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SOME NEW RESULTS ON PRODUCTS OF THE APOSTOL-GENOCCHI POLYNOMIALS

YUAN HE

ABSTRACT. We perform a further investigation for the Apostol-Genocchi polynomials numbers. By making use of the generating function methods and summation transform techniques, we establish some new formulae for products of any arbitrary number of the Apostol-Genocchi polynomials and numbers. The results presented here are the corresponding generalizations of some known formulae on the classical Genocchi polynomials and numbers.

1. INTRODUCTION

The classical Bernoulli polynomials $B_n(x)$ and the classical Genocchi polynomials $G_n(x)$ are usually defined by means of the following generating functions:

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad(|t|<2\pi) \tag{1.1}$$

and

$$\frac{2te^{xt}}{e^t+1}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}\quad(|t|<\pi). \tag{1.2}$$

The rational numbers B_n and G_n given by

$$B_n=B_n(0)\quad\text{and}\quad G_n=G_n(0) \tag{1.3}$$

are called the classical Bernoulli numbers and the classial Genocchi numbers, respectively. These polynomials and numbers play important roles in different areas of mathematics such as number theory, combinatorics, special functions and analysis. Numerous interesting properties for them can be found in many books and papers; see for example, [7, 14, 17, 18, 19, 26, 27, 29, 30, 31].

We now turn to some widely-investigated analogues of the classical Bernoulli polynomials $B_n(x)$ and the classical Genocchi polynomials $G_n(x)$, i.e., the Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ and the Apostol-Genocchi polynomials $\mathcal{G}_n(x;\lambda)$. They are usually defined by means of the following generating functions (see, e.g., [20, 21, 24]):

$$\frac{te^{xt}}{\lambda e^t-1}=\sum_{n=0}^{\infty}\mathcal{B}_n(x;\lambda)\frac{t^n}{n!} \tag{1.4}$$

$$(|t|<2\pi\text{ when }\lambda=1;\ |t|<|\log\lambda|\text{ when }\lambda\neq1)$$

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and

$$\frac{2te^{xt}}{\lambda e^t+1}=\sum_{n=0}^{\infty}\mathcal{G}_n(x;\lambda)\frac{t^n}{n!} \tag{1.5}$$

$$(|t|<\pi \text{ when } \lambda=1;\ |t|<|\log(-\lambda)| \text{ when } \lambda\neq 1)$$

In particular, $\mathcal{B}_n(\lambda)$ and $\mathcal{G}_n(\lambda)$ given by

$$\mathcal{B}_n(\lambda)=\mathcal{B}_n(0;\lambda) \quad \text{and} \quad \mathcal{G}_n(\lambda)=\mathcal{G}_n(0;\lambda) \tag{1.6}$$

are called the Apostol-Bernoulli numbers and the Apostol-Genocchi numbers, respectively. Obviously, $\mathcal{B}_n(x;\lambda)$ and $\mathcal{G}_n(x;\lambda)$, respectively, reduces to $B_n(x)$ and $G_n(x)$ when $\lambda=1$. It is worth mentioning that the Apostol-Bernoulli polynomials were firstly introduced by Apostol [3] (see also Srivastava [28] for a systematic study) in order to evaluate the value of the Hurwitz-Lerch zeta function. For some related results on the Apostol type polynomials and numbers, one can consult to [6, 8, 11, 16, 22, 24, 33].

The idea of the present paper stems from the work of Agoh [1, 2]. We establish some new formulae of products of any arbitrary number of the Apostol-Genocchi polynomials and numbers by making use of the generating function methods and summation transform techniques. It turns out that some results presented here are the corresponding generalizations of several known formulae including the recent ones discovered by Agoh [2] on the classical Genocchi polynomials and numbers.

2. The statement of results

Let n be a positive integer and let $m_1,\ldots,m_n$ be non-negative integers. In the following we denote by $[t_1^{m_1}\cdots t_n^{m_n}]f(t_1,\ldots,t_n)$ the coefficients of $t_1^{m_1}\cdots t_n^{m_n}$ in $f(t_1,\ldots,t_n)$. We first recall the elementary and beautiful idea contributed to Euler, namely (see, e.g., [4, 5])

$$(1+x_1)(1+x_2)(1+x_3)\cdots=(1+x_1)+x_2(1+x_1)+x_3(1+x_1)(1+x_2)+\cdots. \tag{2.1}$$

Obviously, the finite form of (2.1) can be expressed as

$$\begin{aligned}(1+x_1)(1+x_2)\cdots(1+x_n)&=(1+x_1)+x_2(1+x_1)+\cdots\\&\quad+x_n(1+x_1)(1+x_2)\cdots(1+x_{n-1}).\end{aligned} \tag{2.2}$$

We shall make use of (2.2) to establish some new formulae for products of any arbitrary number of the Apostol-Genocchi polynomials and numbers. It is easily seen that for $1\leq r\leq n$, substituting x_r-1 for x_r in (2.2) gives

$$x_1\cdots x_n-1=\sum_{r=1}^{n}(x_r-1)x_1\cdots x_{r-1}, \tag{2.3}$$

where the product $x_1\cdots x_{r-1}$ is considered to be equal to 1 when $r=1$. If we take $x_r=-\lambda_r e^{t_r}$ for $1\leq r\leq n$ in (2.3) then we have

$$(-1)^n\lambda_1\cdots\lambda_n e^{t_1+\cdots+t_n}-1=\sum_{r=1}^{n}(-1)^r(\lambda_r e^{t_r}+1)\prod_{k=1}^{r-1}\lambda_k e^{t_k}. \tag{2.4}$$

It follows from (2.4) that

$$\prod_{i=1}^{n}\frac{2t_ie^{x_it_i}}{\lambda_ie^{t_i}+1}=\frac{1}{(-1)^n\lambda_1\cdots\lambda_ne^{t_1+\cdots+t_n}-1}\times\sum_{r=1}^{n}(-1)^r(\lambda_re^{t_r}+1)\prod_{k=1}^{r-1}\lambda_ke^{t_k}\prod_{i=1}^{n}\frac{2t_ie^{x_it_i}}{\lambda_ie^{t_i}+1}. \quad (2.5)$$

Observe that

$$(\lambda_re^{t_r}+1)\prod_{k=1}^{r-1}\lambda_ke^{t_k}\prod_{i=1}^{n}\frac{2t_ie^{x_it_i}}{\lambda_ie^{t_i}+1}=2t_re^{x_r(t_1+\cdots+t_n)}\prod_{i=1}^{r-1}\lambda_i\frac{2t_ie^{(x_i-x_r+1)t_i}}{\lambda_ie^{t_i}+1}\prod_{i=r+1}^{n}\frac{2t_ie^{(x_i-x_r)t_i}}{\lambda_ie^{t_i}+1}. \quad (2.6)$$

Hence, by applying (2.6) to (2.5), we get

$$\prod_{i=1}^{n}\frac{2t_ie^{x_it_i}}{\lambda_ie^{t_i}+1}=\sum_{r=1}^{n}(-1)^r\frac{2t_re^{x_r(t_1+\cdots+t_n)}}{(-1)^n\lambda_1\cdots\lambda_ne^{t_1+\cdots+t_n}-1}\times\prod_{i=1}^{r-1}\lambda_i\frac{2t_ie^{(x_i-x_r+1)t_i}}{\lambda_ie^{t_i}+1}\prod_{i=r+1}^{n}\frac{2t_ie^{(x_i-x_r)t_i}}{\lambda_ie^{t_i}+1}, \quad (2.7)$$

which means

$$\begin{aligned}&\left[\frac{t_1^{m_1}}{m_1!}\cdots\frac{t_n^{m_n}}{m_n!}\right]\left(\prod_{i=1}^{n}\frac{2t_ie^{x_it_i}}{\lambda_ie^{t_i}+1}\right)\\&=m_1!\cdots m_n!\sum_{r=1}^{n}(-1)^r[t_1^{m_1}\cdots t_{r-1}^{m_{r-1}}t_r^{m_r-1}t_{r+1}^{m_{r+1}}\cdots t_n^{m_n}]\\&\times\left(\frac{2e^{x_r(t_1+\cdots+t_n)}}{(-1)^n\lambda_1\cdots\lambda_ne^{t_1+\cdots+t_n}-1}\prod_{i=1}^{r-1}\lambda_i\frac{2t_ie^{(x_i-x_r+1)t_i}}{\lambda_ie^{t_i}+1}\right.\\&\left.\times\prod_{i=r+1}^{n}\frac{2t_ie^{(x_i-x_r)t_i}}{\lambda_ie^{t_i}+1}\right). \quad (2.8)\end{aligned}$$

It is trivial to get

$$\left[\frac{t_1^{m_1}}{m_1!}\cdots\frac{t_n^{m_n}}{m_n!}\right]\left(\prod_{i=1}^{n}\frac{2t_ie^{x_it_i}}{\lambda_ie^{t_i}+1}\right)=\prod_{i=1}^{n}\mathcal{G}_{m_i}(x_i;\lambda_i). \quad (2.9)$$

We next consider the right hand side of (2.8). Since $\mathcal{B}_0(x;\lambda)=0$ when $\lambda\neq 1$ and $\mathcal{G}_0(x;\lambda)=0$ (see, e.g., [20, 23]) , then by (1.3) we have

$$\frac{e^{xt}}{\lambda e^t-1}=\sum_{n=0}^{\infty}\frac{\mathcal{B}_{n+1}(x;\lambda)}{n+1}\frac{t^n}{n!}\quad(\lambda\neq 1), \quad (2.10)$$

and

$$\frac{2e^{xt}}{\lambda e^t+1}=\sum_{n=0}^{\infty}\frac{\mathcal{G}_{n+1}(x;\lambda)}{n+1}\frac{t^n}{n!}. \quad (2.11)$$

Notice that for non-negative integer N (see, e.g., [32]),

$$(t_1+\cdots+t_n)^N=\sum_{\substack{k_1+\cdots+k_n=N\\ k_1,\ldots,k_n\geq 0}}\binom{N}{k_1,\ldots,k_n}t_1^{k_1}\cdots t_n^{k_n}, \tag{2.12}$$

where $\binom{n}{r_1,\ldots,r_k}$ denotes by the multinomials coefficient given by

$$\binom{n}{r_1,\ldots,r_k}=\frac{n!}{r_1!\cdots r_k!}\quad (n,r_1,\ldots,r_k\geq 0). \tag{2.13}$$

So from (2.10) and (2.11), we obtain that for even integer n and $\lambda_1\cdots\lambda_n\neq 1$,

$$\begin{aligned}\frac{e^{x_r(t_1+\cdots+t_n)}}{(-1)^n\lambda_1\cdots\lambda_n e^{t_1+\cdots+t_n}-1}&\\ =\sum_{N=0}^{\infty}\frac{\mathcal{B}_{N+1}(x_r;\lambda_1\cdots\lambda_n)}{N+1}&\sum_{\substack{k_1+\cdots+k_n=N\\ k_1,\ldots,k_n\geq 0}}\frac{t_1^{k_1}}{k_1!}\cdots\frac{t_n^{k_n}}{k_n!},\end{aligned} \tag{2.14}$$

and for odd integer n,

$$\begin{aligned}\frac{2e^{x_r(t_1+\cdots+t_n)}}{(-1)^n\lambda_1\cdots\lambda_n e^{t_1+\cdots+t_n}-1}&\\ =-\sum_{N=0}^{\infty}\frac{\mathcal{G}_{N+1}(x_r;\lambda_1\cdots\lambda_n)}{N+1}&\sum_{\substack{k_1+\cdots+k_n=N\\ k_1,\ldots,k_n\geq 0}}\frac{t_1^{k_1}}{k_1!}\cdots\frac{t_n^{k_n}}{k_n!}.\end{aligned} \tag{2.15}$$

It follows from (1.3), (1.4), (2.8), (2.9), (2.14) and (2.15) that if n is an even integer, then for positive integers $m_1,\ldots,m_n$ and $\lambda_1\cdots\lambda_n\neq 1$,

$$\begin{aligned}&\mathcal{G}_{m_1}(x_1;\lambda_1)\mathcal{G}_{m_2}(x_2;\lambda_2)\cdots\mathcal{G}_{m_n}(x_n;\lambda_n)\\ &=2\sum_{r=1}^{n}(-1)^r\sum_{k_1,\cdots,k_{r-1},k_{r+1},\cdots,k_n\geq 0}\frac{m_1!\cdots m_n!}{k_1!\cdots k_{r-1}!\cdot(m_r-1)!\cdot k_{r+1}!\cdots k_n!}\\ &\quad\times\frac{\mathcal{B}_{k_1+\cdots+k_{r-1}+(m_r-1)+k_{r+1}+\cdots+k_n+1}(x_r;\lambda_1\cdots\lambda_n)}{k_1+\cdots+k_{r-1}+(m_r-1)+k_{r+1}+\cdots+k_n+1}\\ &\quad\times\prod_{i=1}^{r-1}\lambda_i\frac{\mathcal{G}_{m_i-k_i}(x_i-x_r+1;\lambda_i)}{(m_i-k_i)!}\prod_{i=r+1}^{n}\frac{\mathcal{G}_{m_i-k_i}(x_i-x_r;\lambda_i)}{(m_i-k_i)!}.\end{aligned} \tag{2.16}$$

and if n is an odd integer, then for positive integers $m_1,\ldots,m_n$,

$$\begin{aligned}&\mathcal{G}_{m_1}(x_1;\lambda_1)\mathcal{G}_{m_2}(x_2;\lambda_2)\cdots\mathcal{G}_{m_n}(x_n;\lambda_n)\\ &=\sum_{r=1}^{n}(-1)^{r-1}\sum_{k_1,\cdots,k_{r-1},k_{r+1},\cdots,k_n\geq 0}\frac{m_1!\cdots m_n!}{k_1!\cdots k_{r-1}!\cdot(m_r-1)!\cdot k_{r+1}!\cdots k_n!}\\ &\quad\times\frac{\mathcal{G}_{k_1+\cdots+k_{r-1}+(m_r-1)+k_{r+1}+\cdots+k_n+1}(x_r;\lambda_1\cdots\lambda_n)}{k_1+\cdots+k_{r-1}+(m_r-1)+k_{r+1}+\cdots+k_n+1}\\ &\quad\times\prod_{i=1}^{r-1}\lambda_i\frac{\mathcal{G}_{m_i-k_i}(x_i-x_r+1;\lambda_i)}{(m_i-k_i)!}\prod_{i=r+1}^{n}\frac{\mathcal{G}_{m_i-k_i}(x_i-x_r;\lambda_i)}{(m_i-k_i)!}.\end{aligned} \tag{2.17}$$

Thus, by replacing k_i by $m_i - k_i$ for $i \neq r$ in (2.16) and (2.17), we obtain the following formulae for products of an arbitrary number of the Apostol-Genocchi polynomials.

Theorem 2.1. *Let $m_1, \cdots, m_n$ be n positive integers. If n is an even integer, then*

$$\begin{aligned}
&\mathcal{G}_{m_1}(x_1;\lambda_1)\mathcal{G}_{m_2}(x_2;\lambda_2)\cdots\mathcal{G}_{m_n}(x_n;\lambda_n)\\
&\quad = 2\sum_{\substack{k_1+\cdots+k_n=m_1+\cdots+m_n\\ k_1,\ldots,k_n\geq 0}}\sum_{r=1}^{n}(-1)^r\frac{m_r}{k_r}\mathcal{B}_{k_r}(x_r;\lambda_1\cdots\lambda_n)\\
&\qquad\qquad \times\prod_{i=1}^{r-1}\binom{m_i}{k_i}\lambda_i\mathcal{G}_{k_i}(x_i-x_r+1;\lambda_i)\\
&\qquad\qquad\quad \times\prod_{i=r+1}^{n}\binom{m_i}{k_i}\mathcal{G}_{k_i}(x_i-x_r;\lambda_i)\quad(\lambda_1\cdots\lambda_n\neq 1).
\end{aligned}\tag{2.18}$$

If n is an odd integer, then

$$\begin{aligned}
&\mathcal{G}_{m_1}(x_1;\lambda_1)\mathcal{G}_{m_2}(x_2;\lambda_2)\cdots\mathcal{G}_{m_n}(x_n;\lambda_n)\\
&\quad = \sum_{\substack{k_1+\cdots+k_n=m_1+\cdots+m_n\\ k_1,\ldots,k_n\geq 0}}\sum_{r=1}^{n}(-1)^{r-1}\frac{m_r}{k_r}\mathcal{G}_{k_r}(x_r;\lambda_1\cdots\lambda_n)\\
&\qquad \times\prod_{i=1}^{r-1}\binom{m_i}{k_i}\lambda_i\mathcal{G}_{k_i}(x_i-x_r+1;\lambda_i)\prod_{i=r+1}^{n}\binom{m_i}{k_i}\mathcal{G}_{k_i}(x_i-x_r;\lambda_i).
\end{aligned}\tag{2.19}$$

It follows that we show some special cases of Theorem 2.1. Since the Apostol-Genocchi polynomials satisfy the following difference equation (see, e.g., [23]):

$$\lambda\mathcal{G}_n(x+1;\lambda)+\mathcal{G}_n(x;\lambda)=2nx^{n-1}\quad(n\geq 0),\tag{2.20}$$

by taking $n=2$ in Theorem 2.1, we get that for positive integers m,n and $\lambda\mu\neq 1$,

$$\begin{aligned}
\mathcal{G}_m(x;\lambda)\mathcal{G}_n(y;\mu) &= 2n\sum_{k=0}^{m}\binom{m}{k}\{2k(x-y)^{k-1}-\mathcal{G}_k(x-y;\lambda)\}\frac{\mathcal{B}_{m+n-k}(y;\lambda\mu)}{m+n-k}\\
&\quad -2m\sum_{k=0}^{n}\binom{n}{k}\mathcal{G}_k(y-x;\mu)\frac{\mathcal{B}_{m+n-k}(x;\lambda\mu)}{m+n-k}.
\end{aligned}\tag{2.21}$$

The identity (2.21) can be also found in [12] where it was further considered the case $\lambda\mu = 1$. We also refer to [9, 10, 35] for some similar formulae to (2.21). If we take $n = 3$ in Theorem 2.1, in light of the symmetric relation for the Apostol-Genocchi polynomials (see, e.g., [23]):

$$\lambda\mathcal{G}_n(1-x;\lambda)=(-1)^{n+1}\mathcal{G}_n\left(x;\frac{1}{\lambda}\right)\quad(n\geq 0),\tag{2.22}$$

we obtain that for positive integers m_1, m_2, m_3,

$$\begin{aligned}
&\mathcal{G}_{m_1}(x_1;\lambda_1)\mathcal{G}_{m_2}(x_2;\lambda_2)\mathcal{G}_{m_3}(x_3;\lambda_3)\\
&=\sum_{\substack{k_1+k_2+k_3=m_1+m_2+m_3\\ k_1,k_2,k_3\geq 0}}\left\{\frac{m_1}{k_1}\binom{m_2}{k_2}\binom{m_3}{k_3}\mathcal{G}_{k_1}(x_1;\mu)\mathcal{G}_{k_2}(x_2-x_1;\lambda_2)\mathcal{G}_{k_3}(x_3-x_1;\lambda_3)\right.\\
&\quad+\frac{m_2}{k_2}\binom{m_1}{k_1}\binom{m_3}{k_3}(-1)^{k_1}\mathcal{G}_{k_2}(x_2;\mu)\mathcal{G}_{k_1}(x_2-x_1;1/\lambda_1)\mathcal{G}_{k_3}(x_3-x_2;\lambda_3)\\
&\quad\quad+\frac{m_3}{k_3}\binom{m_1}{k_1}\binom{m_2}{k_2}(-1)^{k_1+k_2}\mathcal{G}_{k_3}(x_3;\mu)\mathcal{G}_{k_1}(x_3-x_1;1/\lambda_1)\\
&\quad\quad\quad\left.\times\mathcal{G}_{k_2}(x_3-x_2;1/\lambda_2)\right\}\quad(\mu=\lambda_1\lambda_2\lambda_3). \qquad (2.23)
\end{aligned}$$

Remark 2.2. *Note that (2.19) does not require the condition $\lambda_1\cdots\lambda_n\neq 1$. However, we were unable to get the formula analogous (2.18) in the case $\lambda_1\cdots\lambda_n=1$.*

We next give some higher-order convolution formulae for the Apostol-Genocchi polynomials, which are the corresponding generalization of Agoh's convolution formula on the classical Genocchi polynomials presented in [1, 12]. Clearly, by substituting k for n and u_it for t_i with $u_1+u_2+\cdots+u_k=1$ in (2.7), we discover that for positive integer k,n,

$$\begin{aligned}
\left[\frac{t^n}{n!}\right]\left(\prod_{i=1}^{k}\frac{2u_ite^{x_iu_it}}{\lambda_ie^{u_it}+1}\right)&=\sum_{r=1}^{k}(-1)^r\left[\frac{t^n}{n!}\right]\left(\frac{2u_rte^{x_rt}}{(-1)^k\lambda_1\cdots\lambda_ke^t-1}\right.\\
&\quad\left.\times\prod_{i=1}^{r-1}\lambda_i\frac{2u_ite^{(x_i-x_r+1)u_it}}{\lambda_ie^{u_it}+1}\prod_{i=r+1}^{k}\frac{2u_ite^{(x_i-x_r)u_it}}{\lambda_ie^{u_it}+1}\right). \qquad (2.24)
\end{aligned}$$

It is easy to see from (1.4) that the left hand side of (2.24) can be rewritten as

$$\begin{aligned}
\left[\frac{t^n}{n!}\right]\left(\prod_{i=1}^{k}\frac{2u_ite^{x_iu_it}}{\lambda_ie^{u_it}+1}\right)&=\sum_{\substack{j_1+j_2\cdots+j_k=n\\ j_1,j_2,\ldots,j_k\geq 0}}\frac{n!\cdot u_1^{j_1}u_2^{j_2}\cdots u_k^{j_k}}{j_1!\cdot j_2!\cdots j_k!}\\
&\quad\times\mathcal{G}_{j_1}(x_1;\lambda_1)\mathcal{G}_{j_2}(x_2;\lambda_2)\cdots\mathcal{G}_{j_k}(x_k;\lambda_k), \qquad (2.25)
\end{aligned}$$

and the right hand side of (2.24) can be rewritten in the following ways: if k is an even integer then

$$\begin{aligned}
&\left[\frac{t^n}{n!}\right]\left(\prod_{i=1}^{k}\frac{u_ite^{x_iu_it}}{\lambda_ie^{u_it}-1}\right)\\
&=-2\sum_{r=1}^{k}\sum_{\substack{j_1+j_2\cdots+j_k=n\\ j_1,j_2,\ldots,j_k\geq 0}}\frac{n!\cdot u_1^{j_1}u_2^{j_2}\cdots u_{r-1}^{j_{r-1}}u_ru_{r+1}^{j_{r+1}}\cdots u_k^{j_k}}{j_1!\cdot j_2!\cdots j_k!}\mathcal{B}_{j_r}(x_r;\lambda_1\lambda_2\cdots\lambda_k)\\
&\quad\times\prod_{i=1}^{r-1}\{-\lambda_i\mathcal{G}_{j_i}(x_i-x_r+1;\lambda_i)\}\prod_{i=r+1}^{k}\mathcal{G}_{j_i}(x_i-x_r;\lambda_i), \qquad (2.26)
\end{aligned}$$

and if k is an odd integer then

$$\begin{aligned}
&\left[\frac{t^n}{n!}\right]\left(\prod_{i=1}^{k}\frac{u_i t e^{x_i u_i t}}{\lambda_i e^{u_i t}-1}\right)\\
&=\sum_{r=1}^{k}\sum_{\substack{j_1+j_2\cdots+j_k=n\\ j_1,j_2,\ldots,j_k\geq 0}}\frac{n!\cdot u_1^{j_1}u_2^{j_2}\cdots u_{r-1}^{j_{r-1}}u_r u_{r+1}^{j_{r+1}}\cdots u_k^{j_k}}{j_1!\cdot j_2!\cdots j_k!}\mathcal{G}_{j_r}(x_r;\lambda_1\lambda_2\cdots\lambda_k)\\
&\qquad\times\prod_{i=1}^{r-1}\{-\lambda_i\mathcal{G}_{j_i}(x_i-x_r+1;\lambda_i)\}\prod_{i=r+1}^{k}\mathcal{G}_{j_i}(x_i-x_r;\lambda_i). \qquad (2.27)
\end{aligned}$$

Since for positive integer $k\geq 2$ and complex numbers $\alpha_1,\alpha_2,\ldots,\alpha_k$ with $\mathrm{Re}(\alpha_j)>-1$ for $j=1,2,\ldots,k$, (see, e.g., [2, 34])

$$\begin{aligned}
&\int_0^1\int_0^{1-u_1}\cdots\int_0^{1-u_1-\cdots-u_{k-2}}u_1^{\alpha_1}u_2^{\alpha_2}\cdots u_k^{\alpha_k}du_1d_2\cdots du_{k-1}\\
&\qquad=\frac{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\cdots\Gamma(\alpha_k+1)}{\Gamma(\alpha_1+\alpha_2+\cdots+\alpha_k+k)}\quad(u_1+u_2+\cdots+u_k=1). \qquad (2.28)
\end{aligned}$$

by equating (2.25), (2.26) and (2.27) and making the above integral operation, with the help of (2.28), we get that if k is an even integer then

$$\begin{aligned}
&(n+k)\sum_{\substack{j_1+j_2\cdots+j_k=n\\ j_1,j_2,\ldots,j_k\geq 0}}\mathcal{G}_{j_1}(x_1;\lambda_1)\mathcal{G}_{j_2}(x_2;\lambda_2)\cdots\mathcal{G}_{j_k}(x_k;\lambda_k)\\
&=-2\sum_{r=1}^{k}\sum_{\substack{j_1+j_2\cdots+j_k=n\\ j_1,j_2,\ldots,j_k\geq 0}}\binom{n+k}{j_r}\mathcal{B}_{j_r}(x_r;\lambda_1\lambda_2\cdots\lambda_k)\prod_{i=1}^{r-1}\{-\lambda_i\mathcal{G}_{j_i}(x_i-x_r+1;\lambda_i)\}\\
&\qquad\times\prod_{i=r+1}^{k}\mathcal{G}_{j_i}(x_i-x_r;\lambda_i), \qquad (2.29)
\end{aligned}$$

and if k is an odd integer then

$$\begin{aligned}
&(n+k)\sum_{\substack{j_1+j_2\cdots+j_k=n\\ j_1,j_2,\ldots,j_k\geq 0}}\mathcal{G}_{j_1}(x_1;\lambda_1)\mathcal{G}_{j_2}(x_2;\lambda_2)\cdots\mathcal{G}_{j_k}(x_k;\lambda_k)\\
&=\sum_{r=1}^{k}\sum_{\substack{j_1+j_2\cdots+j_k=n\\ j_1,j_2,\ldots,j_k\geq 0}}\binom{n+k}{j_r}\mathcal{G}_{j_r}(x_r;\lambda_1\lambda_2\cdots\lambda_k)\prod_{i=1}^{r-1}\{-\lambda_i\mathcal{G}_{j_i}(x_i-x_r+1;\lambda_i)\}\\
&\qquad\times\prod_{i=r+1}^{k}\mathcal{G}_{j_i}(x_i-x_r;\lambda_i). \qquad (2.30)
\end{aligned}$$

Notice that from (2.20) we have

$$\begin{aligned}
&\prod_{i=1}^{r-1}\{-\lambda_i\mathcal{G}_{j_i}(x_i-x_r+1;\lambda_i)\}\\
&\qquad=\sum_{T\subseteq\{1,\ldots,r-1\}}\prod_{i\in T}\mathcal{G}_{j_i}(x_i-x_r;\lambda_i)\times\prod_{i\in\overline{T}}\{-2j_i(x_i-x_r)^{j_i-1}\}. \qquad (2.31)
\end{aligned}$$

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Thus, by applying (2.31) to the right hand sides of (2.29) and (2.30) and then taking $x_1 = x_2 = \cdots = x_k = x$, we immediately obtain the following result.

Theorem 2.3. *Let k, n be positive integers. If k is an even integer, then*

$$\begin{aligned}(n+k) &\sum_{\substack{j_1+j_2\cdots+j_k=n\\ j_1,j_2,\ldots,j_k\geq 0}} \mathcal{G}_{j_1}(x;\lambda_1)\mathcal{G}_{j_2}(x;\lambda_2)\cdots\mathcal{G}_{j_k}(x;\lambda_k)\\ &= \sum_{r=1}^{k}\binom{k}{r-1}(-2)^{k-r+1}\sum_{\substack{j_1+j_2\cdots+j_r=n-k+r\\ j_1,j_2,\ldots,j_r\geq 0}}\binom{n+k}{j_r}\mathcal{B}_{j_r}(x;\lambda_1\lambda_2\cdots\lambda_k)\\ &\qquad\times \mathcal{G}_{j_1}(\lambda_1)\mathcal{G}_{j_2}(\lambda_2)\cdots\mathcal{G}_{j_{r-1}}(\lambda_{r-1}). \qquad (2.32)\end{aligned}$$

If k is an odd integer, then

$$\begin{aligned}(n+k) &\sum_{\substack{j_1+j_2\cdots+j_k=n\\ j_1,j_2,\ldots,j_k\geq 0}} \mathcal{G}_{j_1}(x;\lambda_1)\mathcal{G}_{j_2}(x;\lambda_2)\cdots\mathcal{G}_{j_k}(x;\lambda_k)\\ &= \sum_{r=1}^{k}\binom{k}{r-1}(-2)^{k-r}\sum_{\substack{j_1+j_2\cdots+j_r=n-k+r\\ j_1,j_2,\ldots,j_r\geq 0}}\binom{n+k}{j_r}\mathcal{G}_{j_r}(x;\lambda_1\lambda_2\cdots\lambda_k)\\ &\qquad\times \mathcal{G}_{j_1}(\lambda_1)\mathcal{G}_{j_2}(\lambda_2)\cdots\mathcal{G}_{j_{r-1}}(\lambda_{r-1}). \qquad (2.33)\end{aligned}$$

It becomes obvious that setting $k = 2$ in Theorem 2.3 gives that for positive integer n,

$$\begin{aligned}\sum_{k=0}^{n}\mathcal{G}_k(x;\lambda)\mathcal{G}_{n-k}(x;\mu) + \frac{4}{n+2}\sum_{k=0}^{n}\binom{n+2}{k}\mathcal{B}_k(x;\lambda\mu)\mathcal{G}_{n-k}(\lambda)\\ = \frac{2n(n+1)}{3}\mathcal{B}_{n-1}(x;\lambda\mu). \qquad (2.34)\end{aligned}$$

Since the classical Genocchi polynomials can be expressed in terms of the classical Bernoulli polynomials, as follows,

$$G_n(x) = 2B_n(x) - 2^{n+1}B_n\left(\frac{x}{2}\right) \quad (n \geq 0), \qquad (2.35)$$

by $B_1 = -1/2$, we have $G_0 = 0$ and $G_1 = 1$. Hence, the case $\lambda = \mu = 1$ in (2.34) gives the convolution identity on the classical Genocchi polynomials due to Agoh [1, 12], namely

$$\sum_{k=1}^{n-1}G_k(x)G_{n-k}(x) + \frac{4}{n+2}\sum_{k=0}^{n-2}\binom{n+2}{k}B_k(x)G_{n-k} = 0 \quad (n \geq 2). \qquad (2.36)$$

It is worth noticing that $x = 0$ in (2.36) can give the result (see, e.g., [1]):

$$\sum_{k=2}^{n-2}G_kG_{n-k} + 4\sum_{k=2}^{n-2}\binom{n+1}{k-1}\frac{B_kG_{n-k}}{k} = -\frac{4}{n+2}G_n \quad (n \geq 4), \qquad (2.37)$$

which is very analogous to the convolution identity on the classical Bernoulli numbers due to Matiyasevich [25], in an equivalent form, as follows,

$$\sum_{k=2}^{n-2}B_kB_{n-k} - 2\sum_{k=2}^{n-2}\binom{n+1}{k-1}\frac{B_kB_{n-k}}{k} = \frac{n(n+1)}{n+2}B_n \quad (n \geq 4). \qquad (2.38)$$

For some similar convolution formulae to (2.37) and (2.38), one is referred to [12, 13, 15]. If we take $k = 3$ in Theorem 2.3, we obtain that for positive integer $n \geq 2$,

$$\begin{aligned}\sum_{j_1+j_2+j_3=n} &\mathcal{G}_{j_1}(x;\lambda_1)\mathcal{G}_{j_2}(x;\lambda_2)\mathcal{G}_{j_3}(x;\lambda_3)\\ &-\frac{3}{n+3}\sum_{j_1+j_2+j_3=n}\binom{n+3}{j_3}\mathcal{G}_{j_1}(\lambda_1)\mathcal{G}_{j_2}(\lambda_2)\mathcal{G}_{j_3}(x;\mu)\\ &+\frac{6}{n+3}\sum_{k=0}^{n-1}\binom{n+3}{k}\mathcal{G}_k(x;\mu)\mathcal{G}_{n-1-k}(\lambda_1)\\ &=\frac{4}{n+3}\binom{n+3}{5}\mathcal{G}_{n-2}(x;\mu) \quad (\mu=\lambda_1\lambda_2\lambda_3). \qquad (2.39)\end{aligned}$$

The case $\lambda_1 = \lambda_2 = \lambda_3 = 1$ in (2.39) gives the corresponding formula of products of the classical Genocchi polynomials, as follows,

$$\begin{aligned}\sum_{j_1+j_2+j_3=n} &G_{j_1}(x)G_{j_2}(x)G_{j_3}(x)-\frac{3}{n+3}\sum_{j_1+j_2+j_3=n}\binom{n+3}{j_3}G_{j_1}G_{j_2}G_{j_3}(x)\\ &+\frac{6}{n+3}\sum_{k=0}^{n-1}\binom{n+3}{k}G_k(x)G_{n-1-k}=\frac{4}{n+3}\binom{n+3}{5}G_{n-2}(x), \qquad (2.40)\end{aligned}$$

which is very analogous to the convolution identity on the classical Euler polynomials presented in [2, Corollary 3].

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References

[1] T. Agoh, Convolution identities for Bernoulli and Genocchi polynomials, Electronic J. Combin., **21** (2014), Article ID P1.65.
[2] T. Agoh, K. Dilcher, Higher-order convolutions for Bernoulli and Euler polynomials, J. Math. Anal. Appl., **419** (2014), 1235–1247.
[3] T.M. Apostol, On the Lerch zeta function, Pacific J. Math., **1** (1951), 161–167.
[4] G.E. Andrews, Euler's Pentagonal Number Theorem, Math. Maga., **56** (1983), 279–284.
[5] J. Bell, A summary of Euler's work on the Pentagonal Number Theorem, Arch. Hist. Exact Sci., **64** (2010), 301–373.
[6] K.N. Boyadzhiev, Apostol-Bernoulli functions, derivative polynomials and Eulerian polynomials, Adv. Appl. Discrete Math., **1** (2008), 109–122.
[7] H. Cohen, Number Theory, Volume II, Analytic and Modern Tools, Graduate Texts in Math., vol. **240**, Springer, New York, 2007.
[8] R. Dere, Y. Simsek, H.M. Srivastava, A unified presentation of three families of generalized Apostol type polynomials based upon the theory of the umbral calculus and the umbral algebra, J. Number Theory, **133** (2013), 3245–3263.
[9] Y. He, C.P. Wang, Some formulae of products of the Apostol-Bernoulli and Apostol-Euler polynomials, Discrete Dyn. Nat. Soc., **2012** (2012), Article ID 927953.
[10] Y. He, C.P. Wang, Recurrence formulae for Apostol-Bernoulli and Apostol-Euler polynomials, Adv. Differ. Equ., **2012** (2012), Article ID 209.

[11] Y. He, S. Araci, Sums of products of Apostol-Bernoulli and Apostol-Euler polynomials, Adv. Differ. Equ., **2014** (2014), Article ID 155.
[12] Y. He, S. Araci, H.M. Srivastava, M. Acikgoz, Some new identities for the Apostol-Bernoulli polynomials and the Apostol-Genocchi polynomials, Appl. Math. Comput., **262** (2015), 31–41.
[13] D.S. Kim, T. Kim, S.-H. Lee, Y.-H. Kim, Some identities for the products of two Bernoulli and Euler polynomials, Adv. Differ. Equ., **2012** (2012), Article ID 95.
[14] D.S. Kim, T. Kim, Umbral calculus associated with Bernoulli polynomials, J. Number Theory, **147** (2015), 871–882.
[15] D.S. Kim, T. Kim, Identities arising from higher-order Daehee polynomials bases, Open Math., **13** (2015), 196–208.
[16] M.S. Kim, S. Hu, Sums of products of Apostol-Bernoulli numbers, Ramanujan J., **28** (2012), 113–123.
[17] T. Kim, Symmetry p-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials, J. Differ. Equ. Appl., **14** (2008), 1267–1277.
[18] T. Kim, L.C. Jang, Y.H. Kim, Some properties on the p-adic invariant integral on $\mathbb{Z}_p$ associated with Genocchi and Bernoulli polynomials, J. Comput. Anal. Appl., **7** (2011), 1201–1207.
[19] T. Kim, T. Mansour, Umbral calculus associated with Frobenius-type Eulerian polynomials, Russian J. Math. Physics, **21** (2014), 484–493.
[20] Q.-M. Luo, H.M. Srivastava, Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials, J. Math. Anal. Appl., **308** (2005), 290–302.
[21] Q.-M. Luo, H.M. Srivastava, Some relationships between the Apostol-Bernoulli and Apostol-Euler polynomials, Comput. Math. Appl., **51** (2006), 631–642.
[22] Q.-M. Luo, Fourier expansions and integral representations for the Apostol-Bernoulli and Apostol-Euler polynomials, Math. Comput., **78** (2009), 2193–2208.
[23] Q.-M. Luo, Extension for the Genocchi polynomials and its Fourier expansions and integral representations, Osaka J. Math., **48** (2011), 291–309.
[24] Q.-M. Luo, H.M. Srivastava, Some generalizations of the Apostol-Genocchi polynomials and the Stirling numbers of the second kind, Appl. Math. Comput., **217** (2011), 5702–5728.
[25] Y. Matiyasevich, Identities with Bernoulli numbers, http://logic.pdmi.ras.ru/ yumat/Journal/Bernoulli/bernulli.htm, 1997.
[26] N. Nielsen, Traité élémentaire des nombres de Bernoulli, Gauthier-Villars, Paris, 1923.
[27] S.H. Rim, E.J. Moon, J.H. Jin, S.J. Lee, On the symmetric properties for the generalized Genocchi polynomials, J. Comput. Anal. Appl., **13** (2011), 1240–1245.
[28] H.M. Srivastava, Some formulas for the Bernoulli and Euler polynomials at rational arguments, Math. Proc. Camb. Philos. Soc., **129** (2000), 77–84.
[29] H.M. Srivastava, Á. Pintér, Remarks on some relationships between the Bernoulli and Euler polynomials, Appl. Math. Letters, **17** (2004), 375–380.
[30] H.M. Srivastava, B. Kurt, Y. Simsek, Some families of Genocchi type polynomials and their interpolation functions, Integral Trans. Special Funct., **23** (2012), 919–938; see also Corrigendum, Integral Trans. Special Funct., **23** (2012), 939–940.
[31] H.M. Srivastava, J. Choi, Zeta and q-Zeta Functions and Associated Series and Integrals, Elsevier Sci. Pub., Amsterdam, London and New York, 2012.
[32] R.P. Stanley, Enumerative Combinatorics, Vol. **1**, Cambridge Univ. Press, 1997.
[33] R. Tremblay, S. Gaboury, B.-J. Fugère, A new class of generalized Apostol-Bernoulli polynomials and some analogues of the Srivastava-Pintér addition theorem, Appl. Math. Letters, **24** (2011), 1888–1893.
[34] S. Waldron, A generalised beta integral and the limit of the Bernstein-Durrmeyer operator with Jacobi weights, J. Approx. Theory, **122** (2003), 141–150.
[35] J.Z. Wang, New recurrence formulae for the Apostol-Bernoulli and Apostol-Euler polynomials, Adv. Differ. Equ., **2013** (2013), Article ID 247.

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FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES

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Abstract. In this paper, we solve the following additive functional inequality

$$N(f(x+y)-f(x)-f(y),t) \quad \geq \quad N\left(f\left(\frac{x+y}{2}\right)-\frac{1}{2}f(x)-\frac{1}{2}f(y),t\right) \tag{0.1}$$

and the following quadratic functional inequality

$$\begin{aligned} &N(f(x+y)+f(x-y)-2f(x)-2f(y),t) \\ &\geq N\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y),t\right) \end{aligned} \tag{0.2}$$

in fuzzy normed spaces.

Using the fixed point method, we prove the Hyers-Ulam stability of the additive functional inequality (0.1) and the quadratic functional inequality (0.2) in fuzzy Banach spaces.

1. Introduction and preliminaries

Katsaras [21] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [13, 24, 52]. In particular, Bag and Samanta [2], following Cheng and Mordeson [8], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [23]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 28, 29] to investigate the Hyers-Ulam stability of a quadratic functional inequality in fuzzy Banach spaces.

Definition 1.1. [2, 28, 29, 30] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x,t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N_3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.
(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed vector space*.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [27, 28].

Definition 1.2. [2, 28, 29, 30] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.

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Definition 1.3. [2, 28, 29, 30] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [3]).

The stability problem of functional equations originated from a question of Ulam [51] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [17] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [40] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. The functional equation $f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called the *Jensen equation*.

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [50] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [9] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. Czerwik [10] proved the Hyers-Ulam stability of the quadratic functional equation. The functional equation $f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called a *Jensen type quadratic equation*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4, 18, 20, 25, 36, 37, 38, 41, 42, 44, 45, 46, 47, 48, 49]).

Gilányi [15] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x-y)\| \leq \|f(x+y)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x+y) + f(x-y).$$

See also [43]. Fechner [12] and Gilányi [16] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [35] investigated the Cauchy additive functional inequality

$$\|f(x) + f(y) + f(z)\| \leq \|f(x+y+z)\| \tag{1.2}$$

and the Cauchy-Jensen additive functional inequality

$$\|f(x) + f(y) + 2f(z)\| \leq \left\|2f\left(\frac{x+y}{2} + z\right)\right\| \tag{1.3}$$

and proved the Hyers-Ulam stability of the functional inequalities (1.2) and (1.3) in Banach spaces.

Park [33, 34] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.4. [5, 11] Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \ge n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [19] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [6, 7, 22, 27, 31, 32, 38, 39]).

In Section 2, we solve the additive functional inequality (0.1) and prove the Hyers-Ulam stability of the additive functional inequality (0.1) in fuzzy Banach spaces by using the fixed point method.

In Section 3, we solve the quadratic functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic functional inequality (0.2) in fuzzy Banach spaces by using the fixed point method.

Throughout this paper, assume that X is a real vector space and (Y, N) is a fuzzy Banach space.

2. Additive functional inequality (0.1)

In this section, we prove the Hyers-Ulam stability of the additive functional inequality (0.1) in fuzzy Banach spaces. We need the following lemma to prove the main results.

Lemma 2.1. *Let $f : X \to Y$ be a mapping such that*

$$N(f(x+y) - f(x) - f(y), t) \quad \ge \quad N\left(f\left(\frac{x+y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y), t\right) \tag{2.1}$$

for all $x, y \in X$ and all $t > 0$. Then f is Cauchy additive, i.e., $f(x+y) = f(x) + f(y)$ for all $x, y \in X$.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $N(f(0), t) = N(0, t) = 1$. So $f(0) = 0$.

Letting $y = x$ in (2.1), we get $N(f(2x) - 2f(x), t) \ge N(0, t) = 1$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} N(f(x+y)-f(x)-f(y),t) &\geq N\left(f\left(\frac{x+y}{2}\right)-\frac{1}{2}f(x)-\frac{1}{2}f(y),t\right) \\ &= N\left(\frac{1}{2}(f(x+y)-f(x)-f(y)),t\right) \\ &= N\left(f(x+y)-f(x)-f(y),2t\right) \end{aligned}$$

for all $t>0$. By (N_5) and (N_6), $N(f(x+y)-f(x)-f(y),t)=1$ for all $t>0$. It follows from (N_2) that

$$f(x+y)=f(x)+f(y)$$

for all $x,y\in X$. $\square$

Theorem 2.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi(x,y) \leq \frac{L}{2}\varphi\left(2x,2y\right)$$

for all $x,y\in X$. Let $f:X\to Y$ be an odd mapping satisfying

$$\begin{aligned} &N\left(f(x+y)-f(x)-f(y),t\right) \\ &\quad \geq \min\left\{N\left(f\left(\frac{x+y}{2}\right)-\frac{1}{2}f(x)-\frac{1}{2}f(y),t\right),\frac{t}{t+\varphi(x,y)}\right\} \end{aligned} \tag{2.3}$$

for all $x,y\in X$ and all $t>0$. Then $A(x):=N\text{-}\lim_{n\to\infty}2^n f\left(\frac{x}{2^n}\right)$ exists for each $x\in X$ and defines an additive mapping $A:X\to Y$ such that

$$N\left(f(x)-A(x),t\right) \geq \frac{(2-2L)t}{(2-2L)t+L\varphi(x,x)} \tag{2.4}$$

for all $x\in X$ and all $t>0$.

Proof. Letting $y=x$ in (2.3), we get

$$N\left(f\left(2x\right)-2f(x),t\right) \geq \frac{t}{t+\varphi(x,x)} \tag{2.5}$$

for all $x\in X$.

Consider the set

$$S:=\{g:X\to Y\}$$

and introduce the generalized metric on S:

$$d(g,h)=\inf\left\{\mu\in\mathbb{R}_+ : N(g(x)-h(x),\mu t)\geq \frac{t}{t+\varphi(x,x)},\ \forall x\in X,\forall t>0\right\},$$

where, as usual, $\inf\phi=+\infty$. It is easy to show that (S,d) is complete (see [26, Lemma 2.1]).

Now we consider the linear mapping $J:S\to S$ such that

$$Jg(x):=2g\left(\frac{x}{2}\right)$$

for all $x\in X$.

Let $g,h\in S$ be given such that $d(g,h)=\varepsilon$. Then

$$N(g(x)-h(x),\varepsilon t)\geq \frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{2}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\varphi(x,x)} = \frac{t}{t + \varphi(x,x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.5) that

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), \frac{L}{2}t\right) \geq \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.4, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{2.6}$$

for all $x \in X$. Since $f : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - A(x), \mu t) \geq \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality

$$d(f, A) \leq \frac{L}{2 - 2L}.$$

This implies that the inequality (2.4) holds.

By (2.3),

$$\begin{aligned} &N\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right), 2^n t\right) \\ &\geq \min\left\{N\left(2^n f\left(\frac{x+y}{2^{n+1}}\right) - 2^{n-1} f\left(\frac{x}{2^n}\right) - 2^{n-1} f\left(\frac{y}{2^n}\right), 2^n t\right), \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}\right\} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$N\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right), t\right)$$
$$\geq \min\left\{N\left(2^n f\left(\frac{x+y}{2^{n+1}}\right) - 2^{n-1} f\left(\frac{x}{2^n}\right) - 2^{n-1} f\left(\frac{y}{2^n}\right), t\right), \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n}\varphi(x,y)}\right\}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$N\left(A(x+y) - A(x) - A(y), t\right) \geq N\left(A\left(\frac{x+y}{2}\right) - \frac{1}{2}A(x) - \frac{1}{2}A(y), t\right)$$

for all $x, y \in X$ and all $t > 0$. By Lemma 2.1, the mapping $A : X \to Y$ is Cauchy additive. □

Corollary 2.3. *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying*

$$N\left(f(x+y) - f(x) - f(y), t\right)$$
$$\geq \min\left\{N\left(f\left(\frac{x+y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y), t\right), \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}\right\}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{1-p}$, and we get the desired result. □

Theorem 2.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying (2.3). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{(2 - 2L)t}{(2 - 2L)t + \varphi(x, x)} \tag{2.7}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (2.5) that

$$N\left(f(x) - \frac{1}{2}f(2x), \frac{1}{2}t\right) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in X$. So $d(f, Jf) \leq \frac{1}{2}$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\| \cdot \|$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\begin{aligned} &N\left(f(x+y) - f(x) - f(y), t\right) \\ &\quad \geq \min\left\{ N\left(f\left(\frac{x+y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y), t\right), \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \right\} \end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{(2-2^p)t}{(2-2^p)t + 2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-1}$, and we get the desired result. □

3. Quadratic functional inequality (0.2)

In this section, we prove the Hyers-Ulam stability of the quadratic functional inequality (0.2) in fuzzy Banach spaces. We need the following lemma to prove the main results.

Lemma 3.1. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) \\ &\quad \geq N\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y), t\right) \end{aligned} \tag{3.1}$$

for all $x, y \in X$ and all $t > 0$. Then f is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = x$ in (3.1), we get $N(f(2x) - 4f(x), t) \geq N(0, t) = 1$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} &N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) \\ &\quad \geq N\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y), t\right) \\ &\quad = N\left(\frac{1}{2}\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right), t\right) \\ &\quad = N\left(f(x+y) + f(x-y) - 2f(x) - 2f(y), 2t\right) \end{aligned}$$

for all $t > 0$. By (N_5) and (N_6), $N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) = 1$ for all $t > 0$. It follows from (N_2) that $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$. □

Theorem 3.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{4}\varphi(2x, 2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\begin{aligned} &N(f(x+y)+f(x-y)-2f(x)-2f(y),t) \\ &\geq \min\left\{N\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y),t\right),\frac{t}{t+\varphi(x,y)}\right\} \end{aligned} \tag{3.3}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x)-Q(x),t\right) \geq \frac{(4-4L)t}{(4-4L)t+L\varphi(x,x)} \tag{3.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = x$ in (3.3), we get

$$N\left(f\left(2x\right)-4f(x),t\right) \geq \frac{t}{t+\varphi(x,x)} \tag{3.5}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x)-h(x),\mu t) \geq \frac{t}{t+\varphi(x,x)},\ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [26, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x)-h(x),\varepsilon t) \geq \frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x)-Jh(x),L\varepsilon t) &= N\left(4g\left(\frac{x}{2}\right)-4h\left(\frac{x}{2}\right),L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right)-h\left(\frac{x}{2}\right),\frac{L}{4}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{4}}{\frac{Lt}{4}+\varphi\left(\frac{x}{2},\frac{x}{2}\right)} \geq \frac{\frac{Lt}{4}}{\frac{Lt}{4}+\frac{L}{4}\varphi(x,x)} = \frac{t}{t+\varphi(x,x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg,Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (3.5) that $N\left(f(x)-4f\left(\frac{x}{2}\right),\frac{L}{4}t\right) \geq \frac{t}{t+\varphi(x,x)}$ for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{4}$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{3.6}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (3.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{L}{4 - 4L}.$$

This implies that the inequality (3.4) holds.

By (3.3),

$$\begin{aligned} & N\left(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right), 4^n t\right) \\ & \geq \min\left\{N\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right), 4^n t\right), \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}\right\} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\begin{aligned} & N\left(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right), t\right) \\ & \geq \min\left\{N\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right), t\right), \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x, y)}\right\} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$\begin{aligned} & N\left(Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y), t\right) \\ & \geq N\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y), t\right) \end{aligned}$$

for all $x, y \in X$ and all $t > 0$. By Lemma 3.1, the mapping $Q : X \to Y$ is quadratic, as desired. □

Corollary 3.3. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} & N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) \\ & \geq \min\left\{N\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y), t\right), \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}\right\} \end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 2\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{2-p}$, and we get the desired result. $\square$

Theorem 3.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.3). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \ge \frac{(4-4L)t}{(4-4L)t + \varphi(x,x)} \tag{3.7}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.2.

It follows from (3.5) that

$$N\left(f(x) - \frac{1}{4}f(2x), \frac{1}{4}t\right) \ge \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4} g(2x)$$

for all $x \in X$. So $d(f, Jf) \le \frac{1}{4}$.

The rest of the proof is similar to the proof of Theorem 3.2. $\square$

Corollary 3.5. *Let $\theta \ge 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) \\ &\ge \min\left\{N\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y), t\right), \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}\right\} \end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \ge \frac{(4-2^p)t}{(4-2^p)t + 2\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-2}$, and we get the desired result. $\square$

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

[2] T. Bag and S.K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math. **11** (2003), 687–705.

[3] T. Bag and S.K. Samanta, *Fuzzy bounded linear operators*, Fuzzy Sets and Systems **151** (2005), 513–547.

[4] L. Cădariu, L. Găvruta, P. Găvruta, On the stability of an affine functional equation, J. Nonlinear Sci. Appl. **6** (2013), 60–67.

[5] L. Cădariu and V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math. **4**, no. 1, Art. ID 4 (2003).
[6] L. Cădariu and V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber. **346** (2004), 43–52.
[7] L. Cădariu and V. Radu, *Fixed point methods for the generalized stability of functional equations in a single variable*, Fixed Point Theory and Applications **2008**, Art. ID 749392 (2008).
[8] S.C. Cheng and J.M. Mordeson, *Fuzzy linear operators and fuzzy normed linear spaces*, Bull. Calcutta Math. Soc. **86** (1994), 429–436.
[9] P.W. Cholewa, *Remarks on the stability of functional equations*, Aequationes Math. **27** (1984), 76–86.
[10] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg **62** (1992), 59–64.
[11] J. Diaz and B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305–309.
[12] W. Fechner, *Stability of a functional inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Math. **71** (2006), 149–161.
[13] C. Felbin, *Finite dimensional fuzzy normed linear spaces*, Fuzzy Sets and Systems **48** (1992), 239–248.
[14] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
[15] A. Gilányi, *Eine zur Parallelogrammgleichung äquivalente Ungleichung*, Aequationes Math. **62** (2001), 303–309.
[16] A. Gilányi, *On a problem by K. Nikodem*, Math. Inequal. Appl. **5** (2002), 707–710.
[17] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.
[18] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.
[19] G. Isac and Th.M. Rassias, *Stability of ψ-additive mappings: Appications to nonlinear analysis*, Internat. J. Math. Math. Sci. **19** (1996), 219–228.
[20] S. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press lnc., Palm Harbor, Florida, 2001.
[21] A.K. Katsaras, *Fuzzy topological vector spaces II*, Fuzzy Sets and Systems **12** (1984), 143–154.
[22] J. Kim, G. A. Anastassiou and C. Park, *Additive ρ-functional inequalities in fuzzy normed spaces,* J. Comput. Anal. Appl. (to appear).
[23] I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica **11** (1975), 326–334.
[24] S.V. Krishna and K.K.M. Sarma, *Separation of fuzzy normed linear spaces*, Fuzzy Sets and Systems **63** (1994), 207–217.
[25] J. Lee, C. Park and D. Shin, *An AQCQ-functional equation in matrix normed spaces*, Results Math. **27** (2013), 305–318.
[26] D. Miheţ and V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008), 567–572.
[27] M. Mirzavaziri and M.S. Moslehian, *A fixed point approach to stability of a quadratic equation*, Bull. Braz. Math. Soc. **37** (2006), 361–376.
[28] A.K. Mirmostafaee, M. Mirzavaziri and M.S. Moslehian, *Fuzzy stability of the Jensen functional equation*, Fuzzy Sets and Systems **159** (2008), 730–738.
[29] A.K. Mirmostafaee and M.S. Moslehian, *Fuzzy versions of Hyers-Ulam-Rassias theorem*, Fuzzy Sets and Systems **159** (2008), 720–729.
[30] A.K. Mirmostafaee and M.S. Moslehian, *Fuzzy approximately cubic mappings*, Inform. Sci. **178** (2008), 3791–3798.
[31] C. Park, *Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras*, Fixed Point Theory and Applications **2007**, Art. ID 50175 (2007).
[32] C. Park, *Generalized Hyers-Ulam-Rassias stability of quadratic functional equations: a fixed point approach*, Fixed Point Theory and Applications **2008**, Art. ID 493751 (2008).
[33] C. Park, *Additive ρ-functional inequalities and equations,* J. Math. Inequal. **9** (2015), 17–26.
[34] C. Park, *Additive ρ-functional inequalities in non-Archimedean normed spaces,* J. Math. Inequal. **9** (2015), 397–407.

[35] C. Park, Y. Cho and M. Han, *Stability of functional inequalities associated with Jordan-von Neumann type additive functional equations*, J. Inequal. Appl. **2007**, Art. ID 41820 (2007).
[36] C. Park, K. Ghasemi, S. G. Ghaleh and S. Jang, *Approximate n-Jordan $*$-homomorphisms in C^*-algebras,* J. Comput. Anal. Appl. **15** (2013), 365-368.
[37] C. Park, A. Najati and S. Jang, *Fixed points and fuzzy stability of an additive-quadratic functional equation,* J. Comput. Anal. Appl. **15** (2013), 452–462.
[38] C. Park and Th.M. Rassias, *Fixed points and generalized Hyers-Ulam stability of quadratic functional equations,* J. Math. Inequal. **1** (2007), 515–528.
[39] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.
[40] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[41] Th.M. Rassias, *Problem 16; 2*, Report of the 27^{th} International Symp. on Functional Equations, Aequationes Math. **39** (1990), 292–293; 309.
[42] Th.M. Rassias and K. Shibata, *Variational problem of some quadratic functionals in complex analysis*, J. Math. Anal. Appl. **228** (1998), 234–253.
[43] J. Rätz, *On inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Math. **66** (2003), 191–200.
[44] L. Reich, J. Smítal and M. Štefánková, *Singular solutions of the generalized Dhombres functional equation*, Results Math. **65** (2014), 251–261.
[45] S. Schin, D. Ki, J. Chang and M. Kim, *Random stability of quadratic functional equations: a fixed point approach*, J. Nonlinear Sci. Appl. **4** (2011), 37–49.
[46] S. Shagholi, M. Bavand Savadkouhi and M. Eshaghi Gordji, *Nearly ternary cubic homomorphism in ternary Fréchet algebras,* J. Comput. Anal. Appl. **13** (2011), 1106–1114.
[47] S. Shagholi, M. Eshaghi Gordji and M. Bavand Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras,* J. Comput. Anal. Appl. **13** (2011), 1097–1105.
[48] D. Shin, C. Park and Sh. Farhadabadi, *On the superstability of ternary Jordan C^*-homomorphisms,* J. Comput. Anal. Appl. **16** (2014), 964–973.
[49] D. Shin, C. Park and Sh. Farhadabadi, *Stability and superstability of J^*-homomorphisms and J^*-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.
[50] F. Skof, *Proprietà locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.
[51] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.
[52] J.Z. Xiao and X.H. Zhu, *Fuzzy normed spaces of operators and its completeness*, Fuzzy Sets and Systems **133** (2003), 389–399.

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A generalization of Simpson type inequality via differentiable functions using extended $(s,m)_\phi$-preinvex functions

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Abstract

Some new class of preinvex functions which called (s,m)-preinvex, extended (s,m)-preinvex, $(s,m)_\phi$-preinvex and extended $(s,m)_\phi$-preinvex function are introduced respectively in this paper. An integral identity is established, and then we prove some Simpson type integral inequalities, dealing with the existing similar type integral inequalities in a relatively uniform frame. In particular, we also show some results obtained by these inequalities for extended $(s,m)_\phi$-preinvex under some suitable conditions, which improve the previously known results.

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1 Introduction

The following notations are used throughout this paper. I is an interval on the real line $\mathbb{R}$, $\mathbb{R}_0 = [0,\infty)$. $\mathbb{R}^n$ is used to denote a generic n-dimensional vector space, $\mathbb{R}^n_0$ denotes an n-dimensional nonegative vector space, and $\mathbb{R}^n_+$ denotes an n-dimensional positive vector space. For any subset $K \subseteq \mathbb{R}^n$, $L_1[a,b]$ is the set of integrable functions over the interval $[a,b]$. Let us firstly recall some definitions of various convex type functions.

Definition 1.1 *([6]) A function $f : I \subseteq \mathbb{R} \to \mathbb{R}_0$ is said to be a Godunova-Levin function if f is nonnegative and for all $x, y \in I$, $\lambda \in (0,1)$ we have that*

$$f\big(\lambda x + (1-\lambda)y\big) \le \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda}.$$

Definition 1.2 *([5]) For some $(s,m) \in (0,1]^2$, a function $f : [0,b] \to \mathbb{R}$ is said to be (s,m)-convex in the second sense if for every $x, y \in [0,b]$ and $\lambda \in (0,1]$ we have that*

$$f\big(\lambda x + m(1-\lambda)y\big) \le \lambda^s f(x) + m(1-\lambda)^s f(y).$$

Definition 1.3 *([36]) For some $s \in [-1,1]$ and $m \in (0,1]$, a function $f : [0,b] \to \mathbb{R}_0$ is said to be extended (s,m)-convex if for all $x, y \in [0,b]$ and $\lambda \in (0,1)$ we have that*

$$f\big(\lambda x + m(1-\lambda)y\big) \le \lambda^s f(x) + m(1-\lambda)^s f(y).$$

Definition 1.4 *([1]) A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the map $\eta : K \times K \to \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.*

Notice that every convex set is invex with respect to the map $\eta(y,x) = y-x$, but the converse is not necessarily true. For more details please refer to [1, 37] and the references therein.

Definition 1.5 *([1]) Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$, for every $x, y \in K$, the η-path $P_{x\nu}$ joining the points x and $\nu = x + \eta(y,x)$ is defined by*

$$P_{x\nu} = \big\{z | z = x + t\eta(y,x),\ t \in [0,1]\big\}.$$

Definition 1.6 *([27]) The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect to η if for every $x, y \in K$ and $t \in [0,1]$ we have that*

$$f\big(x + t\eta(y,x)\big) \le (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the map $\eta(y,x) = y - x$, but the converse is not true.

Definition 1.7 *([13]) The function f defined on the invex set $K \subseteq [0,b^*]$ with $b^* > 0$ is said to be m-preinvex with respect to η if for all $x, y \in K$, $t \in [0,1]$ and for some fixed $m \in (0,1]$, we have that*

$$f\big(x + t\eta(y,x)\big) \le (1-t)f(x) + mtf\Big(\frac{y}{m}\Big).$$

Remark 1.1 *Notice that if $y \in [0,b^*]$, then for any $0 < m < 1$, $\dfrac{y}{m}$ could be greater than b^*, which is not in the domain of f. Thus, the right hand side of the inequality in this definition could be meaningless. To fix this flaw, we suggest to replace $[0,b^*]$ by the half real line $\mathbb{R}_0$.*

Definition 1.8 *([14]) Let $K \subseteq \mathbb{R}_0$ be an invex set with respect to η. A function $f : K \to \mathbb{R}$ is said to be s-preinvex with respect to η, if for all $x, y \in K$, $t \in [0,1]$ and some fixed $s \in (0,1]$ we have that*

$$f\big(x + t\eta(y,x)\big) \le (1-t)^s f(x) + t^s f(y).$$

Definition 1.9 *([21]) The set $K_{\phi\eta} \subseteq \mathbb{R}^n$ is said to be ϕ-invex at u with respect to $\phi(.)$, if there exists a bifunction $\eta(.,.) : K_{\phi\eta} \times K_{\phi\eta} \to \mathbb{R}^n$, such that*

$$u + te^{i\phi}\eta(v,u) \in K_{\phi\eta}, \quad \forall u, v \in K_{\phi\eta}, t \in [0,1].$$

The ϕ-invex set $K_{\phi\eta}$ is also called $\phi\eta$-connected set. Note that the convex set with $\phi = 0$ and $\eta(v,u) = v - u$ is a ϕ-invex set, but the converse is not true (see [21]).

Definition 1.10 *([22]) For some fixed $s \in (0,1]$, a function f on the set $K_{\phi\eta}$ is said to be s_φ-preinvex function with respect to ϕ and η, if*

$$f\big(u + te^{i\phi}\eta(v,u)\big) \le (1-t)^s f(u) + t^s f(v), \quad \forall u, v \in K_{\phi\eta}, t \in [0,1].$$

Definition 1.11 *([22]) A function f on the set C_ϕ is said to be ϕ-convex function with respect to ϕ, if and only if*

$$f\big(u + te^{i\phi}(v-u)\big) \le (1-t)f(u) + tf(v), \quad \forall u, v \in C_\phi, t \in [0,1].$$

The following inequality is very remarkable and well known in the literature as Simpson type inequality, which plays an important role in analysis. Particularly, it is well applied in numerical integration.

Theorem 1.1 *([4]) Let $f : [a,b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $||f^{(4)}||_\infty = \sup_{x\in(a,b)} |f^{(4)}(x)| < \infty$. Then the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f(\frac{a+b}{2})\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{1}{2880}||f^{(4)}||_\infty (b-a)^4. \quad (1.1)$$

In recent decades, a lot of inequalities of Simpson type and Hadamard type for various kinds of convex functions have been established and developed by many scholars, some of them may be reformulated as follows.

Theorem 1.2 *([30]) Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1[a,b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$, then*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a) + f(b) + 4f\Big(\frac{a+b}{2}\Big)\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\ &\le \frac{(s-4)6^{s+1} + 2\times 5^{s+2} - 2\times 3^{s+2} + 2}{6^{s+2}(s+1)(s+2)}(b-a)\Big[|f'(a)| + |f'(b)|\Big]. \quad (1.2)\end{aligned}$$

Theorem 1.3 *([4]) Let $f : [a, b] \to \mathbb{R}$ is a differentiable mapping whose derivative is continuous on (a, b) and $||f'||_1 = \int_a^b |f'(x)|\mathrm{d}x < \infty$. Then we have the inequality:*

$$\Big| \int_a^b f(x)\mathrm{d}x - \frac{b-a}{3}\Big[\frac{f(a)+f(b)}{2} + 2f\Big(\frac{a+b}{2}\Big)\Big]\Big| \le \frac{1}{3}||f'||_1(b-a)^2. \tag{1.3}$$

Theorem 1.4 *([35]) Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I$ with $a < b$, $f' \in L_1[a, b]$ and $0 \le \lambda, \mu \le 1$. If $|f'(x)|^q$ for $q \ge 1$ is an extended s-convex on $[a, b]$ for some $s \in [-1, 1]$, specially, when $q = 1$ and $s = -1$, the following inequality holds:*

$$\Big| f\Big(\frac{a+b}{2}\Big) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \Big| \le (b-a)\ln 2\Big(|f'(a)| + |f'(b)|\Big). \tag{1.4}$$

Theorem 1.5 *([2, 29]) Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $|f'|$ is preinvex on K, then, for every $a, b \in K$ with $\eta(b, a) \neq 0$ we have that:*

$$\begin{aligned} &\Big| \frac{f(a) + f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \Big| \\ &\le \frac{|\eta(b,a)|}{8}\Big(|f'(a)| + |f'(b)|\Big) \end{aligned} \tag{1.5}$$

and

$$\Big| f\Big(\frac{2a+\eta(b,a)}{2}\Big) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \Big| \le \frac{|\eta(b,a)|}{8}\Big(|f'(a)| + |f'(b)|\Big). \tag{1.6}$$

Theorem 1.6 *([33]) Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. If $q > 1$, $q \ge r$, $s \ge 0$ and $|f'|$ is preinvex on A, then for every $a, b \in A$ with $\eta(a, b) \neq 0$, we have that*

$$\begin{aligned} &\Big| f\Big(\frac{2b+\eta(a,b)}{2}\Big) - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\mathrm{d}x \Big| \\ &\le \frac{|\eta(a,b)|}{4}\Bigg\{\Big(\frac{1}{r+1}\Big)^{\frac{1}{q}}\Big(\frac{q-1}{2q-r-1}\Big)^{1-\frac{1}{q}}\Big[\frac{(r+1)|f'(a)|^q + (r+3)|f'(b)|^q}{2(r+2)}\Big]^{\frac{1}{q}} \\ &\quad + \Big(\frac{1}{s+1}\Big)^{\frac{1}{q}}\Big(\frac{q-1}{2q-s-1}\Big)^{1-\frac{1}{q}}\Big[\frac{(s+3)|f'(a)|^q + (s+1)|f'(b)|^q}{2(s+2)}\Big]^{\frac{1}{q}}\Bigg\}. \end{aligned}$$

Corollary 1.1 *([33]) Under the conditions of Theorem 1.6, when $r = s = 0$,* *the following inequality holds:*

$$\begin{aligned}
&\left|f\left(\frac{2b+\eta(a,b)}{2}\right)-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\mathrm{d}x\right|\\
&\leq\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\frac{|\eta(a,b)|}{4}\left[\left(\frac{1}{4}|f'(a)|^q+\frac{3}{4}|f'(b)|^q\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\frac{3}{4}|f'(a)|^q+\frac{1}{4}|f'(b)|^q\right)^{\frac{1}{q}}\right]. \qquad (1.7)
\end{aligned}$$

Theorem 1.7 *([22]) Let $I\subseteq\mathbb{R}$ be an open ϕ-invex set with respect to $\eta: I\times I\to\mathbb{R}$. Suppose that $f:I\to\mathbb{R}$ is a differentiable function such that $f'\in L_1[a,a+e^{i\phi}\eta(b,a)]$. If $|f'|$ is ϕ-preinvex on I , then, for $\eta(b,a)>0$,*

$$\begin{aligned}
&\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2}-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)}f(x)\mathrm{d}x\right|\\
&\leq\frac{|e^{i\phi}\eta(b,a)|}{8}\Big(|f'(a)|+|f'(b)|\Big). \qquad (1.8)
\end{aligned}$$

Currently, the Simpson type inequalities concerning different kinds of preinvex and ϕ-convex functions are still interesting research topics to many researchers in the field of convex analysis. For more information please refer to [7–12, 15, 17–20, 23–26, 31, 32, 34] and references cited therein.

Motivated by the inspiring idea in [3, 13, 21, 28] and based on our previous works [16, 38], in this paper we are mainly going to introduce the $(s,m)_\phi$-preinvex function and the extended $(s,m)_\phi$-preinvex function, and then we will establish some Simpson type integral inequalities for extended $(s,m)_\phi$-preinvex functions. In Section 2, we will introduce new definitions and an integral identity. Section 3 will be devoted of presenting the main results.

2 New definitions and an integral identity

We now mainly introduce some new concepts about preinvex function. The class of $(s,m)_\phi$-preinvex function is quite a general and unifying one. This is one of the main motivation of this paper.

Definition 2.1 *Let $K\subseteq\mathbb{R}_0^n$ be an open invex set with respect to $\eta:K\times K\to\mathbb{R}_+^n$. For $f:K\to\mathbb{R}$ and some fixed $(s,m)\in(0,1]\times(0,1]$, if*

$$f\Big(x+\lambda\eta(y,x)\Big)\leq(1-\lambda)^sf(x)+m\lambda^sf\Big(\frac{y}{m}\Big) \qquad (2.1)$$

is valid for all $x,y\in K$, $\lambda\in[0,1]$, then we say that $f(x)$ is an (s,m)-preinvex function with respect to η.

Definition 2.2 *Let $K \subseteq \mathbb{R}_0^n$ be an open invex set with respect to $\eta : K \times K \to \mathbb{R}_+^n$. For $f : K \to \mathbb{R}_0$ and some fixed $(s,m) \in [-1,1] \times (0,1]$, if*

$$f\Big(x + \lambda\eta(y,x)\Big) \leq (1-\lambda)^s f(x) + m\lambda^s f\Big(\frac{y}{m}\Big) \tag{2.2}$$

is valid for all $x, y \in K$, $\lambda \in [0,1]$, then we say that $f(x)$ is an extended (s,m)-preinvex function with respect to η.

Remark 2.1 *In Definition 2.1, if $s = 1$ then one obtains the usual definition of m-preinvex function. If $m = 1$ then one obtains the usual definition of s-preinvex function. It is also worthwhile to note that every (s,m)-preinvex function is (s,m)-convex and every extended (s,m)-preinvex functions is extended (s,m)-convex with respect to $\eta(y,x) = y - x$ respectively.*

Definition 2.3 *A function f on the set $K_{\phi\eta} \subseteq \mathbb{R}_0^n$ is said to be $(s,m)_\varphi$-preinvex function with respect to $\phi(.)$ and $\eta(.,.)$. For $f : K_{\phi\eta} \to \mathbb{R}$ and some fixed $(s,m) \in (0,1] \times (0,1]$, if*

$$f\Big(x+\lambda e^{i\phi}\eta(y,x)\Big) \leq (1-\lambda)^s f(x)+m\lambda^s f\Big(\frac{y}{m}\Big), \quad \forall x,y \in K_{\phi\eta},\ \lambda \in [0,1]. \tag{2.3}$$

Remark 2.2 *In Definition 2.3, if $\phi = 0$ then it reduces to the definition for (s,m)-preinvex function. If $m = 1$ then it reduces to the definition for s_φ-preinvex function. Also, it is obvious that Definition 2.3 is the ϕ-convex function when $\eta(y,x) = y - x$ and $s = m = 1$.*

Definition 2.4 *A function f on the set $K_{\phi\eta} \subseteq \mathbb{R}_0^n$ is said to be extended $(s,m)_\varphi$-preinvex function with respect to $\phi(.)$ and $\eta(.,.)$. For $f : K_{\phi\eta} \to \mathbb{R}_0$ and some fixed $(s,m) \in [-1,1] \times (0,1]$, if*

$$f\Big(x+\lambda e^{i\phi}\eta(y,x)\Big) \leq (1-\lambda)^s f(x)+m\lambda^s f\Big(\frac{y}{m}\Big), \quad \forall x,y \in K_{\phi\eta},\ \lambda \in [0,1]. \tag{2.4}$$

In order to establish some new Simpson type integral inequalities, we need the following key integral identity, which will be used in the sequel.

Lemma 2.1 *Let $K_{\phi\eta} \subseteq \mathbb{R}$ be a ϕ-invex subset with respect to $\phi(.)$ and $\eta : K_{\phi\eta} \times K_{\phi\eta} \subseteq \mathbb{R}$, $a, b \in K_{\phi\eta}$ with $a < a + \eta(b,a)$. If $k, t \in \mathbb{R}$, $f : K_{\phi\eta} \to \mathbb{R}$ is a differentiable function and $f' \in L[a, a + e^{i\phi}\eta(b,a)]$ we have that*

$$\begin{aligned}
&tf(a) + (1-k)f\Big(a + e^{i\phi}\eta(b,a)\Big) + (k-t)f\Big(a + \frac{e^{i\phi}\eta(b,a)}{2}\Big) \\
&\quad -\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x \\
&= e^{i\phi}\eta(b,a)\Big[\int_0^{\frac{1}{2}}(\lambda - t)f'\Big(a + \lambda e^{i\phi}\eta(b,a)\Big)\mathrm{d}\lambda \\
&\quad + \int_{\frac{1}{2}}^1 (\lambda - k)f'\Big(a + \lambda e^{i\phi}\eta(b,a)\Big)\mathrm{d}\lambda\Big].
\end{aligned} \tag{2.5}$$

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Proof. Set

$$J = e^{i\phi}\eta(b,a)\Big[\textstyle\int_0^{\frac{1}{2}}(\lambda - t)f'\Big(a + \lambda e^{i\phi}\eta(b,a)\Big)\mathrm{d}\lambda$$
$$+\textstyle\int_{\frac{1}{2}}^{1}(\lambda - k)f'\Big(a + \lambda e^{i\phi}\eta(b,a)\Big)\mathrm{d}\lambda\Big].$$

Since $a, b \in K_{\phi\eta}$ and $K_{\phi\eta}$ is ϕ-invex subset with respect to ϕ and η, for every $t \in [0,1]$, we have $a + \lambda e^{i\phi}\eta(b,a) \in K_{\phi\eta}$. Integrating by part, it yields that

$$\begin{aligned}
J = e^{i\phi}\eta(b,a)\Bigg\{&\frac{1}{e^{i\phi}\eta(b,a)}\Bigg[(\lambda - t)f\Big(a + \lambda e^{i\phi}\eta(b,a)\Big)\Big|_0^{\frac{1}{2}} \\
&- \int_0^{\frac{1}{2}} f\Big(a + \lambda e^{i\phi}\eta(b,a)\Big)\mathrm{d}\lambda\Bigg] \\
&+ \frac{1}{e^{i\phi}\eta(b,a)}\Bigg[(\lambda - k)f\Big(a + \lambda e^{i\phi}\eta(b,a)\Big)\Big|_{\frac{1}{2}}^{1} - \int_{\frac{1}{2}}^{1} f\Big(a + \lambda e^{i\phi}\eta(b,a)\Big)\mathrm{d}\lambda\Bigg]\Bigg\} \\
= &\Big(\frac{1}{2} - t\Big)f\Bigg(a + \frac{e^{i\phi}\eta(b,a)}{2}\Bigg) + tf(a) - \int_0^{\frac{1}{2}} f\Big(a + \lambda e^{i\phi}\eta(b,a)\Big)\mathrm{d}\lambda \\
&+ (1-k)f\Big(a + e^{i\phi}\eta(b,a)\Big) - \Big(\frac{1}{2} - k\Big)f\Bigg(a + \frac{e^{i\phi}\eta(b,a)}{2}\Bigg) \\
&- \int_{\frac{1}{2}}^{1} f\Big(a + \lambda e^{i\phi}\eta(b,a)\Big)\mathrm{d}\lambda \\
= &\,tf(a) + (1-k)f\Big(a + e^{i\phi}\eta(b,a)\Big) + (k - t)f\Bigg(a + \frac{e^{i\phi}\eta(b,a)}{2}\Bigg) \\
&- \int_0^1 f\Big(a + \lambda e^{i\phi}\eta(b,a)\Big)\mathrm{d}\lambda.
\end{aligned}$$

Let $x = a + \lambda e^{i\phi}\eta(b,a)$, then $\mathrm{d}x = e^{i\phi}\eta(b,a)\mathrm{d}\lambda$ and we have

$$\begin{aligned}
J = &\,tf(a) + (1-k)f\Big(a + e^{i\phi}\eta(b,a)\Big) + (k - t)f\Bigg(a + \frac{e^{i\phi}\eta(b,a)}{2}\Bigg) \\
&- \frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x,
\end{aligned}$$

which is required.

Remark 2.1 *Clearly, applying Lemma 2.1 for* $\phi = 0$, $\eta(b,a) = b - a$, $t = \frac{1}{6}$, *and* $k = \frac{5}{6}$, *then we obtain the Lemma 2.1 in ([28], 2013).*

3 Some Simpson type integral inequalities

In what follows, we establish another refinement of the Simpson's inequality for extended $(s,m)_\phi$-preinvex functions in the second sense.

Theorem 3.1 *Let $A_{\phi\eta} \subseteq \mathbb{R}_0$ be an open ϕ-invex subset with respect to $\phi(.)$ and $\eta : A_{\phi\eta} \times A_{\phi\eta} \to \mathbb{R}_0$, $a, b \in A_{\phi\eta}$ with $a < a + e^{i\phi}\eta(b,a)$, $a < b$. Let $k, t \in \mathbb{R}$. Suppose that $f : A_{\phi\eta} \to \mathbb{R}_0$ is a differentiable function and f' is integrable on $[a, a + e^{i\phi}\eta(b,a)]$. If $|f'|$ is extended $(s,m)_\phi$-preinvex on $A_{\phi\eta}$ for some fixed $(s,m) \in [-1,1] \times (0,1]$ then the following inequality holds:*
1. when $s \in (-1,1]$, we have

$$\begin{aligned}
&\bigg|tf(a) + (1-k)f\Big(a + e^{i\phi}\eta(b,a)\Big) + (k-t)f\bigg(a + \frac{e^{i\phi}\eta(b,a)}{2}\bigg) \\
&\quad - \frac{1}{e^{i\phi}\eta(b,a)} \int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\bigg| \\
&\le \big|e^{i\phi}\eta(b,a)\big| \bigg[\nu_1|f'(a)| + m\nu_2\Big|f'\Big(\frac{b}{m}\Big)\Big|\bigg], \qquad (3.1)
\end{aligned}$$

where

$$v_1 = \frac{2(1-t)^{s+2} + 2(1-k)^{s+2} + \big[2(k+t)(s+2) - 2(s+3)\big]\frac{1}{2^{s+2}} + (ts + 2t - 1)}{(s+1)(s+2)}$$

and

$$v_2 = \frac{2t^{s+2} + 2k^{s+2} + \big[2(s+1) - 2(s+2)(k+t)\big]\frac{1}{2^{s+2}} + (s + 1 - ks - 2k)}{(s+1)(s+2)};$$

2. when $s = -1$, $t = 0$ and $k = 1$, we have

$$\begin{aligned}
&\bigg|f\bigg(a + \frac{e^{i\phi}\eta(b,a)}{2}\bigg) - \frac{1}{e^{i\phi}\eta(b,a)} \int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\bigg| \\
&\le \big|e^{i\phi}\eta(b,a)\big| \ln 2\bigg[|f'(a)| + m\Big|f'\Big(\frac{b}{m}\Big)\Big|\bigg]. \qquad (3.2)
\end{aligned}$$

Proof. 1. When $-1 < s \le 1$, by Lemma 2.1 and using the extended $(s,m)_{\phi\eta}$-preinvexity of $|f'|$ on $A_{\phi\eta}$, we have

$$\begin{aligned}
&\bigg|tf(a) + (1-k)f\Big(a + e^{i\phi}\eta(b,a)\Big) + (k-t)f\bigg(a + \frac{e^{i\phi}\eta(b,a)}{2}\bigg) \\
&\quad - \frac{1}{e^{i\phi}\eta(b,a)} \int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\bigg| \\
&\le \big|e^{i\phi}\eta(b,a)\big| \bigg[\int_0^{\frac{1}{2}} |\lambda - t| \Big|f'\Big(a + e^{i\phi}\lambda\eta(b,a)\Big)\Big|\mathrm{d}\lambda \\
&\quad + \int_{\frac{1}{2}}^1 |\lambda - k| \Big|f'\Big(a + \lambda e^{i\phi}\eta(b,a)\Big)\Big|\mathrm{d}\lambda\bigg]
\end{aligned}$$

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$$
\begin{aligned}
&\le \left|e^{i\phi}\eta(b,a)\right|\bigg\{\int_0^{\frac12}|\lambda-t|(1-\lambda)^s|f'(a)|\mathrm{d}\lambda+m\int_0^{\frac12}|\lambda-t|\lambda^s\Big|f'\Big(\frac{b}{m}\Big)\Big|\mathrm{d}\lambda\\
&\quad+\int_{\frac12}^1|\lambda-k|(1-\lambda)^s|f'(a)|\mathrm{d}\lambda+m\int_{\frac12}^1|\lambda-k|\lambda^s\Big|f'\Big(\frac{b}{m}\Big)\Big|\mathrm{d}\lambda\bigg\}\\
&=\left|e^{i\phi}\eta(b,a)\right|\bigg\{\bigg[\int_0^{\frac12}|\lambda-t|(1-\lambda)^s\mathrm{d}\lambda+\int_{\frac12}^1|\lambda-k|(1-\lambda)^s\mathrm{d}\lambda\bigg]\left|f'(a)\right|\\
&\quad+\bigg[m\int_0^{\frac12}|\lambda-t|\lambda^s\mathrm{d}\lambda+m\int_{\frac12}^1|\lambda-k|\lambda^s\mathrm{d}\lambda\bigg]\Big|f'\Big(\frac{b}{m}\Big)\Big|\bigg\}.
\end{aligned}
$$

Using the fact that

$$
\begin{aligned}
&\int_0^{\frac12}|\lambda-t|(1-\lambda)^s\mathrm{d}\lambda+\int_{\frac12}^1|\lambda-k|(1-\lambda)^s\mathrm{d}\lambda\\
&=\frac{2(1-t)^{s+2}+2(1-k)^{s+2}+\big[2(k+t)(s+2)-2(s+3)\big]\frac{1}{2^{s+2}}+(ts+2t-1)}{(s+1)(s+2)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^{\frac12}|\lambda-t|\lambda^s\mathrm{d}\lambda+\int_{\frac12}^1|\lambda-k|\lambda^s\mathrm{d}\lambda\\
&=\frac{2t^{s+2}+2k^{s+2}+\big[2(s+1)-2(s+2)(k+t)\big]\frac{1}{2^{s+2}}+(s+1-ks-2k)}{(s+1)(s+2)},
\end{aligned}
$$

the desired inequality (3.1) is established.
2. When $s=-1$, $t=0$, and $k=1$, utilizing Lemma 2.1 again and the extended $(-1,m)_{\phi\eta}$-preinvexity of $|f'|$ on $A_{\phi\eta}$, we have that

$$
\begin{aligned}
&\left|f\Big(a+\frac{e^{i\phi}\eta(b,a)}{2}\Big)-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)}f(x)\mathrm{d}x\right|\\
&\le\left|e^{i\phi}\eta(b,a)\right|\bigg[\int_0^{\frac12}|\lambda|\Big|f'\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\Big|\mathrm{d}\lambda\\
&\quad+\int_{\frac12}^1|\lambda-1|\Big|f'\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\Big|\mathrm{d}\lambda\bigg]\\
&\le\left|e^{i\phi}\eta(b,a)\right|\bigg\{\int_0^{\frac12}\Big[\frac{\lambda}{1-\lambda}|f'(a)|+m\frac{\lambda}{\lambda}\Big|f'\Big(\frac{b}{m}\Big)\Big|\Big]\mathrm{d}\lambda
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\frac{1}{2}}^{1}\left[\frac{1-\lambda}{1-\lambda}|f'(a)|+m\frac{1-\lambda}{\lambda}\left|f'\Big(\frac{b}{m}\Big)\right|\right]\mathrm{d}\lambda\Big\}\\
&=\left|e^{i\phi}\eta(b,a)\right|\ln 2\Big[|f'(a)|+m\left|f'\Big(\frac{b}{m}\Big)\right|\Big].
\end{aligned}
$$

This proves as required.
Direct computation yields the following corollaries.

Corollary 3.1 *Under the conditions of Theorem 3.1 and* $s\in(-1,1]$,
1. if $t=\dfrac{1}{6}$ *and* $k=\dfrac{5}{6}$, *we have*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+f\Big(a+e^{i\phi}\eta(b,a)\Big)+4f\left(a+\frac{e^{i\phi}\eta(b,a)}{2}\right)\right]\right.\\
&\quad\left.-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)}f(x)\mathrm{d}x\right|\\
&\le\frac{\big[(s-4)6^{s+1}+2\times5^{s+2}-2\times3^{s+2}+2\big]}{6^{s+2}(s+1)(s+2)}\\
&\quad\times\left|e^{i\phi}\eta(b,a)\right|\left[|f'(a)|+m\left|f'\Big(\frac{b}{m}\Big)\right|\right]; \qquad (3.3)
\end{aligned}
$$

2. if $\phi=0$, $\eta(b,a)=b-a$, *and* $m=1$ *in inequality (3.3), we have*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\
&\le\frac{(s-4)6^{s+1}+2\times5^{s+2}-2\times3^{s+2}+2}{6^{s+2}(s+1)(s+2)}(b-a)\Big[|f'(a)|+|f'(b)|\Big]; \qquad (3.4)
\end{aligned}
$$

3. if $t=k=\dfrac{1}{2}$ *and* $s=m=1$ *in inequality (3.1), we have*

$$
\begin{aligned}
&\left|\frac{f(a)+f\big(a+e^{i\phi}\eta(b,a)\big)}{2}-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)}f(x)\mathrm{d}x\right|\\
&\le\frac{|e^{i\phi}\eta(b,a)|}{8}\Big(|f'(a)|+|f'(b)|\Big); \qquad (3.5)
\end{aligned}
$$

4. if $\phi=0$ *in inequality (3.5), we have*

$$
\begin{aligned}
&\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\right|\\
&\le\frac{|\eta(b,a)|}{8}\Big(|f'(a)|+|f'(b)|\Big). \qquad (3.6)
\end{aligned}
$$

Remark 3.1 *Inequality (3.4) is the same as inequality of (1.2) presented by Sarikaya et al. in ([30], 2010). Inequality (3.5) is the same as inequality of (1.8) presented by Noor et al. in ([22], 2015). Inequality (3.6) is the same as inequality of (1.5) established by Barani et al. in ([2], 2012). Thus, inequality (3.1) is a generalization of these inequalities.*

Corollary 3.2 *The upper bound of the midpoint inequality for the first derivative is developed as follows:*
1. By putting $f(a) = f\Big(a + e^{i\phi}\eta(b,a)\Big) = f\Big(a + \frac{e^{i\phi}\eta(b,a)}{2}\Big)$ *in inequality (3.1), we have:*

$$\begin{aligned}&\left|f\Big(a + \frac{e^{i\phi}\eta(b,a)}{2}\Big) - \frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\right| \\ &\leq \left|e^{i\phi}\eta(b,a)\right|\Big[\nu_1|f'(a)| + m\nu_2\Big|f'\Big(\frac{b}{m}\Big)\Big|\Big], \end{aligned} \tag{3.7}$$

where v_1 *and* v_2 *are defined in Theorem 3.1.*
2. Putting $\phi = 0$, $s = 1$, $m = 1$, $t = \dfrac{1}{6}$, *and* $k = \dfrac{5}{6}$ *in the above inequality (3.7), it yields that*

$$\left|f\Big(\frac{2a+\eta(b,a)}{2}\Big) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x\right| \leq \frac{5|\eta(b,a)|}{72}\Big(|f'(a)| + |f'(b)|\Big). \tag{3.8}$$

Remark 3.2 *It is noted that the above midpoint inequality (3.8) is better than the inequality (1.6) presented by Sarikaya et al. in ([29], 2012).*

Corollary 3.3 *Under the conditions of Theorem 3.1 and* $s = -1$, *if* $\phi = 0$, $\eta(b,a) = b - a$, *we have*

$$\left|f\Big(\frac{a+b}{2}\Big) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq (b-a)\ln 2\Big[|f'(a)| + m\Big|f'\Big(\frac{b}{m}\Big)\Big|\Big]. \tag{3.9}$$

Remark 3.3 *When applying* $m = 1$ *to inequality (3.9), then we get inequality (1.4). Thus, Theorem 3.1 and its consequences generalize the main result in ([35], 2015).*

Theorem 3.2 *Let* f *be defined as in Theorem 3.1 with* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $|f'|^q$ *for* $q > 1$ *is extended* $(s,m)_{\phi\eta}$*-preinvex on* $A_{\phi\eta}$ *for some fixed* $(s,m) \in (-1,1] \times (0,1]$

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then the following inequality holds:

$$\begin{aligned}
&\bigg|tf(a)+(1-k)f\Big(a+e^{i\phi}\eta(b,a)\Big)+(k-t)f\bigg(a+\frac{e^{i\phi}\eta(b,a)}{2}\bigg)\\
&\quad-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)}f(x)\mathrm{d}x\bigg|\\
&\le\frac{|e^{i\phi}\eta(b,a)|}{(p+1)^{\frac{1}{p}}(s+1)^{\frac{1}{q}}}\Bigg\{\bigg[t^{p+1}+\Big(\frac{1}{2}-t\Big)^{p+1}\bigg]^{\frac{1}{p}}\\
&\quad\times\bigg[\Big(1-\Big(\frac{1}{2}\Big)^{s+1}\Big)\Big|f'(a)\Big|^q+m\Big(\frac{1}{2}\Big)^{s+1}\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\bigg]^{\frac{1}{q}}\\
&\quad+\bigg[\Big(k-\frac{1}{2}\Big)^{p+1}+(1-k)^{p+1}\bigg]^{\frac{1}{p}}\\
&\quad\times\bigg[\Big(\frac{1}{2}\Big)^{s+1}\Big|f'(a)\Big|^q+m\Big(1-\Big(\frac{1}{2}\Big)^{s+1}\Big)\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\bigg]^{\frac{1}{q}}\Bigg\}. \qquad (3.10)
\end{aligned}$$

Proof. By Lemma 2.1 and using the famous Hölder's inequality, we have

$$\begin{aligned}
&\bigg|tf(a)+(1-k)f\Big(a+e^{i\phi}\eta(b,a)\Big)+(k-t)f\bigg(a+\frac{e^{i\phi}\eta(b,a)}{2}\bigg)\\
&\quad-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)}f(x)\mathrm{d}x\bigg|\\
&\le\Big|e^{i\phi}\eta(b,a)\Big|\bigg[\int_0^{\frac{1}{2}}|\lambda-t|\Big|f'\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\Big|\mathrm{d}\lambda\\
&\quad+\int_{\frac{1}{2}}^1|\lambda-k|\Big|f'\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\Big|\mathrm{d}\lambda\bigg]\\
&\le\Big|e^{i\phi}\eta(b,a)\Big|\Bigg\{\bigg(\int_0^{\frac{1}{2}}|\lambda-t|^p\mathrm{d}\lambda\bigg)^{\frac{1}{p}}\bigg[\int_0^{\frac{1}{2}}\Big|f'\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\Big|^q\mathrm{d}\lambda\bigg]^{\frac{1}{q}}\\
&\quad+\bigg(\int_{\frac{1}{2}}^1|\lambda-k|^p\mathrm{d}\lambda\bigg)^{\frac{1}{p}}\bigg[\int_{\frac{1}{2}}^1\Big|f'\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\Big|^q\mathrm{d}\lambda\bigg]^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

Also, making use of the extended $(s,m)_{\phi\eta}$-convexity of $|f'|^q$, it follows that

$$\begin{aligned}
&\bigg|tf(a)+(1-k)f\Big(a+e^{i\phi}\eta(b,a)\Big)+(k-t)f\bigg(a+\frac{e^{i\phi}\eta(b,a)}{2}\bigg)\\
&\quad-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)}f(x)\mathrm{d}x\bigg|
\end{aligned}$$

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$$
\begin{aligned}
&\leq \left|e^{i\phi}\eta(b,a)\right|\Bigg\{\bigg(\int_0^{\frac{1}{2}} |\lambda - t|^p \mathrm{d}\lambda\bigg)^{\frac{1}{p}} \\
&\times\bigg[\int_0^{\frac{1}{2}} \bigg((1-\lambda)^s\Big|f'(a)\Big|^q + m\lambda^s\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\bigg)\mathrm{d}\lambda\bigg]^{\frac{1}{q}} \\
&+\bigg(\int_{\frac{1}{2}}^1 |\lambda - k|^p \mathrm{d}\lambda\bigg)^{\frac{1}{p}}\bigg[\int_{\frac{1}{2}}^1 \bigg((1-\lambda)^s\Big|f'(a)\Big|^q + m\lambda^s\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\bigg)\mathrm{d}\lambda\bigg]^{\frac{1}{q}}\Bigg\}.
\end{aligned}
$$

Direct calculation yields that

$$
\int_0^{\frac{1}{2}} |\lambda - t|^p \mathrm{d}\lambda = \frac{t^{p+1} + \left(\frac{1}{2} - t\right)^{p+1}}{p+1}, \quad \int_{\frac{1}{2}}^1 |\lambda - k|^p \mathrm{d}\lambda = \frac{\left(k - \frac{1}{2}\right)^{p+1} + (1-k)^{p+1}}{p+1},
$$

similarly, we have

$$
\int_0^{\frac{1}{2}} (1-\lambda)^s \mathrm{d}\lambda = \int_{\frac{1}{2}}^1 \lambda^s \mathrm{d}\lambda = \frac{1 - \left(\frac{1}{2}\right)^{s+1}}{s+1}, \quad \int_0^{\frac{1}{2}} \lambda^s \mathrm{d}\lambda = \int_{\frac{1}{2}}^1 (1-\lambda)^s \mathrm{d}\lambda = \frac{\left(\frac{1}{2}\right)^{s+1}}{s+1}.
$$

Therefore, combining the above four equalities can lead to the desired result. The statement in Theorem 3.2 is proved.

Corollary 3.4 *Under the condition of Theorem 3.2,*
1. when $s = 1$, we have

$$
\begin{aligned}
&\Bigg|tf(a) + (1-k)f\Big(a + e^{i\phi}\eta(b,a)\Big) + (k-t)f\bigg(a + \frac{e^{i\phi}\eta(b,a)}{2}\bigg) \\
&-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\Bigg| \\
&\leq \frac{|e^{i\phi}\eta(b,a)|}{2^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\Bigg\{\bigg[t^{p+1} + \Big(\frac{1}{2} - t\Big)^{p+1}\bigg]^{\frac{1}{p}}\bigg[\frac{3|f'(a)|^q}{4} + \frac{m|f'(\frac{b}{m})|^q}{4}\bigg]^{\frac{1}{q}} \\
&\quad+\bigg[\Big(k - \frac{1}{2}\Big)^{p+1} + (1-k)^{p+1}\bigg]^{\frac{1}{p}}\bigg[\frac{|f'(a)|^q}{4} + \frac{3m|f'(\frac{b}{m})|^q}{4}\bigg]^{\frac{1}{q}}\Bigg\}; \quad (3.11)
\end{aligned}
$$

2. when $\phi = 0$, $k = 1$, $t = 0$, and $m = 1$ in inequality (3.11), we can get

$$
\begin{aligned}
&\left|f\bigg(a + \frac{\eta(b,a)}{2}\bigg) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x\right| \\
&\leq \bigg(\frac{1}{p+1}\bigg)^{\frac{1}{p}}\frac{|\eta(b,a)|}{4}\bigg[\bigg(\frac{3}{4}|f'(a)|^q + \frac{1}{4}|f'(b)|^q\bigg)^{\frac{1}{q}} \\
&\quad+\bigg(\frac{1}{4}|f'(a)|^q + \frac{3}{4}|f'(b)|^q\bigg)^{\frac{1}{q}}\bigg]. \quad (3.12)
\end{aligned}
$$

Remark 3.4 *As* $p = \dfrac{q}{q-1}$, *exchange* a *and* b *in inequality (3.12), then we can deduce the inequality (1.7).*

In the following corollary, we have the midpoint inequality for powers in terms of the first derivative.

Corollary 3.5 *By substituting* $f(a) = f\Big(a + e^{i\phi}\eta(b,a)\Big) = f\Big(a + \frac{e^{i\phi}\eta(b,a)}{2}\Big)$, $t = \dfrac{1}{6}$, *and* $k = \dfrac{5}{6}$ *in Theorem 3.2, we have*

$$\begin{aligned}
&\left|\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x - f\left(a + \frac{e^{i\phi}\eta(b,a)}{2}\right)\right| \\
&\le \frac{|e^{i\phi}\eta(b,a)|}{2^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left[\left(\frac{1}{6}\right)^{p+1} + \left(\frac{1}{3}\right)^{p+1}\right]^{\frac{1}{p}} \\
&\times\left\{\left[\frac{3|f'(a)|^q}{4} + \frac{m|f'(\frac{b}{m})|^q}{4}\right]^{\frac{1}{q}} + \left[\frac{|f'(a)|^q}{4} + \frac{3m|f'(\frac{b}{m})|^q}{4}\right]^{\frac{1}{q}}\right\}. \quad (3.13)
\end{aligned}$$

In the following theorem, we obtain another form of Simpson type inequality for powers in term of the first derivative.

Theorem 3.3 *Let* f *be defined as in Theorem 3.1. If the mapping* $|f'|^q$ *for* $q \ge 1$ *is extended* $(s,m)_{\phi\eta}$*-preinvex on* $A_{\phi\eta}$ *for some fixed* $(s,m) \in (-1,1]\times(0,1]$ *then*

$$\begin{aligned}
&\Bigg|tf(a) + (1-k)f\Big(a + e^{i\phi}\eta(b,a)\Big) + (k-t)f\left(a + \frac{e^{i\phi}\eta(b,a)}{2}\right) \\
&\quad - \frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x\Bigg| \\
&\le \left|e^{i\phi}\eta(b,a)\right|\Bigg\{\left(t^2 - \frac{1}{2}t + \frac{1}{8}\right)^{1-\frac{1}{q}}\left[\xi_1|f'(a)|^q + m\xi_2\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}} \\
&\quad + \left(k^2 - \frac{3}{2}k + \frac{5}{8}\right)^{1-\frac{1}{q}}\left[\xi_3|f'(a)|^q + m\xi_4\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}\Bigg\}, \quad (3.14)
\end{aligned}$$

where

$$\xi_1 = \frac{t(s+2) - 1 + 2(1-t)^{s+2} + (2ts + 4t - s - 3)\frac{1}{2^{s+2}}}{(s+1)(s+2)},$$

$$\xi_2 = \frac{2t^{s+2} + (s + 1 - 2ts - 4t)\frac{1}{2^{s+2}}}{(s+1)(s+2)},$$

$$\xi_3 = \frac{2(1-k)^{s+2} + (2ks + 4k - s - 3)\frac{1}{2^{s+2}}}{(s+1)(s+2)},$$

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and

$$\xi_4=\frac{2k^{s+2}+(s+1-2ks-4k)\dfrac{1}{2^{s+2}}+(s+1-ks-2k)}{(s+1)(s+2)}.$$

Proof. By Lemma 2.1 and power-mean inequality, it follows that

$$\begin{aligned}
&\Bigg|tf(a)+(1-k)f\Big(a+e^{i\phi}\eta(b,a)\Big)+(k-t)f\left(a+e^{i\phi}\frac{\eta(b,a)}{2}\right)\\
&\quad-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)}f(x)\mathrm{d}x\Bigg|\\
&\le\left|e^{i\phi}\eta(b,a)\right|\bigg[\int_0^{\frac12}|\lambda-t|\Big|f'\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\Big|\mathrm{d}\lambda\\
&\quad+\int_{\frac12}^1|\lambda-k|\Big|f'\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\Big|\mathrm{d}\lambda\bigg]\\
&\le\left|e^{i\phi}\eta(b,a)\right|\Bigg\{\left(\int_0^{\frac12}|\lambda-t|\mathrm{d}\lambda\right)^{1-\frac1q}\left[\int_0^{\frac12}|\lambda-t|\Big|f'\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\Big|^q\mathrm{d}\lambda\right]^{\frac1q}\\
&\quad+\left(\int_{\frac12}^1|\lambda-k|\mathrm{d}\lambda\right)^{1-\frac1q}\left[\int_{\frac12}^1|\lambda-k|\Big|f'\Big(a+\lambda e^{i\phi}\eta(b,a)\Big)\Big|^q\mathrm{d}\lambda\right]^{\frac1q}\Bigg\}.
\end{aligned}$$

Using the extended $(s,m)_{\phi\eta}$-convexity of $|f'|^q$, we have that

$$\begin{aligned}
&\Bigg|tf(a)+(1-k)f\Big(a+e^{i\phi}\eta(b,a)\Big)+(k-t)f\left(a+e^{i\phi}\frac{\eta(b,a)}{2}\right)\\
&\quad-\frac{1}{e^{i\phi}\eta(b,a)}\int_a^{a+e^{i\phi}\eta(b,a)}f(x)\mathrm{d}x\Bigg|\\
&\le\left|e^{i\phi}\eta(b,a)\right|\Bigg\{\left(\int_0^{\frac12}|\lambda-t|\mathrm{d}\lambda\right)^{1-\frac1q}\\
&\quad\times\left[\int_0^{\frac12}|\lambda-t|\left((1-\lambda)^s\Big|f'(a)\Big|^q+m\lambda^s\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\right)\mathrm{d}\lambda\right]^{\frac1q}+\left(\int_{\frac12}^1|\lambda-k|\mathrm{d}\lambda\right)^{1-\frac1q}\\
&\quad\times\left[\int_{\frac12}^1|\lambda-k|\left((1-\lambda)^s\Big|f'(a)\Big|^q+m\lambda^s\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\right)\mathrm{d}\lambda\right]^{\frac1q}\Bigg\}.
\end{aligned}$$

By simple calculations, we can get

$$\int_0^{\frac12}|\lambda-t|\mathrm{d}\lambda=t^2-\frac12t+\frac18,\quad\int_{\frac12}^1|\lambda-k|\mathrm{d}\lambda=k^2-\frac32k+\frac58,\tag{3.15}$$

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$$\int_0^{\frac{1}{2}} |\lambda - t|(1-\lambda)^s \mathrm{d}\lambda = \frac{t(s+2) - 1 + 2(1-t)^{s+2} + (2ts + 4t - s - 3)\frac{1}{2^{s+2}}}{(s+1)(s+2)}, \quad (3.16)$$

$$\int_0^{\frac{1}{2}} |\lambda - t|\lambda^s \mathrm{d}\lambda = \frac{2t^{s+2} + (s + 1 - 2ts - 4t)\frac{1}{2^{s+2}}}{(s+1)(s+2)}, \quad (3.17)$$

$$\int_{\frac{1}{2}}^1 |\lambda - k|(1-\lambda)^s \mathrm{d}\lambda = \frac{2(1-k)^{s+2} + (2ks + 4k - s - 3)\frac{1}{2^{s+2}}}{(s+1)(s+2)}, \quad (3.18)$$

and

$$\int_{\frac{1}{2}}^1 |\lambda - k|\lambda^s \mathrm{d}\lambda = \frac{2k^{s+2} + (s + 1 - 2ks - 4k)\frac{1}{2^{s+2}} + (s + 1 - ks - 2k)}{(s+1)(s+2)}. \quad (3.19)$$

Thus, our desired result can be obtained by combining equalities (3.15)-(3.19), the proof is completed.

Corollary 3.6 *Let f be defined as in Theorem 3.3, if $s = 1$, $t = \frac{1}{6}$, and $k = \frac{5}{6}$, the inequality holds for m-convex functions:*

$$\begin{aligned} &\left| \frac{1}{6}\left[f(a) + f\Big(a + e^{i\phi}\eta(b,a)\Big) + 4f\Big(a + \frac{e^{i\phi}\eta(b,a)}{2}\Big)\right] \right. \\ &\quad \left. - \frac{1}{e^{i\phi}\eta(b,a)} \int_a^{a+e^{i\phi}\eta(b,a)} f(x)\mathrm{d}x \right| \\ &\leq \left|e^{i\phi}\eta(b,a)\right| \Big(\frac{5}{72}\Big)^{1-\frac{1}{q}} \left[\Big(\frac{61}{1296}|f'(a)|^q + \frac{29m}{1296}\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\Big)^{\frac{1}{q}} \right. \\ &\quad \left. + \Big(\frac{29}{1296}|f'(a)|^q + \frac{61m}{1296}\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\Big)^{\frac{1}{q}} \right]. \quad (3.20) \end{aligned}$$

In particular, if $m = 1$, $\phi = 0$, and $\eta(b,a) = b - a$ in inequality (3.20), the inequality holds for convex function. If $|f'(x)| \leq Q$, $\forall x \in I$, then we have

$$\left| \frac{1}{6}\left[f(a) + 4f\Big(\frac{b+a}{2}\Big) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \right| \leq \frac{5(b-a)}{36}Q. \quad (3.21)$$

Remark 3.5 *It is observed that the inequality (3.21) is an improvement compared with inequality (1.3). Thus, Theorem 3.3 and its consequences generalize the main results in ([4], 2000).*

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References

[1] T. Antczak, Mean value in invexity analysis, *Nonlinear Anal.*, **60**, no. 8, 1473-1484 (2005); Available online at http://dx.doi.org/10.1016/j.na.2004.11.005.

[2] A. Barani, A. G. Ghazanfari, and S. S. Dragomir, Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex, *J. Inequal. Appl.*, 2012:**247**, 9 pages (2012); Available online at http://dx.doi.org/10.1186/1029-242X-2012-247.

[3] L. Chun and F. Qi, Inequalities of Simpson type for functions whose third derivatives are extended s-convex functions and applications to means, *J. Comput. Anal. Appl.*, **19**, no. 3, 555-569 (2015).

[4] S. S. Dragomir, R. P. Agarwal, and P. Cerone, On Simpson's inequality and applications, *J. Inequal. Appl.*, **5**, no. 6, 533-579 (2000); Available online at http://dx.doi.org/10.1155/S102558340000031X.

[5] N. Eftekhari, Some remarks on (s, m)-convexity in the second sense, *J. Math. Inequal.*, **8**, no. 3, 489-495 (2014); Available online at http://dx.doi.org/10.7153/jmi-08-36.

[6] E. K. Godunova and V. I. Levin, Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions, *Numer. Math. Math. Phys.*, **166**, 138-142 (1985).

[7] İ. İşcan, Hermite-Hadamard's inequalities for preinvex function via fractional integrals and related fractional inequalities, *American J. Math. Anal.*, **1**, no. 3, 33-38 (2013); Available online at http://arxiv.org/abs/1204.0272v2.

[8] W.-D. Jing, D.-W. Niu, and F. Qi, Some inequalities of Hermite-Hadamard type for r-ϕ-preinvex functions, *Tamkang J. Math.*, **45**, no. 1, 31-38 (2014); Available online at http://dx.doi.org/10.5556/j.tkjm.45.2014.1261.

[9] M. A. Latif, On Hermite-Hadamard type integral inequalities for n-times differentiable preinvex functions with applications, *Stud. Univ. Babeş-Bolyai Math.*, **58**, no. 3, 325-343 (2013).

[10] M. A. Latif, Some Hermite-Hadamard type inequalities for functions whose partial derivatives in absloute value are preinvex on the co-ordinates, *Facta Universitatis (NIŠ) Ser. Math. Inform.*, **28**, no. 3, 257-270 (2013).

[11] M. A. Latif and S. S. Dragomir, Some weighted integral inequalities for differentiable preinvex and prequsiinvex functions with applications, *J. Inequal. Appl.*, 2013:**575**, 19 pages (2013); Available online at http://dx.doi.org/10.1186/1029-242X-2013-575.

[12] M. A. Latif, S. S. Dragomir, and E. Momoniat, Some weighted Hermite-Hadamard-Noor type inequalities for differentiable preinvex and quasi preinvex functions, *Punjab Univ. J. Math.*, **47**, no. 1, 57-72 (2015).

[13] M. A. Latif and M. Shoaib, Hermite-Hadamard type integral inequalities for differentiable m-preinvex and (α, m)-preinvex functions, *J. Egypt. Math. Soc.*, **23**, 236-241(2015); Available online at http://dx.doi.org/10.1016/j.joems.2014.06.006.

[14] J.-Y. Li, On Hadamard-type inequalities for s-preinvex functions, *J. Chongqing Norm. Univ. (Natural Science) China*, **27**, no. 4, 5-8 (2010); Available online at http://dx.doi.org/10.3969/J.ISSN.1672-6693.2010.04.002.

[15] T.-Y. Li and G.-H. Hu, On the strictly G-preinvex function, *J. Inequal. Appl.*, 2014:**210**, 9 pages (2014); Available online at http://dx.doi.org/10.1186/1029-242X-2014-210.

[16] Y.-J. Li and T.-S. Du, On Simpson type inequalities for functions whose derivatives are extended (s, m)-GA-convex functions, *Pure. Appl. Math. China*, **31**, no. 5, 487-497 (2015); Available online at http://dx.doi.org/10.3969/j.issn.1008-5513.2015.05.008.

[17] M. Matłoka, On some Hadamard-type inequalities for (h_1, h_2)-preinvex functions on the co-ordinates, *J. Inequal. Appl.*, 2013:**227**, 12 pages (2013); Available online at http://dx.doi.org/10.1186/1029-242X-2013-227.

[18] M. Matłoka, Inequalities for h-preinvex functions, *Appl. Math. Comput.*, **234**, 52-57 (2014); Available online at http://dx.doi.org/10.1016/j.amc.2014.02.030.

[19] M. A. Noor, Hadamard integral inequalities for product of two preinvex functions, *Nonlinear Anal. Forum*, **14**, 167-173 (2009); Available online at http://prof.ks.ac.kr/bslee/naf/table/vol-1400/NAF1418.pdf

[20] M. A. Noor, Hermite-Hadamard integral inequalities for log-ϕ-convex functions, *Nonlinear Anal. Forum*, **13**, no. 2, 119-124 (2008); Available online at http://prof.ks.ac.kr/bslee/naf/table/vol-1302/NAF130201.pdf

[21] M. A. Noor and K. I. Noor, Generalized preinvex functions and their properties, *J. Appl. Math. Stochastic Anal.*, Article ID 12736, 13 pages (2006); Available online at http://dx.doi.org/10.1155/JAMSA/2006/12736.

[22] M. A. Noor, K. I. Noor, M. U. Awan, and S. Khan, Hermite-Hadamard type inequalities for differentiable h_ϕ-preinvex functions, *Arab. J. Math.*, **4**, 63-76 (2015); Available online at http://dx.doi.org/10.1007/s40065-014-0124-3.

[23] M. A. Noor, K. I. Noor, M. A. Ashraf, M. U. Awan, and B. Bashir, Hermite-Hadamard inequalities for h_ϕ-convex functions, *Nonlinear Anal. Formum*, **18**, 65-76 (2013).

[24] J. Park, Simpson-like and Hermite-Hadamard-like type integral inequalities for twice differentiable preinvex functions, *International J. Pure. Appl. Math.*, **79**, no. 4, 623-640 (2012).

[25] J. Park, Hermite-Hadamard-like inequalities for functions whose derivatives of n-th order are preinvex and logarithmic preinvex, *Appl. Math. Sci.*, **7**, no. 122, 6053-6068 (2013); Available online at http://dx.doi.org/10.12988/ams.2013.39520.

[26] J. Park, Hermite-Hadamard-like type integral inequalities for functions whose derivatives of n-th order are preinvex, *Appl. Math. Sci.*, **7**, no. 133, 6637-6650 (2013); Available online at http://dx.doi.org/10.12988/ams.2013.310569.

[27] R. Pini, Invexity and generalized convexity, *Optimization*, **22**, no.4, 513-525 (1991); Available online at http://dx.doi.org/10.1080/02331939108843693.

[28] S. Qaisar, C.-J. He, and S. Hussain, A generalizations of Simpson's type inequality for differentiable functions using (α, m)-convex functions and applications, *J. Inequal. Appl.*, 2013:**158**, 13 pages (2013); Available online at http://dx.doi.org/10.1186/1029-242X-2013-158.

[29] M. Z. Sarikaya, N. Alp, and H. Bozkurt, On Hermite-Hadamard type integral inequalities for preinvex and log-preinvex functions, *Contemporary Anal. Appl. Math.*, **1**, No. 2, 237-252(2013); Available online at http://caam.fatih.edu.tr/docs/articles/171.pdf

[30] M. Z. Sarikaya, E. Set, and M. E. Özdemir, On new inequalities of Simpson's type for s-convex functions, *Comput. Math. Appl.*, **60**, no. 8, 2191-2199 (2010); Available online at http://dx.doi.org/10.1016/j.camwa.2010.07.033.

[31] S.-H. Wang and F. Qi, Hermite-Hadamard type inequalities for n-times differentiable and preinvex functions, *J. Inequal. Appl.*, 2014:**49**, 9 pages (2014); Available online at http://dx.doi.org/10.1186/1029-242X-2014-49.

[32] Y. Wang, S.-H. Wang, and F. Qi, Simpson type integral inequalities in which the power of the absolute value of the first derivative of the integrand is s-preinvex, *Facta Univ., Ser. Math. Inform.*, **28**, no. 2, 151-159(2013).

[33] Y. Wang, B.-Y. Xi, and F. Qi, Hermite-Hadamard type integral inequalities when the power of the absolute value of the first derivative of the integrand is preinvex, *Le. Mat.*, **69**, no. 1, 89-96 (2014); Available online at http://dx.doi.org/10.4418/2014.69.1.6.

[34] Y. Wang, M.-M. Zheng, and F. Qi, Integral inequalities of Hermite-Hadamard type for functions whose derivatives are α-preinvex, *J. Inequal. Appl.*, 2014:**97**, 10 pages (2014); Available online at http://dx.doi.org/10.1186/1029-242X-2014-97.

[35] B.-Y. Xi and F. Qi, Inequalities of Hermite-Hadamard type for extended s-convex functions and applications to means, *J. Nonlinear Convex Anal.*, **16**, no. 5, 873-890 (2015).

[36] B.-Y. Xi, Y. Wang, and F. Qi, Some integral inequalities of Hermite-Hadamard type for extended (s, m)-convex functions, *Transylv. J. Math. Mech.*, **5**, no. 1, 69-84 (2013).

[37] X.-M. Yang, X.-Q. Yang, and K. L. Teo, Generalized invexity and generalized invariant monotonicity, *J. Optimiz. Theory App.*, **117**, no. 3, 607-625 (2003).

[38] Z.-Q. Yang, Y.-J. Li, and T.-S. Du, A generalization of Simpson type inequality via differentiable functions using (s, m)-convex functions, *Italian J. Pure. Appl. Maths.*, **35**, 327-338(2015).

ISOMETRIC EQUIVALENCE OF LINEAR OPERATORS ON SOME SPACES OF ANALYTIC FUNCTIONS

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Abstract. In this paper, we are interested in the isometric equivalence problem for the weighted composition operator $W_{u,\varphi}$ and the composition operator C_φ on the Hardy and the Dirichlet space. We show what properties the operators must satisfy to insure that they are isometric equivalent.

1. INTRODUCTION

Let $\mathbb{D}$ be the unit disk in the complex plane, and $S(\mathbb{D})$ be the set of analytic self-maps of $\mathbb{D}$. The algebra of all holomorphic functions with domain $\mathbb{D}$ will be denoted by $H(\mathbb{D})$.

Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$, the multiplication operator M_u is defined by $(M_u f)(z) = u(z)f(z)$, and the weighted composition operator $W_{u,\varphi}$ induced by u and φ is defined by $(W_{u,\varphi}f)(z) = u(z)f(\varphi(z))$ for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$. If let $u \equiv 1$, then $W_{u,\varphi} = C_\varphi$, which is often called composition operator. As is well known, the weighted composition operator can be regarded as a generalization of a multiplication operator and a composition operator.

Let X and Y be two Banach spaces and T_1 and T_2 are bounded linear operators on X and Y respectively . We say that T_1 and T_2 are isometrically equivalent if there exists surjective isometries U_X and U_Y on X and Y respectively such that $U_X T_1 = T_2 U_Y$. For $X = Y$, two operators T_1 and T_2 are said to be similar if there is a bounded invertible operator S such that $ST_2 = T_1 S$. If S could be chosen to be an isometry as well, then T_1 and T_2 are said to be isometrically isomorphic. If X is a Hilbert space as well as a Banach space, then isometric isomorphism on X is referred to as unitary equivalence.

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The "unitary equivalence" problem for operators on a Hilbert space has received a lot of attention over the yeas. The analogous problem for operators on Banach spaces, due to the lack of inner product structure, requires different techniques directly related to the specific settings under consideration. The "isometric equivalence problem" arises as to what properties the operators must satisfy to insure that they are isometrically equivalent. There has been some recent work on the isometric equivalence problem in spaces of analytic functions. In [1], Wright investigated the isometric equivalence of composition operators for $X = Y = H^p(\mathbb{D})$ for $1 \leq p < \infty$ and $p \neq 2$, he obtained that if two composition operators C_{φ_1} and C_{φ_2} are isometrically equivalent on Hardy space H^p, then $\varphi_1(z) = e^{i\theta}\varphi_2(e^{-i\theta}z)$. In [2, 3], Hornor and Jamison studied isometric equivalence of composition operators on several important Banach spaces of analytic function spaces on the unit disk $\mathbb{D}$. In [4], Jamison studied isometric equivalence of composition operators for $X = Y = \mathcal{B}$, where $\mathcal{B}$ is a Bloch space. He obtained that if two composition operators C_{φ_1} and C_{φ_2} are isometrically equivalent on $\mathcal{B}$, then there is an automorphism φ such that $\varphi_1(\varphi(z)) = \varphi(\varphi_2(z))$; he also investigated the isometric equivalence problem of certain operators on some specific types of Banach spaces. In [5], Nadia studied isometric equivalence of differentiated composition operators on some analytic function spaces. He obtained that two operators DC_{φ_1} and $DC_{\varphi_2} : H^p \to H^q (1 < p, q < \infty$, and $p, q \neq 2)$, then $DC_{\varphi_1}W_p = W_q DC_{\varphi_2}$ if and only if $\varphi_1(z) = e^{-i\theta_p}\varphi_2(e^{i\theta_q}z)$, here $DC_\varphi : H^p \to H^q$ is defined to be $DC_\varphi f = (f \circ \varphi)'$, W_p and W_q are surjective isometries on H^p and H^q.

Building on those foundation, the present paper continues this line of research. More precisely, we first investigated the case of weighted composition operators on the Hardy spaces.

2. Isometric equivalence of weighted composition operators on Hardy spaces

Let $H^\infty(\mathbb{D})$ denote the space of bounded holomorphic functions f on the unit disk with the supremum

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|.$$

The surjective linear isometries of $H^\infty(\mathbb{D})$ were determined in [6]. It was proven that, a surjective linear isometry T of $H^\infty(\mathbb{D})$ is of the form:

$$Tf = \alpha f(\tau) \tag{1}$$

for every $f \in H^\infty(\mathbb{D})$. Where τ is a conformal map of $\mathbb{D}$ and α is a unimodular complex number.

The Hardy space $H^p(\mathbb{D})$ for $1 \leq p < \infty$ is defined to be the Banach space of analytic functions in $\mathbb{D}$ such that

$$\|f\|_{H^p} = \sup_{0<r<1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta \right\}^{1/p} < \infty.$$

In [7], it was proved that if $p \neq 2$, a surjective isometry T of H^p is of the form:

$$Tf = \alpha(\tau')^{1/p} f(\tau) \tag{2}$$

where $|\alpha| = 1$ and τ is a conformal map of the disk.

In the following theorem, we are interested in the isometric equivalence for the weighted composition operators on the space $H^\infty(\mathbb{D})$:

Lemma 1. *If φ_1 and φ_2 are analytic functions of the disk into itself, then C_{φ_1} and C_{φ_2} are isometrically equivalent on $H^\infty(\mathbb{D})$ if and only if $\varphi_1(\tau) = \tau(\varphi_2)$; M_{u_1} and M_{u_2} are isometrically equivalent on $H^\infty(\mathbb{D})$ if and only if $u_1(\tau) = u_2$, where $u_1, u_2 \in H(\mathbb{D})$ and τ is a conformal map of the disk.*

Proof. Suppose C_{φ_1} and C_{φ_2} are isometrically equivalent on $H^\infty(\mathbb{D})$, then there is a surjective isometry T on $H^\infty(\mathbb{D})$ such that $TC_{\varphi_1} = C_{\varphi_2}T$. From (1),we have

$$\alpha f(\varphi_1(\tau)) = \alpha f(\tau(\varphi_2))$$

for any $f \in H^\infty(\mathbb{D})$, so $\varphi_1(\tau) = \tau(\varphi_2)$. For the converse, if $\varphi_1(\tau) = \tau(\varphi_2)$, then

$$TC_{\varphi_1} f = C_{\varphi_2} T f$$

for any $f \in H^\infty(\mathbb{D})$.

Similarly, suppose M_{u_1} and M_{u_2} are isometrically equivalent on $H^\infty(\mathbb{D})$, then $TM_{u_1} f = M_{u_2} T f$ for any $f \in H^\infty(\mathbb{D})$. It's easy to get that

$$\alpha u_1(\tau) f(\tau) = \alpha u_2 f(\tau),$$

so $u_1(\tau) = u_2$. For the converse, if $u_1(\tau) = u_2$, then

$$\alpha u_1(\tau) f(\tau) = \alpha u_2 f(\tau)$$

for any $f \in H^\infty(\mathbb{D})$. This implies that M_{u_1} and M_{u_2} are isometrically equivalent on $H^\infty(\mathbb{D})$. The converse is obviously. $\square$

Theorem 2. *Let $u_1, u_2 \in H(\mathbb{D})$ and $\varphi_1, \varphi_2 \in S(\mathbb{D})$. Two weighted composition operators W_{u_1,φ_1} and W_{u_2,φ_2} are isometrically equivalent on $H^\infty(\mathbb{D})$ if and only if C_{φ_1} and C_{φ_2}, M_{u_1} and M_{u_2} are isometrically equivalent on $H^\infty(\mathbb{D})$ respectively .*

Proof. For the sufficiently, suppose C_{φ_1} and C_{φ_2}, M_{u_1} and M_{u_2} are isometrically equivalent, then there exists surjective isometric T on $H^\infty(\mathbb{D})$ such that

$$TC_{\varphi_2} = C_{\varphi_1} T, \quad \text{and} \quad TM_{u_2} = M_{u_1} T.$$

It follows from (1) that

$$TW_{u_2,\varphi_2} f = TM_{u_2} C_{\varphi_2} f = M_{u_1} T T^{-1} C_{\varphi_1} T f = M_{u_1} C_{\varphi_1} T f = W_{u_1,\varphi_1} T f.$$

For the necessity, now suppose $TW_{u_1,\varphi_1} f = W_{u_2,\varphi_2} T f$, where T is a surjective isometry on $H^\infty(\mathbb{D})$. It follows from (1) that

$$\alpha u_1(\tau) f(\varphi_1(\tau)) = \alpha u_2 f(\tau(\varphi_2)) \tag{3}$$

for any $f \in H^\infty(\mathbb{D})$.

Let $f = z$ in (3), then $\varphi_1(\tau) = \tau(\varphi_2)$. By Lemma 1, we know that C_{φ_1} and C_{φ_2} are isometrically equivalent on $H^\infty(\mathbb{D})$.

Let $f = 1$in (3), then $u_1(\tau) = u_2$. Consequently, from Lemma 1, this implies that M_{u_1} and M_{u_2} are isometrically equivalent on $H^\infty(\mathbb{D})$. The proof of this theorem is completed. □

Theorem 3. *Suppose M_{u_1} and M_{u_2} are two multiplication operators on $H^p(\mathbb{D})$,$1 \leq p < \infty$ and $p \neq 2$, then M_{u_1} and M_{u_2} are isometrically equivalent if and only if there exists a conformal map τ of the unit disk onto itself such that $u_1 = u_2(\tau)$.*

Proof. Suppose that M_{u_1} and M_{u_2} are isometrically equivalent, and T is an isometry of $H^p(\mathbb{D})$ onto itself. Then

$$M_{u_1}Tf = TM_{u_2}f$$

for any $f \in H^p(\mathbb{D})$. It follows from (2) that

$$bu_1 f(\tau)(\tau')^{1/p} = bu_2(\tau)f(\tau)(\tau')^{1/p}.$$

So $u_1 = u_2(\tau)$. The converse is obvious. □

Theorem 4. *Let C_{φ_1} is a composition operator on $H^p(\mathbb{D})$,$1 \leq p < \infty$ and $p \neq 2$, C_{φ_2} is a composition operator on H^∞ then C_{φ_1} and C_{φ_2} are isometrically equivalent if and only if φ_1 and φ_2 are constants or $\varphi_1(z) = e^{i\theta}\varphi_2(\tau(z))$, here τ is a conformal map of the disk.*

Proof. Suppose C_{φ_1} and C_{φ_2} are isometrically equivalent, then there exists an isometry T_1 on H^p and an isometry T_2 on H^∞ such that $C_{\varphi_1}T_1 f = T_2 C_{\varphi_2} f$. By (1) and (2), from the expression of T_1 and T_2, we have

$$\alpha(\tau_1'(\varphi_1))^{1/p} f(\tau(\varphi_1)) = \beta f(\varphi_2(\tau_2)), \tag{4}$$

where α and β are unimodular complex numbers; $\tau_i, i = 1, 2$ is conformal map of D onto itself.

Let $f = 1$ in (4), then $\alpha(\tau_1'(\varphi_1))^{1/p} = \beta$.

Let $f = z$ in (4), then $\alpha(\tau_1'(\varphi_1))^{1/p}\tau_1(\varphi_1) = \beta\varphi_2(\tau_2)$. Here $\tau_i(z) = \lambda_i \frac{w_i - z}{1 - \bar{w}_i z}$, where $|\lambda_i| = 1$, $|w_i| < 1$.

If $w_i \neq 0, i = 1, 2$, we can get φ_1 and φ_2 are constant functions. If $w_i = 0, i = 1, 2$, we can get $\lambda_1\varphi_1 = \varphi_2(\lambda_2 z)$, that is: $\varphi_1(z) = e^{i\theta_1}\varphi_2(e^{i\theta_2}z)$. If φ_1 and φ_2 are not constant functions, from $\alpha(\tau_1'(\varphi_1))^{1/p} = \beta$, we can get $w_1 = 0$ and $\alpha(\lambda_1)^{1/p} = \beta$, then $\lambda_1\varphi_1(z) = \varphi_2(\tau_2(z))$, that is $\varphi_1(z) = e^{i\theta}\varphi_2(\tau_2(z))$

The converse is obvious. □

3. Isometric equivalence of composition operators on Dirichlet space

For $1 \leq p < \infty$, let L_a^p denote the Bergman space of the unit disk $\mathbb{D}$ and $\|\cdot\|_p$ denote the usual norm. $\mathcal{D}^p$ will denote the space of analytic functions on $\mathbb{D}$ for which $f' \in L_a^p$. The norm on $\mathcal{D}^p$ is defined as the following:

$$\|f\|_{\mathcal{D}^p} = (|f(0)|^p + \|f'\|_p)^{1/p}.$$

In [8], the linear isometry T of $\mathcal{D}^p$ $(p \neq 2)$ onto itself is given by:

$$Tf(z) = \lambda[f(0) + \mu \int_0^z [\phi'(\xi)]^{2/p} f'(\phi(\xi))d\xi] \tag{5}$$

where $|\lambda| = |\mu| = 1$ and ϕ is a conformal map of the disk.

We have the following theorem for isometric equivalence of composition operators on $\mathcal{D}^p$:

Theorem 5. *Let φ_1 and φ_2 be analytic maps of the disk.The composition operators C_{φ_1} and C_{φ_2} are isometrically equivalent, as operators on $\mathcal{D}^p$ $(1 < p < \infty$ and $p \neq 2)$ if and only if $\varphi_1(z) = \varphi_2(z) = z$.*

Proof. First assume that C_{φ_1} and C_{φ_2} are isometrically equivalent then for any $f \in \mathcal{D}^p$ and T a surjective linear isometry on $\mathcal{D}^p$,we have

$$TC_{\varphi_1}f = C_{\varphi_2}Tf$$

By (5):

$$TC_{\varphi_1}f(z) = \lambda[f(\varphi_1(0)) + \mu \int_0^z [\phi'(\xi)]^{2/p} f'(\varphi_1(\phi(\xi)))\varphi_1'(\phi(\xi))d\xi]$$

and

$$C_{\varphi_2}Tf(z) = \lambda[f(0) + \mu \int_0^{\varphi_2(z)} [\phi'(\xi)]^{2/p} f'(\phi(\xi))d\xi]$$

From $TC_{\varphi_1}f = C_{\varphi_2}Tf$, we can get:

$$\begin{aligned}&\lambda[f(\varphi_1(0)) + \mu \int_0^z [\phi'(\xi)]^{2/p} f'(\varphi_1(\phi(\xi)))\varphi_1'(\phi(\xi))d\xi] \\ &= \lambda[f(0) + \mu \int_0^{\varphi_2(z)} [\phi'(\xi)]^{2/p} f'(\phi(\xi))d\xi]\end{aligned}$$

for any $f \in \mathcal{D}^p$. Derivative with respect to z on both sides of above equation, we can get:

$$[\phi'(z)]^{2/p} f'(\varphi_1(\phi(z)))\varphi_1'(\phi(z)) = [\phi'(\varphi_2(z))]^{2/p} f'(\phi(\varphi_2(z)))\varphi_2'(z).$$

Let $f = z$,we get

$$[\phi'(z)]^{2/p}\varphi_1'(\phi(z)) = [\phi'(\varphi_2(z))]^{2/p}\varphi_2'(z); \tag{6}$$

Let$f = z^2$,we get

$$[\phi'(z)]^{2/p}\varphi_1(\phi(z))\varphi_1'(\phi(z)) = [\phi'(\varphi_2(z))]^{2/p}\phi(\varphi_2(z))\varphi_2'(z);$$

Let$f = z^n$,we can get

$$[\phi'(z)]^{2/p}(\varphi_1(\phi(z)))^{n-1}\varphi_1'(\phi(z)) = [\phi'(\varphi_z(z))]^{2/p}(\phi(\varphi_2(z)))^{n-1}\varphi_2'(z).$$

For $p \neq 1$, we can get:

$$\varphi_1(\phi(z)) = \phi(\varphi_2(z)). \tag{7}$$

Derivative with respect to z, we can get:

$$\varphi_1'(\phi(z))\phi'(z) = \phi'(\varphi_2(z))\varphi_2'(z) \tag{8}$$

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The following equations can be obtained from (6),(7) and (8)

$$\phi'(z) = \phi'(\varphi_2(z)) \quad and \quad \varphi_2'(z) = \varphi_1'(\phi(z)). \tag{9}$$

The conformal map ϕ of the unit disk can be written as the form $\phi(z) = \lambda\frac{z-a}{1-\bar{a}z}$, where $|\lambda| = 1, |a| < 1$, so

$$\phi'(z) = \lambda\frac{1-|a|^2}{(1-\bar{a}z)^2}. \tag{10}$$

Then $\varphi_1(z) = \varphi_2(z) = z$ can be got from (7),(9)and (10).

The sufficient condition is obvious. □

References

[1] R. C. Wright, Equivalent composition operators, Integr.edu.oper.theory, 14(1991), 775-786.

[2] W. Hornor and J. E. Jamison, Isometrically equivalent composition operators, Contemporary Math., 213(1998), 65-72.

[3] W. Hornor and J. E. Jamison, Isometrically equivalent composition operators on spaces of analytic vector-valued functions, Glasq. Math. J., 41(3)(1999), 441-451.

[4] J. E. Jamison, Isometric equivalence of certain operators on Banach spaces, Integr.edu.oper.theory, 56(2006), 469-482.

[5] Nadia J. Gal, Isometric equivalence of differentiated composition operators between spaces of analytic functions, Houston Jouranl of Mathematics, 33(3)(2007),921-926.

[6] K. deLeeuw, W. Rudin, and J. Werner, The isometries of some function spaces, Proc. Amer. Math. Soc. 11(1960), 694-698.

[7] F. Forelli, The isometries of H^p, Can. J. Math. 16(1964), 721-728.

[8] William Hornor and James E. Jamison, Isometries of some Banach spaces of analytic functions, Integr.edu.oper.theory, 41(2001),410-425.

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S-fuzzy subalgebras and their S-products in BE-algebras

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Abstract. Using the s-norm S, the notions of an S-fuzzy subalgebra of BE-algebras are introduced, and some properties are investigated. The S-product of S-fuzzy subalgebras is discussed.

1. Introduction

In [5], H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra. S. S. Ahn and K. S. So [3,4] introduced the notion of ideals in BE-algebras. S. S. Ahn et al. [1] fuzzified the concept of BE-algebras, investigated some of their properties. Y. B. Jun and S. S. Ahn ([6]) provided several degrees in defining a fuzzy filter and a fuzzy implicative filter. It is a generalization of a fuzzy filter in BE-algebras.

In this paper, we introduce the notion of an S-fuzzy subalgebra of BE-algebras over an s-norm S, and we investigate some related properties. We also discuss the S-product of S-fuzzy subalgebras of BE-algebras.

2. Preliminaries

An algebra $(X;*,1)$ of type (2, 0) is called a BE-*algebra* ([5]) if

(BE1) $x*x=1$ for all $x\in X$;
(BE2) $x*1=1$ for all $x\in X$;
(BE3) $1*x=x$ for all $x\in X$;
(BE4) $x*(y*z)=y*(x*z)$ for all $x,y,z\in X$ (*exchange*)

We introduce a relation "$\leq$" on a BE-algebra X by $x\leq y$ if and only if $x*y=1$. A non-empty subset S of a BE-algebra X is said to be a *subalgebra* of X if it is closed under the operation "$*$". Noticing that $x*x=1$ for all $x\in X$, it is clear that $1\in S$. A BE-algebra $(X;*,1)$ is said to be *self distributive* if $x*(y*z)=(x*y)*(x*z)$ for all $x,y,z\in X$. A mapping

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$f : X \to Y$ of BE-algebras is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for any $x, y \in X$. A homomorphism f of BE-algebras is called an *epimorphism* if f is onto. Note that if f is a homomorphism of BE-algebras, then $f(1) = 1$.

Proposition 2.1([5])**.** *Let $(X; *, 1)$ be a self distributive BE-algebra. Then the following hold: for any $x, y, z \in X$,*

(i) *if $x \leq y$, then $z * x \leq z * y$ and $y * z \leq x * z$;*
(ii) *$y * z \leq (z * x) * (y * z)$;*
(iii) *$y * z \leq (x * y) * (x * z)$.*

A BE-algebra $(X; *, 1)$ is said to be *transitive* if it satisfies Proposition 2.1(iii).

We now review some fuzzy logic concepts. Let X be a non-empty set. A fuzzy set μ in X is a function $\mu : X \to [0, 1]$. Given a fuzzy set μ in X and $\alpha \in [0, 1]$, the set

$$U(\mu; \alpha) := \{x \in X | \mu(x) \geq \alpha\} (\text{resp. } L(\mu; \alpha) := \{x \in X | \mu(x) \leq \alpha\})$$

is called an *upper* (resp. *lower*) *level subset* of μ.

Definition 2.2([1])**.** Let μ be a fuzzy set in a BE-algebra X. Then μ is called a *fuzzy BE-algebra* of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

Definition 2.3([6]). A binary operation S on $[0, 1]$ is called a *t-conorm* if

(S1) boundary condition: $S(x, 0) = x$;
(S2) commutativity: $S(x, y) = S(y, x)$;
(S3) associativity: $S(x, S(y, z)) = S(S(x, y), z)$;
(S4) monotonicity: $S(x, y) \leq S(x, z)$ whenever $y \leq z$, for all $x, y, z \in [0, 1]$.

We call such a t-conorm an *s-norm* in this paper.

Note that $\max\{x, y\} \leq S(x, y)$ for all $x, y \in [0, 1]$. Moreover, $([0, 1]; S)$ is a commutative semigroup with 0 as the neutral element. In particular,

$$S(S(x, y), S(z, t)) = S(S(x, z), S(y, t))$$

holds for all $x, y, z, t \in [0, 1]$.

The set of all idempotents with respect to S, i.e., the set

$$E_S := \{x \in [0, 1] | S(x, x) = x\}$$

is a subsemigroup of $([0, 1]; S)$. If $Im(\mu) \subseteq E_S$, then the fuzzy set μ is said to be *idempotent*.

3. S-fuzzy subalgebras

In what follows, let S and X denote an s-norm and a BE-algebra respectively, unless otherwise specified.

Definition 3.1. A fuzzy set μ in X is called a *fuzzy subalgebra* of X over S (briefly, an S*-fuzzy subalgebra* of X) if it satisfies
(SF_0) $\mu(x * y) \leq S(\mu(x), \mu(y))$
for all $x, y \in X$. An S-fuzzy subalgebra μ of X is said to be *idempotent* if $Im(\mu) \subseteq E_S$, where $E_S = \{x \in [0,1] | S(x,x) = x\}$.

Example 3.2. Let $X := \{1, a, b, c, d\}$ be a BE-algebra([5]) with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Let $S_m : [0,1] \times [0,1] \to [0,1]$ be a function defined by $S_m(x,y) := \min(x+y, 1)$ for all $x, y \in [0,1]$. Then it is easy to see that S_m is an s-norm. Define a fuzzy set μ in X by $\mu(1) = 0, \mu(a) = \mu(b) = 0.5$ and $\mu(c) = \mu(d) = 1$. Then μ is an S_m-fuzzy subalgebra of X, which is not idempotent, since $0.5 \in Im(\mu)$ and $0.5 \notin E_{S_m}$.

Let $S_M : [0,1] \times [0,1] \to [0,1]$ be a function defined by $S_M(x,y) := \max(x,y)$ for all $x, y \in [0,1]$. Then S_M is also an s-norm. It follows that μ is an idempotent S_M-fuzzy subalgebra of X.

Proposition 3.3. *Let S_m be the s-norm defined in Example 3.2 and let S be a subalgebra of X. Then the fuzzy set μ in X defined by*

$$\mu(x) := \begin{cases} 0 & \text{if } x \in S \\ 1 & \text{otherwise,} \end{cases}$$

is an idempotent S_m-fuzzy subalgebra of X.

Proof. Let $x, y \in X$. If $x, y \in S$, then $x * y \in S$ and so $\mu(x * y) = 0 \leq S_m(\mu(x), \mu(y))$. If $x \notin S$ and $y \notin S$, then $\mu(x) = 1 = \mu(y)$. Hence $S_m(\mu(x), \mu(y)) = \min\{1 + 1, 1\} = 1 \geq \mu(x * y)$. If exactly one of x and y belongs to S, then exactly one of $\mu(x)$ and $\mu(y)$ is equal to 0. It follows that $S_m(\mu(x), \mu(y)) = min\{1 + 0, 1\} = 1 \geq \mu(x * y)$. Therefore μ is an S_m-fuzzy subalgebra of X. Obviously, $Im(\mu) \subseteq E_{S_m}$, completing the proof. $\square$

Proposition 3.4. *If μ is an idempotent S-fuzzy subalgebra of X, then $\mu(1) \leq \mu(x)$ and $\mu(x*y) \leq \max\{\mu(x), \mu(y)\}$ for all $x, y \in X$.*

Proof. For any $x, y \in X$, we have $\mu(1) = \mu(x*x) \leq S(\mu(x), \mu(x)) = \mu(x)$ and $\max\{\mu(x), \mu(y)\} = S(\max\{\mu(x), \mu(y)\}, \max\{\mu(x), \mu(y)\}) \geq S(\mu(x), \mu(y)) \geq \mu(x*y)$. This competes the proof. □

Proposition 3.5. *Let μ be a fuzzy set of X. If every non-empty lower level subset $L(\mu;\alpha)$ of μ is a subalgebra of X, then μ is an S-fuzzy subalgebra of X.*

Proof. Assume that there exist $a, b \in X$ such that $\mu(a*b) > S(\mu(a), \mu(b))$. If we take $m_0 := \frac{1}{2}(\mu(a*b) + S(\mu(a), \mu(b)))$, then $\mu(a*b) > m_0 > S(\mu(a), \mu(b)) \geq \max(\mu(a), \mu(b))$. Hence $a, b \in L(\mu; m_0)$, but $a*b \notin L(\mu; m_0)$. This is a contradiction and so μ satisfies the inequality $\mu(x*y) \leq S(\mu(x), \mu(y))$ for all $x, y \in X$. This completes the proof. □

The converse of Proposition 3.5 may not be true as seen in the following example.

Example 3.6. In Example 3.2, define a fuzzy set ν in X by $\nu(1) = 0, \nu(b) = \nu(d) = 0.5$ and $\nu(a) = \nu(c) = 1$. Let S_m be the s-norm in Example 3.2. Then it is easy to see that ν is an S_m-fuzzy subalgebra of X, but the lower level subset $L(\nu; 0.5) = \{1, b, d\}$ is not a subalgebra of X, since $b*d = c \notin L(\nu; 0.5)$.

Proposition 3.7. *Let μ be an idempotent S-fuzzy subalgebra of X. Then the non-empty lower level subset $L(\mu;\alpha)$ of μ is a subalgebra of X.*

Proof. Let $x, y \in L(\mu;\alpha)$, where $\alpha \in [0,1]$. Then $\mu(x) \leq \alpha$ and $\mu(y) \leq \alpha$. Hence $\mu(x*y) \leq S(\mu(x), \mu(y)) \leq S(\alpha, \alpha) = \alpha$ and so $x*y \in L(\mu;\alpha)$. Thus $L(\mu;\alpha)$ is a subalgebra of X. □

Proposition 3.8. *Let μ be an S-fuzzy subalgebra of X. If there is a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} S(\mu(x_n), \mu(x_n)) = 0$, then $\mu(1) = 0$.*

Proof. For any $x \in X$, we have $\mu(1) = \mu(x*x) \leq S(\mu(x), \mu(x))$. Therefore $\mu(1) \leq S(\mu(x_n), \mu(x_n))$ for each $n \in \mathbb{N}$ and so $0 \leq \mu(1) \leq lim_{n\to\infty} S(\mu(x_n), \mu(x_n)) = 0$. It follows that $\mu(1) = 0$. □

Let f be a mapping defined on X and let μ be a fuzzy set in $f(X)$. The fuzzy set $f^{-1}(\mu)$ in X defined by $[f^{-1}(\mu)](x) := \mu(f(x))$ for all $x \in X$ is called the *preimage* of μ under f.

Theorem 3.9. *Let $f : X \to Y$ be an epimorphism of BE-algebras and let μ be an S-fuzzy subalgebra of Y. Then the preimage $f^{-1}(\mu)$ of μ under f is also an S-fuzzy subalgebra of X.*

Proof. Assume that μ is an S-fuzzy subalgebra of Y. Let $x, y \in X$. Then

$$\begin{aligned}[f^{-1}(\mu)](x*y)) =& \mu(f(x*y)) = \mu(f(x)*f(y)) \\ \leq& S(\mu(f(x)), \mu(f(y))) = S([f^{-1}(\mu)](x), [f^{-1}(\mu)](y)).\end{aligned}$$

Hence $f^{-1}(\mu)$ is an S-fuzzy subalgebra of X. □

Let μ be a fuzzy set in X and let f be a mapping defined on X. The fuzzy set μ^f in $f(X)$ defined by $\mu^f(y) := \inf_{x\in f^{-1}(y)} \mu(x)$ for all $y \in f(X)$ is called the *anti-image* of μ under f.

Definition 3.10. An s-norm S on $[0,1]$ is said to be *continuous* if S is a continuous function from $[0,1]\times[0,1]$ to $[0,1]$ with respect to the usual topology.

Theorem 3.11 *Let S be a continuous s-norm and let $f : X \to Y$ be an epimorphism of BE-algebras. If μ is an S-fuzzy subalgebra of X, then anti-image μ^f is also an S-fuzzy subalgebra of Y.*

Proof. Let $A_1 := f^{-1}(y_1), A_2 := f^{-1}(y_2)$ and $A_{12} := f^{-1}(y_1 * y_2)$, where $y_1, y_2 \in Y$. Consider the set $A_1 * A_2 := \{x \in X | x = a_1 * a_2 \text{ for some } a_1 \in A_1, a_2 \in A_2\}$. If $x \in A_1 * A_2$, then $x = x_1 * x_2$ for some $x_1 \in A_1, x_2 \in A_2$ and so $f(x) = f(x_1 * x_2) = f(x_1) * f(x_2) = y_1 * y_2$, i.e., $x \in f^{-1}(y_1 * y_2) = A_{12}$. Hence $A_1 * A_2 \subseteq A_{12}$. It follows that

$$\begin{aligned}\mu^f(y_1 * y_2) &= \inf_{x\in f^{-1}(y_1*y_2)} \mu(x) = \inf_{x\in A_{12}} \mu(x)\\ &\leq \inf_{x\in A_1*A_2} \mu(x) = \inf_{x\in A_1, x_2\in A_2} \mu(x_1 * x_2)\\ &\leq \inf_{x\in A_1, x_2\in A_2} S(\mu(x_1), \mu(x_2)).\end{aligned}$$

Since S is continuous, if ϵ is any positive number, then there exists a number $\delta > 0$ such that $S(x_1^*, x_2^*) \subseteq S(\inf_{x_1\in A_1} \mu(x_1), \inf_{x_2\in A_2} \mu(x_2)) + \epsilon$, whenever $x_1^* \leq \inf_{x_1\in A_1} \mu(x_1) + \delta$ and $x_2^* \leq \inf_{x_2\in A_2} \mu(x_2) + \delta$. Choose $a_1 \in A_1, a_2 \in A_2$ such that $\mu(a_1) \leq \inf_{x_1\in A_1} \mu(x_1) + \delta$ and $\mu(a_2) \leq \inf_{x_2\in A_2} \mu(x_2) + \delta$. Then $S(\mu(a_1), \mu(a_2)) \leq S(\inf_{x_1\in A_1} \mu(x_1), \inf_{x_2\in A_2} \mu(x_2)) + \epsilon$. Hence we have

$$\begin{aligned}\mu^f(y_1 * y_2) &\leq \inf_{x_1\in A_1, x_2\in A_2} S(\mu(x_1), \mu(x_2))\\ &\leq S(\inf_{x_1\in A_1} \mu(x_1), \inf_{x_2\in A_2} \mu(x_2))\\ &= S(\mu^f(y_1), \mu^f(y_2)).\end{aligned}$$

Thus μ^f is an S-fuzzy subalgebra of Y. □

Theorem 3.12. *Let μ be an idempotent S-fuzzy subalgebra of X. Then the set*

$$X_\mu := \{x \in X | \mu(x) = \mu(1)\}$$

is a subalgebra of X.

Proof. Noticing that $\mu(1) \leq \mu(x)$ for all $x \in X$, we have $L(\mu; \mu(1)) = \{x \in X | \mu(x) \leq \mu(1)\} = \{x \in X | \mu(x) = \mu(1)\} = X_\mu$. By Proposition 3.7, X_μ is a subalgebra of X. □

Proposition 3.13. *Let μ, ν be idempotent S-fuzzy subalgebras of X. If $\mu \subset \nu$ and $\mu(1) = \nu(1)$, then $X_\mu \subset X_\nu$.*

Proof. Assume that $\mu \subset \nu$ and $\mu(1) = \nu(1)$. Let $x \in X_\mu$. Then $\nu(x) > \mu(x) = \mu(1) = \nu(1)$. Noticing $\nu(x) \leq \nu(1)$ for all $x \in X$, we have $\nu(x) = \nu(1)$, i.e., $x \in X_\nu$. This completes the proof. □

Theorem 3.14. *Let S_M be an s-norm defined in Example 3.2. Let μ be an S_M-fuzzy subalgebra of X and let $f : [\mu(1), 1] \to [0, 1]$ be an increasing function. Define a fuzzy set $\mu_f : X \to [0, 1]$ by*

$$\mu_f(x) := f(\mu(x))$$

for all $x \in X$. Then μ_f is an S_M-fuzzy subalgebra of X. Furthermore, if $f(\alpha) \geq \alpha$ for all $\alpha \in [\mu(1), 1]$, then $\mu \subseteq \mu_f$.

Proof. Let $x, y \in X$. Then

$$\begin{aligned} \mu_f(x * y) =& f(\mu(x * y)) \leq f(S_M(\mu(x), \mu(y))) \\ \leq& S_M(f(\mu(x)), f(\mu(y))) = S_M(\mu_f(x), \mu_f(y)). \end{aligned}$$

Hence μ_f is an S_M-fuzzy subalgebra of X. Assume that $f(\alpha) \geq \alpha$ for all $\alpha \in [\mu(1), 1]$. Then $\mu_f(x) = f(\mu(x)) \geq \mu(x)$ for all $x \in X$, which proves that $\mu \subset \mu_f$. □

4. Direct products and s-normed products

Definition 4.1. Let μ and ν be fuzzy sets of X and let S be an s-norm of X. Then the S*-product* of μ and ν is defined by

$$[\mu \cdot \nu]_S(x) := S(\mu(x), \nu(x))$$

for all $x \in X$ and we denote it by $[\mu \cdot \nu]_S$.

Theorem 4.2. *Let μ, ν be two S-fuzzy subalgebras of X and let S^* be an s-norm which dominates S, i.e.,*

$$S^*(S(a, b), S(c, d)) \leq S(S^*(a, c), S^*(b, d))$$

for all a, b, c and $d \in [0, 1]$. Then the S^*-product $[\mu \cdot \nu]_{S^*}$ of μ and ν is an S-fuzzy subalgebra of X.

Proof. For any $x, y \in X$, we have

$$\begin{aligned} [\mu \cdot \nu]_{S^*}(x * y) &= S^*\{\mu(x * y), \nu(x * y)\} \\ &\leq S^*\{S\{\mu(x), \mu(y)\}, S\{\nu(x), \nu(y)\}\} \\ &\leq S\{S^*\{\mu(x), \nu(x)\}, S^*\{\mu(y), \nu(y)\}\} \\ &= S\{[\mu \cdot \nu]_{S^*}(x), [\mu \cdot \nu]_{S^*}(y)\}. \end{aligned}$$

Hence $[\mu \cdot \nu]_{S^*}$ is an S-fuzzy subalgebra of X. □

Let $f : X \to Y$ be an epimorphism of BE-algebras. If μ and ν are S-fuzzy subalgebras of Y, then the S^*-product $[\mu \cdot \nu]_{S^*}$ of μ and ν is also an S-fuzzy subalgebra of Y whenever S^* dominates S. Since every epimorphic preimage of an S-fuzzy subalgebra is also an S-fuzzy subalgebra, the preimages $f^{-1}(\mu), f^{-1}(\nu)$ and $f^{-1}([\mu\cdot\nu]_{S^*})$ are S-fuzzy subalgebras. The next theorem provides the relation between $f^{-1}([\mu\cdot\nu]_{S^*})$ and the S^*-product $[f^{-1}(\mu)\cdot f^{-1}(\nu)]_{S*}$ of $f^{-1}(\mu)$ and $f^{-1}(\nu)$.

Proposition 4.3. *Assume that $f : X \to Y$ is an epimorphism of BE-algebras and S, S^* are s-norms such that S^* dorminates S. For any S-fuzzy subalgebras μ and ν of Y, we have*

$$f^{-1}([\mu\cdot\nu]_{S^*}) = [f^{-1}(\mu)\cdot f^{-1}(\nu)]_{S^*}.$$

Proof. For any $x \in X$, we obtain

$$\begin{aligned}\{f^{-1}([\mu\cdot\nu]_{S*})\}(x) =&[\mu\cdot\nu]_{S^*}(f(x))\\ =&S^*\{\mu(f(x)),\nu(f(x))\}\\ =&S^*([f^{-1}(\mu)](x),[f^{-1}(\nu)](x))\\ =&[f^{-1}(\mu)\cdot f^{-1}(\nu)]_{S^*}(x),\end{aligned}$$

completing the proof. □

Let $(X_1, *_1, 1_1)$ and $(X_2, *_2, 1_2)$ be BE-algebras. Define a binary operation "$*$" on $X_1\times X_2$ by

$$(x_1, x_2) * (y_1, y_2) := (x_1 *_1 x_2, y_1 *_2 y_2)$$

for all $(x_1, x_2), (y_1, y_2) \in X$. Then $(X, *, 1)$ is a BE-algebra, where $1 = (1_1, 1_2)$.

Theorem 4.4. *Let $X = X_1 \times X_2$ be the direct product of BE-algebras X_1 and X_2. If μ_1 (resp., μ_2) is an S-fuzzy subalgebra of X_1(resp., X_2), then $\mu := \mu_1 \times \mu_2$ is an S-fuzzy subalgebra of X defined by*

$$\mu(x_1, x_2) = (\mu_1 \times \mu_2)(x_1, x_2) = S(\mu_1(x_1), \mu_2(x_2))$$

for all $(x_1, x_2) \in X_1 \times X_2$.

Proof. Let $x = (x_1, x_2), y = (y_1, y_2) \in X$. Then we have

$$\begin{aligned}\mu(x*y) =&\mu((x_1,x_2)*(y_1,y_2))\\ =&\mu(x_1*y_1,x_2*y_2)\\ =&S(\mu_1(x_1*y_1),\mu_2(x_2*y_2))\\ \le&S(S(\mu_1(x_1),\mu_1(y_1)),S(\mu_2(x_2),\mu_2(y_2)))\\ =&S(S(\mu_1(x_1),\mu_2(x_2)),S(\mu_1(y_1),\mu_2(y_2)))\\ =&S(\mu(x_1,x_2),\mu(y_1,y_2))\\ =&S(\mu(x),\mu(y)).\end{aligned}$$

Hence $\mu = \mu_1 \times \mu_2$ is an S-fuzzy subalgebra of X. □

Now, we generalize the idea to the product of S-fuzzy subalgebras. We first need to generalize the domain of an s-norm to $\prod_{i=1}^{n}[0,1]$ as follows.

Definition 4.5. We define a map $S_n : \prod_{i=1}^{n}[0,1] \to [0,1]$ is defined by $S_n(\alpha_1, \alpha_2, \cdots, \alpha_n) := S(\alpha_i, S_{n-1}(\alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_n))$ for all $1 \le i \le n$, where $S_2 = S$ and $S_1 = \mathrm{id}_{[0,1]}$.

Using the induction on n, we have following two lemmas:

Lemma 4.6. *For an s-norm S and every α_i, β_i, where $1 \le i \le n$ and $n \ge 2$, we have*

$$\begin{aligned} S_n(S(\alpha_1, \beta_1),&S(\alpha_2, \beta_2), \cdots, S(\alpha_n, \beta_n) \\ &= S(S_n(\alpha_1, \alpha_2, \cdots, \alpha_n), S_n(\beta_1, \beta_2, \cdots, \beta_n)). \end{aligned}$$

Lemma 4.7. *For an s-norm S and every $\alpha_1, \alpha_2, \cdots, \alpha_n \in [0,1]$, where $n \ge 2$, we have*

$$\begin{aligned} S_n(\alpha_1, \alpha_2, \cdots, \alpha_n) &= S(\cdots, S(S(S(\alpha_1, \alpha_2), \alpha_3, \alpha_4), \cdots, \alpha_n) \\ &= S(\alpha_1, S(\alpha_2, S(\alpha_3, \cdots, S(\alpha_{n-1}, \alpha_n) \cdots))). \end{aligned}$$

Theorem 4.8. *Let $X := \prod_{i=1}^{n} X_i$ be the direct product of BE-algebras $\{X_i\}_{i=1}^{n}$. If μ_i is an S-fuzzy subalgebra of X_i, where $1 \le i \le n$, then $\mu = \prod_{i=1}^{n} \mu_i$ defined by*

$$\mu(x_1, \cdots, x_n) = (\prod_{i=1}^{n} \mu_i)(x_1, \cdots, x_n) = S(\mu(x_1), \cdots, \mu(x_n))$$

is an S-fuzzy subalgebra of X.

Proof. Let $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n)$ be any elements of X. Using Lemmas 4.6 and 4.7, we have

$$\begin{aligned} \mu(x * y) &= \mu((x_1 * y_1), (x_2 * y_2), \cdots, (x_n * y_n)) \\ &= S_n(\mu_1(x_1 * y_1), \cdots, \mu_n(x_n * y_n)) \\ &\le S_n(S(\mu_1(x_1), \mu_1(y_1)), \cdots, S(\mu_n(x_n), \mu_n(y_n))) \\ &= S(S_n(\mu_1(x_1), \mu_2(x_2), \cdots, \mu_n(x_n)), \\ &\quad S_n(\mu_1(y_1), \mu_2(y_2), \cdots, \mu_n(y_n))) \\ &= S(\mu((x_1, \cdots, x_n) * (y_1, \cdots, y_n))) \\ &= S(\mu(x * y)). \end{aligned}$$

Hence $\mu = \prod_{i=1}^{n} \mu_i$ is an S-fuzzy subalgebra of X. □

Theorem 4.9. *Let S be a continuous s-norm and $f : X \to Y$ be an epimorphism of BE-algebras, and let μ and ν be S-fuzzy subalgebras of X. If an s-norm S^* dominates S, then*

$$([\mu \cdot \nu]_{S^*})^f \subseteq [\mu^f \cdot \nu^f]_{S^*}.$$

Proof. By Theorems 4.2 and 3.11, the S^*-product $[\mu \cdot \nu]_{S^*}$ is an S-fuzzy subalgebra of X, and the S^*-product $[\mu^f \cdot \nu^f]_{S^*}$ is an S-fuzzy subalgebra of Y. Moreover, for each $y \in Y$, we have

$$\begin{aligned}
([\mu \cdot \nu]_{S^*})^f(y) &= \inf_{x \in f^{-1}(y)} [\mu \cdot \nu]_{S^*}(x) \\
&= \inf_{x \in f^{-1}(y)} S^*(\mu(x), \nu(x)) \\
&\leq S^*(\inf_{x \in f^{-1}(y)} \mu(x), \inf_{x \in f^{-1}(y)} \nu(x)) \\
&= S^*(\mu^f(y), \nu^f(y)) \\
&= ([\mu^f \cdot \nu^f]_{S^*})(y),
\end{aligned}$$

proving that $([\mu \cdot \nu]_{S^*})^f \subseteq [\mu^f \cdot \nu^f]_{S^*}$. □

References

[1] S. S. Ahn, Y. H. Kim and K. S. So, *Fuzzy BE-algebras*, J. Appl. Math. and Informatics **29** (2011), 1049-1057.
[2] S. S. Ahn and J. M. Ko, *On vague filters in BE-algebras*, Commun. Korean Math. Soc. **26** (2011), 417-425.
[3] S. S. Ahn and K. K. So, *On ideals and upper sets in BE-algebras*, Sci. Math. Jpn. **68** (2008), 279-285.
[4] S. S. Ahn and K. K. So, *On generalized upper sets in BE-algebras,* Bull. Korean Math. Soc. **46** (2009), 281-287.
[5] H. S. Kim and Y. H. Kim, *On BE-algebras,* Sci. Math. Jpn. **66** (2007), 113-116.
[6] E. P. Klement, R. Mesiar and E. Pap, *Triangular norms. Position paper I: basic anaytical and algebraic propertis*, Fuzzy Sets and Systems **143**(2004), 5-26.
[7] Y. B. Jun and S. S. Ahn, *Fuzzy implicative filters of BE-algebras with degrees in the interval* $(0, 1]$, J. Computational Analsis and Applications **15**(2013), 1456-1466.
[8] L. A. Zadeh, *Fuzzy sets*, Inform. Control **56** (2008), 338-353.

Fixed point results for generalized g-quasi-contractions of Perov-type in cone metric spaces over Banach algebras without the assumption of normality

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Abstract: In this paper, we introduce the concept of generalized g-quasi-contractions of Perov-type in the setting of cone metric spaces over Banach algebras. By omitting the assumption of normality of the cone we establish common fixed point theorems for generalized g-quasi-contractions of Perov-type with the spectral radius $r(\lambda)$ of the g-quasi-contractive constant vector λ satisfying $r(\lambda) \in [0,\ 1)$ in the setting of cone metric spaces over Banach algebras. The main results generalize, extend and unify several well-known comparable results in the literature. As a result, we extend the famous Ćirić fixed point theorem to the version in the setting of cone metric spaces over Banach algebras.

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1 Introduction

It is known that the modern metric fixed point theory was motivated from Banach contraction principle (see, e.g., [1]) which plays an important role in various fields of applied mathematical analysis. In 1922, Polish mathematician proved the following classical Banach contraction principle:

Theorem 1.1 ([1]) Let $T : X \to X$ be a contraction on a complete metric space (X, d). Then T possesses exactly one fixed point $x^* \in X$. Moreover, for any point $x \in X$, the sequence $\{T^n(x) : n = 0, 1, 2, ...\}$ converges to $x^* \in X$. That is $\lim_{n\to\infty} T^n(x) = x^*$, for each $x \in X$, where T^n denotes the n-fold composition of T.

Since 1922, many authors have obtained all kinds of versions to extend the famous Banach contraction principle. In general, people did such extensions by means of two methods. One is to extend Banach contraction to other more general mapping or mappings (for example, when two or more mappings are involved and discussed, the common fixed point(s) is(are) usually investigated). The other is to extend classical metric space to more general spaces (usually called abstract spaces). There are many generalizations of the concept metric space in the literature. In 1964, Perov [34] introduced vector valued metric space, instead of general metric space, and obtained a Banach type fixed point theorem on such a complete generalized metric space. Later on, following Perov, many authors studied fixed point results of Perov-type in more general abstract spaces, such as cone metric spaces, etc (see [35]-[39]). Among them, Cvetković and Rakočević [36] introduced the concept of f-quasi-contraction of Perov-type and obtained fixed point results for such kind mappings, which is a generalization of the famous Ćirić mappings. Let $(X,\, d)$ be a complete metric space. Recall that a mapping $T : X \to X$ is called a quasi-contraction if, for some $k \in [0,\, 1)$ and for all $x,\, y \in X$, one has

$$d(Tx,\, Ty) \leqslant k \max\{d(x,\, y),\, d(x,\, Tx),\, d(y,\, Ty),\, d(x,\, Ty),\, d(y,\, Tx)\}.$$

Ćirić [13] introduced and studied quasi-contractions as one of the most general classes of contractive-type mappings. He proved the well-known theorem that any quasi-contraction T has a unique fixed point. Recently, many authors obtained various similar results on cone b-metric spaces (some authors call such spaces cone metric type spaces) and cone metric spaces. See, for instance, [7]-[15].

Since 2010, some authors have investigated the problem of whether cone metric spaces are equivalent to metric spaces in terms of the existence of the fixed points of the mappings involved. They used to establish the equivalence between some fixed point results in metric and in (topological vector spaces valued) cone metric spaces by means of the nonlinear scalarization function ξ_e where e denotes the vector in the internal of the underlying solid cone (see [16]-[19]). Very recently, based on the concept of cone metric spaces, Liu and Xu [21] studied cone metric spaces with Banach algebras, replacing Banach spaces by Banach algebras as the underlying spaces of cone metric spaces. In [21], the authors proved some fixed point theorems of quasi-contractions in cone metric spaces over Banach algebras, but the proof relied strongly on the assumption that the underlying cone is normal. We need state that it is significant to study cone metric spaces with Banach algebras (which we would like to call in this paper cone metric spaces over Banach algebras). This is because there are examples to show that one is unable to conclude that the cone metric space $(X,\, d)$ over a Banach algebra $\mathcal{A}$ discussed is equivalent to the metric space $(X,\, d^*)$, where the metric d^* is defined by $d^* = \xi_e \circ d$, here the nonlinear scalarization function $\xi_e : \mathcal{A} \to \mathbb{R}$ ($e \in \mathrm{int}P$) is defined by

$$\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - P\}. \tag{1.1}$$

See [20] for more details.

In the present paper we introduce the concept of generalized g-quasi-contractions of Perov-type in cone metric spaces over Banach algebras and obtain common fixed point theorems for two weakly compatible self-mappings satisfying g-quasi-contractive condition in the case of g-quasi-contractive constant vector with $r(\lambda) \in [0, 1/s)$ in cone metric spaces without the assumption of normality. Our main results extend the fixed point theorem of quasi-contractions of Das-Naik in metric spaces to the case in cone metric spaces over Banach algebras. As consequences, we obtain the versions of Ćirić fixed point theorem and Banach contraction principle in the setting of cone metric spaces over Banach algebras. Our main results generalize and extend the relevant results in the literature (see, for example, [3]-[9], [13], [15], [21], [23], [25], [27]).

In addition, we give an example to show that the main results are genuine generalizations of the corresponding results in the literature.

2 Preliminaries

Let $\mathcal{A}$ always be a real Banach algebra. That is, $\mathcal{A}$ is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for all $x,\ y,\ z \in \mathcal{A},\ \alpha \in \mathbb{R}$):

1. $(xy)z = x(yz)$;
2. $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;
3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
4. $\|xy\| \leqslant \|x\|\,\|y\|$.

Throughout this paper, we shall assume that a Banach algebra $\mathcal{A}$ has a unit (i.e., a multiplicative identity) e such that $ex = xe = x$ for all $x \in \mathcal{A}$. An element $x \in \mathcal{A}$ is said to be invertible if there is an inverse element $y \in \mathcal{A}$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}. For more details, we refer to [28].

The following proposition is well known (see [28]).

Proposition 2.1 Let $\mathcal{A}$ be a Banach algebra with a unit e, and $x \in \mathcal{A}$. If the spectral radius $r(x)$ of x is less than 1, i.e.,

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geqslant 1} \|x^n\|^{\frac{1}{n}} < 1,$$

then $e - x$ is invertible. Actually,

$$(e-x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

Now let us recall the concepts of cone and semi-order for a Banach algebra $\mathcal{A}$. A subset P of $\mathcal{A}$ is called a cone if

1. P is non-empty closed and $\{\theta,\ e\} \subset P$;
2. $\alpha P + \beta P \subset P$ for all non-negative real numbers $\alpha,\ \beta$;
3. $P^2 = PP \subset P$;
4. $P \cap (-P) = \{\theta\}$,

where θ denotes the null of the Banach algebra $\mathcal{A}$. For a given cone $P \subset \mathcal{A}$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \mathrm{int}P$, where $\mathrm{int}P$ denotes the interior of P.

The cone P is called normal if there is a number $M > 0$ such that for all $x,\, y \in \mathcal{A}$,

$$\theta \preceq x \preceq y \Rightarrow \|x\| \leqslant M\|y\|.$$

The least positive number satisfying above is called the normal constant of P.

In the following we always assume that P is a cone in Banach algebra $\mathcal{A}$ with $\mathrm{int}P \neq \emptyset$ and $\preceq$ is the partial ordering with respect to P.

Definition 2.1 (See [2], [3], [20], [21]) Let X be a non-empty set. Suppose the mapping $d:\ X \times X \to \mathcal{A}$ satisfies

1. $0 \preceq d(x,\, y)$ for all $x,\, y \in X$ and $d(x,\, y) = 0$ if and only if $x = y$;
2. $d(x,\, y) = d(y,\, x)$ for all $x,\, y \in X$;
3. $d(x,\, y) \preceq d(x,\, z) + d(z,\, x)$ for all $x,\, y,\, z \in X$.

Then d is called a cone metric on X, and $(X,\, d)$ is called a cone metric space over a Banach algebra $\mathcal{A}$.

Definition 2.2 (See [2], [3], [20], [21]) Let $(X,\, d)$ be a cone metric space with a solid cone P over a Banach algebra $\mathcal{A}$, $x \in X$ and $\{x_n\}$ a sequence in X. Then

1. $\{x_n\}$ converges to x whenever for each $c \in \mathcal{A}$ with $\theta \ll c$ there is a natural number N such that $d(x_n,\, x) \ll c$ for all $n \geqslant N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.
2. $\{x_n\}$ is a Cauchy sequence whenever for each $c \in \mathcal{A}$ with $\theta \ll c$ there is a natural number N such that $d(x_n,\, x_m) \ll c$ for all $n,\, m \geqslant N$.
3. $(X,\, d)$ is a complete cone metric space if every Cauchy sequence is convergent.

Now, we shall appeal to the following lemmas in the sequel.

Lemma 2.1 (See [12]) If E is a real Banach space with a cone P and if $a \preceq \lambda a$ with $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.

Lemma 2.2 (See [27]) If E is a real Banach space with a solid cone P and if $\theta < u \ll c$ for each $\theta \ll c$, then $u = \theta$.

Lemma 2.3 (See [27]) If E is a real Banach space with a solid cone P and if $\|x_n\| \to 0 (n \to \infty)$, then for any $\theta \ll \epsilon$, there exists $N \in \mathbb{N}$ such that for any $n > N$, we have $x_n \ll \epsilon$.

Finally, let us recall the concept of quasi-contraction defining on the cone metric spaces over Banach algebras, which is introduced in [21].

Definition 2.3 (See [21]) Let $(X,\, d)$ be a cone metric space over a Banach algebra $\mathcal{A}$. A mapping $T : X \to X$ is called a quasi-contraction if for some $k \in P$ with $r(k) < 1$ and for all $x,\, y \in X$, one has

$$d(Tx,\, Ty) \preceq ku,$$

where

$$u \in \{d(x,\, y),\, d(x,\, Tx),\, d(y,\, Ty),\, d(x,\, Ty),\, d(y,\, Tx)\}.$$

Remark 2.1 (See [29]) If $r(k) < 1$, then $\|k^m\| \to 0 (m \to \infty)$.

Lemma 2.4 (See [4]-[6], [23], [32]) Let $\preceq$ be the partial ordering with respect to P, where P is the given solid cone P of the Banach algebra $\mathcal{A}$. The following properties are often used while dealing with cone metric spaces where the underlying cone is solid but not necessarily normal.

(1) If $u \ll v$ and $v \preceq w$, then $u \ll w$.

(2) If $\theta \preceq u \ll c$ for each $c \in \mathrm{int}P$, then $u = \theta$.

(3) If $a \preceq b + c$ for each $c \in \mathrm{int}P$, then $a \preceq b$.

(4) If $c \in \mathrm{int}P$ and $a_n \to \theta$, then there exists $n_0 \in \mathbb{N}$ such that $a_n \ll c$ for all $n > n_0$.

(5) Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$, $x \in X$ and $\{x_n\}$ be a sequence in X. If $d(x_n, x) \preceq b_n$ and $b_n \to \theta$, then $x_n \to x$.

Lemma 2.5 (See [3]) The limit of a convergent sequence in cone metric space is unique.

Definition 2.6 (See [4], [11]) The mappings $f, g : X \to X$ are called weakly compatible, if for every $x \in X$ holds $fgx = gfx$ whenever $fx = gx$.

Definition 2.7 (See [4], [11], [14]) Let f and g be self-maps of a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Lemma 2.6 (See (See [4], [11], [14]) Let f and g be weakly compatible self-maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.

Definition 2.8 (See [23] Let (X, d) be a cone metric space. A mapping $f : X \to X$ such that, for some constant $\lambda \in [0, 1)$ and for every $x, y \in X$, there exists an element

$$u \in C(g; x, y) = \{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\}$$

for which $d(fx, fy) \preceq \lambda u$ is called a g-quasi-contraction, where $g : X \to X$, $f(X) \subset g(X)$.

Definition 2.9 Let $(X,\ d)$ be a cone metric space over a Banach algebra $\mathcal{A}$. A mapping $f : X \to X$ is called a generalized g-quasi-contractions of Perov-type, if there exist a mapping $g : X \to X$ with $f(X) \subset g(X)$ and some $\lambda \in P$ with $r(\lambda) \in [0,\ 1)$, for all $x,\ y \in X$, one has

$$d(fx, fy) \preceq \lambda u, \tag{2.1}$$

where

$$u \in C(g; x, y) = \{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\}.$$

Definition 2.10 (See [30], [31]) Let P be a solid cone in a Banach space $\mathcal{A}$. A sequence $\{u_n\} \subset P$ is a c-sequence if for each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for $n \geq n_0$.

It is easy to show the following propositions.

Proposition 2.2 (See [30]) Let P be a solid cone in a Banach space $\mathcal{A}$ and let $\{u_n\}$ and $\{v_n\}$ be sequences in P. If $\{u_n\}$ and $\{v_n\}$ are c-sequences and $\alpha, \beta > 0$, then $\{\alpha u_n + \beta v_n\}$ is a c-sequence.

In addition to Proposition 2.2 above, the following propositions are crucial to the proof of our main results.

Proposition 2.3 (See [30]) Let P be a solid cone in a Banach algebra $\mathcal{A}$ and let $\{u_n\}$ be a sequence in P. Then the following conditions are equivalent.

(1) $\{u_n\}$ is a c-sequence.

(2) For each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \prec c$ for $n \geq n_0$.

(3) For each $c \gg \theta$ there exists $n_1 \in \mathbb{N}$ such that $u_n \preceq c$ for $n \geq n_1$.

Proposition 2.4 (See [30]) Let P be a solid cone in a Banach algebra $\mathcal{A}$ and let $\{u_n\}$ be a sequence in P. Suppose that $k \in P$ is an arbitrarily given vector and $\{u_n\}$ is a c-sequence in P. Then $\{ku_n\}$ is a c-sequence.

Proposition 2.5 Let $\mathcal{A}$ be a Banach algebra with a unit e, P be a cone in $\mathcal{A}$ and $\preceq$ be the semi-order be yielded by the cone P. Let $\lambda \in P$. If the spectral radius $r(\lambda)$ of λ is less than 1, then the following assertions hold true.

(i) We have $(e-\lambda)^{-1} \succ \theta$. In addition, we have $\theta \preceq \lambda^n \preceq (e-\lambda)^{-1}\lambda^n \preceq (e-\lambda)^{-1}\lambda$ for any integer $n \geq 1$.

(ii) For any $u \succ \theta$, we have $u \not\preceq \lambda u$. Moreover, we have $u \not\preceq \lambda^n u$ for any integer $n \geq 1$.

Proposition 2.6 (See [29]) Let (X, d) be a complete cone metric space over a Banach algebra $\mathcal{A}$ and let P be the underlying solid cone in Banach algebra $\mathcal{A}$. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to $x \in X$, then we have

(i) $\{d(x_n, x)\}$ is a c-sequence;

(ii) for any $p \in \mathbb{N}$, $\{d(x_n, x_{n+p})\}$ is a c-sequence.

3 Main results

In this section, without the assumption of normality of the underlying cone, we give some common fixed point theorems for generalized g-quasi-contractions of Perov-type with the spectral radius $r(\lambda)$ of the g-quasi-contractive constant vector λ satisfying $r(\lambda) \in [0,\ 1)$ in the setting of cone metric spaces over Banach algebras.

Theorem 3.1 Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$ and the underlying solid cone P. Let the mapping $f : X \to X$ be the g-quasi-contractions of Perov-type with the spectral radius $r(\lambda)$ of the g-quasi-contractive constant vector λ satisfying $r(\lambda) \in [0,\ 1)$. If the range of g contains the range of f and $g(X)$ or $f(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.

We begin the proof of Theorem 3.1 with a useful lemma. For each $x_0 \in X$, set $gx_1 = fx_0$ and $gx_{n+1} = fx_n$. We will prove that $\{gx_n\}$ is a Cauchy sequence. First, we shall show the following lemmas. Note that for these lemmas, we suppose that all the conditions in Theorem 3.1 are satisfied.

Lemma 3.1 For any $N \geq 2$ and $1 \leq m \leq N-1$, one has that

$$d(gx_N, gx_m) \preceq \lambda(e-\lambda)^{-1} d(gx_1, gx_0). \tag{3.1}$$

Proof. We now prove Lemma 3.1 by induction. When $N = 2, m = 1$, since $f : X \to X$ is a generalized g-quasi-contractions of Perov-type satisfying (2.1), there exists

$$u_1 \in C(g; x_1, x_0) = \{d(gx_1, gx_0), d(gx_1, gx_2), d(gx_0, gx_1), d(gx_1, gx_1), d(gx_0, gx_2)\}$$

such that

$$d(gx_2, gx_1) \preceq \lambda u_1.$$

Hence, $u_1 = d(gx_1, gx_0)$ or $u_1 = d(gx_0, gx_2)$. (Note that it is obvious that $u_1 \neq d(gx_1, gx_2)$ since $d(gx_2, gx_1) \npreceq \lambda d(gx_1, gx_2)$ and $u_1 \neq d(gx_1, gx_1)$ since $d(gx_1, gx_2) \neq \theta$.)

When $u_1 = d(gx_1, gx_0)$, we have

$$d(gx_2, gx_1) \preceq \lambda d(gx_0, gx_1) \preceq \lambda(e-\lambda)^{-1} d(gx_1, gx_0).$$

When $u_1 = d(gx_2, gx_0)$, then we have

$$d(gx_2, gx_1) \preceq \lambda d(gx_2, gx_0) \preceq \lambda[d(gx_2, gx_1) + d(gx_1, gx_0)].$$

So we get

$$(e-\lambda) d(gx_2, gx_1) \preceq \lambda d(gx_1, gx_0),$$

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which implies that

$$d(gx_2, gx_1) \preceq \lambda(e-\lambda)^{-1} d(gx_1, gx_0).$$

Hence, (3.1) holds for $N = 2$ and $m = 1$.

Suppose that for some $N \geq 2$ and for any $2 \leq p \leq N$ and $1 \leq n \leq p$, one has

$$d(gx_p, gx_n) \preceq \lambda(e-\lambda)^{-1} d(gx_1, gx_0). \tag{3.2}$$

That is,

$$d(gx_p, gx_1) \preceq \lambda(e-\lambda)^{-1} d(gx_1, gx_0), \tag{3.2.1}$$

$$d(gx_p, gx_2) \preceq \lambda(e-\lambda)^{-1} d(gx_1, gx_0), \tag{3.2.2}$$

......

$$d(gx_p, gx_{p-1}) \preceq \lambda(e-\lambda)^{-1} d(gx_1, gx_0). \tag{3.2.$p-1$}$$

Then, we need to prove that for $N+1 \geq 2$ and any $1 \leq n < N+1$, one has

$$d(gx_{N+1}, gx_n) \preceq \lambda(e-\lambda)^{-1} d(gx_1, gx_0). \tag{3.3}$$

That is,

$$d(gx_{N+1}, gx_1) \preceq \lambda(e-\lambda)^{-1} d(gx_1, gx_0), \tag{3.3.1}$$

$$d(gx_{N+1}, gx_2) \preceq \lambda(e-\lambda)^{-1} d(gx_1, gx_0), \tag{3.3.2}$$

......

$$d(gx_{N+1}, gx_{N-1}) \preceq \lambda(e-\lambda)^{-1} d(gx_1, gx_0), \tag{3.3.$N-1$}$$

$$d(gx_{N+1}, gx_N) \preceq \lambda(e-\lambda)^{-1} d(gx_1, gx_0). \tag{3.3.N}$$

In fact, since $f : X \to X$ is a g-quasi-contraction, there exists

$$u_1 \in C(g; x_N, x_0) = \{d(gx_N, gx_0), d(gx_N, gx_{N+1}), d(gx_0, gx_1), d(gx_N, gx_1), d(gx_0, gx_{N+1})\}$$

such that

$$d(gx_{N+1}, gx_1) \preceq \lambda u_1.$$

If $u_1 = d(gx_N, gx_1)$, then by (3.2.1) we have

$$d(gx_{N+1}, gx_1) \preceq \lambda^2(e-\lambda)^{-1} d(gx_1, gx_0) \preceq \lambda^2(e-\lambda)^{-1} d(gx_1, gx_0) \preceq \lambda(e-\lambda)^{-1} d(gx_1, gx_0).$$

If $u_1 = d(gx_0, gx_1)$, then we have

$$d(gx_{N+1}, gx_1) \preceq \lambda d(gx_1, gx_0) \preceq \lambda d(gx_1, gx_0) \preceq \lambda(e-\lambda)^{-1} d(gx_1, gx_0).$$

If $u_1 = d(gx_N, gx_0)$, then by (3.2.1) we have

$$\begin{aligned} d(gx_{N+1}, gx_1) &\preceq \lambda d(gx_N, gx_0) \preceq \lambda(d(gx_N, gx_1) + d(gx_1, gx_0)) \\ &\preceq \lambda(\lambda(e-\lambda)^{-1} d(gx_1, gx_0) + d(gx_1, gx_0)) \\ &= \lambda(\lambda(e-\lambda)^{-1} + e) d(gx_1, gx_0) \\ &= \lambda(e-\lambda)^{-1} d(gx_1, gx_0). \end{aligned}$$

If $u_1 = d(gx_0, gx_{N+1})$, then we have

$$d(gx_{N+1}, gx_1) \preceq \lambda d(gx_0, gx_{N+1}) \preceq \lambda(d(gx_0, gx_1) + d(gx_1, gx_{N+1})).$$

Hence, we see

$$(e-\lambda) d(gx_{N+1}, gx_1) \preceq \lambda d(gx_0, gx_1),$$

which implies that

$$d(gx_{N+1}, gx_1) \preceq (e-\lambda)^{-1} \lambda d(gx_0, gx_1).$$

Without loss of generality, suppose that $u_1 = d(gx_N, gx_{N+1})$. Since $f : X \to X$ is a g-quasi-contraction, there exists $u_2 \in C(g; x_{N-1}, x_N)$ such that

$$u_1 = d(gx_N, gx_{N+1}) \preceq \lambda u_2,$$

where

$$\begin{aligned} C(g; x_{N-1}, x_N) = \{&d(gx_{N-1}, gx_N), d(gx_{N-1}, gx_N), d(gx_N, gx_{N+1}), \\ &d(gx_{N-1}, gx_{N+1}), d(gx_N, gx_N)\}. \end{aligned}$$

So, we get

$$d(gx_{N+1}, gx_1) \preceq \lambda u_1 \preceq \lambda^2 u_2.$$

Similarly, it is easy to see that $u_2 \neq d(gx_N, gx_N)$ since $u_2 \neq \theta$ and $u_2 \neq d(gx_N, gx_{N+1})$ since $d(gx_N, gx_{N+1}) \not\preceq \lambda^2 d(gx_N, gx_{N+1})$.

If $u_2 = d(gx_{N-1}, gx_N)$, then by the induction assumption (3.2) we have

$$\begin{aligned} d(gx_{N+1}, gx_1) &\preceq \lambda^2 u_2 \preceq \lambda^3 (e-\lambda)^{-1} d(gx_1, gx_0) \\ &\preceq \lambda^3 (e-\lambda)^{-1} d(gx_1, gx_0) \\ &\preceq \lambda (e-\lambda)^{-1} d(gx_1, gx_0). \end{aligned}$$

Without loss of generality, suppose that $u_2 = d(gx_{N-1}, gx_{N+1})$. There exists $u_3 \in C(g; x_{N-2}, x_N)$ such that

$$u_2 = d(gx_{N-1}, gx_{N+1}) \preceq \lambda u_3,$$

where

$$\begin{aligned} C(g; x_{N-2}, x_N) = \{&d(gx_{N-2}, gx_N), d(gx_{N-2}, gx_{N-1}), d(gx_N, gx_{N+1}), \\ &d(gx_{N-2}, gx_{N+1}), d(gx_N, gx_{N-1})\}. \end{aligned}$$

In general, suppose that $u_{i-1} = d(gx_{N-i+2}, gx_{N+1})$. Since $f : X \to X$ is a g-quasi-contraction, by the similar arguments above, there exists $u_i \in C(g; x_{N-i+1}, x_N)$ such that

$$u_{i-1} = d(gx_{N-i+2}, gx_{N+1}) \preceq \lambda u_i,$$

for which we obtain

$$d(gx_{N+1}, gx_1) \preceq \lambda u_1 \preceq \lambda^2 u_2 \preceq \cdots \preceq \lambda^i u_i,$$

where

$$\begin{aligned} C(g; x_{N-i+1}, x_N) = \{&d(gx_{N-i+1}, gx_N), d(gx_{N-i+1}, gx_{N-i+2}), d(gx_N, gx_{N+1}), \\ &d(gx_{N-i+1}, gx_{N+1}), d(gx_N, gx_{N-i+2})\}. \end{aligned}$$

Similarly, it is easy to see that $u_i \neq d(gx_N, gx_{N+1})$. This is because by Proposition 2.5(iii) we have

$$u_1 = d(gx_N, gx_{N+1}) \not\preceq \lambda^{i-1} d(gx_N, gx_{N+1}).$$

So we know that if $u_i = d(gx_{N-i+1}, gx_N)$ or $u_i = d(gx_{N-i+1}, gx_{N-i+2})$ or $u_i = d(gx_N, gx_{N-i+2})$ then by the induction assumption (3.2) we have $u_i \preceq \lambda(e-\lambda)^{-1} d(gx_1, gx_0)$. Hence,

$$\begin{aligned} d(gx_{N+1}, gx_1) &\preceq \lambda^i u_i \preceq \lambda^{i+1}(e-\lambda)^{-1} d(gx_1, gx_0) \\ &\preceq (\lambda)^{i+1}(e-\lambda)^{-1} d(gx_1, gx_0) \\ &\preceq \lambda(e-\lambda)^{-1} d(gx_1, gx_0), \end{aligned}$$

which means that (3.3.1) holds true. Without loss of generality, suppose that $u_i = d(gx_{N-i+1}, gx_{N+1})$. Then by the similar arguments as above we have $u_i \preceq \lambda u_{i+1}$, where $u_{i+1} \in C(g; x_{N-i}, x_N)$. Hence, there is a sequence $\{u_n\}$ such that

$$d(gx_{N+1}, gx_1) \preceq \lambda u_1 \preceq \lambda^2 u_2 \preceq \cdots \preceq \lambda^{N-1} u_{N-1} \preceq \lambda^N u_N,$$

where

$$u_{N-1} = d(gx_2, gx_{N+1}) \preceq \lambda u_N$$

and

$$u_N \in C(g; x_1, x_N) = \{d(gx_1, gx_N), d(gx_1, gx_2), d(gx_N, gx_{N+1}), d(gx_N, gx_2), d(gx_1, gx_{N+1})\}.$$

Obviously, $u_N \neq d(gx_1, gx_{N+1})$ and $u_N \neq d(gx_N, gx_{N+1})$. On the contrary, if $u_N = d(gx_1, gx_{N+1})$, then $u_N \preceq \lambda^N u_N$, a contradiction. If $u_N = d(gx_N, gx_{N+1}) = u_1$, then we have

$$u_1 = d(gx_N, gx_{N+1}) \preceq \lambda^2 u_2 \preceq \cdots \preceq \lambda^{N-1} u_{N-1} \preceq \lambda^{N-1} u_1,$$

a contradiction. Hence, it follows that $u_N = d(gx_1, gx_N)$, $u_N = d(gx_1, gx_2)$ or $u_N = d(gx_N, gx_2)$. By the induction assumption (3.2), in any case, we have

$$u_N \preceq \lambda(e-\lambda)^{-1} d(gx_1, gx_0). \tag{3.4}$$

Therefore, we get

$$\begin{aligned} d(gx_{N+1}, gx_1) &\preceq \lambda u_1 \preceq \lambda^2 u_2 \preceq \cdots \preceq \lambda^N u_N \\ &\preceq \lambda^N (e-\lambda)^{-1} \lambda d(gx_1, gx_0) \\ &\preceq (\lambda)^{N+1} (e-\lambda)^{-1} d(gx_1, gx_0) \\ &\preceq \lambda(e-\lambda)^{-1} d(gx_1, gx_0). \end{aligned} \tag{3.5}$$

That is to say, (3.3.1) is true. By (3.5), we have

$$u_1 \preceq \lambda^{N-1} \lambda (e-\lambda)^{-1} d(gx_1, gx_0).$$

Thus,

$$\begin{aligned} d(gx_N, gx_{N+1}) = u_1 &\preceq \lambda^{N-1} \lambda (e-\lambda)^{-1} d(gx_1, gx_0) \\ &\preceq (\lambda)^N (e-\lambda)^{-1} d(gx_1, gx_0) \\ &\preceq \lambda (e-\lambda)^{-1} d(gx_1, gx_0), \end{aligned}$$

which implies that (3.3.N) is true. Similarly, since

$$u_2 = d(gx_{N-1}, gx_{N+1}), \ldots, u_i = d(gx_{N-i+1}, gx_{N+1}), \ldots,$$

by (3.4) and (3.5) we get

$$u_i \preceq \lambda^{N-i} u_N \preceq \lambda^{n-i+1} (e-\lambda)^{-1} d(gx_1, gx_0). \tag{3.6}$$

Hence, it follows from (3.6) that (3.3.2)-(3.3.$N-1$) are all true. That is, (3.3) is true. Therefore, we conclude that Lemma 3.1 holds true.

By Lemma 3.1, we immediately obtain the following result.

Lemma 3.2 For all $i, j \in \mathbb{N}_+$, one has

$$d(gx_i, gx_j) \preceq \lambda(e-\lambda)^{-1} d(gx_0, gx_1). \tag{3.7}$$

Now, we begin to prove Theorem 3.1. First, we need to show that $\{gx_n\}$ is a Cauchy sequence. For all $n > m$, there exists

$$\begin{aligned}\nu_1 \in C(g; x_{n-1}, x_{m-1}) =& \{d(gx_{n-1}, gx_{m-1}), d(gx_{n-1}, gx_n),\\ & d(gx_{m-1}, gx_m), d(gx_{n-1}, gx_m), d(gx_{m-1}, gx_n)\}\end{aligned}$$

such that

$$d(fx_{n-1}, fx_{m-1}) \preceq \lambda\nu_1.$$

Using the g-quasi-contractive condition repeatedly, we easily show by induction that there must exist

$$\nu_k \in \{d(gx_i, gx_j) : 0 \le i < j \le n\} \ (k = 2, 3, \ldots, m)$$

such that

$$\nu_k \preceq \lambda\nu_{k+1} \ (k = 1, 2, \ldots, m-1). \tag{3.8}$$

For convenience, we write $\nu_m = d(gx_i, gx_j)$ where $0 \le i < j \le n$.

Using the triangular inequality, we have

$$d(gx_i, gx_j) \preceq d(gx_i, gx_0) + d(gx_0, gx_j) \quad (0 \le i, j \le n),$$

and by Lemma 3.2 we obtain

$$\begin{aligned}d(gx_n, gx_m) &= d(fx_{n-1}, fx_{m-1}) \preceq \lambda\nu_1 \preceq \lambda^2\nu_2 \preceq \cdots \preceq \lambda^m\nu_m\\ &\preceq \lambda^m d(gx_i, gx_j)\\ &= \lambda^{m+1}(e-\lambda)^{-1} d(gx_1, gx_0).\end{aligned}$$

Since $r(\lambda) < 1$, by Remark 2.1 we have that $\lambda^{m+1}(e-\lambda)^{-1}d(gx_1, gx_0) \to \theta$ as $m \to \infty$, so by Proposition 2.4, it is easy to see that for any $c \in \mathrm{int}P$, there exists $n_0 \in \mathbb{N}$ such that for all $n > m > n_0$,

$$d(gx_n, gx_m) \preceq \lambda^{m+1}(e-\lambda)^{-1}d(gx_1, gx_0) \ll c.$$

So $\{gx_n\}$ is a Cauchy sequence in $g(X)$. If $g(X) \subset X$ is complete, there exist $q \in g(X)$ and $p \in X$ such that $gx_n \to q$ as $n \to \infty$ and $gp = q$.

Now, from (2.1) we get

$$d(fx_n, fp) \preceq \lambda\nu$$

where

$$\nu \in C(g; x_n, p) = \{d(gx_n, gp), d(gx_n, fx_n), d(gp, fp), d(gx_n, fp), d(fx_n, gp)\}.$$

Clearly at least one of the following five cases holds for infinitely many n.

(1) $d(fx_n, fp) \preceq \lambda d(gx_n, gp) \preceq \lambda d(gx_{n+1}, gp) + \lambda d(gx_{n+1}, gx_n)$;

(2) $d(fx_n, fp) \preceq \lambda d(gx_n, fx_n) = \lambda d(gx_n, gx_{n+1})$;

(3) $d(fx_n, fp) \preceq \lambda d(gp, fp) \preceq \lambda d(gx_{n+1}, gp) + \lambda d(gx_{n+1}, fp)$,
that is, $d(fx_n, fp) \preceq \lambda(e-\lambda)^{-1}d(gx_{n+1}, gp)$;

(4) $d(fx_n, fp) \preceq \lambda d(gx_n, fp) \preceq \lambda d(gx_{n+1}, fp) + \lambda d(gx_{n+1}, gx_n)$,
that is, $d(fx_n, fp) \preceq \lambda(e-\lambda)^{-1}d(gx_{n+1}, gx_n)$;

(5) $d(fx_n, fp) \preceq \lambda d(fx_n, gp) = \lambda d(gx_{n+1}, gp)$.

As $\lambda \preceq \lambda(e-\lambda)^{-1}$ (since $\theta \preceq \lambda$ and $r(\lambda) < 1$), we obtain that

$$d(gx_{n+1}, fp) \preceq \lambda(e-\lambda)^{-1}[d(gx_{n+1}, gx_n) + d(gx_{n+1}, q)].$$

Since $gx_n \to q$ as $n \to \infty$, we get that for any $c \in \mathrm{int}P$, there exists $n_1 \in \mathbb{N}$ such that for all $n > n_1$, one has

$$d(gx_{n+1}, fp) \ll c.$$

By Lemmas 2.4 and 2.5, we have $gx_n \to fp$ as $n \to \infty$ and $q = fp$.

Now if w is another point such that $gu = fu = w$, hence

$$d(w, q) = d(fu, fp) \preceq \lambda\nu,$$

where $r(\lambda) \in [0,\ 1)$ and

$$\nu \in C(g; u, p) = \{d(gu, gp), d(gu, fu), d(gp, fp), d(gu, fp), d(fu, gp)\}.$$

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It is obvious that $d(w,q)=\theta$, i.e., $w=q$. Therefore, q is the unique point of coincidence of f and g in X. Moreover, the mappings f and g are weakly compatible, by Lemma 2.6 we know that q is the unique common fixed point of f and g.

Similarly, if $f(X)$ is complete, the above conclusion is also established.

According to Das-Naik version of the known theorem in the setting of metric spaces from [33], we have following result similar to Theorem 3.1.

Theorem 3.2 Suppose one of the following conditions holds:

(1) As in Theorem 3.1, let $(X,\, d)$ be a complete cone metric space over a Banach algebra $\mathcal{A}$. Assume one of $f\,(X)$ or $g\,(X)$ is closed and the other conditions in Theorem 3.1 are not changeable;

(2) As in Theorem 3.1, let (X,d) be a complete cone metric space over a Banach algebra $\mathcal{A}$. Assume f,g are cone compatible and both continuous and the other conditions in Theorem 3.1 are not changeable;

(3) As in Theorem 3.1, let (X,d) be a complete cone metric space over a Banach algebra $\mathcal{A}$. Assume f commutes with g, f or g is continuous (see Theorem 3.2 in Cvetković-Rakočević [36]) and the other conditions in Theorem 3.1 are not changeable.

Then the conclusions of Theorem 3.1 are also true.

Proof. (1) The proof of this case is the same as that in Theorem 3.1.

(2) The sequence $y_n = fx_n = gx_{n+1}, y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$ converges to some $z \in X$ as $n \to \infty$. Further, since $fx_n \to z$ and $gx_n \to z$ we get that

$$d\,(fz,gz) \leq d\,(fz,fgx_n) + d\,(fgx_n,gfx_n) + d\,(gfx_n,gz) \to 0+0+0=0.$$

Hence $fz=gz=\omega$. Hence, f,g has (a unique) point of coincidence. Since f and g are compatible then they are weakly compatible. Therefore by standard result they have a unique common fixed point (in this case it is ω).□

(3) Let g be continuous.

Then we get $gy_n \to gz$ and $fy_n \to gz$ since f commutes with g. Indeed, $fy_n = fgx_{n+1} = gfy_{n+1} \to gz$.

So we get

$$\begin{aligned} d(fz, gz) &\preceq d(fz, fy_n) + d(fy_n, gz) \\ &\preceq \lambda u + d(fy_n, gz), \end{aligned}$$

where

$$u \in \{d(gz, gy_n), d(gz, fz), d(gy_n, fy_n), d(gz, fy_n), d(gy_n, fz) + d(fy_n, gz)\}.$$

If $u = d(gz, gy_n)$ or $u = d(gy_n, fy_n)$ or $d(gz, fy_n)$, then we obtain that $\lambda u + d(fy_n, gz)$ is a c-sequence. This mens that $fz = gz$. If $u = d(gz, fz)$ or $u = d(gy_n, fz) \preceq d(gy_n, gz) + d(gz, fz)$ we get

$$d(fz, gz) \preceq \lambda d(gz, fz) + d(fy_n, gz)$$

or

$$d(fz, gz) \preceq \lambda d(gz, fz) + \lambda d(gy_n, gz) + d(fy_n, gz).$$

In both cases we have that

$$d(fz, gz) \preceq (e - \lambda)^{-1} c_n,$$

where c_n is a c-sequence. Hence, f, g have a unique point of coincidence. Since f commutes with g then they are weakly compatible and by known result have a unique fixed point.

Now let f be continuous.

Again, $fy_n \to fz$ and $gy_n = gfx_n = fgx_n = fy_{n-1} \to fz$.

Further we get

$$d(fz., z) \preceq d(fz, fy_n) + d(fy_n, y_n) + d(y_n, y).$$

Since $d(fz, fy_n) + d(y_n, y) = c_n$ is c-sequence it is sufficient to estimate $d(fy_n, y_n)$.

We have

$$d(fy_n, y_n) = d(fy_n, fx_n) \preceq \lambda u,$$

where

$$\begin{aligned} u &\in \{d(gy_n, gx_n), d(gy_n, fy_n), d(gx_n, fx_n), d(gy_n, fx_n), d(gx_n, fy_n)\} \\ &= \{d(fy_{n-1}, y_{n-1}), d(fy_{n-1}, fy_n), d(fy_{n-1}, y_n), d(y_{n-1}, fy_n), d(y_{n-1}, y_n)\}. \end{aligned}$$

Now we get the following cases:

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I) $u = d(fy_{n-1}, y_{n-1})$. Then

$$\begin{aligned} d(fz., z) &\preceq c_n + \lambda d(fy_{n-1}, y_{n-1}) \\ &\preceq c_n + \lambda(d(fy_{n-1}, fy) + d(fy, y) + d(y, y_{n-1})) \end{aligned}$$

or

$$\begin{aligned} d(fz., z) &\preceq (e-\lambda)^{-1} c_n + (e-\lambda)^{-1} \lambda d(fy_{n-1}, fy) + (e-\lambda)^{-1} \lambda d(y, y_{n-1}) \\ &= d_n \text{ where } d_n \text{ is a new } c\text{-sequence.} \end{aligned}$$

II) $u = d(fy_{n-1}, fy_{n-1})$
III) $u = d(fy_{n-1}, y_n)$
IV) $u = d(y_{n-1}, fy_n)$
V) $u = d(y_{n-1}, y_n)$
In all cases we obtained that $fz = z$.
For details see Theorem 3.2 in Cvetković-Rakočević [36].

Corollary 3.1 Let (X, d) be a complete cone metric space over a Banach algebra $\mathcal{A}$ and let P be the underlying cone with $k \in P$. If the mapping $T : X \to X$ is a quasi-contraction, then T has a unique fixed point in X. And for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Proof. Set $g = I_X$, the identity mapping from X to X. It is obvious to see that Theorem 3.1 yields Corollary 3.1.

Remark 3.1 Corollary 3.1 does not need to require the assumption of normality of the cone P. So Corollary 3.1 improves and generalizes Theorem 9 in [21].

Remark 3.2 From the proof of Lemma 3.1, we note that the technique of induction appearing in Theorem 3.1 is somewhat different from that in Theorem 9 from [21], and also different from that in Theorem 2.6 from [11], which is more interesting and easily to understood. In addition, the proof of Theorem 3.1 is a valuable addition to [9] since Theorem 3.1 is a generalization of Theorem 3 from [9] but some main results in the proof of Theorem 3 from [9] were not proved in general.

Remark 3.3 Taking $E = \mathbb{R}$, $P = [0, +\infty)$, $\|.\| = |.|$, $\lambda \in [0, 1)$ in Theorem 3.1, we get Das-Naik's result from [33]; if $g = I_X$ we get Ćirić's result from [13], both in the setting of metric spaces.

The following corollary is the Jungck's result in the setting of cone metric spaces over Banach algebras.

Corollary 3.2 Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$ with the underlying solid cone P. Let the mappings $f,\ g : X \to X$ satisfy the condition that for $\lambda \in P$ with $r(\lambda) \in [0,\ 1)$ and for every $x, y \in X$ holds $d(fx, fy) \preceq \lambda d(gx, gy)$. If $g(X) \subset f(X)$ and $g(X)$ or $f(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

The next result is the Banach contraction principle in the setting of cone metric spaces over Banach algebras.

Corollary 3.3 (see [29]) Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$ the underlying solid cone P. Let the mapping $f : X \to X$ satisfy the condition that for $\lambda \in P$ with $r(\lambda) \in [0,\ 1)$ and for every $x, y \in X$ holds $d(fx, fy) \preceq \lambda d(x, y)$ (namely, f is a generalized Lipschitz contraction). If $f(X)$ is a complete subspace of X, then f has a unique point in X.

We will present an example to show that the results presented above are real generalizations of the corresponding results in the literature.

Example 3.1 Let $X = [1, \infty)$ and $\mathcal{A}$ be a set of all real valued function on $[0, 1]$ which also have continuous derivates on $[0, 1]$ with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and the usual multiplication. Let $P = \{x \in \mathcal{A} : x(t) \geq 0, t \in [0, 1]\}$. It is clear that P is a nonrmal cone and $\mathcal{A}$ is a Banach algebra with a unit $e = 1$. Define a mapping

$$d : X \times X \to \mathcal{A}$$

by

$$d(x, y)(t) := |x - y|\, e^t.$$

We make a conclusion that (X, d) is a complete cone metric space over Banach algebra $\mathcal{A}$. Now define the mappings $f, g : X \to X$ by $f(x) = 3x - 2, g(x) = 4x - 3$. Choose

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$\lambda(t) = \frac{1}{12}t + \frac{3}{4}$. Since $f(X) \subseteq g(X)$ and $r(\lambda) = \frac{5}{6}$, thus, all the conditions of Theorem 3.1 are satisfied and consequently f and g have a unique comon fixed point $x = 1$. Indeed, for $x, y \in X$ we can putting $u(x, y) = d(g(x), g(y)) = 4|x - y|$. In this case we have

$$d(f(x), f(y)) = 3|x-y|e^t \leq \left(\frac{1}{12}t + \frac{3}{4}\right) 4|x-y|e^t \Leftrightarrow \frac{3}{4} \leq \frac{1}{12}t + \frac{3}{4},$$

which is indeed true. On the other hand, we see that

$$f(g(x)) = f(4x-3) = 3(4x-3) - 2 = 12x - 11 = 4(3x-2) - 3 = g(f(x)),$$

that is, f commutes with g and other words f, g are weakly compatible.

Now let us estimate $r(\lambda) = \lim_{n\to\infty} \|\lambda^n\|^{\frac{1}{n}}$. Since

$$\lambda^n(t) = \left(\frac{1}{12}t + \frac{3}{4}\right)^n, (\lambda^n(t))' = \frac{n}{12}\left(\frac{1}{12}t + \frac{3}{4}\right)^{n-1},$$

we have $(t = 1)$

$$\|\lambda\|_\infty + \|\lambda'\|_\infty = \left(\frac{5}{6}\right)^n + \frac{n}{12}\left(\frac{5}{6}\right)^{n-1} = \frac{n}{12}\left(\frac{5}{6}\right)^{n-1}\left(\frac{12}{n} \cdot \frac{5}{6} + 1\right) = \frac{n}{12}\left(\frac{5}{6}\right)^{n-1}\left(1 + \frac{10}{n}\right).$$

Further we get

$$\|\lambda^n\|^{\frac{1}{n}} = \left(\frac{n}{12}\right)^{\frac{1}{n}} \left(\frac{5}{6}\right)^{\frac{n-1}{n}} \left(1 + \frac{10}{n}\right)^{\frac{1}{n}} \to \frac{5}{6} < 1.$$

However, both f and g are not quasi-contraction. Indeed, for $x = 2, y = 1$ and for all λ with $r(\lambda) \in [0, 1)$, we get

$$d(f2, f1)(t) = d(4, 1)(t) = 3e^t > \lambda(t)u,$$

for all

$$\begin{aligned} u &\in \left\{d(2,1)e^t, d(2,f2)e^t, d(1,f1)e^t, d(2,f1)e^t, d(1,f2)e^t\right\} \\ &= \left\{e^t, 2e^t, 0, e^t, 3e^t\right\}, \end{aligned}$$

and similarly

$$d(g2, g1)(t) = d(5, 1)(t) = 4e^t > \lambda(t)u$$

for all

$$\begin{aligned} u &\in \left\{d(2,1)e^t, d(2,g2)e^t, d(1,g1)e^t, d(2,g1)e^t, d(1,g2)e^t\right\} \\ &= \left\{e^t, 3e^t, 0, e^t, 4e^t\right\}. \end{aligned}$$

Hence, Theorem 3.1 is a genuine generalization of Theorem 9 from [21].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors contribute equally and significantly in writing this paper. All the authors read and approve the final manuscript.

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References

[1] S. Banach, *Sur les operations dans les ensembles abstrait et leur application aux equations, integrals* . Fundam. Math., (3)1922, 13.

[2] P. P. Zabrejko, *K-metric and K-normed linear spaces: survey.* Collect. Math., 48(1997), 825-859.

[3] L.-G. Huang and X. Zhang, *Cone metric space and fixed point theorems of contractive mappings*. J. Math. Anal. Appl., 332(2007), 1468-1476.

[4] M. Abbas and G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric spaces*. J. Math. Anal. Appl. 341(2008), 416-420.

[5] S. Radenović, V. Rakočević and Sh. Rezapour, Common fixed points for (g, f) type maps in cone metric spaces. Appl. Math. Comput., 218(2011), 480-491.

[6] S. H. Cho and J. S. Bae, *Common fixed points theorems for mappings satisfying property (E, A) on cone metric spaces*. Math. Comput. Modell., 53(2011), 945-951.

[7] D. Ilić and V. Rakočević, *Quasi-contraction on a cone metric space*. Appl. Math. Lett., 22: 5(2009), 728-731.

[8] Z. Kadelburg, S. Radenović and V. Rakočević, *Remarks on "Quasi-contraction on a cone metric space"*. Appl. Math. Lett., 22: 11(2009), 1674-1679.

[9] Lj. Gajić and V. Rakočević, *Quasi-contractions on a nonnormal cone metric space*. Funct. Anal. Appl., 46: 1(2012), 75-79.

[10] N. Hussian and M. H. Shah, *KKM mappings in cone b-metric spaces*. Comput. Math. Appl., 62(2011), 1677-1684.

[11] L. Shi and S. Xu, *Common fixed point theorems for two weakly compatible self-mappings in cone b-metric spaces*. Fixed Point Theory Appl., 2013, 2013: 120.

[12] Z. Kadelburg, M. Ravlović and S. Radenović, *Common Fixed point theorems for ordered contractions and quasi-contractions in ordered cone metric spaces*, Comput. Math. Appl., 59(2010), 3148-3159.

[13] Lj. B. Ćirić, *A generalization of Banach's contraction priciple*. Proc. Amer. Math. Soc., 45(1974), 267-273.

[14] M.H. Shah, S. Simić, N. Hussain, A. Sretenović and S. Radenović, *Common fixed points theorems for occasionally weakly compatible pairs on cone metric type spaces*. J. Comput. Anal. Appl., 14(2) (2012), 290-297.

[15] M.P. Stanić, A.S. Cvetković, S. Simić and S. Dimitrijević, *Common fixed point under contractive condition of Ćirić type on cone metric type spaces*. Fixed Point Theory Appl., 2012, 2012:35.Volume 2011, Article ID 589725.

[16] W. S. Du, *A note on cone metric fixed point theory and its equivalence*. Nonl. Anal., 72: 5 (2010) 2259-2261.

[17] Y. Feng and W. Mao, *The equivalence of cone metric spaces and metric spaces*. Fixed Point Theory 11:2(2010), 259-264.

[18] Z. Kadelburg, S. Radenović and V. Rakočević, *A note on the equivalence of some metric and cone metric fixed point results*. Appl. Math. Lett., 24(2011), 370-374.

[19] H. Çakallı, A. Sönmez and Ç. Genç, *On an equivalence of topological vector space valued cone metric spaces and metric spaces*. Appl. Math. Lett., 25 (2012) 429-433.

[20] H. Liu and S. Xu, *Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings*. Fixed Point Theory Appl., 2013, 2013: 320.

[21] H. Liu and S. Xu, *Fixed point theorem of quasi-contractions on cone metric spaces with Banach algebras*, Abstract Appl. Anal., Volume 2013, Article ID 187348, 5 pages.

[22] I. Marian and A. Branga, *Common fixed point results in b-cone metric spaces over topological vector spaces*. General Mathematics., Vol. 20, No. 1(2012).

[23] D. llić and V. Rakočević, *Common fixed points for maps on cone metric space*, J. Math. Anal. Appl., 341(2008), 876-882.

[24] Sh. Rezapour and R. Hamlbarani, *Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings"*, J. Math. Anal. Appl., 345(2008) 719-724.

[25] S. Janković, Z. Golubović and S. Radenović, *Compatible and weakly compatible mappings in cone metric spaces*. Math. Comput. Modell., 52(2010), 1728-1738.

[26] M. Jovanović, Z. Kadelburg and S. Radenović, *Common Fixed Point Results in Metric-Type Spaces*. Fixed Point Theory Appl., Volume 2010, Article ID 978121.

[27] S. Radenović and B. E. Rhoades, *Fixed point theorem for two non-self mappings in cone metric spaces*. Comput. Math. Appl., 57(2009), 1701-1707.

[28] Walter Rudin, *Functional Analysis*, 2nd ed., McGraw-Hill, 1991.

[29] S. Xu and S. Radenović, *Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality.* Fixed Point Theory Appl., 2014, 2014:102.

[30] Z. Kadelburg, and S. Radenović, *A note on various types of cones and fixed point results in cone metric spaces.* Asian J. Math. Appl., Volume 2013, Article ID ama0104, 7 pages.

[31] M. Dordević, D. Dorić, Z. Kadelburg, S. Radenović and D. Spasić, *Fixed point results under c-distance in tvs-cone metric spaces.* Fixed Point Theory Appl., 2011, 2011: 29.

[32] S. Janković, Z. Kadelburg and S. Radenović, *On the cone metric space: A survey.* Nonl. Anal., 74(2011), 2591-2601.

[33] K. M. Das, K. Naik, *Common fixed point theorems for commuting maps on a metric space.* Proc. Am. Math. Soc., 77 (1979), 369-373.

[34] A. I. Perov, *On Cauchy problem for a system of ordinary differential equations,* (in Russian) Priblizhen. Metody Reshen. Difer. Uravn. 2 (1964) 115-134.

[35] A. Szilard, *A note on Perov's fixed point theorem,* Fixed Point Theory, Volume 4, No. 1, 2003, 105-108.

[36] M. Cvetković, V. Rakočević, *Common fixed point results for mappings of Perov type*, Math. Nachr. 1C18 (2015) / DOI 10.1002/mana.201400098.

[37] M. Cvetković, V. Rakočević, *Extensions of Perov theorem*, Carpat. J. Math. **31** (2015), 2, 181-188.

[38] M. Cvetković, V. Rakočević, *Quasi-contraction of Perov type,* Appl. Math. Comput. **235** (2014) 712-722.

[39] Z. Li and S. Jing, *Quasi-contractions restricted with linear bounded mappings.* Fixed Point Theory Appl., 2014, 2014:87.

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On some inequalities of the Bateman's $G-$function

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Abstract

In the paper, we prove that the Bateman's $G-$function satisfies the double inequality

$$\sum_{n=1}^{2m} \frac{(2^n-1)B_{2n}}{nx^{2n}} < G(x) - \frac{1}{x} < \sum_{n=1}^{2m-1} \frac{(2^n-1)B_{2n}}{nx^{2n}}, \qquad m \in \mathbb{N}$$

with best bounds, where $B_r's$ are the Bernoulli numbers and we study the monotonicity of some functions involving the function $G(x)$. Also, we present some estimates for the error term of a class of the alternating series, which improve and generalize some recent resutls and we prove the increasing monotonicity of a sequence arising from computation of the intersecting probability between a plane couple and a convex body.

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Key Words: Digamma function, Bateman's G-function, sharp inequality, monotonicity, alternating series, sequence.

1 Introduction.

The ordinary gamma function is defined by [3]

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \qquad x > 0$$

and the derivative of $\log\Gamma(x)$ is called the digamma function and is denoted by $\psi(x)$. We can considered to the gamma function, the digamma function and the the Riemann zeta function as the most important special functions [5]. For more details on bounding the gamma function and its logarithmic derivatives, please refer to the papers [2]-[5], [7], [8], [14]-[20] [22]-[26], [35]-[41]

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and plenty of references therein.

The Bateman's $G-$function is defined by Erdélyi [6] as

$$G(x) = \psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right), \qquad x \neq 0, -1, -2, ... \tag{1}$$

which satisfies [6]:

$$G(1+x) + G(x) = \frac{2}{x} \tag{2}$$

and

$$G(1-x) + G(x) = 2\pi \csc(\pi x). \tag{3}$$

The function $G(x)$ can be defined by the hypergeometric function as

$$G(x) = \frac{2}{x}\, {}_2F_1(1, x; 1+x; -1).$$

From the integral representation of the function $\psi(z)$ [3]

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right) dt, \qquad x > 0$$

we obtain the following integral representation

$$G(x) = \int_0^\infty \frac{2\, e^{-xt}}{1+e^{-t}} dt, \qquad x > 0. \tag{4}$$

The function $G(x)$ is very useful in summing and estimating certain numerical and algebraic series [27] . For example:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{s\, k + u} = \frac{1}{2s} G\left(\frac{u}{s}\right), \qquad u \neq 0, -s, -2s, ... \tag{5}$$

and its n^{th} partial sum is given by

$$\sum_{k=0}^{n} \frac{(-1)^k}{s\, k + u} = \frac{1}{2s}\left[G\left(\frac{u}{s}\right) + (-1)^n G\left(\frac{u}{s} + n + 1\right)\right], \qquad u \neq 0, -s, -2s, ... \tag{6}$$

Qiu and Vuorinen [43] deduced the inequality

$$\frac{4(1.5 - \log 4)}{x^2} < G(x) - \frac{1}{x} < \frac{1}{2x^2}, \qquad x > \frac{1}{2}. \tag{7}$$

Mahmoud and Agarwal [16] presented an asymptotic formula for Bateman's G-function $G(x)$ and deduced the double inequality

$$\frac{1}{2x^2 + 1.5} < G(x) - \frac{1}{x} < \frac{1}{2x^2}, \qquad x > 0 \tag{8}$$

which improve the lower bound of the inequality (7) and they posed a sharp double inequality of the function $G(x)$ as a conjecture. Mortici [21] established the inequality

$$0 < \psi(x+u) - \psi(x) \le \psi(u) + \gamma + \frac{1}{u} - u \qquad x \ge 1;\ u \in (0,1), \tag{9}$$

where γ is the Euler constant, which also improves the result of Qiu and Vuorinen. Also, Alzer presented the double inequality [2]

$$\frac{1}{x} - A_n(u;x) - \delta_n(u;x) < \psi(x+u) - \psi(x) < \frac{1}{x} - A_n(u;x),$$

where $n \ge 0$ be an integer, $x > 0$, $u \in (0,1)$,

$$A_n(u;x) = (1-u)\left[\frac{1}{u+n+1} + \sum_{i=0}^{n-1} \frac{1}{(x+i+1)(x+i+u)}\right]$$

and

$$\delta_n(u;x) = \frac{1}{x+n+u} \log \frac{(x+n)^{(x+n)(1-u)}(x+n+1)^{(x+n+1)u}}{(x+n+u)^{x+n+u}}.$$

In this paper, we prove the conjecture posed by Mahmoud and Agarwal [16] about a sharp double inequality of the function $G(x)$. We will study the completely monotonicity property of some functions involving the Bateman's $G-$function. Our results generalize and improve some inequalities about the error term of a class of alternating series and will prove the main result of [9] about the increasing monotonicity of a certain sequence .

2 Main results.

Theorem 1. *The Bateman's $G-$function satisfies*

$$G(x) = \frac{1}{x} + \sum_{n=1}^{m} \frac{(2^{2n}-1)B_{2n}}{nx^{2n}} + \frac{(2^{2m+2}-1)B_{2m+2}}{(m+1)x^{2m+2}}\theta_1, \qquad m = 1,2,3,... \tag{10}$$

where $B_i's$ are Bernoulli numbers, θ_1 is independent of x and $0 < \theta_1 < 1$.

Proof. Using the integral representation of the function $G(x)$ and the formula [1]

$$\frac{1}{x^s} = \frac{1}{(s-1)!}\int_0^\infty t^{s-1}e^{-xt}dt, \qquad s \in \mathbb{N}$$

we get

$$G(x) - \frac{1}{x} = \int_0^\infty \tanh(t/2)\, e^{-xt}dt. \tag{11}$$

We will apply a technique which used later by Qi and Guo [34]. By the expansion [1]

$$\tanh(t/2) = \sum_{k=1}^{\infty} \frac{4t}{t^2 + \pi^2(2k-1)^2}$$

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and the identity

$$\frac{4t}{t^2+\pi^2(2k-1)^2} = \sum_{n=1}^{m} \frac{4(-1)^{n-1}t^{2n-1}}{\pi^{2n}(2k-1)^{2n}} + \frac{4(-1)^m t^{2m+1}}{\pi^{2m}(2k-1)^{2m}} \frac{1}{t^2+\pi^2(2k-1)^2}, \quad m \in \mathbb{N}$$

we obtain

$$G(x) - \frac{1}{x} = \int_0^\infty \sum_{k=1}^{\infty} \left(\sum_{n=1}^{m} \frac{4(-1)^{n-1}t^{2n-1}}{\pi^{2n}(2k-1)^{2n}} + \frac{4(-1)^m t^{2m+1}}{\pi^{2m}(2k-1)^{2m}} \frac{1}{t^2+\pi^2(2k-1)^2} \right) e^{-xt}dt \quad m \in \mathbb{N}.$$

Now

$$\sum_{k=1}^{\infty} \sum_{n=1}^{m} \frac{4(-1)^{n-1}t^{2n-1}}{\pi^{2n}} \frac{1}{(2k-1)^{2n}} = \sum_{n=1}^{m} \frac{4(-1)^{n-1}t^{2n-1}}{\pi^{2n}} (1-2^{-2n})\zeta(2n),$$

where $\zeta(t)$ is the Riemann zeta function which satisfies [3]

$$\zeta(2s) = \frac{(-1)^{s-1}\pi^{2s}2^{2s-1}}{(2s)!} B_{2s}, \qquad s \in \mathbb{N}.$$

Then

$$\sum_{k=1}^{\infty} \sum_{n=1}^{m} \frac{4(-1)^{n-1}t^{2n-1}}{\pi^{2n}} \frac{1}{(2k-1)^{2n}} = \sum_{n=1}^{m} \frac{2(2^{2n}-1)B_{2n}}{(2n)!} t^{2n-1}, \qquad m \in \mathbb{N}. \tag{12}$$

Also,

$$\sum_{k=1}^{\infty} \frac{4(-1)^m t^{2m+1}}{\pi^{2m}(2k-1)^{2m}} \frac{1}{t^2+\pi^2(2k-1)^2} = \frac{4(-1)^m t^{2m+1}}{\pi^{2m+2}} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2m+2}} \frac{1}{\left(\frac{t}{\pi(2k-1)}\right)^2+1} \quad m \in \mathbb{N}$$

and hence

$$\sum_{k=1}^{\infty} \frac{4(-1)^m t^{2m+1}}{\pi^{2m}(2k-1)^{2m}} \frac{1}{t^2+\pi^2(2k-1)^2} = \frac{4(-1)^m t^{2m+1}}{\pi^{2m+2}} \theta(t) \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2m+2}}, \quad m \in \mathbb{N}$$

where $0 < \theta(t) < 1$. Then

$$\sum_{k=1}^{\infty} \frac{4(-1)^m t^{2m+1}}{\pi^{2m}(2k-1)^{2m}} \frac{1}{t^2+\pi^2(2k-1)^2} = \frac{2(2^{2m+2}-1)t^{2m+1}B_{2m+2}}{(2m+2)!} \theta(t), \qquad 0 < \theta(t) < 1; \ m \in \mathbb{N}. \tag{13}$$

Now

$$G(x) - \frac{1}{x} = \sum_{n=1}^{m} \frac{2(2^n-1)B_{2n}}{(2n)!} \int_0^\infty t^{2n-1}e^{-xt}dt + \frac{2(2^{2m+2}-1)B_{2m+2}}{(2m+2)!} \int_0^\infty \theta(t)t^{2m+1}e^{-xt}dt. \tag{14}$$

Using the ordinary gamma function and its functional equation $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$, we get

$$G(x) - \frac{1}{x} = \sum_{n=1}^{m} \frac{(2^{2n}-1)B_{2n}}{nx^{2n}} + \frac{(2^{2m+2}-1)B_{2m+2}}{(m+1)x^{2m+2}} \theta_1, \qquad m \in \mathbb{N}$$

where θ_1 is independent of x and $0 < \theta_1 < 1$. □

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Theorem 2 ([16], Conjecture 1)**.** *The Bateman's G−function satisfies the following double inequality*

$$\sum_{n=1}^{2m} \frac{(2^{2n}-1)B_{2n}}{nx^{2n}} < G(x) - \frac{1}{x} < \sum_{n=1}^{2m-1} \frac{(2^{2n}-1)B_{2n}}{nx^{2n}} \qquad m = 1, 2, 3, ... \tag{15}$$

with sharp bounds, where $B_i's$ are Bernoulli numbers.

Proof. The inequality (15) satisfies from the relation (10) and the following property of Bernoulli constants [12]:

$$B_{2r+2} < 0 \quad and \quad B_{2r+4} > 0 \quad for\ r = 1, 3, 5, ... \tag{16}$$

Now, we will prove the sharpness of the inequality (15) using Mortici's technique [25]. From the definition [11], the asymptotic expansion of a function $T(x)$ of the form

$$T(x) = K(x) + b_0 + \sum_{k=1}^{\infty} \frac{b_k}{x^k}$$

satisfies for every fixed r, that

$$\lim_{x\to\infty} x^r \left[T(x) - \left(K(x) + b_0 + \sum_{k=1}^{r} \frac{b_k}{x^k} \right) \right] = 0.$$

Using the relation (10), we have

$$\lim_{x\to\infty} x^{2m} \left[G(x) - \frac{1}{x} - \sum_{n=1}^{m-1} \frac{(2^{2n}-1)B_{2n}}{nx^{2n}} \right] = \frac{(2^{2m}-1)B_{2m}}{m}, \quad m = 1, 2, 3, ... \tag{17}$$

If we have other constants $h_2, h_4, h_6, ...$ satisfy

$$\sum_{i=1}^{2} \frac{h_{2i}}{x^{2i}} < G(x) - \frac{1}{x} < \sum_{i=1}^{1} \frac{h_{2i}}{x^{2i}},$$

$$\sum_{i=1}^{4} \frac{h_{2i}}{x^{2i}} < G(x) - \frac{1}{x} < \sum_{i=1}^{3} \frac{h_{2i}}{x^{2i}},$$

$$\sum_{i=1}^{6} \frac{h_{2i}}{x^{2i}} < G(x) - \frac{1}{x} < \sum_{i=1}^{5} \frac{h_{2i}}{x^{2i}},$$

etc. Then these inequalities give us that

$$\begin{gathered} \lim_{x\to\infty} x^2 \left[G(x) - \tfrac{1}{x}\right] = h_2, \\ \lim_{x\to\infty} x^4 \left[G(x) - \tfrac{1}{x} - \tfrac{h_2}{x^2}\right] = h_4, \\ \lim_{x\to\infty} x^6 \left[G(x) - \tfrac{1}{x} - \tfrac{h_2}{x^2} - \tfrac{h_4}{x^4}\right] = h_6, \end{gathered} \tag{18}$$

etc. Comparing the relations (17) and (18), gives us that

$$h_{2j} = \frac{(2^{2j}-1)B_{2j}}{j}, \qquad \forall\, j \in N. \tag{19}$$

This means that the constants $\frac{(2^{2j}-1)B_{2j}}{j}$ in the inequality (15) are the best. Also, the constant one in the function $G(x) - \frac{1}{x}$ can not be improved whatsoever, see [16]. □

Remark 1. As a special case of the inequality (15), we get

$$\frac{1}{2x^2} - \frac{1}{4x^4} < G(x) - \frac{1}{x} < \frac{1}{2x^2} - \frac{1}{4x^4} + \frac{1}{2x^6}, \tag{20}$$

which improve the right hand side of the inequality (8) for $x > 0$ and its left hand side for $x > \sqrt{\frac{3}{2}}$.

Lemma 2.1. *For $m \in \mathbb{N}$, the functions*

$$F_m(x) = G(x) - \frac{1}{x} - \sum_{n=1}^{2m} \frac{(2^{2n}-1)B_{2n}}{nx^{2n}}$$

and

$$H_m(x) = -G(x) + \frac{1}{x} + \sum_{n=1}^{2m-1} \frac{(2^{2n}-1)B_{2n}}{nx^{2n}}$$

are strictly completely monotonic.

Proof. Using the relation (14), we have

$$G(x) - \frac{1}{x} - \sum_{n=1}^{m} \frac{(2^{2n}-1)B_{2n}}{nx^{2n}} = \frac{2(2^{2m+2}-1)B_{2m+2}}{(2m+2)!} \int_0^\infty \theta(t) t^{2m+1} e^{-xt} dt.$$

Then

$$(-1)^k \frac{d^k}{dx^k}\left(G(x) - \frac{1}{x} - \sum_{n=1}^{m} \frac{(2^{2n}-1)B_{2n}}{nx^{2n}} \right) = \frac{2(2^{2m+2}-1)B_{2m+2}}{(2m+2)!} \int_0^\infty \theta(t) t^{2m+k+1} e^{-xt} dt.$$

Using the Bernoulli number's property (16), we get

$$(-1)^k \frac{d^k}{dx^k}\left(G(x) - \frac{1}{x} - \sum_{n=1}^{2m} \frac{(2^{2n}-1)B_{2n}}{nx^{2n}} \right) > 0$$

and

$$(-1)^k \frac{d^k}{dx^k}\left(G(x) - \frac{1}{x} - \sum_{n=1}^{2m-1} \frac{(2^{2n}-1)B_{2n}}{nx^{2n}} \right) < 0.$$

□

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Corollary 2.2. *For odd k, we have*

$$\sum_{n=1}^{2m-1} \frac{(2^{2n}-1)(2n)(2n+1)...(2n+k-1)B_{2n}}{nx^{2n}} < G^{(k)}(x) - \frac{k!}{x^{k+1}}$$

$$< \sum_{n=1}^{2m} \frac{(2^{2n}-1)(2n)(2n+1)...(2n+k-1)B_{2n}}{nx^{2n}}; \qquad m = 1, 2, 3, ...$$

and the inequality will reverse for even k's.

3 Applications

3.1 Bounds of the error of some alternating series

A series of the form

$$\sum_{r=1}^{\infty} (-1)^r a_r$$

where $a_r > 0$ for all r, is called an alternating series. By Leibnitz's Theorem [11], the alternating series converges if a_r decreases monotonically and $a_r \to 0$ as $r \to \infty$. Moreover, let S denote the sum of the series and S_n its n^{th} partial sum, then

$$|S_n - S| < a_{n+1}, \qquad n \in \mathbb{N}.$$

For further details about finding estimates for the error $|S_n - S|$, please refer to [13], [28]-[33]. The alternating series [10]

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \ln 2 \quad \textit{and} \quad \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} = \frac{\pi}{4}$$

presented early important results of the calculus. Kazarinoff [10] deduced the following error estimates

$$\frac{1}{4n+2} < \left| \sum_{r=1}^{n} \frac{(-1)^r}{2r-1} - \frac{\pi}{4} \right| < \frac{1}{4n-2}, \qquad n \in \mathbb{N} \tag{21}$$

and

$$\frac{1}{2(n+1)} < \left| \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r} - \ln 2 \right| < \frac{1}{2n}, \qquad n \in \mathbb{N} \tag{22}$$

by studying the function

$$E_n = \int_0^{\pi/4} \tan^n \theta d\theta, \qquad n \in \mathbb{N}.$$

Tóth [32] improved Kazarinoff's estimates by

$$\frac{1}{4n+2\sqrt{19}-8} < \left| \sum_{r=1}^{n} \frac{(-1)^r}{2r-1} - \frac{\pi}{4} \right| < \frac{1}{4n}, \qquad n \in \mathbb{N} \tag{23}$$

and

$$\frac{1}{2n+2\sqrt{7}-4} < \left|\sum_{r=1}^{n}\frac{(-1)^{r+1}}{r} - \ln 2\right| < \frac{1}{2n+1}, \qquad n \in \mathbb{N}. \tag{24}$$

Also, Tóth and Bukor [33] shown that the best constants a and b such that the inequalities

$$\frac{1}{2n+a} \leq \left|\sum_{r=1}^{n}\frac{(-1)^{r+1}}{r} - \ln 2\right| < \frac{1}{2n+b}, \qquad n \geq 1 \tag{25}$$

hold are $a = \frac{2\ln 2-1}{1-\ln 2}$ and $b = 1$.

Koumandos [13] refined Kazarinoff's estimate (21) by

$$\frac{1}{4n+c} \leq \left|\sum_{r=1}^{n}\frac{(-1)^{r}}{2r-1} - \frac{\pi}{4}\right| < \frac{1}{4n+d}, \qquad n \in \mathbb{N} \tag{26}$$

where the constants $c = \frac{4}{4-\pi} - 4$ and $d = 0$ are the best possible.

In [16], Mahmoud and Agarwal presented the following generalization

$$\frac{4(l+n)^2+10(l+n)+9}{2(l+n+1)\left[4(l+n)^2+8(l+n)+7\right]} < \left|\sum_{r=n+1}^{\infty}\frac{(-1)^{r-1}}{r+l}\right| < \frac{2(l+n)+3}{4(l+n+1)^2}, \tag{27}$$

where $l > -n-1$ and $-l \notin \mathbb{N}$. The double inequality (27) improved the two inequalities (25) and (26) for $n > 1$.

Now, using (5) and (6), we have

$$\left|\sum_{r=n+1}^{\infty}\frac{(-1)^{r-1}}{r+l}\right| = \left|\frac{(-1)^n}{2}G\left(l+n+1\right)\right| = \frac{1}{2}G\left(l+n+1\right), \qquad -l \notin \mathbb{N}. \tag{28}$$

Then our double inequality (15) will give us sharp bounds of the the error $\left|\sum_{r=n+1}^{\infty}\frac{(-1)^{r-1}}{r+l}\right|$, for $-l \notin \mathbb{N}$.

Lemma 3.1.

$$\frac{2}{n} + \sum_{r=1}^{2n}\frac{2(2^{2r}-1)B_{2r}}{r(l+n+1)^{2r}} < \left|\sum_{r=n+1}^{\infty}\frac{(-1)^{r-1}}{r+l}\right| < \frac{2}{n} + \sum_{r=1}^{2n-1}\frac{2(2^{2r}-1)B_{2r}}{r(l+n+1)^{2r}} \qquad n \in \mathbb{N} \tag{29}$$

with sharp bounds, where $l > -n-1$ and $-l \notin \mathbb{N}$.

Remark 2. The inequality (29) improve the inequalities (25) and (26) for special values of the parameter l. Also, it is a generalization of the inequality (27).

3.2 New proof of the increasing monotonicity of a sequence arising from computation of the intersecting probability between a plane couple and a convex body

The increasing of the sequence

$$P_k = \frac{k-1}{2}\left(\int_0^{\pi/2} \sin^{k-1} v \, dv\right)^2, \qquad k \in \mathbb{N}$$

was a question arises from computation of the intersecting probability between a plane couple and a convex body [9]. To prove the increasing monotonicity of the sequences P_k, Guo and Qi [9] studied equivalently the increasing monotonicity of the sequence

$$Q_k = \frac{1}{k}\frac{\Gamma^2\left(\frac{k+1}{2}\right)}{\Gamma^2\left(\frac{k}{2}\right)} \qquad k \in \mathbb{N}.$$

Qi, Mortici and Guo [42] investigated an asymptotic formula for the function

$$\phi(t) = 2\left(\log\Gamma\left(\frac{t+1}{2}\right) - \log\Gamma\left(\frac{t}{2}\right)\right) - \log t \qquad t > 0$$

and proved some properties of the sequence Q_k. Also, Mahmoud [17] generalized some properties of the function $\phi(t)$ and answered about the two posed questions in [42] about the sequence Q_k.

The first derivative of the function $\phi(t)$ can be represented by

$$\phi'(t) = G(t) - \frac{1}{t}$$

and then the function $\phi'(t)$ is strictly completely monotonic, that is

$$(-1)^r\left(\phi'(t)\right)^{(r)} > 0, \qquad r = 0, 1, 2, \dots .$$

Hence the function $\phi(t)$ is increasing and also the function

$$Q(t) = \frac{1}{t}\frac{\Gamma^2\left(\frac{t+1}{2}\right)}{\Gamma^2\left(\frac{t}{2}\right)} \qquad t > 0$$

since $Q'(t) = Q(t)\phi'(t)$. Then $Q(t)$ is increasing function and hence the sequence Q_k is increasing sequence, which is the main result of [9].

References

[1] M. Abramowitz, I. A. Stegun (Eds.), Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, Dover, New York, 1965.

[2] H. Alzer, On some inequalities for the gamma and psi function, Math. Comput., 66, 217, 373-389, 1997.

[3] G. E. Andrews, R.Askey and R.Roy, Special Functions, Cambridge Univ. Press, 1999.

[4] N. Batir, Inequalities for the gamma function, Arch. Math (Basel), 91, 554-563, 2008.

[5] N. Batir, An approximation formula for $n!$, Proyecciones Journal of Mathematics Vol. 32, No 2, 173-181, 2013.

[6] A. Erdélyi et al., Higher Transcendental Functions Vol. I-III, California Institute of Technology - Bateman Manuscript Project, 1953-1955 McGraw-Hill Inc., reprinted by Krieger Inc. 1981.

[7] B.-N. Guo and F. Qi, A class of completely monotonic functions involving divided differences of the psi and tri-gamma functions and some applications, J. Korean Math. Soc. 48, 655-667, 2011.

[8] B.-N. Guo and F. Qi, Sharp inequalities for the psi function and harmonic numbers, Analysis–International mathematical journal of analysis and its applications 34 , no. 2, 201208, 2014.

[9] B.-N. Guo and F. Qi, On the increasing monotonicity of a sequence originating from Computation of the probability of intersecting between a plane couple and a convex body, Turkish Journal of Analysis and Number Theory, vol. 3, no. 1, 21-23, 2015.

[10] N. D. Kazarinoff, Analytic Inequalities, Holt, Rhinehart and Winston, New York, 1961.

[11] K. Knopp, Theory and application of infinite series, Dover, New York, 1990.

[12] S. Koumandos, Remarks on some completely monotonic functions, Journal of Mathematical Analysis and Applications 324, no. 2, 1458-1461, 2006.

[13] S. Koumandos, Monotonicity of some functions involving the Psi function and sharp estimates for the remainders of certain series, Int. J. Contemp. Math. Sci., Vol. 2, no. 15, 713 - 720, 2007.

[14] M. Mahmoud, M. A. Alghamdi and R. P Agarwal, New upper bounds of $n!$, Journal of Inequalities and Application, 2012, 2012:27.

[15] M. Mahmoud, A new proof of an inequality for the logarithm of the gamma function and its sharpness, Turkish Journal of Analysis and Number Theory, Vol. 2, No. 4, 147-151, 2014.

[16] M. Mahmoud and R. P. Agarwal, Bounds for Bateman's $G-$function and its applications, Georgian Mathematical Journal, To appear 2016.

[17] M. Mahmoud, Some properties of a function related to a sequence originating from computation of the probability of intersecting between a plane couple and a convex body, Submitted for publication.

[18] M. Mansour and M. A. Obaid, Bounds of q-factorial [n]q!, Ars Combinatoria, Vol. CII, 313-319, 2011.

[19] M. Mansour, On quicker convergence towards Euler's constant, J. Comput. Anal. Appl. 17, No. 4, 632-638, 2014.

[20] M. Mansour, Refinement of some inequalities of q-gamma function, J. Computational Analysis and Applications, Vol. 18, No.4, 645-654, 2015.

[21] C. Mortici, A sharp inequality involving the psi function, Acta Universitatis Apulensis, 41-45, 2010.

[22] C. Mortici, A quicker convergence toward the γ constant with the logarithm term involving the constant e, Carpathian J. Math. 26(1), 86-91, 2010.

[23] C. Mortici, Estimating gamma function in terms of digamma function, Math. Comput. Model., 52, no. 5-6, 942-946, 2010.

[24] C. Mortici, New approximation formulas for evaluating the ratio of gamma functions. Math. Comp. Modell. 52(1-2), 425-433, 2010.

[25] C. Mortici, A new representation formula for the factorial function, Thai Journal of Mathematics, Vol. 8, no. 2, 249-254, 2010.

[26] C. Mortici, Accurate estimates of the gamma function involving the psi function, Numer. Funct. Anal. Optim., 32, no. 4, 469-476, 2011.

[27] K. Oldham, J. Myland and J. Spanier, An Atlas of Functions, 2nd edition. Springer, 2008.

[28] A. Sîntămărina, A new proof for estimating the remainder of the alternating harmonic series, Creat. Math. Inform., 21, 2, 221-225, 2012.

[29] A. Sîntămărina, Sharp estimates regarding the remainder of the alternating harmonic series, Mathematical Inequalities & Applications, Volume 18, Number 1, 347352 2015.

[30] V. Timofte, On Leibniz series defined by convex functions, J. Math. Anal. Appl. 300, no. 1, 160-171, 2004.

[31] V. Timofte, Integral estimates for convergent positive series, J. Math. Anal. Appl. 303, no. 1, 90-102, 2005.

[32] L. Tóth, On a class of Leibniz series, Rev. Anal. Numér. Théor. Approx., 21 , 195-199, 1992.

[33] L. Tóth and J. Bukor, On the alternating series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$, J. Math. Anal. Appl. 282, 1, 21-25, 2003.

[34] F. Qi and Bai-Ni Guo, Monotonicity and convexity of ratio between gamma functions to different powers, Journal of the Indonesian Mathematical Society (MIHMI) 11, no. 1, 39-49, 2005.

[35] F. Qi and H. M Srivastava, Some uniqueness results for the nontrivially complete monotonicity of a class of functions involving the polygamma and related functions, Integral Transforms Spec. Funct. 21 , 849-858, 2010.

[36] F. Qi, R.-Q. Cui, C.-P. Chen and B.-N. Guo, Some completely monotonic functions involving polygamma functions and an application, Journal of Mathematical Analysis and Applications 310, no. 1, 303-308, 2005.

[37] F. Qi and B.-N. Guo, Completely monotonic functions involving divided differences of the di- and tri-gamma functions and some applications, Commun. Pure Appl. Anal. 8, 1975-1989, 2009.

[38] F. Qi and B.-N. Guo, Necessary and sufficient conditions for functions involving the tri- and tetra-gamma functions to be completely monotonic, Adv. in Appl. Math. 44, 71-83, 2010.

[39] F. Qi and Q.-M. Luo, Bounds for the ratio of two gamma functions—From Wendel's and related inequalities to logarithmically completely monotonic functions, Banach Journal of Mathematical Analysis 6, no. 2, 132-158, 2012.

[40] F. Qi and Q.-M. Luo, Bounds for the ratio of two gamma functions: from Wendel's asymptotic relation to Elezović-Giordano-Pečarić's theorem, Journal of Inequalities and Applications 2013, 2013:542, 20 pages.

[41] F. Qi , Bounds for the ratio of two gamma functions, J. Inequal. Appl. 2010 (2010), Article ID 493058, 84 pages.

[42] F. Qi, C. Mortici, and B.-N. Guo, Some properties of a sequence arising from computation of the intersecting probability between a plane couple and a convex body, ResearchGate Research, available online at http://dx.doi.org/10.13140/RG.2.1.1176.0165.

[43] S.-L. Qiu and M. Vuorinen, Some properties of the gamma and psi functions with applications, Math. Comp., 74, no. 250, 723-742, 2004.

3-VARIABLE ADDITIVE ρ-FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES

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Abstract. In this paper, we introduce and investigate the following additive ρ-functional inequalities

$$\begin{aligned} &N(f(x+y+z)-f(x)-f(y)-f(z),t) \\ &\qquad \geq N\left(\rho\left(2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-2f(z)\right),t\right), \\ &N\left(2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-2f(z),t\right) \\ &\qquad \geq N\left(\rho\left(2f\left(\frac{x+y+z}{2}\right)-f(x)-f(y)-f(z)\right),t\right), \\ &N(f(x+y+z)-f(x)-f(y)-f(z),t) \\ &\qquad \geq N\left(\rho\left(2f\left(\frac{x+y+z}{2}\right)-f(x)-f(y)-f(z)\right),t\right) \end{aligned}$$

in fuzzy normed spaces.

Furthermore, we prove the Hyers-Ulam stability of the above additive ρ-functional inequalities in fuzzy Banach spaces.

1. Introduction and preliminaries

Katsaras [16] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [10, 20, 44]. In particular, Bag and Samanta [2], following Cheng and Mordeson [8], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [19]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 24, 25] to investigate the Hyers-Ulam stability of additive ρ-functional inequalities in fuzzy Banach spaces.

Definition 1.1. [2, 24, 25, 26] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x,t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N_3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.
(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed vector space.*

We know that $N(-x,t) = N(x,t)$ for all $x \in X$ by (N_3).

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The other properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [23, 24].

Definition 1.2. [2, 24, 25, 26] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N\text{-}\lim_{n\to\infty} x_n = x$.

Definition 1.3. [2, 24, 25, 26] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [3]).

The stability problem of functional equations originated from a question of Ulam [43] concerning the stability of group homomorphisms. The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [36] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [11] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. The functional equation $f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called the *Jensen equation*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [7, 13, 15, 17, 18, 21, 31, 32, 33, 34, 37, 38, 39, 40, 41, 42]).

Park [29, 30] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.4. [4, 9] Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [14] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 23, 27, 28, 34, 35]).

Lemma 1.5. *Let $N(x,t) \geq N(\lambda x, t)$ for all $t > 0$. Assume that λ is a fixed real with $|\lambda| < 1$. Then $x = 0$.*

Proof. Putting $\frac{t}{|\lambda|^{n-1}}$ instead of t, we get

$$N\left(x, \frac{t}{|\lambda|^{n-1}}\right) \geq N\left(|\lambda| x, \frac{t}{|\lambda|^{n-1}}\right) \geq N\left(x, \frac{t}{|\lambda|^{n}}\right)$$

So we get

$$N(x,t) \quad \geq \quad N\left(x, \frac{t}{|\lambda|^{n}}\right)$$

for all positive integers n. Passing the limit $n \to \infty$, we get $N(x,t) = 1$ by (N_5), and so $x = 0$ by (N_2). $\square$

In this paper, we introduce and investigate additive ρ-functional inequalities associated with the following additive functional equations

$$\begin{aligned} f(x+y+z) - f(x) - f(y) - f(z) &= 0 \\ 2f\left(\frac{x+y}{2} + z\right) - f(x) - f(y) - 2f(z) &= 0 \\ 2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z) &= 0 \end{aligned}$$

in fuzzy normed spaces.

Furthermore, we prove the Hyers-Ulam stability of the additive ρ-functional inequalities in fuzzy Banach spaces.

Throughout this paper, assume that X is a real fuzzy normed space with norm $N(\cdot, t)$ and that Y is a fuzzy Banach space with norm $N(\cdot, t)$.

2. Additive ρ-functional inequality I

In this section, we investigate the additive ρ-functional inequality

$$\begin{aligned} & N\left(f(x+y+z) - f(x) - f(y) - f(z), t\right) \\ & \geq N\left(\rho\left(2f\left(\frac{x+y}{2} + z\right) - f(x) - f(y) - 2f(z)\right), t\right) \end{aligned} \tag{2.1}$$

in fuzzy normed spaces. Assume that ρ is a fixed real number with $|\rho| < \frac{1}{2}$.

Lemma 2.1. *Let $f : X \to Y$ be a mapping satisfying* (2.1) *for all $x, y, z \in X$. Then $f : X \to Y$ is additive.*

Proof. Letting $x = y = z = 0$ in (2.1), we get $N(2f(0), t) \geq N(2\rho f(0), t)$ and so $f(0) = 0$ by Lemma 1.5.

Replacing y by x and z by $-x$ in (2.1), we get $N(f(x) + f(-x), t) \geq N(2\rho(f(x) + f(-x)), t)$ and so $f(-x) = -f(x)$ for all $x \in X$ by Lemma 1.5.

Replacing y by x and z by $-2x$ in (2.1), we get

$$N(f(2x) - 2f(x), t) \geq N(2\rho(f(2x) - 2f(x)), t)$$

and so $f(2x) = 2f(x)$ for all $x \in X$ by Lemma 1.5.

Replacing z by $-x-y$ in (2.1), we get

$$N\left(f(x+y) - f(x) - f(y), t\right) \geq N\left(\rho\left(f(x+y) - f(x) - f(y)\right), t\right)$$

and so $f(x+y) = f(x) + f(y)$ for all $x, y \in X$ by Lemma 1.5.

Hence $f : X \to Y$ is additive. $\square$

We prove the Hyers-Ulam stability of the additive ρ-functional inequality (2.1) in fuzzy Banach spaces.

Theorem 2.2. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ satisfying*

$$\varphi(x, y, z) \le \frac{L}{2}\varphi(2x, 2y, 2z), \quad \varphi(0,0,0) = 0$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\begin{aligned} (2.2) \qquad & N(f(x+y+z) - f(x) - f(y) - f(z), t) \\ & \ge \min\left\{N\left(\rho\left(2f\left(\frac{x+y}{2}+z\right) - f(x) - f(y) - 2f(z)\right), t\right), \frac{t}{t+\varphi(x,y,z)}\right\} \end{aligned}$$

for all $x, y, z \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x), t) \ge \frac{(2-2L)t}{(2-2L)t + L\varphi(x,x,0)} \tag{2.3}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $x = y = z = 0$ in (2.2), we get $N(2f(0), t) \ge N(2\rho f(0), t)$ and so $f(0) = 0$ by Lemma 1.5.

Replacing y by x and z by 0 in (2.2), we get

$$N(f(2x) - 2f(x), t) \ge \frac{t}{t+\varphi(x,x,0)} \tag{2.4}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \ge \frac{t}{t+\varphi(x,x,0)}, \ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [22, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \ge \frac{t}{t+\varphi(x,x,0)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{2}\varepsilon t\right) \\ &\ge \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right)} \ge \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\varphi(x,x,0)} = \frac{t}{t+\varphi(x,x,0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.4) that $N\left(f(x) - 2f\left(\frac{x}{2}\right), \frac{L}{2}t\right) \ge \frac{t}{t+\varphi(x,x,0)}$ for all $x \in X$ and all $t > 0$. So $d(f, Jf) \le \frac{L}{2}$.

By Theorem 1.4, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e., $A\left(\frac{x}{2}\right) = \frac{1}{2}A(x)$ for all $x \in X$. Since $f : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0,\infty)$ satisfying $N(f(x) - A(x), \mu t) \ge \frac{t}{t+\varphi(x,x,0)}$ for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \le \frac{1}{1-L} d(f, Jf)$, which implies the inequality $d(f, A) \le \frac{L}{2-2L}$. This implies that the inequality (2.3) holds.

By (2.2),

$$\begin{aligned} &N\left(2^n\left(f\left(\frac{x+y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right), 2^n t\right) \\ &\ge \min\left\{N\left(2^n\rho\left(2f\left(\frac{x+y+2z}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - 2f\left(\frac{z}{2^n}\right)\right), 2^n t\right),\right. \\ &\qquad\qquad \left.\frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right)}\right\} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$.

Replacing t by $\frac{t}{2^n}$, we get

$$\begin{aligned} &N\left(2^n\left(f\left(\frac{x+y+z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right), t\right) \\ &\ge \min\left\{N\left(2^n\rho\left(2f\left(\frac{x+y+2z}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - 2f\left(\frac{z}{2^n}\right)\right), t\right),\right. \\ &\qquad\qquad \left.\frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n}\varphi(x,y,z)}\right\} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n}\varphi(x,y,z)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$\begin{aligned} &N\left(A(x+y+z) - A(x) - A(y) - A(z), t\right) \\ &\quad \ge N\left(\rho\left(2A\left(\frac{x+y}{2} + z\right) - A(x) - A(y) - 2A(z)\right), t\right) \end{aligned}$$

for all $x, y \in X$ and all $t > 0$. By Lemma 2.1, the mapping $A : X \to Y$ is Cauchy additive, as desired. $\square$

Theorem 2.3. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $L < 1$ satisfying*

$$\varphi(x,y,z) \le 2L\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right), \quad \varphi(0,0,0)=0$$

for all $x,y,z \in X$. Let $f : X \to Y$ be a mapping satisfying (2.2). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N(f(x)-A(x),t) \ge \frac{(2-2L)t}{(2-2L)t+\varphi(x,x,0)} \tag{2.5}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (2.4) that $f(0)=0$ and

$$N(f(2x)-2f(x),t) \ge \frac{t}{t+\varphi(x,x,0)}$$

for all $x \in X$. Consider the linear mapping $J : S \to S$ such that $Jg(x) := \frac{1}{2}g(2x)$.

$$N(Jf(x)-f(x),t) \ge \frac{t}{t+\frac{1}{2}\varphi(x,x,0)}$$

So, we can get $d(Jf,f) \ge \frac{1}{2}$

The rest of the proof is similar to the proof of Theorem 2.2. □

Lemma 2.4. *Let $f : X \to Y$ be a mapping satisfying*

$$f(x+y+z)-f(x)-f(y)-f(z) = \rho\left(2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-2f(z)\right) \tag{2.6}$$

for all $x,y,z \in X$. Then $f : X \to Y$ is additive.

Proof. Letting $x=y=z=0$ in (2.6), we get $-2f(0) = -2\rho f(0)$ and so $f(0)=0$.

Replacing y by x and letting $z=0$ in (2.6), we get $f(2x)-2f(x)=0$ and so $f(2x)=2f(x)$ for all $x \in X$.

Letting $z=0$ in (2.6), we get

$$f(x+y)-f(x)-f(y) = \rho\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right) = \rho(f(x+y)-f(x)-f(y))$$

and so $f(x+y)=f(x)+f(y)$ for all $x,y \in X$. □

Now, we prove the Hyers-Ulam stability of an additive ρ-functional inequality associated with (2.6) in fuzzy Banach spaces.

Theorem 2.5. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $L < 1$ satisfying*

$$\varphi(x,y,z) \le \frac{L}{2}\varphi(2x,2y,2z), \quad \varphi(0,0,0)=0$$

for all $x,y,z \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\begin{aligned} N\Big((f(x+y+z)-f(x)-f(y)-f(z) \\ -\rho\left(2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-2f(z)\right),t\Big) \ge \frac{t}{t+\varphi(x,y,z)} \end{aligned} \tag{2.7}$$

for all $x,y,z \in X$ and all $t > 0$. Then there exists an unique additive mapping $A : X \to Y$ satisfying (2.3).

Proof. Letting $x = y = z = 0$ in (2.7), we get $N(2(1-\rho)f(0), t) = 1$. So $f(0) = 0$.
Replacing y by x and z by 0 in (2.7), we get

$$N(f(2x) - 2f(x), t) \geq \frac{t}{t + \varphi(x, x, 0)} \tag{2.8}$$

for all $x \in X$. So

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) \geq \frac{t}{t + \varphi(\frac{x}{2}, \frac{x}{2}, 0)} \geq \frac{t}{t + \frac{L}{2}\varphi(x, x, 0)}$$

for all $x \in X$. Consider the linear mapping $J : S \to S$ such that $Jg(x) = 2g\left(\frac{x}{2}\right)$.

$$N(f(x) - Jf(x), t) \geq \frac{t}{t + \frac{L}{2}\varphi(x, x, 0)}$$

and so $d(f, Jf) \leq \frac{L}{2}$
The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 2.6. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ satisfying*

$$\varphi(x, y, z) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right), \varphi(0, 0, 0) = 0$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying (2.7). Then there exists an unique additive mapping $A : X \to Y$ satisfying (2.5).

Proof. It follows from (2.8) that $f(0) = 0$ and

$$N(f(2x) - 2f(x), t) \geq \frac{t}{t + \varphi(x, x, 0)}$$

for all $x \in X$. Consider the linear mapping $J : S \to S$ such that $Jg(x) = \frac{1}{2}g(2x)$.

$$N(Jf(x) - f(x), t) \geq \frac{t}{t + \frac{1}{2}\varphi(x, x, 0)}$$

So, we can get $d(f, Jf) \geq \frac{1}{2}$
The rest of the proof is similar to the proof of Theorem 2.2. □

3. Additive ρ-functional inequality II

In this section, we investigate the additive ρ-functional inequality

$$\begin{aligned} N\left(2f\left(\frac{x+y}{2} + z\right) - f(x) - f(y) - 2f(z), t\right) \\ \geq N\left(\rho\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z)\right), t\right) \end{aligned} \tag{3.1}$$

in fuzzy normed spaces. Assume that ρ is a fixed real with $|\rho| < 1$.

Lemma 3.1. *Let $f : X \to Y$ be a mapping satisfying (3.1) for all $x, y, z \in X$. Then $f : X \to Y$ is additive.*

Proof. Letting $x = y = z = 0$ in (3.1), we get $N(2f(0), t) \geq N(\rho f(0), t)$ and so $f(0) = 0$ by Lemma 1.5.

Replacing z by x and letting $y = 0$ in (3.1), we get $N\left(2f\left(\frac{3x}{2}\right) - 3f(x), t\right) = 1$ and so $f(\frac{3x}{2}) = \frac{3}{2}f(x)$ for all $x \in X$ by Lemma 1.5.

Replacing y by x and z by x in (3.1), we get $N(2f(2x) - 4f(x), t) = 1$ and so $f(2x) = 2f(x)$ for all $x \in X$ by Lemma 1.5.

Replacing y by $-x$ and z by y in (3.1), we get

$$N\left(f(x)+f(-x),t\right) \geq N\left(\rho\left(f(x)+f(-x)\right),t\right)$$

and so $f(-x) = -f(x)$ for all $x \in X$ by Lemma 1.5.

Replacing z by $-x-y$ in (3.1), we get

$$N\left(f(x+y)-f(x)-f(y),t\right) \geq N\left(\rho\left(f(x+y)-f(x)-f(y)\right),t\right)$$

and so $f(x+y) = f(x)+f(y)$ for all $x, y \in X$ by Lemma 1.5. So $f : X \to Y$ is additive. □

We prove the Hyers-Ulam stability of the additive ρ-functional inequality (3.1) in fuzzy Banach spaces.

Theorem 3.2. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $L < 1$ satisfying*

$$\varphi(x,y,z) \leq \frac{2}{3}L\varphi\left(\frac{3}{2}x, \frac{3}{2}y, \frac{3}{2}z\right), \quad \varphi(0,0,0) = 0$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\begin{aligned} &N\left(2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-2f(z),t\right) \\ &\quad \geq \min\left\{N\left(\rho\left(2f\left(\frac{x+y+z}{2}\right)-f(x)-f(y)-f(z)\right),t\right), \frac{t}{t+\varphi(x,y,z)}\right\} \end{aligned} \tag{3.2}$$

for all $x, y, z \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x)-A(x),t\right) \geq \frac{(3-3L)t}{(3-3L)t+L\varphi(x,0,x)} \tag{3.3}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $x = y = z = 0$ in (3.2), we get $N(2f(0),t) \geq N(\rho f(0),t)$ and so $f(0) = 0$ by Lemma 1.5.

Replacing y by 0 and z by x, we get

$$N\left(2f\left(\frac{3}{2}x\right)-3f(x),t\right) \geq \frac{t}{t+\varphi(x,0,x)} \tag{3.4}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x)-h(x),\mu t) \geq \frac{t}{t+\varphi(x,0,x)},\ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is known that (S,d) is complete.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{3}{2}g\left(\frac{2}{3}x\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then $N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t+\varphi(x,0,x)}$ for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(\frac{3}{2}g\left(\frac{2}{3}x\right) - \frac{3}{2}h\left(\frac{2}{3}x\right), L\varepsilon t\right) = N\left(g\left(\frac{2}{3}x\right) - h\left(\frac{2}{3}x\right), \frac{2}{3}L\varepsilon t\right) \\ &\geq \frac{\frac{2Lt}{3}}{\frac{2Lt}{3} + \varphi\left(\frac{2}{3}x, 0, \frac{2}{3}x\right)} \geq \frac{\frac{2Lt}{3}}{\frac{2Lt}{3} + \frac{2L}{3}\varphi(x,0,x)} = \frac{t}{t+\varphi(x,0,x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq L d(g,h)$$

for all $g, h \in S$.

It follows from (3.4) that

$$N\left(f(x) - \frac{3}{2}f\left(\frac{2}{3}x\right), \frac{L}{3}t\right) \geq \frac{t}{t+\varphi(x,0,x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{3}$.

By Theorem 1.4, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e., $A\left(\frac{2}{3}x\right) = \frac{2}{3}A(x)$ for all $x \in X$. Since $f : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying $N(f(x) - A(x), \mu t) \geq \frac{t}{t+\varphi(x,0,x)}$ for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} \left(\frac{3}{2}\right)^n f\left(\left(\frac{2}{3}\right)^n x\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality $d(f, A) \leq \frac{L}{3-3L}$. This implies that the inequality (3.3) holds.

By (3.2),

$$\begin{aligned} & N\left(\left(\frac{3}{2}\right)^n \left[2f\left(\left(\frac{2}{3}\right)^n \left(\frac{x+y}{2} + z\right)\right) - f\left(\left(\frac{2}{3}\right)^n x\right) - f\left(\left(\frac{2}{3}\right)^n y\right) - 2f\left(\left(\frac{2}{3}\right)^n z\right)\right], \right. \\ & \qquad \left. \left(\frac{3}{2}\right)^n t\right) \\ & \geq \min\left\{N\left(\left(\frac{3}{2}\right)^n \rho\left[2f\left(\left(\frac{2}{3}\right)^n \frac{x+y+z}{2}\right) - f\left(\left(\frac{2}{3}\right)^n x\right)\right], \left(\frac{3}{2}\right)^n t\right), \right. \\ & \qquad \left. \frac{t}{t+\varphi\left(\left(\frac{2}{3}\right)^n x, \left(\frac{2}{3}\right)^n y, \left(\frac{2}{3}\right)^n z\right)}\right\} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$.

Replacing t by $\frac{t}{2^n}$, we get that

$$N\left(\left(\frac{3}{2}\right)^n\left[2f\left(\left(\frac{2}{3}\right)^n\left(\frac{x+y}{2}+z\right)\right)-f\left(\left(\frac{2}{3}\right)^n x\right)-f\left(\left(\frac{2}{3}\right)^n y\right)-2f\left(\left(\frac{2}{3}\right)^n z\right)\right],t\right)$$
$$\geq \min\left\{N\left(\left(\frac{3}{2}\right)^n\rho\left[2f\left(\left(\frac{2}{3}\right)^n\frac{x+y+z}{2}\right)-f\left(\left(\frac{2}{3}\right)^n x\right)\right],t\right),\frac{\left(\frac{2}{3}\right)^n t}{\left(\frac{2}{3}\right)^n t+\left(\frac{2L}{3}\right)^n\varphi(x,y,z)}\right\}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty}\frac{\left(\frac{2}{3}\right)^n t}{\left(\frac{2}{3}\right)^n t+\left(\frac{2L}{3}\right)^n\varphi(x,y,z)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$N\left(2A\left(\frac{x+y}{2}+z\right)-A(x)-A(y)-2A(z),t\right)$$
$$\geq N\left(\rho\left(2A\left(\frac{x+y+z}{2}\right)-A(x)-A(y)-A(z)\right),t\right)$$

for all $x, y \in X$ and all $t > 0$. By Lemma 2.1, the mapping $A : X \to Y$ is Cauchy additive, as desired.

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 3.3. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $L < 1$ satisfying*

$$\varphi(x,y,z) \leq \frac{3}{2}L\varphi\left(\frac{2}{3}x,\frac{2}{3}y,\frac{2}{3}z\right), \quad \varphi(0,0,0) = 0$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying (3.2). Then $A(x) := N$-$\lim_{n\to\infty}\left(\frac{2}{3}\right)^n f\left(\left(\frac{3}{2}\right)^n x\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N(f(x)-A(x),t) \geq \frac{(3-3L)t}{(3-3L)t+\varphi(x,0,x)} \tag{3.5}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.2. It follows from (3.4) that $f(0) = 0$ and

$$N\left(2f\left(\frac{3}{2}x\right)-3f(x),t\right) \geq \frac{t}{t+\varphi(x,0,x)}$$

for all $x \in X$. Consider the linear mapping $J : S \to S$ such that $Jg(x) := \frac{2}{3}g\left(\frac{3}{2}x\right)$.

$$N(Jf(x)-f(x),t) \geq \frac{t}{t+\frac{1}{3}\varphi(x,x,0)}$$

So, we can get $d(Jf, f) \geq \frac{1}{3}$

The rest of the proof is similar to the proof of Theorem 3.2. □

From now on, we investigate another additive ρ-functional inequality

$$N\left(2f\left(\frac{x+y+z}{2}\right)-f(x)-f(y)-f(z),t\right) \tag{3.6}$$
$$\geq N\left(\rho\left(2f\left(\frac{x+y}{2}+z\right)-f(x)-f(y)-2f(z)\right),t\right)$$

in fuzzy normed spaces. Assume that ρ is a fixed real with $|\rho| < \frac{1}{2}$.

Lemma 3.4. *Let $f : X \to Y$ be a mapping satisfying* (3.6) *for all $x, y, z \in X$. Then $f : X \to Y$ is additive.*

Proof. Letting $x = y = z = 0$ in (3.6), we get $N(f(0), t) \geq N(2|\rho|f(0), t)$. and so $f(0) = 0$ by Lemma 1.5.

Letting $x = y = 0$ in (3.6), we get $N\left(2f\left(\frac{z}{2}\right) - f(z), t\right) = 1$ and so $f(\frac{x}{2}) = \frac{1}{2}f(x)$ for all $x \in X$.

Replacing z by $-x - y$ in (3.6), we get

$$N\left(f(-x-y) + f(x) + f(y), t\right) \geq N\left(\rho\left(f(-x-y) + f(x) + f(y)\right), t\right)$$

and so

$$f(-x-y) = -f(x) - f(y)$$

for all $x, y \in X$.

Letting $y = 0$ in (3.6), we get $f(-x) = -f(x)$ for all $x \in X$.

Thus $f(x) + f(y) = -f(-x-y) = f(x+y)$ for all $x, y \in X$. Hence $f : X \to Y$ is additive. $\square$

4. Additive ρ-functional inequality III

In this section, we investigate the additive ρ-functional inequality

$$\begin{aligned} N\left(f(x+y+z) - f(x) - f(y) - f(z), t\right) \\ \geq N\left(\rho\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z)\right), t\right) \end{aligned} \tag{4.1}$$

in fuzzy normed spaces. Assume that ρ is a fixed real with $|\rho| < 1$.

Lemma 4.1. *Let $f : X \to Y$ be a mapping satisfying* (4.1) *for all $x, y, z \in X$. Then $f : X \to Y$ is additive.*

Proof. Letting $x = y = z = 0$ in (4.1), we get $N(2f(0), t) \geq N(\rho f(0), t)$. and so $f(0) = 0$ by Lemma 1.5.

Replacing z by $-x - y$ in (4.1), we get

$$N(f(x) + f(y) + f(-x-y), t) \geq N(\rho(f(x) + f(y) + f(-x-y)), t)$$

and so

$$f(x) + f(y) + f(-x-y) = 0 \tag{4.2}$$

for all $x, y \in X$.

Letting $y = -x$ in (4.2), we get $f(-x) = -f(x)$ for all $x \in X$.

Thus $f(x) + f(y) = -f(-x-y) = f(x+y)$ for all $x, y \in X$. So $f : X \to Y$ is additive. $\square$

We prove the Hyers-Ulam stability of the additive ρ-functional inequality (4.1) in fuzzy Banach spaces.

Theorem 4.2. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ satisfying*

$$\varphi(x, y, z) \leq \frac{L}{2}\varphi\left(2x, 2y, 2z\right), \varphi(0, 0, 0) = 0$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\begin{aligned} N\left(f\left(x+y+z\right) - f(x) - f(y) - f(z), t\right) \\ \geq \min\left\{N\left(\rho\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z)\right), t\right), \frac{t}{t + \varphi(x, y, z)}\right\} \end{aligned} \tag{4.3}$$

for all $x, y, z \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{(2-2L)t}{(2-2L)t + L\varphi(x,x,0)} \tag{4.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $x = y = z = 0$ in (4.3), we get $N(2f(0), t) \geq N(\rho f(0), t)$ and so $f(0) = 0$ by Lemma 1.5.

Replacing y by x and letting $z = 0$ in (4.3), we get

$$N\left(f\left(2x\right) - 2f(x), t\right) \geq \frac{t}{t + \varphi(x,x,0)} \tag{4.5}$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Consider the linear mapping $J : S \to S$ such that $Jg(x) = 2g\left(\frac{x}{2}\right)$. Then

$$N(f(x) - Jf(x), t) \geq \frac{t}{t + \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right)} \geq \frac{t}{t + \frac{L}{2}\varphi(x,x,0)}$$

So $d(f, Jf) \leq \frac{L}{2}$.

The rest of the proof is similar to the proof of the Theorem 2.2. □

Theorem 4.3. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ satisfying*

$$\varphi(x, y, z) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right), \quad \varphi(0,0,0) = 0$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying (4.3). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{(2-2L)t}{(2-2L)t + \varphi(x,x,x)} \tag{4.6}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (4.5) that $f(0) = 0$ and

$$N\left(f\left(2x\right) - 2f(x), t\right) \geq \frac{t}{t + \varphi(x,x,0)}$$

for all $x \in X$.

Consider the linear mapping $J : S \to S$ such that $Jg(x) = \frac{1}{2}g(2x)$. Then

$$N(f(x) - Jf(x), t) \geq \frac{2t}{2t + \varphi(x,x,0)} \geq \frac{t}{t + \frac{1}{2}\varphi(x,x,0)}$$

So $d(f, Jf) \leq \frac{1}{2}$. The rest of the proof is similar to the proof of the Theorem 4.2. □

Lemma 4.4. *Let $f : X \to Y$ be a mapping satisfying*

$$f(x+y+z) - f(x) - f(y) - f(z) = \rho\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z)\right) \tag{4.7}$$

for all $x, y, z \in X$. Then $f : X \to Y$ is additive.

Proof. Letting $x = y = z = 0$ in (4.7), we get $-2f(0) = -\rho f(0)$ and so $f(0) = 0$.

Replacing y by x and letting $z = 0$ in (4.7), we get $f(2x) - 2f(x) = 0$ and so $f(2x) = 2f(x)$ for all $x \in X$.

Letting $z = 0$ in (4.7), we get

$$f(x+y) - f(x) - f(y) = \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right) = \rho(f(x+y) - f(x) - f(y))$$

and so $f(x+y) = f(x) + f(y)$ for all $x, y \in X$. □

Now, we prove the Hyers-Ulam stability of an additive ρ-functional inequality associated with (4.7) in fuzzy Banach spaces.

Theorem 4.5. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ satisfying*

$$\varphi(x, y, z) \le \frac{L}{2}\varphi(2x, 2y, 2z), \quad \varphi(0, 0, 0) = 0$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\begin{aligned} N\Big((f(x+y+z) - f(x) - f(y) - f(z) \\ -\rho\left(2f\left(\frac{x+y+z}{2}\right) - f(x) - f(y) - f(z)\right), t\Big) \ge \frac{t}{t + \varphi(x, y, z)} \end{aligned} \tag{4.8}$$

for all $x, y, z \in X$ and all $t > 0$. Then there exists an unique additive mapping $A : X \to Y$ satisfying (4.4).

Proof. Letting $x = y = z = 0$ in (4.8), we get $N((2-\rho)f(0), t) = 1$ and so $f(0) = 0$.

Replacing y by x and letting $z = 0$ in (4.8), we get

$$N(f(2x) - 2f(x), t) \ge \frac{t}{t + \varphi(x, x, 0)} \tag{4.9}$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Let $Jg(x) = 2g\left(\frac{x}{2}\right)$. Then

$$N(f(x) - Jf(x), t) \ge \frac{t}{t + \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right)} \ge \frac{t}{t + \frac{L}{2}\varphi(x, x, 0)}$$

So $d(f, Jf) \le \frac{L}{2}$.

The rest of the proof is similar to the proof of Theorem 4.2. □

Theorem 4.6. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ satisfying*

$$\varphi(x, y, z) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right), \quad \varphi(0, 0, 0) = 0$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying (4.8)*. Then there exists an unique additive mapping $A : X \to Y$ satisfying* (4.6).

Proof. It follows from (4.9) that $f(0) = 0$ and

$$N(f(2x) - 2f(x), t) \ge \frac{t}{t + \varphi(x, x, 0)}$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Consider the linear mapping $J : S \to S$ such that $Jg(x) = \frac{1}{2}g(2x)$. Then

$$N(f(x) - Jf(x), t) \ge \frac{2t}{2t + \varphi(x, x, 0)} \ge \frac{t}{t + \frac{1}{2}\varphi(x, x, 0)}$$

So $d(f, Jf) \leq \frac{1}{2}$.

The rest of the proof is similar to the proof of Theorem 4.2. □

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
[2] T. Bag and S.K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math. **11** (2003), 687–705.
[3] T. Bag and S.K. Samanta, *Fuzzy bounded linear operators*, Fuzzy Sets and Systems **151** (2005), 513–547.
[4] L. Cădariu and V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math. **4**, no. 1, Art. ID 4 (2003).
[5] L. Cădariu and V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber. **346** (2004), 43–52.
[6] L. Cădariu and V. Radu, *Fixed point methods for the generalized stability of functional equations in a single variable*, Fixed Point Theory and Applications **2008**, Art. ID 749392 (2008).
[7] I. Chang and Y. Lee, *Additive and quadratic type functional equation and its fuzzy stability*, Results Math. **63** (2013), 717–730.
[8] S.C. Cheng and J.M. Mordeson, *Fuzzy linear operators and fuzzy normed linear spaces*, Bull. Calcutta Math. Soc. **86** (1994), 429–436.
[9] J. Diaz and B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305–309.
[10] C. Felbin, *Finite dimensional fuzzy normed linear spaces*, Fuzzy Sets and Systems **48** (1992), 239–248.
[11] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
[12] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.
[13] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.
[14] G. Isac and Th.M. Rassias, *Stability of ψ-additive mappings: Appications to nonlinear analysis*, Internat. J. Math. Math. Sci. **19** (1996), 219–228.
[15] S. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press lnc., Palm Harbor, Florida, 2001.
[16] A.K. Katsaras, *Fuzzy topological vector spaces II*, Fuzzy Sets and Systems **12** (1984), 143–154.
[17] H. Kim, M. Eshaghi Gordji, A. Javadian and I. Chang, *Homomorphisms and derivations on unital C^*-algebras related to Cauchy-Jensen functional inequality,* J. Math. Inequal. **6** (2012), 557–565.
[18] H. Kim, J. Lee and E. Son, *Approximate functional inequalities by additive mappings,* J. Math. Inequal. **6** (2012), 461–471.
[19] I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica **11** (1975), 326–334.
[20] S.V. Krishna and K.K.M. Sarma, *Separation of fuzzy normed linear spaces*, Fuzzy Sets and Systems **63** (1994), 207–217.
[21] J. Lee, C. Park and D. Shin, *An AQCQ-functional equation in matrix normed spaces*, Results Math. **27** (2013), 305–318.
[22] D. Miheţ and V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008), 567–572.
[23] M. Mirzavaziri and M.S. Moslehian, *A fixed point approach to stability of a quadratic equation*, Bull. Braz. Math. Soc. **37** (2006), 361–376.
[24] A.K. Mirmostafaee, M. Mirzavaziri and M.S. Moslehian, *Fuzzy stability of the Jensen functional equation*, Fuzzy Sets and Systems **159** (2008), 730–738.
[25] A.K. Mirmostafaee and M.S. Moslehian, *Fuzzy versions of Hyers-Ulam-Rassias theorem*, Fuzzy Sets and Systems **159** (2008), 720–729.
[26] A.K. Mirmostafaee and M.S. Moslehian, *Fuzzy approximately cubic mappings*, Inform. Sci. **178** (2008), 3791–3798.
[27] C. Park, *Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras*, Fixed Point Theory and Applications **2007**, Art. ID 50175 (2007).
[28] C. Park, *Generalized Hyers-Ulam-Rassias stability of quadratic functional equations: a fixed point approach*, Fixed Point Theory and Applications **2008**, Art. ID 493751 (2008).
[29] C. Park, *Additive ρ-functional inequalities and equations,* J. Math. Inequal. **9** (2015), 17–26.
[30] C. Park, *Additive ρ-functional inequalities in non-Archimedean normed spaces,* J. Math. Inequal. **9** (2015), 397–407.

[31] C. Park, Y. Cho and M. Han, *Stability of functional inequalities associated with Jordan-von Neumann type additive functional equations*, J. Inequal. Appl. **2007**, Art. ID 41820 (2007).
[32] C. Park, K. Ghasemi, S. G. Ghaleh and S. Jang, *Approximate n-Jordan $*$-homomorphisms in C^*-algebras,* J. Comput. Anal. Appl. **15** (2013), 365-368.
[33] C. Park, A. Najati and S. Jang, *Fixed points and fuzzy stability of an additive-quadratic functional equation,* J. Comput. Anal. Appl. **15** (2013), 452–462.
[34] C. Park and Th.M. Rassias, *Fixed points and generalized Hyers-Ulam stability of quadratic functional equations,* J. Math. Inequal. **1** (2007), 515–528.
[35] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.
[36] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[37] L. Reich, J. Smítal and M. Štefánková, *Singular solutions of the generalized Dhombres functional equation*, Results Math. **65** (2014), 251–261.
[38] S. Schin, D. Ki, J. Chang and M. Kim, *Random stability of quadratic functional equations: a fixed point approach*, J. Nonlinear Sci. Appl. **4** (2011), 37–49.
[39] S. Shagholi, M. Bavand Savadkouhi and M. Eshaghi Gordji, *Nearly ternary cubic homomorphism in ternary Fréchet algebras,* J. Comput. Anal. Appl. **13** (2011), 1106–1114.
[40] S. Shagholi, M. Eshaghi Gordji and M. Bavand Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras,* J. Comput. Anal. Appl. **13** (2011), 1097–1105.
[41] D. Shin, C. Park and Sh. Farhadabadi, *On the superstability of ternary Jordan C^*-homomorphisms,* J. Comput. Anal. Appl. **16** (2014), 964–973.
[42] D. Shin, C. Park and Sh. Farhadabadi, *Stability and superstability of J^*-homomorphisms and J^*-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.
[43] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.
[44] J.Z. Xiao and X.H. Zhu, *Fuzzy normed spaces of operators and its completeness*, Fuzzy Sets and Systems **133** (2003), 389–399.

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An Approach to Separability of Integrable Hamiltonian System

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Abstract

Directly from Benenti's theorem, which characterizes separability with one single Killing tensor, we adopt an algorithm to execute the task of separability test. The algorithm is applied to generalized quartic and quintic polynomial potentials as well as some multi-separable systems on (pseudo)-Euclidean spaces. It yields many well-known integrable systems in a unified and straight manner, in contrast to some complicated techniques employed in the literature to derive them.

Keywords: completely integrable system; separable system; Killing two-tensor; Hénon-Heiles systems

1. Introduction

Finite dimensional completely integrable system has always attracted much attention. Recently authors adopted various methods or perspectives to investigate them. Prominent of all, are the separability theory of Hamilton-Jacobi equation [1], the approach of Lax representations [2], and the bi-Hamiltonian theory [3, 4] among others (see e.g. the references above and therein).

For a given Hamilton system finding canonical separation coordinates is very non-trivial. The above approaches can partly solve this problem. Sklyanin developed a method based on a Lax pair [5]. The separable coordinates are obtained from the spectrum of the Lax operator. Another approach is based on the existence of a bi-Hamiltonian representation [6, 4]. The separable variables, called Darboux-Nijenhuis coordinates, are related with the recursion operator constructed from the Poisson pencil.

For a generic system there is no intrinsic criterion of the existence of a Lax or bi-Hamiltonian formulation. Benenti [7, 8] has developed an intrinsic characterization for a Hamiltonian system being separable (see Theorem 1). It is based on geometric properties of the Killing tensors corresponding to the first integrals of the system. We can make a comparison of these approaches to the separability of the Hamilton-Jacobi equation. While the technique based on the Lax or bi-Hamiltonian formulation may be more effective in studying particular examples (for which such formulation has been found beforehand), the Benenti approach is more rigorous from the mathematical point of view.

Though integrability does not necessarily imply separability, the separable class constitute the vital examples among all integrable systems. Directly from Benenti's theorem, we can adopt a strategy to cope with the problem of separability test. We present this method as an executable algorithm, which are especially applicable to families of Hamiltonian systems containing some numeric constants. In this paper we employ this algorithm to test several natural systems, recovering some known models obtained by other approaches such as Painlevé analysis or differential Galois theory [9].

This paper is organised as follows. In Section 2, some basic concepts in Killing tensors method of H-J separability are reviewed, then based on it we suggest an executable algorithm to make concrete separability test. The algorithm is, in Section 3, applied to test several potentials, including inhomogeneous quartic polynomial potential, homogeneous quintic potential, as well as some multi-separable systems on Euclidean and Minkowski planes. These will yield many well-known integrable systems in a unified and straight manner, in contrast to complicated techniques employed in the literature. The last section is devoted to some concluding remarks.

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2. The Geometric Method to Variables Separation and an Executable Algorithm

We briefly recall some necessary facts about the separation of variables method, considered in the framework of symplectic geometry. Let $(\mathcal{M}, \omega)$ be a symplectic manifold with symplectic form ω. Then the Poisson bivector is $P = \omega^{-1}$. Note here we view ω (and P) as transformation taking vector field to one-form (and vice-versa, respectively). It is well known that the Schouten bracket vanishes, $[P, P] = 0$.

Let $(\boldsymbol{q}, \boldsymbol{p})$, $\boldsymbol{q} = (q_1, \ldots, q_n)$, $\boldsymbol{p} = (p_1, \ldots, p_n)$ be the (local) canonical coordinates on $\mathcal{M}$, then the Poisson bivector is $P = \sum_{i=1}^{n} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}$. The *Hamiltonian vector field* corresponding to a smooth function $H = H(\boldsymbol{q}, \boldsymbol{p})$ is defined as $X_H = P\, \mathrm{d}H$. The triple (M, P, H) is called a *Hamiltonian system*.

In this paper we will focus on Hamiltonian system in the setting of Riemannian geometry. That is to say, the phase space is the cotangent bundle T^*M of some (pseudo)-Riemannian manifold (M, g). We remind that we are finding separable coordinates related to the original physics coordinates $(\boldsymbol{q}, \boldsymbol{p})$ via a point transformation. In the setting of Riemannian geometry a natural Hamiltonian $H = T + V$ takes the form as follows

$$H = \sum_{i,j=1}^{n} \frac{1}{2} G^{ij}(\boldsymbol{q})\, p_i\, p_j + V(\boldsymbol{q}) \tag{2.1}$$

where G^{ij} is the inverse of metric g and $V(\boldsymbol{q})$ the potential. The classical *Hamilton-Jacobi equation* reads

$$\sum_{i,j=1}^{n} \frac{1}{2}\, G^{ij}\, \partial_i W\, \partial_j W + V = E \tag{2.2}$$

where E is the constant of conserved energy (Hamiltonian). It is a first-order partial differential equation of the unknown W.

Definition 1. *The Hamiltonian H (2.1) is* separable *in the canonical variable $(\boldsymbol{q}, \boldsymbol{p})$, if the Hamilton-Jacobi equation (2.2) admits a complete integral of additive form*

$$W(\boldsymbol{q}, \boldsymbol{\alpha}) = \sum\nolimits_{i=1}^{n} W_i(q^i, \boldsymbol{\alpha}), \tag{2.3}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ are integration constants, such that $\det\left[\frac{\partial^2 W}{\partial q_i\, \partial \alpha_j}\right] \neq 0$. *The variables $(\boldsymbol{q}, \boldsymbol{p})$ are called* separable variables.

It is well known that the n first integrals obtained by solving the Hamilton-Jacobi equation (2.2) are either quadratic or linear in momenta and thus correspond to Killing 2-tensors or Killing vectors, respectively.

Definition 2. *A* Killing tensor *K of valence p defined on (M, g) is a symmetric $(p, 0)$-tensor satisfying the Killing tensor equation*

$$[K, G] = 0, \tag{2.4}$$

where $[\ ,\ ]$ denotes the Schouten bracket.

All Killing p-tensors constitute a vector space $\mathcal{K}^p(M)$. For manifold of constant curvature its dimension attains the maximum, see e.g. [10].

Separability of the natural Hamiltonian $H = T + V$ depends on the Killing 2-tensor of the underlying manifold (M, g). This idea, due to Eisenhart, has been extensively exploited by many authors, see e.g. [11, 12]. The intrinsic criterion given by Benenti [7, 8] allows one to characterize separability by a single Killing 2-tensor (orthogonal case), or a Killing 2-tensor together with an abelian algebra of Killing vectors (non-orthogonal case). Here we focus on the orthogonal case since it is more common.

Theorem 1 (Benenti)**.** *A Hamiltonian $H = T + V$ is separable in some orthogonal coordinates if and only if there exists a Killing 2-tensor $\mathbf{K}$ with pointwise simple and real eigenvalues, orthogonally integrable eigenvectors and such that*

$$\mathrm{d}\,(\mathbf{K}\, \mathrm{d}V) = 0. \tag{2.5}$$

The $(0,2)$-tensor $\mathbf{K}$ is called *characteristic Killing tensor*. On a Riemannian manifold it can be viewed as $(2,0)$- or $(1,1)$-tensor by lowering or raising indices via metric g or G. Here $\mathbf{K}$ in (2.5) is seen as a $(1,1)$-tensor which takes a one-form to another. In local coordinates (2.5) entails the following one-form is closed,

$$\mathbf{K}\,\mathrm{d}V = \sum\nolimits_{i,j,l} g_{ij}\,\mathbf{K}^{jl}\,\partial_l V\,\mathrm{d}q^i\,. \tag{2.6}$$

This theorem elegantly and intrinsically characterizes the orthogonal separability of the natural Hamiltonian (2.1). Often in the literature one is faced with a general system with some parameters involved in the Hamiltonian. Many sophisticated methods have to be invented and applied to identify the rare cases which are separable, or integrable. We will revisit some of these examples in later sections.

From the Benenti's theorem 1 we come up with an approach to deal with the problem of searching for separable case of parameters, presented by an Algorithm as below:

Algorithm. Let H be a natural Hamiltonian with potential $V(\boldsymbol{q}; a_i)$ defined on some pseudo-Riemannian manifold (M, g), where a_i's are some constant parameters. The special values of parameters, that guarantees the system H is H-J separable, are achieved during execution of the algorithm.

Begin.

Step 1. For the pseudo-Riemannian manifold (M, g), using the Killing tensor equation (2.4) to calculate the general Killing 2-tensor K. All of them constitute a vector space $\mathcal{K}^2(M)$ of dimension d. The expression of a general Killing 2-tensor is $\mathbf{K} = \sum_{i=1}^{d} C_i\mathbf{K}_i$, where $(\mathbf{K}_i)$ is the basis, $C_i \in \mathbb{R}$.

If $d < \dim(M)$, then by theorems due to Kalnins & Miller [11] there exists no separable potential — Stop.

Essentially, this step is a pure problem of differential geometry.

Step 2. The Killing tensor $\mathbf{K}$ obtained above is of covariant $(2,0)$-type. Using metric g, transform it to a $(1,1)$-tensor $\widehat{\mathbf{K}}$ which can be regarded as an endomorphism of the cotangent bundle T^*M.

By abuse of notation we use $\mathbf{K}$ to denote the new tensor below. Note that in matrix form $\mathbf{K}$ is always symmetric, $\widehat{\mathbf{K}}$ is not so in general.

Step 3. Insert the $(1,1)$-tensor into the core equation (2.5). The vanishing of form $\mathrm{d}(\mathbf{K}\mathrm{d}V)$ entails the vanishing of all its coefficients. Thus a system of equations involving variables q_k and constants a_i, C_j follows, which are usually (or can be transformed to) polynomials of q_k.

Simplify this system of equations, eventually we obtain algebraic equations of the parameters a_i, C_j only. Solve the system of algebraic equations. The obtained solutions are candidates of separable cases.

Step 4. Substitute the C_j's back into the general expression of Killing tensor. Calculate its eigenvalues and eigenvectors. Check whether they satisfy the additional condition in Benenti's theorem. These gives the complete set of all separable cases.

Step 5 (Optional). For a separable case, by using the eigenvalues and eigenvectors obtained in step 4, we can figure out which concrete coordinate system permits the separability of the corresponding system. (see e.g. [12])

End.

3. Applications

In this section, we first review the Killing vectors (tensors) of $\mathbb{E}^2$ (see e.g. [13]), then we use them to make a detailed analysis of several systems involved with some constants.

In the situation of $\mathbb{E}^2$ we will write the familiar (x, y) for (q^1, q^2), and (p_x, p_y) for (p_1, p_2), here (x, y) is the usual Cartesian coordinate. The space of Killing vectors has dim $\mathcal{K}^1(\mathbb{E}^2) = 3$ with a basis [13] being

$$\partial_x, \partial_y \quad \text{(two translations)}, \qquad y\partial_x - x\partial_y \quad \text{(rotation)} \tag{3.1}$$

where we adopt the notation $\partial_x = \frac{\partial}{\partial x}$, $\partial_y = \frac{\partial}{\partial y}$. The second space $\mathcal{K}^2(\mathbb{E}^2)$ has a basis [13]

$$\begin{aligned} &\mathbf{K}_1 = \partial_x^2, \qquad \mathbf{K}_2 = \partial_y^2, \qquad \mathbf{K}_3 = \partial_x\partial_y + \partial_y\partial_x \quad (= G)\ , \\ &\mathbf{K}_4 = -2y\,\partial_x^2 + x\,\partial_x\partial_y + x\,\partial_y\partial_x, \quad \mathbf{K}_5 = -2x\,\partial_y^2 + y\,\partial_x\partial_y + y\,\partial_y\partial_x, \\ &\mathbf{K}_6 = y^2\,\partial_x^2 + x^2\,\partial_y^2 - xy\,\partial_x\partial_y - xy\,\partial_y\partial_x. \end{aligned} \tag{3.2}$$

So the expression for a general Killing 2-tensor is

$$\begin{aligned} \mathbf{K} = \sum\nolimits_{i=1}^{6} C_i\mathbf{K}_i = (C_6 y^2 - 2C_4\,y + C_1)\,\partial_x^2 + (C_6 x^2 - 2C_5\,x + C_2)\,\partial_y^2 \,+ \\ + (-C_6\,xy + C_4\,x + C_5\,y + C_3)\,(\partial_x\partial_y + \partial_y\partial_x) \end{aligned} \tag{3.3}$$

or, in matrix form

$$(\mathbf{K}^{ij}) = \begin{pmatrix} C_6 y^2 - 2C_4\,y + C_1 & -C_6\,xy + C_4\,x + C_5\,y + C_3 \\ -C_6\,xy + C_4\,x + C_5\,y + C_3 & C_6 x^2 - 2C_5\,x + C_2 \end{pmatrix} \tag{3.4}$$

where C_i are constants.

At last, we mention a special non-separable situation, that is,

$$\textbf{(NS)} \qquad C_1 = C_2, \quad C_3 = C_4 = C_5 = C_6 = 0.$$

In such a case the matrix $\mathbf{K} = C_1\,I_2$, where I_2 denotes the identity matrix. It is not simple as it admits two coincident eigenvalues. This means the characteristic tensor $\mathbf{K}$ does not exist, hence the system is not separable. Such a special case arises several times during our arguments later.

3.1. System with a General Quartic Potential

We shall use these general results to several specified systems defined on $\mathbb{E}^2$. In this section we consider a system with a quartic polynomial potential, whose Hamiltonian is

$$H = \frac{1}{2}(p_x^2 + p_y^2) + (\lambda\,x^2 + \mu\,y^2) + (c\,x^4 + b\,x^2y^2 + a\,y^4) \tag{3.5}$$

in which a, b, c, λ, μ are constants. Note that the system (3.5) is called Yang-Mills-type system in [16]. In general, the system with arbitrary parameters are non-integrable and display chaotic behaviour.

After applying the celebrated Painlevé analysis or differential Galois theory, several special cases for values of constants are identified, which turn out to be integrable (see e.g. [14, 15, 16]). These cases are given by

(i) $b = 0$, all other parameters are arbitrary,
(ii) $a : b : c = 1 : 2 : 1$, λ and μ arbitrary,
(iii) $a : b : c = 1 : 6 : 1$, $\lambda = \mu$,
(iv) $a : b : c = 1 : 12 : 16$, $\lambda = 4\mu$,
(v) $a : b : c = 1 : 6 : 8$, $\lambda = 4\mu$, (proved to be the only non-separable case below)

Cases (ii)–(v) are well-known integrable Hénon-Heiles systems [17]. For each of these cases there exists a second integral of motion K independent of H [17].

Remark 1. One may note that the four cases given above are not symmetric for λ, μ whereas the Hamiltonian is so. Actually there are not only five integrable cases as above, but more. The additional cases

(iv)′ $a:b:c = 16:12:1, \quad \lambda = \mu/4,$
(v)′ $a:b:c = 8:6:1, \quad \lambda = \mu/4,$

are symmetric (thus equivalent) to the cases (iv) and (v), respectively. We can make an assumption $a \le c$ to eliminate these isomorphisms.

Our main result in this subsection is the following

Theorem 2. *For Hamiltonian system with quartic potential (3.5), there exist exact four cases of values of constants, which guarantee the corresponding H-J equation being additively separable. These cases are exactly the first four cases (i)—(iv) in the list above.*

This shows case (v) is the only integrable, but non-separable case.

Proof. Notice now (2.6) reads $\mathbf{K}\,\mathrm{d}V = \sum_{i,j=1}^{2} \mathbf{K}^{ij}\,\partial_j V\,\mathrm{d}q^i$, whose explicit expressions is messy. After taking exterior differentiation

$$\mathrm{d}(\mathbf{K}\,\mathrm{d}V) = Z\;\mathrm{d}x \wedge \mathrm{d}y,$$

its coefficient is a polynomial of x and y, which after collecting the same entries reads

$$\begin{aligned} Z &= (24aC_6 - 12bC_6)\,y^3x + (12bC_6 - 24cC_6)\,yx^3 + (-12aC_4 + 16bC_4)\,y^2x \\ &+(-16bC_5 + 12cC_5)\,yx^2 + (-2bC_4 + 24cC_4)\,x^3 + (-24aC_5 + 2bC_5)\,y^3 \\ &+(-12aC_3 + 2bC_3)\,y^2 + (-4bC_1 + 4bC_2 + 8\mu C_6 - 8\lambda C_6)\,xy + (-2bC_3 + 12cC_3)\,x^2 \\ &+(-2\mu C_4 + 8\lambda C_4)\,x + (-8\mu C_5 + 2\lambda C_5)\,y - 2\mu C_3 + 2\lambda C_3. \end{aligned} \tag{3.6}$$

The vanishing of two-form $\mathrm{d}(\mathbf{K}\mathrm{d}V)$ means its coefficient Z vanishes. All the parameters $a, b, c, \lambda, \mu, C_i$ in (3.6) are constants. In turn Z vanishes identically entails all its coefficients of x, y vanishes. A system of algebraic equations follows,

$$C_3(\lambda - \mu) = C_4(4\lambda - \mu) = C_5(\lambda - 4\mu) = 0 \tag{3.7a}$$
$$C_3(6a - b) = C_3(b - 6c) = 0 \tag{3.7b}$$
$$C_4(b - 12c) = C_4(3a - 4b) = 0 \tag{3.7c}$$
$$C_5(12a - b) = C_5(4b - 3c) = 0 \tag{3.7d}$$
$$C_6(2a - b) = C_6(b - 2c) = 0 \tag{3.7e}$$
$$(C_1 - C_2)b + 2C_6(\lambda - \mu) = 0 \tag{3.7f}$$

This is the system of algebraic equations we want to solve in detail. First we notice when $b = 0$, the parameters $C_1 - C_2 \neq 0$, $C_i = 0, i = 3, 4, 5, 6$, directly solve the above system. The matrix corresponding to Killing tensor $\mathbf{K}$ has distinct eigenvalues C_1, C_2. Hence this is a separable case, which corresponds to case (i) in our list.

To proceed we will always assume $b \neq 0$ below. It can be seen $b \neq 0$ implies $a, c \neq 0$. Otherwise constants C_i are exactly in the non-separable situation (**NS**). To solve the system (3.7), we observe that (3.7b) and (3.7e) imply

$$C_3(a - c) = C_6(a - c) = 0 \tag{3.8}$$

Based on this observation one can make classification as below:

- $\mathbf{a \neq c}$. Equations (3.8) imply that $C_3 = C_6 = 0$. Substituting this into (3.7), one has

$$\begin{aligned} C_4(4\lambda - \mu) = C_5(\lambda - 4\mu) &= 0 \\ C_4(b - 12c) = C_4(3a - 4b) &= 0 \\ C_5(12a - b) = C_5(4b - 3c) &= 0 \\ C_1 - C_2 &= 0 \end{aligned} \tag{3.9}$$

The second equation implies $C_4(a-16c)=0$. As $a-16c \neq 0$ (otherwise $c/a = 1/16 < 1$), it holds that $C_4 = 0$. We claim $C_5 \neq 0$, otherwise the constants C_k are in non-separable situation (**NS**). Combing all the results above, we arrive at

$$a : b : c = 1 : 12 : 16, \qquad \lambda = 4\mu,$$

which recovers the case (iv) in the list.

- $\mathbf{a} = \mathbf{c}$. Again we have $C_4(a-16c) = 0$, which reduces to $C_4 a = 0$. As $a \neq 0$ hence $C_4 = 0$. Similarly $C_5 = 0$. Substituting $C_4 = C_5 = 0, a = c$ into the system (3.7), it reduces to

$$\begin{aligned} &C_3(\lambda-\mu) = 0 \\ &C_3(6a-b) = C_6(2a-b) = 0 \\ &(C_1 - C_2)b + 2C_6(\lambda-\mu) = 0 \end{aligned} \tag{3.10}$$

  We discuss its possible solutions:

  – $a = c$, $\lambda = \mu$. The condition $\lambda = \mu$ reduces the system (3.10) further to

  $$C_3(6a-b) = C_6(2a-b) = 0, \qquad C_1 - C_2 = 0.$$

  One of C_3 and C_6 should be non-zero (otherwise the situation (**NS**) arise again), which implies $a : b : c = 1 : 6 : 1$, or $a : b : c = 1 : 2 : 1$. They corresponds to the case (iii) and (ii), respectively.

  – $a = c$, $\lambda \neq \mu$. The system (3.10) is reduced to

  $$C_3 = C_6(2a-b) = (C_1 - C_2)b + 2C_6(\lambda-\mu) = 0$$

  Once more $C_6 \neq 0$ to avoid the situation (**NS**), which gives $a : b : c = 1 : 2 : 1$. It is case (ii) in the list. This completes the proof of Theorem 2.

□

Remark 2. One could use the Killing tensor to figure out the coordinate system in which the H-J equation separates. For example, let us consider the case (ii). According to Step 4 in Algorithm, taking $b = 2a, c = a \neq 0$ back into the original system (3.7) one find the following to be a solution of the system (3.7):

$$C_1 = \frac{\mu-\lambda}{a}\ , \ \ C_2 = C_3 = C_4 = C_5 = 0, \ \ C_6 = 1$$

Note that $(\mu - \lambda)$ is in general not zero. The characteristic tensor (3.4) turns out to be

$$\mathbf{K} = \begin{pmatrix} y^2 + \frac{\mu-\lambda}{a} & -xy \\ -xy & x^2 \end{pmatrix} . \tag{3.11}$$

Its characteristic equation is

$$\Lambda^2 - (x^2 + y^2 + \frac{\mu-\lambda}{a})\Lambda + \frac{\mu-\lambda}{a}x^2 = 0 \tag{3.12}$$

or in equivalent form

$$\frac{x^2}{\Lambda} + \frac{ay^2}{a\Lambda - (\mu-\lambda)} = 0. \tag{3.13}$$

For the case of $\lambda \neq \mu$, the above equation (3.13) defines just well-known elliptic-hyperbolic coordinates in the Euclidean plane. The eigenvalues λ^1, λ^2, i.e. the solutions of the equation (3.12) or (3.13), are the variables of separation for the dynamical system. Hence we conclude the system is separable in the elliptic coordinates (λ^1, λ^2), determined by $\frac{\mu-\lambda}{a}$.

For the case of $\lambda = \mu$, the solutions of (3.12) are $\lambda^1 = 0$, $\lambda^2 = x^2 + y^2$, one of which is constant. This implies the system separates in degenerated elliptic coordinates. The Hamiltonian is

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \lambda(x^2 + y^2) + a(x^4 + 2x^2y^2 + y^4)$$

with potential $V = \lambda r^2 + a r^4$ depending on r only. It is easy to see the system separates in the standard polar coordinates (r, θ).

3.2. System with a Homogeneous Quintic Potential

Next we consider a family of Hamiltonian systems with a quintic polynomial potential,

$$H = \frac{1}{2}(p_x^2 + p_y^2) + ay^5 + b\, y^3x^2 + c\, yx^4, \tag{3.14}$$

where a, b, c are scalar constants. Note they are not the most general quintic potential. Our main result regarding this potential is the following

Theorem 3. *For the Hamiltonian (3.14), there exists exact 3 cases of parameters for the corresponding H-J equation to be additively separable. These cases are:*

(i) $b = c = 0$, *a arbitrary;* *(ii)* $a : b : c = 16 : 16 : 3$; *(iii)* $a : b : c = 1 : 10 : 5$.

Remark 3. Case (i) is trivial in that it corresponds to Hamiltonian $H = (p_x^2 + p_y^2)/2 + ay^5$, which trivially separates in Cartesian coordinates. Case (ii) has already appeared in literature [14] where the authors obtained it by using the (weak) Painlevé method. Case (iii) appeared in Perelomov's book [18, p.81]

Proof of Theorem 3. We apply the algorithm again, now to the potential (3.14). Using the general Killing tensor (3.4), it follows that the 2-form $\mathrm{d}(\mathbf{K}\mathrm{d}V) = Z\mathrm{d}x \wedge \mathrm{d}y$, with coefficient Z given by

$$\begin{aligned} Z &= (-6bC_4 + 32cC_4)yx^3 + (-20aC_4 + 20bC_4)y^3x + (21bC_6 - 28cC_6)y^2x^3 \\ &+ (35aC_6 - 14bC_6)y^4x + (-6bC_1 + 6bC_2)y^2x + (-6bC_3 + 12cC_3)yx^2 \\ &+ (-27bC_5 + 12cC_5)y^2x^2 + (-4cC_1 + 4cC_2)x^3 + (-35aC_5 + 2bC_5)y^4 \\ &+ 7cx^5C_6 + (-20aC_3 + 2bC_3)y^3 - 11cx^4C_5. \end{aligned}$$

The form $\mathrm{d}(\mathbf{K}\mathrm{d}V)$ vanishes entails Z also vanishes. Again, Z is a polynomial of variable x, y, hence all of its coefficients are zero. Thus we arrive at a system of algebraic equations

$$\begin{aligned} c(C_1 - C_2) = b(C_1 - C_2) = 0, \\ C_3(b - 2c) = C_3(10a - b) = C_4(3b - 16c) = 0, \\ C_4(a - b) = C_5(9b - 4c) = C_5(35a - 2b) = 0, \\ C_6(3b - 4c) = C_6(5a - 2b) = 0, \qquad cC_5 = cC_6 = 0, \end{aligned} \tag{3.15}$$

We analyze the solution of this family:

- $c = 0$. Substitute this into the system (3.15) to produce

$$\begin{aligned} b(C_1 - C_2) = 0 \\ C_3(10a - b) = C_4(a - b) = 0, \\ C_5(35a - 2b) = C_6(5a - 2b) = 0, \\ C_3b = C_4b = C_5b = C_6b = 0, \end{aligned} \tag{3.16}$$

  One easily sees that b is also zero. (Otherwise the new system leads to the (**NS**) case). Hence the above system (3.16) further reduces to $C_3a = C_4a = C_5a = C_6a = 0$. For any a, it admits a solution $C_1 - C_2 \neq 0$, $C_3 = C_4 = C_5 = C_6 = 0$. Thus we have a separability corresponding to the case (i). In fact we can obviously see this from the original potential (3.14). In the special case of $b = c = 0$, the Hamiltonian is additively itself, implying it separates in the canonical Cartesian system.

- $c \neq 0$. The system (3.15) can be simplified to be

$$\begin{aligned} C_1 - C_2 = C_5 = C_6 = 0, \\ C_3(b-2c) = C_3(10a-b) = 0, \\ C_4(3b-16c) = C_4(a-b) = 0, \end{aligned}$$

  One can see one of C_3, C_4 is not zero. (Otherwise $C_1 - C_2 = C_3 = C_4 = C_5 = C_6 = 0$ — dissatisfies the basic theorem 1). So there exists two possibilities:

  - $C_3 \neq 0$, which gives $a : b : c = 1 : 10 : 5$ corresponding to case (iii).
  - $C_4 \neq 0$, which gives $a : b : c = 16 : 16 : 3$ corresponding to case (ii). We thus reproduce all the cases in the theorem.

□

3.3. Multi-Separable Potentials on Euclidean and Minkowski Planes

We now apply our Algorithm to identify some multi-separable systems which are defined on (pseudo)-Euclidean spaces. We remind that a Hamiltonian system is *multi-separable* if it separates in several distinct coordinate systems.

Theorem 4. *For the system*

$$H = \frac{1}{2}(p_x^2 + p_y^2) + x^2 + ay^2 + \frac{b}{x^2} \tag{3.17}$$

with potential defined on Euclidean plane $\mathbb{E}^2$*, where* a, b *are two constants, there exists exact three cases of parameters such that* H *is multi-separable. They are given by*

(i) $a = 1/4,\ b = 0$; *(ii)* $a = 1,\ b$ *arbitrary;* *(iii)* $a = 4,\ b$ *arbitrary.*

Remark 4. Here we consider the multi-separability, i.e. 2^{nd}-order super-integrability for the system (3.17). For any integer $a = k^2, k \in \mathbb{N}$, it admits an additional first integral which is a k^{th}-order polynomial in momenta [19], implying its (higher order) super-integrability. For such potentials there exist much more super-integrable cases than multi-separable ones.

Note that case (iii) is the celebrated Smorodinsky-Winternitz I potential [20], thus by using our Algorithm we reproduce this system quite straightforwardly.

Proof of Theorem 4. According to the algorithm we apply $\mathbf{K}$ (3.4) to $\mathrm{d}V$ where $V = x^2 + ay^2 + b/x^2$. After exterior derivative one has

$$\mathrm{d}\,(\mathbf{K}\,\mathrm{d}V) = Z\,\mathrm{d}x \wedge \mathrm{d}y \tag{3.18}$$

with the coefficient Z given by

$$\begin{aligned} Z = \frac{2}{x^4} \cdot \big(4(a-1)x^5 y\, C_6 + (1-4a)x^4 y\, C_5 \\ +(4-a)x^5 C_4 + (1-a)x^4 C_3 + 3byC_5 + 3bC_3\big) \end{aligned} \tag{3.19}$$

In Z's expression, C_k, a, b are constants. One notice Z is not polynomial in (x, y), but rational functions. Nevertheless, we can transform it to be a polynomial as below. The form $\mathrm{d}(\mathbf{K}\mathrm{d}V)$ vanishes equivalents to $Z \equiv 0$, which, in turn, equivalents to the vanishing of the polynomial $\tilde{Z} = Z \cdot x^4/2$. So we obtain a system of algebraic equations

$$\begin{aligned} &bC_3 = bC_5 = 0, \\ &C_4(a-4) = C_3(a-1) = 0, \\ &C_6(a-1) = C_5(4a-1) = 0 \end{aligned} \tag{3.20}$$

Since C_1, C_2 do not arise in the equations, all of $C_k = 0$ except that $C_1 - C_2 \neq 0$ solves the system above. This implies the Hamiltonian is separable in the Cartesian coordinates. For the system to be multi-separability, it suffices to find another solution linearly independent of the trivial solution given above.

A new solution to equations (3.20) exists if and only if one of the following cases occurs:

- $C_6 \neq 0 \Rightarrow a = 1,\ b$ arbitrary;
- $C_4 \neq 0 \Rightarrow a = 4,\ b$ arbitrary;
- $C_5 \neq 0 \Rightarrow a = \frac{1}{4},\ b = 0$;
- $C_3 \neq 0 \Rightarrow a = 1, b = 0$.

Observe that the last case is only a subcase of the first case. Thus we obtain exact three multi-separable cases, corresponding to cases (ii), (iii), (i) in the theorem, respectively. □

The next configuration space we consider is a Minkowski case $\mathbb{M}^2$ whose metric is $g = \mathrm{d}x^2 - \mathrm{d}y^2$. We compare the two planes $\mathbb{M}^2$ and $\mathbb{E}^2$. Both are of constat curvature (zero), hence the dimensions of vector spaces of their Killing tensor attain the maxima: $\mathcal{K}^1(\mathbb{M}^2)$, $\mathcal{K}^2(\mathbb{M}^2)$ are of dimension three and six, respectively.

Nevertheless, the basis of Killing tensors (hence the entire spaces) are not identical. The Minkowski $\mathbb{M}^2$ has the basis of Killing vectors (compare with (3.1))

$$\partial_x,\ \partial_y \quad \text{(two translations)}, \qquad y\partial_x + x\partial_y \quad \text{(Minkowski “rotation”)} \tag{3.21}$$

The basis of Killing 2-tensors are the following (compare with (3.2))

$$\begin{aligned}
&\mathbf{K}_1 = \partial_x^2, \qquad \mathbf{K}_2 = \partial_y^2, \qquad \mathbf{K}_3 = \partial_x\partial_y + \partial_y\partial_x, \\
&\mathbf{K}_4 = 2y\,\partial_x^2 + x\,\partial_x\partial_y + x\,\partial_y\partial_x, \quad \mathbf{K}_5 = 2x\,\partial_y^2 + y\,\partial_x\partial_y + y\,\partial_y\partial_x, \\
&\mathbf{K}_6 = y^2\partial_x^2 + xy\,\partial_x\partial_y + xy\,\partial_y\partial_x + x^2\partial_y^2,
\end{aligned} \tag{3.22}$$

Carrying out an analysis similar to that for the Euclidean plane (Theorem 4), we arrive at

Theorem 5. *For the system*

$$H = \frac{1}{2}(p_x^2 - p_y^2) + x^2 + ay^2 + \frac{b}{x^2} \tag{3.23}$$

with potential defined on Minkowski $\mathbb{M}^2$, a, b are constants, there exists exact three cases such that H is multi-separable. They are given by

(i) $a = -1/4,\ b = 0$; *(ii) $a = -1,\ b$ arbitrary;* *(iii) $a = -4,\ b$ arbitrary.*

4. Concluding Discussions

Based on Benenti's classical theorem 1, we have suggested an Algorithm and applied it to detect H-J separability of several families of two-dimensional natural systems. This method has the advantage of having a clear procedure and not depending on intricate techniques which can be seen in lots of literatures, thus executable in a computer-like environment.

However, the applications we make here are merely preliminary. There are several directions one can take into account to improve and extend its scope. For example, one may consider some nontrivial (pseudo)-Riemannian spaces such as spaces of constant curvature S^n, H^n etc., or surfaces of revolution. These manifolds are easy to handle as their Killing tensor are much investigated. The crucial task in step 1 in our Algorithm is thus solved.

Another line is to generalize the potentials under discussion to more general ones, which may be involved some arbitrary functions. This can greatly enlarge the families of separable systems. Proceeding the analysis as above may yield some well-known or novel models. It is natural that the calculations are much more complicated, with the aid of computer symbolic system sometimes being a necessity.

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References

[1] V. I. Arnol'd, Mathematical Methods of Classical Mechanics, Springer-Verlag, New-York 1988.

[2] J. Moser, Finitely many mass points on the line under the influence of an exponential potential—an integrable system, in Dynamical Systems, Theory and Applications (1974, Rencontres, BattelleRes. Inst., Seattle, Wash.), Lecture Notes in Phys., Vol. 38, Berlin, Springer, 1975, 467–497.

[3] M. Błaszak, Multi-Hamiltonian Theory of Dynamical Systems, Springer-Verlag, New York 1998.

[4] A. Ibort, F. Magri and G. Marmo, Bihamiltonian structures and Stäckel separability, J. Geom. Phys. 33 (2000) 210–228.

[5] E. K. Sklyanin, Separation of variables. New trends, Progr. Theor. Phys. Suppl. 118 (1995), 35–60.

[6] G. Falqui, F. Magri, M. Pedroni and J. P. Zubelli, A Bi-Hamiltonian Theory for Stationary KdV Flows and Their Separability, Regular and Chaotic Dynamics, 2000, V.5, 33–51.

[7] S. Benenti. Orthogonal separable dynamical systems. In: O.Kowalski, D.Krupka (Eds.) Proceedings of the 5th International Conference on Differential Geometry and Its Applications, Silesian University at Opava, August 24–28, 1992. Diff. Geom. Appl. 1993. V. 1. P. 163–184.

[8] S. Benenti, Intrinsic characterization of the variable separation in the Hamilton-Jacobi equation, J. Math. Phys. 38 (1997), 6578–6602.

[9] J. J. Morales-Ruiz, Differential Galois Theory and Non-Integrability of Hamiltonian Systems, Progress in Mathematics, vol. 179, Birkhäuser Verlag, Basel (1999)

[10] G. Thompson, Killing tensors in spaces of constant curvature, J. Math. Phys. 27 (1986), 2693–2699.

[11] E. G. Kalnins and W. Miller, Jr., Killing tensors and variable separation for Hamilton-Jacobi and Helmholtz equations, SIAM J. Math. Anal. 11 (1980), 1011–1026.

[12] R. G. McLenaghan, R. G. Smirnov and D. The, An extension of the classical theory of algebraic invariants to pseudo-Riemannian geometry and Hamiltonian mechanics, Journal of Mathematical Physics 45, 1079 (2004)

[13] A. T. Bruce, R. G. McLenaghan and R. G. Smirnov, A systematic study of the Toda lattice in the context of the Hamilton-Jacobi theory, Z. angew. Math. Phys. ZAMP 52 (2001) 171–190

[14] A. Ramani, B. Dorozzi, and B. Grammaticos, Painlevé Conjecture Revisited, Phys. Rev. Lett., (1982), 49 1539–1541.

[15] T. Bountis, H. Segur, and F. Vivaldi, Integrable Hamiltonian systems and the Painlevé property, Phys. Rev. A 25 1257–1264, (1982)

[16] M. E. Sansaturio, I. Vigo-Aguiar and J. M. Ferrándiz, Non-integrability of some Hamiltonian systems in polar coordinates, J. Phys. A: Math. Gen. 30 (1997) 5869

[17] R. Conte, M. Musette and C. Verhoeven, Explicit integration of the Hénon-Heiles Hamiltonians. J. of Nonlin. Math. Phys., (2005) 12(Supplement 1): 212–227. [arXiv:nlin.SI/0412057]

[18] A.M. Perelomov, Integrable Systems of Classical Mechanics and Lie Algebras, Birkäuser Verlag, Basel (1990)

[19] J. F. Cariñena, G. Marmo and M. F. Rañada, Non-symplectic symmetries and bi-Hamiltonian structures of the rational harmonic oscillator, J. Phys. A 35, L679–L686 (2002).

[20] T.I. Friš, V. Mandrosov, Y.A. Smorodinsky, M. Uhlíř and P. Winternitz, On higher symmetries in quantum mechanics, Phys. Lett., 16 (1965), 354-356

Cross-entropy for generalized hesitant fuzzy sets and their use in multi-criteria decision making

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Abstract

In this paper, the cross-entropy for generalized hesitant fuzzy sets (GHFSs) is developed by integrating the cross-entropy for intuitionistic fuzzy sets (IFSs) and hesitant fuzzy sets (HFSs). First, several measurement formulas are discussed and their properties are studied. Then, two approaches, which are based on the developed generalized hesitant fuzzy cross-entropy, are proposed for solving multi-criteria decision making (MCDM) problems under an generalized hesitant fuzzy environment. Finally, an example is provided to illustrate the practicality and effectiveness of the developed approaches.

1 Introduction

The cross-entropy measures are mainly used to measure the discrimination information, and then it is an important measure in decision making, pattern recognition and other real-world problems. Lots of studies on this issue have been extended and developed to fuzzy and its extended environments. For instance, Vlachos and Sergiadis [14] introduced the concepts of discrimination information and cross-entropy for intuitionistic fuzzy sets (IFSs), and revealed the connection between the notions of entropies for fuzzy sets and IFSs in terms of fuzziness and intuitionism. Hung and Yang [6] constructed J-divergence of IFSs and introduced some useful distance and similarity measures between two IFSs, and applied them to clustering analysis and pattern recognition. Based

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on which, Xia and Xu [17] proposed some cross-entropy and entropy formulas for IFSs and applied them to group decision making. Ye [21] proposed a method of fault diagnosis based on the vague cross-entropy. He [22] also introduced the cross-entropy for IFSs and interval-valued intuitionistic fuzzy sets (IVIFSs) and utilized then to solve multi-criteria decision making (MCDM) problems. Wang and Li [15] provided two improved methods for solving MCDM problems, which were based on the cross-entropy for IFSs. Hung et al. [5] introduced the discrimination information and cross-entropy for IFSs and also used them to improve the fault diagnosis of turbine problems. Mao et al. [9] introduced the cross-entropy and entropy measures for IFSs. Zang and Yu [28] constructed a series of mathematical programming models, which were based on an interval-valued intuitionistic fuzzy cross-entropy, in order to determine the criteria weights and applied them to MCDM problems. Xia and Xu [17] proposed two methods for determining the optimal weights of criteria and developed two pairs of entropy and cross-entropy measures for intuitionistic fuzzy values. The relationships among the entropy, cross-entropy and similarity measures have also attracted many attentions. For example, Liu [8] gave the axiomatic definitions of entropy, distance measure, and similarity measure of fuzzy sets and discussed their basic relations. Zeng and Li [25] discussed the relationship between the similarity measure and the entropy of interval-valued fuzzy sets. Zang and Jiang [27] proposed the entropy and cross-entropy for IVIFSs and discussed the connections among some important information measures. Xu and Xia [19] introduced the concepts of entropy and cross-entropy for hesitant fuzzy sets (HFSs), analyzed the relationships among the entropy, cross-entropy and similarity measures, and developed two multi-attribute decision making methods.

Qjan et al. [10], recently, introduced the concept of generalized hesitant fuzzy sets (GHFSs), extending the element of HFSs from real numbers to intuitionistic fuzzy values, which can arise in group decision making problem. GHFS is fit for the situation when decision maker have a hesitation among several possible memberships with uncertainties. GHFS can reflect the human's hesitance more objectively than other extensions of fuzzy set (IFS, IVIFS and HFS), and thus it is necessary to develop some theories about GHFSs. In this paper, we discuss the cross-entropy for generalized hesitant information. To do this, Section 2 reviews some related preliminaries such as IFSs, HFSs and GHFSs. In Section 3, we propose some cross-entropy formulas for generalized hesitant fuzzy elements, obtain some important conclusions, and provide an example to illustrate the application of cross-entropy in MCDM problem. Finally, Section 4 gives the concluding remarks.

2 Basic concepts

Intuitionistic fuzzy sets introduced by Atanassov [1] have been proven to be highly useful to deal with uncertainty and vagueness.

Definition 1. [1] Let X be ordinary non-empty set. An intuitionistic fuzzy set (IFS) A in X is defined as

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}, \tag{1}$$

where $\mu_A, \nu_A : X \to [0,1]$ denote, respectively, the membership and non-membership functions of A with the condition: $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.

For an IFS A, $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ represents the degree of hesitation or intuitionistic index of x to A. For a fuzzy set, the degree of hesitation $\pi_A(x) = 0$. Thus for each x, $\mu_A(x)$ and $\nu_A(x)$ define an interval $[\mu_A(x), 1 - \nu_A(x)]$. This interval is the vague value of value set by Gau and Buethrer [4] (Bustince and Burillo [3] proved that vague sets are equivalent to IFSs). Further, the interval can also represent an interval-valued fuzzy set [10]. Hence Xu [18] concluded that IFSs are also equivalent to interval-valued fuzzy sets, and replaced Eq. (1) with

$$A = \{\langle x, [\mu_A(x), 1 - \nu_A(x)]\rangle | x \in X\}. \tag{2}$$

The ordered pair $\alpha(x) = (\mu_\alpha(x), \nu_\alpha(x))$ is referred to an intuitionistic fuzzy value (IFV) [18], where $\mu_\alpha(x), \nu_\alpha(x) \in [0,1]$ and $\mu_\alpha(x) + \nu_\alpha(x) \leq 1$. Associated with the degree of hesitation, an IFV can also be equivalently denoted by $\alpha(x) = (\mu_\alpha(x), \nu_\alpha(x), \pi_\alpha(x))$, where $\mu_\alpha(x), \nu_\alpha(x), \pi_\alpha(x) \in [0,1]$ and $\mu_\alpha(x) + \nu_\alpha(x) + \pi_\alpha(x) = 1$. In the rest of this paper, for a certain x in X, IFV $a = (\mu, \nu, \pi)$ is abbreviated as $a = (\mu, \nu)$ when no misunderstanding raises. Since an IFV represent an interval, an interval $[\mu, 1 - \nu]$ in $[0,1]$ will be directly transformed into (μ, ν).

Definition 2. [5, 6, 14, 17, 22] Let $\alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1})$ and $\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2})$ be IFVs, then the cross-entropy α_1 and α_2, denoted as $\mathrm{CE}(\alpha_1, \alpha_2)$, should satisfy the following properties:

(1) $\mathrm{CE}(\alpha_1, \alpha_2) \geq 0$;

(2) $\mathrm{CE}(\alpha_1, \alpha_2) = 0$ if $\alpha_1 = \alpha_2$;

(3) $\mathrm{CE}(\alpha_1^c, \alpha_2^c) = \mathrm{CE}(\alpha_1, \alpha_2)$, where $\alpha_i^c = (\nu_{\alpha_i}, \mu_{\alpha_i})$ is the complement of α_i $(i = 1, 2)$.

In the following, some intuitionistic fuzzy cross-entropy and symmetric intuitionistic fuzzy cross-entropy formulas are reviewed.

Vlachos and Sergiadis [14] developed

$$\mathrm{CE}_1(\alpha_1, \alpha_2) = \mu_{\alpha_1} \ln \frac{2\mu_{\alpha_1}}{\mu_{\alpha_1} + \mu_{\alpha_2}} + \nu_{\alpha_1} \ln \frac{2\nu_{\alpha_1}}{\nu_{\alpha_1} + \nu_{\alpha_2}}, \tag{3}$$

and

$$\begin{aligned}\mathrm{CE}_2(\alpha_1, \alpha_2) = 2\Bigg(&\frac{\mu_{\alpha_1} \ln \mu_{\alpha_1} + \mu_{\alpha_2} \ln \mu_{\alpha_2}}{2} - \frac{\mu_{\alpha_1} + \mu_{\alpha_2}}{2} \ln \frac{\mu_{\alpha_1} + \mu_{\alpha_2}}{2} \\ &+ \frac{\nu_{\alpha_1} \ln \nu_{\alpha_1} + \nu_{\alpha_2} \ln \nu_{\alpha_2}}{2} - \frac{\nu_{\alpha_1} + \nu_{\alpha_2}}{2} \ln \frac{\nu_{\alpha_1} + \nu_{\alpha_2}}{2}\Bigg).\end{aligned} \tag{4}$$

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Hung and Yang [6] defined

$$\begin{aligned}\mathrm{CE}_3(\alpha_1,\alpha_2) = 2\Bigg(&\frac{\mu_{\alpha_1}\ln\mu_{\alpha_1}+\mu_{\alpha_2}\ln\mu_{\alpha_2}}{2} - \frac{\mu_{\alpha_1}+\mu_{\alpha_2}}{2}\ln\frac{\mu_{\alpha_1}+\mu_{\alpha_2}}{2}\\ &+\frac{\nu_{\alpha_1}\ln\nu_{\alpha_1}+\nu_{\alpha_2}\ln\nu_{\alpha_2}}{2} - \frac{\nu_{\alpha_1}+\nu_{\alpha_2}}{2}\ln\frac{\nu_{\alpha_1}+\nu_{\alpha_2}}{2}\\ &+\frac{\pi_{\alpha_1}\ln\nu_{\alpha_1}+\nu_{\alpha_2}\ln\nu_{\alpha_2}}{2} - \frac{\nu_{\alpha_1}+\nu_{\alpha_2}}{2}\ln\frac{\nu_{\alpha_1}+\nu_{\alpha_2}}{2}\Bigg). \qquad (5)\end{aligned}$$

Ye [22] proposed

$$\begin{aligned}\mathrm{CE}_4(\alpha_1,\alpha_2) = &\frac{\mu_{\alpha_1}+1-\nu_{\alpha_1}}{2}\log_2\frac{2(\mu_{\alpha_1}+1-\nu_{\alpha_1})}{2+\mu_{\alpha_1}-\nu_{\alpha_1}+\mu_{\alpha_2}-\nu_{\alpha_2}}\\ &+\frac{\nu_{\alpha_1}+1-\mu_{\alpha_1}}{2}\log_2\frac{2(\nu_{\alpha_1}+1-\mu_{\alpha_1})}{2-\mu_{\alpha_1}+\nu_{\alpha_1}-\mu_{\alpha_2}+\nu_{\alpha_2}}. \qquad (6)\end{aligned}$$

Hung et al. [5] developed

$$\begin{aligned}\mathrm{CE}_5(\alpha_1,\alpha_2) = &\mu_{\alpha_1}\log_2\frac{2\mu_{\alpha_1}}{\mu_{\alpha_1}+\mu_{\alpha_2}} + \nu_{\alpha_1}\log_2\frac{2\nu_{\alpha_1}}{\nu_{\alpha_1}+\nu_{\alpha_2}}\\ &+\pi_{\alpha_1}\log_2\frac{2\pi_{\alpha_1}}{\pi_{\alpha_1}+\pi_{\alpha_2}}. \qquad (7)\end{aligned}$$

Xia and Xu [17] proposed

$$\begin{aligned}\mathrm{CE}_6(\alpha_1,\alpha_2) = &\frac{1}{1-2^{1-q}}\left(\frac{\mu^q_{\alpha_1}+\mu^q_{\alpha_2}}{2} - \left(\frac{\mu_{\alpha_1}+\mu_{\alpha_2}}{2}\right)^q + \frac{\nu^q_{\alpha_1}+\nu^q_{\alpha_2}}{2}\right.\\ &\left.-\left(\frac{\nu_{\alpha_1}+\nu_{\alpha_2}}{2}\right)^q + \frac{\pi^q_{\alpha_1}+\pi^q_{\alpha_2}}{2} - \left(\frac{\pi_{\alpha_1}+\pi_{\alpha_2}}{2}\right)^q\right), \qquad (8)\end{aligned}$$

where $1 < q \le 2$.

For the symmetric property, it is necessary to modify Eqs. (3)-(8) to obtain a symmetric discrimination information measures for IFVs ([11, 26]):

$$\mathrm{CE}^*_k(\alpha_1,\alpha_2) = \mathrm{CE}_k(\alpha_1,\alpha_2) + \mathrm{CE}_k(\alpha_2,\alpha_1),\ k = 1,2,\ldots,6. \qquad (9)$$

The hesitant fuzzy set [12, 13], as a generalization of fuzzy set, permits the membership degree of an element to a set presented as several possible values between 0 and 1, which can better describe the situations where people have hesitancy in providing their preferences over objects in process of decision making.

Definition 3. [12, 13] Given a fixed set X, a hesitant fuzzy set (HFS) on X in terms of function h is that when applied to X returns a subset of $[0, 1]$, which can be represented as the following mathematical symbol:

$$E = \{\langle x, h(x)\rangle | x \in X\}, \qquad (10)$$

where $h(x)$ is a set of the some values in $[0,1]$, denoting the possible membership degrees of the element $x \in X$ to the set E. For convenience, Xia and Xu [16] called $h(x)$ a hesitant fuzzy element (HFE) and the set of all HFEs is denoted by HFES.

Definition 4. [19] Let h_1 and h_2 be two HFEs, then the cross-entropy of h_1 and h_2, denoted as $\mathrm{CE}(h_1, h_2)$, should satisfy the following properties:
(1) $\mathrm{CE}(h_1, h_2) \geq 0$;
(2) $\mathrm{CE}(h_1, h_2) = 0$ if and only if $h_1^{\sigma(i)} = h_2^{\sigma(i)}$ for all $i = 1, 2, \ldots, l$.

Based on Definition 4, $l = l(h_1) = l(h_2)$ and denote the number of elements in h_1 and h_2. The elements are arranged in increasing order in h_1 and h_2, respectively, and $h_1^{\sigma(i)}$ $(i = 1, 2, \ldots, l(h_1))$ and $h_2^{\sigma(i)}$ $(i = 1, 2, \ldots, l(h_2))$ are the ith smallest values in h_1 and h_2, respectively. Xu and Xia [19] constructed several cross-entropy for HFEs:

$$\begin{aligned}
&\mathrm{CE}_1(h_1, h_2) \\
&\quad = \frac{1}{lT} \sum_{i=1}^{l} \left(\frac{(1 + q h_1^{\sigma(i)}) \ln(1 + q h_1^{\sigma(i)}) + (1 + q h_2^{\sigma(i)}) \ln(1 + q h_2^{\sigma(i)})}{2} \right. \\
&\quad - \frac{2 + q h_1^{\sigma(i)} + q h_2^{\sigma(i)}}{2} \ln \frac{2 + q h_1^{\sigma(i)} + q h_2^{\sigma(i)}}{2} + \frac{(1 + q(1 - h_1^{\sigma(l-i+1)}))}{2} \\
&\quad \times \ln(1 + q(1 - h_1^{\sigma(l-i+1)})) + \frac{(1 + q(1 - h_2^{\sigma(l-i+1)})) \ln(1 + q(1 - h_2^{\sigma(l-i+1)}))}{2} \\
&\quad - \frac{2 + q(1 - h_1^{\sigma(l-i+1)} + 1 - h_2^{\sigma(l-i+1)})}{2} \\
&\quad \left. \times \ln \frac{2 + q(1 - h_1^{\sigma(l-i+1)} + 1 - h_2^{\sigma(l-i+1)})}{2} \right), \qquad (11)
\end{aligned}$$

where $T = (1+q)\ln(1+q) - (2+q)(\ln(2+q) - \ln 2)$ and $q > 0$.

$$\begin{aligned}
&\mathrm{CE}_2(h_1, h_2) \\
&\quad = \frac{1}{(1 - 2^{1-p})l} \sum_{i=1}^{l} \left(\frac{(h_1^{\sigma(i)})^p + (h_2^{\sigma(i)})^p}{2} + \frac{(1 - h_1^{\sigma(l-i+1)})^p + (1 - h_2^{\sigma(l-i+1)})^p}{2} \right. \\
&\quad \left. - \left(\frac{h_1^{\sigma(i)} + h_2^{\sigma(i)}}{2} \right)^p + \left(\frac{1 - h_1^{\sigma(l-i+1)} + 1 - h_2^{\sigma(l-i+1)}}{2} \right)^p \right), \quad p > 1. \qquad (12)
\end{aligned}$$

For the symmetric property, it is necessary to modify Eqs. (11) and (12) to obtain a symmetric discrimination information measure for HFEs:

$$\mathrm{CE}_k^*(h_1, h_2) = \mathrm{CE}_k(h_1, h_2) + \mathrm{CE}_k(h_2, h_1), \ k = 1, 2. \qquad (13)$$

Note that Eqs. (11) and (12) are all defined under the assumption that two HFEs are of the same length. If the corresponding HFEs are not equal in length, then the shorter one should be extended to be the same size as the longer one by adding the same value repeatedly.

3 Generalized hesitant fuzzy sets and their cross-entropy measures

3.1 Generalized hesitant fuzzy sets

During the evaluating process, several possible memberships of an alternative satisfying a certain criterion may be not only crisp values but also interval values in $[0, 1]$. In order to handle this kind of assessment in decision making, Qjan et al. [10] extended HFSs by using IFSs to modify Definition 3.

Definition 5. [10] Given a set of N membership functions:

$$M = \{\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i}) | 0 \leq \mu_{\alpha_i}, \nu_{\alpha_i} \leq 1,\ \mu_{\alpha_i} + \nu_{\alpha_i} \leq 1,\ i = 1, 2, \ldots, N\} \quad (14)$$

the generalized hesitant fuzzy set (GHFS) associated with M, that is $\tilde{h}_M$, is defined as follows:

$$\tilde{h}_M(x) = \cup_{(\mu_{\alpha_i}, \nu_{\alpha_i}) \in M} \{(\mu_{\alpha_i}(x), \nu_{\alpha_i}(x))\}. \quad (15)$$

Note that HFSs, IFSs and fuzzy sets are special cases of GHFSs redefined here. In fact, if $\mu_{\alpha_i} + \nu_{\alpha_i} = 1$, for $i = 1, 2, \ldots, N$, then GHFSs reduce to HFSs. If $N = 1$ or union of N IFSs, i.e. $\cup_{i=1}^{N} \alpha_i$, in Eq. (14) is convex set in $[0, 1]$, then GHFSs reduce to IFSs. If $N = 1$ and $\mu_{\alpha_N} + \nu_{\alpha_N} = 1$, then GHFSs reduce to FSs. Thus GHFSs are not only the generalization of HFSs but also the generalized representation of fuzzy sets, IFSs and HFSs. For the sake of convenience, given a certain $x \in X$, α represents an IFS in $\tilde{h}(x)$. Notice that α is represented an interval as well. Similar to [16], $\tilde{h}_M(x)$, abbreviated as $\tilde{h}(x)$, is called a generalized hesitant fuzzy element (GHFE) and the set of all GHFEs is denoted by GHFES.

Let $l(\tilde{h})$ be the number of elements of a GHFE $\tilde{h}$. In most cases of two GHFEs $\tilde{h}_1$ and $\tilde{h}_2$, the numbers of elements of $\tilde{h}_1$ and $\tilde{h}_2$ may be different, i.e. $l(\tilde{h}_1) \neq l(\tilde{h}_2)$, and for convenience, let $l = \max\{l(\tilde{h}_1), l(\tilde{h}_2)\}$. To operate correctly, we should extend the shorter ones, until both of them have the same length when we compare them. To extend the shorter one, the best way is to add the same values several times in it. In fact, we can extend the shorter one by adding any values in it. The selection of this value mainly depends on the decision makers' risk preferences. Optimists anticipate desirable outcomes and may add the maximum value, while pessimists expect unfavorable outcomes and may add the minimum value. In this paper, we assume the GHFEs $\tilde{h}_1$ and $\tilde{h}_2$ should have the same length l when we compare them.

Some useful operations on GHFEs are as follows:

Definition 6. [10] Let $\tilde{h}$, $\tilde{h}_1$ and $\tilde{h}_2$ be three GHFEs and $\lambda > 0$, then

(1) $\tilde{h}_1 \cup \tilde{h}_2 = \cup_{\alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2} \{\alpha_1 \cup \alpha_2\} = \cup_{\alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2} \{(\max\{\mu_{\alpha_1}, \mu_{\alpha_2}\}, \min\{\nu_{\alpha_1}, \nu_{\alpha_2}\})\}$;
(2) $\tilde{h}_1 \cap \tilde{h}_2 = \cup_{\alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2} \{\alpha_1 \cap \alpha_2\} = \cup_{\alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2} \{(\min\{\mu_{\alpha_1}, \mu_{\alpha_2}\}, \max\{\nu_{\alpha_1}, \nu_{\alpha_2}\})\}$;
(3) $\tilde{h}^c = \cup_{\alpha \in \tilde{h}} \{\alpha^c\} = \cup_{\alpha \in \tilde{h}} \{(\nu_\alpha, \mu_\alpha)\}$;
(4) $\tilde{h}_1 \oplus \tilde{h}_2 = \cup_{\alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2} \{\alpha_1 \oplus \alpha_2\} = \cup_{\alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2} \{(\mu_{\alpha_1} + \mu_{\alpha_2} - \mu_{\alpha_1}\mu_{\alpha_2}, \nu_{\alpha_1}\nu_{\alpha_2})\}$;
(5) $\tilde{h}_1 \otimes \tilde{h}_2 = \cup_{\alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2} \{\alpha_1 \otimes \alpha_2\} = \cup_{\alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2} \{(\mu_{\alpha_1}\mu_{\alpha_2}, \nu_{\alpha_1} + \nu_{\alpha_2} - \nu_{\alpha_1}\nu_{\alpha_2})\}$;
(6) $\lambda \tilde{h} = \cup_{\alpha \in \tilde{h}} \{\lambda \alpha\} = \cup_{\alpha \in \tilde{h}} \{(1 - (1 - \mu_\alpha)^\lambda, \nu_\alpha^\lambda)\}$;
(7) $\tilde{h}^\lambda = \cup_{\alpha \in \tilde{h}} \{\alpha^\lambda\} = \cup_{\alpha \in \tilde{h}} \{(\mu_\alpha^\lambda, 1 - (1 - \nu_\alpha)^\lambda)\}$.

Definition 7. Let $\tilde{h}_i$ $(i = 1, 2, \ldots, n)$ be a collection of GHFEs, and let $\mathrm{GHFWA} : \mathrm{GHFES}^n \to \mathrm{GHFES}$, if

$$\mathrm{GHFWA}_w(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n) = w_1\tilde{h}_1 \oplus w_2\tilde{h}_2 \oplus \cdots \oplus w_n\tilde{h}_n, \tag{16}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $\tilde{h}_i$ $(i = 1, 2, \ldots, n)$ with $w_i \geq 0$ and $\sum_{i=1}^n w_i = 1$, then GHFWA is called the generalized hesitant fuzzy weighted averaging (GHFWA) operator.

Based on operations (4)-(7) of GHFEs described in Definition 6, we can derive the following result.

Theorem 1. Let $\tilde{h}_i = \cup_{\alpha_i \in \tilde{h}_i} \{\alpha_i\}$ $(i = 1, 2, \ldots, n)$ be a collection of GHFEs, and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $\tilde{h}_i$ $(i = 1, 2, \ldots, n)$ with $w_i \geq 0$ and $\sum_{i=1}^n w_i = 1$. Then the aggregated value, by using the GHFWA operator, is also a GHFE, and

$$\begin{aligned} &\mathrm{GHFWA}_w(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n) \\ &= \bigcup_{\alpha_1 \in \tilde{h}_1, \alpha_2 \in \tilde{h}_2, \ldots, \alpha_n \in \tilde{h}_n} \left\{ \left(1 - \prod_{i=1}^n (1 - \mu_{\alpha_i})^{w_i}, \prod_{i=1}^n \nu_{\alpha_i}^{w_i} \right) \right\}. \end{aligned} \tag{17}$$

Theorem 1 can be proved by using the mathematical induction and then the process is omitted here.

Definition 8. Let $\tilde{h}_i$ $(i = 1, 2, \ldots, n)$ be a collection of GHFEs, and let $\mathrm{GHFWG} : \mathrm{GHFES}^n \to \mathrm{GHFES}$, if

$$\mathrm{GHFWG}_w(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n) = \tilde{h}_1^{w_1} \otimes \tilde{h}_2^{w_2} \otimes \cdots \otimes \tilde{h}_n^{w_n}, \tag{18}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $\tilde{h}_i$ $(i = 1, 2, \ldots, n)$ with $w_i \geq 0$ and $\sum_{i=1}^n w_i = 1$, then GHFWG is called the generalized hesitant fuzzy weighted geometric (GHFWG) operator.

Theorem 2. Let $\tilde{h}_i = \cup_{\alpha_i \in \tilde{h}_i} \{\alpha_i\}$ $(i = 1, 2, \ldots, n)$ be a collection of GHFEs, and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $\tilde{h}_i$ $(i = 1, 2, \ldots, n)$ with $w_i \geq 0$ and $\sum_{i=1}^n w_i = 1$. Then the aggregated value, by using the GHFWG

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operator, is also a GHFE, and

$$\begin{aligned}&\mathrm{GHFWG}_w(\tilde{h}_1,\tilde{h}_2,\ldots,\tilde{h}_n)\\&=\bigcup_{\alpha_1\in\tilde{h}_1,\alpha_2\in\tilde{h}_2,\ldots,\alpha_n\in\tilde{h}_n}\left\{\left(\prod_{i=1}^{n}\mu_{\alpha_i}^{w_i},1-\prod_{i=1}^{n}(1-\nu_{\alpha_i})^{w_i}\right)\right\}.\end{aligned}\tag{19}$$

Theorem 2 can be also proved by using the mathematical induction and then the process is omitted here.

3.2 Cross-entropy measures of GHFEs

Definition 9. Let $\tilde{h}_1,\tilde{h}_2\in$ GHFES and CE : GHFES $\times$ GHFES $\to R$, then the cross-entropy of $\tilde{h}_1$ and $\tilde{h}_2$, denoted as $\mathrm{CE}(\tilde{h}_1,\tilde{h}_2)$, should satisfy the following properties:

(1) $\mathrm{CE}(\tilde{h}_1,\tilde{h}_2)\geq 0$;

(2) If $\tilde{h}_1=\tilde{h}_2$, then $\mathrm{CE}(\tilde{h}_1,\tilde{h}_2)=0$;

(3) $\mathrm{CE}(\tilde{h}_1^c,\tilde{h}_2^c)=\mathrm{CE}(\tilde{h}_1,\tilde{h}_2)$, where $\tilde{h}_i^c$ is the complement of $\tilde{h}_i$ defined in Definition 6.

On the basis of Definition 9, we can construct several cross-entropy for GHFEs:

$$\begin{aligned}\mathrm{CE}_1(\tilde{h}_1,\tilde{h}_2)=&\frac{1}{l(\tilde{h}_1)}\sum_{\alpha_1\in\tilde{h}_1}\left(\frac{1}{l(\tilde{h}_2)}\sum_{\alpha_2\in\tilde{h}_2}\left(\mu_{\alpha_1}\log_2\frac{2\mu_{\alpha_1}}{\mu_{\alpha_1}+\mu_{\alpha_2}}\right)\right)\\&+\frac{1}{l(\tilde{h}_1)}\sum_{\alpha_1\in\tilde{h}_1}\left(\frac{1}{l(\tilde{h}_2)}\sum_{\alpha_2\in\tilde{h}_2}\left(\nu_{\alpha_1}\log_2\frac{2\nu_{\alpha_1}}{\nu_{\alpha_1}+\nu_{\alpha_2}}\right)\right);\end{aligned}\tag{20}$$

$$\begin{aligned}\mathrm{CE}_2(\tilde{h}_1,\tilde{h}_2)=&\sqrt[p]{\frac{1}{l(\tilde{h}_1)}\sum_{\alpha_1\in\tilde{h}_1}\left(\frac{1}{l(\tilde{h}_2)}\sum_{\alpha_2\in\tilde{h}_2}\left(\mu_{\alpha_1}\log_2\frac{2\mu_{\alpha_1}}{\mu_{\alpha_1}+\mu_{\alpha_2}}\right)\right)^p}\\&+\sqrt[p]{\frac{1}{l(\tilde{h}_1)}\sum_{\alpha_1\in\tilde{h}_1}\left(\frac{1}{l(\tilde{h}_2)}\sum_{\alpha_2\in\tilde{h}_2}\left(\nu_{\alpha_1}\log_2\frac{2\nu_{\alpha_1}}{\nu_{\alpha_1}+\nu_{\alpha_2}}\right)\right)^p},\end{aligned}\tag{21}$$

where $p\geq 1$;

$$\begin{aligned}&\mathrm{CE}_3(\tilde{h}_1,\tilde{h}_2)\\&=\frac{1}{l(\tilde{h}_1)}\sum_{\alpha_1\in\tilde{h}_1}\left(\frac{1}{l(\tilde{h}_2)}\sum_{\alpha_2\in\tilde{h}_2}\left(\frac{\mu_{\alpha_1}+1-\nu_{\alpha_1}}{2}\log_2\frac{2(\mu_{\alpha_1}+1-\nu_{\alpha_1})}{2+\mu_{\alpha_1}-\nu_{\alpha_1}+\mu_{\alpha_2}-\nu_{\alpha_2}}\right)\right)\\&+\frac{1}{l(\tilde{h}_1)}\sum_{\alpha_1\in\tilde{h}_1}\left(\frac{1}{l(\tilde{h}_2)}\sum_{\alpha_2\in\tilde{h}_2}\left(\frac{1-\mu_{\alpha_1}+\nu_{\alpha_1}}{2}\log_2\frac{2(1-\mu_{\alpha_1}+\nu_{\alpha_1})}{2-\mu_{\alpha_1}+\nu_{\alpha_1}-\mu_{\alpha_2}+\nu_{\alpha_2}}\right)\right);\end{aligned}\tag{22}$$

$$\begin{aligned}
&\mathrm{CE}_4(\tilde{h}_1,\tilde{h}_2)\\
&=\sqrt[p]{\frac{1}{l(\tilde{h}_1)}\sum_{\alpha_1\in\tilde{h}_1}\left(\frac{1}{l(\tilde{h}_2)}\sum_{\alpha_2\in\tilde{h}_2}\left(\frac{\mu_{\alpha_1}+1-\nu_{\alpha_1}}{2}\log_2\frac{2(\mu_{\alpha_1}+1-\nu_{\alpha_1})}{2+\mu_{\alpha_1}-\nu_{\alpha_1}+\mu_{\alpha_2}-\nu_{\alpha_2}}\right)\right)^p}\\
&+\sqrt[p]{\frac{1}{l(\tilde{h}_1)}\sum_{\alpha_1\in\tilde{h}_1}\left(\frac{1}{l(\tilde{h}_2)}\sum_{\alpha_2\in\tilde{h}_2}\left(\frac{1-\mu_{\alpha_1}+\nu_{\alpha_1}}{2}\log_2\frac{2(1-\mu_{\alpha_1}+\nu_{\alpha_1})}{2-\mu_{\alpha_1}+\nu_{\alpha_1}-\mu_{\alpha_2}+\nu_{\alpha_2}}\right)\right)^p}, \quad (23)
\end{aligned}$$

where $p \geq 1$;

$$\begin{aligned}
&\mathrm{CE}_5(\tilde{h}_1,\tilde{h}_2)\\
&=\frac{1}{1-2^{1-q}}\left(\frac{1}{l(\tilde{h}_1)}\sum_{\alpha_1\in\tilde{h}_1}\left(\frac{1}{l(\tilde{h}_2)}\sum_{\alpha_2\in\tilde{h}_2}\left(\frac{\mu_{\alpha_1}^q+\mu_{\alpha_2}^q}{2}-\left(\frac{\mu_{\alpha_1}+\mu_{\alpha_2}}{2}\right)^q\right)\right)\right.\\
&\left.+\frac{1}{l(\tilde{h}_1)}\sum_{\alpha_1\in\tilde{h}_1}\left(\frac{1}{l(\tilde{h}_2)}\sum_{\alpha_2\in\tilde{h}_2}\left(\frac{\nu_{\alpha_1}^q+\nu_{\alpha_2}^q}{2}-\left(\frac{\nu_{\alpha_1}+\nu_{\alpha_2}}{2}\right)^q\right)\right)\right), \quad (24)
\end{aligned}$$

where $1 < q \leq 2$;

$$\begin{aligned}
&\mathrm{CE}_6(\tilde{h}_1,\tilde{h}_2)\\
&=\sqrt[p]{\frac{1}{l(\tilde{h}_1)}\sum_{\alpha_1\in\tilde{h}_i}\frac{1}{1-2^{1-q}}\left(\frac{1}{l(\tilde{h}_2)}\sum_{\alpha_2\in\tilde{h}_2}\left(\frac{\mu_{\alpha_1}^q+\mu_{\alpha_2}^q}{2}-\left(\frac{\mu_{\alpha_1}+\mu_{\alpha_2}}{2}\right)^q\right)\right)^p}\\
&+\sqrt[p]{\frac{1}{l(\tilde{h}_1)}\sum_{\alpha_1\in\tilde{h}_i}\frac{1}{1-2^{1-q}}\left(\frac{1}{l(\tilde{h}_2)}\sum_{\alpha_2\in\tilde{h}_2}\left(\frac{\nu_{\alpha_1}^q+\nu_{\alpha_2}^q}{2}-\left(\frac{\nu_{\alpha_1}+\nu_{\alpha_2}}{2}\right)^q\right)\right)^p}, \quad (25)
\end{aligned}$$

where $p \geq 1$ and $1 < q \leq 2$.

For the symmetric property, it is necessary to modify Eqs. (20)-(25) to a symmetric discrimination information measure for GHFEs as follows:

$$\mathrm{CE}_k^*(\tilde{h}_1,\tilde{h}_2)=\mathrm{CE}_k(\tilde{h}_1,\tilde{h}_2)+\mathrm{CE}_k(\tilde{h}_2,\tilde{h}_1),\ k=1,2,\ldots,6. \quad (26)$$

Example 1. Let $\tilde{h}_i$ $(i = 1,2,3)$ and $\tilde{h}$ be three patterns and a sample. They are denoted by GHFEs as follows: $\tilde{h}_1 = \{(0.5,0.4),(0.6,0.3)\}$, $\tilde{h}_2 = \{(0.4,0.5),(0.8,0.1)\}$, $\tilde{h}_3 = \{(0.3,0.4),(0.7,0.2)\}$ and $\tilde{h} = \{(0.5,0.4),(0.7,0.2)\}$. Given the sample $\tilde{h}$, which pattern does this sample $\tilde{h}$ most probably belong to ?

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For convenience, let $p = q = 2$. By (20)-(25), we have

$$\begin{aligned}
&\mathrm{CE}_1^*(\tilde{h}_1, \tilde{h}) = 0.0275,\ \mathrm{CE}_1^*(\tilde{h}_2, \tilde{h}) = 0.0976,\ \mathrm{CE}_1^*(\tilde{h}_3, \tilde{h}) = 0.0693;\\
&\mathrm{CE}_2^*(\tilde{h}_1, \tilde{h}) = 0.2601,\ \mathrm{CE}_2^*(\tilde{h}_2, \tilde{h}) = 0.4285,\ \mathrm{CE}_2^*(\tilde{h}_3, \tilde{h}) = 0.3982;\\
&\mathrm{CE}_3^*(\tilde{h}_1, \tilde{h}) = 0.0241,\ \mathrm{CE}_3^*(\tilde{h}_2, \tilde{h}) = 0.0838,\ \mathrm{CE}_3^*(\tilde{h}_3, \tilde{h}) = 0.0541;\\
&\mathrm{CE}_4^*(\tilde{h}_1, \tilde{h}) = 0.2609,\ \mathrm{CE}_4^*(\tilde{h}_2, \tilde{h}) = 0.4298,\ \mathrm{CE}_4^*(\tilde{h}_3, \tilde{h}) = 0.3856;\\
&\mathrm{CE}_5^*(\tilde{h}_1, \tilde{h}) = 0.0300,\ \mathrm{CE}_5^*(\tilde{h}_2, \tilde{h}) = 0.1000,\ \mathrm{CE}_5^*(\tilde{h}_3, \tilde{h}) = 0.0800;\\
&\mathrm{CE}_6^*(\tilde{h}_1, \tilde{h}) = 0.0239,\ \mathrm{CE}_6^*(\tilde{h}_2, \tilde{h}) = 0.0707,\ \mathrm{CE}_6^*(\tilde{h}_3, \tilde{h}) = 0.0620.
\end{aligned}$$

From this data, the proposed symmetric discrimination information measures CE_k^* $(k = 1, 2, 3, 4, 5, 6)$ show the same classification according to the principle of the minimum degree of symmetric discrimination information measure for GHFEs. Thus, the sample $\tilde{h}$ belongs to the pattern $\tilde{h}_1$.

4 Two MCDM approaches based on the cross-entropy measures of GHFEs

For a MCDM problem, let $X = \{x_1, x_2, \ldots, x_m\}$ be a set of m alternatives, and $Y = \{y_1, y_2, \ldots, y_n\}$ be a set of n criteria, whose weight vector is $w = (w_1, w_2, \ldots, w_n)^T$, satisfying $w_j > 0$, $j = 1, 2, \ldots, n$ and $\sum_{j=1}^n w_j = 1$, where w_j denotes the importance degree of the criterion y_j. Decision makers evaluate the performance of alternatives with respect to criteria based on their knowledge and experience. One decision maker could give several evaluation values. However, in the case where two or more decision makers give the same value, it is counted only once. The performance of the alternative x_i with respect to the criterion y_j is measured by a GHFE $\tilde{e}_{ij} = \{\beta_{ij} = (\mu_{\beta_{ij}}, \nu_{\beta_{ij}})|\beta_{ij} \in \tilde{e}_{ij}\}$, where $\mu_{\beta_{ij}}$ indicates the degree that the alternative x_i satisfies the criterion y_j, $\nu_{\beta_{ij}}$ indicates the degree that the alternative x_i does not satisfy the criterion y_j, such that $\mu_{\beta_{ij}} \in [0, 1]$, $\nu_{\beta_{ij}} \in [0, 1]$, $\mu_{\beta_{ij}} + \nu_{\beta_{ij}} \leq 1$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$. All $\tilde{e}_{ij}$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ are contained in the generalized hesitant fuzzy decision matrix $E = (\tilde{e}_{ij})_{m\times n}$ (see Table 1).

Table 1: The generalized hesitant fuzzy decision matrix

	y_1	y_2	$\cdots$	y_n
x_1	$\tilde{e}_{11}$	$\tilde{e}_{12}$	$\cdots$	$\tilde{e}_{in}$
x_2	$\tilde{e}_{21}$	$\tilde{e}_{22}$	$\cdots$	$\tilde{e}_{2n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_m	$\tilde{e}_{m1}$	$\tilde{e}_{m2}$	$\cdots$	$\tilde{e}_{mn}$

In what follows, we develop two approaches to multi-criteria decision making under generalized hesitant fuzzy environment.

Approach I.

Step 1. Normalize the performance values and then construct the normalized generalized hesitant fuzzy decision matrix.

If all the criteria y_j $(j = 1, 2, \dots, n)$ are of the same type, then the performance values do not need normalization. Whereas there are, generally, benefit criteria (the bigger the performance values the better) and cost criteria (the smaller the performance values the better) in multi-criteria decision making, in such case, we may transform the performance values of the cost type into the performance values of the benefit type. Then, $E = (\tilde{e}_{ij})_{m\times n}$ can be transformed into the matrix $F = (\tilde{h}_{ij})_{m\times n}$, where

$$\tilde{h}_{ij} = \cup_{\alpha_{ij}\in\tilde{h}_{ij}}\{\alpha_{ij}\} = \begin{cases} \cup_{\beta_{ij}\in\tilde{d}_{ij}}\{\beta_{ij}\}, & \text{for benefit criterion } y_j; \\ \cup_{\beta_{ij}\in\tilde{d}_{ij}}\{\beta^c_{ij}\}, & \text{for cost criterion } y_j, \end{cases}$$
$$i = 1, 2, \dots, m;\ j = 1, 2, \dots, n, \tag{27}$$

where β^c_{ij} is the complement of $\beta_{ij} = (\mu_{\beta_{ij}}, \nu_{\beta_{ij}})$ such that $\beta^c_{ij} = (\nu_{\beta_{ij}}, \mu_{\beta_{ij}})$.

Step 2. Calculate the separation degree of each component $\tilde{h}_{ij}$ to positive ideal solution and negative ideal solution.

The positive ideal solution (PIS) and negative ideal solution (NIS) can be denoted as $\tilde{h}^+ = \{(1, 0)\}$ and $\tilde{h}^- = \{(0, 1)\}$, respectively, within the generalized hesitant fuzzy environment. The separation between alternatives can be calculated by cross-entropies. For the convenience of both calculation and analysis, only one cross-entropy (21) is selected. The separation degrees, G^+_{ij} and G^-_{ij}, of each $\tilde{h}_{ij}$ $(i = 1, 2, \dots, m; j = 1, 2, \dots, n)$ to the PIS $\tilde{h}^+$ and NIS $\tilde{h}^-$, respectively, are derived from Eq. (21):

$$\begin{aligned} G^+_{ij} &= \mathrm{CE}^*_2(\tilde{h}_{ij}, \tilde{h}^+) = \mathrm{CE}_2(\tilde{h}_{ij}, \tilde{h}^+) + \mathrm{CE}_2(\tilde{h}^+, \tilde{h}_{ij}) \\ &= \sqrt[p]{\frac{1}{l(\tilde{h}_{ij})}\sum_{\alpha_{ij}\in\tilde{h}_{ij}}\left(\mu_{\alpha_{ij}}\log_2\frac{2\mu_{\alpha_{ij}}}{\mu_{\alpha_{ij}}+1}\right)^p} + \sqrt[p]{\frac{1}{l(\tilde{h}_{ij})}\sum_{\alpha_{ij}\in\tilde{h}_{ij}}\nu^p_{\alpha_{ij}}} \\ &\quad + \sqrt[p]{\left(\frac{1}{l(\tilde{h}_{ij})}\sum_{\alpha_{ij}\in\tilde{h}_{ij}}\left(\log_2\frac{2}{1+\mu_{\alpha_{ij}}}\right)\right)^p} \end{aligned} \tag{28}$$

and

$$\begin{aligned} G^-_{ij} &= \mathrm{CE}^*_1(\tilde{h}_{ij}, \tilde{h}^-) = \mathrm{CE}_1(\tilde{h}_{ij}, \tilde{h}^-) + \mathrm{CE}_1(\tilde{h}^-, \tilde{h}_{ij}) \\ &= \sqrt[p]{\frac{1}{l(\tilde{h}_{ij})}\sum_{\alpha_{ij}\in\tilde{h}_{ij}}\mu^p_{\alpha_{ij}}} + \sqrt[p]{\frac{1}{l(\tilde{h}_{ij})}\sum_{\alpha_{ij}\in\tilde{h}_{ij}}\left(\nu_{\alpha_{ij}}\log_2\frac{2\nu_{\alpha_{ij}}}{\nu_{\alpha_{ij}}+1}\right)^p} \\ &\quad + \sqrt[p]{\left(\frac{1}{l(\tilde{h}_{ij})}\sum_{\alpha_{ij}\in\tilde{h}_{ij}}\left(\log_2\frac{2}{1+\nu_{\alpha_{ij}}}\right)\right)^p}. \end{aligned} \tag{29}$$

Step 3. Calculate the closeness degree of the alternatives to the NIS.

The closeness degree $G(x_i)$ of each alternative x_i $(i = 1, 2, \ldots, m)$ to the NIS can be obtained by following:

$$G(x_i) = \sum_{j=1}^{n} w_j G_{ij}, \text{ where } G_{ij} = \frac{G_{ij}^-}{G_{ij}^+ + G_{ij}^-}. \tag{30}$$

Step 4. Rank the alternatives.

The bigger the closeness degree $G(x_i)$, the better the alternative x_i will be, as the alternative x_i is closer to the PIS $\tilde{h}^+$. Therefore, the alternatives x_i $(i = 1, 2, \ldots, m)$ can be ranked in descending order according to the closeness degrees so that the best alternative can be selected.

Approach II.

Step 1. For this step, see Approach I.

Step 2. Calculate the overall aggregated values of each alternative.

Utilize the GHFWA operator (17) (or the GHFWG operator (18)):

$$\begin{aligned}\tilde{h}_i &= \mathrm{GHFWA}_w(\tilde{h}_{i1}, \tilde{h}_{i2}, \ldots, \tilde{h}_{in}) \\ &= \bigcup_{\alpha_{i1}\in\tilde{h}_{i1}, \alpha_{i2}\in\tilde{h}_{i2}, \ldots, \alpha_{in}\in\tilde{h}_{in}} \left\{ \left(1 - \prod_{j=1}^{n} (1-\mu_{\alpha_{ij}})^{w_j}, \prod_{j=1}^{n} \nu_{\alpha_{ij}}^{w_j} \right) \right\} \end{aligned} \tag{31}$$

or

$$\begin{aligned}\tilde{h}_i &= \mathrm{GHFWG}_w(\tilde{h}_{i1}, \tilde{h}_{i2}, \ldots, \tilde{h}_{in}) \\ &= \bigcup_{\alpha_{i1}\in\tilde{h}_{i1}, \alpha_{i2}\in\tilde{h}_{i2}, \ldots, \alpha_{in}\in\tilde{h}_{in}} \left\{ \left(\prod_{j=1}^{n} \mu_{\alpha_{ij}}^{w_j}, 1 - \prod_{j=1}^{n} (1-\nu_{\alpha_{ij}})^{w_j} \right) \right\} \end{aligned} \tag{32}$$

to aggregate all the performance values $\tilde{h}_{ij}$ $(j = 1, 2, \ldots, n)$ of the ith line and get the overall performance value $\tilde{h}_i$ corresponding to the alternatives x_i.

Step 3. Calculate the closeness degree of the alternatives to the PIS.

Utilize the cross-entropy (21) between the overall performance value $\tilde{h}_i$ $(i = 1, 2, \ldots, m)$ and the PIS $\tilde{h}^+ = \{(1, 0)\}$ to get closeness degree of each alternative x_i $(i = 1, 2, \ldots, m)$ to the PIS $\tilde{h}^+$:

$$\begin{aligned} G(x_i) &= \mathrm{CE}_2^*(\tilde{h}_i, \tilde{h}^+) = \mathrm{CE}_2(\tilde{h}_i, \tilde{h}^+) + \mathrm{CE}_2^*(\tilde{h}^+, \tilde{h}_i) \\ &= \sqrt[p]{\frac{1}{l(\tilde{h}_i)} \sum_{\alpha_i\in\tilde{h}_i} \left(\mu_{\alpha_i} \log_2 \frac{2\mu_{\alpha_i}}{\mu_{\alpha_i}+1} \right)^p} + \sqrt[p]{\frac{1}{l(\tilde{h}_i)} \sum_{\alpha_i\in\tilde{h}_i} \nu_{\alpha_i}^p} \\ &\quad + \sqrt[p]{\left(\frac{1}{l(\tilde{h}_i)} \sum_{\alpha_i\in\tilde{h}_i} \left(\log_2 \frac{2}{1+\mu_{\alpha_i}} \right) \right)^p}. \end{aligned} \tag{33}$$

Step 4. Rank the alternatives.

The smaller the closeness degree $G(x_i)$, the better the alternative x_i will be, as the alternative x_i is closer to the PIS $\tilde{h}^+$. Therefore, the alternatives x_i ($i = 1, 2, \ldots, m$) can be ranked in ascending order according to the closeness degrees so that the best alternative can be selected.

5 An illustrative example

In this section, a generalized hesitant fuzzy MCDM problem of selecting an investment is used to illustrate the proposed methods.

A city is planning to build a municipal library. One of the problems facing the city development commissioner is to determine what kind of air-conditioning system should be installed in the library (adapted from [20]). The contractor offers five feasible alternatives x_i ($i = 1, 2, 3, 4, 5$), which might be adapted to the physical structure of the library. Suppose that three criteria y_1 (economic), y_2 (functional), and y_3 (operational) are taken into consideration in the installation problem, and the weight vector of the criteria y_j ($j = 1, 2, 3$) is $w = (0.3, 0.5, 0.2)^T$. Assume that the characteristics of the alternatives x_i ($i = 1, 2, 3, 4, 5$) with respect to the criterion y_j ($j = 1, 2, 3$) are represented by the GHFEs $\tilde{h}_{ij} = \{\alpha_{ij} = (\mu_{\alpha_{ij}}, \nu_{\alpha_{ij}})|\alpha_{ij} \in \tilde{h}_{ij}\}$, where $\mu_{\alpha_{ij}}$ indicates the degree that the alternative x_i satisfies the criterion y_j and $\nu_{\alpha_{ij}}$ indicates the degree that the alternative x_i does not satisfy the criterion y_j, such that $\mu_{\alpha_{ij}}, \nu_{\alpha_{ij}} \in [0, 1]$ and $\mu_{\alpha_{ij}} + \nu_{\alpha_{ij}} \leq 1$. All $\tilde{h}_{ij} = \{\alpha_{ij} = (\mu_{\alpha_{ij}}, \nu_{\alpha_{ij}})|\alpha_{ij} \in \tilde{h}_{ij}\}$ ($i = 1, 2, 3, 4, 5$; $j = 1, 2, 3$) are contained in the generalized hesitant fuzzy decision matrix $E = (\tilde{h}_{ij})_{5\times 3}$ (see Table 2).

Table 2: The generalized hesitant fuzzy decision matrix

	y_1	y_2	y_3
x_1	$\{(0.3, 0.2), (0.3, 0.4)\}$	$\{(0.7, 0.2), (0.5, 0.2)\}$	$\{(0.5, 0.2), (0.6, 0.3)\}$
x_2	$\{(0.5, 0.2), (0.6, 0.2)\}$	$\{(0.3, 0.1), (0.4, 0.2)\}$	$\{(0.7, 0.1), (0.8, 0.1)\}$
x_3	$\{(0.3, 0.4), (0.4, 0.5)\}$	$\{(0.7, 0.2), (0.8, 0.1)\}$	$\{(0.4, 0.3), (0.4, 0.4)\}$
x_4	$\{(0.2, 0.6), (0.2, 0.7)\}$	$\{(0.8, 0.1), (0.7, 0.2)\}$	$\{(0.7, 0.2), (0.8, 0.1)\}$
x_5	$\{(0.8, 0.1), (0.7, 0.2)\}$	$\{(0.6, 0.3), (0.7, 0.2)\}$	$\{(0.2, 0.5), (0.2, 0.6)\}$

To select the best air-conditioning system, we utilize above-mentioned two approaches to find the decision result(s).

Approach I.

Step 1. Considering that all the attributes y_j ($j = 1, 2, 3$) are benefit type attributes, the performance values of the alternatives x_i ($i = 1, 2, 3, 4, 5$) do not need normalization.

Step 2. Utilize Eqs. (28) and (29) (let $p = 2$) to calculate the separation degree G_{ij} of each component $\tilde{h}_{ij}$ ($i = 1, 2, 3, 4, 5$; $j = 1, 2, 3$) to PIS $\tilde{h}^+ =$

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$\{(1,0)\}$ and NIS $\tilde{h}^- = \{(0,1)\}$ and then we get the following results:

$$
\begin{aligned}
&G_{11}^+ = 1.2724, G_{12}^+ = 0.7737, G_{13}^+ = 0.8951, G_{21}^+ = 0.8401, G_{22}^+ = 1.0549,\\
&G_{23}^+ = 0.4620, G_{31}^+ = 1.3496, G_{32}^+ = 0.5201, G_{33}^+ = 1.1911, G_{41}^+ = 1.7059,\\
&G_{42}^+ = 0.5201, G_{43}^+ = 0.5201, G_{51}^+ = 0.5201, G_{52}^+ = 0.7573, G_{53}^+ = 1.6062,\\
&G_{11}^- = 1.2458, G_{12}^- = 1.6622, G_{13}^- = 1.5574, G_{21}^- = 1.6062, G_{22}^- = 1.4369,\\
&G_{23}^- = 1.8601, G_{31}^- = 1.1264, G_{32}^- = 1.8351, G_{33}^- = 1.2969, G_{41}^- = 0.7023,\\
&G_{42}^- = 1.8351, G_{43}^- = 1.8351, G_{51}^- = 1.8351, G_{52}^- = 1.6571, G_{53}^- = 0.8401.
\end{aligned}
$$

Step 3. Utilize the weight vector $w = (0.3, 0.5, 0.2)^T$ of the criteria y_j $(j = 1,2,3)$ and Eq. (30) to calculate the closeness degree $G(x_i)$ of the alternatives x_i $(i = 1,2,3,4,5)$ to the NIS:

$$G(x_1) = 0.6166, G(x_2) = 0.6455, G(x_3) = 0.6303, G(x_4) = 0.6329, G(x_5) = 0.6456.$$

Using this, we rank all alternatives x_i $(i = 1,2,3,4,5)$ in descending order in accordance with the values $G(x_i)$ $(i = 1,2,3,4,5)$:

$$x_5 \succ x_2 \succ x_4 \succ x_3 \succ x_1.$$

Therefore, the best alternative is x_5.

Approach II.
Step 1. For this step, see Approach I.

Step 2. Utilize the GHFWA operator (31) to aggregate all the performance values $\tilde{h}_{ij}$ $(i = 1,2,3,4,5; j = 1,2,3)$ of the ith line and get the overall performance value $\tilde{h}_i$ corresponding to the alternatives x_i $(i = 1,2,3,4,5)$;

$$
\begin{aligned}
\tilde{h}_1 = \{&(0.5716, 0.2000), (0.5903, 0.2169), (0.4469, 0.2000), (0.4710, 0.2169),\\
&(0.5716, 0.2462), (0.5903, 0.2670), (0.4469, 0.2462), (0.4710, 0.2670)\};\\
\tilde{h}_2 = \{&(0.4658, 0.1231), (0.5075, 0.1231), (0.5055, 0.1741), (0.5440, 0.1741),\\
&(0.5004, 0.1231), (0.5393, 0.1231), (0.5375, 0.1741), (0.5735, 0.1741)\};\\
\tilde{h}_3 = \{&(0.5557, 0.2670), (0.5557, 0.2828), (0.6372, 0.1888), (0.6372, 0.2000),\\
&(0.5757, 0.2855), (0.5757, 0.3024), (0.6536, 0.2018), (0.6536, 0.2138)\};\\
\tilde{h}_4 = \{&(0.6712, 0.1966), (0.6969, 0.1711), (0.5974, 0.2781), (0.6287, 0.2421),\\
&(0.6712, 0.2059), (0.6969, 0.1793), (0.5974, 0.2912), (0.6287, 0.2535)\};\\
\tilde{h}_5 = \{&(0.6268, 0.2390), (0.6268, 0.2479), (0.6768, 0.1951), (0.6768, 0.2024),\\
&(0.5785, 0.2942), (0.5785, 0.3051), (0.6350, 0.2402), (0.6350, 0.2491)\}.
\end{aligned}
$$

Step 3. Utilize Eq. (33) to calculate the closeness degree $G(x_i)$ of each alternative x_i $(i = 1,2,3,4,5)$ to the PIS $\tilde{h}^+ = \{(1,0)\}$:

$$G(x_1) = 0.9146, G(x_2) = 0.8291, G(x_3) = 0.8103, G(x_4) = 0.7350, G(x_5) = 0.7799.$$

Using this, we rank all alternatives x_i $(i = 1, 2, 3, 4, 5)$ in ascending order according to the values $G(x_i)$ $(i = 1, 2, 3, 4, 5)$:

$$x_4 \succ x_5 \succ x_3 \succ x_2 \succ x_1.$$

Therefore, the best alternative is x_4.

If we utilize the GHFWG operator (32) in Step 2 of Approach II, then the closeness degree $G(x_i)$ of each alternative x_i $(i = 1, 2, 3, 4, 5)$ to the PIS is calculated:

$$G(x_1) = 0.9835, G(x_2) = 0.9117, G(x_3) = 0.9700, G(x_4) = 1.0543, G(x_5) = 0.9597.$$

and so the ranking of all alternatives x_i $(i = 1, 2, 3, 4, 5)$ in ascending order according to the values $G(x_i)$ $(i = 1, 2, 3, 4, 5)$ is obtained:

$$x_2 \succ x_5 \succ x_3 \succ x_1 \succ x_4.$$

Therefore, the best alternative is x_2.

From the above analysis, we know that the results obtained by the proposed approaches are different. Each of methods has its advantages and disadvantages and none of them can always perform better than the others in any situations. It perfectly depends on how we look at things, and not on how they are themselves. As we can see, in Approach II, depending on aggregation operators used, the ranking of the alternatives is different. Therefore, the results may lead to different decisions.

6 Conclusions

In this paper, we developed cross-entropy under generalized hesitant fuzzy environment. Axiomatic definition about this information measure have been given for GHFEs. Two approaches, based on the developed generalized hesitant fuzzy cross-entropy, of generalized hesitant fuzzy MCDM are developed which permits the decision maker to provide several possible IFVs for an alternative under the given criterion, which is consistent with humans' hesitant thinking. The illustrative example demonstrated the validity and practicability of the developed approaches.

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References

[1] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986), 87-96.

[2] K. Atanassov and G. Gargov, Interval-valued intuitionistic fuzzy sets, Fuzzy Sets and Systems 31 (1989), 343-349.

[3] P. Burillo and H. Bustince, Vague sets are intuitionistic fuzzy sets, Fuzzy Sets and Systems 79 (1996), 403-405.

[4] W.L. Gau and D.J. Bueherer, Vague sets, IEEE Transactions on Systems, Man and Cybernetics 23 (1994), 610-614.

[5] K.C. Hung, K.P. Lin and C.C. Weng, Fault diagnosis of turbine using an improved intuitionistic fuzzy croos entropy approach, in: 2011 International Conference on Fuzzy System, Taipei, Taiwan, 2011, pp. 590-594.

[6] W.L. Hung and M.S. Yang, On the J-divergence og intuitionistic fuzzy sets with its application to pattern recognition, Information Sciences 178 (2008), 1641-1650.

[7] J.Q. Li, G.N. Deng, H.X. Li and W.Y. Zeng, The relationship between similarity measure and entropy of intuitionisticfuzzy sets, Information Sciences 188 (2012), 314-321.

[8] X.C. Liu, Entropy, distance measure and similarity measure of fuzzy sets and their relations, Fuzzy Sets and Systems 52 (1992), 305-318.

[9] J.J. Mao, D.B. Yao and C.C. Wang, A novel cross-entropy and entropy measures of IFSs and their applications, Knowledge-Based Systems 48 (2013), 37-45.

[10] G. Qjan, H. Wang, X.Q. Feng, Generalized hesitant fuzzy sets and their application in decision support system, Knowledge-Based Systems 37 (2013), 357-365.

[11] X.G. Shang and W.S. Jiang, A note on fuzzy information measure, Pattern Recognition Letter 18 (1997), 425-432.

[12] V. Torra, Hesitant fuzzy sets, International Journal of Intelligent Systems 25 (2010), 529-539.

[13] V. Torra, Y. Narukawa, On hesitant fuzzy sets and decision, in: The 18th IEEE International Conference on Fuzzy Systems, Jeju Island, Korea, 2009. pp. 1378-1382.

[14] I.K. Vlachos and Sergiadis, Intuitionistic fuzzy information application to pattern recognition, Pattern Recognition Letter 28 (2007), 197-206.

[15] J.Q. Wang and J.J. Li, Multi-criteria fuzzy decision making method based on cross entropy and score functions, Expert Systems with Applications 38 (2011), 1032-1038.

[16] M.M. Xia, Z.S. Xu, Hesitant fuzzy information aggregation in decision making, International Journal of Approximate Reasoning 52 (2011), 395-407.

[17] M.M. Xia and Z.S. Xu, Entropy/cross entropy based group decision making under intuitionistic fuzzy environment, Information Fusion 13 (2012), 31-47.

[18] Z.S. Xu, Intuitionistic fuzzy aggregation operators, IEEE Transactions on Fuzzy Systems 15 (2007), 1179-1187.

[19] Z.S. Xu and M.M. Xia, Hesitant fuzzy entropy and cross-entropy and their use in multiattribute decision-making, International Journal of IntelligentSystems 27 (2012), 799-822.

[20] Z.S. Xu and R.R. Yager, Intuitionistic fuzzy Bonferroni means, IEEE Transactions on Systems, Man and Cybernetics 41 (2011), 568-578.

[21] J. Ye, Fault diagnosis of turbine based on fuzzy cross entropy of vague sets, Expert Systems with Applications 36 (2009), 3103-3106.

[22] J. Ye, Multicriteria fuzzy decision-making method based on intuitionistic fuzzy cross-entropy, in: Proceedings of the 2009 International Conference on Intelligent Human-Machine and Cybernetics, Hangzhou, China, 2009, pp. 59-61.

[23] J. Ye, Fuzzy cross entropy of interval-valued intuitionistic fuzzy sets and its optimal decision-making method based on the weights of alternatives, Expert Systems with Applications 38 (2011), 6179-6183.

[24] L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965), 338-353.

[25] Q.S. Zeng and H.X. Li, Relationship between simialrity measure and entropy of interval-valued fuzzy sets, Fuzzy Sets and Systems 157 (2006), 1477-1484.

[26] Q.S. Zhang and S.Y. Jiang, A note on information entropy measure for vague sets and its application, Information Sciences 178 (2008), 4184-4191.

[27] Q.S. Zhang and S.Y. Jiang, Relationships between entropy and simialrity measure of interval-valued intuitionistic fuzzy sets, International Journal of Intelligent Systems 25 (2010), 1121-1140.

[28] H.M. Zhang and L.Y. Yu, MADM method based on cross-entropy and extended TOPSIS with interval-valued intuitionistic fuzzy sets, Knowledge-Based Systems 30 (2012), 115-120.

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ON HARMONIC QUASICONFORMAL MAPPINGS WITH FINITE AREA

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ABSTRACT. In this paper, we study the class of harmonic K-quasiconformal mappings of the unit disk $\mathbf{U}$ with finite Euclidean areas $|f(\mathbf{U})|_{euc}$. We first give the Schwarz-pick lemma (cf. [8]) for this class of mappings as follows:

$$|f_z(z)| \leq \sqrt{\frac{|f(\mathbf{U})|_{euc}}{\pi(1-k^2)}}\frac{1}{1-|z|}, \quad z \in \mathbf{U},$$

where $k = \frac{K-1}{K+1}$. Furthermore, we obtain the sharp coefficient estimates of this class of mappings. As an application, for harmonic mappings $f \in S_H^0$ with finite $|f(\mathbf{U})|_{euc}$ we obtain sharp coefficient estimates.

1. INTRODUCTION

Let $\mathbf{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$. The classical Schwarz lemma says that if an analytic function φ of $\mathbf{U}$ satisfies that $|\varphi(z)| < 1$ and $\varphi(0) = 0$. Then $|\varphi(z)| \leq |z|$ and $|\varphi'(0)| \leq 1$. The equality occurs if and only if $\varphi(z) = e^{i\alpha}z$, where α is a real constant. The classical Schwarz lemma is a cornerstone in complex analysis and attracts one to give various versions of its generalization.

A complex-valued function $f(z)$ of class C^2 is said to be a harmonic mapping if it satisfies $f_{z\bar{z}} = 0$. It is known that every harmonic mapping $f(z)$ defined in $\mathbf{U}$ admits a canonical decomposition $f(z) = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic in $\mathbf{U}$ with $g(0) = 0$. One can refer to [5] and the references therein for more details about harmonic mappings.

For $z \in \mathbf{U}$, let

$$\Lambda_f(z) = \max_{0\leq\theta\leq 2\pi} |f_z(z) + e^{-2i\theta} f_{\bar{z}}(z)| = |f_z(z)| + |f_{\bar{z}}(z)|, \tag{1}$$

and

$$\lambda_f(z) = \min_{0\leq\theta\leq 2\pi} |f_z(z) + e^{-2i\theta} f_{\bar{z}}(z)| = ||f_z(z)| - |f_{\bar{z}}(z)||. \tag{2}$$

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The Lewy theorem [7] tells us that a harmonic mapping f is locally univalent and sense-preserving in $\mathbf{U}$ if and only if its Jacobian satisfies the following condition

$$J_f(z) = \lambda_f(z)\Lambda_f(z) = |f_z(z)|^2 - |f_{\bar{z}}(z)|^2 > 0 \quad \text{for} \quad z \in \mathbf{U}.$$

Suppose that $f(z)$ is a sense-preserving univalent harmonic mapping of $\mathbf{U}$. Then $f(z)$ is a K-quasiconformal mapping if and only if

$$K(f) := \sup_{z\in\mathbf{U}} \frac{|f_z(z)| + |f_{\bar{z}}(z)|}{|f_z(z)| - |f_{\bar{z}}(z)|} \leq K.$$

Harmonic quasiconformal mappings are natural generalizations of conformal mappings. Recently, many mathematicians have studied such an active topic and obtained many interesting results (cf.[1], [6], [10], [8], [12], [13], [14], [15]).

In 2007, M.Knežević and M.Mateljević [8] proved the following Schwarz-Pick lemma for harmonic quasiconformal mappings.

Theorem A. *Let f be a harmonic K-quasiconformal mapping of $\mathbf{U}$ into itself. Then*

$$|f_z(z)| \leq \frac{(K+1)(1-|f(z)|^2)}{2(1-|z|^2)}$$

holds for all $z \in \mathbf{U}$, and

$$d_\lambda(f(z_1), f(z_2)) \leq K d_\lambda(z_1, z_2)$$

holds for any $z_1, z_2 \in \mathbf{U}$, where d_λ is the hyperbolic distance.

Furthermore, they obtained the opposite inequalities in Theorem A as $|f_z(z)| \geq \frac{(K+1)(1-|f(z)|^2)}{2K(1-|z|^2)}$ and $d_\lambda(f(z_1), f(z_2)) \geq \frac{1}{K} d_\lambda(z_1, z_2)$ by assuming that f is onto. Such an assumption is necessary since that $|f_z(z)|$ will bounded below by a positive constant (see [8] for more details).

In 2010, X. Chen and A. Fang [2] improved the above results as follows.

Theorem B. *Let $\Omega \subset \mathbb{C}$ be a simply connected convex domain of hyperbolic type and λ_Ω be its hyperbolic metric density with the Gaussian curvature -4. If f is a harmonic K-quasiconformal mapping of $\mathbf{U}$ onto Ω, then the inequalities*

$$\frac{(K+1)\lambda_{\mathbf{U}}(z)}{2K\lambda_\Omega(f(z))} \leq |f_z(z)| \leq \frac{(K+1)\lambda_{\mathbf{U}}(z)}{2\lambda_\Omega(f(z))}$$

hold for all $z \in \mathbf{U}$. Moreover, the above estimates are sharp.

We point out that the composition of a harmonic mapping f with a conformal mapping φ is still harmonic. Hence we can fix the defined domain as $\mathbf{U}$ for harmonic mappings. However, $\varphi \circ f$ is not harmonic in general. This implies that the Schwarz-Pick lemma for harmonic quasiconformal mappings will closely relate to its target domain. Instead of the assumption that harmonic quasiconformal mappings have convex or bounded ranges, we study the class of harmonic mappings with finite Euclidean areas. Example 1 shows there exists a harmonic mapping with an unbounded range but finite Euclidean area.

Assume that $f(z) = h(z) + \overline{g(z)}$ is a harmonic K-quasiconformal mapping of $\mathbf{U}$ with a finite Euclidean area $|f(\mathbf{U})|_{euc}$, where

$$h(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n$$

are analytic in $\mathbf{U}$. Under the assumption of finite Euclidean areas, we obtain a new version of the Schwarz-Pick lemma for the class of harmonic K-quasiconformal mappings as follows

$$|f_z(z)| \le \sqrt{\frac{|f(\mathbf{U})|_{euc}}{\pi(1-k^2)}} \frac{1}{1-|z|}, \quad z \in \mathbf{U}, \tag{3}$$

where $k = \frac{K-1}{K+1}$. See Theorem 1 for details.

Furthermore, we obtain the sharp coefficient estimates for $f(z)$

$$|a_n|^2 + |b_n|^2 \le \frac{(K+1/K)|f(\mathbf{U})|_{euc}}{2n\pi} \quad (n = 1, 2, \ldots). \tag{4}$$

This result is given at Theorem 2.

Denote by S_H the family of all sense-preserving univalent harmonic mappings defined in $\mathbf{U}$ which admit a canonical representation $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{5}$$

are analytic in $\mathbf{U}$. The class S_H^0 is the subclass of S_H with $g'(0) = 0$.

A well-known result of the classical analytic univalent functions is the Bieberbach conjecture which was posed by Ludwig Bieberbach in 1916 and was finally proven by Louis de Branges [4]. This result has many important geometric applications. In 1984, T.Sheil-Small [3] published a landmark paper which pointed out that many classical results of conformal mappings have analogues of harmonic mappings. One of the famous results is the coefficients conjecture of S_H^0. As an application of Theorem 2, we obtain the coefficients estimate for $f \in S_H^0$ which is given by (9).

2. Main results and their proofs

Theorem 1. *Let $K \ge 1$ be a constant. If $f(z) = h(z) + \overline{g(z)}$ is a harmonic K-quasiconformal mapping of the unit disk $\mathbf{U}$ such that its Euclidean area $|f(\mathbf{U})|_{euc}$ is finite, then*

$$|f_z(z)| \le \sqrt{\frac{|f(\mathbf{U})|_{euc}}{\pi(1-k^2)}} \frac{1}{1-|z|}, \quad z \in \mathbf{U},$$

where $k = \frac{K-1}{K+1}$.

Proof. Since $f(z)$ is a harmonic K-quasiconformal mapping, we obtain that

$$\sup_{z\in\mathbf{U}} \left| \frac{g'(z)}{h'(z)} \right| \le k = \frac{K-1}{K+1}.$$

Then

$$\begin{aligned} |f(\mathbf{U})|_{euc} &= \int\int_{\mathbf{U}} (|h'(z)|^2 - |g'(z)|^2)\, d\sigma \\ &= \int\int_{\mathbf{U}} |h'(z)|^2 \left(1 - \frac{|g'(z)|^2}{|h'(z)|^2}\right) d\sigma \\ &\geq (1-k^2)\int\int_{\mathbf{U}} |h'(z)|^2\, d\sigma. \end{aligned}$$

This implies that

$$\int\int_{\mathbf{U}} |h'(z)|^2\, d\sigma \leq \frac{|f(\mathbf{U})|_{euc}}{1-k^2}. \tag{6}$$

By [11, Corollary 2.6.4 and Theorem 2.4.1], we obtain $|h'(z)|^2$ is subharmonic in $\mathbf{U}$. Thus, for $r \in [0, 1-|z|)$, it follows

$$|h'(z)|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} |h'(z+re^{i\theta})|^2\, d\theta. \tag{7}$$

Utilizing the inequality (6), we get

$$\begin{aligned} \pi(1-|z|)^2|h'(z)|^2 &\leq \int_0^{2\pi}\int_0^{1-|z|} r|h'(z+re^{i\theta})|^2\, dr\, d\theta \\ &= \int\int_{D(z)} |h'(\zeta)|^2\, d\sigma \\ &\leq \int\int_{\mathbf{U}} |h'(\zeta)|^2\, d\sigma \leq \frac{|f(\mathbf{U})|_{euc}}{1-k^2}, \end{aligned}$$

where $D(z) := \{\zeta \in \mathbb{C}, |\zeta - z| < 1-|z|\} \subseteq \mathbf{U}$. Then $|h'(z)|^2 \leq \frac{|f(\mathbf{U})|_{euc}}{\pi(1-|z|)^2(1-k^2)}$.

This completes the proof. □

Remark 1. *The Euclidean area of $f(\mathbf{U})$ is finite doesn't imply that f is bounded. The following Example* 1 *shows that Theorem* 1 *is not covered by Theorem A and Theorem B. Furthermore, let $f(z) = e^{i\alpha}z$ be a conformal mapping of $\mathbf{U}$ onto itself, where α is a real constant. Then $|f_z(z)| = 1$ and $|f(\mathbf{U})|_{euc} = \pi$. This shows that* (3) *is sharp at $z = 0$.*

Example 1. *Let $\Omega_1 = \{\zeta \in \mathbb{C} : 0 \leq \operatorname{Im}\zeta \leq 1, \operatorname{Re}\zeta \geq 0\}$ and $\varphi_1(z) : \mathbf{U} \mapsto \Omega_1$ be a conformal mapping. Let $\varphi_2(\zeta) := \frac{1}{2i}\ln\frac{1+\zeta i}{1-\zeta i}$ be a conformal mapping of Ω_1 onto Ω_2. Then $w(z) = \varphi_2 \circ \varphi_1(z)$ is a conformal mapping of $\mathbf{U}$ onto Ω_2. Here Ω_2 is an unbounded (and not convex) domain with the boundary curves $\{c_1 : \operatorname{Im} w = 0, 0 \leq \operatorname{Re} w \leq \frac{\pi}{2}\}$, $\{c_2 : \operatorname{Re} w = 0, \operatorname{Im} w \in \mathbb{R}\}$ and $\{c_3 : w(t+i) = \frac{1}{2i}\ln\frac{ti}{2-ti},\ t \in [0,\infty)\}$ which is shown by figure* 1. *Then $|w(\mathbf{U})|_{euc} = \frac{\pi \ln 2}{8}$ is finite. This shows that Theorem* 1 *is not covered by Theorem A and Theorem B.*

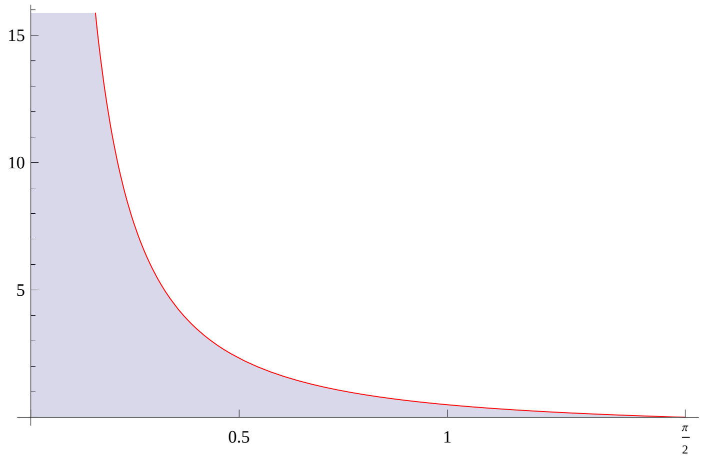

FIGURE 1. Image of the domain Ω_2

Theorem 2. *Let $K \geq 1$ be a constant. If $f(z) = h(z) + \overline{g(z)}$ is a harmonic K-quasiconformal mapping of $\mathbf{U}$ such that $|f(\mathbf{U})|_{euc}$ is finite, where*

$$h(z) = \sum_{n=0}^{\infty} a_n z^n \quad and \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{8}$$

are analytic in $\mathbf{U}$. Then

$$|a_n|^2 + |b_n|^2 \leq \frac{(K + 1/K)|f(\mathbf{U})|_{euc}}{2n\pi} \quad (n = 1, 2, \ldots).$$

The above coefficient estimates are sharp for all $n = 1, 2, \ldots,$ with the extremal functions $f_n(z) = \frac{a}{\sqrt{n}} z^n + \frac{ka}{\sqrt{n}} \overline{z^n}$ where $k = \frac{K-1}{K+1}$ and $a \in \mathbb{C}$ is a constant.

Proof. For every $z = re^{i\theta} \in \mathbf{U}$,

$$f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta} + \sum_{n=1}^{\infty} \overline{b_n} r^n e^{-in\theta}.$$

Hence $h'(re^{i\theta}) = \sum\limits_{n=1}^{\infty} n a_n r^{n-1} e^{i(n-1)\theta}$ and $g'(re^{i\theta}) = \sum\limits_{n=1}^{\infty} n b_n r^{n-1} e^{i(n-1)\theta}$. Applying the Parseval identity, we obtain

$$\int\int_{\mathbf{U}} (|h'(z)|^2 + |g'(z)|^2) d\sigma = \pi \sum_{n=1}^{\infty} n(|a_n|^2 + |b_n|^2).$$

Since $f(z)$ is a K-quasiconformal mapping, we have

$$|h'(z)|^2 + |g'(z)|^2 \leq \frac{1}{2}(K + 1/K) \left(|h'(z)|^2 - |g'(z)|^2\right).$$

This implies that

$$\begin{aligned}\pi\sum_{n=1}^{\infty}n(|a_n|^2+|b_n|^2) &\leq \int\int_{\mathbf{U}}\frac{1}{2}(K+1/K)(|h'(z)|^2-|g'(z)|^2)d\sigma\\ &= \frac{1}{2}(K+1/K)|f(\mathbf{U})|_{euc}.\end{aligned}$$

Then

$$|a_n|^2+|b_n|^2\leq\frac{(K+1/K)|f(\mathbf{U})|_{euc}}{2n\pi}\quad(n=1,2,\ldots).$$

Let $f_n(z)=\frac{a}{\sqrt{n}}z^n+\frac{ka}{\sqrt{n}}\overline{z^n}$, then $|f_n(z)|_{euc}=\int\int_{\mathbf{U}}J_{f_n}(z)d\sigma=(1-k^2)\pi|a|^2$. This shows that $|a_n|^2+|b_n|^2=\frac{(1+k^2)|a|^2}{n}=\frac{(K+1/K)|f(\mathbf{U})|_{euc}}{2n\pi}$. Hence, the estimates are sharp.

This completes the proof. $\square$

Theorem 3. *If $f=h+\bar{g}\in S_H^0$ satisfies that $|f(\mathbf{U})|_{euc}$ is finite, where h, g are given by* (5) *with $b_1=0$. Then*

$$|a_n|^2+|b_n|^2\leq s(n,t_0),\quad(n=2,3,\ldots),$$

where $s(n,t_0)$ is given by (10).

Proof. Let $F(\zeta):=\frac{f(t\zeta)}{t}$, where $f\in S_H^0$, $\zeta\in\mathbf{U}$ and $0<t<1$. Then $\Lambda_F(\zeta)=\Lambda_f(t\zeta)$ holds for all $\zeta\in\mathbf{U}$. According to (5) we see that

$$\begin{aligned}F(\zeta) &= \zeta+\sum_{n=2}^{\infty}a_nt^{n-1}\zeta^n+\sum_{n=2}^{\infty}\overline{b_n}t^{n-1}\overline{\zeta^n}\\ &= \zeta+\sum_{n=2}^{\infty}A_n\zeta^n+\sum_{n=2}^{\infty}\overline{B_n\zeta^n},\end{aligned}$$

where $A_n=t^{n-1}a_n$ and $B_n=t^{n-1}b_n$. Let $\omega(z)=\frac{\overline{f_{\bar{z}}(z)}}{f_z(z)}$. Then $\omega(z)$ is holomorphic in $\mathbf{U}$ satisfying $\omega(0)=0$ and $|\omega(z)|<1$. By the Schwarz lemma we know that $|\omega(z)|\leq|z|$ for $z\in\mathbf{U}$. Therefore, for any $0<t<1$ and $\mathbf{U}_t:=\{z:|z|<t\}$ we have

$$\frac{\Lambda_f(z)}{\lambda_f(z)}=\frac{1+|\omega(z)|}{1-|\omega(z)|}\leq\frac{1+t}{1-t}:=K_t.$$

This implies that F is a K_t-quasiconformal mapping of $\mathbf{U}$. Furthermore,

$$|F(\zeta)|_{euc}=\int\int_{\mathbf{U}_t}J_f(\zeta)d\sigma\leq|f(\mathbf{U})|_{euc}.$$

Applying Theorem 2, we have

$$|A_n|^2+|B_n|^2\leq\frac{(1+t^2)|f(\mathbf{U})|_{euc}}{n\pi(1-t^2)},\quad(n=2,3,\ldots).$$

Hence,

$$|a_n|^2+|b_n|^2 \le \frac{(1+t^2)|f(\mathbf{U})|_{euc}}{n\pi t^{2n-2}(1-t^2)} := s(n,t), \quad (n=2,3,\ldots). \tag{9}$$

Since $\lim_{t\to 0} s(n,t) = \infty = \lim_{t\to 1} s(n,t)$, we see that $\min_{0<t<1} s(n,t)$ exists. Choose the minimal point $t_0 = \sqrt{\frac{n-1}{\sqrt{(n-1)^2+1}+1}}$, then

$$\begin{aligned} |a_n|^2+|b_n|^2 &\le s(n,t_0) \\ &= \left(\frac{1+\sqrt{(n-1)^2+1}}{n-1}\right)^{n-1}\left(\frac{\sqrt{(n-1)^2+1}+n}{\sqrt{(n-1)^2+1}+2-n}\right)\frac{|f(\mathbf{U})|_{euc}}{n\pi}. \end{aligned} \tag{10}$$

The proof is completed. □

Remark 2. *By direct calculation, we see that $s(n,t_0)$ is an increasing function of n and $\lim_{n\to\infty} s(n,t_0) = \frac{2e|f(\mathbf{U})|_{euc}}{\pi}$. This implies that $s(n,t_0) \le \frac{2e|f(\mathbf{U})|_{euc}}{\pi}$. Therefore,*

$$|a_n|+|b_n| \le \sqrt{2(|a_n|^2+|b_n|^2)} \le 2\sqrt{\frac{e|f(\mathbf{U})|_{euc}}{\pi}}.$$

References

1. Sh.Chen, S.Ponnusamy and X.Wang, Landau's Theorem and marden constant for harmonic v-Bloch mappings, *Bull. Aust. Math. Soc.* **84**(2011), 19–22.
2. X.D. Chen and A.N. Fang, A Schwarz-Pick inequality for harmonic quasiconformal mappings and its applications, *J. Math. Anal. Appl.* **369**(2010), 22–28.
3. J.G.Clunie and T.Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. A. I.* **9**(1984), 3–25.
4. L. de Branges, A proof of the Bieberbach conjecture, *Acta Math.* **154**(1985), no. 1-2, 137-152.
5. P.Duren, Harmonic mappings in the plane, *Cambridge University Press*, New York, 2004.
6. D.Kalaj, Quasiconformal and harmonic mappings between Jordan domains, *Math. Z.*, **(11)**(2008), 237–252.
7. H. Lewy, On the non-vanishing of the Jacobian in certain one to one mapings, *Bull. Amer. Math. Soc.* **42**(1936) 689-692.
8. M.Knežević and M.Mateljević, On the quasi-isometries of harmonic quasiconformal mappings, *J. Math. Anal. Appl.* **334**(2007), 404–413.
9. D.Partyka and K.Sakan, On bi-Lipschitz type inequalitites for quasiconformal harmonic mappings, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **32**(2007), 579–594.
10. M.Pavlović, Boundary correspondence under harmonic quasiconformal homeomorphism of the unit disk, *Ann. Acad. Sci. Fenn. (Series A1):Math.* **27**(2)(2002), 365–372.
11. T. Ransford, Potential Theory in the Complex Plane, *London Mathematical Society Student Texts 28, Cambridge University Press,* 1995.
12. J.F. Zhu and X.M. Zeng, Estimate for Heinz inequality in the small dilatation of harmonic quasiconformal mappings, *J. Compu. Analy. Appl.* **13**(2011), 1081-1087.
13. J.F. Zhu, Some estimates for harmonic mappings with given boundary function, *J. Math. Anal. Appl.* **411**(2014), 631–638.
14. J.F. Zhu, Landau theorem for planar harmonic mappings, *Complex Anal. Oper. Theory,* March, 2015, DOI: 10.1007/s11785-015-0449-8.
15. J.F. Zhu, Coefficients estimate for harmonic v- Bloch mappings and harmonic K- quasiconformal mappings, *Bull. Malaysian Math. Sci. Soc.* July, 2015, DOI: 10.1007/s40840-015-0175-4.

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Weak Estimates of the Multidimensional Finite Element and Their Applications

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In this article we first introduce interpolation operator of projection type in multidimensional spaces. Then we derive weak estimates for tensor-product block finite elements. Finally, the applications of the weak estimates in superconvergent properties are discussed.

1 Introduction

Superconvergence of the finite element approximation for second order elliptic boundary value problems has been an active research topic (see [1–7]). It is well known that the weak estimates for the finite element and the estimates for the discrete Green's function play important roles in the superconvergence study (see [8–15]). In this article we focus on the study of the weak estimates and we will derive the weak estimates for the multidimensional finite element.

we shall use the symbol C to denote a generic constant, which is independent from the discretization parameter h and which may not be the same in each occurrence and also use the standard notations for the Sobolev spaces and their norms.

We consider the following Poisson equation:

$$\mathcal{L}u \equiv -\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathcal{R}^d$ $(d \geq 2)$ is a bounded polytopic domain. The weak formulation of (1.1) reads,

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ satisfying} \\ a(u\,,\,v) = (f\,,\,v) \text{ for all } v \in H_0^1(\Omega). \end{cases}$$

where

$$a(u\,,\,v) \equiv \int_\Omega \nabla u \cdot \nabla v \, dX,$$

and

$$(f\,,\,v) \equiv \int_\Omega f v \, dX.$$

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Let $\{\mathcal{T}^h\}$ be a regular family of rectangular partitions of $\bar{\Omega}$. Denote by $S^h(\Omega)$ a continuous piecewise tensor-product m-degree polynomials space regarding this kind of partitions and let $S_0^h(\Omega) = S^h(\Omega) \cap H_0^1(\Omega)$. Discretizing the above weak formulation using $S_0^h(\Omega)$ as approximating space means,

$$\begin{cases} \text{Find } u_h \in S_0^h(\Omega) \text{ satisfying} \\ a(u_h\,,\, v) = (f\,,\, v) \text{ for all } v \in S_0^h(\Omega). \end{cases}$$

Thus, the following Galerkin orthogonality relation holds.

$$a(u - u_h\,,\, v) = 0 \ \ \forall\, v \in S_0^h(\Omega). \tag{1.2}$$

2 Weak Estimates for the Finite Element

In this section, we first introduce an interpolation operator of projection type in multidimensional spaces, and then derive the weak estimates for the finite element by using the interpolation operator of projection type.

Let element

$$\begin{aligned} e &= (x_{1,e} - h_{1,e}, x_{1,e} + h_{1,e}) \times (x_{2,e} - h_{2,e}, x_{2,e} + h_{2,e}) \\ &\quad \times \cdots \times (x_{d,e} - h_{d,e}, x_{d,e} + h_{d,e}) \\ &\equiv I_1 \times I_2 \times \cdots \times I_d, \end{aligned} \tag{2.1}$$

and let $\{l_{1,j}(x_1)\}_{j=0}^{\infty}$, $\{l_{2,j}(x_2)\}_{j=0}^{\infty}$,$\cdots$, $\{l_{d,j}(x_d)\}_{j=0}^{\infty}$ be the normalized orthogonal Legendre polynomial systems on $L^2(I_1)$, $L^2(I_2)$,$\cdots$, $L^2(I_d)$, respectively. Now let $\partial_{x_1}\partial_{x_2}\cdots\partial_{x_d} u \in L^2(e)$. Then we have the following expansion:

$$\partial_{x_1}\partial_{x_2}\cdots\partial_{x_d} u = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_d=0}^{\infty} \alpha_{i_1 i_2 \cdots i_d} l_{1,i_1}(x_1) l_{2,i_2}(x_2)\cdots l_{d,i_d}(x_d), \tag{2.2}$$

where

$$\alpha_{i_1 i_2 \cdots i_d} = \int_e \partial_{x_1}\partial_{x_2}\cdots\partial_{x_d} u\, l_{1,i_1}(x_1) l_{2,i_2}(x_2)\cdots l_{d,i_d}(x_d)\, dX. \tag{2.3}$$

Set

$$\omega_{k,0}(x_k) = 1,\ \omega_{k,j+1}(x_k) = \int_{x_{k,e}-h_{k,e}}^{x_k} l_{k,j}(\xi)\, d\xi,\ k = 1, \cdots, d,\ j \geq 0.$$

By the Parseval equality, we have for $X = (x_1, x_2, \cdots, x_d) \in e$

$$u(X) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_d=0}^{\infty} \beta_{i_1 i_2 \cdots i_d} \omega_{1,i_1}(x_1)\omega_{2,i_2}(x_2)\cdots\omega_{d,i_d}(x_d), \tag{2.4}$$

where

$$\beta_{00\cdots 0} = u(x_{1,e} - h_{1,e}, x_{2,e} - h_{2,e}, \cdots, x_{d,e} - h_{d,e}),$$

$$\beta_{i_1 00\cdots 0} = \int_{I_1} \partial_{x_1} u(x_1, x_{2,e} - h_{2,e}, \cdots, x_{d,e} - h_{d,e}) l_{1,i_1-1}(x_1)\, dx_1,$$

$$\begin{aligned}\beta_{i_1 i_2 0\cdots 0} &= \int_{I_1\times I_2} \partial_{x_1}\partial_{x_2} u(x_1, x_2, x_{3,e} - h_{3,e}, \cdots, x_{d,e} - h_{d,e}) \\ &\quad l_{1,i_1-1}(x_1) l_{2,i_2-1}(x_2)\, dx_1 dx_2,\end{aligned}$$

$$\begin{aligned}\beta_{i_1 i_2\cdots i_d} &= \int_e \partial_{x_1}\partial_{x_2}\cdots\partial_{x_d} u(X) \\ &\quad l_{1,i_1-1}(x_1) l_{2,i_2-1}(x_2)\cdots l_{d,i_d-1}(x_d)\, dX,\end{aligned}$$

where $i_k \geq 1,\ k = 1, \cdots, d$. Similarly, the other coefficients can also be given.

We introduce a standard tensor-product polynomial spaces of degree $m \geq 1$ denoted by T_m, i.e.,

$$q(X) = \sum_{(i_1, i_2, \cdots, i_d)\in I} a_{i_1 i_2\cdots i_d} x_1^{i_1} x_2^{i_2} \cdots x_d^{i_d},\ q \in T_m,$$

where the indexing set I is as follows:

$$I = \{(i_1, i_2, \cdots, i_d) | 0 \leq i_1, i_2, \cdots, i_d \leq m\}.$$

Define the tensor-product interpolation operator of projection type by Π_m^e: $H^d(e) \to T_m(e)$ such that

$$\Pi_m^e u(X) = \sum_{(i_1, i_2, \cdots, i_d)\in I} \beta_{i_1 i_2\cdots i_d} \omega_{1,i_1}(x_1)\omega_{2,i_2}(x_2)\cdots\omega_{d,i_d}(x_d). \tag{2.5}$$

By the definitions of the finite element space $S_0^h(\Omega)$ and Π_m^e, we have the interpolation operator of project type

$$\Pi_m : H^d(\Omega) \cap H_0^1(\Omega) \to S_0^h(\Omega),$$

where $(\Pi_m u)|_e = \Pi_m^e u$.

In addition, the function $\omega_{k,i}(x_k)$ has the following properties (see [5]):

$$\begin{array}{ll} a. & \omega_{k,i}(x_{k,e} \pm h_{k,e}) = 0,\ i \geq 2, \\ b. & \omega_{k,i}(x_{k,e} - (x_k - x_{k,e})) = (-1)^i \omega_{k,i}(x_{k,e} + (x_k - x_{k,e})),\ i \geq 2, \\ c. & (\omega_{k,i}, p_m) = 0,\ \forall\, p_m \in P_m(I_k),\ i \geq m+3, \\ d. & (\omega_{k,i}, \omega_{k,j}) = 0,\ i, j \geq 2,\ i \neq j,\ \text{and } |i-j| \neq 2, \end{array} \tag{2.6}$$

where $k = 1, \cdots, d$. For simplicity, we write

$$\lambda_{i_1 i_2\cdots i_d} = \beta_{i_1 i_2\cdots i_d} \omega_{1,i_1}(x_1)\omega_{2,i_2}(x_2)\cdots\omega_{d,i_d}(x_d).$$

From (2.4) and (2.5),

$$\begin{aligned} R &= u - \Pi_m^e u \\ &= \Big(\sum_{i_1=0}^{m}\sum_{i_2=0}^{m}\cdots\sum_{i_{d-1}=0}^{m}\sum_{i_d=m+1}^{\infty} \\ &\quad + \sum_{i_1=0}^{m}\sum_{i_2=0}^{m}\cdots\sum_{i_{d-1}=m+1}^{\infty}\sum_{i_d=0}^{\infty} \\ &\quad + \cdots + \sum_{i_1=0}^{m}\sum_{i_2=m+1}^{\infty}\sum_{i_3=0}^{\infty}\cdots\sum_{i_d=0}^{\infty} \\ &\quad + \sum_{i_1=m+1}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_{d-1}=0}^{\infty}\sum_{i_d=0}^{\infty}\Big)\lambda_{i_1 i_2\cdots i_d}, \end{aligned} \tag{2.7}$$

which is called remainder of interpolation. Next, we will derive the weak estimates for the finite element.

Theorem 2.1 *Let $\{\mathcal{T}^h\}$ be a regular family of rectangular partitions of $\bar{\Omega}$, $u \in W^{m+2,\infty}(\Omega) \cap H_0^1(\Omega)$, and $v \in S_0^h(\Omega)$. Then, the m-degree interpolation operator of projection type Π_m satisfies the following weak estimates:*

$$|a(u - \Pi_m u, v)| \le Ch^{m+1}\|u\|_{m+2,\,\infty,\,\Omega}|v|_{1,\,1,\,\Omega},\ \ m \ge 1, \tag{2.8}$$

and

$$|a(u - \Pi_m u, v)| \le Ch^{m+2}\|u\|_{m+2,\,\infty,\,\Omega}|v|^h_{2,\,1,\,\Omega},\ \ m \ge 2. \tag{2.9}$$

where $|v|^h_{2,\,1,\,\Omega} = \sum_{e\in\mathcal{T}^h}|v|_{2,\,1,\,e}$.

Proof. By the properties of $\omega_{k,i}(x_k)$ (see (2.6), c) as well as the orthogonality of the Legendre polynomial system, we have

$$\int_e \nabla R\cdot\nabla v\,dX = \int_e \nabla r\cdot\nabla v\,dX \equiv I_e\ \ \forall e \in \mathcal{T}^h,$$

where

$$\begin{aligned} r &= \Big(\sum_{i_1=0}^{m}\sum_{i_2=0}^{m}\cdots\sum_{i_{d-1}=0}^{m}\sum_{i_d=m+1}^{m+2} \\ &\quad + \sum_{i_1=0}^{m}\sum_{i_2=0}^{m}\cdots\sum_{i_{d-1}=m+1}^{m+2}\sum_{i_d=0}^{m+2} \\ &\quad + \cdots + \sum_{i_1=0}^{m}\sum_{i_2=m+1}^{m+2}\sum_{i_3=0}^{m+2}\cdots\sum_{i_d=0}^{m+2} \\ &\quad + \sum_{i_1=m+1}^{m+2}\sum_{i_2=0}^{m+2}\cdots\sum_{i_{d-1}=0}^{m+2}\sum_{i_d=0}^{m+2}\Big)\lambda_{i_1 i_2\cdots i_d}. \end{aligned} \tag{2.10}$$

Obviously, r only contains finite terms.

Among the indices i_k, $k = 1, 2, \cdots, d$, when some $i_k = m+1$ or $m+2$, and the others are zero, we have by the orthogonality of the Legendre polynomial system

$$\int_e \nabla\lambda_{i_1 i_2\cdots i_d}\cdot\nabla v\,dX = 0. \tag{2.11}$$

When only two of the indices i_k, $k = 1, 2, \cdots, d$ are nonzero, and the others are zero, without loss of generality, we assume $i_1 \ne 0$, $i_2 \ne 0$, and $i_3 = i_4 = \cdots = i_d = 0$. It is easy to see that $i_1 + i_2 \ge m+2$. Thus, the integration by parts yields

$$\beta_{i_1 i_2 0\cdots 0} = \int_{I_1\times I_2}\partial_{x_1}\partial_{x_2}u(x_1, x_2, x_{3,e} - h_{3,e}, \cdots, x_{d,e} - h_{d,e})$$

$$
\begin{aligned}
& l_{1,i_1-1}(x_1)l_{2,i_2-1}(x_2)\,dx_1dx_2 \\
= \;& (-1)^{s+t}\int_{I_1\times I_2}\partial_{x_1}^{s+1}\partial_{x_2}^{t+1}u(x_1,x_2,x_{3,e}-h_{3,e},\cdots,x_{d,e}-h_{d,e}) \\
& D^{-s}l_{1,i_1-1}(x_1)D^{-t}l_{2,i_2-1}(x_2)\,dx_1dx_2,
\end{aligned}
$$

where $0\le s\le i_1-1,\ 0\le t\le i_2-1,\ s+t=m,$. The operator $D^{-n}(n\ge 1)$ denotes the integration operator of order n such that

$$\frac{d^n}{dx_i^n}(D^{-n}\varphi(x_i))=\varphi(x_i).$$

In particular, when $n=0$,

$$D^{-n}\varphi(x_i)=\varphi(x_i).$$

Thus,

$$|\beta_{i_1i_20\cdots 0}|\le Ch^{m+1}\|u\|_{m+2,\,\infty,\,e}. \tag{2.12}$$

In addition

$$
\begin{aligned}
\left|\int_e\nabla\lambda_{i_1i_20\cdots 0}\cdot\nabla v\,dX\right| &\le |\beta_{i_1i_20\cdots 0}|\left|\int_e\nabla\left(\omega_{1,i_1}(x_1)\omega_{2,i_2}(x_2)\right)\cdot\nabla v\,dX\right| \\
&\le C|\beta_{i_1i_20\cdots 0}|\int_e|\nabla v|\,dX
\end{aligned}
$$

Further, from (2.12), we have

$$\left|\int_e\nabla\lambda_{i_1i_20\cdots 0}\cdot\nabla v\,dX\right|\le Ch^{m+1}\|u\|_{m+2,\,\infty,\,e}|v|_{1,\,1,\,e}. \tag{2.13}$$

Similar to the arguments as above, without loss of generality, when $i_k\ne 0,\ k=1,2,\cdots,j$ and $i_{j+1}=i_{j+2}=\cdots=i_d=0$, we have

$$\left|\int_e\nabla\lambda_{i_1i_2\cdots i_j0\cdots 0}\cdot\nabla v\,dX\right|\le Ch^{m+1}\|u\|_{m+2,\,\infty,\,e}|v|_{1,\,1,\,e}. \tag{2.14}$$

Finally, we consider the case of $i_k\ne 0,\ k=1,2,\cdots,d$. Obviously, $\sum_{k=1}^d i_k\ge m+d$. We have by the integration by parts

$$
\begin{aligned}
\beta_{i_1i_2\cdots i_d} &= \int_e\partial_{x_1}\partial_{x_2}\cdots\partial_{x_d}u(X) \\
& \quad l_{1,i_1-1}(x_1)l_{2,i_2-1}(x_2)\cdots l_{d,i_d-1}(x_d)\,dX \\
&= (-1)^{s_1+s_2+\cdots+s_d}\int_e\partial_{x_1}^{s_1+1}\partial_{x_2}^{s_2+1}\cdots\partial_{x_d}^{s_d+1}u(X) \\
& \quad D^{-s_1}l_{1,i_1-1}(x_1)D^{-s_2}l_{2,i_2-1}(x_2)\cdots D^{-s_d}l_{d,i_d-1}(x_d)\,dX,
\end{aligned}
$$

where $0\le s_k\le i_k-1,\ k=1,\cdots,d$ and $\sum_{k=1}^d s_k=m+2-d$. Thus,

$$|\beta_{i_1i_2\cdots i_d}|\le Ch^{m+2-\frac{d}{2}}\|u\|_{m+2,\,\infty,\,e}. \tag{2.15}$$

Obviously,

$$\begin{aligned}\left|\int_e \nabla\lambda_{i_1i_2\cdots i_d}\cdot\nabla v\,dX\right| &\leq |\beta_{i_1i_2\cdots i_d}|\left|\int_e \nabla\left(\omega_{1,i_1}(x_1)\omega_{2,i_2}(x_2)\cdots\omega_{d,i_d}(x_d)\right)\cdot\nabla v\,dX\right|\\ &\leq Ch^{\frac{d-2}{2}}|\beta_{i_1i_2\cdots i_d}|\int_e|\nabla v|\,dX\end{aligned}$$

Further, from (2.15), we have

$$\left|\int_e \nabla\lambda_{i_1i_2\cdots i_d}\cdot\nabla v\,dX\right|\leq Ch^{m+1}\|u\|_{m+2,\,\infty,\,e}|v|_{1,\,1,\,e}. \tag{2.16}$$

Combining (2.10), (2.11), (2.13), (2.14), and (2.16) yields

$$|I_e|\leq Ch^{m+1}\|u\|_{m+2,\,\infty,\,e}|v|_{1,\,1,\,e}. \tag{2.17}$$

Summing over all elements proves the result (2.8). In the following, we will prove the result (2.9).

If $m\geq 2$, without loss of generality, we assume $i_k\neq 0$, $k=1,2,\cdots,j$ and $i_{j+1}=i_{j+2}=\cdots=i_d=0$.

$$\begin{aligned}I_{i_1i_2\cdots i_j0\cdots 0} &\equiv \textstyle\int_e \nabla\lambda_{i_1i_2\cdots i_j0\cdots 0}\cdot\nabla v\,dX\\ &= \beta_{i_1i_2\cdots i_j0\cdots 0}\textstyle\int_e \nabla\left(\omega_{1,i_1}(x_1)\omega_{2,i_2}(x_2)\cdots\omega_{j,i_j}(x_j)\right)\cdot\nabla v\,dX\\ &= \beta_{i_1i_2\cdots i_j0\cdots 0}\textstyle\int_e \partial_{x_1}\left(\omega_{1,i_1}(x_1)\omega_{2,i_2}(x_2)\cdots\omega_{j,i_j}(x_j)\right)\partial_{x_1}v\,dX\\ &\quad+\beta_{i_1i_2\cdots i_j0\cdots 0}\textstyle\int_e \partial_{x_2}\left(\omega_{1,i_1}(x_1)\omega_{2,i_2}(x_2)\cdots\omega_{j,i_j}(x_j)\right)\partial_{x_2}v\,dX\\ &\quad+\cdots+\beta_{i_1i_2\cdots i_j0\cdots 0}\textstyle\int_e \partial_{x_j}\left(\omega_{1,i_1}(x_1)\omega_{2,i_2}(x_2)\cdots\omega_{j,i_j}(x_j)\right)\partial_{x_j}v\,dX\\ &= I_1+I_2+\cdots+I_j.\end{aligned} \tag{2.18}$$

We assume $i_1\geq m+1$, thus $i_1\geq m+1\geq 3$. By the orthogonality of the Legendre polynomial system,

$$I_1=\beta_{i_1i_2\cdots i_j0\cdots 0}\int_e l_{1,i_1-1}(x_1)\omega_{2,i_2}(x_2)\cdots\omega_{j,i_j}(x_j)\partial_{x_1}v\,dX=0. \tag{2.19}$$

In addition

$$\begin{aligned}I_2 &= \beta_{i_1i_2\cdots i_j0\cdots 0}\textstyle\int_e \omega_{1,i_1}(x_1)l_{2,i_2-1}(x_2)\cdots\omega_{j,i_j}(x_j)\partial_{x_2}v\,dX\\ &= -\beta_{i_1i_2\cdots i_j0\cdots 0}\textstyle\int_e D^{-1}\omega_{1,i_1}(x_1)l_{2,i_2-1}(x_2)\cdots\omega_{j,i_j}(x_j)\partial_{x_1}\partial_{x_2}v\,dX.\end{aligned} \tag{2.20}$$

Similar to the arguments of (2.12), we get

$$|\beta_{i_1i_2\cdots i_j0\cdots 0}|\leq Ch^{m+2-\frac{j}{2}}\|u\|_{m+2,\,\infty,\,e}. \tag{2.21}$$

In fact

$$D^{-1}\omega_{1,i_1}(x_1)l_{2,i_2-1}(x_2)\cdots\omega_{j,i_j}(x_j)=\mathcal{O}(h^{\frac{j}{2}}). \tag{2.22}$$

Combining (2.20)–(2.22) yields

$$|I_2|\leq Ch^{m+2}\|u\|_{m+2,\,\infty,\,e}|v|_{2,\,1,\,e}. \tag{2.23}$$

Similarly, we have

$$|I_k| \leq Ch^{m+2}\|u\|_{m+2,\,\infty,\,e}|v|_{2,\,1,\,e},\ k=3,\cdots,j. \tag{2.24}$$

From (2.18), (2.19), (2.23), and (2.24),

$$|I_{i_1 i_2\cdots i_j 0\cdots 0}| \leq Ch^{m+2}\|u\|_{m+2,\,\infty,\,e}|v|_{2,\,1,\,e},\ k=3,\cdots,j. \tag{2.25}$$

When each $i_k \neq 0$, $k=1,\cdots,d$, similar to the above arguments, we easily get

$$\left|\int_e \nabla\lambda_{i_1 i_2\cdots i_d}\cdot\nabla v\,dX\right| \leq Ch^{m+2}\|u\|_{m+2,\,\infty,\,e}|v|_{2,\,1,\,e}. \tag{2.26}$$

From (2.10), (2.11), (2.25), and (2.26),

$$|I_e| \leq Ch^{m+2}\|u\|_{m+2,\,\infty,\,e}|v|_{2,\,1,\,e}.$$

Summing over all elements proves the result (2.9).

3 Superconvergence of the Finite Element

In this section, we will give applications of the weak estimates. Some applications may be found in the published literatures. First we need to give the definitions of the discrete Green's function and the discrete derivative Green's function. For every $Z\in\Omega$, we define the discrete derivative Green's function $\partial_{Z,\ell}G_Z^h \in S_0^h$ and the discrete Green's function $G_Z^h \in S_0^h$ such that (see [7])

$$a(\partial_{Z,\ell}G_Z^h,\,v) = \partial_\ell v(Z)\ \ \forall\, v\in S_0^h(\Omega), \tag{3.1}$$

$$a(G_Z^h,\,v) = v(Z)\ \ \forall\, v\in S_0^h(\Omega), \tag{3.2}$$

where $\ell\in\mathcal{R}^d$ and $|\ell|=1$. $\partial_\ell v(Z)$ stands for the onesided directional derivative

$$\partial_\ell v(Z) = \lim_{|\Delta Z|\to 0}\frac{v(Z+\Delta Z)-v(Z)}{|\Delta Z|},\ \ \Delta Z = |\Delta Z|\ell.$$

As for $\partial_{Z,\ell}G_Z^h$ and G_Z^h, we have obtained some estimates (see [8–15]).

Using the weak estimates (see (2.8) and (2.9)) and the estimates for $\partial_{Z,\ell}G_Z^h$ and G_Z^h, we give superconvergent estimates of the multidimensional tensor-product m-degree finite element as following:

- In the case of $d=3$, we have (see [8–10])

$$\left|\partial_{Z,\ell}G_Z^h\right|_{1,1} = \mathcal{O}(|\ln h|^{\frac{4}{3}}), \tag{3.3}$$

$$\left|\partial_{Z,\ell}G_Z^h\right|_{2,1}^h = \mathcal{O}(h^{-1}), \tag{3.4}$$

$$\left|G_Z^h\right|_{2,1}^h = \mathcal{O}(|\ln h|^{\frac{2}{3}}). \tag{3.5}$$

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Thus, from (1.2), (2.8), (2.9), and (3.1)–(3,5), we get superconvergent estimates

$$|u_h - \Pi_m u|_{1,\,\infty,\,\Omega} \leq Ch^{m+1}|\ln h|^{\frac{4}{3}}\|u\|_{m+2,\,\infty,\,\Omega}, \;\; m=1, \quad \text{(see [9])}$$

$$|u_h - \Pi_m u|_{1,\,\infty,\,\Omega} \leq Ch^{m+1}\|u\|_{m+2,\,\infty,\,\Omega}, \;\; m \geq 2, \quad \text{(see [9])}$$

and

$$|u_h - \Pi_m u|_{0,\,\infty,\,\Omega} \leq Ch^{m+2}|\ln h|^{\frac{2}{3}}\|u\|_{m+2,\,\infty,\,\Omega}, \;\; m \geq 2. \;\; \text{(see [10])}$$

• In the case of $d=4$, we have

$$\left|\partial_{Z,\ell} G_Z^h\right|_{1,1} = \mathcal{O}(|\ln h|^{\frac{5}{4}}), \quad \text{(see [11])} \tag{3.6}$$

$$\left|\partial_{Z,\ell} G_Z^h\right|_{2,1}^h = \mathcal{O}(h^{-1}|\ln h|^{\frac{1}{2}}), \quad \text{(see [12])} \tag{3.7}$$

$$\left|G_Z^h\right|_{2,1}^h = \mathcal{O}(|\ln h|^{\frac{1}{2}}). \tag{3.8}$$

Thus, from (1.2), (2.8), (2.9), (3.1), (3.2), and (3.6)–(3,8), we get superconvergent estimates

$$|u_h - \Pi_m u|_{1,\,\infty,\,\Omega} \leq Ch^{m+1}|\ln h|^{\frac{5}{4}}\|u\|_{m+2,\,\infty,\,\Omega}, \;\; m=1,$$

$$|u_h - \Pi_m u|_{1,\,\infty,\,\Omega} \leq Ch^{m+1}|\ln h|^{\frac{1}{2}}\|u\|_{m+2,\,\infty,\,\Omega}, \;\; m \geq 2, \quad \text{(see [12])}$$

and

$$|u_h - \Pi_m u|_{0,\,\infty,\,\Omega} \leq Ch^{m+2}|\ln h|^{\frac{1}{2}}\|u\|_{m+2,\,\infty,\,\Omega}, \;\; m \geq 2.$$

• In the case of $d=5$, we have (see [13, 14])

$$\left|\partial_{Z,\ell} G_Z^h\right|_{1,1} = \mathcal{O}(|\ln h|^{\frac{7}{5}}), \tag{3.9}$$

$$\left|G_Z^h\right|_{2,1}^h = \mathcal{O}(|\ln h|^{\frac{9}{5}}). \tag{3.10}$$

Thus, from (1.2), (2.8), (2.9), (3.1), (3.2), (3.9), and (3,10), we get superconvergent estimates

$$|u_h - \Pi_m u|_{1,\,\infty,\,\Omega} \leq Ch^{m+1}|\ln h|^{\frac{7}{5}}\|u\|_{m+2,\,\infty,\,\Omega}, \;\; m \geq 1,$$

and

$$|u_h - \Pi_m u|_{0,\,\infty,\,\Omega} \leq Ch^{m+2}|\ln h|^{\frac{9}{5}}\|u\|_{m+2,\,\infty,\,\Omega}, \;\; m \geq 2.$$

• In the case of $d=6$, we have

$$\left|\partial_{Z,\ell} G_Z^h\right|_{1,1} = \mathcal{O}(|\ln h|^{\frac{4}{3}}), \quad \text{(see [15])} \tag{3.11}$$

$$\left|G_Z^h\right|_{2,1}^h = \mathcal{O}(|\ln h|^{\frac{4}{3}}). \tag{3.12}$$

Remark 1. The result (3.12) was submitted in JOCAAA.

Thus, from (1.2), (2.8), (2.9), (3.1), (3.2), (3.11), and (3,12), we get superconvergent estimates

$$|u_h - \Pi_m u|_{1,\,\infty,\,\Omega} \leq Ch^{m+1}|\ln h|^{\frac{4}{3}}\|u\|_{m+2,\,\infty,\,\Omega},\ \ m \geq 1,\ \ \text{(see [15])}$$

and

$$|u_h - \Pi_m u|_{0,\,\infty,\,\Omega} \leq Ch^{m+2}|\ln h|^{\frac{4}{3}}\|u\|_{m+2,\,\infty,\,\Omega},\ \ m \geq 2.$$

- In the case of $d \geq 7$, we only have

$$\left|\partial_{Z,\ell} G_Z^h\right|_{1,1} = \mathcal{O}(h^{\frac{2-d}{2}}), \tag{3.13}$$

$$\left|G_Z^h\right|_{2,1}^h = \mathcal{O}(h^{\frac{4-d}{2}}). \tag{3.14}$$

Remark 2. According to the results (3.13) and (3.14), we can not obtain the pointwise superconvergent estimates in the case of $d \geq 7$.

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References

1. I. Babuška and T. Strouboulis, The finite element method and its reliability, Numerical Mathematics and Scientific Computation, Oxford Science Publications, 2001.

2. C. M. Chen, Construction theory of superconvergence of finite elements (in Chinese), Hunan Science and Technology Press, Changsha, China, 2001.

3. C. M. Chen and Y. Q. Huang, High accuracy theory of finite element methods (in Chinese), Hunan Science and Technology Press, Changsha, China, 1995.

4. Q. Lin and N. N. Yan, Construction and analysis of high efficient finite elements (in Chinese), Hebei University Press, Baoding, China, 1996.

5. Q. Lin and Q. D. Zhu, The preprocessing and postprocessing for the finite element method (in Chinese), Shanghai Scientific and Technical Publishers, Shanghai, China, 1994.

6. L. B. Wahlbin, Superconvergence in Galerkin finite element methods, Springer Verlag, Berlin, 1995.

7. Q. D. Zhu and Q. Lin, Superconvergence theory of the finite element methods (in Chinese), Hunan Science and Technology Press, Changsha, China, 1989.

8. J. H. Liu and Q. D. Zhu, The estimate for the $W^{1,1}$-seminorm of discrete derivative Green's function in three dimensions (in Chinese), J. Hunan Univ. Arts Sci. 16 (2004), 1–3.

9. J. H. Liu and Q. D. Zhu, Pointwise supercloseness of tensor-product block finite elements, Numer. Meth. Part. Differ. Equa. 25 (2009) 990–1008.

10. J. H. Liu, B. Jia, and Q. D. Zhu, An estimate for the three-dimensional discrete Green's function and applications, J. Math. Anal. Appl. 370 (2010), 350–363.

11. J. H. Liu and G. Hu, An estimate for the discrete derivative Greens function for the 4D variable coefficient elliptic equation, J. Comp. Anal. Appl. 14 (2012), 165–172.

12. J. H. Liu and Y. S. Jia, An estimate for the four-dimensional discrete derivative Green's function and its applications in FE superconvergence, J. Comp. Anal. Appl. 19 (2015), 462–469.

13. J. H. Liu and Q. D. Zhu, The $W^{1,1}$-Seminorm estimate for the discrete derivative Greens function for the 5D Poisson equation, J. Comp. Anal. Appl. 13 (2011), 1143–1156.

14. J. H. Liu and Y. S. Jia, Five-dimensional discrete Green's function and its estimates, J. Comp. Anal. Appl. 18 (2015), 620–627.

15. J. H. Liu, Q. D. Zhu, and Y. S. Jia, Superconvergence of the six-dimensional linear and tensor-product k ($k \geq 1$)-order finite elements (in Chinese), Sci Sin Math. 45 (2015), 1337–1344.

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NEW INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE FOR OPERATOR m-CONVEX AND (α, m)-CONVEX FUNCTIONS

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ABSTRACT: In this paper, authors introduce the concepts of operator m-convex function and operator (α, m)-convex function, and establish some new integral inequalities of Hermite-Hadamard type for operator m-convex and (α, m)-convex functions.

KEY WORDS: Integral inequality; operator m-convex function; operator (α, m)-convex function.

2010 Mathematics Subject Classification: 15A39, 26A51, 26D15, 47A63.

1. Introduction

Throughout this paper, we adopt the notations: $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_0 = [0, \infty)$.
We firstly list the definition of convex functions.

Definition 1.1. The function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex function if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

One of the most important integral inequalities for convex functions is the Hadamard inequality (or the Hermite-Hadamard inequality). The following double inequality is well known as the Hadamard inequality in the literature. If any f is convex function on $[a, b] \subseteq \mathbb{R}$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

Both inequalities hold in the reversed direction if f is concave on $[a, b]$. We note that the Hermite-Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality.

In the literature, the concepts of m-convexity and $(\alpha; m)$-convexity are well known. The concept of m-convexity was first introduced by G. Toader in [11] (see also [1]) and it is defined as follows:

Definition 1.2 ([11])**.** The function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be m-convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$, we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y). \tag{1.3}$$

The class of (α, m)-convex functions was also first introduced in [8] and it is defined as follows:

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Definition 1.3 ([8])**.** The function $f : [0, b] \to \mathbb{R}$, $b > 0$ is said to be (α, m)-convex, where $(\alpha, m) \in [0, 1]^2$, if we have

$$f(tx + m(1-t)y) \le t^\alpha f(x) + m(1 - t^\alpha)f(y) \tag{1.4}$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

In [1], S. S. Dragomir and G. Toader proved the following Hadamard type inequalities for m-convex functions.

Theorem 1.1 ([1])**.** *Let $f : \mathbb{R}_0 \to \mathbb{R}$ be an m-convex function with $m \in (0, 1]$. if $0 \le a < b < \infty$ and $f \in L_1[a, b]$, then the following inequality holds*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \min\left\{\frac{f(a) + mf\left(\frac{b}{m}\right)}{2}, \frac{f(b) + mf\left(\frac{a}{m}\right)}{2}\right\}. \tag{1.5}$$

In [2], S. S. Dragomir established new Hadamard-type inequalities for m-convex functions.

Theorem 1.2 ([2])**.** *Let $f : \mathbb{R}_0 \to \mathbb{R}$ be an m-convex function with $m \in (0, 1]$. if $0 \le a < b < \infty$ and $f \in L_1[am, b]$, then the following inequality holds*

$$\frac{1}{m+1}\left[\frac{1}{mb-a}\int_a^{mb} f(x)\,\mathrm{d}x + \frac{1}{b-ma}\int_{ma}^b f(x)\,\mathrm{d}x\right] \le \frac{f(a)+f(b)}{2}. \tag{1.6}$$

In [10], E. Set et al. proved the following Hadamard type inequalities for (α, m)-convex functions.

Theorem 1.3 ([10])**.** *Let $f : \mathbb{R}_0 \to \mathbb{R}$ be an (α, m)-convex function with $(\alpha, m) \in (0, 1]^2$. if $0 \le a < b < \infty$ and $f \in L_1[a, b]$, then the following inequality holds*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \le & \frac{1}{2^\alpha(b-a)}\int_a^b \left[f(x) + m(2^\alpha - 1)f\left(\frac{x}{m}\right)\right]\mathrm{d}x \\ \le & \frac{1}{2^{\alpha+1}(\alpha+1)}\left\{f(a) + f(b) + m(\alpha + 2^\alpha - 1)\left[f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)\right]\right. \\ & \left. + \alpha m^2(2^\alpha - 1)\left[f\left(\frac{a}{m^2}) + f(\frac{b}{m^2}\right)\right]\right\}\mathrm{d}x. \end{aligned} \tag{1.7}$$

Some generalizations of this result can be found in [12] and [13].

In recent years, several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of operator convex functions introduced by S. S. Dragomir in [5].

Now we review the operator order in $B(H)$ and the continuous functional calculus for a bounded self-adjoint operator. For self-adjoint operators $A, B \in B(H)$, we write $A \le B$ if $\langle Ax, x\rangle \le \langle Bx, x\rangle$ for every vector $x \in H$, we call it the operator order.

Let A be a bounded self-adjoint linear operator on a complex Hilbert space $(H; \langle ., .\rangle)$. The Gelfand map establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all continuous complex-valued functions defined on the spectrum of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [6], p.3). For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$, we have

(*i*) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

(*ii*) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(f^*) = \Phi(f)^*$;

(*iii*) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f|$;

(*iv*) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$ for $t \in Sp(A)$.

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With this notation, we define

$$f(A) := \Phi(f) \quad \text{for all} \quad f \in C(Sp(A)) \tag{1.8}$$

and we call it the continuous functional calculus for a bounded self-adjoint operator A.

If A is a bounded self-adjoint operator and f is a real-valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on H. Moreover, if both f and g are real-valued functions on $Sp(A)$ such that $f(t) \leq g(t)$ for any $t \in Sp(A)$, then $f(A) \leq f(B)$ in the operator order in $B(H)$.

We denoted by $B(H)^+$ the set of all positive operators in $B(H)$ and

$$C(H) := \{A \in B(H)^+ : AB + BA \geq 0 \quad \text{for all} \quad B \in B(H)^+\}. \tag{1.9}$$

It is obvious that $C(H)$ is a closed convex cone in $B(H)$.

A real valued continuous function f on an interval $I \subseteq \mathbb{R}$ is said to be operator convex (operator concave) if the operator inequality

$$f((1-t)A + tB) \leq (\geq)(1-t)f(A) + tf(B) \tag{1.10}$$

holds in the operator order in $B(H)$, for all $t \in [0,1]$ and for every bounded self-adjoint operators A and B in $B(H)$ whose spectra are contained in I.

In [5], S. S. Dragomir gave the operator version of the Hermite-Hadamard inequality for operator convex functions.

Theorem 1.4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be an operator convex function on the interval I. Then for any self-adjoint operators A and B with spectra in I, we have the inequality*

$$\begin{aligned} &f\left(\frac{A+B}{2}\right) \leq \frac{1}{2}\left[f\left(\frac{3A+B}{4}\right) + f\left(\frac{A+3B}{4}\right)\right] \\ &\leq \int_0^1 f(tA + (1-t)B)\,\mathrm{d}t \leq \frac{1}{2}\left[f\left(\frac{A+B}{2}\right) + \frac{f(A)+f(B)}{2}\right] \leq \frac{f(A)+f(B)}{2}. \end{aligned} \tag{1.11}$$

For recent results related to Hermite-Hadamard type inequalities are given in [3], [4], [6], [7], and plenty of references therein.

The goal of this paper is to obtain new inequalities like those given in Theorems 1.1, 1.2, 1.3, but now for operator m-convex and (α, m)-convex functions.

2. Operator m-convex and (α, m)-convex functions

In order to verify our main results, the following preliminary definitions and lemmas are necessary.

Definition 2.1. Let $[0, b] \subseteq \mathbb{R}_0$ with $b > 0$ and K be a convex set of $B(H)^+$. A continuous function $f : [0, b] \to \mathbb{R}$ is said to be operator m-convex on $[0, b]$ for operators in K, if

$$f(tA + m(1-t)B) \leq tf(A) + m(1-t)f(B) \tag{2.1}$$

in the operator order in $B(H)$, for all $t \in [0, 1]$ and every positive operators A and B in K whose spectra are contained in $[0, b]$ and for some fixed $m \in [0, 1]$.

Remark 2.1. For $m = 1$, we recapture the concept of operator convex functions defined on $[0, b]$ and for $m = 0$ we get the concept of operator starshaped functions on $[0, b]$, namely, we call $f : [0, b] \to \mathbb{R}$ to be operator starshaped if

$$f(tA) \leq tf(A) \tag{2.2}$$

for all $t \in [0, 1]$ and every positive operators A in $B(H)^+$ whose spectra are contained in $[0, b]$.

Lemma 2.1. *If f is operator m-convex, then $f(0) \leq 0$, where 0 is the zero operator on H.*

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Proof. Taking $A = 0$ and $B = 0$ in the inequality (2.1), then

$$(1-t)(1-m)f(0) \leq 0.$$

Also by $t, m \in [0, 1]$, we get $f(0) \leq 0$. □

Lemma 2.2. *If f is operator m-convex, then f is operator starshaped.*

Proof. For all $t \in [0, 1]$ and positive operators $A \in B(H)^+$ whose spectra is contained in $[0, b]$, we write

$$f(tA) = f(tA + m(1-t)0) \leq tf(A) + m(1-t)f(0) \leq tf(A).$$

□

Lemma 2.3. *If f is operator m_1-convex and $0 \leq m_2 < m_1 \leq 1$, then f is operator m_2-convex.*

Proof. For all $t \in [0, 1]$ and positive operators $A, B \in B(H)^+$ whose spectra are contained in $[0, b]$, we drive

$$\begin{aligned} f(tA + m_2(1-t)B) &= f\Big(tA + m_1(1-t)\Big(\frac{m_2}{m_1}\Big)B\Big) \leq tf(A) + m_1(1-t)f\Big(\frac{m_2}{m_1}B\Big) \\ &\leq tf(A) + m_1(1-t)\frac{m_2}{m_1}f(B) = tf(A) + m_2(1-t)f(B). \end{aligned}$$

□

Definition 2.2. Let $[0, b] \subseteq \mathbb{R}_0$ with $b > 0$ and K be a convex set of $B(H)^+$. A continuous function $f : [0, b] \to \mathbb{R}$ is said to be operator (α, m)-convex on $[0, b]$ for operators in K, if

$$f(tA + m(1-t)B) \leq t^\alpha f(A) + m(1-t^\alpha)f(B) \tag{2.3}$$

in the operator order in $B(H)$, for all $t \in [0, 1]$ and every positive operators A and B in K whose spectra are contained in $[0, b]$ and for some fixed $(\alpha, m) \in [0, 1]^2$.

Remark 2.2. It can be easily seen that for $(\alpha, m) \in \{(0, 0), (1, 1), (1, m)\}$ one obtains the following classes of functions: operator increasing, operator convex and operator m-convex functions respectively.

Similarly to the proof of Lemma 2.1, the following result is valid.

Lemma 2.4. *If f is operator (α, m)-convex, then $f(0) \leq 0$, where 0 is the zero operator on H.*

Lemma 2.5 ([9])**.** *Let $A, B \in B(H)^+$. Then $AB + BA$ is positive if and only if $f(A + B) \leq f(A) + f(B)$ for all non-negative operator monotone functions f on $\mathbb{R}_0$.*

Now, we give an example of operator m-convex function.

Example 2.1. Since for every positive operator $A, B \in C(H)$, $AB + BA \geq 0$. Utilizing Lemma 2.5, we obtain

$$[tA + m(1-t)B]^s \leq t^s A^s + m^s(1-t)^s B^s \leq tA^s + m(1-t)B^s.$$

Therefore, the continuous function $f(t) = t^s (0 < s \leq 1)$ is operator m-convex on $\mathbb{R}_0$ for operators in $C(H)$.

Remark 2.3. We can consider the same continuous function $f(t) = t^s (0 < s \leq 1)$ as an example of operator (α, m)-convex function for $\alpha = 1$.

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3. Some new Hermite-Hadamard type inequalities

We will now point out some new results of the Hermite-Hadamard type.

Theorem 3.1. *Let the continuous function $f:\mathbb{R}_0\to\mathbb{R}$ be operator (α,m)-convex for operators in $K\subseteq B(H)^+$ with $(\alpha,m)\in[0,1]\times(0,1]$. Then for all positive operator $A,B\in K$ with spectra in $\mathbb{R}_0$, the following inequality holds:*

$$\int_0^1 f(tA+(1-t)B)\,\mathrm{d}t \le min\left\{\frac{f(A)+\alpha mf\left(\frac{B}{m}\right)}{\alpha+1},\frac{f(B)+\alpha mf\left(\frac{A}{m}\right)}{\alpha+1}\right\}. \tag{3.1}$$

Proof. For $x\in H$ with $\|x\|=1$ and $m,t\in(0,1]$, we have

$$\big\langle(tA+m(1-t)B)x,x\big\rangle = t\langle Ax,x\rangle+m(1-t)\langle Bx,x\rangle\in\mathbb{R}_0, \tag{3.2}$$

since $\langle Ax,x\rangle\in Sp(A)\subseteq\mathbb{R}_0$ and $\langle Bx,x\rangle\in Sp(B)\subseteq\mathbb{R}_0$.

Continuity of f and (3.2) imply that the operator-valued integral $\int_0^1 f(tA+m(1-t)B)\,\mathrm{d}t$ exists.

Since f is operator (α,m)-convex, therefore for $(\alpha,m)\in[0,1]\times(0,1]$ and $A,B\in K$, we show

$$f(tA+(1-t)B)\le t^\alpha f(A)+m(1-t^\alpha)f\left(\frac{B}{m}\right)$$

and

$$f(tB+(1-t)A)\le t^\alpha f(B)+m(1-t^\alpha)f\left(\frac{A}{m}\right)$$

for all $t\in[0,1]$.

Integrating over t on $[0,1]$, we obtain

$$\int_0^1 f(tA+(1-t)B)\,\mathrm{d}t\le\frac{f(A)+\alpha mf\left(\frac{B}{m}\right)}{\alpha+1}$$

and

$$\int_0^1 f(tB+(1-t)A)\,\mathrm{d}t\le\frac{f(B)+\alpha mf\left(\frac{A}{m}\right)}{\alpha+1}.$$

However

$$\int_0^1 f(tA+(1-t)B)\,\mathrm{d}t=\int_0^1 f(tB+(1-t)A)\,\mathrm{d}t,$$

and the inequality (3.1) is obtained, which completes the proof. □

Corollary 3.1.1. *Under the assumptions of Theorem 3.1, choosing $\alpha=1$, we get the inequality for operator m-convex functions:*

$$\int_0^1 f(tA+(1-t)B)\,\mathrm{d}t\le min\left\{\frac{f(A)+mf\left(\frac{B}{m}\right)}{2},\frac{f(B)+mf\left(\frac{A}{m}\right)}{2}\right\}. \tag{3.3}$$

Furthermore, for $\alpha,m=1$ we have

$$\int_0^1 f(tA+(1-t)B)\,\mathrm{d}t\le\frac{f(A)+f(B)}{2}. \tag{3.4}$$

Theorem 3.2. *Let the continuous function $f:\mathbb{R}_0\to\mathbb{R}$ be operator (α,m)-convex for operators in $K\subseteq B(H)^+$ with $(\alpha,m)\in[0,1]\times(0,1]$. Then for all positive operator $A,B\in K$ with spectra in $\mathbb{R}_0$, the following inequalities hold:*

$$f\left(\frac{A+B}{2}\right)\le\frac{1}{2^\alpha}\int_0^1\left[f(tA+(1-t)B)+m(2^\alpha-1)f\left(\frac{(1-t)A+tB}{m}\right)\right]\mathrm{d}t$$

$$\begin{aligned}\le&\frac{1}{2^{\alpha+1}(\alpha+1)}\bigg\{f(A)+f(B)+m(\alpha+2^{\alpha}-1)\bigg[f\bigg(\frac{A}{m}\bigg)+f\bigg(\frac{B}{m}\bigg)\bigg]\\&+\alpha m^{2}(2^{\alpha}-1)\bigg[f\bigg(\frac{A}{m^{2}}\bigg)+f\bigg(\frac{B}{m^{2}}\bigg)\bigg]\bigg\}.\end{aligned}\tag{3.5}$$

Proof. By operator (α, m)-convexity of f, we give

$$\begin{aligned}f\bigg(\frac{A+B}{2}\bigg)&\le\frac{1}{2^{\alpha}}f(tA+(1-t)B)+m\bigg(1-\frac{1}{2^{\alpha}}\bigg)f\bigg(\frac{(1-t)A+tB}{m}\bigg)\\&=\frac{1}{2^{\alpha}}\bigg[f(tA+(1-t)B)+m(2^{\alpha}-1)f\bigg(\frac{(1-t)A+tB}{m}\bigg)\bigg],\end{aligned}\tag{3.6}$$

where $(\alpha, m) \in [0, 1] \times (0, 1], t \in [0, 1]$ and $A, B \in K$ with spectra in $\mathbb{R}_0$.

Integrating over $t \in [0, 1]$, we drive the first inequality in (3.5).

Next, from operator (α, m)-convexity of f, we also deduce

$$\begin{aligned}&\frac{1}{2^{\alpha}}\bigg[f(tA+(1-t)B)+m(2^{\alpha}-1)f\bigg(\frac{(1-t)A+tB}{m}\bigg)\bigg]\\\le&\frac{1}{2^{\alpha}}\bigg\{t^{\alpha}f(A)+m(1-t^{\alpha})f\bigg(\frac{B}{m}\bigg)+m(2^{\alpha}-1)\bigg[t^{\alpha}f\bigg(\frac{B}{m}\bigg)+m(1-t^{\alpha})f\bigg(\frac{A}{m^{2}}\bigg)\bigg]\bigg\}.\end{aligned}\tag{3.7}$$

Integrating over t on $[0, 1]$, we get

$$\begin{aligned}&\frac{1}{2^{\alpha}}\int_{0}^{1}\bigg[f(tA+(1-t)B)+m(2^{\alpha}-1)f\bigg(\frac{(1-t)A+tB}{m}\bigg)\bigg]\,\mathrm{d}t\\=&\frac{1}{2^{\alpha}(\alpha+1)}\bigg\{f(A)+m(\alpha+2^{\alpha}-1)f\bigg(\frac{B}{m}\bigg)+\alpha m^{2}(2^{\alpha}-1)f\bigg(\frac{A}{m^{2}}\bigg)\bigg].\end{aligned}\tag{3.8}$$

Similarly, taking into account that

$$\int_{0}^{1}f(tA+(1-t)B)\,\mathrm{d}t=\int_{0}^{1}f(tB+(1-t)A)\,\mathrm{d}t$$

and changing the roles of A and B, we obtain

$$\begin{aligned}&\frac{1}{2^{\alpha}}\int_{0}^{1}\bigg[f(tA+(1-t)B)+m(2^{\alpha}-1)f\bigg(\frac{(1-t)A+tB}{m}\bigg)\bigg]\,\mathrm{d}t\\=&\frac{1}{2^{\alpha}(\alpha+1)}\bigg\{f(B)+m(\alpha+2^{\alpha}-1)f\bigg(\frac{A}{m}\bigg)+\alpha m^{2}(2^{\alpha}-1)f\bigg(\frac{B}{m^{2}}\bigg)\bigg].\end{aligned}\tag{3.9}$$

Summing the inequalities (3.8) and (3.9) and dividing by 2, we get the second inequality in (3.5). The proof thus is complete. $\square$

Corollary 3.2.1. *With the conditions of Theorem 3.2, taking $\alpha = 1$, we obtain the inequalities for operator m-convex functions:*

$$\begin{aligned}f\bigg(\frac{A+B}{2}\bigg)\le&\frac{1}{2}\int_{0}^{1}\bigg[f(tA+(1-t)B)+mf\bigg(\frac{(1-t)A+tB}{m}\bigg)\bigg]\,\mathrm{d}t\\\le&\frac{1}{8}\bigg\{f(A)+f(B)+2m\bigg[f\bigg(\frac{A}{m}\bigg)+f\bigg(\frac{B}{m}\bigg)\bigg]+m^{2}\bigg[f\bigg(\frac{A}{m^{2}}\bigg)+f\bigg(\frac{B}{m^{2}}\bigg)\bigg]\bigg\}.\end{aligned}\tag{3.10}$$

In addition, if $\alpha, m = 1$, we have

$$f\bigg(\frac{A+B}{2}\bigg)\le\int_{0}^{1}f(tA+(1-t)B)\,\mathrm{d}t\le\frac{f(A)+f(B)}{2}.\tag{3.11}$$

Theorem 3.3. *Let the continuous function $f : \mathbb{R}_0 \to \mathbb{R}$ be operator (α, m)-convex for operators in $K \subseteq B(H)^+$ with $(\alpha, m) \in [0,1] \times (0,1]$. Then for all positive operator $A, B \in K$ with spectra in $\mathbb{R}_0$, the following inequality holds:*

$$\int_0^1 f(tA + (1-t)B)\,\mathrm{d}t \le \frac{f(A) + f(B) + \alpha m\big[f\big(\frac{A}{m}\big) + f\big(\frac{B}{m}\big)\big]}{2(\alpha+1)}. \tag{3.12}$$

Proof. Using operator (α, m)-convexity of f, we can write

$$f(tA + (1-t)B) \le t^\alpha f(A) + m(1-t^\alpha) f\left(\frac{B}{m}\right)$$

and

$$f(tB + (1-t)A) \le t^\alpha f(B) + m(1-t^\alpha) f\left(\frac{A}{m}\right)$$

for all $t \in [0,1]$ and some fixed $(\alpha, m) \in [0,1] \times (0,1]$.

Adding the above inequalities and integrating over t on $[0,1]$, we have

$$\int_0^1 f(tA + (1-t)B)\,\mathrm{d}t + \int_0^1 f(tB + (1-t)A)\,\mathrm{d}t \le \frac{f(A) + f(B) + \alpha m\big[f\big(\frac{A}{m}\big) + f\big(\frac{B}{m}\big)\big]}{\alpha+1}.$$

As it is easy to see that

$$\int_0^1 f(tA + (1-t)B)\,\mathrm{d}t = \int_0^1 f(tB + (1-t)A)\,\mathrm{d}t,$$

we deduce the desired result. The proof of Theorem 3.3 is complete. □

Corollary 3.3.1. *Under the assumptions of Theorem 3.3, letting $\alpha = 1$, we get the inequality for operator m-convex functions:*

$$\int_0^1 f(tA + (1-t)B)\,\mathrm{d}t \le \frac{f(A) + f(B) + m\big[f\big(\frac{A}{m}\big) + f\big(\frac{B}{m}\big)\big]}{4}. \tag{3.13}$$

In addition, for $\alpha, m = 1$, we have

$$\int_0^1 f(tA + (1-t)B)\,\mathrm{d}t \le \frac{f(A) + f(B)}{2}. \tag{3.14}$$

Theorem 3.4. *Let the continuous function $f : \mathbb{R}_0 \to \mathbb{R}$ be operator (α, m)-convex for operators in $K \subseteq B(H)^+$ with $(\alpha, m) \in [0,1] \times (0,1]$. Then for all positive operator $A, B \in K$ with spectra in $\mathbb{R}_0$, the following inequality holds:*

$$\int_0^1 \big[f(tA + m(1-t)B) + f(tB + m(1-t)A)\big]\,\mathrm{d}t \le \frac{(1+m\alpha)[f(A) + f(B)]}{\alpha+1}. \tag{3.15}$$

Proof. By operator (α, m)-convexity of f, we can obtain

$$f(tA + m(1-t)B) \le t^\alpha f(A) + m(1-t^\alpha) f(B),$$
$$f((1-t)A + mtB) \le (1-t)^\alpha f(A) + m(1-(1-t)^\alpha) f(B),$$
$$f(tB + m(1-t)A) \le t^\alpha f(B) + m(1-t^\alpha) f(A),$$

and

$$f((1-t)B + mtA) \le (1-t)^\alpha f(B) + m(1-(1-t)^\alpha) f(A)$$

for all $t \in [0,1]$ and some fixed $(\alpha, m) \in [0,1] \times (0,1]$.

Adding the above inequalities with each other, we get

$$\begin{aligned} &f(tA + m(1-t)B) + f((1-t)A + mtB) + f(tB + m(1-t)A) + f((1-t)B + mtA) \\ &\le \big[t^\alpha + (1-t)^\alpha + m(1-t^\alpha) + m(1-(1-t)^\alpha)\big]\big[f(A) + f(B)\big]. \end{aligned}$$

Now integrating over $t \in [0,1]$ and taking into account that

$$\int_0^1 f(tA + m(1-t)B)\,\mathrm{d}t = \int_0^1 f((1-t)A + mtB)\,\mathrm{d}t$$

and

$$\int_0^1 f(tB + m(1-t)A)\,\mathrm{d}t = \int_0^1 f((1-t)B + mtA)\,\mathrm{d}t$$

we obtain the inequality (3.15). The proof of Theorem 3.4 is complete. □

Corollary 3.4.1. *Under the assumptions of Theorem 3.4, choosing $\alpha = 1$, we get the inequality for operator m-convex functions:*

$$\int_0^1 \big[f(tA + m(1-t)B) + f(tB + m(1-t)A)\big]\,\mathrm{d}t \le \frac{(1+m)[f(A)+f(B)]}{2}. \tag{3.16}$$

Moreover, for $\alpha, m = 1$, we have

$$\int_0^1 f(tA + (1-t)B)\,\mathrm{d}t \le \frac{f(A)+f(B)}{2}. \tag{3.17}$$

Theorem 3.5. *Let the continuous function $f : \mathbb{R}_0 \to \mathbb{R}$ be operator (α, m)-convex for operators in $K \subseteq B(H)^+$ with $(\alpha, m) \in [0,1] \times (0,1]$. Then for all positive operator $A, B \in K$ with spectra in $\mathbb{R}_0$, the following inequalities hold:*

$$\begin{aligned}
&f\Big(\frac{2-m}{2}B + \frac{m}{2}(mA)\Big)\\
\le&\frac{1}{2^\alpha}\int_0^1\bigg[f\big(t(2-m)B + (1-t)m^2A\big) + m(2^\alpha - 1)f\bigg(\frac{(1-t)(2-m)B + tm^2A}{m}\bigg)\bigg]\,\mathrm{d}t\\
\le&\frac{1}{2^\alpha(\alpha+1)}\bigg[f((2-m)B) + m\big(\alpha + 2^\alpha - 1\big)f(mA)\\
&+ m^2\alpha(2^\alpha - 1)f\bigg(\frac{(2-m)B}{m^2}\bigg)\bigg].
\end{aligned} \tag{3.18}$$

Proof. From operator (α, m)-convexity of f, we can deduce

$$\begin{aligned}
&f\Big(\frac{2-m}{2}B + \frac{m}{2}(mA)\Big)\\
\le&\frac{1}{2^\alpha}f\big(t(2-m)B + (1-t)m^2A\big) + m\Big(1 - \frac{1}{2^\alpha}\Big)f\bigg(\frac{(1-t)(2-m)B + tm^2A}{m}\bigg)\\
=&\frac{1}{2^\alpha}\bigg[f\big(t(2-m)B + (1-t)m^2A\big) + m(2^\alpha - 1)f\bigg(\frac{(1-t)(2-m)B + tm^2A}{m}\bigg)\bigg],
\end{aligned} \tag{3.19}$$

where $(\alpha, m) \in [0,1] \times [0,1], t \in (0,1]$ and $A, B \in K$ with spectra in $\mathbb{R}_0$.

Integrating over $t \in [0,1]$, we drive the first inequality in (3.18).

Next, by operator (α, m)-convexity of f, we also write

$$f\big(t(2-m)B + (1-t)m^2A\big) \le t^\alpha f((2-m)B) + m(1-t^\alpha)f(mA) \tag{3.20}$$

and

$$f\bigg(\frac{(1-t)(2-m)B + tm^2A}{m}\bigg) \le t^\alpha f(mA) + m(1-t^\alpha)f\bigg(\frac{(2-m)B}{m^2}\bigg). \tag{3.21}$$

Submitting the inequalities (3.20) and (3.21) into the inequality (3.19), we get

$$\frac{1}{2^\alpha}\bigg[f\big(t(2-m)B + (1-t)m^2A\big) + m(2^\alpha - 1)f\bigg(\frac{(1-t)(2-m)B + tm^2A}{m}\bigg)\bigg]$$

$$\begin{aligned}&\leq \frac{1}{2^\alpha}\bigg\{t^\alpha f((2-m)B) + m\big[1-t^\alpha + t^\alpha(2^\alpha-1)\big]f(mA)\\&\quad + m^2(2^\alpha-1)(1-t^\alpha)f\bigg(\frac{(2-m)B}{m^2}\bigg)\bigg\}.\end{aligned} \tag{3.22}$$

Integrating over t on $[0,1]$, we deduce the second inequality in (3.18). This completes the proof of the Theorem 3.5. □

Corollary 3.5.1. *Under the assumptions of Theorem 3.5, letting $\alpha = 1$, we get the inequalities for operator m-convex functions:*

$$\begin{aligned}&f\bigg(\frac{2-m}{2}B + \frac{m}{2}(mA)\bigg)\\&\leq \frac{1}{2}\int_0^1 \bigg[f\big(t(2-m)B + (1-t)m^2A\big) + mf\bigg(\frac{(1-t)(2-m)B + tm^2A}{m}\bigg)\bigg]\,\mathrm{d}t\\&\leq \frac{1}{4}\bigg[f((2-m)B) + 2mf(mA) + m^2 f\bigg(\frac{(2-m)B}{m^2}\bigg)\bigg].\end{aligned} \tag{3.23}$$

In addition, for $\alpha, m = 1$, we drive

$$f\bigg(\frac{A+B}{2}\bigg) \leq \int_0^1 f(tA + (1-t)B)\,\mathrm{d}t \leq \frac{f(A)+f(B)}{2}. \tag{3.24}$$

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References

[1] S. S. Dragomir and G. Toader,Some inequalities for m-convex functions, *Studia Univ. Babes-Bolyai, Mathematica*, 38:1, 21–28 (1993).

[2] S. S. Dragomir, On some new inequalities of Hermite-Hadamard type for m-convex functions, *Tamkang J. Math.*, 3:1, 45–55 (2002).

[3] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure Appl. Math.*, 3:2, 775–788 (2002); Available online at `http://jipam.vu.edu.cn`.

[4] S. S. Dragomir, An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure Appl. Math.*, 3:3, Article 35 (2002); Available online at `http://jipam.vu.edu.cn`.

[5] S. S. Dragomir, Hermite-Hadamard's type inequalities for operator convex functions, *Applied Mathematics and Computation*, 218, 766–772 (2011); Available online at `http://www.elsevier.com/locate/amc`.

[6] T. Furuta, J. M. Hot, J. Pečarić, and Y. Seo, *Mond-Pečarić method in operator inequalities, Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[7] A. G. Ghazanfari,The Hermite-Hadamard type inequalities for operator s-convex functions, *Journal of Advanced Research in Pure Mathematics*, 6:3, 52–61 (2014); Available online at `http://dx.doi.org/10.5373/jarpm.1876.110613`.

[8] V. G. A. Mihesan, *A generalization of the convexity, Seminar on Functional Equations, Approx. and Convex.*, Cluj-Napoca, Romania (1993).

[9] M. S. Moslehian and H. Najafi, Around operator monotone functions, *Integr. Equ. Oper. Theory.*, 71, 575–582 (2011); Available online at `http://dx.doi.org/10.1007/s00020-011-1921-0`.

[10] E. Set, M. Z. Sardari, M. E. Ozdemir, and J. Rooin, On generalizations of the Hadamard inequality for (α, m)-convex functions, *RGMIA Res. Rep. Coll.*, 12:4, Article 4 (2009).

[11] G. Toader,Some generalizations of the convexity, *Proc. Colloq. Approx. Opt. ClujNapoca*, 329–338 (1984)

[12] S. H. Wang, B. Y. Xi, and F. Qi, Some new inequalities of Hermite-Hadamard type n-time differentiable functions which are m-convex, *Analysis*, 32, 247–262 (2012); Available online at `http://dx.doi.org/10.1524/anly.2012.1167`.

[13] S. H. Wang, B. Y. Xi, and F. Qi, On Hermite-Hadamard type inequalities for $(\alpha; m)$-Convex Functions, *Int. J. Open Problems Comput. Math.*, 5:4, 46–57 (2012); Available online at `http://www.i-csrs.org`.

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Non-periodic Multivariate Stochastic Fourier Sine Approximation and Uncertainty Analysis *

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Abstract. In data analysis, one needs to study Fourier sine analysis on the unit cube. However, for this kind of non-periodic case, no exact result is available. In this paper, firstly, based on our multivariate function decomposition, we deduce an asymptotic formula of Fourier sine coefficients of continuously differentiable functions f on $[0,1]^d$. Secondly, we deduce an asymptotic formula of hyperbolic cross approximations of Fourier sine series of f on $[0,1]^d$. By this way we can reconstruct high-dimensional signals by using fewest Fourier sine coefficients. Thirdly, we extend these results to Fourier sine analysis of stochastic processes and give uncertainty of stochastic Fourier sine approximation, i.e., we obtain expectations and variances of stochastic Fourier sine coefficients and stochastic Fourier sine approximation errors. Finally, we discuss some known stochastic processes.

Key words: asymptotic behavior, multivariate decomposition, stochastic approximation, hyperbolic cross truncation

1. Introduction

It is well known that Fourier sine analysis on $[0,1]^d$ is an important tool for signal processing. Based our decomposition of multivariate continuous functions on the cube [11], we first deduce an asymptotic formula of Fourier sine coefficients of continuously differentiable function f on $[0,1]^d$ and obtain a necessary and sufficient condition:

$$c_{\mathbf{n}}(f) = o\left(\frac{1}{n_1 \cdots n_d}\right)$$

for each $n_k \to \infty$. Next we deduce an asymptotic formula of hyperbolic cross truncations of the Fourier sine series of f. Thirdly, we extend these results to the case of stochastic processes. In detail, we will obtain the following three asymptotic behaviors of stochastic Fourier sine analysis.

Suppose that ξ is a continuously differentiable stochastic process on $[0,1]^d$.

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(i) The expectation of Fourier sine coefficients $c_{\mathbf{n}}(\xi)$ satisfy

$$E[\, c_{\mathbf{n}}(\xi)\,] = \left(\prod_{j=1}^{d} \frac{2}{\pi n_j}\right)(\alpha_{\mathbf{n}}(\xi)+o(1)) \qquad (\mathbf{n}=(n_1,...,n_d)\in \mathbb{Z}_+^d)$$

for each $n_k\to\infty$, where $\alpha_{\mathbf{n}}(\xi)$ is an algebraic sum of expectation of ξ at vertexes of the cube $[0,1]^d$.

(ii) The variance of Fourier sine coefficients $c_{\mathbf{n}}(\xi)$ satisfy

$$\mathrm{Var}\,(c_{\mathbf{n}}(\xi)) = \left(\prod_{j=1}^{d} \frac{4}{\pi^2 n_j^2}\right)(\theta_{\mathbf{n}}(\xi)+o(1))$$

for each $n_k\to\infty$, where $\theta_{\mathbf{n}}(\xi)$ is an algebraic sum of covariance of $\xi(\lambda)$ and $\xi(\lambda')$, where λ and λ' are any two vertexes of $[0,1]^d$.

(iii) The mean square error of hyperbolic cross truncations $S_N^{(h)}(\xi)$ (see (6.1)) of the Fourier sine series of ξ satisfies

$$E[\| S_N^{(h)}(\xi)-\xi \|_2^2] = W_N(1+o(1)) \qquad (N\to\infty),$$

the principal part W_N is equal to

$$W_N = \left(\frac{1}{\pi^2}\right)^d \left(\sum_{\mathbf{p}\in\Theta_N} \frac{1}{p_1^2p_2^2\cdots p_d^2}\right)\left(\sum_{\lambda\in\{0,1\}^d} E[|\xi(\lambda)|^2]\right)$$

and

$$W_N \sim \frac{\log^{d-1} N}{N},$$

where $\{0,1\}^d$ is the set of vertexes of the cube $[0,1]^d$. The number of Fourier sine coefficients in the Nth hyperbolic cross truncation $S_N^{(h)}(\xi)$ satisfies $N_c\sim N\log^{d-1}N$. So

$$E[\| \xi - S_N^{(h)}(\xi) \|_2^2] \sim \frac{\log^{2d-2} N_c}{N_c}.$$

However, the number of Fourier sine coefficients in the Nth ordinary partial sum satisfies $N_c = N^d$. So

$$E[\| \xi - S_N(\xi) \|_2^2] \sim \frac{1}{N_c^{\frac{1}{d}}}.$$

Finally, we discuss some known stochastic processes.

2. Preliminaries

Denote the set of vertexes of the unit cube $[0,1]^d$ by $\{0,1\}^d$ and the boundary of $[0,1]^d$ by $\partial([0,1]^d)$, and $\mathbb{Z}_+^d=\{(n_1,...,n_d)|$ each $n_k\in\mathbb{Z}_+\}$ and $\mathbb{Z}_+$ is the set of natural numbers.

If f is a function defined on $[0,1]^d$ and $\frac{\partial^d f}{\partial t_1\cdots\partial t_d}$ continuous on $[0,1]^d$, we say $f\in W([0,1]^d)$. If ξ is a stochastic process defined on $[0,1]^d$ and $\frac{\partial^d \xi}{\partial t_1\cdots\partial t_d}$ continuous on $[0,1]^d$, we say $\xi\in SW([0,1]^d)$. Denote the expectation and variance of a stochastic variable η by $E[\eta]$ and $\mathrm{Var}(\eta)$, respectively. Denote the covariance and correlation of two stochastic variable ξ,η by $\mathrm{Cov}(\xi,\eta)$ and $R(\xi,\eta)$, respectively.

2.1. Projection operators and fundamental polynomials

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We always assume e_1 and e_2 are two disjoint subsets of the set $\{1, 2, ..., d\}$. Define a projection operator Q_{e_1,e_2} from $[0,1]^d$ to $\partial([0,1]^d)$ as

$$Q_{e_1,e_2}(t_1, ..., t_d) = (v_1, ..., v_d), \tag{2.1}$$

where

$$v_k = \begin{cases} 0, & k \in e_1, \\ 1, & k \in e_2, \\ t_k, & k \in e \quad (e = \{1, ..., d\} \setminus (e_1 \bigcup e_2)). \end{cases}$$

The fundamental polynomial $P^{(e_1,e_2)}(\mathbf{t})$ is defined as

$$P^{(e_1,e_2)}(\mathbf{t}) = \prod_{k \in e_1} (1 - t_k) \prod_{k \in e_2} t_k. \tag{2.2}$$

For example, consider the case $d = 3$. If $e_1 = \{1, 3\}$ and $e_2 = 2$, then

$$Q_{e_1,e_2}(\mathbf{t}) = (0, 1, 0),$$

$$P^{(e_1,e_2)}(\mathbf{t}) = t_2(1 - t_1)(1 - t_3),$$

where $\mathbf{t} = (t_1, t_2, t_3) \in [0,1]^3$. If $e_1 = \emptyset$ and $e_2 = \{1, 2\}$, then

$$Q_{e_1,e_2}(\mathbf{t}) = (1, 1, t_3),$$

$$P^{(e_1,e_2)}(\mathbf{t}) = t_1 t_2,$$

where $\mathbf{t} = (t_1, t_2, t_3) \in [0,1]^3$.

2.2. Decompositions of continuous functions on $[0,1]^d$

Any continuous function f on the cube $[0,1]^d$ can be decomposed into [11]

$$f = \sum_{\nu=1}^{d+1} h_\nu, \tag{2.3}$$

where

$$h_1 = \sum_{|e_1|+|e_2|=d} f(Q_{e_1,e_2}) P^{(e_1,e_2)}.$$

$$h_\nu = \sum_{|e_1|+|e_2|=d-\nu+1} f_{\nu-1}(Q_{e_1,e_2}) P^{(e_1,e_2)} \qquad (2 \le \nu \le d),$$

$$h_{d+1} = f - h_1 - \cdots - h_d,$$

and

$$f_0 = f,$$

$$f_{\nu-1} = f_{\nu-2} - h_{\nu-1} \qquad (2 \le \nu \le d),$$

$$f_d = f_{d-1} - h_d.$$

and the cardinality of a set F is denoted by $|F|$, and

$$\sum_{|e_1|+|e_2|=k} A_{e_1,e_2} := \sum_{\substack{e_1,e_2 \in \{1,...,d\} \\ e_1 \bigcap e_2 = \emptyset \\ |e_1|+|e_2|=k}} A_{e_1,e_2}.$$

3

The following proposition shows the structure of each h_ν.

Proposition 2.1 [11]. If f is a $d-$variate continuous function on the unit cube $[0,1]^d$, then

(i) h_1 is a $d-$variate polynomial and each $f(Q_{e_1,e_2}\mathbf{t})$ is a constant and it is the value of f at a vertex of the cube $[0,1]^d$. Precisely say,

$$f(Q_{e_1,e_2}\mathbf{t}) = f(\lambda_1, ..., \lambda_d),$$

where

$$\lambda_k = \begin{cases} 0, & k \in e_1, \\ 1, & k \in e_2 \quad (e_1 \bigcup e_2 = \{1, ..., d\}). \end{cases}$$

(ii) for each $2 \le \nu \le d$, h_ν is a sum of products of a $(\nu-1)-$variate function $f_{\nu-1}(Q_{e_1,e_2}\cdot)$ on $[0,1]^{\nu-1}$ and $(d-\nu+1)-$variate polynomial $P^{(e_1,e_2)}$, where each product is of separation of variables. Moreover,

$$f_{\nu-1}(Q_{e_1,e_2}\cdot) = 0 \quad \text{on} \quad \partial([0,1]^{\nu-1});$$

(iii) h_{d+1} is a $d-$variate function on $[0,1]^d$ and $h_{d+1}(\cdot) = 0$ on $\partial([0,1]^d)$.

If ξ is a d-variate continuous stochastic process on $[0,1]^d$, then the above decomposition and Proposition 2.1 are still valid.

For example, if ξ is a bivariate continuous function on $[0,1]^2$, then $\xi = h_1 + h_2 + h_3$ and

$$h_1(\mathbf{t}) = \xi(0,0)(1-t_1)(1-t_2) + \xi(0,1)(1-t_1)t_2 + \xi(1,0)t_1(1-t_2) + \xi(1,1)t_1t_2,$$

$$h_2(\mathbf{t}) = \xi_1(1,t_2)t_1 + \xi_1(0,t_2)(1-t_1) + \xi_1(t_1,1)t_2 + \xi_1(t_1,0)(1-t_2) \qquad (\xi_1 = \xi - h_1),$$

$$h_3(\mathbf{t}) = \xi_1(\mathbf{t}) - h_2(\mathbf{t}).$$

We see from this decomposition that h_1 is a polynomial determined by values of f at four vertexes of $[0,1]^2$, h_2 is a sum of products of separation of variables, and $h_2(\mathbf{t}) = 0$ at four vertexes of $[0,1]^2$, and the bivariate function $h_3(\mathbf{t})$ vanishes on the boundary of $[0,1]^2$.

2.3. Fourier sine series of stochastic processes

Let a probability space (Ω, F, P) be given. A stochastic variable ξ is defined as a function ξ from Ω to $\mathbb{R}$ or $\mathbb{C}$. In this paper we always assume that ξ satisfies $E[|\xi|^2] < \infty$, i.e., assume that ξ is a second-order stochastic variable. For a stochastic process $\xi(\mathbf{t})$ on $[0,1]^d$, its autocorrelation function and covariance function are defined respectively as:

$$R_\xi(\mathbf{t},\mathbf{s}) := E[\xi(\mathbf{t})\xi(\mathbf{s})]$$

$$\mathrm{Cov}(\xi(\mathbf{t}),\ \xi(\mathbf{s})) := \mathbf{E}[(\xi(\mathbf{t}) - \mathbf{E}[\xi(\mathbf{t})])(\xi(\mathbf{s}) - \mathbf{E}[\xi(\mathbf{s})])]$$

We recall some known concepts in stochastic calculus [14, 15].

Let $\{\xi_n\}_1^\infty$ be a sequence of second-order stochastic variables and ξ be a second-order stochastic variable. If $\lim_{n\to\infty} E[|\xi_n - \xi|^2] = 0$, then we say $\{\xi_n\}_1^\infty$ converges to ξ in the mean square sense. Based on the above concepts, one can derive the concept of continuous and the concepts of the derivatives and the integrals of stochastic processes [3].

For a stochastic process ξ on $[0,1]^d$, the derivative and the expectation can be exchanged, the integral and the expectation can be exchanged, and Newton-Leibnitz formula holds. For a product of a stochastic process and a deterministic function, differential formula of products holds and the integration by parts also holds.

Let $\xi(\mathbf{t})$ be a stochastic process on $[0,1]^d$ and

$$\int_{[0,1]^d} E[\xi^2(\mathbf{t})]\mathrm{d}\mathbf{t} < \infty.$$

Then $\xi(\mathbf{t})$ can be expanded into the stochastic Fourier sine series

$$\xi(\mathbf{t}) = \sum_{\mathbf{n}\in\mathbb{Z}_+^d} c_{\mathbf{n}}(\xi)\, T_{\mathbf{n}}(\mathbf{t}) \qquad (\, T_{\mathbf{n}}(\mathbf{t}) = \prod_{j=1}^{d} \sin \pi n_j t_j \,) \tag{2.4}$$

in mean square sense, where the Fourier sine coefficients are stochastic variables and

$$c_{\mathbf{n}}(\xi) = 2^d \int_{[0,1]^d} \xi(\mathbf{t})\, T_{\mathbf{n}}(\mathbf{t})\, \mathrm{d}\mathbf{t} \ \ (\mathbf{n} \in \mathbb{Z}_+^d).$$

3. Asymptotic behavior of Fourier sine coefficients

Let f be a continuous function on the unit cube $[0,1]^d$. By (2.3), f can be decomposed as

$$f = \sum_{\nu=1}^{d+1} h_\nu,$$

where $\{h_\nu\}_1^{d+1}$ are stated in Section 2.1 and

$$f_0 = f,$$

$$f_{\nu-1} = f_{\nu-2} - h_{\nu-1} \qquad (2 \le \nu \le d).$$

If $f \in W([0,1]^d)$, we easily prove

$$f_{\nu-1} \in W([0,1]^d) \qquad (\nu = 2, ..., d). \tag{3.1}$$

Based on this decomposition, we give an asymptotic representation of Fourier sine coefficients of f.

Theorem 3.1. If f is a continuous function on $[0,1]^d$ and $f \in W([0,1]^d)$, then its Fourier sine coefficients $c_{\mathbf{n}}(f)$ possess the asymptotic behavior:

$$c_{\mathbf{n}}(f) = \left(\prod_{j=1}^{d} \frac{2}{\pi n_j} \right) (K_{\mathbf{n}}^d(f) + \eta_1 + \cdots + \eta_d),$$

where $\eta_k \to 0$ as $n_k \to \infty \, (k = 1, ., , , d)$ and

$$K_{\mathbf{n}}^d(f) = \sum_{\lambda\in\{0,1\}^d} f(\lambda)\epsilon_{\mathbf{n}}(\lambda),$$

$$\epsilon_{\mathbf{n}}(\lambda) = \begin{cases} \prod\limits_{j\in G_\lambda} (-1)^{n_j+1}, & G_\lambda \ne \emptyset, \\ 1, & G_\lambda = \emptyset, \end{cases} \tag{3.2}$$

where $\lambda = (\lambda_1, ..., \lambda_d) \in \{0,1\}^d$ and $\mathbf{n} = (n_1, ..., n_d)$, and $G_\lambda = \{j \in \{1, ..., d\}, \ \lambda_j = 1\}$.

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From this, we see that $\epsilon_{\mathbf{n}}(\lambda)=\pm 1$ and $K_{\mathbf{n}}^d(f)$ is an algebraic sum of values of f at vertexes of $[0,1]^d$. From Theorem 3.1, we deduce the following corollary. This corollary plays an important role in the proof of Theorem 4.1.

Corollary 3.2. If f is a continuous function on $[0,1]^d$ and $f\in W([0,1]^d)$, then its Fourier sine coefficients $c_{\mathbf{n}}(f)$ satisfy

(i) $\sum_{\mathbf{q}\in\{0,1\}^d}|c_{2\mathbf{p}+\mathbf{q}}(f)|^2=\left(\frac{2}{\pi^2}\right)^d\frac{1}{p_1^2p_2^2\cdots p_d^2}\left(\sum\limits_{\lambda\in\{0,1\}^d}|f(\lambda)|^2+\eta_1'+\cdots+\eta_d'\right)$;

(ii) $c_{\mathbf{n}}(f)=\left(\prod\limits_{j=1}^d\frac{1}{n_j}\right)(\eta_1+\cdots+\eta_d)$ if and only if $f(\lambda)=0$ $(\lambda\in\{0,1\}^d)$ ($\eta_k\to 0$ as $n_k\to\infty$).

From Corollary 3.2, we deduce immediately that $c_{\mathbf{n}}(f)=o\left(\frac{1}{n_1\cdots n_d}\right)$ for each $n_k\to\infty$ if and only if $f(\lambda)=0$ $(\lambda\in\{0,1\}^d)$.

For example, consider the case $d=3$. If $f\in W([0,1]^3)$, then Fourier sine coefficients $c_{n_1,n_2,n_3}(f)$ possess the asymptotic behavior:

$$\begin{aligned}c_{n_1,n_2,n_3}(f)=\frac{8}{n_1n_2n_3\pi^3}(\,&f(0,0,0)-(-1)^{n_1}f(1,0,0)-(-1)^{n_2}f(0,1,0)\\&-(-1)^{n_3}f(0,0,1)+(-1)^{n_1+n_2}f(1,1,0)+(-1)^{n_1+n_3}f(1,0,1)\\&+(-1)^{n_2+n_3}f(0,1,1)-(-1)^{n_1+n_2+n_3}f(1,1,1)+\eta_1+\eta_2+\eta_3\,),\end{aligned}$$

where $\eta_k\to 0$ as $n_k\to\infty$ $(k=1,2,3)$.

Proof of Theorem 3.1. By (2.3), the Fourier sine coefficients $c_{\mathbf{n}}(f)$ satisfy

$$c_{\mathbf{n}}(f)=\sum_{\nu=1}^{d+1}c_{\mathbf{n}}(h_\nu),\tag{3.3}$$

where $c_{\mathbf{n}}(h_\nu)=2^d\int\limits_{[0,1]^d}h_\nu(\mathbf{t})T_{\mathbf{n}}(\mathbf{t})\mathrm{d}\mathbf{t}$.

First, we compute $c_{\mathbf{n}}(h_1)$. By (2.4), we have

$$c_{\mathbf{n}}(h_1)=2^d\sum_{|e_1|+|e_2|=d}\int\limits_{[0,1]^d}f(Q_{e_1,e_2}\mathbf{t})P^{(e_1,e_2)}(\mathbf{t})T_{\mathbf{n}}(\mathbf{t})\mathrm{d}\mathbf{t}.$$

By Proposition 2.1 (i), $f(Q_{e_1,e_2}\mathbf{t})=f(\lambda)$ is a constant independent of $\mathbf{t}$. So

$$c_{\mathbf{n}}(h_1)=2^d\sum_{|e_1|+|e_2|=d}f(\lambda_1,...,\lambda_d)\int\limits_{[0,1]^d}P^{(e_1,e_2)}(\mathbf{t})T_{\mathbf{n}}(\mathbf{t})\mathrm{d}\mathbf{t},\tag{3.4}$$

where $\lambda_k=\begin{cases}0, & k\in e_1,\\ 1, & k\in e_2\ (e_1\bigcup e_2=\{1,...,d\}).\end{cases}$ Since

$$P^{(e_1,e_2)}(\mathbf{t})=\prod_{j\in e_1}(1-t_j)\prod_{j\in e_2}t_j,$$

$$T_{\mathbf{n}}(\mathbf{t})=\prod_{j=1}^d\sin(\pi n_jt_j),\qquad \mathbf{t}=(t_1,...,t_d),$$

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a direct computation shows that

$$\begin{aligned}
\int\limits_{[0,1]^d} P^{(e_1,e_2)}(\mathbf{t})T_{\mathbf{n}}(\mathbf{t})\mathrm{d}\mathbf{t} &= \left(\prod_{j\in e_1}\int\limits_0^1(1-t_j)\sin(\pi n_j t_j)\mathrm{d}t_j\right)\left(\prod_{j\in e_2}\int\limits_0^1 t_j\sin(\pi n_j t_j)\mathrm{d}t_j\right)\\
&= \left(\prod_{j\in e_1}\tfrac{1}{\pi n_j}\right)\left(\prod_{j\in e_2}\tfrac{(-1)^{n_j+1}}{\pi n_j}\right)\\
&= \left(\prod_{j\in(e_1\bigcup e_2)}\tfrac{1}{\pi n_j}\right)\prod_{j\in e_2}(-1)^{n_j+1}\\
&= \left(\prod_{j=1}^{d}\tfrac{1}{\pi n_j}\right)\prod_{j\in e_2}(-1)^{n_j+1}\qquad (e_1\bigcup e_2=\{1,...,d\}).
\end{aligned}$$

From this and (3.4), we get

$$c_{\mathbf{n}}(h_1)=2^d\left(\sum_{|e_1|+|e_2|=d} f(\lambda_1,...,\lambda_d)\prod_{j\in e_2}(-1)^{n_j+1}\right)\prod_{j=1}^{d}\frac{1}{\pi n_j}.$$

Since $e_1\bigcup e_2=\{1,...,d\}$ and $e_1=\{j\in\{1,...,d\},\ \lambda_j=0\}$, and $e_2=\{j\in\{1,...,d\},\ \lambda_j=1\}=: G_\lambda$,

$$c_{\mathbf{n}}(h_1)=2^d\,K_{\mathbf{n}}^d(f)\left(\prod_{j=1}^{d}\frac{1}{\pi n_j}\right), \tag{3.5}$$

where $K_{\mathbf{n}}^d(f)=\sum\limits_{\lambda\in\{0,1\}^d} f(\lambda)\prod\limits_{j\in G_\lambda}(-1)^{n_j+1}$.

Next, we compute $c_{\mathbf{n}}(h_\nu)$ $(2\le\nu\le d)$. By (2.3) and (2.4),

$$c_{\mathbf{n}}(h_\nu)=2^d\sum_{|e_1|+|e_2|=d-\nu+1}\ \int\limits_{[0,1]^d} f_{\nu-1}(Q_{e_1,e_2}\mathbf{t})\,P^{(e_1,e_2)}(\mathbf{t})T_{\mathbf{n}}(\mathbf{t})\mathrm{d}\mathbf{t}. \tag{3.6}$$

Since $|e_1|+|e_2|=d-\nu+1$, we may denote $e_1\bigcup e_2=\{\beta_1,...,\beta_{d-\nu+1}\}$. By (2.2), the fundamental polynomial $P^{(e_1,e_2)}(\mathbf{t})$ only depends on $t_{\beta_1},...,t_{\beta_{d-\nu+1}}$, write

$$P^{(e_1,e_2)}(\mathbf{t})=P^{(e_1,e_2)}(t_{\beta_1},...,t_{\beta_{d-\nu+1}}).$$

Since $e=\{1,...,d\}\setminus(e_1\bigcup e_2)$ and $|e|=\nu-1$, we may denote $e=\{\alpha_1,...,\alpha_{\nu-1}\}$. Then $f_{\nu-1}(Q_{e_1,e_2}\mathbf{t})$ only depends on $t_{\alpha_1},...,t_{\alpha_{\nu-1}}$, write

$$f_{\nu-1}(Q_{e_1,e_2}\mathbf{t})=f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},...,t_{\alpha_{\nu-1}}). \tag{3.7}$$

So each product $f_{\nu-1}(Q_{e_1,e_2}\mathbf{t})P^{(e_1,e_2)}(\mathbf{t})$ in (3.6) is of separated variable type, and so

$$\int\limits_{[0,1]^d} f_{\nu-1}(Q_{e_1,e_2}\mathbf{t})\,P^{(e_1,e_2)}(\mathbf{t})T_{\mathbf{n}}(\mathbf{t})\mathrm{d}\mathbf{t}=L_{\nu,\mathbf{n}}^{(1)}(e_1,e_2)L_{\nu,\mathbf{n}}^{(2)}(e_1,e_2), \tag{3.8}$$

where

$$L_{\nu,\mathbf{n}}^{(1)}(e_1,e_2)=\int\limits_{[0,1]^{d-\nu+1}} P^{(e_1,e_2)}(t_{\beta_1},...,t_{\beta_{d-\nu+1}})\prod_{j=1}^{d-\nu+1}\sin(\pi n_{\beta_j}t_{\beta_j})\mathrm{d}t_{\beta_1}\cdots\mathrm{d}t_{\beta_{d-\nu+1}},$$

$$L^{(2)}_{\nu,\mathbf{n}}(e_1,e_2)=\int\limits_{[0,1]^{\nu-1}} f^{e_1,e_2}_{\nu-1}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})\prod_{j=1}^{\nu-1}\sin(\pi n_{\alpha_j}t_{\alpha_j})\mathrm{d}t_{\alpha_1}\cdots \mathrm{d}t_{\alpha_{\nu-1}}.$$

We compute $L^{(2)}_{\nu,\mathbf{n}}(e_1,e_2)$. We rewrite it as follows:

$$L^{(2)}_{\nu,\mathbf{n}}(e_1,e_2)=\int\limits_{[0,1]^{\nu-2}}\left(\int\limits_0^1 f^{e_1,e_2}_{\nu-1}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})\sin(\pi n_{\alpha_1}t_{\alpha_1})\mathrm{d}t_{\alpha_1}\right)\prod_{j=2}^{\nu-1}\sin(\pi n_{\alpha_j}t_{\alpha_j})\mathrm{d}t_{\alpha_2}\cdots \mathrm{d}t_{\alpha_{\nu-1}}. \tag{3.9}$$

From (3.1) and (3.7), we know that $f^{e_1,e_2}_{\nu-1}\in W([0,1]^{\nu-1})$. Using integration by parts, the part inside brackets:

$$\int\limits_0^1 f^{e_1,e_2}_{\nu-1}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})\sin(\pi n_{\alpha_1}t_{\alpha_1})\mathrm{d}t_{\alpha_1} = -f^{e_1,e_2}_{\nu-1}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})\frac{\cos(\pi n_{\alpha_1}t_{\alpha_1})}{\pi n_{\alpha_k}}\bigg|_{t_{\alpha_1}=0}^1$$

$$+\frac{1}{\pi n_{\alpha_1}}\int\limits_0^1\frac{\partial f^{e_1,e_2}_{\nu-1}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})}{\partial t_{\alpha_1}}\cos(\pi n_{\alpha_1}t_{\alpha_1})\mathrm{d}t_{\alpha_1}.$$

By (3.7) and Proposition 2.1 (ii), we have

$$f^{e_1,e_2}_{\nu-1}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})\big|_{t_{\alpha_k}=0}^1=0 \quad (k=1,...,\nu-1). \tag{3.10}$$

Therefore,

$$\int\limits_0^1 f^{e_1,e_2}_{\nu-1}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})\sin(\pi n_{\alpha_1}t_{\alpha_1})\mathrm{d}t_{\alpha_1} = \frac{1}{\pi n_{\alpha_1}}\int\limits_0^1\frac{\partial f^{e_1,e_2}_{\nu-1}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})}{\partial t_{\alpha_1}}\cos(\pi n_{\alpha_1}t_{\alpha_1})\mathrm{d}t_{\alpha_1}.$$

From this and (3.9), we get

$$L^{(2)}_{\nu,\mathbf{n}}(e_1,e_2)=\frac{1}{\pi n_{\alpha_1}}\int\limits_{[0,1]^{\nu-2}}\left(\int\limits_0^1\frac{\partial f^{e_1,e_2}_{\nu-1}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})}{\partial t_{\alpha_1}}\sin(\pi n_{\alpha_2}t_{\alpha_2})\,\mathrm{d}t_{\alpha_2}\right)\prod_{j=3}^{\nu-1}\sin(\pi n_{\alpha_j}t_{\alpha_j})$$

$$\cos(\pi n_{\alpha_1}t_{\alpha_1})\,\mathrm{d}t_{\alpha_1}\mathrm{d}t_{\alpha_3}\cdots\mathrm{d}t_{\alpha_{\nu-1}}. \tag{3.11}$$

Using integration by parts, the part inside brackets:

$$\int\limits_0^1\frac{\partial}{\partial t_{\alpha_1}}f^{e_1,e_2}_{\nu-1}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})\,\sin(\pi n_{\alpha_2}t_{\alpha_2})\,\mathrm{d}t_{\alpha_2}$$

$$=-\frac{\partial f^{e_1,e_2}_{\nu-1}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})}{\partial t_{\alpha_1}}\frac{\cos(\pi\alpha_2 t_{\alpha_2})}{\pi n_{\alpha_2}}\bigg|_{t_{\alpha_2}=0}^1$$

$$+\frac{1}{\pi n_{\alpha_2}}\int\limits_{[0,1]^{\nu-1}}\frac{\partial^2 f^{e_1,e_2}_{\nu-1}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})}{\partial t_{\alpha_1}\partial t_{\alpha_2}}\cos(\pi n_{\alpha_2}t_{\alpha_2})\,\mathrm{d}t_{\alpha_2}.$$

By Proposition 2.1 (ii),

$$f_{\nu-1}(t_{\alpha_1},0,t_{\alpha_3},...,t_{\alpha_{\nu-1}})=f_{\nu-1}(t_{\alpha_1},1,t_{\alpha_3},...,t_{\alpha_{\nu-1}})=0,$$

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and so

$$\frac{\partial f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},0,...,t_{\alpha_{\nu-1}})}{\partial t_{\alpha_1}}=\frac{\partial f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},1,...,t_{\alpha_{\nu-1}})}{\partial t_{\alpha_1}}=0.$$

Therefore,

$$\int_0^1 \frac{\partial}{\partial t_{\alpha_1}} f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})\,\sin(\pi n_{\alpha_2}t_{\alpha_2})\,\mathrm{d}t_{\alpha_2}$$

$$=\frac{1}{\pi n_{\alpha_2}}\int_{[0,1]^{\nu-1}} \frac{\partial^2 f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})}{\partial t_{\alpha_1}\partial t_{\alpha_2}}\,\cos(\pi n_{\alpha_2}t_{\alpha_2})\,\mathrm{d}t_{\alpha_2}.$$

From this and (3.11), we get

$$L_{\nu,\mathbf{n}}^{(2)}(e_1,e_2)=\frac{1}{(\pi n_{\alpha_1})(\pi n_{\alpha_2})}\int_{[0,1]^{\nu-2}}\left(\int_0^1 \frac{\partial^2 f_{\nu-1}^{e_1,e_2}(t_{\alpha_1,...,t_{\alpha_{\nu-1}}})}{\partial t_{\alpha_1}\partial t_{\alpha_2}}\,\sin(\pi n_{\alpha_3}t_{\alpha_3})\,\mathrm{d}t_{\alpha_3}\right)$$

$$\bullet\prod_{j=4}^{\nu-1}\sin(\pi n_{\alpha_j}t_{\alpha_j})\cos(\pi n_{\alpha_1}t_{\alpha_1})\,\cos(\pi n_{\alpha_2}t_{\alpha_2})\,\mathrm{d}t_{\alpha_1}\mathrm{d}t_{\alpha_2}\mathrm{d}t_{\alpha_4}\cdots\mathrm{d}t_{\alpha_{\nu-1}}.$$

Continuing this procedure, we deduce finally that

$$L_{\nu,\mathbf{n}}^{(2)}(e_1,e_2)=\left(\prod_{j=1}^{\nu-1}\frac{1}{\pi n_{\alpha_j}}\right)\int_{[0,1]^{\nu-1}}\frac{\partial^{\nu-1} f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},...,t_{\alpha_{\nu-1}})}{\partial t_{\alpha_1}\cdots\partial t_{\alpha_{\nu-1}}}\left(\prod_{j=1}^{\nu-1}\cos(\pi n_{\alpha_j}t_{\alpha_j})\right)\,\mathrm{d}t_{\alpha_1}\cdots\mathrm{d}t_{\alpha_{\nu-1}}.$$

Since

$$\frac{\partial^{\nu-1} f_{\nu-1}^{e_1,e_2}(t_{\alpha_1},\cdots,t_{\alpha_{\nu-1}})}{\partial t_{\alpha_1}\cdots\partial t_{\alpha_{\nu-1}}}\in C([0,1]^{\nu-1})$$

and $e=\{\alpha_1,...,\alpha_{\nu-1}\}$, applying the Riemann-Lebesgue lemma, we get

$$L_{\nu,\mathbf{n}}^{(2)}(e_1,e_2)=o\left(\prod_{j=1}^{\nu-1}\frac{1}{n_{\alpha_j}}\right)=\left(\prod_{j\in e}\frac{1}{n_j}\right)\epsilon_e, \tag{3.12}$$

where $\epsilon_e\to 0$ as $n_j\to\infty$ $(j\in e)$.

We compute $L_{\nu,\mathbf{n}}^{(1)}(e_1,e_2)$. Notice that $e_1\bigcup e_2=\{\beta_1,...,\beta_{d-\nu+1}\}$. We assume in which

$$e_1=\{\gamma_1,...,\gamma_{m_1}\},$$

$$e_2=\{\delta_1,...,\delta_{m_2}\},$$

where $m_1+m_2=d-\nu+1$. By (2.2), we get

$$L_{\nu,\mathbf{n}}^{(1)}(e_1,e_2)$$

$$=\int_{[0,1]^{d-\nu+1}} P^{(e_1,e_2)}(t_{\beta_1},...,t_{\beta_{d-\nu+1}})\prod_{j=1}^{d-\nu+1}\sin(\pi n_{\beta_j}t_{\beta_j})\mathrm{d}t_{\beta_1}\cdots\mathrm{d}t_{\beta_{d-\nu+1}},$$

$$=\int_{[0,1]^{m_1}}\prod_{j=1}^{m_1}(1-t_{\gamma_j})\,\sin(\pi n_{\gamma_j}t_{\gamma_j})\,\mathrm{d}t_{\gamma_1}\cdots\mathrm{d}t_{\gamma_{m_1}}\int_{[0,1]^{m_2}}\prod_{j=1}^{m_2}t_{\delta_j}\sin(\pi n_{\delta_j}t_{\delta_j})\,\mathrm{d}t_{\delta_1}\cdots\mathrm{d}t_{\delta_{m_2}}.$$

A direct computation shows that

$$L^{(1)}_{\nu,\mathbf{n}}(e_1,e_2)=\left(\prod_{j=1}^{m_1}\frac{1}{\pi n_{\gamma_j}}\right)\left(\prod_{j=1}^{m_2}\frac{(-1)^{n_{\delta_j}+1}}{\pi n_{\delta_j}}\right)=O\left(\prod_{j\in e_1\bigcup e_2}\frac{1}{n_j}\right).$$

By (3.12) and (3.6), we have

$$\int\limits_{[0,1]^d} f_{\nu-1}(Q_{e_1,e_2}\mathbf{t})\,P^{(e_1,e_2)}(\mathbf{t})T_{\mathbf{n}}(\mathbf{t})\mathrm{d}\mathbf{t}=\left(\prod_{j=1}^{d}\frac{1}{n_j}\right)\epsilon_e$$

and

$$c_{\mathbf{n}}(h_\nu)=2^d\left(\prod_{j=1}^{d}\frac{1}{n_j}\right)\left(\sum_{|e_1|+|e_2|=d-\nu+1}\epsilon_e\right),$$

where $e=\{1,...,d\}\setminus(e_1\bigcup e_2)$ and $\epsilon_e\to 0$ as $n_j\to\infty\,(j\in e)$. From this, we can deduce that

$$c_{\mathbf{n}}(h_\nu)=\left(\prod_{j=1}^{d}\frac{1}{n_j}\right)(\eta_1^\nu+\cdots+\eta_d^\nu)\qquad(\nu=2,...,d),\tag{3.13}$$

where $\eta_k^{\prime\prime}\to 0$ as $n_k\to\infty$ $(k=1,...,d)$. Finally, we compute $c_{\mathbf{n}}(h_{d+1})$. Since

$$c_{\mathbf{n}}(h_{d+1})=2^d\int\limits_{[0,1]^d} h_{d+1}(t_1,...,t_d)\prod_{j=1}^{d}\sin(\pi n_j t_j)\,\mathrm{d}t_1\cdots\mathrm{d}t_d,$$

by Proposition 2.1 (iii) and (3.3), applying the integration by parts and the Riemann-Lebesgue lemma, we get

$$\begin{aligned}c_{\mathbf{n}}(h_{d+1}) &= 2^d\prod_{j=1}^{d}\tfrac{1}{\pi n_j}\int\limits_{[0,1]^d}\tfrac{\partial^d h_{d+1}(t_1,...,t_d)}{\partial t_1\cdots\partial t_d}\prod_{j=1}^{d}\cos(\pi n_j t_j)\,\mathrm{d}t_1\cdots\mathrm{d}t_d\\ &= o\left(\prod_{j=1}^{d}\tfrac{1}{n_j}\right)=\left(\prod_{j=1}^{d}\tfrac{1}{n_j}\right)\epsilon,\end{aligned}$$

where $\epsilon\to 0$ as $n_j\to\infty$ $(j=1,...,d)$. From this and (3.5), and (3.13), it follows by (3.3) that

$$c_{\mathbf{n}}(f)=c_{\mathbf{n}}(h_1)+\sum_{\nu=2}^{d}c_{\mathbf{n}}(h_\nu)+c_{\mathbf{n}}(h_{d+1})=2^d\left(\prod_{j=1}^{d}\frac{1}{\pi n_j}\right)\left(K^d_{\mathbf{n}}(f)+\eta_1+\cdots+\eta_d\right),$$

where $K^d_{\mathbf{n}}(f)$ is stated in (3.5) and $\eta_k\to 0$ as $n_k\to\infty\,(k=1,...,d)$. Theorem 3.1 is proved. □

Proof of Corollary 3.2. Let $n=2\,\mathbf{p}+\mathbf{q}$ ($\mathbf{p}\in Z^d_+$, $\mathbf{q}\in\{0,1\}^d$). Then, for each $\mathbf{q}=(q_1,...,q_d)$ and $\mathbf{p}=(p_1,...,p_d)$, by Theorem 3.1, we have

$$c_{2\mathbf{p}+\mathbf{q}}(f)=2^d\left(\prod_{j=1}^{d}\frac{1}{\pi(2p_j+q_j)}\right)(K^d_{2\mathbf{p}+\mathbf{q}}(f)+o(\eta_1+\cdots+\eta_d))$$

and

$$K^d_{2\mathbf{p}+\mathbf{q}}(f) = \sum_{\lambda\in\{0,1\}^d} f(\lambda) \prod_{j\in G_\lambda} (-1)^{q_j+1} =: K^d_{\mathbf{q}}(f).$$

From this, we see that $K^d_{2\mathbf{p}+\mathbf{q}}$ only depends on $\mathbf{q}$. So, for $\mathbf{q} \in \{0,1\}^d$, we have

$$c_{2\mathbf{p}+\mathbf{q}}(f) = \left(\prod_{j=1}^d \frac{1}{\pi p_j}\right) (K^d_{\mathbf{q}}(f) + \hat{\eta}_1 + \cdots + \hat{\eta}_d, \tag{3.14}$$

where $\hat{\eta}_k \to 0$ as $p_k \to \infty\,(k = 1, ..., d)$ and

$$\sum_{\mathbf{q}\in\{0,1\}^d} |c^2_{2\mathbf{p}+\mathbf{q}}(f)| = \left(\prod_{j=1}^d \frac{1}{\pi^2 p_j^2}\right) \left(\sum_{\mathbf{q}\in\{0,1\}^d} \left(K^d_{\mathbf{q}}(f)\right)^2 + \eta'_1 + \cdots + \eta'_d\right), \tag{3.15}$$

where $\eta'_k \to 0$ as $p_k \to \infty\,(k = 1, ..., d)$. By (3.2), we have

$$\sum_{q\in\{0,1\}^d} (K^d_{\mathbf{q}}(f))^2 = \sum_{\lambda\in\{0,1\}^d} \sum_{\lambda'\in\{0,1\}^d} f(\lambda)f(\lambda') \sum_{q\in\{0,1\}^d} \epsilon_q(\lambda)\epsilon_q(\lambda').$$

where

$$\epsilon_q(\lambda)\epsilon_q(\lambda') = \prod_{j\in G_\lambda} (-1)^{q_j+1} \prod_{j\in G_{\lambda'}} (-1)^{q_j+1} = (-1)^{\sum\limits_{j\in G_\lambda} q_j + \sum\limits_{j\in G_{\lambda'}} q_j + |G_\lambda| + |G_{\lambda'}|}.$$

When $\lambda \neq \lambda'$, without loss of generality, we assume that $i \in G_\lambda,\ i \notin G_{\lambda'}$. So we have

$$\sum_{q\in\{0,1\}^d} \epsilon_q(\lambda)\epsilon_q(\lambda') = \sum_{q_1=0}^1 \cdots \sum_{q_{i-1}=0}^1 \sum_{q_{i+1}=0}^1 \cdots \sum_{q_d=0}^1 (-1)^{\sum\limits_{\substack{j\in G_\lambda \\ j\neq i}} q_j + \sum\limits_{j\in G_{\lambda'}} q_j + |G_\lambda| + |G_{\lambda'}|} \sum_{q_i=0}^1 (-1)^{q_i} = 0. \tag{3.16}$$

When $\lambda = \lambda'$. Then

$$\sum_{q\in\{0,1\}^d} \epsilon_q(\lambda)\epsilon_q(\lambda') = \sum_{q_1=0}^1 \cdots \sum_{q_d=0}^1 (-1)^{2\sum\limits_{j\in G_\lambda} q_j + 2|G_\lambda|} = 2^d. \tag{3.17}$$

From this, we get

$$\sum_{\mathbf{q}\in\{0,1\}^d} (K^d_{\mathbf{q}}(f))^2 = \left(\sum_{\substack{\lambda,\lambda'\in\{0,1\}^d \\ \lambda=\lambda'}} + \sum_{\substack{\lambda,\,\lambda'\in\{0,1\}^d \\ \lambda\neq\lambda'}}\right) f(\lambda)f(\lambda') \sum_{q\in\{0,1\}^d} \epsilon_q(\lambda)\epsilon_q(\lambda') = 2^d \sum_{\lambda\in\{0,1\}^d} f^2(\lambda).$$

From this and (3.15), we get (i). If

$$c_{\mathbf{n}}(f) = \left(\prod_{j=1}^d \frac{1}{n_j}\right) (\eta_1 + \cdots + \eta_d), \tag{3.18}$$

we have

$$\sum_{\mathbf{q}\in\{0,1\}^d} |c_{2\mathbf{p}+\mathbf{q}}|^2 = \left(\prod_{j=1}^d \frac{1}{p_j^2}\right) (\eta_1 + \cdots + \eta_d).$$

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By (i), the later is equivalent to

$$\sum_{\lambda\in\{0,1\}^d} f^2(\lambda)=0$$

which is equivalent to $f(\lambda)=0$ ($\lambda\in\{0,1\}^d$). Conversely, if $f(\lambda)=0$ ($\lambda\in\{0,1\}^d$), then, by Theorem 3.1, we deduce (3.18). So we get (ii). Corollary 3.2 is proved. □

4. Asymptotic behaviors of hyperbolic cross truncation approximation

Approximation rate of multivariate functions by partial sum of Fourier sine series deteriorates rapidly as the dimension d increases.

Let f be a continuous function on $[0,1]^d$. Denote the partial sums of its Fourier sine series by $S_N(f)$:

$$S_N(f;\mathbf{t})=\sum_{n_1,...,n_d=1}^{N} c_{\mathbf{n}}(f)\,T_{\mathbf{n}}(\mathbf{t}) \qquad \mathbf{n}=(n_1,...,n_d),$$

where $c_{\mathbf{n}}(f)=2^d\int\limits_{[0,1]^d} f(\mathbf{t})T_{\mathbf{n}}(\mathbf{t})\,\mathrm{d}\mathbf{t}$ and $T_{\mathbf{n}}(\mathbf{t})$ is stated in (2.4).

If f is a continuous function on $[0,1]^d$ and $f\in W([0,1]^d)$, then, by the Parseval identity, the partial sum $S_N(f)$ of the Fourier sine series of f satisfies

$$\begin{aligned}
2^d \parallel S_N(f)-f\parallel_2^2 &= \left(\sum_{n_1,...,n_d=1}^{\infty}-\sum_{n_1,...,n_d=1}^{N}\right)c^2_{n_1,...,n_d}(f)\\
&= O(1)\left(\sum_{n_1,...,n_d=1}^{\infty}-\sum_{n_1,...,n_d=1}^{N}\right)\frac{1}{n_1^2n_2^2\cdots n_d^2}\\
&= O(1)\left(\sum_{\nu=1}^{d}\frac{d!}{\nu!(d-\nu)!}\left(\sum_{n_1,...,n_\nu=N+1}^{\infty}\frac{1}{n_1^2\cdots n_\nu^2}\right)\left(\sum_{n_{\nu+1},...,n_d=1}^{N}\frac{1}{n_{\nu+1}^2\cdots n_d^2}\right)\right)\\
&= O(1)\left(\sum_{\nu=1}^{d}\left(\sum_{k=N+1}^{\infty}\frac{1}{k^2}\right)^{\nu}\left(\sum_{k=1}^{N}\frac{1}{k^2}\right)^{d-\nu+1}\right)=O\left(\tfrac{1}{N}\right).
\end{aligned} \tag{4.1}$$

In the partial sum of Fourier sine series, the number of its Fourier sine coefficients:

$$N_c=N^d.$$

So, by (4.1), it follows that for $f\in W([0,1]^d)$, the partial sums $S_N(f)$ satisfy

$$\parallel f-S_N(f)\parallel_2^2=O\left(\frac{1}{N}\right)=O\left(\frac{1}{N_c^{\frac{1}{d}}}\right).$$

Consider hyperbolic cross truncations of the Fourier sine series of f on $[0,1]^d$. The Fourier sine series of f can be rewritten in the form

$$f(\mathbf{t})=\sum_{p_1,...,p_d=1}^{\infty}\sum_{\mathbf{q}\in\{0,1\}^d} c_{2\mathbf{p}+\mathbf{q}}T_{2\mathbf{p}+\mathbf{q}}(\mathbf{t}) \qquad (\mathbf{p}=(p_1,...,p_d)\in\mathbb{Z}_+).$$

Define the hyperbolic cross truncations of Fourier sine series of f are

$$S_N^{(h)}(f;\mathbf{t}) = \sum_{\substack{1\le|\mathbf{p}|\le N-1\\ 1\le p_1,\dots,p_d\le N-1}} \sum_{\mathbf{q}\in\{0,1\}^d} c_{2\mathbf{p}+\mathbf{q}} T_{2\mathbf{p}+\mathbf{q}}(\mathbf{t}),$$

where $|\mathbf{p}| = \prod_{k=1}^{d} p_k$.

Based on asymptotic formula, we deduce that the asymptotic formula of hyperbolic cross truncations of Fourier sine series

Denote

$$\Theta_N = \left\{ \mathbf{p} = (p_1,...,p_d) \in \mathbb{Z}_+^d \quad p_1 \in \mathbb{Z}_+, \quad 1 \le p_2,...,p_d \le N-1, \quad \prod_{k=1}^{d} p_k \ge N \right\}. \tag{4.2}$$

The difference $f(\mathbf{t}) - S_N^{(h)}(f;\mathbf{t})$ is equal to

$$\sum_{\mathbf{p}\in\Theta_N} \sum_{\mathbf{q}\in\{0,1\}^d} c_{2\mathbf{p}+\mathbf{q}} T_{2\mathbf{p}+\mathbf{q}}(\mathbf{t}) + \left(\sum_{p_1,\dots,p_d=1}^{\infty} - \sum_{p_1=1}^{\infty} \sum_{p_2,\dots,p_d=1}^{N} \right) \sum_{\mathbf{q}\in\{0,1\}^d} c_{2\mathbf{p}+\mathbf{q}} T_{2\mathbf{p}+\mathbf{q}}(\mathbf{t}).$$

By the Parseval identity, we deduce that

$$\begin{aligned} 2^d \parallel f - S_N^{(h)}(f) \parallel_2^2 &= \sum_{\mathbf{p}\in\Theta_N} \sum_{\mathbf{q}\in\{0,1\}^d} c_{2\mathbf{p}+\mathbf{q}}^2 \\ &+ \left(\sum_{p_1,\dots,p_d=1}^{\infty} - \sum_{p_1=1}^{\infty} \sum_{p_2,\dots,p_d=1}^{N} \right) \sum_{\mathbf{q}\in\{0,1\}^d} c_{2\mathbf{p}+\mathbf{q}}^2 \\ &= P_N(f) + Q_N(f). \end{aligned} \tag{4.3}$$

By Corollary 3.2 and (4.1),

$$\begin{aligned} Q_N(f) &\le \left(\sum_{p_1,\dots,p_d=1}^{\infty} - \sum_{p_1,\dots,p_d=1}^{N} \right) \sum_{\mathbf{q}\in\{0,1\}^d} c_{2\mathbf{p}+\mathbf{q}}^2 \\ &= O(1) \left(\sum_{p_1,\dots,p_d=1}^{\infty} - \sum_{p_1,\dots,p_d=1}^{N} \right) \frac{1}{p_1^2\cdots p_d^2} = O\left(\frac{1}{N}\right). \end{aligned} \tag{4.4}$$

By Corollary 3.2, it follows that

$$\begin{aligned} P_N(f) &= \left(\frac{2}{\pi^2}\right)^d \sum_{\mathbf{p}\in\Theta_N} \frac{1}{p_1^2\cdots p_d^2} \left(\sum_{\lambda\in\{0,1\}^d} |f(\lambda)|^2 + \eta_1' + \cdots + \eta_d' \right) \\ &= P_N^{(1)} + P_N^{(2)}, \end{aligned} \tag{4.5}$$

where

$$P_N^{(1)} = \left(\frac{2}{\pi^2}\right)^d \sum_{\mathbf{p}\in\Theta_N} \frac{1}{p_1^2\cdots p_d^2} \sum_{\lambda\in\{0,1\}^d} |f(\lambda)|^2,$$

$$P_N^{(2)} = O(1) \sum_{\mathbf{p}\in\Theta_N} \frac{1}{p_1^2\cdots p_d^2}(\eta_1' + \cdots \eta_d'),$$

and $\eta_k \to 0$ as $p_k \to \infty$ $(k = 1, ..., d)$.

We estimate the order of $P_N^{(1)} \to 0$ as $N \to \infty$.

Notice that

$$P_N^{(1)} \sim \sum_{\mathbf{p}\in\Theta_N} \frac{1}{p_1^2\cdots p_d^2} \sim \int_{x_1,\dots,x_d\geq N} \frac{\mathrm{d}x_1\cdots\mathrm{d}x_d}{x_1^2\cdots x_d^2} =: R_N$$

and

$$R_N = \int_1^N \mathrm{d}x_1 \int_1^{\frac{N}{x_1}} \mathrm{d}x_2 \cdots \int_1^{\frac{N}{x_1\cdots x_k}} \mathrm{d}x_{k+1} \cdots \int_{\frac{N}{x_1\cdots x_{d-1}}}^{\infty} \frac{\mathrm{d}x_d}{x_1^2\cdots x_d^2}.$$

A direct computation shows that

$$\begin{aligned}\int_{\frac{N}{x_1\cdots x_{d-1}}}^{\infty} \frac{\mathrm{d}x_d}{x_1^2\cdots x_d^2} &= \frac{1}{x_1^2\cdots x_{d-1}^2}\int_{\frac{N}{x_1\cdots x_{d-1}}}^{\infty} \frac{\mathrm{d}x_d}{x_d^2} \\ &= \frac{1}{Nx_1\cdots x_{d-1}},\end{aligned}$$

$$\begin{aligned}\int_1^{\frac{N}{x_1\cdots x_{d-2}}} \mathrm{d}x_{d-1} \int_{\frac{N}{x_1\cdots x_{d-1}}}^{\infty} \frac{\mathrm{d}x_d}{x_1^2\cdots x_d^2} &= \frac{1}{Nx_1\cdots x_{d-2}} \int_1^{\frac{N}{x_1\cdots x_{d-2}}} \frac{\mathrm{d}x_{d-1}}{x_{d-1}} \\ &= \frac{1}{Nx_1\cdots x_{d-2}} \log\frac{N}{x_1\cdots x_{d-2}},\end{aligned}$$

and

$$\begin{aligned}&\int_1^{\frac{N}{x_1\cdots x_{d-3}}} \mathrm{d}x_{d-2} \int_1^{\frac{N}{x_1\cdots x_{d-2}}} \mathrm{d}x_{d-1} \int_{\frac{N}{x_1\cdots x_{d-1}}}^{\infty} \frac{\mathrm{d}x_d}{x_1^2\cdots x_d^2} \\ &= \frac{1}{Nx_1\cdots x_{d-3}} \int_1^{\frac{N}{x_1\cdots x_{d-3}}} \frac{1}{x_{d-2}} \log\frac{N}{x_1\cdots x_{d-2}}\, \mathrm{d}x_{d-2} \\ &= \frac{1}{Nx_1\cdots x_{d-3}} \int_1^{\frac{N}{x_1\cdots x_{d-3}}} \frac{\log u}{u}\mathrm{d}u \\ &\sim \frac{1}{Nx_1\cdots x_{d-3}} \log^2\frac{N}{x_1\cdots x_{d-3}}.\end{aligned}$$

Continuing this procedure, we deduce that

$$P_N^{(1)} \sim R_N \sim \frac{1}{N}\int_1^N \frac{1}{x_1}\log^{d-2}\frac{N}{x_1}\,\mathrm{d}x_1 \sim \frac{\log^{d-1}N}{N}. \tag{4.6}$$

We estimate $P_N^{(2)}$.

Let

$$S_N^{(k)} = \sum_{\mathbf{p}\in\Theta_N} \frac{\eta_k'}{p_1^2\cdots p_d^2} \qquad (k = 1, ..., d).$$

From this and (4.6), we deduce that

$$S_N^{(1)} = \sum_{\mathbf{p}\in\Theta_N} \frac{\eta_1'}{p_1^2\cdots p_d^2} = O\left(\frac{1}{N}\right)\int_1^N \frac{1}{x_1}\log^{d-2}\frac{N}{x_1}\,\eta_1'\,\mathrm{d}x_1.$$

Since $\eta_1' \to 0$ as $x_1 \to \infty$ and $\int_1^\infty \frac{1}{x_1} \log^{d-2} \frac{N}{x_1} \mathrm{d}x_1 = \infty$, by a known result in Calculus and (4.6),

$$S_N^{(1)} = o\left(\frac{1}{N}\right) \int_1^N \frac{1}{x_1} \log^{d-2} \frac{N}{x_1} \,\mathrm{d}x_1 = o\left(\frac{\log^{d-1} N}{N}\right).$$

An argument similar to $S_N^{(1)}$ shows that for each k,

$$S_N^{(k)} = o\left(\frac{\log^{d-1} N}{N}\right).$$

From this and (4.2)-(4.6), we get

$$2^d \parallel f - S_N^{(h)} \parallel_2^2 = P_N^{(1)}(1 + o(1)).$$

Theorem 4.1. Let $f \in W([0,1]^d)$. Then hyperbolic cross truncations $S_N^{(h)}(f; \mathbf{t})$ satisfy

$$\parallel f - S_N^{(h)}(f) \parallel_2^2 = \widetilde{P}_N^{(1)}(1 + o(1)) \qquad (N \to \infty),$$

where

$$\widetilde{P}_N^{(1)} = \left(\frac{1}{\pi^2}\right)^d \left(\sum_{\mathbf{p} \in \Theta_N} \frac{1}{p_1^2 \cdots p_d^2}\right) \left(\sum_{\lambda \in \{0,1\}^d} |f(\lambda)|^2\right)$$

and

$$\Theta_N = \{\mathbf{p} = (p_1, ..., p_d): \quad p_1 \in \mathbb{Z}_+, \quad 1 \le p_2, ..., p_d \le N - 1, \quad p_1, ..., p_d \ge N\} \tag{4.7}$$

and

$$P_N^{(1)} \sim \frac{\log^{d-1} N}{N}.$$

We easily see that the number of Fourier sine coefficients in hyperbolic cross truncations satisfies

$$N_c \sim N \log^{d-1} N.$$

In fact,

$$\begin{aligned} N_c &= \sum_{\mathbf{p} \in \Theta_N} \sum_{\mathbf{q} \in \{0,1\}^d} 1 \\ &\sim \int_1^N \mathrm{d}x_1 \int_1^{\frac{N}{x_1}} \mathrm{d}x_2 \cdots \int_1^{\frac{N}{x_1 \cdots x_{d-1}}} \mathrm{d}x_d \\ &\sim \int_1^N \mathrm{d}x_1 \int_1^{\frac{N}{x_1}} \mathrm{d}x_2 \cdots \int_1^{\frac{N}{x_1 \cdots x_{d-2}}} \frac{N}{x_1 \cdots x_{d-1}} \mathrm{d}x_{d-1} \\ &\sim N \log^{d-1} N. \end{aligned}$$

Therefore, by Theorem 4.1,

$$\parallel f - S_N^{(h)}(f) \parallel_2^2 \sim \frac{\log^{2d-2} N_c}{N_c}.$$

From this, we see that for $f \in W([0,1]^d)$, the hyperbolic cross approximation of Fourier sine series is a better approximation tool than ordinary partial sum approximation.

5. Asymptotic behaviors of stochastic Fourier sine coefficients

We extend the results in Sections 3-4 to stochastic processes. Let $\xi(\mathbf{t})$ be a continuous stochastic process on $[0,1]^d$. Then $\xi(\mathbf{t})$ can be expanded into the stochastic Fourier sine series

$$\xi(\mathbf{t}) = \sum_{\mathbf{n}\in\mathbb{Z}_+^d} c_{\mathbf{n}}(\xi)\, T_{\mathbf{n}}(\mathbf{t}) \tag{5.1}$$

in mean square sense, where

$$c_{\mathbf{n}}(\xi) = 2^d \int_{[0,1]^d} \xi(\mathbf{t}) T_{\mathbf{n}}(\mathbf{t})\, \mathrm{d}\mathbf{t}$$

and $T_{\mathbf{n}}(\mathbf{t})$ is stated in (2.4). The Fourier sine coefficients are stochastic variables. We discuss their expectations, second-order moments, and variances.

Theorem 5.1. If ξ is a stochastic process on $[0,1]^d$ and $\xi \in SW([0,1]^d)$, then the expectations, second-order moments, and variances of its Fourier sine coefficients possess the following asymptotic behaviors:

(i) $E[\, c_{\mathbf{n}}(\xi)\,] = \left(\prod_{j=1}^{d} \frac{2}{\pi n_j} \right) (\alpha_{\mathbf{n}}(\xi) + r_1 + \cdots + r_d)$ and

$$\alpha_{\mathbf{n}}(\xi) = \sum_{\lambda\in\{0,1\}^d} (\, E[\, \xi(\lambda)\,] \epsilon_{\mathbf{n}}(\lambda)\,)\,,$$

where $\alpha_{\mathbf{n}}(\xi)$ is an algebraic sum of expectation of ξ at vertexes of the cube $[0,1]^d$ and $r_k \to 0$ as $n_k \to \infty$, and $\epsilon_{\mathbf{n}}(\lambda)$ is stated in (3.2).

(ii) $E[\, c_{\mathbf{n}}^2(\xi)\,] = \left(\prod_{j=1}^{d} \frac{4}{\pi^2 n_j^2} \right) (\beta_{\mathbf{n}}(\xi) + r_1' + \cdots + r_d')$ and

$$\beta_{\mathbf{n}}(\xi) = \sum_{\lambda\in\{0,1\}^d} \sum_{\lambda'\in\{0,1\}^d} E[\, \xi(\lambda)\xi(\lambda')\,] \epsilon_{\mathbf{n}}(\lambda)\epsilon_{\mathbf{n}}(\lambda'),$$

where $r_k' \to 0$ as $n_k \to \infty$.

(iii) $\mathrm{Var}\,[\, c_{\mathbf{n}}(\xi)\,] = \left(\prod_{j=1}^{d} \frac{4}{\pi^2 n_j^2} \right) (\theta_{\mathbf{n}}(\xi) + r_1'' + \cdots + r_d'')$, where

$$\theta_{\mathbf{n}}(\xi) = \sum_{\lambda\in\{0,1\}^d} \sum_{\lambda'\in\{0,1\}^d} \mathrm{Cov}(\xi(\lambda),\ \xi(\lambda'))\, \epsilon_{\mathbf{n}}(\lambda)\, \epsilon_{\mathbf{n}}(\lambda').$$

and $r_k'' \to 0$ as $n_k \to \infty$.

For example, consider the case $d = 2$. Assume that a stochastic process $\xi \in SW([0,1]^2)$. Then

$$E[\, c_{\mathbf{n}}(\xi)\,] = \frac{4}{n_1 n_2 \pi^2} \left(\, E[\, \xi(0,0)\,] - (-1)^{n_1} E[\, \xi(1,0)\,] - (-1)^{n_2} E[\, \xi(0,1)\,] + E[\, \xi(1,1)\,] + r_1 + r_2\,\right)$$

and

$$\begin{aligned} \mathrm{Var}\,(c_{\mathbf{n}}(\xi)) \quad &= \quad \tfrac{16}{n_1^2 n_2^2 \pi^4} (\eta_{0,0} - (-1)^{n_1}\eta_{0,1} - (-1)^{n_2}\eta_{0,2} + \eta_{0,3} - (-1)^{n_1}\eta_{1,0} \\ &+ \quad \eta_{1,1} + (-1)^{n_1+n_2}\eta_{1,2} - (-1)^{n_1}\eta_{1,3} - (-1)^{n_2}\eta_{2,0} + (-1)^{n_1+n_2}\eta_{2,1} \\ &+ \quad \eta_{2,2} - (-1)^{n_2}\eta_{2,3} + \eta_{3,0} - (-1)^{n_1}\eta_{3,1} - (-1)^{n_2}\eta_{3,2} + \eta_{3,3} + r_1' + r_2'), \end{aligned}$$

where

$$\eta_{\lambda_1+2\lambda_2,\lambda_1'+2\lambda_2'} = \mathrm{Cov}\,(\xi(\lambda_1,\lambda_2),\ \xi(\lambda_1',\ \lambda_2')) \qquad (\lambda_1,\lambda_2,\lambda_1',\lambda_2' = 0 \ \text{ or } \ 1)$$

and $r_1,\, r_1' \to 0$ as $n_1 \to \infty$ and $r_2,\, r_2' \to 0$ as $n_2 \to \infty$.

Proof of Theorem 5.1. Exchanging the expectation and integral, we deduce from (5.1) that

$$E[\, c_{\mathbf{n}}(\xi)\,] = 2^d \int\limits_{[0,1]^d} E[\,\xi(\mathbf{t})\,]\,\mathbf{T}_{\mathbf{n}}(\mathbf{t})\,\mathrm{d}\mathbf{t} = c_{\mathbf{n}}(\,E[\,\xi(\mathbf{t})\,]\,),$$

i.e., $E[\,c_{\mathbf{n}}(\xi)\,]$ is the Fourier sine coefficients of the deterministic function $E[\,\xi(\mathbf{t})\,]$. Exchanging the expectation and partial derivative, we deduce from $\xi \in SW([0,1]^d)$ that

$$E[\,\xi(\mathbf{t})\,] \in W([0,1]^d).$$

Using Theorem 3.1, we get (i).

By (5.1), we get $|c_{\mathbf{n}}(\xi)|^2 = 2^{2d} \int\limits_{[0,1]^d} \int\limits_{[0,1]^d} \xi(\mathbf{t})\xi(\mathbf{s})\, T_{\mathbf{n}}(\mathbf{t})T_{\mathbf{n}}(\mathbf{s})\mathrm{d}\mathbf{t}\,\mathrm{d}\mathbf{s}$, and so

$$E[\,|c_{\mathbf{n}}(\xi)|^2\,] = 2^{2d} \int\limits_{[0,1]^d} \int\limits_{[0,1]^d} R_\xi(\widetilde{\mathbf{t}})\, T_{\mathbf{n}}(\mathbf{t})T_{\mathbf{n}}(\mathbf{s})\,\mathrm{d}\mathbf{t}\,\mathrm{d}\mathbf{s}, \tag{5.2}$$

where the autocorrelation function $R_\xi(\widetilde{\mathbf{t}}) = E[\,\xi(\mathbf{t})\xi(\mathbf{s})\,]$ is a $2d-$variate deterministic function and

$$\mathbf{t} = (t_1, ..., t_d), \qquad\qquad \mathbf{s} = (s_1, ..., s_d),$$

$$\widetilde{\mathbf{t}} = (t_1, ..., t_{2d}) \qquad (t_{d+i} = s_i,\ i = 1, ..., d).$$

Let $n_{d+j} = n_j$ $(j = 1, ..., d)$. Then (5.2) can be rewritten into

$$E[\,|c_{n_1,...,n_d}(\xi)|^2\,] = 2^{2d} \int\limits_{[0,1]^{2d}} R_\xi(\widetilde{\mathbf{t}}) \prod_{j=1}^{2d} \sin(\pi n_j t_j)\mathrm{d}\widetilde{\mathbf{t}}.$$

From the definition of Fourier sine coefficients, we see that $E[\,|c_{n_1,...,n_d}(\xi)|^2\,]$ is the Fourier sine coefficient of $2d-$variate function R_ξ, that is,

$$E[\,|c_{n_1,...,n_d}(\xi)|^2\,] = c_{n_1,...,n_{2d}}(R_\xi) \qquad (n_{d+j} = n_j,\ j = 1, ..., d). \tag{5.3}$$

By the assumption $\xi \in SW([0,1]^d)$, we deduce that $R_\xi \in W([0,1]^{2d})$. In Theorem 3.1, replacing f by R_ξ and d by $2d$ and letting $n_{d+j} = n_j$ $(j = 1, ..., d)$, we obtain

$$c_{n_1,...,n_{2d}}(R_\xi) = \left(\prod_{j=1}^{2d} \frac{2}{\pi n_j} \right) (\,\beta_{n_1,...,n_d}(\xi) + r_1 + \cdots + r_d\,)\,, \tag{5.4}$$

where

$$\beta_{n_1,...,n_d}(\xi) = K^{2d}_{n_1,...,n_{2d}}(R_\xi) = \sum_{\widetilde{\lambda} \in \{0,1\}^{2d}} R_\xi(\widetilde{\lambda}) \left(\prod_{j \in G_{\widetilde{\lambda}}} (-1)^{n_j+1} \right)$$

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and

$$\widetilde{\lambda} = (\lambda_1, ..., \lambda_{2d}) = (\lambda_1, ..., \lambda_d, \lambda'_1, ..., \lambda'_d),$$
$$G_{\widetilde{\lambda}} = \{j \in \{1, ..., 2d\}, \ \ \lambda_j = 1\}.$$

By $n_{d+j} = n_j$ $(j = 1, ..., d)$, we have

$$\begin{aligned}\prod_{j\in G_{\widetilde{\lambda}}} (-1)^{n_j+1} &= \left(\prod_{j\in(G_{\widetilde{\lambda}}\bigcap\{1,...,d\})} (-1)^{n_j+1}\right)\left(\prod_{j\in(G_{\widetilde{\lambda}}\bigcap\{d+1,...,2d\})} (-1)^{n_j+1}\right) \\ &= \prod_{j\in G_{\lambda}} (-1)^{n_j+1} \prod_{j\in G_{\lambda'}} (-1)^{n_j+1},\end{aligned} \tag{5.5}$$

where

$$G_{\lambda} = \{j \in \{1, ..., d\}, \ \ \lambda_j = 1\},$$
$$G_{\lambda'} = \{j \in \{1, ..., d\}, \ \ \lambda'_j = 1\}.$$

Since

$$R_{\xi}(\lambda_1, ..., \lambda_{2d}) = E[\xi(\lambda_1, ..., \lambda_d)\xi(\lambda'_1, ..., \lambda'_d)] \qquad (\lambda_{d+j} = \lambda'_j),$$

by (5.5), we deduce by (5.4) that

$$\beta_{n_1,...,n_d}(\xi) = \sum_{\lambda\in\{0,1\}^d} \sum_{\lambda'\in\{0,1\}^d} E[\,\xi(\lambda_1, ..., \lambda_d)\xi(\lambda'_1, ..., \lambda'_d)\,] \prod_{j\in G_{\lambda}} (-1)^{n_j+1} \prod_{j\in G_{\lambda'}} (-1)^{n_j+1}. \tag{5.6}$$

From this and (5.3)-(5.4), we get (ii). From (i),

$$E[c^2_{\mathbf{n}}(\xi)] = \prod_{j=1}^{d} \frac{4}{\pi^2 n_j^2}(\alpha^2_{\mathbf{n}}(\xi) + \widetilde{r}_1 + \cdots + \widetilde{r}_d),$$

where each $\widetilde{r}_k \to 0$ as $n_k \to \infty$. Again, by (ii),

$$\mathrm{Var}(c_{\mathbf{n}}(\xi)) = E[c^2_{\mathbf{n}}(\xi)] - |E[c_{\mathbf{n}}(\xi)]|^2 = \prod_{j=1}^{d} \frac{4}{\pi^2 n_j^2}(\beta_{\mathbf{n}}(\xi) - \alpha^2_{\mathbf{n}}(\xi) + r''_1 + \cdots + r''_d),$$

where $r''_k \to 0$ as $n_k \to \infty$. Noticing that

$$\begin{aligned}\beta_{\mathbf{n}}(\xi) - \alpha^2_{\mathbf{n}}(\xi) &= \sum_{\lambda,\lambda'\in\{0,1\}^d} \left(E[\xi(\lambda)\xi(\lambda')] - E[\xi(\lambda)]E[\xi(\lambda')]\right)\epsilon_{\mathbf{n}}(\lambda)\epsilon_{\mathbf{n}}(\lambda') \\ &= \sum_{\lambda,\lambda'\in\{0,1\}^d} \mathrm{Cov}(\xi(\lambda),\, \xi(\lambda'))\epsilon_{\mathbf{n}}(\lambda)\epsilon_{\mathbf{n}}(\lambda'),\end{aligned}$$

we get (iii). Theorem 5.1 is proved. □

6. Asymptotic behavior of hyperbolic cross hyperbolic cross approximations of stochastic Fourier sine series

Let $\xi(\mathbf{t})$ be a continuous stochastic process on $[0,1]^d$. The hyperbolic cross truncation of its Fourier sine series is

$$S^{(h)}_N(\xi, \mathbf{t}) = \sum_{|\mathbf{p}|\leq N-1} \sum_{\mathbf{q}\in\{0,1\}^d} c_{2\mathbf{p}+\mathbf{q}}\, T_{2\mathbf{p}+\mathbf{q}}(\mathbf{t}), \tag{6.1}$$

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where $|\mathbf{p}| = \prod_{k=1}^{d} p_k$ ($\mathbf{p} = (p_1, ..., p_d) \in \mathbb{Z}_+^d$), is a stochastic sine polynomial. We give an asymptotic behavior of the hyperbolic cross approximation.

Theorem 6.1. Let ξ be a stochastic process on $[0,1]^d$. If $\xi \in SW([0,1]^d)$. Then the hyperbolic cross truncations $S_N^{(h)}(\xi)$ of the stochastic Fourier sine series of ξ satisfy

$$E[\parallel S_N^{(h)}(\xi) - \xi \parallel_2^2] = W_N(\xi)(1 + o(1)) \qquad (N \to \infty),$$

where

$$W_N(\xi) = \left(\frac{1}{\pi^2}\right)^d \sum_{\mathbf{p}\in\Theta_N} \frac{1}{p_1^2 \cdots p_d^2} \sum_{\lambda\in\{0,1\}^d} E[|\xi(\lambda)|^2]$$

and Θ_N is stated in (4.2) and

$$P_N^{(1)}(\xi) \sim \frac{\log^{d-1} N}{N}.$$

Proof. By using an argument similar to Section 4, we deduce that (4.3) is still valid when f is replaced by ξ. Taking expectation on both sides,

$$\begin{aligned}
&2^d E[\| \xi - S_N^{(h)}(\xi) \|_2^2] \\
&= \sum_{\mathbf{p}\in\Theta_N} \sum_{\mathbf{q}\in\{0,1\}^d} E[c^2_{2\mathbf{p}+\mathbf{q}}(\xi)] \\
&\quad + \left(\sum_{p_1,\dots,p_d=1}^{\infty} - \sum_{p_1=1}^{\infty} \sum_{p_2,\dots,p_d=1}^{N} \right) \sum_{\mathbf{q}\in\{0,1\}^d} E[c^2_{2\mathbf{p}+\mathbf{q}}(\xi)] \\
&= P_N(\xi) + Q_N(\xi).
\end{aligned}$$

By Theorem 5.1,

$$E[c^2_{2\mathbf{p}+\mathbf{q}}(\xi)] = O\left(\frac{1}{p_1^2 \cdots p_d^2}\right).$$

This implies that $Q_N(\xi) = O\left(\frac{1}{N}\right)$. By Theorem 5.1 (ii), it follows that

$$\begin{aligned}
&2^d E[\parallel \xi - S_N^{(h)}(\xi) \parallel_2^2] \\
&= \tfrac{1}{\pi^{2d}} \sum_{\mathbf{p}\in\Theta_N} \tfrac{1}{p_1^2\cdots p_d^2} \left(\sum_{\mathbf{q}\in\{0,1\}^d} \beta_{2\mathbf{p}+\mathbf{q}} + r_1'' + \cdots + r_d'' \right) \qquad (6.2) \\
&=: P_{\mathbf{n}}^{(1)}(\xi) + P_N^{(2)}(\xi) + O\left(\tfrac{1}{N}\right),
\end{aligned}$$

where Θ_N is stated in (4.7) and each $r_k'' \to 0$ as $n_k \to \infty$, and

$$P_N^{(1)}(\xi) = \frac{1}{\pi^{2d}} \sum_{\mathbf{p}\in\Theta_N} \frac{1}{p_1^2\cdots p_d^2} \left(\sum_{\mathbf{q}\in\{0,1\}^d} \beta_{2\mathbf{p}+\mathbf{q}} \right),$$

$$P_N^{(2)}(\xi) = \frac{1}{\pi^{2d}} \sum_{\mathbf{p}\in\Theta_N} \frac{1}{p_1^2\cdots p_d^2} \left(\sum_{\mathbf{q}\in\{0,1\}^d} (r_1'' + \cdots + r_d'') \right).$$

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By (3.17),

$$\sum_{\mathbf{q}\in\{0,1\}^d} \beta_{2\mathbf{p}+\mathbf{q}} = \frac{1}{\pi^{2d}} \sum_{\lambda,\lambda'\in\{0,1\}^d} E[\xi(\lambda)\xi(\lambda')] \left(\sum_{\mathbf{q}\in\{0,1\}^d} \epsilon_{2\mathbf{p}+\mathbf{q}}(\lambda)\epsilon_{2\mathbf{p}+\mathbf{q}}(\lambda') \right)$$

$$= \frac{2^d}{\pi^{2d}} \sum_{\lambda\in\{0,1\}^d} E[\xi^2(\lambda)].$$

So

$$P_N^{(1)}(\xi) = \left(\frac{2}{\pi^2}\right)^d \sum_{\mathbf{p}\in\Theta_N} \frac{1}{p_1^2\cdots p_d^2} \sum_{\lambda\in\{0,1\}^d} E[|\xi(\lambda)|^2].$$

Similar to the argument of Theorem 4.1,

$$P_N^{(1)}(\xi) \sim \sum_{\mathbf{p}\in\Theta_N} \frac{1}{p_1^2\cdots p_d^2} \sim \frac{\log^{d-1} N}{N}$$

and

$$P_N^{(2)}(\xi) = o\left(\frac{\log^{d-1} N}{N}\right).$$

From this and (6.2), we deduce the desired result. □

7. Examples

In data analysis, the following three stochastic processes are often used [16].

(i) Gaussian stochastic process $\xi_{SE}(\mathbf{t})$ with mean $\mathbf{0}$ and square exponential covariance function:

$$K_{SE}(\mathbf{t},\mathbf{t}') = e^{\left(-\frac{\|\mathbf{t}-\mathbf{t}'\|_2^2}{2l^2}\right)}, \tag{7.1}$$

where $\mathbf{t} = (t_1, ..., t_d)$ and $\mathbf{t}' = (t_1', ..., t_d')$, and $\| \mathbf{t} \|_2^2 = \sum_{k=1}^{d} t_k^2$, and $l > 0$.

(ii) Gaussian stochastic process $\xi_{RQ}(\mathbf{t})$ with mean $\mathbf{0}$ and rational quadratic covariance function:

$$K_{RQ}(\mathbf{t},\mathbf{t}') = (1+ \| \mathbf{t}-\mathbf{t}' \|^2)^{-\alpha},$$

where $\mathbf{t} = (t_1, ..., t_d)$ and $\mathbf{t}' = (t_1', ..., t_d')$, and $\alpha \geq 0$.

(iii) Gaussian stochastic process $\xi_L(\mathbf{t})$ with mean $\mathbf{0}$ and linear covariance function:

$$K_L(\mathbf{t},\mathbf{t}') = <\mathbf{t},\ \mathbf{t}'>,$$

where $\mathbf{t} = (t_1, ..., t_d)$ and $\mathbf{t}' = (t_1', ..., t_d')$, and $<\mathbf{t},\ \mathbf{t}'> = \sum_{k=1}^{d} t_k t_k'$.

Since these stochastic processes are differentiable [14, 15], we can use the theorems in Sections 5-6 to research their Fourier sine expansions, including variance estimates of Fourier sine coefficients and asymptotic formulas of hyperbolic cross truncation approximation.

We expand ξ_{SE} into a Fourier sine series on $[0,1]^d$ as follows:

$$\xi_{SE}(\mathbf{t}) = \sum_{\mathbf{n}\in\mathbb{Z}_+^d} c_{\mathbf{n}}(\xi_{SE})T_{\mathbf{n}}(\mathbf{t}),$$

where $c_{\mathbf{n}}(\xi_{SE}) = 2^d \int\limits_{[0,1]^d} \xi_{SE}(\mathbf{t}) T_{\mathbf{n}}(\mathbf{t}) \mathrm{d}\mathbf{t}$.

Since $E[\xi_{SE}(\mathbf{t})] = 0\,(\mathbf{t} \in [0,1]^d)$, by Theorem 5.1 (i), we have

$$E[c_{\mathbf{n}}(\xi_{SE})] = o\left(\frac{1}{n_1 \cdots n_d}\right) \quad \text{as each} \quad n_k \to \infty \qquad (\mathbf{n} = (n_1, ..., n_d)).$$

Let $\mathbf{t} = (t_1, ..., t_d)$ and $\mathbf{t}' = (t_1', ..., t_d')$. Then

$$\| \mathbf{t} - \mathbf{t}' \|_2^2 = \sum_{k=1}^{d} (t_k - t_k')^2,$$

and by (7.1),

$$\mathrm{Cov}(\xi_{SE}(\mathbf{t}),\ \xi_{SE}(\mathbf{t}')) = \prod_{k=1}^{d} e^{-\frac{1}{2l^2}(t_k - t_k')^2}.$$

By Theorem 5.1 (iii), we obtain that the Fourier sine coefficients $c_{\mathbf{n}}(\xi_{SE})$ satisfy,

$$\mathrm{Var}(c_{\mathbf{n}}(\xi_{SE})) = \prod_{j=1}^{d} \frac{4}{\pi^2 n_j^2} (\theta_{\mathbf{n}}(\xi_{SE}) + r_1'' + \cdots + r_d'') \qquad \text{and} \quad r_k'' \to 0 \text{ as } n_k \to \infty.$$

where

$$\theta_{\mathbf{n}}(\xi_{SE}) = \sum_{\lambda \in \{0,1\}^d} \sum_{\lambda' \in \{0,1\}^d} \left(\prod_{k=1}^{d} e^{-\frac{1}{2l^2}(\lambda_k - \lambda_k')^2} \right) \varepsilon_{\mathbf{n}}(\lambda) \varepsilon_{\mathbf{n}}(\lambda'),$$

and $\varepsilon_{\mathbf{n}}(\lambda)$ is stated in (3.2) and $\lambda = (\lambda_1, ..., \lambda_d)$ and $\lambda' = (\lambda_1', ..., \lambda_d')$.

Next we find the asymptotic formula of the hyperbolic cross truncation approximation of Fourier sine series of ξ_{SE}. The hyperbolic cross truncation is

$$S_N^{(h)}(\xi_{SE}, \mathbf{t}) = \sum_{p_1 \cdots p_d \le N-1} \sum_{\mathbf{q} \in \{0,1\}^d} c_{2\mathbf{p}+\mathbf{q}}(\xi_{SE}) T_{2\mathbf{p}+\mathbf{q}}(\mathbf{t}) \qquad (p_i \in \mathbb{Z}_+,\ i = 1, ..., d).$$

Note that $E[|\xi_{SE}(\mathbf{t})|^2] = R_{SE}(0) = 1$ and $\sum\limits_{\lambda \in \{0,1\}^d} 1 = 2^d$. By theorem 6.1, we have

$$E[\| S_N^{(h)}(\xi_{SE}) - \xi_{SE} \|_2^2] = W_N(\xi_{SE})(1 + o(1)) \qquad (N \to \infty),$$

where

$$W_N(\xi_{SE}) = \left(\frac{1}{\pi^2}\right)^d \sum_{\mathbf{p} \in \Theta_N} \frac{1}{p_1^2 \cdots p_d^2} \sum_{\lambda \in \{0,1\}^d} 1 = \left(\frac{4}{\pi^2}\right)^d \sum_{\mathbf{p} \in \Theta_N} \frac{1}{p_1^2 \cdots p_d^2}.$$

So we get the asymptotic formula as follows:

$$E[\| S_N^{(h)}(\xi_{SE}) - \xi_{SE} \|_2^2] = \left(\frac{2}{\pi^2}\right)^d \left(\sum_{\mathbf{p} \in \Theta_N} \frac{1}{p_1^2 \cdots p_d^2} \right) (1 + o(1)) \qquad (N \to \infty),$$

where Θ_N is stated in (4.2) and the number N_c of Fourier sine coefficients in the hyperbolic cross truncation $S_N^{(h)}(\xi_{SE})$ is equivalent to $N \log^{d-1} N$.

Similarly, for the stochastic processes ξ_{RQ} and ξ_L, using the same method as above, we can give the variance estimates of their Fourier sine coefficients and asymptotic formulas of hyperbolic cross truncation approximations.

References

[1] V. Barthelmann, E. Novak, and K. Ritter, High dimensional polynomial interpolation on sparse grids, Advances in Computational Mathematics, 12(4) (2000), 273-288.

[2] M. Griebel and J. Hamaekers, Sparse grids for the Schrödinger equation, ESALM Mathematical Modelling and Numerical Analysis, 41 (2007), 215-247.

[3] E. C. Klebaner, Introduction to stochastic calculus with application, World scientific publishing, 2012.

[4] C. Lanczos, Discourse on Fourier series, Hafner Pub. Co. New York, 1966.

[5] J. Shen and L. Wang, Sparse spectral approximations of high-dimensional problems based on hyperbolic cross, SIAM Journal on Numerical Analysis, 48 (3) (2010), 1087-1109.

[6] J. Shen and H. Yu, Efficient spectral sparse grid methods and applications to high-dimensional elliptic problems, SIAM Journal on Scientific Computing, 32(6) (2010), 3228-3250.

[7] E. M. Stein and G. Weiss, Introduction to Fourier analysis on Euclidean spaces, Princeton Univ. Press, 1971.

[8] T. Szabados, An elementary introduction to the Wiener process and stochastic integrals. Studia Sci. Math. Hung. 31 (1996), 249-297.

[9] A. F. Timan, Theory at approximation of functions of a real variable, Pergamon, 1963.

[10] L. Villafuerte and B. M. Chen-Charpentier, A random differential transform method: Theory and applications, Appl. Math. Letters, 25(10) (2012), 1490-1494.

[11] Z. Zhang, Decomposition and approximation of multivariate functions on the cube, Acta. Math. Sinica, 29(1) (2013), 119-136.

[12] A. Zygmund, Trigonometric series, I, II, 2nd Edition, Cambridge, 1968.

[13] Z. Zhang, Approximation of bivariate functions via smooth extensions, The Scientific World Journal, vol. 2014, Article ID 102062, pp 1-16, 2014.

[14] B. Hajek, An Exploration of Random Processes for Engineers, 2012, see: http://www.ifp.illinois.edu/ Hajek/Papers/randomprocDec11.pdf

[15] J.-P. Kahane, Some random series of functions, Cambridge Studies in Advanced Mathematics, Volume 5, 2rd, Cambridge University Press, Cambridge, 1994

[16] C. E. Rasmussen, C.K.I. Williams, Gaussian Processes for Machine Learning, MIT Press, 2006.

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