Volume 23, Number 7 November 30, 2017
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(fifteen times annually)

Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, Madison,WI,USA.

Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $750, Electronic OPEN ACCESS. Individual:Print $380. For any other part of the world add $140 more(handling and postages) to the above prices for Print. No credit card payments.

JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

Editorial Board
Associate Editors of Journal of Computational Analysis and Applications

George Cybenko
Thayer School of Engineering
Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Juan J. Trujillo
University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es
Fractional: Differential Equations-Operators-Fourier Transforms, Special functions, Approximations, and Applications

Ram Verma
International Publications
1200 Dallas Drive #824 Denton,
TX 76205, USA
Verma99@msn.com
Applied Nonlinear Analysis, Numerical Analysis, Variational Inequalities, Optimization Theory, Computational Mathematics, Operator Theory

Xiang Ming Yu
Department of Mathematical Sciences
Southwest Missouri State University
Springfield, MO 65804-0094
417-836-5931
xmy944f@missouristate.edu
Classical Approximation Theory, Wavelets

Lotfi A. Zadeh
Professor in the Graduate School and Director, Computer Initiative, Soft Computing (BISC)
Computer Science Division
University of California at Berkeley
Berkeley, CA 94720
Office: 510-642-4959
Sec: 510-642-8271
Home: 510-526-2569
FAX: 510-642-1712
zadeh@cs.berkeley.edu
Fuzzyness, Artificial Intelligence, Natural language processing, Fuzzy logic

Richard A. Zalik
Department of Mathematics
Auburn University
Auburn University, AL 36849-5310
USA.
Tel 334-844-6557 office
678-642-8703 home
Fax 334-844-6555
zalik@auburn.edu
Approximation Theory, Chebychev Systems, Wavelet Theory

Ahmed I. Zayed
Department of Mathematical Sciences
DePaul University
2320 N. Kenmore Ave.
Chicago, IL 60614-3250
773-325-7808
e-mail: azayed@condor.depaul.edu
Shannon sampling theory, Harmonic analysis and wavelets, Special functions and orthogonal polynomials, Integral transforms

Ding-Xuan Zhou
Department Of Mathematics
City University of Hong Kong
83 Tat Chee Avenue
Kowloon, Hong Kong
852-2788 9708,Fax:852-2788 8561
e-mail: mazhou@cityu.edu.hk
Approximation Theory, Spline functions, Wavelets

Xin-long Zhou
Fachbereich Mathematik, Fachgebiet Informatik
Gerhard-Mercator-Universitat Duisburg
Lotharstr.65, D-47048 Duisburg, Germany
e-mail:Xzhou@informatik.uni-duisburg.de
Fourier Analysis, Computer-Aided Geometric Design, Computational Complexity, Multivariate Approximation Theory, Approximation and Interpolation Theory

Instructions to Contributors
Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou

Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .
Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).
If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.
References should include (in the following order):
initials of first and middle name, last name of author(s)
title of article,

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

<u>Journal Article</u>

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

<u>Book</u>

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

<u>Contribution to a Book</u>

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus <u>homepage</u>.

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

DIFFERENTIAL EQUATIONS ASSOCIATED WITH MODIFIED DEGENERATE BERNOULLI AND EULER NUMBERS

TAEKYUN KIM, DAE SAN KIM, HYUCK IN KWON, AND JONG JIN SEO

Abstract. In this paper, we consider some ordinary differential equations associated with modified degenerate Euler and Bernoulli numbers and give some new identities for these numbers arising from our differential equations.

1. Introduction

As is well known, Bernoulli numbers are defined by the generating function

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!},\quad (\text{see [1–12]}), \tag{1.1}$$

and the Euler numbers are given by generating function

$$\frac{2}{e^t+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!},\quad (\text{see [7, 8]}). \tag{1.2}$$

In [2], L. Carlitz considered the degenerate Bernoulli and Euler numbers which are defined by the generating functions

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}=\sum_{n=0}^{\infty}\beta_n(\lambda)\frac{t^n}{n!}, \tag{1.3}$$

and

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}=\sum_{n=0}^{\infty}\mathcal{E}_n(\lambda)\frac{t^n}{n!}. \tag{1.4}$$

Note that $\lim_{\lambda\to 0}\beta_n(\lambda)=B_n$ and $\lim_{\lambda\to 0}\mathcal{E}_n(\lambda)=E_n$, $(n\geq 0)$.

Now, we define the modified degenerate Bernoulli and Euler numbers which are slightly different from the Carlitz degenerate Bernoulli and Euler numbers as follows:

$$\frac{t}{(1+\lambda)^{\frac{t}{\lambda}}-1}=\sum_{n=0}^{\infty}\tilde{\beta}_n(\lambda)\frac{t^n}{n!},\quad (\text{see [3]}), \tag{1.5}$$

and

$$\frac{2}{(1+\lambda)^{\frac{t}{\lambda}}+1}=\sum_{n=0}^{\infty}\tilde{\mathcal{E}}_n(\lambda)\frac{t^n}{n!},\quad (\text{see [9]}). \tag{1.6}$$

2010 Mathematics Subject Classification : 05A19, 11B37, 11B68, 11B83, 34A30.

Key words and phrases: modified degenerate Bernoulli and Euler numbers, differential equations.

From (1.5) and (1.4), we easily note that

$$\lim_{\lambda\to 0}\tilde{\beta}_n(\lambda)=B_n \quad \text{and} \quad \lim_{\lambda\to 0}\tilde{\mathcal{E}}_n(\lambda)=E_n, \quad (n\geq 0). \tag{1.7}$$

For $r\in\mathbb{N}$, the higher-order modified Bernoulli and Euler numbers are also defined by the generating functions

$$\left(\frac{t}{(1+\lambda)^{\frac{t}{\lambda}}-1}\right)^r=\sum_{n=0}^{\infty}\tilde{\beta}_n^{(r)}(\lambda)\frac{t^n}{n!}, \tag{1.8}$$

and

$$\left(\frac{2}{(1+\lambda)^{\frac{t}{\lambda}}+1}\right)^r=\sum_{n=0}^{\infty}\tilde{\mathcal{E}}_n^{(r)}(\lambda)\frac{t^n}{n!}. \tag{1.9}$$

Recall that the higher order Bernoulli and Euler numbers are given by the generating functions

$$\left(\frac{t}{e^t-1}\right)^r=\sum_{n=0}^{\infty}B_n^{(r)}\frac{t^n}{n!}, \tag{1.10}$$

and

$$\left(\frac{2}{e^t+1}\right)^r=\sum_{n=0}^{\infty}E_n^{(r)}\frac{t^n}{n!}, \quad (\text{see [6, 11]}). \tag{1.11}$$

From (1.8), (1.9), (1.10) and (1.11), we note that

$$\lim_{\lambda\to 0}\tilde{\beta}_n^{(r)}(\lambda)=B_n^{(r)} \quad \text{and} \quad \lim_{\lambda\to 0}\tilde{\mathcal{E}}_n^{(r)}(\lambda)=E_n^{(r)}.$$

In [1], Bayad-Kim studied the following nonlinear differential equations:

$$F_q^N=\frac{1}{(N-1)!}\sum_{k=1}^{N}a_k(N)F_q^{(k-1)}, \quad (N\in\mathbb{N}), \tag{1.12}$$

where $F^{(k)}=F^{(k)}(t)=\left(\frac{d}{dt}\right)^kF$.

For $F_q(t)=\frac{1}{qe^t\pm 1}$, Bayad-Kim gave explicit formulae for Apostol-Bernoulli and Apostol-Euler numbers and polynomials which are derived from (1.12).

In [4], Guo-Qi obtained the following results

$$\left(\frac{d}{dt}\right)^k\left(\frac{1}{\lambda e^{\alpha t}-1}\right)=(-1)^k\alpha^k\sum_{m=1}^{k+1}(m-1)!S_2(k+1,m)\left(\frac{1}{\lambda e^{\alpha t}-1}\right)^m, \tag{1.13}$$

and

$$\left(\frac{1}{\lambda e^{\alpha t}-1}\right)^k=\frac{1}{(k-1)!}\sum_{m=1}^{k}\frac{(-1)^{m-1}}{\alpha^{m-1}}S_1(k,m)\left(\frac{d}{dt}\right)^{m-1}\left(\frac{1}{\lambda e^{\alpha t}-1}\right), \tag{1.14}$$

where $k\in\mathbb{N}$, and $S_1(k,m)$ and $S_2(k,m)$ are respectively the Stirling numbers of the first kind and of the second kind (see [4, 10]). However, the results of Guo-Qi are immediately obtained from the paper of Bayad-Kim in [1] by replacing q by λ and t by αt (α =constnat).

Recently, Kim-Kim studied the nonlinear differential equations given by

$$\left(\frac{d}{dt}\right)^N\left(\frac{1}{(1+\lambda t)^{\frac{1}{\lambda}}\pm 1}\right)=\frac{(-1)^N}{(1+\lambda t)^N}\sum_{i=1}^{N+1}a_i(N,\lambda)F^i, \tag{1.15}$$

where

$$F = F(t) = \frac{1}{(1+\lambda t)^{\frac{1}{\lambda}} \pm 1} \quad (\text{see [7]}).$$

From (1.15), we derived some new identities involving degenerate Euler and Bernoulli polynomials.

In this paper, along the same line as [7] we study some ordinary differential equations arising from the generating functions of the modified degenerate Bernoulli and Euler numbers. From those equations, we derive some new identities for the modified degenerate Bernoulli and Euler numbers.

2. Differential equations associated with modified degenerate Bernoulli and Euler numbers

Let

$$F = F(t) = \left((1+\lambda)^{\frac{t}{\lambda}} \pm 1\right)^{-1}. \tag{2.1}$$

Then, by (2.1), we get

$$\begin{aligned} F^{(1)} = \frac{dF}{dt} &= -\left((1+\lambda)^{\frac{t}{\lambda}} \pm 1\right)^{-2}(1+\lambda)^{\frac{t}{\lambda}}\frac{1}{\lambda}\log(1+\lambda) \\ &= -\frac{1}{\lambda}\log(1+\lambda)\left((1+\lambda)^{\frac{t}{\lambda}} \pm 1\right)^{-2}\left((1+\lambda)^{\frac{t}{\lambda}} \pm 1 \mp 1\right) \\ &= -\frac{1}{\lambda}\log(1+\lambda)\left(F \mp F^2\right), \end{aligned} \tag{2.2}$$

$$\begin{aligned} F^{(2)} &= \frac{dF^{(1)}}{dt} \\ &= -\frac{1}{\lambda}\log(1+\lambda)\left(F^{(1)} \mp 2FF^{(1)}\right) \\ &= -\frac{1}{\lambda}\log(1+\lambda)(1 \mp 2F)F^{(1)} \\ &= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^2(1 \mp 2F)\left(F \mp F^2\right) \\ &= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^2\left(F \mp 3F^2 + 2F^3\right). \end{aligned} \tag{2.3}$$

Thus we are led to put

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t) \\ &= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^N \sum_{i=1}^{N+1} a_{i-1}^{\pm}(N) F^i, \quad (N = 0, 1, 2, \dots), \end{aligned} \tag{2.4}$$

where $a_{i-1}^{+}(N)$ corresponds to $\left((1+\lambda)^{\frac{t}{\lambda}} + 1\right)^{-1}$ and $a_{i-1}^{+}(N)$ does to $\left((1+\lambda)^{\frac{t}{\lambda}} - 1\right)^{-1}$.

Now, from (2.4), we have

$$\begin{aligned} &F^{(N+1)} \\ &= \frac{d}{dt}F^{(N)} \end{aligned} \tag{2.5}$$

$$
\begin{aligned}
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N}\sum_{i=1}^{N+1} a_{i-1}^{\pm}(N)\, iF^{i-1}F^{(1)} \\
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N+1}\left\{\sum_{i=1}^{N+1} ia_{i-1}^{\pm}(N)\,F^{i} \mp \sum_{i=2}^{N+2}(i-1)\,a_{i-2}^{\pm}(N)\,F^{i}\right\} \\
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N+1}\Big\{a_{0}^{\pm}(N)\,F \mp (N+1)\,a_{N}^{\pm}(N)\,F^{N+2} \\
&\quad + \sum_{i=2}^{N+1}\left(ia_{i-1}^{\pm}(N) \mp (i-1)\,a_{i-2}^{\pm}(N)\right)F^{i}\Big\}.
\end{aligned}
$$

On the other hand, by replacing N by $N+1$ in (2.4), we get

$$
F^{(N+1)} = \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N+1}\sum_{i=1}^{N+2} a_{i-1}^{\pm}(N+1)\,F^{i}. \tag{2.6}
$$

Comparing the coefficients on both sides of (2.5) and (2.6), we obtain

$$
a_{0}^{\pm}(N+1) = a_{0}^{\pm}(N), \tag{2.7}
$$

$$
a_{N+1}^{\pm}(N+1) = \mp(N+1)\,a_{N}^{\pm}(N), \tag{2.8}
$$

and

$$
a_{i-1}^{\pm}(N+1) = ia_{i-1}^{\pm}(N) \mp (i-1)\,a_{i-2}^{\pm}(N), \tag{2.9}
$$

for $2 \le i \le N+1$.

Also, by (1.12), we get

$$
F = F^{(0)} = a_{0}^{\pm}(0)\,F. \tag{2.10}
$$

Thus, by (2.10), we see that

$$
a_{0}^{\pm}(0) = 1. \tag{2.11}
$$

It is easy to show that

$$
\begin{aligned}
F^{(1)} &= -\frac{1}{\lambda}\log(1+\lambda)\sum_{i=1}^{2} a_{i-1}^{\pm}(1)\,F^{i} \\
&= -\frac{1}{\lambda}\log(1+\lambda)\left(a_{0}^{\pm}(1)\,F + a_{1}^{\pm}(1)\,F^{2}\right) \\
&= -\frac{1}{\lambda}\log(1+\lambda)\left(F \mp F^{2}\right).
\end{aligned} \tag{2.12}
$$

Thus, by comparing the coefficients on both sides of (2.12), we have

$$
a_{0}^{\pm}(1) = 1, \quad a_{1}^{\pm}(1) = \mp 1. \tag{2.13}
$$

From (2.7) and (2.8), we note that

$$
a_{0}^{\pm}(N+1) = a_{0}^{\pm}(N) = \cdots = a_{0}^{\pm}(0) = 1, \tag{2.14}
$$

and

$$
\begin{aligned}
a_{N+1}^{+}(N+1) &= -(N+1)\,a_{N}^{+}(N) \\
&= (-1)^{2}(N+1)\,Na_{N-1}^{+}(N-1) \\
&\ \ \vdots
\end{aligned} \tag{2.15}
$$

$$= (-1)^{N+1}(N+1)!a_0^+(0)$$
$$= (-1)^{N+1}(N+1)!,$$

$$a_{N+1}^-(N+1) = (N+1)a_N^-(N) = (N+1)Na_{N-1}^-(N-1) \\ \vdots \\ = (N+1)!a_0^-(0) = (N+1)!. \tag{2.16}$$

By (2.15) and (2.16), we easily get

$$a_{N+1}^{\pm}(N+1) = (\mp 1)^{N+1}(N+1)!. \tag{2.17}$$

Observe also that the matrix $\left(a_i^+(j)\right)_{0\le i,j\le N}$ and $\left(a_i^-(j)\right)_{0\le i,j\le N}$ are as follows:

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \\ N \end{array}\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & & N \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ & (-1)1! & & & & \\ & & (-1)^2 2! & & & \\ & & & (-1)^3 3! & & \\ & & 0 & & \ddots & \\ & & & & & (-1)^N N! \end{bmatrix} \end{array} = \left(a_i^+(j)\right)_{0\le i,j\le N}$$

and

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \\ N \end{array}\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & & N \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ & 1! & & & & \\ & & 2! & & & \\ & & & 3! & & \\ & 0 & & & \ddots & \\ & & & & & N! \end{bmatrix} \end{array} = \left(a_i^-(j)\right)_{0\le i,j\le N}$$

For $i=2$ in (2.9), we have

$$\begin{aligned} & a_1^{\pm}(N+1) \\ &= \mp a_0^{\pm}(N) + 2a_1^{\pm}(N) \\ &= \mp a_0^{\pm}(N) + 2\left(\mp a_0^{\pm}(N-1) + 2a_1^{\pm}(N-1)\right) \\ &= \mp\left(a_0^{\pm}(N) + 2a_0^{\pm}(N-1)\right) + 2^2 a_1^{\pm}(N-1) \\ &= \mp\left(a_0^{\pm}(N) + 2a_0^{\pm}(N-1)\right) + 2^2\left(\mp a_0^{\pm}(N-2) + 2a_1^{\pm}(N-2)\right) \\ &= \mp\left(a_0^{\pm}(N) + 2a_0^{\pm}(N-1) + 2^2 a_0^{\pm}(N-2)\right) + 2^3 a_1^{\pm}(N-2) \\ & \vdots \end{aligned} \tag{2.18}$$

$$= \mp \sum_{i=0}^{N-1} 2^i a_0^{\pm}(N-i) + 2^N a_1^{\pm}(1) = \mp \sum_{i=0}^{N} 2^i a_0^{\pm}(N-i).$$

Let us take $i = 3$ in (2.9). Then, we note that

$$\begin{aligned} &a_2^{\pm}(N+1) \\ &= \mp 2a_1^{\pm}(N) + 3a_2^{\pm}(N) \\ &= \mp 2a_1^{\pm}(N) + 3\left(\mp 2a_1^{\pm}(N-1) + 3a_2^{\pm}(N-1)\right) \\ &= \mp 2\left(a_1^{\pm}(N) + 3a_1^{\pm}(N-1)\right) + 3^2 a_2^{\pm}(N-1) \\ &= \mp 2\left(a_1^{\pm}(N) + 3a_1^{\pm}(N-1)\right) + 3^2\left(\mp 2a_1^{\pm}(N-2) + 3a_2^{\pm}(N-2)\right) \\ &= \mp 2\left(a_1^{\pm}(N) + 3a_1^{\pm}(N-1) + 3^2 a_1^{\pm}(N-2)\right) + 3^3 a_2^{\pm}(N-2) \\ &\vdots \\ &= \mp 2\sum_{i=0}^{N-2} 3^i a_1^{\pm}(N-i) + 3^{N-1} a_2^{\pm}(2) \\ &= \mp 2\sum_{i=0}^{N-1} 3^i a_1^{\pm}(N-i). \end{aligned} \tag{2.19}$$

For $i = 4$ in (2.9), we have

$$\begin{aligned} &a_3^{\pm}(N+1) \\ &= \mp 3a_2^{\pm}(N) + 4a_3^{\pm}(N) \\ &= \mp 3a_2^{\pm}(N) + 4\left(\mp 3a_2^{\pm}(N-1) + 4a_3^{\pm}(N-1)\right) \\ &= \mp 3\left(a_2^{\pm}(N) + 4a_2^{\pm}(N-1)\right) + 4^2 a_3^{\pm}(N-1) \\ &= \mp 3\left(a_2^{\pm}(N) + 4a_2^{\pm}(N-1)\right) + 4^2\left(\mp 3a_2^{\pm}(N-2) + 4a_3^{\pm}(N-2)\right) \\ &= \mp 3\left(a_2^{\pm}(N) + 4a_2^{\pm}(N-1) + 4^2 a_2^{\pm}(N-2)\right) + 4^3 a_3^{\pm}(N-2) \\ &\vdots \\ &= \mp 3\sum_{i=0}^{N-3} 4^i a_2^{\pm}(N-i) + 4^{N-2} a_3^{\pm}(3) \\ &= \mp \sum_{i=0}^{N-2} 4^i a_2^{\pm}(N-i). \end{aligned} \tag{2.20}$$

Continuing this process, we can deduce that

$$a_j^{\pm}(N+1) = \mp j \sum_{i=0}^{N-j+1} (j+1)^i a_{j-1}^{\pm}(N-i), \tag{2.21}$$

for $1 \leq j \leq N$.

Now, we give explicit expression for $a_j^{\pm}(N+1)$, $(1 \leq j \leq N)$.

$$a_1^{\pm}(N+1) = \mp \sum_{i_1=0}^{N} 2^{i_1}, \tag{2.22}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.7, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

$$a_2^{\pm}(N+1) = \mp 2\sum_{i_2=0}^{N-1} 3^{i_2} a_1^{\pm}(N-i_2) = \mp 2\sum_{i_2=0}^{N-1} 3^{i_2}(\mp 1)\sum_{i_1=0}^{N-i_2-1} 2^{i_1} = (\mp 1)^2 2!\sum_{i_2=0}^{N-1}\sum_{i_1=0}^{N-1-i_2} 3^{i_2}2^{i_1}, \tag{2.23}$$

and, by (2.23), we get

$$\begin{aligned} a_3^{\pm}(N+1) &= \mp 3\sum_{i_3=0}^{N-2} 4^{i_3} a_2^{\pm}(N-i_3) \\ &= \mp 3\sum_{i_3=0}^{N-2} 4^{i_3}(\mp 1)^2 2!\sum_{i_2=0}^{N-i_3-2}\sum_{i_1=0}^{N-i_3-i_2-2} 3^{i_2}2^{i_1} \\ &= (\mp 1)^3 3!\sum_{i_3=0}^{N-2}\sum_{i_2=0}^{N-2-i_3}\sum_{i_1=0}^{N-2-i_3-i_2} 4^{i_3}3^{i_2}2^{i_1}. \end{aligned} \tag{2.24}$$

So, we can deduce that

$$a_j^{\pm}(N+1) = (\mp 1)^j j!\sum_{i_j=0}^{N-j+1}\sum_{i_{j-1}=0}^{N-j+1-i_j}\cdots\sum_{i_1=0}^{N-j+1-i_j-\cdots-i_2} (j+1)^{i_j} j^{i_{j-1}}\cdots 2^{i_1}, \tag{2.25}$$

where $1 \le j \le N$.

Remark. Observe that $a_{N+1}^{\pm}(N+1) = (\mp 1)^{N+1}(N+1)!$ is the same as the above expression with $j = N+1$. Therefore, by (2.4) and (2.25), we obtain the following theorem.

Theorem 1. *The ordinary differential equations*

$$F^{(N)} = \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^N \sum_{i=1}^{N+1} a_{i-1}^{-}(N) F^i, \quad (N = 0, 1, 2, \dots),$$

have a solution $F = F(t) = \frac{1}{(1+\lambda)^{\frac{t}{\lambda}} - 1}$, *where* $a_0^{-}(N) = 1$,

$$a_j^{-}(N) = j!\sum_{i_j=0}^{N-j}\sum_{i_{j-1}=0}^{N-j-i_j}\cdots\sum_{i_1=0}^{N-j-i_j-\cdots-i_2} (j+1)^{i_j} j^{i_{j-1}}\cdots 2^{i_1},$$

for $1 \le j \le N$.

Theorem 2. *The ordinary differential equations*

$$F^{(N)} = \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^N \sum_{i=1}^{N+1} a_{i-1}^{+}(N) F^i, \quad (N = 0, 1, 2, \dots),$$

have a solution $F = F(t) = \frac{1}{(1+\lambda)^{\frac{t}{\lambda}}+1}$, *where* $a_0^+(N) = 1$,

$$a_j^+(N) = (-1)^j j! \sum_{i_j=0}^{N-j} \sum_{i_{j-1}=0}^{N-j-i_j} \cdots \sum_{i_1=0}^{N-j-i_j-\cdots-i_2} (j+1)^{i_j} j^{i_{j-1}} \cdots 2^{i_1},$$

for $1 \le j \le N$.

Now, we observe that

$$\begin{aligned} &\sum_{k=0}^{\infty} \tilde{\mathcal{E}}_{k+N}(\lambda) \frac{t^k}{k!} \\ &= \left(\sum_{k=0}^{\infty} \tilde{\mathcal{E}}_k(\lambda) \frac{t^k}{k!} \right)^{(N)} \\ &= 2\left(\frac{1}{(1+\lambda)^{\frac{t}{\lambda}}+1} \right)^{(N)} \\ &= 2\left(-\frac{1}{\lambda}\log(1+\lambda) \right)^N \sum_{i=1}^{N+1} a_{i-1}^+(N) \left(\frac{1}{(1+\lambda)^{\frac{t}{\lambda}}+1} \right)^i \\ &= \left(-\frac{1}{\lambda}\log(1+\lambda) \right)^N \sum_{i=1}^{N+1} a_{i-1}^+(N) 2^{1-i} \left(\frac{2}{(1+\lambda)^{\frac{t}{\lambda}}+1} \right)^i \\ &= \sum_{k=0}^{\infty} \left(\left(-\frac{1}{\lambda}\log(1+\lambda) \right)^N \sum_{i=1}^{N+1} 2^{1-i} a_{i-1}^+(N) \tilde{\mathcal{E}}_k^{(i)}(\lambda) \right) \frac{t^k}{k!}. \end{aligned} \tag{2.26}$$

Thus, by comparing the coefficients on both sides of (2.26), we get

$$\tilde{\mathcal{E}}_{k+N}(\lambda) = \left(-\frac{1}{\lambda}\log(1+\lambda) \right)^N \sum_{i=1}^{N+1} 2^{1-i} a_{i-1}^+(N) \tilde{\mathcal{E}}_k^{(i)}(\lambda), \tag{2.27}$$

for $k, N = 0, 1, 2, \ldots$.

Therefore, by (2.27), we obtain the following theorem.

Theorem 3. *For* $k, N = 0, 1, 2, \ldots$, *we have*

$$\tilde{\mathcal{E}}_{k+N}(\lambda) = \left(-\frac{1}{\lambda}\log(1+\lambda) \right)^N \sum_{i=1}^{N+1} 2^{1-i} a_{i-1}^+(N) \tilde{\mathcal{E}}_k^{(i)}(\lambda),$$

where $a_0^+(N) = 1$,

$$a_j^+(N) = (-1)^j j! \sum_{i_j=0}^{N-j} \sum_{i_{j-1}=0}^{N-j-i_j} \cdots \sum_{i_1=0}^{N-j-i_j-\cdots-i_2} (j+1)^{i_j} j^{i_{j-1}} \cdots 2^{i_1}, \tag{2.28}$$

where $1 \le j \le N$.

Corollary 4. $\tilde{\mathcal{E}}_N(x) = \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^N \sum_{i=1}^{N+1} 2^{1-i} a_{i-1}^+(N)$.

Replacing t by $\frac{t}{\lambda}\log(1+\lambda)$ in (1.11), we obtain

$$\sum_{n=0}^{\infty} \tilde{\mathcal{E}}_n^{(r)}(\lambda) \frac{t^n}{n!} = \left(\frac{2}{(1+\lambda)^{\frac{t}{\lambda}}+1} \right)^r \tag{2.29}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.7, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

$$= \sum_{n=0}^{\infty} E_n^{(r)} \frac{\left(\frac{1}{\lambda}\log(1+\lambda)\, t\right)^n}{n!}.$$

Thus, by (2.29), we get

$$\tilde{\mathcal{E}}_n^{(r)}(\lambda) = \left(\frac{1}{\lambda}\log(1+\lambda)\right)^n E_n^{(r)}, \quad (n \geq 0). \tag{2.30}$$

From (2.30), we obtain the following corollary.

Corollary 5. *For $k, N = 0, 1, 2, \dots,$ we have*

$$E_{k+N} = (-1)^N \sum_{i=1}^{N+1} 2^{1-i} a_{i-1}^{+}(N)\, E_k^{(i)},$$

where $a_j^{+}(N)(0 \leq j \leq N)$ are as in (2.28).

From (1.3), we note that

$$\begin{aligned} &\frac{1}{(1+\lambda)^{\frac{t}{\lambda}} - 1} \\ &= \sum_{k=0}^{\infty} \tilde{\beta}_k(\lambda) \frac{t^{k-1}}{k!} \\ &= \sum_{k=1}^{\infty} \tilde{\beta}_k(\lambda) \frac{t^{k-1}}{k!} + \tilde{\beta}_0(\lambda) \frac{1}{t} \\ &= \sum_{k=0}^{\infty} \tilde{\beta}_{k+1}(\lambda) \frac{t^k}{(k+1)!} + \frac{\lambda}{\log(1+\lambda)} t^{-1}. \end{aligned} \tag{2.31}$$

Thus, by (2.31), we get

$$\begin{aligned} &\left(\frac{1}{(1+\lambda)^{\frac{t}{\lambda}} - 1}\right)^{(N)} \\ &= \sum_{k=N}^{\infty} \tilde{\beta}_{k+1}(\lambda) \frac{(k)_N}{(k+1)!} t^{k-N} \\ &\quad + (-1)^N N! \frac{\lambda}{\log(1+\lambda)} t^{-N-1}. \end{aligned} \tag{2.32}$$

From (2.32), we note that

$$\begin{aligned} &t^{N+1}\left(\frac{1}{(1+\lambda)^{\frac{t}{\lambda}} - 1}\right)^{(N)} \\ &= \sum_{k=N}^{\infty} \tilde{\beta}_{k+1}(\lambda) \frac{(k)_N}{(k+1)!} t^{k+1} + (-1)^N N! \frac{\lambda}{\log(1+\lambda)} \\ &= \sum_{k=N+1}^{\infty} \tilde{\beta}_k(\lambda) (k-1)_N \frac{t^k}{k!} + (-1)^N N! \frac{\lambda}{\log(1+\lambda)}. \end{aligned} \tag{2.33}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.7, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

On the other hand, by Theorem 1, we get

$$
\begin{aligned}
(2.34)\qquad & t^{N+1}\left(\frac{1}{(1+\lambda)^{\frac{t}{\lambda}}-1}\right)^{(N)} \\
&= t^{N+1}\left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N}\sum_{i=1}^{N+1} a_{i-1}^{-}(N)\left(\frac{1}{(1+\lambda)^{\frac{t}{\lambda}}-1}\right)^{i} \\
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N}\sum_{i=1}^{N+1} a_{i-1}^{-}(N)\, t^{N+1-i}\left(\frac{t}{(1+\lambda)^{\frac{t}{\lambda}}-1}\right)^{i} \\
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N}\sum_{i=1}^{N+1} a_{i-1}^{-}(N)\, t^{N+1-i}\sum_{l=0}^{\infty}\tilde{\beta}_{l}^{(i)}(\lambda)\frac{t^{l}}{l!} \\
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N}\sum_{i=0}^{N} a_{N-i}^{-}(N)\sum_{l=0}^{\infty}\tilde{\beta}_{l}^{(N+1-i)}(\lambda)\frac{t^{l+i}}{l!} \\
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N}\sum_{i=0}^{N}\sum_{l=0}^{\infty} a_{N-i}^{-}(N)\,\tilde{\beta}_{l}^{(N+1-i)}(\lambda)\frac{t^{l+i}}{l!} \\
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N}\sum_{i=0}^{N}\sum_{k=i}^{\infty} a_{N-i}^{-}(N)\,\tilde{\beta}_{k-i}^{(N+1-i)}(\lambda)\frac{t^{k}}{(k-i)!}
\end{aligned}
$$

From (2.34), we have

$$
\begin{aligned}
(2.35)\qquad & t^{N+1}\left(\frac{1}{(1+\lambda)^{\frac{t}{\lambda}}-1}\right)^{(N)} \\
&= \left(-\frac{1}{\lambda}\log(1+\lambda)\right)^{N} \\
&\quad\times\left\{\sum_{k=0}^{N}\sum_{i=0}^{k} a_{N-i}^{-}(N)\,\tilde{\beta}_{k-i}^{(N+1-i)}(\lambda)(k)_{i}\frac{t^{k}}{k!}\right. \\
&\quad\left.+\sum_{k=N+1}^{\infty}\sum_{i=0}^{N} a_{N-i}^{-}(N)\,\tilde{\beta}_{k-i}^{(N+1-i)}(\lambda)(k)_{i}\frac{t^{k}}{k!}\right\}.
\end{aligned}
$$

Comparing (2.33) and (2.35), we obtain the following theorem.

Theorem 6. *Let N be a positive integer. Then*

(i) $\tilde{\beta}_k(\lambda) = \frac{1}{(k-1)_N}\left(-\frac{1}{\lambda}\log(1+\lambda)\right)^N \sum_{i=0}^{N} a_{N-i}^{-}(N)\,\tilde{\beta}_{k-i}^{(N+1-i)}(\lambda)(k)_i$, *where* $k \geq N+1$, $(k)_N = k(k-1)\cdots(k-N+1)$ *for* $N \geq 1$, *and* $(k)_0 = 1$.

(ii) *For* $1 \leq k \leq N$, *we have*

$$\sum_{i=0}^{k} a_{N-i}^{-}(N)\,\tilde{\beta}_{k-i}^{(N+1-i)}(\lambda)(k)_i = 0,$$

where $a_0^{-}(N) = 1$,

$$(2.36)\qquad a_j^{-}(N) = j!\sum_{i_j=0}^{N-j}\sum_{i_{j-1}=0}^{N-j-i_j}\cdots\sum_{i_1=0}^{N-j-i_j-\cdots i_2}(j+1)^{i_j}\, j^{i_{j-1}}\cdots 2^{i_1},$$

$(1 \leq j \leq N)$.

Replacing t by $\frac{t}{\lambda}\log(1+\lambda)$ in (1.10), we get

$$\left(\frac{t}{(1+\lambda)^{\frac{t}{\lambda}}-1}\right)^r = \sum_{n=0}^{\infty} B_n^{(r)} \left(\frac{1}{\lambda}\log(1+\lambda)\right)^{n-r} \frac{t^n}{n!}. \tag{2.37}$$

Thus, from (2.37), we have

$$\tilde{\beta}_n^{(r)}(\lambda) = \left(\frac{1}{\lambda}\log(1+\lambda)\right)^{n-r} B_n^{(r)}, \quad \text{for } n \geq 0. \tag{2.38}$$

From (2.38), we obtain the following corollary.

Corollary 7. *Let N be any positive integer. Then*

(i) $B_k = \frac{(-1)^N}{(k-1)_N} \sum_{i=0}^{N} a_{N-i}^{-}(N) B_{k-i}^{(N+1-i)}(k)_i$, *for $k \geq N+1$,*

(ii) $\sum_{i=0}^{k} a_{N-i}^{-}(N) B_{k-i}^{(N+1-i)}(k)_i = 0$, *for $1 \leq k \leq N$, where $a_j^{-}(N)$ $(0 \leq j \leq N)$ are as in (2.36).*

References

1. A. Bayad and T. Kim, *Higher recurrences for Apostol-Bernoulli-Euler numbers*, Russ. J. Math. Phys. **19** (2012), no. 1, 1–10. MR 2892600
2. L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers*, Utilitas Math. **15** (1979), 51–88. MR 531621
3. D. V. Dolgy, T. Kim, H.-I. Kwon, and J. J. Seo, *On the modified degenerate Bernoulli polynomials*, Adv. Stud. Contemp. Math. **272** (2016), 251–257.
4. B.-N. Guo and F. Qi, *Explicit formulae for computing Euler polynomials in terms of Stirling numbers of the second kind*, J. Comput. Appl. Math. **272** (2014), 251–257. MR 3227382
5. L.-C. Jang, C. S. Ryoo, J. J. Seo, and H. I. Kwon, *Some properties of the twisted Changhee polynomials and their zeros*, Appl. Math. Comput. **274** (2016), 169–177. MR 3433125
6. D. Kang, J. Jeong, S.-J. Lee, and S.-H. Rim, *A note on the Bernoulli polynomials arising from a non-linear differential equation*, Proc. Jangjeon Math. Soc. **16** (2013), no. 1, 37–43. MR 3059283
7. T. Kim and D. S. Kim, *Identities involving degenerate Euler numbers and polynomials arising from non-linear differential equations*, J. Nonlinear Sci. Appl. **9** (2016), 2086–2098.
8. ______, *A Note on Nonlinear Changhee Differential Equations*, Russ. J. Math. Phys. **23** (2016), no. 1, 1–5.
9. H. I. Kwon, T. Kim, and J. J. Seo, *modified degenerate Euler polynomials*, Adv. Stud. Contemp. Math. **26** (2016), no. 1, 204–209.
10. F. Qi and B.-N. Guo, *Viewing some nonlinear ode's and their solutions from the angel of derivative polynomials*, (preprint), see: http://www.kms.or.kr.
11. S.-H. Rim, J. Jeong, and J.-W. Park, *Some identities involving Euler polynomials arising from a non-linear differential equation*, Kyungpook Math. J. **53** (2013), no. 4, 553–563. MR 3150547
12. Y. Simsek, *Interpolation functions of the Eulerian type polynomials and numbers*, Adv. Stud. Contemp. Math. (Kyungshang) **23** (2013), no. 2, 301–307. MR 3088760

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.7, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin, 300387, China, Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
E-mail address: tkkim@kw.ac.kr

Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea
E-mail address: dskim@sogang.ac.kr

Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
E-mail address: sura@kw.ac.kr

Department of Applied Mathematics, Pukyong National University, Busan, Republic of Korea(Corresponding author)
E-mail address: seo2011@pknu.ac.kr

ADDITIVE-QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN BANACH SPACES

SUNGSIK YUN[1], JUNG RYE LEE[2*], CHOONKIL PARK[3*], AND DONG YUN SHIN[4*]

Abstract. Let

$$\begin{aligned} M_1 f(x,y) : &= \frac{3}{4}f(x+y) - \frac{1}{4}f(-x-y) \\ &+\frac{1}{4}f(x-y) + \frac{1}{4}f(y-x) - f(x) - f(y), \end{aligned}$$

$$M_2 f(x,y) := 2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) - f(x) - f(y).$$

We solve the additive-quadratic ρ-functional inequalities

$$\|M_1 f(x,y)\| \le \|\rho M_2 f(x,y)\|, \tag{0.1}$$

where ρ is a fixed complex number with $|\rho| < \frac{1}{2}$ and

$$\|M_2 f(x,y)\| \le \|\rho M_1 f(x,y)\|, \tag{0.2}$$

where ρ is a fixed complex number with $|\rho| < 1$.

Using the direct method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequalities (0.1) and (0.2) in complex Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [23] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [9] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [15] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [22] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 3, 4, 6, 7, 10, 13, 14, 16, 17, 18, 19, 20, 21, 24, 25]).

In Section 2, we solve the additive-quadratic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.1) in Banach spaces.

2010 *Mathematics Subject Classification.* Primary 39B62, 39B52.

Key words and phrases. Hyers-Ulam stability; additive-quadratic ρ-functional inequality; Banach space.

*Corresponding authors.

In Section 3, we solve the additive-quadratic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.2) in Banach spaces.

In this paper, assume that X is a complex normed space and that Y is a complex Banach space.

2. Additive-quadratic ρ-functional inequality (0.1) in Banach spaces

Throughout this section, assume that ρ is a complex number with $|\rho| < \frac{1}{2}$.

We solve and investigate the additive-quadratic ρ-functional inequality (0.1) in normed spaces.

Lemma 2.1.
(i) *If a mapping $f : X \to Y$ satisfies $M_1 f(x,y) = 0$, then $f = f_o + f_e$, where $f_o(x) := \frac{f(x)-f(-x)}{2}$ is the Cauchy additive mapping and $f_e(x) := \frac{f(x)+f(-x)}{2}$ is the quadratic mapping.*
(ii) *If a mapping $f : X \to Y$ satisfies $M_2 f(x,y) = 0$, then $f = f_o + f_e$, where $f_o(x) := \frac{f(x)-f(-x)}{2}$ is the Cauchy additive mapping and $f_e(x) := \frac{f(x)+f(-x)}{2}$ is the quadratic mapping.*

Proof. (i)

$$M_1 f_o(x,y) = f_o(x+y) - f_o(x) - f_o(y) = 0$$

for all $x, y \in X$. So f_o is the Cauchy additive mapping.

$$M_1 f_e(x,y) = \frac{1}{2} f_e(x+y) + \frac{1}{2} f_e(x-y) - f_e(x) - f_e(y) = 0$$

for all $x, y \in X$. So f_o is the quadratic mapping.

(ii)

$$M_2 f_o(x,y) = 2 f_o\left(\frac{x+y}{2}\right) - f_o(x) - f_o(y) = 0$$

for all $x, y \in X$. Since $M_2 f(0,0) = 0$, $f(0) = 0$ and f_o is the Cauchy additive mapping.

$$M_2 f_e(x,y) = 2 f_e\left(\frac{x+y}{2}\right) + 2 f_e\left(\frac{x-y}{2}\right) - f_e(x) - f_e(y) = 0$$

for all $x, y \in X$. Since $M_2 f(0,0) = 0$, $f(0) = 0$ and f_e is the quadratic mapping.

Therefore, the mapping $f : X \to Y$ is the sum of the Cauchy additive mapping and the quadratic mapping. □

Lemma 2.2.
(i) *If an odd mapping $f : X \to Y$ satisfies*

$$\|M_1 f(x,y)\| \le \|\rho M_2 f(x,y)\| \tag{2.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.
(ii) *If an even mapping $f : X \to Y$ satisfies (2.1), then $f : X \to Y$ is quadratic.*

Proof. (i) Assume that $f : X \to Y$ satisfies (2.1).

Since f is an odd mapping, $f(0) = 0$.

Letting $y = x$ in (2.1), we get

$$\|f(2x) - 2f(x)\| \le 0$$

and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &\leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| \\ &= |\rho|\|f(x+y) - f(x) - f(y)\| \end{aligned}$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

(ii) Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get

$$\|f(0)\| \leq \|2\rho f(0)\|.$$

So $f(0) = 0$.

Letting $y = x$ in (2.1), we get

$$\|\frac{1}{2}f(2x) - 2f(x)\| \leq 0$$

and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{2.3}$$

for all $x \in X$.

It follows from (2.1) and (2.3) that

$$\begin{aligned} &\left\|\frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) - f(x) - f(y)\right\| \\ &\leq \left\|\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \\ &= |\rho|\left\|\frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) - f(x) - f(y)\right\| \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (2.1) in complex Banach spaces for an odd mapping case.

Theorem 2.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be an odd mapping such that*

$$\Psi(x, y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty, \tag{2.4}$$

$$\|M_1 f(x, y)\| \leq \|\rho M_2 f(x, y)\| + \varphi(x, y) \tag{2.5}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{2}\Psi(x, x) \tag{2.6}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.5), we get

$$\|f(2x) - 2f(x)\| \le \varphi(x,x) \tag{2.7}$$

for all $x \in X$. So

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \le \varphi\left(\frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1} \left\|2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \sum_{j=l}^{m-1} 2^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \end{aligned}\tag{2.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.8) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Since f is an odd mapping, A is an odd mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.8), we get (2.6).

It follows from (2.4) and (2.5) that

$$\begin{aligned}\|A(x+y) - A(x) - A(y)\| &= \lim_{n\to\infty} \left\|2^n \left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\ &\le \lim_{n\to\infty} \left\|2^n \rho\left(2f\left(\frac{x+y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\ &+ \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \\ &= \left\|\rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\| \end{aligned}$$

for all $x, y \in X$. So

$$\|A(x+y) - A(x) - A(y)\| \le \left\|\rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\|$$

for all $x, y \in X$. By Lemma 2.2, the mapping $A : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (2.6). Then we have

$$\begin{aligned}\|A(x) - T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\ &\le \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| + \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| \\ &\le 2^q \Psi\left(\frac{x}{2^q}, \frac{x}{2^q}\right), \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A, as desired. □

Corollary 2.4. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping such that*

$$\|M_1 f(x,y)\| \le \|\rho M_2 f(x,y)\| + \theta(\|x\|^r + \|y\|^r) \tag{2.9}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{2^r - 2}\|x\|^r$$

for all $x \in X$.

Theorem 2.5. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be an odd mapping satisfying* (2.5) *and*

$$\Psi(x,y) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y) < \infty \tag{2.10}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2}\Psi(x,x) \tag{2.11}$$

for all $x \in X$.

Proof. It follows from (2.7) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{2}\varphi(x,x)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^j x\right) - \frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\le \sum_{j=l}^{m-1}\frac{1}{2^{j+1}}\varphi(2^j x, 2^j x) \end{aligned} \tag{2.12}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.12) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.12), we get (2.11).

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 2.6. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping satisfying* (2.9). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2\theta}{2 - 2^r}\|x\|^r \tag{2.13}$$

for all $x \in X$.

Now, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (2.1) in complex Banach spaces for an even mapping case.

Theorem 2.7. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$, (2.5) and*

$$\Psi(x,y) : \;=\; \sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{2.14}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{2}\Psi(x,x) \tag{2.15}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.5), we get

$$\left\|\frac{1}{2}f(2x) - 2f(x)\right\| \le \varphi(x,x) \tag{2.16}$$

for all $x \in X$. So

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \le 2\varphi\left(\frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1} \left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \sum_{j=l}^{m-1} \frac{4^{j+1}}{2}\varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \end{aligned} \tag{2.17}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.17) that the sequence $\{4^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{4^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} 4^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Since f is an even mapping, Q is an even mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.17), we get (2.15).

It follows from (2.5) and (2.14) that

$$\begin{aligned}&\left\|\frac{1}{2}Q\left(\frac{x+y}{2}\right) + \frac{1}{2}Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right\| \\ &= \lim_{n\to\infty}\left\|4^n\left(\frac{1}{2}f\left(\frac{x+y}{2^n}\right) + \frac{1}{2}f\left(\frac{x-y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\ &\le \lim_{n\to\infty}\left\|4^n\rho\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| + \lim_{n\to\infty} 4^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \\ &= \left\|\rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\right\|\end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned}&\left\|\frac{1}{2}Q\left(\frac{x+y}{2}\right) + \frac{1}{2}Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right\| \\ &\qquad \le \left\|\rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\right\|\end{aligned}$$

for all $x, y \in X$. By Lemma 2.2, the mapping $Q : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.15). Then we have

$$\begin{aligned}\|Q(x) - T(x)\| &= \left\|4^q Q\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right)\right\| \\ &\le \left\|4^q Q\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\| + \left\|4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\| \\ &\le 4^q \Psi\left(\frac{x}{2^q}, \frac{x}{2^q}\right),\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q, as desired. □

Corollary 2.8. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.9). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{4\theta}{2^r - 4}\|x\|^r$$

for all $x \in X$.

Theorem 2.9. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$, (2.5) and*

$$\Psi(x, y) := \sum_{j=0}^{\infty} \frac{1}{4^j}\varphi(2^j x, 2^j y) < \infty \tag{2.18}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{2}\Psi(x, x) \tag{2.19}$$

for all $x \in X$.

Proof. It follows from (2.16) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{1}{2}\varphi(x, x)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f\left(2^j x\right) - \frac{1}{4^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\le \sum_{j=l}^{m-1}\frac{1}{2\cdot 4^j}\varphi(2^j x, 2^j x)\end{aligned} \tag{2.20}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.20) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.20), we get (2.19).

The rest of the proof is similar to the proof of Theorem 2.7. □

Corollary 2.10. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.9). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{4\theta}{4 - 2^r}\|x\|^r \tag{2.21}$$

for all $x \in X$.

Remark 2.11. If ρ is a real number such that $-\frac{1}{2} < \rho < \frac{1}{2}$ and Y is a real Banach space, then all the assertions in this section remain valid.

3. Additive-quadratic ρ-functional inequality (0.2) in complex Banach spaces

Throughout this section, assume that ρ is a complex number with $|\rho| < 1$.

We solve and investigate the additive-quadratic ρ-functional inequality (0.2) in complex normed spaces.

Lemma 3.1.
(i) *If an odd mapping $f : X \to Y$ satisfies*

$$\|M_2 f(x, y)\| \le \|\rho M_1 f(x, y)\| \tag{3.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.
(ii) *If an even mapping $f : X \to Y$ satisfies $f(0) = 0$ and (3.1), then $f : X \to Y$ is quadratic.*

Proof. (i) Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \le 0 \tag{3.2}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{2}f(x)$ for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}\|f(x+y) - f(x) - f(y)\| &= \left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \\ &\le |\rho|\|f(x+y) - f(x) - f(y)\|\end{aligned}$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

(ii) Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \le 0 \tag{3.3}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$ for all $x \in X$.

It follows from (3.1) and (3.3) that

$$\begin{aligned}\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\| \\ =\left\|2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right\| \\ \leq |\rho|\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\|\end{aligned}$$

and so

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (3.1) in complex Banach spaces for an odd mapping case.

Theorem 3.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be an odd mapping satisfying*

$$\Psi(x,y) := \sum_{j=0}^{\infty} 2^j \varphi\left(\frac{x}{2^j},\frac{y}{2^j}\right) < \infty,$$

$$\|M_2 f(x,y)\| \leq \|\rho M_1 f(x,y)\| + \varphi(x,y) \tag{3.4}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \leq \Psi(x,0) \tag{3.5}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.4), we get

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\| = \left\|2f\left(\frac{x}{2}\right)-f(x)\right\| \leq \varphi(x,0) \tag{3.6}$$

for all $x \in X$. So

$$\begin{aligned}\left\|2^l f\left(\frac{x}{2^l}\right)-2^m f\left(\frac{x}{2^m}\right)\right\| &\leq \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right)-2^{j+1}f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\leq \sum_{j=l}^{m-1} 2^j\varphi\left(\frac{x}{2^j},0\right)\end{aligned} \tag{3.7}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.7) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Since f is an odd mapping, A is an odd mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.7), we get (3.5).

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 3.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping satisfying*

$$\|M_2 f(x,y)\| \le \|\rho M_1 f(x,y)\| + \theta(\|x\|^r + \|y\|^r) \tag{3.8}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2^r\theta}{2^r - 2}\|x\|^r$$

for all $x \in X$.

Theorem 3.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be an odd mapping satisfying* (3.4) *and*

$$\Psi(x,y) := \sum_{j=1}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \Psi(x,0) \tag{3.9}$$

for all $x \in X$.

Proof. It follows from (3.6) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{2}\varphi(2x,0)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^j x\right) - \frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\le \sum_{j=l+1}^{m}\frac{1}{2^j}\varphi(2^j x, 0) \end{aligned} \tag{3.10}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.10) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.10), we get (3.9).

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 3.5. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be an odd mapping satisfying* (3.8)*. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2^r\theta}{2 - 2^r}\|x\|^r$$

for all $x \in X$.

Now, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (3.1) in complex Banach spaces for an even mapping case.

Theorem 3.6. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$, (3.4) and*

$$\Psi(x, y) : \;=\; \sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \Psi(x, 0) \tag{3.11}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.4), we get

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| = \left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \le \varphi(x, 0) \tag{3.12}$$

for all $x \in X$. So

$$\begin{aligned} \left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1} \left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \sum_{j=l}^{m-1} 4^j \varphi\left(\frac{x}{2^j}, 0\right) \end{aligned} \tag{3.13}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.13) that the sequence $\{4^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{4^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} 4^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Since f is an even mapping, Q is an even mapping. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.13), we get (3.11).

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 3.7. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.8). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{2^r \theta}{2^r - 4} \|x\|^r$$

for all $x \in X$.

Theorem 3.8. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$, (3.4) and*

$$\Psi(x, y) := \sum_{j=1}^{\infty} \frac{1}{4^j} \varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \Psi(x, 0) \tag{3.14}$$

for all $x \in X$.

Proof. It follows from (3.12) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{1}{4}\varphi(2x,0)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f\left(2^j x\right) - \frac{1}{4^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\le \sum_{j=l+1}^{m}\frac{1}{4^j}\varphi(2^j x,0) \qquad (3.15)\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.15) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.15), we get (3.14).

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 3.9. *Let $r < 2$ and θ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$, (3.8). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{2^r\theta}{4-2^r}\|x\|^r$$

for all $x \in X$.

Remark 3.10. If ρ is a real number such that $-1 < \rho < 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

Acknowledgments

This research was supported by Hanshin University Research Grant.

References

[1] M. Adam, *On the stability of some quadratic functional equation*, J. Nonlinear Sci. Appl. **4** (2011), 50–59.
[2] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
[3] L. Cădariu, L. Găvruta and P. Găvruta, *On the stability of an affine functional equation,* J. Nonlinear Sci. Appl. **6** (2013), 60–67.
[4] A. Chahbi and N. Bounader, *On the generalized stability of d'Alembert functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 198–204.
[5] P. W. Cholewa, *Remarks on the stability of functional equations,* Aequationes Math. **27** (1984), 76–86.
[6] Z. Daróczy and Gy. Maksa, *A functional equation involving comparable weighted quasi-arithmetic means*, Acta Math. Hungar. **138** (2013), 147–155.
[7] G. Z. Eskandani and P. Găvruta, *Hyers-Ulam-Rassias stability of pexiderized Cauchy functional equation in 2-Banach spaces,* J. Nonlinear Sci. Appl. **5** (2012), 459–465.
[8] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
[9] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.

[10] C. Park, *Orthogonal stability of a cubic-quartic functional equation*, J. Nonlinear Sci. Appl. **5** (2012), 28–36.
[11] C. Park, *Additive ρ-functional inequalities and equations*, J. Math. Inequal. **9** (2015), 17–26.
[12] C. Park, *Additive ρ-functional inequalities in non-Archimedean normed spaces*, J. Math. Inequal. **9** (2015), 397–407.
[13] C. Park, K. Ghasemi, S. G. Ghaleh and S. Jang, *Approximate n-Jordan $*$-homomorphisms in C^*-algebras,* J. Comput. Anal. Appl. **15** (2013), 365–368.
[14] C. Park, A. Najati and S. Jang, *Fixed points and fuzzy stability of an additive-quadratic functional equation,* J. Comput. Anal. Appl. **15** (2013), 452–462.
[15] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[16] K. Ravi, E. Thandapani and B. V. Senthil Kumar, *Solution and stability of a reciprocal type functional equation in several variables,* J. Nonlinear Sci. Appl. **7** (2014), 18–27.
[17] S. Schin, D. Ki, J. Chang and M. Kim, *Random stability of quadratic functional equations: a fixed point approach*, J. Nonlinear Sci. Appl. **4** (2011), 37–49.
[18] S. Shagholi, M. Bavand Savadkouhi and M. Eshaghi Gordji, *Nearly ternary cubic homomorphism in ternary Fréchet algebras,* J. Comput. Anal. Appl. **13** (2011), 1106–1114.
[19] S. Shagholi, M. Eshaghi Gordji and M. Bavand Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras,* J. Comput. Anal. Appl. **13** (2011), 1097–1105.
[20] D. Shin, C. Park and Sh. Farhadabadi, *On the superstability of ternary Jordan C^*-homomorphisms,* J. Comput. Anal. Appl. **16** (2014), 964–973.
[21] D. Shin, C. Park and Sh. Farhadabadi, *Stability and superstability of J^*-homomorphisms and J^*-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.
[22] F. Skof, *Propriet locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.
[23] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.
[24] C. Zaharia, *On the probabilistic stability of the monomial functional equation,* J. Nonlinear Sci. Appl. **6** (2013), 51–59.
[25] S. Zolfaghari, *Approximation of mixed type functional equations in p-Banach spaces*, J. Nonlinear Sci. Appl. **3** (2010), 110–122.

[1]Department of Financial Mathematics,
Hanshin University, Gyeonggi-do 18101,
Republic of Korea
E-mail address: ssyun@hs.ac.kr

[2]Department of Mathematics,
Daejin University, Kyunggi 11159,
Republic of Korea
E-mail address: jrlee@hdaejin.ac.kr

[3]Research Institute for Natural Sciences,
Hanyang University, Seoul 04763,
Republic of Korea
E-mail address: baak@hanyang.ac.kr

[4]Department of Mathematics,
University of Seoul, Seoul 02504,
Republic of Korea
E-mail address: dyshin@uos.ac.kr

STABILITY OF ADDITIVE-QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN BANACH SPACES

CHOONKIL PARK[1], JUNG RYE LEE[2*], AND SUNG JIN LEE[3*]

Abstract. Let

$$\begin{aligned} M_1f(x,y): \quad &= \quad \frac{3}{4}f(x+y) - \frac{1}{4}f(-x-y) \\ &\quad +\frac{1}{4}f(x-y) + \frac{1}{4}f(y-x) - f(x) - f(y), \end{aligned}$$

$$M_2f(x,y) := 2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) - f(x) - f(y).$$

We solve the additive-quadratic ρ-functional inequalities

$$\|M_1f(x,y)\| \le \|\rho M_2f(x,y)\|, \tag{0.1}$$

where ρ is a fixed complex number with $|\rho| < \frac{1}{2}$ and

$$\|M_2f(x,y)\| \le \|\rho M_1f(x,y)\|, \tag{0.2}$$

where ρ is a fixed complex number with $|\rho| < 1$.

Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequalities (0.1) and (0.2) in complex Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [31] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [23] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [11] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [30] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [8] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The stability problems of various functional equations have

2010 *Mathematics Subject Classification.* Primary 39B62, 47H10, 39B52.

Key words and phrases. Hyers-Ulam stability; additive-quadratic ρ-functional inequality; fixed point; Banach space.

*Corresponding authors.

been extensively investigated by a number of authors (see [1, 3, 7, 10, 17, 18, 19, 20, 21, 24, 25, 26, 27, 28, 29, 32, 33]).

We recall a fundamental result in fixed point theory.

Theorem 1.1. [4, 9] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [13] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 15, 16, 22]).

In Section 2, we solve the additive-quadratic ρ-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.1) in Banach spaces by using the fixed point method.

In Section 3, we solve the additive-quadratic ρ-functional inequality (0.2) and prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.2) in Banach spaces by using the fixed point method.

In this paper, assume that X is a complex normed space and that Y is a complex Banach space.

2. Additive-quadratic ρ-functional inequality (0.1) in Banach spaces

Throughout this section, assume that ρ is a complex number with $|\rho| < \frac{1}{2}$.

We solve and investigate the additive-quadratic ρ-functional inequality (0.1) in complex normed spaces.

Lemma 2.1.
(i) *If a mapping $f : X \to Y$ satisfies $M_1 f(x, y) = 0$, then $f = f_o + f_e$, where $f_o(x) := \frac{f(x)-f(-x)}{2}$ is the Cauchy additive mapping and $f_e(x) := \frac{f(x)+f(-x)}{2}$ is the quadratic mapping.*
(ii) *If a mapping $f : X \to Y$ satisfies $M_2 f(x, y) = 0$, then $f = f_o + f_e$, where $f_o(x) := \frac{f(x)-f(-x)}{2}$ is the Cauchy additive mapping and $f_e(x) := \frac{f(x)+f(-x)}{2}$ is the quadratic mapping.*

Proof. (i)

$$M_1 f_o(x, y) = f_o(x + y) - f_o(x) - f_o(y) = 0$$

for all $x, y \in X$. So f_o is the Cauchy additive mapping.

$$M_1 f_e(x, y) = \frac{1}{2} f_e(x + y) + \frac{1}{2} f_e(x - y) - f_e(x) - f_e(y) = 0$$

for all $x, y \in X$. So f_o is the quadratic mapping.

(ii)

$$M_2 f_o(x,y) = 2f_o\left(\frac{x+y}{2}\right) - f_o(x) - f_o(y) = 0$$

for all $x, y \in X$. Since $M_2 f(0,0) = 0$, $f(0) = 0$ and f_o is the Cauchy additive mapping.

$$M_2 f_e(x,y) = 2f_e\left(\frac{x+y}{2}\right) + 2f_e\left(\frac{x-y}{2}\right) - f_e(x) - f_e(y) = 0$$

for all $x, y \in X$. Since $M_2 f(0,0) = 0$, $f(0) = 0$ and f_e is the quadratic mapping.

Therefore, the mapping $f : X \to Y$ is the sum of the Cauchy additive mapping and the quadratic mapping. □

Lemma 2.2.
(i) *If an odd mapping $f : X \to Y$ satisfies*

$$\|M_1 f(x,y)\| \le \|\rho M_2 f(x,y)\| \tag{2.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.
(ii) *If an even mapping $f : X \to Y$ satisfies* (2.1), *then $f : X \to Y$ is quadratic.*

Proof. (i) Assume that $f : X \to Y$ satisfies (2.1).

Since f is an odd mapping, $f(0) = 0$.

Letting $y = x$ in (2.1), we get $\|f(2x) - 2f(x)\| \le 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &\le \left\|\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right)\right\| \\ &= |\rho|\|f(x+y) - f(x) - f(y)\| \end{aligned}$$

and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$.

(ii) Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $\|f(0)\| \le \|2\rho f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.1), we get $\|\frac{1}{2} f(2x) - 2f(x)\| \le 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4} f(x) \tag{2.3}$$

for all $x \in X$.

It follows from (2.1) and (2.3) that

$$\begin{aligned}\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\| \\ &\le \left\|\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\| \\ &= |\rho|\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\|\end{aligned}$$

and so

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x, y \in X$. □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (2.1) in complex Banach spaces.

Theorem 2.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \le \frac{L}{2}\varphi(x, y) \tag{2.4}$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$\|M_1f(x, y) - \rho M_2f(x, y)\| \le \varphi(x, y) \tag{2.5}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{L}{2(1 - L)}\varphi(x, x) \tag{2.6}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.5), we get

$$\|f(2x) - 2f(x)\| \le \varphi(x, x) \tag{2.7}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \le \mu\varphi(x, x), \ \forall x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [14]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \le \varepsilon\varphi(x, x)$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.7, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

for all $x \in X$. Hence

$$\begin{aligned} \|Jg(x) - Jh(x)\| &= \left\|2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right)\right\| \leq 2\varepsilon\varphi\left(\frac{x}{2}, \frac{x}{2}\right) \\ &\leq 2\varepsilon\frac{L}{2}\varphi(x, x) = L\varepsilon\varphi(x, x) \end{aligned}$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (2.7) that

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \leq \frac{L}{2}\varphi(x, x)$$

for all $x \in X$. So $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A(x) = 2A\left(\frac{x}{2}\right) \tag{2.8}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.8) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - A(x)\| \leq \mu\varphi(x, x)$$

for all $x \in X$;

(2) $d(J^l f, A) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L}d(f, Jf)$, which implies

$$\|f(x) - A(x)\| \leq \frac{L}{2(1-L)}\varphi(x, x)$$

for all $x \in X$.

It follows from (2.4) and (2.5) that

$$\begin{aligned} &\left\|A(x+y) - A(x) - A(y) - \rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)\right\| \\ &= \lim_{n\to\infty}\left\|2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right. \\ &\qquad \left. -2^n\rho\left(2f\left(\frac{x+y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\ &\leq \lim_{n\to\infty} 2^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \end{aligned}$$

for all $x, y \in X$. So

$$A(x+y) - A(x) - A(y) = \rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)$$

for all $x, y \in X$. By Lemma 2.2, the mapping $A : X \to Y$ is additive. □

Corollary 2.4. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping satisfying*

$$\|M_1 f(x,y) - \rho M_2 f(x,y)\| \le \theta(\|x\|^r + \|y\|^r) \tag{2.9}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f_o(x) - A(x)\| \le \frac{2\theta}{2^r - 2}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.3 by taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{1-r}$ and we get the desired result. □

Theorem 2.5. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \le \frac{L}{4}\varphi(x,y) \tag{2.10}$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.5). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f_e(x) - Q(x)\| \le \frac{L}{2(1-L)}\varphi(x,x) \tag{2.11}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.5) for f_e, we get

$$\left\|\frac{1}{2}f(2x) - 2f(x)\right\| \le \varphi(x,x) \tag{2.12}$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 2.3.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \le \varepsilon\varphi(x,x)$$

for all $x \in X$. Hence

$$\begin{aligned}\|Jg(x) - Jh(x)\| &= \left\|4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right)\right\| \le 4\varepsilon\varphi\left(\frac{x}{2}, \frac{x}{2}\right) \\ &\le 4\varepsilon\frac{L}{4}\varphi(x,x) = L\varepsilon\varphi(x,x)\end{aligned}$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.12) that

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \leq 2\varphi\left(\frac{x}{2}, \frac{x}{2}\right) \leq \frac{L}{2}\varphi(x,x)$$

for all $x \in X$. So $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(x\right) = 4Q\left(\frac{x}{2}\right) \tag{2.13}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.13) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - Q(x)\| \quad \leq \quad \mu\varphi\left(x, x\right)$$

for all $x \in X$;

(2) $d(J^l f, Q) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$, which implies

$$\|f(x) - Q(x)\| \leq \frac{L}{2(1-L)}\varphi\left(x, x\right)$$

for all $x \in X$.

It follows from (2.4) and (2.5) that

$$\begin{aligned}
&\left\|\frac{1}{2}Q\left(\frac{x+y}{2}\right) + \frac{1}{2}Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right. \\
&\left. - \rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\right\| \\
&= \lim_{n\to\infty} \left\|4^n\left(\frac{1}{2}f\left(\frac{x+y}{2^n}\right) + \frac{1}{2}f\left(\frac{x-y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right. \\
&\left. - 4^n\rho\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\
&\leq \lim_{n\to\infty} 4^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0
\end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned}
&\frac{1}{2}Q\left(\frac{x+y}{2}\right) + \frac{1}{2}Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y) \\
&\qquad = \rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)
\end{aligned}$$

for all $x, y \in X$. By Lemma 2.2, the mapping $Q : X \to Y$ is quadratic. □

Corollary 2.6. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.9). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{4\theta}{2^r - 4}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.5 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{2-r}$ and we get the desired result. □

Theorem 2.7. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.5). Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{2(1 - L)}\varphi(x, x)$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.3.

It follows from (2.7) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \leq \frac{1}{2}\varphi(x, x)$$

for all $x \in X$.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 2.8. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping satisfying (2.9). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^r}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.7 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{r-1}$ and we get desired result. □

Theorem 2.9. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.5). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{2(1 - L)}\varphi(x, x)$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.7, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.3.

It follows from (2.12) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{1}{2} \varphi(x, x)$$

for all $x \in X$.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4} g\,(2x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.5. □

Corollary 2.10. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.9). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \quad \leq \quad \frac{4\theta}{4 - 2^r} \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.9 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{r-2}$ and we get desired result. □

Remark 2.11. If ρ is a real number such that $-\frac{1}{2} < \rho < \frac{1}{2}$ and Y is a real Banach space, then all the assertions in this section remain valid.

3. Additive-quadratic ρ-functional inequality (0.2) in complex Banach spaces

Throughout this section, assume that ρ is a complex number with $|\rho| < 1$.

We solve and investigate the additive-quadratic ρ-functional inequality (0.2) in complex normed spaces.

Lemma 3.1.
(i) *If an odd mapping $f : X \to Y$ satisfies*

$$\|M_2 f(x, y)\| \leq \|\rho M_1 f(x, y)\| \tag{3.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.
(ii) *If an even mapping $f : X \to Y$ satisfies $f(0) = 0$ and (3.1), then $f : X \to Y$ is quadratic.*

Proof. (i) Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$\left\| 2f\left(\frac{x}{2}\right) - f(x) \right\| \leq 0 \tag{3.2}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{2} f(x)$ for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}\|f(x+y)-f(x)-f(y)\| &= \left\|2f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\| \\ &\leq |\rho|\|f(x+y)-f(x)-f(y)\|\end{aligned}$$

and so

$$f(x+y)=f(x)+f(y)$$

for all $x, y \in X$.

(ii) Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get

$$\left\|4f\left(\frac{x}{2}\right)-f(x)\right\| \leq 0 \tag{3.3}$$

and so $f\left(\frac{x}{2}\right)=\frac{1}{4}f(x)$ for all $x \in X$.

It follows from (3.1) and (3.3) that

$$\begin{aligned}&\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\| \\ &= \left\|2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right\| \\ &\leq |\rho|\left\|\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)-f(x)-f(y)\right\|\end{aligned}$$

and so

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x, y \in X$. □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional equation (3.1) in complex Banach spaces.

Theorem 3.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2},\frac{y}{2}\right) \leq \frac{L}{2}\varphi(x,y)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$\|M_2f(x,y)-\rho M_1f(x,y)\| \leq \varphi(x,y) \tag{3.4}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ a such that

$$\|f(x)-A(x)\| \leq \frac{1}{1-L}\varphi(x,0)$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.4), we get

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\| = \left\|2f\left(\frac{x}{2}\right)-f(x)\right\| \leq \varphi(x,0) \tag{3.5}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0)=0\}$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.7, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

and introduce the generalized metric on S:

$$d(g,h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu\varphi(x,0), \ \forall x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [14]).

We consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 3.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping satisfying*

$$\|M_2 f(x,y) - \rho M_1 f(x,y)\| \leq \theta(\|x\|^r + \|y\|^r) \tag{3.6}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2^r\theta}{2^r - 2}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{1-r}$ and we get desired result. □

Theorem 3.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{4}\varphi(x,y)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.4). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{1-L}\varphi(x,0)$$

for all $x \in X$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 3.2.

Letting $y = 0$ in (3.4), we get

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| = \left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq \varphi(x,0) \tag{3.7}$$

for all $x \in X$.

We consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.5. □

Corollary 3.5. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and* (3.6). *Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2^r\theta}{2^r - 4}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x, y) = \theta(\|x\|^r+\|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{2-r}$ and we get desired result. □

Theorem 3.6. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (3.4). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{L}{1 - L}\varphi(x, 0)$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.2.

It follows from (3.5) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \leq \frac{1}{2}\varphi(2x, 0) \leq L\varphi(x, 0)$$

for all $x \in X$.

We consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 3.7. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be an odd mapping satisfying* (3.6). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2^r\theta}{2 - 2^r}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.6 by taking $\varphi(x, y) = \theta(\|x\|^r+\|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{r-1}$ and we get desired result. □

Theorem 3.8. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.4). *Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{L}{1 - L}\varphi(x, 0)$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.2.

It follows from (3.7) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \leq \frac{1}{4} \varphi(2x, 0) \leq L\varphi(x, 0)$$

for all $x \in X$.

We consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4} g(2x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.5. □

Corollary 3.9. *Let $r < 2$ and θ be positive real numbers, and let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.6). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2^r \theta}{4 - 2^r} \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.8 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{r-2}$ and we get desired result. □

Remark 3.10. If ρ is a real number such that $-1 < \rho < 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

References

[1] M. Adam, *On the stability of some quadratic functional equation*, J. Nonlinear Sci. Appl. **4** (2011), 50–59.
[2] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
[3] L. Cădariu, L. Găvruta, P. Găvruta, *On the stability of an affine functional equation,* J. Nonlinear Sci. Appl. **6** (2013), 60–67.
[4] L. Cădariu, V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math. **4**, no. 1, Art. ID 4 (2003).
[5] L. Cădariu, V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber. **346** (2004), 43–52.
[6] L. Cădariu, V. Radu, *Fixed point methods for the generalized stability of functional equations in a single variable*, Fixed Point Theory Appl. **2008**, Art. ID 749392 (2008).
[7] A. Chahbi, N. Bounader, *On the generalized stability of d'Alembert functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 198–204.
[8] P. W. Cholewa, *Remarks on the stability of functional equations,* Aequationes Math. **27** (1984), 76–86.
[9] J. Diaz, B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. **74** (1968), 305–309.
[10] G. Z. Eskandani, P. Găvruta, *Hyers-Ulam-Rassias stability of pexiderized Cauchy functional equation in 2-Banach spaces,* J. Nonlinear Sci. Appl. **5** (2012), 459–465.
[11] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
[12] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.
[13] G. Isac, Th. M. Rassias, *Stability of ψ-additive mappings: Appications to nonlinear analysis*, Internat. J. Math. Math. Sci. **19** (1996), 219–228.

[14] D. Miheţ, V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008), 567–572.

[15] C. Park, *Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras*, Fixed Point Theory Appl. **2007**, Art. ID 50175 (2007).

[16] C. Park, *Generalized Hyers-Ulam-Rassias stability of quadratic functional equations: a fixed point approach*, Fixed Point Theory Appl. **2008**, Art. ID 493751 (2008).

[17] C. Park, *Orthogonal stability of a cubic-quartic functional equation*, J. Nonlinear Sci. Appl. **5** (2012), 28–36.

[18] C. Park, *Additive ρ-functional inequalities and equations*, J. Math. Inequal. **9** (2015), 17–26.

[19] C. Park, *Additive ρ-functional inequalities in non-Archimedean normed spaces*, J. Math. Inequal. **9** (2015), 397–407.

[20] C. Park, K. Ghasemi, S. G. Ghaleh, S. Jang, *Approximate n-Jordan $*$-homomorphisms in C^*-algebras,* J. Comput. Anal. Appl. **15** (2013), 365–368.

[21] C. Park, A. Najati, S. Jang, *Fixed points and fuzzy stability of an additive-quadratic functional equation,* J. Comput. Anal. Appl. **15** (2013), 452–462.

[22] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.

[23] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[24] K. Ravi, E. Thandapani, B. V. Senthil Kumar, *Solution and stability of a reciprocal type functional equation in several variables,* J. Nonlinear Sci. Appl. **7** (2014), 18–27.

[25] S. Schin, D. Ki, J. Chang, M. Kim, *Random stability of quadratic functional equations: a fixed point approach*, J. Nonlinear Sci. Appl. **4** (2011), 37–49.

[26] S. Shagholi, M. Bavand Savadkouhi, M. Eshaghi Gordji, *Nearly ternary cubic homomorphism in ternary Fréchet algebras,* J. Comput. Anal. Appl. **13** (2011), 1106–1114.

[27] S. Shagholi, M. Eshaghi Gordji, M. Bavand Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras,* J. Comput. Anal. Appl. **13** (2011), 1097–1105.

[28] D. Shin, C. Park, d Sh. Farhadabadi, *On the superstability of ternary Jordan C^*-homomorphisms,* J. Comput. Anal. Appl. **16** (2014), 964–973.

[29] D. Shin, C. Park, Sh. Farhadabadi, *Stability and superstability of J^*-homomorphisms and J^*-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.

[30] F. Skof, *Propriet locali e approssimazione di operatori*, Rend. Sem. Mat. Fis. Milano **53** (1983), 113–129.

[31] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.

[32] C. Zaharia, *On the probabilistic stability of the monomial functional equation,* J. Nonlinear Sci. Appl. **6** (2013), 51–59.

[33] S. Zolfaghari, *Approximation of mixed type functional equations in p-Banach spaces*, J. Nonlinear Sci. Appl. **3** (2010), 110–122.

[1]Research Institute for Natural Sciences,
Hanyang University, Seoul 04763,
Republic of Korea
E-mail address: baak@hanyang.ac.kr

[2]Department of Mathematics,
Daejin University, Kyunggi 11159,
Republic of Korea
E-mail address: jrlee@hdaejin.ac.kr

[3]Department of Mathematics,
Daejin University, Kyunggi 11159,
Republic of Korea
E-mail address: hyper@daejin.ac.kr

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.7, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Global Attractivity and the Periodic Nature of Third Order Rational Difference Equation

E. M. Elsayed[1,2], Faris Alzahrani[1], and H. S. Alayachi[1]
[1]Mathematics Department, Faculty of Science,
King Abdulaziz University,
P. O. Box 80203, Jeddah 21589, Saudi Arabia.
[2]Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura 35516, Egypt.
E-mails: emmelsayed@yahoo.com, faris.kau@hotmail.com,
hazas2010@hotmail.com.

ABSTRACT

The main target of our study to cover the solutions behavior of the following difference equation

$$x_{n+1} = ax_n + bx_{n-1} + \frac{c + dx_{n-2}}{e + fx_{n-2}}, \quad n = 0, 1, ...,$$

where the parameters $a,\ b,\ c,\ d,\ e$ and f are positive real numbers and the initial conditions $x_{-2},\ x_{-1}$ and x_0 are positive real numbers.

Keywords: stability, boundedness, periodicity, global attractor, difference equations.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

Our objective in this research is to study character of global stability and the periodicity of the solutions of the recursive sequence

$$x_{n+1} = ax_n + bx_{n-1} + \frac{c + dx_{n-2}}{e + fx_{n-2}}, \tag{1}$$

where the following parameters $a,\ b,\ c,\ d,\ e$ and f are defined as positive real numbers and the initial conditions $x_{-2},\ x_{-1}$ and x_0 are also defined as positive real numbers.

The theory of discrete dynamical systems and difference equations developed greatly during the last twenty-five years of the twentieth century. Applications of discrete dynamical systems and difference equations have appeared recently in many areas. The theory of difference equations occupies a central position in applicable analysis. There is no doubt that the theory of difference equations will continue to play an important role in mathematics as a whole. Nonlinear difference equations of order greater than one are of paramount importance in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, physiology, physics, engineering, economics, probability theory, genetics, psychology and resource management [12]. It is very interesting to investigate the behavior of solutions of a higher-order rational difference equation and to discuss the local asymptotic stability of its equilibrium points. Rational difference equations have been studied by several authors. Especially there has been a great interest in the study of the attractivity of the solutions of such equations. For more results for the rational difference equations, we refer the interested reader to [1–30].

The study of the nonlinear rational difference equations of a higher order is quite challenging and rewarding, and the results about these equations offer prototypes towards the development of the basic theory of the global behavior of nonlinear difference equations of a big order, recently, many researchers have investigated the behavior of the solution of difference equations for example: Abo-Zeid and Al-Shabi [1] investigated the global stability, and periodic nature of the positive solutions of the difference equation

$$x_{n+1} = \frac{A+Bx_n}{C+Dx_n x_{n-2}}.$$

Belhannache et al. [5] studied the global behavior of positive solutions of the following third order difference equation

$$x_{n+1} = \frac{A+Bx_{n-1}}{C+Dx_n^p x_{n-2}^q}.$$

Dehghan and Rastegar [11], deal with the qualitative behavior of solutions of the higher-order non-linear difference equation

$$x_{n+1} = \frac{p+qx_n+rx_{n-k}}{1+x_{n-k}}.$$

Din [14] investigated the local asymptotic stability, global stability, the periodic character, semicycle analysis and the boundedness nature of the following rational difference equation

$$x_{n+1} = \frac{A+Bx_n+Cx_{n-k}}{1+x_n+x_{n-k}}.$$

In [16] Elabbasy et al. investigated the global stability character, boundedness and the periodicity of solutions of the difference equation

$$x_{n+1} = \frac{\alpha x_n+\beta x_{n-1}+\gamma x_{n-2}}{Ax_n+Bx_{n-1}+Cx_{n-2}}.$$

Elsayed [22] investigated the local and global stability, boundedness character and obtained the solution of some special cases of the following recursive sequence

$$x_{n+1} = ax_{n-1} + \frac{bx_n x_{n-1}}{cx_n+dx_{n-2}}.$$

A. El-Moneam, and Zayed [20]-[21] studied the periodicity, the boundedness and the global stability of the positive solutions of the following nonlinear difference equations

$$\begin{aligned} x_{n+1} &= Ax_n + Bx_{n-k} + Cx_{n-l} + \frac{bx_{n-k}}{dx_{n-k}-ex_{n-l}}. \\ x_{n+1} &= Ax_n + Bx_{n-k} + Cx_{n-l} + +Dx_{n-\sigma} + \frac{bx_{n-k}+hx_{n-l}}{dx_{n-k}+ex_{n-l}}. \end{aligned}$$

Su and Li [52] studied the global asymptotic stability of the nonlinear difference equation

$$x_{n+1} = \frac{\alpha+\beta x_n}{A+Bx_n+Cx_{n-1}}.$$

Yalçınkaya et al. [54] considered the dynamics of the difference equation

$$x_{n+1} = \frac{ax_{n-k}}{b+cx_n^p}.$$

For some related work see [31–57].

2. SOME BASIC PROPERTIES AND DEFINITIONS

Here, we recall some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let $F : I^{k+1} \to I$, be a continuously differentiable function. Then for every set of initial conditions $x_{-k},\ x_{-k+1},\ ...,\ x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1. (Equilibrium Point) A point $\overline{x} \in I$ is called an equilibrium point of Eq.(2) if

$$\overline{x} = F(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of Eq.(2), or equivalently, $\overline{x}$ is a fixed point of F.

Definition 2. (Periodicity) A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Definition 3. (Stability)

(i) The equilibrium point $\overline{x}$ of Eq.(2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(2) is locally asymptotically stable if $\overline{x}$ is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have $\lim_{n\to\infty} x_n = \overline{x}$.

(iii) The equilibrium point $\overline{x}$ of Eq.(2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(2).

(v) The equilibrium point $\overline{x}$ of Eq.(2) is unstable if $\overline{x}$ is not locally stable.

The linearized equation of Eq.(2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}. \tag{3}$$

Theorem A. [47] Assume that $p, q \in R$ and $k \in \{0, 1, 2, ...\}$. Then $|p| + |q| < 1$, is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, ...$$

Remark: Theorem A can be easily extended to a general linear equations of the form

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, ... \tag{4}$$

where $p_1, p_2, ..., p_k \in R$ and $k \in \{1, 2, ...\}$. Then Eq. (4) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1.$$

Theorem B. [48] Let $g : [a, b]^{k+1} \to [a, b]$, be a continuous function, where k is a positive integer, and where $[a, b]$ is an interval of real numbers. Consider the difference equation

$$x_{n+1} = g(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ... \tag{5}$$

Suppose that g satisfies the following conditions.

(1) For each integer i with $1 \leq i \leq k+1$; the function $g(z_1, z_2, ..., z_{k+1})$ is weakly monotonic in z_i for fixed $z_1, z_2, ..., z_{i-1}, z_{i+1}, ..., z_{k+1}$.

(2) If m, M is a solution of the system

$$m = g(m_1, m_2, ..., m_{k+1}), \qquad M = g(M_1, M_2, ..., M_{k+1}),$$

then $m = M$, where for each $i = 1, 2, ..., k+1$, we set

$$\begin{aligned} m_i &= \left\{ \begin{matrix} m, & \text{if } g \text{ is non-decreasing in } z_i, \\ M, & \text{if } g \text{ is non-increasing in } z_i, \end{matrix} \right\}, \\ Mi &= \left\{ \begin{matrix} M, & \text{if } g \text{ is non-decreasing in } z_i, \\ m, & \text{if } g \text{ is non-increasing in } z_i. \end{matrix} \right\}. \end{aligned}$$

Then there exists exactly one equilibrium point $\bar{x}$ of Equation (5), and every solution of Equation (5) converges to $\bar{x}$.

3. LOCAL STABILITY OF THE EQUILIBRIUM POINT OF EQ.(1)

This section deals with study the local stability character of the equilibrium point of Eq.(1)

Eq.(1) has equilibrium point and is given by

$$\overline{x} = a\overline{x} + b\overline{x} + \frac{c + d\overline{x}}{e + f\overline{x}} \qquad \Rightarrow \qquad \overline{x}(1 - a - b) = \frac{c + d\overline{x}}{e + f\overline{x}},$$

$$f(1-a-b)\overline{x}^2 + [e(1-a-b) - d]\,\overline{x} - c = 0$$

If $d > e(1-a-b) > 0$, then the only positive equilibrium point of Eq.(1) is given by

$$\overline{x} = \frac{[d - e(1-a-b)] + \sqrt{[d - e(1-a-b)]^2 + 4fc(1-a-b)}}{2f(1-a-b)}.$$

Let $f : (0, \infty)^3 \longrightarrow (0, \infty)$ be a continuous function defined by

$$f(u, v, w) = au + bv + \frac{c + dw}{e + fw}. \tag{6}$$

Therefore it follows that

$$\frac{\partial f(u, v, w)}{\partial u} = a, \quad \frac{\partial f(u, v, w)}{\partial v} = b, \quad \frac{\partial f(u, v, w)}{\partial w} = \frac{(de - fc)}{(e + fw)^2}.$$

Then we see that

$$\frac{\partial f(\overline{x}, \overline{x}, \overline{x})}{\partial u} = a = -a_2, \quad \frac{\partial f(\overline{x}, \overline{x}, \overline{x})}{\partial v} = b = -a_1, \quad \frac{\partial f(\overline{x}, \overline{x}, \overline{x})}{\partial w} = \frac{de - fc}{(e + f\overline{x})^2} = -a_0.$$

Then the linearized equation of Eq.(1) about $\overline{x}$ is

$$y_{n+1} + a_2 y_n + a_1 y_{n-1} + a_0 y_{n-2} = 0, \tag{7}$$

whose characteristic equation is

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0. \tag{8}$$

Theorem 1. Assume that

$$\frac{|de-fc|}{(e+f\overline{x})^2}<1-a-b.$$

Then the positive equilibrium point of Eq.(1) is locally asymptotically stable.

Proof: It follows by Theorem A that, Eq.(7) is asymptotically stable if all roots of Eq.(8) lie in the open disc $|\lambda|<1$ that is if

$$|a_2|+|a_1|+|a_0|<1 \qquad \Rightarrow \qquad |a|+|b|+\left|\frac{de-fc}{(e+f\overline{x})^2}\right|<1,$$

and so

$$a+b+\frac{|de-fc|}{(e-f\overline{x})^2}<1,$$

or

$$\frac{|de-fc|}{(e+f\overline{x})^2}<1-a-b.$$

The proof is complete.

4. BOUNDEDNESS OF SOLUTIONS OF EQ.(1)

Here we study the boundedness nature of solutions of Eq.(1).

Theorem 2. Every solution of Eq.(1) is bounded if $a+b+\dfrac{d}{e}<1$.

Proof: Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(1). It follows from Eq.(1) that

$$x_{n+1}=ax_n+bx_{n-1}+\frac{c+dx_{n-2}}{e+fx_{n-2}}\le ax_n+bx_{n-1}+\frac{c+dx_{n-2}}{e}.$$

Then

$$x_{n+1}\le ax_n+bx_{n-1}+\frac{d}{e}x_{n-2}+\frac{c}{e} \quad \text{for all} \quad n\ge 1.$$

By using a comparison, we can write the right hand side as follows

$$y_{n+1}=ay_n+by_{n-1}+\frac{d}{e}y_{n-2}+\frac{c}{e},$$

and this equation is locally asymptotically stable if $a+b+\frac{d}{e}<1$, and converges to the equilibrium point $\overline{y}=\frac{c}{e\left(1-a-b-\frac{d}{e}\right)}$. Therefore

$$\limsup_{n\to\infty} x_n\le\frac{c}{e\left(1-a-b-\frac{d}{e}\right)}.$$

Thus the solution is bounded.

Theorem 3. Every solution of Eq.(1) is unbounded if $a>1$ (*or* $b>1$).

Proof: Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(1). Then from Eq.(1) we see that

$$x_{n+1}=ax_n+bx_{n-1}+\tfrac{c+dx_{n-2}}{e+fx_{n-2}}>ax_n \quad \text{for all} \quad n\ge 1.$$

We see that the right hand side can write as follows

$$y_{n+1}=ay_n \quad \Rightarrow \quad y_n=a^n y_0,$$

and this equation is unstable because $a>1$, and $\lim\limits_{n\to\infty} y_n=\infty$. Then by using ratio test $\{x_n\}_{n=-2}^{\infty}$ is unbounded from above (when $b>1$ is similar).

5. EXISTENCE OF PERIOD TWO SOLUTIONS

In this section we study the existence of periodic solutions of Eq.(1). The following theorem states the necessary and sufficient conditions that this equation has periodic solutions of prime period two.

Theorem 4. Eq.(1) has positive prime period two solutions if and only if

$$(i)\ (eB-d)^2B^2f^2-4aBf^2(e^2(1-b)B-ed(1-b)-acf)>0,\quad B=b-a-1.$$

Proof: First suppose that there exists a prime period two solution $..., p, q, p, q, ...,$ of Eq.(1). We will prove that Condition (i) holds. We see from Eq.(1) that

$$\begin{aligned} p &= aq+bp+\frac{c+dq}{e+fq}, \qquad q=ap+bq+\frac{c+dp}{e+fp}. \\ p(1-b)-aq &= \frac{c+dq}{e+fq}, \qquad q(1-b)-ap=\frac{c+dp}{e+fp}. \end{aligned}$$

Then

$$ep(1-b)+pqf(1-b)-aeq-afq^2=c+dq,$$

and

$$eq(1-b)+pqf(1-b)-aep-afp^2=c+dp.$$

Then

$$ep(1-b)+pqf(1-b)-afq^2=c+(d+ae)q, \tag{9}$$

and

$$eq(1-b)+pqf(1-b)-afp^2=c+(d+ae)p. \tag{10}$$

Subtracting (9) from (10) gives

$$e(1-b)(p-q)+af(p-q)(p+q)=-(d+ae)(p-q).$$

Since $p\neq q$, it follows that

$$\begin{aligned} e(1-b)+af(p+q) &= -(d+ae), \\ p+q &= \frac{e(b-1-a)-d}{af}. \end{aligned}$$

or

$$p+q=\frac{eB-d}{af}, \qquad B=b-a-1. \tag{11}$$

Again, adding (9) and (10) yields

$$\begin{aligned} &e(1-b)(p+q)+2pqf(1-b)-af(p^2+q^2)=2c+(d+ae)(p+q), \\ &2pqf(1-b)-af((p+q)^2-2pq)=2c+(p+q)(d+ae-e(1-b)). \end{aligned} \tag{12}$$

It follows by (11), (12) and the relation

$$p^2+q^2=(p+q)^2-2pq \quad \text{for all } p,q\in R,$$

that

$$2pqf(1-b)+2afpq=af(p+q)^2+2c+(p+q)(d+e(a-1+b)).$$

and

$$2pqf((1-b)+a)=2c+(p+q)\left\{d+e(a-1+b)+af(p+q)\right\}.$$

From Eq. (11) we have

$$\begin{aligned} 2pqf((1-b)+a) &= 2c+(p+q)\left\{d+e(a-1+b)+e(b-1-a)-d\right\}, \\ 2pqf((1-b+a)) &= 2c+(p+q)\left\{-2e+2eb\right\}, \end{aligned}$$

$$\begin{aligned} pqf(-B) &= c+(p+q)e(b-1) \\ pqfB &= e(1-b)\left(\frac{eB-d}{af}\right)-c. \end{aligned}$$

Thus

$$pq=\frac{e^2(1-b)B-ed(1-b)-af}{aBf^2}. \tag{13}$$

Now it is clear from Eq.(11) and Eq.(13) that p and q are the two distinct roots of the quadratic equation

$$\begin{aligned} t^2-\left(\tfrac{eB-d}{af}\right)t+\left(\tfrac{e^2(1-b)B-ed(1-b)-acf}{aBf^2}\right) &= 0, \\ aBf^2t^2-(eB-d)Bft+\left(e^2(1-b)B-ed(1-b)-acf\right) &= 0, \end{aligned} \tag{14}$$

and so

$$(eB-d)^2B^2f^2>4aBf^2\left(e^2(1-b)B-ed(1-b)-acf\right),$$

or

$$(eB-d)^2B^2f^2-4aBf^2(e^2(1-b)B-ed(1-b)-acf)>0.$$

Therefore Inequality (i) holds.

Second suppose that Inequality (i) is true. We will show that Eq.(1) has a prime period two solution. Assume that

$$p=\frac{(eB-d)Bf+\sqrt{\zeta}}{2aBf^2},\quad q=\frac{(eB-d)Bf-\sqrt{\zeta}}{2aBf^2},$$

where $\zeta=(eB-d)^2B^2f^2-4aBf^2(e^2(1-b)B-ed(1-b)-acf)$.
We see from Inequality (i) that

$$(eB-d)^2B^2f^2-4aBf^2(e^2(1-b)B-ed(1-b)-acf)>0,$$

which equivalents to

$$(eB-d)^2B^2f^2>4aBf^2\left(e^2(1-b)B-ed(1-b)-acf\right).$$

Therefore p and q are distinct real numbers. Set $x_{-2}=p$, $x_{-1}=q$ and $x_0=p$. We wish to show that $x_1=x_{-1}=q$ and $x_2=x_0=p$. It follows from Eq.(1) that

$$x_1=ap+bq+\frac{c+dp}{e+fp}=\tfrac{a(eB-d)Bf+a\sqrt{\zeta}}{2aBf^2}+\tfrac{b(eB-d)Bf-b\sqrt{\zeta}}{2aBf^2}+\frac{c+\left(\frac{d(eB-d)Bf+d\sqrt{\zeta}}{2aBf^2}\right)}{e+\left(\frac{(eB-d)Bf^2+f\sqrt{\zeta}}{2aBf^2}\right)}.$$

Multiplying the denominator and numerator by $2aBf^2$ gives

$$x_1=a(eB-d)Bf+a\sqrt{\zeta}+b(eB-d)Bf-b\sqrt{\zeta}+\frac{2acBf^2+\left(d(eB-d)Bf+d\sqrt{\zeta}\right)}{2aeBf^2+\left((eB-d)Bf^2+f\sqrt{\zeta}\right)}.$$

By simple computations we can see that

$$x_1=\frac{(eB-d)Bf+\sqrt{\zeta}}{2aBf^2}=q.$$

Similarly as before one can easily show that $x_2=p$. Then it follows by induction that

$$x_{2n}=p\quad\text{and}\quad x_{2n+1}=q\quad\text{for all}\quad n\geq-2.$$

Thus Eq.(1) has the prime period two solution ...,p,q,p,q,..., where p and q are the distinct roots of the quadratic equation (14) and the proof is complete.

6. GLOBAL ATTRACTIVITY OF THE EQUILIBRIUM POINT OF EQ.(1)

In this section we investigate the global asymptotic stability of Eq.(1).

Theorem 5. The equilibrium point $\overline{x}$ is a global attractor of Eq.(1) if one of the following statements holds

$$de \geq fc \text{ and } (1-a-b)\, e \geq d. \tag{15}$$
$$de < fc \text{ and } (1-a-b) \geq 0. \tag{16}$$

Proof: Let α and β be a real numbers and assume that $g:[\alpha,\beta]^3 \longrightarrow [\alpha,\beta]$ be a function defined by

$$g(u,v,w) = au + bv + \frac{c+dw}{e+fw}.$$

Then

$$\frac{\partial g(u,v,w)}{\partial u} = a, \quad \frac{\partial g(u,v,w)}{\partial v} = b, \quad \frac{\partial g(u,v,w)}{\partial w} = \frac{de-fc}{(e+fw)^2}.$$

We consider the two cases:-

Case (1): Assume that (15) is true, then we can easily see that the function $g(u,v,w)$ increasing in u, v and w.

Suppose that (m, M) is a solution of the system $M = g(M,M,M)$ *and* $m = g(m,m,m)$. Then from Eq.(1), we see that

$$M = aM + bM + \frac{c+dM}{de+fM}, \quad m = am + bm + \frac{c+dm}{e+fm},$$
$$M(1-a-b) = \frac{c+dM}{e+fM}, \quad m(1-a-b) = \frac{c+dm}{e+fm},$$

then

$$MAe + AfM^2 = c + dM, \quad mAe + Afm^2 = c + dm, \quad A = 1-a-b.$$

Subtracting this two equations we obtain

$$(M-m)\{Ae + Af(M+m) - d\} = 0,$$

under the conditions $Ae \geq d,\ a < 1$, we see that $M = m$. It follows by Theorem B that $\overline{x}$ is a global attractor of Eq.(1) and then the proof is complete.

Case (2): Assume that (16) is true, then we can easily see that the function $g(u,v,w)$ increasing in u, v and decreasing in w.

Suppose that (m, M) is a solution of the system $M = g(M,M,m)$ *and* $m = g(m,m,M)$.Then from Eq.(1), we see that

$$M = aM + bM + \frac{c+dm}{e+fm}, \quad m = am + bm + \frac{c+dM}{e+fM},$$
$$MA = \frac{c+dm}{e+fm}, \quad mA = \frac{c+dM}{e+fM},$$

then

$$MAe + MAfm = c + dm, \quad mAe + fMmA = c + dM.$$

Subtracting we obtain

$$(M-m)\,(Ae+d) = 0,$$

under the conditions $(1-a-b) > 0$, we see that $M = m$. Also, from Theorem B, we see that $\overline{x}$ is a global attractor of Eq.(1) and then the proof is complete.

7. NUMERICAL EXAMPLES

For confirming the results of this paper, we consider numerical examples which represent different types of solutions to Eq. (1).

Example 1. We assume $x_{-2} = .5,\ x_{-1} = 3,\ x_0 = 9,\ a = .2,\ b = .7,\ c = .2,\ d = .6,\ e = 1.3,\ f = 5.3$**.** See Fig. 1.

Example 2. See Fig. 2, since $x_{-2} = .5,\ x_{-1} = 3,\ x_0 = 9,\ a = .4,\ b = .6,\ c = .2,\ d = .6,\ e = 1.3,\ f = 5.3$.

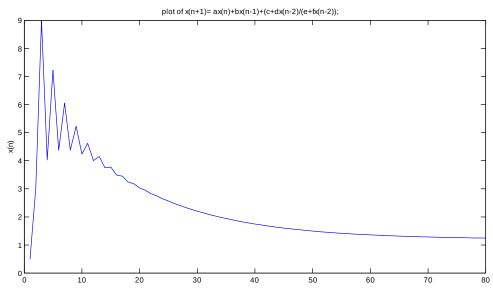

Figure 1.

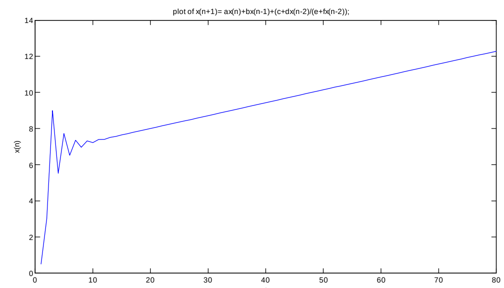

Figure 2.

Example 3. We consider $x_{-2} = 2.5,\ x_{-1} = 3,\ x_0 = 9,\ a = .4,\ b = .5,\ c = 2,\ d = 6,\ e = 3,\ f = 5$. See Fig. 3.

Example 4. See Fig. 4, since $x_{-2} = 2.5,\ x_{-1} = 3,\ x_0 = 9,\ a = 1,\ b = .5,\ c = 2,\ d = 6,\ e = 3,\ f = 5$.

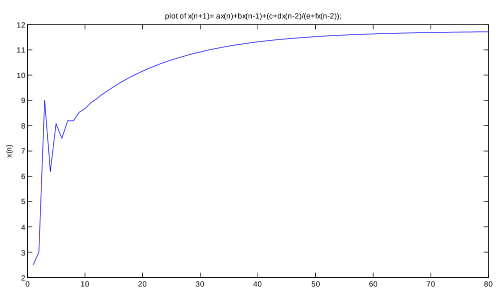

Figure 3.

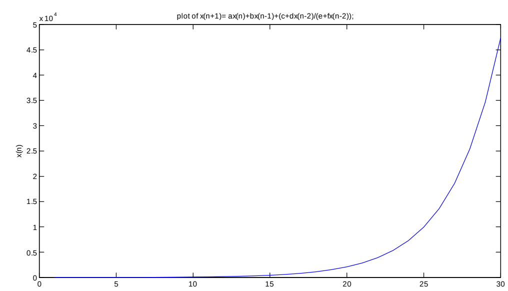

Figure 4.

Example 5. Fig. 5. shows the solutions when $a = .7,\ b = .5,\ c = .2,\ d = .1,\ e = .3,\ f = .5,\ x_{-2} = 2.5,\ x_{-1} = .3,\ x_0 = .9$.

Example 6. Fig. 6. shows the period two solutions when $a = .6,\ b = .5,\ c = .82,\ d = .7,\ e = .3,\ f = .5,\ x_{-2} = p,\ x_{-1} = q,\ x_0 = p$. $\left(\text{Since } p, q = \frac{(eB-d)Bf \pm \sqrt{\zeta}}{2aBf^2}\right)$.

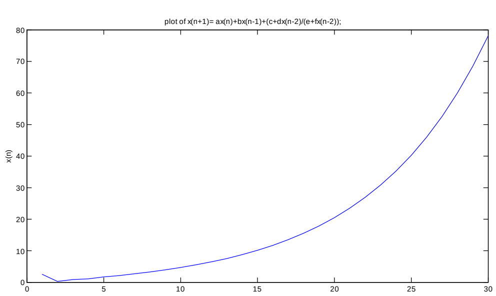

Figure 5.

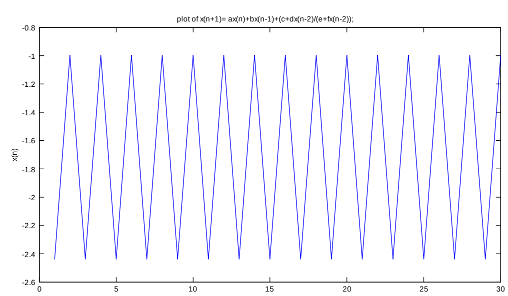

Figure 6.

References

[1] R. Abo-Zeid, M A. Al-Shabi, Global behavior of a third order difference equation, Tamkang Journal of Mathematics, 43 (3) (2012), 375–383.

[2] R. Abo-Zeid, Global Behavior of a Higher Order Rational Difference Equation, International Journal of Difference Equations, 10 (1) (2015), 1–11.

[3] M. Aloqeili, Dynamics of a rational difference equation, Appl. Math. Comp., 176 (2) (2006), 768-774.

[4] A. Asiri, E. M. Elsayed, and M. M. El-Dessoky, On the Solutions and Periodic Nature of Some Systems of Difference Equations, J. Comput. Theor. Nanos., 12 (10) (2015), 3697-3704.

[5] F. Belhannache, N. Touafek and R. Abo-Zeid, Dynamics of a third-order rational difference equation, Bull. Math. Soc. Sci. Math. Roumanie, Tome 59 (107) (1) (2016), 13–22.

[6] A. Brett, E. J. Janowski, and M. R. S. Kulenovic, Global Asymptotic Stability for Linear Fractional Difference Equation, Journal of Difference Equations, Volume 2014, Article ID 275312, 11 pages.

[7] E. Bedford and K. Kim, Dynamics of rational surface automorphisms: linear fractional recurrences, Journal of Geometric Analysis, 19 (3) (2009), 553–583.

[8] C. Cinar, T. Mansour, and I. Yalcinkaya, On the difference equation of higher order, Utilitas Mathematica, 92 (2013), 161-166.

[9] O. H. Criner, W. E. Taylor, and J. L. Williams, On the Solutions of a System of Nonlinear Difference Equations, International Journal of Difference Equations, 10 (2) (2015), 161–166.

[10] S. E. Das and M. Bayram, On a System of Rational Difference Equations, World Applied Sciences Journal 10 (11) (2010), 1306-1312.

[11] M. Dehghan, N. Rastegar, On the global behavior of a high-order rational difference equation, Computer Physics Communications, 180, (2009), 873–878.

[12] Q. Din, Qualitative nature of a discrete predator-prey system, Contemporary Methods in Mathematical Physics and Gravitation, l (1) (2015), 27-42.

[13] Q. Din, Global character of a rational difference equation, Thai J. Math., 12 (1) (2014), 55–70.

[14] Q. Din, Global behavior of a rational difference equation, Acta Univ. Apul., 30 (2012), 35-49.

[15] E. M. Elabbasy, and A. A. El-biaty, On the nonlinear rational difference equation, International Journal of Scientific & Engineering Research, 7 (1) (2016), 433-436.

[16] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, Global attractivity and periodic character of a fractional difference equation of order three, Yokohama Math. J., 53 (2007), 89-100.

[17] M. M. El-Dessoky, and E. M. Elsayed, On the solutions and periodic nature of some systems of rational difference equations, J. Comput. Anal. Appl., 18 (2) (2015), 206-218.

[18] H. El-Metwally and E. M. Elsayed, Qualitative Behavior of some Rational Difference Equations, Journal of Computational Analysis and Applications, 20 (2) (2016), 226-236.

[19] H. El-Metwally, I. Yalçınkaya, C. Cinar, Global Stability of an Economic Model, Utilitas Mathematica, 95 (2014), 235-244.

[20] M.A. El-Moneam, and E. M. E. Zayed, On the dynamics of the nonlinear rational difference equation $x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-l} + \frac{bx_{n-k}}{dx_{n-k} - ex_{n-l}}$, Journal of the Egyptian Mathematical Society, 23 (3) (2015), 494–499.

[21] M. A. El-Moneam, and E. M. E. Zayed, Dynamics of the Rational Difference Equation, Information Sciences Letters, 3 (2) (2014), 45-53.

[22] E. M. Elsayed, Solution and attractivity for a rational recursive sequence, Discrete Dynamics in Nature and Society, Volume 2011, Article ID 982309, 17 pages.

[23] E. M. Elsayed, Behavior and expression of the solutions of some rational difference equations, Journal of Computational Analysis and Applications 15 (1) (2013), 73-81.

[24] E. M. Elsayed, Solution for systems of difference equations of rational form of order two, Computational and Applied Mathematics 33 (3) (2014), 751-765.

[25] E. M. Elsayed, On the solutions and periodic nature of some systems of difference equations, International Journal of Biomathematics 7 (6) (2014), 1450067, (26 pages).

[26] E. M. Elsayed, New method to obtain periodic solutions of period two and three of a rational difference equation, Nonlinear Dynamics 79 (1) (2015), 241-250.

[27] E. M. Elsayed, Dynamics and Behavior of a Higher Order Rational Difference Equation, The Journal of Nonlinear Science and Applications, 9 (4) (2016), 1463-1474.

[28] E. M. Elsayed and A. M. Ahmed, Dynamics of a three-dimensional systems of rational difference equations, Mathematical Methods in The Applied Sciences, 39 (5) (2016), 1026–1038.

[29] E. M. Elsayed and A. Alghamdi, Dynamics and Global Stability of Higher Order Nonlinear Difference Equation, Journal of Computational Analysis and Applications, 21 (3) (2016), 493-503.

[30] E. M. Elsayed and M. M. El-Dessoky, Dynamics and global behavior for a fourth-order rational difference equation, Hacettepe Journal of Mathematics and Statistics 42 (5) (2013), 479–494.

[31] E. M. Elsayed, M. M. El-Dessoky and E. O. Alzahrani, The Form of The Solution and Dynamics of a Rational Recursive Sequence, J. Comput. Anal. Appl., 17 (1) (2014), 172–186.

[32] E. M. Elsayed, M. M. El-Dessoky and Asim Asiri, Dynamics and Behavior of a Second Order Rational Difference equation, J. Comput. Anal. Appl., 16 (4) (2014), 794–807.

[33] E. M. Elsayed and H. El-Metwally, Stability and solutions for rational recursive sequence of order three, J. Comput. Anal. Appl. 17 (2) (2014), 305-315.

[34] E. M. Elsayed and H. El-Metwally, Global behavior and periodicity of some difference equations, Journal of Computational Analysis and Applications 19 (2) (2015), 298-309.

[35] E. M. Elsayed and T. F. Ibrahim, Solutions and periodicity of a rational recursive sequences of order five, Bulletin of the Malaysian Mathematical Sciences Society, 38 (1) (2015), 95-112.

[36] E. M. Elsayed and T. F. Ibrahim, Periodicity and solutions for some systems of nonlinear rational difference equations, Hacettepe Journal of Mathematics and Statistics, 44 (6) (2015), 1361–1390.

[37] M. E. Erdoğan, C. Cinar, I. Yalçınkaya, On the dynamics of the recursive sequence, Mathematical and Computer Modelling, 54 (2011), 1481-1485.

[38] Y. Halim, and M. Bayram, On the solutions of a higher-order difference equation in terms of generalized Fibonacci sequences, Mathematical Methods in the Applied Sciences, Article first published online: 26 OCT 2015, DOI: 10.1002/mma.3745.

[39] T. F. Ibrahim, Boundedness and stability of a rational difference equation with delay, Rev. Roum. Math. Pures Appl., 57, (2012), 215-224.

[40] T. F. Ibrahim, Periodicity and Global Attractivity of Difference Equation of Higher Order. J. Comput. Anal. Appl., 16 (2014), 552-564.

[41] T. F. Ibrahim and N. Touafek, On a third order rational difference equation with variable coefficients, Dyn. Cont. Disc. Impu. Syst., Appl. Algo., 20 (2013) 251-264.

[42] D. Jana and E. M. Elsayed, Interplay between strong Allee effect, harvesting and hydra effect of a single population discrete - time system, International Journal of Biomathematics, 9 (1) (2016), 1650004, (25 pages).

[43] S. Kalabusic, M. R. S. Kulenovic and M. Mehuljic, Global Dynamics and Bifurcations of Two Quadratic Fractional Second Order Difference Equations, Journal of Computational Analysis and Applications, 21 (1) (2016), 132-143.

[44] A. Khaliq, and E. M. Elsayed, The Dynamics and Solution of some Difference Equations, The Journal of Nonlinear Science and Applications (JNSA), 9 (3) (2016), 1052-1063.

[45] A. Q. Khan, and M. N. Qureshi, Global dynamics of a competitive system of rational difference equations, Mathematical Methods in the Applied Sciences, 38 (18) (2015), 4786–4796.

[46] A. Q. Khan, M. N. Qureshi and Q. Din, Asymptotic behavior of an anti-competitive system of rational difference equations, Life Science Journal 11 (7s) (2014), 16-20.

[47] V. L. Kocic and G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic Publishers, Dordrecht, 1993.

[48] M. R. S. Kulenovic and G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman & Hall / CRC Press, 2001.

[49] A. S. Kurbanli, C. Cinar, I. Yalçınkaya, On the behavior of positive solutions of the system of rational difference equations, Mathematical and Computer Modelling, 53 (2011), 1261-1267.

[50] O. Ocalan, Global dynamics of a non autonomous rational difference equation, J. Appl. Math. & Informatics, 32 (5-6) (2014), 843 - 848.

[51] M. Saleh, M. Aloqeili, On the difference equation $y_{n+1} = A + \frac{y_n}{y_{n-k}}$, Appl. Math. Comput. 176 (1) (2006) 359 363.

[52] Y. H. Su, W. T. Li, Global asymptotic stability of a second-order nonlinear difference equation, Applied Mathematics and Computation, 168 (2005), 981–989.

[53] N. Touafek, On a second order rational difference equation, Hacettepe Journal of Mathematics and Statistics, 41 (6) (2012), 867–874.

[54] I. Yalcinkaya and C. Cinar On the dynamics of difference equation $\frac{ax_{n-k}}{b+cx_n^p}$, Fasciculi Mathematici (42) (2009), 141-148.

[55] Y. Yazlik, On the solutions and behavior of rational difference equations, Journal of Computational Analysis and Applications, 17 (3) (2014), 584-594.

[56] Y. Yazlik, E. M. Elsayed and N. Taskara, On the Behaviour of the Solutions of Difference Equation Systems, Journal of Computational Analysis and Applications, 16 (5) (2014), 932–941.

[57] E. M. E. Zayed and M. A. El-Moneam, On the rational recursive sequence $x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-k}}$, Communications on Applied Nonlinear Analysis, 15 (2008), 47-57.

Asymptotically stability of solutions of fuzzy differential equations in the quotient space of fuzzy numbers

Dong Qiu*, Yumei Xing, Lihong Zhang
School of science,
Chongqing University of Posts and Telecommunications,
Nanan, Chongqing, 400065, P. R. China

Abstract

In this paper, we investigate essentially stability theory for the fuzzy differential equations in the quotient space of fuzzy numbers by Lyapunov-like functions. By using the differential inequalities and the comparison principle for Lyapunov-like functions, we give some sufficient criterias for the asymptotically stability, equi-asymptotically stability and uniformly asymptotically stability of the trivial solution of the fuzzy differential equations.

Keywords: Fuzzy number; Quotient space; Fuzzy differential equation; Asymptotically stability

1 Introduction

Recently, the study of fuzzy differential equations has been gained importance due to its application. Subsequently, the existence and uniqueness of solutions of the initial value ptoblems for fuzzy differential equations under kinds of conditions were studied in [8, 9, 11, 14, 18, 24] and the relationship between a solution and its approximate solutions to fuzzy differential equations were established in [19, 25, 26]. Further, the essentially stability theory for fuzzy differential equations by Lyapunov-like functions were investigated in [2, 12, 28]. In particular, Hien [4] researched the asymptotic stability of solutions of fuzzy differential equations by Lyapunovs second method.

The above these results of fuzzy differential equations based on well known and widely used Hukuhara difference [6] and the H-differentiability of Puri and Ralescu [20]. But in many applications the Hukuhara difference appears to have several limitations and to be very restrictive [1, 8]. In [15, 16], Mareš presented a natural equivalence relation between fuzzy quantities. This equivalence relation can be used to partition of the set of fuzzy quantities into equivalence classes having the desired group properties for the addition operation [7, 17, 27]. Hong and Do [5] defined a more refined equivalence relation than Mareš [15] and improved Mareš's results. In [21], Qiu et al. showed that the method of finding the inverse operation of fuzzy numbers in the sense of Mareš is very intuitive. As an application of the main results, it is shown that if we identify every fuzzy number with the corresponding equivalence class, there wound be more differentiable fuzzy functions than what is found in the literature. After that, the fuzzy differential equations in the quotient space of fuzzy numbers were investigated [23, 22]. In this paper, we shall study the stability of the trivial solution of the fuzzy differential equations in the quotient space of fuzzy numbers by Lyapunov's second method.

2 Preliminaries

A fuzzy set $\widetilde{x}$ of $\mathbb{R}$ is characterized by a membership function $\mu_{\widetilde{x}} : \mathbb{R} \to [0, 1]$. For each such fuzzy set $\widetilde{x}$, we denote by $[\widetilde{x}]^\alpha = \{x \in \mathbb{R} : \mu_{\widetilde{x}}(x) \geq \alpha\}$ for any $\alpha \in (0, 1]$, its α-level set. We define the set

*Corresponding author. Tel.:+86-15123126186; Fax:+86-23-62471796; E-mail: dongqiumath@163.com (D. Qiu).

$[\widetilde{x}]^0$ by $[\widetilde{x}]^0 = \overline{\bigcup_{\alpha\in(0,1]} [\widetilde{x}]^\alpha}$, where $\overline{A}$ denotes the closure of a crisp set A. A fuzzy set $\widetilde{x}$ is said to be a fuzzy number if it satisfies the following conditions [3]:

(1) $\widetilde{x}$ is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $\mu_{\widetilde{x}}(x_0) = 1$;
(2) $\widetilde{x}$ is convex, i.e., $\mu_{\widetilde{x}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\widetilde{x}}(x_1), \mu_{\widetilde{x}}(x_2)\}$, for all $x_1, x_2 \in \mathbb{R}$ and $\lambda \in (0,1)$;
(3) $\widetilde{x}$ is upper semi-continuous;
(4) $[\widetilde{x}]^0$ is compact.

Equivalently, a fuzzy number $\widetilde{x}$ is a fuzzy set with non-empty bounded closed level sets $[\widetilde{x}]^\alpha = [\widetilde{x}_L(\alpha), \widetilde{x}_R(\alpha)]$ for all $\alpha \in [0,1]$, where $[\widetilde{x}_L(\alpha), \widetilde{x}_R(\alpha)]$ denotes a closed interval with the left end point $\widetilde{x}_L(\alpha)$ and the right end point $\widetilde{x}_R(\alpha)$. We denote the class of fuzzy numbers by $\mathscr{F}$. We say that a fuzzy number $\widetilde{s} \in \mathscr{F}$ is symmetric [15], if $\mu_{\widetilde{s}}(x) = \mu_{\widetilde{s}}(-x)$, for all $x \in \mathbb{R}$, i.e., $\widetilde{s} = -\widetilde{s}$. The set of all symmetric fuzzy numbers will be denoted by $\mathscr{S}$.

Definition 2.1 *[5] Let $\widetilde{x}, \widetilde{y} \in \mathscr{F}$. We say that $\widetilde{x}$ is equivalent to $\widetilde{y}$ and write $\widetilde{x} \sim \widetilde{y}$ if and only if there exist symmetric fuzzy numbers $\widetilde{s_1}, \widetilde{s_2} \in \mathscr{S}$ such that $\widetilde{x} + \widetilde{s_1} = \widetilde{y} + \widetilde{s_2}$.*

The equivalence relation defined above is reflexive, symmetric and transitive [15]. Let $\langle\widetilde{x}\rangle$ denote the equivalence class containing the element $\widetilde{x}$ and denote the set of equivalence classes by $\mathscr{F}/\mathscr{S}$.

Definition 2.2 *[10] Let $f : [a,b] \to \mathbb{R}$. f is said be of bounded variation if there exists a $C > 0$ such that*

$$\sum_{i=1}^{n} |f(x_{i-1}) - f(x_i)| \leq C$$

for every partition $a = x_0 < x_1 < \cdots < x_n = b$ on $[a,b]$. The total variation of f on $[a,b]$ is defined by

$$V_a^b(f) = \sup_p \sum_{i=1}^{n} |f(x_{i-1}) - f(x_i)|,$$

where p represents all partitions of $[a,b]$. The set of all functions of bounded variation on $[a,b]$ is denoted by $BV[a,b]$.

Definition 2.3 *[7] For a fuzzy number $\widetilde{x}$, we define a function $\widetilde{x}_M : [0,1] \to \mathbb{R}$ by assigning the midpoint of each α-level set to $\widetilde{x}_M(\alpha)$ for all $\alpha \in [0,1]$, i.e.,*

$$\widetilde{x}_M(\alpha) = \frac{\widetilde{x}_L(\alpha) + \widetilde{x}_R(\alpha)}{2}.$$

Then the function $\widetilde{x}_M : [0,1] \to \mathbb{R}$ will be called the midpoint function of the fuzzy number $\widetilde{x}$.

Lemma 2.1 *[21] For any $\tilde{x} \in \mathscr{F}$, the midpoint function $\tilde{x}_M$ is continuous from the right at 0 and continuous from the left on $[0,1]$. Furthermore it is a function of bounded variation on $[0,1]$.*

Definition 2.4 *[16] Let $\widetilde{x} \in \mathscr{F}$ and let $\widehat{x}$ be a fuzzy number such that $\widetilde{x} = \widehat{x} + \widetilde{s}$ for some $\widetilde{s} \in \mathscr{S}$, if $\widehat{x} = \widetilde{y} + \widetilde{s}_1$ for some $\widetilde{y} \in \mathscr{F}$ and $\widetilde{s}_1 \in \mathscr{S}$, then $\widetilde{s}_1 = \widetilde{0}$. Then the fuzzy number $\widehat{x}$ will be called the Mareš core of the fuzzy number $\widetilde{x}$.*

Definition 2.5 *[22] Define $d_{\sup} : \mathscr{F}/\mathscr{S} \times \mathscr{F}/\mathscr{S} \to \mathbb{R}^+ \cup \{0\}$ by*

$$d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle) = \sup_{\alpha\in[0,1]} \left|M_{\langle\widetilde{x}\rangle}(\alpha) - M_{\langle\widetilde{y}\rangle}(\alpha)\right|,$$

for any $\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle \in \mathscr{F}/\mathscr{S}$.

We know that $(\mathscr{F}/\mathscr{S}, d_{\sup})$ is a metric space [21].

3 Main results

Definition 3.1 *[22] For each $m(t) \in C[J,\mathbb{R}]$, where J is a subinterval of $(0,+\infty)$, we will define $d^+ : C[J,\mathbb{R}] \to \mathbb{R}$ by*

$$d^+m(t) = \overline{\lim_{h\to 0^+}} \frac{1}{h}(m(t+h) - m(t)).$$

Definition 3.2 *[23] A mapping $F : J \to \mathscr{F}/\mathscr{S}$ is differentiable at $t_0 \in J$ if for small $|h| > 0$, there exists an $F'(t_0) \in \mathscr{F}/\mathscr{S}$ such that*

$$\lim_{h\to 0} d_{\sup}\left(\frac{F(t_0+h) - F(t_0)}{h}, F'(t_0)\right) = 0.$$

Definition 3.3 *[23] A mapping $F : J \to \mathscr{F}/\mathscr{S}$ is measurable if F is measurable with respect to $d_{\sup}$.*

A mapping $F : J \to \mathscr{F}/\mathscr{S}$ is called integrably bounded if there exists an integrable function $h : J \to \mathbb{R}^+ \cup \{0\}$ such that $|M_{F(t)}(\alpha)| \le h(t)$ for all $t \in J$ and $\alpha \in [0,1]$; a mapping $F : J \to \mathscr{F}/\mathscr{S}$ is said to be of uniformly bounded variation with respect to $\alpha \in [0,1]$ (for short, of uniformly bounded variation) if there exists a constant $K > 0$ such that $V_0^1\left(M_{F(t)}\right) \le K$, for each $t \in J$ [23].

Definition 3.4 *[23] Let $F : J \to \mathscr{F}/\mathscr{S}$ be measurable. The integral of F over J, denoted $\int_J F(t)dt$, is a mapping $M_{\int_J F(t)dt} : [0,1] \to \mathbb{R}$, which is defined by the equation*

$$M_{\int_J F(t)dt}(\alpha) = \int_J M_{F(t)}(\alpha)dt$$

for each $\alpha \in [0,1]$. The mapping F is said to be integrable over J if there exists an $\langle \widetilde{x}_0 \rangle \in \mathscr{F}/\mathscr{S}$ such that $M_{\int_J F(t)dt} = M_{\langle \widetilde{x}_0 \rangle}$. In this case, we denote the integral by

$$\int_J F(t)dt = \langle \widetilde{x}_0 \rangle .$$

Assume that $f : \mathbb{R}_+ \times S(\rho) \to \mathscr{F}/\mathscr{S}$ is continuous and of uniformly bounded variation, where $S(\rho) = \{\langle \widetilde{x} \rangle \in \mathscr{F}/\mathscr{S} : d_{\sup}(\langle \widetilde{x} \rangle, \langle \widetilde{0} \rangle) < \rho\}$. We consider the initial value problem for the fuzzy differential equation

$$x'(t) = f(t, x(t)), \qquad x(t_0) = x_0. \tag{1}$$

We assume that $f(t, \langle \widetilde{0} \rangle) = \langle \widetilde{0} \rangle$ so that we have the trivial solution $x(t) = \langle \widetilde{0} \rangle$ for (1).

We shall discuss some simple asymptotically stability results of solutions of (1) by Lyapunov's second method. First, we give some notions of concerning the stability of the trivial solution of (1). Let $x(t) = x(t, t_0, x_0)$ be any solution of (1) existing on $[t_0, +\infty)$. Denote $\mathcal{K} = \{\omega \in C[\mathbb{R}_+, \mathbb{R}_+], \omega(0) = 0, \omega(\cdot)$ is increasing $\}$.

Definition 3.5 *The trivial solution $x(t) = \langle \widetilde{0} \rangle$ of (1) is said to be*

$(S1)$ *stable, if for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that if $d_{\sup}(x_0, \langle \widetilde{0} \rangle) < \delta$ then*

$$d_{\sup}(x(t), \langle \widetilde{0} \rangle) < \varepsilon, \qquad t \ge t_0;$$

$(S2)$ *uniformly stable, if δ in $(S1)$ is independent of t_0;*

$(S3)$ *asymptotically stable, if it is stable and for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0) > 0$ and $T = T(t_0, x_0, \varepsilon) > 0$ such that if $d_{\sup}(x_0, \langle \widetilde{0} \rangle) < \delta$ then*

$$d_{\sup}(x(t), \langle \widetilde{0} \rangle) < \varepsilon, \qquad t \ge t_0 + T;$$

$(S4)$ *equi-asymptotically stable, if T in $(S3)$ is independent of x_0;*

$(S5)$ *uniformly asymptotically stable, if it is uniformly stable and δ and T in $(S4)$ are independent of t_0.*

Lemma 3.1 *[13] Suppose that $g(t,\varphi)$ be a continuous function on $\mathbb{R}_+^2$ and $r(t) = r(t,t_0,\varphi_0)$, $\varphi(t_0) = \varphi_0$ be the maximal solution of the scalar differential equation:*

$$\frac{d\varphi}{dt} = g(t,\varphi), \quad \varphi(t_0) = \varphi_0 \geq 0, \tag{2}$$

existing on $[t_0,+\infty)$. Let $m(t)$ be a continuous function on $\mathbb{R}_+$ satisfies

$$d^+m(t) = \overline{\lim_{h\to 0^+}} \frac{m(t+h)-m(t)}{h} \leq g(t,m(t)), \qquad t \geq t_0.$$

Then $m(t) \leq r(t)$, for each $t \geq t_0$ if $m(t_0) \leq \varphi_0$.

Let $V(t,\langle\widetilde{x}\rangle) : \mathbb{R}_+ \times S(\rho) \to \mathbb{R}$ be a given function. Then we define

$$D_f^+V(t,\langle\widetilde{x}\rangle) = \overline{\lim_{h\to 0^+}} \frac{1}{h}\left(V(t+h,\langle\widetilde{x}\rangle + hf(t,\langle\widetilde{x}\rangle)) - V(t,\langle\widetilde{x}\rangle)\right),$$

where $f(\cdot)$ is the right-hand side of (1). Note that, if $V(t,x)$ is Lipchitzian in x, then we have

$$d^+V(t,x(t)) \leq D_f^+V(t,x(t)).$$

Lemma 3.2 *[22] Suppose that*
(1) $|V(t,\langle\widetilde{x}\rangle) - V(t,\langle\widetilde{y}\rangle)| \leq L(t)d_{\sup}(\langle\widetilde{x}\rangle,\langle\widetilde{y}\rangle)$, $V(\cdot,\cdot) \in C[\mathbb{R}_+ \times S(\rho),\mathbb{R}_+]$ *and* $L(\cdot) \in C[\mathbb{R}_+,\mathbb{R}_+]$;
(2) $D_f^+V(t,\langle\widetilde{x}\rangle) \leq g(t,V(t,\langle\widetilde{x}\rangle))$, $g(\cdot,\cdot) \in C[\mathbb{R}_+^2,\mathbb{R}]$.
If $x(t) = x(t,t_0,x_0)$ is any solution of (1) through (t_0,x_0) existing on $[t_0,+\infty)$ such that $V(t_0,x_0) \leq \varphi_0$, then we have

$$V(t,x(t)) \leq r(t,t_0,\varphi_0), \qquad t \geq t_0,$$

where $r(t,t_0,\varphi_0)$ is the maximal solution of the scalar differential equation (2) existing on $[t_0,+\infty)$.

Lemma 3.3 *Suppose that*
(1) $|V(t,\langle\widetilde{x}\rangle) - V(t,\langle\widetilde{y}\rangle)| \leq L(t)d_{\sup}(\langle\widetilde{x}\rangle,\langle\widetilde{y}\rangle)$, $V(\cdot,\cdot) \in C[\mathbb{R}_+ \times S(\rho),\mathbb{R}_+]$ *and* $L(\cdot) \in C[\mathbb{R}_+,\mathbb{R}_+]$;
(2) $D_f^+V(t,\langle\widetilde{x}\rangle) \leq -\omega(h(t,\langle\widetilde{x}\rangle)) + g(t,V(t,\langle\widetilde{x}\rangle))$, $h(\cdot,\cdot) \in C[\mathbb{R}_+ \times S(\rho),\mathbb{R}_+]$, $\omega(\cdot) \in \mathcal{K}$ *and* $g(t,\varphi) \in C[\mathbb{R}_+^2,\mathbb{R}]$ *is nondecreasing with respect to φ for each $t \in \mathbb{R}_+$.*

If $x(t) = x(t,t_0,x_0)$ is any solution of (1) through (t_0,x_0) existing on $[t_0,+\infty)$ such that $V(t_0,x_0) \leq \varphi_0$, then we have

$$V(t,x(t)) + \int_{t_0}^{t} \omega(h(s,x(s)))ds \leq r(t,t_0,\varphi_0), \qquad t \geq t_0,$$

where $r(t,t_0,\varphi_0)$ is the maximal solution of the scalar differential equation (2) existing on $[t_0,+\infty)$.

Proof. Let $m(t) = V(t,x(t)) + \int_{t_0}^{t} \omega(h(s,x(s)))ds \geq V(t,x(t))$ for each $t \geq t_0$. Then $m(t_0) = V(t_0,x_0) \leq \varphi_0$ and for small $h > 0$,

$$\begin{aligned}
m(t+h) - m(t) &= V(t+h,x(t+h)) + \int_{t_0}^{t+h} \omega(h(s,x(s)))ds \\
&- V(t,x(t)) - \int_{t_0}^{t} \omega(h(s,x(s)))ds \\
&= V(t+h,x(t+h)) - V(t+h,x(t)+hf(t,x(t))) \\
&+ V(t+h,x(t)+hf(t,x(t))) - V(t,x(t)) + \int_{t}^{t+h} \omega(h(s,x(s)))ds \\
&\leq L(t+h)d_{\sup}(x(t+h),x(t)+hf(t,x(t))) \\
&+ V(t+h,x(t)+hf(t,x(t))) - V(t,x(t)) + \int_{t}^{t+h} \omega(h(s,x(s)))ds.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
d^+m(t) &= \overline{\lim_{h\to 0^+}} \frac{m(t+h)-m(t)}{h} \\
&\le D_f^+V(t,x(t)) + \overline{\lim_{h\to 0^+}} \frac{1}{h}\int_t^{t+h} \omega(h(s,x(s)))ds \\
&+ L(t)\overline{\lim_{h\to 0^+}} \frac{1}{h} d_{\sup}(x(t+h), x(t)+hf(t,x(t))) \\
&= D_f^+V(t,x(t)) + \omega(h(t,x(t))) \\
&+ L(t)\overline{\lim_{h\to 0^+}} d_{\sup}\left(\frac{x(t+h)-x(t)}{h}, f(t,x(t))\right) \\
&= D_f^+V(t,x(t)) + \omega(h(t,x(t))) + L(t)d_{\sup}\left(x'(t), f(t,x(t))\right) \\
&= D_f^+V(t,x(t)) + \omega(h(t,x(t))) \le g(t,V(t,x(t))),
\end{aligned}$$

for each $t \ge t_0$. By the monotonicity of $g(t,\varphi)$ with respect to φ for each $t \ge t_0$, we have

$$d^+m(t) \le g(t,V(t,x(t))) \le g(t,m(t)),$$

for each $t \ge t_0$. By Lemma 3.1, we obtain

$$V(t,x(t)) + \int_{t_0}^t \omega(h(s,x(s)))ds = m(t) \le r(t,t_0,\varphi_0), \qquad t \ge t_0.$$

□

Theorem 3.1 *Suppose that there exists a function $V(t,\langle\widetilde{x}\rangle)$ satisfies the following conditions:*
(1) $|V(t,\langle\widetilde{x}\rangle) - V(t,\langle\widetilde{y}\rangle)| \le L(t)d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle)$, $V(\cdot,\cdot) \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ *and* $L(\cdot) \in C[\mathbb{R}_+,\mathbb{R}_+]$*;*
(2) $\omega(d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle)) \le V(t,\langle\widetilde{x}\rangle)$, $V(t,\langle\widetilde{0}\rangle) = 0$, $\omega(\cdot) \in \mathcal{K}$*;*
(3) $D_f^+V(t,\langle\widetilde{x}\rangle) \le g(t,V(t,\langle\widetilde{x}\rangle))$, $g(\cdot,\cdot) \in C[\mathbb{R}_+^2, \mathbb{R}], g(t,0) = 0$.
If the solution $\varphi(t) = 0$ of (2) is asymptotically stable, then the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is asymptotically stable.

Proof. If the solution $\varphi(t) = 0$ of (2) is asymptotically stable, then by (S3) of Definition 3.5, we have it is stable. Thus, by Theorem 3.1 in [22], we get that the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is stable.

Since for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0, x_0, \varepsilon)$ such that if $0 \le \varphi_0 < \delta_0$ then

$$|\varphi(t,t_0,\varphi_0)| < \omega(\varepsilon), \qquad t \ge t_0 + T.$$

Since $V(t,\langle\widetilde{0}\rangle) = 0$, we have

$$V(t_0,\langle\widetilde{x}\rangle) = |V(t_0,\langle\widetilde{x}\rangle) - V(t_0,\langle\widetilde{0}\rangle)| \le L(t_0)d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle),$$

for each $\langle\widetilde{x}\rangle \in S(\rho)$. Thus, there exists $\delta = \delta(t_0)$ such that if $d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle) < \delta$, then $V(t_0,\langle\widetilde{x}\rangle) < \delta_0$.

Let $x(t) = x(t,t_0,x_0)$ be any solution of (1) through (t_0,x_0) existing on $[t_0,+\infty)$. Next, we shall show that if $d_{\sup}(x_0,\langle\widetilde{0}\rangle) < \delta$ then $d_{\sup}(x(t),\langle\widetilde{0}\rangle) < \varepsilon$ for each $t \ge t_0 + T$. By the conditions (1), (3) and Lemma 3.2, we get

$$V(t,x(t)) \le r(t,t_0,V(t_0,x_0)), \qquad t \ge t_0 + T,$$

where $r(t,t_0,V(t_0,x_0))$ is the maximal solution of the scalar differential equation (2) existing on $[t_0,+\infty)$. Since $V(t_0,x_0) < \delta_0$, we have $r(t,t_0,V(t_0,x_0)) < \omega(\varepsilon)$ for each $t \ge t_0 + T$ and therefore

$$V(t,x(t)) \le r(t,t_0,V(t_0,x_0)) < \omega(\varepsilon), \qquad t \ge t_0 + T.$$

By the condition (2), we get

$$\omega(d_{\sup}(x(t), \langle\widetilde{0}\rangle)) \leq V(t, x(t)) < \omega(\varepsilon), \qquad t \geq t_0 + T.$$

By the monotonicity of ω, we have

$$d_{\sup}(x(t), \langle\widetilde{0}\rangle) < \varepsilon, \qquad t \geq t_0 + T.$$

Hence, the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is asymptotically stable. □

Theorem 3.2 *Suppose that there exists a function $V(t, \langle\widetilde{x}\rangle)$ satisfies the conditions (1), (2) and (3) of Theorem 3.1. If the solution $\varphi(t) = 0$ of (2) is equi-asymptotically stable, then the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is equi-asymptotically stable.*

Proof. In fact, we can show Theorem 3.2 by a similar method of Theorem 3.1. □

Theorem 3.3 *Suppose that there exists a function $V(t, \langle\widetilde{x}\rangle)$ satisfies the following conditions:*
(1) $|V(t, \langle\widetilde{x}\rangle) - V(t, \langle\widetilde{y}\rangle)| \leq L(t)d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle)$, $V(\cdot,\cdot) \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ *and* $L(\cdot) \in C[\mathbb{R}_+, \mathbb{R}_+]$;
(2) $\omega_1(d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle)) \leq V(t, \langle\widetilde{x}\rangle) \leq \omega_2(t, d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle))$, $\omega_1(\cdot), \omega_2(t,\cdot) \in \mathcal{K}$;
(3) $D_f^+ V(t, \langle\widetilde{x}\rangle) \leq -\beta V(t, \langle\widetilde{x}\rangle), \beta > 0$.
Then the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is equi-asymptotically stable.

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of (1) through (t_0, x_0) existing on $[t_0, +\infty)$. By Theorem 3.2 in [22], we get that the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is stable. Thus, taking $\varepsilon = \rho$, there exists a $\delta = \delta(t_0, \rho)$ such that if $d_{\sup}(x_0, \langle\widetilde{0}\rangle) < \delta$, then

$$d_{\sup}(x(t), \langle\widetilde{0}\rangle) < \rho, \qquad t \geq t_0.$$

Let the function $g(t, \varphi) = -\beta\varphi$, $(t, \varphi) \in \mathbb{R}_+^2$ and $\varphi_0 = V(t_0, x_0)$ in Lemma 3.2. Then we know that

$$r(t, t_0, \varphi_0) = V(t_0, x_0)e^{-\beta(t-t_0)}, \qquad t \geq t_0,$$

is the unique solution of the scalar differential equation (2). Thus, by Lemma 3.2, we obtain

$$V(t, x(t)) \leq V(t_0, x_0)e^{-\beta(t-t_0)}, \qquad t \geq t_0.$$

For any given $\varepsilon > 0$, we take $T = T(t_0, \varepsilon) = \frac{1}{\beta}\ln\frac{\omega_2(t_0, \delta)}{\omega_1(\varepsilon)} + 1$. Then, by the condition (2), we get

$$\begin{aligned}
\omega_1(d_{\sup}(x(t), \langle\widetilde{0}\rangle)) &\leq V(t, x(t)) \leq V(t_0, x_0)e^{-\beta(t-t_0)} \\
&\leq e^{-\beta}\omega_2(t_0, d_{\sup}(x_0, \langle\widetilde{0}\rangle))\frac{\omega_1(\varepsilon)}{\omega_2(t_0, \delta)} \\
&\leq e^{-\beta}\omega_2(t_0, \delta))\frac{\omega_1(\varepsilon)}{\omega_2(t_0, \delta)} \\
&= e^{-\beta}\omega_1(\varepsilon) < \omega_1(\varepsilon),
\end{aligned}$$

which implies that

$$d_{\sup}(x(t), \langle\widetilde{0}\rangle) < \varepsilon, \qquad t \geq t_0 + T.$$

Hence, the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is equi-asymptotically stable. □

Theorem 3.4 *Suppose that there exists a function $V(t, \langle\widetilde{x}\rangle)$ satisfies the following conditions:*
(1) $|V(t, \langle\widetilde{x}\rangle) - V(t, \langle\widetilde{y}\rangle)| \leq L(t)d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle)$, $V(\cdot,\cdot) \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ *and* $L(\cdot) \in C[\mathbb{R}_+, \mathbb{R}_+]$;
(2) $\omega_1(d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle)) \leq V(t, \langle\widetilde{x}\rangle) \leq \omega_2(d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle))$, $\omega_1(\cdot), \omega_2(\cdot) \in \mathcal{K}$;
(3) $D_f^+ V(t, \langle\widetilde{x}\rangle) \leq g(t, V(t, \langle\widetilde{x}\rangle))$, $g(\cdot,\cdot) \in C[\mathbb{R}_+^2, \mathbb{R}], g(t, 0) = 0$.
If the solution $\varphi(t) = 0$ of (2) is uniformly asymptotically stable, then the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is uniformly asymptotically stable.

Proof. If the solution $\varphi(t) = 0$ of (2) is uniformly asymptotically stable, then by (S5) of Definition 3.5, we have it is uniformly stable. Thus, by Theorem 3.3 in [22], we get that the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is uniformly stable.

Since for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta_0 > 0$ and $T = T(\varepsilon)$ such that if $0 \leq \varphi_0 < \delta_0$ then

$$|\varphi(t, t_0, \varphi_0)| < \omega_1(\varepsilon), \qquad t \geq t_0 + T.$$

Since $\omega_1(\cdot), \omega_2(\cdot) \in \mathcal{K}$, there exist a $\delta > 0$ such that $\omega_2(\delta) < \omega_1(\delta_0)$.

Let $x(t) = x(t, t_0, x_0)$ be any solution of (1) through (t_0, x_0) existing on $[t_0, +\infty)$. Next, we shall show that if $d_{\sup}(x_0, \langle\widetilde{0}\rangle) < \delta$ then $d_{\sup}(x(t), \langle\widetilde{0}\rangle) < \varepsilon$ for each $t \geq t_0 + T$. By the conditions (1), (3) and Lemma 3.2, we get

$$V(t, x(t)) \leq r(t, t_0, \omega_1^{-1}(V(t_0, x_0))), \qquad t \geq t_0 + T,$$

where $r(t, t_0, \omega_1^{-1}(V(t_0, x_0)))$ is the maximal solution of the scalar differential equation (2) existing on $[t_0, +\infty)$. By the condition (2), we have

$$V(t_0, x_0) \leq \omega_2(d_{\sup}(x_0, \langle\widetilde{0}\rangle)) \leq \omega_2(\delta) < \omega_1(\delta_0).$$

Thus, by the monotonicity of ω_1, we have $\omega_1^{-1}(V(t_0, x_0)) \leq \delta_0$, which implies that

$$r(t, t_0, \omega_1^{-1}(V(t_0, x_0))) < \omega_1(\varepsilon), \qquad t \geq t_0 + T$$

and therefore

$$V(t, x(t)) \leq r(t, t_0, \omega_1^{-1}(V(t_0, x_0))) < \omega_1(\varepsilon), \qquad t \geq t_0 + T.$$

By the condition (2), we get

$$\omega_1(d_{\sup}(x(t), \langle\widetilde{0}\rangle)) \leq V(t, x(t)) < \omega_1(\varepsilon), \qquad t \geq t_0 + T.$$

By the monotonicity of ω_1, we have

$$d_{\sup}(x(t), \langle\widetilde{0}\rangle) < \varepsilon, \qquad t \geq t_0 + T.$$

Hence, the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is uniformly asymptotically stable. □

Theorem 3.5 *Suppose that there exists a function $V(t, \langle\widetilde{x}\rangle)$ satisfies the following conditions:*

(1) $|V(t, \langle\widetilde{x}\rangle) - V(t, \langle\widetilde{y}\rangle)| \leq L(t) d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle)$, $V(\cdot, \cdot) \in C[\mathbb{R}_+ \times S(\rho), \mathbb{R}_+]$ *and* $L(\cdot) \in C[\mathbb{R}_+, \mathbb{R}_+]$;
(2) $\omega_1(d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle)) \leq V(t, \langle\widetilde{x}\rangle) \leq \omega_2(d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle))$, $\omega_1(\cdot), \omega_2(\cdot) \in \mathcal{K}$;
(3) $D_f^+ V(t, \langle\widetilde{x}\rangle) \leq -\omega_3(d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle))$, $\omega_3(\cdot) \in \mathcal{K}$.

Then the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is uniformly asymptotically stable.

Proof. Let $x(t) = x(t, t_0, x_0)$ be any solution of (1) through (t_0, x_0) existing on $[t_0, +\infty)$. By Theorem 3.4 in [22], we get that the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is uniformly stable. Thus, taking $\varepsilon = \rho$, there exists a $\delta = \delta(\rho)$ such that if $d_{\sup}(x_0, \langle\widetilde{0}\rangle) < \delta$, then

$$d_{\sup}(x(t), \langle\widetilde{0}\rangle) < \rho, \qquad t \geq t_0.$$

Let the function $g(t, \varphi) \equiv 0$, $(t, \varphi) \in \mathbb{R}_+^2$ and $\varphi_0 = V(t_0, x_0)$ in Lemma 3.3. Then we know that $r(t, t_0, \varphi_0) \equiv V(t_0, x_0)$ is the unique solution of the scalar differential equation (2). Thus, by Lemma 3.3, we obtain

$$V(t, x(t)) + \int_{t_0}^{t} \omega_3(d_{\sup}(x(s), \langle\widetilde{0}\rangle)) ds \leq V(t_0, x_0), \qquad t \geq t_0.$$

For any given $\varepsilon > 0$, we take $T = T(\varepsilon) = \dfrac{\omega_2(\delta)}{\omega_3\omega_2^{-1}\omega_1(\varepsilon)} + 1$. Suppose that $d_{\sup}(x(t), \langle\widetilde{0}\rangle) \geq \omega_2^{-1}\omega_1(\varepsilon)$ for each $t \in [t_0, t_0 + T]$. Then, by the condition (2), we get

$$\begin{aligned} V(t, x(t)) &= V(t_0, x_0) - \int_{t_0}^{t} \omega_3(d_{\sup}(x(s), \langle\widetilde{0}\rangle))ds \\ &\leq \omega_2(d_{\sup}(x_0, \langle\widetilde{0}\rangle)) - \omega_3\omega_2^{-1}\omega_1(\varepsilon)(t - t_0) \\ &< \omega_2(\delta) - \omega_3\omega_2^{-1}\omega_1(\varepsilon)(t - t_0), \end{aligned}$$

for each $t \in [t_0, t_0 + T]$. Thus, we obtain

$$0 \leq V(t_0 + T, x(t_0 + T)) < \omega_2(\delta) - \omega_3\omega_2^{-1}\omega_1(\varepsilon)T = -\omega_3\omega_2^{-1}\omega_1(\varepsilon) < 0.$$

This is a contradiction, thus there exists a $t^* \in [t_0, t_0 + T]$ such that

$$d_{\sup}(x(t^*), \langle\widetilde{0}\rangle) < \omega_2^{-1}\omega_1(\varepsilon).$$

Since $D_f^+ V(t, \langle\widetilde{x}\rangle) \leq -\omega_3(d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle)) \leq 0$, we have

$$V(t, x(t)) \leq V(t^*, x(t^*)), \qquad t \geq t^*.$$

Then, by the condition (2), we get

$$\begin{aligned} \omega_1(d_{\sup}(x(t), \langle\widetilde{0}\rangle)) &\leq V(t, x(t)) \leq V(t^*, x(t^*)) \\ &\leq \omega_2(d_{\sup}(x(t^*), \langle\widetilde{0}\rangle)) \\ &< \omega_2\omega_2^{-1}\omega_1(\varepsilon) = \omega_1(\varepsilon), \end{aligned}$$

which implies that $d_{\sup}(x(t), \langle\widetilde{0}\rangle) < \varepsilon$ for each $t \geq t^*$. Hence, we obtain

$$d_{\sup}(x(t), \langle\widetilde{0}\rangle) < \varepsilon, \qquad t \geq t_0 + T.$$

Consequently, the trivial solution $x(t) = \langle\widetilde{0}\rangle$ of (1) is uniformly asymptotically stable. □

Example 3.1 *Define $F : \mathbb{R}_+ \to \mathscr{F}/\mathscr{S}$ by the α-level sets of the fuzzy mapping*

$$\left[\widehat{F(t)}\right]^\alpha = \left[-\frac{2e^{-\alpha}}{1+t}, 0\right], \qquad \alpha \in [0, 1],$$

where $\widehat{F(t)}$ is the Mareš core of $F(t)$, for each $t \in \mathbb{R}_+$. Thus, we have

$$M_{F(t)}(\alpha) = -\frac{e^{-\alpha}}{1+t}, \qquad \alpha \in [0, 1],$$

for each $t \in \mathbb{R}_+$. It is obvious that $M_{F(t)}(\alpha)$ is continuous from the right at 0 and continuous from the left on $[0, 1]$ with respect to α. Since $M_{F(t)}(\alpha)$ is increasing with respect to α, we get

$$V_0^1(M_{F(t)}) = \frac{1 - e^{-1}}{1+t} \leq 1 - e^{-1}, \qquad t \in \mathbb{R}_+.$$

Thus, we obtain that $F(t)$ is of uniformly bounded variation. Since $M_{F(t)}(\alpha)$ is uniformly continuous with respect to $t \in \mathbb{R}_+$, we get that $F(t)$ is continuous with respect to $d_{\sup}$. Define $f : \mathbb{R}_+ \times \mathscr{F}/\mathscr{S} \to \mathscr{F}/\mathscr{S}$ by

$$f(t, \langle\widetilde{x}\rangle) = F(t)\langle\widetilde{x}\rangle.$$

It is obvious that f is continuous with respect to $d_{\sup}$ and of uniformly bounded variation.

Consider a Lyapunov function $V(t, \langle\widetilde{x}\rangle) = d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle)$. *Then* $V(t, \langle\widetilde{0}\rangle) = d_{\sup}(\langle\widetilde{0}\rangle, \langle\widetilde{0}\rangle) = 0$ *and*

$$|V(t, \langle\widetilde{x}\rangle) - V(t, \langle\widetilde{y}\rangle)| = \left| d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle) - d_{\sup}(\langle\widetilde{y}\rangle, \langle\widetilde{0}\rangle) \right| \leq d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{y}\rangle),$$

for any $(t, \langle\widetilde{x}\rangle), (t, \langle\widetilde{y}\rangle) \in \mathbb{R}_+ \times \mathscr{F}/\mathscr{S}$. *By Definition 2.9, for a small* $h > 0$, *we have*

$$\begin{aligned} V(t+h, \langle\widetilde{x}\rangle + hf(t, \langle\widetilde{x}\rangle)) &= d_{\sup}(\langle\widetilde{x}\rangle + hf(t, \langle\widetilde{x}\rangle), \langle\widetilde{0}\rangle) = d_{\sup}(\langle\widetilde{x}\rangle + hF(t)\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle) \\ &= \sup_{\alpha\in[0,1]} \left| M_{\langle\widetilde{x}\rangle}(\alpha) + hM_{F(t)}(\alpha)M_{\langle\widetilde{x}\rangle}(\alpha) \right| \\ &\leq \sup_{\alpha\in[0,1]} \left| M_{\langle\widetilde{x}\rangle}(\alpha) \right| \left(1 + h \sup_{\alpha\in[0,1]} M_{F(t)}(\alpha) \right) \\ &= \left(1 - \frac{he^{-1}}{1+t} \right) d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle). \end{aligned}$$

Hence, we get

$$D_f^+ V(t, \langle\widetilde{x}\rangle) = \overline{\lim_{h\to 0^+}} \frac{1}{h} \left(V(t+h, \langle\widetilde{x}\rangle + hf(t, \langle\widetilde{x}\rangle)) - V(t, \langle\widetilde{x}\rangle) \right) \leq -\frac{e^{-1}}{1+t} d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle).$$

Let $g(t, \varphi) = -\frac{e^{-1}}{1+t}\varphi$. *Then, we have*

$$D_f^+ V(t, \langle\widetilde{x}\rangle) \leq g(t, d_{\sup}(\langle\widetilde{x}\rangle, \langle\widetilde{0}\rangle)) = g(t, V(t, \langle\widetilde{x}\rangle)).$$

It's easy to show that the solution $\varphi = 0$ *of* (2) *is asymptotically stable. Hence, by Theorem 3.1, the trivial solution* $x(t) = \langle\widetilde{0}\rangle$ *of* (1) *is asymptotically stable.*

Acknowledgements

The authors thank the anonymous reviewers for their valuable comments. This work was supported by The National Natural Science Foundation of China (Grant no. 11671001), The Graduate Teaching Reform Research Program of Chongqing Municipal Education Commission (No.YJG143010).

References

[1] *G.A. Anastassiou*, On H-fuzzy differentiation, Math. Balk. **16** (2002) 155-193.

[2] *P. Diamond*, Stability and periodicity in fuzzy differential equations, IEEE Trans. Fuzzy Systems **8** (2000) 583-590.

[3] *D. Dubois and H. Prade*, Fuzzy Sets and Systems, Academic Press, New York, 1980.

[4] *L.V. Hien*, A note on the asymptotic stability of fuzzy differential equations, Ukrainian Math. J., **57** (2005) 1066-1076.

[5] *D. Hong and H. Do*, Additive decomposition of fuzzy quantities, Inf. Sci. **88** (1996) 201-207.

[6] *M. Hukuhara*, Integration des applications measurables dont la valeur est un compact convexe, Funkc. Ekvacioj **10** (1967) 205-223.

[7] *K.D. Jamison*, A normed space of fuzzy number equivalence classes, UCD/CCM Report No. 112, October 1997.

[8] *O. Kaleva*, Fuzzy Differential Equations, Fuzzy Sets Syst. **24** (1987) 301-317.

[9] *O. Kaleva*, The Cauchy problem for fuzzy differential equations, Fuzzy Sets Syst. **35** (1990) 389-396.

[10] *A. N. Kolmogorov and S.V. Fomin*, Introductory Real Analysis, Dover Publications, New York, 1975.

[11] *N. Kumaresan, J. Kavikumar, M. Kumudthaa and K. Ratnavelu*, Solution of Fuzzy Differential Equation under Generalized Differentiability by Genetic Programming, World Academy of Science, Eng. Tech. **5** (2011) 375-380.

[12] *V. Lakshmikantham, S. Leela and J. Vasundhara Devi*, Stability theory for set differential equations, Math. Anal. **11** (2004) 181-189.

[13] *V. Lakshmikantham and S. Leela*, Differential and Integral Inequalities, Vol. I , Academic Press, New York 1996.

[14] *V. Lakshmikantham and A.S. Vatsala*, Existence of fixed points of fuzzy mappings via theory of fuzzy differential equations, J. Comput. Appl. Math. **113** (2000) 195-200.

[15] *M. Mareš*, Addition of fuzzy quantities: Disjunction-conjunction approach, Kybernetika **25** (1989) 104-116.

[16] *M. Mareš*, Additive decomposition of fuzzy quantities, Fuzzy Sets Syst. **47** (1992) 341-346.

[17] *G. Panda, M. Panigrahi and S. Nanda*, Equivalence class in the set of fuzzy numbers and its application in decision-making problems, Int. J. Math. Math. Sci. **19** (2006) 741-765.

[18] *J.Y. Park, Y.C. Kwun, and J.U. Jeong*, Existence of solutions of fuzzy integral equations in Banach spaces, Fuzzy Sets Syst. **72** (1995) 373-378.

[19] *J.Y. Park and H.K. Han*, Fuzzy differential equations, Fuzzy Sets Syst. **110** (2000) 69-77.

[20] *M.L. Purl and D.A. Ralescu*, Differentials of fuzzy functions, J. Math. Anal. Appl. **91** (1983) 552-558.

[21] *D. Qiu, C.X. Lu, W. Zhang and Y.Y. Lan*, Algebraic properties and topological properties of the quotient space of fuzzy numbers based on Mareš equivalence relation, Fuzzy Sets Syst. **245** (2014) 63-82.

[22] *D. Qiu, C.X. Lu, W. Zhang, Q. Zhang and C. Mu*, Basic theorems for fuzzy difference equations in the quotient space of fuzzy numbers, Advances in Difference Equations **2014**, 2014:303

[23] *D. Qiu, W. Zhang and C.X. Lu*, On fuzzy differential equtions in the quotient space of fuzzy numbers, Fuzzy Sets Syst. **295** (2016) 72-98.

[24] *S.J. Song, L. Guo and C.B. Feng*, Global existence of solution to fuzzy diffrential equations, Fuzzy Sets Syst. **115** (2000) 371-376.

[25] *S.J. Song, C.X. Wu and X.P. Xue*, Existence and uniqueness theorem to the Cauchy problem of fuzzy diffrential equations under dissipative conditions, Comput. Math. Appl. **51** (2006) 1483-1492.

[26] *C.X. Wu, S.J. Song and E.S. Lee*, Approximate solutions and existence and uniqueness theorem to the Cauchy problem of fuzzy diffrential equations, J. Math. Anal. Appl. **202** (1996) 629-644.

[27] *C.X. Wu and Z. Zhao*, Some notes on the characterization of compact sets of fuzzy sets with L_p metric, Fuzzy Sets Syst. **159** (2008) 2104-2115.

[28] *T. Yoshizawa*, Stability Theory by Lyapunov's second method, The mathematical Society of Japan, Tokyo, 1966.

ON DIFFERENTIAL EQUATIONS ASSOCIATED WITH SQUARED HERMITE POLYNOMIALS

[1]TAEKYUN KIM, [2]DAE SAN KIM, [3]LEE-CHAE JANG, [4]HYUCK IN KWON

[1]Department of Mathematics, College of Science, Tianjin Polytechnic University,
Tianjin City, 300387, China
E-mail : tkkim@kw.ac.kr
[1]Department of Mathematics, Kwangwoon University,
Seoul 139-701, Republic of Korea
E-mail : tkkim@kw.ac.kr

[2]Department of Mathematics, Sogang University,
Seoul 121-742, Republic of Korea
E-mail : dskim@sogang.ac.kr

[3]Graduate School of Education, Konkuk University,
Seoul 143-701, Republic of Korea
E-mail : lcjang@konkuk.ac.kr

[4]Department of Mathematics, Kwangwoon University,
Seoul 139-701, Republic of Korea
E-mail : sura@kw.ac.kr

Abstract. In this paper, we investigate differential equations associated with squared Hermite polynomials and derive some new and explicit identities for these polynomials arising from the differential equations.

1. Introduction

As a method of obtaining new identities for special polynomials and numbers, in [8] T. Kim initiated a remarkable idea of using ordinary differential equations. Namely, he derived a family of nonlinear differential equations, indexed by positive integers, satisfied by the generating function of the Frobenius-Euler numbers and used them in order to get an interesting identity expressing higher-order Frobenius-Euler numbers in terms of (ordinary) Frobenius-Euler numbers. Here, more precisely, the differential equations are satisfied not by the generating function of the Frobenius-Euler numbers but by a constant multiple of that.

This method turned out to be very fruitful and can be applied to many interesting special polynomials and numbers (see [5, 8–11]). For example, linear differential equations are derived for Bessel polynomials, Changhee polynomials, actuarial polynomials, Meixner polynomials of the first kind, Poisson-Charlier polynomials, Laguerre polynomials, Hermite polynomials, and Stirling polynomials, while nonlinear ones are obtained for Bernoulli numbers of the

2010 Mathematics Subject Classification : 05A19, 11B37, 11B83, 34A30.
Key words and phrases: squared Hermite polynomials, differential equations.

second, Boole numbers, Chebyshev polynomials of the first, second, third, and fourth kind, degenerate Euler numbers, degenerate Eulerian polynomials, Korobov numbers, and Legendre polynomials.

To be specific, we will illustrate the results in the case of Bernoulli numbers of the second kind (see [5]). Firstly, it is shown that the function $F = F(t) = \frac{1}{log(1+t)}$ satisfies the family of nonlinear differential equations

$$F^{(N)}(t) = \frac{(-1)^N}{(1+t)^N} \sum_{j=2}^{N+1} (j-1)!(N-1)! H_{N-1,j-2} F^j \qquad (N = 1, 2, \cdots), \tag{1}$$

where H_N are the generalized harmonic numbers defined by

$$\begin{aligned} &H_{N,0} = 1, \quad \text{for all } N, \\ &H_{N,1} = \frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{1}, \\ &H_{N,j} = \frac{H_{N-1,j-1}}{N} + \frac{H_{N-1,j-1}}{N-1} + \cdots + \frac{H_{j-1,j-1}}{j} \quad (N \geq j \geq 2). \end{aligned} \tag{2}$$

Recall that the Bernoulli numbers of the second b_n are given by the generating function

$$\frac{t}{log(1+t)} = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!} \qquad \text{(see [5])}. \tag{3}$$

More generally, the Bernoulli numbers of the second $b_n^{(r)}$ of order r are defined by the generating function

$$\left(\frac{t}{log(1+t)}\right)^r = \sum_{n=0}^{\infty} b_n^{(r)} \frac{t^n}{n!} \qquad \text{(see [5])}. \tag{4}$$

Then, secondly the family of differential equations in (1) are used to derive the following interesting identities: for $N = 1, 2, \cdots$ and $n = 0, 1, \cdots$, we have

$$\begin{aligned} &(-1)^n \sum_{j=0}^{\min\{n,N-1\}} (N-j)!(N-1)! H_{N-1,N-1-j} (n)_j b_{n-j}^{(N+1-j)} \\ &= \begin{cases} (-1)^N N!(N)_n & \text{if} \quad 0 \leq n \leq N, \\ \sum_{l=0}^{n-N-1} \binom{N}{l} \frac{b_{n-l}}{n-l} (n)_{l+N+1} & \text{if} \quad n \geq N+1. \end{cases} \end{aligned} \tag{5}$$

As a generalization of the usual factorial $n!$, the double factorial of a positive integer n is defined by

$$n!! = \begin{cases} n \cdot (n-2) \cdots 5 \cdot 3 \cdot 1 & \text{if} \quad n > 0 \text{ odd}, \\ n \cdot (n-2) \cdots 6 \cdot 4 \cdot 2 & \text{if} \quad n > 0, \text{ even}, \\ 1 & \text{if} \quad n = -1, \ 0. \end{cases} \tag{6}$$

(see [1]).

Throughout this paper, the double factorials will be used.

The Hermite polynomials are classical orthogonal polynomials used such diverse areas as combinatorics, numerical analysis, probability, finite element methods, systems theory and quantum mechanics (see [2–4, 6, 7, 12–14]).

With the Roman's definition of Hermite polynomials $H_n(x)$ as

$$H_n(x) = e^{xt - t^2/2}, \tag{7}$$

we see from ([3], p.250) that

$$(1-t^2)^{-1/2}e^{x[t/(1+t)]} = \sum_{n=0}^{\infty}[H_n(\sqrt{x})]^2\frac{t^n}{n!}. \tag{8}$$

For brevity, we denote $[H_n(\sqrt{x})]^2$ by $SH_n(x)$, and hence

$$(1-t^2)^{-1/2}e^{x[t/(1+t)]} = \sum_{n=0}^{\infty}SH_n(x)\frac{t^n}{n!}. \tag{9}$$

In this paper, we would like to derive a family of linear differential equations satisfied by the generating function of the squared Hermite polynomials in (9) and use them in order to get an interesting identity for those polynomials. As an easy consequence of this result, we will have an expression for the squared Hermite polynomials.

2. Differential equations for the squared Hermite polynomials

In this paper, all differentiations are taken with respect to t, while x being fixed.
Let

$$\begin{aligned} F = F(t;x) &= \left(1-t^2\right)^{-\frac{1}{2}} e^{x\left(\frac{t}{t+1}\right)} \\ &= (1-t)^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}} e^{x\left(\frac{t}{t+1}\right)}. \end{aligned} \tag{10}$$

Then

$$\begin{aligned} F^{(1)} &= \frac{1}{2}(1-t)^{-\frac{3}{2}}(1+t)^{-\frac{1}{2}}e^{x\left(\frac{t}{t+1}\right)} - \frac{1}{2}(1-t)^{-\frac{1}{2}}(1+t)^{-\frac{3}{2}}e^{x\left(\frac{t}{t+1}\right)} \\ &\quad +(1-t)^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}}(1+t)^{-2}xe^{x\left(\frac{t}{t+1}\right)} \\ &= \left\{\frac{1}{2}(1-t)^{-1} - \frac{1}{2}(1+t)^{-1} + x(1+t)^{-2}\right\}F. \end{aligned} \tag{11}$$

$$\begin{aligned} F^{(2)} &= \left\{\frac{1}{2}(1-t)^{-2} + \frac{1}{2}(1+t)^{-2} - 2x(1+t)^{-3}\right\}F \\ &\quad + \left\{\frac{1}{2}(1-t)^{-1} - \frac{1}{2}(1+t)^{-1} + x(1+t)^{-2}\right\}^2 F \\ &= \left\{\frac{1}{2}(1-t)^{-2} + \frac{1}{2}(1+t)^{-2} - 2x(1+t)^{-3}\right\}F \\ &\quad + \left\{\frac{1}{4}(1-t)^{-2} + \frac{1}{4}(1+t)^{-2} + x^2(1+t)^{-4}\right. \\ &\quad \left. -\frac{1}{2}(1-t)^{-1}(1+t)^{-1} - x(1+t)^{-3} + x(1-t)^{-1}(1+t)^{-2}\right\}F \\ &= \left\{\frac{3}{4}(1-t)^{-2} - \frac{1}{2}(1-t)^{-1}(1+t)^{-1} + x(1-t)^{-1}(1+t)^{-2}\right. \\ &\quad \left. +\frac{3}{4}(1+t)^{-2} - 3x(1+t)^{-3} + x^2(1+t)^{-4}\right\}F. \end{aligned} \tag{12}$$

So, we are led to put

$$F^{(N)} = \left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} a_{i,j}(N,x)(1-t)^{-i}(1+t)^{-j}\right)F. \tag{13}$$

Here $a_{i,j}(N,x)$ are polynomials in x.

$$\begin{aligned}
F^{(N+1)} &= \left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} i a_{i,j}(N,x)(1-t)^{-(i+1)}(1+t)^{-j}\right)F \\
&\quad - \left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} j a_{i,j}(N,x)(1-t)^{-i}(1+t)^{-(j+1)}\right)F \\
&\quad + \left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} a_{i,j}(N,x)(1-t)^{-i}(1+t)^{-j}\right. \\
&\quad \left.\times\left\{\frac{1}{2}(1-t)^{-1}-\frac{1}{2}(1+t)^{-1}+x(1+t)^{-2}\right\}\right)F \\
&= \left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} (i+\frac{1}{2}) a_{i,j}(N,x)(1-t)^{-(i+1)}(1+t)^{-j}\right)F \\
&\quad - \left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} (j+\frac{1}{2}) a_{i,j}(N,x)(1-t)^{-i}(1+t)^{-(j+1)}\right)F \\
&\quad + \left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} x a_{i,j}(N,x)(1-t)^{-i}(1+t)^{-(j+2)}\right)F \\
&= \left(\sum_{i=1}^{N+1}\sum_{j=N+1-i}^{2(N+1-i)} (i-\frac{1}{2}) a_{i-1,j}(N,x)(1-t)^{-i}(1+t)^{-j}\right)F \\
&\quad - \left(\sum_{i=0}^{N}\sum_{j=N-i+1}^{2(N-i)+1} (j-\frac{1}{2}) a_{i,j-1}(N,x)(1-t)^{-i}(1+t)^{-j}\right)F \\
&\quad + \left(\sum_{i=0}^{N}\sum_{j=N-i+2}^{2(N-i)+2} x a_{i,j-2}(N,x)(1-t)^{-i}(1+t)^{-j}\right)F. \qquad (14)
\end{aligned}$$

On the other hand,

$$F^{(N+1)} = \left(\sum_{i=0}^{N+1}\sum_{j=N+1-i}^{2(N+1-i)} a_{i,j}(N+1,x)(1-t)^{-i}(1+t)^{-j}\right)F. \qquad (15)$$

In order to add the sums in (14), we decompose them as follows:

$$\begin{aligned}
\sum_{i=1}^{N+1}\sum_{j=N+1-i}^{2(N+1-i)} &= \sum_{i=1}^{N}\sum_{j=N+2-i}^{2(N-i)+1} + \sum_{i=1}^{N}\sum_{j=N+1-i} \\
&\quad + \sum_{i=1}^{N}\sum_{j=2(N+1-i)} + \sum_{i=N+1}\sum_{j=0}; \qquad (16)
\end{aligned}$$

$$\begin{aligned}
\sum_{i=0}^{N}\sum_{j=N-i+1}^{2(N-i)+1} &= \sum_{i=1}^{N}\sum_{j=N-i+2}^{2(N-i)+1} + \sum_{i=1}^{N}\sum_{j=N-i+1} \\
&\quad + \sum_{i=0}\sum_{j=N+2}^{2N+1} + \sum_{i=0}\sum_{j=N+1}; \qquad (17)
\end{aligned}$$

$$\begin{aligned}\sum_{i=0}^{N}\sum_{j=N-i+2}^{2(N-i)+2} &= \sum_{i=1}^{N}\sum_{j=N-i+2}^{2(N-i)+1}+\sum_{i=1}^{N}\sum_{j=2(N-i)+2}\\ &+\sum_{i=0}\sum_{j=N+2}^{2N+1}+\sum_{i=0}\sum_{j=2N+2}. \end{aligned} \tag{18}$$

Now, the sum in (14) can be rewritten as

$$\begin{aligned} &F^{(N+1)}\\ =\ & \sum_{i=1}^{N}\sum_{j=N+2-i}^{2(N-i)+1}\left\{(i-\frac{1}{2})a_{i-1,j}(N,x)-(j-\frac{1}{2})a_{i,j-1}(N,x)+xa_{i,j-2}(N,x)\right\}\\ &\qquad\times(1-t)^{-i}(1+t)^{-j}F\\ &+\sum_{i=1}^{N}\left\{(i-\frac{1}{2})a_{i-1,N-i+1}(N,x)-(N-i+\frac{1}{2})a_{i,N-i}(N,x)\right\}\\ &\qquad\times(1-t)^{-i}(1+t)^{-(N-i+1)}F\\ &+\sum_{i=1}^{N}\left\{(i-\frac{1}{2})a_{i-1,2(N+1-i)}(N,x)+xa_{i,2(N-i)}(N,x)\right\}(1-t)^{-i}(1+t)^{-2(N+1-i)}F\\ &+\sum_{j=N+2}^{2N+1}\left\{-(j-\frac{1}{2})a_{0,j-1}(N,x)+xa_{0,j-2}(N,x)\right\}(1+t)^{-j}F\\ &-(N+\frac{1}{2})a_{0,N}(N,x)(1+t)^{-(N+1)}F+xa_{0,2N}(N,x)(1+t)^{-(2N+2)}F\\ &+(N+\frac{1}{2})a_{N,0}(N,x)(1-t)^{-(N+1)}F. \end{aligned} \tag{19}$$

Comparing (15) and (19), we obtain: for $1\le i\le N$, $N-i+2\le j\le 2(N-i)+1$,

$$a_{i,j}(N+1,x)=(i-\frac{1}{2})a_{i-1,j}(N,x)-(j-\frac{1}{2})a_{i,j-1}(N,x)+xa_{i,j-2}(N,x); \tag{20}$$

for $1\le i\le N$,

$$a_{i,N-i+1}(N+1,x)=(i-\frac{1}{2})a_{i-1,N-i+1}(N,x)-(N-i+\frac{1}{2})a_{i,N-i}(N,x); \tag{21}$$

for $1\le i\le N$,

$$a_{i,2(N+1-i)}(N+1,x)=(i-\frac{1}{2})a_{i-1,2(N+1-i)}(N,x)+xa_{i,2(N-i)}(N,x); \tag{22}$$

for $N+2\le j\le 2N+1$,

$$a_{0,j}(N+1,x)=-(j-\frac{1}{2})a_{0,j-1}(N,x)+xa_{0,j-2}(N,x); \tag{23}$$

$$a_{0,N+1}(N+1,x)=-(N+\frac{1}{2})a_{0,N}(N,x); \tag{24}$$

$$a_{0,2N+2}(N+1,x)=xa_{0,2N}(N,x); \tag{25}$$

$$a_{N+1,0}(N+1,x)=(N+\frac{1}{2})a_{N,0}(N,x). \tag{26}$$

Note here that all of these recurrence relations can be merged into one relation (20), for $0\le i\le N+1$, $N-i+1\le j\le 2(N-i+1)$, with the understanding that

$$a_{i,j}(N,x)=0, \tag{27}$$

unless $0 \leq i \leq N,\ N-i \leq j \leq 2(N-i)$. In addition to these, we have the following initial conditions:

$$F = F^{(0)} = a_{0,0}(0,x)F \longrightarrow a_{0,0}(0,x) = 1, \tag{28}$$

$$\begin{aligned} F^{(1)} &= \left(\sum_{i=0}^{1} \sum_{j=1-i}^{2(1-i)} a_{i,j}(1,x)(1-t)^{-i}(1+t)^{-j} \right) F \\ &= \left(a_{0,1}(1,x)(1+t)^{-1} + a_{0,2}(1,x)(1+t)^{-2} + a_{1,0}(1,x)(1-t)^{-1} \right) F \\ &= \left(\frac{1}{2}(1-t)^{-1} - \frac{1}{2}(1+t)^{-1} + x(1+t)^{-2} \right) F \\ &\longrightarrow \ a_{1,0}(1,x) = \frac{1}{2},\ a_{0,1}(1,x) = -\frac{1}{2},\ a_{0,2}(1,x) = x. \end{aligned} \tag{29}$$

As easy consequences, from (24)-(26) we get

$$\begin{aligned} a_{N+1,0}(N+1,x) &= \left(N+\frac{1}{2}\right) a_{N,0}(N,x) \\ &= \left(N+\frac{1}{2}\right)\left(N-\frac{1}{2}\right) a_{N-1,0}(N-1,x) \\ &= \cdots \\ &= \left(N+\frac{1}{2}\right)\left(N-\frac{1}{2}\right) \cdots \frac{3}{2} a_{1,0}(1,x) \\ &= \left(\frac{1}{2}\right)^{N+1} (2N+1)!! \end{aligned} \tag{30}$$

$$\begin{aligned} a_{0,N+1}(N+1,x) &= -\left(N+\frac{1}{2}\right) a_{0,N}(N,x) \\ &= (-1)^2 \left(N+\frac{1}{2}\right)\left(N-\frac{1}{2}\right) a_{0,N-1}(N-1,x) \\ &= \cdots \\ &= (-1)^N \left(N+\frac{1}{2}\right)\left(N-\frac{1}{2}\right) \cdots \frac{3}{2} a_{0,1}(1,x) \\ &= \left(-\frac{1}{2}\right)^{N+1} (2N+1)!! \end{aligned} \tag{31}$$

$$\begin{aligned} a_{0,2N+2}(N+1,x) &= x a_{0,2N}(N,x) = x^2 a_{0,2(N-1)}(N-1,x) \\ &= x^N a_{0,2}(1,x) = x^{N+1} a_{0,0}(0,x) = x^{N+1}. \end{aligned} \tag{32}$$

Let $N+2 \leq j \leq 2N+1$. Then, from (23), we have

$$a_{0,j}(N+1,x) = x a_{0,j-2}(N,x) - \left(j-\frac{1}{2}\right) a_{0,j-1}(N,x). \tag{33}$$

For $j = N+2$, we get the following:

$$\begin{aligned} & a_{0,N+2}(N+1,x) \\ =\ & x a_{0,N}(N,x) - \left(N+\frac{3}{2}\right) a_{0,N+1}(N,x) \\ =\ & x a_{0,N}(N,x) - \left(N+\frac{3}{2}\right)\left(x a_{0,N-1}(N-1,x) - \left(N+\frac{1}{2}\right) a_{0,N}(N-1,x) \right) \\ =\ & x \left(a_{0,N}(N,x) - (N+\frac{3}{2}) a_{0,N-1}(N-1,x) \right) \\ & +(-1)^2 \left(N+\frac{3}{2}\right)\left(N+\frac{1}{2}\right)\left(x a_{0,N-2}(N-2,x) - \left(N-\frac{1}{2}\right) a_{0,N-1}(N-2,x) \right) \end{aligned}$$

$$
\begin{aligned}
&= \cdots \\
&= x\sum_{k=0}^{N-1}(-1)^k\left(N+\frac{3}{2}\right)\left(N+\frac{1}{2}\right)\cdots\left(N-k+\frac{5}{2}\right)a_{0,N-k}(N-k,x) \\
&\quad +(-1)^N\left(N+\frac{3}{2}\right)\left(N+\frac{1}{2}\right)\cdots\frac{5}{2}a_{0,2}(1,x) \\
&= x\sum_{k=0}^{N}\left(-\frac{1}{2}\right)^k(2N+3)(2N+1)\cdots(2N-2k+5)a_{0,N-k}(N-k,x) \\
&= x\sum_{k=0}^{N}\left(-\frac{1}{2}\right)^k\frac{(2N+3)!!}{(2N-2k+3)!!}a_{0,N-k}(N-k,x). \qquad (34)
\end{aligned}
$$

For $j=N+3$, we obtain the following:

$$
\begin{aligned}
&a_{0,N+3}(N+1,x) \\
&= xa_{0,N+1}(N,x)-\left(N+\frac{5}{2}\right)a_{0,N+2}(N,x) \\
&= xa_{0,N+1}(N,x)-\left(N+\frac{5}{2}\right)\left(xa_{0,N}(N-1,x)-\left(N+\frac{3}{2}\right)a_{0,N+1}(N-1,x)\right) \\
&= x\left(a_{0,N+1}(N,x)-\left(N+\frac{5}{2}\right)a_{0,N}(N-1,x)\right) \\
&\quad (-1)^2\left(N+\frac{5}{2}\right)\left(N+\frac{3}{2}\right)\left(xa_{0,N-1}(N-2,x)-\left(N+\frac{1}{2}\right)a_{0,N}(N-2,x)\right) \\
&= \cdots \\
&= x\sum_{k=0}^{N-2}(-1)^k\left(N+\frac{5}{2}\right)\left(N+\frac{3}{2}\right)\cdots\left(N-k+\frac{7}{2}\right)a_{0,n-k+1}(N-k,x) \\
&\quad +(-1)^{N-1}\left(N+\frac{5}{2}\right)\left(N+\frac{3}{2}\right)\cdots\frac{9}{2}a_{0,4}(2,x) \\
&= x\sum_{k=0}^{N-1}(-1)^k\left(N+\frac{5}{2}\right)\left(N+\frac{3}{2}\right)\cdots\left(N-k+\frac{7}{2}\right)a_{0,n-k+1}(N-k,x) \\
&= x\sum_{k=0}^{N-1}\left(-\frac{1}{2}\right)^k\frac{(2N+5)!!}{(2N-2k+5)!!}a_{0,N-k+1}(N-k,x). \qquad (35)
\end{aligned}
$$

Continuing this process, we can deduce that, for $N+2\le j\le 2N+1$,

$$a_{0,j}(N+1,x)=x\sum_{k=0}^{2N+2-j}\left(-\frac{1}{2}\right)^k\frac{(2j-1)!!}{(2j-2k-1)!!}a_{0,j-k-2}(N-k,x). \qquad (36)$$

Let $1\le i\le N$. Then, from (21), we have

$$a_{i,N-i+1}(N+1,x)=\left(i-\frac{1}{2}\right)a_{i-1,N-i+1}(N,x)-\left(N-i+\frac{1}{2}\right)a_{i,N-i}(N,x). \qquad (37)$$

For $i=1$, we obtain the following:

$$
\begin{aligned}
&a_{1,N}(N+1,x) \\
&= \frac{1}{2}a_{0,N}(N,x)-\left(N-\frac{1}{2}\right)a_{1,N-1}(N,x) \\
&= \frac{1}{2}a_{0,N}(N,x)-\left(N-\frac{1}{2}\right)\left(\frac{1}{2}a_{0,N-1}(N-1,x)-(N-\frac{3}{2})a_{1,N-2}(N-1,x)\right) \\
&= \frac{1}{2}\left(a_{0,N}(N,x)-\left(N-\frac{1}{2}\right)a_{0,N-1}(N-1,x)\right)
\end{aligned}
$$

$$
\begin{aligned}
& +(-1)^2\left(N-\frac{1}{2}\right)\left(N-\frac{3}{2}\right)\left(\frac{1}{2}a_{0,N-2}(N-2,x)-\left(N-\frac{5}{2}\right)a_{1,N-3}(N-2,x)\right)\\
= & \cdots\\
= & \frac{1}{2}\sum_{k=0}^{N-1}(-1)^k\left(N-\frac{1}{2}\right)\left(N-\frac{3}{2}\right)\cdots\left(N-\frac{2k-1}{2}\right)a_{0,N-k}(N-k,x)\\
& +(-1)^N\left(N-\frac{1}{2}\right)\cdots\frac{1}{2}a_{1,0}(1,x)\\
= & \frac{1}{2}\sum_{k=0}^{N}(-1)^k\left(N-\frac{1}{2}\right)\left(N-\frac{3}{2}\right)\cdots\left(N-\frac{2k-1}{2}\right)a_{0,N-k}(N-k,x)\\
= & \frac{1}{2}\sum_{k=0}^{N}\left(-\frac{1}{2}\right)^k\frac{(2N-1)!!}{(2N-2k-1)!!}a_{0,N-k}(N-k,x). \qquad (38)
\end{aligned}
$$

For $i=2$, we get the following:

$$
\begin{aligned}
& a_{2,N-1}(N+1,x)\\
= & \frac{3}{2}a_{1,N-1}(N,x)-\left(N-\frac{3}{2}\right)a_{2,N-2}(N,x)\\
= & \frac{3}{2}a_{1,N-1}(N,x)-\left(N-\frac{3}{2}\right)\left(\frac{3}{2}a_{1,N-2}(N-1,x)-\left(N-\frac{5}{2}\right)a_{2,N-3}(N-1,x)\right)\\
= & \frac{3}{2}\left(a_{1,N-1}(N,x)-\left(N-\frac{3}{2}\right)a_{1,N-2}(N-1,x)\right)\\
& +(-1)^2\left(N-\frac{3}{2}\right)\left(N-\frac{5}{2}\right)\left(\frac{3}{2}a_{1,N-3}(N-2,x)-\left(N-\frac{7}{2}\right)a_{2,N-4}(N-2,x)\right)\\
= & \cdots\\
= & \frac{3}{2}\sum_{k=0}^{N-2}(-1)^k\left(N-\frac{3}{2}\right)\left(N-\frac{5}{2}\right)\cdots\left(N-\frac{2k+1}{2}\right)a_{1,N-k-1}(N-k,x)\\
& +(-1)^{N-1}\left(N-\frac{3}{2}\right)\left(N-\frac{5}{2}\right)\cdots\frac{1}{2}a_{2,0}(2,x)\\
= & \frac{3}{2}\sum_{k=0}^{N-1}(-1)^k\left(N-\frac{3}{2}\right)\left(N-\frac{5}{2}\right)\cdots\left(N-\frac{2k+1}{2}\right)a_{1,N-k-1}(N-k,x)\\
= & \frac{3}{2}\sum_{k=0}^{N-1}\left(-\frac{1}{2}\right)^k\frac{(2N-3)!!}{(2N-2k-3)!!}a_{1,N-k-1}(N-k,x). \qquad (39)
\end{aligned}
$$

Continuing this process, we can deduce that, for $1\le i\le N$,

$$
\begin{aligned}
& a_{i,N-i+1}(N+1,x)\\
= & \frac{2i-1}{2}\sum_{k=0}^{N-i+1}\left(-\frac{1}{2}\right)^k\frac{(2N-2i+1)!!}{(2N-2k-2i+1)!!}a_{i-1,N-k-i+1}(N-k,x). \qquad (40)
\end{aligned}
$$

Let $1\le i\le N$. Then, from (22), we have

$$
\begin{aligned}
& a_{i,2(N+1-i)}(N+1,x)\\
= & \left(i-\frac{1}{2}\right)a_{i-1,2(N+1-i)}(N,x)+xa_{i,2(N-i)}(N,x). \qquad (41)
\end{aligned}
$$

Then, proceeding analogously to the case of (37), we can deduce that, for $1\le i\le N$,

$$
a_{i,2(N+1-i)}(N+1)=\frac{2i-1}{2}\sum_{k=0}^{N-i+1}x^k a_{i-1,2(N-k-i+1)}(N-k,x), \qquad (42)
$$

For $1 \leq i \leq N,\ N-i+2 \leq j \leq 2(N-i)+1$, from (20) we have

$$\begin{aligned} & a_{i,j}(N+1,x) \\ = & \left(i-\frac{1}{2}\right) a_{i-1,j}(N,x) - \left(j-\frac{1}{2}\right) a_{i,j-1}(N,x) + x a_{i,j-2}(N,x). \end{aligned} \tag{43}$$

Let $i=1$, Then, with $N+1 \leq j \leq 2N-1$, (43) becomes

$$a_{1,j}(N+1,x) = \frac{1}{2} a_{0,j}(N,x) + x a_{1,j-2}(N,x) - \left(j-\frac{1}{2}\right) a_{1,j-1}(N,x). \tag{44}$$

For $j = N+1$, we get the following:

$$\begin{aligned} & a_{1,N+1}(N+1,x) \\ = & \frac{1}{2} a_{0,N+1}(N,x) + x a_{1,N-1}(N,x) - \left(N+\frac{1}{2}\right) a_{1,N}(N,x) \\ = & \frac{1}{2} a_{0,N+1}(N,x) + x a_{1,N-1}(N,x) \\ & - \left(N+\frac{1}{2}\right)\left(\frac{1}{2} a_{0,N}(N-1,x) + x a_{1,N-2}(N-1,x) - \left(N-\frac{1}{2}\right) a_{1,N-1}(N-1,x)\right) \\ = & \frac{1}{2}\left(a_{0,N+1}(N,x) - \left(N+\frac{1}{2}\right) a_{0,N}(N-1,x)\right) \\ & + x\left(a_{1,N-1}(N,x) - \left(N+\frac{1}{2}\right) a_{1,N-2}(N-1,x)\right) + (-1)^2\left(N+\frac{1}{2}\right)\left(N-\frac{1}{2}\right) \\ & \times\left(\frac{1}{2} a_{0,N-1}(N-2,x) + x a_{1,N-3}(N-2,x) - \left(N-\frac{3}{2}\right) a_{1,N-2}(N-2,x)\right) \\ & = \cdots \\ & = \frac{1}{2}\sum_{k=0}^{N-2}(-1)^k\left(N+\frac{1}{2}\right)\left(N-\frac{1}{2}\right)\cdots\left(N-\frac{2k-3}{2}\right) a_{0,N-k+1}(N-k,x) \\ & + x\sum_{k=0}^{N-2}(-1)^k\left(N+\frac{1}{2}\right)\left(N-\frac{1}{2}\right)\cdots\left(N-\frac{2k-3}{2}\right) a_{1,N-k-1}(N-k,x) \\ & + (-1)^{N-1}\left(N+\frac{1}{2}\right)\left(N-\frac{1}{2}\right)\cdots\left(\frac{5}{2}\right) a_{1,2}(2,x) \\ & = \sum_{k=0}^{N-1}\left(-\frac{1}{2}\right)^k \frac{(2N+1)!!}{(2N-2k+1)!!}\left(\frac{1}{2} a_{0,N-k+1}(N-k,x) + x a_{1,N-k-1}(N-k,x)\right). \end{aligned} \tag{45}$$

For $j = N+2$, we obtain the following:

$$\begin{aligned} & a_{1,N+2}(N+1,x) \\ = & \frac{1}{2} a_{0,N+2}(N,x) + x a_{1,N}(N,x) - \left(N+\frac{3}{2}\right) a_{1,N+1}(N,x) \\ & = \frac{1}{2} a_{0,N+2}(N,x) + x a_{1,N}(N,x) \\ & - \left(N+\frac{3}{2}\right)\left(\frac{1}{2} a_{0,N+1}(N-1,x) + x a_{1,N-1}(N-1,x) - \left(N+\frac{1}{2}\right) a_{1,N}(N-1,x)\right) \\ & = \frac{1}{2}\left(a_{0,N+2}(N,x) - \left(N+\frac{3}{2}\right) a_{0,N+1}(N-1,x)\right) \\ & + x\left(a_{1,N}(N,x) - \left(N+\frac{3}{2}\right) a_{1,N-1}(N-1,x)\right) \\ & + (-1)^2\left(N+\frac{3}{2}\right)\left(N+\frac{1}{2}\right) \end{aligned}$$

$$\begin{aligned}
&\times\left(\frac{1}{2}a_{0,N}(N-2,x)+xa_{1,N-2}(N-2,x)-\left(N-\frac{1}{2}\right)a_{1,N-1}(N-2,x)\right)\\
&=\cdots\\
&=\frac{1}{2}\sum_{k=0}^{N-3}(-1)^k\left(N+\frac{3}{2}\right)\left(N+\frac{1}{2}\right)\cdots\left(N-\frac{2k-5}{2}\right)a_{0,N-k+2}(N-k,x)\\
&\quad+x\sum_{k=0}^{N-3}(-1)^k\left(N+\frac{3}{2}\right)\left(N+\frac{1}{2}\right)\cdots\left(N-\frac{2k-5}{2}\right)a_{1,N-k}(N-k,x)\\
&\quad+(-1)^{N-2}\left(N+\frac{3}{2}\right)\left(N+\frac{1}{2}\right)\cdots\frac{9}{2}a_{1,4}(3,x)\\
&=\sum_{k=0}^{N-2}\left(-\frac{1}{2}\right)^k\frac{(2N+3)!!}{(2N-2k+3)!!}\left(\frac{1}{2}a_{0,N-k+2}(N-k,x)+xa_{1,N-k}(N-k,x)\right). \qquad (46)
\end{aligned}$$

Continuing this process, we can deduce that, for $N+1\le j\le 2N-1$,

$$\begin{aligned}
&a_{1,j}(N+1,x)\\
=&\sum_{k=0}^{2N-j}\left(-\frac{1}{2}\right)^k\frac{(2j-1)!!}{(2j-2k-1)!!}\left(\frac{1}{2}a_{0,j-k}(N-k)+xa_{1,j-k-2}(N-k,x)\right). \qquad (47)
\end{aligned}$$

Let $i=2$. Then, with $N\le j\le 2N-3$, (43) becomes

$$\begin{aligned}
&a_{2,j}(N+1,x)\\
=&\ \frac{3}{2}a_{1,j}(N,x)+xa_{2,j-2}(N,x)-\left(j-\frac{1}{2}\right)a_{2,j-1}(N,x). \qquad (48)
\end{aligned}$$

Then, proceeding analogously to the case of (44), we can deduce that, for $N\le j\le 2N-3$,

$$\begin{aligned}
&a_{2,j}(N+1,x)\\
=&\sum_{k=0}^{2N-j-2}\left(-\frac{1}{2}\right)^k\frac{(2j-1)!!}{(2j-2k-1)!!}\left(\frac{3}{2}a_{1,j-k}(N-k,x)+xa_{2,j-k-2}(N-k,x)\right) \qquad (49)
\end{aligned}$$

Thus we can deduce that, for $1\le i\le N$, $N-i+2\le j\le 2(N-i)+1$,

$$\begin{aligned}
&a_{i,j}(N+1,x)\\
=&\sum_{k=0}^{2N-j-2i+2}\left(-\frac{1}{2}\right)^k\frac{(2j-1)!!}{(2j-2k-1)!!}\\
&\times\left(\frac{2i-1}{2}a_{i-1,j-k}(N-k,x)+xa_{i,j-k-2}(N-k,x)\right) \qquad (50)
\end{aligned}$$

Our results can be summarized as:

$$\begin{aligned}
&a_{0,0}(0,x)=1;\\
&a_{N+1,0}(N+1,x)=\left(-\frac{1}{2}\right)^{N+1}(2N+1)!!;\\
&a_{0,N+1}(N+1,x)=\left(-\frac{1}{2}\right)^{N+1}(2N+1)!!;\\
&a_{0,2N+2}(N+1,x)=x^{N+1};\\
&a_{0,j}(N+1,x)=x\sum_{k=0}^{2N+2-j}\left(-\frac{1}{2}\right)^k\frac{(2j-1)!!}{(2j-2k-1)!!}a_{0,j-k-2}(N-k,x)\\
&\quad\text{for } N+2\le j\le 2N+1;
\end{aligned}$$

$$a_{i,N-i+1}(N+1,x) = \frac{2i-1}{2}\sum_{k=0}^{N-i+1}\left(-\frac{1}{2}\right)^k \frac{(2N-2i+1)!!}{(2N-2k-2i+1)!!}a_{i-1,N-k-i+1}(N-k,x)$$

for $1 \le i \le N$;

$$a_{i,2(N+1-i)}(N+1,x) = \frac{2i-1}{2}\sum_{k=0}^{N-i+1} x^k a_{i-1,2(N-k-i+1)}(N-k,x),$$

for $1 \le i \le N$;

$$a_{i,j}(N+1,x) = \sum_{k=0}^{2N-j-2i+2}\left(-\frac{1}{2}\right)^k \frac{(2j-1)!!}{(2j-2k-1)!!}\left(\frac{2i-1}{2}a_{i-1,j-k}(N-k,x) + x a_{i,j-k-2}(N-k,x)\right),$$

for $1 \le i \le N, N-i+2 \le j \le 2(N-i)+1$. (51)

From these, we can conclude that, for $0 \le i \le N+1$, $N+1-i \le j \le 2(N+1-i)$,

$$a_{i,j}(N+1,x) = \sum_{k=0}^{2N-j-2i+2}\left(-\frac{1}{2}\right)^k \frac{(2j-1)!!}{(2j-2k-1)!!} \times \left(\frac{2i-1}{2}a_{i-1,j-k}(N-k,x) + x a_{i,j-k-2}(N-k,x)\right), \quad (52)$$

with $a_{0,0}(0,x) = 1$, $a_{1,0}(1,x) = \frac{1}{2}$, $a_{0,1}(1,x) = -\frac{1}{2}$, $a_{0,2}(1,x) = x$, except for $i = 0$ and $j = N+1$, in which case

$$a_{0,N+1}(N+1,x) = \left(-\frac{1}{2}\right)^{N+1}(2N+1)!!. \quad (53)$$

Our results can now be stated as the following theorem.

Theorem 1. The ordinary differential equations

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F = \left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} a_{i,j}(N,x)(1-t)^{-i}(1+t)^{-j}\right)F, \quad (54)$$

$(N = 0, 1, 2, \cdots,)$ have a solution $F = F(t,x) = (1-t^2)^{-\frac{1}{2}}e^{x\left(\frac{t}{1+t}\right)}$, where, for $0 \le i \le N$, $N-i \le j \le 2(N-i)$,

$$a_{i,j}(N,x) = \sum_{k=0}^{2N-j-2i}\left(-\frac{1}{2}\right)^k \frac{(2j-1)!!}{(2j-2k-1)!!} \times \left(\frac{2i-1}{2}a_{i-1,j-k}(N-k-1,x) + x a_{i,j-k-2}(N-k-1,x)\right), \quad (55)$$

with $a_{0,0}(0,x) = 1$, $a_{1,0}(1,x) = \frac{1}{2}$, $a_{0,1}(1,x) = -\frac{1}{2}$, $a_{0,2}(1,x) = x$, except for $i = 0$ and $j = N$, in which case

$$a_{0,N}(N,x) = \left(-\frac{1}{2}\right)^N (2N-1)!!. \quad (56)$$

3. Applications of differential equations

We recall from (9) that the squared Hermite polynomials $SH_k(x)$ are given by the generating function

$$F = F(t;x) = (1-t^2)^{-\frac{1}{2}} e^{\left(\frac{t}{1+t}\right)} = \sum_{k=0}^{\infty} SH_k(x)\frac{t^k}{k!}. \tag{57}$$

Here we derive some new and explicit identities for the squared Hermite polynomials from the differential equations in Theorem 1. Now, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} SH_{k+N}(x)\frac{t^k}{k!} &= \left(\sum_{k=0}^{\infty} SH_k(x)\frac{t^k}{k!}\right)^{(N)} \\
&= \left((1-t^2)^{-\frac{1}{2}} e^{x\left(\frac{t}{1+t}\right)}\right)^{(N)} \\
&= \left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} a_{i,j}(N,x)(1-t)^{-i}(1+t)^{-j}\right) F \\
&= \sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)} a_{i,j}(N,x)\sum_{l=0}^{\infty}(i+l-1)_l\frac{t^l}{l!} \\
&\quad \times \sum_{m=0}^{\infty}(-1)^m(j+m-1)_m\frac{t^m}{m!}\sum_{n=0}^{\infty}SH_n(x)\frac{t^n}{n!} \\
&= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)}\sum_{l+m+n=k}\binom{k}{l,m,n}\right. \\
&\quad \left.\times (-1)^m(i+l-1)_l(j+m-1)_m a_{i,j}(N,x)SH_n(x)\right)\frac{t^k}{k!}.
\end{aligned} \tag{58}$$

From this, we have, for $k, N = 0, 1, 2, \cdots$

$$\begin{aligned}
SH_{k+N}(x) &= \sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)}\sum_{l+m+n=k}\binom{k}{l,m,n} \\
&\quad \times(-1)^m(i+l-1)_l(j+m-1)_m a_{i,j}(N,x)SH_n(x).
\end{aligned} \tag{59}$$

Thus we obtain the following theorem.

Theorem 2. For $k, N = 0, 1, 2, \cdots$

$$\begin{aligned}
SH_{k+N}(x) &= \sum_{i=0}^{N}\sum_{j=N-i}^{2(N-i)}\sum_{l+m+n=k}\binom{k}{l,m,n} \\
&\quad \times(-1)^m(i+l-1)_l(j+m-1)_m a_{i,j}(N,x)SH_n(x),
\end{aligned}$$

where $a_{i,j}(N,x)$ are as in Theorem 1.

Letting $k = 0$ in (59), we obtain the following result giving expressions for the squared Hermite polynomials $SH_N(x)$.

Theorem 3. For $N = 0, 1, 2, \cdots$

$$SH_N(x) = \sum_{i=0}^{N} \sum_{j=N-i}^{2(N-i)} a_{i,j}(N, x),$$

where $a_{i,j}(N, x)$ are as in Theorem 1.

References

[1] G. Arfken, *Mathematical Methods for Physicists 3rd ed.*, Orlando, FL: Academic Press, 1985.

[2] O. S. Berlyand, R.I. Gavrilova, A.P. Prudnikov, *Tables of integral error functions and Hermite polynomials. Translated by Prasenjit Basu.* A Pergamon Press Book The Macmillan Co., New York 1962. vi+163 pp.

[3] Erdelyi, A. (ed.), *Higher Transcendental Function,* The Bateman Manuscript Project. Vol. III, McGraw-Hill, New York, 1953.

[4] Y. Kakizawa, *Some integrals involving multivariate Hermite polynomials: Application to evaluating higher-order local powers,* Statist. Probab. Lett. 110 (2016), 162-168.

[5] D. S. Kim, T. Kim, *Some identities for Bernoulli numbers of the second kind arising from a non-linear differential equation* B. Korean Math. Soc. 52(2015), no. 6, 2001–2010.

[6] D. S. Kim, T. Kim, *A note on the Hermite numbers and polynomials.* Math. Inequal. Appl. 16 (2013), no. 4, 1115-1122.

[7] D. S. Kim, T. Kim. S.-H. Rim, D. V. Dolgy, *Barnes' multiple Bernoulli and Hermite mixed-type polynomials.* Proc. Jangjeon Math. Soc. 18 (2015), no. 1, 7-19.

[8] T. Kim, T. Mansour, *Umbral calculus associated with Frobenius-type Eulerian polynomials,* Russ. J. Math. Phys. 21 (2014), no. 4, 484-493.

[9] T. Kim, D. S. Kim, *Identities involving degenerate Euler numbers and polynomials arising from non-linear differential equations,* J. Nonlinear Sci. Appl. 9(2016), 2086-2098.

[10] T. Kim, D. S. Kim, *A note on Changhee differential equations,* Russ. J. Math. Phys., to appear.

[11] T. Kim, D. S. Kim, T. Mansour, J.-j. Seo, *Linear differential equations for families of polynomials,* J. Ineq. Appl. (2016), 2016:95.

[12] J.I. Marcum, *Tables of Hermite polynomials and the derivatives of the error function, Report P-90,* The RAND Corporation, Santa Monica, Calif., (1948).

[13] S. Roman, *The umbral calculus*, Pure and Applied Mathematics, vol. 111, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984.

[14] V.V. Shustov, *Approximation of functions by asymmetric two-point Hermite polynomials and its optimization,*, Comput. Math. Math. Phys. 55 (2015), no. 12, 1960-1974.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.7, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Quenching for the discrete heat equation with a singular absorption term on finite graphs

Qiao Xin[a,*], Dengming Liu[b]

[a] *College of Mathematics and Statistics, Yili Normal University, Yining Xinjiang, 835000, P. R. China*

[b] *School of Mathematics and Computational Science, Hunan University of Science and Technology, Xiangtan, 411201 P. R. China*

Abstract

We study the quenching for the discrete semi-linear heat equation with singular absorption $u_t = \Delta_\omega u - \lambda u^{-p}$ on finite graph with Dirichlet boundary condition and the positive initial condition $u_0(x)$. When $\lambda^{-p} \geq \max_{x\in S} u_0(x)$, we prove that the solution will quench in finite time by comparison principal. Meanwhile, we study the quenching rate. Moreover, we also prove that there exists a critical exponent λ^* such that the problem admits a global solution for all $\lambda \leq \lambda^*$. Finally, a numerical experiment on two finite graphs is given to illustrate our results.

Keywords: Discrete heat equation; singular absorption; quenching; graphs.
MSC: 35B05, 35B33, 45G05

1 Introduction

Let G be a graph with vertex set V and edge set E, where the vertex set is divided into the boundary vertices ∂S and the interior vertices S which is connected, and we always assume G is a finite, connected, simple (without multiple edges and loops) graph in the following context. In this paper, we mainly study the quenching phenomena for the following semi-linear discrete heat equation with singular absorption on finite graph

*Corresponding author. E-mail address:xinqiaoylsy@163.com(Qiao Xin).

G

$$\begin{cases} u_t = \Delta_\omega u - \lambda u^{-p}, & x \in S \text{ and } t \in (0,T), \\ u(x,t) = 1, & x \in \partial S \text{ and } t \in (0,T), \\ u(x,0) = u_0(x), & x \in S, \end{cases} \tag{1}$$

here p, λ and T are positive constants, the initial value $u_0(x) \in C(V)$ and satisfies $0 < u_0(x) \leq 1$ for any $x \in S$. The function space $C(V)$ denotes the set of all functions which are definite on the vertices V of the graph G, and Δ_ω denotes the discrete Laplacian operator on finite graph, which is defined as follows (see [1]),

$$\Delta_\omega u(x) = \sum_{y \in V} [u(y) - u(x)] \cdot \omega(x,y),$$

where the function $\omega(x,y)$ is called the weighted function, and satisfies

$$\begin{aligned} &(i)\ \omega(x,x) = 0, \text{ for any } x \in V, \\ &(ii)\ \omega(x,y) = \omega(y,x) \geq 0, \text{ for any } x, y \in V, \\ &(iii)\ \omega(x,y) = 0, \text{ if and only if } (x,y) \notin E. \end{aligned}$$

Moreover, $d_\omega(x) = \sum\limits_{x \in V} \omega(x,y)$ denotes the degree of the node $x \in V$ of the weighted graph G, and we assume that $d_\omega(x) \leq 1$ for any $x \in S$.

By introducing $v(x,t) = 1 - u(x,t)$, it is not difficult to verify that the function $v(x,t)$ satisfies the following initial boundary value problem

$$\begin{cases} v_t = \Delta_\omega v + \lambda (1-v)^{-p}, & x \in S \text{ and } t \in (0,T), \\ v(x,t) = 0, & x \in \partial S \text{ and } t \in (0,T), \\ v(x,0) = 1 - u_0(x), & x \in S. \end{cases} \tag{2}$$

In the continuous case including the local and nonlocal diffusion equation likes (1) or (2), its quenching phenomena has attracted much attention from the work of H. Kawarada [2] in 1975. This type of the diffusion equation with a singular absorption term (or a reaction term) comes form the polarization phenomena in ionic conductors [2], and can be considered as a limiting case of models in chemical catalyst kinetics or models of in enzyme kinetics [4, 5, 3, 6]. The detailed researches on the quenching phenomena can be found in [9, 6, 7, 8] and the references therein. Especially, for the nonlinear diffusion equation

$$u_t - u_{xx} = -u^{-p}, -l < x < l$$

with non-homogeneous Dirichlet boundary condition and the positive initial value, its quenching occurs in finite time for sufficiently large l in[2, 7]. Moreover, the quenching of the semilinear parabolic equation

$$u_t - \Delta u = g(u)$$

with homogeneous Dirichlet boundary condition and the positive initial value was also studied, the readers can refer to [10, 11]. On the other hand, the authors of [9] considered the quenching behaviour of the following nonlocal diffusion equation

$$u_t = J * u - u - \lambda u^{-p},$$

the critical parameter λ^* and the quenching rate and the quenching set were also given.

Recently, the ω-harmonic function and the $\omega-$heat equation were considered by many authors since the discrete heat equation has been widely applied to the fields of heat and energy transfer, electrical networks, image processing and so on [1, 12, 13]. In [14], Y.S. Chung, Y.S. Lee et.al considered the extinction and positivity of the discrete heat equation with absorption on network

$$u_t = \Delta_\omega u - u^p,$$

where $p > 0$. Furthermore, the extinction and positivity for the p, ω-heat equation with absorption was also studied in [16, 15]. Blow-up for the $\omega-$heat equation with a reaction term on graphs

$$u_t = \Delta_\omega u + \lambda u^p,$$

where $p > 0$ was researched in [17, 18]. The asymptotic behavior of solutions for the ω-heat equation with reaction and absorption term was considered in [19].

Motivated by the above works, the purpose of this paper is to discuss the quenching phenomenons for the discrete heat equation with singular absorption term and the non-homonomous Dirichlet boundary conditions. The local existence and uniqueness of solutions are obtained in the next section. In the third section, we will show the comparison principal for the discrete heat equation (1). The sufficient conditions on quenching and quenching rate are proved in the section 4. In the section 5, we mainly discuss the existence of the global solution. In the last section, we give some numerical experiments to illustrate our results.

2 Local existence and uniqueness of solutions

Lemma 2.1 *Suppose $0 < u_0(x) \leq 1$, then, there exists a unique solution $u \in C[0, T) \times C(V)$ for the problem (1). Moreover, if T is finite, then*

$$\lim_{t \to T^-} u(x, t) = 0 \tag{3}$$

for some $x \in S$.

Proof. Since $0 < u_0(x) \leq 1$, there exists a positive constant ε, such that $2\varepsilon < u_0(x) \leq 1$. Set

$$X_0 = \{u \in C[0,t_0] \times C(V), \varepsilon \leq u \leq K \text{ and } u(x) \equiv 1 \text{ for any } x \in \partial S\},$$

where $K > 1$ and

$$t_0 < \min\left\{\frac{K-1}{K}, \frac{\varepsilon}{K+\lambda\varepsilon^{-p}}, \frac{1}{2+\lambda p\varepsilon^{-p-1}}\right\}. \tag{4}$$

Now, we define the operator as follows:

$$T_{u_0}[u](x,t) = \begin{cases} u_0(x) + \int_0^t \Delta_\omega u(x,s)ds - \lambda \int_0^t u^{-p}(x,s)ds, & x \in S, 0 \leq t \leq t_0, \\ 1, & x \in \partial S, 0 \leq t \leq t_0, \end{cases}$$

and the norm of the Banach space X_0

$$\|u(x,t)\|_{X_0} = \max_{x\in V} \max_{t\in[0,t_0]} |u(x,t)|$$

for any $u(x,t) \in X_0$.

First, we prove that the operator T_{u_0} maps X_0 into X_0. It is easy to verify that $T_{u_0}[u](x,t)$ is continuous about the time t for any fixed node $x \in V$. On the other hand, for any $u(x,t) \in X_0$, we have

$$T_{u_0}[u](x,t) \geq 2\varepsilon - (K + \lambda\varepsilon^{-p})t_0 \geq \varepsilon, \tag{5}$$

moreover, we also have

$$T_{u_0}[u](x,t) \leq 1 + Kt_0 = K(\frac{1}{K} + t_0) \leq K. \tag{6}$$

Next, we show that T_{u_0} is a strict contraction in X_0. That is to say, for any $u, v \in X_0$, we get

$$\begin{aligned} \|u - v\|_{X_0} \leq & \left\| \int_0^t \sum_{y\in V} [u(y,s) - v(y,s)]\, \omega(x,y)ds \right\|_{X_0} \\ & + \left\| \int_0^t [u(x,s) - v(x,s)]ds \right\|_{X_0} + \lambda \left\| \int_0^t [v^{-p}(x,s) - u^{-p}(x,s)]ds \right\|_{X_0} \\ \leq & \; 2t_0\|u - v\|_{X_0} + \lambda p \left\| \int_0^t |\xi|^{-p-1}|u(x,s) - v(x,s)|ds \right\|_{X_0} \\ \leq & \; t_0(2 + \lambda p\varepsilon^{-p-1})\|u - v\|_{X_0} < \|u - v\|_{X_0}. \end{aligned}$$

Hence, by Banach fixed point theorem, there exists a unique $u \in X_0$ such that $u = T_{u_0(x)}[u]$, so, for any $x \in S$, we have

$$u(x,t) = \begin{cases} u_0(x) + \int_0^t \Delta_\omega u(x,s)ds - \lambda \int_0^t u^{-p}(x,s)ds, & x \in S \\ 1, & x \in \partial S, \end{cases} \tag{7}$$

thus, we can get $u(x,t)$ is the unique solution to the problem (1) in $t \in [0,t_0]$. Now, if $u(x,t_0) > 0$, we can continue the above procedure, and then, the solution can be extend to the time interval $[t_0,t_1]$. This procedure can be continued again and again until $\lim_{t\to T^-} u(x,t) \to 0$ for some time T which may be infinite.

3 Comparison principle

In this section, we mainly show a comparison principal. To do this, we begin with the definition of the super-solution and sub-solution to the problem (1).

Definition 3.1 *A function $\overline{u} \in C(V) \times C[0,T)$ is a super-solution to the problem* (1) *if $\overline{u}$ is a positive function and satisfies*

$$\begin{cases} \overline{u}_t \geq \Delta_\omega \overline{u} - \lambda \overline{u}^{-p}, & x \in S \text{ and } t \in (0,T), \\ \overline{u}(x,t) \geq 0, & x \in \partial S \text{ and } t \in (0,T), \\ \overline{u}(x,0) \geq u_0(x), & x \in S, \end{cases} \tag{8}$$

Analogously, we say that $\underline{u} \in C(V) \times C[0,T)$ is a sub-solution if it satisfies the reverses above inequalities.

Now, we have the following comparison principle.

Theorem 3.1 ***(Comparison principle)*** *Suppose $\overline{u}$ and $\underline{u}$ be a super-solution and a sub-solution to the problem* (1.1)*, respectively, then $\overline{u} \geq \underline{u}$ in $(x,t) \in V \times [0,T)$.*

Proof. For any $0 < t_0 < T$, set $m = \min\limits_{S\times[0,t_0]} \{\overline{u}, \underline{u}\}$ and $M = \max\limits_{S\times[0,t_0]} \{\overline{u}, \underline{u}\}$, thus, we know that m, M are the positive constants. And then, suppose $v(x,t) = \underline{u} - \overline{u}$. Notice that $v(x,0) > 0$ for any $x \in S$. By the definitions of the super-solution and the sub-solution, we can get

$$v_t \geq \Delta_\omega v - \lambda(\underline{u}^{-p} - \overline{u}^{-p}), \tag{9}$$

let $v^+(x,t) = \max\{v(x,t), 0\} \geq 0$. Thus, multiplying v^+ both sides of the above inequality, and integrating on S, we obtain

$$\begin{aligned} &\frac{1}{2}\left(\int_{x\in S} (v^+(x,t))^2\right)_t \\ &\leq \int_{x\in S} \Delta_\omega v(x,t) v^+(x,t) + \int_{x\in S} (\underline{u}^{p(x)} - \overline{u}^{p(x)}) v^+(x,t), \end{aligned} \tag{10}$$

For the first term of the right part of the above inequality, we have

$$\int_{x\in S} \Delta_\omega v(x,t) v^+(x,t) \leq 0. \tag{11}$$

In fact, let $J(t) = \{x \in V : v(x,t) > 0\}$, if $J(t)$ is empty set, we have the desired results. Now, assume $J(t)$ is not an empty set. Due to $\underline{u}(x,t) \leq 0$, $\overline{u}(x,t) \geq 0$ for any $x \in \partial S$ and $0 \leq t \leq t_0$, so $v(x,t) = \underline{u}(x,t) - \overline{u}(x,t) \leq 0$ for any $x \in \partial S$ and $0 \leq t \leq t_0$.

Now, we get $J(t) \subset S$. Thus, if $x \in J(t)$ and $y \in V \setminus J(t)$, we have $v(x,t) > 0$ and $v(y,t) - v(x,t) < 0$, hence, we have

$$\sum_{x\in J(t)} \sum_{y\in V\setminus J(t)} v(x,t)[v(y,t) - v(x,t)]\omega(x,y) < 0.$$

Furthermore, we get

$$\begin{aligned}
&\sum_{x\in S}\sum_{y\in V} v^+(x,t)[v(y,t) - v(x,t)]\omega(x,y)\\
&= \sum_{x\in J(t)} \sum_{y\in J(t)} v(x,t)[v(y,t) - v(x,t)]\omega(x,y)\\
&+ \sum_{x\in J(t)} \sum_{y\in V\setminus J(t)} v(x,t)[v(y,t) - v(x,t)]\omega(x,y)\\
&= -\frac{1}{2} \sum_{x\in J(t)} \sum_{y\in J(t)} [v(y,t) - v(x,t)]^2\omega(x,y)\\
&+ \sum_{x\in J(t)} \sum_{y\in V\setminus J(t)} v(x,t)[v(y,t) - v(x,t)]\omega(x,y) < 0.
\end{aligned} \tag{12}$$

On the other hand, for any fixed $x \in S$, by mean value theorem, we have

$$\underline{u}^{-p}(x,t) - \overline{u}^{-p}(x,t) = -p\xi^{-p-1}(x,t)v(x,t),$$

where $\xi(x,t) = \theta(x)\underline{u}(x,t) + (1-\theta(x))\overline{u}(x,t)$ and $0 \le \theta(x) \le 1$. And then, we have $m \le \xi(x,t) \le M$. Thus, for the second term of the right part of the inequality (10), we also have

$$\int_{x\in S} (\underline{u}^{p(x)} - \bar{u}^{p(x)})v^+(x,t) \le -m^{-p-1} \int_{x\in S} (v^+(x,t))^2. \tag{13}$$

Combine the inequalities (10), (12) and (13), we obtain

$$\left(\int_{x\in V} (v^+(x,t))^2\right)_t < 0. \tag{14}$$

There exists a contradiction. Hence $J(t) = \emptyset$. By the arbitrariness of t_0, we obtain $\overline{u}(x,t) \ge \underline{u}(x,t)$, for $(x,t) \in V \times [0,T)$.

4 Quenching phenomena and quenching rate

In this section, similar to the method used in [9], we mainly propose the quenching condition and quenching rate. Before the discussions and proofs, we firstly give some notes about the initial value condition and also the boundary condition. Since the absorption term is singular at points which satisfy $u(x) = 0$, we need the initial value $u_0(x) > 0$. Moreover, if $\max_{x\in S} u_0(x) > 1$, we can set

$$U(t) = (\lambda p)^{\frac{1}{p+1}}(A-t)^{\frac{1}{p+1}},$$

where $A = \max_{x\in S} u_0(x)$, and then, it is easy to verify that $U(t)$ is a super-solution to the discrete diffusion equation (1) when $U(t) \geq 1$. Thus, by the comparison principle, there exists t_0 such that $1 \geq U(t_0) \geq u(x, t_0)$. Hence, we can discuss the quenching pheromone to the problem (1) with the large initial value beginning with the initial time time $t = t_0$. The following proof can be similarly done. Finally, if we choose the homogenous Dirichlet boundary condition, i.e. set $u(x, t) = 0$ for any $x \in \partial S$, and then, we can also get $U(t)$ is also a super-solution to the problem (1) for any $t < A$, and then, we have $u(x, t)$ always quenches in finite time, i.e. the solution to the problem (1) is not global.

Next, we give the proof of the quenching phenomena about the problem (1), we mainly have the following two results.

Theorem 4.1 *If the initial value $u_0(x)$ satisfies that*

$$\max_{x\in S} u_0(x) \leq \lambda^{\frac{1}{p}} < 1, \tag{15}$$

and then, the solution to the problem (1) *quenches in finite time T.*

Proof. It is easy to verify that

$$v(x,t) = \begin{cases} \lambda^{\frac{1}{p}}, & x \in S, \\ 1, & x \in \partial S, \end{cases} \tag{16}$$

is the super-solution to the problem (1), thus, by the comparison principle, we have $u(x, t) \leq \lambda^{\frac{1}{p}}$ for any $x \in S$ and $t \in [0, T)$.

Now, assume $u(x, t)$ attains its minimum value at the nodes x^* for any fix time t. At this point, we have

$$\begin{aligned} u_t(x^*, t) &= \sum_{y\in V} u(y, t)\omega(x^*, y) - d^* u(x^*, t) - \lambda u^{-p}(x^*, t) \\ &\leq d^* - d^* u(x^*, t) - \lambda u^{-p}(x^*, t) \\ &\leq -d^* u(x^*, t), \end{aligned} \tag{17}$$

where $d^* = d_\omega(x^*)$. Integrating both sides of the above inequality in $[0, t]$, we can get

$$u(x^*, t) \leq u_0(x^*)e^{-d^*t} \leq \lambda^{\frac{1}{p}} e^{-d^*t}. \tag{18}$$

Thus, for the equality in (17), note that the function $-s^{-p}$ is increasing, hence, choose $t_0 \geq \frac{ln(2d^*)}{pd^*}$, and then, for any $t \geq t_0$, we can also get

$$\begin{aligned} u_t(x^*, t) &\leq d^* - d^* u(x^*, t) - \lambda u^{-p}(x^*, t) \\ &\leq d^* - \frac{\lambda}{2} u^{-p}(x^*, t) - \frac{\lambda}{2} u^{-p}(x^*, t) \\ &\leq d^* - \frac{1}{2} e^{pd^*t} - \frac{\lambda}{2} u^{-p}(x^*, t) \\ &\leq -\frac{\lambda}{2} u^{-p}(x^*, t), \end{aligned} \tag{19}$$

7

Integrating both sides of the above inequality in $[t_0, t]$, we can obtain

$$\begin{aligned} &u^{p+1}(u(x^*,t)) \\ &\le u^{p+1}(u(x^*,t_0)) - \frac{(p+1)\lambda}{2}(t-t_0) \\ &\le u_0^{p+1}(x^*)e^{-d^*(p+1)t_0} - \frac{(p+1)\lambda}{2}(t-t_0), \end{aligned}$$

from this inequality, we have $u(x,t)$ quenches at finite time T, moreover, we have

$$T \le t_0 + \frac{(p+1)\lambda}{2}u_0^{p+1}(x^*)e^{-d^*(p+1)t_0}. \tag{20}$$

Theorem 4.2 *If $\lambda \ge 1$, then the solution to the problem* (1) *also quenches in finite time.*

Proof. Since $\lambda \ge 1$ and $0 < u_0(x) \le 1$, we have $\max_{x\in V} u_0(x) \le \lambda^{\frac{1}{p+1}}$. Now, it is easy to verify that $v(x,t) \equiv \lambda^{\frac{1}{p+1}}, x \in V$ is a super-solution to the problem (1). Thus, by the comparison principle, we also have $u(x,t) \le \lambda^{\frac{1}{p+1}}$ for any $x \in V$.

Now, also assume $u(x,t)$ attains its minimum value at the nodes x^* for any fix time t. At this point, we have

$$\begin{aligned} u_t(x^*,t) &= \sum_{y\in V} u(y,t)\omega(x^*,y) - d^*u(x^*,t) - \lambda u^{-p}(x^*,t) \\ &\le \lambda^{\frac{1}{p+1}} - d^*u(x^*,t) - \lambda u^{-p}(x^*,t) \\ &\le -d^*u(x^*,t), \end{aligned} \tag{21}$$

Integrating both sides of the above inequality on $[0,t]$, we can get

$$u(x^*,t) \le u_0(x^*)e^{-d^*t}. \tag{22}$$

Thus, for any $t \ge t_0$, from the inequality in (21) and by choosing $t_0 \ge \frac{\ln 2 - \frac{p\ln\lambda}{p+1}}{pd^*}$, it follows that

$$\begin{aligned} u_t(x^*,t) &\le \lambda^{\frac{1}{p+1}} - d^*u(x^*,t) - \lambda u^{-p}(x^*,t) \\ &= \lambda^{\frac{1}{p+1}} - d^*u(x^*,t) - \frac{\lambda}{2}u^{-p}(x^*,t) - \frac{\lambda}{2}u^{-p}(x^*,t) \\ &\le \lambda^{\frac{1}{p+1}} - \frac{\lambda}{2}e^{pd^*t} - \frac{\lambda}{2}u^{-p}(x^*,t) \\ &\le -\frac{\lambda}{2}u^{-p}(x^*,t), \end{aligned} \tag{23}$$

Integrating both sides of the above inequality on $[t_0,t]$, we can obtain

$$\begin{aligned} &u^{p+1}(u(x^*,t)) \\ &\le u^{p+1}(u(x^*,t_0)) - \frac{(p+1)\lambda}{2}(t-t_0) \\ &\le u_0^{p+1}(x^*)e^{-(p+1)d^*t_0} - \frac{(p+1)\lambda}{2}(t-t_0), \end{aligned}$$

by this inequality, we get $u(x,t)$ quenches at finite time T, moreover, we also have

$$T \leq t_0 + \frac{(p+1)\lambda}{2} u_0^{p+1}(x^*) e^{-(p+1)d^* t_0}. \tag{24}$$

Theorem 4.3 ***(The quenching rate)*** *If the solution $u(x,t)$ to the problem (1) quenches in finite time T at the node x^*, and then, we have*

$$\lim_{t\to T^-} (T-t)^{\frac{-1}{p+1}} u(x^*,t) = [(p+1)\lambda]^{\frac{1}{p+1}}.$$

Proof. Since $0 < u_0(x) \leq 1$, and then, it is easy to verify that $v(x,t) \equiv 1$ is a super-solution to the problem (1), by the comparison principle, we know that $0 < u(x,t) \leq 1$ for any $x \in V$ and $t \in [0,T)$.

Now, multiply u^p on the both sides of the discrete heat equation in the problem (1), and then, we get

$$u^p u_t = u^p \Delta_\omega u - \lambda, x \in S, t \in [0,T). \tag{25}$$

Next, we establish the upper bound of the quenching rate. Due to $0 < u(x,t) \leq 1$, we have

$$\begin{aligned} u^p u_t &= u^p \Delta_\omega u - \lambda \\ &= u^p \sum_{y\in V} u(y,t)\omega(x,y) - d_\omega(x) u^{p+1} - \lambda \\ &\geq -u^{p+1} - \lambda \geq -1 - \lambda \end{aligned} \tag{26}$$

for any $x \in S, t \in [0,T)$. Assume that $u(x,t)$ quenches in finite time T at the node x^*, and then, integrating the inequality $u^p u_t \geq -1-\lambda$ on the time t on $[t,T]$ on the node x^*, due to $u(x,t) \to 0$ when $t \to T^-$, we can get

$$u^{p+1}(x^*,t) \leq (p+1)(\lambda+1)(T-t).$$

Moreover, due to the inequality $u^p u_t \geq -u^{p+1} - \lambda$, thus, at the quenching node x^*, we also have

$$u^p u_t(x^*,t) \geq -(p+1)(\lambda+1)(T-t) - \lambda, \tag{27}$$

Integrating again in the time interval $[t,T]$, we have

$$-\frac{1}{p+1} u^{p+1}(x^*,t) \geq \frac{1}{2}(p+1)(\lambda+1)(T-t)^2 - \lambda(T-t), \tag{28}$$

thus, we get

$$\frac{u^{p+1}(x^*,t)}{T-t} \leq (p+1)\lambda \left(-(p+1)\frac{2(\lambda+1)}{\lambda}(T-t) + 1 \right). \tag{29}$$

Now, we establish the lower bound of the quenching rate. By the equation (31) and $0 < u(x,t) \leq 1$, we also have

$$u^p u_t = u^p \sum_{y \in V} u(y,t)\omega(x,y) - d_\omega(x)u^{p+1} - \lambda \leq u^p - \lambda.$$

Thus, by the inequality (26), at the quenching node x^*, we can obtain the following inequality

$$u^p u_t \leq [(p+1)(\lambda+1)(T-t)]^{\frac{p}{p+1}} - \lambda.$$

Integrating in the time interval $[t, T]$, we have

$$\frac{u^{p+1}(x^*,t)}{T-t} \geq (p+1)\lambda \left(-\frac{(p+1)^{\frac{2p+1}{p+1}}(\lambda+1)^{\frac{p}{p+1}}}{(2p+1)\lambda}(T-t) + 1 \right). \tag{30}$$

Combine the inequalities (29) and (30), and let $t \to T^-$, we can obtain the need results.

5 The existence of a global solution

In this section, we investigate the existence of a global solution to the problem (1) with the initial value $u_0(x) \equiv 1$ for any $x \in S$. To do this, we begin with the following lemma.

Lemma 5.1 *There exists a small nonnegative constant λ^*, such that if $\lambda \leq \lambda^*$, then the eigenvalue problem*

$$\begin{cases} \Delta_\omega u(x) = \lambda u^{-p}(x), & x \in S, \\ u(x) = 1, & x \in \partial S, \end{cases} \tag{31}$$

exists at least one solution.

Proof. Let $C(V)$ denotes the set of all the functions which are defined on the finite graph G with its nodes V, and then, the norm on $C(V)$ is as follows:

$$\|v\|_{C(V)} = \max_{x \in V} v(x). \tag{32}$$

Furthermore, set $C_0(V) = \{v(x) \in C(V) \text{ and } v(x) \equiv 0 \text{ for any } x \in \partial S\}$ and assume that $A = \{v \in C_0(V) : -\varepsilon < v(x) < 1\}$ is a open subset of $C_0(V)$, the nonlinear function $F(\lambda, v) : (-\varepsilon, \varepsilon) \times A \to C(S)$ is defined as

$$F(\lambda, v) = \Delta_\omega v + \lambda(1-v)^{-p}, \tag{33}$$

where ε is a small enough constant.

It is obviously that $F(\lambda, v)$ is differentiable function and $F(0,0) = 0$. Moreover, the Fréchet derivative of $F(\lambda, v)$ at $(0,0)$ is

$$F_v(0,0)[z(x)] = \Delta_\omega z(x) \tag{34}$$

is a continuous linear operator for any $z(x) \in A$. In fact, for any sequence $z_m(x) \to z(x)$, we have $\|\Delta_\omega[z_m(x) - z(x)]\|_{C(V)} \leq |V|\|z_m - z\|_{C(V)}$, so $F_v(0,0)$ is a continuous operator. Moreover, its kernel is the function $z = 0$ (see [1]), and then, it is injective. On the other hand, $F_v(0,0)$ is a linear transformation on finite dimensional space, and then, it is also a compact linear operator, hence, it is also bijective. By the Open-Mapping Theorem we deduce that $F_v(0,0)$ is a linear homeomorphism of $C_0(V)$ into $C_0(V)$. By the Implicit Function Theorem in the appendix A of [20], there exists a neighborhoods $U \in (-\varepsilon, \varepsilon)$ of $\lambda = 0$ and $W \in A$ of $v(x) \equiv 0$ such that $F(\lambda, v_\lambda) = 0$ for any $\lambda \in U$, and $v_\lambda \in W$ is unique. Thus, for any $\lambda < \lambda^* \in U$, suppose $u_\lambda(x) = 1 - v_\lambda(x)$, it is easy to verify that $u_\lambda(x)$ is a solution to the equation (31).

Based on the above lemma, we have the following theorem on the existence of the global solution to the problem (1) with $u_0(x) = 1$.

Theorem 5.1 *There exists a constat λ^*, such that $\lambda \leq \lambda*$, the problem (1) with the initial value $u_0(x) = 1$ has a global solution, while for $\lambda > \lambda^*$, then no global solution exists.*

Proof. Firstly, from the proofs of Theorem 4.1 and 4.2, we have the solution to the problem (1) with the initial value $u_0(x) = 1$ quenching in infinite time is impossible. Moreover, set $w(x,t) = u_t(x,t)$, and then, we get w satisfies

$$\begin{cases} w_t = \Delta_\omega w + p\lambda u^{-p-1} w, & (x,t) \in S \times (0,T), \\ w(x,t) = 0, & (x,t) \in \partial S \times (0,T), \\ w(x,0) = -\lambda, & x \in S. \end{cases} \tag{35}$$

Then, by comparison principle, we obtain that $w = u_t \leq 0$. On the other hand, by the Lemma 5.1, we have λ is small enough, the equation (31) exists a positive solution $v_\lambda(x)$, in fact, it is also a sub-solution to the problem (1) with the initial value $u_0(x) = 1$. Hence, the solution of (1) with the initial value $u_0(x) = 1$ satisfies that, either it quenches in finite time, or it converges to a stationary solution

Next, we discuss the critical exponent of the quenching and the global existence. In fact, If $u(x,t)$ is a global solution to the problem (1), then, we know that $u(x,t) \to u_\infty$ as $t \to \infty$ and u_∞ is a solution the the problem (31), is a stationary solution to the equation (31). Moreover, for any fix constant λ_1, if there exists a solution $v_{\lambda_1}(x)$ to the problem (31), i.e. $v_{\lambda_1}(x)$ satisfies

$$\Delta_\omega v_{\lambda_1}(x) = \lambda v_{\lambda_1}^{-p}(x), \tag{36}$$

furthermore, it is easy to verify that $v_{\lambda_1}(x)$ is a sub-solution to the problem (1) with the initial value $u_0(x) = 1$ and $\lambda \leq \lambda_1$. Thus, the solution to the problem (1) with the initial value $u_0(x) = 1$ is global when $\lambda \leq \lambda_1$. By this monotonicity property given

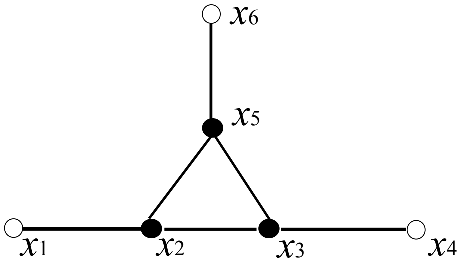

Figure 1: The graph G_1

above discussion, set $\lambda^* = \sup\limits_{\lambda\in B} \lambda$, where the set $B = \{\lambda : v_\lambda(x)$ exists to (36)$\}$. This completes the proof.

6 Numerical experiments

In this section, we consider a graph G_1 (as shown in Figure 1), which has six nodes $x_1, x_2, \cdots, x_6$, where x_2, x_3, x_5 are interior and x_1, x_4, x_6 are the boundary. Moreover, we only consider the weight function $\omega \equiv \frac{1}{3}$. Thus, the discrete heat equation in (1) is

$$\begin{cases} u_t(x_2,t) = \frac{1}{3} + \frac{1}{3}u(x_3,t) + \frac{1}{3}u(x_5,t) - u(x_2,t) - \lambda u^{-p}(x_2,t) \\ u_t(x_3,t) = \frac{1}{3} + \frac{1}{3}u(x_2,t) + \frac{1}{3}u(x_5,t) - u(x_3,t) - \lambda u^{-p}(x_3,t) \\ u_t(x_5,t) = \frac{1}{3} + \frac{1}{3}u(x_2,t) + \frac{1}{3}u(x_3,t) - u(x_5,t) - \lambda u^{-p}(x_5,t) \end{cases} \tag{37}$$

Now, we also suppose that the exponent $p = 1.2, \lambda = 0.8$. Moreover, the discrete Laplacian operator Δ_ω on the graph G_1 is as follows:

$$\Delta_\omega = -\frac{1}{3}\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} \tag{38}$$

Thus, set $U(t) = (u(x_2,t), u(x_3,t), u(x_5,t))^T$, and then, we have the equation (37) can be rewrote as follows:

$$U_t = \frac{\mathbf{1}}{3} + \Delta_\omega * U(t) - 0.8U^{-2}(t), \text{ with } U(0) = (0.3, 0.35, 0.4)^T, \tag{39}$$

where $\mathbf{1} = (1,1,1)^T$.

By Theorem 4.1, we get $U(t)$ quenches in finite time, moreover, U_t blows up in finite time. Since the system (39) is nonlinear, it is difficult to compute its analytic solutions. Hence, we consider its numerical solutions. The explicit difference scheme to the system (39) is as follows:

$$U_{n+1} = U_n + \Delta t(\frac{\mathbf{1}}{3} + \Delta_\omega * U_n - 0.8U_n^{-2}), \text{ with } U_0 = (0.3, 0.35, 0.4)^T, \tag{40}$$

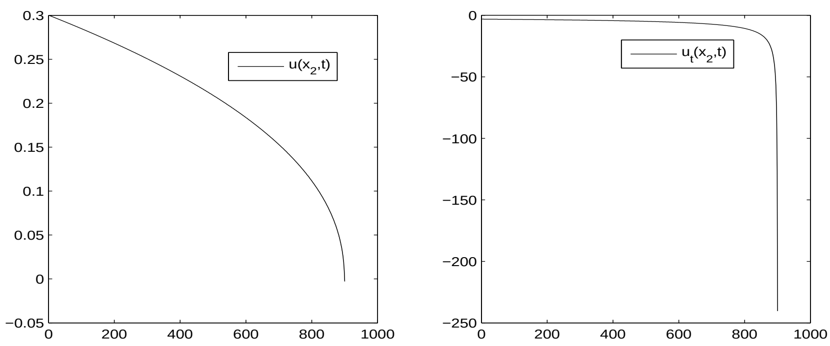

Figure 2: Quenching of $u(x_2,t)$ and Blow-up of $u_t(x_2,t)$ in finite time

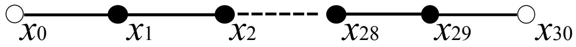

Figure 3: The graph G_2

where U_n denotes $U(n\Delta t)$ for $n = 1, 2, 3, \cdots$ and Δt is the time step which taking as $0.043/n$ in the numerical experiment. The numerical experiment result is shown in Figure 2. From this numerical experiment, we know that the solution $U(t)$ quenches and U_t blows up in finite time.

At the end of this section, we give another example. Now, we consider the discrete heat equation (1) on the following finite graph G_2 (as shown in Figure 3), which has six nodes $x_0, x_2, \cdots, x_{30}$, where $x_1, x_2, \cdots, x_{29}$ are interior and x_0, x_{30} are the boundary. Moreover, we only consider the weight function $\omega(x_i, x_j) \equiv \frac{1}{4}$. Thus, the discrete heat equation in (1) is

$$\begin{cases} u_t(x_1,t) = \frac{1}{4}(1 + u(x_2,t) - 2u(x_1,t)) - \lambda u^{-p}(x_1,t), \\ u_t(x_i,t) = \frac{1}{4}(u(x_{i-1}) + u(x_{i+1},t) - 2u(x_i,t)) - \lambda u^{-p}(x_i,t), 1 \le i \le 28, \\ u_t(x_{29},t) = \frac{1}{4}(1 + u(x_{28},t) - 2u(x_{29},t)) - \lambda u^{-p}(x_{29},t), \end{cases} \tag{41}$$

where $\lambda = 1$, $p = 1.2$, and then, let the initial value $u_0(x_i) = 1 - 0.9\sin\left(\frac{i}{30}\pi\right)$, where $1 \le i \le 29$ and $u(x_0,t) = u(x_{30},t) = 1$. Thus, by the theorem 4.2, we have the solution $u(x_i,t)$ will quench in finite time. Also since the nonlinear of the system (41), we consider the following difference scheme:

$$V_{n+1} = V_n + \Delta t\left(B + \Delta_\omega V_n - \lambda V_n^{-p}\right), n = 0, 1, 2, \cdots, \tag{42}$$

where $V_n = (u(x_1, n\Delta t), u(x_2, n\Delta t), \cdots, u(x_{29}, n\Delta t))^T$, $B = (1/4, 0, \cdots, 0, 1/4)^T$ is a $29-$ dimensions vector, $\Delta t = 0.0001/n$ is the time step, and the discrete Laplacian

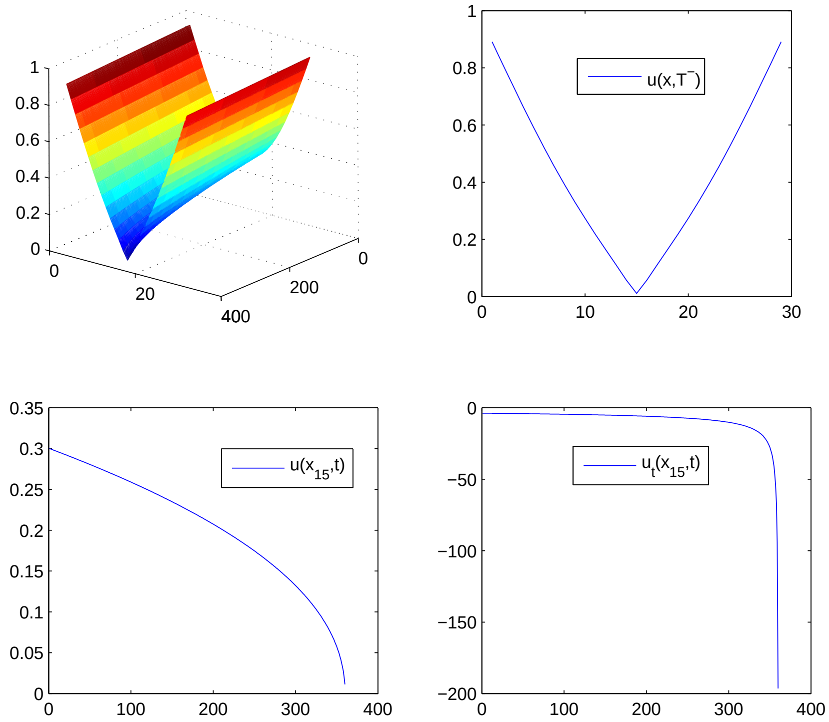

Figure 4: Quenching of $u(x,t)$ and Blow-up of $u_t(x_{15},t)$ in finite time

operator on the graph G_2 is as follows:

$$\Delta_\omega = \frac{1}{4}\begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & 0 & 0 & 1 & -2 \end{pmatrix}_{29\times 29}. \tag{43}$$

Moreover, the initial value $V_0 = (u_0(x_1), u_0(x_2)), \cdots, u_0(x_{29}))$. The numerical experiment results can be found in Figure 4.

7 Conclusion

In this paper, we mainly consider the quenching problem and the global solution of the discrete heat equation with a singular absorption, the quenching time, quenching rate and the critical exponent were also given. We only prove the existence of the critical exponent, its upper and lower bounds may be established by the Kaplan's method in the further work.

Acknowledgments

This work is supported by the Natural Science Foundation of Uygur Autonomous Region of Xinjiang(Grant No.201442137-30) and the National Natural Science Foundation of China (Grant No.11461075).

References

[1] Chung, S. Y.; Berenstein, C.A. $\omega-$harmonic functions and inverse conductivity problems on networks. *SIAM J. Appl. Math.*, *65*, 1200-1226, **2005**.

[2] Kawarada, H. On solutions fo the initial-boundary value problem for $u_t = u_{xx} + \frac{1}{1-u}$. *Publ. Res. Inst. Math. Soc.*, *10*, 729-736, **1975**.

[3] Fila, M.; Levine, H. A.; Vazquez, J. A. Stabilization of solutions weakly singular quenching problmes. *Proc. Amer. Math. Soc.*, *119*, 555-559, **1993**.

[4] Aris, R. The mathemcatical theory of diffusion and reaction in permeable catalysts. Clarendon Press, Oxford, 1975.

[5] Diaz, J. L. Nonliear partial differential equations and free boundaries, Vol. I, Elliptic Equations, Pitman Res. Notes Math. Ser., vol. 106, Longman Sci. Tech., Harlow, 1985.

[6] Phillips, D. Existence of solutions to a quenching problem. *Appl. Anal.*, *24*, 253-264, **1987**.

[7] Salin, T. On quenching with logarithmic singularity. *Nonlinear Anal.-Theor.*, *52*, 261-289, **2003**.

[8] Zhou, S. M.; Mu, C. L.; Zeng, R. Quenching for a non-local diffusion equaiton with a singular absorption term and Neumann boundary condition. *Z. Angew. Math. Phys.*, *62*, 483-493, **2011**.

[9] Ferreira, R. Quenching phenomena for a non-local diffusion equation with a singular absorption. *Israel J. Math.*, *184*, 387-402, **2011**.

[10] Acker, A.; Walter, W. The Quenching Problem for Nonlinear Parabolic Differential Equations, in *Ordinary and Partial Differential Equations*, Everitt W. N., Sleeman B. D.; Springer-Verlag: New York, 2006; pp. 1-12

[11] Dai, Q. Y.; Gu, Y. G. A short note on quenching phenomena for semilinear parabolic equation. *J. Diff. Equ.*, *137*, 240-250, **1997**.

[12] Curtis, E. B.; Morrow, J. A. Determining the resistors in a network. *SIAM J. Appl. Math.*, *50*, 918-930, **1990**.

[13] Elmoataz, A.; L*é*zoray, O.; Bougleux, S. Nonlocal discrete regularization on weighted graphs: A framework for iamge and manifold processing. *IEEE Trans. Image Prcoess.*, *17*, 1047-1060, **2009**.

[14] Chung, Y. S.; Lee, Y. S.; Chung, S. Y. Extinction and positivity of the solutions of the heat equations with absorption on networks. *J. Math. Anal. Appl.*, *380*, 642-652, **2011**.

[15] Xin, Q.; Mu, C. L.; Liu, D. M. Extinction and positivity of the solutions for a $p-$laplacian equation with absorption on graphs. *J. Appl. Math.*, *2011*, pp. 1-12, **2011**.

[16] Chung, S. Y.; Park, J.H. Extinction and positivity of the solutions for a $p-$Laplacian equation with absorption on networks. *Comput. Math. Appl.*, *69*, 223-234, **2015**.

[17] Xin, Q.; Xu, L.; Mu, C. L. Blow-up for the $\omega-$heat equation with Dirichlet boundary conditions and a reaction term on graphs. *Appl. Anal.* , *93*, 1691-1701, **2013**.

[18] Zhou, W. C.; Chen, M. M.; Liu, W. J. Critical exponent and blow-up rate for the ω-diffusion equations on graphs with dirichlet boundary conditions. *Electron. J. Diff. Equ.*, *2014*, pp. 1-13, **2014**.

[19] Liu, W. J; Chen, K. W; Yu, J. Extinction and asymptotic behavior of solutions for the ω-heat equation on graphs with source and interior absorption. *J. Math. Anal. Appl.*, *435*, 112-132, **2016**.

[20] Crandaill, M. G. Bifurcation from simple eigenvalues. *J. Func. Anal.*, *8*, 321-340, **1971**.

Nonlocal fractional-order boundary value problems with generalized Riemann-Liouville integral boundary conditions

Bashir Ahmad [a], Sotiris K. Ntouyas [b,a], Jessada Tariboon [c,d]

[a] Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
e-mail: bashirahmad_qau@yahoo.com

[b] Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece
e-mail: sntouyas@uoi.gr

[c] Nonlinear Dynamic Analysis Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, 10800 Thailand

[d] Centre of Excellence in Mathematics, CHE, Sri Ayutthaya Rd., Bangkok 10400, Thailand
e-mail: jessada.t@sci.kmutnb.ac.th

Abstract

In this paper, we study existence and uniqueness of solutions for nonlocal boundary value problems of Caputo fractional differential equations equipped with generalized Riemann-Liouville integral boundary conditions. A variety of fixed point theorems such as Banach's fixed point theorem, nonlinear contractions, Krasnoselskii's fixed point theorem, Schaefer's fixed point theorem, Leray-Schauder's nonlinear alternative and Leray-Schauder degree theory are applied to obtain the desired results. Several examples are discussed for illustration of the obtained results.

Key words and phrases: Caputo fractional derivative; generalized Riemann-Liouville integral; nonlocal boundary conditions; fixed point theorems.
AMS (MOS) Subject Classifications: 26A33; 34A08

1 Introduction

We investigate the sufficient criteria for existence of solutions for the following Caputo fractional differential equation

$$D^q x(t) = f(t, x(t)), \qquad 0 < t < T, \tag{1}$$

subject to nonlocal generalized Riemann-Liouville fractional integral boundary conditions of the form

$$\begin{aligned} x(0) &= \gamma \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\zeta} \frac{s^{\rho-1} x(s)}{(\zeta^\rho - s^\rho)^{1-\alpha}} ds := \gamma \ {}^\rho I^\alpha x(\zeta), \\ x(T) &= \delta \frac{\rho^{1-\beta}}{\Gamma(\beta)} \int_0^{\xi} \frac{s^{\rho-1} x(s)}{(\xi^\rho - s^\rho)^{1-\beta}} ds := \delta \ {}^\rho I^\beta x(\xi), \quad 0 < \zeta, \xi < T, \end{aligned} \tag{2}$$

where D^q denote the Caputo fractional derivative of order q, ${}^\rho I^z$, $z \in \{\alpha, \beta\}$, is the generalized Riemman-Liouville fractional integral of order $z > 0$, $\rho > 0$, ζ, ξ arbitrary, with $\zeta, \xi \in (0, T)$, $\gamma, \delta \in \mathbb{R}$ and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

As a second problem, we study Caputo fractional differential equation (1) supplemented with a combination of Riemman-Liouville and generalized Riemman-Liouville integral boundary conditions:

$$\begin{aligned} x(0) &= \gamma \frac{1}{\Gamma(\alpha)} \int_0^{\zeta} (\zeta - s)^{\alpha-1} x(s) ds := \gamma \ J^{\alpha} x(\zeta), \\ x(T) &= \delta \frac{\rho^{1-\beta}}{\Gamma(\beta)} \int_0^{\xi} \frac{s^{\rho-1} x(s)}{(t^{\rho} - s^{\rho})^{1-\beta}} ds := \delta \ {}^{\rho}I^{\beta} x(\xi), \quad 0 < \zeta, \xi < T, \end{aligned} \tag{3}$$

where J^q is the Riemman-Liouville fractional integral of order $q > 0$ while ${}^{\rho}I^{\beta}$ denote generalized Riemman-Liouville fractional integral of order $\beta > 0$, $\rho > 0$.

The subject of fractional differential equations has evolved into an interesting and popular field of research during the last few decades. The surge in developing several aspects of fractional calculus owes to its extensive applications in several branches of engineering and technical sciences such as physics, chemical technology, population dynamics, biotechnology, biosciences, control theory and economics. The nonlocal nature of fractional derivatives, which takes into account memory and hereditary properties of various materials and processes, has played a key role in improving the mathematical modeling based on integer-order derivatives, for instance, see [1, 2, 3, 4].

Fractional-order boundary value problems supplemented with different kinds of boundary conditions have been studied by several researchers. In particular, integral boundary conditions involving classical, Riemann-Liouville or Hadamard or Erdélyi-Kober type integral operators have received significant attention. In [5], Riemann-Liouville and Hadamard fractional integrals are jointly represented by a single integral, which is called generalized Riemann-Liouville fractional integral (see Definition 2.2). For some recent works on the topic we refer the reader to a series of papers [6]-[20] and the references cited therein.

The purpose of the present study is to develop the existence theory for problems (1)-(2) and (1)-(3) by means of standard tools of fixed point theory. In Section 2 we recall some preliminary facts that we need in the sequel. In Section 3 we present our main results, while Section 4 contains examples illustrating the results obtained in Section 3.

2 Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [2, 3] and present preliminary results needed in our proofs later.

Definition 2.1 *The Riemann-Liouville fractional integral of order $q > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by*

$$J^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds,$$

provided the right-hand side is point-wise defined on $(0, \infty)$.

Definition 2.2 *[5] The generalized Riemann-Liouville fractional integral of order $q > 0$ and $\rho > 0$ of a function $f(t)$ for all $0 < t < \infty$, is defined as*

$${}^{\rho}I^q f(t) = \frac{\rho^{1-q}}{\Gamma(q)} \int_0^t \frac{s^{\rho-1} f(s)}{(t^{\rho} - s^{\rho})^{1-q}} ds,$$

provided the right-hand side is point-wise defined on $(0, \infty)$.

Remark 2.3 *From the above definition it follows that when $\rho = 1$ we arrive at the standard Riemann-Liouville fractional integral, which is used to define both the Riemann-Liouville and Caputo fractional derivatives, while when $\rho \to 0$ we have*

$$\lim_{\rho \to 0} {}^{\rho}I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \left(\log \frac{t}{s} \right)^{q-1} \frac{f(s)}{s} ds,$$

which is the famous Hadamard fractional integral. See [5].

Definition 2.4 *The Riemann-Liouville fractional derivative of order $q > 0$, $n-1 < q < n$, $n \in \mathbb{N}$, is defined as*

$$D^q_{0+}f(t) = \frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1}f(s)ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 2.5 *The Caputo derivative of order q for a function $f : [0,\infty) \to \mathbb{R}$ can be written as*

$${}^cD^q f(t) = D^q\left(f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right), \quad t > 0, \quad n-1 < q < n.$$

Remark 2.6 *If $f(t) \in C^n[0,\infty)$, then*

$${}^cD^q f(t) = \frac{1}{\Gamma(n-q)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}}ds = I^{n-q}f^{(n)}(t),\ t > 0,\ n-1 < q < n.$$

Lemma 2.7 *Let constants $q > 0$ and $p > 0$. Then:*

$${}^\rho I^q t^p = \frac{\Gamma\left(\frac{p+\rho}{\rho}\right)}{\Gamma\left(\frac{p+\rho q+\rho}{\rho}\right)}\frac{t^{p+\rho q}}{\rho^q}. \tag{4}$$

Proof. By Definition 2.2, we have

$$\begin{aligned} {}^\rho I^q t^p &= \frac{\rho^{1-q}}{\Gamma(q)}\int_0^t \frac{s^{\rho-1}s^p}{(t^\rho - s^\rho)^{1-q}}\,ds = \frac{\rho^{1-q}}{\Gamma(q)}\frac{t^{p+\rho q}}{\rho}\int_0^1 \frac{u^{\frac{p}{\rho}}}{(1-u)^{1-q}}\,du \\ &= \frac{\rho^{1-q}}{\Gamma(q)}\frac{t^{p+\rho q}}{\rho}B\left(\frac{p+\rho}{\rho}, q\right) = \frac{t^{p+\rho q}}{\rho^q}\frac{\Gamma\left(\frac{p+\rho}{\rho}\right)}{\Gamma\left(\frac{p+\rho q+\rho}{\rho}\right)}. \end{aligned}$$

This completes the proof. □

Lemma 2.8 *For any $y \in AC([0,T],\mathbb{R})$, x is a solution of the linear fractional boundary value problem*

$$\begin{cases} {}^cD^q x(t) = y(t), \quad 1 < q \le 2, \\ x(0) = \gamma\ {}^\rho I^\alpha x(\zeta), \quad x(T) = \delta\ {}^\rho I^\beta x(\xi), \quad 0 < \zeta, \xi < T, \end{cases} \tag{5}$$

if and only if

$$x(t) = J^q y(t) + \frac{\gamma}{\Lambda}(v_4 - tv_3)\ {}^\rho I^\alpha J^q y(\zeta) + \frac{1}{\Lambda}(v_2 + tv_1)\Big(\delta\ {}^\rho I^\beta J^q y(\xi) - J^q y(T)\Big), \tag{6}$$

where

$$\begin{aligned} v_1 &= 1 - \gamma\frac{\zeta^{\rho\alpha}}{\rho^\alpha}\frac{1}{\Gamma(\alpha+1)}, \qquad v_2 = \gamma\frac{\zeta^{\rho\alpha+1}}{\rho^\alpha}\frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho\alpha+\rho}{\rho})}, \\ v_3 &= 1 - \delta\frac{\xi^{\rho\beta}}{\rho^\beta}\frac{1}{\Gamma(\beta+1)}, \qquad v_4 = T - \delta\frac{\xi^{\rho\beta+1}}{\rho^\beta}\frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho\beta+\rho}{\rho})}, \end{aligned} \tag{7}$$

and

$$\Lambda = v_1 v_4 + v_2 v_3 \neq 0. \tag{8}$$

Proof. For arbitrary constants $c_0, c_1 \in \mathbb{R}$, the general solution of the fractional differential equation in (5) can be written as [2]

$$x(t) = c_0 + c_1 t + J^q y(t). \tag{9}$$

Applying the generalized fractional integral operator on (9) and using Lemma 2.7, we get

$$^\rho I^z x(t) = {}^\rho I^z J^q y(t) + c_0 \frac{t^{\rho z}}{\rho^z} \frac{1}{\Gamma(z+1)} + c_1 \frac{t^{\rho z+1}}{\rho^z} \frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho z+\rho}{\rho})}. \tag{10}$$

Using (9) and (10) in boundary conditions of (5), we get the system

$$\begin{aligned} \left(1-\gamma\frac{\zeta^{\rho\alpha}}{\rho^\alpha}\frac{1}{\Gamma(\alpha+1)}\right)c_0 - \gamma\frac{\zeta^{\rho\alpha+1}}{\rho^\alpha}\frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho\alpha+\rho}{\rho})}c_1 &= \gamma\ {}^\rho I^\alpha J^q y(\zeta), \\ \left(1-\delta\frac{\xi^{\rho\beta}}{\rho^\beta}\frac{1}{\Gamma(\beta+1)}\right)c_0 + \left(T-\delta\frac{\xi^{\rho\beta+1}}{\rho^\beta}\frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho\beta+\rho}{\rho})}\right)c_1 &= \delta\ {}^\rho I^\beta J^q y(\xi) - J^q y(T). \end{aligned} \tag{11}$$

Solving (11) together with the notations (7) and (8), we find that

$$c_0 = \frac{1}{\Lambda}\Big\{\gamma v_4\ {}^\rho I^\alpha J^q y(\zeta) + v_2\Big(\delta\ {}^\rho I^\beta J^q y(\xi) - J^q y(T)\Big)\Big\},$$

$$c_1 = \frac{1}{\Lambda}\Big\{v_1\Big(\delta\ {}^\rho I^\beta J^q y(\xi) - J^q y(T)\Big) - \gamma v_2\ {}^\rho I^\alpha J^q y(\zeta)\Big\}.$$

Substituting the values of c_0 and c_1 in (9) yields the solution (6). Conversely, it can easily be shown by direct computation that the integral equation (6) satisfies the problem (5). This completes the proof.□

Our next lemma deals with the linear variant of (1)-(3). We do not provide the proof of this result as it is similar to the preceding one.

Lemma 2.9 *For any $y \in AC([0,T],\mathbb{R})$, x is a solution of the linear fractional boundary value problem*

$$\begin{cases} {}^cD^q x(t) = y(t), \quad 1 < q \le 2, \\ x(0) = \gamma\ J^\alpha x(\zeta), \quad x(T) = \delta\ {}^\rho I^\beta x(\xi), \quad 0 < \zeta, \xi < T, \end{cases} \tag{12}$$

if and only if

$$x(t) = J^q y(t) + \frac{\gamma}{\Lambda_1}(u_4 - tu_3)\ J^{q+\alpha} y(\zeta) + \frac{1}{\Lambda_1}(u_2 + tu_1)\Big(\delta\ {}^\rho I^\beta J^q y(\xi) - J^q y(T)\Big), \tag{13}$$

where

$$u_1 = 1 - \gamma\frac{\zeta^\alpha}{\Gamma(\alpha+1)}, \quad u_2 = \gamma\frac{\zeta^{\alpha+1}}{\Gamma(\alpha+2)}, \quad u_3 = 1 - \delta\frac{\xi^{\rho\beta}}{\rho^\beta}\frac{1}{\Gamma(\beta+1)}, \quad u_4 = T - \delta\frac{\xi^{\rho\beta+1}}{\rho^\beta}\frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho\beta+\rho}{\rho})}, \tag{14}$$

and

$$\Lambda_1 = u_1 u_4 + u_2 u_3 \neq 0. \tag{15}$$

3 Existence results

Let us denote by $\mathcal{C} = C([0,T],\mathbb{R})$ the Banach space of all continuous functions from $[0,T] \to \mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by $\|x\| = \sup\{|x(t)| : t \in [0,T]\}$. By $L^1([0,T],\mathbb{R})$ we mean the Banach space of measurable functions $x : [0,T] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^T |x(t)|dt$.

In view of Lemma 2.8, we introduce operators $\mathcal{Q}, \widehat{\mathcal{Q}} : \mathcal{C} \to \mathcal{C}$ associated with problems (1)-(2) and (1)-(3) respectively by

$$\begin{aligned}(\mathcal{Q}x)(t) &= J^q f(s,x(s))(t) + \frac{\gamma}{\Lambda}(v_4 - tv_2)\ {}^\rho I^\alpha J^q f(s,x(s))(\zeta) \\ &+\frac{1}{\Lambda}(v_2+tv_1)\Big(\delta\ {}^\rho I^\beta J^q f(s,x(s))(\xi) - J^q f(s,x(s))(T)\Big),\ t\in[0,T],\end{aligned} \tag{16}$$

$$\begin{aligned}(\widehat{\mathcal{Q}}x)(t) &= J^q f(s,x(s))(t) + \frac{\gamma}{\Lambda_1}(u_4 - tu_3)\ J^{q+\alpha} f(s,x(s))(\zeta) \\ &+\tfrac{1}{\Lambda_1}(u_2+tu_1)\Big(\delta\ {}^\rho I^\beta J^q (s,x(s))(\xi) - J^q f(s,x(s))(T)\Big),\quad t\in[0,T].\end{aligned} \tag{17}$$

In the sequel, we use the following expression:

$${}^\rho I^h f(s,x(s))(y) = \frac{\rho^{1-h}}{\Gamma(h)}\int_0^y \frac{s^{\rho-1}f(s,x(s))}{(y^\rho - s^\rho)^{1-h}}ds, \quad h\in\{\alpha,\beta\}.$$

Further, we set the constants

$$\begin{aligned}\Omega : &= \frac{T^q}{\Gamma(q+1)} + \frac{|\gamma|(|v_4|+T|v_2|)\zeta^{q+\rho\alpha}}{|\Lambda|\rho^\alpha\Gamma(q+1)}\frac{\Gamma\Big(\frac{q+\rho}{\rho}\Big)}{\Gamma\Big(\frac{q+\rho\alpha+\rho}{\rho}\Big)} \\ &+\frac{(|v_2|+T|v_1|)}{|\Lambda|}\left(\frac{|\delta|\xi^{q+\rho\beta}}{\rho^\beta\Gamma(q+1)}\frac{\Gamma\Big(\frac{q+\rho}{\rho}\Big)}{\Gamma\Big(\frac{q+\rho\beta+\rho}{\rho}\Big)} + \frac{T^q}{\Gamma(q+1)}\right).\end{aligned} \tag{18}$$

$$\begin{aligned}\Omega_1 : &= \frac{T^q}{\Gamma(q+1)} + \frac{|\gamma|(|u_4|+T|u_2|)\zeta^{\alpha+q}}{|\Lambda_1|\Gamma(\alpha+q+1)} \\ &+\frac{(|u_2|+T|u_1|)}{|\Lambda_1|}\left(\frac{|\delta|\xi^{q+\rho\beta}}{\rho^\beta\Gamma(q+1)}\frac{\Gamma\Big(\frac{q+\rho}{\rho}\Big)}{\Gamma\Big(\frac{q+\rho\beta+\rho}{\rho}\Big)} + \frac{T^q}{\Gamma(q+1)}\right).\end{aligned} \tag{19}$$

In the following subsections, we establish several existence and uniqueness results for problems (1)-(2) and (1)-(3) by applying a variety of fixed point theorems. We present in details the proofs for problem (1)-(2), while the proofs for problem (1)-(3) are omitted as they are similar to the ones obtained for problem (1)-(2).

3.1 Existence and uniqueness result via Banach's fixed point theorem

Theorem 3.1 *Assume that:*

(H_1) *there exists a positive constant L such that $|f(t,x) - f(t,y)| \le L|x-y|$, for each $t\in[0,T]$ and $x,y\in\mathbb{R}$.*

If

$$L\Omega < 1, \tag{20}$$

where Ω is defined by (18), then the boundary value problem (1)-(2) has a unique solution on $[0,T]$.

Proof. Observe that a fixed point problem equivalent to problem (1)-(2) is $x = \mathcal{Q}x$, where the operator $\mathcal{Q}$ is defined by (16), and that the existence of a fixed point of the operator $\mathcal{Q}$ implies the existence of a solution for problem (1)-(2). Applying the Banach contraction mapping principle, we shall show that $\mathcal{Q}$ has a unique fixed point. For that we let $\sup_{t\in[0,T]}|f(t,0)| = M < \infty$ and choose $r \ge \frac{M\Omega}{1-L\Omega}$. To show that $\mathcal{Q}B_r \subset B_r$, where $B_r = \{x\in\mathcal{C} : \|x\| \le r\}$, we have for any $x\in B_r$ that

$$|(\mathcal{Q}x)(t)| \le \sup_{t\in[0,T]}\Bigg\{J^q|f(s,x(s))|(t) + \frac{|\gamma|}{|\Lambda|}(|v_4|+T|v_2|)\ {}^\rho I^\alpha J^q|f(s,x(s))|(\zeta)$$

$$
\begin{aligned}
&+\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(\delta\ {}^\rho I^\beta J^q|f(s,x(s))|(\xi)+J^q|f(s,x(s))|(T)\Big)\\
\leq\ & J^q(|f(s,x(s))-f(s,0)|+|f(s,0)|)(T)\\
&+\frac{|\gamma|}{|\Lambda|}(|v_4|+T|v_2|)\ {}^\rho I^\alpha J^q(|f(s,x(s))-f(s,0)|+|f(s,0)|)(\zeta)\\
&+\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q(|f(s,x(s))-f(s,0)|+|f(s,0)|)(\xi)\\
&+J^q(|f(s,x(s))-f(s,0)|+|f(s,0)|)(T)\Big)\\
\leq\ & (L\|x\|+M)J^q(1)(T)+(L\|x\|+M)\frac{|\gamma|}{|\Lambda|}(|v_4|+T|v_2|)\ {}^\rho I^\alpha J^q(1)(\zeta)\\
&+(L\|x\|+M)\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q(1)(\xi)+J^q(1)(T)\Big)\\
\leq\ & (Lr+M)\left\{\frac{T^q}{\Gamma(q+1)}+\frac{|\gamma|(|v_4|+T|v_2|)\zeta^{q+\rho\alpha}}{|\Lambda|\rho^\alpha\Gamma(q+1)}\frac{\Gamma\Big(\frac{q+\rho}{\rho}\Big)}{\Gamma\Big(\frac{q+\rho\alpha+\rho}{\rho}\Big)}\right.\\
&\left.+\frac{(|v_2|+T|v_1|)}{|\Lambda|}\left(\frac{|\delta|\xi^{q+\rho\beta}}{\rho^\beta\Gamma(q+1)}\frac{\Gamma\Big(\frac{q+\rho}{\rho}\Big)}{\Gamma\Big(\frac{q+\rho\beta+\rho}{\rho}\Big)}+\frac{T^q}{\Gamma(q+1)}\right)\right\}\\
\leq\ & (Lr+M)\Omega\leq r,
\end{aligned}
$$

which implies that $\mathcal{Q}B_r\subset B_r$.

Next, we let $x,y\in\mathcal{C}$. Then for $t\in[0,T]$, we have

$$
\begin{aligned}
|\mathcal{Q}x(t)-\mathcal{Q}y(t)|\ \leq\ & \sup_{t\in[0,T]}\Big\{J^q|f(s,x(s))-f(s,y(s))|(t)\\
&+\frac{|\gamma|}{|\Lambda|}(|v_4|+T|v_2|)\ {}^\rho I^\alpha J^q|f(s,x(s))-f(s,y(s))|(\zeta)\\
&+\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(\delta\ {}^\rho I^\beta J^q|f(s,x(s))-f(s,y(s))|(\xi)\\
&+J^q|f(s,x(s))-f(s,y(s))|(T)\Big)\\
\leq\ & L\|x-y\|J^q(1)(T)+L\|x-y\|\frac{|\gamma|}{|\Lambda|}(|v_4|+T|v_2|)\ {}^\rho I^\alpha J^q(1)(\zeta)\\
&+L\|x-y\|\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q(1)(\xi)+J^q(1)(T)\Big)\\
=\ & L\Omega\|x-y\|,
\end{aligned}
$$

which leads to $\|\mathcal{Q}x-\mathcal{Q}y\|\leq L\Omega\|x-y\|$. As $L\Omega<1$, $\mathcal{Q}$ is a contraction. Therefore, it follows by the Banach's contraction mapping principle that $\mathcal{Q}$ has a fixed point which in fact is the unique solution of problem (1)-(2). The proof is completed. □

Theorem 3.2 *Assume that* (H_1) *holds. If*

$$L\Omega_1<1, \tag{21}$$

where Ω_1 *is defined by (19), then the boundary value problem (1)-(3) has a unique solution on* $[0,T]$.

3.2 Existence result via Krasnoselskii's fixed point theorem

Lemma 3.3 *(Krasnoselskii's fixed point theorem) [21]. Let* M *be a closed, bounded, convex and nonempty subset of a Banach space* X. *Let* A,B *be the operators such that* (a) $Ax+Bx\in M$ *whenever*

$x, y \in M$; (b) *A is compact and continuous;* (c) *B is a contraction mapping. Then there exists* $z \in M$ *such that* $z = Az + Bz$.

Theorem 3.4 *Let* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function satisfying* (H_1). *In addition we assume that*

(H_2) $|f(t,x)| \le \varphi(t), \quad \forall (t,x) \in [0,T] \times \mathbb{R}$, *and* $\varphi \in C([0,T], \mathbb{R}^+)$.

Then the problem (1)-(2) has at least one solution on $[0,T]$ *provided*

$$L\left\{\frac{|\gamma|(|v_4|+T|v_2|)\zeta^{q+\rho\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(q+1)}\frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\alpha+\rho}{\rho}\right)} + \frac{(|v_2|+T|v_1|)}{|\Lambda|}\left(\frac{|\delta|\xi^{q+\rho\beta}}{\rho^{\beta}\Gamma(q+1)}\frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\beta+\rho}{\rho}\right)} + \frac{T^q}{\Gamma(q+1)}\right)\right\} < 1. \tag{22}$$

Proof. Define the operators $\mathcal{Q}_1, \mathcal{Q}_2 : \mathcal{C} \to \mathcal{C}$ as follows

$$\begin{aligned}
\mathcal{Q}_1 x(t) &= J^q f(s,x(s))(t), \quad t \in [0,T], \\
\mathcal{Q}_2 x(t) &= \frac{\gamma}{\Lambda}(v_4 - tv_2)\, {}^{\rho}I^{\alpha} J^q f(s,x(s))(\zeta) \\
&\quad + \frac{1}{\Lambda}(v_2 + tv_1)\Big(\delta\, {}^{\rho}I^{\beta} J^q f(s,x(s))(\xi) - J^q f(s,x(s))(T)\Big), \quad t \in [0,T].
\end{aligned}$$

Setting $\sup_{t\in[0,T]} \varphi(t) = \|\varphi\|$ and choosing $\rho \ge \|\varphi\|\Omega$, where Ω is defined by (18), we consider $B_\rho = \{x \in \mathcal{C} : \|x\| \le \rho\}$. For any $x, y \in B_\rho$, we have

$$\begin{aligned}
|\mathcal{Q}_1 x(t) + \mathcal{Q}_2 y(t)| &\le \sup_{t\in[0,T]}\Bigg\{ J^q|f(s,x(s))|(t) + \frac{|\gamma|}{|\Lambda|}(|v_4|+T|v_2|)\, {}^{\rho}I^{\alpha} J^q |f(s,x(s))|(\zeta) \\
&\quad + \frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(|\delta|\, {}^{\rho}I^{\beta} J^q|f(s,x(s))|(\xi) + J^q|f(s,x(s))|(T)\Big)\Bigg\} \\
&\le \|\varphi\|\Bigg\{\frac{T^q}{\Gamma(q+1)} + \frac{|\gamma|(|v_4|+T|v_2|)\zeta^{q+\rho\alpha}}{|\Lambda|\rho^{\alpha}\Gamma(q+1)}\frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\alpha+\rho}{\rho}\right)} \\
&\quad + \frac{(|v_2|+T|v_1|)}{|\Lambda|}\left(\frac{|\delta|\xi^{q+\rho\beta}}{\rho^{\beta}\Gamma(q+1)}\frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\beta+\rho}{\rho}\right)} + \frac{T^q}{\Gamma(q+1)}\right)\Bigg\} \\
&= \|\varphi\|\Omega \le \rho.
\end{aligned}$$

This shows that $\mathcal{Q}_1 x + \mathcal{Q}_2 y \in B_\rho$. Using (22), it ca easily be established that $\mathcal{Q}_2$ is a contraction.

Continuity of f implies that the operator $\mathcal{Q}_1$ is continuous. Also, $\mathcal{Q}_1$ is uniformly bounded on B_ρ as

$$\|\mathcal{Q}_1 x\| \le \frac{T^q}{\Gamma(q+1)}\|\varphi\|.$$

Now we prove the compactness of the operator $\mathcal{Q}_1$.

We define $\sup_{(t,x)\in[0,T]\times B_\rho} |f(t,x)| = \bar{f} < \infty$, and consequently, for $t_1, t_2 \in [0,T]$, $t_1 < t_2$, we have

$$\begin{aligned}
|\mathcal{Q}_1 x(t_2) - \mathcal{Q}_1 x(t_1)| &= \Big| J^q f(s,x(s))(t_2) - J^q f(s,x(s))(t_1) \Big| \\
&\le \frac{\bar{f}}{\Gamma(q)}\left|\int_0^{t_1} [(t_2-s)^{q-1} - (t_1-s)^{q-1}]ds + \int_{t_1}^{t_2} (\tau_2 - s)^{q-1} ds\right|
\end{aligned}$$

$$\leq \frac{\bar{f}}{\Gamma(q+1)}[|t_2^q - t_1^q| + |t_2 - t_1|^q],$$

which tends to zero as $t_2 - t_1 \to 0$ is independent of x. Thus, $\mathcal{Q}_1$ is equicontinuous. So $\mathcal{Q}_1$ is relatively compact on B_ρ. Hence, by the Arzelá-Ascoli theorem, $\mathcal{Q}_1$ is compact on B_ρ. Thus all the assumptions of Lemma 3.3 are satisfied. So the conclusion of Lemma 3.3 implies that problem (1)-(2) has at least one solution on $[0, T]$ □

Theorem 3.5 *Assume that (H_1) and (H_2) hold. Then the problem (1)-(3) has at least one solution on $[0, T]$ provided*

$$L\left\{\frac{|\gamma|(|u_4| + T|u_2|)\zeta^{\alpha+q}}{|\Lambda_1|\Gamma(\alpha+q+1)} + \frac{(|v_2| + T|v_1|)}{|\Lambda_1|}\left(\frac{|\delta|\xi^{q+\rho\beta}}{\rho^\beta\Gamma(q+1)}\frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\beta+\rho}{\rho}\right)} + \frac{T^q}{\Gamma(q+1)}\right)\right\} < 1. \tag{23}$$

3.3 Existence and uniqueness result via nonlinear contractions

Definition 3.6 *Let E be a Banach space and let $\mathcal{F} : E \to E$ be a mapping. $\mathcal{F}$ is said to be a nonlinear contraction if there exists a continuous nondecreasing function $\Theta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Theta(0) = 0$ and $\Theta(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$ with the property:*

$$\|\mathcal{F}x - \mathcal{F}y\| \leq \Theta(\|x - y\|), \qquad \forall x, y \in E.$$

Lemma 3.7 *(Boyd and Wong)[22]. Let E be a Banach space and let $\mathcal{F} : E \to E$ be a nonlinear contraction. Then $\mathcal{F}$ has a unique fixed point in E.*

Theorem 3.8 *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption:*

(H_3) $|f(t, x) - f(t, y)| \leq z(t)\dfrac{|x - y|}{A^* + |x - y|}$, *for $t \in [0, T]$, $x, y \geq 0$, where $z : [0, T] \to \mathbb{R}^+$ is continuous and A^* is the constant given by*

$$A^* := J^q z(T) + \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q z(\zeta) + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big\{|\delta|\ {}^\rho I^\beta J^q z(\xi) + J^q z(T)\Big\}.$$

Then the problem (1)-(2) has a unique solution on $[0, T]$.

Proof. Consider the operator $\mathcal{Q} : \mathcal{C} \to \mathcal{C}$ defined by (16) and a continuous nondecreasing function $\Theta : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\Theta(\varepsilon) = \frac{A^*\varepsilon}{A^* + \varepsilon}, \qquad \forall \varepsilon \geq 0.$$

Note that the function Θ satisfies $\Theta(0) = 0$ and $\Theta(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$.

For any $x, y \in \mathcal{C}$ and for each $t \in [0, T]$, we have

$$\begin{aligned}
&|\mathcal{Q}x(t) - \mathcal{Q}y(t)| \\
\leq\ & \sup_{t\in[0,T]}\Bigg\{J^q|f(s, x(s)) - f(s, y(s))|(t) + \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q|f(s, x(s)) - f(s, y(s))|(\zeta) \\
&+\frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q|f(s, x(s)) - f(s, y(s))|(\xi) + J^q|f(s, x(s)) - f(s, y(s))|(T)\Big)\Bigg\} \\
\leq\ & J^q\left(z(s)\frac{|x - y|}{A^* + |x - y|}\right)(T) + \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q\left(z(s)\frac{|x - y|}{A^* + |x - y|}\right)(\zeta)
\end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\left\{|\delta|\ {}^\rho I^\beta J^q\left(z(s)\frac{|x-y|}{A^*+|x-y|}\right)(\xi)+J^q\left(z(s)\frac{|x-y|}{A^*+|x-y|}\right)(T)\right\} \\
\leq\ & \frac{\Theta(\|x-y\|)}{A^*}\Big[J^q z(T)+\frac{|\gamma|}{|\Lambda|}(|v_4|+T|v_2|)\ {}^\rho I^\alpha J^q z(\zeta)+\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big\{|\delta|\ {}^\rho I^\beta J^q z(\xi)+J^q z(T)\Big\}\Big] \\
=\ & \Theta(\|x-y\|).
\end{aligned}
$$

This implies that $\|\mathcal{Q}x-\mathcal{Q}y\|\leq\Theta(\|x-y\|)$. Therefore $\mathcal{Q}$ is a nonlinear contraction. Hence, by Lemma 3.7 the operator $\mathcal{Q}$ has a unique fixed point which is the unique solution of the problem (1)-(2). This completes the proof. □

Theorem 3.9 *Let $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying the assumption:*

$(H_3)'$ $|f(t,x)-f(t,y)|\leq z(t)\dfrac{|x-y|}{A_1^*+|x-y|}$, *for $t\in[0,T]$, $x,y\geq 0$, where $z:[0,T]\to\mathbb{R}^+$ is continuous and A_1^* is the constant given by*

$$A_1^* := J^q z(T)+\frac{|\gamma|}{|\Lambda|}(|u_4|+T|u_2|)\ J^{\alpha+q}z(\zeta)+\frac{1}{|\Lambda|}(|u_2|+T|u_1|)\Big\{|\delta|\ {}^\rho I^\beta J^q z(\xi)+J^q z(T)\Big\}.$$

Then the problem (1)-(3) has a unique solution on $[0,T]$.

3.4 Existence result via Schaefer fixed point theorem

Lemma 3.10 *[23] Let X be a Banach space. Assume that $T:X\to X$ is a completely continuous operator and the set $V=\{u\in X\mid u=\mu Tu, 0<\mu<1\}$ is bounded. Then T has a fixed point in X.*

Theorem 3.11 *Assume that there exists a positive constant L_1 such that $|f(t,x)|\leq L_1$ for $t\in[0,1]$, $x\in\mathbb{R}$. Then the boundary value problem (1)-(2) has at least one solution on $[0,T]$.*

Proof. As a first step, it will be shown that the operator $\mathcal{Q}$ defined by (16) is completely continuous. Observe that continuity of $\mathcal{Q}$ follows from the continuity of f. For a positive constant r, let $B_r=\{x\in\mathcal{C}:\|x\|\leq r\}$ be a bounded ball in $\mathcal{C}$. Then for $t\in[0,T]$ we have

$$
\begin{aligned}
|\mathcal{Q}x(t)| \leq\ & J^q|f(s,x(s))|(t)+\frac{|\gamma|}{|\Lambda|}(|v_4|+T|v_2|)\ {}^\rho I^\alpha J^q|f(s,x(s))|(\zeta) \\
& +\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q|f(s,x(s))|(\xi)+J^q|f(s,x(s))|(T)\Big) \\
\leq\ & L_1 J^q(1)(T)+L_1\frac{|\gamma|}{|\Lambda|}(|v_4|+T|v_2|)\ {}^\rho I^\alpha J^q(1)(\zeta) \\
& +L_1\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q(1)(\xi)+J^q(1)(T)\Big), \\
\leq\ & L_1\left\{\frac{T^q}{\Gamma(q+1)}+\frac{|\gamma|(|v_4|+T|v_2|)\zeta^{q+\rho\alpha}}{|\Lambda|\rho^\alpha\Gamma(q+1)}\frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\alpha+\rho}{\rho}\right)}\right. \\
& \left.+\frac{(|v_2|+T|v_1|)}{|\Lambda|}\left(\frac{|\delta|\xi^{q+\rho\beta}}{\rho^\beta\Gamma(q+1)}\frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\beta+\rho}{\rho}\right)}-\frac{T^q}{\Gamma(q+1)}\right)\right\} \\
=\ & L_1\Omega.
\end{aligned}
$$

Now, for $\tau_1,\tau_2\in[0,1]$ with $\tau_1<\tau_2$, we get

$$|\mathcal{Q}x(\tau_2)-\mathcal{Q}x(\tau_1)| \leq |J^q f(s,x(s))(\tau_2)-J^q f(s,x(s))(\tau_1)|+\frac{|\gamma||v_2||\tau_2-\tau_1|}{|\Lambda|}\ {}^\rho I^\alpha J^q|f(s,x(s))|(\zeta)$$

$$
\begin{aligned}
& +\frac{|v_1||\tau_2-\tau_1|}{|\Lambda|}\Big(|\delta|\ {}^\rho I^\beta J^q|f(s,x(s))|(\xi)+J^q|f(s,x(s))|(T)\Big)\\
\leq\ & \frac{L_1}{\Gamma(q)}\left|\int_0^{\tau_1}[(\tau_2-s)^{q-1}-(\tau_1-s)^{q-1}]ds+\int_{\tau_1}^{\tau_2}(\tau_2-s)^{q-1}ds\right|\\
& +\frac{L_1|\gamma||v_2||\tau_2-\tau_1|}{|\Lambda|}\ {}^\rho I^\alpha J^q(\zeta)+\frac{L_1|v_1||\tau_2-\tau_1|}{|\Lambda|}\Big(|\delta|\ {}^\rho I^\beta J^q(\xi)+J^q(T)\Big).
\end{aligned}
$$

As $\tau_2-\tau_1\to 0$, the right-hand side of the above inequality tends to zero independently of $x\in B_r$. Therefore by the Arzelá-Ascoli theorem the operator $\mathcal{Q}:\mathcal{C}\to\mathcal{C}$ is completely continuous.

Next, we consider the set $V=\{x\in\mathcal{C}: x=\mu\mathcal{Q}x,\ 0<\mu<1\}$. In order to show that V is bounded, let $x\in V$ and $t\in[0,T]$. Then

$$
\begin{aligned}
\|x\| \leq\ & L_1\left\{\frac{T^q}{\Gamma(q+1)}+\frac{|\gamma|(|v_4|+T|v_2|)\zeta^{q+\rho\alpha}}{|\Lambda|\rho^\alpha\Gamma(q+1)}\frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\alpha+\rho}{\rho}\right)}\right.\\
& \left.+\frac{(|v_2|+T|v_1|)}{|\Lambda|}\left(\frac{|\delta|\xi^{q+\rho\beta}}{\rho^\beta\Gamma(q+1)}\frac{\Gamma\left(\frac{q+\rho}{\rho}\right)}{\Gamma\left(\frac{q+\rho\beta+\rho}{\rho}\right)}-\frac{T^q}{\Gamma(q+1)}\right)\right\}\\
=\ & L_1\Omega.
\end{aligned}
$$

Therefore, V is bounded. Hence, by Lemma 3.10, the boundary value problem (1)-(2) has at least one solution. □

Theorem 3.12 *Assume that there exists a positive constant L_1 such that $|f(t,x)|\leq L_1$ for $t\in[0,1]$, $x\in\mathbb{R}$. Then the boundary value problem (1)-(3) has at least one solution on $[0,T]$.*

3.5 Existence result via Leray-Schauder's Degree Theory

Theorem 3.13 *Let $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Suppose that*

(H_4) *there exist constants $0\leq\nu<\Omega^{-1}$, and $M>0$ such that*

$$|f(t,x)|\leq\nu|x|+M \quad \text{for all} \quad (t,x)\in[0,T]\times\mathbb{R},$$

where Ω is defined by (18).

Then the boundary value problem (1)-(2) has at least one solution on $[0,T]$.

Proof. In view of the fixed point problem

$$x=\mathcal{Q}x, \tag{24}$$

where the operator $\mathcal{Q}:\mathcal{C}\to\mathcal{C}$ is given by (16), we have to establish that there exists at least one solution $x\in C[0,T]$ satisfying (24). Set a ball $B_R\subset C[0,T]$ with a constant radius $R>0$ as

$$B_R=\{x\in\mathcal{C}:\max_{t\in[0,T]}|x(t)|<R\}.$$

Then we have to show that the operator $\mathcal{Q}:\overline{B}_R\to C[0,T]$ satisfies the condition

$$x\neq\theta\mathcal{Q}x,\quad \forall x\in\partial B_R,\quad \forall\theta\in[0,1]. \tag{25}$$

Next, we introduce

$$H(\theta,x)=\theta\mathcal{Q}x,\quad x\in\mathcal{C},\quad \theta\in[0,1].$$

As shown in Theorem 3.16 we have that the operator $\mathcal{Q}$ is continuous, uniformly bounded and equicontinuous. Then, by the Arzelá-Ascoli theorem, a continuous map h_θ defined by $h_\theta(x)=x-H(\theta,x)=$

$x - \theta\mathcal{Q}x$ is completely continuous. If (25) holds, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned} \deg(h_\theta, B_R, 0) &= \deg(I - \theta\mathcal{Q}, B_R, 0) = \deg(h_1, B_R, 0) \\ &= \deg(h_0, B_R, 0) = \deg(I, B_R, 0) = 1 \neq 0, \quad 0 \in B_R, \end{aligned}$$

where I denotes the unit operator. By the nonzero property of Leray-Schauder degree, we have $h_1(x) = x - \mathcal{Q}x = 0$ for at least one $x \in B_R$. Let us assume that $x = \theta\mathcal{Q}x$ for some $\theta \in [0,1]$ and for all $t \in [0,T]$. Then

$$\begin{aligned} |x(t)| &= |\theta\mathcal{Q}x(t)| \\ &\le J^q|f(s,x(s))|(t) + \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q|f(s,x(s))|(\zeta) \\ &\quad + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q|f(s,x(s))|(\xi) + J^q|f(s,x(s))|(T)\Big) \\ &\le (\nu|x| + M)J^q p(s)(T) + (\nu|x| + M)\frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q(1)(\zeta) \\ &\quad + (\nu|x| + M)\frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q(1)(\xi) + J^q(1)(T)\Big) \\ &= (\nu|x| + M)\Omega, \end{aligned}$$

which, on taking the norm $\sup_{t\in[0,T]}|x(t)| = \|x\|$ and solving for $\|x\|$, yields

$$\|x\| \le \frac{M\Omega}{1 - \nu\Omega}.$$

If $R = \dfrac{M\Omega}{1-\nu\Omega} + 1$, (25) holds. This completes the proof. □

Theorem 3.14 *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that*

$(H_4)'$ *there exist constants $0 \le \nu < \Omega_1^{-1}$, and $M > 0$ such that*

$$|f(t,x)| \le \nu|x| + M \quad \text{for all} \quad (t,x) \in [0,T] \times \mathbb{R},$$

where Ω_1 is defined by (19).

Then the boundary value problem (1)-(3) has at least one solution on $[0,T]$.

3.6 Existence result via Leray-Schauder's nonlinear alternative

Lemma 3.15 *(Nonlinear alternative for single valued maps [24]). Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $\mathcal{A} : \bar{U} \to C$ is a continuous, compact (that is, $\mathcal{A}(\bar{U})$ is a relatively compact subset of C) map. Then either*

(i) $\mathcal{A}$ has a fixed point in $\bar{U}$, or

(ii) there is a $x \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $x = \lambda\mathcal{A}(x)$.

Theorem 3.16 *Assume that*

(H_5) *there exists a continuous nondecreasing function $\Phi : [0,\infty) \to (0,\infty)$ and a function $p \in L^1([0,T],\mathbb{R}^+)$ such that*

$$|f(t,x)| \le p(t)\Phi(\|x\|) \quad \text{for each} \quad (t,x) \in [0,T] \times \mathbb{R};$$

(H_6) *there exists a constant* $N > 0$ *such that*

$$\frac{N}{\Phi(N)\Big\{J^q p(s)(T) + A_1 + A_2\Big\}} > 1,$$

where

$$\begin{aligned} A_1 &= \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q p(s)(\zeta), \\ A_2 &= \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q p(s)(\xi) + J^q p(s)(T)\Big). \end{aligned}$$

Then the boundary value problem (1)-(2) has at least one solution on $[0, T]$.

Proof. Let the operator $\mathcal{Q}$ be defined by (16). We first show that $\mathcal{Q}$ *maps bounded sets (balls) into bounded sets in* $C([0, T], \mathbb{R})$. For a positive constant r, let $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$ be a bounded ball in $\mathcal{C}$. Then for $t \in [0, T]$ we have

$$\begin{aligned} |\mathcal{Q}x(t)| &\leq J^q|f(s, x(s))|(t) + \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q|f(s, x(s))|(\zeta) \\ &\quad + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q|f(s, x(s))|(\xi) + J^q|f(s, x(s))|(T)\Big) \\ &\leq \Phi(\|x\|)J^q p(s)(T) + \Phi(\|x\|)\frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q p(s)(\zeta) \\ &\quad + \Phi(\|x\|)\frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q p(s)(\xi) + J^q p(s)(T)\Big), \end{aligned}$$

and consequently,

$$\begin{aligned} \|\mathcal{Q}x\| &\leq \Phi(r)\Big\{J^q p(s)(T) + \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q p(s)(\zeta) \\ &\quad + \frac{1}{|\Lambda|}(|v_2| + T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q p(s)(\xi) + J^q p(s)(T)\Big)\Big\}. \end{aligned}$$

Next we will show that *the operator* $\mathcal{Q}$ *maps bounded sets into equicontinuous sets of* $\mathcal{C}$. Let $\tau_1, \tau_2 \in [0, T]$ with $\tau_1 < \tau_2$ and $x \in B_r$. Then we have

$$\begin{aligned} |\mathcal{Q}x(\tau_2) - \mathcal{Q}x(\tau_1)| &\leq |J^q f(s, x(s))(\tau_2) - J^q f(s, x(s))(\tau_1)| + \frac{|\alpha||v_2||\tau_2 - \tau_1|}{|\Lambda|}\ {}^\rho I^\alpha J^q|f(s, x(s))|(\zeta) \\ &\quad + \frac{|v_1||\tau_2 - \tau_1|}{|\Lambda|}\Big(|\delta|\ {}^\rho I^\beta J^q|f(s, x(s))|(\xi) + J^q|f(s, x(s))|(T)\Big) \\ &\leq \frac{\Phi(r)}{\Gamma(q)}\left|\int_0^{\tau_1}[(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}]p(s)ds + \int_{\tau_1}^{\tau_2}(\tau_2 - s)^{q-1}p(s)ds\right| \\ &\quad + \frac{\Phi(r)|\gamma||v_2||\tau_2 - \tau_1|}{|\Lambda|}\ {}^\rho I^\alpha J^q p(s)(T) \\ &\quad + \frac{\Phi(r)|v_1||\tau_2 - \tau_1|}{|\Lambda|}\Big(|\delta|\ {}^\rho I^\beta J^q p(s)(\xi) + J^q p(s)(T)\Big). \end{aligned}$$

As $\tau_2 - \tau_1 \to 0$, the right-hand side of the above inequality tends to zero independently of $x \in B_r$. Therefore by the Arzelá-Ascoli theorem the operator $\mathcal{Q} : \mathcal{C} \to \mathcal{C}$ is completely continuous.

Finally, we show that there exists an open set $U \subset \mathcal{C}$ with $x \neq \theta\mathcal{P}x$ for $\theta \in (0, 1)$ and $x \in \partial U$.

Let x be a solution. Then, for $t \in [0, T]$, and following the similar computations as in the first step, we have

$$|x(t)| \leq \Phi(\|x\|)\Big\{J^q p(s)(T) + \frac{|\gamma|}{|\Lambda|}(|v_4| + T|v_2|)\ {}^\rho I^\alpha J^q p(s)(\zeta)$$

$$+\frac{1}{|\Lambda|}(|v_2|+T|v_1|)\Big(|\delta|\ {}^\rho I^\beta J^q p(s)(\xi)+J^q p(s)(T)\Big)\Big\}$$

which leads to

$$\frac{\|x\|}{\Phi(\|x\|)\Big\{J^q p(s)(T)+A_1+A_2\Big\}}\le 1.$$

In view of (H_6), there exists N such that $\|x\|\neq N$. Let us set

$$\mathcal{U}=\{x\in C([0,T],\mathbb{R})\,:\,\|x\|<N\}.$$

We see that the operator $\mathcal{Q}:\overline{\mathcal{U}}\to C([0,T],\mathbb{R})$ is continuous and completely continuous. From the choice of $\mathcal{U}$, there is no $x\in\partial\mathcal{U}$ such that $x=\theta\mathcal{Q}x$ for some $\theta\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type, we deduce that $\mathcal{Q}$ has a fixed point $x\in\overline{\mathcal{U}}$ which is a solution of the boundary value problem (1)-(2). This completes the proof. □

Theorem 3.17 *Assume that (H_5) holds. In addition we suppose that:*

$(H_6)'$ *there exists a constant $N'>0$ such that*

$$\frac{N'}{\Phi_1(N')\Big\{J^q p(s)(T)+A_1'+A_2'\Big\}}>1, \tag{26}$$

where

$$\begin{aligned} A_1' &= \frac{|\gamma|}{|\Lambda_1|}(|u_4|+T|u_2|)J^{\alpha+q}p(s)(\zeta),\\ A_2' &= \frac{1}{|\Lambda_1|}(|u_2|+T|u_1|)\Big(|\delta|\ {}^\rho I^\beta J^q p(s)(\xi)+J^q p(s)(T)\Big). \end{aligned}$$

Then the boundary value problem (1)-(3) has at least one solution on $[0,T]$.

4 Examples

In this section, we present some examples to illustrate our results.

Example 4.1 *Consider the following nonlocal boundary value problem involving generalized Riemann-Liouville fractional integral boundary conditions*

$$\begin{cases} D^{\frac{3}{2}}x(t)=\dfrac{3}{25}\left(\dfrac{4x^2(t)+5|x(t)|}{3+4|x(t)|}\right)e^{-2t}+\dfrac{1}{2}\cos^2 t+1, & t\in\left[0,\dfrac{5}{3}\right],\\ x(0)=\dfrac{1}{2}\,{}^{\frac{\sqrt{3}}{2}}I^{\frac{4}{\sqrt{3}}}x\left(\dfrac{2}{3}\right), \quad x\left(\dfrac{5}{3}\right)=\dfrac{3}{4}\,{}^{\frac{\sqrt{3}}{2}}I^{\frac{\pi}{2}}x\left(\dfrac{4}{3}\right), \end{cases} \tag{27}$$

where $q=3/2$, $T=5/3$, $\gamma=1/2$, $\rho=\sqrt{3}/2$, $\alpha=4/\sqrt{3}$, $\zeta=2/3$, $\delta=3/4$, $\beta=\pi/2$, $\xi=4/3$ and $f(t,x)=(3/25)((4x^2+5|x|)/(3+4|x|))e^{-2t}+(1/2)\cos^2 t+1$. Using given information, we find that $v_1=0.8856776719$, $v_2=0.02007036728$, $v_3=0.0060494642$, $v_4=1.202612652$, $\Lambda=1.065248589$ and $\Omega=4.304419870$. Also $|f(t,x)-f(t,y)|\le(1/5)|x-y|$. Thus the condition (H_1) is satisfied with $L=1/5$ and $L\Omega=0.8608839740<1$. Therefore, by Theorem 3.1, problem (27) has a unique solution on $[0,5/3]$.

Example 4.2 *Consider the following nonlocal boundary value problem*

$$\begin{cases} D^{\frac{5}{3}}x(t)=\dfrac{5}{48}(1+\sin^2 t)\dfrac{|x(t)|}{1+|x(t)|}+3t^2+\dfrac{2}{3}, & t\in\left[0,\dfrac{7}{4}\right],\\ x(0)=\dfrac{3}{2}\,{}^{\frac{5}{6}}I^{\frac{e}{\sqrt{2}}}x\left(\dfrac{5}{4}\right), \quad x\left(\dfrac{7}{4}\right)=\dfrac{4}{5}\,{}^{\frac{5}{6}}I^{\frac{11}{13}}x\left(\dfrac{3}{4}\right). \end{cases} \tag{28}$$

Here $q = 5/3$, $T = 7/4$, $\gamma = 3/2$, $\rho = 5/6$, $\alpha = e/\sqrt{2}$, $\zeta = 5/4$, $\delta = 4/5$, $\beta = 11/13$, $\xi = 3/4$ and $f(t,x) = (5(1+\sin^2 t)/48)(|x|/(1+|x|)) + 3t^2 + (2/3)$. Using the given data, we obtain $v_1 = -0.633695322$, $v_2 = 0.5982054854$, $v_3 = 0.1931118977$, $v_4 = 1.448388097$, $\Lambda = -0.8023161650 \neq 0$. As $|f(t,x) - f(t,y)| \leq (5/24)|x-y|$, we have that (H_1) is satisfied with $L = 5/24$. Further, we have $\Omega_2 = 0.9828570350 < 1$. Also

$$|f(t,x)| \leq \frac{5}{48}(1+\sin^2 t) + 3t^2 + \frac{2}{3} := \varphi(t),$$

which implies that the condition (H_2) holds true. In consequence, the conclusion of Theorem 3.4 applies and problem (28) has at least one solution on $[0, 7/4]$.

Example 4.3 *Consider the following nonlocal boundary value problem*

$$\begin{cases} D^{\frac{4}{3}}x(t) = \frac{1}{4}(t^{\frac{1}{3}}+1)\left(\frac{|x(t)|}{1+|x(t)|}\right) + \frac{3}{2}t + \frac{1}{3}, & t \in \left[0, \frac{1}{2}\right], \\ x(0) = \frac{2}{\sqrt{\pi}}\, {}^{\frac{1}{\sqrt{3}}}I^{\frac{7}{4}}x\left(\frac{1}{4}\right), \quad x\left(\frac{1}{2}\right) = \frac{3}{e^2}\, {}^{\frac{1}{\sqrt{3}}}I^{\frac{8}{13}}x\left(\frac{1}{8}\right). \end{cases} \tag{29}$$

Here $q = 4/3$, $T = 1/2$, $\gamma = 2/\sqrt{\pi}$, $\rho = 1/\sqrt{3}$, $\alpha = 7/4$, $\zeta = 1/4$, $\delta = 3/e^2$, $\beta = 8/13$, $\xi = 1/8$ and $f(t,x) = ((t^{1/3}+1)/4)(|x|/(1+|x|)) + (3/2)t + (1/3)$. Using the previous information, we have $v_1 = 0.5478797820$, $v_2 = 0.02539640314$, $v_3 = 0.6962686485$, $v_4 = 0.4808910650$ and $\Lambda = 0.2811532112$. Choosing $z(t) = (t^{1/3}+1)/4$, find that $A^* = 0.2768779852$ and also

$$|f(t,x) - f(t,y)| \leq \frac{1}{4}(t^{\frac{1}{3}}+1)\frac{|x-y|}{0.2768779852 + |x-y|}.$$

Therefore, all assumptions of Theorem 3.8 are satisfied. Hence the problem (29) has at least one solution on $[0, 1/2]$.

Example 4.4 *Consider the following nonlocal boundary value problem with both Riemann-Liouville and generalized Riemann-Liouville fractional integral boundary conditions*

$$\begin{cases} D^{\frac{5}{4}}x(t) = \tan^{-1}\left(\frac{x^4(t)+3x^2(t)}{1+|x(t)|}\right)\left(e^{\frac{3}{2}-t}+1\right) + 3\pi, & t \in \left[0, \frac{3}{2}\right], \\ x(0) = \frac{4}{\sqrt{7}} J^{\frac{5}{\sqrt{3}}}x\left(\frac{1}{2}\right), \quad x\left(\frac{3}{2}\right) = \frac{\pi}{2}\, {}^{\frac{2}{7}}I^{\frac{3}{8}}x\left(\frac{5}{4}\right). \end{cases} \tag{30}$$

Here $q = 5/4$, $T = 3/2$, $\gamma = 4/\sqrt{7}$, $\alpha = 5/\sqrt{3}$, $\zeta = 1/2$, $\delta = \pi/2$, $\rho = 2/7$, $\beta = 3/8$, $\xi = 5/4$ and $f(t,x) = \tan^{-1}((x^4+3x^2)/(1+|x|))(e^{(3/2)-t}+1) + 3\pi$. From the given constants, we have $u_1 = 0.9607949552$, $u_2 = 0.005043420754$, $u_3 = -1.895136694$, $u_4 = -0.378780447$ and $\Lambda_1 = -0.3734883143 \neq 0$. As $f(t,x) \leq 4\pi := L_1$ for all $x \in \mathbb{R}$, therefore from Theorem 3.11, the problem 30 has at least one solution on $[0, 3/2]$.

Example 4.5 *Consider the following nonlocal boundary value problem subjected to both Riemann-Liouville and generalized Riemann-Liouville fractional integral boundary conditions*

$$\begin{cases} D^{\frac{8}{5}}x(t) = \frac{1}{(t^{\frac{1}{2}}+10)^2}\left(\frac{10x^2(t)+1}{3+|x(t)|}\right) + e^{-|x(t)|} + \frac{1}{3}, & t \in [0, \pi], \\ x(0) = \frac{\log 2}{\sqrt{3}} J^{\frac{3}{4}}x\left(\frac{\pi}{2}\right), \quad x(\pi) = \frac{\log 3}{\sqrt{8}}\, {}^{\frac{5}{\sqrt{7}}}I^{\frac{3}{\sqrt{e}}}x\left(\frac{\pi}{3}\right). \end{cases} \tag{31}$$

Here $q = 8/5$, $T = \pi$, $\gamma = \log 2/\sqrt{3}$, $\alpha = 3/4$, $\zeta = \pi/2$, $\delta = \log 3/\sqrt{8}$, $\rho = 5/\sqrt{7}$, $\beta = 3/\sqrt{e}$, $\xi = \pi/3$ and $f(t,x) = (1/(t^{1/2}+10)^2)((10x^2+1)/(3+|x|)) + e^{-|x|} + (1/3)$. By direct computation of given constants, we obtain $u_1 = 0.9607949552$, $u_2 = 0.2381638392$, $u_3 = 0.9635754531$, $u_4 = 3.121155944$ and $\Lambda_1 = 3.228279714 \neq 0$. In addition, we can find that $\Omega_1 = 8.997039531$. It is easy to see that

$$|f(t,x)| \leq \frac{1}{10}|x| + \frac{4}{3},$$

which leads to $\nu := 1/10 < \Omega_1^{-1} = 0.1111476721$ and $M := 4/3 > 0$. Applying the conclusion of Theorem 3.13, we get that the problem (31) has at least one solution on $[0, \pi]$.

Example 4.6 *Consider the following nonlocal boundary value problem supplemented with both Riemann-Liouville and generalized Riemann-Liouville fractional integral boundary conditions*

$$\begin{cases} D^{\frac{7}{4}}x(t) = \dfrac{(\sqrt{t}+1)}{12}\left(\dfrac{x^2(t)\sin^2 x(t)}{3(1+|x(t)|)} + e^{-t}\cos^2 t\right), & t \in \left[0, \dfrac{12}{5}\right], \\ x(0) = \dfrac{1}{\sqrt{3}} J^{\frac{7}{9}} x\left(\dfrac{8}{5}\right), \quad x\left(\dfrac{12}{5}\right) = \dfrac{3}{16}\, {}^{\frac{1}{\sqrt{\pi}}} I^{\frac{1}{\sqrt{e}}} x\left(\dfrac{11}{5}\right), \end{cases} \tag{32}$$

where $q = 7/4$, $T = 12/5$, $\gamma = 1/\sqrt{3}$, $\alpha = 7/9$, $\zeta = 8/5$, $\delta = 3/16$, $\rho = 1/\sqrt{\pi}$, $\beta = 1/\sqrt{e}$, $\xi = 11/15$ and $f(t,x) = ((\sqrt{t}+1)/12)((x^2\sin^2 x)/(3(1+|x|)) + e^{-t}\cos^2 t)$. By the given values, we get $u_1 = 0.1010372543$, $u_2 = 0.8090664711$, $u_3 = 0.6114216572$, $u_4 = 1.970342759$, $\Lambda_1 = 0.6937587849 \neq 0$. Since

$$|f(t,x)| \leq \frac{(\sqrt{t}+1)}{12}\left(\frac{1}{3}|x| + 1\right) := p(t)\Phi_1(|x|),$$

the condition (H_4) is satisfied. Also $A_1' = 0.4202876316$, $A_2' = 0.7604168186$. Clearly condition (26) is satisfied for $N' > 3.560603169$. Therefore, by Theorem 3.17, problem (32) has at least one solution on $[0, 12/5]$.

Acknowledgement:

This research is partially supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand.

References

[1] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[2] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[3] Y. Zhou, Basic Theory of Fractional Differential Equations, World Scientific Publishing Co. Pte. Ltd., Hackensack, New Jersey, 2014.

[4] A. B. Malinowska, T. Odzijewicz, D. F. M. Torres, *Advanced Methods in the Fractional Calculus of Variations*, Springer, 2015.

[5] U. N. Katugampola, New Approach to a generalized fractional integral, *Appl. Math. Comput.* **218** (2011), 860-865.

[6] J. T. Machado, V. Kiryakova F. Mainardi, Recent history of fractional calculus, *Commun. Nonlinear Sci. Numer. Simulat.* **16** (2011), 1140-1153.

[7] B. Ahmad, J.J. Nieto, Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions, *Bound. Value Probl.* 2011:36 (2011).

[8] J.R. Graef, L. Kong, Q. Kong, Application of the mixed monotone operator method to fractional boundary value problems, *Fract. Calc. Differ. Calc.* **2** (2011), 554-567.

[9] Z.B. Bai, W. Sun, Existence and multiplicity of positive solutions for singular fractional boundary value problems, *Comput. Math. Appl.* **63** (2012), 1369-1381.

[10] B. Ahmad, S.K. Ntouyas, A. Alsaedi, A study of nonlinear fractional differential equations of arbitrary order with Riemann-Liouville type multistrip boundary conditions, *Math. Probl. Eng.* 2013 (2013), Art. ID 320415, 9 pp.

[11] L. Zhang, B. Ahmad, G. Wang, R.P. Agarwal, Nonlinear fractional integro-differential equations on unbounded domains in a Banach space, *J. Comput. Appl. Math.* **249** (2013), 51-56. (2013).

[12] J. Henderson, N. Kosmatov, Eigenvalue comparison for fractional boundary value problems with the Caputo derivative, *Fract. Calc. Appl. Anal.* **17** (2014), 872-880.

[13] J.R. Wang, Y. Zhou, M. Feckan, On the nonlocal Cauchy problem for semilinear fractional order evolution equations, *Cent. Eur. J. Math.* **12** (2014), 911-922.

[14] X. Liu, Z. Liu, X. Fu, Relaxation in nonconvex optimal control problems described by fractional differential equations, *J. Math. Anal. Appl.* **409** (2014), 446-458.

[15] S.K. Ntouyas, S. Etemad, On the existence of solutions for fractional differential inclusions with sum and integral boundary conditions, *Appl. Math. Comp.* **266** (2015), 235-243.

[16] S.K. Ntouyas, S. Etemad, J. Tariboon, Existence of solutions for fractional differential inclusions with integral boundary conditions, *Bound. Value Prob.* **2015:92** (2015)

[17] S.K. Ntouyas, S. Etemad, J. Tariboon, Existence results for multi-term fractional differential inclusions, *Adv. Differ. Equ.* **2015:140** (2015).

[18] B. Ahmad, S.K. Ntouyas, Some fractional-order one-dimensional semi-linear problems under nonlocal integral boundary conditions, *Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Math. RACSAM* **110** (2016), 159-172.

[19] C. Thaiprayoon, S.K. Ntouyas, J. Tariboon, On the nonlocal Katugampola fractional integral conditions for fractional Langevin equation, *Adv. Differ. Equ.* (2015) **2015:374**.

[20] L. Zhang, Z. Xuan, Multiple positive solutions for a second-order boundary value problem with integral boundary conditions, *Bound. Value Probl.* 2016:60 (2016).

[21] M.A. Krasnoselskii, Two remarks on the method of successive approximations, *Uspekhi Mat. Nauk* **10** (1955), 123-127.

[22] D.W. Boyd, J.S.W. Wong, On nonlinear contractions, *Proc. Amer. Math. Soc.* **20** (1969), 458-464.

[23] D.R. Smart, *Fixed Point Theorems*, Cambridge University Press, 1980.

[24] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

On entire function sharing a small function CM with its high order forward difference operator

Jie Zhang, Hai Yan Kang *

College of Science, China University of Mining and Technology, Xuzhou 221116, PR China

Email: zhangjie1981@cumt.edu.cn, haiyankang@cumt.edu.cn

Liang Wen Liao

Department of Mathematics, Nanjing University, Nanjing 210093, PR China

Email: maliao@nju.edu.cn

August 31, 2016

Abstract: In this paper, we investigate the uniqueness of an entire function of finite order sharing a small entire function with its high order forward difference operator. The results obtained extend some known theorems and also show the exact solutions of some certain difference equations.

Key words and phrases: uniqueness; entire function; difference equation; differential equation; small function.

2000 Mathematics Subject Classification: 30D35; 34M10.

1 Introduction and main results

In this paper, a meromorphic function always means it is meromorphic in the whole complex plane C. We assume that the reader is familiar with the standard notations in the Nevanlinna theory. We use the standard notations such as $T(r,f), m(r,f), N(r,f), \overline{N}(r,f)$ in value distribution theory (see [11, 18, 19]). And we denote by $S(r,f)$ any quantity satisfying $S(r,f) = o\{T(r,f)\}$, as $r \to \infty$, possibly outside of a set E with finite linear or logarithmic measure, not necessarily the same at each occurrence. A meromorphic function a is said to be a small function with respect to f if and only if $T(r,a) = S(r,f)$. We use $\lambda(f)$ and $\sigma(f)$ to denote the exponent of convergence of zeros of f and the order of f respectively. We say that two meromorphic functions f and g share a value a IM (ignoring multiplicities) if $f-a$ and $g-a$ have the same zeros. If $f-a$ and $g-a$ have the same zeros with the same multiplicities, then we say that they share the value a CM (counting multiplicities). We define the forward difference operator $\Delta f = f(z+1) - f(z)$ and the high order forward difference operator $\Delta^n f = \Delta^{n-1}(\Delta f)$ by recurrence. Moreover, $\Delta^n f = \sum_{j=0}^{n} C_n^j (-1)^{n-j} f(z+j)$.

*Corresponding author. This research was supported by the Fundamental Research Funds for the Central Universities (No. 2015QNA52) and NSF of China (No. 11601507).

In 1976, L. Rubel and C.C. Yang [7] studied the uniqueness of an entire function sharing two values with its derivative and they proved the following classical result.

Theorem 1 *Let f be a nonconstant entire function. If f and f' share two distinct finite values CM, then $f \equiv f'$.*

In 1996, R. Brück [2] studied the uniqueness theory about an entire function sharing one value with its first derivative and posed the following interesting conjecture.

Conjecture 1 *Let f be nonconstant entire function satisfying that the super order $\sigma_2(f) < \infty$ is not a positive integer. If f and f' share one finite value a CM, then $f' - a = c(f - a)$ holds for some nonzero constant c.*

It is well known that Δf can be considered as the difference counterpart of f'. So regarding Theorem A and Conjecture, it is natural to ask that what can be said about the relationship between Δf and f if they share one or two values CM. The difference analogue of the lemma on the logarithmic derivative and Nevanlinna theory for the difference operator have been founded recently (see [3, 8, 9]), which brings about a number of papers focusing on such uniqueness problems. The authors in [17, 16, 20], for example, obtained the following results by considering the special case of entire functions of order less than 1 or 2 respectively.

Theorem 2 *[17] Let f be a transcendental entire function such that $\sigma(f) < 1$, n be a positive integer and η be a nonzero complex number. If f and $\Delta_\eta^n f$ share a finite value a CM, then $\Delta_\eta^n f - a = c(f - a)$ holds for some nonzero complex number c.*

Theorem 3 *[16] Let f be a transcendental entire function of order $\sigma(f) < 2$ and $\eta \neq 0$ be a complex number that is not a period of f. If f and $\Delta_\eta^n f$ share the value 0 CM, then $\Delta_\eta^n f/f$ reduces to a nonzero constant.*

Theorem 4 *[20] Let f be a transcendental entire function such that $\sigma(f) < 2$ and $\lambda(f) < \sigma(f)$. If f and $\Delta^n f$ share the value 0 CM, then f must be form of $f(z) = Ae^{\alpha z}$, where A and α are two nonzero constants.*

In this paper, we deal with the general case of entire function of finite order and obtain the following results which extend Theorem 2 and Theorem 4.

Theorem 5 *Let f be a transcendental entire function such that $\sigma(f) < \infty$, let $a \not\equiv 0$ be an entire function such that $\sigma(a) < 1$ and $\lambda(f - a) < \sigma(f)$. If f and $\Delta^n f$ share a CM, then a must reduce to a polynomial with degree at most $n - 1$ and f must be form of*

$$f(z) = a + bae^{\beta z},$$

where b and β are two nonzero constants such that $e^\beta = 1$.

2

Theorem 6 *Let f be a transcendental entire function such that $\lambda(f) < \sigma(f) < \infty$, let $a \not\equiv 0$ be an entire function such that $\sigma(a) < \sigma(f)$. If f and $\Delta^n f$ share a CM, then f must be form of $f(z) = be^{\beta z}$, where b and β are two nonzero constants such that $(e^{\beta} - 1)^n = 1$.*

Theorem 7 *Let f be a transcendental entire function such that $\lambda(f) < \max\{\sigma(f) - 1, 1\} < \infty$. If $f(z)$ and $\Delta^n f$ share the value 0 CM, then f must be form of $f(z) = he^{\beta z}$, where h and β are two nonzero constants.*

2 Some lemmas

Lemma 1 *(see[3]) Let f be a transcendental meromorphic function with finite order σ and η be a nonzero complex number, then for each $\varepsilon > 0$, we have*

$$T(r, f(z+\eta)) = T(r, f) + O(r^{\sigma-1+\varepsilon}) + O(\log r),$$

$$i.e.,\ T(r, f(z+\eta)) = T(r, f) + S(r, f).$$

Lemma 2 *(see[3]) Let f be a transcendental meromorphic function with finite order σ. Then for each $\varepsilon > 0$, we have*

$$m\big(r, \frac{f(z+c)}{f(z)}\big) = O(r^{\sigma-1+\varepsilon}).$$

Lemma 3 *(see[3]) Let η be a nonzero complex number and f be a meromorphic function of finite order σ. Let $\varepsilon > 0$ be given, then there exists a subset $E \subset (1, \infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin E \cup [0, 1]$, we have*

$$e^{-r^{\sigma-1+\varepsilon}} \leq \left|\frac{f(z+\eta)}{f(z)}\right| \leq e^{r^{\sigma-1+\varepsilon}}.$$

Lemma 4 *(see [4]) Let f be a nonconstant meromorphic function of order $\sigma < \infty$, and let λ' and λ'' be, respectively, the exponent of convergence of the zeros and poles of f. Then for any given $\varepsilon > 0$, there exists a set $E \subset (1, +\infty)$ of $|z| = r$ of finite logarithmic measure, so that*

$$2\pi i n_{z,\eta} + \log\frac{f(z+\eta)}{f(z)} = \eta\frac{f'(z)}{f(z)} + O(r^{\beta+\varepsilon}), \tag{1}$$

or equivalently,

$$\frac{f(z+\eta)}{f(z)} = e^{\eta\frac{f'(z)}{f(z)} + O(r^{\beta+\varepsilon})},$$

holds for $r \notin E \cup [0, 1]$, where $n_{z,\eta}$ in (1) is an integer depending on both z and η, $\beta = \max\{\sigma - 2, 2\lambda - 2\}$ if $\lambda < 1$ and $\beta = \max\{\sigma - 2, \lambda - 1\}$ if $\lambda \geq 1$ and $\lambda = \max\{\lambda', \lambda''\}$.

Lemma 5 *(see [5]) Suppose that $P(z) = (\alpha+i\beta)z^n + \cdots$ (α, β are real numbers, $|\alpha| + |\beta| \neq 0$) is a polynomial with degree $n \geq 1$, that $A(z) \not\equiv 0$ is an entire function with $\sigma(A) < n$. Set $g(z) = A(z)e^{P(z)}, z = re^{i\theta}, \delta(P,\theta) = \alpha\cos n\theta - \beta\sin n\theta$. Then for any given $\varepsilon > 0$, there exists a set $H_1 \subset [0, 2\pi)$ that has the linear measure zero, such that for any $\theta \in [0, 2\pi) \setminus (H_1 \cup H_2)$, there is $R > 0$ such that for $|z| = r > R$, we have*
(i) if $\delta(P,\theta) > 0$, then

$$\exp\{(1-\varepsilon)\delta(P,\theta)r^n\} < |g(re^{i\theta})| < \exp\{(1+\varepsilon)\delta(P,\theta)r^n\};$$

(ii) if $\delta(P,\theta) < 0$, then

$$\exp\{(1+\varepsilon)\delta(P,\theta)r^n\} < |g(re^{i\theta})| < \exp\{(1-\varepsilon)\delta(P,\theta)r^n\},$$

where $H_2 = \{\theta \in [0, 2\pi); \delta(P,\theta) = 0\}$ is a finite set.

Lemma 6 *(see [1]) Let g be a transcendental function of order less than 1, and h be a positive constant. Then there exists an ε set E such that*

$$\frac{g'(z+\eta)}{g(z+\eta)} \to 0, \ \frac{g(z+\eta)}{g(z)} \to 1, \mathit{as}\ z \to \infty\ \mathit{in}\ C\backslash E$$

uniformly in η for $|\eta| \leq h$. Further, the set E may be chosen so that if $z \notin E$ and $|z|$ is sufficiently large, the function g has no zeroes or poles in $|\zeta - z| \leq h$.

Remark 1 *According to Hayman [12], an ε set is defined to be a countable union of open discs not containing the origin and subtending angles at the origin whose sum is finite. Suppose E is an ε set, then the set of $r \geq 1$ for which the circle $S(0,r)$ meets E has finite logarithmic measure and for almost all real θ the intersection of E with the ray $\arg z = \theta$ is bounded.*

Lemma 7 *(see [18]) Suppose that $f_1, f_2, \cdots, f_n$ ($n \geq 2$) are meromorphic functions and $g_1, g_2, \cdots, g_n$ are entire functions satisfying the following conditions:*

(i) $\sum_{j=1}^{n} f_j e^{g_j} \equiv 0;$
(ii) $g_j - g_k$ *are not constants for* $1 \leq j < k \leq n;$
(iii) For $1 \leq j \leq n, 1 \leq h < k \leq n,\ T(r, f_j) = o\{T(r, e^{g_h - g_k})\}(r \to \infty, r \notin E).$

Then $f_j \equiv 0$ ($j = 1, 2, \cdots, n$).

Lemma 8 *(see [6]) Let w be a transcendental meromorphic function with $\sigma < \infty$. Let $\Gamma = \{(k_1, j_1), \cdots, (k_m, j_m)\}$ be a finite set of distinct pairs of integers satisfying $k_i > j_i \geq 0$ for $i = 1, 2, \cdots, m$. Also let $\varepsilon > 0$ be a given constant, then there exists a set $E \subset (1, +\infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| \notin E \cup [0, 1]$ and for all $(k, j) \in \Gamma$, one has*

$$\left|\frac{w^{(k)}(z)}{w^{(j)}(z)}\right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}.$$

Lemma 9 *(see[18]) Let f be a nonconstant meromorphic function in the complex plane and $R(f) = p(f)/q(f)$, where $p(f) = \sum_{k=0}^{p} a_k f^k$ and $q(f) = \sum_{j=0}^{q} b_j f^j$ are two mutually prime polynomials in f. If the coefficients a_k, b_j are small functions of f and $a_k \not\equiv 0, b_j \not\equiv 0$, then*

$$T(r, R(f)) = \max\{p, q\} T(r, f) + S(r, f).$$

Lemma 10 *Let g be polynomial of degree at lest two. Then*

$$m\big(r, \sum_{j=0}^{n} a_j e^{g(z+j)-g(z)}\big) = m(r, e^{g(z+n)-g(z)}) + S(r, e^{g(z+n)-g(z)}),$$

where the coefficients a_j are small meromorphic functions of $e^{g(z+n)-g(z)}$.
Proof. Set $g(z) = a_l z^l + a_{l-1} z^{l-1} + \ldots + a_0$, $a_l \neq 0$, $l \geq 2$ and $H(z) = e^{l a_l z^{l-1}}$. Then we get $g(z+j) - g(z) = j l a_l z^{l-1} + \cdots$, and then $e^{g(z+j)-g(z)} = b_j e^{j l a_l z^{l-1}}$, where $\sigma(b_j) \leq l - 2$. So we have

$$\sum_{j=0}^{n} a_j e^{g(z+j)-g(z)} = \sum_{j=0}^{n} \tilde{a}_j e^{j l a_l z^{l-1}} = \sum_{j=0}^{n} \tilde{a}_j H^j,$$

where $\tilde{a}_j = a_j b_j$ are small function of H. Application Lemma 9 to the equation above gives our conclusion immediately.

Lemma 11 *Let f be a transcendental entire function such that $2 \leq \sigma(f) < \infty$, also let $a \not\equiv 0$ be an entire function such that $\sigma(a) < \sigma(f)$ and $\lambda(f - a) < \sigma(f)$. If the difference equation*

$$\Delta^n f - a = (f - a) e^Q \tag{2}$$

holds, where Q is a nonconstant entire function, then Q is a polynomial such that $\deg Q = \sigma(f) - 1$.

Proof. From our assumption and Lemma 1, it is obvious for us to get that Q is a polynomial and

$$F := f - a = h e^g \tag{3}$$

holds, where g is a polynomial with degree l satisfying $l = \sigma(f) \geq 2$, and h is an entire function originated from the canonical product of $f - a$ satisfying $\lambda(h) = \sigma(h) < \sigma(f)$. Set $g(z) = a_l z^l + a_{l-1} z^{l-1} + \ldots + a_0$ and $Q(z) = b_s z^s + a_{s-1} z^{s-1} + \ldots + b_0$ respectively. Substitution (3) into (2) yields

$$e^Q = \frac{\Delta^n f - a}{f - a} = \sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{F(z+j)}{F(z)} + \frac{\Delta^n a - a}{F(z)}. \tag{4}$$

First of all, we estimate the first term $\sum_{j=0}^{n} C_n^j (-1)^{n-j} F(z+j)/F(z)$ on the right side of (4). Employing the definition of F, it turns out that $\sigma(F) = \sigma(f) =$

5

$l \geq 2$ and $\lambda(F) = \sigma(h) < \sigma(f)$. By applying Lemma 4 to F, for any given $\varepsilon > 0$ small enough, there exists a set E with finite logarithmic measure such that

$$\frac{F(z+j)}{F(z)} = e^{j\frac{F'(z)}{F(z)} + O(r^{\beta+\varepsilon})}, \text{ as } r \to \infty, \text{ not in } E \cup [0,1], \tag{5}$$

where $\beta = \sigma(f) - 2$ if $\sigma(h) < 1$ or $\beta = \max\{\sigma(f) - 2, \sigma(h) - 1\}$ if $\sigma(h) \geq 1$. Combining the fact $\sigma(h) < \sigma(f) = l$, we get $\beta < \sigma(f) - 1 = l - 1$. By Lemma 8, we see, for any given $\varepsilon > 0$ small enough, that

$$\left|\frac{h'(z)}{h(z)}\right| \leq r^{\sigma(h)-1+\varepsilon} = o(r^{l-1}) \tag{6}$$

holds for $|z| = r \notin E$. Thus from (3) and (6), we obtain

$$\frac{F'(z)}{F(z)} = g'(z) + \frac{h'(z)}{h(z)} = la_l z^{l-1}(1 + o(1)) \tag{7}$$

as $|z| = r \to \infty$ not in E. So from (5) and (7), we obtain

$$\frac{F(z+j)}{F(z)} = e^{jla_l z^{l-1}(1+o(1))}, \quad r \notin E. \tag{8}$$

Secondly, we estimate the second term $(\Delta^n a - a)/F$ on the right side of (4). It is easy to see $N := \sigma(\Delta^n a - a) \leq \sigma(a) < \sigma(f) = l$ in a similar way by Lemma 1, which gives, for any given $\varepsilon > 0$, that

$$M(r, \Delta^n a - a) < e^{r^{N+\varepsilon}} \tag{9}$$

holds for all r large sufficiently. Let $\delta(\theta) = \cos((l-1)\theta + \arg a_l)$, $\delta(g,\theta) = \cos(l\theta + \arg a_l)$ and $z = re^{i\theta}$. It follows Lemma 5 that for any given $\varepsilon > 0$, there exists a set $H \subset [0, 2\pi)$ that has the linear measure zero, such that for any $\theta \in [0, 2\pi) \setminus H$, there is $R > 0$ such that for $|z| = r > R$, we have

$$\exp\{(1-\varepsilon)|a_l|\delta(g,\theta)r^l\} < |F(re^{i\theta})| \tag{10}$$

if $\delta(g,\theta) > 0$. So by (10) and (9), we see $(\Delta^n a - a)/F \to 0$, as $z = re^{i\theta} \to \infty$ such that $\delta(g,\theta) > 0$. By Lemma 3, for any for any given $\varepsilon > 0$ small enough, we have

$$e^{-r^{\sigma(h)-1+\varepsilon}} \leq \left|\frac{h(z+c)}{h(z)}\right| \leq e^{r^{\sigma(h)-1+\varepsilon}} \tag{11}$$

holds for all sufficient large $r \notin E$.

Lastly, we take such $z = re^{i\theta}$ that $\theta \in [0, 2\pi) \setminus H; \delta(g,\theta) > 0$ and consider three cases separately in the next section.

Case 1 If $\delta(\theta) < 0$, then

$$|e^{jla_l z^{l-1}(1+o(1))}| = e^{jl|a_l|r^{l-1}\delta(\theta)(1+o(1))} \to 0, \text{ as } r \to \infty.$$

By (4), (9), (11) and the equation above, we obtain $e^{Q(z)} = (-1)^n + o(1)$. It means Q is bounded on such θ and $r \notin E$, which implies Q is a constant. And then by (3) and (4), we obtain

$$k := e^Q = (-1)^n + \sum_{j=1}^{n} C_n^j (-1)^{n-j} \frac{h(z+j)}{h(z)} e^{g(z+j)-g(z)} + \frac{\Delta^n a - a}{h(z)e^{g(z)}}. \tag{12}$$

If $\Delta^n a - a \not\equiv 0$, then by (11), (12), and the fact $\sigma((\Delta^n a - a)/h) < \sigma(e^g)$, we see

$$\begin{aligned} &\tfrac{|a_l|}{\pi} r^l (1+o(1)) + S(r, e^g) = m(r, e^{-g}) + S(r, e^g) = m(r, \tfrac{\Delta^n a - a}{he^g}) \\ &\le \sum_{j=1}^{n} m(r, \tfrac{h(z+j)}{h(z)}) + \sum_{j=1}^{n} m(r, e^{g(z+j)-g(z)}) \\ &\le r^{\sigma(h)-1+\varepsilon} + \tfrac{n(n+1)}{2} \tfrac{|a_l l|}{\pi} r^{l-1}(1+o(1)), r \notin E, \end{aligned}$$

which is impossible. If $\Delta^n a - a \equiv 0$, then by (12), we see

$$k = (-1)^n + \sum_{j=1}^{n} C_n^j (-1)^{n-j} \frac{h(z+j)}{h(z)} e^{g(z+j)-g(z)}. \tag{13}$$

Employing representation $\sigma(h) < \deg g(z) = l$ and (11), we see

$$\left| \frac{h(z+j)}{h(z)} e^{g(z+j)-g(z)} \right| = e^{jl|a_l| r^{l-1} \delta(\theta)(1+o(1))}.$$

holds for $r \notin E$. And then in this situation, $(h(z+n)/h(z))e^{g(z+n)-g(z)}$ is the only maximal magnitude of module term in (13) by taking such z that $\delta(\theta) > 0$, which is also impossible.
Case 2 If $\delta(\theta) > 0$, then by (4), (8),(9) and (10), we obtain

$$e^{|b_s| r^s \cos(\arg b_s + s\theta)(1+o(1))} = |e^Q| = (1+o(1)) e^{nl|a_l| r^{l-1} \delta(\theta)(1+o(1))} \to \infty.$$

It means $s = l-1$ on such θ and $r \notin E$, which yields $s = l-1$.
Case 3 $\delta(\theta) = 0$. Since the set $\{\theta : \delta(\theta) = 0\}$ is just a finite set and $\delta(g, \theta)$ is a continuous function of θ, so we can chose another $\tilde{\theta}$ near θ, possibly outside of a set with the linear measure zero, such that $\delta(g, \tilde{\theta}) > 0$ and $\delta(\tilde{\theta}) \neq 0$, and then this case can be transformed into case 1 or case 2.

Using the similar method in Lemma 11, we can prove the following lemma.

Lemma 12 *Let f be a transcendental entire function such that $2 \le \sigma(f) < \infty$ and $\lambda(f) < \sigma(f)$, let $a \not\equiv 0$ be an entire function such that $\sigma(a) < \sigma(f)$. If the difference equation $\Delta^n f - a = (f-a)e^Q$ holds, where Q is a nonconstant entire function, then Q is a polynomial such that* $\deg Q = \sigma(f) - 1$.

Lemma 13 *Let a be an entire function of order less than 1. If a satisfies the difference equation $\Delta^n a - a = 0$, then $a \equiv 0$.*

Proof. Suppose on the contrary $a \not\equiv 0$. Then by Lemma 6, we see

$$1 = \frac{\Delta^n a}{a} = \sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{a(z+j)}{a} \to \sum_{j=0}^{n} C_n^j (-1)^{n-j} = (1-1)^n = 0$$

as $r \to +\infty, r \notin E_\varepsilon$, where E_ε is an ε set. It is impossible.

Lemma 14 *Let a be an entire function of order less than 1. Then a satisfies the difference equation $\Delta^n a = 0$ implies a is a polynomial of degree at most $n-1$.*

Proof. Set $H_i := \Delta^{n-i} a, \quad j = 0, 1, \ldots, n$. Then $H_1(z+1) - H_1(z) = \Delta H_1 = H_0 = \Delta^n a = 0$. If H_1 is a nonconstant entire function, then it is easy to see that $z_k = k \in Z$ are some different zeros of $H_1(z) - H_1(0)$, which implies

$$\overline{N}(r, \frac{1}{H_1(z) - H_1(0)}) \geq r(1 + o(1)).$$

So $\sigma(H_1) \geq 1$, which is a contradiction. Thus H_1 is a constant, and then $0 = H_1{}' = (\Delta H_2)' = \Delta H_2'$. By a similar discussion, we see H_2' is a constant and then $H_2'' = 0$. Repeating this process, we can obtain $a^{(n)} = H_n^{(n)} = 0$. Thus a is a polynomial whose degree is at most $n-1$.

3 The proofs of main theorems

1. Proof of theorem 5.
Since $\Delta^n f$ and f share the function a CM, so there exists a polynomial Q by Lemma 1 such that

$$\Delta^n f - a = (f - a)e^Q. \tag{14}$$

It follows $\lambda(f - a) < \sigma(f)$ that

$$f - a = he^g, \tag{15}$$

where g is a polynomial whose degree l satisfying $l = \sigma(f) \geq 1$, and h is an entire function originated from the canonical product of $f - a$ satisfying $\lambda(h) = \sigma(h) < \sigma(f) = l$. By substituting (15) into (14), we can obtain

$$[\Delta^n a - a] + \sum_{j=0}^{n} C_n^j (-1)^{n-j} h(z+j) e^{g(z+j)} = h(z)e^{g(z)+Q(z)}. \tag{16}$$

In what follows, we shall consider two cases separately to our discussion.

Case 1 $\sigma(f) \geq 2$. We rewrite (16) as the following form

$$[\Delta^n a - a] + [\sum_{j=0}^{n} C_n^j (-1)^{n-j} h(z+j) e^{g(z+j)-g(z)} - h(z)e^{Q(z)}]e^{g(z)} = 0. \tag{17}$$

By applying Lemma 11 to (14), we see $\deg Q = l-1$. Applying Lemma 7 to (17) and invoking the relation $\deg Q = l-1$, it turns out that $\Delta^n a - a = 0$, which mans $a \equiv 0$ by Lemma 13. Thus we get a contradiction with our assumption.

Case 2 $l = \deg g = \sigma(f) < 2$, in other worlds, $\sigma(f) = 1$. Thus without loss of generality, we can rewrite (15) as the form of $f - a = he^{\beta z}$, where β is a nonzero constant. By (14), we see $\deg(Q) \le \sigma(f) = 1$, and then we shall consider two subcases in this case respectively as follows.

Case 2.1 Q is a constant. Then we can rewrite (17) as the following form

$$[\Delta^n a - a] + [H_n - he^{Q}]e^{\beta z} = 0, \tag{18}$$

where $H_n = \sum\limits_{j=0}^{n} C_n^j (-1)^{n-j} h(z+j)k^j, \quad k = e^{\beta}$. It follows (18) and Lemma 7 that $\Delta^n a - a = 0$, which leads to a contradiction with our assumption similarly.

Case 2.2 $\deg(Q) = 1$. Set $Q(z) = \gamma z + d$, where γ is a nonzero constant. By substituting $Q(z) = \gamma z + d$ into (16), we see

$$[\Delta^n a - a] + H_n e^{\beta z} = e^d h e^{(\beta+\gamma)z}. \tag{19}$$

If $\beta + \gamma \neq 0$, then by (19) and Lemma 7, we get $h \equiv 0$, which is a contradiction. If $\beta + \gamma = 0$, then (19) reduces to

$$[\Delta^n a - a] + H_n e^{\beta z} = e^d h. \tag{20}$$

Then by (20) and Lemma 7, we see

$$H_n = \sum_{j=0}^{n} C_n^j (-1)^{n-j} h(z+j) k^j = 0 \tag{21}$$

and

$$[\Delta^n a - a] = e^d h. \tag{22}$$

Employing representation (21) and Lemma 6, it turns out that

$$0 = \sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{h(z+j)}{h} k^j \to \sum_{j=0}^{n} C_n^j (-1)^{n-j} k^j = (k-1)^n$$

as $z \to \infty$ not in an ε set. Thus we obtain $k = e^{\beta} = 1$ from the equation above. Substituting $k = 1$ into (21), we see $H_n = \sum\limits_{j=0}^{n} C_n^j (-1)^{n-j} h(z+j) = \Delta^n h = 0$. By Lemma 14 and the equation above, we see that h is a polynomial whose degree is at most $n-1$. If a is a transcendental function, and we take z such that $|z| = r$ and $|a(z)| = M(r,a)$, then we have

$$\lim_{z\to\infty} e^d \frac{h(z)}{a(z)} = 0.$$

However, we have by (22) that

$$e^d\frac{h}{a}=\frac{\Delta^n a}{a}-1=\sum_{j=0}^{n}C_n^j(-1)^{n-j}\frac{a(z+j)}{a}-1\to\sum_{j=0}^{n}C_n^j(-1)^{n-j}-1=-1$$

as $z\to\infty$ in $z\in\{z:|a(z)|=M(r,a)\}\backslash E_\varepsilon$, where E_ε is an ε set, which is impossible. Thus a is a polynomial and then $\deg(a)=\deg(\Delta^n a-a)=\deg e^d h=\deg h$, which leads to that a is a polynomial with degree at most $n-1$. Furthermore we get $\Delta^n a=0$ and $-a=e^d h$ from (22) and then f must be form of

$$f(z)=a(z)+ba(z)e^{\beta z},$$

where $b:=-e^{-d}$ and β are two nonzero constants such that $e^\beta=1$.

2. Proof of Theorem 6.

Using the same method as in Theorem 1, we see

$$\Delta^n f-a=(f-a)e^Q \tag{23}$$

and

$$f=he^g, \tag{24}$$

where g is a polynomial of degree l satisfying $l=\sigma(f)\geq 1$, h is an entire function originated from the canonical product of f satisfying $\lambda(h)=\sigma(h)<\sigma(f)=l$, and Q is a polynomial of degree at most l. From (23)-(24), we obtain

$$\sum_{j=0}^{n}C_n^j(-1)^{n-j}h(z+j)e^{g(z+j)}=h(z)e^{g(z)+Q(z)}+a(z)-a(z)e^{Q(z)}. \tag{25}$$

In the next section, we shall consider two cases separately.

Case 1 $\sigma(f)\geq 2$. We rewrite (25) as the following form

$$[\sum_{j=0}^{n}C_n^j(-1)^{n-j}h(z+j)e^{g(z+j)-g(z)}-h(z)e^{Q(z)}]e^{g(z)}=a(z)-a(z)e^{Q(z)}. \tag{26}$$

From Lemma 12, we see $\deg Q=l-1\geq 1$. Then by (26) and Lemma 7, we obtain $a-ae^Q=0$. Thus $e^Q\equiv 1$ or $a\equiv 0$, which is impossible.

Case 2 $l=\deg g=\sigma(f)<2$, in other words, $\sigma(f)=1$. Thus without loss of generality, we can rewrite (24) as the form of $f=he^{\beta z}$, where β is a nonzero constant. It is easy to see $\deg(Q)\leq 1$.We shall consider two subcases.

Case 2.1 Q is a constant. Then by (26), we see $e^Q=1$ and

$$\sum_{j=0}^{n}C_n^j(-1)^{n-j}h(z+j)k^j-h(z)=0, \tag{27}$$

where $k=e^\beta$. From (27), we see

$$1=\sum_{j=0}^{n}C_n^j(-1)^{n-j}k^j\frac{h(z+j)}{h}\to\sum_{j=0}^{n}C_n^j(-1)^{n-j}k^j=(k-1)^n \tag{28}$$

10

as $z \to \infty$ not in an ε set. It means $(k-1)^n = 1$ and then

$$\sum_{j=0}^{n} C_n^j (-1)^{n-j} k^j = 1. \tag{29}$$

By (27) and (29), we see

$$\sum_{j=0}^{n} C_n^j (-1)^{n-j} k^j [h(z+j) - h(z)] = 0. \tag{30}$$

Set $B(z) = \Delta h = h(z+1) - h(z)$, then from Lemma 1, it is easy for us to see $\sigma(B) \le \sigma(h) < 1$. From the definition of $B(z)$. Using the same method in Theorem 4 [20], we can proof $B(z) \equiv 0$. That is $h(z+1) = h(z)$. So we get h is a nonzero constant using the same method as in Lemma 14, and then f must be form of $f(z) = be^{\beta z}$, where $b := h$ and β are two nonzero constants such that $(e^\beta - 1)^n = 1$.

Case 2.2 $\deg(Q) = 1$. Set $Q(z) = \gamma z + d$, where γ is a nonzero constant. Then (25) becomes

$$\sum_{j=0}^{n} C_n^j (-1)^{n-j} k^j h(z+j) e^{\beta z} - a = e^d h(z) e^{(\beta+\gamma)z} - e^d a e^{\gamma z}. \tag{31}$$

If $\beta + \gamma \neq 0$ and $\beta - \gamma \neq 0$, then by (31) and Lemma 7, we get $a \equiv 0$ and $h \equiv 0$, which is a contradiction. If $\beta - \gamma = 0$, then (31) becomes

$$\{[\sum_{j=0}^{n} C_n^j (-1)^{n-j} h(z+j) k^j] + ae^d\} e^{\beta z} - a = e^d e^{2\beta z},$$

and we also get a contradiction by applying Lemma 7 to the equation above. If $\beta + \gamma = 0$, then (31) becomes

$$\{\sum_{j=0}^{n} C_n^j (-1)^{n-j} h(z+j) k^j\} e^{2\beta z} = (e^d h(z) + a) e^{\beta z} - ae^d,$$

we can get a contradiction in a same way.

3. Proof of theorem 7.

We shall consider the following three cases separately to our discussion.

Case 1 $\sigma(f) < 1$. By Theorem 2, we get $\Delta^n f = cf$ holds for some nonzero complex number c. Then by Lemma 6, we get

$$c = \frac{\Delta^n f}{f} = \sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{f(z+j)}{f(z)} \to \sum_{j=0}^{n} C_n^j (-1)^{n-j} = (1-1)^n = 0$$

as $z \to \infty$, possibly outside of a ε set. Therefore $c = 0$, which is a contradiction.

Case 2 $1 \le \sigma(f) < 2$ and $\lambda(f) < 1$. Then we can get our conclusion immediately by Theorem 4.

Case 3 $\sigma(f) \geq 2$ and $\lambda(f) < \sigma(f) - 1$. Using the same method as in Theorem 5, we see

$$\Delta^n f = f e^Q \tag{32}$$

and

$$f = h e^g, \tag{33}$$

where $g(z) = a_l z^l + a_{l-1} z^{l-1} + \ldots + a_0$, $Q(z) = b_s z^s + a_{s-1} z^{s-1} + \ldots + b_0$, $l \geq 2$, $s \leq k$, are polynomials, h is an entire function originated from the canonical product of f satisfying $\lambda(h) = \sigma(h) < \sigma(f) - 1 = l - 1$. From (32)-(33), we obtain

$$\sum_{j=0}^{n} C_n^j (-1)^{n-j} \frac{h(z+j)}{h(z)} e^{g(z+j)-g(z)} = e^{Q(z)}. \tag{34}$$

Recall $g(z+j) - g(z) = j a_l l z^{l-1}(1 + o(1))$. By (34), Lemma 1 and 10, we see

$$\begin{aligned} \tfrac{|b_s|}{\pi} r^s \sim m(r, e^Q) \quad &= m\big(r, \sum_{j=0}^{n} C_n^j (-1)^{n-j} \tfrac{h(z+j)}{h(z)} e^{g(z+j)-g(z)}\big) \\ &= m\big(r, e^{g(z+n)-g(z)}\big) + S(r, e^{g(z+n)-g(z)}) \sim \tfrac{nl|a_l|}{\pi} r^{l-1}. \end{aligned}$$

It means $s = l - 1$ and $|b_s| = nl|a_l|$. We can rewrite (34) as the following form

$$\sum_{j=0}^{n-1} C_n^j (-1)^{n-j} \frac{h(z+j)}{h(z)} e^{j a_l l z^{l-1}(1+o(1))} + \frac{h(z+n)}{h(z)} e^{A_n} e^{n a_l l z^{l-1}} = e^B e^{b_{l-1} z^{l-1}}, \tag{35}$$

where A_n, B are two polynomials with degree at most $l - 2$. Recalling (11) and taking any θ such that $\delta(\theta) = \cos((l-1)\theta + \arg a_l) > 0$, then we get $\tilde{\delta}(\theta) = \cos((l-1)\theta + \arg b_{l-1}) > 0$ by (35), and then

$$e^{nl|a_l| r^{l-1} \delta(\theta)(1+o(1))} = e^{|b_{l-1}| r^{l-1} \tilde{\delta}(\theta)(1+o(1))}.$$

That means $\delta(\theta) = \tilde{\delta}(\theta)$. By the arbitrariness of θ, we see $\arg a_l = \arg b_{l-1}$. Thus we obtain $b_s = nla_l$, and then (35) becomes

$$\sum_{j=0}^{n-1} C_n^j (-1)^{n-j} \frac{h(z+j)}{h(z)} e^{j a_l l z^{l-1}(1+o(1))} = e^B (1 - \frac{h(z+n)}{h(z)} e^{A_n - B}) e^{n l a_l z^{l-1}}. \tag{36}$$

It is obvious $\sigma(e^B(1 - (h(z+n)/h) e^{A_n - B})) \leq \max\{\sigma(h), l-2\} < l - 1$. If $e^B - (h(z+n)/h) e^{A_n} \not\equiv 0$, then from (36) and Lemma 10, we see

$$\frac{nl|a_l|}{\pi} r^{l-1} \sim T(r, e^B (1 - \frac{h(z+n)}{h(z)} e^{A_n - B}) e^{n l a_l z^{l-1}}) \sim \frac{(n-1)l|a_l|}{\pi} r^{l-1},$$

which is impossible. If $e^B - (h(z+n)/h(z)) e^{A_n} \equiv 0$, then (36) becomes

$$\sum_{j=0}^{n-1} C_n^j (-1)^{n-j} \frac{h(z+j)}{h(z)} e^{j a_l l z^{l-1}(1+o(1))} = 0, \tag{37}$$

12

however $(h(z+n-1)/h(z))e^{(n-1)a_l l z^{l-1}}$ is the only maximal magnitude of module term in (37) when taking $\delta(\theta) > 0$, which is impossible.

References

[1] W. Bergweiler and J. K. Langley, Zeros of differences of meromorphic functions, *Math. Proc. Cambridge Philos. Soc*, 142 (2007), 133-147.

[2] R. Brück, On entire functions which share one value CM with their first derivative, *Results Math*, 30 (1996), 21-24.

[3] Y. M. Chiang, S. J. Feng, On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane, *Ramanujian J*, 16 (2008), 105-129.

[4] Y. M. Chiang, S. J. Feng, On the growth of logarithmic differences, difference quotients and logarithmic derivatives of meromorphic functions, *Trans. Amer. Math. Soc*, 361 (2009), 3767-3791.

[5] Z. X. Chen, The growth of solutions of $f'' + e^{-z}f' + Q(z)f = 0$ where the order $(Q) = 1$, *Science in China (Series A)*, 45(3) 2002, 290-300.

[6] G. Gundersen, Finite order solutions of second order linear differential equations, *Trans. Amer. Math. Soc*, 305 (1988), 415-429.

[7] L. Rubel and C. C. Yang, Values shared by an entire function and its derivative, in: *Complex Analysis, Kentucky*, 1976. Pro. Conf. in: *Lecture Notes in Math*, Springer-Verlag, Berlin, (599) 1977, 101-103.

[8] R. G. Halburd and R. J. Korhonen Nevanlinna theory for the difference operator, *Ann. Acad. Sci. Fenn. Math*, 31(2) 2006, 463-478.

[9] R. G. Halburd and R. J. Korhonen, Difference analogue of the lemma on the logarithmic derivative with applications to the difference equations, *J. Math. Anal. Appl*, 314 (2006), 477-487.

[10] R. G. Halburd and R. J. Korhonen, Meromorphic solutions of difference equations, integrability and the discrete, *Journal of Physics A: Mathematical and Theoretical*, 40(6), 1-38.

[11] W. K. Hayman, Meromorphic Functions, *Clarendon Press*, Oxford, 1964.

[12] W. K. Hayman, Slowly growing integral and subharmonic functions, *Comment. Math. Helv*, 34 (1960), 75-84.

[13] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo, J. Zhang, Value sharing results for shifts of meromorphic functions, and sufficient conditions for periodicity, *J. Math. Anal. Appl*, 355 (2009), 352-363.

[14] I. Laine, Nevanlinna theory and complex differential equations, *Studies in Math*, vol 15, de Gruyter, Berlin, 1993.

[15] K. Liu, Zeros of difference polynomials of meromorphic functions, *Res Math*, 57 (2010), 365-376.

[16] K. Liu, I. Laine, A note on a value distribution of difference polynomials, *Bull. Aust. Math. Soc*, 81 (2010), 353-360.

[17] K. Liu and L. Z. Yang, Value distribution of the difference operator, *Arch. Math*, 92 (2009), 270-278.

[18] C. C. Yang and H. X. Yi, Uniqueness theory of meromorphic functions, *Science Press*, Beijing, Second Printed in 2006.

[19] L. Yang, Value Distribution Theory, *Springer-Verlag & Science Press*, Berlin, 1993.

[20] J. Zhang, J. J. Zhang and L. W. Liao, Entire functions sharing zero CM with their high order difference operators, *Taiwanese J. Math*, 18(3) (2014), 701-709.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO.7, 2017, COPYRIGHT 2017 EUDOXUS PRESS, LLC

Global Attractivity for Nonautonomous Difference Equation via Linearization

Arzu Bilgin and M. R. S. Kulenović[1]

Department of Mathematics
University of Rhode Island, Kingston, Rhode Island 02881-0816, USA

Abstract. Consider the difference equation

$$\vec{x}_{n+1} = f(n, \vec{x}_n, ..., \vec{x}_{n-k}), \quad n = 0, 1, \ldots,$$

where $k \in \{0, 1, \ldots\}$ and the initial conditions are real vectors. We investigate the asymptotic behavior of the solutions of the considered equation. We give some effective conditions for the global stability and global asymptotic stability of the zero or positive equilibrium of this equation. Our results are based on application of the linearizations technique. We illustrate our results with many examples that include some equations from mathematical biology.

Keywords: attractivity, difference equations, discrete dynamical system, global, linear fractional, rational, stability

AMS 2000 Mathematics Subject Classification: 39A10, 39A20, 37B25, 37D10, 37M99.

1 Introduction and preliminaries

Consider the difference equation

$$\vec{x}_{n+1} = f(n, \vec{x}_n, ..., \vec{x}_{n-k}), \quad n = 0, 1, \ldots \tag{1}$$

where $k \in \{0, 1, \ldots\}$ and the initial conditions are real vectors in $\mathbb{R}^p, p \geq 2$. In many cases we investigate equation(1) by embedding equation(1) into a higher iteration of the form

$$\vec{x}_{n+l} = F(n, \vec{x}_{n+l-1}, ..., \vec{x}_{n-k}), \quad n = 0, 1, \ldots \tag{2}$$

where $l \in \{1, 2, \ldots\}$, see [4, 5, 8]. By linearizing equation (2) and bringing it to the form

$$\vec{x}_{n+1} = \sum_{i=1-l}^{k} g_i \vec{x}_{n-i}, \tag{3}$$

where g_i in general, depend on n and the state variables $\vec{x}_k$ we can prove very general attractivity and asymptotic stability results for both autonomous and nonautonomous difference equations. The functions g_i are in general matrices but they can also be the scalars as well, see Section 3. This approach was used to get effective and applicable global asymptotic and global attractivity results for linear fractional difference equation, see [2] and quadratic fractional difference equation, see [3] with both constant and nonconstant coefficients. Furthermore, this approach produced global asymptotic and global attractivity results for nonautonomous difference equations with very general coefficients which can be discontinuous functions of n or state variables, see [4, 5, 8]. See [1, 7, 10, 11] for the case of monotone systems, where more precise results were obtained.

In this paper we use method of linearization to extend some of the results about the global attractivity and asymptotic stability of scalar equation from [4] to the case of vector equation (2). We illustrate our results with many examples that include some transition functions from mathematical biology such as linear, Beverton-Holt, sigmoid Beverton-Holt, etc., see [6, 7, 9, 11, 12] for related results. The rest of this section contains some definitions and preliminary results. Second section contains our main results on global attractivity in the case when the sum of the norms of g_i is less than 1. The third section

[1]Corresponding author, *e-mail: mkulenovic@uri.edu*

gives some results on global attractivity in the delicate case when the sum of the scalar functions g_i is 1. The fourth section provides several examples which illustrate our results.

Denote by $\|\vec{x}\|$ any norm in $\mathbb{R}^p$.

Definition 1 *The zero equilibrium of equation* (3) *is stable if for* $(\forall \epsilon > 0)(\exists \delta > 0, N)$*:*

$$\|\vec{x}_i\| < \delta, i = -k, \ldots, 0 \Longrightarrow \|\vec{x}_n\| < \epsilon, \text{ for all } n \geq N.$$

The zero equilibrium is asymptotically stable if it is stable and

$$\lim_{n\to\infty} \vec{x}_n = \vec{0}.$$

Lemma 1 *Let* $\mathbf{I} - \sum_{i=0}^{k} g_i$ *be invertible for* $n = 1, 2, \ldots,$ *where* $\mathbf{I}$ *is identity matrix.Then equation* (3) *has no nonzero equilibrium.*

Proof. Otherwise, equation (3) has the equilibrium $\bar{\mathbf{x}} \neq \vec{0}$. By pluging $\vec{x}_n = \bar{\mathbf{x}}$ in equation (3) we get

$$(\mathbf{I} - \sum_{i=0}^{k} g_i)\bar{\mathbf{x}} = \vec{0},$$

which implies $\bar{\mathbf{x}} = \vec{0}$, which is a contradiction. □

Remark 1 The matrix $\mathbf{I} - \sum_{i=0}^{k} g_i$ is invertible if the condition

$$\|\sum_{i=0}^{k} g_i\| < 1 \tag{4}$$

is satisfied in which case we have

$$(\mathbf{I} - \sum_{i=0}^{k} g_i)^{-1} = \sum_{k=0}^{\infty}\sum_{i=0}^{k} g_i. \tag{5}$$

The condition (4) is implied by more applicable condition

$$\sum_{i=0}^{k} \|g_i\| < 1. \tag{6}$$

Remark 2 Equation (1) admits the following generalized identity

$$\vec{x}_{n+1} - \sum_{i=0}^{k} g_i\vec{K} = \sum_{i=0}^{k} g_i(\vec{x}_{n-i} - \vec{K}), \tag{7}$$

where $\vec{K}$ is an arbitrary vector. Generalized identity (7) implies

$$\|\vec{x}_{n+1} - \sum_{i=0}^{k} g_i\vec{K}\| \leq \sum_{i=0}^{k} \|g_i\|\|\vec{x}_{n-i} - \vec{K}\|. \tag{8}$$

Furthermore by taking $\vec{K} = \vec{0}$ in equation (8), we obtain another useful inequality

$$\|\vec{x}_{n+1}\| - L\sum_{i=0}^{k} \|g_i\| \leq \sum_{i=0}^{k} \|g_i\|(\|\vec{x}_{n-i}\| - L), \tag{9}$$

where L is an arbitrary constant.

Lemma 2 *Suppose that equation* (1) *has the linearization* (3) *and the functions* $g_i : R^{p+1} \to M_{p\times p}$, *where* $M_{p\times p}, p \geq 1$ *is the set of all real* $p \times p$ *matrices, are such that*

$$\sum_{i=0}^{k} \|g_i\| \leq 1, \quad n = 0, 1, \ldots$$

Then if equation (1) *has the zero equilibrium it is a stable fixed point.*

Proof. Assume that equation (1) has the zero equilibrium and the linearization (3). By taking $\vec{K} = \vec{0}$ in equation (8) we have

$$\|\vec{x}_{n+1}\| \leq \sum_{i=0}^{k} \|g_i\|\|\vec{x}_{n-i}\|.$$

Assume that $\sum_{i=0}^{k} \|\vec{x}_{-i}\| < \delta$. Take $\delta = \epsilon$. Then $\|\vec{x}_{-i}\| < \delta$ for $i = 0, 1, \ldots$. Hence

$$\|\vec{x}_1\| \leq \sum_{i=0}^{k} \|g_i\|\|\vec{x}_{-i}\| < \delta \sum_{i=0}^{k} \|g_i\| \leq \delta = \epsilon,$$

$$\|\vec{x}_2\| \leq \sum_{i=0}^{k} \|g_i\|\|\vec{x}_{1-i}\| < \delta \sum_{i=0}^{k} \|g_i\| \leq \delta = \epsilon$$

and so by induction $\|\vec{x}_n\| < \epsilon$ for $n \geq -k$. □

2 Main results

In this section we present our main results on global attractivity and global asymptotic stability of the equilibrium solutions of equation (1) which has the linearization (3).

Theorem 1 *Let* $l \in \{1, 2, \ldots\}$. *Suppose that equation* (1) *has the linearization* (3) *subject to the condition*

$$\sum_{i=1-l}^{k} \|g_i\| \leq 1, n = 0, 1, \ldots \tag{10}$$

Let $M_0 = \max\{\|\vec{x}_{l-1}\|, \ldots, \|\vec{x}_{-k}\|\}$. *Then every solution of equation* (1) *is bounded. In particular* $\|\vec{x}_n\| \leq M_0$ *for* $n \geq -k$.

Proof. Let $L \in R$. Then equation (9) implies

$$\|\vec{x}_{n+l}\| - L \sum_{i=1-l}^{k} \|g_i\| \leq \sum_{i=1-l}^{k} \|g_i\|(\|\vec{x}_{n-i}\| - L), \quad n = 0, 1, \ldots \tag{11}$$

By taking $L = M_0$ and $n = 0$ in equation (11), we obtain

$$\|\vec{x}_l\| - M_0 \sum_{i=1-l}^{k} \|g_i\| \leq \|g_{1-l}\|(\|\vec{x}_{l-1}\| - M_0) + \ldots + \|g_k\|(\|\vec{x}_{-k}\| - M_0) \leq 0,$$

which in view of equation (10) implies $\|x_l\| \leq M_0$. By using induction, we obtain

$$\|\vec{x}_{n+l}\| - M_0 \sum_{i=1-l}^{k} \|g_i\| \leq \|g_{1-l}\|(\|\vec{x}_{n+l-1}\| - M_0) + \ldots + \|g_k\|(\|\vec{x}_{n-k}\| - M_0) \leq 0, \quad n = 0, 1, \ldots$$

and so

$$\|\vec{x}_{n+l}\| \leq M_0 \sum_{i=1-l}^{k} \|g_i\| \leq M_0, \quad n = 0, 1, \ldots$$

Thus $\|\vec{x}_{n+l}\| \leq M_0$ for $n \geq -k$. □

Theorem 2 *Let $l \in \{1, 2, \ldots\}$. Suppose that equation* (1) *has the linearization* (3) *where the functions $g_i : R^{k+1} \to M_{p\times p}$ are such that*

$$\sum_{i=1-l}^{k} \|g_i\| \leq a < 1, \quad n = 0, 1, \ldots \tag{12}$$

Then

$$\lim_{n\to\infty} \vec{x}_n = \vec{0}.$$

Proof. Let $L \in R$. Then every solution of equation (3) satisfies the inequality (11). Let $\gamma = l + k$. Define $M_N = \max\{\|\vec{x}_{\gamma N+l-1}\|, \ldots, \|\vec{x}_{\gamma N-k}\|\}$ for $N = 0, 1, \ldots$. Observe that if $\|\vec{x}_{\gamma N+l-1}\| = \ldots = \|\vec{x}_{\gamma N-k}\| = \vec{0}$ for some $N \geq 0$, then by (11) with $L = 0$ we get that

$$\|\vec{x}_{\gamma N+l+j}\| = \vec{0}, \quad j = 0, 1, \ldots$$

and so $\lim_{n\to\infty} \vec{x}_n = \vec{0}$.
Assume that $M_N > 0$ for all $N \geq 0$. By using (11) with $L = M_N$ and $n = \gamma N$ we obtain

$$\|\vec{x}_{\gamma N+l}\| - \sum_{i=1-l}^{k} \|g_i\| M_N \leq \|g_{1-l}\|(\|\vec{x}_{\gamma N+l-1}\| - M_N) + \ldots + \|g_k\|(\|\vec{x}_{\gamma N-k}\| - M_N) \leq 0$$

and so

$$\|\vec{x}_{\gamma N+l}\| \leq \sum_{i=1-l}^{k} \|g_i\| M_N \leq a M_N < M_N.$$

Similarly, by taking $n = \gamma N + 1$ in (11) we obtain

$$\|\vec{x}_{\gamma N+l+1}\| - \sum_{i=1-l}^{k} \|g_i\| M_N \leq \|g_{1-l}\|(\|\vec{x}_{\gamma N+l}\| - M_N) + \ldots + \|g_k\|(\|\vec{x}_{\gamma N-k+1}\| - M_N) \leq 0$$

and so

$$\|\vec{x}_{\gamma N+l+1}\| \leq \sum_{i=1-l}^{k} \|g_i\| M_N \leq a M_N < M_N.$$

Hence by induction we have that

$$\|\vec{x}_{\gamma N+l+j}\| \leq \sum_{i=1-l}^{k} \|g_i\| M_N \leq a M_N < M_N.$$

Thus

$$M_{N+1} \leq a M_N < M_N, \tag{13}$$

and so the sequence $\{M_N\}_{N=0}^{\infty}$ is decreasing sequence bounded below by zero. Furthermore (13) implies

$$M_N \leq a^{N+1} M_0 \to 0 \qquad as \qquad N \to \infty.$$

Hence

$$0 \leq \lim_{N\to\infty} \vec{x}_{\gamma N-j} \leq \lim_{N\to\infty} M_N = 0, \quad j = 1-l,\ldots,k.$$

Therefore

$$\lim_{n\to\infty} \vec{x}_n = \vec{0}.$$

□

Corollary 1 *Suppose that equation* (1) *has the linearization* (3), *where* $l = 1$ *and the functions* $g_i : R^{k+1} \to M_{p\times p}$ *are such that*

$$\sum_{i=0}^{k} \|g_i\| \leq a < 1, \quad n = 0,1,\ldots.$$

Then if equation (1) *has a zero equilibrium it is globally asymptotically stable.*

Assuming that f is differentiable in some neighborhood of the equilibrium solution $\bar{x}$, by applying Theorem 2 and Lemma 2 to the standard linearization of equation (1) about the equilibrium solution $\bar{x}$

$$\vec{x}_{n+1} = \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}}(\bar{x},\ldots,\bar{x})\vec{x}_{n-i}, \quad n = 0,1,\ldots, \tag{14}$$

where $\frac{\partial f}{\partial x_{n-i}}(\bar{x},\ldots,\bar{x})$ is the Jacobian matrix evaluated at the equilibrium point, we obtain the following result, which is local in the nature because of the fact that the standard linearization is local.

Corollary 2 *Assume that* f *is differentiable in some neighborhood of the equilibrium solution* $\bar{x}$. *The equilibrium* $\bar{x}$ *of equation* (1) *is locally asymptotically stable if*

$$\sum_{i=0}^{k} \|\frac{\partial f}{\partial x_{n-i}}(\bar{x},\ldots,\bar{x})\| \leq a < 1.$$

3 The case when g_i are scalar functions

In this section we consider the case when all g_i are scalar functions. In this case the linearization (3) is equivalent to p scalar equations of the form

$$x^m_{n+1} = \sum_{i=1-l}^{k} g_i x^m_{n-i}, \quad n = 0,1,\ldots; m = 1,\ldots,p. \tag{15}$$

For instance, in the case of second order difference equation in $\mathbb{R}^2$, we have that vector equation

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = g_0 \begin{bmatrix} x_n \\ y_n \end{bmatrix} + g_1 \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} \quad n = 0,1,\ldots \qquad g_0, g_1 \geq 0 \tag{16}$$

is equivalent to the system

$$\begin{aligned} x_{n+1} &= g_0 x_n + g_1 x_{n-1} \\ y_{n+1} &= g_0 y_n + g_1 y_{n-1}. \end{aligned} \tag{17}$$

The next results apply to a special linearization (3) of equation (1), where all g_i are scalar functions.

Theorem 3 *Let $l \in \{1, 2, \ldots\}$. Suppose that equation (1) has the linearization (3), where the functions $g_i : \mathbb{R}^{k+1} \to [0, \infty)$ are such that*

$$\sum_{i=1-l}^{k} g_i \geq a > 1, \quad n \geq 0.$$

Then if for some $n \geq 0$

(a) $\vec{x}_{n+l-1}, \ldots, \vec{x}_{n-k} > 0$, then $\lim_{n\to\infty} \vec{x}_n = \infty$, componentwise;

(b) $\vec{x}_{n+l-1}, \ldots, \vec{x}_{n-k} < 0$, then $\lim_{n\to\infty} \vec{x}_n = -\infty$, componentwise.

Proof. Proof follows from Theorem 2 in [4] applied to equation(15). □

A delicate case when

$$\sum_{i=1-l}^{k} g_i = 1, \quad n = 0, 1, \ldots. \tag{18}$$

is treated in the following three results.

Theorem 4 *Suppose that on some interval I equation (1) has the linearization (3), where the functions $g_i : \mathbb{R}^{k+1} \to [0, \infty)$ are such that (18) is satisfied. Then there exists $A > 0$ such that for $n \geq 0$ every positive g_i satisfies*

$$A \leq g_i \leq 1, \quad n = 0, 1, \ldots. \tag{19}$$

Proof. Proof follows from Proposition 3 in [4] applied to equation (15). □

Theorem 5 *Suppose that on some interval I equation (1) has the linearization (3), where the functions $g_i : \mathbb{R}^{k+1} \to [0, \infty)$ are such that (18) is satisfied. Assume that there exists $A > 0$ such that*

$$g_{1-l} \geq A, \quad n = 0, 1, \ldots. \tag{20}$$

Then if $\vec{x}_{-k}, \ldots, \vec{x}_0 \in I$

$$\lim_{n\to\infty} \vec{x}_n = \vec{L},$$

where $\vec{L} \in I^p$ is a constant vector

Proof. Proof follows from Theorem 4 in [4] applied to equation (15). □

Theorem 6 *Suppose that on some interval $I \subset \mathbb{R}$ equation (1) has the linearization (3), where the functions $g_i : \mathbb{R}^{k+1} \to [0, \infty)$ are such that (18) is satisfied. Assume that there exists $A > 0$ such that for some $j \in \{2 - l, \ldots, k - 1\}$*

$$g_j \geq A, g_{j+1} \geq A, \quad n = 0, 1, \ldots. \tag{21}$$

If $\vec{x}_{l-1}, \ldots, \vec{x}_{-k} \in I$, then

$$\lim_{n\to\infty} \vec{x}_n = \vec{L},$$

where $\vec{L} \in I^p$ is a constant vector

Proof. Proof follows from Theorem 5 in [4] applied to equation (15). □

4 Examples

In this section we present some examples that illustrate our results.

Example 1 Every solution of the vector equation in $\mathbb{R}^2$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{a}{1+x_n} & b_n \\ c_n & \frac{d}{1+y_n} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, n = 0, 1, \dots,$$

where $a, d > 0, b_n, c_n \geq 0, x_0, y_0 \geq 0, n = 0, 1, \dots$, converges to the zero equilibrium if $\max\{a + U_c, d + U_b\} < 1$ is satisfied, where U_b and U_c are upper bounds of sequences $\{b_n\}$ and $\{c_n\}$ respectively. Indeed, in this case if $\|x\|$ denotes the L_1 norm we have

$$\|g_0\| = \left\| \begin{bmatrix} \frac{a}{1+x_n} & b_n \\ c_n & \frac{d}{1+y_n} \end{bmatrix} \right\| = \max\left\{ \frac{a}{1+x_n} + c_n, \frac{d}{1+y_n} + b_n \right\} \leq \max\{a + U_c, d + U_b\} < 1,$$

that is $U_c < 1 - a, U_b < 1 - d$, and the result follows from Theorem 2 and Corollary 1. Thus in this case the zero equilibrium is globally asymptotically stable. If we use L_2 norm we have that the zero equilibrium is globally asymptotically stable if $\max\{a + U_b, d + U_c\} < 1$ is satisfied.

Example 2 Every solution of the vector equation in $\mathbb{R}^2$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{a}{1+x_n} & b \\ c & \frac{d}{1+y_n} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, n = 0, 1, \dots, \tag{22}$$

where $a, b, c, d > 0, x_0, y_0 \geq 0$, converges to the zero equilibrium if $\max\{a + c, b + d\} < 1$ is satisfied. Indeed, in this case if $\|x\|$ denotes the L_1 norm we have that

$$\|g_0\| = \left\| \begin{bmatrix} \frac{a}{1+x_n} & b \\ c & \frac{d}{1+y_n} \end{bmatrix} \right\| = \max\left\{ \frac{a}{1+x_n} + c, \frac{d}{1+y_n} + b \right\} \leq \max\{a + c, b + d\} < 1$$

and the result follows from Theorem 2 and Corollary 1. Thus in this case the zero equilibrium is globally asymptotically stable. If we use L_2 norm we have that $\max\{a + b, c + d\} < 1$ implies that the zero equilibrium is globally asymptotically stable.

Next, consider the positive equilibrium $E(\bar{x}, \bar{y})$. Then we have that the positive equilibrium $E(\bar{x}, \bar{y})$ of system (22) satisfies the system

$$\begin{aligned} \bar{x} &= a\frac{\bar{x}}{1+\bar{x}} + b\bar{y} \\ \bar{y} &= c\bar{x} + d\frac{\bar{y}}{1+\bar{y}}. \end{aligned} \tag{23}$$

which implies

$$\begin{aligned} \bar{x}\frac{1+\bar{x}-a}{1+\bar{x}} &= b\bar{y} \\ \bar{y}\frac{1+\bar{y}-d}{1+\bar{y}} &= c\bar{x}. \end{aligned}$$

Thus the positive equilibrium exists if

$$\bar{x} > a - 1, \bar{y} > d - 1. \tag{24}$$

Linearizing system (22) about the positive equilibrium E gives the following system

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{a}{(1+\bar{x})(1+x_n)} & b \\ c & \frac{d}{(1+\bar{y})(1+y_n)} \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix}, n = 0, 1, \dots, \tag{25}$$

where $u_n = x_n - \bar{x}, v_n = y_n - \bar{y}$. By using Theorem 2 and Corollary 1 with L_1 norm, we obtain that the condition for global asymptotic stability of $E(\bar{x}, \bar{y})$ to be

$$\bar{x} > \frac{a + c - 1}{1 - c} \quad \text{if} \quad c < 1 < a + c, \quad \bar{y} > \frac{b + d - 1}{1 - b} \quad \text{if} \quad b < 1 < b + d.$$

If we use L_2 norm we obtain sufficient condition for global asymptotic stability of $E(\bar{x},\bar{y})$ to be

$$\bar{x} > \frac{a+b-1}{1-b} \quad \text{if} \quad b<1<a+b, \quad \bar{y} > \frac{c+d-1}{1-c} \quad \text{if} \quad c<1<c+d.$$

Example 3 Every solution of the vector equation in $\mathbb{R}^2$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{a}{1+x_n} & \frac{b}{1+y_n} \\ \frac{c}{1+x_n} & \frac{d}{1+y_n} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, n=0,1,\ldots, \tag{26}$$

where $a,b,c,d>0, x_0, y_0 \geq 0, n=0,1,\ldots$, converges to the zero equilibrium if $\max\{a+c, b+d\}<1$ is satisfied. Indeed, in this case if $\|x\|_1$ denotes the L_1 norm we have

$$\|g_0\|_1 = \left\| \begin{bmatrix} \frac{a}{1+x_n} & \frac{b}{1+y_n} \\ \frac{c}{1+x_n} & \frac{d}{1+y_n} \end{bmatrix} \right\|_1 = \max\left\{ \frac{a}{1+x_n} + \frac{c}{1+x_n}, \frac{b}{1+y_n} + \frac{d}{1+y_n} \right\} \leq \max\{a+c, b+d\} < 1$$

and the result follows from Theorem 2 and Corollary 1. Thus in this case the zero equilibrium is globally asymptotically stable.

In the case if $\|x\|_2$ denotes the L_2 norm we have

$$\|g_0\|_2 = \left\| \begin{bmatrix} \frac{a}{1+x_n} & \frac{b}{1+y_n} \\ \frac{c}{1+x_n} & \frac{d}{1+y_n} \end{bmatrix} \right\|_2 = \max\left\{ \frac{a}{1+x_n} + \frac{b}{1+y_n}, \frac{c}{1+x_n} + \frac{d}{1+y_n} \right\} \leq \max\{a+b, c+d\} < 1.$$

In this case the condition for global asymptotic stability of the zero equilibrium becomes $\max\{a+b, c+d\}<1$.

Now, consider global attractivity of the positive equilibrium $E(\bar{x},\bar{y})$ of system (26). The positive equilibrium of system (26) satisfies the system

$$\begin{aligned} \bar{x} &= a\frac{\bar{x}}{1+\bar{x}} + b\frac{\bar{y}}{1+\bar{y}} \\ \bar{y} &= c\frac{\bar{x}}{1+\bar{x}} + d\frac{\bar{y}}{1+\bar{y}}. \end{aligned} \tag{27}$$

Adding two equations in (27) we obtain

$$\bar{x}+\bar{y} = (a+c)\frac{\bar{x}}{1+\bar{x}} + (b+d)\frac{\bar{y}}{1+\bar{y}},$$

which implies

$$\frac{\bar{x}}{1+\bar{x}}(1+\bar{x}-a-c) = \frac{\bar{y}}{1+\bar{y}}(b+d-1-\bar{y})$$

and so we obtain that the positive equilibrium satisfies

$$\bar{x} > a+c-1 \Leftrightarrow \bar{y} < b+d-1. \tag{28}$$

Linearizing system (26) about the positive equilibrium E gives the following system

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{a}{(1+\bar{x})(1+x_n)} & \frac{b}{(1+\bar{y})(1+y_n)} \\ \frac{c}{(1+\bar{x})(1+x_n)} & \frac{d}{(1+\bar{y})(1+y_n)} \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix}, n=0,1,\ldots,$$

where $u_n = x_n - \bar{x}, v_n = y_n - \bar{y}$. By using Theorem 2 and Corollary 1 with L_1 norm, we obtain that the condition

$$\bar{x} > a+c-1, \bar{y} > b+d-1. \tag{29}$$

is sufficient for the global asymptotic stability of the positive equilibrium solution. The condition (29) contradicts condition (28). If we use L_2 norm we obtain sufficient condition for the global asymptotic stability of the positive equilibrium solution to be

$$\begin{aligned} b\bar{x} + a\bar{y} &< 1-a-b \\ d\bar{x} + c\bar{y} &< 1-c-d. \end{aligned}$$

8

Example 4 Every solution of the vector equation in $\mathbb{R}^n$

$$\vec{x}_{n+1} = A_n \vec{x}_n \tag{30}$$

where

$$\vec{x}_n = \begin{bmatrix} x_n^1 \\ x_n^2 \\ \vdots \\ x_n^k \end{bmatrix}, \quad A_n = \begin{bmatrix} \frac{a_{11}}{1+x_n^1} & \frac{a_{12}}{1+x_n^2} & \cdots & \frac{a_{1k}}{1+x_n^k} \\ \frac{a_{21}}{1+x_n^1} & \frac{a_{22}}{1+x_n^2} & \cdots & \frac{a_{2k}}{1+x_n^k} \\ \vdots & & & \\ \frac{a_{k1}}{1+x_n^1} & \frac{a_{k2}}{1+x_n^2} & \cdots & \frac{a_{kk}}{1+x_n^k} \end{bmatrix}$$

where $a_{ij} > 0, i, j = 0, 1, \ldots \quad x_0, y_0 \geq 0, n = 0, 1, \ldots,$, converges to the zero equilibrium if

$$\|g_0\|_1 = \left\| \begin{bmatrix} \frac{a_{11}}{1+x_n^1} & \frac{a_{12}}{1+x_n^2} & \cdots & \frac{a_{1k}}{1+x_n^k} \\ \frac{a_{21}}{1+x_n^1} & \frac{a_{22}}{1+x_n^2} & \cdots & \frac{a_{2k}}{1+x_n^k} \\ \vdots & & & \\ \frac{a_{k1}}{1+x_n^1} & \frac{a_{k2}}{1+x_n^2} & \cdots & \frac{a_{kk}}{1+x_n^k} \end{bmatrix} \right\|_1$$

$$\begin{aligned} &= \max\left\{ \frac{a_{11}}{1+x_n^1} + \frac{a_{21}}{1+x_n^1} + \ldots + \frac{a_{k1}}{1+x_n^1}, \ldots, \frac{a_{1k}}{1+x_n^1} + \frac{a_{2k}}{1+x_n^1} + \ldots + \frac{a_{kk}}{1+x_n^1} \right\} \\ &\leq \max\{a_{11} + a_{21} + \ldots + a_{k1}, \ldots, a_{1k} + a_{2k} + \ldots + a_{kk}\} \\ &= \max_{1 \leq j \leq n} \{\sum_{i=1}^{k} a_{ij}\} < 1, \end{aligned}$$

which follows from Theorem 2 and Corollary 1. Thus in this case the zero equilibrium is globally asymptotically stable.

Now, consider global attractivity of the positive equilibrium of system (30). The positive equilibrium satisfies the system

$$(A_n(\bar{\vec{x}}) - \mathbf{I})\bar{\vec{x}} = \vec{0},$$

where

$$A_n(\bar{\vec{x}}) = \begin{bmatrix} \frac{a_{11}}{1+\bar{x}^1} & \frac{a_{12}}{1+\bar{x}^2} & \cdots & \frac{a_{1k}}{1+\bar{x}^k} \\ \frac{a_{21}}{1+\bar{x}^1} & \frac{a_{22}}{1+\bar{x}^2} & \cdots & \frac{a_{2k}}{1+\bar{x}^k} \\ \vdots & & & \\ \frac{a_{k1}}{1+\bar{x}^1} & \frac{a_{k2}}{1+\bar{x}^2} & \cdots & \frac{a_{kk}}{1+\bar{x}^k} \end{bmatrix}.$$

Linearizing system (30) about the positive equilibrium E gives the following system

$$\vec{u}_{n+1} = \begin{bmatrix} \frac{a_{11}}{(1+\bar{x})(1+x_n^1)} & \frac{a_{12}}{(1+\bar{x})(1+x_n^2)} & \cdots & \frac{a_{1k}}{(1+\bar{x})(1+x_n^k)} \\ \frac{a_{21}}{(1+\bar{x})(1+x_n^1)} & \frac{a_{22}}{(1+\bar{x})(1+x_n^2)} & \cdots & \frac{a_{2k}}{(1+\bar{x})(1+x_n^k)} \\ \vdots & & & \\ \frac{a_{k1}}{(1+\bar{x})(1+x_n^1)} & \frac{a_{k2}}{(1+\bar{x})(1+x_n^2)} & \cdots & \frac{a_{kk}}{(1+\bar{x})(1+x_n^k)} \end{bmatrix} \vec{u}_n, \quad n = 0, 1, \ldots,$$

where $\vec{u}_n = \vec{x}_n - \bar{\vec{x}}$. By using Theorem 2 and Corollary 1 with L_1 norm, we obtain that the condition

$$\|g_0\|_1 = \left\| \begin{bmatrix} \frac{a_{11}}{(1+\bar{x})(1+x_n^1)} & \frac{a_{12}}{(1+\bar{x})(1+x_n^2)} & \cdots & \frac{a_{1k}}{(1+\bar{x})(1+x_n^k)} \\ \frac{a_{21}}{(1+\bar{x})(1+x_n^1)} & \frac{a_{22}}{(1+\bar{x})(1+x_n^2)} & \cdots & \frac{a_{2k}}{(1+\bar{x})(1+x_n^k)} \\ \vdots & & & \\ \frac{a_{k1}}{(1+\bar{x})(1+x_n^1)} & \frac{a_{k2}}{(1+\bar{x})(1+x_n^2)} & \cdots & \frac{a_{kk}}{(1+\bar{x})(1+x_n^k)} \end{bmatrix} \right\|_1$$

$$\begin{aligned}
&= \max\left\{\tfrac{a_{11}}{(1+\bar{x})(1+x_n^1)} + \ldots + \tfrac{a_{k1}}{(1+\bar{x})(1+x_n^1)}, \ldots, \tfrac{a_{1k}}{(1+\bar{x})(1+x_n^k)} + \tfrac{a_{2k}}{(1+\bar{x})(1+x_n^k)} + \ldots + \tfrac{a_{kk}}{(1+\bar{x})(1+x_n^k)}\right\}\\
&\leq \max\{\tfrac{1}{1+\bar{x}}(a_{11}+a_{21}+\ldots+a_{k1},\ldots,a_{1k}+a_{2k}+\ldots+a_{kk})\}\\
&= \tfrac{1}{1+\bar{x}} \max_{1\leq j\leq n}\{\sum_{i=1}^{k} a_{ij}\}\\
&< 1
\end{aligned}$$

implies the global asymptotic stability of the positive equilibrium solution. By using Theorem 2 and Corollary 1 with L_1 norm, we obtain that the condition for the global asymptotic stability of the positive equilibrium solution is

$$1+\bar{x} > \sum_{i=1}^{k} a_{ij} \Longleftrightarrow \bar{x} > \sum_{i=1}^{k} a_{ij} - 1.$$

Example 5 The cooperative system

$$\begin{bmatrix} x_{n+1}\\ y_{n+1}\end{bmatrix} = \begin{bmatrix} a & \frac{b}{1+y_n}\\ \frac{c}{1+x_n} & d\end{bmatrix}\begin{bmatrix} x_n\\ y_n\end{bmatrix}, n=0,1,\ldots, \tag{31}$$

where $a, b, c, d > 0$, $x_0, y_0 \geq 0$ was considered in [1]. The equilibrium solutions are the zero equilibrium $E_0(0,0)$ and when $a<1, d<1$ the unique positive equilibrium solution $E_+(\bar{x},\bar{y})$, is given as

$$\bar{x} = \frac{b}{1-a}\frac{\bar{y}}{1+\bar{y}}, \quad \bar{y} = \frac{bc-(1-d)(1-a)}{(1-d)(b+1-a)},$$

when

$$(1-a)(1-d) < bc. \tag{32}$$

The local stability of system (31) is described with the following result, see [1]

Claim 1 *Consider system* (31).

1.) *The positive equilibrium $E_+(\bar{x},\bar{y})$ of system (31) is locally asymptotically stable when (32) holds.*

2.) *The zero equilibrium $E_0(0,0)$ of system (31) is locally asymptotically stable if $bc < (1-a)(1-d)$; it is a saddle point if $bc > (1-a)(1-d)$; it is a nonhyperbolic equilibrium if $bc = (1-a)(1-d)$.*

The global dynamics of system (31) is described with the following result, see [1]:

Theorem 7 *Consider system* (31).

1.) *If $a \geq 1$ then $\lim_{n\to\infty} x_n = \infty$ and $\lim_{n\to\infty} y_n = \infty$ if $d \geq 1$ and $\lim_{n\to\infty} y_n = \frac{c}{1-d}$, if $d < 1$.*

2.) *If $d \geq 1$ then $\lim_{n\to\infty} y_n = \infty$ and $\lim_{n\to\infty} x_n = \infty$ if $a \geq 1$ and $\lim_{n\to\infty} x_n = \frac{b}{1-a}$, if $a < 1$.*

3.) *The positive equilibrium $E_+(\bar{x},\bar{y})$ of system (31) is globally asymptotically stable when (32) holds.*

4.) *The zero equilibrium $E_+(\bar{x},\bar{y})$ of system (31) is globally asymptotically stable when $a<1, d<1$ and*

$$bc \leq (1-a)(1-d) \tag{33}$$

holds.

Theorem 2 and Corollary 1 implies that any of two conditions $\max\{a+c, b+d\} < 1$ or $\max\{a+b, c+d\} < 1$ provides the global asymptotic stability of the zero equilibrium. Both of these conditions imply (33) which is clearly the necessary and sufficient condition for the global asymptotic stability of the zero equilibrium..

Linearizing system (31) about the positive equilibrium $E(\bar{x},\bar{y})$ gives the following system

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = \begin{bmatrix} a & \frac{b}{(1+\bar{y})(1+y_n)} \\ \frac{c}{(1+\bar{x})(1+x_n)} & d \end{bmatrix} \begin{bmatrix} u_n \\ v_n \end{bmatrix}, \quad n = 0,1,\ldots,$$

where $u_n = x_n - \bar{x}, v_n = y_n - \bar{y}$. By using Theorem 2 and Corollary 1 with L_1 or L_2 norm, we obtain that the condition

$$\max\left\{a + \frac{c}{1+\bar{x}}, \frac{b}{1+\bar{y}} + d\right\} < 1 \quad \text{or} \quad \max\left\{a + \frac{b}{1+\bar{y}}, \frac{c}{1+\bar{x}} + d\right\} < 1 \tag{34}$$

implies that the positive equilibrium $E(\bar{x},\bar{y})$ is globally asymptotically stable. Condition (34) implies condition (32) which is clearly the necessary and sufficient condition for the global asymptotic stability of the positive equilibrium.

Example 6 Every solution of the vector equation in $\mathbb{R}^2$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{an}{1+n^2} & \frac{cn}{1+n^3} \\ \frac{bn}{1+n^2} & \frac{dn}{1+n^3} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} \frac{An}{1+n} & \frac{Cn}{1+n^2} \\ \frac{Bn}{1+n} & \frac{Dn}{1+n^2} \end{bmatrix} \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}, n = 0,1,\ldots,$$

where $a,b,c,d,A,B,C,D > 0, x_{-1}, y_{-1}, x_0, y_0 \geq 0, n = 0,1,\ldots,$ converges to the zero equilibrium if $\max\{\frac{a+b}{2}, \frac{2(c+d)}{32^{1/3}}\} + \max\{A+B, \frac{C+D}{2}\} < 1$ is satisfied. Indeed, in this case if $\|x\|$ denotes the L_1 norm we have

$$\|g_0\| = \left\| \begin{bmatrix} \frac{an}{1+n^2} & \frac{cn}{1+n^3} \\ \frac{bn}{1+n^2} & \frac{dn}{1+n^3} \end{bmatrix} \right\| = \max\left\{\frac{(a+b)n}{1+n^2}, \frac{(c+d)n}{1+n^3}\right\} \leq \max\left\{\frac{a+b}{2}, \frac{2(c+d)}{32^{1/3}}\right\}$$

and

$$\|g_1\| = \left\| \begin{bmatrix} \frac{An}{1+n} & \frac{Cn}{1+n^2} \\ \frac{Bn}{1+n} & \frac{Dn}{1+n^2} \end{bmatrix} \right\| = \max\left\{\frac{(A+B)n}{1+n}, \frac{(C+D)n}{1+n^2}\right\} \leq \max\left\{A+B, \frac{C+D}{2}\right\}$$

and the result follows from Theorem 2 and Corollary 1. Thus in this case the zero equilibrium is globally asymptotically stable.

Example 7 The vector equation in $\mathbb{R}^2$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \frac{ax_n}{1+x_n} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \frac{a}{1+x_n} \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix}, n = 0,1,\ldots \tag{35}$$

is equivalent to the system

$$\begin{aligned} x_{n+1} &= \tfrac{ax_n}{1+x_n} x_n + \tfrac{a}{1+x_n} x_{n-1} \\ y_{n+1} &= \tfrac{ax_n}{1+x_n} y_n + \tfrac{a}{1+x_n} y_{n-1}, \quad n = 0,1,\ldots, \end{aligned}$$

where $a > 0$. Since $g_0 + g_1 = a$ for all $n = 0,1,\ldots$ we have the following result which proof follows from Theorems 2, 3, 5 and Corollary 1.

Proposition 1 *The following trichotomy holds for equation* (35)*:*

(a) *if $a < 1$ then the zero equilibrium of* (35) *is globally asymptotically stable.*

(b) *if $a = 1$ then every nonnegative constant vector $\vec{L}$ is an equilibrium of* (35) *and every solution of* (35) *converges to some constant vector.*

(a) *if $a > 1$ then every set of positive (resp. negative) initial conditions generates the solution which component-wise tends to ∞ (resp. $-\infty$).*

Proposition 1 can be extended to the case of corresponding vector equation in $\mathbb{R}^p$.

Acknowledgements. M.R.S. Kulenović is supported in part by Maitland P. Simmons Foundation.

References

[1] A. Bilgin and M. R. S. Kulenović, Global Asymptotic Stability for Discrete Single Species Biological Models, (to appear).

[2] A. M. Brett, E. J. Janowski and M. R. S. Kulenović, Global Asymptotic Stability for Linear Fractional Difference equation, *J. Difference equations*, 1(2014), 12 p.

[3] M. DiPippo, E. J. Janowski and M. R. S. Kulenović, Global Asymptotic Stability for Quadratic Fractional Difference equation, *Adv. Difference Equ.* 2015, 2015:179, 13 pp.

[4] E. J. Janowski and M. R. S. Kulenović, Attractivity and global stability for linearizable difference equations, *Comput. Math. Appl.* 57 (2009), no. 9, 1592–1607.

[5] E. J. Janowski, M. R. S. Kulenović and E. Silić, Periodic Solutions of Linearizable Difference equations, *International J. Difference Equ.*, 6(2011), 113–125.

[6] V. L. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference equations of Higher Order with Applications*, Kluwer Academic Publishers, Dordrecht, 1993.

[7] U. Krause, *Positive dynamical systems in discrete time. Theory, models, and applications.* De Gruyter Studies in Mathematics, 62. De Gruyter, Berlin, 2015.

[8] M. R. S. Kulenović and M. Mehuljić, Global Behavior of Some Rational Second Order Difference equations, *International J. Difference Equ.*, 7(2012), 151–160.

[9] M. R. S. Kulenović and O. Merino, *Discrete Dynamical Systems and Difference equations with Mathematica*, Chapman and Hall/CRC, Boca Raton, London, 2002.

[10] M. R. S. Kulenović and O. Merino, A global attractivity result for maps with invariant boxes. *Discrete Contin. Dyn. Syst. Ser. B*, 6(2006), 97–110.

[11] R. Nussbaum, Global Stability, Two Conjectures, and Maple, *Nonlinear Analysis, Theory, Method and Applications*, 66(2007), 1064–1090.

[12] H. Sedaghat, *Nonlinear Difference equations, Theory with applications to social science models,. Mathematical Modelling: Theory and Applications*, 15. Kluwer Academic Publishers, Dordrecht, 2003.

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 23, NO. 7, 2017