Volume 31, Number 3 September 2023
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(published quarterly) www.eudoxuspress.com.

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Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

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Generalization of Interval Jacobi and Gauss-Seidel Methods for Interval Linear System

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Abstract

The paper presents iterative methods for solving interval linear system of equations. We present a generalization of interval Jacobi method and interval Gauss-Seidel method by generalizing interval diagonal matrices to band interval matrices, and discuss the convergence analysis of the proposed methods. More specifically, we prove that both the proposed methods converge for any initial approximation if the coefficient interval matrix of the system is either an *interval strictly diagonally dominant matrix*, or *interval M-matrix* or *interval H-matrix*. Numerical experiment are carried out to assess the effectiveness of the proposed methods.

Keywords: Convergence, Generalized interval Jacobi method, Generalized interval Gauss-Seidel method, Linear interval systems
2020 Mathematics Subject Classification: 15A30, 65G40, 65H10, 65G30

1 Introduction

Many practical problems involving uncertainties, such as uncertainty in engineering or design problems, global optimization, mathematical programming problems etc., get reduced to solving system of interval equations. We refer to [12, 15, 16, 17, 20, 26] to find the literature on interval analysis dealing with uncertainty. It is worthwhile to mention that the solution set enclosure of interval linear systems, plays a significant role as data are impacted by uncertainty in many real-world problems that involves interval linear systems. However, it is well-known that the interval computations are NP-hard problems. In other words, one cannot expect an algorithm for computing all computations for the interval in less than exponential running time. So the research has been driven

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for finding a solution set enclosure for the interval linear equations with less computations.

Interval Jacobi, interval Gauss-Seidel, Bauer-Skeel, Hansen-Bliek-Rohn, Krawczyk iteration methods are among the oldest well-known iterative methods for solving interval linear systems. In [2, 7, 19, 21], authors proved that the above mentioned methods may not produce an optimal enclosure. However Hladík [19] in 2014, proposed a new interval operator that generalizes the interval Gauss-Seidel method, by introducing a new parameter. He further proved both theoretically and numerically that incorporation of such parameter is more effective than the Gauss-Seidel method, and provides tightening solution set enclosure of interval linear systems. Parametric interval linear system of equations were investigated in [18, 6]. In [21], author studied the Hansen-Bliek-Rohn method and the Bauer-Skeel method and their modification based on the preconditioning of the system and on the residual form. The paper aims to develop the iterative methods and their convergency to solve interval linear systems with uncertain coefficients. More specifically, we generalize the interval Jacobi method and interval Gauss-Seidel method for solving interval linear systems and analyze the convergence of these methods.

Throughout the paper, the sets of all real intervals, the set of n-dimensional real interval vectors, and the set of $m\times n$ real interval matrices are denoted by $\mathbb{IR}$, $\mathbb{IR}^n$ and $\mathbb{IR}^{m,n}$, respectively. We write bold letters to represent interval matrices/vectors, whereas normal letters are used to represent real matrices/vectors. A real interval matrix of order $m \times n$ for two real matrices $\underline{A}$ and $\overline{A}$, is defined as $\mathbf{A} = \{A \in \mathbb{R}^{m,n} : \underline{A} \leq A \leq \overline{A}\}$, with componentwise inequality $\underline{A} \leq \overline{A}$.

Consider the following system of interval linear systems of equations

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{1}$$

with $\mathbf{A} \in \mathbb{IR}^{n,n}$ and $\mathbf{b} \in \mathbb{IR}^n$ are given interval matrix and interval vector respectively, $\mathbf{x} \in \mathbb{IR}^n$ is unknown. The solution set of (1) is enclosed by

$$\Sigma(\mathbf{A}, \mathbf{b}) := \{\tilde{x} \in \mathbb{R}^n : \tilde{A}\tilde{x} = \tilde{b} \text{ for some } \tilde{A} \in \mathbf{A},\ \tilde{b} \in \mathbf{b}\}$$

The smallest interval enclosure of $\Sigma(\mathbf{A}, \mathbf{b})$ with regard to inclusion is represented by the interval $\mathbf{\Sigma} := \square\Sigma(\mathbf{A}, \mathbf{b}) = [\inf(\Sigma(\mathbf{A}, \mathbf{b})), \sup(\Sigma(\mathbf{A}, \mathbf{b}))]$, is known as the interval hull of the solution set $\Sigma(\mathbf{A}, \mathbf{b})$.

Let $\mathbf{D}$, $-\mathbf{E}$ and $-\mathbf{F}$ be respectively, represent the diagonal part, strictly lower triangular and strictly upper triangular parts of the interval matrix $\mathbf{A}$, so that $\mathbf{A} = \mathbf{D} - \mathbf{E} - \mathbf{F}$. If $0 \notin \mathbf{A}_{ii}$, then the interval Jacobi method and interval Gauss-Seidel method (see [3, 30]) for solving (1) are respectively, given by

$$\begin{aligned} \mathbf{x}^{(k+1)} &= \mathbf{D}^{-1}(\mathbf{E} + \mathbf{F})\mathbf{x}^{(k)} + \mathbf{D}^{-1}\mathbf{b} \\ \mathbf{x}^{(k+1)} &= (\mathbf{D} - \mathbf{E})^{-1}\mathbf{F}\mathbf{x}^{(k)} + (\mathbf{D} - \mathbf{E})^{-1}\mathbf{b}, \end{aligned} \tag{2}$$

We write $\mathbf{H}_J = \mathbf{D}^{-1}(\mathbf{E}+\mathbf{F})$ and $\mathbf{H}_{GS} = (\mathbf{D} - \mathbf{E})^{-1}\mathbf{F}$ to represent the iteration matrices for the interval Jacobi and interval Gauss-Seidel method, respectively. Details of interval Jacobi and Gauss-Seidel method for solving interval linear equations can be found in [3].

It is known from the literature that M-matrices, L-matrices, strictly diagonally dominant (SDD) and symmetric positive definite (SPD) matrices are among the classes of matrices for which both Jacobi and Gauss-Seidel methods converge for any initial guess for a given linear system of equations $Ax = b$. In [4], Salkuyeh generalized the Jacobi and Gauss-Seidel method by generalizing the diagonal matrix to a band matrix, which are given by the following iteration relations

$$x^{(k+1)} = T_m^{-1}(E_m + F_m)x^{(k)} + T_m^{-1}b \tag{3}$$
$$x^{(k+1)} = (T_m - E_m)^{-1}F_m x^{(k)} + (T_m - E_m)^{-1}b \tag{4}$$

where $m \geq 0$ and $F_m = A - T_m - E_m$ and

$$T_m = \begin{bmatrix} a_{1,1} & \dots & a_{1,m+1} & 0 \\ \vdots & \ddots & \ddots & \\ a_{m+1,1} & & & a_{n-m,n} \\ & \ddots & \ddots & \\ 0 & & a_{n,n-m} & a_{n,n} \end{bmatrix}, \quad E_m = \begin{bmatrix} 0 & \dots & 0 \\ -a_{m+2,1} & & \\ \vdots & \ddots & \vdots \\ -a_{n,1} & \dots & -a_{n-m-1,n} \end{bmatrix}$$

Salkuyeh proved that if the coefficient matrix of a system of linear equations is either SDD or an M-matrix, then the generalized Jacobi (GJ) and generalized Gauss-Seidel (GGS) methods converge. In [22], authors proved the convergence of GJ and GGS methods for H-matrices, however both methods may fail to converge for SPD and for L-matrices.

In this paper we generalize the interval Gauss-Seidel and interval Jacobi methods similar to equation (3) and (4), respectively, to obtain a tighter enclosure of the solution set. As mentioned earlier, the iteration schemes defined in (3) and (4) converge for SDD, M-matrices and for H-matrices, so motivated by these results we verify the convergence criteria of both generalized interval Jacobi and interval Gauss-Seidel methods for these classes of interval coefficient matrices.

This paper is organized as follows: In Section 2, we provide the notations and basic definitions related to interval analysis and define various classes of interval matrices under consideration. We also listed a few well-known results that are used in our study. Section 3 introduces the generalization of interval Jacobi method and discuss the convergence analysis of the method for solving (1) for various classes of interval coefficient matrices. In section 4, we describe the generalized interval Gauss-Seidel method and its convergence analysis for various classes of interval coefficient matrices. Numerical experiments are carried out for the proposed methods in Section 5. Finally, Section 6 ends with some concluding remarks.

2 Notation and preliminaries

In accordance with the standard notations, intervals are marked by boldface throughout this article. To represent the lower and upper bounds of intervals respectively, the underscores and overscores notations are used. So, any interval $\mathbf{x}$ is written as $\mathbf{x} = [\underline{x}, \overline{x}]$. For the interval $\mathbf{x}$, *magnitude* and *mignitude* are defined respectively as, $|\mathbf{x}| := \max\{|x| : x \in \mathbf{x}\} = \max\{|\underline{x}|, |\overline{x}|\}$ and $\langle \mathbf{x} \rangle := \min\{|x| : x \in \mathbf{x}\} = \min\{|\underline{x}|, |\overline{x}|\}$. *Magnitude* and *mignitude* of interval matrix $\mathbf{A}$ are defined componentwise, and denoted by corresponding notations as defined for intervals. For a given interval matrix $\mathbf{A} = (\mathbf{A}_{ij}) \in \mathbb{IR}^{n,n}$, we denote $|\mathbf{A}| := (|\mathbf{A}|_{ij}) \in \mathbb{R}^{n,n}$, and the comparison matrix of $\mathbf{A}$ is represented by $\langle \mathbf{A} \rangle$, which has entries $\langle \mathbf{A} \rangle_{ii} = \langle \mathbf{A}_{ii} \rangle$ and $\langle \mathbf{A} \rangle_{ij} = -|\mathbf{A}_{ij}|$, for $i \neq j$. Note that both $|\mathbf{A}|$ and $\langle \mathbf{A} \rangle$ are real matrices. Next we provide some properties of the interval matrices, and define various classes of interval matrices under consideration.

Definition 2.1. [3, 10, 11] An interval matrix $\mathbf{A} \in \mathbb{IR}^{n,n}$ is said to be *regular* if every $A \in \mathbf{A}$ is nonsingular. For a *regular* $\mathbf{A}$, the inverse $\mathbf{A}^{-1}$ is defined as

$$\mathbf{A}^{-1} := \Box\{A^{-1} : A \in \mathbf{A}\}$$

where $\Box\Sigma := [\inf \Sigma,\ \sup \Sigma]$ denotes the hull of Σ, which is the tightest enclosure for Σ. It is to be noted that the smallest interval matrix $\mathbf{A}^{-1}$ includes the set $\{A^{-1} : A \in \mathbf{A}\}$.

Definition 2.2. [3] For any real intervals $\mathbf{x} = [\underline{x}, \overline{x}], \mathbf{y} = [\underline{y}, \overline{y}]$, interval addition, subtraction and multiplication are defined as

(i) $\mathbf{x} + \mathbf{y} = [\underline{x} + \underline{y},\ \overline{x} + \overline{y}]$

(ii) $\mathbf{x} - \mathbf{y} = [\underline{x} - \overline{y},\ \overline{x} - \underline{y}]$

(iii) The interval multiplication $\mathbf{xy}$ is displayed in the following table:

$*$	$\mathbf{y} \geq 0$	$\mathbf{y} \ni 0$	$\mathbf{y} \leq 0$
$\mathbf{x} \geq 0$	$[\underline{x}\underline{y}, \overline{x}\overline{y}]$	$[\overline{x}\underline{y}, \overline{x}\overline{y}]$	$[\overline{x}\underline{y}, \underline{x}\overline{y}]$
$\mathbf{x} \ni 0$	$[\underline{x}\overline{y}, \overline{x}\overline{y}]$	$[\min\{\underline{x}\overline{y}, \overline{x}\underline{y}\}, \max\{\underline{x}\underline{y}, \overline{x}\overline{y}\}]$	$[\overline{x}\underline{y}, \underline{x}\underline{y}]$
$\mathbf{x} \leq 0$	$[\underline{x}\overline{y}, \overline{x}\underline{y}]$	$[\underline{x}\overline{y}, \underline{x}\underline{y}]$	$[\overline{x}\overline{y}, \underline{x}\underline{y}]$

Definition 2.3. [3] If $\mathbf{A}, \mathbf{B} \in \mathbb{IR}^{m,n}$, addition and subtraction for interval matrices are defined as

(i) $\mathbf{A} + \mathbf{B} = \Box\{A + B :\ A \in \mathbf{A},\ B \in \mathbf{B}\}$

(ii) $\mathbf{A} - \mathbf{B} = \Box\{A - B :\ A \in \mathbf{A},\ B \in \mathbf{B}\}$

If $\mathbf{A} = [\underline{A}, \overline{A}]$ and $\mathbf{B} = [\underline{B}, \overline{B}]$, then

$$\mathbf{A} + \mathbf{B} = [\underline{A} + \underline{B},\ \overline{A} + \overline{B}] \quad \text{and} \quad \mathbf{A} - \mathbf{B} = [\underline{A} - \overline{B},\ \overline{A} - \underline{B}\ ]$$

Definition 2.4. [3] If $\mathbf{A} \in \mathbb{IR}^{m,n}$ and $\mathbf{B} \in \mathbb{IR}^{n,p}$, then $\mathbf{AB} \in \mathbb{IR}^{m,p}$ is defined as

$$\mathbf{AB} = \square\{\tilde{A}\tilde{B} : \tilde{A} \in \mathbf{A},\ \tilde{B} \in \mathbf{B}\}$$

If $\mathbf{A} = (\mathbf{A}_{ij})$ and $\mathbf{B} = (\mathbf{B}_{ij})$, then $(\mathbf{AB})_{ik} = \sum\limits_{j=1}^{n} \mathbf{A}_{ij}\mathbf{B}_{jk}$

Definition 2.5. [3, 14] Let $\mathbf{A} \in \mathbb{IR}^{n,n}$ and $0 \notin \mathbf{A}_{ii}$ for all i. If the comparison matrix $\langle \mathbf{A} \rangle$ of $\mathbf{A}$ is strictly diagonally dominant,that is, if for all i, $\langle \mathbf{A}_{ii} \rangle > \sum\limits_{j \neq i} |\mathbf{A}_{ij}|$ then $\mathbf{A}$ is known to be an interval *strictly diagonally dominant* (SDD) matrix.

Definition 2.6. [1, 3] A real matrix $A \in \mathbb{R}^{n,n}$ is called an *L-matrix* if it has positive diagonal entries and nonpositive off-diagonal entries. An interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$ is an *interval L-matrix* if each $A \in \mathbf{A}$ is an L-matrix, equivalently, if $\underline{A}_{ii} > 0$ for all i and $\overline{A}_{ij} \leq 0$, for $i \neq j$.

Definition 2.7. [1] A matrix $A \in \mathbb{R}^{n,n}$ is said to be a *Z-matrix* if A has nonpositive off-diagonal entries. If a Z-matrix A can be written as $A = \alpha I - B$, where $\alpha > \rho(B)$, the spectral radius of B, then A is called an *M-matrix*. Instead of nonsingular M-matrix we write M-matrix for convenience. A *Z-matrix* A becomes an *M-matrix* if and only if there exists a $u > 0$ such that $Au > 0$.

We now state the characterization of M-matrices.

Theorem 2.8. [1] Let $A \in \mathbb{R}^{n,n}$ be a Z-matrix. Then following equivalent conditions hold:

(i) A is an M-matrix.

(ii) $A^{-1} \geq 0$.

(iii) There exists $u > 0$ such that $Au > 0$.

Definition 2.9. [3] An *interval M-matrix* is a square interval matrix $\mathbf{A} \in \mathbb{IR}^{n,n}$ such that $\mathbf{A}_{ik} \leq 0$, that is, every element in $\mathbf{A}_{ik}$ is nonpositive, for all $i \neq k$ and $\mathbf{A}u > 0$ for some real $u > 0$.

Definition 2.10. [3] An *interval H-matrix* $\mathbf{A} \in \mathbb{IR}^{n,n}$ is an interval matrix whose comparison matrix $\langle \mathbf{A} \rangle$ is an M-matrix. Equivalently, we say that $\mathbf{A}$ is an interval H-matrix if and only if $\langle \mathbf{A} \rangle u > 0$ for some $u > 0$.

Definition 2.11. [1] A splitting of a real $n \times n$ matrix A is defined as $A = M - N$, with nonsingular M. A splitting $A = M - N$ of the matrix A is said to be

(i) *regular* if $M^{-1} \geq 0$ and $N \geq 0$.

(ii) *weak regular* if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$.

(iii) *M-splitting* if M is a M-matrix and $N \geq 0$.

Definition 2.12. A *splitting* of a square interval matrix $\mathbf{A} \in \mathbb{IR}^{n,n}$ is defined as $\mathbf{A} = \mathbf{M} - \mathbf{N}$, with regular $\mathbf{M}$.

Next we state few basic results on matrices, that are required to establish our results in the subsequent sections.

Proposition 1. [3, 23] If $A, B \in \mathbb{R}^{n,n}$, and $|A| \leq B$, then $\rho(A) \leq \rho(B)$.

Theorem 2.13. [1] Let A be an M-matrix and let $A = M - N$ be a regular or weak regular splitting of A, then $\rho(M^{-1}N) < 1$.

Theorem 2.14. [27] Let M-splitting of A be $A = M - N$. Then $\rho(M^{-1}N) < 1$ if and only if A is a nonsingular M-matrix.

Theorem 2.15. [1, 13, 23] Let A be a nonnegative matrix. Then

(i) If $\alpha \in \mathbb{R}$ and $Ax \geq \alpha x$, for some positive $x \in \mathbb{R}^n$, then $\rho(A) \geq \alpha$.

(ii) If $Ax \leq \alpha x$ for some $x \geq (\neq)0$, then $\rho(A) \leq \alpha$.

Theorem 2.16. [23] If A is nonnegative matrix, then $\rho(A)$ is an eigenvalue of A and there is a nonnegative nonzero vector x such that $Ax = \rho(A)x$.

For convenience we have provided some well-known results on interval matrices that will be used to check the convergence of the mentioned methods in next sections of this paper.

Theorem 2.17. [3] Let $\mathbf{A}, \mathbf{B} \in \mathbb{IR}^{n,n}$. Then following conditions hold:

(i) If $\mathbf{A}$ is an M-matrix and $\mathbf{B} \subseteq \mathbf{A}$, then $\mathbf{B}$ is an M-matrix. Each $\tilde{A} \in \mathbf{A}$ in particular, is an M-matrix.

(ii) $\mathbf{A} = [\underline{A},\ \overline{A}]$ is an M-matrix if and only if $\underline{A}$ and $\overline{A}$ are M-matrices.

(iii) Every M-matrix $\mathbf{A} = [\underline{A},\ \overline{A}]$ is regular with $\mathbf{A}^{-1} = [\overline{A}^{-1}, \underline{A}^{-1}] \geq 0$ and $|\mathbf{A}^{-1}| = \langle \mathbf{A} \rangle^{-1}$.

Theorem 2.18. [14] For an interval matrix $\mathbf{A}$ we have

(i) if $\mathbf{A}$ is interval triangular (lower/upper) matrix, then $\mathbf{A}$ is an interval H-matrix.

(ii) if $\mathbf{A}$ is an interval H-matrix, then $|\mathbf{A}^{-1}| \leq \langle \mathbf{A} \rangle^{-1}$. Equality holds if $\mathbf{A}$ is an interval M-matrix.

Proposition 2. [3] For $\mathbf{A}, \mathbf{B} \in \mathbb{IR}^{n,n}$ and $\mathbf{C} \in \mathbb{IR}^{n,q}$, the properties listed below hold:

(i) $\langle \mathbf{A} \rangle = \langle \tilde{A} \rangle$, for some $\tilde{A} \in \mathbf{A}$.

(ii) $|\mathbf{AB}| \leq |\mathbf{A}||\mathbf{B}|$

(iii) $\langle \mathbf{A} \pm \mathbf{B} \rangle \geq \langle \mathbf{A} \rangle - |\mathbf{B}|$.

(iv) $|\mathbf{A}| - |\mathbf{B}| \leq |\mathbf{A} \pm \mathbf{B}| \leq |\mathbf{A}| + |\mathbf{B}|$.

(v) $|\mathbf{AC}| \geq \langle \mathbf{A} \rangle |\mathbf{C}|$.

Theorem 2.19. [14] Let $\mathbf{C}, \mathbf{D} \in \mathbb{IR}^{n,n}$ satisfy $\rho(|\mathbf{C}||\mathbf{D}|) < 1$. Then for any $\mathbf{g} \in \mathbb{IR}^n$, the following statements hold:

(i) The equation $\mathbf{x} = \mathbf{C}(\mathbf{Dx} + \mathbf{g})$ has a unique solution $\mathbf{x} \in \mathbb{IR}^n$

(ii) For any initial vector $\mathbf{x}^0 \in \mathbb{IR}^n$, the iteration

$$\mathbf{x}^{(k+1)} = \mathbf{C}(\mathbf{Dx}^{(k)} + \mathbf{g}),\ k = 0, 1, ...$$

converges to the solution $\mathbf{x}$ of the equation $\mathbf{x} = \mathbf{C}(\mathbf{Dx} + \mathbf{g})$.

3 Generalized Interval Jacobi method

In this section, we propose a generalization of the interval Jacobi method for solving interval linear system similar to that of generalized Jacobi method introduced by Salkuyeh [5] and Saha *et al.* [22] for solving linear systems of matrix equations. Furthermore, we study the convergence properties of the proposed method for solving interval linear system with the coefficient matrix as either an interval SDD matrix, an interval M-matrix or an interval H-matrix.

In Section 2 the splitting of $\mathbf{A}$ for interval Jacobi method for solving (1) is given in equation (2) as $\mathbf{A} = \mathbf{M} - \mathbf{N}$ with $\mathbf{M} = \mathbf{D},\ \mathbf{N} = \mathbf{E} + \mathbf{F}$.

We now propose generalized interval Jacobi (GIJ) method for solving interval linear system similar to (3) and (4), which was introduced by Salkuyeh [4] for general matrices.

Let $\mathbf{A} = [\underline{a_{ij}}, \overline{a_{ij}}]$ be a square interval matrix of order n. Consider an interval band matrix $\mathbf{T}_m = [\underline{t_{ij}}, \overline{t_{ij}}]$ of $2m + 1$ bandwidth, which is characterized as

$$\mathbf{t}_{ij} = \begin{cases} [\underline{a_{ij}}, \overline{a_{ij}}], & \text{if } |i - j| \leq m \\ 0, & \text{elsewhere} \end{cases}$$

For $1 \leq m < n$, interval matrix $\mathbf{A}$ is decomposed as $\mathbf{A} = \mathbf{T}_m - \mathbf{E}_m - \mathbf{F}_m$, with strict lower part -$\mathbf{E}_m$ and strict upper part -$\mathbf{F}_m$ of $\mathbf{A}$. The interval matrices $\mathbf{T}_m,\ \mathbf{E}_m$ and $\mathbf{F}_m$ are expressed as follows

$$\mathbf{T}_m = \begin{bmatrix} & [\underline{a_{11}}, \overline{a_{11}}] & \cdots & [\underline{a_{1,m+1}}, \overline{a_{1,m+1}}] & & 0 \\ & \vdots & \ddots & & \ddots & \\ [\underline{a_{m+1,1}}, \overline{a_{m+1,1}}] & & & & & [\underline{a_{n-m,n}}, \overline{a_{n-m,n}}] \\ & & \ddots & & \ddots & \\ & 0 & & [\underline{a_{n,n-m}}, \overline{a_{n,n-m}}] & & [\underline{a_{n,n}}, \overline{a_{n,n}}] \end{bmatrix},$$

$$\mathbf{E}_m = \begin{bmatrix} & 0 & \cdots & & 0 \\ -[\underline{a_{m+2,1}}, \overline{a_{m+2,1}}] & & & & \\ & \vdots & \ddots & & \vdots \\ & -[\underline{a_{n,1}}, \overline{a_{n,1}}] & \cdots & & -[\underline{a_{n-m-1,n}}, \overline{a_{n-m-1,n}}] \end{bmatrix},$$

$$\mathbf{F}_m = \begin{bmatrix} 0 & -[\underline{a_{1,m+2}}, \overline{a_{1,m+2}}] & & \cdots - & [\underline{a_{1,n}}, \overline{a_{1,n}}] \\ \vdots & & \ddots & & \vdots \\ 0 & & \cdots & & -[\underline{a_{n-m-1,n}}, \overline{a_{n-m-1,n}}] \\ & & & & \end{bmatrix} \tag{5}$$

Definition 3.1. Let $\mathbf{T}_m$, $\mathbf{E}_m$ and $\mathbf{F}_m$ be the interval matrices specified in (5). For any $1 \leq m < n$ decompose $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{T}_m - \mathbf{E}_m - \mathbf{F}_m \tag{6}$$

which is corresponding to splitting

$$\mathbf{A} = \mathbf{M}_m - \mathbf{N}_m \tag{7}$$

with $\mathbf{M}_m = \mathbf{T}_m$ and $\mathbf{N}_m = \mathbf{E}_m + \mathbf{F}_m$. Then *generalized interval Jacobi* (GIJ) method to solve (1) is defined as,

$$\mathbf{x}^{(k+1)} = \mathbf{M}_m^{-1}\left(\mathbf{N}_m\mathbf{x}^{(k)} + \mathbf{b}\right) \tag{8}$$

For GIJ method $\mathbf{L} = \mathbf{T}_m^{-1}(\mathbf{E}_m + \mathbf{F}_m)$ is the iteration interval matrix. By decomposing $\mathbf{T}_m = \mathbf{D} + \mathbf{R}_m$, $\mathbf{A}$ can also be written as

$$\mathbf{A} = \mathbf{D} + \mathbf{R}_m - \mathbf{E}_m - \mathbf{F}_m \tag{9}$$

Remark 3.2. From (6), we can decompose $\langle\mathbf{A}\rangle$ as

$$\langle\mathbf{A}\rangle = \langle\mathbf{T}_m\rangle - |\mathbf{E}_m| - |\mathbf{F}_m| = \langle\mathbf{D}\rangle - |\mathbf{R}_m| - |\mathbf{E}_m| - |\mathbf{F}_m| \tag{10}$$

and is associated with the splitting

$$\langle\mathbf{A}\rangle = \langle\mathbf{M}_m\rangle - |\mathbf{N}_m| \tag{11}$$

with $\widetilde{M}_1 = \langle\mathbf{M}_m\rangle = \langle\mathbf{T}_m\rangle = \langle\mathbf{D}\rangle - |\mathbf{R}_m|$ and $\widetilde{N}_1 = |\mathbf{N}_m| = |\mathbf{E}_m| + |\mathbf{F}_m|$.

Notation. Throughout the paper following notations are used:

$$\mathbf{R}_m = (\mathbf{R}_{ij}), \quad \mathbf{E}_m = (\mathbf{E}_{ij}), \quad \mathbf{F}_m = (\mathbf{F}_{ij})$$
$$\widetilde{R}_i = \sum_{\substack{j=1\\ j\neq i}}^{n} |\mathbf{R}_{ij}|, \quad \widetilde{E}_i = \sum_{\substack{j=1\\ j\neq i}}^{n} |\mathbf{E}_{ij}|, \quad \widetilde{F}_i = \sum_{\substack{j=1\\ j\neq i}}^{n} |\mathbf{F}_{ij}|$$

3.1 Convergence analysis of GIJ method

In this section we discuss the convergence criterion of GIJ method for solving interval linear system (1) with various classes of coefficient interval matrices. In particular we show that the GIJ method is convergent for interval SDD matrices, interval M-matrices and interval H-matrices using the idea of interval splitting as well as the various characterizations of interval M- and interval H-matrices.

Throughout the section we consider the splitting of $\mathbf{A}$ defined in (7) and (10). More specifically, we write $\mathbf{M}_m := \mathbf{T}_m$ and $\mathbf{N}_m := \mathbf{E}_m + \mathbf{F}_m$. It is known that GIJ method converges if $\rho(|\mathbf{M}_m^{-1}||\mathbf{N}_m|) < 1$ due to Theorem 2.19. Since computing the inverse of interval matrix is NP-hard, so we use the matrix $\widetilde{L}_m = \langle \mathbf{M}_m \rangle^{-1} |\mathbf{N}_m| = \tilde{M}_1^{-1}\tilde{N}_1$ to check the convergence of GIJ method. The following theorem provides a relation between the spectral radius of the iteration matrix $\widehat{L}_m = |\mathbf{M}_m^{-1}||\mathbf{N}_m|$ with $\widetilde{L}_m = \langle \mathbf{M}_m \rangle^{-1} |\mathbf{N}_m| = \tilde{M}_1^{-1}\tilde{N}_1$.

Theorem 3.3. Let $\widehat{L}_m = |\mathbf{M}_m^{-1}||\mathbf{N}_m|$ and $\widetilde{L}_m = \langle \mathbf{M}_m \rangle^{-1} |\mathbf{N}_m| = \widetilde{M}_1^{-1}\widetilde{N}_1$. If $\mathbf{M}_m$ is an interval H-matrix (or interval M-matrix) then the following results hold

(i) $\widehat{L}_m \leq \widetilde{L}_m$ (equality holds if $\mathbf{M}_m$ is an interval M-matrix)

(ii) $\rho(\widehat{L}_m) \leq \rho(\widetilde{L}_m)$.

Proof. (i) As $\mathbf{M}_m$ is given an interval H-matrix, from Theorem 2.18 we have that
$$\widehat{L}_m = |\mathbf{M}_m^{-1}||\mathbf{N}_m| \leq \langle \mathbf{M}_m \rangle^{-1} |\mathbf{N}_m| = \widetilde{L}_m$$

(ii) It follows from (i) and from Proposition 1. □

Remark 3.4. The above theorem shows that if $\rho(\widetilde{L}_m) < 1$, then GIJ converges. This idea has been used to check the convergence of GIJ method in case $\mathbf{M}_m$ is an interval H-(or M-) matrix.

Remark 3.5. [3, 29] Interval SDD matrices are a special case of interval H-matrices, that is, interval SDD matrices $\mathbf{A} = (\mathbf{a}_{ij})$ that satisfy for all i, $\langle \mathbf{A}_{ii} \rangle > \sum_{j\neq i} |\mathbf{A}_{ij}|$, are H-matrices.

Following theorem gives an upper bound for the spectral radius of the matrix $\widetilde{L}_m$ of GIJ method to solve linear interval equations with interval SDD matrix as coefficient interval matrix.

Theorem 3.6. For any $1 \leq i \leq n$, let $\langle \mathbf{T} \rangle_i = \langle \mathbf{D}_{ii} \rangle - \widetilde{R}_i > 0$. Then

$$\rho(\widetilde{L}_m) \leq \max_{i \in \mathbb{N}} \frac{\widetilde{E}_i + \tilde{F}_i}{\langle \mathbf{T} \rangle_i}$$

Proof. Suppose that λ is an eigenvalue of the matrix $\widetilde{L}_m$ satisfying

$$|\lambda| > \max_{i \in \mathbb{N}} \frac{\widetilde{E}_i + \widetilde{F}_i}{\langle \mathbf{T} \rangle_i} = \frac{\widetilde{E}_i + \widetilde{F}_i}{\langle \mathbf{D}_{ii} \rangle - \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} |\mathbf{R}_{ij}|}, \text{ for all } 1 \leq i \leq n \tag{12}$$

which implies that

$$|\lambda \langle \mathbf{D}_{ii} \rangle| = |\lambda| \langle \mathbf{D}_{ii} \rangle > \widetilde{E}_i + \widetilde{F}_i + |\lambda| \sum_{\substack{j=1 \\ j \neq i}}^{n} |\mathbf{R}_{ij}| \tag{13}$$

Now $\det(\lambda I - \widetilde{L}_m) = 0$ implies $\det(\langle \mathbf{T}_m \rangle^{-1}) \det(C_1) = 0$ where $C_1 = \lambda \langle \mathbf{T}_m \rangle - |\mathbf{E}_m| - |\mathbf{F}_m|$, which again implies that $\det(C_1) = 0$. This is a contradiction to the fact that C_1 is SDD and hence nonsingular. Thus,

$$\rho(\widetilde{L}_m) \leq \max_{1 \leq i \leq n} \frac{\tilde{E}_i + \tilde{F}_i}{\langle \mathbf{T} \rangle_i}$$

□

Following results show the convergence of GIJ for interval SDD matrices.

Theorem 3.7. For interval SDD matrix $\mathbf{A}$, GIJ method (8) converges for any initial guess.

Proof. Let $\mathbf{A}$ be an interval SDD matrix. Decompose $\mathbf{A}$ as in (6). As $\mathbf{A}$ is a SDD matrix, so is the matrix $\mathbf{M}_m$, that is

$$\langle \mathbf{D}_{ii} \rangle > \sum_{j \neq i} |\mathbf{R}_{ij}|$$

Then $\mathbf{M}_m$ is an interval SDD matrix and hence an H-matrix due to the Remark 3.5.

Suppose λ is an eigenvalue of $\widetilde{L}_m$ and $|\lambda| \geq 1$. Then we have that

$$\begin{aligned}
\det(\lambda I - \widetilde{L}_m) = 0 &\Rightarrow \det(\lambda \langle \mathbf{M}_m \rangle - |\mathbf{N}_m|) = 0 \\
&\Rightarrow \det(\lambda \langle \mathbf{D} \rangle - \lambda |\mathbf{R}_m| - |\mathbf{N}_m|) = 0 \\
&\Rightarrow \det\left(\langle \mathbf{D} \rangle - |\mathbf{R}_m| - \frac{1}{\lambda} |\mathbf{N}_m|\right) = 0 \\
&\Rightarrow \det\left(\langle \mathbf{D} \rangle - |\mathbf{R}_m| - \frac{1}{\lambda} |\mathbf{E}_m| - \frac{1}{\lambda} |\mathbf{F}_m|\right) = 0
\end{aligned}$$

This shows that the matrix $Q = (\langle \mathbf{D} \rangle - |\mathbf{R}_m| - \frac{1}{\lambda}|\mathbf{E}_m| - \frac{1}{\lambda}|\mathbf{F}_m|)$ is singular. As $|\lambda| \geq 1$, that is, $\frac{1}{|\lambda|} \leq 1$ and hence $\mathbf{A}$ is SDD implies that Q is SDD, a contradiction to the fact that Q is singular. Thus $\rho(\widetilde{L}_m) < 1$ and the result holds for interval SDD matrix. □

Next two theorems provide the convergence criteria of GIJ method for two important classes under consideration, namely, the classes of interval M-matrices and interval H-matrices.

Theorem 3.8. GIJ method converges for interval M-matrix $\mathbf{A}$, for any $m \leq n$.

Proof. Let $\mathbf{A}$ be an interval M-matrix of order n. Then by Theorem 2.17 (iii), $\langle \mathbf{A} \rangle^{-1} \geq 0$. Since $\langle \mathbf{A} \rangle$ is a Z-matrix, Theorem 2.8 implies that $\langle \mathbf{A} \rangle$ is an M-matrix. Let $\langle \mathbf{A} \rangle = \widetilde{M}_1 - \widetilde{N}_1$ be the splitting of $\langle \mathbf{A} \rangle$ defined in (11). As $\langle \mathbf{A} \rangle$ is M-matrix, there exists $u > 0$ such that $\langle \mathbf{A} \rangle u > 0$ which implies that $\langle \mathbf{M}_m \rangle u > 0$, that is, $\widetilde{M}_1 u > 0$. Thus $\widetilde{M}_1$ is an M-matrix, by Theorem 2.8. Also, $\widetilde{N}_1 \geq 0$. Therefore, $\langle \mathbf{A} \rangle = \widetilde{M}_1 - \widetilde{N}_1$ is regular splitting, and hence by Theorem 2.13, $\rho(\widetilde{L}_m) < 1$. It is obvious that $\mathbf{M}_m$ is an interval M-matrix and hence Remark 3.4 implies that GIJ converges, for any initial guess. □

Theorem 3.9. GIJ method for solving interval linear system converges for interval H-matrix $\mathbf{A}$.

Proof. Let $\mathbf{A}$ be an interval H-matrix, so that the matrix $\langle \mathbf{A} \rangle$ is an M-matrix. Then as shown in Theorem 3.8, it can be proved that $\rho(\widetilde{L}_m) < 1$, which implies that GIJ method converges for interval H-matrix. □

4 Generalized interval Gauss-Seidel method

The interval Gauss-Seidel method for solving system of interval linear equations was introduced by Neumaier [3] and Moore [25]. In [4, 22], authors considered generalized Gauss-Seidel method (a particular case of AOR method) and discussed convergence properties thoroughly for various classes of matrices, like, SDD, SPD, M-matrices, L-matrices and for H-matrices as the coefficient matrices of the linear system. Using the similar approach we now propose generalized version of interval Gauss-Seidel (GIGS) method for solving interval linear systems. It is shown in [22] that generalized Gauss-Seidel may not converge for SPD and for L-matrices. Since general matrices are particular case of interval matrices, so this section is emphasised on checking the convergence of generalized interval Gauss-Seidel (GIGS) method, only for interval SDD matrices, interval M-matrices and for interval H-matrices. We now begin with defining iteration steps for generalized interval Gauss-Seidel method.

Definition 4.1. Consider the decomposition of $\mathbf{A}$, defined in equation (7) and the splitting

$$\mathbf{A} = \mathbf{M}_m - \mathbf{N}_m \tag{14}$$

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where $\mathbf{M}_m = \mathbf{T}_m - \mathbf{E}_m$ and $\mathbf{N}_m = \mathbf{F}_m$. Then the iteration step for GIGS method to solve the interval linear system (1), is defined as,

$$\mathbf{x}^{(k+1)} = \quad \mathbf{M}_m^{-1}\left(\mathbf{N}_m\mathbf{x}^{(k)} + \mathbf{b}\right) \tag{15}$$

Further decompose $\langle\mathbf{A}\rangle$ same as in (10) and consider the associated splitting (11) where

$$M_1 = \langle\mathbf{M}_m\rangle = \langle\mathbf{T}_m\rangle - |\mathbf{E}_m|, \quad N_1 = |\mathbf{N}_m| = |\mathbf{F}_m| \tag{16}$$

Now we emphasize on the convergence of GIGS method for interval SDD-matrices, interval matrices H-matrices and for interval M-matrices.

4.1 Convergence analysis of GIGS method

Convergence analysis of GIGS method is similar to that of GIJ method discussed in Section 3. For interval Gauss-Seidel method the splitting of $\mathbf{A}$ is considered as $\mathbf{A} = \mathbf{M} - \mathbf{N}$ with $\mathbf{M} = \mathbf{D} - \mathbf{E}$ and $\mathbf{N} = \mathbf{F}$ and the iteration matrix is given by $\mathbf{M}^{-1}\mathbf{N}$. If $\acute{G} := |\mathbf{M}^{-1}||\mathbf{N}|$ and $C := \langle\mathbf{M}\rangle^{-1}|\mathbf{N}|$, then from Theorem 2.19, it is known that interval Gauss-Seidel method converges if $\rho(\acute{G}) < 1$.

Following two results are immediate consequences of Theorem 2.18 and hence the proofs are skipped.

Theorem 4.2. If $\mathbf{M}$ is interval H-matrix then the following results hold

(i) $\acute{G} \le C$.

(ii) $\rho(\acute{G}) \le \rho(C)$ (equality holds if $\mathbf{M}$ is an interval M-matrix).

Theorem 4.3. Consider the splitting as given in equation (14) and (16). Let $\widetilde{G} = |{\mathbf{M}_m}^{-1}| \cdot |\mathbf{N}_m|$ and $C_m = \langle\mathbf{M}_m\rangle^{-1}|\mathbf{N}_m| = M_1^{-1}N_1$. If $\mathbf{M}_m$ is an interval H-matrix (or interval M-matrix) then the following results hold:

(i) $\widetilde{G} \le C_m$ (equality holds if $\mathbf{M}_m$ is an interval M-matrix).

(ii) $\rho(\widetilde{G}) \le \rho(C_m)$.

Remark 4.4. From the above results, it is obvious that if $\rho(C_m) < 1$ (with $C_0 = C$), then GIGS method converges, which will be used to prove the convergence of GIGS method in the succeeding results.

All through this section, we stick to the following notations:

(i) $\acute{G} := |\mathbf{M}^{-1}||\mathbf{N}|$, with $\mathbf{M} = \mathbf{D} - \mathbf{E}$ and $\mathbf{N} = \mathbf{F}$

(ii) $C := \langle\mathbf{M}\rangle^{-1}|\mathbf{N}|$.

(iii) $\widetilde{G} := |{\mathbf{M}_m}^{-1}| \cdot |\mathbf{N}_m|$, with $\mathbf{M}_m = \mathbf{T}_m - \mathbf{E}_m$ and $\mathbf{N}_m = \mathbf{F}_m$

(iv) $C_m := \langle\mathbf{M}_m\rangle^{-1}|\mathbf{N}_m| = M_1^{-1}N_1$

We begin our results with the following theorem that presents a spectral bound of C_m and hence for $\widetilde{G}$.

Theorem 4.5. Let $\mathbf{A}$ be an interval SDD matrix. Suppose $C_m = \langle \mathbf{T}_m - \mathbf{E}_m\rangle^{-1}|\mathbf{F}_m|$. Let $\widetilde{R}_i, \widetilde{E}_i, \widetilde{F}_i$ be defined same as in Notation 3, and, let $\langle \mathbf{T}\rangle_i = \langle \mathbf{D}_{ii}\rangle - \widetilde{R}_i$. If $\langle \mathbf{T}\rangle_i > \widetilde{E}_i,\ \forall i \in N$ then $\rho(C_m) \leq \max\limits_{i\in\langle n\rangle}\left(\dfrac{\widetilde{F}_i}{\langle \mathbf{T}\rangle_i - \widetilde{E}_i}\right)$, where $\langle n\rangle = \{1, 2, \ldots, n\}$.

Proof. Let λ be an eigenvalue C_m. Choose $x \neq 0 \in \mathbb{R}^n$ such that

$$\begin{aligned} C_m x = \lambda x \quad &\Rightarrow \quad \langle \mathbf{T}_m - \mathbf{E}_m\rangle^{-1}|\mathbf{F}_m|x = \lambda x \\ &\Rightarrow \quad (\lambda\langle \mathbf{T}_m\rangle - \lambda|\mathbf{E}_m| - |\mathbf{F}_m|)x = 0 \\ &\Rightarrow \quad [(\lambda\langle \mathbf{D}\rangle - \lambda|\mathbf{R}_m| - \lambda|\mathbf{E}_m| - |\mathbf{F}_m|]\, x = 0 \\ &\Rightarrow \quad \left[\langle \mathbf{D}\rangle - |\mathbf{R}_m| - |\mathbf{E}_m| - \frac{1}{\lambda}|\mathbf{F}_m|\right] x = 0 \end{aligned}$$

Therefore the matrix $Q = \langle \mathbf{D}\rangle - |\mathbf{R}_m| - |\mathbf{E}_m| - \frac{1}{\lambda}|\mathbf{F}_m|$ is singular, which implies Q is not SDD. Hence there exists an $i \in N$ such that

$$\langle \mathbf{D}_{ii}\rangle \leq \widetilde{R}_i + \widetilde{E}_i + \left|\frac{1}{\lambda}\right| \widetilde{F}_i$$

After simplification we get

$$|\lambda| \leq \max_{i\in\langle n\rangle} \frac{\widetilde{F}_i}{(|\mathbf{D}_{ii}| - \widetilde{R}_i) - \widetilde{E}_i}$$

which implies that

$$\rho(C_m) \leq \max_{i\in\langle n\rangle} \frac{\widetilde{F}_i}{\langle \mathbf{T}\rangle_i - \widetilde{E}_i}$$

□

Note that above theorem provides a spectral upper bound of C_m and hence of the iteration matrix $\widetilde{G}$ of the GIGS method.

Lemma 4.6. If $\mathbf{A} = \mathbf{T}_m - \mathbf{E}_m - \mathbf{F}_m$ be an interval SDD-matrix, then $\mathbf{M}_m = \mathbf{T}_m - \mathbf{E}_m$ is an interval H-matrix.

Proof. As $\mathbf{A}$ is an interval SDD matrix, the comparison matrix $\langle \mathbf{A}\rangle$ of $\mathbf{A}$ satisfies $\langle \mathbf{A}_{ii}\rangle > \sum\limits_{j\neq i} |\mathbf{A}_{ij}|$, for all i. Then

$$\langle \mathbf{A}_{ii}\rangle > \sum_{j\neq i} |\mathbf{A}_{ij}| \geq \sum_{\substack{j\neq i \\ |i-j|\leq m}} |\mathbf{A}_{ij}| + \sum_{i>j+m} |\mathbf{A}_{ij}|$$

which shows that $\mathbf{M}_m$ is an interval SDD matrix, hence an interval H-matrix by Remark 3.5. □

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Theorem 4.7. GIGS method given in (2) converges for interval SDD matrix $\mathbf{A}$, for any initial guess.

Proof. Let $\mathbf{A}$ be an interval SDD matrix. Consider the splitting of $\mathbf{A}$ as mentioned in equation (14). Then $\mathbf{M}_m = \mathbf{T}_m - \mathbf{E}_m$ is an interval H-matrix by Lemma 4.6. Thus it suffices to show $\rho(C_m) < 1$.

Suppose that λ is an eigenvalue of C_m and $|\lambda| \geq 1$. Take $Q = \langle \mathbf{D} \rangle - |\mathbf{R}_m| - |\mathbf{E}_m| - \frac{1}{\lambda}|\mathbf{F}_m|$. Now $\frac{1}{|\lambda|} \leq 1$ and $\mathbf{A}$ is SDD imply that Q is SDD. Again as shown in Theorem 4.5, we can prove that $\det(Q) = 0$, which contradicts the fact that Q is SDD. Hence $\rho(C_m) < 1$ and thus the result holds for interval SDD matrix. □

The successive two theorems analyze the convergence of GIGS method for interval M-matrices.

Theorem 4.8. GIGS method for solving (1) converges for interval M-matrix $\mathbf{A}$.

Proof. Let $\mathbf{A}$ be an interval M-matrix, then $\langle \mathbf{A} \rangle$ is an M-matrix by Theorem 2.17(iii). Consider the decomposition of $\mathbf{A}$ and $\langle \mathbf{A} \rangle$ respectively, defined in (6) and (10). As $\mathbf{A} = \mathbf{M}_m - \mathbf{N}_m$ is interval M-matrix, there exists $v > 0$ such that $\mathbf{A}v > 0$ which signifies that $\mathbf{M}_m v > 0$, because $\mathbf{N}_m \geq 0$. Hence $\mathbf{M}_m$ is an interval M-matrix. We need to show $\rho(\widetilde{G}) < 1$.

Since $\langle \mathbf{A} \rangle = M_1 - N_1$, where M_1 and N_1 mentioned in equation (16), is an M-matrix, we can choose $u > 0$ such that $\langle \mathbf{A} \rangle u > 0$ which leads to $M_1 u > 0$ that shows M_1 is an M-matrix. By Definition 2.11, $\langle \mathbf{A} \rangle = M_1 - N_1$ is an M-splitting with nonsingular M_1, so Theorem 2.14 gives $\rho(C_m) < 1$. Also using Theorem 4.3 for interval M-matrix $\mathbf{M}_m$ we have $\rho(\widetilde{G}) = \rho(C_m)$. Thus we get $\rho(\widetilde{G}) = \rho(C_m) < 1$. □

Theorem 4.9. Let $\mathbf{A}$ be an interval M-matrix. Then $\rho(C_m) \leq \rho(C)$, for any $m \geq 1$.

Proof. By Lemma 4.6, $C = \langle \mathbf{M} \rangle^{-1} |\mathbf{N}|$ is a nonnegative matrix and hence by Perron-Frobenius Theorem $\rho(C)$ is an eigenvalue of C and there is an $x \geq (\neq) 0$ such that $Cx = \lambda x$, that is, $\lambda \langle \mathbf{M} \rangle x = |\mathbf{N}| x$.

Let us write $|\mathbf{E}| = |\mathbf{E}_m| + |\mathbf{R}_m^E|$ and $|\mathbf{F}| = |\mathbf{F}_m| + |\mathbf{R}_m^F|$. We now have that

$$
\begin{aligned}
\lambda \langle \mathbf{M}_m \rangle x &= \lambda \left(\langle \mathbf{D} \rangle - |\mathbf{E}_m| \right) x \\
&= \lambda \left(\langle \mathbf{D} \rangle - |\mathbf{E}| + |\mathbf{R}_m^E| \right) x \\
&= \lambda \langle \mathbf{M} \rangle x + \lambda |\mathbf{R}_m^E| x \\
&\geq |\mathbf{N}| x = |\mathbf{F}| x \\
&= \left(|\mathbf{F}_m| + |\mathbf{R}_m^F| \right) x \geq |\mathbf{F}_m| x = |\mathbf{N}_m| x
\end{aligned}
$$

As $\langle \mathbf{M}_m \rangle^{-1} \geq 0$, so we have that $\lambda x \geq \langle \mathbf{M}_m \rangle^{-1} |\mathbf{N}_m| x$ and hence by Theorem 2.15 we get

$$\rho(\mathbf{C}_m) = \rho\left(\langle \mathbf{M}_m \rangle^{-1} |\mathbf{N}_m|\right) \leq \lambda = \rho(C).$$

□

Note that Theorem 4.9 leads to the fact that if interval GS method converges, then GIGS method converges for any choice of bandwidth m.

Next theorem is for special case of interval M-matrices, which provides a comparison of spectral radii of iterative matrices of GIGS for different bandwidth. A similar result for AOR method for linear system was presented by Salkuyeh [5].

Theorem 4.10. Let $\mathbf{A}$ be an interval M-matrix. If $C_k = \langle \mathbf{M}_k \rangle^{-1} |\mathbf{N}_k|$, then for any $m \geq p$, $\rho(C_m) \leq \rho(C_p)$.

Proof. As $\mathbf{A}$ is interval M-matrix and so is $\langle \mathbf{M}_p \rangle$, so C_p is nonnegative matrix. Thus by Perron-Frobenius theorem we can choose an eigenvector $x \geq (\neq)0$ associated with the eigenvalue $\lambda = \rho(C_p)$, so that $C_p x = \lambda x$, that is, $(|\mathbf{N}_p| - \lambda \langle \mathbf{M}_p \rangle) x = 0$.

If we write $\mathbf{T}_p = \mathbf{D} + \mathbf{R}_p$ and $\mathbf{T}_m = \mathbf{D} + \mathbf{R}_m$, then

$$|\mathbf{A}| = |\mathbf{D}| + |\mathbf{R}_m| - |\mathbf{E}_m| - |\mathbf{F}_m| = |\mathbf{D}| + |\mathbf{R}_p| - |\mathbf{E}_p| - |\mathbf{F}_p|$$

which implies that $R + L + U = 0$, where $R = |\mathbf{R}_m| - |\mathbf{R}_p|$, $L = |\mathbf{E}_p| - |\mathbf{E}_m|$ and $U = |\mathbf{F}_p| - |\mathbf{F}_m|$. Since $m \geq p$, so we must have $R \leq 0,\ L \geq 0$ and $U \geq 0$. We now have that

$$\begin{aligned}
C_m x - \lambda x &= \langle \mathbf{M}_m \rangle^{-1} \left(|\mathbf{N}_m| - \lambda \langle \mathbf{M}_m \rangle\right) x \\
&= \langle \mathbf{M}_m \rangle^{-1} \left(|\mathbf{F}_m| - \lambda \langle \mathbf{D} \rangle + \lambda |\mathbf{R}_m| + \lambda |\mathbf{E}_m|\right) x \\
&= \langle \mathbf{M}_m \rangle^{-1} \left(|\mathbf{F}_p| - U - \lambda \langle \mathbf{D} \rangle + \lambda R + \lambda |\mathbf{R}_p| + \lambda |\mathbf{E}_p| - \lambda L\right) x \\
&= \langle \mathbf{M}_m \rangle^{-1} \left(|\mathbf{N}_p| - \lambda \langle \mathbf{M}_p \rangle\right) x - \langle \mathbf{M}_m \rangle^{-1} \left(U + \lambda L - \lambda R\right) x \\
&= -\langle \mathbf{M}_m \rangle^{-1} \left(U + \lambda L - \lambda R\right) x \\
&\leq 0
\end{aligned}$$

Hence $C_m x \leq \lambda x$ and hence $\rho(C_m) \leq \lambda = \rho(C_p)$ by Theorem 2.15. □

Following example validates the above two theorems.

Example 4.11. Consider the interval M-matrix

$$\mathbf{A} = \begin{pmatrix} 4 & [-1,0] & [-1,0] & [-1,0] \\ [-1,0] & 5 & [-1,0] & [-1,0] \\ [-1,0] & [-1,0] & 4 & [-1,0] \\ [-1,0] & [-1,0] & [-1,0] & 5 \end{pmatrix}$$

Then we have that $\rho(C) = 0.4640$, $\rho(C_1) = 0.2749 < 1$ and $\rho(C_2) = 0.1111 < 1$ that is $\rho(C_2) < \rho(C_1) < \rho(C) < 1$ which shows that the above results hold.

We conclude the section by checking the convergence property of GIGS method for interval linear system with coefficient matrix as interval H-matrices.

Theorem 4.12. GIGS method converges for interval H-matrices.

Proof. Let $\mathbf{A}$ be an interval H-matrix, then $\langle \mathbf{A} \rangle$ is an M-matrix. It suffices to prove $\rho(\widetilde{G}) < 1$, where $\widetilde{G} = |\mathbf{M}_m{}^{-1}| \cdot |\mathbf{N}_m|$.

As $\langle \mathbf{A} \rangle = M_1 - N_1$, with M_1 and N_1 are defined in (16), is an M-matrix, so as shown in Theorem 4.8, we can find $u > 0$ such that $M_1 u > 0$. Thus $\langle \mathbf{A} \rangle = M_1 - N_1$ is regular splitting with $M_1^{-1} \geq 0$ and $N_1 > 0$. Hence by Theorem 2.13, $\rho(M_1^{-1} N_1) < 1$. Since $\mathbf{M}_m$ is an interval H-matrix, we have $\rho(\widetilde{G}) \leq \rho(C_m)$ due to Theorem 4.3. Thus we get $\rho(\widetilde{G}) \leq \rho(C_m) < 1$ by Remark 3.4. Hence GIGS method converges for any initial guess. $\square$

However, for interval L-matrices GIGS and GIJ methods may not converge. For simplicity we consider examples of interval L-matrix with constant entries to see the convergence behavior of both the methods for interval L-matrices.

Example 4.13. Consider the interval L-matrix (with constant entries)

$$\mathbf{A} = \begin{pmatrix} 2 & -3 & -6 \\ -3 & 1 & -4 \\ -4 & -5 & 3 \end{pmatrix}$$

If $m = 1$ then for GIGS method we get $\rho(\widetilde{G}) = 1.0459 > 1$ and for GIJ method we get $\rho(\widehat{L}_m) = 2.0725 > 1$. Thus it shows GIGS method and GIJ method do not converge for $\mathbf{A}$.

Example 4.14. Consider the interval L-matrix (with constant entries)

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -1 \\ -3 & 2 & -3 \\ -2 & -1 & 2 \end{pmatrix}$$

If $m = 1$ then GIGS method gives $\rho(\widetilde{G}) = 0.6364 < 1$ but GIJ method gives $\rho(\widehat{L}_m) = 1$. In this case GIGS converges but GIJ method doesn't converge for $\mathbf{A}$.

5 Numerical Illustration

In this section numerical examples are considered to compare the convergence of generalized interval Jacobi method and generalized interval Gauss-Seidel method. In particular, examples are considered with coefficient matrix $\mathbf{A}$ as an interval SDD matrix, interval M-matrix, and an interval H-matrix. The computations are carried out in MATLAB(2021b) with the interval toolbox INTLAB v12 [28] and on a PC-Intel(R) Core(TM) i7-5700U CPU @1.80 GHz, 8 GB RAM. The computations are rounding to four digits and the stopping criteria is chosen as $\|\mathsf{qdist}(\mathbf{x}^{(k+1)}, \mathbf{x}^{(k)})\| \le 10^{-6}$, where $\mathsf{qdist}(\mathbf{x}, \mathbf{y}) := \max\{|\underline{x} - \underline{y}|, |\overline{x} - \overline{y}|\}$ represents a measure of distance between the intervals $\mathbf{x} = [\underline{x}, \overline{x}]$ and $\mathbf{y} = [\underline{y}, \overline{y}]$, and in case $\mathbf{x},\ \mathbf{y} \in \mathbb{IR}^n$, then $\mathsf{qdist}(\mathbf{x}, \mathbf{y}) = [\mathsf{qdist}(\mathbf{x}_1, \mathbf{y}_1), \ldots, \mathsf{qdist}(\mathbf{x}_n, \mathbf{y}_n)] \in \mathbb{R}^n$, where $\mathbf{x}_i$ represents the i-th entry of the interval vector $\mathbf{x}$.

We now furnish examples for various classes of interval coefficient matrices, and provide comparisons of our proposed methods with IJ and IGS methods , in terms of no. of iterations, time (in seconds) and $r_k = \|(\mathsf{qdist}(\mathbf{x}^{(\mathsf{k})}, \mathbf{x})\|$, where $\mathbf{x}$ is the enclosure obtained by using verifylss.

Example 5.1. Consider the interval linear system (1) with the coefficient interval matrix as interval strictly diagonally dominant matrix mentioned in Neumaier book [3] .

$$\mathbf{A} := \begin{pmatrix} 3 & [-2, 2] & 0 \\ 0 & 3 & [-2, 2] \\ [-2, 2] & 0 & 3 \end{pmatrix}$$

and

$$\mathbf{b} := \begin{pmatrix} [-1, 1] \\ [-1, 1] \\ 2 \end{pmatrix}$$

Then the function verifylss from the package INTLAB produces the enclosure

$$\mathbf{x} = ([-1.2282, 1.2282], [-1.3423, 1.3423], [-0.1599, 1.4932])^T$$

Taking the initial guess as $\mathbf{x}_0 = ([0, 3], [0, 3], [0, 3])^T$, generalized interval Jacobi converges after 13 iterations and yields the enclosure

$$\mathbf{x}_{\mathrm{GIJ}} = ([-1.2106, 1.2106], [-1.3158, 1.3158], [-0.1404, 1.4737])^T$$

whereas the generalized interval Gauss-Seidel converges after 1 iteration and produces the enclosure

$$\mathbf{x}_{\mathrm{GIGS}} = ([-1.2398, 1.2398], [-1.3481, 1.3481], [-0.1714, 1.5048])^T$$

Table 1 compares GIJ and GIGS methods (taking $m = 1$) with IJ and IGS methods (with $m = 0$) with the initial guess taken as $\mathbf{x}_0 = ([0, 3], [0, 3], [0, 3])^T$.

Iterative method	No. of iterations	r	time in second
GIJ	13	0.0372	0.0225
GIGS	1	0.0173	0.0147
IJ	35	0.0372	0.0437
IGS	24	0.0372	0.0237

Table 1: Numerical result for the interval SDD-matrix with $m = 1$

From the above table we can see that the generalized interval Jacobi method gives the tightest solution set enclosure.

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Example 5.2. Consider the interval linear system (1) with the coefficient interval M-matrix

$$\mathbf{A} := \begin{pmatrix} 4 & [-2,0] & [-1,0] \\ [-1,0] & [3,4] & -1 \\ [-2,-1] & [-1,0] & 5 \end{pmatrix}$$

and

$$b := \begin{pmatrix} 1.2 \\ 1.5 \\ 5 \end{pmatrix}$$

then the verifylss function yields the enclosure

$$\mathbf{x} = ([-0.2935, 1.6658], [0.0677, 1.7258], [0.5795, 2.0115])^T$$

Taking initial guess as $\mathbf{x}_0 = ([-1,2], 1, [2,3])^T$, generalized interval Jacobi converges after 16 iterations and provides the enclosure

$$\mathbf{x}_{\text{GIJ}} = ([-0.2935, 1.6658], [0.0677, 1.7258], [0.5795, 2.0115])^T$$

whereas generalized interval Gauss-Seidel converges after 9 iterations and gives the enclosure

$$\mathbf{x}_{\text{GIGS}} = ([-0.2936, 1.6658], [0.0676, 1.7259], [0.5795, 2.0116])^T$$

Table 2 provides a comparison of GIJ and GIGS methods (taking $m = 1$) with IJ and IGS methods (with $m = 0$) with the initial guess taken as $\mathbf{x}_0 = ([-1,2], 1, [2,3])^T$.

Iterative method	No. of iterations	r	time in seconds
GIJ	16	3.23×10^{-5}	0.0272
GIGS	9	1.25×10^{-4}	0.0211
IJ	30	2.38×10^{-5}	0.0443
IGS	18	2.46×10^{-5}	0.0236

Table 2: Numerical result for the interval M-matrix with $m = 1$

This shows that in case of interval M-matrices, generalized interval Jacobi method gives the tightest enclosure of the solution set.

Example 5.3. Consider the interval linear system (1) with the following coefficient interval H-matrix

$$\mathbf{A} := \begin{pmatrix} [4,5] & [-2,2] & [-1,0] \\ [0,1] & [3,5] & [-1,1] \\ [-1,1] & [1,3] & 5 \end{pmatrix}$$

and

$$\mathbf{b} := \begin{pmatrix} [0.1, 0.5] \\ [-0.3, -0.1] \\ [0.3, 0.4] \end{pmatrix}$$

Then the function verifylss from the package INTLAB generates the enclosure

$$\mathbf{x} = ([-0.2425, 0.3967], [-0.3567, 0.2375], [-0.1857, 0.3734])^T$$

Taking the initial guess as $\mathbf{x}_0 = (1, 5, 4)^T$, generalized interval Jacobi converges after 15 iteartions and yields the enclosure

$$\mathbf{x}_{\text{GIJ}} = ([-0.2426, 0.3968], [-0.3567, 0.2375], [-0.1857, 0.3734])^T$$

whereas the generalized interval Gauss-Seidel converges after 9 iterations and produces the enclosure

$$\mathbf{x}_{\text{GIGS}} = ([-0.2425, 0.3967], [-0.3567, 0.2374], [-0.1857, 0.3734])^T$$

We now produce a comparison in Table 3 of GIJ and GIGS methods (taking $m = 1$) with IJ and IGS methods (with $m = 0$) with the initial guess taken as $\mathbf{x}_0 = (1, 5, 4)^T$.

Iterative method	No. of iterations	r	time in seconds
GIJ	15	3.96×10^{-5}	0.0247
GIGS	9	9.73×10^{-6}	0.0209
IJ	45	1.11×10^{-5}	0.0475
IGS	24	1.27×10^{-5}	0.0294

Table 3: Numerical result for the interval H-matrix with $m = 1$

For this example the most tightest enclosure of solution set provided by the generalized interval Gauss-Seidel method.

6 Conclusion

In this paper, we proposed a generalized interval Jacobi (GIJ) method and generalized interval Gauss-Seidel method (GIGS). These methods are generalization of interval Jacobi and interval Gauss-Seidel methods, discussed by Neumaier [3, 25] to solve interval linear system. The GIJ and GIGS methods are proposed similar to that introduced by Salkuyeh in [4], by generalizing the diagonal interval matrix to a band interval matrix. We proved that both the proposed methods converge for interval SDD matrix, interval M-matrix, and for interval H-matrix. Further we found that for interval M-matrices, GIGS method converges for any choice of bandwidth m if interval GS method converges. At last we consider numerical examples to observe that GIJ gives a tighter enclosure for interval M- coefficient matrices, whereas GIGS provides a tighter enclosure of the solution set for interval H-matrices. This leads to the open problem that the same can be concluded in general.

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Extended Jacobi Elliptic Function Technique: A Tool for Solving Nonlinear Wave Equations with emblematic Software

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Abstract

The Extended Jacobi Elliptic Function Technique (EJET) is a powerful technique for finding the solutions for traveling waves form coming from Non-Linear Waveguides (NLWs). As a result, solitary and shock-wave profiles are obtained simultaneously with corresponding amplitudes and speeds by this method for three types of nonlinear wave equations. A class of nonlinear wave equations of particular interest in mathematical physics have been used to investigate the legality and credibility of this technique. A short script is considered a symbolic software package that calculates traveling wave solutions in exact form.

Key words: Extended Jacobi Elliptic Function Technique; Traveling Waves; solitary and shock-wave profiles; Symbolic Software
Mathematics Subject Classification(2010): 35M10, 65Z05.

[1]

1 Introduction

Nonlinearity is a mesmerizing component of nature, with nonlinear wave phenomena appearing in one way or another in nearly all scientific and engineering fields such as physics (Plasma and Fluid), Ocean Engineering, Chemical Dynamics, Geochemistry and mathematical biology (Population Dynamics) [[1]-[5]] .The nonlinear equations appear in different scenarios in daily real-life situations and very difficult to solve it [[6]-[8]]. Manny method are used to find the solutions (solitary and shock-wave solution) of nonlinear wave phenomena like Tanh-Coth Method [[5],[9]], Expansion method [[10]-[13]] the decomposition method with Integral transformation [[14]-[16]] and so on.
The development of the present paper is as follows. In Section 2, we have outline

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of EJET for solving NLW. In Section-3 (Application 3.1), we apply this Technique to the second order nonlinear partial differential equations (SONLPDE). And also applied in Section 4 (Application 3.2) K.G. equation and In Section 5 (Application 3.3) Population Dynamics equation. In section 6 discussion and numerical Sketch and in section 6 result and conclusions.

2 Outline of Extended Jacobi Elliptic Function Technique

We now present a brief strategy of the technique. Given non-linear wave equation

$$\Re\left(v, v_t, v_x, v_{tt}, v_{xx} \ldots.\right)=0 \tag{2.1}$$

can be converted to an ordinary differential equation (ODE)

$$P\left(v, v', v'', v''' \ldots.\right)=0 \tag{2.2}$$

upon using a wave variable $z=\alpha\left(x-ct\right)$where α and c are wave number and wave speed respectively. Introducing a new independent variable

$$v(x,t)=v\left(\alpha(x-ct)\right)=v(z)$$

By the Jacobi elliptic function expansion method, $v\left(z\right)$ can be expressed as a finite series in the form of Jacobi elliptic functions[[10],[17]-[18]] ,

$$u\left(z\right)=\sum_{i=0}^{n} \lambda_i (\psi\left(z\right))^i \tag{2.3}$$

is prepared and its highest degree is $O\left\{\psi\left(z\right)\right\}=n.$ where

$$\psi=\psi\left(z\right)$$

satisfies the eq.(2.1) the following auxiliary equation:

$$\frac{\partial}{\partial z}\left(\psi\left(z\right)\right)=\psi'\left(z\right)=\kappa\sqrt{p\,\psi^4(z)+q\,\psi^2\left(z\right)+r} \tag{2.4}$$

Where $\kappa=\pm 1$ and p, q and r are constants. It holds for $\psi\left(z\right)$ as

$$\left.\begin{array}{l}
\frac{\partial^2}{\partial z^2}\left(\psi\left(z\right)\right)=\psi''=2p\,\psi^3(z)+q\,\psi(z)\\
\frac{\partial^3}{\partial z^3}\left(\psi\left(z\right)\right)=\psi'''=\left(6p\,\psi^2(z)+q\,\right)\psi'(z)\\
\frac{\partial^4}{\partial z^4}\left(\psi\left(z\right)\right)=\psi''''(z)=24p^2\psi^5(z)+20p\,q\,\psi^3(z)+\left(12pr+q^2\right)\psi(z)\\
\frac{\partial^5}{\partial z^5}\left(\psi\left(z\right)\right)=\psi'''''(z)=\left(120p^2\psi^4(z)+60p\,q\,\psi^2(z)+12pr+q^2\right)\psi'(z)\\
\ldots
\end{array}\right\} \tag{2.5}$$

we present many closed solutions for eq.(2.4) . In fact, these solutions as $\psi(z)$ = Jacobi elliptic functions, can be casted to hypothesis for more solutions eq.(2.1).$sn(\xi) = sn(\xi, m)$, $dn(\xi) = dn(\xi, m)$and $cn(\xi) = cn(\xi, m)$are the Jacobi elliptic function with modulus m, where $0 < m < 1$. These functions are considerable the resulting formulas:

$$sn^2(\xi) + cn^2(\xi) = 1,\ dn^2(\xi) + m^2 sn^2(\xi) = 1$$
$$sn(\xi) = \frac{1}{ns(\xi)},\ cn(\xi) = \frac{1}{nc(\xi)},\ dn(\xi) = \frac{1}{nd(\xi)}$$
$$cs(\xi) = \frac{sn(\xi)}{cn(\xi)},\ ds(\xi) = \frac{dn(\xi)}{sn(\xi)},\ sd(\xi) = \frac{ns(\xi)}{dn(\xi)}$$
$$sn'(\xi) = \frac{d(sn(\xi))}{d\xi} = cn(\xi)\, dn(\xi), dn'(\xi) = \frac{d(dn(\xi))}{d\xi} - m^2 cn(\xi)\, sn(\xi),$$
$$cn'(\xi) = \frac{d(cn(\xi))}{d\xi} - dn(\xi)\, sn(\xi)$$

when $m \to 1$; These functions convert to hyperbolic functions as follows:

$$sn(\xi) \to \tanh(\xi),\ \{cn(\xi), dn(\xi)\} \to \mathrm{sech}(\xi),$$
$$\{sc(\xi), sd(\xi)\} \to \sinh(\xi), \{cd(\xi),\ dc(\xi)\} \to 1$$
$$\{ds(\xi), cs(\xi)\} \to \cos ech(\xi), \{nc(\xi), nd(\xi)\} \to \cosh(\xi), \{ns(\xi)\} \to \coth(\xi)$$

when $m \to 0$ These functions convert to trigonometric functions as follows:

$$\{sn(\xi), sd(\xi)\} \to \sin(\xi),\ \{cn(\xi), cd(\xi)\} \to \cos(\xi),$$
$$\{sc(\xi)\} \to \tan(\xi), \{dn(\xi),\ nd(\xi)\} \to 1$$
$$\{ns(\xi), ds(\xi)\} \to Cosec(\xi), \{cs(\xi)\} \to \cot(\xi) \{nc(\xi), dc(\xi)\} \to \mathrm{Sec}(\xi)$$

Its balancing the highest order derivative term and the nonlinear term and find the value of n in eq. (2.3).

3 Application -3.1

We consider the second order nonlinear partial differential equations with combination Kortewegde Vries (KdV) Equation and BenjaminBonaMahony equation (BBM) Equation of two famous and fundamental nonlinear wave equations. This is as

$$\theta_t + a\,\theta_x + \theta\theta_x + b^2\theta_{x\,x\,x} - c^2\theta_{x\,x\,t} = 0 \tag{3.1}$$

Where $\theta = \theta(x, t)$ unknow wave function with space variable x and time variable t. a, b and c are arbitrary real constant. If $a = 0, c = 0$ then eq. (3.1) is Kortewegde Vries (KdV) Equation, this is one of the most famous non-linear wave equations, it was derived in fluid mechanics to describe shallow water waves in a rectangular channel [[1],[28]]. If $b = 0$ then eq. (3.1) is Benjamin-BonaMahony equation (BBM) Equation, also called regularized long-wave equation (RLWE), this serves as an approximate model in studying the dynamics of small-amplitude surface water waves propagating unidirectionally [[1]]. Suppose that the travelling wave solutions for eq. (3.1) are of the forms as follows

$$\theta(x, t) = \theta(z) = \theta(k(x - \omega t)$$

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where k and ω area constant, put in eq. (3.1) then

$$k\,(a-\omega)\;\theta' + k\,\theta\,\theta' + k^3\left(b^2 + c^2\omega\right)\theta''' = 0$$

Integral one time, take constant zero

$$k\,(a-\omega)\;\theta + \frac{k^2}{2}\theta^2 + k^3\left(b^2 + c^2\omega\right)\theta'' = 0 \tag{3.2}$$

Balancing θ'' with θ^2 in eq. (3.2) gives $2n = n+2$ i.e.,$n = 2$, then

$$\theta\,(z) = \sum_{i=0}^{2} \lambda_i(\psi\,(z))^i = \lambda_0 + \lambda_1\,\psi\,(z) + \lambda_2\,(\psi\,(z))^2 = \lambda_0 + \lambda_1\,\psi + \lambda_2\,\psi^{\,2}$$

$$\theta''\,(z) = \lambda_1\,\psi'' + 2\lambda_2\,\left\{\psi'^{\,2} + \psi\,\psi''\right\}$$

Put these values in eq. (3.2) with eq. (2.5)

$$\begin{aligned}&(a-\omega)\left(\lambda_0 + \lambda_1\,\psi + \lambda_2\,\psi^2\right)\\ &+\frac{k^2}{2}\left(\lambda_0^2 + \lambda_1^2\,\psi^2 + \lambda_2^2\,\psi^2 + 2\lambda_0\lambda_1\,\psi + 2\lambda_1\lambda_2\,\psi^3 + 2\lambda_0\lambda_2\,\psi^2\right)\\ &+k^3\left(b^2 + c^2\omega\right)\left\{2\lambda_2 R + \lambda_1\,q\,\psi + 2\lambda_2 q\,\psi^2 + \psi^3\,(2\lambda_1 p + 2\lambda_2 q) + \psi^4 6\lambda_2 p\right\} = 0\end{aligned} \tag{3.3}$$

equating all terms with the powers in ψ , and setting each of the obtained coefficients for ψ to zero, yields the following set of algebraic equations for $\lambda_0, \lambda_1, \lambda_2, k, \omega, a, b\; and\, c$

$$\begin{aligned}\psi^0:&\quad \lambda_0 k\,(a-\omega) + \tfrac{k^2\lambda_0^2}{2} + 2\lambda_2\,r\,k^3\left(b^2 + c^2\omega\right) = 0\\ \psi^1:&\quad \lambda_1 k\,(a-\omega) + \tfrac{k^2\lambda_0\lambda_1}{2} + \lambda_1\,q\,k^3\left(b^2 + c^2\omega\right) = 0\\ \psi^2:&\quad \lambda_2 k\,(a-\omega) + \tfrac{k^2\left(\lambda_1^2 + 2\lambda_0\lambda_2\right)}{2} + 2\lambda_2\,q\,k^3\left(b^2 + c^2\omega\right) = 0\\ \psi^3:&\quad 2\lambda_2\lambda_1 + k^3\left(b^2 + c^2\omega\right)(2\lambda_1 p + 2\lambda_2 q) = 0\\ \psi^4:&\quad \lambda_2^2 + 6\lambda_2 p k^3\left(b^2 + c^2\omega\right) = 0\end{aligned}$$

One obtains solution

$$\lambda_0 = -\tfrac{k^2\left(b^2 + c^2\omega\right)q + (a-\omega)}{k},\quad \lambda_1 = k^2\left(b^2 + c^2\omega\right)\sqrt{-12p\,q}$$

$$\lambda_2 = -6k^3\left(b^2 + c^2\omega\right)p$$

then

$$\begin{aligned}\theta\,(z) = &-\frac{k^2\left(b^2 + c^2\omega\right)q + (a-\omega)}{k}\\ &+\left\{k^2\left(b^2 + c^2\omega\right)\sqrt{-12p\,q}\right\}\,\psi(z)\\ &-\left\{6k^3\left(b^2 + c^2\omega\right)p\right\}\,(\psi(z)\,)^2\end{aligned}$$

We choose p q and r from [[17],[18]], such that
Solution -1 $p\;:\;\; m^2;\;\; q:\;\; -\left(1+m^2\right)\;\;\; then\; \psi(z) = \;sn(z)$ thus

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$$\theta(z) = -\frac{k^2\left(b^2+c^2\omega\right)\left(1+m^2\right)-(a-\omega)}{k} + \left\{k^2\left(b^2+c^2\omega\right)\sqrt{12m^2\left(1+m^2\right)}\right\} sn(z) - \left\{6k^3\left(b^2+c^2\omega\right)m^2\right\}\left(sn(z)\right)^2$$

Solution -2 $p\ :\ -m^2\ ,\ q:\ \left(2m^2-1\right),\ then\ \psi(z) = cn(z)$ thus

$$\theta(z) = -\frac{k^2\left(b^2+c^2\omega\right)\left(2m^2-1\right)+(a-\omega)}{k} + \left\{k^2\left(b^2+c^2\omega\right)\sqrt{12m^2\left(2m^2-1\right)}\right\} cn(z) + \left\{6k^3\left(b^2+c^2\omega\right)m^2\right\}\left(cn(z)\right)^2$$

Solution -3 $p\ :\ -\frac{1}{4}\ ,\ q:\ \left(\frac{1+m^2}{2}\right),\ r:\ \left(\frac{1-m^2}{2}\right)^2,\ then\ \psi(z) = mcn(z) \pm dn(z)$ thus

$$\theta(z) = -\frac{k^2\left(b^2+c^2\omega\right)\left(1+m^2\right)+2(a-\omega)}{2k} + \left\{k^2\left(b^2+c^2\omega\right)\sqrt{3\left(\frac{1+m^2}{2}\right)}\right\}\left(mcn(z) \pm dn(z)\right) + \left\{\frac{3k^3\left(b^2+c^2\omega\right)}{2}\right\}\left(mcn(z) \pm dn(z)\right)^2$$

Solution -4 $p\ :\ \frac{m^2}{4}\ ,\ q:\ \left(\frac{m^2-2}{2}\right),\ then\ \psi(z) = sn(z) + i\,cn(z)$ thus

$$\theta(z) = -\frac{k^2\left(b^2+c^2\omega\right)\left(m^2-2\right)+2(a-\omega)}{2k} + \left\{k^2\left(b^2+c^2\omega\right)\sqrt{3m^2\left(\frac{2-m^2}{2}\right)}\right\}\left\{sn(z) \pm i\,cn(z)\right\} - \left\{\frac{3m^2k^3\left(b^2+c^2\omega\right)}{2}\right\}\left(sn(z) \pm i\,cn(z)\right)^2$$

4 Application -3.2

We consider nonlinear KleinGordon (NKG) [[19]-[20]]. The Klein-Gordon equations play a significant role in solid state physics, plasma physics, nonlinear optics and quantum field theory

$$\theta_{tt} - \theta_{xx} + \theta + \beta|\theta|^2\theta = 0 \tag{4.1}$$

the travelling wave solutions for Eq. (4.1) are of the forms as follows:

$$\theta(x,t) = \theta(z)\,e^{i(\gamma\ (\omega x-t))} = \theta\left(k(x-\omega t\right)e^{i(\gamma\ (\omega x-t))}$$

where k and ω area constant, put in eq. (2.1) then

$$\left(k^2\omega^2 - k^2\right)\theta'' + \left\{\gamma^2\left(\omega^2-1\right)+1\right\}\theta + \beta\theta^3 = 0 \tag{4.2}$$

Balancing θ'' with θ^3 in eq. (4.2) gives $3n = n+2$ i.e.,$n = 1$, then

$$\theta(z) = \sum_{i=0}^{1}\lambda_i(\psi(z))^i = \lambda_0 + \lambda_1\,\psi(z) = \lambda_0 + \lambda_1\,\psi$$

Put these values in eq. (4.2)

$$\theta''(z) = \lambda_1 \, \psi''$$

Using eq. (2.5) and equating all terms with the powers in ψ , and setting each of the obtained coefficients for ψ to zero, yields set of algebraic equations for $\lambda_0, \lambda_1 k, \omega \, and \, \gamma$,

One obtains solution

$$\lambda_0^2 = -\frac{k^2\left(\omega^2-1\right)q+\left\{\gamma^2\left(\omega^2-1\right)+1\right\}}{3\beta}, \quad \lambda_1^2 = -\frac{2k^2\left(\omega^2-1\right)p}{\beta}$$

Then

$$\theta(z) = \sqrt{-\frac{k^2(\omega^2-1)q+\{\gamma^2(\omega^2-1)+1\}}{3\beta}} + \left\{\sqrt{-\frac{2k^2(\omega^2-1)p}{\beta}}\right\} \psi(z)$$

We choose p, q and r from [[17],[18]], such that

Solution -2.1 $p \ : \ m^2 \ , \quad q: \ -\left(1+m^2\right) \ , \ then \ \psi(z) = \ sn(z)$ thus

$$\theta(z) = \left\{ \begin{array}{l} \sqrt{-\dfrac{\{\gamma^2(\omega^2-1)+1\} - k^2(\omega^2-1)(1+m^2)}{3\beta}} \\ +\left\{\sqrt{-\dfrac{2k^2(\omega^2-1)m^2}{\beta}}\right\} sn\left(k(x-\omega t)\right) \end{array} \right\} e^{i(\gamma \ (\omega x - t))}$$

Solution -2.2 $p \ : \ -m^2 \ , \quad q: \ \left(2m^2-1\right) \ , \ then \ \psi(z) = \ cn(z)$ thus

$$\theta(z) = \left\{ \begin{array}{l} \sqrt{-\dfrac{k^2(\omega^2-1)(2m^2-1) + \{\gamma^2(\omega^2-1)+1\}}{3\beta}} \\ +\left\{\sqrt{-\dfrac{2k^2(1-\omega^2)m^2}{\beta}}\right\} cn\left(k(x-\omega t)\right) \end{array} \right\} e^{i(\gamma \ (\omega x - t))}$$

Solution -2.3 $p \ : \ \left(\frac{1-m^2}{4}\right) \ , \quad q: \ \left(\frac{1+m^2}{2}\right) \ , \ , \ then \ \psi(z) = \frac{cn(z)}{1 \pm \, sn(z)}$ thus

$$\theta(z) = \left\{ \begin{array}{l} \sqrt{-\dfrac{k^2(\omega^2-1)(1+m^2) \, + 2\{\gamma^2(\omega^2-1)+1\}}{6\beta}} \\ +\left\{\sqrt{-\dfrac{2k^2(\omega^2-1)(1-m^2)}{4\beta}}\right\} \left(\dfrac{cn\left(k(x-\omega t)\right)}{1 \pm \ sn(k(x-\omega t))}\right) \end{array} \right\} e^{i(\gamma \ (\omega x - t))}$$

Solution -2.4 $p \ : \ \frac{m^2}{4} \ , \quad q: \ \left(\frac{m^2-2}{2}\right), \ then \ \psi(z) = \ sn(z) + i \, cn(z)$ thus

$$\theta(z) = \left\{ \begin{array}{l} \sqrt{-\dfrac{\{2\gamma^2(\omega^2-1)+1\} + k^2(\omega^2-1)(m^2-2)}{6\beta}} \\ +\left\{\sqrt{-\dfrac{2k^2(\omega^2-1)m^2}{4\beta}}\right\} (\, sn(z) + i \, cn(z)) \end{array} \right\} e^{i(\gamma \ (\omega x - t))}$$

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5 Application -3.3

We consider Fisher equation

$$\theta_t = \delta_1 \, \theta_{xx} + \delta_2 \, \theta \, (1 - \theta) \tag{5.1}$$

introduced by Fisher [[21]] to describe the propagation of a virile mutant in an infinitely long habitat. It also represents a model equation for the evolution of a neutron population in a nuclear reactor [[22]-[23]] and a prototype model for a spreading flame [[24]-[25]]. The travelling wave solutions for Eq. (5.1) are of the forms as follows:

$$\theta\,(x,t) = \theta\,(z) = \theta\,(k(x - \omega t)$$

where k and area constant, put in eq. (5.1) then

$$k\omega\theta' + k^2\delta_1 \, \theta'' + \delta_2 \, \theta - \delta_2 \, \theta^2 = 0 \tag{5.2}$$

Balancing θ'' with θ^2 in eq. (5.2) gives $2n = n + 2$ i.e.,$n = 2$, then

$$\theta\,(z) = \sum_{i=0}^{2} \lambda_i (\psi\,(z))^i = \lambda_0 + \lambda_1 \, \psi\,(z) + \lambda_2 \, (\psi\,(z))^2 = \lambda_0 + \lambda_1 \, \psi + \lambda_2 \, \psi^{\,2}$$

$$\theta''\,(z) = \lambda_1 \, \psi'' + 2\lambda_2 \, \left\{\psi'^{\,2} + \psi \, \psi''\right\}$$

Put these values in eq. (5.2)

$$(k\omega + \delta_2) \, \left(\lambda_0 + \lambda_1 \, \psi + \lambda_2 \, \psi^{\,2}\right) + k^2\delta_1 \, \left(\lambda_1 \, \psi'' + 2\lambda_2 \, \left\{\psi'^{\,2} + \psi \, \psi''\right\}\right)$$
$$- \delta_2 \left(\lambda_0 + \lambda_1 \, \psi + \lambda_2 \, \psi^{\,2}\right)^2 = 0 \tag{5.3}$$

Using eq. (2.5) and collecting the coefficients of the same power $\psi^i\,(z)(\psi'(z))^j$ $(j = 0, 1 \; i = 0, 1, 2, 3, 4...)$ and setting each of the attained coefficients to be zero we have a set of over determined algebraic equations for $\lambda_0, \lambda_1, \lambda_2, k, \omega, \delta_1 \; and \, \delta_2$. One obtains solution

$$\lambda_0 = \tfrac{4k^2\delta_1 q + \delta_2}{2\delta_2}, \quad \lambda_2 = \tfrac{4k^2\delta_1 p}{\delta_2}, \quad \lambda_1 = 2\lambda_2$$

Then

$$\theta\,(z) = \tfrac{4k^2\delta_1 q + \delta_2}{2\delta_2} + \tfrac{4k^2\delta_1 p}{\delta_2} \, \psi(z) + \tfrac{4k^2\delta_1 p}{2\delta_2} \, \psi^2(z)$$

We choose p q and r from [[17]-[18]], such that
Solution -3.1 $p \; : \; m^2 \;, \quad q : \; -\left(1 + m^2\right) \;, \; then \; \psi(z) = \; sn(z)$ thus

$$\theta\,(z) = \tfrac{\delta_2 - 4k^2\delta_1\left(1 + m^2\right)}{2\delta_2} + \tfrac{4k^2\delta_1 m^2}{\delta_2} \, sn(z) + \tfrac{4k^2\delta_1 m^2}{2\delta_2} \, sn^2(z)$$

Solution -3.2 $p \; : \; -m^2 \;, \quad q : \; \left(2m^2 - 1\right) \;, \; then \; \psi(z) = \; cn(z)$ thus

$$\theta\,(z) = \tfrac{4k^2\delta_1\left(2m^2 - 1\right) + \delta_2}{2\delta_2} - \tfrac{4k^2\delta_1 m^2}{\delta_2} \, cn(z) - \tfrac{4k^2\delta_1 m^2}{2\delta_2} \, cn^2(z)$$

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Solution -3.3 $p\ :\ \left(\frac{1}{4}\right)\ ,\ \ q:\ \left(\frac{1-2m^2}{2}\right)\ ,\ \ ,\ then\ \psi(z) = msn(z) \pm i\, dn(z)$thus

$$\theta\left(z\right) = \frac{2k^2\delta_1\left(1-2m^2\right)+\delta_2}{2\delta_2} + \frac{k^2\delta_1}{\delta_2}\left\{m\, sn(z) \pm i\, dn(z)\right\} + \frac{k^2\delta_1}{2\delta_2}\left\{m\, sn(z) \pm i\, dn(z)\right\}^2$$

Solution -3.4 $p\ :\ 1\ ,\ \ q:\ \left(2-4m^2\right)\ ,\ \ ,\ then\ \psi(z) = \frac{sn(z)dn(z)}{cn(z)}$thus

$$\theta\left(z\right) = \frac{8k^2\delta_1\left(1-2m^2\right)+\delta_2}{2\delta_2} + \frac{4k^2\delta_1}{\delta_2}\frac{sn(z)dn(z)}{cn(z)} + \frac{4k^2\delta_1}{2\delta_2}\left\{\frac{sn(z)dn(z)}{cn(z)}\right\}^2$$

6 Discussion and Numerical Sketch

It should be noted that, although many exact solutions are obtained in this work, it has been proved that some solutions in applications 3.1, 3.2 and 3.3 are equivalent to the solution of in the literature. like solution for 2.1 of application 3.2 [[26]] and solution for 3.2 of application 3.3 [[27]].

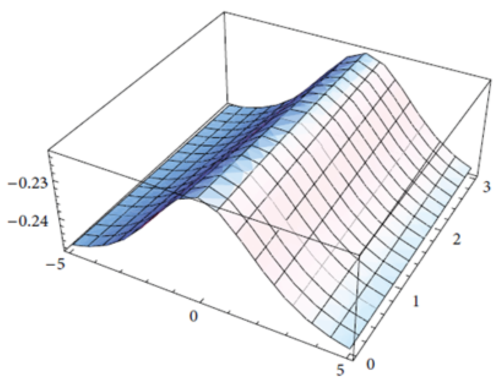


Figure 1: Travelling waves solution for 2 of 3.1 are plotted: bright solitary waves $m \to 1$

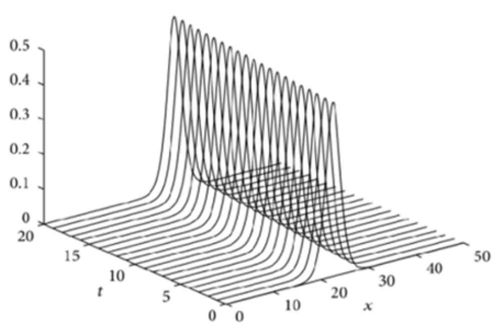
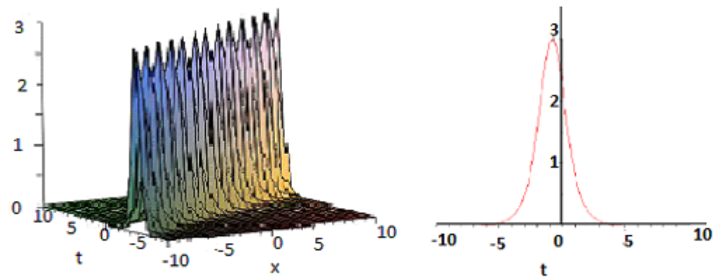


Figure 2: Soliton solution for 2.1 of 3.2 are plotted: solitary waves,$m \to 1$



(a) Shape of in 3D (b) Shape of in 2D

Figure 3: Soliton solution for 3.1 of 3.3 are plotted: solitary waves,$m \to 1$

7 Result and Conclusions

The Extended Jacobi Elliptic Function Technique has been successfully applied to obtain exact solution for three nonlinear wave equations. Moreover, the soliton-like solutions and trigonometric-function solutions have been also obtained as limiting cases on Jacobi Elliptic Function as $m \to 1$ and $m \to 0$.All solutions were verified by Maple package program and fig. (1), fig. (2) and fig. (3) are also new solitary wave solution for eq. (6), eq. (9) and eq. (11) respectively.

The main advantage of this method over other methods is that it provides exact solutions for all types, including Jacobian-elliptic functions. Finally, it is pertinent to mention that the proposed method is also a straightforward, short, promising and powerful method for other nonlinear evolution equations in mathematical physics. The algorithm of the method is very applicative and influential to investigate many solutions.

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Laplace Variational Iteration Method for Solving fractional Wave like Equations

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30. december 2022

This paper introduces the latest procedure for explaining certain types of fractional wave equations using the variation iteration method (VIM) and Laplace transform. The Laplace variation iteration method is a type of semi-analytical technique applied to both linear and non-linear equations without requiring linearization, discretization, or perturbation. It is not a time-consuming method and converses the solution rapidly with the exact and less error solution. This approach is delineated and then explained through several example cases. The outcomes demonstrate that this alternate strategy yields reliable outcomes and the results are displayed graphically.

1 Introduction

Mathematics, engineering, and sciences are full of amazing phenomena that can be precisely described by using mathematical techniques from fractional calculus, such as the perception of fractional order derivatives and integrals [6,14,15,19]. Differential equations of fractional order [25,26,27,28] have been gaining a lot of attention newly owing to the precise understanding of nonlinear phenomena.

The Wave equations are the linear partial differential equations of the second order. This equation describes the waves, which are a common part of classic physics. These include water waves, sound waves, and light waves. Over the last few years, there has been a new application of wave-like models to physical problems. These models can be used in different fields [2,12,16,17]. Due to the importance of wave-like equations, many researchers [2,20,21] have considered solutions to these equations. For the current issue, we take into consideration the following fractional wave equations with variable

$$D_t^\alpha v(x,t) = F(x',y',z')\frac{\partial^2 v}{\partial x'^2} + G(x',y',z')\frac{\partial^2 v}{\partial y'^2} + H(x',y',z')\frac{\partial^2 v}{\partial z'^2}; \quad 1 < \alpha \leq 2 \tag{1}$$

with the initial conditions

$$u(x',y',z',0) = h(x',y',z'), u_t(x',y',z',0) = m(x',y',z') \tag{2}$$

An analytical approach that is more powerful than the traditional variational technique is called the "Variational iteration method" (VIM). It was initially recommended by He [8]. The "Laplace variational iteration method" is a combination of the "Laplace transform" and "variational iteration method." Applications of VIM to fractional differential equations are slow to converge, mainly because they directly use the Lagrange multipliers of ordinary differential equations (ODEs) [23]. Wu and Baleanu [24] pointed out that it can be difficult to apply integrals by parts of the Riemann-Liouville (RL) integral resulting from the constructed correction function. To overcome this shortcoming, they proposed to identify generalized Lagrange multipliers via the Laplace transform. This method has been utilized by many authors to solve several difficulties [1,3,7]. The novelty of this work lies in applying the "Laplace variational iteration method" (LVIM) for solving heat equations of fractional order.

2 Preliminaries

Definition 1 The Caputo derivative of arbitrary order [4] of function v(x,t) is presented as

$$D_t^{\alpha} v\,(\,x,t\,) = \frac{1}{\Gamma(m-\mu} \int_0^t (t-\delta)^{m-\mu-1}\, v^{(m)}\,(\,x,\delta\,)\,d\delta = J_t^{m-\mu} D^m u(\,v,t\,); \quad m-1<\mu\leq\ m, m\in\ N, \tag{3}$$

where $\frac{d^{\alpha}}{dt^{\alpha}}$ and J_t^{α} shows the Riemann- Liouville integral operator of fractional order [19], $\alpha > 0$

$$J_t^{\alpha} v(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\delta)^{\alpha-1} v(x,\delta)\,d\delta\ ; m-1<\mu\leq m, m\in N \tag{4}$$

Definition 2 The Laplace Transform [18,19] of f(t), $t > 0$ is defined as

$$L[f(t)] = F(s) = \int_0^t e^{-st} f(t)\,dt \tag{5}$$

Definition 3 The Laplace transform of $D_t^{\alpha} v(x,t)$ is explained as [18,19]

$$L[D_t^{\alpha} v(x,t)] = L[v(x,t)] - \sum_{n=0}^{m-1} v^n(x,0) s^{\alpha-n-1}; m-1<\alpha\leq m, m\in N \tag{6}$$

Definition 4 The Mittag-Leffler function is explained as [18]

$$E_{\alpha}\,(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma\,(\alpha n+1)} \tag{7}$$

$$E_{\alpha,\beta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n+\beta)} \tag{8}$$

3 Variational Iteration Method

He [10] established a method VIM for solving problems. This is a common technique used to evaluate solutions for linear and non-linear problems. Illustrate the VIM model, we take into consideration the subsequent non-linear equation with given constraints::

$$Pv(x,t) + Qv(x,t) = f(x,t) \tag{9}$$

where $'v'$ is the unknown function, 'P', 'Q' are linear and nonlinear operators, and f is the source term. The correction functional for (7) is given as follows:

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda[Pv_n(\xi,t) + Qv_n(\xi,t) - f(\xi,t)]d\xi \tag{10}$$

where λ is a general Lagrange multiplier that can be identified optimally via the variation theory. The subscript n indicates the nth approximation and u_n is considered as a restricted variation $\delta u_n = 0$.

3.1 Laplace Variational Iteration Method (LVIM)

To demonstrate the elementary purpose of (LVIM), deliberate a general fractional non-linear nonhomogeneous partial differential equation through the primary situations of the type

$$D_t^\alpha v(x,t) + Pv(x,t) + Qv(x,t) = f(x,t); m-1 < \alpha \leq m, m \in N \tag{11}$$

$$v_n(x,0) = h_k(x); n = 0,1,2,3,...m-1 \tag{12}$$

where D_t^α is the Caputo derivative. P and Q are linear and nonlinear operators, respectively, and f is the source term. By applying Laplace transform pertaining to t, on both sides of (9), we get

$$L[v(x,t)] = \frac{1}{s^\alpha}\sum_{n=0}^{m-1} v^n(x,0)s^{\alpha-n-1} + \frac{1}{s^\alpha}L[f(x,t)] - \frac{1}{s^\alpha}L[Pv(x,t)+Qv(x,t)] \tag{13}$$

taking inverse Laplace transform on equation (13)

$$v(x,t) = L^{-1}\left[\frac{1}{s^{\alpha}}\sum_{n=0}^{m-1} s^{\alpha-1-n}v^n(x,0) + \frac{1}{s^{\alpha}}L[f(x,t)]\right] - L^{-1}\left[\frac{1}{s^{\alpha}}L[Pv(x,t)+Qv(x,t)]\right] \tag{14}$$

by differentiating (14), concerning t, we get

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial}{\partial t}\left\{L^{-1}\left[\frac{1}{s^{\alpha}}\sum_{n=0}^{m-1} s^{\alpha-1-n}v^n(x,0) + \frac{1}{s^{\alpha}}L[f(x,t)]\right] - L^{-1}\left[\frac{1}{s^{\alpha}}L[Pv(x,t)+Qv(x,t)]\right]\right. \tag{15}$$

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The correction functional for (15)

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda \left[\frac{\partial v_n(x,\varepsilon)}{\partial \varepsilon} - \frac{\partial}{\partial \varepsilon}\left\{L^{-1}\left[\frac{1}{s^{\alpha}}\sum_{n=0}^{m-1} s^{\alpha-1-n} v^n(x,0) + \frac{1}{s^{\alpha}} L[f(x,t)]\right]\right\} - L^{-1}\left[\frac{1}{s^{\alpha}} L[Pv(x,\varepsilon) + Qv(x,\varepsilon)]\right]\right] d\varepsilon \tag{16}$$

The general Lagrange multiplier for (16) can be identified optimally via variation theory to get

$$1 + (\lambda)_{\epsilon=t} = 0 \tag{17}$$

From (17), we get

$$\lambda = -1 \tag{18}$$

Substituting $\lambda = -1$ into (16), then the iterative formula for $n = 0, 1, 2, \ldots$, as follows:

$$v_{n+1}(x,t) = v_n(x,t) - \int_0^t \left[\frac{\partial v_n(x,\varepsilon)}{\partial \varepsilon} - \frac{\partial}{\partial \varepsilon}\left\{L^{-1}\left[\frac{1}{s^{\alpha}}\sum_{n=0}^{m-1} s^{\alpha-1-n} v^n(x,0) + \frac{1}{s^{\alpha}} L[f(x,t)]\right] - L^{-1}\left[\frac{1}{s^{\alpha}} L[Pv(x,\varepsilon) + Qv(x,\varepsilon)]\right]\right] d\varepsilon \tag{19}$$

Begin with the early iteration

$$v_0(x,t) = v(x,0) + t v_t(x,0) \tag{20}$$

As a limit of the subsequent approximations, the exact answer is provided $v_n(x,t)$, $n = 0, 1, 2, \ldots$; alternatively in other words

$$v(x,t) = \lim_{n\to\infty} v_n(x,t) \tag{21}$$

3.2 Applications of LVIM for Solving fractional wave-like equations

Problem 1 Deliberate the succeeding 1-D fractional wave-like equation:

$$D_t^{\alpha} v(x,t) = \frac{1}{2} x^2 \frac{\partial^2 v}{\partial x^2}; 1 < \alpha \leq 2 \tag{22}$$

initial condition:

$$v_0(x,y,0) = 0, v_t(x,0) = x^2 \tag{23}$$

taking the Laplace transformation on (22) and result specified by equation (23) we obtain

$$L\left[v(\ x,t\ )\right] = \frac{x}{s} + \frac{x^2}{s^2} + \frac{1}{2s^{\alpha}} x^2 L\left[\frac{\partial^2 u}{\partial x^2}\right] \tag{24}$$

applying inverse Laplace transformation to the Equation (24), we have

$$v(\ x,t\ ) = x + x^2 t + L^{-1}\left[\frac{1}{2s^{\alpha}} x^2 L\left[\frac{\partial^2 v}{\partial x^2}\right]\right] \tag{25}$$

differentiating Equation (25) concerning t, we have

$$\frac{\partial v}{\partial t} = x^2 + \frac{\partial}{\partial t} L^{-1}\left[\frac{1}{2s^{\alpha}} x^2 L\left[\frac{\partial^2 v}{\partial x^2}\right]\right] \tag{26}$$

the correction functional for $\lambda = -1$ is offered by

$$v_{n+1}(\ x,t\ ) = v_n(\ x,t\ ) - \int_0^t \left[\frac{\partial v_n(\ x,\varepsilon\ )}{\partial \varepsilon} - x^2 - \frac{\partial}{\partial \varepsilon} L^{-1}\left\{\frac{1}{2s^{\alpha}} x^2 L\left(\frac{\partial^2 v_n}{\partial x^2}\right)\right\}\right] d\varepsilon \tag{27}$$

the initial iteration

$$v_0(x,0) = x + x^2 t \tag{28}$$

using the equation in equation (26), we have

$$v_0(x,t) = v_0(x,0) = x + x^2 t \tag{29}$$

$$v_1(x,t) = x + x^2 t + x^2 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \tag{30}$$

$$v_2(x,t) = x + x^2 t + x^2 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + x^2 \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \tag{31}$$

therefore it is expected that the general term in the successive approximation

$$v_n(x,t) = x + x^2[t + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + ...] \tag{32}$$

assumed that the solution was in closed form by

$$v(x,t) = \lim_{n\to\infty} v_n(x,t) = x + x^2 t E_{\alpha,2}(t^\alpha) \tag{33}$$

where $E_{\alpha,2}(t^\alpha)$ is the Mittag- Laffler Function defined in equation (6) letting $\alpha = 2$ then

$$v(x,t) == x + x^2 t + \frac{sinht}{t} \tag{34}$$

Numerical and Graphical discussion

In this part we found a record for numerical explanation of equation (31) and plot some graphs for $\alpha = 0.25, 0.5, 0.75, 1$.

Table 1: The values of $v(x,t)$ for $\alpha = 0.25$

$\alpha = 0.25$					
t	X=1	X=3	X=5	X=7	X=9
0	2	12	30	56	90
2	22.907618	50.042141	135.672615	263.118327	432.379275
4	13.010819	111.097371	305.270476	595.530113	981.876342
6	21.343969	186.09572	513.599243	1003.854451	1656.861548
8	30.896476	272.068286	752.411907	1471.927338	2430.614579
10	41.483595	367.352360	1017.089089	1990.696185	3288.171245

Table 2: The values of $v(x,t)$ for $\alpha = 0.50$

$\alpha = 0.50$					
t	X=1	X=3	X=5	X=7	X=9
0	2	12	30	56	90
2	49.149229	58.14922947	133.192304	258.256916	424.343065
4	16.018022	138.162200	380.450555	742.883089	1225.459800
6	31.055812	273.502315	756.395319	1479.734827	2443.520836
8	51.021537	453.193835	1255.538432	2458.055328	4060.744521
10	75.788321	676.094894	1874.708039	3671.627756	6066.854046

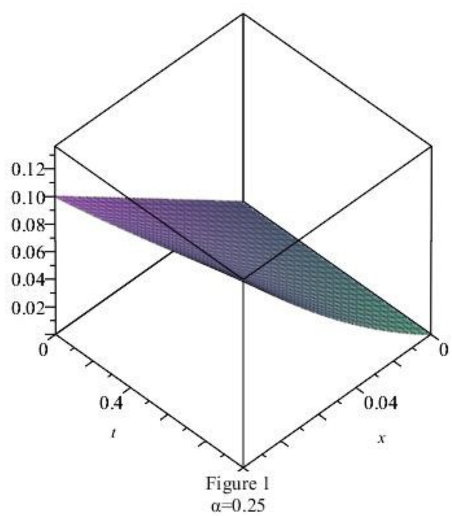
Table 3:The values of $v(x,t)$ for $\alpha = 0.75$

$\alpha = 0.75$					
t	X=1	X=3	X=5	X=7	X=9
0	2	12	30	56	90
2	5.239601	41.156416	110.990046	214.740490	352.407748
4	19.958726	173.628535	478.968154	935.977582	1544.656820
6	57.104733	507.942604	1407.618347	2756.131960	4553.483443
8	129.591614	1160.324531	3219.790365	6307.989114	10424.92078
10	251.599886	2258.398981	6269.997168	12286.39445	20307.59083

Table 4:The values of $v(x,t)$ for $\alpha = 1$

$\alpha = 1.0$					
t	X=1	X=3	X=5	X=7	X=9
0	2	12	30	56	90
2	5.33	42	113.33	219.33	360
4	20.66	180	496.66	970.66	1602
6	56	498	1380	2702	4464
8	119.33	1068	2963.33	5805.33	9594
10	218.66	1962	5446.66	10672.66	17640

The solution is graphically presented in Figures 1,2,3, and 4 for various fractional orders of α



Figur 1: The behaviour of $v(x,t)$ w.r.t. x and t for $\alpha = 0.25$

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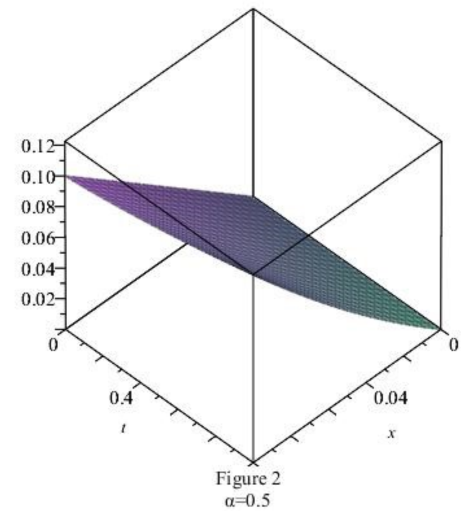
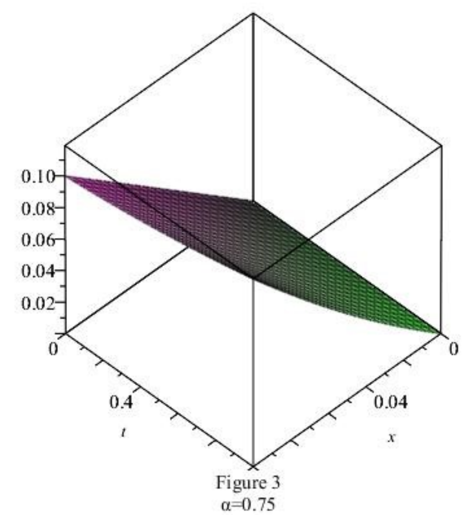
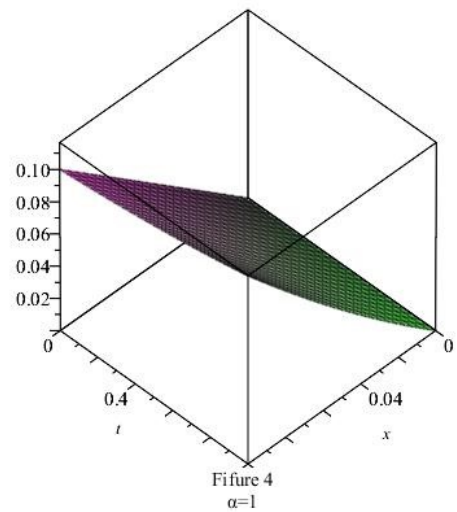


Figur 2: The behaviour of $v(x,t)$ w.r.t. x and t for $\alpha = 0.50$



Figur 3: The behaviour of $v(x,t)$ w.r.t. x and t for $\alpha = 0.75$



Figur 4: The behaviour of $v(x,t)$ w.r.t x and t for $\alpha = 1$

Problem 2 Deliberate the succeeding 2-D fractional wave-like equation:

$$D_t^{\alpha} v(x,y,t) = \frac{1}{2}[x^2 \frac{\partial^2 v}{\partial x^2} + y^2 \frac{\partial^2 v}{\partial y^2}], 1 < \alpha \leq 2 \tag{35}$$

initial condition:

$$v_0(x,y,0) = x^4, v_t(x,y,0) = y^4 \tag{36}$$

taking the Laplace transformation on (34) and using the result specified by (35), we achieve,

$$v(\ x,y,t\ ) = x^4 + y^4 t + L^{-1}\left[\frac{1}{12s^{\alpha}} L\left[x^2 \frac{\partial^2 v}{\partial x^2} + y^2 \frac{\partial^2 v}{\partial y^2}\right]\right] \tag{37}$$

apply inverse Laplace transform we have

$$\frac{\partial v}{\partial t} = y^4 + \frac{\partial}{\partial t} L^{-1}\left[\frac{1}{12s^{\alpha}} L\left[x^2 \frac{\partial^2 v}{\partial x^2} + y^2 \frac{\partial^2 v}{\partial y^2}\right]\right] \tag{38}$$

the correction functional for $\lambda = -1$ is given as follows

$$v_{n+1}(x,y,t) = v_n(x,y,t) - \int_0^t \left[\frac{\partial v_n(x,y,\varepsilon)}{\partial \varepsilon} - y^4 - \right. \\ \left. \frac{\partial}{\partial \varepsilon} L^{-1}\left\{\frac{1}{12s^\alpha} L\left(x^2\frac{\partial^2 v_n}{\partial x^2} + y^2\frac{\partial^2 v_n}{\partial y^2}\right)\right\} d\varepsilon \tag{39}$$

the initial iteration

$$v_0(x,y,0) = x^4 + y^4 t \tag{40}$$

using the equation in equation (38), we have

$$v_0(x,y,t) = x^4 + y^4 t \tag{41}$$

$$v_1(x,y,t) = x^4 + y^4 t + x^4 \frac{t^{\alpha+1}}{\Gamma(\alpha+1)} + y^4 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \tag{42}$$

$$v_2(x,y,t) = x^4 + y^4 t + x^4 \frac{t^{\alpha+1}}{\Gamma(\alpha+1)} + y^4 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + x^4 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + y^4 \frac{t^{2\alpha+1}}{\Gamma(\alpha+2)} \tag{43}$$

assumed that the solution was in closed form by

$$v(x,y,t) = \lim_{n\to\infty} v_n(x,y,t) = x^4 E_\alpha(t^\alpha) + y^4 E_{\alpha,2}(t^\alpha) \tag{44}$$

where $E_{\alpha,2}(t^\alpha)$ is the Mittag- Laffler Function defined in equation (6) letting $\alpha = 2$ then

$$v(x,y,t) == x^4 cosht + y^4 sinht \tag{45}$$

Numerical and Graphical discussion

In this part we found a record for numerical explanation of equation (42) and plot some graphs for $\alpha = 0.25, 0.5, 0.75, 1$.

Table 5: The values of $v(x,y,t)$ for $\alpha = 0.25$

$\alpha = 0.25$					
t	X=1	X=3	X=5	X=7	X=9
0	16	81	625	2401	6561
2	10.723461	322.384991	2448.587477	9388.799277	25645.15124
4	13.010819	111.097371	305.270476	595.530113	981.876342
6	30.834622	470.086827	3457.001819	13208.40076	36049.51540
8	42.943473	524.703270	3814.269888	14553.73738	39709.24682
10	56.013753	578.426352	4130.832026	15728.39172	42893.84687

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Table 6:The values of $v(x, y, t)$ for $\alpha = 0.50$

$\alpha = 0.50$					
t	X=1	X=3	X=5	X=7	X=9
0	16	81	625	2401	6561
2	10.723461	378.384991	2878.483393	11040.56935	30158.96889
4	25.274780	605.815447	4553.491981	17441.49478	47629.60945
6	44.819765	825.936021	6137.526560	23478.30743	64096.35272
8	69.213075	1044.536135	7676.732939	29328.90486	80045.70396
10	98.356569	1263.816429	9188.943467	35062.15232	95666.06497

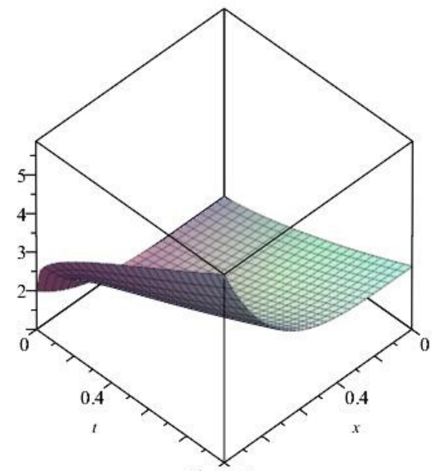
Table 7:The values of $v(x, y, t)$ for $\alpha = 0.75$

$\alpha = 0.75$					
t	X=1	X=3	X=5	X=7	X=9
0	16	81	625	2401	6561
2	9.535236	406.142637	3103.072959	11907.75724	32531.34205
4	23.880947	831.523787	6323.495098	24253.16615	66250.59382
6	44.109711	1342.276394	10169.80983	38989.11017	106493.7576
8	70.420350	1926.202514	14545.52123	55743.88528	152244.5578
10	103.057275	2575.614244	19389.00163	74279.76634	202852.7288

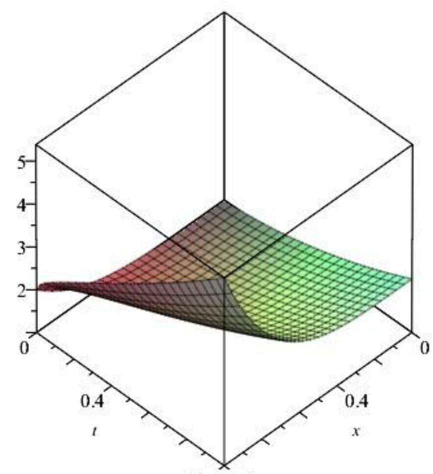
Table 8:The values of $v(x, y, t)$ for $\alpha = 1$

$\alpha = 1.0$					
t	X=1	X=3	X=5	X=7	X=9
0	16	81	625	2401	6561
2	10.33	410.33	3130.33	12010.33	32810.33
4	35.66	1075.66	8147.66	31235.66	85315.66
6	85	2085	15685	60085	164085
8	166.33	3446.33	25750.33	98566.33	269126.33
10	287.66	5167.66	38351.66	146687.66	400447.66

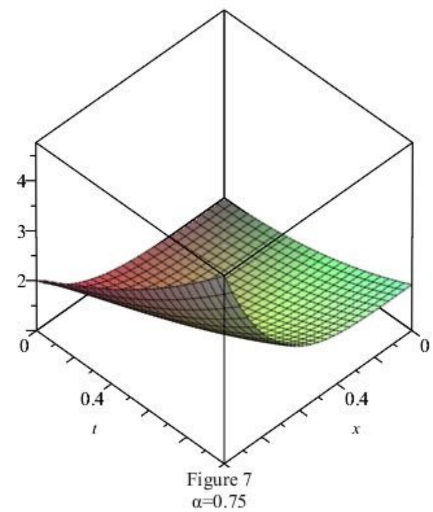
The solution is graphically presented in Figures 5,6,7, and 8 for various fractional orders of α



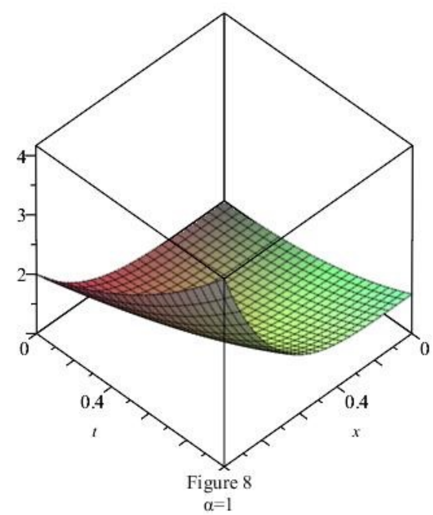
Figur 5: The behaviour of $v(x,y,t)$ w.r.t. x and t for $\alpha = 0.25$



Figur 6: The behaviour of $v(x,y,t)$ w.r.t. x and t for $\alpha = 0.50$



Figur 7: The behaviour of $v(x, y, t)$ w.r.t. x and t for $\alpha = 0.75$



Figur 8: The behaviour of $v(x, y, t)$ w.r.t.x and t for $\alpha = 1$

Problem 3 Deliberate the succeeding 3-D fractional wave-like equation:

$$D_t^{\alpha} v(\, x, y, z, t\,) = x^2 + y^2 + z^2 + \frac{1}{2}\left[x^2 \frac{\partial^2 v}{\partial x^2} + y^2 \frac{\partial^2 v}{\partial y^2} + z^2 \frac{\partial^2 v}{\partial z^2}\right]; 1 < \alpha \leq 2 \quad (46)$$

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initial condition:

$$v_0(x,y,z,0)=0, v_t(x,y,z,0)=x^2+y^2-z^2 \tag{47}$$

taking the laplace transform of equation (45) and result obtained by equation (46) we obtain

$$\begin{aligned} L\left[v(\ x,y,z,t\ )\right] = \frac{x^2+y^2-z^2}{s^2} + \frac{1}{s^\alpha}L\left(x^2+y^2+z^2\right) + \\ \frac{1}{2s^\alpha}L\left[x^2\frac{\partial^2 v}{\partial x^2}+y^2\frac{\partial^2 v}{\partial y^2}+z^2\frac{\partial^2 v}{\partial z^2}\right] \end{aligned} \tag{48}$$

apply inverse Laplace transform we have

$$\begin{aligned} v(\ x,y,z,t\ ) = t\left(x^2+y^2-z^2\right) + \left(x^2+y^2+z^2\right)\frac{t^\alpha}{\Gamma(\alpha+1)} + \\ L^{-1}\left[\frac{1}{2s^\alpha}L\left[x^2\frac{\partial^2 v}{\partial x^2}+y^2\frac{\partial^2 v}{\partial y^2}+z^2\frac{\partial^2 v}{\partial z^2}\right]\right] \end{aligned} \tag{49}$$

differentiating Equation (48) regarding t, we have

$$\begin{aligned} \frac{\partial v}{\partial t} = \left(x^2+y^2-z^2\right) + \left(x^2+y^2+z^2\right)\alpha\frac{t^{\alpha-1}}{\Gamma(\alpha+1)} + \\ \frac{\partial}{\partial t}\left\{L^{-1}\left[\frac{1}{2s^\alpha}L\left[x^2\frac{\partial^2 v}{\partial x^2}+y^2\frac{\partial^2 v}{\partial y^2}+z^2\frac{\partial^2 v}{\partial z^2}\right]\right]\right\} \end{aligned} \tag{50}$$

the correction functional for $\lambda=-1$ is given as follows

$$\begin{aligned} v_{n+1}(x,y,z,t) = v_n(x,y,z,t) - \int_0^t \left[\frac{\partial v_n(x,y,z,\varepsilon)}{\partial \varepsilon} - \left(x^2+y^2-z^2\right) - \right. \\ \left. \left(x^2+y^2+z^2\right)\frac{\alpha\varepsilon^{\alpha-1}}{\Gamma(\alpha+1)} - \frac{\partial}{\partial \varepsilon}\left\{L^{-1}\left[\frac{1}{2s^\alpha}L\left[x^2\frac{\partial^2 v_n}{\partial x^2}+y^2\frac{\partial^2 v_n}{\partial y^2}+z^2\frac{\partial^2 v_n}{\partial z^2}\right]\right]\right\} d\varepsilon \right. \end{aligned} \tag{51}$$

the initial iteration

$$v_0(x,y,z,0) = (x^2+y^2-z^2)t + (x^2+y^2+z^2\ )\frac{t^\alpha}{\Gamma(\alpha+1)} \tag{52}$$

then, we have

$$v_1(\ x,y,z,t\ ) = v_0(\ x,y,z,t\ ) - \int_0^t \left[\frac{\partial v_0(\ x,y,z,\ \varepsilon\ )}{\partial \varepsilon} - \left(x^2+y^2-z^2\right) - \left(x^2+y^2+z^2\right) \frac{\alpha \varepsilon^{\alpha-1}}{\Gamma\left(\alpha+1\right)} - \frac{\partial}{\partial \varepsilon}\left\{ L^{-1}\left[\frac{1}{2s^\alpha} L \left[x^2 \frac{\partial^2 v_0}{\partial x^2} + y^2 \frac{\partial^2 v_0}{\partial y^2} + z^2 \frac{\partial^2 v_0}{\partial z^2} \right]\right]\right\} d\varepsilon \tag{53}$$

$$v_1(\ x,y,z,t\ ) = t\left(x^2+y^2-z^2\right) + \left(x^2+y^2+z^2\right)\frac{t^\alpha}{\Gamma(\alpha+1)} + \left(x^2+y^2-z^2\right) \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \left(x^2+y^2+z^2\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \tag{54}$$

$$v_2(\ x,y,z,t\ ) = v_1(\ x,y,z,t\ ) - \int_0^t \left[\frac{\partial v_1(x,y,z,\varepsilon)}{\partial \varepsilon} - \left(x^2+y^2-z^2\right) - \left(x^2+y^2+z^2\right) \frac{\alpha \varepsilon^{\alpha-1}}{\Gamma\left(\alpha+1\right)} - \frac{\partial}{\partial \varepsilon}\left\{ L^{-1}\left[\frac{1}{2s^\alpha} L \left[x^2 \frac{\partial^2 v_1}{\partial x^2} + y^2 \frac{\partial^2 v_1}{\partial y^2} + z^2 \frac{\partial^2 v_1}{\partial z^2} \right]\right]\right\} d\varepsilon \tag{55}$$

$$v_2(\ x,y,z,t\ ) = t\left(x^2+y^2-z^2\right) + \left(x^2+y^2+z^2\right)\frac{t^\alpha}{\Gamma(\alpha+1)} + \left(x^2+y^2-z^2\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \left(x^2+y^2+z^2\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \left(x^2+y^2-z^2\right) \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \left(x^2+y^2+z^2\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \tag{56}$$

assumed that the solution was in closed form by

$$v(\ x,y,z,t\ ) = limn \to \infty v_n(\ x,y,z,t\ ) = \ t\left(x^2+y^2-z^2\right) E_{\alpha,2}\left(t^\alpha\right) + \left(x^2+y^2+z^2\right)\left[E_\alpha\left(t^\alpha\right)-1\right] \tag{57}$$

where $E_\alpha\left(t^\alpha\right)$ and $E_{\alpha,2}(t^\alpha)$ are the Mittag-Laffer Function defined in equations (7) and (8) letting $\alpha = 2$ then

$$v(x,y,z,t) = (x^2+y^2)e^t + z^2 e^{-t} - (x^2+y^2+z^2) \tag{58}$$

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Numerical and Graphical discussion

In this part we found a record for numerical explanation of equation (51) and plot some graphs for different values of $\alpha = 0.25, 0.5, 0.75, 1$.

Table 9:The values of $v(x, y, z, t)$ for $\alpha = 0.25$

$\alpha = 0.25$					
t	X=1	X=3	X=5	X=7	X=9
0	0	0	0	0	0
2	8.513426	60.620834	164.835650	321.157875	529.587508
4	13.992796	109.935171	301.819920	589.647043	973.416542
6	20.636626	169.729635	467.915654	915.194682	36049.51540
8	28.188647	237.697827	656.716187	1285.243727	2123.280446
10	36.506662	312.559961	864.666558	1692.826455	27107.039651

Table 10:The values of $v(x, y, z, t)$ for $\alpha = 0.50$

$\alpha = 0.50$					
t	X=1	X=3	X=5	X=7	X=9
0	0	0	0	0	0
2	9.127692	66.149229	180.192304	351.256916	579.343065
4	21.018022	173.162200	477.450555	933.883089	1542.459800
6	38.05581275	326.502315	903.395319	23478.30743	64096.35272
8	60.021537	524.193835	1452.538432	2845.055328	4701.744521
10	86.788321	765.094894	2121.708039	4156.62775	6869.854046

Table 11 The values of $v(x, y, z, t)$ for $\alpha = 0.75$

$\alpha = 0.75$					
t	X=1	X=3	X=5	X=7	X=9
0	0	0	0	0	0
2	8.793468	63.1412156	171.836710	334.879951	552.270940
4	23.663151	196.968364	543.578790	1063.494429	1756.71528
6	49.835450	432.519053	1197.886259	2345.937068	3876.671480
8	89.129443	786.164995	2180.236099	4271.342754	7059.484963
10	143.116946	1272.052519	3529.923663	6916.730379	903.395319

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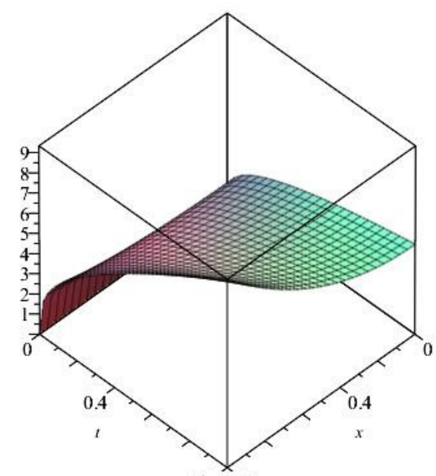
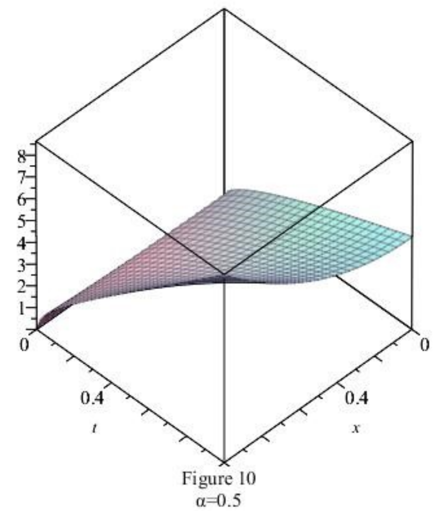
Table 12:The values of $v(x, y, z, t)$ for $\alpha = 1$

$\alpha = 1.0$					
t	X=1	X=3	X=5	X=7	X=9
0	0	0	0	0	0
2	8.3333	59	160.33	312.33	515
4	25.33	215	593.66	1161.66	1919
6	63	551	1527	2991	4943
8	128.33	1139	3160.33	6192.33	10235
10	229.66	2051	5693.66	11157.66	18443

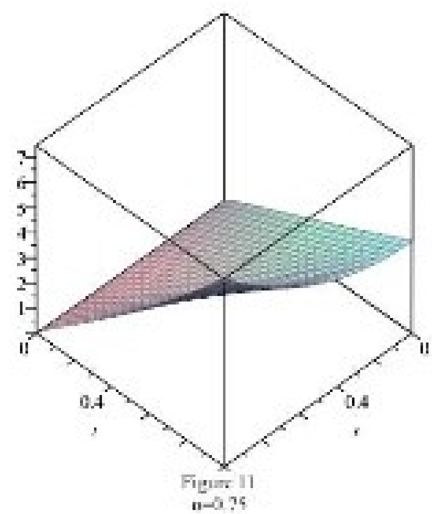
The solution is graphically presented in Figures 9,10,11, and 12 for various fractional orders of α



Figur 9: The behaviour of $v(x, y, z, t)$ w.r.t. x and t for $\alpha = 0.25$



Figur 10: The behaviour of $v(x, y, z, t)$ w.r.t. x and t for $\alpha = 0.50$



Figur 11: The behaviour of $v(x, y, z, t)$ w.r.t. x and t for $\alpha = 0.75$

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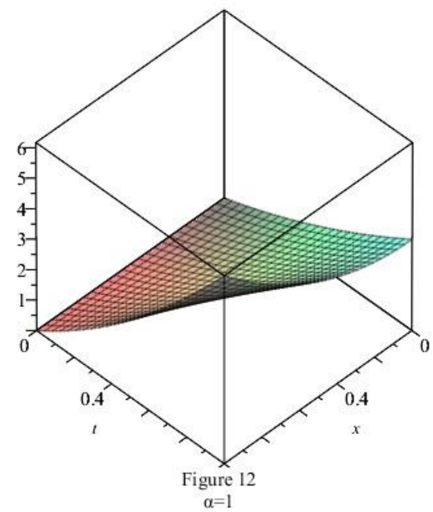

Figur 12: The behaviour of $v(x, y, z, t)$ w.r.t. x and t for $\alpha = 1$

4 Conclusion:

We review the Laplace variational iteration method to show why it works well for obtaining approximate analytical solutions of nonlinear equations governing nonlinear phenomena. In the conferred document, the "Laplace Variational Iteration Method" is productively executed for the fractional wave equation, wherever we put in the fractional derivative in form of Caputo sense. The analytical, consequent, and comprehensive outcomes have been specified in expressions of a power series that come together to the exact solutions. The graphical consequences of the analysis are also manifested. In the future authors and scholars may use this paper for reference purposes and different values for parameters may be used for the graphical presentation so that the related phenomena may well be understood.

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Boros Integral Involving the Product of Family of Polynomials and the Incomplete I-Function

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Abstract

The current manuscript's goal is to determine the Boros integral with three parameters, which comprises of the multiplication of the incomplete I-function and a family of polynomials. Some interesting corollaries are provided as a specific case of our primary findings.

Keywords: Incomplete Gamma function, Incomplete I-function, Mellin-Barnes integrals contour, Boros integral, Generalized family of polynomials.
MSC 2010: 33B20, 33C05, 33C60, 33E12.

1 Introduction and Preliminaries

Due to new hurdles in applied science and technology in the present period, the popularity of special functions is growing by the day. Special functions have been used widely in the varity of fields of fluid problems, biological problems, communication and other probelms of physics (see [1, 6–9, 13, 16, 21–24]). However, it has been noted that there are many issues in the fields of astrophysics and heat conduction for which the answers provided by the most prominent groups of special functions are insufficient. In this instance, the illustration makes use of the definition of incomplete gamma functions and its generalisations. So, The investigation of incomplete hypergeometric functions, incomplete H-functions, and incomplete $\overline{H}$-functions has been made possible by the use of incomplete type of gamma functions. For more details, one can see [2, 3] about incomplete functions and their recent applications.

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Jangid et al. [10] recently introduced a new category of incomplete I- functions ${}^{\gamma}I_{p,\,q}^{m,\,n}(y)$ and ${}^{\Gamma}I_{p,\,q}^{m,\,n}(y)$, which is the generalization of Rathie's I-function [18] and it is described as:

$$\begin{aligned}
{}^{\gamma}I_{p,\,q}^{m,\,n}(y) &= {}^{\gamma}I_{p,\,q}^{m,\,n}\left[y \left| \begin{array}{c} (f_1,\varsigma_1;\mathcal{F}_1:t),(f_2,\varsigma_2;\mathcal{F}_2),\cdots,(f_p,\varsigma_p;\mathcal{F}_p) \\ (g_1,\varrho_1;\mathcal{G}_1),\cdots,(g_q,\varrho_q;\mathcal{G}_q) \end{array} \right.\right] \\
&= {}^{\gamma}I_{p,\,q}^{m,\,n}\left[y \left| \begin{array}{c} (f_1,\varsigma_1;\mathcal{F}_1:t),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,p} \\ (g_j,\varrho_j;\mathcal{G}_j)_{1,\,q} \end{array} \right.\right] = \frac{1}{2\pi i}\int_{\pounds} \phi(r,t)\; y^r\; dr, \qquad (1)
\end{aligned}$$

and

$$\begin{aligned}
{}^{\Gamma}I_{p,\,q}^{m,\,n}(y) &= {}^{\Gamma}I_{p,\,q}^{m,\,n}\left[y \left| \begin{array}{c} (f_1,\varsigma_1;\mathcal{F}_1:t),(f_2,\varsigma_2;\mathcal{F}_2),\cdots,(f_p,\varsigma_p;\mathcal{F}_p) \\ (g_1,\varrho_1;\mathcal{G}_1),\cdots,(g_q,\varrho_q;\mathcal{G}_q) \end{array} \right.\right] \\
&= {}^{\Gamma}I_{p,\,q}^{m,\,n}\left[y \left| \begin{array}{c} (f_1,\varsigma_1;\mathcal{F}_1:t),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,p} \\ (g_j,\varrho_j;\mathcal{G}_j)_{1,\,q} \end{array} \right.\right] = \frac{1}{2\pi i}\int_{\pounds} \Phi(r,t)\; y^r\; dr, \qquad (2)
\end{aligned}$$

$\forall\ y \neq 0$, where

$$\phi(r,t) = \frac{\{\gamma(1-f_1+\varsigma_1 r,\, t)\}^{\mathcal{F}_1} \prod\limits_{j=1}^{m}\{\Gamma(g_j-\varrho_j r)\}^{\mathcal{G}_j} \prod\limits_{j=2}^{n}\{\Gamma(1-f_j+\varsigma_j r)\}^{\mathcal{F}_j}}{\prod\limits_{j=n+1}^{p}\{\Gamma(f_j-\varsigma_j r)\}^{\mathcal{F}_j} \prod\limits_{j=m+1}^{q}\{\Gamma(1-g_j+\varrho_j r)\}^{\mathcal{G}_j}}, \qquad (3)$$

and

$$\Phi(r,t) = \frac{\{\Gamma(1-f_1+\varsigma_1 r,\, t)\}^{\mathcal{F}_1} \prod\limits_{j=1}^{m}\{\Gamma(g_j-\varrho_j r)\}^{\mathcal{G}_j} \prod\limits_{j=2}^{n}\{\Gamma(1-f_j+\varsigma_j r)\}^{\mathcal{F}_j}}{\prod\limits_{j=n+1}^{p}\{\Gamma(f_j-\varsigma_j r)\}^{\mathcal{F}_j} \prod\limits_{j=m+1}^{q}\{\Gamma(1-g_j+\varrho_j r)\}^{\mathcal{G}_j}}, \qquad (4)$$

where, $\gamma(.,t)$ and $\Gamma(.,t)$ are the lower and upper incomplete gamma functions described in (6) and (7).
The incomplete I-functions ${}^{\gamma}I_{p,\,q}^{m,\,n}(y)$ and ${}^{\Gamma}I_{p,\,q}^{m,\,n}(y)$ exist $\forall\ t \geq 0$ in accordance with Rathie's parameters and contour mentioned in [18] with,

$$\Delta > 0,\ |arg(y)| < \Delta\frac{\pi}{2},$$

where

$$\Delta = \sum_{j'=1}^{m} \mathcal{G}_{j'}\,\varrho_{j'} - \sum_{j'=m+1}^{q} \mathcal{G}_{j'}\,\varrho_{j'} + \sum_{j'=1}^{n} \mathcal{F}_{j'}\,\varsigma_{j'} - \sum_{j'=n+1}^{p} \mathcal{F}_{j'}\,\varsigma_{j'}\,.$$

For $\mathcal{F}_1 = 1$, the following relation is satisfied by the incomplete I-functions:

$${}^{\gamma}I_{p,\,q}^{m,\,n}(y) + {}^{\Gamma}I_{p,\,q}^{m,\,n}(y) = I_{p,\,q}^{m,\,n}(y), \qquad (5)$$

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for the well known Rathie's I-function [18]. Some additional properties regarding the incomplete I- function can be found in [4].

The incomplete gamma functions $\gamma(r,t)$ and $\Gamma(r,t)$ are described in the following way:

$$\gamma(r,t)=\int_0^t u^{r-1}\,e^{-u}\,du, \qquad \big(t \geqq 0; \Re(r)>0\big), \tag{6}$$

and

$$\Gamma(r,t)=\int_t^\infty u^{r-1}\,e^{-u}\,du, \qquad \big(t \geqq 0;\ \Re(r)>0 \quad \text{when} \quad t=0\big), \tag{7}$$

recognized as the lower and upper incomplete gamma functions respectively. The following relation is satisfied by the incomplete gamma functions.

$$\gamma(r,t)+\Gamma(r,t)=\Gamma(r), \qquad \big(\Re(r)>0\big). \tag{8}$$

A general class of polynomials was studied by the Srivastava [19, 20], defined in the following way:

$$S_V^U[t]=\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{V,R}\,t^R\,, \tag{9}$$

where $U\in\mathbf{Z}^+$ and $A_{V,\,R}$ are real or complex numbers arbitrary constant. The notations $[k]$ indicates the Floor function and $(\kappa)_\mu$ denote the Pochhammer symbol described by:

$$(\kappa)_0=1 \qquad \textit{and} \qquad (\kappa)_\mu=\frac{\Gamma(\kappa+\mu)}{\Gamma(\kappa)} \qquad\qquad (\mu\in\mathcal{C}),$$

in the form of the Gamma function.

Lemma 1. *Let $b>0$, $c\geq 0$, $a>-\sqrt{bc}$ and $P>\frac{1}{2}$, we have the integral depending upon the three parameters, see Boros and Molls [5, 14].*

$$\int_0^\infty\left[\frac{h^2}{bh^4+2ah^2+c}\right]^P dh=\frac{B\left(P-\frac{1}{2},\,\frac{1}{2}\right)}{2^{P+1/2}\sqrt{b}\left[a+\sqrt{bc}\right]^{P-1/2}}, \tag{10}$$

where $B(m,\,n)$ denotes the Beta function. Equation (10) can also be expressed in the following way, by using the relation $B(m,n)=\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$.

$$\int_0^\infty\left[\frac{h^2}{bh^4+2ah^2+c}\right]^P dh=\frac{\sqrt{\pi}\,\Gamma\left(P-\frac{1}{2}\right)}{\Gamma(P)\sqrt{b}\,2^{P+\frac{1}{2}}(a+\sqrt{bc})^{P-\frac{1}{2}}}. \tag{11}$$

Concerning the proof, see Boros and Moll [5] and Quershi et al. [17].

Hundreds of special functions have been employed in applied mathematics and computing sciences for many centuries due to their outstanding features and wide range of applications. The application of image formulas involving one or more variable special functions under various definite integrals is crucial from the perspective of the usefulness of these consequences in the evaluation of generalised integrals, applied physics, and many engineering areas. A variety of improper integrals involving incomplete I-functions and the family of polynomials have been examined in this study, primarily motivated by various applications of these findings. A significant amount of additional findings can be constructed as special instances from our main results because of the unified character of our results.

2 Main Results

For $X = \frac{h^2}{bh^4+2ah^2+c}$, the following is the outcomes:

Theorem 1. *For $c \geq 0, b > 0\,, a > -\sqrt{bc}$, $P > \frac{1}{2}$ then we have the following result:*

$$\int_0^\infty \left(\frac{h^2}{bh^4+2ah^2+c}\right)^P {}^{\Gamma}I_{p,\,q}^{m,\,n}(X^e\, y)dh = \frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}}$$

$$\times\ {}^{\Gamma}I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(a+\sqrt{bc})]^{-e}\ \middle|\ \begin{matrix} (f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P,\,e;\,1\right),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,p} \\ (1-P,\,e;\,1),(g_j,\varrho_j;\mathcal{G}_j)_{1,\,q}\end{matrix}\right]. \tag{12}$$

Proof. The LHS of equation (12) is:

$$G^{'} = \int_0^\infty \left(\frac{h^2}{bh^4+2ah^2+c}\right)^P {}^{\Gamma}I_{p,\,q}^{m,\,n}(X^e\, y)dh. \tag{13}$$

Replace the incomplete I- function ${}^{\Gamma}I_{p,\,q}^{m,\,n}(y)$ by (2), we get:

$$G^{'} = \int_0^\infty X^P \frac{1}{2\pi i}\int_{\pounds} \Phi(r,t)\ (X^e\, y)^r\ dh\, dr, \tag{14}$$

where $\Phi(r,t)$ is given by (4).
Interchange the integration order in the above equation gives:

$$G^{'} = \frac{1}{2\pi i}\int_{\pounds} \frac{\{\Gamma(1-f_1+\varsigma_1 r,t)\}^{\mathcal{F}_1}\prod_{j=1}^{m}\{\Gamma(g_j-\varrho_j r)\}^{\mathcal{G}_j}\prod_{j=2}^{n}\{\Gamma(1-f_j+\varsigma_j r)\}^{\mathcal{F}_j}}{\prod_{j=n+1}^{p}\{\Gamma(f_j-\varsigma_j r)\}^{\mathcal{F}_j}\prod_{j=m+1}^{q}\{\Gamma(1-g_j+\varrho_j r)\}^{\mathcal{G}_j}} y^r$$

$$\times \int_0^\infty X^{P+er}\, dh\, dr. \tag{15}$$

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Now with the help of Lemma 1, evaluate the integral, we get:

$$\int_0^\infty X^{P+er} dh = \frac{\sqrt{\pi}\Gamma(P+er-\frac{1}{2})}{\Gamma(P+er)\sqrt{b}\,2^{P+er+\frac{1}{2}}(a+\sqrt{bc})^{P+er-\frac{1}{2}}}. \tag{16}$$

Put (16) in (15), we get:

$$G' = \frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}} \times \frac{1}{2\pi i}$$

$$\int_{\pounds} \frac{\{\Gamma(1-f_1+\varsigma_1 r, t)\}^{\mathcal{F}_1}\{\Gamma\left(P+er-\frac{1}{2}\right)\}\prod_{j=1}^{m}\{\Gamma(g_j-\varrho_j r)\}^{\mathcal{G}_j}\prod_{j=2}^{n}\{\Gamma(1-f_j+\varsigma_j r)\}^{\mathcal{F}_j}}{\prod_{j=n+1}^{p}\{\Gamma(f_j-\varsigma_j r)\}^{\mathcal{F}_j}\prod_{j=m+1}^{q}\{\Gamma(1-g_j+\varrho_j r)\}^{\mathcal{G}_j}\{\Gamma\left(P+er\right)\}}$$

$$\times\, y^r[2(a+\sqrt{bc})]^{-er}\,dr. \tag{17}$$

Now convert equation (17) in incomplete I- function to obtain the desired result.
□

Theorem 2. *For $b>0,\ c\geq 0,\ a>-\sqrt{bc},\ P>\frac{1}{2}$ then we have the following result:*

$$\int_0^\infty \left(\frac{h^2}{h^4+2ah^2+c}\right)^P {}^{\gamma}I_{p,\,q}^{m,\,n}(X^e\,y)dh = \frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}}$$

$$\times\ {}^{\gamma}I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(a+\sqrt{bc})]^{-e}\ \middle|\ \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P;\,e,\,1\right),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,p}\\ (1-P,\,e;\,1),(g_j,\varrho_j;\mathcal{G}_j)_{1,\,q}\end{matrix}\right]. \tag{18}$$

Theorem 2 is proved in the same way as theorem 1 with the same conditions.

Theorem 3. *For $b>0,\ c\geq 0,\ a>-\sqrt{bc},\ P>\frac{1}{2}$ and the coefficient $A_{V,R}$ are real or complex arbitrary constants and $U\in Z^+$ then we have the following result:*

$$\int_0^\infty \left(\frac{h^2}{bh^4+2ah^2+c}\right)^P S_V^U[wX]\ {}^{\Gamma}I_{p,\,q}^{m,\,n}(X^e\,y)dh =$$

$$\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}} \times \sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{V,R}\,w^R.\,\frac{1}{[2(a+\sqrt{bc})]^R}$$

$$\times\ {}^{\Gamma}I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(a+\sqrt{bc})]^{-e}\ \middle|\ \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P-R,\,e,\,1\right),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,p}\\ (1-P-R,\,e,\,1),(g_j,\varrho_j;\mathcal{G}_j)_{1,\,q}\end{matrix}\right]. \tag{19}$$

Proof. The LHS of (19) is:

$$G = \int_0^\infty \left(\frac{h^2}{bh^4 + 2ah^2 + c}\right)^P S_V^U[wX] \; {}^\Gamma I_{p,\,q}^{m,\,n}(X^e\, y)dh. \tag{20}$$

Replace the Srivastava Polynomial $S_V^U[t]$ and incomplete I- function ${}^\Gamma I_{p,\,q}^{m,\,n}(y)$ by (9) and (2) respectively, we get:

$$G = \int_0^\infty X^P \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{V,R}\,(w\,X)^R \frac{1}{2\pi i}\int_{\pounds} \Phi(r,t)\,(X^e\,y)^r\;dh\,dr, \tag{21}$$

where $\Phi(r,t)$ is given by (4).
Interchange the integration order in the above equation gives:

$$\sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{V,R}\, w^R \frac{1}{2\pi i}\int_{\pounds} \frac{\{\Gamma(1-f_1+\varsigma_1 r,t)\}^{\mathcal{F}_1}\prod_{j=1}^{m}\{\Gamma(g_j-\varrho_j r)\}^{\mathcal{G}_j}\prod_{j=2}^{n}\{\Gamma(1-f_j+\varsigma_j r)\}^{\mathcal{F}_j}}{\prod_{j=n+1}^{p}\{\Gamma(f_j-\varsigma_j r)\}^{\mathcal{F}_j}\prod_{j=m+1}^{q}\{\Gamma(1-g_j+\varrho_j r)\}^{\mathcal{G}_j}}$$

$$\times\, y^r \times \int_0^\infty X^{P+R+er}\,dh\,dr. \tag{22}$$

Now with the help of Lemma 1, evaluate the integral, we get:

$$\int_0^\infty X^{P+R+er}dh = \frac{\sqrt{\pi}\,\Gamma(P+R+er-\frac{1}{2})}{\Gamma(P+R+er)\sqrt{b}\,2^{P+R+er+\frac{1}{2}}(a+\sqrt{bc})^{P+R+er-\frac{1}{2}}}. \tag{23}$$

Put (23) in (22), we get:

$$\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}} \times \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{V,R}\, w^R \cdot \frac{1}{[2(a+\sqrt{bc})]^R} \times \frac{1}{2\pi i}$$

$$\int_{\pounds} \frac{\{\Gamma(1-f_1+\varsigma_1 r,t)\}^{\mathcal{F}_1}\{\Gamma\left(P+R+er-\frac{1}{2}\right)\}\prod_{j=1}^{m}\{\Gamma(g_j-\varrho_j r)\}^{\mathcal{G}_j}\prod_{j=2}^{n}\{\Gamma(1-f_j+\varsigma_j r)\}^{\mathcal{F}_j}}{\prod_{j=n+1}^{p}\{\Gamma(f_j-\varsigma_j r)\}^{\mathcal{F}_j}\prod_{j=m+1}^{q}\{\Gamma(1-g_j+\varrho_j r)\}^{\mathcal{G}_j}\{\Gamma\left(P+R+er\right)\}}$$

$$\times\, y^r[2(a+\sqrt{bc})]^{-er}\,dr. \tag{24}$$

Now convert equation (24) in incomplete I- function to obtain the desired result. □

Theorem 4. *For $b > 0$, $c \geq 0$, $a > -\sqrt{bc}$, $P > \frac{1}{2}$ and the coefficient $A_{V,R}$ are real or complex arbitrary constants and $U \in Z^+$ then we have the following*

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result:

$$\int_0^\infty \left(\frac{h^2}{bh^4+2ah^2+c}\right)^P S_V^U[wX]\ {}^{\gamma}I_{p,\,q}^{m,\,n}(X^e\,y)dh =$$
$$\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}} \times \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{V,R}\, w^R \frac{1}{[2(a+\sqrt{bc})]^R}$$
$$\times\ {}^{\gamma}I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(a+\sqrt{bc})]^{-e}\ \middle|\ \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P-R,\,e,\,1\right),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,p}\\ (1-P-R,\,e,\,1),(g_j,\varrho_j;\mathcal{G}_j)_{1,\,q}\end{matrix}\right]. \tag{25}$$

Theorem (4) is proved in the same way as theorem (3) with the same conditions.
Remark: If we set $U=1, A_{V,0}=1$ and $A_{V,R}=0\ \forall\ R\neq 0$ in theorem 3 and 4 then the result is same as that of theorem 1 and 2.

3 Special Case

In this section, as a particular instance of Theorem 3 and Theorem 4, we establish the Boros integral for the multiplication of Srivastava polynomial with the incomplete $\bar{I}$- function and the incomplete $\bar{H}$- function. Further, some special value will be given to Srivastava polynomial in order to get the outcomes in the form of Hermite and Laguerre polynomials. If we provide the parameter of particular features, we get the following special cases to delineate the use of fundamental outcomes.
(i) Incomplete $\bar{I}$- function: If we set $\mathcal{G}_j=1$ for $1\leq j\leq m$ in (2) and making use of the connection, that is (see [11])

$$\begin{aligned}{}^{\Gamma}\overline{I}_{p,\,q}^{m,\,n}(y) &= {}^{\Gamma}\overline{I}_{p,\,q}^{m,\,n}\left[y\ \middle|\ \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,p}\\ (g_j,\varrho_j;1)_{1,\,m},(g_j,\varrho_j;\mathcal{G}_j)_{m+1,\,q}\end{matrix}\right]\\ &= \frac{1}{2\pi i}\int_{\pounds}\overline{\phi}(r,t)y^r\,dr,\end{aligned} \tag{26}$$

where,

$$\overline{\phi}(r,t) = \frac{\{\Gamma(1-f_1+\varsigma_1 r,t)\}^{\mathcal{F}_1}\prod_{j=1}^{m}\Gamma(g_j-\varrho_j r)\prod_{j=2}^{n}\{\Gamma(1-f_j+\varsigma_j r)\}^{\mathcal{F}_j}}{\prod_{j=n+1}^{p}\{\Gamma(f_j-\varsigma_j r)\}^{\mathcal{F}_j}\prod_{j=m+1}^{q}\{\Gamma(1-g_j+\varrho_j r)\}^{\mathcal{G}_j}}, \tag{27}$$

in (19) and (25), then we obtain the corollaries as follows:

Corollary 1. *For $b>0$, $c\geq 0$, $a>-\sqrt{bc}$, $P>\frac{1}{2}$ and the coefficient $A_{V,R}$ are real or complex arbitrary constants and $U\in Z^+$ then we have the following*

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result:

$$\int_0^\infty \left(\frac{h^2}{bh^4+2ah^2+c}\right)^P S_V^U[wX]\ {}^{\Gamma}\bar{I}_{p,\,q}^{m,\,n}(X^e\,y)dh =$$

$$\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}} \times \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{V,R}\, w^R\, \frac{1}{[2(a+\sqrt{bc})]^R}$$

$$\times\ {}^{\Gamma}\bar{I}_{p+1,\,q+1}^{m,\,n+1}\left[y[2(a+\sqrt{bc})]^{-e}\ \middle|\ \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P-R,\,e;\,1\right),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,p}\\ (1-P-R,\,e;\,1),(g_j,\varrho_j;\,1)_{1,\,m},\,(g_j,\varrho_j;\mathcal{G}_j)_{m+1,\,q}\end{matrix}\right]. \tag{28}$$

Corollary 2. *For $b>0$, $c\geq 0$, $a>-\sqrt{bc}$, $P>\frac{1}{2}$ and the coefficient $A_{V,R}$ are real or complex arbitrary constants and $U\in Z^+$ then we have the following result:*

$$\int_0^\infty \left(\frac{h^2}{bh^4+2ah^2+c}\right)^P S_V^U[wX]\ {}^{\gamma}\bar{I}_{p,\,q}^{m,\,n}(X^e\,y)dh =$$

$$\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}} \times \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{V,R}\, w^R.\, \frac{1}{[2(a+\sqrt{bc})]^R}$$

$$\times\ {}^{\gamma}\bar{I}_{p+1,\,q+1}^{m,\,n+1}\left[y[2(a+\sqrt{bc})]^{-e}\ \middle|\ \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P-R,\,e;\,1\right),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,p}\\ (1-P-R,\,e;\,1),(g_j,\varrho_j;\,1)_{1,\,m},\,(g_j,\varrho_j;G_j)_{m+1,\,q}\end{matrix}\right]. \tag{29}$$

(ii) Incomplete $\bar{H}$- function: If we set $\mathcal{F}_j=1$ for $n+1\leq j\leq p$ and $\mathcal{G}_j=1$ for $1\leq j\leq m$ in (2) and making use of the connection, that is (see [12], [15])

$$\begin{aligned}\overline{\Gamma}_{p,\,q}^{m,\,n}(y) &= \overline{\Gamma}_{p,\,q}^{m,\,n}\left[y\ \middle|\ \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,n},(f_j,\varsigma_j;1)_{n+1,\,p}\\ (g_j,\varrho_j;1)_{1,\,m},(g_j,\varrho_j;\mathcal{G}_j)_{m+1,\,q}\end{matrix}\right]\\ &= \overline{\Gamma}_{p,\,q}^{m,\,n}\left[y\ \middle|\ \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,n},(f_j,\varsigma_j)_{n+1,\,p}\\ (g_j,\varrho_j)_{1,\,m},(g_j,\varrho_j;\mathcal{G}_j)_{m+1,\,q}\end{matrix}\right]\\ &= \frac{1}{2\pi i}\int_{\pounds}\overline{\psi}(r,t)y^r\,dr,\end{aligned} \tag{30}$$

where,

$$\overline{\psi}(r,t)=\frac{\{\Gamma(1-f_1+\varsigma_1 r,t)\}^{\mathcal{F}_1}\prod_{j=1}^{m}\Gamma(g_j-\varrho_j r)\prod_{j=2}^{n}\{\Gamma(1-f_j+\varsigma_j r)\}^{\mathcal{F}_j}}{\prod_{j=n+1}^{p}\Gamma(f_j-\varsigma_j r)\prod_{j=m+1}^{q}\{\Gamma(1-g_j+\varrho_j r)\}^{\mathcal{G}_j}}, \tag{31}$$

in (19) and (25), then we obtain the corollaries as follows.

Corollary 3. *For $b > 0$, $c \geq 0$, $a > -\sqrt{bc}$, $P > \frac{1}{2}$ and the coefficient $A_{V,R}$ are real or complex arbitrary constants and $U \in Z^+$ then we have the following result:*

$$\int_0^\infty \left(\frac{h^2}{bh^4 + 2ah^2 + c}\right)^P S_V^U[wX]\ \bar{\Gamma}_{p,\,q}^{m,\,n}(X^e\,y)dh$$
$$= \frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}} \times \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{V,R}\, w^R\, \frac{1}{[2(a+\sqrt{bc})]^R}$$
$$\times\ \bar{\Gamma}_{p+1,\,q+1}^{m,\,n+1}\left[y[2(a+\sqrt{bc})]^{-e}\ \middle|\ \begin{matrix} (f_1, \varsigma_1; \mathcal{F}_1 : t), \left(\frac{3}{2} - P - R,\, e,\, 1\right), \\ (1 - P - R,\, e,\, 1), (g_j, \varrho_j; 1)_{1,\,m}, \end{matrix}\right.$$
$$\left.\begin{matrix} (f_j, \varsigma_j; \mathcal{F}_j)_{2,\,n},\ (f_j, \varsigma_j; 1)_{n+1,\,p} \\ (g_j, \varrho_j; \mathcal{G}_j)_{m+1,\,q} \end{matrix}\right]. \tag{32}$$

Corollary 4. *For $b > 0$, $c \geq 0$, $a > -\sqrt{bc}$, $P > \frac{1}{2}$ and the coefficient $A_{V,R}$ are real or complex arbitrary constants and $U \in Z^+$ then we have the following result:*

$$\int_0^\infty \left(\frac{h^2}{bh^4 + 2ah^2 + c}\right)^P S_V^U[wX]\ \bar{\gamma}_{p,\,q}^{m,\,n}(X^e\,y)dh$$
$$= \frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}} \times \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{V,R}\, w^R.\, \frac{1}{[2(a+\sqrt{bc})]^R}$$
$$\times\ \bar{\gamma}_{p+1,\,q+1}^{m,\,n+1}\left[y[2(a+\sqrt{bc})]^{-e}\ \middle|\ \begin{matrix} (f_1, \varsigma_1; \mathcal{F}_1 : t), \left(\frac{3}{2} - P - R,\, e,\, 1\right), \\ (1 - P - R,\, e,\, 1), (g_j, \varrho_j; 1)_{1,\,m}, \end{matrix}\right.$$
$$\left.\begin{matrix} (f_j, \varsigma_j; \mathcal{F}_j)_{2,\,n},\ (f_j, \varsigma_j; 1)_{n+1,\,p} \\ (g_j, \varrho_j; \mathcal{G}_j)_{m+1,\,q} \end{matrix}\right]. \tag{33}$$

(iii) Hermite Polynomial: If we set $A_{V,\,R} = (-1)^R$ and $U = 2$ in (9) then $S_V^2[t] \to t^{V/2} H_V\left(\frac{1}{2\sqrt{t}}\right)$ and making use of the connection, that is (see [20]):

$$H_V(t) = \sum_{R=0}^{[V/2]} (-1)^R \frac{V!}{R!(V-2R)!} (2t)^{V-2R}, \tag{34}$$

in (19) and (25), then we obtain the corollaries as follows.

Corollary 5. *For $b > 0$, $c \geq 0$, $a > -\sqrt{bc}$ and $P > \frac{1}{2}$ then we have the following*

result:

$$\int_0^\infty \left(\frac{h^2}{bh^4+2ah^2+c}\right)^P (Xw)^{\frac{V}{2}} H_V\left(\frac{1}{2\sqrt{Xw}}\right) {}^{\Gamma}I_{p,\,q}^{m,\,n}(X^e\,y)dh =$$

$$\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}} \times \sum_{R=0}^{[V/2]} \frac{(-1)^R\,V!}{R!\,(V-2R)!}\,w^R \cdot \frac{1}{[2(a+\sqrt{bc})]^R}$$

$$\times\ {}^{\Gamma}I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(a+\sqrt{bc})]^{-e}\ \middle|\ \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P-R,\,e,\,1\right),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,p}\\ (1-P-R,\,e,\,1),(g_j,\varrho_j;\mathcal{G}_j)_{1,\,q}\end{matrix}\right]. \tag{35}$$

Corollary 6. *For* $b>0,\ c\geq 0,\ a>-\sqrt{bc}$ *and* $P>\frac{1}{2}$ *then we have the following result:*

$$\int_0^\infty \left(\frac{h^2}{bh^4+2ah^2+c}\right)^P (Xw)^{\frac{V}{2}} H_V\left(\frac{1}{2\sqrt{Xw}}\right) {}^{\gamma}I_{p,\,q}^{m,\,n}(X^e\,y)dh =$$

$$\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}} \times \sum_{R=0}^{[V/2]} \frac{(-1)^R\,V!}{R!\,(V-2R)!}\,w^R \cdot \frac{1}{[2(a+\sqrt{bc})]^R}$$

$$\times\ {}^{\gamma}I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(a+\sqrt{bc})]^{-e}\ \middle|\ \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P-R,\,e,\,1\right),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,p}\\ (1-P-R,\,e,\,1),(g_j,\varrho_j;\mathcal{G}_j)_{1,\,q}\end{matrix}\right]. \tag{36}$$

(iv) Laguerre Polynomial: If we set $A_{V,\,R}=\binom{V+\alpha}{V}\frac{1}{(\alpha+1)_R}$ and $U=1$ in (9) then $S_V^1[t]\rightarrow L_V^{(\alpha)}(t)$ and making use of the connection, that is (see [20]).

$$L_V^{\alpha}(t)=\sum_{R=0}^{V}\binom{V+\alpha}{V-R}\frac{(-t)^R}{R!}, \tag{37}$$

in (19) and (25), then we obtain the corollaries as follows.

Corollary 7. *For* $b>0,\ c\geq 0,\ a>-\sqrt{bc}$ *and* $P>\frac{1}{2}$ *then we have the following result:*

$$\int_0^\infty \left(\frac{h^2}{bh^4+2ah^2+c}\right)^P L_V^{(\alpha)}[wX]\ {}^{\Gamma}I_{p,\,q}^{m,\,n}(X^e\,y)dh =$$

$$\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}} \times \sum_{R=0}^{V}\binom{V+\alpha}{V-R}\frac{(-w)^R}{R!}\,\frac{1}{[2(a+\sqrt{bc})]^R}$$

$$\times\ {}^{\Gamma}I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(a+\sqrt{bc})]^{-e}\ \middle|\ \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P-R,\,e,\,1\right),(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,p}\\ (1-P-R,\,e,\,1),(g_j,\varrho_j;\mathcal{G}_j)_{1,\,q}\end{matrix}\right]. \tag{38}$$

Corollary 8. *For $b > 0$, $c \geq 0$, $a > -\sqrt{bc}$ and $P > \frac{1}{2}$ then we have the following result:*

$$\int_0^\infty \left(\frac{h^2}{bh^4+2ah^2+c}\right)^P L_V^{(\alpha)}[wX]\ {}^{\gamma}I_{p,\,q}^{m,\,n}(X^e\,y)dh =$$

$$\frac{\sqrt{\pi}}{2\sqrt{b}\left[2\left(a+\sqrt{bc}\right)\right]^{P-\frac{1}{2}}} \times \sum_{R=0}^{V} \binom{V+\alpha}{V-R} \frac{(-w)^R}{R!} \frac{1}{[2(a+\sqrt{bc})]^R}$$

$$\times\ {}^{\gamma}I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(a+\sqrt{bc})]^{-e}\ \middle|\ \begin{array}{c}(f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P-R,\,e,\,1\right)(f_j,\varsigma_j;\mathcal{F}_j)_{2,\,p}\\ (1-P-R,\,e,\,1),(g_j,\varrho_j;\mathcal{G}_j)_{1,\,q}\end{array}\right]. \tag{39}$$

Remark: If we set $U = 1, A_{V,0} = 1$ and $A_{V,R} = 0\ \forall\ R \neq 0$ for the the first four corollaries (Corollary 1- Corollary 4) then it becomes the special case for Theorem 1 and 2.

4 Concluding Remarks

In this article, we obtain the Boros integral with three parameter for the incomplete I- function which is the extension of the I- function investigated by Jangid et al. [10] and we also study Boros integral for the product of incomplete I- function and Srivastava Polynomial. As the incomplete I- function generalize varity of incomplete functions like: I- function, Meijer G- function, hypergeometric function, H-function, $\bar{I}$- function and many other functions.

Also, Srivastava Polynomial generalize various other polynomial like: Hermite polynomial, Jacobi polynomial, Laguerre polynomial, Gegenbauer polynomial, Legendre polynomial, Tchebycheff polynomial, Gould-Hopper Polynomial and several other polynomials. Our main findings are therefore important and can be used to count the many Boros integral forms associated with various special functions and polynomials.

Acknowledgements

The authors would like to thank anonymous referees for their useful critical comments and suggestions for improving the research paper.

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Two-State Retrial Queueing Model with Catastrophe

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December 29, 2022

Abstract

The present paper discusses a two-state retrial queueing system with catastrophe. If a customer on arrival finds the server free then it is served immediately. Such customer is known as primary customer. Moreover, if the server is busy then the customer joins virtual queue and retries for service after a random amount of time. This customer is called a secondary customer. Primary and secondary customers both follow Poisson processes. Inter arrival times and service times both follow exponential distribution. Catastrophe occurs on a busy server and its occurrence follows a Poisson process. Catastrophe causes the failure of server and so the server is sent for repair after occurrence of catastrophe. The repair times are also exponentially distributed. Time dependent probabilities for exact number of arrivals in the system and departures from the system when the server is idle or busy are obtained by using recursive approach. The probability of server being under repair is also obtained. Verification of results is done. Numerical results are generated and represented graphically to study the effect of various parameters.

Keywords: Retrial, Arrivals, Departures, Catastrophe, Repair.

1 Introduction

In many real life situations it has been observed that a customer does not get the service instantly on arrival. So he tries again for the service after a random period of time which is popularly known as retrial. The queueing systems with these repeated attempts have been used in many fields such as telecommunication, computer networks, data transmission, etc. Analysis of such systems developed a new class of queueing systems which is known as retrial queueing

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systems. Retrial queue is a model of the system with finite capacity where if the arrival finds a free server, it is served immediately. However, if the server is not free, the customer leaves the service area and joins the virtual queue known as orbit. Thereafter it retries from the orbit after a random amount of time to get service.

Call centers serve as a basic example to retrial systems where call agent is the server and a person who is calling is the customer. If a customer is able to connect the call agent immediately after making a call, he is answered else he has to repeat the call.

[6] is the early work done on retrial queues. [16] discussed some important single server retrial queuing models and represented analytic results. [10] analyzed the single server retrial queue with finite number of sources and established customer's arrival distribution, busy period and waiting time process. An explanation of the retrial queueing system is shown in the following diagram:

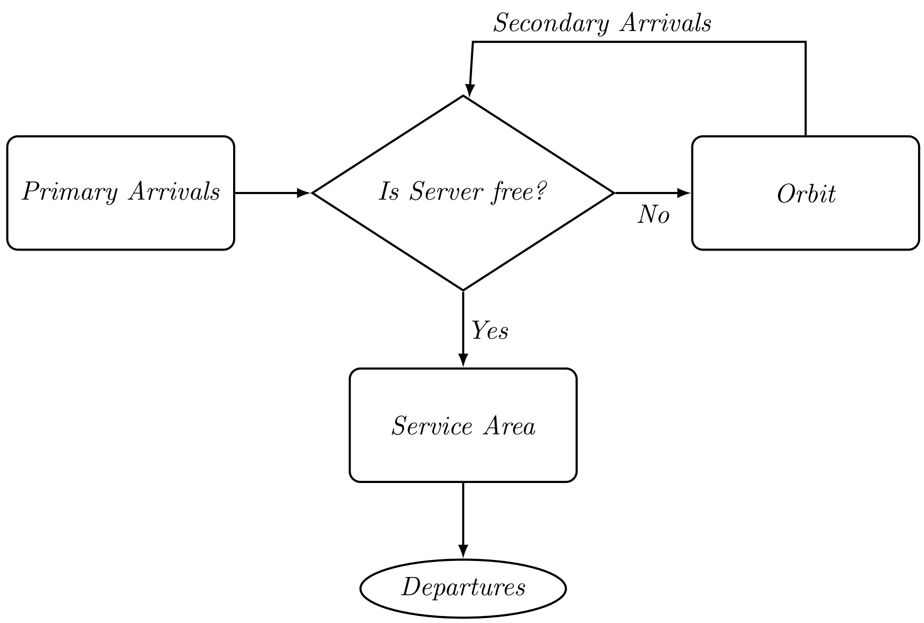

Figure 1. Basic Structure of a Retrial Queueing System

[13] was the first who introduced the concept of two-state in 'Some New Results for the M/M/1 Queue'. In this paper they obtained a closed form solution for the probability that exactly i arrivals and j services occur over a time interval of length t. [14] studied the two-state single server retrial queuing model in which the time dependent probabilities of exact number of arrivals and departures in the system are obtained when the server is free or busy. [15] developed 'A two-state retrial queueing model with feedback having two identical parallel servers' in which transient solution is obtained for the retrial queueing model.

In recent years many researchers have shown interest towards the concept of catastrophe. Catastrophe is a sudden, unexpected failure in a machine, computer network, electronic system, communication system, etc. Catastrophe occurs at random, deletes all the customers present in the system and inactivates the service facilities for a short period of time. Catastrophe resets the system from current state to zero state at random time intervals. Catastrophe may come from outside the system or from another service station. Retrial queueing system with catastrophe can be seen in call centers, computer networks and in telecommunication networks. In population dynamics, catastrophe can be considered as the natural disasters such as floods, storms, etc. On the other hand in the queueing models, catastrophe makes the system empty and causes server's breakdown.

For example: In call centers with the occurrence of catastrophic events like power failure, virus attacks will result in loss of all the calls present at that time and breakdown of the network.

[8], [5] are the works done on Catastrophe occurring in a simple Markovian queue. [11] discussed the asymptotic behavior of the probability of server being free. [12] worked on M/M/1 queuing system with catastrophes. Transient solution is obtained for system with server failure and non-zero repair time.

Furthermore, the server is sent for repair immediately when the failure occurs. After getting repaired, the server comes back to its working position and the system becomes ready to serve new customers. [4] proposed $M/M/\infty$ queueing system with catastrophe and repairable servers.

Following diagram shows the basic structure of retrial queueing system with catastrophe.

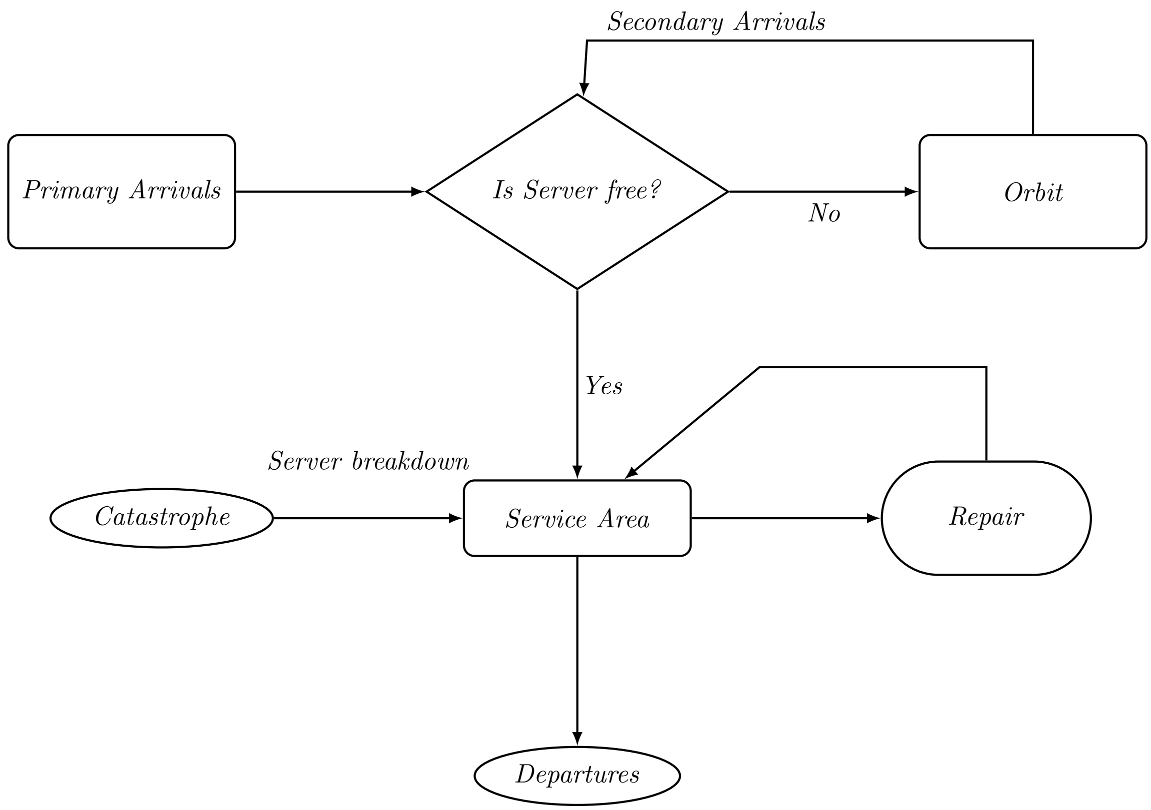

Figure 2. Basic Structure of a Retrial Queue With Catastrophe.

[1] studied the transient behaviour of two-processor heterogeneous system with catastrophes, server failures and repairs. [7] studied a fractional $M/M/1$ queue with catastrophe. [3] obtained transient solution of markovin queues with catastrophe having infinite servers.

The transient solutions are used to study the dynamic behaviour of a system. They are useful to study the characteristics of a system on different time points. Therefore, transient analysis of queueing systems is extremely important from theoretical and practical perspective.

In this paper, we derive two-state time-dependent probabilities for exact number of arrivals to the system and departures from the system by time t when the server is idle or busy for a single server retrial queueing system. The factor two-state makes the results well-quantified as in the case of [13]. Also we obtain the probability of server being under repair when the server fails due to catastrophic events at random time intervals. Besides these theoretical solutions, we present some numerical results graphically to study the effect of various parameters and the behaviour of probabilities with respect to average service times.

The paper has been organized in the following sections:

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In section 2 the complete mathematical description of the model is defined. Also, the difference-differential equations are derived in this section. Solution of the model is given in section 3 in which we obtained the transient state probabilities and the probability of server being under repair. In section 4 verification of results is done. The numerical results are obtained and represented graphically in section 5. In section 6 the busy period probabilities of system and the server are obtained numerically and presented graphically. Section 7 discusses the conclusion and in section 8 acknowledgment is given. Finally the references are listed.

2 Model Description

In this paper, we are considering a two-state single server retrial queueing system with catastrophe. In this system, customers arrive according to Poisson process. If on arrival customer finds the server busy, he joins the orbit and retries from the orbit. These retrials are considered to be secondary arrivals. Catastrophe occurs according to Poisson process. We are assuming in our model that catastrophe occurs only when the system is non-empty and when the server is busy. It has no effect on the system when the system is empty. Catastrophe makes system empty and also causes server's breakdown by deleting all the customers present in the system. Once the system becomes empty or when the server breaks down, it is sent for repair immediately. Further, it is assumed that during the repair time no arrival can take place. The detailed description of the model is given as follows:

- Arrival Process: The primary customers arrive at the system according to Poisson process with mean arrival rate λ.
- Retrial Process: The secondary customers arrive at the system according to Poisson process with mean retrial rate θ.
- Service Process: The service times are exponentially distributed with parameter μ.
- Catastrophe: Catastrophe follow Poisson process with mean rate ξ.
- Repair: The repair time is exponentially distributed with parameter τ.

Also, the primary and secondary arrivals, inter-arrival times, service times, departures and catastrophes are mutually independent.

Laplace transformation $\bar{f}(s)$ of $f(t)$ is given by:

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t) dt; \quad Re(s) > 0$$

The Laplace inverse of

$$\frac{Q(p)}{P(p)} = \sum_{k=1}^{n}\sum_{l=1}^{m_k} \frac{t^{m_k-l}e^{a_k t}}{(m_k-l)!(l-1)!} \times \frac{d^{l-1}}{dp^{l-1}}\left(\frac{Q(p)}{P(p)}\right)(p-a_k)^{m_k} \quad \forall p = a_k, \quad a_i \neq a_k \; for \; i \neq k$$

where

$$P(p) = (p-a_1)^{m_1}(p-a_2)^{m_2}.......(p-a_n)^{m_n}$$

Q(p) is a polynomial of degree $< m_1 + m_2 + m_3 + ... + m_n - 1$

$$If \;\; L^{-1}\{f(s)\} = F(t) \;\; and \;\; L^{-1}\{g(s)\} = G(t) \;\; then$$
$$L^{-1}\{f(s)g(s)\} = \int_0^t F(u)G(t-u)du = F * G$$

$F * G$ is called the convolution of F and G.

2.1 The Two-Dimensional State Model

2.1.1 Notations

$P_{i,j,0}(t)$ = Probability that there are exactly i number of arrivals in the system and j number of departures from the system by time t and the server is free.

$P_{i,j,1}(t)$ = Probability that there are exactly i arrivals, j departures from the system by time t and the server is busy.

$Q(t)$=Probability that the server is under repair by time t.

$P_{i,j}(t)$ = Probability that there are exactly i arrivals in the system and j departures from the system by time t.

$$P_{i,j}(t) = P_{i,j,0}(t) + P_{i,j,1}(t) \qquad \forall\, i,j \;\; i \geq j;$$
$$and \qquad P_{i,j,1}(t) = 0, \; i \leq j; \qquad P_{i,j,0}(t) = 0, \quad i < j;$$

Initially

$$P_{0,0,0}(0) = 1; \qquad P_{i,j,0}(0) = 0, \quad P_{i,j,1}(0) = 0, \; i,j \neq 0; \qquad Q(0) = 0;$$

2.1.2 The Difference-Differential equations governing the system are:

$$\frac{d}{dt}P_{i,j,0}(t) = -(\lambda + (i-j)\theta)P_{i,j,0}(t) + \mu P_{i,j-1,1}(t) \qquad i \geq j > 0 \tag{1}$$

$$\frac{d}{dt}P_{0,0,0}(t) = -\lambda P_{0,0,0}(t) + \tau Q(t) \tag{2}$$

$$\frac{d}{dt}P_{i,j,1}(t) = -(\lambda + \mu + \xi)P_{i,j,1}(t) + \lambda P_{i-1,j,0}(t) + \lambda P_{i-1,j,1}(t)(1 - \delta_{i-1,j}) + (i-j)\theta P_{i,j,0}(t) \qquad i > j \geq 0 \tag{3}$$

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$$\frac{d}{dt}Q(t) = -\tau Q(t) + \xi \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} P_{i,j,1}(t) \tag{4}$$

where

$$\delta_{i-1,j} = \begin{cases} 1 & \text{when } i-1=j \\ 0 & \text{otherwise} \end{cases}$$

Using the Laplace transformation $\bar{f}(s)$ of $f(t)$ given by

$$\bar{f}(s) = \int_0^{\infty} e^{-st} f(t)dt, \quad Re(s) > 0$$

in the equations (1)-(4) along with the initial conditions, we have

$$(s+\lambda+(i-j)\theta)\bar{P}_{i,j,0}(s) = \mu\bar{P}_{i,j-1,1}(s) \qquad i \geq j > 0 \tag{5}$$

$$\begin{aligned}(s+\lambda+\mu+\xi)\bar{P}_{i,j,1}(s) = {} & \lambda\bar{P}_{i-1,j,0}(s) + \lambda\bar{P}_{i-1,j,1}(s)(1-\delta_{i-1,j}) + \\ & (i-j)\theta\bar{P}_{i,j,0}(s) \qquad i > j \geq 0\end{aligned} \tag{6}$$

$$(s+\lambda)\bar{P}_{0,0,0}(s) = 1 + \tau\bar{Q}(s) \tag{7}$$

$$(s+\tau)\bar{Q}(s) = \xi \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \bar{P}_{i,j,1}(s) \tag{8}$$

where

$$\delta_{i-1,j} = \begin{cases} 1 & \text{when } i-1=j \\ 0 & \text{otherwise} \end{cases}$$

3 Solution of the Problem

Solving equations (5)-(8) recursively, we have

$$\bar{P}_{0,0,0}(s) = \frac{1}{s+\lambda} + \frac{\tau}{s+\lambda}\bar{Q}(s) \tag{9}$$

$$\bar{P}_{i,0,1}(s) = \left(\frac{\lambda}{s+\lambda+\mu+\xi}\right)^i \left(\frac{1}{s+\lambda} + \frac{\tau}{s+\lambda}\bar{Q}(s)\right) \qquad i \geq 1 \tag{10}$$

$$\bar{P}_{i,1,0}(s) = \left[\frac{\mu}{s+\lambda+(i-1)\theta}\left(\frac{\lambda}{s+\lambda+\mu+\xi}\right)^i \left(\frac{1}{s+\lambda} + \frac{\tau}{s+\lambda}\bar{Q}(s)\right)\right] \quad i \geq 1 \tag{11}$$

$$\bar{P}_{i,i,0}(s) = \frac{\mu}{s+\lambda}\left[\frac{\lambda}{s+\lambda+\mu+\xi}\bar{P}_{i-1,i-1,0}(s) + \frac{\theta}{s+\lambda+\mu+\xi}\bar{P}_{i,i-1,0}(s)\right] \quad i \geq 2 \tag{12}$$

$$\bar{P}_{i,i-1,1}(s) = \frac{\lambda}{s+\lambda+\mu+\xi}\bar{P}_{i-1,i-1,0}(s) + \frac{\theta}{s+\lambda+\mu+\xi}\bar{P}_{i,i-1,0}(s) \qquad i \geq 2 \tag{13}$$

$$\bar{P}_{i,j,0}(s) = \frac{\mu}{s+\lambda+(i-j)\theta}\Bigg[\sum_{k=1}^{i-j+1}\left(\frac{\lambda}{s+\lambda+\mu+\xi}\right)^{i-j-k+1}\eta_k'(s)\bar{P}_{j+k-1,j-1,0}(s) + \left(\frac{\lambda}{s+\lambda+\mu+\xi}\right)^{i-j}\bar{P}_{j,j-1,1}(s)\Bigg] \qquad i>j>1 \tag{14}$$

where

$$\eta_k' = \begin{cases} 1 & \text{if } k=1 \\ 1+\dfrac{k\theta}{s+\lambda+\mu+\xi} & \text{if } k=2 \ to \ i-j \\ \dfrac{k\theta}{s+\lambda+\mu+\xi} & \text{if } k=i-j+1 \end{cases}$$

$$\bar{P}_{i,j,1}(s) = \sum_{k=1}^{i-j}\Bigg[\left(\frac{\lambda}{s+\lambda+\mu+\xi}\right)^{i-j-k}\eta_k'(s)\bar{p}_{j+k,j,0}(s)\Bigg] + \left(\frac{\lambda}{s+\lambda+\mu+\xi}\right)^{i-j-1}\bar{P}_{j+1,1,1}(s) \qquad i\geq j+2, j\geq 1 \tag{15}$$

where

$$\eta_k' = \begin{cases} 1 & \text{if } k=1 \\ 1+\dfrac{k\theta}{s+\lambda+\mu+\xi} & \text{if } k=2 \ to \ i-j-1 \\ \dfrac{k\theta}{s+\lambda+\mu+\xi} & \text{if } k=i-j \end{cases}$$

$$\bar{Q}(s) = \frac{\xi}{s+\tau}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\bar{P}_{i,j,1}(s) \tag{16}$$

Taking the Inverse Laplace transform of equations $(9)-(16)$, we have

$$P_{0,0,0}(t) = e^{-\lambda t} + \tau e^{-\lambda t} * Q(t) \tag{17}$$

$$P_{i,0,1}(t) = \lambda^i e^{-\lambda t}\Bigg[\frac{1}{(\mu+\xi)^i} - e^{-(\mu+\xi)t}\sum_{r=0}^{i-1}\frac{t^r}{r!}\frac{1}{(\mu+\xi)^{i-r}}\Bigg] + \tau\lambda^i \Bigg[\frac{1}{(\mu+\xi)^i} - e^{-(\mu+\xi)t}\sum_{r=0}^{i-1}\frac{t^r}{r!}\frac{1}{(\mu+\xi)^{i-r}}\Bigg] * Q(t) \qquad i\geq 1 \tag{18}$$

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$$P_{i,1,0}(t) = \mu e^{-(\lambda+(i-1)\theta)t} * P_{i,0,1}(t) \qquad i \geq 1 \tag{19}$$

$$P_{i,i,0}(t) = \mu\lambda e^{-\lambda t}\left[\frac{1}{\mu+\xi} - \frac{e^{-(\mu+\xi)t}}{\mu+\xi}\right] * P_{i-1,i-1,0}(t) + \mu\theta e^{-\lambda t} \left[\frac{1}{\mu+\xi} - \frac{e^{-(\mu+\xi)t}}{\mu+\xi}\right] * P_{i,i-1,0}(t) \qquad i > 1 \tag{20}$$

$$P_{i,i-1,1}(t) = \lambda e^{-(\lambda+\mu+\xi)t} * P_{i-1,i-1,0}(t) + \theta e^{-(\lambda+\mu+\xi))t} * P_{i,i-1,0}(t) \qquad i > 1 \tag{21}$$

$$\begin{aligned} P_{i,j,0}(t) &= \mu\lambda^{i-j} e^{-(\lambda+(i-j)\theta)t}\left[\frac{1}{(\mu+\xi)^{i-j}} - e^{-(\mu+\xi)t}\sum_{r=0}^{i-j-1}\frac{t^r}{r!}\frac{1}{(\mu+\xi)^{i-j-r}}\right] * P_{j,j-1,0}(t) \\ &+ e^{-(\lambda+(i-j)\theta)t}\sum_{k=2}^{i-j}\mu\lambda^{i-j-k+1}\left[\frac{1}{(\mu+\xi)^{i-j-k+1}} - e^{-(\mu+\xi)t}\sum_{r=0}^{i-j-k}\frac{t^r}{r!}\frac{1}{(\mu+\xi)^{i-j-k-r+1}}\right] \\ &* P_{j+k-1,j-1,0}(t) + e^{-(\lambda+(i-j)\theta)t}\sum_{k=2}^{i-j}(\mu k\theta)\lambda^{i-j-k+1}\left[\frac{1}{(\mu+\xi)^{i-j-k+2}}\right. \\ &\left. - e^{-(\mu+\xi)t}\sum_{r=0}^{i-j-k+1}\frac{t^r}{r!}\frac{1}{\mu^{i-j-k-r+2}}\right] * P_{j+k-1,j-1,0}(t) + e^{-(\lambda+(i-j)\theta)t} \\ &((i-j+1)\mu\theta)\left[\frac{1}{\mu+\xi} - \frac{e^{-(\mu+\xi)t}}{\mu+\xi}\right] * P_{i,j-1,0}(t) + \mu\lambda^{i-j} e^{-(\lambda+(i-j)\theta)t} \\ &\left[\frac{1}{(\mu+\xi)^{i-j}} - e^{-(\mu+\xi)t}\sum_{r=0}^{i-j-1}\frac{t^r}{r!}\frac{1}{(\mu+\xi)^{i-j-r}}\right] * P_{j,j-1,1}(t) \qquad i > j > 1 \end{aligned} \tag{22}$$

$$\begin{aligned} P_{i,j,1}(t) &= \lambda^{i-j-1} e^{-(\lambda+\mu+\xi)t}\frac{t^{i-j-2}}{(i-j-2)!} * P_{j+1,j,0}(t) + e^{-(\lambda+\mu+\xi)t}\sum_{k=2}^{i-j-1}\lambda^{i-j-k} \\ &\frac{t^{i-j-k-1}}{(i-j-k-1)!} * P_{j+k,j,0}(t) + e^{-(\lambda+\mu+\xi)t}\sum_{k=2}^{i-j-1}k\theta\lambda^{i-j-k}\frac{t^{i-j-k}}{(i-j-k)!} * P_{j+k,j,0}(t) + \\ &(i-j)\theta e^{-(\lambda+\mu+\xi)t} * P_{i,j,0}(t) + \lambda^{i-j-1}e^{-(\lambda+\mu+\xi)t}\frac{t^{i-j-2}}{(i-j-2)!} * P_{j+1,j,1}(t) \\ & \qquad i \geq j+2, j \geq 1 \end{aligned} \tag{23}$$

$$Q(t) = \xi e^{-\tau t}\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}P_{i,j,1}(t) \tag{24}$$

4 Verification of Results

- Summing equations (9)-(16) over i and j we get,

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\left[\bar{P}_{i,j,0}(s)+\bar{P}_{i,j,1}(s)\right]+\bar{Q}(s)=\frac{1}{s}$$

  and hence

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\left[P_{i,j,0}(t)+P_{i,j,1}(t)\right]+Q(t)=1$$

  which is the verification of our results.

- Define $U_{n,l}(t)$= Probability that there are n customers in the orbit at time t. The server is idle when $l=0$ and server is busy when $l=1$.
  When the server is idle, it is represented as $U_{n,0}(t)$:

$$U_{n,0}(t)=\sum_{j=0}^{\infty}P_{j+n,j,0}(t)$$

  where n is the number of customers in the orbit, which can be calculated by using the following formula:
  n=(number of arrivals - number of departures).
  When the server is busy, it is represented as $U_{n,1}(t)$:

$$U_{n,1}(t)=\sum_{j=0}^{\infty}P_{j+n+1,j,1}(t)$$

  In this case:
  n= (number of arrivals - number of departures - 1).
  Using the above definitions in (1)-(4) and let $\xi=0$, $\tau=0$ the equations in statistical equilibrium are:

$$(\lambda+n\theta)\,U_{n,0}=\mu\,U_{n,1} \qquad n\geq 0$$

$$(\lambda+\mu)\,U_{n,1}=\lambda(U_{n,0}+U_{n-1,1})+(n+1)\theta\,U_{(n+1),0} \quad n\geq 2$$

  which coincides with the results (1.5) and (1.6) of [9]

5 Numerical Solution and Graphical Representation

The Numerical results are generated using MATLAB programming for the case $\rho=\left(\frac{\lambda}{\mu}\right)$=0.5, $\eta=\left(\frac{\theta}{\mu}\right)$=0.6, $\tau'=\left(\frac{\tau}{\mu}\right)$=0.4, $\xi'=\left(\frac{\xi}{\mu}\right)$=0.3. In following tables, we observe some significant probabilities at various time instants whose sum approaches to 1.

Table I. At time t =1

t	$P_{0,0,0}$	$P_{1,1,0}$	$P_{2,1,0}$	$P_{2,2,0}$	$P_{3,1,0}$	$P_{3,2,0}$	$P_{3,3,0}$	$P_{4,1,0}$	$P_{4,2,0}$	$P_{1,0,1}$	$P_{2,0,1}$
1	0.6118	0.1029	0.0117	0.0043	0.0012	0.0007	0.0001	0.0001	0.0001	0.1702	0.0335

$P_{2,1,1}$	$P_{3,0,1}$	$P_{3,1,1}$	Sum
0.0153	0.0049	0.0032	0.96

Table II. At time t =5

t	$P_{0,0,0}$	$P_{1,1,0}$	$P_{2,1,0}$	$P_{2,2,0}$	$P_{4,1,0}$	$P_{4,2,0}$	$P_{4,3,0}$	$P_{4,4,0}$	$P_{5,1,0}$	$P_{5,2,0}$
5	0.1769	0.1584	0.0205	0.099	0.0009	0.0058	0.0128	0.0096	0.0004	0.0024

$P_{5,3,0}$	$P_{5,4,0}$	$P_{5,5,0}$	$P_{1,0,1}$	$P_{2,0,1}$	$P_{2,1,1}$	$P_{3,0,1}$	$P_{3,1,1}$	$P_{3,2,1}$	$P_{4,0,1}$
0.0063	0.0073	0.0032	0.0546	0.0174	0.0553	0.0057	0.0258	0.0332	0.0019

$P_{4,1,1}$	$P_{4,2,1}$	$P_{4,3,1}$	$P_{5,0,1}$	$P_{5,1,1}$	$P_{5,2,1}$	$P_{5,3,1}$	$P_{5,4,1}$	$Q(t)$	Sum
0.0098	0.0182	0.0118	0.0008	0.0047	0.0104	0.0105	0.0041	0.1663	0.934

Table III. At time t =15

t	$P_{0,0,0}$	$P_{1,1,0}$	$P_{2,1,0}$	$P_{2,2,0}$	$P_{3,1,0}$	$P_{3,2,0}$	$P_{3,3,0}$	$P_{5,1,0}$	$P_{5,2,0}$	$P_{5,3,0}$
15	0.1544	0.0866	0.0109	0.0575	0.002	0.011	0.0433	0.0001	0.0006	0.0025

$P_{5,4,0}$	$P_{5,5,0}$	$P_{6,5,0}$	$P_{6,6,0}$	$P_{7,3,0}$	$P_{7,4,0}$	$P_{7,5,0}$	$P_{7,6,0}$	$P_{7,7,0}$	$P_{1,0,1}$
0.0086	0.0271	0.0079	0.0186	0.0002	0.0008	0.0024	0.0061	0.0108	0.043

$P_{2,0,1}$	$P_{2,1,1}$	$P_{3,0,1}$	$P_{3,1,1}$	$P_{3,2,1}$	$P_{4,0,1}$	$P_{4,1,1}$	$P_{4,2,1}$	$P_{4,3,1}$	$P_{6,0,1}$
0.0119	0.0277	0.0033	0.012	0.0199	0.0009	0.0043	0.0104	0.0157	0.0001

$P_{6,1,1}$	$P_{6,2,1}$	$P_{6,3,1}$	$P_{6,4,1}$	$P_{6,5,1}$	$P_{8,2,1}$	$P_{8,4,1}$	$P_{8,5,1}$	$P_{8,6,1}$	$P_{8,7,1}$	$Q(t)$
0.0005	0.0016	0.004	0.0078	0.0102	0.0003	0.0026	0.0056	0.0095	0.0102	0.1914

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$P_{5,0,1}$	$P_{5,1,1}$	$P_{5,2,1}$	$P_{5,3,1}$	$P_{5,4,1}$	$P_{7,3,1}$	$P_{7,4,1}$	$P_{7,5,1}$	$P_{7,6,1}$	Sum
0.0003	0.0014	0.0042	0.0089	0.0129	0.0016	0.0037	0.0065	0.0073	0.8911

Table IV. At time t =25

t	$P_{0,0,0}$	$P_{1,1,0}$	$P_{2,2,0}$	$P_{3,3,0}$	$P_{4,4,0}$	$P_{5,5,0}$	$P_{6,6,0}$	$P_{7,7,0}$	$P_{8,8,0}$	$P_{1,0,1}$	$P_{5,0,1}$
25	0.1391	0.0798	0.0525	0.0369	0.027	0.0204	0.0158	0.0125	0.1515	0.0389	0.0002

$P_{7,5,1}$	$P_{7,6,1}$	$P_{6,4,1}$	$P_{5,3,1}$	$P_{5,4,0}$	$P_{7,6,0}$	$P_{7,5,0}$	$P_{5,4,1}$	$P_{4,1,1}$	$P_{6,5,1}$
0.0048	0.0059	0.0061	0.0077	0.007	0.0044	0.0017	0.0099	0.004	0.0076

$P_{7,4,1}$	$P_{2,1,1}$	$P_{4,2,1}$	$P_{5,2,1}$	$P_{6,5,0}$	$P_{2,0,1}$	$P_{3,2,1}$	$P_{6,3,1}$	$Q(t)$	Sum
0.0029	0.0257	0.0095	0.0039	0.0056	0.0109	0.0181	0.0034	0.1694	0.8831

Table V. At time t =40

t	$P_{6,6,0}$	$P_{7,7,0}$	$P_{5,5,0}$	$P_{5,4,0}$	$P_{3,2,0}$	$P_{7,6,0}$	$P_{4,4,0}$	$P_{6,5,1}$	$P_{7,3,1}$	$P_{7,4,1}$	$P_{7,6,1}$
40	0.0131	0.0101	0.0171	0.0059	0.0085	0.0037	0.0227	0.0064	0.0011	0.0024	0.0049

$P_{0,0,0}$	$P_{1,1,0}$	$P_{1,0,1}$	$P_{2,1,1}$	$P_{3,1,1}$	$P_{4,3,1}$	$P_{4,2,1}$	$P_{5,3,1}$	$P_{5,4,1}$	$P_{7,5,1}$	$P_{2,1,0}$
0.1153	0.0661	0.0323	0.0213	0.0093	0.0111	0.0079	0.0064	0.0083	0.0024	0.0083

$P_{2,2,0}$	$P_{6,4,0}$	$P_{3,3,0}$	$Q(t)$	$P_{3,1,0}$	$P_{4,3,0}$	$P_{5,2,0}$	$P_{5,3,0}$	$P_{8,8,0}$	$P_{2,0,1}$	$P_{3,0,1}$
0.0436	0.0017	0.0309	0.1406	0.0015	0.0073	0.0005	0.0019	0.295	0.0.009	0.0025

$P_{3,2,1}$	$P_{4,0,1}$	$P_{4,1,1}$	Sum
0.0151	0.0.0007	0.0033	0.9382

The probabilities against time are represented graphically in the following figures.

12

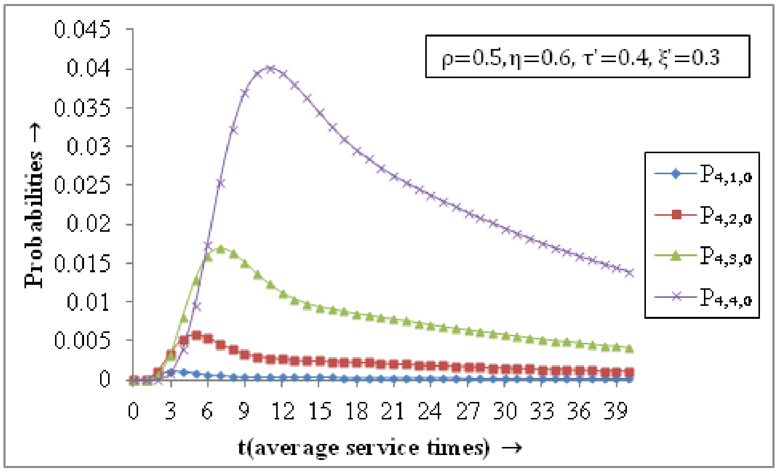

Figure 3. Probabilities $P_{4,1,0}$, $P_{4,2,0}$, $P_{4,3,0}$ and $P_{4,4,0}$ against average service times

In figure 3, the probabilities $P_{4,1,0}$, $P_{4,2,0}$, $P_{4,3,0}$ and $P_{4,4,0}$ are plotted against time t for the given case. It is observed that all the probabilities increase initially and then decrease. Also it can be seen that the probabilities attain higher values for greater number of departures.

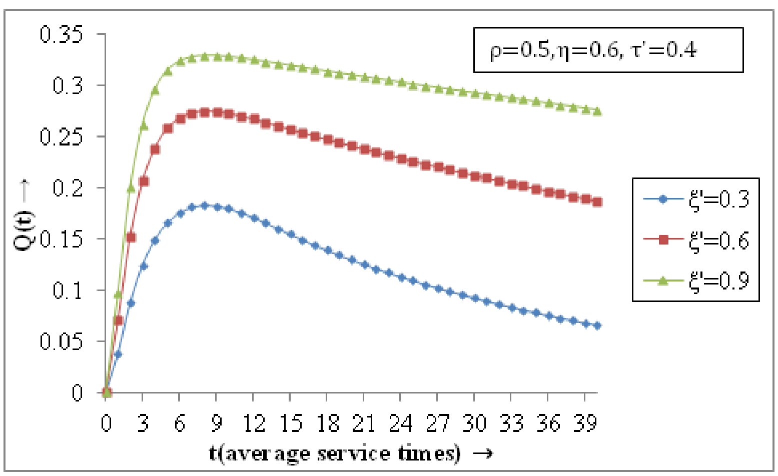

Figure 4. Effect of change in $\xi^{'}$ on the probability $Q(t)$

In figure 4, we study the effect of change in $\xi^{'}$(catastrophe rate per unit service time) on the probability $Q(t)$(probability of server being under repair). From the graph it can be seen that whenever the catastrophe rate per unit service time increases, the probability $Q(t)$ also increases which is as desired.

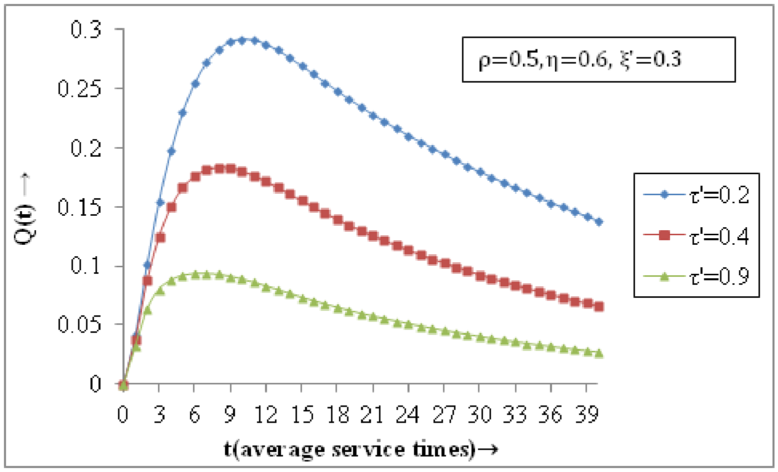

Figure 5. Effect of change in $\tau^{'}$ on the probability $Q(t)$

In figure 5, the effect of change in $\tau^{'}$(repair rate per unit service time) on the probability $Q(t)$ is studied. From the graph it is clearly visible that whenever the repair rate per unit service time increases, the probability $Q(t)$ decreases.

6 Busy Period Probabilities

In this section we discuss the busy period probabilities of the server and the system.
The Probability of busy server is given by:

$$P(Server\ is\ busy) = \sum_{i>j\geq 0} P_{i,j,1}(t) \tag{25}$$

The Probability of busy system is given by:

$$P(System\ is\ busy\ ) = \sum_{i>j\geq 0} \Big(P_{i,j,0}(t) + P_{i,j,1}(t)\Big) + Q(t) \tag{26}$$

6.1 Numerical and Graphical Representation of Busy Period Probabilities

The numerical results are obtained using MATLAB programming and following [2]. The Probabilities of system busy and server busy are obtained for different values of ρ keeping other parameters constant and are presented in the table given below.

Table VI. Probabilities of System Busy and Server Busy

t	Probability(System Busy)			Probability(Server Busy)		
	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
0	0	0	0	0	0	0
1	0.2788	0.369	0.4485	0.2286	0.2967	0.3546
2	0.3858	0.5009	0.5956	0.2626	0.3298	0.3827
3	0.4432	0.5683	0.6651	0.2678	0.3315	0.3785
4	0.4779	0.6067	0.7017	0.2689	0.3286	0.3685
5	0.5005	0.6311	0.724	0.2684	0.3226	0.3544
6	0.5171	0.6494	0.7405	0.2662	0.3137	0.338
7	0.5314	0.6654	0.7545	0.2622	0.3026	0.3208
8	0.5453	0.6804	0.7671	0.2565	0.2902	0.3042
9	0.5594	0.6949	0.7786	0.2495	0.2773	0.2885
10	0.5741	0.7086	0.7891	0.2416	0.2645	0.2741

The probabilities of system busy and server busy are also represented graphically.

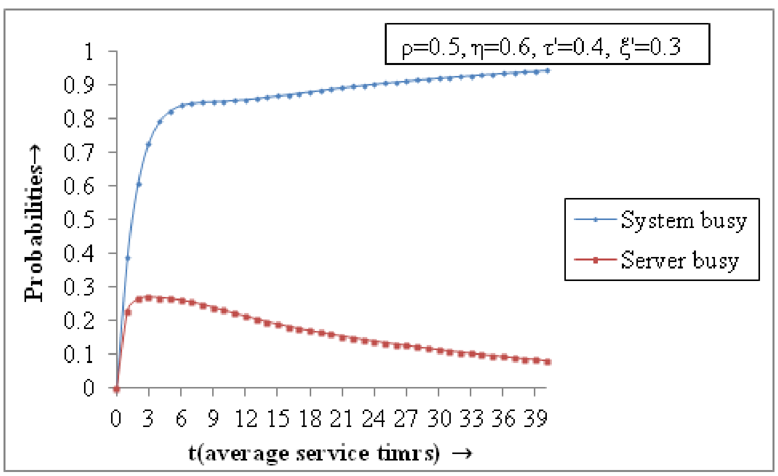

Figure 6. Probabilities of system busy and server busy against average service times

In Figure 6, the probabilities of system busy and server busy are plotted against time t for the case $\rho = 0.7$, $\eta = 0.6$, $\tau^{'}$=0.4, $\xi^{'}$=0.3. It is clear from the graph that probability of system busy is higher than probability of server busy. The probability of system busy increases rapidly with the increase in time. However, the probability of server busy increases first and then decreases gradually with time.

7 Conclusion

In this paper, we studied a two-state single server retrial queueing system with catastrophe. The catastrophes have significant impact on businesses, computer networks, etc. It is very important to manage the risk of catastrophe for the smooth functioning of the system. Moreover, the two-dimensional state queueing model has been proven to be a viable tool for understanding and quantifying factors. The proposed method is highly applicable in modeling many practical situations like in submitting any application online, ticket booking services using telephone facility, withdrawing cash at an ATM, manufacturing sectors, call centers, etc. In this paper, the transient state probabilities and the probability of server being under repair are obtained. Numerical results and graphical representations are also given.

8 Acknowledgment

The authors are highly thankful to the DST for developing computing facilities in the department of Statistics. The authors have made use of these facilities while doing this research work.

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Managing Tourism in North East India using Fuzzy Linear Programming

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Abstract

The best allocation of limited resources to activities with the aim of accomplishing the desired goal, such as maximization of profit or minimization of cost, is the focus of linear programming. The relationships between activities in linear programming models satisfy the proportionality and additivity requirements since they are linear interactions. This feature of linear programming is extended to the tourism industry which is one of the major industries in the global economy with respect to invested capital and earnings of foreign currencies. In todays era of design thinking, automation, and met averse, there is a very close margin between solutions to similar real-world problems. This is where the need and demand for fuzzy and imprecise linear programming arises. The membership functions provide the model developer the freedom to grade the imprecision as per his/her preference, thereby enabling a unique solution for each problem. While considering the tourism problem a number of factors like natural resources, people, history, culture, security, accommodation, entertainment, political stability, cost of services, tour operator, tour information, and advertisement play an important role in enhancing the sector to a large extent. Combined and classified they come under the following categories like leisure tourism, therapeutic/spa tourism, conference tourism, political tourism, sports/recreational tourism, cultural tourism, social tourism, conference tourism, recreational tourism, sports tourism, religious tourism, health tourism, etc. Each of these sectors requires careful investment on the part of the Government and other stake holders for development purpose. i.e. only a proper marketing mix will ensure a better return for the state. Taking this aspect into consideration the authors decided to introduce a hypothetical LP maximization model for the tourism industry in North East. Cost and space allocation are the constraints in the model. Among the various categories, three tourism forms (heritage, eco-tourism, and pilgrimage tourism) are considered which are used to optimize the allocation of the States marketing budget in tourism in such a way that the appropriate sector provides the greatest likelihood of producing the strongest return on investment. The maximization of profit has been done using various methods, like Werner's, Verdegay's, and Zimmermans. The authors have concluded the results based on the sensitivity analysis that has been done in the process of maximizing profit along with a maximal satisfaction level.

Keywords: Tourism, fuzzy linear programming, optimization, decision making

1 Introduction

The travel, tourism, and hospitality sector in India have the potential to promote grassroots sustainable development and promote economic expansion. Over 39.3 million jobs were created in the sector in 2013, which also brought USD 18.13 billion to foreign currency profits and INR 2.178 lakh crore to India's GDP. However, India only accounts for 0.64% of global tourist visits, despite having enormous potential. According to Public and Social Policies Management (PSPM), YES BANK, December 2014, India is likely the only nation that offers a variety of tourism options. These include beach tourism (India has the longest coastline in the East), spiritual tourism, Ayurveda and other types of Indian medicine, heritage tourism, Eco-tourism, mountain tourism, forest tourism, and adventure tourism. On the basis of their occurrence and the results of engaging in tourism, Professors Bernecker and Kaspar(183) list some more categories of tourism as follows: leisure tourism, therapeutic/spa tourism, conference tourism, political tourism, and sports/recreational tourism. The following additional categories of tourism can also be created depending on the preferences and expectations of the travellers: nature tourism, cultural tourism, social tourism, conference tourism, leisure tourism, sports tourism, religious tourism, health tourism, etc. [Paylos, 2013]. The tourism industry plays a crucial role in the expansion of other vital industries with high growth and employment potential, including those in the healthcare, infrastructure, and education sectors, as well as the alignment of macroeconomic policies with issues of regional development. Important initiatives like the e-Visa, the opening of new airports and rail stations, infrastructure, insurance, and real estate sectors have created viable impetus, essential for continuing critical mass momentum and investment in the under-leveraged inbound segment given the rapid evolution of global travel dynamics. When seen from the perspective of tourism, the North East is certainly a wonderland. For international visitors seeking peace and quiet, the waterfalls, forests, rhinoceroses, colorful birds, nature paths, the sun sinking over the mountains, lush tea gardens, and golf courses with helipads are travelers. The aesthetics and vibrant festivals that take place all year long will be a bonus harvest for them. This massive influx of visitors has profound commercial ramifications in addition to spiritual ones. Following significant government initiatives, a variety of product offers, a growing economy, rising levels of disposable income, and a rise in international tourist inflow, East and North East India have seen an unprecedented 27% gain in foreign tourist inflow. The enormous potential of this resource-rich region is still largely unrealized due to issues with its law and order, poor infrastructure and connection, unemployment, and slow economic growth, among other things.

2 Literature Review

As part of a larger development planning process for changing the area's tourist development, the authors of [5] offer a formulation for linear programming and a vector analysis that assess the available tourism forms in the Dirfis area in Greece. The purpose of the article is to examine how three different types of tourismconference, ecotourism, and pilgrimagecontribute to the local tourism industry in light of the available resources. The authors [6] provide a strategic plan that can aid in the growth of sustainable tourism in popular tourist spots. The A'SWOT (AHP-SWOT) hybrid method was developed by combining the AHP and the SWOT (Strengths, Weaknesses, Opportunities, and Threats) analysis (Analytic Hierarchy Process). The AHP approach was used to prioritise these aspects after a SWOT analysis was conducted to identify the key strategic factors. The researchers [9] investigated the potential applications of LP in the hotel sector. A straightforward optimization problem was attempted to be graphically solved in a hotel's F&B production division, and an ideal solution was obtained. In [11], concepts of revenue optimization are explained with regard to the tourism and hotel industries. Also, deterministic linear programming models of airlines and hotels are presented, and the solution is proposed through a genetic algorithm. Again in [7], a linear programming model is presented as a means to support the formulation of tourist policy in the case of the West Frisian Islands. A model is constructed that calculates the maximum employment effect that can be reached by different levels of government and shows the optimal combination of policy tools in order to achieve this maximum. The researchers discuss the quality of the tourism industry and the programming for its development in Iran in [12]. Based on a case study of all elements of Iran's tourism industry system, this study employs a unified assessment of the industry's quality. SWOT analysis aided in determining the weaknesses and threats, aiming to raise the quality of the indicators. In addition, linear programming from the standpoint of internal and external relations with the national economy has been applied. In [17], the levels of sustainable tourism and environmental sustainability were practically measured in different cities of Kerman Province using a composite indicator, a linear programming model, the Delphi method, and the questionnaire technique. The results of this study showed that the tourism opportunities were not used appropriately in these cities and tourist destinations, and those environmental aspects had very bad situations compared to social and economic aspects. In other words, environmental health had the lowest level of sustainability. The researchers in [19] discussed the concept of over-tourism. It aims to provide more clarity with regard to what tourism entails by placing the concept in a historical context and presenting results from a qualitative investigation among 80 stakeholders in 13 European cities. Seven over-tourism myths are identified that may inhibit a well-rounded understanding of the concept. The researchers in [8] created a model to investigate tourist preferences that used ten attributes of tourist destinations. Fuzzy set theory [2] was adopted as the main analysis method to find the tourists preferences. In [10], a numerical method for solving fuzzy linear programming problems with fuzzy decision variables is proposed. The purpose of this work is to derive the analytic formula of error estimation regarding the approximate optimal solution. In [3], fuzzy set theory is used as a case study in the e-commerce industry for the city of Shiraz. An electronic system in the form of a website is developed, which tourists can use to find appropriate accommodation by inputting data related to their interests and needs. In light of the above literature, the present study is an attempt in this direction to analyze the contribution of different forms of tourism to the overall revenue of the government. Once proper sectors are identified by the authority, proper investment could be made for their further development. Using fuzzy linear programming, the problem is formulated and solved using fuzzy programming techniques. In recent literature very few formulations of single objective linear programming problem under fuzzy environment and application in tourism. The novelty is to improve the quality of policy decisions via optimization approach and allow decision-makers to tap the great tourism resource potentials like geographical, climate, natural attractions, cultural, and ancient heritages. Basically how we will profit from tourism in a remote area via theoretical and then practical ways.The study concludes that the tourist will have approximately 99% satisfaction if parameters are changed from 5% to 20%.This will have very less impact on net profit earned through tourism.

3 Preliminaries

Let X denotes a universal set. Then a fuzzy (Zadeh(1965)) subset A of X is defined by its membership function $\mu_A : X \rightarrow [0, 1]$ which assign to each element x in X a real number μ_A in the interval $[0, 1]$,where the value of μ_A at x represents the grade of membership of x in A. The nearer the values of μ_A is unity, the higher the grade of membership of x in A.

3.1 Fuzzy Number

Fuzzy Number [22] is a fuzzy set A on R must possess at least the following three properties that A must be a normal fuzzy set; must be a closed interval for every α_A must be a closed interval for every $\alpha \in (0, 1]$; The support of $A, 0+_A$ must be bounded.

4 Linear Programming framework for the Tourism Development Problem

4.1 Model Developments

It is assumed that the State wants to develop a particular area and therefore a proper investment plan has to be decided upon. Based on the available tourist resources, the government or the various stakeholders must design its development policy [5]. For demonstrating the management of the tourism scenario in the state the following assumptions are made:
Decision Variable:

- Number of Heritage and culture sites infrastructures x_1
- Number of eco-tourism sites infrastructures x_2
- Number of infrastructures at pilgrimage locations x_3

Profits per sector are:

- Heritage and culture Tourism: 6 monetary units Eco Tourism: $4mu$
- Pilgrimage Tourism: $3mu$
- The goal of the state is to maximize revenue i.e. $maximize : 6x_1 + 4x_2 + 3x_3$
- The region that can be used to build the "logistics" infrastructure is $50000m2$.

Constraints:

- The prerequisites for each category are as follows:

- Heritage and Culture Tourism: 800 $\tau\mu$.
- Eco-Tourism: 600 $\tau\mu$.
- Pilgrimage Tourism: 500 $\tau\mu$.

The overall cost of property 39;s maintenance should not be more than the mu. The real cost for each category is: Heritage and culture sites: Tourism: $10mu$ Eco Eco-tourism Tourism: $8mu$ Pilgrimage Tourism: $3mu$ The model can thus be developed now as it has been simplified to a simple linear programming problem. The model can help maximize profit from ecotourism, pilgrimage tourism, and, conference tourism while optimally using important limited resources like land area. The companys objective maximize its gain, i.e.,

$$
\begin{aligned}
\text{Maximize Z} \quad & 6x_1 + 4x_2 + 3x_3 \\
\text{subject to} \quad & 8x_1 + 6x_2 + 5x_3 \leq 500 \\
& 10x_1 + 8x_2 + 3x_3 \leq 360 \\
& x_1, x_2, x_3 \geq 0
\end{aligned} \tag{1}
$$

Two slack variables x_4 and x_5 are introductions for the maximization of the following linear programming problem:

$$
\begin{aligned}
f(x_1, x_2, x_3, x_4, x_5) = \quad & 6x_1 + 4x_2 + 3x_3 + 0x_4 + 0x_5 \\
\text{where} \quad & 8x_1 + 6x_2 + 5x_3 + x_4 + 0x_5 = 500 \\
& 10x_1 + 8x_2 + 3x_3 + 0x_4 + x_5 = 360 \\
& x_i \geq 0 \qquad i = 1, 2, 3, 4, 5.
\end{aligned}
$$

Using the simplex algorithm, profit maximization occurs when $x_1 = 11.5$ (i.e. when there are 11 heritage and culture sites), $X_2 = 0$ (i.e. there are no place for eco-tourism) and $X_3 = 81.5$ (i.e 81 pilgrimage areas). Max $Z = 313.5$. Heritage venues won't help maximize profits, thus they are consequently not thought to be relevant. While certainty, reliability and precision are frequently illusory concepts in real-world applications, linear programming models represent real-world situations with some sets of parameters that are determined by experts and decision makers. As a result, experts and decision-makers are often unable to determine the precise value of parameters or may not be able to precisely specify the objective functions or constraints. The use of fuzzy linear programming has the advantage that the decision-maker can model the issue in accordance with the current state of knowledge because it is typically impractical to describe the restrictions and the goal function in precise terms. Many real-world problems find their solution in traditional theory. In original LPP, coefficients and right-hand sides must be well defined. The use of deterministic and stochastic models to model real-world situations necessitates a lot of data processing. In todays era of design thinking, automation and met averse, these is very close margin between solutions of similar real-world problems. Thus the need and demand of fuzzy and imprecise linear programming arises here. Some model parameters can only be approximated roughly in the event of genuine problems. While imprecise input is substituted by average data in classical models, fuzzy models allow decision makers to model their subjective imaginations as exactly as they can explain them. Therefore, the classical LP are not applicable, instead, the Fuzzy Linear Programming [1, 13] is used to model such situations. By introducing fuzziness to LP, such problems can be overcome. When a decision must be made in a fuzzy environment, LP may be modified in one of the three ways listed below. The objective function should not be maximised, to start. In other words, a level of aspiration that cannot be clearly defined as optimal must be reached. Second, the limitations might be ambiguous. The $\leq$ sign might not have a traditional definition or be used in a strictly mathematical sense, but there might be some room for error. When the limitations represent aspirational levels that are not well defined, this can occur. Last but not least, data may be inaccurate due to a lack of precision or some ambiguity in the data collection technique.

4.2 DECISION MAKING IN A FUZZY SCENARIO [18]

Decision making under fuzzy context is the confluence of fuzzy constraints and fuzzy objective functions . The distinction between constraint and goal function vanishes as a fuzzy environment attains ultimate symmetry. Zimmermann (1978) was the first to categorize fuzzy mathematical programming into symmetric and non-symmetric models. Subsequently, it has also been classified by Leung (1998) into four categories: - crisp objective and fuzzy constraints, fuzzy objective and crisp constraints, fuzzy objective and fuzzy constraints and robust programming. Linear programming can also be classified as follows: i. Linear programming problem with uncertain resources ii. A nonsymmetric model by Verdegay iii. Werner's methodology iv. Zimmermann's Model v. Chana's Methodology: A Nonsymmetric Model [4]. In this paper we have considered the objective function, constraints and both constraints and objective fuzzy.The approaches of Werner,Verdegay and Zimmermann depict fuzziness in the model in objectives and constraints.

4.2.1 Werners Method [16]

Werner proposed that the objective function is taken to be fuzzy as the total fuzzy resources or fuzzy inequality constraints. Tolerances given by p_i are fuzzy and given. The construction of membership function μ_0 for objective function is as:

$$\mu_0(x) = \begin{cases} 1 & if cx > Z^1 \\ \dfrac{1-(Z^1 - cx)}{Z^1 - Z^0} & if Z_0 < cx < Z_1 \\ 0 & if Z_0 < cx \end{cases}$$

where Z_1 and Z_0 are the values obtained after maximization and minimization of the single objective function.
A symmetric model is as

$$\begin{aligned} \text{Max} \quad & \alpha \\ \text{subject to} \quad & \mu_0(x) \geq \alpha \\ & \mu_i(x) \geq \alpha, \quad \forall\; i, \alpha \in [0,1] \; and \; x \geq 0 \end{aligned} \tag{2}$$

4.2.2 Chanas [2] & Verdegays Approach [15, 14]

In the Verdegays approach for non-symmetric model, the constraint is fuzzy while the objective function is not fuzzy. This means that the value of constraint is between 0 and 1 while the objective function has crisp value. The model can be understood as equivalent parametric programming:

$$\begin{aligned} &Maximize \quad cx \\ &\text{subject to} \quad (Ax)_j \leq b_i + (1-\alpha)p_i \quad \forall\, i, \alpha \in [0,1] \ and \ x \geq 0 \end{aligned} \tag{3}$$

where p is tolerance parameter and b is basic value.

4.2.3 Solution of Zimmermann [21, 20]

By the Zimmermanns approach,the linear programming problem is solved by adding the objective function cx as a fuzzy goal to the constraints. It is not certain that Zimmermann method (ZM) will give the "best" option when this new LP has alternate optimal solutions (AOS). There are two possibilities: cx might have distinct bounded values for the AOS or it could be unbounded. But since most of the AOS may have same solution, it's possible that we don't offer the best possible solution to the decision maker (DM) unless we check the value of cx for all AOS; it's possible that cx is unbounded yet ZM presents a bounded solution as the best. Zimmermans Approach for solving the fuzzy LPP takes into account a direct relation between α and θ. Further, it takes the variance of tolerance parameter and the graph for same can be obtained.

Let $\alpha = 1 - \theta$ then equaion becomes

$$\begin{aligned} &Min \quad \theta \\ &\text{subject to} \quad cx \geq b_0 - \theta\, p_0 \\ &\qquad (Ax)_i \leq b_i + \theta\, p_i, \qquad \forall\, i, \theta \in [0,1] \ and \ x \geq 0 \end{aligned} \tag{4}$$

The above formulated LPP is solved using Verdegay's and Werner approach.

5 PROBLEM WITH VARYING TOLERANCES AND GRAPHICAL INTERPRETATIONS USING WERNERS METHOD

$$\begin{aligned} MaxZ \quad & 6x_1 + 4x_2 + 3x_3 \\ \text{subject to} \quad & 8x_1 + 6x_2 + 3x_3 \leq 500 \\ & 10x_1 + 8x_2 + 3x_3 \leq 360 \\ & x_1, x_2, x_3 \geq 0 \end{aligned} \tag{5}$$

we get,

$$Z = 313.81 \qquad x_1 = 11.54 \qquad x_2 = 0.00 \qquad x_3 = 81.54$$

Tolerance set at $P_i = 5\%$

$$\begin{aligned} MaxZ \quad & 6x_1 + 4x_2 + 3x_3 \\ \text{subject to} \quad & 8x_1 + 6x_2 + 3x_3 \leq 500 + 50\theta \\ & 10x_1 + 8x_2 + 3x_3 \leq 360 + 18\theta \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

$$\mu_1(x) = \begin{cases} 1 & if g_1(x) < 500 \\ 1 - \dfrac{(g_1(x) - 500)}{25} & if 500 < g_1(x) < 525 \\ 0 & if g_1(x) > 525 \end{cases}$$

$$\mu_2(x) = \begin{cases} 1 & if g_2(x) < 360 \\ 1 - \dfrac{(g_2(x) - 360)}{18} & if 360 < g_2(x) < 378 \\ 0 & if g_2(x) > 378 \end{cases}$$

Table showing the tolerance level set at 5%

θ	x_1	x_2	x_3	Z
0	11.54	0.00	81.54	313.85
0.1	11.60	0.00	81.95	315.42
0.2	11.65	0.00	82.35	316.98
0.3	11.71	0.00	82.76	318.55
0.4	11.77	0.00	83.17	320.12
0.5	11.83	0.00	83.58	321.69
0.6	13.42	0.00	81.52	325.11
0.7	11.94	0.00	84.39	324.83
0.8	12.00	0.00	84.80	326.40
0.9	12.06	0.00	85.21	327.97
1.0	12.12	0.00	85.62	329.54

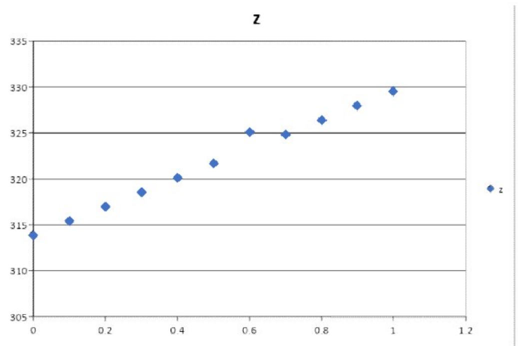

Figure 1: Theta versus z at 5% tolerance parameter.

Tolerance set at $P_i = 10\%, P_1 = 50, P_2 = 36$

$$\begin{aligned} MaxZ \quad & 6x_1 + 4x_2 + 3x_3 \\ \text{subject to} \quad & 8x_1 + 6x_2 + 3x_3 \leq 500 + 50\theta \\ & 10x_1 + 8x_2 + 3x_3 \leq 360 + 36\theta \\ & x_1, x_2, x_3 \geq 0 \end{aligned} \tag{6}$$

θ	x_1	x_2	x_3	Z
0	11.54	0.00	81.54	313.85
0.1	11.65	0.00	82.35	316.98
0.2	11.77	0.00	83.17	320.12
0.3	11.88	0.00	83.98	323.26
0.4	12.00	0.00	84.80	326.40
0.5	14.97	0.00	76.10	318.12
0.6	12.23	0.00	86.43	332.68
0.7	12.35	0.00	87.25	335.82
0.8	12.46	0.00	88.06	338.95
0.9	12.58	0.00	88.88	342.09
1.0	12.69	0.00	89.69	345.23

Table 1: Table showing the tolerance level set at 5%

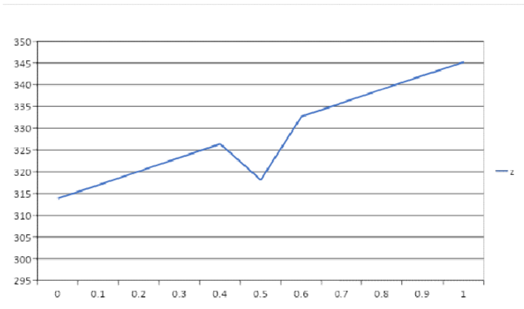

Figure 2: Theta versus z at 10% tolerance parameter.

The next tolerance level is set at $P_i = 15\%, P_1 = 75, P_2 = 54$

$$\begin{aligned} MaxZ \quad & 6x_1 + 4x_2 + 3x_3 \\ \text{subject to} \quad & 8x_1 + 6x_2 + 3x_3 \leq 500 + 75\theta \\ & 10x_1 + 8x_2 + 3x_3 \leq 360 + 54\theta \\ & x_1, x_2, x_3 \geq 0 \end{aligned} \tag{7}$$

θ	x_1	x_2	x_3	Z
0	11.54	0.00	81.54	313.85
0.1	11.71	0.00	82.76	318.55
0.2	71.31	0.00	251.95	323.26
0.3	12.06	0.00	85.21	327.97
0.4	12.23	0.00	86.43	332.62
0.5	12.40	0.00	87.65	337.38
0.6	12.58	0.00	88.88	342.09
0.7	12.75	0.00	90.10	346.80
0.8	12.92	0.00	91.32	351.51
0.9	13.10	0.00	92.55	356.22
1.0	14.13	0.00	90.88	357.46

Table 2: Table showing the tolerance level set at 15%

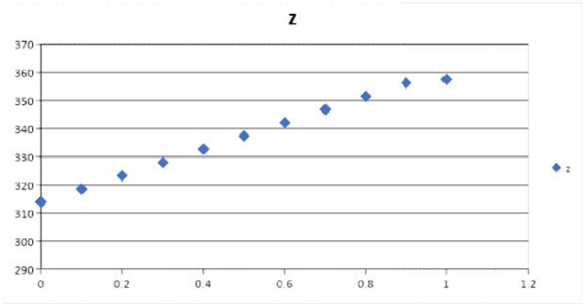

Figure 3: Theta versus z at 15% tolerance parameter.

The next tolerance level is set at $P_i = 20\%, P_1 = 100, P_2 = 72$

$$\begin{aligned} MaxZ \quad & 6x_1 + 4x_2 + 3x_3 \\ \text{subject to} \quad & 8x_1 + 6x_2 + 3x_3 \leq 500 + 100\theta \\ & 10x_1 + 8x_2 + 3x_3 \leq 360 + 72\theta \\ & x_1, x_2, x_3 \geq 0 \end{aligned} \tag{8}$$

θ	x_1	x_2	x_3	Z
0	0.00	0.00	0.00	0.00
0.1	11.77	0.00	83.17	320.12
0.2	12.00	0.00	84.80	326.40
0.3	12.23	0.00	84.43	332.68
0.4	12.46	0.00	88.06	338.95
0.5	12.92	0.00	89.69	345.23
0.6	12.58	0.00	81.52	351.51
0.7	13.15	0.00	92.95	357.78
0.8	13.38	0.00	94.58	364.06
0.9	13.62	0.00	96.22	370.34
1.0	13.85	0.00	97.85	376.62

Table 3: Table showing the tolerance level set at 20%

5.1 Zimmermann Approach [13]

The formulation is that both the objective function and constraints are fuzzy:

$$\begin{aligned} Min \quad & \theta \\ \text{subject to} \quad & cx \geq b_0 - \theta p_0 \\ & (Ax)_i \leq b_i + \theta p_i, \qquad \forall\, i, \theta \in [0,1]\ and x \geq 0 \end{aligned}$$

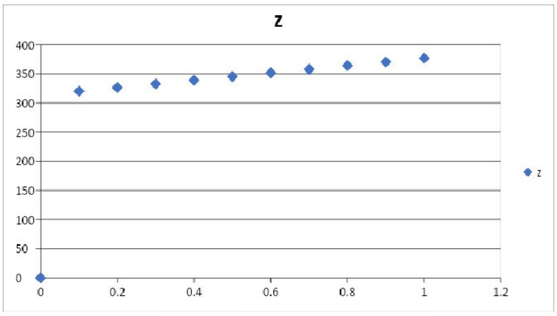

Figure 4: Theta versus z at 20% tolerance parameter.

The value of $\theta \in [0, 1]$ and non -negativity condition $x \geq 0$.
We obtain the formulation for our problem as

$$\begin{aligned} Min \quad & \theta \\ \text{subject to} \quad & 6x_1 + 13x_2 + 10x_3 + 20x_4 + 25x_5 + 21.25\theta \geq 21.25 \\ & 15x_1 + 3x_2 + 5x_3 + 6x_4 + 10x_5 + 15\theta \geq 15 \\ & x_1 + 3x_2 + 2x_3 + 5x_4 + x_5 + 4\theta \geq 4 \\ & 15x_1 + 3x_2 + 5x_3 + 6x_4 + 10x_5 - 0.75\theta \leq 15 \\ & x_1 + 3x_2 + 2x_3 + 5x_4 + x_5 - 0.2\theta \leq 4 \\ & x_5 - 0.125\theta \leq 0.25 \\ & x_1 + x_2 + x_3 + x_4 + x_5 - 0.50\theta \leq 1 \end{aligned} \tag{9}$$

On solving the above formulation we obtain the table values as:

θ	x_1	x_2	x_3	$\alpha = 1 - \theta$	z
0	11.54	0	81.54	1	313.85
0.1	11.6	0	81.95	0.9	315.42
0.2	11.65	0	82.35	0.8	316.98
0.3	11.71	0	82.76	0.7	318.55
0.4	11.77	0	83.17	0.6	320.12
0.5	11.83	0	83.58	0.5	321.69
0.6	13.42	0	81.52	0.4	325.11
0.7	11.94	0	84.39	0.3	324.83
0.8	12.00	0	84.80	0.2	326.4
0.9	12.06	0	85.21	0.1	327.97
1.0	12.12	0	85.62	0	329.54

Table 4: Values obtained by Zimmermanns method

6 RESULTS AND DISCUSSIONS

As the value of p increases, the consistency of all other parameters increases. At 20% tolerance parameter, the satisfaction level is 100% & the value of $Z = 376.62$. At 15%

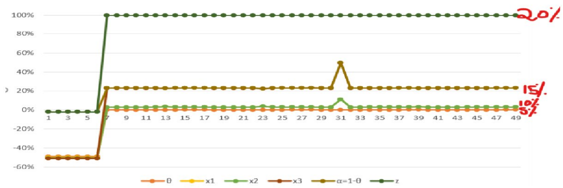

Figure 5: Solution by Zimmermanns approach and values for various tolerance parameters.

tolerance parameter the satisfaction level is 100% & the value of $Z = 357.46$. At 10% tolerance parameter the satisfaction level is 100% & the value of $Z = 345.23$. At 5% tolerance parameter the satisfaction level is 100% & the value of $Z = 329.54$. The decision maker can alter some circumstances whenever he wants to alter the original model and end the solution procedure whenever he is satisfied.

7 Conclusion

A simple linear programming model for revenue optimization of tourism industry is presented in this paper. Henceforth fuzziness is incorporated and their solution obtained through fuzzy linear programming approach. Fuzziness can help to offer a more natural description of uncertain data which depicts the real world phenomenon. Many real-world problems can be solved using general approaches for linear programming, but due to human nature, fuzziness and imprecision make a significant difference to these difficulties. Taking into account the vagueness, Fuzzy Linear Mathematical Programming is able to solve more problems with incremented levels of exactness. They are the mathematical tools which have immense potential for handling uncertainty inherent in real time data. Werners method and Verdegays method do not allow the freedom to solve for desired goal and objective like Zimmermann and Chanas does. In the case study that has been carried out, it can be understood that for various tolerance parameters, the maximum value of objective function changes. Hence, the value of tolerance which yields maximum results is selected. Thus the implementation of this approach can encourage tourism stakeholders such as national and local government, tourism businesses, and local communities to play a guiding role. This technique can improve the quality of policy decisions and allow decision-makers to take advantage of the great tourism resource potential, including geographical, climate, natural attractions, cultural, and ancient heritages. The future scope lies in multiobjective approach to tourism problem with fuzzy linear and non linear membership functions for objectives, constraints or both.

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A homotopy based computational scheme for local fractional Helmholtz and Laplace equations

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December 31, 2022

In this work, we investigate solutions for the local fractional Helmholtz and Laplace equations on Cantor set having importance in electrostatics, gravitation and fluid dynamics. To find exact solutions, the q-local fractional homotopy analysis transform method (q-LFHATM) has been used. The numerical results computed with the aid of the applied scheme shows that it is an efficient and accurate tool to solve differential equations with local fractional derivatives.

Keywords: Local fractional derivative operator; Partial differential equations; Laplace equation; Helmholtz equation; q-local fractional homotopy analysis transform method.

1 Introduction

The concept of local fractional calculus (LFC) has been used to model and analyze numerous fractal equations some of which are Fokker-Planck equation [1, 2], fractal wave equations [3], fractal-time dynamical systems [4, 5], the local fractional stress strain relations [6], the local fractional heat conduction model [7], local fractional Tricomi equation [8], local fractional Laplace equations [9], the Helmholtz equation associated with local fractional operator [10], fractal signals [11, 12], fractal Fourier analysis [13], Yang Fourier transform [14, 15, 16], Yang-Laplace transform [15, 17], fractal vehicular traffic flow [18], local fractional modelling in growths of population [19], and Boussinesq equation containing local fractional operator [20], etc. Some recent outcomes of different authors on local fractional methods involving local fractional integral transforms can be seen in a series of articles [21, 22, 23].

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This work presents a very useful scheme known as the q-local fractional homotopy analysis transform method (q-LFHATM), which is a combination of q-HAM and the local fractional Laplace transform (LFLT). The proposed q-LFHATM is implemented to analyze the local fractional Helmholtz and Laplace equations. The merger of q-HAM and LFLT resulted in lesser C.P.U time (RAM-1 GB or more and Processor 2.65 GHz or more) while solving fractional-order nonlinear problems. The ability of the proposed method to achieve the series solution of local fractional Helmholtz and Laplace equations over a vast domain by picking approximate values for parameters is one of its advantages. El-Tavil & Huseen proposed the q-HAM [24, 25] which is a smooth generalization of the HAM. The HAM was first proposed and used by Liao [26, 27] to solve several problems found in engineering, science and finance.
The rest of the article is organized as follows: Section 2 reports basic definitions and formulae of LFC and LFLT. Section 3, the working of q-LFHATM is explained. The q-LFHATM is utilized to derive the solutions of local fractional Helmholtz and Laplace equations under different fractal conditions in Section 4. Section 5 presents the glimpse of numerical simulation for fractal order $\rho = \ln 2/\ln 3$. At the end, conclusion is reported in Section 6.

2 Preliminaries

Here, we provide certain important concepts of LFC and LFLT.
Definition 2.1. If we have a relation [12, 28]

$$|\theta(t) - \theta(t_0)| < \varepsilon^{\alpha}, 0 < \alpha \le 1, \tag{1}$$

with $|\theta(t) - \theta(t_0)| < \delta$, for $\varepsilon, \alpha \in R$, then the function $\theta(t)$ is said to be local fractional continuous (LFC) at $t = t_0$ and is indicated by $\lim_{t\to t_0} \theta(t) = \theta(t_0)$. Here, $\theta(t)$ is called LFC on (a, b) and is expressed as

$$\theta(t) \in C_{\alpha}(a, b). \tag{2}$$

Definition 2.2. A function $\theta(t)$ is a nondifferentiable function of exponentα $(0 < \alpha \le 1)$ if it satisfies the Hölder function of the exponent α. Then for $t, s \in T$, we have [12, 28]

$$|\theta(t) - \theta(s)| < C|t - s|^{\alpha}. \tag{3}$$

Definition 2.3. $\theta(t)$ is said to be continuous of α, $0<\alpha \le 1$, or α continuous if there exists the following condition [12] $|\theta(t) - \theta(t_0)| < \varepsilon^{\alpha}$,

$$\theta(t) - \theta(t_0) = o((t - t_0)^{\alpha}). \tag{4}$$

In view of (4), Eq. (1) presents the standard form of local fractional continuity.
Definition 2.4. If $\theta(t) \in C_{\alpha}(a, b)$, then local fractional derivative (LFD) of $\theta(t)$ of order α at $x = x_0$ is written as [12, 28]:

$$\theta^{(\alpha)}(t_0) = \frac{d^{\alpha}}{dt^{\alpha}}\theta(t)|_{t=t_0} = \lim_{t\to t_0} \frac{\Delta^{\alpha}(\theta(t) - \theta(t_0))}{(t - t_0)^{\alpha}}, \tag{5}$$

where $\Delta^{\alpha}(\theta(t)-\theta(t_0)) \cong \Gamma(\alpha+1)(\theta(t)-\theta(t_0))$.
For any $t \in (a,b)$, we have $\theta^{(\alpha)}(t) = D_t^{\alpha}\theta(t)$. The LFD of $m\alpha$ order is expressed as:

$$\theta^{(m\alpha)} \equiv \overbrace{D_t^{\alpha}....D_t^{\alpha}\theta(t)}^{\text{m times}},$$

whereas the local fractional partial derivative (LFPD) of $m\alpha$ order is expressed as:

$$\frac{\partial^{m\alpha}\theta(t)}{\partial t^{m\alpha}} \equiv \overbrace{\frac{\partial^{\alpha}}{\partial t^{\alpha}}....\frac{\partial^{\alpha}}{\partial t^{\alpha}}\theta(t)}^{\text{m times}}.$$

Definition 2.5. Let $\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}|\theta(t)|(dt)^{\alpha} < m < \infty$. Then the Yang-Laplace (YL) transform [29, 30] of $\theta(t)$ is defined as:

$$L_{\alpha}\{\theta(t)\} = \theta_s^{L,\alpha}(s) = \frac{1}{\Gamma(1+\alpha)}\int_0^{\infty} E_{\alpha}(-s^{\alpha}t^{\alpha})\,\theta(t)\,(dt)^{\alpha}, 0 < \alpha \leq 1. \quad (6)$$

Here, the latter integral converges and $s^{\alpha} \in R^{\alpha}$.
Definition 2.6. The inverse of YL transform of $\theta(t)$ is stated as

$$L_{\alpha}^{-1}\left(\theta_s^{L,\alpha}(s)\right) = \theta(t) = \frac{1}{(2\pi)^{\alpha}}\int_{v-i\omega}^{v+i\omega} E_{\alpha}(s^{\alpha}t^{\alpha})\,\theta_s^{L,\alpha}(s)\,(ds)^{\alpha}, 0 < \alpha \leq 1, \quad (7)$$

where $s^{\alpha} = v^{\alpha} + i^{\alpha}\omega^{\alpha}$; fractal imaginary unit i^{α} and $Re(s) = \alpha > 0$.
Some useful formulae of LFLT [11, 12] are mentioned here:

$$L_{\alpha}\{a\theta(t) + b\phi(t)\} = a\theta_s^{L,\alpha}(s) + b\phi_s^{L,\alpha}(s), \quad (8)$$

$$L_{\alpha}\{E_{\alpha}(c^{\alpha}t^{\alpha})\,\theta(t)\} = \theta_s^{L,\alpha}(s-c), \quad (9)$$

$$L_{\alpha}\left\{\theta^{(m\alpha)}(t)\right\} = s^{m\alpha}\theta_s^{L,\alpha}(s)$$

$$-s^{(m-1)\alpha}\theta(0) - s^{(m-2)\alpha}\theta^{(\alpha)}(0) - \cdots - \theta^{((m-1)\alpha)}(0), \quad (10)$$

$$L_{\alpha}\{E_{\alpha}(c^{\alpha}t^{\alpha})\} = \frac{1}{s^{\alpha}-c^{\alpha}}, \quad (11)$$

$$L_{\alpha}\{\sin_{\alpha}(c^{\alpha}t^{\alpha})\} = \frac{c^{\alpha}}{s^{2\alpha}+c^{2\alpha}}, \quad (12)$$

$$L_{\alpha}\{t^{m\alpha}\} = \frac{\Gamma(1+m\alpha)}{s^{(m+1)\alpha}}. \quad (13)$$

Definition 2.7. The Mittag-Leffler function (MLF) is formulated as [12, 28]

$$E_{\alpha}(t^{\alpha}) = \sum_{m=0}^{\infty}\frac{t^{m\alpha}}{\Gamma(1+m\alpha)}, 0 < \alpha \leq 1. \quad (14)$$

The following results hold true;

$$\sin_\alpha\left(t^\alpha\right)=\sum_{m=0}^{\infty}(-1)^m\frac{t^{(2m+1)\alpha}}{\Gamma\left(1+(2m+1)\alpha\right)},$$

$$\cos_\alpha\left(t^\alpha\right)=\sum_{m=0}^{\infty}(-1)^m\frac{t^{2m\alpha}}{\Gamma\left(1+2m\alpha\right)},0<\alpha\leq 1. \tag{15}$$

Certain fundamental formulas and results used in the work are presented below:

$$\frac{d^\alpha t^{m\alpha}}{dt^\alpha}=\frac{\Gamma\left(1+m\alpha\right)t^{(m-1)\alpha}}{\Gamma\left(1+(m-1)\alpha\right)}, \tag{16}$$

$$\frac{d^\alpha E_\alpha\left(t^\alpha\right)}{dt^\alpha}=E_\alpha\left(t^\alpha\right), \tag{17}$$

$$\frac{d^\alpha E_\alpha\left(mt^\alpha\right)}{dt^\alpha}=mE_\alpha\left(mt^\alpha\right). \tag{18}$$

3 Working plan of q-LFHATM

To elucidate the procedure of q-LFHATM, the following nonlinear local fractional partial differential equation (LFPDE) is investigated

$$Ł_\alpha\,\varepsilon(\eta,\kappa)+R_\alpha\,\varepsilon(\eta,\kappa)+N_\alpha\varepsilon(\eta,\kappa)=h(\eta,\kappa),\quad n-1<\alpha\leq n, \tag{19}$$

where $Ł_\alpha$ indicates the linear local fractional operator, R_α indicates the remaining local linear differential operator, N_α stands for the nonlinear differential operator and $h(\eta,\kappa)$ signifies the source term.
Operating the LFLT on Eq. (19), we obtain

$$\begin{aligned}L_\alpha\left[\varepsilon(\eta,\kappa)\right]-s^{-\alpha}\varepsilon(\eta,0)-s^{-2\alpha}\varepsilon^{(\alpha)}(\eta,0)-\cdots-s^{-t\alpha}\varepsilon^{((t-1)\alpha)}(\eta,0)\\+s^{-t\alpha}\,L_\alpha\left[R_\alpha\varepsilon(\eta,\kappa)+N_\alpha\varepsilon(\eta,\kappa)-h(\eta,\kappa)\right]=0.\end{aligned} \tag{20}$$

We describe the nonlinear operator as:

$$\begin{aligned}N[\psi(\eta,\kappa;l)]=L_\alpha\left[\psi(\eta,\kappa;l)\right]-s^{-\alpha}\psi(\eta,\kappa;l)(0^+)-s^{-2\alpha}\psi^{(\sigma)}(\eta,\kappa;l)(0^+)-\cdots\\-s^{-t\alpha}\psi^{((t-1)\alpha)}(\eta,\kappa;l)(0^+)+s^{-t\alpha}L_\alpha\left[R_\alpha\varepsilon(\eta,\kappa)+N_\alpha\varepsilon(\eta,\kappa)-h(\eta,\kappa)\right].\end{aligned} \tag{21}$$

In Eq. (21), $l\in[0,\,1/n]$ and $\psi(\eta,\kappa;l)$ is a real valued function of η,κ & l. Now, the homotopy is framed as:

$$(1-nl)\,L_\alpha\left[[\psi(\eta,\kappa;l)-\varepsilon_0(\eta,\kappa)\right]=\hbar\,l\,N\left[\varepsilon(\eta,\kappa)\right]. \tag{22}$$

In Eq. (22), L_α stands for the LFLT operator, $n\geq 1$,$l\in[0,\,1/n]$ is an embedding variable, $\hbar\neq 0$ stands for an auxiliary parameter, $\varepsilon_0(\eta,\kappa)$ denotes initial guess (IG) of $\varepsilon(\eta,\kappa)$ and $\psi(\eta,\kappa;l)$ is an unidentified function. Clearly, for $l=0$ and $l=\frac{1}{n}$, the results obtained are

$$\psi(\eta,\kappa;0)=\varepsilon_0(\eta,\kappa),\qquad\psi(\eta,\kappa\,;\frac{1}{n})=\varepsilon(\eta,\kappa), \tag{23}$$

4

respectively. Therefore, when l approaches from 0 to $\frac{1}{n}$, $\psi(\eta,\kappa;l)$ changes from the IG $\varepsilon_0(\eta,\kappa)$ to solution $\varepsilon(\eta,\kappa)$. Taylor series expansion of $\psi(\eta,\kappa;l)$ provides

$$\psi(\eta,\kappa;l)=\sum_{m=0}^{\infty}\varepsilon_m(\eta,\kappa)l^{\,m}. \tag{24}$$

where

$$\varepsilon_m(\eta,\kappa)=\frac{1}{m\,!}\frac{\partial^m\psi(\eta,\kappa;l)}{\partial l^m}\,|_{l=0}. \tag{25}$$

For proper values of $u_0(x,t)$,n and $\hbar$, the series (10) converges for $l=\frac{1}{n}$, then we obtain

$$\varepsilon(\eta,\kappa)=\sum_{m=0}^{\infty}\varepsilon_m(\eta,\kappa)(\frac{1}{n})^m. \tag{26}$$

Now, the set of vectors is characterized as

$$\overrightarrow{\varepsilon}_m=\{\varepsilon_0(\eta,\kappa),\ \varepsilon_1(\eta,\kappa),\cdots,\ \varepsilon_m(\eta,\kappa)\}. \tag{27}$$

Next, the mth-order deformation equation is composed as

$$L_\alpha\left[\varepsilon_m(\eta,\kappa)-\chi_m\varepsilon_{m-1}(\eta,\kappa)\right]=\hbar\,\Re_m(\overrightarrow{\varepsilon}_{m-1}). \tag{28}$$

Operating the inverse LFLT on Eq. (28), we obtain

$$\varepsilon_m(\eta,\kappa)=\chi_m\varepsilon_{m-1}(\eta,\kappa)+\hbar\,{L_\alpha}^{-1}[\Re_m(\overrightarrow{\varepsilon}_{m-1})]. \tag{29}$$

In Eq. (29), the value of $\Re_k(\vec{\phi}_{k-1})$ and χ_k are presented below

$$\Re_m(\overrightarrow{\varepsilon}_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}N\left[\varepsilon(\eta,\kappa;l)\right]}{\partial l^{m-1}}\,|_{l=0}, \tag{30}$$

and

$$\chi_m=\begin{cases}0, & m\le 1,\\ n, & m>1.\end{cases} \tag{31}$$

4 Illustrative examples

Example 4.1 Let us take the local fractional Helmholtz equation [10] as follows

$$\frac{\partial^{2\rho}\phi(u,v)}{\partial u^{2\rho}}+\frac{\partial^{2\rho}\phi(u,v)}{\partial v^{2\rho}}=\phi(u,v),0<\rho\le 1 \tag{32}$$

with initial-boundary conditions given as:

$$\phi\left(0,v\right)=0,\frac{\partial^\rho\phi(0,v)}{\partial u^\rho}=E_\rho\left(v^\rho\right). \tag{33}$$

Applying LFLT on Eq. (32), we obtain

$$L_\rho\left\{\phi(u,v)\right\}-s^\rho\phi(0,v)-s^{-2\rho}\frac{\partial^\rho\phi(0,v)}{\partial u^\rho}+s^{-2\rho}L_\rho\left\{\frac{\partial^{2\rho}\phi(u,v)}{\partial^{2\rho}v}-\phi(u,v)\right\}=0,$$

5

or

$$L_\rho\left\{\phi(u,v)\right\}-s^{-2\rho}E_\rho\left(y^\rho\right)+s^{-2\rho}L_\rho\left\{\frac{\partial^{2\rho}\phi(u,v)}{\partial^{2\rho}v}-\phi(u,v)\right\}=0. \tag{34}$$

The nonlinear operator is defined as

$$N[\psi(\eta,\kappa;l)]=L_\rho\left[\psi(\eta,\kappa;l)\right]-s^{-2\rho}E_\rho\left(y^\rho\right)+s^{-2\rho}L_\beta\left\{\frac{\partial^{2\rho}\psi(\eta,\kappa;l)}{\partial^{2\rho}v}-\psi(\eta,\kappa;l)\right\}, \tag{35}$$

and so

$$\Re_m\left(\vec{\phi}_{m-1}\left(u,v\right)\right)=L_\rho\left\{\phi_{m-1}\left(u,v\right)\right\}$$

$$-\left(1-\frac{\chi_m}{n}\right)s^{-2\rho}E_\rho\left(v^\rho\right)+s^{-2\rho}L_\rho\left[\frac{\partial^{2\rho}\phi_{m-1}\left(u,v\right)}{\partial v^{2\rho}}-\phi_{m-1}\left(u,v\right)\right]. \tag{36}$$

The mth-order deformation equation is built as

$$L_\rho\left\{\phi_m\left(u,v\right)-\chi_m\phi_{m-1}\left(u,v\right)\right\}=\hbar\Re_m\left(\vec{\phi}_{m-1}\left(u,v\right)\right). \tag{37}$$

Applying inverse LFLT on Eq. (37), we obtain

$$\phi_m\left(u,v\right)=\chi_m\phi_{m-1}\left(u,v\right)+\hbar L_\rho^{-1}\left\{\Re_m\left(\vec{\phi}_{m-1}\left(u,v\right)\right)\right\}. \tag{38}$$

For $m=1$, we have

$$\phi_1\left(u,v\right)=\chi_1\phi_0\left(u,v\right)+\hbar L_\rho^{-1}\left\{\Re_1\left(\vec{\phi}_0\left(u,v\right)\right)\right\}$$

or

$$\phi_1\left(u,v\right)=-\hbar E_\rho\left(v^\rho\right)\frac{u^\rho}{\Gamma\left(1+\rho\right)}. \tag{39}$$

For $m=2$, we have

$$\phi_2\left(u,v\right)=\chi_2\phi_1\left(u,v\right)+\hbar L_\rho^{-1}\left\{\Re_2\left(\vec{\phi}_1\left(u,v\right)\right)\right\},$$

or

$$\phi_2\left(u,v\right)=-\left(n+\hbar\right)\hbar E_\rho\left(v^\rho\right)\frac{u^\rho}{\Gamma\left(1+\rho\right)}. \tag{40}$$

For $m=3$, we have

$$\phi_3\left(u,v\right)=\chi_3\phi_2\left(u,v\right)+\hbar L_\rho^{-1}\left\{\Re_3\left(\vec{\phi}_2\left(u,v\right)\right)\right\},$$

or

$$\phi_3\left(u,v\right)=-\left(n+\hbar\right)\hbar^2 E_\rho\left(v^\rho\right)\frac{u^\rho}{\Gamma\left(1+\rho\right)}, \tag{41}$$

& so on.

6

Hence, the nondifferentiable solution is expressed as

$$\phi(u,v)=\sum_{m=0}^{\infty}\phi_m(u,v)(\frac{1}{n})^m.$$

$$\phi(u,v)=\phi_0(u,v)+\frac{\phi_1(u,v)}{n}+\frac{\phi_2(u,v)}{n^2}+\frac{\phi_3(u,v)}{n^3}+\cdots$$

or

$$\phi(u,v)=\frac{1}{n}\left[\hbar E_\rho\left(v^\rho\right)\frac{u^\rho}{\Gamma\left(1+\rho\right)}\right]-\frac{1}{n^2}\left[\left(n+\hbar\right)\hbar E_\rho\left(v^\rho\right)\frac{u^\rho}{\Gamma\left(1+\rho\right)}\right]$$
$$-\frac{1}{n^3}\left[\left(n+\hbar\right)\hbar^2 E_\rho\left(v^\rho\right)\frac{u^\rho}{\Gamma\left(1+\rho\right)}\right]+\cdots. \tag{42}$$

On letting $\hbar=-1$ and $n=1$, one can achieve the result

$$\phi\left(u,v\right)=\frac{u^\rho}{\Gamma\left(1+\rho\right)}E_\rho\left(v^\rho\right). \tag{43}$$

which is the solution of fractal problem (32).

Example 4.2 Take the following local fractional Laplace equation [9]

$$\frac{\partial^{2\rho}\phi(u,v)}{\partial u^{2\rho}}+\frac{\partial^{2\rho}\phi(u,v)}{\partial v^{2\rho}}=0,0<\rho\leq 1, \tag{44}$$

with initial-boundary conditions given as:

$$\phi\left(0,v\right)=-E_\rho\left(v^\rho\right),\frac{\partial^\rho\phi(0,v)}{\partial u^\rho}=0. \tag{45}$$

Applying LFLT on Eq. (44), we obtain

$$L_\rho\left\{\phi(u,v)\right\}-s^{-\rho}\phi(0,v)-s^{-2\rho}\frac{\partial^\rho\phi(0,v)}{\partial u^\rho}+s^{-2\rho}L_\rho\left\{\frac{\partial^{2\rho}\phi(u,v)}{\partial^{2\rho}v}\right\}=0,$$

or

$$L_\rho\left\{\phi(u,v)\right\}+s^{-\rho}E_\rho\left(y^\rho\right)+s^{-2\rho}L_\rho\left\{\frac{\partial^{2\rho}\phi(u,v)}{\partial^{2\rho}v}\right\}=0. \tag{46}$$

The nonlinear operator is

$$N[\psi(\eta,\kappa;l)]=L_\rho\left[\psi(\eta,\kappa;l)\right]+s^{-\rho}E_\rho\left(y^\rho\right)+s^{-2\rho}L_\beta\left\{\frac{\partial^{2\rho}\psi(\eta,\kappa;l)}{\partial^{2\rho}v}\right\}, \tag{47}$$

and so

$$\Re_m\left(\vec{\phi}_{m-1}(u,v)\right)=L_\rho\left\{\phi_{m-1}(u,v)\right\}$$
$$+\left(1-\frac{\chi_m}{n}\right)s^{-\rho}E_\rho\left(v^\rho\right)+s^{-2\rho}L_\rho\left[\frac{\partial^{2\rho}\phi_{m-1}(u,v)}{\partial v^{2\rho}}\right]. \tag{48}$$

7

The mth-order deformation equation is constituted as:

$$L_\rho \left\{ \phi_m (u,v) - \chi_m \phi_{m-1} (u,v) \right\} = \hbar \Re_m \left(\vec{\phi}_{m-1}(u,v) \right). \tag{49}$$

Operating the inverse LFLT, we obtain

$$\phi_m (u,v) = \chi_m \phi_{m-1} (u,v) + \hbar L_\rho^{-1} \left\{ \Re_m \left(\vec{\phi}_{m-1}(u,v) \right) \right\}. \tag{50}$$

Taking $m = 1, 2, 3, \ldots$, we get

For $m = 1$

$$\phi_1 (u,v) = -\hbar E_\rho (v^\rho) \frac{u^{2\rho}}{\Gamma(1+2\rho)}. \tag{51}$$

For $m = 2$

$$\phi_2 (u,v) = -(n+\hbar) \hbar E_\rho (v^\rho) \frac{u^{2\rho}}{\Gamma(1+2\rho)} - \hbar^2 E_\rho (v^\rho) \frac{u^{4\rho}}{\Gamma(1+4\rho)}. \tag{52}$$

For $m = 3$

$$\begin{aligned} \phi_3 (u,v) = & -(n+\hbar)^2 \hbar E_\rho (v^\rho) \frac{u^{2\rho}}{\Gamma(1+2\rho)} - 2(n+\hbar) \hbar^2 E_\rho (v^\rho) \frac{u^{4\rho}}{\Gamma(1+4\rho)} \\ & -\hbar^3 E_\rho (v^\rho) \frac{u^{6\rho}}{\Gamma(1+6\rho)}, \end{aligned} \tag{53}$$

& so on.

Hence, the nondifferentiable solution is presented as

$$\begin{aligned} \phi(u,v) = & -\frac{1}{n} \left[\hbar E_\rho (v^\rho) \frac{u^{2\rho}}{\Gamma(1+2\rho)} \right] - \frac{1}{n^2} \left[(n+\hbar) \hbar E_\rho (v^\rho) \frac{u^{2\rho}}{\Gamma(1+2\rho)} - \hbar^2 E_\rho (v^\rho) \frac{u^{4\rho}}{\Gamma(1+4\rho)} \right] \\ & -\frac{1}{n^3} \left[(n+\hbar)^2 \hbar E_\rho (v^\rho) \frac{u^{2\rho}}{\Gamma(1+2\rho)} + 2(n+\hbar) \hbar^2 E_\rho (v^\rho) \frac{u^{4\rho}}{\Gamma(1+4\rho)} + \hbar^3 E_\rho (v^\rho) \frac{u^{6\rho}}{\Gamma(1+6\rho)} \right] + \cdots. \end{aligned} \tag{54}$$

Setting $\hbar = -1$ and $n = 1$, one can have

$$\phi (u,v) = E_\rho (v^\rho) \left(-1 + \frac{u^{2\rho}}{\Gamma(1+2\rho)} - \frac{u^{4\rho}}{\Gamma(1+4\rho)} + \frac{u^{6\rho}}{\Gamma(1+6\rho)} - \cdots \right).$$

It can be written as

$$\phi (u,v) = E_\rho (v^\rho) \left(\sum_{m=0}^{\infty} (-1)^m \frac{u^{2m\rho}}{\Gamma(1+2m\rho)} \right).$$

The solution of Eq. (44) is constituted as

$$\phi (u,v) = E_\rho (v^\rho) \cos_\rho (u^\rho). \tag{55}$$

Example 4.3 Finally, the following Laplace equation with LFD [9] is investigated

$$\frac{\partial^{2\rho}\phi(u,v)}{\partial u^{2\rho}}+\frac{\partial^{2\rho}\phi(u,v)}{\partial v^{2\rho}}=0, 0<\rho\le 1 \tag{56}$$

with initial-boundary conditions given as:

$$\phi\left(0,v\right)=0, \frac{\partial^{\rho}\phi(0,v)}{\partial u^{\rho}}=-E_{\rho}\left(y^{\rho}\right). \tag{57}$$

Applying LFLT on Eq. (56), we obtain

$$L_{\rho}\left\{\phi(u,v)\right\}-s^{-\rho}\phi(0,v)-s^{-2\rho}\frac{\partial^{\rho}\phi(0,v)}{\partial u^{\rho}}+s^{-2\rho}L_{\rho}\left\{\frac{\partial^{2\rho}\phi(u,v)}{\partial^{2\rho}v}\right\}=0$$

or

$$L_{\rho}\left\{\phi(u,v)\right\}+s^{-2\rho}E_{\rho}\left(y^{\rho}\right)+s^{-2\rho}L_{\rho}\left\{\frac{\partial^{2\rho}\phi(u,v)}{\partial^{2\rho}v}\right\}=0. \tag{58}$$

The nonlinear operator is constituted as

$$N[\psi(\eta,\kappa;l)]=L_{\rho}\left[\psi(\eta,\kappa;l)\right]+s^{-2\rho}E_{\rho}\left(y^{\rho}\right)+s^{-2\rho}L_{\beta}\left\{\frac{\partial^{2\rho}\psi(\eta,\kappa;l)}{\partial^{2\rho}v}\right\}, \tag{59}$$

and so

$$\Re_{m}\left(\vec{\phi}_{m-1}\left(u,v\right)\right)=L_{\rho}\left\{\phi_{m-1}\left(u,v\right)\right\}$$

$$+\left(1-\frac{\chi_{m}}{n}\right)s^{-2\rho}E_{\rho}\left(v^{\rho}\right)+s^{-2\rho}L_{\rho}\left[\frac{\partial^{2\rho}\phi_{m-1}\left(u,v\right)}{\partial v^{2\rho}}\right]. \tag{60}$$

Next, we present the mth-order deformation equation as

$$L_{\rho}\left\{\phi_{m}\left(u,v\right)-\chi_{m}\phi_{m-1}\left(u,v\right)\right\}=\hbar\Re_{m}\left(\vec{\phi}_{m-1}\left(u,v\right)\right). \tag{61}$$

Applying the inverse LFLT, we obtain

$$\phi_{m}\left(u,v\right)=\chi_{m}\phi_{m-1}\left(u,v\right)+\hbar L_{\rho}^{-1}\left\{\Re_{m}\left(\vec{\phi}_{m-1}\left(u,v\right)\right)\right\}. \tag{62}$$

Taking $m=1,2,3,\ldots$, we get
For $m=1$, we have

$$\phi_{1}\left(u,v\right)=\hbar E_{\rho}\left(v^{\rho}\right)\frac{u^{\rho}}{\Gamma\left(1+\rho\right)}. \tag{63}$$

For $m=2$, we obtain

$$\phi_{2}\left(u,v\right)=\left(n+\hbar\right)\hbar E_{\rho}\left(v^{\rho}\right)\frac{u^{\rho}}{\Gamma\left(1+\rho\right)}+\hbar^{2}E_{\rho}\left(v^{\rho}\right)\frac{u^{3\rho}}{\Gamma\left(1+3\rho\right)}. \tag{64}$$

For $m=3$, we find

$$\phi_{3}\left(u,v\right)=\left(n+\hbar\right)^{2}\hbar E_{\rho}\left(v^{\rho}\right)\frac{u^{\rho}}{\Gamma\left(1+\rho\right)}$$

$$+2\left(n+\hbar\right)\hbar^{2}E_{\rho}\left(v^{\rho}\right)\frac{u^{3\rho}}{\Gamma\left(1+3\rho\right)}+\hbar^{3}E_{\rho}\left(v^{\rho}\right)\frac{u^{5\rho}}{\Gamma\left(1+5\rho\right)},\tag{65}$$

Hence, the nondifferentiable solution is

$$\phi(u,v)=\sum_{m=0}^{\infty}\phi_{m}(u,v)(\frac{1}{n})^{m}.$$

or

$$\phi(u,v)=\frac{1}{n}\left[\hbar E_{\rho}\left(v^{\rho}\right)\frac{u^{\rho}}{\Gamma\left(1+\rho\right)}\right]+\frac{1}{n^{2}}\left[\left(n+\hbar\right)\hbar E_{\rho}\left(v^{\rho}\right)\frac{u^{\rho}}{\Gamma\left(1+\rho\right)}-\hbar^{2}E_{\rho}\left(v^{\rho}\right)\left(\frac{u^{3\rho}}{\Gamma\left(1+3\rho\right)}\right)\right]$$

$$+\frac{1}{n^{3}}\left[\left(n+\hbar\right)^{2}\hbar E_{\rho}\left(v^{\rho}\right)\frac{u^{\rho}}{\Gamma\left(1+\rho\right)}+2\left(n+\hbar\right)\hbar^{2}E_{\rho}\left(v^{\rho}\right)\frac{u^{3\rho}}{\Gamma\left(1+3\rho\right)}+\hbar^{3}E_{\rho}\left(v^{\rho}\right)\frac{u^{5\rho}}{\Gamma\left(1+5\rho\right)}\right]+\cdots.\tag{66}$$

On using the values $\hbar=-1$ and $n=1$, we have

$$\phi\left(u,v\right)=E_{\rho}\left(v^{\rho}\right)\left(-\frac{u^{\rho}}{\Gamma\left(1+\rho\right)}+\frac{u^{3\rho}}{\Gamma\left(1+3\rho\right)}-\frac{u^{5\rho}}{\Gamma\left(1+5\rho\right)}+\cdots\right).$$

The solution of Eq. (56) in closed form is expressed as

$$\phi\left(u,v\right)=E_{\rho}\left(v^{\rho}\right)\left(\sum_{m=0}^{\infty}\left(-1\right)^{m}\frac{u^{(2m+1)\rho}}{\Gamma\left(1+\left(2m+1\right)\rho\right)}\right).$$

or

$$\phi\left(u,v\right)=E_{\rho}\left(v^{\rho}\right)\sin_{\rho}\left(u^{\rho}\right).\tag{67}$$

5 Numerical simulation

This section presents numerical outcomes for fractal problem given in Examples 4.1-4.3 under fractal initial-boundary conditions. The 3D graphs for the local fractional Helmholtz and Laplace equations are demonstrated on the Cantor set for the fractal order $\rho=\ln 2/\ln 3$ via MATLAB. The graphics authenticates that the achieved solutions for Examples 4.1-4.3 depend on the fractal order ρ of the LFD. The 3D graphical visuals show the fractal pattern of the nondifferentiable function $\phi\left(u,\, v\right)$ in Examples 4.1-4.3.

6 Conclusions

In this work, the q-LFHATM is utilized to obtain the nondifferentiable solutions for the Helmholtz and Laplace equation in fractal media. The computed results establish the reliability and efficiency of the proposed technique and the applied method can be used to solve many other LFPDEs arising in fractal media. Finally, the computer simulations are also presented for fractal analysis of local fractional Helmholtz and Laplace models.

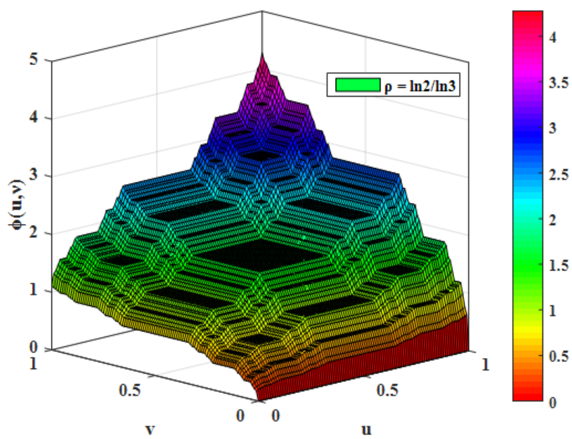

Figure 1: 3D nature of $\phi(u,v)$ w.r.t. u and v for Example 4.1

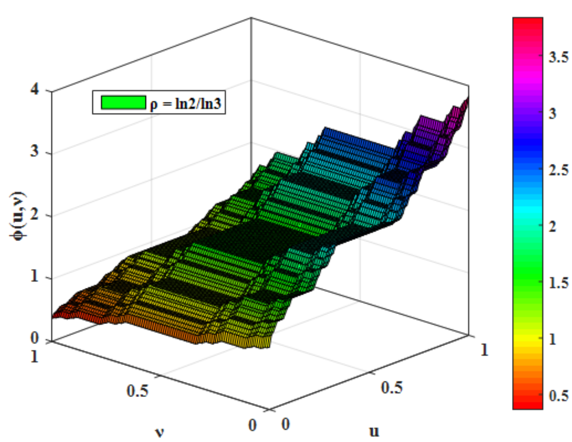

Figure 2: 3D behavior of $\phi(u,v)$ w.r.t. u and v for Example 4.2

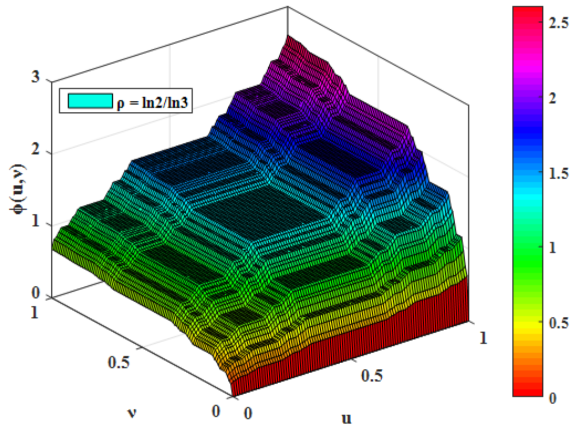

Figure 3: 3D pattern of $\phi(u,v)$ w.r.t. u and v for Example 4.3

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 31, NO. 3, 2023, COPYRIGHT 2023 EUDOXUS PRESS, LLC

On asymptotic behavior of a quadratic functional equation

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December 9, 2022

Abstract

The main goal of this paper is to investigate the stability problems for the following quadratic functional equation

$$f(x+y+z)+f(x+y-z)+f(x-y+z)+f(-x+y+z) = 4f(x)+4f(y)+4f(z)$$

on an unbounded restricted domain. As a consequence, we can apply the obtained results to obtain some asymptotic behaviors of that equation in normed spaces. Moreover, we introduce a new inequality that characterizes the inner product spaces.

1 Introduction

In 1940, Stanisław Marcin Ulam proposed the following problem [25]:

> Let $(G_1, .)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(.,.)$. Given a real number $\varepsilon > 0$, does there exist a real number $\delta > 0$ such that if a mapping $h\colon G_1 \to G_2$ satisfies the inequality $d(h(x.y), h(x) * h(y)) \le \delta$ for all $x, y \in G_1$, then there is a homomorphism $H\colon G_1 \to G_2$ with $d(h(x), H(x)) \le \varepsilon$ for all $x \in G_1$?

This problem gave rise to what we now call Ulam's stability of functional equations.

In a later year, an affirmative answer to the Ulam stability problem was given by D. H. Hyers for Banach spaces (see [13]). Several generalizations of this result are discussed. T. Aoki [5] for additive maps and by T.M. Rassias

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[19] for linear maps considering an unbounded Cauchy difference. P. Găvruţă [12] provided a further generalization of the Rassias' theorem by using a general control function. During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors (see, [1, 19, 20, 21, 23]).

Throughout the paper, let $(G,+)$ be an Abelian group and Y be a linear space on the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Let us note that a mapping $q\colon G \to Y$ is called quadratic if q satisfies the well-known quadratic functional equation

$$q(x+y) + q(x-y) = 2q(x) + 2q(y), \quad x, y \in G. \tag{1.1}$$

Quadratic functional equation (1.1) was used by Jordan and von Neumann [14] to characterize inner product spaces. Several other functional equations are used for this characterization. Maurice Fréchet in [11] obtained a characterization of the inner product spaces among normed linear spaces by using the following functional equation

$$f(x+y+z) + f(x) + f(y) + f(z) = f(x+y) + f(y+z) + f(x+z).$$

Theorem 1.1. *[11] Let $(X, \|.\|)$ be a normed linear space. Then X is an inner product space with respect to $\|.\|$ if and only if*

$$\|x+y+z\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2 = \|x+y\|^2 + \|x+z\|^2 + \|y+z\|^2, \quad x, y, z \in X.$$

Motivated by this idea, we deal with the following functional equation:

$$\begin{aligned} f(x+y+z) + f(x+y-z) + f(x-y+z) + f(-x+y+z) \\ = 4f(x) + 4f(y) + 4f(z). \end{aligned} \tag{1.2}$$

This equation was first introduced and solved by S. Jung [15]. In fact, he proved the following theorem

Theorem 1.2 ([15], Theorem 2.1.)**.** *Let X and Y be vector spaces over fields of characteristic different from 2, respectively. If $f\colon X \to Y$ satisfies the functional equations*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \tag{1.3}$$

$$f(x-y-z) + f(x) + f(y) + f(z) = f(x-y) + f(y+z) + f(z-x) \tag{1.4}$$

and

$$\begin{aligned} f(x+y+z) + f(x+y-z) + f(x-y+z) + f(-x+y+z) \\ = 4f(x) + 4f(y) + 4f(z), \end{aligned} \tag{1.5}$$

then each of the equations (1.3), (1.4)*, and* (1.5) *is equivalent to the other.*

Recently, EL-Fassi *et al.* [9] treated the Ulam-type stability of (1.2) in the class of functions from an Abelian group into a Banach space. The method used in [9, Theorem 4] based on a fixed point theorem [7, Theorem 1] and the argument presented is all on the whole set. This aspect of the domain is very important. However if we consider a subset of the domain which does not present all the possibilities when we have on the whole set, is the stability of the equation still valid? Many others studies this question about the stability on restricted domains of some functional equations (see [24, 22]).

Inspired by the works of Hyers [13], Brzdęk [8] and Park [17], and by a direct method, we investigate the stability of (1.2) on a restricted unbounded domain. Then, using these results, we study an asymptotic behavior of this functional equation. We also obtain a new criterion on characterization of inner product spaces by involving our functional equation. Before we state it, let us recall the definitions of quasi-norm and quasi-normed Abelian group.

We recall some basic facts concerning quasi-norm and quasi-normed Abelian group.

Definition 1.3. [4] Let $(G,+)$ be an Abelian group. A function $\rho\colon G \to \mathbb{R}$ is called a quasi-norm on G if:

1. $0 \leq \rho(x) \leq +\infty$ for all $x \in G$ (positive definite);
2. $\rho(x) = \rho(-x)$ for all $x \in G$ (even);
3. $\rho(x+y) \leq \rho(x) + \rho(y)$ for all $x, y \in G$ (subadditivity);
4. $\rho(0) = 0$.

If $\rho(x) < +\infty$ for all $x \in G$ we say that ρ is a finite quasi-norm. The pair (G, ρ) is called quasi-normed Abelian group if ρ is a quasi-norm on G.

A triplet $(G,+,\delta)$ is called metric Abelian group if $(G,+)$ is an Abelian group and δ is a translation invariant metric on G. This metric can be turned into a quasi-norm $\|.\|_\delta \colon G \to \mathbb{R}$, via $\|x\|_\delta = \delta(x,0)$ and the pair $(G, \|.\|_\delta)$ is a quasi-normed Abelian group. For a more detailed definition of such terminology, one can refer to [10, 18].

In this paper, assume that $(G,+)$ be an Abelian group, Y be a linear space on the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For a given mapping $f\colon G \longrightarrow Y$, we define the function $\Delta_f \colon G \times G \times G \longrightarrow Y$ by

$$\begin{aligned}\Delta_f(x,y,z) := {} & f(x+y+z) + f(x+y-z) + f(x-y+z) \\ & + f(-x+y+z) - 4f(x) - 4f(y) - 4f(z), \quad x,y,z \in G.\end{aligned}$$

2 Stability in restricted domains

In this section we study the stability problem of the functional equation (1.2).

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Theorem 2.1. *Let $(G, +, \delta)$ be a metric Abelian group, $\|.\|_\delta$ be the induced quasi-norm of δ and $(Y, \|.\|)$ be a Banach space. Let $\varepsilon \geq 0$, $d > 0$ be arbitrary real numbers. Suppose that $f : G \to Y$ is a function satisfies*

$$\|\Delta_f(x, y, z)\| \leq \varepsilon, \quad \|x + y + z\|_\delta \geq d. \tag{2.1}$$

Then, there exists a unique quadratic function $Q : G \to Y$ such that

$$\|Q(x) - f(x)\| \leq \frac{5\varepsilon}{8}, \qquad x \in G. \tag{2.2}$$

Proof. Let $f : G \to Y$ be a function fulfilling (2.1). Taking $x = y = z$ in (2.1), we get

$$\|f(3x) - 9f(x)\| \leq \varepsilon, \quad \|3x\|_\delta \geq d,$$

witch implies

$$\|f(3x) - 9f(x)\| \leq \varepsilon, \quad \|x\|_\delta \geq d.$$

Therefore,

$$\left\| \frac{f(3^{n+1}x)}{9^{n+1}} - \frac{f(3^m x)}{9^m} \right\| \leq \sum_{k=m}^{k=n} \frac{\varepsilon}{9^{k+1}}, \tag{2.3}$$

for all natural numbers ; $n \geq m$, and $\|x\|_\delta \geq d$. Therefore, $\left\{ \frac{f(3^n x)}{9^n} \right\}_{n=0}^{\infty}$ is a Cauchy sequence for each $x \in G$ with $\|x\|_\delta \geq d$. It is easily to infer that the sequence $\left\{ \frac{f(3^n x)}{9^n} \right\}_n$ is Cauchy in the whole G. As Y is Banach space, this Cauchy sequence is convergent. We define $Q : G \to Y$ by

$$Q(x) := \lim_{n \to +\infty} \frac{f(3^n x)}{9^n}, \quad x \in G. \tag{2.4}$$

For $x \in G \setminus \{0\}$, we choose $N \in \mathbb{N}$ so large that for all $n \geq N$, $\|3^n x\|_\delta \geq d$. By (2.4), we see that

$$\begin{aligned} \|\Delta_Q(x, y, z)\| = \lim_{n \to +\infty} \frac{1}{9^n} \Big\| & f(3^n x + 3^n y + 3^n z) + f(3^n x + 3^n y - 3^n z) \\ & + f(3^n x - 3^n y + 3^n z) + f(-3^n x + 3^n y + 3^n z) \\ & - 4[f(3^n x) + f(3^n y) + f(3^n z)] \Big\| \\ \leq & \lim_{n \to +\infty} \frac{\varepsilon}{9^n} = 0. \end{aligned}$$

Hence, Q fulfills equation (1.2) for all $x \in G \setminus \{0\}$.

Since

$$Q(0) = \lim_{n \to +\infty} \frac{f(0)}{9^n} = 0,$$

the function Q fulfills equation (1.2) for all $x \in G$. Since Q is a solution of (1.2), we infer that Q is a quadratic function in G.

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Taking the limit as $n \to +\infty$ and putting $m = 0$, we get from (2.3)

$$\|Q(x) - f(x)\| \leq \frac{\varepsilon}{8}, \quad \|x\|_\delta \geq d. \tag{2.5}$$

Next, we extend (2.5) to the whole G. Let $z \in G$ and choose $\|x\|_\delta \geq \|z\|_\delta + d$ such that $\|y\|_\delta \geq \|z\|_\delta + d$ and $\|x+y\|_\delta \geq \|z\|_\delta + d$. Clearly, $\|x+y\|_\delta \geq d$, $\|x+z\|_\delta \geq d$ and $\|x+y+z\|_\delta \geq d$.Then by (2.5), we get

$$\begin{aligned}
&\|Q(x+y+z) - f(x+y+z)\| \leq \frac{\varepsilon}{8};\\
&\|Q(x+y-z) - f(x+y-z)\| \leq \frac{\varepsilon}{8};\\
&\|Q(x-y+z) - f(x-y+z)\| \leq \frac{\varepsilon}{8};\\
&\|Q(-x+y+z) - f(-x+y+z)\| \leq \frac{\varepsilon}{8};\\
&\|-4Q(x) + 4f(x)\| \leq \frac{4\varepsilon}{8};\\
&\|-4Q(y) + 4f(y)\| \leq \frac{4\varepsilon}{8}.
\end{aligned}$$

Adding these inequalities and applying (2.5) and (2.1), we get

$$\|4Q(z) - 4f(z)\| \leq \varepsilon + \frac{3\varepsilon}{2}.$$

Therefore

$$\|Q(z) - f(z)\| \leq \frac{5\varepsilon}{8},$$

for $z \in G$.

It remains to prove the uniqueness of Q. Assume that $Q' : G \to Y$ is another quadratic function that satisfies inequality (2.2). Then we have

$$\begin{aligned}
\|Q(x) - Q'(x)\| &\leq \|Q(x) - f(x)\| + \|Q'(x) - f(x)\|\\
&\leq \frac{5\varepsilon}{4}, \quad x \in G,
\end{aligned}$$

Since Q and Q' are quadratic, the last inequality implies that

$$\begin{aligned}
\|Q(x) - Q'(x)\| &= \frac{1}{9^n} \|Q(3^n x) - Q'(3^n x)\|\\
&\leq \frac{1}{9^n} \times \frac{5\varepsilon}{4}, \qquad x \in G, n \in \mathbb{N} \setminus \{0\}.
\end{aligned}$$

Taking the limit as $n \longrightarrow \infty$, we obtain $Q(x) = Q'(x)$ for all $x \in G$. This completes the proof. $\square$

3 Asymptotic behavior of the equation

As a consequences, we can prove some corollaries concerning the asymptotic behaviors of the functional equation (1.2).

Corollary 3.1. *Let $(G, +, \delta)$ be a metric Abelian group, $\|.\|_\delta$ be the induced quasi-norm of δ and $(Y, \|.\|)$ be a normed space. If a mapping $f : G \to Y$ satisfies*

$$\limsup_{\|x+y+z\|_\delta \to +\infty} \Delta_f(x, y, z) = 0, \tag{3.1}$$

then f is a quadratic function on X.

Proof. Let $f : G \to Y$ be a mapping satisfies (3.1). Then, there exists a sequence $\{d_n\}_{n=1}^{\infty}$ of positive real numbers such that

$$\|\Delta_f(x, y, z)\| \leq \frac{1}{n}, \quad \|x + y + z\|_\delta \geq d_n, \quad n > 1.$$

Let $\widetilde{Y}$ be the completion of Y. By Theorem 2.1, there exists a unique quadratic function $Q_n : G \to \widetilde{Y}$ solution of (1.2) and such that

$$\|Q_n(x) - f(x)\| \leq \frac{5}{8n}, \quad x \in G, \quad n > 1. \tag{3.2}$$

Let l and m be integers satisfying $m > l > 0$. From (3.2) we obtain

$$\|Q_m(x) - f(x)\| \leq \frac{5}{8m} \leq \frac{5}{8l}, \quad x \in G.$$

Hence, the uniqueness of Q_n implies that $Q_l = Q_m$ holds for any $l, m \in \mathbb{N}$. Taking the limit as $n \longrightarrow \infty$ in (3.2), we infer that f is quadratic. Then the result follows. ☐

Using Theorem 2.1, we obtain the results.

Corollary 3.2. *Let $(G, +, \delta)$ be a metric Abelian group, $\|.\|_\delta$ be the induced quasi-norm of δ and $(Y, \|.\|)$ be a Banach space. Let $\psi : G \times G \times G \to [0, +\infty)$. If a mapping $f : G \to Y$ satisfies*

$$\begin{cases} \lim\limits_{\|x+y+z\|_\delta \to +\infty} \psi(x, y, z) = +\infty; \\ \limsup\limits_{\|x+y+z\|_\delta \to +\infty} \psi(x, y, z) \, \|\Delta_f(x, y, z)\| < \infty, \end{cases} \tag{3.3}$$

then f is a quadratic function on G.

Proof. It follows from (3.3) that there exist constants $s > 0$ and R such that

$$\psi(x, y, z) \, \|\Delta_f(x, y, z)\| < R, \quad \|x + y + z\|_\delta \geq s.$$

Since $\lim_{\|x+y+z\|_\delta \to +\infty} \psi(x,y,z) = +\infty$, then for an arbitrary $\varepsilon > 0$ there is $M > 0$ such that

$$\psi(x,y,z) \geq \frac{R}{\varepsilon}, \quad \|x+y+z\|_\delta \geq M.$$

Then,

$$\|\Delta_f(x,y,z)\| < \varepsilon, \quad \|x+y+z\|_\delta \geq \max\{s, M\}.$$

Let $\widetilde{Y}$ be the completion of Y. Using theorem 2.1, there exists a unique quadratic function $Q : G \to Y$ solution of (1.2) and such that

$$\|Q(x) - f(x)\| \leq \frac{5\varepsilon}{8}, \quad x \in G.$$

Since ε is arbitrary, we infer that $Q(x) = f(x)$ for all $x \in G$. $\square$

Corollary 3.3. *Let $(G, +, \delta)$ be a metric Abelian group, $\|.\|_\delta$ be the induced quasi-norm of δ and $(Y, \|.\|)$ be a Banach space. Let $p < 0$ and $\lambda > 0$ be arbitrary real numbers. If a mapping $f : G \to Y$ satisfies*

$$\|\Delta_f(x,y,z)\| \leq \lambda \|x+y+z\|_\delta^p, \quad x,y,z \in G \setminus \{0\}.$$

Then f is a quadratic function on $G \setminus \{0\}$.

4 Application

Several functional equations were used to characterize inner product spaces from normed spaces, for instance, see [14, 2, 6, 3, 16]. Quadratic functional equation was used to characterize inner product spaces by making use the parallelogram equality [14]:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

This characterization gave rise to what we now call Jordan-von Neumann characterization. Other characterization is given by Fréchet in [11], he proved that a normed space $(X, \|.\|)$ is an inner product space if and only if

$$\|x+y+z\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2 = \|x+y\|^2 + \|x+z\|^2 + \|y+z\|^2, \quad x,y,z \in X.$$

Now, we can apply the functional equation (1.2) in a characterizations of inner product spaces.
Let $\mathbb{K}$ be the field of real or complex numbers. Let $(X, \|.\|)$ be a normed space over $\mathbb{K}$ and $X_0 := X \setminus \{0\}$. Write

$$D(x,y,z) = \|x+y+z\|^2 + \|x+y-z\|^2 + \|x-y+z\|^2 + \|-x+y+z\|^2 - 4\|x\|^2 - 4\|y\|^2 - 4\|z\|^2.$$

Theorem 4.1. *Let $(X, \|.\|)$ be a normed space over $\mathbb{K}$. Suppose that*

$$D(x, y, z) = 0, \quad x, y, z \in X.$$

Then X is an inner product space.

Proof. Let $X \neq \{0\}$ be a normed space over $\mathbb{K}$ such that

$$\|x + y + z\|^2 + \|x + y - z\|^2 + \|x - y + z\|^2 + \|-x + y + z\|^2 \\ - 4\|x\|^2 - 4\|y\|^2 - 4\|z\|^2 = 0, \quad (4.1)$$

for $x, y, z \in X$. Putting $z = 0$ in (4.1), we get

$$\|x + y\|^2 + \|x + y\|^2 + \|x - y\|^2 + \|-x + y\|^2 - 4\|x\|^2 - 4\|y\|^2 = 0,$$

for $x, y \in X$, then the Jordan-von Neumann characterization holds

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Consequently, X is an inner product space. □

Theorem 4.2. *Let $(X, \|.\|)$ be a normed space over $\mathbb{K}$. Suppose that*

$$\sup_{x,y,z \in X} \frac{|D(x, y, z)|}{\lambda \|x + y + z\|^p} < \infty, \quad x + y + z \in X_0, \quad p < 0, \quad \lambda > 0.$$

Then X is an inner product space.

Proof. Write $f(x) = \|x\|$ for $x \in X$. From Corollary 3.3 we easily derive that f is a quadratic function, which yields the statement. □

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