L if and only if it satisfies the following conditions:*

$$(\forall x, y \in L)\,(h(x) \subseteq h(y \to x))\,, \tag{3.8}$$

$$(\forall x, a, b \in L)\,(h(a) \cap h(b) \subseteq h((a \to (b \to x)) \to x))\,. \tag{3.9}$$

Proof. Assume that h is a hesitant fuzzy filter of L. Using (2.1), (3.3) and (3.4), we have

$$h(y \to x) \supseteq h(x) \cap h(x \to (y \to x)) = h(x) \cap h(1) = h(x)$$

for all $x, y \in L$. Using (3.6) and (3.7), we get

$$h((a \to (b \to x)) \to x) \supseteq h((a \to (b \to x)) \to (b \to x)) \cap h(b) \supseteq h(a) \cap h(b)$$

for all $a, b, x \in L$.

Conversely, let h be a hesitant fuzzy set on L satisfying two conditions (3.8) and (3.9). If we take $y := x$ in (3.8), then $h(x) \subseteq h(x \to x) = h(1)$ for all $x \in L$. Using (3.9) induces

$$h(y) = h(1 \to y) = h((x \to y) \to (x \to y)) \to y) \supseteq h(x \to y) \cap h(x)$$

for all $x, y \in L$. Therefore h is a hesitant fuzzy filter of L by Lemma 3.4. □

Theorem 3.9. *A hesitant fuzzy set h on L is a hesitant fuzzy filter of L if and only if the set*

$$h_\tau := \{x \in L \mid \tau \subseteq h(x)\}$$

is a filter of L for all $\tau \in \mathscr{P}([0,1])$ with $h_\tau \neq \emptyset$.

Proof. Assume that h is a hesitant fuzzy filter of L. Let $x, y \in L$ and $\tau \in \mathscr{P}([0,1])$ be such that $x \in h_\tau$ and $x \to y \in h_\tau$. Then $\tau \subseteq h(x)$ and $\tau \subseteq h(x \to y)$. It follows from (3.3) and (3.4) that $h(1) \supseteq h(x) \supseteq \tau$ and $h(y) \supseteq h(x) \cap h(x \to y) \supseteq \tau$ and so that $1 \in h_\tau$ and $y \in h_\tau$. Hence h_τ is a filter of L by Proposition 2.4.

Conversely, suppose that h_τ is a filter of L for all $\tau \in \mathscr{P}([0,1])$ with $h_\tau \neq \emptyset$. For any $x \in L$, let $h(x) = \delta$. Then $x \in h_\delta$ and h_δ is a filter of L. Hence $1 \in h_\delta$ and so $h(x) = \delta \subseteq h(1)$. For

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any $x, y \in L$, let $h(x) = \delta_x$ and $h(x \to y) = \delta_{x\to y}$. If we take $\delta = \delta_x \cap \delta_{x\to y}$, then $x \in h_\delta$ and $x \to y \in h_\delta$ which imply that $y \in h_\delta$. Thus

$$h(x) \cap h(x \to y) = \delta_x \cap \delta_{x\to y} = \delta \subseteq h(y).$$

Therefore h is a hesitant fuzzy filter of L by Lemma 3.4. □

Theorem 3.10. *For a hesitant fuzzy set h on L, let $\tilde{h}$ be a hesitant fuzzy set on L defined by*

$$\tilde{h} : L \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} h(x) & \text{if } x \in h_\tau, \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\tau \in \mathscr{P}([0,1]) \setminus \{\emptyset\}$. If h is a hesitant fuzzy filter of L, then so is $\tilde{h}$.

Proof. Suppose that h is a hesitant fuzzy filter of L. Then h_τ is a filter of L for all $\tau \in \mathscr{P}([0,1])$ with $h_\tau \neq \emptyset$ by Theorem 3.9. Thus $1 \in h_\tau$, and so $\tilde{h}(1) = h(1) \supseteq h(x) \supseteq \tilde{h}(x)$ for all $x \in L$. Let $x, y \in L$. If $x \in h_\tau$ and $x \to y \in h_\tau$, then $y \in h_\tau$. Hence

$$\tilde{h}(x) \cap \tilde{h}(x \to y) = h(x) \cap h(x \to y) \subseteq h(y) = \tilde{h}(y).$$

If $x \notin h_\tau$ or $x \to y \notin h_\tau$, then $\tilde{h}(x) = \emptyset$ or $\tilde{h}(x \to y) = \emptyset$. Thus

$$\tilde{h}(x) \cap \tilde{h}(x \to y) = \emptyset \subseteq \tilde{h}(y).$$

Therefore $\tilde{h}$ is a hesitant fuzzy filter of L. □

Theorem 3.11. *If h is a hesitant fuzzy filter of L, then the set*

$$\Gamma_a := \{x \in L \mid h(a) \subseteq h(x)\}$$

is a filter of L for every $a \in L$.

Proof. Since $h(1) \supseteq h(a)$ for all $a \in L$, we have $1 \in \Gamma_a$. Let $x, y \in L$ be such that $x \in \Gamma_a$ and $x \to y \in \Gamma_a$. Then $h(x) \supseteq h(a)$ and $h(x \to y) \supseteq h(a)$. Since h is a hesitant fuzzy filter of L, it follows from (3.4) that

$$h(y) \supseteq h(x) \cap h(x \to y) \supseteq h(a)$$

so that $y \in \Gamma_a$. Hence Γ_a is a filter of L by Proposition 2.4. □

Theorem 3.12. *Let $a \in L$ and let h be a hesitant fuzzy set on L. Then*

(1) *If Γ_a is a filter of L, then h satisfies the following condition:*

$$(\forall x, y \in L)\, (h(a) \subseteq h(x) \cap h(x \to y) \ \Rightarrow\ h(a) \subseteq h(y)). \tag{3.10}$$

(2) *If h satisfies (3.3) and (3.10), then Γ_a is a filter of L.*

Proof. (1) Assume that Γ_a is a filter of L. Let $x, y \in L$ be such that

$$h(a) \subseteq h(x) \cap h(x \to y).$$

Then $x \to y \in \Gamma_a$ and $x \in \Gamma_a$. Using (2.15), we have $y \in \Gamma_a$ and so $h(y) \supseteq h(a)$.

(2) Suppose that h satisfies (3.3) and (3.10). From (3.3) it follows that $1 \in \Gamma_a$. Let $x, y \in L$ be such that $x \in \Gamma_a$ and $x \to y \in \Gamma_a$. Then $h(a) \subseteq h(x)$ and $h(a) \subseteq h(x \to y)$, which imply that $h(a) \subseteq h(x) \cap h(x \to y)$. Thus $h(a) \subseteq h(y)$ by (3.10), and so $y \in \Gamma_a$. Therefore Γ_a is a filter of L by Proposition 2.4. □

Definition 3.13 ([14])**.** A nonempty subset F of L is called a *G-filter* of L if it is a filter of L that satisfies the following condition:

$$(\forall x, y \in L)\,((x \odot x) \to y \in F \;\Rightarrow\; x \to y \in F)\,. \tag{3.11}$$

We consider the hesitant fuzzification of G-filters.

Definition 3.14. A hesitant fuzzy set h on L is called a *hesitant fuzzy G-filter* of L if it is a hesitant fuzzy filter of L that satisfies:

$$(\forall x, y \in L)\,(h((x \odot x) \to y) \subseteq h(x \to y))\,. \tag{3.12}$$

Note that the condition (3.12) is equivalent to the following condition:

$$(\forall x, y \in L)\,(h(x \to (x \to y)) \subseteq h(x \to y))\,. \tag{3.13}$$

Example 3.15. The hesitant fuzzy filter h in Example 3.3 is a hesitant fuzzy G-filter of L.

Lemma 3.16. *Every hesitant fuzzy filter h of L satisfies the following condition:*

$$(\forall x, y, z \in L)\,(h(x \to (y \to z)) \cap h(x \to y) \subseteq h(x \to (x \to z)))\,. \tag{3.14}$$

Proof. Let $x, y, z \in L$. Using (2.4) and (2.6), we have

$$x \to (y \to z) = y \to (x \to z) \le (x \to y) \to (x \to (x \to z)).$$

It follows from Theorem 3.5 that

$$h(x \to (y \to z)) \cap h(x \to y) \subseteq h(x \to (x \to z)).$$

This completes the proof. □

Theorem 3.17. *Let h be a hesitant fuzzy set on L. Then h is a hesitant fuzzy G-filter of L if and only if it is a hesitant fuzzy filter of L that satisfies the following condition:*

$$(\forall x, y, z \in L)\,(h(x \to (y \to z)) \cap h(x \to y) \subseteq h(x \to z))\,. \tag{3.15}$$

Proof. Assume that h is a hesitant fuzzy G-filter of L. Then h is a hesitant fuzzy filter of L. Note that $x \leq 1 = (x \to y) \to (x \to y)$, and thus $x \to y \leq x \to (x \to y)$ for all $x, y \in L$. It follows from (3.1) that $h(x \to y) \subseteq h(x \to (x \to y))$. Combining this and (3.13), we have

$$h(x \to y) = h(x \to (x \to y)) \tag{3.16}$$

for all $x, y \in L$. Using (3.14) and (3.16), we have

$$h(x \to (y \to z)) \cap h(x \to y) \subseteq h(x \to z)$$

for all $x, y, z \in L$.

Conversely, let h be a hesitant fuzzy filter of L that satisfies the condition (3.15). If we put $y = x$ and $z = y$ in (3.15) and use (2.1) and (3.3), then

$$\begin{aligned} h(x \to y) &\supseteq h(x \to (x \to y)) \cap h(x \to x) \\ &= h(x \to (x \to y)) \cap h(1) \\ &= h(x \to (x \to y)) \end{aligned}$$

for all $x, y \in L$. Therefore h is a hesitant fuzzy G-filter of L. □

Theorem 3.18. *Let h be a hesitant fuzzy filter of L. Then h is a hesitant fuzzy G-filter of L if and only if the following condition holds:*

$$(\forall x \in L)\, (h(x \to (x \odot x)) = h(1))\,. \tag{3.17}$$

Proof. Assume that h satisfies the condition (3.17) and let $x, y \in L$. Since

$$x \to (x \to y) = (x \odot x) \to y \leq (x \to (x \odot x)) \to (x \to y)$$

by (2.4) and (2.6), it follows from (3.1) that

$$h(x \to (x \to y)) \subseteq h((x \to (x \odot x)) \to (x \to y)).$$

Hence, we have

$$\begin{aligned} h(x \to y) &\supseteq h((x \to (x \odot x)) \to (x \to y)) \cap h(x \to (x \odot x)) \\ &\supseteq h(x \to (x \to y)) \cap h(x \to (x \odot x)) \\ &= h(x \to (x \to y)) \cap h(1) \\ &= h(x \to (x \to y)) \end{aligned}$$

by using (3.4), (3.17) and (3.3). Hence h is a hesitant fuzzy G-filter of L. □

Theorem 3.19. (Extension property) *Let h and g be hesitant fuzzy filters of L such that $h \subseteq g$, i.e., $h(x) \subseteq g(x)$ for all $x \in L$ and $h(1) = g(1)$. If h is a hesitant fuzzy G-filter of L, then so is g.*

Proof. Assume that h is a hesitant fuzzy G-filter of L. Using (2.4) and (2.1), we have

$$x \to (x \to ((x \to (x \to y)) \to y)) = (x \to (x \to y)) \to (x \to (x \to y)) = 1$$

for all $x, y \in L$. Thus

$$\begin{aligned} g(x \to ((x \to (x \to y)) \to y)) &\supseteq h(x \to ((x \to (x \to y)) \to y)) \\ &= h(x \to (x \to ((x \to (x \to y)) \to y))) \\ &= h(1) = g(1) \end{aligned}$$

by hypotheses and (3.16), and so

$$g(x \to ((x \to (x \to y)) \to y)) = g(1)$$

for all $x, y \in L$ by (3.3). Since g is a hesitant fuzzy filter of L, it follows from (3.4), (2.4) and (3.3) that

$$\begin{aligned} g(x \to y) &\supseteq g(x \to (x \to y)) \cap g((x \to (x \to y)) \to (x \to y)) \\ &= g(x \to (x \to y)) \cap g(x \to ((x \to (x \to y)) \to y)) \\ &= g(x \to (x \to y)) \cap g(1) \\ &= g(x \to (x \to y)) \end{aligned}$$

for all $x, y \in L$. Therefore g is a hesitant fuzzy G-filter of L. □

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