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# HIGHER-ORDER DEGENERATE BERNOULLI POLYNOMIALS

DAE SAN KIM AND TAEKYUN KIM

ABSTRACT. Carlitz introduced the degenerate Bernoulli polynomials and derived, among other things, the so-called degenerate Staudt-Clausen theorem for the degenerate Bernoulli numbers as an analogue of the classical Staudt-Clausen theorem. In this paper, we consider the higher-order Carlitz's degenerate Bernoulli polynomials with umbral calculus viewpoint and derive new identities and properties of those polynomials associated with special polynomials which are derived from umbral calculus.

## 1. INTRODUCTION

The degenerate Bernoulli polynomials $\beta_n(\lambda, x)$ $(\lambda \neq 0)$ are defined by Carlitz to be

$$(1.1) \qquad \frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(\lambda, x) \frac{t^n}{n!}, \quad (\lambda \neq 0), \quad (\text{see } [3, 4]).$$

Ustinov rediscovered these polynomials in [18], which are called Korobov polynomials of the second kind and denoted by $k_n^{(\lambda)}(x)$.

When $x = 0$, $\beta_n(\lambda) = \beta_n(\lambda, 0)$ are called the degenerate Bernoulli numbers. Now, we observe that

$$(1.2) \qquad \lim_{\lambda \to 0} \beta_n(\lambda, x) = \beta_n(0, x) = B_n(x), \quad \lim_{\lambda \to \infty} \lambda^{-n} \beta_n(\lambda, \lambda x) = b_n(x),$$

where $B_n(x)$ and $b_n(x)$ are the Bernoulli polynomials of the first kind and of the second kind.

The first few degenerate Bernoulli polynomials are given by $\beta_0(\lambda, x) = 1$, $\beta_1(\lambda, x) = x - \frac{1}{2} + \frac{1}{2}\lambda$, $\beta_2(\lambda, x) = x^2 - x + \frac{1}{6} - \frac{1}{6}\lambda^2$, $\beta_3(\lambda, x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x - \frac{3}{2}\lambda x^2 + \frac{3}{2}\lambda x + \frac{1}{4}\lambda^3 - \frac{1}{4}\lambda, \ldots$. As an analogue of the classical Staudt-Clausen theorem for Bernoulli numbers, Carlitz proved the so called degenerate Staudt-Clausen theorem for $\beta_n(\lambda)$, $(\lambda$ a rational number) (see [3, 19, 20]). The generalized falling factorials $(x|\lambda)_n$ for any $\lambda \in \mathbb{C}$ are defined as

$$(x|\lambda)_0 = 1, \quad (x|\lambda)_n = x(x - \lambda) \cdots (x - \lambda(n-1)), \quad (\text{for } n > 0).$$

Carlitz also found in [4] the following relation expressing sums of generalized falling factorals in terms of degenerate Bernoulli polynomials: for integers $l, m$ with $l \geq 1, m \geq 0$,

$$(1.3) \qquad \sum_{i=0}^{l-1} (i|\lambda)_m = \frac{1}{m+1} (\beta_{m+1}(\lambda, l) - \beta_{m+1}(\lambda)),$$

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which, by letting $\lambda \to 0$, becomes the familiar relation

$$(1.4) \qquad \sum_{i=0}^{l-1} i^m = \frac{1}{m+1} \left( B_{m+1}(l) - B_{m+1} \right).$$

For $r \in \mathbb{N}$, the Bernoulli polynomials of the second kind of order $r$ are defined by the generating function to be

$$(1.5) \qquad \left( \frac{t}{\log(1+t)} \right)^r (1+t)^x = \sum_{n=0}^{\infty} b_n^{(r)}(x) \frac{t^n}{n!}, \qquad (\text{see } [16]),$$

and the Bernoulli polynomials of order $r$ are given by

$$(1.6) \qquad \left( \frac{t}{e^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \qquad (\text{see } [2, 5\text{--}7, 9]).$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$, $b_n^{(r)} = b_n^{(r)}(0)$ are called the Bernouli numbers of the first kind of order $r$ and of the second kind of order $r$. For $\mu \in \mathbb{C}$ with $\mu \neq 1$, the Frobenius-Euler polynomials with order $s \in \mathbb{N}$ are defined by the generating function to be

$$(1.7) \qquad \left( \frac{1-\mu}{e^t - \mu} \right)^s e^{xt} = \sum_{n=0}^{\infty} H_n^{(s)}(x|\mu) \frac{t^n}{n!}, \qquad (\text{see } [1, 10\text{--}12]).$$

When $x = 0$, $H_n^{(s)}(\mu) = H_n^{(s)}(0|\mu)$ are called the Frobenius-Euler numbers of order $s$. As is well known, the Stirling number of the second kind is defined by the generating function to be

$$(1.8) \qquad \left( e^t - 1 \right)^n = n! \sum_{l=n}^{\infty} S_2(l, m) \frac{t^l}{l!}, \qquad (n \in \mathbb{Z}_{\geq 0}), \qquad (\text{see } [16, 17]).$$

For $n \geq 0$, the Stirling number of the first kind is given by

$$(x)_n = x(x-1)\cdots(x-(n-1)) = \sum_{l=0}^{n} S_1(n, l) x^l, \qquad (\text{see } [13, 15, 16, 21]).$$

Let $\mathcal{F}$ be the set of all formal power series in the variable $t$:

$$(1.9) \qquad \mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \,\middle|\, a_k \in \mathbb{C} \right\}.$$

Let $\mathbb{P} = \mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L | p(x) \rangle$ denotes the action of the linear functional $L$ on $p(x)$ which satisfies $\langle L + M | p(x) \rangle = \langle L | p(x) \rangle + \langle M | p(x) \rangle$, and $\langle cL | p(x) \rangle = c \langle L | p(x) \rangle$, where $c$ is a complex constant. The linear functional $\langle f(t) | \cdot \rangle$ on $\mathbb{P}$ is defined as

$$(1.10) \qquad \langle f(t) | x^n \rangle = a_n, \qquad (n \geq 0), \qquad \text{where } f(t) \in \mathcal{F}.$$

Thus, by (1.9) and (1.10), we get

$$(1.11) \qquad \langle t^k | x^n \rangle = n! \delta_{n,k}, \qquad (n, k \geq 0), \qquad (\text{see } [14, 16]),$$

where $\delta_{n,k}$ is the Kronecker symbol.

Let $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L | x^k \rangle}{k!} t^k$. Then, by (1.11), we get $\langle f_L(t) | x^n \rangle = \langle L | x^n \rangle$. So, the map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in $t$ and the vector space of all

linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the umbral algebra and the umbral calculus is the study of umbral algebra. The order $o(f(t))$ of a power series $f(t) \neq 0$ is the smallest integer $k$ for which the coefficient of $t^k$ does not vanish (see [14, 16]). If $o(f(t)) = 0$, then $f(t)$ is called an invertible series; if $o(f(t)) = 1$, then $f(t)$ is called a delta series. Let $f(t), g(t)$ be a delta series and an invertible series, respectively. Then there exists a unique sequence $s_n(x)(\deg s_n(x) = n)$ such that $\left\langle g(t)f(t)^k \mid s_n(x) \right\rangle = n!\delta_{n,k}$, for $k \geq 0$. Such a sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$which is denoted by $s_n(x) \sim (g(t), f(t))$ (see [14, 16]). The sequence $s_n(x)$ is Sheffer for $(g(t), f(t))$ if and only if

$$(1.12) \qquad \frac{1}{g(\overline{f}(t))}e^{y\overline{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!}t^k, \quad (y \in \mathbb{C}), \quad (\text{see } [11, 17]),$$

where $\overline{f}(t)$ is the compositional inverse of $f(t)$ with $\overline{f}(f(t)) = f(\overline{f}(t)) = t$.

Let $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. Then we see that

$$(1.13) \qquad f(t) = \sum_{k=0}^{\infty} \left\langle f(t) \mid x^k \right\rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \left\langle t^k \mid p(x) \right\rangle \frac{x^k}{k!}.$$

From (1.13), we have

$$(1.14) \qquad t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \quad e^{yt}p(x) = p(x+y).$$

By (1.14), we get $\left\langle e^{yt} \mid p(x) \right\rangle = p(y)$.

For $s_n(x) \sim (g(t), f(t))$, we have the following equations ([16]):

$$(1.15) \quad f(t)s_n(x) = ns_{n-1}(x), \ (n \geq 1), \quad s_n(x+y) = \sum_{j=0}^{n} \binom{n}{j}s_j(x)p_{n-j}(y),$$

where $p_n(x) = g(t)s_n(x)$,

$$(1.16)$$
$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right)\frac{1}{f'(t)}s_n(x), \quad s_n(x) = \sum_{j=0}^{n}\frac{1}{j!}\left\langle g(\overline{f}(t))^{-1}\overline{f}(t)^j \mid x^n \right\rangle x^j,$$

and

$$\left\langle f(t) \mid xp(x) \right\rangle = \left\langle \partial_t f(t) \mid p(x) \right\rangle,$$
$$(1.17)$$
$$\frac{d}{dx}s_n(x) = \sum_{l=0}^{n-1}\binom{n}{l}\left\langle \overline{f}(t) \mid x^{n-l} \right\rangle s_l(x), \quad (n \geq 1).$$

In particular, for $p_n(x) \sim (1, f(t))$, $q_n(x) \sim (1, g(t))$, we note that

$$(1.18) \qquad q_n(x) = x\left(\frac{f(t)}{g(t)}\right)^n x^{-1}p_n(x), \quad (n \geq 1).$$

Let us assume that $s_n(x) \sim (g(t), f(t))$, $r_n(x) \sim (h(t), l(t))$. Then we have

$$(1.19) \qquad s_n(x) = \sum_{m=0}^{n} C_{n,m}r_m(x), \quad (n \geq 0),$$

where

$$(1.20) \qquad C_{n,m} = \frac{1}{m!} \left\langle \frac{h\left(\overline{f}\left(t\right)\right)}{g\left(\overline{f}\left(t\right)\right)} \left(l\left(\overline{f}\left(t\right)\right)\right)^m \Big| x^n \right\rangle, \quad (\text{see } [16]).$$

In this paper, we consider, for any positive integer $r$, the degenerate Bernoulli polynomials $\beta_n^{(r)}(\lambda, x)$ of order $r$ which are defined by the generating function to be

$$(1.21) \qquad \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}\right)^r (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n^{(r)}(\lambda, x) \frac{t^n}{n!}, \quad (r \in \mathbb{Z}_{\geq 0}).$$

From (1.20) and (1.21), we note that

$$(1.22) \qquad \beta_n^{(r)}(\lambda, x) \sim \left(\left(\frac{\lambda(e^t - 1)}{e^{\lambda t} - 1}\right)^r, \frac{1}{\lambda}\left(e^{\lambda t} - 1\right)\right).$$

That is, $\beta_n^{(r)}(\lambda, x)$ is the Sheffer polynomial for the pair

$$\left(g(t) = \left(\frac{\lambda(e^t - 1)}{e^{\lambda t} - 1}\right)^r, f(t) = \frac{1}{\lambda}\left(e^{\lambda t} - 1\right)\right).$$

The purpose of this paper is to give new identities and properties of the higher-order degenerate Bernoulli polynomials associated with special polynomials which are derived from umbral calculus.

## 2. Higher-order degenerate Bernoulli polynomials

For $n \geq 0$, we note that

$$x^n \sim (1, t), \quad \left(\frac{\lambda(e^t - 1)}{e^{\lambda t} - 1}\right)^r \beta_n^{(r)}(\lambda, x) \sim \left(1, \frac{1}{\lambda}\left(e^{\lambda t} - 1\right)\right),$$

From (1.18), we can derive the following equation:

$$(2.1) \qquad \left(\frac{\lambda(e^t - 1)}{e^{\lambda t} - 1}\right)^r \beta_n^{(r)}(\lambda, x) = x\left(\frac{\lambda t}{e^{\lambda t} - 1}\right)^n x^{-1} x^n = x\left(\frac{\lambda t}{e^{\lambda t} - 1}\right)^n x^{n-1}$$

$$= \sum_{l=0}^{n-1} \binom{n-1}{l} \lambda^l B_l^{(n)} x^{n-l}, \quad (n \geq 1).$$

Thus, by (2.1), we get

$$(2.2) \qquad \beta_n^{(r)}(\lambda, x)$$

$$= \sum_{l=0}^{n-1} \binom{n-1}{l} \lambda^l B_l^{(n)} \left(\frac{e^{\lambda t} - 1}{\lambda(e^t - 1)}\right)^r x^{n-l}$$

$$= \sum_{l=0}^{n-1} \binom{n-1}{l} \lambda^l B_l^{(n)} \left(\frac{t}{e^t - 1}\right)^r \left(\frac{e^{\lambda t} - 1}{\lambda t}\right)^r x^{n-l}$$

$$= \sum_{l=0}^{n-1} \binom{n-1}{l} \lambda^l B_l^{(n)} \left(\frac{t}{e^t - 1}\right)^r \left(r! \sum_{k=0}^{\infty} S_2(k+r, r) \frac{\lambda^k}{(k+r)!} t^k\right) x^{n-l}$$

$$= \sum_{l=0}^{n-1} \binom{n-1}{l} \lambda^l B_l^{(n)} \left(\frac{t}{e^t - 1}\right)^r \sum_{k=0}^{n-l} \frac{\binom{n-l}{k}}{\binom{k+r}{r}} S_2(k+r, r) \lambda^k x^{n-l-k}$$

$$= \sum_{l=0}^{n-1} \sum_{k=0}^{n-l} \frac{\binom{n-1}{l}\binom{n-l}{k}}{\binom{k+r}{r}} S_2(k+r,r) \lambda^{k+l} B_l^{(n)} \left(\frac{t}{e^t-1}\right)^r x^{n-l-k}$$

$$= \sum_{l=0}^{n-1} \sum_{k=0}^{n-l} \frac{\binom{n-1}{l}\binom{n-l}{k}}{\binom{k+r}{r}} S_2(k+r,r) \lambda^{k+l} B_l^{(n)} B_{n-l-k}^{(r)}(x).$$

Therefore, by (2.2), we obtain the following theorem.

**Theorem 2.1.** *For $n \geq 1$, we have*

$$\beta_n^{(r)}(\lambda, x) = \sum_{l=0}^{n-1} \sum_{k=0}^{n-l} \frac{\binom{n-1}{l}\binom{n-l}{k}}{\binom{k+r}{r}} S_2(k+r,r) \lambda^{k+l} B_l^{(n)} B_{n-k-l}^{(r)}(x).$$

*Remark.* When $x = 0$ and $r = 1$, we get

$$(2.3) \qquad \beta_n(\lambda) = \sum_{l=0}^{n-1} \sum_{k=0}^{n-l} \frac{1}{k+1} \binom{n-1}{l}\binom{n-l}{k} \lambda^{k+l} B_l^{(n)} B_{n-k-l}.$$

From (1.18), (1.22) and

$$(x|\lambda)_n = \lambda^n \left(\frac{x}{\lambda}\right)_n = \sum_{m=0}^{n} S_1(n,m) \lambda^{n-m} x^m \sim \left(1, \frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right).$$

We note that

$$(2.4) \quad \beta_n^{(r)}(\lambda, x) = \sum_{m=0}^{n} S_1(n,m) \lambda^{n-m} \left(\frac{e^{\lambda t}-1}{\lambda(e^t-1)}\right)^r x^m$$

$$= \sum_{m=0}^{n} S_1(n,m) \lambda^{n-m} \left(\frac{t}{e^t-1}\right)^r \left(\frac{e^{\lambda t}-1}{\lambda t}\right)^r x^m$$

$$= \sum_{m=0}^{n} S_1(n,m) \lambda^{n-m} \left(\frac{t}{e^t-1}\right)^r \sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{k+r}{r}} S_2(k+r,r) \lambda^k x^{m-k}$$

$$= \lambda^n \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{k+r}{r}} S_1(n,m) S_2(k+r,r) \lambda^{k-m} B_{m-k}^{(r)}(x).$$

Therefore, by (2.4), we obtain the following theorem.

**Theorem 2.2.** *For $n \geq 0$, we have*

$$\beta_n^{(r)}(\lambda, x) = \lambda^n \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{\binom{m}{k}}{\binom{k+r}{r}} S_1(n,m) S_2(k+r,r) \lambda^{k-m} B_{m-k}^{(r)}(x).$$

*Remark.* For $r = 1$ and $x = 0$, we get an expression for the degenerate Bernoulli numbers:

$$(2.5) \qquad \beta_n(\lambda) = \lambda^n \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{1}{k+1} \binom{m}{k} S_1(n,m) \lambda^{k-m} B_{m-k}.$$

Here we use the conjugation representation.

For $\beta_n^{(r)}(\lambda, x) \sim \left( g(t) = \left( \frac{\lambda(e^t-1)}{e^{\lambda t}-1} \right)^r, f(t) = \frac{1}{\lambda}\left( e^{\lambda t}-1 \right) \right)$, we observe that

(2.6)
$$\left\langle g\left( \overline{f}(t) \right)^{-1} \overline{f}(t)^j \,\middle|\, x^n \right\rangle$$

$$= \left\langle \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1} \right)^r \left( \frac{1}{\lambda}\log(1+\lambda t) \right)^j \,\middle|\, x^n \right\rangle$$

$$= \lambda^{-j} \left\langle \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1} \right)^r \,\middle|\, j! \sum_{l=j}^{\infty} S_1(l,j) \frac{\lambda^l}{l!} t^l x^n \right\rangle$$

$$= j!\lambda^{-j} \sum_{l=j}^{n} \binom{n}{l} S_1(l,j) \lambda^l \left\langle \sum_{m=0}^{\infty} \beta_m^{(r)}(\lambda) \frac{t^m}{m!} \,\middle|\, x^{n-l} \right\rangle$$

$$= j!\lambda^{-j} \sum_{l=j}^{n} \binom{n}{l} S_1(l,j) \lambda^l \beta_{n-l}^{(r)}(\lambda).$$

Therefore, by (1.16) and (2.6), we obtain the following theorem.

**Theorem 2.3.** *For $n \geq 0$, $r \geq 1$, we have*

$$\beta_n^{(r)}(\lambda, x) = \sum_{j=0}^{n} \lambda^{-j} \left( \sum_{l=j}^{n} \binom{n}{l} S_1(l,j) \lambda^l \beta_{n-l}^{(r)}(\lambda) \right) x^j.$$

*Remark.* Recall that

(2.7)  $\left( \frac{\lambda(e^t-1)}{e^{\lambda t}-1} \right)^r \beta_n^{(r)}(\lambda, x) \sim \left( 1, \frac{1}{\lambda}\left( e^{\lambda t}-1 \right) \right)$,   $(x|\lambda)_n \sim \left( 1, \frac{1}{\lambda}\left( e^{\lambda t}-1 \right) \right)$.

Thus, by (2.7), we get

(2.8)   $\left( \frac{\lambda(e^t-1)}{e^{\lambda t}-1} \right)^r \beta_n^{(r)}(\lambda, x) = (x|\lambda)_n$,   and   $\frac{e^{\lambda t}-1}{\lambda}(x|\lambda)_n = n(x|\lambda)_{n-1}$.

From (2.8), we have

(2.9)            $\left( e^t-1 \right)^r \beta_n^{(r)}(\lambda, x) = \left( \frac{e^{\lambda t}-1}{\lambda} \right)^r (x|\lambda)_n$

$$= \begin{cases} (n)_r\,(x|\lambda)_{n-r} & , \text{ if } r \leq n \\ 0 & , \text{ if } r > n. \end{cases}$$

By (2.9), we get

(2.10)   $t^r \beta_n^{(r)}(\lambda, x) = \begin{cases} (n)_r\,\lambda^{n-r} \left( \frac{t}{e^t-1} \right)^r \left( \frac{x}{\lambda} \right)_{n-r} & , \text{ if } r \leq n \\ 0 & , \text{ if } r > n \end{cases}$

$$= \begin{cases} (n)_r\,\lambda^{n-r} \sum_{m=0}^{n-r} S_1(n-r,m)\lambda^{-m} B_m^{(r)}(x) & , \text{ if } r \leq n \\ 0 & , \text{ if } r > n. \end{cases}$$

Therefore, from (1.14) and (2.10), we have

(2.11)
$$\left( \frac{d}{dx} \right)^r \beta_n^{(r)}(\lambda, x) = \begin{cases} (n)_r\,\lambda^{n-r} \sum_{m=0}^{n-r} S_1(n-r,m)\lambda^{-m} B_m^{(r)}(x) & , \text{ if } r \leq n \\ 0 & , \text{ if } r > n. \end{cases}$$

In particular,

$$(2.12) \qquad \frac{d}{dx}\beta_n(\lambda, x) = \begin{cases} n\lambda^{n-1}\sum_{m=0}^{n-1} S_1(n-1,m)\lambda^{-m}B_m(x) & , \text{ if } r \le n \\ 0 & , \text{ if } r > n. \end{cases}$$

To proceed further, we recall that the $\lambda$-Daehee polynomials $D_{n,\lambda}^{(r)}(x)$ of order $r$ are given by

$$(2.13) \qquad \left(\frac{\lambda \log(1+t)}{(1+t)^\lambda - 1}\right)^r (1+t)^x = \sum_{n=0}^\infty D_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}, \quad (\text{see } [12, 16]).$$

From (1.5), (1.11) and (2.13), we have

$$(2.14) \qquad \beta_n^{(r)}(\lambda, y)$$

$$= \left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}\right)^r (1+\lambda t)^{\frac{y}{\lambda}} \,\middle|\, x^n \right\rangle$$

$$= \sum_{l=0}^n \binom{n}{l}\lambda^l b_l^{(r)}\left(\frac{y}{\lambda}\right) \left\langle \left(\frac{\log(1+\lambda t)}{\lambda\left((1+\lambda t)^{\frac{1}{\lambda}} - 1\right)}\right)^r \,\middle|\, x^{n-l} \right\rangle$$

$$= \sum_{l=0}^n \binom{n}{l}\lambda^l b_l^{(r)}\left(\frac{y}{\lambda}\right) \left\langle \sum_{m=0}^\infty D_{m,\frac{1}{\lambda}}^{(r)}\lambda^m\frac{t^m}{m!} \,\middle|\, x^{n-l} \right\rangle$$

$$= \sum_{l=0}^n \binom{n}{l}\lambda^l b_l^{(r)}\left(\frac{y}{\lambda}\right) D_{n-l,\frac{1}{\lambda}}^{(r)}\lambda^{n-l}$$

$$= \lambda^n \sum_{l=0}^n \binom{n}{l} D_{n-l,\frac{1}{\lambda}}^{(r)} b_l^{(r)}\left(\frac{y}{\lambda}\right),$$

and

$$(2.15) \qquad \beta_n^{(r)}(\lambda, y)$$

$$= \left\langle \sum_{l=0}^\infty \beta_l^{(r)}(\lambda, y)\frac{t^l}{l!} \,\middle|\, x^n \right\rangle$$

$$= \left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}\right)^r (1+\lambda t)^{\frac{y}{\lambda}} \,\middle|\, x^n \right\rangle$$

$$= \left\langle \left(\frac{\lambda t}{\log(1+\lambda t)}\right)^r \,\middle|\, \left(\frac{\log(1+\lambda t)}{\lambda\left((1+\lambda t)^{\frac{1}{\lambda}} - 1\right)}\right)^r (1+\lambda t)^{\frac{y}{\lambda}} x^n \right\rangle$$

$$= \left\langle \left(\frac{\lambda t}{\log(1+\lambda t)}\right)^r \,\middle|\, \sum_{l=0}^\infty D_{l,\lambda^{-1}}^{(r)}\left(\frac{y}{\lambda}\right)\lambda^l\frac{t^l}{l!} x^n \right\rangle$$

$$= \sum_{l=0}^n \binom{n}{l}\lambda^l D_{l,\lambda^{-1}}^{(r)}\left(\frac{y}{\lambda}\right) \left\langle \sum_{m=0}^\infty b_m^{(r)}\lambda^m\frac{t^m}{m!} \,\middle|\, x^{n-l} \right\rangle$$

$$= \sum_{l=0}^n \binom{n}{l}\lambda^l D_{l,\lambda^{-1}}^{(r)}\left(\frac{y}{\lambda}\right) b_{n-l}^{(r)}\lambda^{n-l}$$

$$= \lambda^n \sum_{l=0}^{n} \binom{n}{l} b_{n-l}^{(r)} D_{l,\lambda^{-1}}^{;(r)} \left( \frac{y}{\lambda} \right).$$

Therefore, by (2.14) and (2.15), we obtain the following theorem.

**Theorem 2.4.** *For $n \geq 0$, we have*

$$\sum_{l=0}^{n} \binom{n}{l} D_{n-l,\frac{1}{\lambda}}^{(r)} b_l^{(r)} \left( \frac{x}{\lambda} \right) = \sum_{l=0}^{n} \binom{n}{l} b_{n-l}^{(r)} D_{l,\lambda^{-1}}^{(r)} \left( \frac{x}{\lambda} \right) = \lambda^{-n} \beta_n^{(r)} (\lambda, x).$$

Recalling that

$$\beta_n^{(r)} (\lambda, x) \sim \left( g(t) = \left( \frac{\lambda (e^t - 1)}{e^{\lambda t} - 1} \right)^r , f(t) = \frac{1}{\lambda} \left( e^{\lambda t} - 1 \right) \right),$$

we observe that
(2.16)

$$\left\langle g \left( \overline{f}(t) \right)^{-1} \overline{f}(t)^j \middle| x^n \right\rangle$$

$$= j! \lambda^{-j} \sum_{l=j}^{n} S_1(l,j) \binom{n}{l} \lambda^l \left\langle \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r \middle| x^{n-l} \right\rangle$$

$$= j! \lambda^{-j} \sum_{l=j}^{n} S_1(l,j) \binom{n}{l} \lambda^l \left\langle \left( \frac{\log(1+\lambda t)}{\lambda \left( (1+\lambda t)^{\frac{1}{\lambda}} - 1 \right)} \right)^r \middle| \left( \frac{\lambda t}{\log(1+\lambda t)} \right)^r x^{n-l} \right\rangle$$

$$= j! \lambda^{-j} \sum_{l=j}^{n} S_1(l,j) \binom{n}{l} \lambda^l \left\langle \left( \frac{\log(1+\lambda t)}{\lambda \left( (1+\lambda t)^{\frac{1}{\lambda}} - 1 \right)} \right)^r \middle| \sum_{m=0}^{\infty} b_m^{(r)} \frac{\lambda^m}{m!} t^m x^{n-l} \right\rangle$$

$$= j! \lambda^{-j} \sum_{l=j}^{n} S_1(l,j) \binom{n}{l} \lambda^l \sum_{m=0}^{n-l} \binom{n-l}{m} \lambda^m b_m^{(r)} \left\langle \left( \frac{\log(1+\lambda t)}{\lambda \left( (1+\lambda t)^{\frac{1}{\lambda}} - 1 \right)} \right)^r \middle| x^{n-l-m} \right\rangle$$

$$= j! \lambda^{-j} \sum_{l=j}^{n} S_1(l,j) \binom{n}{l} \lambda^l \sum_{m=0}^{n-l} \binom{n-l}{m} \lambda^m b_m^{(r)} D_{n-l-m,\lambda^{-1}}^{(r)} \lambda^{n-l-m}$$

$$= j! \lambda^n \sum_{l=j}^{n} \sum_{m=0}^{n-l} \binom{n}{l} \binom{n-l}{m} S_1(l,j) \lambda^{-j} b_m^{(r)} D_{n-l-m,\lambda^{-1}}^{(r)}.$$

From (1.16) and (2.16), we have
(2.17)

$$\beta_n^{(r)} (\lambda, x) = \lambda^n \sum_{j=0}^{n} \left\{ \sum_{l=j}^{n} \sum_{m=0}^{n-l} \binom{n}{l} \binom{n-l}{m} S_1(l,j) \lambda^{-j} b_m^{(r)} D_{n-l-m,\lambda^{-1}}^{(r)} \right\} x^j.$$

*Remark.* We have

$$\lim_{\lambda \to 0} \beta_n^{(r)} (\lambda, x) = \beta_n^{(r)} (0, x) = B_n^{(r)} (x),$$

$$\lim_{\lambda \to 0} D_{n,\lambda}^{(r)} (x) = (x)_n,$$

$$\lim_{\lambda \to \infty} \lambda^{-n} \beta_n^{(r)} (\lambda, \lambda x) = b_n^{(r)} (x),$$

$$\lim_{\lambda \to \infty} \lambda^{-n} D_{n,\lambda}^{(r)} (\lambda x) = B_n^{(r)} (x).$$

where $r > 0$.

From (1.22), we note that

$$(2.18) \qquad p_n (x) = \left( \frac{\lambda (e^t - 1)}{e^{\lambda t} - 1} \right)^r \beta_n^{(r)} (\lambda, x) = (x|\lambda)_n \sim \left( 1, \frac{1}{\lambda} \left( e^{\lambda t} - 1 \right) \right).$$

By (2.18) and (1.15), we get

$$(2.19) \qquad \beta_n^{(r)} (\lambda, x + y) = \sum_{j=0}^{n} \binom{n}{j} \beta_j^{(r)} (\lambda, x) (y|\lambda)_{n-j},$$

and, by (1.14) and (1.15), we get

$$(2.20) \qquad \frac{1}{\lambda} \left( e^{\lambda t} - 1 \right) \beta_n^{(r)} (\lambda, x) = n \beta_{n-1}^{(r)} (\lambda, x).$$

From (2.20), we have

$$(2.21) \qquad \beta_n^{(r)} (\lambda, x + \lambda) - \beta_n^{(r)} (\lambda, x) = n\lambda \beta_{n-1}^{(r)} (\lambda, x).$$

Therefore, by (2.17), (2.19) and (2.20), we obtain the following theorem.

**Theorem 2.5.** *For $n \geq 0$, we have*

$$\beta_n^{(r)} (\lambda, x) = \lambda^n \sum_{j=0}^{n} \left\{ \sum_{l=j}^{n} \sum_{m=0}^{n-l} \binom{n}{l} \binom{n-l}{m} S_1 (l, j) \lambda^{-j} b_m^{(r)} D_{n-l-m, \lambda^{-1}}^{(r)} \right\} x^j,$$

*and*

$$\beta_n^{(r)} (\lambda, x + \lambda) = \sum_{j=0}^{n} \binom{n}{j} \beta_j^{(r)} (\lambda, x) (\lambda|\lambda)_{n-j}$$

$$= n\lambda \beta_{n-1}^{(r)} (\lambda, x) + \beta_n^{(r)} (\lambda, x).$$

For $\beta_n^{(r)} (\lambda, x) \sim \left( g(t) = \left( \frac{\lambda (e^t - 1)}{e^{\lambda t} - 1} \right)^r, f(t) = \frac{1}{\lambda} \left( e^{\lambda t} - 1 \right) \right)$, we note that

$$(2.22) \qquad \frac{g'(t)}{g(t)}$$

$$= (\log g(t))'$$

$$= r \left( \log \lambda + \log \left( e^t - 1 \right) - \log \left( e^{\lambda t} - 1 \right) \right)'$$

$$= \frac{r}{t} \left( \sum_{l=0}^{\infty} B_l (1) \frac{t^l}{l!} - \sum_{l=0}^{\infty} B_l (1) \frac{\lambda^l t^l}{l!} \right)$$

$$= \frac{r}{t} \sum_{l=1}^{\infty} B_l (1) \left( 1 - \lambda^l \right) \frac{t^l}{l!}$$

$$= r \sum_{l=0}^{\infty} B_{l+1} (1) \left( 1 - \lambda^{l+1} \right) \frac{t^l}{(l+1)!}.$$

By (2.22), we get

$$(2.23) \qquad \frac{g'(t)}{g(t)} \beta_n^{(r)} (\lambda, x)$$

$$=r\sum_{l=0}^{\infty} B_{l+1}\left(1\right)\left(1-\lambda^{l+1}\right)\frac{t^l}{(l+1)!}\lambda^n$$

$$\times \sum_{m=0}^{n}\sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{k+r}{r}}S_1\left(n,m\right)S_2\left(k+r,r\right)\lambda^{k-m}B^{(r)}_{m-k}\left(x\right)$$

$$=\lambda^n r\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{l=0}^{m-k} S_1\left(n,m\right)S_2\left(k+r,r\right)\lambda^{k-m}$$

$$\times \sum_{l=0}^{m-k} B_{l+1}\left(1\right)\left(1-\lambda^{l+1}\right)\frac{1}{(l+1)!}\left(m-k\right)_l B^{(r)}_{m-k-l}\left(x\right)$$

$$=\lambda^n r\sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{l=0}^{m-k}\frac{1}{m-k-l+1}\frac{\binom{m}{k}\binom{m-k}{l}}{\binom{k+r}{r}}\left(\lambda^{k-m}-\lambda^{1-l}\right)$$

$$\times S_1\left(n,m\right)S_2\left(k+r,r\right)B_{m-k-l+1}\left(1\right)B^{(r)}_{l}\left(x\right)$$

$$=r\sum_{l=0}^{n}\sum_{m=l}^{n}\sum_{k=l}^{m}\frac{1}{k-l+1}\frac{\binom{m}{k}\binom{k}{l}}{\binom{m-k+r}{r}}\left(\lambda^{n-k}-\lambda^{n-l+1}\right)S_1\left(n,m\right)$$

$$\times S_2\left(m-k+r,r\right)B_{k-l+1}\left(1\right)B^{(r)}_{l}\left(x\right).$$

From (1.16) and (2.23), we have

$$(2.24)\qquad \beta^{(r)}_{n+1}\left(\lambda,x\right)$$

$$=x\beta^{(r)}_{n}\left(\lambda,x-\lambda\right)-e^{-\lambda t}\frac{g'\left(t\right)}{g\left(t\right)}\beta^{(r)}_{n}\left(\lambda,x\right)$$

$$=x\beta^{(r)}_{n}\left(\lambda,x-\lambda\right)-r\sum_{l=0}^{n}\left(\sum_{m=l}^{n}\sum_{k=l}^{m}\frac{1}{k-l+1}\frac{\binom{m}{k}\binom{k}{l}}{\binom{m-k+r}{r}}\left(\lambda^{n-k}-\lambda^{n-l+1}\right)\right.$$

$$\times S_1\left(n,m\right)S_2\left(m-k+r,r\right)B_{k-l+1}\left(1\right)\bigg)B^{(r)}_{l}\left(x-\lambda\right).$$

Therefore, by (2.24), we obtain the following theorem.

**Theorem 2.6.** *For $n\geq 0$, we have*

$$\beta^{(r)}_{n+1}\left(\lambda,x\right)$$

$$=x\beta^{(r)}_{n}\left(\lambda,x-\lambda\right)-r\sum_{l=0}^{n}\left(\sum_{m=l}^{n}\sum_{k=l}^{m}\frac{1}{k-l+1}\frac{\binom{m}{k}\binom{k}{l}}{\binom{m-k+r}{r}}\left(\lambda^{n-k}-\lambda^{n-l+1}\right)\right.$$

$$\times S_1\left(n,m\right)S_2\left(m-k+r,r\right)B_{k-l+1}\left(1\right)\bigg)B^{(r)}_{l}\left(x-\lambda\right).$$

By $\beta^{(r)}_{n}\left(\lambda,x\right)\sim\left(g\left(t\right)=\left(\frac{\lambda\left(e^t-1\right)}{e^{\lambda t}-1}\right)^r,f\left(t\right)=\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right)$, we get

$$(2.25)\qquad\qquad \left\langle \overline{f}\left(t\right)\big|\, x^{n-l}\right\rangle$$

$$=\left\langle \frac{1}{\lambda}\log\left(1+\lambda t\right)\bigg|\, x^{n-l}\right\rangle$$

$$=\lambda^{-1}\left\langle \sum_{m=1}^{\infty}\left(-1\right)^{m-1}\lambda^m\left(m-1\right)!\frac{t^m}{m!}\bigg|\, x^{n-l}\right\rangle$$

$$= \lambda^{-1} (-1)^{n-l-1} \lambda^{n-l} (n-l-1)!$$
$$= (-\lambda)^{n-l-1} (n-l-1)!.$$

From (1.17) and (2.25), we have

$$(2.26) \qquad \frac{d}{dx} \beta_n^{(r)} (\lambda, x) = n! \sum_{l=0}^{n-1} \frac{(-\lambda)^{n-l-1}}{l! (n-l)} \beta_l^{(r)} (\lambda, x).$$

Let $n \geq 1$. Then, by (1.11) and (1.17), we get

$$(2.27) \qquad \beta_n^{(r)} (\lambda, y)$$

$$= \left\langle \sum_{l=0}^{\infty} \beta_l^{(r)} (\lambda, y) \frac{t^l}{l!} \middle| x^n \right\rangle$$

$$= \left\langle \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r (1+\lambda t)^{\frac{y}{\lambda}} \middle| x^n \right\rangle$$

$$= \left\langle \partial_t \left( \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r (1+\lambda t)^{\frac{y}{\lambda}} \right) \middle| x^{n-1} \right\rangle$$

$$= \left\langle \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r \partial_t (1+\lambda t)^{\frac{y}{\lambda}} \middle| x^{n-1} \right\rangle$$

$$+ \left\langle \left( \partial_t \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r \right) (1+\lambda t)^{\frac{y}{\lambda}} \middle| x^{n-1} \right\rangle.$$

The first term of (2.27) is

$$(2.28) \qquad y \left\langle \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r (1+\lambda t)^{\frac{y-\lambda}{\lambda}} \middle| x^{n-1} \right\rangle = y \beta_{n-1}^{(r)} (\lambda, y - \lambda).$$

For the second term of (2.27), we observe that

$$(2.29) \qquad \partial_t \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r = r \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^{r-1} \partial_t \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right),$$

where

$$(2.30) \qquad \partial_t \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)$$

$$= \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1 - t (1+\lambda t)^{\frac{1}{\lambda} - 1}}{\left( (1+\lambda t)^{\frac{1}{\lambda}} - 1 \right)^2}$$

$$= \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1 - t \left\{ (1+\lambda t)^{\frac{1}{\lambda} - 1} - (1+\lambda t)^{-1} \right\} - t (1+\lambda t)^{-1}}{\left( (1+\lambda t)^{\frac{1}{\lambda}} - 1 \right)^2}$$

$$(2.31) \qquad = -\frac{1}{1+\lambda t} \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} + \frac{1}{t}$$

$$\times \left\{ \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1} - \frac{1}{(1+\lambda t)}\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^2 \right\}.$$

Thus, by $(2.29)$ and $(2.30)$, we get

$$(2.32) \qquad \partial_t \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r$$

$$= -\frac{r}{(1+\lambda t)}\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r$$

$$+ \frac{r}{t}\left\{\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r - \frac{1}{1+\lambda t}\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r+1}\right\}.$$

From $(2.32)$, we note that the second term of $(2.27)$ is

$$(2.33)$$

$$-r\left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r (1+\lambda t)^{\frac{y-\lambda}{\lambda}} \,\middle|\, x^{n-1}\right\rangle$$

$$+r\left\langle (1+\lambda t)^{\frac{y}{\lambda}} \,\middle|\, \frac{1}{t}\left\{\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r - \frac{1}{1+\lambda t}\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r+1}\right\} x^{n-1}\right\rangle$$

$$= -r\beta_{n-1}^{(r)}(\lambda, y-\lambda)$$

$$+ \frac{r}{n}\left\langle (1+\lambda t)^{\frac{y}{\lambda}} \,\middle|\, \left\{\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r - \frac{1}{1+\lambda t}\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r+1}\right\} x^{n}\right\rangle$$

$$= -r\beta_{n-1}^{(r)}(\lambda, y-\lambda)$$

$$+ \frac{r}{n}\left\langle \left(\frac{t}{(1+t)^{\frac{1}{\lambda}}-1}\right)^r (1+\lambda t)^{\frac{y}{\lambda}} \,\middle|\, x^{n}\right\rangle$$

$$- \frac{r}{n}\left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r+1} (1+\lambda t)^{\frac{y-\lambda}{\lambda}} \,\middle|\, x^{n}\right\rangle$$

$$= -r\beta_{n-1}^{(r)}(\lambda, y-\lambda) + \frac{r}{n}\beta_n^{(r)}(\lambda, y) - \frac{r}{n}\beta_n^{(r+1)}(\lambda, y-\lambda).$$

By $(2.27)$, $(2.28)$ and $(2.33)$, we get

$$(2.34) \qquad \left(1-\frac{r}{n}\right)\beta_n^{(r)}(\lambda, x) = (x-r)\beta_{n-1}^{(r)}(\lambda, x-\lambda) - \frac{r}{n}\beta_n^{(r+1)}(\lambda, x-\lambda).$$

Therefore, by $(2.34)$, we obtain the following theorem.

**Theorem 2.7.** *For $n \geq 1$, we have*

$$\beta_n^{(r+1)}(\lambda, x-\lambda) = \left(1-\frac{n}{r}\right)\beta_n^{(r)}(\lambda, x) + \left(\frac{n}{r}x - n\right)\beta_{n-1}^{(r)}(\lambda, x-\lambda).$$

Here we compute

$$(2.35) \qquad \left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r \left(\frac{1}{\lambda}\log(1+\lambda t)\right)^m \,\middle|\, x^n\right\rangle$$

in two different ways.

On one hand, it is equal to

$$(2.36) \qquad \lambda^{-m} \left\langle \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r (\log(1+\lambda t))^m \middle| x^n \right\rangle$$

$$= \lambda^{-m} \left\langle \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r \middle| m! \sum_{l=m}^{\infty} S_1(l,m) \frac{\lambda^l}{l!} t^l x^n \right\rangle$$

$$= m! \lambda^{-m} \sum_{l=m}^{n} \binom{n}{l} S_1(l,m) \lambda^l \left\langle \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r \middle| x^{n-l} \right\rangle$$

$$= m! \lambda^{-m} \sum_{l=m}^{n} \binom{n}{l} S_1(l,m) \lambda^l \left\langle \sum_{k=0}^{\infty} \beta_k^{(r)}(\lambda) \frac{t^k}{k!} \middle| x^{n-l} \right\rangle$$

$$= m! \lambda^{-m} \sum_{l=m}^{n} \binom{n}{l} S_1(l,m) \lambda^l \beta_{n-l}^{(r)}(\lambda).$$

On the other hand, it is equal to

$$(2.37) \qquad \left\langle \partial_t \left( \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r \left( \frac{1}{\lambda} \log(1+\lambda t) \right)^m \right) \middle| x^{n-1} \right\rangle$$

$$= \left\langle \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r \partial_t \left( \frac{1}{\lambda} \log(1+\lambda t) \right)^m \middle| x^{n-1} \right\rangle$$

$$+ \left\langle \left( \partial_t \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r \right) \left( \frac{1}{\lambda} \log(1+\lambda t) \right)^m \middle| x^{n-1} \right\rangle.$$

The first term of $(2.37)$ is

$$(2.38)$$

$$m \left\langle \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r \left( \frac{1}{\lambda} \log(1+\lambda t) \right)^{m-1} (1+\lambda t)^{-1} \middle| x^{n-1} \right\rangle$$

$$= m \lambda^{-(m-1)} \left\langle \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r (1+\lambda t)^{-1} \middle| (\log(1+\lambda t))^{m-1} x^{n-1} \right\rangle$$

$$= m \lambda^{-(m-1)} \left\langle \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r (1+\lambda t)^{-1} \middle| (m-1)! \sum_{l=m-1}^{\infty} S_1(l,m-1) \frac{\lambda^l t^l}{l!} x^{n-1} \right\rangle$$

$$= m! \lambda^{-(m-1)} \sum_{l=m-1}^{n-1} \binom{n-1}{l} S_1(l,m-1) \lambda^l \left\langle \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r (1+\lambda t)^{-\frac{\lambda}{\lambda}} \middle| x^{n-1-l} \right\rangle$$

$$= m! \lambda^{-(m-1)} \sum_{l=m-1}^{n-1} \binom{n-1}{l} S_1(l,m-1) \lambda^l \beta_{n-1-l}^{(r)}(\lambda, -\lambda).$$

For the second term of $(2.37)$, we recall that

$$(2.39) \qquad \partial_t \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r$$

$$= -\frac{r}{(1+\lambda t)}\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r}$$

$$+\frac{r}{t}\left\{\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r}-\frac{1}{1+\lambda t}\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r+1}\right\}.$$

Now, the second term of (2.37) is

(2.40)

$$\lambda^{-m}\left\langle \partial_{t}\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r}\Bigg| (\log(1+\lambda t))^{m}\, x^{n-1}\right\rangle$$

$$=m!\lambda^{-m}\sum_{l=m}^{n-1}\binom{n-1}{l}S_{1}(l,m)\,\lambda^{l}\left\langle \partial_{t}\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r}\Bigg| x^{n-1-l}\right\rangle$$

$$=m!\lambda^{-m}\sum_{l=m}^{n-1}\binom{n-1}{l}S_{1}(l,m)\,\lambda^{l}\left\{-r\left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r}(1+\lambda t)^{\frac{-\lambda}{\lambda}}\Bigg| x^{n-l-1}\right\rangle\right.$$

$$+\frac{r}{n-l}\left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r}\Bigg| x^{n-l}\right\rangle$$

$$\left.-\frac{r}{n-l}\left\langle \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r+1}(1+\lambda t)^{-\frac{\lambda}{\lambda}}\Bigg| x^{n-l}\right\rangle\right\}$$

$$=m!\lambda^{-m}\sum_{l=m}^{n-1}\binom{n-1}{l}S_{1}(l,m)\,\lambda^{l}$$

$$\times\left\{-r\beta_{n-1-l}^{(r)}(\lambda,-\lambda)+\frac{r}{n-l}\beta_{n-l}^{(r)}(\lambda)-\frac{r}{n-l}\beta_{n-l}^{(r+1)}(\lambda,-\lambda)\right\}.$$

From (2.35), (2.36), (2.37), and (2.40), we have

$$m!\lambda^{-m}\sum_{l=m}^{n}\binom{n}{l}S_{1}(l,m)\,\lambda^{l}\beta_{n-l}^{(r)}(\lambda)$$

$$=m!\lambda^{-(m-1)}\sum_{l=m-1}^{n-1}\binom{n-1}{l}S_{1}(l,m-1)\,\lambda^{l}\beta_{n-l-1}^{(r)}(\lambda,-\lambda)$$

$$+m!\lambda^{-m}\sum_{l=m}^{n-1}\binom{n-1}{l}S_{1}(l,m)\,\lambda^{l}\left(-r\beta_{n-1-l}^{(r)}(\lambda,-\lambda)+\frac{r}{n-l}\beta_{n-l}^{(r)}(\lambda)\right.$$

$$\left.-\frac{r}{n-l}\beta_{n-l}^{(r+1)}(\lambda,-\lambda)\right),$$

where $n-1\geq m\geq 1$.

After simplification and modification, we get: for $n-1\geq m\geq 1$,

(2.41)         $$\sum_{l=0}^{n-m}\binom{n}{l}S_{1}(n-l,m)\,\lambda^{n-l}\beta_{l}^{(r)}(\lambda)$$

$$
=\lambda \sum_{l=0}^{n-m} \binom{n-1}{l} S_1 (n-l-1, m-1) \lambda^{n-l-1} \beta_l^{(r)} (\lambda, -\lambda)
$$

$$
+ \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1 (n-l-1, m) \lambda^{n-l-1}
$$

$$
\times \left( -r\beta_l^{(r)} (\lambda, -\lambda) + \frac{r}{l+1} \beta_{l+1}^{(r)} (\lambda) - \frac{r}{l+1} \beta_{l+1}^{(r+1)} (\lambda, -\lambda) \right)
$$

$$
=\lambda \sum_{l=0}^{n-m} \binom{n-1}{l} S_1 (n-l-1, m-1) \lambda^{n-l-1} \beta_l^{(r)} (\lambda, -\lambda)
$$

$$
- r \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1 (n-l-1, m) \lambda^{n-l-1} \beta_l^{(r)} (\lambda, -\lambda)
$$

$$
+ \frac{r}{n} \sum_{l=0}^{n-m-1} \binom{n}{l+1} S_1 (n-l-1, m) \lambda^{n-l-1} \beta_{l+1}^{(r)} (\lambda)
$$

$$
- \frac{r}{n} \sum_{l=0}^{n-m-1} \binom{n}{l+1} S_1 (n-l-1, m) \lambda^{n-l-1} \beta_{l+1}^{(r+1)} (\lambda, -\lambda)
$$

$$
=\lambda \sum_{l=0}^{n-m} \binom{n-1}{l} S_1 (n-l-1, m-1) \lambda^{n-l-1} \beta_l^{(r)} (\lambda, -\lambda)
$$

$$
- r \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1 (n-l-1, m) \lambda^{n-l-1} \beta_l^{(r)} (\lambda, -\lambda)
$$

$$
+ \frac{r}{n} \sum_{l=0}^{n-m} \binom{n}{l} S_1 (n-l, m) \lambda^{n-l} \beta_l^{(r)} (\lambda)
$$

$$
- \frac{r}{n} \sum_{l=0}^{n-m} \binom{n}{l} S_1 (n-l, m) \lambda^{n-l} \beta_l^{(r+1)} (\lambda, -\lambda) .
$$

Therefore, by (2.41), we obtain the following theorem.

**Theorem 2.8.** *For $n - 1 \geq m \geq 1$, we have*

$$
\left( 1 - \frac{r}{n} \right) \sum_{l=0}^{n-m} \binom{n}{l} S_1 (n-l, m) \lambda^{n-l} \beta_l^{(r)} (\lambda)
$$

$$
=\lambda \sum_{l=0}^{n-m} \binom{n-1}{l} S_1 (n-l-1, m-1) \lambda^{n-l-1} \beta_l^{(r)} (\lambda, -\lambda)
$$

$$
- r \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1 (n-l-1, m) \lambda^{n-l-1} \beta_l^{(r)} (\lambda, -\lambda)
$$

$$
- \frac{r}{n} \sum_{l=0}^{n-m} \binom{n}{l} S_1 (n-l, m) \lambda^{n-l} \beta_l^{(r+1)} (\lambda, -\lambda) .
$$

For $r > s \geq 1$, by (1.19), (1.20) and (1.22), we get

$$(2.42) \qquad \beta_n^{(r)}(\lambda, x) = \sum_{m=0}^{n} C_{n,m} \beta_m^{(s)}(\lambda, x),$$

where

$$(2.43) \qquad C_{n,m} = \frac{1}{m!} \left\langle \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^{r-s} t^m \middle| x^n \right\rangle$$

$$= \frac{1}{m!} \left\langle \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^{r-s} \middle| t^m x^n \right\rangle$$

$$= \binom{n}{m} \left\langle \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^{r-s} \middle| x^{n-m} \right\rangle$$

$$= \binom{n}{m} \left\langle \sum_{l=0}^{\infty} \beta_l^{(r-s)}(\lambda) \frac{t^l}{l!} \middle| x^{n-m} \right\rangle$$

$$= \binom{n}{m} \beta_{n-m}^{(r-s)}(\lambda).$$

Therefore, by (2.42) and (2.43), we obtain the following theorem.

**Theorem 2.9.** *For $r > s \geq 1$, we have*

$$\beta_n^{(r)}(\lambda, x) = \sum_{m=0}^{n} \binom{n}{m} \beta_{n-m}^{(r-s)}(\lambda) \beta_m^{(s)}(\lambda, x).$$

*Remark.* Replacing $x$ by $x + \lambda$ in Theorem 2.7, we have

$$(2.44) \qquad \beta_n^{(r+1)}(\lambda, x) = \left(1 - \frac{n}{r}\right) \beta_n^{(r)}(\lambda, x + \lambda) + \left(\frac{n}{r} x + \frac{n}{r} \lambda - n\right) \beta_{n-1}^{(r)}(\lambda, x).$$

From Theorem 2.5, we note that

$$(2.45) \qquad \beta_n^{(r)}(\lambda, x + \lambda) = \sum_{j=0}^{n} \binom{n}{j} \beta_j^{(r)}(\lambda, x) (\lambda|\lambda)_{n-j}$$

$$= n\lambda \beta_{n-1}^{(r)}(\lambda, x) + \beta_n^{(r)}(\lambda, x).$$

Substituting (2.45) into (2.44), we get
$$(2.46)$$
$$\beta_n^{(r+1)}(\lambda, x) = \left(1 - \frac{n}{r}\right) \beta_n^{(r)}(\lambda, x) + \frac{n}{r} (x + (\lambda - 1) r - (n - 1) \lambda) \beta_{n-1}^{(r)}(\lambda, x).$$

By using this and induction on $r$, it is shown in [[19]] that

$$(2.47) \qquad \beta_n^{(r)}(\lambda, x) = r \binom{n}{r} \sum_{k=0}^{r-1} (-1)^{r-1-k} \sigma_{r-1,k}(\lambda, x, n) \frac{\beta_{n-k}(\lambda, x)}{n - k},$$

where

$$\sigma_{r,k}(\lambda, x, n) = \sum_{1 \leq i_k < i_{k-1} < \cdots < i_1 \leq r} \prod_{j=1}^{k} (x + (\lambda - 1) i_j - (n - j) \lambda).$$

For $r > s \geq 1$, by (1.19), (1.20) and (1.22), we have

$$(2.48) \qquad \beta_n^{(s)}(\lambda, x) = \sum_{m=0}^{n} C_{n,m} \beta_m^{(r)}(\lambda, x),$$

where

$$(2.49) \qquad C_{n,m} = \frac{1}{m!} \left\langle \left( \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} \right)^{r-s} t^m \middle| x^n \right\rangle$$

$$= \binom{n}{m} \left\langle \left( \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} \right)^{r-s} \middle| x^{n-m} \right\rangle.$$

Observe here that

$$(2.50) \qquad \left( \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} \right)^{r-s}$$

$$= \left( \frac{e^{\frac{1}{\lambda} \log(1+\lambda t)} - 1}{t} \right)^{r-s}$$

$$= \frac{1}{t^{r-s}} \left( e^{\frac{1}{\lambda} \log(1+\lambda t)} - 1 \right)^{r-s}$$

$$= \frac{1}{t^{r-s}} (r-s)! \sum_{l=r-s}^{\infty} S_2(l, r-s) \frac{(\log(1+\lambda t))^l}{\lambda^l l!}$$

$$= (r-s)! \sum_{l=r-s}^{\infty} \frac{S_2(l, r-s)}{l!} \left( \frac{\log(1+\lambda t)}{\lambda t} \right)^l t^{l-(r-s)}$$

$$= (r-s)! \sum_{l=0}^{\infty} \frac{S_2(l+r-s, r-s)}{(l+r-s)!} t^l \left( \frac{\log(1+\lambda t)}{\lambda t} \right)^{l+r-s}$$

$$= (r-s)! \sum_{l=0}^{\infty} \frac{S_2(l+r-s, r-s)}{(l+r-s)!} t^l (l+r-s)!$$

$$\times \sum_{k=0}^{\infty} S_1(k+l+r-s, l+r-s) \frac{(\lambda t)^k}{(k+l+r-s)!}$$

$$= (r-s)! \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} S_1(k+l+r-s, l+r-s)$$

$$\times S_2(l+r-s, r-s) \frac{\lambda^k}{(k+l+r-s)!} t^{k+l}$$

$$= (r-s)! \sum_{j=0}^{\infty} \sum_{k+l=j} S_1(k+l+r-s, l+r-s)$$

$$\times S_2(l+r-s, r-s) \frac{\lambda^k}{(k+l+r-s)!} t^{k+l}$$

$$= (r-s)! \sum_{j=0}^{\infty} \sum_{k=0}^{j} S_1(j+r-s, j-k+r-s)$$

$$\times S_2 \left( j - k + r - s, r - s \right) \frac{\lambda^k}{(j+r-s)!} t^j.$$

From (2.49) and (2.50), we have

(2.51)

$$C_{n,m}$$

$$= \binom{n}{m} (r-s)!$$

$$\times \left\langle \sum_{j=0}^{\infty} \left( \sum_{k=0}^{j} S_1 \left( j + r - s, j - k + r - s \right) S_2 \left( j - k + r - s, r - s \right) \frac{\lambda^k j!}{(j+r-s)!} \right) \frac{t^j}{j!} \middle| x^{n-m} \right\rangle$$

$$= \binom{n}{m} (r-s)! \sum_{k=0}^{n-m} S_1 \left( n - m + r - s, n - m - k + r - s \right)$$

$$\times S_2 \left( n - m - k + r - s, r - s \right) \frac{\lambda^k (n-m)!}{(n-m+r-s)!}$$

$$= \frac{\binom{n}{m}}{\binom{n-m+r-s}{r-s}} \sum_{k=0}^{n-m} S_1 \left( n - m + r - s, n - m - k + r - s \right) S_2 \left( n - m - k + r - s, r - s \right) \lambda^k.$$

Therefore, by (2.48) and (2.51), we obtain the following theorem.

**Theorem 2.10.** *For $r > s \geq 1$, we have*

$$\beta_n^{(s)} (\lambda, x)$$

$$= \sum_{m=0}^{n} \left\{ \frac{\binom{n}{m}}{\binom{n-m+r-s}{r-s}} \sum_{k=0}^{n-m} S_1 \left( n - m + r - s, n - m - k + r - s \right) \right.$$

$$\left. \times S_2 \left( n - m - k + r - s, r - s \right) \lambda^k \right\} \beta_m^{(r)} (\lambda, x)$$

$$= \sum_{m=0}^{n} \left\{ \frac{\binom{n}{m}}{\binom{n-m+r-s}{r-s}} \sum_{k=0}^{n-m} S_1 \left( n - m + r - s, k + r - s \right) \right.$$

$$\left. \times S_2 \left( k + r - s, r - s \right) \lambda^{n-m-k} \right\} \beta_m^{(r)} (\lambda, x) .$$

For $(x|\lambda)_n \sim \left( 1, \frac{1}{\lambda} \left( e^{\lambda t} - 1 \right) \right)$, by (1.19), (1.20) and (1.22), we get

(2.52) $$\beta_n^{(r)} (\lambda, x) = \sum_{m=0}^{n} C_{n,m} (x|\lambda)_m ,$$

where

(2.53) $$C_{n,m} = \frac{1}{m!} \left\langle \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r t^m \middle| x^n \right\rangle$$

$$= \frac{1}{m!} \left\langle \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r \middle| t^m x^n \right\rangle$$

$$= \binom{n}{m} \left\langle \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r \middle| x^{n-m} \right\rangle$$

$$= \binom{n}{m} \left\langle \sum_{k=0}^{\infty} \beta_k^{(r)} (\lambda) \frac{t^k}{k!} \middle| x^{n-m} \right\rangle$$

$$= \binom{n}{m} \beta_{n-m}^{(r)} (\lambda).$$

Therefore, by (2.52) and (2.53), we obtain the following theorem.

**Theorem 2.11.** *For $n \geq 0$, we have*

$$\beta_n^{(r)} (\lambda, x) = \sum_{m=0}^{n} \binom{n}{m} \beta_{n-m}^{(r)} (\lambda) (x|\lambda)_m.$$

*Remark.* For $n \geq 0$, we get
(2.54)
$$(x|\lambda)_n = \sum_{m=0}^{n} \left\{ \frac{\binom{n}{m}}{\binom{n-m+r}{r}} \sum_{k=0}^{n-m} S_1 (n-m+r, k+r) S_2 (k+r, r) \lambda^{n-m-k} \right\} \beta_m^{(r)} (\lambda, x).$$

By (1.6) and (1.12), we easily get

(2.55)
$$B_n^{(s)} (x) \sim \left( \left( \frac{e^t - 1}{t} \right)^s, t \right), \quad (s \in \mathbb{N}).$$

From (1.19), (1.20), (1.22) and (2.55), we have

(2.56)
$$B_n^{(s)} (x) = \sum_{m=0}^{n} C_{n,m} \beta_m^{(r)} (\lambda, x),$$

where
(2.57)

$$C_{n,m} = \frac{1}{m!} \left\langle \frac{\left( \frac{\lambda (e^t - 1)}{e^{\lambda t} - 1} \right)^r}{\left( \frac{e^t - 1}{t} \right)^s} \left( \frac{1}{\lambda} (e^{\lambda t} - 1) \right)^m \middle| x^n \right\rangle$$

$$= \frac{1}{m! \lambda^m} \left\langle \left( \frac{\lambda t}{e^{\lambda t} - 1} \right)^r \left( \frac{e^t - 1}{t} \right)^r \left( \frac{t}{e^t - 1} \right)^s (e^{\lambda t} - 1)^m \middle| x^n \right\rangle$$

$$= \frac{1}{m! \lambda^m} \left\langle \left( \frac{\lambda t}{e^{\lambda t} - 1} \right)^r \left( \frac{e^t - 1}{t} \right)^r \left( \frac{t}{e^t - 1} \right)^s \middle| m! \sum_{l=m}^{\infty} S_2 (l, m) \frac{\lambda^l}{l!} t^l x^n \right\rangle$$

$$= \lambda^{-m} \sum_{l=m}^{n} \binom{n}{l} S_2 (l, m) \lambda^l \left\langle \left( \frac{\lambda t}{e^{\lambda t} - 1} \right)^r \left( \frac{e^t - 1}{t} \right)^r \left( \frac{t}{e^t - 1} \right)^s \middle| x^{n-l} \right\rangle$$

$$= \lambda^{-m} \sum_{l=m}^{n} \binom{n}{l} S_2 (l, m) \lambda^l \left\langle \left( \frac{e^t - 1}{t} \right)^r \left( \frac{t}{e^t - 1} \right)^s \middle| \sum_{k=0}^{\infty} B_k^{(r)} \frac{\lambda^k}{k!} t^k x^{n-l} \right\rangle$$

$$= \lambda^{-m} \sum_{l=m}^{n} \binom{n}{l} S_2 (l, m) \lambda^l \sum_{k=0}^{n-l} \binom{n-l}{k} B_k^{(r)} \lambda^k \left\langle \left( \frac{e^t - 1}{t} \right)^r \left( \frac{t}{e^t - 1} \right)^s \middle| x^{n-l-k} \right\rangle.$$

*Case 1.* For $r > s \geq 1$, we have

(2.58)
$$\left\langle \left( \frac{e^t - 1}{t} \right)^r \left( \frac{t}{e^t - 1} \right)^s \middle| x^{n-l-k} \right\rangle$$

$$= \left\langle \left( \frac{e^t - 1}{t} \right)^{r-s} \middle| x^{n-l-k} \right\rangle$$

$$= \left\langle (r-s)! \sum_{j=0}^{\infty} \frac{S_2(j+r-s, r-s)\, j!\, t^j}{(j+r-s)!}\, \frac{}{j!} \middle| x^{n-l-k} \right\rangle$$

$$= (r-s)! \frac{S_2(n-l-k+r-s, r-s)(n-l-k)!}{(n-l-k+r-s)!}$$

$$= S_2(n-l-k+r-s, r-s) \Big/ \binom{n-l-k+r-s}{r-s}$$

*Case 2.* For $r = s \geq 1$, we get

(2.59) $\quad \left\langle \left( \frac{e^t - 1}{t} \right)^r \left( \frac{t}{e^t - 1} \right)^s \middle| x^{n-l-k} \right\rangle = \left\langle 1 \middle| x^{n-l-k} \right\rangle = \delta_{0,n-l-k} = \delta_{k,n-l}.$

*Case 3.* For $s > r \geq 1$, we have

(2.60) $\quad \left\langle \left( \frac{e^t - 1}{t} \right)^r \left( \frac{t}{e^t - 1} \right)^s \middle| x^{n-l-k} \right\rangle = \left\langle \left( \frac{t}{e^t - 1} \right)^{s-r} \middle| x^{n-l-k} \right\rangle = B_{n-l-k}^{(s-r)}.$

Therefore, by (2.56), (2.57), (2.58), (2.59) and (2.60), we obtain the following theorem.

**Theorem 2.12.** *Let $n \geq 0$. Then we have*

$$B_n^{(s)}(x) = \begin{cases}
\displaystyle\sum_{m=0}^{n} \left\{ \lambda^{-m} \sum_{l=m}^{n} \sum_{k=0}^{n-l} \frac{\binom{n}{l}\binom{n-l}{k}}{\binom{n-l-k+r-s}{r-s}} S_2(l,m) \right. \\
\displaystyle\left. \times S_2(n-l-k+r-s, r-s)\, \lambda^{k+l} B_k^{(r)} \right\} \beta_m^{(r)}(\lambda, x), & \text{if } r > s \geq 1, \\[4mm]
\displaystyle\lambda^n \sum_{m=0}^{n} \left\{ \lambda^{-m} \sum_{l=m}^{n} \binom{n}{l} S_2(l,m) B_{n-l}^{(r)} \right\} \beta_m^{(r)}(\lambda, x), & \text{if } r = s \geq 1, \\[4mm]
\displaystyle\sum_{m=0}^{n} \lambda^{-m} \left\{ \sum_{l=m}^{n} \sum_{k=0}^{n-l} \binom{n}{l}\binom{n-l}{k} S_2(l,m) \right. \\
\displaystyle\left. \times \lambda^{k+l} B_k^{(r)} B_{n-l-k}^{(s-r)} \right\} \beta_m^{(r)}(\lambda, x), & \text{if } s > r \geq 1.
\end{cases}$$

*Remark.* Let $r > s \geq 1$. Then we get
(2.61)

$$\beta_n^{(r)}(\lambda, x) = \sum_{m=0}^{n} \left\{ \lambda^{-m} \sum_{l=m}^{n} \sum_{k=0}^{n-l} \binom{n}{l}\binom{n-l}{k} S_1(l,m)\, \lambda^{k+l} b_k^{(s)} \beta_{n-l-k}^{(r-s)}(\lambda) \right\} B_m^{(s)}(x).$$

For $r = s \geq 1$, we have

(2.62) $\qquad \beta_n^{(r)}(\lambda, x) = \lambda^n \sum_{m=0}^{n} \lambda^{-m} \left\{ \sum_{l=m}^{n} \binom{n}{l} S_1(l,m)\, b_{n-l}^{(s)} \right\} B_m^{(s)}(x).$

If $s > r \geq 1$, then we note that

(2.63) $\quad \beta_n^{(r)}(\lambda, x)$

$$= \lambda^n \sum_{m=0}^{n} \left\{ \lambda^{-m} \sum_{l=m}^{n} \sum_{k=0}^{n-l} \sum_{i=0}^{n-l-k} \frac{\binom{n}{l}\binom{n-l}{k}}{\binom{n-l-k+s-r}{s-r}} S_1(l,m) \right.$$

$$\times S_1 \left( n - l - k + s - r, i + s - r \right) S_2 \left( i + s - r, s - r \right) \lambda^{-i} b_k^{(s)} \Big\} B_m^{(s)} \left( x \right).$$

From (1.7) and (1.12), we get

(2.64)
$$H_n^{(s)} \left( x | \mu \right) \sim \left( \left( \frac{e^t - \mu}{1 - \mu} \right)^s, t \right).$$

By (1.19), (1.20), (1.22) and (2.64), we have

(2.65)
$$\beta_n^{(r)} \left( \lambda, x \right) = \sum_{m=0}^{n} C_{n,m} H_m^{(s)} \left( x | \mu \right),$$

where
(2.66)

$$C_{n,m}$$

$$= \frac{1}{m!} \left\langle \frac{\left( e^{\frac{1}{\lambda} \log(1 + \lambda t)} - \mu / 1 - \mu \right)^s}{\left( \lambda \left( e^{\frac{1}{\lambda} \log(1 + \lambda t)} - 1 \right) / e^{\log(1 + \lambda t)} - 1 \right)^r} \left( \frac{1}{\lambda} \log \left( 1 + \lambda t \right) \right)^m \Bigg| x^n \right\rangle$$

$$= \frac{1}{m! \lambda^m \left( 1 - \mu \right)^s} \left\langle \left( \left( 1 + \lambda t \right)^{\frac{1}{\lambda}} - \mu \right)^s \left( \frac{t}{\left( 1 + \lambda t \right)^{\frac{1}{\lambda}} - 1} \right)^r \left( \log \left( 1 + \lambda t \right) \right)^m \Bigg| x^n \right\rangle$$

$$= \frac{1}{\lambda^m \left( 1 - \mu \right)^s} \sum_{l=m}^{n} \binom{n}{l} S_1 \left( l, m \right) \lambda^l \left\langle \left( \left( 1 + \lambda t \right)^{\frac{1}{\lambda}} - \mu \right)^s \Bigg| \left( \frac{t}{\left( 1 + \lambda t \right)^{\frac{1}{\lambda}} - 1} \right)^r x^{n-l} \right\rangle$$

$$= \frac{1}{\lambda^m \left( 1 - \mu \right)^s} \sum_{l=m}^{n} \binom{n}{l} S_1 \left( l, m \right) \lambda^l \sum_{k=0}^{n-l} \binom{n-l}{k} \beta_k^{(r)} \left( \lambda \right) \left\langle \left( \left( 1 + \lambda t \right)^{\frac{1}{\lambda}} - \mu \right)^s \Bigg| x^{n-l-k} \right\rangle.$$

It is easy to show that

(2.67)
$$\left\langle \left( \left( 1 + \lambda t \right)^{\frac{1}{\lambda}} - \mu \right)^s \Bigg| x^{n-l-k} \right\rangle$$

$$= \sum_{i=0}^{s} \binom{s}{i} \left( -\mu \right)^{s-i} \left\langle \sum_{j=0}^{\infty} \left( i | \lambda \right)_j \frac{t^j}{j!} \Bigg| x^{n-l-k} \right\rangle$$

$$= \sum_{i=0}^{s} \binom{s}{i} \left( -\mu \right)^{s-i} \left( i | \lambda \right)_{n-l-k}.$$

From (2.66) and (2.67), we have
(2.68)

$$C_{n,m}$$

$$= \frac{1}{\lambda^m \left( 1 - \mu \right)^s} \sum_{l=m}^{n} \binom{n}{l} S_1 \left( l, m \right) \lambda^l \sum_{k=0}^{n-l} \binom{n-l}{k} \beta_k^{(r)} \left( \lambda \right) \sum_{i=0}^{s} \binom{s}{i} \left( -\mu \right)^{s-i} \left( i | \lambda \right)_{n-l-k}$$

$$= \frac{1}{\lambda^m \left( 1 - \mu \right)^s} \sum_{l=m}^{n} \sum_{k=0}^{n-l} \sum_{i=0}^{s} \binom{n}{l} \binom{n-l}{k} \binom{s}{i} S_1 \left( l, m \right) \lambda^l \left( -\mu \right)^{s-i} \beta_k^{(r)} \left( \lambda \right) \left( i | \lambda \right)_{n-l-k}.$$

Therefore, by (2.65) and (2.68), we obtain the following theorem.

**Theorem 2.13.** *For $\mu \in \mathbb{C}$ with $\mu \neq 1$, $n \geq 0$, we have*

$$\beta_n^{(r)}(\lambda, x)$$

$$= \frac{1}{(1-\mu)^s} \sum_{m=0}^{n} \left\{ \lambda^{-m} \sum_{l=m}^{n} \sum_{k=0}^{n-l} \sum_{i=0}^{s} \binom{n}{l} \binom{n-l}{k} \binom{s}{i} S_1(l,m) \lambda^l (-\mu)^{s-i} \right.$$

$$\left. \times \ \beta_k^{(r)}(\lambda)(i|\lambda)_{n-l-k} \right\} H_m^{(s)}(x|\mu).$$

*Remark.* For $n \geq 0$, we have

$$H_n^{(s)}(x|\mu)$$

$$= \sum_{m=0}^{n} \left\{ \frac{1}{\lambda^m} \sum_{l=0}^{n-m} \sum_{k=m}^{n-l} \binom{n}{l} \binom{n-l}{k} S_2(k,m) \lambda^{k+l} B_l^{(r)} H_{n-l-k-j}^{(r)}(\mu) \right\} \beta_m^{(r)}(\lambda, x).$$

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# KOROBOV POLYNOMIALS OF THE SEVENTH KIND AND OF THE EIGHTH KIND

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ABSTRACT. In this paper, we consider the Korobov polynomials of the seventh kind and of the eighth kind. We present several explicit formulas and recurrence relations for these polynomials. In addition, we establish connections between our polynomials and several known families of polynomials.

## 1. INTRODUCTION

The *degenerate Bernoulli polynomials* are the degenerate version of Bernoulli polynomials introduced by Calitz [3, 4]. On the other hand, the Korobov polynomials of the first kind are the first degenerate version of the Bernoulli polynomials of the second kind, see [13, 14].

In recent years, many researchers studied various kinds of degenerate versions of families polynomials like Bernoulli polynomials, Euler polynomials, falling factorial polynomials, Bell polynomials and their variants, see [6–10] and references therein. Along this line of research, we introduced in [8, 9] four kinds of new degenerate versions of Bernoulli polynomials of the second kind, called the *Korobov polynomials of the third, fourth, fifth, and sixth kind.*

Here, we will discuss two other degenerate versions of Bernoulli polynomials of the second kind, namely, the *Korobov polynomials of the seventh and eighth kind.* We will investigate some properties, explicit expressions, recurrence relations, and connections with other families polynomials with the help of umbral calculus (see [10, 15, 16]). To do that, we recall some families polynomials. The *Bernoulli polynomials of the second kind* $b_n(x)$ are given by the generating function

$$(1.1) \qquad \frac{t}{\log(1+t)}(1+t)^x = \sum_{n \geq 0} b_n(x)\frac{t^n}{n!}.$$

For $x = 0$, $b_n = b_n(0)$ are called the *Bernoulli numbers of the second kind.* The *Daehee polynomials* $D_n(x)$ are defined by the generating function

$$(1.2) \qquad \frac{\log(1+t)}{t}(1+t)^x = \sum_{n \geq 0} D_n(x)\frac{t^n}{n!}.$$

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When $x = 0$, $D_n = D_n(0)$ are called the *Daehee numbers*. The *Krobov polynomials* $K_n(\lambda, x)$ of the first kind are given by

$$(1.3) \qquad \frac{\lambda t}{(1+t)^\lambda - 1}(1+t)^x = \sum_{n \geq 0} K_n(\lambda, x)\frac{t^n}{n!}.$$

When $x = 0$, $K_n(\lambda) = K_n(\lambda, 0)$ are called the *Korobov numbers of the first kind*. The *degenerate falling factorial polynomials* $(x)_{n,\lambda}$ were defined in [7] by the generating function

$$(1.4) \qquad (1+\lambda)^{\frac{x}{\lambda}\frac{(1+t)^\lambda - 1}{\lambda}} = \sum_{n \geq 0}(x)_{n,\lambda}\frac{t^n}{n!}.$$

Clearly, $\lim_{\lambda \to 0}(x)_{n,\lambda} = (x)_n$, the $n$th falling factorial polynomial. These polynomials can be defined as $(x)_{n,\lambda} \sim (1, f(t))$, where

$$(1.5) \qquad f(t) = \left(1 + \frac{\lambda^2 t}{\log(1+\lambda)}\right)^{\frac{1}{\lambda}} - 1 \text{ and } \bar{f}(t) = \frac{\log(1+\lambda)}{\lambda}\frac{(1+t)^\lambda - 1}{\lambda}.$$

Note that we write $s_n(x) \sim (g(t), f(t))$ if $\sum_{n \geq 0} s_n(x)\frac{t^n}{n!} = \frac{1}{g(\bar{f}(t))}e^{x\bar{f}(t)}$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$, see [15, 16]. The *degenerate Stirling numbers of the first kind* $S_1(n, k \mid \lambda)$, $n \geq k \geq 0$, were given in [7] by the generating function

$$(1.6) \qquad \frac{1}{k!}\left(\frac{(1+t)^\lambda - 1}{\lambda}\right)^k = \sum_{n \geq k} S_1(n, k \mid \lambda)\frac{t^n}{n!},$$

so that, in the notation of umbral calculus, $S_1(n, k \mid \lambda) = \frac{1}{k!}\left\langle \left(\frac{(1+t)^\lambda - 1}{\lambda}\right)^k \mid x^n \right\rangle$. Then, it was shown in [7] that

$$(x)_{n,\lambda} = \sum_{k=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^k S_1(n, k \mid \lambda)x^k$$

with

$$S_1(n, k \mid \lambda) = \sum_{m=k}^{n} S_1(n, m)S_2(m, k)\lambda^{m-k},$$

where $\lim_{\lambda \to 0} S_1(n, k \mid \lambda) = S_1(n, k)$ is the Stirling number of the first kind.

Here, we introduce *Korobov polynomials of the seventh kind* $K_{n,7}(\lambda, x)$ and of the eighth kind $K_{n,9}(\lambda, x)$, respectively given by

$$(1.7) \qquad \frac{\log(1+\lambda t)}{\lambda \log(1+t)}(1+\lambda)^{\frac{x}{\lambda}\frac{(1+t)^\lambda - 1}{\lambda}} = \sum_{n \geq 0} K_{n,7}(\lambda, x)\frac{t^n}{n!},$$

$$(1.8) \qquad \frac{\log(1+\lambda t)}{(1+t)^\lambda - 1}(1+\lambda)^{\frac{x}{\lambda}\frac{(1+t)^\lambda - 1}{\lambda}} = \sum_{n \geq 0} K_{n,8}(\lambda, x)\frac{t^n}{n!}.$$

When $x = 0$, $K_{n,7}(\lambda) = K_{n,7}(\lambda, 0)$ and $K_{n,8}(\lambda) = K_{n,8}(\lambda, 0)$ are called the *Korobov numbers of the seventh kind and of the eighth kind*, respectively. We observe that

$$\lim_{\lambda \to 0} \frac{\lambda t}{(1+t)^\lambda - 1}(1+t)^x = \lim_{\lambda \to 0} \frac{\log(1 + \lambda t)}{\lambda \log(1+t)}(1+\lambda)^{\frac{x}{\lambda}\frac{(1+t)^\lambda - 1}{\lambda}}$$

$$= \lim_{\lambda \to 0} \frac{\log(1 + \lambda t)}{(1+t)^\lambda - 1}(1+\lambda)^{\frac{x}{\lambda}\frac{(1+t)^\lambda - 1}{\lambda}} = \frac{t}{\log(1+t)}(1+t)^x,$$

which implies that $\lim_{\lambda \to 0} K_n(\lambda, x) = \lim_{\lambda \to 0} K_{n,7}(\lambda, x) = \lim_{\lambda \to 0} K_{n,8}(\lambda, x) = b_n(x)$. It is immediate to see that $K_{n,7}(\lambda, x)$ and $K_{n,8}(\lambda, x)$ are Sheffer sequences (see [15, 16]) for the respective pairs $\left( \frac{\lambda \log(1 + f(t))}{\log(1 + \lambda f(t))}, f(t) \right)$ and $\left( \frac{(1 + f(t))^\lambda - 1}{\log(1 + \lambda f(t))}, f(t) \right)$, where $f(t)$ is given in (1.5). Thus, (1.7) and (1.8) can be presented as

$$(1.9) \qquad K_{n,7}(\lambda, x) \sim \left( \frac{\lambda \log(1 + f(t))}{\log(1 + \lambda f(t))}, f(t) \right) = \left( \frac{\log\left(1 + \frac{\lambda^2 t}{\log(1+\lambda)}\right)}{\log(1 + \lambda f(t))}, f(t) \right),$$

$$(1.10) \qquad K_{n,8}(\lambda, x) \sim \left( \frac{(1 + f(t))^\lambda - 1}{\log(1 + \lambda f(t))}, f(t) \right) = \left( \frac{\frac{\lambda^2 t}{\log(1+\lambda)}}{\log(1 + \lambda f(t))}, f(t) \right).$$

In the next two sections, we will use umbral calculus in order to study some properties, explicit formulas, recurrence relations and identities about the Korobov polynomials of the seventh kind and of the eighth kind. In last section, we present connections between our polynomials and several known families of polynomials.

## 2. EXPLICIT EXPRESSIONS

In this section, we present several explicit formulas for the Korobov polynomials of the seventh kind and of the eighth kind, namely $K_{n,7}(\lambda, x)$ and $K_{n,8}(\lambda, x)$.

**Theorem 2.1.** *For all $n \geq 0$,*

$$K_{n,7}(\lambda, x) = \sum_{k=0}^{n} \sum_{\ell=k}^{n} \binom{n}{\ell} \frac{\log^k(1 + \lambda)}{\lambda^k} S_1(\ell, k | \lambda) K_{n-\ell,7}(\lambda) x^k$$

$$= \sum_{k=0}^{n} \sum_{\ell=k}^{n} \sum_{m=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{m} \frac{\log^k(1 + \lambda)}{\lambda^k} S_1(\ell, k | \lambda) b_m D_{n-\ell-m} \lambda^{n-\ell-m} x^k,$$

$$K_{n,8}(\lambda, x) = \sum_{k=0}^{n} \sum_{\ell=k}^{n} \binom{n}{\ell} \frac{\log^k(1 + \lambda)}{\lambda^k} S_1(\ell, k | \lambda) K_{n-\ell,8}(\lambda) x^k$$

$$= \sum_{k=0}^{n} \sum_{\ell=k}^{n} \sum_{m=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{m} \frac{\log^k(1 + \lambda)}{\lambda^k} S_1(\ell, k | \lambda) K_m(\lambda) D_{n-\ell-m} \lambda^{n-\ell-m} x^k.$$

*Proof.* We proceed the proof by using the conjugation representation for Sheffer sequences (see [15,16]): $s_n(x) = \sum_{k=0}^{n} \frac{1}{k!} \langle g(\bar{f}(t))^{-1} \bar{f}(t)^k | x^n \rangle x^k$, for any $s_n(x) \sim (g(t), f(t))$.

Thus, by (1.9), we have

$$K_{n,7}(\lambda, x) = \sum_{k=0}^{n} \frac{1}{k!} \left\langle \frac{\log(1+\lambda t)}{\lambda \log(1+t)} \frac{\log^k(1+\lambda)((1+t)^\lambda - 1)^k}{\lambda^{2k}} | x^n \right\rangle x^k$$

$$= \sum_{k=0}^{n} \frac{\log^k(1+\lambda)}{\lambda^k} \left\langle \frac{\log(1+\lambda t)}{\lambda \log(1+t)} | \frac{((1+t)^\lambda - 1)^k}{k! \lambda^k} x^n \right\rangle x^k,$$

which, by (1.6) and (1.7), implies

$$K_{n,7}(\lambda, x) = \sum_{k=0}^{n} \frac{\log^k(1+\lambda)}{\lambda^k} \left\langle \frac{\log(1+\lambda t)}{\lambda \log(1+t)} | \sum_{\ell \geq k} S_1(\ell, k|\lambda) \frac{t^\ell}{\ell!} x^n \right\rangle x^k$$

(2.1)
$$= \sum_{k=0}^{n} \sum_{\ell=k}^{n} \binom{n}{\ell} \frac{\log^k(1+\lambda)}{\lambda^k} S_1(\ell, k|\lambda) \left\langle \frac{\log(1+\lambda t)}{\lambda \log(1+t)} | x^{n-\ell} \right\rangle x^k$$

$$= \sum_{k=0}^{n} \sum_{\ell=k}^{n} \binom{n}{\ell} \frac{\log^k(1+\lambda)}{\lambda^k} S_1(\ell, k|\lambda) K_{n-\ell,7}(\lambda) x^k.$$

On the other hand, by (2.1), we have

$$K_{n,7}(\lambda, x) = \sum_{k=0}^{n} \sum_{\ell=k}^{n} \binom{n}{\ell} \frac{\log^k(1+\lambda)}{\lambda^k} S_1(\ell, k|\lambda) \left\langle \frac{\log(1+\lambda t)}{\lambda t} | \frac{t}{\log(1+t)} x^{n-\ell} \right\rangle x^k,$$

which, by (1.1) and (1.2), we obtain

$$K_{n,7}(\lambda, x)$$

$$= \sum_{k=0}^{n} \sum_{\ell=k}^{n} \binom{n}{\ell} \frac{\log^k(1+\lambda)}{\lambda^k} S_1(\ell, k|\lambda) \left\langle \frac{\log(1+\lambda t)}{\lambda t} | \sum_{m \geq 0} b_m \frac{t^m}{m!} x^{n-\ell} \right\rangle x^k$$

$$= \sum_{k=0}^{n} \sum_{\ell=k}^{n} \sum_{m=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{m} \frac{\log^k(1+\lambda)}{\lambda^k} S_1(\ell, k|\lambda) b_m \left\langle \frac{\log(1+\lambda t)}{\lambda t} | x^{n-\ell-m} \right\rangle x^k$$

$$= \sum_{k=0}^{n} \sum_{\ell=k}^{n} \sum_{m=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{m} \frac{\log^k(1+\lambda)}{\lambda^k} S_1(\ell, k|\lambda) b_m \left\langle \sum_{j \geq 0} D_j \lambda^j \frac{t^j}{j!} | x^{n-\ell-m} \right\rangle x^k$$

$$= \sum_{k=0}^{n} \sum_{\ell=k}^{n} \sum_{m=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{m} \frac{\log^k(1+\lambda)}{\lambda^k} S_1(\ell, k|\lambda) b_m D_{n-\ell-m} \lambda^{n-\ell-m} x^k,$$

which completes the proof of formulas for $k_{n,7}(\lambda, x)$.

Now let us deal with the case $K_{n,8}(\lambda, x)$. Similarly, by using the conjugation representation for Sheffer sequences, (1.10) and (1.6), we obtain

(2.2)     $$K_{n,8}(\lambda, x) = \sum_{k=0}^{n} \sum_{\ell=k}^{n} \binom{n}{\ell} \frac{\log^k(1+\lambda)}{\lambda^k} S_1(\ell, k|\lambda) \left\langle \frac{\log(1+\lambda t)}{(1+t)^\lambda - 1} | x^{n-\ell} \right\rangle x^k$$

$$= \sum_{k=0}^{n} \sum_{\ell=k}^{n} \binom{n}{\ell} \frac{\log^k(1+\lambda)}{\lambda^k} S_1(\ell, k|\lambda) K_{n-\ell,8}(\lambda) x^k.$$

On the other hand, by (2.2), we have

$$K_{n,8}(\lambda, x) = \sum_{k=0}^{n} \sum_{\ell=k}^{n} \binom{n}{\ell} \frac{\log^k(1+\lambda)}{\lambda^k} S_1(\ell, k|\lambda) \left\langle \frac{\log(1+\lambda t)}{\lambda t} \Big| \frac{\lambda t}{(1+t)^\lambda - 1} x^{n-\ell} \right\rangle x^k,$$

which, by (1.3) and (1.2), we obtain

$$K_{n,8}(\lambda, x)$$
$$= \sum_{k=0}^{n} \sum_{\ell=k}^{n} \binom{n}{\ell} \frac{\log^k(1+\lambda)}{\lambda^k} S_1(\ell, k|\lambda) \left\langle \frac{\log(1+\lambda t)}{\lambda t} \Big| \sum_{m \geq 0} K_m(\lambda) \frac{t^m}{m!} x^{n-\ell} \right\rangle x^k$$
$$= \sum_{k=0}^{n} \sum_{\ell=k}^{n} \sum_{m=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{m} \frac{\log^k(1+\lambda)}{\lambda^k} S_1(\ell, k|\lambda) K_m(\lambda) \left\langle \frac{\log(1+\lambda t)}{\lambda t} \Big| x^{n-\ell-m} \right\rangle x^k$$
(2.3)
$$= \sum_{k=0}^{n} \sum_{\ell=k}^{n} \sum_{m=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{m} \frac{\log^k(1+\lambda)}{\lambda^k} S_1(\ell, k|\lambda) K_m(\lambda) D_{n-\ell-m} \lambda^{n-\ell-m} x^k,$$

which completes the proof. □

Now, we express our polynomials in terms of the degenerate falling factorial polynomials.

**Theorem 2.2.** *For all $n \geq 0$,*

$$K_{n,7}(\lambda, x) = \sum_{\ell=0}^{n} \binom{n}{\ell} \left( \sum_{m=0}^{n-\ell} \binom{n-\ell}{m} \lambda^{n-\ell-m} b_m D_{n-\ell-m} \right) (x)_{\ell,\lambda},$$

$$K_{n,8}(\lambda, x) = \sum_{\ell=0}^{n} \binom{n}{\ell} \left( \sum_{m=0}^{n-\ell} \binom{n-\ell}{m} \lambda^{n-\ell-m} K_m(\lambda) D_{n-\ell-m} \right) (x)_{\ell,\lambda}.$$

*Proof.* By (1.9), we have

$$K_{n,7}(\lambda, y) = \left\langle \frac{\log(1+\lambda t)}{\lambda \log(1+t)} (1+\lambda)^{\frac{y}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}} \Big| x^n \right\rangle = \left\langle \frac{\log(1+\lambda t)}{\lambda \log(1+t)} \Big| (1+\lambda)^{\frac{y}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}} x^n \right\rangle,$$

which, by (1.4), implies

$$K_{n,7}(\lambda, y) = \left\langle \frac{\log(1+\lambda t)}{\lambda \log(1+t)} \Big| \sum_{\ell \geq 0} (y)_{\ell,\lambda} \frac{t^\ell}{\ell!} x^n \right\rangle = \sum_{\ell=0}^{n} \binom{n}{\ell} (y)_{\ell,\lambda} \left\langle \frac{\log(1+\lambda t)}{\lambda \log(1+t)} \Big| x^{n-\ell} \right\rangle.$$

Therefore, by (2.1), we obtain

$$K_{n,7}(\lambda, y) = \sum_{\ell=0}^{n} \binom{n}{\ell} \left( \sum_{m=0}^{n-\ell} \binom{n-\ell}{m} \lambda^{n-\ell-m} b_m D_{n-\ell-m} \right) (y)_{\ell,\lambda},$$

which completes the proof for $K_{n,7}(\lambda, y)$.

By using similar arguments as above together with (1.10) and (1.4), we obtain

$$K_{n,8}(\lambda, y) = \sum_{\ell=0}^{n} \binom{n}{\ell} (y)_{\ell,\lambda} \left\langle \frac{\log(1+\lambda t)}{(1+t)^\lambda - 1} \Big| x^{n-\ell} \right\rangle.$$

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Therefore, by (2.2) and (2.3), we have

$$K_{n,8}(\lambda, y) = \sum_{\ell=0}^{n} \binom{n}{\ell} \left( \sum_{m=0}^{n-\ell} \binom{n-\ell}{m} \lambda^{n-\ell-m} K_m(\lambda) D_{n-\ell-m} \right) (y)_{\ell,\lambda},$$

which completes the proof.                                                □

In the next theorem, we find explicit formulas for the coefficient of $x^j$ in $K_{n,7}(\lambda, x)$ and $K_{n,8}(\lambda, x)$.

**Theorem 2.3.** *For all $n \geq 0$ and $s = 7, 8$,*

$$K_{n,s}(\lambda, x)$$
$$= \sum_{j=0}^{n} \left( \sum_{k=j}^{n} \sum_{\ell=0}^{k} \sum_{m=0}^{\ell} \frac{(-1)^{\ell-m}}{\ell!} \binom{\ell}{m} \binom{k}{j} (m|\lambda)_{k-j} \frac{\log^j(1+\lambda)}{\lambda^j} S_1(n, k|\lambda) K_{\ell,s}(\lambda) \right) x^j.$$

*Proof.* By (1.4) and (1.9), we have

$$\frac{\lambda \log(1+f(t))}{\log(1+\lambda f(t))} K_{n,7}(\lambda, x) = (x)_{n,\lambda} = \sum_{k=0}^{n} \frac{\log^k(1+\lambda)}{\lambda^k} S_1(n, k|\lambda) x^k \sim (1, f(t)).$$

Thus,

$$K_{n,7}(\lambda, x) = \sum_{k=0}^{n} \frac{\log^k(1+\lambda)}{\lambda^k} S_1(n, k|\lambda) \frac{\log(1+\lambda f(t))}{\lambda \log(1+f(t))} x^k$$

(2.4)
$$= \sum_{k=0}^{n} \sum_{\ell=0}^{k} \frac{\log^k(1+\lambda)}{\lambda^k} S_1(n, k|\lambda) \frac{K_{\ell,7}(\lambda)}{\ell!} (f(t))^\ell x^k.$$

Note that

$$(f(t))^\ell x^k = \sum_{m=0}^{\ell} \binom{\ell}{m} (-1)^{\ell-m} \left( 1 + \lambda^2 t / \log(1+\lambda) \right)^{m/\lambda} x^k$$
$$= \sum_{m=0}^{\ell} \sum_{j=0}^{k} \binom{\ell}{m} (-1)^{\ell-m} (m|\lambda)_j (\lambda / \log(1+\lambda))^j \binom{k}{j} x^{k-j}.$$

Therefore,

$$K_{n,7}(\lambda, x)$$
$$= \sum_{k=0}^{n} \sum_{\ell=0}^{k} \sum_{m=0}^{\ell} \sum_{j=0}^{k} (-1)^{\ell-m} (m|\lambda)_{k-j} \frac{\log^j(1+\lambda)}{\lambda^j} S_1(n, k|\lambda) \frac{K_{\ell,7}(\lambda)}{\ell!} \binom{\ell}{m} \binom{k}{j} x^j$$
$$= \sum_{j=0}^{n} \left( \sum_{k=j}^{n} \sum_{\ell=0}^{k} \sum_{m=0}^{\ell} \frac{(-1)^{\ell-m}}{\ell!} \binom{\ell}{m} \binom{k}{j} (m|\lambda)_{k-j} \frac{\log^j(1+\lambda)}{\lambda^j} S_1(n, k|\lambda) K_{\ell,7}(\lambda) \right) x^j.$$

By (1.4) and (1.10), we have

$$\frac{(1+f(t))^\lambda - 1}{\log(1+\lambda f(t))} K_{n,8}(\lambda, x) = (x)_{n,\lambda} = \sum_{k=0}^{n} \frac{\log^k(1+\lambda)}{\lambda^k} S_1(n, k|\lambda) x^k \sim (1, f(t)).$$

Thus, by using the above arguments, we obtain

$$K_{n,8}(\lambda, x)$$

$$= \sum_{j=0}^{n} \left( \sum_{k=j}^{n} \sum_{\ell=0}^{k} \sum_{m=0}^{\ell} \frac{(-1)^{\ell-m}}{\ell!} \binom{\ell}{m} \binom{k}{j} (m|\lambda)_{k-j} \frac{\log^{j}(1+\lambda)}{\lambda^{j}} S_1(n, k|\lambda) K_{\ell,8}(\lambda) \right) x^{j},$$

which completes the proof. $\qquad\square$

In the next theorem, we express Korobov polynomials of seventh and eighth kinds in terms of Korobov polynomials of fifth and sixth kinds.

**Theorem 2.4.** *For all $n \geq 0$ and $s = 7, 8$,*

$$K_{n,s}(\lambda, x) = \sum_{\ell=0}^{n} \binom{n}{\ell} D_{n-\ell} \lambda^{n-\ell} K_{\ell,s-2}(\lambda, x).$$

*Proof.* Recall that Korobov polynomials of the fifth kind (see [9]) is given by

$$\frac{t}{\log(1+t)} (1+\lambda)^{\frac{x}{\lambda} \frac{(1+t)^{\lambda}-1}{\lambda}} = \sum_{n\geq 0} K_{n,5}(\lambda, x) \frac{t^{n}}{n!}.$$

So, by (1.7), we have

$$K_{n,7}(\lambda, y) = \left\langle \frac{\log(1+\lambda t)}{\lambda t} \Big| \frac{t}{\log(1+t)} (1+\lambda)^{\frac{y}{\lambda} \frac{(1+t)^{\lambda}-1}{\lambda}} x^{n} \right\rangle$$

$$= \sum_{\ell=0}^{n} \binom{n}{\ell} K_{\ell,5}(\lambda, y) \left\langle \frac{\log(1+\lambda t)}{\lambda t} \Big| x^{n-\ell} \right\rangle,$$

which, by (1.2), implies $K_{n,7}(\lambda, y) = \sum_{\ell=0}^{n} \binom{n}{\ell} K_{\ell,5}(\lambda, y) D_{n-\ell} \lambda^{n-\ell}$.

Recall that Korobov polynomials of the sixth kind (see [9]) is defined by

$$\frac{\lambda t}{(1+t)^{\lambda}-1} (1+\lambda)^{\frac{x}{\lambda,y} \frac{(1+t)^{\lambda}-1}{\lambda}} = \sum_{n\geq 0} K_{n,6}(\lambda, x) \frac{t^{n}}{n!}.$$

Similarly, by (1.2) and (1.8), we obtain $K_{n,8}(\lambda, y) = \sum_{\ell=0}^{n} \binom{n}{\ell} K_{\ell,6}(\lambda, y) D_{n-\ell} \lambda^{n-\ell}$, as claimed. $\qquad\square$

In the next theorem, we express our polynomials $K_{n,7}(\lambda, x)$ and $K_{n,8}(\lambda, x)$ in terms of *degenerate Bernoulli numbers* $\beta_{\ell}^{(n)}(\lambda)$ of order $n$, which are given by the generating function

$$(2.5) \qquad \frac{t^{n}}{((1+\lambda t)^{1/\lambda}-1)^{n}} = \sum_{\ell\geq 0} \beta_{\ell}^{(n)}(\lambda) \frac{t^{\ell}}{\ell!}.$$

**Theorem 2.5.** *For all $n \geq 1$ and $s = 7, 8$,*

$$K_{n,s}(\lambda, x)$$

$$\sum_{j=0}^{n} \left( \sum_{k=j}^{n} \sum_{\ell=0}^{k} \sum_{m=0}^{\ell} \frac{(-1)^{\ell-m} \binom{n-1}{k-1} \binom{\ell}{m} \binom{k}{j}}{\ell!} (m|\lambda)_{k-j} \left( \frac{\log(1+\lambda)}{\lambda} \right)^{j} \beta_{n-k}^{(n)}(\lambda) K_{\ell,s}(\lambda) \right) x^{j}.$$

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*Proof.* It is not hard to see that $\frac{\log^n(1+\lambda)x^n}{\lambda^n} \sim (1, \lambda t/\log(1+\lambda))$. Thus, by (1.9), we have

$$\frac{\lambda\log(1+f(t))}{\log(1+\lambda f(t))}K_{n,7}(\lambda, x) = x\left(\frac{\frac{\lambda t}{\log(1+\lambda)}}{(1+\lambda^2 t/\log(1+\lambda))^{1/\lambda}-1}\right)^n x^{-1}\frac{\log^n(1+\lambda)x^n}{\lambda^n}$$

$$= \frac{\log^n(1+\lambda)}{\lambda^n}x\left(\frac{r}{(1+\lambda r)^{1/\lambda}-1}\right)^n\Big|_{r=\lambda t/\log(1+\lambda)} x^{n-1},$$

which, by (2.5), implies

$$\frac{\lambda\log(1+f(t))}{\log(1+\lambda f(t))}K_{n,7}(\lambda, x) = \frac{\log^n(1+\lambda)}{\lambda^n}x\sum_{k\geq 0}\beta_k^{(n)}(\lambda)\frac{1}{k!}\left(\frac{\lambda t}{\log(1+\lambda)}\right)^k x^{n-1}$$

$$= \sum_{k=0}^{n-1}\binom{n-1}{k}\beta_k^{(n)}(\lambda)\left(\frac{\log(1+\lambda)}{\lambda}\right)^{n-k}x^{n-k}$$

$$= \sum_{k=0}^{n}\binom{n-1}{k-1}\beta_{n-k}^{(n)}(\lambda)\left(\frac{\log(1+\lambda)}{\lambda}\right)^k x^k.$$

On the other hand, by (2.4), we have

$$\frac{\log(1+\lambda f(t))}{\lambda\log(1+f(t))}x^k$$

$$= \sum_{\ell=0}^{k}\sum_{m=0}^{\ell}\sum_{j=0}^{k}\frac{K_{\ell,7}(\lambda)}{\ell!}\binom{\ell}{m}\binom{k}{j}(-1)^{\ell-m}(m|\lambda)_{k--j}\frac{\lambda^{k-j}}{\log^{k-j}(1+\lambda)}x^j.$$

Therefore, the polynomials $K_{n,7}(\lambda, x)$ is given by

$$\sum_{j=0}^{n}\left(\sum_{k=j}^{n}\sum_{\ell=0}^{k}\sum_{m=0}^{\ell}\frac{(-1)^{\ell-m}\binom{n-1}{k-1}\binom{\ell}{m}\binom{k}{j}}{\ell!}(m|\lambda)_{k-j}\left(\frac{\log(1+\lambda)}{\lambda}\right)^j\beta_{n-k}^{(n)}(\lambda)K_{\ell,7}(\lambda)\right)x^j.$$

By using similar argument as above with using (1.10), we obtain the formula for the $n$th Korobov polynomial $k_{n,8}(\lambda, x)$ of the eighth kind (we leave the details for the interested reader). □

## 3. RECURRENCES

In this section, we present several recurrences for the Korobov polynomials of the seventh kind and of the eighth kind. Note that, by (1.9), (1.10) and the fact that $(x)_{n,\lambda} \sim (1, f(t))$, we obtain $K_{n,d}(\lambda, x+y) = \sum_{j=0}^{n}\binom{n}{j}K_{j,d}(\lambda, x)(y)_{n-j,\lambda}$, for $d = 7, 8$.

**Proposition 3.1.** *For all $n \geq 1$ and $s = 7, 8$,*

$$K_{n,s}(\lambda, x) + nK_{n-1,s}(\lambda, x)$$

$$= \sum_{m=0}^{n}\left(\sum_{k=m}^{n}\sum_{\ell=k}^{n}\binom{n}{\ell}\binom{k}{m}(1|\lambda)_{k-m}\frac{\log^m(1+\lambda)}{\lambda^m}S_1(\ell, k|\lambda)K_{n-\ell,s}(\lambda)\right)x^m.$$

*Proof.* It is well-known that if $s_n(x) \sim (g(t), f(t))$, then we have $f(t)s_n(x) = ns_{n-1}(x)$ (see [15, 16]). Thus, by (1.9) and (1.10), we obtain $\left(\left(1 + \frac{\lambda^2 t}{\log(1+\lambda)}\right)^{\frac{1}{\lambda}} - 1\right) K_{n,s}(\lambda, x) =$

$nK_{n-1,s}(\lambda, x)$, which implies $K_{n,s}(\lambda, x) + nK_{n-1,s}(\lambda, x) = \left(1 + \frac{\lambda^2 t}{\log(1+\lambda)}\right)^{\frac{1}{\lambda}} K_{n,s}(\lambda, x)$. By Theorem 2.1 we have

$$K_{n,s}(\lambda, x) + nK_{n-1,s}(\lambda, x)$$

$$= \sum_{k=0}^{n} \sum_{\ell=k}^{n} \binom{n}{\ell} \frac{\log^k(1+\lambda)}{\lambda^k} S_1(\ell, k|\lambda) K_{n-\ell,s}(\lambda) \left(1 + \frac{\lambda^2 t}{\log(1+\lambda)}\right)^{\frac{1}{\lambda}} x^k$$

$$= \sum_{k=0}^{n} \sum_{\ell=k}^{n} \sum_{m=0}^{k} \binom{n}{\ell} \frac{\log^{k-m}(1+\lambda)}{\lambda^{k-m}} S_1(\ell, k|\lambda) K_{n-\ell,s}(\lambda)(1|\lambda)_m \frac{t^m}{m!} x^k$$

$$= \sum_{k=0}^{n} \sum_{\ell=k}^{n} \sum_{m=0}^{k} \binom{n}{\ell}\binom{k}{m} \frac{\log^{k-m}(1+\lambda)}{\lambda^{k-m}} S_1(\ell, k|\lambda) K_{n-\ell,s}(\lambda)(1|\lambda)_m x^{k-m}$$

$$= \sum_{m=0}^{n} \left(\sum_{k=m}^{n} \sum_{\ell=k}^{n} \binom{n}{\ell}\binom{k}{m} (1|\lambda)_{k-m} \frac{\log^m(1+\lambda)}{\lambda^m} S_1(\ell, k|\lambda) K_{n-\ell,s}(\lambda)\right) x^m.$$

which completes the proof. $\qquad\qquad\square$

In the next result, we express $\frac{d}{dx}K_{n,7}(\lambda, x)$ and $\frac{d}{dx}K_{n,8}(\lambda, x)$ in terms of $K_{n,7}(\lambda, x)$ and $K_{n,8}(\lambda, x)$, respectively.

**Proposition 3.2.** *For all $n \geq 0$ and $s = 7, 8$,*

$$\frac{d}{dx}K_{n,s}(\lambda, x) = \frac{\log(1+\lambda)}{\lambda^2} \sum_{\ell=0}^{n-1} \binom{n}{\ell} (\lambda)_{n-\ell} K_{\ell,s}(\lambda, x).$$

*Proof.* Note that $\frac{d}{dx}s_n(x) = \sum_{\ell=0}^{n-1} \binom{n}{\ell}\langle \bar{f}(t)|x^{n-\ell}\rangle s_\ell(x)$, for all $s_n(x) \sim (g(t), f(t))$, see [15, 16]. So, for $s_n(x) = K_{n,s}(\lambda, x)$, it remains to compute $A = \langle \bar{f}(t)|x^{n-\ell}\rangle$. By (1.9) and (1.10), we have $A = \frac{\log(1+\lambda)}{\lambda^2}\langle \sum_{j\geq 1}(\lambda)_j \frac{t^j}{j!}|x^{n-\ell}\rangle = \frac{\log(1+\lambda)}{\lambda^2}(\lambda)_{n-\ell}$, which completes the proof. $\qquad\square$

**Theorem 3.3.** *For all $n \geq 1$ and $s = 7, 8$,*

$$K_{n,s}(\lambda, x) = \frac{x\log(1+\lambda)}{\lambda} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (\lambda-1)_{n-1-\ell} K_{\ell,s}(\lambda, x)$$

$$+ \frac{1}{n} \sum_{\ell=0}^{n} \sum_{m=0}^{n-\ell} \binom{n}{\ell} (n-\ell)_{n-\ell-m} u_s(\ell, m),$$

*where*

$$u_7(\ell) = b_\ell \left\{(-\lambda)^{n-\ell-m}(x)_{m,\lambda} - (-1)^{n-\ell-m}K_{m,7}(\lambda, x)\right\},$$

$$u_8(\ell) = K_\ell(\lambda)\left\{(-\lambda)^{n-\ell-m}(x)_{m,\lambda} - \binom{\lambda-1}{n-\ell-m}K_{m,8}(\lambda, x)\right\}.$$

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*Proof.* Since the similarity between $K_{n,7}(\lambda, x)$ and $K_{n,8}(\lambda, x)$ (see (1.9) and (1.10)), we omit the proof of the case $K_{n,8}(\lambda, x)$ and give only the details of the case $K_{n,7}(\lambda, x)$. By (1.9), we have

$$(3.1) \qquad K_{n,7}(\lambda, y) = \left\langle \frac{d}{dt} \left( \frac{\log(1+\lambda t)}{\lambda \log(1+t)} (1+\lambda)^{\frac{y}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}} \right) | x^{n-1} \right\rangle = A + B,$$

where $B = \left\langle \frac{d}{dt} \frac{\log(1+\lambda t)}{\lambda \log(1+t)} (1+\lambda)^{\frac{y}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}} | x^{n-1} \right\rangle$ and $A = \left\langle \frac{\log(1+\lambda t)}{\lambda \log(1+t)} \frac{d}{dt} (1+\lambda)^{\frac{y}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}} | x^{n-1} \right\rangle$.

First, we compute the term $B$.

$$B = \left\langle \frac{\log(1+\lambda t)}{\lambda \log(1+t)} (1+\lambda)^{\frac{y}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}} \frac{\log(1+\lambda)}{\lambda} y(1+t)^{\lambda-1} | x^{n-1} \right\rangle$$

$$= \frac{y \log(1+\lambda)}{\lambda} \left\langle \frac{\log(1+\lambda t)}{\lambda \log(1+t)} (1+\lambda)^{\frac{y}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}} | (1+t)^{\lambda-1} x^{n-1} \right\rangle$$

$$= \frac{y \log(1+\lambda)}{\lambda} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (\lambda-1)_\ell \left\langle \frac{\log(1+\lambda t)}{\lambda \log(1+t)} (1+\lambda)^{\frac{y}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}} | x^{n-1-\ell} \right\rangle$$

$$= \frac{y \log(1+\lambda)}{\lambda} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (\lambda-1)_\ell K_{n-1-\ell,7}(\lambda, y)$$

$$= \frac{y \log(1+\lambda)}{\lambda} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (\lambda-1)_{n-1-\ell} K_{\ell,7}(\lambda, y).$$

Now, we compute the first term $A$,

$$A = \left\langle \frac{t}{\log(1+t)} \frac{1}{t} \left\{ \frac{1}{1+\lambda t} - \frac{\log(1+\lambda t)}{\lambda \log(1+t)} (1+t)^{-1} \right\} (1+\lambda)^{\frac{y}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}} | x^{n-1} \right\rangle$$

$$= \frac{1}{n} \left\langle \left\{ \frac{1}{1+\lambda t} - \frac{\log(1+\lambda t)}{\lambda \log(1+t)} (1+t)^{-1} \right\} (1+\lambda)^{\frac{y}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}} | \frac{t}{\log(1+t)} x^n \right\rangle$$

$$= \frac{1}{n} \left\langle \left\{ \frac{1}{1+\lambda t} - \frac{\log(1+\lambda t)}{\lambda \log(1+t)} (1+t)^{-1} \right\} (1+\lambda)^{\frac{y}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}} | \sum_{\ell \geq 0} b_\ell \frac{t^\ell}{\ell!} x^n \right\rangle.$$

Note that $\frac{1}{1+\lambda t} - \frac{\log(1+\lambda t)}{\lambda \log(1+t)}(1+t)^{-1}$ has order at least one. Thus,

$$A = \frac{1}{n} \sum_{\ell=0}^{n} \binom{n}{\ell} b_\ell \left\{ \left\langle (1+\lambda)^{\frac{y}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}} | \frac{1}{1+\lambda t} x^{n-\ell} \right\rangle \right.$$

$$\left. - \left\langle \frac{\log(1+\lambda t)}{\lambda \log(1+t)} (1+\lambda)^{\frac{y}{\lambda} \frac{(1+t)^\lambda - 1}{\lambda}} | \frac{1}{1+t} x^{n-\ell} \right\rangle \right\}$$

$$= \frac{1}{n} \sum_{\ell=0}^{n} \binom{n}{\ell} b_\ell \left\{ \sum_{m=0}^{n-\ell} (-\lambda)^m (n-\ell)_m \left\langle \sum_{k \geq 0} (y)_{k,\lambda} \frac{t^k}{k!} | x^{n-\ell-m} \right\rangle \right.$$

$$\left. - \sum_{m=0}^{n-\ell} (-1)^m (n-\ell)_m \left\langle \sum_{k \geq 0} K_{k,7}(\lambda, y) \frac{t^k}{k!} | x^{n-\ell-m} \right\rangle \right\},$$

which implies

$$A = \frac{1}{n} \sum_{\ell=0}^{n} \sum_{m=0}^{n-\ell} \binom{n}{\ell} b_\ell \left\{ (-\lambda)^m (n-\ell)_m (y)_{n-\ell-m,\lambda} - (-1)^m (n-\ell)_m K_{n-\ell-m,7}(\lambda, y) \right\},$$

Hence, by substituting the expressions of $A$ and $B$ in (3.1), we complete the proof.   $\square$

## 4. CONNECTIONS WITH FAMILIES OF POLYNOMIALS

In this section, we present some examples on the connections with families of polynomials. To do that, we recall for any two Sheffer sequences $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), \ell(t))$, we have that $s_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x)$, where (see [15, 16])

$$(4.1) \qquad C_{n,m} = \frac{1}{m!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} (\ell(\bar{f}(t)))^m | x^n \right\rangle.$$

We start with the connection to *Bernoulli polynomials* $B_n^{(s)}(x)$ *of order* $s$. Recall that the *Bernoulli polynomials* $B_n^{(s)}(x)$ *of order* $s$ are defined by the generating function $\left(\frac{t}{e^t-1}\right)^s e^{xt} = \sum_{n\geq 0} B_n^{(s)}(x) \frac{t^n}{n!}$, equivalently,

$$(4.2) \qquad B_n^{(s)}(x) \sim \left( \left(\frac{e^t-1}{t}\right)^s, t \right)$$

(see [2, 5, 15]). In the next result, we express our polynomials in terms of Bernoulli polynomials of order $s$.

**Theorem 4.1.** *Let* $d = 7, 8$. *For all* $n \geq 0$, $K_{n,d}(\lambda, x) = \sum_{k=0}^{n} C_{n,m} B_m^{(s)}(x)$, *where*

$$C_{n,m} = \sum_{\ell=0}^{n-m} \sum_{k=0}^{n-\ell-m} \frac{\binom{n}{\ell}\binom{k+m}{m}}{\binom{k+s}{s}} K_{\ell,d}(\lambda) S_2(k+s,s) \frac{\log^{k+m}(1+\lambda)}{\lambda^{k+m}} S_1(n-\ell, k+m|\lambda).$$

*Proof.* Since the similarity between $K_{n,7}(\lambda, x)$ and $K_{n,8}(\lambda, x)$ (see (1.9) and (1.10)), we omit the proof of the case $K_{n,8}(\lambda, x)$ and give only the details of the case $K_{n,7}(\lambda, x)$. Let $K_{n,7}(\lambda, x) = \sum_{m=0}^{n} C_{n,m} B_m^{(s)}(x)$. So, by (1.9) and (4.2), we have

$$
\begin{aligned}
C_{n,m} &= \frac{1}{m!} \left\langle \frac{(e^{\bar{f}(t)}-1)^s}{\bar{f}^s(t)} \frac{\log(1+\lambda t)}{\lambda \log(1+t)} \bar{f}^m(t) | x^n \right\rangle \\
&= \frac{1}{m!} \left\langle \frac{(e^{\bar{f}(t)}-1)^s}{\bar{f}^s(t)} \bar{f}^m(t) \Big| \frac{\log(1+\lambda t)}{\lambda \log(1+t)} x^n \right\rangle \\
&= \frac{1}{m!} \left\langle \frac{(e^{\bar{f}(t)}-1)^s}{\bar{f}^s(t)} \bar{f}^m(t) \Big| \sum_{\ell \geq 0} K_{\ell,7}(\lambda) \frac{t^\ell}{\ell!} x^n \right\rangle \\
&= \frac{1}{m!} \sum_{\ell=0}^{n} \binom{n}{\ell} K_{\ell,7}(\lambda) \left\langle s! \sum_{k \geq 0} S_2(k+s,s) \frac{\bar{f}^{k+m}(t)}{(k+s)!} | x^{n-\ell} \right\rangle.
\end{aligned}
$$

Thus,

$$C_{n,m} = \frac{s!}{m!} \sum_{\ell=0}^{n-m} \sum_{k=0}^{n-\ell-m} \binom{n}{\ell} K_{\ell,7}(\lambda) S_2(k+s,s) \left\langle \frac{\bar{f}^{k+m}(t)}{(k+s)!} \middle| x^{n-\ell} \right\rangle$$

$$= \frac{s!}{m!} \sum_{\ell=0}^{n-m} \sum_{k=0}^{n-\ell-m} \binom{n}{\ell} K_{\ell,7}(\lambda) S_2(k+s,s) \frac{\log^{k+m}(1+\lambda)}{(k+s)!\lambda^{k+m}} \left\langle \frac{((1+t)^\lambda-1)^{k+m}}{\lambda^{k+m}} \middle| x^{n-\ell} \right\rangle$$

$$= \frac{s!}{m!} \sum_{\ell=0}^{n-m} \sum_{k=0}^{n-\ell-m} \binom{n}{\ell} K_{\ell,7}(\lambda) S_2(k+s,s) \frac{\log^{k+m}(1+\lambda)}{(k+s)!\lambda^{k+m}} (k+m)! S_1(n-\ell,k+m|\lambda).$$

Therefore,

$$C_{n,m} = \sum_{\ell=0}^{n-m} \sum_{k=0}^{n-\ell-m} \frac{\binom{n}{\ell}\binom{k+m}{m}}{\binom{k+s}{s}} K_{\ell,7}(\lambda) S_2(k+s,s) \frac{\log^{k+m}(1+\lambda)}{\lambda^{k+m}} S_1(n-\ell,k+m|\lambda),$$

as required.                                                                                       $\square$

Similar techniques as in the proof of the previous theorem, we can express our polynomials $K_{n,7}(\lambda,x), K_{n,8}(\lambda,x)$ in terms of other families. Below we present two examples, where we leave the proofs to the interested reader. In the first example, we express our polynomials in terms of Frobenius-Euler polynomials. Note that the *Frobenius-Euler polynomials* $H_n^{(s)}(x|\mu)$ of order $s$ are defined by the generating function $\left(\frac{1-\mu}{e^t-\mu}\right)^s e^{xt} = \sum_{n\geq 0} H_n^{(s)}(x|\mu)\frac{t^n}{n!}, (\mu \neq 1)$, or equivalently, $H_n^{(s)}(x|\mu) \sim \left(\left(\frac{e^t-\mu}{1-\mu}\right)^s, t\right)$ (see [1, 11, 12]).

**Theorem 4.2.** *For all $n \geq 0$ and $d = 7,8$, $K_{n,d}(\lambda,x) = \sum_{m=0}^n C_{n,m} H_m^{(s)}(x|\mu)$, where*

$$C_{n,m} = \sum_{\ell=0}^{n-m} \sum_{k=0}^s \sum_{j=k}^{n-\ell-m} \frac{k!\binom{j+m}{m}\binom{n}{\ell}\binom{s}{k}}{(1-\mu)^k} \frac{\log^{j+m}(1+\lambda)}{\lambda^{j+m}} S_1(n-\ell,j+m|\lambda) S_2(j,k) K_{\ell,d}(\lambda).$$

For what follows, we define the associated sequence for $1 - (1 + \lambda^2 t/\log(1+\lambda))^{-1/\lambda}$, namely $(x)^{(n,\lambda)}$. Thus,

$$(x)^{(n,\lambda)} \sim (1, 1 - (1 + \lambda^2 t/\log(1+\lambda))^{-1/\lambda}).$$

Recall here that $(x)_n \sim (1, e^t - 1)$, $(x)^{(n)} \sim (1, 1 - e^{-t})$, $(x)_{n,\lambda} \sim (1, (1 + \lambda^2 t/\log(1+\lambda))^{1/\lambda} - 1)$ and $(1 + \lambda^2 t/\log(1+\lambda))^{1/\lambda} - 1 \to e^t - 1$, as $\lambda \to 0$. Now, we ready to present our second example.

**Theorem 4.3.** *For all $n \geq 0$ and $d = 7,8$, $K_{n,d}(\lambda,x) = \sum_{m=0}^n C_{n,m}(x)^{(m,\lambda)}$, where*

$$C_{n,m} = \sum_{\ell=0}^n (-1)^{n-\ell-m} \binom{n}{\ell} \binom{n-\ell}{m} (n-1-\ell)_{n-\ell-m} K_{\ell,d}(\lambda).$$

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# Some identities on the higher-order twisted $q$-Euler numbers and polynomials

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**Abstract :** In this paper we investigate some interesting symmetric identities for twisted $q$-Euler polynomials of higher order in complex field.

**Key words :** Symmetric properties, power sums, Euler numbers and polynomials, twisted $q$-Euler numbers and polynomials.

**2000 Mathematics Subject Classification :** 11B68, 11S40, 11S80.

## 1. Introduction

Euler numbers and polynomials possess many interesting properties and arising in many areas of mathematics, mathematical physics and statistical physics. Many mathematicians have studied in the area of the $q$- extension of Euler numbers and polynomials(see [1, 2, 3, 5, 6, 7, 8, 9, 11, 13]). Recently, Y. Hu studied several identities of symmetry for Carlitz's $q$-Bernoulli numbers and polynomials in complex field(see [3]). D. Kim *et al.*[4] derived some identities of symmetry for $(h, q)$-extension of higher-order Euler numbers and polynomials. D. V. Dolgy *et al.*[2] derived some identities of symmetry for higher-order generalized $q$-Euler polynomials. In this paper, we establish some interesting symmetric identities for twisted $q$-Euler polynomials of higher order in complex field. The purpose of this paper is to present a systemic study of the twisted $q$-Euler numbers and polynomials of higher-order by using the multiple $q$-Euler zeta function. Throughout this paper, the notations $\mathbb{N}, \mathbb{Z}, \mathbb{R},$ and $\mathbb{C}$ denote the sets of positive integers, integers, real numbers, and complex numbers, respectively, and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. We assume that $q \in \mathbb{C}$ with $|q| < 1$. Throughout this paper we use the notation:

$$[x]_q = \frac{1 - q^x}{1 - q} \ (\text{cf. } [1, 2, 3, 5]) \ .$$

Note that $\lim_{q \to 1}[x] = x$. Let $\varepsilon$ be the $p^N$-th root of unity(see [10, 12, 13]).

In [5], T. Kim introduced the multiple $q$-Euler zeta function which interpolates higher-order $q$-Euler polynomials at negative integers as follows:

$$\zeta_{q,r}(s, x) = [2]_q^r \sum_{m_1, \cdots, m_r = 0}^{\infty} \frac{(-1)^{\sum_{j=1}^r m_j} q^{\sum_{j=1}^r m_j}}{[m_1 + \cdots + m_r + x]_q^s}, \tag{1}$$

where $s \in \mathbb{C}$ and $x \in \mathbb{R}$, with $x \neq 0, -1, -2, \ldots$.

Recently, D. V. Dolgy *et al.*[2] considered some symmetric identities for higher-order generalized $q$-Euler polynomials. The Euler polynomials of order $r \in \mathbb{N}$ attached to $\chi$ are also defined by the generating function:

$$\left(2 \sum_{l=0}^{d-1} \frac{\chi(l)(-1)^l e^{(x+l)t}}{e^{dt} + 1}\right)^r = \sum_{m=0}^{\infty} E_{m,\chi}^{(r)}(x) \frac{t^m}{m!}. \tag{2}$$

When $x = 0, E_{n,\chi}^{(r)} = E_{n,\chi}^{(r)}(0)$ are called the Euler numbers $E_{n,\chi}^{(r)}$ attached to $\chi$(see [2, 4]).

For $h \in \mathbb{Z}, \alpha, k \in \mathbb{N}$, and $n \in \mathbb{Z}_+$, we introduced the higher order twisted $q$-Euler polynomials with weight $\alpha$ as follows(see [7]):

$$\widetilde{E}_{n,q,\varepsilon}^{(\alpha)}(k|x) = \frac{[2]_q^k}{(1-q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{\alpha l x}}{(1+\varepsilon q^{\alpha l+h}) \cdots (1+\varepsilon q^{\alpha l+h-k+1})}.$$

In the special case, $x = 0$, $\widetilde{E}_{n,q,w}^{(\alpha)}(k|0) = \widetilde{E}_{n,q,w}^{(\alpha)}(k)$ are called the higher-order twisted $q$-Euler numbers with weight $\alpha$.

We consider the higher order $q$-Euler polynomials of order $r$ attached to $\chi$ twisted by ramified roots of unity as follows(see [10]):

$$\sum_{n=0}^{\infty} E_{n,\chi,\varepsilon,q}^{(r)}(x) \frac{t^n}{n!} = 2^r \sum_{m_1,\ldots,m_r=0}^{\infty} (-\varepsilon)^{\sum_{j=0}^{r} m_j} \left( \prod_{i=1}^{r} \chi(m_i) \right) e^{[x+\sum_{j=1}^{r} m_j]_q t}.$$

In the special case $x = 0$, the sequence $E_{n,\chi,\varepsilon,q}^{(r)}(0) = E_{n,\chi,\varepsilon,q}^{(r)}$ are called the $n$-th $q$-Euler numbers of order $r$ attached to $\chi$ twisted by ramified roots of unity.

As is well known, the higher-order twisted $q$-Euler polynomials $E_{n,q,\varepsilon}^{(k)}(x)$ are defined by the following generating function to be

$$\widetilde{F}_{q,\varepsilon}^{(k)}(t,x) = [2]_q^k \sum_{m_1,\cdots,m_k=0}^{\infty} (-1)^{m_1+\cdots+m_k} \varepsilon^{m_1+\cdots+m_k} e^{[m_1+\cdots+m_k+x]_q t}$$

$$= \sum_{n=0}^{\infty} E_{n,q,\varepsilon}^{(k)}(x) \frac{t^n}{n!}, \tag{3}$$

where $k \in \mathbb{N}$. When $x = 0, E_{n,q,\varepsilon}^{(k)} = E_{n,q,\varepsilon}^{(k)}(0)$ are called the higher-order twisted $q$-Euler numbers $E_{n,q,\varepsilon}^{(k)}$. Observe that if $q \to 1, \varepsilon \to 1$, then $E_{n,q,\varepsilon}^{(k)} \to E_n^{(k)}$ and $E_{n,q,\varepsilon}^{(k)}(x) \to E_{n,\chi}^{(k)}(x)$.

By using (3) and Cauchy product, we have

$$E_{n,q,\varepsilon}^{(k)}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{lx} E_{l,q,\varepsilon}^{(k)} [x]_q^{n-l}$$

$$= (q^x E_{q,\varepsilon}^{(k)} + [x]_q)^n, \tag{4}$$

with the usual convention about replacing $(E_{q,\varepsilon}^{(k)})^n$ by $E_{n,q,\varepsilon}^{(k)}$.

By using complex integral and (3), we can also obtain the multiple twisted $q$-$l$-function as follows:

$$l_{q,\varepsilon}^{(k)}(s,x) = \frac{1}{\Gamma(s)} \int_0^{\infty} \widetilde{F}_{q,\varepsilon}^{(k)}(-t,x) t^{s-1} dt$$

$$= [2]_q^k \sum_{m_1,\cdots,m_k=0}^{\infty} \frac{(-1)^{\sum_{j=1}^{k} m_j} \varepsilon^{\sum_{j=1}^{k} m_j}}{[m_1+\cdots+m_k+x]_q^s}, \tag{5}$$

where $s \in \mathbb{C}$ and $x \in \mathbb{R}$, with $x \neq 0, -1, -2, \ldots$.

By using Cauchy residue theorem, the value of multiple twisted $q$-$l$-function at negative integers is given explicitly by the following theorem:

**Theorem 1.** Let $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$. We obtain

$$l_{q,\varepsilon}^{(k)}(-n,x) = E_{n,q,\varepsilon}^{(k)}(x).$$

The purpose of this paper is to obtain some interesting identities of the power sums and the higher-order twisted $q$-Euler polynomials $E_{n,q,\varepsilon}^{(k)}(x)$ using the symmetric properties for multiple

twisted $q$-$l$-function. In this paper, if we take $\varepsilon = 1$ in all equations of this article, then [2] are the special case of our results.

## 2. Symmetry identities for multiple twisted $q$-$l$-function

In this section, by using the similar method of [2, 3, 4], expect for obvious modifications, we investigate some symmetric identities for higher-order twisted $q$-Euler polynomials $E_{n,q,\varepsilon}^{(k)}(x)$. We assume that $\varepsilon$ be the $p^N$-th root of unity. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $h \in \mathbb{Z}, k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain certain symmetry identities for multiple twisted $q$-$l$-function.

Observe that $[xy]_q = [x]_{q^y}[y]_q$ for any $x, y \in \mathbb{C}$. In (5), we derive next result by substitute $w_2 x + \dfrac{w_2}{w_1}(j_1 + \cdots + j_k)$ for $x$ in and replace $q$ and $\varepsilon$ by $q^{w_1}$ and $\varepsilon^{w_1}$, respectively.

$$
\frac{1}{[2]_{q^{w_1}}^k} l_{q^{w_1},\varepsilon^{w_1}}^{(k)}\left(s, w_2 x + \frac{w_2}{w_1}(j_1 + \cdots + j_k)\right)
$$

$$
= \sum_{m_1,\cdots,m_k=0}^{\infty} \frac{(-1)^{\sum_{j=1}^k m_j} \varepsilon^{w_1 \sum_{j=1}^k m_j}}{[m_1 + \cdots + m_k + w_2 x + \frac{w_2}{w_1}(j_1 + \cdots + j_k)]_{q^{w_1}}^s}
$$

$$
= \sum_{m_1,\cdots,m_k=0}^{\infty} \frac{(-1)^{\sum_{j=1}^k m_j} \varepsilon^{w_1 \sum_{j=1}^k m_j}}{\left[\dfrac{w_1(m_1 + \cdots + m_k) + w_1 w_2 x + w_2(j_1 + \cdots + j_k)}{w_1}\right]_{q^{w_1}}^s}
$$

$$
= \sum_{m_1,\cdots,m_k=0}^{\infty} \frac{(-1)^{\sum_{j=1}^k m_j} \varepsilon^{w_1 \sum_{j=1}^k m_j}}{\dfrac{[w_1(m_1 + \cdots + m_k) + w_1 w_2 x + w_2(j_1 + \cdots + j_k)]_q^s}{[w_1]_q^s}}
$$

$$
= [w_1]_q^s \sum_{m_1,\cdots,m_k=0}^{\infty} \frac{(-1)^{\sum_{j=1}^k m_j} \varepsilon^{w_1 \sum_{j=1}^k m_j}}{[w_1(m_1 + \cdots + m_k) + w_1 w_2 x + w_2(j_1 + \cdots + j_k)]_q^s} \qquad (6)
$$

$$
= [w_1]_q^s \sum_{m_1,\cdots,m_k=0}^{\infty} \sum_{i_1,\cdots,i_k=0}^{w_2-1} \frac{(-1)^{\sum_{j=1}^k m_j} \varepsilon^{w_1 \sum_{j=1}^k m_j}}{[w_1(m_1 + \cdots + m_k) + w_1 w_2 x + w_2(j_1 + \cdots + j_k)]_q^s}
$$

$$
= [w_1]_q^s \sum_{m_1,\cdots,m_k=0}^{\infty} \sum_{i_1,\cdots,i_k=0}^{w_2-1} (-1)^{\sum_{j=1}^k (dw_2 m_j + i_j)}
$$

$$
\times \varepsilon^{w_1 \sum_{j=1}^k (dw_2 m_j + i_j)}
$$

$$
\times \left([w_1(dw_2 m_1 + i_1) + \cdots + w_1(dw_2 m_k + i_k) + w_1 w_2 x + w_2(j_1 + \cdots + j_k)]_q^s\right)^{-1}
$$

$$
= [w_1]_q^s \sum_{m_1,\cdots,m_k=0}^{\infty} \sum_{i_1,\cdots,i_k=0}^{w_2-1} (-1)^{\sum_{j=1}^k m_j} (-1)^{\sum_{j=1}^k i_j}
$$

$$
\times \varepsilon^{dw_1 w_2 \sum_{j=1}^k m_j} \varepsilon^{w_1 \sum_{j=1}^k i_j}
$$

$$
\times \left([w_1 w_2(x + dm_1 + \cdots + dm_k) + w_1(i_1 + \cdots + i_k) + w_2(j_1 + \cdots + j_k)]_q^s\right)^{-1}
$$

Thus, from (6), we can derive the following equation.

$$
\frac{[w_2]_q^s}{[2]_{q^{w_1}}^k} \sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^k j_l} \varepsilon^{w_2 \sum_{l=1}^k j_l}
$$

$$
\times l_{q^{w_1},\varepsilon^{w_1}}^{(k)}\left(s, w_2 x + \frac{w_2}{w_1}(j_1 + \cdots + j_k)\right)
$$

$$
= [w_1]_q^s[w_2]_q^s \sum_{m_1,\cdots,m_k=0}^{\infty} \sum_{i_1,\cdots,i_k=0}^{w_2-1} \sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^k (j_l + i_l + m_l)} \qquad (7)
$$

$$
\times \varepsilon^{dw_1 w_2 \sum_{l=1}^k m_l} \varepsilon^{w_1 \sum_{l=1}^k i_l} \varepsilon^{w_2 \sum_{l=1}^k j_l}
$$

$$
\times \left([w_1 w_2(x + dm_1 + \cdots + dm_k) + w_1(i_1 + \cdots + i_k) + w_2(j_1 + \cdots + j_k)]_q^s\right)^{-1}
$$

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By using the same method as (7), we have

$$
\frac{[w_1]_q^s}{[2]_{q^{w_2}}^k} \sum_{j_1,\cdots,j_k=0}^{w_2-1} (-1)^{\sum_{l=1}^k j_l} \varepsilon^{w_1 \sum_{l=1}^k j_l}
$$

$$
\times l_{q^{w_2},\varepsilon^{w_2}}^{(k)}\left(s, w_1 x + \frac{w_1}{w_2}(j_1 + \cdots + j_k)\right)
$$

$$
= [w_1]_q^s [w_2]_q^s \sum_{m_1,\cdots,m_k=0}^{\infty} \sum_{j_1,\cdots,j_k=0}^{w_2-1} \sum_{i_1,\cdots,i_k=0}^{w_1-1} (-1)^{\sum_{l=1}^k (j_l+i_l+m_l)}
$$

$$
\times \varepsilon^{dw_1 w_2 \sum_{l=1}^k m_l} \varepsilon^{w_2 \sum_{l=1}^k i_l} \varepsilon^{w_1 \sum_{l=1}^k j_l}
$$

$$
\times \left([w_1 w_2(x + dm_1 + \cdots + dm_k) + w_1(j_1 + \cdots + j_k) + w_2(i_1 + \cdots + i_k)]_q^s\right)^{-1}
$$

$$(8)$$

Therefore, by (7) and (8), we have the following theorem.

**Theorem 2.** Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $h \in \mathbb{Z}$, we obtain

$$
[w_2]_q^s [2]_{q^{w_2}}^k \sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^k j_l} \varepsilon^{w_2 \sum_{l=1}^k j_l} l_{q^{w_1},\varepsilon^{w_1}}^{(k)}\left(s, w_2 x + \frac{w_2}{w_1}(j_1 + \cdots + j_k)\right)
$$

$$
= [w_1]_q^s [2]_{q^{w_1}}^k \sum_{j_1,\cdots,j_k=0}^{w_2-1} (-1)^{\sum_{l=1}^k j_l} \varepsilon^{w_1 \sum_{l=1}^k j_l} l_{q^{w_2},\varepsilon^{w_2}}^{(k)}\left(s, w_1 x + \frac{w_1}{w_2}(j_1 + \cdots + j_k)\right)
$$

$$(9)$$

By (9) and Theorem 1, we obtain the following theorem.

**Theorem 3.** Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $h \in \mathbb{Z}, k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$
[w_2]_q^s [2]_{q^{w_2}}^k \sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^k j_l} \varepsilon^{w_2 \sum_{l=1}^k j_l} E_{n,q^{w_1},\varepsilon^{w_1}}^{(k)}\left(w_2 x + \frac{w_2}{w_1}(j_1 + \cdots + j_k)\right)
$$

$$
= [w_1]_q^s [2]_{q^{w_1}}^k \sum_{j_1,\cdots,j_k=0}^{w_2-1} (-1)^{\sum_{l=1}^k j_l} \varepsilon^{w_1 \sum_{l=1}^k j_l} E_{n,q^{w_2},\varepsilon^{w_2}}^{(k)}\left(w_1 x + \frac{w_1}{w_2}(j_1 + \cdots + j_k)\right).
$$

$$(10)$$

From (4), we note that

$$
E_{n,q,\varepsilon}^{(k)}(x + y) = (q^{x+y} E_{n,q,\varepsilon}^{(k)} + [x+y]_q)^n
$$

$$
= \sum_{i=0}^n \binom{n}{i} q^{xi} E_{i,q,\varepsilon}^{(k)}(y)[x]_q^{n-i}.
$$

$$(11)$$

with the usual convention about replacing $(E_{q,\varepsilon}^{(k)})^n$ by $E_{n,q,\varepsilon}^{(k)}$.

By (11), we have

$$
\sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^k j_l} \varepsilon^{w_2 \sum_{l=1}^k j_l} E_{n,q^{w_1},\varepsilon^{w_1}}^{(k)}\left(w_2 x + \frac{w_2}{w_1}(j_1 + \cdots + j_k)\right)
$$

$$
= \sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^k j_l} \varepsilon^{w_2 \sum_{l=1}^k j_l}
$$

$$
\times \sum_{i=0}^n \binom{n}{i} q^{w_2 i(j_1+\cdots+j_k)} E_{i,q^{w_1},\varepsilon^{w_1}}^{(k)}(w_2 x) \left[\frac{w_2}{w_1}(j_1 + \cdots + j_k)\right]_{q^{w_1}}^{n-i}
$$

$$(12)$$

$$
= \sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^k j_l} \varepsilon^{w_2 \sum_{l=1}^k j_l}
$$

$$
\times \sum_{i=0}^n \binom{n}{i} q^{w_2(n-i)\sum_{l=1}^k j_l} E_{n-i,q^{w_1},\varepsilon^{w_1}}^{(k)}(w_2 x) \left[\frac{w_2}{w_1}(j_1 + \cdots + j_k)\right]_{q^{w_1}}^i
$$

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Hence we have the following theorem.

**Theorem 4.** Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$, $w_2 \equiv 1 \pmod{2}$. For $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$\sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^k j_l} \varepsilon^{w_2 \sum_{l=1}^k j_l} E_{n,q^{w_1},\varepsilon^{w_1}}^{(k)} \left( w_2 x + \frac{w_2}{w_1}(j_1 + \cdots + j_k) \right)$$

$$= \sum_{i=0}^n \binom{n}{i} [w_2]_q^i [w_1]_q^{-i} E_{n-i,q^{w_1},\varepsilon^{w_1}}^{(k)} (w_2 x) \sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^k j_l} \varepsilon^{w_2 \sum_{l=1}^k j_l} [j_1 \cdots + j_k]_{q^{w_2}}^i.$$

For each integer $n \geq 0$, let

$$\mathcal{S}_{n,i,q,\varepsilon}^{(k)}(w) = \sum_{j_1,\cdots,j_k=0}^{w-1} (-1)^{\sum_{l=1}^k j_l} \varepsilon^{\sum_{l=1}^k j_l} [j_1 \cdots + j_k]_q^i.$$

The above sum $\mathcal{S}_{n,i,q,\varepsilon}^{(k)}(w)$ is called the alternating $q$-power sums.

By Theorem 4, we have

$$[2]_{q^{w_2}}^k [w_1]_q^n \sum_{j_1,\cdots,j_k=0}^{w_1-1} (-1)^{\sum_{l=1}^k j_l} \varepsilon^{w_2 \sum_{l=1}^k j_l} E_{n,q^{w_1},\varepsilon^{w_1}}^{(k)} \left( w_2 x + \frac{w_2}{w_1}(j_1 + \cdots + j_k) \right)$$

$$= [2]_{q^{w_2}}^k \sum_{i=0}^n \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} E_{n-i,q^{w_1},\varepsilon^{w_1}}^{(k)} (w_2 x) \mathcal{S}_{n,i,q^{w_2},\varepsilon^{w_2}}^{(k)} (w_1) \tag{13}$$

By using the same method as in (13), we have

$$[2]_{q^{w_1}}^k [w_2]_q^n \sum_{j_1,\cdots,j_k=0}^{w_2-1} (-1)^{\sum_{l=1}^k j_l} \varepsilon^{w_1 \sum_{l=1}^k j_l} E_{n,q^{w_2},\varepsilon^{w_2}}^{(k)} \left( w_1 x + \frac{w_1}{w_2}(j_1 + \cdots + j_k) \right)$$

$$= [2]_{q^{w_1}}^k \sum_{i=0}^n \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} E_{n-i,q^{w_2},\varepsilon^{w_2}}^{(k)} (w_1 x) \mathcal{S}_{n,i,q^{w_1},\varepsilon^{w_1}}^{(k)} (w_2) \tag{14}$$

Therefore, by (13) and (14) and Theorem 3, we have the following theorem.

**Theorem 5.** Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$, $w_2 \equiv 1 \pmod{2}$. For $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$[2]_{q^{w_2}}^k \sum_{i=0}^n \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} E_{n-i,q^{w_1},\varepsilon^{w_1}}^{(k)} (w_2 x) \mathcal{S}_{n,i,q^{w_2},\varepsilon^{w_2}}^{(k)} (w_1)$$

$$= [2]_{q^{w_1}}^k \sum_{i=0}^n \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} E_{n-i,q^{w_2},\varepsilon^{w_2}}^{(k)} (w_1 x) \mathcal{S}_{n,i,q^{w_1},\varepsilon^{w_1}}^{(k)} (w_2).$$

By Theorem 5, we obtain the interesting symmetric identity for the higher-order twisted $q$-Euler numbers $E_{n,q,\varepsilon}^{(k)}$ in complex field.

**Corollary 6.** Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$, $w_2 \equiv 1 \pmod{2}$. For $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, we obtain

$$[2]_{q^{w_2}}^k \sum_{i=0}^n \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} \mathcal{S}_{n,i,q^{w_2},\varepsilon^{w_2}}^{(k)} (w_1) E_{n-i,q^{w_1},\varepsilon^{w_1}}^{(k)}$$

$$= [2]_{q^{w_1}}^k \sum_{i=0}^n \binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} \mathcal{S}_{n,i,q^{w_1},\varepsilon^{w_1}}^{(k)} (w_2) E_{n-i,q^{w_2},\varepsilon^{w_2}}^{(k)}.$$

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# UMBRAL CALCULUS ASSOCIATED WITH NEW DEGENERATE BERNOULLI POLYNOMIALS

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ABSTRACT. In this paper, we introduce new degenerate Bernoulli polynomials which are derived from umbral calculus and investigate some interesting properties of those polynomials.

## 1. INTRODUCTION

The Bernoulli polynomials are defined by the generating function

$$(1.1) \qquad \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (\text{see } [1\text{--}14]).$$

When $x = 0$, $B_n = B_n(0)$ are called the ordinary Bernoulli numbers. From (1.1), we note that

$$(1.2) \qquad B_n(x) = \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l}, \quad (n \geq 0), \quad (\text{see } [13]).$$

Thus, by (1.2), we get

$$(1.3) \qquad \frac{d}{dx} B_n(x) = n B_{n-1}(x), \quad (n \in \mathbb{N}).$$

In [2], L. Carlitz introduced the degenerate Bernoulli polynomials which are given by the generating function

$$(1.4) \qquad \frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(x \mid \lambda) \frac{t^n}{n!}.$$

When $x = 0$, $\beta_n(0 \mid \lambda) = \beta_n(\lambda)$ are called Carlitz's degenerate Bernoulli numbers (see [2]).

Thus, by (1.4), we get

$$(1.5) \qquad \beta_n(x \mid \lambda) = \sum_{l=0}^{n} \binom{n}{l} \beta_l(\lambda) (x \mid \lambda)_{n-l}, \quad (n \geq 0),$$

where $(x \mid \lambda)_n = x(x - \lambda) \cdots (x - \lambda(n - 1))$.

Let $\mathbb{C}$ be the field of complex numbers and let $\mathcal{F}$ be the set of all formal power series in the variable $t$ over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \,\middle|\, a_k \in \mathbb{C} \right\}.$$

Let $\mathbb{P} = \mathbb{C}[x]$ and $\mathbb{P}^*$ denotes the vector space of all linear functionals on $\mathbb{P}$. The action of the linear functional $L \in \mathbb{P}^*$ on a polynomial $p(x)$ is denoted by $\langle L | p(x) \rangle$, and linearly extended as $\langle cL + c'L' | p(x) \rangle = c \langle L | p(x) \rangle + c' \langle L' | p(x) \rangle$, where $c$ and $c' \in \mathbb{C}$.

For $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \in \mathcal{F}$, we define a linear functional on $\mathbb{P}$ by setting

(1.6)
$$\langle f(t) | x^n \rangle = a_n$$

for all $n \geq 0$, (see [1, 5, 13]).

Thus, by (1.6), we get

(1.7)
$$\langle t^k | x^n \rangle = n! \delta_{n,k}, \quad (n, k \geq 0), \quad (\text{see } [7, 13]),$$

where $\delta_{n,k}$ is the Kronecker's symbol.

Let $f_L(t) = \sum_{k=0}^{\infty} \langle L | x^k \rangle \frac{t^k}{k!}$. Then we have $\langle f_L(t) | x^n \rangle = \langle L | x^n \rangle$ ($n \geq 0$). The mapping $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ will denote both the algebra of formal power series in $t$ and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power seires and a linear functional. We shall call $\mathcal{F}$ the umbral algebra. The umbral calculus is the study of umbral algebra and can be also desribed as a systematic study of the class of Sheffer sequences. The order $o(f)$ of the non-zero power series $f(t)$ is the smallest integer $k$ for which the coefficient of $t^k$ does not vanish (see [12, 13]).

For $f(t), g(t) \in \mathcal{F}$ with $o(f) = 1$ and $o(g) = 0$, there exists a unique sequence $s_n(x)$ of polynomials such that

$$\left\langle g(t) f(t)^k \middle| s_n(x) \right\rangle = n! \delta_{n,k}, \quad (n, k \geq 0).$$

The sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$ (see [10, 13]).

Let $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. Then by (1.7), we get

(1.8)
$$\left\langle e^{yt} \middle| p(x) \right\rangle = p(y), \quad \langle f(t) g(t) | p(x) \rangle = \langle g(t) | f(t) p(x) \rangle,$$

and

(1.9)
$$f(t) = \sum_{k=0}^{\infty} \langle f(t) | x^k \rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \langle t^k | p(x) \rangle \frac{x^k}{k!}, \quad (\text{see } [13]).$$

By (1.9), we easily get

(1.10)
$$p^{(k)}(0) = \langle t^k | p(x) \rangle = \left\langle 1 \middle| p^{(k)}(x) \right\rangle, \quad (k \geq 0),$$

where $p^{(k)}(0)$ denotes the $k$-th derivative of $p(x)$ with respect to $x$ at $x = 0$.

From (1.10), we have

(1.11)
$$t^k p(x) = p^{(k)}(x).$$

In [13], it is known that

(1.12)
$$s_n(x) \sim (g(t), f(t)) \iff \frac{1}{g(\overline{f}(t))} e^{x\overline{f}(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!},$$

where $\overline{f}(t)$ is the compositional inverse of $f(t)$ such that $f(\overline{f}(t)) = \overline{f}(f(t)) = t$.

From (1.7), we can easily derive

(1.13)
$$e^{yt} p(x) = p(x + y), \quad \text{where } p(x) \in \mathbb{P} = \mathbb{C}[x].$$

For $p(x) \in \mathbb{P}$, we have

$$\left\langle \left. \frac{e^{yt} - 1}{t} \right| p(x) \right\rangle = \int_0^y p(u)\, du, \quad \langle f(t) | xp(x) \rangle = \langle \partial_t f(t) | p(x) \rangle.$$

Let $f_1(t), f_2(t), \ldots, f_m(t) \in \mathcal{F}$. Then we have

$$(1.14) \quad \langle f_1(t) f_2(t) \cdots f_m(t) | x^n \rangle = \sum \binom{n}{i_1, \ldots, i_m} \langle f_1(t) | x^{i_1} \rangle \cdots \langle f_m(t) | x^{i_m} \rangle$$

where the sum is over all nonnegative integers $i_1, \ldots, i_m$ such that $i_1 + \cdots + i_m = n$.

In this paper, we introduce new degenerate Bernoulli polynomials which are different Carlitz's degenerate Bernoulli polynomials and investigate some interesting properties of those polynomials.

## 2. Umbral calculus and degenerate Bernoulli polynomials

From (1.1) and (1.13), we have

$$(2.1) \qquad\qquad B_n(x) \sim \left( \frac{e^t - 1}{t}, t \right), \quad (n \geq 0).$$

Now, we introduce the new degenerate Bernoulli polynomials which are derived from Sheffer sequence as follows:

$$(2.2) \qquad\qquad \beta_{n,\lambda}(x) \sim \left( \frac{(1 + \lambda t)^{\frac{1}{\lambda}} - 1}{t}, t \right), \quad (n \geq 0).$$

From (1.12) and (2.2), we have

$$(2.3) \qquad\qquad \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!} = \frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} e^{xt}.$$

When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers.

Note that

$$(2.4) \qquad \sum_{n=0}^{\infty} \lim_{\lambda \to 0} \beta_{n,\lambda}(x) \frac{t^n}{n!} = \lim_{\lambda \to 0} \frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} e^{xt}$$

$$= \frac{t}{e^t - 1} e^{xt}$$

$$= \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Thus, by (2.4), we get

$$(2.5) \qquad\qquad \lim_{\lambda \to 0} \beta_{n,\lambda}(x) = B_n(x), \quad (n \geq 0).$$

From (2.3), we have

$$(2.6) \qquad \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!} = \left( \sum_{l=0}^{\infty} \beta_{l,\lambda} \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} \frac{x^m}{m!} t^m \right)$$

$$= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} \beta_{l,\lambda} x^{n-l} \right) \frac{t^n}{n!}.$$

Thus, by (2.6), we get

$$\beta_{n,\lambda}(x) = \sum_{l=0}^{n} \binom{n}{l} \beta_{l,\lambda} x^{n-l}, \quad (n \geq 0),$$

and

$$(2.7) \qquad \frac{d}{dx} \beta_{n,\lambda}(x) = \sum_{l=1}^{n} \binom{n}{l} \beta_{l,\lambda}(n-l) x^{n-l-1}$$

$$= n \sum_{l=0}^{n-1} \binom{n-1}{l} \beta_{l,\lambda} x^{n-1-l}$$

$$= n \beta_{n-1,\lambda}(x).$$

From (1.11) and (2.3), we have

$$(2.8) \qquad \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} x^n = \beta_{n,\lambda}(x), \quad (n \geq 0),$$

and

$$(2.9) \qquad t\beta_{n,\lambda}(x) = \frac{d}{dx} \beta_{n,\lambda}(x) = n\beta_{n-1,\lambda}(x), \quad (n \geq 1).$$

Thus, by (2.8) and (2.9), we get

$$(2.10) \qquad \int_x^{x+y} \beta_{n,\lambda}(u)\, du$$

$$= \frac{1}{n+1} \{\beta_{n+1,\lambda}(x+y) - \beta_{n+1,\lambda}(x)\}$$

$$= \frac{e^{yt} - 1}{t} \beta_{n,\lambda}(x)$$

$$= \sum_{k=1}^{\infty} \frac{y^k}{k!} t^{k-1} \beta_{n,\lambda}(x).$$

From (2.9), we have

$$(2.11) \qquad \beta_n(x) = t \left\{ \frac{1}{n+1} \beta_{n+1,\lambda}(x) \right\}.$$

Thus, by (2.11), we get

$$(2.12) \qquad \left\langle \frac{e^{yt} - 1}{t} \middle| \beta_{n,\lambda}(x) \right\rangle = \left\langle e^{yt} - 1 \middle| \frac{1}{n+1} \beta_{n+1,\lambda}(x) \right\rangle$$

$$= \int_0^y \beta_{n,\lambda}(u)\, du.$$

Therefore, by (2.12), we obtain the following theorem.

**Theorem 1.** *For $n \geq 0$, we have*

$$\left\langle \frac{e^{yt} - 1}{t} \middle| \beta_{n,\lambda}(x) \right\rangle = \int_0^y \beta_{n,\lambda}(u)\, du.$$

                                       

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For $r \in \mathbb{N}$, the degenerate Bernoulli polynomials of order $r$ are defined by the generating function

$$(2.13) \qquad \left( \frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(r)} (x) \frac{t^n}{n!}.$$

When $x = 0$, $\beta_{n,\lambda}^{(r)} = \beta_{n,\lambda}^{(r)} (0)$ are called the higher order degenerate Bernoulli numbers.

Indeed, $\lim_{\lambda \to 0} \beta_{n,\lambda}^{(r)} (x) = B_n^{(r)} (x)$, where $B_n^{(r)} (x)$ are the higher-order Bernoulli polynomials which are defined by the generating function

$$\left( \frac{t}{e^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)} (x) \frac{t^n}{n!}.$$

From (2.13), we have

$$(2.14) \qquad \beta_{n,\lambda}^{(r)} (x) = \sum_{l=0}^{n} \binom{n}{l} \beta_{l,\lambda}^{(r)} x^{n-l}, \quad (n \geq 0),$$

and

$$(2.15) \qquad \frac{d}{dx} \beta_{n,\lambda}^{(r)} (x) = \sum_{l=0}^{n} \binom{n}{l} \beta_{l,\lambda}^{(r)} (n - l) x^{n-l-1}$$

$$= n \sum_{l=0}^{n-1} \binom{n-1}{l} \beta_{l,\lambda}^{(r)} x^{n-l-1}$$

$$= n \beta_{n-1,\lambda}^{(r)} (x), \quad (n \geq 1).$$

By (2.8) and (2.13), we easily get

$$(2.16) \qquad \beta_{n,\lambda}^{(r)} = \sum_{l_1 + \cdots + l_r = n} \binom{n}{l_1, \ldots, l_r} \beta_{l_1,\lambda} \cdots \beta_{l_r,\lambda}.$$

Thus, by (2.14) and (2.16), we see that $\beta_{n,\lambda}^{(r)} (x)$ is a monic polynomial of degree $n$ with coefficients in $\mathbb{Q} (\lambda)$.

From (2.14) and (2.15), we can derive

$$(2.17) \qquad \int_x^{x+y} \beta_{n,\lambda}^{(r)} (u) \, du = \frac{1}{n+1} \left\{ \beta_{n+1,\lambda}^{(r)} (x + y) - \beta_{n+1,\lambda}^{(r)} (x) \right\}$$

$$= \frac{e^{yt} - 1}{t} \beta_{n,\lambda}^{(r)} (x).$$

If $s_n (x) \sim (g (t), t)$, then $s_n (x)$ is called an Appell sequence.

From (1.12) and (2.13), we have

$$(2.18) \qquad \beta_{n,\lambda}^{(r)} (x) \sim \left( \left( \frac{(1 + \lambda t)^{\frac{1}{\lambda}} - 1}{t} \right)^r, t \right), \quad (n \geq 0).$$

Thus, by (2.18), we note that $\beta_{n,\lambda}^{(r)} (x)$ is the Appell sequence for $\left( \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} \right)^r$.

From (2.18), we have

$$(2.19) \qquad \left( \frac{(1 + \lambda t)^{\frac{1}{\lambda}} - 1}{t} \right)^r \beta_{n,\lambda}^{(r)} (x) \sim (1, t), \quad x^n \sim (1, t), \quad (n \geq 0).$$

Thus, by (2.19), we get

$$
(2.20) \qquad x^n = \left( \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} \right)^r \beta_{n,\lambda}^{(r)}(x), \quad (n \geq 0).
$$

We observe that

$$
(2.21) \quad \left( \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} \right)^r
$$

$$
= \frac{1}{t^r} \left( e^{\frac{1}{\lambda} \log(1+\lambda t)} - 1 \right)^r
$$

$$
= \frac{1}{t^r} r! \sum_{l=r}^{\infty} S_2(l,r) \lambda^{-l} \frac{(\log(1+\lambda t))^l}{l!}
$$

$$
= \frac{1}{t^r} r! \sum_{l=0}^{\infty} S_2(l+r,r) \lambda^{-(l+r)} \frac{1}{(l+r)!} (l+r)! \sum_{n=l+r}^{\infty} S_1(n,l+r) \frac{\lambda^n t^n}{n!}
$$

$$
= \frac{1}{t^r} r! \sum_{l=0}^{\infty} S_2(l+r,r) \lambda^{-(l+r)} \sum_{n=l}^{\infty} S_1(n+r,l+r) \frac{\lambda^{n+r}}{(n+r)!} t^{n+r}
$$

$$
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} S_2(l+r,r) S_1(n+r,l+r) \lambda^{n-l} \frac{1}{\binom{n+r}{r}} \right) \frac{t^n}{n!}
$$

By (2.20) and (2.21), we get
(2.22)

$$
x^m = \sum_{n=0}^{m} \sum_{l=0}^{n} S_2(l+r,r) S_1(n+r,l+r) \lambda^{n-l} \frac{\binom{m}{n}}{\binom{n+r}{r}} \beta_{m-n,\lambda}^{(r)}(x), \quad (m \geq 0).
$$

Therefore, by (2.22), we obtain the following theorem.

**Theorem 2.** *For $m \geq 0$, we have*

$$
x^m = \sum_{n=0}^{m} \sum_{l=0}^{n} S_2(l+r,r) S_1(n+r,l+r) \lambda^{n-l} \frac{\binom{m}{n}}{\binom{n+r}{r}} \beta_{m-n,\lambda}^{(r)}(x),
$$

*where $S_1(m,n)$ and $S_2(m,n)$ are the Stirling numbers of the first kind and of the second kind defined by*

$$
(x)_n = \sum_{l=0}^{n} S_1(n,l) x^l,
$$

$$
x^n = \sum_{l=0}^{n} S_2(n,l) (x)_l.
$$

From (1.11) and (2.18), we have

$$
(2.23) \qquad t\beta_{n,\lambda}^{(r)}(x) = t \left\{ \frac{1}{n+1} \beta_{n+1,\lambda}^{(r)}(x) \right\}, \quad (n \geq 0),
$$

and

$$
(2.24) \qquad \left\langle \frac{e^{yt}-1}{t} \,\middle|\, \beta_{n,\lambda}^{(r)}(x) \right\rangle = \left\langle e^{yt}-1 \,\middle|\, \frac{1}{n+1} \beta_{n+1,\lambda}^{(r)}(x) \right\rangle
$$

$$= \int_0^y \beta_{n,\lambda}^{(r)}(u)\, du.$$

Moreover,

(2.25) $\beta_{n,\lambda}^{(r)}$

$$= \left\langle \left( \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \right)^r \middle| x^n \right\rangle$$

$$= \sum_{n=i_1+\cdots+i_r} \binom{n}{i_1,\ldots,i_r} \left\langle \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \middle| x^{i_1} \right\rangle \cdots \left\langle \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \middle| x^{i_r} \right\rangle$$

and

(2.26) $$\beta_{n,\lambda} = \left\langle \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \middle| x^n \right\rangle, \quad (n \geq 0).$$

Therefore, by (2.24), (2.25) and (2.26), we obtain the following theorem.

**Theorem 3.** *For $n \geq 0$, we have*

$$\left\langle \frac{e^{yt} - 1}{t} \middle| \beta_{n,\lambda}^{(r)}(x) \right\rangle = \int_0^y \beta_{n,\lambda}^{(r)}(u)\, du,$$

*and*

$$\beta_{n,\lambda}^{(r)} = \sum_{n=i_1+\cdots+i_r} \binom{n}{i_1,\ldots,i_r} \beta_{i_1,\lambda} \cdots \beta_{i_r,\lambda}.$$

Let $\mathbb{P}_n = \{ p(x) \in \mathbb{C}[x] |\, \deg p(x) \leq n \}, \quad (n \geq 0)$. For $p(x) \in \mathbb{P}_n$, we assume that

(2.27) $$p(x) = \sum_{k=0}^n b_k \beta_{k,\lambda}(x).$$

From (2.2), we have

(2.28) $$\left\langle \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} t^k \middle| \beta_{n,\lambda}(x) \right\rangle = n! \delta_{n,k}, \quad (n, k \geq 0).$$

Thus, by (2.27) and (2.28), we get

(2.29) $$\left\langle \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} t^k \middle| p(x) \right\rangle = \sum_{l=0}^n b_l \left\langle \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} t^k \middle| \beta_{l,\lambda}(x) \right\rangle$$

$$= \sum_{l=0}^n b_l l! \delta_{l,k} = k! b_k.$$

Hence,

(2.30) $$b_k = \frac{1}{k!} \left\langle \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} t^k \middle| p(x) \right\rangle$$

$$= \frac{1}{k!} \left\langle \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} \middle| p^{(k)}(x) \right\rangle,$$

where $p^{(k)}(x) = \frac{d^k}{dx^k} p(x)$.

Therefore, by (2.27) and (2.30), we obtain the following theorem.

**Theorem 4.** *Let $p(x) \in \mathbb{P}_n$. Then we have*

$$p(x) = \sum_{k=0}^{n} b_k \beta_{k,\lambda}(x),$$

*where*

$$b_k = \frac{1}{k!} \left\langle \left. \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} \right| p^{(k)}(x) \right\rangle.$$

Let $p(x) \in \mathbb{P}_n$ with $p(x) = \beta_{n,\lambda}^{(r)}(x)$. Then, we have

$$(2.31) \qquad p^{(k)}(x) = \left(\frac{d}{dx}\right)^k \beta_{n,\lambda}^{(r)}(x) = k! \binom{n}{k} \beta_{n-k,\lambda}^{(r)}(x).$$

Let us assume that

$$(2.32) \qquad p(x) = \beta_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{n} b_k \beta_{k,\lambda}(x).$$

Then, by Theorem 5, we get

$$
\begin{aligned}
(2.33) \qquad b_k &= \frac{1}{k!} \left\langle \left. \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} \right| p^{(k)}(x) \right\rangle \\
&= \binom{n}{k} \left\langle \left. \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} \right| \beta_{n-k,\lambda}^{(r)}(x) \right\rangle \\
&= \binom{n}{k} \left\langle \left. \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} \right| \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}\right)^r x^{n-k} \right\rangle \\
&= \binom{n}{k} \left\langle 1 \left| \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}\right)^{r-1} x^{n-k} \right. \right\rangle \\
&= \binom{n}{k} \beta_{n-k,\lambda}^{(r-1)}.
\end{aligned}
$$

Therefore, by (2.32) and (2.33), we obtain the following theorem.

**Theorem 5.** *For $r \in \mathbb{N}$ and $n \geq 0$, we have*

$$\beta_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{n} \binom{n}{k} \beta_{n-k,\lambda}^{(r-1)} \beta_{k,\lambda}(x).$$

Let $p(x) \in \mathbb{P}_n$ with $p(x) = \sum_{k=0}^{n} b_k^{(r)} \beta_{k,\lambda}^{(r)}(x)$. By (2.18), we get

$$
\begin{aligned}
(2.34) \qquad & \left\langle \left. \left(\frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t}\right)^r t^k \right| p(x) \right\rangle = \sum_{l=0}^{n} b_l^{(r)} \left\langle \left. \left(\frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t}\right)^r t^k \right| \beta_{l,\lambda}^{(r)}(x) \right\rangle \\
& \qquad\qquad = \sum_{l=0}^{n} b_l^{(r)} l! \delta_{l,k} = k! b_k^{(r)}.
\end{aligned}
$$

Thus, by (2.34), we get

$$(2.35) \qquad b_k^{(r)} = \frac{1}{k!} \left\langle \left( \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} \right)^r t^k \, \middle| \, p(x) \right\rangle.$$

**Theorem 6.** *For $p(x) \in \mathbb{P}_n$, we have*

$$p(x) = \sum_{k=0}^{n} b_k^{(r)} \beta_{k,\lambda}^{(r)}(x),$$

*where*

$$b_k^{(r)} = \frac{1}{k!} \left\langle \left( \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} \right)^r t^k \, \middle| \, p(x) \right\rangle$$

$$= \frac{1}{k!} \left\langle \left( \frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} \right)^r \, \middle| \, p^{(k)}(x) \right\rangle,$$

*where $p^{(k)}(x) = \left( \frac{d}{dx} \right)^k p(x)$.*

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# Regularization Smoothing Approximation of Fuzzy Parametric Variational Inequality Constrained Stochastic Optimization

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**Abstract.** This work is motivated by the fact that very little is known about the fuzzy parametric variational inequalities constrained stochastic optimization problems in finite dimension real numeral spaces, which are studied more difficult because of the existence of random variable and fuzzified version. Based on the notion of quasi-Monte Carlo estimate and method of centres with entropic regularization, we develop a class of new regularization smoothing approximation approaches to discretize the stochastic optimization problem with continuous random variable, and construct a centre iterative algorithm for approximating the optimal solutions of the stochastic optimization problems. Further, we give some comprehensive convergence theorems of optimal solutions for the resulting optimization problem. Finally, a numerical illustration is analyzed.

**Key Words and Phrases.** Regularization smoothing approximation, fuzzy parametric variational inequality, Stochastic optimization problem, centre iterative algorithm with quasi-Monte Carlo estimate, comprehensive convergence.

**AMS Subject Classification.** 49J40, 65K05, 90C30, 90C33

## 1 Introduction

As all we know, mathematical program with equilibrium constraints is a constrained optimization problem in which the essential constraints are defined by a parametric variational inequality. This class of problems can be regarded as a generalization of a bilevel programming problem and it therefore plays an important role in many fields such as transportation, communication networks, structural mechanics, economic equilibrium, multilevel game, and mathematical programming itself. See, for example, [1–7] and the reference therein. Moreover, in order to describe the uncertainties, Monica [5] considered the Bochner integrability setting, a measure space of indices and use random fuzzy mappings, and presented random fixed point theorems with random fuzzy mappings, extensions of the ones with random data.

In this paper, we study approximation of optimal solutions for the following fuzzy parametric variational inequality constrained stochastic optimization problem in $n$-dimension real numeral set $\mathbb{R}^n$:

$$
\begin{aligned}
\min_{x,y(\cdot)} \quad & E_\omega[f(x,y(\omega),\omega)] \\
\text{s.t.} \quad & x \in U \subset \mathbb{R}^n, \\
& y(\omega) \in C(x,\omega) \\
& \langle F(x,y(\omega),\omega), z(\omega) - y(\omega) \rangle \gtreqless 0, \text{ for all } z(\omega) \in C(x,\omega) \text{ and a.e. } \omega \in \Omega,
\end{aligned} \tag{1.1}
$$

where $E_\omega$ denotes the mathematical expectation with respect to the random variable $\omega \in \Omega$ on probability space $(\Omega, \mathcal{A}, \Gamma)$, $f : \mathbb{R}^{n+m} \times \Omega \to \mathbb{R}$ and $F : \mathbb{R}^{n+m} \times \Omega \to \mathbb{R}^m$ are two nonlinear random functions, $C : \mathbb{R}^n \times \Omega \to 2^{\mathbb{R}^m}$ is a multi-valued random function, $\langle F(x,y(\omega),\omega), z(\omega) - y(\omega) \rangle \gtreqless 0$ are fuzzy inequalities (also called fuzzy stochastic variational inequality problems, in short, $\widetilde{VI_\omega}$), "$\gtreqless$"

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denotes the fuzzified version of "$\geq$" with the linguistic interpretation "approximately greater than or equal to", and "a.e." is the abbreviation for almost everywhere".

**Remark 1.1.** Problem (1.1) is brand new in the literature including and can be thought as a generalized version of some problems, includes a number of stochastic mathematical program with equilibrium constraints (SMPEC), mathematical programs with fuzzy equilibrium constraints (MPFEC) mathematical program with equilibrium constraints (MPEC) and mathematical program with complementarity constraints (MPCC) have been studied by many authors as special cases. See, for example, [1–4, 6, 8–14] and the references therein, and the following examples.

**Example 1.1** If $\Omega$ is a singleton, then problem (1.1) reduces to the following MPFEC:

$$\begin{aligned}
\min \quad & g(x, u) \\
\text{s.t.} \quad & x \in U, \\
& u \text{ solves } \widetilde{VI}\left(G(x, \cdot), D(x)\right),
\end{aligned} \tag{1.2}$$

where $g : \mathbb{R}^{n+m} \to \mathbb{R}$ and $G : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is a continuously differentiable function, $D : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is a set valued function, and $u$ solves $\widetilde{VI}(G(x, \cdot), D(x))$ if and only if $u \in D(x)$ and $\langle G(x, u), z - u \rangle \gtrsim 0$ for all $z \in D(x)$. Problem (1.2) was introduced and studied by Hu and Liu [12] and Lan et al. [13]. Moreover, Hu and Liu [12] pointed out "although a powerful theory has been developed for variational inequalities, the parameterized setting in MPEC makes these problems very difficult to solve, and due to the vagueness involved in real world problems, the MPEC problem in a fuzzy environment becomes an important problem both in theory and in practice", and "problem (1.2) is a constrained optimization problem whose constraints include some fuzzy parametric variational inequalities.

In 2013, inspired by the works of Hu and Liu [12] and other researchers, Lan et al. [13] constructed an iterative algorithm for finding a solution of a class of mathematical program problems with fuzzy parametric variational inequality constraints by using a new smoothing approach based on a version of the method of centres with entropic regularization techniques. In fact, the tolerance approach and entropic regularization technique have been successfully proposed in solving various problems, which are important numerical methods for solving fuzzy variational inequalities in a fuzzy environment and nonlinear semi-infinite programming problems. See, for example, [3, 8, 14–21] and the references therein.

**Example 1.2.** Since a solution satisfying a fuzzy inequality system to a membership degree close to 1 is a near optimal solution to the corresponding regular inequality problem [22], if $y(\omega) \equiv v$ for all $\omega \in \Omega$ and the degree for the fuzzy inequalities in (1.1) is close to 1, then problem (1.1) is equivalent to the following SMPEC:

$$\begin{aligned}
\min \quad & E_\omega[f(x, v, \omega)] \\
\text{s.t.} \quad & x \in U, \ \omega \in \Omega \\
& v \text{ solves } VI(F(x, \cdot, \omega), C(x, \omega)),
\end{aligned} \tag{1.3}$$

where $VI(F(x, \cdot, \omega), C(x, \omega))$ denotes the variational inequality problem defined by the pair $(F(x, \cdot, \omega), C(x, \omega))$ for all $x \in \mathbb{R}^n$ and $\omega \in \Omega$. In 2003, Lin et al. [9] considered problem (1.3) and showed that SMPEC can be thought as a generalization of MPEC, and proposed a smoothing implicit programming method to establish a comprehensive convergence theory for the lower-level wait-and-see model. Further, there are many stochastic formulations of MPEC proposed in the recent discussions. For related works, we refer readers to [1, 3, 6, 8, 10, 11]. However, there has been very little study on applications of these theories and approaches to (1.1).

Over years of development, optimization approaches have become one of the most promising techniques for engineering applications and an MPEC is a hard problem because its constraints fail to satisfy a standard constraint qualification at any feasible point [23]. However, since the existence of the random variable $\omega$ and the fuzzified version "$\gtrsim$" mean that (1.1) involves multiple complementarity-type constraints, it is more difficult to solve problem (1.1) than to solve an ordinary MPCC, MPEC, MPFEC or SMPEC generally. Therefore, our focus in this paper is to develop a class of new regularization smoothing approximation approaches to define some parameters of the objective function fuzzy yielded by fuzzy constraints, and consider the approximation-solvability for an equivalent stochastic parametric optimization problem of problem (1.1).

<center>2</center>

Motivated and inspired by the above works, we shall give some preliminaries needed throughout the whole paper in Section 2. Specially, by using the notion of tolerance approach and the fuzzy set theory, we show that the fuzzy parametric variational inequality constrained stochastic optimization problem (1.1) and a fuzzy complementarity constrained optimization problem can be converted to a regular nonlinear parametric optimization problem. In Sections 3, we will construct a centre iterative algorithm and develop a class of new regularization smoothing approximation approach for solving the stochastic fuzzy optimization based on quasi-Monte Carlo estimate, and establish comprehensive convergence theorems of the solution. We also report some numerical simulation analysis results in Section 4.

## 2    Preliminaries

Throughout in this paper, we assumption that $(\Omega, \mathcal{A}, \Gamma)$ is a complete $\sigma$-finite measure space and the probability measure $\Gamma$ of our considered space $(\Omega, \mathcal{A}, \Gamma)$ is non-atomic. Let $\mathcal{B}(\mathbb{R}^m)$ be the class of Borel $\sigma$-fields in $\mathbb{R}^m$ and $P(U)$ denote the power set of a vector space $U$.

**Definition 2.1.** (i) A function $y : \Omega \to \mathbb{R}^m$ is said to be measurable, if for any $B \in \mathcal{B}(\mathbb{R}^m)$, $\{\omega \in \Omega : y(\omega) \in B\} \in \mathcal{A})$.

(ii) The multi-valued function $\Psi : \Omega \to P(U)$ is called said to be measurable, if for any $B \in \mathcal{B}(U)$, $\Psi^{-1}(B) = \{\omega \in \Omega : \Psi(\omega) \cap B \neq \emptyset\} \in \mathcal{A}$.

(iii) A multi-valued random function $\Phi : \mathbb{R}^n \times \Omega \to 2^{\mathbb{R}^m}$ is said to be measurable, if for any $x \in \mathbb{R}^n$, $\Phi(x, \cdot)$ is measurable.

(iv) $F : \mathbb{R}^{n+m} \times \Omega \to \mathbb{R}^m$ is called a random and continuously differentiable function, when $F(x, z, \omega) = \zeta(\omega)$ is measurable for any $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$, and $F(\cdot, \cdot, \omega)$ is continuously differentiable for all $\omega \in \Omega$.

**Definition 2.2.** Let $C, C^* : \mathbb{R}^n \times \Omega \to 2^{\mathbb{R}^m}$ be two multi-valued random function. Then

(i) $C(x, \omega)$ is said to be convex cone, if $C(x, \cdot)$ is convex cone for every $x \in \mathbb{R}^n$, that is,

$$\alpha y(\cdot) + \beta w(\cdot) \in C(x, \cdot) \text{ for any positive scalars } \alpha, \beta \text{ and all measurable function } y(\cdot), w(\cdot) \in C(x, \cdot);$$

(ii) $C^*(x, \omega)$ is called polar (dual) cone of $C(x, \omega) \subset \mathbb{R}^m$ for $x \in \mathbb{R}^n$ and $\omega \in \Omega$, if $C^*(x, \cdot)$ is polar (dual) cone for every $x \in \mathbb{R}^n$, i.e.

$$\langle \xi, \nu(\cdot) \rangle \geq 0 \quad \forall \xi \in \mathbb{R}^m \text{ and for each measurable function } \nu(\cdot) \in C(x, \cdot).$$

In other words, the polar (dual) cone $C^*(x, \omega)$ can be expressed as follows:

$$C^*(x, \omega) = \{\xi \in \mathbb{R}^m \| \langle \xi, \nu(\omega) \rangle \geq 0 \quad \forall \nu(\omega) \in C(x, \omega)\}.$$

**Definition 2.3.** Let $\sigma > 0$ and $\varsigma > 0$ be constants. a function $M : \mathbb{R}^n \to \mathbb{R}^m$ is said to be Hölder continuous on $K \subset \mathbb{R}^m$ with order $\sigma$ and Hölder constant $\varsigma$ if

$$\|M(u) - M(v)\| \leq \varsigma \|u - v\|^\sigma, \quad \forall u, v \in K.$$

holds for all $u$ and $v$ in $K$.

**Remark 2.1.** If $\sigma = 1$, then the definition of Hölder continuity reduces to definition of Lipschitz continuity. We note that for two different positive numbers $\sigma$ and $\sigma'$, Hölder continuous functions with order $\sigma$ and those with order $\sigma'$ constitute different subclasses. For example, the function $M(u) := \sqrt{\|u\|}$ for all $u \in K \subset \mathbb{R}^m$ is Hölder continuous with order $\sigma = \frac{1}{2}$, but not Lipschitz continuous.

In the sequel, we give some preparations needed later to approximating the optimal solutions of problem (1.1). First, we propose discretization of the stochastic objective function in (1.1) with continuous random variable.

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**Lemma 2.1.** Let $\zeta : \Omega \to [0, +\infty)$ be the continuous probability density function of $\omega$. Then the objective function in (1.1) can be represented as

$$E_\omega(f(x, y(\omega), \omega) = \frac{1}{L} \sum_{\omega \in \Omega_L} f(x, y(\omega), \omega)\zeta(\omega), \qquad (2.1)$$

where $\Omega_L := \{\omega_1, \omega_2, \cdots, \omega_L\}$ is a uniformly distributed sample set from $\Omega$.

*Proof.* Let $\Omega$ be a sample space, which is usually denoted using set notation, and the possible outcomes are listed as elements in the set. If $\Omega$ is unbounded, under some mild conditions, we can approximate the problem by a sequence of programs with bounded sampling spaces (see [11]) for more details. In the sequel, let $\Omega$ be a bounded rectangle. In particular, without loss of generality, we assume that $\Omega = [0, 1]^\kappa$. Let $\zeta : \Omega \to [0, +\infty)$ be the continuous probability density function of $\omega$. Then the objective function in (1.1) can be represented as

$$E_\omega(f(x, y(\omega), \omega) = \int_\Omega f(x, y(\omega), \omega)\zeta(\omega)d\omega.$$

Based on quasi-Monte Carlo method in [24], now we estimate numerical integration to the objective function in problem (1.1). Roughly speaking, given a function $\phi : \Omega \to \mathbb{R}$, the quasi-Monte Carlo estimate for $E_\omega[\phi(\omega)]$ is obtained by taking a uniformly distributed sample set $\Omega_L := \{\omega_1, \omega_2, \cdots, \omega_L\}$ from $\Omega$ and letting $E_\omega[\phi(\omega)] \approx \frac{1}{L} \sum_{\omega \in \Omega_L} \phi(\omega)$. This implies that (2.1) holds. $\square$

Next, we consider the random membership functions of each fuzzy stochastic inequality and stochastic fuzzy objective yielded by the fuzzy constraints in (1.1).

Let the membership function for each fuzzy stochastic inequality $\langle F(x, y(\omega), \omega), z - y(\omega) \rangle \gtrless 0$ as follows: for all $x \in \mathbb{R}^n$ and any $z \in C(x, \omega)$,

$$\mu_{\tilde{\Omega}_z}(x, y(\omega), \omega) = \begin{cases} 1, & \text{if } \langle F(x, y(\omega), \omega), z - y(\omega) \rangle \geq 0, \\ \mu_z(\langle F(x, y(\omega), \omega), z - y(\omega) \rangle), & \text{if } \langle F(x, y(\omega), \omega), z - y(\omega) \rangle \in [-t_z, 0), \quad (2.2) \\ 0, & \text{if } \langle F(x, y(\omega), \omega), z - y(\omega) \rangle < -t_z, \end{cases}$$

specify the degree to which the regular inequality $\langle F(x, y(\omega), \omega), z - y(\omega) \rangle \geq 0$ is satisfied, where $\tilde{\Omega}_z$ is a fuzzy set actually determined by the fuzzy stochastic inequality in $\mathbb{R}^{n+m} \times \Omega$, $t_z \geq 0$ is the tolerance level which can be tolerated by decision makers in the accomplishment of the fuzzy stochastic inequality $\langle F(x, y(\omega), \omega), z - y(\omega) \rangle \gtrless 0$. We usually assume that $\mu_z(\langle F(x, y(\omega), \omega), z - y(\omega) \rangle) \in [0, 1]$ and it is continuous and strictly increasing over $[-t_z, 0)$. Fig. 1 shows different shapes of such membership fu
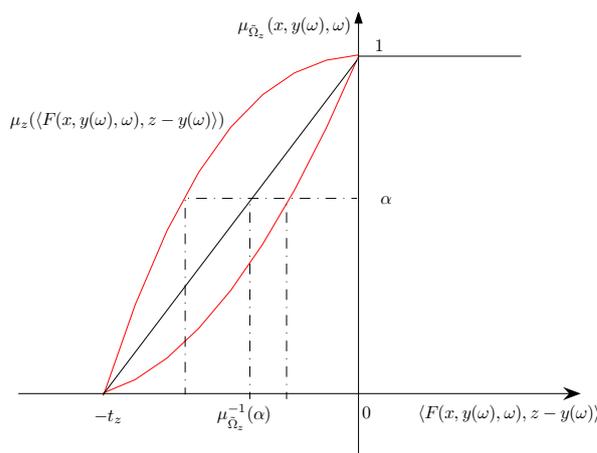


Figure 1: The membership function $\mu_{\tilde{\Omega}_z}(x, y(\omega), \omega)$.

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Similarly, the random membership function of the objective, $\mu_{\tilde{S}_0}(x, y(\omega), \omega)$, is defined as follows:

$$\mu_{\tilde{S}_0}(x, y(\omega), \omega) = \begin{cases} 1, & \text{if } E_\omega[f(x, y(\omega), \omega)] < \underline{f}, \\ \mu_0(E_\omega[f(x, y(\omega), \omega)]), & \text{if } E_\omega[f(x, y(\omega), \omega)] \in [\underline{f}, \bar{f}), \\ 0, & \text{if } E_\omega[f(x, y(\omega), \omega)] \geq \bar{f}, \end{cases} \quad (2.3)$$

where $\underline{f}$ and $\bar{f}$ are two parameters defined as follows:

$$\begin{aligned}
\bar{f} = \min \quad & E_\omega[f(x, y(\omega), \omega)] \\
\text{s.t.} \quad & x \in U, \ \omega \in \Omega, \\
& \langle F(x, y(\omega), \omega), z - y(\omega) \rangle \geq 0, \quad \forall z \in C(x, \omega)
\end{aligned} \quad (2.4)$$

and

$$\begin{aligned}
\underline{f} = \min \quad & E_\omega[f(x, y(\omega), \omega)] \\
\text{s.t.} \quad & x \in U, \ \omega \in \Omega, \\
& \langle F(x, y(\omega), \omega), z - y(\omega) \rangle \geq -t_z, \quad \forall z \in C(x, \omega).
\end{aligned} \quad (2.5)$$

By [22, 25], one can know that studying such a problem (1.1) is related to finding "almost optimal" solutions for a general convex minimization problem (see also [13, 14, 17]). Thus, we extend the idea and have the following result.

**Lemma 2.2.** Let $C(x, \omega)$ be a convex cone for all $x \in \mathbb{R}^n$ and $\omega \in \Omega$. Then the problem $\widetilde{VI_\omega}$, i.e., finding $y(\omega) \in C(x, \omega)$ such that

$$\langle F(x, y(\omega), \omega), z(\omega) - y(\omega) \rangle \underset{\sim}{\geqq} 0, \quad \forall z(\omega) \in C(x, \omega), \quad (2.6)$$

is equivalent to the fuzzy complementarity problem of finding $y(\omega) \in \mathbb{R}^m$ such that

$$y(\omega) \in C(x, \omega), \quad \langle F(x, y(\omega), \omega), y(\omega) \rangle \underset{\sim}{\overline{=}} 0, \quad F(x, y(\omega), \omega) \underset{\sim}{\overset{\in}{\sim}} C^*(x, \omega), \quad (2.7)$$

where "$\underset{\sim}{=}$" denotes the fuzzified version of "$=$" with the linguistic interpretation "approximately equal to", "$\underset{\sim}{\in}$" denotes the fuzzified version of "$\in$" with the linguistic interpretation "approximately in" and $C^*(x, \omega)$ is a polar (dual) cone of $C(x, \omega) \subset \mathbb{R}^m$ for all $x \in \mathbb{R}^n$ and $\omega \in \Omega$.

*Proof.* We start by showing that problem (2.7)$\subset$ problem (2.6). For any $x \in \mathbb{R}^n$ and $\omega \in \Omega$, suppose that $y^*(\omega)$ is a solution of problem (2.7), then we have

$$\langle F(x, y^*(\omega), \omega), y^*(\omega) \rangle \underset{\sim}{\overline{=}} 0 \quad (2.8)$$

and

$$\langle F(x, y^*(\omega), \omega), \upsilon(\omega) \rangle \underset{\sim}{\geqq} 0, \quad \forall \upsilon(\omega) \in C(x, \omega). \quad (2.9)$$

Combining (2.8) and (2.9), we have $\langle F(x, y^*(\omega), \omega), \upsilon(\omega) - y^*(\omega) \rangle \underset{\sim}{\geqq} 0$ for all $\upsilon(\omega) \in C(x, \omega)$. Thus, $y^*(\omega)$ is also a solution of problem (2.6) for all $\omega \in \Omega$.

Now we show that problem (2.6) $\subset$ problem (2.7). Let $y^*(\omega)$ be the solution of problem (2.6) with the membership degree $\alpha \in [0, 1]$ for every $\omega \in \Omega$. According to the tolerance approach [15, 21], by (2.2), we have

$$\langle F(x, y^*(\omega), \omega), \upsilon(\omega) - y^*(\omega) \rangle \geq \mu_{\tilde{\Omega}_z}^{-1}(\alpha) \geq -t_z, \quad \forall \upsilon(\omega) \in C(x, \omega), \quad (2.10)$$

where for all $\upsilon(\omega) \in C(x, \omega)$ and any $\omega \in \Omega$, $\mu_{\tilde{\Omega}_z}^{-1}$ is the inverse functions of $\mu_{\tilde{\Omega}_z}(x, \cdot, \omega)$ and $t_z > 0$ is the tolerance level which a decision maker can tolerate in the accomplishment of the fuzzy inequality $\langle F(x, y(\omega), \omega), \upsilon(\omega) - y(\omega) \rangle \underset{\sim}{\geqq} 0$. Suppose that for $\bar{t}_z \geq 0$ and $\hat{t}_z < 0$, either

$$\langle F(x, y^*(\omega), \omega), y^*(\omega) \rangle > \bar{t}_z \quad \text{or} \quad \langle F(x, y^*(\omega), \omega), y^*(\omega) \rangle < \hat{t}_z$$

is true. For any $x \in \mathbb{R}^n$ and each $\omega \in \Omega$, since $C(x, \omega)$ is a convex cone, we have $\langle F(x, y^*(\omega), \omega), y^*(\omega) \rangle \geq \frac{-t_z}{\lambda-1}$, $t_z \geq 0$ when $\upsilon(\omega) = \lambda y^*(\omega)$ with $\lambda > 1$, and $\langle F(x, y^*(\omega), \omega), y^*(\omega) \rangle \leq t_z$, when $\upsilon(\omega) = 0$. If

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$\langle F(x, y^*(\omega), \omega), y^*(\omega)\rangle > \bar{t}_z$ for $\bar{t}_z \geq 0$, This leads to a contradiction. If $\langle F(x, y^*(\omega), \omega), y^*(\omega)\rangle < \hat{t}_z$ for $\hat{t}_z < 0$, then $t_z \leq \hat{t}_z$. There lies a contradiction. Therefore,

$$\hat{t}_z \leq \langle F(x, y^*(\omega), \omega), y^*(\omega)\rangle \leq \bar{t}_z,$$

for $\bar{t}_z \geq 0$ and $\hat{t}_z < 0$, that is, $\langle F(x, y^*(\omega), \omega), y^*(\omega)\rangle \overset{=}{\sim} 0$. Furthermore, from (2.10), we have for any $v(\omega) \in C(x, \omega)$,

$$\langle F(x, y^*(\omega), \omega), v(\omega)\rangle \geq \langle F(x, y^*(\omega), \omega), y^*(\omega)\rangle - t_z \geq \hat{t}_z - t_z.$$

This implies that $\langle F(x, y^*(\omega), \omega), v(\omega)\rangle \overset{\geq}{\sim} 0$ for all $v(\omega) \in C(x, \omega)$. Hence, we have $F(x, y^*(\omega), \omega) \overset{\in}{\sim} C^*(x, \omega)$. Therefore, $y^*(\omega)$ for any $\omega \in \Omega$ is also a solution of problem (2.7). This completes the proof. □

Based on Lemma 2.2 and the work of [21], we have the following results.

**Lemma 2.3.** Let $C(x, \omega)$ is a convex cone with polar (dual) cone $C^*(x, \omega)$ for all $(x, \omega) \in \mathbb{R}^n \times \Omega$ and $\alpha$ be a new variable. Then the stochastic optimization problem (1.1) can eventually be expressed as the following regular semi-infinite optimization problem with finitely many variables $x, y(\omega), \omega$ and $\alpha$:

$$\begin{aligned} \max \quad & \alpha \\ \text{s.t.} \quad & \mu_{\tilde{S}_0}(x, y(\omega), \omega) \geq \alpha, \\ & \mu_{\tilde{\Omega}_z}(x, y(\omega), \omega) \geq \alpha, \\ & (x, y(\omega), \omega) \in S, \\ & 0 \leq \alpha \leq 1, \end{aligned} \tag{2.11}$$

where $S = \{(x, y(\omega), \omega) \in \mathbb{R}^{n+m} \times \Omega \,|\, x \in U, \omega \in \Omega, F(x, y(\omega), \omega) \overset{\in}{\sim} C^*(x, \omega)\}$, and the random membership functions $\mu_{\tilde{S}_0}$ and $\mu_{\tilde{\Omega}_z}$ are the same as in (2.3) and (2.2), respectively.

*Proof.* It follows from Lemma 2.1 that, in order to find a solution to the stochastic optimization problem (1.1) with $C(x, \omega)$ being a convex cone for $(x, \omega) \in \mathbb{R}^n \times \Omega$, we should consider the following stochastic fuzzy complementarity constrained optimization problem:

$$\begin{aligned} \min \quad & E_\omega[f(x, y(\omega), \omega)] \\ \text{s.t.} \quad & x \in U, \omega \in \Omega, y(\omega) \in C(x, \omega), \\ & \langle F(x, y(\omega), \omega), y(\omega)\rangle \overset{=}{\sim} 0, \\ & F(x, y(\omega), \omega) \overset{\in}{\sim} C^*(x, \omega). \end{aligned} \tag{2.12}$$

Since a global minimum is often required for practical problems, by the work of [21] and the description of the fuzzy stochastic inequalities (2.6), and a solution of problem (2.12) can be taken as the solution with the highest membership in the fuzzy decision set and eventually obtained by solving the following regular nonlinear parametric optimization problem:

$$\max_{(x, y(\omega), \omega) \in S} \min \left\{ \mu_{\tilde{S}_0}(x, y(\omega), \omega), \mu_{\tilde{\Omega}_z}(x, y(\omega), \omega) \right\},$$

which implies that the result holds for the new variable $\alpha$. □

**Remark 2.2.** Moreover, if the membership functions $\mu_{\tilde{S}_0}$ and $\mu_{\tilde{\Omega}_z}$ in Lemma 2.3 are invertible, then from (2.11), we get

$$\begin{aligned} \max \quad & \alpha \\ \text{s.t.} \quad & (x, y(\omega), \omega)_0 \geq \mu_{\tilde{S}_0}^{-1}(\alpha), \\ & (x, y(\omega), \omega)_{C^*} \geq \mu_{\tilde{\Omega}_z}^{-1}(\alpha), \\ & (x, y(\omega), \omega) \in S, \\ & 0 \leq \alpha \leq 1, \end{aligned} \tag{2.13}$$

where $(x, y(\omega), \omega)_0$ and $(x, y(\omega), \omega)_{C^*}$ can be followed by (2.3) and (2.2), respectively.

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# 3   Regularization smoothing approximation algorithms

In this section, based on the "method of centres" with entropic regularization, we develop a class of new smoothing approach and construct a centre iterative algorithm for solving the stochastic fuzzy optimization (1.1), and give the solution theorems.

In the sequel, we first give the following assumption ($\mathbf{H}_C$) for convenience: Define

$$C(x,\omega) := \quad \{v(\omega) \in \mathbb{R}^m \mid D(x)v(\omega) \geq 0,\ D(x) = [d_i(x)] \text{ is an } l \times m \\ \text{matrix}, d_i(x) \text{ is the } i\text{th row of } D(x),\ \forall i = 1, 2, \cdots, l\}. \tag{3.1}$$

**STEP I.** The random membership function of the fuzzy stochastic inequalities in (1.1) can be specified under condition ($\mathbf{H}_C$).

From (3.1), it is easy to see that the multi-valued operator $C(x,\omega)$ is a convex cone for any $(x,\omega) \in \mathbb{R}^n \times \Omega$, and can be shown that $F(x,y(\omega),\omega) \in C^*(x,\omega)$ if and only if there exists a nonnegative random vector $r(\omega) = (r_1(\omega), r_2(\omega), \cdots, r_l(\omega))^T \in \mathbb{R}^l$ such that

$$F(x,y(\omega),\omega) = r_1(\omega)d_1^T(x) + r_2(\omega)d_2^T(x) + \cdots + r_l(\omega)d_l^T(x) = D^T(x)r(\omega), \tag{3.2}$$

that is, for every $i = 1, 2, \cdots, l$, $d_i'(x)F(x,y(\omega),\omega) \geq 0$, where $d_i'(x)$ is normal to $d_i(x)$ (see [17, 26]).

It follows that the fuzzy stochastic optimization problem (2.12) can be rewritten as the following generalized stochastic optimization problem with fuzzy stochastic inequality constraints:

$$
\begin{aligned}
\min \quad & E_\omega[f(x,y(\omega),\omega)] \\
\text{s.t.} \quad & x \in U,\ \omega \in \Omega, \\
& d_i(x)y(\omega) \geq 0,\ i = 1, 2, \cdots, l, \\
& \langle F(x,y(\omega),\omega), y(\omega)\rangle \gtreqless 0, \\
& \langle -F(x,y(\omega),\omega), y(\omega)\rangle \gtreqless 0, \\
& d_i'(x)F(x,y(\omega),\omega) \gtreqless 0,\ i = 1, 2, \cdots, l,
\end{aligned}
\tag{3.3}
$$

and each fuzzy stochastic inequality in (3.3) can be represented by a fuzzy set $\tilde{S}_j$ (i.e., represent of $\tilde{\Omega}_z$ in (2.11) or (2.13)) with corresponding random membership function $\mu_{\tilde{S}_j}(x,y(\omega),\omega)$ for $j = 1, 2, \cdots, l+2$. To specify the membership functions $\mu_{\tilde{S}_j}$, $j = 1, 2, \cdots, l+2$, similar treatment to (2.2), we define the membership functions as follows:

$$
\mu_{\tilde{S}_1}(x,y(\omega),\omega) = 
\begin{cases}
1, & \text{if } \langle F(x,y(\omega),\omega), y(\omega)\rangle \geq 0, \\
\mu_1(\langle F(x,y(\omega),\omega), y(\omega)\rangle), & \text{if } \langle F(x,y(\omega),\omega), y(\omega)\rangle \in [-t_1, 0), \\
0, & \text{if } \langle F(x,y(\omega),\omega), y(\omega)\rangle < -t_1,
\end{cases}
$$

$$
\mu_{\tilde{S}_2}(x,y(\omega),\omega) = 
\begin{cases}
1, & \text{if } \langle -F(x,y(\omega),\omega), y(\omega)\rangle \geq 0, \\
\mu_2(\langle -F(x,y(\omega),\omega), y(\omega)\rangle), & \text{if } \langle -F(x,y(\omega),\omega), y(\omega)\rangle \in [-t_2, 0), \\
0, & \text{if } \langle -F(x,y(\omega),\omega), y(\omega)\rangle < -t_2,
\end{cases}
\tag{3.4}
$$

$$
\mu_{\tilde{S}_{i+2}}(x,y(\omega),\omega) = 
\begin{cases}
1, & \text{if } d_i'F(x,y(\omega),\omega) \geq 0, \\
\mu_{i+2}(d_i'F(x,y(\omega),\omega)), & \text{if } d_i'F(x,y(\omega),\omega) \in [-t_{i+2}, 0), \\
0, & \text{if } d_i'F(x,y(\omega),\omega) < -t_{i+2},
\end{cases}
$$

where $t_{i+2} \geq 0$ for $i = 1, 2, \cdots, l$, is the tolerance level which one decision maker can tolerate in the accomplishment of the fuzzy stochastic inequalities in (3.3).

**STEP II.** Discrete approximation of problem (1.1) with condition ($\mathbf{H}_C$) need to be given.

By Lemma 2.1 and **STEP I**, we have the following problem as an appropriate discrete approximation of problem (3.3):

$$
\begin{aligned}
\min \quad & \frac{1}{L} \sum_{\omega \in \Omega_L} f(x,y(\omega),\omega)\zeta(\omega) \\
\text{s.t.} \quad & x \in U \subset \mathbb{R}^n,\ \omega \in \Omega_L, \\
& d_i(x)y(\omega) \geq 0,\ i = 1, 2, \cdots, l, \\
& \langle F(x,y(\omega),\omega), y(\omega)\rangle \gtreqless 0, \\
& \langle -F(x,y(\omega),\omega), y(\omega)\rangle \gtreqless 0, \\
& d_i'(x)F(x,y(\omega),\omega) \gtreqless 0,\ i = 1, 2, \cdots, l.
\end{aligned}
\tag{3.5}
$$

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We note that the sample set $\Omega_L$ is chosen to be asymptotically dense in $\Omega$. Especially, it follows from (2.4), (2.5) and (3.5) that the appropriate discrete approximation of two parameter $\bar{f}$ and $\underline{f}$ can be shown as follows, respectively:

$$
\begin{aligned}
\bar{f} = \min \quad & \tfrac{1}{L} \sum_{\omega \in \Omega_L} f(x, y(\omega), \omega) \zeta(\omega) \\
\text{s.t.} \quad & x \in U \subset \mathbb{R}^n, \ \omega \in \Omega_L, \\
& d_i(x) y(\omega) \geq 0, \ i = 1, 2, \cdots, l, \\
& \langle F(x, y(\omega), \omega), y(\omega) \rangle \geq 0, \\
& \langle -F(x, y(\omega), \omega), y(\omega) \rangle \geq 0, \\
& d_i'(x) F(x, y(\omega), \omega) \geq 0, \ i = 1, 2, \cdots, l
\end{aligned}
\tag{3.6}
$$

and

$$
\begin{aligned}
\underline{f} = \min \quad & \tfrac{1}{L} \sum_{\omega \in \Omega_L} f(x, y(\omega), \omega) \zeta(\omega) \\
\text{s.t.} \quad & x \in U \subset \mathbb{R}^n, \ \omega \in \Omega_L, \\
& d_i(x) y(\omega) \geq 0, \ i = 1, 2, \cdots, l, \\
& \langle F(x, y(\omega), \omega), y(\omega) \rangle \geq -t_1, \\
& \langle -F(x, y(\omega), \omega), y(\omega) \rangle \geq -t_2, \\
& d_i'(x) F(x, y(\omega), \omega) \geq -t_{i+2}, \ i = 1, 2, \cdots, l.
\end{aligned}
\tag{3.7}
$$

**STEP III.** A new centre iterative method for solving problem (3.8) should be adopt.

It follows from (2.13), (2.3) and (3.4)-(3.7) that an optimal solution of the stochastic optimization problem (1.1) can be obtained by approximating for the following stochastic parametric optimization problem:

$$
\left\{
\begin{aligned}
\max \quad & \alpha \\
\text{s.t.} \quad & \mu_{\tilde{S}_0}^{-1}(\alpha) - \tfrac{1}{L} \sum_{\omega \in \Omega_L} f(x, y(\omega), \omega) \zeta(\omega) \geq 0, \\
& \mu_{\tilde{S}_1}^{-1}(\alpha) - \langle F(x, y(\omega), \omega), y(\omega) \rangle \geq 0, \\
& \mu_{\tilde{S}_2}^{-1}(\alpha) + \langle F(x, y(\omega), \omega), y(\omega) \rangle \geq 0, \\
& -\mu_{\tilde{S}_j}^{-1}(\alpha) + d_{j-2}' F(x, y(\omega), \omega) \geq 0, \quad j = 3, 4, \cdots, l+2, \\
& d_i(x) y(\omega) \geq 0, \quad i = 1, 2, \cdots, l, \\
& 0 \leq \alpha \leq 1, \quad x \in U, \ \omega \in \Omega_L.
\end{aligned}
\right.
\tag{3.8}
$$

It is interested in developing an efficient algorithm to solve (3.8) based on a framework of centre iterations. This iterative approach can be traced back to Huard's work [28]. The basic concepts are easy to understand and very adaptive to new developments. To describe the approach, we denote the feasible domain of (3.8) by a set $V$ and define some terminologies. A general assumption for this approach is that $V$ is bounded and convex, and the interior of $V$ is nonempty.

**Definition 3.1** For any given point $(x, y(\omega), \omega, \alpha)$ in the convex domain $V$, we define the "distance $\mathcal{L}$ of $(x, y(\omega), \omega, \alpha)$ to the boundary of $V$" by a continuous function

$$
\begin{aligned}
\mathcal{L}((x, y(\omega), \omega, \alpha), V) \quad = \min_{\substack{i=1,2,\cdots,l \\ j=3,4,\cdots,l+2}} \Big\{ & \alpha, \ 1 - \alpha, \ \mu_{\tilde{S}_0}^{-1}(\alpha) - \tfrac{1}{L} \sum_{\omega \in \Omega_L} f(x, y(\omega), \omega) \zeta(\omega), \\
& \mu_{\tilde{S}_1}^{-1}(\alpha) - \langle F(x, y(\omega), \omega), y(\omega) \rangle, \mu_{\tilde{S}_2}^{-1}(\alpha) + \langle F(x, y(\omega), \omega), y(\omega) \rangle, \\
& -\mu_{\tilde{S}_j}^{-1}(\alpha) + d_{j-2}' F(x, y(\omega), \omega), \ d_i(x) y(\omega) \Big\}.
\end{aligned}
$$

**Definition 3.2** Let a distance function $\mathcal{L}((x, y(\omega), \omega, \alpha), V)$ be defined on a convex domain $V$. Then a point $(\bar{x}, \bar{y}(\omega), \omega, \bar{\alpha}) \in V$ is called the "centre of $V$", if it maximizes the distance function $\mathcal{L}((x, y(\omega), \omega, \alpha), V)$, i.e.,

$$
(\bar{x}, \bar{y}(\omega), \omega, \bar{\alpha}): \quad \mathcal{L}((\bar{x}, \bar{y}(\omega), \omega, \bar{\alpha}), V) = \max \{ \mathcal{L}((x, y(\omega), \omega, \alpha), V) \mid (x, y(\omega), \omega, \alpha) \in V \}.
$$

Thus, a new centre iterative method for problem (3.8) could be described as follows.

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**Algorithm 3.1.** *Step* 1. Taking a point $(x^k, y^k(\omega), \omega, \alpha^k)$ in $V$, then we consider the distance $\mathcal{L}$ in convex domain $W_k = V \cap \{(x, y(\omega), \omega, \alpha) | \alpha \geq \alpha^k\}$.

*Step* 2. Solving the maximal problem $\max\{\mathcal{L}((x, y(\omega), \omega, \alpha), W_k) | (x, y(\omega), \omega, \alpha) \in W_k\}$ and denoting the new iterative point $(x^{k+1}, y^{k+1}(\omega), \omega, \alpha^{k+1})$ as a centre of $W_k$, then we have

$$(x^{k+1}, y^{k+1}(\omega), \omega, \alpha^{k+1}):$$
$$\mathcal{L}((x^{k+1}, y^{k+1}(\omega), \omega, \alpha^{k+1}), W_k) = \max\{\mathcal{L}((x, y(\omega), \omega, \alpha), W_k) | (x, y(\omega), \omega, \alpha) \in W_k\},$$

where

$$\mathcal{L}((x, y(\omega), \omega, \alpha), W_k) = \min_{\substack{i=1,2,\cdots,l \\ j=3,4,\cdots,l+2}} \Big\{ \alpha - \alpha^k,\ \alpha,\ 1 - \alpha,\ \mu_{\tilde{S}_0}^{-1}(\alpha) - \frac{1}{L} \sum_{\omega \in \Omega_L} f(x, y(\omega), \omega)\zeta(\omega),$$
$$\mu_{\tilde{S}_1}^{-1}(\alpha) - \langle F(x, y(\omega), \omega), y(\omega) \rangle,\ \mu_{\tilde{S}_2}^{-1}(\alpha) + \langle F(x, y(\omega), \omega), y(\omega) \rangle,$$
$$-\mu_{\tilde{S}_j}^{-1}(\alpha) + d'_{j-2}F(x, y(\omega), \omega),\ d_i(x)y(\omega) \Big\}$$

is the distance function defined on the convex domain $W_k$.

*Step* 3. Start working again with $(x^{k+1}, y^{k+1}(\omega), \omega, \alpha^{k+1})$ instead of $(x^k, y^k(\omega), \omega, \alpha^k)$ and go to *Step* 1.

It follows from the properties introduced in [28, Lemma 2.2] and Algorithm 3.1 that the major computational work lies in the determination of the centres required, i.e., at the $k$th iteration, the following "min-max problem" should be solved:

$$-\min_{x, y(\omega), \omega, \alpha} \mathcal{L}((x, y(\omega), \omega, \alpha), W_k) = \min_{x, y(\omega), \omega, \alpha} \max_{\substack{i=1,2,\cdots,l \\ j=3,4,\cdots,l+2}} \Big\{ \alpha^k - \alpha,\ -\alpha,\ \alpha - 1,$$
$$\frac{1}{L} \sum_{\omega \in \Omega_L} f(x, y(\omega), \omega)\zeta(\omega) - \mu_{\tilde{S}_0}^{-1}(\alpha),$$
$$\langle F(x, y(\omega), \omega), y(\omega) \rangle - \mu_{\tilde{S}_1}^{-1}(\alpha), \tag{3.9}$$
$$-\langle F(x, y(\omega), \omega), y(\omega) \rangle - \mu_{\tilde{S}_2}^{-1}(\alpha),$$
$$-d'_{j-2}F(x, y(\omega), \omega) + \mu_{\tilde{S}_j}^{-1}(\alpha),\ -d_i(x)y(\omega) \Big\}.$$

**STEP IV** A class of new regularization smoothing approximation algorithms is developed under condition $(\mathbf{H}_C)$.

Since the maximal membership function (see [12]) in the "min-max" problem (3.9) is non-differentiability, it is easy to see that one major difficulty encountered is to develop a class of new smoothing approximation methods, which are based on the notion of newly proposed "entropic regularization procedure" (see [18]).

**Algorithm 3.2.**

*Step* 1. Set $k = 0$, give the initial iterate $(x^0, y^0(\omega), \omega, \alpha^0)$ which is an interior point of $V$ defined by (3.8), a sufficiently small constant $\epsilon > 0$, and an upper bound $Q$ which is the maximum number of unconstrained minimizations to be performed.

*Step* 2. Starting from $(x^k, y^k(\omega), \omega, \alpha^k)$, apply a standard quasi-Newton line search of MATLAB software to solve the unconstrained smooth convex program (3.6), (3.7) and the following unconstrained smooth convex program:

$$-\min_{x, y(\omega), \omega, \alpha} \mathcal{L}_\gamma((x, y(\omega), \omega, \alpha), W_k)$$
$$= \frac{1}{\gamma} \ln \Big\{ \exp[\gamma(\alpha^k - \alpha)] + \exp[\gamma(-\alpha)] + \exp[\gamma(\alpha - 1)]$$
$$+ \exp[\gamma(\frac{1}{L} \sum_{\omega \in \Omega_L} f(x, y(\omega), \omega)\zeta(\omega) - \mu_{\tilde{S}_0}^{-1}(\alpha))]$$
$$+ \exp[\gamma(\langle F(x, y(\omega), \omega), y(\omega) \rangle - \mu_{\tilde{S}_1}^{-1}(\alpha))]$$
$$+ \exp[\gamma(-\langle F(x, y(\omega), \omega), y(\omega) \rangle - \mu_{\tilde{S}_2}^{-1}(\alpha))] \tag{3.10}$$
$$+ \sum_{j=3}^{l+2} \exp[\gamma(-d'_{j-2}F(x, y(\omega), \omega) + \mu_{\tilde{S}_j}^{-1}(\alpha))]$$
$$+ \sum_{i=1}^{l} \exp[\gamma(-d_i(x)y)] \Big\}$$

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with a sufficiently large $\gamma$. Denote its solution by $(x^{k+1}, y^{k+1}(\omega), \omega, \alpha^{k+1})$ in the light of Algorithm 3.1.

*Step* 3. If $k > 1$ and $\|(x^{k+1}, y^{k+1}(\omega), \omega, \alpha^{k+1}) - (x^k, y^k(\omega), \omega, \alpha^k)\|_2 \le \epsilon$, then the computation terminates with $(x^{k+1}, y^{k+1}(\omega), \omega, \alpha^{k+1})$ as the solution. If $k > Q$, then the computation terminates with a failure.

*Step* 4. $k \leftarrow k + 1$ and go to *Step* 2.

From Algorithm 3.2, it follows that $\min_{x, y(\omega), \omega, \alpha} \mathcal{L}_\gamma((x, y(\omega), \omega, \alpha), W_k)$ provides a centre of $W_k$, as $\gamma \to \infty$. By using a moderately large $\gamma$, we can obtain an accurate approximation. Also because of the special "log-exponential" form of $\mathcal{L}_\gamma((x, y(\omega), \omega, \alpha), W_k)$, we can avoid most overflow problems in computation. Moreover, since problem (3.10) is an unconstrained, smooth, and convex optimization program, the commonly used solution methods, such as the quasi-Newton line search of MATLAB software, can be readily applied.

**Remark 3.1.** We note that Algorithm 3.1 appears in *Step* 2 of Algorithm 3.2. It is the fuzzy constraints in (1.1) that yields a fuzzy objective. Hence, a class of new and interesting regularization smoothing approximation approaches must be chosen to define two parameters in (3.6) and (3.7), and to employ for solving problem (3.10) which is equivalent to the stochastic parametric optimization problem (3.8).

**STEP V** Comprehensive convergence theorems based on Algorithm 3.2 should be proved.

In the sequel, we first give the following lemmas and results.

**Lemma 3.1.** Let the function $\varphi : \Omega \to \mathbb{R}$ be continuous. Then we have

$$\lim_{L \to \infty} \frac{1}{L} \sum_{\omega \in \Omega_L} \varphi(\omega)\zeta(\omega) = \int_\Omega \varphi(\omega)\zeta(\omega)d\omega.$$

*Proof.* Taking $N = L$, $\bar{I}^s = \Omega$, $J = \Omega_L$, $x_i = \omega_i$ $(i = 1, 2, \cdots)$ and $f = \varphi\zeta$, then from the results (2.2) and (2.3) given in Chapter 2 of [24, pp. 13-14], the result holds. This completes the proof. □

**Remark 3.2.** By Lemma 3.1, we know immediately that

$$\lim_{L \to \infty} \frac{1}{L} \sum_{\omega \in \Omega_L} f(x, y(\omega), \omega)\zeta(\omega) = \int_\Omega f(x, y(\omega), \omega)\zeta(\omega)d\omega \quad (3.11)$$

and particularly,

$$\lim_{L \to \infty} \frac{1}{L} \sum_{\omega \in \Omega_L} \zeta(\omega) = \int_\Omega \zeta(\omega)d\omega = 1. \quad (3.12)$$

**Lemma 3.2.** If $\psi(x)$ is continuous, strictly increasing and linear over a convex set $U$ in $\mathbb{R}^n$, then its inverse $\psi^{-1}$ is linear.

**Theorem 3.1.** Suppose that condition $(\mathbf{H}_C)$ holds, the set $U \subset \mathbb{R}^n$ is nonempty and bounded, $F : \mathbb{R}^{n+m} \times \Omega \to \mathbb{R}^m$ is continuously differentiable, and $f : \mathbb{R}^{n+m} \times \Omega \to \mathbb{R}$ is Hölder continuous in $(x, y(\cdot))$ on $U \times \mathbb{R}^m$ with order $\sigma > 0$ and Hölder constant $\varsigma(\omega) > 0$ for all $\omega \in \Omega_L$ satisfying

$$\int_\Omega \varsigma(\omega)d\zeta(\omega) < +\infty.$$

Then

(i) problem (3.6) has at least one optimal solution when $L$ is large enough;

(ii) $(x^*, y^*(\cdot))$ is an optimal solution of problem (2.4) when $x^*$ is an accumulation point of the sequence $\{x^L\}$ and $y^*(\cdot)$ is defined by

$$y^*(\omega) := \max_{i=1,2,\cdots,l}\{-F(x^*, 0, \omega), -d_i'(x^*)F(x^*, 0, \omega), 0\}, \quad \omega \in \Omega. \quad (3.13)$$

*Proof.* (i) Let $\mathcal{F}_L$ be the feasible region of problem (3.6). It is not difficult to see that $\mathcal{F}_L$ is a nonempty and closed set and the objective function of problem (3.6) is bounded below on $\mathcal{F}_L$. Thus,

there exists a sequence $\{(x^k, y^k(\omega))_{\omega \in \Omega_L}\} \subset \mathcal{F}_L$ such that

$$\lim_{k \to \infty} \frac{1}{L} \sum_{\omega \in \Omega_L} f(x^k, y^k(\omega), \omega)\zeta(\omega)$$

$$= \inf_{(x, y(\omega), \omega)_{\omega \in \Omega_L}} \frac{1}{L} \sum_{\omega \in \Omega_L} f(x, y(\omega), \omega)\zeta(\omega). \tag{3.14}$$

It follows from the boundedness of $U$ and the Hölder continuity of $f$ that the sequence $\{x^k\}$ and the function $f$ are bounded.

On the other hand, noting that $(x^k, y^k(\omega))_{\omega \in \Omega_L} \in \mathcal{F}_L$ for every $k$, we have

$$0 \le y^k(\omega) \perp \frac{1}{L} \sum_{\omega \in \Omega_L} F(x^k, y^k(\omega), \omega)\zeta(\omega) \ge 0, \tag{3.15}$$

where the symbol $\perp$ means the two vectors are perpendicular to each other. Assume that the sequence $\{y^k(\omega)\}_{\omega \in \Omega_L}$ is unbounded. Taking a subsequence if necessary, let

$$\lim_{\substack{k \to \infty \\ \omega \in \Omega_L}} \|y^k(\omega)\| = +\infty, \lim_{\substack{k \to \infty \\ \omega \in \Omega_L}} \frac{y^k(\omega)}{\|y^k(\omega)\|} = \bar{y}(\omega), \|\bar{y}(\omega)\| = 1. \tag{3.16}$$

Then, for all $\omega \in \Omega_L$, dividing (3.15) by $\|y^k(\omega)\|$ and letting $k \to +\infty$, we have for any $x \in U$,

$$0 \le \bar{y}(\omega) \perp \frac{1}{L} \sum_{\omega \in \Omega_L} F(x, \bar{y}(\omega), \omega)\zeta(\omega) \ge 0.$$

This contradicts (3.16) by the continuous differentiability of $F$, and so $\{y^k(\omega)\}$ is bounded for each $\omega \in \Omega_L$ with $\zeta(\omega) > 0$. For any $\omega \in \Omega_L$ with $\zeta(\omega) = 0$, we redefine $y^k(\omega)$ by

$$y^k(\omega) := \max_{i=1,2,\cdots,l} \{-F(x^k, 0, \omega), -d_i'(x^k)F(x^k, 0, \omega), 0\}.$$

Hence, the sequence $\{(x^k, y^k(\omega))_{\omega \in \Omega_L}\}$ is bounded and (3.14) remains valid. Therefore, the closeness of $\mathcal{F}_L$ implies that any accumulation point of $\{(x^k, y^k(\omega))_{\omega \in \Omega_L}\}$ must be an optimal solution of problem (3.6).

(ii) By the assumptions, the sequence $\{x^L\}$ contains a subsequence converging to $x^*$. Without loss of generality, we suppose $\lim_{L \to \infty} x^L = x^*$.

Firstly, we prove that $(x^*, y^*(\cdot))$ is feasible to problem (3.6). To this end, we define

$$\hat{y}^L(\omega) := \max_{i=1,2,\cdots,l} \{-F(x^L, 0, \omega), -d_i'(x^L)F(x^L, 0, \omega), 0\}, \quad \omega \in \Omega. \tag{3.17}$$

It is obvious that $(x^*, \hat{y}^L(\omega))_{\omega \in \Omega_L}$ is feasible to problem (3.6) for every $L$. Since $F(x^*, y^*(\omega), \omega) \ge 0$ by the definition (3.13), it is sufficient to show that

$$(y^*(\omega))^T F(x^*, y^*(\omega), \omega) = 0, \quad \omega \in \Omega. \tag{3.18}$$

Let $\bar{\omega} \in \Omega$ be fixed. Since the sample set $\Omega_L$ is chosen to be asymptotically dense in $\Omega$, there exists a sequence $\{\bar{\omega}_L\}$ of samples such that $\bar{\omega}_L \in \Omega_L$ for each $L$ and $\lim_{L \to \infty} \bar{\omega}_L = \bar{\omega}$. Thus, we obtain

$$(\hat{y}^L(\bar{\omega}_L))^T F(x^L, \hat{y}^L(\bar{\omega}_L), \bar{\omega}_L) = 0, \quad L = 1, 2, \cdots.$$

Letting $L \to +\infty$ and taking the continuity of the functions $F(x, y(\cdot), \cdot)$ on the compact set $\Omega$ into account, we have

$$(y^*(\bar{\omega}))^T F(x^*, y^*(\bar{\omega}), \bar{\omega}) = 0.$$

By the arbitrariness of $\bar{\omega} \in \Omega$, now we know that (3.18) immediately holds. This completes the proof of the feasibility of $(x^*, y^*(\cdot))$ in (3.6).

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Next, let $(x, y(\cdot))$ be an arbitrary feasible solution of (3.6). It follows from the results of (i) and obvious that $(x, y(\omega), \omega)_{\omega \in \Omega_L}$ is feasible to problem (3.6) for any $L$. Moreover, from the Höcontinuity of $f$, we have

$$\frac{1}{L} \sum_{\omega \in \Omega_L} \left[ f(x^L, y^L(\omega), \omega) - f(x^L, \hat{y}^L(\omega), \omega) \right] \zeta(\omega)$$

$$= \frac{1}{L} \sum_{\omega \in \Omega_L} \left[ f(x^L, y^L(\omega), \omega) - f(x^L, \hat{y}^L(\omega), \omega) \right] \zeta(\omega)$$

$$\leq \frac{1}{L} \sum_{\omega \in \Omega_L} \left[ \|y^L(\omega) - \hat{y}^L(\omega)\| \varsigma(\omega) \right] \cdot \zeta(\omega) \to 0 \quad \text{as} \quad k \to \infty, \quad \text{w.p.1.}$$

which along with Lemma 3.1 yields

$$\lim_{L \to \infty} \frac{1}{L} \sum_{\omega \in \Omega_L} f(x^k, y^k(\omega), \omega) = \lim_{L \to \infty} \frac{1}{L} \sum_{\omega \in \Omega_L} f(x^*, y^*(\omega), \omega) = E_\omega[f(x^*, y^*(\omega), \omega)] \quad \text{w.p.1,}$$

which indicates that $(x^*, y^*(\cdot))$ is an optimal solution of problem (1.1) with probability one and the feasibility of $(x^L, y^L(\omega), \omega)_{\omega \in \Omega_L}$ in (3.6) that $(x^L, \hat{y}^L(\omega), \omega)_{\omega \in \Omega_L}$ is also an optimal solution of problem (3.6). Thus, since $f$ is Hölder continuous in $(x, y(\cdot))$ on $U \times \mathbb{R}^m$, we obtain

$$\frac{1}{L} \sum_{\omega \in \Omega_L} \left[ f(x^*, y^*(\omega), \omega) - f(x, y(\omega), \omega) \right] \zeta(\omega)$$

$$\leq \frac{1}{L} \sum_{\omega \in \Omega_L} \left[ f(x^*, y^*(\omega), \omega) - f(x^L, \hat{y}^L(\omega), \omega) \right] \zeta(\omega)$$

$$\leq \frac{1}{L} \sum_{\omega \in \Omega_L} \left| f(x^*, y^*(\omega), \omega) - f(x^L, \hat{y}^L(\omega), \omega) \right| \zeta(\omega)$$

$$\leq \frac{1}{L} \sum_{\omega \in \Omega_L} \zeta(\omega) \cdot \left[ \|x^L - x^*\| + \|\hat{y}^L(\omega) - y^*(\omega)\| \right] \varsigma(\omega). \tag{3.19}$$

It follows from (3.12) that the sequence $\left\{ \frac{1}{L} \sum_{\omega \in \Omega_L} \zeta(\omega) \right\}$ is bounded. This yields

$$\lim_{L \to \infty} \frac{1}{L} \sum_{\omega \in \Omega_L} \left| f(x^*, y^*(\omega), \omega) - f(x^L, \hat{y}^L(\omega), \omega) \right| \zeta(\omega) = 0. \tag{3.20}$$

Thus, by letting $L \to +\infty$ in (3.19) and taking (3.11) and (3.20) into account, we have

$$\int_\Omega f(x^*, y^*(\omega), \omega) \zeta(\omega) d\omega \leq \int_\Omega f(x, y(\omega), \omega) \zeta(\omega) d\omega,$$

which implies that $x^*$ together with $y^*(\cdot)$ constitutes an optimal solution of problem (3.6). This completes the proof. □

Similarly, by Lemma 3.1, (3.11), (3.12) and proof of Theorem 3.1, we have the following result.

**Theorem 3.2.** Assume that condition $(\mathbf{H}_C)$ holds, and $f$, $F$ and $U$ are the same as in Theorem 3.1. Then

(i) problem (3.7) has at least one optimal solution when $L$ is large enough;

(ii) $(x^*, y^*(\cdot))$ is an optimal solution of problem (2.5) when $x^*$ is an accumulation point of the sequence $\{x^L\}$ and $y^*(\cdot)$ is defined by

$$y^*(\omega) := \max_{i=1,2,\cdots,l} \{ -t_1 - F(x^*, 0, \omega), -t_2 + F(x^*, 0, \omega), -t_{i+2} - d_i'(x^*) F(x^*, 0, \omega), 0 \}, \quad \omega \in \Omega.$$

Now, consider the case that the membership function of each fuzzy stochastic inequality and the objective function $E_\omega[f(x, y(\omega), \omega)]$ in (3.3) is continuous, strictly increasing, and linear over the

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corresponding tolerance interval. A commonly used example in fuzzy set theory is that $\psi(x) = 1 - bx^\beta$ with $b > 0$ and $\beta > 1$. In this case, from the theory of convex analysis [27], Lemma 3.2, and Theorems 3.1 and 3.2, we have the following simple result.

**Theorem 3.3.** Suppose that condition $(\mathbf{H}_C)$ holds. If $F : \mathbb{R}^{n+m} \times \Omega \to \mathbb{R}^m$ is monotone in the second variable, and $\mu_{\tilde{\Omega}_z}(x, y(\omega), \omega)$ is continuous, strictly increasing and linear for all $z \in C(x, \omega)$ and any $(x, y(\omega), \omega) \in \mathbb{R}^{n+m} \times \Omega$, then we can find an optimal solution $(x^*, y^*(\cdot))$ of the stochastic optimization problem (1.1) by solving the following stochastic parametric optimization problem: (3.8), which can be readily approximated by the iterative sequence $\{(x^{k+1}, y^{k+1}(\omega), \omega, \alpha^{k+1})\}$ generated by Algorithm 3.2.

# 4   Simulation analysis

In this section, we shall give an example to illustrate the validity of our approaches.

Taking $n = 2, m = 3, l = 2, U = [1, 14] \times [1, 14], d_1(x) = (-1, -1, 3), d_2(x) = (-n2, 1, -1),$

$$f(x, y(\omega), \omega) = (x_1 - y_1)^2 + x_2 y_2 + 2\omega, \ F(x, y(\omega), \omega) = \begin{pmatrix} -2x_1 + y_1 - 3y_2 + \omega \\ x_1 + x_2 + 3y_1 - y_3 - 2\omega \\ -2x_2 + y_2 + 2y_3 - \omega \end{pmatrix},$$

$$C(x, \omega) = \left\{ y(\omega) = (y_1, y_2, y_3)^T \in \mathbb{R}^3 \ \middle| \ \begin{pmatrix} d_1(x) \\ d_2(x) \end{pmatrix} y(\omega) \geq 0 \right\},$$

and Letting $d_1'(x) = (0, 3, 1)$ and $d_2'(x) = (1, 2, 0)$ in (3.3), and $\zeta(\omega_\ell) = p_\ell \ (\ell = 1, 2, \cdots, L)$ in (3.5), then we have

$$\varphi(x, y(\omega), \omega) := E_\omega[(x_1 - y_1)^2 + x_2 y_2 + 2\omega] = \frac{1}{L} \sum_{\ell=1}^{L} [(x_1 - y_1)^2 + x_2 y_2 + 2\omega_\ell] p_\ell,$$

$$f_1(x, y(\omega), \omega) = \langle F(x, y(\omega), \omega), y(\omega) \rangle = -2x_1 y_1 + x_1 y_2 + x_2 y_2 - 2x_2 y_3 + y_1^2$$
$$+ \omega y_1 - 2\omega y_2 + 2y_3^2 - \omega y_3,$$

$$f_2(x, y(\omega), \omega) = \langle -F(x, y(\omega), \omega), y(\omega) \rangle = 2x_1 y_1 - x_1 y_2 - x_2 y_2 + 2x_2 y_3 - y_1^2$$
$$- \omega y_1 + 2\omega y_2 - 2y_3^2 + \omega y_3,$$

$$f_3(x, y(\omega), \omega) = d_1'(x) F(x, y(\omega), \omega) = 3x_1 + x_2 + 9y_1 + y_2 - y_3 - 7\omega,$$

$$f_4(x, y(\omega), \omega) = d_2'(x) F(x, y(\omega), \omega) = 2x_2 + 7y_1 - 3y_2 - 2y_3 - 3\omega.$$

Thus, problem (3.5) is equivalent to the following generalized fuzzy stochastic inequality constrained optimization program:

$$\begin{array}{ll} \min & \varphi(x, y(\omega), \omega) \\ \text{s.t.} & 1 \leq x_1, x_2 \leq 14, \quad y_1, y_2, y_3 \geq 0, \\ & -y_1 - y_2 + y_3 \geq 0, \quad -2y_1 + y_2 - y_3 \geq 0, \\ & f_\iota(x, y(\omega), \omega) \gtreqqless 0, \quad \iota = 1, 2, 3, 4, \end{array} \tag{4.1}$$

with the membership function $\mu_{\tilde{S}_\tau}(x, y(\omega), \omega) \ (\tau = 1, 2, 3, 4)$, being specified as $t_1 = 9, t_2 = 2, t_3 = 6, t_4 = 10,$

$$\mu_{\tilde{S}_1}(x, y(\omega), \omega) = \begin{cases} 1, & \text{if} \quad f_1(x) \geq 0, \\ 1 - \frac{f_1(x, y(\omega), \omega)}{9}, & \text{if} \quad f_1(x, y(\omega), \omega) \in [-9, 0), \\ 0, & \text{if} \quad f_1(x, y(\omega), \omega) < -9, \end{cases}$$

$$\mu_{\tilde{S}_2}(x, y(\omega), \omega) = \begin{cases} 1, & \text{if} \quad f_2(x, y(\omega), \omega) \geq 0, \\ 1 - \frac{f_2(x, y(\omega), \omega)}{2}, & \text{if} \quad f_2(x, y(\omega), \omega) \in [-2, 0), \\ 0, & \text{if} \quad f_2(x) < -2, \end{cases}$$

$$\mu_{\tilde{S}_3}(x, y(\omega), \omega) = \begin{cases} 1, & \text{if} \quad f_3(x, y(\omega), \omega) \geq 0, \\ 1 - \frac{f_3(x, y(\omega), \omega)}{6}, & \text{if} \quad f_3(x, y(\omega), \omega) \in [-6, 0), \\ 0, & \text{if} \quad f_3(x, y(\omega), \omega) < -6, \end{cases}$$

$$\mu_{\tilde{S}_4}(x, y(\omega), \omega) = \begin{cases} 1, & \text{if} \quad f_4(x) \geq 0, \\ 1 - \frac{f_4(x, y(\omega), \omega)}{10}, & \text{if} \quad f_4(x, y(\omega), \omega) \in [-10, 0), \\ 0, & \text{if} \quad f_4(x) < -10, \end{cases}$$

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and

$$\mu_{\tilde{S}_0}(x, y(\omega), \omega) = \begin{cases} 1, & \text{if} \quad \varphi(x, y(\omega), \omega) < \underline{f}, \\ \frac{\bar{f} - \varphi(x, y(\omega), \omega)}{\bar{f} - \underline{f}}, & \text{if} \quad \varphi(x, y(\omega), \omega) \in [\underline{f}, \bar{f}), \\ 0, & \text{if} \quad \langle \varphi(x, y(\omega), \omega) \geq \bar{f}, \end{cases}$$

where

$$\begin{aligned} \bar{f} = \min \quad & \varphi(x, y(\omega), \omega) \\ \text{s.t.} \quad & 1 \leq x_1, x_2 \leq 14, \quad y_1, y_2, y_3 \geq 0, \\ & -y_1 - y_2 + 3y_3 \geq 0, \quad -2y_1 + y_2 - y_3 \geq 0, \\ & f_\iota(x, y(\omega), \omega) \geq 0, \quad \iota = 1, 2, 3, 4, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \underline{f} = \min \quad & \varphi(x, y(\omega), \omega) \\ \text{s.t.} \quad & 1 \leq x_1, x_2 \leq 14, \quad y_1, y_2, y_3 \geq 0,, \\ & -y_1 - y_2 + 3y_3 \geq 0, \quad -2y_1 + y_2 - y_3 \geq 0, \\ & f_1(x, y(\omega), \omega) \geq -9, \quad f_2(x, y(\omega), \omega) \geq -2, \\ & f_3(x, y(\omega), \omega) \geq -6, \quad f_4(x, y(\omega), \omega) \geq -10. \end{aligned} \tag{4.3}$$

By Bellman and Zadeh's method of fuzzy decision making [15] and Algorithm 3.2, now we know that the conditions of Theorem 3.3 hold, and so an optimal solution of the problem (4.1) can be obtained by solving the following unconstrained and smooth nonlinear parametric optimization problem:

$$\begin{aligned} \min_{x, y(\omega), \omega, \alpha} \frac{1}{\gamma} \ln \Big\{ & \exp[\gamma(\alpha^k - \alpha)] + \exp[\gamma(-\alpha)] + \exp[\gamma(\alpha - 1)] \\ & + \exp[\gamma(\varphi(x, y(\omega), \omega) - (\bar{f} - \alpha(\bar{f} - \underline{f})))] \\ & + \exp[\gamma(f_1(x, y(\omega), \omega) - 9(1 - \alpha))] \\ & + \exp[\gamma(f_2(x, y(\omega), \omega) - 2(1 - \alpha))] \\ & + \exp[\gamma(-f_3(x, y(\omega), \omega) + 6(1 - \alpha))] \\ & + \exp[\gamma(-f_4(x, y(\omega), \omega) + 10(1 - \alpha))] \\ & + \exp[\gamma(y_1 + y_2 - 3y_3)] + \exp[\gamma(2y_1 - y_2 + y_3)] \\ & + \exp[\gamma(1 - x_1)] + \exp[\gamma(1 - x_2)] + \exp[\gamma(x_1 - 14)] \\ & + \exp[\gamma(x_2 - 14)] + \exp[\gamma(-y_1)] + \exp[\gamma(-y_2)] + \exp[\gamma(-y_3)] \Big\} \end{aligned} \tag{4.4}$$

with $\gamma$ being sufficiently large, where the optimal values of $\bar{f}$ and $\underline{f}$ are obtained by computing (4.2) and (4.3), respectively.

Choosing $x^0 = (4.0000, 2.0000)$, $y^0(\omega) = (1.0547, 1.0564, 0.1574)$ and $\alpha^0 = 0.2$ and setting $L = 3$ with the probability $p_1 = 0.1590, p_2 = 0.6821, p_3 = 0.1589$, and $\epsilon = 10^{-5}$, $Q = 10^6$ and fixed $\gamma = 12$, then for each iteration of Algorithm 3.2, we first generate the random variable $\omega$ by using normrnd function (that is, normal distribution function) of MATLAB 7.0 software. Secondly, we solve from problems (4.2) and (4.3) to problem (4.4) in turn by the commonly used quasi-Newton line search of MATLAB software 7.0.

Here, the first layer iteration searching optimization is to solve problems (4.2) and (4.3), respectively. And the second optimizing process is to find the optimal solution of problem (4.1) via solving the unconstrained and smooth nonlinear parametric optimization problem (4.4). We only present four optimal solution $(x^*, y^*(\cdot))$ with respect to the random variable $\omega$ and the corresponding membership degree $\alpha^*$ for whole stochastic optimization problem, which is listed in Table 1. Further, Table 2 show that each iteration calculation results including iterative solutions with the random variable $\omega = 0.128808$ for the second optimizing process to this problem. The results for every step in Table 2 (i.e., $k = 0, 1, 2, \cdots, 11$) come from the first layer iteration process, which are too much and so they are omitted.

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Table 1: The optimal solution with the random variable and membership degree

| $k$ $(x^*, y^*(\omega), \omega)$ | $\alpha^*$ |
|---|---|
| 1 (0.984674, 1.158197, 0.036645, 0.228989, 0.128690, 0.128808) | 0.954363 |
| 2 (0.987781, 1.219403, 0.024577, 0.187961, 0.097964, 0.345629) | 0.949652 |
| 3 (0.984337, 1.156955, 0.036586, 0.232004, 0.127346, 0.191080) | 0.953185 |
| 4 (0.988056, 1.226168, 0.023580, 0.184104, 0.095786, 0.353731) | 0.949391 |

Table 2: Data for Computational results with the random variable $\omega = 0.128808$

| $k$ | $(x^k, y^k)$ | $\alpha^k$ | Iterations No. |
|---|---|---|---|
| 0 | (4.0000, 2.0000, 1.0547, 1.0564, 0.1574) | 0.2 | 46 |
| 1 | (0.989779,1.252773,0.027679,0.173089,0.106607) | 0.871896 | 16 |
| 2 | (0.985831,1.166544,0.034618,0.217474,0.125437) | 0.924577 | 13 |
| 3 | (0.984928,1.159731,0.036190,0.226463,0.128032) | 0.945141 | 15 |
| 4 | (0.984726,1.158504,0.036552,0.228473,0.128555) | 0.951749 | 14 |
| 5 | (0.984687,1.158268,0.036623,0.228867,0.128658) | 0.953643 | 9 |
| 6 | (0.984677,1.158241,0.036639,0.228955,0.128681) | 0.954166 | 4 |
| 7 | (0.984676,1.158237,0.036638,0.228959,0.128678) | 0.954309 | 3 |
| 8 | (0.984676,1.158233,0.036638,0.228962,0.128678) | 0.954348 | 5 |
| 9 | (0.984674,1.158198,0.036645,0.228989,0.128690) | 0.954359 | 1 |
| 10 | (0.984674,1.158197,0.036645,0.228989,0.128690) | 0.954362 | 1 |
| 11 | (0.984674,1.158197,0.036645,0.228989,0.128690) | 0.954363 | |

# 5    Concluding remarks

In this paper, by developing a class of new regularization smoothing approximation approaches, we investigated approximation solvability of the following fuzzy parametric variational inequality constrained stochastic optimization problems in $n$-dimension real numeral set $\mathbb{R}^n$:

$$\begin{aligned}
\min_{x,y(\cdot)} \quad & E_\omega[f(x, y(\omega), \omega)] \\
\text{s.t.} \quad & x \in U, \\
& y(\omega) \in C(x, \omega), \\
& \langle F(x, y(\omega), \omega), z(\omega) - y(\omega) \rangle \gtreqless 0, \quad \forall z(\omega) \in C(x, \omega),
\end{aligned} \tag{5.1}$$

which has been very little studied by right of the known theories and approaches in the literature. It is because the existence of the random variable and the fuzzified version mean that (5.1) involves multiple complementarity-type constraints, and solving problem (5.1) is more difficult than solving an ordinary mathematical program with (fuzzy) equilibrium constraints or stochastic mathematical program with equilibrium constraints.

Based on the notion of tolerance approach with entropic regularization and fuzzy set theory, we first showed that solving the stochastic optimization problem with fuzzy parametric variational inequality constraints is equivalent to solving a fuzzy complementarity constrained stochastic optimization problem, which can be converted to a regular nonlinear parametric optimization problem with continuous random variables. Then, we constructed a centre iterative algorithm and developed a class of new regularization smoothing approximation approaches for solving a problem with continuous random variables based on quasi-Monte Carlo estimate and entropic regularization technique, and discussed a comprehensive convergence theory for approximating the resulting optimization problem. Finally, numerical example was provided to illustrate our main results applying quasi-Newton line search of MATLAB software.

We remark that in the paper, based on the concept that fuzzy constraints should yield a fuzzy

objective, we must choose a class of new regularization smoothing approximation approaches to define the objective function value of two optimization problems as the parameters in an equivalent stochastic parametric optimization problem. Hence, the problem presented in this paper is brand new and the method is also new and interesting.

Whether the corresponding results of Theorem 3.3 hold when the objective function is a fuzzy stochastic function, the constraints are fuzzy implicit variational inequalities (such as in [14, 17]) or elliptic inequalities subject to physical phenomenon, and the numerical testing is some large-scale applications, which are still **open questions** to be solved in further research.

# Acknowledgments

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# The Split Common Fixed Point Problem for Demicontractive Mappings in Banach Spaces

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**Abstract.** In this paper, based on the work by Moudafi and inspired by Takahashi and Xu, we try to investigate the split common fixed point problems for the class of demicontractive mappings in the setting of two Banach spaces, and obtain the strong and weak convergence theorems. The results presented in the paper improve and extend some recent well-known corresponding results.

**Keywords**: split common fixed point problem; demicontractive mapping; Demiclosed principle, weak and strong convergence theorems.

**2010 AMS Subject Classification**: 47H09, 49J25.

## 1 Introduction and Preliminaries

The split common fixed point problem was introduced by Moudafi [1] in 2010. Moudafi proposed an iteration scheme and obtained a weak convergence theorem of the split common fixed point problem for demicontractive mappings in the setting of two Hilbert spaces. Since then, many authors investigated the split common fixed point problems of other nonlinear mappings in the setting of two Hilbert spaces (see [2-7]). At the beginning of 2015, Takahashi [8] first attempted to introduce and consider the split feasibility problem and split common null point problem in the setting of one Hilbert space and one Banach space. By using hybrid methods and Halperns type methods under suitable conditions, some strong and weak convergence theorems for such problems are obtained. The results presented in [8] seem to be the first outside Hilbert spaces. This naturally brings us to solve the split common fixed point problem for demicontractive mappings in the setting of two Banach space.

Let $E_1$ and $E_2$ be two real Banach spaces, and $A : E_1 \rightarrow E_2$ be a bounded linear operator such that $A \neq 0$. The split common fixed point problem (SCFP) for nonlinear mappings $S$ and $T$ is to find a point $x \in E_1$ such thst

$$x \in F(S) \quad \text{and} \quad Ax \in F(T), \tag{1.1}$$

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where $F(S)$ and $F(T)$ denote the sets of fixed points of $S$ and $T$, respectively. We use $\Gamma$ to denote the set of solutions of SCFP for mappings $S$ and $T$, that is,

$$\Gamma = \{x \in F(S)| \ Ax \in F(T)\}.$$

In this paper, we use the following algorithm to approximate a split common fixed point of demicontractive mappings in the setting of two Banach spaces.

**Algorithm:** Let $E_1$ and $E_2$ be two real Banach spaces, $A : E_1 \to E_2$ be a bounded linear operator, $A^*$ be the adjoint operator of $A$ and $J_i$ be the normalized duality mapping from $E_i$ to $2^{E_i^*}$, $\ i = 1, 2$. Now, we define the iterative scheme $\{x_n\}$:
Let $x_1 \in E_1$ be arbitrary, for all $n \geq 1$, set

$$y_n = x_n + \gamma J_1^{-1} A^* J_2 (T - I) A x_n, \tag{1.2}$$

$$x_{n+1} = (1 - \alpha_n)y_n + \alpha_n S(y_n), \tag{1.3}$$

where $S : E_1 \to E_1$ and $T : E_2 \to E_2$ are two demicontractive mappings.

Under some suitable conditions, the iterative scheme $\{x_n\}$ is shown to converge weakly and strongly to a split common fixed point of demicontractive mappings $T$ and $S$. Our result extends the split common fixed point problem from Hilbert spaces to Banach spaces.

In order to solve this problem mentioned above, we recall the following concepts and results.

Let $E$ be a real Banach space with norm $\| \cdot \|$ and let $E^*$ be the dual space of $E$, We denote the value of $y^* \in E$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in $E$, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$.

We recall that $T : E \to E$ is demicontractive (see for example [9]) if there exists a constant $\eta \in [0, 1)$ such that

$$\|Tx - q\|^2 \leq \|x - q\|^2 + \eta \|x - Tx\|^2, \ \ \forall (x, q) \in E \times F(T). \tag{1.4}$$

An operator satisfying (1.4) will be referred to as a $\eta$-demicontractive mapping.

It is worth noting that the class of demicontractive maps contains important operators such as the quasi-nonexpansive maps and the strictly pseudocontractive maps with fixed points.

A mapping $T : E \to E$ is called quasi-nonexpansive, if

$$\|Tx - q\| \leq \|x - q\|$$

for all $(x, q) \in E \times F(T)$. A mapping $T : E \to E$ is strictly pseudocontractive, if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \beta \|x - y - (Tx - Ty)\|^2$$

for all $(x, y) \in E \times E$ and for some $\beta \in [0, 1)$.

A mapping $T : E \to E$ is called demiclosed at zero, if for any sequence $\{x_n\} \subset E$ and $x \in E$, we have

$$x_n \rightharpoonup x, \ (I - T)(x_n) \to 0 \Rightarrow x \in F(T).$$

A mapping $S : E \to E$ is said to be semi-compact, if for any sequence $\{x_n\}$ in $E$ such that $\|x_n - Sx_n\| \to 0, (n \to \infty)$, there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $x^* \in E$.

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The normalized duality mapping $J$ from $E$ to $2^{E^*}$ is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in E.$$

Let $U = \{x \in E : \|x\| = 1\}$. The norm of $E$ is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \quad \text{exists.}$$

In the case, $E$ is called smooth. $E$ is smooth if and only if $J$ is single-valued. We denote the single-valued normalized duality mapping by $J$.

The modulus of convexity of $E$ is defined by

$$\delta_E(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon \right\},$$

for every $\epsilon$ with $0 \leq \epsilon \leq 2$. A Banach space $E$ is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. $E$ is said to be $p$-uniformly convex, if there exists a constant $a > 0$ such that $\delta_E(\epsilon) \geq a\epsilon^p$ for all $0 < \epsilon \leq 2$.

Let $\rho_E : [0, \infty) \to [0, \infty)$ be the modulus of smoothness of $E$ defined by

$$\rho_E(t) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x \in U, \|y\| \leq t\}.$$

A Banach space $E$ is said to be uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$. Let $q$ be a fixed real number with $q > 1$. Then a Banach space $E$ is said to be $q$-uniformly smooth if there exists a constant $b > 0$ such that $\rho_E(t) \leq bt^q$ for all $t > 0$. It is well known that every $q$-uniformly smooth Banach space is uniformly smooth.

A Banach space $E$ is said to satisfy the Opial's condition [10] if for any sequence $\{x_n\} \subset E$, $x_n \rightharpoonup x$ implies

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|,$$

for all $y \in E$ with $y \neq x$.

**Lemma 1.1.** [11] *Let $E$ be a 2-uniformly convex Banach space. Then the following inequality holds:*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)c\|x - y\|^2, \quad \forall x, y \in E, \quad (1.5)$$

*where $0 \leq \lambda \leq 1$, $c = \mu(1) > 0$,*

$$
\begin{aligned}
\mu(t) : \quad &= \quad \inf \left\{ \frac{\lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \|\lambda x + (1 - \lambda)y\|^2}{\lambda(1 - \lambda)} : \right. \\
&\qquad \left. 0 < \lambda < 1, x, y \in E \ \text{ and } \ \|x - y\| = t \right\} \\
&> \quad 0.
\end{aligned}
$$

**Lemma 1.2.** [11] *Let $E$ be a 2-uniformly smooth Banach space with the best smoothness constants $\kappa > 0$. Then the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jy \rangle + 2\|\kappa y\|^2,$$

*for all $x, y \in E$.*

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## 2    Main Results

**Lemma 2.1.** *Let $E_1$ be a real 2-uniformly convex and 2-uniformly smooth Banach spaces with the best smoothness constant $\kappa$ satisfying $0 < \kappa < \frac{1}{\sqrt{2}}$ , $E_2$ be a real Banach space, and $A : E_1 \to E_2$ be a bounded linear operator. Let $S : E_1 \to E_1$ be $\beta$-demicontractive and $T : E_2 \to E_2$ be $\eta$-demicontractive with $F(S) \neq \emptyset$ and $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by algorithm (1.2)-(1.3) is Féjer-monotone with respect to $\Gamma = \{x \in F(S) | Ax \in F(T)\}$, that is, for every $z \in \Gamma$,*

$$\|x_{n+1} - z\| \leq \|x_n - z\|, \ \ \forall n \in N,$$

*where $0 < \gamma < \min\left\{\frac{1-\eta}{\|A\|^2}, \frac{1-2\kappa}{\|A\|^2}\right\}$ and $\alpha_n \in (0, 1 - \frac{\beta}{c}]$, $\beta < c = \mu(1)$.*

**Proof.** Let $z \in \Gamma$. Then $z \in F(S)$ and $Az \in F(T)$. It follows from Lemma 1.1 and (1.3) that

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &= \|(1 - \alpha_n)y_n + \alpha_n S(y_n) - z\|^2 \\
&= \|(1 - \alpha_n)(y_n - z) + \alpha_n(S(y_n) - z)\|^2 \\
&\leq (1 - \alpha_n)\|y_n - z\|^2 + \alpha_n\|S(y_n) - z\|^2 \\
&\quad -\alpha_n(1 - \alpha_n)c\|S(y_n) - y_n\|^2 \qquad\qquad (2.1) \\
&\leq (1 - \alpha_n)\|y_n - z\|^2 + \alpha_n\|y_n - z\|^2 + \alpha_n\beta\|S(y_n) - y_n\|^2 \\
&\quad -\alpha_n(1 - \alpha_n)c\|S(y_n) - y_n\|^2 \\
&\leq \|y_n - z\|^2 - \alpha_n(c - \beta - c\alpha_n)\|S(y_n) - y_n\|^2,
\end{aligned}
$$

where $c = \mu(1)$.

On the other hand, It follows from (1.2) and Lemma 1.2 that

$$
\begin{aligned}
\|y_n - z\|^2 &= \|x_n + \gamma J_1^{-1} A^* J_2(T - I)Ax_n - z\|^2 \\
&= \|x_n - z + \gamma J_1^{-1} A^* J_2(T - I)Ax_n\|^2 \\
&\leq \|\gamma J_1^{-1} A^* J_2(T - I)Ax_n\|^2 + 2\gamma\langle x_n - z, A^* J_2(T - I)Ax_n\rangle \\
&\quad +2\kappa^2\|x_n - z\|^2 \\
&\leq \gamma^2\|A\|^2\|(T - I)Ax_n\|^2 + 2\gamma\langle Ax_n - Az, J_2(T - I)Ax_n\rangle \\
&\quad +2\kappa^2\|x_n - z\|^2 \\
&= \gamma^2\|A\|^2\|(T - I)Ax_n\|^2 + 2\kappa^2\|x_n - z\|^2 \\
&\quad 2\gamma\langle Ax_n - TAx_n + TAx_n - Az, J_2(T - I)Ax_n\rangle \\
&\leq \gamma^2\|A\|^2\|(T - I)Ax_n\|^2 + 2\kappa^2\|x_n - z\|^2 \\
&\quad -2\gamma\|(T - I)Ax_n\|^2 + 2\gamma\langle TAx_n - Az, J_2(T - I)Ax_n\rangle \\
&\leq (\gamma^2\|A\|^2 - 2\gamma)\|(T - I)Ax_n\|^2 + 2\kappa^2\|x_n - z\|^2 \\
&\quad +\gamma(\|TAx_n - Az\|^2 + \|(T - I)Ax_n\|^2) \\
&\leq 7(\gamma^2\|A\|^2 - 2\gamma)\|(T - I)Ax_n\|^2 + 2\kappa^2\|x_n - z\|^2 \\
&\quad +\gamma(\|Ax_n - Az\|^2 + \eta(Ax_n - TAx_n\|^2 + \|(T - I)Ax_n\|^2) \\
&\leq (2\kappa^2 + \gamma\|A\|^2)\|x_n - z\|^2 - \gamma(1 - \eta - \gamma\|A\|^2)\|(T - I)Ax_n\|^2,
\end{aligned}
$$

where $A^*$ is the adjoint operator of $A$ and $J_i$ is the normalized duality mapping from $E_i$ to $2^{E_i^*}, i = 1, 2$.

In addition, since $0 < \kappa < \frac{1}{\sqrt{2}}$ and $0 < \gamma < \frac{1-2\kappa^2}{\|A\|^2}$, $0 < \gamma\|A\|^2 + 2\kappa^2 < 1$, so we have

$$\|y_n - z\|^2 \leq \|x_n - z\|^2 - \gamma(1 - \eta - \gamma\|A\|^2)\|(T - I)Ax_n\|^2. \qquad (2.2)$$

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It follows from (2.1) and (2.2) that

$$\|x_{n+1} - z\|^2 \;\leq\; \|x_n - z\|^2 - \gamma(1 - \eta - \gamma\|A\|^2)\|(T - I)Ax_n\|^2 \qquad (2.3)$$
$$- \alpha_n(c - \beta - c\alpha_n)\|S(y_n) - y_n\|^2.$$

Finally, by the assumptions on $\gamma$ and $\alpha_n$, we obtain the desired result.

**Theorem 2.2.** *Let $E_1$ be a real 2-uniformly convex and 2-uniformly smooth Banach space satisfying Opial's condition with the best smoothness constant $\kappa$ satisfying $0 < \kappa < \frac{1}{\sqrt{2}}$, and $E_2$ be a real Banach space. Let $A : E_1 \to E_2$ be a bounded linear operator, $S : E_1 \to E_1$ and $T : E_2 \to E_2$ be two demicontractive mappings with constants $\beta$ and $\eta$ with $F(S) \neq \emptyset$ and $F(T) \neq \emptyset$, respectively. Assume that $I - S$ and $I - T$ are demiclosed at zero. If $\Gamma \neq \emptyset$, then the sequence $\{x_n\}$ generated by algorithm (1.2)-(1.3) converges weakly to a split common fixed point $x \in \Gamma$, for $0 < \gamma < \min\left\{\frac{1-\eta}{\|A\|^2}, \frac{1-2\kappa}{\|A\|^2}\right\}$, $\alpha_n \in (\delta, 1 - \frac{\beta}{c} - \delta)$, $\beta < c = \mu(1)$, and for small enough $\delta > 0$.*

**Proof.** From (2.3) and the fact that $0 < \gamma < \min\left\{\frac{1-\eta}{\|A\|^2}, \frac{1-2\kappa}{\|A\|^2}\right\}$ and $\alpha_n \in (\delta, 1 - \frac{\beta}{c} - \delta)$, we obtain that the sequence $\{\|x_n - z\|\}$ is monotonically decreasing and thus converges to some positive real limit $l(z)$. From (2.3), we have

$$\gamma(1 - \eta - \gamma\|A\|^2)\|(I - T)Ax_n\|^2 \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2.$$

Therefore,

$$\lim_{n \to \infty} \|(I - T)Ax_n\| = 0. \qquad (2.4)$$

From the *Féjer*-monotonicity of $\{x_n\}$, it follows that the sequence is bounded. Denoting by $x$ a weak-cluster point of $\{x_n\}$. Let $k = 0, 1, 2, \ldots$ be the sequence of indices, such that $x_{n_k} \rightharpoonup x$, as $k \to \infty$. Then from (2.4) and demiclosedness of $I - T$ at zero, we obtain $T(Ax) = Ax$, that is, $Ax \in F(T)$.

Now, by setting $y_n = x_n + \gamma J_1^{-1}A^* J_2(I - T)Ax_n$, it follows that $y_{n_k} \rightharpoonup x$. Again from (2.3), we obtain

$$\alpha_n(c - \beta - c\alpha_n)\|y_n - S(y_n)\|^2 \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2.$$

Using the convergence of the sequence $\{\|x_n - z\|\}$, we get

$$\lim_{n \to \infty} \|y_n - S(y_n)\| = 0, \qquad (2.5)$$

which combined with the demiclosedness of $I - S$ at zero and the weak convergence of $\{y_{n_k}\}$ to $y$ yields $S(x) = x$. Hence, $x \in F(S)$ and therefore $x \in \Gamma$. Since $E_1$ satises Opial's condition, we know that $\{x_n\}$ converges weakly to $x \in \Gamma$.

**Theorem 2.3.** *Let $E_1$ be a real 2-uniformly convex and 2-uniformly smooth Banach space satisfying Opial's condition with the best smoothness constant $\kappa$ satisfying $0 < \kappa < \frac{1}{\sqrt{2}}$ , and $E_2$ be a real Banach space. Let $A : E_1 \to E_2$ be a bounded linear operator, $S : E_1 \to E_1$ and $T : E_2 \to E_2$ be two demicontractive mappings with constants $\beta$ and $\eta$ with $F(S) \neq \emptyset$ and $F(T) \neq \emptyset$, respectively. Assume that $I - S$ and $I - T$ are demiclosed at zero. If $\Gamma \neq \emptyset$ and $S$ is semi-compact, then the sequence $\{x_n\}$ generated by algorithm (1.2)-(1.3) converges strongly to a split common fixed point $x \in \Gamma$, for $0 < \gamma < min\{\frac{1-\eta}{\|A\|^2}, \frac{1-2\kappa}{\|A\|^2}\}$, $\alpha_n \in (\delta, 1 - \frac{\beta}{c} - \delta)$, $\beta < c = \mu(1)$, and for a small enough $\delta > 0$.*

**Proof.** It follows from (1.2) that

$$\|x_n - y_n\| = \|J_1(x_n - y_n)\| = \|\gamma A^* J_2(I - T)Ax_n\|,$$

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and so, from (2.4) we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{2.6}$$

Since $S$ is semi-compact, from (2.5), there exist subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that $\{y_{n_j}\}$ converges strongly to $x^* \in E_1$. Using (2.6), we know that $\{x_{n_j}\}$ converges strongly to $x^*$. By Theorem 2.2, we know that $\{x_n\}$ converges weakly to $x$, so we have $x^* = x$. Since $\lim_{n\to\infty} \|x_n - x\|$ exists and $\lim_{j\to\infty} \|x_{n_j} - x\| = 0$, we know that $\{x_n\}$ converges strongly to $x \in \Gamma$.

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6

# Iterated binomial transform of the $k$-Lucas sequence

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**Abstract**

In this study, we apply "$r$" times the binomial transform to $k$-Lucas sequence. Also, the Binet formula, summation, generating function of this transform are found using recurrence relation. Finally, we give the properties of iterated binomial transform with classical Lucas sequence.

**Keywords**: $k$-Lucas sequence, iterated binomial transform, Pell sequence.

*Ams Classification: 11B65, 11B83.*

## 1  Introduction and Preliminaries

There are so many studies in the literature that concern about the special number sequences such as Fibonacci, Lucas and generalized Fibonacci anad Lucas numbers (see, for example [1]-[3], and the references cited therein). In Fibonacci and Lucas numbers, there clearly exists the term Golden ratio which is defined as the ratio of two consecutive of these numbers that converges to $\alpha = \frac{1+\sqrt{5}}{2}$. It is also clear that the ratio has so many applications in, specially, Physics, Engineering, Architecture, etc.[4]. Also, many generalizations of the Fibonacci sequence have been introduced and studied matrix applications of this sequence in [13]-[16].

For $n \geq 1$, $k$-Lucas sequence is defined by the recursive equation:

$$L_{k,n+1} = kL_{k,n} + L_{k,n-1}, \qquad L_{k,0} = 2 \text{ and } L_{k,1} = k. \tag{1.1}$$

In addition, some matrix-based transforms can be introduced for a given sequence. Binomial transform is one of these transforms and there are also other ones such as rising and falling binomial transforms(see [5]-[12]). Given

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an integer sequence $X = \{x_0, x_1, x_2, \ldots\}$, the binomial transform $B$ of the sequence $X$, $B(X) = \{b_n\}$, is given by

$$b_n = \sum_{i=0}^{n} \binom{n}{i} x_i.$$

In [10], authors gave the application of the several class of transforms to the $k$-Lucas sequence. For example, for $n \geq 1$, authors obtained recurrence relation of the binomial transform for $k$-Lucas sequence

$$b_{k,n+1} = (2+k)\, b_{k,n} - k b_{k,n-1}, \quad b_{k,0} = 2 \text{ and } b_{k,1} = k + 2.$$

Falcon [11] studied the iterated application of some Binomial transforms to the $k$-Fibonacci sequence. For example, author obtained recurrence relation of the iterated binomial transform for $k$-Fibonacci sequence

$$c_{k,n+1}^{(r)} = (2r+k)\, c_{k,n}^{(r)} - \left(r^2 + kr - 1\right) c_{k,n-1}^{(r)}, \quad c_{k,0}^{(r)} = 0 \text{ and } c_{k,1}^{(r)} = 1.$$

Motivated by [11, 12], the goal of this paper is to apply iteratively the binomial transform to the $k$-Lucas sequence. Also, the properties of this transform are found by recurrence relation. Finally, the relation of between the transform and the iterated binomial transform of $k$-Fibonacci sequence by deriving new formulas are illustrated.

## 2 Iterated Binomial Transform of $k$-Lucas Sequences

In this section, we will mainly focus on iterated binomial transforms of $k$-Lucas sequences to get some important results. In fact, we will also present the recurrence relation, Binet formula, summation, generating function of the transform and relationships betweeen of the transform and iterated binomial transform of $k$-Fibonacci sequence.

The iterated binomial transform of the $k$-Lucas sequences is demonstrated by $B_k^{(r)} = \left\{ b_{k,n}^{(r)} \right\}$, where $b_{k,n}^{(r)}$ is obtained by applying "$r$" times the binomial transform to $k$-Lucas sequence. It is obvious that $b_{k,0}^{(r)} = 2$ and $b_{k,1}^{(r)} = 2r + k$.

The following lemma will be the key proof of the next theorems.

**Lemma 2.1** *For $n \geq 0$ and $r \geq 1$, the following equality hold:*

$$b_{k,n+1}^{(r)} = b_{k,n}^{(r)} + \sum_{j=0}^{n} \binom{n}{j} b_{k,j+1}^{(r-1)}.$$

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**Proof.** By using definition of binomial transform and the well known binomial equality

$$\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1},$$

we obtain

$$
\begin{aligned}
b_{k,n+1}^{(r)} &= \sum_{j=0}^{n+1} \binom{n+1}{j} b_{k,j}^{(r-1)} \\
&= \sum_{j=1}^{n+1} \binom{n+1}{j} b_{k,j}^{(r-1)} + b_{k,0}^{(r-1)}
\end{aligned}
$$

$$
\begin{aligned}
b_{k,n+1}^{(r)} &= \sum_{j=1}^{n+1} \binom{n}{j} b_{k,j}^{(r-1)} + \sum_{j=1}^{n+1} \binom{n}{j-1} b_{k,j}^{(r-1)} + b_{k,0}^{(r-1)} \\
&= \sum_{j=0}^{n+1} \binom{n}{j} b_{k,j}^{(r-1)} + \sum_{j=0}^{n+1} \binom{n}{j-1} b_{k,j}^{(r-1)} \\
&= \sum_{j=0}^{n} \binom{n}{j} b_{k,j}^{(r-1)} + \sum_{j=-1}^{n} \binom{n}{j} b_{k,j+1}^{(r-1)} \\
&= b_{k,n}^{(r)} + \sum_{j=0}^{n} \binom{n}{j} b_{k,j+1}^{(r-1)}
\end{aligned}
$$

which is desired result. ∎

In [10], the authors obtained the following equality for binomial transform of $k$-Lucas sequences. However, in here, we obtain the equality in terms of iterated binomial transform of the $k$-Lucas sequences as a consequence of Lemma 2.1. To do that we take $r = 1$ in Lemma 2.1:

$$b_{k,n+1} = b_{k,n} + \sum_{j=0}^{n} \binom{n}{j} L_{k,j+1}.$$

**Theorem 2.1** *For $n \geq 0$ and $r \geq 1$, the recurrence relation of sequence $\left\{ b_{k,n}^{(r)} \right\}$ is*

$$b_{k,n+1}^{(r)} = (2r + k)\, b_{k,n}^{(r)} - \left( r^2 + kr - 1 \right) b_{k,n-1}^{(r)}, \qquad (2.1)$$

*with initial conditions $b_{k,0}^{(r)} = 2$ and $b_{k,1}^{(r)} = 2r + k$.*

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**Proof.** The proof will be done by induction steps on $r$ and $n$.

First of all, for $r = 1$, from the equality 2.2 in [10], it is true $b_{k,n+1} = (2 + k)\, b_{k,n} - k b_{k,n-1}$.

Let us consider definition of iterated binomial transform, then we have

$$b_{k,2}^{(r)} = k^2 + 2rk + 2r^2 + 2.$$

The initial conditions are

$$b_{k,0}^{(r)} = 2 \text{ and } b_{k,1}^{(r)} = 2r + k.$$

Hence, for $n = 1$, the Eq. (2.1) is true, that is $b_{k,2}^{(r)} = (2r + k)\, b_{k,1}^{(r)} - \left(r^2 + kr - 1\right) b_{k,0}^{(r)}$.

Actually, by assuming the Eq. (2.1) holds for all $(r - 1, n)$ and $(r, n - 1)$, that is,

$$b_{k,n+1}^{(r-1)} = (2r - 2 + k)\, b_{k,n}^{(r-1)} - \left((r - 1)^2 + k\,(r - 1) - 1\right) b_{k,n-1}^{(r-1)},$$

and

$$b_{k,n}^{(r)} = (2r + k)\, b_{k,n-1}^{(r)} - \left(r^2 + kr - 1\right) b_{k,n-2}^{(r)}.$$

Now, by taking into account Lemma 2.1, we obtain

$$
\begin{aligned}
b_{k,n+1}^{(r)} &= b_{k,n}^{(r)} + \sum_{j=0}^{n} \binom{n}{j} b_{k,j+1}^{(r-1)} \\
&= \sum_{j=0}^{n} \binom{n}{j} b_{k,j}^{(r-1)} + \sum_{j=0}^{n} \binom{n}{j} b_{k,j+1}^{(r-1)} \\
&= \sum_{j=1}^{n} \binom{n}{j} \left( b_{k,j}^{(r-1)} + b_{k,j+1}^{(r-1)} \right) + b_{k,0}^{(r-1)} + b_{k,1}^{(r-1)}.
\end{aligned}
$$

By reconsidering our assumption, we write

$$
\begin{aligned}
b_{k,n+1}^{(r)} &= \sum_{j=1}^{n} \binom{n}{j} \left( b_{k,j}^{(r-1)} + (2r - 2 + k)\, b_{k,j}^{(r-1)} - \left(r^2 - 2r + kr - k\right) b_{k,j-1}^{(r-1)} \right) + b_{k,0}^{(r-1)} + b_{k,1}^{(r-1)} \\
&= (2r + k - 1) \sum_{j=1}^{n} \binom{n}{j} b_{k,j}^{(r-1)} - \left(r^2 - 2r + kr - k\right) \sum_{j=1}^{n} \binom{n}{j} b_{k,j-1}^{(r-1)} + b_{k,0}^{(r-1)} + b_{k,1}^{(r-1)} \\
&= (2r + k - 1) \sum_{j=0}^{n} \binom{n}{j} b_{k,j}^{(r-1)} - \left(r^2 - 2r + kr - k\right) \sum_{j=1}^{n} \binom{n}{j} b_{k,j-1}^{(r-1)} + b_{k,0}^{(r-1)} + b_{k,1}^{(r-1)} \\
&\quad - (2r + k - 1)\, b_{k,0}^{(r-1)} \\
&= (2r + k - 1)\, b_{k,n}^{(r)} - \left(r^2 - 2r + kr - k\right) \sum_{j=1}^{n} \binom{n}{j} b_{k,j-1}^{(r-1)} + (2 - 2r - k)\, b_{k,0}^{(r-1)} + b_{k,1}^{(r-1)}.
\end{aligned}
$$

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Then we have

$$b_{k,n+1}^{(r)} - (2r + k - 1) b_{k,n}^{(r)} = -\left(r^2 - 2r + kr - k\right) \sum_{j=1}^{n} \binom{n}{j} b_{k,j-1}^{(r-1)} + 4 - 2r - k.$$

$$(2.2)$$

By taking $n \to n - 1$, it is

$$
\begin{aligned}
b_{k,n}^{(r)} &= (2r + k - 1) b_{k,n-1}^{(r)} - \left(r^2 - 2r + kr - k\right) \sum_{j=1}^{n-1} \binom{n-1}{j} b_{k,j-1}^{(r-1)} + 4 - 2r - k \\
&= (2r + k - 1) b_{k,n-1}^{(r)} - \left(r^2 - 2r + kr - k\right) \sum_{j=1}^{n} \left[ \binom{n}{j} - \binom{n-1}{j-1} \right] b_{k,j-1}^{(r-1)} + 4 - 2r - k \\
&= (2r + k - 1) b_{k,n-1}^{(r)} - \left(r^2 - 2r + kr - k\right) \sum_{j=1}^{n} \binom{n}{j} b_{k,j-1}^{(r-1)} \\
&\quad + \left(r^2 - 2r + kr - k\right) \sum_{j=1}^{n} \binom{n-1}{j-1} b_{k,j-1}^{(r-1)} + 4 - 2r - k
\end{aligned}
$$

$$
\begin{aligned}
b_{k,n}^{(r)} &= (2r + k - 1) b_{k,n-1}^{(r)} - \left(r^2 - 2r + kr - k\right) \sum_{j=1}^{n} \binom{n}{j} b_{k,j-1}^{(r-1)} \\
&\quad + \left(r^2 - 2r + kr - k\right) \sum_{j=0}^{n-1} \binom{n-1}{j} b_{k,j}^{(r-1)} + 4 - 2r - k \\
&= (2r + k - 1) b_{k,n-1}^{(r)} - \left(r^2 - 2r + kr - k\right) \sum_{j=1}^{n} \binom{n}{j} b_{k,j-1}^{(r-1)} \\
&\quad + \left(r^2 - 2r + kr - k\right) b_{k,n-1}^{(r)} + 4 - 2r - k \\
&= \left(r^2 + kr - 1\right) b_{k,n-1}^{(r)} - \left(r^2 - 2r + kr - k\right) \sum_{j=1}^{n} \binom{n}{j} b_{k,j-1}^{(r-1)} + 4 - 2r - k.
\end{aligned}
$$

Hence, we have

$$b_{k,n}^{(r)} - \left(r^2 + kr - 1\right) b_{k,n-1}^{(r)} = -\left(r^2 - 2r + kr - k\right) \sum_{j=1}^{n} \binom{n}{j} b_{k,j-1}^{(r-1)} + 4 - 2r - k.$$

If last expression put in place in the equation (2.2), then we get

$$
\begin{aligned}
b_{k,n+1}^{(r)} &= (2r + k - 1) b_{k,n}^{(r)} + b_{k,n}^{(r)} - \left(r^2 + kr - 1\right) b_{k,n-1}^{(r)} \\
&= (2r + k) b_{k,n}^{(r)} - \left(r^2 + kr - 1\right) b_{k,n-1}^{(r)}
\end{aligned}
$$

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which completed the proof of this theorem. ∎

The characteristic equation of sequence $\left\{b_{k,n}^{(r)}\right\}$ in (2.1) is $\lambda^2 - (2r + k)\lambda + r^2 + kr - 1 = 0$. Let $\lambda_1$ and $\lambda_2$ be the roots of this equation. Then, Binet's formulas of sequence $\left\{b_{k,n}^{(r)}\right\}$ can be expressed as

$$b_{k,n}^{(r)} = \left(\frac{k + \sqrt{k^2 + 4}}{2} + r\right)^n + \left(\frac{k - \sqrt{k^2 + 4}}{2} + r\right)^n. \qquad (2.3)$$

In here, we obtain the equalities given in [10] in terms of iterated binomial transform of the $k$-Lucas sequences as a consequence of Theorem 2.1. To do that we take $r = 1$ in Theorem 2.1 and the Eq. (2.3):

$$b_{k,n+1} = (2 + k)\, b_{k,n} - kb_{k,n-1},$$

and

$$b_{k,n} = \left(\frac{k + 2 + \sqrt{k^2 + 4}}{2}\right)^n + \left(\frac{k + 2 - \sqrt{k^2 + 4}}{2}\right)^n.$$

Now, we give the sum of iterated binomial transform for $k$-Lucas sequences.

**Theorem 2.2** *Sum of sequence* $\left\{b_{k,n}^{(r)}\right\}$ *is*

$$\sum_{i=0}^{n-1} b_{k,i}^{(r)} = \frac{\left(r^2 + kr - 1\right) b_{k,n-1}^{(r)} - b_{k,n}^{(r)} - k - 2r + 2}{r^2 + kr - k - 2r}.$$

**Proof.** By considering Eq. (2.3), we have

$$\sum_{i=0}^{n-1} b_{k,i}^{(r)} = \sum_{i=0}^{n-1} \left(\lambda_1^i + \lambda_2^i\right).$$

Then we obtain

$$\sum_{i=0}^{n-1} b_{k,i}^{(r)} = \left(\frac{\lambda_1^n - 1}{\lambda_1 - 1}\right) + \left(\frac{\lambda_2^n - 1}{\lambda_2 - 1}\right).$$

Afterward, by taking into account equations $\lambda_1.\lambda_2 = r^2 + kr - 1$ and $\lambda_1 + \lambda_2 = k + 2r$, we conclude

$$\sum_{i=0}^{n-1} b_{k,i}^{(r)} = \frac{\left(r^2 + kr - 1\right) b_{k,n-1}^{(r)} - b_{k,n}^{(r)} - k - 2r + 2}{r^2 + kr - k - 2r}.$$

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■

Note that, if we take $r = 1$ in Theorem 2.2, we obtain the summation of binomial transform for $k$-Lucas sequence:

$$\sum_{i=0}^{n-1} b_{k,i} = b_{k,n} - k b_{k,n-1} + k$$

**Theorem 2.3** *The generating function of the iterated binomial transform for* $\{L_{k,n}\}$ *is*

$$\sum_{i=0}^{\infty} b_{k,i}^{(r)} x^i = \frac{2 - (2r + k) x}{1 - (2r + k) x + (r^2 + kr - 1) x^2}.$$

**Proof.** Assume that $b(k, x, r) = \sum_{i=0}^{\infty} b_{k,i}^{(r)} x^i$ is the generating function of the iterated binomial transform for $\{L_{k,n}\}$. From Theorem 2.1, we obtain

$$
\begin{aligned}
b(k, x, r) &= b_{k,0}^{(r)} + b_{k,1}^{(r)} x + \sum_{i=2}^{\infty} \left( (2r + k) b_{k,i-1}^{(r)} - \left(r^2 + kr - 1\right) b_{k,i-2}^{(r)} \right) x^i \\
&= b_{k,0}^{(r)} + b_{k,1}^{(r)} x - (2r + k) b_{k,0}^{(r)} x + (2r + k) x \sum_{i=0}^{\infty} b_{k,i}^{(r)} x^i \\
&\quad - \left(r^2 + kr - 1\right) x^2 \sum_{i=0}^{\infty} b_{k,i}^{(r)} x^i \\
&= b_{k,0}^{(r)} + \left( b_{k,1}^{(r)} - (2r + k) b_{k,0}^{(r)} \right) x + (2r + k) x b(k, x, r) \\
&\quad - \left(r^2 + kr - 1\right) x^2 b(k, x, r).
\end{aligned}
$$

Now rearrangement of the equation implies that

$$b(k, x, r) = \frac{b_{k,0}^{(r)} + \left( b_{k,1}^{(r)} - (2r + k) b_{k,0}^{(r)} \right) x}{1 - (2r + k) x + (r^2 + kr - 1) x^2},$$

which equals to the $\sum_{i=0}^{\infty} b_{k,i}^{(r)} x^i$ in theorem. Hence, the result. ■

In here, we obtain the generating function given in [10] in terms of iterated binomial transform of the $k$-Lucas sequences as a consequence of Theorem 2.3. To do that we take $r = 1$ in Theorem 2.3:

$$\sum_{i=0}^{\infty} b_{k,i} x^i = \frac{2 - (2 + k) x}{1 - (2 + k) x + k x^2}.$$

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In the following theorem, we present the relationship between the iterated binomial transform of $k$-Lucas sequence and iterated binomial transform of $k$-Fibonacci sequence.

**Theorem 2.4** *For $n > 0$, the relationship of between the transforms $\left\{ b_{k,n}^{(r)} \right\}$ and $\left\{ c_{k,n}^{(r)} \right\}$ is illustrated by following way:*

$$b_{k,n}^{(r)} = c_{k,n+1}^{(r)} - \left( r^2 + kr - 1 \right) c_{k,n-1}^{(r)}, \tag{2.4}$$

*where $b_{k,n}^{(r)}$ is the iterated binomial transform of $k$-Lucas sequence and $c_{k,n}^{(r)}$ is the iterated binomial transform of $k$-Fibonacci sequence.*

**Proof.** By using the Eq.(2.4), let be

$$b_{k,n}^{(r)} = X c_{k,n+1}^{(r)} + Y c_{k,n-1}^{(r)}.$$

If we take $n = 1$ and 2, we have the system

$$\begin{cases} b_{k,1}^{(r)} = X c_{k,2}^{(r)} + Y c_{k,0}^{(r)}, \\ b_{k,2}^{(r)} = X c_{k,3}^{(r)} + Y c_{k,1}^{(r)}. \end{cases}$$

By considering definition of the iterated binomial transforms for $k$-Lucas, $k$-Fibonacci sequence and Cramer rule for the system, we obtain

$$\begin{cases} 2r + k = (2r + k) X, \\ k^2 + 2rk + 2r^2 + 2 = \left( 3r^2 + 3rk + k^2 + 1 \right) X + Y \end{cases}$$

and

$$X = 1 \text{ and } Y = - \left( r^2 + kr - 1 \right)$$

which is completed the proof of this theorem. ∎

Note that, if we take $r = 1$ in Theorem 2.4, we obtain the relationship of between the binomial transform for $k$-Lucas sequence and the binomial transform for $k$-Fibonacci sequence:

$$b_{k,n} = c_{k,n+1} - k c_{k,n-1}.$$

**Corollary 2.1** *We should note that choosing $k = 1$ in the all results of section 2, it is actually obtained some properties of the iterated binomial transform for classical Lucas sequence such that the recurrence relation, Binet formula, summation, generating function and relationship of between binomial transforms for Fibonacci and Lucas sequences.*

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**Corollary 2.2** *We should note that choosing $k = 2$ in the all results of section 2, it is actually obtained some properties of the iterated binomial transform for classical Pell-Lucas sequence such that the recurrence relation, Binet formula, summation, generating function and relationship of between binomial transforms for Pell and Pell-Lucas sequences.*

**Conclusion 2.1** *In this paper, we define the iterated binomial transform for k-Lucas sequence and present some properties of this transform. By the results in Sections 2 of this paper, we have a great opportunity to compare and obtain some new properties over this transform. This is the main aim of this paper. Thus, we extend some recent result in the literature.*

*In the future studies on the iterated binomial transform for number sequences, we expect that the following topics will bring a new insight.*

**(1)** *It would be interesting to study the iterated binomial transform for Fibonacci and Lucas matrix sequences,*

**(2)** *Also, it would be interesting to study the iterated binomial transform for Pell and Pell-Lucas matrix sequences.*

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# Nielsen fixed point theory for digital images

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September 1, 2015

### Abstract

In this paper, we introduce the Nielsen fixed point theory in digital images. We also deal with some important properties of the Nielsen number and calculate the Nielsen number of some digital images. We get some new results using digital covering maps and Nielsen number.

**Keywords:** Fixed point, Nielsen number, digital homotopy.
**2010 Mathematics Subject Classification.** 47H10, 54H25, 68U10.

## 1 Introduction

Digital topology is often used in computer graphics, pattern recognition and image processing. This topic has been studied by important researchers such as Rosenfeld, Kong, Kopperman, Boxer, Karaca, Han, etc. Their goal is to determine not only similarities but also differences between digital images and topology.

Fixed point theory with applications is an important area in topology. This theory continues to develop with new computations and come out of new invariants. Nielsen fixed point theorem is a notable theorem in this theory because it gives a way to count fixed points. One of the main goals in digital topology is to classify digital images. For this reason, we use the Nielsen number which is a powerful invariant for digital images.

In 1920s, Jakop Nielsen introduced the Nielsen theory and the Nielsen number. He focused on both the existence problem of fixed points and the problem of determining the minimal number of fixed points in the homotopy classes. He did this by introducing the Nielsen number of a self map. This number is a homotopy invariant lower bound for the number of fixed points of the map. In this area, there are significant works [10, 11, 12, 16, 17, 18, 19].

Boxer [6] introduces the digital covering space and showed that the existence of digital universal covering spaces. Boxer and Karaca [7] classify digital covering spaces using the conjugacy class corresponding to a digital covering space. Boxer and Karaca [8] study digital versions of some properties of covering spaces from algebraic topology. Karaca and Ege [20] get some results related to the simplicial homology groups of $2D$ digital images. Ege and Karaca [13] give characteristic properties of the simplicial homology groups.

This paper is organized as follows. The second section provides the general notions of digital images, digital homotopy, digital covering spaces and digital homology groups. In Section 3 we present the Nielsen fixed point theorem for digital images, give some examples and properties. In Section 4 we discuss about the relation between Nielsen theory and digital universal covering space. We finally make some conclusions about this topic.

## 2 Preliminaries

A *digital image* consists of a pair $(X, \kappa)$, where $\mathbb{Z}$ is the set of integers, $X \subset \mathbb{Z}^n$ for some positive integer $n$, and $\kappa$ indicates an adjacency relation for the members of $X$.

**Definition 2.1.** [3]. For a positive integer $l$ with $1 \le l \le n$ and two distinct points $p = (p_1, p_2, \ldots, p_n)$, $q = (q_1, q_2, \ldots, q_n) \in \mathbb{Z}^n$, $p$ and $q$ are $c_l$-*adjacent*, if

(1) there are at most $l$ indices $i$ such that $|p_i - q_i| = 1$, and

(2) for all other indices $j$ such that $|p_j - q_j| \ne 1$, $p_j = q_j$.

The notation $c_l$ represents the number of points $q \in \mathbb{Z}^n$ that are adjacent to a given point $p \in \mathbb{Z}^n$. Thus, in $\mathbb{Z}$, we have $c_1 = 2$-adjacency; in $\mathbb{Z}^2$, we have $c_1 = 4$-adjacency and $c_2 = 8$-adjacency; in $\mathbb{Z}^3$, we have $c_1 = 6$-adjacency, $c_2 = 18$-adjacency, and $c_3 = 26$-adjacency [5]. A $\kappa$-neighbor of $p \in \mathbb{Z}^n$ [3] is a point of $\mathbb{Z}^n$ that is $\kappa$-adjacent to $p$.

A digital interval [2] is defined by $[a,b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \leq z \leq b\}$ where $a, b \in \mathbb{Z}$ and $a < b$. A digital image $X \subset \mathbb{Z}^n$ is $\kappa$-connected [15] if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, \ldots, x_r\}$ of points of a digital image $X$ such that $x = x_0$, $y = x_r$ and $x_i$ and $x_{i+1}$ are $\kappa$-neighbors where $i = 0, 1, \ldots, r-1$.

**Definition 2.2.** [3]. Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with $\kappa_0$-adjacency and $\kappa_1$-adjacency, respectively. A function $f : X \longrightarrow Y$ is said to be $(\kappa_0, \kappa_1)$-continuous if for every $\kappa_0$-connected subset $U$ of $X$, $f(U)$ is a $\kappa_1$-connected subset of $Y$. We say that such a function is digitally continuous.

**Proposition 2.3.** *[3]. Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with $\kappa_0$-adjacency and $\kappa_1$-adjacency, respectively. The function $f : X \to Y$ is $(\kappa_0, \kappa_1)$-continuous if and only if for every $\kappa_0$-adjacent points $\{x_0, x_1\}$ of $X$, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are $\kappa_1$-adjacent in $Y$.*

In a digital image $X$, if there is a $(2, \kappa)$-continuous function $f : [0, m]_{\mathbb{Z}} \to X$ such that $f(0) = x$ and $f(m) = y$, then there exists a digital $\kappa$-path [6] from $x$ to $y$. If $f(0) = f(m)$, then we say that $f$ is digital $\kappa$-loop and the point $f(0)$ is the base point of the loop $f$. When a digital loop $f$ is a constant function, it is said to be a trivial loop.

**Definition 2.4.** Let $(X, \kappa_0) \subset \mathbb{Z}^{n_0}$ and $(Y, \kappa_1) \subset \mathbb{Z}^{n_1}$ be digital images. A function $f : X \to Y$ is called a $(\kappa_0, \kappa_1)$-isomorphism [2] if $f$ is $(\kappa_0, \kappa_1)$-continuous and bijective and $f^{-1} : Y \to X$ is $(\kappa_1, \kappa_0)$-continuous.

**Definition 2.5.** [3]. Let $(X, \kappa_0) \subset \mathbb{Z}^{n_0}$ and $(Y, \kappa_1) \subset \mathbb{Z}^{n_1}$ be digital images. We say that two $(\kappa_0, \kappa_1)$-continuous functions $f, g : X \to Y$ are digitally $(\kappa_0, \kappa_1)$-homotopic in $Y$ if there is a positive integer $m$ and a function $H : X \times [0, m]_{\mathbb{Z}} \to Y$ such that

- for all $x \in X$, $H(x, 0) = f(x)$ and $H(x, m) = g(x)$;
- for all $x \in X$, the induced function $H_x : [0, m]_{\mathbb{Z}} \to Y$ defined by

$$H_x(t) = H(x, t) \quad \text{for all } t \in [0, m]_{\mathbb{Z}},$$

is $(2, \kappa_1)$-continuous; and
- for all $t \in [0, m]_{\mathbb{Z}}$, the induced function $H_t : X \to Y$ defined by

$$H_t(x) = H(x, t) \quad \text{for all } x \in X,$$

is $(\kappa_0, \kappa_1)$-continuous.

The function $H$ is called a digital $(\kappa_0, \kappa_1)$-homotopy between $f$ and $g$. If these functions are digitally $(\kappa_0, \kappa_1)$-homotopic, it is denoted $f \simeq_{(\kappa_0, \kappa_1)} g$. The digital $(\kappa_0, \kappa_1)$-homotopy relation [3] is equivalence among digitally continuous functions $f : (X, \kappa_0) \to (Y, \kappa_1)$.

If $f : [0, m_1]_{\mathbb{Z}} \to X$ and $g : [0, m_2]_{\mathbb{Z}} \to X$ are digital $\kappa$-paths with $f(m_1) = g(0)$, then define the product $(f * g) : [0, m_1 + m_2]_{\mathbb{Z}} \to X$ [3] by

$$(f * g)(t) = \begin{cases} f(t), & t \in [0, m_1]_{\mathbb{Z}} \\ g(t - m_1), & t \in [m_1, m_1 + m_2]_{\mathbb{Z}}. \end{cases}$$

Let $f$ and $f'$ be $\kappa$-loops in a digital image $(X, x_0)$. We say $f'$ is a trivial extension of $f$ [3] if there are sets of $\kappa$-paths $\{f_1, \ldots, f_r\}$ and $\{F_1, \ldots, F_p\}$ in $X$ such that:
(1) $r \leq p$,
(2) $f = f_1 * \ldots * f_r$,
(3) $f' = F_1 * \ldots * F_p$,
(4) There are indices $1 \leq i_1 < i_2 < \ldots < i_r \leq p$ such that $F_{i_j} = f_j$, $1 \leq j \leq r$ and $i \neq \{i_1, \ldots, i_r\}$ implies $F_i$ is a trivial loop.

If $f, g : [0, m]_{\mathbb{Z}} \to X$ are $\kappa$-paths such that $f(0) = g(0)$ and $f(m) = g(m)$, then a homotopy

$$H : [0, m]_{\mathbb{Z}} \times [0, M]_{\mathbb{Z}} \to X$$

between $f$ and $g$ such that for all $t \in [0, M]_{\mathbb{Z}}$, $H(0, t) = f(0)$ and $H(m, t) = f(m)$, holds the endpoints fixed.

Two loops $f, f_0$ with the same base point $x_0 \in X$ belong to the same loop class $[f]_X$ if they have trivial extensions that can be joined by a homotopy that holds the endpoints fixed (see [4]).

Let $(E, \kappa)$ be a digital image and let $\varepsilon$ be a positive integer. The $\kappa$-neighborhood of $e_0 \in E$ with radius $\varepsilon$ is the set

$$N_\kappa(e_0, \varepsilon) = \{e \in E \mid l_\kappa(e_0, e) \leq \varepsilon\} \cup \{e_0\},$$

where $l_\kappa(e_0, e)$ is the length of a shortest $\kappa$-path from $e_0$ to $e$ in $E$ (see [14]).

**Definition 2.6.** [6]. Let $(E, \kappa_0)$ and $(B, \kappa_1)$ be digital images. A map $p : E \to B$ is called a $(\kappa_0, \kappa_1)$-covering map if the followings are true:

1. $p$ is a $(\kappa_0, \kappa_1)$-continuous surjection.

2. for each $b \in B$, there exists an indexing set $M$ such that $p^{-1}(b)$ can be indexed as $p^{-1}(b) = \{e_i | i \in M\}$ and the following conditions hold:

- $p^{-1}(N_{\kappa_1}(b, 1)) = \bigcup_{i \in M} N_{\kappa_0}(e_i, 1)$,
- if $i, j \in M$, $i \neq j$, then $N_{\kappa_0}(e_i, 1) \cap N_{\kappa_0}(e_j, 1) = \emptyset$,
- the restriction map $p|_{N_{\kappa_0}(e_i, 1)} : N_{\kappa_0}(e_i, 1) \to N_{\kappa_1}(b, 1)$ is a $(\kappa_0, \kappa_1)$-isomorphism for all $i \in M$.

Let $(E, \kappa_0)$, $(B, \kappa_1)$ and $(X, \kappa_2)$ be digital images, let $p : E \to B$ be a $(\kappa_0, \kappa_1)$-covering map, and $f : X \to B$ be $(\kappa_2, \kappa_1)$-continuous. A lifting of $f$ with respect to $p$ is a $(\kappa_2, \kappa_0)$-continuous function $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$ (see [14]).

**Definition 2.7.** [6]. Let $(E_0, p_0, B)$ be a $(\kappa_0, \kappa_B)$-covering. Suppose $\mathcal{C}$ is a set of $(\kappa_E, \kappa_B)$-coverings of $B$ such that for every $(E, p, B) \in \mathcal{C}$, there is a $(\kappa_0, \kappa_E)$-covering $(E_0, p_E, E)$. Then the pair $(E_0, p_0)$ is a universal covering space of $B$ for the set $\mathcal{C}$.

**Definition 2.8.** [21]. Let $S$ be a set of nonempty subset of a digital image $(X, \kappa)$. Then the members of $S$ are called *simplexes* of $(X, \kappa)$, if the followings hold:

- If $p$ and $q$ are distinct points of $s \in S$, then $p$ and $q$ are $\kappa$-adjacent,
- If $s \in S$ and $\emptyset \neq t \subset s$, then $t \in S$.

An $m$-simplex is a simplex $S$ such that $|S| = m + 1$. For a digital $m$-simplex $P$, if $P'$ is a nonempty proper subset of $P$, then $P'$ is called *a face* of $P$.

**Definition 2.9.** [1]. Let $(X, \kappa)$ be a finite collection of digital $m$-simplices, $0 \leq m \leq d$ for some non-negative integer $d$. If the followings hold, then $(X, \kappa)$ is called *a finite digital simplicial complex*:

- If $P$ belongs to $X$, then every face of $P$ also belongs to $X$.
- If $P, Q \in X$, then $P \cap Q$ is either empty or a common face of $P$ and $Q$.

**Definition 2.10.** [1]. Let $(X, \kappa) \subset \mathbb{Z}^n$ be a digital oriented simplicial complex with $m$-dimension. $C_q^\kappa(X)$ is a free abelian group with basis all digital $(\kappa, q)$-simplices in $X$. A homomorphism $\partial_q : C_q^\kappa(X) \longrightarrow C_{q-1}^\kappa(X)$ called *the boundary operator*. If $\sigma = [v_0, \ldots, v_q]$ is an oriented simplex with $0 < q \leq m$, $\partial_q$ is defined by

$$\partial_q \sigma = \partial_q[v_0, \ldots, v_q] = \sum_{i=0}^{q} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_q]$$

where $\hat{v}_i$ means the vertex $v_i$ is to be deleted from the array.

We remark that for $q < 0, m < q$, since $C_q^\kappa(X)$ is the trivial group, the operator $\partial_q$ is the trivial homomorphism for $q \leq 0$, $m < q$. We notice that $\partial_{q-1} \circ \partial_q = 0$ [1] for $q \geq 0$.

**Definition 2.11.** [1]. Let $(X, \kappa) \subset \mathbb{Z}^n$ be a digital oriented simplicial complex with $m$-dimension.

- $Z_q^\kappa(X) = \text{Ker } \partial_q$ is called the group of digital simplicial $q$-cycles.
- $B_q^\kappa(X) = \text{Im } \partial_{q+1}$ is called the group of digital simplicial $q$-boundaries.
- $H_q^\kappa(X) = Z_q^\kappa(X)/B_q^\kappa(X)$ is called the $q$th digital simplicial homology group.

# 3    Nielsen Theory for Digital Images

Let $(X, \kappa)$ be a digital image and let $f : X \to X$ be a digital map. The fixed point set of $f$ is $Fix(f) = \{x \in X : f(x) = x\}$. The main object of study in topological fixed point theory is the minimum number of fixed points which is denoted by $M[f]$ among all digital maps $(\kappa, \kappa)$-homotopic to $f$. For example, $M[f] = 0$ means that there is a digital map $g$ which is $(\kappa, \kappa)$-homotopic to $f$ such that $g(x) \neq x$ for all $x \in X$.

To calculate $M[f]$ we have to examine the fixed point sets of every map homotopic to $f$. In the fixed point theory, it is made use of a homotopy invariant, called the Nielsen number of $f$. Its computation requires only a knowledge of the map $f$ itself.

**Definition 3.1.** Let $f : (X, \kappa_1) \longrightarrow (Y, \kappa_2)$ be a $(\kappa_1, \kappa_2)$-continuous map where $(X, \kappa_1)$ and $(Y, \kappa_2)$ are digital images. Then $f$ induces homomorphisms $f_* : H_*^{\kappa_1}(X) \longrightarrow H_*^{\kappa_2}(Y)$ and $f_*$ can be thought of as a homomorphisms of the integers. The integer $deg(f)$ to which the number 1 gets sent is called the degree of the map $f$.

**Definition 3.2.** Let $(X, \kappa)$ be a digital image, $A \subset X$ and $f : A \to X$ a digital map. We define the *fixed point index* of $f$ as $ind(f) = deg(F)$ where $F(x) = x - f(x)$ and $x \in X$.

Some properties of fixed point index can be given as follows. We don't prove these because they are proved similarly in Algebraic Topology.

1. (Homotopy invariance) Let $A \subset X \times [0,m]_{\mathbb{Z}}$ be digital image with $\kappa$-adjacency and $F : A \to X$ be a digital map such that
$$Fix(F) = \{(x,t) \in A : F(x,t) = x\}.$$
Then $ind(f_0) = ind(f_m)$, where $f_t = F(-,t)$ for $0 \le t < m$ and a positive integer $m$.

2. (Commutativity) Let $(X, \kappa_1)$ and $(Y, \kappa_2)$ be digital images and let $f : A \to Y$ and $g : B \to X$ be digital $(\kappa_1, \kappa_2)$-continuous maps, respectively, where $A \subset X$, $B \subset Y$. Then $Fix(gf) = Fix(fg)$ and $ind(fg) = ind(gf)$.

Now we define Nielsen number for digital images.

**Definition 3.3.** Let $(X, \kappa)$ be a digital image and $f : (X, \kappa) \to (X, \kappa)$ a self-map. Two fixed points $x, y \in Fix(f)$ are Nielsen related if and only if there is a $\kappa$-path $c : [0, m]_{\mathbb{Z}} \to X$ satisfying $c(0) = x$, $c(m) = y$ and the $\kappa$-paths $c$, $f \circ c$ are fixed end point homotopic, i.e. there is a digital map $H : [0, n]_{\mathbb{Z}} \times [0, m]_{\mathbb{Z}} \to X$ satisfying
$$H(t,0) = c(t), \quad H(t,m) = f \circ c(t), \quad H(0,s) = x, \quad H(n,s) = y.$$

This is an equivalence relation, hence $Fix(f)$ splits into disjoint Nielsen classes. A fixed point class $F$ is essential if its index is nonzero. The number of essential fixed point classes is called the *Nielsen number* of $f$, denoted $N(f)$.

We give some characteristic examples about the Nielsen number.

**Example 3.4.** Let $(X, \kappa)$ be a digital image. If $f : X \to X$ is a constant digital map, then $N(f) = 1$.

Since the boundary $Bd(I^{n+1})$ of an $(n+1)$-cube $I^{n+1}$ is homeomorphic to $n$-sphere $S^n$, we can represent a digital sphere by using the boundary of a digital cube. Boxer [5] defines sphere-like digital image as $S_n = [-1,1]_{\mathbb{Z}}^{n+1} \setminus \{0_{n+1}\} \subset \mathbb{Z}^{n+1}$, where $0_n$ denotes the origin of $\mathbb{Z}^n$.

**Example 3.5.** $S_1 = \{c_0 = (1,0), c_1 = (1,1), c_2 = (0,1), c_3 = (-1,1), c_4 = (-1,0), c_5 = (-1,-1), c_6 = (0,-1), c_7 = (1,-1)\}$ is digital 1-sphere with 4-adjacency in $\mathbb{Z}^2$. Let $f : (S_1, 4) \to (S_1, 4)$ be a digital map of degree 1. Then $f$ can be considered as identity map and is $(4,4)$-homotopic to a fixed point free map. Thus we have $N(f) = 0$.

Let's give some important properties of Nielsen number for digital images.

**Theorem 3.6.** *Let $(X, \kappa)$ be any digital image. If $f \simeq_{(\kappa, \kappa)} g : X \to X$, then $N(f) = N(g)$.*

*Proof.* We must show that there is a bijection between sets of essential classes of $f$ and $g$. Let $H(t,s)$ be a digital $(\kappa, \kappa)$-homotopy from $f$ and $g$. For every Nielsen class $A \subset Fix(f)$, there is one $A' \subset Fix(H)$ containing $A$. Let
$$B = \{x \in X \mid (x,m) \in A'\}.$$
So $B$ is a Nielsen class of $g$ or is empty. From homotopy invariance index property, we have $ind(f, A) = ind(g, B)$. If $A$ is essential, then $B$ is essential. As a result, we find a map from the set of essential classes of $f$ to the set of essential classes of $g$.

On the other hand, $H(x, m - t)$ gives the inverse map. Consequently, we get $N(f) = N(g)$. $\square$

**Theorem 3.7.** *Let $(X, \kappa)$ be a digital image and $f : X \to X$ be a digital map. Any digital map $g$ digital $(\kappa, \kappa)$-homotopic to $f$ has at least $N(f)$ fixed points.*

*Proof.* Using Theorem 3.6, we have $N(f) = N(g)$. Since each essential Nielsen class of $g$ is nonempty, we get $M[g] \ge N(g)$ where
$$M[g] = \min\{\#Fixg \mid g \simeq_{(\kappa, \kappa)} f : X \to X\}.$$
$\square$

**Theorem 3.8.** *Let $(X, \kappa)$ and $(Y, \kappa')$ be any digital images, $f : X \to Y$ and $g : Y \to X$ be digital maps. Then $N(g \circ f) = N(f \circ g)$.*

*Proof.* For digital maps $f$ and $g$, if we use commutativity property of fixed point index, i.e. $Fix(f \circ g) = Fix(g \circ f)$ and $ind(f \circ g) = ind(g \circ f)$, we have a bijection which preserves index between the sets of essential Nielsen classes. As a result, we have $N(g \circ f) = N(f \circ g)$. $\square$

**Lemma 3.9.** *Let $A \subset X$ be digital image with $\kappa$-adjacency and $f : X \to X$ be digital map such that $f(X) \subset A$, where $X$ is any digital image with $\kappa$-adjacency. If $f_A : A \to A$ is the restriction of $f$, then $N(f_A) = N(f)$.*

*Proof.* Let $i : A \to X$ be inclusion map. Assume that $g : X \to A$ is given by $g(x) = f(x)$. Using Theorem 3.8, we conclude

$$N(f) = N(i \circ g) = N(g \circ i) = N(f_A)$$

because $i \circ g = f$ and $g \circ i = f_A$. □

**Theorem 3.10.** *Let $(X, \kappa)$ and $(Y, \kappa^{'})$ be any two digital images. Suppose that $h : X \to Y$ is a digital $(\kappa, \kappa^{'})$-homotopy equivalence and let the diagram*

$$
\begin{array}{ccc}
X & \xrightarrow{f} & X \\
\downarrow{h} & & \downarrow{h} \\
Y & \xrightarrow{g} & Y
\end{array}
$$

*be digital $(\kappa, \kappa^{'})$-homotopy commutative, i.e. $h \circ f \simeq_{(\kappa, \kappa^{'})} g \circ h$. Then $N(f) = N(g)$.*

*Proof.* Assume that the digital $(\kappa^{'}, \kappa)$-homotopy inverse of $h$ is $m : Y \to X$. Then $m \circ h \simeq_{(\kappa, \kappa)} 1_X$ and $h \circ m \simeq_{(\kappa^{'}, \kappa^{'})} 1_Y$. By Theorem 3.6 and Theorem 3.8, we have

$$
\begin{aligned}
N(f) = N(f(mh)) = N((fm)h) &= N(h(fm)) \\
&= N((hf)m) \\
&= N((gh)m) \\
&= N(g(hm)) \\
&= N(g).
\end{aligned}
$$

□

**Theorem 3.11.** *Let $(X, \kappa)$ be a digital image and $f : (X, \kappa) \to (X, \kappa)$ be a digital map. $N(f)$ is a lower bound for the number of fixed points in the homotopy class of $f$, i.e.*

$$0 \leq N(f) \leq M[f] := \min\{\#Fixg \mid g \simeq_{(\kappa, \kappa)} f : X \to X\}.$$

*Proof.* By the definition of Nielsen number, we have $N(f) \geq 0$. On the other hand, since we know that each Nielsen class contains at least one fixed point of $f$, we conclude that $0 \leq N(f) \leq M[f]$. □

# 4    Nielsen Theory and Digital Universal Covering Spaces

Since there is a connection between the digital fundamental group and the digital universal covering of a space, same results can be also obtained by the lifts of the considered maps to the digital universal coverings. Let $(X, \kappa)$ be a digital image and $p : \tilde{X} \to \tilde{X}$ be the digital universal covering of $X$, with group $\pi$ of covering transformations. Let $\tilde{f} : \tilde{X} \to \tilde{X}$ be a lifting of $f$, i.e., have a commutative diagram

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X} \\
\downarrow{p} & & \downarrow{p} \\
X & \xrightarrow{f} & X
\end{array}
$$

If $\tilde{f}^{'}$ is another lifting of $\tilde{f}$, then $\tilde{f}^{'} = \alpha \circ \tilde{f}$ for some $\alpha \in \pi$. The set of all liftings of $f$ is $\{\alpha \circ \tilde{f} \mid \alpha \in \pi\}$.

For any $\alpha \in \pi$, $\tilde{f} \circ \alpha$ is a lifting of $f$ and so we have $\alpha^{'} \circ \tilde{f} = \tilde{f} \circ \alpha$ for some $\alpha^{'} \in \pi$. This defines a homomorphism $\varphi : \pi \to \pi$ given by $\varphi(\alpha) = \alpha^{'}$. Define the Reidemeister action of $\pi$ on $\pi$ as follows:

$$\pi \times \pi \to \pi$$
$$(\gamma, \alpha) \mapsto \gamma \alpha \varphi(\gamma)^{-1},$$

where $\gamma, \alpha \in \pi$. This defines an equivalence relation. We say that the Reidemeister classes of its equivalence classes. The set of the Reidemeister classes determined by $\varphi$ is denoted by $\mathcal{R}[\varphi] = \{[\alpha] \mid \alpha \in \pi\}$.

**Theorem 4.1.** *Let $(X, \kappa)$ be a digital image, $f : X \to X$ be a digital map and $\tilde{f} : \tilde{X} \to \tilde{X}$ be a lifting of $f$. Then $[\alpha] = [\alpha^{'}]$ if and only if $p(Fix(\alpha \circ \tilde{f})) = p(Fix(\alpha^{'} \circ \tilde{f}))$, where $p : \tilde{X} \to X$ is a digital covering map of $f$ and $\alpha, \alpha^{'} \in \pi$.*

*Proof.* For necessary condition, since the fixed point sets of any two digital homotopic maps are same, we conclude that

$$[\alpha] = [\alpha'] \Rightarrow \alpha \simeq_{(\tilde{\kappa}, \tilde{\kappa})} \alpha'$$
$$\Rightarrow \alpha \circ \tilde{f} \simeq_{(\tilde{\kappa}, \tilde{\kappa})} \alpha' \circ \tilde{f}$$
$$\Rightarrow Fix(\alpha \circ \tilde{f}) = Fix(\alpha' \circ \tilde{f})$$
$$\Rightarrow p(Fix(\alpha \circ \tilde{f})) = p(Fix(\alpha' \circ \tilde{f})).$$

For sufficient condition, let $a \in p(Fix(\alpha \circ \tilde{f})) = p(Fix(\alpha' \circ \tilde{f}))$. Then we have $p(\tilde{a}) = a$ and $p(\tilde{a'}) = a$. Moreover, we get

$$\alpha \circ \tilde{f}(\tilde{a}) = \tilde{a} = 1_{\tilde{X}}(\tilde{a}) \quad \text{and} \quad \alpha' \circ \tilde{f}(\tilde{a}) = \tilde{a'} = 1_{\tilde{X}}(\tilde{a'})$$

Finally, we can say

$$\alpha(\tilde{f}(\tilde{a})) = \alpha'(\tilde{f}(\tilde{a})) \quad \Rightarrow \quad \alpha = \alpha' \quad \Rightarrow \quad \alpha \simeq_{(\tilde{\kappa}, \tilde{\kappa})} \alpha',$$

where $\tilde{a}$ is any point of $\tilde{X}$. As a result, we have $[\alpha] = [\alpha']$.  □

**Corollary 4.2.** *If $p(Fix(\alpha\tilde{f}))$ is any fixed point class, then*

$$Fix(f) = \coprod_{[\alpha] \in \mathcal{R}[\varphi]} p(Fix(\alpha\tilde{f})),$$

*where $[\alpha]$ is a Reidemeister class.*

**Lemma 4.3.** *Let $(X, \kappa), (\tilde{X}, \tilde{\kappa})$ be digital images, $f : X \to X$ be a digital map and $\tilde{f} : \tilde{X} \to \tilde{X}$ be any lifting of $f$. Then any two points in $p(Fix(\tilde{f})) \subset Fix(f)$ are Nielsen related. We have also*

$$Fix(f) = \bigcup_{\tilde{f'}} p(Fix(\tilde{f'}))$$

*where $\tilde{f'}$ is a lifting of $f$.*

*Proof.* Let $a$ and $b$ be any two points in $p(Fix(\tilde{f}))$. We say that

$$p(\tilde{a}) = \tilde{a}, \quad p(\tilde{b}) = \tilde{b}, \quad \tilde{f}(\tilde{a}) = \tilde{a}, \quad \tilde{f}(\tilde{b}) = \tilde{b},$$

for some $\tilde{a}, \tilde{b} \in Fix(\tilde{f})$. We denote a $\tilde{\kappa}$-path from $\tilde{a}$ to $\tilde{b}$ in $\tilde{X}$ by $\tilde{\theta}$. There is a digital homotopy $\tilde{H}$ such that

$$\tilde{H} : \tilde{X} \times [0, m]_{\mathbb{Z}} \to \tilde{X}$$

$$\tilde{H}(\tilde{x}, 0) = \tilde{\theta} \quad \text{and} \quad \tilde{H}(\tilde{x}, m) = \tilde{f} \circ \tilde{\theta}$$

because $\tilde{X}$ is $\tilde{\kappa}$-connected digital image. Then we have $p \circ \tilde{H} = H : X \times [0, m']_{\mathbb{Z}} \to X$ is a digital homotopy between two $\kappa$-paths

$$\theta = p \circ \tilde{\theta} \quad \text{and} \quad f \circ \theta = p \circ (\tilde{f} \circ \tilde{\theta})$$

which join two points $a, b \in Fix(f)$. As a result, $a$ and $b$ are Nielsen related points.

Now we prove the latter statement. Let $u \in Fix(f)$ and $\tilde{u} \in p^{-1}(u)$. Then $f(u) = u$ and $p^{-1}(u) = \tilde{u}$. Moreover, $\tilde{f'}(\tilde{u}) = \tilde{u}$ because $\tilde{f'}$ is a lifting of $f$. We can say the following result.

$$\tilde{f'}(\tilde{u}) = \tilde{u} \quad \Rightarrow \quad p \circ \tilde{f'}(\tilde{u}) = p(\tilde{u}) = u \quad \Rightarrow \quad u \in p(Fix(\tilde{f'})).$$

Consequently, we have $Fix(f) = \bigcup_{\tilde{f'}} p(Fix(\tilde{f'}))$.  □

Let $p : \tilde{X} \to X$ be a digital universal covering map. Let

$$\mathcal{O}_X = \{\alpha \in \tilde{X} \to \tilde{X} : p \circ \alpha = p\}$$

denote the group of deck transformations of this digital covering map.

**Lemma 4.4.** *Let $\mathcal{C}$ be the set of liftings of $f$ and $\tilde{f}, \tilde{f'} \in \mathcal{C}$. If $p(Fix(\tilde{f})) = p(Fix(\tilde{f'})) \neq \emptyset$, then there is an $\alpha \in \mathcal{O}_X$ such that $\alpha \circ \tilde{f} = \tilde{f'} \circ \alpha$.*

*Proof.* If $p(Fix(\tilde{f})) = p(Fix(\tilde{f'})) \neq \emptyset$, then there are two points $\tilde{x}, \tilde{x'}$ such that $p(\tilde{x}) = p(\tilde{x'})$ where

$$\tilde{x} \in Fix(\tilde{f}) \quad \Rightarrow \quad \tilde{f}(\tilde{x}) = \tilde{x} \quad \text{and} \quad \tilde{x'} \in Fix(\tilde{f'}) \quad \Rightarrow \quad \tilde{f'}(\tilde{x'}) = \tilde{x'}.$$

Since $\alpha \in \mathcal{O}_X$, i.e. $p \circ \alpha = p$, we have $\alpha(\tilde{x}) = \tilde{x'}$. We conclude that

$$\tilde{f'} \circ \alpha(\tilde{x}) = \tilde{f'}(\tilde{x'}) = \tilde{x'} = \alpha(\tilde{x}) = \alpha(\tilde{f}(\tilde{x})) = \alpha \circ \tilde{f}(\tilde{x}).$$

As a result, we have $\alpha \tilde{f} = \tilde{f'} \alpha$.          $\square$

**Lemma 4.5.** *Let* $\tilde{f'} = \alpha \tilde{f} \alpha^{-1}$ *for an* $\alpha \in \mathcal{O}_X$. *Then* $p(Fix(\tilde{f})) = p(Fix(\tilde{f'}))$.

*Proof.* By assumption, we have $p(\alpha(\tilde{x})) = p(\tilde{x})$. If $u \in p(Fix(\tilde{f}))$, then $p(\tilde{x}) = u$ and $\tilde{f}(\tilde{x}) = \tilde{x}$. Since

$$p \circ \tilde{f}(\tilde{x}) = u \quad \Rightarrow \quad p \circ \alpha^{-1} \circ \tilde{f'} \circ \alpha(\tilde{x}) = p \circ \tilde{f'}(\tilde{x'}) = u,$$

we have $u \in p(Fix(\tilde{f'}))$. As a result, $p(Fix(\tilde{f})) = p(Fix(\tilde{f'}))$.          $\square$

# 5    Conclusion

The essential aim of this paper is to determine fixed point properties for a digital image. This work can play an important role in digital images because Nielsen theory gives an information about the number of fixed points of a map. Since the Nielsen number is a powerful invariant in digital images, we think that this work will be useful for fixed point theory, especially Nielsen theory.

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# A FIXED POINT THEOREM AND STABILITY OF ADDITIVE-CUBIC FUNCTIONAL EQUATIONS IN MODULAR SPACES

CHANG IL KIM, GILJUN HAN*, AND SEONG-A SHIM

ABSTRACT. In this paper, we investigate a fixed point theorem for a mapping without the condition of bounded orbit in a modular space, whose induced modular is lower semi-continunous. Using this fixed point theorem, we prove the generalized Hyers-Ulam stability for an additive-cubic functional equation in modular spaces without $\triangle_2$-conditions and the convexity.

## 1. INTRODUCTION AND PRELIMINARIES

The question of stability for a generic functional equation was originated in 1940 by Ulam [14]. Concerning a group homomorphism, Ulam posted the question asking how likely to an automorphism a function should behave in order to guarantee the existence of an automorphism near such functions. Hyers [3] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [1] for additive mappings and by Rassias [12] for linear mappings by considering an unbounded Cauchy difference, the latter of which has influenced many developments in the stability theory. This area is then referred to as the generalized Hyers-Ulam stability. In 1994, P. Găvruta [2] generalized these theorems for approximate additive mappings controlled by the unbounded Cauchy difference with regular conditions.

A problem that mathematicians has dealt with is "how to generalize the classical function space $L^p$". A first attempt was made by Birnhaum and Orlicz in 1931. This generalization found many applications in differential and intergral equations with kernls of nonpower types. The more abstract generalization was given by Nakano [10] in 1950 based on replacing the particular integral form of the functional by an abstract one that satisfies some good properties. This functional was called *modular*. This idea was refined, generalized by Musielak and Orlicz [8] in 1959 and studied by many authors ([4], [7], [11], [19]).

Recently, Sadeghi [13] presented a fixed point method to prove the generalized Hyers-Ulam stability of functional equations in modular spaces with the $\triangle_2$-condition and Wongkum, Chaipunya, and Kumam [15] proved the fixed point theorem and the generalized Hyers-Ulam stability for quadratic mappings in a modular space whose modular is convex, lower semi-continuous but do not satisfy the $\triangle_2$-condition.

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In this paper, we investigate a fixed point theorem in modular spaces, whose induced modular is lower semi-continuous, for a mapping with some conditions in place of the condition of bounded orbit and using this fixed point theorem, we will prove the generalized Hyers-Ulam stability for the following additive-cubic functional equation

$$(1.1) \qquad f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 2f(2x) + 4f(x) = 0$$

in modular spaces without $\triangle_2$-conditions and the convexity.

In fact, the equation (1.1) has been studied in various spaces. For example, in quasi-Banach spaces ([9]), in F-spaces ([16]), in non-Archimedean fuzzy normed spaces ([17]), and in intuitionistic fuzzy normed spaces ([18]), etc. Unlike Banach spaces and F-spaces, due to the absence of the triangle inequality in modular spaces, we need subtle caculations in the proofs of Lemma 1.4 and Theorem 2.2.

**Definition 1.1.** Let $X$ be a vector space over a field $\mathbb{K}(\mathbb{R}$ or $\mathbb{C})$.
(1) A generalized functional $\rho : X \longrightarrow [0, \infty]$ is called *a modular* if
    (M1) $\rho(x) = 0$ if and only if $x = 0$,
    (M2) $\rho(\alpha x) = \rho(x)$ for every scalar $\alpha$ with $|\alpha| = 1$, and
    (M3) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ for all $x, y \in X$ and for all nonnegative real numbers $\alpha, \beta$ with $\alpha + \beta = 1$.
(2) If (M3) is replaced by
    (M4) $\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y)$
for all $x, y \in X$ and for all nonnegative real numbers $\alpha, \beta$ with $\alpha + \beta = 1$, then we say that $\rho$ is a *convex modular*.

**Remark 1.2.** Let $\rho$ be a modular on a vector space $X$. Then by (M1) and (M3), we can easily show that for any positive real number $\delta$ with $\delta < 1$,

$$\rho(\delta x) \le \rho(x)$$

for all $x \in X$.

For any modular $\rho$ on $X$, the modular space $X_\rho$ is defined by

$$X_\rho := \{x \in X \mid \rho(\lambda x) \to 0 \ as \ \lambda \to 0\}.$$

Let $X_\rho$ be a modular space and let $\{x_n\}$ be a sequence in $X_\rho$. Then (i) $\{x_n\}$ is called $\rho$-*convergent* to a point $x \in X_\rho$ if $\rho(x_n - x) \to 0$ as $n \to \infty$, (ii) $\{x_n\}$ is called $\rho$-*Cauchy* if for any $\epsilon > 0$, there is a $k \in \mathbb{N}$ such that $\rho(x_n - x_m) < \epsilon$ for all $m, n \in \mathbb{N}$ with $n, m \ge k$, and (iii) a subset $K$ of $X_\rho$ is called $\rho$-*complete* if each $\rho$-Cauchy sequence is $\rho$-convergent to an element of $K$.

Another unnatural behavior one usually encounter is that the convergence of a sequence $\{x_n\}$ to $x$ does not imply that $\{cx_n\}$ converges to $cx$ for some $c \in \mathbb{K}$. Thus, many mathematicians imposed some additional conditions for a modular to meet in order to make the multiples of $\{x_n\}$ converge naturally. Such preferences are referred to mostly under the term related to the $\triangle_2$-conditions.

A modular space $X_\rho$ is said to *satisfy the $\triangle_2$-condition* if there exists $k \ge 2$ such that $\rho(2x) \le k\rho(x)$ for all $x \in X_\rho$. Some authors varied the notion so that only $k > 0$ is required and called it *the $\triangle_2$-type condition*. In fact, one may see that these two notions coincide. There are still a number of equivalent notions related to the $\triangle_2$-conditions.

In [5], Khamsi proved a series of fixed point theorems in modular spaces where the modulars do not satisfy $\triangle_2$-conditions. His results exploit one unifying hypothesis in which the boundedness of an orbit is assumed.

**Lemma 1.3.** *(see* [5]*) Let $X_\rho$ be a modular space whose induced modular is lower semi-continuous and let $C \subseteq X_\rho$ be a $\rho$-complete subset. If $T : C \longrightarrow C$ is a $\rho$-contraction, that is, there is a constant $L \in [0,1)$ such that*

$$\rho(Tx - Ty) \leq L\rho(x - y), \ \ \forall x, y \in C$$

*and $T$ has a bounded orbit at a point $x_0 \in C$, that is,*

$$\sup\{\rho(T^n x_0 - T^m x_0) \mid n, m \in \mathbb{N} \cup \{0\}\} < \infty$$

*then the sequence $\{T^n x_0\}$ is $\rho$-convergent to a point $w \in C$.*

Now, we will prove a fixed point theorem in modular spaces where the map $T$ do not assume to be the boundedness of an orbit. Our results exploit one unifying hypothesis in which some conditions are assumed.

**Lemma 1.4.** *Let $X_\rho$ be a modular space whose induced modular is lower semi-continuous and let $C \subseteq X_\rho$ be a $\rho$-complete subset. Let $T : C \longrightarrow C$ be a mapping such that*

(1.2) $$2Tx = T2x, \ \ \forall x \in C.$$

*Suppose that there is a constant $L \in [0,1)$ with*

(1.3) $$\rho(2Tx - 2Ty) \leq L\rho(x - y), \ \ \forall x, y \in C$$

*and $\rho(Tx_o - x_o) < \infty$ at $x_o \in C$. Then the sequence $\{T^n \frac{x_0}{4}\}$ is $\rho$-convergent to some point $w \in C$ and*

(1.4) $$\rho(\frac{x_0}{4} - w) \leq \frac{2}{1-L}\rho(Tx_0 - x_0).$$

*Proof.* By (M1) and (M3), we have $\rho(Tx - Ty) \leq \rho(2Tx - 2Ty)$ and so, by (1.3), $T$ is a $\rho$-contration. Hence we have

$$\rho(\frac{1}{2}T^2 x_0 - \frac{1}{2}x_0) \leq \rho(T^2 x_0 - Tx_0) + \rho(Tx_0 - x_0)$$
$$\leq (L+1)\rho(Tx_0 - x_0).$$

Let $Gx = 2Tx$ for all $x \in C$. By (1.3), we have

$$\rho(\frac{1}{2}T^n x_0 - \frac{1}{2}x_0) \leq \rho(T^n x_0 - Tx_0) + \rho(Tx_0 - x_0)$$
$$= \rho\Big(\frac{1}{2}G(T^{n-1} x_0) - \frac{1}{2}Gx_0\Big) + \rho(Tx_0 - x_0)$$
$$\leq L\rho\Big(\frac{1}{2}T^{n-1} x_0 - \frac{1}{2}x_0\Big) + \rho(Tx_0 - x_0)$$

for all $n \in \mathbb{N}$ with $n \geq 2$ and by induction, we have

$$\rho(\frac{1}{2}T^n x_0 - \frac{1}{2}x_0) \leq \Sigma_{k=0}^{n-1} L^k \rho(Tx_0 - x_0) \leq \frac{1}{1-L}\rho(Tx_0 - x_0)$$

for all $n \in \mathbb{N}$. For any non-negative integers $m, n$ with $m > n$,

$$\rho(\frac{1}{4}T^n x_0 - \frac{1}{4}T^m x_0) \le \rho(\frac{1}{2}T^n x_0 - \frac{1}{2}x_0) + \rho(\frac{1}{2}T^m x_0 - \frac{1}{2}x_0)$$

$$(1.5)$$

$$\le \frac{2}{1-L}\rho(T x_0 - x_0).$$

By (1.2), $T$ has a bounded orbit at a point $\frac{x_0}{4} \in C$ and thus by Lemma 1.3, $\{T^n \frac{x_0}{4}\}$ is $\rho$-convergent to a point $\omega \in C$. Since $\rho$ is lower semi-continuous, by taking $n = 0$ and $m \to \infty$ in (1.5), we have (1.4). $\qquad\square$

If $\rho$ is convex, then Lemma 1.4 can be replaced by the following lemma.

**Lemma 1.5.** *All conditions in Lemma 1.4 are assumed. Suppose that $\rho$ is convex and $0 \le L < 2$. Then the sequence $\{T^n \frac{x_0}{4}\}$ is $\rho$-convergent to some point $w \in C$ and*

$$(1.6) \qquad\qquad \rho(\frac{x_0}{4} - w) \le \frac{1}{2-L}\rho(T x_0 - x_0).$$

*Proof.* By (M1) and (M4), we have $\rho(Tx - Ty) \le \frac{1}{2}\rho(2Tx - 2Ty)$ and since $0 \le L < 2$, by (1.3), $T$ is a $\rho$-contration. Hence by (M4), we have

$$\rho(\frac{1}{2}T^2 x_0 - \frac{1}{2}x_0) \le \frac{1}{2}\rho(T^2 x_0 - T x_0) + \frac{1}{2}\rho(T x_0 - x_0)$$

$$\le (\frac{1}{4}L + \frac{1}{2})\rho(T x_0 - x_0).$$

Let $Gx = 2Tx$ for all $x \in C$. By (1.3), we have

$$\rho(\frac{1}{2}T^n x_0 - \frac{1}{2}x_0) \le \frac{1}{2}\rho(T^n x_0 - T x_0) + \frac{1}{2}\rho(T x_0 - x_0)$$

$$= \rho\left(\frac{1}{2}G(T^{n-1}x_0) - \frac{1}{2}Gx_0\right) + \frac{1}{2}\rho(T x_0 - x_0)$$

$$\le \frac{1}{2}L\rho\left(\frac{1}{2}T^{n-1}x_0 - \frac{1}{2}x_0\right) + \frac{1}{2}\rho(T x_0 - x_0)$$

for all $n \in \mathbb{N}$ with $n \ge 2$ and by induction, we have

$$\rho(\frac{1}{2}T^n x_0 - \frac{1}{2}x_0) \le \Sigma_{k=0}^{n-1}\frac{L^k}{2^{k+1}}\rho(T x_0 - x_0) \le \frac{1}{2-L}\rho(T x_0 - x_0)$$

for all $n \in \mathbb{N}$. For any non-negative integers $m, n$ with $m > n$,

$$\rho(\frac{1}{4}T^n x_0 - \frac{1}{4}T^m x_0) \le \frac{1}{2}\rho(\frac{1}{2}T^n x_0 - \frac{1}{2}x_0) + \frac{1}{2}\rho(\frac{1}{2}T^m x_0 - \frac{1}{2}x_0)$$

$$\le \frac{1}{2-L}\rho(T x_0 - x_0).$$

The rest of the proof is similar to Lemma 1.4. $\qquad\square$

Let $\rho$ be a modular on $X$, $V$ a linear space. Define a set $\mathbb{M}$ by

$$\mathbb{M} := \{g : V \longrightarrow X_\rho \mid g(0) = 0\}$$

and a generalized function $\widetilde{\rho}$ on $\mathbb{M}$ by

$$\widetilde{\rho}(g) := \inf\{c > 0 \mid \rho(g(x)) \le c\psi(x, x), \ \forall x \in V\},$$

for each $g \in \mathbb{M}$, where $\psi : V^2 \longrightarrow [0, \infty)$ a mapping. Then $\mathbb{M}$ is a linear space, $\widetilde{\rho}$ is a modular on $\mathbb{M}$. Furthermore, if $\rho$ is convex, then $\widetilde{\rho}$ is also convex([15]).

**Lemma 1.6.** *Let $V$ be a linear space, $X_\rho$ a $\rho$-complete modular space, where $\rho$ is lower semi-continuous and $f : V \longrightarrow X_\rho$ a mapping with $f(0) = 0$. Let $\psi : V^2 \longrightarrow [0, \infty)$ be a mapping. Then we have the following :*
*(1) $\mathbb{M}_{\widetilde{\rho}} = \mathbb{M}$ and $\mathbb{M}_{\widetilde{\rho}}$ is $\widetilde{\rho}$-complete.*
*(2) $\widetilde{\rho}$ is lower semi-continuous.*

*Proof.* (1) By the definition of $\mathbb{M}_{\widetilde{\rho}}$, $\mathbb{M}_{\widetilde{\rho}} = \mathbb{M}$. Let $\epsilon > 0$ be given. Take any $\widetilde{\rho}$-Cauchy sequence $\{g_n\}$ in $\mathbb{M}_{\widetilde{\rho}}$. Then there is an $l \in \mathbb{N}$ such that for $n, m \in \mathbb{N}$ with $n, m \geq l$,

$$(1.7) \qquad\qquad \rho(g_n(x) - g_m(x)) \leq \epsilon\psi(x, x)$$

for all $x \in V$. Hence $\{g_n(x)\}$ is a $\rho$-Cauchy sequence in $X_\rho$ for all $x \in X$. Since $X_\rho$ is $\rho$-complete, there is a mapping $g : V \longrightarrow X_\rho$ such that $\rho(g_n(x) - g(x)) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $x \in X$. Then there is an $m \in \mathbb{N}$ such that

$$\rho(g_m(0) - g(0)) = \rho(g(0)) \leq \epsilon$$

and hence $g \in \mathbb{M}_{\widetilde{\rho}}$. Since $\rho$ is a lower semi-continuous, by (1.7), we have

$$\rho(g_n(x) - g(x)) \leq \liminf_{m \longrightarrow \infty} \rho(g_n(x) - g_m(x)) \leq \epsilon\psi(x, x)$$

for all $x \in X$. Hence $\mathbb{M}_{\widetilde{\rho}}$ is $\widetilde{\rho}$-complete.

(2) Suppose that $\{g_n\}$ is a sequence in $\mathbb{M}_{\widetilde{\rho}}$ which is $\widetilde{\rho}$-convergent to $g \in \mathbb{M}_{\widetilde{\rho}}$. Let $\epsilon > 0$. Then for any $n \in \mathbb{N}$, there is a positive real number $c_n$ such that

$$\widetilde{\rho}(g_n) \leq c_n \leq \widetilde{\rho}(g_n) + \epsilon$$

and so

$$\rho(g(x)) \leq \liminf_{n \to \infty} \rho(g_n(x)) \leq \liminf_{n \to \infty} c_n\psi(x, x) \leq \left( \liminf_{n \to \infty} \rho(g_n(x)) + \epsilon \right)\psi(x, x)$$

for all $x \in X$. Hence $\widetilde{\rho}$ is lower semi-continuous. $\qquad\qquad\square$

## 2. The generalized Hyers-Ulam stability for (1.1) in modular spaces

Throughout this section, we assume that every modular is lower semi-continuous. In this section, we will prove the generalized Hyers-Ulam stability for (1.1) by using our fixed point theorem. We can easily show the following lemma.

**Lemma 2.1.** *Let $X$ and $Y$ be vector spaces. Let $f : X \longrightarrow Y$ satisfies (1.1) and $f(0) = 0$. Then we have :*
*(1) $f$ is additive if and only if $f(2x) = 2f(x)$ for all $x \in X$.*
*(2) $f$ is cubic if and only if $f(2x) = 8f(x)$ for all $x \in X$.*

For any mapping $g : X \longrightarrow Y$, let

$$g_a(x) = \frac{1}{4}(g(2x) - 8g(x)), \ \ g_c(x) = g(2x) - 2g(x)$$

and

$$Dg(x, y) = g(2x + y) + g(2x - y) - 2g(x + y) - 2g(x - y) - 2g(2x) + 4g(x).$$

**Theorem 2.2.** *Let $V$ be a linear space, $X_\rho$ a $\rho$-complete modular space. Suppose that $f : V \longrightarrow X_\rho$ satisfies $f(0) = 0$ and*

(2.1)
$$\rho(Df(x,y)) \leq \phi(x,y)$$

*for all $x, y \in V$, where $\phi : V^2 \longrightarrow [0, \infty)$ is a mapping such that*

(2.2)
$$\phi(2x, 2y) \leq L\phi(x,y)$$

*for some $L$ with $0 \leq L < 1$ and for all $x, y \in V$. Then there exists a unique additive-cubic mapping $F : V \longrightarrow X_\rho$ such that*

(2.3)
$$\rho(F(x) - \frac{3}{16}f(x)) \leq \frac{4}{1-L}\psi(x,x)$$

*for all $x \in V$, where $\psi(x,y) = \phi(x, 2y) + \phi(x,y) + \phi(0,y)$.*

*Proof.* Let $\psi(x,y) = \phi(x, 2y) + \phi(x,y) + \phi(0,y)$. Then by Lemma 1.6, $\mathbb{M}_{\widetilde{\rho}} = \mathbb{M}$ is $\widetilde{\rho}$-complete and $\widetilde{\rho}$ is lower semi-continuous.

Define $T_a : \mathbb{M}_{\widetilde{\rho}} \longrightarrow \mathbb{M}_{\widetilde{\rho}}$ by $T_a g(x) = \frac{1}{2}g(2x)$ for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and all $x \in V$. Let $g, h \in \mathbb{M}_{\widetilde{\rho}}$. Suppose that $\widetilde{\rho}(g - h) \leq c$ for some positive real number $c$. Then by (2.3) and Remark 1.2, we have

$$\rho(T_a g(x) - T_a h(x)) \leq \rho(g(2x) - h(2x)) \leq Lc\psi(x,x)$$

for all $x \in V$ and so $\widetilde{\rho}(T_a g - T_a h) \leq L\widetilde{\rho}(g - h)$. Hence $T_a$ is a $\widetilde{\rho}$-contraction. By (2.1), we get

(2.4)
$$\rho(f(x) + f(-x)) \leq \phi(0, x),$$

(2.5)
$$\rho(f(3x) - 4f(2x) + 5f(x)) \leq \phi(x,x),$$

and

(2.6)
$$\rho(f(4x) - 2f(3x) - 2f(2x) - 2f(-x) + 4f(x)) \leq \phi(x, 2x),$$

for all $x \in V$. By (2.4), (2.5), and (2.6), we obtain

$$\rho(T_a f_a(x) - f_a(x)) = \rho(\frac{1}{2}f_a(2x) - f_a(x)) \leq \phi(x, 2x) + \phi(x,x) + \phi(0,x) = \psi(x,x)$$

for all $x \in V$ and hence we have

(2.7)
$$\widetilde{\rho}(T_a f_a - f_a) \leq 1.$$

Let $Gg = 2T_a g$ for all $g \in \mathbb{M}_{\widetilde{\rho}}$. Then

$$Gg(x) = g(2x)$$

for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and for all $x \in V$. Suppose that $\widetilde{\rho}(g - h) \leq c$ for some positive real number $c$, where $g, h \in \mathbb{M}_{\widetilde{\rho}}$. Then $\rho(g(x) - h(x)) \leq c\psi(x,x)$ for all $x \in V$ and by (2.2), we have

$$\rho(Gg(x) - Gh(x)) = \rho(g(2x) - h(2x)) \leq c\psi(2x, 2x) \leq cL\psi(x,x)$$

for all $x \in V$. Hence $\widetilde{\rho}(Gg - Gh) \leq cL$ and so

$$\widetilde{\rho}(Gg - Gh) \leq L\widetilde{\rho}(g - h).$$

Since $T_a$ is linear, by Lemma 1.4, there is an $A \in \mathbb{M}_{\widetilde{\rho}}$ such that $\{T_a^n \frac{f_a}{4}\}$ is $\widetilde{\rho}$-convergent to $A$. In fact, we get

(2.8)
$$\lim_{n\to\infty} \rho(\frac{1}{2^{n+2}} f_a(2^n x) - A(x)) = 0$$

for all $x \in V$. Since $\widetilde{\rho}$ is lower semi-continuous, we get

$$\widetilde{\rho}(T_a A - A) \le \liminf_{n\to\infty} \widetilde{\rho}(T_a A - T_a^{n+1} \frac{f_a}{4}) \le \liminf_{n\to\infty} L\widetilde{\rho}(A - T_a^n \frac{f_a}{4}) = 0$$

and hence $A$ is a fixed point of $T_a$ in $\mathbb{M}_{\widetilde{\rho}}$. Replacing $x$ and $y$ by $2^n x$ and $2^n y$ in (2.1), respectively, by (2.2), we have

$$\rho(\frac{1}{2^{n+2}} Df_a(2^n x, 2^n y))$$
$$\le \rho(\frac{1}{2^{n+3}} Df(2^{n+1}x, 2^{n+1}y)) + \rho(\frac{1}{2^n} Df(2^n x, 2^n y))$$
$$\le L^{n+1}\phi(x, y) + L^n\phi(x, y)$$

for all $x, y \in V$ and for all $n \in \mathbb{N}$. Hence we get

(2.9)
$$\lim_{n\to\infty} \rho\Big(\frac{1}{2^{n+2}} Df_a(2^n x, 2^n y)\Big) = 0$$

for all $x, y \in V$. Note that

$$\rho\Big(\frac{1}{2^{n+10}} Df_a(2^n x, 2^n y) - \frac{1}{2^8} DA(x, y)\Big)$$
$$\le \rho\Big(\frac{1}{2^{n+9}} f_a(2^n(2x + y)) - \frac{1}{2^7} A(2x + y)\Big) + \rho\Big(\frac{1}{2^{n+8}} f_a(2^n(2x - y)) - \frac{1}{2^6} A(2x - y)\Big)$$
$$+ \rho\Big(\frac{1}{2^{n+7}} 2f_a(2^n(x + y)) - \frac{1}{2^5} 2A(x + y)\Big) + \rho\Big(\frac{1}{2^{n+6}} 2f_a(2^n(x - y)) - \frac{1}{2^4} 2A(x - y)\Big)$$
$$+ \rho\Big(\frac{1}{2^{n+5}} 2f_a(2^n 2x) - \frac{1}{2^3} 2A(2x)\Big) + \rho\Big(\frac{1}{2^{n+5}} 2f_a(2^n(x - y)) - \frac{1}{2^3} 4A(x - y)\Big)$$

for all $x, y \in V$ and for all $n \in \mathbb{N}$. Hence we have

(2.10)
$$\lim_{n\to\infty} \rho\Big(\frac{1}{2^{n+10}} Df_a(2^n x, 2^n y) - \frac{1}{2^8} DA(x, y)\Big) = 0$$

for all $x, y \in V$ . Since

$$\rho\Big(\frac{1}{2^9} DA(x, y)\Big) \le \rho\Big(\frac{1}{2^{n+10}} Df_a(2^n x, 2^n y) - \frac{1}{2^8} DA(x, y)\Big) + \rho\Big(\frac{1}{2^{n+10}} Df_a(2^n x, 2^n y)\Big)$$

for all $x, y \in V$ and for all $n \in \mathbb{N}$, by (2.9) and (2.10), we get

(2.11)
$$DA(x, y) = 0$$

for all $x, y \in V$. By (1.4) in Lemma 1.4, we get

(2.12)
$$\widetilde{\rho}(A - \frac{1}{4} f_a) \le \frac{2}{1 - L}.$$

Define $T_c : \mathbb{M}_{\widetilde{\rho}} \longrightarrow \mathbb{M}_{\widetilde{\rho}}$ by $T_c g(x) = \frac{1}{8} g(2x)$ for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and all $x \in V$. By (2.4), (2.5), and (2.6), we obtain

$$\rho(\frac{1}{2^3} f_c(2x) - f_c(x)) \le \psi(x, x)$$

for all $x \in V$ and hence

(2.13) $$\widetilde{\rho}(T_c f_c - f_c) \leq 1.$$

Let $Hg = 2T_c g$ for all $g \in \mathbb{M}_{\widetilde{\rho}}$. Then

$$Hg(x) = \frac{1}{4}g(2x).$$

for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and for all $x \in V$. Let $g, h \in \mathbb{M}_{\widetilde{\rho}}$. Suppose that $\widetilde{\rho}(g - h) \leq c$ for some positive real number $c$. Then $\rho(g(x) - h(x)) \leq c\psi(x, x)$ for all $x \in V$ and by (2.3), we get

$$\rho(Hg(x) - Hh(x)) = \rho(\frac{1}{4}g(2x) - \frac{1}{4}h(2x)) \leq c\psi(2x, 2x) \leq cL\psi(x, x)$$

for all $x \in V$. Hence $\widetilde{\rho}(Hg - Hh) \leq cL$ and so

$$\widetilde{\rho}(Hg - Hh) \leq L\widetilde{\rho}(g - h).$$

Since $T_c$ is linear, by Lemma 1.4, there is a $C \in \mathbb{M}_{\widetilde{\rho}}$ such that $\{T_c^n \frac{1}{4}f_c\}$ is $\widetilde{\rho}$-convergent to $C$. Since $\widetilde{\rho}$ is lower semi-continuous, we get

$$\widetilde{\rho}(T_c C - C) \leq \liminf_{n \to \infty} \widetilde{\rho}(T_c C - T_c^{n+1}\frac{1}{4}f_c) \leq \liminf_{n \to \infty} L\widetilde{\rho}(C - T_c^n \frac{1}{4}f_c) = 0$$

and hence $C$ is a fixed point of $T_c$ in $\mathbb{M}_{\widetilde{\rho}}$. Replacing $x$ and $y$ by $2^n x$ and $2^n y$ in (2.1), respectively, by (2.2), we have

$$\rho(\frac{1}{2^{3n+2}}Df_c(2^n x, 2^n y))$$
$$\leq \rho(\frac{1}{2^{3n+1}}Df(2^{n+1}x, 2^{n+1}y)) + \rho(\frac{1}{2^{3n}}Df(2^n x, 2^n y))$$
$$\leq L^{n+1}\phi(x, y) + L^n\phi(x, y)$$

for all $x, y \in V$. Hence we get

$$\lim_{n \to \infty} \rho\Big(\frac{1}{2^{3n+2}}Df_c(2^n x, 2^n y)\Big) = 0$$

for all $x, y \in V$. Similar to $A$, we have

(2.14) $$DC(x, y) = 0$$

for all $x, y \in V$ and by (1.4) in Lemma 1.4, we get

$$\rho(C(x) - \frac{1}{4}f_c(x)) \leq \frac{2}{1 - L}\psi(x, x)$$

for all $x \in X$. Hence we have

(2.15) $$\widetilde{\rho}(C - \frac{1}{4}f_c) \leq \frac{2}{1 - L}.$$

Let $F = \frac{1}{8}C - \frac{1}{2}A$. Since $A$ is a fixed point of $T_a$, $A(2x) = 2A(x)$ for all $x \in X$ and similarly, $C(2x) = 8C(x)$ for all $x \in X$. By Lemma 2.1, $A$ is additive and $C$ is cubic. Hence $F$ is an additive-cubic mapping. Since $f(x) = \frac{1}{6}f_c(x) - \frac{2}{3}f_a(x)$, we have

$$\widetilde{\rho}(F - \frac{3}{16}f) \leq \widetilde{\rho}(A - \frac{1}{4}f_a) + \widetilde{\rho}(\frac{1}{4}C - \frac{1}{16}f_c) \leq \widetilde{\rho}(A - \frac{1}{4}f_a) + \widetilde{\rho}(C - \frac{1}{4}f_c),$$

and hence by (2.12) and (2.15), we have (2.3).

To prove the uniquness of $F$, let $K : V \longrightarrow X_\rho$ be another additive-cubic mapping with (2.3). By (2.3), we get

$$\rho(\frac{1}{4}K(x) - \frac{1}{4}F(x)) \le \rho(K(x) - \frac{3}{16}f(x)) + \rho(F(x) - \frac{3}{16}f(x)))$$

$$\le \frac{8}{1-L}\psi(x,x)$$

for all $x \in V$ and so

$$\rho(\frac{1}{16}K_a(x) - \frac{1}{16}F_a(x)) \le \rho(\frac{1}{32}K(2x) - \frac{1}{32}F(2x)) + \rho(\frac{1}{4}K(x) - \frac{1}{4}F(x)))$$

$$\le \frac{8(1+L)}{1-L}\psi(x,x)$$

for all $x \in V$. Since $F_a$ and $K_a$ are fixed points of $T_a$, we have

$$\rho(\frac{1}{16}K_a(x) - \frac{1}{16}F_a(x)) = \rho(\frac{1}{16}T_a^n K_a(x) - \frac{1}{16}T_a^n F_a(x))$$

$$\le \frac{8(1+L)}{1-L}L^n\psi(x,x)$$

for all $x \in V$ and for all $n \in \mathbb{N}$. Letting $n \to \infty$ in the last inequality, we have $F_a = K_a$ and similarly, we have $F_c = K_c$. Thus $F = K$. $\square$

Comparing the results in a modular and a convex modular, we may see that the coefficient in the case of convex modular is smaller.

**Theorem 2.3.** *Suppose that every assumption of Theorem 2.2 holds, $\rho$ is convex and $0 \le L < 2$. Then there exists a unique additive-cubic mapping $F : V \longrightarrow X_\rho$ such that*

$$(2.16) \qquad \rho(F(x) - \frac{3}{16}f(x)) \le \frac{5}{32(2-L)}\psi(x,x)$$

*for all $x \in V$, where $\psi(x,y) = \phi(x,2y) + \phi(x,y) + \phi(0,y)$.*

*Proof.* Define $T_a : \mathbb{M}_{\widetilde{\rho}} \longrightarrow \mathbb{M}_{\widetilde{\rho}}$ by $T_a g(x) = \frac{1}{2}g(2x)$ for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and all $x \in V$. Let $g, h \in \mathbb{M}_{\widetilde{\rho}}$. Suppose that $\widetilde{\rho}(g - h) \le c$ for some positive real number $c$. Then by (2.3) and (M4), we have

$$\rho(T_a g(x) - T_a h(x)) \le \frac{1}{2}\rho(g(2x) - h(2x)) \le \frac{1}{2}Lc\psi(x,x)$$

for all $x \in V$ and so $\widetilde{\rho}(T_a g - T_a h) \le \frac{1}{2}L\widetilde{\rho}(g - h)$. Hence $T_a$ is a $\widetilde{\rho}$-contraction. By (2.1), we get

$$(2.17) \qquad \rho(f(x) + f(-x)) \le \phi(0,x),$$

$$(2.18) \qquad \rho(f(3x) - 4f(2x) + 5f(x)) \le \phi(x,x),$$

and

$$(2.19) \qquad \rho(f(4x) - 2f(3x) - 2f(2x) - 2f(-x) + 4f(x)) \le \phi(x,2x),$$

for all $x \in V$. By (2.17), (2.18), and (2.19), we obtain

$$\rho(\frac{1}{2}f_a(2x) - f_a(x)) \le \frac{1}{8}\phi(x,2x) + \frac{1}{4}\phi(x,x) + \frac{1}{4}\phi(0,x) \le \frac{1}{4}\psi(x,x)$$

for all $x \in V$ and hence

$$\rho(T_a f_a(x) - f_a(x)) \leq \frac{1}{4}\psi(x,x)$$

for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and all $x \in V$. Hence we have

(2.20) $$\widetilde{\rho}(T_a f_a - f_a) \leq \frac{1}{4}.$$

Let $Gg = 2T_a g$ for all $g \in \mathbb{M}_{\widetilde{\rho}}$. Then

$$Gg(x) = g(2x)$$

for all $x \in V$. Suppose that $\widetilde{\rho}(g - h) \leq c$ for some positive real number $c$. Then $\rho(g(x) - h(x)) \leq c\psi(x,x)$ for all $x \in V$ and by (2.2), we have

$$\rho(Gg(x) - Gh(x)) = \rho(g(2x) - h(2x)) \leq c\psi(2x, 2x) \leq cL\psi(x,x)$$

for all $x \in V$. Hence $\widetilde{\rho}(Gg - Gh) \leq cL$ and so

$$\widetilde{\rho}(Gg - Gh) \leq L\widetilde{\rho}(g - h).$$

Since $T_a$ is linear, by Lemma 1.5, there is an $A \in \mathbb{M}_{\widetilde{\rho}}$ such that $\{T_a^n \frac{f_a}{4}\}$ is $\widetilde{\rho}$-convergent to $A$. In fact, we get

$$\lim_{n \to \infty} \rho(\frac{1}{2^{n+2}} f_a(2^n x) - A(x)) = 0$$

for all $x \in V$. Since $\widetilde{\rho}$ is lower semi-continuous, we get

$$\widetilde{\rho}(T_a A - A) \leq \liminf_{n \to \infty} \widetilde{\rho}(T_a A - T_a^{n+1}\frac{f_a}{4}) \leq \liminf_{n \to \infty} L\widetilde{\rho}(A - T_a^n \frac{f_a}{4}) = 0$$

and hence $A$ is a fixed point of $T_a$ in $\mathbb{M}_{\widetilde{\rho}}$. Similar to Theorem 2.2, we have

(2.21) $$DA(x,y) = 0$$

for all $x, y \in V$ and by (1.6) in Lemma 1.5 and (2.20), we get

(2.22) $$\widetilde{\rho}(A - \frac{1}{4}f_a) \leq \frac{1}{4(2 - L)}.$$

Define $T_c : \mathbb{M}_{\widetilde{\rho}} \longrightarrow \mathbb{M}_{\widetilde{\rho}}$ by $T_c g(x) = \frac{1}{8}g(2x)$ for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and all $x \in V$. By (2.17), (2.18), and (2.19), we obtain

$$\rho(\frac{1}{2^3} f_c(2x) - f_c(x)) \leq \frac{1}{4}\psi(x,x)$$

for all $x \in V$ and hence

(2.23) $$\widetilde{\rho}(T_c f_c - f_c) \leq \frac{1}{4}.$$

Let $Hg = 2T_c g$ for all $g \in \mathbb{M}_{\widetilde{\rho}}$. Then

$$Hg(x) = \frac{1}{4}g(2x).$$

for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and for all $x \in V$. Let $g, h \in \mathbb{M}_{\widetilde{\rho}}$. Suppose that $\widetilde{\rho}(g - h) \leq c$ for some positive real number $c$. Then $\rho(g(x) - h(x)) \leq c\psi(x,x)$ for all $x \in V$ and by (2.3), we get

$$\rho(Hg(x) - Hh(x)) = \rho(\frac{1}{4}g(2x) - \frac{1}{4}h(2x)) \le c\psi(2x, 2x) \le cL\psi(x, x)$$

for all $x \in V$. Hence $\widetilde{\rho}(Gg - Gh) \le cL$ and so

$$\widetilde{\rho}(Hg - Hh) \le L\widetilde{\rho}(g - h).$$

Since $T_c$ is linear, by Lemma 1.5, there is a $C \in \mathbb{M}_{\widetilde{\rho}}$ such that $\{T_c^n \frac{1}{4}f_c\}$ is $\widetilde{\rho}$-convergent to $C$. Since $\widetilde{\rho}$ is lower semi-continuous, we get

$$\widetilde{\rho}(T_c C - C) \le \liminf_{n \to \infty} \widetilde{\rho}(T_c C - T_c^{n+1}\frac{1}{4}f_c) \le \liminf_{n \to \infty} L\widetilde{\rho}(C - T_c^n \frac{1}{4}f_c) = 0$$

and hence $C$ is a fixed point of $T_c$ in $\mathbb{M}_{\widetilde{\rho}}$. Similar to Theorem 2.2, we get

$$(2.24) \qquad\qquad\qquad DC(x, y) = 0$$

for all $x, y \in V$ and by (1.6) in Lemma 1.5, we get

$$\rho(C(x) - \frac{1}{4}f_c(x)) \le \frac{1}{4(2 - L)}\psi(x, x)$$

for all $x \in X$. Hence we have

$$(2.25) \qquad\qquad\qquad \widetilde{\rho}(C - \frac{1}{4}f_c) \le \frac{1}{4(2 - L)}.$$

Let $F = \frac{1}{8}C - \frac{1}{2}A$. Since $A$ is a fixed point of $T_a$, $A(2x) = 2A(x)$ for all $x \in X$ and similarly, $C(2x) = 8C(x)$ for all $x \in X$. By Lemma 2.1, $A$ is additive and $C$ is cubic. Hence $F$ is an additive-cubic mapping. Since $f(x) = \frac{1}{6}f_c(x) - \frac{2}{3}f_a(x)$, by (2.22) and (2.25), we have

$$\widetilde{\rho}(F - \frac{3}{16}f) \le \frac{1}{2}\widetilde{\rho}(A - \frac{1}{4}f_a) + \frac{1}{2}\widetilde{\rho}(\frac{1}{4}C - \frac{1}{16}f_c) \le \frac{1}{2}\widetilde{\rho}(A - \frac{1}{4}f_a) + \frac{1}{8}\widetilde{\rho}(C - \frac{1}{4}f_c).$$

and hence we have (2.16). The rest of the proof is similar to Theorem 2.2. $\square$

**Remark 2.4.** Sadeghi [13] proved the generalized Hyers-Ulam stability of functional equations in modular spaces with the $\triangle_2$-condition and in [15], authors proved the stability of mappings $f : V \longrightarrow X_\rho$ and $\phi : V^2 \longrightarrow [0, \infty)$ satisfying $f(0) = 0$,

$$(2.26) \qquad \lim_{n \to \infty} \frac{\phi(2^n x, 2^n y)}{4^n} = 0, \; \phi(2x, 2x) \le 4L\phi(x, x), \; \forall x, y \in V,$$

and

$$\rho(4f(x + y) + 4f(x - y) - 8f(x) - 8f(y)) \le \phi(x, y), \; \forall x, y \in V$$

for some real number $L$ with $0 \le L < \frac{1}{2}$ whose codomain is equipped with a convex and lower semi-continuous modular without $\triangle_2$-conditions. Our results guarantee the stability of an additive-cubic mapping, whose induced modular is lower semi-continuous without the convexity and $\triangle_2$-conditions if $0 \le L < \frac{1}{4}$. Further, in [15], authors left whether the multiple of 4 on the left side of the inequality (6) can be dropped as a problem. We can solve the problem by using Lemma 1.4 and its proof is similar to the proof in Theorem 2.2.

In fact, suppose that $\phi : V^2 \longrightarrow [0, \infty)$ is a mapping with (2.26) and that $0 \le L < \frac{1}{4}$. Let $f : V \longrightarrow X_\rho$ be a mapping such that $f(0) = 0$ and

$$(2.27) \qquad \rho(f(x + y) + f(x - y) - 2f(x) - 2f(y)) \le \phi(x, y)$$

for all $x, y \in X$. Let $\psi(x, y) = \phi(x, y)$ for all $x, y \in V$, $f_0(x) = 4f(x)$, and $Tg(x) = \frac{1}{4}g(2x)$. Then by (2.27), we have

$$\rho(Tf_0(x) - f_0(x)) = \rho(\frac{1}{4}f_0(2x) - f_0(x)) \leq \phi(x, x)$$

for all $x \in V$ and so

(2.28) $$\widetilde{\rho}(Tf_0 - f_0) \leq 1.$$

Moreover, by (2.26), we have

(2.29) $$\widetilde{\rho}(2Tg - 2Th) \leq 4L\widetilde{\rho}(g - h).$$

Since $0 \leq L < \frac{1}{4}$, by Lemma 1.4, there is a fixed point $Q \in \mathbb{M}_{\widetilde{\rho}}$ such that $\{T^n \frac{f_0}{4}\} = \{T^n f\}$ converges to $Q$ in $X_\rho$ and

(2.30) $$\rho(Q(x) - f(x)) \leq \frac{2}{1 - 4L}\phi(x, x)$$

for all $x \in V$. We can show that $Q$ is a quadratic mapping ([15]).

Further, suppose that $\rho$ is convex and $0 \leq L < 1$. Then (2.28) and (2.29) can be replaced by

$$\widetilde{\rho}(Tf_0 - f_0) \leq \frac{1}{4}$$

and

$$\widetilde{\rho}(2Tg - 2Th) \leq 2L\widetilde{\rho}(g - h),$$

respectively. By Lemma 1.5 and (1.6),

$$\rho(Q(x) - f(x)) \leq \frac{1}{4(2 - 2L)}\phi(x, x) = \frac{1}{8(1 - L)}\phi(x, x)$$

for all $x \in V$.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

## Acknowledgment

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# Results on value-shared of admissible function and non-admissible function in the unit disc *

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**Abstract**

In this paper, we consider the uniqueness problem of admissible functions and non-admissible functions sharing some values in the unit disc. We obtain: If $f_1$ is admissible and $f_2$ is inadmissible satisfying $\lim_{r \to 1^-} T(r, f_2) = \infty$, $a_j (j = 1, 2, \ldots; q)$ be $q$ distinct complex numbers. Then

(i) $f_1(z), f_2(z)$ can share at most three values $a_1, a_2, a_3$ $IM$;

(ii) $f_1(z), f_2(z)$ can share at most five values $a_j (j = 1, 2, \ldots; 5)$ with reduced weight 1. Our results of this paper are improvement of the uniqueness theorems of meromorphic functions sharing some values on the whole complex plane which given by Yi and Cao.

**Key words:** uniqueness; meromorphic function; admissible; non-admissible.

**Mathematical Subject Classification (2010):** 30D 35.

## 1 Introduction and Main Results

In what follows, we shall assume that reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions such as the proximity function $m(r, f)$, counting function $N(r, f)$, characteristic function $T(r, f)$, the first and second main theorems, lemma on the logarithmic derivatives etc. of Nevalinna theory, (see Hayman [7] , Yang [16] and Yi and Yang [19]). For a meromorphic function $f$, $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ for all $r$ outside a possible exceptional set of finite logarithmic measure.

We use $\mathbb{C}$ to denote the open complex plane, $\widehat{\mathbb{C}} := \mathbb{C} \bigcup \{\infty\}$ to denote the extended complex plane, and $\mathbb{D} = \{z : |z| < 1\}$ to denote the unit disc.

Let $f, g$ be two non-constant meromorphic functions in $\mathbb{D}$ and $a \in \widehat{\mathbb{C}}$. If $E(a, \mathbb{D}, f) = E(a, \mathbb{D}, g)$, we say $f$ and $g$ share $a$ $CM$(counting multiplicities) in $\mathbb{D}$. If $\overline{E}(a, \mathbb{D}, f) =$

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$\overline{E}(a, \mathbb{D}, g)$, we say $f$ and $g$ share $a$ $IM$(ignoring multiplicities) in $\mathbb{D}$. If $\mathbb{D}$ is replaced by $\mathbb{C}$, we give the simple notation as before, $E(a, f), \overline{E}(a, f)$ and so on(see [16]).

R.Nevanlinna [12] proved the following well-known theorems.

**Theorem 1.1** (see [12]) *If $f$ and $g$ are two non-constant meromorphic functions that share five distinct values $a_1, a_2, a_3, a_4, a_5$ IM in $\mathbb{C}$, then $f(z) \equiv g(z)$.*

**Theorem 1.2** (see [12]) *If $f$ and $g$ are two distinct non-constant meromorphic functions that share four distinct values $a_1, a_2, a_3, a_4$ CM in $\mathbb{C}$, then $f$ is a Möbius transformation of $g$ , two of the shared values, say $a_1$ and $a_2$ are Picard values, and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.*

After their very work, the uniqueness of meromorphic functions with shared values in the whole complex plane attracted many investigations (see [16]). In 1987 and 1988, Yi [17, 18] dealt with the problems of multiple values and uniquness of meromorphic functions sharing some values in the whole complex plane by adopting L.Yang's method and obtained some results which improved the concerning theorems due to Gopalakrishna and Bhoosnurmath's [6], Ueda [14]. To state the theorems, we will explain some notations as follows.

Let $f(z)$ be a non-constant meromorphic function, an arbitrary complex number $a \in \hat{\mathbb{C}}$, and $k$ be a positive integer. We use $\overline{E}_{k)}(a, f)$ to denote the set of zeros of $f - a$, with multiplicities no greater than $k$, in which each zero counted only once. We say that $f(z)$ and $g(z)$ share the value $a$ with reduced weight $k$, if $\overline{E}_{k)}(a, f) = \overline{E}_{k)}(a, g)$.

In 1987, Yi [17] obtained the uniqueness theorems concerning multiple values of meromorphic functions as follows.

**Theorem 1.3** *(see [17, 19, Theorem 3.15]). Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, $a_j(j = 1, 2, \ldots, q)$ be $q$ distinct complex numbers, and let $k_j(j = 1, 2, \ldots, q)$ be positive integers or $\infty$ satisfying*

(1)
$$k_1 \geq k_2 \geq \cdots \geq k_q \geq 1.$$

*If*

$$\overline{E}_{k_j)}(a_j, f) = \overline{E}_{k_j)}(a_j, g) \quad (j = 1, 2, \ldots, q)$$

*and*

(2)
$$\sum_{j=3}^{q} \frac{k_j}{k_j + 1} > 2,$$

*then $f(z) \equiv g(z)$.*

In recent, it is an interesting topic to investigate the uniqueness with shared values in the subregion of the complex plane such as the unit disc, an angular domain, see [1, 2, 9, 10, 11, 15, 20, 21, 22]. In 1999, Fang [5] studied the uniqueness problem of admissible meromorphic functions in the unit disc $\mathbb{D}$ sharing two sets and three sets. Later, there were some results of uniqueness of meromorphic function in the unit disc concerning admissible functions. To state some uniqueness theorems of meromorphic functions in the unit disc $\mathbb{D}$, we need the following basic notations and definitions of meromorphic functions in $\mathbb{D}$(see [3], [4], [8]).

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**Definition 1.1** *Let $f$ be a meromorphic function in $\mathbb{D}$ and $\lim_{r\to1^-} T(r,f) = \infty$. Then*

$$\alpha(f) := \limsup_{r\to1^-} \frac{T(r,f)}{-\log(1-r)}$$

*is called the index of inadmissibility of $f$. If $\alpha(f) = \infty$, $f$ is called admissible.*

**Definition 1.2** *Let $f$ be a meromorphic function in $\mathbb{D}$ and $\lim_{r\to1^-} T(r,f) = \infty$. Then*

$$\rho(f) := \limsup_{r\to1^-} \frac{\log^+ T(r,f)}{-\log(1-r)}$$

*is called the order (of growth) of $f$.*

For admissible functions, the following theorem plays a very important role in studies the uniqueness problems of meromorphic functions in the unit disc.

**Theorem 1.4** (see [13, Theorem 3]). *Let $f$ be an admissible meromorphic function in $\mathbb{D}$, $q$ be a positive integer and $a_1, a_2, \ldots, a_q$ be pairwise distinct complex numbers. Then, for $r \to 1^-$, $r \notin E$,*

$$(q-2)T(r,f) \le \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f-a_j}\right) + S(r,f),$$

*where $E \subset (0,1)$ is a possibly occurring exceptional set with $\int_E \frac{dr}{1-r} < \infty$. If the order of $f$ is finite, the remainder $S(r,f)$ is a $O\left(\log \frac{1}{1-r}\right)$ without any exceptional set.*

In 2005, Titzhoff [13] investigated the uniqueness of two admissible functions in the unit disc $\mathbb{D}$ by using the Second Main Theorem for admissible functions (Theorem 1.4) and obtained the five values theorem in the unit disc $\mathbb{D}$ as follows.

**Theorem 1.5** (see [13]). *If two admissible function $f, g$ share five distinct values, then $f \equiv g$.*

In 2009, Mao and Liu [11] gave a different method to investigate the uniqueness problem of meromorphic functions in unit disc and obtained the following results.

**Theorem 1.6** (see [11]). *Let $f, g$ be two meromorphic functions in $\mathbb{D}$, $a_j \in \widehat{\mathbb{C}}(j = 1, 2, \ldots, 5)$ be five distinct values, and $\Delta(\theta_0, \delta)(0 < \delta < \pi)$ be an angular domain such that for some $a \in \widehat{\mathbb{C}}$,*

$$(3) \qquad \limsup_{r\to1^-} \frac{\log n(r, \Delta(\theta_0, \delta/2), f(z) = a)}{\log \frac{1}{1-r}} = \tau > 1.$$

*If $f$ and $g$ share $a_j(j = 1, 2, \ldots, 5)$ IM in $\Delta(\theta_0, \delta)$, then $f(z) \equiv g(z)$.*

**Remark 1.1** *In fact, the condition (3) implies that $f$ is admissible in the unit disc. Therefore, Theorem 1.6 is one result of uniqueness of admissible functions in the unit disc.*

For admissible functions in the unit disc $\mathbb{D}$, from Theorem 1.4, using the same argument as in the proofs of Theorem 1.3, we can easily get the following results.

**Theorem 1.7** *Let $f_1(z)$ and $f_2(z)$ be two admissible meromorphic functions in $\mathbb{D}$, $a_j(j = 1, 2, \ldots, q)$ be $q$ distinct complex numbers, and let $k_j(j = 1, 2, \ldots, q)$ be positive integers or $\infty$ satisfying (1). If $f_1(z)$ and $f_2(z)$ satisfy*

$$(4) \qquad \overline{E}_{k_j)}(a_j, \mathbb{D}, f_1) = \overline{E}_{k_j)}(a_j, \mathbb{D}, f_2) \quad (j = 1, 2, \ldots, q)$$

*and (2), then $f_1(z) \equiv f_2(z)$, where $\overline{E}_{k)}(a, \mathbb{D}, f)$ to denote the set of zeros of $f - a$ in $\mathbb{D}$, with multiplicities no greater than $k$, in which each zero counted only once.*

**Remark 1.2** *For $a \in \widehat{\mathbb{C}}$ and a positive integer $k$, we can say that $f_1(z), f_2(z)$ share the value $a$ in $\mathbb{D}$ with reduced weight $k$, if $\overline{E}_{k)}(a, \mathbb{D}, f_1) = \overline{E}_{k)}(a, \mathbb{D}, f_2)$.*

Similar to the corollary of Theorem 1.3 (see [19, Corollary,pp.181.]), we can get the following corollary.

**Corollary 1.1** *Let $f_1(z)$ and $f_2(z)$ be two admissible meromorphic functions in $\mathbb{D}$, $a_j(j = 1, 2, \ldots, q)$ be $q$ distinct complex numbers, and let $k_j(j = 1, 2, \ldots, q)$ be positive integers or $\infty$ satisfying (1) and (4).*
  *(i) if $q = 7$, then $f_1(z) \equiv f_2(z)$;*
  *(ii) if $q = 6$ and $k_3 \geq 2$, then $f_1(z) \equiv f_2(z)$;*
  *(iii) if $q = 5$, $k_3 \geq 3$ and $k_5 \geq 2$, then $f_1(z) \equiv f_2(z)$;*
  *(iv) if $q = 5$ and $k_4 \geq 4$, then $f_1(z) \equiv f_2(z)$;*
  *(v) if $q = 5$, $k_3 \geq 5$ and $k_4 \geq 3$, then $f_1(z) \equiv f_2(z)$;*
  *(vi) if $q = 5$, $k_3 \geq 6$ and $k_4 \geq 2$, then $f_1(z) \equiv f_2(z)$.*

**Remark 1.3** *In Theorem 1.5, the conclusion $f(z) \equiv g(z)$ holds when $q = 5$ and $k_j = \infty$ $(j = 1, 2, \ldots, 5)$. From Corollary 1.1, we can get that $f_1(z) \equiv f_2(z)$ when $q = 5$ and $k_j$ $(j = 1, 2, \ldots, 5)$ satisfy any of the four conditions (i)-(iv). Hence, Corollary 1.1 is an improvement of Theorem 1.5.*

For non-admissible functions, the following theorem also plays a very important role in studies theirs uniqueness problems.

**Theorem 1.8** (see [13, Theorem 2]). *Let $f$ be a meromorphic function in $\mathbb{D}$ and $\lim_{r \to 1^-} T(r, f) = \infty$, $q$ be a positive integer and $a_1, a_2, \ldots, a_q$ be pairwise distinct complex numbers. Then, for $r \to 1^-$, $r \notin E$,*

$$(q - 2)T(r, f) \leq \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f - a_j}\right) + \log \frac{1}{1 - r} + S(r, f).$$

**Remark 1.4** *(i) In contrast to admissible functions, the term $\log \frac{1}{1-r}$ in Theorem 1.8 does not necessarily enter the remainder $S(r, f)$ because the non-admissible function $f$ may have $T(r, f) = O\left(\log \frac{1}{1-r}\right)$.*

  *(ii) If $0 < \alpha(f) < \infty$, we can see that $S(r, f) = o\left(\log \frac{1}{1-r}\right)$ holds in Theorem 1.8 without a possible exception set.*

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From Theorem 1.8 and Remark 1.4, we can see that the uniqueness of non-admissible functions is more intricate than the case of admissible functions.

In this paper, we will deal with the uniqueness problem of non-admissible functions in $\mathbb{D}$. We use $\Upsilon_\alpha$ to denote the class of non-admissible functions satisfying the condition: $\alpha(f) = \alpha(0 < \alpha < \infty)$ for $f \in \Upsilon_\alpha$. For the class $\Upsilon_\alpha$, we get the following results

**Theorem 1.9** *Let $f(z) \in \Upsilon_\alpha$, $a_j(j = 1, 2, \ldots, q)$ are $q$ distinct complex numbers. If $q = 5 + [\frac{2}{k} + \frac{k+1}{k\alpha}]$, then there does not exist $g(z)(\not\equiv f(z)) \in \Upsilon_\alpha$ satisfying*

$$(5) \qquad \overline{E}_{k)}(a_j, \mathbb{D}, f) = \overline{E}_{k)}(a_j, \mathbb{D}, g), \quad (j = 1, 2, \ldots, q),$$

*where $[x]$ denotes the largest integer less than or equal to $x$.*

**Corollary 1.2** *Let $f(z) \in \Upsilon_\alpha$. Then $f(z)$ is uniquely determined in $\Upsilon_\alpha$ by one of the following cases:*
    *(i) if $f$ have seven point-sets $\overline{E}_{1)}(a_j, \mathbb{D}, f)(j = 1, 2, \ldots, 7)$ and $\alpha > 1$;*
    *(ii) if $f$ have six point-sets $\overline{E}_{2)}(a_j, \mathbb{D}, f)(j = 1, 2, \ldots, 6)$ and $\alpha > \frac{3}{2}$;*
    *(iii) if $f$ have five point-sets $\overline{E}_{3)}(a_j, \mathbb{D}, f)(j = 1, 2, \ldots, 5)$ and $\alpha > 4$.*

**Remark 1.5** *For Corollary 1.1, we can see that the conclusion (iii) in Corollary 1.1 is an improvement of Theorem 1.6. In fact, the conclusion of Theorem 1.6 is that non-constant meromorphic function $f$ is uniquely determined in $\mathbb{D}$ by five point-sets $\overline{E}_{\infty)}(a_j, \mathbb{D}, f)(j = 1, 2, \ldots, 5)$ and $\alpha(f) = \infty$.*

**Theorem 1.10** *Let $\alpha > 12$ and $f(z) \in \Upsilon_\alpha$, $a_j(j = 1, 2, \ldots, 5)$ be five distinct complex numbers. Then $f(z)$ is uniquely determined in $\Upsilon$ by three point-sets $\overline{E}_{3)}(a_j, \mathbb{D}, f)(j = 1, 2, 3)$ and two point-sets $\overline{E}_{2)}(a_j, \mathbb{D}, f)(j = 4, 5)$.*

Furthermore, for the uniqueness of regular inadmissibility functions we obtain the following theorems

**Theorem 1.11** *Let $a_j(j = 1, 2, \ldots, q)$ be $q$ distinct complex numbers, and let $k_j(j = 1, 2, \ldots, q)$ be positive integers or $\infty$ satisfying (1). If $f_1(z), f_2(z)$ be non-constant regular inadmissibility functions satisfying $0 < \alpha(f_1), \alpha(f_2) < \infty$, (4) and*

$$(6) \qquad \sum_{j=3}^{q} \frac{k_j}{k_j + 1} - 2 > \frac{2}{\alpha(f_1) + \alpha(f_2)},$$

*then $f_1(z) \equiv f_2(z)$.*

From Theorem 1.11, similar to Corollary 1.1, we can get the following results easily.

**Corollary 1.3** *Let $a_j(j = 1, 2, \ldots, q)$ be $q$ distinct complex numbers, and let $k_j(j = 1, 2, \ldots, q)$ be positive integers or $\infty$ satisfying (1), $\alpha := \min\{\alpha(f_1), \alpha(f_2)\}$. And let $f_1(z), f_2(z)$ be non-constant regular inadmissibility functions satisfying $0 < \alpha(f_1), \alpha(f_2) < \infty$ and (4),*
    *(i) if $\alpha > 1$, $q = 7$ and $k_7 \geq 2$, then $f_1(z) \equiv f_2(z)$;*
    *(ii) if $\alpha > 1$, $q = 6$ and $k_6 \geq 4$, then $f_1(z) \equiv f_2(z)$;*

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$(iii)$ *if* $\alpha > 2$ *and* $q = 7$, *then* $f_1(z) \equiv f_2(z)$;
$(iv)$ *if* $\alpha > 3$, $q = 6$ *and* $k_3 \geq 2$, *then* $f_1(z) \equiv f_2(z)$;
$(v)$ *if* $\alpha > 6$, $q = 5$, $k_3 \geq 3$ *and* $k_5 \geq 2$, *then* $f_1(z) \equiv f_2(z)$;
$(vi)$ *if* $\alpha > 10$, $q = 5$ *and* $k_4 \geq 4$, *then* $f_1(z) \equiv f_2(z)$;
$(vii)$ *if* $\alpha > 12$, $q = 5$, $k_3 \geq 5$ *and* $k_4 \geq 3$, *then* $f_1(z) \equiv f_2(z)$;
$(viii)$ *if* $\alpha > 42$, $q = 5$, $k_3 \geq 6$ *and* $k_4 \geq 2$, *then* $f_1(z) \equiv f_2(z)$.

**Remark 1.6** *In Corollary 1.1,* $f_1(z), f_2(z)$ *are all admissible functions, that is,* $\alpha(f_1) = \infty$ *and* $\alpha(f_2) = \infty$. *From the conclusions of Corollary 1.3, we see that* $f_1(z) \equiv f_2(z)$ *holds when* $f_1(z), f_2(z)$ *are non-constant regular inadmissibility functions with* $\min\{\alpha(f_1), \alpha(f_2)\} > \zeta$ *and* $\zeta$ *a positive constant. Hence, Corollary 1.3 is an improvement of Corollary 1.1.*

The following theorem will show that an admissible function can share sufficiently many values concerning multiple values with another inadmissible function.

**Theorem 1.12** *If* $f_1$ *is admissible and* $f_2$ *is inadmissible satisfying* $\lim_{r \to 1^-} T(r, f_2) = \infty$, $a_j(j = 1, 2, \ldots, q)$ *be* $q$ *distinct complex numbers, and let* $k_j(j = 1, 2, \ldots, q)$ *be positive integers or* $\infty$ *satisfying (1). Then*

$$(7) \qquad \sum_{j=2}^{q} \frac{k_j}{k_j + 1} - 2 > 0$$

*and (4) do not hold at same time.*

**Corollary 1.4** *If* $f_1$ *is admissible and* $f_2$ *is inadmissible satisfying* $\lim_{r \to 1^-} T(r, f_2) = \infty$, $a_j(j = 1, 2, \ldots, q)$ *be* $q$ *distinct complex numbers. Then*
   *(i)* $f_1(z), f_2(z)$ *can share at most three values* $a_1, a_2, a_3$ *IM;*
   *(ii)* $f_1(z), f_2(z)$ *can share at most five values* $a_j(j = 1, 2, \ldots, 5)$ *with reduced weight 1;*
*And any one of the following cases can not hold*
   *(iii)* $q = 4$ *and* $\overline{E}_{k_1)}(a_1, \mathbb{D}, f_1) = \overline{E}_{k_1)}(a_1, \mathbb{D}, f_2)$ $(k_1 \geq 6)$, $\overline{E}_{6)}(a_2, \mathbb{D}, f_1) = \overline{E}_{6)}(a_2, \mathbb{D}, f_2)$, $\overline{E}_{2)}(a_3, \mathbb{D}, f_1) = \overline{E}_{2)}(a_3, \mathbb{D}, f_2)$ *and* $\overline{E}_{1)}(a_4, \mathbb{D}, f_1) = \overline{E}_{1)}(a_4, \mathbb{D}, f_2)$;
   *(iv)* $q = 4$ *and* $\overline{E}_{k_1)}(a_1, \mathbb{D}, f_1) = \overline{E}_{k_1)}(a_1, \mathbb{D}, f_2)$ $(k_1 \geq 3)$, $\overline{E}_{3)}(a_2, \mathbb{D}, f_1) = \overline{E}_{3)}(a_2, \mathbb{D}, f_2)$, $\overline{E}_{2)}(a_3, \mathbb{D}, f_1) = \overline{E}_{2)}(a_3, \mathbb{D}, f_2)$ *and* $\overline{E}_{2)}(a_4, \mathbb{D}, f_1) = \overline{E}_{2)}(a_4, \mathbb{D}, f_2)$;
   *(v)* $q = 5$ *and* $\overline{E}_{k)}(a_i, \mathbb{D}, f_1) = \overline{E}_{k)}(a, \mathbb{D}, f_2)$ $(k \geq 2, i = 1, 2)$, $\overline{E}_{1)}(a_j, \mathbb{D}, f_1) = \overline{E}_{1)}(a_j, \mathbb{D}, f_2)$ $(j = 3, 4, 5)$.

## 2   Some Lemmas

To prove our results, we will require the following lemmas.

**Lemma 2.1** (see [13, Lemma 1]). *Let* $f(z), g(z)$ *satisfy* $\lim_{r \to 1^-} T(r, f) = \infty$ *and* $\lim_{r \to 1^-} T(r, g) = \infty$. *If there is a* $K \in (0, \infty)$ *with*

$$T(r, f) \leq KT(r, g) + S(r, f) + S(r, g),$$

*then each* $S(r, f)$ *is also an* $S(r, g)$.

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**Lemma 2.2** (see [19, Lemma 3.4]). *Let $f(z)$ be a non-constant meromorphic function, $a$ be an arbitrary complex number, and $k$ be a positive integer. Then*

$$\overline{N}\left(r, \frac{1}{f-a}\right) \leq \frac{k}{k+1} \overline{N}_{k)}\left(r, \frac{1}{f-a}\right) + \frac{1}{k+1} N\left(r, \frac{1}{f-a}\right),$$

*and*

$$\overline{N}\left(r, \frac{1}{f-a}\right) \leq \frac{k}{k+1} \overline{N}_{k)}\left(r, \frac{1}{f-a}\right) + \frac{1}{k+1} T(r, f) + O(1),$$

*where $\overline{N}_{k)}\left(r, \frac{1}{f-a}\right)$ are denoted by the zeros of $f-a$ in $|z| \leq r$, whose multiplicities are not greater than $k$ and are counted only once.*

From Lemma 2.2 and Theorems 1.4 and 1.8, we can get the following Lemma

**Lemma 2.3** *Let $f(z)$ be a meromorphic function in $\mathbb{D}$ and $\lim_{r \to 1^-} T(r, f) = \infty$, $a_j (j = 1, 2, \ldots, q)$ be $q$ distinct complex numbers, and $k_j (j = 1, 2, \ldots, q)$ be positive integers or $\infty$. If $f$ is an admissible function, then*

$$\left(q - 2 - \sum_{j=1}^{q} \frac{1}{k_j + 1}\right) T(r, f) \leq \sum_{j=1}^{q} \frac{k_j}{k_j + 1} \overline{N}_{k_j)}\left(r, \frac{1}{f-a_j}\right) + S(r, f);$$

*If $f$ is a non-admissible function, then*

$$\left(q - 2 - \sum_{j=1}^{q} \frac{1}{k_j + 1}\right) T(r, f) \leq \sum_{j=1}^{q} \frac{k_j}{k_j + 1} \overline{N}_{k_j)}\left(r, \frac{1}{f-a_j}\right) + \log \frac{1}{1-r} + S(r, f),$$

*where $S(r, f)$ is stated as in Theorem 1.4.*

# 3   Proofs of Theorems 1.9 and 1.10

## 3.1   The Proof of Theorem 1.9

Suppose that there exists $g(z) \in \Upsilon_\alpha$ satisfying (5) and $f(z) \not\equiv g(z)$. Without loss of generality, we can assume that $a_j (j = 1, 2, \ldots, q)$ are all finite numbers, otherwise, a suitable linear transformation will be done. Since $f(z), g(z) \in \Upsilon_\alpha$, from Lemma 2.3, we have

$$(8) \quad \left(q - 2 - \frac{q}{k+1}\right) T(r, f) \leq \frac{k}{k+1} \sum_{j=1}^{q} \overline{N}_{k)}\left(r, \frac{1}{f-a_j}\right) + \log \frac{1}{1-r} + S(r, f).$$

If follows form (5) that

$$(9) \quad \sum_{j=1}^{q} \overline{N}_{k)}\left(r, \frac{1}{f-a_j}\right) \leq N\left(r, \frac{1}{f-g}\right) \leq T(r, f) + T(r, g) + O(1).$$

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From (8) and (9), we have

$$\left(\frac{qk}{k+1} - \frac{3k+2}{k+1}\right) T(r,f) \le \frac{k}{k+1} T(r,g) + \log\frac{1}{1-r} + S(r,f).$$

Similarly, we have

$$\left(\frac{qk}{k+1} - \frac{3k+2}{k+1}\right) T(r,g) \le \frac{k}{k+1} T(r,f) + \log\frac{1}{1-r} + S(r,g).$$

Combining the above two inequalities, we get

(10) $$\left(\frac{qk}{k+1} - \frac{4k+2}{k+1}\right) \{T(r,f) + T(r,g)\} \le 2\log\frac{1}{1-r} + S(r,f) + S(r,g).$$

Since $f(z), g(z) \in \Upsilon_\alpha$ and $0 < \alpha < \infty$, from the definition of index and $q = 5 + [\frac{2}{k} + \frac{k+1}{k\alpha}]$, we have for $0 < \varepsilon < \alpha - \frac{k+1}{qk-4k-2}$, there exists a sequence $\{r_m\} \to 1^-$ such that

(11) $$T(r_m, f) > (\alpha - \varepsilon)\log\frac{1}{1-r_m}, \quad T(r_m, g) > (\alpha - \varepsilon)\log\frac{1}{1-r_m},$$

for all $m \to \infty$. From $f(z), g(z) \in \Upsilon_\alpha$ and the assumptions of Theorem 1.9, we can see that $S(r,f) = o\left(\log\frac{1}{1-r}\right)$ and $S(r,g) = o\left(\log\frac{1}{1-r}\right)$. From this fact and (10)-(11), we have

(12) $$\left\{2\left(\frac{qk}{k+1} - \frac{4k+2}{k+1}\right)(\alpha - \varepsilon) - 2\right\}\log\frac{1}{1-r_m} < o\left(\log\frac{1}{1-r_m}\right).$$

From (12) and $2\left(\frac{qk}{k+1} - \frac{4k+2}{k+1}\right)(\alpha - \varepsilon) - 2 > 0$, we can get a contradiction. Hence, we have $f(z) \equiv g(z)$.

Thus, this completes the proof of Theorem 1.9.

### 3.2 The Proof of Theorem 1.10

Suppose that there exists $g(z) \in \Upsilon_\alpha$ satisfying $f(z) \not\equiv g(z)$ and

(13) $$\overline{E}_{3)}(a_j, \mathbb{D}, f) = \overline{E}_{3)}(a_j, \mathbb{D}, g), \quad (j = 1, 2, 3)$$
$$\overline{E}_{2)}(a_j, \mathbb{D}, f) = \overline{E}_{2)}(a_j, \mathbb{D}, g), \quad (j = 4, 5).$$

Without loss of generality, we can assume that $a_j (j = 1, 2, \ldots, 5)$ are all finite numbers, otherwise, a suitable linear transformation will be done. Since $f(z), g(z) \in \Upsilon_\alpha$, from Lemma 2.3, we have

(14) $$\left(5 - 2 - \frac{3}{4} - \frac{2}{3}\right) T(r,f)$$

$$\le \frac{3}{4}\sum_{j=1}^{3}\overline{N}_{3)}\left(r, \frac{1}{f-a_j}\right) + \frac{2}{3}\sum_{j=4}^{5}\overline{N}_{2)}\left(r, \frac{1}{f-a_j}\right) + \log\frac{1}{1-r} + S(r,f)$$

$$\le \frac{3}{4}\left(\sum_{j=1}^{3}\overline{N}_{3)}\left(r, \frac{1}{f-a_j}\right) + \sum_{j=4}^{5}\overline{N}_{2)}\left(r, \frac{1}{f-a_j}\right)\right) + \log\frac{1}{1-r} + S(r,f).$$

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From (13), we have

$$\sum_{j=1}^{3} \overline{N}_{3)}\left(r, \frac{1}{f-a_j}\right) + \sum_{j=4}^{5} \overline{N}_{2)}\left(r, \frac{1}{f-a_j}\right) \leq N\left(r, \frac{1}{f-g}\right) \leq T(r,f) + T(r,g) + O(1).$$

From this inequality and (14), we have

$$(15) \qquad \frac{5}{6}T(r,f) \leq \frac{3}{4}T(r,g) + \log\frac{1}{1-r} + S(r,f).$$

Similarly, we have

$$(16) \qquad \frac{5}{6}T(r,g) \leq \frac{3}{4}T(r,f) + \log\frac{1}{1-r} + S(r,g).$$

Since $f(z), g(z) \in \Upsilon_\alpha$ and $\alpha > 12$, from the definition of index, we have for any $\varepsilon(0 < \varepsilon < \alpha - 12)$, there exists a sequence $\{r_m\} \to 1^-$ satisfying (11) for all $m \to \infty$. From this fact and (15)-(16), we have

$$(17) \qquad \left(\frac{1}{6}(\alpha-\varepsilon) - 2\right)\log\frac{1}{1-r_m} < o\left(\log\frac{1}{1-r_m}\right).$$

Since $\alpha > 12$ and $0 < \varepsilon < \alpha - 12$, we have $\frac{1}{6}(\alpha-\varepsilon) - 2\frac{1}{6}(\alpha-\varepsilon) - 2 > 0$, a contradiction. Hence, we have $f(z) \equiv g(z)$.

Thus, this completes the proof of Theorem 1.10.

## 4    Proof of Theorem 1.11

Without loss of generality, we may assume that all $a_j(j = 1, 2, \ldots, q)$ are finite, otherwise, a suitable Möbius transformation will be done. From Lemma 2.3, we have

$$(18) \qquad \left(q - 2 - \sum_{j=1}^{q}\frac{1}{k_j+1}\right)T(r,f_1) \leq \sum_{j=1}^{q}\frac{k_j}{k_j+1}\overline{N}_{k_j)}\left(r, \frac{1}{f_1-a_j}\right) + \log\frac{1}{1-r}$$
$$+ S(r, f_1).$$

From (1), we have

$$(19) \qquad \frac{1}{2} \leq \frac{k_q}{k_q+1} \leq \cdots \leq \frac{k_2}{k_2+1} \leq \frac{k_1}{k_1+1} \leq 1.$$

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From (18) and (19), we have

$$
(20) \quad \left( \sum_{j=1}^{q} \frac{k_j}{k_j+1} - 2 \right) T(r, f_1)
$$

$$
\leq \frac{k_3}{k_3+1} \sum_{j=1}^{q} \overline{N}_{k_j)} \left( r, \frac{1}{f_1 - a_j} \right) + \sum_{j=1}^{2} \left( \frac{k_j}{k_j+1} - \frac{k_3}{k_3+1} \right) \overline{N}_{k_j)} \left( r, \frac{1}{f_1 - a_j} \right)
$$

$$
+ \log \frac{1}{1-r} + S(r, f_1)
$$

$$
\leq \frac{k_3}{k_3+1} \sum_{j=1}^{q} \overline{N}_{k_j)} \left( r, \frac{1}{f_1 - a_j} \right) + \sum_{j=1}^{2} \left( \frac{k_j}{k_j+1} - \frac{k_3}{k_3+1} \right) T(r, f_1)
$$

$$
+ \log \frac{1}{1-r} + S(r, f_1).
$$

If $f_1(z) \not\equiv f_2(z)$, from the assumptions of Theorem 1.11, we have

$$
(21) \quad \sum_{j=1}^{q} \overline{N}_{k_j)} \left( r, \frac{1}{f_1 - a_j} \right) \leq N \left( r, \frac{1}{f_1 - f_2} \right) \leq T(r, f_1) + T(r, f_2) + O(1).
$$

From this inequality, we have

$$
(22) \quad \left( \sum_{j=3}^{q} \frac{k_j}{k_j+1} + \frac{k_3}{k_3+1} - 2 \right) T(r, f_1) \leq \frac{k_3}{k_3+1} T(r, f_2) + \log \frac{1}{1-r} + S(r, f_1).
$$

Similarly, we have

$$
(23) \quad \left( \sum_{j=3}^{q} \frac{k_j}{k_j+1} + \frac{k_3}{k_3+1} - 2 \right) T(r, f_2) \leq \frac{k_3}{k_3+1} T(r, f_1) + \log \frac{1}{1-r} + S(r, f_2).
$$

Since $0 < \alpha(f_1), \alpha(f_2) < \infty$, we have $S(r, f_1) = o\left( \log \frac{1}{1-r} \right), S(r, f_2) = o\left( \log \frac{1}{1-r} \right)$. And from the definition of index, for any $\varepsilon$ satisfying

$$
0 < 2\varepsilon < \min \left\{ \alpha(f_1), \alpha(f_2), \alpha(f_1) + \alpha(f_2) - \frac{2}{\sum_{j=3}^{q} \frac{k_j}{k_j+1}} \right\},
$$

there exists a sequence $\{r_m\} \to 1^-$ such that

$$
(24) \quad T(r_m, f_1) > (\alpha(f_1) - \varepsilon) \log \frac{1}{1 - r_m}, \quad T(r_m, f_2) > (\alpha(f_2) - \varepsilon) \log \frac{1}{1 - r_m},
$$

for all $m \to \infty$. From (22)-(24), we have

$$
(25) \quad \left( (\alpha(f_1) + \alpha(f_2) - 2\varepsilon) \sum_{j=3}^{q} \frac{k_j}{k_j+1} - 2 \right) \log \frac{1}{1 - r_m} < o\left( \log \frac{1}{1 - r_m} \right).
$$

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Since $0 < 2\varepsilon < \alpha(f_1) + \alpha(f_2) - \frac{2}{\sum_{j=3}^{q} \frac{k_j}{k_j+1}}$, we have $(\alpha(f_1) + \alpha(f_2) - 2\varepsilon) \sum_{j=3}^{q} \frac{k_j}{k_j+1} - 2 > 0$, a contradiction. Hence, we have $f_1(z) \equiv f_2(z)$.

Thus, this completes the proof of Theorem 1.11.

# 5 Proofs of Theorem 1.12 and Corollary 1.4

## 5.1 The Proof of Theorem 1.12

We will employ the proof by contradiction, that is, suppose that (4) and (7) hold at the same time. Since $f_1(z)$ is admissible, from Lemma 2.3, and by using the same argument as in Theorem 1.11, we can easily get

$$\left( \sum_{j=3}^{q} \frac{k_j}{k_j+1} + \frac{2k_2}{k_2+1} - 2 \right) T(r, f_1) \leq \frac{k_2}{k_2+1} (T(r, f_1) + T(r, f_2)) + S(r, f_1),$$

that is,

$$(26) \qquad \left( \sum_{j=2}^{q} \frac{k_j}{k_j+1} - 2 \right) T(r, f_1) \leq \frac{k_2}{k_2+1} T(r, f_2) + S(r, f_1).$$

Set $K = \sum_{j=2}^{q} \frac{k_j}{k_j+1} - 2$. If $K > 0$, from (26), we have

$$(27) \qquad T(r, f_1) \leq K' T(r, f_2) + S(r, f_1),$$

where $K' = \frac{1}{K} \frac{k_2}{k_2+1}$. Since $k_j > 0 (j = 1, 2, \ldots, q)$, we have $K' > 0$ as $K > 0$. From this and Lemma 2.1, we can get that each $S(r, f_1)$ is also an $S(r, f_2)$. Since $f_1(z)$ is admissible and $f_2(z)$ is non-admissible, we can get $T(r, f_2) = S(r, f_1)$. Thus, we have

$$(28) \qquad T(r, f_2) = S(r, f_1) = S(r, f_2) = o(T(r, f_2)).$$

Since $\lim_{r \to 1^-} T(r, f_2) = \infty$ and (28), we can get a contradiction.

Hence, we prove that (4) and (7) do not hold at the same time.

# 6 The Proof of Corollary 1.4

(i) Suppose that $f_1(z), f_2(z)$ share four values $a_j (j = 1, 2, 3, 4)$ $IM$, that is, $k_j = \infty (j = 1, 2, 3, 4)$. Since $f_1(z)$ is admissible, from Theorem 1.4, we have

$$(29) \qquad 2T(r, f_1) \leq \sum_{j=1}^{4} N\left( r, \frac{1}{f_1 - a_j} \right) + S(r, f_1).$$

Since $f_1(z), f_2(z)$ share four values $a_j (j = 1, 2, 3, 4)$ $IM$, we have

$$(30) \qquad \sum_{j=1}^{4} N\left( r, \frac{1}{f_1 - a_j} \right) \leq N\left( r, \frac{1}{f_1 - f_2} \right) \leq T(r, f_1) + T(r, f_2) + O(1).$$

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From (29) and (30), we have

(31) $$T(r, f_1) \leq T(r, f_2) + S(r, f_1).$$

By Lemma 2.1, similar to the proof of Theorem 1.12, we have

$$T(r, f_2) = S(r, f_1) = S(r, f_2) = o(T(r, f_2)).$$

From this and $\lim_{r \to 1^-} T(r, f_2) = \infty$, we can get a contradiction.

Thus, this completes (i) of Corollary 1.4.

Similar to the proof of Corollary 1.4 (i), we can prove (iii),(iv) and (v) of Corollary 1.4 easily. Here we omit the detail.

(ii) Suppose that $f_1, f_2$ share six values $a_j (j = 1, 2, \ldots, 6)$ with reduced weight 1, that is,

(32) $$\overline{E}_{1)}(a_j, \mathbb{D}, f_1) = \overline{E}_{1)}(a_j, \mathbb{D}, f_2), \quad (j = 1, 2, \ldots, 6),$$

and $k_1 = k_2 = \cdots = k_6 = 1$. Then, we can deduce that

$$\sum_{j=2}^{6} \frac{k_j}{k_j + 1} - 2 = 5 \times \frac{1}{2} - 2 = \frac{1}{2} > 0.$$

From this and the conclusion of Theorem 1.12, we get a contradiction.

Thus, this completes the proof of Corollary 1.4.

## Competing interests

The authors declare that they have no competing interests.

## Author's contributions

HYX, LZY and CFY completed the main part of this article, HYX corrected the main theorems. All authors read and approved the final manuscript.

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# COMPOSITIONS INVOLVING SCHUR HARMONICALLY CONVEX FUNCTIONS

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ABSTRACT. The decision theorem of the Schur harmonic convexity for the compositions involving Schur harmonically convex functions is established and used to determine the Schur harmonic convexity of some symmetric functions.

**2010 Mathematics Subject Classification:** Primary 26D15; 05E05; 26B25

**Keywords:** Schur harmonically convex function; harmonically convex function; composite function; symmetric function

## 1. INTRODUCTION

Throughout the article, $\mathbb{R}$ denotes the set of real numbers, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ denotes $n$-tuple ($n$-dimensional real vectors), the set of vectors can be written as

$$\mathbb{R}^n = \left\{ \boldsymbol{x} = (x_1, x_2, \ldots, x_n) : x_i, \in \mathbb{R}, i = 1, 2, \ldots, n \right\},$$

$$\mathbb{R}^n_{++} = \left\{ \boldsymbol{x} = (x_1, x_2, \ldots, x_n) : x_i > 0, i = 1, 2, \ldots, n \right\},$$

$$\mathbb{R}^n_{+} = \left\{ \boldsymbol{x} = (x_1, x_2, \ldots, x_n) : x_i \geq 0, i = 1, 2, \ldots, n \right\}.$$

In particular, the notations $\mathbb{R}$, $\mathbb{R}_{++}$ and $\mathbb{R}_{+}$ denote $\mathbb{R}^1$, $\mathbb{R}^1_{++}$ and $\mathbb{R}^1_{+}$, respectively.

The following conclusion is proved in reference [1, p. 91], [2, p. 64-65].

**Theorem A.** *Let the interval* $[a, b] \subset \mathbb{R}$, $\varphi : \mathbb{R}^n \to \mathbb{R}$, $f : [a, b] \to \mathbb{R}$ *and*

$$\psi(x_1, x_2, \ldots, x_n) = \varphi(f(x_1), f(x_2), \ldots, f(x_n)) : [a, b]^n \to \mathbb{R}.$$

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$(i)$ If $\varphi$ is increasing and Schur-convex and $f$ is convex, then $\psi$ is Schur-convex.

$(ii)$ If $\varphi$ is increasing and Schur-concave and $f$ is concave, then $\psi$ is Schur-concave.

$(iii)$ If $\varphi$ is decreasing and Schur-convex and $f$ is concave, then $\psi$ is Schur-convex.

$(iv)$ If $\varphi$ is increasing and Schur-convex and $f$ is increasing and convex, then $\psi$ is increasing and Schur-convex.

$(v)$ If $\varphi$ is decreasing and Schur-convex and $f$ is decreasing and concave, then $\psi$ is increasing and Schur-convex.

$(vi)$ If $\varphi$ is increasing and Schur-convex and $f$ is decreasing and convex, then $\psi$ is decreasing and Schur-convex.

$(vii)$ If $\varphi$ is decreasing and Schur-convex and $f$ is increasing and concave, then $\psi$ is decreasing and Schur-convex.

$(viii)$ If $\varphi$ is decreasing and Schur-concave and $f$ is decreasing and convex, then $\psi$ is increasing and Schur-concave.

Theorem A is very effective for determine of the Schur-convexity of the composite functions.

The Schur harmonically convex functions were proposed by Chu et al. [3, 4, 5] in 2009. The theory of majorization was enriched and expanded by using this concepts. Regarding the Schur harmonically convex functions, the aim of this paper is to establish the following theorem which is similar to Theorem A.

**Theorem 1.** *Let the interval* $[a,b] \subset \mathbb{R}_{++}$, $\varphi : \mathbb{R}_{++}^n \to \mathbb{R}_{++}$, $f : [a,b] \to \mathbb{R}_{++}$ *and*
$\psi(x_1, x_2, \ldots, x_n) = \varphi(f(x_1), f(x_2), \ldots, f(x_n)) : [a,b]^n \to \mathbb{R}_{++}$.

(i) *If $\varphi$ is increasing and Schur harmonically convex and $f$ is harmonically convex, then $\psi$ is Schur harmonically convex.*

(ii) *If $\varphi$ is increasing and Schur harmonically concave and $f$ is harmonically concave, then $\psi$ is Schur harmonically concave.*

(iii) *If $\varphi$ is decreasing and Schur harmonically convex and $f$ is harmonically concave, then $\psi$ is Schur harmonically convex.*

(iv) *If $\varphi$ is increasing and Schur harmonically convex and $f$ is increasing and harmonically convex, then $\psi$ is increasing and Schur harmonically convex.*

(v) *If $\varphi$ is decreasing and Schur harmonically convex and $f$ is decreasing and harmonically concave, then $\psi$ is increasing and Schur harmonically convex.*

(vi) *If $\varphi$ is increasing and Schur harmonically convex and $f$ is decreasing and harmonically convex, then $\psi$ is decreasing and Schur harmonically convex.*

(vii) *If $\varphi$ is decreasing and Schur harmonically convex and $f$ is increasing and harmonically concave, then $\psi$ is decreasing and Schur harmonically convex.*

(viii) *If $\varphi$ is decreasing and Schur harmonically concave and $f$ is decreasing and harmonically convex, then $\psi$ is increasing and Schur harmonically concave.*

## 2. Definitions and lemmas

In order to prove our results, in this section we will recall useful definitions and lemmas.

**Definition 1.** [1, 2] Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

(i) $\boldsymbol{x} \geq \boldsymbol{y}$ means $x_i \geq y_i$ for all $i = 1, 2, \ldots, n$.

(ii) Let $\Omega \subset \mathbb{R}^n$, $\varphi$: $\Omega \to \mathbb{R}$ is said to be increasing if $\boldsymbol{x} \geq \boldsymbol{y}$ implies $\varphi(\boldsymbol{x}) \geq \varphi(\boldsymbol{y})$. $\varphi$ is said to be decreasing if and only if $-\varphi$ is increasing.

**Definition 2.** [1, 2] Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

We say $\boldsymbol{y}$ majorizes $\boldsymbol{x}$ ($\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$), denoted by $\boldsymbol{x} \prec \boldsymbol{y}$, if $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, where $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

**Definition 3.** [1, 2] Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

(i) A set $\Omega \subset \mathbb{R}^n$ is said to be a convex set if

$$\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} = (\alpha x_1 + (1-\alpha)y_1, \alpha x_2 + (1-\alpha)y_2, \ldots, \alpha x_n + (1-\alpha)y_n) \in \Omega$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, and $\alpha \in [0, 1]$.

(ii) Let $\Omega \subset \mathbb{R}^n$ be convex set. A function $\varphi \colon \Omega \to \mathbb{R}$ is said to be a convex function on $\Omega$ if

$$\varphi\left(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}\right) \leq \alpha\varphi(\boldsymbol{x}) + (1-\alpha)\varphi(\boldsymbol{y})$$

holds for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, and $\alpha \in [0, 1]$. $\varphi$ is said to be a concave function on $\Omega$ if and only if $-\varphi$ is convex function on $\Omega$.

(iii) Let $\Omega \subset \mathbb{R}^n$. A function $\varphi \colon \Omega \to \mathbb{R}$ is said to be a Schur-convex function on $\Omega$ if $\boldsymbol{x} \prec \boldsymbol{y}$ on $\Omega$ implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. A function $\varphi$ is said to be a Schur-concave function on $\Omega$ if and only if $-\varphi$ is Schur-convex function on $\Omega$.

**Lemma 1.** (*Schur-convex function decision theorem*)[1, 2] : *Let $\Omega \subset \mathbb{R}^n$ be symmetric and have a nonempty interior convex set. $\Omega^0$ is the interior of $\Omega$. $\varphi :$ $\Omega \to \mathbb{R}$ is continuous on $\Omega$ and differentiable in $\Omega^0$. Then $\varphi$ is the Schur $-$*

*convex (or Schur − concave, respectively) function if and only if $\varphi$ is symmetric*

*on $\Omega$ and*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \geq 0 (or \ \leq 0, \ respectively) \tag{1}$$

*holds for any $\boldsymbol{x} \in \Omega^0$.*

**Definition 4.** [6] Let $\Omega \subset \mathbb{R}^n_{++}$.

(i) A set $\Omega$ is said to be a harmonically convex set if $\dfrac{\boldsymbol{xy}}{\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y}} \in \Omega$

for every $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\lambda \in [0,1]$, where $\boldsymbol{xy} = \sum_{i=1}^{n} x_i y_i$ and $\dfrac{1}{\boldsymbol{x}} = $

$\left(\dfrac{1}{x_1}, \dfrac{1}{x_2}, \ldots, \dfrac{1}{x_n}\right)$.

(ii) Let $\Omega \subset \mathbb{R}^n_{++}$ be a harmonically convex set. A function $\varphi : \Omega \to \mathbb{R}_{++}$ be

a continuous function, then $\varphi$ is called a harmonically convex (or concave,

respectively) function, if

$$\varphi\left(\frac{1}{\frac{\alpha}{\boldsymbol{x}} + \frac{1-\alpha}{\boldsymbol{y}}}\right) \leq (or \geq, respectively) \frac{1}{\frac{\alpha}{\varphi(\boldsymbol{x})} + \frac{1-\alpha}{\varphi(\boldsymbol{y})}}$$

holds for any $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, and $\alpha \in [0,1]$.

(iii) A function $\varphi : \Omega \to \mathbb{R}_{++}$ is said to be a Schur harmonically convex (or

concave, respectively) function on $\Omega$ if $\dfrac{1}{\boldsymbol{x}} \prec \dfrac{1}{\boldsymbol{y}}$ implies $\varphi(\boldsymbol{x}) \leq$ (or $\geq$,

respectively) $\varphi(\boldsymbol{y})$.

By Definition 4, it is not difficult to prove the following propositions.

**Proposition 1.** Let $\Omega \subset \mathbb{R}^n_{++}$ be a set, and let $\dfrac{1}{\Omega} = \{\left(\dfrac{1}{x_1}, \dfrac{1}{x_2}, \ldots, \dfrac{1}{x_n}\right) :$

$(x_1, x_2, \ldots, x_n) \in \Omega\}$. Then $\varphi : \Omega \to \mathbb{R}_{++}$ is a Schur harmonically convex (or

concave, respectively) function on $\Omega$ if and only if $\varphi(\dfrac{1}{\boldsymbol{x}})$ is a Schur-convex (or

concave, respectively) function on $\dfrac{1}{\Omega}$.

In fact, for any $\boldsymbol{u}, \boldsymbol{v} \in \dfrac{1}{\Omega}$, there exist $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ such that $\boldsymbol{u} = \dfrac{1}{\boldsymbol{x}}, \boldsymbol{v} = \dfrac{1}{\boldsymbol{y}}$. Let $\boldsymbol{u} \prec \boldsymbol{v}$, that is $\dfrac{1}{\boldsymbol{x}} \prec \dfrac{1}{\boldsymbol{y}}$, if $\varphi : \Omega \to \mathbb{R}_{++}$ is a Schur harmonically convex (or concave, respectively) function on $\Omega$, then $\varphi(\boldsymbol{x}) \leq$ (or $\geq$, respectively)$\varphi(\boldsymbol{y})$, namely, $\varphi(\dfrac{1}{\boldsymbol{u}}) \leq$ (or $\geq$, respectively)$\varphi(\dfrac{1}{\boldsymbol{v}})$, this means that $\varphi(\dfrac{1}{\boldsymbol{x}})$ is a Schur-convex (or concave, respectively) function on $\dfrac{1}{\Omega}$. The necessity is proved. The sufficiency can be similar to proof.

**Proposition 2.** $f : [a, b](\subset \mathbb{R}_{++}) \to \mathbb{R}_{++}$ is harmonically convex (or concave, respectively) if and only if $g(x) = \dfrac{1}{f(\frac{1}{x})}$ is concave (or convex, respectively) on $\left[\dfrac{1}{b}, \dfrac{1}{a}\right]$.

In fact, for any $x, y \in \left[\dfrac{1}{b}, \dfrac{1}{a}\right]$, then $\dfrac{1}{x}, \dfrac{1}{y} \in [a, b]$. If $f : [a, b](\subset \mathbb{R}_{++}) \to \mathbb{R}_{++}$ is harmonically convex (or concave, respectively), then

$$f\left(\frac{1}{\alpha x + (1 - \alpha)y}\right) \leq (or \geq, respectively)\frac{1}{\frac{\alpha}{f(\frac{1}{x})} + \frac{1-\alpha}{f(\frac{1}{y})}},$$

this is

$$\frac{1}{f(\frac{1}{\alpha x + (1-\alpha)y})} \geq (or \leq, respectively)\frac{\alpha}{f(\frac{1}{x})} + \frac{1 - \alpha}{f(\frac{1}{y})},$$

this means that $g(x) = \dfrac{1}{f(\frac{1}{x})}$ is concave (or convex, respectively) on $\left[\dfrac{1}{b}, \dfrac{1}{a}\right]$. The necessity is proved. The sufficiency can be similar to proof.

**Lemma 2.** (*Schur harmonically convex function decision theorem*)[5] *Let $\Omega \subset \mathbb{R}^n_{++}$ be a symmetric and harmonically convex set with inner points, and let $\varphi : \Omega \to \mathbb{R}_{++}$ be a continuously symmetric function which is differentiable on interior $\Omega^0$. Then $\varphi$ is Schur harmonically convex (or Schur harmonically concave, respectively) on $\Omega$ if and only if*

$$(x_1 - x_2)\left(x_1^2 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_1} - x_2^2 \frac{\partial \varphi(\boldsymbol{x})}{\partial x_2}\right) \geq 0 \quad (or \leq 0, \ respectively), \quad \boldsymbol{x} \in \Omega^0. \quad (2)$$

## 3. Proof of main results

**Proof of Theorem 1.** We only give the proof of Theorem 1 $(vi)$ in detail. Similar argument leads to the proof of the rest part.

If $\varphi$ is increasing and Schur harmonically convex and $f$ is decreasing and harmonically convex, then by Proposition 1, it follows that $\varphi(\dfrac{1}{x_1}, \dfrac{1}{x_2}, \ldots, \dfrac{1}{x_n})$ is decreasing and Schur convex, and by Proposition 2, it follows that $g(x) = \dfrac{1}{f(\frac{1}{x})}$ is decreasing and concave on $\left[\dfrac{1}{b}, \dfrac{1}{a}\right]$. And then from Theorem A $(iii)$, it follows that

$$\varphi\left(\frac{1}{g(x_1)}, \frac{1}{g(x_2)}, \ldots, \frac{1}{g(x_n)}\right) = \varphi\left(f(\frac{1}{x_1}), f(\frac{1}{x_2}), \ldots, f(\frac{1}{x_n})\right)$$

is increasing and Schur-convex. Again by Proposition 1, it follows that

$$\psi(x_1, x_2, \ldots, x_n) = \varphi(f(x_1), f(x_2), \ldots, f(x_n))$$

is decreasing and Schur harmonically convex.

## 4. Applications

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Its elementary symmetric functions are

$$E_r(\boldsymbol{x}) = E_r(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} \prod_{j=1}^{r} x_{i_j}, \ r = 1, 2, \ldots, n,$$

and defined $E_0(\boldsymbol{x}) = 1$, and $E_r(\boldsymbol{x}) = 0$ for $r < 0$ or $r > n$. The dual forms of the elementary symmetric functions are

$$E_r^*(\boldsymbol{x}) = E_r^*(x_1, x_2, \ldots, x_n) = \prod_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} \sum_{j=1}^{r} x_{i_j}, \ r = 1, 2, \ldots, n,$$

and defined $E_0^*(\boldsymbol{x}) = 1$, and $E_r^*(\boldsymbol{x}) = 0$ for $r < 0$ or $r > n$.

It is well-known that $E_r(\boldsymbol{x})$ is a increasing and Schur-concave function on $\mathbb{R}_+^n$[1].

In [7, 6], Shi proved that $E_r^*(\boldsymbol{x})$ is a increasing and Schur-concave function on $\mathbb{R}_+^n$.

**Theorem 2.** *For $r = 1, 2, \ldots, n, n \geq 2$, $E_r(\boldsymbol{x})$ and $E_r^*(\boldsymbol{x})$ are Schur harmonically convex function on $\mathbb{R}_{++}^n$.*

*Proof.* Noting that

$$E_r(\boldsymbol{x}) = x_1 x_2 E_{r-2}(x_3, x_4, \ldots, x_n) + (x_1 + x_2) E_{r-1}(x_3, x_4, \ldots, x_n)$$

$$+ E_r(x_3, x_4, \ldots, x_n), \qquad r = 1, 2, \ldots, n,$$

then

$$(x_1 - x_2) \left( x_1^2 \frac{\partial E_r(\boldsymbol{x})}{\partial x_1} - x_2^2 \frac{\partial E_r(\boldsymbol{x})}{\partial x_2} \right)$$

$$= (x_1 - x_2)[x_1^2(x_2 E_{r-2}(x_3, x_4, \ldots, x_n) + E_{r-1}(x_3, x_4, \ldots, x_n)) -$$

$$x_2^2(x_1 E_{r-2}(x_3, x_4, \ldots, x_n) + E_{r-1}(x_3, x_4, \ldots, x_n))]$$

$$= (x_1 - x_2)^2 [x_1 x_2 E_{r-2}(x_3, x_4, \ldots, x_n) + (x_1 + x_2) E_{r-1}(x_3, x_4, \ldots, x_n)] \geq 0,$$

by Lemma 2, it follows that $E_r(\boldsymbol{x})$ is Schur harmonically convex on $\mathbb{R}_{++}^n$.

By a direct, though tedious, calculation, and according to Lemma 2, $E_1^*(\boldsymbol{x})$, $E_2^*(\boldsymbol{x})$ is Schur harmonically convex on $\mathbb{R}_{++}^n$. When $r > 2$, it is easy to see that

$$E_r^*(\boldsymbol{x}) = E_r^*(x_1, x_2, \ldots, x_n) = E_r^*(x_2, x_3, \ldots, x_n) \times \prod_{2 \leq i_1 < i_2 < \cdots < i_{r-1} \leq n} \left( x_1 + \sum_{j=1}^{r-1} x_{i_j} \right),$$

then

$$\log E_r^*(\boldsymbol{x}) = \log E_r^*(x_2, x_3, \ldots, x_n) + \sum_{2 \leq i_1 < i_2 < \cdots < i_{r-1} \leq n} \log \left( x_1 + \sum_{j=1}^{r-1} x_{i_j} \right).$$

Now, it leads to

$$\frac{1}{E_r^*(\boldsymbol{x})} \frac{\partial E_r^*(\boldsymbol{x})}{\partial x_1} = \sum_{2 \leq i_1 < i_2 < \cdots < i_{r-1} \leq n} \frac{1}{x_1 + \sum_{j=1}^{r-1} x_{i_j}},$$

and then

$$\frac{\partial E_r^*(\boldsymbol{x})}{\partial x_1} = E_r^*(\boldsymbol{x}) \times$$

$$\left[ \sum_{3 \le i_1 < i_2 < \cdots < i_{r-1} \le n} \frac{1}{x_1 + \sum_{j=1}^{r-1} x_{i_j}} + \sum_{3 \le i_1 < i_2 < \cdots < i_{r-2} \le n} \frac{1}{x_1 + x_2 + \sum_{j=1}^{r-2} x_{i_j}} \right].$$

By the same arguments,

$$\frac{\partial E_r^*(\boldsymbol{x})}{\partial x_2} = E_r^*(\boldsymbol{x}) \times$$

$$\left[ \sum_{3 \le i_1 < i_2 < \cdots < i_{r-1} \le n} \frac{1}{x_2 + \sum_{j=1}^{r-1} x_{i_j}} + \sum_{3 \le i_1 < i_2 < \cdots < i_{r-2} \le n} \frac{1}{x_1 + x_2 + \sum_{j=1}^{r-2} x_{i_j}} \right].$$

Thus,

$$(x_1 - x_2) \left( x_1^2 \frac{\partial E_r^*(\boldsymbol{x})}{\partial x_1} - x_2^2 \frac{\partial E_r^*(\boldsymbol{x})}{\partial x_2} \right)$$

$$= (x_1 - x_2) E_r^*(\boldsymbol{x}) \times \left[ \sum_{3 \le i_1 < i_2 < \cdots < i_{r-1} \le n} \left( \frac{x_1^2}{x_1 + \sum_{j=1}^{r-1} x_{i_j}} - \frac{x_2^2}{x_2 + \sum_{j=1}^{r-1} x_{i_j}} \right) + \right.$$

$$\left. (x_1^2 - x_2^2) \cdot \sum_{3 \le i_1 < i_2 < \cdots < i_{r-2} \le n} \frac{1}{x_1 + x_2 + \sum_{j=1}^{r-2} x_{i_j}} \right]$$

$$= (x_1 - x_2)^2 E_r^*(\boldsymbol{x}) \times \left[ \sum_{3 \le i_1 < i_2 < \cdots < i_{r-1} \le n} \frac{x_1 x_2 + (x_1 + x_2) \sum_{j=1}^{r-1} x_{i_j}}{(x_1 + \sum_{j=1}^{r-1} x_{i_j})(x_2 + \sum_{j=1}^{r-1} x_{i_j})} + \right.$$

$$\left. (x_1 + x_2) \cdot \sum_{3 \le i_1 < i_2 < \cdots < i_{r-2} \le n} \frac{1}{x_1 + x_2 + \sum_{j=1}^{r-2} x_{i_j}} \right] \ge 0,$$

by Lemma 2, it follows that $E_r^*(\boldsymbol{x})$ is Schur harmonically convex on $\mathbb{R}_{++}^n$.    $\square$

For $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, the complete symmetric functions $c_n(\boldsymbol{x}, r)$ are

defined as

$$c_n(\boldsymbol{x}, r) = \sum_{i_1 + i_2 + \cdots + i_n = r} \prod_{j=1}^n x_j^{i_j}, \ r = 1, 2, \ldots, n,$$

where $c_0(\boldsymbol{x}, r) = 1$, $r \in \{1, 2, \ldots, n\}$, $i_1, i_2, \ldots, i_n$ are non-negative integers.

The dual forms of the complete symmetric functions $c_n^*(\boldsymbol{x}, r)$ are

$$c_n^*(\boldsymbol{x}, r) = \prod_{i_1+i_2+\cdots+i_n=r} \sum_{j=1}^{n} i_j x_j, \ r = 1, 2, \ldots, n,$$

where $i_j (j = 1, 2, \ldots, n)$ are non-negative integers.

Guan [8] discussed the Schur-convexity of $c_n(\boldsymbol{x}, r)$ and proved that $c_n(\boldsymbol{x}, r)$ is increasing and Schur-convex on $\mathbb{R}_{++}^n$. Subsequently, Chu et al. [5] proved that $c_n(\boldsymbol{x}, r)$ is Schur harmonically convex on $\mathbb{R}_{++}^n$.

Zhang and Shi [9] proved that $c_n^*(\boldsymbol{x}, r)$ is increasing, Schur-concave and Schur harmonically convex on $\mathbb{R}_{++}^n$.

In the following, we prove that the Schur harmonic convexity of the composite functions involving the above symmetric functions and their dual form by using Theorem 1.

**Theorem 3.** *The following symmetric functions are increasing and Schur harmonically convex on $(0,1)^n, r = 1, 2, \ldots, n$,*

$$E_r\left(\frac{1+\boldsymbol{x}}{1-\boldsymbol{x}}\right) = \sum_{1\leq i_1<i_2<\cdots<i_r\leq n} \prod_{j=1}^{r} \frac{1+x_{i_j}}{1-x_{i_j}}, \tag{3}$$

$$E_r^*\left(\frac{1+\boldsymbol{x}}{1-\boldsymbol{x}}\right) = \prod_{1\leq i_1<i_2<\cdots<i_r\leq n} \sum_{j=1}^{r} \frac{1+x_{i_j}}{1-x_{i_j}}, \tag{4}$$

$$c_n\left(\frac{1+\boldsymbol{x}}{1-\boldsymbol{x}}, r\right) = \sum_{i_1+i_2+\cdots+i_n=r} \prod_{j=1}^{n} \left(\frac{1+x_j}{1-x_j}\right)^{i_j} \tag{5}$$

*and*

$$c_n^*\left(\frac{1+\boldsymbol{x}}{1-\boldsymbol{x}}, r\right) = \prod_{i_1+i_2+\cdots+i_n=r} \sum_{j=1}^{n} i_j \left(\frac{1+x_j}{1-x_j}\right). \tag{6}$$

*Proof.* Let $f(x) = \dfrac{1+x}{1-x}, x \in (0,1)$. Then $f(x) > 0$, $f'(x) = \dfrac{2}{(1-x)^2} > 0$, so $f$ is increasing on $(0,1)$.

And let $g(x) = \dfrac{1}{f(\frac{1}{x})} = \dfrac{x-1}{x+1}$. Then $g''(x) = -\dfrac{4}{(x+1)^3} < 0$, this means that $\dfrac{1}{f(\frac{1}{x})}$ is concave on $(1, \infty)$, by Proposition 2, it follows that $f$ is harmonically convex on $(0, 1)$. Since $E_r(\boldsymbol{x})$, $E_r^*(\boldsymbol{x})$, $c_n(\boldsymbol{x}, r)$ and $c_n^*(\boldsymbol{x}, r)$ are all increasing and Schur harmonically convex function on $\mathbb{R}_{++}^n$, by Theorem 1 $(iv)$, it follows that Theorem 3 holds. $\qquad\qquad\square$

*Remark* 1. By Lemma 2, Xia and Chu [10] proved that $E_r\left(\dfrac{1+\boldsymbol{x}}{1-\boldsymbol{x}}\right)$ is Schur harmonically convex on $(0, 1)^n$. By the properties of Schur harmonically convex function, Shi and Zhang [11] proved that $E_r^*\left(\dfrac{1+\boldsymbol{x}}{1-\boldsymbol{x}}\right)$ is Schur harmonically convex on $(0, 1)^n$. By Theorem 1, we give a new proof.

**Theorem 4.** *The following symmetric functions are increasing and Schur harmonically convex on* $\mathbb{R}_{++}^n, r = 1, 2, \ldots, n,$

$$E_r\left(\boldsymbol{x}^{\frac{1}{r}}\right) = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} \prod_{j=1}^{r} x_{i_j}^{\frac{1}{r}}, \qquad (7)$$

$$E_r^*\left(\boldsymbol{x}^{\frac{1}{r}}\right) = \prod_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} \sum_{j=1}^{r} x_{i_j}^{\frac{1}{r}}, \qquad (8)$$

$$c_n\left(\boldsymbol{x}^{\frac{1}{r}}, r\right) = \sum_{i_1 + i_2 + \cdots + i_n = r} \prod_{j=1}^{n} x_j^{\frac{i_j}{r}} \qquad (9)$$

*and*

$$c_n^*\left(\boldsymbol{x}^{\frac{1}{r}}, r\right) = \prod_{i_1 + i_2 + \cdots + i_n = r} \sum_{j=1}^{n} i_j x_j^{\frac{1}{r}}. \qquad (10)$$

*Proof.* For $r \geq 1$, let $p(x) = x^{\frac{1}{r}}, x \in \mathbb{R}_{++}$. Then $p'(x) = \frac{1}{r} x^{\frac{1}{r}-1} > 0$, so $p$ is increasing on $\mathbb{R}_{++}$.

And let $q(x) = \dfrac{1}{p(\frac{1}{x})} = x^{\frac{1}{r}} = p(x)$. Then $q''(x) = \frac{1}{r}(\frac{1}{r} - 1)x^{\frac{1}{r}-2} \leq 0$, this means that $\dfrac{1}{p(\frac{1}{x})}$ is concave on $\mathbb{R}_{++}$, by Proposition 2, it follows that $p$ is harmonically convex on $\mathbb{R}_{++}$. Since $E_r(\boldsymbol{x})$, $E_r^*(\boldsymbol{x})$, $c_n(\boldsymbol{x}, r)$ and $c_n^*(\boldsymbol{x}, r)$ are all increasing and

Schur harmonically convex function on $\mathbb{R}_{++}^n$, by Theorem 1 $(iv)$, it follows that Theorem 4 holds.                                                                      □

*Remark* 2. By Lemma 2, Chu and Lv [3] proved that the Hamy's symmetric function $E_r\left(\boldsymbol{x}^{\frac{1}{r}}\right)$ is Schur harmonically convex on $\mathbb{R}_{++}^n$. Later, K. Z. Guan and R. K. Guan [12] further studied the harmonic convexity of the generalized Hamy symmetric function.

By Lemma 2, Meng et al. [13] proved that the dual form of the Hamy's symmetric function $E_r^*\left(\boldsymbol{x}^{\frac{1}{r}}\right)$ is Schur harmonically convex on $\mathbb{R}_{++}^n$.

By Lemma 2, Chu and Sun [4] proved that $c_n\left(\boldsymbol{x}^{\frac{1}{r}}, r\right)$ is Schur harmonically convex on $\mathbb{R}_{++}^n$ .

By Theorem 1, we give a new proof.

Since $f(x) = \dfrac{1+x}{1-x}$ is increasing and harmonically convex on $(0,1)$, from Theorem 1 $(iv)$ and Theorem 4, it follows

**Theorem 5.** *The following symmetric functions are increasing and Schur harmonically convex on* $(0,1)^n, r = 1, 2, \ldots, n,$

$$E_r\left(\left(\frac{1+\boldsymbol{x}}{1-\boldsymbol{x}}\right)^{\frac{1}{r}}\right) = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} \prod_{j=1}^{r} \left(\frac{1+x_{i_j}}{1-x_{i_j}}\right)^{\frac{1}{r}}, \qquad (11)$$

$$E_r^*\left(\left(\frac{1+\boldsymbol{x}}{1-\boldsymbol{x}}\right)^{\frac{1}{r}}\right) = \prod_{1 \le i_1 < i_2 < \cdots < i_r \le n} \sum_{j=1}^{r} \left(\frac{1+x_{i_j}}{1-x_{i_j}}\right)^{\frac{1}{r}}, \qquad (12)$$

$$c_n\left(\left(\frac{1+\boldsymbol{x}}{1-\boldsymbol{x}}\right)^{\frac{1}{r}}, r\right) = \sum_{i_1+i_2+\cdots+i_n=r} \prod_{j=1}^{n} \left(\frac{1+x_j}{1-x_j}\right)^{\frac{i_j}{r}} \qquad (13)$$

*and*

$$c_n^*\left(\left(\frac{1+\boldsymbol{x}}{1-\boldsymbol{x}}\right)^{\frac{1}{r}}, r\right) = \prod_{i_1+i_2+\cdots+i_n=r} \sum_{j=1}^{n} i_j \left(\frac{1+x_j}{1-x_j}\right)^{\frac{1}{r}}. \qquad (14)$$

*Remark* 3. By Lemma 2, Long and Chu [14] proved that $E_r^* \left( \left( \frac{1+\boldsymbol{x}}{1-\boldsymbol{x}} \right)^{\frac{1}{r}} \right)$ is Schur harmonically convex on $(0,1)^n$. By Theorem 1, we give a new proof.

**Theorem 6.** *The following symmetric functions are increasing and Schur harmonically convex on* $(0,1)^n, r = 1, 2, \ldots, n$,

$$E_r \left( \frac{\boldsymbol{x}}{1-\boldsymbol{x}} \right) = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} \prod_{j=1}^{r} \frac{x_{i_j}}{1 - x_{i_j}}, \tag{15}$$

$$E_r^* \left( \frac{\boldsymbol{x}}{1-\boldsymbol{x}} \right) = \prod_{1 \le i_1 < i_2 < \cdots < i_r \le n} \sum_{j=1}^{r} \frac{x_{i_j}}{1 - x_{i_j}}, \tag{16}$$

$$c_n \left( \frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r \right) = \sum_{i_1 + i_2 + \cdots + i_n = r} \prod_{j=1}^{n} \left( \frac{x_j}{1 - x_j} \right)^{i_j} \tag{17}$$

*and*

$$c_n^* \left( \frac{\boldsymbol{x}}{1-\boldsymbol{x}}, r \right) = \prod_{i_1 + i_2 + \cdots + i_n = r} \sum_{j=1}^{n} i_j \left( \frac{x_j}{1 - x_j} \right). \tag{18}$$

*Proof.* Let $h(x) = \frac{x}{1-x}, x \in (0,1)$. Then $h'(x) = \frac{1}{(1-x)^2} > 0$, so $h$ is increasing on $(0,1)$.

And let $k(x) = \frac{1}{h(\frac{1}{x})} = x - 1$. Then $k''(x) = 0$, this means that $\frac{1}{h(\frac{1}{x})}$ is concave on $(1, \infty)$, by Proposition 2, it follows that $h$ is harmonically convex on $(0,1)$. Since $E_r(\boldsymbol{x}), E_r^*(\boldsymbol{x}), c_n(\boldsymbol{x}, r)$ and $c_n^*(\boldsymbol{x}, r)$ are all increasing and Schur harmonically convex function on $\mathbb{R}_{++}^n$, by Theorem 1 $(iv)$, it follows that Theorem 5 holds. $\square$

*Remark* 4. By the judgement theorem of Schur harmonic convexity for a class of symmetric functions, Shi and Zhang [15] proved that $E_r \left( \frac{\boldsymbol{x}}{1-\boldsymbol{x}} \right)$ is Schur harmonically convex on $(0,1)^n$. Here by Theorem 1, we give a new proof.

By the properties of Schur harmonically convex function, Shi and Zhang [11] proved that $E_r^* \left( \frac{\boldsymbol{x}}{1-\boldsymbol{x}} \right)$ is Schur harmonically convex on $\left[ \frac{1}{2}, 1 \right)^n$. By Theorem 1, this conclusion is extended to the collection $(0,1)^n$.

By Lemma 2, Sun et al. [16] proved that $c_n\left(\dfrac{\boldsymbol{x}}{1-\boldsymbol{x}}, r\right)$ is Schur harmonically convex on $[0,1)^n$, here by Theorem 1, we give a new proof.

Since $f(x) = \dfrac{x}{1-x}$ is increasing and harmonically convex on $(0,1)$, from Theorem 1 $(iv)$ and Theorem 4, it follows

**Theorem 7.** *The following symmetric functions are increasing and Schur harmonically convex on* $(0,1)^n, r = 1, 2, \ldots, n,$

$$E_r\left(\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}\right)^{\frac{1}{r}}\right) = \sum_{1\le i_1 < i_2 < \cdots < i_r \le n} \prod_{j=1}^{r} \left(\frac{x_{i_j}}{1-x_{i_j}}\right)^{\frac{1}{r}}, \qquad (19)$$

$$E_r^*\left(\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}\right)^{\frac{1}{r}}\right) = \prod_{1\le i_1 < i_2 < \cdots < i_r \le n} \sum_{j=1}^{r} \left(\frac{x_{i_j}}{1-x_{i_j}}\right)^{\frac{1}{r}}, \qquad (20)$$

$$c_n\left(\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}\right)^{\frac{1}{r}}, r\right) = \sum_{i_1+i_2+\cdots+i_n=r} \prod_{j=1}^{n} \left(\frac{x_j}{1-x_j}\right)^{\frac{i_j}{r}} \qquad (21)$$

*and*

$$c_n^*\left(\left(\frac{\boldsymbol{x}}{1-\boldsymbol{x}}\right)^{\frac{1}{r}}, r\right) = \prod_{i_1+i_2+\cdots+i_n=r} \sum_{j=1}^{n} i_j \left(\frac{x_j}{1-x_j}\right)^{\frac{1}{r}}. \qquad (22)$$

*Remark* 5. By Lemma 2, Sun [17] proved that $E_r\left(\left(\dfrac{\boldsymbol{x}}{1-\boldsymbol{x}}\right)^{\frac{1}{r}}\right)$ is Schur harmonically convex on $[0,1)^n$. Here by Theorem 1, we give a new proof.

### Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

### Acknowledgments

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# A NOTE ON DEGENERATE GENERALIZED $q$-GENOCCHI POLYNOMIALS

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ABSTRACT. In this paper,we consider degenerate generalized $q$-Genocchi polynomials arising from $p$-adic fermionic $q$-integral on $\mathbb{Z}_p$. We found some interesting identities of these polynomials.

## 1. INTRODUCTION

Let $p$ be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$. Let $q$ be an indeterminate in $\mathbb{C}_p$ such that $|q - 1|_p < p^{-\frac{1}{p-1}}$.

Let $f(x)$ be a continuous function on $\mathbb{Z}_p$. Then the *p-adic fermionic $q$-integral on $\mathbb{Z}_p$* is defined by Kim to be

$$
\int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N - 1} f(x) \mu_{-q}\left(x + p^N \mathbb{Z}_p\right)
$$
$$
= \frac{[2]_q}{2} \lim_{N \to \infty} \sum_{x=0}^{p^N - 1} f(x)(-1)^x q^x, \text{ (see [9])}.
\tag{1.1}
$$

Thus, by (1.1), we get

$$
q \int_{\mathbb{Z}_p} f(x+1) d\mu_{-q}(x) + \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = [2]_q f(0),
\tag{1.2}
$$

and

$$
q^n \int_{\mathbb{Z}_p} f(x+n) d\mu_{-q}(x) + (-1)^{n-1} \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = [2]_q \sum_{l=0}^{n-1} f(l) q^l (-1)^{n-1-l}, \tag{1.3}
$$

where $n \in \mathbb{N}$ (see [5-10, 12]).

It is known that the *$q$-Euler polynomials* are given by the generating function as follows:

$$
\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-q}(y) = \frac{[2]_q}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}.
\tag{1.4}
$$

When $x = 0$, $E_{n,q} = E_{n,q}(0)$ are called *$q$-Euler numbers* (see [5, 9, 12]).

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Recently, degenerate $q$-Euler polynomials are introduced by he generating function as follows:

$$\frac{[2]_q}{q(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}\mathcal{E}_{n,q,\lambda}(x)\frac{t^n}{n!}, \ (\text{see } [9]). \qquad (1.5)$$

It is known that the $q$-*Genocchi polynomials* are given by the generating function as follows:

$$\int_{\mathbb{Z}_p} te^{(x+y)t}d\mu_{-q}(y) = \frac{t[2]_q}{qe^t+1}e^{xt} = \sum_{n=0}^{\infty}G_{n,q}(x)\frac{t^n}{n!}. \qquad (1.6)$$

When $x = 0$, $G_{n,q} = E_{n,q}(0)$ are called $q$-*Genocchi numbers* (see [1, 2, 4, 6-8]).

Now, the degenerate $q$-Genocchi polynomials are introduced by the generating function as follows:

$$\frac{t[2]_q}{q(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}\mathcal{G}_{n,q,\lambda}(x)\frac{t^n}{n!}. \qquad (1.7)$$

Note that $\lim_{\lambda\to 0}\mathcal{G}_{n,q,\lambda}(x) = G_{n,q}(x)$, $(n \geq 0)$, (see [3]).

For $d \in \mathbb{N}$ with $d \equiv (\text{mod } 2)$ and $(d,p) = 1$, we set

$$X = \varprojlim_{N}\mathbb{Z}/dp^N\mathbb{Z}, \ X^* = \bigcup_{0<a<dp, \ a\nmid p}(a+dp\mathbb{Z}_p),$$

and

$$a+dp^N\mathbb{Z}_p = \left\{x \in X \mid x \equiv a \ (\text{mod } dp^N)\right\},$$

where $a \in \mathbb{Z}$ with $0 \leq a < dp^N - 1$.

For $d \in \mathbb{N}$ with $d \equiv 1 \ (\text{mod } 2)$, let us assume that $\chi$ is a Dirichlet character with conductor $d$. Now, we consider the generalized $q$-Genocchi polynomials attached to $\chi$ which are given by the generating function to be

$$\int_X \chi(y)te^{(x+y)t}d\mu_{-q}(y) = \left(\frac{t[2]_q}{q^de^{dt}+1}\sum_{a=0}^{d-1}q^q(-1)^a\chi(a)e^{at}\right)e^{xt}$$
$$= \sum_{n=0}^{\infty}G_{n,q,\chi}(x)\frac{t^n}{n!}, \ (\text{see } [4\text{-}6, 8]). \qquad (1.8)$$

When $x = 0$, $G_{n,q,\chi} = G_{n,q,\chi}(0)$ are called *generalized $q$-Genocchi numbers attached to $\chi$*.

One of the most recent papers on the theory of Genocchi polynomials and numbers is the paper T. Kim(see [6-8]), which deals mainly with the theory of Genocchi polynomials and numbers. Facts on Bernoulli polynomials and Euler polynomials, to which Genocchi polynomials may be related, has been derived in Volkenborn integral (see [3]). While a lot of the properties of Genocchi polynomials bear a striking resemblance to the properties of Beroulli and Euler polynomials, some properties are rather different. Note that Genocchi polynomials occur naturally in the areas of elementary number theory, complex analytic number theory, homotopy theory, differential topology, theory of modular forms, $p$-adic analytic number theory, quantum physics (see [1-13]).

In the viewpoint of (1.8), we consider degenerate generalized $q$-Genocchi polynomials which are derived from the fermionic $q$-integral on $\mathbb{Z}_p$. The purpose of this paper is to investigate some properties and identities of degenerate generalized $q$-Genocchi polynomials.

## 2. Degenerate generalized $q$-Genocchi polynomials

In this section, we assume that $\lambda, t \in \mathbb{C}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$. For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, let $\chi$ be a Dirichlet's character with conductor $d$.

In the viewpoint of (1.8), we consider degenerate generalized $q$-Genocchi polynomials which are given by the generating function to be

$$\int_X \chi(y) t (1 + \lambda t)^{\frac{x+y}{\lambda}} d\mu_{-q}(y)$$
$$= \left( \frac{t[2]_q}{q^d (1 + \lambda t)^{\frac{d}{\lambda}} + 1} \sum_{a=0}^{d-1} q^a (-1)^a \chi(a)(1 + \lambda t)^{\frac{a}{\lambda}} \right) (1 + \lambda t)^{\frac{x}{\lambda}} \qquad (2.1)$$
$$= \sum_{n=0}^{\infty} \mathcal{G}_{n,\chi,q,\lambda}(x) \frac{t^n}{n!},$$

where $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$.

From (1.5) and (2.1), we have

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,\chi,q,\lambda}(x) \frac{t^n}{n!} = \left( \frac{t[2]_q}{q^d (1 + \lambda t)^{\frac{d}{\lambda}} + 1} \sum_{a=0}^{d-1} q^a (-1)^a \chi(a)(1 + \lambda t)^{\frac{a+x}{\lambda}} \right)$$
$$= \frac{[2]_q}{[2]_{q^d}} \sum_{a=0}^{d-1} q^a (-1)^a \chi(a) \left( \frac{t[2]_{q^d}}{q^d (1 + \lambda t)^{\frac{d}{\lambda}} + 1} (1 + \lambda t)^{\frac{d}{\lambda} \frac{a+x}{d}} \right) \qquad (2.2)$$
$$= \sum_{n=0}^{\infty} \left( \frac{[2]_q}{[2]_{q^d}} \sum_{a=0}^{d-1} q^a (-1)^a \chi(a) \mathcal{G}_{n,q^d,\frac{\lambda}{d}} \left( \frac{a+x}{d} \right) \frac{d^n t^{n+1}}{n!} \right).$$

Thus, by (2.2), we get

$$\mathcal{G}_{n,\chi,q,\lambda}(x) = \frac{n d^{n-1} [2]_q}{[2]_{q^d}} \sum_{a=0}^{d-1} q^a (-1)^a \chi(a) \mathcal{G}_{n-1,q^d,\frac{\lambda}{d}} \left( \frac{a+x}{d} \right), \quad (n \ge 0). \qquad (2.3)$$

Therefore, by (2.3), we obtain the following theorem.

**Theorem 2.1.** *For $d \in \mathbb{N}$ with $d \equiv 1 \ (mod\ 2)$, $n \ge 0$, we have*

$$\mathcal{G}_{n,\chi,q,\lambda}(x) = \int_X \chi(y) t (x + y|\lambda)_n d\mu_{-q}(y)$$
$$= \frac{n[2]_q}{[2]_{q^d}} d^{n-1} \sum_{a=0}^{d-1} \chi(a) q^a (-1)^a \mathcal{G}_{n-1,q^d,\frac{\lambda}{d}} \left( \frac{a+x}{d} \right),$$

*where*

$$(x|\lambda)_n = x(x-\lambda) \cdots (x - \lambda(n-1))$$
$$= \lambda^n \left( \frac{x}{\lambda} \right)_n.$$

For $n \ge 0$, we observe that

$$(x + y|\lambda)_n = \lambda^n \left( \frac{x+y}{\lambda} \right)_n = \lambda^n \sum_{l=0}^{n} S_1(n,l) \left( \frac{x+y}{\lambda} \right)^l$$
$$= \sum_{l=0}^{n} S_1(n,l) \lambda^{n-l} (x+y)^l. \qquad (2.4)$$

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By (2.4), we get

$$
\int_X \chi(y)t(x+y|\lambda)_n d\mu_{-q}(y) = \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l} \int_X \chi(y)t(x+y)^l d\mu_{-q}(y)
$$
$$
= \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l} G_{l,q,\chi}(x), \ (n \geq 0),
$$
(2.5)

where $S_1(n,l)$ is the Stirling number of the first kind. Therefore, by (2.5), we obtain the following theorem.

**Theorem 2.2.** *For $n \geq 0$, we have*

$$
\mathcal{G}_{n,\chi,q,\lambda}(x) = \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l} G_{l,q,\chi}(x).
$$

By replacing $t$ by $\frac{1}{\lambda}(e^{\lambda t} - 1)$ in (2.1), we get

$$
\int_X \chi(y)\frac{1}{\lambda}(e^{\lambda t} - 1)e^{(x+y)t} d\mu_{-q}(y) = \sum_{m=0}^{\infty} \mathcal{G}_{m,\chi,q,\lambda}(x)\frac{1}{m!}\left(\frac{1}{\lambda}(e^{\lambda t} - 1)\right)^m
$$
$$
= \sum_{m=0}^{\infty} \mathcal{G}_{m,\chi,q,\lambda}(x)\lambda^{-m} \sum_{n=m}^{\infty} S_2(n,m)\frac{\lambda^n}{n!}t^n
$$
(2.6)
$$
= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \lambda^{n-m} S_2(n,m)\mathcal{G}_{m,\chi,q,\lambda}(x)\right)\frac{t^n}{n!},
$$

where $S_2(n,m)$ is the Stirling number of the second kind.

From (1.8), we note that

$$
\int_X \chi(y)te^{(x+y)t} d\mu_{-q}(y) = \left(\frac{t[2]_q}{q^d e^{dt} + 1}\right) \sum_{a=0}^{d-1} q^a(-1)^a \chi(a)e^{(a+x)t}
$$
$$
= \sum_{n=0}^{\infty} G_{n,q,\chi}(x)\frac{t^n}{n!}.
$$
(2.7)

By multiplying $t$ on the both side (2.6), we get

$$
\int_X \chi(y)te^{(x+y)t} d\mu_{-q}(y) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{1}{e^{\lambda t} - 1}\lambda^{n-m+1} S_2(n,m)\mathcal{G}_{m,\chi,q,\lambda}(x)\right)\frac{t^{n+1}}{n!}.
$$
(2.8)

Therefore, by (2.6), (2.7) and (2.8), we obtain the following theorem.

**Theorem 2.3.** *For $n \geq 0$, we have*

$$
G_{n,q,\chi}(x) = \sum_{m=0}^{n-1} \frac{n}{e^{\lambda t} - 1}\lambda^{n-m} S_2(n-1,m)\mathcal{G}_{m,\chi,q,\lambda}(x).
$$

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. From (1.3), we have

$$
q^d \int_X (x+d|\lambda)_n \chi(x) d\mu_{-q}(x) + \int_X (x|\lambda)_n \chi(x) d\mu_{-q}(x)
$$
$$
= [2]_q \sum_{a=0}^{d-1} \chi(a)q^q(-1)^a (a|\lambda)_n, \ (n \geq 0).
$$
(2.9)

Therefore, by Theorem 2.1 and (2.8), we obtain the following theorem.

**Theorem 2.4.** *For $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, $n \geq 0$, we have*

$$q^d \mathcal{G}_{n,\chi,q,\lambda}(d) + \mathcal{G}_{n,\chi,q,\lambda} = t[2]_q \sum_{a=0}^{d-1} \chi(a) q^q (-1)^a (a|\lambda)_n,$$

*where $\mathcal{G}_{n,\chi,q,\lambda} = \mathcal{G}_{n,\chi,q,\lambda}(0)$ are called degenerate generalized $q$-Genocchi numbers attached to $\chi$.*

Now, we observe that

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,\chi,q,\lambda}(x) \frac{t^n}{n!} = \left( \frac{t[2]_q \sum_{a=0}^{d-1} q^a (-1)^a \chi(a)(1+\lambda t)^{\frac{a}{\lambda}}}{q^d (1+\lambda t)^{\frac{d}{\lambda}} + 1} \right) (1+\lambda t)^{\frac{x}{\lambda}}$$

$$= \left( \sum_{m=0}^{\infty} \mathcal{G}_{n,\chi,q,\lambda} \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} (x|\lambda)_l \frac{t^l}{l!} \right) \qquad (2.10)$$

$$= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \binom{n}{m} \mathcal{G}_{m,\chi,q,\lambda}(x|\lambda)_{n-m} \right) \frac{t^n}{n!}.$$

Thus, by comparing the coefficients on the both sides, we obtain the following theorem.

**Theorem 2.5.** *For $n \geq 0$, we have*

$$\mathcal{G}_{n,\chi,q,\lambda}(x) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{G}_{m,\chi,q,\lambda}(x|\lambda)_{n-m}.$$

Now, we observe that

$$\frac{(-1)^n}{n!} \mathcal{G}_{n,\chi,q,\lambda} = \frac{(-1)^n}{n!} \int_X \chi(x) t(x|\lambda)_n d\mu_{-q}(x)$$

$$= \lambda^n \int_X \binom{-\frac{x}{\lambda} + n - 1}{n} \chi(x) t d\mu_{-q}(x)$$

$$= \lambda^n \sum_{l=0}^{n} \binom{n-1}{l-1} \frac{(-1)^l}{\lambda^l l!} \int_X \chi(x) t(x|-\lambda)_l d\mu_{-q}(x) \qquad (2.11)$$

$$= \sum_{l=0}^{n} \binom{n-1}{l-1} \lambda^{n-l} (-1)^l \frac{1}{l!} \mathcal{G}_{l,\chi,q,-\lambda}$$

$$= \sum_{l=1}^{n} \binom{n-1}{l-1} \lambda^{n-l} (-1)^l \frac{\mathcal{G}_{l,\chi,q,-\lambda}}{l!}.$$

Therefore, by (2.11), we obtain the following theorem.

**Theorem 2.6.** *For $n \geq 0$, we have*

$$\frac{(-1)^n}{n!} \mathcal{G}_{n,\chi,q,\lambda} = \sum_{l=1}^{n} \binom{n-1}{l-1} \lambda^{n-l} (-1)^l \frac{\mathcal{G}_{l,\chi,q,-\lambda}}{l!}.$$

Note that

$$\lim_{\lambda \to 0} \mathcal{G}_{n,\chi,q,\lambda}(x) = G_{n,q,\chi}(x), \quad (n \geq 0).$$

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# Cubic soft ideals in $BCK/BCI$-algebras

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**Abstract.** The notions of cubic soft ∘-subalgebras and (closed) cubic soft ideals in $BCK/BCI$-algebras are introduced, and related properties are investigated. Relations between cubic soft subalgebras, cubic soft ∘-subalgebras and (closed) cubic soft ideals are discussed. Conditions for a cubic soft subalgebras to be a (closed) cubic soft ideals are provided. Characterizations of cubic soft ideals are considered. R-union and R-intersection of cubic soft ideals are discussed.

## 1. INTRODUCTION

To solve complicated problems in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [8]. Maji et al. [5] and Molodtsov [8] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [8] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [5] described the application of soft set theory to a decision making problem. Maji et al. [6] also studied several operations on the theory of soft sets. Jun et al. [2, 4] applied the notion of soft sets to $BCK/BCI$-algebras and $d$-algebras.

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Combining cubic sets and soft sets, Muhiuddin and Al-roqi [10] introduced the notions of (internal, external) cubic soft sets, P-cubic (resp. R-cubic) soft subsets, R-union (resp. R-intersection, P-union, P-intersection) of cubic soft sets, and the complement of a cubic soft set. They investigated several related properties, and applied the notion of cubic soft sets to $BCK/BCI$-algebras. In [9], Muhiuddin et al. considered several basic operations of cubic soft sets, namely, "AND" operation and "OR" operation based on the P-order and the R-order. They provided an example to illustrate that the R-union of two internal cubic soft sets might not be internal. They also discussed conditions for the R-union of two internal cubic soft sets to be an internal cubic soft set, and investigated several properties of cubic soft subalgebras in $BCK/BCI$-algebras based on a parameter.

In this paper, we introduce the notions of cubic soft ∘-subalgebras and (closed) cubic soft ideals in $BCK/BCI$-algebras, and investigate related properties. We consider relations between cubic soft subalgebras, cubic soft ∘-subalgebras and (closed) cubic soft ideals, and provide conditions for a cubic soft subalgebras to be a (closed) cubic soft ideals. We discuss characterizations of cubic soft ideals. We show that the R-intersection of cubic soft ideals is a cubic soft ideal. We also show that if parameter sets are mutually disjoint then the R-union of cubic soft ideals is a cubic soft ideal. We provide an example to show that the R-union of cubic soft ideals is not a cubic soft ideal when parameter sets are not disjoint.

## 2. Preliminaries

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a $BCI$-algebra if it satisfies the following axioms:

(I) $(\forall x, y, z \in X) \, (((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in X) \, ((x * (x * y)) * y = 0)$,
(III) $(\forall x \in X) \, (x * x = 0)$,
(IV) $(\forall x, y \in X) \, (x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a $BCI$-algebra $X$ satisfies the following identity:

(V) $(\forall x \in X) \, (0 * x = 0)$,

then $X$ is called a $BCK$-algebra. Any $BCK/BCI$-algebra $X$ satisfies the following conditions:

(a1) $(\forall x \in X) \, (x * 0 = x)$,
(a2) $(\forall x, y, z \in X) \, (x * y = 0 \Rightarrow (x * z) * (y * z) = 0, \, (z * y) * (z * x) = 0)$,
(a3) $(\forall x, y, z \in X) \, ((x * y) * z = (x * z) * y)$,
(a4) $(\forall x, y, z \in X) \, (((x * z) * (y * z)) * (x * y) = 0)$.

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. A $BCK$-algebra $X$ is said to be with *condition (S)* if, for all $x, y \in X$, the set $\{z \in X \mid z * x \leq y\}$ has a greatest element, written $x \circ y$. A $BCI$-algebra $X$ is said to be *p-semisimple* if its $BCK$-part is equal to $\{0\}$. In a $p$-semisimple $BCI$-algebra, the following conditions are valid:

(a5) $(\forall x, y \in X) \, (0 * (x * y) = y * x)$.
(a6) $(\forall x, y \in X) \, (x * (x * y) = y)$.

Cubic soft ideals in $BCK/BCI$-algebras

A nonempty subset $S$ of a $BCK/BCI$-algebra $X$ is called a *subalgebra* of $X$ if $x * y \in S$ for all $x, y \in S$. A subset $I$ of a $BCK/BCI$-algebra $X$ is called an *ideal* of $X$ if it satisfies, for all $x, y \in X$, the following conditions:

(b1) $0 \in I$,

(b2) $x * y \in I, \ y \in I \ \Rightarrow \ x \in I$.

An ideal $I$ of a $BCI$-algebra $X$ is said to be *closed* if $0 * x \in I$ for all $x \in I$. We refer the reader to the books [1, 7] for further information regarding $BCK/BCI$-algebras.

By an *interval number* we mean a closed subinterval $\tilde{a} = [a^-, a^+]$ of $I$, where $0 \leq a^- \leq a^+ \leq 1$. Denote by $[I]$ the set of all interval numbers. Let us define what is known as *refined minimum* and *refined maximun* (briefly, rmin and rmax) of two elements in $[I]$. We also define the symbols "$\succeq$", "$\preceq$", "$=$" in case of two elements in $[I]$. Consider two interval numbers $\tilde{a}_1 := \left[a_1^-, a_1^+\right]$ and $\tilde{a}_2 := \left[a_2^-, a_2^+\right]$. Then

$$\text{rmin}\left\{\tilde{a}_1, \tilde{a}_2\right\} = \left[\min\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right],$$

$$\text{rmax}\left\{\tilde{a}_1, \tilde{a}_2\right\} = \left[\max\left\{a_1^-, a_2^-\right\}, \max\left\{a_1^+, a_2^+\right\}\right],$$

$$\tilde{a}_1 \succeq \tilde{a}_2 \text{ if and only if } a_1^- \geq a_2^- \text{ and } a_1^+ \geq a_2^+,$$

and similarly we may have $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 = \tilde{a}_2$. To say $\tilde{a}_1 \succ \tilde{a}_2$ (resp. $\tilde{a}_1 \prec \tilde{a}_2$) we mean $\tilde{a}_1 \succeq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$ (resp. $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$).

Let $X$ be a nonempty set. A function $A : X \rightarrow [I]$ is called an *interval-valued fuzzy set* (briefly, an *IVF set*) in $X$. Let $[I]^X$ stand for the set of all IVF sets in $X$. For every $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the *degree* of membership of an element $x$ to $A$, where $A^- : X \rightarrow I$ and $A^+ : X \rightarrow I$ are fuzzy sets in $X$ which are called a *lower fuzzy set* and an *upper fuzzy set* in $X$, respectively. For simplicity, we denote $A = [A^-, A^+]$.

Molodtsov [8] defined the soft set in the following way: Let $U$ be an initial universe set and $E$ be a set of parameters. Let $\mathcal{P}(U)$ denotes the power set of $U$ and $A \subset E$.

**Definition 2.1** ([8])**.** A pair $(F, A)$ is called a *soft set* over $U$, where $F$ is a mapping given by

$$F : A \rightarrow \mathcal{P}(U).$$

In other words, a soft set over $U$ is a parameterized family of subsets of the universe $U$. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of $\varepsilon$-approximate elements of the soft set $(F, A)$. Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [8].

**Definition 2.2** ([3])**.** Let $U$ be a universe. By a *cubic set* in $U$ we mean a structure

$$\mathscr{A} = \{\langle x, A(x), \lambda(x)\rangle \mid x \in U\}$$

in which $A$ is an IVF set in $U$ and $\lambda$ is a fuzzy set in $U$.

In what follows, a cubic set $\mathscr{A} = \{\langle x, \bar{\mu}_A(x), \lambda_A(x)\rangle \mid x \in U\}$ is simply denoted by $\mathscr{A} = \langle \bar{\mu}_A, \lambda_A \rangle$, and denote by $\mathcal{C}^U$ the collection of all cubic sets in $U$.

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**Definition 2.3** ([10]). Let $U$ be an initial universe set and let $E$ be a set of parameters. A *cubic soft set* over $U$ is defined to be a pair $(\mathscr{F}, A)$ where $\mathscr{F}$ is a mapping from $A$ to $\mathcal{C}^U$ and $A \subset E$. Note that the pair $(\mathscr{F}, A)$ can be represented as the following set:

$$(\mathscr{F}, A) := \{\mathscr{F}(\varepsilon) \mid \varepsilon \in A\} \text{ where } \mathscr{F}(\varepsilon) = \langle \bar{\mu}_{\mathscr{F}(\varepsilon)}, \lambda_{\mathscr{F}(\varepsilon)} \rangle.$$

## 3. Cubic soft ideals

In what follows, let $U$ be an initial universe set which is a $BCK/BCI$-algebra unless otherwise specified.

**Definition 3.1** ([10]). A cubic soft set $(\mathscr{F}, A)$ over $U$ is said to be a *cubic soft BCK/BCI-algebra* over $U$ based on a parameter $\varepsilon$ (briefly, $\varepsilon$-*cubic soft subalgebra* over $U$) if there exists a parameter $\varepsilon \in A$ such that

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y) \succeq \text{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\} \tag{3.1}$$

$$\lambda_{\mathscr{F}(\varepsilon)}(x * y) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x), \lambda_{\mathscr{F}(\varepsilon)}(y)\} \tag{3.2}$$

for all $x, y \in U$. If $(\mathscr{F}, A)$ is an $\varepsilon$-cubic soft subalgebra over $U$ for all $\varepsilon \in A$, we say that $(\mathscr{F}, A)$ is a *cubic soft subalgebra* over $U$.

**Definition 3.2.** Let $U$ be a $BCK$-algebra with the condition (S). Given a parameter $\varepsilon \in A$, a cubic soft set $(\mathscr{F}, A)$ over $U$ is said to be a *cubic soft $\circ$-subalgebra* over $U$ based on $\varepsilon$ (briefly, $\varepsilon$-*cubic soft $\circ$-subalgebra* over $U$) if it satisfies the following conditions:

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x \circ y) \succeq \text{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\} \tag{3.3}$$

$$\lambda_{\mathscr{F}(\varepsilon)}(x \circ y) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x), \lambda_{\mathscr{F}(\varepsilon)}(y)\} \tag{3.4}$$

for all $x, y \in U$. If $(\mathscr{F}, A)$ is an $\varepsilon$-cubic soft $\circ$-subalgebra over $U$ for all $\varepsilon \in A$, we say that $(\mathscr{F}, A)$ is a *cubic soft $\circ$-subalgebra* over $U$.

**Definition 3.3.** Given a parameter $\varepsilon \in A$, a cubic soft set $(\mathscr{F}, A)$ over $U$ is said to be a *cubic soft ideal* over $U$ based on $\varepsilon$ (briefly, $\varepsilon$-*cubic soft ideal* over $U$) if it satisfies the following conditions:

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(0) \succeq \bar{\mu}_{\mathscr{F}(\varepsilon)}(x), \ \lambda_{\mathscr{F}(\varepsilon)}(0) \leq \lambda_{\mathscr{F}(\varepsilon)}(x), \tag{3.5}$$

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x) \succeq \text{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\} \tag{3.6}$$

$$\lambda_{\mathscr{F}(\varepsilon)}(x) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x * y), \lambda_{\mathscr{F}(\varepsilon)}(y)\} \tag{3.7}$$

for all $x, y \in U$. If $(\mathscr{F}, A)$ is an $\varepsilon$-cubic soft ideal over $U$ for all $\varepsilon \in A$, we say that $(\mathscr{F}, A)$ is a *cubic soft ideal* over $U$.

**Example 3.4.** Let $(Y, *, 0)$ be a $BCI$-algebra and consider the adjoint $BCI$-algebra $(\mathbb{Z}, -, 0)$ of the additive group $(\mathbb{Z}, +, 0)$ of integers. Then the direct product $U := Y \times \mathbb{Z}$ of $Y$ and $\mathbb{Z}$ is a $BCI$-algebra (see [1]). For any $\varepsilon \in A$, let $(\mathscr{F}, A)$ be a soft set over $U$ defined by

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x) = \begin{cases} \tilde{a} = [a^-, a^+](\neq [0,0]) & \text{if } x \in Y \times \mathbb{N}_0, \\ [0,0] & \text{otherwise,} \end{cases}$$

<center>Cubic soft ideals in $BCK/BCI$-algebras</center>

$$\lambda_{\mathscr{F}(\varepsilon)}(x) = \begin{cases} s & \text{if } x \in Y \times \mathbb{N}_0, \\ t & \text{otherwise,} \end{cases}$$

where $\mathbb{N}$ is the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $s, t \in [0, 1]$ with $s < t$. Then $(\mathscr{F}, A)$ is an $\varepsilon$-cubic soft ideal over $U$.

**Example 3.5.** Let $U = \{0, a, b, c\}$ be a $BCI$-algebra with the following Cayley table 1:

<center>TABLE 1. Cayley table of the operation $*$</center>

| $*$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $c$ |
| $a$ | $a$ | $0$ | $0$ | $c$ |
| $b$ | $b$ | $b$ | $0$ | $c$ |
| $c$ | $c$ | $c$ | $c$ | $0$ |

Let $(\mathscr{F}, A)$ be a cubic soft set over $U$, where $A = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, with the tabular representation in Table 2.

<center>TABLE 2. Tabular representation of the cubic soft set $(\mathscr{F}, A)$</center>

| | $\varepsilon_1$ | $\varepsilon_2$ | $\varepsilon_3$ |
|---|---|---|---|
| $0$ | $\langle [0.5, 0.8], \ 0.6 \rangle$ | $\langle [0.8, 1.0], \ 0.1 \rangle$ | $\langle [0.4, 0.6], \ 0.7 \rangle$ |
| $a$ | $\langle [0.8, 0.9], \ 0.7 \rangle$ | $\langle [0.3, 0.7], \ 0.8 \rangle$ | $\langle [0.1, 0.2], \ 0.7 \rangle$ |
| $b$ | $\langle [0.1, 0.7], \ 0.5 \rangle$ | $\langle [0.3, 0.7], \ 0.8 \rangle$ | $\langle [0.1, 0.7], \ 0.3 \rangle$ |
| $c$ | $\langle [0.2, 0.6], \ 0.9 \rangle$ | $\langle [0.3, 0.7], \ 0.8 \rangle$ | $\langle [0.3, 0.6], \ 0.2 \rangle$ |

Then $(\mathscr{F}, A)$ is not an $\varepsilon_1$-cubic soft ideal over $U$ since

$$\bar{\mu}_{\mathscr{F}(\varepsilon_1)}(0) = [0.5, 0.8] \not\succeq [0.8, 0.9] = \bar{\mu}_{\mathscr{F}(\varepsilon_1)}(a).$$

We know that $(\mathscr{F}, A)$ is an $\varepsilon_2$-cubic soft ideal over $U$. $(\mathscr{F}, A)$ is not an $\varepsilon_3$-cubic soft ideal over $U$ since

$$\bar{\mu}_{\mathscr{F}(\varepsilon_3)}(a) = [0.1, 0.2] \not\succeq [0.3, 0.6] = \text{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon_3)}(a * c), \bar{\mu}_{\mathscr{F}(\varepsilon_3)}(c)\}.$$

**Proposition 3.6.** *If $(\mathscr{F}, A)$ is an $\varepsilon$-cubic soft ideal over $U$, then*

$$(\forall x, y \in U) \left( x \leq y \ \Rightarrow \ \bar{\mu}_{\mathscr{F}(\varepsilon)}(y) \preceq \bar{\mu}_{\mathscr{F}(\varepsilon)}(x), \ \lambda_{\mathscr{F}(\varepsilon)}(y) \geq \lambda_{\mathscr{F}(\varepsilon)}(x) \right).$$

*Proof.* Let $x, y \in U$ be such that $x \leq y$. Then $x * y = 0$, and so

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x) \succeq \text{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\}$$
$$= \text{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(0), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\} = \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)$$

and

$$\lambda_{\mathscr{F}(\varepsilon)}(x) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x * y), \lambda_{\mathscr{F}(\varepsilon)}(y)\}$$
$$= \max\{\lambda_{\mathscr{F}(\varepsilon)}(0), \lambda_{\mathscr{F}(\varepsilon)}(y)\} = \lambda_{\mathscr{F}(\varepsilon)}(y).$$

This completes the proof. $\square$

**Proposition 3.7.** *Let $(\mathscr{F}, A)$ be an $\varepsilon$-cubic soft ideal over $U$ for a parameter $\varepsilon \in A$. If the inequality $x * y \leq z$ holds in $U$, then*

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x) \succeq \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(z)\}$$

*and $\lambda_{\mathscr{F}(\varepsilon)}(x) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(y), \lambda_{\mathscr{F}(\varepsilon)}(z)\}$.*

*Proof.* Assume that $x * y \leq z$ for all $x, y, z \in U$. Then

$$
\begin{aligned}
\bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y) &\succeq \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}((x * y) * z), \bar{\mu}_{\mathscr{F}(\varepsilon)}(z)\} \\
&= \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(0), \bar{\mu}_{\mathscr{F}(\varepsilon)}(z)\} = \bar{\mu}_{\mathscr{F}(\varepsilon)}(z)
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
\lambda_{\mathscr{F}(\varepsilon)}(x * y) &\leq \max\{\lambda_{\mathscr{F}(\varepsilon)}((x * y) * z), \lambda_{\mathscr{F}(\varepsilon)}(z)\} \\
&= \max\{\lambda_{\mathscr{F}(\varepsilon)}(0), \lambda_{\mathscr{F}(\varepsilon)}(z)\} = \lambda_{\mathscr{F}(\varepsilon)}(z),
\end{aligned}
\tag{3.9}
$$

which implies from (3.6) and (3.7) that

$$
\begin{aligned}
\bar{\mu}_{\mathscr{F}(\varepsilon)}(x) &\succeq \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\} \\
&\succeq \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(z)\}
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_{\mathscr{F}(\varepsilon)}(x) &\leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x * y), \lambda_{\mathscr{F}(\varepsilon)}(y)\} \\
&\leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(y), \lambda_{\mathscr{F}(\varepsilon)}(z)\}.
\end{aligned}
$$

This completes the proof. $\qquad\square$

**Theorem 3.8.** *In a BCK-algebra $U$ with the condition (S), every $\varepsilon$-cubic soft ideal $(\mathscr{F}, A)$ over $U$ is an $\varepsilon$-cubic soft $\circ$-subalgebra over $U$ for all $\varepsilon \in A$.*

*Proof.* Let $\varepsilon \in A$. Since $U$ has the condition (S), we have $(x \circ y) * x \leq y$ for all $x, y \in U$. Hence (3.6), (3.7) and Proposition 3.6 imply that

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x \circ y) \succeq \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}((x \circ y) * x), \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)\} \succeq \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\}$$

and

$$\lambda_{\mathscr{F}(\varepsilon)}(x \circ y) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}((x \circ y) * x), \lambda_{\mathscr{F}(\varepsilon)}(x)\} \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x), \lambda_{\mathscr{F}(\varepsilon)}(y)\}$$

for all $x, y \in U$. Therefore $(\mathscr{F}, A)$ is an $\varepsilon$-cubic soft $\circ$-subalgebra over $U$ for all $\varepsilon \in A$. $\quad\square$

**Theorem 3.9.** *In a BCK-algebra $U$, if $(\mathscr{F}, A)$ is an $\varepsilon$-cubic soft ideal over $U$, then it is an $\varepsilon$-cubic soft subalgebra over $U$ for all $\varepsilon \in A$.*

*Proof.* Let $(\mathscr{F}, A)$ be an $\varepsilon$-cubic soft ideal over $U$ where $\varepsilon \in A$. For any $x, y \in U$, we have

$$
\begin{aligned}
\bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y) &\succeq \mathrm{rmin}\left\{\bar{\mu}_{\mathscr{F}(\varepsilon)}((x * y) * x), \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)\right\} \\
&= \mathrm{rmin}\left\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(0), \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)\right\} \\
&= \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)
\end{aligned}
$$

and $\lambda_{\mathscr{F}(\varepsilon)}(x * y) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}((x * y) * x), \lambda_{\mathscr{F}(\varepsilon)}(x)\} = \max\{\lambda_{\mathscr{F}(\varepsilon)}(0), \lambda_{\mathscr{F}(\varepsilon)}(x)\} = \lambda_{\mathscr{F}(\varepsilon)}(x)$. Therefore $(\mathscr{F}, A)$ is an $\varepsilon$-cubic soft subalgebra over $U$. $\qquad\square$

Theorem 3.9 is not true in a $BCI$-algebra. In fact, the $\varepsilon$-cubic soft ideal $(\mathscr{F}, A)$ in Example 3.4 is not an $\varepsilon$-cubic soft subalgebra over $U$ since

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}\left((0,0) * (0,1)\right) = \bar{\mu}_{\mathscr{F}(\varepsilon)}(0,-1) = [0,0] \not\succeq \tilde{a} = [a^-, a^+]$$
$$= \mathrm{rmin}\left\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(0,0), \bar{\mu}_{\mathscr{F}(\varepsilon)}(0,1)\right\}$$

and/or

$$\lambda_{\mathscr{F}(\varepsilon)}\left((0,0) * (0,1)\right) = \lambda_{\mathscr{F}(\varepsilon)}(0,-1) = t \not\leq s = \mathrm{max}\left\{\lambda_{\mathscr{F}(\varepsilon)}(0,0), \lambda_{\mathscr{F}(\varepsilon)}(0,1)\right\}.$$

**Definition 3.10.** Let $U$ be a $BCI$-algebra and $\varepsilon \in A$. An $\varepsilon$-cubic soft ideal $(\mathscr{F}, A)$ over $U$ is said to be *closed* if $\bar{\mu}_{\mathscr{F}(\varepsilon)}(0 * x) \succeq \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)$ and $\lambda_{\mathscr{F}(\varepsilon)}(0 * x) \leq \lambda_{\mathscr{F}(\varepsilon)}(x)$ for all $x \in U$.

**Example 3.11.** The $\varepsilon_2$-cubic soft ideal $(\mathscr{F}, A)$ in Example 3.5 is closed.

**Theorem 3.12.** *In a $BCI$-algebra, every closed cubic soft ideal is a cubic soft subalgebra.*

*Proof.* Let $(\mathscr{F}, A)$ be a closed cubic soft ideal over $U$. Then $\bar{\mu}_{\mathscr{F}(\varepsilon)}(0 * x) \succeq \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)$ and $\lambda_{\mathscr{F}(\varepsilon)}(0 * x) \leq \lambda_{\mathscr{F}(\varepsilon)}(x)$ for all $x \in U$. It follows from (a3), (3.6) and (3.7) that

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y) \succeq \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}((x * y) * x), \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)\}$$
$$= \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(0 * y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)\}$$
$$\succeq \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)\}$$

and

$$\lambda_{\mathscr{F}(\varepsilon)}(x * y) \succeq \mathrm{rmin}\{\lambda_{\mathscr{F}(\varepsilon)}((x * y) * x), \lambda_{\mathscr{F}(\varepsilon)}(x)\}$$
$$= \mathrm{max}\{\lambda_{\mathscr{F}(\varepsilon)}(0 * y), \lambda_{\mathscr{F}(\varepsilon)}(x)\}$$
$$\leq \mathrm{max}\{\lambda_{\mathscr{F}(\varepsilon)}(y), \lambda_{\mathscr{F}(\varepsilon)}(x)\}$$

for all $x, y \in U$. Therefore $(\mathscr{F}, A)$ is a cubic soft ideal over $U$. $\square$

We provide a condition for a cubic soft subalgebra over $U$ to be a (closed) cubic soft ideal over $U$.

**Theorem 3.13.** *In a $p$-semisimple $BCI$-algebra $U$, every cubic soft subalgebra over $U$ is a closed cubic soft ideal over $U$.*

*Proof.* Let $(\mathscr{F}, A)$ be a cubic soft subalgebra over a $p$-semisimple $BCI$-algebra $U$ and let $\varepsilon \in A$ be a parameter. For every $x \in U$, we have

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(0) = \bar{\mu}_{\mathscr{F}(\varepsilon)}(x * x) \succeq \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x), \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)\} = \bar{\mu}_{\mathscr{F}(\varepsilon)}(x),$$
$$\lambda_{\mathscr{F}(\varepsilon)}(0) = \lambda_{\mathscr{F}(\varepsilon)}(x * x) \leq \mathrm{max}\{\lambda_{\mathscr{F}(\varepsilon)}(x), \lambda_{\mathscr{F}(\varepsilon)}(x)\} = \lambda_{\mathscr{F}(\varepsilon)}(x). \tag{3.10}$$

Using (3.1), (3.2) and (3.10), we get

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(0 * x) \succeq \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(0), \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)\} = \bar{\mu}_{\mathscr{F}(\varepsilon)}(x),$$
$$\lambda_{\mathscr{F}(\varepsilon)}(0 * x) \leq \mathrm{max}\{\lambda_{\mathscr{F}(\varepsilon)}(0), \lambda_{\mathscr{F}(\varepsilon)}(x)\} = \lambda_{\mathscr{F}(\varepsilon)}(x). \tag{3.11}$$

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For any $x, y \in U$, we have

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x) = \bar{\mu}_{\mathscr{F}(\varepsilon)}(y * (y * x)) \succeq \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y * x)\}$$
$$= \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(0 * (x * y))\}$$
$$\succeq \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\}$$

and

$$\lambda_{\mathscr{F}(\varepsilon)}(x) = \lambda_{\mathscr{F}(\varepsilon)}(y * (y * x)) \le \max\{\lambda_{\mathscr{F}(\varepsilon)}(y), \lambda_{\mathscr{F}(\varepsilon)}(y * x)\}$$
$$= \max\{\lambda_{\mathscr{F}(\varepsilon)}(y), \lambda_{\mathscr{F}(\varepsilon)}(0 * (x * y))\}$$
$$\le \max\{\lambda_{\mathscr{F}(\varepsilon)}(x * y), \lambda_{\mathscr{F}(\varepsilon)}(y)\}$$

by using (a6), (3.1), (3.2), (a5) and (3.11). Therefore $(\mathscr{F}, A)$ is a closed cubic soft ideal over $U$. □

**Corollary 3.14.** *If a BCI-algebra $U$ satisfies any one of the following conditions:*

- $U = \{0 * x \mid x \in U\}$,
- *every element of $U$ is minimal,*
- $(\forall x, y \in U) \ (x * (0 * y) = y * (0 * x))$,
- $(\forall x \in U) \ (0 * x = 0 \ \Rightarrow \ x = 0)$,
- $(\forall x, y \in U) \ ((x * y) * z = x * (y * z))$,
- $(\forall x, y \in U) \ (x * y = y * x)$,
- $(\forall x \in U) \ (0 * x = x)$,
- $(\forall x, y, z \in U) \ ((x * y) * (x * z) = z * y)$,

*then every cubic soft subalgebra over $U$ is a closed cubic soft ideal over $U$.*

**Theorem 3.15.** *For a cubic soft set $(\mathscr{F}, A)$ over a BCK-algebra $U$ with condition $(S)$ and a parameter $\varepsilon \in A$, the following are equivalent.*

(i) *$(\mathscr{F}, A)$ is an $\varepsilon$-cubic soft ideal over $U$.*

(ii) *For every $x, y, z \in U$, if $x \le y \circ z$, then $\bar{\mu}_{\mathscr{F}(\varepsilon)}(x) \succeq \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(z)\}$ and $\lambda_{\mathscr{F}(\varepsilon)}(x) \le \max\{\lambda_{\mathscr{F}(\varepsilon)}(y), \lambda_{\mathscr{F}(\varepsilon)}(z)\}$.*

*Proof.* Assume that $(\mathscr{F}, A)$ is an $\varepsilon$-cubic soft ideal over $U$ and $x \le y \circ z$ for all $x, y, z \in U$. Then

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x) \succeq \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x * (y \circ z)), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y \circ z)\}$$
$$= \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(0), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y \circ z)\}$$
$$= \bar{\mu}_{\mathscr{F}(\varepsilon)}(y \circ z)$$
$$\succeq \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(z)\}$$

and

$$\lambda_{\mathscr{F}(\varepsilon)}(x) \le \max\{\lambda_{\mathscr{F}(\varepsilon)}(x * (y \circ z)), \lambda_{\mathscr{F}(\varepsilon)}(y \circ z)\}$$
$$= \max\{\lambda_{\mathscr{F}(\varepsilon)}(0), \lambda_{\mathscr{F}(\varepsilon)}(y \circ z)\}$$
$$= \lambda_{\mathscr{F}(\varepsilon)}(y \circ z)$$
$$\le \max\{\lambda_{\mathscr{F}(\varepsilon)}(y), \lambda_{\mathscr{F}(\varepsilon)}(z)\}.$$

<div align="center">Cubic soft ideals in $BCK/BCI$-algebras</div>

Conversely suppose that (ii) is valid. Since $0 \leq x \circ x$ for all $x \in U$, it follows from (ii) that

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(0) \succeq \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x), \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)\} = \bar{\mu}_{\mathscr{F}(\varepsilon)}(x)$$

and

$$\lambda_{\mathscr{F}(\varepsilon)}(0) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x), \lambda_{\mathscr{F}(\varepsilon)}(x)\} = \lambda_{\mathscr{F}(\varepsilon)}(x)$$

for all $x \in U$. Since $x \leq (x * y) \circ y$ for all $x, y \in U$, we have

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x) \succeq \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\}$$

and

$$\lambda_{\mathscr{F}(\varepsilon)}(x) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x * y), \lambda_{\mathscr{F}(\varepsilon)}(y)\}$$

for all $x, y \in U$. Therefore $(\mathscr{F}, A)$ is an $\varepsilon$-cubic soft ideal over $U$. $\qquad\square$

**Theorem 3.16.** *Given a parameter $\varepsilon \in A$, a cubic soft set $(\mathscr{F}, A)$ over $U$ is an $\varepsilon$-cubic soft ideal over $U$ if and only if the nonempty sets*

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}^{\Leftarrow}[\delta_1, \delta_2] := \left\{ x \in U \mid \bar{\mu}_{\mathscr{F}(\varepsilon)}(x) \succeq [\delta_1, \delta_2] \right\}$$

*and*

$$\lambda_{\mathscr{F}(\varepsilon)}^{\rightarrow}(t) := \left\{ x \in U \mid \lambda_{\mathscr{F}(\varepsilon)}(x) \leq t \right\}$$

*are ideals of $U$ for all $[\delta_1, \delta_2] \in [I]$ and $t \in [0, 1]$.*

*Proof.* Assume that a cubic soft set $(\mathscr{F}, A)$ over $U$ is an $\varepsilon$-cubic soft ideal over $U$. Suppose that $\bar{\mu}_{\mathscr{F}(\varepsilon)}^{\Leftarrow}[\delta_1, \delta_2] \cap \lambda_{\mathscr{F}(\varepsilon)}^{\rightarrow}(t) \neq \emptyset$ for all $[\delta_1, \delta_2] \in [I]$ and $t \in [0, 1]$. Obviously, $0 \in \bar{\mu}_{\mathscr{F}(\varepsilon)}^{\Leftarrow}[\delta_1, \delta_2] \cap \lambda_{\mathscr{F}(\varepsilon)}^{\rightarrow}(t)$. Let $x$ and $y$ be elements of $U$ such that $x * y \in \bar{\mu}_{\mathscr{F}(\varepsilon)}^{\Leftarrow}[\delta_1, \delta_2]$ and $y \in \bar{\mu}_{\mathscr{F}(\varepsilon)}^{\Leftarrow}[\delta_1, \delta_2]$. Then $\bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y) \succeq [\delta_1, \delta_2]$ and $\bar{\mu}_{\mathscr{F}(\varepsilon)}(y) \succeq [\delta_1, \delta_2]$. It follows from (3.6) that

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x) \succeq \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\} \succeq \mathrm{rmin}\{[\delta_1, \delta_2], [\delta_1, \delta_2]\} = [\delta_1, \delta_2].$$

Hence $x \in \bar{\mu}_{\mathscr{F}(\varepsilon)}^{\Leftarrow}[\delta_1, \delta_2]$. Now if $x * y, y \in \lambda_{\mathscr{F}(\varepsilon)}^{\rightarrow}(t)$, then $\lambda_{\mathscr{F}(\varepsilon)}(x * y) \leq t$ and $\lambda_{\mathscr{F}(\varepsilon)}(y) \leq t$. Using (3.7), we have $\lambda_{\mathscr{F}(\varepsilon)}(x) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x * y), \lambda_{\mathscr{F}(\varepsilon)}(y)\} \leq t$, and so $x \in \lambda_{\mathscr{F}(\varepsilon)}^{\rightarrow}(t)$. Therefore $\bar{\mu}_{\mathscr{F}(\varepsilon)}^{\Leftarrow}[\delta_1, \delta_2]$ and $\lambda_{\mathscr{F}(\varepsilon)}^{\rightarrow}(t)$ are ideals of $U$.

Conversely, suppose that $\bar{\mu}_{\mathscr{F}(\varepsilon)}^{\Leftarrow}[\delta_1, \delta_2]$ and $\lambda_{\mathscr{F}(\varepsilon)}^{\rightarrow}(t)$ are ideals of $U$ for all $[\delta_1, \delta_2] \in [I]$ and $t \in [0, 1]$. Assume that there exists $a \in U$ such that $\bar{\mu}_{\mathscr{F}(\varepsilon)}(0) \nsucceq \bar{\mu}_{\mathscr{F}(\varepsilon)}(a)$ or $\lambda_{\mathscr{F}(\varepsilon)}(0) > \lambda_{\mathscr{F}(\varepsilon)}(a)$. Let $\bar{\mu}_{\mathscr{F}(\varepsilon)}(0) = [0^-, 0^+]$ and $\bar{\mu}_{\mathscr{F}(\varepsilon)}(a) = [a^-, a^+]$. Then $0^- < a^-$ and $0^+ < a^+$ which imply that $0^- < \delta_1 < a^-$ and $0^+ < \delta_2 < a^+$, that is,

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(0) = [0^-, 0^+] < [\delta_1, \delta_2] < [a^-, a^+]$$

by taking $[\delta_1, \delta_2] := \left[\frac{1}{2}(0^- + a^-), \frac{1}{2}(0^+ + a^+)\right]$. Hence $0 \notin \bar{\mu}_{\mathscr{F}(\varepsilon)}^{\Leftarrow}[\delta_1, \delta_2]$. Also $0 \notin \lambda_{\mathscr{F}(\varepsilon)}^{\rightarrow}(a_t)$ where $a_t = \lambda_{\mathscr{F}(\varepsilon)}(a)$. This is a contradiction, and so (3.5) is valid. Assume that there exist $a, b \in U$ such that

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(a) \nsucceq \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(a * b), \bar{\mu}_{\mathscr{F}(\varepsilon)}(b)\} \tag{3.12}$$

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or

$$\bar{\lambda}_{\mathscr{F}(\varepsilon)}(a) > \max\{\bar{\lambda}_{\mathscr{F}(\varepsilon)}(a * b), \bar{\lambda}_{\mathscr{F}(\varepsilon)}(b)\}. \tag{3.13}$$

For the case (3.12), let $\bar{\mu}_{\mathscr{F}(\varepsilon)}(a) = [\delta_1, \delta_2]$, $\bar{\mu}_{\mathscr{F}(\varepsilon)}(a * b) = [\gamma_1, \gamma_2]$ and $\bar{\mu}_{\mathscr{F}(\varepsilon)}(b) = [\gamma_3, \gamma_4]$. Then

$$[\delta_1, \delta_2] \prec \mathrm{rmin}\{[\gamma_1, \gamma_2], [\gamma_3, \gamma_4]\} = [\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}].$$

Hence $\delta_1 < \min\{\gamma_1, \gamma_3\}$ and $\delta_2 < \min\{\gamma_2, \gamma_4\}$. Taking

$$[\tau_1, \tau_2] = \tfrac{1}{2}\left(\bar{\mu}_{\mathscr{F}(\varepsilon)}(a) + \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(a * b), \bar{\mu}_{\mathscr{F}(\varepsilon)}(b)\}\right)$$

implies that

$$\begin{aligned}
[\tau_1, \tau_2] &= \tfrac{1}{2}\left([\delta_1, \delta_2] + [\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}]\right) \\
&= \left[\tfrac{1}{2}\left(\delta_1 + \min\{\gamma_1, \gamma_3\}\right), \tfrac{1}{2}\left(\delta_2 + \min\{\gamma_2, \gamma_4\}\right)\right].
\end{aligned}$$

It follows that

$$\begin{aligned}
\min\{\gamma_1, \gamma_3\} &> \tau_1 = \tfrac{1}{2}\left(\delta_1 + \min\{\gamma_1, \gamma_3\}\right) > \delta_1, \\
\min\{\gamma_2, \gamma_4\} &> \tau_2 = \tfrac{1}{2}\left(\delta_2 + \min\{\gamma_2, \gamma_4\}\right) > \delta_2,
\end{aligned}$$

and so that

$$[\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}] \succ [\tau_1, \tau_2] \succ [\delta_1, \delta_2] = \bar{\mu}_{\mathscr{F}(\varepsilon)}(a).$$

Therefore $a \notin \bar{\mu}_{\mathscr{F}(\varepsilon)}^{\Leftarrow}[\tau_1, \tau_2]$. On the other hand, we know that

$$\begin{aligned}
\bar{\mu}_{\mathscr{F}(\varepsilon)}(a * b) &= [\gamma_1, \gamma_2] \succeq [\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}] \succ [\tau_1, \tau_2], \\
\bar{\mu}_{\mathscr{F}(\varepsilon)}(b) &= [\gamma_3, \gamma_4] \succeq [\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}] \succ [\tau_1, \tau_2],
\end{aligned}$$

which imply that $a * b, b \in \bar{\mu}_{\mathscr{F}(\varepsilon)}^{\Leftarrow}[\tau_1, \tau_2]$. This is a contradiction, and so

$$\bar{\mu}_{\mathscr{F}(\varepsilon)}(x) \succeq \mathrm{rmin}\{\bar{\mu}_{\mathscr{F}(\varepsilon)}(x * y), \bar{\mu}_{\mathscr{F}(\varepsilon)}(y)\}$$

for all $x, y \in U$. Now, (3.13) implies that there exists $t_0 \in (0, 1)$ such that

$$\lambda_{\mathscr{F}(\varepsilon)}(a) \geq t_0 > \max\{\lambda_{\mathscr{F}(\varepsilon)}(a * b), \lambda_{\mathscr{F}(\varepsilon)}(b)\}.$$

Hence $a * b, b \in \lambda_{\mathscr{F}(\varepsilon)}^{\rightarrow}(t_0)$ but $a \notin \lambda_{\mathscr{F}(\varepsilon)}^{\rightarrow}(t_0)$. This is a contradiction, and therefore

$$\lambda_{\mathscr{F}(\varepsilon)}(x) \leq \max\{\lambda_{\mathscr{F}(\varepsilon)}(x * y), \lambda_{\mathscr{F}(\varepsilon)}(y)\}$$

for all $x, y \in U$. Consequently, $(\mathscr{F}, A)$ is an $\varepsilon$-cubic soft ideal over $U$. $\square$

**Definition 3.17** ([10]). The *R-union* of cubic soft sets $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ over $U$ is a cubic soft set $(\mathscr{H}, C)$ where $C = A \cup B$ and

$$\mathscr{H}(\varepsilon) = \begin{cases} \mathscr{F}(\varepsilon) & \text{if } \varepsilon \in A \setminus B, \\ \mathscr{G}(\varepsilon) & \text{if } \varepsilon \in B \setminus A, \\ \mathscr{F}(\varepsilon) \cup_R \mathscr{G}(\varepsilon) & \text{if } \varepsilon \in A \cap B \end{cases}$$

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for all $\varepsilon \in C$. This is denoted by $(\mathscr{H}, C) = (\mathscr{F}, A) \uplus_R (\mathscr{G}, B)$. Also the *R-intersection* of cubic soft sets $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ over $U$ is a cubic soft set $(\mathscr{H}, C)$ where $C = A \cup B$ and

$$\mathscr{H}(\varepsilon) = \begin{cases} \mathscr{F}(\varepsilon) & \text{if } \varepsilon \in A \setminus B, \\ \mathscr{G}(\varepsilon) & \text{if } \varepsilon \in B \setminus A, \\ \mathscr{F}(\varepsilon) \cap_R \mathscr{G}(\varepsilon) & \text{if } \varepsilon \in A \cap B \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $(\mathscr{H}, C) = (\mathscr{F}, A) \cap_R (\mathscr{G}, B)$.

**Theorem 3.18.** *If $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ are cubic soft ideals over $U$, then so is the R-intersection $(\mathscr{H}, C) = (\mathscr{F}, A) \cap_R (\mathscr{G}, B)$ of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$.*

*Proof.* Straightforward. □

**Theorem 3.19.** *Let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be cubic soft ideals over $U$. If $A$ and $B$ are disjoint, then the R-union of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ is a cubic soft ideal over $U$.*

*Proof.* By means of Definition 3.17, we can write $(\mathscr{F}, A) \uplus_R (\mathscr{G}, B) = (\mathscr{H}, C)$, where $C = A \cup B$ and for all $\varepsilon \in C$,

$$\mathscr{H}(\varepsilon) = \begin{cases} \mathscr{F}(\varepsilon) & \text{if } \varepsilon \in A \setminus B, \\ \mathscr{G}(\varepsilon) & \text{if } \varepsilon \in B \setminus A, \\ \mathscr{F}(\varepsilon) \cup_R \mathscr{G}(\varepsilon) & \text{if } \varepsilon \in A \cap B \end{cases}$$

Since $A \cap B = \emptyset$, either $\varepsilon \in A \setminus B$ or $\varepsilon \in B \setminus A$ for all $\varepsilon \in C$. If $\varepsilon \in A \setminus B$, then $\mathscr{H}(\varepsilon) = \mathscr{F}(\varepsilon)$ is a cubic soft ideal over $U$. If $\varepsilon \in B \setminus A$, then $\mathscr{H}(\varepsilon) = \mathscr{G}(\varepsilon)$ is a cubic soft ideal over $U$. Hence $(\mathscr{H}, C) = (\mathscr{F}, A) \uplus_R (\mathscr{G}, B)$ is a cubic soft ideal over $U$. □

The following example shows that Theorem 3.19 is not valid if $A$ and $B$ are not disjoint.

**Example 3.20.** Let $U = \{0, a, b, c\}$ be a $BCI$-algebra with the Cayley table in Table 3.

TABLE 3. Cayley table of the operation $*$

| $*$ | $0$ | $a$ | $b$ | $c$ |
|-----|-----|-----|-----|-----|
| $0$ | $0$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $0$ | $c$ | $b$ |
| $b$ | $b$ | $c$ | $0$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $0$ |

Consider sets of parameters $A := \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ and $B := \{\varepsilon_3, \varepsilon_4\}$. Then $A$ and $B$ are not disjoint. Let $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ be cubic soft sets over $U$ with the tabular representations in Table 4 and Table 5, respectively.
Then $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ are cubic soft ideals over $U$, and the R-union $(\mathscr{H}, C) = (\mathscr{F}, A) \uplus_R (\mathscr{G}, B)$ of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ is represented by Table 6.
Note that

$$\bar{\mu}_{\mathscr{H}(\varepsilon_3)}(c) = [0.4, 0.7] \not\succeq [0.6, 0.8] = \mathrm{rmin}\{\bar{\mu}_{\mathscr{H}(\varepsilon_3)}(c * a), \bar{\mu}_{\mathscr{H}(\varepsilon_3)}(a)\}$$

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TABLE 4. Tabular representation of the cubic soft set $(\mathscr{F}, A)$

|   | $\varepsilon_1$ | $\varepsilon_2$ | $\varepsilon_3$ |
|---|---|---|---|
| 0 | $\langle[0.6, 0.8], 0.3\rangle$ | $\langle[0.5, 0.9], 0.4\rangle$ | $\langle[0.7, 0.9], 0.3\rangle$ |
| $a$ | $\langle[0.3, 0.7], 0.5\rangle$ | $\langle[0.2, 0.5], 0.7\rangle$ | $\langle[0.6, 0.8], 0.8\rangle$ |
| $b$ | $\langle[0.3, 0.7], 0.5\rangle$ | $\langle[0.3, 0.6], 0.7\rangle$ | $\langle[0.4, 0.7], 0.8\rangle$ |
| $c$ | $\langle[0.3, 0.7], 0.5\rangle$ | $\langle[0.2, 0.5], 0.6\rangle$ | $\langle[0.4, 0.7], 0.5\rangle$ |

TABLE 5. Tabular representation of the cubic soft set $(\mathscr{G}, B)$

|   | $\varepsilon_3$ | $\varepsilon_4$ |
|---|---|---|
| 0 | $\langle[0.7, 1.0], 0.2\rangle$ | $\langle[0.4, 0.8], 0.1\rangle$ |
| $a$ | $\langle[0.3, 0.7], 0.7\rangle$ | $\langle[0.2, 0.6], 0.3\rangle$ |
| $b$ | $\langle[0.6, 0.8], 0.4\rangle$ | $\langle[0.2, 0.6], 0.6\rangle$ |
| $c$ | $\langle[0.3, 0.7], 0.7\rangle$ | $\langle[0.4, 0.8], 0.6\rangle$ |

TABLE 6. Tabular representation of the cubic soft set $(\mathscr{H}, C)$

|   | $\varepsilon_1$ | $\varepsilon_2$ | $\varepsilon_3$ | $\varepsilon_4$ |
|---|---|---|---|---|
| 0 | $\langle[0.6, 0.8], 0.3\rangle$ | $\langle[0.5, 0.9], 0.4\rangle$ | $\langle[0.7, 1.0], 0.2\rangle$ | $\langle[0.4, 0.8], 0.1\rangle$ |
| $a$ | $\langle[0.3, 0.7], 0.5\rangle$ | $\langle[0.2, 0.5], 0.7\rangle$ | $\langle[0.6, 0.8], 0.7\rangle$ | $\langle[0.2, 0.6], 0.3\rangle$ |
| $b$ | $\langle[0.3, 0.7], 0.5\rangle$ | $\langle[0.3, 0.6], 0.7\rangle$ | $\langle[0.6, 0.8], 0.4\rangle$ | $\langle[0.2, 0.6], 0.6\rangle$ |
| $c$ | $\langle[0.3, 0.7], 0.5\rangle$ | $\langle[0.2, 0.5], 0.6\rangle$ | $\langle[0.4, 0.7], 0.5\rangle$ | $\langle[0.4, 0.8], 0.6\rangle$ |

and/or $\lambda_{\mathscr{H}(\varepsilon_3)}(a) = 0.7 \not\leq 0.5 = \max\{\lambda_{\mathscr{H}(\varepsilon_3)}(a * b), \lambda_{\mathscr{H}(\varepsilon_3)}(b)\}$. Hence the R-union $(\mathscr{H}, C) = (\mathscr{F}, A) \uplus_R (\mathscr{G}, B)$ of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ is not a cubic soft ideal over $U$.

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# Hyers-Ulam stability of the delayed homogeneous matrix difference equation with constructive method

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### Abstract

We prove Hyers-Ulam stability of the first order delayed homogeneous matrix difference equation $\vec{x}_{i+p} = \mathbf{A}(i)\vec{x}_i$ for all integers $i$.

## 1 Introduction

Throughout this paper, we denote by $\mathbb{C}$, $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{Z}$ the set of all complex numbers, of all positive integers, of all nonnegative integers, and the set of all integers, respectively. Given a fixed positive integer $n$, let $(\mathbb{C}^n, \|\cdot\|_n)$ be a complex normed space, each of whose elements is a column vector, and let $\mathbb{C}^{n\times n}$ be a vector space consisting of all $(n \times n)$ complex matrices. We choose a norm $\|\cdot\|_{n\times n}$ on $\mathbb{C}^{n\times n}$ which is compatible with $\|\cdot\|_n$, *i.e.*, both norms obey

$$\|\mathbf{AB}\|_{n\times n} \leq \|\mathbf{A}\|_{n\times n}\|\mathbf{B}\|_{n\times n} \quad \text{and} \quad \|\mathbf{A}\vec{x}\|_n \leq \|\mathbf{A}\|_{n\times n}\|\vec{x}\|_n \tag{1.1}$$

for all $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n\times n}$ and $\vec{x} \in \mathbb{C}^n$.

A matrix difference equation is a difference equation with matrix coefficients in which the value of vector at one point depends on the values of preceding points.

In this paper, we prove Hyers-Ulam stability of the first order delayed homogeneous matrix difference equation

$$\vec{x}_{i+p} = \mathbf{A}(i)\vec{x}_i \tag{1.2}$$

for all integers $i \in \mathbb{Z}$, where each transition matrix $\mathbf{A}(i)$ is nonsingular and $p$ is a fixed integer larger than 1. More precisely, we prove that if a vector sequence $\{\vec{y}_i\}_{i\in\mathbb{Z}}$ of $\mathbb{C}^n$ satisfies the inequality

$$\|\vec{y}_{i+p} - \mathbf{A}(i)\vec{y}_i\|_n \leq \varepsilon$$

for all $i \in \mathbb{Z}$, then there exists a solution $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ to the delayed matrix difference equation (1.2) such that the bound for $\|\vec{y}_i - \vec{x}_i\|_n$ depends on $\varepsilon$ and the transition matrices $\mathbf{A}(i)$ only. We refer the reader to [1, 2, 3, 4, 6] for the exact definition of Hyers-Ulam stability.

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[0]Key words and phrases: difference equation; matrix difference equation; delayed matrix difference equation; Hyers-Ulam stability; approximation.
[0]2010 Mathematics Subject Classification: Primary 39A45, 39B82; Secondary 39A06, 39B42.

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## 2    Preliminaries

Throughout this paper, the transition matrix $\mathbf{A}(i)$ of $\mathbb{C}^{n \times n}$ is defined by

$$\mathbf{A}(i) = \begin{pmatrix} a_{11}(i) & a_{12}(i) & \cdots & a_{1n}(i) \\ a_{21}(i) & a_{22}(i) & \cdots & a_{2n}(i) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(i) & a_{n2}(i) & \cdots & a_{nn}(i) \end{pmatrix}$$

for any integer $i$. We moreover assume that every $\mathbf{A}(i)$ is nonsingular. We will use the following abbreviation.

$$\Phi(j,k) := \begin{cases} \displaystyle\prod_{i=k}^{j-1} \mathbf{A}(i) = \mathbf{A}(j-1)\mathbf{A}(j-2)\cdots\mathbf{A}(k) & \text{(for } j > k), \\[2em] \mathbf{I}_{n \times n} & \text{(for } j = k), \end{cases} \tag{2.1}$$

where we set $\Phi(j,k) := \big(\Phi(k,j)\big)^{-1} = \mathbf{A}(j)^{-1}\mathbf{A}(j+1)^{-1}\cdots\mathbf{A}(k-1)^{-1}$ for $j < k$ and $\mathbf{I}_{n \times n}$ denotes the $(n \times n)$ identity matrix. Sometimes, we use $\Phi(j)$ and $\Phi^{-1}(k,j)$ instead of $\Phi(j,0)$ and $\big(\Phi(k,j)\big)^{-1}$, respectively.

In the following lemma, we introduce some properties of $\Phi(j,k)$ without proof.

**Lemma 2.1** *Assume that $n$ is a fixed positive integer. If the transition matrix $\mathbf{A}(i)$ of $\mathbb{C}^{n \times n}$ is nonsingular for any integer $i$, then it holds that*

(i)  $\Phi(j+1,k) = \mathbf{A}(j)\Phi(j,k)$;

(ii)  $\Phi^{-1}(j,k+1) = \mathbf{A}(k)\Phi^{-1}(j,k)$;

(iii)  $\mathbf{A}(k-1)^{-1}\Phi^{-1}(j,k) = \Phi^{-1}(j,k-1)$

*for all integers $j$ and $k$.*

## 3    Hyers-Ulam stability of $\vec{x}_{i+p} = \mathbf{A}(i)\vec{x}_i$

We now prove our main theorem concerning Hyers-Ulam stability of the delayed homogeneous matrix difference equation (1.2). Obviously, our theorem is a generalization and an improvement of [5, Theorem 2.1].

**Theorem 3.1** *Assume that $n > 0$ and $p > 1$ are fixed integers and $\varepsilon$ is a nonnegative real number. For all integers $i$, assume that $\mathbf{A}(i)$ is a nonsingular $(n \times n)$ complex-valued matrix for which there exists a constant $K > 0$ such that*

$$\sum_{j=0}^{\infty} \left\| \left( \prod_{k=0}^{j} \mathbf{A}(i+kp) \right)^{-1} \right\|_{n \times n} \leq K \tag{3.1}$$

*for all integers $i$. If a sequence $\{\vec{y}_i\}_{i\in\mathbb{Z}}$ of $\mathbb{C}^n$ satisfies the inequality*

$$\left\|\vec{y}_{i+p} - \mathbf{A}(i)\vec{y}_i\right\|_n \leq \varepsilon \tag{3.2}$$

*for all integers $i$, then there exists a unique solution $\{\vec{T}_i\}_{i\in\mathbb{Z}}$ to the first order delayed homogeneous matrix difference equation (1.2) such that*

$$\left\|\vec{T}_i - \vec{y}_i\right\|_n \leq K\varepsilon \tag{3.3}$$

*for each integer $i$.*

**Proof.** In view of (3.2), there exists a sequence $\{\vec{\varepsilon}_i\}_{i\in\mathbb{Z}}$ of $\mathbb{C}^n$ such that

$$\vec{y}_{i+p} - \mathbf{A}(i)\vec{y}_i = \vec{\varepsilon}_i \tag{3.4}$$

for all integers $i$ and

$$\sup_{i\in\mathbb{Z}}\left\|\vec{\varepsilon}_i\right\|_n \leq \varepsilon. \tag{3.5}$$

First, we use the induction on $m$ to prove

$$\vec{y}_{i+mp} = \left(\prod_{k=0}^{m-1}\mathbf{A}(i+kp)\right)\vec{y}_i + \sum_{j=0}^{m-1}\left(\prod_{k=j+1}^{m-1}\mathbf{A}(i+kp)\right)\vec{\varepsilon}_{i+jp} \tag{3.6}$$

for all $i\in\mathbb{Z}$ and $m\in\mathbb{N}_0$. Obviously, the equality (3.6) is true for $m\in\{0,1\}$. Assume that the equality (3.6) is true for some positive integer $m$. It then follows from (3.4) and (3.6) that

$$\begin{aligned}
\vec{y}_{i+(m+1)p} &= \mathbf{A}(i+mp)\vec{y}_{i+mp} + \vec{\varepsilon}_{i+mp} \\
&= \left(\prod_{k=0}^{m}\mathbf{A}(i+kp)\right)\vec{y}_i + \sum_{j=0}^{m-1}\left(\prod_{k=j+1}^{m}\mathbf{A}(i+kp)\right)\vec{\varepsilon}_{i+jp} + \vec{\varepsilon}_{i+mp} \\
&= \left(\prod_{k=0}^{m}\mathbf{A}(i+kp)\right)\vec{y}_i + \sum_{j=0}^{m}\left(\prod_{k=j+1}^{m}\mathbf{A}(i+kp)\right)\vec{\varepsilon}_{i+jp},
\end{aligned}$$

which follows from (3.6) by replacing $m$ with $m+1$.

If we set

$$\vec{T}_i(m) := \left(\prod_{k=0}^{m}\mathbf{A}(i+kp)\right)^{-1}\vec{y}_{i+(m+1)p}$$

for all $i\in\mathbb{Z}$ and $m\in\mathbb{N}_0$, then it follows from (3.6) that

$$\vec{T}_i(m) = \vec{y}_i + \sum_{j=0}^{m}\left(\prod_{k=0}^{j}\mathbf{A}(i+kp)\right)^{-1}\vec{\varepsilon}_{i+jp}. \tag{3.7}$$

Let $m$ and $n$ be nonnegative integers with $n > m$. Then, by (3.7), we have

$$\vec{T}_i(n) - \vec{T}_i(m) = \sum_{j=m+1}^{n} \left( \prod_{k=0}^{j} \mathbf{A}(i + kp) \right)^{-1} \vec{\varepsilon}_{i+jp}$$

for any fixed integer $i$. In view of (3.5) and (3.1), we further get

$$\begin{aligned}
\left\| \vec{T}_i(n) - \vec{T}_i(m) \right\|_n &= \left\| \sum_{j=m+1}^{n} \left( \prod_{k=0}^{j} \mathbf{A}(i + kp) \right)^{-1} \vec{\varepsilon}_{i+jp} \right\|_n \\
&\leq \sum_{j=m+1}^{n} \left\| \left( \prod_{k=0}^{j} \mathbf{A}(i + kp) \right)^{-1} \vec{\varepsilon}_{i+jp} \right\|_n \\
&\leq \sum_{j=m+1}^{n} \left\| \left( \prod_{k=0}^{j} \mathbf{A}(i + kp) \right)^{-1} \right\|_{n \times n} \left\| \vec{\varepsilon}_{i+jp} \right\|_n \\
&\leq \varepsilon \sum_{j=m+1}^{n} \left\| \left( \prod_{k=0}^{j} \mathbf{A}(i + kp) \right)^{-1} \right\|_{n \times n} \\
&\to 0, \quad \text{as} \quad m \to \infty,
\end{aligned}$$

for every $i \in \mathbb{Z}$. Hence, $\{\vec{T}_i(m)\}_{m \in \mathbb{N}_0}$ is a Cauchy sequence for each fixed $i \in \mathbb{Z}$, and we can define

$$\vec{T}_i := \lim_{m \to \infty} \vec{T}_i(m) \tag{3.8}$$

for each $i \in \mathbb{Z}$.

By (3.4), (3.7), and (3.8), we obtain

$$\begin{aligned}
\vec{T}_{i+p} - \mathbf{A}(i)\vec{T}_i &= \vec{y}_{i+p} + \sum_{j=0}^{\infty} \left( \prod_{k=0}^{j} \mathbf{A}\big(i + (k+1)p\big) \right)^{-1} \vec{\varepsilon}_{i+(j+1)p} \\
&\quad - \mathbf{A}(i)\vec{y}_i - \sum_{j=0}^{\infty} \left( \prod_{k=1}^{j} \mathbf{A}(i + kp) \right)^{-1} \vec{\varepsilon}_{i+jp} \\
&= \vec{y}_{i+p} + \sum_{j=0}^{\infty} \left( \prod_{k=1}^{j+1} \mathbf{A}(i + kp) \right)^{-1} \vec{\varepsilon}_{i+(j+1)p} \\
&\quad - \mathbf{A}(i)\vec{y}_i - \sum_{j=0}^{\infty} \left( \prod_{k=1}^{j} \mathbf{A}(i + kp) \right)^{-1} \vec{\varepsilon}_{i+jp} \\
&= \vec{y}_{i+p} + \sum_{j=1}^{\infty} \left( \prod_{k=1}^{j} \mathbf{A}(i + kp) \right)^{-1} \vec{\varepsilon}_{i+jp} \\
&\quad - \mathbf{A}(i)\vec{y}_i - \sum_{j=0}^{\infty} \left( \prod_{k=1}^{j} \mathbf{A}(i + kp) \right)^{-1} \vec{\varepsilon}_{i+jp} \\
&= \vec{0}
\end{aligned}$$

for all $i \in \mathbb{Z}$. Moreover, it follows from (3.5), (3.1), (3.7), and (3.8) that

$$
\begin{aligned}
\left\| \vec{T}_i - \vec{y}_i \right\|_n &= \left\| \sum_{j=0}^{\infty} \left( \prod_{k=0}^{j} \mathbf{A}(i+kp) \right)^{-1} \vec{\varepsilon}_{i+jp} \right\|_n \\
&\leq \sum_{j=0}^{\infty} \left\| \left( \prod_{k=0}^{j} \mathbf{A}(i+kp) \right)^{-1} \vec{\varepsilon}_{i+jp} \right\|_n \\
&\leq \sum_{j=0}^{\infty} \left\| \left( \prod_{k=0}^{j} \mathbf{A}(i+kp) \right)^{-1} \right\|_{n \times n} \left\| \vec{\varepsilon}_{i+jp} \right\|_n \\
&\leq K\varepsilon
\end{aligned}
$$

for all $i \in \mathbb{Z}$.

Finally, we prove the uniqueness of the sequence $\{\vec{T}_i\}_{i \in \mathbb{Z}}$. Assume that $\{\vec{U}_i\}_{i \in \mathbb{Z}}$ is another solution to the difference equation (1.2). By applying the induction on $m$, we prove that

$$
\vec{U}_i = \left( \prod_{k=0}^{m} \mathbf{A}(i+kp) \right)^{-1} \vec{U}_{i+(m+1)p} \tag{3.9}
$$

for any $m \in \mathbb{N}_0$. Obviously, (3.9) is true for $m = 0$. Assume now that (3.9) is true for some integer $m \geq 0$. It then follows from (1.2) and (3.9) that

$$
\begin{aligned}
\vec{U}_i &= \left( \prod_{k=0}^{m} \mathbf{A}(i+kp) \right)^{-1} \vec{U}_{i+(m+1)p} \\
&= \left( \prod_{k=0}^{m+1} \mathbf{A}(i+kp) \right)^{-1} \mathbf{A}\big(i+(m+1)p\big) \vec{U}_{i+(m+1)p} \\
&= \left( \prod_{k=0}^{m+1} \mathbf{A}(i+kp) \right)^{-1} \vec{U}_{i+(m+2)p},
\end{aligned}
$$

which can be obtained from (3.9) by replacing $m$ with $m+1$. Thus, by (3.1), (3.3), and (3.9), we have

$$
\begin{aligned}
\left\| \vec{T}_i - \vec{U}_i \right\|_n &= \left\| \left( \prod_{k=0}^{m} \mathbf{A}(i+kp) \right)^{-1} \big( \vec{T}_{i+(m+1)p} - \vec{U}_{i+(m+1)p} \big) \right\|_n \\
&\leq \left\| \left( \prod_{k=0}^{m} \mathbf{A}(i+kp) \right)^{-1} \right\|_{n \times n} \left\| \vec{T}_{i+(m+1)p} - \vec{y}_{i+(m+1)p} \right\|_n \\
&\quad + \left\| \left( \prod_{k=0}^{m} \mathbf{A}(i+kp) \right)^{-1} \right\|_{n \times n} \left\| \vec{y}_{i+(m+1)p} - \vec{U}_{i+(m+1)p} \right\|_n \\
&\leq 2K\varepsilon \left\| \left( \prod_{k=0}^{m} \mathbf{A}(i+kp) \right)^{-1} \right\|_{n \times n} \\
&\to 0, \quad \text{as} \quad m \to \infty,
\end{aligned}
$$

for all $i \in \mathbb{Z}$, which implies the uniqueness of $\{\vec{T}_i\}_{i \in \mathbb{Z}}$.                                    $\square$

## 4   Examples

At a glance, the condition (3.1) would seem too strong so that we could seldom find practical examples. But we get rid of such a misunderstanding through introducing a few examples for the sequence $\{\mathbf{A}(i)\}_{i\in\mathbb{Z}}$ of transition matrices which satisfy the condition (3.1).

**Example 4.1** *Let us set $n = 1$ and $p = 3$. If $\mathbf{A}(i) = 2^3$ is a $(1 \times 1)$ matrix for every integer $i$, then we have*

$$\sum_{j=0}^{\infty} \left| \left( \prod_{k=0}^{j} \mathbf{A}(i+3k) \right)^{-1} \right| = \sum_{j=0}^{\infty} \big( \underbrace{2^3 \cdot 2^3 \cdots 2^3}_{j+1} \big)^{-1} = \sum_{j=0}^{\infty} 2^{-3(j+1)} = \frac{1}{7},$$

*i.e., the condition (3.1) is satisfied with $K = \frac{1}{7}$.*

     *Assume that a sequence $\{y_i\}_{i\in\mathbb{Z}}$ of complex numbers satisfies the inequality*

$$\left| y_{i+3} - 2^3 y_i \right| \leq \varepsilon$$

*for all integers $i$, where $\varepsilon$ is an arbitrarily given nonnegative real number. Then, according to Theorem 3.1, there exists a unique sequence $\{x_i\}_{i\in\mathbb{Z}}$ of complex numbers such that*

$$x_{i+3} = 2^3 x_i \tag{4.1}$$

*and*

$$|x_i - y_i| \leq \frac{1}{7}\varepsilon$$

*for all integers $i$.*

     *Indeed, the delayed difference equation (4.1) is strongly related to the nonlinear difference equation*

$$x_{i+1} = 2^{i+1} - \frac{2^{2i+1}}{x_i}.$$

**Example 4.2** *We consider the difference equation with two variables given as*

$$\begin{pmatrix} u_{i+1} \\ v_{i+1} \end{pmatrix} = \begin{pmatrix} \dfrac{-2^{2i+1}}{(i^2+2i+2)(i^2+1)v_i - (i^2+2i+2)2^{i+1}} \\[2ex] \dfrac{2^{i+2}}{i^2+2i+2} - \dfrac{2^{2i+1}}{(i^2+2i+2)(i^2+1)u_i} \end{pmatrix}$$

*for all integers $i$, where $\{u_i\}_{i\in\mathbb{Z}}$ and $\{v_i\}_{i\in\mathbb{Z}}$ are sequences of complex numbers. By a straightforward calculation, we show that*

$$\begin{pmatrix} u_{i+2} \\ v_{i+2} \end{pmatrix} = \begin{pmatrix} \dfrac{4(i^2+1)}{i^2+4i+5} & 0 \\[2ex] 0 & \dfrac{4(i^2+1)}{i^2+4i+5} \end{pmatrix} \begin{pmatrix} u_i \\ v_i \end{pmatrix}$$

*for all integers $i$. We now define the $(2 \times 2)$ matrix $\mathbf{A}(i)$ by*

$$\mathbf{A}(i) := \begin{pmatrix} \dfrac{4(i^2+1)}{i^2+4i+5} & 0 \\ 0 & \dfrac{4(i^2+1)}{i^2+4i+5} \end{pmatrix} = \frac{4(i^2+1)}{(i+2)^2+1}\,\mathbf{I}_{2\times 2},$$

*for every integer $i$, where $\mathbf{I}_{2\times 2}$ denotes the $(2 \times 2)$ identity matrix. Then we have*

$$\left(\prod_{k=0}^{j}\mathbf{A}(i+2k)\right)^{-1} = \frac{(i+2j+2)^2+1}{(i^2+1)4^{j+1}}\,\mathbf{I}_{2\times 2}$$

*for all nonnegative integers $j$. Hence, we see that*

$$\sum_{j=0}^{\infty}\left\|\left(\prod_{k=0}^{j}\mathbf{A}(i+2k)\right)^{-1}\right\|_{\infty} = \sum_{j=0}^{\infty}\frac{(i+2j+2)^2+1}{(i^2+1)4^{j+1}}$$

$$\leq \sum_{j=0}^{\infty}\frac{(|i|+2j+2)^2+1}{(i^2+1)4^{j+1}}$$

$$\leq \sum_{j=0}^{\infty}\frac{1}{4^{j+1}} + \sum_{j=0}^{\infty}\frac{1}{2}\frac{j+1}{4^j} + \sum_{j=0}^{\infty}\frac{(j+1)^2}{4^j}$$

$$\leq \frac{1}{3} + \sum_{j=0}^{\infty}\frac{1}{2}\frac{1}{2^j} + \sum_{j=0}^{\infty}\frac{9}{4}\frac{1}{2^j}$$

$$= \frac{35}{6},$$

*i.e., the condition (3.1) is satisfied with $K = \frac{35}{6}$.*

*Let $\varepsilon$ is an arbitrarily given nonnegative real number. Assume that a sequence $\{\vec{y}_i\}_{i\in\mathbb{Z}}$ of $\mathbb{C}^2$ satisfies the inequality*

$$\left\|\vec{y}_{i+2} - \mathbf{A}(i)\vec{y}_i\right\|_{\infty} \leq \varepsilon$$

*for all integers $i$. Then, due to Theorem 3.1 with $n = 2$, $p = 2$, and $K = \frac{35}{6}$, there exists a unique solution $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ to the delayed homogeneous matrix difference equation (1.2) such that*

$$\left\|\vec{x}_i - \vec{y}_i\right\|_{\infty} \leq \frac{35}{6}\varepsilon$$

*for any integer $i$.*

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# Mathematical analysis of $(n + 3)$-dimensional virus dynamics model

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**Abstract**

An $(n + 3)$-dimensional nonlinear mathematical model for the virus dynamics with humoral immunity and $n$-stages of infected cells is proposed and analyzed. Two threshold parameters, the basic reproduction number, $R_0^M$ and the humoral immunity number, $R_1^M$ are derived. Utilizing Lyapunov functions and LaSalle's invariance principle, the global asymptotic stability of all steady states of the model is obtained. An example is presented and some numerical simulations are conducted in order to illustrate the dynamical behavior.

**Keywords:** Virus dynamics; global stability; humoral immunity; Lyapunov function.

## 1    Introduction

During the past decades many human viruses have been found such as HIV, HBV, HCV and HTLV-I. To understand the virus dynamics, several mathematical models for virus dynamics have been proposed and analyzed (see e.g. [1]-[16]). One of the most important features of mathematical models is the global stability of steady states which gives us a detailed information and enhances our understanding about the virus dynamics. Therefore several researchers studied the global stability of virus dynamics models (see e.g. [5], [6], [7], [8], [9], [11], [12], [13], [14], [19], [20]). Some of these papers consider a single-infected stage for infected cells (see e.g. [5], [6], [7], [11], [12] and [14]). Other works consider double-infected stages for infected cells, the first stage is the latently infected cells which contain viruses but do not produce it and the second stage is the actively infected cells which produce new viruses (see e.g. [8] [9], [19] and [20]). As reported in [21], [22] and [23], due to ongoing viral replication in the virus dynamics process such as HIV, the time from the contact of viruses and uninfected target cells to the death of the cells modeled by dividing the process into $n$ short stages $y_1 \to y_2 \to .... \to y_n$. Georgescu and Hsieh [20] have proposed a virus dynamics model with multi-staged infected cells. However, the model does not consider the immune response.

It should be pointed out that the immune response plays an important role in controlling the disease progression. There are two main responses for immune system, Cytotoxic T Lymphocyte (CTL) immune response and humoral immune response. The function of the CTL cells is to kill the infected cells. The humoral immunity is based on the B cells which produce antibodies to attack the viruses [1]. It is mentioned in [24] that, in malaria, the antibodies are more effective than CTL cells [24]. Several works incorporate the humoral immune response into the virus dynamics models (see e.g. [25]-[31]). Elaiw and AlShamrani [29], [30] studied the global stability of virus dynamics models with double-infected stages for infected cells.

The aim of this paper is to study a general virus dynamics model with multi-staged infected cells and humoral immunity. Our model is an improvement of the model presented in [20] by taking into account the humoral immune response, and by assuming a more general incidence rate which includes the form given in [20]. We use Lyapunov functions and LaSalle's invariance principle to prove the global stability of all the steady states of the model. We show that there exist two bifurcation parameters, the basic reproduction number $R_0^M$

and the humoral immunity number $R_1^M$. We establish a set of sufficient conditions which guarantee the global stability of all steady states of the model.

## 2 The model

In this section we propose the following model:

$$\dot{x} = \lambda - dx - g(x, v), \tag{1}$$

$$\dot{y}_1 = g(x, v) - a_1 \phi_1(y_1), \tag{2}$$

$$\dot{y}_i = \tilde{a}_{i-1} \phi_{i-1}(y_{i-1}) - a_i \phi_i(y_i), \quad i = 2, 3, ..., n, \tag{3}$$

$$\dot{v} = \tilde{a}_n \phi_n(y_n) - pzv - uv, \tag{4}$$

$$\dot{z} = rzv - bz. \tag{5}$$

All parameters and variables have the same identifications given in Section 1. The model is a generalization of several existing model by considering general functions for: (i) the incidence rate of infection $g(x, v)$; (ii) the production rates of infected cells $g(x, v)$ and $\tilde{a}_{i-1} \phi_{i-1}(y_{i-1})$, $i = 2, ..., n$; (iii) the removal rate of infected cells $a_i \phi_i(y_i)$, $i = 1, ..., n$; (iii) the production rate of viruses $\tilde{a}_n \phi_n(y_n)$. Functions $g$ and $\phi_i$ are continuously differentiable and satisfy the following conditions:

**Condition C1.** (i) $g(x, v) > 0$, $g(0, v) = g(x, 0) = 0$ for all $x, v > 0$ and
(ii) $\frac{\partial g(x,v)}{\partial x} > 0$, $\frac{\partial g(x,v)}{\partial v} > 0$, $\frac{\partial g(x,0)}{\partial v} > 0$ for all $x, v > 0$.
**Condition C2.** (i) $g(x, v) \leq v \frac{\partial g(x,0)}{\partial v}$ for all $x, v > 0$ and
(ii) $\left( \frac{\partial g(x,0)}{\partial v} \right)' > 0$ for all $x, v > 0$.
**Condition C3.** (i) $\phi_i(y_i) > 0$ for all $y_i > 0$, $\phi_i(0) = 0$, $i = 1, 2, ..., n$,
(ii) $\phi_i'(y_i) > 0$ for all $y_i > 0$, $i = 1, 2, ..., n$, and
(iii) there is $\alpha_i > 0$, $i = 1, ..., n$ such that $\phi_i(y_i) \geq \alpha_i y_i$ for all $y_i > 0$.

## 3 Properties of solutions

In this section, we study some properties of the solutions of the model such as the non-negativity and boundedness.

**Proposition 1.** Suppose that Conditions C1 and C3 are hold. Then there exist positive numbers $M_j$, $j = 1, 2, ..., n + 2$, such that the compact set

$$\Theta = \left\{ (x, y_1, ..., y_n, v, z) \in \mathbb{R}_{\geq 0}^{n+3} : 0 \leq x \leq M_1, 0 \leq y_i \leq M_i, 0 \leq v \leq M_{n+1}, 0 \leq z \leq M_{n+2}, \ i = 1, ..., n \right\}$$

is positively invariant.

**Proof.** Since

$$\dot{x} \mid_{x=0} = \lambda > 0,$$
$$\dot{y}_1 \mid_{y_1=0} = g(x, v) \geq 0 \qquad \text{for all } x, v \in [0, \infty),$$
$$\dot{y}_i \mid_{y_i=0} = \tilde{a}_{i-1} \phi_{i-1}(y_{i-1}) \geq 0 \qquad \text{for all } y_{i-1} \in [0, \infty), \ i = 2, 3, ..., n,$$
$$\dot{v} \mid_{v=0} = \tilde{a}_n \phi_n(y_n) \geq 0 \qquad \text{for all } y_n \in [0, \infty),$$
$$\dot{z} \mid_{z=0} = 0,$$

Then, the orthant $\mathbb{R}_{\geq 0}^{n+3}$ is positively invariant for system (1)-(5).

To show the boundedness of the solutions we let $G_1(t) = x(t) + y_1(t)$, then

$$\dot{G}_1 = \lambda - dx - a_1 \phi_1(y_1) \leq \lambda - dx - a_1 \alpha_1 y_1 \leq \lambda - \delta_1 (x + y_1) \leq \lambda - \delta_1 G_1,$$

where $\delta_1 = \min\{d, a_1\alpha_1\}$. It follows that,

$$G_1(t) \le e^{-\delta_1 t}\left(G_1(0) - \frac{\lambda}{\delta_1}\right) + \frac{\lambda}{\delta_1}.$$

Hence, $0 \le G_1(t) \le M_1$ if $G_1(0) \le M_1$ for $t \ge 0$ where $M_1 = \frac{\lambda}{\delta_1}$. The non-negativity of $x$ and $y_1$ implies that, $0 \le x(t), y_1(t) \le M_1$ if $x(0) + y_1(0) \le M_1$. From Eq. (3) and Condition C3, we have

$$\dot{y}_2 = \tilde{a}_1\phi_1(y_1) - a_2\phi_2(y_2) \le \tilde{a}_1\phi_1(M_1) - a_2\alpha_2 y_2.$$

It follows that, $0 \le y_2(t) \le M_2$ if $y_2(0) \le M_2$, where $M_2 = \dfrac{\tilde{a}_1\phi_1(M_1)}{a_2\alpha_2}$. Similarly, we can show $0 \le y_i(t) \le M_i$ if $y_i(0) \le M_i$, where $M_i = \dfrac{\tilde{a}_{i-1}\phi_{i-1}(M_{i-1})}{a_i\alpha_i}$ $i = 3, ..., n$. Finally, we let $G_2(t) = v(t) + \frac{p}{r}z(t)$, then

$$\dot{G}_2 = \tilde{a}_n\phi_n(y_n) - uv - \frac{pb}{r}z$$
$$\le \tilde{a}_n\phi_n(M_n) - \delta_2\left(v + \frac{p}{r}z\right) = \tilde{a}_n\phi_n(M_n) - \delta_2 G_2,$$

where $\delta_2 = \min\{u, b\}$. It follows that, $0 \le G_2(t) \le M_{n+1}$ if $G_2(0) \le M_{n+1}$, where $M_{n+1} = \dfrac{\tilde{a}_n\phi_n(M_n)}{\delta_2}$. Since $v(t)$ and $z(t)$ are non-negative, then $0 \le v(t) \le M_{n+1}$ and $0 \le z(t) \le M_{n+2}$ if $v(0) + \frac{p}{r}z(0) \le M_{n+1}$, where $M_{n+2} = \frac{r}{p}M_{n+1}$. Therefore, all the variables of the model are bounded and the region $\Theta$ is positively invariant with respect to model (1)-(5). $\square$

# 4   The steady states and biological bifurcations

In this section, we prove the existence of the steady states of system (1)-(5) and derive two bifurcation parameters.

**Lemma 1.** Assume that Conditions C1-C3 are satisfied, then there exist two bifurcation parameters $R_0^M > R_1^M > 0$ such that

(i) if $R_0^M \le 1$, then the system has only one positive steady state $Q_0 \in \Theta$.

(ii) if $R_1^M \le 1 < R_0^M$, then then the system has only two positive steady states $Q_0 \in \Theta$ and $Q_1 \in \Theta$, and

(iii) if $R_1^M > 1$, then then the system has three positive steady states $Q_0 \in \Theta$, $Q_1 \in \Theta$ and $Q_2 \in \overset{\circ}{\Theta}$.

**Proof**. At any steady state $E(x, y_1, ..., y_n, v, z)$, the following equations hold:

$$\lambda - dx - g(x, v) = 0, \tag{6}$$
$$g(x, v) - a_1\phi_1(y_1) = 0, \tag{7}$$
$$\tilde{a}_{i-1}\phi_{i-1}(y_{i-1}) - a_i\phi_i(y_i) = 0, \quad i = 2, ..., n, \tag{8}$$
$$\tilde{a}_n\phi_n(y_n) - uv - pzv = 0, \tag{9}$$
$$(rv - b)z = 0. \tag{10}$$

Eq. (10) has two possibilities, $z = 0$ and $v = \dfrac{b}{r}$. When $z = 0$, then from Eqs. (6)-(9) we get

$$\lambda - dx = g(x, v) = \left(\prod_{j=1}^{i}\frac{a_j}{\tilde{a}_j}\right)\tilde{a}_i\phi_i(y_i) = \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)uv, \quad i = 1, ..., n, \tag{11}$$

The continuity and strictly increasing properties of $\phi_i$ imply that $\phi_i^{-1}$ exists and it is also continuous and strictly increasing [32]. Define $f_i(v) = \phi_i^{-1}\left(\left(\prod_{j=1}^{i}\frac{\tilde{a}_j}{a_j}\right)\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)\frac{uv}{\tilde{a}_i}\right)$, $i = 1, 2, ..., n$, then $f_i(0) = 0$ and $f_i(v) > 0$ for all $v > 0$. From Eq. (11), we get

$$y_i = f_i(v), \qquad x = x_0 - \frac{1}{d}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)uv, \tag{12}$$

and

$$g\left(x_0 - \frac{1}{d}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)uv, v\right) - \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)uv = 0, \tag{13}$$

where $x_0 = \lambda/d$. Condition C1 implies that Eq. (13) has two possible solutions $v = 0$ and $v \neq 0$. If $v = 0$, then from Eq. (12), we get the disease-free steady state $Q_0 = (x_0, \overbrace{0, ..., 0, 0}^{n+2})$. Let us consider the case $v \neq 0$. Define

$$\Psi_1(v) = g\left(x_0 - \frac{1}{d}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)uv, v\right) - \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)uv = 0.$$

We have, $\Psi_1(0) = 0$, and $\Psi_1(\hat{v}) = -\lambda < 0$.where $\hat{v} = \frac{\lambda}{u}\left(\prod_{j=1}^{n}\frac{\tilde{a}_j}{a_j}\right)$. Moreover,

$$\Psi_1'(0) = -\frac{u}{d}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)\frac{\partial g(x_0, 0)}{\partial x} + \frac{\partial g(x_0, 0)}{\partial v} - \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)u.$$

From Condition C1 we have $\frac{\partial g(x_0, 0)}{\partial x} = 0$, then

$$\Psi_1'(0) = u\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)\left(\frac{1}{u}\left(\prod_{j=1}^{n}\frac{\tilde{a}_j}{a_j}\right)\frac{\partial g(x_0, 0)}{\partial v} - 1\right).$$

Therefore, if $\frac{1}{u}\left(\prod_{j=1}^{n}\frac{\tilde{a}_j}{a_j}\right)\frac{\partial g(x_0, 0)}{\partial v} > 1$, then $\Psi_1'(0) > 0$ and there exists a $v_1 \in (0, \hat{v})$ such that $\Psi_1(v_1) = 0$. Substituting $v = v_1$ in Eq. (6) and letting

$$\Psi_2(x) = \lambda - dx - g(x, v_1) = 0.$$

According to Condition C1, $\Psi_2$ is a strictly decreasing, $\Psi_2(0) = \lambda > 0$ and $\Psi_2(x_0) = -g(x_0, v_1) < 0$. Thus, there exists a unique $x_1 \in (0, x_0)$ such that $\Psi_2(x_1) = 0$. On the other hand, from Eq. (12) we have $y_{i,1} = f_i(v_1) > 0$, $i = 1, ..., n$. It follows that, a endemic steady state without humoral immune response $Q_1 = (x_1, y_{1,1}, ..., y_{n,1}, v_1, 0)$ exists when $\frac{1}{u}\left(\prod_{j=1}^{n}\frac{\tilde{a}_j}{a_j}\right)\frac{\partial g(x_0, 0)}{\partial v} > 1$. Let us define the basic reproduction number as:

$$R_0^M = \frac{1}{u}\left(\prod_{j=1}^{n}\frac{\tilde{a}_j}{a_j}\right)\frac{\partial g(x_0, 0)}{\partial v}.$$

The other possibility of Eq. (10) is $v = v_2 = \frac{b}{r}$. Let

$$\Psi_3(x) = \lambda - dx - g(x, v_2) = 0.$$

Clearly, $\Psi_3$ is a strictly decreasing, $\Psi_3(0) = \lambda > 0$ and $\Psi_3(x_0) = -g(x_0, v_2) < 0$. Thus, there exists a unique $x_2 \in (0, x_0)$ such that $\Psi_3(x_2) = 0$. It follows that,

$$y_{i,2} = \phi_i^{-1}\left(\left(\prod_{j=1}^{i}\frac{\tilde{a}_j}{a_j}\right)\frac{g(x_2, v_2)}{\tilde{a}_i}\right) > 0.$$

Further, $z_2 = \frac{u}{p}(R_1^M - 1)$, where

$$R_1^M = \frac{1}{u}\left(\prod_{j=1}^{n}\frac{\tilde{a}_j}{a_j}\right)\frac{g(x_2, v_2)}{v_2}$$

represents the humoral immunity number. It follows that, if $R_1^M > 1$, then there exists a endemic steady state with humoral immune response $Q_2 = (x_2, y_{1,2}, ..., y_{n,2}, v_2, z_2)$.

Now we show that $Q_0, Q_1 \in \Theta$ and $Q_2 \in \overset{\circ}{\Theta}$. Clearly, $Q_0 \in \Theta$. We have $x_1 \in (0, x_0)$, then

$$0 < x_1 < \frac{\lambda}{d} \leq \frac{\lambda}{\delta_1} = M_1.$$

From Eq. (11), we get

$$a_1 \alpha_1 y_{1,1} \leq a_1 \phi_1(y_{1,1}) = \lambda - dx_1 < \lambda \Rightarrow 0 < y_{1,1} < \frac{\lambda}{a_1 \alpha_1} \leq M_1.$$

Also, from Eq. (8), we have

$$a_2 \alpha_2 y_{2,1} \leq a_2 \phi_2(y_{2,1}) = \tilde{a}_1 \phi_1(y_{1,1}) < \tilde{a}_1 \phi_1(M_1) \Rightarrow 0 < y_{2,1} < \frac{\tilde{a}_1 \phi_1(M_1)}{a_2 \alpha_2} = M_2.$$

Consequently, for $i = 3, ..., n$, we have

$$a_i \alpha_i y_{i,1} \leq a_i \phi_i(y_{i,1}) = \tilde{a}_{i-1} \phi_{i-1}(y_{i-1,1}) < \tilde{a}_{i-1} \phi_{i-1}(M_{i-1}) \Rightarrow 0 < y_{i,1} < \frac{\tilde{a}_{i-1} \phi_{i-1}(M_{i-1})}{a_i \alpha_i} = M_i.$$

Eq. (9) implies that,

$$uv_1 = \tilde{a}_n \phi_n(y_{n,1}) < \tilde{a}_n \phi_n(M_n) \Rightarrow 0 < v_1 < \frac{\tilde{a}_n \phi_n(M_n)}{u} \leq \frac{\tilde{a}_n \phi_n(M_n)}{\delta_2} = M_{n+1}.$$

We have also $z_1 = 0$, then $Q_1 \in \Theta$. Similarly, one can show that $0 < x_2 < M_1$ and $0 < y_{i,2} < M_i$, $i = 1, ..., n$. Now we show that if $R_1^M > 1$, then $0 < v_2 < M_{n+1}$ and $0 < z_2 < M_{n+2}$. From Eq. (9) we have

$$uv_2 + pv_2 z_2 = \tilde{a}_n \phi_n(y_{n,2}).$$

Then

$$uv_2 < \tilde{a}_n \phi_n(y_{n,2}) < \tilde{a}_n \phi_n(M_n) \Rightarrow 0 < v_2 < \frac{\tilde{a}_n \phi_n(M_n)}{u} \leq M_{n+1},$$

$$pv_2 z_2 < \tilde{a}_n \phi_n(y_{n,2}) < \tilde{a}_n \phi_n(M_n) \Rightarrow 0 < z_2 < \frac{r\tilde{a}_n \phi_n(M_n)}{pb} \leq M_{n+2}.$$

Then, $Q_2 \in \overset{\circ}{\Theta}$. Clearly from Condition C2, we have

$$R_1^M = \frac{1}{u} \left( \prod_{j=1}^{n} \frac{\tilde{a}_j}{a_j} \right) \frac{g(x_2, v_2)}{v_2} \leq \frac{1}{u} \left( \prod_{j=1}^{n} \frac{\tilde{a}_j}{a_j} \right) \frac{\partial g(x_2, 0)}{\partial v} < \frac{1}{u} \left( \prod_{j=1}^{n} \frac{\tilde{a}_j}{a_j} \right) \frac{\partial g(x_0, 0)}{\partial v} = R_0^M. \square$$

## 5   Global stability analysis

In this section, we study the global stability of system (1)-(5) by constructing suitable Lyapunov functionals. The stability of the disease-free steady state $Q_0$ will be given in the following result.

**Theorem 1.** Let Conditions C1-C3 hold true and $R_0^M \leq 1$, then $Q_0$ is globally asymptotically stable (GAS) in $\Theta$.

**Proof.** Define

$$V_0(x, y_1, ..., y_n, v, z) = x - x_0 - \int_{x_0}^{x} \lim_{v \to 0^+} \frac{g(x_0, v)}{g(\eta, v)} d\eta + \sum_{i=1}^{n} \left( \prod_{j=1}^{i-1} \frac{a_j}{\tilde{a}_j} \right) y_i + \left( \prod_{j=1}^{n} \frac{a_j}{\tilde{a}_j} \right) v + \frac{p}{r} \left( \prod_{j=1}^{n} \frac{a_j}{\tilde{a}_j} \right) z, \quad (14)$$

where $\prod_{j=1}^{0} \frac{a_j}{\tilde{a}_j} = 1$. It is seen that, $V_0(x, y_1, ..., y_n, v, z) > 0$ for all $x, y_1, ..., y_n, v, z > 0$, while $V_0(x_0, \overbrace{0, ..., 0, 0}^{n+2}) = 0$.

We calculate $\frac{dV_0}{dt}$ along the solutions of model (1)-(5) as:

$$\frac{dV_0}{dt} = \left( 1 - \lim_{v \to 0^+} \frac{g(x_0, v)}{g(x, v)} \right) \dot{x} + \sum_{i=1}^{n} \left( \prod_{j=1}^{i-1} \frac{a_j}{\tilde{a}_j} \right) \dot{y}_i + \left( \prod_{j=1}^{n} \frac{a_j}{\tilde{a}_j} \right) \dot{v} + \frac{p}{r} \left( \prod_{j=1}^{n} \frac{a_j}{\tilde{a}_j} \right) \dot{z}. \quad (15)$$

We have

$$\sum_{i=1}^{n} \left( \prod_{j=1}^{i-1} \frac{a_j}{\tilde{a}_j} \right) \dot{y}_i = g(x,v) - a_1 \phi_1(y_1) + \sum_{i=2}^{n} \left( \prod_{j=1}^{i-1} \frac{a_j}{\tilde{a}_j} \right) (\tilde{a}_{i-1} \phi_{i-1}(y_{i-1}) - a_i \phi_i(y_i))$$

$$= g(x,v) - \left( \prod_{j=1}^{n} \frac{a_j}{\tilde{a}_j} \right) \tilde{a}_n \phi_n(y_n).$$

Then

$$\frac{dV_0}{dt} = dx_0 \left( 1 - \lim_{v \to 0^+} \frac{g(x_0,v)}{g(x,v)} \right) \left( 1 - \frac{x}{x_0} \right) + g(x,v) \lim_{v \to 0^+} \frac{g(x_0,v)}{g(x,v)} - u \left( \prod_{j=1}^{n} \frac{a_j}{\tilde{a}_j} \right) v - \frac{pb}{r} \left( \prod_{j=1}^{n} \frac{a_j}{\tilde{a}_j} \right) z$$

$$= dx_0 \left( 1 - \frac{\partial g(x_0,0)/\partial v}{\partial g(x,0)/\partial v} \right) \left( 1 - \frac{x}{x_0} \right) + u \left( \prod_{j=1}^{n} \frac{a_j}{\tilde{a}_j} \right) \left( \frac{1}{u} \left( \prod_{j=1}^{n} \frac{\tilde{a}_j}{a_j} \right) \frac{g(x,v)}{v} \frac{\partial g(x_0,0)/\partial v}{\partial g(x,0)/\partial v} - 1 \right) v$$

$$- \frac{pb}{r} \left( \prod_{j=1}^{n} \frac{a_j}{\tilde{a}_j} \right) z. \tag{16}$$

From (i) of Condition C2, we have

$$\frac{dV_0}{dt} \le dx_0 \left( 1 - \frac{\partial g(x_0,0)/\partial v}{\partial g(x,0)/\partial v} \right) \left( 1 - \frac{x}{x_0} \right) + u \left( \prod_{j=1}^{n} \frac{a_j}{\tilde{a}_j} \right) (R_0^M - 1) v - \frac{pb}{r} \left( \prod_{j=1}^{n} \frac{a_j}{\tilde{a}_j} \right) z. \tag{17}$$

From (ii) of Condition C1, we get

$$\left( 1 - \frac{\partial g(x_0,0)/\partial v}{\partial g(x,0)/\partial v} \right) \left( 1 - \frac{x}{x_0} \right) \le 0,$$

where the equality occurs at $x = x_0$. Therefore, if $R_0^M \le 1$, then $\frac{dV_0}{dt} \le 0$ for all $x, v, z > 0$. One can easily show that $\frac{dV_0}{dt} = 0$ occurs at $Q_0$. Using LaSalle's invariance principle, we derive that $Q_0$ is GAS. $\square$

To prove the global stability of the two steady states $Q_1$ and $Q_2$, we need the following condition on the incidence rate function.

**Condition C4.**
$$\left( 1 - \frac{g(x,v_i)}{g(x,v)} \right) \left( \frac{g(x,v)}{g(x,v_i)} - \frac{v}{v_i} \right) \le 0, \quad x, v > 0, \ i = 1, 2$$

**Lemma 2.** Suppose that Conditions C1-C4 are satisfied and $R_0^M > 1$. Then $x_1, x_2, v_1, v_2$ exist satisfying

$$sgn(x_2 - x_1) = sgn(v_1 - v_2) = sgn(R_1^M - 1).$$

**Proof.** From Condition C1, for $x_1, x_2, v_1, v_2 > 0$, we have

$$(g(x_2, v_2) - g(x_1, v_2))(x_2 - x_1) > 0, \tag{18}$$

$$(g(x_1, v_2) - g(x_1, v_1))(v_2 - v_1) > 0. \tag{19}$$

Using Condition C4 with $i = 1$, $x = x_1$ and $v = v_2$ we get

$$(g(x_1, v_2)v_1 - g(x_1, v_1)v_2)(g(x_1, v_2) - g(x_1, v_1)) < 0. \tag{20}$$

It follows from inequality (19) that

$$(g(x_1, v_2)v_1 - g(x_1, v_1)v_2)(v_1 - v_2) > 0. \tag{21}$$

First, we claim $sgn(x_2 - x_1) = sgn(v_1 - v_2)$. Suppose this is not true, i.e., $sgn(x_2 - x_1) = sgn(v_2 - v_1)$. Using the conditions of the steady states $Q_1$ and $Q_2$ we have

$$(\lambda - dx_2) - (\lambda - dx_1) = g(x_2, v_2) - g(x_1, v_1)$$

$$= (g(x_2, v_2) - g(x_1, v_2)) + (g(x_1, v_2) - g(x_1, v_1)).$$

Therefore, from inequalities (18) and (19) we get:

$$sgn\,(x_1 - x_2) = sgn\,(x_2 - x_1),$$

which leads to a contradiction. Thus, $sgn\,(x_2 - x_1) = sgn\,(v_1 - v_2)$. Using the steady state conditions for $Q_1$ we have $\frac{1}{u}\left(\prod_{j=1}^{n}\frac{\tilde{a}_j}{a_j}\right)\frac{g(x_1,v_1)}{v_1} = 1$, then

$$
\begin{aligned}
R_1^M - 1 &= \frac{1}{u}\left(\prod_{j=1}^{n}\frac{\tilde{a}_j}{a_j}\right)\frac{g(x_2,v_2)}{v_2} - \frac{1}{u}\left(\prod_{j=1}^{n}\frac{\tilde{a}_j}{a_j}\right)\frac{g(x_1,v_1)}{v_1} \\
&= \frac{1}{u}\left(\prod_{j=1}^{n}\frac{\tilde{a}_j}{a_j}\right)\left[\frac{1}{v_2}\left(g(x_2,v_2) - g(x_1,v_2)\right) + \frac{1}{v_1 v_2}\left(g(x_1,v_2)v_1 - g(x_1,v_1)v_2\right)\right].
\end{aligned}
$$

Thus, from inequalities (18) and (21) we get $sgn(R_1^M - 1) = sgn(v_1 - v_2)$. $\square$

**Theorem 2.** Assume that Conditions C1-C4 are satisfied. If $R_1^M \le 1 < R_0^M$, then $Q_1$ is GAS in $\Theta$.

**Proof.** Define:

$$
V_1(x, y_1, ..., y_n, v, z) = x - x_1 - \int_{x_1}^{x}\frac{g(x_1,v_1)}{g(\eta,v_1)}d\eta + \sum_{i=1}^{n}\left(\prod_{j=1}^{i-1}\frac{a_j}{\tilde{a}_j}\right)\left(y_i - y_{i,1} - \int_{y_{i,1}}^{y_i}\frac{\phi_i(y_{i,1})}{\phi_i(\eta)}d\eta\right)
$$
$$
+ \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)v_1 H\left(\frac{v}{v_1}\right) + \frac{p}{r}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)z. \tag{22}
$$

We note that, $V_1$ is positive and reaches its global minimum at $Q_1$. Calculating the time derivative of $V_1$ along the trajectories of system (1)-(5), we obtain

$$
\begin{aligned}
\frac{dV_1}{dt} &= \left(1 - \frac{g(x_1,v_1)}{g(x,v_1)}\right)(\lambda - dx - g(x,v)) + \left(1 - \frac{\phi_1(y_{1,1})}{\phi_1(y_1)}\right)(g(x,v) - a_1\phi_1(y_1)) \\
&+ \sum_{i=2}^{n}\left(\prod_{j=1}^{i-1}\frac{a_j}{\tilde{a}_j}\right)\left(1 - \frac{\phi_i(y_{i,1})}{\phi_i(y_i)}\right)(\tilde{a}_{i-1}\phi_{i-1}(y_{i-1}) - a_i\phi_i(y_i)) \\
&+ \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)\left(1 - \frac{v_1}{v}\right)(\tilde{a}_n\phi_n(y_n) - uv - pzv) + \frac{p}{r}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)(rzv - bz). \tag{23}
\end{aligned}
$$

We have

$$
\sum_{i=2}^{n}\left(\prod_{j=1}^{i-1}\frac{a_j}{\tilde{a}_j}\right)(\tilde{a}_{i-1}\phi_{i-1}(y_{i-1}) - a_i\phi_i(y_i)) = a_1\phi_1(y_1) - \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)\tilde{a}_n\phi_n(y_n). \tag{24}
$$

Then,

$$
\begin{aligned}
\frac{dV_1}{dt} &= \left(1 - \frac{g(x_1,v_1)}{g(x,v_1)}\right)(\lambda - dx) + g(x,v)\frac{g(x_1,v_1)}{g(x,v_1)} - \frac{\phi_1(y_{1,1})g(x,v)}{\phi_1(y_1)} \\
&+ a_1\phi_1(y_{1,1}) - \sum_{i=2}^{n}\left(\prod_{j=1}^{i-1}\frac{a_j}{\tilde{a}_j}\right)\frac{\tilde{a}_{i-1}\phi_i(y_{i,1})\phi_{i-1}(y_{i-1})}{\phi_i(y_i)} \\
&+ \sum_{i=2}^{n}\left(\prod_{j=1}^{i-1}\frac{a_j}{\tilde{a}_j}\right)a_i\phi_i(y_{i,1}) - \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)uv - \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)\tilde{a}_n\frac{v_1\phi_n(y_n)}{v} \\
&+ \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)uv_1 + \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)pv_1 z - \frac{pb}{r}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)z. \tag{25}
\end{aligned}
$$

Using the steady state conditions for $Q_1$:

$$\lambda = dx_1 + g(x_1, v_1),$$

$$g(x_1, v_1) = \left( \prod_{j=1}^{i} \frac{a_j}{\tilde{a}_j} \right) \tilde{a}_i \phi_i(y_{i,1}) = \left( \prod_{j=1}^{n} \frac{a_j}{\tilde{a}_j} \right) uv_1, \ i = 1, ..., n.$$

We obtain

$$
\begin{aligned}
\frac{dV_1}{dt} &= \left( 1 - \frac{g(x_1, v_1)}{g(x, v_1)} \right) (dx_1 - dx) + g(x_1, v_1) \left( 1 - \frac{g(x_1, v_1)}{g(x, v_1)} \right) + g(x_1, v_1) \frac{g(x, v)}{g(x, v_1)} \\
&\quad - g(x_1, v_1) \frac{\phi_1(y_{1,1}) g(x, v)}{\phi_1(y_1) g(x_1, v_1)} + (n+1) g(x_1, v_1) - g(x_1, v_1) \sum_{i=2}^{n} \frac{\phi_i(y_{i,1}) \phi_{i-1}(y_{i-1})}{\phi_i(y_i) \phi_{i-1}(y_{i-1,1})} \\
&\quad - g(x_1, v_1) \frac{v}{v_1} - g(x_1, v_1) \frac{v_1 \phi_n(y_n)}{v \phi_n(y_{n,1})} + p \left( \prod_{j=1}^{n} \frac{a_j}{\tilde{a}_j} \right) \left( v_1 - \frac{b}{r} \right) z.
\end{aligned}
\tag{26}
$$

We can rewrite Eq. (26) as follows

$$
\begin{aligned}
\frac{dV_1}{dt} &= dx_1 \left( 1 - \frac{g(x_1, v_1)}{g(x, v_1)} \right) \left( 1 - \frac{x}{x_1} \right) + g(x_1, v_1) \left[ \frac{g(x, v)}{g(x, v_1)} - \frac{v}{v_1} \right] \\
&\quad + g(x_1, v_1) \left[ (n+2) - \frac{g(x_1, v_1)}{g(x, v_1)} - \frac{\phi_1(y_{1,1}) g(x, v)}{\phi_1(y_1) g(x_1, v_1)} \right. \\
&\quad \left. - \sum_{i=2}^{n} \frac{\phi_i(y_{i,1}) \phi_{i-1}(y_{i-1})}{\phi_i(y_i) \phi_{i-1}(y_{i-1,1})} - \frac{v_1 \phi_n(y_n)}{v \phi_n(y_{n,1})} \right] + p \left( \prod_{j=1}^{n} \frac{a_j}{\tilde{a}_j} \right) (v_1 - v_2) z \\
&= dx_1 \left( 1 - \frac{g(x_1, v_1)}{g(x, v_1)} \right) \left( 1 - \frac{x}{x_1} \right) + g(x_1, v_1) \left( 1 - \frac{g(x, v_1)}{g(x, v)} \right) \left( \frac{g(x, v)}{g(x, v_1)} - \frac{v}{v_1} \right) \\
&\quad + g(x_1, v_1) \left[ (n+3) - \frac{g(x_1, v_1)}{g(x, v_1)} - \frac{\phi_1(y_{1,1}) g(x, v)}{\phi_1(y_1) g(x_1, v_1)} - \sum_{i=2}^{n} \frac{\phi_i(y_{i,1}) \phi_{i-1}(y_{i-1})}{\phi_i(y_i) \phi_{i-1}(y_{i-1,1})} \right. \\
&\quad \left. - \frac{v_1 \phi_n(y_n)}{v \phi_n(y_{n,1})} - \frac{v g(x, v_1)}{v_1 g(x, v)} \right] + p \left( \prod_{j=1}^{n} \frac{a_j}{\tilde{a}_j} \right) (v_1 - v_2) z.
\end{aligned}
\tag{27}
$$

From Conditions C1 and C4, we get that, the first and second terms of Eq. (27) are less than or equal to zero. Since the geometrical mean is less than or equal to the arithmetical mean, then $(n+3) \leq \frac{g(x_1, v_1)}{g(x, v_1)} + \frac{\phi_1(y_{1,1}) g(x, v)}{\phi_1(y_1) g(x_1, v_1)} + \sum_{i=2}^{n} \frac{\phi_i(y_{i,1}) \phi_{i-1}(y_{i-1})}{\phi_i(y_i) \phi_{i-1}(y_{i-1,1})} + \frac{v_1 \phi_n(y_n)}{v \phi_n(y_{n,1})} + \frac{v g(x, v_1)}{v_1 g(x, v)}$. Lemma 2 implies that, if $R_1^M \leq 1$, then $v_1 \leq v_2$. It follows that, $\frac{dV_1}{dt} \leq 0$ for all $x, y_i, v, z > 0$, $i = 1, ..., n$. The solutions of system (1)-(5) are limited to $\Omega$, the largest invariant subset of $\left\{ (x, y_1, ..., y_n, v, z) : \frac{dV_1}{dt} = 0 \right\}$. We have $\frac{dV_1}{dt} = 0$ at the singleton $\{Q_1\}$. Thus, the global asymptotic stability of the endemic steady state without humoral immune response $Q_1$ follows from LaSalle's invariance principle. $\square$

**Theorem 3.** Let Conditions C1-C4 are satisfied and $R_1^M > 1$, then $Q_2$ is GAS in $\overset{\circ}{\Theta}$.

**Proof.** We construct a Lyapunov functional as follows:

$$
\begin{aligned}
V_2(x, y_1, ..., y_n, v, z) &= x - x_2 - \int_{x_2}^{x} \frac{g(x_2, v_2)}{g(\eta, v_2)} d\eta + \sum_{i=1}^{n} \left( \prod_{j=1}^{i-1} \frac{a_j}{\tilde{a}_j} \right) \left( y_i - y_{i,2} - \int_{y_{i,2}}^{y_i} \frac{\phi_i(y_{i,2})}{\phi_i(\eta)} d\eta \right) \\
&\quad + \left( \prod_{j=1}^{n} \frac{a_j}{\tilde{a}_j} \right) v_2 H \left( \frac{v}{v_2} \right) + \frac{p}{r} \left( \prod_{j=1}^{n} \frac{a_j}{\tilde{a}_j} \right) z_2 H \left( \frac{z}{z_2} \right).
\end{aligned}
\tag{28}
$$

Note that $V_2 > 0$ for all $x, y_1, ..., y_n, v, z > 0$ and $V_2(x_2, y_{1,2}, ..., y_{n,2}, v_2, z_2) = 0$. Function $V_2$ satisfies:

$$\frac{dV_2}{dt} = \left(1 - \frac{g(x_2, v_2)}{g(x, v_2)}\right)(\lambda - dx - g(x, v)) + \left(1 - \frac{\phi_1(y_{1,2})}{\phi_1(y_1)}\right)(g(x, v) - a_1\phi_1(y_1))$$

$$+ \sum_{i=2}^{n}\left(\prod_{j=1}^{i-1}\frac{a_j}{\tilde{a}_j}\right)\left(1 - \frac{\phi_i(y_{i,2})}{\phi_i(y_i)}\right)(\tilde{a}_{i-1}\phi_{i-1}(y_{i-1}) - a_i\phi_i(y_i))$$

$$+ \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)\left(1 - \frac{v_2}{v}\right)(\tilde{a}_n\phi_n(y_n) - uv - pzv) + \frac{p}{r}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)\left(1 - \frac{z_2}{z}\right)(rzv - bz). \tag{29}$$

Using Eq. (24), we get

$$\frac{dV_2}{dt} = \left(1 - \frac{g(x_2, v_2)}{g(x, v_2)}\right)(\lambda - dx) + g(x, v)\frac{g(x_2, v_2)}{g(x, v_2)} - \frac{\phi_1(y_{1,2})g(x, v)}{\phi_1(y_1)} + a_1\phi_1(y_{1,2})$$

$$- \sum_{i=2}^{n}\left(\prod_{j=1}^{i-1}\frac{a_j}{\tilde{a}_j}\right)\tilde{a}_{i-1}\frac{\phi_i(y_{i,2})\phi_{i-1}(y_{i-1})}{\phi_i(y_i)} + \sum_{i=2}^{n}\left(\prod_{j=1}^{i-1}\frac{a_j}{\tilde{a}_j}\right)a_i\phi_i(y_{i,2})$$

$$- \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)uv - \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)\tilde{a}_n\frac{v_2\phi_n(y_n)}{v} + \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)uv_2$$

$$+ \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)pv_2z - \frac{pb}{r}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)z - p\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)vz_2 + \frac{pb}{r}\left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)z_2. \tag{30}$$

Using the steady state conditions for $Q_2$:

$$\lambda = dx_2 + g(x_2, v_2), \quad v_2 = \frac{b}{r},$$

$$g(x_2, v_2) = \left(\prod_{j=1}^{i}\frac{a_j}{\tilde{a}_j}\right)\tilde{a}_i\phi_i(y_{i,2}) = \left(\prod_{j=1}^{n}\frac{a_j}{\tilde{a}_j}\right)(uv_2 + pv_2z_2), \quad i = 1, ..., n,$$

we get

$$\frac{dV_2}{dt} = \left(1 - \frac{g(x_2, v_2)}{g(x, v_2)}\right)(dx_2 - dx) + g(x_2, v_2)\left(1 - \frac{g(x_2, v_2)}{g(x, v_2)}\right) + g(x_2, v_2)\frac{g(x, v)}{g(x, v_2)}$$

$$- g(x_2, v_2)\frac{\phi_1(y_{1,2})g(x, v)}{\phi_1(y_1)g(x_2, v_2)} + (n+1)g(x_2, v_2) - g(x_2, v_2)\sum_{i=2}^{n}\frac{\phi_i(y_{i,2})\phi_{i-1}(y_{i-1})}{\phi_i(y_i)\phi_{i-1}(y_{i-1,2})}$$

$$- g(x_2, v_2)\frac{v}{v_2} - g(x_2, v_2)\frac{v_2\phi_n(y_n)}{v\phi_n(y_{n,2})}. \tag{31}$$

We can rewrite Eq. (31) as follows:

$$\frac{dV_2}{dt} = dx_2\left(1 - \frac{g(x_2, v_2)}{g(x, v_2)}\right)\left(1 - \frac{x}{x_2}\right) + g(x_2, v_2)\left(1 - \frac{g(x, v_2)}{g(x, v)}\right)\left(\frac{g(x, v)}{g(x, v_2)} - \frac{v}{v_2}\right)$$

$$+ g(x_2, v_2)\left[(n+3) - \frac{g(x_2, v_2)}{g(x, v_2)} - \frac{\phi_1(y_{1,2})g(x, v)}{\phi_1(y_1)g(x_2, v_2)} - \sum_{i=2}^{n}\frac{\phi_i(y_{i,2})\phi_{i-1}(y_{i-1})}{\phi_i(y_i)\phi_{i-1}(y_{i-1,2})}\right.$$

$$\left. - \frac{v_2\phi_n(y_n)}{v\phi_n(y_{n,2})} - \frac{vg(x, v_2)}{v_2g(x, v)}\right]. \tag{32}$$

We note from Conditions C1 and C4 and the relationship between the arithmetical and geometrical means that, we obtain $\frac{dV_2}{dt} \leq 0$ for all $x, y_1, ..., y_n, v, z > 0$. The solutions of model (1)-(5) are limited to $\Lambda$, the largest invariant subset of $\{(x, y_1, ..., y_n, v, z) : \frac{dV_2}{dt} = 0\}$. It is easy to see that $\frac{dV_2}{dt} = 0$ occurs at $Q_2$. The global asymptotic stability of $Q_2$ follows from LaSalle's invariance principle. $\square$

# 6   Example and numerical simulations

In this section, we introduce an example and perform some numerical simulations to confirm our theoretical results. By using the Lyapunov direct method, we have established a set of conditions on the functions $g(x, v)$ and $\phi_i(y_i)$ and on the parameters $R_0^M$ and $R_1^M$ ensuring the global asymptotic stability of the steady states of model (1)-(5). We consider the following model with two stages (i.e. $n = 2$):

$$\dot{x} = \lambda - dx - \frac{\pi xv}{(1 + \gamma x)(1 + \delta v)}, \tag{33}$$

$$\dot{y}_1 = \frac{\pi xv}{(1 + \gamma x)(1 + \delta v)} - a_1 y_1, \tag{34}$$

$$\dot{y}_2 = \tilde{a}_1 y_1 - a_2 y_2, \tag{35}$$

$$\dot{v} = \tilde{a}_2 y_2 - pzv - uv, \tag{36}$$

$$\dot{z} = rzv - bz, \tag{37}$$

where $\pi \in (0, \infty)$ and $\gamma, \delta \in [0, \infty)$. In this example we have

$$\phi_i(y_i) = y_i, \ i = 1, ..., n, \ g(x, v) = \frac{\pi xv}{(1 + \gamma x)(1 + \delta v)},$$

which guarantee that Condition C3 holds true. Now, we verify Conditions C1, C2 and C4. Clearly, $g(x, v) > 0$, $g(0, v) = g(x, 0) = 0$ for all $x, v \in (0, \infty)$, and

$$\frac{\partial g(x, v)}{\partial x} = \frac{\pi v}{(1 + \gamma x)^2 (1 + \delta v)}, \qquad \frac{\partial g(x, v)}{\partial v} = \frac{\pi x}{(1 + \gamma x)(1 + \delta v)^2}, \qquad \frac{\partial g(x, 0)}{\partial v} = \frac{\pi x}{1 + \gamma x}.$$

Then, for all $x, v \in (0, \infty)$, we have $\frac{\partial g(x,v)}{\partial x} > 0$, $\frac{\partial g(x,v)}{\partial v} > 0$ and $\frac{\partial g(x,0)}{\partial v} > 0$. Therefore Condition C1 is satisfied. We have also

$$g(x, v) = \frac{\pi xv}{(1 + \gamma x)(1 + \delta v)} \leq \frac{\pi xv}{1 + \gamma x} = v\frac{\partial g(x, 0)}{\partial v},$$

$$\left(\frac{\partial g(x, 0)}{\partial v}\right)' = \frac{\pi}{(1 + \gamma x)^2} > 0 \text{ for all } x > 0.$$

It follows that, C2 is satisfied. Moreover,

$$\left(1 - \frac{g(x, v_i)}{g(x, v)}\right)\left(\frac{g(x, v)}{g(x, v_i)} - \frac{v}{v_i}\right) = -\frac{\delta (v - v_i)^2}{v_i (1 + \delta v)(1 + \delta v_i)} < 0 \text{ for all } v, v_i \in (0, \infty), \ i = 1, 2.$$

Thus, C4 is satisfied and the global stability results demonstrated in Theorems 1-3 are guaranteed. The parameters $R_0^M$ and $R_1^M$ are given by:

$$R_0^M = \frac{\tilde{a}_1 \tilde{a}_2 \pi}{a_1 a_2 u}\frac{x_0}{1 + \gamma x_0}, \qquad R_1^M = \frac{\tilde{a}_1 \tilde{a}_2 \pi}{a_1 a_2 u}\frac{x_2}{(1 + \gamma x_2)(1 + \delta v_2)}. \tag{38}$$

Now, we will perform some numerical simulations for the model (33)-(37). The values of some parameters of the example are listed in Table 1. The other parameters $\pi$, $r$ and $\gamma$ will be varied. All computations are carried out by MATLAB.

We are interested to study the following cases:

**Case (A): Effect of $\pi$ and $r$ on the stability of steady states:**

In this case, we have chosen three different initial conditions:

**IC(1):** $x(0) = 400$, $y_1(0) = y_2(0) = 1$, $v(0) = 0.2$ and $z(0) = 0.5$,

**IC(2):** $x(0) = 600$, $y_1(0) = y_2(0) = 2$, $v(0) = 0.5$ and $z(0) = 1$,

**IC(3):** $x(0) = 800$, $y_1(0) = 5$, $y_2(0) = 3$, $v(0) = 0.9$ and $z(0) = 1.5$.

The evolution of the dynamics of model (33)-(37) was observed over a time interval $[0, 500]$. We fix the value of $\gamma = 0.5$ and change the values of parameters $\pi$ and $r$ to get three sets as follows:

Table 1: The values of the parameters of model (33)-(37).

| Parameter | Value | Parameter | Value | Parameter | Value |
|-----------|-------|-----------|-------|-----------|-------|
| $\lambda$ | 10 | $a_1$ | 1 | $p$ | 0.5 |
| $d$ | 0.01 | $a_2$ | 1.5 | $r$ | Varied |
| $\beta$ | Varied | $\tilde{a}_1$ | 0.5 | $b$ | 0.3 |
| $\gamma$ | Varied | $\tilde{a}_2$ | 1 | | |
| $\delta$ | 0.1 | $u$ | 3 | | |

**Set (I)**: We choose, $\pi = 4$ and $r = 0.3$. Using the values of the parameters given in Table 1, we compute $R_0^M = 0.89 < 1$ and $R_1^M = 0.80 < 1$, which means that the system has a disease-free steady state $Q_0$ and it is GAS based on Theorem 1. Evidently, Figures 1-5 show that, the states of the system eventually approach $Q_0 = (1000, 0, 0, 0, 0)$ for the three initial conditions IC(1)-IC(3). This case corresponds to the healthy state where the viruses are cleared.

**Set (II)**: We take $\pi = 5$ and $r = 0.3$. With such choice we have, $R_1^M = 0.99 < 1 < R_0^M = 1.11$. Consequently, Lemma 1 and Theorem 2 state that, $Q_1$ exists and it is GAS. Figures 1-5 show that the numerical simulations illustrate our theoretical results given in Theorem 2. We observe that, the trajectory of the system will converge to $Q_1 = (140.43, 8.60, 2.87, 0.96, 0)$ for the three initial conditions IC(1)-IC(3). This case corresponds to a chronic infection but with inactive immune response.

**Set (III)**: We choose, $\pi = 5$ and $r = 1$. Then, we calculate $R_0^M = 1.11 > 1$ and $R_1^M = 1.08 > 1$, this means that, the system has three steady states $Q_0$, $Q_1$ and $Q_2$. Thus, from Theorem 3, $Q_2$ is GAS. From Figures 1-5, we observe a consistency between the numerical results and theoretical results of Theorem 3. We observe that, the trajectory of the system show oscillating behavior for a period before reaching $Q_2 = (709.56, 2.90, 0.97, 0.3, 0.45)$, in the same time frame for the three initial conditions IC(1)-IC(3).

**Case (B): Effect of $\gamma$ on the stability of the steady states**

Let us consider $\pi$ and $r$ be fixed. In this case, we take the values of $\pi = 5$ and $r = 1$, and consider different values of $\gamma$. Here we take the initial condition as given in IC(1), while the evolution of the dynamics of model (33)-(37) was observed over a time interval $[0, 600]$. Table 2 contains the values of the bifurcation parameters $R_0^M$ and $R_1^M$ with different values of $\gamma$ of model (33)-(37).

Table 2: The values of the threshold parameters $R_0^M$ and $R_1^M$ with different values of $\gamma$ of model (33)-(37).

| Different values of $\gamma$ | $R_0^M$ | $R_1^M$ | The equilibria |
|------------------------------|---------|---------|----------------|
| 0.30 | 1.85 | 1.79 | $Q_2 = (517.67, 4.82, 1.61, 0.3, 4.72)$ |
| 0.40 | 1.39 | 1.34 | $Q_2 = (637.35, 3.63, 1.21, 0.3, 2.06)$ |
| 0.54 | 1.03 | 0.996 | $Q_1 = (763.16, 2.37, 0.79, 0.26, 0)$ |
| 0.55 | 1.01 | 0.98 | $Q_1 = (926.89, 0.73, 0.24, 0.08, 0)$ |
| 0.60 | 0.92 | 0.90 | $Q_0 = (1000, 0, 0, 0, 0)$ |
| 0.70 | 0.79 | 0.77 | $Q_0 = (1000, 0, 0, 0, 0)$ |

Table 2 and Figures 6-10 show that, when $\gamma$ is increased, the infection rate is decreased which leads to an increase in the concentration of the uninfected cells and a decrease on the concentrations of the (first/second) stage of infected cells, free viruses and B cells.

**Case (C): Effect of the multiple stages of infected cells on the dynamics of virus dynamics:**

To show the effect of multiple stages of infected cells on the dynamical behavior of the virus, we consider

the following model with single stage of infected cells and compare it with model (33)-(37):

$$\dot{x} = \lambda - dx - \frac{\pi x v}{(1 + \gamma x)(1 + \delta v)}, \tag{39}$$

$$\dot{y}_1 = \frac{\pi x v}{(1 + \gamma x)(1 + \delta v)} - a_1 y_1, \tag{40}$$

$$\dot{v} = \tilde{a}_1 y_1 - pzv - uv, \tag{41}$$

$$\dot{z} = rzv - bz. \tag{42}$$

Consequently, the bifurcation parameters for this system are given by:

$$R_0^{single} = \frac{\tilde{a}_1 \pi}{a_1 u} \frac{x_0}{1 + \gamma x_0}, \qquad R_1^{single} = \frac{\tilde{a}_1 \pi}{a_1 u} \frac{x_2}{(1 + \gamma x_2)(1 + \delta v_2)}. \tag{43}$$

Since $\tilde{a}_i < a_i$, then from Eqs. (38) and (43) we have

$$R_0^M = \frac{\tilde{a}_1 \tilde{a}_2 \pi}{a_1 a_2 u} \frac{x_0}{1 + \gamma x_0} < \frac{\tilde{a}_1 \pi}{a_1 u} \frac{x_0}{1 + \gamma x_0} = R_0^{single},$$

$$R_1^M = \frac{\tilde{a}_1 \tilde{a}_2 \pi}{a_1 a_2 u} \frac{x_2}{(1 + \gamma x_2)(1 + \delta v_2)} < \frac{\tilde{a}_1 \pi}{a_1 u} \frac{x_2}{(1 + \gamma x_2)(1 + \delta v_2)} = R_1^{single}.$$

Here we consider the following initial condition: $x(0) = 400$, $y_1(0) = 0.5$, $y_2(0) = 1$, $v(0) = 0.2$ and $z(0) = 0.5$. The evolution of the dynamics of models (33)-(37) and (39)-(42) was observed over a time interval $[0, 600]$. Let us consider the values of parameters listed in Table 1 and choose the values $\pi = 3.5$, $r = 1.5$ and $\gamma = 0.5$. By calculating the bifurcation parameters for systems (33)-(37) and (39)-(42), we obtain

$$R_0^M = 0.78 < 1.16 = R_0^{single}, \qquad R_1^M = 0.76 < 1.14 = R_1^{single}.$$

Therefore, with the same values of the parameters, the steady state $Q_0$ is stable for system (33)-(37) but unstable for system (39)-(42). The presence of multiple stages of infected cells reduces the infection progress. Figures 11-14 show a comparison between the evolution of the uninfected cells, infected cells, free virus particles and B cells of the two systems (33)-(37) and (39)-(42). We observe that, the concentration of uninfected cells of the model with three stages of infected cells is larger than that of system with only one single stage of infected cells (see Figures 11), while the concentrations of first stage of infected cells, viruses and B cells with three stages are less than that of system with a single stage of infected cells (see Figures 12-14). From a biological point of view, the multiple stages of infected cells plays a similar role as antiviral treatment in eliminating the virus. We observe that, if the number of stages of infected cells is increased, then the viral replication is suppressed and the viruses can be cleared from the body. This give us some suggestions on new drugs to increase the number of stages of infected cells.

# 7    Conclusion

We have studied a general virus dynamics model with humoral immunity. We have assumed that the infected cells passes through $n$-stages to produce mature viruses. We have obtained two bifurcation parameters, the basic reproduction number and the humoral immunity number. We have established a set of sufficient conditions which guarantee the global stability of the model. The global asymptotic stability of the three steady states, $Q_0$, $Q_1$ and $Q_2$ has been investigated by constructing Lyapunov functionals and using LaSalle's invariance principle. To support our theoretical results, we have presented an example and conducted some numerical simulations.

# 8    Acknowledgment

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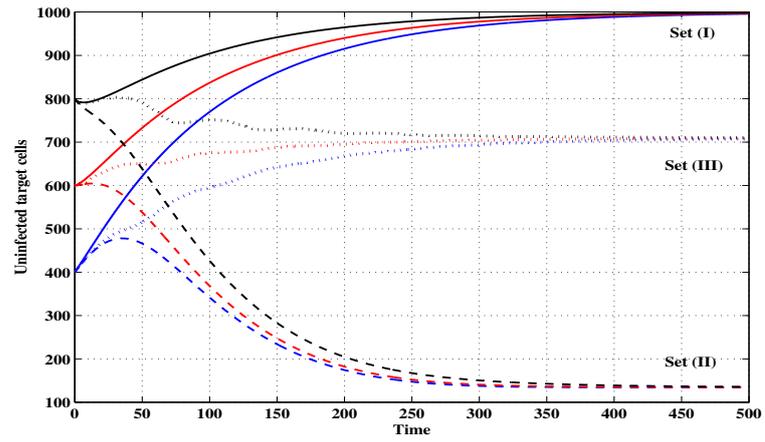
Figure 1: The uninfected cells for model (33)-(37).
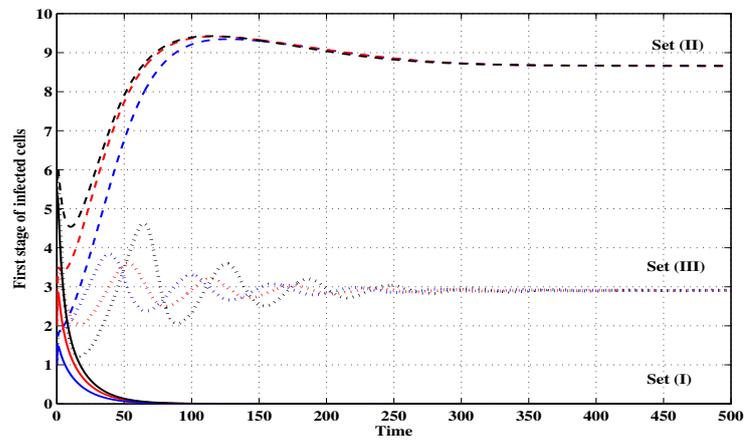


Figure 2: The first stage infected cells for model (33)-(37).
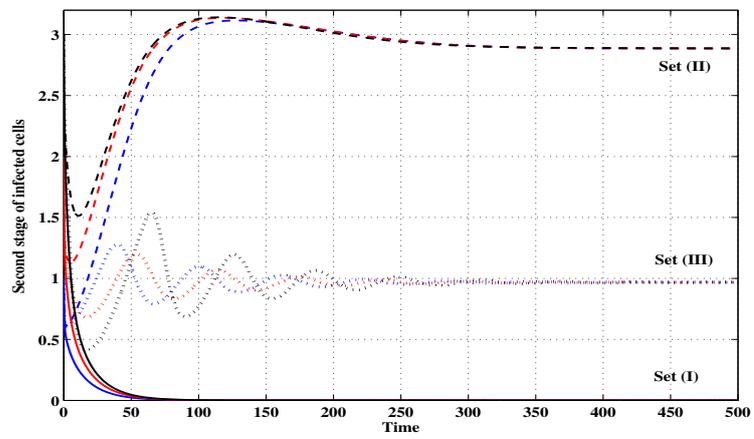


Figure 3: The second stage infected cells for model (33)-(37).
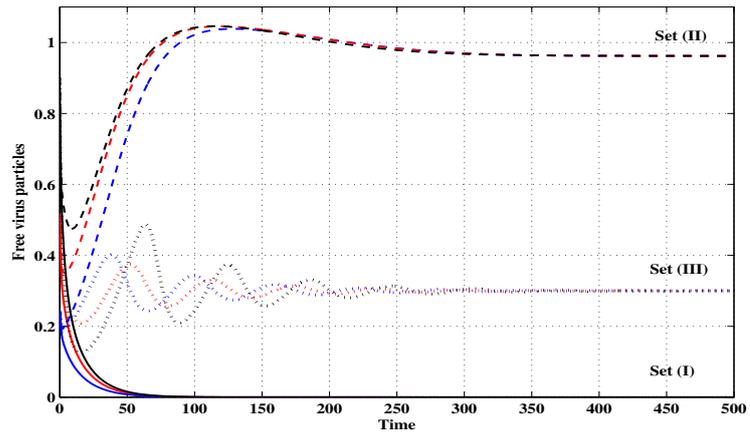
Figure 4: The free virus particles for model (33)-(37).
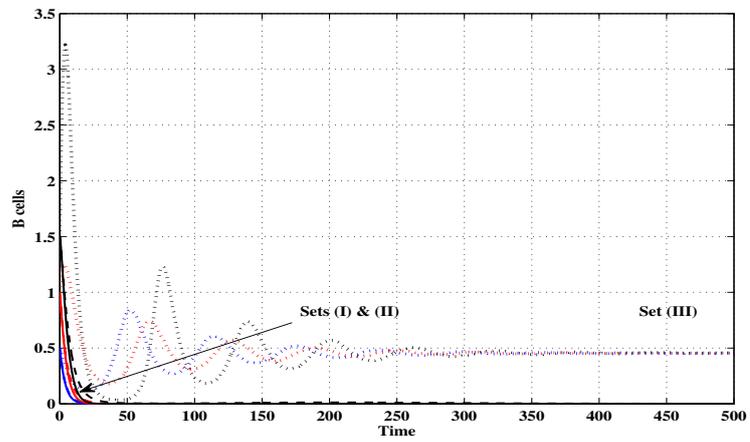


Figure 5: The B cells for model (33)-(37).
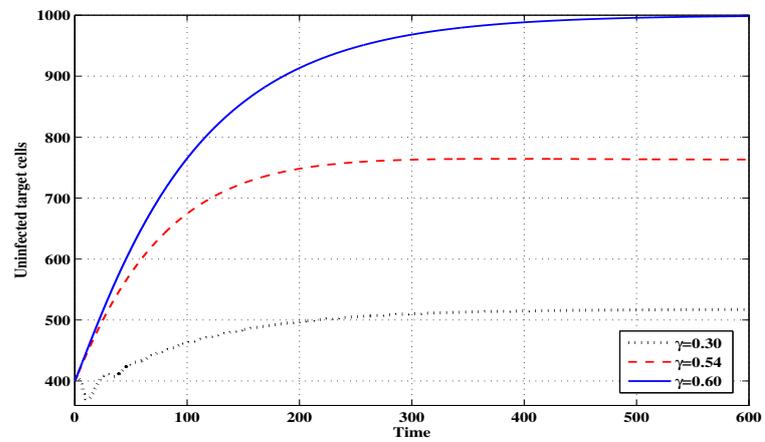


Figure 6: The uninfected target cells for model (33)-(37) under different values of $\gamma$.

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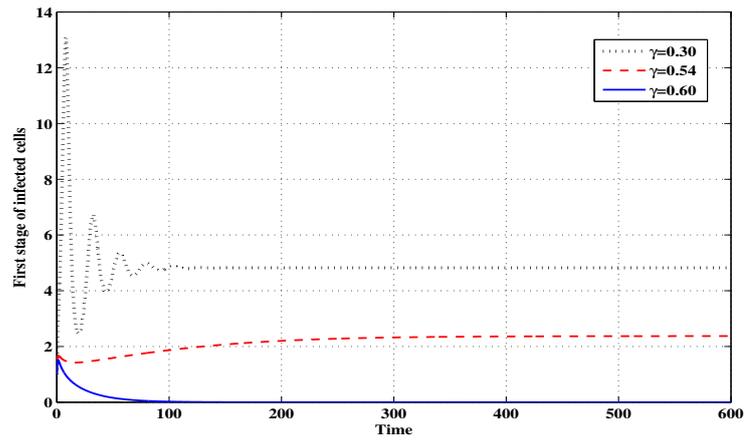
Figure 7: The first stage infected cells for model (33)-(37) under different values of $\gamma$.
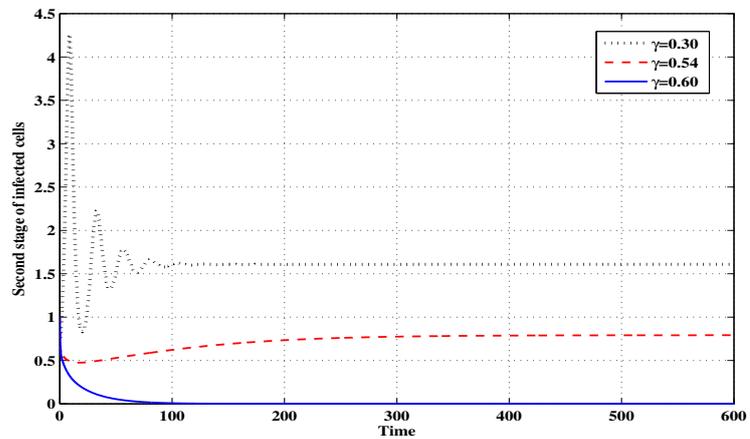


Figure 8: The second stage infected cells for model (33)-(37) under different values of $\gamma$.
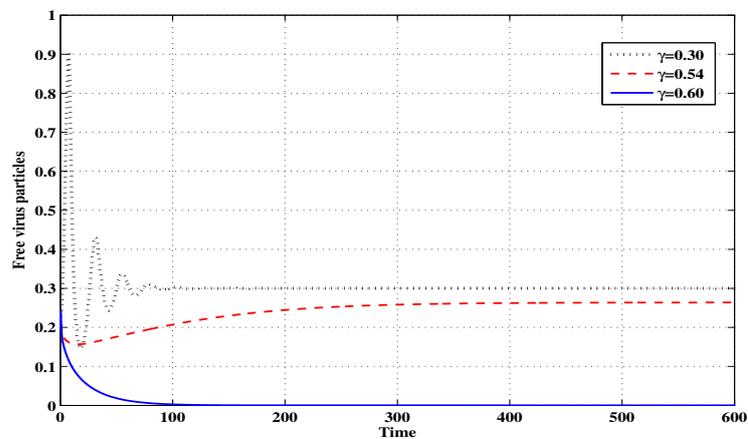


Figure 9: The free virus particles for model (33)-(37) under different values of $\gamma$.
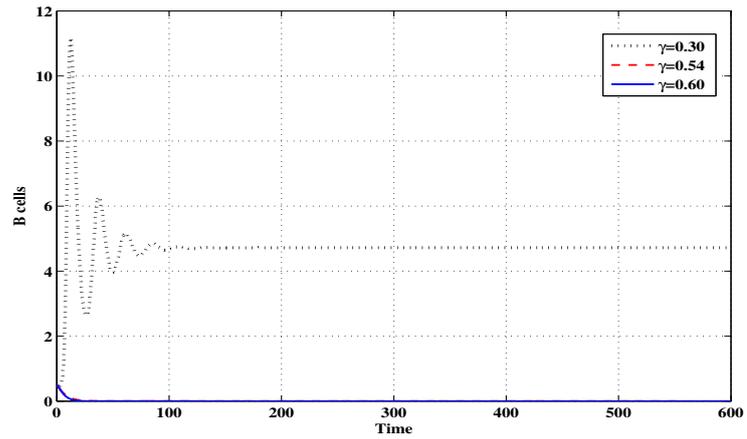
Figure 10: The B cells for model (33)-(37) under different values of $\gamma$.
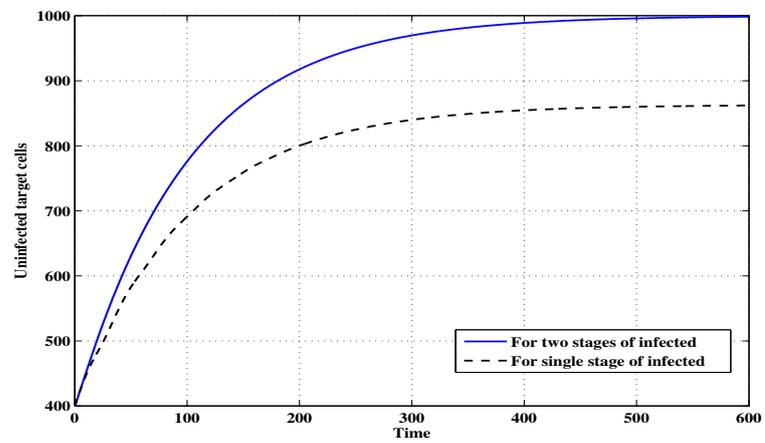


Figure 11: Comparison on the concentration of the uninfected cells for systems (33)-(37) and (39)-(42).
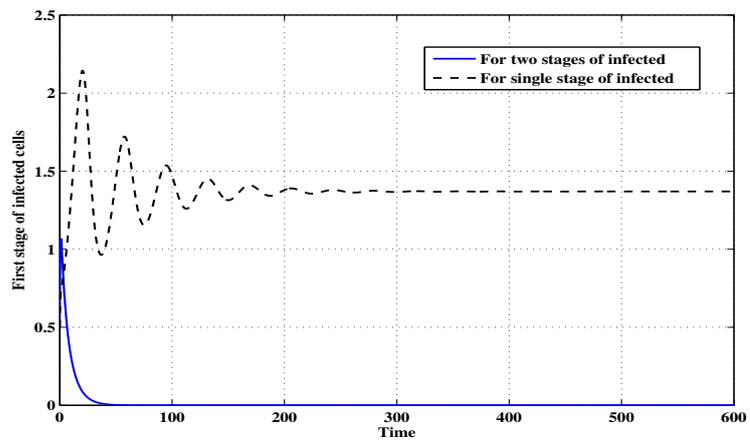


Figure 12: Comparisons on the concentration of the first stage of infected cells for systems (33)-(37) and (39)-(42).
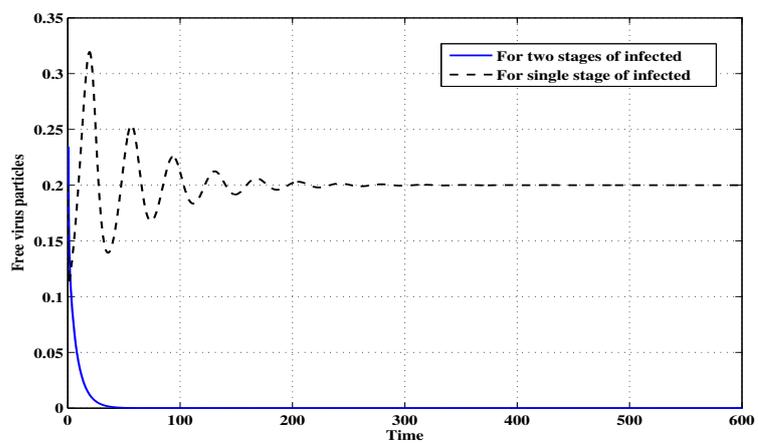
Figure 13: Comparisons on the concentration of the free virus particles for systems (33)-(37) and (39)-(42).
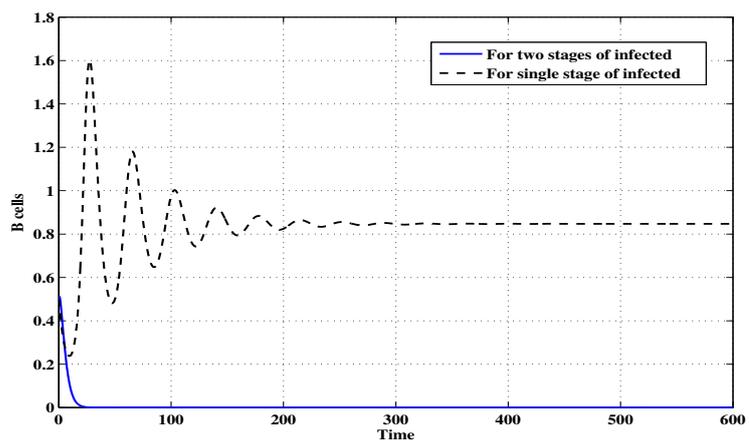


Figure 14: Comparisons on the concentration of the B cells for systems (33)-(37) and (39)-(42).

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# On the dynamics of a certain four-order fractional difference equations

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**Abstract:** This paper is concerned with the following rational recursive sequences

$$x_{n+1} = \frac{x_{n-1}x_{n-2}}{A + By_{n-3}}, \; y_{n+1} = \frac{y_{n-1}y_{n-2}}{C + Dx_{n-3}}, \, n = 0, 1, \cdots,$$

where the parameters $A, B, C, D$ are positive constants. The initial condition $x_{-3}, x_{-2}, x_{-1}, x_0$ and $y_{-3}, y_{-2}, y_{-1}, y_0$ are arbitrary nonnegative real numbers. We give sufficient conditions under which the equilibrium $(0,0)$ of the system is globally asymptotically stable, which extends and includes corresponding results obtained in the cited references [12-17]. Moreover, the asymptotic behavior of others equilibrium points is also studied. Our approach to the problem is based on new variational iteration method for the more general nonlinear difference equations and inequality skills as well as the linearization techniques.

**Keywords:** recursive sequences; equilibrium point; asymptotical stability; positive solutions.

## 1. Introduction

Nonlinear Difference equations have been studied because they model numerous real life problems in biology, ecology, physics, economics and so forth [1-5]. Today, with the dramatically development of computer-based computational techniques, difference equations are found to be much appropriate mathematical representations for computer simulation, experiment and computation, which play an important role in realistic applications [6]. Therefore, recently there has been an increasing interest in the study of qualitative analysis of rational difference equations. And the present cardinal problem of

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asymptotic behavior of solutions for a rational difference equation has received extensive attention from researchers (see, e.g., [7-11] and the references therein).

Elabbasy [12] obtained the form of the solutions of the following rational difference system

$$x_{n+1} = \frac{x_{n-1}}{\pm 1 + x_{n-1}y_n}, \quad y_{n+1} = \frac{y_{n-1}}{\mp 1 + y_{n-1}x_n} \tag{1.1}$$

with nonzero real number initial conditions.

In particular, Clark and Kulenovic [13, 14] discussed the global stability properties and asymptotic behavior of solutions for the recursive sequence

$$x_{n+1} = \frac{x_n}{a + cy_n}, \quad y_{n+1} = \frac{y_n}{b + dx_n}, \quad n = 0,1,\cdots, \tag{1.2}$$

where $a,b,c,d \in (0,\infty)$ and the initial conditions $x_0$ and $y_0$ are arbitrary nonnegative numbers.

In 2012, Zhang et al. [15] investigated the stability character and asymptotic behavior of the solution for the system of difference equations

$$x_{n+1} = \frac{x_{n-2}}{B + y_{n-2}y_{n-1}y_n}, \quad y_{n+1} = \frac{y_{n-2}}{A + x_{n-2}x_{n-1}x_n}, \quad n = 0,1,\cdots, \tag{1.3}$$

where $A, B \in (0,\infty)$, and the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0 \in (0,\infty)$.

Recently, the following nonlinear two-dimensional difference systems

$$x_{n+1} = \varphi(x_{n-t_1}, y_{n-s_1}), \quad y_{n+1} = \psi(y_{n-s_2}, x_{n-t_2}), \tag{1.4}$$

where $t_1, s_1, s_2, t_2$ are all positive integers, was studied by Liu et al. [16], in which they gave some sufficient conditions such that every positive solution of this equation converges to the unique equilibrium point.

More recently, in [17] the authors studied analogous results for the system of difference equations

$$x_{n+1} = ax_n + by_{n-1}e^{-x_n}, \quad y_{n+1} = cy_n + dx_{n-1}e^{-y_n}, \tag{1.5}$$

where $a,b,c,d$ are positive constants and the initial values $x_1, x_0, y_1, y_0$ are positive numbers. For more related work, one can refer to [18-22] and references therein.

Inspired by the above works, the essential problem we consider in this paper is the asymptotic behavior of the solution for the difference equation

$$x_{n+1} = \frac{x_{n-1}x_{n-2}}{A + By_{n-3}}, \quad y_{n+1} = \frac{y_{n-1}y_{n-2}}{C + Dx_{n-3}}, \quad n = 0, 1, \cdots, \tag{1.6}$$

where the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty)$, $y_{-3}, y_{-2}, y_{-1}, y_0 \in (0, \infty)$ and $A, B, C, D$ are positive constants.

This paper proceeds as follows. In Section 2, we introduce some definitions and preliminary results. The main results and their proofs are given in Section 3.

## 2. Preliminaries

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Let $I_x, I_y$ be some intervals of real numbers and $f: I_x^4 \times I_y^4 \to I_x$, $g: I_x^4 \times I_y^4 \to I_y$ be continuously differentiable functions. Then for every initial conditions $(x_i, y_i) \in I_x \times I_y$, $(i = -3, -2, -1, 0)$, the system of difference equations

$$\begin{cases} x_{n+1} = f(x_n, x_{n-1}, x_{n-2}, x_{n-3}, y_n, y_{n-1}, y_{n-2}, y_{n-3}), \\ y_{n+1} = g(x_n, x_{n-1}, x_{n-2}, x_{n-3}, y_n, y_{n-1}, y_{n-2}, y_{n-3}), \end{cases} \quad n = 0, 1, 2, \cdots, \quad (2.1)$$

has a unique solution $\{(x_n, y_n)\}_{n=-3}^{\infty}$. A point $(\overline{x}, \overline{y}) \in I_x \times I_y$ is called an equilibrium point of (2.1) if $\overline{x} = f(\overline{x}, \overline{x}, \overline{x}, \overline{x}, \overline{y}, \overline{y}, \overline{y}, \overline{y})$, $\overline{y} = g(\overline{x}.\overline{x}, \overline{x}, \overline{x}, \overline{y}, \overline{y}, \overline{y}, \overline{y})$, i. e., $(x_n, y_n) = (\overline{x}, \overline{y})$ for all $n \geq 0$.

Interval $I_x \times I_y$ is called invariant for system (2.1) if, for all $n > 0$, $x_n \in I_x$, $y_n \in I_y$ when the initial conditions $x_{-3}, x_{-2} x_{-1}, x_0 \in I_x$, $y_{-3}, y_{-2}, y_{-1}, y_0 \in I_y$.

**Definition 2.1** Assume that $(\overline{x}, \overline{y})$ is a fixed point of (2.1). Then

(*i*) $(\overline{x}, \overline{y})$ is said to be stable relative to $I_x \times I_y$ if for every $\varepsilon > 0$, there exits $\delta > 0$ such that for any initial conditions $(x_i, y_i) \in I_x \times I_y$ $(i = -3, -2, -1, 0)$, with $\sum_{i=-3}^{0} |x_i - \overline{x}| < \delta$, $\sum_{i=-3}^{0} |y_i - \overline{y}| < \delta$, implies $|x_n - \overline{x}| < \varepsilon$, $|y_n - \overline{y}| < \varepsilon$.

(*ii*) $(\overline{x}, \overline{y})$ is called an attractor relative to $I_x \times I_y$ if for all $(x_i, y_i) \in I_x \times I_y$ $(i = -3, -2, -1, 0)$, $lim_{n \to \infty} x_n = \overline{x}$, $lim_{n \to \infty} y_n = \overline{y}$.

(*iii*) $(\overline{x}, \overline{y})$ is called asymptotically stable relative to $I_x \times I_y$ if it is stable and an attractor.

(*iv*) Unstable if it is not stable.

**Theorem 2.1** Assume that $X(n+1) = F(X(n)), n = 0, 1, \cdots$, is a system of difference equations and $\overline{X}$ is the equilibrium point of this system i.e., $F(\overline{X}) = \overline{X}$.

(*i*) If all eigenvalues of the Jacobian matrix $J_F$, evaluated at $\overline{X}$ lie inside the open unit disk $|\lambda| < 1$, then $\overline{X}$ is locally asymptotically stable.

(*ii*) If all eigenvalues of the Jacobian matrix $J_F$, evaluated at $\overline{X}$ has modulus greater than one then $\overline{X}$ is unstable.

**Definition 2.2** Let $p$, $q$, $s$, $t$ be four nonnegative integers such that $p + q = s + t = n$. Splitting $(x, y) = (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)$ into $(x, y) = ([x]_p, [x]_q, [y]_s, [y]_t)$, where $[x]_\sigma$ denotes a vector with $\sigma$-components of $x$, we say that the function $f(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)$ possesses a mixed monotone property in subsets $I^{2n}$ of $R^{2n}$ if $f([x]_p, [x]_q, [y]_s, [y]_t)$ is monotone nondecreasing in each component of $[x]_p, [y]_s$ and is monotone nonincreasing in each component of $[x]_q, [y]_t$ for $(x, y) \in I^{2n}$. In particular, if $q = t = 0$, then it is said to be monotone nondecreasing

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in $I^{2n}$.

## 3. The Main Results

In this section, we investigate the asymptotic behavior of the equilibrium points of the systems (1.6). It is easy to know that the systems (1.6) have four equilibrium points $(0,0)$, $(0,C)$, $(A,0)$, and $((A+BC)/(1-BD),(C+AD)/(1-BD))$.

**Theorem 3.1** The equilibrium point $(0,0)$ of (1.6) is locally asymptotically stable.

**Proof.** We can easily obtain that the linearized system of (1.6) about the equilibrium point $(0,0)$ is

$$\varphi_{n+1} = D\varphi_n \qquad (3.1)$$

where

$$\varphi_n = \begin{bmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ x_{n-3} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \qquad (3.2)$$

Thus, the characteristic equation of (3.2) is

$$f(\lambda) = \lambda^8 = 0.$$

This shows that all the roots of characteristic equation lie inside unit disk. So the equilibrium $(0,0)$ is locally asymptotically stable.

**Theorem 3.2** Let

$$\begin{cases} x_{n+1} = f(x_n, x_{n-1}, \cdots, x_{n-k}, y_n, y_{n-1}, \cdots, y_{n-k}), \\ y_{n+1} = g(x_n, x_{n-1}, \cdots, x_{n-k}, y_n, y_{n-1}, \cdots, y_{n-k}), \end{cases} \quad n = 0,1,\cdots, \qquad (3.3)$$

$[a,b]$ be an interval of real numbers and assume that $f:[a,b]^{k+1}\times[c,d]^{k+1} \to [a,b]$ and $g:[a,b]^{k+1}\times[c,d]^{k+1} \to [c,d]$ are two continuous functions satisfying the mixed monotone property. If there exit

$$m_0 \le \min\{x_{-k}, x_{-k+1}, \cdots, x_0\} \le \max\{x_{-k}, x_{-k+1}, \cdots, x_0\} \le M_0,$$

and

$$n_0 \le \min\{y_{-k}, y_{-k+1}, \cdots, y_0\} \le \max\{y_{-k}, y_{-k+1}, \cdots, y_0\} \le N_0$$

such that

$$m_0 \le f([m_0]_p, [M_0]_q, [n_0]_s, [N_0]_t) \le f([M_0]_p, [m_0]_q, [N_0]_s, [n_0]_t) \le M_0, \qquad (3.4)$$

and

$$n_0 \le g([m_0]_p, [M_0]_q, [n_0]_s, [N_0]_t) \le g([M_0]_p, [m_0]_q, [N_0]_s, [n_0]_t) \le N_0, \qquad (3.5)$$

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then there exit $(m, M) \in [m_0, M_0]^2$ and $(n, N) \in [n_0, N_0]^2$ satisfying

$$M = f([M]_p, [m]_q, [N]_s, [n]_t), \quad m = f([m]_p, [M]_q, [n]_s, [N]_t), \qquad (3.6)$$

and

$$N = g([M]_p, [m]_q, [N]_s, [n]_t), \quad n = g([m]_p, [M]_q, [n]_s, [N]_t). \qquad (3.7)$$

Moreover, if $m = M$ and $n = N$, then the system (3.3) has a unique equilibrium point $(\overline{x}, \overline{y}) \in [m_0, M_0] \times [n_0, N_0]$ and every solution of (3.3) converges to $(\overline{x}, \overline{y})$.

**Proof.** Using $m_0, M_0$ and $n_0, N_0$ as two couples of initial iteration, we construct four sequences $\{m_i\}, \{M_i\}, \{n_i\}$ and $\{N_i\}$ $(i = 1, 2, \cdots)$ from the following equations

$$m_i = f([m_{i-1}]_p, [M_{i-1}]_q, [n_{i-1}]_s, [N_{i-1}]_t), \quad M_i = f([M_{i-1}]_p, [m_{i-1}]_q, [N_{i-1}]_s, [n_{i-1}]_t),$$

and

$$n_i = g([m_{i-1}]_p, [M_{i-1}]_q, [n_{i-1}]_s, [N_{i-1}]_t), \quad N_i = g([M_{i-1}]_p, [m_{i-1}]_q, [N_{i-1}]_s, [n_{i-1}]_t).$$

It is obvious from the mixed monotone property of functions $f$ and $g$ that the sequences $\{m_i\}, \{M_i\}, \{n_i\}$ and $\{N_i\}$ $(i = 1, 2, \cdots)$ possess the following monotone property

$$m_0 \leq m_1 \leq \cdots \leq m_i \leq \cdots \leq M_i \leq \cdots \leq M_1 \leq M_0, \qquad (3.8)$$

and

$$n_0 \leq n_1 \leq \cdots \leq n_i \leq \cdots \leq N_i \leq \cdots \leq N_1 \leq N_0, \qquad (3.9)$$

where $i = 0, 1, 2, \cdots$.

Moreover, one has

$$m_i \leq x_l \leq M_i \qquad \text{for } l \geq (k+1)i + 1, i = 0, 1, 2, \cdots. \qquad (3.10)$$

and

$$n_i \leq y_l \leq N_i \qquad \text{for } l \geq (k+1)i + 1, i = 0, 1, 2, \cdots. \qquad (3.11)$$

Set

$$m = \lim_{i \to \infty} m_i, \ M = \lim_{i \to \infty} M_i, \ n = \lim_{i \to \infty} n_i, \ N = \lim_{i \to \infty} N_i, \qquad (3.12)$$

then

$$m \leq \liminf_{i \to \infty} x_i \leq \limsup_{i \to \infty} x_i \leq M, \quad n \leq \liminf_{i \to \infty} y_i \leq \limsup_{i \to \infty} y_i \leq N. \qquad (3.13)$$

By the continuity of $f$ and $g$, we have

$$M = f([M]_p, [m]_q, [N]_s, [n]_t), m = f([m]_p, [M]_q, [n]_s, [N]_t), \qquad (3.14)$$

and

$$N = g([M]_p, [m]_q, [N]_s, [n]_t), n = g([m]_p, [M]_q, [n]_s, [N]_t). \qquad (3.15)$$

Moreover, if $m = M, n = N$, then $m = M = \lim_{i \to \infty} x_i = \overline{x}, n = N = \lim_{i \to \infty} y_i = \overline{y}$, and then the proof is complete.

**Theorem 3.3** If $A = C, B = D$, the equilibrium point $(0, 0)$ of the systems (1.6) is a global attractor for any initial conditions

$$(x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0) \in (0, A)^8.$$

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**Proof.** Let $(f,g):(0,\infty)^4 \times (0,\infty)^4 \to (0,\infty)\times(0,\infty)$ be a function defined by

$$f(x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0) = \frac{x_{n-1}x_{n-2}}{A+By_{n-3}},$$

and

$$g(x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0) = \frac{y_{n-1}y_{n-2}}{A+Bx_{n-3}}.$$

We can easily see that the functions $f$ and $g$ possess a mixed monotone property in subsets $(0,A)^8$ of $R^8$.

Let

$$M_0 = N_0 = \max\{x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0\}, \quad \frac{M_0 - A}{B} < n_0 = m_0 < 0.$$

We have

$$m_0 \le \frac{m_0^2}{A+BN_0} \le \frac{M_0^2}{A+Bn_0} \le M_0, \tag{3.16}$$

$$n_0 \le \frac{n_0^2}{A+BM_0} \le \frac{N_0^2}{A+Bm_0} \le N_0, \tag{3.17}$$

Then from (1.6) and Theorem 3.2, there exit $m, M \in [m_0, M_0]$, $n, N \in [n_0, N_0]$ satisfying

$$m = \frac{m^2}{A+BN}, \qquad M = \frac{M^2}{A+Bn}, \tag{3.18}$$

$$n = \frac{n^2}{A+BM}, \qquad N = \frac{N^2}{A+Bm}. \tag{3.19}$$

In view of

$$m < M < M_0 < A+Bn_0 < A+Bn < A+BN,$$

and

$$n < N < N_0 < A+Bm_0 < A+Bm < A+BM,$$

thus, one has

$$M = m = N = n = 0. \tag{3.20}$$

It follows by Theorem 3.2 that the equilibrium point $(0,0)$ of (1.6) is a global attractor. The proof is complete.

**Theorem 3.4** The equilibrium point $(0,0)$ of the system (1.6) is global asymptotically stability for any initial conditions

$$(x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0) \in (0,A)^8.$$

**Proof.** The result follows from Theorems 3.1 and 3.3.

**Theorem 3.5** The equilibrium point $(0,C)$, $(A,0)$ of the system (1.6) is unstable.

**Proof.** We can easily obtain that the linearized system of the system (1.6) about the equilibrium $(0,C)$ is

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$$\varphi_{n+1} = D^* \varphi_n , \qquad (3.21)$$

where

$$\varphi_n = \begin{bmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ x_{n-3} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \end{bmatrix}, \quad D^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -D & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The characteristic equation of the systems (3.20) is

$$P(\lambda) = \lambda^5 (\lambda^3 - \lambda - 1). \qquad (3.22)$$

It is obvious that $P(1) = -1, P(2) = 160$. It follows by the intermediate value theorem for continuous function that there exists $\lambda > 1$ so that $P(\lambda) = 0$. Therefore, one of the roots of characteristic equation (3.22) lies outside unit disk. According to Theorem 2.1, the equilibrium $(0, C)$ is unstable.

Similarly, we can obtain that the unique equilibrium $(A, 0)$ is unstable.

**Theorem 3.6** If $BD < 1$, the equilibrium point $(\bar{x}, \bar{y}) = (\dfrac{A + BC}{1 - BD}, \dfrac{C + AD}{1 - BD})$ is locally asymptotically stable. If $BD > 1$, the equilibrium point $(\bar{x}, \bar{y})$ is unstable.

**Proof.** We can easily obtain that the linearized system of (1.6) about the equilibrium $(\bar{x}, \bar{y})$ is

$$\varphi_{n+1} = D^* \varphi_n , \qquad (3.23)$$

where

$$\varphi_n = \begin{bmatrix} x_n \\ x_{n-1} \\ x_{n-2} \\ x_{n-3} \\ y_n \\ y_{n-1} \\ y_{n-2} \\ y_{n-3} \end{bmatrix}, \quad D^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -B \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -D & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The characteristic equation of (3.23) is

$$P(\lambda) = \lambda^8 - BD = 0. \qquad (3.24)$$

In view of $BD < 1$, this shows that all the roots of characteristic equation lie inside unit disk, so the unique equilibrium $(\bar{x}, \bar{y})$ is locally asymptotically stable. If $BD > 1$, one of the roots of characteristic equation lie outside unit disk, so the unique equilibrium

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$(\overline{x}, \overline{y})$ is unstable.

## 4. Conclusions

This paper presents the use of a variational iteration method for systems of nonlinear difference equations. This technique is a powerful tool for solving various difference equations and can also be applied to other nonlinear differential equations in mathematical physics. The variational iteration method provides an efficient method to handle the nonlinear structure. We have dealt with the problem of global asymptotic stability analysis for a class of nonlinear difference equations. The general sufficient conditions have been obtained to ensure the existence, uniqueness and global asymptotic stability of the equilibrium point $(0,0)$ for the nonlinear difference equation. These criteria generalize and improve some known results in [12-17]. Moreover, the asymptotic behavior of others equilibrium points is also studied. In addition, the sufficient conditions that we obtained are very simple, which provide flexibility for the application and analysis of nonlinear difference equation.

**Remark:** Our model and results are different from the existence ones such as those of References [12-17]. In particular, the new variational iteration method can be applied to the models of References [12-17] and the more general nonlinear difference equations. In some sense, we enrich the theoretical results of the difference equations.

## Acknowledgements

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