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Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

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Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
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TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
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Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

Christodoulos A. Floudas
Department of Chemical Engineering
Princeton University
Princeton,NJ 08544-5263
609-258-4595(x4619 assistant)
e-mail: floudas@titan.princeton.edu
Optimization Theory&Applications, Global Optimization

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de

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Journal of Computational Analysis and Applications

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Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

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Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
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PERIODIC ORBITS OF SINGULAR RADIALLY SYMMETRIC SYSTEMS

SHENGJUN LI [1,2], WULAN LI [3] AND YIPING FU [1]

ABSTRACT. We study the existence of periodic orbits of planar radially symmetric systems with a singularity. These orbits have periods which are large integer multiples of the period of the forcing, and rotate exactly once around the origin in their period time. The proof is based on the use of topological degree theory and a fixed point theorem in cones.

1. INTRODUCTION

In the paper [11], Fonda and J.Ureña have studied the periodic, subharmonic and quasi-periodic orbits for the radially symmetric system

$$\ddot{x} + f(t, |x|)\frac{x}{|x|} = 0, \quad x \in \mathbb{R}^2 \setminus \{0\}, \tag{1.1}$$

where $f \in C((\mathbb{R}/T\mathbb{Z}) \times (0, \infty), \mathbb{R})$ may be singular at the origin. As mentioned in [10], many phenomena of the nature obey to laws of (1.1), such as the Newtonian equation for the motion of a particle subjected to the gravitational attraction of a sun which lies at the origin. Setting $\rho(t) = |x(t)|$, they proved the following result:

Theorem 1.1 Suppose that $f(t, \rho) > 0$ for $t \in [0, T], \rho > 0$ and satisfies the following conditions:

(A$_1$) $\lim_{\rho\to\infty} f(t, \rho)/\rho = 0$, for *a.e.* $t \in \mathbb{R}$.

(A$_2$) There exists some function $h \in L^1_{loc}(\mathbb{R})$ and some number $r_0 > 0$ such that

$$|f(t, \rho)| \le h(t)\rho, \text{ on } \mathbb{R} \times [r_0, +\infty].$$

Then, there exists a connect set $\mathcal{C}$ of T-radially periodic solutions of (1.1) which goes from zero to infinity.

We look for solutions $x(t) \in \mathbb{R}^2$ which never attain the singularity, in the sense that

$$x(t) \neq 0, \quad \text{for every } t \in \mathbb{R}.$$

Using the same idea in [8], we may write the solutions of (1.1) in polar coordinates

$$x(t) = \rho(t)(\cos\varphi(t), \sin\varphi(t)).$$

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Key words and phrases. Periodic orbits, singular radially symmetric systems, topological degree theory, fixed point theorem in cone.

Then we have the collisionless orbits if $\rho(t) > 0$ for every t. Moreover, equation (1.1) is equivalent to the following system

$$\begin{cases} \ddot{\rho} + f(t,\rho) - \frac{\mu^2}{\rho^3} = 0, \\ \rho^2 \dot{\varphi} = \mu, \end{cases} \tag{1.2}$$

where μ is the angular momentum of $x(t)$. Recall that μ is constant in time along any solution.

If x is a T-radially periodic, then ρ must be T-periodic. We will prove the existence of a T-periodic solution ρ of the first equation in (1.2). We thus consider the boundary value problem

$$\begin{cases} \ddot{\rho} + f(t,\rho) = \frac{\mu^2}{\rho^3}, \\ \rho(0) = \rho(T), \quad \dot{\rho}(0) = \dot{\rho}(T). \end{cases} \tag{1.3}$$

Let $\mu = 0$, (1.3) can be written the singular T-periodic problem

$$\ddot{\rho} + f(t,\rho) = 0. \tag{1.4}$$

The question about the existence of non-collision periodic orbits for scalar equations and dynamical systems with singularities has attracted much attention of many researchers over many years. See[5, 7, 12, 13, 15, 24]. Usually, the proof is based on variational approach [1, 2, 6, 16, 22], the method of upper and lower solutions [3, 21], some fixed point theorems [19, 26, 27, 28, 29] or the topological degree theory [17, 18, 23, 30]. In particular, several existence results for the following scalar differential equation

$$\ddot{x} + a(t)x = f(t,x) \tag{1.5}$$

has been established in [23, 25, 27]. Note that (1.5) is a nonlinear perturbation of Hill equation

$$\ddot{x} + a(t)x = 0.$$

Moreover, it has been found that a particular case of (1.5), the Ermakov–Pinney equation, plays an important role in studying the Lyapunov stability of periodic solutions of Lagrangian equations [20].

Our main motivation is to obtain by the above papers [9, 17, 27], by the use of topological degree theory and a well-known fixed point theorem in cones, we prove the existence of large-amplitude periodic orbits whose minimal period is an integer multiple of T, and rotate exactly once around the origin in their period time.

The rest of this paper is organized as follows. In Section 2, some preliminaries are given. In Section 3, , we give the main results.

2. Preliminaries

We say that the scale linear equation

$$\ddot{x} + a(t)x = 0 \tag{2.1}$$

is nonresonant if its unique T-periodic solution is the trivial one. When (2.1) is nonresonant, as a consequence of Fredholm's alternative, the nonhomogeneous equation

$$\ddot{x} + a(t)x = h(t)$$

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admits a unique T-periodic solution which can be written as

$$x(t) = \int_0^T G(t,s)h(s)ds,$$

where $G(t,s)$ is the Green's function of (2.1), associated with periodic boundary conditions

$$x(0) = x(T), \quad \dot{x}(0) = \dot{x}(T). \tag{2.2}$$

Throughout this paper, we always assume that the following standing hypothesis is satisfied:

(H) $a(t)$ is a continuous T-function and the Green's function of (2.1) is positive for all $(t,s) \in [0,T] \times [0,T]$.

In other words, the strict anti-maximum principle holds for (2.1)-(2.2). It is prove in [25] that if $a(t)$ satisfies $a \succ 0$ and $\underline{\lambda}_1(a) > 0$, then condition (H) is satisfied; here the notation $a \succ 0$ means that $a(t) \geq 0$ for all $t \in [0,T]$ and $a(t) > 0$ for t in a subset of positive measure, $\underline{\lambda}_1(a)$ denotes the first anti-periodic eigenvalue of

$$x'' + (\lambda + a(t))x = 0$$

subject to the anti-periodic boundary conditions

$$x(0) + x(T) = 0, \quad \dot{x}(0) + \dot{x}(T) = 0.$$

Now we make condition (H) clear. When $a(t) \equiv k^2$, condition (H) is equivalent to saying that $0 < k^2 \leq \lambda_1 = (\pi/T)^2$, where λ_1 is the first eigenvalue of the homogeneous equation $x'' + k^2x = 0$ with Dirichlet boundary conditions $x(0) = x(T) = 0$. For a non-constant function $a(t)$, there is an L^p-criterion proved in [25]. To describe these, we use $\|\cdot\|_q$ to denote the usual L^q-norm over $(0,T)$ for any given exponent $q \in [1,\infty]$. The conjugate exponent of q is denoted by $p : \frac{1}{p} + \frac{1}{q} = 1$. Let $\mathbf{M}(q)$ denote the best Sobolev constant in the following inequality

$$C\|u\|_q^2 \leq \|u'\|_2^2 \qquad \text{for all } u \in H_0^1(0,T).$$

The explicit formula for $\mathbf{M}(q)$ is

$$\mathbf{M}(q) = \begin{cases} \frac{2\pi}{qT^{1+2/q}} \left(\frac{2}{q+2}\right)^{1-2/q} \left(\frac{\Gamma(1/q)}{\Gamma(1/2+1/q)}\right)^2, & \text{for } 1 \leq q < \infty, \\ \frac{4}{T}, & \text{for } q = \infty, \end{cases}$$

where $\Gamma(\cdot)$ is the Gamma function of Euler. Let us define

$$\mathcal{A} = \{a \in L^p[0,T] : a \succ 0, \|a\|_p < \mathbf{M}(2q) \ \text{ for some } 1 \leq p \leq +\infty\} \tag{2.3}$$

Lemma 2.1[25] Assume that $a(t) \in \mathcal{A}$, then (2.1) satisfies the standing hypothesis (H), i.e, $G(t,s) > 0$ for all $(t,s) \in [0,T] \times [0,T]$.

Remark 2.2 If $p = 1$, condition $\|a\|_p < \mathbf{M}(2q)$ can be weakened to $\|a\|_1 \leq \mathbf{M}(\infty) = 4$ by the celebrated stability criterion of Lyapunov. In case $p = \infty$, condition $\|a\|_p < \mathbf{M}(2q)$ reads as $\|a\|_\infty < \mathbf{M}(2) = \pi^2$, which is a well known criterion for the anti-maximum principle used in related literature. In this case,$\|a\|_p < \mathbf{M}(2q)$ can be weakened to $a(t) \prec \pi^2$.

Under hypothesis (H), we always denote

$$M = \max_{0\leq s,t\leq T} G(t,s), \qquad m = \min_{0\leq s,t\leq T} G(t,s), \qquad \sigma = \frac{m}{M}. \tag{2.4}$$

Thus $M > m > 0$ and $0 < \sigma < 1$.

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In order to prove our results, we need two preliminary results. The first one is a well-known fixed point theorem in cones, which can be found in [14].

Theorem 2.3 Let X be a Banach space and K a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1, \bar{\Omega}_2 \backslash \Omega_2$. Let

$$T : K \cap (\bar{\Omega}_2 \backslash \Omega_1) \to K$$

be a continuous and completely continuous operator such that

(i) $\|Tx\| \leq \|x\|$ for $x \in K \cap \partial\Omega_1$;

(ii) There exist $\psi \in K\ \{0\}$ such that $x \neq Tx + \lambda\psi$ for $x \in K \cap \partial\Omega_2$ and $\lambda > 0$.

Then T has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$. The same conclusion remains valid if (i) holds on $K \cap \partial\Omega_2$ and (ii) holds $K \cap \partial\Omega_1$.

In applications below, we take $X = C[0, T]$ with the supremum norm $\| \cdot \|$ and define

$$K = \{x \in X : x(t) \geq 0 \text{ for all } t \in [0, T] \text{ and } \min_{0 \leq t \leq T} \geq \sigma\|x\|\}.$$

where σ is as in (2.4).

One can readily verify that K is a cone in X. Define an operator $T : X \to X$ by

$$(Tx)(t) = \int_0^T G(t, s)F(s, x(s))ds$$

for $x \in X$ and $t \in [0, T]$, where $F : [0, T] \times \mathbb{R} \to [0, \infty)$ is continuous and $G(t, s)$ is the Green's function of (2.1).

Lemma 2.4 T is well defined and maps X into K. Moreover, T is continuous and completely continuous.

Proof It is easy to see that T is continuous and completely continuous since F is a continuous function. Thus, we only need to show that $T(X) \subset K$. Let $x \in X$, then we have

$$\begin{aligned}
\min_{0 \leq x \leq T}(Tx)(t) &= \min_{0 \leq x \leq T} \int_0^T G(t, s)F(s, x(s))ds \\
&\geq m \int_0^T F(s, x(s))ds \\
&= \sigma M \int_0^T F(s, x(s))ds \\
&\geq \sigma \max_{0 \leq x \leq T} \int_0^T G(t, s)F(s, x(s))ds \\
&= \sigma\|Tx\|.
\end{aligned}$$

This implies that $T(X) \subset K$ and the proof is completed.

To state the second preliminary result, we recall some notation and terminology from [9]. Let X be a Banach space of functions, such that $C^1([0, T]) \subseteq X \subseteq C([0, T])$, with continuous immersions, and set $X_* = \{\rho \in X : \min \rho > 0\}$.

Define the following two operators:

$$D(L) = \{\rho \in W^{2,1}(0, T) : \rho(0) = \rho(T), \dot{\rho}(0) = \dot{\rho}(T)\},$$

$$L : D(L) \subset X \to L^1(0, T), \quad L\rho = \ddot{\rho}, \tag{2.5}$$

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and

$$N : X_* \to L^1(0,T), \quad (N\rho)(t) = -f(t,\rho(t))$$

Taking $\sigma \in \mathbb{R}$ not belonging to the spectrum of L, (1.5) can be translated to the fixed problem

$$\rho = (L - \sigma I)^{-1}(N - \sigma I)\rho.$$

We will say that a set $\Omega \subseteq X$ is uniformly positively bounded below if there is a constant $\delta > 0$ such that $\min \rho \geq \delta$ for every $\rho \in \Omega$. we need the following theorem, which has been proved in [9].

Theorem 2.5 Let Ω be an open bounded subset of X, uniformly positively bounded below. Assume that there is no solution of (1.5), on the boundary $\partial\Omega$, and that

$$deg(I - (L - \sigma I)^{-1}(N - \sigma I), \Omega, 0) \neq 0.$$

Then, there exists a $k_1 \geq 1$ such that, for every integer $k \geq k_1$, systems (1.1) has a periodic solution $x_k(t)$ with minimal period kT, which makes exactly one revolution around the origin in the period time kT. The function $|x_k(t)|$ is T-periodic and, when restricted to $[0,T]$, it belongs to Ω. Moreover, if μ_k denotes the angular momentum associated to $x_k(t)$, then

$$\lim_{k\to\infty} \mu_k = 0.$$

3. Main Results

In this section, we state and prove the main results. First we recall that $\mathcal{A}$ denotes the set defined by (2.3).

Theorem 3.1 Suppose that there exist $a(t) \in \mathcal{A}$ and $0 < r < R$ such that

(H_1) $-a(t)\rho \leq f(t,\rho) \leq \sigma/r - 1/\sigma r$, $\forall \rho \in [\sigma r, r]$,

(H_2) $f(t,\rho) \geq 0$, $\forall \rho \in [\sigma R, R]$.

Then, equation(1.4) has a T-periodic solution, and there exists a $k_1 \geq 1$ such that, for every integer $k \geq k_1$, system (1.1) has a periodic solution with minimal period kT, which makes exactly one revolution around the origin in the period time kT. Moreover, exist constant $C > 0$ (independent of μ and k) such that

$$\frac{1}{C} < |x_k(t)| < C, \quad \text{for every } t \in \mathbb{R} \text{ and every } k \geq k_1,$$

and, if μ_k denotes the angular momentum associated to $x_k(t)$ then

$$\lim_{k\to\infty} \mu_k = 0.$$

Now we begin by showing that Theorem 3.1, and use topological degree theory. To this end, we deform (1.4) to a simpler singular autonomous equation

$$\ddot{\rho} + \frac{1}{r^2}\rho - \frac{1}{\rho} = 0.$$

where r is as in Theorem 3.1.

In order to apply Theorem 2.5, we consider the $\mu = 0$ and study for $\tau \in [0,1]$, the following homotopy equation

$$\ddot{\rho} + f(t,\rho;\tau) = 0, \quad \tau \in [0,1], \tag{3.1}$$

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associated to periodic boundary conditions

$$\rho(0) = \rho(T), \qquad \dot\rho(0) = \dot\rho(T),$$

where

$$f(t,\rho;\tau) = \tau f(t,\rho) + (1-\tau)(\frac{\rho}{r^2} - \frac{1}{\rho}).$$

Note that $f(t,\rho;\tau)$ satisfies the conditions:

(H_1') $f(t,\rho;\tau) + a(t)\rho \geq 0,\ \forall \rho \in [\sigma r, R]$,

(H_2') $f(t,\rho;\tau) \leq 0, \forall \rho \in [\sigma r, r]$ and $f(t,\rho;\tau) \geq 0,\ \forall \rho \in [\sigma R, R]$.

uniformly with respect to $\tau \in [0,1]$. We need to find a priori estimates for the possible positive T-periodic solutions of (3.1). The important point to be proved is the following.

Proposition 3.2 Suppose that there exist $a \in \mathcal{A}$ and $0 < r < R$ such that $f(t,\rho;\tau)$ satisfies (H_1') and (H_2'). Then, equation (3.1) has at least one T-periodic solution.

Proof The existence is established using Theorem 2.3. To do so, let us write equation (3.1) as

$$\ddot\rho + a(t)\rho = f(t,\rho;\tau) + a(t)\rho.$$

Define the open sets

$$\Omega_1 = \{\rho \in X : \|\rho\| < r, \quad \Omega_2 = \{\rho \in X : \|\rho\| < R\}.$$

Let K be a cone defined by (2.5) and define an operator on K by

$$(\Phi\rho)(t) = \int_0^T G(t,s)\,[f(s,\rho(s);\tau) + a(t)\rho]\,ds.$$

Clearly, $\Phi : K \cap (\bar\Omega_R \backslash \Omega_r) \to C[0,T]$ is continuous and completely continuous since $f : [0,T] \times [\sigma r, R] \times [0,1]$ is continuous. Also we have $\Phi(K) \subset K$.

By the first inequality of condition (H_1'), we have $f(t,\rho;\tau) + a(t)\rho \geq a(t)\rho,\ \forall \rho \in [\sigma r, r]$. Let $\psi \equiv 1$, so $\psi \in K$. Now we prove that

$$\rho \neq \Phi\rho + \lambda\rho,\ \forall \rho \in K \cap \partial\Omega_r \text{ and } \lambda > 0. \tag{3.2}$$

Suppose not, that is, suppose there exist $\rho_0 \in K \cap \partial\Omega_r$ and $\lambda_0 > 0$ such that $\rho_0 = \Phi\rho_0 + \lambda_0\psi$. Now since $\rho_0 \in K \cap \partial\Omega_r$, then $\rho_0(t) \geq \sigma\|\rho_0\| = \sigma r$. Let $\mu = \min_{t\in[0,T]} \rho_0(t)$. Then we have

$$\begin{aligned}
\rho_0(t) &= (\Phi\rho_0)(t) + \lambda_0 \\
&= \int_0^T G(t,s)[f(s,\rho_0(s);\tau) + a(s)\rho_0(s)]ds + \lambda_0 \\
&\geq \int_0^T G(t,s)a(s)\rho_0(s)ds + \lambda_0 \\
&\geq \mu \int_0^T G(t,s)a(s)ds + \lambda_0 = \mu + \lambda_0,
\end{aligned}$$

note $\int_0^T G(t,s)a(s)ds = 1$. This implies $\mu \geq \mu + \lambda_0$, a contradiction. Therefore, (3.2) holds.

On the other hand, by the second inequality of condition (H_2'), we have

$$f(t,\rho;\tau) + a(t)\rho \leq a(t)\rho, \quad \forall \rho \in [\sigma R, R].$$

Now we prove that

$$\|\Phi x\| \le \|x\|, \quad x \in K \cap \partial\Omega_R. \tag{3.3}$$

In fact, for any $\rho \in K \cap \partial\Omega_R$, we have

$$\begin{aligned}(\Phi x)(t) &= \int_0^T G(t,s)\left[f(s,\rho_0(s);\tau) + a(s)\rho(s)\right] ds \\ &\le \int_0^T G(t,s)a(s)\rho(s)ds \\ &\le \int_0^T G(t,s)a(s)ds \cdot \max_{t\in[0,T]} \rho(t) = \|\rho\|.\end{aligned}$$

Therefore, $\|\Phi\rho\| \le \|\rho\|$, that is, (3.3) holds.

It follows from Theorem 3.3, (3.2) and (3.3) that Φ has a fixed point $\rho \in K \cap (\bar{\Omega}_R \backslash \Omega_r)$, the proof is finished. □

Proof of Theorem 3.1 Now, from Proposition 3.2, this fixed point is a positive solution of (3.1) satisfying $r \le \|x\| \le R$.

Notice the boundary condition $\dot\rho(0) = \dot\rho(T)$. Integrate (3.1) from 0 to T, we get

$$\int_0^T \ddot\rho(t)dt = -\int_0^T f(t,\rho(t);\tau)dt = 0.$$

Thus $\|f(t,\rho(t);\tau)\|_1 = 2\|f^+(t,\rho(t);\tau)\|_1$. Since $\rho(0) = \rho(T)$, there exists $t_1 \in [0,T]$ such that $\dot\rho(t_1) = 0$. Therefore

$$\begin{aligned}\|\dot\rho\| &= \max_{0\le t\le T} |\dot\rho(t)| = \max_{0\le t\le T}\left|\int_{t_1}^t \ddot\rho(s)ds\right| \\ &\le \int_0^T |f(s,\rho(s);\tau)|\,ds = 2\int_0^T \left|f^+(s,\rho(s);\tau)\right| ds \\ &\le 2\int_0^T |a(s)\rho(s)|\,ds \\ &\le 2R\|a\|_1 := H.\end{aligned}$$

where $f^+(t,\rho(t);\tau) = \max\{f(t,\rho(t);\tau),0\}$.

Define the linear operator L as in (2.5) and the Nemytzkii operator

$$N_\tau : X_* \to L^1(0,T),$$

$$(N_\tau\rho)(t) = -f(t,\rho(t);\tau),$$

(3.1) also can be translated to the fixed problem

$$\rho = (L-\sigma I)^{-1}(N_\tau - \sigma I)\rho, \tag{3.4}$$

since $L - \sigma I$ is invertible.

Take $C = \max\{1/r, R, H\}$ and let the open bounded in X be

$$\Omega = \{\rho \in X : \frac{1}{C} < \rho(t) < C \quad \text{and} \quad |\dot\rho(t)| < C \quad \text{for all } t \in [0,T]\}.$$

Obviously, Ω is an open subset of $C^1[0,T]$, and equation (3.4) has no solutions on $\partial\Omega$.

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In order to compute the degree, we consider equation (3.1). By homotopy invariance of degree, the degree has to be the same for every $\tau \in [0,1]$. Therefore, we consider the equation (3.1) with $\tau = 0$, that is the equation

$$\ddot{\rho} + \frac{1}{r^2}\rho - \frac{1}{\rho} = 0,$$

which is equivalent to the system

$$\dot{Y} = F(Y) = \left(u, \frac{\rho}{r^2} - \frac{1}{\rho}\right),$$

where $Y = (\rho, u)$.

It is easy to know that F has a unique zero (ρ_0, u_0) and the determinant of Jacobian matrix satisfies $|J_F(\rho_0, u_0)| > 0$. By Lemma the result of Capietto, Mawhin and Zanolin [4], the Leray-Schauder degree of $I - L^{-1}N(\mu, \cdot)$ is equal to the Brouwer degree of F, i.e.,

$$d_L(I - L^{-1}N(\mu, \cdot,), \Omega, 0) = d_B(F, (\frac{1}{C}, C) \times (-C, C)) = 1.$$

By Theorem 2.1, the proof of Theorem 3.1 is thus completed.

It is a direct consequence of Theorem 3.1 taking r and R small and big enough, respectively. We obtain

Corollary 3.3 Assume that the following two conditions hold:

(H_3) $\lim_{\rho \to 0^+} f(t, \rho)/\rho = -\infty$, uniformly for $t \in [0, T]$,

(H_4) $\lim_{\rho \to +\infty} f(t, \rho)/\rho = +\infty$, uniformly for $t \in [0, T]$

Then problem (1.1) has the same conclusion of Theorem 3.1.

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[1] College of Information Sciences and Technology, Hainan University, Haikou, 570228, China

[2] School of Matheematical Sciences and Computing Technology, Central South University, Changsha, 410083, China

[3] College of Information Science and Computer Engineering, Wenzhou Medical University, Wenzhou, 325035, China

E-mail address: shjli626@126.com (S. Li)
E-mail address: liwulan@163.com (W. Li)
E-mail address: pinger80@126.com (Y. Fu)

Approximate ternary Jordan ring homomorphisms in ternary Banach algebras

M. Eshaghi Gordji[1], Vahid Keshavarz[2], Jung Rye Lee[3], Dong Yun Shin[4*] and Choonkil Park[5]

[1,2]**Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran**

[3]Department of Mathematics, Daejin University, Kyeonggi 487-711, Korea

[4]Department of Mathematics, University of Seoul, Seoul 130-743, Korea

[5]Department of Mathematics, Hanyang Universityl, Seoul 133-791, Korea

e-mail: m.eshaghi@semnan.ac.ir, v.keshavarz68@yahoo.com, jrlee@daejin.ac.kr, dyshin@uos.ac.kr, baak@hanyang.ac.kr

Abstract. Let A and B be real ternary Banach algebras. An additive mapping $\Im : (A, [\ ]_A) \longrightarrow (B, [\ ]_B)$ is called a ternary Jordan homomorphism if $\Im([x, x, x]_A) = [\Im(x), \Im(x), \Im(x)]_B$ for all $x \in A$.

In this paper, we investigate the stability and superstability of ternary Jordan ring homomorphisms in ternary Banach algebras by using the fixed point method.

1. Introduction

Ternary algebraic operations were considered in the 19th century by several mathematicians. Cayley [5] who introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii in 1990 [13]. As an application in physics, the quark model inspired a particular brand of ternary algebraic systems. There are also some applications, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics (the anyons), supersymmetric theories, Yang-Baxter equation, etc, (cf. [1, 26]).

The comments on physical applications of ternary structures can be found in [2, 8, 9, 21, 22, 23, 26].

Let A and B be ternary Banach algebras. An additive mapping $\Im : (A, [\ ]_A) \to (B, [\ ]_B)$ is called a ternary ring homomorphism if

$$\Im([x, y, z]_A) = [\Im(x), \Im(y), \Im(z)]_B$$

for all $x, y, z \in A$. An additive mapping $\Im : (A, [\ ]_A) \to (B, [\ ]_B)$ is called a ternary Jordan ring homomorphism if

$$\Im([x, x, x]_A) = [\Im(x), \Im(x), \Im(x)]_B$$

for all $x \in A$.

We say that a functional equation (Q) is stable if any function g satisfying the equation (Q) approximately is near to true solution of (Q). Also, we say that a functional equation is superstable if every approximately solution is an exact solution of it.

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[0]*Corresponding uuthor: dyshin@uos.ac.kr (Dong Yun Shin).

Approximate ternary Jordan ring homomorphisms

The study of stability problems originated from a famous talk given by Ulam [25] in 1940: "Under what condition does there exist a homomorphism near an approximate homomorphism?" In 1941, Hyers [12] answered affirmatively the question of Ulam for additive mappings between Banach spaces. A generalized version of the theorem of Hyers for approximately additive maps was given by Rassias [20] in 1978. For more details about various results concerning such problems the reader is referred to [3, 6, 7, 10, 11, 15, 16, 17, 18, 19, 24].

We need the following fixed point theorem.

Theorem 1.1. [14] *Suppose that (Ω, d) is a complete generalized metric space and $T : \Omega \to \Omega$ is a strictly contractive mapping with the Lipschitz constant L. Then, for any $x \in \Omega$, either*

$$d(T^n x, T^{n+1}x) = \infty, \quad \forall n \geq 0,$$

or there exists a positive integer n_0 such that

(1) *$d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;*

(2) *the sequence $\{T^n x\}$ is convergent to a fixed point y^* of T;*

(3) *y^* is the unique fixed point of T in $\Lambda = \{y \in \Omega : d(T^{n_0}x, y) < \infty\}$;*

(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Lambda$.*

In this paper, we prove the stability and superstability of ternary Jordan ring homomorphisms in ternary Banach algebras by using the fixed point method.

2. Stability of ternary Jordan ring homomorphisms

In this section, we establish the stability ternary Jordan ring homomorphisms in ternary Banach algebras.

Throughout this section, assume that A and B are ternary Banach algebras.

Lemma 2.1. [9] *Let $f : A \to B$ be an additive mapping. Then the following assertions are equivalent*

$$f([a, a, a]) = [f(a), f(a), f(a)] \tag{2.1}$$

for all $a \in A$, and

$$f([a, b, c] + [b, c, a] + [c, a, b]) = [f(a), f(b), f(c)] + [f(b), f(c), f(a)] + [f(c), f(a), f(b)] \tag{2.2}$$

for all $a, b, c \in A$.

Theorem 2.2. *Let $f : A \to B$ be a mapping for which there exist functions $\varphi : A \times A \to [0, \infty)$ and $\psi : A \times A \times A \to [0, \infty)$ such that*

$$\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y), \tag{2.3}$$

$$\|f([x, y, z] + [y, z, x] + [z, x, y]) - [f(x), f(y), f(z)] - [f(y), f(z), f(x)] - [f(z), f(x), f(y)]\| \leq \psi(x, y, z) \tag{2.4}$$

for all $x, y, z \in A$. If there exists a constant $0 < L < 1$ such that

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{2}\varphi(x, y), \tag{2.5}$$

$$\psi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \le \frac{L}{2^3}\psi(x, y, z) \tag{2.6}$$

for all $x, y, z \in A$, then there exists a unique ternary Jordan ring homomorphism $\Im : A \to B$

$$\|f(x) - \Im(x)\| \le \frac{L}{2(1-L)}\varphi(x, x) \tag{2.7}$$

for all $x \in A$.

Proof. It follows from (2.5) and (2.6) that

$$\lim_{n\to\infty} 2^n\varphi\Big(\frac{x}{2^n}, \frac{y}{2^n}\Big) = 0, \tag{2.8}$$

$$\lim_{n\to\infty} 2^{3n}\psi\Big(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\Big) = 0 \tag{2.9}$$

for all $x, y, z \in A$. By (2.5), $\lim_{n\to\infty} 2^n\varphi(0, 0) = 0$ and so $\varphi(0, 0) = 0$. Letting $x = y = 0$ in (2.3), we get $f(0) \le \varphi(0, 0) = 0$ and so $f(0) = 0$.

Let $\Omega = \{g : A \to B,\ g(0) = 0\}$. We introduce a generalized metric on Ω as follows:

$$d(g, h) = d_\varphi(g, h) = \inf\{K \in (0, \infty) :\ \|g(x) - h(x)\| \le K\varphi(x, x),\ \forall x \in A\} .$$

It is easy to show that (Ω, d) is a generalized complete metric space.

Now, we consider the mapping $T : \Omega \to \Omega$ defined by $Tg(x) = 2g(\frac{x}{2})$ for all $x \in A$ and $g \in \Omega$. Note that, for all $g, h \in \Omega$ and $x \in A$,

$$\begin{aligned} d(g, h) < K &\Rightarrow \|g(x) - h(x)\| \le K\varphi(x, x) \\ &\Rightarrow \|2g(\frac{x}{2}) - 2h(\frac{x}{2})\| \le 2\ K\ \varphi(\frac{x}{2}, \frac{x}{2}) \\ &\Rightarrow \|2g(\frac{x}{2}) - 2h(\frac{x}{2})\| \le L\ K\ \varphi(x, x) \\ &\Rightarrow d(Tg, Th) \le L\ K. \end{aligned}$$

Hence we show that

$$d(Tg, Th) \le L\ d(g, h)$$

for all $g, h \in \Omega$, that is, T is a strictly contractive mapping of Ω with the Lipschitz constant L.

Putting $y = x$ in (2.3), we get

$$\|f(2x) - 2f(x)\| \le \varphi(x, x)$$

for all $x \in A$. So

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \le \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \le \frac{L}{2}\ \varphi(x, x)$$

for all $x \in A$, that is, $d(f, Tf) \le \frac{L}{2} < \infty$.

Let us denote

$$\Im(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$ since $\lim_{n\to\infty} d(T^n f, \Im) = 0$. By the result in [4] , $\Im$ is a ternary Jordan mapping and so it follows from the definition of $\Im$, (2.4) and (2.9) that

$$\begin{aligned}
&\|\Im([x,y,z]+[y,z,x]+[z,x,y]) - [\Im(x),\Im(y),\Im(z)] - [\Im(y),\Im(z),\Im(x)] - [\Im(z),\Im(x),\Im(y)]\| \\
&= \lim_{n\to\infty} \Big\| f\left(2^{3n}[\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}] + 2^{3n}[\frac{z}{2^n},\frac{y}{2^n},\frac{x}{2^n}] + 2^{3n}[\frac{z}{2^n},\frac{x}{2^n},\frac{y}{2^n}]\right) \\
&- 2^{3n}[f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right), f\left(\frac{z}{2^n}\right)] - 2^{3n}[f\left(\frac{y}{2^n}\right), f\left(\frac{z}{2^n}\right), f\left(\frac{x}{2^n}\right)] - 2^{3n}[f\left(\frac{z}{2^n}\right), f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right)]\Big\| \\
&\leq \lim_{n\to\infty} 2^{3n}\psi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right) = 0
\end{aligned}$$

and so

$$\Im([x,y,z]+[y,z,x]+[z,x,y]) = [\Im(x),\Im(y),\Im(z)] + [\Im(y),\Im(z),\Im(x)] + [\Im(z),\Im(x),\Im(y)]$$

for all $x \in A$.

According to Theorem 1.1, since $\mathfrak{F}$ is the unique fixed point of T in the set $\Lambda = \{g \in \Omega : d(f,g) < \infty\}$, $\mathfrak{F}$ is the unique mapping such that

$$\|f(x) - \mathfrak{F}(x)\| \leq K\ \varphi(x,x)$$

for all $x \in A$ and $K > 0$. Again, using Theorem 1.1, we have

$$d(f,\mathfrak{F}) \leq \frac{1}{1-L} d(f,Tf) \leq \frac{L}{2(1-L)}$$

and so

$$\|f(x) - \mathfrak{F}(x)\| \leq \frac{L}{2(1-L)}\ \varphi(x,x)$$

This completes the proof. □

Corollary 2.3. *Let θ, p be nonnegative real numbers with $r,p > 1$ and $\frac{r-3p}{2} \geq 1$. Suppose that $f : A \to B$ is a mapping such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^r + \|y\|^r),$$

$$\|f([x,y,z]+[y,z,x]+[z,x,y]) - [f(x),f(y),f(z)] - [f(y),f(z),f(x)] - [f(z),f(x),f(y)]\| \leq \theta(\|x\|^p.\|y\|^p.\|z\|^p)$$

for all $x,y,z \in A$. Then there exists a unique ternary Jordan ring homomorphism $\mathfrak{F} : A \to B$ satisfying

$$\|f(x) - \mathfrak{F}(x)\| \leq \frac{\theta}{(2^r - 2)}\|x\|^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.2 by taking

$$\varphi(x,y) := \theta(\|x\|^r + \|y\|^r), \quad \psi(x,y,z) := \theta(\|x\|^p.\|y\|^p.\|z\|^p)$$

for all $x,y \in A$. Then we can choose $L = 2^{1-r}$ and so the desired conclusion follows. □

Remark 2.4. Let $f : A \to B$ be a mapping with $f(0) = 0$ such that there exist functions $\varphi : A \times A \to [0, \infty)$ and $\psi : A \times A \times A \to [0, \infty)$ satisfying (2.3) and (2.4). Let $0 < L < 1$ be a constant such that

$$\varphi(2x, 2y) \le 2L\varphi(x, y), \quad \psi(2x, 2y, 2z) \le 2^3 L\psi(x, y, z)$$

for all $x, y, z \in A$. By the similar method as in the proof of Theorem 2.2, one can show that there exists a unique ternary Jordan ring homomorphism $\mathfrak{F} : A \to X$ satisfying

$$\|f(x) - \mathfrak{F}(x)\| \le \frac{1}{2(1-L)}\varphi(x, x)$$

for all $x \in A$. For the case

$$\varphi(x, y) := \delta + \theta(\|x\|^r + \|y\|^r), \quad \psi(x, y, z) := \delta + \theta(\|x\|^p \cdot \|y\|^p \cdot \|z\|^p)$$

(where θ, δ are nonnegative real numbers and $r > 0, p < 1$ and $\frac{r-3p}{2} \ge 1$), there exists a unique ternary Jordan ring homomorphism $\Im : A \to X$ satisfying

$$\|f(x) - \mathfrak{F}(x)\| \le \frac{\delta}{(2-2^r)} + \frac{\theta}{(2-2^r)}\|x\|^r$$

for all $x \in A$.

3. Superstability of ternary Jordan ring homomorphisms

In this section, we formulate and prove the superstability of ternary Jordan ring homomorphisms.

Theorem 3.1. *Suppose that there exist function $\psi : A \times A \times A \to [0, \infty)$ and a constant $0 < L < 1$ such that*

$$\psi\Big(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\Big) \le \frac{L}{2^3}\psi(x, y, z)$$

for all $x, y, z \in A$. Moreover, if $f : A \to B$ is an additive mapping such that

$$\|f([x, y, z] + [y, z, x] + [z, x, y]) - [f(x), f(y), f(z)] - [f(y), f(z), f(x)] - [f(z), f(x), f(y)]\| \le \psi(x, y, z)$$

for all $x, y, z \in A$, then f is a ternary Jordan ring homomorphism.

Proof. The proof of this theorem is omitted as similar to the proof of Theorem 2.2. □

Corollary 3.2. *Let θ, r, s be nonnegative real numbers with $r > 1$ and $s > 3$. If $f : A \to B$ is an additive mapping such that*

$$\|f([x, y, z] + [y, z, x] + [z, x, y]) - [f(x), f(y), f(z)] - [f(y), f(z), f(x)] - [f(z), f(x), f(y)]\| \le \theta(\|x\|^s + \|y\|^s + \|z\|^s)$$

for all $x, y, z \in A$, then f is a ternary Jordan ring homomorphism.

Remark 3.3. Let θ, r be nonnegative real numbers with $r < 1$. Suppose that there exists a function $\psi : A \times A \times A \to [0, \infty)$ and a constant $0 < L < 1$ such that

$$\psi(2x, 2y, 2z) \leq 2^3 L\psi(x, y, z)$$

for all $x, y, z \in A$. If $f : A \to B$ is an additive mapping such that

$$\|f([x, y, z] + [y, z, x] + [z, x, y]) - [f(x), f(y), f(z)] - [f(y), f(z), f(x)] - [f(z), f(x), f(y)]\| \leq \psi(x, y, z)$$

for all $x, y, z \in A$, then f is a ternary Jordan ring homomorphism.

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Approximate controllability of fractional impulsive stochastic functional differential inclusions with infinite delay and fractional sectorial operators

Zuomao Yan* and Xiumei Jia

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Abstract: In this paper, the approximate controllability of fractional impulsive stochastic functional differential inclusions with infinite delay and fractional sectorial operators is considered. By using the stochastic analysis, the fractional sectorial operators and a fixed-point theorem for multi-valued maps, a new set of necessary and sufficient conditions are formulated which guarantees the approximate controllability of the fractional impulsive stochastic system. The results are obtained under the assumption that the associated linear system is approximately controllable. Finally, an example is also given to illustrate the obtained theory.

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1 Introduction

The notion of controllability has played a central role throughout the history of modern control theory. Moreover, approximate controllable systems are more prevalent and very often approximate controllability is completely adequate in applications; see [1-3]. Therefore, various approximate controllability problems for different kinds of dynamical systems have been investigated in many publications; see [4,5] and references therein. The fractional differential equations has received a great deal of attention, and they play an important role in many applied fields, including viscoelasticity, electrochemistry, control, porous media, electromagnetic and so on. In recent years, several papers have studied the approximate controllability of semilinear fractional differential systems without delay and infinite delay (see [6-9]). As a result of its widespread use, the approximate controllability of stochastic systems have received extensive attention. More recently, there are very few contributions regarding the approximate controllability of fractional stochastic control system. For example, Sakthivel et al. [10], Kerboua et al. [11], Muthukumar and Rajivganthi [12], Farahi and Guendouzi [13].

Impulsive partial functional differential equations or inclusions have become an active area of investigation due to their applications in fields such as mechanics, electrical engineering, medicine biology (see [14]). Recently, the approximate controllability for some fractional impulsive semilinear differential systems have been studied in several papers. For example, Liu and Bin [15] studied the approximate controllability for a class of Riemann-Liouville fractional impulsive differential inclusions. Balasubramaniam et al. [16] derived sufficient conditions for the approximate controllability of impulsive fractional integro-differential systems with nonlocal conditions in Hilbert space. Chalishajar et al. [17] discussed the approximate controllability of abstract impulsive fractional neutral evolution equations with infinite delay in Banach spaces. However, besides impulse effects and delays, stochastic effects likewise exist in real systems. For semilinear impulsive stochastic control systems in Hilbert spaces, there are several papers devoted to the approximate controllability (see [18,19]). Zang and Li [20] obtained the approximate controllability of fractional impulsive neutral stochastic differential equations with nonlocal conditions by using Krasnoselskii-Schaefer-type fixed point theorem.

Motivated by the researches mentioned previously, in this paper we consider the approximate controllability of a class of fractional impulsive stochastic functional differential inclusions with infinite delay in Hilbert spaces of the form

$$^cD^\alpha N(x_t) \in AN(x_t) + Bu(t) + F(t, x_t)\frac{dw(t)}{dt}, \qquad (1)$$
$$\alpha \in (0,1), t \in J = [0,b], t \neq t_k,$$

$$x_0 = \varphi \in \mathcal{B}, \qquad (2)$$

$$\Delta x(t_k) = I_k(x_{t_k}), \quad k = 1, \ldots, m, \qquad (3)$$

where the state $x(\cdot)$ takes values in a separable real Hilbert space H with inner product $\langle\cdot,\cdot\rangle_H$ and norm $\|\cdot\|_H$. Here $^cD^\alpha$ is the Caputo fractional derivative of the order $\alpha \in (0,1)$ with the lower limit zero, A is a fractional sectorial operator defined on $(H, \|\cdot\|_H)$. Let K be another separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_K$ and norm $\|\cdot\|_K$. Suppose $\{w(t) : t \geq 0\}$ is a given K-valued Wiener process with a covariance operator $Q > 0$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which is generated by the Wiener process w. The control function $u \in L^p_{\mathcal{F}}(J, U)$, a Hilbert space of admissible control functions, $p \geq 2$ be an integer, and B is a bounded linear operator from a Banach space U to H. The time history $x_t : (-\infty, 0] \to H$ given by $x_t(\theta) = x(t+\theta)$ belongs to some abstract phase space $\mathcal{B}$ defined axiomatically; $F, G, I_k(k = 1, \ldots, m), N(\psi) = \psi(0) - G(t, \psi), \psi \in \mathcal{B}$, are given functions to be specified later. Moreover, let $0 < t_1 < \cdots < t_m < b$, are prefixed points and the symbol $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^-)$ and $x(t_k^+)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. The initial data $\{\varphi(t) : -\infty < t \leq 0\}$ is an $\mathcal{F}_0$-adapted, $\mathcal{B}$-valued random variable independent of the Wiener process w with finite second moment.

To the best of our knowledge, the approximate controllability of fractional impulsive stochastic functional differential inclusions with infinite delay and fractional sectorial operators and the form (1)-(3) is an untreated topic in the literature. To close the gap in this paper, we study this interesting problem. Sufficient conditions for the approximate controllability results are derived by a fixed-point theorem for multi-valued maps combined with the stochastic analysis and the fractional sectorial operators. The known results appeared in [15-20] are generalized to the fractional impulsive stochastic systems settings and the case of infinite delay.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and necessary preliminaries. In Section 3, we give our main results. In Section 4, an example is given to illustrate our results.

2 Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with probability measure P on Ω and a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Let H, K be two real separable Hilbert spaces and we denote by $\langle\cdot,\cdot\rangle_H, \langle\cdot,\cdot\rangle_K$ their inner products and by $\|\cdot\|_H, \|\cdot\|_K$ their vector norms, respectively. $L(H, K)$ be the space of bounded linear operators mapping K into H equipped with the usual norm $\|\cdot\|_H$ and $L(H)$ denotes the Banach space of bounded linear operators from H to H. Let $\{w(t) : t \geq 0\}$ denote an K-valued Wiener process defined on the probability space $(\Omega, \mathcal{F}, P)$ with covariance operator Q. We assume that there exists a complete orthonormal system $\{e_n\}_{n=1}^{\infty}$ in K, a bounded sequence of nonnegative real numbers $\{\lambda_n\}_{n=1}^{\infty}$ such that $Qe_n = \lambda_n e_n, n = 1, 2, \ldots$, and a sequence β_n of independent Brownian motions such that

$$\langle w(t), e\rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\langle e_n, e\rangle\beta_n(t), \ e \in K, t \in J,$$

and $\mathcal{F}_t = \mathcal{F}_t^w$, where $\mathcal{F}_t^w$ is the σ-algebra generated by $\{w(s) : 0 \leq s \leq t\}$. Let $L_2^0 = L_2(K_0, H)$ be the space of all Hilbert-Schmidt operators from K_0 to H with the norm $\| \psi \|_{L_2^0}^2 = \mathrm{Tr}((\psi Q^{1/2})(\psi Q^{1/2})^*)$ for any $\psi \in L_2^0$. Clearly for any bounded operators $\psi \in L(K, H)$ this norm reduces to $\| \psi \|_{L_2^0}^2 = \mathrm{Tr}(\psi Q\psi^*)$. Let $L^p(\mathcal{F}_b, H)$ be the Banach space of all $\mathcal{F}_b$-measurable pth power integrable random variables with values in the Hilbert space H. Let $C([0, b]; L^p(\mathcal{F}, H))$ be the Banach space of continuous maps from $[0, b]$ into $L^p(\mathcal{F}, H)$ satisfying the condition $\sup_{t\in J} E \| x(t) \|_H^p < \infty$.

We use the notations $\mathcal{P}(H)$ is the family of all subsets of H. Let us introduce the following notations:

$$\mathcal{P}_{cl}(H) = \{x \in \mathcal{P}(H) : x \text{ is closed}\}, \quad \mathcal{P}_{bd}(H) = \{x \in \mathcal{P}(H) : x \text{ is bounded}\},$$

$$\mathcal{P}_{cv}(H) = \{x \in \mathcal{P}(H) : x \text{ is convex}\}, \quad \mathcal{P}_{cp}(H) = \{x \in \mathcal{P}(H) : x \text{ is compact}\}.$$

Consider $H_d : \mathcal{P}(H) \times \mathcal{P}(H) \to R^+ \cup \{\infty\}$ given by

$$H_d(\widetilde{A}, \widetilde{B}) = \max\left\{ \sup_{\tilde{a}\in\widetilde{A}} d(\tilde{a}, \widetilde{B}), \sup_{\tilde{b}\in\widetilde{B}} d(\widetilde{A}, \tilde{a}) \right\},$$

where $d(\widetilde{A}, \tilde{b}) = \inf_{\tilde{a}\in\widetilde{A}} d(\tilde{a}, \tilde{b}), d(\tilde{a}, \widetilde{B}) = \inf_{\tilde{b}\in\widetilde{B}} d(\tilde{a}, \tilde{b})$. Then, $(\mathcal{P}_{bd,cl}(H), H_d)$ is a metric space and $(\mathcal{P}_{cl}(H), H_d)$ is a generalized metric space. In what follows, we briefly introduce some facts on multi-valued analysis. For more details, one can see [21,22].

A multi-valued map $\Phi : J \to \mathcal{P}_{cl}(H)$ is said to be measurable if for each $x \in H$, the function $Y : J \to R^+$ defined by $Y(t) = d(x, \Phi(t)) = \inf\{d(x, z) : z \in \Phi(t)\}$ is measurable.

Φ has a fixed point if there is $x \in H$ such that $x \in \Phi(x)$. The set of fixed points of the multi-valued operator Φ will be denoted by FixΦ.

Definition 2.1. A multi-valued operator $\Phi : H \to P_{cl}(H)$ is called:

(a) γ-Lipschitz if there exists $\gamma > 0$ such that

$$H_d(\Phi(x), \Phi(y)) \leq \gamma d(x, y), \;\; x, y \in H.$$

(b) a contraction if it is γ-Lipschitz with $\gamma < 1$.

In this paper, we assume that the phase space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a seminormed linear space of $\mathcal{F}_0$-measurable functions mapping $(-\infty, 0]$ into H, and satisfying the following fundamental axioms due to Hale and Kato (see e.g., in [23]).

(A) If $x : (-\infty, \sigma + b] \to H$, $b > 0$, is such that $x|_{[\sigma,\sigma+b]} \in C([\sigma, \sigma + b], H)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + b]$ the following conditions hold:

  (i) x_t is in $\mathcal{B}$;

  (ii) $\| x(t) \|_H \leq \tilde{H} \| x_t \|_{\mathcal{B}}$;

  (iii) $\| x_t \|_{\mathcal{B}} \leq K(t - \sigma) \sup\{\| x(s) \|_H : \sigma \leq s \leq t\} + M(t - \sigma) \| x_\sigma \|_{\mathcal{B}}$, where $\tilde{H} \geq 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, K is continuous and M is locally bounded, and $\tilde{H}, K, M$ are independent of $x(\cdot)$.

(B) For the function $x(\cdot)$ in (A), the function $t \to x_t$ is continuous from $[\sigma, \sigma+b]$ into $\mathcal{B}$.

(C) The space $\mathcal{B}$ is complete.

The next result is a consequence of the phase space axioms.

Lemma 2.1. Let $x : (-\infty, b] \to H$ be an $\mathcal{F}_t$-adapted measurable process such that the $\mathcal{F}_0$-adapted process $x_0 = \varphi(t) \in L_2^0(\Omega, \mathcal{B})$ and $x|_{[0,b]} \in \mathcal{PC}([0, b], H)$, then

$$\| x_s \|_{\mathcal{B}} \leq M_b E \| \varphi \|_{\mathcal{B}} + K_b \sup_{0\leq s\leq b} E \| x(s) \|_H,$$

where $K_b = \sup\{K(t) : 0 \leq t \leq b\}, M_b = \sup\{M(t) : 0 \leq t \leq b\}$.

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We introduce the space $\mathcal{PC}$ formed by all $\mathcal{F}_t$-adapted measurable, H-valued stochastic processes $\{x(t) : t \in [0,b]\}$ such that x is continuous at $t \neq t_k$, $x(t_k) = x(t_k^-)$ and $x(t_k^+)$ exists for all $k = 1, ..., m$. In this paper, we always assume that $\mathcal{PC}$ is endowed with the norm

$$\| x \|_{\mathcal{PC}} = (\sup_{0\leq t\leq b} E \| x(t) \|_H^p)^{\frac{1}{p}}.$$

Then $(\mathcal{PC}, \| \cdot \|_{\mathcal{PC}})$ is a Banach space.

Definition 2.2 ([24]). The fractional integral of order γ with the lower limit zero for a function $h \in L^1(J,H)$ is defined as

$$I_t^\gamma h(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{h(s)}{(t-s)^{1-\gamma}} ds, \quad t > 0, \gamma > 0$$

provided the right side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.3 ([24]). The Riemann-Liouville derivative of order γ with the lower limit zero for a function $h \in L^1(J,H)$ can be written as

$$D_t^\gamma h(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{h(s)}{(t-s)^{\gamma+1-n}} ds, \quad t > 0, n-1 < \gamma < n.$$

Definition 2.4 ([24]). The Caputo derivative of order γ for a function $h \in L^1(J,H)$ can be written as

$$D_t^\gamma h(t) = D_t^\gamma (h(t) - h(0)), \quad t > 0, 0 < \gamma < 1.$$

Next, we are ready to recall some facts of fractional Cauchy problem.

$$^cD_t^\alpha x(t) = Ax(t), \quad t \geq 0, \tag{4}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{5}$$

where A is linear closed and $D(A)$ is dense.

Definition 2.5 ([25]). A family $\{S_\alpha(t) : t \geq 0\} \subset L(H)$ is called a solution operator for (4)-(5) if the following conditions are satisfied:

(a) $S_\alpha(t)$ is strongly continuous for $t \geq 0$ and $S_\alpha(0) = I$.

(b) $S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)\varphi = S_\alpha(t)A\varphi$ for all $\varphi \in D(A), t \geq 0$.

(c) $S_\alpha(t)\varphi$ is a solution of (4)-(5) for all $\varphi \in D(A), t \geq 0$.

Definition 2.6 ([24]). An operator A is said to be belong to $e^\alpha(M,\omega)$ if the solution operator $S_\alpha(\cdot)$ of (4)-(5) satisfies

$$\| S_\alpha(t) \|_{L(H)} \leq Me^{\omega t}, \quad t \geq 0$$

for some constants $M \geq 1$ and $\omega \geq 0$.

Definition 2.7 ([24]). A solution operator $S_\alpha(\cdot)$ of (4)-(5) is called analytic if it admits an analytic extension to a sector $\Sigma_{\theta_0} = \{\lambda \in \mathbf{C} - \{0\} : \arg\lambda < \theta_0\}$ for some $\theta_0 \in (0, \frac{\pi}{2}]$. An analytic solution operator is said to be of analyticity type (θ_0, ω_0) if for each $\theta < \theta_0$ and $\omega > \omega_0$ there is an $M = M(\theta, \omega)$ such that

$$\| S_\alpha(t) \|_{L(H)} \leq M e^{\omega \mathrm{Re} t}, \quad t \in \Sigma_\theta.$$

Set

$$e^\alpha(\omega) = \bigcup\{e^\alpha(M, \omega) : M \geq 1\}, e^\alpha := \bigcup\{e^\alpha(\omega) : \omega \geq 0\},$$

and $A^\alpha(\theta_0, \omega_0) = \{A \in e^\alpha : A$ generates an analytic solution operator S_α of type $(\theta_0, \omega_0)\}$.

Remark 2.3 ([25, Theorem 2.14]). Let $\alpha \in (0, 2)$. A linear closed densely defined operator A belongs to $A^\alpha(\theta_0, \omega_0)$ if and only if $\lambda^\alpha \in \rho(A)$ for each $\lambda \in \Sigma_{\theta_0 + \frac{\pi}{2}}(\omega_0) = \{\mathbf{C} - \{0\} : |\arg(\lambda - \omega_0)| < \theta_0 + \frac{\pi}{2}\}$ and for any $\omega > \omega_0, \theta < \theta_0$ there is a constant $C = C(\theta, \omega)$ such that

$$\| \lambda^{\alpha-1} R(\lambda^\alpha, A) \|_{L(H)} \leq \frac{C}{|\lambda - \omega|}$$

for $\lambda \in \Sigma_{\theta_0 + \frac{\pi}{2}}$.

According to the proof of Theorem 2.14 in [25], if $A \in A^\alpha(\theta_0, \omega_0)$ for some $\theta_0 \in (0, \pi)$ and $\omega_0 \in R$, the solution operator for the Eq. (4)-(5) is given by

$$S_\alpha(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \lambda^{\alpha-1} R(\lambda^\alpha, A) d\lambda$$

for a suitable path Γ. Next, a mild solution of the Cauchy problem

$${}^c D_t^\alpha x(t) = Ax(t) + f(t), \quad t \in J,$$

$$x_0 = \varphi \in \mathcal{B},$$

can be defined by

$$x(t) = S_\alpha(t)\varphi + \int_0^t T_\alpha(t-s) f(s) ds,$$

where

$$T_\alpha(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} R(\lambda^\alpha, A) d\lambda$$

for a suitable path Γ and $f : J \to H$ is continuous.

Lemma 2.2 ([25]). If $A \in A^\alpha(\theta_0, \omega_0)$ then

$$\| S_\alpha(t) \|_{L(H)} \leq M e^{\omega t}, \quad \| T_\alpha(t) \|_{L(H)} \leq C e^{\omega t}(1 + t^{\alpha-1})$$

for every $t > 0, \omega > \omega_0$. So putting

$$\tilde{M}_S := \sup_{0 \leq t \leq b} \| S_\alpha(t) \|_{L(H)}, \quad \tilde{M}_T := \sup_{0 \leq t \leq b} C e^{\omega t}(1 + t^{\alpha-1}),$$

we get

$$\| S_\alpha(t) \|_{L(H)} \leq \tilde{M}_S, \quad \| T_\alpha(t) \|_{L(H)} \leq t^{\alpha-1}\tilde{M}_T.$$

Based on the above consideration, we introduce the definition of mild solution for (1)-(3).

Definition 2.8. Let $A \in A^\alpha(\theta_0, \omega_0)$ with $\theta_0 \in (0, \frac{\pi}{2}]$ and $\omega_0 \in R$. An $\mathcal{F}_t$-adapted stochastic process $x : (-\infty, b] \to H$ is called a mild solution of the system (1)-(3) if $x_0 = \varphi \in \mathcal{B}$ satisfying $x_0 \in L_2^0(\Omega, H), x|_{[0,b]} \in \mathcal{PC}$, and

$$x(t) = \begin{cases} S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, x_t) \\ \quad + \int_0^t T_\alpha(t-s)Bu(s)ds \\ \quad + \int_0^t T_\alpha(t-s)f(s)dw(s), & t \in [0, t_1], \\ S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, x_t) \\ \quad + S_\alpha(t-t_1)I_1(x_{t_1}) \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)Bu(s)ds \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)f(s)dw(s), & t \in (t_1, t_2], \\ \vdots \\ S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, x_t) \\ \quad + \sum_{k=1}^m S_\alpha(t-t_k)I_k(x_{t_k}) \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)Bu(s)ds \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)f(s)dw(s), & t \in (t_m, b], \end{cases}$$

where $f \in S_{F,x} = \{f \in L^p(J, L_2^0) : f(t) \in F(t, x_t) \text{ a.e. } t \in J\}$.

Let $x(t; \varphi, u)$ denotes state value of the system (1)-(3) at time t corresponding to the control $u \in L^p_{\mathcal{F}}(J, U)$. In particular, the state of system (1)-(3) at $t = b, x(b; \varphi, u)$ is called the terminal state with control u and the initial value φ. Introduce the set $\mathcal{B}(b; \varphi, u) = \{x(b; \varphi, u), u(\cdot) \in L^p_{\mathcal{F}}(J, U)\}$ is called the reachable set of the system (1)-(3), where $L^p_{\mathcal{F}}(J, U)$ is the closed subspace of $L^p_{\mathcal{F}}(J \times \Omega, U)$, consisting of all $\mathcal{F}_t$-adapted, U-valued stochastic processes.

Definition 2.9. The system (1)-(3) is said to be approximately controllable on the interval J if $\overline{\mathcal{B}(b; \varphi, u)} = L^p(\mathcal{F}_b, H)$, where $\overline{\mathcal{B}(b; \varphi, u)}$ is the closure of the reachable set.

It is convenient at this point to define operators

$$\Gamma_\tau^b = \int_\tau^b \mathcal{S}_\alpha(b-s)BB^*\mathcal{S}_\alpha^*(b-s)ds, \quad 0 \leq \tau < b,$$

$$\Gamma_0^b = \int_0^b \mathcal{S}_\alpha(b-s)BB^*\mathcal{S}_\alpha^*(b-s)ds,$$

$$R(a, \Gamma_\tau^b) = (aI + \Gamma_\tau^b)^{-1}, R(a, \Gamma_0^b) = (aI + \Gamma_0^b)^{-1} \text{ for } a > 0,$$

where B^* denotes the adjoint of B and $\mathcal{S}_\alpha^*(t)$ is the adjoint of $\mathcal{S}_\alpha(t)$. It is straightforward that the operator Γ_τ^b is a linear bounded operator.

Lemma 2.4 ([3]). For any $\tilde{x}_b \in L^p(\mathcal{F}_b, H)$ there exists $\tilde{\phi} \in L^p_{\mathcal{F}}(\Omega; L^2(0, b; L_2^0))$ such that $\tilde{x}_b = E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s)$.

Now for any $a > 0$ and $\tilde{x}_b \in L^p(\mathcal{F}_b, H)$ we define the control function

$$u_x^a(t) = \begin{cases} S^* T_\alpha^*(b-t)(aI + \Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s) \\ \quad -S_\alpha(b)[\varphi(0) - G(0,\varphi)] - G(b, x_b)\Big] \\ \quad -B^* T_\alpha^*(b-t)\int_0^b (aI + \Gamma_s^b)^{-1}\mathcal{S}_\alpha(b-s)f(s)dw(s), \quad t \in [0, t_1], \\ S^* T_\alpha^*(b-t)(aI + \Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s) \\ \quad -S_\alpha(b)[\varphi(0) - G(0,\varphi)] - G(b, x_b)\Big] \\ \quad -B^* T_\alpha^*(b-t)(aI + \Gamma_s^b)^{-1}S_\alpha(t-t_1)I_1(x_{t_1}) \\ \quad -B^* T_\alpha^*(b-t)\int_0^b (aI + \Gamma_s^b)^{-1}\mathcal{S}_\alpha(b-s)f(s)dw(s), \quad t \in (t_1, t_2], \\ \vdots \\ S^* T_\alpha^*(b-t)(aI + \Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s) \\ \quad -S_\alpha(b)[\varphi(0) - G(0,\varphi)] - G(b, x_b)\Big] \\ \quad -B^* T_\alpha^*(b-t)\sum_{k=1}^m (aI + \Gamma_s^b)^{-1}S_\alpha(t-t_k)I_k(x_{t_k}) \\ \quad -B^* T_\alpha^*(b-t)\int_0^b (aI + \Gamma_s^b)^{-1}\mathcal{S}_\alpha(b-s)f(s)dw(s), \quad t \in (t_m, b], \end{cases}$$

where $f \in S_{F,x} = \{f \in L^p(J, L_2^0) : f(t) \in F(t, x_t) \text{ a.e. } t \in J\}$.
Lemma 2.5 ([26]). For any $p \geq 1$ and for arbitrary L_2^0-valued predictable process $\phi(\cdot)$ such that

$$\sup_{s\in[0,t]} E\left\| \int_0^s \phi(v)dw(v)\right\|_H^{2p} \leq (p(2p-1))^p \left(\int_0^t (E \parallel \phi(s) \parallel_{L_2^0}^{2p})^{1/p} ds\right)^p, t \in [0, \infty).$$

In the rest of this paper, we denote by $M_1 = \parallel B \parallel_H, C_p = (p(p-1)/2)^{p/2}$.

Our main results are based on the following lemma.
Lemma 2.6 ([27]). Let (H, d) be a complete metric space. If $\Phi : H \to P_{cl}(H)$ is a contraction, then Fix $\Phi \neq \emptyset$.

3 Main results

In this section we shall present and prove our main results. Let us list the following hypotheses.

(H1) The function $G : J \times \mathcal{B} \to H$ is continuous, and there exists a positive constant L_G such that

$$E \parallel G(t, \psi_1) - G(t, \psi_2) \parallel_H^p \leq L_G \parallel \psi_1 - \psi_2 \parallel_{\mathcal{B}}^p$$

for $t \in J, \psi_1, \psi_2 \in \mathcal{B}$.

(H2) The function $F : J \times \mathcal{B} \to \mathcal{P}_{cp}(L_2^0)$ is a multifunction such that $(\cdot, \phi) \to F(t, \phi)$ is measurable for each $\phi \in \mathcal{B}$.

(H3) There exists a function $l(t) \in L^{\frac{1}{q}}(J, R^+), q \in (0, \alpha)$ such that

$$EH_d^p(F(t,\phi_1), F(t,\phi_2)) \leq l(t) \parallel \phi_1 - \phi_2 \parallel_{\mathcal{B}}^p$$

for $t \in J, \phi_1, \phi_2 \in \mathcal{B}$, and

$$d^p(0, F(t,0)) \leq l(t)$$

for a.e. $t \in J$.

(H4) The functions $I_k : \mathcal{B} \to H$ are continuous and there exist constants c_k such that

$$E \parallel I_k(\psi_1) - I_k(\psi_2) \parallel_H^p \leq c_k \parallel \psi_1 - \psi_2 \parallel_{\mathcal{B}}^p$$

for $\psi_1, \psi_2 \in \mathcal{B}, k = 1, \ldots, m$.

(H5) For each $0 \leq t < b$, the operator $aR(a, \Gamma_\tau^b) \to 0$ in the strong operator topology as $a \to 0^+$ i.e., the linear differential Cauchy problem corresponding to system (1)-(3) is approximately controllable on J.

Theorem 3.1. *Let $A \in A^\alpha(\theta_0, \omega_0)$ with $\theta_0 \in (0, \frac{\pi}{2}]$ and $\omega_0 \in R$. If the assumptions* (H1)-(H4) *are satisfied, then the system* (1)-(3) *has at least one mild solution on J, provided that*

$$4^{p-1}K_b^p\Bigg[L_G + m^{p-1}\tilde{M}_S^p\sum_{i=1}^m c_i + C_p\tilde{M}_T^p\left(\frac{1-q}{p(1-\alpha)+1-q}\right)^{1-q} \times b^{p(\alpha-1/2)-q} \parallel l \parallel_{L^{\frac{1}{q}}(J,R^+)}\Bigg]\left[1 + 3^{p-1}\tilde{M}_T^{2p}M_1^{2p}\frac{1}{a^p}\frac{b^{p(2\alpha-1)}}{2p(\alpha-1)+1}\right] < 1.$$

Proof. We introduce the space $\mathcal{B}_b$ of all functions $x : (-\infty, b] \to H$ such that $x_0 \in \mathcal{B}$ and the the restriction $x|_{[0,b]} \in \mathcal{PC}$. Let$\| \cdot \|_b$ be a seminorm in $\mathcal{B}_b$ defined by

$$\parallel x \parallel_b = \parallel x_0 \parallel_{\mathcal{B}} + (\sup_{0\leq s\leq b} \parallel x(s) \parallel_H^p)^{\frac{1}{p}}, \quad x \in \mathcal{B}_b.$$

We consider the multi-valued map $\Phi : \mathcal{B}_b \to \mathcal{P}(\mathcal{B}_b)$ by Φx the set of $\rho \in \mathcal{B}_b$ such that

$$\rho(t) = \begin{cases} S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, x_t) \\ \quad + \int_0^t T_\alpha(t-s)Bu_x^a(s)ds + \int_0^t T_\alpha(t-s)f(s)dw(s), \quad t \in [0, t_1], \\ S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, x_t) \\ \quad + S_\alpha(t-t_1)I_1(x_{t_1}) \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)Bu_x^a(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)f(s)dw(s), \quad t \in (t_1, t_2], \\ \vdots \\ S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, x_t) \\ \quad + \sum_{k=1}^m S_\alpha(t-t_k)I_k(x_{t_k}) \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)Bu_x^a(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)f(s)dw(s), \quad t \in (t_m, b], \end{cases}$$

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where $f \in S_{F,x} = \{f \in L^p(J, L_2^0) : f(t) \in F(t, x_t) \text{ a.e. } t \in J\}$.

For $\varphi \in \mathcal{B}$, we define $\tilde{\varphi}$ by

$$\tilde{\varphi}(t) = \begin{cases} \varphi(t), & -\infty < t \leq 0, \\ S_\alpha(t)\varphi(0), & 0 \leq t \leq b, \end{cases}$$

then $\tilde{\varphi} \in \mathcal{B}_b$. Set $x(t) = y(t) + \tilde{\varphi}(t), -\infty < t \leq b$. It is clear to see that x satisfies Definition 2.8 if and only if y satisfies $y_0 = 0$ and

$$y(t) = \begin{cases} S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, y_t + \tilde{\varphi}_t) \\ \quad + \int_0^t T_\alpha(t-s)Bu_y^a(s)ds + \int_0^t T_\alpha(t-s)f(s)dw(s), & t \in [0, t_1], \\ S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, y_t + \tilde{\varphi}_t) \\ \quad + S_\alpha(t-t_1)I_1(y_{t_1} + \tilde{\varphi}_{t_1}) \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)Bu_y^a(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)f(s)dw(s), & t \in (t_1, t_2], \\ \vdots \\ S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, y_t + \tilde{\varphi}_t) \\ \quad + \sum_{k=1}^m S_\alpha(t-t_k)I_k(y_{t_k} + \tilde{\varphi}_{t_k}) \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)Bu_y^a(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)f(s)dw(s), & t \in (t_m, b], \end{cases}$$

where

$$u_y^a(t) = \begin{cases} S^*T_\alpha^*(b-t)(aI + \Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s) \\ \quad -S_\alpha(b)[\varphi(0) - G(0,\varphi)] - G(b, x_b)\Big] \\ \quad -B^*T_\alpha^*(b-t)\int_0^b (aI + \Gamma_s^b)^{-1}T_\alpha(b-s)f(s)dw(s), & t \in [0, t_1], \\ S^*T_\alpha^*(b-t)(aI + \Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s) \\ \quad -S_\alpha(b)[\varphi(0) - G(0,\varphi)] - G(b, x_b)\Big] \\ \quad -B^*T_\alpha^*(b-t)(aI + \Gamma_s^b)^{-1}S_\alpha(t-t_1)I_1(x_{t_1}) \\ \quad -B^*T_\alpha^*(b-t)\int_0^b (aI + \Gamma_s^b)^{-1}T_\alpha(b-s)f(s)dw(s), & t \in (t_1, t_2], \\ \vdots \\ S^*T_\alpha^*(b-t)(aI + \Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s) \\ \quad -S_\alpha(b)[\varphi(0) - G(0,\varphi)] - G(b, x_b)\Big] \\ \quad -B^*T_\alpha^*(b-t)\sum_{k=1}^m (aI + \Gamma_s^b)^{-1}S_\alpha(t-t_k)I_k(x_{t_k}) \\ \quad -B^*T_\alpha^*(b-t)\int_0^b (aI + \Gamma_s^b)^{-1}T_\alpha(b-s)f(s)dw(s), & t \in (t_m, b], \end{cases}$$

and $f \in S_{F,y} = \{f \in L^p(J, L_2^0) : f(t) \in F(t, y_s + \tilde{\varphi}_s) \text{ a.e. } t \in J\}$.

Let $\mathcal{B}_b^0 = \{y \in \mathcal{B}_b : y_0 = 0 \in \mathcal{B}\}$. For any $y \in \mathcal{B}_b^0$,

$$\| y \|_b = \| y_0 \|_{\mathcal{B}} + (\sup_{0 \leq s \leq b} \| y(s) \|_H^p)^{\frac{1}{p}} = (\sup_{0 \leq s \leq b} \| y(s) \|_H^p)^{\frac{1}{p}},$$

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thus $(\mathcal{B}_b^0, \|\cdot\|_b)$ is a Banach space. Define the multi-valued map $\bar{\Phi} : \mathcal{B}_b^0 \to \mathcal{P}(\mathcal{B}_b^0)$ by $\bar{\Phi}y$ the set of $\bar{\rho} \in \mathcal{B}_b^0$ such that $\bar{\rho}(t) = 0, t \in [-\infty, 0]$ and

$$\bar{\rho}(t) = \begin{cases} -S_\alpha(t)G(0,\varphi) + G(t, y_t + \tilde{\varphi}_t) \\ \quad + \int_0^t T_\alpha(t-s)Bu_y^a(s)ds + \int_0^t T_\alpha(t-s)f(s)dw(s), & t \in [0, t_1], \\ -S_\alpha(t)\varphi(0) + G(t, y_t + \tilde{\varphi}_t) \\ \quad + S_\alpha(t-t_1)I_1(y_{t_1} + \tilde{\varphi}_{t_1}) \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)Bu_y^a(s)ds + \int_0^t T_\alpha(t-s)f(s)dw(s), & t \in (t_1, t_2], \\ \vdots \\ -S_\alpha(t)\varphi(0) + G(t, y_t + \tilde{\varphi}_t) \\ \quad + \sum_{k=1}^m S_\alpha(t-t_k)I_k(y_{t_k} + \tilde{\varphi}_{t_k}) \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)Bu_y^a(s)ds + \int_0^t T_\alpha(t-s)f(s)dw(s), & t \in (t_m, b], \end{cases}$$

where $f \in S_{F,y}$. Obviously, the operator Φ has a fixed point if and only if operator $\bar{\Phi}$ has a fixed point, to prove which we shall employ Lemma 2.6. For better readability, we break the proof into a sequence of steps.

Step 1. We show that $(\bar{\Phi}y)(t) \in \mathcal{P}_{cl}(\mathcal{B}_b^0)$.

Indeed, let $y^{(n)}(t) \to y^*(t)$, $(\bar{\rho}_n)_{n\geq 0} \in (\bar{\Phi}y)(t)$ such that $\bar{\rho}_n(t) \to \bar{\rho}_*(t)$ in $\mathcal{B}_b^0$. Then $\bar{\rho}_*(t) \in \mathcal{B}_b^0$ and there exists $f_n \in S_{F,y^{(n)}}$ such that, for each $t \in [0, t_1]$,

$$\bar{\rho}_n(t) = -S_\alpha(t)G(0,\varphi) + G(t, y_t^{(n)} + \tilde{\varphi}_t) + \int_0^t T_\alpha(t-s)Bu_{y^{(n)}}^a(s)ds + \int_0^t T_\alpha(t-s)f_n(s)dw(s),$$

where

$$u_{y^{(n)}}^a(t) = B^*T_\alpha^*(b-t)(aI + \Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s) - S_\alpha(b)[\varphi(0) - G(0,\varphi)] - G(b, y_b^{(n)} + \tilde{\varphi}_b)\Big] - B^*T_\alpha^*(b-t)\int_0^b (aI + \Gamma_s^b)^{-1}T_\alpha(b-s)f_n(s)dw(s).$$

Using the fact that F has compact values and (H3) holds, we may pass to a subsequence if necessary to obtain that f_n converges to f_* in $L^p([0, t_1], L_2^0)$, hence, $f_* \in S_{F,y^*}$. Then, for each $t \in [0, t_1]$,

$$\bar{\rho}_n(t) \to \bar{\rho}_*(t) = -S_\alpha(t)G(0,\varphi) + G(t, y_t^* + \tilde{\varphi}_t) + \int_0^t T_\alpha(t-s)Bu_{y^*}^a(s)ds + \int_0^t T_\alpha(t-s)f_n(s)dw(s),$$

where

$$u_{y^*}^a(t) = S^*T_\alpha^*(b-t)(aI + \Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s)$$

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$$-S_\alpha(b)[\varphi(0) - G(0,\varphi)] - G(b, y_b^* + \tilde{\varphi}_b)\Big]$$
$$-B^*T_\alpha^*(b-t)\int_0^b (aI + \Gamma_s^b)^{-1}\mathcal{S}_\alpha(b-s)f_*(s)dw(s).$$

Similarly, for each $t \in (t_k, t_{k+1}], k = 1, \ldots, m$, we have

$$\bar{\rho}_n(t) = -S_\alpha(t)G(0,\varphi) + G(t, y_t^{(n)} + \tilde{\varphi}_t) + \sum_{i=1}^{k} S_\alpha(t-t_i)I_i(y_{t_i}^{(n)} + \tilde{\varphi}_{t_i})$$
$$+ \int_0^t T_\alpha(t-s)Bu_{y^{(n)}}^a(s)ds + \int_0^t T_\alpha(t-s)f_n(s)dw(s),$$

where

$$u_{y^{(n)}}^a(t) = S^*T_\alpha^*(b-t)(aI + \Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s)$$
$$-S_\alpha(b)[\varphi(0) - G(0,\varphi)] - G(b, y_b^{(n)} + \tilde{\varphi}_b)\Big]$$
$$-B^*T_\alpha^*(b-t)(aI + \Gamma_s^b)^{-1}\sum_{i=1}^{k} S_\alpha(t-t_i)I_i(y_{t_i}^{(n)} + \tilde{\varphi}_{t_i})$$
$$-B^*T_\alpha^*(b-t)\int_0^b (aI + \Gamma_s^b)^{-1}T_\alpha(b-s)f_n(s)dw(s).$$

Using the fact that F has compact values and (H3) holds, we may pass to a subsequence if necessary to obtain that f_n converges to f_* in $L^p([t_k, t_{k+1}], L_2^0)$, hence, $f_* \in S_{F,y^*}$. Then, for each $t \in [t_k, t_{k+1}], k = 1, \ldots, m$,

$$\bar{\rho}_n(t) \to \bar{\rho}_*(t) = -S_\alpha(t)G(0,\varphi) + G(t, y_t^* + \tilde{\varphi}_t)$$
$$+ \sum_{i=1}^{k} S_\alpha(t-t_i)I_i(y_{t_i}^* + \tilde{\varphi}_{t_i})$$
$$+ \int_0^t T_\alpha(t-s)Bu_{y^*}^a(s)ds + \int_0^t T_\alpha(t-s)f_n(s)dw(s),$$

where

$$u_{y^*}^a(t) = S^*T_\alpha^*(b-t)(aI + \Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s)$$
$$-S_\alpha(b)[\varphi(0) - G(0,\varphi)] - G(b, y_b^* + \tilde{\varphi}_b)\Big]$$
$$-B^*T_\alpha^*(b-t)(aI + \Gamma_s^b)^{-1}\sum_{i=1}^{k} S_\alpha(t-t_i)I_i(y_{t_i}^* + \tilde{\varphi}_{t_i})$$
$$-B^*T_\alpha^*(b-t)\int_0^b (aI + \Gamma_s^b)^{-1}T_\alpha(b-s)f_*(s)dw(s).$$

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Therefore, $\bar{\rho}_*(t) \in (\bar{\Phi}y)(t)$ and $(\bar{\Phi}y)(t) \in \mathcal{P}_{cl}(\mathcal{B}_b^0)$.

Step 2. We show that $(\bar{\Phi}y)(t)$ is a contractive multi-valued map for each $y(t) \in \mathcal{B}_b^0$.

Let $t \in [0, t_1]$ and $y(t), \hat{y}(t) \in \mathcal{B}_b^0$ and let $\bar{\rho}(t) \in (\bar{\Phi}y)(t)$. Then there exists $f \in S_{F,y}$ such that

$$\begin{aligned}\bar{\rho}(t) &= -S_\alpha(t)G(0,\varphi) + G(t, y_t + \tilde{\varphi}_t)\\ &\quad + \int_0^t T_\alpha(t-s)Bu_y^a(s)ds + \int_0^t T_\alpha(t-s)f(s)dw(s).\end{aligned}$$

From (H3), there exists $v(t) \in F(t, \hat{y}_t + \tilde{\varphi}_t)$ such that

$$E \parallel f(t) - v(t) \parallel_{L_2^0}^p \leq l(t) \parallel y_t - \hat{y}_t \parallel_{\mathcal{B}}^p .$$

Consider $\Lambda : [0, t_1] \to \mathcal{P}(L_2^0)$, given by

$$\Lambda(t) = \{v(t) \in H : E \parallel f(t) - v(t) \parallel_{L_2^0}^p \leq l(t) \parallel y_t - \hat{y}_t \parallel_{\mathcal{B}}^p\}.$$

Since the multi-valued operator $W(t) = \Lambda(t) \cap F(t, \hat{y}_t + \tilde{\varphi}_t)$ is measurable (see [28], Proposition III.4), there exists a function $\hat{f}(t)$, which is a measurable selection for W. So, $\hat{f}(t) \in F(t, \hat{y}_t + \tilde{\varphi}_t)$ and

$$E \parallel f(t) - \hat{f}(t) \parallel_{L_2^0}^p \leq l(t) \parallel y_t - \hat{y}_t \parallel_{\mathcal{B}}^p .$$

For each $t \in [0, t_1]$, we define

$$\begin{aligned}\hat{\rho}(t) &= -S_\alpha(t)G(0,\varphi) + G(t, \hat{y}_t + \tilde{\varphi}_t)\\ &\quad + \int_0^t T_\alpha(t-s)Bu_{\hat{y}}^a(s)ds + \int_0^t T_\alpha(t-s)\hat{f}(s)dw(s).\end{aligned}$$

Then, for each $t \in [0, t_1]$, we have

$$\begin{aligned}&E \parallel \bar{\rho}(t) - \hat{\rho}(t) \parallel_H^p\\ &\quad \leq 3^{p-1}E \parallel G(t, y_t + \tilde{\varphi}_t) - G(t, \hat{y}_t + \tilde{\varphi}_t) \parallel_H^p\\ &\qquad + 3^{p-1}E\left\| \int_0^t T_\alpha(t-s)B[u_y^a(s) - u_{\hat{y}}^a(s)]ds \right\|_H^p\\ &\qquad + 3^{p-1}E\left\| \int_0^t T_\alpha(t-s)[f(s) - \hat{f}(s)]dw(s) \right\|_H^p\\ &\quad \leq 3^{p-1}L_G \parallel y_t - \hat{y}_t \parallel_{\mathcal{B}}^p\\ &\qquad + 3^{p-1}\tilde{M}_T^p t_1^{p-1} \int_0^t (t-s)^{p(\alpha-1)} E \parallel B[u_y^a(s) - u_{\hat{y}}^a(s)] \parallel_H^p ds\\ &\qquad + 3^{p-1}C_p\tilde{M}_T^p \left[\int_0^t \left[(t-s)^{p(\alpha-1)} E \parallel f(s) - \hat{f}(s) \parallel_{L_2^0}^p \right]^{2/p} ds \right]^{p/2}\\ &\quad \leq 3^{p-1}L \parallel y_t - \hat{y}_t \parallel_{\mathcal{B}}^p\end{aligned}$$

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$$
\begin{aligned}
&+6^{p-1}\tilde{M}_T^{2p}M_1^{2p}\frac{1}{a^p}t_1^{p-1}\int_0^t[(t-s)(b-s)]^{p(\alpha-1)}\Big[L\parallel y_t-\hat{y}_t\parallel_{\mathcal{B}}^p\\
&+C_pt_1^{p/2-1}\tilde{M}_T^p\int_0^b(b-\tau)^{p(\alpha-1)}l(\tau)\parallel y_\tau-\hat{y}_\tau\parallel_{\mathcal{B}}^p d\tau\Big]ds\\
&+3^{p-1}C_pt_1^{p/2-1}\tilde{M}_T^p\int_0^t(t-s)^{p(\alpha-1)}l(s)\parallel y_s-\hat{y}_s\parallel_{\mathcal{B}}^p ds\\
\le\ & 3^{p-1}K_b^pL_G\parallel y-\hat{y}\parallel_b^p\\
&+6^{p-1}K_b^p\tilde{M}_T^{2p}M_1^{2p}\frac{1}{a^p}t_1^{p-1}\int_0^t[(t-s)(b-s)]^{p(\alpha-1)}\Big[L_G\\
&+C_pt_1^{p/2-1}\tilde{M}_T^p\Big(\int_0^b(b-\tau)^{\frac{p(\alpha-1)}{1-q}}d\tau\Big)^{1-q}\parallel l\parallel_{L^{\frac{1}{q}}(J,R^+)}\Big]ds\parallel y-\hat{y}\parallel_b^p\\
&+3^{p-1}K_b^pC_pt_1^{p/2-1}\tilde{M}_T^p\Big(\int_0^t(t-s)^{\frac{p(\alpha-1)}{1-q}}ds\Big)^{1-q}\\
&\times\parallel l\parallel_{L^{\frac{1}{q}}(J,R^+)}\parallel y-\hat{y}\parallel_b^p\\
\le\ & 3^{p-1}K_b^p\Big(L_G+2^{p-1}\tilde{M}_T^{2p}M_1^{2p}\frac{1}{a^p}t_1^{p-1}\frac{1}{2p(\alpha-1)+1}b^{2p(\alpha-1)+1}\Big[L_G\\
&+C_pt_1^{p/2-1}\tilde{M}_T^p\Big(\frac{1-q}{p(1-\alpha)+1-q}\Big)^{1-q}b^{p(\alpha-1)+1-q}\parallel l\parallel_{L^{\frac{1}{q}}(J,R^+)}\Big]\\
&+3^{p-1}K_b^pC_pt_1^{p/2-1}\tilde{M}_T^p\Big(\frac{1-q}{p(1-\alpha)+1-q}\Big)^{1-q}\\
&\times t_1^{p(\alpha-1)+1-q}\parallel l\parallel_{L^{\frac{1}{q}}(J,R^+)}\Big)\parallel y-\hat{y}\parallel_b^p.
\end{aligned}
$$

Similarly, for each $t\in(t_k,t_{k+1}], k=1,\ldots,m$. Let $y(t),\hat{y}(t)\in\mathcal{B}_b^0$ and let $\bar{\rho}(t)\in(\bar{\Phi}y)(t)$. Then there exists $f\in S_{F,y}$ such that

$$
\begin{aligned}
\bar{\rho}(t)=&-S_\alpha(t)G(0,\varphi)+G(t,y_t+\tilde{\varphi}_t)+\sum_{i=1}^kS_\alpha(t-t_i)I_i(y_{t_i}+\tilde{\varphi}_{t_i})\\
&+\int_0^tT_\alpha(t-s)Bu_y^a(s)ds+\int_0^tT_\alpha(t-s)f(s)dw(s).
\end{aligned}
$$

From (H3), there exists $v(t)\in F(t,\hat{y}_t+\tilde{\varphi}_t)$ such that

$$E\parallel f(t)-v(t)\parallel_{L_2^0}^p\le l(t)\parallel y_t-\hat{y}_t\parallel_{\mathcal{B}}^p.$$

Consider $\Lambda:(t_k,t_{k+1}]\to\mathcal{P}(L_2^0)$, given by

$$\Lambda(t)=\{v(t)\in H:E\parallel f(t)-v(t)\parallel_{L_2^0}^p\le l(t)\parallel y_t-\hat{y}_t\parallel_{\mathcal{B}}^p\}.$$

Since the multi-valued operator $W(t)=\Lambda(t)\cap F(t,\hat{y}_t+\tilde{\varphi}_t)$ is measurable (see [28], Proposition III.4), there exists a function $\hat{f}(t)$, which is a measurable se-

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lection for W. So, $\hat{f}(t) \in F(t, \hat{y}_t + \tilde{\varphi}_t)$ and

$$E \parallel f(t) - \hat{f}(t) \parallel_{L_2^0}^p \leq l(t) \parallel y_t - \hat{y}_t \parallel_{\mathcal{B}}^p .$$

For each $t \in (t_k, t_{k+1}], k = 1, \ldots, m$, we define

$$\begin{aligned}\hat{\rho}(t) &= -S_\alpha(t)G(0,\varphi) + G(t, \hat{y}_t + \tilde{\varphi}_t) + \sum_{i=1}^{k} S_\alpha(t-t_i)I_i(\hat{y}_{t_i} + \tilde{\varphi}_{t_i}) \\ &\quad + \int_0^t T_\alpha(t-s)Bu_{\hat{y}}^a(s)ds + \int_0^t T_\alpha(t-s)\hat{f}(s)dw(s).\end{aligned}$$

Then, for each $t \in (t_k, t_{k+1}], k = 1, \ldots, m$, we have

$$\begin{aligned}
&E \parallel \bar{\rho}(t) - \hat{\rho}(t) \parallel_H^p \\
&\quad \leq 4^{p-1}E \parallel G(t, y_t + \tilde{\varphi}_t) - G(t, \hat{y}_t + \tilde{\varphi}_t) \parallel_H^p \\
&\qquad +4^{p-1}E\left\| \sum_{i=1}^{k} S_\alpha(t-t_i)[I_i(y_{t_i} + \tilde{\varphi}_{t_i}) - I_i(\hat{y}_{t_i} + \tilde{\varphi}_{t_i})] \right\|_H^p \\
&\qquad +4^{p-1}E\left\| \int_0^t T_\alpha(t-s)B[u_y^a(s) - u_{\hat{y}}^a(s)]ds \right\|_H^p \\
&\qquad +4^{p-1}E\left\| \int_0^t T_\alpha(t-s)[f(s) - \hat{f}(s)]dw(s) \right\|_H^p \\
&\quad \leq 4^{p-1}L_G \parallel y_t - \hat{y}_t \parallel_{\mathcal{B}}^p \\
&\qquad +4^{p-1}k^{p-1}\tilde{M}_S^p \sum_{i=1}^{k} E \parallel I_i(y_{t_i} + \tilde{\varphi}_{t_i}) - I_i(\hat{y}_{t_i} + \tilde{\varphi}_{t_i}) \parallel_H^p \\
&\qquad +3^{p-1}\tilde{M}_T^p(t_{k+1} - t_k)^{p-1} \int_0^t (t-s)^{p(\alpha-1)} E \parallel B[u_y^a(s) - u_{\hat{y}}^a(s)] \parallel_H^p ds \\
&\qquad +4^{p-1}C_p\tilde{M}_T^p\left[\int_0^t \left[(t-s)^{p(\alpha-1)} E \parallel f(s) - \hat{f}(s) \parallel_{L_2^0}^p \right]^{2/p} ds \right]^{p/2} \\
&\quad \leq 4^{p-1}L_G \parallel y_t - \hat{y}_t \parallel_{\mathcal{B}}^p + 4^{p-1}k^{p-1}\tilde{M}_S^p \sum_{i=1}^{k} c_i \parallel y_{t_i} - \hat{y}_{t_i} \parallel_{\mathcal{B}}^p \\
&\qquad +12^{p-1}\tilde{M}_T^{2p}M_1^{2p}\frac{1}{a^p}(t_{k+1} - t_k)^{p-1} \int_0^t [(t-s)(b-s)]^{p(\alpha-1)} \\
&\qquad \times\Big[L_G \parallel y_t - \hat{y}_t \parallel_{\mathcal{B}}^p + k^{p-1}\tilde{M}_S^p \sum_{i=1}^{k} c_i \parallel y_{t_i} - \hat{y}_{t_i} \parallel_{\mathcal{B}}^p \\
&\qquad +C_p(t_{k+1} - t_k)^{p/2-1}\tilde{M}_T^p \int_0^b (b-\tau)^{p(\alpha-1)} l(\tau) \parallel y_\tau - \hat{y}_\tau \parallel_{\mathcal{B}}^p d\tau \Big] ds \\
&\qquad +4^{p-1}C_p(t_{k+1} - t_k)^{p/2-1}\tilde{M}_T^p \int_0^t (t-s)^{p(\alpha-1)} l(s) \parallel y_s - \hat{y}_s \parallel_{\mathcal{B}}^p ds
\end{aligned}$$

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$$\begin{aligned}
&\leq 4^{p-1}K_b^pL_G \parallel y-\hat{y}\parallel_b^p +4^{p-1}K_b^pk^{p-1}\tilde{M}_S^p\sum_{i=1}^{k}c_i \parallel y-\hat{y}\parallel_b^p \\
&\quad +12^{p-1}K_b^p\tilde{M}_T^{2p}M_1^{2p}\frac{1}{a^p}(t_{k+1}-t_k)^{p-1}\int_0^t[(t-s)(b-s)]^{p(\alpha-1)} \\
&\quad \times\bigg[L_G+k^{p-1}\tilde{M}_S^p\sum_{i=1}^{k}c_i+C_p(t_{k+1}-t_k)^{p/2-1}\tilde{M}_T^p \\
&\quad \times\bigg(\int_0^b(b-\tau)^{\frac{p(\alpha-1)}{1-q}}d\tau\bigg)^{1-q}\parallel l\parallel_{L^{\frac{1}{q}}(J,R^+)}\bigg]ds\parallel y-\hat{y}\parallel_b^p \\
&\quad +4^{p-1}K_b^pC_p(t_{k+1}-t_k)^{p/2-1}\tilde{M}_T^p\bigg(\int_0^t(t-s)^{\frac{p(\alpha-1)}{1-q}}ds\bigg)^{1-q} \\
&\quad \times\parallel l\parallel_{L^{\frac{1}{q}}(J,R^+)}\parallel y-\hat{y}\parallel_b^p \\
&\leq 4^{p-1}K_b^p\bigg(L_G+k^{p-1}\tilde{M}_S^p\sum_{i=1}^{k}c_i+3^{p-1}\tilde{M}_T^{2p}M_1^{2p}\frac{1}{a^p}(t_{k+1}-t_k)^{p-1} \\
&\quad \times\frac{1}{2p(\alpha-1)+1}b^{2p(\alpha-1)+1}\bigg[L_G+k^{p-1}\tilde{M}_S^p\sum_{i=1}^{k}c_i \\
&\quad +C_p(t_{k+1}-t_k)^{p/2-1}\tilde{M}_T^p\bigg(\frac{1-q}{p(1-\alpha)+1-q}\bigg)^{1-q} \\
&\quad \times b^{p(\alpha-1)+1-q}\parallel l\parallel_{L^{\frac{1}{q}}(J,R^+)}\bigg] \\
&\quad +K_b^pC_p(t_{k+1}-t_k)^{p/2-1}\tilde{M}_T^p\bigg(\frac{1-q}{p(1-\alpha)+1-q}\bigg)^{1-q} \\
&\quad \times(t_{k+1}-t_k)^{p(\alpha-1)+1-q}\parallel l\parallel_{L^{\frac{1}{q}}(J,R^+)}\bigg)\parallel y-\hat{y}\parallel_b^p .
\end{aligned}$$

Thus, for all $t\in[0,b]$, we have

$$\parallel \bar{\rho}-\hat{\rho}\parallel_b^p\leq \tilde{L}\parallel y-\hat{y}\parallel_b^p,$$

and

$$H_d^p(\bar{\Phi}y,\bar{\Phi}\hat{y})\leq \tilde{L}\parallel y-\hat{y}\parallel_b^p,$$

where

$$\begin{aligned}
\tilde{L}=4^{p-1}K_b^p&\bigg[L_G+m^{p-1}\tilde{M}_S^p\sum_{i=1}^{m}c_i+C_p\tilde{M}_T^p\bigg(\frac{1-q}{p(1-\alpha)+1-q}\bigg)^{1-q} \\
&\times b^{p(\alpha-1/2)-q}\parallel l\parallel_{L^{\frac{1}{q}}(J,R^+)}\bigg]\bigg[1+3^{p-1}\tilde{M}_T^{2p}M_1^{2p}\frac{1}{a^p}\frac{b^{p(2\alpha-1)}}{2p(\alpha-1)+1}\bigg]<1.
\end{aligned}$$

Hence, $\bar{\Phi}$ is a contraction on $\mathcal{B}_b^0$. In view of Lemma 2.6, we conclude that $\bar{\Phi}$ has at least one fixed point $y^*\in\mathcal{B}_b^0$. Let $x(t)=y^*(t)+\tilde{\varphi}(t), t\in(-\infty,b]$. Then, x

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is a fixed point of the operator Φ, which implies that x is a mild solution of the problem (1)-(3) and the proof of Theorem 3.1 is complete.

Theorem 3.2. *Assume that assumptions of Theorem* 3.1 *and* (H5) *are satisfied and* $\{T_\alpha(t) : t \geq 0\}$ *is compact. Moreover, if* F *is uniformly-bounded, then the system* (1)-(3) *is approximately controllable on* J.

Proof. Let $x^a(\cdot)$ be a fixed point of Φ in $\mathcal{B}_b$. By Theorem 3.1, any fixed point of Φ is a mild solution of the system (1)-(3). This means that there is $x^a \in \Phi(x^a)$, that is, there is $f \in S_{F,x^a}$ such that

$$x^a(t) = \begin{cases} S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, x_t^a) \\ \quad + \int_0^t T_\alpha(t-s)Bu_{x^a}^a(s)ds + \int_0^t T_\alpha(t-s)f(s)dw(s), \quad t \in [0, t_1], \\ S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, x_t^a) + S_\alpha(t-t_1)I_1(x_{t_1}^a) \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)Bu_{x^a}^a(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)f(s)dw(s), \quad t \in (t_1, t_2], \\ \vdots \\ S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, x_t^a) \\ \quad + \sum_{k=1}^m S_\alpha(t-t_k)I_k(x_{t_k}^a) \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)Bu_{x^a}^a(s)ds + \int_0^t \mathcal{S}_\alpha(t-s)f(s)dw(s), \quad t \in (t_m, b], \end{cases}$$

where

$$u_x^a(t) = \begin{cases} S^*T_\alpha^*(b-t)(aI + \Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s) \\ \quad -S_\alpha(b)[\varphi(0) - G(0,\varphi)] - G(b, x_b)\Big] \\ \quad -B^*T_\alpha^*(b-t)\int_0^b (aI + \Gamma_s^b)^{-1}\mathcal{S}_\alpha(b-s)f(s)dw(s), \quad t \in [0, t_1], \\ S^*T_\alpha^*(b-t)(aI + \Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s) \\ \quad -S_\alpha(b)\varphi(0) - G(0,\varphi)] - G(b, x_b)\Big] \\ \quad -B^*T_\alpha^*(b-t)(aI + \Gamma_s^b)^{-1}S_\alpha(t-t_1)I_1(x_{t_1}) \\ \quad -B^*T_\alpha^*(b-t)\int_0^b (aI + \Gamma_s^b)^{-1}\mathcal{S}_\alpha(b-s)f(s)dw(s), \quad t \in (t_1, t_2], \\ \vdots \\ S^*T_\alpha^*(b-t)(aI + \Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s) \\ \quad -S_\alpha(b)\varphi(0) - G(0,\varphi)] - G(b, x_b)\Big] \\ \quad -B^*T_\alpha^*(b-t)\sum_{k=1}^m (aI + \Gamma_s^b)^{-1}S_\alpha(t-t_k)I_k(x_{t_k}) \\ \quad -B^*T_\alpha^*(b-t)\int_0^b (aI + \Gamma_s^b)^{-1}\mathcal{S}_\alpha(b-s)f(s)dw(s), \quad t \in (t_m, b]. \end{cases}$$

By using the stochastic Fubini theorem, it is easy to see that

$$\begin{aligned} x^a(b) &= S_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(b, x_b^a) + \sum_{k=1}^m S_\alpha(b-t_k)I_k(x_{t_k}^a) \\ &\quad + \int_0^b \mathcal{S}_\alpha(b-s)Bu_{x^a}^a(s)ds + \int_0^b \mathcal{S}_\alpha(t-s)f(s)dw(s) \end{aligned}$$

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$$
\begin{aligned}
&= \tilde{x}_b - a(aI + \Gamma_0^b)^{-1}\Big[E\tilde{x}_b + \int_0^b \tilde{\phi}(s)dw(s) - S_\alpha(b)[\varphi(0) - G(0,\varphi)] \\
&\quad -G(b, \bar{x}_b^a)\Big] - a\sum_{k=1}^m (aI + \Gamma_s^b)^{-1} S_\alpha(b - t_k) I_k(x_{t_k}) \\
&\quad -a\int_0^b (aI + \Gamma_s^b)^{-1}\mathcal{S}_\alpha(b - s) f(s) dw(s).
\end{aligned}
$$

By the assumption that the sequences $\{f(s)\}$ is uniformly bounded on J. Thus there is a subsequence, still denoted by $\{f(s)\}$ that converge weakly to say $f^{**}(s)$ in L_2^0. Now, the compactness of $T_\alpha(t), t > 0$ which implies that $T_\alpha(b - s)[f(s) - f^{**}(s)] \to 0$. Also, by (H5), for all $t \in J$, $a(aI + \Gamma_s^b)^{-1} \to 0$ strongly as $a \to 0^+$ and $\| a(aI+\Gamma_s^b)^{-1} \| \leq 1$. Thus, for $t \in [0, b]$, by the Lebesque dominated convergence theorem it follows that

$$
\begin{aligned}
&E \parallel x^a(b) - \tilde{x}_b \parallel_H^p \\
&\quad \leq 5^{p-1} E \parallel a(aI + \Gamma_0^b)^{-1}[E\tilde{x}_b - S_\alpha(b)[\varphi(0) - G(0,\varphi,0)] - G(b, \bar{x}_b^a)] \parallel_H^p \\
&\quad +5^{p-1}E\Big\| \sum_{k=1}^m a(aI + \Gamma_s^b)^{-1} S_\alpha(b - t_k) I_k(x_{t_k})\Big\|_H^p \\
&\quad +5^{p-1}E\Big(\int_0^b \parallel a(aI + \Gamma_0^b)^{-1}\tilde{\phi}(s) \parallel_H^2 ds\Big)^{p/2} \\
&\quad +5^{p-1}E\Big(\int_0^b \parallel a(aI + \Gamma_s^b)^{-1} T_\alpha(b - s)[f(s) - f^{**}(s)] \parallel_H^2 ds\Big)^{p/2} \\
&\quad +5^{p-1}E\Big(\int_0^b \parallel a(aI + \Gamma_s^b)^{-1} T_\alpha(b - s) f^{**}(s) \parallel_H^2 ds\Big)^{p/2} \\
&\quad \to 0 \quad \text{as } a \to 0^+.
\end{aligned}
$$

So $x^a(b) \to \tilde{x}_b$ holds, which shows that the system (1)-(3) is approximately controllable and the proof is complete.

4 Application

Consider the fractional impulsive partial stochastic neutral functional differential inclusions in the following form

$$
\begin{aligned}
D_t^\alpha N(z_t)(x) \in \frac{\partial^2}{\partial x^2} N(z_t)(x) + \tilde{u}(t,x) \\
+ \int_{-\infty}^t \tilde{b}_1(t, s-t, x, z(s,x)) ds \frac{w(t)}{dt}, \qquad (6) \\
0 \leq t \leq b, 0 \leq x \leq \pi,
\end{aligned}
$$

$$
z(t,0) = z(t,\pi) = 0, \quad 0 \leq t \leq b, \qquad (7)
$$

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$$z(\tau,x)=\varphi(\tau,x),\quad \tau\leq 0, 0\leq x\leq \pi, \tag{8}$$

$$\triangle z(t_k,x)=\int_{-\infty}^{t_k}\eta_k(s-t_k)z(s,x)ds,\quad k=1,2,\ldots,m, \tag{9}$$

where D_t^{α} is a Caputo fractional partial derivative of order $0<\alpha<1$, and $\tilde{u}(\cdot)$ is a real function of bounded variation on $[0,b]$. $w(t)$ denotes a standard cylindrical Wiener process in H defined on a stochastic space $(\Omega,\mathcal{F},P)$. In this system,

$$N(z_t)(x)=z(t,x)-\int_{-\infty}^{t}b_1(s-t)z(s,x)ds.$$

Let $H=L^2([0,\pi])$ with the norm $\|\cdot\|$ and define the operators $A:D(A)\subseteq H\to H$ by $A\omega=\omega''$ with the domain

$$D(A):=\{\omega\in H:\omega,\omega' \text{ are absolutely continuous}, \omega''\in H, \omega(0)=\omega(\pi)=0\}.$$

It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ in H. Furthermore, A has a discrete spectrum with eigenvalues of the form $-n^2, n\in\mathbf{N}$ and corresponding normalized eigenfunctions are given by $x_n(z)=\sqrt{\frac{2}{\pi}}\sin(nz)$. In addition $\{x_n:n\in\mathbf{N}\}$ is an orthonormal basis for H, $T(t)y=\sum_{n=1}^{\infty}e^{-n^2t}(y,x_n)x_n$ for all $y\in H$, and every $t>0$. From these expressions it follows that $(T(t))_{t\geq 0}$ is a uniformly bounded compact semigroup, so that $R(\lambda,A)=(\lambda-A)^{-1}$ is a compact operator for all $\lambda\in\rho(A)$ i.e. $A\in A^{\alpha}(\theta^0,\omega^0)$.

Let $r\geq 0, 1\leq p<\infty$ and let $\tilde{h}:(-\infty,-r]\to R$ be a nonnegative measurable function which satisfies the conditions (h-5), (h-6) in the terminology of Hino et al. [29]. Briefly, this means that $\tilde{h}$ is locally integrable and there is a non-negative, locally bounded function γ on $(-\infty,0]$ such that $\tilde{h}(\xi+\tau)\leq\gamma(\xi)\tilde{h}(\tau)$ for all $\xi\leq 0$ and $\theta\in(-\infty,-r)\setminus N_{\xi}$, where $N_{\xi}\subseteq(-\infty,-r)$ is a set whose Lebesgue measure zero. We denote by $\mathcal{PC}_r\times L^p(\tilde{h},H)$ the set consists of all classes of functions $\varphi:(-\infty,0]\to H$ such that $\varphi_{|_{[-r,0]}}\in\mathcal{PC}([-r,0],H)$, $\varphi(\cdot)$ is Lebesgue measurable on $(-\infty,-r)$, and $\tilde{h}\parallel\varphi\parallel^p$ is Lebesgue integrable on $(-\infty,-r)$. The seminorm is given by

$$\parallel\varphi\parallel_{\mathcal{B}}=\sup_{-r\leq\tau\leq 0}\parallel\varphi(\tau)\parallel+\left(\int_{-\infty}^{-r}\tilde{h}(\tau)\parallel\varphi\parallel^p d\tau\right)^{1/p}.$$

The space $\mathcal{B}=\mathcal{PC}_r\times L^p(\tilde{h},H)$ satisfies axioms (A)-(C). Moreover, when $r=0$ and $p=2$, we can take $\tilde{H}=1$, $M(t)=\gamma(-t)^{1/2}$ and $K(t)=1+(\int_{-t}^{0}\tilde{h}(\tau)d\tau)^{1/2}$, for $t\geq 0$ (see [29, Theorem 1.3.8] for details).

Additionally, we will assume that

(i) The function $b_1:R\to R$, is continuous, and $\tilde{L}_1=(\int_{-\infty}^{0}\frac{(b_1(s))^2}{\tilde{h}(s)}ds)^{\frac{1}{2}}<\infty$,

(ii) The function $\tilde{b}_1:R^4\to R$, is continuous and there exist continuous functions $a_j:R\to R, j=1,2$, such that

$$|\tilde{b}_1(t,s,x,y)|\leq a_1(t)a_2(s)|y|,\quad (t,s,x,y)\in R^4,$$

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and

$$|\tilde{b}_1(t,s,x,y_1) - \tilde{b}_1(t,s,x,y_2)| \\ \leq a_1(t)a_2(s)|y_1 - y_2|, \quad (t,s,x,y_1),(t,s,x,y_2) \in R^4$$

with $\hat{L}_1 = (\int_{-\infty}^{0} \frac{(a_2(s))^2}{\tilde{h}(s)} ds)^{\frac{1}{2}} < \infty$.

(iii) The functions $\eta_k : R \to R, k = 1,2,\dots,m$, are continuous, and $L_k = (\int_{-\infty}^{0} \frac{(\eta_k(s))^2}{\tilde{h}(s)} ds)^{\frac{1}{2}} < \infty$ for every $k = 1,2,\dots,m$,

Take $\varphi \in \mathcal{B} = \mathcal{PC}_0 \times L^2(\tilde{h}, H)$ with $\varphi(\theta)(x) = \varphi(\theta, x), (\theta, x) \in (-\infty, 0] \times \mathcal{B}$. Let $G : [0,b] \times \mathcal{B} \to H$, $F : [0,b] \times \mathcal{B} \to \mathcal{P}(H)$ be the operators defined by

$$N(\psi)(x) = \psi(0,x) - G(t,\psi)(x),$$

$$G(t,\psi)(x) = \int_{-\infty}^{0} b_1(s)\psi(s,x)ds,$$

$$F(t,\psi)(x) = \int_{-\infty}^{0} \tilde{b}_1(t,s,x,\psi(s,x))ds.$$

Also defining the maps I_k and B by

$$I_k(\psi)(x) = \int_{-\infty}^{0} \eta_k(s)\psi(s,x)ds, \ (Bu)(t)(x) = \tilde{u}(t,x).$$

Using these definitions, we can represent the system (6)-(9) in the abstract form (1)-(3). Moreover, for any $t \in [0,b], \psi, \psi_1 \in \mathcal{B}$, we have that $E \parallel G(t,\psi) - G(t,\psi_1) \parallel^p \leq L_G \parallel \psi - \psi_1 \parallel_{\mathcal{B}}^p$, $E \parallel F(t,\psi) - F(t,\psi_1) \parallel^p \leq L_F \parallel \psi - \psi_1 \parallel_{\mathcal{B}}^p$, $E \parallel I_k(\psi) - I_k(\psi_1) \parallel^p \leq (L_k)^p \parallel \psi - \psi_1 \parallel_{\mathcal{B}}^p, k = 1,2,\dots,m$, and F is bounded linear operators with $E \parallel F \parallel_{L(\mathcal{B},H)}^p \leq L_F$, where $L_G = (\tilde{L}_1)^p, L_F = (\parallel a_1 \parallel_\infty \hat{L}_1)^p$. Further, we can impose some suitable conditions on the above-defined functions to verify the assumptions on Theorem 3.2. Hence by Theorems 3.2, the system (6)-(9) is approximately controllable on $[0,b]$.

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Zuomao Yan and Xiumei Jia
Department of Mathematics,
Hexi University,
Zhangye, Gansu
734000, P.R. China
* Corresponding author.
E-mail address: yanzuomao@163.com

HYERS-ULAM STABILITY OF GENERAL ADDITIVE MAPPINGS IN C^*-ALGEBRA

GANG LU, GUOXIAN CAI, YUANFENG JIN*, AND CHOONKIL PARK

ABSTRACT. In this paper, we prove that the generalized Hyers-Ulam stability of homomorphisms in C^*-algebras and Lie C^*-algebras and also of derivations on C^*-algebras and Lie C^*-algebras for an 4-variable additive functional equation

1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations originated from a question of Ulam [28] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [19] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [9] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. The stability problems for several functional equations or inequations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2]–[8],[10], [12]–[14], [18]–[21],[22]-[27],[29]).

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) d(x,y)=0 if and only if x=y;
(2) d(x,y)=d(y,x) for all $x, y \in X$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.1 (see[6],[7]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for*

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*Corresponding author: jinyuanfeng1976@hotmail.com(Y. Jin).

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each given element $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty \tag{1.1}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) *$d(J^n x, J^{n+1} x) < \infty$, fir all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{N_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

By the using fixed point method, the stability problems of several functional equations have been extensively investigated by a number of authors(see[5][6][16][17]).

This paper, using the fixed point method, we prove the generalized Hyers-Ulam stability of homomorphisms in C^*-algebras and of derivations on C^*-algebras for the general Jensen-type functional equation. And, we prove that the generalized Hyers-Ulam stability of homomorphisms in Lie C^*-algebras and of derivations on Lie C^*-algebras for the following additive functional equation:

$$\begin{aligned} & f(dx_1 + ax_2 + bx_3 + cx_4) + f(ax_1 + dx_2 + cx_3 + bx_4) \\ & + f(bx_1 + cx_2 + dx_3 + ax_4) + f(cx_1 + bx_2 + ax_3 + dx_4) \\ & = (d + a + b + c) f(x_1 + x_2 + x_3 + x_4) \end{aligned} \tag{1.2}$$

Here a, b, c and d are real numbers with $a+b+c+d \neq 0$. Throughout the paper, assume that k is $a+b+c+d$.

2. Stability of Homomorphisms and Derivations in C^*-Algebras

Throughout this section, assume that X is a C^*-algebras with norm $\|\cdot\|_X$ and that Y is a C^*-algebra with norm $\|\cdot\|_Y$.

For a given mapping $f : X \to Y$, we define

$$\begin{aligned} & F_\mu f(x_1, x_2, x_3, x_4) := \\ & \mu f(dx_1 + ax_2 + bx_3 + cx_4) + \mu f(ax_1 + dx_2 + cx_3 + bx_4) \\ & + \mu f(bx_1 + cx_2 + dx_3 + ax_4) + \mu f(cx_1 + bx_2 + ax_3 + dx_4) \\ & - (d + a + b + c) f(\mu(x_1 + x_2 + x_3 + x_4)) \end{aligned} \tag{2.1}$$

for all $\mu \in \mathbb{T}^1 := \{v \in \mathbb{C} : |v| = 1\}$ and $x_1, \cdots, x_m \in X$.

Note that a $\mathbb{C}$-linear mapping $H : X \to Y$ is called a *homomorphism* in C^*-algebras if H satisfies $H(xy) = H(x)H(y)$ and $H(x^*) = H(x)^*$ for all $x, y \in X$. Now we prove the Hyers-Ulam stability of homomorphisms in C^*-algebras for the functional equation $F_\mu f(x, y) = 0$.

Theorem 2.1. *Let a, b, c and d be fixed nonzero real numbers. Let $f : X \to Y$ be an mapping for which there exists a function $\varphi : X^4 \to [0, \infty)$, such that*

$$\|F_\mu f(x_1, x_2, x_3, x_4)\|_Y \leq \varphi(x_1, x_2, x_3, x_4), \tag{2.2}$$

$$\|f(xy) - f(x)f(y)\|_Y \leq \varphi(x, x, y, y), \tag{2.3}$$

$$\|f(x^*) - f(x)^*\|_Y \leq \varphi(x, x, x, x) \tag{2.4}$$

for all $\mu \in \mathbb{T}^1 := \{v \in \mathbb{C} : |v| = 1\}$ and all $x_1, x_2, x_3, x_4, x, y \in X$. If there exists an $0 < L < 1$ such that $\varphi(x_1, x_2, x_3, x_4) \leq \frac{|k|}{4} L\varphi\left(\frac{4}{k}x_1, \frac{4}{k}x_2, \frac{4}{k}x_3, \frac{4}{k}x_4\right)$ for all $x_1, x_2, x_3, x_4 \in X$, $d, a, b, c, \alpha_4 \in \mathbb{R}$ with $4 < |k|$, then there exists a unique C^*-algebra homomorphism $H : X \to Y$ such that

$$\|f(x) - H(x)\|_Y \leq \frac{4}{|k|(1 - L)} \varphi\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}, \frac{x}{4}\right) \tag{2.5}$$

for all $x \in X$.

Proof. It follows that $\varphi(x_1, x_2, x_3, x_4) \leq \frac{|k|}{4} L\varphi\left(\frac{4}{k}x_1, \frac{4}{k}x_2, \frac{4}{k}x_3, \frac{4}{k}x_4\right)$ that

$$\lim_{j\to\infty} \frac{4^j}{|k|^j} \varphi\left(\frac{(k)^j}{4^j}x_1, \frac{(k)^j}{4^j}x_2, \frac{(k)^j}{4^j}x_3, \frac{(k)^j}{4^j}x_4\right) = 0 \tag{2.6}$$

for all $x, y \in X$.

Consider the set

$$A := \{g : X \to Y\} \tag{2.7}$$

and introduce the *generalized metric* on A:

$$d(g, h) = \inf\{C \in \mathbb{R}_+ : \|g(x) - h(x)\|_Y \leq C\varphi\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}, \frac{x}{4}\right), \forall x \in X\}. \tag{2.8}$$

It is easy to show that (A, d) is complete.

Now we consider the linear mapping $J : A \to A$ such that

$$Jg(x) := \frac{4}{|k|} g\left(\frac{k}{4}x\right) \tag{2.9}$$

for all $x \in X$.

By Theorem 3.1 of [6]

$$d(Jg, Jh) \leq Ld(g, h) \tag{2.10}$$

for all $g, h \in A$.

Letting $\mu = 1$ and $x_1 = x_2 = x_3 = x_4 = x$ in (2.2), we get

$$\left\|\frac{4}{k} f\left(\frac{k}{4}x\right) - f(x)\right\| \leq \frac{1}{|k|}\varphi(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}, \frac{x}{4}) \leq \frac{4}{|k|}\varphi(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}, \frac{x}{4})$$

for all $x \in X$.

Hence $d(f, Jf) \leq \frac{4}{|k|}$.

By Theorem 1.1, there exists a mapping $H : X \to Y$ such that

(1) H is a fixed point of J, that is,

$$\frac{4}{|k|}H\left(\frac{k}{4}x\right) = H(x) \tag{2.11}$$

for all $x \in X$. The mapping H is a unique fixed point of J in the set

$$B = \{g \in A : d(f, g) < \infty\}. \tag{2.12}$$

This implies that H is a unique mapping satisfying (2.24) such that there exists $C \in (0, \infty)$ satisfying

$$\|H(x) - f(x)\|_Y \leq C\varphi\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}, \frac{x}{4}\right) \tag{2.13}$$

for all $x \in X$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the inequality

$$\lim_{n\to\infty} \frac{4^n}{|k|^n} f\left(\frac{(k)^n x}{4^n}\right) = H(x) \tag{2.14}$$

for all $x \in X$.

(3) $d(f, H) \leq \frac{1}{1-L} d(f, Jf)$, which implies the ineqality

$$d(f, H) \leq \frac{4}{|k|(1 - L)}. \tag{2.15}$$

This implies that the inequality (2.5) holds.

Next, we show that $H(x)$ is additive map.

$$\begin{aligned}
&\|H(dx_1 + ax_2 + bx_3 + cx_4) + H(ax_1 + dx_2 + cx_3 + bx_4)\\
&+H(bx_1 + cx_2 + dx_3 + ax_4)\\
&+H(cx_1 + bx_2 + ax_3 + dx_4) - (d + a + b + c)H(x_1 + x_2 + x_3 + x_4)\|\\
&= \lim_{l\to\infty} \left\| \frac{4^l}{|k|^l} f\left(\frac{(k)^l}{4^l}(dx_1 + ax_2 + bx_3 + cx_4)\right)\right.\\
&+\frac{4^l}{|k|^l} f\left(\frac{(k)^l}{4^l}(ax_1 + dx_2 + cx_3 + bx_4)\right)\\
&+\frac{4^l}{|k|^l} f\left(\frac{(k)^l}{4^l}(bx_1 + cx_2 + dx_3 + ax_4)\right)\\
&+\frac{4^l}{|k|^l} f\left(\frac{(k)^l}{4^l}(cx_1 + bx_2 + ax_3 + dx_4)\right)\\
&\left.-(d + a + b + c)\frac{4^l}{|k|^l} f\left(\frac{(k)^l}{4^l}(x_1 + x_2 + x_3 + x_4)\right)\right\|\\
&\leq \lim_{l\to\infty} \frac{4^l}{|k|^l}\varphi\left(\frac{(k)^l}{4^l}x_1, \frac{(k)^l}{4^l}x_2, \frac{(k)^l}{4^l}x_3, \frac{(k)^l}{4^l}x_4\right) = 0
\end{aligned}$$

Therefore, the mapping $H : X \to Y$ is Cauchy additive.

By a similar method with above, we may get

$$\mu H(x) = H(\mu x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in X$. Thus one can show that the mapping $H : X \to Y$ is $\mathbb{C}$-linear.

It follows from (2.3)that

$$\begin{aligned}
&\|H(xy) - H(x)H(y)\|_Y \\
&= \lim_{n\to\infty} \left|\frac{4}{k}\right|^{2n} \left\| f\left(\frac{(k)^{2n}xy}{4^{2n}}\right) - f\left(\frac{(k)^n x}{4^n}\right) f\left(\frac{(k)^n y}{4^n}\right) \right\| \\
&\le \lim_{n\to\infty} \left|\frac{4}{k}\right|^{n} \left\| f\left(\frac{(k)^{2n}xy}{4^{2n}}\right) - f\left(\frac{(k)^n x}{4^n}\right) f\left(\frac{(k)^n y}{4^n}\right) \right\| \\
&\le \lim_{n\to\infty} \left|\frac{4}{k}\right|^{n} \varphi\left(\frac{(k)^n x}{4^n}, \frac{(k)^n x}{4^n}, \frac{(k)^n y}{4^n}, \frac{(k)^n y}{4^n}\right) = 0
\end{aligned}$$

for all $x, y \in X$. So

$$H(xy) = H(x)H(y) \tag{2.16}$$

for all $x, y \in X$.

It follows from (2.4) that

$$\begin{aligned}
\|H(x^*) - H(x)^*\|_Y &= \lim_{n\to\infty} \left|\frac{4}{k}\right|^{n} \left\| f\left(\frac{(k)^n x^*}{4^n}\right) - f\left(\frac{(k)^n x}{4^n}\right)^* \right\|_Y \\
&\le \lim_{n\to\infty} \left|\frac{4}{k}\right|^{n} \varphi\left(\frac{(k)^n x}{4^n}, \frac{(k)^n x}{4^n}, \frac{(k)^n x}{4^n}, \frac{(k)^n x}{4^n}\right) = 0
\end{aligned}$$

for all $x \in X$. So

$$H(x^*) = H(x)^*$$

for all $x \in X$.

Thus $H : X \to Y$ is C^*-algebra homomorphism satisfying (2.5), as desired.

□

Theorem 2.2. *Let a, b, c and d be fixed nonzero real numbers. Let $f : X \to Y$ be a mapping for which there exists a function $\varphi : X^4 \to [0, \infty)$ satisfying (2.2),(2.3) and (2.4). If there exists an $L < 1$ such $\varphi(x_1, x_2, x_3, x_4) \le \frac{4}{|k|} L\varphi\left(\frac{k}{4}x_1, \frac{k}{4}x_2, \frac{k}{4}x_3, \frac{k}{4}x_4\right)$ for all $x_1, x_2, x_3, x_4 \in X$ with $|k| < 4$, then there exists a unique C^*-algebra homomorphism $H : X \to Y$ such that*

$$\|f(x) - H(x)\| \le \frac{1}{4 - 4L}\varphi\left(\frac{x}{k}, \frac{x}{k}, \frac{x}{k}, \frac{x}{k}\right)$$

for all $x \in X$.

Proof. Consider the set

$$A := \{g : X \to Y\}$$

and introduce the generalized metric on A:

$$d(g,h) = \inf\{C \in \mathbb{R}_+ \|g(x) - h(x)\|_Y \le C\varphi\left(\frac{x}{k}, \frac{x}{k}, \frac{x}{k}, \frac{x}{k}\right), \forall x \in X\} \tag{2.17}$$

We consider the linear mapping $J : A \to A$ such that

$$Jg(x) := \frac{|k|}{4} g\left(\frac{4}{k}\right) \tag{2.18}$$

for all $x \in X$.

It follow from (2.2) that

$$\left\| f(x) - \frac{k}{4}\left(\frac{4}{k}x\right)\right\| \le \frac{1}{4}\varphi\left(\frac{x}{k}, \frac{x}{k}, \frac{x}{k}, \frac{x}{k}\right) \tag{2.19}$$

for all $x \in X$. Hence $d(f, Jf) \le \frac{1}{4}$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Recall that a $\mathbb{C}$-linear mapping $\delta : X \to Y$ is called a *derivation* on X satisfies $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in X$.

Theorem 2.3. *Let a, b, c, d be the fixed real numbers. Let $f : X \to Y$ be an mapping for which there exists a function $\varphi : X^4 \to [0, \infty)$, such that*

$$\|F_\mu f(x_1, x_2, x_3, x_4)\|_Y \le \varphi(x_1, x_2, x_3, x_4), \tag{2.20}$$

$$\|f(xy) - f(x)y - xf(y)\|_Y \le \varphi(x, x, y, y), \tag{2.21}$$

for all $\mu \in \mathbb{T}^1 := \{v \in \mathbb{C} : |v| = 1\}$ and all $x_1, x_2, x_3, x_4, x, y \in X$. If there exists an $L < 1$ such that $\varphi(x_1, x_2, x_3, x_4) \le \frac{|k|}{4} L\varphi\left(\frac{4}{k}x_1, \frac{4}{k}x_2, \frac{4}{k}x_3, \frac{4}{k}x_4\right)$ for all $x_1, x_2, x_3, x_4 \in X$ with $|k| > 4$, then there exists a unique derivation $\delta : X \to X$ such that

$$\|f(x) - \delta(x)\|_Y \le \frac{1}{|k|(1-L)}\varphi\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}, \frac{x}{4}\right) \tag{2.22}$$

for all $x \in X$.

Proof. It follows from $\varphi(x_1, x_2, x_3, x_4) \le \frac{|k|}{4} L\varphi\left(\frac{4}{k}x_1, \frac{4}{k}x_2, \frac{4}{k}x_3, \frac{4}{k}x_4\right)$ that

$$\lim_{j\to\infty} \left|\frac{4}{|k|}\right|^j \varphi\left(\left(\frac{k}{4}\right)^j x_1, \left(\frac{k}{4}\right)^j x_2, \left(\frac{k}{4}\right)^j x_3, \left(\frac{k}{4}\right)^j x_4\right) = 0$$

for all $x_1, x_2, x_3, x_4 \in X$.

Consider the set

$$A := \{g : X \to X\}$$

and introduce the generalized metric on A:

$$d(g,h) = \inf\{C \in \mathbb{R}_+ : \|g(x) - h(x)\| \le C\varphi\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}, \frac{x}{4}\right), \forall x \in X\}.$$

It is easy to show that (A, d) is complete.

Now we consider the linear mapping $J : A \to A$ such that

$$Jg(x) := \frac{4}{k} g\left(\frac{k}{4}x\right)$$

for all $x \in X$.

By Theorem 3.1 of [6]

$$d(Jg, Jh) \leq Ld(g, h) \tag{2.23}$$

for all $g, h \in A$.

Letting $\mu = 1$ and $x_1 = x_2 = x_3 = x_4 = x$ in (2.2), we get

$$\left\| \frac{4}{k} f\left(\frac{k}{4}x\right) - f(x) \right\| \leq \frac{1}{|k|}\varphi(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}, \frac{x}{4})$$

for all $x \in X$.

Hence $d(f, Jf) \leq \frac{1}{|k|}$.

By Theorem 1.1, there exists a mapping $\delta : X \to Y$ such that

(1) δ is a fixed point of J, that is,

$$\frac{4}{k}\delta\left(\frac{k}{4}x\right) = \delta(x) \tag{2.24}$$

for all $x \in X$. The mapping δ is a unique fixed point of J in the set

$$B = \{g \in A : d(f, g) < \infty\}. \tag{2.25}$$

This implies that δ is a unique mapping satisfying (2.24) such that there exists $C \in (0, \infty)$ satisfying

$$\|\delta(x) - f(x)\|_Y \leq C\varphi\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}, \frac{x}{4}\right) \tag{2.26}$$

for all $x \in X$.

(2) $d(J^n f, \delta) \to 0$ as $n \to \infty$. This implies the inequality

$$\lim_{n\to\infty} \frac{4^n}{|k|^n} f\left(\frac{(k)^n x}{4^n}\right) = \delta(x) \tag{2.27}$$

for all $x \in X$.

(3) $d(f, \delta) \leq \frac{1}{1-L} d(f, Jf)$, which implies the ineqality

$$d(f, H) \leq \frac{1}{|k|(1 - L)}. \tag{2.28}$$

This implies that the inequality (2.22) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.4. *Let a, b, c, d be the fixed real numbers. Let $f : X \to Y$ be an mapping for which there exists a function $\varphi : X^4 \to [0, \infty)$, such that*

$$\|F_\mu f(x_1, x_2, x_3, x_4)\|_Y \le \varphi(x_1, x_2, x_3, x_4), \tag{2.29}$$

$$\|f(xy) - f(x)y - xf(y)\|_Y \le \varphi(x, x, y, y), \tag{2.30}$$

for all $\mu \in \mathbb{T}^1 := \{v \in \mathbb{C} : |v| = 1\}$ and all $x_1, x_2, x_3, x_4, x, y \in X$. If there exists an $L < 1$ such that $\varphi(x_1, x_2, x_3, x_4) \le \frac{4}{|k|} L\varphi\left(\frac{k}{4}x_1, \frac{k}{4}x_2, \frac{k}{4}x_3, \frac{k}{4}x_4\right)$ for all $x_1, x_2, x_3, x_4 \in X$ with $|k| < 4$, then there exists a unique derivation $\delta : X \to X$ such that

$$\|f(x) - \delta(x)\|_Y \le \frac{1}{4(1-L)}\varphi\left(\frac{x}{k}, \frac{x}{k}, \frac{x}{k}, \frac{x}{k}\right) \tag{2.31}$$

for all $x \in X$.

Proof. The proof is similar to the proof of 2.3. □

3. Stability of homomorphisms in Lie C^*-algebras

A C^*-algebra $\mathcal{C}$, endowed with the Lie product

$$[x, y] := \frac{xy - yx}{2}$$

on $\mathcal{C}$, is called a Lie C^*-algebras(see[5],[15]).

Definition 3.1. Let X and Y be Lie C^*-algebras. A $\mathbb{C}$-linear mapping $H : X \to Y$ is called a Lie C^**-algebras homomorphism* if $H([x, y]) = [H(x), H(y)]$ for all $x, y \in X$.

Throughout this section, assume that X is a Lie C^*-algebras with a norm $\|\cdot\|_X$ and B is a Lie C^*-algebras with a norm $\|\cdot\|_Y$.

Now, we prove the generalized Hyers-Ulam stability of homomorphisms in Lie C^*-algebras for the functional equation $D_\mu f(xx_1, x_2, x_3, x_4) = 0$.

Theorem 3.2. *Let a, b, c, d be the fixed real numbers. Let $f : X \to Y$ be an mapping for which there exists a function $\varphi : X^4 \to [0, \infty)$, such that*

$$\|F_\mu f(x_1, x_2, x_3, x_4)\|_Y \le \varphi(x_1, x_2, x_3, x_4), \tag{3.1}$$

$$\|f([x, y]) - [f(x), f(y)]\|_Y \le \varphi(x, x, y, y), \tag{3.2}$$

for all $\mu \in \mathbb{T}^1 := \{v \in \mathbb{C} : |v| = 1\}$ and all $x_1, x_2, x_3, x_4, x, y \in X$. If there exists an $0 < L < 1$ such that $\varphi(x_1, x_2, x_3, x_4) \le \frac{|k|}{4} L\varphi\left(\frac{4}{k}x_1, \frac{4}{k}x_2, \frac{4}{k}x_3, \frac{4}{k}x_4\right)$ for all $x_1, x_2, x_3, x_4 \in X$ with $|k| > 4$, then there exists a unique derivation $H : X \to X$ such that

$$\|f(x) - H(x)\|_Y \le \frac{1}{|k|(1-L)}\varphi\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}, \frac{x}{4}\right) \tag{3.3}$$

for all $x \in X$.

Proof. By the same method as in the proof of Theorem 2.1, we can get the mapping $H : X \to Y$ given by

$$H(x) = \lim_{n\to\infty} \frac{4^n}{|k|^n} f\left(\frac{k^n}{4^n}x\right)$$

for all $x \in X$. Thus it follows from 3.2that

$$\begin{aligned}\|H([x,y]) - [H(x),H(y)]\|_Y &= \lim_{n\to\infty} \frac{4^{2n}}{|k|^{2n}} \left\| f\left(\frac{k^{2n}}{4^{2n}}[x,y]\right) - \left[f\left(\frac{k^n}{4^n}x\right), f\left(\frac{k^n}{4^n}x\right)\right]\right\|_Y \\ &\le \lim_{n\to\infty} \frac{4^{2n}}{|k|^{2n}} \varphi\left(\frac{k^n}{4^n}x, \frac{k^n}{4^n}y\right) = 0\end{aligned}$$

for all $x, y \in X$, and so

$$H([x,y]) = [H(x),H(y)]$$

for all $x, y \in X$. Therefore, $H : X \to Y$ is a Lie C^*-algebras homomorphism satisfying 3.3. This completes the proof. □

Theorem 3.3. *Let a, b, c, d be the fixed real numbers. Let $f : X \to Y$ be an mapping for which there exists a function $\varphi : X^4 \to [0,\infty)$, such that*

$$\|F_\mu f(x_1,x_2,x_3,x_4)\|_Y \le \varphi(x_1,x_2,x_3,x_4), \tag{3.4}$$

$$\|f([x,y]) - [f(x),f(y)]\|_Y \le \varphi(x,x,y,y), \tag{3.5}$$

for all $\mu \in \mathbb{T}^1 := \{v \in \mathbb{C} : |v| = 1\}$ and all $x_1, x_2, x_3, x_4, x, y \in X$. If there exists an $0 < L < 1$ such that $\varphi(x_1,x_2,x_3,x_4) \le \frac{4}{|k|}L\varphi\left(\frac{k}{4}x_1, \frac{k}{4}x_2, \frac{k}{4}x_3, \frac{k}{4}x_4\right)$ for all $x_1, x_2, x_3, x_4 \in X$ with $|k| < 4$, then there exists a unique derivation $H : X \to X$ such that

$$\|f(x) - H(x)\|_Y \le \frac{1}{4(1-L)}\varphi\left(\frac{x}{k}, \frac{x}{k}, \frac{x}{k}, \frac{x}{k}\right) \tag{3.6}$$

for all $x \in X$.

Proof. By the same method as in the proof of Theorem 2.1, we can get the mapping $H : X \to Y$ given by

$$H(x) = \lim_{n\to\infty} \frac{|k|^n}{4^n} f\left(\frac{4^n}{k^n}x\right)$$

for all $x \in X$. Thus it follows from 3.5that

$$\begin{aligned}\|H([x,y]) - [H(x),H(y)]\|_Y &= \lim_{n\to\infty} \frac{|k|^{2n}}{4^{2n}} \left\| f\left(\frac{4^{2n}}{k^{2n}}[x,y]\right) - \left[f\left(\frac{4^n}{k^n}x\right), f\left(\frac{4^n}{k^n}x\right)\right]\right\|_Y \\ &\le \lim_{n\to\infty} \frac{|k|^{2n}}{4^{2n}} \varphi\left(\frac{4^n}{k^n}x, \frac{4^n}{k^n}y\right) = 0\end{aligned}$$

for all $x, y \in X$, and so

$$H([x,y]) = [H(x),H(y)]$$

for all $x, y \in X$. Therefore, $H : X \to Y$ is a Lie C^*-algebras homomorphism satisfying 3.6. This completes the proof. □

4. Stability of derivations in Lie C^*-algebras

Definition 4.1. Let X be a Lie C^*-algebra. A $\mathcal{C}$-linear mapping $\delta : X \to X$ is called a *Lie derivation* if $\delta([x,y]) = [\delta(x), y] + [x, \delta(y)]$ for all $x, y \in X$.

Throughout this section, assume that X is a Lie C^*-algebra with a norm $\|\cdot\|_X$.

Finally, we prove the generalized Hyers-Ulam stability of derivations on Lie C^*-algebras for the functional equation $D_\mu f(x_1, x_2, x_3, x_4) = 0$.

Theorem 4.2. *Let a, b, c, d be the fixed real numbers. Let $f : X \to Y$ be an mapping for which there exists a function $\varphi : X^4 \to [0, \infty)$, such that*

$$\|F_\mu f(x_1, x_2, x_3, x_4)\|_Y \leq \varphi(x_1, x_2, x_3, x_4), \tag{4.1}$$

$$\|f([x,y]) - [f(x), y] - [x, f(y)]\|_Y \leq \varphi(x, x, y, y), \tag{4.2}$$

for all $\mu \in \mathbb{T}^1 := \{v \in \mathbb{C} : |v| = 1\}$ and all $x_1, x_2, x_3, x_4, x, y \in X$. If there exists an $0 < L < 1$ such that $\varphi(x_1, x_2, x_3, x_4) \leq \frac{|k|}{4} L \varphi\left(\frac{4}{k}x_1, \frac{4}{k}x_2, \frac{4}{k}x_3, \frac{4}{k}x_4\right)$ for all $x_1, x_2, x_3, x_4 \in X$ with $|k| > 4$, then there exists a unique derivation $\delta : X \to X$ such that

$$\|f(x) - \delta(x)\|_Y \leq \frac{1}{|k|(1-L)} \varphi\left(\frac{x}{4}, \frac{x}{4}, \frac{x}{4}, \frac{x}{4}\right) \tag{4.3}$$

for all $x \in X$.

Proof. By the same method as in the proof of Theorem 2.1, we can get the mapping $\delta : X \to Y$ given by

$$\delta(x) = \lim_{n\to\infty} \frac{4^n}{|k|^n} f\left(\frac{k^n}{4^n}x\right)$$

for all $x \in X$. Thus it follows from 4.2that

$$\begin{aligned}
&\|\delta([x,y]) - [\delta(x), y] - [x, \delta(y)]\|_Y \\
&= \lim_{n\to\infty} \frac{4^{2n}}{|k|^{2n}} \left\| f\left(\frac{k^{2n}}{4^{2n}}[x,y]\right) - \left[f\left(\frac{k^n}{4^n}x\right), \frac{k^n}{4^n}y\right] - \left[\frac{k^n}{4^n}x, f\left(\frac{k^n}{4^n}y\right)\right]\right\|_Y \\
&\leq \lim_{n\to\infty} \frac{4^{2n}}{|k|^{2n}} \varphi\left(\frac{k^n}{4^n}x, \frac{k^n}{4^n}y\right) = 0
\end{aligned}$$

for all $x, y \in X$, and so

$$\delta([x,y]) = [\delta(x), y] + [x, \delta(y)]$$

for all $x, y \in X$. Therefore, $\delta : X \to X$ is a Lie C^*-algebras homomorphism satisfying 4.3. This completes the proof. □

Theorem 4.3. *Let a, b, c, d be the fixed real numbers. Let $f : X \to Y$ be an mapping for which there exists a function $\varphi : X^4 \to [0, \infty)$, such that*

$$\|F_\mu f(x_1, x_2, x_3, x_4)\|_Y \leq \varphi(x_1, x_2, x_3, x_4), \tag{4.4}$$

$$\|f([x,y]) - [f(x), y] - [x, f(y)]\|_Y \leq \varphi(x, x, y, y), \tag{4.5}$$

for all $\mu \in \mathbb{T}^1 := \{v \in \mathbb{C} : |v| = 1\}$ *and all* $x_1, x_2, x_3, x_4, x, y \in X$*. If there exists an* $0 < L < 1$ *such that* $\varphi(x_1, x_2, x_3, x_4) \leq \frac{4}{|k|} L\varphi\left(\frac{k}{4}x_1, \frac{k}{4}x_2, \frac{k}{4}x_3, \frac{k}{4}x_4\right)$ *for all* $x_1, x_2, x_3, x_4 \in X$ *with* $|k| < 4$*, then there exists a unique derivation* $\delta : X \to X$ *such that*

$$\|f(x) - \delta(x)\|_Y \leq \frac{1}{4(1-L)}\varphi\left(\frac{x}{k}, \frac{x}{k}, \frac{x}{k}, \frac{x}{k}\right) \tag{4.6}$$

for all $x \in X$*.*

Proof. By the same method as in the proof of Theorem 2.1, we can get the mapping $\delta : X \to Y$ given by

$$\delta(x) = \lim_{n\to\infty} \frac{|k|^n}{4^n} f\left(\frac{4^n}{k^n}x\right)$$

for all $x \in X$. Thus it follows from 4.5 that

$$\begin{aligned}
&\|\delta([x, y]) - [H(x), y] - [x, H(y)]\|_Y \\
&= \lim_{n\to\infty} \frac{|k|^{2n}}{4^{2n}} \left\| \left(\frac{4^{2n}}{k^{2n}}[x, y]\right) - \left[f\left(\frac{4^n}{k^n}x\right), \frac{4^n}{k^n}y\right] - \left[\frac{4^n}{k^n}x, f\left(\frac{4^n}{k^n}x\right)\right] \right\|_Y \\
&\leq \lim_{n\to\infty} \frac{|k|^{2n}}{4^{2n}} \varphi\left(\frac{4^n}{k^n}x, \frac{4^n}{k^n}y\right) = 0
\end{aligned}$$

for all $x, y \in X$, and so

$$\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$$

for all $x, y \in X$. Therefore, $\delta : X \to X$ is a Lie C^*-algebras homomorphism satisfying 4.6. This completes the proof. □

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Gang Lu
1.Department of Mathematics, School of Science, ShenYang University of Technology, Shenyang 110870, P.R. China
2.Department of Mathematics, Zhejiang University, Hangzhou 310027,Peoples Republic of China
E-mail address: lvgang1234@hanmail.net

Guoxian Cai
Department of Mathematics,Ajou University, Suwon 443-749, Korea
E-mail address: cai110@naver.com

Yuanfeng Jin
Department of Mathematics, Yanbian University, Yanji 133001, P.R. China P.R. China
E-mail address: jinyuanfeng1976@hotmail.com

Choonkil Park
Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Korea
E-mail address: baak@hanyang.ac.kr

A Higher Order Multi-step Iterative Method for Computing the Numerical Solution of Systems of Nonlinear Equations Associated with Nonlinear PDEs and ODEs

Malik Zaka Ullah[a,b,*], S. Serra-Capizzano[b], Fayyaz Ahmad[c], Arshad Mahmood[a], Eman S. Al-Aidarous[a]

[a] Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia
[b] Dipartimento di Scienza e Alta Tecnologia, universita dell'Insubria, Via Valleggio 11, 22100 Como, Italy
[c] Dept. de Física i Enginyeria Nuclear, universitat Politècnica de Catalunya, Comte d'Urgell 187, 08036 Barcelona, Spain

Abstract

The main focus of research in the current article is to address the construction of an efficient higher order multi-step iterative methods to solve systems of nonlinear equations associated with nonlinear partial differential equations (PDEs) and ordinary differential equations (ODEs). The construction includes second order Frechet derivatives. The proposed multi-step iterative method uses two Jacobian evaluations at different points and requires only one inversion (in the sense of LU-factorization) of Jacobian. The enhancement of convergence-order (CO) is hidden in the formation of matrix polynomial. The cost of matrix vector multiplication is expensive computationally. We developed a matrix polynomial of degree two for base method and degree one to perform multi-steps so we need just one matrix vector multiplication to perform each further step. The base method has convergence order four and each additional step enhance the CO by three. The general formula for CO is $3s - 2$ for $s \geq 2$ and 2 for $s = 1$ where s is the step number. The number of function evaluations including Jacobian are $s + 2$ and number of matrix vectors multiplications are s. For s-step iterative method we solve s upper and lower triangular systems when right hand side is a vector and 1 pair of triangular systems when right hand side is a matrix. It is shown that the computational cost is almost same for Jacobian and second order Frechet derivative associated with systems of nonlinear equations due to PDEs and ODEs. The accuracy and validity of proposed multi-step iterative method is checked with different PDEs and ODEs.

Keywords: Multi-step, Iterative methods, Systems of nonlinear equations, Nonlinear partial differential equations, Nonlinear ordinary differential equations

1. Introduction

A valuable discussion can be found about Frechet derivatives in [1]. We will show that why higher order Frechet derivatives are avoided in the construction of iterative methods for general systems of nonlinear equations and why there are suitable with for a particular class of systems of nonlinear equations associated with ODEs and PDEs. To make things simpler, consider a system of three nonlinear equations

$$\mathbf{F}(\mathbf{y}) = [f_1(\mathbf{y}), f_2(\mathbf{y}), f_3(\mathbf{y})]^T = 0, \tag{1}$$

*Corresponding author
Email addresses: mzhussain@kau.edu.sa (Malik Zaka Ullah), stefano.serrac@uninsubria.it (S. Serra-Capizzano), fayyaz.ahmad@upc.edu (Fayyaz Ahmad), mahmood.arshad35@yahoo.com (Arshad Mahmood), ealaidarous@kau.edu.sa (Eman S. Al-Aidarous)

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where $\mathbf{y} = [y_1,\ y_2,\ y_3]^T$. The first order Frechet derivative (Jacobian) of (1) is

$$\mathbf{F}'(\mathbf{y}) = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \frac{\partial f_1}{\partial y_3} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \frac{\partial f_2}{\partial y_3} \\ \frac{\partial f_3}{\partial y_1} & \frac{\partial f_3}{\partial y_2} & \frac{\partial f_3}{\partial y_3} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \tag{2}$$

Next we proceed for the calculation of second-order Frechet derivative. Suppose $\mathbf{h} = [h_1,\ h_2,\ h_3]^T$ is a constant vector.

$$\mathbf{F}'(\mathbf{y})\mathbf{h} = \begin{bmatrix} h_1 f_{11} + h_2 f_{12} + h_3 f_{13} \\ h_1 f_{21} + h_2 f_{22} + h_3 f_{23} \\ h_1 f_{31} + h_2 f_{32} + h_3 f_{33} \end{bmatrix}, \tag{3}$$

$$\mathbf{F}''(\mathbf{y})\mathbf{h}^2 = \begin{bmatrix} f_{111} & f_{122} & f_{133} \\ f_{211} & f_{222} & f_{233} \\ f_{311} & f_{322} & f_{333} \end{bmatrix} \begin{bmatrix} h_1^2 \\ h_2^2 \\ h_3^2 \end{bmatrix} + 2 \begin{bmatrix} f_{121} & f_{113} & f_{123} \\ f_{212} & f_{213} & f_{223} \\ f_{312} & f_{313} & f_{323} \end{bmatrix} \begin{bmatrix} h_1 h_2 \\ h_1 h_3 \\ h_2 h_3 \end{bmatrix}. \tag{4}$$

Clearly the computational cost for second-order Frechet derivative is high in the case of general systems of nonlinear equations. Many systems of nonlinear equations associated with PDEs and ODEs can be written as

$$\begin{cases} \mathbf{F}(y) = L(y) + f(y) + w = 0, \\ \mathbf{F}(\mathbf{y}) = \mathbf{A}\mathbf{y} + f(\mathbf{y}) + \mathbf{w} = 0, \end{cases} \tag{5}$$

where A is the discrete approximation to linear differential operator $L(\cdot)$ and $f(\cdot)$ is the nonlinear function. If we write down the second-order Frechet derivative of (5) by using (4) we get

$$\mathbf{F}''(\mathbf{y})\mathbf{h}^2 = \begin{bmatrix} f''(y_1) & 0 & 0 & \cdots & 0 \\ 0 & f''(y_2) & 0 & \cdots & 0 \\ 0 & 0 & f''(y_3) & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & f''(y_n) \end{bmatrix} \begin{bmatrix} h_1^2 \\ h_2^2 \\ h_3^2 \\ \vdots \\ h_n^2 \end{bmatrix} \tag{6}$$

For the further analysis , we introduce some notation. If $\boldsymbol{a} = [a_1,\ a_2,\ ,\cdots, a_n]^T$ and $\boldsymbol{b} = [b_1,\ b_2,\ ,\cdots, b_n]^T$ are vectors then the diagonal matrix of a vector and point-wise product we define as

$$\mathrm{diag}(\boldsymbol{a}) = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix},\ \boldsymbol{a} \odot \boldsymbol{b} = \mathrm{diag}(\boldsymbol{a})\,\boldsymbol{b} = [a_1 b_1,\ a_2 b_2, \cdots, a_n b_n]^T. \tag{7}$$

For the motivation of readers we list some famous nonlinear ODEs and PDEs and their first- and second-order derivatives in scalar and vectorial forms (Frechet derivatives). Let $\mathbf{D}_x$ and $\mathbf{D}_t$ are the discrete approximations of differential operators in spatial and temporal dimensions and u is the function of spatial variables and in some cases temporal variable is also taken. We also introduce a function h which is independent from u and I_t, I_x are identity matrices of the size number of nodes in temporal and spatial dimensions respectively.

1.1. Bratu problem

The Bratu problem is discussed in [2] and it is stated as

$$\begin{cases} f(u) = u'' + \lambda e^u = 0, \qquad u(0) = u(1) = 0, \\ \dfrac{df(u)}{du}h = h'' + \lambda e^u h, \\ \dfrac{d^2 f(u)}{du^2}h^2 = \lambda e^u h^2, \\ \mathbf{F}(\mathbf{u}) = \mathbf{D}_x^2\mathbf{u} + \lambda e^{\mathbf{u}} = \mathbf{0}, \\ \mathbf{F}'\mathbf{h} = \mathbf{D}_x^2\mathbf{h} + \lambda e^{\mathbf{u}} \odot \mathbf{h}, \\ \mathbf{F}' = \mathbf{D}_x^2 + \lambda \operatorname{diag}(e^{\mathbf{u}}), \\ \mathbf{F}''\mathbf{h}^2 = \lambda e^{\mathbf{u}} \odot \mathbf{h}^2. \end{cases} \tag{8}$$

The closed form solution of Bratu problem can be written as

$$\begin{cases} u(x) = -2log\left(\dfrac{cosh((x-0.5)(0.5\theta))}{cosh(0.25\theta)}\right), \\ \theta = \sqrt{2\lambda} cosh(0.25\theta). \end{cases} \tag{9}$$

The critical value of λ satisfies $4 = \sqrt{4\lambda_c} sinh(0.25\theta_c)$. The Bratu problem has two solution, unique solution and no solution if $\lambda < \lambda_c$, $\lambda = \lambda_c$ and $\lambda > \lambda_c$ respectively. The critical value $\lambda_c = 3.51383071912516$.

1.2. Frank-Kamenetzkii problem

The Frank-Kamenetzkii problem [3] is written as

$$\begin{cases} u'' + \dfrac{1}{x}u' + \lambda e^u = 0, \qquad u'(0) = u(1) = 0, \\ \mathbf{F}(\mathbf{u}) = \mathbf{D}_x^2\mathbf{u} + \dfrac{1}{\mathbf{x}} \odot \mathbf{D}_x\mathbf{u} + \lambda e^{\mathbf{u}} = \mathbf{0}, \\ \mathbf{F}'\mathbf{h} = \mathbf{D}_x^2\mathbf{h} + \dfrac{1}{\mathbf{x}} \odot \mathbf{D}_x\mathbf{h} + \lambda e^{\mathbf{u}} \odot \mathbf{h}, \\ \mathbf{F}' = \mathbf{D}_x^2 + \operatorname{diag}\left(\dfrac{1}{\mathbf{x}}\right)\mathbf{D}_x + \lambda \operatorname{diag}(e^{\mathbf{u}}), \\ \mathbf{F}''\mathbf{h}^2 = \lambda e^{\mathbf{u}} \odot \mathbf{h}^2. \end{cases} \tag{10}$$

The Frank-Kamenetzkii problem has no solution ($\lambda > 2$), ($\lambda = 2$) and two solution ($\lambda < 2$). The closed form solution of (10) is given as

$$\begin{cases} c_1 = log\left(2(4-\lambda) \pm 4\sqrt{2(2-\lambda)}\right), \\ c_2 = log\left(\dfrac{4-\lambda) \pm 2\sqrt{2(2-\lambda)}}{2\lambda^2}\right), \\ u(x) = log\left(\dfrac{16e^{c_1}}{(2\lambda + e^{c_1}x^2)^2}\right), \\ u(x) = log\left(\dfrac{16e^{c_1}}{(1 + 2\lambda e^{c_2}x^2)^2}\right). \end{cases} \tag{11}$$

1.3. Lane-Emden equation

The Lane-Emden equation is classical equation [4] which is introduced in 1870 by Lane and later Emden (1907) studied it. Lane-Emden equation deals with mass density distribution inside a spherical star when it is in hydrostatic equilibrium. The lane-Emden equation for index $n = 5$ can be written as

$$\begin{cases} u'' + \dfrac{2}{x}u' + u^5 = 0, & u(0) = 1,\ u'(0) = 0, \\ \mathbf{F}(\mathbf{u}) = \mathbf{D}_x^2\mathbf{u} + \dfrac{1}{\mathbf{x}} \odot \mathbf{D}_x\mathbf{u} + \mathbf{u}^5, \\ \mathbf{F}'\mathbf{h} = \mathbf{D}_x^2\mathbf{h} + \dfrac{1}{\mathbf{x}} \odot \mathbf{D}_x\mathbf{h} + 5\mathbf{u}^4 \odot \mathbf{h}, \\ \mathbf{F}' = \mathbf{D}_x^2 + \mathrm{diag}\left(\dfrac{1}{\mathbf{x}}\right)\mathbf{D}_x + 5\ \mathrm{diag}(\mathbf{u}^4), \\ \mathbf{F}''\mathbf{h}^2 = 20\ \mathbf{u}^3 \odot \mathbf{h}^2. \end{cases} \tag{12}$$

The closed form solution of (12) can be written as

$$u(x) = \left(1 + \frac{x^2}{3}\right)^{-\frac{1}{2}}. \tag{13}$$

1.4. Klien-Gordan equation

Klien-Gordan equation is discussed and solved in [5].

$$\begin{cases} u_{tt} - c^2u_{xx} + f(u) = p, & -\infty < x < \infty,\ t > 0 \\ \mathbf{F}(\mathbf{u}) = (\mathbf{D}_t^2 - c^2\mathbf{D}_x^2)\mathbf{u} + f(\mathbf{u}) - \boldsymbol{p}, \\ \mathbf{F}'\mathbf{h} = (\mathbf{D}_t^2 - c^2\mathbf{D}_x^2)\mathbf{h} + f'(\mathbf{u}) \odot \boldsymbol{h}, \\ \mathbf{F}' = \mathbf{D}_t^2 - c^2\mathbf{D}_x^2 + \mathrm{diag}(f'(\mathbf{u})), \\ \mathbf{F}''\mathbf{h}^2 = f''(\mathbf{u}) \odot \mathbf{h}^2, \end{cases} \tag{14}$$

where $f(u)$ is the odd function of u and initial conditions are

$$\begin{cases} u(x,0) = g_1(x), \\ u_t(x,0) = g_2(x). \end{cases} \tag{15}$$

We have calculated the second-order Frechet derivatives of four different nonlinear ODEs and PDEs. Clearly the computational cost of second-order Frechet derivatives are not higher than first-order Frechet derivatives or Jacobians. So we insist that the second-order Frechet derivatives for particular class of ODEs and PDEs are not expensive as they are in the case of general systems of nonlinear equations. The main source of information about iterative methods is the manuscript written by J. F. Traub [6] in 1964. Recently many researchers have contributed in the area of iterative method for systems of nonlinear equations [7–16]. The major part of work is devoted for the construction iterative methods for the single variable nonlinear equations[17]. According to Traub's conjecture if we use n function evaluations, then the maximum CO is 2^n in the case of single variable nonlinear equation but for multi-variable case we do not have such claim. In the case of systems of nonlinear equations the multi-steps iterative methods are interesting because with minimum computational cost we are aimed to construct higher-order convergence iterative methods. For the better understanding we can divide multi-steps iterative methods in two parts one is called base method and second part is called multi-steps. In the base method we construct an iterative method in way that it provides maximum enhancement in the convergence-order with minimum computational cost when we perform multi-steps. Malik et. al.

[18] proposed the following multi-step iterative method (MZ_1) :

$$
MZ_1 = \begin{cases} \text{Number of steps} & = m \geq 2 \\ \text{CO} & = 2m \\ \text{Function evaluations} & = m+1 \\ \text{Inverses} & = 2 \\ \text{Matrix vector multiplications} & = 1 \\ \text{Number of solutions of systems} & \\ \text{of linear equations} & \\ \text{when right hand side is matrix} & = 1 \\ \text{when right hand side is vector} & = m-1 \end{cases}
\left\{ \begin{array}{l} \text{Base-Method} \rightarrow \begin{cases} \mathbf{F}'(\mathbf{x})\boldsymbol{\phi}_1 = \mathbf{F}(\mathbf{x}) \\ \mathbf{y}_1 = \mathbf{x} - \frac{2}{3}\boldsymbol{\phi}_1 \\ \mathbf{W} = \frac{1}{2}\left(3\mathbf{F}'(\mathbf{y}_1) - \mathbf{F}'(\mathbf{x})\right) \\ \mathbf{WT} = 3\mathbf{F}'(\mathbf{y}_1) + \mathbf{F}'(\mathbf{x}) \\ \mathbf{y}_2 = \mathbf{x} - \frac{1}{4}T\boldsymbol{\phi}_1 \end{cases} \\ (m-2)\text{-steps} \rightarrow \begin{cases} \text{for } s = 1, m-2 \\ \quad \mathbf{W}\boldsymbol{\phi}_{s+1} = \mathbf{F}(\mathbf{y}_{s+1}), \\ \quad \mathbf{y}_{s+2} = \mathbf{y}_{s+1} - \boldsymbol{\phi}_{s+1}, \\ \text{end} \end{cases} \end{array} \right.
$$

In [19] F. Soleymani and co-researchers constructed an other multi-step iterative method (FS):

$$
FS = \begin{cases} \text{Number of steps} & = m \geq 2 \\ \text{CO} & = 2m \\ \text{Function evaluations} & = m+1 \\ \text{Inverses} & = 2 \\ \text{Matrix vector multiplications} & = 2m-3 \\ \text{Number of solutions of systems} & \\ \text{of linear equations} & \\ \text{when right hand side is matrix} & = 1 \\ \text{when right hand side is vector} & = m-1 \end{cases}
\left\{ \begin{array}{l} \text{Base-Method} \rightarrow \begin{cases} \mathbf{F}'(\mathbf{x})\boldsymbol{\phi}_1 = \mathbf{F}(\mathbf{x}) \\ \mathbf{y}_1 = \mathbf{x} - \frac{2}{3}\boldsymbol{\phi}_1 \\ \mathbf{W} = \frac{1}{2}\left(3\mathbf{F}'(\mathbf{y}_1) - \mathbf{F}'(\mathbf{x})\right) \\ \mathbf{WT} = 3\mathbf{F}'(\mathbf{y}_1) + \mathbf{F}'(\mathbf{x}) \\ \mathbf{y}_2 = \mathbf{x} - \mathbf{T}\boldsymbol{\phi}_1 \end{cases} \\ (m-2)\text{-steps} \rightarrow \begin{cases} \text{for } s = 1, m-2 \\ \quad \mathbf{F}'(\mathbf{x})\boldsymbol{\phi}_{s+1} = \mathbf{F}(\mathbf{y}_{s+1}), \\ \quad \mathbf{y}_{s+2} = \mathbf{y}_{s+1} - \mathbf{T}^2\boldsymbol{\phi}_{s+1}, \\ \text{end} \end{cases} \end{array} \right.
$$

H. Montazeri et. al. [20] developed the more efficient multi-step iterative methods (HM):

$$
HM = \begin{cases} \text{Number of steps} & = m \geq 2 \\ \text{CO} & = 2m \\ \text{Function evaluations} & = m+1 \\ \text{Inverses} & = 1 \\ \text{Matrix vector multiplications} & = m \\ \text{Number of solutions of systems} & \\ \text{of linear equations} & \\ \text{when right hand side is matrix} & = 1 \\ \text{when right hand side is vector} & = m-1 \end{cases}
\left\{ \begin{array}{l} \text{Base-Method} \rightarrow \begin{cases} \mathbf{F}'(\mathbf{x})\boldsymbol{\phi}_1 = \mathbf{F}(\mathbf{x}) \\ \mathbf{y}_1 = \mathbf{x} - \frac{2}{3}\boldsymbol{\phi}_1 \\ \mathbf{F}'(\mathbf{x})\mathbf{T} = \mathbf{F}'(\mathbf{y}_1) \\ \mathbf{y}_2 = \mathbf{x} - \left(\frac{23}{8}\mathbf{I} - 3\mathbf{T} + \frac{9}{8}\mathbf{T}^2\right)\boldsymbol{\phi}_1 \end{cases} \\ (m-2)\text{-steps} \rightarrow \begin{cases} \text{for } s = 1, m-2 \\ \quad \mathbf{F}'(\mathbf{x})\boldsymbol{\phi}_{s+1} = \mathbf{F}(\mathbf{y}_{s+1}), \\ \quad \mathbf{y}_{s+2} = \mathbf{y}_{s+1} - \left(\frac{5}{2}\mathbf{I} - \frac{3}{2}\mathbf{T}\right)\boldsymbol{\phi}_{s+1}, \\ \text{end} \end{cases} \end{array} \right.
$$

2. The proposed new multi-step iterative method

We proposed a new multi-step iterative method (MZ_2):

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$$\mathrm{MZ}_2 = \begin{cases} \text{Number of steps} & = m \geq 2 \\ \text{CO} & = 3m-2 \\ \text{Function evaluations} & = m+2 \\ \text{Inverses} & = 1 \\ \text{Matrix vector} & \\ \text{multiplications} & = m \\ \text{Number of solutions} & \\ \text{of systems of linear} & \\ \text{ equations when} & \\ \text{right hand side is matrix} & = 1 \\ \text{right hand side is vector} & = m \end{cases} \quad \begin{cases} \text{Base-Method} \rightarrow \begin{cases} \mathbf{F}'(\mathbf{x})\boldsymbol{\phi}_1 = \mathbf{F}(\mathbf{x}) \\ \mathbf{F}'(\mathbf{x})\boldsymbol{\phi}_2 = \mathbf{F}''\left(\mathbf{x} - \frac{4}{9}\boldsymbol{\phi}_1\right)\boldsymbol{\phi}_1^2 \\ \mathbf{y}_1 = \mathbf{x} - \left(\boldsymbol{\phi}_1 + \frac{3}{2}\boldsymbol{\phi}_2\right) \\ \mathbf{F}'(\mathbf{x})\mathbf{T} = \mathbf{F}'(\mathbf{y}_1) \\ \mathbf{y}_2 = \mathbf{x} - \left(\frac{7}{2}\mathbf{I} - 6\mathbf{T} + \frac{7}{2}\mathbf{T}^2\right)\left(\boldsymbol{\phi}_1 + \frac{3}{2}\boldsymbol{\phi}_2\right) \end{cases} \\ (m-2)\text{-steps} \rightarrow \begin{cases} \text{for } s = 1, m-2 \\ \quad \mathbf{F}'(\mathbf{x})\boldsymbol{\phi}_{s+2} = \mathbf{F}(\mathbf{y}_{s+1}), \\ \quad \mathbf{y}_{s+2} = \mathbf{y}_{s+1} - \left(2\mathbf{I} - \mathbf{T}\right)\boldsymbol{\phi}_{s+2}, \\ \text{end} \end{cases} \end{cases}$$

We claim that the convergence-order of our proposed multi-step iterative method is

$$\mathrm{CO} = \begin{cases} 2 & m = 1, \\ 3m-2 & m \geq 2, \end{cases} \tag{16}$$

where m is the number of steps of MZ_2. The computational costs of MZ_1 and FS are high because both methods use two inversions of matrices. The multi-step iterative method HM use only one inversion of Jacobian and hence is a good candidate for the performance comparison. For further discussion we will not consider MZ_1 and FS methods. We presented comparison between MZ_2 and HM in Table1 and 2. The Table 1 tells us if the number of function evaluations and number of solutions of system of linear equations are equal then the performance of MZ_2 in terms of convergence-order is better than HM when number of step of MZ_2 are grater or equal to four. When the convergence-orders of both iterative methods are equal then we can see from Table 2 that the computation effort of HM is always more than that of MZ_2 for $m \geq 2$. The performance index to measure the efficiency of an iterative method to solve systems of nonlinear equation is defined as

$$\rho = CO^{\frac{1}{flops}}. \tag{17}$$

In Table 3 we provided the computational cost of different operation and Table 4 shows the performance index as defined in (20) for a particular case when HM and MZ_2 have the same convergence-order. Clearly the performance index of MZ_2 is better than that of HM.

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Table 1: Comparison between multi-steps iterative method MZ_2 and HM if number of function evaluations and solutions of system of linear equations are equal.

	MZ_2 $(m \geq 2)$	HM $(m \geq 2)$	MZ_2 $(m = 2)$	HM $(m = 3)$	MZ_2 $(m = m_1)$	HM $(m = m_1 + 1)$	Difference $MZ_2 - HM$
Number of steps	m	m	2	3	m_1	$m_1 + 1$	1
Convergence-order	$3m - 2$	$2m$	4	6	$3m_1 - 2$	$2(m_1 + 1)$	$m_1 - 4$
Function evaluations	$m + 2$	$m + 1$	4	4	$m_1 + 2$	$m_1 + 2$	0
Solution of system of linear equations when right hand side is vector	m	$m - 1$	2	2	m_1	m_1	0
Solution of system of linear equations when right hand side is matrix	1	1	1	1	1	1	0
Matrix vector multiplications	m	m	2	3	m_1	$m_1 + 1$	-1

Table 2: Comparison between multi-steps iterative method MZ_2 and HM if convergence-orders are equal.

	MZ_2 $(m \geq 1)$	HM $(m \geq 1)$	Difference $HM - MZ_2$
Number of steps	$2m$	$3m - 1$	$m - 1$
Convergence-order	$6m - 2$	$6m - 2$	0
Function evaluations	$2m + 2$	$3m$	$m - 2$
Solution of system of linear equations when right hand side is vector	$2m$	$3m$	m
Solution of system of linear equations when right hand side is matrix	1	1	0
Matrix vector multiplications	$2m$	$3m - 1$	$m - 1$

Table 3: Computational cost of different operations (the computational cost of a division is three times to multiplication).

LU decomposition		
Multiplications $\frac{n(n-1)(2n-1)}{6}$	Divisions $\frac{n(n-1)}{2}$	Total cost $\frac{n(n-1)(2n-1)}{6} + 3\frac{n(n-1)}{2}$
Two triangular systems (if right hand side is a vector)		
Multiplications $n(n-1)$	Divisions n	Total cost $n(n-1) + 3n$
Two triangular systems (if right hand side is a matrix)		
Multiplications $n^2(n-1)$	Divisions n^2	Total cost $n^2(n-1) + 3n^2$
Matrix vector multiplication		
n^2		

Table 4: Comparison of performance index between multi-steps iterative methods MZ_2 and HM.

Iterative methods	HM	MZ_2
Number of steps	5	4
Rate of convergence	10	10
Number of functional evaluations	$6n$	$6n$
The classical efficiency index	$2^{1/(6n)}$	$2^{1/(6n)}$
Number of Lu factorizations	1	1
Cost of Lu factorizations	$\frac{n(n-1)(2n-1)}{6} + 3\frac{n(n-1)}{2}$	$\frac{n(n-1)(2n-1)}{6} + 3\frac{n(n-1)}{2}$
Cost of linear systems	$4(n(n-1) + 3n) + n^2(n-1) + 3n^2$	$4(n(n-1) + 3n) + n^2(n-1) + 3n^2$
Matrix vector multiplications	$5n^2$	$4n^2$
Flops-like efficiency index	$10^{1/\left(\frac{4n^3}{3}+12n^2+\frac{38}{3}n\right)}$	$10^{1/\left(\frac{4n^3}{3}+11n^2+\frac{38}{3}n\right)}$

3. Convergence Analysis

In this section, we will prove that the local convergence-order of MZ_2 is seven for $m = 3$ and later we will establish a proof for the convergence-order of multi-step iterative scheme MZ_2, by using mathematical induction.

Theorem 3.1. *Let* $\mathbf{F} : \Gamma \subseteq \mathbb{R}^n \to \mathbb{R}^n$ *be sufficiently Frechet differentiable on an open convex neighborhood* Γ *of* $\mathbf{x}^* \in \mathbb{R}^n$ *with* $\mathbf{F}(\mathbf{x}^*) = 0$ *and* $det(\mathbf{F}'(\mathbf{x}^*)) \neq 0$*. Then the sequence* $\{\mathbf{x}_k\}$ *generated by the iterative scheme* MZ_2 *converges to* $\mathbf{x}^*$ *with local order of convergence seven, and produces the following error equation*

$$\mathbf{e}_{k+1} = \mathbf{L}\mathbf{e}_k{}^7 + O(\mathbf{e}_k{}^8), \tag{18}$$

where $\mathbf{e}_k = \mathbf{x}_k - \mathbf{x}^*$, $\mathbf{e}_k{}^p = \overbrace{(\mathbf{e}_k, \mathbf{e}_k, \cdots, \mathbf{e}_k)}^{p\text{-times}}$ *and* $\mathbf{L} = -2060\mathbf{C}_2^6 - 618\mathbf{C}_3\mathbf{C}_2^4 + 260/9\mathbf{C}_2^3\mathbf{C}_4 + 26/3\mathbf{C}_3\mathbf{C}_2\mathbf{C}_4 - 30\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2 - 6\mathbf{C}_3\mathbf{C}_2^2\mathbf{C}_3 - 100\mathbf{C}_2^3\mathbf{C}_3\mathbf{C}_2 - 20\mathbf{C}_2^4\mathbf{C}_3$ *is a p-linear function i.e.* $\mathbf{L} \in \mathbb{L}\,\overbrace{(\mathbb{R}^n, \mathbb{R}^n, \cdots, \mathbb{R}^n)}^{p\text{-times}}$ *and* $L\mathbf{e}_k{}^p \in \mathbb{R}^n$.

Proof. Let $\mathbf{F} : \Gamma \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be sufficiently Frechet differentiable function in Γ. The *qth* Frechet derivative of $\mathbf{F}$ at $v \in \mathbb{R}^n$, $q \geq 1$, is the $q-$ *linear* function $\mathbf{F}^{(q)}(v) : \overbrace{\mathbb{R}^n\mathbb{R}^n \cdots \mathbb{R}^n}^{\text{q-times}}$ such that $\mathbf{F}^{(q)}(v)(u_1, u_2, \cdots, u_q) \in \mathbb{R}^n$. The Taylor's series expansion of $\mathbf{F}(\mathbf{x}_k)$ around $\mathbf{x}^*$ can be written as:

$$\mathbf{F}(\mathbf{x}_k) = \mathbf{F}(\mathbf{x}^* + \mathbf{x}_k - \mathbf{x}^*) = \mathbf{F}(\mathbf{x}^* + \mathbf{e}_k), \tag{19}$$

$$= \mathbf{F}(\mathbf{x}^*) + \mathbf{F}'(\mathbf{x}^*)\mathbf{e}_k + \frac{1}{2!}\mathbf{F}''(\mathbf{x}^*)\mathbf{e}_k{}^2 + \frac{1}{3!}F^{(3)}(\mathbf{x}^*)\mathbf{e}_k{}^3 + O\left(\mathbf{e}_k^4\right), \tag{20}$$

$$= \mathbf{F}'(\mathbf{x}^*)\Big(\mathbf{e}_k + \frac{1}{2!}\mathbf{F}'(\mathbf{x}^*)^{-1}\mathbf{F}''(\mathbf{x}^*)\mathbf{e}_k{}^2 + \frac{1}{3!}\mathbf{F}'(\mathbf{x}^*)^{-1}F^{(3)}(\mathbf{x}^*)\mathbf{e}_k{}^3 + O\left(\mathbf{e}_k^4\right)\Big), \tag{21}$$

$$= \mathbf{C}_1\Big(\mathbf{e}_k + \mathbf{C}_2\mathbf{e}_k{}^2 + \mathbf{C}_3\mathbf{e}_k{}^3 + O\left(\mathbf{e}_k^4\right)\Big), \tag{22}$$

where $\mathbf{C}_1 = \mathbf{F}'(\mathbf{x}^*)$ and $\mathbf{C}_s = \dfrac{1}{s!}\mathbf{F}'(\mathbf{x}^*)^{-1}F^{(s)}(\mathbf{x}^*)$ for $s \geq 2$. From (22), we can calculate the Frechet derivative of $\mathbf{F}$:

$$\mathbf{F}'(\mathbf{x}_k) = \mathbf{C}_1\Big(\mathbf{I} + 2\mathbf{C}_2\mathbf{e}_k + 3\mathbf{C}_3\mathbf{e}_k{}^2 + 4\mathbf{C}_3\mathbf{e}_k{}^3 + O\left(\mathbf{e}_k^4\right)\Big), \tag{23}$$

where $\mathbf{I}$ is the identity matrix. Furthermore, we calculate the inverse of the Jacobian matrix

$$\begin{aligned}\mathbf{F}'(\mathbf{x}_k)^{-1} = \Big(&\mathbf{I} - 2\mathbf{C}_2\mathbf{e}_k + \left(4\mathbf{C}_2^2 - 3\mathbf{C}_3\right)\mathbf{e}_k^2 + \left(6\mathbf{C}_3\mathbf{C}_2 + 6\mathbf{C}_2\mathbf{C}_3 - 8\mathbf{C}_2^3 - 4\mathbf{C}_4\right)\mathbf{e}_k^3 + \Big(8\mathbf{C}_4\mathbf{C}_2 + 9\mathbf{C}_3^2 + 8\mathbf{C}_2\mathbf{C}_4 - 5\mathbf{C}_5 - \\ &12\mathbf{C}_3\mathbf{C}_2^2 - 12\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2 - 12\mathbf{C}_2^2\mathbf{C}_3 + 16\mathbf{C}_2^4\Big)\mathbf{e}_k^4 + \Big(24\mathbf{C}_3\mathbf{C}_2^3 + 24\mathbf{C}_2^3\mathbf{C}_3 + 24\mathbf{C}_2^2\mathbf{C}_3\mathbf{C}_2 + 24\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2^2 + \\ &10\mathbf{C}_5\mathbf{C}_2 + 12\mathbf{C}_4\mathbf{C}_3 + 12\mathbf{C}_3\mathbf{C}_4 + 10\mathbf{C}_2\mathbf{C}_5 - 6\mathbf{C}_6 - 16\mathbf{C}_4\mathbf{C}_2^2 - 18\mathbf{C}_3^2\mathbf{C}_2 - 18\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3 - 16\mathbf{C}_2\mathbf{C}_4\mathbf{C}_2 - \\ &18\mathbf{C}_2\mathbf{C}_3^2 - 16\mathbf{C}_2^2\mathbf{C}_4 - 32\mathbf{C}_2^5\Big)\mathbf{e}_k^5 + \Big(32\mathbf{C}_4\mathbf{C}_2^3 + 64\mathbf{C}_2^6 - 48\mathbf{C}_3\mathbf{C}_2^4 + 12\mathbf{C}_2\mathbf{C}_6 + 16\mathbf{C}_4^2 + 15\mathbf{C}_3\mathbf{C}_5 + \\ &15\mathbf{C}_5\mathbf{C}_3 + 12\mathbf{C}_6\mathbf{C}_2 - 24\mathbf{C}_4\mathbf{C}_2\mathbf{C}_3 - 24\mathbf{C}_4\mathbf{C}_3\mathbf{C}_2 - 20\mathbf{C}_2^2\mathbf{C}_5 - 24\mathbf{C}_2\mathbf{C}_3\mathbf{C}_4 - 24\mathbf{C}_2\mathbf{C}_4\mathbf{C}_3 + 32\mathbf{C}_2^3\mathbf{C}_4 - \\ &20\mathbf{C}_2\mathbf{C}_5\mathbf{C}_2 + 36\mathbf{C}_2^2\mathbf{C}_3^2 - 20\mathbf{C}_5\mathbf{C}_2^2 + 32\mathbf{C}_2^2\mathbf{C}_4\mathbf{C}_2 + 32\mathbf{C}_2\mathbf{C}_4\mathbf{C}_2^2 + 36\mathbf{C}_2\mathbf{C}_3^2\mathbf{C}_2 + 36\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3 + \\ &36\mathbf{C}_3^2\mathbf{C}_2^2 - 7\mathbf{C}_7 - 24\mathbf{C}_3\mathbf{C}_2\mathbf{C}_4 - 27\mathbf{C}_3^3 - 24\mathbf{C}_3\mathbf{C}_4\mathbf{C}_2 + 36\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2 + 36\mathbf{C}_3\mathbf{C}_2^2\mathbf{C}_3 - 48\mathbf{C}_2^2\mathbf{C}_3\mathbf{C}_2^2 - \\ &48\mathbf{C}_2^3\mathbf{C}_3\mathbf{C}_2 - 48\mathbf{C}_2^4\mathbf{C}_3 - 48\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2^3\Big)\mathbf{e}_k^6 + O\left(\mathbf{e}_k^7\right)\Big)\mathbf{C}_1^{-1}\end{aligned} \tag{24}$$

By multiplying $\mathbf{F}'(\mathbf{x}_k)^{-1}$ and $\mathbf{F}(\mathbf{x}_k)$, we obtain $\boldsymbol{\phi}_1$:

$$\begin{aligned}\boldsymbol{\phi}_1 = \;&\mathbf{e}_k - \mathbf{C}_2\mathbf{e}_k^2 + \left(2\mathbf{C}_2^2 - 2\mathbf{C}_3\right)\mathbf{e}_k^3 + \left(-3\mathbf{C}_4 - 4\mathbf{C}_2^3 + 3\mathbf{C}_3\mathbf{C}_2 + 4\mathbf{C}_2\mathbf{C}_3\right)\mathbf{e}_k^4 + \Big(-4\mathbf{C}_5 - 6\mathbf{C}_3\mathbf{C}_2^2 - 6\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2 - \\ &8\mathbf{C}_2^2\mathbf{C}_3 + 8\mathbf{C}_2^4 + 4\mathbf{C}_4\mathbf{C}_2 + 6\mathbf{C}_3^2 + 6\mathbf{C}_2\mathbf{C}_4\Big)\mathbf{e}_k^5 + \Big(-5\mathbf{C}_6 + 12\mathbf{C}_3\mathbf{C}_2^3 + 16\mathbf{C}_2^3\mathbf{C}_3 + 12\mathbf{C}_2^2\mathbf{C}_3\mathbf{C}_2 + \\ &12\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2^2 - 8\mathbf{C}_4\mathbf{C}_2^2 - 9\mathbf{C}_3^2\mathbf{C}_2 - 12\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3 - 8\mathbf{C}_2\mathbf{C}_4\mathbf{C}_2 - 12\mathbf{C}_2\mathbf{C}_3^2 - 12\mathbf{C}_2^2\mathbf{C}_4 - 16\mathbf{C}_2^5 + 5\mathbf{C}_5\mathbf{C}_2 + \\ &8\mathbf{C}_4\mathbf{C}_3 + 9\mathbf{C}_3\mathbf{C}_4 + 8\mathbf{C}_2\mathbf{C}_5\Big)\mathbf{e}_k^6 + O\left(\mathbf{e}_k^7\right).\end{aligned} \tag{25}$$

The expression for $\boldsymbol{\phi}_2$ is the following:

$$\boldsymbol{\phi}_2 = 2\mathbf{C}_2\mathbf{e}_k^2 + \left(-8\mathbf{C}_2^2 + 10/3\mathbf{C}_3\right)\mathbf{e}_k^3 + \left(26\mathbf{C}_2^3 - 38/3\mathbf{C}_3\mathbf{C}_2 - 12\mathbf{C}_2\mathbf{C}_3 + 100/27\mathbf{C}_4\right)\mathbf{e}_k^4 + \left(-364/27\mathbf{C}_2\mathbf{C}_4 - 18\mathbf{C}_3^2 - 416/27\mathbf{C}_4\mathbf{C}_2 + 116/3\mathbf{C}_2^2\mathbf{C}_3 + 36\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2 + 122/3\mathbf{C}_3\mathbf{C}_2^2 + 2500/729\mathbf{C}_5 - 76\mathbf{C}_2^4\right)\mathbf{e}_k^5 + \left(-106\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2^2 - 298/3\mathbf{C}_2^2\mathbf{C}_3\mathbf{C}_2 - 344/3\mathbf{C}_2^3\mathbf{C}_3 + 1282/27\mathbf{C}_2^2\mathbf{C}_4 + 140/3\mathbf{C}_2\mathbf{C}_3^2 + 1106/27\mathbf{C}_2\mathbf{C}_4\mathbf{C}_2 - 118\mathbf{C}_3\mathbf{C}_2^3 + 1364/27\mathbf{C}_4\mathbf{C}_2^2 - 10664/729\mathbf{C}_2\mathbf{C}_5 - 520/27\mathbf{C}_3\mathbf{C}_4 - 544/27\mathbf{C}_4\mathbf{C}_3 - 12290/729\mathbf{C}_5\mathbf{C}_2 + 54\mathbf{C}_3^2\mathbf{C}_2 + 184/3\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3 + 6250/2187\mathbf{C}_6 + 208\mathbf{C}_2^5\right)\mathbf{e}_k^6 + O\left(\mathbf{e}_k^7\right). \quad (26)$$

The expressions for $\mathbf{y}_1$, T, $\mathbf{y}_2$ and $\mathbf{y}_3$ in order are

$$\mathbf{y}_1 - \mathbf{x}^* = -2\mathbf{C}_2\mathbf{e}_k^2 + \left(10\mathbf{C}_2^2 - 3\mathbf{C}_3\right)\mathbf{e}_k^3 + \left(-23/9\mathbf{C}_4 - 35\mathbf{C}_2^3 + 16\mathbf{C}_3\mathbf{C}_2 + 14\mathbf{C}_2\mathbf{C}_3\right)\mathbf{e}_k^4 + (-278/243\mathbf{C}_5 - 55\mathbf{C}_3\mathbf{C}_2^2 - 48\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2 - 50\mathbf{C}_2^2\mathbf{C}_3 + 106\mathbf{C}_2^4 + 172/9\mathbf{C}_4\mathbf{C}_2 + 21/9\mathbf{C}_3^2 + 128\mathbf{C}_2\mathbf{C}_4\big)\mathbf{e}_k^5 + \left(147\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2^2 + 137\mathbf{C}_2^2\mathbf{C}_3\mathbf{C}_2 + 156\mathbf{C}_2^3\mathbf{C}_3 - 533/9\mathbf{C}_2^2\mathbf{C}_4 - 58\mathbf{C}_2\mathbf{C}_3^2 - 481/9\mathbf{C}_2\mathbf{C}_4\mathbf{C}_2 + 165\mathbf{C}_3\mathbf{C}_2^3 - 610/9\mathbf{C}_4\mathbf{C}_2^2 + 3388/243\mathbf{C}_2\mathbf{C}_5 + 179/9\mathbf{C}_3\mathbf{C}_4 + 200/9\mathbf{C}_4\mathbf{C}_3 + 4930/243\mathbf{C}_5\mathbf{C}_2 - 72\mathbf{C}_3^2\mathbf{C}_2 - 80\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3 + 520/729\mathbf{C}_6 - 296\mathbf{C}_2^5\right)\mathbf{e}_k^6 + O\left(\mathbf{e}_k^7\right). \quad (27)$$

$$\mathbf{T} = \mathbf{I} - 2\mathbf{C}_2\mathbf{e}_k - 3\mathbf{C}_3\mathbf{e}_k^2 + \left(6\mathbf{C}_3\mathbf{C}_2 - 4\mathbf{C}_4 + 20\mathbf{C}_2^3\right)\mathbf{e}_k^3 + \left(12\mathbf{C}_3\mathbf{C}_2^2 + 20\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2 + 28\mathbf{C}_2^2\mathbf{C}_3 - 110\mathbf{C}_2^4 + 8\mathbf{C}_4\mathbf{C}_2 + 9\mathbf{C}_3^2 + 26/9\mathbf{C}_2\mathbf{C}_4 - 5\mathbf{C}_5\right)\mathbf{e}_k^4 + \left(-180\mathbf{C}_3\mathbf{C}_2^3 - 156\mathbf{C}_2^3\mathbf{C}_3 - 136\mathbf{C}_2^2\mathbf{C}_3\mathbf{C}_2 - 134\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2^2 + 18\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3 + 200/9\mathbf{C}_2\mathbf{C}_4\mathbf{C}_2) + 24\mathbf{C}_2\mathbf{C}_3^2 + 68/3\mathbf{C}_2^2\mathbf{C}_4 + 432\mathbf{C}_2^5 + 10\mathbf{C}_5\mathbf{C}_2 + 12\mathbf{C}_4\mathbf{C}_3 + 12\mathbf{C}_3\mathbf{C}_4 + 1874/243\mathbf{C}_2\mathbf{C}_5 - 6\mathbf{C}_6\right)\mathbf{e}_k^5 + \left(-112\mathbf{C}_4\mathbf{C}_2^3 - 1456\mathbf{C}_2^6 + 1050\mathbf{C}_3\mathbf{C}_2^4 + 9788/729\mathbf{C}_2\mathbf{C}_6 + 16\mathbf{C}_4^2 + 15\mathbf{C}_3\mathbf{C}_5 + 15\mathbf{C}_5\mathbf{C}_3 + 12\mathbf{C}_6\mathbf{C}_2 - 24\mathbf{C}_4\mathbf{C}_3\mathbf{C}_2 + 3028/243\mathbf{C}_2^2\mathbf{C}_5 + 142/9\mathbf{C}_2\mathbf{C}_3\mathbf{C}_4 + 184/9\mathbf{C}_2\mathbf{C}_4\mathbf{C}_3 - 1474/9\mathbf{C}_2^3\mathbf{C}_4 + 5000/243\mathbf{C}_2\mathbf{C}_5\mathbf{C}_2 - 164\mathbf{C}_2^2\mathbf{C}_3^2 - 454/3\mathbf{C}_2^2\mathbf{C}_4\mathbf{C}_2 - 1220/9\mathbf{C}_2\mathbf{C}_4\mathbf{C}_2^2 - 144\mathbf{C}_2\mathbf{C}_3^2\mathbf{C}_2 - 196\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3 - 222\mathbf{C}_3^2\mathbf{C}_2^2 - 7\mathbf{C}_7 + 20/3\mathbf{C}_3\mathbf{C}_2\mathbf{C}_4 - 26/3\mathbf{C}_3\mathbf{C}_4\mathbf{C}_2 - 240\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2 - 258\mathbf{C}_3\mathbf{C}_2^2\mathbf{C}_3 + 562\mathbf{C}_2^2\mathbf{C}_3\mathbf{C}_2^2 + 546\mathbf{C}_2^3\mathbf{C}_3\mathbf{C}_2 + 624\mathbf{C}_2^4\mathbf{C}_3 + 690\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2^3\right)\mathbf{e}_k^6 + O\left(\mathbf{e}_k^7\right). \quad (28)$$

$$\mathbf{y}_2 - \mathbf{x}^* = \left(-5\mathbf{C}_3\mathbf{C}_2 + 13/9\mathbf{C}_4 - 103\mathbf{C}_2^3 - \mathbf{C}_2\mathbf{C}_3\right)\mathbf{e}_k^4 + \left(-104/9\mathbf{C}_2\mathbf{C}_4 - 21/2\mathbf{C}_3^2 - 80/9\mathbf{C}_4\mathbf{C}_2 - 148\mathbf{C}_2^2\mathbf{C}_3 - 100\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2 - 109\mathbf{C}_3\mathbf{C}_2^2 + 937/243\mathbf{C}_5 + 666\mathbf{C}_2^4\right)\mathbf{e}_k^5 + \left(869\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2^2 + 873\mathbf{C}_2^2\mathbf{C}_3\mathbf{C}_2 + 954\mathbf{C}_2^3\mathbf{C}_3 - 1133/9\mathbf{C}_2^2\mathbf{C}_4 - 124\mathbf{C}_2(\mathbf{C}_3^2) - 895/9\mathbf{C}_2\mathbf{C}_4\mathbf{C}_2 + 1074\mathbf{C}_3\mathbf{C}_2^3 - 1114/9\mathbf{C}_4\mathbf{C}_2^2 - 715/27\mathbf{C}_2\mathbf{C}_5 - 238/9\mathbf{C}_3\mathbf{C}_4 - 178/9\mathbf{C}_4\mathbf{C}_3 - 3575/243\mathbf{C}_5\mathbf{C}_2 - 75\mathbf{C}_3^2\mathbf{C}_2 - 158\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3 + 4894/729\mathbf{C}_6 - 1990\mathbf{C}_2^5\right)\mathbf{e}_k^6 + \left(3632/3\mathbf{C}_4\mathbf{C}_2^3 + 420\mathbf{C}_2^6 - 4958\mathbf{C}_3\mathbf{C}_2^4 - 30616/729\mathbf{C}_2\mathbf{C}_6 - 404/9\mathbf{C}_4^2 - 7343/162\mathbf{C}_3\mathbf{C}_5 - 16001/486\mathbf{C}_5\mathbf{C}_3 - 15620/729\mathbf{C}_6\mathbf{C}_2 - 1580/9\mathbf{C}_4\mathbf{C}_2\mathbf{C}_3 - 580/9\mathbf{C}_4\mathbf{C}_3\mathbf{C}_2 - 18334/243\mathbf{C}_2^2\mathbf{C}_5 - 761/9\mathbf{C}_2\mathbf{C}_3\mathbf{C}_4 - 847/9\mathbf{C}_2\mathbf{C}_4\mathbf{C}_3 + 1074\mathbf{C}_2^3\mathbf{C}_4 - 19556/243\mathbf{C}_2\mathbf{C}_5\mathbf{C}_2 + 1118\mathbf{C}_2^2\mathbf{C}_3^2 - 35410/243\mathbf{C}_5\mathbf{C}_2^2) + 8924/9\mathbf{C}_2^2\mathbf{C}_4\mathbf{C}_2 + 3038/3\mathbf{C}_2\mathbf{C}_4\mathbf{C}_2^2 + 1040\mathbf{C}_2\mathbf{C}_3^2\mathbf{C}_2 + 1262\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3 + 1390\mathbf{C}_3^2\mathbf{C}_2^2 + 63418/6561\mathbf{C}_7 - 919/9\mathbf{C}_3\mathbf{C}_2\mathbf{C}_4 - 165/2\mathbf{C}_3^3 - 589/9\mathbf{C}_3\mathbf{C}_4\mathbf{C}_2 + 1331\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2 + 1542\mathbf{C}_3\mathbf{C}_2^2\mathbf{C}_3 - 2678\mathbf{C}_2^2\mathbf{C}_3\mathbf{C}_2^2 - 2886\mathbf{C}_2^3\mathbf{C}_3\mathbf{C}_2 - 2881\mathbf{C}_2^4\mathbf{C}_3 - 3871\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2^3\right)\mathbf{e}_k^7 + O\left(\mathbf{e}_k^8\right). \quad (29)$$

$$\mathbf{y}_3 - \mathbf{x}^* = \left(-2060\mathbf{C}_2^6 - 618\mathbf{C}_3\mathbf{C}_2^4 + 260/9\mathbf{C}_2^3\mathbf{C}_4 + 26/3\mathbf{C}_3\mathbf{C}_2\mathbf{C}_4 - 30\mathbf{C}_3\mathbf{C}_2\mathbf{C}_3\mathbf{C}_2 - 6\mathbf{C}_3\mathbf{C}_2^2\mathbf{C}_3 - 100\mathbf{C}_2^3\mathbf{C}_3\mathbf{C}_2 - 20\mathbf{C}_2^4\mathbf{C}_3\right)\mathbf{e}_k^7 + O\left(\mathbf{e}_k^8\right). \quad (30)$$

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□

Theorem 3.2. *The multi-step iterative scheme MZ_2 has the local convergence-order $3m-2$, using $m(\geq 2)$ evaluations of a sufficiently differentiable function* $\mathbf{F}$*, two first-order Frechet derivatives* $\mathbf{F}'$ *and one second-order Frechet derivate* $\mathbf{F}''$ *per full-cycle.*

Proof. The proof is established from mathematical induction. For $m = 1,\ 2,\ 3$ the convergence-orders are two, four and seven from (27), (29) and (30) respectively. Consequently our claim concerning the convergence-order $3m-2$ is true for $m = 2,\ 3$.

We assume that our claim is true for $m = q > 3$, i.e., the convergence-order of MZ_2 is $3q-2$. The qth-step and $(q-1)th$-step of iterative scheme MZ_2 can be written as:

$$\text{Frozen-factor} = (2\mathbf{I}-\mathbf{T})\mathbf{F}'(\mathbf{x})^{-1}, \tag{31}$$

$$\mathbf{y}_{q-1} = \mathbf{y}_{q-2} - (\text{Frozen-factor})\,\mathbf{F}(\mathbf{y}_{q-2}), \tag{32}$$

$$\mathbf{y}_{q} = \mathbf{y}_{q-1} - (\text{Frozen-factor})\,\mathbf{F}(\mathbf{y}_{q-1}). \tag{33}$$

The enhancement in the convergence-order of MZ_2 from $(q-1)th$-step to qth-step is $(3q-2)-(3(q-1)-2) = 3$. Now we write the $(q+1)th$-step of MZ_2:

$$\mathbf{y}_{q+1} = \mathbf{y}_{q} - (\text{Frozen-factor})\,\mathbf{F}(\mathbf{y}_{q}). \tag{34}$$

The increment in the convergence-order of MZ_2, due to $(q+1)th$-step, is exactly three, because the use of the Frozen-factor adds an additive constant in the convergence-order[19]. Finally the convergence-order after the addition of the $(q+1)th$-step is $3q-2+3 = 3q+1 = 3(q+1)-2$, which completes the proof. □

4. Numerical Testing

For the verification of convergence-order, we use the following definition for the computational convergence-order (COC):

$$\text{COC} \approx \frac{log\Big(\|\mathbf{x}_{q+2}-\mathbf{x}^*\|_\infty/\|\mathbf{x}_{q+1}-\mathbf{x}^*\|_\infty\Big)}{log\Big(\|\mathbf{x}_{q+1}-\mathbf{x}^*\|_\infty/\|\mathbf{x}_{q}-\mathbf{x}^*\|_\infty\Big)}, \tag{35}$$

where $Max(|\mathbf{x}_{q+2}-\mathbf{x}^*|)$ is the maximum absolute error. The number of solutions of systems of linear equations are same in both iterative methods when right hand side is a matrix so we will not mention it in comparison tables. The main benefit of multi-step iterative methods is that we invert Jacobian once and then use it again and again in multi-steps part to get better convergence-order for a single cycle of iterative method. We have conducted numerical tests for four different problems to show the accuracy and validity of our proposed multi-step iterative method MZ_2. For the purpose of comparison we adopt two ways (i) when both iterative methods have same number of function evaluations and solution of systems of linear equations (ii) when both schemes have same convergence order. Tables 5, 7 and 8 show that when we number of function evaluations and solutions of systems of linear equation are equal and the convergence order of MZ_2 is higher than ten then our proposed scheme show better accuracy in less execution time. On the other hand if convergence-order of MZ_2 is less than ten then the performance of HM is relatively better. For the second cases when we equate the convergence-orders the execution time of MZ_2 are always less than that of HM because HM performs more steps to achieve the same convergence-order. Tables 6, 9 and 10 shows that MZ_2 achieve better or almost equal accuracy with less execution time. We have also simulated one PDE Klein-Gordon and results are depicted in Table 11. As we have commented if the convergence-order is less ten the performance of HM is better and it is clearly evident in Table 11 but the accuracy of MZ_2 is comparable with HM. The numerical error in solution due to MZ_2 is shown in Figure 1 and Figure 2 corresponds to numerical solution of Klein-Gordon PDE. In the case of Klein-Gordon equation by keeping the mesh size fix, if we increase the number of iterations or either number of steps both iterative method can not improve the accuracy.

Table 5: Comparison of performances for different multi-step methods in the case of the Bratu problem when number of function evaluations and number of solutions of systems of linear equations are equal in both iterative methods.

Iterative methods		MZ_2	HM
Number of iterations		1	1
Size of problem		200	200
Number of steps		32	33
Theoretical convergence-order(CO)		94	66
Number of function evaluations per iteration		34	34
Solutions of system of linear equations per iteration		32	32
Number of matrix vector multiplication per iteration		32	33
	λ		
$\|\|\mathbf{x}_q - \mathbf{x}^*\|\|_\infty$	1	$3.62e-156$	$7.55e-110$
	2	$4.78e-142$	$2.31e-98$
	3	$3.91e-50$	$4.05e-35$
Execution time		23.48	24.0

Table 6: Comparison of performances for different multi-step methods in the case of the Bratu problem when convergence orders are equal in both itrative methods.

Iterative methods	MZ_2	HM
Number of iterations	1	1
Size of problem	250	250
Number of steps	120	179
Theoretical convergence-order(CO)	358	358
Number of function evaluations per iteration	122	180
Solutions of system of linear equations per iteration	120	178
Number of matrix vector multiplication per iteration	120	179
$\|\|\mathbf{x}_q - \mathbf{x}^*\|\|_\infty$, $(\lambda = 1)$	$3.98e-235$	$3.98e-235$
Execution time	59.67	70.22

Table 7: Comparison of performances for different multi-step methods in the case of the Bratu problem when number of function evaluations and number of solutions of systems of linear equations are equal in both iterative methods.

Iterative methods	MZ_2	HM
Number of iterations	3	3
Size of problem	250	250
Number of steps	3	4
Theoretical convergence-order(CO)	7	8
Computational convergence-order(COC)	6.75	7.81
Number of function evaluations per iteration	5	5
Solutions of system of linear equations per iteration	3	3
Number of matrix vector multiplication per iteration	3	4
$\|\|\mathbf{x}_q - \mathbf{x}^*\|\|_\infty$	$8.44e-150$	$3.92e-161$
Execution time	63.75	64.66

Table 8: Comparison of performances for different multi-step methods in the case of the Frank Kamenetzkii problem when number of function evaluations and number of solutions of systems of linear equations are equal in both iterative methods.

Iterative methods	MZ_2	HM
Number of iterations	3	3
Size of problem	150	150
Number of steps	3	4
Theoretical convergence-order(CO)	7	8
Computational convergence-order(COC)	7.39	8.64
Number of function evaluations per iteration	5	5
Solutions of system of linear equations per iteration	3	3
Number of matrix vector multiplication per iteration	3	4
$\|\|\mathbf{x}_q - \mathbf{x}^*\|\|_\infty$	$4.21e-126$	$3.21e-149$
Execution time	16.10	16.68

Table 9: Comparison of performances for different multi-step methods in the case of the Frank Kamenetzkii problem when convergence orders are equal in both iterative methods.

Iterative methods	MZ_2	HM
Number of iterations	1	1
Size of problem	150	150
Number of steps	80	119
Theoretical convergence-order(CO)	238	238
Number of function evaluations per iteration	82	120
Solutions of system of linear equations per iteration	80	118
Number of matrix vector multiplication per iteration	80	119
$\|\|\mathbf{x}_k - \mathbf{x}^*\|\|_\infty$, $(\lambda = 1)$	$6.46e-116$	$3.95e-99$
Execution time	19.89	28.21

Table 10: Comparison of performances for different multi-step methods in the case of the Lane-Emden equation when convergence orders are equal.

Iterative methods	MZ_2	HM
Number of iterations	1	1
Size of problem	100	100
Number of steps	30	44
Theoretical convergence-order(CO)	88	88
Number of function evaluations per iteration	32	45
Solutions of system of linear equations per iteration	30	43
Number of matrix vector multiplication per iteration	30	44
$\|\mathbf{x}_q - \mathbf{x}^*\|_\infty$	$1.95e-34$	$2.64e-37$
Execution time	3.01	3.53

Table 11: Comparison of performances for different multi-step methods in the case of the Klien Gordon equation , initial guess $u(x_i, t_j) = 0$, $u(x,t) = \delta sech(\kappa(x - \nu t))$, $\kappa = \sqrt{\frac{k}{c^2 - \nu^2}}$, $\delta = \sqrt{\frac{2k}{\gamma}}$, $c = 1$, $\gamma = 1$, $\nu = 0.5$, $k = 0.5$, $n_x = 170$, $n_t = 26$, $x \in [-22,\ 22]$, $t \in [0,\ 0.5]$.

Iterative methods		MZ_2	HM
Number of iterations		1	1
Size of problem		4420	4420
Number of steps		4	4
Theoretical convergence-order(CO)		10	8
Number of function evaluations per iteration		6	5
Solutions of system of linear equations per iteration		4	3
Number of matrix vector multiplication per iteration		4	4
	Steps		
$\|\mathbf{x}_q - \mathbf{x}^*\|_\infty$	1	$3.24e-1$	$4.11e-1$
	2	$7.51e-3$	$2.62e-3$
	3	$2.70e-5$	$2.63e-5$
	4	$5.59e-7$	$4.39e-7$
Execution time		94.13	80.18

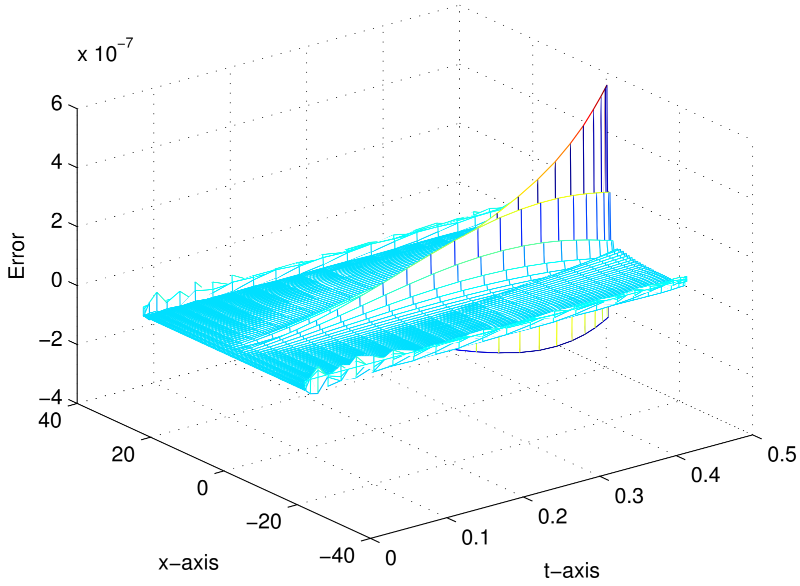

Figure 1: Absolute error plot for multi-step method MZ_2 in the case of the Klien Gordon equation , initial guess $u(x_i, t_j) = 0$, $u(x,t) = \delta sech(\kappa(x - vt)$, $\kappa = \sqrt{\frac{k}{c^2 - v^2}}$, $\delta = \sqrt{\frac{2k}{\gamma}}$, $c = 1$, $\gamma = 1$, $v = 0.5$, $k = 0.5$, $n_x = 170$, $n_t = 26$, $x \in [-22,\ 22]$, $t \in [0,\ 0.5]$.

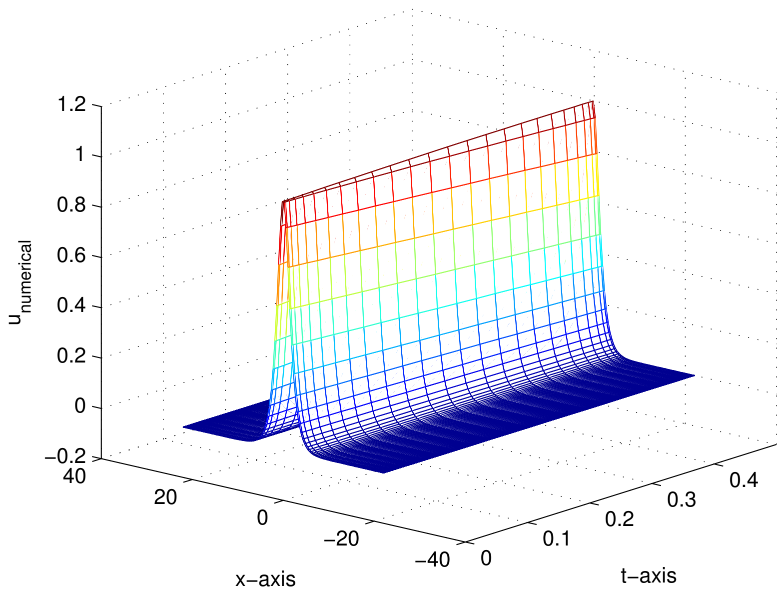

Figure 2: Numerical solution of the Klien Gordon equation , $x \in [-22,\ 22]$, $t \in [0,\ 0.5]$.

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5. Conclusions

The inversion of Jacobian is computationally expensive and multi-step iterative methods can provide remedy to it by offering good convergence-order with relatively less computational cost. The best way to construct a multi-step method is to reduce the number of Jacobian and function evaluations, inversion of Jacobian, matrix-vector and vector-vector multiplications. Higher-order Frechet derivatives are computationally expensive when use them for the solution of systems of nonlinear equations but for a particular of ODEs and PDEs we could use them because they are just diagonal matrices. Our proposed scheme MZ_2 shows good accuracy when we perform more and more multi-steps and it also depends on the nature of problem sometime. The computational convergence-order of MZ_2 is also calculated in some examples and it agrees with theoretical proved convergence-order.

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QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES

JI-HYE KIM AND CHOONKIL PARK*

Abstract. In this paper, we solve the following quadratic ρ-functional inequalities

$$N(f(x+y)+f(x-y)-2f(x)-2f(y),t) \geq N\left(\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right),t\right) \tag{0.1}$$

where ρ is a fixed real number with $|\rho|<1$, and

$$N\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y),t\right) \geq N\left(\rho\left(f(x+y)+f(x-y)-2f(x)-2f(y)\right),t\right) \tag{0.2}$$

where ρ is a fixed real number with $|\rho|<\frac{1}{2}$.

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequalities (0.1) and (0.2) in fuzzy Banach spaces.

1. Introduction and preliminaries

Katsaras [21] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [13, 24, 52]. In particular, Bag and Samanta [2], following Cheng and Mordeson [8], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [23]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 28, 29] to investigate the Hyers-Ulam stability of quadratic ρ-functional inequalities in fuzzy Banach spaces.

Definition 1.1. [2, 28, 29, 30] Let X be a real vector space. A function $N: X\times\mathbb{R}\to[0,1]$ is called a *fuzzy norm* on X if for all $x,y\in X$ and all $s,t\in\mathbb{R}$,

(N_1) $N(x,t)=0$ for $t\leq 0$;
(N_2) $x=0$ if and only if $N(x,t)=1$ for all $t>0$;
(N_3) $N(cx,t)=N(x,\frac{t}{|c|})$ if $c\neq 0$;
(N_4) $N(x+y,s+t)\geq\min\{N(x,s),N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty}N(x,t)=1$.
(N_6) for $x\neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X,N) is called a *fuzzy normed vector space*.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [27, 28].

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*Corresponding author: Choonkil Park (email: baak@hanyang.ac.kr).

Definition 1.2. [2, 28, 29, 30] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.

Definition 1.3. [2, 28, 29, 30] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [3]).

The stability problem of functional equations originated from a question of Ulam [51] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [17] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [40] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [50] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [9] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. Czerwik [10] proved the Hyers-Ulam stability of the quadratic functional equation. The functional equation $f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called a *Jensen type quadratic equation*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4, 18, 20, 25, 36, 37, 38, 41, 42, 44, 45, 46, 47, 48, 49]).

Gilányi [15] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x-y)\| \leq \|f(x+y)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x+y) + f(x-y).$$

See also [43]. Fechner [12] and Gilányi [16] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [35] investigated the Cauchy additive functional inequality

$$\|f(x) + f(y) + f(z)\| \leq \|f(x+y+z)\| \tag{1.2}$$

and the Cauchy-Jensen additive functional inequality

$$\|f(x) + f(y) + 2f(z)\| \leq \left\|2f\left(\frac{x+y}{2} + z\right)\right\| \tag{1.3}$$

and proved the Hyers-Ulam stability of the functional inequalities (1.2) and (1.3) in Banach spaces.

Park [33, 34] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.4. [5, 11] Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \ge n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [19] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [6, 7, 22, 27, 31, 32, 38, 39]).

In Section 2, we solve the quadratic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.1) in fuzzy Banach spaces by using the fixed point method.

In Section 3, we solve the quadratic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.2) in fuzzy Banach spaces by using the fixed point method.

Throughout this paper, assume that X is a real vector space and (Y, N) is a fuzzy Banach space.

2. Quadratic ρ-functional inequality (0.1)

In this section, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.1) in fuzzy Banach spaces. Let ρ be a real number with $|\rho| < 1$. We need the following lemma to prove the main results.

Lemma 2.1. *Let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} &N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) \\ &\ge N\left(\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right), t\right) \end{aligned} \tag{2.1}$$

for all $x, y \in X$ and all $t > 0$. Then f is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $N(2f(0), t) \ge N(\rho(2f(0)), t) = N\left(2f(0), \frac{t}{|\rho|}\right)$ for all $t > 0$. By (N_5) and (N_6), $N(f(0), t) = 1$ for all $t > 0$.
It follows from (N_2) that $f(0) = 0$.

Letting $y = x$ in (2.1), we get $N(f(2x) - 4f(x), t) \geq N(0, t) = 1$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} &N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) \\ &\geq N\left(\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right), t\right) \\ &= N\left(\frac{1}{2}\rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right), t\right) \\ &= N\left(f(x+y) + f(x-y) - 2f(x) - 2f(y), \frac{2t}{|\rho|}\right) \end{aligned}$$

for all $t > 0$. By (N_5) and (N_6), $N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) = 1$ for all $t > 0$. It follows from (N_2) that $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$. □

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{4}\varphi(2x, 2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and

$$\begin{aligned} &N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) \\ &\geq \min\left\{N\left(\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right), t\right), \frac{t}{t + \varphi(x, y)}\right\} \end{aligned} \tag{2.3}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(4-4L)t}{(4-4L)t + L\varphi(x, x)} \tag{2.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = x$ in (2.3), we get

$$N(f(2x) - 4f(x), t) \geq \frac{t}{t + \varphi(x, x)} \tag{2.5}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(x, x)},\ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [26, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{4}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \frac{L}{4}\varphi(x, x)} = \frac{t}{t + \varphi(x, x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (2.5) that $N\left(f(x) - 4f\left(\frac{x}{2}\right), \frac{L}{4}t\right) \geq \frac{t}{t+\varphi(x,x)}$ for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{4}$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{2.6}$$

for all $x \in X$. Since $f : X \to Y$ is even, $Q : X \to Y$ is a even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{L}{4 - 4L}.$$

This implies that the inequality (2.4) holds.

By (2.3),

$$\begin{aligned} &N\left(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right), 4^n t\right) \\ &\geq \min\left\{N\left(\rho\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right), 4^n t\right), \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}\right\} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$N\left(4^n\left(f\left(\frac{x+y}{2^n}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right),t\right)$$
$$\geq \min\left\{N\left(\rho\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right)+2f\left(\frac{x-y}{2^{n+1}}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)\right),t\right),\frac{\frac{t}{4^n}}{\frac{t}{4^n}+\frac{L^n}{4^n}\varphi(x,y)}\right\}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{4^n}}{\frac{t}{4^n}+\frac{L^n}{4^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$N(Q(x+y)+Q(x-y)-2Q(x)-2Q(y),t)$$
$$\geq N\left(\rho\left(2Q\left(\frac{x+y}{2}\right)+2Q\left(\frac{x-y}{2}\right)-Q(x)-Q(y)\right),t\right)$$

for all $x, y \in X$ and all $t > 0$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic, as desired. □

Corollary 2.3. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying*

$$N(f(x+y)+f(x-y)-2f(x)-2f(y),t)$$
$$\geq \min\left\{N\left(\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right),t\right),\frac{t}{t+\theta(\|x\|^p+\|y\|^p)}\right\}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x)-Q(x),t) \geq \frac{(2^p-4)t}{(2^p-4)t+2\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{2-p}$, and we get the desired result. □

Theorem 2.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq 4L\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.3). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x)-Q(x),t) \geq \frac{(4-4L)t}{(4-4L)t+\varphi(x,x)} \tag{2.7}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (2.5) that

$$N\left(f(x)-\frac{1}{4}f(2x),\frac{1}{4}t\right) \geq \frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{1}{4}$. Hence $d(f,Q) \leq \frac{1}{4-4L}$, which implies that the inequality (2.7) holds.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying*

$$N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) \geq \min\left\{N\left(\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right), t\right), \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}\right\}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(4-2^p)t}{(4-2^p)t + 2\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-2}$, and we get the desired result. □

3. Quadratic ρ-functional inequality (0.2)

In this section, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.2) in fuzzy Banach spaces. Let ρ be a real number with $|\rho| < \frac{1}{2}$. We need the following lemma to prove the main results.

Lemma 3.1. *Let $f : X \to Y$ be a mapping such that*

$$N\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y), t\right) \geq N(\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)), t) \tag{3.1}$$

for all $x, y \in X$ and all $t > 0$. Then f is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $N(2f(0), t) \geq N(\rho(2f(0)), t) = N\left(2f(0), \frac{t}{|\rho|}\right)$ for all $t > 0$. By (N_5) and (N_6), $N(f(0), t) = 1$ for all $t > 0$.
It follows from (N_2) that $f(0) = 0$.

Letting $y = 0$ in (3.1), we get $N\left(4f\left(\frac{x}{2}\right) - f(x), t\right) \geq N(0, t) = 1$ for all $t > 0$ and so

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} N&\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y), t\right) \\ &= N\left(\frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) - f(x) - f(y), t\right) \\ &= N(f(x+y) + f(x-y) - 2f(x) - 2f(y), 2t) \\ &\geq N(\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)), t) \\ &= N\left(f(x+y) + f(x-y) - 2f(x) - 2f(y), \frac{t}{|\rho|}\right) \end{aligned}$$

for all $t > 0$. By (N_5) and (N_6), $N(f(x+y) + f(x-y) - 2f(x) - 2f(y), t) = 1$ for all $t > 0$. It follows from (N_2) that $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$. □

Theorem 3.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le \frac{L}{4}\varphi(2x,2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying

$$N\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y), t\right) \tag{3.3}$$
$$\ge \min\left\{N\left(\rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right), t\right), \frac{t}{t+\varphi(x,y)}\right\}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \ge \frac{(1-L)t}{(1-L)t + \varphi(x,0)} \tag{3.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $x = y = 0$ in (3.3), we get $N(2f(0), t) \ge N(\rho(2f(0)), t) = N\left(2f(0), \frac{t}{|\rho|}\right)$ for all $t > 0$. So $f(0) = 0$.

Letting $y = 0$ in (3.3), we get

$$N\left(4f\left(\frac{x}{2}\right) - f(x), t\right) \ge \frac{t}{t+\varphi(x,0)} \tag{3.5}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \ge \frac{t}{t+\varphi(x,0)},\ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [26, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \ge \frac{t}{t+\varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{4}\varepsilon t\right) \\ &\ge \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \varphi\left(\frac{x}{2}, 0\right)} \ge \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \frac{L}{4}\varphi(x,0)} = \frac{t}{t+\varphi(x,0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in S$.

It follows from (3.5) that

$$N\left(f(x) - 4f\left(\frac{x}{2}\right), t\right) \geq \frac{t}{t + \varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq 1$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{3.6}$$

for all $x \in X$. Since $f : X \to Y$ is even, $Q : X \to Y$ is an even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (3.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(x,0)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{1}{1-L}.$$

This implies that the inequality (3.4) holds.

By (3.3),

$$\begin{aligned}
&N\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right), 4^n t\right) \\
&\geq \min\left\{N\left(\rho\left(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right)\right), 4^n t\right), \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}\right\}
\end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\begin{aligned}
&N\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right), t\right) \\
&\geq \min\left\{N\left(\rho\left(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right)\right), t\right), \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x,y)}\right\}
\end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$\begin{aligned}
&N\left(2Q\left(\frac{x+y}{2}\right) + 2\left(\frac{x-y}{2}\right) - Q(x) - Q(y), t\right) \\
&\geq N\left(\rho\left(Q\left(x+y\right) + Q\left(x-y\right) - 2Q(x) - 2Q(y)\right), t\right)
\end{aligned}$$

for all $x, y \in X$ and all $t > 0$. By Lemma 3.1, the mapping $Q : X \to Y$ is quadratic, as desired. □

Corollary 3.3. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying*

$$N\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y),t\right)$$
$$\geq \min\left\{N\left(\rho\left(f(x+y)+f(x-y)-2f(x)-2f(y)\right),t\right), \frac{t}{t+\theta(\|x\|^p+\|y\|^p)}\right\}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x)-Q(x),t\right) \geq \frac{(2^p-4)t}{(2^p-4)t+2^p\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x,y) := \theta(\|x\|^p+\|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{2-p}$, and we get the desired result. □

Theorem 3.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq 4L\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.3). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f\left(2^n x\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x)-Q(x),t\right) \geq \frac{(1-L)t}{(1-L)t+\varphi(x,0)} \tag{3.7}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.2.

It follows from (3.5) that

$$N\left(f(x)-\frac{1}{4}f(2x),Lt\right) \geq \frac{t}{t+\varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq L$. Hence

$$d(f,Q) \leq \frac{1}{1-L},$$

which implies that the inequality (3.7) holds.

The rest of the proof is similar to the proof of Theorem 3.2. □

Corollary 3.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying*

$$N\left(2f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)-f(x)-f(y),t\right)$$
$$\geq \min\left\{N\left(\rho\left(f\left(x+y\right)+f\left(x-y\right)-2f(x)-2f(y)\right),t\right), \frac{t}{t+\theta(\|x\|^p+\|y\|^p)}\right\}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x)-Q(x),t\right) \geq \frac{(4-2^p)t}{(4-2^p)t+2^p\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-2}$, and we get the desired result. □

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Ji-hye Kim
Department of Mathematics, Hanyang University, Seoul 133-791, Korea
E-mail address: `saharin@hanyang.ac.kr`

Choonkil Park
Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Korea
E-mail address: `baak@hanyang.ac.kr`

The Quadrature rules of the fuzzy Henstock – Stieltjes integral on a infinite interval†

Ling Wang*
School of Information Engineering, Gansu Institute of Political Science and Law, Lanzhou, Gansu 730070, P.R. China

Abstract: In this paper, the calculating methods for the fuzzy Henstock-Stieltjes integral on a infinite interval are proposed. It includes quadrature rules and the error estimates such as the midpoint-type rule, trapezoidal-type rule, Simpson's formula, $\delta-$fine quadrature rules, their error estimates, and so on. Finally, an example is given to illuminate the effectiveness the methods proposed in this paper.
Keywords: Fuzzy numbers; Fuzzy Henstock-Stieltjes integral; calculating methods
AMS subject classifications. 26E50; 28E10.

1 Introduction

It is well known that the notion of the Stieltjes integral for fuzzy-number-valued functions was originally proposed by Nanda [1] in 1989. Many generalizations of the fuzzy Riemann-Stieltjes integral were considered by scholars [2, 3, 4]. In 1998, Wu [5] proposed the concept of fuzzy Riemann-Stieltjes integral by means of the representation theorem of fuzzy-number-valued functions, whose membership function could be obtained by solving a nonlinear programming problem, but it is difficult to calculate and extend to the higher-dimensional space. In 2006, Ren et al. introduced the concept of two kinds of fuzzy Riemann-Stieltjes integral for fuzzy-number-valued functions [3, 4] and showed that a continuous fuzzy-number-valued function was fuzzy Riemann-Stieltjes integrable with respect to a real-valued increasing function. To overcome the limitations of the existing studies and to characterize continuous linear functionals on the space of Henstock integrable fuzzy-number-valued functions, the concept of the Henstock-Stieltjes integral for fuzzy-number-valued functions was defined and discussed in 2012, and some useful results for this integral were shown, such as the integrability, the continuity and the differentiability of the primitive, numerical calculus of the integration, the convergence theorems, and so on. The integral for fuzzy-number-valued functions on a infinite interval, as a expectation of fuzzy random variable, was originally investigated by Puri and Ralescu in 1986 [6]. In their opinion, a fuzzy random variable as a fuzzy-number-valued function and the expectation $E(X)$ of a fuzzy random variable X equals to a fuzzy integral $E(X) = \int X$ or set-valued integral of X_λ. In 2007, the concept of the fuzzy Henstock integral on infinite interval was proposed and discussed in order to solve the expectation $E(X)$ of a fuzzy random variable X which distribution function has some kinds of discontinuity or non-integrability by Gong and Wang [7]. After that, the Henstock-Stieltjes integral for fuzzy-number-valued functions on infinite interval which is an extension of the usual fuzzy Riemann-Stieltjes integral on infinite interval was investigated by Duan in 2014 [8], and several necessary and sufficient conditions of the integrability for fuzzy-number-valued functions are given by means of the Henstock-Stieltjes integral of real-valued functions on infinite interval and Henstock integral of fuzzy-number-valued functions on infinite interval. In this paper, we shall discuss the calculating methods for the fuzzy Henstock-Stieltjes integral on a infinite interval: one is to calculate directly by the fuzzy Henstock-Stieltjes integral on a infinite interval, including quadrature rules and the error estimates such as the midpoint-type rule, trapezoidal-type rule, Simpson's formula, $\delta-$fine quadrature rules and their error estimates; another is to calculate by using the equivalent characteristic of fuzzy Henstock-Stieltjes integrability, whose membership function could be obtained by solving a nonlinear programming problem.

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* Tel.:+8618693076970, E-mail: 18693076970@163.com

2 Preliminaries

Fuzzy set $\tilde{u} \in E^1$ is called a fuzzy number if $\tilde{u}$ is a normal, convex fuzzy set, upper semi-continuous and supp $u = \overline{\{x \in \mathbb{R} \mid u(x) > 0\}}$ is compact. Here $\bar{A}$ denotes the closure of A. We use E^1 to denote the fuzzy number space [1-6].

Let $\tilde{u}, \tilde{v} \in E^1, k \in \mathbb{R}$, the addition and scalar multiplication are defined by

$$[\tilde{u} + \tilde{v}]_\lambda = [\tilde{u}]_\lambda + [\tilde{v}]_\lambda, \qquad [k\tilde{u}]_\lambda = k[\tilde{u}]_\lambda,$$

respectively, where $[\tilde{u}]_\lambda = \{x : u(x) \geqslant \lambda\} = [u^-_\lambda, u^+_\lambda]$, for any $\lambda \in [0, 1]$.

We use the Hausdorff distance between fuzzy numbers given by $D : E^1 \times E^1 \to [0, +\infty)$ as follows [1-6]:

$$D(\tilde{u}, \tilde{v}) = \sup_{\lambda\in[0,1]} d([\tilde{u}]_\lambda, [\tilde{v}]_\lambda) = \sup_{\lambda\in[0,1]} \max\{|u^-_\lambda - v^-_\lambda|, |u^+_\lambda - v^+_\lambda|\},$$

where d is the Hausdorff metric. $D(\tilde{u}, \tilde{v})$ is called the distance between $\tilde{u}$ and $\tilde{v}$.

Recall, also, that a function $\tilde{f} : [a, b] \to E^1$ is said to be bounded if there exists $M \in \mathbb{R}$ such that $\|\tilde{f}(x)\| = D(\tilde{f}(x), \tilde{0}) \leqslant M$ for any $x \in [a, b]$. Notice that here $\|\tilde{f}(x_0)\|$ does not stand for the norm of E^1.

Definition 2.1 [7,8,9]. $\overline{R}$ denote the generalized real line, for $\tilde{f}$ defined on $[a, +\infty]$, we define $\tilde{f}(+\infty) = \tilde{0}$, and $\tilde{0} \cdot (+\infty) = \tilde{0}$.

Let $\delta : [a, +\infty] \to R^+$ be a positive real function. A division $P = \{[x_{i-1}, x_i]; \xi_i\}$ is said to be δ-fine, if the following conditions are satisfied:
(1)$a = x_0 < x_1 < ... < x_{n-1} = b < x_n = +\infty$;
(2)$\xi_i \in [x_{i-1}, x_i] \subset O(\xi_i), i = 1, 2, ..., n$;
where $O(\xi_i) = (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for $i = 1, 2, ..., n - 1$, and $O(\xi_n) = [b, +\infty)$.

For brevity, we write $T = \{[u, v]; \xi\}$, where $[u, v]$ denotes a typical interval in T and ξ is the associated point of $[u, v]$.

Definition 2.2 [8]. Let $\alpha : [a, +\infty] \to \mathbb{R}$ be an increasing function. A fuzzy-number-valued function $\tilde{f}(x)$ is said to be fuzzy Henstock-Stieltjes integrable with respect to α on $[a, +\infty]$ if there exists a fuzzy number $\tilde{H} \in E^1$ such that for every $\varepsilon > 0$, there is a function $\delta(x) > 0$ on $[a, +\infty]$ such that for any δ-fine division $T = \{[x_{i-1}, x_i]; \xi_i\}_{i=1}^n$, we have

$$D(\sum_{i=1}^n [\alpha(x_i) - \alpha(x_{i-1})]\tilde{f}(\xi_i), \tilde{H}) < \varepsilon.$$

We write $(FHS) \int_a^{+\infty} \tilde{f}(x)\mathrm{d}\alpha = \tilde{H}$ and $(\tilde{f}, \alpha) \in FHS[a, +\infty]$.

The definition of $\tilde{f} \in FHS(-\infty, a]$ is similar. Naturally, we define $\tilde{f} \in FHS(-\infty, +\infty)$ iff $\tilde{f} \in FHS(-\infty, a]$ and $\tilde{f} \in FHS[a, +\infty)$, and furthermore

$$(FHS) \int_{-\infty}^{+\infty} \tilde{f}(x)d\mathrm{d}\alpha = (FHS) \int_{-\infty}^{a} \tilde{f}(x)d\mathrm{d}\alpha + (FHS) \int_a^{+\infty} \tilde{f}(x)d\mathrm{d}\alpha.$$

For brevity, we always assume that $\alpha : [a, +\infty] \to \mathbb{R}$ is an increasing function.

Lemma 2.1[8]. Let $\alpha : [a, +\infty] \to \mathbb{R}$ be an increasing function and let $\tilde{f} : [a, +\infty] \to E^1$. Then the following statements are equivalent:

(1) $(\tilde{f}, \alpha) \in FHS[a, +\infty]$ and $(FHS) \int_a^{+\infty} \tilde{f}(x)\mathrm{d}\alpha = \tilde{A}$;

(2) for any $\lambda \in [0, 1]$, f^-_λ and f^+_λ are Henstock-Stieltjes integrable with respect to α on $[a, +\infty]$ for any $\lambda \in [0, 1]$ uniformly ($\delta(x)$ is independent of $\lambda \in [0, 1]$), and

$$[(FHS) \int_a^{+\infty} \tilde{f}(x)\mathrm{d}\alpha]_\lambda = [(HS) \int_a^{+\infty} f^-_\lambda(x)\mathrm{d}\alpha, (HS) \int_0^{+\infty} f^+_\lambda(x)\mathrm{d}\alpha].$$

(3) For any $b > a$, $\tilde{f} \in FHS[a, b]$, $\lim\limits_{b\to+\infty} \int_a^b \tilde{f}(x)\mathrm{d}\alpha$ as a fuzzy number exists and

$$\lim_{b\to+\infty} \int_a^b \tilde{f}(x)\mathrm{d}\alpha = \int_a^{+\infty} \tilde{f}(x)\mathrm{d}\alpha.$$

3 Quadrature rules of the Henstock-Stieltjes integral for fuzzy-number-valued functions on infinite interval

We shall use the modulus of oscillation for a fuzzy-valued function to discuss the quadrature rules of expectations for fuzzy random variables in this section. For the numerical calculus of fuzzy integral, there were some discussions by the fuzzy Riemann integral, improper fuzzy Riemann integral, using the probabilistic Monte Carlo method, and the quadrature rules for fuzzy Henstock integral on a finite interval[1, 3, 4]. However, the calculus above will be restricted when the distribution function of a random variable on $(-\infty,+\infty)$ or the distribution function of a random variable has some kind of discontinuity or non-integrability. Furthermore, fuzzy Henstock integral is convenient for numerical calculus since it is a Riemann-type integral. Since a fuzzy random variable is a measurable fuzzy-valued function $\tilde{f}:(-\infty,+\infty)\to E^1$, therefore without loss of the generality, we only discuss the quadrature rules of Henstock integrals for the measurable fuzzy-valued functions on $[a,+\infty)$. For a fuzzy-valued function, since its Henstock integrability implies measurability, for brevity we always assume that the fuzzy-valued functions discussed are measurable throughout this section.

Definition 3.1[7, 10]. Let $\tilde{f}:[a,+\infty)\to E^1$ be a bounded mapping. Then the function $\omega_{[a,+\infty)}(\tilde{f},\cdot): R^+\bigcup\{0\}\to R^+$,

$$\omega_{[a,+\infty)}(\tilde{f},\delta)=\sup\{D(\tilde{f}(x),\tilde{f}(y)):x,y\in[a,+\infty),|x-y|\le\delta\}$$

is called the modulus of oscillation of $\tilde{f}$ on $[a,+\infty)$.

Theorem 3.1[7] Obviously, the following statements hold:

(i) $D(\tilde{f}(x),\tilde{f}(y))\le\omega_{[a,+\infty)}(\tilde{f},|x-y|),\forall x,y\in[a,+\infty)$ for any $x,y\in[a,+\infty)$;

(ii) $\omega_{[a,+\infty)}(\tilde{f},\delta)$ is nondecreasing mapping in δ and nonincreasing in a;

(iii) $\omega_{[a,+\infty)}(\tilde{f},0)=0$;

(iv) $\omega_{[a,+\infty)}(\tilde{f},\delta_1+\delta_2)\le\omega_{[a,+\infty)}(\tilde{f},\delta_1)+\omega_{[a,+\infty)}(\tilde{f},\delta_2)$ for any $\delta_1,\delta_2\ge0$;

(v) $\omega_{[a,+\infty)}(\tilde{f},n\delta)\le n\omega_{[a,+\infty)}(\tilde{f},\delta)$ for any $\delta\ge0,n\in N$;

(vi) $\omega_{[a,+\infty)}(\tilde{f},\lambda\delta)\le(\lambda+1)\omega_{[a,+\infty)}(\tilde{f},\delta)$ for any $\delta\ge0,\lambda\ge0$.

Theorem 3.2 Let $\tilde{f}\in FHS[a,+\infty)$ be a bounded function, and $\alpha:[a,+\infty]\to\mathbb{R}$ an increasing function. Then for any division $T: a=x_0<x_1<...<x_{n-1}=b<x_n=+\infty$ and any point $\xi_i\in[x_{i-1},x_i], i=1,2,3,...,n-1$, and $\xi_n=+\infty$, we have

$$D(\int_a^{+\infty}\tilde{f}(x)d\alpha,\sum_{i=1}^{n}[\alpha(x_i)-\alpha(x_{i-1})]\tilde{f}(\xi_i))\le\sum_{i=1}^{n-1}[\alpha(x_i)-\alpha(x_{i-1})]\omega_{[x_{i-1},x_i]}(\tilde{f},x_i-x_{i-1})+\alpha_b,$$

where α_b stands for $\|\int_b^{+\infty}\tilde{f}(x)d\alpha\|_{E^1}$ or $\int_b^{+\infty}\|\tilde{f}(x)\|_{E^1}d\alpha$, and $\alpha_b\to0$ $(b\to+\infty)$.

Proof. The subinterval including $+\infty$ is denoted by $[b,+\infty](x_{n-1}=b,x_n=+\infty)$, according to the additivity of interval for fuzzy Henstock-Stieltjes integral, we have $\int_a^{+\infty}\tilde{f}(x)d\alpha=\int_a^b\tilde{f}(x)d\alpha+$

$\int_b^{+\infty}\tilde{f}(x)d\alpha$, and

$$\begin{aligned}
&D(\int_a^{+\infty}\tilde{f}(x)d\alpha,\sum_{i=1}^{n}[\alpha(x_i)-\alpha(x_{i-1})]\tilde{f}(\xi_i))\\
&\leq D(\int_a^{b}\tilde{f}(x)d\alpha,\sum_{i=1}^{n-1}[\alpha(x_i)-\alpha(x_{i-1})]\tilde{f}(\xi_i))+D(\int_b^{+\infty}\tilde{f}(x)d\alpha,\int_b^{+\infty}\tilde{f}(\xi_n)d\alpha,)\\
&\leq \sum_{i=1}^{n-1}[\alpha(x_i)-\alpha(x_{i-1})]\omega_{[x_{i-1},x_i]}(\tilde{f},x_i-x_{i-1})+D(\int_b^{+\infty}\tilde{f}(x),\tilde{0})d\alpha\\
&\leq \sum_{i=1}^{n}[\alpha(x_i)-\alpha(x_{i-1})]\omega_{[x_{i-1},x_i]}(\tilde{f},x_i-x_{i-1})+\alpha_b,
\end{aligned}$$

where $\tilde{f}(\xi_n)=\tilde{0}$. By Lemma 2.1, $\alpha_b\to 0$ when $b\to+\infty$.
The proof is complete.

Taking in Theorem 3.2 $n=2, x_1=\xi_1=\xi_2=x$; $n=2, x_1=x, \xi_1=u, \xi_2=v$ and $n=4, x_1=\alpha, x_2=\beta, \xi_1=u, \xi_2=v, \xi_3=w$ respectively, we obtain the midpoint-type, trapezoidal-type and Simpson's inequalities in some sense with its error estimations as follows.

Corollary 3.1 Let $\tilde{f}\in FHS[a,+\infty)$ be a bounded function. Then
(i)

$$D(\int_a^{+\infty}\tilde{f}(x)d\alpha,[\alpha(b)-\alpha(a)]\tilde{f}(x))\leq[\alpha(x)-\alpha(a)]\omega_{[a,x]}(\tilde{f},x-a)+[\alpha(b)-\alpha(x)]\omega_{[x,b]}(\tilde{f},b-x)+\alpha_b$$

for any $b\geq a$ and $x\in[a,b]$;
(ii)

$$\begin{aligned}
&D(\int_a^{+\infty}\tilde{f}(x)d\alpha,[\alpha(x)-\alpha(a)]\tilde{f}(u)+[\alpha(b)-\alpha(x)]\tilde{f}(v))\\
&\leq[\alpha(x)-\alpha(a)]\omega_{[a,x]}(\tilde{f},x-a)+[\alpha(b)-\alpha(x)]\omega_{[x,b]}(\tilde{f},b-x)+\alpha_b
\end{aligned}$$

for any $b\geq a$ and $x\in[a,b], u\in[a,x], v\in[x,b]$;
(iii)

$$\begin{aligned}
&D(\int_a^{+\infty}\tilde{f}(x)d\alpha,[\alpha(\beta_1)-\alpha(a)]\tilde{f}(u)+[\alpha(\beta_2)-\alpha(\beta_1)]\tilde{f}(v)+[\alpha(b)-\alpha(\beta_2)]\tilde{f}(w))\\
&\leq[\alpha(\beta_1)-\alpha(a)]\omega_{[a,\alpha]}(\tilde{f},\beta_1-a)+[\alpha(\beta_2)-\alpha(\beta_1)]\omega_{[\beta_1,\beta_2]}(\tilde{f},\beta_2-\beta_1)\\
&+[\alpha(b)-\alpha(\beta_2)]\omega_{[\beta_2,b]}(\tilde{f},b-\beta_2)+\alpha_b
\end{aligned}$$

for any $b\geq a$, $\alpha,\beta\in[a,b]$, and $u\in[a,\beta_1]$, $v\in[\beta_1,\beta_2]$, $w\in[\beta_2,b]$, where α_b stands for $\|\int_b^{+\infty}\tilde{f}(x)d\alpha\|_{E^1}$ or $\int_b^{+\infty}\|\tilde{f}(x)\|_{E^1}d\alpha$, and $\alpha_b\to 0$ $(b\to+\infty)$.

Corollary 3.2 Let $\tilde{f}\in FHS[a,+\infty)$ be a bounded function. Then
(i)

$$\begin{aligned}
&D(\int_a^{+\infty}\tilde{f}(x)d\alpha,[\alpha(b)-\alpha(a)]\tilde{f}(\frac{a+b}{2}))\\
&\leq[\alpha(b)-\alpha(a)]\omega_{[a,b]}(\tilde{f},\frac{b-a}{2})+\alpha_b\\
&\leq[\alpha(b)-\alpha(a)]\omega_{[a,+\infty)}(\tilde{f},\frac{b-a}{2})+\alpha_b;
\end{aligned}$$

(ii)

$$D(\int_a^{+\infty}\tilde{f}(x)d\alpha,[\alpha(\frac{b+a}{2})-\alpha(a)]\tilde{f}(a)+[\alpha(b)-\alpha(\frac{b+a}{2})]\tilde{f}(b))$$
$$\leq[\alpha(\frac{b+a}{2})-\alpha(a)]\omega_{[a,\frac{b+a}{2}]}(\tilde{f},\frac{b-a}{2})+[\alpha(b)-\alpha(\frac{b+a}{2})]\omega_{[\frac{b+a}{2},b]}(\tilde{f},\frac{b-a}{2})+\alpha_b;$$

(iii)

$$D(\int_a^{+\infty}\tilde{f}(x)d\alpha,[\alpha(\frac{2a+b}{3})-\alpha(a)]\tilde{f}(a)+[\alpha(\frac{a+2b}{3})-\alpha(\frac{2a+b}{3})]\tilde{f}(\frac{a+b}{2})+[\alpha(b)-\alpha(\frac{a+2b}{3})]\tilde{f}(b))$$
$$\leq[\alpha(\frac{2a+b}{3})-\alpha(a)]\omega_{[a,\frac{2a+b}{3}]}(\tilde{f},\frac{b-a}{3})+[\alpha(\frac{a+2b}{3})-\alpha(\frac{2a+b}{3})]\omega_{[\frac{2a+b}{3},[\frac{a+2b}{3}]}(\tilde{f},\frac{b-a}{3})$$
$$+[\alpha(b)-\alpha(\frac{a+2b}{3})]\omega_{[\frac{a+2b}{3},b]}(\tilde{f},\frac{b-a}{3})+\alpha_b,$$

where α_b stands for $\|\int_b^{+\infty}\tilde{f}(x)d\alpha\|_{E^1}$ or $\int_b^{+\infty}\|\tilde{f}(x)\|_{E^1}d\alpha$, and $\alpha_b\to 0$ $(b\to+\infty)$.

Using Theorem 3.2, we can also obtain another numerical calculous of Henstock-Stieltjes integrals with error estimations.

Corollary 3.3 Let $\tilde{f}\in FHS[a,+\infty)$ be a bounded function. Then

(1)

$$D(\int_a^{+\infty}\tilde{f}(x)d\alpha,\sum_{i=1}^{n}[\alpha(x_i)-\alpha(x_{i-1})]\tilde{f}(\xi_i))$$
$$\leq[\alpha(b)-\alpha(a)]\omega_{[a,b]}(\tilde{f},\|T\|)+\alpha_b$$
$$\leq[\alpha(b)-\alpha(a)]\omega_{[a,+\infty)}(\tilde{f},\|T\|)+\alpha_b;$$

(2)

$$D(\int_a^{+\infty}\tilde{f}(x)d\alpha,\sum_{i=1}^{n}[\alpha(x_i)-\alpha(x_{i-1})]\tilde{f}(\xi_i))$$
$$\leq\|\alpha(T)\|\sum_{i=1}^{n-1}\omega_{[a,b]}(\tilde{f},x_i-x_{i-1})+\alpha_b$$
$$\leq\|\alpha(T)\|\sum_{i=1}^{n-1}\omega_{[a,+\infty)}(\tilde{f},x_i-x_{i-1})+\alpha_b;$$

(3) If $\alpha:[a,b]\to\mathbb{R}$ is an increasing function satisfying $\alpha\in C^1[a,+\infty]$, then

$$D(\int_a^{+\infty}\tilde{f}(x)d\alpha,\sum_{i=1}^{n}[\alpha(x_i)-\alpha(x_{i-1})]\tilde{f}(\xi_i))$$
$$\leq M\|T\|\sum_{i=1}^{n-1}\omega_{[a,b]}(\tilde{f},x_i-x_{i-1})+\alpha_b$$

for any division $T: a=x_0<x_1<...<x_{n-1}=b<x_n=+\infty$ and any point $\xi_i\in[x_{i-1},x_i], i=1,2,...n-1$ $\xi_n=+\infty$, where α_b stands for $\|\int_b^{+\infty}\tilde{f}(x)d\alpha\|_{E^1}$ or $\int_b^{+\infty}\|\tilde{f}(x)\|_{E^1}d\alpha$, $\alpha_b\to 0$ $(b\to+\infty)$, $\|T\|=\max\{x_i-x_{i-1}: i=1,2,...n-1\}$ denotes the modulus of division T, $\|\alpha(T)\|=\max\{\alpha(x_i)-\alpha(x_{i-1}): i=1,2,...n-1\}$, and M is the bound of α on $[a,b]$.

Proof. By using Theorem 3.2 and Theorem 3.1, (1) and (2) are obvious. We only prove that (3) holds. In fact, we have

$$\begin{aligned}
&D(\int_a^{+\infty} \tilde{f}(x)d\alpha, \sum_{i=1}^{n}[\alpha(x_i) - \alpha(x_{i-1})]\tilde{f}(\xi_i)) \\
&\leq \sum_{i=1}^{n-1}[\alpha(x_i - \alpha(x_{i-1})]\omega_{[x_{i-1},x_i]}(\tilde{f}, x_i - x_{i-1}) + \alpha_b \\
&= \sum_{i=1}^{n-1}\alpha'(\zeta_i)(x_i - x_{i-1})\omega_{[x_{i-1},x_i]}(\tilde{f}, x_i - x_{i-1}) + \alpha_b \\
&\leq M\|T\| \sum_{i=1}^{n-1}\omega_{[a,b]}(\tilde{f}, x_i - x_{i-1}) + \alpha_b \\
&\leq M\|T\| \sum_{i=1}^{n-1}\omega_{[a,+\infty)}(\tilde{f}, x_i - x_{i-1}) + \alpha_b.
\end{aligned}$$

4. $\delta-$ fine quadrature rules of the Henstock-Stieltjes integral for fuzzy-number-valued functions on infinite interval

Definition 4.1 Let $S_n = \sum_{i=1}^{n}[\alpha(x_i) - \alpha(x_{i-1})]\tilde{f}(\xi_i)$ be a quadrature rule and $\delta : [a, +\infty] \to R^+$. S_n is said to be a $\delta-$fine quadrature rule, if $\xi_i \in [x_{i-1}, x_i] \subset O(\xi_i), i = 1, 2, ..., n$, where $O(\xi_i) = (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ for $i = 1, 2, ..., n-1$, and $O(\xi_n) = [b, +\infty)$.

We can deduce expressions for the remainder of $\delta-$fine quadrature rules by using Theorem 3.2 and Theorem 3.1(ii,v) as follows.

Theorem 4.1 Let $\tilde{f} \in FHS[a, +\infty)$ be a bounded function. If $S_n = \sum_{i=1}^{n}[\alpha(x_i) - \alpha(x_{i-1})]\tilde{f}(\xi_i)$ is a $\delta-$fine quadrature rule, then

$$D(\int_a^{+\infty} \tilde{f}(x)d\alpha, S_n) \leq 2\sum_{i=1}^{n-1}[\alpha(x_i) - \alpha(x_{i-1})]\delta(\xi_i)\omega_{[x_{i-1}x_i]}(\tilde{f}, \delta(\xi_i)) + \alpha_b.$$

Here α_b stands for $\| \int_b^{+\infty} \tilde{f}(x)d\alpha\|_{E^1}$ or $\int_b^{+\infty} \|\tilde{f}(x)\|_{E^1}d\alpha$, and $\alpha_b \to 0$ $(b \to +\infty)$.

Theorem 4.2 Let $\tilde{f} \in FHS[a, +\infty)$ be a bounded function, and $\alpha : [a, b] \to \mathbb{R}$ be an increasing function such that $\alpha \in C^1[a, +\infty]$. If $S_n = \sum_{i=1}^{n}[\alpha(x_i) - \alpha(x_{i-1})]\tilde{f}(\xi_i)$ is a $\delta-$fine quadrature rule, then

$$D(\int_a^{+\infty} \tilde{f}(x)d\alpha, S_n) \leq 4M\sum_{i=1}^{n-1}\delta(\xi_i)\omega_{[x_{i-1}x_i]}(\tilde{f}, \delta(\xi_i)) + \alpha_b.$$

Here M is the bound of α' on $[a, b]$, α_b stands for $\| \int_b^{+\infty} \tilde{f}(x)d\alpha\|_{E^1}$ or $\int_b^{+\infty} \|\tilde{f}(x)\|_{E^1}d\alpha$, and $\alpha_b \to 0$ $(b \to +\infty)$.

Proof By using Theorem 3.2 and Theorem 3.1(ii,v), we have

$$\begin{aligned}
&D(\int_a^{+\infty}\tilde{f}(x)d\alpha, S_n)\\
&\leq 2\sum_{i=1}^{n-1}[\alpha(x_i)-\alpha(x_{i-1})]\delta(\xi_i)\omega_{[x_{i-1}x_i]}(\tilde{f},\delta(\xi_i))+\alpha_b\\
&\leq 2\sum_{i=1}^{n-1}\alpha'(\zeta_i)(x_i-x_{i-1})\delta(\xi_i)\omega_{[x_{i-1}x_i]}(\tilde{f},\delta(\xi_i))+\alpha_b\\
&\leq 4M\sum_{i=1}^{n-1}\delta(\xi_i)\omega_{[x_{i-1}x_i]}(\tilde{f},\delta(\xi_i))+\alpha_b.
\end{aligned}$$

Corollary 4.1 Let $\tilde{f}\in FHS[a,+\infty)$ be a bounded function, $\alpha:[a,b]\to\mathbb{R}$ an increasing function such that $\alpha\in C^1[a,+\infty]$, M the bound of α' on $[a,b]$. Then
(i)

$$\begin{aligned}
&D(\int_a^{+\infty}\tilde{f}(x)d\alpha,[\alpha(b)-\alpha(a)]\tilde{f}(x))\\
&\leq 4M\delta(x)\omega_{[a,b]}(\tilde{f},\delta(x))+\alpha_b;
\end{aligned}$$

for any $x\in[a,b]$ such that the quadrature rule $[\alpha(b)-\alpha(a)]\tilde{f}(x)$ is $\delta-$fine;
(ii)

$$\begin{aligned}
&D(\int_a^{+\infty}\tilde{f}(x)d\alpha,[\alpha(x)-\alpha(a)]\tilde{f}(u)+[\alpha(b)-\alpha(x)]\tilde{f}(v))\\
&\leq 4M[\delta(u)\omega_{[a,x]}(\tilde{f},\delta(u)+\delta(v)\omega_{[x,b]}(\tilde{f},\delta(v))]+\alpha_b;
\end{aligned}$$

for any $x\in[a,b], u\in[a,x]$ and $v\in[x,b]$ such that the trapezoidal-type quadrature rule $[\alpha(x)-\alpha(a)]\tilde{f}(u)+[\alpha(b)-\alpha(x)]\tilde{f}(v))$ is $\delta-$fine;
(iii)

$$\begin{aligned}
&D(\int_a^{+\infty}\tilde{f}(x)d\alpha,[\alpha(\beta_1)-\alpha(a)]\tilde{f}(u)+[\alpha(\beta_2)-\alpha(\beta_1)]\tilde{f}(v)+[\alpha(b)-\alpha(\beta_2)]\tilde{f}(w))\\
&\leq 4M[\delta(u)\omega_{[a,b]}(\tilde{f},\delta(u))+\delta(v)\omega_{[a,b]}(\tilde{f},\delta(v))+\delta(w)\omega_{[a,b]}(\tilde{f},\delta(w))]+\alpha_b
\end{aligned}$$

for any $\beta_1,\beta_2\in[a,b]$, and $u\in[a,\beta_1], v\in[\beta_1,\beta_2], w\in[\beta_2,b]$, such that Simpson's formula is $\delta-$fine. Here α_b stands for $\|\int_b^{+\infty}\tilde{f}(x)d\alpha\|_{E^1}$ or $\int_b^{+\infty}\|\tilde{f}(x)\|_{E^1}d\alpha$, and $\alpha_b\to 0$ $(b\to+\infty)$.

The following theorem shows that $\delta-$fine quadrature rules converge for the bounded Henstock-Stieltjes integrable functions.

Theorem 4.3 Let $\tilde{f}\in FHS[a,+\infty)$ be a bounded function. Then there exist functions δ_n : $[a,+\infty]\to R^+$ and a sequence of δ_n-fine quadrature rules $S_n=\sum_{i=1}^{m_n}[\alpha(x_i)-\alpha(x_{i-1})]\tilde{f}(\xi_i)$ such that S_n converges to $\int_a^{+\infty}\tilde{f}(x)d\alpha$.

Proof. From the definition of Henstock-Stieltjes integrability on infinite interval for all $\varepsilon>0$ there exists a function δ such that for any $\delta-$fine division(which can be interpreted as a $\delta-$fine quadrature rule), we have

$$D(\int_a^{+\infty}\tilde{f}(x)d\alpha, S_n)<\varepsilon.$$

Taking $\varepsilon=\frac{1}{n}$ in the inequality we obtain that the statement of the theorem holds. The proof is complete.

Corollary 4.2 Let $\tilde{f} \in FHS[a, +\infty)$ be a bounded function. Then for any natural number n, there exist functions $\delta_n : [a, +\infty] \to R^+$, $b_n \geq a$, and a sequence of δ_n-fine quadrature rules $S_n = \sum_{i=1}^{m_n}[\alpha(x_i) - \alpha(x_{i-1})]\tilde{f}(\xi_i)$ such that

$$D(\int_a^{+\infty} \tilde{f}(x)d\alpha, S_n) < \varepsilon.$$

5. Examples

Example 5.1 Let $\tilde{f} : [1, +\infty] \to E^1$ be given by

$$\tilde{f}(x,s) = \begin{cases} s, & s \in [0,1], x \text{ is rational}, \\ 0, & s \in (-\infty, 0) \cup (1, +\infty), x \text{ is rational}, \\ 1 - \frac{s}{e^{-x^2}}, & s \in [0, e^{-x^2}], x \text{ is irrational}, \\ 0, & s \in (-\infty, 0) \cup (e^{-x^2}, +\infty), x \text{ is irrational}, \\ 1, & s = 0, x = +\infty, \\ 0, & s \in (-\infty, 0) \cup (0, +\infty), x = +\infty, \end{cases}$$

and $\alpha(x) = x$.

We could prove that $\tilde{f}$ is (FHS) integrable on $[0, +\infty)$ according to the equivalence of fuzzy (HS) integrability and uniform (HS) integrability of f_λ^- and f_λ^+. Furthermore, $\int_0^{+\infty} \tilde{f}(x)d\alpha = \tilde{H}$, and $\delta-$fine quadrature rule $S_n = \sum_{i=1}^{n}[\alpha(x_i) - \alpha(x_{i-1})]\tilde{f}(\xi_i)$ converges to fuzzy number $\tilde{H}$ which relationship function is defined by

$$H(s) = \begin{cases} 1 - \frac{2}{\sqrt{\pi}}s, & s \in [0, \frac{\sqrt{\pi}}{2}], \\ 0, & x \text{ is others}, \end{cases}$$

That is to say, $H_\lambda^- = 0$, $H_\lambda^+ = (1-\lambda)\frac{\sqrt{\pi}}{2}$. In fact, we note that

$$f_\lambda^-(x) = \begin{cases} \lambda, & x \text{ is rational}, \\ 0, & x = +\infty, \\ 0, & x \text{ is irrational}, \end{cases} \quad f_\lambda^+(x) = \begin{cases} 1, & x \text{ is rational}, \\ 0, & x = +\infty, \\ (1-\lambda)e^{-x^2}, & x \text{ is irrational}. \end{cases}$$

Since $f_\lambda^+(x) \leq e^{-x^2}$, f_λ^-, f_λ^+ are Henstock integrable uniformly for $\lambda \in [0, 1]$ and

$$\int_0^{+\infty} f_\lambda^-(x)d\alpha = 0(+\infty) = 0, \int_0^{+\infty} f_\lambda^+(x)d\alpha = \lim_{b\to+\infty}\int_0^b f_\lambda^+(x)d\alpha = (1-\lambda)\frac{\sqrt{\pi}}{2}.$$

It follows that

$$\int_0^{+\infty} \tilde{f}(x)d\alpha = \tilde{H}.$$

For any $\varepsilon > 0$, we define

$$\delta(\xi) = \begin{cases} \frac{\varepsilon}{2^{i+2}}, & \xi = r_i, \\ \frac{\varepsilon}{4}\xi, & otherwise, \end{cases}$$

where $Q = \{r_1, r_2, r_3, ...\}$ stands for the set of all rational numbers on $[0, +\infty)$ and for any $\delta-$fine division $T : 1 = x_0 < x_1 < ... < x_{n-1} = b < x_n = +\infty(\xi_n = +\infty, \tilde{f}(\xi_n) = (0,0,0))$, then $S_n = \sum_{i=1}^{n}[\alpha(x_i) - \alpha(x_{i-1})]\tilde{f}(\xi_i)$ a any $\delta-$fine quadrature rule. Note that $\omega_{[x_{i-1}x_i]}(\tilde{f}, \delta(\xi_i)) = 1$. Then we have the following results.

(1) According to Theorem 3.2, we have

$$\begin{aligned}
&D(S_n, \tilde{H}) \\
&\leq \sum_{i=1}^{n-1}[\alpha(x_i) - \alpha(x_{i-1})]\omega_{[x_{i-1},x_i]}(\tilde{f}, x_i - x_{i-1}) + \alpha_b \\
&= \sum_{i=1}^{n-1}[\alpha(x_i) - \alpha(x_{i-1})] + \alpha_b \\
&= b + \alpha_b,
\end{aligned}$$

where $\alpha_b = \| \int_b^{+\infty} \tilde{f}(x)d\alpha\|_{E^1}$ or $\int_b^{+\infty} \|\tilde{f}(x)\|_{E^1}d\alpha$, and $\alpha_b \to 0$ $(b \to +\infty)$. Indeed,

$$\alpha_b = \int_b^{+\infty} \|\tilde{f}(x)\|_{E^1}d\alpha = \sup_{\lambda\in[0,1]}\{\int_b^{+\infty}(1-\lambda)e^{-x^2}\},$$

and $\alpha_b \to 0$.

(2) According to Theorem 4.2, we have

$$\begin{aligned}
&D(S_n, \tilde{H}) \\
&\leq \sum_{i=1}^{n-1}\delta(\xi_i)\omega_{[x_{i-1}x_i]}(\tilde{f}, \delta(\xi_i)) + \| \int_b^{+\infty} \tilde{f}(x)dx\|_{E^1} \\
&=\leq 4\sum_{i=1}^{n-1}\delta(\xi_i) + \alpha_b.
\end{aligned}$$

(iii) According to Corollary 4.1, we have

(i)

$$D((b-0)\tilde{f}(x), \tilde{H}) \leq 4\delta(x) + \alpha_b$$

for any $x \in [0, b]$ such that the quadrature rule $(b-0)\tilde{f}(x)$ is $\delta-$fine;

(ii)

$$D((x-0)\tilde{f}(u) + (b-x)\tilde{f}(v), \tilde{H}) \leq 4(\delta(u) + \delta(v)) + \alpha_b$$

for any $x \in [0, b], u \in [0, x]$ and $v \in [x, b]$ such that the trapezoidal-type quadrature rule $(x-0)\tilde{f}(u) + (b-x)\tilde{f}(v)$ is $\delta-$fine;

(iii)

$$\begin{aligned}
&D((\beta_1 - 0)\tilde{f}(u) + (\beta_2 - \beta_1)\tilde{f}(v) + (b - \beta_2)\tilde{f}(w), \tilde{H}) \\
&\leq 4(\delta(u) + \delta(v) + \delta(w)) + \alpha_b
\end{aligned}$$

for any $\alpha, \beta \in [0, b]$, and $u \in [0, \alpha], v \in [\alpha, \beta], w \in [\beta, b]$, such that Simpson's formula is $\delta-$fine, where $\alpha_b = \| \int_b^{+\infty} \tilde{f}(x)d\alpha\|_{E^1}$ or $\int_b^{+\infty} \|\tilde{f}(x)\|_{E^1}d\alpha$, and $\alpha_b \to 0$ $(b \to +\infty)$. Indeed,

$$\alpha_b = \int_b^{+\infty} \|\tilde{f}(x)\|_{E^1}d\alpha = \sup_{\lambda\in[0,1]}\{\int_b^{+\infty}(1-\lambda)e^{-x^2}\},$$

and $\alpha_b \to 0$.

5. Conclusion

We have discussed the numerical calculus of the fuzzy Henstock-Stieltjes integral for fuzzy-valued functions on $[a, +\infty)$. It is well known that the quadrature rules and numerical calculus are restricted when the distribution function of a random variable is unbounded, defined on $(-\infty, +\infty)$ or have some kind of non-integrability in the previous papers, however, applying the methods proposed in this paper, the problems mentioned above are solved. It includes quadrature rules and the error estimates, such as the midpoint-type rule, trapezoidal-type rule, Simpson's rule, $\delta-$ fine formula and their error estimates, and so on.

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CUBIC AND QUARTIC ρ-FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES

JOOHO ZHIANG, JEONGHUN CHU*, GEORGE A. ANASTASSIOU, AND CHOONKIL PARK*

ABSTRACT. In this paper, we solve the following cubic ρ-functional inequality

$$\begin{aligned} & N\left(f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x),t\right) \\ & \geq N\left(\rho\left(4f\left(x+\frac{y}{2}\right)+4f\left(x-\frac{y}{2}\right)-f(x+y)-f(x-y)-6f(x)\right),t\right) \end{aligned} \tag{0.1}$$

in fuzzy normed spaces, where ρ is a fixed real number with $|\rho|<2$, and the following quartic ρ-functional inequality

$$\begin{aligned} & N\left(f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y),t\right) \\ & \geq N\left(\rho\left(8f\left(x+\frac{y}{2}\right)+8f\left(x-\frac{y}{2}\right)-2f(x+y)-2f(x-y)-12f(x)+3f(y)\right),t\right) \end{aligned} \tag{0.2}$$

in fuzzy normed spaces, where ρ is a fixed real number with $|\rho|<2$.

Using the fixed point method, we prove the Hyers-Ulam stability of the cubic ρ-functional inequality (0.1) and the quartic ρ-functional inequality (0.2) in fuzzy Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

Katsaras [20] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [11, 24, 50]. In particular, Bag and Samanta [2], following Cheng and Mordeson [8], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [23]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 29, 30] to investigate the Hyers-Ulam stability of cubic ρ-functional inequalities and quartic ρ-functional inequalities in fuzzy Banach spaces.

Definition 1.1. [2, 29, 30, 31] Let X be a real vector space. A function $N : X\times\mathbb{R}\to[0,1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t\in\mathbb{R}$,

(N_1) $N(x,t)=0$ for $t\leq 0$;
(N_2) $x=0$ if and only if $N(x,t)=1$ for all $t>0$;
(N_3) $N(cx,t)=N(x,\frac{t}{|c|})$ if $c\neq 0$;
(N_4) $N(x+y,s+t)\geq\min\{N(x,s),N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty}N(x,t)=1$.
(N_6) for $x\neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X,N) is called a *fuzzy normed vector space.*

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [28, 29].

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*Corresponding author.

Definition 1.2. [2, 29, 30, 31] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N\text{-}\lim_{n\to\infty} x_n = x$.

Definition 1.3. [2, 29, 30, 31] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [3]).

The stability problem of functional equations originated from a question of Ulam [49] concerning the stability of group homomorphisms. Hyers [15] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [41] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [12] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [7, 16, 19, 21, 22, 25, 37, 38, 39, 43, 44, 45, 46, 47, 48]).

In [18], Jun and Kim considered the following cubic functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x). \tag{1.1}$$

It is easy to show that the function $f(x) = x^3$ satisfies the functional equation (1.1), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*.

In [26], Lee et al. considered the following quartic functional equation

$$f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y). \tag{1.2}$$

It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.2), which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*.

Gilányi [13] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x - y)\| \leq \|f(x + y)\| \tag{1.3}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x + y) + f(x - y).$$

See also [42]. Fechner [10] and Gilányi [14] proved the Hyers-Ulam stability of the functional inequality (1.3). Park, Cho and Han [36] investigated the Cauchy additive functional inequality

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\| \tag{1.4}$$

and the Cauchy-Jensen additive functional inequality

$$\|f(x)+f(y)+2f(z)\| \leq \left\|2f\left(\frac{x+y}{2}+z\right)\right\| \tag{1.5}$$

and proved the Hyers-Ulam stability of the functional inequalities (1.4) and (1.5) in Banach spaces.

Park [34, 35] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.4. [4, 9] Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [17] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 28, 32, 33, 39, 40]).

In Section 2, we solve the cubic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the cubic ρ-functional inequality (0.1) in fuzzy Banach spaces by using the fixed point method.

In Section 3, we solve the quartic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the quartic ρ-functional inequality (0.2) in fuzzy Banach spaces by using the fixed point method.

Throughout this paper, assume that ρ is a fixed real number with $|\rho| < 2$.

2. Cubic ρ-functional inequality (0.1)

In this section, we solve and investigate the cubic ρ-functional inequality (0.1) in fuzzy Banach spaces.

Lemma 2.1. *Let (Y, N) be a fuzzy normed vector space. Let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} &N((f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x),t) \\ &\geq N\left(\rho\left(4f\left(x+\frac{y}{2}\right)+4f\left(x-\frac{y}{2}\right)-f(x+y)-f(x-y)-6f(x)\right),t\right) \end{aligned} \tag{2.1}$$

for all $x, y \in X$ and all $t > 0$. Then $f : X \to Y$ is cubic.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $N(-14f(0), t) \geq 1$. So $f(0) = 0$.

Letting $y = 0$ in (2.1), we get $N(2f(2x) - 16f(x), t) \geq 1$ and so $f(2x) = 8f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{8}f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} &N\left(f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x), t\right) \\ &\geq N\left(\rho\left(4f\left(x+\frac{y}{2}\right) + 4f\left(x-\frac{y}{2}\right) - f(x+y) - f(x-y) - 6f(x)\right), t\right) \\ &= N\left(f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x), \frac{2t}{|\rho|}\right) \end{aligned}$$

for all $t > 0$. By (N_5) and (N_6),

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x)$$

for all $x, y \in X$, since $|\rho| < 2$. So $f : X \to Y$ is cubic. □

We prove the Hyers-Ulam stability of the cubic ρ-functional inequality (2.1) in fuzzy Banach spaces.

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{8}\varphi(2x, 2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying

$$\begin{aligned} &N\left(f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x), t\right) \qquad (2.3) \\ &\geq min(N\left(\rho\left(4f\left(x+\frac{y}{2}\right) + 4f\left(x-\frac{y}{2}\right) - f(x+y) - f(x-y) - 6f(x)\right), t\right), \frac{t}{t+\varphi(x,y)})) \end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Then $C(x) := N\text{-}\lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$N\left(f(x) - C(x), t\right) \geq \frac{(16-16L)t}{(16-16L)t + L\varphi(x, 0)} \tag{2.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = 0$ in (2.3), we get

$$N(2f(2x) - 16f(x), t) \geq \frac{t}{t + \varphi(x, 0)} \tag{2.5}$$

and so $N\left(f(x) - 8f\left(\frac{x}{2}\right), \frac{t}{2}\right) \geq \frac{t}{t+\varphi\left(\frac{x}{2},0\right)}$ for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(x, 0)},\ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [27, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 8g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(8g\left(\frac{x}{2}\right) - 8h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{8}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{8}}{\frac{Lt}{8} + \varphi\left(\frac{x}{2}, 0\right)} \geq \frac{\frac{Lt}{8}}{\frac{Lt}{8} + \frac{L}{8}\varphi(x, 0)} = \frac{t}{t + \varphi(x, 0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (2.5) that

$$N\left(f(x) - 8f\left(\frac{x}{2}\right), \frac{L}{16}t\right) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{16}$.

By Theorem 1.4, there exists a mapping $C : X \to Y$ satisfying the following:

(1) C is a fixed point of J, i.e.,

$$C\left(\frac{x}{2}\right) = \frac{1}{8}C(x) \tag{2.6}$$

for all $x \in X$. The mapping C is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that C is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - C(x), \mu t) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$;

(2) $d(J^n f, C) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right) = C(x)$$

for all $x \in X$;

(3) $d(f, C) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, C) \leq \frac{L}{16 - 16L}.$$

This implies that the inequality (2.4) holds.

By (2.3),

$$N\left(8^n\left(f\left(\frac{2x+y}{2^n}\right)+f\left(\frac{2x-y}{2^n}\right)-2f\left(\frac{x+y}{2^n}\right)-2f\left(\frac{x-y}{2^n}\right)-12f\left(\frac{x}{2^n}\right)\right),8^nt\right)$$
$$\geq \min\left\{N\left(8^n\rho\left(4f\left(\frac{x+\frac{y}{2}}{2^n}\right)+4f\left(\frac{x-\frac{y}{2}}{2^n}\right)-f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x-y}{2^n}\right)-6f\left(\frac{x}{2^n}\right)\right),8^nt\right),\right.$$
$$\left.\frac{t}{t+\varphi(\frac{x}{2^n},\frac{y}{2^n})}\right\}$$

for all $x,y\in X$, all $t>0$ and all $n\in\mathbb{N}$. So

$$N\left(8^n\left(f\left(\frac{2x+y}{2^n}\right)+f\left(\frac{2x-y}{2^n}\right)-2f\left(\frac{x+y}{2^n}\right)-2f\left(\frac{x-y}{2^n}\right)-12f\left(\frac{x}{2^n}\right)\right),t\right)$$
$$\geq \min\left\{N\left(8^n\rho\left(4f\left(\frac{x+\frac{y}{2}}{2^n}\right)+4f\left(\frac{x-\frac{y}{2}}{2^n}\right)-f\left(\frac{x+y}{2^n}\right)-f\left(\frac{x-y}{2^n}\right)-6f\left(\frac{x}{2^n}\right)\right),t\right),\right.$$
$$\left.\frac{\frac{t}{8^n}}{\frac{t}{8^n}+\frac{L^n}{8^n}\varphi(x,y)}\right\}$$

Since $\lim_{n\to\infty}\frac{\frac{t}{8^n}}{\frac{t}{8^n}+\frac{L^n}{8^n}\varphi(x,y)}=1$ for all $x,y\in X$ and all $t>0$,

$$N(C(2x+y)+C(2x-y)-2C(x+y)-2C(x-y)-12C(x),t)$$
$$\geq N(\rho(4C(x+\frac{y}{2})+4C(x-\frac{y}{2})-C(x+y)-C(x-y)-6C(x),t)$$

for all $x,y\in X$ and all $t>0$. By Lemma 2.1, the mapping $C:X\to Y$ is cubic, as desired. □

Corollary 2.3. *Let $\theta\geq 0$ and let p be a real number with $p>3$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f:X\to Y$ be a mapping satisfying*

$$N\left(f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x),t\right) \tag{2.7}$$
$$\geq min\left\{N\left(\rho\left(4f\left(x+\frac{y}{2}\right)+4f\left(x-\frac{y}{2}\right)-f(x+y)-f(x-y)-6f(x)\right),t\right),\right.$$
$$\left.\frac{t}{t+\theta(\|x\|^p+\|y\|^p)}\right\}$$

for all $x,y\in X$ and all $t>0$. Then $C(x):=N\text{-}\lim_{n\to\infty}8^nf(\frac{x}{2^n})$ exists for each $x\in X$ and defines a cubic mapping $C:X\to Y$ such that

$$N\left(f(x)-C(x),t\right)\geq\frac{2(2^p-8)t}{2(2^p-8)t+\theta\|x\|^p}$$

for all $x\in X$ and all $t>0$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x,y):=\theta(\|x\|^p+\|y\|^p)$ for all $x,y\in X$. Then we can choose $L=2^{3-p}$, and we get the desired result. □

Theorem 2.4. *Let $\varphi:X^2\to[0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi(x,y)\leq 8L\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying (2.3)*. Then $C(x) := N\text{-}\lim_{n\to\infty} \frac{1}{8^n} f(2^n x)$ exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that*

$$N(f(x) - C(x), t) \geq \frac{(16-16L)t}{(16-16L)t + \varphi(x,0)} \tag{2.8}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (2.5) that

$$N\left(f(x) - \frac{1}{8}f(2x), \frac{1}{16}t\right) \geq \frac{t}{t + \varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{8}g(2x)$$

for all $x \in X$. Then $d(f, Jf) \leq \frac{1}{16}$. Hence

$$d(f, C) \leq \frac{1}{16-16L},$$

which implies that the inequality (2.8) holds.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 3$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying* (2.7)*. Then $C(x) := N\text{-}\lim_{n\to\infty} \frac{1}{8^n} f(2^n x)$ exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that*

$$N(f(x) - C(x), t) \geq \frac{2(8-2^p)t}{2(8-2^p)t + \theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-3}$, and we get the desired result. □

3. Quartic ρ-functional inequality (0.2)

In this section, we solve and investigate the quartic ρ-functional inequality (0.2) in fuzzy Banach spaces.

Lemma 3.1. *Let (Y, N) be a fuzzy normed vector space. A mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\begin{aligned} & N(f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y), t) \\ & \geq N\left(\rho\left(8f\left(x+\frac{y}{2}\right) + 8f\left(x-\frac{y}{2}\right) - 2f(x+y) - 2f(x-y) - 12f(x) + 3f(y)\right), t\right) \end{aligned} \tag{3.1}$$

for all $x, y \in X$ and all $t > 0$. Then f is quartic.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get $N(2f(2x) - 32f(x), t) \geq N(0, t) = 1$ and so

$$f\left(\frac{x}{2}\right) = \frac{1}{16}f(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}
&N\left(f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y),t\right)\\
&\geq N\left(\rho\left(8f\left(x+\frac{y}{2}\right)+8f\left(x-\frac{y}{2}\right)-2f\left(x+y\right)-2f\left(x-y\right)-12f\left(x\right)+3f\left(y\right)\right),t\right)\\
&= N\left(\frac{\rho}{2}\left(f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y)\right),t\right)\\
&= N\left(f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y),\frac{2t}{|\rho|}\right)
\end{aligned}$$

for all $t>0$ and all $x,y\in X$. By (N_5) and (N_6),

$$N\left(f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y),t\right)=1$$

for all $t>0$ and all $x,y\in X$. It follows from (N_2) that

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y)$$

for all $x,y\in X$. □

We prove the Hyers-Ulam stability of the quartic ρ-functional inequality (3.1) in fuzzy Banach spaces.

Theorem 3.2. *Let $\varphi: X^2\to[0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi(x,y)\leq\frac{L}{16}\varphi\left(2x,2y\right)$$

for all $x,y\in X$. Let $f:X\to Y$ be a mapping satisfying $f(0)=0$ and

$$\begin{aligned}
&N\left(f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y),t\right) &(3.3)\\
&\geq\min\left\{N\left(\rho\left(8f\left(x+\frac{y}{2}\right)+8f\left(x-\frac{y}{2}\right)-2f\left(x+y\right)-2f\left(x-y\right)-12f\left(x\right)+3f\left(y\right)\right),t\right),\right.\\
&\qquad\left.\frac{t}{t+\varphi(x,0)}\right\}
\end{aligned}$$

for all $x,y\in X$ and all $t>0$. Then $Q(x):=N\text{-}\lim_{n\to\infty}16^nf\left(\frac{x}{2^n}\right)$ exists for each $x\in X$ and defines a quartic mapping $Q:X\to Y$ such that

$$N\left(f(x)-Q(x),t\right)\geq\frac{(32-32L)t}{(32-32L)t+L\varphi(x,0)} \tag{3.4}$$

for all $x\in X$ and all $t>0$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 2.2.

Letting $y=0$ in (3.3), we get

$$N\left(2f(2x)-32f\left(x\right),t\right)=N\left(32f\left(x\right)-2f(2x),t\right)\geq\frac{t}{t+\varphi(x,0)} \tag{3.5}$$

for all $x\in X$. Now we consider the linear mapping $J:S\to S$ such that

$$Jg(x):=16g\left(\frac{x}{2}\right)$$

for all $x\in X$.

Let $g,h\in S$ be given such that $d(g,h)=\varepsilon$. Then

$$N(g(x)-h(x),\varepsilon t)\geq\frac{t}{t+\varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(16g\left(\frac{x}{2}\right) - 16h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{16}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{16}}{\frac{Lt}{16} + \varphi\left(\frac{x}{2}, 0\right)} \geq \frac{\frac{Lt}{16}}{\frac{Lt}{16} + \frac{L}{16}\varphi(x, 0)} = \frac{t}{t + \varphi(x, 0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (3.5) that

$$N\left(f(x) - 16f\left(\frac{x}{2}\right), \frac{L}{32}t\right) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{32}$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{16}Q(x) \tag{3.6}$$

for all $x \in X$. Since $f : X \to Y$ is even, $Q : X \to Y$ is an even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (3.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 16^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{L}{32 - 32L}.$$

This implies that the inequality (3.4) holds.

By the same method as in the proof of Theorem 2.2, it follows from (3.3) that

$$\begin{aligned} &N\left(Q(2x + y) + Q(2x - y) - 4Q(x + y) - 4Q(x - y) - 24Q(x) + 6Q(y), t\right) \\ &\geq N\left(\rho\left(8Q\left(x + \frac{y}{2}\right) + 8Q\left(x - \frac{y}{2}\right) - 2Q(x + y) - 2Q(x - y) - 12Q(x) + 3Q(y)\right), t\right) \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. By Lemma 3.1, the mapping $Q : X \to Y$ is quartic. □

Corollary 3.3. *Let $\theta \geq 0$ and let p be a real number with $p > 4$. Let X be a normed vector space with norm $\| \cdot \|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} & N\left(f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y),t\right) \qquad (3.7) \\ & \geq \min\left\{N\left(\rho\left(8f\left(x+\frac{y}{2}\right)+8f\left(x-\frac{y}{2}\right)-2f\left(x+y\right)-2f\left(x-y\right)-12f\left(x\right)+3f\left(y\right)\right),t\right),\right. \\ & \qquad \left.\frac{t}{t+\theta(\|x\|^p+\|y\|^p)}\right\} \end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 16^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that

$$N\left(f(x)-Q(x),t\right) \geq \frac{2(2^p-16)t}{2(2^p-16)t+\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{4-p}$, and we get the desired result. □

Theorem 3.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq 16L\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.3). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{16^n} f(2^n x)$ exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that

$$N\left(f(x)-Q(x),t\right) \geq \frac{(32-32L)t}{(32-32L)t+\varphi(x,0)} \qquad (3.8)$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (3.5) that

$$N\left(f(x)-\frac{1}{16}f(2x),\frac{1}{32}t\right) \geq \frac{t}{t+\varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{16}g\left(2x\right)$$

for all $x \in X$. Then $d(f, Jf) \leq \frac{1}{32}$. Hence

$$d(f,Q) \leq \frac{1}{32-32L},$$

which implies that the inequality (3.8) holds.

The rest of the proof is similar to the proof of Theorem 3.2. □

Corollary 3.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 4$. Let X be a normed vector space with norm $\| \cdot \|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.7). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{16^n} f(2^n x)$ exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that*

$$N\left(f(x)-Q(x),t\right) \geq \frac{2(16-2^p)t}{2(16-2^p)t+\theta\|x\|^p}$$

for all $x \in X$*.*

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-4}$, and we get the desired result. □

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Jooho Zhiang, Jeonghun Chu
Mathematics Branch, Seoul Science High School, Seoul 110-530, Korea
E-mail address: vjoohov@naver.com

George A. Anastassiou
Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA
E-mail address: ganastss@memphis.edu

Choonkil Park
Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Korea
E-mail address: baak@hanyang.ac.kr

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A RIGHT PARALLELISM RELATION FOR MAPPINGS TO POSETS

HEE SIK KIM, J. NEGGERS AND KEUM SOOK SO*

Abstract. In this paper, we study mappings $f, g : X \to P$, where P is a poset and X is a set, under the relation $f \,||\, g$, of right parallelism, $f(a) \le f(b)$ implies $g(a) \le g(b)$, which is reflexive and transitive but not necessarily symmetric. We prove several results of the type: if f has property P and $f \,||\, g$, then g has property P as well, or of the converse type. Doing so permits us to observe several conditions on mappings and/or groupoids $(X, *)$, upon which mappings may act in particular ways, which are of interest in their own right also. The special case $f(x) = x$ with $f \,||\, g$ yielding increasing/non-decreasing mappings $g : X \to P$ brings into focus a number of well-known situations seen from a different perspective.

1. Introduction.

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([4, 5]).

In the study of groupoids $(X, *)$ defined on a set X, it has also proven useful to investigate the semigroups $(Bin(X), \Box)$ where $Bin(X)$ is the set of all binary systems (groupoids) $(X, *)$ along with an associative product operation $(X, *)\Box(X, \bullet) = (X, \Box)$ such that $x \Box y = (x * y) \bullet (y * x)$ for all $x, y \in X$. Thus, e.g., it becomes possible to recognize that the left-zero-semigroup $(X, *)$ with $x * y = x$ for all $x, y \in X$ acts as the identity of this semigroup ([2]). H. F. Fayoumi ([1]) introduced the notion of the center $ZBin(X)$ in the semigroup $Bin(X)$ of all binary systems on a set X, and showed that a groupoid $(X, \bullet) \in ZBin(X)$ if and only if it is a locally-zero groupoid.

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$*$ Correspondence: Tel.: +82 33 248 2011, Fax: +82 33 256 2011 (K. S. So).

In this paper, we study mappings $f, g : X \to P$ where P is a poset and relations of the type $f \,||\, g$, i.e., f is right-(left-) parallel to g provided that $f(a) \leq f(b)$ implies $g(a) \leq g(b)$ as well, for all $a, b \in X$, where this condition applies. Since no assumptions about any order relation on X are made, this relation is a generalization of the special case $X = P$ and $f(x) = x$, where $a \leq b$ implies $g(a) \leq g(b)$, i.e., g is an order-preserving mapping. Even in this most general format it is possible to extract information concerning properties of $||$, i.e., $f \,||\, f$, and $f \,||\, g, g \,||\, h$ implies $f \,||\, h$ and the fact that $f \,||\, g$ does not imply $g \,||\, f$, to demonstrate the one-sided-ness of $f \,||\, g$. At the same time through the introduction of the groupoid structures $(X, *)$ as elements of $(Bin(X), \Box)$, the semigroup of binary systems (groupoids) on X, mappings f may acquire many different kinds of properties, such as $f(x * y) \leq f(x)$ for all $x, y \in X$ (left shrinking), for example, which then implies $g(x * y) \leq g(x)$ for all $x, y \in X$, so that this property is preserved by parallelism. If $P = [0, 1]$ with the usual order, then $f, g : X \to P$ yields the mappings f, g as fuzzy subsets of X and then the condition $f(x * y) \geq \min\{f(x), f(y)\}$ implies that if $f \,||\, g$, then $g(x * y) \geq \min\{g(x), g(y)\}$ as well, i.e., if f is a fuzzy subgroupoid of $(X, *)$ and $f \,||\, g$, then g is a fuzzy subgroupoid also. From these examples it should be clear that many other similar conclusions can be obtained in this setting, several of which we have provided in the following.

2. Preliminaries.

Let $(X, <)$ be a poset (partially ordered set), i.e., a set equipped with a relation $<$ where $x < y$ implies $y \not< x$ and $x < y, y < z$ implies $x < z$. The relation $\leq$ as usual means $x = y$ or $x < y$. For details on the theory of posets we refer the reader to [3, 4]. In these texts further references are supplied as well.

Given a non-empty set X, we let $Bin(X)$ denote the collection of all groupoids $(X, *)$, where $* : X \times X \to X$ is a map and where $*(x, y)$ is written in the usual product form. Given elements $(X, *)$ and $(X, \bullet)$ of $Bin(X)$, define a product "$\Box$" on these groupoids as follows:

$$(X, *) \Box (X, \bullet) = (X, \Box)$$

where

$$x \Box y = (x * y) \bullet (y * x)$$

for any $x, y \in X$. Using that notion, H. S. Kim and J. Neggers proved the following theorem.

Theorem 2.1. ([2]) *$(Bin(X), \square)$ is a semigroup, i.e., the operation "$\square$" as defined in general is associative. Furthermore, the left- zero-semigroup is the identity for this operation.*

The notion of BCK/BCI-algebras was introduced by Y. Imai and K. Iséki. An algebra $(X, *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if for any $x, y, z \in X$, it satisfies the following conditions:

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $x * 0 = x$,
(III) $x * y = 0$ and $y * x = 0$ imply $x = y$.

A BCI-algebra $(X, *, 0)$ is said to be *p-semisimple* if $0 * (0 * x) = x$ for all $x \in X$.

Theorem 2.2. ([5]) *Let X be a BCI-algebra. Then the following are equivalent: for all $x, y, z \in X$,*

(1) *X is p-semisimple,*
(II) $(x * y) * (z * u) = (x * z) * (y * u)$,
(III) $0 * (x * y) = y * x$.

For further reference on BCK/BCI-algebras, we refer to [5].

3. Right(left)-parallelism.

Given a set X and a poset P, we shall consider the set P^X consisting of all functions $f : X \to P$, i.e.,

$$P^X := \{\varphi \mid \varphi : X \to P : \text{ a map}\}$$

If the order relation on P is denoted by $\leq$, then P^X has an induced order $f \leq g$, provided $f(x) \leq g(x)$ for all $x \in X$.

Let $f, g \in P^X$. A map g is said to be *right-parallel* to f if any $a, b \in X$ with $f(a) \leq f(b)$ it is true that $g(a) \leq g(b)$, and we denote this fact by $f \,||\, g$. In this case, f is said to be *left-parallel* to g.

The following hold for parallelism: for any $f, g, h \in P^X$,

(i) $f \,||\, f$,
(ii) if $f \,||\, g$ and $g \,||\, h$, then $f \,||\, h$

Thus, the relation of "right-parallelism" is both reflexive and transitive in all cases.

Remark. The relation "$||$" is not symmetric. If $P := \mathbf{R}$, the real numbers with the regular order, then $f(x) := x^3$ and $g(x) := \max\{0, x\}$ implies $f \,||\, g$, but not $g \,||\, f$, i.e., $a^3 \leq b^3$ implies $\max\{0, a\} \leq \max\{0, b\}$

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but $a = -1, b = -2$ implies $\max\{0, -1\} = \max\{0, -2\}$ and $(-2)^3 < (-1)^3$.

Proposition 3.1. *Let $f(\in P^X)$ be a constant mapping. If g is right-parallel to f, then g is also a constant mapping.*

Proof. Given $a, b \in X$, since f is constant, we have $f(a) \leq f(b)$. It follows that $g(a) \leq g(b)$, since $f \,||\, g$. Similarly, $f(b) \leq f(a)$ implies $g(b) \leq g(a)$, proving the proposition. □

Proposition 3.2. *Any function f is left-parallel to a constant function $g : X \to P$.*

Proof. Straightforward. □

Proposition 3.3. *Let $f : P \to P$ be the identity function. If g is right-parallel to f, then $g : P \to P$ is monotonically increasing.*

Proof. Let $f : P \to P$ be an identity function and let g be right-parallel to f. Then, for any $a, b \in P$ with $a \leq b$, we have $f(a) \leq f(b)$. Since $f \,||\, g$, $g(a) \leq g(b)$. This proves that g is monotonically increasing. □

Proposition 3.4. *Let $f : P \to P$ be the identity function. If f is right-parallel to g, then $g(a) \leq g(b)$ implies $a \leq b$.*

Proof. Given $a, b \in P$, since f is right-parallel to g, $g(a) \leq g(b)$ implies $f(a) \leq f(b)$. Since f is an identity function, $g(a) \leq g(b)$ implies $a \leq b$, □

Proposition 3.5. *If g is right-parallel to f and $f(a) = f(b)$ for some $a, b \in X$, then $g(a) = g(b)$.*

Proof. $f(a) = f(b)$ implies $f(a) \leq f(b)$. Since g is right-parallel to f, we have $g(a) \leq g(b)$. Similarly, $f(b) \leq f(a)$ implies $g(b) \leq g(a)$, proving the proposition. □

Proposition 3.6. *Let $\varphi : P \to Q$ be an order-preserving mapping of posets satisfying the condition: for any $\alpha, \beta \in P$,*

$$\varphi(\alpha) \leq \varphi(\beta) \quad \textit{implies} \quad \alpha \leq \beta$$

If $f, g : X \to P$ are mappings with $f \,||\, g$, then $\varphi \circ f \,||\, \varphi \circ g$.

Proof. Assume that $\varphi(f(a)) \leq \varphi(f(b))$ for some $a, b \in X$. Then $f(a) \leq f(b)$. It follows from $f \,||\, g$ that $g(a) \,||\, g(b)$. Since φ is order-preserving, we obtain $\varphi(g(a)) \leq \varphi(g(b))$, proving the proposition. □

Proposition 3.7. *Let P, Q be posets, and let $f, k : X \to P$ be maps with $f \,||\, k$ and let $g, h : Y \to Q$ be maps with $g \,||\, h$. If we define $f \times g : X \times Y \to P \times Q$ by $(f \times g)(x, y) := (f(x), g(y))$ and*

$k\times h : X\times Y \to P\times Q$ by $(k\times h)(x, y) := (k(x), h(y))$, then $f\times g \parallel k\times h$.

Proof. Suppose that $(f\times g)(a, b) \leq (f\times g)(a', b')$ for some $(a, b), (a', b') \in X \times Y$. Then $(f(a), g(b)) \leq (f(a'), g(b'))$ in $P \times Q$. It follows that $f(a) \leq f(a')$ and $g(b) \leq g(b')$. Since $f \parallel k$ and $g \parallel h$, we have $f \times g \parallel k \times h$. □

Proposition 3.8. *Let $f, g : X \to P$ be mappings with $f \parallel g$. If f_A, g_A are restrictions of f and g respectively, where $A \subseteq X$, then $f_A \parallel g_A$.*

4. Right-parallel-property.

Let $f : X \to P$ be a map. A property α is said to be a *right-parallel-property* for f if $f \parallel g$, then g also has the same property.

Proposition 4.1. *Constancy is a right-parallel-property.*

Proof. See Proposition 3.1. □

Let $X := \mathbf{R}$ be the set of all real numbers and let P be a poset. A map $f : X \to P$ is said to be *periodic of period* p if

$$f(x + p) = f(x)$$

for all $x \in X$.

Proposition 4.2. *Periodicity is a right-parallel-property.*

Proof. Assume that f is periodic of period p and $f \parallel g$. If $f(x+p) = f(x)$ for all $x \in X$, then $f(x + p) \leq f(x)$. Since $f \parallel g$, we have $g(x + p) \leq g(x)$. Similarly, $f(x) \leq f(x + p)$ implies $g(x) \leq g(x + p)$, proving that $g(x + p) = g(x)$ for all $x \in X$. □

Let $(X, *)$ be a groupoid. A map $f : X \to P$ is said to be a *rank subalgebra* of $(X, *)$ if for all $x, y \in X$,

$$f(x * y) \geq \min\{f(x), f(y)\}$$

In this case, $(X, *)$ is said to be a *rank-characteristic-groupoid* for the mapping $f : X \to P$. Note that $f(x)$ and $f(y)$ may not be "comparable" in a general poset. We need to also consider the case $f(x*y) \geq f(x)$ or $f(x * y) \geq f(y)$ for further investigation.

Proposition 4.3. *Rank-subalgebra is a right-parallel-property.*

Proof. Assume that f is a rank-subalgebra of a groupoid $(X, *)$ and $f \parallel g$. Then $f(x * y) \geq \min\{f(x), f(y)\}$ for all $x, y \in X$. Without loss of generality, we let $f(x * y) \geq f(x)$. Since $f \parallel g$, we have $g(x * y) \geq g(x)$, proving the proposition. □

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Proposition 4.4. *Let $(X, *)$ and $(X, \star)$ be rank-characteristic-groupoids for $f : X \to P$. If $(X, \Box) := (X, *)\Box(X, \star)$, then $(X, \Box)$ is also a rank-characteristic-groupoid for f.*

Proof. Since $(X, \Box) := (X, *)\Box(X, \star)$, $x\Box y = (x * y) \star (y * x)$ for all $x, y \in X$. It follows that

$$\begin{aligned} f(x\Box y) &= f((x * y) \star (y * x)) \\ &\geq \min\{f(x * y), f(y * x)\} \\ &\geq \min\{f(x), f(y)\}, \end{aligned}$$

showing that $(X, \Box)$ is also a rank-characteristic-groupoid for f. □

This shows that *the rank-characteristic-groupoids for $f : X \to P$ form a subsemigroup with respect to the product $\Box$ of the semigroup $(Bin(X), \Box)$.*

Proposition 4.5. *The left-zero-semigroup $(X, *)$ is a rank-characteristic-groupoid for any map $f : X \to P$.*

Proof. Given $x, y \in X$, we have

$$f(x * y) = f(x) \geq \min\{f(x), f(y)\},$$

for any map $f : X \to P$. □

A map $f : X \to P$ is said to be *strongly bounded above* if there exists an $x_1 \in X$ such that $f(x) \leq f(x_1)$ for all $x \in X$.

Proposition 4.6. *Strongly bounded above is a right-parallel-property.*

Proof. Let $f : X \to P$ be strongly bounded above and let $f \mid\mid g$. Then there exists an $x_1 \in X$ such that $f(x) \leq f(x_1)$ for all $x \in X$. Since $f \mid\mid g$, we obtain $g(x) \leq g(x_1)$ for all $x \in X$, proving that g is strongly bounded above. □

A map $f : X \to P$ is said to be *strongly bounded below* if there exists an $x_0 \in X$ such that $f(x_0) \leq f(x)$ for all $x \in X$.

Proposition 4.6′. *Strongly bounded below is a right-parallel-property.*

A map $f : X \to P$ is said to be *P-compact* if there exist $x_0, x_1 \in X$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in X$.

Proposition 4.7. *P-compact is a right-parallel-property.*

5. Left-shrinking.

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Given a groupoid $(X,*)$, a mapping $f : X \to P$ is said to be *left-shrinking* if, for all $x, y \in X$,

$$f(x * y) \le f(x)$$

Similarly, a mapping $f : X \to P$ is said to be *right-shrinking* if, for all $x, y \in X$, $f(x * y) \le f(y)$. Note that we need to consider the case $f(x * y) \le f(x)$ or $f(x * y) \le f(y)$ for all $x, y \in X$ for further investigation.

Proposition 5.1. *If $f : X \to P$ is left-shrinking and $f \mid\mid g$, then $g : X \to P$ is also left-shrinking.*

Proof. If $f : X \to P$ is left-shrinking, then $f(x * y) \le f(x)$ for all $x, y \in X$. Since $f \mid\mid g$, we have $g(x*y) \le g(x)$ for all $x, y \in X$, proving that g is left-shrinking. □

Proposition 5.2. *Let $(X,*)$ and $(X,\star)$ be groupoids and let $(X,\square) := (X,*)\square(X,\star)$. If $f : X \to P$ is left-shrinking for $(X,*)$ and $(X,\star)$, then f is also left-shrinking for $(X,\square)$.*

Proof. Given $x, y \in X$, we have

$$\begin{aligned} f(x\square y) &= f((x * y) \star (y * x)) \\ &\le f(x * y) \\ &\le f(x), \end{aligned}$$

proving that f is left-shrinking for $(X,\square)$. □

Proposition 5.3. *Let $(P,\le),(Q,\le)$ be posets and let $(X,*),(Y,\bullet)$ be groupoids. Define a binary operation $\diamond$ on $X \times Y$ by*

$$(x, y) \diamond (x', y') := (x * x', y \bullet y').$$

If we define $f \times g$ as in Proposition 3.7 for any left-shrinking maps $f : X \to P$ and $g : Y \to Q$, then $f \times g$ is also left-shrinking for $(X \times Y, \diamond)$.

Proof. Given $(x, y), (x', y') \in X \times Y$, we have

$$\begin{aligned} (f \times g)((x, y) \diamond (x', y')) &= (f \times g)(x * x', y \bullet y')) \\ &= (f(x * x'), g(y \bullet y')) \\ &\le (f(x), g(y)) \\ &= (f \times g)(x, y), \end{aligned}$$

proving the proposition. □

Proposition 5.4. *Let $(X,*)$ be a left-zero-semigroup. If $f : X \to P$ is right-shrinking, then f is a constant mapping.*

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Proof. Given $x, y \in X$, we have $f(x) = f(x * y) \leq f(y)$, proving that $f(x) = f(y)$ for all $x, y \in X$. □

Proposition 5.5. *Let $\varphi : (X, *) \to (Y, \bullet)$ be a homomorphism of groupoids and let $f : (Y, \bullet) \to P$ be left-shrinking. Then $f \circ \varphi : (X, *) \to P$ is also left-shrinking.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned}(f \circ \varphi)(x * y) &= f(\varphi(x * y)) \\ &= f(\varphi(x) \bullet \varphi(y)) \\ &\leq f(\varphi(x)) \\ &= (f \circ \varphi)(x),\end{aligned}$$

proving that $f \circ \varphi : (X, *) \to P$ is also left-shrinking. □

Proposition 5.6. *Let $f : (X, *) \to P$ be left-shrinking. If $\varphi : P \to Q$ is order-preserving, then $\varphi \circ f : (X, *) \to Q$ is left-shrinking.*

Proof. If $f : (X, *) \to P$ is left-shrinking, then $f(x * y) \leq f(x)$ for all $x, y \in X$. It follows from φ is order-preserving that $\varphi(f(x * y)) \leq \varphi(f(x))$ for all $x, y \in X$, proving the proposition. □

Proposition 5.7. *Let $\varphi : (X, *) \to (Y, \bullet)$ be a homomorphism of groupoids and let $f, g : (Y, \bullet) \to P$ be mappings. If f is left-parallel to g, then $f \circ \varphi$ is left-parallel to $g \circ \varphi$.*

Proof. Let $a, b \in X$ such that $(f \circ \varphi)(a) \leq (f \circ \varphi)(b)$. Then $f(\varphi(a)) \leq f(\varphi(b))$. Since $f \mid\mid g$, we have $g(\varphi(a)) \leq g(\varphi(b))$, proving the proposition. □

Given a groupoid $(X, *)$, a mapping $f : X \to P$ is said to be *left-expanding* (resp., *right-expanding*) if, for all $x, y \in X$, $f(x * y) \geq f(y)$ (resp., $f(x * y) \geq f(x)$). Note that f is expanding if and only if f is both left-expanding and right-expanding.

6. $(X, *)[f](X, \bullet)$.

Let $(X, *), (X, \bullet) \in Bin(X)$. Given a map $f : (X, *) \to P$, we define a relation $(X, *)[f](X, \bullet)$ if $f(x * y) \leq f(x \bullet y)$ for all $x, y \in X$.

Proposition 6.1. *If $(X, *)[f](X, \bullet)$ and $(X, \bullet)[f](X, *)$, then $f(x * y) = f(x \bullet y)$ for all $x, y \in X$.*

In fact, $f(x) \neq f(y)$ is possible.

Proposition 6.2. *Let $(X, *)[f](X, \bullet)$. If f is left-shrinking for $(X, \bullet)$, then f is also left-shrinking for $(X, *)$.*

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Proof. For any $x, y \in X$, we have

$$f(x * y) \leq f(x \bullet y) \leq f(x)$$

□

Proposition 6.3. *Let $(X, *)[f](X, \bullet)$. If $f \,||\, g$, then $(X, *)[g](X, \bullet)$.*

Proof. Let $(X, *)[f](X, \bullet)$. Then $f(x * y) \leq f(x \bullet y)$ for all $x, y \in X$. Since $f \,||\, g$, we obtain $g(x * y) \leq g(x \bullet y)$, proving the proposition. □

Theorem 6.4. *If $(X, \bullet)[f](X, \diamond)$, then*

$$(X, *)\square(X, \bullet)[f](X, *)\square(X, \diamond)$$

for all $(X, *) \in Bin(X)$.

Proof. Let $(X, \square_1) := (X, *)\square(X, \bullet)$, i.e., $x\square_1 y = (x*y)\bullet(y*x)$ for all $x, y \in X$, and let $(X, \square_2) := (X, *)\square(X, \diamond)$, i.e., $x\square_2 y = (x * y) \diamond (y * x)$ for all $x, y \in X$. If $(X, \bullet)[f](X, \diamond)$, then $f(x \bullet y) \leq f(x \diamond y)$ for all $x, y \in X$. It follows that

$$\begin{aligned} g(x\square_1 y) &= f((x * y) \bullet (y * x)) \\ &\leq f((x * y) \diamond (y * x)) \\ &= f(x\square_2 y), \end{aligned}$$

proving that $(X, *)\square(X, \bullet)[f](X, *)\square(X, \diamond)$. □

7. Groupoids parallelism.

Let $(X, *), (X, \bullet) \in Bin(X)$ and let $(X, \leq)$ be a poset. A groupoid $(X, \bullet)$ is said to be *right parallel* to a groupoid $(X, *)$ with respect to the poset $(X, \leq)$ if $*(a, b) \leq *(a', b')$ implies $\bullet(a, b) \leq \bullet(a', b')$, i.e., $a * b \leq a' * b'$ implies $a \bullet b \leq a' \bullet b'$. We denote it by $(X, *) \,||\, (X, \bullet)$. Note that $(X, *) \,||\, (X, \bullet)$ and $(X, \bullet) \,||\, (X, \triangledown)$ implies $(X, *) \,||\, (X, \triangledown)$.

Example 7.1. Let $(X, \bullet)$ be a trivial groupoid, i.e., $x \bullet y = t$ for some $t \in X$, for all $x, y \in X$. Then $(X, *) \,||\, (X, \bullet)$ for all $(X, *) \in Bin(X)$. In fact, if $a * b \leq a' * b'$, then $a \bullet b = t \leq t = a' \bullet b'$.

A groupoid $(X, *)$ is said to be $\leq$*-commutative* if $a * b \leq a' * b'$ then $b * a \leq b' * a'$. Clearly, if $(X, *)$ is commutative, i.e., $x * y = y * x$ for all $x, y \in X$, then it is $\leq$-commutative. A groupoid $(X, *)$ is said to be *strictly* $\leq$*-commutative* if $a * b = a' * b'$ then $b * a = b' * a'$.

Example 7.2. Let $\leq$ be the diagonal relation, i.e., $x \leq y$ if and only if $x = y$, for all $x, y \in X$. If a groupoid $(X, *)$ is strictly $\leq$-commutative, then it is $\leq$-commutative.

A groupoid $(X,*)$ is said to be $\leq$-*ordering* if $x \leq x'$ and $y \leq y'$ implies $x*y \leq x'*y'$ (and $y*x \leq y'*x'$ also). Let $\leq$ be the diagonal relation on X. Then $x \leq x', y \leq y'$ means $x = x', y = y'$ and thus for any groupoid $(X,\bullet)$ whatsoever we have $x\bullet y = x'\bullet y'$ and $x\bullet y \leq x'\bullet y'$, whence $(X,\bullet)$ is $\leq$-ordering.

Example 7.3. Let $(X,*,0)$ be a p-semisimple BCI-algebra (or a medial groupoid). Then $(X,*)$ is $\leq$-ordering. In fact, if $x \leq x'$ and $y \leq y'$, then $x*x' = 0 = y*y'$. It follows that $(x*y)*(x'*y') = (x*x')*(y*y') = 0$, proving that $x*y \leq x'*y'$.

Proposition 7.4. *Let $(X,*)$ be a $\leq$-ordering groupoid. If $(X,*)\,||\,(X,\bullet)$, then $(X,\bullet)$ is also $\leq$-ordering.*

Proof. Let $x \leq x'$ and $y \leq y'$. Since $(X,*)$ is $\leq$-ordering, we have $x*y \leq x'*y'$. It follows that $x\bullet y \leq x'\bullet y'$, since $(X,*)\,||\,(X,\bullet)$. □

Proposition 7.5. *Let $(X,*),(X,\bullet)$ be $\leq$-ordering groupoids. If $(X,\Box) = (X,*)\Box(X,\bullet)$, then $(X,\Box)$ is also $\leq$-ordering.*

Proof. Let $x \leq x'$ and $y \leq y'$. Then $x*y \leq x'*y'$, $y*x \leq y'*x'$, since $(X,*)$ is $\leq$-ordering. Since $(X,\bullet)$ is $\leq$-ordering, we obtain $(x*y)\bullet(y*x) \leq (x'*y')\bullet(y'*x')$, i.e., $x\Box y \leq y\Box x$. This proves that $(X,\Box)$ is also $\leq$-ordering. □

Theorem 7.6. *Let $(X,*)$ be a $\leq$-commutative groupoid and let $(X,\bullet)$ be a $\leq$-ordering groupoid. If $(X,\Box) = (X,*)\Box(X,\bullet)$, then $(X,*)\,||\,(X,\Box)$.*

Proof. If $a*b \leq a'*b'$, then $b*a \leq b'*a'$, since $(X,*)$ is $\leq$-commutative. Since $(X,\bullet)$ is a $\leq$-ordering groupoid, we obtain $(a*b)\bullet(b*a) \leq (a'*b')\bullet(b'*a')$, i.e., $a\Box b \leq a'\Box b'$, proving the theorem. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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Hee Sik Kim
Department of Mathematics
Hanyang University
Seoul 133-791, Korea
heekim@hanyang.ac.kr

J. Neggers
Department of Mathematics
University of Alabama
Tuscaloosa, AL 35487-0350
U. S. A.
jneggers@as.ua.edu

Keum Sook So
Department of Mathematics
Hallym University
Chuncheon, 200-702, Korea
ksso@hallym.ac.kr

Existence results for nonlinear generalized three-point boundary value problems for fractional differential equations and inclusions

Mohamed Abdalla Darwish[a] **and Sotiris K. Ntouyas**[b]

[a]Department of Mathematics, Sciences Faculty for Girls, King Abdulaziz University, Jeddah, Saudi Arabia
e-mail: dr.madarwish@gmail.com

[b]Department of Mathematics, University of Ioannina
451 10 Ioannina, Greece
e-mail: sntouyas@uoi.gr

Abstract

This paper studies boundary value problems of nonlinear fractional differential equations and inclusions, of order $q \in (1, 2]$ with generalized three-point boundary conditions. Some existence and uniqueness results are obtained by using a variety of fixed point theorems. Some illustrative examples are also discussed.

Key words and phrases: Fractional differential equations; Fractional differential inclusions; three-point generalized boundary conditions; existence; contraction principle; Krasnoselskii's fixed point theorem; Leray-Schauder degree.
AMS (MOS) Subject Classifications: 26A33, 34A12, 34B15, 34A60.

1 Introduction

Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes, see [27]. These characteristics of the fractional derivatives make the fractional-order models more realistic and practical than the classical integer-order models. As a matter of fact, fractional differential equations arise in many engineering and scientific disciplines such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. [22, 27, 28, 29]. In recent years, there are many papers dealing with the existence of solutions to various fractional differential equations. For some recent development on the topic, see [1]-[13] and the references therein.

Recently, the existence of positive solutions was studied for generalized second order three-point boundary value problems for equations or systems, see [14], [20], [21], [26], and the references cited therein.

Here, in the first part of this paper, we discuss the existence and uniqueness of solutions for a boundary value problem of nonlinear fractional differential equations of order $q \in (1, 2]$ with generalized three-point boundary conditions given by

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \quad 0 < t < 1, \quad 1 < q \le 2, \\ x(0) = \beta x(\eta), \quad x(1) = \alpha x(\eta), \end{cases} \tag{1.1}$$

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where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous, and α, β, η are constants with $0 < \eta < 1$ and $1 - \beta + (\beta - \alpha)\eta \neq 0$. Here, $\mathcal{C} = C([0,1],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,1] \to \mathbb{R}$ endowed with a topology of uniform convergence with the norm denoted by $\|\cdot\|$.

Some new existence and uniqueness results are proved for the boundary value problem (1.1), by using a variety of fixed point theorems. Thus, in Theorem 3.1 we prove an existence and uniqueness result by using Banach's contraction principle, in Theorem 3.3 we prove the existence of a solution by using Krasnoselskii's fixed point theorem, while in Theorem 3.6 we prove the existence of a solution via Leray-Schauder nonlinear alternative. In Theorem 3.9 we prove an existence and uniqueness result by using a fixed point theorem of Boyd and Wong [15] for nonlinear contractions. Some illustrative examples are also discussed.

In the second part of this paper, we study the following generalized three-point boundary value problem for fractional differential inclusions

$$\begin{cases} {}^cD^q x(t) \in F(t, x(t)), \quad 0 < t < 1, \quad 1 < q \le 2, \\ x(0) = \beta x(\eta), \quad x(1) = \alpha x(\eta), \end{cases} \tag{1.2}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all non-empty subsets of $\mathbb{R}$, and α, β, η are as in problem (1.1).

For the problem (1.2), the aim here is to establish existence results when the right hand side is convex as well as nonconvex valued. In the first result, Theorem 4.8, we prove the existence of solutions for the problem (1.2), when the right hand side has convex values, via Leray-Schauder nonlinear alternative for Kakutani maps and F satisfying a Carathéodory condition. In the second result, Tleorem 4.16, we shall combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values. Finally, in the third result, Theorem 4.20, we shall use the fixed point theorem for contraction multivalued maps due to Covitz and Nadler.

It is worth mentioning that, the methods used are standard, however their exposition in the framework of problems (1.1) and (1.2) is new.

2 Preliminaries

Let us recall some basic definitions of fractional calculus [22, 29].

Definition 2.1 *For a continuous function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as*

$${}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \quad n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2 *The Riemann-Liouville fractional integral of order q is defined as*

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

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provided the integral exists.

Definition 2.3 *The Riemann-Liouville fractional derivative of order q for a continuous function $g(t)$ is defined by*

$$D^q g(t) = \frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n \int_0^t \frac{g(s)}{(t-s)^{q-n+1}}ds, \quad n=[q]+1,$$

provided the right hand side is pointwise defined on $(0,\infty)$.

Lemma 2.4 *For a given $g \in C([0,1],\mathbb{R})$ the unique solution of the boundary value problem*

$$\begin{cases} {}^cD^q x(t) = g(t), \quad 0<t<1, \quad 1<q\le 2, \\ x(0)=\beta x(\eta), \quad x(1)=\alpha x(\eta), \end{cases}$$

is given by

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}g(s)ds + \frac{(\beta-1)t-\beta\eta}{\Delta\Gamma(q)}\int_0^1 (1-s)^{q-1}g(s)ds \\ &+\frac{\beta+(\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}g(s)ds, \quad 0\le t\le 1, \end{aligned} \tag{2.1}$$

where $\Delta = 1-\beta+(\beta-\alpha)\eta \neq 0$.

Proof. For some constants $c_0, c_1 \in \mathbb{R}$, we have

$$x(t) = I^q g(t) - c_0 - c_1 t = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}g(s)ds - c_0 - c_1 t. \tag{2.2}$$

We have $x(0) = -c_0$, $x(\eta) = \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}g(s)ds - c_0 - c_1\eta$ and thus from the first boundary condition we have

$$(\beta-1)c_0 + \beta\eta c_1 = \beta\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}g(s)ds. \tag{2.3}$$

Also from the second boundary condition we get

$$(\alpha-1)c_0 + (\alpha\eta-1)c_1 = \alpha\int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)}g(s)ds - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}g(s)ds. \tag{2.4}$$

From (2.3), (2.4) we find c_0, c_1 and substituting in (2.2) we obtain the solution (2.1). □

3 Existence results-Differential Equations

In view of Lemma 2.4, we define an operator $F : \mathcal{C} \to \mathcal{C}, \mathcal{C} = C([0,1],\mathbb{R})$ by

$$\begin{aligned} (Fx)(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(t,x(s))ds \\ &+\frac{(\beta-1)t-\beta\eta}{\Delta\Gamma(q)}\int_0^1 (1-s)^{q-1}f(t,x(s))ds \\ &+\frac{\beta+(\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}f(t,x(s))ds, \quad 0\le t\le 1, \end{aligned} \tag{3.1}$$

For convenience, let us set

$$\lambda_1 = \frac{1}{|\Delta|} \sup_{t\in[0,1]} |(\beta-1)t - \beta\eta|, \quad \lambda_2 = \frac{1}{|\Delta|} \sup_{t\in[0,1]} |\beta + (\alpha-\beta)t|$$

and

$$\Lambda = \frac{1}{\Gamma(q+1)}(1+\lambda_1+\lambda_2\eta^q). \tag{3.2}$$

3.1 Existence result via Banach's fixed point theorem

Theorem 3.1 *Assume that $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is a continuous function and satisfies the assumption*

(A_1) $|f(t,x) - f(t,y)| \le L|x-y|, \forall t \in [0,1],\ L > 0,\ x,y \in \mathbb{R}$

with $L < 1/\Lambda$, where Λ is given by (3.2). Then the boundary value problem (1.1) has a unique solution.

Proof. Setting $\sup_{t\in[0,1]} |f(t,0)| = M$ and choosing $\rho \ge \frac{\Lambda M}{1-L\Lambda}$, we show that $FB_\rho \subset B_\rho$, where $B_\rho = \{x \in \mathcal{C} : \|x\| \le \rho\}$ and F defined in (3.1). For $x \in B_\rho$, we have

$$\begin{aligned}
\|(Fx)(t)\| &\le \sup_{t\in[0,1]} \left\{ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s,x(s))| ds + \frac{\lambda_1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} f(s,x(s))| ds \right.\\
&\quad \left. + \frac{\lambda_2}{\Gamma(q)} \int_0^\eta (\eta-s)^{q-1} |f(s,x(s))| ds \right\}\\
&\le \sup_{t\in[0,1]} \left\{ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} (|f(s,x(s)) - f(s,0)| + |f(s,0)|)\ \| ds \right.\\
&\quad + \frac{\lambda_1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} (|f(s,x(s)) - f(s,0)| + |f(s,0)|) ds\\
&\quad \left. + \frac{\lambda_2}{\Gamma(q)} \int_0^\eta (\eta-s)^{q-1} (|f(s,x(s)) - f(s,0)| + |f(s,0)|) ds \right\}\\
&\le (L\rho + M) \sup_{t\in[0,1]} \left\{ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds + \frac{\lambda_1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} ds \right.\\
&\quad \left. + \frac{\lambda_2}{\Gamma(q)} \int_0^\eta (\eta-s)^{q-1} ds \right\}\\
&\le \frac{(L\rho+M)}{\Gamma(q+1)} (1+\lambda_1+\lambda_2\eta^q) = (L\rho+M)\,\Lambda \le \rho.
\end{aligned}$$

Now, for $x,y \in \mathcal{C}$ and for each $t \in [0,1]$, we obtain

$$\begin{aligned}
\|(Fx)(t) - (Fy)(t)\| &\le \sup_{t\in[0,1]} \left\{ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(t,x(s)) - f(s,y(s))| ds \right.\\
&\quad + \frac{\lambda_1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} |f(t,x(s)) - f(s,y(s))| ds
\end{aligned}$$

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$$
\begin{aligned}
&+\frac{\lambda_2}{\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}|f(t,x(s))-f(s,y(s))|ds\Big\} \\
\leq\ & L\|x-y\| \sup_{t\in[0,1]}\left\{\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds+\frac{\lambda_1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}ds\right. \\
&\left.+\frac{\lambda_2}{\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}ds\right\} \\
\leq\ & \frac{L}{\Gamma(q+1)}\left(1+\lambda_1+\lambda_1\eta^q\right)\|x-y\| = L\Lambda\|x-y\|,
\end{aligned}
$$

where Λ is given by (3.2). Observe that Λ depends only on the parameters involved in the problem. As $L < 1/\Lambda$, therefore F is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Example 3.2 *Consider the following generalized three-point fractional boundary value problem*

$$
\begin{cases}
{}^cD^{1/2}x(t) = \dfrac{1}{(t+9)^2}\dfrac{|x|}{1+|x|}, \quad t\in[0,1], \\
x(0) = \dfrac{1}{2}x\left(\dfrac{1}{4}\right), \ x(1) = 2x\left(\dfrac{1}{4}\right).
\end{cases}
\tag{3.3}
$$

Here, $q=3/2,\ \beta=1/2,\ \alpha=2,\ \eta=1/4$, and $f(t,x)=\dfrac{1}{(t+9)^2}\dfrac{|x|}{1+|x|}$. We find $\Delta=\dfrac{1}{8}, \lambda_1 = 5, \lambda_2=16$ and $\Lambda=\dfrac{32}{3\sqrt{\pi}}$. As $|f(t,x)-f(t,y)|\leq\dfrac{1}{81}|x-y|$, therefore, (A_1) is satisfied with $L=\dfrac{1}{81}$. Further, $L\Lambda=\dfrac{32}{243\sqrt{\pi}}<1$. Thus, by the conclusion of Theorem 3.1, the boundary value problem (3.3) has a unique solution on $[0,1]$.

3.2 Existence result via Krasnoselskii's fixed point theorem

Theorem 3.3 *(Krasnoselskii's fixed point theorem)[24]. Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Let A, B be the operators such that: (i) $Ax+By\in M$ whenever $x,y\in M$; (ii) A is compact and continuous; (iii) B is a contraction mapping. Then there exists $z\in M$ such that $z=Az+Bz$.*

Theorem 3.4 *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function and the assumption (A_1) holds. In addition we assume that*

(A_2) $|f(t,x)|\leq\mu(t),\quad \forall(t,x)\in[0,1]\times\mathbb{R},$ *and* $\mu\in C([0,1],\mathbb{R}^+)$.

If

$$
\frac{L}{\Gamma(q+1)}(\lambda_1+\lambda_2\eta^q)<1,
\tag{3.4}
$$

then the boundary value problem (1.1) has at least one solution on $[0,1]$.

Proof. Letting $\sup_{t\in[0,1]}|\mu(t)| = \|\mu\|$, we fix

$$\overline{r} \geq \frac{\|\mu\|}{\Gamma(q+1)}(1+\lambda_1+\lambda_2\eta^q),$$

and consider $B_{\overline{r}} = \{x \in \mathcal{C} : \|x\| \leq \overline{r}\}$. We define the operators $\mathcal{P}$ and $\mathcal{Q}$ on $B_{\overline{r}}$ as

$$\begin{aligned}
(\mathcal{P}x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds, \quad 0 \leq t \leq 1,\\
(\mathcal{Q}x)(t) &= \frac{(\beta-1)t-\beta\eta}{\Delta\Gamma(q)} \int_0^1 (1-s)^{q-1} f(t,x(s))ds\\
&\quad + \frac{\beta+(\alpha-\beta)t}{\Delta\Gamma(q)} \int_0^\eta (\eta-s)^{q-1} f(t,x(s))ds, \quad 0 \leq t \leq 1.
\end{aligned}$$

For $x, y \in B_{\overline{r}}$, we find that $\|\mathcal{P}x + \mathcal{Q}y\| \leq \frac{\|\mu\|}{\Gamma(q+1)}(1+\lambda_1+\lambda_2\eta^q) \leq \overline{r}$.

Thus, $\mathcal{P}x + \mathcal{Q}y \in B_{\overline{r}}$. It follows from the assumption (A_1) together with (3.4) that $\mathcal{Q}$ is a contraction mapping. Continuity of f implies that the operator $\mathcal{P}$ is continuous. Also, $\mathcal{P}$ is uniformly bounded on $B_{\overline{r}}$ as $\|\mathcal{P}x\| \leq \frac{\|\mu\|}{\Gamma(q+1)}$. Now we prove the compactness of the operator $\mathcal{P}$.

In view of (A_1), we define $\sup_{(t,x)\in[0,1]\times B_{\overline{r}}}|f(t,x)| = \overline{f}$, and consequently we have

$$\begin{aligned}
\|(\mathcal{P}x)(t_1) - (\mathcal{P}x)(t_2)\| &= \left\| \frac{1}{\Gamma(q)} \int_0^{t_1} [(t_2-s)^{q-1} - (t_1-s)^{q-1}] f(t,x(s))ds \right.\\
&\quad \left. + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2-s)^{q-1} f(t,x(s))ds \right\|\\
&\leq \frac{\overline{f}}{\Gamma(q+1)} |2(t_2-t_1)^q + t_1^q - t_2^q|,
\end{aligned}$$

which is independent of x. Thus, $\mathcal{P}$ is equicontinuous. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{P}$ is compact on $B_{\overline{r}}$. Thus all the assumptions of Theorem 3.3 are satisfied. So the conclusion of Theorem 3.3 implies that the boundary value problem (1.1) has at least one solution on $[0,1]$. □

3.3 Existence result via Leray-Schauder Alternative

Theorem 3.5 *(Nonlinear alternative for single valued maps)[19]. Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either (i) F has a fixed point in $\overline{U}$, or (ii)]there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

Theorem 3.6 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that:*

(A_3) *There exist a function $p \in C([0,1],\mathbb{R}^+)$, and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ nondecreasing such that $|f(t,x)| \leq p(t)\psi(\|x\|), \quad \forall(t,x) \in [0,1] \times \mathbb{R}$;*

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(A_4) *There exists a constant* $M > 0$ *such that*

$$\frac{M}{\frac{\|p\|\psi(M)}{\Gamma(q+1)}\{1+\lambda_1+\lambda_2\eta^q\}} > 1.$$

Then the boundary value problem (1.1) has at least one solution on $[0,1]$.

Proof. Consider the operator $F : \mathcal{C} \to \mathcal{C}$ defined by (3.1).

We show that F maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$. For a positive number ρ, let $B_\rho = \{x \in C([0,1],\mathbb{R}) : \|x\| \le \rho\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then,

$$\begin{aligned}
|(Fx)(t)| &\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(t,x(s))|ds + \frac{\lambda_1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}|f(t,x(s))|ds \\
&\quad + \frac{\lambda_2}{\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}|f(t,x(s))|ds \\
&\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}p(s)\psi(\|x\|)\,ds + \frac{\lambda_1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}p(s)\psi(\|x\|)\,ds \\
&\quad + \frac{\lambda_2}{\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}p(s)\psi(\|x\|)\,ds \\
&\le \frac{\|p\|\psi(\|x\|)}{\Gamma(q+1)}\{1+\lambda_1+\lambda_2\eta^q\}.
\end{aligned}$$

Hence

$$\|Fx\| \le \frac{\|p\|\psi(\rho)}{\Gamma(q+1)}\{1+\lambda_1+\lambda_2\eta^q\}.$$

Next we show that F maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$. Let $t', t'' \in [0,1]$ with $t' < t''$ and $x \in B_\rho$, where B_ρ is a bounded set of $C([0,1],\mathbb{R})$. Then we have

$$\begin{aligned}
|(Fx)(t'') - (Fx)(t')| &= \left|\frac{1}{\Gamma(q)}\int_0^{t''} (t''-s)^{q-1}f(t,x(s))ds - \frac{1}{\Gamma(q)}\int_0^{t'} (t'-s)^{q-1}f(t,x(s))ds\right| \\
&\quad + \frac{|(\beta-1)||t''-t'|}{\Delta\Gamma(q)}\int_0^1 (1-s)^{q-1}f(t,x(s))ds \\
&\quad + \frac{|\alpha-\beta||t''-t'|}{\Delta\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}f(t,x(s))ds \\
&\le \frac{\|p\|\psi(\rho)}{\Gamma(q)}\int_0^{t'} |(t''-s)^{q-1} - (t'-s)^{q-1}|ds + \frac{\|p\|\psi(\rho)}{\Gamma(q)}\int_{t'}^{t''} (t''-s)^{q-1}ds \\
&\quad + \frac{\|p\|\psi(\rho)|(\beta-1)||t''-t'|}{\Delta\Gamma(q+1)} + \frac{\|p\|\psi(\rho)|\alpha-\beta||t''-t'|}{\Delta\Gamma(q+1)}.
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_\rho$ as $t'' - t' \to 0$. As F satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that $F : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is completely continuous.

Let x be a solution. Then for $t \in [0,1]$, and using the computations in proving that F is bounded, we have

$$\begin{aligned} |x(t)| &= |\lambda(Fx)(t)| \\ &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(t,x(s))|ds + \frac{\lambda_1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}|f(t,x(s))|ds \\ &\quad + \frac{\lambda_2}{\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}|f(t,x(s))|ds \\ &\leq \frac{\|p\|\psi(\|x\|)}{\Gamma(q+1)}\{1+\lambda_1+\lambda_2\eta^q\} \end{aligned}$$

and consequently

$$\frac{\|x\|}{\frac{\|p\|\psi(\|x\|)}{\Gamma(q+1)}\{1+\lambda_1+\lambda_2\eta^q\}} \leq 1.$$

In view of (A_4), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in C([0,1],\mathbb{R}) : \|x\| < M+1\}.$$

Note that the operator $F : \overline{U} \to C([0,1],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda F x$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 3.5), we deduce that F has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1). This completes the proof. □

3.4 Existence result via nonlinear contractions

Definition 3.7 *Let E be a Banach space and let $F : E \to E$ be a mapping. F is said to be a nonlinear contraction if there exists a continuous nondecrasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Psi(0) = 0$ and $\Psi(\xi) < \xi$ for all $\xi > 0$ with the property: $\|Fx - Fy\| \leq \Psi(\|x-y\|)$, $\forall x, y \in E$.*

Lemma 3.8 *(Boyd and Wong)[15]. Let E be a Banach space and let $F : E \to E$ be a nonlinear contraction. Then F has a unique fixed point in E.*

Theorem 3.9 *Assume that:*

(A_5) $|f(t,x) - f(t,y)| \leq h(t)\dfrac{|x-y|}{H^* + |x-y|}$, $t \in [0,1]$, $x, y \geq 0$, *where* $h : [0,1] \to \mathbb{R}^+$ *is continuous and*

$$H^* = \frac{1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}h(s)ds + \frac{\lambda_1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}h(s)ds + \frac{\lambda_2}{\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}h(s)ds.$$

Then the boundary value problem (1.1) has a unique solution.

Proof. Consider the operator $F : \mathcal{C} \to \mathcal{C}$ given by (3.1). Let the continuous nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\Psi(0) = 0$ and $\Psi(\xi) < \xi$ for all $\xi > 0$ defined by $\Psi(\xi) =$

$\dfrac{H^*\xi}{H^*+\xi}$, $\forall \xi \geq 0$. Let $x, y \in C([0,1],\mathbb{R})$. Then $|f(s,x(s)) - f(s,y(s))| \leq \dfrac{h(s)}{H^*}\Psi(\|x-y\|)$ so that

$$\begin{aligned} |Fx(t)-Fy(t)| &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h(s)\frac{|x(s)-y(s)|}{H^*+|x(s)-y(s)|}ds \\ &+\frac{\lambda_1}{\Gamma(q)}\int_0^1 (1-s)^{q-1}h(s)\frac{|x(s)-y(s)|}{H^*+|x(s)-y(s)|}ds \\ &+\frac{\lambda_2}{\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}h(s)\frac{|x(s)-y(s)|}{H^*+|x(s)-y(s)|}ds, \end{aligned}$$

for $t \in [0,1]$. Then $\|Fx - Fy\| \leq \Psi(\|x-y\|)$ and F is a nonlinear contraction and it has a unique fixed poitn in $C([0,1],\mathbb{R})$, by Lemma 3.8. □

Example 3.10 *Let us consider the boundary value problem*

$$\begin{cases} {}^cD^{3/2}x(t) = \dfrac{t|x|}{1+|x|}, \quad 0<t<1, \\ x(0) = \dfrac{1}{2}x\left(\dfrac{1}{4}\right), \ x(1) = 2x\left(\dfrac{1}{4}\right). \end{cases} \tag{3.5}$$

Here, $q = 3/2, \beta = 1/2, \alpha = 2, \eta = 1/4$ and $f(t,x) = \dfrac{t|x|}{1+|x|}$. We choose $h(t) = 1+t$ and find that $H^* = 7.97$. Clearly $\Big|f(t,x) - f(t,y)\Big| = \Big|\dfrac{t(|x|-|y|)}{1+|x|+|y|+|x||y|}\Big| \leq \dfrac{(1+t)|x-y|}{7.97+|x-y|}$. Thus, the conclusion of Theorem 3.9 applies and problem (3.5) has a unique solution.

4 Existence results-Differential Inclusions

Definition 4.1 *A function $x \in C^2([0,1],\mathbb{R})$ is a solution of the problem (1.2) if $x(0) = \beta x(\eta)$, $x(1) = \alpha x(\eta)$, and exists a function $f \in L^1([0,1],\mathbb{R})$ such that $f(t) \in F(t,x(t))$ a.e. on $[0,1]$ and*

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s)ds + \frac{(\beta-1)t-\beta\eta}{\Delta\Gamma(q)}\int_0^1 (1-s)^{q-1}f(s)ds \\ &+\frac{\beta+(\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}f(s)ds. \end{aligned}$$

4.1 The Carathéodory case

In this subsection, we are concerned with the existence of solutions for the problem (1.2) when the right hand side has convex values. We first recall some preliminary facts.

For a normed space $(X, \|\cdot\|)$, let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

Definition 4.2 *A multi-valued map $G : X \to \mathcal{P}(X)$:*

(i) is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$;

(ii) is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in P_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$);

(iii) is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$;

(iv) is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(X)$;

(v) has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$.

Remark 4.3 *It is known that, if the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$.*

Definition 4.4 *A multivalued map $G : [0; 1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function*

$$t \longmapsto d(y, G(t)) = \inf\{\|y - z\| : z \in G(t)\}$$

is measurable.

Definition 4.5 *A multivalued map $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if*

(i) *$t \longmapsto F(t, x)$ is measurable for each $x \in X$;*

(ii) *$x \longmapsto F(t, x)$ is upper semicontinuous for almost all $t \in [0, 1]$;*

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) *for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1([0, 1], \mathbb{R}^+)$ such that*

$$\|F(t, x)\| = \sup\{|v| : v \in F(t, x)\} \le \varphi_\alpha(t)$$

for all $\|x\| \le \alpha$ and for a. e. $t \in [0, 1]$.

For each $y \in C([0, 1], \mathbb{R})$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0, 1], \mathbb{R}) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in [0, 1]\}.$$

The consideration of this subsection is based on the following fixed point theorem ([19]).

Theorem 4.6 *(Nonlinear alternative for Kakutani maps).[19]. Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{cp,c}(C)$ is a upper semicontinuous compact map. Then either (i) F has a fixed point in $\overline{U}$, or (ii) there is a $u \in \partial U$ and $\lambda \in (0, 1)$ with $u \in \lambda F(u)$.*

The following lemma will be used in the sequel.

Lemma 4.7 *([25]) Let X be a Banach space. Let $F : [0,T] \times X \to \mathcal{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,1],X)$ to $C([0,1],X)$. Then the operator*

$$\Theta \circ S_F : C([0,1],X) \to P_{cp,c}(C([0,1],X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0,1],X) \times C([0,1],X)$.

Theorem 4.8 *Assume that (A_4) holds. In addition we suppose that the following conditions*

(H_1) *$F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is Carathéodory and has nonempty compact convex values;*

(H_2) *there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C([0,1],\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|) \quad \text{for each } (t,x) \in [0,1] \times \mathbb{R},$$

are satisfied. Then the boundary value problem (1.2) has at least one solution on $[0,1]$.

Proof. In order to transform boundary value problem (1.2) into a fixed point problem, consider the multivalued operator $\Omega : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ defined by

$$\Omega(x) = \left\{ \begin{array}{l} h \in C([0,1],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} f(s)ds \\ + \dfrac{(\beta-1)t - \beta\eta}{\Delta\Gamma(q)} \displaystyle\int_0^1 (1-s)^{q-1} f(s)ds \\ + \dfrac{\beta + (\alpha-\beta)t}{\Delta\Gamma(q)} \displaystyle\int_0^\eta (\eta-s)^{q-1} f(s)ds, \quad 0 \le t \le 1, \end{array} \right. \end{array} \right\}$$

for $f \in S_{F,x}$. Clearly, according to Lemma 2.4, the fixed points of Ω are solutions to boundary value problem (1.2). We will show that Ω satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps.

As a first step, *we show that Ω is convex for each $x \in C([0,1],\mathbb{R})$*. This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore we omit the proof.

Next, we show that Ω *maps bounded sets into bounded sets in $C([0,1],\mathbb{R})$*. For a positive number ρ, let $B_\rho = \{x \in C([0,1],\mathbb{R}) : \|x\| \le \rho\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then, for each $h \in \Omega(x), x \in B_\rho$, there exists $f \in S_{F,x}$ such that

$$\begin{aligned} h(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds + \frac{(\beta-1)t - \beta\eta}{\Delta\Gamma(q)} \int_0^1 (1-s)^{q-1} f(s)ds \\ &\quad + \frac{\beta + (\alpha-\beta)t}{\Delta\Gamma(q)} \int_0^\eta (\eta-s)^{q-1} f(s)ds, \end{aligned}$$

Then, as in Theorem 3.6, we have

$$|h(t)| \le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s)|ds + \frac{\lambda_1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} |f(s)|ds$$

$$
\begin{aligned}
&+\frac{\lambda_2}{\Gamma(q)}\int_0^\eta (\eta-s)^{q-1}|f(s)|ds \\
\le\ & \frac{\|p\|\psi(\|x\|)}{\Gamma(q+1)}\left\{1+\lambda_1+\lambda_2\eta^q\right\}.
\end{aligned}
$$

Thus,

$$
\|h\| \le \frac{\|p\|\psi(\rho)}{\Gamma(q+1)}\left\{1+\lambda_1+\lambda_2\eta^q\right\}.
$$

Now we show that Ω *maps bounded sets into equicontinuous sets of* $C([0,1],\mathbb{R})$. Let $t', t'' \in [0,1]$ with $t' < t''$ and $x \in B_\rho$, where B_ρ is a bounded set of $C([0,1],\mathbb{R})$. For each $h \in \Omega(x)$, we obtain, as in Theorem 3.6,

$$
\begin{aligned}
|h(t'')-h(t')| \le\ & \left|\frac{1}{\Gamma(q)}\int_0^{t'}[(t''-s)^{q-1}-(t'-s)^{q-1}]f(s)ds+\frac{1}{\Gamma(q)}\int_{t'}^{t''}(t''-s)^{q-1}f(s)ds\right| \\
&+\frac{\|p\|\psi(\rho)|(\beta-1)||t''-t'|}{\Delta\Gamma(q+1)}+\frac{\|p\|\psi(\rho)|\alpha-\beta||t''-t'|}{\Delta\Gamma(q+1)}.
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_{r'}$ as $t''-t' \to 0$. As Ω satisfies the above three assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\Omega : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ is completely continuous. In our next step, we show that Ω *has a closed graph.* Let $x_n \to x_*, h_n \in \Omega(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \Omega(x_*)$. Associated with $h_n \in \Omega(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for each $t \in [0,1]$,

$$
\begin{aligned}
h_n(t) =\ & \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f_n(s)ds+\frac{(\beta-1)t-\beta\eta}{\Delta\Gamma(q)}\int_0^1(1-s)^{q-1}f_n(s)ds \\
&+\frac{\beta+(\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}f_n(s)ds, \quad t\in[0,1].
\end{aligned}
$$

Thus we have to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [0,1]$,

$$
\begin{aligned}
h_*(t) =\ & \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f_*(s)ds+\frac{(\beta-1)t-\beta\eta}{\Delta\Gamma(q)}\int_0^1(1-s)^{q-1}f_*(s)ds \\
&+\frac{\beta+(\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}f_*(s)ds.
\end{aligned}
$$

Let us consider the continuous linear operator $\Theta : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ given by

$$
\begin{aligned}
f \mapsto \Theta(f) =\ & \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(s)ds+\frac{(\beta-1)t-\beta\eta}{\Delta\Gamma(q)}\int_0^1(1-s)^{q-1}f(s)ds \\
&+\frac{\beta+(\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}f(s)ds.
\end{aligned}
$$

Observe that

$$
\|h_n(t)-h_*(t)\| = \left\|\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}(f_n(s)-f_*(s))ds\right.
$$

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$$+\frac{(\beta-1)t-\beta\eta}{\Delta\Gamma(q)}\int_0^1(1-s)^{q-1}(f_n(s)-f_*(s))ds$$
$$+\frac{\beta+(\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}(f_n(s)-f_*(s))ds\Bigg\| \to 0,$$

as $n\to\infty$. Thus, it follows by Lemma 4.7 that $\Theta\circ S_F$ is a closed graph operator. Further, we have $h_n(t)\in\Theta(S_{F,x_n})$. Since $x_n\to x_*$, therefore, we have

$$\begin{aligned} h_*(t) &= \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f_*(s)ds+\frac{(\beta-1)t-\beta\eta}{\Delta\Gamma(q)}\int_0^1(1-s)^{q-1}f_*(s)ds \\ &\quad +\frac{\beta+(\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}f_*(s)ds, \end{aligned}$$

for some $f_*\in S_{F,x_*}$.

Finally, *we discuss a priori bounds on solutions.* Let x be a solution of (1.2). Then there exists $f\in L^1([0,1],\mathbb{R})$ with $f\in S_{F,x}$ such that, for $t\in[0,1]$, we have

$$\begin{aligned} h(t) &= \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s)ds+\frac{(\beta-1)t-\beta\eta}{\Delta\Gamma(q)}\int_0^1(1-s)^{q-1}f(s)ds \\ &\quad +\frac{\beta+(\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}f(s)ds. \end{aligned}$$

In view of (H_2), and using the computations in second step above, for each $t\in[0,1]$, we obtain

$$\begin{aligned} |h(t)| &\le \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}|f(s)|ds+\frac{\lambda_1}{\Gamma(q)}\int_0^1(1-s)^{q-1}|f(s)|ds \\ &\quad +\frac{\lambda_2}{\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}|f(s)|ds \\ &\le \frac{\|p\|\psi(\|x\|)}{\Gamma(q+1)}\left\{1+\lambda_1+\lambda_2\eta^q\right\}. \end{aligned}$$

Consequently, we have

$$\frac{\|x\|}{\frac{\|p\|\psi(\|x\|)}{\Gamma(q+1)}\left\{1+\lambda_1+\lambda_2\eta^q\right\}}\le 1.$$

In view of (A_4), there exists M such that $\|x\|\ne M$. Let us set

$$U=\{x\in C([0,1],\mathbb{R}):\|x\|<M+1\}.$$

Note that the operator $\Omega:\overline{U}\to\mathcal{P}(C([0,1],\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $x\in\partial U$ such that $x\in\mu\Omega(x)$ for some $\mu\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 4.6), we deduce that Ω has a fixed point $x\in\overline{U}$ which is a solution of the problem (1.2). This completes the proof. □

Example 4.9 *Consider the following fractional boundary value problem*

$$\begin{cases} {}^cD^{3/2}x(t) \in F(t, x(t)), \quad 0 < t < 1, \\ x(0) = \dfrac{1}{2}x\left(\dfrac{1}{4}\right), \; x(1) = 2x\left(\dfrac{1}{4}\right). \end{cases} \tag{4.1}$$

Here, $q = 3/2, \beta = 1/2, \; \alpha = 2, \; \eta = 1/4$ and $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by $x \to F(t,x) = \left[\dfrac{|x|^3}{|x|^3+3} + 2t^3 + 1, \dfrac{|x|}{|x|+1} + t + 1\right]$. For $f \in F$, we have $|f| \leq \max\left(\dfrac{|x|^3}{|x|^3+3} + 2t^3 + 1, \dfrac{|x|}{|x|+1} + t + 1\right) \leq 4, \quad x \in \mathbb{R}$. Thus, $\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leq 4 = p(t)\psi(\|x\|), \quad x \in \mathbb{R}$, with $p(t) = 1, \; \psi(\|x\|) = 4$. Further, using the condition (A_4) we find that $M > 21.092278$. Clearly, all the conditions of Theorem 4.8 are satisfied. So there exists at least one solution of the problem (4.1) on $[0,1]$.

4.2 The lower semi-continuous case

As a next result, we study the case when F is not necessarily convex valued. Our strategy to deal with this problems is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo [16] for lower semi-continuous maps with decomposable values.

Definition 4.10 *Let X be a nonempty closed subset of a Banach space E and $G : X \to \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E.*

Definition 4.11 *Let A be a subset of $[0,1] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,1]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$.*

Definition 4.12 *A subset $\mathcal{A}$ of $L^1([0,1],\mathbb{R})$ is decomposable if for all $x, y \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0,1] = J$, the function $x\chi_{\mathcal{J}} + y\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.*

Definition 4.13 *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator. We say N has a property (BC) if N is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.*

Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F} : C([0,1] \times \mathbb{R}) \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ associated with F as

$$\mathcal{F}(x) = \{w \in L^1([0,1],\mathbb{R}) : w(t) \in F(t, x(t)) \text{ for a.e. } t \in [0,1]\},$$

which is called the Nemytskii operator associated with F.

Definition 4.14 *Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.*

Lemma 4.15 *([16]) Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0,1],\mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Theorem 4.16 *Assume that $(A_4), (H_2)$ and the following condition holds:*

(H_3) *$F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that*

- (**a**) *$(t,x) \longmapsto F(t,x)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,*
- (**b**) *$x \longmapsto F(t,x)$ is lower semicontinuous for each $t \in [0,1]$.*

Then the boundary value problem (1.2) has at least one solution on $[0,1]$.

Proof. It follows from (H_3) and (H_2) that F is of l.s.c. type. Then from Lemma 4.15, there exists a continuous function $f : C^2([0,1],\mathbb{R}) \to L^1([0,1],\mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0,1],\mathbb{R})$.

Consider the problem

$$\begin{cases} {}^cD^q x(t) = f(x(t)), \quad 0 < t < 1, \quad 1 < q \le 2, \\ x(0) = \beta x(\eta), \quad x(1) = \alpha x(\eta) \end{cases} \tag{4.2}$$

in the space $C^2([0,1],\mathbb{R})$. It is clear that if $x \in C^2([0,1],\mathbb{R})$ is a solution of the problem (4.2), then x is a solution to the problem (1.2). In order to transform the problem (4.2) into a fixed point problem, we define the operator $\overline{\Omega}$ as

$$\begin{aligned} \overline{\Omega}x(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(x(s))ds + \frac{(\beta-1)t - \beta\eta}{\Delta\Gamma(q)}\int_0^1 (1-s)^{q-1} f(x(s))ds \\ &\quad + \frac{\beta + (\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta (\eta-s)^{q-1} f(x(s))ds, \quad 0 \le t \le 1. \end{aligned}$$

It can easily be shown that $\overline{\Omega}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 4.8. So we omit it. This completes the proof. □

4.3 The Lipschitz case

Now we prove the existence of solutions for the problem (1.2) with a nonconvex valued right hand side by applying a fixed point theorem for multivalued map due to Covitz and Nadler [18].

Let (X,d) be a metric space induced from the normed space $(X; \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by $H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}$, where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [23]).

Definition 4.17 *A multivalued operator $N : X \to P_{cl}(X)$ is called:*

(a) *$\gamma-$Lipschitz if and only if there exists $\gamma > 0$ such that*

$$H_d(N(x), N(y)) \le \gamma d(x,y) \text{ for each } x, y \in X;$$

(b) *a contraction if and only if it is $\gamma-$Lipschitz with $\gamma < 1$.*

Lemma 4.18 *(Covitz-Nadler) [18]. Let (X, d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Definition 4.19 *A measurable multi-valued function $F : [0,1] \to \mathcal{P}(X)$ is said to be integrably bounded if there exists a function $h \in L^1([0,1], X)$ such that for all $v \in F(t)$, $\|v\| \le h(t)$ for a.e. $t \in [0,1]$.*

Theorem 4.20 *Assume that the following conditions hold:*

(H_4) $F : [0,1] \times \mathbb{R} \to P_{cp}(\mathbb{R})$ is such that $F(\cdot, x) : [0,1] \to P_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$;

(H_5) $H_d(F(t,x), F(t,\bar{x})) \le m(t)|x - \bar{x}|$ for almost all $t \in [0,1]$ and $x, \bar{x} \in \mathbb{R}$ with $m \in C([0,1], \mathbb{R}^+)$ and $d(0, F(t,0)) \le m(t)$ for almost all $t \in [0,1]$.

Then the boundary value problem (1.2) has at least one solution on $[0,1]$ if

$$\frac{\|m\|}{\Gamma(q+1)} \{1 + \lambda_1 + \lambda_2 \eta^q\} < 1.$$

Proof. We transform the problem (1.2) into a fixed point problem. Consider the set-valued map $\Omega : C([0,1], \mathbb{R}) \to \mathcal{P}(C([0,1], \mathbb{R}))$ defined at the begining of the proof of Theorem 4.8. It is clear that the fixed point of Ω are solutions of the problem (1.2).

Note that, by the assumption (H_4), since the set-valued map $F(\cdot, x)$ is measurable, it admits a measurable selection $f : [0,1] \to \mathbb{R}$ (see Theorem III.6 [17]). Moreover, from assumption (H_5) $|f(t)| \le m(t) + m(t)|x(t)|$, i.e. $f(\cdot) \in L^1([0,1], X)$. Therefore the set $S_{F,x}$ is nonempty. Also note that since $S_{F,x} \neq \emptyset$, $\Omega(x) \neq \emptyset$ for any $x \in C([0,1], \mathbb{R})$.

Now we show that the operator Ω satisfies the assumptions of Lemma 4.18. To show that $\Omega(x) \in \mathcal{P}_{cl}((C[0,1], \mathbb{R}))$ for each $x \in C([0,1], \mathbb{R})$, let $\{u_n\}_{n \ge 0} \in \Omega(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $C([0,1], \mathbb{R})$. Then $u \in C([0,1], \mathbb{R})$ and there exists $v_n \in S_{F,x}$ such that, for each $t \in [0,1]$,

$$\begin{aligned} u_n(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v_n(s) ds + \frac{(\beta - 1)t - \beta\eta}{\Delta\Gamma(q)} \int_0^1 (1-s)^{q-1} v_n(s) ds \\ &\quad + \frac{\beta + (\alpha - \beta)t}{\Delta\Gamma(q)} \int_0^\eta (\eta - s)^{q-1} v_n(s) ds. \end{aligned}$$

As F has compact values, we may pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([0,1], \mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0,1]$,

$$u_n(t) \to u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v(s) ds + \frac{(\beta - 1)t - \beta\eta}{\Delta\Gamma(q)} \int_0^1 (1-s)^{q-1} v(s) ds$$

$$+\frac{\beta+(\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}v(s)ds.$$

Hence, $u\in\Omega(x)$ and $\Omega(x)$ is closed.

Next we show that Ω is a contraction on $C([0,1],\mathbb{R})$, i.e. there exists $\gamma<1$ such that

$$H_d(\Omega(x),\Omega(\bar{x}))\leq\gamma\|x-\bar{x}\| \quad \text{for each} \quad x,\bar{x}\in C([0,1],\mathbb{R}).$$

Let $x,\bar{x}\in C([0,1],\mathbb{R})$ and $h_1\in\Omega(x)$. Then there exists $v_1(t)\in F(t,x(t))$ such that, for each $t\in[0,1]$,

$$\begin{aligned} h_1(t) &= \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}v_1(s)ds+\frac{(\beta-1)t-\beta\eta}{\Delta\Gamma(q)}\int_0^1(1-s)^{q-1}v_1(s)ds \\ &\quad +\frac{\beta+(\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}v_1(s)ds. \end{aligned}$$

By (H_6), we have $H_d(F(t,x),F(t,\bar{x}))\leq m(t)|x(t)-\bar{x}(t)|$. So, there exists $w\in F(t,\bar{x}(t))$ such that $|v_1(t)-w|\leq m(t)|x(t)-\bar{x}(t)|, \quad t\in[0,1]$.

Define $U:[0,1]\to\mathcal{P}(\mathbb{R})$ by $U(t)=\{w\in\mathbb{R}:|v_1(t)-w|\leq m(t)|x(t)-\bar{x}(t)|\}$. Since the multivalued operator $U(t)\cap F(t,\bar{x}(t))$ is measurable (Proposition III.4 [17]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t)\in F(t,\bar{x}(t))$ and for each $t\in[0,1]$, we have $|v_1(t)-v_2(t)|\leq m(t)|x(t)-\bar{x}(t)|$.

For each $t\in[0,1]$, let us define

$$\begin{aligned} h_2(t) &= \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}v_2(s)ds+\frac{(\beta-1)t-\beta\eta}{\Delta\Gamma(q)}\int_0^1(1-s)^{q-1}v_2(s)ds \\ &\quad +\frac{\beta+(\alpha-\beta)t}{\Delta\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}v_2(s)ds. \end{aligned}$$

Thus,

$$\begin{aligned} |h_1(t)-h_2(t)| &\leq \frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}|v_1(s)-v_2(s)|ds+\frac{\lambda_1}{\Gamma(q)}\int_0^1(1-s)^{q-1}|v_1(s)-v_2(s)|ds \\ &\quad +\frac{\lambda_2}{\Gamma(q)}\int_0^\eta(\eta-s)^{q-1}|v_1(s)-v_2(s)|ds \\ &\leq \frac{\|m\|\|x-\overline{x}\|}{\Gamma(q+1)}\left\{1+\lambda_1+\lambda_2\eta^q\right\}. \end{aligned}$$

Hence,

$$\|h_1-h_2\|\leq\frac{\|m\|\|x-\overline{x}\|}{\Gamma(q+1)}\left\{1+\lambda_1+\lambda_2\eta^q\right\}.$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$H_d(\Omega(x),\Omega(\bar{x}))\leq\gamma\|x-\bar{x}\|\leq\frac{\|m\|\|x-\overline{x}\|}{\Gamma(q+1)}\left\{1+\lambda_1+\lambda_2\eta^q\right\}.$$

Since Ω is a contraction, it follows by Lemma 4.18 that Ω has a fixed point x which is a solution of (1.2). This completes the proof. □

Remark 4.21 *The results of this paper can easily to be generalized to boundary value problems for fractional differential equations and inclusions with deviating arguments and generalized three point boundary conditions. Thus we can study, by similar methods and obvious modifications, the following boundary value problem for fractional differential equations*

$$\begin{cases} {}^cD^q x(t) = f(t, x(\sigma(t))), \quad 0 < t < 1, \quad 1 < q \le 2, \\ x(t) = \beta x(\eta), \quad -r \le t \le 0 \\ x(1) = \alpha x(\eta), \end{cases} \tag{4.3}$$

where ${}^cD^q$ *denotes the Caputo fractional derivative of order* q, $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ *is continuous,* $-r = \min_{t\in[0,1]} \sigma(t)$, $\sigma : [0,1] \to [-r, 1]$ *is continuous with* $\sigma(t) \le t, \forall t \in [0,1]$ *and* α, β, η *are constants with* $0 < \eta < 1$ *and* $1 - \beta + (\beta - \alpha)\eta \neq 0$, *or the corresponding boundary value problem for fractional differential inclusions*

$$\begin{cases} {}^cD^q x(t) \in F(t, x(\sigma(t))), \quad 0 < t < 1, \quad 1 < q \le 2, \\ x(t) = \beta x(\eta), \quad -r \le t \le 0 \\ x(1) = \alpha x(\eta), \end{cases} \tag{4.4}$$

where ${}^cD^q$ *denotes the Caputo fractional derivative of order* q, *and* $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a multivalued map,* $\mathcal{P}(\mathbb{R})$ *is the family of all subsets of* $\mathbb{R}$.

Remark 4.22 *It is obvious that the methods used in this paper can be applied to other types of nonlocal boundary value problems. For example for the following four point boundary value problem*

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \quad 0 < t < 1, \quad 1 < q \le 2, \\ x(t) = \alpha x(\xi), \quad x(1) = \beta x(\eta), \end{cases} \tag{4.5}$$

where α, β, ξ, η *are constants with* $0 < \xi, \eta < 1$ *and* $\Delta := \alpha(\beta\eta - 1) - (\beta - 1)(\alpha\xi - 1) \neq 0$. *The solution of the problem (4.5) is given by*

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s))\, ds + \frac{\alpha[(\beta-1)t - \beta\eta + 1]}{\Delta} \int_0^\xi \frac{(\xi-s)^{q-1}}{\Gamma(q)} f(s, x(s))\, ds \\ &+ \frac{\beta[\alpha\xi - 1 - \alpha t]}{\Delta} \int_0^\eta \frac{(\eta-s)^{q-1}}{\Gamma(q)} f(s, x(s))\, ds \\ &+ \frac{\alpha t - \alpha\xi + 1}{\Delta} \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s, x(s))\, ds, \quad 0 \le t \le 1. \end{aligned}$$

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QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN FUZZY BANACH SPACES

CHOONKIL PARK AND SUN YOUNG JANG*

Abstract. In this paper, we solve the following quadratic ρ-functional inequalities

$$N\left(f(x+y)+f(x-y)-2f(x)-2f(y)-\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right),t\right) \geq \frac{t}{t+\varphi(x,y)}, \quad (0.1)$$

where ρ is a fixed real number with $\rho \neq 2$, and

$$N\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)-\rho\left(f(x+y)+f(x-y)-2f(x)-2f(y)\right),t\right) \geq \frac{t}{t+\varphi(x,y)}, \quad (0.2)$$

where ρ is a fixed real number with $\rho \neq \frac{1}{2}$.

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequalities (0.1) and (0.2) in fuzzy Banach spaces.

1. Introduction and preliminaries

Katsaras [21] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [13, 24, 52]. In particular, Bag and Samanta [2], following Cheng and Mordeson [8], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [23]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 28, 29] to investigate the Hyers-Ulam stability of quadratic ρ-functional inequalities in fuzzy Banach spaces.

Definition 1.1. [2, 28, 29, 30] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x,t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N_3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.
(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X,N) is called a *fuzzy normed vector space.*

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [27, 28].

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*Corresponding author: Sung Young Jang (email: jsym@ulsan.ac.kr).

Definition 1.2. [2, 28, 29, 30] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.

Definition 1.3. [2, 28, 29, 30] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [3]).

The stability problem of functional equations originated from a question of Ulam [51] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [17] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [40] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [50] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [9] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. Czerwik [10] proved the Hyers-Ulam stability of the quadratic functional equation. The functional equation $f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called a *Jensen type quadratic equation*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4, 18, 20, 25, 36, 37, 38, 41, 42, 44, 45, 46, 47, 48, 49]).

Gilányi [15] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x-y)\| \leq \|f(x+y)\| \tag{1.1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x+y) + f(x-y).$$

See also [43]. Fechner [12] and Gilányi [16] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [35] investigated the Cauchy additive functional inequality

$$\|f(x) + f(y) + f(z)\| \leq \|f(x+y+z)\| \tag{1.2}$$

and the Cauchy-Jensen additive functional inequality

$$\|f(x) + f(y) + 2f(z)\| \leq \left\|2f\left(\frac{x+y}{2} + z\right)\right\| \tag{1.3}$$

and proved the Hyers-Ulam stability of the functional inequalities (1.2) and (1.3) in Banach spaces.

Park [33, 34] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.4. [5, 11] Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \ge n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [19] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [6, 7, 22, 27, 31, 32, 38, 39]).

In Section 2, we solve the quadratic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.1) in fuzzy Banach spaces by using the fixed point method.

In Section 3, we solve the quadratic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.2) in fuzzy Banach spaces by using the fixed point method.

Throughout this paper, assume that X is a real vector space and (Y, N) is a fuzzy Banach space.

2. Quadratic ρ-functional inequality (0.1)

In this section, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.1) in fuzzy Banach spaces. Let ρ be a real number with $\rho \ne 2$. We need the following lemma to prove the main results.

Lemma 2.1. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) = \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right) \quad (2.1)$$

for all $x, y \in X$. Then $f : X \to Y$ is quadratic.

Proof. Replacing y by x in (2.1), we get $f(2x) - 4f(x) = 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$\begin{aligned} f(x+y) + f(x-y) - 2f(x) - 2f(y) &= \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right) \\ &= \frac{\rho}{2} (f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned}$$

and so $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$, as desired. □

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le \frac{L}{4} \varphi(2x, 2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$N(f(x+y) + f(x-y) - 2f(x) - 2f(y) \\ -\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right), t) \geq \frac{t}{t+\varphi(x,y)} \tag{2.2}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x) - Q(x), t\right) \geq \frac{(4-4L)t}{(4-4L)t + L\varphi(x,x)} \tag{2.3}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = x$ in (2.2), we get

$$N\left(f\left(2x\right) - 4f(x), t\right) \geq \frac{t}{t+\varphi(x,x)} \tag{2.4}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t+\varphi(x,x)}, \ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [26, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{4}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \frac{L}{4}\varphi(x,x)} = \frac{t}{t+\varphi(x,x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.4) that $N\left(f(x) - 4f\left(\frac{x}{2}\right), \frac{L}{4}t\right) \geq \frac{t}{t+\varphi(x,x)}$ for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{4}$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{2.5}$$

for all $x \in X$. Since $f : X \to Y$ is even, $Q : X \to Y$ is a even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.5) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{L}{4 - 4L}.$$

This implies that the inequality (2.3) holds.

By (2.2),

$$\begin{aligned} N\Bigg(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right) \\ - \rho\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right), 4^n t\Bigg) \geq \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\begin{aligned} N\Bigg(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right) \\ - \rho\left(4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right), t\Bigg) \geq \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x, y)} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y) = \rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)$$

for all $x, y \in X$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic, as desired. □

Corollary 2.3. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} N\left(f(x+y) + f(x-y) - 2f(x) - 2f(y) - \rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right), t\right) \\ \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \end{aligned} \tag{2.6}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 2\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{2-p}$, and we get the desired result. □

Theorem 2.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.2). Then $Q(x) := N$-$\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \ge \frac{(4-4L)t}{(4-4L)t + \varphi(x,x)} \tag{2.7}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4} g(2x)$$

for all $x \in X$.

It follows from (2.4) that

$$N\left(f(x) - \frac{1}{4}f(2x), \frac{1}{4}t\right) \ge \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \le \frac{1}{4}$. Hence $d(f, Q) \le \frac{1}{4-4L}$, which implies that the inequality (2.7) holds.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $\theta \ge 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be amapping satisfying $f(0) = 0$ and (2.6). Then $Q(x) := N$-$\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \ge \frac{(4-2^p)t}{(4-2^p)t + 2\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-2}$, and we get the desired result. □

3. Quadratic ρ-functional inequality (0.2)

In this section, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (0.2) in fuzzy Banach spaces. Let ρ be a real number with $\rho \ne \frac{1}{2}$. We need the following lemma to prove the main results.

Lemma 3.1. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) = \rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right) \tag{3.1}$$

for all $x, y \in X$. Then $f : X \to Y$ is quadratic.

Proof. Letting $y = 0$ in (3.1), we get $4f\left(\frac{x}{2}\right) - f(x) = 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$\begin{aligned}\frac{1}{2}f(x+y) - \frac{1}{2}f(x-y) - f(x) - f(y) &= 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \\ &= \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\end{aligned}$$

and so $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$, as desired. □

Theorem 3.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le \frac{L}{4}\varphi(2x,2y)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\begin{aligned}N\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) - \rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right), t\right) \\ \ge \frac{t}{t + \varphi(x,y)} \qquad (3.2)\end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \ge \frac{(1-L)t}{(1-L)t + \varphi(x,0)} \qquad (3.3)$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = 0$ in (3.2), we get

$$N\left(4f\left(\frac{x}{2}\right) - f(x), t\right) \ge \frac{t}{t + \varphi(x,0)} \qquad (3.4)$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \ge \frac{t}{t + \varphi(x,0)},\ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [26, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \ge \frac{t}{t + \varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{4}\varepsilon t\right) \\ &\ge \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \varphi\left(\frac{x}{2},0\right)} \ge \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \frac{L}{4}\varphi(x,0)} = \frac{t}{t + \varphi(x,0)}\end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in S$.

It follows from (3.4) that

$$N\left(f(x) - 4f\left(\frac{x}{2}\right), t\right) \ge \frac{t}{t + \varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \le 1$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{3.5}$$

for all $x \in X$. Since $f : X \to Y$ is even, $Q : X \to Y$ is an even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (3.5) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \ge \frac{t}{t + \varphi(x,0)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \le \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, Q) \le \frac{1}{1-L}.$$

This implies that the inequality (3.4) holds.

By (3.2),

$$\begin{aligned} N\Bigg(& 4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right) \\ & -\rho\left(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right)\right), 4^n t\Bigg) \ge \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\begin{aligned} N\Bigg(& 4^n\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right) \\ & -\rho\left(4^n\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right)\right), t\Bigg) \ge \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x,y)} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$2Q\left(\frac{x+y}{2}\right) + 2\left(\frac{x-y}{2}\right) - Q(x) - Q(y) = \rho\left(Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y)\right)$$

for all $x, y \in X$=. By Lemma 3.1, the mapping $Q : X \to Y$ is quadratic, as desired. $\square$

Corollary 3.3. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$N(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \tag{3.6}$$
$$-\rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right), t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}$$

for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x) - Q(x), t\right) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 2^p\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{2-p}$, and we get the desired result. □

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.2). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f\left(2^n x\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x) - Q(x), t\right) \geq \frac{(1 - L)t}{(1 - L)t + \varphi(x, 0)} \tag{3.7}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g\left(2x\right)$$

for all $x \in X$.

It follows from (3.4) that

$$N\left(f(x) - \frac{1}{4}f(2x), Lt\right) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq L$. Hence

$$d(f, Q) \leq \frac{1}{1 - L},$$

which implies that the inequality (3.7) holds.

The rest of the proof is similar to the proof of Theorem 3.2. □

Corollary 3.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.6). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(4 - 2^p)t}{(4 - 2^p)t + 2^p\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-2}$, and we get the desired result. □

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Choonkil Park
Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
E-mail address: `baak@hanyang.ac.kr`

Sung Young Jang
Department of Mathematics, University of Ulsan, Ulsan 44610, Korea
E-mail address: `jsym@ulsan.ac.kr`

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Remarks on common fixed point results for cyclic contractions in ordered b-metric spaces

Huaping Huang[1], Stojan Radenović[2], Tatjana Aleksic Lampert[3]
[1]School of Mathematics and Statistics, Hubei Normal University, Huangshi, 435002, China
[2]State University of Novi Pazar, Serbia
[3]Faculty of Science, Department of Mathematics and Informatics
Radoja Domanovica 12, 34 000 Kragujevac, Serbia

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Abstract. The purpose of this paper is to prove that some common fixed point theorems for cyclic contractions are equivalent to the counterpart of noncyclic contractions in the same setting. Our results improve and complement several results for cyclic contractions established in [Fixed Point Theory Appl., 2013: 256]. Furthermore, an application to the existence and uniqueness of solution for a class of integral equations is given to illustrate the superiority of the obtained assertions.

Keywords: (A, B)-weakly increasing, common fixed point, altering distance function, regular
MSC: 47H10, 54H25.

1. INTRODUCTION AND PRELIMINARIES

Since Banach fixed point theorem (see [1]) appeared in the world, there have been overwhelming trend in mathematical activities. This theorem presents numerous applications. For instance, it gives the conditions under which maps (single or multivalued) have solutions. Fixed point theory itself is a beautiful mixture of analysis (pure and applied), topology, and geometry. It has been revealed as a very powerful and important tool in the study of nonlinear phenomena. Over the last several decades, scholars have generalized this theorem greatly from several directions. Whereas, one of most influential generalizations is from spaces. Wherein, the fact from usual metric spaces to b-metric spaces is very popular. b-metric spaces, also called metric type spaces, were introduced in [2] and [3]. Afterwards, a large number of fixed point theorems have been presented in such spaces (see [4-15]). Recently, scholars cultivate some interests in fixed point theorems for cyclic contractions (see [15-19]). However, the authors of this paper find that many fixed point results for cyclic contractions are actually equivalent to those of noncyclic contractions in the same spaces. Throughout this paper, we obtain some equivalences between cyclic contractions and noncyclic contractions in the setting of b-metric spaces. Moreover, we obtain some common fixed point theorems without considering cyclic contractions. Further, as an applications, we cope with the existence and uniqueness of solutions of integral equations.

For the sake of the reader, we recall some well-known concepts and results as follows.

Definition 1.1([9]) Let X be a (nonempty) set and $s \geq 1$ a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric on X if, for all $x, y, z \in X$, the following conditions hold:

(b1) $d(x, y) = 0$ if and only if $x = y$;

(b2) $d(x, y) = d(y, x)$;

[1]E-mail addresses: mathhhp@163.com (H. Huang); fixedpoint50@gmail.com, radens@beotel.rs (S. Radenović); taleksic@kg.ac.rs (T.-A. Lampert)

(b3) $d(x,z) \leq s[d(x,y)+d(y,z)]$.
In this case, the pair (X,d) is called a b-metric space or metric type space. If $(X,\preceq)$ is still a partially ordered set, then $(X,\preceq,d)$ is called an ordered b-metric space.

Otherwise, for some other definitions in b-metric spaces such as convergence, Cauchy sequence, completeness, see [8-15] and the references therein.

Definition 1.2([22]) A function $\varphi:[0,\infty)\to[0,\infty)$ is called an altering distance function if the following properties hold:

(1) φ is continuous and nondecreasing;

(2) $\varphi(t)=0$ if and only if $t=0$.

Definition 1.3([21]) Let $(X,\preceq)$ be a partially ordered set, and let A and B be closed subsets of X with $A\cup B=X$. Let $f,g:X\to X$ be two mappings. The pair (f,g) is said to be (A,B)-weakly increasing if $fx\preceq gfx$ for all $x\in A$ and $gy\preceq fgy$ for all $y\in B$. In particular, (f,g) is said to be weakly increasing if $fx\preceq gfx$ and $gx\preceq fgx$ for all $x\in X$.

Definition 1.4([13]) An ordered b-metric space $(X,\preceq,d)$ is called regular if for any non-decreasing sequence $\{x_n\}$ in X such that $x_n\to x$ $(n\to\infty)$, one has $x_n\preceq x$ for all $n\in\mathbb{N}$.

Definition 1.5([16]) Let A and B be nonempty subsets of a metric space (X,d) and $T:A\cup B\to A\cup B$. Then T is called a cyclic map if $T(A)\subseteq B$ and $T(B)\subseteq A$.

Shatanawi and Postolache proved the following common fixed point results for cyclic contractions in the framework of ordered metric spaces.

Theorem 1.6([19]) Let $(X,\preceq,d)$ be a complete ordered metric space, and let A,B be closed nonempty subsets of X with $X=A\cup B$. Let $f,g:X\to X$ be (A,B)-weakly increasing mappings with respect to $\preceq$. Suppose that

(a) $X=A\cup B$ is a cyclic representation of X with respect to the pair (f,g), *i.e.*, $f(A)\subseteq B$ and $g(B)\subseteq A$;

(b) there exist $0<\delta<1$ and an altering distance function ψ such that for any comparable elements $x,y\in X$ with $x\in A$ and $y\in B$, we have

$$\psi(d(fx,gy))\leq\delta\psi\left(\max\left\{d(x,y),d(x,fx),d(y,gy),\frac{1}{2}(d(x,gy)+d(y,fx))\right\}\right);$$

(c) f or g is continuous, or

(c') $(X,\preceq,d)$ is regular.

Then f and g have a common fixed point.

It should be noted that cyclic contractions (unlike Banach-type contractions) need not to be continuous. This concept is an interesting increase in nonlinear analysis. In addition, Hussain *et al.* [15] introduced the notion of ordered cyclic weakly (ψ,φ,L,A,B)-contraction and proved the following fixed point results.

Definition 1.7 Let $(X,\preceq,d)$ be an ordered b-metric space, let $f,g:X\to X$ be two mappings, and let A and B be nonempty closed subsets of X. The pair (f,g) is called an ordered cyclic weakly (ψ,φ,L,A,B)-contraction if

(1) $X=A\cup B$ is a cyclic representation of X with respect to the pair (f,g);

(2) there exist two altering distance functions ψ,φ and a constant $L\geq 0$, such that for arbitrary comparable elements $x,y\in X$ with $x\in A$ and $y\in B$, we have

$$\psi\left(s^2d(fx,gy)\right)\leq\psi(M_s(x,y))-\varphi(M_s(x,y))+L\psi(N(x,y)),$$

where

$$M_s(x,y)=\max\left\{d(x,y),d(x,fx),d(y,gy),\frac{1}{2s}(d(x,gy)+d(y,fx))\right\} \tag{1.1}$$

and

$$N(x,y)=\min\{d(y,gy),d(x,gy),d(y,fx)\}. \tag{1.2}$$

Theorem 1.8 Let $(X,\preceq,d)$ be a complete ordered b-metric space, and let A and B be closed subsets of X. Let $f,g:X\to X$ be (A,B)-weakly increasing mappings with respect to $\preceq$. Suppose that

(a) the pair (f,g) is an ordered cyclic weakly (ψ,φ,L,A,B)-contraction;

(b) f or g is continuous.

Then f and g have a common fixed point $u\in A\cap B$.

Theorem 1.9 Let the hypothesis of Theorem 1.8 be satisfied, except that condition (b) is replaced by the following assumption:

(b') $(X,\preceq,d)$ is regular.

Then f and g have a common fixed point $u\in A\cap B$.

The following lemmas will be utilized in the proof of our main results.

Lemma 1.10([20]) If some ordinary fixed point theorem in the setting of complete metric spaces has a true cyclic-type extension, then these both theorems are equivalent.

Lemma 1.11([5]) Let $\{y_n\}$ be a sequence in a b-metric space (X,d) with $s\geq 1$ such that

$$d(y_n,y_{n+1})\leq \lambda d(y_{n-1},y_n)$$

for some $\lambda\in[0,\frac{1}{s})$, and each $n=1,2,\ldots$. Then $\{y_n\}$ is a Cauchy sequence in (X,d).

2. MAIN RESULTS

In this section, following the trend mentioned above, we extend such considerations to the simpler equivalent results so that we can enlarge, in a unified manner, the class of problems that can be investigated.

Theorem 2.1 Let $(X,\preceq,d)$ be a complete ordered metric space, and let $f,g:X\to X$ be the weakly increasing mappings. Suppose that

(a) there exist $0<\delta<1$ and an altering distance function ψ such that for any comparable elements $x,y\in X$, we have that

$$\psi(d(fx,gy))\leq\delta\psi(\max\{d(x,y),d(x,fx),d(y,gy),\frac{1}{2}(d(x,gy)+d(y,fx))\});$$

(b) f or g is continuous, or

(c) $(X,\preceq,d)$ is regular.

Then f and g have a common fixed point.

The proof of Theorem 2.1 is trivial because we have the following:

Theorem 2.2 Theorem 1.6 is equivalent with Theorem 2.1.

Proof Putting $A=B=X$ in Theorem 1.6, we obtain Theorem 2.1. In other words, Theorem 1.6 implies Theorem 2.1. The proof for the converse is same as in [20-21]. Namely, we depend on Lemma 1.10. □

In the sequel, we announce the following noncyclic case result.

Theorem 2.3 Let $(X,\preceq,d)$ be a complete ordered b-metric space, and let $f,g:X\to X$ be the weakly increasing mappings. Suppose that there exist altering distance function ψ and φ, and the constants $\varepsilon>1$, $L\geq 0$ such that

$$\psi(s^{\varepsilon}d(fx,gy))\leq\psi(M_s(x,y))-\varphi(M_s(x,y))+L\psi(N(x,y)) \tag{2.1}$$

for all comparable $x,y\in X$, where $M_s(x,y)$ and $N(x,y)$ are given by (1.1) and (1.2), respectively. If either f or g is continuous, or the space $(X,\preceq,d)$ is regular, then f and g have a common fixed point.

Proof Choose $x_0 \in X$ and construct a sequence $\{x_n\}$ as follows:

$$x_{2n+1} = fx_{2n}, \qquad x_{2n+2} = gx_{2n+1}.$$

Since (f, g) is weakly increasing, then

$$x_1 \preceq x_2 \preceq x_3 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots .$$

If $x_{2n} = x_{2n+1}$ or $x_{2n+1} = x_{2n+2}$ for some n, then the proof is trivial and hence we omit it. Now we assume that $x_n \neq x_{n+1}$ for all n. We shall only prove that

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n), \tag{2.2}$$

for all $n = 1, 2, ...$, where $\lambda \in [0, \frac{1}{s})$. Indeed, by (2.1), it establishes that

$$\begin{aligned}\psi(s^{\varepsilon} d(x_{2n+1}, x_{2n+2})) &= \psi(s^{\varepsilon} d(fx_{2n}, gx_{2n+1})) \\ &\leq \psi(M_s(x_{2n}, x_{2n+1})) + L\psi(N(x_{2n}, x_{2n+1})),\end{aligned}$$

where $M_s(x_{2n}, x_{2n+1}) = \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}$ and $N(x_{2n}, x_{2n+1}) = 0$. Hence, it is not hard to verify that

$$s^{\varepsilon} d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}). \tag{2.3}$$

Similarly, we obtain that

$$s^{\varepsilon} d(x_{2n}, x_{2n+1}) \leq d(x_{2n-1}, x_{2n}). \tag{2.4}$$

Uniting (2.3) and (2.4), ones have (2.2).

Now by Lemma 1.11, we demonstrate that $\{x_n\}$ is a Cauchy sequence and therefore there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$. Thus

$$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} fx_{2n} = x. \tag{2.5}$$

In view of $x_{2n} \to x$, without loss of generality, assume that f is continuous. Then

$$\lim_{n\to\infty} fx_{2n} = fx. \tag{2.6}$$

It follows immediately from (2.5) and (2.6) that $x = fx$.

Further, by using $x \preceq x$ we can prove that the condition (2.1) implies the existence of common fixed point of f and g. Indeed, put $x = y$ in (2.1) it follows that

$$\psi(s^{\varepsilon} d(fx, gx)) \leq \psi(M_s(x, x)) - \varphi(M_s(x, x)) + L\psi(N(x, x)).$$

Now that $M_s(x, x) = d(x, gx)$ and $N(x, y) = 0$, one has

$$\psi(s^{\varepsilon} d(fx, gx)) \leq \psi(d(x, gx)) - \varphi(d(x, gx)) + L \cdot 0 \leq \psi(d(x, gx)),$$

which means that

$$s^{\varepsilon} d(fx, gx) = s^{\varepsilon} d(x, gx) \leq d(x, gx),$$

Consequently, $x = gx$ (because $\varepsilon > 1$).

The assumption of continuity of one of the mappings f or g can be replaced by the condition that b-metric space $(X, \preceq, d)$ is regular.

In fact, let $(X, \preceq, d)$ be regular. Via the mentioned above, we can construct an increasing sequence $\{x_n\}$ in X such that $x_n \to x$ $(n \to \infty)$ for some $x \in X$. Then $x_n \preceq x$ for all $n \in \mathbb{N}$. We shall have to show that $fx = gx = x$.

First, we have

$$\frac{1}{s} d\left(x, gx\right) \leq d\left(x, x_{2n+1}\right) + d\left(fx_{2n}, gx\right). \tag{2.7}$$

By (2.1) we get

$$\psi\left(s^{\varepsilon} d\left(fx_{2n}, gx\right)\right) \leq \psi\left(M_s\left(x_{2n}, x\right)\right) + L\psi\left(N\left(x_{2n}, x\right)\right),$$

where

$$M_s\left(x_{2n}, x\right) = \max\left\{d\left(x_{2n}, x\right), d\left(x_{2n}, x_{2n+1}\right), d\left(x, gx\right), \frac{d\left(x_{2n}, gx\right) + d\left(x, x_{2n+1}\right)}{2s}\right\} \tag{2.8}$$

and

$$N\left(x_{2n}, x\right) = \min\left\{d\left(x, gx\right), d\left(x_{2n}, gx\right), d\left(x, x_{2n+1}\right)\right\}. \tag{2.9}$$

Letting $n \to \infty$ in (2.8) and (2.9) and using

$$\frac{d\left(x_{2n}, gx\right) + d\left(x, x_{2n+1}\right)}{2s} \leq \frac{d\left(x_{2n}, x\right) + d\left(x, gx\right)}{2} + \frac{d\left(x, x_{2n+1}\right)}{2s},$$

we obtain $\lim_{n\to\infty} M_s\left(x_{2n}, x\right) = d\left(x, gx\right)$ and $\lim_{n\to\infty} N\left(x_{2n}, x\right) = 0$. Further, we deduce that

$$\overline{\lim_{n\to\infty}}\, \psi\left(s^{\varepsilon} d\left(fx_{2n}, gx\right)\right) \leq \psi\left(\overline{\lim_{n\to\infty}}\, M_s\left(x_{2n}, x\right)\right) + L \cdot \psi\left(0\right) = \psi\left(d\left(x, gx\right)\right).$$

Since ψ is nondecreasing, we arrive at

$$\overline{\lim_{n\to\infty}}\, s^{\varepsilon} d\left(fx_{2n}, gx\right) \leq d\left(x, gx\right). \tag{2.10}$$

Now (2.7) and (2.10) imply that $gx = x$. Similarly, we claim that $fx = x$. □

Remark 2.4 Theorem 2.3 improves and generalizes the main results of [15] (also see Theorem 1.8 and Theorem 1.9) in several directions. For one thing, the constant $\varepsilon > 1$ is arbitrary and is not only limited to $\varepsilon = 2$ stated by Theorem 1.8 and Theorem 1.9. This probably brings us more convenience in applications. For another thing, Theorem 2.3 dismisses the cyclic representation. In addition, the proof Theorem 2.3 is much simpler than the one of Theorem 1.8 and Theorem 1.9.

Finally we announce the main result of this paper:

Theorem 2.5 Theorem 1.8 together with Theorem 1.9 is equivalent to Theorem 2.3 in case of $\varepsilon = 2$.

Proof For all details and explanations see [20], [21] and the proof of Theorem 2.1.

3. APPLICATION

By using Theorem 2.3, we shall consider the existence of solutions for the following integral equation with an unknown function u:

$$u\left(t\right) = \int_0^T G\left(t, s\right) f\left(s, u\left(s\right)\right) ds,\ t \in [0, T], \tag{3.1}$$

where $T > 0$ is a constant, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$, $G : [0, T] \times [0, T] \to [0, \infty)$ are given continuous functions.

Denote $X = C[0,T]$ be the set of real continuous functions on $[0,T]$ and let $d : X \times X \to \mathbb{R}^+$ be given by

$$d(u,v) = \max_{0\le t\le T} |u(t) - v(t)|^2, \quad \forall u,v \in X.$$

It is easy to check that (X,d) is a complete b-metric space with parameter $s = 2$. We endow X with the partial order given by

$$x \preceq y \Leftrightarrow x(t) \le y(t) \text{ for all } t \in [0,T].$$

Validly, $(X, \preceq, d)$ is regular.

Define a mapping $T : X \to X$ by

$$Tu(t) := \int_0^T G(t,z) f(z,u(z))\, dz, \quad t \in [0,T],$$

then u is a solution of the given equation (3.1) if and only if it is a fixed point of T. We shall prove that T has a fixed point under the following assumptions.

(i) For all $z \in [0,T]$, $f(z,.)$ is a decreasing function, that is, $x,y \in \mathbb{R}, x \ge y$ implies $f(z,x) \le f(z,y)$;

(ii) There exists a constant $\gamma > 0$ such that

$$\max_{0\le t\le T} \int_0^T G(t,z)\, dz \le \frac{10}{21\sqrt{\gamma}};$$

(iii) For all $z \in [0,T]$ and for all comparable $x,y \in X$,

$$\begin{aligned} 0 &\le |f(z,x(z)) - f(z,y(z))| \\ &\le \Bigg(\gamma \max\Bigg\{ |x(z)-y(z)|^2, |x(z)-Tx(z)|^2, |y(z)-Ty(z)|^2, \\ &\quad \frac{|x(z)-Ty(z)|^2 + |y(z)-Tx(z)|^2}{4} \Bigg\}\Bigg)^{\frac{1}{2}}. \end{aligned} \tag{3.2}$$

(iv) There exists a constant $\varepsilon \in \left(1, \frac{2\ln 2.1}{\ln 2}\right)$.

Theorem 3.1 Under the conditions (i)-(iv), the equation (3.1) has a solution $x^* \in X$.

Proof First of all, if $x \preceq y$, then by (i), we have

$$Ty(t) - Tx(t) = \int_0^T G(t,z)\left[f(z,y(z)) - f(z,x(z))\right] dz \ge 0, \quad t \in [0,T].$$

That is, $Tx \preceq Ty$. This means that T is increasing.

By virtue of (3.2), we have that

$$\begin{aligned} &[f(z,x) - f(z,y)]^2 \\ &\le \gamma \max\Bigg\{ d(x,y), d(x,Tx), d(y,Ty), \\ &\quad \frac{d(x,Ty) + d(y,Tx)}{4} \Bigg\}. \end{aligned} \tag{3.3}$$

Then for all $t \in [0, T]$ and all comparable $x, y \in X$, by (ii) and (3.3), we speculate that

$$
\begin{aligned}
d(Tx, Ty) &= \max_{t\in[0,T]} |Tx(t) - Ty(t)|^2 \\
&= \max_{t\in[0,T]} \left(\int_0^T G(t,z) \left[f(z, x(z)) - f(z, y(z)) \right] dz \right)^2 \\
&\leq \frac{100}{441} \max \left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{4} \right\}.
\end{aligned}
$$

By (iv), it follows that $\frac{100}{441} < \frac{1}{2^{\varepsilon}} = \frac{1}{s^{\varepsilon}}$, thus all the conditions of Theorem 2.3 are satisfied where ψ, φ are identity mappings and $T = f = g, L = 0$. So T has a fixed point $u(t) \in X$, that is, the integral equation (3.1) has a solution $u(t) \in X = C[0, T]$. □

Remark 3.2 In the above application we use ordinary fixed point theorem, while Corollary 2 of [15] uses cyclical-type fixed point result. Actually, these both results are equivalent, then our approach has an advantage because we use the conditions (i)-(iv), while in Corollary 2 of [15] authors utilize the conditions (4.2)-(4.7) as well as two subsets A_1 and A_2. Also, our application shows that their main result is not applicable.

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A FIXED POINT METHOD TO THE STABILITY OF A JENSEN FUNCTIONAL EQUATION IN INTUITIONISTIC FUZZY 2-BANACH SPACES

CHOONKIL PARK, EHSAN MOVAHEDNIA*, GEORGE A. ANASTASSIOU, AND SUNGSIK YUN*

Abstract. In this paper, we recall the notion of intuitionistic fuzzy 2-normed space introduced in [1] and using the fixed point method, we investigate the Hyers-Ulam stability of the following functional equation

$$2f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)+f\left(\frac{y-x}{2}\right)=f(x)+f(y) \qquad (1)$$

in intuitionistic fuzzy 2-Banach spaces.

1. Introduction

The concept of the stability for functional equations was introduced for the first time by Ulam in 1940 [2]. He proposed the famous Ulam stability problem for a metric group homomorphism. In 1941, Hyers [3] solved this stability problem for additive mappings subject to the Hyers condition on approximately additive mappings in Banach spaces. In 1951, Bourgin [4] treated the Ulam stability problem for additive mappings. Subsequently the result of Hyers was generalized by Rassias [5] for linear mapping by considering an unbounded Cauchy difference.

The functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. The Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [6] for mappings $f:\mathcal{X}\to\mathcal{Y}$, where $\mathcal{X}$ is a normed space and $\mathcal{Y}$ is a Banach space.

In 1984, Katsaras [7] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space. Later, some mathematicians have defined fuzzy norms on a linear space from various points of view [8, 9]. In particular, in 2003, Bag and Samanta [10], following Cheng and Mordeson [11], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [12]. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces. Recently, considerable attention has been increasing to the problem of fuzzy stability of functional equations. Several various stability results concerning Cauchy, Jensen, simple quadratic, and cubic functional equations have been investigated [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24].

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*Corresponding author.

Quite recently, the stability results in the setting of intuitionistic fuzzy normed space have been studied in [25, 26, 27, 28]; respectively, while the idea of intuitionistic fuzzy normed space was introduced in [29].

2. Preliminaries

Definition 2.1. *Let $\mathcal{X}$ be a real linear space of dimension greater than one and let $\|\cdot,\cdot\|$ be a real-valued function on $\mathcal{X}\times\mathcal{X}$ satisfying the following condition:*
(1) $\|x,y\|=\|y,x\|$ *for all* $x,y\in\mathcal{X}$;
(2) $\|x,y\|=0$ *if and only if* x,y *are linearly dependent;*
(3) $\|\alpha x,y\|=|\alpha|\|x,y\|$ *for all* $x,y\in\mathcal{X}$ *and* $\alpha\in\mathbb{R}$;
(4) $\|x,y+z\|\leq\|x,y\|+\|x,z\|$ *for all* $x,y,z\in\mathcal{X}$.
Then the function $\|\cdot,\cdot\|$ is called a 2-norm on X and the pair $(X,\|\cdot,\cdot\|)$ is called a 2-normed linear space.

Definition 2.2. *A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if $*$ satisfies the following conditions:*
(1) $*$ *is commutative and associative;*
(2) $*$ *is continuous;*
(3) $a*1=a$ *for all* $a\in[0,1]$;
(4) $a*b\leq c*d$, *whenever* $a\leq c$ *and* $b\leq d$ *for all* $a,b,c,d\in[0,1]$.

Example 2.1. *An example of continuous t-norm is*

$$a*b=\min\{a,b\}.$$

Definition 2.3. *A binary operation $\diamond:[0,1]\times[0,1]\to[0,1]$ is a continuous t-conorm if $\diamond$ satisfies the following conditions:*
(1) $\diamond$ *is commutative and associative;*
(2) $\diamond$ *is continuous;*
(3) $a\diamond 0=a$ *for all* $a\in[0,1]$;
(4) $a\diamond b\leq c\diamond d$, *whenever* $a\leq c$ *and* $b\leq d$ *for all* $a,b,c,d\in[0,1]$.

Example 2.2. *An example of continuous t-conorm is*

$$a\diamond b=\max\{a,b\}.$$

Definition 2.4. *Let $\mathcal{X}$ be a real linear space. A fuzzy subset μ of $\mathcal{X}\times\mathcal{X}\times\mathbb{R}$ is called a fuzzy 2-norm on $\mathcal{X}$ if and only if for all $x,y,z\in\mathcal{X}$, and $t,s,c\in\mathbb{R}$,*
(1) $\mu(x,y,t)=0$ *for all* $t\leq 0$.
(2) $\mu(x,y,t)=1$ *if and only if* x,y *are linearly dependent for all* $t>0$.
(3) $\mu(x,y,t)$ *is invariant under any permutation of* x,y.
(4) $\mu(x,cy,t)=\mu(x,y,\frac{t}{|c|})$ *for all* $t>0$ *and* $c\neq 0$.
(5) $\mu(x+z,y,t+s)\geq\mu(x,y,t)*\mu(z,y,s)$ *for all* $t,s>0$.
(6) $\mu(x,y,.)$ *is a non-decreasing function on* $\mathbb{R}$ *and*

$$\lim_{t\to\infty}\mu(x,y,t)=1$$

Then μ is said to be a fuzzy 2-norm on a linear space $\mathcal{X}$, and the pair $(\mathcal{X},\mu)$ is called a fuzzy 2-normed linear space.

Example 2.3. *Let $(\mathcal{X}, \|\cdot,\cdot\|)$ be a 2-normed linear space. Define*

$$\mu(x,y,t) = \begin{cases} \frac{t}{t+\|x,y\|} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0, \end{cases}$$

where $x, y \in \mathcal{X}$ and $t \in \mathbb{R}$. Then $(\mathcal{X}, \mu)$ is a fuzzy 2-normed linear space.

Definition 2.5. *Let $(\mathcal{X}, \mu)$ be a fuzzy 2-normed linear space. Let $\{x_n\}$ be a sequence in $\mathcal{X}$. Then $\{x_n\}$ is said to be convergent if there exists $x \in \mathcal{X}$ such that*

$$\lim_{n\to\infty} \mu(x_n - x, y, t) = 1$$

for all $t > 0$ and all $y \in \mathcal{X}$.

Definition 2.6. *Let $(\mathcal{X}, \mu)$ be a fuzzy 2-normed linear space. Let $\{x_n\}$ be a sequence in $\mathcal{X}$. Then $\{x_n\}$ is said to be a Cauchy sequence if*

$$\lim_{n\to\infty} \mu(x_{n+p} - x_n, y, t) = 1$$

for all $t > 0$, all $y \in \mathcal{X}$ and $p = 1, 2, 3, \cdots$.

Let $(\mathcal{X}, \mu)$ be a fuzzy 2-normed linear space and $\{x_n\}$ be a Cauchy sequence in $\mathcal{X}$. If $\{x_n\}$ is convergent in $\mathcal{X}$ then $(\mathcal{X}, \mu)$ is said to be a fuzzy 2-Banach space.

Definition 2.7. *Let $\mathcal{X}$ be a real linear space. A fuzzy subset ν of $\mathcal{X} \times \mathcal{X} \times \mathbb{R}$ such that for all $x, y, z \in \mathcal{X}$, and $t, s, c \in \mathbb{R}$,*

(1) *$\nu(x, y, t) = 1$ for all $t \leq 0$.*
(2) *$\nu(x, y, t) = 0$ if and only if x, y are linearly dependent for all $t > 0$.*
(3) *$\nu(x, y, t)$ is invariant under any permutation of x, y.*
(4) *$\nu(x, cy, t) = \nu(x, y, \frac{t}{|c|})$ for all $t > 0$, $c \neq 0$.*
(5) *$\nu(x, y+z, t+s) \leq \nu(x, y, t) \diamond \nu(x, z, s)$ for all $s, t > 0$*
(6) *$\nu(x, y, .)$ is a nonincreasing function and*

$$\lim_{t\to\infty} \nu(x, y, t) = 0$$

Then ν is said to be an anti fuzzy 2-norm on a linear space $\mathcal{X}$ and the pair $(\mathcal{X}, \nu)$ is called an anti fuzzy 2-normed linear space.

Definition 2.8. *Let $(\mathcal{X}, \nu)$ be an anti fuzzy 2-normed linear space and $\{x_n\}$ be a sequence in $\mathcal{X}$. Then $\{x_n\}$ is said to be convergent if there exists $x \in \mathcal{X}$ such that*

$$\lim_{n\to\infty} \nu(x_n - x, y, t) = 0$$

for all $t > 0$ and all $y \in \mathcal{X}$.

Definition 2.9. *Let $(\mathcal{X}, \nu)$ be an anti fuzzy 2-normed linear space and $\{x_n\}$ be a sequence in $\mathcal{X}$. Then $\{x_n\}$ is said to be a Cauchy sequence if*

$$\lim_{n\to\infty} \nu(x_{n+p} - x_n, y, t) = 0$$

for all $t > 0$, all $y \in \mathcal{X}$ and $p = 1, 2, 3, \cdots$.

Let $(\mathcal{X},\nu)$ be an anti fuzzy 2-normed linear space and $\{x_n\}$ be a Cauchy sequence in $\mathcal{X}$. If $\{x_n\}$ is convergent in $\mathcal{X}$ then $(\mathcal{X},\nu)$ is said to be an anti fuzzy 2-Banach space.

The following lemma is easy to prove and we will omit it.

Lemma 2.1. *Consider the set L^* and operation $\leq_{L^*}$ defined by*

$$L^* = \{(x_1,x_2) : (x_1,x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\}$$

$$(x_1,x_2) \leq_{L^*} (y_1,y_2) \Longleftrightarrow x_1 \leq y_1\ ,\ x_2 \geq y_2$$

for all $(x_1,x_2),(y_1,y_2) \in L^*$. Then $(L^*,\leq_{L^*})$ is a complete lattice.

Definition 2.10. *A continuous t-norm τ on $L=[0,1]^2$ is said to be continuous t-representable if there exist a continuous t-norm $*$ and a continuous t-conorm $\diamond$ on $[0,1]$ such that, for all $x=(x_1,x_2), y=(y_1,y_2) \in L$,*

$$\tau(x,y) = (x_1 * y_1, x_2 \diamond y_2).$$

Definition 2.11. *Let $\mathcal{X}$ be a set. A function $d : \mathcal{X}\times\mathcal{X} \to [0,\infty]$ is called a generalized metric on $\mathcal{X}$ if and only if d satisfies:*

(M_1) $d(x,y)=0 \Leftrightarrow x=y\ \forall x,y \in \mathcal{X}$
(M_2) $d(x,y)=d(y,x)\ \forall x,y \in \mathcal{X}$
(M_3) $d(x,z) \leq d(x,y)+d(y,z)\ \forall x,y,z \in \mathcal{X}$

Theorem 2.1. ([30]) *Let $(\mathcal{X},d)$ be a complete generalized metric space and $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then, for all $x \in \mathcal{X}$, either*

$$d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(a) *$d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) < \infty$ for all $n \geq n_0$;*
(b) *the sequence $\{\mathcal{J}^n x\}$ converges to a fixed point y^* of $\mathcal{J}$;*
(c) *y^* is the unique fixed point of $\mathcal{J}$ in the set $\mathcal{Y} = \{y \in \mathcal{X} : d(\mathcal{J}^{n_0}x, y) < \infty\}$;*
(d) *$d(y,y^*) \leq \frac{1}{1-L} d(y,\mathcal{J}y)$ for all $y \in \mathcal{Y}$.*

This theorem was used by Cădariu and Radu (see [31, 32, 33, 34]) and then others to obtain the applications of fixed point theory in stability problems (cf. [24, 35, 36, 37, 38, 39, 40, 41, 42, 43]).

Definition 2.12. *A 3-tuple $(\mathcal{X},\rho_{\mu,\nu},\tau)$ is said to be an intuitionistic fuzzy 2-normed space(for short, IF2NS) if $\mathcal{X}$ is a real linear space, and μ and ν are a fuzzy 2-norm and an anti fuzzy 2-norm, respectively, such that $\nu(x,y,t)+\mu(x,y,t) \leq 1$, τ is continuous t-representable, and*

$$\rho_{\mu,\nu} : \mathcal{X}\times\mathcal{X}\times\mathbb{R} \to L^*$$

$$\rho_{\mu,\nu}(x,y,t) = (\mu(x,y,t),\nu(x,y,t))$$

is a function satisfying the following conditions, for all $x,y,z \in \mathcal{X}$, and $t,s,\alpha \in \mathbb{R}$,

(1) *$\rho_{\mu,\nu}(x,y,t) = (0,1) = 0_{L^*}$ for all $t \leq 0$.*
(2) *$\rho_{\mu,\nu}(x,y,t) = (1,0) = 1_{L^*}$ if and only if x,y are linearly dependent, for all $t>0$.*
(3) *$\rho_{\mu,\nu}(\alpha x,y,t) = \rho_{\mu,\nu}(x,y,\frac{t}{|\alpha|})$ for all $t>0$ and $\alpha \neq 0$*
(4) *$\rho_{\mu,\nu}(x,y,t)$ is invariant under any permutation of x,y.*
(5) *$\rho_{\mu,\nu}(x+z,y,t+s) \geq_{L^*} \tau(\rho_{\mu,\nu}(x,y,t),\rho_{\mu,\nu}(z,y,s))$ for all $t,s>0$.*

(6) *$\rho_{\mu,\nu}(x,y,.)$ is continuous and*

$$\lim_{t\to 0}\rho_{\mu,\nu}(x,y,t)=0_{L^*} \text{ and } \lim_{t\to\infty}\rho_{\mu,\nu}(x,y,t)=1_{L^*}$$

Then $\rho_{\mu,\nu}$ is said to be an intuitionstic fuzzy 2-norm on a real linear space $\mathcal{X}$.

Example 2.4. *Let $(\mathcal{X},\|\cdot,\cdot\|)$ be a 2-normed space,*

$$\tau(a,b)=(a_1b_1,\min(a_2+b_2,1))$$

be continuous t-representable for all $a=(a_1,a_2), b=(b_1,b_2)\in L^*$ and μ,ν be a fuzzy and an anti fuzzy 2-norm, respectively. We define

$$\rho_{\mu,\nu}(x,y,t)=\left(\frac{t}{t+m\|x,y\|},\frac{\|x,y\|}{t+m\|x,y\|}\right)$$

for all $t\in\mathbb{R}^+$ and $m>1$. Then $(\mathcal{X},\rho_{\mu,\nu},\tau)$ is an IF2NS.

Definition 2.13. *A sequence $\{x_n\}$ in an IF2NS $(\mathcal{X},\rho_{\mu,\nu},\tau)$ is said to be convergent to a point $x\in\mathcal{X}$ if*

$$\lim_{n\to\infty}\rho_{\mu,\nu}(x_n-x,y,t)=1_{L^*}$$

for all $t>0$ and all $y\in\mathcal{X}$.

Definition 2.14. *A sequence $\{x_n\}$ in an IF2NS $(\mathcal{X},\rho_{\mu,\nu},\tau)$ is said to be a Cauchy sequence if for any $0<\epsilon<1$ and $t>0$, there exists $n_0\in\mathcal{N}$ such that*

$$\rho_{\mu,\nu}(x_n-x_m,y,t)\geq_{L^*}(1-\epsilon,\epsilon)$$

for all $n,m\geq n_0$ and all $y\in\mathcal{X}$.

Definition 2.15. *An IF2NS space $(\mathcal{X},\rho_{\mu,\nu},\tau)$ is said to be complete if every Cauchy sequence in $(\mathcal{X},\rho_{\mu,\nu},\tau)$ is convergent. A complete intuitionistic fuzzy 2-normed space is called an intuitionistic fuzzy 2-Banach space.*

3. Hyers-Ulam stability of the functional equation (1) in IF2NS: an odd mapping case

Using the fixed point alternative approach, we prove the Hyers-Ulam stability of the functional equation (1) in intuitionistic fuzzy 2-Banach spaces for an odd mapping case.

Let $\mathcal{X},\mathcal{Y}$ be real linear spaces. For a given mapping $f:\mathcal{X}\to\mathcal{Y}$, we define

$$Df(x,y):=2f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)+f\left(\frac{y-x}{2}\right)-f(x)-f(y).$$

Lemma 3.1. *Let $\mathcal{X},\mathcal{Y}$ be real linear spaces. An odd mapping $f:\mathcal{X}\to\mathcal{Y}$ satisfies*

$$2f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)+f\left(\frac{y-x}{2}\right) = f(x)+f(y) \tag{2}$$

if and only if it is Jensen additive.

Proof. Assume that $f:\mathcal{X}\to\mathcal{Y}$ satisfies (2). Since f is odd, we have $f(-x)=-f(x)$ for all $x,y\in\mathcal{X}$. It follows from (2) that $2f\left(\frac{x+y}{2}\right)=f(x)+f(y)$ for all $x,y\in\mathcal{X}$.

Conversely, assume that $f:\mathcal{X}\to\mathcal{Y}$ is Jensen additive. Then it is easy to show that f satisfies (2). □

Theorem 3.1. *Let $\mathcal{X}$ be a real linear space, $(\mathcal{Z}, \rho'_{\mu,\nu}, \tau')$ an intuitionistic fuzzy 2-normed space and let $\phi : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}, \varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}$ be mappings such that for some $0 < \alpha^2 < 2$*

$$\rho'_{\mu,\nu}\left(\phi(2x, 2y), \varphi(2x, 2y), t\right) \geq_{L^*} \rho'_{\mu,\nu}\left(\alpha\phi(x, y), \varphi(x, y), t\right) \tag{3}$$

for all $x, y \in \mathcal{X}$ and $t \in \mathbb{R}^+$. Let $(\mathcal{Y}, \rho_{\mu,\nu}, \tau)$ be a complete intuitionistic fuzzy 2-normed space. If $\xi : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ is a mapping such that $\xi(2x, 2y) = \frac{1}{\alpha}\xi(x, y)$ for all $x, y \in \mathcal{X}$ and $f : \mathcal{X} \to \mathcal{Y}$ is an odd mapping such that

$$\rho_{\mu,\nu}(Df(x, y), \xi(x, y), t) \geq_{L^*} \rho'_{\mu,\nu}(\phi(x, y), \varphi(x, y), t) \tag{4}$$

for all $x, y \in \mathcal{X}, t > 0$, then there is a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\rho_{\mu,\nu}(f(x) - A(x), \xi(x, 0), t) \geq_{L^*} \rho'_{\mu,\nu}\left(\phi(x, 0), \varphi(x, 0), \frac{2 - \alpha^2}{\alpha^2} t\right) \tag{5}$$

Proof. Putting $y = 0$ in (4), we have

$$\rho_{\mu,\nu}\left(2f\left(\frac{x}{2}\right) - f(x), \xi(x, 0), t\right) \geq_{L^*} \rho'_{\mu,\nu}\left(\phi(x, 0), \varphi(x, 0), t\right). \tag{6}$$

Replacing x by $2x$ in (6), we have

$$\rho_{\mu,\nu}\left(2f(x) - f(2x), \xi(2x, 0), t\right) \geq_{L^*} \rho'_{\mu,\nu}\left(\phi(2x, 0), \varphi(2x, 0), t\right). \tag{7}$$

It follows from (3), (7) and the property of ξ that

$$\begin{aligned} \rho_{\mu,\nu}\left(f(x) - \frac{f(2x)}{2}, \xi(x, 0), t\right) &\geq_{L^*} \rho'_{\mu,\nu}\left(\phi(2x, 0), \varphi(2x, 0), \frac{2}{\alpha} t\right) \\ &\geq_{L^*} \rho'_{\mu,\nu}\left(\frac{\alpha^2}{2}\phi(x, 0), \varphi(x, 0), t\right) \end{aligned}$$

for all $x \in \mathcal{X}$ and $t > 0$.

Consider the set $\Omega = \{g : \mathcal{X} \to \mathcal{Y}\}$ and define a generalized metric d on Ω by

$$d(g, h) = \inf\left\{c \in \mathbb{R}^+ : \rho_{\mu,\nu}(g(x) - h(x), \xi(x, 0), t) \geq_{L^*} \rho'_{\mu,\nu}(c\phi(x, 0), \varphi(x, 0), t)\right\}$$

for all $x \in \mathcal{X}$ and $t > 0$ with $\inf \emptyset = \infty$. It is easy to show that (Ω, d) is complete (see [44]).

Define $J : \mathcal{X} \to \mathcal{X}$ by $Jg(x) = \dfrac{g(2x)}{2}$ for all $x \in \mathcal{X}$. Now, we prove that J is strictly contractive mapping of Ω with the Lipschitz constant $\frac{\alpha^2}{2}$.
Let $g, h \in E$ be given such that $d(g, h) < \epsilon$. Then

$$\rho_{\mu,\nu}\left(g(x) - h(x), \xi(x, 0), t\right) \geq_{L^*} \rho'_{\mu,\nu}(\epsilon\phi(x, 0), \varphi(x, 0), t)$$

for all $x \in \mathcal{X}$ and $t > 0$. So

$$\rho_{\mu,\nu}\left(Jg(x) - Jh(x), \xi(x, 0), t\right) = \rho_{\mu,\nu}\left(g(2x) - h(2x), \xi(2x, 0), \frac{2}{\alpha} t\right)$$

$$\geq_{L^*} \rho'_{\mu,\nu}\left(\epsilon\phi(2x, 0), \varphi(2x, 0), \frac{2}{\alpha} t\right) =_{L^*} \rho'_{\mu,\nu}\left(\frac{\alpha^2}{2}\epsilon\phi(x, 0), \varphi(x, 0), t\right).$$

Then $d(Jg, Jh) \leq \frac{\alpha^2}{2}\, d(g, h)$ for all $g, h \in \Omega$. It follows from (7) that

$$d(f, Jf) \leq \frac{\alpha^2}{2} < \infty$$

It follows from Theorem 2.1 that there exists a mapping $A : \mathcal{X} \to \mathcal{Y}$ satisfying the following:

(1) A is a fixed point of J, that is,

$$A(2x) = 2A(x) \tag{8}$$

(2) The mapping A is a unique fixed point of J in the set

$$\Delta = \{h \in \Omega : d(g, h) < \infty\}$$

This implies that A is a unique mapping satisfying (8).

(3) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies that

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = A(x)$$

for all $x \in X$.

(4) $d(f, A) \leq \frac{1}{1-L} d(f, Jf)$ with $f \in \Delta$, which implies the inequality $d(f, A) \leq \frac{\alpha^2}{2-\alpha^2}$. So

$$\rho_{\mu,\nu}\left(f(x) - A(x), \xi(x, 0), t\right) \geq_{L^*} \rho'_{\mu,\nu}\left(\phi(x, 0), \varphi(x, 0), \frac{2-\alpha^2}{\alpha^2} t\right).$$

This implies that the inequality (5) holds.

It remains to show that A is an additive mapping. Replacing x and y by $2^n x$ and $2^n y$ in (4), respectively, we get

$$\rho_{\mu,\nu}\left(\frac{1}{2^n} Df(2^n x, 2^n y), \xi(2^n x, 2^n y), \frac{t}{2^n}\right) \geq_{L^*} \rho'_{\mu,\nu}\left(\phi(2^n x, 2^n y), \varphi(2^n x, 2^n y), t\right).$$

By the property of $\xi(x, y)$, we have

$$\rho_{\mu,\nu}\left(\frac{1}{2^n} Df(2^n x, 2^n y), \frac{1}{\alpha^n}\xi(x, y), \frac{t}{2^n}\right) \geq_{L^*} \rho'_{\mu,\nu}\left(\phi(2^n x, 2^n y), \varphi(2^n x, 2^n y), t\right).$$

Thus

$$\rho_{\mu,\nu}\left(\frac{1}{2^n} Df(2^n x, 2^n y), \xi(x, y), t\right) \geq_{L^*} \rho'_{\mu,\nu}\left(\phi(2^n x, 2^n y), \varphi(2^n x, 2^n y), \frac{2^n\ t}{\alpha^n}\right)$$
$$\geq_{L^*} \rho'_{\mu,\nu}\left(\alpha^n \phi(x, y), \varphi(x, y), \frac{2^n\ t}{\alpha^n}\right) =_{L^*} \rho'_{\mu,\nu}\left(\frac{\alpha^{2n}}{2^n}\phi(x, y), \varphi(x, y), t\right).$$

As $n \to \infty$, we have

$$\rho_{\mu,\nu}(DA(x, y), \xi(x, y), t) \geq_{L^*} 1_{L^*}.$$

Thus A is an additive mapping, as desired. □

Corollary 3.1. *Let X be a linear space, $(\mathcal{Z}, \rho'_{\mu,\nu}, \tau')$ be an IF2N-space, $(\mathcal{Y}, \rho_{\mu,\nu}, \tau)$ be a complete IF2N-space, p be real number and $z_0, z_1 \in \mathcal{Z}$. If $\xi : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ is a mapping such that $\xi(2x, 2y) = \frac{1}{2^p}\xi(x, y)$ for all $x, y \in \mathcal{X}$ and $f : \mathcal{X} \to \mathcal{Y}$ is an odd mapping such that*

$$\rho_{\mu,\nu}(Df(x, y), \xi(x, y), t) \geq_{L^*} \rho'_{\mu,\nu}((\|x\|^p + \|y\|^p)z_0, z_1, t)$$

for all $x, y \in \mathcal{X}$, $t > 0$ and $0 < p < \frac{1}{2}$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\rho_{\mu,\nu}(f(x) - A(x), \xi(x, 0), t) \geq_{L^*} \rho'_{\mu,\nu}\left(\|x\|^p z_0, z_1, \frac{2 - 2^{2p}}{2^{2p}}t\right)$$

for all $x \in \mathcal{X}$ and $t > 0$.

Proof. Let $\phi, \varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}$ be defined by $\phi(x, y) = (\|x\|^p + \|y\|^p) z_0$ and $\varphi(x, y) = z_1$. Then the result follows from Theorem 3.1 by taking $\alpha = 2^p$. □

Corollary 3.2. *Let X be a linear space, $(\mathcal{Z}, \rho'_{\mu,\nu}, \tau')$ be an IF2N-space, $(Y, \rho_{\mu,\nu}, \tau)$, be a complete IF2N-space and let $z_0, z_1 \in \mathcal{Z}$. If $f : \mathcal{X} \to \mathcal{Y}$ is an odd mapping such that*

$$\rho_{\mu,\nu}(Df(x, y), \xi(x, y), t) \geq_{L^*} \rho'_{\mu,\nu}(\epsilon z_0, z_1, t)$$

for all $x, y \in \mathcal{X}, t > 0$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\rho_{\mu,\nu}(f(x) - A(x), \xi(x, 0), t) \geq_{L^*} \rho'_{\mu,\nu}(\epsilon z_0, z_1, t)$$

for all $x \in \mathcal{X}$ and $t > 0$.

Proof. Let $\phi, \varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}$ be defined by $\phi(x, y) = z_0$ and $\varphi(x, y) = z_1$. Then the result follows from Theorem 3.1 by taking $\alpha = 1$. □

4. Hyers-Ulam stability of the functional equation (1) in IF2NS: an even mapping case

Using the fixed point alternative approach, we prove the Hyers-Ulam stability of the functional equation (1) in intuitionistic fuzzy 2-Banach spaces for an even mapping case.

Lemma 4.1. *Let $\mathcal{X}, \mathcal{Y}$ be real linear spaces. An even mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies*

$$2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) = f(x) + f(y) \tag{9}$$

if and only if it is Jensen quadratic.

Proof. Assume that $f : \mathcal{X} \to \mathcal{Y}$ satisfies (9). Since f is even, we have $f(-x) = f(x)$ for all $x, y \in \mathcal{X}$. It follows from (9) that $2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = f(x) + f(y)$ for all $x, y \in \mathcal{X}$.

Conversely, assume that $f : \mathcal{X} \to \mathcal{Y}$ is Jensen quadratic. Then it is easy to show that f satisfies (9). □

Theorem 4.1. *Let $\mathcal{X}$ be a real linear space, $(\mathcal{Z}, \rho'_{\mu,\nu}, \tau')$ an intuitionistic fuzzy 2-normed space and let $\phi : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}, \varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}$ be mappings such that for some $0 < \alpha^2 < 4$*

$$\rho'_{\mu,\nu}\left(\phi(2x, 2y), \varphi(2x, 2y), t\right) \geq_{L^*} \rho'_{\mu,\nu}\left(\alpha\phi(x, y), \varphi(x, y), t\right) \tag{10}$$

for all $x, y \in \mathcal{X}$ and $t \in \mathbb{R}^+$. Let $(\mathcal{Y}, \rho_{\mu,\nu}, \tau)$ be a complete intuitionistic fuzzy 2-normed space. If $\xi : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ is a mapping such that $\xi(2x, 2y) = \frac{1}{\alpha}\xi(x, y)$ for all $x, y \in \mathcal{X}$ and $f : \mathcal{X} \to \mathcal{Y}$

is an even mapping satisfying $f(0) = 0$ *and* (4), *then there is a unique quadratic mapping* $Q : \mathcal{X} \to \mathcal{Y}$ *such that*

$$\rho_{\mu,\nu}(f(x) - Q(x), \xi(x, 0), t) \geq_{L^*} \rho'_{\mu,\nu}\left(\phi(x, 0), \varphi(x, 0), \frac{4-\alpha^2}{\alpha^2}t\right) \tag{11}$$

for all $x \in \mathcal{X}$ *and* $t > 0$.

Proof. Putting $y = 0$ in (4), we have

$$\rho_{\mu,\nu}\left(4f\left(\frac{x}{2}\right) - f(x), \xi(x, 0), t\right) \geq_{L^*} \rho'_{\mu,\nu}\left(\phi(x, 0), \varphi(x, 0), t\right). \tag{12}$$

Replacing x by $2x$ in (12), we have

$$\rho_{\mu,\nu}\left(4f(x) - f(2x), \xi(2x, 0), t\right) \geq_{L^*} \rho'_{\mu,\nu}\left(\phi(2x, 0), \varphi(2x, 0), t\right). \tag{13}$$

It follows from (10), (13) and the property of ξ that

$$\begin{aligned}\rho_{\mu,\nu}\left(f(x) - \frac{f(2x)}{4}, \xi(x, 0), t\right) &\geq_{L^*} \rho'_{\mu,\nu}\left(\phi(2x, 0), \varphi(2x, 0), \frac{4}{\alpha}t\right)\\ &\geq_{L^*} \rho'_{\mu,\nu}\left(\frac{\alpha^2}{4}\phi(x, 0), \varphi(x, 0), t\right)\end{aligned}$$

for all $x \in \mathcal{X}$ and $t > 0$.

Consider the set $\Omega = \{g : \mathcal{X} \to \mathcal{Y}\}$ and define a generalized metric d on Ω as in Theorem 3.1.

Define $J : \mathcal{X} \to \mathcal{X}$ by $Jg(x) = \dfrac{g(2x)}{4}$ for all $x \in \mathcal{X}$. Now, we prove that J is strictly contractive mapping of Ω with the Lipschitz constant $\frac{\alpha^2}{4}$.
Let $g, h \in E$ be given such that $d(g, h) < \epsilon$. Then

$$\rho_{\mu,\nu}\left(g(x) - h(x), \xi(x, 0), t\right) \geq_{L^*} \rho'_{\mu,\nu}(\epsilon\phi(x, 0), \varphi(x, 0), t)$$

for all $x \in \mathcal{X}$ and $t > 0$. So

$$\rho_{\mu,\nu}\left(Jg(x) - Jh(x), \xi(x, 0), t\right) = \rho_{\mu,\nu}\left(g(2x) - h(2x), \xi(2x, 0), \frac{4}{\alpha}t\right)$$
$$\geq_{L^*} \rho'_{\mu,\nu}\left(\epsilon\phi(2x, 0), \varphi(2x, 0), \frac{4}{\alpha}t\right) =_{L^*} \rho'_{\mu,\nu}\left(\frac{\alpha^2}{4}\epsilon\phi(x, 0), \varphi(x, 0), t\right).$$

Then $d(Jg, Jh) \leq \dfrac{\alpha^2}{4}\, d(g, h)$ for all $g, h \in \Omega$. It follows from (13) that $d(f, Jf) \leq \dfrac{\alpha^2}{4} < \infty$.

It follows from Theorem 2.1 that there exists a mapping $Q : \mathcal{X} \to \mathcal{Y}$ satisfying the following:

(1) Q is a fixed point of J, that is,

$$Q(2x) = 4Q(x) \tag{14}$$

(2) The mapping Q is a unique fixed point of J in the set

$$\Delta = \{h \in \Omega : d(g, h) < \infty\}$$

This implies that Q is a unique mapping satisfying (14).

(3) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies that

$$\lim_{n\to\infty} \frac{f(2^n x)}{4^n} = Q(x)$$

for all $x \in X$.

(4) $d(f,Q) \leq \dfrac{1}{1-L} d(f, Jf)$ with $f \in \Delta$, which implies the inequality $d(f,Q) \leq \frac{\alpha^2}{4-\alpha^2}$. So

$$\rho_{\mu,\nu}\left(f(x) - Q(x), \xi(x,0), t\right) \geq_{L^*} \rho'_{\mu,\nu}\left(\phi(x,0), \varphi(x,0), \frac{4-\alpha^2}{\alpha^2} t\right).$$

This implies that the inequality (11) holds.

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 4.1. *Let X be a linear space, $(\mathcal{Z}, \rho'_{\mu,\nu}, \tau')$ be an IF2N-space, $(\mathcal{Y}, \rho_{\mu,\nu}, \tau)$ be a complete IF2N-space, p be real number and $z_0, z_1 \in \mathcal{Z}$. If $\xi : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ is a mapping such that $\xi(2x, 2y) = \frac{1}{2^p}\xi(x,y)$ for all $x, y \in \mathcal{X}$ and $f : \mathcal{X} \to \mathcal{Y}$ is an even mapping satisfying $f(0) = 0$ and*

$$\rho_{\mu,\nu}(Df(x,y), \xi(x,y), t) \geq_{L^*} \rho'_{\mu,\nu}((\|x\|^p + \|y\|^p) z_0, z_1, t)$$

for all $x, y \in \mathcal{X}$, $t > 0$ and $0 < p < 1$, then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\rho_{\mu,\nu}(f(x) - Q(x), \xi(x,0), t) \geq_{L^*} \rho'_{\mu,\nu}\left(\|x\|^p z_0, z_1, \frac{4-4^p}{4^p} t\right)$$

for all $x \in \mathcal{X}$ and $t > 0$.

Proof. Let $\phi, \varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}$ be defined by $\phi(x,y) = (\|x\|^p + \|y\|^p) z_0$ and $\varphi(x,y) = z_1$. Then the result follows from Theorem 4.1 by taking $\alpha = 2^p$. □

Corollary 4.2. *Let X be a linear space, $(\mathcal{Z}, \rho'_{\mu,\nu}, \tau')$ be an IF2N-space, $(Y, \rho_{\mu,\nu}, \tau)$, be a complete IF2N-space and let $z_0, z_1 \in \mathcal{Z}$. If $f : \mathcal{X} \to \mathcal{Y}$ is an even mapping satisfying $f(0) = 0$ and*

$$\rho_{\mu,\nu}(Df(x,y), \xi(x,y), t) \geq_{L^*} \rho'_{\mu,\nu}(\epsilon z_0, z_1, t)$$

for all $x, y \in \mathcal{X}, t > 0$, then there exists a unique quadratic mapping $Q : \mathcal{X} \to \mathcal{Y}$ such that

$$\rho_{\mu,\nu}(f(x) - Q(x), \xi(x,0), t) \geq_{L^*} \rho'_{\mu,\nu}(\epsilon z_0, z_1, 3t)$$

for all $x \in \mathcal{X}$ and $t > 0$.

Proof. Let $\phi, \varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{Z}$ be defined by $\phi(x,y) = z_0$ and $\varphi(x,y) = z_1$. Then the result follows from Theorem 4.1 by taking $\alpha = 1$. □

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Choonkil Park
Department of Mathematics, Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Korea
E-mail address: baak@hanyang.ac.kr

Ehsan Movahednia
Department of Mathematics, Behbahan Khatam Al-Anbia University of Technology, Behbahan, Iran
E-mail address: movahednia@bkatu.ac.ir

George A. Anastassiou
Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA
E-mail address: ganastss@memphis.edu

Sungsik Yun
Department of Financial Mathematics, Hanshin University, Gyeonggi-do 447-791, Korea
E-mail address: ssyun@hs.ac.kr

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Characterization of modular spaces

Manuel De la Sen[1], Donal O'Regan[2], Reza Saadati[3]

[1]Institute of Research and Development of Processes

University of the Basque Country Campus of Leioa

Barrio Sarriena 48940 (Bizkaia), Spain

manuel.delasen@ehu.eus

[2]School of Mathematics Statistics and Applied Mathematics

National University of Ireland, Galway, Ireland.

donal.oregan@nuigalway.ie

[3]Department of Mathematics

Iran University of Science and Technology, Tehran, Iran.

rsaadati@iust.ac.ir

Abstract

In this paper we study the structure of modular spaces and random normed spaces and we show that a modular could induce a random norm

[0]Tel/Fax: +981212263650

and vice versa. Also we prove the topology generated by a modular (with a certain property) coincides with the topology generated by a random norm, and so in some situations the study of modular spaces reduces to the study of random normed spaces.

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Keywords: modular spaces; random normed spaces; topology.

1 Introduction

Orlicz and Birnbaum generalized the Lebesgue function spaces L^p and the theory of Orlicz spaces inspired Nakano [1] to develop the theory of modular spaces. This was generalized by Musielak and Orlicz [2]. For a good introduction to the theory of Orlicz spaces we refer the reader to Krasnoselskii and Rutickii [3]. In this paper, we show that a modular could induce a random norm and vice versa and also we show that the topology generated by a modular (with a certain property) coincides with the topology generated by a random norm.

2 Modular spaces

We start with a brief introduction to modular spaces (see [4–6, 8, 9]).

Let X be a vector space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). A functional $\rho : X \to [0, \infty]$ is called a modular, if for f, $g \in X$, we have for any $\alpha \in \mathbb{F}$:

(i) $\rho(f) = 0$ if and only if $f = 0$;

(ii) $\rho(\alpha f) = \rho(f)$ whenever $|\alpha| = 1$;

(iii) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ whenever $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.

If ρ is a modular in X, then the set defined by

$$X_\rho = \{h \in X : \lim_{\lambda \to 0} \rho(\lambda h) = 0\} \tag{2.1}$$

is called *a modular space.*

Definition 2.1. Let X_ρ be a modular space. The sequence $\{f_n\}_{n\in\mathbb{N}}$ in X_ρ is said to be ρ–convergent to $f \in X_\rho$ if $\rho(f_n - f) \to 0$, as $n \to \infty$.

The following definition plays an important role in the theory of modular function spaces.

Definition 2.2. Let X_ρ be a modular space. We say that ρ has the Ω-property if $\rho(x_n) \to 0$ implies $\rho(\lambda x_n) \to 0$ for $\lambda > 0$; here x_n is a sequence in X_ρ.

For example it is easy to see that $\rho(x) = \ln(1 + \|x\|)$ and $\rho(x) = \exp(\|x\|) - 1$ have the Ω-property (see [4]).

3 Random normed spaces

Definition 3.1. A *triangular norm* (shorter *t-norm*) is a binary operation on the unit interval $[0, 1]$, i.e., a function $T : [0, 1] \times [0, 1] \to [0, 1]$ such that for all

$a, b, c \in [0,1]$ the following four axioms are satisfied:

(T1) $T(a,b) = T(b,a)$ (: commutativity);

(T2) $T(a,(T(b,c))) = T(T(a,b),c)$ (: associativity);

(T3) $T(a,1) = a$ (: boundary condition);

(T4) $T(a,b) \leq T(a,c)$ whenever $b \leq c$ (: monotonicity).

The commutativity of (T1), the monotonicity (T4), and the boundary condition (T3) imply that, for any t-norm T and $x \in [0,1]$, the following boundary conditions are also satisfied:

$$T(x,1) = T(1,x) = x,$$

$$T(x,0) = T(0,x) = 0,$$

and so all the t-norms coincide on the boundary of the unit square $[0,1]^2$.

The monotonicity of a t-norm T in its second component (T4) is, together with the commutativity (T1), equivalent to the (joint) monotonicity in both components, i.e., to

$$T(x_1,y_1) \leq T(x_2,y_2) \text{ whenever } x_1 \leq x_2 \text{ and } y_1 \leq y_2. \tag{3.1}$$

Basic examples are the Łukasiewicz t-norm T_L:

$$T_L(a,b) = \max(a+b-1,0), \ \forall a,b \in [0,1]$$

and the t-norms T_P, T_M, T_D, where

$$T_P(a,b) := ab,$$

$$T_M(a,b) := \min\{a,b\},$$

$$T_D(a,b) := \begin{cases} \min(a,b), & \text{if max(a,b)=1;} \\ 0, & \text{otherwise.} \end{cases}$$

If, for two t-norms T_1 and T_2, the inequality $T_1(x,y) \leq T_2(x,y)$ holds for all $(x,y) \in [0,1]^2$, then we say that T_1 is *weaker* than T_2 or, equivalently, that T_2 is stronger than T_2.

As a result of (3.1), we obtain

$$T(x,y) \leq T(x,1) = x,$$

$$T(x,y) \leq T(1,y) = y$$

for each $(x,y) \in [0,1]^2$. Since trivially $T(x,y) \geq 0 = T_D(x,y)$ for all $(x,y) \in (0,1)^2$, for an arbitrary t-norm T, we get

$$T_D \leq T \leq T_M,$$

i.e., T_D is weaker and T_M is stronger than any other t-norm, and also since $T_L < T_P$ we obtain the following ordering for the four basic t-norms

$$T_D < T_L < T_P < T_M.$$

Throughout this paper, Δ^+ is the space of distribution functions that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \longrightarrow [0,1]$, such that F is left-continuous, non-decreasing on $\mathbb{R}$ and $F(0) = 0$.

Now D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- F(x)$ denotes the left limit of the function f at the point

x, that is, $l^- F(x) = \lim_{t \to x^-} F(t)$. In particular for any $a \geq 0$, ε_a is the specific distribution function defined by

$$\varepsilon_a(t) = \begin{cases} 0 & t \leq a \\ 1 & t > a. \end{cases}$$

Definition 3.2. [10] A *random normed space* (briefly, RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t-norm, and μ is a mapping from X into D^+ such that, if μ_x denotes the value of μ at $x \in X$, the following conditions hold:

(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;

(RN2) $\mu_{\alpha x}(t) = \mu_x\left(\frac{t}{|\alpha|}\right)$ for all $x \in X$, $t > 0$, $\alpha \neq 0$;

(RN3) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

Definition 3.3. Let (X, μ, T) be an RN-space. A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.

Definition 3.4. Let (X, μ, T) be an RN-space. We say that μ has the $\Omega^\star$-property if $\mu_{x_n}(1) \to 1$ implies $\mu_{x_n}(t) \to 1$ for $t > 0$; here x_n is a sequence in X.

Theorem 3.5. [11] If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n \to \infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.

Example 3.6. [12] Let $(X, \|.\|)$ be a normed linear space. Define

$$\mu_x(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ \frac{t}{t+\|x\|}, & \text{if } t > 0. \end{cases}$$

Then (X, μ, T_P) is a random normed space.

Example 3.7. [12] Let $(X, \|.\|)$ be a normed linear space. Define

$$\mu_x(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ e^{-\left(\frac{\|x\|}{t}\right)}, & \text{if } t > 0. \end{cases}$$

Then (X, μ, T_P) is a random normed space.

Example 3.8. [13] Let $(X, \|\cdot\|)$ be a normed linear space. Define

$$\mu_x(t) = \begin{cases} \max\{1 - \frac{\|x\|}{t}, 0\}, & \text{if } \ t > 0; \\ 0, & \text{if } \ t \leq 0. \end{cases}$$

Then $(X, \mu, \ T_L)$ is a RN-space (this was essentially proved by Musthari in [14]; see also [15]).

Definition 3.9. Let (X, μ, T) be an RN-space. We say that μ has the Ω^1-property if $\mu_x(1) = 1$ implies $x = 0$.

It is easy to see that that the RN-spaces in Examples 3.6, 3.7, 3.8 have the Ω^1-property (and also the $\Omega^\star$-property).

For more results on RN-spaces and similar spaces refer [16]– [21].

4 Main results

Theorem 4.1. *Let (X, μ, T) be a RN-space with the Ω^1-property. Define a function*

$$\varphi : [0,1] \longrightarrow [0,+\infty]$$

such that

(1) *φ is continuous and $\varphi(0) = +\infty$ and $\varphi(1) = 0$;*

(2) *φ is strictly decreasing on $[0,1]$;*

(3) *$\varphi(T(a,b)) \leq \varphi(a) + \varphi(b)$ for all $a, b \in [0,1]$.*

Let $\rho(x) = \phi(\mu_x(1))$ for $x \in X$. Then, X_ρ is a modular space.

Proof. Let (X, μ, T) be an RN-space with the Ω^1-property and let φ be a function satisfying (1)–(3).

(i) Let $x \in X$. The Ω^1-property of μ together with (RN1) imply

$$0 = \rho(x) = \varphi(\mu_x(1)) \Longleftrightarrow \mu_x(1) = 1 \Longleftrightarrow x = 0.$$

(ii) is clear. (iii) Let $x, y \in X$ and $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$. Then (note (RN3), (2) and then (RN2),(3))

$$\begin{aligned}
\rho(\alpha x + \beta y) &= \varphi(\mu_{\alpha x + \beta y}(1)) \\
&\leq \varphi[T(\mu_{\alpha x}(\alpha), \mu_{\beta y}(\beta))] \\
&\leq \varphi(\mu_x(1)) + \varphi(\mu_y(1)) \\
&= \rho(x) + \rho(y).
\end{aligned}$$

Now, for $x \in X$ (note (RN2)),

$$\lim_{t\to 0}\rho(tx) = \lim_{t\to 0}\varphi(\mu_{tx}(1)) = \lim_{t\to 0}\varphi\left(\mu_x\left(\frac{1}{|t|}\right)\right) = \varphi(1) = 0,$$

so, X_ρ is a modular space. □

Example 4.2. Let X be a normed linear space and let (X, μ, T_P) be the random normed space in Example 3.6. Let

$$\varphi(u) = \begin{cases} +\infty, & \text{if } u = 0; \\ \ln\frac{1}{u}, & \text{if } 0 < u \leq 1. \end{cases}$$

The function φ satisfies conditions (1)–(3) in Theorem 4.1. Now Theorem 4.1 guarantees that $\phi(\mu_x(1)) = \ln(1 + \|x\|)$ is a modular (note it is also easy to check this directly).

Theorem 4.3. *Let X_ρ be a modular space. Let T be a continuous t-norm. Define a function*

$$\psi : [0, +\infty] \longrightarrow [0, 1]$$

such that

(1) *ψ is continuous and $\psi(0) = 1$ and $\psi(+\infty) = 0$;*

(2) *ψ is strictly decreasing on $[0, +\infty]$;*

(3) *$\psi(a + b) \geq T(\psi(a), \psi(b))$ for all $a, b \in [0, +\infty)$.*

Let

$$\mu_x(t) = \begin{cases} 0, & \text{if } \quad t \leq 0; \\ \psi\left(\rho\left(\frac{x}{t}\right)\right), & \text{if } \quad t > 0. \end{cases}$$

Then, (X, μ, T) is a RN-space.

Proof. (RN1). For $t > 0$ and $x \in X$ we have $\mu_x(t) = 1$ iff $\psi(\rho\left(\frac{x}{t}\right)) = 1$ iff $\rho\left(\frac{x}{t}\right) = 0$ iff $x = 0$.

(RN2). For $t > 0$ and $x \in X$ we have for $\alpha \neq 0$ (note (ii))

$$\mu_{\alpha x}(t) = \psi\left(\rho\left(\frac{\alpha x}{t}\right)\right) = \psi\left(\rho\left(\frac{x}{t/|\alpha|}\right)\right) = \mu_x\left(\frac{t}{|\alpha|}\right).$$

(RN3). For $t, s > 0$ and $x, y \in X$ we have (note (iii) and (3))

$$\begin{aligned}
\mu_{x+y}(t+s) &= \psi\left(\rho\left(\frac{x+y}{t+s}\right)\right) \\
&= \psi\left(\rho\left(\frac{1}{1+\frac{s}{t}}\left(\frac{x}{t}\right) + \frac{1}{1+\frac{t}{s}}\left(\frac{y}{s}\right)\right)\right) \\
&\geq \psi\left[\rho\left(\frac{x}{t}\right) + \rho\left(\frac{y}{s}\right)\right] \\
&\geq T\left(\psi\left(\rho\left(\frac{x}{t}\right)\right), \psi\left(\rho\left(\frac{y}{s}\right)\right)\right) \\
&= T(\mu_x(t), \mu_y(s)).
\end{aligned}$$

□

Example 4.4. Let X be a normed linear space. Consider the modular

$$\rho(x) = \ln(1 + \|x\|),$$

for $x \in X$. Let $\psi(t) = \exp(-t)$ for $t \in (-\infty, +\infty)$. Then the function satisfies conditions (1)–(3) in Theorem 4.3. Consider the t-norm T_P and

$$\mu_x(\lambda) = \begin{cases} 0, & \text{if } \lambda \leq 0; \\ \frac{\lambda}{\lambda + \|x\|} = \psi\left(\rho\left(\frac{x}{\lambda}\right)\right), & \text{if } 0 < \lambda. \end{cases}$$

Now Theorem 4.3 guarantees that (X, μ, T_P) is an RN-space.

Now, we consider the topology induced by a modular.

Theorem 4.5. (1). *Let (X, μ, T_P) be a RN-space with the $\Omega^\star$-property and the Ω^1-property. Let τ_μ be the topology induced by the random norm μ. Then, there exists a modular which induces a topology which coincides with τ_μ on X.*

(2). *Let (X_ρ, ρ) be a modular space with the Ω-property and let τ_ρ be the topology induced by the modular ρ. Then there exists a random norm μ which induces a topology which coincides with τ_ρ on X.*

Proof. (1). Let (X, μ, T_P) be a RN-space. Let φ be as in Example 4.2 and let $\rho(x) = \phi(\mu_x(1))$ for $x \in X$. Then, from Theorem 4.1, ρ is a modular. Now, let $\{x_n\}$ be a sequence in (X, μ, T_p) converging to x in X, i.e., $\mu_{x_n - x}(t)$ tends to 1 for $t > 0$ (so in particular $\mu_{x_n-x}(1)$ tends to 1). Then, $\rho(x_n - x) = \varphi(\mu_{x_n-x}(1))$ tends to 0, i.e., $\{x_n\}$ converges to x in the sense of Definition 2.1.

Next let $\{x_n\}$ be a sequence converging to x in X in the sense of Definition 2.1 with modular ρ (here φ is as in Example 4.2 and $\rho(x) = \phi(\mu_x(1))$ for $x \in X$) i.e., $\varphi(\mu_{x_n-x}(1))$ tends to 0. Then $\mu_{x_n-x}(1)$ tends to 1. Now since μ has the $\Omega^\star$-property, then for $t > 0$ we have that $\mu_{x_n-x}(t)$ tends to 1 i.e., $\{x_n\}$ converges to x in the sense of Definition 3.3.

Now, let A be an open set in (X, μ, T_P). Put $B = A^c$. We show B is a closed set in (X_ρ, ρ). Let x be an element in the closure of B in (X_ρ, ρ). Then there exists a sequence $\{x_n\}$ in B with x_n converging to x in the sense of Definition 2.1 with modular ρ. Now from the above x_n converges to x in the sense of Definition

3.3. Now since B is a closed set in (X, μ, T_P) then $x \in B$. Thus B is a closed set in (X_ρ, ρ) so A is an open set in (X_ρ, ρ). A similar argument show that if C is an open set in (X_ρ, ρ) then C is an open set in (X, μ, T_P).

(2). Let (X_ρ, ρ) be a modular space with the Ω-property. Let ψ be as in Example 4.4. Then Theorem 4.3 guarantees that (X, μ, T_P) is a RN-space (here μ is as in Theorem 4.3). Now, let $\{x_n\}$ be a sequence in (X_ρ, ρ) converging to x in X, i.e., $\rho(x_n - x)$ tends to 0. Now since ρ has the Ω-property, then for $t > 0$ we have that $\mu_{x_n - x}(t) = \psi\left(\rho\left(\frac{x_n - x}{t}\right)\right)$ tends to 1, i.e., $\{x_n\}$ converges to x in the sense of Definition 3.3.

Next let $\{x_n\}$ be a sequence converging to x in X in the sense of Definition 3.3 i.e., $\mu_{x_n - x}(t) = \psi\left(\rho\left(\frac{x_n - x}{t}\right)\right)$ tends to 1 for $t > 0$ (here ψ is as in Example 4.4 and μ is as in Theorem 4.3). Then $\rho\left(\frac{x_n - x}{t}\right)$ tends to 0 for $t > 0$ so in particular $\rho(x_n - x)$ tends to 0 i.e., $\{x_n\}$ converges to x in the sense of Definition 2.1.

Now, let A be an open set in (X_ρ, ρ). Put $B = A^c$. We show B is a closed set in (X, μ, T_P). Let x be an element in the closure of B in (X, μ, T_P). Then there exists a sequence $\{x_n\}$ in B with x_n converging to x in the sense of Definition 3.3 with random norm μ. Now from the above x_n converges to x in the sense of Definition 2.1. Now since B is a closed set in (X_ρ, ρ) then $x \in B$. Thus B is a closed set in (X, μ, T_P) so A is an open set in (X, μ, T_P). A similar argument show that if C is an open set in (X, μ, T_P) then C is an open set in (X_ρ, ρ). □

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Ulam-Hyers stability of 3-Jordan homomorphisms in C^*-ternary algebras

Madjid Eshaghi Gordji[1], **Vahid Keshavarz**[1], **Choonkil Park**[2*] **and Sun Young Jang**[3*]

[1]Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran

[2]Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Korea

[3]Department of Mathematics, University of Ulsan, Ulsan 680-749, Korea

e-mail: m.eshaghi@semnan.ac.ir, v.keshavarz68@yahoo.com, baak@hanyang.ac.kr, jsym@ulsan.ac.kr

Abstract. In this paper, we investigate the Ulam-Hyers stability of C^*-ternary 3-Jordan homomorphisms for the functional equation

$$f(x_1 + x_2, y_1 + y_2, z_1 + z_2) = \sum_{1\le i,j,k\le 2} f(x_i, y_j, z_k)$$

in C^*-ternary algebras.

1. Introduction

Ternary algebraic operations were considered in the 19th century by several mathematicians and physicists such as Cayley [8] who introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii [14]. As an application in physics, the quark model inspired a particular brand of ternary algebraic systems. The so-called Nambu mechanics which has been proposed by Nambu [11] in 1973, is based on such structures. There are also some applications, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics (the anyons), supersymmetric theories, Yang-Baxter equation, etc, (cf. [15, 16, 26]).

The comments on physical applications of ternary structures can be found in [1, 6, 14].

A C^*-ternary algebra is a complex Banach space, equipped with a ternary product $(x, y, z) \to [x, y, z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $\Big[x, y, [z, u, v]\Big] = \Big[x, [y, z, u], v\Big] = \Big[[x, y, z], u, v\Big]$, and satisfies $\|[x, y, z]\| \le \|x\| \cdot \|y\| \cdot \|z\|$, $\|[x, x, x]\| = \|x\|^3$ (see [3, 28]).

Every left Hilbert C^*-module is a C^*-ternary algebra via the ternary product $[x, y, z] := \langle x, y\rangle z$.

Let A and B be two Banach ternary algebras. An additive mapping $H : (A, [\ ]_A) \to (B, [\ ]_B)$ is called a ternary ring homomorphism if

$$H([x, y, z]_A) = [H(x), H(y), H(z)]_B$$

for all $x, y, z \in A$. An additive mapping $H : (A, [\ ]_A) \to (B, [\ ]_B)$ is called a Jordan homomorphism if

$$H([x, x, x]_A) = [H(x), H(x), H(x)]_B$$

for all $x \in A$.

Definition 1.1. Let A and B be C^*-ternary algebras. A 3-linear mapping $H : A \times A \times A \to B$ over $\mathbb{C}$ is called a C^*-ternary 3-homomorphism if it satisfies

$$H([x_1, y_1, z_1], [x_2, y_2, z_2], [x_3, y_3, z_3]) \quad = [H(x_1, x_2, x_3), H(y_1, y_2, y_3), H(z_1, z_2, z_3)]$$

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[0*]Corresponding author.

for all $x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3 \in A$. A 3-linear mapping $H : A \times A \times A \to B$ over C is called a C^*-ternary algebra 3-Jordan homomorphism if it satisfies

$$H([x, x, x], [y, y, y], [z, z, z]) = [H(x, x, x), H(y, y, y), H(z, z, z)]$$

for all $x, y, z \in A$

The study of stability problems originated from a famous talk given by Ulam [27] in 1940: "Under what condition does there exist a homomorphism near an approximate homomorphism?" In the next year 1941, Hyers [13] answered affirmatively the question of Ulam for additive mappings between Banach spaces. Then, Aoki [4] considered the stability problem with unbounded Cauchy differences. A generalized version of the theorem of Hyers for approximately additive maps was given by Rassias [20] in 1978. Let X and Y be real or complex vector spaces. For a mapping $f : X \times X \times X \to Y$, consider the functional equation:

$$f(x_1 + x_2, y_1 + y_2, z_1 + z_2) = \sum_{1 \le i,j,k \le 2} f(x_i, y_j, z_k) \tag{1.1}$$

In 2006, Park and Bae [19] showed that a mapping $f : X \times X \times X \to Y$ satisfies the equation (1.1) if and only if the mapping f is 3-additive. We investigate the Ulam-Hyers stability in C^*-ternary algebras for the 3-additive mappings satisfying (1.1). The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2, 7, 9, 10, 17, 18, 21, 22, 23, 24, 25, 29, 30]).

2. Ulam-Hyers stability of C^*-ternary 3-Jordan homomorphisms

The following lemma was proved in [5].

Lemma 2.1. *Let X and Y be real or complex vector spaces. Let $f : X \times X \times X \to Y$ be a 3-additive mapping such that $f(\lambda x, \mu y, \nu z) = \lambda\mu\nu f(x, y, z)$for all $\lambda, \mu, \nu \in T_1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x, y, z \in X$. Then f is 3-linear over $\mathbb{C}$.*

Using the above lemma, one can obtain the following result.

The following lemma was proved in [5].

Lemma 2.2. *Let X and Y be complex vector spaces and let $f : X \times X \times X \to Y$ be a mapping such that*

$$f(\lambda x_1 + \lambda x_2, \mu y_1 + \mu y_2, \nu z_1 + \nu z_2) = \lambda\mu\nu \sum_{1 \le i,j,k \le 2} f(x_i, y_j, z_k) \tag{2.1}$$

for all $\lambda, \mu, \nu \in T_1$ and all $x_1, x_2, y_1, y_2, z_1, z_2 \in X$. Then f is 3-linear over $\mathbb{C}$.

Lemma 2.3. *Let A and B be two Banach ternary algebras. Let $f : A \to B$ be an additive mapping. Then the following assertions are equivalent*

$$H([x, x, x], [y, y, y], [z, z, z]) = [H(x, x, x), H(y, y, y), H(z, z, z)] \tag{2.2}$$

for all $x, y, z \in A$.

$$\begin{aligned} \|f\Big(&([x_1, x_2, x_3] + [x_2, x_3, x_1] + [x_3, x_1, x_2]), ([y_1, y_2, y_3] + [y_2, y_3, y_1] + [y_3, y_1, y_2]),\\ &([z_1, z_2, z_3] + [z_2, z_3, z_1] + [z_3, z_1, z_2])\Big)\\ = \Big(&f([x_1, x_2, x_3] + [x_2, x_3, x_1] + [x_3, x_1, x_2]), f([y_1, y_2, y_3] + [y_2, y_3, y_1] + [y_3, y_1, y_2]),\\ &f([z_1, z_2, z_3] + [z_2, z_3, z_1] + [z_3, z_1, z_2])\Big)\|_B, \end{aligned} \tag{2.3}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$.

Proof. The proof is similar to the proof of [12, Lemma 2.1]. If we replace x, y, z by $x_1 + x_2 + x_3, y_1 + y_2 + y_3, z_1 + z_2 + z_3$ in (2.2), respectively, then we can easily obtain (2.3).

For the converse, if we replace x_1, x_2, x_3 by x, y_1, y_2, y_3 by y and z_1, z_2, z_3 by z in (2.3), we can easily obtain (2.2). □

From now on, assume that A is a C^*-ternary algebra with norm $\|.\|_A$ and that B is a C^*-ternary algebra with norm $\|.\|_B$. For a given mapping $f : A \times A \times A \to B$, we define

$$D_{\lambda,\mu,\nu} f(x_1, x_2, y_1, y_2, z_1, z_2) := f(\lambda x_1 + \lambda x_2, \mu y_1 + \mu y_2, \nu z_1 + \nu z_2) - \lambda\mu\nu \sum_{1\le i,j,k\le 2} f(x_i, y_j, z_k). \tag{2.4}$$

Theorem 2.4. *Let $p, q, r \in (0,\infty)$ with $p + q + r < 3$ and $\theta \in (0,\infty)$, and let $f : A \times A \times A \to B$ be a mapping such that*

$$\|D_{\lambda,\mu,\nu} f(x_1, x_2, y_1, y_2, z_1, z_2)\|_B \le \theta \cdot \max\{\|x_1\|_A, \|x_2\|_A\}^p \cdot \max\{\|y_1\|_A, \|y_2\|_A\}^q \cdot \max\{\|z_1\|_A, \|z_2\|_A\}^r, \tag{2.5}$$

$$\begin{aligned} &\|f\Big(([x_1, x_2, x_3] + [x_2, x_3, x_1] + [x_3, x_1, x_2]), ([y_1, y_2, y_3] + [y_2, y_3, y_1] + [y_3, y_1, y_2]), ([z_1, z_2, z_3] + [z_2, z_3, z_1] + [z_3, z_1, z_2])\Big) \\ &-\Big(f([x_1, x_2, x_3] + [x_2, x_3, x_1] + [x_3, x_1, x_2]), f([y_1, y_2, y_3] + [y_2, y_3, y_1] + [y_3, y_1, y_2]), \\ &\quad f([z_1, z_2, z_3] + [z_2, z_3, z_1] + [z_3, z_1, z_2])\Big)\|_B \le \theta \sum_{i=1}^{3} (\|x_i\|_A^p \cdot \|y_i\|_A^q \cdot \|z_i\|_A^r) \end{aligned} \tag{2.6}$$

for all $\lambda, \mu, \nu \in T_1$ and all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. Then there exists a unique C^*-ternary 3-Jordan homomorphism $H : A \times A \times A \to B$

$$\|f(x, y, z) - H(x, y, z)\|_B \le \frac{\theta}{8 - 2^{p+q+r}} \|x\|_A^p \cdot \|y\|_A^q \cdot \|z\|_A^r \tag{2.7}$$

for all $x, y, z \in A$.

Proof. By the same reasoning as in the proof of [5, Theorem 2.3], there exists a unique 3-additive mapping $H : A\times A\times A \to B$ satisfying (2.7). By Lemma 2.1, the 3-linear mapping $H : A \times A \times A \to B$ is given by

$$H(\lambda x, \mu y, \nu z) := \lim_{n\to\infty} \frac{1}{8^n} f(2^n \lambda x, 2^n \mu y, 2^n \nu z) = \lim_{n\to\infty} \lambda\mu\nu \frac{1}{8^n} f(2^n x, 2^n y, 2^n z) = \lambda\mu\nu H(x, y, z)$$

for all $\lambda, \mu, \nu \in T_1$ and all $x, y, z \in A$.

It follows from (2.6) that

$$\begin{aligned} &\|H\Big(([x_1, x_2, x_3] + [x_2, x_3, x_1] + [x_3, x_1, x_2]), ([y_1, y_2, y_3] + [y_2, y_3, y_1] + [y_3, y_1, y_2]), ([z_1, z_2, z_3] + [z_2, z_3, z_1] + [z_3, z_1, z_2])\Big) \\ &- \Big(H([x_1, x_2, x_3] + [x_2, x_3, x_1] + [x_3, x_1, x_2]), H([y_1, y_2, y_3] + [y_2, y_3, y_1] + [y_3, y_1, y_2]), H([z_1, z_2, z_3] + [z_2, z_3, z_1] + [z_3, z_1, z_2])\Big)\|_B \\ &= \lim_{n\to\infty} \frac{1}{8^n} \|f\Big(([2^n x_1, 2^n x_2, 2^n x_3] + [2^n x_2, 2^n x_3, 2^n x_1] + [2^n x_3, 2^n x_1, 2^n x_2]), \\ &([2^n y_1, 2^n y_2, 2^n y_3] + [2^n y_2, 2^n y_3, 2^n y_1] + [2^n y_3, 2^n y_1, 2^n y_2]), ([2^n z_1, 2^n z_2, 2^n z_3] + [2^n z_2, 2^n z_3, 2^n z_1] + [2^n z_3, 2^n z_1, 2^n z_2])\Big) \\ &- \Big(f([2^n x_1, 2^n x_2, 2^n x_3] + [2^n x_2, 2^n x_3, 2^n x_1] + [2^n x_3, 2^n x_1, 2^n x_2]), \\ &f([2^n y_1, 2^n y_2, 2^n y_3] + [2^n y_2, 2^n y_3, 2^n y_1] + [2^n y_3, 2^n y_1, 2^n y_2]), f([2^n z_1, 2^n z_2, 2^n z_3] + [2^n z_2, 2^n z_3, 2^n z_1] + [2^n z_3, 2^n z_1, 2^n z_2])\Big)\|_B \\ &\le \lim_{n\to\infty} \frac{\theta}{8^n} \sum_{i=1}^{3} \|x_i\|_A^p \cdot \|y_i\|_A^q \cdot \|z_i\|_A^r = 0 \end{aligned}$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. So

$$H\Big(([x_1,x_2,x_3]+[x_2,x_3,x_1]+[x_3,x_1,x_2]),([y_1,y_2,y_3]+[y_2,y_3,y_1]+[y_3,y_1,y_2]),([z_1,z_2,z_3]+[z_2,z_3,z_1]+[z_3,z_1,z_2])\Big)$$
$$= \Big(H([x_1,x_2,x_3]+[x_2,x_3,x_1]+[x_3,x_1,x_2]),H([y_1,y_2,y_3]+[y_2,y_3,y_1]+[y_3,y_1,y_2]),H([z_1,z_2,z_3]+[z_2,z_3,z_1]+[z_3,z_1,z_2])\Big)\|_B$$

for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$.

Now, let $T : A \times A \times A \to B$ be another 3-additive mapping satisfying (2.7). Then we have

$$\|H(x,y,z)-T(x,y,z)\|_B = \frac{1}{8^n}\|H(2^n x, 2^n y, 2^n z) - T(2^n x, 2^n y, 2^n z)\|_B$$
$$\le \frac{1}{8^n}\|H(2^n x, 2^n y, 2^n z) - f(2^n x, 2^n y, 2^n z)\|_B + \frac{1}{8^n}\|f(2^n x, 2^n y, 2^n z) - T(2^n x, 2^n y, 2^n z)\|_B$$
$$\le \frac{2^{(p+q+r-3)n+1}\theta}{8-2^{p+q+r}}\|x\|_A^p \cdot \|y\|_A^q \cdot \|z\|_A^r,$$

which tends to zero as $n \to \infty$ for all $x, y, z \in A$. So we can conclude that $H(x,y,z) = T(x,y,z)$ for all $x, y, z \in A$. This proves the uniqueness of H.

Thus the mapping $H : A \to B$ is a unique C^*-ternary 3-Jordan homomorphism satisfying (2.7). □

Putting $p = q = r = 0$ and $\theta = \varepsilon$ in Theorem 2.3, we obtain the Ulam stability for the 3-additive functional equation (1.1).

Corollary 2.5. *Let $\varepsilon \in (0,\infty)$ and let $f : A \times A \times A \to B$ be a mapping satisfying*

$$\|D_{\lambda,\mu,\nu} f(x_1,x_2,y_1,y_2,z_1,z_2)\|_B \le \varepsilon,$$

$$\|f\Big(([x_1,x_2,x_3]+[x_2,x_3,x_1]+[x_3,x_1,x_2]),([y_1,y_2,y_3]+[y_2,y_3,y_1]+[y_3,y_1,y_2]),([z_1,z_2,z_3]+[z_2,z_3,z_1]+[z_3,z_1,z_2])\Big)$$
$$- \Big(f([x_1,x_2,x_3]+[x_2,x_3,x_1]+[x_3,x_1,x_2]),f([y_1,y_2,y_3]+[y_2,y_3,y_1]+[y_3,y_1,y_2]),$$
$$f([z_1,z_2,z_3]+[z_2,z_3,z_1]+[z_3,z_1,z_2])\Big)\|_B \le 3\varepsilon$$

for all $\lambda, \mu, \nu \in T_1$ and all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. Then there exists a unique C^*-ternary 3-Jordan homomorphism $H : A \times A \times A \to B$

$$\|f(x,y,z) - H(x,y,z)\|_B \le \frac{\varepsilon}{7}$$

for all $x, y, z \in A$.

Theorem 2.6. *Let $p \in (0,3)$ and $\theta \in (0,8)$, and let $f : A \times A \times A \to B$ be a mapping such that*

$$\|D_{\lambda,\mu,\nu} f(x_1,x_2,y_1,y_2,z_1,z_2)\|_B \le \theta \sum_{i=1}^{2}(\|x_i\|_A^p + \|y_i\|_A^q + \|z_i\|_A^r), \tag{2.8}$$

$$\|f\Big(([x_1,x_2,x_3]+[x_2,x_3,x_1]+[x_3,x_1,x_2]),([y_1,y_2,y_3]+[y_2,y_3,y_1]+[y_3,y_1,y_2]),([z_1,z_2,z_3]+[z_2,z_3,z_1]+[z_3,z_1,z_2])\Big)$$
$$- \Big(f([x_1,x_2,x_3]+[x_2,x_3,x_1]+[x_3,x_1,x_2]),f([y_1,y_2,y_3]+[y_2,y_3,y_1]+[y_3,y_1,y_2]),f([z_1,z_2,z_3]+[z_2,z_3,z_1]+[z_3,z_1,z_2])\Big)\|_B$$
$$\le \theta \sum_{i=1}^{3}(\|x_i\|_A^p + \|y_i\|_A^q + \|z_i\|_A^r) \tag{2.9}$$

for all $\lambda, \mu, \nu \in T_1$ and all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. Then there exists a unique C^*-ternary 3-Jordan homomorphism $H : A \times A \times A \to B$

$$\|f(x,y,z) - H(x,y,z)\|_B \le \frac{2\theta}{8-2^p}(\|x\|_A^p + \|y\|_A^q + \|z\|_A^r)$$

for all $x, y, z \in A$.

Proof. The proof is similar to the proof of Theorem 2.4. □

Theorem 2.7. *Let* $p, q, r \in (0, \infty)$ *with* $p+q+r < 3$, $s \in (0,3)$ *and* $\theta, \eta \in (0, \infty)$, *and let* $f : A \times A \times A \to B$ *be a mapping such that*

$$\|D_{\lambda,\mu,\nu} f(x_1, x_2, y_1, y_2, z_1, z_2)\|_B \leq \theta \cdot \max\{\|x_1\|_A, \|x_2\|_A\}^p \cdot \max\{\|y_1\|_A, \|y_2\|_A\}^q \cdot \max\{\|z_1\|_A, \|z_2\|_A\}^r + \eta \sum_{i=1}^{2} (\|x_i\|_A^s + \|y_i\|_A^s + \|z_i\|_A^s), \tag{2.10}$$

$$\begin{aligned} &\|f\Big(([x_1, x_2, x_3] + [x_2, x_3, x_1] + [x_3, x_1, x_2]), ([y_1, y_2, y_3] + [y_2, y_3, y_1] + [y_3, y_1, y_2]), ([z_1, z_2, z_3] + [z_2, z_3, z_1] + [z_3, z_1, z_2])\Big) \\ &-\Big(f([x_1, x_2, x_3] + [x_2, x_3, x_1] + [x_3, x_1, x_2]), f([y_1, y_2, y_3] + [y_2, y_3, y_1] + [y_3, y_1, y_2]), \\ &f([z_1, z_2, z_3] + [z_2, z_3, z_1] + [z_3, z_1, z_2])\Big)\|_B \leq \theta \sum_{i=1}^{3} (\|x_i\|_A^p \cdot \|y_i\|_A^q \cdot \|z_i\|_A^r) + \eta \sum_{i=1}^{2} (\|x_i\|_A^s + \|y_i\|_A^s + \|z_i\|_A^s) \end{aligned} \tag{2.11}$$

for all $\lambda, \mu, \nu \in T_1$ *and all* $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$. *Then there exists a unique* C^**-ternary 3-Jordan homomorphism* $H : A \times A \times A \to B$

$$\|f(x, y, z) - H(x, y, z)\|_B \leq \frac{\theta}{8 - 2^{p+q+r}} \|x\|_A^p \cdot \|y\|_A^q \cdot \|z\|_A^r + \frac{2\eta}{8 - 2^s} (\|x_i\|_A^s + \|y_i\|_A^s + \|z_i\|_A^s)$$

for all $x, y, z \in A$.

Proof. The proof is similar to the proof of Theorem 2.4. □

Theorem 2.8. *Let* $p \in (0,3)$ *and* $\theta \in (0, 8)$, *and let* $f : A \times A \times A \to B$ *be a mapping satisfying (2.8), (2.9) and* $f(0,0,0) = 0$. *Then there exists a unique* C^**-ternary 3-Jordan homomorphism* $H : A \times A \times A \to B$ *such that*

$$\|f(x, y, z) - H(x, y, z)\|_B \leq \frac{2\theta}{2^p - 8} (\|x\|_A^p + \|y\|_A^q + \|z\|_A^r)$$

for all $x, y, z \in A$.

Theorem 2.9. *Letp,* $q, r \in (0, \infty)$ *with* $p + q + r > 3$, $s \in (0, 3)$ *and* $\theta, \eta \in (0, \infty)$, *and let* $f : A \times A \times A \to B$ *be a mapping satisfying (2.10), (2.11) and* $f(0, 0, 0) = 0$. *Then there exists a unique* C^**-ternary 3-Jordan homomorphism* $H : A \times A \times A \to B$ *such that*

$$\|f(x, y, z) - H(x, y, z)\|_B \leq \frac{\theta}{2^{p+q+r-8}} \|x\|_A^p \cdot \|y\|_A^q \cdot \|z\|_A^r + \frac{2\eta}{2^s - 8} (\|x_i\|_A^s + \|y_i\|_A^s + \|z_i\|_A^s)$$

for all $x, y, z \in A$.

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