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The Naimark-Sacker bifurcation and symptotic approximation of the invariant curve of a certain difference equation

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Abstract

We compute the direction of the Naimark-Sacker bifurcation for the difference equation $x_{n+1} = p + \frac{x_n^2}{x_{n-1}^2}$ where p is a positive number and the initial conditions x_{-1} and x_0 are positive numbers. Furthermore, we give the asymptotic approximation of the invariant curve.

Keywords: difference equation, Naimark-Sacker bifurcation, normal form. invariant curve, stability.
AMS 2010 Mathematics Subject Classification: 39A10, 39A20, 65L20

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1 Introduction and Preliminaries

In this paper we consider the difference equation

$$x_{n+1} = p + \frac{x_n^2}{x_{n-1}^2}, \quad n = 0, 1, \dots, \tag{1}$$

where the parameter a is positive number and the initial conditions x_{-1} and x_0 are positive numbers. Clearly equation (1) has the unique equilibrium point $\bar{x} = p + 1$. Linear fractional version of equation (1)

$$x_{n+1} = p + \frac{x_n}{x_{n-1}}, \quad n = 0, 1, \dots, \tag{2}$$

was considered in [3], where we proved that the unique equilibrium $\bar{x} = p+1$ of equation (2) is globally asymptotically stable. Introduction of quadratic terms into equation (2) changes local stability analysis and consequently the global dynamics as well. In particular, quadratic terms introduces the possibility of Naimark-Sacker bifurcation and the existence of locally stable periodic solution, see [6] for several similar examples.

The linearized equation of equation (2) at the equilibrium point $\bar{x} = p + 1$ is

$$z_{n+1} = \frac{2}{p+1} z_n - \frac{2}{p+1}, \quad n = 0, 1, \dots,$$

with the characteristic equation

$$\lambda^2 - \frac{2}{p+1}\lambda + \frac{2}{p+1} = 0,$$

and the characteristic roots

$$\lambda_{\pm} = \frac{1 \pm i\sqrt{2p+1}}{p+1}.$$

Since

$$|\lambda_{\pm}| = \sqrt{\frac{2}{p+1}}$$

it is clear that that the equilibrium point $\bar{x} = p + 1$ is asymptotically stable if $p > 1$, non-hyperbolic if $p = 1$ and unstable if $p < 1$. In all cases the eigenvalues are complex conjugate numbers which indicates the presence of the Naimark-Sacker bifurcation at $p = 1$. We will prove that indeed the equilibrium point $\bar{x} = p + 1$ is globally asymptotically stable if $p > \sqrt{2}$ and that the Naimark-Sacker bifurcation takes the place at $p = 1$. Our tool in proving global asymptotic stability of equation (2) is the result in [3, 5]. We conjecture that the equilibrium point $\bar{x} = p + 1$ is globally asymptotically stable if $a > 1$. Furthermore, we give some numeric values of parameter a with corresponding periodic solutions. Our bifurcation diagram indicates a complicated behavior and possible chaos for the values $p < 1$.

Now, for the sake of completness we give the basic facts about the Naimark-Sacker bifurcation.

The Hopf bifurcation is well known phenomenon for a system of ordinary differential equations in two or more dimension, whereby, when some parameter is varied, a pair of complex conjugate eigenvalues of the Jacobian matrix at a fixed point crosses the imaginary axis, so that the fixed point changes its behavior from stable to unstable and a limit cycle appears. In the discrete setting, the Naimark-Sacker bifurcation is the discrete analogue of the Hopf bifurcation.

The Naimark-Sacker bifurcation occurs for a discrete system depending on a parameter, λ say, with a fixed point whose Jacobian has a pair of complex conjugate $\mu(\lambda)$, $\bar{\mu}(\lambda)$ which cross the unit transversally at $\lambda = \lambda_0$.

The following result is referred as the Neimark-Sacker bifurcation Theorem [1, 4, 7, 8, 11].

Theorem 1 (Naimark-Sacker bifurcation) *Let*

$$\mathbf{F} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2; \quad (\lambda, x) \to \mathbf{F}(\lambda, \mathbf{x})$$

be a C^4 map depending on real parameter λ satisfying the following conditions:

(i) $F(\lambda, \mathbf{0}) = 0$ *for λ near some fixed λ_0;*

(ii) $DF(\lambda, \mathbf{0})$ *has two non-real eigenvalues $\mu(\lambda)$ and $\bar{\mu}(\lambda)$ for λ near λ_0 with $|\mu(\lambda_0)| = 1$;*

(iii) $\frac{d}{d\lambda}|\mu(\lambda)| = d(\lambda_0) < 0$ *at $\lambda = \lambda_0$ (transversality condition);*

(iv) $\mu^k(\lambda_0) \neq 1$ *for $k = 1, 2, 3, 4$. (nonresonance condition).*

Then there is a smooth λ-dependent change of coordinate bringing F into the form

$$F(\lambda, \mathbf{x}) = \mathcal{F}(\lambda, \mathbf{x}) + O(\| \mathbf{x} \|^5)$$

and there are smooth function $a(\lambda)$, $b(\lambda)$, and $\omega(\lambda)$ so that in polar coordinates the function $\mathcal{F}(\lambda, x)$ is given by

$$\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} |\mu(\lambda)|r + a(\lambda)r^3 \\ \theta + \omega(\lambda) + b(\lambda)r^2 \end{pmatrix}. \tag{3}$$

If $a(\lambda_0) < 0$, then there is a neighborhood U of the origin and a $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$ and $x_0 \in U$, then ω-limit set of x_0 is the origin if $\lambda > \lambda_0$ and belongs to a closed invariant C^1 curve $\Gamma(\lambda)$ encircling the origin if $\lambda < \lambda_0$. Furthermore, $\Gamma(\lambda_0) = 0$.
If $a(\lambda_0) > 0$, then there is a neighborhood U of the origin and a $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$ and $x_0 \in U$, then α-limit set of x_0 is the origin if $\lambda < \lambda_0$ and belongs to a closed invariant C^1 curve $\Gamma(\lambda)$ encircling the origin if $\lambda > \lambda_0$. Furthermore, $\Gamma(\lambda_0) = 0$.

Consider a general map $\mathbf{F}(\lambda_0, \mathbf{x})$ that has a fixed point at the origin with complex eigenvalues $\mu(\lambda_0) = \alpha(\lambda_0) + i\beta(\lambda_0)$ and $\bar{\mu}(\lambda_0) = \alpha(\lambda_0) - i\beta(\lambda_0)$ satisfying $\alpha(\lambda_0)^2 + \beta(\lambda_0)^2 = 1$ and $\beta(\lambda_0) \neq 0$. Assume that

$$\mathbf{F}(\lambda_0, \mathbf{x}) = \mathbf{A}(\lambda_0)\mathbf{x} + \mathbf{G}(\lambda_0, \mathbf{x}) \tag{4}$$

where $\mathbf{A}$ is Jacobian matrix of $\mathbf{F}$ evaluated at fixed point $(0, 0)$, and

$$\mathbf{G}(\lambda_0, \mathbf{x}) := \begin{pmatrix} g_1(\lambda_0, x_1, x_2) \\ g_2(\lambda_0, x_1, x_2) \end{pmatrix}.$$

Here we donate $\mu(\lambda_0) = \mu$, $\mathbf{A}(\lambda_0) = \mathbf{A}$ and $\mathbf{G}(\lambda_0, x) = \mathbf{G}(\mathbf{x})$. We let $\mathbf{p}$ and $\mathbf{q}$ be eigenvectors of A associated with μ satisfying

$$\mathbf{Aq} = \mu\mathbf{q}, \quad \mathbf{p}A = \mu\mathbf{p}, \quad \mathbf{pq} = 1$$

and $\mathbf{\Phi} = (\mathbf{q}, \bar{\mathbf{q}})$. Assume that

$$\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right) = \frac{1}{2}(\mathbf{g}_{20}z^2 + 2\mathbf{g}_{11}z\bar{z} + \mathbf{g}_{02}\bar{z}^2) + O(|z|^3)$$

and

$$\begin{array}{l} \mathbf{K}_{20} = (\mu^2 I - A)^{-1}\mathbf{g}_{20} \\ \mathbf{K}_{11} = (I - A)^{-1}\mathbf{g}_{11} \quad . \\ \mathbf{K}_{02} = (\bar{\mu}^2 I - A)^{-1}\mathbf{g}_{02} \end{array} \tag{5}$$

Let

$$\begin{aligned} \mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} + \frac{1}{2}(\mathbf{K}_{20}\xi^2 + 2\mathbf{K}_{11}\xi\bar{\xi} + \mathbf{K}_{02}\bar{\xi}^2)\right) \\ = \frac{1}{2}(\mathbf{g}_{20}\xi^2 + 2\mathbf{g}_{11}\xi\bar{\xi} + \mathbf{g}_{02}\bar{\xi}^2) \\ + \frac{1}{6}(\mathbf{g}_{30}\xi^3 + 3\mathbf{g}_{21}\xi^2\bar{\xi} + 3\mathbf{g}_{12}\xi\bar{\xi}^2 + \mathbf{g}_{03}\bar{\xi}^3) + O(|\xi|^4), \end{aligned} \tag{6}$$

then

$$a(\lambda_0) = \frac{1}{2}Re(\mathbf{p}\mathbf{g}_{21}\bar{\mu}).$$

Corollary 1 ([9]) *Assume $a(\lambda_0) \neq 0$ and $\lambda = \lambda_0 + \eta$ where η is a sufficient small parameter. If $\bar{\mathbf{x}}$ is fixed point of F then invariant curve $\Gamma(\lambda)$ from Theorem 1 can be approximated by*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \bar{\mathbf{x}} + 2\rho_0 Re\left(\mathbf{q}e^{i\theta}\right) + \rho_0^2\left(Re\left(\mathbf{K}_{20}e^{2i\theta}\right) + \mathbf{K}_{11}\right),$$

where

$$d = \frac{d}{d\eta}|\mu(\lambda)|\Big|_{\lambda=\lambda_0}, \quad \rho_0 = \sqrt{-\frac{d}{a}\eta}, \quad \theta \in \mathbb{R}.$$

Here "Re" represents the real parts of those complex numbers.

The second section of the paper gives global asymptotic stability result for the values of parameter $p > \sqrt{2}$ and the third section gives the reduction to the normal form and computation of the coefficients of the Naimark-Sacker bifurcation and the asymptotic approximation of the invariant curve. Our computational method is based on the computational algorithm developed in [9] rather than more often used computational algorithm in [10]. The advantage of the computational algorithm of [9] lies in the fact that this algorithm computes also the approximate equation of the invariant curve in Naimark-Sacker theorem, which is not provided by Wan's algorithm. Here we give numeric and visual eveidence that the approximate equation of the invariant curve is accurate. See Figure 4.

2 Global Asymptotic Stability

We use the method of embedding [2]. By substituting

$$x_n = p + \left(\frac{x_{n-1}}{x_{n-2}}\right)^2$$

in equation (1) we get:

$$x_{n+1} = p + \left(\frac{p}{x_{n-1}} + \frac{x_{n-1}}{x_{n-2}^2}\right)^2.$$

Now by substituting for x_{n-1} in the term $\frac{x_{n-1}}{x_{n-2}^2}$ of the last equation we we obtain

$$x_{n+1} = p + \left(\frac{p}{x_{n-1}} + \frac{p}{x_{n-2}^2} + \frac{1}{x_{n-3}^2}\right)^2. \qquad (7)$$

From equation (7) we observe that $p<x_n<p+(1+\frac{1}{p}+\frac{1}{p^2})^2$ for $n \geq 4$.

Also from (1) and (7) we have:

$$\begin{cases} x_{n+1} - p = \left(\frac{x_n}{x_{n-1}}\right)^2 \\ x_{n+1} - p = \left(\frac{p}{x_{n-1}} + \frac{p}{x_{n-2}^2} + \frac{1}{x_{n-3}^2}\right)^2 \end{cases}.$$

Consequently

$$\left(\frac{x_n}{x_{n-1}}\right)^2 = \left(\frac{p}{x_{n-1}} + \frac{p}{x_{n-2}^2} + \frac{1}{x_{n-3}^2}\right)^2,$$

which implies:

$$x_{n+1} = p + \frac{px_n}{x_{n-1}^2} + \frac{x_n}{x_{n-2}^2}. \qquad (8)$$

Replacing x_n in (8) by $p + \left(\frac{x_{n-1}}{x_{n-2}}\right)^2$ we obtain the equation

$$x_{n+1} = p + \frac{a^2}{x_{n-1}^2} + \frac{p + x_n}{x_{n-2}^2}. \qquad (9)$$

Observe now that every solution of equation (1) is also a solution of equation (9), with initial values x_{-2}, x_{-1} and $x_0 = p + (\frac{x_0}{x_{-1}})^2$.

Observe also that it is of the form $x_{n+1} = f(x_n, x_{n-1}, x_{n-2})$ where :

$$f(u, v, w) = p + \frac{p^2}{v^2} + \frac{p+u}{w^2}$$

.

Theorem 2 *If $p > \sqrt{2}$ then the equilibrium of equation (1) is globally asymptotically stable.*

Proof. First we show that every interval I of the form $[p, \mathcal{U}]$ where $\mathcal{U} \geq \frac{p(p^2+p+1)}{(p^2-1)}$ with $p > 1$ is invariant for the function f.

Let $\mathcal{U} > p$ then $I = [p, \mathcal{U}]$ is invariant if and only if for all $u, v, w \in I, f(u, v, w) \in I$ that is:

$$p \leq p + \frac{p^2}{v^2} + \frac{p+u}{w^2} \leq \mathcal{U}.$$

As $p \leq u, v, w \leq \mathcal{U}$ we have that: $p \leq f(u, v, w) \leq p + 1 + \frac{1}{p} + \frac{\mathcal{U}}{p^2}$. We also know that if $\mathcal{U}$ satisfies: $p + 1 + \frac{1}{p} + \frac{\mathcal{U}}{p^2} \leq \mathcal{U}$ then we have

$$f(u, v, w) \leq \mathcal{U}.$$

It follows that given $p > 1$ such $\mathcal{U}$ exists and therefore I is invariant for f where $\mathcal{U} \geq \frac{p(p^2+p+1)}{(p^2-1)}$. In the following we may assume $p > 1$ and $\mathcal{U} = \frac{p(p^2+p+1)}{(p^2-1)}$, so I is invariant by f.

Next, we prove that I is an attracting interval, that is every solution of equation (8) must enter the interval I. Observe that given the initial values x_{-2}, x_{-1} and x_0 for equation (8), we have $x_n > p$ for $n \geq 1$.

Now if $x_3 \leq \mathcal{U}$ then $x_n \in [p, \mathcal{U}]$ for all $n \geq 3$. Otherwise, from equation (4) given that x_{n-2} , $x_{n-3} > p$ we have

$$x_n < p + 1 + \frac{1}{p} + \frac{x_{n-1}}{p^2},$$

that is if we set $A = p + 1 + \frac{1}{p}$

$$x_n < A + \frac{x_{n-1}}{p^2}.$$

Thus by induction we can conclude that

$$x_n < A \frac{1 - (\frac{1}{p^2})^{n-3}}{1 - \frac{1}{p^2}} + \frac{x_3}{(p^2)^{n-3}}. \tag{10}$$

It is straightforward to check that when $x_3 > \mathcal{U}$ the right hand side of (10) is a decreasing sequence that converges to $A\ (\frac{1}{1-\frac{1}{p^2}})$. This limit is in fact $\mathcal{U} = \frac{p(p^2+p+1)}{(p^2-1)}$. It follows that there must exist $k > 3$ such that: $a < x_k < \mathcal{U}$ Otherwise x_n must converge to $\mathcal{U}$ which is impossible.

Thus we have $x_{k-1}, x_{k-2} > p$ and $x_k \leq \mathcal{U}$, hence $x_{k+1} \in [a, \mathcal{U}]$ it follows by induction that $x_n \in [p, \mathcal{U}]$ for $n \geq k$.

Consequently every solution of equation (8) must enter the interval $[p, \mathcal{U}]$.

Now that we have an invariant and attracting interval we check the conditions of Theorem A.0.5 [3]:

$$\begin{cases} f(M, m, m) = M \\ f(m, M, M) = m \end{cases} \Leftrightarrow \begin{cases} M = p + \frac{p^2+p+M}{m^2} \\ m = p + \frac{p^2+p+m}{M^2} \end{cases} .$$

From the second equation we get

$$M^2 = \frac{p^2 + p + m}{m - p}. \tag{11}$$

On the other hand the system is equivalent to:

$$\begin{cases} (M - p)m^2 = p^2 + p + M \\ (m - p)M^2 = p^2 + p + m \end{cases} \Leftrightarrow \begin{cases} Mm^2 = pm^2 + p^2 + p + M \\ mM^2 = pM^2 + p^2 + p + m \end{cases}$$

By subtracting the second equation from the first we obtain:

$$Mm(m-M) = p(m-M)(m+M) - (m-M)$$

and given that $m \neq M$ we have:

$$Mm = p(m+M) - 1$$

which implies:

$$M = \frac{pm-1}{m-p}. \tag{12}$$

Equations (11) and (12) yield

$$\frac{(pm-1)^2}{(m-p)^2} = \frac{p^2+p+m}{m-p},$$

which implies:

$$(pm-1)^2 = (p^2+p+m)(m-p).$$

This leads to the following quadratic equation:

$$m^2(p^2-1) - m(p^2+2p) + p^2(p+1) + 1 = 0,$$

which discriminant is

$$\Delta = (p^2+2p)^2 - 4(p^2-1)(p^2(p+1)+1)$$

and

$$\Delta = -4p^5 - 3p^4 + 8p^3 + 4p^2 + 4 = (\sqrt{2}-p)(4p^4 + (3+4\sqrt{2})p^3 + 3\sqrt{2}p^2 + 2p + 2\sqrt{2}).$$

It is clear that when $a > \sqrt{2}$ there is no real solutions. and when $p = \sqrt{2}$ there is one unique solution $m = p+1 = M$. Consequently if $a \geq \sqrt{2}$ the conditions of Theorem A.0.5 [3] or Theorem 1 [5] are fully satisfied and therefore every solution must converge to the unique equilibrium $(p+1)$ □

Conjecture 1 *The equilibrium point* $\bar{x} = p+1$ *of equation* (2) *is globally asymptotically stable if* $p > 1$.

Remark 1 It could have been easier to prove the fact if we restrict the set of solutions of equation (4) to the ones satisfied by equation (1) as the solutions must oscillate about the equilibrium $(p+1)$ that is there exist k such that: $p < x_k < p+1 < \mathcal{U}$.

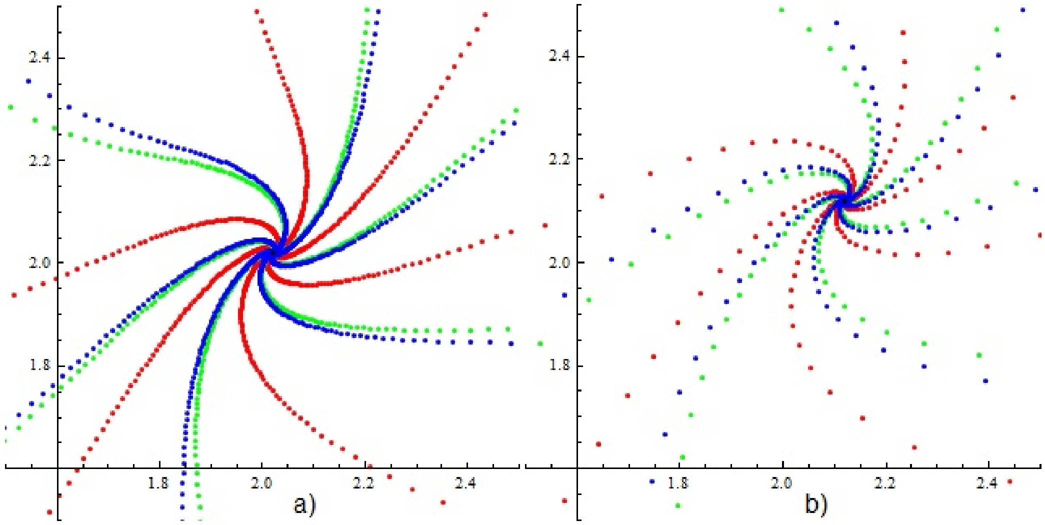

Figure 1: a) Phase diagrams when $n = 10,000$ and a) $p = 1.02$ b) $p = 1.12$

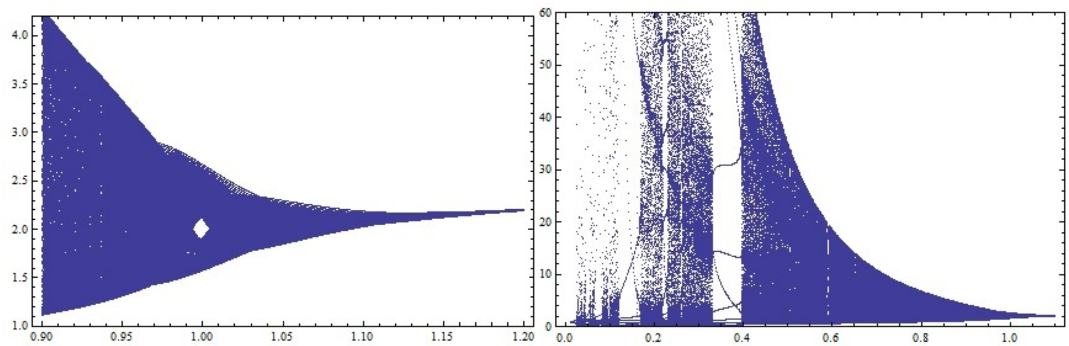

Figure 2: Bifurcation diagrams in $(p - x)$ plane.

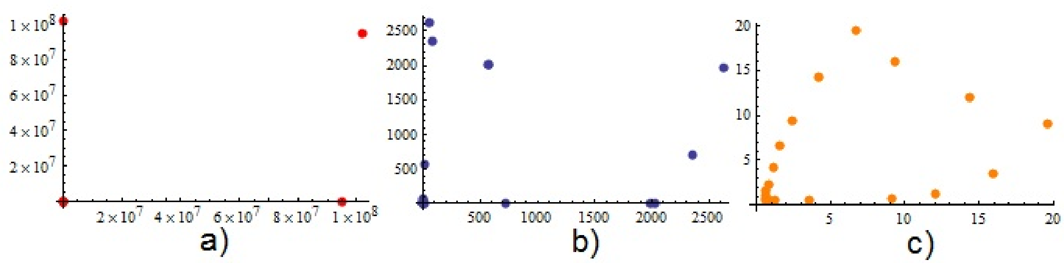

Figure 3: Periodic orbit for a) $p = 0.01$ b) $p = 0.15$ c) $p = 0.5901$ (See Table 2).

3 Reduction to the normal form

If we make a change of variable $y_n = x_n - \bar{x}$, then the transformed equation is given by

$$y_{n+1} = \frac{(p + y_n + 1)^2}{(p + y_{n-1} + 1)^2} - 1, \quad n = 0, 1, \dots. \tag{13}$$

a	Period of the sol.	Solution
0.01	8	$\{0.877631, 0.01, 0.0101298, 1.03613, 10462.3, 1.01959 \times 10^8, 9.49713 \times 10^7, 0.877631\}$
0.15	20	$\{574.846, 2023.71, 12.5435, 0.150038, 0.150143, 1.1514, 58.9583, 2622.2, 1978.22, 0.719138, 0.15, 0.193507, 1.81422, 88.0493, 2355.59, 715.88, 0.242359, 0.15, 0.533058, 12.7789\}$
0.5901	19	$\{0.804816, 0.597988, 1.14217, 4.23826, 14.3595, 12.0691, 1.29653, 0.60164, 0.805431, 2.38228, 9.33854, 15.9565, 3.50965, 0.638479, 0.623195, 1.5428, 6.71883, 19.5558, 9.06166\}$

Table 1: Periodic solutions for some values of p.

Set

$$u_n = y_{n-1} \text{ and } v_n = y_n \text{ for } n = 0, 1, \ldots$$

and write Eq.(1) in the equivalent form:

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= \frac{(p+v_n+1)^2}{(p+u_n+1)^2} - 1. \end{aligned} \tag{14}$$

Let F be the corresponding map defined by:

$$\mathbf{F}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{(p+v+1)^2}{(p+u+1)^2} - 1 \end{pmatrix}. \tag{15}$$

Then $\mathbf{F}$ has the unique fixed point $(0,0)$ and the Jacobian matrix of $\mathbf{F}$ at $(0,0)$ is given by

$$Jac_{\mathbf{F}}(0,0) = \begin{pmatrix} 0 & 1 \\ -\frac{2}{p+1} & \frac{2}{p+1} \end{pmatrix}$$

It is easy to see that

$$\mathbf{F}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{2}{p+1} & \frac{2}{p+1} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \mathbf{F}_1\begin{pmatrix} u \\ v \end{pmatrix}, \tag{16}$$

where

$$\mathbf{F}_1\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{(p+v+1)^2}{(p+u+1)^2} + \frac{2u}{p+1} - \frac{2v}{p+1} - 1 \end{pmatrix}.$$

The eigenvalues of $Jac_{\mathbf{F}}(0,0)$ are $\mu(p)$ and $\overline{\mu(p)}$ where

$$\mu(p) = \frac{1 + i\sqrt{2p+1}}{p+1}, \quad |\mu(p)| = \sqrt{\frac{2}{p+1}}.$$

One can prove that for $p = p_0 = 1$ we obtain $\mu(p_0)| = 1$ and

$$\mu(p_0) = \frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad \mu^2(p_0) = -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad \mu^3(p_0) = -1, \quad \mu^4(p_0) = -\frac{1}{2} - \frac{i\sqrt{3}}{2},$$

from which follows that $\mu^k(p_0) \neq 1$ for $k = 1, 2, 3, 4$. Furthermore, we get

$$\frac{d}{dp}|\mu(p)| = -\frac{1}{\sqrt{2}}\left(\frac{1}{p+1}\right)^{3/2}, \quad \frac{d|\mu(p)|}{dp}\bigg|_{p=p_0} = -\frac{1}{4} < 0.$$

The eigenvectors of corresponding to $\mu(p)$ and $\overline{\mu(p)}$ are $\mathbf{q}(p)$ and $\overline{\mathbf{q}(p)}$, where

$$\mathbf{q}(p) = \left(\frac{1 - i\sqrt{2p+1}}{p+1}, 1\right)^T.$$

Substituting $p = p_0 = 1$ into (16) we get

$$\mathbf{F}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix} + \mathbf{G}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{17}$$

where

$$\mathbf{A} = Jac_{\mathbf{F}}(0,0)|_{p=1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \text{ and } \mathbf{G}\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} 0 \\ \frac{(v+2)^2}{(u+2)^2} + u - v - 1 \end{pmatrix}.$$

Hence, for $p = p_0$ system (14) is equivalent to

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \mathbf{A}\begin{pmatrix} u_n \\ v_n \end{pmatrix} + \mathbf{G}\begin{pmatrix} u_n \\ v_n \end{pmatrix}. \tag{18}$$

Define the basis of $\mathbb{R}^2$ by $\mathbf{\Phi} = (\mathbf{q}, \bar{\mathbf{q}})$, where $\mathbf{q} = \mathbf{q}(p_0)$, then we can represent (u, v) as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = (\mathbf{q}z + \bar{\mathbf{q}}\bar{z}) = \begin{pmatrix} \frac{1}{2}\left(1 + i\sqrt{3}\right)\bar{z} + \frac{1}{2}\left(1 - i\sqrt{3}\right)z \\ \bar{z} + z \end{pmatrix}.$$

By using this, we have

$$\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ \frac{(\bar{z}+z+2)^2}{\left(\frac{1}{2}\left(1+i\sqrt{3}\right)\bar{z}+\frac{1}{2}\left(1-i\sqrt{3}\right)z+2\right)^2} + \frac{1}{2}\left(-1 + i\sqrt{3}\right)\bar{z} - \frac{1}{2}\left(1 + i\sqrt{3}\right)z - 1 \end{pmatrix} \tag{19}$$

Thus we obtain that

$$\begin{aligned}
\mathbf{g}_{20} &= \frac{\partial^2}{\partial z^2}\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)\bigg|_{z=0} = \begin{pmatrix} 0 \\ \frac{1}{4}i\left(\sqrt{3} + 5i\right) \end{pmatrix} \\
\mathbf{g}_{11} &= \frac{\partial^2}{\partial z \partial \bar{z}}\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)\bigg|_{z=0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\mathbf{g}_{02} &= \frac{\partial^2}{\partial \bar{z}^2}\mathbf{G}\left(\mathbf{\Phi}\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)\bigg|_{z=0} = \begin{pmatrix} 0 \\ -\frac{1}{4}i\left(\sqrt{3} - 5i\right) \end{pmatrix},
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
\mathbf{K}_{20} &= (\mu^2 I - A)^{-1}\mathbf{g}_{20} = \begin{pmatrix} -\frac{1}{2} - \frac{i\sqrt{3}}{4} \\ \frac{5}{8} - \frac{i\sqrt{3}}{8} \end{pmatrix} \\
\mathbf{K}_{11} &= (I - A)^{-1}\mathbf{g}_{11} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\mathbf{K}_{02} &= (\bar{\mu}^2 I - A)^{-1}\mathbf{g}_{02} = \overline{\mathbf{K}_{20}}
\end{aligned} \tag{21}$$

By using $\mathbf{K}_{20}$, $\mathbf{K}_{11}$ and $\mathbf{K}_{02}$ we have that

$$\mathbf{g}_{21} = \frac{\partial^3}{\partial z^2 \partial \bar{z}} \mathbf{G} \left(\mathbf{\Phi} \begin{pmatrix} z \\ \bar{z} \end{pmatrix} + \frac{1}{2} \mathbf{K}_{20} z^2 + \mathbf{K}_{11} z\bar{z} + \frac{1}{2} \mathbf{K}_{02} \bar{z}^2 \right) \Big|_{z=0} = \begin{pmatrix} 0 \\ -\frac{i\sqrt{3}}{8} \end{pmatrix}. \tag{22}$$

It is easy to see that $\mathbf{pA} = \mu \mathbf{p}$ and $\mathbf{pq} = 1$ where

$$\mathbf{p} = \left(\frac{i}{\sqrt{3}}, \frac{1}{6} \left(3 - i\sqrt{3} \right) \right)$$

and

$$a(p_0) = \frac{1}{2} Re(\mathbf{pg}_{21} \bar{\mu}) = -\frac{1}{16} < 0.$$

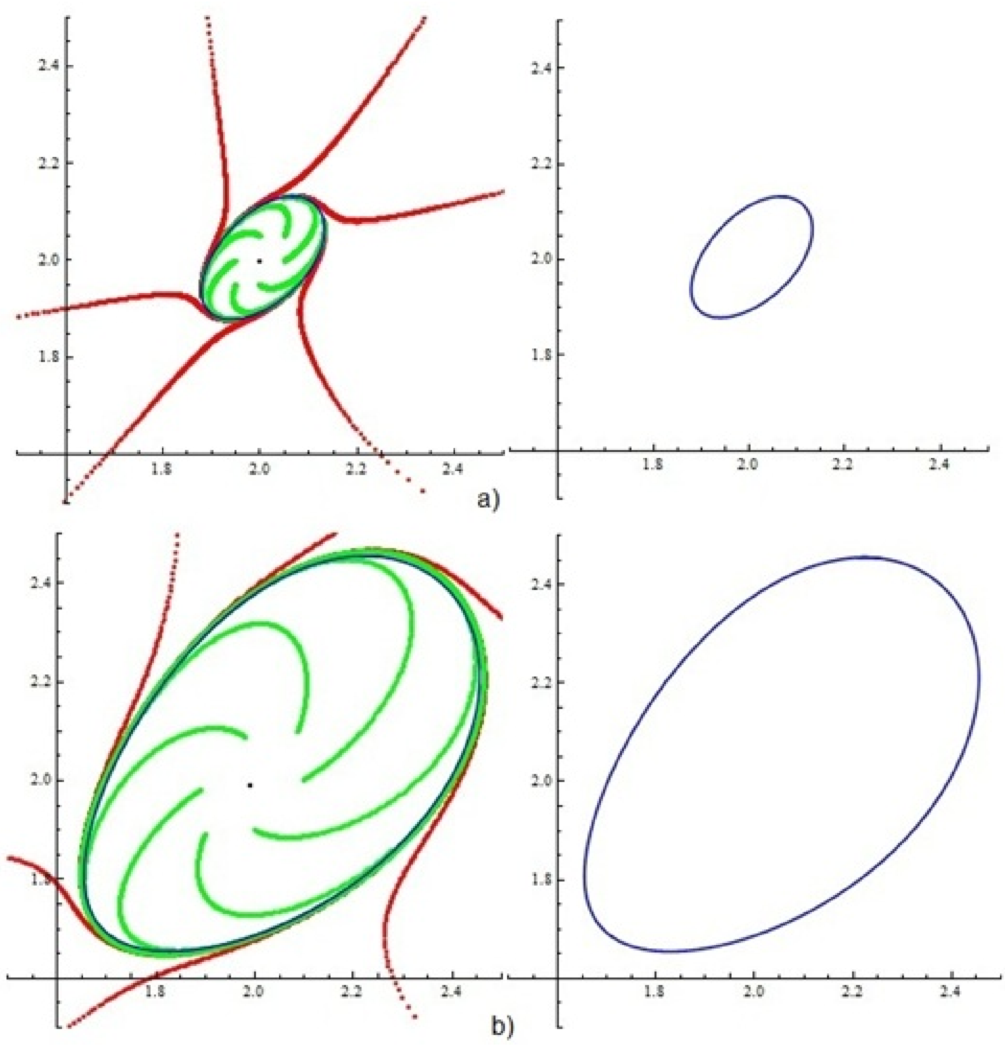

Figure 4: Trajectories and invariant curve for a) $p = 0.999$ b) $p = 0.99$.

Thus we prove the following result:

Theorem 3 *Let $\bar{x} = p + 1$. Then there is a neighborhood U of the equilibrium point $\bar{x}$ and a $\rho > 0$ such that for $|p - 1| < \rho$ and $x_0, x_{-1} \in U$, then ω-limit set of solution of Eq(1), with initial condition x_0, x_{-1} is equilibrium point $\bar{x}$ if $p > 1$ and belongs to a closed invariant C^1 curve $\Gamma(p)$ encircling the equilibrium point $\bar{x}$ if $p < 1$. Furthermore, $\Gamma(1) = 0$ and invariant curve $\Gamma(p)$ can be approximated by*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \begin{pmatrix} p + 1 + 2\sqrt{1-p}\left(\sqrt{3}\sin\theta + \cos\theta\right) - (p-1)\left(\sqrt{3}\sin 2\theta - 2\cos 2\theta + 4\right) \\ p + 1 + 4\sqrt{1-p}\cos\theta - \frac{1}{2}(p-1)\left(\sqrt{3}\sin 2\theta + 5\cos 2\theta + 8\right) \end{pmatrix}$$

Proof. The proof follows from above discussion and Theorem 1 and Corollary 1. □

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Triple reverse order law for Moore-Penrose inverse of operator product *

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Abstract

In this paper, we study the reverse order law for the Moore-Penrose inverse of an operator product $T_1T_2T_3$. In particular, using the matrix form of a bounded linear operator we derive some necessary and sufficient conditions for the reverse order law $(T_1T_2T_3)^\dagger = T_3^\dagger T_2^\dagger T_1^\dagger$. Moreover, some finite dimensional results are extended to infinite dimensional settings.

Keywords: Moore-Penrose inverse; Reverse order law; Bounded linear operator; Operator product; Hilbert space.

AMS(MOS) Subject Classifications: 47A05; 15A09; 15A24.

1 Introduction

Throughout this paper, "an operator" means "a bounded linear operator over Hilbert space". Let $\mathbb{H}$, $\mathbb{I}$, $\mathbb{J}$ and $\mathbb{K}$ denote arbitrary Hilbert spaces. We use $L(\mathbb{H},\mathbb{K})$ to denote the set of all bounded linear operators from $\mathbb{H}$ to $\mathbb{K}$. Especially, $L(\mathbb{H})$=$L(\mathbb{H},\mathbb{H})$. For an operator $T \in L(\mathbb{H},\mathbb{K})$, the symbols $R(T)$, $N(T)$ and T^* denote the range, the null-space and the adjoint of T, respectively. I denotes the unit operator over Hilbert space and O is the zero operator over Hilbert space. An operator $T \in L(\mathbb{H})$ is a Hermitian operator if and only if $T^* = T$. An operator $T \in L(\mathbb{H})$ is an invertible operator if and only if there is a operator $U \in L(\mathbb{H})$, such that $TU = UT = I$. If such operator U exists, we denotes it by T^{-1}.

Recall that an operator $X \in L(\mathbb{K},\mathbb{H})$ is called the Moore-Penrose inverse of $T \in L(\mathbb{H},\mathbb{K})$, if X satisfies the following four operator equations [16],

$$(1)\ TXT = T, \quad (2)\ XTX = X, \quad (3)\ (TX)^* = TX, \quad (4)\ (XT)^* = XT.$$

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If such operator X exists then it is unique and is denoted by $T^{\dagger}$. It is well known that the Moore-Penrose inverse of T exists if and only if $R(T)$ is closed [5, 8].

For a subset $\{i, j, \cdots, k\}$ of the set $\{1, 2, 3, 4\}$, the set of operators satisfying the equations (i), (j), $\cdots$, (k) from among equations (1)-(4) is denoted by $T\{i, j, \cdots, k\}$. An operator in $T\{i, j, \cdots, k\}$ is called an $\{i, j, \cdots, k\}$-inverse of T and is denoted by $T^{(i,j,\cdots,k)}$. For example, an operator X of the set $T\{1\}$ is called a $\{1\}$-inverse or a g-inverse of T and denoted by $X = T^{(1)}$. One usually denotes any $\{1, 3\}$-inverse of the set $T\{1, 3\}$ as $T^{(1,3)}$ which is also called a least squares g-inverse of T. Any $\{1, 4\}$-inverse of the set $T\{1, 4\}$ is denoted by $T^{(1,4)}$ which is also called a minimum norm g-inverse of T. The unique $\{1, 2, 3, 4\}$-inverse of T is the Moore-Penrose inverse of T. We refer the reader to [1, 14] for basic results on the generalized inverses of bounded linear operators.

If s is a semigroup with the unit 1 and if $a_i \in s$, $i = 1, 2, 3$, are invertible, then the equality $(a_1a_2a_3)^{-1} = a_3^{-1}a_2^{-1}a_1^{-1}$ is called the reverse order law for the ordinary inverse. Let T_i, $i = 1, 2, 3$, be three operators over Hilbert space such that the product $T_1T_2T_3$ is meaningful. If each of the three operators is invertible, then the product $T_1T_2T_3$ is invertible too, and the ordinary inverse of $T_1T_2T_3$ satis es the reverse order law $(T_1T_2T_3)^{-1} = T_3^{-1}T_2^{-1}T_1^{-1}$. However, this so-called reverse order law is not necessarily true for other kind generalized inverses. An interesting problem is, for given $\{i, j, \cdots, k\}$-inverses and operators T_i, $i = 1, 2, 3$, with $T_1T_2T_3$ is meaningful, when

$$(T_1T_2T_3)\{i, j, \cdots, k\} = T_3\{i, j, \cdots, k\}T_2\{i, j, \cdots, k\}T_1\{i, j, \cdots, k\}?$$

The reverse order laws for generalized inverses of operator product yield a class of interesting problems that are fundamental in the theory of generalized inverses of operator, see [1, 10, 21]. Theory and computations of the reverse order laws for generalized inverses of operator product are important subjects in many branches of applied science, such as nonlinear control theory, operator theory, operator algebra, global analysis and approximation theory, see [1, 6, 20, 21]. Suppose T_i, $i = 1, 2, 3$, and are bounded linear operators over Hilbert space. The least squares technique (LS):

$$\min_{Y} \|(T_1T_2T_3)Y \quad \|_2,$$

is used in many practical scienti c problems. Any solution Y of the above LS problem can be expressed as $Y = (T_1T_2T_3)^{(1,3)}$. If the LS problem is consistent, then the minimum norm solution Y has the form $Y = (T_1T_2T_3)^{(1,4)}$. The unique minimal norm least square solution Y of the LS problem is $Y = (T_1T_2T_3)^{\dagger}$. One such problem concerned with the above LS problem is, under what conditions, $(T_1T_2T_3)^{(i,j,\cdots,k)} = T_3^{(i,j,\cdots,k)}T_2^{(i,j,\cdots,k)}T_1^{(i,j,\cdots,k)}$?

Since the middle 1960s, the reverse order law for generalized inverses have attracted considerable attention, and a signi cant number of paper treat the su cient or equivalent conditions such that the reverse order law holds in some sense. It is a classical result of Greville [10], that $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ if and only if $R(A^*AB) \subseteq R(B)$ and $R(BB^*A^*) \subseteq R(A^*)$, in this case when A and B are complex matrices. This result is extended to bounded linear operators on Hilbert space, by Bouldin [2] and Izumino [12]. In [13] the reverse order law for the Moore-Penrose inverse is proved in rings with involutions. In [4] D.S.Cvetkovic-IIic studied this reverse order law in C^*-algebra. Then, in [7], the reverse order law for the Moore-Penrose inverse is obtained as a consequence of some set equalities. The reader can nd some interesting and related results in [7, 15, 17, 18, 19, 22].

In 1986, R.E.Hartwig [11] rst discussed the reverse order law for Moore-Penrose inverse of three matrices product. In the paper [9] D.S. Djordjevic et al., extended the results of [11] to the bounded linear operators on Hilbert space, using some algebraic method. In this paper, we revisit this reverse order law by applying the technique of matrix form of bounded linear operators [3]. Let $T_1 \in L(\mathbb{J},\mathbb{K})$, $T_2 \in L(\mathbb{I},\mathbb{J})$ and $T_3 \in L(\mathbb{H},\mathbb{I})$ such that T_1, T_2, T_3 and $T_1T_2T_3$ have closed ranges. Then using the technique of matrix form of a bounded linear operator [3] and the solving operator equations, we will revisit the following reverse order law $(T_1T_2T_3)^\dagger = T_3^\dagger T_2^\dagger T_1^\dagger$. Some new simpler equivalent conditions for this reverse order law are obtained.

We rst mention the following results, which will be used in this paper.

Lemma 1.1. [3, 7, 8] *Let $T \in L(\mathbb{H},\mathbb{K})$ have a closed range. Let H_1 and H_2 be closed and mutually orthogonal subspace of $\mathbb{H}$, such that $H_1 \bigoplus H_2 = \mathbb{H}$. Let K_1 and K_2 be closed and mutually orthogonal subspace of $\mathbb{K}$, such that $\mathbb{K} = K_1 \bigoplus K_2$. Then the operator T has the following matrix representations with respect to the orthogonal sums of subspaces $\mathbb{H} = H_1 \bigoplus H_2 = R(T^*) \bigoplus N(T)$ and $\mathbb{K} = K_1 \bigoplus K_2 = R(T) \bigoplus N(T^*)$:*

(1) $T = \begin{pmatrix} T_{11} & T_{12} \\ O & O \end{pmatrix} : \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \to \begin{pmatrix} R(T) \\ N(T^*) \end{pmatrix}$ *and* $T^\dagger = \begin{pmatrix} T_{11}^*E^{-1} & O \\ T_{12}^*E^{-1} & O \end{pmatrix} : \begin{pmatrix} R(T) \\ N(T^*) \end{pmatrix} \to \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$,
where $E = T_{11}T_{11}^* + T_{12}T_{12}^*$ is invertible on $R(T)$;

(2) $T = \begin{pmatrix} T_{11} & O \\ T_{21} & O \end{pmatrix} : \begin{pmatrix} R(T^*) \\ N(T) \end{pmatrix} \to \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$ *and* $T^\dagger = \begin{pmatrix} F^{-1}T_{11}^* & F^{-1}T_{12}^* \\ O & O \end{pmatrix} : \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \to \begin{pmatrix} R(T^*) \\ N(T) \end{pmatrix}$,
where $F = T_{11}^*T_{11} + T_{21}^*T_{21}$ is invertible on $R(T^*)$;

(3) $T = \begin{pmatrix} T_{11} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T^*) \\ N(T) \end{pmatrix} \to \begin{pmatrix} R(T) \\ N(T^*) \end{pmatrix}$ *and* $T^\dagger = \begin{pmatrix} T_{11}^{-1} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T) \\ N(T^*) \end{pmatrix} \to \begin{pmatrix} R(T^*) \\ N(T) \end{pmatrix}$,
where T_{11} is invertible.

Lemma 1.2. [1] *Let $T \in L(\mathbb{H},\mathbb{K})$ and $N \in L(\mathbb{K},\mathbb{H})$ have closed ranges. Then,*

(1) $TT^\dagger N = N \Leftrightarrow R(N) \subseteq R(T)$;

(2) $NT^\dagger T = N \Leftrightarrow R(N^*) \subseteq R(T^*)$.

2 The triple reverse order law for Moore-Penrose inverse of operator product

Let $T_1 \in L(\mathbb{J},\mathbb{K})$, $T_2 \in L(\mathbb{I},\mathbb{J})$ and $T_3 \in L(\mathbb{H},\mathbb{I})$, such that T_1, T_2, T_3 and $T_1T_2T_3$ have closed ranges. In this section, we will give necessary and su cient conditions for the triple reverse order law of the Moore-Penrose inverse of the operator product $T_1T_2T_3$. First of all let us de ne

$$E = T_1^\dagger T_1, \quad F = T_3T_3^\dagger, \quad P = ET_2F, \quad Q = FT_2^\dagger E, \quad M = T_1T_2T_3, \quad G = T_3^\dagger T_2^\dagger T_1^\dagger. \tag{2.1}$$

In terms of these, we get the following results.

Theorem 2.1. *Let $T_1 \in L(\mathbb{J},\mathbb{K})$, $T_2 \in L(\mathbb{I},\mathbb{J})$ and $T_3 \in L(\mathbb{H},\mathbb{I})$, such that T_1, T_2, T_3 and $T_1T_2T_3$ have closed ranges. Then the following statements are equivalent:*

(1) $(T_1T_2T_3)^\dagger = T_3^\dagger T_2^\dagger T_1^\dagger$;

(2) $Q \in P\{1,2\}$, *and* $T_1^*T_1PQ$, $QPT_3T_3^*$ *are two Hermitian operators*;

(3) $MGM = G$, *and* $GMG = G$, *and* $(MG)^* = MG$, *and* $(GM)^* = GM$.

Proof. (1)$\Leftrightarrow$ (3): Obvious.

Next, we will prove (2)$\Leftrightarrow$ (3). From Lemma 1.1, we know that the operators T_1, T_2, T_3, $T_1T_2T_3$ and $T_3^\dagger T_2^\dagger T_1^\dagger$ have the following matrix form with respect to the orthogonal sum of subspaces:

$$T_1 = \begin{pmatrix} T_1^{11} & T_1^{12} \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix} \to \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix}, \tag{2.2}$$

$$T_1^\dagger = \begin{pmatrix} (T_1^{11})^*D^{-1} & O \\ (T_1^{12})^*D^{-1} & O \end{pmatrix} : \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \to \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix}, \tag{2.3}$$

where $D = T_1^{11}(T_1^{11})^* + T_1^{12}(T_1^{12})^*$ is invertible on $R(T_1)$.

$$T_2 = \begin{pmatrix} T_2^{11} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix} \to \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix}, \tag{2.4}$$

$$T_2^\dagger = \begin{pmatrix} (T_2^{11})^{-1} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2) \\ N(T_2^*) \end{pmatrix} \to \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix}, \tag{2.5}$$

where T_2^{11} is invertible.

$$T_3 = \begin{pmatrix} T_3^{11} & O \\ T_3^{21} & O \end{pmatrix} : \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix} \to \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix}, \tag{2.6}$$

$$T_3^\dagger = \begin{pmatrix} S^{-1}(T_3^{11})^* & S^{-1}(T_3^{21})^* \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_2^*) \\ N(T_2) \end{pmatrix} \to \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix}, \tag{2.7}$$

where $S = (T_3^{11})^*T_3^{11} + (T_3^{21})^*T_3^{21}$ is invertible on $R(T_3^*)$.

Let $M = T_1T_2T_3$ and $G = T_3^\dagger T_2^\dagger T_1^\dagger$, then form (2.2)$\sim$(2.7), we have

$$M = T_1T_2T_3 = \begin{pmatrix} T_1^{11}T_2^{11}T_3^{11} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix} \to \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \tag{2.8}$$

and

$$G = T_3^\dagger T_2^\dagger T_1^\dagger = \begin{pmatrix} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} & O \\ O & O \end{pmatrix} : \begin{pmatrix} R(T_1) \\ N(T_1^*) \end{pmatrix} \to \begin{pmatrix} R(T_3^*) \\ N(T_3) \end{pmatrix}. \tag{2.9}$$

According to the formulas (2.1)$\sim$(2.7), we have

$$Q = \begin{pmatrix} T_3^{11} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} T_1^{11} & T_3^{11} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} T_1^{12} \\ T_3^{21} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} T_1^{11} & T_3^{21} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} T_1^{12} \end{pmatrix} \tag{2.10}$$

and

$$P = \begin{pmatrix} (T_1^{11})^* D^{-1} T_1^{11} T_2^{11} T_3^{11} S^{-1}(T_3^{11})^* & (T_1^{11})^* D^{-1} T_1^{11} T_2^{11} T_3^{11} S^{-1}(T_3^{21})^* \\ (T_1^{12})^* D^{-1} T_1^{11} T_2^{11} T_3^{11} S^{-1}(T_3^{11})^* & (T_1^{12})^* D^{-1} T_1^{11} T_2^{11} T_3^{11} S^{-1}(T_3^{21})^* \end{pmatrix}. \tag{2.11}$$

From (2.2), (2.6), (2.10) and (2.11), we get

$$T_1^* T_1 PQ = \begin{pmatrix} {}_{11} & {}_{12} \\ {}_{21} & {}_{22} \end{pmatrix}, \text{where} \tag{2.12}$$

$$\begin{aligned} {}_{11} &= (T_1^{11})^* T_1^{11} T_2^{11} T_3^{11} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} T_1^{11}, \\ {}_{12} &= (T_1^{11})^* T_1^{11} T_2^{11} T_3^{11} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} T_1^{12}, \\ {}_{21} &= (T_1^{12})^* T_1^{11} T_2^{11} T_3^{21} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} T_1^{11}, \\ {}_{22} &= (T_1^{12})^* T_1^{11} T_2^{11} T_3^{21} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} T_1^{12}, \end{aligned}$$

and

$$QPT_3T_3^* = \begin{pmatrix} {}_{11} & {}_{12} \\ {}_{21} & {}_{22} \end{pmatrix}, \text{where} \tag{2.13}$$

$$\begin{aligned} {}_{11} &= T_3^{11} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} T_1^{11} T_2^{11} T_3^{11} (T_3^{11})^*, \\ {}_{12} &= T_3^{11} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} T_1^{11} T_2^{11} T_3^{11} (T_3^{21})^*, \\ {}_{21} &= T_3^{21} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} T_1^{11} T_2^{11} T_3^{11} (T_3^{11})^*, \\ {}_{22} &= T_3^{21} S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* D^{-1} T_1^{11} T_2^{11} T_3^{11} (T_3^{21})^*. \end{aligned}$$

Combining (2.8) with (2.9), we know that $G = M^\dagger$ (i.e. $T_3^\dagger T_2^\dagger T_1^\dagger = (T_1T_2T_3)^\dagger$), if and only if

$$(I)\ MGM = M, \quad (II)\ GMG = G, \quad (III)\ (MG)^* = MG, \quad (IV)\ (GM)^* = GM. \tag{2.14}$$

From the formulas (2.10)$\sim$(2.13), we know that the statement (2) of Theorem 2.1 can be rewrited as

$$(a)\ PQP = P,\ (b)\ QPQ = Q,\ (c)\ (T_1^*T_1PQ)^* = T_1^*T_1PQ,\ (d)\ (QPT_3T_3^*)^* = QPT_3T_3^*. \tag{2.15}$$

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In the rest of this section, we will prove (2.14) is equivalent to (2.15). That is the conditions (2) in Theorem 2.1 is equal to the conditions (3) in Theorem 2.1.

(I)⇔(a): From (2.8) and (2.9), we have

$$\begin{aligned} MGM &= (T_1T_2T_3)(T_3^{\dagger}T_2^{\dagger}T_1^{\dagger})(T_1T_2T_3) \\ &= \begin{pmatrix} T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11} & O \\ O & O \end{pmatrix}. \end{aligned} \tag{2.16}$$

Then from (2.8) and (2.16), we know that the inclusion $MGM = M$ is equivalent to

$$T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11} = T_1^{11}T_2^{11}T_3^{11}. \tag{2.17}$$

By the formulas (2.10) and (2.11), we have

$$PQP = \begin{pmatrix} {}_{11} & {}_{12} \\ {}_{21} & {}_{22} \end{pmatrix}, \text{where} \tag{2.18}$$

$$\begin{aligned} {}_{11} &= (T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*, \\ {}_{12} &= (T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^*, \\ {}_{21} &= (T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*, \\ {}_{22} &= (T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^*. \end{aligned}$$

From (2.11) and (2.18), we know that the inclusion $PQP = P$ is equivalent to

$$\begin{aligned} & (T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^* \\ = \; & (T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*, \end{aligned} \tag{2.19}$$

$$\begin{aligned} & (T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^* \\ = \; & (T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^*, \end{aligned} \tag{2.20}$$

$$\begin{aligned} & (T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^* \\ = \; & (T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*, \end{aligned} \tag{2.21}$$

$$\begin{aligned} & (T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^* \\ = \; & (T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^*. \end{aligned} \tag{2.22}$$

If the equation (2.17) holds, we have the equations (2.19)∼(2.22) hold too. That is (I)⇒(a).

On the other hand, if the equations (2.19)∼(2.22) hold, we have

$$\begin{aligned} & T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*T_3^{11} \\ = \; & T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*T_3^{11}, \end{aligned} \tag{2.23}$$

$$\begin{aligned}&T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^*T_3^{21}\\&=T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^*T_3^{21},\end{aligned}\tag{2.24}$$

$$\begin{aligned}&T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*T_3^{11}\\&=T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*T_3^{11},\end{aligned}\tag{2.25}$$

$$\begin{aligned}&T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^*T_3^{21}\\&=T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{21})^*T_3^{21}.\end{aligned}\tag{2.26}$$

Combining (2.23), (2.24) with the de nition of S in (2.7), we have

$$\begin{aligned}&T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}\\=\;&T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}.\end{aligned}\tag{2.27}$$

Combining (2.25), (2.26) with the de nition of D in (2.3), we have

$$\begin{aligned}&T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}\\=\;&T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}.\end{aligned}\tag{2.28}$$

From the results in (2.27) and (2.28), we have

$$T_1^{11}T_2^{11}T_3^{11}=T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}T_2^{11}T_3^{11}.\tag{2.29}$$

That is (a)⇒(I).

(II)⇔(b): With the same method of the proof of (I)⇔(a), the condition $GMG = G$ is easily seen to be equivalent to $QPQ = Q$.

(III)⇔(c): From (2.8) and (2.9), we have

$$MG=(T_1T_2T_3)(T_3^\dagger T_2^\dagger T_1^\dagger)=\begin{pmatrix}T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1} & O\\ O & O\end{pmatrix}.\tag{2.30}$$

Since S and D are Hermitian operators, then the inclusion $(MG)^*=MG$ is equivalent to

$$T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}=D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*.\tag{2.31}$$

By the formulas (2.12), we have that the inclusion $(T_1^*T_1PQ)^*=T_1^*T_1PQ$ is equivalent to

$$\begin{aligned}&(T_1^{11})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}\\=\;&(T_1^{11})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*T_1^{11},\end{aligned}\tag{2.32}$$

$$\begin{aligned}&(T_1^{11})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{12}\\=\;&(T_1^{11})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*T_1^{12},\end{aligned}\tag{2.33}$$

$$\begin{aligned}&(T_1^{12})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}\\=\ &(T_1^{12})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*T_1^{11},\end{aligned}\tag{2.34}$$

$$\begin{aligned}&(T_1^{12})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{12}\\=\ &(T_1^{12})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*T_1^{12}.\end{aligned}\tag{2.35}$$

If the equation (2.31) holds, we have the equations (2.32)∼(2.35) hold too. That is (III)⇒(c).

On the other hand, if the equations (2.32)∼(2.35) hold, we have

$$\begin{aligned}&T_1^{11}(T_1^{11})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}(T_1^{11})^*\\=\ &T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*T_1^{11}(T_1^{11})^*,\end{aligned}\tag{2.36}$$

$$\begin{aligned}&T_1^{11}(T_1^{11})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{12}(T_1^{12})^*\\=\ &T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*T_1^{12}(T_1^{12})^*,\end{aligned}\tag{2.37}$$

$$\begin{aligned}&T_1^{12}(T_1^{12})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{11}(T_1^{11})^*\\=\ &T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*T_1^{11}(T_1^{11})^*,\end{aligned}\tag{2.38}$$

$$\begin{aligned}&T_1^{12}(T_1^{12})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1}T_1^{12}(T_1^{12})^*\\=\ &T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*T_1^{12}(T_1^{12})^*.\end{aligned}\tag{2.39}$$

Combining (2.36), (2.37) with the de nition of $D = T_1^{11}(T_1^{11})^* + T_1^{12}(T_1^{12})^*$ in (2.3), we have

$$\begin{aligned}&T_1^{11}(T_1^{11})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*\\=\ &T_1^{11}(T_1^{11})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*D.\end{aligned}\tag{2.40}$$

Combining (2.38), (2.39) with the de nition of D, we have

$$\begin{aligned}&T_1^{12}(T_1^{12})^*T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*\\=\ &T_1^{12}(T_1^{12})^*D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*D.\end{aligned}\tag{2.41}$$

Finally, from (3.40), (3.41) and the de nition of D, we have

$$DT_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^* = T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*D. \tag{2.42}$$

Since $D = (T_1^{11})(T_1^{11})^* + (T_1^{12})(T_1^{12})^*$ is invertible on $R(T_1)$, then (2.42) can be rewrited as

$$T_1^{11}T_2^{11}T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^{-1}(T_1^{11})^*D^{-1} = D^{-1}T_1^{11}((T_2^{11})^{-1})^*T_3^{11}S^{-1}(T_3^{11})^*(T_2^{11})^*(T_1^{11})^*. \tag{2.43}$$

That is (c)⇒(III).

(IV)⇔(d): With the same method of the proof of (III)⇔(c), we can get the result that the condition $(GM)^* = GM$ is equivalent to $(QPT_3T_3^*)^* = QPT_3T_3^*$ without the proof.

From the above proof, the formulas (2.14) is equivalent to (2.15). We then complete the proof of the theorem. ■

Be the same as (2.1), $Q = FT_2^\dagger E$ and $P = ET_2F$, next we will derive some other equivalent conditions for the triple reverse order law $(T_1T_2T_3)^\dagger = T_3^\dagger T_2^\dagger T_1^\dagger$.

Theorem 2.2. *Let* $T_1 \in L(\mathbb{J}, \mathbb{K})$, $T_2 \in L(\mathbb{I}, \mathbb{J})$ *and* $T_3 \in L(\mathbb{H}, \mathbb{I})$, *such that* T_1, T_2, T_3 *and* $T_1T_2T_3$ *have closed ranges. Then the following statements are equivalent:*

(1) $(T_1T_2T_3)^\dagger = T_3^\dagger T_2^\dagger T_1^\dagger$;

(2) $Q \in P\{1, 2\}$ and $T_1^*T_1PQ$, $QPT_3T_3^*$ are two Hermitian operators;

(3) $Q \in P\{1\}$ and $R(T_1^*T_1P) = R(Q^*)$ and $R(T_3T_3^*P^*) = R(Q)$;

(4) $(PQ)^2 = PQ$ and $R(T_1^*T_1P) = R(Q^*)$ and $R(T_3T_3^*P^*) = R(Q)$.

Proof. (1)⇔(2): By the results in Theorem 2.1, we know that (1)⇔(2).

(2)⇒(3): According to the de nitions of the generalized inverses of operators, we have

$$Q \in P\{1, 2\} \Rightarrow Q \in P\{1\}. \tag{2.44}$$

By the de nitions of the ranges of operators and the formula (2.44), we have

$$R(T_1^*T_1P) = R(T_1^*T_1PQP) \subseteq R(T_1^*T_1PQ) \subseteq R(T_1^*T_1P). \tag{2.45}$$

That is

$$R(T_1^*T_1P) = R(T_1^*T_1PQ). \tag{2.46}$$

If $T_1^*T_1PQ$ is a Hermitian operator, then

$$R(T_1^*T_1P) = R(T_1^*T_1PQ) = R(Q^*P^*T_1^*T_1) = R(Q^*P^*T_1^\dagger T_1). \tag{2.47}$$

Since $Q^*P^*T_1^\dagger T_1 = Q^*P^*$, then from (2.44) and (2.47), we have

$$R(T_1^*T_1P) = R(Q^*P^*T_1^\dagger T_1) = R(Q^*P^*) = R(Q^*). \tag{2.48}$$

Similarly, if $QPT_3T_3^*$ is a Hermitian operator, we have

$$R(T_3^*T_3P^*) = R(T_3^*T_3P^*Q^*) = R(QPT_3T_3^*) = R(QP) = R(Q). \tag{2.49}$$

Combining (2.44), (2.48) with (2.49), we have the result (2)⇒(3).

(3)⇒(4): Obvious.

(4)⇒(2): Firstly, we will prove that if the statement (4) in Theorem 2.2 is true, then $PQP = P$. Since $P = PT_3T_3^\dagger$ and $R(T_3T_3^*P^*) = R(Q)$, then we have

$$R(P) = R(PT_3) = R(PT_3T_3^*P^*) = R(PQ). \tag{2.50}$$

Combining (2.50) with $(PQ)^2 = PQ$, we have

$$PQP = P \text{ and } (QP)^2 = QP. \tag{2.51}$$

Secondly, we will prove that if the statement (4) in Theorem 2.2 is true, then $QPQ = Q$. From the statement (4) in Theorem 2.2 and the de nitions of Q and P, we have

$$\begin{aligned} R(Q^*) &= R(T_1^*T_1P) = R(T_1^*T_1PP^*T_1^*T_1) = R(T_1^*T_1PP^*T_1^\dagger T_1) \\ &= R(T_1^*T_1PP^*) = R(Q^*P^*). \end{aligned} \tag{2.52}$$

Combining (2.52) with $(Q^*P^*)^2 = Q^*P^*$, we have

$$Q^*P^*Q^* = Q^* \text{ i.e. } QPQ = Q. \tag{2.53}$$

Thirdly, we will prove that if the statement (4) in Theorem 2.2 is true, then $T_1^*T_1PQ$ is a Hermitian operator. Since $R(T_1^*T_1P) = R(Q^*)$ and $R(Q^*P^*) = R(Q^*)$, then we have

$$Q^*P^*T_1^*T_1P = T_1^*T_1P. \tag{2.54}$$

From (2.54), we have

$$Q^*P^*T_1^*T_1PQ = T_1^*T_1PQ = (T_1^*T_1PQ)^*. \tag{2.55}$$

Fourthly, we will prove that if the statement (4) in Theorem 2.2 is true, then $QPT_3T_3^*$ is a Hermitian operator. Since $R(T_3T_3^*P^*) = R(Q)$ and $QPQ = Q$, then we have

$$R(QP) = R(Q) \text{ and } QPT_3T_3^*P^* = T_3T_3^*P^*. \tag{2.56}$$

From (2.56), we have

$$QPT_3T_3^*P^*Q^* = T_3T_3^*P^*Q^* = (QPT_3T_3^*)^* = QPT_3T_3^*. \tag{2.57}$$

Combining the formulas (2.51), (2.53), (2.55) with (2.57), we immediately obtain the result (4)⇒(2). We then complete the proof of the theorem. ■

Let us now see how some of the special cases come out of the conditions of Theorem 2.2.

Corollary 2.1. *Let $T_1 \in L(\mathbb{J}, \mathbb{K})$, $T_2 \in L(\mathbb{I}, \mathbb{J})$ and $T_3 \in L(\mathbb{H}, \mathbb{I})$, such that T_1, T_2, T_3 and $T_1T_2T_3$ have closed ranges. If $R(T_2) \subseteq R(T_1^*)$ and $R(T_2^*) \subseteq R(T_3)$, then*

$$(T_1T_2T_3)^\dagger = T_3^\dagger T_2^\dagger T_1^\dagger \Leftrightarrow R(T_1^*T_1T_2) \subseteq R(T_2) \text{ and } R(T_3T_3^*T_2^*) \subseteq R(T_2^*).$$

Proof. According to the hypothesis $R(T_2) \subseteq R(T_1^*)$ and $R(T_2^*) \subseteq R(T_3)$ and the results in Lemma 1.2, we have

$$Q = FT_2^\dagger E = T_2^\dagger, \quad P = ET_2F = T_2. \tag{2.58}$$

$\Rightarrow$: If $(T_1T_2T_3)^\dagger = T_3^\dagger T_2^\dagger T_1^\dagger$, then from Theorem 2.1 and Theorem 2.2 , we have $(PQ)^2 = PQ$ and $R(T_1^*T_1P) = R(Q^*)$ and $R(T_3T_3^*P^*) = R(Q)$. So, we get

$$R(T_1^*T_1T_2) = R((T_2^\dagger)^*) \subseteq R(T_2) \text{ and } R(T_3T_3^*T_2^*) = R(T_2^\dagger) \subseteq R(T_2^*). \tag{2.59}$$

$\Leftarrow$: From (2.58), we have $PQP = P$ and $QPQ = Q$. That is

$$Q \in P\{1, 2\}. \tag{2.60}$$

By (2.58), we also have

$$T_1^*T_1PQ = T_1^*T_1T_2T_2^\dagger \text{ and } QPT_3T_3^* = T_2^\dagger T_2T_3T_3^*. \tag{2.61}$$

Combining the hypothesis $R(T_1^*T_1T_2) \subseteq R(T_2)$ with results in Lemma 1.2, we have

$$T_2T_2^\dagger T_1T_1^*T_2T_2^\dagger = T_1T_1T_2^*T_2^\dagger = (T_1T_1T_2^*T_2^\dagger)^*. \tag{2.62}$$

Combining the hypothesis $R(T_3T_3^*T_2) \subseteq R(T_2^*)$ with results in Lemma 1.2, we have

$$T_2^\dagger T_2T_3T_3^*T_2^*(T_2^*)^\dagger = T_3T_3^*T_2^*(T_2^*)^\dagger = (T_3T_3^*T_2^*(T_2^*)^\dagger)^* = T_2^\dagger T_2T_3T_3^* = (T_2^\dagger T_2T_3T_3^*)^*. \tag{2.63}$$

According to the formulas (2.59), (2.60), (2.62), (2.63) and the statement (2) in Theorem 2.2, we immediately obtain the results of Corollary 2.1. ■

Corollary 2.2. *Let $T_1 \in L(\mathbb{J}, \mathbb{K})$, $T_2 \in L(\mathbb{I}, \mathbb{J})$ and $T_3 \in L(\mathbb{H}, \mathbb{I})$, such that T_2 and $T_1T_2T_3$ have closed ranges. If $T_1^\dagger T_1 = I$ and $T_3T_3^\dagger = I$ (i.e. T_1 and T_3 are invertible operators), then*

$$(T_1T_2T_3)^\dagger = T_3^{-1}T_2^\dagger T_1^{-1} \Leftrightarrow R(T_1^*T_1T_2) \subseteq R(T_2) \text{ and } R(T_3T_3^*T_2^*) \subseteq R(T_2^*).$$

Corollary 2.3. *Let $T_1 \in L(\mathbb{J}, \mathbb{K})$, $T_2 \in L(\mathbb{I}, \mathbb{J})$ and $T_3 \in L(\mathbb{H}, \mathbb{I})$, such that T_1, T_2, T_3, $T_1T_2T_3$ and $T_1^\dagger T_1T_2T_3T_3^\dagger$ have closed ranges. If $T_1^\dagger T_1 = T_1$ and $T_3T_3^\dagger = T_3$, then*

$$(T_1T_2T_3)^\dagger = T_3^\dagger T_2^\dagger T_1^\dagger \Leftrightarrow T_3T_3^\dagger T_2^\dagger T_1^\dagger T_1 = (T_1^\dagger T_1T_2T_3T_3^\dagger)^\dagger.$$

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DIFFERENTIAL EQUATIONS ARISING FROM CERTAIN SHEFFER SEQUENCE

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ABSTRACT. In this paper, we study some differential equations arising from certain Sheffer sequence and investigate some identities for the Sheffer sequence of polynomials which is related to the theory of hyperbolic differential equations.

1. Introduction

A partial differential equation of the second-order

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0,$$

is called hyperbolic if the matrix is

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 0, \text{ (see [6])}.$$

The wave equation is an example of a hyperbolic partial differential equation. A sequence $S_n(x)$ is called a Sheffer sequence if the generating function has the form

$$\sum_{k=0}^{\infty} S_k(x)\frac{t^k}{k!} = A(t)e^{xB(t)},$$

where

$$A(t) = A_0 + A_1 t + A_2 t^2 + \cdots$$
$$B(t) = B_1 t + B_2 t^2 + \cdots, \qquad \text{with} \quad A_0 \neq 0, \ B_0 \neq 0 \text{ (see [12])}.$$

If $f(t)$ is a delta series and $g(t)$ is an invertible series, there exists a uniquen sequence $S_n(x)$ of Sheffer polynomials such that the orthogonality condition $< g(t)f(t)^k|S_n(x) >= \delta_{n,k}$ holds, where $\delta_{n,k}$ is the Kronecker delta (see [8-11]).

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In this paper, we consider the Sheffer sequence given by the pair $\left(\frac{1}{1+t}, 1-(1+t)^{-2}\right)$, namely

$$F(t,x) = \frac{1}{\sqrt{1-t}} e^{x\left(\frac{1}{\sqrt{1-t}}-1\right)} = \sum_{n=0}^{\infty} h_n(x) \frac{t^n}{n!}. \tag{1.1}$$

In [5], Erdélyi also considered a Sheffer sequence which is related to $h_n(x)$. Indeed, his sequence is given by $g_n(x) = \frac{1}{n!} h_n(x)$. Also, we note that

$$h_n(x) = xe^{-x} \left[\frac{d}{dx^2}\right]^n (x^{2n-1}e^x), \text{ (see [5]).} \tag{1.2}$$

The polynomials $h_n(x)$ have applications to the theory of hyperbolic differential equations (see [1-4]). From (1.1), by replacing t by $1-e^{-2t}$, we can derive the following equation:

$$\begin{aligned} e^t e^{x(e^t-1)} &= \sum_{n=0}^{\infty} (-1)^n h_n(x) \frac{1}{n!} (e^{-2t}-1)^n \\ &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} (-1)^{n+m} h_n(x) 2^m S_2(n,m) \right) \frac{t^m}{m!}, \end{aligned} \tag{1.3}$$

where $S_2(n,m)$ is the Stirling number of the second kind.
As is well known, the Bell polynomials are defined by the generating function

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x) \frac{t^n}{n!}, \text{ (see [7]).} \tag{1.4}$$

By (1.3), we get

$$\begin{aligned} e^t e^{x(e^t-1)} &= \left(\sum_{l=0}^{\infty} \frac{t^l}{l!} \right) \left(\sum_{n=0}^{\infty} Bel_n(x) \frac{t^n}{n!} \right) \\ &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} \binom{m}{n} Bel_n(x) \right) \frac{t^m}{m!}. \end{aligned} \tag{1.5}$$

From (1.3) and (1.5), we have

$$\sum_{n=0}^{m} \binom{m}{n} Bel_n(x) = \sum_{n=0}^{m} (-1)^{n+m} h_n(x) 2^m S_2(n,m), \ (m \geq 0). \tag{1.6}$$

In this paper, we study some differential equations arising from certain sheffer sequence and investigate some identities for the Sheffer sequence of polynomials which is related to the theory of hyperbolic differential equations.

2. Differential equations arising from certain Sheffer sequence

Let

$$F = F(t,x) = (1-t)^{-\frac{1}{2}} e^{x\left((1-t)^{-\frac{1}{2}}-1\right)} \tag{2.1}$$

Then, we have

$$\begin{aligned} F^{(1)} =& \frac{dF(t,x)}{dt} = (1-t)^{-\frac{1}{2}} e^{x\left((1-t)^{-\frac{1}{2}}-1\right)} \left(\tfrac{1}{2}(1-t)^{-1} + \frac{1}{2}x(1-t)^{-\frac{3}{2}}\right) \\ =& \left(\frac{1}{2}(1-t)^{-1} + \frac{1}{2}x(1-t)^{-\frac{3}{2}}\right) F, \end{aligned} \tag{2.2}$$

$$F^{(2)} = \frac{dF^{(1)}}{dt} = \left(\frac{3}{4}(1-t)^{-2} + \frac{5}{4}x(1-t)^{-\frac{5}{2}} + \frac{1}{4}x^2(1-t)^{-3}\right) F, \tag{2.3}$$

and

$$F^{(3)} = \left(\frac{15}{8}(1-t)^{-3} + \frac{33}{8}x(1-t)^{-\frac{7}{2}} + \frac{12}{8}x^2(1-t)^{-4} + \frac{1}{8}x^3(1-t)^{-\frac{9}{2}}\right) F.$$

Thus, we are let to put

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t,x) = \left(\sum_{i=0}^{N} a_i(N) x^i (1-t)^{-N-\frac{1}{2}i}\right) F, \tag{2.4}$$

where $N = 0, 1, 2, \cdots$.

Taking the derivative of (2.4) with respect to t, we have

$$\begin{aligned} F^{(N+1)} =& \frac{dF^{(N)}}{dt} = \left(\sum_{i=0}^{N}(N+\tfrac{1}{2}i) a_i(N) x^i (1-t)^{-N-1-\frac{1}{2}i}\right) F \\ & + \left(\sum_{i=0}^{N} a_i(N) x^i (1-t)^{-N-\frac{1}{2}i}\right) F^{(1)} \\ =& \left(\sum_{i=0}^{N}(N+\tfrac{1}{2}i) a_i(N) x^i (1-t)^{-N-1-\frac{1}{2}i}\right) F \\ & + \left(\sum_{i=0}^{N} a_i(N) x^i (1-t)^{-N-\frac{1}{2}i}\right) \left(\frac{1}{2}(1-t)^{-1} + \frac{1}{2}x(1-t)^{-\frac{3}{2}}\right) F \end{aligned} \tag{2.5}$$

$$= \left(\sum_{i=0}^{N} \left(N + \tfrac{1}{2}i + \tfrac{1}{2}\right) a_i(N) x^i (1-t)^{-N-1-\frac{1}{2}i} + \sum_{i=0}^{N} \frac{1}{2} a_i(N) x^{i+1} (1-t)^{-N-\frac{3}{2}-\frac{1}{2}i}\right) F$$

$$= \left(\sum_{i=0}^{N} \left(N + \tfrac{1}{2}i + \tfrac{1}{2}\right) a_i(N) x^i (1-t)^{-N-1-\frac{1}{2}i} + \sum_{i=1}^{N+1} \frac{1}{2} a_{i-1}(N) x^i (1-t)^{-N-1-\frac{1}{2}i}\right) F.$$

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On the other hand, by replacing N by $N+1$ in (2.4), we get

$$F^{(N+1)} = \left(\sum_{i=0}^{N+1} a_i(N+1) x^i (1-t)^{-N-1-\frac{1}{2}i} \right) F. \tag{2.6}$$

Comparing the coefficients on both sides of (2.5) and (2.6), we obtain the following recurrence relations:

$$a_0(N+1) = (N+\tfrac{1}{2})a_0(N), \;\; a_{N+1}(N+1) = \frac{1}{2}a_N(N), \tag{2.7}$$

and

$$a_i(N+1) = \frac{1}{2}a_{i-1}(N) + \left(N + \tfrac{1}{2}i + \tfrac{1}{2}\right) a_i(N), \;\; (1 \le i \le N). \tag{2.8}$$

In addition, we note that

$$F = F^{(0)} = a_0(0)F. \tag{2.9}$$

Thus, by (2.9), we easily get

$$a_0(0) = 1. \tag{2.10}$$

For $N = 1$ in (1.5) and (1.2), it is not difficult to show that

$$\begin{aligned} &\left(\frac{1}{2}(1-t)^{-1} + \frac{1}{2}x(1-t)^{-3/2} \right) F = F^{(1)} \\ &= \left(a_0(1)(1-t)^{-1} + a_1(x)x(1-t)^{-3/2} \right) F. \end{aligned} \tag{2.11}$$

By comparing the coefficients on both sides of (2.11), we easily get

$$a_0(1) = \frac{1}{2}, \quad a_1(1) = \frac{1}{2}. \tag{2.12}$$

From (2.7), we can easily derive the following equations:

$$\begin{aligned} &a_{N+1}(N+1) = \frac{1}{2}a_N(N) = \left(\frac{1}{2}\right)^2 a_{N-1}(N-1) = \cdots = \left(\frac{1}{2}\right)^{N+1}, \\ &a_0(0) = \left(\frac{1}{2}\right)^{N+1}, \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} a_0(N+1) =& (N+\tfrac{1}{2})a_0(N) = (N+\tfrac{1}{2})(N-\tfrac{1}{2})a_0(N-1) = \cdots \\ =& (N+\tfrac{1}{2})(N-\tfrac{1}{2}) \cdots \tfrac{3}{2} \cdot \tfrac{1}{2} a_0(0) = (N+\tfrac{1}{2})_{N+1}, \end{aligned} \tag{2.14}$$

where

$$(x)_n = x(x-1)\cdots(x-n+1), \;\; (n \ge 1), \quad (x)_0 = 1.$$

The matrix $(a_i(j))$ $(0 \leq i, j \leq N)$ is given by

$$\big(a_i(j)\big) = \begin{pmatrix} 1 & \frac{1}{2} & \left(\frac{3}{2}\right)_2 & \left(\frac{5}{2}\right)_3 & \cdots & \left(N-\frac{1}{2}\right)_N \\ 0 & \frac{1}{2} & \cdots & \cdots & \cdots & \cdot \\ 0 & 0 & \left(\frac{1}{2}\right)^2 & \cdots & \cdots & \cdot \\ 0 & 0 & 0 & \left(\frac{1}{2}\right)^3 & \cdots & \cdot \\ \vdots & \vdots & \vdots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \left(\frac{1}{2}\right)^N \end{pmatrix}$$

For $i = 1, 2, 3$ in (2.8), we have

$$\begin{aligned} a_1(N+1) =& \frac{1}{2}a_0(N) + (N+1)a_1(N) \\ =& \frac{1}{2}\Big(a_0(N) + (N+1)a_0(N-1)\Big) + (N+1)Na_1(N-1) \\ =& \frac{1}{2}\Big(a_0(N) + (N+1)a_0(N-1) + (N+1)Na_0(N-2)\Big) \\ &+ (N+1)N(N-1)a_1(N-2) \\ =& \cdots \\ =& \frac{1}{2}\sum_{k=0}^{N-1}(N+1)_k a_0(N-k) + (N+1)_N a_1(1) \\ =& \frac{1}{2}\sum_{k=0}^{N}(N+1)_k a_0(N-k), \end{aligned} \tag{2.15}$$

$$\begin{aligned} a_2(N+1) =& \frac{1}{2}\sum_{k=0}^{N-2}\left(N+\frac{3}{2}\right)_k a_1(N-k) + \left(N+\frac{3}{2}\right)_{N-1} a_2(2) \\ =& \frac{1}{2}\sum_{k=0}^{N-1}\left(N+\frac{3}{2}\right)_k a_1(N-k), \end{aligned}$$

and

$$\begin{aligned} a_3(N+1) =& \frac{1}{2}\sum_{k=0}^{N-3}(N+2)_k a_2(N-k) + (N+2)_{N-2}a_3(3) \\ =& \frac{1}{2}\sum_{k=0}^{N-2}(N+2)_k a_2(N-k). \end{aligned}$$

Continuing this process, we have

$$a_i(N+1)=\frac{1}{2}\sum_{k=0}^{N-i+1}\left(N+\tfrac{1}{2}i+\tfrac{1}{2}\right)_k a_{i-1}(N-k),\ (1\le i\le N). \tag{2.16}$$

Now, we give explicit expressions for $a_i(N+1)$, $(1\le i\le N)$. From (2.16), we note that

$$a_1(N+1)=\frac{1}{2}\sum_{k_1=0}^{N}(N+1)_{k_1}a_0(N-k_1)=\frac{1}{2}\sum_{k_1=0}^{N}(N+1)_{k_1}(N-k_1-\tfrac{1}{2})_{N-k_1}, \tag{2.17}$$

$$\begin{aligned}a_2(N+1)&=\frac{1}{2}\sum_{k_2=0}^{N-1}\left(N+\tfrac{3}{2}\right)_{k_2}a_1(N-k_2)\\&=\left(\frac{1}{2}\right)^2\sum_{k_2=0}^{N-1}\sum_{k_1=0}^{N-k_2-1}\left(N+\tfrac{3}{2}\right)_{k_2}(N-k_2)_{k_1}(N-k_2-k_1-\tfrac{3}{2})_{N-k_2-k_1-1},\end{aligned} \tag{2.18}$$

$$\begin{aligned}a_3(N+1)=&\left(\frac{1}{2}\right)^3\sum_{k_3=0}^{N-2}\sum_{k_2=0}^{N-2-k_3}\sum_{k_1=0}^{N-2-k_3-k_2}(N+2)_{k_3}(N-k_3+\tfrac{1}{2})_{k_2}\\&\times(N-k_3-k_2-1)_{k_1}(N-k_3-k_2-k_1-\tfrac{5}{2})_{N-k_3-k_2-k_1-2},\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}a_4(N+1)=&\left(\frac{1}{2}\right)^4\sum_{k_4=0}^{N-3}\sum_{k_3=0}^{N-3-k_4}\sum_{k_2=0}^{N-3-k_4-k_3}\sum_{k_1=0}^{N-3-k_4-k_3-k_2}(N+\tfrac{5}{2})_{k_4}\\&\times(N-k_4+1)_{k_3}(N-k_4-k_3-\tfrac{1}{2})_{k_2}(N-k_4-k_3-k_2-2)_{k_1}\\&\times(N-k_4-k_3-k_2-k_1-\tfrac{7}{2})_{N-k_4-k_3-k_2-k_1-3}.\end{aligned} \tag{2.20}$$

So, we can deduce that, for $1\le i\le N$,

$$\begin{aligned}&a_i(N+1)\\&=\left(\frac{1}{2}\right)^i\sum_{k_i=0}^{N-i+1}\sum_{k_{i-1}=0}^{N-i+1-k_i}\cdots\sum_{k_1=0}^{N-i+1-k_i-\cdots-k_2}\prod_{l=1}^{i}\Big(N+\tfrac{3}{2}l+\tfrac{1}{2}-i-\sum_{j=l+1}^{i}k_j\Big)_{k_l}\\&\times\big(N+\tfrac{1}{2}-i-\sum_{j=1}^{i}k_j\big)_{N+1-i-\sum_{j=1}^{i}k_j}.\end{aligned} \tag{2.21}$$

Therefore, by (2.21), we obtain the following theorem.

Theorem 1. For $N = 0, 1, 2 \cdots$, the following family of differential equations

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t,x) = \left(\sum_{i=0}^{N} a_i(N) x^i (1-t)^{-N-\frac{1}{2}i}\right) F$$

have a solution

$$F = F(t,x) = (1-t)^{-1/2} e^{x((1-t)^{-1/2}-1)},$$

where

$$a_0(N) = \left(N - \tfrac{1}{2}\right)_N,$$

$$a_i(N) = \left(\frac{1}{2}\right)^i \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-i-k_i} \cdots \sum_{k_1=0}^{N-i-k_i-\cdots-k_2} \prod_{l=1}^{i} \left(N + \tfrac{3}{2}l - \tfrac{1}{2} - i - \sum_{j=l+1}^{i} k_j\right)_{k_l}$$
$$\times \left(N - \tfrac{1}{2} - i - \sum_{j=1}^{i} k_j\right)_{N-i-\sum_{j=1}^{i} k_j}.$$

From (1.1), we note that

$$\begin{aligned}
\sum_{k=0}^{\infty} h_{k+N}(x)\frac{t^k}{k!} &= F^{(N)} = \left(\sum_{i=0}^{N} a_i(N) x^i (1-t)^{-N-\frac{1}{2}i}\right) F \\
&= \sum_{i=0}^{N} a_i(N) x^i \sum_{l=0}^{\infty} \left(N + \tfrac{1}{2}i + l - 1\right)_l \frac{t^l}{l!} \sum_{m=0}^{\infty} h_m(x) \frac{t^m}{m!} \\
&= \sum_{i=0}^{N} a_i(N) x^i \sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} \binom{k}{l} \left(N + \tfrac{1}{2}i + l - 1\right)_l h_{k-l}(x)\right) \frac{t^k}{k!} \\
&= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{N} \sum_{l=0}^{k} \binom{k}{l} \left(N + \tfrac{1}{2}i + l - 1\right)_l a_i(N) x^i h_{k-l}(x)\right) \frac{t^k}{k!}.
\end{aligned} \tag{2.22}$$

Thus, by comparing the coefficients on both sides of (2.22), we obtain the following theorem.

Theorem 2. For $k, N = 0, 1, 2, \cdots$, we have

$$h_{k+N}(x) = \sum_{i=0}^{N} \sum_{l=0}^{k} \binom{k}{l} \left(N + \tfrac{1}{2}i + l - 1\right)_l a_i(N) x^i h_{k-l}(x) \tag{2.23}$$

Letting $k = 0$ in (2.23), we obtain the following corollary.

Corollary 3. For $N = 0, 1, 2, \cdots$, we have

$$h_N(x) = \sum_{i=0}^{N} a_i(N)x^i. \tag{2.24}$$

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Hyers-Ulam stability of the first order inhomogeneous matrix difference equation

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Abstract

We prove Hyers-Ulam stability of the first order linear inhomogeneous matrix difference equation $\vec{x}_{i+1} = \mathbf{A}(i)\vec{x}_i + \vec{g}(i)$ for all integers $i \in \mathbb{Z}$. Moreover, we show Hyers-Ulam stability of the nth order linear difference equation as a corollary.

1 Introduction

Throughout this paper, we denote by $\mathbb{C}$, $\mathbb{N}$, $\mathbb{N}_0$, and $\mathbb{Z}$ the set of all complex numbers, of all positive integers, of all nonnegative integers, and the set of all integers, respectively. Given a fixed positive integer n, let $(\mathbb{C}^n, \|\cdot\|_n)$ be a complex normed space, each of whose elements is a column vector, and let $\mathbb{C}^{n\times n}$ be a vector space consisting of all $(n \times n)$ complex matrices. We choose a norm $\|\cdot\|_{n\times n}$ on $\mathbb{C}^{n\times n}$ which is compatible with $\|\cdot\|_n$, *i.e.*, both norms obey

$$\|\mathbf{AB}\|_{n\times n} \le \|\mathbf{A}\|_{n\times n}\|\mathbf{B}\|_{n\times n} \quad \text{and} \quad \|\mathbf{A}\vec{x}\|_n \le \|\mathbf{A}\|_{n\times n}\|\vec{x}\|_n \tag{1.1}$$

for all $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n\times n}$ and $\vec{x} \in \mathbb{C}^n$.

A matrix difference equation is a difference equation with matrix coefficients in which the value of vector at one point depends on the values of preceding (succeeding) points.

In this paper, we prove Hyers-Ulam stability of the first order linear inhomogeneous matrix difference equation

$$\vec{x}_{i+1} = \mathbf{A}(i)\vec{x}_i + \vec{g}(i) \tag{1.2}$$

for all integers $i \in \mathbb{Z}$, where the transition matrices $\mathbf{A}(i)$ are nonsingular. More precisely, we prove that if a vector sequence $\{\vec{y}_i\}_{i\in\mathbb{Z}}$ of $\mathbb{C}^n$ satisfies the inequality

$$\|\vec{y}_{i+1} - \mathbf{A}(i)\vec{y}_i - \vec{g}(i)\|_n \le \varepsilon_{i+1}$$

for all $i \in \mathbb{Z}$, then there exists a solution $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ to the first order matrix difference equation (1.2) such that the bound for $\|\vec{y}_i - \vec{x}_i\|_n$ depends on the sequence $\{\varepsilon_i\}_{i\in\mathbb{Z}}$ and the transition

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matrices $\mathbf{A}(i)$ only. Moreover, we investigate Hyers-Ulam stability of the nth order linear inhomogeneous difference equation of the form

$$a(i+1) = p_1(i)a(i) + p_2(i)a(i-1) + \cdots + p_n(i)a(i-n+1) + r(i), \tag{1.3}$$

where $p_j, r : \mathbb{Z} \to \mathbb{C}$ are given functions with $p_n(i) \neq 0$ for all $i \in \mathbb{Z}$. We refer the reader to [7, 8, 9, 12, 20] for the exact definition of Hyers-Ulam stability.

2 Preliminaries

In this section, we investigate the general solution to the first order linear inhomogeneous matrix difference equation (1.2) for all integers $i \in \mathbb{Z}$, where

$$\vec{x}_i = \begin{pmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{in} \end{pmatrix} \in \mathbb{C}^n \quad \text{and} \quad \mathbf{A}(i) = \begin{pmatrix} a_{11}(i) & a_{12}(i) & \cdots & a_{1n}(i) \\ a_{21}(i) & a_{22}(i) & \cdots & a_{2n}(i) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(i) & a_{n2}(i) & \cdots & a_{nn}(i) \end{pmatrix} \in \mathbb{C}^{n\times n}.$$

Throughout this paper, we use the following abbreviation.

$$\Phi(n,m) := \begin{cases} \displaystyle\prod_{k=m}^{n-1} \mathbf{A}(k) = \mathbf{A}(n-1)\mathbf{A}(n-2)\cdots\mathbf{A}(m) & (\text{for } n > m), \\ \mathbf{I} & (\text{for } n = m), \end{cases} \tag{2.1}$$

where we set $\Phi(n,m) := \big(\Phi(m,n)\big)^{-1} = \mathbf{A}(n)^{-1}\mathbf{A}(n+1)^{-1}\cdots\mathbf{A}(m-1)^{-1}$ for $n < m$ and $\mathbf{I}$ denotes the identity matrix. Sometimes, we use $\Phi(n)$ and $\Phi^{-1}(m,n)$ instead of $\Phi(n,0)$ and $\big(\Phi(m,n)\big)^{-1}$, respectively.

In the following lemma, we introduce some properties of $\Phi(n,m)$ without proof.

Lemma 2.1 *Given a fixed positive integer n, assume that every transition matrix $\mathbf{A}(i) \in \mathbb{C}^{n\times n}$ is nonsingular. It holds that*

(i) $\Phi(i+1,k) = \mathbf{A}(i)\Phi(i,k)$;

(ii) $\Phi^{-1}(i,k+1) = \mathbf{A}(k)\Phi^{-1}(i,k)$;

(iii) $\mathbf{A}(k-1)^{-1}\Phi^{-1}(i,k) = \Phi^{-1}(i,k-1)$

for all integers $i,k \in \mathbb{Z}$.

In the following lemma, we give the general solution to the first order linear inhomogeneous matrix difference equation (1.2).

Lemma 2.2 *Given a fixed positive integer n, assume that every transition matrix $\mathbf{A}(i) \in \mathbb{C}^{n\times n}$ is nonsingular and the vectors $\vec{g}(i) \in \mathbb{C}^n$ are given. A vector sequence $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ of $\mathbb{C}^n$*

is a solution to the first order linear inhomogeneous matrix difference equation (1.2) *if and only if the sequence* $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ *is given in the form of*

$$\vec{x}_i := \begin{cases} \Phi(i,0)\vec{x}_0 + \sum_{k=0}^{i-1} \Phi(i,k+1)\vec{g}(k) & (\textit{for } i \geq 0), \\ \Phi^{-1}(0,i)\vec{x}_0 - \sum_{k=1}^{-i} \Phi^{-1}(i+k,i)\vec{g}(i+k-1) & (\textit{for } i < 0), \end{cases} \tag{2.2}$$

where $\vec{x}_0 \in \mathbb{C}^n$ *is an arbitrarily given vector.*

Proof. First, we assume that the sequence $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ is given in the form of (2.2) and we prove that the sequence $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ is a solution to the first order linear inhomogeneous matrix difference equation (1.2).

If i is a nonnegative integer, then it follows from the first formula of (2.2) and Lemma 2.1 (i) that

$$\begin{aligned} \vec{x}_{i+1} &= \Phi(i+1,0)\vec{x}_0 + \sum_{k=0}^{i} \Phi(i+1,k+1)\vec{g}(k) \\ &= \mathbf{A}(i)\Phi(i,0)\vec{x}_0 + \sum_{k=0}^{i} \mathbf{A}(i)\Phi(i,k+1)\vec{g}(k) \\ &= \mathbf{A}(i)\left(\Phi(i,0)\vec{x}_0 + \sum_{k=0}^{i-1} \Phi(i,k+1)\vec{g}(k) \right) + \vec{g}(i) \\ &= \mathbf{A}(i)\vec{x}_i + \vec{g}(i) \end{aligned}$$

for any integer $i \geq 0$.

If $i = -1$, then we use (2.2) to get

$$\vec{x}_{i+1} = \vec{x}_0$$

and

$$\vec{x}_i = \vec{x}_{-1} = \Phi^{-1}(0,-1)\vec{x}_0 - \Phi^{-1}(0,-1)\vec{g}(-1) = \mathbf{A}(-1)^{-1}\vec{x}_0 - \mathbf{A}(-1)^{-1}\vec{g}(-1).$$

Hence, we have

$$\vec{x}_{i+1} = \mathbf{A}(i)\vec{x}_i + \vec{g}(i)$$

for $i = -1$.

If i is an integer less than -1, then it follows from the second formula of (2.2) and Lemma

2.1 (ii) that

$$\begin{aligned}
\vec{x}_{i+1} &= \Phi^{-1}(0, i+1)\vec{x}_0 - \sum_{k=1}^{-i-1} \Phi^{-1}(i+1+k, i+1)\vec{g}(i+k) \\
&= \mathbf{A}(i)\Phi^{-1}(0,i)\vec{x}_0 - \sum_{k=1}^{-i-1} \mathbf{A}(i)\Phi^{-1}(i+k+1, i)\vec{g}(i+k) \\
&= \mathbf{A}(i)\Phi^{-1}(0,i)\vec{x}_0 - \sum_{j=2}^{-i} \mathbf{A}(i)\Phi^{-1}(i+j, i)\vec{g}(i+j-1) \\
&= \mathbf{A}(i)\Phi^{-1}(0,i)\vec{x}_0 - \sum_{k=1}^{-i} \mathbf{A}(i)\Phi^{-1}(i+k, i)\vec{g}(i+k-1) + \mathbf{A}(i)\Phi^{-1}(i+1, i)\vec{g}(i) \\
&= \mathbf{A}(i)\vec{x}_i + \vec{g}(i)
\end{aligned}$$

for all integers $i < -1$.

Now, we assume that the sequence $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ is a solution to the first order linear inhomogeneous matrix difference equation (1.2) and we prove that the sequence $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ has the form of (2.2). We can easily show that the first formula of (2.2) holds for $i = 0$. We now assume that the first formula of (2.2) holds for some nonnegative integer i. Then, by using Lemma 2.1 (i), we obtain

$$\begin{aligned}
\vec{x}_{i+1} &= \mathbf{A}(i)\vec{x}_i + \vec{g}(i) \\
&= \mathbf{A}(i)\left(\Phi(i,0)\vec{x}_0 + \sum_{k=0}^{i-1} \Phi(i, k+1)\vec{g}(k)\right) + \vec{g}(i) \\
&= \Phi(i+1, 0)\vec{x}_0 + \sum_{k=0}^{i-1} \Phi(i+1, k+1)\vec{g}(k) + \vec{g}(i) \\
&= \Phi(i+1, 0)\vec{x}_0 + \sum_{k=0}^{i} \Phi(i+1, k+1)\vec{g}(k)
\end{aligned}$$

by replacing i with $i+1$ in the first formula of (2.2).

Finally, we assume that the sequence $\{\vec{x}_i\}$ is a solution to (1.2) and we will prove that $\vec{x}_i$ is expressed by the second formula of (2.2) for every negative integer i. If we set $i = -1$ in (1.2), then we get

$$\vec{x}_0 = \mathbf{A}(-1)\vec{x}_{-1} + \vec{g}(-1) \quad \text{or} \quad \vec{x}_{-1} = \mathbf{A}(-1)^{-1}\vec{x}_0 - \mathbf{A}(-1)^{-1}\vec{g}(-1),$$

which we obtain from the second formula of (2.2) by setting $i = -1$. We now assume that $\vec{x}_i$ is expressed as the second formula of (2.2) for some negative integer i. Then, it follows from (1.2), the second formula of (2.2), and Lemma 2.1 (iii) that

$$\vec{x}_i = \mathbf{A}(i-1)\vec{x}_{i-1} + \vec{g}(i-1)$$

or

$$\begin{aligned}\vec{x}_{i-1} &= \mathbf{A}(i-1)^{-1}\vec{x}_i - \mathbf{A}(i-1)^{-1}\vec{g}(i-1) \\ &= \mathbf{A}(i-1)^{-1}\left(\Phi^{-1}(0,i)\vec{x}_0 - \sum_{k=1}^{-i}\Phi^{-1}(i+k,i)\vec{g}(i+k-1)\right) - \mathbf{A}(i-1)^{-1}\vec{g}(i-1) \\ &= \Phi^{-1}(0,i-1)\vec{x}_0 - \sum_{k=0}^{-i}\Phi^{-1}(i+k,i-1)\vec{g}(i+k-1) \\ &= \Phi^{-1}(0,i-1)\vec{x}_0 - \sum_{k=1}^{-i+1}\Phi^{-1}(i+k-1,i-1)\vec{g}(i+k-2),\end{aligned}$$

which is a consequence of the second formula of (2.2) provided we replace i with $i-1$. □

Remark 2.3 *Given a fixed positive integer n, assume that every transition matrix $\mathbf{A}(i) \in \mathbb{C}^{n\times n}$ is nonsingular and the vectors $\vec{g}(i) \in \mathbb{C}^n$ are given. If vector sequences $\{\vec{x}_{i,h}\}_{i\in\mathbb{Z}}$ and $\{\vec{x}_{i,p}\}_{i\in\mathbb{Z}}$ of $\mathbb{C}^n$ are defined by*

$$\vec{x}_{i,h} := \begin{cases} \Phi(i,0)\vec{x}_0 & (\text{for } i \geq 0), \\ \Phi^{-1}(0,i)\vec{x}_0 & (\text{for } i < 0) \end{cases}$$

resp.

$$\vec{x}_{i,p} := \begin{cases} \displaystyle\sum_{k=0}^{i-1}\Phi(i,k+1)\vec{g}(k) & (\text{for } i \geq 0), \\ \displaystyle -\sum_{k=1}^{-i}\Phi^{-1}(i+k,i)\vec{g}(i+k-1) & (\text{for } i < 0), \end{cases}$$

then then the sequence $\{\vec{x}_{i,h}\}_{i\in\mathbb{Z}}$ is a solution to the homogeneous difference equation $\vec{x}_{i+1} = \mathbf{A}(i)\vec{x}_i$ corresponding to (1.2) and the sequence $\{\vec{x}_{i,p}\}_{i\in\mathbb{Z}}$ is a particular solution to the first order linear inhomogeneous matrix difference equation (1.2).

3 Hyers-Ulam stability of $\vec{x}_{i+1} = \mathbf{A}(i)\vec{x}_i + \vec{g}(i)$

We now prove our main theorem concerning the Hyers-Ulam stability of the first order linear inhomogeneous matrix difference equation (1.2). Obviously, our theorem is a generalization and an improvement of [13, Theorem 2.1].

Theorem 3.1 *Given a fixed positive integer n, let $(\mathbb{C}^n, \|\cdot\|_n)$ and $(\mathbb{C}^{n\times n}, \|\cdot\|_{n\times n})$ be complex normed spaces, whose elements are column vectors resp. $(n\times n)$ complex matrices, with the property (1.1). Assume that every transition matrix $\mathbf{A}(i) \in \mathbb{C}^{n\times n}$ is nonsingular, the vectors $\vec{g}(i) \in \mathbb{C}^n$ are given, and that $\{\varepsilon_i\}_{i\in\mathbb{Z}}$ is a sequence of nonnegative real numbers. If a vector sequence $\{\vec{y}_i\}_{i\in\mathbb{Z}}$ of $\mathbb{C}^n$ satisfies the inequality*

$$\|\vec{y}_{i+1} - \mathbf{A}(i)\vec{y}_i - \vec{g}(i)\|_n \leq \varepsilon_{i+1} \tag{3.1}$$

for all $i \in \mathbb{Z}$*, then there exists a solution* $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ *to the first order linear inhomogeneous matrix difference equation* (1.2) *such that*

$$\|\vec{y}_i - \vec{x}_i\|_n \leq \begin{cases} \sum_{k=1}^{i} \varepsilon_k \|\Phi(i,k)\|_{n\times n} + \|\Phi(i,0)\|_{n\times n}\|\vec{y}_0 - \vec{x}_0\|_n & (\text{for } i \geq 0), \\ \sum_{k=1}^{-i} \varepsilon_{i+k} \left\|\Phi^{-1}(i+k,i)\right\|_{n\times n} + \left\|\Phi^{-1}(0,i)\right\|_{n\times n}\|\vec{y}_0 - \vec{x}_0\|_n & (\text{for } i < 0). \end{cases}$$

Proof. First, we assume that $i \geq 0$. In view of Lemma 2.2, the vector sequence $\{\vec{x}_i\}_{i=0,1,\ldots}$ defined by

$$\vec{x}_i = \Phi(i,0)\vec{x}_0 + \sum_{k=0}^{i-1} \Phi(i,k+1)\vec{g}(k) \tag{3.2}$$

satisfies the first order linear inhomogeneous matrix difference equation (1.2) for $i \geq 0$.

We now apply the mathematical induction to prove that

$$\vec{y}_i - \Phi(i,0)\vec{y}_0 - \sum_{k=0}^{i-1} \Phi(i,k+1)\vec{g}(k) = \sum_{k=1}^{i} \Phi(i,k)\big(\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big) \tag{3.3}$$

for all integers $i \geq 0$. It is obvious that the equality (3.3) holds for $i = 0$. We assume that the equality (3.3) holds for some integer $i \geq 0$. Then, it follows from Lemma 2.1 (i) and (3.3) that

$$\begin{aligned} &\vec{y}_{i+1} - \Phi(i+1,0)\vec{y}_0 - \sum_{k=0}^{i} \Phi(i+1,k+1)\vec{g}(k) \\ &= \vec{y}_{i+1} - \mathbf{A}(i)\Phi(i,0)\vec{y}_0 - \sum_{k=0}^{i} \mathbf{A}(i)\Phi(i,k+1)\vec{g}(k) \\ &= \vec{y}_{i+1} - \mathbf{A}(i)\vec{y}_i - \vec{g}(i) + \mathbf{A}(i)\left(\vec{y}_i - \Phi(i,0)\vec{y}_0 - \sum_{k=0}^{i-1} \Phi(i,k+1)\vec{g}(k)\right) \\ &= \sum_{k=1}^{i} \mathbf{A}(i)\Phi(i,k)\big(\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big) + \vec{y}_{i+1} - \mathbf{A}(i)\vec{y}_i - \vec{g}(i) \\ &= \sum_{k=1}^{i+1} \Phi(i+1,k)\big(\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big), \end{aligned}$$

which can be obtained from the equality (3.3) by replacing i with $i+1$. Thus, we conclude by induction that the equality (3.3) holds for all integers $i \geq 0$.

Hence, it follows from (3.1) and (3.3) that

$$\begin{aligned}\left\|\vec{y}_i - \Phi(i,0)\vec{y}_0 - \sum_{k=0}^{i-1} \Phi(i,k+1)\vec{g}(k)\right\|_n &\\ \leq \sum_{k=1}^{i} \|\Phi(i,k)\|_{n\times n} \big\|\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big\|_n & \qquad (3.4)\\ \leq \sum_{k=1}^{i} \varepsilon_k \|\Phi(i,k)\|_{n\times n} &\end{aligned}$$

for $i \geq 0$. In view of (3.2) and (3.4), we have

$$\|\vec{y}_i - \Phi(i,0)\vec{y}_0 + \Phi(i,0)\vec{x}_0 - \vec{x}_i\|_n \leq \sum_{k=1}^{i} \varepsilon_k \|\Phi(i,k)\|_{n\times n}$$

or

$$\|\vec{y}_i - \vec{x}_i\|_n \leq \sum_{k=1}^{i} \varepsilon_k \|\Phi(i,k)\|_{n\times n} + \|\Phi(i,0)\|_{n\times n} \|\vec{y}_0 - \vec{x}_0\|_n$$

for all integers $i \geq 0$.

Now, assume that $i < 0$. By Lemma 2.2, the sequence $\{\vec{x}_i\}_{i=-1,-2,\ldots}$ defined by

$$\vec{x}_i = \Phi^{-1}(0,i)\vec{x}_0 - \sum_{k=1}^{-i} \Phi^{-1}(i+k,i)\vec{g}(i+k-1) \qquad (3.5)$$

satisfies the first order linear inhomogeneous matrix difference equation (1.2) for $i < 0$. Using the mathematical induction, we prove that

$$\begin{aligned}\vec{y}_i - \Phi^{-1}(0,i)\vec{y}_0 + \sum_{k=1}^{-i} \Phi^{-1}(i+k,i)\vec{g}(i+k-1) &\\ = -\sum_{k=i+1}^{0} \Phi^{-1}(k,i)\big(\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big) & \qquad (3.6)\end{aligned}$$

for all integers $i < 0$. It is obvious that the equality (3.6) holds for $i = -1$. We assume that the equality (3.6) holds for some integer $i < 0$. Then, it follows from Lemma 2.1 (*ii*), (*iii*), and (3.6) that

$$\begin{aligned}&\vec{y}_{i-1} - \Phi^{-1}(0,i-1)\vec{y}_0 + \sum_{k=1}^{-i+1} \Phi^{-1}(i+k-1,i-1)\vec{g}(i+k-2)\\ &= \vec{y}_{i-1} - \mathbf{A}(i-1)^{-1}\Phi^{-1}(0,i)\vec{y}_0 + \sum_{k=1}^{-i+1} \mathbf{A}(i-1)^{-1}\Phi^{-1}(i+k-1,i)\vec{g}(i+k-2)\\ &= \mathbf{A}(i-1)^{-1}\left(\mathbf{A}(i-1)\vec{y}_{i-1} - \Phi^{-1}(0,i)\vec{y}_0 + \sum_{k=1}^{-i+1} \Phi^{-1}(i+k-1,i)\vec{g}(i+k-2)\right)\\ &= -\mathbf{A}(i-1)^{-1}\big(\vec{y}_i - \mathbf{A}(i-1)\vec{y}_{i-1} - \vec{g}(i-1)\big)\end{aligned}$$

$$
\begin{aligned}
& + \mathbf{A}(i-1)^{-1}\left(\vec{y}_i - \Phi^{-1}(0,i)\vec{y}_0 + \sum_{k=2}^{-i+1} \Phi^{-1}(i+k-1,i)\vec{g}(i+k-2)\right) \\
= & -\mathbf{A}(i-1)^{-1}\big(\vec{y}_i - \mathbf{A}(i-1)\vec{y}_{i-1} - \vec{g}(i-1)\big) \\
& - \mathbf{A}(i-1)^{-1} \sum_{k=i+1}^{0} \Phi^{-1}(k,i)\big(\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big) \\
= & -\sum_{k=i}^{0} \mathbf{A}(i-1)^{-1}\Phi^{-1}(k,i)\big(\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big) \\
= & -\sum_{k=i}^{0} \Phi^{-1}(k,i-1)\big(\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big),
\end{aligned}
$$

which can be obtained from the equality (3.6) by replacing i with $i-1$. By induction, we conclude that the equality (3.6) holds for any integer $i < 0$.

Therefore, by (3.1) and (3.6), we get

$$
\begin{aligned}
& \left\| \vec{y}_i - \Phi^{-1}(0,i)\vec{y}_0 + \sum_{k=1}^{-i} \Phi^{-1}(i+k,i)\vec{g}(i+k-1) \right\|_n \\
& \leq \sum_{k=i+1}^{0} \|\Phi^{-1}(k,i)\|_{n\times n} \big\|\vec{y}_k - \mathbf{A}(k-1)\vec{y}_{k-1} - \vec{g}(k-1)\big\|_n \\
& \leq \sum_{k=i+1}^{0} \varepsilon_k \|\Phi^{-1}(k,i)\|_{n\times n}
\end{aligned} \tag{3.7}
$$

for any integer $i < 0$. Taking (3.5) and (3.7) into account, we get

$$
\|\vec{y}_i - \Phi^{-1}(0,i)\vec{y}_0 + \Phi^{-1}(0,i)\vec{x}_0 - \vec{x}_i\|_n \leq \sum_{k=i+1}^{0} \varepsilon_k \|\Phi^{-1}(k,i)\|_{n\times n}
$$

or

$$
\begin{aligned}
\|\vec{y}_i - \vec{x}_i\|_n & \leq \sum_{k=i+1}^{0} \varepsilon_k \big\|\Phi^{-1}(k,i)\big\|_{n\times n} + \big\|\Phi^{-1}(0,i)\big\|_{n\times n} \|\vec{y}_0 - \vec{x}_0\|_n \\
& = \sum_{k=1}^{-i} \varepsilon_{i+k} \big\|\Phi^{-1}(i+k,i)\big\|_{n\times n} + \big\|\Phi^{-1}(0,i)\big\|_{n\times n} \|\vec{y}_0 - \vec{x}_0\|_n
\end{aligned}
$$

for all integers $i < 0$. □

4 Applications

In this section, let n be a fixed positive integer. We assume that the nth order linear inhomogeneous difference equation of the form (1.3) is given, where $p_j, r : \mathbb{Z} \to \mathbb{C}$ are given functions with $p_n(i) \neq 0$ for all $i \in \mathbb{Z}$.

If we set

$$
\|\mathbf{A}\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}| \quad \text{and} \quad \|\vec{x}\|_\infty = \max_{1\leq j\leq n} |x_j|
$$

for all $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\vec{x} \in \mathbb{C}^n$, then these norms satisfy the conditions in (1.1).

We now prove Hyers-Ulam stability of the nth order linear inhomogeneous difference equation (1.3).

Theorem 4.1 *Let n be a fixed positive integer and $p_1, \ldots, p_n, r : \mathbb{Z} \to \mathbb{C}$ be given functions with $p_n(i) \neq 0$ for all $i \in \mathbb{Z}$. Assume that a sequence $\{\varepsilon_i\}_{i\in\mathbb{Z}}$ of nonnegative numbers is given. If a sequence $\{a(i)\}_{i\in\mathbb{Z}}$ of complex numbers satisfies the inequality*

$$\big|a(i+1) - p_1(i)a(i) - p_2(i)a(i-1) - \cdots - p_n(i)a(i-n+1) - r(i)\big| \leq \varepsilon_{i+1} \tag{4.1}$$

for all $i \in \mathbb{Z}$, then there exists a sequence $\{c(i)\}_{i\in\mathbb{Z}}$ of complex numbers which is a solution to the nth order linear inhomogeneous difference equation (1.3) such that

$$|a(i) - c(i)| \leq \begin{cases} \displaystyle\sum_{k=1}^{i} \varepsilon_k \|\Phi(i,k)\|_\infty + \|\Phi(i,0)\|_\infty \|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i \geq 0), \\ \displaystyle\sum_{k=1}^{-i} \varepsilon_{i+k} \big\|\Phi^{-1}(i+k,i)\big\|_\infty + \big\|\Phi^{-1}(0,i)\big\|_\infty \|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i < 0), \end{cases}$$

where $\Phi(i,k)$ and $\Phi^{-1}(i,k)$ are defined in (2.1) and (4.2), and where $\vec{y}_0$ and $\vec{x}_0$ are defined in (4.7).

Proof. For any $k \in \{1, 2, \ldots, n-1\}$, we define the complex numbers $b_k(i)$ by

$$\begin{aligned} b_1(i) &= a(i-1), \\ b_2(i) &= b_1(i-1), \\ b_3(i) &= b_2(i-1), \\ &\vdots \\ b_{n-1}(i) &= b_{n-2}(i-1) \end{aligned}$$

for all $i \in \mathbb{Z}$. We further define

$$\mathbf{A}(i) := \begin{pmatrix} p_1(i) & p_2(i) & p_3(i) & \cdots & p_{n-1}(i) & p_n(i) \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \tag{4.2}$$

$$\vec{y}_i := \begin{pmatrix} a(i) \\ b_1(i) \\ b_2(i) \\ \vdots \\ b_{n-1}(i) \end{pmatrix} \quad \text{and} \quad \vec{g}(i) := \begin{pmatrix} r(i) \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{4.3}$$

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for all $i \in \mathbb{Z}$, where $\mathbf{A}(i)$ is an $n \times n$ matrix and $\vec{y}_i, \vec{g}(i)$ are $n \times 1$ vectors.

Using these notations and considering (4.1), the sequence $\{\vec{y}_i\}_{i\in\mathbb{Z}}$ satisfies the inequality

$$\|\vec{y}_{i+1} - \mathbf{A}(i)\vec{y}_i - \vec{g}(i)\|_\infty \le \varepsilon_{i+1}$$

for all $i \in \mathbb{Z}$. Moreover, by the assumption that $p_n(i) \neq 0$ for all $i \in \mathbb{Z}$, we can see that every $\mathbf{A}(i)$ is nonsingular.

According to Theorem 3.1, there exists a solution $\{\vec{x}_i\}_{i\in\mathbb{Z}}$ to the first order linear inhomogeneous matrix difference equation (1.2) such that

$$\|\vec{y}_i - \vec{x}_i\|_\infty \le \begin{cases} \displaystyle\sum_{k=1}^{i} \varepsilon_k \|\Phi(i,k)\|_\infty + \|\Phi(i,0)\|_\infty \|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i \ge 0), \\ \displaystyle\sum_{k=1}^{-i} \varepsilon_{i+k} \left\|\Phi^{-1}(i+k,i)\right\|_\infty + \left\|\Phi^{-1}(0,i)\right\|_\infty \|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i < 0). \end{cases} \tag{4.4}$$

If we set

$$\vec{x}_i := \begin{pmatrix} x_1(i) \\ x_2(i) \\ \vdots \\ x_n(i) \end{pmatrix}, \tag{4.5}$$

then it follows from (1.2) that

$$\begin{aligned} x_1(i+1) &= p_1(i)x_1(i) + p_2(i)x_2(i) + p_3(i)x_3(i) + \cdots + p_n(i)x_n(i) + r(i), \\ x_2(i+1) &= x_1(i), \\ x_3(i+1) &= x_2(i), \\ &\vdots \\ x_n(i+1) &= x_{n-1}(i) \end{aligned} \tag{4.6}$$

for all $i \in \mathbb{Z}$. Moreover, if we define $c(i) := x_1(i)$ for all integers i, then we have

$$\begin{aligned} x_1(i+1) &= c(i+1), \\ x_1(i) &= c(i), \\ x_2(i) &= x_1(i-1) = c(i-1), \\ &\vdots \\ x_n(i) &= x_{n-1}(i-1) = \cdots = x_1(i-n+1) = c(i-n+1). \end{aligned}$$

Hence, by (4.6), the sequence $\{c(i)\}_{i\in\mathbb{Z}}$ is a solution to the nth order linear inhomogeneous difference equation (1.3).

Since

$$\vec{y}_i = \begin{pmatrix} a(i) \\ a(i-1) \\ a(i-2) \\ \vdots \\ a(i-n+1) \end{pmatrix} \quad \text{and} \quad \vec{x}_i = \begin{pmatrix} c(i) \\ c(i-1) \\ c(i-2) \\ \vdots \\ c(i-n+1) \end{pmatrix} \tag{4.7}$$

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for all $i \in \mathbb{Z}$, we get

$$|a(i) - c(i)| \leq \|\vec{y}_i - \vec{x}_i\|_\infty$$

for all $i \in \mathbb{Z}$. In view of (4.4), we complete the proof of this theorem. $\square$

We now consider the second order linear homogeneous difference equation of the form

$$a(i+1) = p_1(i)a(i) + p_2(i)a(i-1) \tag{4.8}$$

for all $i \in \mathbb{Z}$. The solution of (4.8) is called the (extended) Fibonacci numbers when $p_1(i) = p_2(i) \equiv 1$, $a(0) = 1$, and $a(1) = 1$.

If we substitute $n = 2$, $p_1(i) = 1$, $p_2(i) = 1$, and $r(i) = 0$ for all $i \in \mathbb{Z}$ in Theorem 4.1, then we prove the following corollary concerning Hyers-Ulam stability of the Fibonacci difference equation. However, this corollary shows that Theorem 4.1 is not efficient when the transition matrices $\mathbf{A}(i)$ are constant, *i.e.*, $\mathbf{A}(i) = \mathbf{A}$ for all $i \in \mathbb{Z}$. Nevertheless, we introduce this corollary because its proof includes some new properties of the extended Fibonacci numbers. (In general, it is reasonable to apply [21, Theorem 5] when the transition matrices $\mathbf{A}(i)$ are constant.)

Corollary 4.2 *Assume that a sequence $\{\varepsilon_i\}_{i\in\mathbb{Z}}$ of nonnegative numbers is given. If a sequence $\{a(i)\}_{i\in\mathbb{Z}}$ of complex numbers satisfies the inequality*

$$|a(i+1) - a(i) - a(i-1)| \leq \varepsilon_{i+1} \tag{4.9}$$

for all $i \in \mathbb{Z}$, then there exists a sequence $\{c(i)\}_{i\in\mathbb{Z}}$ of complex numbers which is a solution to the Fibonacci difference equation, i.e., the difference equation (4.8) with $p_1(i) = p_2(i) \equiv 1$ such that

$$|a(i) - c(i)| \leq \begin{cases} \displaystyle\sum_{k=1}^{i} \varepsilon_k F(i-k+1) + F(i+1)\|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i \geq 0), \\ \displaystyle\sum_{k=1}^{-i} \varepsilon_{i+k} F(k+1) + F(-i+1)\|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i < 0), \end{cases}$$

where $F(i)$ denotes the ith extended Fibonacci number and

$$\|\vec{y}_0 - \vec{x}_0\|_\infty = \max\big\{|a(0) - c(0)|, |a(-1) - c(-1)|\big\}.$$

Proof. If we set

$$\mathbf{A} := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \vec{y}_i := \begin{pmatrix} a(i) \\ a(i-1) \end{pmatrix},$$

then it follows from (4.9) that

$$\|\vec{y}_{i+1} - \mathbf{A}\vec{y}_i\|_\infty \leq \varepsilon_{i+1}$$

for all $i \in \mathbb{Z}$.

According to Theorem 4.1, there exists a sequence $\{c(i)\}_{i\in\mathbb{Z}}$ of complex numbers which is a solution to the Fibonacci difference equation (4.8) with $p_1(i) = p_2(i) \equiv 1$ such that

$$|a(i) - c(i)| \leq \begin{cases} \displaystyle\sum_{k=1}^{i} \varepsilon_k \big\|\mathbf{A}^{i-k}\big\|_\infty + \big\|\mathbf{A}^i\big\|_\infty \|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i \geq 0), \\ \displaystyle\sum_{k=1}^{-i} \varepsilon_{i+k} \big\|\mathbf{A}^{-k}\big\|_\infty + \big\|\mathbf{A}^i\big\|_\infty \|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i < 0), \end{cases} \tag{4.10}$$

where $\vec{y}_i$ and $\vec{x}_i$ are defined in (4.7) for all $i \in \mathbb{Z}$.

Here, we introduce some (extended) Fibonacci numbers explicitly.

$$\begin{aligned} &\ldots,\ F(-4) = 2,\ F(-3) = -1,\ F(-2) = 1,\ F(-1) = 0, \\ &F(0) = 1,\ F(1) = 1,\ F(2) = 2,\ F(3) = 3,\ F(4) = 5,\ \ldots \end{aligned} \tag{4.11}$$

and we prove that

$$F(i)F(i-1) < 0 \tag{4.12}$$

for any integer $i \leq -2$. If the relation (4.12) were not true, then there would exist an integer $i_0 \leq -2$ such that $F(i_0)F(i_0 - 1) \geq 0$. Then we would have

$$\begin{aligned} -1 &= F(-2)F(-3) \\ &= F(-3)^2 + F(-3)F(-4) \\ &= F(-3)^2 + F(-4)^2 + F(-4)F(-5) \\ &\ \vdots \\ &= F(-3)^2 + F(-4)^2 + \cdots + F(i_0)^2 + F(i_0)F(i_0 - 1) \\ &\geq 0, \end{aligned}$$

which is a contradiction.

We now prove that

$$|F(i)| = |F(-i-2)| \tag{4.13}$$

for any $i \in \mathbb{Z}$. First, we apply the induction to prove that the equality (4.13) holds for all integers $i \geq 0$. In view of (4.11), it is obvious that the equality (4.13) holds for $i \in \{0, 1, 2\}$. Assume that (4.13) holds for all integers $1 \leq i \leq i_0$, where i_0 is an integer not less than 2. In view of (4.11) and (4.12), we further have

$$\begin{aligned} |F(i_0 + 1)| &= |F(i_0) + F(i_0 - 1)| \\ &= |F(i_0)| + |F(i_0 - 1)| \\ &= |F(-i_0 - 2)| + |F(-i_0 - 1)| \\ &= | - F(-i_0 - 2) + F(-i_0 - 1)| \\ &= |F(-i_0 - 3)|, \end{aligned}$$

which can be obtained from (4.13) by replacing i with $i_0 + 1$. Hence, we conclude that the equality (4.13) holds for all integers $i \geq 0$.

Now, we apply an induction to prove that the equality (4.13) holds for all integers $i < 0$. In view of (4.11), we easily see that the equality (4.13) holds for $i \in \{-1, -2\}$. Assume that (4.13) holds for all integers $i_0 \le i \le -3$, where i_0 is an integer less than -2. Then, by (4.12) and (4.13), we have

$$\begin{aligned} |F(i_0 - 1)| &= |F(i_0 + 1) - F(i_0)| \\ &= |F(i_0 + 1)| + |F(i_0)| \\ &= |F(-i_0 - 3)| + |F(-i_0 - 2)| \\ &= |F(-i_0 - 3) + F(-i_0 - 2)| \\ &= |F(-i_0 - 1)|, \end{aligned}$$

which we can obtain from (4.13) by replacing i with $i_0 - 1$. Thus, the equality (4.13) holds for all integers $i < 0$.

Moreover, we apply the mathematical induction to prove

$$\mathbf{A}^i = \begin{pmatrix} F(i) & F(i-1) \\ F(i-1) & F(i-2) \end{pmatrix} \tag{4.14}$$

for any $i \in \mathbb{Z}$. Obviously, the equality (4.14) holds for $i \in \{0, 1\}$. Assume that (4.14) holds for some integer $i \ge 0$. Then, we get

$$\begin{aligned} \mathbf{A}^{i+1} = \mathbf{A}^i \mathbf{A} &= \begin{pmatrix} F(i) & F(i-1) \\ F(i-1) & F(i-2) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} F(i) + F(i-1) & F(i) \\ F(i-1) + F(i-2) & F(i-1) \end{pmatrix} \\ &= \begin{pmatrix} F(i+1) & F(i) \\ F(i) & F(i-1) \end{pmatrix}, \end{aligned}$$

which can be obtained from (4.14) by replacing i with $i + 1$. Similarly, we prove that the equality (4.14) holds for all negative integers i.

Using (4.13) and (4.14), we prove that

$$\left\| \mathbf{A}^i \right\|_\infty = \begin{cases} F(i+1) & (\text{for } i \ge 0), \\ F(-i+1) & (\text{for } i < 0). \end{cases} \tag{4.15}$$

It is obvious that the first equality of (4.15) is true for $i \in \{0, 1\}$. Assume that $i \ge 2$. Then, considering (4.14) and the fact that $i - 2 \ge 0$, we have

$$\begin{aligned} \left\| \mathbf{A}^i \right\|_\infty &= \max\left\{ |F(i)| + |F(i-1)|, |F(i-1)| + |F(i-2)| \right\} \\ &= \max\left\{ F(i) + F(i-1), F(i-1) + F(i-2) \right\} \\ &= \max\left\{ F(i+1), F(i) \right\} \\ &= F(i+1) \end{aligned}$$

for any integer $i \ge 2$.

Now, we prove the equality (4.15) for $i < 0$. It follows from (4.13) and (4.14) that

$$\begin{aligned} \left\| \mathbf{A}^i \right\|_\infty &= \max\left\{ |F(i)| + |F(i-1)|, |F(i-1)| + |F(i-2)| \right\} \\ &= \max\left\{ |F(-i-2)| + |F(-i-1)|, |F(-i-1)| + |F(-i)| \right\} \\ &= \max\left\{ F(-i-2) + F(-i-1), F(-i-1) + F(-i) \right\} \\ &= \max\left\{ F(-i), F(-i+1) \right\} \\ &= F(-i+1) \end{aligned}$$

for any integer $i < 0$.

Finally, by (4.10) and (4.15), we have

$$|a(i) - c(i)| \leq \begin{cases} \displaystyle\sum_{k=1}^{i} \varepsilon_k F(i-k+1) + F(i+1)\|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i \geq 0), \\ \displaystyle\sum_{k=1}^{-i} \varepsilon_{i+k} F(k+1) + F(-i+1)\|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i < 0), \end{cases}$$

which completes our proof. □

According to [16, Theorem 5.1], the following formula is true:

$$\sum_{k=1}^{i} F(k) = F(i+2) - 2 \tag{4.16}$$

for all $i \in \mathbb{N}_0$, where $F(i)$ denotes the ith extended Fibonacci number with the initial values, $F(-1) = 0$, $F(0) = 1$, and $F(1) = 1$.

Remark 4.3 Let ε be an arbitrarily given positive number. Assume that a sequence $\{a(i)\}_{i\in\mathbb{Z}}$ of complex numbers satisfies the inequality

$$|a(i+1) - a(i) - a(i-1)| \leq \varepsilon$$

for all $i \in \mathbb{Z}$. According to Corollary 4.2 and (4.16), there exists a sequence $\{c(i)\}_{i\in\mathbb{Z}}$ of complex numbers which is a solution to the Fibonacci difference equation such that

$$|a(i) - c(i)| \leq \begin{cases} F(i+2)\varepsilon - 2\varepsilon + F(i+1)\|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i > 0), \\ \|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i = 0), \\ F(-i+3)\varepsilon - 3\varepsilon + F(-i+1)\|\vec{y}_0 - \vec{x}_0\|_\infty & (\text{for } i < 0), \end{cases}$$

where $F(i)$ denotes the ith extended Fibonacci number with the initial values, $F(-1) = 0$, $F(0) = 1$, and $F(1) = 1$, and

$$\|\vec{y}_0 - \vec{x}_0\|_\infty = \max\big\{|a(0) - c(0)|, |a(-1) - c(-1)|\big\}.$$

In particular, under strong additional conditions that $a(-1) = c(-1)$ and $a(0) = c(0)$, the last inequality reduces into

$$|a(i) - c(i)| \leq \begin{cases} F(i+2)\varepsilon - 2\varepsilon & (\text{for } i > 0), \\ 0 & (\text{for } i = 0), \\ F(-i+3)\varepsilon - 3\varepsilon & (\text{for } i < 0). \end{cases}$$

Remark 4.4 The Hyers-Ulam stability of the Fibonacci functional equation has been investigated in [1, 10, 11, 14, 15], while Hyers-Ulam stability of the linear difference equations has been investigated in [1, 2, 3, 5, 17, 18, 19]. It should be remarked that many interesting theorems have been proved in [4, 6] concerning the linear (or nonlinear) recurrences. Especially, Hyers-Ulam stability of the first order matrix difference equations with constant matrix has been proved in [21] in the domain $\mathbb{N}_0$.

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Self Adjoint Operator Ostrowski type Inequalities

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Abstract

We present here several self adjoint operator Ostrowski type inequalities to all directions. These are based in the operator order over a Hilbert space.

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Key Words and Phrases: Self adjoint operator, Hilbert space, Ostrowski inequality.

1 Motivation

In 1938, A. Ostrowski [12] proved the following important inequality:

Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(x)\right| \leq \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty ,$$

for any $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

In this article we present self adjoint operator Ostrowski type inequalities on a Hilbert space in the operator order.

2 Background

Let A be a selfadjoint linear operator on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. The Gelfand map establishes a $*-$isometrically isomorphism Φ between the set

1

$C(Sp(A))$ of all continuous functions defind on the spectrum of A, denoted $Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see e.g. [10, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ (the operation composition is on the right) and $\Phi(\overline{f}) = (\Phi(f))^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f), \text{ for all } f \in C(Sp(A)),$$

and we call it the continuous functional calculus for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$ then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued continuous functions on $Sp(A)$ then the following important property holds:

(P) $f(t) \geq g(t)$ for any $t \in Sp(A)$, implies that $f(A) \geq g(A)$ in the operator order of $B(H)$ (the Banach algebra of all bounded linear operators from H into itself).

Equivalently, we use (see [8], pp. 7-8):

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and $\{E_\lambda\}_\lambda$ be its spectral family.

Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following spectral representation in terms of the Riemann-Stieljes integral:

$$\langle f(U)x, y \rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, y \rangle),$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of bounded variation on the interval $[m, M]$, and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y \rangle,$$

for any $x, y \in H$. Furthermore, it is known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is increasing and right continuous on $[m, M]$.

We have also the formula

$$\langle f(U)x, x \rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, x \rangle), \quad \forall\, x \in H.$$

As a symbol we can write

$$f(U)=\int_{m-0}^{M} f(\lambda)\, dE_{\lambda}.$$

Above, $m=\min\{\lambda|\lambda\in Sp(U)\}:=\min Sp(U)$, $M=\max\{\lambda|\lambda\in Sp(U)\}:=\max Sp(U)$. The projections $\{E_{\lambda}\}_{\lambda\in\mathbb{R}}$, are called the spectral family of A, with the properties:

(a) $E_{\lambda}\leq E_{\lambda'}$ for $\lambda\leq\lambda'$;

(b) $E_{m-0}=0_H$ (zero operator), $E_M=1_H$ (identity operator) and $E_{\lambda+0}=E_{\lambda}$ for all $\lambda\in\mathbb{R}$.

Furthermore

$$E_{\lambda}:=\varphi_{\lambda}(U),\ \ \forall\ \lambda\in\mathbb{R},$$

is a projection which reduces U, with

$$\varphi_{\lambda}(s):=\begin{cases}1, & \text{for } -\infty<s\leq\lambda,\\ 0, & \text{for } \lambda<s<+\infty.\end{cases}$$

The spectral family $\{E_{\lambda}\}_{\lambda\in\mathbb{R}}$ determines uniquely the self-adjoint operator U and vice versa.

For more on the topic see [11], pp. 256-266, and for more detalis see there pp. 157-266. See also [7].

Some more basics are given (we follow [8], pp. 1-5):

Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{C}$. A bounded linear operator A defined on H is selfjoint, i.e., $A=A^*$, iff $\langle Ax,x\rangle\in\mathbb{R}$, $\forall\ x\in H$, and if A is selfadjoint, then

$$\|A\|=\sup_{x\in H:\|x\|=1}|\langle Ax,x\rangle|.$$

Let A,B be selfadjoint operators on H. Then $A\leq B$ iff $\langle Ax,x\rangle\leq\langle Bx,x\rangle$, $\forall$ $x\in H$.

In particular, A is called positive if $A\geq 0$.

Denote by

$$\mathcal{P}:=\left\{\varphi(s):=\sum_{k=0}^{n}\alpha_k s^k|n\geq 0,\ \alpha_k\in\mathbb{C},\ 0\leq k\leq n\right\}.$$

If $A\in\mathcal{B}(H)$ is selfadjoint, and $\varphi(s)\in\mathcal{P}$ has real coefficients, then $\varphi(A)$ is selfadjoint, and

$$\|\varphi(A)\|=\max\{|\varphi(\lambda)|,\lambda\in Sp(A)\}.$$

If φ is any function defined on $\mathbb{R}$ we define

$$\|\varphi\|_A:=\sup\{|\varphi(\lambda)|,\lambda\in Sp(A)\}.$$

If A is selfadjoint operator on Hilbert space H and φ is continuous and given that $\varphi(A)$ is selfadjoint, then $\|\varphi(A)\| = \|\varphi\|_A$. And if φ is a continuous real valued function so it is $|\varphi|$, then $\varphi(A)$ and $|\varphi|(A) = |\varphi(A)|$ are selfadjoint operators (by [8], p. 4, Theorem 7).

Hence it holds

$$\||\varphi(A)|\| = \||\varphi|\|_A = \sup\{||\varphi(\lambda)||, \lambda \in Sp(A)\}$$

$$= \sup\{|\varphi(\lambda)|, \lambda \in Sp(A)\} = \|\varphi\|_A = \|\varphi(A)\|,$$

that is

$$\||\varphi(A)|\| = \|\varphi(A)\|.$$

For a selfadjoint operator $A \in \mathcal{B}(H)$ which is positive, there exists a unique positive selfadjoint operator $B := \sqrt{A} \in \mathcal{B}(H)$ such that $B^2 = A$, that is $\left(\sqrt{A}\right)^2 = A$. We call B the square root of A.

Let $A \in \mathcal{B}(H)$, then A^*A is selfadjoint and positive. Define the "operator absolute value" $|A| := \sqrt{A^*A}$. If $A = A^*$, then $|A| = \sqrt{A^2}$.

For a continuous real valued function φ we observe the following:

$$|\varphi(A)| \text{ (the functional absolute value) } = \int_{m-0}^{M} |\varphi(\lambda)| \, dE_\lambda =$$

$$\int_{m-0}^{M} \sqrt{(\varphi(\lambda))^2} dE_\lambda = \sqrt{(\varphi(A))^2} = |\varphi(A)| \text{ (operator absolute value)},$$

where A is a selfadjoint operator.

That is we have

$$|\varphi(A)| \text{ (functional absolute value) } = |\varphi(A)| \text{ (operator absolute value)}.$$

3 Main Results

Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$, $m < M$; $m, M \in \mathbb{R}$.

In the next we obtain Ostrowski type inequalities in the operator order of $\mathcal{B}(H)$ (the Banach algebra of all bounded linear operators from H into itself).

We mention

Theorem 1 *([2], p. 498) Let $f \in C^1([m, M])$, $m < M$, $s \in [m, M]$. Then*

$$\left| \frac{1}{M-m} \int_m^M f(t) \, dt - f(x) \right| \leq \left(\frac{(s-m)^2 + (M-s)^2}{2(M-m)} \right) \|f'\|_\infty. \quad (1)$$

By applying property (P) to (1), we obtain in the operator order the following inequality:

Theorem 2 *Let $f \in C^1([m,M])$. Then*

$$\left|\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)1_H - f(A)\right| \leq \left(\frac{(A-m1_H)^2+(M1_H-A)^2}{2(M-m)}\right)\|f'\|_\infty . \tag{2}$$

We mention

Theorem 3 *([1], p. 191, Cerone-Dragomir) Let $f:[m,M]\to\mathbb{R}$ be a continuous on $[m,M]$ and twice differentiable function on (m,M), whose second derivative $f'':(m,M)\to\mathbb{R}$ is bounded on (m,M). Then*

$$\left|f(s)-\frac{1}{M-m}\int_m^M f(t)\,dt-\left(\frac{f(M)-f(m)}{M-m}\right)\left(s-\frac{m+M}{2}\right)\right| \leq \tag{3}$$

$$\frac{1}{2}\left\{\left[\frac{\left(s-\frac{m+M}{2}\right)^2}{(M-m)^2}+\frac{1}{4}\right]^2+\frac{1}{12}\right\}(M-m)^2\|f''\|_\infty \leq \frac{\|f''\|_\infty}{6}(M-m)^2,$$

$\forall\ s\in[m,M]$.

By applying property (P) to (3), we obtain in the operator order the following inequality:

Theorem 4 *All as in Theorem 3. Then*

$$\left|f(A)-\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)1_H-\left(\frac{f(M)-f(m)}{M-m}\right)\left(A-\left(\frac{m+M}{2}\right)1_H\right)\right| \tag{4}$$

$$\leq \frac{1}{2}\left\{\left[\frac{\left(A-\left(\frac{m+M}{2}\right)1_H\right)^2}{(M-m)^2}+\frac{1}{4}1_H\right]^2+\frac{1}{12}1_H\right\}(M-m)^2\|f''\|_\infty$$

$$\leq \left(\frac{\|f''\|_\infty}{6}(M-m)^2\right)1_H.$$

We mention

Theorem 5 *([3], p. 14) Let $f:[m,M]\to\mathbb{R}$ be 3-times differentiable on $[m,M]$. Assume that f''' is bounded on $[m,M]$. Let any $s\in[m,M]$. Then*

$$\left|f(s)-\frac{1}{M-m}\int_m^M f(t)\,dt-\left(\frac{f(M)-f(m)}{M-m}\right)\left(s-\left(\frac{m+M}{2}\right)\right)-\right.$$

$$\left.\left(\frac{f'(M)-f'(m)}{2(M-m)}\right)\left[s^2-(m+M)s+\left(\frac{m^2+M^2+4mM}{6}\right)\right]\right| \qquad (5)$$
$$\leq \frac{\|f'''\|_\infty}{(M-m)^3}Z(s),$$

where

$$Z(s)=\left[mMs^4-\frac{1}{3}m^2M^3s+\frac{1}{3}m^3Ms^2-mM^2s^3-\frac{1}{3}m^3M^2s+\frac{1}{3}mM^3s^2\right.$$
$$+m^2M^2s^2-m^2Ms^3-\frac{1}{2}ms^5-\frac{1}{2}Ms^5+\frac{1}{6}s^6+\frac{3}{4}m^2s^4+\frac{3}{4}M^2s^4+\frac{1}{3}M^2m^4-$$
$$\frac{2}{3}m^3s^3-\frac{2}{3}M^3s^3-\frac{1}{3}M^3m^3+\frac{5}{12}m^4s^2+\frac{5}{12}M^4s^2+\frac{1}{3}M^4m^2-$$
$$\left.\frac{2}{15}Mm^5-\frac{2}{15}mM^5-\frac{1}{6}m^5s-\frac{1}{6}M^5s+\frac{m^6}{20}+\frac{M^6}{20}\right]. \qquad (6)$$

Using (P) property and (5), (6) we derive

Theorem 6 *Let $f:[m,M]\to\mathbb{R}$ be 3-times differentiable on $[m,M]$. Assume that f''' is bounded on $[m,M]$. Then*

$$\left|f(A)-\left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)1_H-\left(\frac{f(M)-f(m)}{M-m}\right)\left(A-\left(\frac{m+M}{2}\right)1_H\right)\right.$$
$$\left.-\left(\frac{f'(M)-f'(m)}{2(M-m)}\right)\left[A^2-(m+M)A+\left(\frac{m^2+M^2+4mM}{6}\right)1_H\right]\right| \qquad (7)$$
$$\leq \frac{\|f'''\|_\infty}{(M-m)^3}Z(A),$$

where

$$Z(A)=\left[mMA^4-\frac{1}{3}m^2M^3A+\frac{1}{3}m^3MA^2-mM^2A^3-\frac{1}{3}m^3M^2A+\right.$$
$$\frac{1}{3}mM^3A^2+m^2M^2A^2-m^2MA^3-\frac{1}{2}mA^5-\frac{1}{2}MA^5+\frac{1}{6}A^6+\frac{3}{4}m^2A^4+$$
$$\frac{3}{4}M^2A^4+\left(\frac{1}{3}M^2m^4\right)1_H-\frac{2}{3}m^3A^3-\frac{2}{3}M^3A^3-\left(\frac{1}{3}M^3m^3\right)1_H+$$
$$\frac{5}{12}m^4A^2+\frac{5}{12}M^4A^2+\left(\frac{1}{3}M^4m^2\right)1_H-$$
$$\left.\left(\frac{2}{15}Mm^5\right)1_H-\left(\frac{2}{15}mM^5\right)1_H-\frac{1}{6}m^5A-\frac{1}{6}M^5A+\left(\frac{m^6+M^6}{20}\right)1_H\right]. \qquad (8)$$

6

Let $f \in AC([m, M])$ (absolutely continuous functions on $[m, M]$), $0 < \alpha < 1$. Denote the right Caputo fractional derivative by $D_{t-}^{\alpha} f$ (see [4], p. 22) and the left Caputo fractional derivative by $D_{*t}^{\alpha} f$ (see [4], p. 78), $\forall\, t \in [m, M]$.

We need

Theorem 7 *([4], p. 44) Let $0 < \alpha < 1$, $f \in AC([m, M])$, and $\left\| D_{t-}^{\alpha} f \right\|_{\infty,[m,t]}$, $\left\| D_{*t}^{\alpha} f \right\|_{\infty,[t,M]} < \infty$, $\forall\, t \in [m, M]$. Then*

$$\left| \frac{1}{M-m} \int_m^M f(z)\, dz - f(t) \right| \leq$$

$$\frac{1}{(M-m)\,\Gamma(\alpha+2)} \left\{ \left\| D_{t-}^{\alpha} f \right\|_{\infty,[m,t]} (t-m)^{\alpha+1} + \left\| D_{*t}^{\alpha} f \right\|_{\infty,[t,M]} (M-t)^{\alpha+1} \right\} \leq \tag{9}$$

$$\frac{1}{\Gamma(\alpha+2)} \max \left\{ \left\| D_{t-}^{\alpha} f \right\|_{\infty,[m,t]}, \left\| D_{*t}^{\alpha} f \right\|_{\infty,[t,M]} \right\} (M-m)^{\alpha}, \tag{10}$$

$\forall\, t \in [m, M]$.

By property (P) and Theorem 7 we derive

Theorem 8 *Let $0 < \alpha < 1$, $f \in AC([m, M])$, and there exists $K > 0$, such that*

$$\left\| D_{t-}^{\alpha} f \right\|_{\infty,[m,t]}, \left\| D_{*t}^{\alpha} f \right\|_{\infty,[t,M]} \leq K, \quad \forall\, t \in [m, M]. \tag{11}$$

Then

$$\left| \left(\frac{1}{M-m} \int_m^M f(z)\, dz \right) 1_H - f(A) \right| \leq$$

$$\frac{K}{(M-m)\,\Gamma(\alpha+2)} \left\{ (A - m1_H)^{\alpha+1} + (M1_H - A)^{\alpha+1} \right\} \leq \tag{12}$$

$$\left(\frac{K}{\Gamma(\alpha+2)} (M-m)^{\alpha} \right) 1_H. \tag{13}$$

We mention the Fink ([9]) inequality

Theorem 9 *Let $f^{(n-1)}$ be absolutely continuous on $[m, M]$ and $f^{(n)} \in L_\infty(m, M)$, $n \in \mathbb{N}$. Then*

$$\left| f(s) + \sum_{k=1}^{n-1} F_k(s) - \frac{n}{M-m} \int_m^M f(t)\, dt \right| \leq$$

$$\frac{\left\| f^{(n)} \right\|_\infty}{(n+1)!\,(M-m)} \left[(M-s)^{n+1} + (s-m)^{n+1} \right], \quad \forall\, s \in [m, M], \tag{14}$$

where

$$F_k(s) := \left(\frac{n-k}{k!} \right) \left(\frac{f^{(k-1)}(m)\,(s-m)^k - f^{(k-1)}(M)\,(s-M)^k}{M-m} \right). \tag{15}$$

7

If $n = 1$, *then* $\sum_{k=1}^{n-1} = 0$.

Inequality (14) is sharp, in the sense that is attained by an optimal f *for any* $s \in [m, M]$.

By property (P) and Theorem 9 we obtain

Theorem 10 *Let* $f^{(n-1)}$ *be absolutely continuous on* $[m, M]$ *and* $f^{(n)} \in L_\infty(m, M)$, $n \in \mathbb{N}$. *Then*

$$\left| f(A) + \sum_{k=1}^{n-1} F_k(A) - \left(\frac{n}{M-m} \int_m^M f(t)\, dt \right) 1_H \right| \leq \tag{16}$$

$$\frac{\left\| f^{(n)} \right\|_\infty}{(n+1)!\,(M-m)} \left[(M1_H - A)^{n+1} + (A - m1_H)^{n+1} \right],$$

where

$$F_k(A) := \left(\frac{n-k}{k!} \right) \left(\frac{f^{(k-1)}(m)\,(A - m1_H)^k - f^{(k-1)}(M)\,(A - M1_H)^k}{M-m} \right). \tag{17}$$

If $n = 1$, *then* $\sum_{k=1}^{n-1} F_k(A) = 0_H$.

We use here the sequence $\{B_k(t),\ k \geq 0\}$ of Bernoulli polynomials which is uniquely determined by the following identities:

$$\begin{aligned} &B_k'(t) = kB_{k-1}(t),\ \ k \geq 1,\ \ B_0(t) = 1 \\ &\text{and} \\ &B_k(t+1) - B_k(t) = kt^{k-1},\ \ k \geq 0. \end{aligned} \tag{18}$$

The values $B_k = B_k(0)$, $k \geq 0$ are the known Bernoulli numbers.

We mention

Theorem 11 *([3], p. 23) (see also [5]) Let* $f : [m, M] \to \mathbb{R}$ *be such that* $f^{(n-1)}$, $n \in \mathbb{N}$, *is a continuous function and* $f^{(n)}(t)$ *exists and is finite for all but a countable set of* t *in* (m, M) *and that* $f^{(n)} \in L_\infty([m, M])$.

Dentote by

$$\Delta_n(s) := f(s) - \frac{1}{M-m} \int_m^M f(t)\, dt -$$

$$\sum_{k=1}^{n-1} \frac{(M-m)^{k-1}}{k!} B_k\left(\frac{s-m}{M-m} \right) \left[f^{(k-1)}(M) - f^{(k-1)}(m) \right], \tag{19}$$

$\forall\ s \in [m, M]$.

Then

$$\left|\Delta_n(s)\right| \leq \frac{(M-m)^n}{n!}\left(\sqrt{\frac{(n!)^2}{(2n)!}\left|B_{2n}\right| + B_n^2\left(\frac{s-m}{M-m}\right)}\right)\left\|f^{(n)}\right\|_\infty, \tag{20}$$

$\forall\ n \in \mathbb{N}$; $\forall\ s \in [m, M]$.

Using the (P) property and Theorem 11 we derive:

Theorem 12 *All terms and assumptions as in Theorem 11. Denote by*

$$\Delta_n(A) := f(A) - \left(\frac{1}{M-m}\int_m^M f(t)\,dt\right)1_H -$$

$$\sum_{k=1}^{n-1}\frac{(M-m)^{k-1}}{k!}B_k\left(\frac{A-m1_H}{M-m}\right)\left[f^{(k-1)}(M) - f^{(k-1)}(m)\right]. \tag{21}$$

Then

$$\left|\Delta_n(A)\right| \leq \frac{(M-m)^n}{n!}\left(\sqrt{\left(\frac{(n!)^2}{(2n)!}\left|B_{2n}\right|\right)1_H + B_n^2\left(\frac{A-m1_H}{M-m}\right)}\right)\left\|f^{(n)}\right\|_\infty, \tag{22}$$

$\forall\ n \in \mathbb{N}$.

Denote by (see [3], p. 24)

$$I_4(\lambda) := \begin{cases} \frac{16\lambda^5}{5} - 7\lambda^4 + \frac{14}{3}\lambda^3 - \lambda^2 + \frac{1}{30}, & 0 \leq \lambda \leq \frac{1}{2}, \\ -\frac{16\lambda^5}{5} + 9\lambda^4 - \frac{26\lambda^3}{3} + 3\lambda^2 - \frac{1}{10}, & \frac{1}{2} \leq \lambda \leq 1, \end{cases} \tag{23}$$

which is continuous in $\lambda \in [0, 1]$.

Also denote by

$$B := \left(\frac{A-m1_H}{M-m}\right)$$

and

$$I_4\left(\frac{A-m1_H}{M-m}\right) = I_4(B) =$$

$$\begin{cases} \frac{16}{5}B^5 - 7B^4 + \frac{14}{3}B^3 - B^2 + \frac{1}{30}1_H, & 0_H \leq B \leq \frac{1}{2}1_H, \\ -\frac{16}{5}B^5 + 9B^4 - \frac{26B^3}{3} + 3B^2 - \frac{1}{10}1_H, & \frac{1}{2}1_H \leq B \leq 1_H. \end{cases} \tag{24}$$

We mention

Theorem 13 *([3], p. 25) All terms and assumptions as in Theorem 11, case of $n = 4$. For every $s \in [m, M]$ it holds*

$$\left|\Delta_4(s)\right| \leq \frac{(M-m)^4}{24}I_4(\lambda)\left\|f^{(4)}\right\|_\infty,$$

where $I_4(\lambda)$ is given by (23) with

$$\lambda = \frac{s-m}{M-m}. \tag{25}$$

Furthermore we have that

$$|\Delta_4(s)| \leq \frac{(M-m)^4}{720} \left\| f^{(4)} \right\|_{\infty}, \tag{26}$$

$\forall\ s \in [m, M]$.

Using property (P) and Theorem 13 we find

Theorem 14 *All terms and assumptions are according to Theorem 11-13. Then*

$$|\Delta_4(A)| \leq \frac{(M-m)^4}{24} I_4\left(\frac{A - m1_H}{M-m}\right) \left\| f^{(4)} \right\|_{\infty}, \tag{27}$$

where $I_4\left(\frac{A-m1_H}{M-m}\right)$ is given by (24).

Furthermore we have that

$$|\Delta_4(A)| \leq \left(\frac{(M-m)^4}{720} \left\| f^{(4)} \right\|_{\infty}\right) 1_H. \tag{28}$$

Next we follow [6].

Let $(P_n)_{n\in\mathbb{N}}$ be a harmonic sequence of polynomials, that is $P_n' = P_{n-1}$, $P_0 = 1$. Let $f : [m, M] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \in \mathbb{N}$. Setting

$$\overline{F_k} = \frac{(-1)^k (n-k)}{M-m} \left[P_k(m) f^{(k-1)}(m) - P_k(M) f^{(k-1)}(M)\right], \quad k = 1, ..., n-1, \tag{29}$$

and

$$k(t,s) = \begin{cases} t-m, & \text{if } t \in [m,s] \\ t-M, & \text{if } t \in (s,M], \end{cases} \tag{30}$$

we get that

$$\frac{1}{n}\left[f(s) + \sum_{k=1}^{n-1} (-1)^k P_k(s) f^{(k)}(s) + \sum_{k=1}^{n-1} \overline{F_k}\right] - \frac{1}{M-m}\int_m^M f(t)\,dt = \tag{31}$$

$$\frac{(-1)^{n-1}}{n(M-m)} \int_m^M P_{n-1}(t)\, k(t,s)\, f^{(n)}(t)\,dt,$$

$\forall\ s \in [m, M]$. The above sums are defined to be zero for $n = 1$.

For the harmonic sequence of polynomials

$$P_k(t) = \frac{(t-s)^k}{k!}, \quad k \geq 0 \tag{32}$$

identity (31) collapses to the Fink identity, see [9].

We may rewrite generalized Fink identity (31) as follows:

$$f(s) = \sum_{k=1}^{n-1} (-1)^{k+1} P_k(s) f^{(k)}(s) + \tag{33}$$

$$\sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{M-m} \left[P_k(M) f^{(k-1)}(M) - P_k(m) f^{(k-1)}(m) \right] +$$

$$\frac{n}{M-m} \int_m^M f(t)\,dt + \frac{(-1)^{n+1}}{M-m} \int_m^M P_{n-1}(t) k(t,s) f^{(n)}(t)\,dt,$$

$\forall\ s \in [m,M]$, $n \in \mathbb{N}$, when $n=1$ the above sums are zero.

Next we integrate the representation formula (33) against projections E_s to derive the operator representation formula:

$$f(A) = \sum_{k=1}^{n-1} (-1)^{k+1} P_k(A) f^{(k)}(A) + \tag{34}$$

$$\left[\sum_{k=1}^{n-1} \frac{(-1)^k (n-k)}{M-m} \left[P_k(M) f^{(k-1)}(M) - P_k(m) f^{(k-1)}(m) \right] + \right.$$

$$\left. \frac{n}{M-m} \int_m^M f(t)\,dt \right] 1_H + \frac{(-1)^{n+1}}{M-m} \int_{m-0}^M \left(\int_m^M P_{n-1}(t) k(t,s) f^{(n)}(t)\,dt \right) dE_s.$$

The sequence of polynomials

$$P_k(t) = \frac{1}{k!} \left(t - \frac{m+M}{2} \right)^k, \ \ k \geq 0, \tag{35}$$

is also harmonic.

We mention

Theorem 15 *([6]) Let $f:[m,M] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \in \mathbb{N}$ and $f^{(n)} \in L_p([m,M])$, $1 \leq p \leq \infty$. Then*

$$\left| \left[f(s) + \sum_{k=1}^{n-1} (-1)^k P_k(s) f^{(k)}(s) + \sum_{k=1}^{n-1} \overline{F_k} \right] - \frac{n}{M-m} \int_m^M f(t)\,dt \right| \leq \tag{36}$$

$$\frac{1}{M-m} \left\| P_{n-1}(\cdot) k(\cdot,s) \right\|_{p',[m,M]} \left\| f^{(n)} \right\|_p,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

We observe that

$$\int_m^M \left| P_{n-1}(t) k(t,s) \right|^{p'} dt \leq \left\| P_{n-1} \right\|_{\infty,[m,M]}^{p'} \int_m^M |k(t,s)|^{p'} dt = \tag{37}$$

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$$\|P_{n-1}\|_{\infty,[m,M]}^{p'} \left[\int_m^s (t-m)^{p'} dt + \int_s^M (M-t)^{p'} dt \right] =$$

$$\|P_{n-1}\|_{\infty,[m,M]}^{p'} \left[\frac{(s-m)^{p'+1} + (M-s)^{p'+1}}{p'+1} \right].$$

Therefore we obtain

$$\|P_{n-1}(\cdot) k(\cdot,s)\|_{p',[m,M]} \leq \|P_{n-1}\|_{\infty,[m,M]} \left[\frac{(M-s)^{p'+1} + (s-m)^{p'+1}}{p'+1} \right]^{\frac{1}{p'}}. \tag{38}$$

Hence we have

Theorem 16 *Let $f : [m, M] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \in \mathbb{N}$ and $f^{(n)} \in L_p([m, M])$, $1 \leq p \leq \infty$. Then*

$$\left| \left(f(s) + \sum_{k=1}^{n-1} (-1)^k P_k(s) f^{(k)}(s) \right) + \left(\sum_{k=1}^{n-1} \overline{F_k} \right) - \left(\frac{n}{M-m} \int_m^M f(t) dt \right) \right| \leq$$

$$\left(\frac{\|f^{(n)}\|_p}{M-m} \|P_{n-1}\|_{\infty,[m,M]} \right) \left[\frac{(M-s)^{p'+1} + (s-m)^{p'+1}}{p'+1} \right]^{\frac{1}{p'}}, \tag{39}$$

$\forall\ s \in [m, M]$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

We get the following operator inequality:

Theorem 17 *Let $f : [m, M] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \in \mathbb{N}$ and $f^{(n)} \in L_p([m, M])$, $1 \leq p \leq \infty$. Then*

$$\left| \left(f(A) + \sum_{k=1}^{n-1} (-1)^k P_k(A) f^{(k)}(A) \right) + \left(\sum_{k=1}^{n-1} \overline{F_k} \right) 1_H - \right.$$

$$\left. \left(\frac{n}{M-m} \int_m^M f(t) dt \right) 1_H \right| \leq$$

$$\left(\frac{\|f^{(n)}\|_p}{M-m} \|P_{n-1}\|_{\infty,[m,M]} \right) \left[\frac{(M1_H - A)^{p'+1} + (A - m1_H)^{p'+1}}{p'+1} \right]^{\frac{1}{p'}}, \tag{40}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. By (P) property and (39). ■

We give

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Corollary 18 *(to Theorem 16) (see also [6]) We have*

$$\left| \left[f(s) + \sum_{k=1}^{n-1} \frac{(-1)^k}{k!} \left(s - \frac{m+M}{2} \right)^k f^{(k)}(s) + \right. \right.$$

$$\left. \sum_{k=1}^{n-1} \frac{(M-m)^{k-1}(n-k)}{k! 2^k} \left[f^{(k-1)}(m) - (-1)^k f^{(k-1)}(M) \right] \right]$$

$$\left. - \frac{n}{M-m} \int_m^M f(t)\, dt \right| \leq$$

$$\left(\frac{\left\| f^{(n)} \right\|_p (M-m)^{n-2}}{2^{n-1}(n-1)!} \right) \left[\frac{(M-s)^{p'+1} + (s-m)^{p'+1}}{p'+1} \right]^{\frac{1}{p'}}, \tag{41}$$

$\forall\ s \in [m, M]$, *where* $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Set $P_k(t) = \frac{1}{k!}\left(t - \frac{m+M}{2}\right)^k$, $k \geq 0$, in Theorem 16. ∎

We finish with the operator inequality:

Corollary 19 *(to Theorem 17) We have*

$$\left| \left[f(A) + \sum_{k=1}^{n-1} \frac{(-1)^k}{k!} \left(A - \left(\frac{m+M}{2} \right) 1_H \right)^k f^{(k)}(A) + \right. \right.$$

$$\left. \left(\sum_{k=1}^{n-1} \frac{(M-m)^{k-1}(n-k)}{k! 2^k} \left[f^{(k-1)}(m) - (-1)^k f^{(k-1)}(M) \right] \right) 1_H \right]$$

$$\left. - \left(\frac{n}{M-m} \int_m^M f(t)\, dt \right) 1_H \right| \leq$$

$$\left(\frac{\left\| f^{(n)} \right\|_p (M-m)^{n-2}}{2^{n-1}(n-1)!} \right) \left[\frac{(M1_H - A)^{p'+1} + (A - m1_H)^{p'+1}}{p'+1} \right]^{\frac{1}{p'}}, \tag{42}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. By Corollary 18 and (P) property. ∎

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Integer and Fractional Self Adjoint Operator Opial type Inequalities

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Abstract

We present here several integer and fractional self adjoint operator Opial type inequalities to many directions. These are based in the operator order over a Hilbert space.

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Key Words and Phrases: Self adjoint operator, Hilbert space, Opial inequality, fractional derivative.

1 Motivation

In 1960, Z. Opial ([9]) proved the following famous inequality that motivates our work here.

Let $f \in C^1([0,h])$ be such that $f(0) = f(h) = 0$, and $f(t) > 0$ in $(0,h)$. Then

$$\int_0^h |f(t) f'(t)| \, dt \leq \frac{h}{4} \int_0^h (f'(t))^2 \, dt.$$

The constant $\frac{h}{4}$ is the best.

In this article we present integer and fractional self adjoint operator Opial type inequalities on a Hilbert space in the operator order.

2 Background

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$. The Gelfand map establishes a $*-$isometrically isomorphism Φ between the set $C(Sp(A))$ of all continuous functions defined on the spectrum of A, denoted

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$Sp(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see e.g. [6, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha \Phi(f) + \beta \Phi(g)$;

(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ (the operation composition is on the right) and $\Phi(\overline{f}) = (\Phi(f))^*$;

(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;

(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f), \text{ for all } f \in C(Sp(A)),$$

and we call it the continuous functional calculus for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$ then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, i.e. $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued continuous functions on $Sp(A)$ then the following important property holds:

(P) $f(t) \geq g(t)$ for any $t \in Sp(A)$, implies that $f(A) \geq g(A)$ in the operator order of $B(H)$. (the Banach algebra of all bounded linear operators from H into itself).

Equivalently, we use (see [5], pp. 7-8):

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$ with the spectrum $Sp(U)$ included in the interval $[m, M]$ for some real numbers $m < M$ and $\{E_\lambda\}_\lambda$ be its spectral family.

Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following spectral representation in terms of the Riemann-Stieljes integral:

$$\langle f(U)x, y \rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, y \rangle),$$

for any $x, y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of bounded variation on the interval $[m, M]$, and

$$g_{x,y}(m-0) = 0 \text{ and } g_{x,y}(M) = \langle x, y \rangle,$$

for any $x, y \in H$. Furthermore, it is known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is increasing and right continuous on $[m, M]$.

We have also the formula

$$\langle f(U)x, x \rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, x \rangle), \quad \forall\, x \in H.$$

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As a symbol we can write

$$f(U)=\int_{m-0}^{M} f(\lambda)\,dE_{\lambda}.$$

Above, $m=\min\{\lambda|\lambda\in Sp(U)\}:=\min Sp(U)$, $M=\max\{\lambda|\lambda\in Sp(U)\}:=\max Sp(U)$. The projections $\{E_{\lambda}\}_{\lambda\in\mathbb{R}}$, are called the spectral family of A, with the properties:

(a) $E_{\lambda}\leq E_{\lambda'}$ for $\lambda\leq\lambda'$;

(b) $E_{m-0}=0_H$ (zero operator), $E_M=1_H$ (identity operator) and $E_{\lambda+0}=E_{\lambda}$ for all $\lambda\in\mathbb{R}$.

Furthermore

$$E_{\lambda}:=\varphi_{\lambda}(U),\ \ \forall\ \lambda\in\mathbb{R},$$

is a projection which reduces U, with

$$\varphi_{\lambda}(s):=\begin{cases}1, & \text{for } -\infty<s\leq\lambda,\\ 0, & \text{for } \lambda<s<+\infty.\end{cases}$$

The spectral family $\{E_{\lambda}\}_{\lambda\in\mathbb{R}}$ determines uniquely the self-adjoint operator U and vice versa.

For more on the topic see [8], pp. 256-266, and for more detalis see there pp. 157-266. See also [4].

Some more basics are given (we follow [5], pp. 1-5):

Let $(H;\langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{C}$. A bounded linear operator A defined on H is selfjoint, i.e., $A=A^*$, iff $\langle Ax,x\rangle\in\mathbb{R}$, $\forall\ x\in H$, and if A is selfadjoint, then

$$\|A\|=\sup_{x\in H:\|x\|=1}|\langle Ax,x\rangle|.$$

Let A,B be selfadjoint operators on H. Then $A\leq B$ iff $\langle Ax,x\rangle\leq\langle Bx,x\rangle$, $\forall\ x\in H$.

In particular, A is called positive if $A\geq 0$.

Denote by

$$\mathcal{P}:=\left\{\varphi(s):=\sum_{k=0}^{n}\alpha_k s^k|n\geq 0,\ \alpha_k\in\mathbb{C},\ 0\leq k\leq n\right\}.$$

If $A\in\mathcal{B}(H)$ is selfadjoint, and $\varphi(s)\in\mathcal{P}$ has real coefficients, then $\varphi(A)$ is selfadjoint, and

$$\|\varphi(A)\|=\max\{|\varphi(\lambda)|,\lambda\in Sp(A)\}.$$

If φ is any function defined on $\mathbb{R}$ we define

$$\|\varphi\|_A:=\sup\{|\varphi(\lambda)|,\lambda\in Sp(A)\}.$$

If A is selfadjoint operator on Hilbert space H and φ is continuous and given that $\varphi(A)$ is selfadjoint, then $\|\varphi(A)\| = \|\varphi\|_A$. And if φ is a continuous real valued function so it is $|\varphi|$, then $\varphi(A)$ and $|\varphi|(A) = |\varphi(A)|$ are selfadjoint operators (by [5], p. 4, Theorem 7).

Hence it holds

$$\| |\varphi(A)| \| = \| |\varphi| \|_A = \sup \{ || \varphi(\lambda) || , \lambda \in Sp(A) \}$$

$$= \sup \{ |\varphi(\lambda)| , \lambda \in Sp(A) \} = \|\varphi\|_A = \|\varphi(A)\| ,$$

that is

$$\| |\varphi(A)| \| = \|\varphi(A)\| .$$

For a selfadjoint operator $A \in \mathcal{B}(H)$ which is positive, there exists a unique positive selfadjoint operator $B := \sqrt{A} \in \mathcal{B}(H)$ such that $B^2 = A$, that is $\left(\sqrt{A}\right)^2 = A$. We call B the square root of A.

Let $A \in \mathcal{B}(H)$, then A^*A is selfadjoint and positive. Define the "operator absolute value" $|A| := \sqrt{A^*A}$. If $A = A^*$, then $|A| = \sqrt{A^2}$.

For a continuous real valued function φ we observe the following:

$$|\varphi(A)| \text{ (the functional absolute value) } = \int_{m-0}^{M} |\varphi(\lambda)| \, dE_\lambda =$$

$$\int_{m-0}^{M} \sqrt{(\varphi(\lambda))^2} dE_\lambda = \sqrt{(\varphi(A))^2} = |\varphi(A)| \text{ (operator absolute value)},$$

where A is a selfadjoint operator.

That is we have

$$|\varphi(A)| \text{ (functional absolute value) } = |\varphi(A)| \text{ (operator absolute value)}.$$

3 Main Results

Let A be a selfadjoint operator in the Hilbert space H with the spectrum $Sp(A) \subseteq [m, M]$, $m < M$; $m, M \in \mathbb{R}$.

In the next we obtain Opial type inequalities, both integer and fractional cases, in the operator order of $\mathcal{B}(H)$ (the Banach algebra of all bounded linear operators from H into itself).

Let the real valued function $f \in C([m, M])$, and we consider

$$g(t) = \int_m^t f(z) \, dz, \ \ \forall \ t \in [m, M] , \tag{1}$$

then $g \in C([m, M])$.

We denote by

$$\int_{m1_H}^{A} f := \Phi(g) = g(A). \tag{2}$$

We understand and write that $(r > 0)$

$$g^r(A) = \Phi(g^r) =: \left(\int_{m1_H}^{A} f\right)^r.$$

Clearly $\left(\int_{m1_H}^{A} f\right)^r$ is a self adjoint operator on H, for any $r > 0$.

All of our functions in this article will be real valued. From [3] we mention the following basic version of Opial inequality:

Theorem 1 *Let $f \in C^1([m, M])$ with $f(m) = 0$. Then*

$$\int_m^\lambda |f(t)| |f'(t)| \, dt \leq \left(\frac{\lambda - m}{2}\right) \int_m^\lambda (f'(t))^2 \, dt, \quad \forall \, \lambda \in [m, M]. \tag{3}$$

When $f(t) = t - m$, $t \in [m, M]$, inequality (3) becomes equality.

By applying properties (P) and (ii) to (3) we obtain

Theorem 2 *Let $f \in C^1([m, M])$ with $f(m) = 0$. Then*

$$\int_{m1_H}^{A} |ff'| \leq \frac{1}{2}(A - m1_H)\left(\int_{m1_H}^{A} (f')^2\right). \tag{4}$$

We mention

Theorem 3 *([3]) Let $f \in C^1([m, M])$ with $f(m) = 0$, and $1 \leq p \leq 2$. Then*

$$\int_m^\lambda |f(t)|^p |f'(t)|^p \, dt \leq K(p)(\lambda - m)\left(\int_m^\lambda (f'(t))^2 \, dt\right)^p, \quad \forall \, \lambda \in [m, M], \tag{5}$$

where

$$K(p) = \begin{cases} \frac{1}{2}, & p = 1, \\ \frac{4}{\pi^2}, & p = 2, \\ \frac{2-p}{2p}\left(\frac{1}{p}\right)^{2p-2} I^{-p}, & 1 < p < 2, \end{cases} \tag{6}$$

with

$$I = \int_0^1 \left\{1 + \frac{2(p-1)}{2-p} z\right\}^{-2} \{1 + (p-1)z\}^{\frac{1}{p}-1} \, dz.$$

For $p = 1$, equality holds in (5) only for f linear.

By applying properties (P) and (ii) to (5) we derive

Theorem 4 *Here all are as in Theorem 3. It holds*

$$\int_{m1_H}^{A} |ff'|^p \leq K(p)(A - m1_H)\left(\int_{m1_H}^{A} (f')^2\right)^p. \tag{7}$$

We mention

Theorem 5 *([7]) Let $f \in C^1([m, M])$ with $f(m) = 0$, and $p, q \geq 1$. Then*

$$\int_m^\lambda |f(t)|^p |f'(t)|^q \, dt \leq \left(\frac{q}{p+q}\right)(\lambda - m)^p \int_m^\lambda |f'(t)|^{p+q} \, dt, \quad \forall \ \lambda \in [m, M]. \tag{8}$$

By applying properties (P) and (ii) to (8) we find

Theorem 6 *Let $f \in C^1([m, M])$ with $f(m) = 0$, and $p, q \geq 1$. Then*

$$\int_{m1_H}^{A} |f|^p |f'|^q \leq \left(\frac{q}{p+q}\right)(A - m1_H)^p \left(\int_{m1_H}^{A} |f'|^{p+q}\right). \tag{9}$$

We mention

Theorem 7 *([11]) Let $p > -1$. Let $f \in C^1([m, M])$, and $f(m) = 0$. Then*

$$\int_m^\lambda t^p |f(t) f'(t)| \, dt \leq \frac{1}{2\sqrt{p+1}} \int_m^\lambda \left(\lambda^{p+1} - mt^p\right)(f'(t))^2 \, dt \tag{10}$$

$$\leq \frac{1}{2\sqrt{p+1}} \int_m^\lambda \left(M^{p+1} - mt^p\right)(f'(t))^2 \, dt, \quad \forall \ \lambda \in [m, M]. \tag{11}$$

(inequality (11) is our derivation).

By applying properties (P) and (ii) to (10), (11) we obtain

Theorem 8 *Let $p > -1$. Let $f \in C^1([m, M])$ and $f(m) = 0$. Then*

$$\int_{m1_H}^{A} (id)^p |ff'| \leq \frac{1}{2\sqrt{p+1}} \left(\int_{m1_H}^{A} \left(M^{p+1} - m(id)^p\right)(f')^2\right). \tag{12}$$

We mention

Theorem 9 *([1], p. 20) Let $q(t)$ be positive continuous and non-increasing function on $[m, M]$. Further, let $f \in C^1([m, M])$, and $f(m) = 0$. Let $l \geq 0$, $w \geq 1$. Then*

$$\int_m^\lambda q(t) |f(t)|^l |f'(t)|^w \, dt \leq \left(\frac{w}{l+w}\right)(\lambda - m)^l \int_m^\lambda q(t) |f'(t)|^{l+w} \, dt, \tag{13}$$

$\forall \ \lambda \in [m, M]$.

By applying property (P) and (ii) to (13) we obtain

Theorem 10 *All as in Theorem 9. Then*

$$\int_{m1_H}^{A} q\left|f\right|^{l}\left|f'\right|^{w} \leq \left(\frac{w}{l+w}\right)\left(A-m1_H\right)^{l}\int_{m1_H}^{A} q\left|f'\right|^{l+w}. \tag{14}$$

We mention

Theorem 11 *(see [1], p. 68) Let $q(t)$ positive, continuous and non-increasing on $[m,M]$. Further let $f_1, f_2 \in C^1([m,M])$ with $f_1(m) = f_2(m) = 0$. Let $l \geq 0$, $w \geq 1$. Then*

$$\int_{m}^{\lambda} q(t)\left|f_1(t)f_2(t)\right|^{l}\left[\left|f_1(t)f_2'(t)\right|^{w}+\left|f_1'(t)f_2(t)\right|^{w}\right]dt \leq$$

$$\frac{w}{2(l+w)}(\lambda-m)^{2l+w}\int_{m}^{\lambda} q(t)\left[\left(f_1'(t)\right)^{2(l+w)}+\left(f_2'(t)\right)^{2(l+w)}\right]dt, \tag{15}$$

$\forall\ \lambda \in [m,M]$.

By applying property (P) and (ii) to (15) we obtain

Theorem 12 *All as in Theorem 11. Then*

$$\int_{m1_H}^{A} q\left|f_1f_2\right|^{l}\left[\left|f_1f_2'\right|^{w}+\left|f_1'f_2\right|^{w}\right] \leq \tag{16}$$

$$\frac{w}{2(l+w)}\left(A-m1_H\right)^{2l+w}\int_{m1_H}^{A} q\left[\left(f_1'\right)^{2(l+w)}+\left(f_2'\right)^{2(l+w)}\right].$$

We mention

Theorem 13 *([10], p. 308) Let $f \in C^n([m,M])$, $n \in \mathbb{N}$, $f^{(i)}(m) = 0$, for $i = 0,1,2,...,n-1$. Then*

$$\int_{m}^{\lambda}\left|f(t)f^{(n)}(t)\right|dt \leq \frac{(\lambda-m)^n}{2}\int_{m}^{\lambda}\left(f^{(n)}(t)\right)^2 dt, \quad \forall\ \lambda \in [m,M]. \tag{17}$$

Using properties (P) and (ii) on (17) we derive

Theorem 14 *All as in Theorem 13. Then*

$$\int_{m1_H}^{A}\left|f\cdot f^{(n)}\right| \leq \frac{\left(A-m1_H\right)^n}{2}\left(\int_{m1_H}^{A}\left(f^{(n)}\right)^2\right). \tag{18}$$

We mention from [10], p. 309

Theorem 15 *Let $f_1, f_2 \in C^n([m,M])$ such that $f_1^{(k)}(m) = f_2^{(k)}(m) = 0$, for $k = 0,1,...,n-1$, $n \in \mathbb{N}$. Then*

$$\int_m^\lambda \left[\left|f_1(t) f_2^{(n)}(t)\right| + \left|f_2(t) f_1^{(n)}(t)\right|\right] dt \leq$$

$$B(\lambda - m)^n \int_m^\lambda \left[\left(f_1^{(n)}(t)\right)^2 + \left(f_2^{(n)}(t)\right)^2\right] dt, \quad \forall\, \lambda \in [m,M], \tag{19}$$

where

$$B = \frac{1}{2n!}\left(\frac{n}{2n-1}\right)^{\frac{1}{2}}. \tag{20}$$

Using (19) and properties (P) and (ii) we obtain

Theorem 16 *All as in Theorem 15. Then*

$$\int_{m1_H}^A \left[\left|f_1 f_2^{(n)}\right| + \left|f_2 f_1^{(n)}\right|\right] \leq$$

$$B(A - m1_H)^n \left(\int_{m1_H}^A \left(\left(f_1^{(n)}\right)^2 + \left(f_2^{(n)}\right)^2\right)\right). \tag{21}$$

Here we follow [2], p. 8.

Definition 17 *Let $\nu > 0$, $n := [\nu]$ (integral part), and $\alpha := \nu - n$ $(0 < \alpha < 1)$. Let $f \in C([m,M])$ and define*

$$(J_\nu^m f)(z) = \frac{1}{\Gamma(\nu)} \int_m^z (z-t)^{\nu-1} f(t)\, dt, \tag{22}$$

all $m \leq z \leq M$, where Γ is the gamma function, the generalized Riemann-Liouville integral. We define the subspace $C_m^\nu([m,M])$ of $C^n([m,M])$:

$$C_m^\nu([m,M]) := \left\{f \in C^n([m,M]) : J_{1-\alpha}^m f^{(n)} \in C^1([m,M])\right\}. \tag{23}$$

So let $f \in C_m^\nu([m,M])$; we define the generalized ν-fractional derivative (of Canavati type) of f over $[m,M]$ as

$$D_m^\nu f := \left(J_{1-\alpha}^m f^{(n)}\right)'. \tag{24}$$

Notice that

$$\left(J_{1-\alpha}^m f^{(n)}\right)(z) = \frac{1}{\Gamma(1-\alpha)} \int_m^z (z-t)^{-\alpha} f^{(n)}(t)\, dt \tag{25}$$

exists for $f \in C_m^\nu([m,M])$, all $m \leq z \leq M$.

Also notice that $D_m^\nu f \in C([m,M])$.

We need

Theorem 18 *([2], p. 15) Let* $f \in C_m^{\nu}([m,M])$, $\nu \geq 1$ *and* $f^{(i)}(m) = 0$, $i = 0,1,...,n-1$, $n := [\nu]$. *Here* $\lambda \in [m,M]$, *and* $l = 1,...,n-1$. *Let* $p,q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\int_m^{\lambda} \left| f^{(l)}(w) \right| \left| \left(D_m^{\nu} f \right)(w) \right| dw \leq$$

$$\frac{2^{-\frac{1}{q}} (\lambda - m)^{\frac{(\nu p - lp - p + 2)}{p}}}{\Gamma(\nu - l) \left((\nu p - lp - p + 1)(\nu p - lp - p + 2) \right)^{\frac{1}{p}}} \left(\int_m^{\lambda} \left| \left(D_m^{\nu} f \right)(w) \right|^q dw \right)^{\frac{2}{q}}. \tag{26}$$

Using (26), properties (P) and (ii) we get

Theorem 19 *All as in Theorem 18. Then*

$$\int_{m1_H}^{A} \left| f^{(l)} \right| \left| \left(D_m^{\nu} f \right) \right| \leq$$

$$\frac{2^{-\frac{1}{q}} (A - m1_H)^{\frac{(\nu p - lp - p + 2)}{p}}}{\Gamma(\nu - l) \left((\nu p - lp - p + 1)(\nu p - lp - p + 2) \right)^{\frac{1}{p}}} \left(\int_{m1_H}^{A} \left| \left(D_m^{\nu} f \right) \right|^q \right)^{\frac{2}{q}}. \tag{27}$$

We need

Theorem 20 *([2], p. 26) Let* $\gamma_1, \gamma_2 \geq 0$, $\nu \geq 1$ *be such that* $\nu - \gamma_1, \nu - \gamma_2 \geq 1$ *and* $f \in C_m^{\nu}([m,M])$ *with* $f^{(i)}(m) = 0$, $i = 0,1,...,n-1$, $n := [\nu]$. *Here* $\lambda \in [m,M]$. *Let* q *be a nonnegative continuous functions on* $[m,M]$. *Denote*

$$Q(\lambda) := \left(\int_m^{\lambda} (q(w))^2 dw \right)^{\frac{1}{2}}, \quad \forall \ \lambda \in [m,M]. \tag{28}$$

Then

$$\int_m^{\lambda} q(w) \left| D_m^{\gamma_1}(f)(w) \right| \left| D_m^{\gamma_2}(f)(w) \right| dw \leq$$

$$K(q, \gamma_1, \gamma_2, \nu, \lambda, m) \left(\int_m^{\lambda} \left(D_m^{\nu} f(w) \right)^2 dw \right), \tag{29}$$

where

$$K(q, \gamma_1, \gamma_2, \nu, \lambda, m) := \frac{Q(\lambda)}{\sqrt[3]{6}} \frac{1}{\Gamma(\nu - \gamma_1)\Gamma(\nu - \gamma_2)} \cdot$$

$$\frac{(\lambda - m)^{2\nu - \gamma_1 - \gamma_2 - \frac{1}{2}}}{\left(\nu - \gamma_1 - \frac{5}{6}\right)^{\frac{1}{6}} \left(\nu - \gamma_2 - \frac{5}{6}\right)^{\frac{1}{6}} \left(4\nu - 2\gamma_1 - 2\gamma_2 - \frac{7}{3}\right)^{\frac{1}{2}}}. \tag{30}$$

Using (30) and Remark 3.4 of [2], p. 26, and properties (P) and (ii) to obtain

Theorem 21 *All terms and assumptions as in Theorem 20. Then*

$$\int_{m1_H}^{A} q\left|D_m^{\gamma_1}(f)\right|\left|D_m^{\gamma_2}(f)\right| \leq$$

$$K(q,\gamma_1,\gamma_2,\nu,A,m)\left(\int_{m1_H}^{A}\left(D_m^{\nu}f\right)^2\right), \tag{31}$$

where

$$K(q,\gamma_1,\gamma_2,\nu,A,m) := \frac{Q(A)}{\sqrt[3]{6}}\frac{1}{\Gamma(\nu-\gamma_1)\Gamma(\nu-\gamma_2)}\cdot$$

$$\frac{(A-m1_H)^{2\nu-\gamma_1-\gamma_2-\frac{1}{2}}}{\left(\nu-\gamma_1-\frac{5}{6}\right)^{\frac{1}{6}}\left(\nu-\gamma_2-\frac{5}{6}\right)^{\frac{1}{6}}\left(4\nu-2\gamma_1-2\gamma_2-\frac{7}{3}\right)^{\frac{1}{2}}}. \tag{32}$$

We need

Theorem 22 *([2], p. 30) Let $\gamma \geq 0$, $\nu \geq 1$, $\nu-\gamma \geq 1$, let q be a nonnegative continuous function on $[m,M]$. Let $f \in C_m^{\nu}([m,M])$ with $f^{(i)}(m) = 0$, $i = 0,1,...,n-1$, $n := [\nu]$. Let $\lambda \in [m,M]$. Call*

$$Q(\lambda) := \left(\int_m^{\lambda}(q(w))^2(w-m)^{2\nu-2\gamma-1}dw\right)^{\frac{1}{2}}, \tag{33}$$

and

$$K(q,\gamma,\nu,\lambda,m) := \frac{Q(\lambda)}{\sqrt{2(2\nu-2\gamma-1)}\Gamma(\nu-\gamma)}. \tag{34}$$

Then

$$\int_m^{\lambda} q(w)\left|D_m^{\gamma}f(w)\right|\left|D_m^{\nu}f(w)\right|dw \leq K(q,\gamma,\nu,\lambda,m)\left(\int_m^{\lambda}\left(\left(D_m^{\nu}f\right)(w)\right)^2dw\right). \tag{35}$$

Using (33)-(35) and properties (P) and (ii) we derive

Theorem 23 *All as in Theorem 22. Denote by*

$$K(q,\gamma,\nu,A,m) := \frac{Q(A)}{\sqrt{2(2\nu-2\gamma-1)}\Gamma(\nu-\gamma)}. \tag{36}$$

Then

$$\int_{m1_H}^{A} q\left|D_m^{\gamma}f\right|\left|D_m^{\nu}f\right| \leq K(q,\gamma,\nu,A,m)\left(\int_{m1_H}^{A}\left(\left(D_m^{\nu}f\right)\right)^2\right). \tag{37}$$

We need

Theorem 24 *([2], p. 92) Let $\nu \geq 1$, $\gamma_1, \gamma_2 \geq 0$, such that $\nu - \gamma_1 \geq 1$, $\nu - \gamma_2 \geq 1$, and $f_1, f_2 \in C_m^{\nu}([m, M])$ with $f_1^{(i)}(m) = f_2^{(i)}(m) = 0$, $i = 0, 1, ..., n-1$, $n := [\nu]$. Here $\lambda \in [m, M]$. Let $\lambda_\alpha, \lambda_\beta, \lambda_\nu \geq 0$. Set*

$$\rho(\lambda) := \frac{(\lambda - m)^{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\beta - \gamma_2\lambda_\beta + 1)}}{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\beta - \gamma_2\lambda_\beta + 1)(\Gamma(\nu - \gamma_1 + 1))^{\lambda_\alpha}(\Gamma(\nu - \gamma_2 + 1))^{\lambda_\beta}}. \tag{38}$$

Then

$$\int_m^{\lambda} \left[|(D_m^{\gamma_1} f_1)(w)|^{\lambda_\alpha} |(D_m^{\gamma_2} f_2)(w)|^{\lambda_\beta} |(D_m^{\nu} f_1)(w)|^{\lambda_\nu} + |(D_m^{\gamma_2} f_1)(w)|^{\lambda_\beta} |(D_m^{\gamma_1} f_2)(w)|^{\lambda_\alpha} |(D_m^{\nu} f_2)(w)|^{\lambda_\nu} \right] dw \leq$$

$$\frac{\rho(\lambda)}{2} \left[\|D_m^{\nu} f_1\|_\infty^{2(\lambda_\alpha + \lambda_\nu)} + \|D_m^{\nu} f_1\|_\infty^{2\lambda_\beta} + \|D_m^{\nu} f_2\|_\infty^{2\lambda_\beta} + \|D_m^{\nu} f_2\|_\infty^{2(\lambda_\alpha + \lambda_\nu)} \right], \tag{39}$$

all $m \leq \lambda \leq M$.

Using (39) and properties (P) and (ii) we derive

Theorem 25 *All here as in Theorem 24. Set*

$$\rho(A) := \frac{(A - m1_H)^{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\beta - \gamma_2\lambda_\beta + 1)}}{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\beta - \gamma_2\lambda_\beta + 1)(\Gamma(\nu - \gamma_1 + 1))^{\lambda_\alpha}(\Gamma(\nu - \gamma_2 + 1))^{\lambda_\beta}}. \tag{40}$$

Then

$$\int_{m1_H}^{A} \left[|(D_m^{\gamma_1} f_1)|^{\lambda_\alpha} |(D_m^{\gamma_2} f_2)|^{\lambda_\beta} |(D_m^{\nu} f_1)|^{\lambda_\nu} + |(D_m^{\gamma_2} f_1)|^{\lambda_\beta} |(D_m^{\gamma_1} f_2)|^{\lambda_\alpha} |(D_m^{\nu} f_2)|^{\lambda_\nu} \right] \leq$$

$$\frac{\rho(A)}{2} \left[\|D_m^{\nu} f_1\|_\infty^{2(\lambda_\alpha + \lambda_\nu)} + \|D_m^{\nu} f_1\|_\infty^{2\lambda_\beta} + \|D_m^{\nu} f_2\|_\infty^{2\lambda_\beta} + \|D_m^{\nu} f_2\|_\infty^{2(\lambda_\alpha + \lambda_\nu)} \right]. \tag{41}$$

We give

Definition 26 *([2], p. 270) Let $\nu > 0$, $n := \lceil \nu \rceil$ (ceiling of ν), $f \in AC^n([m, M])$ (i.e. $f^{(n-1)}$ is absolutely continuous on $[m, M]$, that is in $AC([m, M])$). We define the Caputo fractional derivative*

$$(D_{*m}^{\nu} f)(z) := \frac{1}{\Gamma(n - \nu)} \int_m^z (z - t)^{n - \nu - 1} f^{(n)}(t) \, dt, \tag{42}$$

which exists almost everywhere for $z \in [m, M]$.

Notice that $D_{*m}^0 f = f$, and $D_{*m}^n f = f^{(n)}$.

We mention

Theorem 27 *([2], p. 397) Let $\nu \geq \gamma+1$, $\gamma \geq 0$. Call $n := \lceil \nu \rceil$ and assume $f \in C^n([m, M])$ such that $f^{(k)}(m) = 0$, $k = 0, 1, ..., n-1$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $m \leq \lambda \leq M$. Then*

$$\int_m^\lambda |(D_{*m}^\gamma f)(w)| \, |(D_{*m}^\nu f)(w)| \, dw \leq$$

$$\frac{(\lambda - m)^{\frac{(p\nu - p\gamma - p + 2)}{p}}}{\left(\sqrt[q]{2}\right) \Gamma(\nu - \gamma) \left((p\nu - p\gamma - p + 1)(p\nu - p\gamma - p + 2)\right)^{\frac{1}{p}}} \left(\int_m^\lambda |D_{*m}^\nu f(w)|^q \, dw \right)^{\frac{2}{q}}. \tag{43}$$

Note: By Proposition 15.114 ([2], p. 388) we have that $D_{*m}^\nu f, D_{*m}^\gamma f \in C([m, M])$.

Using (43) and Properties (P) and (ii) we give

Theorem 28 *All as in Theorem 27. Then*

$$\int_{m1_H}^A |(D_{*m}^\gamma f)| \, |(D_{*m}^\nu f)| \leq$$

$$\frac{(A - m1_H)^{\frac{(p\nu - p\gamma - p + 2)}{p}}}{\left(\sqrt[q]{2}\right) \Gamma(\nu - \gamma) \left((p\nu - p\gamma - p + 1)(p\nu - p\gamma - p + 2)\right)^{\frac{1}{p}}} \left(\int_{m1_H}^A |D_{*m}^\nu f|^q \right)^{\frac{2}{q}}. \tag{44}$$

We need

Theorem 29 *([2], p. 398) Let $\nu \geq 2$, $k \geq 0$, $\nu \geq k+2$. Call $n := \lceil \nu \rceil$ and $f \in C^n([m, M]) : f^{(j)}(m) = 0$, $j = 0, 1, ..., n-1$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $m \leq \lambda \leq M$. Then*

$$\int_m^\lambda \left|\left(D_{*m}^k f\right)(w)\right| \left|\left(D_{*m}^{k+1} f\right)(w)\right| dw \leq$$

$$\frac{(\lambda - m)^{\frac{2(p\nu - pk - p + 1)}{p}}}{2\left(\Gamma(\nu - k)\right)^2 (p\nu - pk - p + 1)^{\frac{2}{p}}} \left(\int_m^\lambda |D_{*m}^\nu f(w)|^q \, dw \right)^{\frac{2}{q}}. \tag{45}$$

Using (45) and Properties (P) and (ii) we find

Theorem 30 *All as in Theorem 29. Then*

$$\int_{m1_H}^A \left|\left(D_{*m}^k f\right)\right| \left|\left(D_{*m}^{k+1} f\right)\right| \leq$$

$$\frac{(A - m1_H)^{\frac{2(p\nu - pk - p + 1)}{p}}}{2\left(\Gamma(\nu - k)\right)^2 (p\nu - pk - p + 1)^{\frac{2}{p}}} \left(\int_{m1_H}^A |D_{*m}^\nu f|^q \right)^{\frac{2}{q}}. \tag{46}$$

We need

Theorem 31 *([2], p. 399) Let* $\gamma_i \geq 0$, $\nu \geq 1$, $\nu - \gamma_i \geq 1$*;* $i = 1, ..., l$, $n := \lceil \nu \rceil$*, and* $f \in C^n([m, M])$ *such that* $f^{(k)}(m) = 0$, $k = 0, 1, ..., n-1$*. Here* $m \leq \lambda \leq M$*;* $q_1(\lambda), q_2(\lambda)$ *continuous functions on* $[m, M]$ *such that* $q_1(\lambda) \geq 0$, $q_2(\lambda) > 0$ *on* $[m, M]$*, and* $r_i > 0 : \sum_{i=1}^{l} r_i = r$*. Let* $s_1, s_1' > 1 : \frac{1}{s_1} + \frac{1}{s_1'} = 1$ *and* $s_2, s_2' > 1 : \frac{1}{s_2} + \frac{1}{s_2'} = 1$*, and* $p > s_2$*.*

Denote by

$$Q_1(\lambda) := \left(\int_m^{\lambda} (q_1(w))^{s_1'} dw \right)^{\frac{1}{s_1'}} \tag{47}$$

and

$$Q_2(\lambda) := \left(\int_m^{\lambda} (q_2(w))^{\frac{-s_2'}{p}} dw \right)^{\frac{r}{s_2'}}, \tag{48}$$

$$\sigma := \frac{p - s_2}{p s_2}. \tag{49}$$

Then

$$\int_m^{\lambda} q_1(w) \prod_{i=1}^{l} \left| D_{*m}^{\gamma_i} f(w) \right|^{r_i} dw \leq$$

$$Q_1(\lambda) Q_2(\lambda) \prod_{i=1}^{l} \left\{ \frac{\sigma^{r_i \sigma}}{(\Gamma(\nu - \gamma_i))^{r_i} (\nu - \gamma_i - 1 + \sigma)^{r_i \sigma}} \right\}.$$

$$\frac{(\lambda - m)^{\left(\sum_{i=1}^{l} (\nu - \gamma_i - 1) r_i + \sigma r\right) + \frac{1}{s_1}}}{\left(\left(\sum_{i=1}^{l} (\nu - \gamma_i - 1) r_i s_1 \right) + r s_1 \sigma + 1 \right)^{\frac{1}{s_1}}} \left(\int_m^{\lambda} q_2(w) \left| D_{*m}^{\nu} f(w) \right|^p dw \right)^{\frac{r}{p}}. \tag{50}$$

Using (50) and properties (P) and (ii) we obtain

Theorem 32 *All here as in Theorem 31. Set*

$$Q_1(A) := \left(\int_{m1_H}^{A} (q_1)^{s_1'} \right)^{\frac{1}{s_1'}} \tag{51}$$

and

$$Q_2(A) := \left(\int_{m1_H}^{A} (q_2)^{\frac{-s_2'}{p}} \right)^{\frac{r}{s_2'}}. \tag{52}$$

Then

$$\int_{m1_H}^{A} q_1 \prod_{i=1}^{l} \left| D_{*m}^{\gamma_i} f \right|^{r_i} \leq$$

$$Q_1(A)\,Q_2(A)\prod_{i=1}^{l}\left\{\frac{\sigma^{r_i\sigma}}{(\Gamma(\nu-\gamma_i))^{r_i}(\nu-\gamma_i-1+\sigma)^{r_i\sigma}}\right\}\cdot$$

$$\frac{(A-m1_H)^{\left(\sum_{i=1}^{l}(\nu-\gamma_i-1)r_i+\sigma r\right)+\frac{1}{s_1}}}{\left(\left(\sum_{i=1}^{l}(\nu-\gamma_i-1)\,r_i s_1\right)+r s_1\sigma+1\right)^{\frac{1}{s_1}}}\left(\int_{m1_H}^{A}q_2\left|D_{*m}^{\nu}f\right|^p\right)^{\frac{r}{p}}. \tag{53}$$

One can give many more operator Opial type (both integer and fractional) inequalities.

We choose to stop here.

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Numerical solution of the generalized Hirota-Satsuma coupled Korteweg-de Vries equation by Fourier Pseudospectral method

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Abstract

In this paper, an approximate solution of the generalized Hirota-Satsuma (HS) coupled Korteweg-de Vries (KdV) equation by the use of Fourier pseudospectral method is presented. A time discrete scheme is constructed by approximating the time derivative using forward difference formula, while the pseudospectral method is used in the space direction. The stability and convergence of the scheme are investigated using the energy method. The numerical results reveal that the Fourier pseudospectral method is a convenient, effective and accurate method to solve the generalized HS coupled KdV equation.

Key words: Generalized Hirota-Satsuma coupled Korteweg-de Vries equation, Fourier pseudospectral method, Stability, Convergence.

1 Introduction

The generalized HS coupled KdV equations are as follows [1, 2]:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^3 u}{\partial x^3} - 3u\frac{\partial u}{\partial x} + 3\frac{\partial}{\partial x}(vw), \qquad x \in \Omega, t \in [0,T], \tag{1.1}$$

$$\frac{\partial v}{\partial t} = -\frac{\partial^3 v}{\partial x^3} + 3u\frac{\partial v}{\partial x}, \qquad x \in \Omega, t \in [0,T], \tag{1.2}$$

$$\frac{\partial w}{\partial t} = -\frac{\partial^3 w}{\partial x^3} + 3u\frac{\partial w}{\partial x}, \qquad x \in \Omega, t \in [0,T] \tag{1.3}$$

with initial conditions

$$u(x,0) = f(x), \qquad v(x,0) = g(x), \qquad w(x,0) = h(x), \qquad x \in \Omega, \tag{1.4}$$

and boundary conditions

$$u(-L,t) = u(L,t) = 0,\ v(-L,t) = v(L,t) = 0,\ w(-L,t) = w(L,t) = 0,\ t \in [0,T], \tag{1.5}$$

where $\Omega = [-L, L]$. Hirota-Satsuma [1] introduced generalized the HS coupled KdV equations in 1976 and these equations are models of shallow water waves. The equations (1.1)–(1.5) have travelling wave solutions and multiple soliton solutions.

The equations (1.1)–(1.5) have attracted the attention of many researchers and a lot of work has already been carried out on solution methods. For example, the homotopy perturbation method (HPM) by Ganji and Rafei [3], homotopy analysis method (HAM) and Adomian's decomposition method (ADM) by Abbasbandy [4], modified extended tanh function method by Ali [5], direct algebraic method by Zhang Huiqun [6]. Rong Jihong et al. [7] used bifurcation theory technique. The auxiliary function method was used by Yang Feng and Hong-Qing [8], analytical technique by Ganji et al. [9], homogenous balance

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method by Adel Raly et al. [10]. Jacobi elliptic functions expansion method by Baojin Hong [11]. Travelling wave solutions of the above equations investigated by Zuo and Zhang [12], Xie and Ding [13], Feng and Li [14]. A differential transform method (DTM) and reduced differential transform method (RDTM) was used by Reze and Malek [15], Hirota's bilinear method and pfaffian techniques by Junchao Chen et al. [16], while the Lie group method was applied by Mina B. et al. [17].

1.1 A brief review of Fourier pseudospectral method

In the last two decades spectral methods have been extensively used in the field of numerical solution of nonlinear partial differential equations. The use of spectral methods for solving partial differential and integro-differential equations have the advantage that its accuracy is higher than other standard numerical methods. Spectral methods retain the exponential rate of convergence when the solutions of the problems is sufficiently smooth. Spectral methods have three different categories namely Galerkin method, collocation method and tau method. The pseudospectral method is a type of spectral method which is easy to apply for nonlinear partial differential equations with periodic boundary value problems. For a more detailed discussion of spectral methods, please see ([18, 19, 20, 21, 22]).

The Fourier pseudospectral method involves two steps. First, the discrete representation of the solution is constructed by using trigonometric polynomial to interpolate the solution at collocation points. Second, the equations for the discrete values of the solution are obtained from the original equations. This second step involves finding an approximation for the differential operator in terms of the discrete values of the solution at collocation points. For detailed, please see ([18, 19, 23, 26]).

1.2 The main aim of the paper

In this paper, a Fourier pseudospectral method is applied to solve the generalized HS coupled KdV equation. A finite difference method is used in the time direction and Fourier pseudospectral method in the space direction. The stability of the time discrete scheme and convergence of the approximate solution is investigated by the energy method [29]. Numerical results are shown to demonstrate the efficiency of the method. It should be noted that Darvishi et al. [27] solved the same equation by pseudospectral method and transformed the partial differential equation to ordinary differential equations. They found the numerical solution by using classical fourth-order Runge-Kutta method. There is no proof of stability and convergence. In our paper, we follow the approach of [23, 28].

The outline of the paper is as follows. In section 2 we present some preliminaries which will be used in next two sections. Section 3 is related to stability of the scheme for generalized Hirota-Satsuma (HS) coupled Korteweg-de Vries (KdV) equation. Convergence of the approximate solution is proved in section 4. Numerical results are presented for the applicability of the method section 5. Finally the conclusion is given in section 6.

2 Preliminaries

The inner product and norm are defined by $(u, v) = \int_\Omega u(x)v(x)dx$ and $\|u\|^2 = (u, u)$ respectively. The maximum norm is denoted by $\|u\|_\infty$. The periodic Sobolev space is defined by [23]:

$$H^1 = \left\{ u \in L^2(R) : \frac{du}{dx} \in L^2(R) \right\}, \qquad H^1_p = \{u \in H^1(R) : u(x-L) = u(x+L)\}.$$

The Sobolev norm and semi-norms are defined by [23]:

$$\|u\| = (u,u)^{1/2}, \quad \|u\|_{H^1} = (\|u\|^2 + \|\frac{\partial u}{\partial x}\|^2)^{1/2}, \quad |u|_k = |u|_{H^k} = \sum_{|\beta|=k} (\int_\Omega (D^\beta u)^2 dx)^{1/2}.$$

We define $t_n = n\tau, \quad n = 0, 1, ..., N$, where $\tau = T/N$ is the step size in time direction. The equation (1.1)–(1.3) is evaluated at the point (x, t_n), $n = 0, 1, \ldots, N$. We denote $u^n = u(x, t_n)$, $v^n = v(x, t_n)$ and

$w^n = w(x, t_n)$, then equation (1.1), (1.2) and (1.3) can be written as:

$$u^{n+1} = u^n + \tau\left(\frac{1}{2}\frac{\partial^3}{\partial x^3}u^n - 3u^n\frac{\partial u^n}{\partial x} + 3\frac{\partial}{\partial x}(v^n w^n)\right) + \tau R_1^n, \tag{2.1}$$

$$v^{n+1} = v^n + \tau\left(-\frac{\partial^3}{\partial x^3}v^n + 3u^n\frac{\partial v^n}{\partial x}\right) + \tau R_2^n, \tag{2.2}$$

$$w^{n+1} = w^n + \tau\left(-\frac{\partial^3}{\partial x^3}w^n + 3u^n\frac{\partial w^n}{\partial x}\right) + \tau R_3^n, \tag{2.3}$$

where R_1^n, R_2^n, and R_3^n are residual of the equation (2.1), (2.2) and (2.3) respectively. Furthermore $|R_1^n| < C_1\tau$, $|R_2^n| < C_2\tau$ and $|R_3^n| < C_3\tau$ for some positive constants C_1, C_2 and C_3. By ignoring the small terms R_1^n, R_2^n and R_3^n in the above equations, the time discrete scheme for the equation (2.1), (2.2) and (2.3) can be obtained as:

$$U^{n+1} = U^n + \tau\left(\frac{1}{2}\frac{\partial^3}{\partial x^3}U^n - 3U^n\frac{\partial U^n}{\partial x} + 3\frac{\partial}{\partial x}(V^n W^n)\right), \tag{2.4}$$

$$V^{n+1} = V^n + \tau\left(-\frac{\partial^3}{\partial x^3}V^n + 3U^n\frac{\partial V^n}{\partial x}\right), \tag{2.5}$$

$$W^{n+1} = W^n + \tau\left(-\frac{\partial^3}{\partial x^3}W^n + 3U^n\frac{\partial W^n}{\partial x}\right), \tag{2.6}$$

where $U^n = U(x, t_n)$, $V^n = V(x, t_n)$ and $W^n = W(x, t_n)$. We present a lemma, which will be useful for the proof of stability and convergence.

Lemma 2.1 ([24]). *If $m \geq 1$, and $u, v \in H^m(\Omega)$, there exists a constant C independent of u, v and N, such that*

$$\|uv\|_m \leq C\|u\|_m \|v\|_m .$$

3 Stability

Assume $U^n(x,t)$ to be the approximate solution of $u^n(x,t)$, $V^n(x,t)$ to be the approximate solution of $v^n(x,t)$ and $W^n(x,t)$ be the approximate solution of $w^n(x,t)$. For simplicity we denote $u^n = u^n(x,t)$ and similarly for other variables. Let

$$\widetilde{u}^n = u^n - U^n, \qquad \widetilde{v}^n = v^n - V^n, \qquad \widetilde{w}^n = w^n - W^n.$$

Subtracting (2.4) from (2.1), (2.5) from (2.2) and (2.6) from (2.3) results in

$$\widetilde{u}^{n+1} = \widetilde{u}^n + \frac{\tau}{2}\frac{\partial^3}{\partial x^3}\widetilde{u}^n - 3\tau\left(u^n\frac{\partial u^n}{\partial x} - U^n\frac{\partial U^n}{\partial x}\right) + 3\tau\frac{\partial}{\partial x}\left(v^n w^n - V^n W^n\right), \tag{3.1}$$

$$\widetilde{v}^{n+1} = \widetilde{v}^n + \tau\left(-\frac{\partial^3}{\partial x^3}\widetilde{v}^n\right) + 3\tau\left(u^n\frac{\partial v^n}{\partial x} - U^n\frac{\partial V^n}{\partial x}\right), \tag{3.2}$$

$$\widetilde{w}^{n+1} = \widetilde{w}^n + \tau\left(-\frac{\partial^3}{\partial x^3}\widetilde{w}^n\right) + 3\tau\left(u^n\frac{\partial w^n}{\partial x} - U^n\frac{\partial W^n}{\partial x}\right). \tag{3.3}$$

Taking the inner product of (3.1), (3.2) and (3.3) with $\widetilde{u}^{n+1}$, $\widetilde{v}^{n+1}$ and $\widetilde{w}^{n+1}$ respectively. By applying Cauchy-Schwartz inequality, algebraic and Young's inequalities, we have

$$\begin{aligned}(1+3\tau)\|\widetilde{u}^{n+1}\|^2 + \tau\left\|\frac{\partial\widetilde{u}^{n+1}}{\partial x}\right\|^2 \leq{}& \|\widetilde{u}^n\|^2 + \tau\left\|\frac{\partial^2\widetilde{u}^n}{\partial x^2}\right\|^2 - 3\tau\left\|u^n\frac{\partial u^n}{\partial x} - U^n\frac{\partial U^n}{\partial x}\right\|^2 \\ &+ 3\tau\|v^n w^n - V^n W^n\|^2,\end{aligned} \tag{3.4}$$

$$(1+3\tau)\|\widetilde{v}^{n+1}\|^2 + \tau\left\|\frac{\partial\widetilde{v}^{n+1}}{\partial x}\right\|^2 \leq \|\widetilde{v}^n\|^2 + \tau\left\|\frac{\partial^2\widetilde{v}^n}{\partial x^2}\right\|^2 + 3\tau\left\|u^n\frac{\partial v^n}{\partial x} - U^n\frac{\partial V^n}{\partial x}\right\|^2, \tag{3.5}$$

$$(1+3\tau)\|\widetilde{w}^{n+1}\|^2+\tau\left\|\frac{\partial\widetilde{w}^{n+1}}{\partial x}\right\|^2\leq\|\widetilde{w}^{n}\|^2+\tau\left\|\frac{\partial^2\widetilde{w}^{n}}{\partial x^2}\right\|^2+3\tau\left\|u^n\frac{\partial w^n}{\partial x}-U^n\frac{\partial W^n}{\partial x}\right\|^2, \tag{3.6}$$

Now we are going to estimate nonlinear terms of (3.4), (3.5) and (3.6). Again we apply Cauchy-Schwartz inequality and lemma 2.1, we get

$$\begin{aligned}\left\|u^n\frac{\partial u^n}{\partial x}-U^n\frac{\partial U^n}{\partial x}\right\| &= \left\|u^n\frac{\partial u^n}{\partial x}-u^n\frac{\partial U^n}{\partial x}+u^n\frac{\partial U^n}{\partial x}-U^n\frac{\partial U^n}{\partial x}\right\|\\ &= \left\|u^n\left(\frac{\partial u^n}{\partial x}-\frac{\partial U^n}{\partial x}\right)+\frac{\partial U^n}{\partial x}(u^n-U^n)\right\|\\ &\leq \|u^n\|_\infty\left\|\frac{\partial u^n}{\partial x}-\frac{\partial U^n}{\partial x}\right\|+\left\|\frac{\partial U^n}{\partial x}\right\|_\infty\|u^n-U^n\|\\ &\leq C_4\left(\left\|\frac{\partial u^n}{\partial x}-\frac{\partial U^n}{\partial x}\right\|+\|u^n-U^n\|\right)\end{aligned}$$

where $C_4=\left(\|\frac{\partial U^n}{\partial x}\|_\infty,\ \|u^n\|_\infty\right)$, we obtain

$$\left\|u^n\frac{\partial u^n}{\partial x}-U^n\frac{\partial U^n}{\partial x}\right\|^2\leq C_4\left(\left\|\frac{\partial\widetilde{u}^n}{\partial x}\right\|^2+\|\widetilde{u}^n\|^2\right)$$

Similarly we can apply Cauchy-Schwartz inequality and lemma 2.1, we get the estimation of nonlinear terms of (3.4), (3.5) and (3.6), we have

$$\begin{aligned}&\|v^nw^n-V^nW^n\|^2\leq C_5\left(\|\widetilde{v}^n\|^2+\|\widetilde{w}^n\|^2\right)\\ &\left\|u^n\frac{\partial v^n}{\partial x}-U^n\frac{\partial U^n}{\partial x}\right\|^2\leq C_6\left(\|\widetilde{u}^n\|^2+\left\|\frac{\partial\widetilde{v}^n}{\partial x}\right\|^2\right),\\ &\left\|u^n\frac{\partial w^n}{\partial x}-U^n\frac{\partial W^n}{\partial x}\right\|^2\leq C_7\left(\|\widetilde{u}^n\|^2+\left\|\frac{\partial\widetilde{w}^n}{\partial x}\right\|^2\right).\end{aligned}$$

where $C_5=\left(\|\frac{\partial V^n}{\partial x}\|_\infty,\ \|u^n\|_\infty\right)$, $C_6=\left(\|\frac{\partial W^n}{\partial x}\|_\infty,\ \|u^n\|_\infty\right)$, where $C_7=(\|v^n\|_\infty,\ \|W^n\|_\infty)$. Substituting the value of above values into (3.4), (3.5) and (3.6). Further more $\widetilde{C}=\max(C_4,C_5,C_6,C_8)$. We get

$$\begin{aligned}&(1-3\tau)\left(\|\widetilde{u}^{n+1}\|^2+\left\|\frac{\partial\widetilde{u}^{n+1}}{\partial x}\right\|^2+\|\widetilde{v}^{n+1}\|^2+\left\|\frac{\partial\widetilde{v}^{n+1}}{\partial x}\right\|^2+\|\widetilde{w}^{n+1}\|^2+\left\|\frac{\partial\widetilde{w}^{n+1}}{\partial x}\right\|^2\right)\\ &\quad\leq(1+3\tau)\widetilde{C}\left(\|\widetilde{u}^{n}\|^2+\left\|\frac{\partial\widetilde{u}^{n}}{\partial x}\right\|^2+\|\widetilde{v}^{n}\|^2+\left\|\frac{\partial\widetilde{v}^{n}}{\partial x}\right\|^2+\|\widetilde{w}^{n}\|^2+\left\|\frac{\partial\widetilde{w}^{n}}{\partial x}\right\|^2\right)\end{aligned} \tag{3.7}$$

$$\begin{aligned}\|\widetilde{u}^{n+1}\|^2_{H^1}+\|\widetilde{v}^{n+1}\|^2_{H^1}+\|\widetilde{w}^{n+1}\|^2_{H^1} &\leq \left(\frac{(1+3\tau)\widetilde{C}}{1-3\tau}\right)\left(\|\widetilde{u}^{n}\|^2_{H^1}+\|\widetilde{v}^{n}\|^2_{H^1}+\|\widetilde{w}^{n}\|^2_{H^1}\right)\\ &\leq \left(\frac{(1+3\tau)\widetilde{C}}{1-3\tau}\right)^2\left(\|\widetilde{u}^{n-1}\|^2_{H^1}+\|\widetilde{v}^{n-1}\|^2_{H^1}+\|\widetilde{w}^{n-1}\|^2_{H^1}\right)\\ &\;\;\vdots\\ &\leq \left(\frac{(1+3\tau)\widetilde{C}}{1-3\tau}\right)^{n+1}\left(\|\widetilde{u}^{0}\|^2_{H^1}+\|\widetilde{v}^{0}\|^2_{H^1}+\|\widetilde{w}^{0}\|^2_{H^1}\right)\end{aligned}$$

Let

$$\lim_{n\longrightarrow\infty}\left(\frac{\widetilde{C}(1+3\tau)}{1-3\tau}\right)^{n+1}=\lim_{n\longrightarrow\infty}\left(\frac{\widetilde{C}(1+\frac{3\tau}{n+1})}{1-\frac{3\tau}{n+1}}\right)^{n+1}=\frac{\widetilde{C}e^{3\tau}}{e^{-3\tau}}=e^{6\widetilde{C}\tau} \tag{3.8}$$

Therefore

$$\|\widetilde{u}^{n+1}\|_{H^1}^2 + \|\widetilde{v}^{n+1}\|_{H^1}^2 + \|\widetilde{w}^{n+1}\|_{H^1}^2 \leq \sqrt{e^{6\widetilde{C}\tau}} \left(\|\widetilde{u}^0\|_{H^1}^2 + \|\widetilde{v}^0\|_{H^1}^2 + \|\widetilde{w}^0\|_{H^1}^2\right)$$

Theorem 1. *Let u_0, v_0 and w_0 belong to $H^1(\Omega)$. Further, let u^n, v^n and w^n be the solution for initial boundary value problem (1.1)–(1.5) and U^n, V^n and W^n be the solution of the time discrete scheme (2.4)–(2.6). If $\tau < 1/3$ then solution of the discrete scheme is stable in H^1 norm*

4 Convergence

In this section we consider the convergence of of approximate solution of generalized HS coupled KdV equation. Define

$$\widetilde{U}^n = u^n - U^n, \qquad \widetilde{V}^n = v^n - V^n, \qquad \widetilde{W}^n = w^n - W^n.$$

From equations (2.1)–(2.3) and (2.4)–(2.6), we obtain

$$\widetilde{U}^{n+1} = \widetilde{U}^n + \frac{\tau}{2}\frac{\partial^3 \widetilde{U}^n}{\partial x^3} + 3\tau\left(u^n\frac{\partial u^n}{\partial x} - U^n\frac{\partial U^n}{\partial x}\right) - 3\tau\frac{\partial}{\partial x}\left(v^n w^n - V^n W^n\right) + \tau R_1^n, \tag{4.1}$$

$$\widetilde{V}^{n+1} = \widetilde{V}^n + \tau\left(-\frac{\partial^3 \widetilde{V}^n}{\partial x^3}\right) + 3\tau\left(u^n\frac{\partial v^n}{\partial x} - U^n\frac{\partial V^n}{\partial x}\right) + \tau R_2^n, \tag{4.2}$$

$$\widetilde{W}^{n+1} = \widetilde{W}^n + \tau\left(-\frac{\partial^3 \widetilde{W}^n}{\partial x^3}\right) + 3\tau\left(u^n\frac{\partial w^n}{\partial x} - U^n\frac{\partial W^n}{\partial x}\right) + \tau R_3^n. \tag{4.3}$$

Taking the inner product of (4.1), (4.2) and (4.3) with $\widetilde{U}^{n+1}$, $\widetilde{V}^{n+1}$ and $\widetilde{W}^{n+1}$ respectively, yields

$$\|\widetilde{U}^{n+1}\|^2 \leq \frac{1}{2}\|\widetilde{U}^n\|^2 - \frac{\tau}{2}\left(\left\|\frac{\partial^2 \widetilde{U}^n}{\partial x^2}\right\|^2 + \left\|\frac{\partial \widetilde{U}^{n+1}}{\partial x}\right\|^2\right) + \tau|R_1^n|\|\widetilde{U}^{n+1}\| + G_1 + G_2, \tag{4.4}$$

$$\|\widetilde{V}^{n+1}\|^2 \leq \frac{1}{2}\|\widetilde{V}^n\|^2 + \frac{\tau}{2}\left(\left\|\frac{\partial^2 \widetilde{V}^n}{\partial x^2}\right\|^2 + \left\|\frac{\partial \widetilde{V}^{n+1}}{\partial x}\right\|^2\right) + \tau|R_2^n|\|\widetilde{V}^{n+1}\| + G_3, \tag{4.5}$$

$$\|\widetilde{W}^{n+1}\|^2 \leq \frac{1}{2}\|\widetilde{W}^n\|^2 + \frac{\tau}{2}\left(\left\|\frac{\partial^2 \widetilde{W}^n}{\partial x^2}\right\|^2 + \left\|\frac{\partial \widetilde{W}^{n+1}}{\partial x}\right\|^2\right) + \tau|R_3^n|\|\widetilde{W}^{n+1}\| + G_4, \tag{4.6}$$

where

$$G_1 = -3\tau\left(u^n\frac{\partial u^n}{\partial x} - U^n\frac{\partial U^n}{\partial x}, \widetilde{U}^{n+1}\right), \qquad G_2 = 3\tau\frac{\partial}{\partial x}\left(v^n w^n - V^n W^n, \widetilde{U}^{n+1}\right),$$

$$G_3 = \tau\left(u^n\frac{\partial v^n}{\partial x} - U^n\frac{\partial V^n}{\partial x}, \widetilde{V}^{n+1}\right), \qquad G_4 = 3\tau\left(u^n\frac{\partial w^n}{\partial x} - U^n\frac{\partial W^n}{\partial x}, \widetilde{W}^{n+1}\right).$$

By using the algebraic inequality and lemma 2.1, we get

$$|G_1| \leq 3\tau\left\|u^n\frac{\partial u^n}{\partial x} - U^n\frac{\partial U^n}{\partial x}\right\|^2 + \|\widetilde{U}^{n+1}\|^2 \leq C_8\left(\left\|\frac{\partial \widetilde{u}^n}{\partial x}\right\|^2 + \|\widetilde{u}^n\|^2\right) + \|\widetilde{U}^{n+1}\|^2, \tag{4.7}$$

$$|G_2| \leq 3\tau\left\|v^n w^n - V^n W^n\right\|^2 + \|\widetilde{U}^{n+1}\|^2 \leq C_9\left(\|\widetilde{v}^n\|^2 + \|\widetilde{w}^n\|^2\right) + \|\widetilde{U}^{n+1}\|^2, \tag{4.8}$$

$$|G_3| \leq 3\tau\left\|u^n\frac{\partial v^n}{\partial x} - U^n\frac{\partial U^n}{\partial x}\right\|^2 + \|\widetilde{V}^{n+1}\|^2 \leq C_{10}\left(\|\widetilde{u}^n\|^2 + \left\|\frac{\partial \widetilde{v}^n}{\partial x}\right\|^2\right) + \|\widetilde{V}^{n+1}\|^2, \tag{4.9}$$

$$|G_4| \leq 3\tau\left\|u^n\frac{\partial w^n}{\partial x} - U^n\frac{\partial W^n}{\partial x}\right\|^2 + \|\widetilde{W}^{n+1}\|^2 \leq C_{11}\left(\|\widetilde{u}^n\|^2 + \left\|\frac{\partial \widetilde{w}^n}{\partial x}\right\|^2\right) + \|\widetilde{W}^{n+1}\|^2, \tag{4.10}$$

where C_8, C_9, C_{10} and C_{11} are constants independent of τ and N. Let $\widetilde{M} = \max(C_8, C_9, C_{10}, C_{11})$ Putting the values of (4.7) and (4.8) in to (4.4). Also substituting the values of (4.9) and (4.10) in to

(4.5) and (4.6) respectively. By using the same technique as in the previous section, we can obtain a equation similar to (3.7).

$$\begin{aligned}(1-3\tau)&\left(\|\widetilde{U}^{n+1}\|^2+\left\|\frac{\partial \widetilde{U}^{n+1}}{\partial x}\right\|^2+\|\widetilde{V}^{n+1}\|^2+\left\|\frac{\partial \widetilde{V}^{n+1}}{\partial x}\right\|^2+\|\widetilde{W}^{n+1}\|^2+\left\|\frac{\partial \widetilde{W}^{n+1}}{\partial x}\right\|^2\right)\\ &\leq (1+3\tau)\widetilde{M}\left(\|\widetilde{U}^{n}\|^2+\left\|\frac{\partial \widetilde{U}^{n}}{\partial x}\right\|^2+\|\widetilde{V}^{n}\|^2+\left\|\frac{\partial \widetilde{V}^{n}}{\partial x}\right\|^2+\|\widetilde{W}^{n}\|^2+\left\|\frac{\partial \widetilde{W}^{n}}{\partial x}\right\|^2\right)\\ &+\tau\vartheta^2|R_1^n|^2+\tau\vartheta^2|R_2^n|^2+\tau\vartheta^2|R_3^n|^2.\end{aligned} \tag{4.11}$$

$$\begin{aligned}\|\widetilde{U}^{n+1}\|_{H^1}^2+\|\widetilde{V}^{n+1}\|_{H^1}^2+\|\widetilde{W}^{n+1}\|_{H^1}^2 \leq \left(\frac{(1+3\tau)\widetilde{M}}{1-3\tau}\right)\Big[\Big(\|\widetilde{U}^{n}\|_{H^1}^2+\|\widetilde{V}^{n}\|_{H^1}^2+\|\widetilde{W}^{n}\|_{H^1}^2\Big)\\ +(\tau\vartheta^2|R_1^n|^2+\tau\vartheta^2|R_2^n|^2+\tau\vartheta^2|R_3^n|^2)\Big]\end{aligned}$$

Let

$$\begin{aligned}\widetilde{E}^{n+1}&=\|\widetilde{U}^{n+1}\|_{H^1}^2+\|\widetilde{V}^{n+1}\|_{H^1}^2+\|\widetilde{W}^{n+1}\|_{H^1}^2\\ \widetilde{R}^{n}&=\tau\vartheta^2(|R_1^n|^2+|R_2^n|^2+|R_3^n|^2)\end{aligned}$$

Then equation (4.11) is written as

$$\begin{aligned}\widetilde{E}^{n+1}&\leq\left(\frac{(1+3\tau)\widetilde{M}}{1-3\tau}\right)\left[\widetilde{E}^{n}+\tau\vartheta^2\widetilde{R}^{n}\right]\\ &\leq\left(\frac{(1+3\tau)\widetilde{M}}{1-3\tau}\right)^2\widetilde{E}^{n-1}+\left(\frac{(1+3\tau)\widetilde{M}}{1-3\tau}\right)\tau\vartheta^2\widetilde{R}^{n-1}+\tau\vartheta^2\widetilde{R}^{n}\\ &\vdots\\ &\leq\left(\frac{(1+3\tau)\widetilde{M}}{1-3\tau}\right)^n\widetilde{E}^{0}+\tau\vartheta^2\sum_{j=0}^{n}\left(\frac{(1+3\tau)\widetilde{M}}{1-3\tau}\right)^j\widetilde{R}^{n-j}\end{aligned}$$

Since $\widetilde{E}^0=0$, we obtain

$$\widetilde{E}^{n+1}\leq(n+1)\tau\vartheta^2\sum_{j=0}^{n}\left(\frac{(1+3\tau)\widetilde{M}}{1-3\tau}\right)^j\widetilde{R}^{n-j}$$

Finally, using the result of (3.8) we get

$$\|u^n-U^n\|+\|v^n-V^n\|+\|w^n-W^n\|\leq(n+1)\tau\vartheta^2e^{6\widetilde{M}t}|R^n|\leq\widetilde{M}\sqrt{\vartheta^2e^{6\widetilde{M}t}}\tau$$

Theorem 2. *Let u^n, v^n and w^n be the solution for initial boundary value problem for (1.1)–(1.5) and let U^n, V^n and W^n be the solution of (2.4)–(2.6) time discrete scheme. If the conditions of Theorem 1 holds. Then the time discrete solution is convergent in H^1 and the convergence rate is $O(\tau)$.*

5 Numerical Results

In this section, we present numerical results to show the efficiency and accuracy of the method, mentioned in previous section. We define maximum error $\|E(u)\|_\infty$, $\|E(v)\|_\infty$ and $\|E(w)\|_\infty$ as follows

$$\begin{aligned}\|E(u)\|_\infty&=\max_{0\leq j\leq N}|u(x_j,t)-U(x_j,t)|,\\ \|E(v)\|_\infty&=\max_{0\leq j\leq N}|v(x_j,t)-V(x_j,t)|,\\ \|E(w)\|_\infty&=\max_{0\leq j\leq N}|w(x_j,t)-W(x_j,t)|,\end{aligned}$$

where u,v,w are the exact solutions of (1.1)–(1.5) and U,V,W are the approximate solutions.

5.1 Example 1

Consider the generalized HS coupled KdV equations (1.1)–(1.5) with the initial conditions [25]:

$$\begin{aligned} u(x,0) &= \frac{\beta - 2\alpha^2}{3} + 2\alpha^2 \tanh^2(\alpha x), \\ v(x,0) &= \frac{4\alpha^2(\beta+\alpha^2)}{3c_1}\left(\frac{c_0}{c_1} - \tanh(\alpha x)\right), \\ w(x,0) &= c_0 + c_1 \tanh(\alpha x) \end{aligned}$$

where c_0, c_1, α and β are arbitrary constants. For practical computation we choose the parameters as $c_0 = 1.5$, $c_1 = 0.1$, $\alpha = 0.1$, $\beta = 1.5$ and $N = 64$.

The absolute error of the U, V and W are given in Table-1, Table-2 and Table-3 respectively. The results of the present method are compared with the results of methods already available in the literature i.e., Reza and Malik [15], Xie and Ding [13] for the variable U, V and W at different values of t. We observe that the absolute error is less than 0.2×10^{-6}. The numerical results of the present method are better than the results obtained by Reza and Malik [15], Xie and Ding [13]. The space-time graphs of U, V and W are given in Figure-1, Figure-2 and Figure-3 respectively. The graph of exact and approximate solution are plotted in Figure-1 to Figure-3 at different values of t.

Table 1: Comparison of numerical results of pseudospectral (present) method for Example-1 with the results obtained from Reza and Malik [15], Xie and Ding [13] for the variable U at different values of t.

t	DTM ([15])	RDTM ([15])	DTM ([13])	Present Method
0.1	3.290e-06	6.719e-10	6.739e-10	2.541e-06
0.4	5.252e-05	1.711e-07	1.719e-07	3.345e-07
0.7	1.597e-04	1.593e-06	1.603e-06	6.144e-07
1.0	3.227e-04	6.574e-06	6.625e-06	8.363e-07

Table 2: Comparison of numerical results of pseudospectral (present) method for Example-1 with the results obtained from Reza and Malik [15], Xie and Ding [13] for the variable V at different values of t.

t	DTM ([15])	RDTM ([15])	DTM ([13])	Present Method
0.1	8.559e-11	3.320e-13	8.828e-11	1.430e-08
0.4	1.698e-10	8.490e-11	3.818e-08	2.234e-08
0.7	8.793e-10	7.951e-10	5.028e-07	5.933e-08
1.0	3.389e-09	3.306e-09	2.689e-06	7.474e-08

Table 3: Comparison of numerical results of pseudospectral (present) method for Example-1 with the results obtained from Reza and Malik [15], Xie and Ding [13] for the variable W at different values of t.

t	DTM ([15])	RDTM ([15])	DTM ([13])	Present Method
0.1	5.349e-08	2.075e-10	4.385e-11	6.095e-08
0.4	1.061e-07	5.306e-08	1.896e-08	7.780e-08
0.7	5.496e-07	4.969e-07	2.497e-07	9.188e-08
1.0	2.118e-06	2.066e-06	1.335e-06	8.989e-08

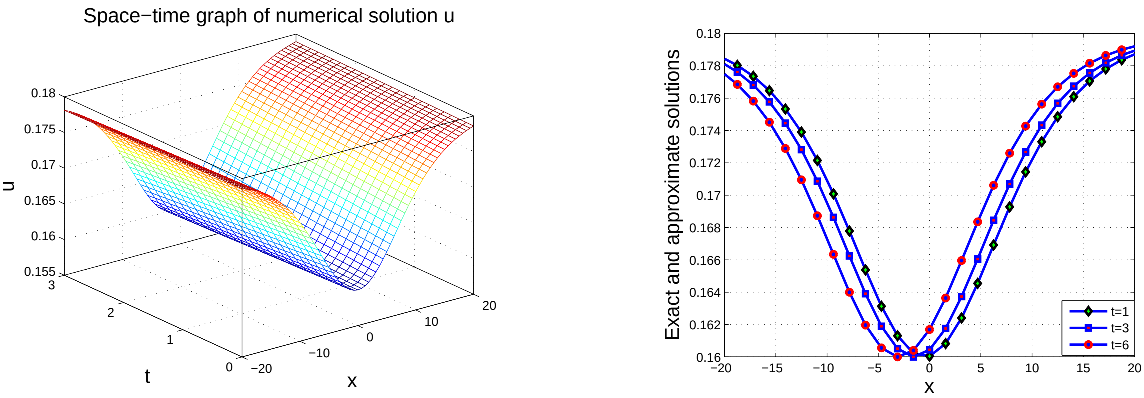

Figure 1: The left figure shows the space-time graphs of U, while the right figure shows the graph of U for different values of t.

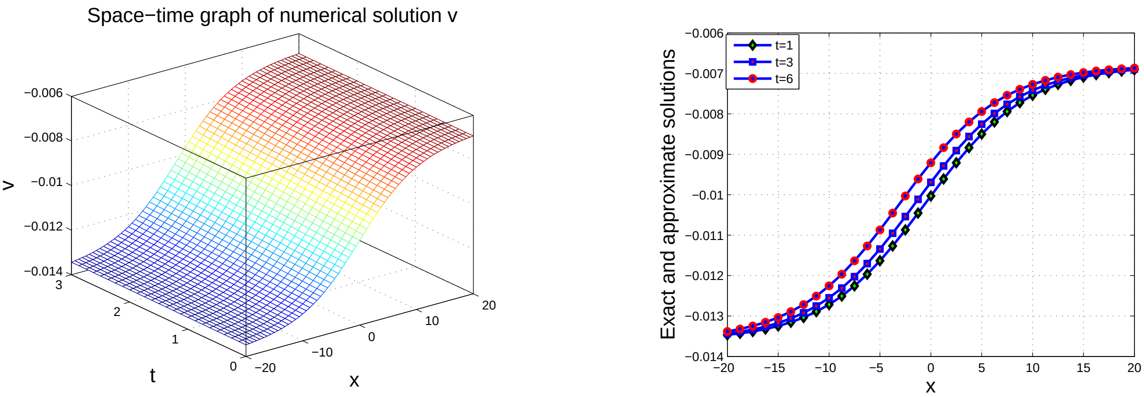

Figure 2: The left figure shows the space-time graphs of V, while the right figure shows the graph of V for different values of t.

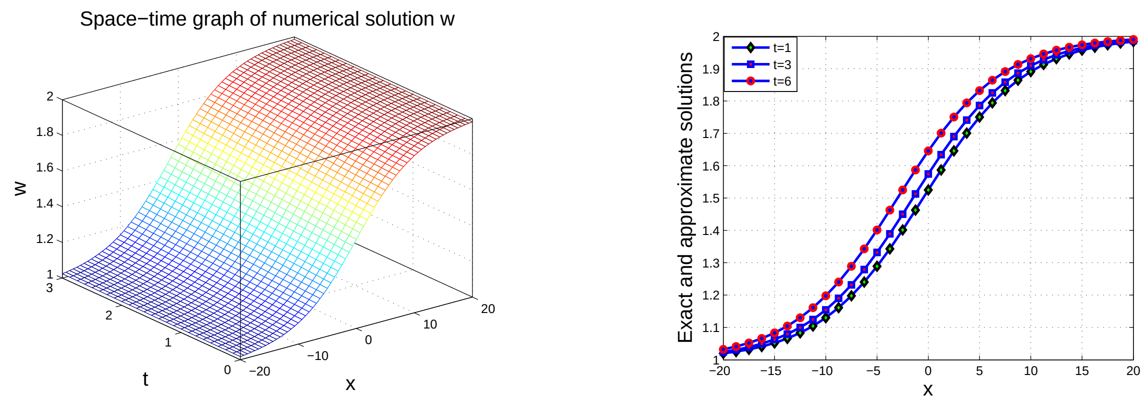

Figure 3: The left figure shows the space-time graphs of W, while the right figure shows the graph of W for different values of t.

5.2 Example 2

We consider the generalized HS coupled KdV equations (1.1)–(1.5) with the initial conditions [25]:

$$u(x,0) = \frac{\beta - 8\alpha^2}{3} + 4\alpha^2 \tanh^2(\alpha x),$$
$$v(x,0) = -\frac{4}{3}\frac{\alpha^2(3\alpha^2 c_0 - 2\beta c_2 + 4\alpha^2 c_2)}{c_2^2} + \left(\frac{4\alpha^2}{c_2}\tanh^2(\alpha x)\right),$$
$$w(x,0) = c_0 + c_2 \tanh^2(\alpha x)$$

where c_0, c_1, c_2, α and β are arbitrary constants. We choose the arbitrary constants for practical computation as, $c_0 = 1.5$, $c_1 = 0.1$, $c_2 = 0.5$, $\alpha = 0.1$, $\beta = 1.5$ and $N = 64$.

The absolute error of U, V and W are given in Table-4, Table-5 and Table-6 respectively. we compare the results of the present method with Reza and Malik [15], Xie and Ding [13] for the variable U, V and W at different value of t. The results are already available in the literature. We observe that the absolute error is less than 0.2×10^{-6}. The numerical results of the present method are comparatively better than the results obtained from Reza and Malik [15], Xie and Ding [13]. The space-time graphs of U, V and W are given in Figure-4, Figure-5 and Figure-6 respectively. The graph of exact and approximate solution are shown in Figure-4 to Figure-6 at different value of t.

Table 4: Comparison of numerical results of pseudospectral (present) method for Example-1 with the results obtained from Reza and Malik [15], Xie and Ding [13] for the variable U at different values of t.

t	DTM ([15])	RDTM ([15])	DTM ([13])	Present Method
0.1	4.279e-09	1.660e-11	2.495e-05	3.762e-09
0.4	8.490e-09	4.245e-09	1.146e-04	4.677e-09
0.7	4.396e-08	3.975e-08	2.293e-04	5.366e-09
1.0	1.694e-07	1.653e-07	3.744e-04	7.595e-09

Table 5: Comparison of numerical results of pseudospectral (present) method for Example-1 with the results obtained from Reza and Malik [15], Xie and Ding [13] for the variable V at different values of t.

t	DTM ([15])	RDTM ([15])	DTM ([13])	Present Method
0.1	8.559e-11	3.320e-13	8.828e-11	1.430e-08
0.4	1.698e-10	8.490e-11	3.818e-08	2.234e-08
0.7	8.793e-10	7.951e-10	5.028e-07	5.933e-08
1.0	3.389e-09	3.306e-09	2.689e-06	7.474e-08

Table 6: Comparison of numerical results of pseudospectral (present) method for Example-1 with the results obtained from Reza and Malik [15], Xie and Ding [13] for the variable W at different values of t.

t	DTM ([15])	RDTM ([15])	DTM ([13])	Present Method
0.1	5.349e-08	2.075e-10	4.385e-11	6.095e-08
0.4	1.061e-07	5.306e-08	1.896e-08	7.780e-08
0.7	5.496e-07	4.969e-07	2.497e-07	9.188e-08
1.0	2.118e-06	2.066e-06	1.335e-06	8.989e-08

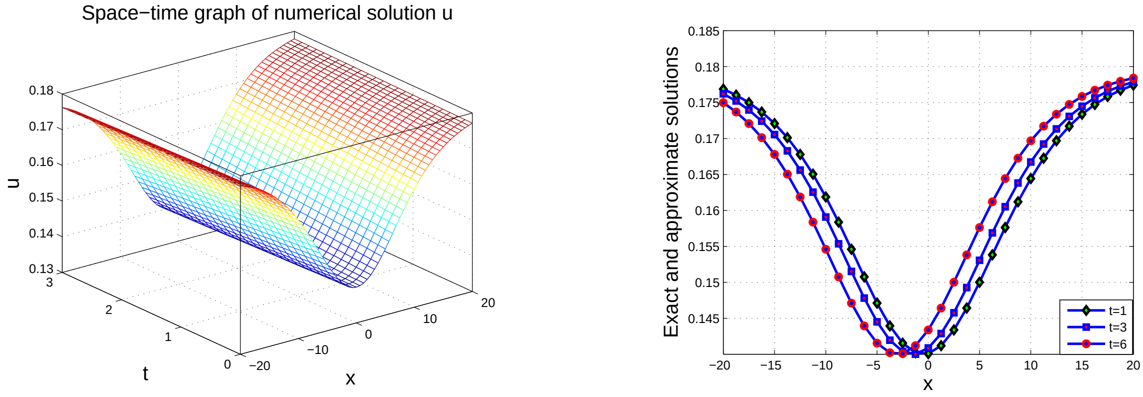

Figure 4: The left figure shows the space-time graphs of U, while the right figure shows the graph of U for different values of t.

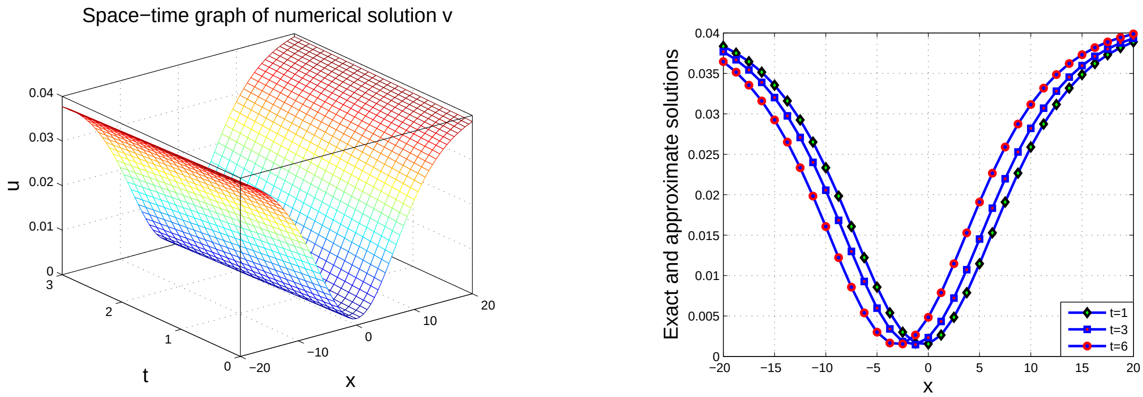

Figure 5: The left figure shows the space-time graphs of V, while the right figure shows the graph of V for different values of t.

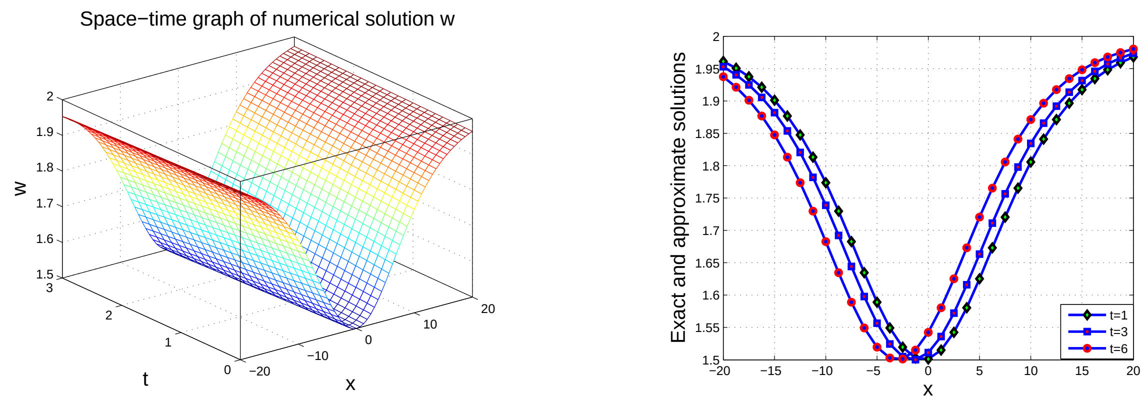

Figure 6: The left figure shows the space-time graphs of W, while the right figure shows the graph of W for different values of t.

6 Conclusion

In this paper, the generalized Hirota-Satsuma (HS) coupled Korteweg-de Vries (KdV) equation is solved numerically using the Fourier pseudospectral method. The time derivative of discrete scheme is approximated by the forward finite difference formula while the pseudospectral method is used in the space direction. The stability and convergence of the discrete scheme are proved by energy estimation method. The obtained solution is presented graphically at various time levels. The numerical results reveal that the Fourier pseudospectral method is convenient, effective and accurate to solve the generalized HS coupled KdV equations.

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