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Journal of Computational Analysis and Applications

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Some properties on non-admissible and admissible functions sharing some sets in the unit disc *

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Abstract

In this paper, we deal with the uniqueness problem of two non-admissible functions sharing some values and sets in the unit disc, and also investigate the problem on an admissible function and a non-admissible function sharing some values and sets. Some theorems of this paper improve the results given by Fang. In addition, the results in this paper analogous version of the uniqueness theorems of meromorphic functions sharing some sets on the whole complex plane which given by Yi and Cao.
Key words: uniqueness; meromorphic function; admissible; non-admissible.
Mathematical Subject Classification (2010): Primary 30D 35.

1 Introduction and main results

We should assume that reader is familiar with the basic results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions (see Hayman [6] , Yang [14] and Yi and Yang [18]). For a meromorphic function f, we use $S(r,f)$ to denote any quantity satisfying $S(r,f)=o(T(r,f))$ for all r outside a possible exceptional set of finite logarithmic measure, and use $\mathbb{C}$ to denote the open complex plane, $\widehat{\mathbb{C}}:=\mathbb{C}\bigcup\{\infty\}$ to denote the extended complex plane, and $\mathbb{D}=\{z:|z|<1\}$ to denote the unit disc.

R. Nevanlinna [10] proved the following well-known theorems.

Theorem 1.1 (see [10]) *If f and g are two non-constant meromorphic functions that share five distinct values a_1,a_2,a_3,a_4,a_5 IM in $\mathbb{C}$, then $f(z)\equiv g(z)$.*

After this work, the uniqueness of meromorphic functions with shared sets and values attracted many investigations (see [18]). Moreover, the uniqueness theory of meromorphic functions is an important subject in the value distribution theory. In this paper, we mainly investigate the uniqueness of meromorphic functions with slow growth sharing some sets in the unit disc.

We firstly introduce the following basic notations and definitions of meromorphic functions in $\mathbb{D}$(see [2, 4, 7, 12, 8, 13, 22]).

Definition 1.1 *(see [12]). Let f be a meromorphic function in $\mathbb{D}$ and $\lim_{r\to 1^-}T(r,f)=\infty$. Then*

$$D(f):=\limsup_{r\to 1^-}\frac{T(r,f)}{-\log(1-r)}$$

is called the (upper) index of inadmissibility of f. If $D(f)=\infty$, f is called admissible.

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Definition 1.2 *(see [12]). Let f be a meromorphic function in $\mathbb{D}$ and $\lim_{r\to 1^-} T(r,f) = \infty$. Then*

$$\rho(f) := \limsup_{r\to 1^-} \frac{\log^+ T(r,f)}{-\log(1-r)}$$

is called the order (of growth) of f.

The Second Main Theorem for admissible functions (see [12, Theorem 3]) is very important in studying the uniqueness of two admissible functions in the unit disc $\mathbb{D}$, which was proved by in 2005.

Theorem 1.2 (see [12, Theorem 3]). *Let f be an admissible meromorphic function in $\mathbb{D}$, q be a positive integer and $a_1, a_2, \ldots, a_q$ be pairwise distinct complex numbers. Then, for $r \to 1^-$, $r \notin E$,*

$$(q-2)T(r,f) \le \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f-a_j}\right) + S(r,f),$$

where $E \subset (0,1)$ is a possibly occurring exceptional set with $\int_E \frac{dr}{1-r} < \infty$. If the order of f is finite, the remainder $S(r,f)$ is a $O\left(\log \frac{1}{1-r}\right)$ without any exceptional set.

In 2005, Titzhoff [12] also obtained the five values theorem for admissible functions in the unit disc $\mathbb{D}$ as follows.

Theorem 1.3 (see [5, 12]). *If two admissible functions f, g share five distinct values, then $f \equiv g$.*

From Theorem 1.2(see [12, Theorem 3]), we can easily obtain a lot of theorems similar to meromorphic functions in the complex plane. In 1999, Fang [5] investigated the uniqueness of admissible functions sharing two sets and three sets and obtained a series of theorems. In 2015, Xu, Yang and Cao [15] investigated the problem on shared values of admissible function and non-admissible function, and obtained some interesting results. Inspired by Xu, Yang and Cao [15] and Fang[5], we further study the problem on shared-sets of admissible function and non-admissible function in the unit disc.

The following theorem also plays a very important role in studies non-admissible functions sharing some sets in this paper.

Theorem 1.4 (see [12, Theorem 2]). *Let f be a meromorphic function in $\mathbb{D}$ and $\lim_{r\to 1^-} T(r,f) = \infty$, q be a positive integer and $a_1, a_2, \ldots, a_q$ be pairwise distinct complex numbers. Then, for $r \to 1^-$, $r \notin E$,*

$$(q-2)T(r,f) \le \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f-a_j}\right) + \log\frac{1}{1-r} + S(r,f).$$

Remark 1.1 *In contrast to admissible functions, the term $\log \frac{1}{1-r}$ in Theorem 1.4 does not necessarily enter the remainder $S(r,f)$ because the non-admissible function f may have $T(r,f) = O\left(\log \frac{1}{1-r}\right)$.*

Remark 1.2 *We can see that $S(r,f) = o\left(\log \frac{1}{1-r}\right)$ holds in Theorem 1.4 without a possible exception set when $0 < D(f) < \infty$.*

The following lemma for non-admissible functions in the unit disc is used in this paper.

Lemma 1.1 *(see [15]). Let $f(z)$ be a meromorphic function in $\mathbb{D}$ and $\lim_{r\to 1^-} T(r,f) = \infty$, $a_j (j = 1, 2, \ldots, q)$ be q distinct complex numbers, and $k_j (j = 1, 2, \ldots, q)$ be positive integers or ∞. If f is a non-admissible function, then*

$$(q-2)T(r,f) < \sum_{j=1}^{q} \frac{k_j}{k_j+1} \overline{N}_{k_j)}\left(r, \frac{1}{f-a_j}\right) + \sum_{j=1}^{q} \frac{1}{k_j+1} N\left(r, \frac{1}{f-a_j}\right) + \log\frac{1}{1-r} + S(r,f),$$

and

$$\left(q-2-\sum_{j=1}^{q}\frac{1}{k_j+1}\right) T(r,f) \le \sum_{j=1}^{q} \frac{k_j}{k_j+1} \overline{N}_{k_j)}\left(r, \frac{1}{f-a_j}\right) + \log\frac{1}{1-r} + S(r,f),$$

where $\overline{n}_{k)}(r, \frac{1}{f-a})$ is used to denote the zeros of $f-a$ in $|z| \le r$, whose multiplicities are no greater than k and are counted only once, $\overline{N}_{k)}(r, \frac{1}{f-a})$ is the corresponding counting functions, and $\frac{k_j}{k_j+1} = 1, \overline{N}_{k_j)}(r, \frac{1}{f-a_j}) = \overline{N}(r, \frac{1}{f-a_j})$ and $\frac{1}{k_j+1} = 0$ if $k_j = \infty$, $S(r,f)$ is stated as in Theorem 1.2.

The main purpose of this paper is to deal with the problem of two non-admissible functions sharing some sets, and an admissible function sharing some sets with an non-admissible function. *Section 2, the uniqueness of two non-admissible functions sharing some sets in $\mathbb{D}$ are investigated and some results showed that the number and weight of sharing sets is related with the index of inadmissibility of functions in $\mathbb{D}$. In section 3, the problem of an admissible function and a non-admissible function sharing some sets is studied, and one of those results shows that admissible function and non-admissible function can share at most five distinct values with reduced weighted 1.*

2 The uniqueness and sharing sets of non-admissible functions in the unit disc

Let S be a set of distinct elements in $\widehat{\mathbb{C}}$ and $\mathbb{X} \subseteq \mathbb{C}$. Define

$$E(S, \mathbb{D}, f) = \bigcup_{a\in S} \{z \in \mathbb{D} | f_a(z) = 0, \quad \textit{counting} \quad \textit{multiplicities}\},$$

$$\overline{E}(S, \mathbb{D}, f) = \bigcup_{a\in S} \{z \in \mathbb{D} | f_a(z) = 0, \quad \textit{ignoring} \quad \textit{multiplicities}\},$$

where $f_a(z) = f(z) - a$ if $a \in \mathbb{C}$ and $f_\infty(z) = 1/f(z)$.

For two non-constant meromorphic functions f, g, we say f and g share the set S CM(counting multiplicities) in $\mathbb{D}$ if $E(S, \mathbb{D}, f) = E(S, \mathbb{D}, g)$; we say f and g share the set S IM(ignoring multiplicities) in $\mathbb{D}$ if $\overline{E}(S, \mathbb{D}, f) = \overline{E}(S, \mathbb{D}, g)$. In particular, as $S = \{a\}$ and $a \in \widehat{\mathbb{C}}$, we say f and g share the value a CM in $\mathbb{D}$ if $E(a, \mathbb{D}, f) = E(a, \mathbb{D}, g)$, and we say f and g share the value a IM in $\mathbb{D}$ if $\overline{E}(a, \mathbb{D}, f) = \overline{E}(a, \mathbb{D}, g)$. We use $\overline{E}_{k)}(a, \mathbb{D}, f)$ to denote the set of zeros of $f-a$ in $\mathbb{D}$, with multiplicities no greater than k, in which each zero counted only once. We say that $f(z)$ and $g(z)$ share the value a with reduced weight k in $\mathbb{D}$, if $\overline{E}_{k)}(a, \mathbb{D}, f) = \overline{E}_{k)}(a, \mathbb{D}, g)$. If $\mathbb{D} = \mathbb{C}$, we have the simple notation as before, $E(S, f), \overline{E}(S, f), \overline{E}_{k)}(a, f)$ and so on(see [18]).

The deficiency of $a \in \widehat{\mathbb{C}}$ with respect to a meromorphic function f on the unit disc $\mathbb{D}$ is defined by

$$\delta(a,f) = \delta(0, f-a) = \liminf_{r\to 1^-} \frac{m(r, \frac{1}{f-a})}{T(r,f)} = 1 - \limsup_{r\to 1^-} \frac{N(r, \frac{1}{f-a})}{T(r,f)},$$

and the reduced deficiency by

$$\Theta(a,f)=\Theta(0,f-a)=1-\limsup_{r\to 1^-}\frac{\overline{N}(r,\frac{1}{f-a})}{T(r,f)}.$$

We now show our main theorems. The first theorem can be called five values theorem of non-admissible functions.

Theorem 2.1 *Let f_1 and f_2 be two non-admissible meromorphic functions in the unit disc $\mathbb{D}$ satisfying $1<D(f_1),D(f_2)<\infty$, and f_1,f_2 share $a_j(j=1,2,3,4,5)$ IM. Then $f_1(z)\equiv f_2(z)$.*

Remark 2.1 *From Theorem 2.1, we can get that $f_1(z)\equiv f_2(z)$ if f_1,f_2 share five distinct values and $D(f_1),D(f_2)>1$. However, the conclusion holds in Theorem 1.3 under the condition which f_1,f_2 are admissible functions, that is, $D(f_1)=\infty$, and $D(f_2)=\infty$. Thus, we can see that Theorem 2.1 is a greatly improvement of Theorem 1.3.*

In order to prove Theorem 2.1, we will prove the following general results of two non-admissible functions sharing some sets.

Theorem 2.2 *Let f_1 and f_2 be two non-admissible meromorphic functions in the unit disc $\mathbb{D}$ satisfying $0<D(f_1),D(f_2)<\infty$. Suppose that*

$$S_j=\{a_j,a_j+b,\ldots,a_j+(l-1)b\},\quad j=1,2,\ldots,q,$$

with $b\neq 0$, $S_i\cap S_j=\emptyset$, $(i\neq j)$ and $q>2+\max\left\{\left[\frac{1}{D(f_1)}\right],\left[\frac{1}{D(f_2)}\right]\right\}$, where $[x]$ denotes the largest integer less than or equal to x. Let k_j $(j=1,2,\ldots,q)$ be positive integers or ∞ satisfying

$$k_1\geq k_2\geq\cdots\geq k_q \tag{1}$$

and

$$\overline{E}_{k_j)}(S_j,\mathbb{D},f_1)=\overline{E}_{k_j)}(S_j,\mathbb{D},f_2),\quad (j=1,2,\ldots,q). \tag{2}$$

Furthermore, let

$$\Theta(f_i)=\sum_a\Theta(0,f_i-a)-\sum_{j=1}^{q}\sum_{s=0}^{l-1}\Theta(0,f_i-(a_j+sb)),(i=1,2),$$

and

$$\begin{aligned}A_1 &= \frac{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1}\delta(0,f_1-(a_j+sb))}{k_m+1}+\sum_{j=m}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta(0,f_1-(a_j+sb))}{k_j+1}\\&\quad+\frac{(lm-3l+1)k_m}{k_m+1}-\frac{(2l-1)k_n}{k_n+1}+\Theta(f_1)-2,\end{aligned}$$

$$\begin{aligned}A_2 &= \frac{\sum_{j=1}^{n-1}\sum_{s=0}^{l-1}\delta(0,f_2-(a_j+sb))}{k_n+1}+\sum_{j=n}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta(0,f_2-(a_j+sb))}{k_j+1}\\&\quad+\frac{(ln-3l+1)k_n}{k_n+1}-\frac{(2l-1)k_m}{k_m+1}+\Theta(f_2)-2,\end{aligned}$$

where m and n are positive integers in $\{1,2,\ldots,q\}$ and a is an arbitrary complex number or ∞. If

$$\min\{A_1,A_2\}\geq\frac{2}{D(f_1)+D(f_2)},\quad and\quad \max\{A_1,A_2\}>\frac{2}{D(f_1)+D(f_2)}. \tag{3}$$

Then $f_1(z)\equiv f_2(z)$.

By letting $l=1$, $q=5$ and $k_1=k_2=\cdots=k_5=\infty$ in Theorem 2.2, we can get Theorem 2.1 easily. Now, we start to prove Theorem 2.2 as follows.

Proof of Theorem 2.2: Suppose that $f_1(z)\not\equiv f_2(z)$. From the second fundamental theorem in the unit disc (Theorem 1.4) we have

$$\begin{aligned}(ql+p-2)T(r,f_1) &< \sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}\left(r,\frac{1}{f_1-(a_j+sb)}\right)+\sum_{k=1}^{p}\overline{N}\left(r,\frac{1}{f_1-d_k}\right)\\ &\quad+\log\frac{1}{1-r}+S(r,f_1).\end{aligned}$$

By definition we have

$$\overline{N}\left(r,\frac{1}{f_1-d_k}\right)<(1-\Theta(0,f_1-d_k))\,T(r,f_1)+S(r,f_1).$$

From Lemma 1.1 and the definition of deficiency, it follows that for $s\in\{0,1,\ldots,l-1\}$

$$\begin{aligned}&\overline{N}\left(r,\frac{1}{f_1-(a_j+sb)}\right)\\ \le\;& \frac{k_j}{k_j+1}\overline{N}_{k_j)}\left(r,\frac{1}{f_1-(a_j+sb)}\right)+\frac{1}{k_j+1}N\left(r,\frac{1}{f_1-(a_j+sb)}\right)\\ <\;& \frac{k_j}{k_j+1}\overline{N}_{k_j)}\left(r,\frac{1}{f_1-(a_j+sb)}\right)+\frac{1}{k_j+1}\left(1-\delta(0,f_1-(a_j+sb))\right)T(r,f_1)\\ &+S(r,f_1).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&(ql+p-2)T(r,f_1)\\ <\;& \left\{\sum_{k=1}^{p}(1-\Theta(0,f_1-d_k))\right\}T(r,f_1)+\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_j}{k_j+1}\overline{N}_{k_j)}(r,\frac{1}{f_1-(a_j+sb)})\\ &+\left\{\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{1}{k_j+1}\left(1-\delta(0,f_1-(a_j+sb))\right)\right\}T(r,f_1)+\log\frac{1}{1-r}+S(r,f_1).\end{aligned}$$

Since $\Theta(0,f-a)\ge 0$ for any meromorphic function f and any complex number $a\in\widehat{\mathbb{C}}$. Without loss of generality, we assume that there exist infinitely many d such that $\Theta(0,f_1-d)>0$ and $d\notin\{a_j+sb: j=1,2,\ldots,q\,\text{and}\,s=0,1,\ldots,l-1\}$. We denote them by d_k $(k=1,2,\ldots,\infty)$. Obviously, $\Theta(f_1)=\Sigma_{k=1}^{\infty}\Theta(0,f_1-d_k)$. Thus there exits a p such that $\Sigma_{k=1}^{p}\Theta(0,f_1-d_k)>\Theta(f_1)-\varepsilon$ holds for any given ε (>0). Noting that

$$1\ge\frac{k_1}{k_1+1}\ge\frac{k_2}{k_2+1}\ge\cdots\ge\frac{k_q}{k_q+1}\ge\frac{1}{2},$$

we can deduce that

$$\begin{aligned}&(ql+p-2)T(r,f_1)\\ <\;& (p-\Theta(f_1)+\varepsilon)\,T(r,f_1)+\frac{k_m}{k_m+1}\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}_{k_j)}\left(r,\frac{1}{f_1-(a_j+sb)}\right)\\ &+\left\{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1}\left(\frac{k_j}{k_j+1}-\frac{k_m}{k_m+1}\right)\left(1-\delta(0,f_1-(a_j+sb))\right)\right\}T(r,f_1)\\ &+\left\{\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{1-\delta(0,f_1-(a_j+sb))}{k_j+1}\right\}T(r,f_1)+\log\frac{1}{1-r},\end{aligned}$$

namely,

$$\left(\frac{l(m-1)k_m}{k_m+1}+B_1-\varepsilon\right)T(r,f_1)<\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_m}{k_m+1}\overline{N}_{k_j)}(r,\frac{1}{f_1-(a_j+sb)})+\log\frac{1}{1-r},$$

where

$$B_1=\frac{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1}\delta(0,f_1-(a_j+sb))}{k_m+1}+\sum_{j=m}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta(0,f_1-(a_j+sb))}{k_j+1}+\Theta(f_1)-2.$$

By a similar discussion as above, we also have

$$\left(\frac{l(n-1)k_n}{k_n+1}+B_2-\varepsilon\right)T(r,f_2)<\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_n}{k_n+1}\overline{N}_{k_j)}\left(r,\frac{1}{f_2-(a_j+sb)}\right)+\log\frac{1}{1-r},$$

where

$$B_2=\frac{\sum_{j=1}^{n-1}\sum_{s=0}^{l-1}\delta(0,f_2-(a_j+sb))}{k_n+1}+\sum_{j=n}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta(0,f_2-(a_j+sb))}{k_j+1}+\Theta(f_2)-2.$$

Hence

$$\begin{aligned}&\left(\frac{l(m-1)k_m}{k_m+1}+B_1-\varepsilon\right)T(r,f_1)+\left(\frac{l(n-1)k_n}{k_n+1}+B_2-\varepsilon\right)T(r,f_2)\\ <&\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_m}{k_m+1}\overline{N}_{k_j)}(r,\frac{1}{f_1-(a_j+sb)})+\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_n}{k_n+1}\overline{N}_{k_j)}(r,\frac{1}{f_2-(a_j+sb)})\\ &+2\log\frac{1}{1-r}.\end{aligned}$$

We now assert that $f_1(z)-f_2(z)\not\equiv sb$, $s=1,2,\ldots,l-1$. Otherwise, we get that a_j $(j=1,2,\ldots,q)$ are the Picard exceptional values of f_1, and that $a_j+(l-1)b$ $(j=1,2,\ldots,q)$ are the Picard exceptional values of f_2. By $q>2+\frac{1}{D(f_1)}$ and Theorem 1.4, we get a contradiction. Similarly, we have $f_2(z)-f_1(z)\not\equiv sb$, $s=1,2,\ldots,l-1$.

By condition (2) and the first fundamental theorem, we have

$$\begin{aligned}&\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}_{k_j)}\left(r,\frac{1}{f_1-(a_j+sb)}\right)\\ &\le\overline{N}\left(r,\frac{1}{f_1-f_2}\right)+\sum_{s=1}^{l-1}\overline{N}\left(r,\frac{1}{f_1-f_2-sb}\right)+\sum_{s=1}^{l-1}\overline{N}\left(r,\frac{1}{f_2-f_1-sb}\right)\\ &\le(2l-1)(T(r,f_1)+T(r,f_2))+O(1).\end{aligned}$$

and

$$\begin{aligned}&\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}_{k_j)}\left(r,\frac{1}{f_2-(a_j+sb)}\right)\\ &\le\overline{N}\left(r,\frac{1}{f_1-f_2}\right)+\sum_{s=1}^{l-1}\overline{N}\left(r,\frac{1}{f_1-f_2-sb}\right)+\sum_{s=1}^{l-1}\overline{N}\left(r,\frac{1}{f_2-f_1-sb}\right)\\ &\le(2l-1)(T(r,f_1)+T(r,f_2))+O(1).\end{aligned}$$

Therefore, from the above discussion we obtain

$$\begin{aligned}&\left(\frac{l(m-1)k_m}{k_m+1}+B_1-\varepsilon\right)T(r,f_1)+\left(\frac{l(n-1)k_n}{k_n+1}+B_2-\varepsilon\right)T(r,f_2)\\ <\;&(2l-1)\left(\frac{k_m}{k_m+1}+\frac{k_n}{k_n+1}\right)(T(r,f_1)+T(r,f_2))+2\log\frac{1}{1-r},\end{aligned}$$

namely,

$$(A_1-\varepsilon)\,T(r,f_1)+(A_2-\varepsilon)\,T(r,f_2)\le 2\log\frac{1}{1-r}. \tag{4}$$

Since $0<D(f_1),D(f_2)<\infty$, we have $S(r,f_1)=o\left(\log\frac{1}{1-r}\right), S(r,f_2)=o\left(\log\frac{1}{1-r}\right)$. And from the definition of index, for any ε satisfying

$$0<2\varepsilon<\min\left\{D(f_1),D(f_2),\max\{A_1,A_2\}-\frac{2}{D(f_1)+D(f_2)}\right\}, \tag{5}$$

there exists a sequence $\{r_t\}\to 1^-$ such that

$$T(r_t,f_1)>(D(f_1)-\varepsilon)\log\frac{1}{1-r_t},\quad T(r_t,f_2)>(D(f_2)-\varepsilon)\log\frac{1}{1-r_t}, \tag{6}$$

for all $t\to\infty$. From (4)-(6), we have

$$[(D(f_1)-\varepsilon)(A_1-\varepsilon)+(D(f_2)-\varepsilon)(A_2-\varepsilon)-2]\log\frac{1}{1-r_t}<o\left(\log\frac{1}{1-r_t}\right). \tag{7}$$

From (7) and ε being arbitrary, the above inequality contradicts to (3). Therefore, the proof of Theorem 2.2 is completed.

We can get the following corollaries from Theorem 2.2.

Corollary 2.1 *Let k_j $(j=1,2,\ldots,q)$ be positive integers or ∞ satisfying (1), and let f_1 and f_2 be two non-admissible meromorphic functions in the unit disc $\mathbb{D}$ satisfying $0<D(f_1),D(f_2)<\infty$ and (2). Suppose that*

$$S_j=\{a_j,a_j+b,\ldots,a_j+(l-1)b\},\quad j=1,2,\ldots,q,$$

with $b\neq 0$, $S_i\cap S_j=\emptyset$, $(i\neq j)$ and $q>2+\max\left\{\left[\frac{1}{D(f_1)}\right],\left[\frac{1}{D(f_2)}\right]\right\}$, where $[x]$ denotes the largest integer less than or equal to x. If

$$\sum_{j=3}^{q}\sum_{s=0}^{l-1}\frac{k_j}{k_j+1}+\frac{(2-2l)k_3}{k_3+1}>2+\frac{2}{D(f_1)+D(f_2)}.$$

Then $f_1(z)\equiv f_2(z)$.

Proof: Let $m=n=3$. Noting that $\Theta(f_i)\ge 0$ and $\delta(0,f_i-(a_j+sb))\ge 0$ for $j=1,2,\ldots,q$ and $i=1,2$, one can deduce from Theorem 2.2 that Corollary 2.1 follows. □

The following corollary is an analog of a result due to H.-X. Yi (Theorem 10.7 in [18], see also [21]) on $\mathbb{C}$.

Corollary 2.2 *Let f_1 and f_2 be two non-admissible meromorphic functions in the unit disc $\mathbb{D}$ satisfying $0<D(f_1),D(f_2)<\infty$. Suppose that*

$$S_j=\{a_j,a_j+b,\ldots,a_j+(l-1)b\},\quad j=1,2,\ldots,q,$$

with $b\neq 0$, $S_i\cap S_j=\emptyset$, $(i\neq j)$ and

$$q>\max\left\{4+\frac{2}{(D(f_1)+D(f_2))l},2+\max\left\{\left[\frac{1}{D(f_1)}\right],\left[\frac{1}{D(f_2)}\right]\right\}\right\}.$$

If $\overline{E}(S_j,\mathbb{D},f_1)=\overline{E}(S_j,\mathbb{D},f_2),(j=1,2,\ldots,q)$. Then $f_1(z)\equiv f_2(z)$.

Proof: Let $k_1 = k_2 = \ldots = k_q = \infty$. One can deduce from Corollary 2.1 that Corollary 2.2 follows immediately. □

Let $l = 1$. Then it is easily derived the following corollary from Corollary 2.1, which is an analog of the Corollary of Theorem 3.15 in [18].

Corollary 2.3 *Let a_j $(j = 1, 2, \ldots, q)$ be q distinct complex numbers in $\widehat{\mathbb{C}}$, and k_j $(j = 1, 2, \ldots, q)$ be positive integers or ∞ satisfying (1), and let f_1 and f_2 be two non-admissible meromorphic functions in the unit disc $\mathbb{D}$ satisfying $0 < D(f_1), D(f_2) < \infty$ and $\overline{E}_{k_j)}(a_j, \mathbb{D}, f_1) = \overline{E}_{k_j)}(a_j, \mathbb{D}, f_2)$. Set $D := \min\{D(f_1), D(f_2)\}$. Then*

(i) if $D > 1$, $q = 7$ and $k_7 \geq 2$, then $f_1(z) \equiv f_2(z)$;
(ii) if $D > 1$, $q = 6$ and $k_6 \geq 4$, then $f_1(z) \equiv f_2(z)$;
(iii) if $D > 2$ and $q = 7$, then $f_1(z) \equiv f_2(z)$;
(iv) if $D > 3$, $q = 6$ and $k_3 \geq 2$, then $f_1(z) \equiv f_2(z)$;
(v) if $D > 6$, $q = 5$, $k_3 \geq 3$ and $k_5 \geq 2$, then $f_1(z) \equiv f_2(z)$;
(vi) if $D > 10$, $q = 5$ and $k_4 \geq 4$, then $f_1(z) \equiv f_2(z)$;
(vii) if $D > 12$, $q = 5$, $k_3 \geq 5$ and $k_4 \geq 3$, then $f_1(z) \equiv f_2(z)$;
(viii) if $D > 42$, $q = 5$, $k_3 \geq 6$ and $k_4 \geq 2$, then $f_1(z) \equiv f_2(z)$.

We now state another main theorem.

Theorem 2.3 *Let f_1 and f_2 be two non-admissible meromorphic functions in the unit disc $\mathbb{D}$ satisfying $0 < D(f_1), D(f_2) < \infty$. Suppose that*

$$S_j = \{c + a_j, c + a_j w, \ldots, c + a_j w^{l-1}\}, \quad j = 1, 2, \ldots, q,$$

with $a_j \neq 0$, $(j = 1, 2, \ldots, q)$, $w = \exp(\frac{2\pi i}{l})$, $S_i \cap S_j = \emptyset$, $(i \neq j)$ and $q > 2 + \max\left\{\left[\frac{1}{D(f_1)}\right], \left[\frac{1}{D(f_2)}\right]\right\}$. Let k_j $(j = 1, 2, \ldots, q)$ be positive integers or ∞ satisfying (1), and

$$\overline{E}_{k_j)}(S_j, \mathbb{D}, f_1) = \overline{E}_{k_j)}(S_j, \mathbb{D}, f_2), \quad (j = 1, 2, \ldots, q). \tag{8}$$

Furthermore, let

$$\Theta(f_i) = \sum_a \Theta(0, f_i - a) - \sum_{j=1}^{q}\sum_{s=0}^{l-1} \Theta(0, f_i - (c + a_j w^s)), (i = 1, 2),$$

and

$$\begin{aligned} A_3 &= \frac{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1} \delta(0, f_1 - (c + a_j w^s))}{k_m + 1} + \sum_{j=m}^{q}\sum_{s=0}^{l-1} \frac{k_j + \delta(0, f_1 - (c + a_j w^s))}{k_j + 1} \\ &\quad + \frac{l(m-2)k_m}{k_m + 1} - \frac{lk_n}{k_n + 1} + \Theta(f_1) - 2, \end{aligned}$$

$$\begin{aligned} A_4 &= \frac{\sum_{j=1}^{n-1}\sum_{s=0}^{l-1} \delta(0, f_2 - (c + a_j w^s))}{k_n + 1} + \sum_{j=n}^{q}\sum_{s=0}^{l-1} \frac{k_j + \delta(0, f_2 - (c + a_j w^s))}{k_j + 1} \\ &\quad + \frac{l(n-2)k_n}{k_n + 1} - \frac{lk_m}{k_m + 1} + \Theta(f_2) - 2, \end{aligned}$$

where m and n are positive integers in $\{1, 2, \ldots, q\}$ and a is an arbitrary complex number or ∞. If

$$\min\{A_3, A_4\} \geq \frac{2}{D(f_1) + D(f_2)}, \quad \text{and} \quad \max\{A_3, A_4\} > \frac{2}{D(f_1) + D(f_2)}. \tag{9}$$

Then $(f_1(z) - c)^l \equiv (f_1(z) - c)^l$.

Proof: We assume that $(f_1(z)-c)^l \not\equiv (f_2(z)-c)^l$. Without loss of generality, we assume that there exist infinitely many d such that $\Theta(0,f_1-d)>0$ and $d\notin\{c+a_jw^s : j=1,2,\ldots,q \text{ and } s=0,1,\ldots,l-1\}$. We denote them by d_k $(k=1,2,\ldots,\infty)$. Obviously, $\Theta(f_1)=\sum_{k=1}^{\infty}\Theta(0,f_1-d_k)$. Thus there exits a p such that $\sum_{k=1}^{p}\Theta(0,f_1-d_k)>\Theta(f_1)-\varepsilon$ holds for any given ε (>0).

Using a similar discussion as in the proof of Theorem 2.2, we obtain

$$\begin{aligned}&\left(\frac{l(m-1)k_m}{k_m+1}+B_3-\varepsilon\right)T(r,f_1)+\left(\frac{l(n-1)k_n}{k_n+1}+B_4-\varepsilon\right)T(r,f_2)\\ <\ &\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_m}{k_m+1}\overline{N}_{k_j)}(r,\frac{1}{f_1-(c+a_jw^s)})+\sum_{j=1}^{q}\sum_{s=0}^{l-1}\frac{k_n}{k_n+1}\overline{N}_{k_j)}(r,\frac{1}{f_2-(c+a_jw^s)})\\ &+2\log\frac{1}{1-r},\end{aligned}$$

where

$$B_3=\frac{\sum_{j=1}^{m-1}\sum_{s=0}^{l-1}\delta(0,f_1-(c+a_jw^s))}{k_m+1}+\sum_{j=m}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta(0,f_1-(c+a_jw^s))}{k_j+1}+\Theta(f_1)-2.$$

$$B_4=\frac{\sum_{j=1}^{n-1}\sum_{s=0}^{l-1}\delta(0,f_2-(c+a_jw^s))}{k_n+1}+\sum_{j=n}^{q}\sum_{s=0}^{l-1}\frac{k_j+\delta(0,f_2-(c+a_jw^s))}{k_j+1}+\Theta(f_2)-2.$$

Furthermore, from condition (8) and the first fundamental theorem, we have

$$\begin{aligned}\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}_{k_j)}(r,\frac{1}{f_1-(c+a_jw^s)})&<\overline{N}(r,\frac{1}{(f_1-c)^l-(f_2-c)^l})\\ &\le l(T(r,f_1)+T(r,f_2))+O(1).\end{aligned}$$

and

$$\begin{aligned}\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}_{k_j)}(r,\frac{1}{f_2-(c+a_jw^s)})&<\overline{N}(r,\frac{1}{(f_1-c)^l-(f_2-c)^l})\\ &\le l(T(r,f_1)+T(r,f_2))+O(1).\end{aligned}$$

Therefore, from the above discussion we obtain

$$\begin{aligned}&\left(\frac{l(m-1)k_m}{k_m+1}+B_3-\varepsilon\right)T(r,f_1)+\left(\frac{l(n-1)k_n}{k_n+1}+B_4-\varepsilon\right)T(r,f_2)\\ <\ &l\left(\frac{k_m}{k_m+1}+\frac{k_n}{k_n+1}\right)(T(r,f_1)+T(r,f_2))+2\log\frac{1}{1-r},\end{aligned}$$

namely,

$$(A_3-\varepsilon)\,T(r,f_1)+(A_4-\varepsilon)\,T(r,f_2)<2\log\frac{1}{1-r}. \tag{10}$$

Since $0<D(f_1),D(f_2)<\infty$, we have $S(r,f_1)=o\left(\log\frac{1}{1-r}\right), S(r,f_2)=o\left(\log\frac{1}{1-r}\right)$. And from the definition of index, for any ε satisfying

$$0<2\varepsilon<\min\left\{D(f_1),D(f_2),\max\{A_3,A_4\}-\frac{2}{D(f_1)+D(f_2)}\right\}, \tag{11}$$

there exists a sequence $\{r_t\} \to 1^-$ such that

$$T(r_t, f_1) > (D(f_1) - \varepsilon)\log\frac{1}{1-r_t}, \quad T(r_t, f_2) > (D(f_2) - \varepsilon)\log\frac{1}{1-r_t}, \tag{12}$$

for all $t \to \infty$. From (10)-(12), we have

$$[(D(f_1) - \varepsilon)(A_3 - \varepsilon) + (D(f_2) - \varepsilon)(A_4 - \varepsilon) - 2]\log\frac{1}{1-r_t} < o\left(\log\frac{1}{1-r_t}\right). \tag{13}$$

From (13) and ε being arbitrary, the above inequality contradicts to (9).

Therefore, the proof of Theorem 2.3 is completed. □

We have an analog of a result due to H.-X. Yi (Theorem 10.8 in [18], see also [21]).

Corollary 2.4 *let f_1 and f_2 be two non-admissible meromorphic functions in the unit disc $\mathbb{D}$ satisfying $0 < D(f_1), D(f_2) < \infty$. Suppose that*

$$S_j = \{c + a_j, c + a_j w, \dots, c + a_j w^{l-1}\}, \quad j = 1, 2, \dots, q,$$

with $a_j \neq 0$, $(j = 1, 2, \dots, q)$, $q > 2 + \frac{2}{l} + \frac{2}{D(f_1)+D(f_2)}$, $w = \exp(\frac{2\pi i}{l})$, $S_i \cap S_j = \emptyset$, $(i \neq j)$. If $\overline{E}(S_j, \mathbb{D}, f_1) = \overline{E}(S_j, \mathbb{D}, f_2)$ for $j = 1, 2, \dots, q$, then $(f_1(z) - c)^l \equiv (f_2(z) - c)^l$.

Proof: Let $m = n = 1$ and $k_1 = k_2 = \ldots = \infty$. Noting that $\Theta(f_i) \geq 0$ and $\delta(0, f_i - (a_j + sb)) \geq 0$ for $j = 1, 2, \dots, q$ and $i = 1, 2$, Then Corollary 2.4 follows immediately from Theorem 2.2. □

3 The problem of sharing sets of admissible function and non-admissible function in the unit disc

We now show that an admissible function can share sufficiently many sets concerning multiple values with another non-admissible function as follows.

Theorem 3.1 *If f_1 is admissible and f_2 is a non-admissible satisfying $\lim_{r\to 1^-} T(r, f_2) = \infty$, $a_j (j = 1, 2, \dots, q)$ be q distinct complex numbers, and let $k_j (j = 1, 2, \dots, q)$ be positive integers or ∞ satisfying (1). Then*

$$\overline{E}_{k_j)}(a_j, \mathbb{D}, f_1) = \overline{E}_{k_j)}(a_j, \mathbb{D}, f_2), \quad (j = 1, 2, \dots, q).$$

and

$$\sum_{j=m+1}^{q}\frac{k_j}{k_j+1} + \frac{(m-1)k_m}{k_m+1} - 2 > 0$$

do not hold at same time.

Theorem 3.2 *If f_1 is admissible and f_2 is a non-admissible satisfying $\lim_{r\to 1^-} T(r, f_2) = \infty$. Suppose that*

$$S_j = \{c + a_j, c + a_j w, \dots, c + a_j w^{l-1}\}, \quad j = 1, 2, \dots, q,$$

with $a_j \neq 0$, $(j = 1, 2, \dots, q)$, $w = \exp(\frac{2\pi i}{l})$, $S_i \cap S_j = \emptyset$, $(i \neq j)$. Then $\overline{E}(S_j, \mathbb{D}, f_1) = \overline{E}(S_j, \mathbb{D}, f_2)$ for $j = 1, 2, \dots, q$, and $q > 1 + \frac{2}{l}$ can not hold at the same time.

To prove the above theorems, we require the following lemmas.

Lemma 3.1 (see [12, Lemma 1]). *Let $f(z), g(z)$ satisfy $\lim_{r\to 1^-} T(r, f) = \infty$ and $\lim_{r\to 1^-} T(r, g) = \infty$. If there is a $K \in (0, \infty)$ with*

$$T(r, f) \leq KT(r, g) + S(r, f) + S(r, g),$$

then each $S(r, f)$ is also an $S(r, g)$.

Lemma 3.2 *If f_1 is admissible and f_2 is a non-admissible satisfying* $\lim_{r\to 1^-} T(r, f_2) = \infty$, $a_j (j = 1, 2, \ldots, q)$ *be q distinct complex numbers, and let $k_j (j = 1, 2, \ldots, q)$ be positive integers or ∞ satisfying (1). Set $A_5 = B_1 + \frac{[(m-3)l+1]k_m}{k_m+1}$. Then (2) and $A_5 > 0$ do not hold at same time, where $B_1, S_j (j = 1, 2, \ldots, q)$ are stated as in Theorem 2.1.*

Proof: Suppose that (2) and $A_5 > 0$ can hold at the same time. Since $f_1(z)$ is an admissible function, using the same argument as in Theorem 2.2 and from Theorem 1.2 and Lemma 1.1, for any $\varepsilon(0 < 2\varepsilon < A_5)$, we have

$$\left(\frac{(m-1)lk_m}{k_m+1} + B_1 - \varepsilon\right) T(r, f_1) < \sum_{j=1}^{q}\sum_{s=0}^{l-1} \frac{k_m}{k_m+1} \overline{N}_{k_j)}(r, \frac{1}{f_1 - (a_j + sb)}) + S(r, f_1),$$

where B_1 is stated as in Section 2.

Since f_1 is admissible and f_2 is non-admissible, we can get that $f_1(z) \not\equiv f_2(z)$. Thus, by condition (2) and the first fundamental theorem, we have

$$\begin{aligned}\sum_{j=1}^{q}\sum_{s=0}^{l-1} \overline{N}_{k_j)}\left(r, \frac{1}{f_1 - (a_j + sb)}\right) \leq& \overline{N}\left(r, \frac{1}{f_1 - f_2}\right) + \sum_{s=1}^{l-1} \overline{N}\left(r, \frac{1}{f_1 - f_2 - sb}\right)\\ &+ \sum_{s=1}^{l-1} \overline{N}\left(r, \frac{1}{f_2 - f_1 - sb}\right)\\ \leq& (2l-1)(T(r, f_1) + T(r, f_2)) + O(1).\end{aligned}$$

From the two above inequality, we get

$$\left(\frac{[(m-3)l+1]k_m}{k_m+1} + B_1 - \varepsilon\right) T(r, f_1) \leq \frac{(2l-1)k_m}{k_m+1} T(r, f_2). \tag{14}$$

Since $0 < \varepsilon < A_5$, we have $\frac{[(m-3)l+1]k_m}{k_m+1} + B_1 - \varepsilon > 0$. From (14), we have

$$T(r, f_1) \leq \frac{1}{A_5 - \varepsilon} \frac{(2l-1)k_m}{k_m+1} T(r, f_2). \tag{15}$$

From Lemma 3.1, (15) and $\frac{1}{A_5-\varepsilon}\frac{(2l-1)k_m}{k_m+1} > 0$, we can get that each $S(r, f_1)$ is also an $S(r, f_2)$. Since $f_1(z)$ is admissible and $f_2(z)$ is non-admissible, we can get $T(r, f_2) = S(r, f_1)$. Thus, we have

$$T(r, f_2) = S(r, f_1) = S(r, f_2) = o(T(r, f_2)).$$

This is a contradiction. Hence, we can get that (2) and $A_5 > 0$ do not hold at the same time. □

Lemma 3.3 *If f_1 is admissible and f_2 is a non-admissible satisfying* $\lim_{r\to 1^-} T(r, f_2) = \infty$, $a_j (j = 1, 2, \ldots, q)$ *be q distinct complex numbers, and let $k_j (j = 1, 2, \ldots, q)$ be positive integers or ∞ satisfying (1). Set $A_6 = B_3 + \frac{(m-2)lk_m}{k_m+1}$. Then (8) and $A_6 > 0$ do not hold at same time, where $B_3, S_j (j = 1, 2, \ldots, q)$ are stated as in Theorem 2.3.*

Proof: Suppose that (8) and $A_6 > 0$ can hold at the same time. Since $f_1(z)$ is an admissible function, using the same argument as in Theorem 2.3 and from Theorem 1.1 and Lemma 1.1, for any $\varepsilon(0 < \varepsilon < A_6)$, we have

$$\left(\frac{(m-1)lk_m}{k_m+1} + B_3 - \varepsilon\right) T(r, f_1) < \sum_{j=1}^{q}\sum_{s=0}^{l-1} \frac{k_m}{k_m+1} \overline{N}_{k_j)}(r, \frac{1}{f_1 - (c + a_j w^s)}) + S(r, f_1),$$

where B_3 is stated as in Section 2.

From the assumptions of Lemma 3.3, we can get that $(f_1(z)-c)^l \not\equiv (f_2(z)-c)^l$. Thus, by condition (8) and the first fundamental theorem, we have

$$\sum_{j=1}^{q}\sum_{s=0}^{l-1}\overline{N}_{k_j)}(r,\frac{1}{f_1-(c+a_jw^s)}) < \overline{N}(r,\frac{1}{(f_1-c)^l-(f_2-c)^l}) \leq l(T(r,f_1)+T(r,f_2))+O(1).$$

From the two above inequality, we get

$$\left(\frac{(m-2)lk_m}{k_m+1}+B_3-\varepsilon\right)T(r,f_1) \leq \frac{lk_m}{k_m+1}T(r,f_2). \qquad (16)$$

Since $0<\varepsilon<A_6$, we have $\frac{(m-2)lk_m}{k_m+1}+B_3-\varepsilon>0$. From (16), we have

$$T(r,f_1) \leq \frac{1}{A_5-\varepsilon}\frac{(2l-1)k_m}{k_m+1}T(r,f_2). \qquad (17)$$

From Lemma 3.1, (17) and $\frac{1}{A_6-\varepsilon}\frac{lk_m}{k_m+1}>0$, we can get that each $S(r,f_1)$ is also an $S(r,f_2)$. Since $f_1(z)$ is admissible and $f_2(z)$ is non-admissible, we can get $T(r,f_2)=S(r,f_1)$. Thus, we have

$$T(r,f_2)=S(r,f_1)=S(r,f_2)=o(T(r,f_2)).$$

This is a contradiction. Hence, we can get that (8) and $A_6>0$ do not hold at the same time.

Thus, the proof of Lemma 3.3 is completed. □

Proof of Theorem 3.1: Let $l=1$, and since $\Theta(f_i)\geq 0$ $(i=1,2)$ and $\delta(0,f_1-a_j)\geq 0$ $(j=1,2,\ldots,q)$, the assertion follows from Lemma 3.2.

Proof of Theorem 3.2: Let $k_1=k_2=\cdots=k_q=\infty$, and since $\Theta(f_i)\geq 0$ $(i=1,2)$ and $\delta(0,f_1-a_j)\geq 0$ $(j=1,2,\ldots,q)$, the assertion follows from Lemma 3.3.

It is very interesting to consider distinct small functions instead of distinct complex numbers (see [9, 11, 17],etc). *Thus it may be interesting to consider the following questions:*

Question 3.1 *What condition on two non-admissible functions in the unit disc $\mathbb{D}$ sharing small functions will guarantee that the two non-admissible functions are identical?*

Question 3.2 *How many small functions can an admissible function and non-admissible function in the unit disc $\mathbb{D}$ share at most?*

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THE FIXED POINT ALTERNATIVE TO THE STABILITY OF AN ADDITIVE (α, β)-FUNCTIONAL EQUATION

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ABSTRACT. In this paper, we solve the additive (α, β)-functional equation

$$f(x) + f(y) + 2f(z) = \alpha f(\beta(x + y + 2z)), \tag{0.1}$$

where α, β are fixed real or complex numbers with $\alpha \neq 4$ and $\alpha\beta = 1$.

Using the fixed point method and the direct method, we prove the Hyers-Ulam stability of the additive (α, β)-functional equation (0.1) in Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations originated from a question of Ulam [24] concerning the stability of group homomorphisms.

The functional equation $f(x + y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [9] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [18] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. See [5, 7, 14, 15, 20, 21, 19, 22, 23, 19, 25] for more information on functional equations.

We recall a fundamental result in fixed point theory.

Theorem 1.1. [2, 6] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*

(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [10] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several

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functional equations have been extensively investigated by a number of authors (see [3, 4, 12, 13, 16, 17]).

In Section 2, we solve the additive (α, β)-functional equation (0.1) in vector spaces and prove the Hyers-Ulam stability of the additive (α, β)-functional equation (0.1) in Banach spaces by using the fixed point method.

In Section 3, we prove the Hyers-Ulam stability of the additive (α, β)-functional equation (0.1) in Banach spaces by using the direct method.

Throughout this paper, assume that X is a normed space and that Y is a Banach space. Let α, β be fixed real or complex numbers with $\alpha \neq 4$ and $\alpha\beta = 1$.

2. Additive (α, β)-functional equation (0.1) in Banach spaces I

We solve the additive (α, β)-functional equation (0.1) in vector spaces.

Lemma 2.1. *Let X and Y be vector spaces. If a mapping $f : X \to Y$ satisfies*

$$f(x) + f(y) + 2f(z) = \alpha f(\beta(x + y + 2z)) \tag{2.1}$$

for all $x, y, z \in X$, then $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = z = 0$ in (2.1), we get $4f(0) = \alpha f(0)$. So $f(0) = 0$.

Letting $y = -x$ and $z = 0$ in (2.1), we get $f(x) + f(-x) = 0$ and so $f(-x) = -f(x)$ for all $x \in X$.

Letting $x = -2z$ and $y = 0$ in (2.1), we get $f(-2z)+2f(z) = 0$ and so $f(2z) = 2f(z)$ for all $z \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2}f(x)$$

for all $x \in X$.

Letting $z = -\frac{x+y}{2}$ in (2.1), we get

$$f(x) + f(y) - f(x + y) = f(x) + f(y) + 2f\left(-\frac{x + y}{2}\right) = 0$$

and so

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in X$. □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive (α, β)-functional equation (2.1) in Banach spaces.

Theorem 2.2. *Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{2}\varphi(x, y, z) \tag{2.2}$$

for all $x, y, z \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\|f(x) + f(y) + 2f(z) - \alpha f(\beta(x + y + 2z))\| \leq \varphi(x, y, z) \tag{2.3}$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{L}{2(1 - L)}(\varphi(x, x, -x) + \varphi(2x, 0, -x)) \tag{2.4}$$

for all $x \in X$.

Proof. Letting $y = x$ and $z = -x$ in (2.3), we get

$$\|2f(x) + 2f(-x)\| \le \varphi(x, x, -x) \tag{2.5}$$

for all $x \in X$.

Replacing x by $2x$ and letting $y = 0$ and $z = -x$ in (2.3), we get

$$\|f(2x) + 2f(-x)\| \le \varphi(2x, 0, -x) \tag{2.6}$$

for all $x \in X$.

It follows from (2.5) and (2.6) that

$$\|f(2x) - 2f(x)\| \le \varphi(x, x, -x) + \varphi(2x, 0, -x) \tag{2.7}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \le \mu(\varphi(x, x, -x) + \varphi(2x, 0, -x)),\ \forall x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [11]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \le \varepsilon(\varphi(x, x, -x) + \varphi(2x, 0, -x))$$

for all $x \in X$. Hence

$$\begin{aligned} \|Jg(x) - Jh(x)\| &= \left\|2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right)\right\| \le 2\varepsilon\left(\varphi\left(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}\right) + \varphi\left(x, 0, -\frac{x}{2}\right)\right) \\ &\le 2\varepsilon\frac{L}{2}(\varphi(x, x, -x) + \varphi(2x, 0, -x)) = L\varepsilon(\varphi(x, x, -x) + \varphi(2x, 0, -x)) \end{aligned}$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g, h)$$

for all $g, h \in S$.

It follows from (2.7) that

$$\begin{aligned} \left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| &\le \varphi\left(\frac{x}{2}, \frac{x}{2}, -\frac{x}{2}\right) + \varphi\left(x, 0, -\frac{x}{2}\right) \\ &\le \frac{L}{2}(\varphi(x, x, -x) + \varphi(2x, 0, -x)) \end{aligned}$$

for all $x \in X$. So $d(f, Jf) \le \frac{L}{2}$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A(x) = 2A\left(\frac{x}{2}\right) \tag{2.8}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.8) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|f(x)-A(x)\| \leq \mu(\varphi(x,x,-x)+\varphi(2x,0,-x))$$

for all $x \in X$;

(2) $d(J^l f, A) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f,A) \leq \frac{1}{1-L} d(f,Jf)$, which implies

$$\|f(x)-A(x)\| \leq \frac{L}{2(1-L)}(\varphi(x,x,-x)+\varphi(2x,0,-x))$$

for all $x \in X$.

It follows from (2.2) and (2.3) that

$$\begin{aligned}&\|A(x)+A(y)+2A(z)-\alpha A(\beta(x+y+2z))\|\\ &= \lim_{n\to\infty} 2^n \left\|f\left(\frac{x}{2^n}\right)+f\left(\frac{y}{2^n}\right)+2f\left(\frac{z}{2^n}\right)-\alpha f\left(\beta\left(\frac{x+y+2z}{2^n}\right)\right)\right\|\\ &\leq \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right)=0\end{aligned}$$

for all $x,y,z \in X$. So

$$A(x)+A(y)+2A(z)-\alpha A(\beta(x+y+2z))=0$$

for all $x,y,z \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive. □

Corollary 2.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying*

$$\|f(x)+f(y)+2f(z)-\alpha f(\beta(x+y+2z))\| \leq \theta(\|x\|^r+\|y\|^r+\|z\|^r) \tag{2.9}$$

for all $x,y,z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \leq \frac{2^r+4}{2^r-2}\theta\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x,y,z)=\theta(\|x\|^r+\|y\|^r+\|z\|^r)$ for all $x,y,z \in X$. Then we can choose $L = 2^{1-r}$ and we get the desired result. □

Theorem 2.4. *Let $\varphi : X^3 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y,z) \leq 2L\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right)$$

for all $x,y,z \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0)=0$ and (2.3). Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \leq \frac{1}{2(1-L)}(\varphi(x,x,-x)+\varphi(2x,0,-x))$$

for all $x \in X$.

Proof. It follows from (2.7) that

$$\left\|f(x)-\frac{1}{2}f(2x)\right\| \le \frac{1}{2}(\varphi(x,x,-x)+\varphi(2x,0,-x))$$

for all $x \in X$.

Let (S,d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.9)*. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x)-A(x)\| \le \frac{4+2^r}{2-2^r}\theta\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x,y,z) = \theta(\|x\|^r+\|y\|^r+\|z\|^r)$ for all $x,y,z \in X$. Then we can choose $L = 2^{r-1}$ and we get desired result. □

3. Additive (α,β)-functional equation (0.1) in Banach spaces II

In this section, using the direct method, we prove the Hyers-Ulam stability of the additive (α,β)-functional equation (2.1) in Banach spaces.

Theorem 3.1. *Let $\varphi : X^3 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\Psi(x,y,z) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j},\frac{y}{2^j},\frac{z}{2^j}\right) < \infty,$$

$$\|f(x)+f(y)+2f(z)-\alpha f(\beta(x+y+2z))\| \le \varphi(x,y,z) \qquad (3.1)$$

for all $x,y,z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x)-A(x)\| \le \frac{1}{2}(\Psi(x,x,-x)+\Psi(2x,0,-x)) \qquad (3.2)$$

for all $x \in X$.

Proof. It follows from (2.7) that

$$\left\|f(x)-2f\left(\frac{x}{2}\right)\right\| \le \varphi\left(\frac{x}{2},\frac{x}{2},-\frac{x}{2}\right)+\varphi\left(x,0,-\frac{x}{2}\right)$$

for all $x \in X$. Hence

$$\left\|2^l f\left(\frac{x}{2^l}\right)-2^m f\left(\frac{x}{2^m}\right)\right\| \le \sum_{j=l}^{m-1}\left\|2^j f\left(\frac{x}{2^j}\right)-2^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\|$$

$$\le \sum_{j=l}^{m-1}\left(2^j\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}},-\frac{x}{2^{j+1}}\right)+2^j\varphi\left(\frac{x}{2^j},0,-\frac{x}{2^{j+1}}\right)\right) \qquad (3.3)$$

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for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.3) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.3), we get (3.2).

Now, let $T : X \to Y$ be another additive mapping satisfying (3.2). Then we have

$$\begin{aligned}\|A(x) - T(x)\| &= \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right)\right\| \\ &\le \left\|2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| + \left\|2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right)\right\| \\ &\le 2^q \Psi\left(\frac{x}{2^q}, \frac{x}{2^q}, -\frac{x}{2^q}\right) + 2^q \Psi\left(\frac{2x}{2^q}, 0, -\frac{x}{2^q}\right),\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.2. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.9). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2^r + 4}{2^r - 2}\theta\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.1 by taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. □

Theorem 3.3. *Let $\varphi : X^3 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$,* (3.1) *and*

$$\Psi(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y, 2^j z) < \infty$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2}(\Psi(x, x, -x) + \Psi(2x, 0, -x)) \tag{3.4}$$

for all $x \in X$.

Proof. It follows from (2.7) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{2}(\varphi(x, x, -x) + \varphi(2x, 0, -x))$$

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for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{2^l}f(2^lx)-\frac{1}{2^m}f(2^mx)\right\| &\le \sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^jx\right)-\frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\le \sum_{j=l}^{m-1}\left(\frac{1}{2^{j+1}}\varphi(2^jx,2^jx,-2^jx)+\frac{1}{2^{j+1}}\varphi(2^{j+1}x,0,-2^jx)\right) \qquad (3.5)\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.5) that the sequence $\{\frac{1}{2^n}f(2^nx)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^nx)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^nx)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.5), we get (3.4).

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.1. □

Corollary 3.4. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.9). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x)-A(x)\| \le \frac{4+2^r}{2-2^r}\theta\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.3 by taking $\varphi(x,y,z) = \theta(\|x\|^r+\|y\|^r+\|z\|^r)$ for all $x, y, z \in X$. □

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The approximation problem of Dirichlet series with regular growth *

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Abstract

By introducing the concept of β_U-order functions, we study the error in approximating Dirichlet series of infinite order in the half plane by Dirichlet polynomials. Some necessary and sufficient conditions on the error and regular growth of finite β_U-order of these functions have been obtained.

Key words: β-order, β_U-order, Regular growth, Dirichlet series.
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1 Introduction and basic notes

Consider Dirichlet series

$$f(s)=\sum_{n=1}^{\infty}a_ne^{\lambda_n s},\qquad s=\sigma+it, \tag{1}$$

where

$$0\leq\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots,\lambda_n\to\infty\ \ as\ \ n\to\infty; \tag{2}$$

$s=\sigma+it$ (σ,t are real variables); a_n are nonzero complex numbers and

$$\limsup_{n\to+\infty}(\lambda_{n+1}-\lambda_n)=h<+\infty, \tag{3}$$

$$\limsup_{n\to+\infty}\frac{\log^+|a_n|}{\lambda_n}=0, \tag{4}$$

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then from (2) and (3), by using the similar method in [19] or [15], we can get

$$\limsup_{n\to\infty}\frac{n}{\lambda_n}=E<+\infty,\qquad \limsup_{n\to\infty}\frac{\log n}{\lambda_n}=0. \tag{5}$$

Then the abscissas of convergence and absolutely convergence is 0, that is, $f(s)$ is an analytic function in the left half plane $H=\{s=\sigma+it:\sigma<0,t\in\mathbb{R}\}$.

We denote D to be the class of all functions $f(s)$ satisfying (2)-(4) and analytic in $Res<0$, denote $\overline{D}_\alpha$ to be the class of all functions $f(s)$ satisfying (2)-(3) and analytic in $Re\le\alpha$ where $-\infty<\alpha<+\infty$. Thus, if $-\infty<\alpha<0$ and $f(s)\in D$, then $f(s)\in\overline{D}_\alpha$; if $0<\alpha<+\infty$ and $f(s)\in\overline{D}_\alpha$, then $f(s)\in D$. We denote Π_k to be the class of all exponential polynomial of degree almost k, that is,

$$\Pi_k=\left\{\sum_{j=1}^{k}b_je^{\lambda_js}:(b_1,b_2,\ldots,b_k)\in\mathbb{C}^k\right\}.$$

For $f(s)\in D$,

$$M(\sigma,f)=\max_{-\infty<t<\infty}|f(\sigma+it)|,\qquad m(\sigma,f)=\max_{n\ge1}\{|a_n|e^{\sigma\lambda_n}\}$$

are called, respectively, the maximum modulus, the maximum term of $f(s)$ for $Res=\sigma<0$.

Definition 1.1 *Let $f(s)\in D$, the order of $f(s)$ can be defined by*

$$\rho=\limsup_{\sigma\to0^-}\frac{\log\log^+M(\sigma,f)}{-\log(-\sigma)},$$

where $\log^+x=\begin{cases}\log x & x\ge1\\ 0 & x<1\end{cases}$

For $\rho=0, 0<\rho<\infty, \rho=\infty$, $f(s)$ can be called, respectively, zero order, finite order, infinite order Dirichlet series. Considerable attention has been paid to the growth and the value distribution of analytic functions defined by Dirichlet series; see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18] for some results.

For $f(s)\in\overline{D}_\alpha, -\infty<\alpha<+\infty$, we denote $E_n(f,\alpha)$ by the error in approximating the function $f(s)$ by exponential polynomials of degree n in uniform norm as

$$E_n(f,\alpha)=\inf_{p\in\Pi_n}\parallel f-p\parallel_\alpha,\quad n=1,2,\ldots,$$

where

$$\parallel f-p\parallel_\alpha=\max_{-\infty<t<+\infty}|f(\alpha+it)-p(\alpha+it)|.$$

In 2010, the authors [17] investigated the relations between the error $E_n(f,\alpha)$ and the growth order of $f(s)$, and obtained some equivalence relation between $E_n(f,\alpha)$ and the regular growth of $f(s)$ with finite order as follows:

Theorem 1.1 (see [17]). *Let $f(s)\in D$ be of finite order ρ, then for any real number $-\infty<\alpha<0$, we have*

$$\lim_{\sigma\to0^-}\frac{\log^+M(\sigma,f)}{U_1(-\frac{1}{\sigma})}=1\Longleftrightarrow\limsup_{n\to+\infty}\frac{\log^+\left[E_n(f,\alpha)e^{-\alpha\lambda_{n+1}}\right]}{BU_1\left(\frac{\lambda_{n+1}}{\log^+\left[E_n(f,\alpha)e^{-\alpha\lambda_{n+1}}\right]}\right)}=1;$$

and there exists a increasing, positive integer sequence $\{n_\nu\}$ satisfying

$$\lim_{\nu\to+\infty}\frac{\log^+\left[E_{n_\nu}(f,\alpha)e^{-\alpha\lambda_{n_\nu+1}}\right]}{BU_1\left(\frac{\lambda_{n_\nu+1}}{\log^+\left[E_{n_\nu}(f,\alpha)e^{-\alpha\lambda_{n_\nu+1}}\right]}\right)}=1,\qquad\lim_{\nu\to+\infty}\frac{\lambda_{n_{\nu+1}}}{\lambda_{n_\nu}}=1,$$

where $B = \frac{(1+\rho)^{1+\rho}}{\rho^\rho}$ *and* $U_1(r) = r^{\rho(r)}$, $\rho(r)$ *satisfies the following conditions:*

(i) there exists a real number $r_0 > 0$, $\rho(r)$ *is nonnegative, continuous, monotone on* $[r_0, +\infty)$, *and tends to* ρ *as* $r \to +\infty$;

(ii) $\lim_{r\to+\infty} \rho'(r) r \log r = 0,$;

(iii) $U_1(kr) = [k^\rho + o(1)]U_1(r) (r \to +\infty)$ *for every positive integer* k, *and* $U_1(r)$ *is an increasing function on* $r \geq r'_0 > r_0$.

Recently, the authors [18] further investigated the relations between the error $E_n(f, \alpha)$ and the growth order of $f(s)$ when $f(s)$ has infinite order, by introducing the concept of β-order.

Theorem 1.2 (see [18]). *Let* $f(s) \in D$ *be of finite* β*-order* ρ_β, *then for any real number* $-\infty < \alpha < 0$, *we have*

$$\limsup_{n\to\infty} \frac{\beta(\lambda_n)}{\log \lambda_n - \log\log^+(E_{n-1}(f, \alpha) e^{-\alpha\lambda_n})} = \rho_\beta.$$

Remark 1.1 *In Theorem 1.2, the definitions of* β*-order and the function* $\beta(x)$ *will be introduced in Section 2.*

Thus, a question arises naturally: what will happen when $\rho_\beta = \infty$ *in Theorem1.2?*

In this paper, we will investigate the above question by using the type functions $U_2(x)$ to enlarge the growth of the denominator $-\log(-\sigma)$ and obtain the main results as follows.

Theorem 1.3 *If Dirichlet series* $f(s) \in D$ *of infinite* β*-order, then we have*

$$\limsup_{\sigma\to 0^-} \frac{\beta(\log^+ M(\sigma, F))}{\log U_2\left(\frac{1}{-\sigma}\right)} = T \Longleftrightarrow \limsup_{\sigma\to 0^+} \frac{\beta(\log^+ m(\sigma, F))}{\log U_2\left(\frac{1}{-\sigma}\right)} = T,$$

where $0 < T < \infty$ *and* $U_2(x) = x^{\rho(x)}$ *satisfies the following conditions*

(i) $\rho(x)$ *is monotone and* $\lim_{x\to\infty} \rho(x) = \infty$;

(ii) $\lim_{x\to\infty} \frac{\log U_2(x')}{\log U_2(x)} = 1$, *where* $x' = x\left(1 + \frac{1}{\log U_2(x)}\right)$.

Remark 1.2 *From Lemma 2.1 and Lemma 1.1 in Section 2, we can prove the conclusion of Theorem 1.3 easily.*

Remark 1.3 *This type function* $U_2(x)$ *is different from the type function* $U_1(x)$ *in Theorem 1.1.*

Remark 1.4 *If Dirichlet series* $f(s)$ *of infinite order has infinite* β*-order and satisfies*

$$\limsup_{\sigma\to 0^-} \frac{\beta(\log^+ M(\sigma, f))}{\log U_2\left(\frac{1}{-\sigma}\right)} = T, \tag{6}$$

then T *is called the* β_U*-order of Dirichlet series* $f(s)$.

Theorem 1.4 *If Dirichlet series* $f(s) \in D$ *with infinite* β*-order, then for any fixed real number* $-\infty < \alpha < 0$, *we have*

$$\limsup_{\sigma\to 0^-} \frac{\beta(\log^+ M(\sigma, f))}{\log U_2\left(\frac{1}{-\sigma}\right)} = T \Longleftrightarrow \limsup_{n\to\infty} \Psi_n(f, \alpha, \lambda_n) = T; \tag{7}$$

where

$$\Psi_n(f, \alpha, \lambda_n) = \frac{\beta(\lambda_n)}{\log U_2\left(\frac{\lambda_n}{\log^+[E_{n-1}(f,\alpha)e^{-\alpha\lambda_n}]}\right)}.$$

Remark 1.5 *From Theorem 1.4, we can see that the type function $U_2(x)$ is more simple then the type function of Wang [16].*

Theorem 1.5 *Under the assumptions of Theorem 1.4, we have*

$$\lim_{\sigma\to 0^-}\frac{\beta(\log^+ M(\sigma,f))}{\log U_2\left(\frac{1}{-\sigma}\right)}=T \Longleftrightarrow \textit{ the right hand of } (7) \textit{ is verified},$$

and there exists a subsequence $\{\lambda_{n(p)}\}\subseteq\{\lambda_n\}$ satisfying

$$\lim_{p\to\infty}\Psi_{n(p)}(f,\alpha,\lambda_{n(p)})=T,\qquad \textit{and}\qquad \lim_{p\to\infty}\frac{\beta(\lambda_{n(p)})}{\beta(\lambda_{n(p+1)})}=1, \tag{8}$$

where

$$\Psi_{n(p)}(f,\alpha,\lambda_{n(p)})=\frac{\beta(\lambda_{n(p)})}{\log U_2\left(\frac{\lambda_{n(p)}}{\log^+[E_{n(p-1)}(f,\alpha)e^{-\alpha\lambda_{n(p)}}]}\right)}.$$

Remark 1.6 *From Theorem 1.5, we get the necessary and sufficient conditions for the limit about the regular growth of $f(s)$, however, Wang [16] only gave the necessary and sufficient conditions for the superior limit. Thus, our results of this paper are more accurate than the previous form [16].*

2 Some Lemmas and the concept of β-order

According to observations, we find that to study the growth of Dirichlet series better, many mathematicians proposed the type functions $U(x)$ to enlarge the growth of the denominator $\log\frac{1}{-\sigma}$ or $-\sigma$ (see [13, 4, 12]), or use some function to control the molecular $M(\sigma,f)$ or $m(\sigma,f)$ in the definition of order. In this paper, we will deal with the growth of Dirichlet series of infinite order by using a class of functions to reduce $M(\sigma,f)$ or $m(\sigma,f)$ which is better than the previous form. So, we firstly give the definition of β-order of Dirichlet series as follows, which is an extension of [10].

Let $\mathfrak{F}$ be the class of all functions $\beta(x)$ satisfies the following conditions:

(i) $\beta(x)$ is defined on $[a,+\infty), a>0$, and positive, strictly increasing, differential and tends to $+\infty$ as $x\to+\infty$;

(ii) $x\beta'(x)=o(1)$ as $x\to+\infty$.

Definition 2.1 *([18]). If Dirichlet series $f(s)$ of infinite order satisfies*

$$\limsup_{\sigma\to 0^+}\frac{\beta(\log^+ M(\sigma,f))}{\log\frac{1}{-\sigma}}=\rho^*,$$

where $\beta(x)\in\mathfrak{F}$, then ρ^* is called the β-order of $f(s)$.

Remark 2.1 *Obviously, the functions $h(x)=\log_p x, p\geq 2, p\in N_+$ satisfy the conditions (i) and (ii), where p is a positive integer, and $\log_1 x=\log x$ and $\log_p x=\log(\log_{p-1}x)$. Thus, p-order is regard as a special case of β-order of Dirichlet series.*

Remark 2.2 *Furthermore, β-order is more precise than p-order to some extent. In fact, for $p(\geq 2)$ is a positive integer, we can find function $\beta(x)\in\mathfrak{F}$ and a positive real function $M(x)$ satisfying*

$$\limsup_{x\to\infty}\frac{\beta(\log M(x))}{\log x}=t,\quad (0<t<\infty),$$

and

$$\limsup_{x\to\infty}\frac{\log_p(\log M(x))}{\log x}=\infty,\quad and\quad \limsup_{x\to\infty}\frac{\log_{p+1}(\log M(x))}{\log x}=0.$$

For example, let

$$M(x)=\exp_{p+1}\{(t\log x)^{1/d}\},\qquad \beta(x)=(\log_{p+1}x)^d,$$

where t is a finite positive real constant and $0<d<1$, we can get that $\rho_p(M)=\infty,\rho_{p+1}(M)=0$ and $\rho_\beta(M)=t$, where $\rho_p(f)$ denote the p-order of f, and $\rho_\beta(f)$ the β-order of f.

Remark 2.3 *If $\rho^*=\infty$ in Definition 2.1, then $f(s)$ is called a Dirichlet series of infinite β-order.*

Lemma 2.1 *(see [16]). Let $\beta(x)\in\mathfrak{F}$ and $\varphi(x)$ be the function satisfying*

$$\limsup_{x\to\infty}\frac{\log^+\varphi(x)}{\log x}=\varrho,\quad (0\le\varrho<\infty),$$

if $M(x)$ satisfies $\limsup_{x\to\infty}\frac{\beta(\log M(x))}{\log x}=\nu(>0)$. Then we have

$$\limsup_{x\to\infty}\frac{\beta(\varphi(x)\log M(x))}{\log x}=\nu.$$

Proof: To prove this lemma, two cases will be considered as follows.

Case 1. If $\varphi(x)$ is not a constant. From the assumptions of Lemma 2.1, we can get that $\varphi(x)\to\infty$ as $x\to\infty$. Then, for sufficiently large x, we have $\varphi(x)>1$. From $\beta(x)\in\mathfrak{F}$, we have $\lim_{x\to\infty}\log M(x)=\infty$. Then from the Cauchy mean value theorem, there exists $\xi(\log M(x)<\xi<\beta(x)\log M(x))$ satisfying

$$\frac{\beta(\varphi(x)\log M(x))-\beta(\log M(x))}{\log(\varphi(x)\log M(x))-\log\log M(x)}=\frac{\beta'(\xi)}{(\log\xi)'}=\xi\beta'(\xi),$$

that is,

$$\beta(\varphi(x)\log M(x))=\beta(\log M(x))+\log\varphi(x)\xi\beta'(\xi).\tag{9}$$

Since $x\beta'(x)=o(1)$ as $x\to+\infty$ and $\limsup_{x\to\infty}\frac{\log\varphi(x)}{\log x}=\varrho,(0\le\varrho<\infty)$, by (9), we can get the conclusion of Lemma 2.1.

Case 2. If $\varphi(x)$ is a constant. By using the same argument as in Case 1, we can prove that Lemma 2.1 is true.

Thus, this completes the proof of Lemma 2.1. □

The following lemma plays an important role to deal with the growth of Dirichlet series, which shows the relation between $M(\sigma,f)$ and $m(\sigma,f)$ of such functions.

Lemma 2.2 ([19]). If Dirichlet series (1) satisfies (2) (3), then for any given $\varepsilon\in(0,1)$ and for $\sigma(<0)$ sufficiently reaching 0, we have

$$m(\sigma,f)\le M(\sigma,f)\le K(\varepsilon)\frac{1}{-\sigma}m((1-\varepsilon)\sigma,f),$$

where $K(\varepsilon)$ is a constant depending on ε and (3).

Lemma 2.3 *If $f(s)\in\overline{D}_\alpha(-\infty<\alpha<+\infty)$, then for any positive integer $n\in\mathbb{N}_+:=\mathbb{N}\backslash\{0\}$, we have*

$$|a_n|e^{\alpha\lambda_n}\le K_2E_{n-1}(f,\alpha),$$

where $K_2>1$ is a real constant.

Proof: From the definition of $E_n(f,\alpha)$, there exists $p(s)\in\Pi_{n-1}$ such that

$$||f-p||_\alpha \le K_2 E_{n-1}(f,\alpha). \tag{10}$$

Since $f(s)\in\overline{D}_\alpha$ and from [19, P.16], for any real numbers $t_0,\vartheta(\neq 0)$, we have

$$\lim_{R\to+\infty}\frac{1}{R}\int_{t_0}^{R} e^{\vartheta it}dt=0 \tag{11}$$

and

$$a_n e^{\alpha\lambda_n}=\lim_{R\to\infty}\frac{1}{R}\int_{t_0}^{R} f(\alpha+it)e^{-\lambda_n it}dt. \tag{12}$$

From (11), for any real number $x\neq 0$, we have

$$\lim_{R\to\infty}\frac{1}{R}\int_{t_0}^{R} e^{x(\alpha+it)}dt=0. \tag{13}$$

Thus, from (12) and (13), for any $p_1(s)\in\Pi_{n-1}$, we have

$$a_n e^{\alpha\lambda_n}=\lim_{R\to\infty}\frac{1}{R}\int_{t_0}^{R}[f(\alpha+it)-p_1(\alpha+it)]e^{-\lambda_n it}dt,$$

that is,

$$|a_n|e^{\alpha\lambda_n}\le ||f-p_1||_\alpha. \tag{14}$$

From (10) and (14), we can prove the conclusion of Lemma 2.3. □

3 The proof of Theorem 1.4

We prove the conclusions of Theorem 1.4 by using the properties of two functions $\beta(x)$ and $U_2(x)$, this method is different from the previous method to some extent.

We first prove " $\Longleftarrow$ " of Theorem 1.4. Suppose that

$$\limsup_{n\to\infty}\Psi_n(f,\alpha,\lambda_n)=\limsup_{n\to\infty}\frac{\beta(\lambda_n)}{\log U_2\left(\frac{\lambda_n}{\log^+[E_{n-1}(f,\alpha)e^{-\alpha\lambda_n}]}\right)}=T. \tag{15}$$

Let

$$A_n=E_{n-1}(f,\alpha)e^{-\alpha\lambda_n},\quad n=1,2,\ldots,$$

then for any positive real number $\tau>0$, for sufficiently large n, we have

$$\lambda_n<\gamma\left((T+\tau)\log U_2\left(\frac{\lambda_n}{\log^+ A_n}\right)\right),$$

where $\gamma(x)$ is the inverse functions of $\beta(x)$. Let $V_2(x)$ and $U_2(x)$ be two reciprocally inverse functions, then we have

$$V_2\left(\exp\left\{\frac{1}{T+\tau}\beta(\lambda_n)\right\}\right)<\frac{\lambda_n}{\log^+ A_n},\quad \log^+ A_n\le\lambda_n\left(V_2\left(\exp\left\{\frac{1}{T+\tau}\beta(\lambda_n)\right\}\right)\right)^{-1}.$$

Thus, we have

$$\log^+(A_n e^{\lambda_n\sigma})\le\lambda_n\left(\left(V_2\left(\exp\left\{\frac{1}{T+\tau}\beta(\lambda_n)\right\}\right)\right)^{-1}+\sigma\right). \tag{16}$$

For any fixed and sufficiently small $\sigma < 0$, set

$$G = \gamma\left((T+\tau)\log U_2\left(\frac{1}{-\sigma} + \frac{1}{-\sigma\log U_2\left(\frac{1}{-\sigma}\right)}\right)\right),$$

that is,

$$\frac{1}{-\sigma} + \frac{1}{-\sigma\log U_2\left(\frac{1}{-\sigma}\right)} = V_2\left(\exp\left\{\frac{1}{T+\tau}\beta(G)\right\}\right). \tag{17}$$

If $\lambda_n \le G$, for sufficiently large n, let $V_2\left(\exp\left\{\frac{1}{T+\tau}\beta(\lambda_n)\right\}\right) \ge 1$, from $\sigma < 0$,(16),(17) and the definition of $U_2(x)$, we have

$$\begin{aligned}\log^+ A_n e^{\lambda_n\sigma} &\le G\left(\left(V_2\left(\exp\left\{\frac{1}{T+\tau}\beta(\lambda_n)\right\}\right)\right)^{-1} + \sigma\right)\\ &\le G = \gamma\left((T+\tau)\log U_2\left(\frac{1}{-\sigma} + \frac{1}{-\sigma\log U_2\left(\frac{1}{-\sigma}\right)}\right)\right)\\ &\le \gamma\left((T+\tau)\log\left[(1+o(1))U_2\left(\frac{1}{-\sigma}\right)\right]\right). \end{aligned}\tag{18}$$

If $\lambda_n > G$, from (16) and (17), we have

$$\begin{aligned}\log^+ A_n e^{\lambda_n\sigma} &\le \lambda_n\left(\left(V_2\left(\exp\left\{\frac{1}{T+\tau}\beta(G)\right\}\right)\right)^{-1} + \sigma\right)\\ &\le \lambda_n\left(\left(\frac{1}{-\sigma} + \frac{1}{-\sigma\log U_2\left(\frac{1}{-\sigma}\right)}\right)^{-1} + \sigma\right) < 0. \end{aligned}\tag{19}$$

For sufficiently large n, from (18) and (19), we have

$$\log^+ A_n e^{\lambda_n\sigma} \le \gamma\left((T+\tau)\log\left[(1+o(1))U_2\left(\frac{1}{-\sigma}\right)\right]\right)$$

Since $A_n = E_{n-1}e^{-\alpha\lambda_n}$ and τ is arbitrary, by Lemma 2.1,Lemma 2.3 and Theorem 1.3, we can get

$$\limsup_{\sigma\to 0^-}\frac{\beta(\log^+ M(\sigma,f))}{\log U_2(\frac{1}{-\sigma})} \le T.$$

Suppose that

$$\limsup_{\sigma\to 0^+}\frac{\beta(\log^+ M(\sigma,f))}{\log U_2(\frac{1}{-\sigma})} = \eta < T.$$

Thus, there exists any real number $\varepsilon(0<\varepsilon<\frac{\eta}{2})$, for any positive integer n and any sufficient small $\sigma < 0$, from Lemma 2.2, we have

$$\log^+ |a_n|e^{\lambda_n\sigma} \le \log M(\sigma,f) \le \gamma\left((T-2\varepsilon)\log U_2(\frac{1}{-\sigma})\right). \tag{20}$$

From (15), there exists a subsequence $\{\lambda_{n(p)}\}$, for sufficiently large p, we have

$$\beta(\lambda_{n(p)}) > (T-\varepsilon)\log U_2\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\right). \tag{21}$$

Take a sequence $\{\sigma_p\}$ satisfying

$$\gamma\left((\eta-2\varepsilon)\log U_2(\frac{1}{-\sigma_p})\right)=\frac{\log^+ A_{n(p)}}{1+\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})}. \tag{22}$$

From (20) and (22), we get

$$\log^+ A_{n(p)}+\lambda_{n(p)}\sigma_p\le\gamma\left((\eta-2\varepsilon)\log U_2(\frac{1}{-\sigma_p})\right)=\frac{\log^+ A_{n(p)}}{1+\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})},$$

that is,

$$\frac{1}{-\sigma_p}\le\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\left(1+\frac{1}{\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})}\right).$$

Thus, we have

$$U_2(\frac{1}{-\sigma_p})\le U_2\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\left(1+\frac{1}{\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})}\right)\right)\le U_2^{1+o(1)}\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\right). \tag{23}$$

From (22) and (23), we have

$$\begin{aligned}\lambda_{n(p)}&=\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\gamma\left((T-2\varepsilon)\log U_2(\frac{1}{\sigma_p})\right)\left(1+\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)\\&=\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\gamma\left((\eta-2\varepsilon)(1+o(1))\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)\left(1+\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right).\end{aligned}$$

Thus, from the Cauchy mean value theorem, there exists a real number ξ between $\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}(1+\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\gamma(\eta-2\varepsilon)(1+o(1))\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})$ and $\gamma(\eta-2\varepsilon)(1+o(1))\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})$ such that

$$\begin{aligned}\beta\left(\lambda_{n(p)}\right)&=\beta\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\left(1+\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)\gamma\left((\eta-2\varepsilon)(1+o(1))\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)\right)\\&=\beta\left(\gamma\left((T-2\varepsilon)(1+o(1))\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)\right)\\&\quad+\log\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\left(1+\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)\right)\xi\beta'(\xi),\end{aligned}$$

Since

$$\lim_{p\to\infty}\frac{\log\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\left(1+\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})\right)\right)}{\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})}=0,$$

then for sufficiently large p, we have

$$\beta\left(\lambda_{n(p)}\right)=(\eta-2\varepsilon)(1+o(1))\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}})+K_2\xi\beta'(\xi)\log U_2(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}), \tag{24}$$

where K_2 is a constant.

From (21),(24) and $\eta < T$, we can get a contradiction. Thus, we can get

$$\limsup_{\sigma\to 0^-} \frac{\beta(\log^+ M(\sigma,f))}{\log U_2(\frac{1}{-\sigma})} = T.$$

Hence, the sufficiency of Theorem 1.4 is completed.

We can prove the necessity of Theorem 1.4 by using the similar argument as in the proof of the sufficiency of Theorem 1.4.

Thus, the proof of Theorem 1.4 is completed.

4 The Proof of Theorem 1.5

We will consider two steps as follows:

Step one: We first prove the sufficiency of Theorem 1.5. From the conditions of Theorem 1.5, for any $\varepsilon(>0)$, there exists a subsequence $\{\lambda_{n(p)}\}$ such that

$$\lambda_{n(p)} \geq \gamma\left((T-\varepsilon)\log U_2\left(\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}}\right)\right), \quad \lim_{p\to\infty}\frac{\beta(\lambda_{n(p)})}{\beta(\lambda_{n(p+1)})} = 1, \tag{25}$$

that is,

$$\frac{\lambda_{n(p)}}{\log^+ A_{n(p)}} \leq V_2\left(\exp\left\{\frac{1}{T-\varepsilon}\beta(\lambda_{n(p)})\right\}\right), \quad \log^+ A_{n(p)} \geq \lambda_{n(p)} V_2\left(\exp\left\{\frac{1}{T-\varepsilon}\beta(\lambda_{n(p)})\right\}\right)^{-1}.$$

Take the sequence $\{\sigma_p\}$ satisfying

$$\lambda_{n(p)} = \gamma\left((T-\varepsilon)\log U_2\left(\frac{1}{-\sigma_p} + \frac{1}{\sigma_p \log U_2(\frac{1}{-\sigma_p})}\right)\right),$$
$$\frac{1}{-\sigma_p} + \frac{1}{\sigma_p \log U_2(\frac{1}{-\sigma_p})} = V_2\left(\exp\left\{\frac{1}{T-\varepsilon}\beta(\lambda_{n(p)})\right\}\right). \tag{26}$$

For any sufficiently small $\sigma < 0$ and $-\infty < \alpha < \sigma < 0$, we have

$$E_{n-1}(f,\alpha) \leq ||f - p_{n-1}||_\alpha \leq \sum_{k=n}^{\infty} |a_k| e^{\lambda_k \alpha} \leq M(\sigma, f)\sum_{k=n}^{\infty} e^{\lambda_n(\alpha-\sigma)}, \tag{27}$$

where $p_{n-1}(s) = \sum_{k=1}^{n-1} a_k e^{\lambda_k s}$. From (3), we take $0 < h' < h$ satisfying $\lambda_{n+1} - \lambda_n \geq h'$ for any integer $n \geq 1$. Thus, for sufficiently small $\sigma < 0$ such that $\sigma \geq \frac{\alpha}{2}$, from (27) we have

$$\begin{aligned}
E_{n-1}(f,\alpha) &\leq M(\sigma,f) e^{\lambda_n(\alpha-\sigma)} \sum_{k=n}^{\infty} e^{(\lambda_k-\lambda_n)(\alpha-\sigma)} \\
&\leq M(\sigma,f) e^{\lambda_n(\alpha-\sigma)} e^{-\frac{\alpha}{2}h'n} \sum_{k=n}^{\infty} e^{\frac{\alpha}{2}h'k} \\
&= M(\sigma,f) e^{\lambda_n(\alpha-\sigma)} \left(1 - e^{\frac{\alpha}{2}h'}\right)^{-1}.
\end{aligned}$$

Then for sufficiently small $\sigma < 0$ and $-\infty < \alpha < \sigma < 0$, we have

$$M(\sigma, f) \geq K_3 E_{n-1}(f,\alpha) e^{-\lambda_n(\alpha-\sigma)} = K_3 A_n e^{\lambda_n \sigma}, \tag{28}$$

where $K_3 = 1 - e^{\frac{\alpha}{2}h'}$. For sufficiently small $\sigma < 0$, we take $\sigma_p \leq \sigma < \sigma_{p+1}$, from (25),(26) and (28), we have

$$\begin{aligned} \log^+ M(\sigma, f) &\geq \log^+ A_{n(p)} + \lambda_{n(p)}\sigma_p + O(1) \qquad (29)\\ &\geq \lambda_{n(p)} \left(V_2 \left(\exp\left\{ \frac{1}{T-\varepsilon}\beta(\lambda_{n(p)}) \right\} \right)^{-1} + \sigma_p \right) + O(1) \\ &\geq \gamma \left((T-\varepsilon) \log U_2 \left(\frac{1}{-\sigma_p} + \frac{1}{\sigma_p \log U_2(\frac{1}{-\sigma_p})} \right) \right) \frac{-\sigma_p}{\log U_2(\frac{1}{-\sigma_p}) - 1} + O(1) \\ &\geq (1+o(1))\gamma \left((T-\varepsilon) \log U_2 \left(\frac{1}{-\sigma_{p+1}} + \frac{1}{\sigma_{p+1} \log U_2(\frac{1}{-\sigma_{p+1}})} \right) \right) \frac{-\sigma_p}{\log U_2(\frac{1}{-\sigma_p}) - 1} \\ &\geq (1+o(1))\gamma \left((T-\varepsilon) \log U_2 \left(\frac{1}{-\sigma} + \frac{1}{\sigma \log U_2(\frac{1}{-\sigma})} \right) \right) \frac{-\sigma}{\log U_2(\frac{1}{-\sigma}) - 1}. \end{aligned}$$

Set

$$\frac{1}{-\sigma} + \frac{1}{\sigma \log U_2(\frac{1}{-\sigma})} = r, \quad r\left(1 + \frac{1}{\log U_2(r)}\right) = R, \quad R\left(1 + \frac{1}{\log U_2(R)}\right) = R',$$

by using a simple calculation, we can get $R' \geq \frac{1}{-\sigma}$. Thus, from the definitions of $U_2(x)$ (ii), we can get

$$\limsup_{\sigma \to 0^-} \frac{\log U_2(r)}{\log U_2(\frac{1}{-\sigma})} = 1. \qquad (30)$$

Since

$$\limsup_{\sigma \to 0^-} \frac{\log \frac{-\sigma}{\log U_2(\frac{1}{-\sigma}) - 1}}{\log U_2(\frac{1}{-\sigma})} = 0,$$

and from Lemma 2.1, (29) and (30), we have

$$\limsup_{\sigma \to 0^-} \frac{\beta(\log^+ M(\sigma, f))}{\log U_2(\frac{1}{-\sigma})} = T.$$

Step two: The necessity of the Theorem 1.5 will be proved as follows. From Theorem 1.4, we can get that the right hand of (7) is verified. Next, we will prove that (8) also holds. We take a positive decreasing sequence $\{\varepsilon_i\}(0 < \varepsilon_i < T)$,$\varepsilon_i \to 0 (i \to \infty)$.

Set

$$F_i = \left\{ n : \Psi_n(f, \alpha, \lambda_n) = \frac{\beta(\lambda_n)}{\log U_2\left(\frac{\lambda_n}{\log^+ A_n}\right)} > T - \varepsilon_i \right\}, \qquad (31)$$

it follows that $\forall i, F_i \neq \Phi$ and $F_i \subset F_{i-1}$. For each i, we arrange the $n (\in F_i)$ in an increasing sequence $\{n^{(i)}(p)\}_{p=1}^{\infty}$, then we consider the two cases in the following.

Case 1. Suppose that $\lim_{\nu \to +\infty} \frac{\beta(\lambda_{n^{(i)}(p+1)})}{\beta(\lambda_{n^{(i)}(p)})} = 1$ for any i. Then there exists $N_i \in F_i (i \in N_+)$, when $n^{(i)}(p) \geq N_i$, we have

$$\frac{\beta\left(\lambda_{n^{(i)}(p+1)}\right)}{\beta\left(\lambda_{n^{(i)}(p)}\right)} \leq 1 + \varepsilon_i. \qquad (32)$$

Note $F_{i+1} \subset F_i$, take $N_{i+1} > N_i$, denote F_i' the subset of F_i

$$F_i' = \{n \in F_i : N_i \leq n \leq N_{i+1}\},$$

thus the elements of F_i' satisfy (31) and (32).

Therefore let $F=\bigcup_{i=1}^{\infty}F_i'$ and arrange the $n(\in E_i')$ in an increasing sequence $\{n_\nu\}$. Thus, the necessity of Theorem 1.5 is proved.

Case 2. If there exists $i\in N_+$ satisfying $\lim_{\nu\to+\infty}\frac{\beta\left(\lambda_{n^{(i)}(p+1)}\right)}{\beta\left(\lambda_{n^{(i)}(p)}\right)}\neq 1$, then since $\lambda_{n^{(i)}(p+1)}>\lambda_{n^{(i)}(p)}$, we get $\lim_{\nu\to+\infty}\frac{\beta(\lambda_{n^{(i)}(p+1)})}{\beta(\lambda_{n^{(i)}(p)})}>1$. Hence there exists $\{n^{(i)}(p_k)\}\subseteq\{n^{(i)}(p)\}$ (still marked with $\{n^{(i)}(p)\}$) and positive real constant $\tau>0$, it follows that

$$\frac{\beta\left(\lambda_{n^{(i)}(p+1)}\right)}{\beta\left(\lambda_{n^{(i)}(p)}\right)}\geq 1+\tau.$$

Let

$$n'(1)=n^{(i)}(1),n'(2)=n^{(i)}(3),\cdots,n'(p)=n^{(i)}(2p-1),\cdots$$
$$n''(1)=n^{(i)}(1),n''(2)=n^{(i)}(4),\cdots,n''(p)=n^{(i)}(2p),\cdots$$

where $\{n'(p)\},\{n''(p)\}$ are two increasing positive integer sequences, and

$$n''(p)<n'(p+1),\quad \beta(\lambda_{n''(p)})>(1+\tau)\beta(\lambda_{n'(p)}),\quad \nu=1,2,\cdots.$$

From (31), for any sufficiently large p, when $n\notin F_i$ satisfies $n'(p)<n<n''(p)$, there exists a positive real number $\delta>0$ such that

$$\lambda_n\leq\gamma\left((T-\delta)\log U_2(\frac{\lambda_n}{\log^+A_n})\right),\quad \frac{\lambda_n}{\log^+A_n}\geq V_2\left(\exp\{\frac{1}{T-\delta}\beta(\lambda_n)\}\right). \tag{33}$$

Thus we have

$$\log^+A_ne^{\sigma\lambda_n}<\lambda_n\left(\frac{1}{V_2\left(\exp\{\frac{1}{T-\delta}\beta(\lambda_n)\}\right)}+\sigma\right). \tag{34}$$

Set

$$G=\gamma\left((T-\delta)\log U_2\left(\frac{1}{-\sigma}+\frac{1}{-\sigma\log U_2\left(\frac{1}{-\sigma}\right)}\right)\right),$$

that is,

$$\frac{1}{-\sigma}+\frac{1}{-\sigma\log U_2\left(\frac{1}{-\sigma}\right)}=V_2\left(\exp\left\{\frac{1}{T-\delta}\beta(G)\right\}\right). \tag{35}$$

If $\lambda_n\geq G$, from (34) and (35), we have

$$\log^+A_ne^{\sigma\lambda_n}\leq\lambda_n\left(\frac{1}{V_2\left(\exp\{\frac{1}{T-\delta}\beta(\lambda_n)\}\right)}+\sigma\right)<0. \tag{36}$$

If $\lambda_n<G$, from (34) and (35), we have

$$\log^+|a_n|e^{\sigma\lambda_n}<G=\gamma\left((T-\delta)\log U_2\left(\frac{1}{-\sigma}+\frac{1}{-\sigma\log U_2\left(\frac{1}{-\sigma}\right)}\right)\right). \tag{37}$$

Choose the sequence $\{\sigma_p\}$ satisfying

$$\sigma_p=-\left[V_2\left(\exp\left\{\frac{1}{T-\delta}\beta(\lambda_{n''(p)})\right\}\right)\right]^{-1}, \tag{38}$$

from the assumptions of the necessity of Theorem 1.5, there exists an integer $N_2 \in N_+$ such that $V_2\left(\exp\left\{\frac{1}{T-\delta}\beta(\lambda_n)\right\}\right) \geq 1$. Then for $n \geq N_2$, we have

$$\log^+ A_n e^{\sigma_p \lambda_n} < \lambda_n \left(V_2\left(\exp\left\{\frac{1}{T-\delta}\beta(\lambda_n)\right\}\right)^{-1} + \sigma_p \right).$$

When $n \geq n''(p)$, it follows $\lambda_n \geq \lambda_{n''(p)}$, and from (38), we have

$$\log^+ A_n e^{\sigma_p \lambda_n} < \lambda_n \left(V_2\left(\exp\left\{\frac{1}{T-\delta}\beta(\lambda_{n''(p)})\right\}\right)^{-1} + \sigma_p \right) = 0. \tag{39}$$

For sufficiently large ν, we have $\lambda_{n'(p)} \geq \lambda_n$ as $N_2 \leq n \leq n'(p)$, and

$$\log^+ A_n e^{\sigma_p \lambda_n} \leq \lambda_{n'(p)} \left(V_2\left(\exp\{\frac{1}{T-\delta}\beta(\lambda_n)\}\right)^{-1} + \sigma_p \right).$$

Since $\lambda_{n'(p)} < \gamma\left(\frac{1}{1+\tau}\beta(\lambda_{n''(p)})\right)$ and $\sigma_p < 0$, from the definition of σ_p, N_2, we can get

$$\log^+ A_n e^{\sigma_\nu \lambda_n} \leq \gamma\left(\frac{1}{1+\tau}\beta(\lambda_{n''(p)})\right) \leq \gamma\left(\frac{T-\delta}{1+\tau}\log U_2\left(\frac{1}{-\sigma_p}\right)\right). \tag{40}$$

Thus, from (36), (37), (39) and (40), we have

$$\log^+ A_n e^{\sigma_p \lambda_n} \leq \gamma\left((T-\delta)\log U_2\left(\frac{1}{-\sigma} + \frac{1}{-\sigma \log U_2\left(\frac{1}{-\sigma}\right)}\right)\right), \quad as \quad n > N_2.$$

By Lemma 2.2, we have

$$\lim_{\sigma_p \to 0^-} \frac{\beta(\log^+ m(\sigma_p, f))}{\log U_2\left(\frac{1}{-\sigma_p}\right)} \leq T - \delta < T. \tag{41}$$

From (41), Theorem 1.3, we can get a contradiction with the following equality

$$\lim_{\sigma \to 0^-} \frac{\beta(\log^+ M(\sigma, f))}{\log U_2\left(\frac{1}{-\sigma}\right)} = T.$$

Thus, the proof of Theorem 1.5 is completed by Step one and Step two.

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On special fuzzy differential subordinations using multiplier transformation

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Abstract

In the present paper we establish several fuzzy differential subordinations regardind the operator $I(m,\lambda,l)$, given by $I(m,\lambda,l):\mathcal{A}\to\mathcal{A}$, $I(m,\lambda,l)f(z)=z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_jz^j$ and $\mathcal{A}=\{f\in\mathcal{H}(U),\ f(z)=z+a_2z^2+\dots,\ z\in U\}$ is the class of normalized analytic functions. A certain fuzzy class, denoted by $SI_{\mathcal{F}}^{\delta}(m,\lambda,l)$, of analytic functions in the open unit disc is introduced by means of this operator. By making use of the concept of fuzzy differential subordination we will derive various properties and characteristics of the class $SI_{\mathcal{F}}^{\delta}(m,\lambda,l)$. Also, several fuzzy differential subordinations are established regarding the operator $I(m,\lambda,l)$.

Keywords: fuzzy differential subordination, convex function, fuzzy best dominant, differential operator.
2000 Mathematical Subject Classification: 30C45, 30A20.

1 Introduction

S.S. Miller and P.T. Mocanu have introduced [10], [11] and developed [12] in the one complex variable functions theory the admissible functions method known as "the differential subordination method" . The application of this method allows to one obtain some special results and to prove easily some classical results from this domain.

G.I. Oros and Gh.Oros [13], [14] wanted to launch a new research direction in mathematics that combines the notions from the complex functions domain with the fuzzy sets theory.

In the same way as mentioned, we can justify that by knowing the properties of a differential expression on a fuzzy set for a function one can be determined the properties of that function on a given fuzzy set. We have analyzed the case of one complex functions, leaving as "open problem" the case of real functions. We are aware that this new research alternative can be realized only through the joint effort of researchers from both domains. The "open problem" statement leaves open the interpretation of some notions from the fuzzy sets theory such that each one interpret them personally according to their scientific concerns, making this theory more attractive.

The notion of fuzzy subordination was introduced in [13]. In [14] the authors have defined the notion of fuzzy differential subordination. In this paper we will study fuzzy differential subordinations obtained with the differential operator studied in [3] using the methods from [4], [5].

Denote by U the unit disc of the complex plane, $U=\{z\in\mathbb{C}:|z|<1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}_n=\{f\in\mathcal{H}(U):f(z)=z+a_{n+1}z^{n+1}+\dots,\ z\in U\}$ with $\mathcal{A}_1=\mathcal{A}$ and $\mathcal{H}[a,n]=\{f\in\mathcal{H}(U):f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\dots,\ z\in U\}$ for $a\in\mathbb{C}$ and $n\in\mathbb{N}$.

Denote by $\mathcal{K}=\left\{f\in\mathcal{A}:\operatorname{Re}\frac{zf''(z)}{f'(z)}+1>0,\ z\in U\right\}$, the class of normalized convex functions in U.

In order to use the concept of fuzzy differential subordination, we remember the following definitions:

Definition 1.1 *[9] A pair (A,F_A), where $F_A:X\to[0,1]$ and $A=\{x\in X:0<F_A(x)\le 1\}$ is called fuzzy subset of X. The set A is called the support of the fuzzy set (A,F_A) and F_A is called the membership function of the fuzzy set (A,F_A). One can also denote $A=\operatorname{supp}(A,F_A)$.*

Remark 1.1 *In the development work we use the following notations for fuzzy sets:*
$F_{f(D)}(f(z)) = supp(f(D), F_{f(D)}\cdot) = \{z \in D : 0 < F_{f(D)}f(z) \leq 1\}$,
$F_{g(D)}(g(z)) = supp(g(D), F_{g(D)}\cdot) = \{z \in D : 0 < F_{g(D)}g(z) \leq 1\}$,
$p(U) = supp(p(U), F_{p(U)}\cdot) = \{z \in U : 0 < F_{p(U)}(p(z)) \leq 1\}$,
$q(U) = supp(q(U), F_{q(U)}\cdot) = \{z \in U : 0 < F_{q(U)}(q(z)) \leq 1\}$,
$h(U) = supp(h(U), F_{h(U)}\cdot) = \{z \in U : 0 < F_{h(U)}(h(z)) \leq 1\}$.

We give a new definition of membership function on complex numbers set using the module notion of a complex number $z = x + iy$, $x, y \in \mathbb{R}$, $|z| = \sqrt{x^2 + y^2} \geq 0$.

Example 1.1 *Let $F : \mathbb{C} \to \mathbb{R}_+$ a function such that $F_{\mathbb{C}}(z) = |F(z)|$, $\forall\ z \in \mathbb{C}$. Denote by $F_{\mathbb{C}}(\mathbb{C}) = \{z \in \mathbb{C} : 0 < F(z) \leq 1\} = \{z \in \mathbb{C} : 0 < |F(z)| \leq 1\} = supp(\mathbb{C}, F_{\mathbb{C}})$ the fuzzy subset of the complex numbers set.*

Remark 1.2 *We call the subset $F_{\mathbb{C}}(\mathbb{C}) = \{z \in \mathbb{C} : 0 < |F(z)| \leq 1\} = U_{\mathcal{F}}(0,1)$ the fuzzy unit disk.*

Example 1.2 *Let $F : \mathbb{C} \to \mathbb{R}_+$, $F(z) = \frac{2-|z|}{2+|z|}$, where $|z| = \sqrt{x^2 + y^2} \geq 0$. A fuzzy subset of the complex numbers set is $A = \{z \in \mathbb{C} : 0 < F_A(z) \leq 1\} = supp(A, F_A) = \{z \in \mathbb{C} : |z| < 2\}$, where $F_A(z) = \begin{cases} F(z), & z \in \{|z| \leq 2\} \\ 0, & z \in \mathbb{C} - \{|z| \leq 2\}. \end{cases}$*

We show that the fuzzy subset is nonempty. Indeed, for $z = 0$, $F_A(0) = F(0) = 1$, so $z = 0 \in A$. More we see that the fuzzy subset A contains all the complex numbers with the properties $|z| < 2$ and all the complex numbers for which $|z| > 2$ not belong to A, i.e. $supp(A, F_A) = \{z \in \mathbb{C} : x^2 + y^2 < 4\}$.

Remark 1.3 *The membership functions can be defined otherwise and we propose that each choose how to define according to their research.*

Definition 1.2 *([13]) Let $D \subset \mathbb{C}$, $z_0 \in D$ be a fixed point and let the functions $f, g \in \mathcal{H}(D)$. The function f is said to be fuzzy subordinate to g and write $f \prec_{\mathcal{F}} g$ or $f(z) \prec_{\mathcal{F}} g(z)$, if are satisfied the conditions:*
1) $f(z_0) = g(z_0)$,
2) $F_{f(D)}f(z) \leq F_{g(D)}g(z)$, $z \in D$.

Definition 1.3 *([14, Definition 2.2]) Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h univalent in U, with $\psi(a, 0; 0) = h(0) = a$. If p is analytic in U, with $p(0) = a$ and satisfies the (second-order) fuzzy differential subordination*

$$F_{\psi(\mathbb{C}^3\times U)}\psi(p(z), zp'(z), z^2p''(z); z) \leq F_{h(U)}h(z), \quad z \in U, \tag{1.1}$$

then p is called a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or more simple a fuzzy dominant, if $F_{p(U)}p(z) \leq F_{q(U)}q(z)$, $z \in U$, for all p satisfying (1.1). A fuzzy dominant $\widetilde{q}$ that satisfies $F_{\widetilde{q}(U)}\widetilde{q}(z) \leq F_{q(U)}q(z)$, $z \in U$, for all fuzzy dominants q of (1.1) is said to be the fuzzy best dominant of (1.1).

Lemma 1.1 *([12, Corollary 2.6g.2, p. 66]) Let $h \in \mathcal{A}$ and $L[f](z) = G(z) = \frac{1}{z}\int_0^z h(t)\,dt$, $z \in U$. If $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) > -\frac{1}{2}$, $z \in U$, then $L(f) = G \in \mathcal{K}$.*

Lemma 1.2 *([15]) Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}^*$ be a complex number with $Re\,\gamma \geq 0$. If $p \in \mathcal{H}[a, n]$ with $p(0) = a$, $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$, $\psi(p(z), zp'(z); z) = p(z) + \frac{1}{\gamma}zp'(z)$ an analytic function in U and $F_{\psi(\mathbb{C}^2\times U)}\left(p(z) + \frac{1}{\gamma}zp'(z)\right) \leq F_{h(U)}h(z)$, i.e. $p(z) + \frac{1}{\gamma}zp'(z) \prec_{\mathcal{F}} h(z)$, $z \in U$, then $F_{p(U)}p(z) \leq F_{g(U)}g(z) \leq F_{h(U)}h(z)$, i.e. $p(z) \prec_{\mathcal{F}} g(z) \prec_{\mathcal{F}} h(z)$, $z \in U$, where $g(z) = \frac{\gamma}{nz^{\gamma/n}}\int_0^z h(t)t^{\gamma/n-1}dt$, $z \in U$. The function q is convex and is the fuzzy best dominant.*

Lemma 1.3 *([15]) Let g be a convex function in U and let $h(z) = g(z) + n\alpha zg'(z)$, $z \in U$, where $\alpha > 0$ and n is a positive integer. If $p(z) = g(0) + p_nz^n + p_{n+1}z^{n+1} + \ldots$, $z \in U$, is holomorphic in U and $F_{p(U)}(p(z) + \alpha zp'(z)) \leq F_{h(U)}h(z)$, i.e. $p(z) + \alpha zp'(z) \prec_{\mathcal{F}} h(z)$, $z \in U$, then $F_{p(U)}p(z) \leq F_{g(U)}g(z)$, i.e. $p(z) \prec_{\mathcal{F}} g(z), z \in U$, and this result is sharp.*

We will study the following differential operator, known as multiplier transformation.

Definition 1.4 *For $f \in \mathcal{A} = \{f \in \mathcal{H}(U) : f(z) = z + a_2 z^2 + \dots,\ z \in U\}$, $m \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$, the operator $I(m, \lambda, l) f(z)$ is defined by the following infinite series $I(m, \lambda, l) f(z) = z + \sum_{j=n+1}^{\infty} \left(\frac{\lambda(j-1)+l+1}{l+1}\right)^m a_j z^j$.*

Remark 1.4 *It follows from the above definition that $(l+1) I(m+1, \lambda, l) f(z) = [l+1-\lambda] I(m, \lambda, l) f(z) + \lambda z (I(m, \lambda, l) f(z))'$, $z \in U$.*

Remark 1.5 *For $l = 0$, $\lambda \geq 0$, the operator $D_\lambda^m = I(m, \lambda, 0)$ was introduced and studied by Al-Oboudi [2], which is reduced to the Sălăgean differential operator [16] for $\lambda = 1$. The operator $I(m, 1, l)$ was studied by Cho and Srivastava [8] and Cho and Kim [7]. The operator $I(m, 1, 1)$ was studied by Uralegaddi and Somanatha [17] and the operator $I(\alpha, \lambda, 0)$ was introduced by Acu and Owa [1]. Cătaş [6] has studied the operator $I_p(m, \lambda, l)$ which generalizes the operator $I(m, \lambda, l)$.*

2 Main results

Using the operator $I(m, \lambda, l)$ we define the class $SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$ and we study fuzzy subordinations.

Definition 2.1 *Let $f(D) = \text{supp}\big(f(D), F_{f(D)}\big) = \{z \in D : 0 < F_{f(D)} f(z) \leq 1\}$, where $F_{f(D)}\cdot$ is the membership function of the fuzzy set $f(D)$ asociated to the function f.*

The membership function of the fuzzy set $(\mu f)(D)$ asociated to the function μf coincide with the membership function of the fuzzy set $f(D)$ asociated to the fuction f, i.e. $F_{(\mu f)(D)}((\mu f)(z)) = F_{f(D)} f(z)$, $z \in D$.

The membership function of the fuzzy set $(f+g)(D)$ asociated to the function $f+g$ coincide with the half of the sum of the membership functions of the fuzzy sets $f(D)$, respectively $g(D)$, asociated to the function f, respectively g, i.e. $F_{(f+g)(D)}((f+g)(z)) = \frac{F_{f(D)} f(z) + F_{g(D)} g(z)}{2}$, $z \in D$.

Remark 2.1 *$F_{(f+g)(D)}((f+g)(z))$ can be defined in other ways.*

Remark 2.2 *Since $0 < F_{f(D)} f(z) \leq 1$ and $0 < F_{g(D)} g(z) \leq 1$, it is evidently that $0 < F_{(f+g)(D)}((f+g)(z)) \leq 1$, $z \in D$.*

Definition 2.2 *Let $\delta \in (0, 1]$, $\lambda, l \geq 0$ and $m \in \mathbb{N}$. A function $f \in \mathcal{A}$ is said to be in the class $SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$ if it satisfies the inequality $F_{(I(m,\lambda,l)f)'(U)}(I(m, \lambda, l) f(z))' > \delta$, $z \in U$.*

Theorem 2.1 *The set $SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$ is convex.*

Proof. Let the functions $f_j(z) = z + \sum_{j=2}^{\infty} a_{jk} z^j$, $k = 1, 2$, $z \in U$, be in the class $SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$. It is sufficient to show that the function $h(z) = \eta_1 f_1(z) + \eta_2 f_2(z)$ is in the class $SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$ with η_1 and η_2 nonnegative such that $\eta_1 + \eta_2 = 1$.

We have $h'(z) = (\mu_1 f_1 + \mu_2 f_2)'(z) = \mu_1 f_1'(z) + \mu_2 f_2'(z)$, $z \in U$, and
$(I(m, \lambda, l) h(z))' = (I(m, \lambda, l)(\mu_1 f_1 + \mu_2 f_2)(z))' = \mu_1 (I(m, \lambda, l) f_1(z))' + \mu_2 (I(m, \lambda, l) f_2(z))'$.

From Definition 2.1 we obtain that

$$F_{(I(m,\lambda,l)h)'(U)}(I(m, \lambda, l) h(z))' = F_{(I(m,\lambda,l)(\mu_1 f_1 + \mu_2 f_2))'(U)}(I(m, \lambda, l)(\mu_1 f_1 + \mu_2 f_2)(z))' =$$
$$F_{(I(m,\lambda,l)(\mu_1 f_1 + \mu_2 f_2))'(U)}\big(\mu_1 (I(m, \lambda, l) f_1(z))' + \mu_2 (I(m, \lambda, l) f_2(z))'\big) =$$
$$\frac{F_{(\mu_1 I(m,\lambda,l) f_1)'(U)}\big(\mu_1 (I(m,\lambda,l) f_1(z))'\big) + F_{(\mu_2 I(m,\lambda,l) f_2)'(U)}\big(\mu_2 (I(m,\lambda,l) f_2(z))'\big)}{2} =$$
$$\frac{F_{(I(m,\lambda,l) f_1)'(U)}(I(m,\lambda,l) f_1(z))' + F_{(I(m,\lambda,l) f_2)'(U)}(I(m,\lambda,l) f_2(z))'}{2}.$$

Since $f_1, f_2 \in SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$ we have $\delta < F_{(I(m,\lambda,l)f_1)'(U)}(I(m, \lambda, l) f_1(z))' \leq 1$ and
$\delta < F_{(I(m,\lambda,l)f_2)'(U)}(I(m, \lambda, l) f_2(z))' \leq 1$, $z \in U$.

Therefore $\delta < \frac{F_{(I(m,\lambda,l) f_1)'(U)}(I(m,\lambda,l) f_1(z))' + F_{(I(m,\lambda,l) f_2)'(U)}(I(m,\lambda,l) f_2(z))'}{2} \leq 1$ and we obtain that
$\delta < F_{(I(m,\lambda,l)h)'(U)}(I(m, \lambda, l) h(z))' \leq 1$, which means that $h \in SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$ and $SI_{\mathcal{F}}^{\delta}(m, \lambda, l)$ is convex. ■

We highlight a fuzzy subset obtained using a convex function. Let the function $h(z) = \frac{1+z}{1-z}$, $z \in U$. After a short calculation we obtain that $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) = Re\frac{1+z}{1-z} > 0$, so $h \in \mathcal{K}$ and $h(U) = \{z \in \mathbb{C} : Re z > 0\}$. We define the membership function for the set $h(U)$ as $F_{h(U)}(h(z)) = Re h(z)$, $z \in U$ and we have $F_{h(U)} h(z) = \text{supp}\big(h(U), F_{h(u)}\big) = \{z \in \mathbb{C} : 0 < F_{h(U)}(h(z)) \leq 1\} = \{z \in U : 0 < Re z \leq 1\}$.

Remark 2.3 *In this case the membership function can be defined otherwise too and we recommend that those interested to make it in accordance with their scientific concern.*

Theorem 2.2 *Let g be a convex function in U and let $h(z) = g(z) + \frac{1}{c+2} z g'(z)$, where $z \in U$, $c > 0$. If $f \in SI^{\delta}_{\mathcal{F}}(m, \lambda, l)$ and $G(z) = I_c(f)(z) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t)\, dt$, $z \in U$, then*

$$F_{(I(m,\lambda,l)f)'(U)} \left(I(m,\lambda,l) f(z)\right)' \le F_{h(U)} h(z), \ \text{ i.e. } \ \left(I(m,\lambda,l) f(z)\right)' \prec_{\mathcal{F}} h(z), \quad z \in U, \tag{2.1}$$

implies $F_{(I(m,\lambda,l)G)'(U)} \left(I(m,\lambda,l) G(z)\right)' \le F_{g(U)} g(z)$, i.e. $\left(I(m,\lambda,l) G(z)\right)' \prec_{\mathcal{F}} g(z)$, $z \in U$, and this result is sharp.

Proof. We obtain that

$$z^{c+1} G(z) = (c+2) \int_0^z t^c f(t)\, dt. \tag{2.2}$$

Differentiating (2.2), with respect to z, we have $(c+1) G(z) + z G'(z) = (c+2) f(z)$ and

$$(c+1) I(m,\lambda,l) G(z) + z \left(I(m,\lambda,l) G(z)\right)' = (c+2) I(m,\lambda,l) f(z), \quad z \in U. \tag{2.3}$$

Differentiating (2.3) we have

$$\left(I(m,\lambda,l) G(z)\right)' + \frac{1}{c+2} z \left(I(m,\lambda,l) G(z)\right)'' = \left(I(m,\lambda,l) f(z)\right)', \quad z \in U. \tag{2.4}$$

Using (2.4), the fuzzy differential subordination (2.1) becomes

$$F_{I(m,\lambda,l)G(U)} \left(\left(I(m,\lambda,l) G(z)\right)' + \frac{1}{c+2} z \left(I(m,\lambda,l) G(z)\right)'' \right) \le F_{g(U)} \left(g(z) + \frac{1}{c+2} z g'(z) \right). \tag{2.5}$$

If we denote

$$p(z) = \left(I(m,\lambda,l) G(z)\right)', \quad z \in U, \tag{2.6}$$

then $p \in \mathcal{H}[1,1]$.

Replacing (2.6) in (2.5) we obtain $F_{p(U)} \left(p(z) + \frac{1}{c+2} z p'(z) \right) \le F_{g(U)} \left(g(z) + \frac{1}{c+2} z g'(z) \right)$, $z \in U$.

Using Lemma 1.3 we have $F_{p(U)} p(z) \le F_{g(U)} g(z)$, $z \in U$, i.e. $F_{(I(m,\lambda,l)G)'(U)} \left(I(m,\lambda,l) G(z)\right)' \le F_{g(U)} g(z)$, $z \in U$, and g is the fuzzy best dominant. We have obtained that $\left(L^m_\alpha G(z)\right)' \prec_{\mathcal{F}} g(z)$, $z \in U$. ■

Example 2.1 *If $f \in SI^{\frac{1}{2}}_{\mathcal{F}}\left(1, \frac{1}{2}, \frac{1}{2}\right)$, then $f'(z) + \frac{1}{3} z f''(z) \prec_{\mathcal{F}} \frac{3-2z}{3(1-z)^2}$ implies $G'(z) + \frac{1}{3} z G''(z) \prec_{\mathcal{F}} \frac{1}{1-z}$, where $G(z) = \frac{3}{z^2} \int_0^z t f(t)\, dt$.*

Theorem 2.3 *Let $h(z) = \frac{1+(2\beta-1)z}{1+z}$, $\beta \in [0,1)$ and $c > 0$. If $\lambda, l \ge 0$, $m \in \mathbb{N}$ and $I_c(f)(z) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t)\, dt$, $z \in U$, then*

$$I_c \left[SI^{\beta}_{\mathcal{F}}(m,\lambda,l) \right] \subset SI^{\beta^*}_{\mathcal{F}}(m,\lambda,l), \tag{2.7}$$

where $\beta^* = 2\beta - 1 + (c+2)(2-2\beta) \int_0^1 \frac{t^{c+1}}{t+1} dt$.

Proof. The function h is convex and using the same steps as in the proof of Theorem 2.2 we get from the hypothesis of Theorem 2.3 that $F_{p(U)} \left(p(z) + \frac{1}{c+2} z p'(z) \right) \le f_{h(U)} h(z)$, where $p(z)$ is defined in (2.6). Using Lemma 1.2 we deduce that $F_{p(U)} p(z) \le F_{g(U)} g(z) \le F_{h(U)} h(z)$, i.e. $F_{(I(m,\lambda,l)G)'(U)} \left(I(m,\lambda,l) G(z)\right)' \le F_{g(U)} g(z) \le F_{h(U)} h(z)$, where $g(z) = \frac{c+2}{z^{c+2}} \int_0^z t^{c+1} \frac{1+(2\beta-1)t}{1+t} dt = 2\beta - 1 + \frac{(c+2)(2-2\beta)}{z^{c+2}} \int_0^z \frac{t^{c+1}}{t+1} dt$. Since g is convex and $g(U)$ is symmetric with respect to the real axis, we deduce

$$F_{I(m,\lambda,l)G(U)} \left(I(m,\lambda,l) G(z)\right)' \ge \min_{|z|=1} F_{g(U)} g(z) = F_{g(U)} g(1) \tag{2.8}$$

and $\beta^* = g(1) = 2\beta - 1 + (c+2)(2-2\beta) \int_0^1 \frac{t^{c+1}}{t+1} dt$.

From (2.8) we deduce inclusion (2.7). ■

Theorem 2.4 *Let g be a convex function, $g(0) = 1$ and let h be the function $h(z) = g(z) + z g'(z)$, $z \in U$. If $\lambda, l \ge 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$ and satisfies the fuzzy differential subordination*

$$F_{(I(m,\lambda,l)f)'(U)} \left(I(m,\lambda,l) f(z)\right)' \le F_{h(U)} h(z), \ \text{ i.e. } \ \left(I(m,\lambda,l) f(z)\right)' \prec_{\mathcal{F}} h(z), \quad z \in U, \tag{2.9}$$

then $F_{I(m,\lambda,l)f(U)} \frac{I(m,\lambda,l) f(z)}{z} \le F_{g(U)} g(z)$, i.e. $\frac{I(m,\lambda,l) f(z)}{z} \prec_{\mathcal{F}} g(z)$, $z \in U$, and this result is sharp.

Proof. Consider $p(z) = \frac{I(m,\lambda,l)f(z)}{z} = \frac{z+\sum_{j=2}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m a_j z^j}{z} = 1 + p_1 z + p_2 z^2 + ...,\quad z \in U$. We deduce that $p \in \mathcal{H}[1,1]$.

Let $I(m,\lambda,l) f(z) = zp(z)$, for $z \in U$. Differentiating we obtain $(I(m,\lambda,l) f(z))' = p(z) + zp'(z)$, $z \in U$. Then (2.9) becomes $F_{p(U)}(p(z) + zp'(z)) \leq F_{h(U)}h(z) = F_{g(U)}(g(z) + zg'(z))$, $z \in U$.

By using Lemma 1.3, we have $F_{p(U)}p(z) \leq F_{g(U)}g(z)$, $z \in U$, i.e. $F_{(I(m,\lambda,l)f)'(U)}\frac{I(m,\lambda,l)f(z)}{z} \leq F_{g(U)}g(z)$, $z \in U$.We obtain that $\frac{I(m,\lambda,l)f(z)}{z} \prec_{\mathcal{F}} g(z)$, $z \in U$, and this result is sharp. ■

Theorem 2.5 *Let h be an holomorphic function which satisfies the inequality* Re $\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, *and $h(0) = 1$. If $\lambda, l \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$ and satisfies the fuzzy differential subordination*

$$F_{(I(m,\lambda,l)f)'(U)}(I(m,\lambda,l) f(z))' \leq F_{h(U)}h(z), \quad i.e. \quad (I(m,\lambda,l) f(z))' \prec_{\mathcal{F}} h(z), \quad z \in U, \tag{2.10}$$

then $F_{I(m,\lambda,l)f(U)}\frac{I(m,\lambda,l)f(z)}{z} \leq F_{q(U)}q(z)$, i.e. $\frac{I(m,\lambda,l)f(z)}{z} \prec_{\mathcal{F}} q(z)$, $z \in U$, where $q(z) = \frac{1}{z}\int_0^z h(t)dt$. The function q is convex and it is the fuzzy best dominant.

Proof. Let $p(z) = \frac{I(m,\lambda,l)f(z)}{z}$, $z \in U$, $p \in \mathcal{H}[1,1]$. Since Re $\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, from Lemma 1.1, we obtain that $q(z) = \frac{1}{z}\int_0^z h(t)dt$ is a convex function and verifies the differential equation asscociated to the fuzzy differential subordination (2.10) $q(z) + zq'(z) = h(z)$, therefore it is the fuzzy best dominant.

Differentiating, we obtain $(I(m,\lambda,l) f(z))' = p(z)+zp'(z)$, $z \in U$ and (2.10) becomes $F_{p(U)}(p(z) + zp'(z)) \leq F_{h(U)}h(z)$, $z \in U$.

Using Lemma 1.3, we have $F_{p(U)}p(z) \leq F_{q(U)}q(z)$, $z \in U$, i.e. $F_{I(m,\lambda,l)f(U)}\frac{I(m,\lambda,l)f(z)}{z} \leq F_{q(U)}q(z)$, $z \in U$. We have obtained that $\frac{I(m,\lambda,l)f(z)}{z} \prec_{\mathcal{F}} q(z)$, $z \in U$. ■

Corollary 2.6 *Let $h(z) = \frac{1+(2\beta-1)z}{1+z}$ a convex function in U, $0 \leq \beta < 1$. If $\lambda, l \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$ and verifies the fuzzy differential subordination*

$$F_{(I(m,\lambda,l)f)'(U)}(I(m,\lambda,l) f(z))' \leq F_{h(U)}h(z), \ i.e. \ (I(m,\lambda,l) f(z))' \prec_{\mathcal{F}} h(z), \quad z \in U, \tag{2.11}$$

then $F_{I(m,\lambda,l)f(U)}\frac{I(m,\lambda,l)f(z)}{z} \leq F_{q(U)}q(z)$, i.e. $\frac{I(m,\lambda,l)f(z)}{z} \prec_{\mathcal{F}} q(z)$, $z \in U$, where q is given by $q(z) = 2\beta - 1 + \frac{2(1-\beta)}{z}\ln(1+z)$, $z \in U$. The function q is convex and it is the fuzzy best dominant.

Proof. We have $h(z) = \frac{1+(2\beta-1)z}{1+z}$ with $h(0) = 1$, $h'(z) = \frac{-2(1-\beta)}{(1+z)^2}$ and $h''(z) = \frac{4(1-\beta)}{(1+z)^3}$, therefore $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) = Re\left(\frac{1-z}{1+z}\right) = Re\left(\frac{1-\rho\cos\theta-i\rho\sin\theta}{1+\rho\cos\theta+i\rho\sin\theta}\right) = \frac{1-\rho^2}{1+2\rho\cos\theta+\rho^2} > 0 > -\frac{1}{2}$.

Following the same steps as in the proof of Theorem 2.5 and considering $p(z) = \frac{I(m,\lambda,l)f(z)}{z}$, the fuzzy differential subordination (2.11) becomes $F_{I(m,\lambda,l)f(U)}(p(z) + zp'(z)) \leq F_{h(U)}h(z)$, $z \in U$.

By using Lemma 1.2 for $\gamma = 1$ and $n = 1$, we have $F_{p(U)}p(z) \leq F_{q(U)}q(z)$, i.e., $F_{I(m,\lambda,l)f(U)}\frac{I(m,\lambda,l)f(z)}{z} \leq F_{q(U)}q(z)$ and $q(z) = \frac{1}{z}\int_0^z h(t)\,dt = \frac{1}{z}\int_0^z \frac{1+(2\beta-1)t}{1+t}dt = 2\beta - 1 + \frac{2(1-\beta)}{z}\ln(1+z)$, $z \in U$. ■

Example 2.2 *Let $h(z) = \frac{1-z}{1+z}$ with $h(0) = 1$, $h'(z) = \frac{-2}{(1+z)^2}$ and $h''(z) = \frac{4}{(1+z)^3}$.*

Since $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) = Re\left(\frac{1-z}{1+z}\right) = Re\left(\frac{1-\rho\cos\theta-i\rho\sin\theta}{1+\rho\cos\theta+i\rho\sin\theta}\right) = \frac{1-\rho^2}{1+2\rho\cos\theta+\rho^2} > 0 > -\frac{1}{2}$, the function h is convex in U.

Let $f(z) = z + z^2$, $z \in U$. For $n = 1$, $m = 1$, $l = 2$, $\lambda = 1$, we obtain $I(1,1,2) f(z) = \frac{2}{3}f(z) + \frac{1}{3}zf'(z) = z + \frac{4}{3}z^2$. Then $(I(1,1,2) f(z))' = 1 + \frac{8}{3}z$ and $\frac{I(1,1,2)f(z)}{z} = 1 + \frac{4}{3}z$. We have $q(z) = \frac{1}{z}\int_0^z \frac{1-t}{1+t}dt = -1 + \frac{2\ln(1+z)}{z}$.

Using Theorem 2.5 we obtain $1 + \frac{8}{3}z \prec_{\mathcal{F}} \frac{1-z}{1+z}$, $z \in U$, induce $1 + \frac{4}{3}z \prec_{\mathcal{F}} -1 + \frac{2\ln(1+z)}{z}$, $z \in U$.

Theorem 2.7 *Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, $z \in U$. If $\lambda, l \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$ and the fuzzy differential subordination*

$$F_{I(m,\lambda,l)f(U)}\left(\frac{zI(m+1,\lambda,l) f(z)}{I(m,\lambda,l) f(z)}\right)' \leq F_{h(U)}h(z), \quad i.e. \quad \left(\frac{zI(m+1,\lambda,l) f(z)}{I(m,\lambda,l) f(z)}\right)' \prec_{\mathcal{F}} h(z), \quad z \in U \tag{2.12}$$

holds, then $F_{I(m,\lambda,l)f(U)}\frac{I(m+1,\lambda,l)f(z)}{I(m,\lambda,l)f(z)} \leq F_{g(U)}g(z)$, i.e. $\frac{I(m+1,\lambda,l)f(z)}{I(m,\lambda,l)f(z)} \prec_{\mathcal{F}} g(z)$, $z \in U$, and this result is sharp.

Proof. Consider $p(z) = \frac{I(m+1,\lambda,l)f(z)}{I(m,\lambda,l)f(z)}$. We have $p'(z) = \frac{(I(m+1,\lambda,l)f(z))'}{I(m,\lambda,l)f(z)} - p(z) \cdot \frac{(I(m+1,\lambda,l)f(z))'}{I(m,\lambda,l)f(z)}$ and we obtain $p(z) + z \cdot p'(z) = \left(\frac{zI(m+1,\lambda,l)f(z)}{I(m,\lambda,l)f(z)}\right)'$.

Relation (2.12) becomes $F_{p(U)}(p(z) + zp'(z)) \leq F_{h(U)}h(z) = F_{g(U)}(g(z) + zg'(z))$, $z \in U$. By using Lemma 1.3, we have $F_{p(U)}p(z) \leq F_{g(U)}g(z)$, $z \in U$, i.e. $F_{I(m,\lambda,l)f(U)} \frac{I(m+1,\lambda,l)f(z)}{I(m,\lambda,l)f(z)} \leq F_{g(U)}g(z)$, $z \in U$. We obtain that $\frac{I(m+1,\lambda,l)f(z)}{I(m,\lambda,l)f(z)} \prec_{\mathcal{F}} g(z)$, $z \in U$. ■

Theorem 2.8 *Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, $z \in U$. If $\lambda, l \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$ and the fuzzy differential subordination* $F_{I(m,\lambda,l)f(U)}\left(\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z) + \left(2 - \frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z)\right) \leq F_{h(U)}h(z)$, *i.e.*

$$\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z) + \left(2 - \frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z) \prec_{\mathcal{F}} h(z), \quad z \in U \tag{2.13}$$

holds, then $F_{I(m,\lambda,l)f(U)}[I(m,\lambda,l)f(z)]' \leq F_{g(U)}g(z)$, *i.e.* $[I(m,\lambda,l)f(z)]' \prec_{\mathcal{F}} g(z)$, $z \in U$. *This result is sharp.*

Proof. Let $p(z) = (I(m,\lambda,l)f(z))'$. We deduce that $p \in \mathcal{H}[1,1]$. We obtain $p(z) + z \cdot p'(z) = I(m,\lambda,l)f(z) + z(I(m,\lambda,l)f(z))' = I(m,\lambda,l)f(z) + \frac{(l+1)I(m+1,\lambda,l)f(z)-(l+1-\lambda)I(m,\lambda,l)f(z)}{\lambda} = \frac{l+1}{\lambda}I(m+1,\lambda,l)f(z) + \left(2 - \frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z)$.

The fuzzy differential subordination becomes $F_{p(U)}(p(z) + zp'(z)) \leq F_{h(U)}h(z) = F_{g(U)}(g(z) + zg'(z))$. By using Lemma 1.3, we have $F_{p(U)}p(z) \leq F_{g(U)}g(z)$, $z \in U$, i.e. $F_{I(m,\lambda,l)f(U)}(I(m,\lambda,l)f(z))' \leq F_{g(U)}g(z)$, $z \in U$, and this result is sharp. ■

Theorem 2.9 *Let h be an holomorphic function which satisfies the inequality* $\operatorname{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}$, $z \in U$, *and $h(0) = 1$. If $\lambda, l \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$ and satisfies the fuzzy differential subordination* $F_{I(m,\lambda,l)f(U)}\left(\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z) + \left(2 - \frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z)\right) \leq F_{h(U)}h(z)$, *i.e.*

$$\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z) + \left(2 - \frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z) \prec_{\mathcal{F}} h(z), \quad z \in U, \tag{2.14}$$

then $F_{I(m,\lambda,l)f(U)}(I(m,\lambda,l)f(z))' \leq F_{q(U)}q(z)$, *i.e.* $(I(m,\lambda,l)f(z))' \prec_{\mathcal{F}} q(z)$, $z \in U$, *where q is given by* $q(z) = \frac{1}{z}\int_0^z h(t)dt$. *The function q is convex and it is the fuzzy best dominant.*

Proof. Since $\operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, from Lemma 1.1, we obtain that $q(z) = \frac{1}{z}\int_0^z h(t)dt$ is a convex function and verifies the differential equation asscociated to the fuzzy differential subordination (2.14) $q(z) + zq'(z) = h(z)$, therefore it is the fuzzy best dominant.
Considering $p(z) = (I(m,\lambda,l)f(z))'$, we obtain $p(z)+zp'(z) = \frac{l+1}{\lambda}I(m+1,\lambda,l)f(z)+\left(2 - \frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z)$, $z \in U$. Then (2.14) becomes $F_{p(U)}(p(z) + zp'(z)) \leq F_{h(U)}h(z)$, $z \in U$.
Since $p \in \mathcal{H}[1,1]$, using Lemma 1.3, we deduce $F_{p(U)}p(z) \leq F_{q(U)}q(z)$, $z \in U$, i.e. $F_{I(m,\lambda,l)f(U)}(I(m,\lambda,l)f(z))' \leq F_{q(U)}q(z)$, $z \in U$. We have obtained that $(I(m,\lambda,l)f(z))' \prec_{\mathcal{F}} q(z)$, $z \in U$. ■

Corollary 2.10 *Let $h(z) = \frac{1+(2\beta-1)z}{1+z}$ be a convex function in U, where $0 \leq \beta < 1$.If $\lambda, l \geq 0$, $m \in \mathbb{N}$, $f \in \mathcal{A}$ and satisfies the differential subordination* $F_{I(m,\lambda,l)f(U)}\left(\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z) + \left(2 - \frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z)\right) \leq F_{h(U)}h(z)$, *i.e.*

$$\frac{l+1}{\lambda}I(m+1,\lambda,l)f(z) + \left(2 - \frac{l+1}{\lambda}\right)I(m,\lambda,l)f(z) \prec_{\mathcal{F}} h(z), \quad z \in U, \tag{2.15}$$

then $F_{I(m,\lambda,l)f(U)}(I(m,\lambda,l)f(z))' \leq F_{q(U)}q(z)$, *i.e.* $(I(m,\lambda,l)f(z))' \prec_{\mathcal{F}} q(z)$, $z \in U$, *where q is given by* $q(z) = 2\beta - 1 + 2(1-\beta)\frac{\ln(1+z)}{z}$, *for $z \in U$. The function q is convex and it is the fuzzy best dominant.*

Proof. Following the same steps as in the proof of Theorem 2.8 and considering $p(z) = (I(m,\lambda,l)f(z))'$, the fuzzy differential subordination (2.15) becomes $F_{p(U)}(p(z) + zp'(z)) \leq F_{h(U)}h(z)$, $z \in U$.

By using Lemma 1.2 for $\gamma = 1$ and $n = 1$, we have $F_{p(U)}p(z) \leq F_{q(U)}q(z)$, i.e., $F_{I(m,\lambda,l)f(U)}(I(m,\lambda,l)f(z))' \leq F_{q(U)}q(z)$, i.e. $(I(m,\lambda,l)f(z))' \prec_{\mathcal{F}} q(z)$, $z \in U$, and $q(z) = \frac{1}{z}\int_0^z h(t)dt = \frac{1}{z}\int_0^z \frac{1+(2\beta-1)t}{1+t}dt = 2\beta - 1 + 2(1-\beta)\frac{1}{z}\ln(z+1)$, $z \in U$. ■

Example 2.3 *Let* $h(z) = \frac{1-z}{1+z}$ *a convex function in* U *with* $h(0) = 1$ *and* $Re\left(\frac{zh''(z)}{h'(z)} + 1\right) > -\frac{1}{2}$ *(see Example 2.2).*

Let $f(z) = z + z^2$, $z \in U$. *For* $n = 1$, $m = 1$, $l = 2$, $\lambda = 1$, *we obtain* $I(1,1,2) f(z) = \frac{2}{3} f(z) + \frac{1}{3} z f'(z) = z + \frac{4}{3} z^2$ *and* $(I(1,1,2) f(z))' = 1 + \frac{8}{3} z$. *We obtain also* $\frac{l+1}{\lambda} I(m+1,\lambda,l) f(z) + \left(2 - \frac{l+1}{\lambda}\right) I(m,\lambda,l) f(z) = 3I(2,1,2) f(z) - I(1,1,2) f(z) = 2z + 4z^2$, *where* $I(2,1,2) f(z) = \frac{2}{3} I(1,1,2) f(z) + \frac{z}{3} (I(1,1,2) f(z))' = 3z + \frac{16}{3} z^2$. *We have* $q(z) = \frac{1}{z} \int_0^z \frac{1-t}{1+t} dt = -1 + \frac{2\ln(1+z)}{z}$.

Using Theorem 2.9 we obtain $2z + 4z^2 \prec_{\mathcal{F}} \frac{1-z}{1+z}$, $z \in U$, *induce* $1 + \frac{8}{3} z \prec_{\mathcal{F}} -1 + \frac{2\ln(1+z)}{z}$, $z \in U$.

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On some differential sandwich theorems involving a multiplier transformation and Ruscheweyh derivative

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Abstract

In this paper we obtain some subordination and superordination results for the operator $IR_{\lambda,l}^{m,n}$ and we establish differential sandwich-type theorems. The operator $IR_{\lambda,l}^{m,n}$ is defined as the Hadamard product of the multiplier transformation $I(m,\lambda,l)$ and Ruscheweyh derivative R^n.

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1 Introduction

Consider $\mathcal{H}(U)$ the class of analytic function in the open unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$, $\mathcal{H}(a,n)$ the subclass of $\mathcal{H}(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$ and $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \dots,\ z \in U\}$ with $\mathcal{A} = \mathcal{A}_1$.

Next we remind the definition of differential subordination and superordination.

Let the functions f and g be analytic in U. The function f is subordinate to g, written $f \prec g$, if there exists a Schwarz function w, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$, for all $z \in U$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h be an univalent function in U. If p is analytic in U and satisfies the second order differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \quad \text{for } z \in U, \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1.1). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of U.

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ and h analytic in U. If p and $\psi\left(p(z), zp'(z), z^2p''(z); z\right)$ are univalent and if p satisfies the second order differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \qquad z \in U, \tag{1.2}$$

then p is a solution of the differential superordination (1.2) (if f is subordinate to F, then F is called to be superordinate to f). An analytic function q is called a subordinant if $q \prec p$ for all p satisfying (1.2). An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant.

Miller and Mocanu [6] obtained conditions h, q and ψ for which the following implication holds $h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z)$.

For two functions $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ and $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$ analytic in the open unit disc U, the Hadamard product (or convolution) of $f(z)$ and $g(z)$, written as $(f * g)(z)$ is defined by $f(z) * g(z) = (f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j$.

We need the following differential operators.

Definition 1.1 *[5] For $f \in \mathcal{A}$, $m \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$, the multiplier transformation $I(m,\lambda,l)f(z)$ is defined by the following infinite series $I(m,\lambda,l)f(z) := z + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{1+l}\right)^m a_j z^j$.*

Remark 1.1 *We have* $(l+1)\, I(m+1,\lambda,l)\, f(z) = (l+1-\lambda)\, I(m,\lambda,l)\, f(z) + \lambda z\, (I(m,\lambda,l)\, f(z))'$, $\quad z \in U$.

Remark 1.2 *For* $l=0$, $\lambda \geq 0$, *the operator* $D_\lambda^m = I(m,\lambda,0)$ *was introduced and studied by Al-Oboudi , which reduced to the Sălăgean differential operator* $S^m = I(m,1,0)$ *for* $\lambda = 1$.

Definition 1.2 *(Ruscheweyh [8]) For* $f \in \mathcal{A}$ *and* $n \in \mathbb{N}$, *the Ruscheweyh derivative* R^n *is defined by* $R^n : \mathcal{A} \to \mathcal{A}$,

$$\begin{aligned} R^0 f(z) &= f(z),\ R^1 f(z) = z f'(z),\ ... \\ (n+1)\, R^{n+1} f(z) &= z\,(R^n f(z))' + n R^n f(z), \quad z \in U. \end{aligned}$$

Remark 1.3 *If* $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, *then* $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$ *for* $z \in U$.

Definition 1.3 *([2]) Let* $\lambda, l \geq 0$ *and* $n, m \in \mathbb{N}$. *Denote by* $IR_{\lambda,l}^{m,n} : \mathcal{A} \to \mathcal{A}$ *the operator given by the Hadamard product of the multiplier transformation* $I(m,\lambda,l)$ *and the Ruscheweyh derivative* R^n, $IR_{\lambda,l}^{m,n} f(z) = (I(m,\lambda,l) * R^n)\, f(z)$, *for any* $z \in U$ *and each nonnegative integers* m, n.

Remark 1.4 *If* $f \in \mathcal{A}$ *and* $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, *then* $IR_{\lambda,l}^{m,n} f(z) = z + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2 z^j$, $z \in U$.

Using simple computation we obtain the following relation.

Proposition 1.1 *[1]For* $m, n \in \mathbb{N}$ *and* $\lambda \geq 0$ *we have*

$$IR_{\lambda,l}^{m+1,n} f(z) = \frac{1+l-\lambda}{l+1} IR_{\lambda,l}^{m,n} f(z) + \frac{\lambda}{l+1} z \left(IR_{\lambda,l}^{m,n} f(z)\right)' \tag{1.3}$$

Definition 1.4 *[7] Denote by* Q *the set of all functions* f *that are analytic and injective on* $\overline{U} \backslash E(f)$, *where* $E(f) = \{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\}$, *and are such that* $f'(\zeta) \neq 0$ *for* $\zeta \in \partial U \backslash E(f)$.

Lemma 1.1 *[7] Let the function* q *be univalent in the unit disc* U *and* θ *and* ϕ *be analytic in a domain* D *containing* $q(U)$ *with* $\phi(w) \neq 0$ *when* $w \in q(U)$. *Set* $Q(z) = z q'(z)\, \phi(q(z))$ *and* $h(z) = \theta(q(z)) + Q(z)$. *Suppose that* Q *is starlike univalent in* U *and* $Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$ *for* $z \in U$. *If* p *is analytic with* $p(0) = q(0)$, $p(U) \subseteq D$ *and* $\theta(p(z)) + zp'(z)\, \phi(p(z)) \prec \theta(q(z)) + zq'(z)\, \phi(q(z))$, *then* $p(z) \prec q(z)$ *and* q *is the best dominant.*

Lemma 1.2 *[4] Let the function* q *be convex univalent in the open unit disc* U *and* ν *and* ϕ *be analytic in a domain* D *containing* $q(U)$. *Suppose that* $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) > 0$ *for* $z \in U$ *and 2.* $\psi(z) = zq'(z)\, \phi(q(z))$ *is starlike univalent in* U. *If* $p(z) \in \mathcal{H}[q(0),1] \cap Q$, *with* $p(U) \subseteq D$ *and* $\nu(p(z)) + zp'(z)\, \phi(p(z))$ *is univalent in* U *and* $\nu(q(z)) + zq'(z)\, \phi(q(z)) \prec \nu(p(z)) + zp'(z)\, \phi(p(z))$, *then* $q(z) \prec p(z)$ *and* q *is the best subordinant.*

2 Main results

We intend to find sufficient conditions for certain normalized analytic functions f such that $q_1(z) \prec \frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec q_2(z)$, $z \in U$, $0 < \delta \leq 1$, where q_1 and q_2 are given univalent functions.

Theorem 2.1 *Let* $\frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}(U)$ *and let the function* $q(z)$ *be analytic and univalent in* U *such that* $q(z) \neq 0$, *for all* $z \in U$. *Suppose that* $\frac{zq'(z)}{q(z)}$ *is starlike univalent in* U. *Let*

$$Re\left(\frac{\xi}{\beta} q(z) + \frac{2\mu}{\beta} q^2(z) + 1 + z\frac{q''(z)}{q'(z)} - z\frac{q'(z)}{q(z)}\right) > 0, \tag{2.1}$$

for $\alpha, \xi, \beta, \mu \in \mathbb{C}$, $\beta \neq 0$, $z \in U$ *and*

$$\psi_{\lambda,l}^{m,n}(\alpha,\xi,\mu,\beta;z) := \alpha + \beta\frac{(l+1)}{\lambda} + \beta\frac{(l+1)}{\lambda}\frac{IR_{\lambda,l}^{m+2,n} f(z)}{IR_{\lambda,l}^{m+1,n} f(z)} - \tag{2.2}$$

$$\beta\frac{(l+1)(1+\delta)}{\lambda}\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z)}+\xi\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}+\mu\frac{z^{2\delta}\left(IR_{\lambda,l}^{m+1,n}f(z)\right)^{2}}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{2+2\delta}}.$$

If q satisfies the following subordination

$$\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)\prec\alpha+\xi q(z)+\mu(q(z))^{2}+\beta\frac{zq'(z)}{q(z)},\tag{2.3}$$

for $\alpha,\xi,\beta,\mu\in\mathbb{C}$, $\beta\neq0$, then $\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\prec q(z)$, and q is the best dominant.

Proof. Consider $p(z):=\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}$, $z\in U$, $z\neq0$, $f\in\mathcal{A}$. Differentiating we obtain $p'(z)=\frac{\delta(1+l)}{\lambda}\frac{z^{\delta-1}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}+\frac{l+1}{\lambda}\frac{z^{\delta-1}IR_{\lambda,l}^{m+2,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}-\frac{(l+1)(1+\delta)}{\lambda}\frac{z^{\delta-1}\left(IR_{\lambda,l}^{m+1,n}f(z)\right)^{2}}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{2+\delta}}$.

By using the identity (1.3), we obtain

$$\frac{zp'(z)}{p(z)}=\frac{\delta(l+1)}{\lambda}+\frac{l+1}{\lambda}\frac{IR_{\lambda,l}^{m+2,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z)}-\frac{(l+1)(1+\delta)}{\lambda}\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z)}.\tag{2.4}$$

By setting $\theta(w):=\alpha+\xi w+\mu w^{2}$ and $\phi(w):=\frac{\beta}{w}$, it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\neq0$, $w\in\mathbb{C}\backslash\{0\}$.

Also, by letting $Q(z)=zq'(z)\phi(q(z))=\beta\frac{zq'(z)}{q(z)}$ and $h(z)=\theta(q(z))+Q(z)=\alpha+\xi q(z)+\mu(q(z))^{2}+\beta\frac{zq'(z)}{q(z)}$, we find that $Q(z)$ is starlike univalent in U.

We get $h'(z)=\xi q'(z)+2\mu q(z)q'(z)+\beta\frac{q'(z)}{q(z)}+\beta z\frac{q''(z)}{q(z)}-\beta z\left(\frac{q'(z)}{q(z)}\right)^{2}$ and $\frac{zh'(z)}{Q(z)}=\frac{\xi}{\beta}q(z)+\frac{2\mu}{\beta}q^{2}(z)+1+z\frac{q''(z)}{q(z)}-z\frac{q'(z)}{q(z)}$.

So we deduce that $Re\left(\frac{zh'(z)}{Q(z)}\right)=Re\left(\frac{\xi}{\beta}q(z)+\frac{2\mu}{\beta}q^{2}(z)+1+z\frac{q''(z)}{q(z)}-z\frac{q'(z)}{q(z)}\right)>0$.

By using (2.4), we obtain $\alpha+\xi p(z)+\mu(p(z))^{2}+\beta\frac{zp'(z)}{p(z)}=$
$\alpha+\beta\frac{(l+1)}{\lambda}+\beta\frac{(l+1)}{\lambda}\frac{IR_{\lambda,l}^{m+2,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z)}-\beta\frac{(l+1)(1+\delta)}{\lambda}\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z)}+\xi\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}+\mu\frac{z^{2\delta}\left(IR_{\lambda,l}^{m+1,n}f(z)\right)^{2}}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{2+2\delta}}$.

By using (2.3), we have $\alpha+\xi p(z)+\mu(p(z))^{2}+\beta\frac{zp'(z)}{p(z)}\prec\alpha+\xi q(z)+\mu(q(z))^{2}+\beta\frac{zq'(z)}{q(z)}$.

Appying Lemma 1.1, we obtain $p(z)\prec q(z)$, $z\in U$, i.e. $\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\prec q(z)$, $z\in U$ and q is the best dominant. ■

Corollary 2.2 *Let $m,n\in\mathbb{N}$, $\lambda,l\geq0$. Assume that (2.1) holds. If $f\in\mathcal{A}$ and $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)\prec\alpha+\xi\frac{1+Az}{1+Bz}+\mu\left(\frac{1+Az}{1+Bz}\right)^{2}+\frac{\beta(A-B)z}{(1+Az)(1+Bz)}$, for $\alpha,\beta,\mu,\xi\in\mathbb{C}$, $\beta\neq0$, $-1\leq B<A\leq1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\prec\frac{1+Az}{1+Bz}$, and $\frac{1+Az}{1+Bz}$ is the best dominant.*

Proof. For $q(z)=\frac{1+Az}{1+Bz}$, $-1\leq B<A\leq1$ in Theorem 2.1 we get the corollary. ■

Corollary 2.3 *Let $m,n\in\mathbb{N}$, $\lambda,l\geq0$. Assume that (2.1) holds. If $f\in\mathcal{A}$ and $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)\prec\alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma}+\frac{2\beta\gamma z}{1-z^{2}}$, for $\alpha,\beta,\mu,\xi\in\mathbb{C}$, $0<\gamma\leq1$, $\beta\neq0$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\prec\left(\frac{1+z}{1-z}\right)^{\gamma}$, and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ is the best dominant.*

Proof. Corollary follows by using Theorem 2.1 for $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $0<\gamma\leq1$. ■

Theorem 2.4 *Let q be analytic and univalent in U such that $q(z)\neq0$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in U. Assume that*

$$Re\left(\frac{\xi}{\beta}q(z)q'(z)+\frac{2\mu}{\beta}q^{2}(z)q'(z)\right)>0,\ \textit{for}\ \xi,\beta,\mu\in\mathbb{C},\ \beta\neq0.\tag{2.5}$$

If $f \in \mathcal{A}$, $\frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$ *is univalent in* U*, where* $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$ *is as defined in (2.2), then*

$$\alpha + \xi q(z) + \mu (q(z))^2 + \frac{\beta z q'(z)}{q(z)} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) \tag{2.6}$$

implies $q(z) \prec \frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}$, $z \in U$*, and* q *is the best subordinant.*

Proof. Consider $p(z) := \frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$.

By setting $\nu(w) := \alpha + \xi w + \mu w^2$ and $\phi(w) := \frac{\beta}{w}$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Since $\frac{\nu'(q(z))}{\phi(q(z))} = \frac{q'(z)q(z)[\xi+2\mu q(z)]}{\beta}$, it follows that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) = Re\left(\frac{\xi}{\beta} q(z) q'(z) + \frac{2\mu}{\beta} q^2(z) q'(z)\right) > 0$, for $\alpha,\beta,\mu \in \mathbb{C}$, $\mu \neq 0$.

By using (2.4) and (2.6) we get $\alpha + \xi q(z) + \mu (q(z))^2 + \frac{\beta z q'(z)}{q(z)} \prec \alpha + \xi p(z) + \mu (p(z))^2 + \frac{\beta z p'(z)}{p(z)}$. Applying Lemma 1.2, we obtain $q(z) \prec p(z) = \frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}$, $z \in U$, and q is the best subordinant. ■

Corollary 2.5 *Let* $m,n \in \mathbb{N}$, $\lambda, l \geq 0$*. Assume that (2.5) holds. If* $f \in \mathcal{A}$, $\frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\alpha + \xi \frac{1+Az}{1+Bz} + \mu \left(\frac{1+Az}{1+Bz}\right)^2 + \frac{\beta(A-B)z}{(1+Az)(1+Bz)} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$, *for* $\alpha,\beta,\xi,\mu \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$*, where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.2), then* $\frac{1+Az}{1+Bz} \prec \frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}$*, and* $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$ in Theorem 2.4 we get the corollary. ■

Corollary 2.6 *Let* $m,n \in \mathbb{N}$, $\lambda, l \geq 0$*. Assume that (2.5) holds. If* $f \in \mathcal{A}$, $\frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\alpha + \xi \left(\frac{1+z}{1-z}\right)^\gamma + \mu \left(\frac{1+z}{1-z}\right)^{2\gamma} + \frac{2\beta\gamma z}{1-z^2} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$, *for* $\alpha,\beta,\mu,\xi \in \mathbb{C}$, $\beta \neq 0$, $0 < \gamma \leq 1$*, where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.2), then* $\left(\frac{1+z}{1-z}\right)^\gamma \prec \frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}$*, and* $\left(\frac{1+z}{1-z}\right)^\gamma$ *is the best subordinant.*

Proof. For $q(z) = \left(\frac{1+z}{1-z}\right)^\gamma$, $0 < \gamma \leq 1$ in Theorem 2.4 we get the corollary. ■

Combining Theorem 2.1 and Theorem 2.4, we state the following sandwich theorem.

Theorem 2.7 *Let* q_1 *and* q_2 *be analytic and univalent in* U *such that* $q_1(z) \neq 0$ *and* $q_2(z) \neq 0$*, for all* $z \in U$*, with* $\frac{zq_1'(z)}{q_1(z)}$ *and* $\frac{zq_2'(z)}{q_2(z)}$ *being starlike univalent. Suppose that* q_1 *satisfies (2.1) and* q_2 *satisfies (2.5). If* $f \in \mathcal{A}$, $\frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$ *is as defined in (2.2) univalent in* U*, then* $\alpha + \xi q_1(z) + \mu (q_1(z))^2 + \frac{\beta z q_1'(z)}{q_1(z)} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) \prec \alpha + \xi q_2(z) + \mu (q_2(z))^2 + \frac{\beta z q_2'(z)}{q_2(z)}$*, for* $\alpha,\beta,\mu,\xi \in \mathbb{C}$, $\beta \neq 0$*, implies* $q_1(z) \prec \frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec q_2(z)$*, and* q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

For $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$, $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 2.8 *Let* $m,n \in \mathbb{N}$, $\lambda, l \geq 0$*. Assume that (2.1) and (2.5) hold. If* $f \in \mathcal{A}$, $\frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\alpha + \xi \frac{1+A_1 z}{1+B_1 z} + \mu \left(\frac{1+A_1 z}{1+B_1 z}\right)^2 + \frac{\beta(A_1-B_1)z}{(1+A_1 z)(1+B_1 z)} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) \prec \alpha + \xi \frac{1+A_2 z}{1+B_2 z} + \mu \left(\frac{1+A_2 z}{1+B_2 z}\right)^2 + \frac{\beta(A_2-B_2)z}{(1+A_2 z)(1+B_2 z)}$*, for* $\alpha,\beta,\mu,\xi \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$*, where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.2), then* $\frac{1+A_1 z}{1+B_1 z} \prec \frac{z^\delta IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec \frac{1+A_2 z}{1+B_2 z}$*, hence* $\frac{1+A_1 z}{1+B_1 z}$ *and* $\frac{1+A_2 z}{1+B_2 z}$ *are the best subordinant and the best dominant, respectively.*

For $q_1(z)=\left(\frac{1+z}{1-z}\right)^{\gamma_1}$, $q_2(z)=\left(\frac{1+z}{1-z}\right)^{\gamma_2}$, where $0<\gamma_1<\gamma_2\le 1$, we have the following corollary.

Corollary 2.9 *Let $m,n\in\mathbb{N}$, $\lambda,l\ge 0$. Assume that (2.1) and (2.5) hold. If $f\in\mathcal{A}$, $\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\in\mathcal{H}[q(0),1]\cap Q$ and $\alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma_1}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma_1}+\frac{2\beta\gamma_1 z}{1-z^2}\prec\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)\prec\alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma_2}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma_2}+\frac{2\beta\gamma_2 z}{1-z^2}$, for $\alpha,\beta,\mu,\xi\in\mathbb{C}$, $\beta\ne 0$, $0<\gamma_1<\gamma_2\le 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\left(\frac{1+z}{1-z}\right)^{\gamma_1}\prec\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\prec\left(\frac{1+z}{1-z}\right)^{\gamma_2}$, hence $\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $\left(\frac{1+z}{1-z}\right)^{\gamma_2}$ are the best subordinant and the best dominant, respectively.*

Changing the functions θ and ϕ we obtain the following results.

Theorem 2.10 *Let $\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\in\mathcal{H}(U)$, $f\in\mathcal{A}$, $z\in U$, $m,n\in\mathbb{N}$, $\lambda,l\ge 0$ and let the function $q(z)$ be convex and univalent in U such that $q(0)=1$, $z\in U$. Assume that*

$$Re\left(\frac{\alpha+\beta}{\beta}+z\frac{q''(z)}{q'(z)}\right)>0, \tag{2.7}$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\ne 0$, $z\in U$, and

$$\psi_{\lambda,l}^{m,n}(\alpha,\beta;z):=\frac{\beta(l+1)}{\lambda}\frac{z^{\delta}IR_{\lambda,l}^{m+2,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}+\left(\alpha+\frac{\beta\delta(l+1)}{\lambda}\right)\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}} \tag{2.8}$$

$$-\frac{\beta(1+\delta)(l+1)}{\lambda}\frac{z^{\delta}\left(IR_{\lambda,l}^{m+1,n}f(z)\right)^2}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{2+\delta}}.$$

If q satisfies the following subordination

$$\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)\prec\alpha q(z)+\beta zq'(z), \tag{2.9}$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\ne 0$, $z\in U$, then $\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\prec q(z)$, $z\in U$, and q is the best dominant.

Proof. Consider $p(z):=\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}$, $z\in U$, $z\ne 0$, $f\in\mathcal{A}$. The function p is analytic in U and $p(0)=1$. Differentiating we get $p'(z)=\frac{\delta(1+l)}{\lambda}\frac{z^{\delta-1}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}+\frac{l+1}{\lambda}\frac{z^{\delta-1}IR_{\lambda,l}^{m+2,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}-\frac{(l+1)(1+\delta)}{\lambda}\frac{z^{\delta-1}\left(IR_{\lambda,l}^{m+1,n}f(z)\right)^2}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{2+\delta}}$.
By using the identity (1.3), we get

$$zp'(z)=\frac{l+1}{\lambda}\frac{z^{\delta}IR_{\lambda,l}^{m+2,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}+\frac{\delta(1+l)}{\lambda}\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}-\frac{(l+1)(1+\delta)}{\lambda}\frac{z^{\delta}\left(IR_{\lambda,l}^{m+1,n}f(z)\right)^2}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{2+\delta}}. \tag{2.10}$$

By setting $\theta(w):=\alpha w$ and $\phi(w):=\beta$, it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\ne 0$, $w\in\mathbb{C}\backslash\{0\}$.

Also, by letting $Q(z)=zq'(z)\phi(q(z))=\beta zq'(z)$, we find that $Q(z)$ is starlike univalent in U.

Let $h(z)=\theta(q(z))+Q(z)=\alpha q(z)+\beta zq'(z)$. We have $Re\left(\frac{zh'(z)}{Q(z)}\right)=Re\left(\frac{\alpha+\beta}{\beta}+z\frac{q''(z)}{q'(z)}\right)>0$.

By using (2.10), we obtain $\alpha p(z)+\beta zp'(z)=\frac{\beta(l+1)}{\lambda}\frac{z^{\delta}IR_{\lambda,l}^{m+2,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}+\left(\alpha+\frac{\beta\delta(l+1)}{\lambda}\right)\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}-\frac{\beta(1+\delta)(l+1)}{\lambda}\frac{z^{\delta}\left(IR_{\lambda,l}^{m+1,n}f(z)\right)^2}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{2+\delta}}$. By using (2.9), we have $\alpha p(z)+\beta zp'(z)\prec\alpha q(z)+\beta zq'(z)$. From Lemma 1.1, we have $p(z)\prec q(z)$, $z\in U$, i.e. $\frac{z^{\delta}IR_{\lambda,l}^{m+1,n}f(z)}{\left(IR_{\lambda,l}^{m,n}f(z)\right)^{1+\delta}}\prec q(z)$, $z\in U$, and q is the best dominant. ■

Corollary 2.11 *Let* $q(z) = \frac{1+Az}{1+Bz}$, $z \in U$, $-1 \le B < A \le 1$, $m, n \in \mathbb{N}$, $\lambda, l \ge 0$. *Assume that (2.7) holds. If* $f \in \mathcal{A}$ *and* $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z) \prec \alpha\frac{1+Az}{1+Bz} + \frac{\beta(A-B)z}{(1+Bz)^2}$, *for* $\alpha, \beta \in \mathbb{C}$, $\beta \ne 0$, $-1 \le B < A \le 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.8), then* $\frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec \frac{1+Az}{1+Bz}$, *and* $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$, in Theorem 2.10 we get the corollary. ■

Corollary 2.12 *Let* $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $m, n \in \mathbb{N}$, $\lambda, l \ge 0$. *Assume that (2.7) holds. If* $f \in \mathcal{A}$ *and* $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z) \prec \alpha\left(\frac{1+z}{1-z}\right)^{\gamma} + \frac{2\beta\gamma z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma}$, *for* $\alpha, \beta \in \mathbb{C}$, $0 < \gamma \le 1$, $\beta \ne 0$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.8), then* $\frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec \left(\frac{1+z}{1-z}\right)^{\gamma}$, *and* $\left(\frac{1+z}{1-z}\right)^{\gamma}$ *is the best dominant.*

Proof. Corollary follows by using Theorem 2.10 for $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $0 < \gamma \le 1$. ■

Theorem 2.13 *Let* q *be convex and univalent in* U *such that* $q(0) = 1$. *Assume that*

$$Re\left(\frac{\alpha}{\beta}q'(z)\right) > 0, \text{ for } \alpha, \beta \in \mathbb{C},\ \beta \ne 0. \tag{2.11}$$

If $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$ *is univalent in* U, *where* $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$ *is as defined in (2.8), then*

$$\alpha q(z) + \beta z q'(z) \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta;z) \tag{2.12}$$

implies $q(z) \prec \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}$, $\delta \in \mathbb{C}$, $\delta \ne 0$, $z \in U$, *and* q *is the best subordinant.*

Proof. Consider $p(z) := \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}$, $z \in U$, $z \ne 0$, $f \in \mathcal{A}$. The function p is analytic in U and $p(0) = 1$.

By setting $\nu(w) := \alpha w$ and $\phi(w) := \beta$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \ne 0$, $w \in \mathbb{C}\backslash\{0\}$.

Since $\frac{\nu'(q(z))}{\phi(q(z))} = \frac{\alpha}{\beta}q'(z)$, it follows that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) = Re\left(\frac{\alpha}{\beta}q'(z)\right) > 0$, for $\alpha, \beta \in \mathbb{C}$, $\beta \ne 0$.

Now, by using (2.12) we obtain $\alpha q(z) + \beta z q'(z) \prec \alpha p(z) + \beta z p'(z)$, $z \in U$. From Lemma 1.2, we have $q(z) \prec p(z) = \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}$, $z \in U$, and q is the best subordinant. ■

Corollary 2.14 *Let* $q(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$, $z \in U$, $m, n \in \mathbb{N}$, $\lambda, l \ge 0$. *Assume that (2.11) holds. If* $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$, *and* $\alpha\frac{1+Az}{1+Bz} + \frac{\beta(A-B)z}{(1+Bz)^2} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$, *for* $\alpha, \beta \in \mathbb{C}$, $\beta \ne 0$, $-1 \le B < A \le 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.8), then* $\frac{1+Az}{1+Bz} \prec \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}$, *and* $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$, in Theorem 2.13 we get the corollary. ■

Corollary 2.15 *Let* $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $m, n \in \mathbb{N}$, $\lambda, l \ge 0$. *Assume that (2.11) holds. If* $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$ *and* $\alpha\left(\frac{1+z}{1-z}\right)^{\gamma} + \frac{2\beta\gamma z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$, *for* $\alpha, \beta \in \mathbb{C}$, $0 < \gamma \le 1$, $\beta \ne 0$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in (2.8), then* $\left(\frac{1+z}{1-z}\right)^{\gamma} \prec \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}}$, *and* $\left(\frac{1+z}{1-z}\right)^{\gamma}$ *is the best subordinant.*

Proof. Corollary follows by using Theorem 2.13 for $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $0 < \gamma \le 1$. ■

Combining Theorem 2.10 and Theorem 2.13, we state the following sandwich theorem.

Theorem 2.16 *Let q_1 and q_2 be convex and univalent in U such that $q_1(z) \neq 0$ and $q_2(z) \neq 0$, for all $z \in U$. Suppose that q_1 satisfies (2.7) and q_2 satisfies (2.11). If $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$, and $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$ is as defined in (2.8) univalent in U, then $\alpha q_1(z) + \beta z q_1'(z) \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta;z) \prec \alpha q_2(z) + \beta z q_2'(z)$, for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, implies $q_1(z) \prec \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec q_2(z)$, $z \in U$, and q_1 and q_2 are respectively the best subordinant and the best dominant.*

For $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$, $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 2.17 *Let $m,n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.7) and (2.11) hold for $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$ and $q_2(z) = \frac{1+A_2 z}{1+B_2 z}$, respectively. If $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$ and $\alpha \frac{1+A_1 z}{1+B_1 z} + \frac{\beta(A_1-B_1)z}{(1+B_1 z)^2} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$ $\prec \alpha \frac{1+A_2 z}{1+B_2 z} + \frac{\beta(A_2-B_2)z}{(1+B_2 z)^2}$, $z \in U$, for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\frac{1+A_1 z}{1+B_1 z} \prec \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec \frac{1+A_2 z}{1+B_2 z}$, $z \in U$, hence $\frac{1+A_1 z}{1+B_1 z}$ and $\frac{1+A_2 z}{1+B_2 z}$ are the best subordinant and the best dominant, respectively.*

For $q_1(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_1}$, $q_2(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, where $0 < \gamma_1 < \gamma_2 \leq 1$, we have the following corollary.

Corollary 2.18 *Let $m,n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.7) and (2.11) hold for $q_1(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $q_2(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, respectively. If $f \in \mathcal{A}$, $\frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \in \mathcal{H}[q(0),1] \cap Q$ and $\alpha \left(\frac{1+z}{1-z}\right)^{\gamma_1} + \frac{2\beta\gamma_1 z}{1-z^2} \left(\frac{1+z}{1-z}\right)^{\gamma_1} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$ $\prec \alpha \left(\frac{1+z}{1-z}\right)^{\gamma_2} + \frac{2\beta\gamma_2 z}{1-z^2} \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, $z \in U$, for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $0 < \gamma_1 < \gamma_2 \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\left(\frac{1+z}{1-z}\right)^{\gamma_1} \prec \frac{z^{\delta} IR_{\lambda,l}^{m+1,n} f(z)}{\left(IR_{\lambda,l}^{m,n} f(z)\right)^{1+\delta}} \prec \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, $z \in U$, hence $\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $\left(\frac{1+z}{1-z}\right)^{\gamma_2}$ are the best subordinant and the best dominant, respectively.*

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FUZZY STABILITY OF A CLASS OF ADDITIVE-QUADRATIC FUNCTIONAL EQUATIONS

CHANG IL KIM AND GILJUN HAN*

Abstract. In this paper, we consider the following functional equation

$$\begin{aligned} &af(x+y)+bf(x-y)+cf(y-x)\\ &=(a+b)f(x)+cf(-x)+(a+c)f(y)+bf(-y) \end{aligned}$$

for a fixed real numbers a, b, c with $a = b + c$ and $a \neq 0$. We study the fuzzy version of the generalized Hyers-Ulam stability for it in the sense of Mirmostafaee and Moslehian.

1. Introduction and preliminaries

In 1940, Ulam proposed the following stability problem (cf. [20]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exists a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [11] gave a partial solution of Ulam's problem for the case of approximate additive mappings. Subsequently, his result was generalized by Aoki [1] for additive mappings, and by Rassias [19] for linear mappings, to consider the stability problem with unbounded Cauchy differences. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians ([5], [6], [7], [10], [18]).

Recently, the stability in fuzzy spaces has been extensively studied ([3], [12], [15], [16], [17]). The concept of fuzzy norm on a linear space was introduced by Katsaras [14] in 1984. Later, Cheng and Mordeson [4] gave a new definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [13]. In 2008, for the first time, Mirmostafaee and Moslehian [16], [17] used the definition of a fuzzy norm in [2] to obtain a fuzzy version of stability for the Cauchy functional equation

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

and the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.2}$$

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We call a solution of (1.1) *an additive mapping* and a solution of (1.2) is called *a quadratic mapping*. Also,

$$f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)=0$$

is called *Drygas functional equation*(see [8], [9] for detail.). It is easy to see that the function $f(x)=px^2+qx$ is a solution of Drygas functional equation and so we can expect that a solution of Drygas functional equation is an additive-quadratic mapping.

Now, we consider the following functional equation

$$\begin{aligned} &af(x+y)+bf(x-y)+cf(y-x)\\ &=(a+b)f(x)+cf(-x)+(a+c)f(y)+bf(-y) \end{aligned} \tag{1.3}$$

for fixed real numbers a,b,c with $a=b+c$ and $a\neq 0$ and show the generalized Hyers-Ulam stability of (1.3) in a fuzzy sense [18].

Definition 1.1. Let X be a real vector space. A function $N:X\times\mathbb{R}\longrightarrow[0,1]$ is called *a fuzzy norm on* X if for all $x,y\in X$ and all $s,t\in\mathbb{R}$,

(N1) $N(x,t)=0$ for $t\leq 0$;
(N2) $x=0$ if and only if $N(x,t)=1$ for all $t>0$;
(N3) $N(cx,t)=N(x,\frac{t}{|c|})$ if $c\neq 0$;
(N4) $N(x+y,s+t)\geq\min\{N(x,s),N(y,t)\}$;
(N5) $N(x,\cdot)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty}N(x,t)=1$;
(N6) for any $x\neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

In this case, the pair (X,N) is called *a fuzzy normed space*.

Let (X,N) be a fuzzy normed space. A sequence $\{x_n\}$ in X is said to be *convergent* in (X,N) if there exists an $x\in X$ such that $\lim_{n\to\infty}N(x_n-x,t)=1$ for all $t>0$. In this case, x is called *the limit of the sequence* $\{x_n\}$ *in* (X,N) and one denotes it by $N-\lim_{n\to\infty}x_n=x$. A sequence $\{x_n\}$ in X is said to be *Cauchy* if for any $\epsilon>0$, there is an $m\in N$ such that for any $n\geq m$ and any positive integer p, $N(x_{n+p}-x_n,t)>1-\epsilon$ for all $t>0$.

It is well known that every convergent sequence in a fuzzy normed space is Cauchy. A fuzzy normed space is said to be *complete* if each Cauchy sequence in it is convergent and a complete fuzzy normed space is called *a fuzzy Banach space*.

2. Solutions and the Generalized Hyers-Ulam stability of (1.3)

In this section, we investigate solutions of (1.3) and prove the generalized Hyers-Ulam stability of (1.3) in fuzzy Banach spaces. Throughout this section, we assume that (X,N) is a fuzzy normed space and (Y,N') is a fuzzy Banach space. In Theorem 2.3, it can be concluded that any solution of (1.3) is additive-quadratic. We start with the following lemma.

Lemma 2.1. *Let $f:X\longrightarrow Y$ be an odd mapping satisfying (1.3). Then f is an additive mapping.*

Proof. Since $a\neq 0$, $f(0)=0$. Since f is an odd mapping, the functional equation (1.3) can be written by

$$af(x+y)+(b-c)f(x-y)=(a+b-c)f(x)+(a-b+c)f(y) \tag{2.1}$$

for all $x, y \in X$. Interchanging x and y in (2.1), we have

$$af(x+y) - (b-c)f(x-y) = (a+b-c)f(y) + (a-b+c)f(x) \tag{2.2}$$

for all $x, y \in X$. By (2.1) and (2.2),

$$af(x+y) = af(x) + af(y)$$

for all $x, y \in X$ and since $a \neq 0$, f is additive. □

Lemma 2.2. *Let $f : X \longrightarrow Y$ be an even mapping satisfying (1.3). Then f is a quadratic mapping.*

Proof. Since $a \neq 0$, $f(0) = 0$. Since f is an even mapping, the functional equation (1.3) can be written by

$$af(x+y) + (b+c)f(x-y) = (a+b+c)f(x) + (a+b+c)f(y) \tag{2.3}$$

for all $x, y \in X$. Letting $y = -y$ in (2.3), we have

$$af(x-y) + (b+c)f(x+y) = (a+b+c)f(x) + (a+b+c)f(y) \tag{2.4}$$

for all $x, y \in X$. Since $a = b + c$, by (2.3) and (2.4), we have

$$2af(x-y) + 2af(x+y) = 4af(x) + 4af(y)$$

for all $x, y \in X$ and since $a \neq 0$, f is a quadratic mapping. □

Combining Lemma 2.1 and Lemma 2.2, we have the following theorem.

Theorem 2.3. *Let $f : X \longrightarrow Y$ be a mapping. If f satisfies (1.3), then f is an additive-quadratic mapping.*

For any mapping $f : X \longrightarrow Y$, we define the difference operator $Df : X^2 \longrightarrow Y$ by

$$Df(x,y) = af(x+y)+bf(x-y)+cf(y-x)-(a+b)f(x)-cf(-x)-(a+c)f(y)-bf(-y)$$

for all $x, y \in X$. For a given $q > 0$, the mapping f is said to be *a fuzzy q-almost additive-quadratic mapping if*

$$N'(Df(x,y), t+s) \geq \min\{N(x, t^q), N(y, s^q)\} \tag{2.5}$$

for all $x, y \in X$ and all positive real numbers t, s.

Theorem 2.4. *Let q be a positive real number with $q \neq 1$, $\frac{1}{2}$ and $f : X \longrightarrow Y$ a fuzzy q-almost additive-quadratic mapping. Then there exists a unique additive-quadratic mapping $F : X \longrightarrow Y$ such that*

(2.6)

$$N(F(x)-f(x), t) \geq \begin{cases} \sup_{s<t}\{N(x, (1-2^{p-1})^q|a|^q s^q)\}, & \text{if } q > 1 \\ \sup_{s<t}\{N(x, (2^{p-1}-1)^q(2-2^{(p-1)})^q|a|^q s^q)\}, & \text{if } \frac{1}{2} < q < 1 \\ \sup_{s<t}\{N(x, (2^{p-1}-2)^q|a|^q s^q\}, & \text{if } 0 < q < \frac{1}{2} \end{cases}$$

holds for all $x \in X$ and all $t > 0$, where $p = \frac{1}{q}$.

Proof. By (2.5), (N2), and (N4), since $a = b + c$, we have

$$N'(Df(0,0),\ t) = N'(f(0),\ \frac{t}{2|a|}) \geq N'(0,\ t^q) = 1$$

for all $t > 0$ and by (N2), $f(0) = 0$.

Case 1. Let $q > 1$ and define a mapping $J_n f : X \longrightarrow Y$ by

$$J_n f(x) = \frac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n} + \frac{f(2^n x) - f(-2^n x)}{2 \cdot 2^n}$$

for all $x \in X$ and all positive integer n. Then we have

$$\begin{aligned} &J_n f(x) - J_{n+1} f(x) \\ &= \frac{2^{n+1} - 1}{a \cdot 2 \cdot 4^{n+1}} Df(-2^n x, -2^n x) - \frac{2^{n+1} + 1}{a \cdot 2 \cdot 4^{n+1}} Df(2^n x, 2^n x) \end{aligned} \tag{2.7}$$

for all $x \in X$ and all positive integer n. By (2.5), (2.7), (N3), and (N4), we have

(2.8)

$$\begin{aligned} &N'(J_m f(x) - J_{m+n} f(x), \sum_{i=m}^{m+n-1} \frac{2^{pi}}{|a| \cdot 2^i} t^p) \\ &= N'(\sum_{i=m}^{m+n-1} [J_i f(x) - J_{i+1} f(x)], \sum_{i=m}^{m+n-1} \frac{2^{pi}}{|a| \cdot 2^i} t^p) \\ &\geq \min\{N'(J_i f(x) - J_{i+1} f(x), \frac{2^{pi}}{|a| \cdot 2^i} t^p) \ \mid\ m \leq i \leq m+n-1\} \\ &\geq \min\{N'(\frac{2^{i+1} - 1}{a \cdot 2 \cdot 4^{i+1}} Df(-2^i x, -2^i x) - \frac{2^{i+1} + 1}{a \cdot 2 \cdot 4^{i+1}} Df(2^i x, 2^i x), \frac{2^{pi}}{|a| \cdot 2^i} t^p) \ \mid \\ &\quad m \leq i \leq m+n-1\} \\ &\geq \min\{\min\{N'(\frac{2^{i+1} + 1}{a \cdot 2 \cdot 4^{i+1}} Df(2^i x, 2^i x), \frac{(2^{i+1} + 1) 2^{pi}}{|a| \cdot 4^{i+1}} t^p), \\ &\quad N'(\frac{2^{i+1} - 1}{a \cdot 2 \cdot 4^{i+1}} Df(-2^i x, -2^i x), \frac{(2^{i+1} - 1) 2^{pi}}{|a| \cdot 4^{i+1}} t^p)\} \ \mid\ m \leq i \leq m+n-1\} \\ &\geq \min\{\min\{N'(Df(2^i x, 2^i x), 2^{pi+1} t^p), N'(Df(-2^i x, -2^i x), 2^{pi+1} t^p)\} | m \leq i \leq m+n-1\} \\ &\geq \min\{\min\{N(2^i x, 2^i t), N(-2^i x, 2^i t)\} \ \mid\ m \leq i \leq m+n-1\} \\ &= N(x, t) \end{aligned}$$

for all $x \in X$, all $t > 0$, and all positive integers m, n. Let $\epsilon > 0$ be given. Since $\lim_{t \longrightarrow \infty} N(x, t) = 1$, there is a t_1 such that $N(x, t_1) > 1 - \epsilon$. Let $t_2 > t_1$. Since $p < 1$, $\sum_{n=0}^{\infty} \frac{2^{pn}}{|a| \cdot 2^n} t_2^p$ is convergent. Let $s > 0$. Then there is a positive integer k such that $\sum_{i=m}^{m+n-1} \frac{2^{pi}}{|a| \cdot 2^i} t_2^p < s$ for $m, n > k$ and so by (2.8), we have

$$\begin{aligned}
&N'(J_mf(x) - J_{m+n}f(x), s)\\
&\geq N'(J_mf(x) - J_{m+n}f(x), \sum_{i=m}^{m+n-1} \frac{2^{pi}}{|a|\cdot 2^i} t_2^p)\\
&\geq N(x, t_2)\\
&\geq 1-\epsilon
\end{aligned}$$

for all $x \in X$. Hence $\{J_nf(x)\}$ is a Cauchy sequence in (Y, N'). Since (Y, N') is a fuzzy Banach space, we can define a mapping $F : X \longrightarrow Y$ by

$$F(x) = N' - \lim_{n\to\infty} J_nf(x)$$

for all $x \in X$. Letting $m = 0$ in (2.8), we have

$$N'(f(x) - J_nf(x), t) \geq N(x, \frac{t^q}{[\sum_{i=0}^{n-1} \frac{2^{pi}}{|a|\cdot 2^i}]^q}) \tag{2.9}$$

for all $x \in X$, all positive integer n, and all $t > 0$. By (N4), we have

$$\begin{aligned}
&N'(DF(x, y), t)\\
&\geq \min\{N'(a[F - J_nf](x + y), \frac{t}{14}), N'(b[F - J_nf](x - y), \frac{t}{14}),\\
&N'(c[F - J_nf](y - x), \frac{t}{14}), N'((a + b)[F - J_nf](x), \frac{t}{14})\\
&- N'(c[F - J_nf](-x), \frac{t}{14}), N'((a + c)[F - J_nf](y), \frac{t}{14})\\
&- N'(b[F - J_nf](-y), \frac{t}{14}), N'(J_nDf(x, y), \frac{t}{2})\}
\end{aligned} \tag{2.10}$$

for all $x, y \in X$ and all positive integer n. The first seven terms on the right-hand of (2.10) tend to 1 as $n \to \infty$ and by (N4), we have

$$\begin{aligned}
&N'(J_nDf(x, y), \frac{t}{2})\\
&\geq \min\{N'(\frac{Df(-2^nx, -2^ny)}{2\cdot 4^n}, \frac{t}{8}), N'(\frac{Df(2^nx, 2^ny)}{2\cdot 4^n}, \frac{t}{8}),\\
&N'(\frac{Df(-2^nx, -2^ny)}{2\cdot 2^n}, \frac{t}{8}), N'(\frac{Df(2^nx, 2^ny)}{2\cdot 2^n}, \frac{t}{8})\}
\end{aligned} \tag{2.11}$$

for all $x, y \in X$, all positive integer n and all $t > 0$. By (N3) and (2.5), we have

$$\begin{aligned}
&N'(\frac{Df(\pm 2^nx, \pm 2^ny)}{2\cdot 4^n}, \frac{t}{8})\\
&= N'(Df(\pm 2^nx, \pm 2^ny, 4^{n-1}t))\\
&\geq \min\{N(2^nx, 2^{q(2n-3)}t^q), N(2^ny, 2^{q(2n-3)}t^q)\}\\
&\geq \min\{N(x, 2^{(2q-1)n-3q}t^q), N(y, 2^{(2q-1)n-3q}t^q)\}
\end{aligned} \tag{2.12}$$

for all $x, y \in X$, all positive integer n, and all $t > 0$. Since $q > 1$, by (2.11) and (2.12), we have

$$\lim_{n\to\infty} N'(J_nDf(x, y), \frac{t}{2}) = 1$$

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and so by (2.10), $N'(DF(x,y),t) = 0$ for all $x, y \in X$ and all $t > 0$. By (N2), $DF(x,y) = 0$ for all $x, y \in X$ and by Theorem 2.3, F is additive-quadraic.

Now we will show that (2.6) holds. Let $x \in X$, $t > 0$, $s > 0$ with $0 < s < t$ and $0 < \epsilon < 1$. Since $F(x) = N' - \lim_{n\to\infty} J_n f(x)$, there is a positive integer n such that

$$N'(F(x) - J_n f(x), t - s) \geq 1 - \epsilon$$

and so by (2.9),

$$\begin{aligned}
&N'(F(x) - f(x), t)\\
&\geq \min\{N'(F(x) - J_n f(x), t - s), N'(J_n f(x) - f(x), s)\}\\
&\geq \min\{1 - \epsilon, N(x, \frac{s^q}{[\sum_{i=0}^{n-1} \frac{2^{pi}}{|a|\cdot 2^i}]^q})\}\\
&\geq \min\{1 - \epsilon, N(x, (1 - 2^{p-1})^q s^q |a|^q)\}.
\end{aligned}$$

and so we have (2.6).

To prove the uniqueness of F, let $F_1 : X \longrightarrow Y$ be another additive-quadratic mapping satisfying (2.6). Then

$$F(x) - F_1(x) = J_n F(x) - J_n F_1(x)$$

for all $x \in X$ and all positive integer n. Hence by (N4), (N5), and (2.6), we have

$$\begin{aligned}
&N'(F(x) - F_1(x), t)\\
&= N'(J_n F(x) - J_n F_1(x), t)\\
&\geq \min\{N'(J_n F(x) - J_n f(x), \frac{t}{2}), N'(J_n F_1(x) - J_n f(x), \frac{t}{2})\}\\
&\geq \min\{N'(\frac{F(2^n x) - f(2^n x)}{2\cdot 4^n}, \frac{t}{8}), N'(\frac{F(-2^n x) - f(-2^n x)}{2\cdot 4^n}, \frac{t}{8}),\\
&\quad N'(\frac{F(2^n x) - f(2^n x)}{2\cdot 2^n}, \frac{t}{8}), N'(\frac{F(-2^n x) - f(-2^n x)}{2\cdot 2^n}, \frac{t}{8}),\\
&\quad N'(\frac{F_1(2^n x) - f(2^n x)}{2\cdot 4^n}, \frac{t}{8}), N'(\frac{F_1(-2^n x) - f(-2^n x)}{2\cdot 4^n}, \frac{t}{8}),\\
&\quad N'(\frac{F_1(2^n x) - f(2^n x)}{2\cdot 2^n}, \frac{t}{8}), N'(\frac{F_1(-2^n x) - f(-2^n x)}{2\cdot 2^n}, \frac{t}{8})\}\\
&\geq \sup_{s<t}\{N(2^n x, (1 - 2^{p-1})^q 2^{(n-3)q} s^q |a|^q)\}\\
&\geq \sup_{s<t}\{N(x, (1 - 2^{p-1})^q |a|^q s^q 2^{(q-1)n-3q})\}
\end{aligned}$$

for all $x, y \in X$, all positive integer n and all $0 < s < t$. Since $q > 1$,

$$\lim_{n\to\infty} \sup_{s<t}\{N(x, (1 - 2^{p-1})^q |a|^q s^q 2^{(q-1)n-3q}\} = 1$$

and so $N'(F(x) - F_1(x), t) = 1$ for all $t > 0$. Hence $F = F_1$.

Case 2. Let $\frac{1}{2} < q < 1$ and define a mapping $J_n f : X \longrightarrow Y$ by

$$J_n f(x) = \frac{f(2^n x) + f(-2^n x)}{2\cdot 4^n} + \frac{2^n}{2}[f(2^{-n} x) - f(-2^{-n} x)]$$

for all $x \in X$ and all positive integer n. Then we have

$$\begin{aligned} &J_n f(x) - J_{n+1} f(x) \\ &= \frac{2^n}{2\cdot a} Df(2^{-(n+1)}x, 2^{-(n+1)}x) - \frac{2^n}{2\cdot a} Df(-2^{-(n+1)}x, -2^{-(n+1)}x) \\ &\quad - \frac{1}{a\cdot 2\cdot 4^{n+1}} Df(2^n x, 2^n x) - \frac{1}{a\cdot 2\cdot 4^{n+1}} Df(-2^n x, -2^n x) \end{aligned} \tag{2.13}$$

for all $x \in X$ and all positive integer n. By (2.5), (2.13), (N3), and (N4), we have

(2.14)

$$\begin{aligned} &N'(J_m f(x) - J_{m+n} f(x), \sum_{i=m}^{m+n-1} [\frac{2^{pi+1}}{|a|\cdot 4^{i+1}} + \frac{2^{1-p(i+1)+i}}{|a|}]t^p) \\ &= N'(\sum_{i=m}^{m+n-1} [J_i f(x) - J_{i+1} f(x)], \sum_{i=m}^{m+n-1} [\frac{2^{pi+1}}{|a|\cdot 4^{i+1}} + \frac{2^{1-p(i+1)+i}}{|a|}]t^p) \\ &\geq \min\{N'(J_i f(x) - J_{i+1} f(x), [\frac{2^{pi+1}}{|a|\cdot 4^{i+1}} + \frac{2^{1-p(i+1)+i}}{|a|}]t^p) \mid m \leq i \leq m+n-1\} \\ &\geq \min\{N'(\frac{1}{a\cdot 2\cdot 4^{i+1}} Df(2^i x, 2^i x) + \frac{1}{a\cdot 2\cdot 4^{i+1}} Df(-2^i x, -2^i x) \\ &\quad - \frac{2^i}{2\cdot a} Df(2^{-(i+1)}x, 2^{-(i+1)}x) + \frac{2^i}{2\cdot a} Df(-2^{-(i+1)}x, -2^{-(i+1)}x), \\ &\quad \frac{2^{pi+1}}{|a|\cdot 4^{i+1}} t^p + \frac{2^{1-p(i+1)+i}}{|a|} t^p) \mid m \leq i \leq m+n-1\} \\ &\geq \min\{\min\{N'(\frac{1}{a\cdot 2\cdot 4^{i+1}} Df(2^i x, 2^i x), \frac{2^{pi+1}}{|a|\cdot 2\cdot 4^{i+1}} t^p), \\ &\quad N'(\frac{1}{a\cdot 2\cdot 4^{i+1}} Df(-2^i x, -2^i x), \frac{2^{pi+1}}{|a|\cdot 2\cdot 4^{i+1}} t^p), \\ &\quad N'(\frac{2^i}{2\cdot a} Df(2^{-(i+1)}x, 2^{-(i+1)}x), \frac{2^{1-p(i+1)+i}}{2\cdot |a|} t^p), \\ &\quad N'(\frac{2^i}{2\cdot a} Df(-2^{-(i+1)}x, -2^{-(i+1)}x), \frac{2^{1-p(i+1)+i}}{2\cdot |a|} t^p)\} \mid m \leq i \leq m+n-1\} \\ &\quad \geq \min\{\min\{N'(Df(2^i x, 2^i x), 2^{pi+1} t^p), N'(Df(-2^i x, -2^i x), 2^{pi+1} t^p), \\ &\quad N'(Df(2^{-(i+1)}x, 2^{-(i+1)}x), 2^{1-p(i+1)} t^p), N'(Df(-2^{-(i+1)}x, -2^{-(i+1)}x), 2^{1-p(i+1)} t^p)\} \mid \\ &\quad m \leq i \leq m+n-1\}] \\ &\geq \min\{\min\{N(2^i x, 2^i t), N(-2^i x, 2^i t), N(2^{-(i+1)}x, 2^{-(i+1)}t), \\ &\quad N(-2^{-(i+1)}x, 2^{-(i+1)}t)\} \mid m \leq i \leq m+n-1\} \\ &= N(x, t) \end{aligned}$$

for all $x \in X$, all $t > 0$, and all positive integers m, n. Let $\epsilon > 0$ be given. Since $\lim_{t\longrightarrow\infty} N(x,t) = 1$, there is a t_1 such that $N(x,t_1) > 1-\epsilon$. Let $t_2 > t_1$. Since $1 < p < 2$, $\sum_{n=0}^{\infty}[\frac{2^{pn+1}}{|a|\cdot 4^{n+1}} + \frac{2^{1-p(n+1)+n}}{|a|}]t_2^p$ is convergent. Let $s > 0$. Then there is a positive integer n such that $\sum_{i=m}^{m+n-1}[\frac{2^{pi+1}}{|a|\cdot 4^{i+1}} + \frac{2^{1-p(i+1)+i}}{|a|}]t_2^p < s$ for $m, n > k$ and so by (2.14), we have

$$\begin{aligned}
&N'(J_mf(x) - J_{m+n}f(x), s)\\
&\geq N'(J_mf(x) - J_{m+n}f(x), \sum_{i=m}^{m+n-1} [\frac{2^{pi+1}}{|a| \cdot 4^{i+1}} + \frac{2^{1-p(i+1)+i}}{|a|}]t_2^p)\\
&\geq N(x, t_2)\\
&\geq 1 - \epsilon
\end{aligned}$$

for all $x \in X$. Hence $\{J_nf(x)\}$ is a Cauchy sequence in (Y, N'). Since (Y, N') is a fuzzy Banach space, we can define a mapping $F : X \longrightarrow Y$ by

$$F(x) = N' - \lim_{n\to\infty} J_nf(x)$$

for all $x \in X$. Letting $m = 0$ in (2.14), we have

$$N'(f(x) - J_nf(x), t) \geq N(x, \frac{t^q}{[\sum_{i=0}^{n-1}(\frac{2^{pi+1}}{|a|\cdot 4^{i+1}} + \frac{2^{1-p(i+1)+i}}{|a|})]^q}) \tag{2.15}$$

for all $x \in X$, all positive integer n, and all $t > 0$. By (N4), we have

$$\begin{aligned}
&N'(DF(x, y), t)\\
&\geq \min\{N'(a[F - J_nf](x + y), \frac{t}{14}), N'(b[F - J_nf](x - y), \frac{t}{14}),\\
&N'(c[F - J_nf](y - x), \frac{t}{14}), N'((a + b)[F - J_nf](x), \frac{t}{14})\\
&- N'(c[F - J_nf](-x), \frac{t}{14}), N'((a + c)[F - J_nf](y), \frac{t}{14})\\
&- N'(b[F - J_nf](-y), \frac{t}{14}), N'(J_nDf(x, y), \frac{t}{2})\}
\end{aligned} \tag{2.16}$$

for all $x, y \in X$ and all positive integer n. The first seven terms on the right-hand of (2.16) tend to 1 as $n \to \infty$ and by (N4), we have

$$\begin{aligned}
&N'(J_nDf(x, y), \frac{t}{2})\\
&\geq \min\{N'(\frac{Df(-2^nx, -2^ny)}{2 \cdot 4^n}, \frac{t}{8}), N'(\frac{Df(2^nx, 2^ny)}{2 \cdot 4^n}, \frac{t}{8}),\\
&N'(2^{n-1}Df(2^{-n}x, 2^{-n}y), \frac{t}{8}), N'(2^{n-1}Df(-2^{-n}x, -2^{-n}y), \frac{t}{8})\}
\end{aligned} \tag{2.17}$$

for all $x, y \in X$, all positive integer n and all $t > 0$. By (N3) and (2.5), we have

$$\begin{aligned}
&N'(\frac{Df(\pm 2^nx, \pm 2^ny)}{2 \cdot 4^n}, \frac{t}{8})\\
&\geq \min\{N(x, 2^{(2q-1)n-3q}t^q), N(y, 2^{(2q-1)n-3q}t^q)\}
\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}
&N'(2^{n-1}Df(\pm 2^{-n}x, \pm 2^{-n}y), \frac{t}{8})\\
&\geq \min\{N(x, 2^{(1-q)n-3q)}t^q), N(y, 2^{(1-q)n-3q)}t^q)\}
\end{aligned} \tag{2.19}$$

for all $x, y \in X$, all positive integer n, and all $t > 0$. Since $\frac{1}{2} < q < 1$, by (2.17), (2.18), and (2.19), we have

$$\lim_{n\to\infty} N'(J_nDf(x, y), \frac{t}{2}) = 1$$

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and so by (2.16), $N'(DF(x,y),t) = 0$ for all $x, y \in X$ and all $t > 0$. By (N2), $DF(x,y) = 0$ for all $x, y \in X$ and by Theorem 2.3, F is additive-quadratic.

Now we will show that (2.6) holds. Let $x \in X$, $t > 0$, $s > 0$ with $0 < s < t$ and $0 < \epsilon < 1$. Since $F(x) = N' - \lim_{n\to\infty} J_n f(x)$, there is a positive integer n such that

$$N'(F(x) - J_n f(x), t - s) \geq 1 - \epsilon$$

and so by (2.15),

$$\begin{aligned}
&N'(F(x) - f(x), t)\\
&\geq \min\{N'(F(x) - J_n f(x), t - s), N'(J_n f(x) - f(x), s)\}\\
&\geq \min\{1 - \epsilon, N(x, \frac{s^q}{[\sum_{i=0}^{n-1}(\frac{2^{pi+1}}{|a|\cdot 4^{i+1}} + \frac{2^{1-p(i+1)+i}}{|a|})]^q})\}\\
&\geq \min\{1 - \epsilon, N(x, (2^{p-1} - 1)^q (2 - 2^{p-1})^q |a|^q s^q)\}.
\end{aligned}$$

and so we have (2.6).

To prove the uniqueness of F, let $F_1 : X \longrightarrow Y$ be another additive-quadratic mapping satisfying (2.6). Then

$$F(x) - J_n F(x) = F_1(x) - J_n F_1(x)$$

for all $x \in X$ and all positive integer n. Hence by (N4), (N5), and (2.6), we have

$$\begin{aligned}
&N'(F(x) - F_1(x), t)\\
&= N'(J_n F(x) - J_n F_1(x), t)\\
&\geq \min\{N'(J_n F(x) - J_n f(x), \frac{t}{2}), N'(J_n F_1(x) - J_n f(x), \frac{t}{2})\}\\
&\geq \min\{N'(\frac{F(2^n x) - f(2^n x)}{2\cdot 4^n}, \frac{t}{8}), N'(\frac{F(-2^n x) - f(-2^n x)}{2\cdot 4^n}, \frac{t}{8}),\\
&\quad N'(2^{n-1}[F(2^{-n}x) - f(2^{-n}x)], \frac{t}{8}), N'(2^{n-1}[F(-2^{-n}x) - f(-2^{-n}x)], \frac{t}{8}),\\
&\quad N'(\frac{F_1(2^n x) - f(2^n x)}{2\cdot 4^n}, \frac{t}{8}), N'(\frac{F_1(-2^n x) - f(-2^n x)}{2\cdot 4^n}, \frac{t}{8}),\\
&\quad N'(2^{n-1}[F_1(2^{-n}x) - f(2^{-n}x)], \frac{t}{8}), N'(2^{n-1}[F_1(-2^{-n}x) - f(-2^{-n}x)], \frac{t}{8})\}\\
&\geq \sup_{s<t}\{N(\pm 2^n x, (2^{p-1} - 1)^q (2 - 2^{p-1})^q 4^{(n-1)q} |a|^q s^q)\}\\
&\geq \sup_{s<t}\{N(x, (2^{p-1} - 1)^q (2 - 2^{p-1})^q 2^{(2q-1)n-2q} |a|^q s^q)\}
\end{aligned}$$

for all $x, y \in X$, all positive integer n and all $t > 0$. Since $\frac{1}{2} < q < 1$, $N'(F(x) - F_1(x), t) = 1$ for all $t > 0$. Hence $F = F_1$.

Case 3. Let $0 < q < \frac{1}{2}$ and define a mapping $J_n f : X \longrightarrow Y$ by

$$J_n f(x) = 2^{2n-1}[f(2^{-n}x) + f(-2^{-n}x)] + 2^{n-1}[f(2^{-n}x) - f(-2^{-n}x)]$$

for all $x \in X$ and all positive integer n. Then we have

(2.20)

$$\begin{aligned}
&J_n f(x) - J_{n+1} f(x)\\
&= \frac{2^{2n-1} + 2^{n-1}}{a} Df(2^{-(n+1)}x, 2^{-(n+1)}x) + \frac{2^{2n-1} - 2^{n-1}}{a} Df(-2^{-(n+1)}x, -2^{-(n+1)}x)
\end{aligned}$$

for all $x \in X$ and all positive integer n. By (2.5), (2.20), (N3), and (N4), we have

$$
\begin{aligned}
& N'(J_m f(x) - J_{m+n} f(x), \sum_{i=m}^{m+n-1} \frac{2^{1-p(i+1)+2i}}{|a|} t^p) \\
&= N'(\sum_{i=m}^{m+n-1} [J_i f(x) - J_{i+1} f(x)], \sum_{i=m}^{m+n-1} \frac{2^{1-p(i+1)+2i}}{|a|} t^p) \\
&\geq \min\{N'(J_i f(x) - J_{i+1} f(x), \frac{2^{1-p(i+1)+2i}}{|a|} t^p) \mid m \leq i \leq m+n-1\} \\
&\geq \min\{N'(\frac{2^{2i-1}+2^{i-1}}{a} Df(2^{-(i+1)}x, 2^{-(i+1)}x) \\
&\quad + \frac{2^{2i-1}-2^{i-1}}{a} Df(-2^{-(i+1)}x, -2^{-(i+1)}x), \frac{2^{1-p(i+1)+2i}}{|a|} t^p)\} \mid m \leq i \leq m+n-1\} \\
&\geq \min\{\min\{N'(\frac{2^{2i-1}+2^{i-1}}{a} Df(2^{-(i+1)}x, 2^{-(i+1)}x), \frac{2^{2i-1}+2^{i-1}}{|a|} 2^{1-p(i+1)} t^p), \\
&\quad N'(\frac{2^{2i-1}-2^{i-1}}{a} Df(-2^{-(i+1)}x, -2^{-(i+1)}x), \frac{2^{2i-1}-2^{i-1}}{|a|} 2^{1-p(i+1)} t^p)\} \\
&\quad \mid m \leq i \leq m+n-1\} \\
&\geq \min\{\min\{N'(Df(2^{-(i+1)}x, 2^{-(i+1)}x), 2^{1-p(i+1)} t^p), \\
&\quad N'(Df(-2^{-(i+1)}x, -2^{-(i+1)}x), 2^{1-p(i+1)} t^p)\} \mid m \leq i \leq m+n-1\} \\
&\geq \min\{\min\{N(2^{-(i+1)}x, 2^{-(i+1)}t), N(-2^{-(i+1)}x, 2^{-(i+1)}t)\} \mid m \leq i \leq m+n-1\} \\
&= N(x,t)
\end{aligned}
$$

for all $x \in X$, all $t > 0$, and all positive integers m, n. Similar to **Case 1.** and **Case 2.**, there is a unique cubic mapping $C : X \longrightarrow Y$ with (2.6). □

We can use Theorem 2.4 to get a classical result in the framework of normed spaces. For example, it is well known that for any normed space $(X, ||\cdot||)$, the mapping $N_X : X \times \mathbb{R} \longrightarrow [0,1]$, defined by

$$
N_X(x,t) = \begin{cases} 0, & \text{if } t < \|x\| \\ 1, & \text{if } t \geq \|x\| \end{cases}
$$

a fuzzy norm on X. In [15], [16] *and* [17], *some examples are provided for the fuzzy norm N_X. Here using the fuzzy norm N_X, we have the following corollary.*

Corollary 2.5. *Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$
\|Df(x,y)\| \leq \|x\|^p + \|y\|^p \tag{2.21}
$$

for a fixed positive number p such that $p \neq 1, 2$. Then there exists a unique additive-quadratic mapping $F : X \longrightarrow Y$ such that the inequality

$$
\|F(x) - f(x)\| \leq \begin{cases} \frac{1}{(1-2^{p-1})|a|} \|x\|^p, & \text{if } 1 < p \\ \frac{1}{(2^{p-1}-1)(2-2^{(p-1)})|a|} \|x\|^p, & \text{if } 1 < p < 2 \\ \frac{1}{(2^{p-1}-2)|a|} \|x\|^p, & \text{if } 2 < p \end{cases}
$$

holds for all $x \in X$.

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Proof. By the definition of N_Y, we have

$$N_Y(Df(x,y), s+t) = \begin{cases} 0, & \text{if } s+t \le \|Df(x,y)\| \\ 1, & \text{if } s+t \ge \|Df(x,y)\|. \end{cases}$$

for all $x, y \in X$ and all $s, t \in \mathbb{R}$. Now, we claim that

$$N_Y(Df(x,y), s+t) \ge \min\{N_X(x, s^q), N_X(y, t^q)\}$$

for all $x, y \in X$ and $s, t > 0$. If $N_Y(Df(x,y), s+t) = 1$, then it is trivial. Suppose that $N_Y(Df(x,y), s+t) = 0$. Then $s+t \le \|Df(x,y)\|$ and by (2.21), either $s \le \|x\|^p$ or $t \le \|y\|^p$. Hence either $N_X(x, s^q) = 0$ or $N_X(y, t^q) = 0$ and thus f is a fuzzy q-almost additive-quadratic mapping. By Theorem 2.4, we have the results. □

The condition $p \ne 1, 2$ in Corollary 2.5 is indispensable. The following example shows that the inequality (2.21) is not stable for $p = 1$, 2, especially in the case of $b = 2$ and $c = -1$. We will give the proof when $p = 1$, and the proof when $p = 2$ is similar. For any $f : X \longrightarrow Y$, let $f_o(x) = \frac{f(x)-f(-x)}{2}$ and $f_e(x) = \frac{f(x)+f(-x)}{2}$.

Example 2.6. Define mappings $t, s : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$t(x) = \begin{cases} x, & \text{if } |x| < 1 \\ -1, & \text{if } x \le -1 \\ 1, & \text{if } 1 \le x, \end{cases}$$

$$s(x) = \begin{cases} x^2, & \text{if } |x| < 1 \\ 1, & \text{ortherwise} \end{cases}$$

and a mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} [\frac{t(2^n x)}{2^n} + \frac{s(2^n x)}{4^n}]$$

We will show that there is a positive integer M such that

$$|D_2 f(x,y)| \le M(|x| + |y|) \tag{2.22}$$

for all $x, y \in \mathbb{R}$, where

$$D_2 g(x,y) = g(x+y) + 2g(x-y) - g(y-x) - 3g(x) + g(-x) - 2g(-y).$$

But there do not exist an additive-quadratic mapping $F : \mathbb{R} \longrightarrow \mathbb{R}$ and a non-negative constant K such that

$$|F(x) - f(x)| \le K|x|^2 \tag{2.23}$$

for all $x \in \mathbb{R}$.

Proof. Note that $s_o(x) = 0$, $t_o(x) = t(x)$, and $|f_o(x)| \le 2$ for all $x \in \mathbb{R}$. First, suppose that $\frac{1}{2} \le |x| + |y|$. Then $|D_2 f_o(x,y)| \le 40(|x| + |y|)$. Now suppose that $\frac{1}{2} > |x| + |y|$. Then there is a non-negative integer m such that

$$\frac{1}{2^{m+2}} \le |x| + |y| < \frac{1}{2^{m+1}}$$

and so $2^m|x| < \frac{1}{2}$, $2^m|y| < \frac{1}{2}$. Hence $\{2^m(x \pm y),\ 2^m x,\ 2^m y\} \subseteq (-1,\ 1)$ and so for any $n = 0, 1, 2, \cdots, m$, $D_2 t_0(2^n x, 2^n y) = 0$ for all $x, y \in X$. Thus

$$D_2 f_o(x,y) = \sum_{n=0}^{\infty} \frac{1}{2^n} D_2 t(2^n x, 2^n y) = \sum_{n=m+1}^{\infty} \frac{1}{2^n} D_2 t(2^n x, 2^n y) \le \frac{40}{2^{m+2}} \le 40(|x|+|y|).$$

Note that $t_e(x) = 0$, $s_e(x) = s(x)$, and $|f_e(x)| \le \frac{4}{3}$ for all $x \in \mathbb{R}$. First, suppose that $\frac{1}{4} \le |x| + |y|$. Then $|D_2 f_e(x,y)| \le \frac{128}{3}(|x| + |y|)$ for all $x, y \in \mathbb{R}$. Now suppose that $\frac{1}{4} > |x| + |y|$. Then there is a non-negative integer k such that

$$\frac{1}{2^{k+2}} \le \Big(|x| + |y|\Big)^{\frac{1}{2}} < \frac{1}{2^{k+1}}.$$

Hence $\{2^k(x \pm y),\ 2^k x,\ 2^k y\} \subseteq (-1,\ 1)$ and so for any $n = 0, 1, 2, \cdots, m$, $D_2 s_e(2^n x, 2^n y) = 0$. Hence

$$D_2 f_e(x,y) = \sum_{n=0}^{\infty} \frac{1}{4^n} D_2 s_e(2^n x, 2^n y) = \sum_{n=k+1}^{\infty} \frac{1}{4^n} D_2 s_e(2^n x, 2^n y) \le \frac{8}{3} \cdot \frac{1}{2^{2k}}.$$

and so we have

$$\Big(D_2 f_e(x,y)\Big)^{\frac{1}{2}} \le 4\Big(\frac{8}{3}\Big)^{\frac{1}{2}} \Big(|x| + |y|\Big)^{\frac{1}{2}}.$$

Thus we have

$$D_2 f_e(x,y) \le \frac{128}{3}(|x| + |y|).$$

and so we have (2.22).

Suppose that there exist an additive mapping $A : \mathbb{R} \longrightarrow \mathbb{R}$, a quadratic mapping $Q : \mathbb{R} \longrightarrow \mathbb{R}$, and a non-negative constant K such that $A+Q$ satisfies (2.23). Since $|f(x)| \le \frac{10}{3}$, by (2.23), we have

$$\frac{10}{3n} - K|x|^2 \le \frac{A(x)}{n} + Q(x) \le \frac{10}{3n} + K|x|^2$$

for all $x \in X$ and all positive integers n and so

$$|Q(x)| \le K|x|^2$$

for all $x \in X$. Hence by (2.23), we have

$$|f - A(x)| \le 2K|x|^2$$

for all $x \in X$.

Since f_o, A are odd and f_e is even,

$$(2.24)\quad |f_e(x)| \le \frac{1}{2}\Big[|f_e(x) + f_o(x) - A(x)| + |f_e(-x) + f_o(-x) - A(-x)|\Big] \le 4K|x|^2$$

for all $x \in X$. Take a positive integer l such that $l > 4K$, and pick $x \in \mathbb{R}$ with $0 < 2^l x < 1$. Then

$$f_e(x) = \sum_{n=0}^{\infty} \frac{s(2^n x)}{4^n} \ge \sum_{n=0}^{l-1} \frac{s(2^n x)}{4^n} \ge l x^2 > 4Kx^2$$

which contradicts to (2.24). □

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Exact controllability for fuzzy differential equations using extremal solutions

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Abstract

In this paper, we devoted study exact controllability for fuzzy differential equations with the control function in credibility spaces. Moreover we study exact controllability for every solutions of fuzzy differential equations. The result is obtained by using extremal solutions.

1 Introduction

The theory of controlled processes is one of the most recent mathematical concepts to enable very important applications in modern engineering. However, actual systems subject to control do not admit a strictly deterministic analysis in view of various random factors that influence their behavior. The theory of controlled processes takes the random nature of a systems behavior into account. Many researchers have studied controlled processes in a credibility space. Arapostathis et al. [1] studied the controllability properties of the class of stochastic differential systems characterized by a linear controlled diffusion perturbed by a

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smooth, bounded, and uniformly Lipschitz nonlinearity. Kwun et al. [8] proved the approximate controllability for fuzzy differential equations driven by Liu process. Lee et al. [10] examined the exact controllability for abstract fuzzy differential equations in a credibility space.

Recently, Kwun et al. [14] studied the existence of extremal solutions for fuzzy differential equations driven by Liu process. Kwun et al. [6, 7] have studied the existence of extremal solutions for fuzzy differential equations in a n-dimensional fuzzy vector space. In this paper, using the extremal solutions, we study the exact controllability for every solutions of fuzzy differential equations in credibility space. We consider the following fuzzy differential equation:

$$\begin{cases} dx(t,\theta) = f(t, x(t,\theta))dC_t + Bu(t), \ t \in [0,T], \\ x(0) = x_0 \in E_N, \end{cases} \tag{1}$$

where the state function $x(t,\theta)$ takes values in $X(\subset E_N)$ and another bounded space $Y(\subset E_N)$. E_N is the set of all upper semi-continuously convex fuzzy numbers on R, $(\Theta, \mathcal{P}, Cr)$ is credibility space, the state function $x : [0,T] \times (\Theta, \mathcal{P}, Cr) \to X$ is a fuzzy process, $f : [0,T] \times X \to X$ is a regular fuzzy function, $u : [0,T] \times (\Theta, \mathcal{P}, Cr) \to Y$ is a control function, B is a linear bounded operator from Y to X. C_t is a standard Liu process, $x_0 \in E_N$ is an initial value.

2 Preliminaries

In this section, we give basic definitions, terminologies, notations and lemmas which are most relevant to our investigated and are needed in later section. All undefined concepts and notions used here are standard.

A fuzzy set of R^n is a function $u : R^n \to [0,1]$. For each fuzzy set u, we denote by $[u]^\alpha = \{x \in R^n : u(x) \geq \alpha\}$ for any $\alpha \in [0,1]$, its α-level set. Let u, v be fuzzy sets of R^n. It is well known that $[u]^\alpha = [v]^\alpha$ for each $\alpha \in [0,1]$ implies $u = v$. Let E^n denote the collection of all fuzzy sets of R^n that satisfies the following conditions:

(1) u is normal, i.e., there exists an $x_0 \in R^n$ such that $u(x_0) = 1$;

(2) u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n$, $0 \leq \lambda \leq 1$;

(3) $u(x)$ is upper semi-continuous, i.e., $u(x_0) \geq \overline{\lim}_{k\to\infty} u(x_k)$ for any $x_k \in R^n$ $(k = 0, 1, 2, \ldots)$, $x_k \to x_0$;

(4) $[u]^0$ is compact.

Definition 2.1. [17] The complete metric D_L on E_N is defined by

$$\begin{aligned} D_L(u,v) &= \sup_{0<\alpha\leq 1} d_L([u]^\alpha, [v]^\alpha) \\ &= \sup_{0<\alpha\leq 1} \max\{|u_l^\alpha - v_l^\alpha|, |u_r^\alpha - v_r^\alpha|\}, \end{aligned}$$

for any $u, v \in E_N$, which satisfies $d_L(u+w, v+w) = d_L(u,v)$.

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Definition 2.2. [5] Let $u, v \in C([0,T], E_N)$. The metric H_1 on $C([0,T], E_N)$ is defined by

$$H_1(u,v) = \sup_{0<t\leq T} D_L(u(t), v(t)).$$

Let Θ be a nonempty set, and let $\mathcal{P}$ the power set of Θ. Each element in $\mathcal{P}$ is called an event. In order to present an axiomatic definition of credibility, it is necessary to assign to each event A a number $Cr\{A\}$ which indicates the credibility that A will occur. In order to ensure that the number $Cr\{A\}$ has certain mathematical properties which we intuitively expect a credibility to have, we accept the following four axioms:

1. (Normality) $Cr\{A\} = 1$.
2. (Monotonicity) Cr is increasing, i.e., $Cr\{A\} \leq Cr\{B\}$ whenever $A \subset B$.
3. (Self-Duality) Cr is self-dual, i.e., $Cr\{A\}+Cr\{A^c\} = 1$ for any $A \in \mathcal{P}(\Theta)$.
4. (Maximality) $Cr\{\cup_i A_i\} = \sup_i Cr\{A_i\}$ for any $\{A_i\}$ with $Cr\{A_i\} \leq 0.5$.

Definition 2.3. [11] Let ξ be a fuzzy variable with the possibility distribution function $\mu : R \to [0,1]$. A fuzzy variable ξ is said to be normal if there exists a real number r such that $\mu(r) = 1$. It is well known that the possibility of $\{\xi \leq r\}$ is defined by

$$\text{Pos}\{\xi \leq r\} = \sup_{u\leq r} \mu(u)$$

while the necessity of $\{\xi \leq r\}$ is defined by

$$\text{Nec}\{\xi \leq r\} = 1 - \text{Pos}\{\xi < r\} = 1 - \sup_{u<r} \mu(u).$$

Definition 2.4. [11] The set function Cr is called a credibility measure if it satisfies above four axioms, and defined as follows:

$$Cr\{A\} = \frac{1}{2}(\text{Pos}\{A\} + \text{Nec}\{A\}),$$

where $\text{Pos}\{A\} = 1 - \text{Nec}\{A^c\}$ with A^c is the complement of A.

Definition 2.5. [12] Let Θ be a nonempty set, $\mathcal{P}$ be the power set of Θ, and let Cr be a credibility measure. Then the triplet $(\Theta, \mathcal{P}, C_r)$ is called a credibility space.

Definition 2.6. [13] A fuzzy variable is a function from a credibility space $(\Theta, \mathcal{P}, C_r)$ to the set of real numbers.

Definition 2.7. [13] Let T be an index set and let $(\Theta, \mathcal{P}, C_r)$ be a credibility space. A fuzzy process is a function from $T \times (\Theta, \mathcal{P}, C_r)$ to the set of real numbers.

That is, a fuzzy process $x(t,\theta)$ is a function of two variables such that the function $x(t^*,\theta)$ is a fuzzy variable for each t^*. For each fixed θ^*, the function $x(t,\theta^*)$ is called a sample path of the fuzzy process. A fuzzy process $x(t,\theta)$ is said to be sample-continuous if the sample path is continuous for almost all θ.

Definition 2.8. Let $(\Theta,\mathcal{P},C_r)$ be a credibility space. For fuzzy random variable $x(t,\theta)$ in a credibility space, for each $\alpha\in(0,1]$, the α-level set $[x(t,\theta)]^\alpha=[x_l^\alpha(t,\theta),x_r^\alpha(t,\theta)]$ is defined by

$$x_l^\alpha(t,\theta)=\inf x^\alpha(t,\theta)=\inf\{a\in R|x(t,\theta)(a)\geq\alpha\},$$
$$x_r^\alpha(t,\theta)=\sup x^\alpha(t,\theta)=\sup\{a\in R|x(t,\theta)(a)\geq\alpha\}.$$

Definition 2.9. [11] Let ξ be a fuzzy variable and r is a real number. Then the expected value of ξ is defined by

$$E\xi=\int_0^{+\infty}Cr\{\xi\geq r\}dr-\int_{-\infty}^0 Cr\{\xi\leq r\}dr$$

provided that at least one of the integrals is finite.

Definition 2.10. [13] A fuzzy process C_t is said to be a Liu process if
(1) $C_0=0$;
(2) C_t has stationary and independent increments;
(3) every increment $C_{t+s}-C_s$ is a normally distributed fuzzy variable with expected value et and variance σ^2t^2, whose membership function is

$$\mu(x)=2\Big(1+\exp\Big(\frac{\pi|x-et|}{\sqrt{6}\sigma t}\Big)\Big)^{-1},\ x\in R.$$

The parameters e and σ are called the *drift* and *diffusion* coefficients, respectively. Liu process is said to be standard if $e=0$ and $\sigma=1$.

Definition 2.11. [3] Let $x(t)$ be a fuzzy process and let C_t be a standard Liu process. For any partition of closed interval $[c,d]$ with $c=t_0<\cdots<t_n=d$, the mesh is written as $\triangle=\max_{1\leq i\leq n}(t_i-t_{i-1})$. Then the fuzzy integral of $x(t)$ with respect to C_t is

$$\int_c^d x(t)dC_t=\lim_{\triangle\to 0}\sum_{i=1}^n x(t_{i-1})(C_{t_i}-C_{t_{i-1}})$$

provided that the limit exists almost surely and is a fuzzy variable.

Lemma 2.1. [3] Let C_t be a standard Liu process. For any given θ with $Cr\{\theta\}>0$, the path C_t is Lipschitz continuous, that is, the following inequality holds

$$|C_{t_1}-C_{t_2}|<K(\theta)|t_1-t_2|,$$

where K is a fuzzy variable called the Lipschitz constant of a Liu process with

$$K(\theta)=\begin{cases}\sup_{0\leq s<t}\frac{|C_t-C_s|}{t-s}, & Cr\{\theta\}>0,\\ \infty, & \text{otherwise},\end{cases}$$

and $E[K^p] < \infty$, $\forall p > 0$.

Lemma 2.2. [3] Let C_t be a standard Liu process, and let $h(t; c)$ be a continuously differentiable function. Define $x_t = h(t; C_t)$. Then we have the following chain rule

$$dx_t = \frac{\partial h(t; C_t)}{\partial t} dt + \frac{\partial h(t; C_t)}{\partial C} dC_t.$$

Lemma 2.3. [3] Let $f(t)$ be continuous fuzzy process, the following inequality of fuzzy integral holds

$$\Big| \int_c^d f(t) dC_t \Big| \leq K \int_c^d |f(t)| dt,$$

where $K = K(\theta)$ is defined in Lemma 2.1.

Definition 2.12. [14] For the partial ordering $\leq_T$, a function $a \in C([0, T] \times (\Theta, \mathcal{P}, C_r), E_N)$ is a $\leq_T$-lower solution for equation (1)($u \equiv 0$) if

$$\begin{cases} a(t, \theta) \leq_T U(t)x_0 + \int_0^t U(t-s)G(s, a(s, \theta))dC(s), \ t \in [0, T], \\ a(0) \leq_T x_0 \in E_N \end{cases} \tag{2}$$

and a function $b \in C([0, T] \times (\Theta, \mathcal{P}, C_r), E_N)$ is a $\leq_T$-upper solution for equation (1)($u \equiv 0$) if

$$\begin{cases} b(t, \theta) \geq_T S(t)x_0 + \int_0^t S(t-s)F(s, b(s, \theta))dC(s), \ t \in [0, T], \\ b(0) \geq_T x_0 \in E_N. \end{cases} \tag{3}$$

Theorem 2.1. [14] Let $a, b \in C([0, T] \times (\Theta, \mathcal{P}, C_r), E_N)$ be, respectively, $\leq_T$-lower and $\leq_T$-upper solutions for equation (1)($u \equiv 0$) on $[0, T]$. Then, there exist monotone sequences $\{a_n\} \uparrow \rho$, $\{b_n\} \downarrow \gamma$ in $C([0, T] \times (\Theta, \mathcal{P}, C_r), E_N)$, where ρ, γ are extremal solutions to equation (1) in the stochastic fuzzy functional interval $[a, b] := \{x \in C([0, T] \times (\Theta, \mathcal{P}, C_r), E_N) | a \leq_T x \leq_T b \text{ on } [0, T]\}$.

3 Exact controllability for fuzzy differential equation using extremal solutions

In this section, we study exact controllability for fuzzy differential equation using extremal solutions (1). In [14], Park et al. proved the existence of extremal solutions for the equation (1). Hence we consider extremal solutions for the equation (1), for each u in Y.

$$\begin{cases} x_t = U(t)x_0 + \int_0^t U(t-s)G(s, x_s)dC_s + \int_0^t U(t-s)Bu_s ds, \\ x(0) = x_0 \in E_N, \end{cases} \tag{4}$$

where $U(t) = e^{-Mt}$ is continuous with $U(0) = I$, $|U(t)| \leq c$, $c > 0$, for all $t \in [0, T]$. And

$$\begin{cases} x_t = S(t)x_0 + \int_0^t S(t-s)F(s, x_s)dC_s + \int_0^t S(t-s)Bu_s ds, \\ x(0) = x_0 \in E_N, \end{cases} \tag{5}$$

where $S(t) = e^{Mt}$ is continuous with $S(0) = I$, $|S(t)| \leq d$, $d > 0$, for all $t \in [0, T]$.

Now we assume the following hypotheses:

(**H1**) For $L_1, L_2 > 0$, $x_0 \in E_N$,

$$d_L\Big([U(t)x_0]^\alpha, [x_0]^\alpha\Big) \leq L_1, \quad d_L\Big([S(t)x_0]^\alpha, [x_0]^\alpha\Big) \leq L_2.$$

(**H2**) For $x(\cdot), y(\cdot) \in C([0, T] \times (\Theta, \mathcal{P}, C_r), E_N)$, $t \in [0, T]$, there exist positive numbers m_1, m_2 such that

$$d_L\Big([G(t, x)]^\alpha, [G(t, y)]^\alpha\Big) \leq m_1 d_L([x]^\alpha, [y]^\alpha),$$
$$d_L\Big([F(t, x)]^\alpha, [F(t, y)]^\alpha\Big) \leq m_2 d_L([x]^\alpha, [y]^\alpha)$$

and $F(0, \mathcal{X}_{\{0\}}(0)) \equiv 0$, $G(0, \mathcal{X}_{\{0\}}(0)) \equiv 0$.

(**H3**) For $L_3 > 0$, $x_0 \in E_N$, $d_L\Big([x_0]^\alpha, [\mathcal{X}_{\{0\}}(0)]^\alpha\Big) \leq L_3$.

(**H4**) For $\varepsilon > 0$, $(L_1 + cm_1KL_3T)e^{cm_1KT} \leq \varepsilon$.

(**H5**) For $\varepsilon > 0$, $(L_2 + dm_2KL_3T)e^{dm_2KT} \leq \varepsilon$.

(**H6**) Let a, b be, respectively, lower solution and upper solution of equation (1)$(u \equiv 0)$, then $[a, b]$ is convex.

We define the controllability concept for a fuzzy differential equation.

Definition 3.1. The equation (1) is said to be controllable on [0,T], if for every $x_0 \in E_N$ there exists a control $u_t \in Y$ such that every solutions $x(\cdot)$ of (1) satisfies a.s. θ, $x_T = x^1 \in X$ (i.e., $[x_T]^\alpha = [x^1]^\alpha$).

Definition 3.2. Define the fuzzy mappings $P_1 : \widetilde{P}(R) \to X$ and $P_2 : \widetilde{P}(R) \to X$ by

$$P_1^\alpha(v) = \begin{cases} \int_0^T U^\alpha(T-s)Bv_s ds, & v \subset \overline{\Gamma}_u, \\ 0, & \text{otherwise}, \end{cases}$$
$$P_2^\alpha(v) = \begin{cases} \int_0^T S^\alpha(T-s)Bv_s ds, & v \subset \overline{\Gamma}_u, \\ 0, & \text{otherwise}, \end{cases}$$

where $\widetilde{P}(R)$ is a nonempty fuzzy subset of R and $\overline{\Gamma}_u$ is the closure of support u. Then there exist $P_{1i}^\alpha, P_{2i}^\alpha (i = l, r)$ such that

$$P_{1l}^\alpha(v_l) = \int_0^T U_l^\alpha(T-s)B(v_s)_l ds, \ (v_s)_l \in [(u_s)_l^\alpha, (u_s)^1],$$
$$P_{1r}^\alpha(v_r) = \int_0^T U_r^\alpha(T-s)B(v_s)_r ds, \ (v_s)_r \in [(u_s)^1, (u_s)_r^\alpha],$$

$$P_{2l}^{\alpha}(v_l) = \int_0^T S_l^{\alpha}(T-s)B(v_s)_l ds,\ (v_s)_l \in [(u_s)_l^{\alpha}, (u_s)^1],$$
$$P_{2r}^{\alpha}(v_r) = \int_0^T S_r^{\alpha}(T-s)B(v_s)_r ds,\ (v_s)_r \in [(u_s)^1, (u_s)_r^{\alpha}].$$

We assume that $\widetilde{P}_{1l}^{\alpha}, \widetilde{P}_{1r}^{\alpha}, \widetilde{P}_{2l}^{\alpha}$ and $\widetilde{P}_{2r}^{\alpha}$ are bijective mappings.

By Definition 3.2, we can introduce α-level set of u_s is

$$\begin{aligned}
[u_s]^{\alpha} &= [(u_s)_l^{\alpha}, (u_s)_r^{\alpha}] \\
&= \frac{1}{2}\Big[(\widetilde{P}_{1l}^{\alpha})^{-1}\Big\{(x^1)_l^{\alpha} - U_l^{\alpha}(T)(x_0)_l^{\alpha} - \int_0^T U_l^{\alpha}(T-s)G_l^{\alpha}(s,(x_s)_l^{\alpha})dC_s\Big\} \\
&\quad +(\widetilde{P}_{2l}^{\alpha})^{-1}\Big\{(x^1)_l^{\alpha} - S_l^{\alpha}(T)(x_0)_l^{\alpha} - \int_0^T S_l^{\alpha}(T-s)F_l^{\alpha}(s,(x_s)_l^{\alpha})dC_s\Big\}, \\
&\quad (\widetilde{P}_{1r}^{\alpha})^{-1}\Big\{(x^1)_r^{\alpha} - U_r^{\alpha}(T)(x_0)_r^{\alpha} - \int_0^T U_r^{\alpha}(T-s)G_r^{\alpha}(s,(x_s)_r^{\alpha})dC_s\Big\} \\
&\quad +(\widetilde{P}_{2r}^{\alpha})^{-1}\Big\{(x^1)_r^{\alpha} - S_r^{\alpha}(T)(x_0)_r^{\alpha} - \int_0^T S_r^{\alpha}(T-s)F_r^{\alpha}(s,(x_s)_r^{\alpha})dC_s\Big\}\Big].
\end{aligned}$$

Theorem 3.1. If Lemma 2.3 and hypotheses (H1)-(H5) are satisfied, then the equation (4) is controllable on $[0,T]$.

Proof By Definition 3.2 and above u_s, substitute the control into the equation (4) yields α-level of $\underline{x}_T$.

$$\begin{aligned}
[\underline{x}_T]^{\alpha} &= \Big[U(T)x_0 + \int_0^T U(T-s)G(s,x_s)dC_s + \int_0^T U(T-s)Bu_s ds\Big]^{\alpha} \\
&= \Big[U_l^{\alpha}(T)(x_0)_l^{\alpha} + \int_0^T U_l^{\alpha}(T-s)G_l^{\alpha}(s,(x_s)_l^{\alpha})dC_s + \int_0^T U_l^{\alpha}(T-s)B \\
&\quad \times\frac{1}{2}\Big[(\widetilde{P}_{1l}^{\alpha})^{-1}\Big\{(x^1)_l^{\alpha} - U_l^{\alpha}(T)(x_0)_l^{\alpha} - \int_0^T U_l^{\alpha}(T-s)G_l^{\alpha}(s,(x_s)_l^{\alpha})dC_s\Big\} \\
&\qquad +(\widetilde{P}_{2l}^{\alpha})^{-1}\Big\{(x^1)_l^{\alpha} - S_l^{\alpha}(T)(x_0)_l^{\alpha} - \int_0^T S_l^{\alpha}(T-s)F_l^{\alpha}(s,(x_s)_l^{\alpha})dC_s\Big\}\Big]ds, \\
&\quad U_r^{\alpha}(T)(x_0)_r^{\alpha} + \int_0^T U_r^{\alpha}(T-s)G_r^{\alpha}(s,(x_s)_r^{\alpha})dC_s + \int_0^T U_r^{\alpha}(T-s)B \\
&\quad \times\frac{1}{2}\Big[(\widetilde{P}_{1r}^{\alpha})^{-1}\Big\{(x^1)_r^{\alpha} - U_r^{\alpha}(T)(x_0)_r^{\alpha} - \int_0^T U_r^{\alpha}(T-s)G_r^{\alpha}(s,(x_s)_r^{\alpha})dC_s\Big\} \\
&\qquad +(\widetilde{P}_{2r}^{\alpha})^{-1}\Big\{(x^1)_r^{\alpha} - S_r^{\alpha}(T)(x_0)_r^{\alpha} - \int_0^T S_r^{\alpha}(T-s)F_r^{\alpha}(s,(x_s)_r^{\alpha})dC_s\Big\}\Big]ds\Big] \\
&= \Big[U_l^{\alpha}(T)(x_0)_l^{\alpha} + \int_0^T U_l^{\alpha}(T-s)G_l^{\alpha}(s,(x_s)_l^{\alpha})dC_s
\end{aligned}$$

$$+\frac{1}{2}P_{1l}^{\alpha}\Big[(\widetilde{P}_{1l}^{\alpha})^{-1}\Big\{(x^1)_l^{\alpha}-U_l^{\alpha}(T)(x_0)_l^{\alpha}-\int_0^T U_l^{\alpha}(T-s)G_l^{\alpha}(s,(x_s)_l^{\alpha})dC_s\Big\}$$

$$+(\widetilde{P}_{2l}^{\alpha})^{-1}\Big\{(x^1)_l^{\alpha}-S_l^{\alpha}(T)(x_0)_l^{\alpha}-\int_0^T S_l^{\alpha}(T-s)F_l^{\alpha}(s,(x_s)_l^{\alpha})dC_s\Big\}\Big],$$

$$U_r^{\alpha}(T)(x_0)_r^{\alpha}+\int_0^T U_r^{\alpha}(T-s)G_r^{\alpha}(s,(x_s)_r^{\alpha})dC_s$$

$$+\frac{1}{2}P_{1r}^{\alpha}\Big[(\widetilde{P}_{1r}^{\alpha})^{-1}\Big\{(x^1)_r^{\alpha}-U_r^{\alpha}(T)(x_0)_r^{\alpha}-\int_0^T U_r^{\alpha}(T-s)G_r^{\alpha}(s,(x_s)_r^{\alpha})dC_s\Big\}$$

$$+(\widetilde{P}_{2r}^{\alpha})^{-1}\Big\{(x^1)_r^{\alpha}-S_r^{\alpha}(T)(x_0)_r^{\alpha}-\int_0^T S_r^{\alpha}(T-s)F_r^{\alpha}(s,(x_s)_r^{\alpha})dC_s\Big\}\Big]\Big]$$

$$=[(x^1)_l^{\alpha},(x^1)_r^{\alpha}]=[x^1]^{\alpha}.$$

Hence this control u_t satisfy a.s. θ, $x_T=x^1$.

Also, using this control, we shall show that the nonlinear operator Φ_1 defined by

$$(\Phi_1 x)_t=U(t)x_0+\int_0^t U(t-s)G(s,x_s)dC_s+\int_0^t U(t-s)B$$
$$\times\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,x_\tau)dC_\tau\Big\}$$
$$+\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,x_\tau)dC_\tau\Big\}\Big]ds,$$

where the fuzzy mappings $(\widetilde{P}_1)^{-1}$ satisfy above statements.

Form hypothesis (H2) and Lemma 2.3, for any given θ with $Cr\{\theta\}>0$, $x(\cdot),\ y(\cdot)\in C([0,T]\times(\Theta,\mathcal{P},Cr),E_N)$, we have

$$d_L\Big([(\Phi_1 x)_t]^{\alpha},[(\Phi_1 y)_t]^{\alpha}\Big)$$
$$=d_L\Big(\Big[U(t)x_0+\int_0^t U(t-s)G(s,x_s)dC_s$$
$$+\int_0^t U(t-s)B\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,x_\tau)dC_\tau\Big\}$$
$$+\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,x_\tau)dC_\tau\Big\}\Big]ds\Big]^{\alpha},$$
$$\Big[U(t)x_0+\int_0^t U(t-s)G(s,y_s)dC_s$$
$$+\int_0^t U(t-s)B\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,y_\tau)dC_\tau\Big\}$$
$$+\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,y_\tau)dC_\tau\Big\}\Big]ds\Big]^{\alpha}\Big)$$
$$\le d_L\Big(\Big[\int_0^t U(t-s)G(s,x_s)dC_s\Big]^{\alpha},\Big[\int_0^t U(t-s)G(s,y_s)dC_s\Big]^{\alpha}\Big)$$

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$$
\begin{aligned}
&+d_L\Big(\Big[\int_0^t U(t-s)B\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,x_\tau)dC_\tau\Big\}\\
&\qquad +\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,x_\tau)dC_\tau\Big\}\Big]ds\Big]^\alpha,\\
&\quad \int_0^t U(t-s)B\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,y_\tau)dC_\tau\Big\}\\
&\qquad +\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,y_\tau)dC_\tau\Big\}\Big]ds\Big]^\alpha\Big)\\
\le\ & d_L\Big(\Big[\int_0^t U(t-s)G(s,x_s)dC_s\Big]^\alpha,\Big[\int_0^t U(t-s)G(s,y_s)dC_s\Big]^\alpha\Big)\\
&+d_L\Big(\Big[\frac{1}{2}P_1\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,x_\tau)dC_\tau\Big\}\\
&\qquad +\frac{1}{2}P_1\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,x_\tau)dC_\tau\Big\}\Big]^\alpha,\\
&\quad \Big[\frac{1}{2}P_1\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,y_\tau)dC_\tau\Big\}\\
&\qquad +\frac{1}{2}P_1\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,y_\tau)dC_\tau\Big\}\Big]^\alpha\Big)\\
\le\ & d_L\Big(\Big[\int_0^t U(t-s)G(s,x_s)dC_s\Big]^\alpha,\Big[\int_0^T U(t-s)G(s,y_s)dC_s\Big]^\alpha\Big)\\
&+d_L\Big(\Big[\int_0^T U(T-s)G(s,x_s)dC_s\Big]^\alpha,\Big[\int_0^t U(T-s)G(s,y_s)dC_s\Big]^\alpha\Big)\\
\le\ & cm_1K\int_0^t d_L\Big([x_s]^\alpha,[y_s]^\alpha\Big)ds+cm_1K\int_0^T d_L\Big([x_s]^\alpha,[y_s]^\alpha\Big)ds.
\end{aligned}
$$

Therefore, by Lemma 2.1, we get

$$
\begin{aligned}
&E\Big(H_1(\Phi_1x,\Phi_1y)\Big)\\
&=E\Big(\sup_{t\in[0,T]}D_L\Big((\Phi_1x)_t,(\Phi_1y)_t\Big)\Big)\\
&=E\Big(\sup_{t\in[0,T]}\sup_{0<\alpha\le1}d_L\Big([(\Phi_1x)_t]^\alpha,[(\Phi_1y)_t]^\alpha\Big)\Big)\\
&\le E\Big(\sup_{t\in[0,T]}\sup_{0<\alpha\le1}cm_1K\Big(\int_0^T d_L\Big([x_s]^\alpha,[y_s]^\alpha\Big)ds+\int_0^T d_L\Big([x_s]^\alpha,[y_s]^\alpha\Big)ds\Big)\Big)\\
&\le E\Big(\sup_{t\in[0,T]}cm_1K\Big(\int_0^t D_L(x_s,y_s)ds+\int_0^T D_L(x_s,y_s)ds\Big)\Big)\\
&\le 2cm_1KTE\Big(H_1(x,y)\Big).
\end{aligned}
$$

We take sufficiently small T, $2cm_1KT<1$. Hence Φ_1 is contraction mapping. By the Banach fixed point theorem, (4) has a unique fixed point. Thus the equation (1) is controllable in $[0,T]$.

Theorem 3.2. If Lemma 2.3 and hypotheses (H1)-(H5) are satisfied, then the equation (5) is controllable on $[0,T]$.

Proof By Definition 3.2 and above u_s, substitute the control into the equation (5) yields α-level of $\overline{x}_T$.

$$
\begin{aligned}
[\overline{x}_T]^\alpha &= \Big[S(T)x_0 + \int_0^T S(T-s)F(s,x_s)dC_s + \int_0^T S(T-s)Bu_s ds\Big]^\alpha \\
&= \Big[S_l^\alpha(T)(x_0)_l^\alpha + \int_0^T S_l^\alpha(T-s)F_l^\alpha(s,(x_s)_l^\alpha)dC_s + \int_0^T S_l^\alpha(T-s)B \\
&\quad \times\frac{1}{2}\Big[(\widetilde{P}_{1l}^\alpha)^{-1}\Big\{(x^1)_l^\alpha - U_l^\alpha(T)(x_0)_l^\alpha - \int_0^T U_l^\alpha(T-s)G_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\} \\
&\qquad +(\widetilde{P}_{2l}^\alpha)^{-1}\Big\{(x^1)_l^\alpha - S_l^\alpha(T)(x_0)_l^\alpha - \int_0^T S_l^\alpha(T-s)F_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\}\Big]ds, \\
&\quad S_r^\alpha(T)(x_0)_r^\alpha + \int_0^T S_r^\alpha(T-s)F_r^\alpha(s,(x_s)_r^\alpha)dC_s + \int_0^T S_r^\alpha(T-s)B \\
&\quad \times\frac{1}{2}\Big[(\widetilde{P}_{1r}^\alpha)^{-1}\Big\{(x^1)_r^\alpha - U_r^\alpha(T)(x_0)_r^\alpha - \int_0^T U_r^\alpha(T-s)G_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\} \\
&\qquad +(\widetilde{P}_{2r}^\alpha)^{-1}\Big\{(x^1)_r^\alpha - S_r^\alpha(T)(x_0)_r^\alpha - \int_0^T S_r^\alpha(T-s)F_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\}\Big]ds\Big] \\
&= \Big[S_l^\alpha(T)(x_0)_l^\alpha + \int_0^T S_l^\alpha(T-s)F_l^\alpha(s,(x_s)_l^\alpha)dC_s \\
&\quad +\frac{1}{2}P_{2l}^\alpha\Big[(\widetilde{P}_{1l}^\alpha)^{-1}\Big\{(x^1)_l^\alpha - U_l^\alpha(T)(x_0)_l^\alpha - \int_0^T U_l^\alpha(T-s)G_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\} \\
&\qquad +(\widetilde{P}_{2l}^\alpha)^{-1}\Big\{(x^1)_l^\alpha - S_l^\alpha(T)(x_0)_l^\alpha - \int_0^T S_l^\alpha(T-s)F_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\}\Big], \\
&\quad S_r^\alpha(T)(x_0)_r^\alpha + \int_0^T S_r^\alpha(T-s)F_r^\alpha(s,(x_s)_r^\alpha)dC_s \\
&\quad +\frac{1}{2}P_{2r}^\alpha\Big[(\widetilde{P}_{1r}^\alpha)^{-1}\Big\{(x^1)_r^\alpha - U_r^\alpha(T)(x_0)_r^\alpha - \int_0^T U_r^\alpha(T-s)G_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\} \\
&\qquad +(\widetilde{P}_{2r}^\alpha)^{-1}\Big\{(x^1)_r^\alpha - S_r^\alpha(T)(x_0)_r^\alpha - \int_0^T S_r^\alpha(T-s)F_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\}\Big]\Big] \\
&= [(x^1)_l^\alpha, (x^1)_r^\alpha] = [x^1]^\alpha.
\end{aligned}
$$

Hence this control u_t satisfy a.s. θ, $x_T = x^1$.

Also, using this control, we shall show that the nonlinear operator Φ_2 defined by

$$
(\Phi_2 x)_t = S(t)x_0 + \int_0^t S(t-s)F(s,x_s)dC_s + \int_0^t S(t-s)B
$$

$$\times\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,x_\tau)dC_\tau\Big\}$$
$$+\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,x_\tau)dC_\tau\Big\}\Big]ds,$$

where the fuzzy mappings $(\widetilde{P}_2)^{-1}$ satisfy above statements.

Form hypothesis (H2) and Lemma 2.3, for any given θ with $Cr\{\theta\}>0$, $x(\cdot),\ y(\cdot)\in C([0,T]\times(\Theta,\mathcal{P},Cr),E_N)$, we have

$$\begin{aligned}
&d_L\Big([(\Phi_2x)_t]^\alpha,[(\Phi_2y)_t]^\alpha\Big)\\
&=d_L\Big(\Big[S(t)x_0+\int_0^t S(t-s)F(s,x_s)dC_s\\
&\quad+\int_0^t S(t-s)B\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,x_\tau)dC_\tau\Big\}\\
&\quad+\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,x_\tau)dC_\tau\Big\}\Big]ds\Big]^\alpha,\\
&\quad\Big[S(t)x_0+\int_0^t S(t-s)F(s,y_s)dC_s\\
&\quad+\int_0^t S(t-s)B\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,y_\tau)dC_\tau\Big\}\\
&\quad+\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,y_\tau)dC_\tau\Big\}\Big]ds\Big]^\alpha\Big)\\
&\le d_L\Big(\Big[\int_0^t S(t-s)F(s,x_s)dC_s\Big]^\alpha,\Big[\int_0^t S(t-s)F(s,y_s)dC_s\Big]^\alpha\Big)\\
&\quad+d_L\Big(\Big[\int_0^t S(t-s)B\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,x_\tau)dC_\tau\Big\}\\
&\quad+\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,x_\tau)dC_\tau\Big\}\Big]ds\Big]^\alpha,\\
&\quad\int_0^t S(t-s)B\frac{1}{2}\Big[\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,y_\tau)dC_\tau\Big\}\\
&\quad+\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,y_\tau)dC_\tau\Big\}\Big]ds\Big]^\alpha\Big)\\
&\le d_L\Big(\Big[\int_0^t S(t-s)F(s,x_s)dC_s\Big]^\alpha,\Big[\int_0^t S(t-s)F(s,y_s)dC_s\Big]^\alpha\Big)\\
&\quad+d_L\Big(\Big[\frac{1}{2}P_2\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,x_\tau)dC_\tau\Big\}\\
&\quad+\frac{1}{2}P_2\widetilde{P}_2^{-1}\Big\{x^1-S(T)x_0-\int_0^T S(T-\tau)F(\tau,x_\tau)dC_\tau\Big\}\Big]^\alpha,\\
&\quad\Big[\frac{1}{2}P_2\widetilde{P}_1^{-1}\Big\{x^1-U(T)x_0-\int_0^T U(T-\tau)G(\tau,y_\tau)dC_\tau\Big\}
\end{aligned}$$

$$+\frac{1}{2}P_2\widetilde{P}_2^{-1}\Big\{x^1 - S(T)x_0 - \int_0^T S(T-\tau)F(\tau, y_\tau)dC_\tau\Big\}\Big]^\alpha\Big)$$
$$\leq d_L\Big(\Big[\int_0^t S(t-s)F(s,x_s)dC_s\Big]^\alpha, \Big[\int_0^T S(t-s)F(s,y_s)dC_s\Big]^\alpha\Big)$$
$$+d_L\Big(\Big[\int_0^T S(T-s)F(s,x_s)dC_s\Big]^\alpha, \Big[\int_0^t S(T-s)F(s,y_s)dC_s\Big]^\alpha\Big)$$
$$\leq dm_2K\int_0^t d_L\Big([x_s]^\alpha, [y_s]^\alpha\Big)ds + dm_2K\int_0^T d_L\Big([x_s]^\alpha, [y_s]^\alpha\Big)ds.$$

Therefore, by Lemma 2.1, we get

$$E\Big(H_1(\Phi_2 x, \Phi_2 y)\Big)$$
$$= E\Big(\sup_{t\in[0,T]} D_L\Big((\Phi_2 x)_t, (\Phi_2 y)_t\Big)\Big)$$
$$= E\Big(\sup_{t\in[0,T]}\sup_{0<\alpha\leq 1} d_L\Big([(\Phi_2 x)_t]^\alpha, [(\Phi_2 y)_t]^\alpha\Big)\Big)$$
$$\leq E\Big(\sup_{t\in[0,T]}\sup_{0<\alpha\leq 1} dm_2K\Big(\int_0^T d_L\Big([x_s]^\alpha, [y_s]^\alpha\Big)ds + \int_0^T d_L\Big([x_s]^\alpha, [y_s]^\alpha\Big)ds\Big)\Big)$$
$$\leq E\Big(\sup_{t\in[0,T]} 3m_2K\Big(\int_0^t D_L(x_s, y_s)ds + \int_0^T D_L(x_s, y_s)ds\Big)\Big)$$
$$\leq 2dm_2KTE\Big(H_1(x,y)\Big).$$

We take sufficiently small T and $2dm_2KT < 1$. Hence Φ_2 is contraction mapping. By the Banach fixed point theorem, (5) has a unique fixed point. Thus the equation (1) is controllable in $[0,T]$.

Theorem 3.3. If Theorems 3.1 and 3.2 and hypotheses (H1)-(H6) are satisfied, then the equation (1) is controllable on $[0,T]$.

Proof For $x_T \in [\underline{x}_T, \overline{x}_T]$, if $[\underline{x}_T, \overline{x}_T]$ is convex, then $x_T = \lambda\underline{x}_T + (1-\lambda)\overline{x}_T, 0 \leq \lambda \leq 1$, we can obtain the following result.

$$[x_T]^\alpha = [\lambda\underline{x}_T + (1-\lambda)\overline{x}_T]^\alpha$$
$$= \Big[\lambda\Big\{U(T)x_0 + \int_0^T U(T-s)G(s,x_s)dC_s + \int_0^T U(T-s)Bu_s ds\Big\}$$
$$+(1-\lambda)\Big\{S(T)x_0 + \int_0^T S(T-s)F(s,x_s)dC_s + \int_0^T S(T-s)Bu_s ds\Big\}\Big]^\alpha$$
$$= \lambda\Big[U(T)x_0 + \int_0^T U(T-s)G(s,x_s)dC_s + \int_0^T U(T-s)Bu_s ds\Big]^\alpha$$
$$+(1-\lambda)\Big[S(T)x_0 + \int_0^T S(T-s)F(s,x_s)dC_s + \int_0^T S(T-s)Bu_s ds\Big]^\alpha$$

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$$
\begin{aligned}
&= \lambda\Big[U_l^\alpha(T)(x_0)_l^\alpha + \int_0^T U_l^\alpha(T-s)G_l^\alpha(s,(x_s)_l^\alpha)dC_s \\
&\quad +\frac{1}{2}P_{1l}^\alpha\Big[(\widetilde{P}_{1l}^\alpha)^{-1}\Big\{(x^1)_l^\alpha - U_l^\alpha(T)(x_0)_l^\alpha - \int_0^T U_l^\alpha(T-s)G_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\} \\
&\qquad +(\widetilde{P}_{2l}^\alpha)^{-1}\Big\{(x^1)_l^\alpha - S_l^\alpha(T)(x_0)_l^\alpha - \int_0^T S_l^\alpha(T-s)F_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\}\Big], \\
&\quad U_r^\alpha(T)(x_0)_r^\alpha + \int_0^T U_r^\alpha(T-s)G_r^\alpha(s,(x_s)_r^\alpha)dC_s \\
&\quad +\frac{1}{2}P_{1r}^\alpha\Big[(\widetilde{P}_{1r}^\alpha)^{-1}\Big\{(x^1)_r^\alpha - U_r^\alpha(T)(x_0)_r^\alpha - \int_0^T U_r^\alpha(T-s)G_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\} \\
&\qquad +(\widetilde{P}_{2r}^\alpha)^{-1}\Big\{(x^1)_r^\alpha - S_r^\alpha(T)(x_0)_r^\alpha - \int_0^T S_r^\alpha(T-s)F_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\}\Big]\Big] \\
&\quad +(1-\lambda)\Big[S_l^\alpha(T)(x_0)_l^\alpha + \int_0^T S_l^\alpha(T-s)F_l^\alpha(s,(x_s)_l^\alpha)dC_s \\
&\quad +\frac{1}{2}P_{2l}^\alpha\Big[(\widetilde{P}_{1l}^\alpha)^{-1}\Big\{(x^1)_l^\alpha - U_l^\alpha(T)(x_0)_l^\alpha - \int_0^T U_l^\alpha(T-s)G_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\} \\
&\qquad +(\widetilde{P}_{2l}^\alpha)^{-1}\Big\{(x^1)_l^\alpha - S_l^\alpha(T)(x_0)_l^\alpha - \int_0^T S_l^\alpha(T-s)F_l^\alpha(s,(x_s)_l^\alpha)dC_s\Big\}\Big], \\
&\quad S_r^\alpha(T)(x_0)_r^\alpha + \int_0^T S_r^\alpha(T-s)F_r^\alpha(s,(x_s)_r^\alpha)dC_s \\
&\quad +\frac{1}{2}P_{2r}^\alpha\Big[(\widetilde{P}_{1r}^\alpha)^{-1}\Big\{(x^1)_r^\alpha - U_r^\alpha(T)(x_0)_r^\alpha - \int_0^T U_r^\alpha(T-s)G_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\} \\
&\qquad +(\widetilde{P}_{2r}^\alpha)^{-1}\Big\{(x^1)_r^\alpha - S_r^\alpha(T)(x_0)_r^\alpha - \int_0^T S_r^\alpha(T-s)F_r^\alpha(s,(x_s)_r^\alpha)dC_s\Big\}\Big]\Big] \\
&= [(x^1)_l^\alpha, (x^1)_r^\alpha] = [x^1]^\alpha.
\end{aligned}
$$

Hence this control u_t satisfy a.s. θ, $x_T = x^1, x_T \in [\underline{x}_T, \overline{x}_T]$. Therefore every solutions of the equation (1) are controllable in $[0, T]$.

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Generalized interval-valued intuitionistic fuzzy soft rough set and its application

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Abstract

In this paper, by integrating interval-valued intuitionistic fuzzy soft set with rough set theory, the concept of generalized interval-valued intuitionistic fuzzy soft rough sets is proposed, which is an extension of generalized intuitionistic fuzzy soft rough sets. Then the properties of this model are investigated. Furthermore, classical representations of generalized interval-valued intuitionistic fuzzy soft rough approximation operators are also introduced. Finally, an approach based on generalized interval-valued intuitionistic fuzzy soft rough sets in decision making is developed, and we provide a practical example to illustrate the validity of this approach.

Key words: Interval-valued intuitionistic fuzzy soft set; Rough set; Generalized interval-valued intuitionistic fuzzy soft rough set; Decision making

1 Introduction

As a framework for the construction of approximations of concepts, rough sets proposed by Pawlak [21,22], is a formal tool for modeling and processing insufficient and incomplete information. In Pawlak's rough set model, the equivalence relation plays an important role, which seems very stringent in daily life. Therefore many researchers have generalized the notion of Pawlak rough set by replacing the equivalence relation with other binary relations. Since the appearance of Pawlak rough set, lots of fruitful results have been achieved [5, 10–12, 15, 16, 25, 28, 29, 31–40, 42, 44–46].

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Soft set theory is presented by Molodtsov [17], which is different from the existing uncertainty theories, such as fuzzy set theory [43], intuitionistic fuzzy set theory [1, 2], interval-valued fuzzy set theory [9, 13, 24], interval-valued intuitionistic fuzzy set theory [3,4], rough set theory [21,22], and so on. In [17], the author pointed out that these theories mentioned above have their inherent difficulties, but soft set has enough parameters so that it is free from inherent difficulties. Therefore, in recent years more and more researchers have joined the ranks of soft set research. For example, Maji et al. [18] initiated the study on hybrid structures involving fuzzy sets and soft sets, and introduced the concept of fuzzy soft sets, which can be viewed as a generalization of soft sets. Subsequently, Maji et al [19] modified the concept of fuzzy soft sets, and proposed a generalized fuzzy soft set theory. Furthermore, Yang et al. [30] extended soft sets to interval-valued fuzzy environment, and first presented the concept of interval-valued fuzzy soft sets by combining interval-valued fuzzy set and soft set. By integrating the intuitionistic fuzzy set with soft set theory, Maji et al. [20] presented the concept of the intuitionistic fuzzy soft set theory. Jiang et al. [14] initiated the concept of interval-valued intuitionistic fuzzy soft sets by the combination of the interval-valued intuitionistic fuzzy sets and soft sets. On the basis of [14], Zhang [46] presented an adjustable approach to interval-valued intuitionistic fuzzy soft sets based decision making by mean of level soft sets of interval-valued intuitionistic fuzzy soft sets. Recently, soft set theory has been developed into hesitant fuzzy environment, and the result is called hesitant fuzzy soft sets [6, 26, 27]. Because it is unreasonable to use hesitant fuzzy soft sets to handle some decision making problems, Zhang et al. [41] extended hesitant fuzzy soft sets to interval-valued hesitant fuzzy environment, and introduced the concept of interval-valued hesitant fuzzy soft sets by combining the interval-valued hesitant fuzzy set and soft set theory. More recently, by combining intuitionistic fuzzy soft set and rough set theory, Zhang et al. [38] introduced the concept of intuitionistic fuzzy soft rough sets, and gave an approach to decision making based on this model. Furthermore, in [42], they pointed out the drawback of the intuitionistic fuzzy soft rough sets, proposed a generalized intuitionistic fuzzy soft rough set model, and then illustrated the validity of this model by a practical example.

As a generalization of fuzzy soft sets, interval-valued fuzzy soft sets and intuitionistic fuzzy soft sets, interval-valued intuitionistic fuzzy soft set is more flexible and effective than other soft set theories to cope with imperfect and imprecise information. Meanwhile, we can note that the final decision results for the decision approach presented by Zhang [46] may be different based on different types of thresholds. That is to say, there actually does not exist a unique or uniform criterion for the evaluation of decision alternatives. That is its limitations and disadvantages. In order to overcome these limitations, we need to define a new interval-valued intuitionistic fuzzy soft set model such that the decision approach based on this model is less affected by subjective factors. In this paper, we mainly devote to the generalization of intuitionistic fuzzy soft rough sets [42] and propose the concept of generalized interval-valued intuitionistic fuzzy soft rough sets by integrating interval-valued intuitionistic fuzzy soft set with rough set. Also its decision making method is given. The most advantage of the decision making method is that it will only use the data information provided by the decision making problem without any additional available information provided by decision makers. Thus it can avoid the effect of subjective factors provided by different experts.

The rest of this paper is organized as follows. Section 2 briefly reviews some preliminaries. In Section 3, an interval-valued intuitionistic fuzzy soft relation is first defined by us. By combining the interval-valued intuitionistic fuzzy soft set and rough sets, then the concept of generalized interval-valued intuitionistic fuzzy soft rough approximation operators is presented and the properties of generalized upper and lower interval-valued intuitionistic fuzzy soft rough approximation operators are examined. Furthermore, classical representations of generalized interval-valued intuitionistic fuzzy soft rough approximation operators are presented. Section 4 is devoted to studying the application of generalized interval-valued intuitionistic fuzzy soft rough sets. Some conclusions and outlooks for further research are given in Section 5.

2 Preliminaries

In this section, we shall briefly recall some basic notions being used in the study.

Before introducing the notion of interval-valued intuitionistic fuzzy soft relation, we first give the concept of soft sets [17] and fuzzy soft sets [18].

Definition 2.1 ([17]) *Let U be an initial universe set and E be a universe set of parameters. A pair (F, E) is called a soft set over U if $F : E \to P(U)$, where $P(U)$ is the set of all subsets of U.*

Definition 2.2 ([18]) *Let U be an initial universe set and E be a universe set of parameters. A pair (F, E) is called a fuzzy soft set over U if $F : E \to F(U)$, where $F(U)$ is the set of all fuzzy subsets of U.*

By using the concepts of soft set and fuzzy soft set, Cagman et al. [7,8] introduced the definitions of crisp soft relation and fuzzy soft relation, respectively.

Definition 2.3 ([7]) *Let (F, E) be a soft set over U. Then a subset of $U \times E$ called a crisp soft relation from U to E is uniquely defined by*

$R = \{< (u, x), \mu_R(u, x) > | (u, x) \in U \times E\}$,

where $\mu_R : U \times E \to \{0, 1\},\ \mu_R(u, x) = \begin{cases} 1, & (u, x) \in R \\ 0, & (u, x) \notin R. \end{cases}$

Definition 2.4 ([8]) *Let (F, E) be a fuzzy soft set over U. Then a fuzzy subset of $U \times E$ called a fuzzy soft relation from U to E is uniquely defined by*

$R = \{< (u,x), \mu_R(u,x) > |(u,x) \in U \times E\}$,
where $\mu_R : U \times E \to [0,1],\ \mu_R(u,x) = \mu_{F(x)}(u)$.

Based on the crisp soft relation proposed by Cagman, Zhang et al. [42] constructed the following crisp soft rough sets.

Definition 2.5 ([42]) *Let U be an initial universe set and E be a universe set of parameters. For an arbitrary crisp soft relation R over $U \times E$, we can define a set-valued function $R_s : U \to P(E)$ by $R_s(u) = \{x \in E|(u,x) \in R\}, u \in U$.*

R is referred to as serial if for all $u \in U, R_s(u) \neq \emptyset$. The pair (U,E,R) is called a crisp soft approximation space. For any $A \subseteq E$, the upper and lower soft approximations of A with respect to (U,E,R), denoted by $\overline{R}(A)$ and $\underline{R}(A)$, are defined, respectively, as follows:

$$\overline{R}(A) = \{u \in U|R_s(u) \cap A \neq \emptyset\},\ \underline{R}(A) = \{u \in U|R_s(u) \subseteq A\}.$$

The pair $(\overline{R}(A), \underline{R}(A))$ is referred to as a crisp soft rough set, and $\overline{R}, \underline{R} : P(E) \to P(U)$ are, referred to as upper and lower crisp soft rough approximation operators, respectively.

Definition 2.6 ([3,4]) *Denote $L = \{(\alpha,\beta)|\alpha = [\alpha_1,\alpha_2] \in Int[0,1], \beta = [\beta_1,\beta_2] \in Int[0,1], \alpha_2 + \beta_2 \leq 1\}$, where $Int[0,1]$ denotes the set of all closed subintervals of $[0,1]$. We define a relation $\leq_L$ on L as follows: $\forall (\alpha,\beta),(\xi,\eta) \in L$,*

$$\begin{aligned}(\alpha,\beta) \leq_L (\xi,\eta) &\Leftrightarrow [\alpha_1,\alpha_2] \leq_{L^I} [\xi_1,\xi_2] \text{ and } [\beta_1,\beta_2] \geq_{L^I} [\eta_1,\eta_2] \\ &\Leftrightarrow \alpha_1 \leq \xi_1, \alpha_2 \leq \xi_2, \beta_1 \geq \eta_1, \text{ and } \beta_2 \geq \eta_2.\end{aligned}$$

Then the relation $\leq_L$ is a partial ordering on L and the pair $(L, \leq_L)$ is a complete lattice with the smallest element $0_L = ([0,0],[1,1])$ and the greatest element $1_L = ([1,1],[0,0])$. The meet operator $\wedge$ and the join operator $\vee$ on $(L, \leq_L)$ which are linked to the ordering $\leq_L$ are, respectively, defined as follows: $\forall (\alpha,\beta),(\xi,\eta) \in L$,
$(\alpha,\beta) \wedge (\xi,\eta) = ([\alpha_1 \wedge \xi_1, \alpha_2 \wedge \xi_2],[\beta_1 \vee \eta_1, \beta_2 \vee \eta_2])$,
$(\alpha,\beta) \vee (\xi,\eta) = ([\alpha_1 \vee \xi_1, \alpha_2 \vee \xi_2],[\beta_1 \wedge \eta_1, \beta_2 \wedge \eta_2])$.

Definition 2.7 ([3,4]) *Let a set U be fixed. The mapping $A : U \to L$ is called an interval-valued intuitionistic fuzzy (IVIF, for short) set on U. An interval-valued intuitionistic fuzzy set A on U can also be denoted by*
$A = \{< x, \mu_A(x), \gamma_A(x) > |x \in U\} = \{< x, [\mu_A^-(x), \mu_A^+(x)], [\gamma_A^-(x), \gamma_A^+(x)] > |x \in U\}$,
where $\mu_A(x) = [\mu_A^-(x), \mu_A^+(x)]$ and $\gamma_A(x) = [\gamma_A^-(x), \gamma_A^+(x)]$ satisfy $0 \leq \mu_A^+(x) + \gamma_A^+(x) \leq 1$ for all $x \in U$, and are, respectively, called the degree of membership and the degree of non-membership of the element $x \in U$ to A.

Let $IVIF(U)$ denotes the family of all interval-valued intuitionistic fuzzy sets on U.

3 Construction of generalized interval-valued intuitionistic fuzzy soft rough sets

In this section, we will present the concept of generalized IVIF soft rough sets by using the IVIF soft relation defined by us.

Definition 3.1 ([14]) *Let U be an initial universe set and E be a universe set of parameters. A pair (F, E) is called an IVIF soft set over U if $F : E \to IVIF(U)$, where $IVIF(U)$ is the set of all IVIF subsets of U.*

In the following, an IVIF soft relation will be presented, which is important for us to construct generalized IVIF soft rough sets.

Definition 3.2 *Let (F, E) be an IVIF soft set over U. Then an IVIF subset of $U \times E$ called an IVIF soft relation from U to E is uniquely defined by*

$R = \{< (u, x), \mu_R(u, x), \gamma_R(u, x) > |(u, x) \in U \times E\}$,

where $\mu_R : U \times E \to Int[0, 1]$ and $\gamma_R : U \times E \to Int[0, 1]$, for all $(u, x) \in U \times E$ such that $\mu_R(u, x) = [\mu^-_R(u, x), \mu^+_R(u, x)]$ and $\gamma_R(u, x) = [\gamma^-_R(u, x), \gamma^+_R(u, x)]$, which satisfy the condition $0 \le \mu^+_R(u, x) + \gamma^+_R(u, x) \le 1$.

Remark 3.3 *In Definition 3.2, if $\mu^-_R(u, x) = \mu^+_R(u, x)$ and $\gamma^-_R(u, x) = \gamma^+_R(u, x)$, namely, $\mu_R : U \times E \to [0, 1]$ and $\gamma_R : U \times E \to [0, 1]$, for all $(u, x) \in U \times E$ such that $0 \le \mu_R(u, x) + \gamma_R(u, x) \le 1$, then R is referred to as an intuitionistic fuzzy soft relation on $U \times E$. If R is an intuitionistic fuzzy soft relation on $U \times E$ and $\mu_R(u, x) + \gamma_R(u, x) = 1$, then R is degenerated to a fuzzy soft relation [8] in Definition 2.4. Hence, among fuzzy soft relation, intuitionistic fuzzy soft relation [42] and IVIF soft relation, the IVIF soft relation is the most generalized one. That is, the IVIF soft relation has included fuzzy soft relation and intuitionistic fuzzy soft relation.*

Let $U = \{u_1, u_2, \cdots, u_m\}$ and $E = \{x_1, x_2, \cdots, x_n\}$. Then the IVIF soft relation R from U to E can be presented by a table as in the following form

R	x_1	x_2	$\cdots$	x_n
u_1	$(\mu_R(u_1, x_1), \gamma_R(u_1, x_1))$	$(\mu_R(u_1, x_2), \gamma_R(u_1, x_2))$	$\cdots$	$(\mu_R(u_1, x_n), \gamma_R(u_1, x_n))$
u_2	$(\mu_R(u_2, x_1), \gamma_R(u_2, x_1))$	$(\mu_R(u_2, x_2), \gamma_R(u_2, x_2))$	$\cdots$	$(\mu_R(u_2, x_n), \gamma_R(u_2, x_n))$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
u_m	$(\mu_R(u_m, x_1), \gamma_R(u_m, x_1))$	$(\mu_R(u_m, x_2), \gamma_R(u_m, x_2))$	$\cdots$	$(\mu_R(u_m, x_n), \gamma_R(u_m, x_n))$

From the above form and the definition of IVIF soft set, we know that every IVIF soft set (F, E) is uniquely characterized by the IVIF soft relation, namely they are mutual determined. It means that an IVIF soft set (F, E) is formally equal to IVIF soft relation.

Therefore, we shall identify any IVIF soft set with IVIF soft relation and view these two concepts as interchangeable. Now, any discussion regard to IVIF soft set could be converted into analysis about IVIF soft relation, which will bring great convenience for our future researches.

In this case, according to the definition of IVIF soft relation, we can construct generalized IVIF soft rough sets as follows.

Definition 3.4 *Let U be an initial universe set and E be a universe set of parameters. For an arbitrary IVIF soft relation R over $U \times E$, the pair (U, E, R) is called an IVIF soft approximation space. For any $A \in IVIF(E)$, we define the upper and lower soft approximations of A with respect to (U, E, R), denoted by $\overline{R}(A)$ and $\underline{R}(A)$, respectively, as follows:*

$$\overline{R}(A) = \{< u, \mu_{\overline{R}(A)}(u), \gamma_{\overline{R}(A)}(u) > | u \in U\}, \quad (1)$$

$$\underline{R}(A) = \{< u, \mu_{\underline{R}(A)}(u), \gamma_{\underline{R}(A)}(u) > | u \in U\}. \quad (2)$$

where

$$\mu_{\overline{R}(A)}(u) = [\bigvee_{x\in E}(\mu_R^-(u,x) \wedge \mu_A^-(x)), \bigvee_{x\in E}(\mu_R^+(u,x) \wedge \mu_A^+(x))],$$
$$\gamma_{\overline{R}(A)}(u) = [\bigwedge_{x\in E}(\gamma_R^-(u,x) \vee \gamma_A^-(x)), \bigwedge_{x\in E}(\gamma_R^+(u,x) \vee \gamma_A^+(x))],$$
$$\mu_{\underline{R}(A)}(u) = [\bigwedge_{x\in E}(\gamma_R^-(u,x) \vee \mu_A^-(x)), \bigwedge_{x\in E}(\gamma_R^+(u,x) \vee \mu_A^+(x))],$$
$$\gamma_{\underline{R}(A)}(u) = [\bigvee_{x\in E}(\mu_R^-(u,x) \wedge \gamma_A^-(x)), \bigvee_{x\in E}(\mu_R^+(u,x) \wedge \gamma_A^+(x))].$$

The pair $(\overline{R}(A), \underline{R}(A))$ is referred to as a generalized IVIF soft rough set of A with respect to (U, E, R).

By $\mu_R^+(u,x) + \gamma_R^+(u,x) \le 1$ and $\mu_A^+(x) + \gamma_A^+(x) \le 1$, it can be easily verified that $\overline{R}(A)$ and $\underline{R}(A) \in IVIF(U)$. So we call $\overline{R}, \underline{R} : IVIF(E) \to IVIF(U)$ generalized upper and lower IVIF soft rough approximation operators, respectively.

Remark 3.5 *If R is an intuitionistic fuzzy soft relation on $U \times E$, then generalized IVIF soft rough approximation operators $\overline{R}(A)$ and $\underline{R}(A)$ in Definition 3.4 degenerate to the following forms:*

$$\overline{R}(A) = \{< u, \mu_{\overline{R}(A)}(u), \gamma_{\overline{R}(A)}(u) > | u \in U\},$$

$$\underline{R}(A) = \{< u, \mu_{\underline{R}(A)}(u), \gamma_{\underline{R}(A)}(u) > | u \in U\}.$$

where

$$\mu_{\overline{R}(A)}(u) = \bigvee_{x\in E}(\mu_R(u,x) \wedge \mu_A(x)), \ \gamma_{\overline{R}(A)}(u) = \bigwedge_{x\in E}(\gamma_R(u,x) \vee \gamma_A(x)),$$

$\mu_{\underline{R}(A)}(u) = \bigwedge_{x\in E}(\gamma_R(u,x) \vee \mu_A(x)),\ \gamma_{\underline{R}(A)}(u) = \bigvee_{x\in E}(\mu_R(u,x) \wedge \gamma_A(x)).$

In that case, the pair $(\overline{R}(A), \underline{R}(A))$ is generated into a generalized IF soft rough set of A with respect to (U, E, R) proposed by Zhang et al. [42]. That is, generalized IVIF soft rough set in Definition 4.4 includes generalized IF soft rough set [42] as a special case.

Remark 3.6 *If R is a fuzzy soft relation on $U \times E$ and $A \in F(E)$, then generalized IVIF soft rough approximation operators $\overline{R}(A)$ and $\underline{R}(A)$ degenerate to the following forms:*

$$\overline{R}(A) = \{< u, \mu_{\overline{R}(A)}(u) > | u \in U\},\quad \underline{R}(A) = \{< u, \mu_{\underline{R}(A)}(u) > | u \in U\}.$$

where $\mu_{\overline{R}(A)}(u) = \bigvee_{x\in E}[\mu_R(u,x) \wedge \mu_A(x)],\ \mu_{\underline{R}(A)}(u) = \bigwedge_{x\in E}[(1 - \mu_R(u,x)) \vee \mu_A(x)].$

In that case, generalized IVIF soft rough approximation operators $\overline{R}(A)$ and $\underline{R}(A)$ are identical with the soft fuzzy rough approximation operators defined by Sun [23]. That is, generalized IVIF soft rough approximation operators in Definition 4.4 are an extension of the soft fuzzy rough approximation operators defined by Sun [23].

In order to better understand the concept of generalized IVIF soft rough approximation operators, let us consider the following example.

Example 3.7 *Suppose that $U = \{u_1, u_2, u_3, u_4, u_5\}$ is the set of five houses under consideration of a decision maker to purchase. Let E be a parameter set, where $E = \{e_1, e_2, e_3, e_4\}$=$\{$expensive; beautiful; size; location$\}$. Mr. X wants to buy the house which qualifies with the parameters of E to the utmost extent from available houses in U. Assume that Mr. X describes the "attractiveness of the houses" by constructing an IVIF soft relation R from U to E. And it is presented by a table as in the following form.*

R	e_1	e_2	e_3	e_4
u_1	$([0.7, 0.8], [0.2, 0.2])$	$([0.3, 0.4], [0.2, 0.5])$	$([0.1, 0.1], [0.7, 0.8])$	$([0.3, 0.4], [0.1, 0.3])$
u_2	$([0.1, 0.2], [0.4, 0.6])$	$([0.6, 0.7], [0.1, 0.2])$	$([0.2, 0.3], [0.5, 0.7])$	$([0.3, 0.6], [0.2, 0.3])$
u_3	$([0.5, 0.6], [0.2, 0.4])$	$([0.3, 0.6], [0.2, 0.3])$	$([0.5, 0.7], [0.1, 0.3])$	$([0.1, 0.8], [0.1, 0.2])$
u_4	$([0.1, 0.3], [0.2, 0.6])$	$([0.5, 0.7], [0.1, 0.2])$	$([0.1, 0.4], [0.3, 0.5])$	$([0.2, 0.3], [0.5, 0.7])$
u_5	$([0.8, 0.9], [0.0, 0.1])$	$([0.3, 0.5], [0.4, 0.5])$	$([0.6, 0.8], [0.1, 0.2])$	$([0.4, 0.6], [0.1, 0.4])$

We can see that the precise evaluation for each object on each parameter is unknown while the lower and upper limits of such an evaluation are given. For example, we can not present the precise membership degree and non-membership degree of how beautiful house u_2 is, however, house u_2 is at least beautiful on the membership degree of 0.6 and it is at most beautiful on the membership degree of 0.7; house u_2 is not at least beautiful on *the non-membership degree of 0.1 and it is not at most beautiful on the non-membership degree of 0.2.*

Now give an IVIF subset A over the parameter set E as follows:

$$A = \{ < e_1, [0.7, 0.8], [0.1, 0.2] >, < e_2, [0.5, 0.7], [0.2, 0.3] >,\\ < e_3, [0.4, 0.6], [0.1, 0.3] >, < e_4, [0.2, 0.6], [0.3, 0.4] > \}.$$

By Equations (1) and (2), we have

$\mu_{\overline{R}(A)}(u_1) = [0.7, 0.8]$, $\gamma_{\overline{R}(A)}(u_1) = [0.2, 0.2]$, $\mu_{\overline{R}(A)}(u_2) = [0.5, 0.7]$,
$\gamma_{\overline{R}(A)}(u_2) = [0.2, 0.3]$, $\mu_{\overline{R}(A)}(u_3) = [0.5, 0.6]$, $\gamma_{\overline{R}(A)}(u_3) = [0.1, 0.3]$,
$\mu_{\overline{R}(A)}(u_4) = [0.5, 0.7]$, $\gamma_{\overline{R}(A)}(u_4) = [0.2, 0.3]$, $\mu_{\overline{R}(A)}(u_5) = [0.7, 0.8]$,
$\gamma_{\overline{R}(A)}(u_5) = [0.1, 0.2]$; $\mu_{\underline{R}(A)}(u_1) = [0.2, 0.6]$, $\gamma_{\underline{R}(A)}(u_1) = [0.3, 0.4]$,
$\mu_{\underline{R}(A)}(u_2) = [0.2, 0.6]$, $\gamma_{\underline{R}(A)}(u_2) = [0.3, 0.4]$, $\mu_{\underline{R}(A)}(u_3) = [0.2, 0.6]$,
$\gamma_{\underline{R}(A)}(u_3) = [0.2, 0.4]$, $\mu_{\underline{R}(A)}(u_4) = [0.4, 0.6]$, $\gamma_{\underline{R}(A)}(u_4) = [0.2, 0.3]$,
$\mu_{\underline{R}(A)}(u_5) = [0.2, 0.6]$, $\gamma_{\underline{R}(A)}(u_5) = [0.3, 0.4]$.

Thus

$$\overline{R}(A) = \{ < u_1, [0.7, 0.8], [0.2, 0.2] >, < u_2, [0.5, 0.7], [0.2, 0.3] >, < u_3, [0.5, 0.6], [0.1, 0.3] >,\\ < u_4, [0.5, 0.7], [0.2, 0.3] >, < u_5, [0.7, 0.8], [0.1, 0.2] > \}$$

and

$$\underline{R}(A) = \{ < u_1, [0.2, 0.6], [0.3, 0.4] >, < u_2, [0.2, 0.6], [0.3, 0.4] >, < u_3, [0.2, 0.6], [0.2, 0.4] >,\\ < u_4, [0.4, 0.6], [0.2, 0.3] >, < u_5, [0.2, 0.6], [0.3, 0.4] > \}.$$

In what follows, we investigate the properties of generalized IVIF soft rough approximation operators.

Theorem 3.8 *Let (U, E, R) be an IVIF soft approximation space. Then the generalized upper and lower IVIF soft rough approximation operators $\overline{R}(A)$ and $\underline{R}(A)$ satisfy the following properties: $\forall A, B \in IVIF(E)$,*

(IVIFSL1) $\underline{R}(A) =\sim \overline{R}(\sim A)$,
(IVIFSU1) $\overline{R}(A) =\sim \underline{R}(\sim A)$;
(IVIFSL2) $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$,
(IVIFSU2) $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$;
(IVIFSL3) $A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$,
(IVIFSU3) $A \subseteq B \Rightarrow \overline{R}(A) \subseteq \overline{R}(B)$;
(IVIFSL4) $\underline{R}(A \cup B) \supseteq \underline{R}(A) \cup \underline{R}(B)$,
(IVIFSU4) $\overline{R}(A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B)$;

Proof. We only prove the properties of the lower IVIF soft rough approximation operator $\underline{R}(A)$. The upper IVIF soft rough approximation operator $\overline{R}(A)$ can be proved similarly.
(IVIFSL1) By Definition 3.4, then we have

$$\begin{aligned}
\sim \underline{R}(\sim A) &= \{< u, \gamma_{\underline{R}(\sim A)}(u), \mu_{\underline{R}(\sim A)}(u) > |u \in U\} \\
&= \{< u, [\bigvee_{x\in E} (\mu_R^-(u,x) \wedge \gamma_{\sim A}^-(x)), \bigvee_{x\in E} (\mu_R^+(u,x) \wedge \gamma_{\sim A}^+(x))], \\
&\quad [\bigwedge_{x\in E} (\gamma_R^-(u,x) \vee \mu_{\sim A}^-(x)), \bigwedge_{x\in E} (\gamma_R^+(u,x) \vee \mu_{\sim A}^+(x))] > |u \in U\} \\
&= \{< u, [\bigvee_{x\in E} (\mu_R^-(u,x) \wedge \mu_A^-(x)), \bigvee_{x\in E} (\mu_R^+(u,x) \wedge \mu_A^+(x))], \\
&\quad [\bigwedge_{x\in E} (\gamma_R^-(u,x) \vee \gamma_A^-(x)), \bigwedge_{x\in E} (\gamma_R^+(u,x) \vee \gamma_A^+(x))] > |u \in U\} \\
&= \{< u, \mu_{\overline{R}(A)}(u), \gamma_{\underline{R}(A)}(u) > |u \in U\} = \overline{R}(A).
\end{aligned}$$

(IVIFSL2) By virtue of Equation (2), we have

$$\begin{aligned}
\underline{R}(A \cap B) &= \{< u, \mu_{\underline{R}(A\cap B)}(u), \gamma_{\underline{R}(A\cap B)}(u) > |u \in U\} \\
&= \{< u, \bigwedge_{x\in E} (\gamma_R(u,x) \vee \mu_{A\cap B}(x)), \bigvee_{x\in E} (\mu_R(u,x) \wedge \gamma_{A\cap B}(x)) > |u \in U\} \\
&= \{< u, [\bigwedge_{x\in E} (\gamma_R^-(u,x) \vee (\mu_A^-(x) \wedge \mu_B^-(x))), \bigwedge_{x\in E} (\gamma_R^+(u,x) \vee (\mu_A^+(x) \wedge \mu_B^+(x)))], \\
&\quad [\bigvee_{x\in E} (\mu_R^-(u,x) \wedge (\gamma_A^-(x) \vee \gamma_B^-(x))), \bigvee_{x\in E} (\mu_R^+(u,x) \wedge (\gamma_A^+(x) \vee \gamma_B^+(x)))] > |u \in U\} \\
&= \{< u, [\mu_{\underline{R}(A)}^-(u) \wedge \mu_{\underline{R}(B)}^-(u), \mu_{\underline{R}(A)}^+(u) \wedge \mu_{\underline{R}(B)}^+(u)], \\
&\quad [\gamma_{\underline{R}(A)}^-(u) \vee \gamma_{\underline{R}(B)}^-(u), \gamma_{\underline{R}(A)}^+(u) \vee \gamma_{\underline{R}(B)}^+(u)] > |u \in U\} \\
&= \{< u, \mu_{\underline{R}(A)}(u) \wedge \mu_{\underline{R}(B)}(u), \gamma_{\underline{R}(A)}(u) \vee \gamma_{\underline{R}(B)}(u) > |u \in U\} = \underline{R}(A) \cap \underline{R}(B).
\end{aligned}$$

(IVIFSL3) It can be easily verified by Definition 3.4.
(IVIFSL4) By (IVIFSL3), it is straightforward. □

In Theorem 3.8, properties (IVIFSL1) and (IVIFSU1) show that the generalized upper lower IVIF soft rough approximation operators $\overline{R}$ and $\underline{R}$ are dual to each other.

Inspired by the concept of cut sets of IF sets in [44,45], we first present the concept of cut sets of IVIF sets before investigating the representing method of the generalized IVIF soft rough approximation operators.

Definition 3.9 *Let* $A = \{< x, \mu_A(x), \gamma_A(x) > |x \in U\} \in IVIF(U)$, *and* $(\alpha, \beta) \in L$, *where* $\alpha = [\alpha_1, \alpha_2], \beta = [\beta_1, \beta_2] \in Int[0,1]$ *with* $\alpha_2 + \beta_2 \leq 1$. *The* (α, β)*-level cut set of* A, *denoted by* A_α^β *, is defined as follows:*

$$\begin{aligned} A_\alpha^\beta &= \{x \in U | \mu_A(x) \geq_{L^I} \alpha, \gamma_A(x) \leq_{L^I} \beta\} \\ &= \{x \in U | \mu_A^-(x) \geq \alpha_1, \mu_A^+(x) \geq \alpha_2, \gamma_A^-(x) \leq \beta_1, \gamma_A^+(x) \leq \beta_2\}. \end{aligned}$$

$$A_\alpha = \{x \in U | \mu_A(x) \geq_{L^I} \alpha\} = \{x \in U | \mu_A^-(x) \geq \alpha_1, \mu_A^+(x) \geq \alpha_2\},$$

and

$$A_{\alpha+} = \{x \in U | \mu_A(x) >_{L^I} \alpha\} = \{x \in U | \mu_A^-(x) > \alpha_1, \mu_A^+(x) > \alpha_2\}$$

are, respectively, called the α*-level cut set and the strong* α*-level cut set of membership generated by* A*. Meanwhile,*

$$A^\beta = \{x \in U | \gamma_A(x) \leq_{L^I} \beta\} = \{x \in U | \gamma_A^-(x) \leq \beta_1, \gamma_A^+(x) \leq \beta_2\}$$

and

$$A^{\beta+} = \{x \in U | \gamma_A(x) <_{L^I} \beta\} = \{x \in U | \gamma_A^-(x) < \beta_1, \gamma_A^+(x) < \beta_2\}$$

are, respectively, referred to as the β*-level cut set and the strong* β*-level cut set of non-membership generated by* A*.*

At the same time, other types of cut sets of the IVIF set A *are denoted as follows:*

$$\begin{aligned} A_{\alpha+}^\beta &= \{x \in U | \mu_A(x) >_{L^I} \alpha, \gamma_A(x) \leq_{L^I} \beta\} \\ &= \{x \in U | \mu_A^-(x) > \alpha_1, \mu_A^+(x) > \alpha_2, \gamma_A^-(x) \leq \beta_1, \gamma_A^+(x) \leq \beta_2\}, \end{aligned}$$

which is called the $(\alpha+, \beta)$*-level cut set of* A*;*

$$\begin{aligned} A_\alpha^{\beta+} &= \{x \in U | \mu_A(x) \geq_{L^I} \alpha, \gamma_A(x) <_{L^I} \beta\} \\ &= \{x \in U | \mu_A^-(x) \geq \alpha_1, \mu_A^+(x) \geq \alpha_2, \gamma_A^-(x) < \beta_1, \gamma_A^+(x) < \beta_2\}, \end{aligned}$$

which is called the $(\alpha, \beta+)$*-level cut set of* A*;*

$$\begin{aligned} A_{\alpha+}^{\beta+} &= \{x \in U | \mu_A(x) >_{L^I} \alpha, \gamma_A(x) <_{L^I} \beta\} \\ &= \{x \in U | \mu_A^-(x) > \alpha_1, \mu_A^+(x) > \alpha_2, \gamma_A^-(x) < \beta_1, \gamma_A^+(x) < \beta_2\}, \end{aligned}$$

which is called the $(\alpha+, \beta+)$*-level cut set of* A*.*

Theorem 3.10 *The cut sets of IVIF sets satisfy the following properties:* $\forall A \in IVIF(U)$, $\alpha = [\alpha_1, \alpha_2], \beta = [\beta_1, \beta_2] \in Int[0,1]$ *with* $\alpha_2 + \beta_2 \leq 1$,

(1) $A_\alpha^\beta = A_\alpha \cap A^\beta$,

(2) $A \subseteq B \Rightarrow A_\alpha^\beta \subseteq B_\alpha^\beta$,

(3) $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$, $(A \cap B)^\beta = A^\beta \cap B^\beta$,

(4) $\alpha \geq_{L^I} \beta, \xi \leq_{L^I} \eta \Rightarrow A_\alpha \subseteq A_\beta, A^\xi \subseteq A^\eta, A_\alpha^\xi \subseteq A_\beta^\eta$.

Proof. By Definition 3.9, (1), (2) and (4) are straightforward.

(3) Since

$$\begin{aligned} A \cap B &= \{< x, \mu_{A\cap B}(x), \gamma_{A\cap B}(x) > | x \in U\} \\ &= \{< x, [\mu_A^-(x) \wedge \mu_B^-(x), \mu_A^+(x) \wedge \mu_B^+(x)], \\ &\qquad [\gamma_A^-(x) \vee \gamma_B^-(x), \gamma_A^+(x) \vee \gamma_B^+(x)] > | x \in U\}, \end{aligned}$$

we have

$$\begin{aligned} (A \cap B)_\alpha &= \{x \in U | \mu_A^-(x) \wedge \mu_B^-(x) \geq \alpha_1, \mu_A^+(x) \wedge \mu_B^+(x) \geq \alpha_2\} \\ &= \{x \in U | \mu_A^-(x) \geq \alpha_1, \mu_B^-(x) \geq \alpha_1, \mu_A^+(x) \geq \alpha_2, \mu_B^+(x) \geq \alpha_2\} \\ &= \{x \in U | \mu_A(x) \geq_{L^I} \alpha, \mu_B(x) \geq_{L^I} \alpha\} = A_\alpha \cap B_\alpha, \end{aligned}$$

and

$$\begin{aligned} (A \cap B)^\beta &= \{x \in U | \gamma_A^-(x) \vee \gamma_B^-(x) \leq \beta_1, \gamma_A^+(x) \vee \gamma_B^+(x) \leq \beta_2\} \\ &= \{x \in U | \gamma_A^-(x) \leq \beta_1, \gamma_B^-(x) \leq \beta_1, \gamma_A^+(x) \leq \beta_2, \gamma_B^+(x) \leq \beta_2\} \\ &= \{x \in U | \gamma_A(x) \leq_{L^I} \beta, \gamma_B(x) \leq_{L^I} \beta\} = A^\beta \cap B^\beta. \end{aligned}$$

Meanwhile, according to (1), we can obtain

$$\begin{aligned} (A \cap B)_\alpha^\beta &= (A \cap B)_\alpha \cap (A \cap B)^\beta \\ &= (A_\alpha \cap A^\beta) \cap (B_\alpha \cap B^\beta) = A_\alpha^\beta \cap B_\alpha^\beta. \end{aligned}$$

□

Assume that R is an IVIF soft relation from U to E, denote

$$R_\alpha = \{(u,x) \in U \times E | \mu_R(u,x) \geq_{L^I} \alpha\} = \{(u,x) \in U \times E | \mu_R^-(u,x) \geq \alpha_1, \mu_R^+(u,x) \geq \alpha_2\},$$

$$R_\alpha(u) = \{x \in E | \mu_R(u,x) \geq_{L^I} \alpha\} = \{x \in E | \mu_R^-(u,x) \geq \alpha_1, \mu_R^+(u,x) \geq \alpha_2\}, \alpha_1, \alpha_2 \in [0,1];$$

$$R_{\alpha+} = \{(u,x) \in U \times E | \mu_R(u,x) >_{L^I} \alpha\} = \{(u,x) \in U \times E | \mu_R^-(u,x) > \alpha_1, \mu_R^+(u,x) > \alpha_2\},$$

$$R_{\alpha+}(u) = \{x \in E | \mu_R(u,x) >_{L^I} \alpha\} = \{x \in E | \mu_R^-(u,x) > \alpha_1, \mu_R^+(u,x) > \alpha_2\}, \alpha_1, \alpha_2 \in [0,1);$$

$$R^\beta = \{(u,x) \in U \times E | \gamma_R(u,x) \leq_{L^I} \beta\} = \{(u,x) \in U \times E | \gamma_R^-(u,x) \leq \beta_1, \gamma_R^+(u,x) \leq \beta_2\},$$

$$R^\beta(u) = \{x \in E | \gamma_R(u,x) \leq_{L^I} \beta\} = \{x \in E | \gamma_R^-(u,x) \leq \beta_1, \gamma_R^+(u,x) \leq \beta_2\}, \beta_1, \beta_2 \in [0,1];$$

$$R^{\beta+} = \{(u,x) \in U \times E | \gamma_R(u,x) <_{L^I} \beta\} = \{(u,x) \in U \times E | \gamma_R^-(u,x) < \beta_1, \gamma_R^+(u,x) < \beta_2\},$$

$$R^{\beta+}(u) = \{x \in E | \gamma_R(u,x) <_{L^I} \beta\} = \{x \in E | \gamma_R^-(u,x) < \beta_1, \gamma_R^+(u,x) < \beta_2\}, \beta_1, \beta_2 \in (0,1].$$

Then R_α, $R_{\alpha+}$, R^β and $R^{\beta+}$ are crisp soft relations on $U \times E$.

The following Theorems 3.12 and 3.13 show that the generalized IVIF soft rough approximation operators can be represented by crisp soft rough approximation operators proposed by Zhang et al. [42].

Theorem 3.11 *Let (U, E, R) be an IVIF soft approximation space, and $A \in IVIF(E)$. Then the generalized upper IVIF soft rough approximation operator can be represented as follows:* $\forall u \in U$, $\overline{a} = [a, a] \in L^I$,

(1)

$$\mu_{\overline{R}(A)}(u) = \bigvee_{\alpha \in L^I} [\alpha \wedge \overline{\overline{R_\alpha}(A_\alpha)(u)}] = \bigvee_{\alpha \in L^I} [\alpha \wedge \overline{\overline{R_\alpha}(A_{\alpha+})(u)}]$$
$$= \bigvee_{\alpha \in L^I} [\alpha \wedge \overline{\overline{R_{\alpha+}}(A_\alpha)(u)}] = \bigvee_{\alpha \in L^I} [\alpha \wedge \overline{\overline{R_{\alpha+}}(A_{\alpha+})(u)}],$$

(2)

$$\gamma_{\overline{R}(A)}(u) = \bigwedge_{\alpha \in L^I} [\alpha \vee \overline{\overline{R^\alpha}(A^\alpha)(u)}] = \bigwedge_{\alpha \in L^I} [\alpha \vee \overline{\overline{R^\alpha}(A^{\alpha+})(u)}]$$
$$= \bigwedge_{\alpha \in L^I} [\alpha \vee \overline{\overline{R^{\alpha+}}(A^\alpha)(u)}] = \bigwedge_{\alpha \in L^I} [\alpha \vee \overline{\overline{R^{\alpha+}}(A^{\alpha+})(u)}]$$

and moreover, for any $\alpha \in L^I$,

(3) $[\overline{R}(A)]_{\alpha+} \subseteq \overline{R_{\alpha+}}(A_{\alpha+}) \subseteq \overline{R_{\alpha+}}(A_\alpha) \subseteq \overline{R_\alpha}(A_\alpha) \subseteq [\overline{R}(A)]_\alpha$,

(4) $[\overline{R}(A)]^{\alpha+} \subseteq \overline{R^{\alpha+}}(A^{\alpha+}) \subseteq \overline{R^{\alpha+}}(A^\alpha) \subseteq \overline{R^\alpha}(A^\alpha) \subseteq [\overline{R}(A)]^\alpha$.

Proof. (1) For any $u \in U$, we have

$$\bigvee_{\alpha \in L^I} [\alpha \wedge \overline{\overline{R_\alpha}(A_\alpha)(u)}] = sup\{\alpha \in L^I | u \in \overline{R_\alpha}(A_\alpha)\} = sup\{\alpha \in L^I | R_\alpha(u) \cap A_\alpha \neq \emptyset\}$$
$$= sup\{\alpha \in L^I | \exists x \in E[x \in R_\alpha(u), x \in A_\alpha]\}$$
$$= sup\{\alpha \in L^I | \exists x \in E[\mu_R(u,x) \geq_{L^I} \alpha, \mu_A(x) \geq_{L^I} \alpha]\}$$
$$= sup\{[\alpha_1, \alpha_2] \in L^I | \exists x \in E[\mu_R^-(u,x) \geq \alpha_1, \mu_R^+(u,x) \geq \alpha_2, \mu_A^-(x) \geq \alpha_1, \mu_A^+(x) \geq \alpha_2]\}$$
$$= sup\{[\alpha_1, \alpha_2] \in L^I | \exists x \in E[\mu_R^-(u,x) \wedge \mu_A^-(x) \geq \alpha_1, \mu_R^+(u,x) \wedge \mu_A^+(x) \geq \alpha_2]\}$$
$$= [\bigvee_{x \in E} (\mu_R^-(u,x) \wedge \mu_A^-(x)), \bigvee_{x \in E} (\mu_R^+(u,x) \wedge \mu_A^+(x))] = \mu_{\overline{R}(A)}(u).$$

Likewise, we can conclude that

$$\mu_{\overline{R}(A)}(u) = \bigvee_{\alpha \in L^I} [\alpha \wedge \overline{\overline{R_\alpha}(A_{\alpha+})(u)}] = \bigvee_{\alpha \in L^I} [\alpha \wedge \overline{\overline{R_{\alpha+}}(A_\alpha)(u)}]$$
$$= \bigvee_{\alpha \in L^I} [\alpha \wedge \overline{\overline{R_{\alpha+}}(A_{\alpha+})(u)}].$$

(2) In terms of Definition 2.5 and notations above, we have

$$\begin{aligned}&\bigwedge_{\alpha\in L^I}[\alpha\vee\overline{\overline{R^\alpha}(A^\alpha)(u)}]=inf\{\alpha\in L^I]|u\in\overline{R^\alpha}(A^\alpha)\}=inf\{\alpha\in L^I|R^\alpha(u)\cap A^\alpha\neq\emptyset\}\\&=inf\{\alpha\in L^I|\exists x\in E[x\in R^\alpha(u),x\in A^\alpha]\}\\&=inf\{\alpha\in L^I|\exists x\in E[\gamma_R(u,x)\leq_{L^I}\alpha,\gamma_A(x)\leq_{L^I}\alpha]\}\\&=inf\{[\alpha_1,\alpha_2]\in L^I|\exists x\in E[\gamma_R^-(u,x)\leq\alpha_1,\gamma_R^+(u,x)\leq\alpha_2,\gamma_A^-(x)\leq\alpha_1,\gamma_A^+(x)\leq\alpha_2]\}\\&=inf\{[\alpha_1,\alpha_2]\in L^I|\exists x\in E[\gamma_R^-(u,x)\vee\gamma_A^-(x)\leq\alpha_1,\gamma_R^+(u,x)\vee\gamma_A^+(x)\leq\alpha_2]\}\\&=[\bigwedge_{x\in E}(\gamma_R^-(u,x)\vee\gamma_A^-(x)),\bigwedge_{x\in E}(\gamma_R^+(u,x)\vee\gamma_A^+(x))]=\gamma_{\overline{R}(A)}(u).\end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}\gamma_{\overline{R}(A)}(u)&=\bigwedge_{\alpha\in L^I}[\alpha\vee\overline{\overline{R^\alpha}(A^{\alpha+})(u)}]=\bigwedge_{\alpha\in L^I}[\alpha\vee\overline{\overline{R^{\alpha+}}(A^\alpha)(u)}]\\&=\bigwedge_{\alpha\in L^I}[\alpha\vee\overline{\overline{R^{\alpha+}}(A^{\alpha+})(u)}].\end{aligned}$$

(3) It is easily verified that $\overline{R_{\alpha+}}(A_{\alpha+})\subseteq\overline{R_{\alpha+}}(A_\alpha)\subseteq\overline{R_\alpha}(A_\alpha)$. We only need to prove that $[\overline{R}(A)]_{\alpha+}\subseteq\overline{R_{\alpha+}}(A_{\alpha+})$ and $\overline{R_\alpha}(A_\alpha)\subseteq[\overline{R}(A)]_\alpha$.

In fact, $\forall u\in[\overline{R}(A)]_{\alpha+}$, we have $\mu_{\overline{R}(A)}(u)>_{L^I}\alpha$. According to Definition 3.4, $\bigvee_{x\in E}[\mu_R^-(u,x)\wedge\mu_A^-(x)]>\alpha_1$ and $\bigvee_{x\in E}[\mu_R^+(u,x)\wedge\mu_A^+(x)]>\alpha_2$. Then $\exists x_0\in E$, such that $\mu_R^-(u,x_0)\wedge\mu_A^-(x_0)>\alpha_1$ and $\mu_R^+(u,x_0)\wedge\mu_A^+(x_0)>\alpha_2$, that is, $\mu_R^-(u,x_0)>\alpha_1,\mu_A^-(x_0)>\alpha_1,\mu_R^+(u,x_0)>\alpha_2$, and $\mu_A^+(x_0)>\alpha_2$. Thus $\mu_R(u,x_0)>_{L^I}\alpha$ and $\mu_A(x_0)>_{L^I}\alpha$, which imply that $x_0\in R_{\alpha+}(u)$ and $x_0\in A_{\alpha+}$. Namely, $R_{\alpha+}(u)\cap A_{\alpha+}\neq\emptyset$. By Definition 2.5, we have $u\in\overline{R_{\alpha+}}(A_{\alpha+})$. Hence $[\overline{R}(A)]_{\alpha+}\subseteq\overline{R_{\alpha+}}(A_{\alpha+})$.

On the other hand, for any $u\in\overline{R_\alpha}(A_\alpha)$, we have $\overline{R_\alpha}(A_\alpha)(u)=1$. Since $\mu_{\overline{R}(A)}(u)=\bigvee_{\beta\in L^I}[\beta\wedge\overline{\overline{R_\beta}(A_\beta)(u)}]\geq_{L^I}\alpha\wedge\overline{\overline{R_\alpha}(A_\alpha)(u)}=\alpha$, we obtain $u\in[\overline{R}(A)]_\alpha$. Hence, $\overline{R_\alpha}(A_\alpha)\subseteq[\overline{R}(A)]_\alpha$.

(4) Similar to the proof of (3), it can be easily verified. □

Theorem 3.12 *Let (U,E,R) be an IVIF soft approximation space, and $A\in IVIF(E)$. Then the generalized lower IVIF soft rough approximation operator can be represented as follows: $\forall u\in U$*

(1)

$$\begin{aligned}\mu_{\underline{R}(A)}(u)&=\bigwedge_{\alpha\in L^I}[\alpha\vee(\overline{1}-\overline{\underline{R^\alpha}(A_{\alpha+})(u)}]=\bigwedge_{\alpha\in L^I}[\alpha\vee(\overline{1}-\overline{\underline{R^\alpha}(A_\alpha)(u)}]\\&=\bigwedge_{\alpha\in L^I}[\alpha\vee(\overline{1}-\overline{\underline{R^{\alpha+}}(A_{\alpha+})(u)}]=\bigwedge_{\alpha\in L^I}[\alpha\vee(\overline{1}-\overline{\underline{R^{\alpha+}}(A_\alpha)(u)}],\end{aligned}$$

(2)

$$\gamma_{\underline{R}(A)}(u) = \bigvee_{\alpha \in L^I} [\alpha \wedge (\bar{1} - \overline{\underline{R_\alpha}(A^{\alpha+})(u)}] = \bigvee_{\alpha \in L^I} [\alpha \wedge (\bar{1} - \overline{\underline{R_\alpha}(A^{\alpha})(u)}]$$
$$= \bigvee_{\alpha \in L^I} [\alpha \wedge (\bar{1} - \overline{\underline{R_{\alpha+}}(A^{\alpha+})(u)}] = \bigvee_{\alpha \in L^I} [\alpha \wedge (\bar{1} - \overline{\underline{R_{\alpha+}}(A^{\alpha})(u)}]$$

and moreover, for any $\alpha \in L^I$,
(3) $[\underline{R}(A)]_{\alpha+} \subseteq \underline{R^\alpha}(A_{\alpha+}) \subseteq \underline{R^{\alpha+}}(A_{\alpha+}) \subseteq \underline{R^{\alpha+}}(A_\alpha) \subseteq [\underline{R}(A)]_\alpha$,
(4) $[\underline{R}(A)]^{\alpha+} \subseteq \underline{R_\alpha}(A^{\alpha+}) \subseteq \underline{R_{\alpha+}}(A^{\alpha+}) \subseteq \underline{R_{\alpha+}}(A^\alpha) \subseteq [\underline{R}(A)]^\alpha$.

Proof. The proof is similar to Theorem 3.12. □

4 Application of IVIF soft rough sets in decision making

In [46], Zhang et al. gave a decision method based on IVIF soft set theory. However, we note that the decision method need to choose the thresholds in advance by decision makers. Thus the decision results will be depend on the threshold values at some degree. Since the thresholds have different kind of subjective preference information, different experts can obtain the different decision results for the same decision problem. So, in order to avoid the effect of the subjective information for the decision results, we only use the data information provided by the decision making problem and don't need any additional available information provided by decision makers. Thus the decision results are more objectively.

Next, we shall develop a new approach to decision making problem based on the generalized IVIF soft rough sets proposed in this paper.

Let (U, E, R) be an IVIF soft approximation space, where U is the universe of the discourse, E is the parameter set, and R is an IVIF soft relation on $U \times E$. Then we can give this decision-making approach based on generalized IVIF soft rough sets with five steps.

First, according to their own needs, the decision makers can construct an IVIF soft relation R from U to E, or IVIF soft set (F, E) over U.

Second, for a ceratin decision evaluation problem, we suppose that one wants to find out the decision alternative in universe with the evaluation value as larger as possible on every evaluate index. On the basis of the assumption, we construct an optimum normal decision object A which is an IVIF set on the evaluation universe E as follows:

$$A = \{< e_i, \max_{1 \le j \le |U|} \mu_R(u_j, e_i), \min_{1 \le j \le |U|} \gamma_R(u_j, e_i) >\},$$

where $|U|$ denotes the cardinality of the universe set U.

Third, by Equations (1) and (2), we can compute the generalized IVIF soft rough approximation operators $\overline{R}(A)$ and $\underline{R}(A)$ of the optimum normal decision object A. Thus, we obtain two most close values $\overline{R}(A)$ and $\underline{R}(A)$ to the decision alternative u_i of the universe set U.

Fourth, Atanassov and Gargov [3,4] introduced the notion of IVIF sets, and gave two operations on two IVIF sets, shown as follows, for all $F, G \in IVIF(U)$,

• Union operation:

$$\begin{aligned} F \cup G = \{< u, &[\mu_F^-(u) \vee \mu_G^-(u), \mu_F^+(u) \vee \mu_G^+(u)], \\ &[\gamma_F^-(u) \wedge \gamma_G^-(u), \gamma_F^+(u) \wedge \gamma_G^+(u)] > | u \in U\}, \end{aligned}$$

• Intersection operation:

$$\begin{aligned} F \cap G = \{< u, &[\mu_F^-(u) \wedge \mu_G^-(u), \mu_F^+(u) \wedge \mu_G^+(u)], \\ &[\gamma_F^-(u) \vee \gamma_G^-(u), \gamma_F^+(u) \vee \gamma_G^+(u)] > | u \in U\}. \end{aligned}$$

In general, the union operation and intersection operation on IVIF sets may result in loss of information in practical decision making problem which affects the accuracy of decision making. Therefore, inspired by the concept of $\oplus$-union operation of intuitionistic fuzzy subset, we also introduce the concept of $\oplus$-union operation of IVIF subset.

Definition 4.1 *Let $F, G \in IVIF(U)$. The $\oplus$-union operation about IVIF sets F and G can be defined as follows:*

$$\begin{aligned} F \oplus G = \{< u, &[\mu_F^-(u) + \mu_G^-(u) - \mu_F^-(u) \cdot \mu_G^-(u), \mu_F^+(u) + \mu_G^+(u) - \mu_F^+(u) \cdot \mu_G^+(u)], \\ &[\gamma_F^-(u) \cdot \gamma_G^-(u), \gamma_F^+(u) \cdot \gamma_G^+(u)] > | u \in U\}. \end{aligned}$$

By using the $\oplus$-union operation rather than the union and intersection operations, we can obtain the choice set as follows

$$\begin{aligned} H = \overline{R}(A) \oplus \underline{R}(A) = \{< u, &[\mu_{\overline{R}(A)}^-(u) + \mu_{\underline{R}(A)}^-(u) - \mu_{\overline{R}(A)}^-(u) \cdot \mu_{\underline{R}(A)}^-(u), \\ &\mu_{\overline{R}(A)}^+(u) + \mu_{\underline{R}(A)}^+(u) - \mu_{\overline{R}(A)}^+(u) \cdot \mu_{\underline{R}(A)}^+(u)], \\ &[\gamma_{\overline{R}(A)}^-(u) \cdot \gamma_{\underline{R}(A)}^-(u), \gamma_{\overline{R}(A)}^+(u) \cdot \gamma_{\underline{R}(A)}^+(u)] > | u \in U\}. \end{aligned}$$

Denote $H = \{< u, \mu_H(u), \gamma_H(u) >\}$.

Finally, define an IVIF value $\lambda = (\mu, \gamma) \in L$, where $\mu = \sup_{1 \le j \le |U|} [\mu_H^-(u_j), \mu_H^+(u_j)]$, $\gamma = \inf_{1 \le j \le |U|} [\gamma_H^-(u_j), \gamma_H^+(u_j)]$. Obviously, IVIF value $\lambda = (\mu, \gamma)$ is the maximum choice value in the choice set H. Hence we take the object u_j in universe U with the maximum choice value as the optimum decision for the given decision making problem. That is to say, if $\mu_H(u_j) \ge_{L^I} \mu$ and $\gamma_H(u_j) \le_{L^I} \gamma$, the optimum decision is u_j.

In general, if there exist two or more objects with the same maximum choice value , then we can take one of them as the optimum decision for the given decision making problem.

To illustrate the new method given above, let us consider the example as follows.

Example 4.2 *Reconsider Example 3.7. Now all the available information on houses under consideration can be formulated as an IVIF soft relation describing attractiveness of house that Mr.X is going to buy. By using the second step of the algorithm for generalized IVIF soft rough sets in decision making presented in this section, we can obtain the optimum normal decision object A as follows*

$$\begin{aligned} A = \{ &< e_1, [0.8, 0.9], [0.0, 0.1] >, < e_2, [0.6, 0.7], [0.1, 0.2] >, \\ &< e_3, [0.6, 0.8], [0.1, 0.2] >, < e_4, [0.4, 0.8], [0.1, 0.2] > \}. \end{aligned}$$

According to Equations (1) and (2), we can conclude that

$$\begin{aligned} \overline{R}(A) = \{ &< u_1, [0.7, 0.8], [0.1, 0.2] >, < u_2, [0.6, 0.7], [0.1, 0.2] >, < u_3, [0.5, 0.8], [0.1, 0.2] >, \\ &< u_4, [0.5, 0.7], [0.1, 0.2] >, < u_5, [0.8, 0.9], [0.0, 0.1] > \} \end{aligned}$$

and

$$\begin{aligned} \underline{R}(A) = \{ &< u_1, [0.4, 0.8], [0.1, 0.2] >, < u_2, [0.4, 0.8], [0.1, 0.2] >, < u_3, [0.4, 0.8], [0.1, 0.2] >, \\ &< u_4, [0.5, 0.7], [0.1, 0.2] >, < u_5, [0.4, 0.8], [0.1, 0.2] > \}. \end{aligned}$$

Now by Definition 4.1, we have

$$\begin{aligned} H = \overline{R}(A) \oplus \underline{R}(A) = \{ &< u_1, [0.82, 0.96], [0.01, 0.04] >, < u_2, [0.76, 0.94], [0.01, 0.04] >, \\ &< u_3, [0.70, 0.96], [0.01, 0.04] >, < u_4, [0.75, 0.91], [0.01, 0.04] >, \\ &< u_5, [0.88, 0.98], [0.00, 0.02] > \}. \end{aligned}$$

Obviously, IVIF value $\lambda = ([0.88, 0.98], [0.00, 0.02])$ *is the maximum choice value in the choice set* H*. Thus the optimal decision is* u_5*. Hence, Mr X will buy the house* u_5*.*

5 Conclusion

Recently, there has been a growing interest in soft set theory. Some extensions of soft sets have been obtained by combining soft set theory with other mathematical models, including fuzzy soft sets, interval-valued fuzzy soft sets, intuitionistic fuzzy soft sets and interval-valued intuitionistic fuzzy soft sets. Among them, the interval-valued intuitionistic fuzzy soft set is the most generalized one. This paper is devoted to the discussion of the combinations of interval-valued intuitionistic fuzzy soft set and rough set. By using an interval-valued intuitionistic fuzzy soft relation, we present a new soft rough set model, called generalized IVIF soft rough sets. Furthermore, the generalized upper and lower IVIF soft rough approximation operators are represented by crisp soft rough approximation operators. Finally, a practical application is provided to illustrate the validity of the generalized IVIF soft rough set.

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GENERALIZATIONS OF HEINZ MEAN OPERATOR INEQUALITIES INVOLVING POSITIVE LINEAR MAP

CHANGSEN YANG AND YINGYA TAO

Abstract. In this paper, we study the Heinz mean inequalities of two positive operators involving positive linear map. We obtain a generalized conclusion based on operator Diaz-Metcalf type inequality. The conclusion is presented as follows: Let Φ be a unital positive linear map, if $0 < {m_1}^2 \leq A \leq {M_1}^2$ and $0 < {m_2}^2 \leq B \leq {M_2}^2$ for some positive real numbers $m_1 \leq M_1$, $m_2 \leq M_2$, then for $\alpha \in [0,1]$ and $p \geq 2$, the following inequality holds :

$$(\frac{M_2 m_2}{M_1 m_1}\Phi(A) + \Phi(B))^p$$
$$\leq 2^{-(p+4)}\left[\frac{M_2 m_2({M_1}^2 + {m_1}^2) + M_1 m_1({M_2}^2 + {m_2}^2)}{\min\{(M_1 m_1)^{\frac{3-\alpha}{2}}(M_2 m_2)^{\frac{1+\alpha}{2}}, (M_1 m_1)^{\frac{2+\alpha}{2}}(M_2 m_2)^{\frac{2-\alpha}{2}}\}}\right]^{2p}\Phi^p(H_\alpha(A,B)).$$

1. Introduction and preliminaries

We represent the set of all bounded operators on $\mathcal{H}$ by $B(\mathcal{H})$. If an operator A satisfies $\langle Ax, x\rangle \geq 0$ for any $x \in \mathcal{H}$, then A is called a positive operator. For two self-adjoint operators A and B, $A \geq B$ means $A - B \geq 0$. The notation $A > 0$ means A is an invertible positive operator.

A linear map Φ: $B(\mathcal{H}) \longrightarrow B(\mathcal{H})$ is called positive (strictly positive), if $\Phi(A) \geq 0$ ($\Phi(A) > 0$) whenever $A \geq 0$ ($A > 0$), and Φ is said to be unital if $\Phi(I) = I$. Take $A, B > 0$ and $\alpha \in [0,1]$, the weighted arithmetic operator mean $A\nabla_\alpha B$, geometric mean $A\sharp_\alpha B$ and harmonic mean $A!_\alpha B$ are defined as follows :

$$A\nabla_\alpha B = (1-\alpha)A + \alpha B,\ A\sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}},\ A!_\alpha B = [(1-\alpha)A^{-1} + \alpha B^{-1}]^{-1}$$

when $\alpha = \frac{1}{2}$, we write $A\nabla B$, $A\sharp B$ and $A!B$ for brevity, respectively. The Heniz mean is defined by $H_\alpha(A,B) = \frac{A\sharp_\alpha B + A\sharp_{1-\alpha} B}{2}$, where $A, B > 0$ and $\alpha \in [0,1]$. Recently, M. S. Moslehian, R. Nakamoto and Y. Seo [1, Theorem 2.1, part (ii)] showed that

Theorem 1.1 Let Φ be positive linear map, if $0 < {m_1}^2 \leq A \leq {M_1}^2$ and $0 < {m_2}^2 \leq B \leq {M_2}^2$ for some positive real numbers $m_1 \leq M_1$ and $m_2 \leq M_2$, we can get operator Diaz-Metcalf type inequality:

$$\frac{M_2 m_2}{M_1 m_1}\Phi(A) + \Phi(B) \leq (\frac{M_2}{m_1} + \frac{m_2}{M_1})\Phi(A\sharp B).$$

Thus $A\sharp B \leq H_\alpha(A,B)$ implies the following.

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Remark 1.2 Let Φ be positive linear map, if $0 < {m_1}^2 \le A \le {M_1}^2$ and $0 < {m_2}^2 \le B \le {M_2}^2$ for some positive real numbers $m_1 \le M_1$ and $m_2 \le M_2$, then for $\alpha \in [0,1]$, the following inequality holds:

$$\frac{M_2 m_2}{M_1 m_1}\Phi(A) + \Phi(B) \le (\frac{M_2}{m_1} + \frac{m_2}{M_1})\Phi(H_\alpha(A,B)).$$

In 2015, Mohammad Sal Moslehian and Xiaohui Fu obtained a second powering of the operator Diaz-Metcalf type inequality:

Theorem 1.3 [9] Let Φ be positive linear map, if $0 < {m_1}^2 \le A \le {M_1}^2$ and $0 < {m_2}^2 \le B \le {M_2}^2$ for some positive real numbers $m_1 \le M_1$ and $m_2 \le M_2$, then the following inequality holds:

$$(\frac{M_2 m_2}{M_1 m_1}\Phi(A) + \Phi(B))^2 \le \left(\frac{(M_1 m_1({M_2}^2 + {m_2}^2) + M_2 m_2({M_1}^2 + {m_1}^2))^2}{8\sqrt{M_1 m_1 M_2 m_2} M_1^2 m_1^2 M_2 m_2}\right)^2 (\Phi(A\sharp B))^2.$$

In the paper we shall give further generalizations of Remark 1.2 in the following section, along with presenting p-th powering of some inequality for Heniz mean based on Remark 1.2 and the following consideration: It is easy to see that the Heniz operator mean interpolates the arithmetic-geometric operator mean inequality: $A!B \le A\sharp B \le H_\alpha(A,B) \le A\nabla B$, and the geometric mean has so-called maximal characterization [2], which says that $\begin{bmatrix} A & A\sharp B \\ A\sharp B & B \end{bmatrix}$ is positive, and moreover, if the operator matrix $\begin{bmatrix} A & X \\ X & B \end{bmatrix}$ is positive with X being self-adjoint, then $A\sharp B \ge X$.

2. Results and Proofs

In order to prove the first main theorem of the paper, first we give the following lemmas.

lemma 2.1. [3] Let Φ be a unital strictly positive linear map and $A > 0$, then $\Phi(A)^{-1} \le \Phi(A^{-1})$.

lemma 2.2. [5] Let $A, B \ge 0$, then the following norm inequality holds : $\|AB\| \le \frac{1}{4}\|A+B\|^2$.

lemma 2.3. [4] Let $A, B \ge 0$, then for $1 \le r < +\infty$, $\|A^r + B^r\| \le \|(A+B)^r\|$.

lemma 2.4. [7] (L-H inequality) If $0 \le \alpha \le 1$, $A \ge B \ge 0$, then $A^\alpha \ge B^\alpha$.

Theorem 2.5. Let Φ be a unital positive linear map, if $0 < {m_1}^2 \le A \le {M_1}^2$ and $0 < {m_2}^2 \le B \le {M_2}^2$ for some positive real numbers $m_1 \le M_1$, $m_2 \le M_2$, then for $\alpha \in [0,1]$ and $p \ge 2$, the following inequality holds :

$$\begin{aligned}&(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B))^p\\ \leq&2^{-(p+4)}\left[\frac{M_2m_2({M_1}^2+{m_1}^2)+M_1m_1({M_2}^2+{m_2}^2)}{\min\{(M_1m_1)^{\frac{3-\alpha}{2}}(M_2m_2)^{\frac{1+\alpha}{2}},(M_1m_1)^{\frac{2+\alpha}{2}}(M_2m_2)^{\frac{2-\alpha}{2}}\}}\right]^{2p}\Phi^p(H_\alpha(A,B)).\end{aligned} \tag{2.1}$$

Proof. Obviously (2.1) is equivalent to

$$\begin{aligned}&\|(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B))^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(H_\alpha(A,B))\|\\ \leq&2^{-(\frac{p}{2}+2)}\left[\frac{M_2m_2({M_1}^2+{m_1}^2)+M_1m_1({M_2}^2+{m_2}^2)}{\min\{(M_1m_1)^{\frac{3-\alpha}{2}}(M_2m_2)^{\frac{1+\alpha}{2}},(M_1m_1)^{\frac{2+\alpha}{2}}(M_2m_2)^{\frac{2-\alpha}{2}}\}}\right]^{p}.\end{aligned}$$

Note that

$$({M_1}^2-A)({m_1}^2-A)A^{-1}\leq 0,$$

implies

$${M_1}^2{m_1}^2A^{-1}-{M_1}^2-{m_1}^2+A\leq 0,$$

therefore

$${M_1}^2{m_1}^2\Phi(A^{-1})+\Phi(A)\leq {M_1}^2+{m_1}^2,$$

which equals to

$$M_1m_1M_2m_2\Phi(A^{-1})+\tfrac{M_2m_2}{M_1m_1}\Phi(A)\leq \tfrac{M_2m_2}{M_1m_1}({M_1}^2+{m_1}^2). \tag{2.2}$$

Similarly, we have

$${M_2}^2{m_2}^2\Phi(B^{-1})+\Phi(B)\leq {M_2}^2+{m_2}^2. \tag{2.3}$$

Since

$$H_\alpha^{-1}(A,B)\leq (A!B)^{-1}=\frac{A^{-1}+B^{-1}}{2},$$

therefore

$$\begin{aligned}&H_\alpha(\frac{A}{M_2m_2M_1m_1},\frac{B}{{M_2}^2{m_2}^2})\\ =&\frac{(\frac{1}{M_2m_2M_1m_1})^{1-\alpha}(\frac{1}{{M_2}^2{m_2}^2})^{\alpha}(A\sharp_\alpha B)+(\frac{1}{M_2m_2M_1m_1})^{\alpha}(\frac{1}{{M_2}^2{m_2}^2})^{1-\alpha}(A\sharp_{1-\alpha}B)}{2}\\ \leq&\max\{(\frac{1}{M_2m_2M_1m_1})^{1-\alpha}(\frac{1}{M_2m_2})^{2\alpha},(\frac{1}{M_2m_2M_1m_1})^{\alpha}(\frac{1}{M_2m_2})^{2-2\alpha}\}H_\alpha(A,B)\\ =&\frac{H_\alpha(A,B)}{\min\{(M_1m_1)^{1-\alpha}(M_2m_2)^{1+\alpha},(M_1m_1)^{\alpha}(M_2m_2)^{2-\alpha}\}}.\end{aligned} \tag{2.4}$$

If we put

$$\beta=\min\{(M_1m_1)^{1-\alpha}(M_2m_2)^{1+\alpha},(M_1m_1)^{\alpha}(M_2m_2)^{2-\alpha}\},$$

then

$$\begin{aligned}
&\beta\Phi^{-1}(H_\alpha(A,B))\\
\le&\Phi^{-1}(H_\alpha(\frac{A}{M_2m_2M_1m_1},\frac{B}{{M_2}^2{m_2}^2}))\\
\le&\Phi({H^{-1}}_\alpha(\frac{A}{M_2m_2M_1m_1},\frac{B}{{M_2}^2{m_2}^2}))\\
\le&\frac{1}{2}\Phi(M_2m_2M_1m_1A^{-1}+{M_2}^2{m_2}^2B^{-1})\\
=&\frac{1}{2}(M_2m_2M_1m_1\Phi(A^{-1})+{M_2}^2{m_2}^2\Phi(B^{-1})).
\end{aligned}$$

By (2.2) and (2.3), we have

$$\begin{aligned}
&\|(\frac{1}{2}(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B)))^{\frac{p}{2}}\beta^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(H_\alpha(A,B))\|\\
\le&\frac{1}{4}\|(\frac{1}{2}(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B)))^{\frac{p}{2}}+\beta^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(H_\alpha(A,B))\|^2\\
\le&\frac{1}{4}\|(\frac{1}{2}(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B))+\beta\Phi^{-1}(H_\alpha(A,B)))^{\frac{p}{2}}\|^2\\
=&\frac{1}{4}\|\frac{1}{2}(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B))+\beta\Phi^{-1}(H_\alpha(A,B))\|^p\\
\le&\frac{1}{4}\|\frac{1}{2}(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B)+M_2m_2M_1m_1\Phi(A^{-1})+{M_2}^2{m_2}^2\Phi(B^{-1}))\|^p\\
\le&2^{-(p+2)}({M_2}^2+{m_2}^2+\frac{M_2m_2}{M_1m_1}({M_1}^2+{m_1}^2))^p.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B))^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(H_\alpha(A,B))\|\\
\le&2^{-(\frac{p}{2}+2)}\left[\frac{M_2m_2({M_1}^2+{m_1}^2)+M_1m_1({M_2}^2+{m_2}^2)}{\min\{(M_1m_1)^{\frac{3-\alpha}{2}}(M_2m_2)^{\frac{1+\alpha}{2}},(M_1m_1)^{\frac{2+\alpha}{2}}(M_2m_2)^{\frac{2-\alpha}{2}}\}}\right]^p.
\end{aligned}$$

Corollary 2.6. In Theorem 2.5, if $1\le p\le 2$, we get

$$\begin{aligned}
&(\frac{M_2m_2}{M_1m_1}\Phi(A)+\Phi(B))^p\\
\le&2^{-3p}\left[\frac{M_2m_2({M_1}^2+{m_1}^2)+M_1m_1({M_2}^2+{m_2}^2)}{\min\{(M_1m_1)^{\frac{3-\alpha}{2}}(M_2m_2)^{\frac{1+\alpha}{2}},(M_1m_1)^{\frac{2+\alpha}{2}}(M_2m_2)^{\frac{2-\alpha}{2}}\}}\right]^{2p}\Phi^p(H_\alpha(A,B)).
\end{aligned}$$

Theorem 2.7. Let Φ be a unital positive linear map, if $0<{m_1}^2\le A\le {M_1}^2$ and $0<{m_2}^2\le B\le {M_2}^2$ for some positive real numbers $m_1\le M_1$, $m_2\le M_2$, then for $\alpha\in[0,1]$ and $p\ge 2$, the following inequality holds :

$$(\Phi(A)\nabla_\alpha\Phi(B))^p\le 2^{-(p+4)}\left[\frac{{M_1}^2+(1-\alpha){m_1}^2+{M_2}^2+\alpha{m_2}^2}{\min\{(M_1m_1)^{1-\alpha}(M_2m_2)^\alpha,(M_1m_1)^\alpha(M_2m_2)^{1-\alpha}\}}\right]^{2p}\Phi^p(H_\alpha(A,B)). \quad (2.5)$$

Proof. Obviously (2.5) is equivalent to

$$\begin{aligned}&\|(\Phi(A)\nabla_\alpha\Phi(B))^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(H_\alpha(A,B))\|\\ \leq&2^{-(\frac{p}{2}+2)}\left[\frac{{M_1}^2+(1-\alpha){m_1}^2+{M_2}^2+\alpha{m_2}^2}{\min\{(M_1m_1)^{1-\alpha}(M_2m_2)^\alpha,(M_1m_1)^\alpha(M_2m_2)^{1-\alpha}\}}\right]^p.\end{aligned}$$

Note that

$$({M_1}^2-(1-\alpha)A)({m_1}^2-A)A^{-1}\leq 0,$$

implies

$${M_1}^2{m_1}^2A^{-1}-{M_1}^2-(1-\alpha){m_1}^2+(1-\alpha)A\leq 0.$$

Therefore

$${M_1}^2{m_1}^2\Phi(A^{-1})+(1-\alpha)\Phi(A)\leq{M_1}^2+(1-\alpha){m_1}^2. \tag{2.6}$$

Similarly, we have

$${M_2}^2{m_2}^2\Phi(B^{-1})+\alpha\Phi(B)\leq{M_2}^2+\alpha{m_2}^2. \tag{2.7}$$

Since

$$H_\alpha^{-1}(A,B)\leq(A!B)^{-1}=\frac{A^{-1}+B^{-1}}{2},$$

and by analogy to (2.4)

$$\begin{aligned}&H_\alpha(\frac{A}{{M_1}^2{m_1}^2},\frac{B}{{M_2}^2{m_2}^2})\\ =&\frac{H_\alpha(A,B)}{\min\{(M_1m_1)^{2-2\alpha}(M_2m_2)^{2\alpha},{(M_1m_1)}^{2\alpha}(M_2m_2)^{2-2\alpha}\}}.\end{aligned}$$

By puting

$$h=\min\{(M_1m_1)^{2-2\alpha}(M_2m_2)^{2\alpha},{(M_1m_1)}^{2\alpha}(M_2m_2)^{2-2\alpha}\},$$

we have

$$\begin{aligned}&h\Phi^{-1}(H_\alpha(A,B))\\ \leq&h\Phi^{-1}(H_\alpha(\frac{A}{M_1^2m_1^2},\frac{B}{{M_2}^2{m_2}^2}))\\ \leq&h\Phi({H^{-1}}_\alpha(\frac{A}{M_1^2m_1^2},\frac{B}{{M_2}^2{m_2}^2}))\\ \leq&\frac{1}{2}\Phi(M_1^2m_1^2A^{-1}+{M_2}^2{m_2}^2B^{-1})\\ =&\frac{1}{2}(M_1^2m_1^2\Phi(A^{-1})+{M_2}^2{m_2}^2\Phi(B^{-1})).\end{aligned}$$

By (2.6) and (2.7), we have

$$
\begin{aligned}
&\|(\frac{1}{2}\Phi(A)\nabla_\alpha\Phi(B))^{\frac{p}{2}}h^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(H_\alpha(A,B))\| \\
\le&\frac{1}{4}\|(\frac{1}{2}\Phi(A)\nabla_\alpha\Phi(B))^{\frac{p}{2}}+h^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(H_\alpha(A,B))\|^2 \\
\le&\frac{1}{4}\|(\frac{1}{2}\Phi(A)\nabla_\alpha\Phi(B)+h\Phi^{-1}(H_\alpha(A,B)))^{\frac{p}{2}}\|^2 \\
=&\frac{1}{4}\|\frac{1}{2}\Phi(A)\nabla_\alpha\Phi(B)+h\Phi^{-1}(H_\alpha(A,B))\|^p \\
\le&\frac{1}{4}\|\frac{1}{2}((1-\alpha)\Phi(A)+\alpha\Phi(B)+M_1^2m_1^2\Phi(A^{-1})+{M_2}^2{m_2}^2\Phi(B^{-1}))\|^p \\
\le&2^{-(p+2)}({M_1}^2+(1-\alpha){m_1}^2+{M_2}^2+\alpha{m_2}^2)^p.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\|(\Phi(A)\nabla_\alpha\Phi(B))^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(H_\alpha(A,B))\| \\
\le&2^{-(\frac{p}{2}+2)}\left[\frac{{M_1}^2+(1-\alpha){m_1}^2+{M_2}^2+\alpha{m_2}^2}{\min\{(M_1m_1)^{1-\alpha}(M_2m_2)^\alpha,(M_1m_1)^\alpha(M_2m_2)^{1-\alpha}\}}\right]^p.
\end{aligned}
$$

Theorem 2.8. Let Φ be a unital positive linear map, if $0<{m_1}^2\le A\le{M_1}^2$ and $0<{m_2}^2\le B\le{M_2}^2$ for some positive real numbers $m_1\le M_1$, $m_2\le M_2$, δ is a arbitrary mean less than or equal to arithmetic mean, then for $\alpha\in[0,1]$ and $p\ge 2$, the following inequality holds :

$$
(\Phi(A)\delta\Phi(B))^p\le 2^{-(2p+4)}\left[\frac{{M_1}^2+{M_2}^2+{m_1}^2+{m_2}^2}{\min\{(M_1m_1)^{1-\alpha}(M_2m_2)^\alpha,(M_1m_1)^\alpha(M_2m_2)^{1-\alpha}\}}\right]^{2p}\Phi^p(H_\alpha(A,B)).
$$

Proof. By the similar method of proofing Theorem 2.7.

Corollary 2.9. In Theorem 2.8, we easily get

$$
{H_\alpha}^p(\Phi(A),\Phi(B))\le 2^{-(2p+4)}\left[\frac{{M_1}^2+{M_2}^2+{m_1}^2+{m_2}^2}{\min\{(M_1m_1)^{1-\alpha}(M_2m_2)^\alpha,(M_1m_1)^\alpha(M_2m_2)^{1-\alpha}\}}\right]^{2p}\Phi^p(H_\alpha(A,B)).
$$

Theorem 2.10. [8] Let $0<m\le A$, $B\le M$, with the scalars $m,M>0$ and σ,τ two arbitrary means beween harmonic and arithmetic means, then for every positive unital linear map Φ, $2\le p<\infty$,

$$
\Phi^p(A\sigma B)\le(\frac{(M+m)^2}{4^{\frac{2}{p}}Mm})^p(\Phi(A)\tau\Phi(B))^p.
$$

By $A!B\le H_\alpha(A,B)\le A\nabla B$, we obtain the following inequality.

Remark 2.11. Let $0<m\le A,B\le M$, then for every positive unital linear map Φ and $0<\alpha<1$, $K(h)=\frac{(h+1)^2}{4h}$, $h=\frac{M}{m}$, $p\ge 2$, the following inequality holds :

$$
\Phi^p(H_\alpha(A,B))\le 2^{2p-4}K^p(h){H_\alpha}^p(\Phi(A),\Phi(B)). \tag{2.8}
$$

lemma 2.12. [6] For any bounded operator X,

$$|X| \leq tI \Longleftrightarrow \|X\| \leq t \Longleftrightarrow \begin{bmatrix} tI & X \\ X^* & tI \end{bmatrix} \geq 0 \ (t \geq 0).$$

Theorem 2.13. Let $0 < m \leq A, B \leq M$, then for every positive unital linear map Φ and $0 < \alpha < 1$, $K(h) = \frac{(h+1)^2}{4h}, h = \frac{M}{m}$, $p \geq 2$, the following inequality holds :

$$\Phi^{\frac{p}{2}}(H_\alpha(A,B)){H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B)) + {H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B))\Phi^{\frac{p}{2}}(H_\alpha(A,B)) \leq 2^{p-1}K^{\frac{p}{2}}(h). \quad (2.9)$$

Proof. By (2.8) we get

$$\|\Phi^{\frac{p}{2}}(H_\alpha(A,B)){H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B))\| \leq 2^{p-2}K^{\frac{p}{2}}(h). \quad (2.10)$$

By (2.10) and Lemma 2.12, we obtain

$$\begin{bmatrix} 2^{p-2}K^{\frac{p}{2}}(h)I & \Phi^{\frac{p}{2}}(H_\alpha(A,B)){H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B)) \\ {H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B))\Phi^{\frac{p}{2}}(H_\alpha(A,B)) & 2^{p-2}K^{\frac{p}{2}}(h)I \end{bmatrix} \geq 0,$$

and

$$\begin{bmatrix} 2^{p-2}K^{\frac{p}{2}}(h)I & {H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B))\Phi^{\frac{p}{2}}(H_\alpha(A,B)) \\ \Phi^{\frac{p}{2}}(H_\alpha(A,B)){H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B)) & 2^{p-2}K^{\frac{p}{2}}(h)I \end{bmatrix} \geq 0.$$

Summing up these two operator matrices above, put

$$2^{p-2}K^{\frac{p}{2}}(h) = t,$$

$$\Phi^{\frac{p}{2}}(H_\alpha(A,B)){H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B)) + {H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B))\Phi^{\frac{p}{2}}(H_\alpha(A,B)) = X.$$

We have

$$\begin{bmatrix} 2tI & X \\ X^* & 2tI \end{bmatrix} \geq 0.$$

Since $\Phi^{\frac{p}{2}}(H_\alpha(A,B)){H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B)) + {H_\alpha}^{-\frac{p}{2}}(\Phi(A),\Phi(B))\Phi^{\frac{p}{2}}(H_\alpha(A,B))$ is self-adioint, (2.9) follows from the maximal characterization of geometric mean.

Corollary 2.14. Let Φ be a unital positive linear map, if $0 < {m_1}^2 \leq A \leq {M_1}^2$ and $0 < {m_2}^2 \leq B \leq {M_2}^2$ for some positive real numbers $m_1 \leq M_1$, $m_2 \leq M_2$, then for $\alpha \in [0,1]$ and $p \geq 2$, the following inequality holds :

$$\begin{aligned} &{H_\alpha}^{\frac{p}{2}}(\Phi(A),\Phi(B))\Phi^{-\frac{p}{2}}(H_\alpha(A,B)) + \Phi^{-\frac{p}{2}}(H_\alpha(A,B)){H_\alpha}^{\frac{p}{2}}(\Phi(A),\Phi(B)) \\ \leq &2^{-(p+1)}\left[\frac{{M_1}^2 + {M_2}^2 + {m_1}^2 + {m_2}^2}{\min\{(M_1m_1)^{1-\alpha}(M_2m_2)^\alpha, (M_1m_1)^\alpha(M_2m_2)^{1-\alpha}\}}\right]^{2p}\Phi^p(H_\alpha(A,B)). \end{aligned}$$

Proof. By Corollary 2.9 and the similar method of proofing Theorem 2.13, we can easily get.

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Existence and uniqueness results of nonlocal fractional sum-difference boundary value problems for fractional difference equations involving sequential fractional difference operators.

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Abstract

In this article, we study some new existence results for a nonlinear fractional difference equation with fractional sum-difference boundary conditions. Our problem containing sequential fractional difference operators that have different orders. The existence and uniqueness results are based on Banach contraction mapping principle and Schaefer's fixed point theorem. Finally, we present some examples to show the importance of these results.

Keywords: Fractional difference equations; boundary value problems; existence.

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1 Introduction

In this paper we consider a fractional sum-difference boundary value problem of a fractional difference equation of the form

$$\begin{cases} \Delta^{\alpha}u(t) = f(t+\alpha-1, u(t+\alpha-1), \Delta^{\mu}\Delta^{\nu}u(t+\alpha-\mu-\nu+1)), \\ u(\alpha-2) = \Delta^{\theta}u(\alpha-\theta-2) = p\,y(u), \\ u(T+\alpha) = q\,\Delta^{-\beta}u(\eta+\beta), \end{cases} \tag{1.1}$$

where $t \in \mathbb{N}_{0,T} := \{0,1,...,T\}$, $p,q > 0$, $2 < \alpha \leq 3$, $0 < \beta,\theta,\mu,\nu \leq 1$, $1 < \mu+\nu \leq 2$, $\eta \in \mathbb{N}_{\alpha-1,T+\alpha-1}$, $f \in (\mathbb{N}_{\alpha-3,T+\alpha} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is a given function, and $y : C\left(\mathbb{N}_{\alpha-3,T+\alpha}, \mathbb{R}\right) \to \mathbb{R}$ is a given functional.

Mathematicians have used this fractional calculus in recent years to model and solve various related problems. In particular, fractional calculus is a powerful tool for the processes which appears in nature, e.g. biology, ecology and other areas.

Fractional difference equations have been interested many researchers since can use for describing many problems in the real-world phenomena such as physics, chemistry,

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mechanics, control systems, flow in porous media, and electrical networks can be found in [1] and [2] and the references therein. An excellent papers dealing with discrete fractional boundary value problems, which has helped to establish some of the basic theory of this field, one may see the papers [3]-[17], and references cited therein.

For example, Kang *et* al. [3] obtained sufficient conditions for the existence of solutions for the nonlocal boundary value problem as follows,

$$\begin{cases} -\Delta^{\mu}y(t) = \lambda h(t+\mu-1)\, f(y(t+\mu-1)), \quad t\in\mathbb{N}_{0,b} := \{0,1,...,b\}, \\ y(\mu-2)=\Psi(y), \quad y(\mu+b)=\Phi(y), \end{cases} \tag{1.2}$$

where $1<\mu\le 2$, $f\in C([0,\infty),[0,\infty))$ and $h\in C(\mathbb{N}_{\mu-1,\mu+b-1},[0,\infty))$ are given functions, and $\Psi,\Phi:\mathbb{R}^{b+3}\to\mathbb{R}$ are given functionals.

Presently, Chasreechai *et* al. [15] examined a Caputo fractional sum-difference equation with nonlocal fractional sum boundary value conditions of the form

$$\begin{cases} \Delta_C^{\alpha}u(t) = f(t+\alpha-1, u(t+\alpha-1), (\Psi^{\beta}u)(t+\alpha-2)), \quad t\in\mathbb{N}_{0,T}, \\ u(\alpha-2) = y(u), \\ u(T+\alpha) = \Delta^{-\gamma}g(T+\alpha+\gamma-3)\, u(T+\alpha+\gamma-3), \end{cases} \tag{1.3}$$

where $1<\alpha\le 2$, $0<\beta\le 1$, $2<\gamma\le 3$. For $U\subseteq\mathbb{R}$, $g\in C(\mathbb{N}_{\alpha-2,T+\alpha},\mathbb{R}^{+}\cap U)$, $f\in C(\mathbb{N}_{\alpha-2,T+\alpha}\times U\times U, U)$ are given functions, $y: C(\mathbb{N}_{\alpha-2,T+\alpha},U)\to U$ is a given functional, and for $\varphi:\mathbb{N}_{\alpha-2,T+\alpha}\times\mathbb{N}_{\alpha-2,T+\alpha}\to[0,\infty)$,

$$(\Psi^{\beta}u)(t) := [\Delta^{-\beta}\varphi\, u](t+\beta) = \frac{1}{\Gamma(\beta)}\sum_{s=\alpha-\beta-2}^{t-\beta}(t-\sigma(s))^{\underline{\beta-1}}\varphi(t,s+\beta)\, u(s+\beta).$$

The plan of this paper is as follows. In Section 2, we recall some definitions and basic lemmas. Also, we derive a representation of the solution to (1.1) by converting the problem to an equivalent fractional sum equation. In Section 3, the existence and uniqueness results of the boundary value problem (1.1) are established by Banach contraction mapping principle and Schaefer's fixed point theorem. An illustrative example is presented in Section 4.

2 Preliminaries

In this section, we introduce notations, definitions, and lemmas that are used in the main results.

Definition 2.1. *We define the generalized falling function by* $t^{\underline{\alpha}} := \dfrac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}$, *for any* t *and* α *for which the right-hand side is defined. If* $t+1-\alpha$ *is a pole of the Gamma function and* $t+1$ *is not a pole, then* $t^{\underline{\alpha}}=0$.

Lemma 2.1. *[10] If $t \le r$, then $t^{\underline{\alpha}} \le r^{\underline{\alpha}}$ for any $\alpha > 0$.*

Definition 2.2. *For $\alpha > 0$ and f defined on $\mathbb{N}_a$, the α-order fractional sum of f is defined by*

$$\Delta^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-\sigma(s))^{\underline{\alpha-1}} f(s),$$

for $t \in \mathbb{N}_{a+\alpha}$ and $\sigma(s) = s+1$.

Definition 2.3. *For $\alpha > 0$ and f defined on $\mathbb{N}_a$, the α-order Riemann-Liouville fractional difference of f is defined by*

$$\Delta^{\alpha} f(t) := \Delta^N \Delta^{-(N-\alpha)} f(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^{t+\alpha} (t-\sigma(s))^{\underline{-\alpha-1}} f(s),$$

where $t \in \mathbb{N}_{a+N-\alpha}$ and $N \in \mathbb{N}$ is chosen so that $0 \le N-1 < \alpha \le N$.

Lemma 2.2. *[10] Let $0 \le N-1 < \alpha \le N$. Then*

$$\Delta^{-\alpha}\Delta^{\alpha} y(t) = y(t) + C_1 t^{\underline{\alpha-1}} + C_2 t^{\underline{\alpha-2}} + \ldots + C_N t^{\underline{\alpha-N}},$$

for some $C_i \in \mathbb{R}$, with $1 \le i \le N$.

To define the solution of the boundary value problem (1.1) we need the following lemma that deals with a linear variant of the boundary value problem (1.1) and gives a representation of the solution.

Lemma 2.3. *Let $\Lambda \neq 0$, $p, q > 0$, $2 < \alpha \le 3$, $0 < \beta, \theta \le 1$, $\eta \in \mathbb{N}_{\alpha-1,\alpha+T-1}$, functions $h : \mathbb{N}_{\alpha-1,\alpha+T-1} \to \mathbb{R}$ and $y : \mathbb{R} \to \mathbb{R}$ be given. Then the problem*

$$\begin{cases} \Delta^{\alpha} u(t) = h(t+\alpha-1), & t \in \mathbb{N}_{0,T}, \\ u(\alpha-2) = \Delta^{\theta} u(\alpha-\theta-2) = p\, y(u), \\ u(T+\alpha) = q\, \Delta^{-\beta} u(\eta+\beta), \end{cases} \tag{2.1}$$

has the unique solution

$$\begin{aligned} u(t) &= -\frac{t^{\underline{\alpha-1}}}{\Lambda\Gamma(\alpha)} \Bigg[\frac{q}{\Gamma(\beta)} \sum_{s=\alpha}^{\eta} \sum_{\xi=0}^{s-\alpha} (\eta+\beta-\sigma(s))^{\underline{\beta-1}} (s-\sigma(\xi))^{\underline{\alpha-1}} h(\xi+\alpha-1) \\ &\quad - \sum_{s=0}^{T} (T+\alpha-\sigma(s))^{\underline{\alpha-1}} h(s+\alpha-1) \Bigg] + \frac{p\, y(u)}{\Gamma(\alpha-1)} \left[t^{\underline{\alpha-2}} - \frac{t^{\underline{\alpha-1}}\Theta}{\Lambda} \right] \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-\sigma(s))^{\underline{\alpha-1}} h(s+\alpha-1), \end{aligned} \tag{2.2}$$

where

$$\Lambda = \frac{q}{\Gamma(\beta)} \sum_{s=0}^{\eta-\alpha+1} (\eta+\beta-s-\alpha)^{\underline{\beta-1}} (s+\alpha-1)^{\underline{\alpha-1}} - \frac{\Gamma(T+\alpha+1)}{\Gamma(T+2)}, \tag{2.3}$$

$$\Theta = \frac{q}{\Gamma(\beta)} \sum_{s=0}^{\eta-\alpha+2} (\eta+\beta-\alpha-s+1)^{\underline{\beta-1}} (s+\alpha-2)^{\underline{\alpha-2}} - \frac{\Gamma(T+\alpha+1)}{\Gamma(T+3)}. \tag{2.4}$$

Proof. From Lemma 2.2, we find that a general solution for (2.1) can be written as

$$u(t) = C_1 t^{\underline{\alpha-1}} + C_2 t^{\underline{\alpha-2}} + C_3 t^{\underline{\alpha-3}} + \Delta^{-\alpha} h(t+\alpha-1), \tag{2.5}$$

for $t \in \mathbb{N}_{\alpha-3,T+\alpha}$.

Using the fractional difference of order $0 < \theta \le 1$ for (2.5), we obtain

$$\begin{aligned}\Delta^{\theta} u(t) \;=\; & \frac{C_1}{\Gamma(-\theta)} \sum_{s=\alpha-1}^{t+\theta} (t-\sigma(s))^{\underline{-\theta-1}} s^{\underline{\alpha-1}} + \frac{C_2}{\Gamma(-\theta)} \sum_{s=\alpha-2}^{t+\theta} (t-\sigma(s))^{\underline{-\theta-1}} s^{\underline{\alpha-2}} \\ & + \frac{C_3}{\Gamma(-\theta)} \sum_{s=\alpha-3}^{t+\theta} (t-\sigma(s))^{\underline{-\theta-1}} s^{\underline{\alpha-3}} \\ & + \frac{1}{\Gamma(-\theta)\Gamma(\alpha)} \sum_{s=\alpha}^{t+\theta} \sum_{\xi=0}^{s-\alpha} (t-\sigma(s))^{\underline{-\theta}} (s-\sigma(\xi))^{\underline{\alpha-1}} h(\xi+\alpha-1),\end{aligned}$$

for $t \in \mathbb{N}_{\alpha-\theta-2,T+\alpha-\theta+1}$.

Applying the condition of (2.1): $u(\alpha-2) = \Delta^{\theta} u(\alpha-\theta-2)$, we have $C_3 = 0$.
So,

$$u(t) = C_1 t^{\underline{\alpha-1}} + C_2 t^{\underline{\alpha-2}} + \Delta^{-\alpha} h(t+\alpha-1). \tag{2.6}$$

From (2.6) and the second condition of (2.1): $u(\alpha-2) = p\, y(u)$, we have

$$C_2 = \frac{p\, y(u)}{\Gamma(\alpha-1)}. \tag{2.7}$$

Hence,

$$u(t) = C_1 t^{\underline{\alpha-1}} + \frac{py(u)}{\Gamma(\alpha-1)} t^{\underline{\alpha-2}} + \Delta^{-\alpha} h(t+\alpha-1), \tag{2.8}$$

for $t \in \mathbb{N}_{\alpha-3,T+\alpha}$.

Using the fractional sum of order $0 < \beta \le 1$ for (2.8), we obtain

$$\Delta^{-\beta} u(t) \;=\; \frac{C_1}{\Gamma(\beta)} \sum_{s=\alpha-1}^{t-\beta} (t-\sigma(s))^{\underline{\beta-1}} s^{\underline{\alpha-1}} + \frac{py(u)}{\Gamma(\beta)\Gamma(\alpha-1)} \sum_{s=\alpha-2}^{t-\beta} (t-\sigma(s))^{\underline{\beta-1}} s^{\underline{\alpha-2}}$$

$$+\frac{1}{\Gamma(\beta)\Gamma(\alpha)}\sum_{s=\alpha}^{t-\beta}\sum_{\xi=0}^{s-\alpha}(t-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}h(\xi+\alpha-1), \tag{2.9}$$

for $t\in\mathbb{N}_{\alpha+\beta-3,T+\alpha+\beta}$.

The third condition of (2.1) implies

$$\begin{aligned}
& q\Delta^{-\beta}u(\eta+\beta)\\
= & \frac{qC_1}{\Gamma(\beta)}\sum_{s=\alpha-1}^{\eta}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}s^{\underline{\alpha-1}}+\frac{p\,q\,y(u)}{\Gamma(\beta)\Gamma(\alpha-1)}\sum_{s=\alpha-2}^{\eta}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}s^{\underline{\alpha-2}}\\
& +\frac{q}{\Gamma(\beta)\Gamma(\alpha)}\sum_{s=\alpha}^{\eta}\sum_{\xi=0}^{s-\alpha}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}h(\xi+\alpha-1)\\
= & C_1(T+\alpha)^{\underline{\alpha-1}}+\frac{p\,y(u)}{\Gamma(\alpha-1)}(T+\alpha)^{\underline{\alpha-2}}+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}h(s+\alpha-1).
\end{aligned}$$

Solving the above equation for the constant C_1, we get

$$\begin{aligned}
C_1 = & \frac{-p\,q\,y(u)}{\Lambda\Gamma(\beta)\Gamma(\alpha-1)}\sum_{s=\alpha-2}^{\eta}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}s^{\underline{\alpha-2}}+\frac{p\,y(u)}{\Lambda\Gamma(\alpha-1)}(T+\alpha)^{\underline{\alpha-2}}\\
& +\frac{1}{\Lambda\Gamma(\alpha)}\sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}h(s+\alpha-1) \qquad (2.10)\\
& -\frac{q}{\Lambda\Gamma(\beta)\Gamma(\alpha)}\sum_{s=\alpha}^{\eta}\sum_{\xi=0}^{s-\alpha}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}h(\xi+\alpha-1),
\end{aligned}$$

where Λ is defined as (2.3). Substituting C_1 into (2.8), we obtain (2.2). □

3 Main Results

In this section, we wish to establish the existence results for problem (1.1). To accomplish this, let $\mathcal{C}=C(\mathbb{N}_{\alpha-3,\alpha+T},\mathbb{R})$ be a Banach space of all function u with the norm defined by

$$\|u\|_{\mathcal{C}}=\max\{\|u\|,\|\Delta^{\mu}\Delta^{\nu}u\|\},$$

where $\|u\|=\max_{t\in\mathbb{N}_{\alpha-3,\alpha+T}}|u(t)|$ and $\|\Delta^{\mu}\Delta^{\nu}u\|=\max_{t\in\mathbb{N}_{\alpha-3,\alpha+T}}|\Delta^{\mu}\Delta^{\nu}u(t-\mu-\nu+2)|$. Also define an operator $F:\mathcal{C}\to\mathcal{C}$ by

$$Fu(t)$$

$$
\begin{aligned}
= & -\frac{t^{\underline{\alpha-1}}}{\Lambda\Gamma(\alpha)}\Bigg[\frac{q}{\Gamma(\beta)}\sum_{s=\alpha}^{\eta}\sum_{\xi=0}^{s-\alpha}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}f(\xi+\alpha-1,u(\xi+\alpha-1), \\
& \Delta^{\mu}\Delta^{\nu}u(\xi+\alpha-\mu-\nu+1))-\sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}f(s+\alpha-1,u(s+\alpha-1), \\
& \Delta^{\mu}\Delta^{\nu}u(s+\alpha-\mu-\nu+1))\Bigg]+\frac{p\,y(u)}{\Gamma(\alpha-1)}\left[t^{\underline{\alpha-2}}-\frac{t^{\underline{\alpha-1}}\Theta}{\Lambda}\right] \qquad (3.1) \\
& +\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}f(s+\alpha-1,u(s+\alpha-1),\Delta^{\mu}\Delta^{\nu}u(s+\alpha-\mu-\nu+1)),
\end{aligned}
$$

for $t \in \mathbb{N}_{\alpha-3,\alpha+T}$, where $\Lambda \neq 0,\ \Theta$ are defined as (2.3),(2.4), respectively. The problem (1.1) has solutions if and only if the operator F has fixed points.

Our first result is based on Banach contraction mapping principle.

Theorem 3.1. *Assume that*

(H_1) *There exist constants $\gamma_1, \gamma_2 > 0$ such that, for each $t \in \mathbb{N}_{\alpha-3,\alpha+T}$ and for all $u, v \in \mathcal{C}$,*

$$
\begin{aligned}
& |f(t,u(t),\Delta^{\mu}\Delta^{\nu}u(t-\mu-\nu+2))-f(t,v(t),\Delta^{\mu}\Delta^{\nu}v(t-\mu-\nu+2))| \\
\leq\ & \gamma_1\,|u(t)-v(t)|+\gamma_2\,|\Delta^{\mu}\Delta^{\nu}u(t-\mu-\nu+2)-\Delta^{\mu}\Delta^{\nu}v(t-\mu-\nu+2)|.
\end{aligned}
$$

(H_2) *There exists a constant $\omega > 0$ such that, for all $u, v \in \mathcal{C}$,*

$$|y(u)-y(v)| \leq \omega|u-v|.$$

(H_3) $\gamma\Omega+\omega\Phi < \frac{(T+2)(T+1)}{(T+\alpha+2)(T+\alpha+1)}$,

where

$$
\begin{aligned}
\gamma &= \max\{\gamma_1+\gamma_2\} && (3.2) \\
\Omega &= \frac{(T+\alpha+2)^{\underline{\alpha-1}}}{|\Lambda|}\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(\alpha+\beta+1)\Gamma(T)}-\frac{(T+\alpha+2)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right|+\frac{(T+\alpha+2)^{\underline{\alpha}}}{\Gamma(\alpha+1)} && (3.3) \\
\Phi &= \frac{p(T+\alpha+2)^{\underline{\alpha-2}}}{\Gamma(\alpha-1)}\left[1+(T+4)\left|\frac{\Theta}{\Lambda}\right|\right]. && (3.4)
\end{aligned}
$$

Then the boundary value problem (1.1) has at least one solution on $\mathbb{N}_{\alpha-3,\alpha+T}$.

Proof. Denote that,

$$\mathcal{H}|u-v|(t)=\big|f(t,u(t),\Delta^{\mu}\Delta^{\nu}u(t-\mu-\nu+2))-f(t,v(t),\Delta^{\mu}\Delta^{\nu}v(t-\mu-\nu+2))\big|.$$

For all $u, v \in \mathcal{C}$, by computing directly, we have

$$\begin{aligned}
&\|Fu - Fv\| \\
= \;& \max_{t\in\mathbb{N}_{\alpha-3,\alpha+T}} \Bigg| -\frac{t^{\underline{\alpha-1}}}{\Lambda\Gamma(\alpha)}\left[\frac{q}{\Gamma(\beta)}\sum_{s=\alpha}^{\eta}\sum_{\xi=0}^{s-\alpha}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}\mathcal{H}|u-v|(\xi)\right. \\
&\left.-\sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}\mathcal{H}|u-v|(s)\right] + \left[t^{\underline{\alpha-2}} - \frac{t^{\underline{\alpha-1}}\Theta}{\Lambda}\right]\frac{p\,|y(u)-y(v)|}{\Gamma(\alpha-1)} \\
&+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}\mathcal{H}|u-v|(s)\Bigg| \\
\le \;& (\gamma\|u-v\|_{\mathcal{C}})\left[\frac{(T+\alpha+2)^{\underline{\alpha}}}{\Gamma(\alpha+1)} + \frac{(T+\alpha+2)^{\underline{\alpha-1}}}{|\Lambda|}\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)} - \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right|\right] \\
&+\left[1+(T+4)\left|\frac{\Theta}{\Lambda}\right|\right]\frac{(\omega\|u-v\|_{\mathcal{C}})\;p\,(T+\alpha+2)^{\underline{\alpha-2}}}{\Gamma(\alpha-1)} \\
= \;& (\gamma\|u-v\|_{\mathcal{C}})\;\Omega + (\omega\|u-v\|_{\mathcal{C}})\;\Phi,
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\Delta^{\mu}\Delta^{\nu}Fu - \Delta^{\mu}\Delta^{\nu}Fv\right\| \\
= \;& \max_{t\in\mathbb{N}_{\alpha-3,\alpha+T}}\left|\left(\Delta^{\mu}\Delta^{\nu}Fu\right)(t-\mu-\nu+2) - \left(\Delta^{\mu}\Delta^{\nu}Fv\right)(t-\mu-\nu+2)\right| \\
< \;& \left(\frac{1}{|\Gamma(-\mu)\Gamma(-\nu)|}\sum_{s=\alpha-\nu}^{T+\alpha-\nu+2}\sum_{\xi=\alpha-1}^{s+\nu}(T+\alpha-\mu-\nu+2-\sigma(s))^{\underline{-\mu-1}}(s-\sigma(\xi))^{\underline{-\nu-1}}\right)\times \\
&(T+\alpha+2)^{\underline{\alpha-1}}\left[\frac{(\gamma\|u-v\|_{\mathcal{C}})}{|\Lambda|}\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)} - \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right|\right. \\
&\left.+\frac{p\;(\omega\|u-v\|_{\mathcal{C}})}{\Gamma(\alpha-1)}\left|\frac{\Theta}{\Lambda}\right|\right] + p\;(\omega\|u-v\|_{\mathcal{C}})\;\frac{(T+\alpha+2)^{\underline{\alpha-2}}}{\Gamma(\alpha-1)}\times \\
&\left(\frac{1}{|\Gamma(-\mu)\Gamma(-\nu)|}\sum_{s=\alpha-\nu}^{T+\alpha-\nu+2}\sum_{\xi=\alpha-2}^{s+\nu}(T+\alpha-\mu-\nu+2-\sigma(s))^{\underline{-\mu-1}}(s-\sigma(\xi))^{\underline{-\nu-1}}\right) \\
&+\left(\frac{1}{|\Gamma(-\mu)\Gamma(-\nu)|}\sum_{s=\alpha-\nu}^{T+\alpha-\nu+2}\sum_{r=\alpha}^{s+\nu}(T+\alpha-\mu-\nu+2-\sigma(s))^{\underline{-\mu-1}}(s-\sigma(r))^{\underline{-\nu-1}}\right)\times \\
&\frac{(\gamma\|u-v\|_{\mathcal{C}})}{\Gamma(\alpha)}\sum_{\xi=0}^{T+2}(T+\alpha+2-\sigma(\xi))^{\underline{\alpha-1}} \\
< \;& \frac{(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1)}\left[\gamma\Omega+\omega\Phi\right]\|u-v\|_{\mathcal{C}}.
\end{aligned}$$

Thus, $\|Fu-Fv\|_{\mathcal{C}} \le \dfrac{(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1)}\left[\gamma\Omega+\omega\Phi\right]\|u-v\|_{\mathcal{C}}.$

By (H_3), we get that F is a contraction mapping, and then Theorem 3.1 implies that boundary value problem (1.1) has unique solution on $\mathbb{N}_{\alpha-3,\alpha+T}$. This completes the proof. □

The second result is based on Schaefer's fixed point theorem.

Theorem 3.2. *(Arzelá-Ascoli Theorem) [18] A set of function in $C[a,b]$ with the sup norm, is relatively compact if and only it is uniformly bounded and equicontinuous on $[a,b]$.*

Theorem 3.3. *[18] If a set is closed and relatively compact then it is compact.*

Theorem 3.4. ***[Schaefer's fixed point theorem]*** *[19] Let X be a Banach space and $T : X \to X$ be a continuous and compact mapping. If the set*

$$\{x \in X \ : \ x = \lambda T(x), \ \textit{for some } \lambda \in (0,1)\}$$

is bounded, then T has a fixed point.

We shall use Schaefer's fixed point theorem to prove that the operator F defined as (3.1), has a fixed point.

Theorem 3.5. *Suppose that there exist constants $L_1, L_2 > 0$ such that, for each $t \in \mathbb{N}_{\alpha-3,\alpha+T}$ and $u \in \mathcal{C}$,*

$$\begin{aligned} |f(t,u(t),\Delta^{\mu}\Delta^{\nu}u(t-\mu-\nu+2))| &\leq L_1 \max\{\|u\|, \|\Delta^{\mu}\Delta^{\nu}u\|\}, \\ |y(u)| &\leq L_2. \end{aligned}$$

Then the problem (1.1) has at least one solution on $\mathbb{N}_{\alpha-3,\alpha+T}$.

Proof. We divide the proof into four steps.

Step I. Verify F map bounded sets into bounded sets in $C(\mathbb{N}_{\alpha-3,\alpha+T})$.
Let $u \in B_L = \{u \in C(\mathbb{N}_{\alpha-3,\alpha+T}) : \|u\|_{\mathcal{C}} \leq L\}$, and choosing a constant

$$L \geq \frac{L_2\Phi(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1) - L_1\Omega\,(T+\alpha+2)(T+\alpha+1)}.$$

Denote that

$$\begin{aligned} \mathcal{H}|u-v|(t) &:= |f(t,u(t),\Delta^{\mu}\Delta^{\nu}u(t-\mu-\nu+2)) - f(t,v(t),\Delta^{\mu}\Delta^{\nu}v(t-\mu-\nu+2))| \\ &\leq \|f(t,u(t),\Delta^{\mu}\Delta^{\nu}u(t-\mu-\nu+2)) - f(t,v(t),\Delta^{\mu}\Delta^{\nu}v(t-\mu-\nu+2))\| \\ &=: \mathcal{H}\|u-v\|(t). \end{aligned}$$

For each $u \in B_L$, we obtain

$$\begin{aligned}
&\|Fu\| \\
= \;& \max_{t\in\mathbb{N}_{\alpha-3,\alpha+T}} \Bigg| -\frac{t^{\underline{\alpha-1}}}{\Lambda\Gamma(\alpha)}\Bigg[\frac{q}{\Gamma(\beta)}\sum_{s=\alpha}^{\eta}\sum_{\xi=0}^{s-\alpha}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}\mathcal{H}|u-v|(\xi) \\
&-\sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}\mathcal{H}|u-v|(s)\Bigg] + \left[t^{\underline{\alpha-2}} - \frac{t^{\underline{\alpha-1}}\Theta}{\Lambda}\right]\frac{p\,|y(u)|}{\Gamma(\alpha-1)} \\
&+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}\mathcal{H}|u-v|(s)\Bigg| \\
\le \;& L_1\|u\|_{\mathcal{C}}\left[\frac{(T+\alpha+2)^{\underline{\alpha}}}{\Gamma(\alpha+1)} + \frac{(T+\alpha+2)^{\underline{\alpha-1}}}{|\Lambda|}\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)} - \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right|\right] \\
&+\left[1+(T+4)\left|\frac{\Theta}{\Lambda}\right|\right]\frac{pL_2\,(T+\alpha+2)^{\underline{\alpha-2}}}{\Gamma(\alpha-1)} \\
\le \;& L_1 L\,\Omega + L_2\,\Phi.
\end{aligned}$$

and

$$\begin{aligned}
&\|\Delta^{\mu}\Delta^{\nu}Fu\| = \max_{t\in\mathbb{N}_{\alpha-3,\alpha+T}}\left|\,(\Delta^{\mu}\Delta^{\nu}Fu)\,(t-\mu-\nu+2)\right| \\
= \;& \max_{t\in\mathbb{N}_{\alpha-3,\alpha+T}}\Bigg\{\frac{1}{|\Gamma(-\mu)\Gamma(-\nu)|}\sum_{s=\alpha-\nu}^{t-\nu+2}\sum_{\xi=\alpha-1}^{s+\nu}(t-\mu-\nu+2-\sigma(s))^{\underline{-\mu-1}}(s-\sigma(\xi))^{\underline{-\nu-1}}\times \\
&\xi^{\underline{\alpha-1}}\Bigg[\frac{(L_1\|u\|_{\mathcal{C}})}{|\Lambda|\Gamma(\alpha)}\Bigg|\frac{q}{\Gamma(\beta)}\sum_{s=\alpha}^{\eta}\sum_{\xi=0}^{s-\alpha}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}} \\
&-\sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}\Bigg| + \frac{pL_2}{\Gamma(\alpha-1)}\left|\frac{\Theta}{\Lambda}\right|\Bigg] \\
&+\frac{1}{|\Gamma(-\mu)\Gamma(-\nu)|}\sum_{s=\alpha-\nu}^{t-\nu+2}\sum_{\xi=\alpha-2}^{s+\nu}(t-\mu-\nu+2-\sigma(s))^{\underline{-\mu-1}}(s-\sigma(\xi))^{\underline{-\nu-1}}\xi^{\underline{\alpha-2}}\times \\
&\left[\frac{pL_2}{\Gamma(\alpha-1)}\,(T-\alpha+2)^{\underline{\alpha-2}}\right] + \frac{(L_1\|u\|_{\mathcal{C}})}{|\Gamma(-\mu)\Gamma(-\nu)|}\sum_{s=\alpha-\nu}^{t-\nu+2}\sum_{r=\alpha}^{s+\nu}(t-\mu-\nu+2-\sigma(s))^{\underline{-\mu-1}}\times \\
&(s-\sigma(r))^{\underline{-\nu-1}}\left[\frac{1}{\Gamma(\alpha)}\sum_{\xi=0}^{r-\alpha}(r-\sigma(\xi))^{\underline{\alpha-1}}\right]\Bigg\} \\
< \;& \left\{\frac{(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1)}\right\}L_1 L\left[\frac{(T+\alpha+2)^{\underline{\alpha}}}{\Gamma(\alpha+1)} + \frac{(T+\alpha+2)^{\underline{\alpha-1}}}{|\Lambda|}\times\right. \\
&\left.\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)} - \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right|\right] + \left\{\frac{(T+\alpha+2)(T+\alpha+1)}{(T+3)(T+2)}\right\}\times
\end{aligned}$$

$$\frac{pL_2}{\Gamma(\alpha-1)}(T+\alpha+2)^{\underline{\alpha-2}}\left[1+(T+4)\left|\frac{\Theta}{\Lambda}\right|\right]$$
$$< \frac{(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1)}\left[L_1L\Omega+L_2\Phi\right].$$

Hence, $\|Fu\|_{\mathcal{C}} \leq L$ where Ω and Φ are defined on 3.3 and 3.4, respectively. Thus F is uniformly bounded.

Step II. Show that F is continuous on B_L.
Let $\epsilon>0$ there exists $\delta=\max\{\delta_1,\delta_2\}>0$ such that, for each $t\in\mathbb{N}_{\alpha-3,\alpha+T}$ and for all $u,v\in B_L$ with

$$\max\{|u(t)-v(t)|,|\Delta^\mu\Delta^\nu u(t-\mu-\nu+2)-\Delta^\mu\Delta^\nu v(t-\mu-\nu+2)|\}<\delta_1,$$

we have

$$\mathcal{H}|u-v|<\frac{\epsilon\,(T+2)(T+1)}{2\Omega\,(T+\alpha+2)(T+\alpha+1)},$$

and for all $u,v\in B_L$ with $|u-v|<\delta_2$, we have

$$|y(u)-y(v)|<\frac{\epsilon\,(T+2)(T+1)}{2\Phi\,(T+\alpha+2)(T+\alpha+1)}.$$

Then, we have

$$\begin{aligned}
&\|Fu(t)-Fv(t)\|\\
=\;&\max_{t\in\mathbb{N}_{\alpha-3,\alpha+T}}\Bigg|-\frac{t^{\underline{\alpha-1}}}{\Lambda\Gamma(\alpha)}\Bigg[\frac{q}{\Gamma(\beta)}\sum_{s=\alpha}^{\eta}\sum_{\xi=0}^{s-\alpha}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}}\mathcal{H}|u-v|(\xi)\\
&-\sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}\mathcal{H}|u-v|(s)\Bigg]+\left[t^{\underline{\alpha-2}}-\frac{t^{\underline{\alpha-1}}\Theta}{\Lambda}\right]\frac{p\,|y(u)-y(v)|}{\Gamma(\alpha-1)}\\
&+\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}\mathcal{H}|u-v|(s)\Bigg|\\
\leq\;&\mathcal{H}\|u-v\|\left[\frac{(T+\alpha+2)^{\underline{\alpha}}}{\Gamma(\alpha+1)}+\frac{(T+\alpha+2)^{\underline{\alpha-1}}}{|\Lambda|}\cdot\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)}-\frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right|\right]\\
&+\|y(u)-y(v)\|\,\frac{p\,(T+\alpha+2)^{\underline{\alpha-2}}}{\Gamma(\alpha-1)}\left[1+(T+4)\left|\frac{\Theta}{\Lambda}\right|\right]\\
=\;&\Omega\,\mathcal{H}\|u-v\|+\Phi\,\|y(u)-y(v)\|.
\end{aligned}$$

Similarly to the proof above and Theorem 3.1, we obtain

$$\left\|\,(\Delta^\mu\Delta^\nu Fu)\,(t-\mu-\nu+2)-(\Delta^\mu\Delta^\nu Fv)\,(t-\mu-\nu+2)\right\|$$

$$< \frac{(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1)}\Big[\Omega\,\mathcal{H}\|u-v\| + \Phi\,\|y(u)-y(v)\|\Big] < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.$$

Hence, $\|Fu - Fv\|_{\mathcal{C}} \le \epsilon$. This means that F is continuous on B_L.

Step III. Examine $F(B_L)$ is equicontinuous with B_L. For any $\epsilon > 0$, there exists $\delta = \max\{\delta_1, \delta_2, \delta_3\} > 0$ such that, for $t_1, t_2 \in \mathbb{N}_{\alpha-3,\alpha+T}$

$$\left|t_2^{\underline{\alpha}} - t_1^{\underline{\alpha}}\right| < \frac{\epsilon\,\Gamma(\alpha+1)\,(T+2)(T+1)}{3L_1(T+\alpha+2)(T+\alpha+1)} \qquad \text{whenever} \quad |t_2 - t_1| < \delta_1,$$

$$\left|t_2^{\underline{\alpha-1}} - t_1^{\underline{\alpha-1}}\right| < \frac{\epsilon\,|\Lambda|\,(T+2)(T+1)}{3(T+\alpha+2)(T+\alpha+1)\left[L_1\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)} - \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha-1)}\right| + \frac{pL_2\,|\Theta|}{\Gamma(\alpha-1)}\right]} \quad \text{whenever} \quad |t_2 - t_1| < \delta_2,$$

$$\left|t_2^{\underline{\alpha-2}} - t_1^{\underline{\alpha-2}}\right| < \frac{\epsilon\Gamma(\alpha-1)(T+2)(T+1)}{3pL_2(T+\alpha+2)(T+\alpha+1)} \qquad \text{whenever} \quad |t_2 - t_1| < \delta_3.$$

Then, we have

$$\begin{aligned}
& |Fu(t_2) - Fu(t_1)| \\
= & \Bigg| -\frac{t_2^{\underline{\alpha-1}} - t_1^{\underline{\alpha-1}}}{\Lambda\Gamma(\alpha)}\Bigg[\frac{q}{\Gamma(\beta)}\sum_{s=\alpha}^{\eta}\sum_{\xi=0}^{s-\alpha}(\eta+\beta-\sigma(s))^{\underline{\beta-1}}(s-\sigma(\xi))^{\underline{\alpha-1}} \times \\
& f(\xi+\alpha-1, u(\xi+\alpha-1), \Delta^{\mu}\Delta^{n}u(\xi+\alpha-\mu-\nu+1)) - \sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}} \times \\
& f(s+\alpha-1, u(s+\alpha-1), \Delta^{\mu}\Delta^{n}u(s+\alpha-\mu-\nu+1))\Bigg] \\
& + \frac{p\,y(u)}{\Gamma(\alpha-1)}\left[\left(t_2^{\underline{\alpha-2}} - t_1^{\underline{\alpha-2}}\right) - \left(t_2^{\underline{\alpha-1}} - t_1^{\underline{\alpha-1}}\right)\frac{\Theta}{\Lambda}\right] \\
& + \frac{1}{\Gamma(\alpha)}\Bigg[\sum_{s=0}^{t_2-\alpha}(t_2-\sigma(s))^{\underline{\alpha-1}} f(s+\alpha-1, u(s+\alpha-1), \Delta^{\mu}\Delta^{n}u(s+\alpha-\mu-\nu+1)) \\
& - \sum_{s=0}^{t_1-\alpha}(t_1-\sigma(s))^{\underline{\alpha-1}} f(s+\alpha-1, u(s+\alpha-1), \Delta^{\mu}\Delta^{n}u(s+\alpha-\mu-\nu+1))\Bigg]\Bigg| \\
\le & \left|t_2^{\underline{\alpha-1}} - t_1^{\underline{\alpha-1}}\right|\left[\frac{L_1}{|\Lambda|}\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)} - \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right| + \frac{pL_2}{\Gamma(\alpha-1)}\left|\frac{\Theta}{\Lambda}\right|\right] \\
& + \frac{L_1}{\Gamma(\alpha)}\left[\sum_{s=0}^{t_2-\alpha}(t_2-\sigma(s))^{\underline{\alpha-1}} + \sum_{s=0}^{t_1-\alpha}(t_1-\sigma(s))^{\underline{\alpha-1}}\right] + \left|t_2^{\underline{\alpha-2}} - t_1^{\underline{\alpha-2}}\right|\frac{pL_2}{\Gamma(\alpha-1)} \\
= & \left|t_2^{\underline{\alpha-1}} - t_1^{\underline{\alpha-1}}\right|\left[\frac{L_1}{|\Lambda|}\left|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)} - \frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right| + \frac{pL_2}{\Gamma(\alpha-1)}\left|\frac{\Theta}{\Lambda}\right|\right]
\end{aligned}$$

$$+\frac{L_1}{\Gamma(\alpha+1)}\big|t_2^{\underline{\alpha}}-t_1^{\underline{\alpha}}\big|+\frac{pL_2}{\Gamma(\alpha-1)}\big|t_2^{\underline{\alpha-2}}-t_1^{\underline{\alpha-2}}\big|.$$

So $\big\|Fu-Fv\big\|<\epsilon$.

Similarly to the proof above and Theorem 3.1, we obtain

$$\begin{aligned}
&\big\|\Delta^\mu\Delta^\nu Fu-\Delta^\mu\Delta^\nu Fu\big\| \\
<\;& \frac{(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1)}\Bigg\{\big|t_2^{\underline{\alpha-1}}-t_1^{\underline{\alpha-1}}\big|\Bigg[\frac{L_1}{|\Lambda|}\bigg|\frac{q\,\Gamma(T+\alpha+\beta)}{\Gamma(T)\Gamma(\alpha+\beta+1)}-\frac{(T+\alpha)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\bigg| \\
&+\frac{pL_2}{\Gamma(\alpha-1)}\bigg|\frac{\Theta}{\Lambda}\bigg|\Bigg]+\frac{L_1}{\Gamma(\alpha+1)}\big|t_2^{\underline{\alpha}}-t_1^{\underline{\alpha}}\big|+\frac{pL_2}{\Gamma(\alpha-1)}\big|t_2^{\underline{\alpha-2}}-t_1^{\underline{\alpha-2}}\big|\Bigg\} \\
<\;& \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon.
\end{aligned}$$

Thus, $\|Fu(t_2)-Fu(t_1)\|_{\mathcal{C}}\leq\epsilon$. This means that $F(B_L)$ is an equicontinuous set.

As a consequence of Steps I to III together with the Arzelá-Ascoli theorem, its imply that $F:C(\mathbb{N}_{\alpha-3,\alpha+T})\to C(\mathbb{N}_{\alpha-3,\alpha+T})$ is completely continuous.

Step IV. *A priori bounds.* We show that the set

$$E=\{u\in C(\mathbb{N}_{\alpha-3,\alpha+T}):u=\lambda Fu\quad\text{for some}\quad 0<\lambda<1\}\;\text{ is bounded.}$$

Let $u\in E$. Then $u(t)=\lambda(Fu)(t)$ for some $0<\lambda<1$. Thus, for each $t\in\mathbb{N}_{\alpha-3,\alpha+T}$, we have

$$|\lambda Fu(t)|<|Fu(t)|<L_1L\,\Omega+L_2\,\Phi:=\Im.$$

So, we have $\big\|\lambda Fu\big\|<\Im$. Similarly to the proof above and Theorem 3.1, we obtain

$$\big\|\lambda\,\Delta^\mu\Delta^\nu Fu\big\|\;<\;\frac{(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1)}\Im\;=:\;\widetilde{\Im}.$$

Hence, $\|\lambda Fu\|_{\mathcal{C}}\leq\widetilde{\Im}$. This shows that E is bounded.

By of the Schaefer's fixed point theorem, we conclude that F has a fixed point which is a solution of the problem (1.1). □

4 Some examples

In this section, in order to illustrate our results, we consider some examples.

Example 4.1. *Consider the following boundary value problem*

$$\Delta^{\frac{5}{2}}u(t) = \frac{e^{-\sin^2(t+\frac{3}{2})}}{\left(t+\frac{15}{2}\right)^2} \cdot \frac{\left|u\left(t+\frac{3}{2}\right)\right| + |\Delta^{\frac{2}{3}}\Delta^{\frac{3}{4}}u\left(t+\frac{25}{12}\right)|}{\left|u\left(t+\frac{3}{2}\right)\right|+1}, \quad t \in \mathbb{N}_{0,4}, \tag{4.1}$$

$$u\left(\frac{1}{2}\right) = \Delta^{\frac{1}{4}}u\left(\frac{1}{4}\right) = \frac{1}{2}\sum_{i=0}^{7} C_i u(t_i), \quad t_i = i - \frac{1}{2}, \tag{4.2}$$

$$u\left(\frac{13}{2}\right) = \frac{1}{3}\Delta^{-\frac{1}{3}}u\left(\frac{29}{6}\right). \tag{4.3}$$

where C_i *are given positive constants with* $\sum_{i=0}^{7} C_i < \frac{1}{10\,e^{20}}$.

Here $p = \frac{1}{2},\ q = \frac{1}{3},\ \theta = \frac{1}{4},\ \alpha = \frac{5}{2},\ \beta = \frac{1}{3},\ \mu = \frac{2}{3},\ \nu = \frac{3}{4},\ \eta = \frac{9}{2},\ T = 4,$ $f\left(t, u(t), \Delta^{\mu}\Delta^{\nu}u\left(t-\mu-\nu+2\right)\right) = \frac{e^{-\sin^2 t}}{(t+6)^2} \cdot \frac{|u(t)|+|\Delta^{\frac{2}{3}}\Delta^{\frac{3}{4}}u\left(t+\frac{7}{12}\right)|}{|u(t)|+1}$ and $y(u) = \sum_{i=0}^{7} C_i u(t_i),\ t_i = i - \frac{1}{2}$.

Let $t \in \mathbb{N}_{-\frac{1}{2},\frac{13}{2}}$ and $u, v \in \mathbb{R}$, then

$$|\Lambda| = 7.781 \neq 0, \quad \Theta = 1.278, \quad \Omega \approx 106.039, \quad \Phi \approx 3.119.$$

Since $\quad |f\left(t, u(t), \Delta^{\mu}\Delta^{\nu}u\left(t-\mu-\nu+2\right)\right) - f\left(t, v(t), \Delta^{\mu}\Delta^{\nu}v\left(t-\mu-\nu+2\right)\right)|$
$\leq \frac{4}{1849}\left|u(t)-v(t)\right| + \frac{4}{1849}\left|\Delta^{\mu}\Delta^{\nu}u\left(t+\frac{7}{12}\right) - \Delta^{\mu}\Delta^{\nu}v\left(t+\frac{7}{12}\right)\right|$
is satisfied with $\gamma = \max\{\gamma_1 + \gamma_2\} = \frac{8}{1849}$.

Also, we get $|y(u)-y(v)| = \left|\sum_{i=0}^{7} C_i u(t_i) - \sum_{i=0}^{7} C_i v(t_i)\right| \leq \sum_{i=0}^{7} C_i |u(t_i) - v(t_i)|$, so (H_2) holds with $\omega = \sum_{i=0}^{7} C_i < \frac{1}{10\,e^{20}}$.

We can show that

$$\frac{(T+\alpha+2)(T+\alpha+1)}{(T+2)(T+1)}\left[\gamma\Omega + \omega\Phi\right] \approx 0.975 < 1.$$

Hence, by Theorem 3.1, the problem (4.1)-(4.3) has unique solution. □

Example 4.2. *Consider the following boundary value problem*

$$\Delta^{\frac{5}{2}}u(t) = \frac{t+\frac{3}{2}}{10\pi}\left[2\sin\left|u\left(t+\frac{3}{2}\right)\right| + \cos\left|\Delta^{\frac{2}{3}}\Delta^{\frac{3}{4}}u\left(t+\frac{25}{12}\right)\right|\right], \ t \in \mathbb{N}_{0,4}, \tag{4.4}$$

$$u\left(\frac{1}{2}\right) = \Delta^{\frac{1}{4}}u\left(\frac{1}{4}\right) = \frac{1}{4}\sum_{i=0}^{7} C_i \frac{|u(t_i)|}{1+|u(t_i)|}, \quad t_i = i - \frac{1}{2}, \tag{4.5}$$

$$u\left(\frac{13}{2}\right) = \frac{1}{5}\Delta^{-\frac{1}{3}}u\left(\frac{29}{6}\right), \tag{4.6}$$

where C_i *are given positive constants with* $\sum_{i=0}^{7} C_i < \frac{1}{e}$.

Here $p = \frac{1}{4},\ q = \frac{1}{5},\ \alpha = \frac{5}{2},\ \beta = \frac{1}{3},\ \theta = \frac{1}{4},\ \mu = \frac{2}{3},\ \nu = \frac{3}{4},\ \eta = \frac{9}{2},\ T = 4,$ $f\left(t, u(t), \Delta^{\mu}\Delta^{\nu}u\left(t-\mu-\nu+2\right)\right) = \frac{t}{10\pi}\left[2\sin|u(t)| + \cos\left|\Delta^{\frac{2}{3}}\Delta^{\frac{3}{4}}u\left(t+\frac{7}{12}\right)\right|\right]$ and $y(u) = \sum_{i=0}^{7} C_i \frac{|u(t_i)|}{1+|u(t_i)|},\quad t_i = i - \frac{1}{2}$. Clearly for $t \in \mathbb{N}_{-\frac{1}{2},\frac{13}{2}}$, we have

$$\begin{aligned} |f(t,u(t),\Delta^{\mu}\Delta^{\nu}u\left(t-\mu-\nu+2\right))| &\leq \frac{13}{20\,\pi}\max\{2,1\} \approx 0.414 \quad \left(L_1 = \frac{13}{20\,\pi}\right) \\ |y(u)| &\leq \sum_{i=0}^{7} C_i \frac{|u(t_i)|}{1+|u(t_i)|} < \frac{1}{e} = L_2. \end{aligned}$$

Hence, by Theorem 3.5, the problem (4.4)-(4.6) has at least one solution. □

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Hesitant fuzzy mighty filters of BE-algebras

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Abstract. The notion of hesitant fuzzy mighty filter of a BE-algebra is introduced and related properties are investigated. We provide conditions for a hesitant fuzzy filter to be a hesitant fuzzy mighty filter. We construct a new quotient structure of a transitive BE-algebra using a hesitant fuzzy filter and study some properties of it.

1. Introduction

In 2007, Kim and Kim [5] introduced the notion of a BE-algebra, and investigated several properties. In [1], Ahn and So introduced the notion of ideals in BE-algebras. They gave several descriptions of ideals in BE-algebras. Song et al. [8] considered the fuzzification of ideals in BE-algebras. They introduced the notion of fuzzy ideals in BE-algebras, and investigated related properties. They gave characterizations of a fuzzy ideal in BE-algebras.

The notions of Atanassov's intuitionistic fuzzy sets, type 2 fuzzy sets and fuzzy multisets etc. are a generalization of fuzzy sets. As another generalization of fuzzy sets, Torra [9] introduced the notion of hesitant fuzzy sets which are a very useful to express peoples hesitancy in daily life. The hesitant fuzzy set is a very useful tool to deal with uncertainty, which can be accurately and perfectly described in terms of the opinions of decision makers. Also, hesitant fuzzy set theory is used in decision making problem etc. (see [3, 7, 11, 12, 13, 14, 15]). In [4], Y. B. Jun and S. S. Ahn introduced the notion of a hesitant fuzzy filter and investigated some properties of it. The authors [2] defined a hesitant fuzzy implicative filter in a BE-algebra and discussed some properties of it.

In this paper, we introduce the notion of hesitant fuzzy mighty filter of a BE-algebra, and investigate some properties of it. We consider characterizations of a hesitant fuzzy mighty filter of a BE-algebra. We provide conditions for a hesitant fuzzy filter to be a hesitant fuzzy mighty filter. We construct a new quotient structure of a transitive BE-algebra using a hesitant fuzzy filter and study some properties of it.

2. Preliminaries

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By a *BE-algebra* ([5]) we mean a system $(X;*,1)$ of type $(2,0)$ which the following axioms hold:

(2.1) $(\forall x \in X)\,(x * x = 1)$,
(2.2) $(\forall x \in X)\,(x * 1 = 1)$,
(2.3) $(\forall x \in X)\,(1 * x = x)$,
(2.4) $(\forall x, y, z \in X)\,(x * (y * z) = y * (x * z)$ (exchange).

We introduce a relation "$\leq$" on X by $x \leq y$ if and only if $x * y = 1$.

A BE-algebra $(X;*,1)$ is said to be *transitive* if it satisfies: for any $x, y, z \in X$, $y * z \leq (x * y) * (x * z)$. A BE-algebra $(X;*,1)$ is said to be *self distributive* if it satisfies: for any $x, y, z \in X$, $x*(y*z) = (x*y)*(x*z)$. Note that every self distributive BE-algebra is transitive, but the converse is not true in general (see [5]).

Every self distributive BE-algebra $(X;*,1)$ satisfies the following properties:

(2.5) $(\forall x, y, z \in X)\,(x \leq y \Rightarrow z * x \leq z * y$ and $y * z \leq x * z)$,
(2.6) $(\forall x, y \in X)\,(x * (x * y) = x * y)$,
(2.7) $(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))$,

Definition 2.1. Let $(X;*,1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is a *filter* of X ([5]) if

(F1) $1 \in F$;
(F2) $(\forall x, y \in X)(x * y, x \in F \Rightarrow y \in F)$.

F is a *mighty filter* ([6]) of X if it satisfies (F1) and

(F3) $(\forall x, y, z \in X)(z * (y * x), z \in F \Rightarrow ((x * y) * y) * x \in F)$.

Theorem 2.2. ([6]) *A filter F of a BE-algebra X is mighty if and only if*

(2.8) $(\forall x, y \in X)(y * x \in F \Rightarrow ((x * y) * y) * x \in F)$.

Definition 2.3. ([9]) Let E be a reference set. A *hesitant fuzzy set* on E is defined in terms of a function that when applied to E returns a subset of $[0,1]$, which can be viewed as the following mathematical representation:

$$H_E := \{(e, h_E(e)) | e \in E\}$$

where $h_E : E \to \mathscr{P}([0,1])$.

Definition 2.4. Given a non-empty subset A of a BE-algebra X, a hesitant fuzzy set

$$H_X := \{(x, h_X(x)) | x \in X\}$$

on satisfying the following condition:

$$h_X(x) = \emptyset \text{ for all } x \notin A$$

is called a *hesitant fuzzy set related to* A (briefly, A-*hesitant fuzzy set*) on X, and is represented by $H_A := \{(x, h_A(x)) \mid x \in X\}$, where h_A is a mapping from X to $\mathscr{P}([0,1])$ with $h_A(x) = \emptyset$ for all $x \notin A$.

For a hesitant set $H_X := \{(x, h_X(x)) \mid x \in X\}$ of a BE-algebra X and a subset γ of $[0,1]$, the hesitant fuzzy γ-inclusive set of H_X, denoted by $H_X(\gamma)$, is defined to be the set

$$H_X(\gamma) := \{x \in X | \gamma \subseteq h_X(x)\}.$$

For any hesitant fuzzy set $H_X = \{(x, h_X(x) | x \in X\}$ and $G_X = \{(x, g_X(x)) | x \in X\}$, we call H_X a *hesitant fuzzy subset* of G_X, denoted by $H_X \widetilde{\subseteq} G_X$, if $h_X(x) \subseteq g_X(x)$ for all $x \in X$.

3. Hesitant fuzzy mighty filters

Definition 3.1. Given a non-empty subset (subalgebra as much as possible) A of a BE-algebra X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X. Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is called a *hesitant fuzzy subalgebra of* X *related to* A (briefly, A-*hesitant fuzzy subalgebra* of X) ([4]) if it satisfies the following condition: $h_A(x) \cap h_A(y) \subseteq h_A(x * y)$ for any $x, y \in A$. An A-hesitant fuzzy subalgebra of X with $A = X$ is called a *hesitant fuzzy subalgebra* of X. An A-hesitant fuzzy set $H_A := \{(x, h_A(x)) \mid x \in X\}$ on X is called a *hesitant fuzzy filter of* X *related to* A (briefly, A-*hesitant fuzzy filter* of X) ([4]) if it satisfies the following condition:

(3.1) $(\forall x \in A)(h_A(x) \subseteq h_A(1))$,

(3.2) $(\forall x, y \in A)(h_A(x * y) \cap h_A(x) \subseteq h_A(y))$.

An A-hesitant fuzzy filter of X with $A = X$ is called a *hesitant fuzzy filter* of X.

Proposition 3.2. ([4]) *Let* $H_A := \{(x, h_A(x)) | x \in X\}$ *be an* A*-hesitant fuzzy filter of a* BE*-algebra* X *where* A *is a subalgebra of* X*. Then the following assertions are valid.*

(i) $(\forall x, y \in A)(x \leq y \Rightarrow h_A(x) \subseteq h_A(y))$,

(ii) $(\forall x, y, z \in A)(z \leq x * y \Rightarrow h_A(y) \supseteq h_A(x) \cap h_A(z))$,

(iii) $(\forall x, y, z \in A)(h_A(x * (y * z)) \cap h_A(y) \subseteq h_A(x * z))$,

(iv) $(\forall a, x \in A)(h_A(a) \subseteq h_A((a * x) * x)$.

Proposition 3.3. *Every hesitant fuzzy filter of a* BE*-algebra* X *is a hesitant fuzzy subalgebra of* X*.*

Proof. Let $H_X = \{(x, h_X(x)) | x \in X\}$ be a hesitant fuzzy filter of X. For any $x, y \in X$, we have $h_X(x) \cap h_X(y) \subseteq h_X(1) \cap h_X(y) = h_X(y * (x * y)) \cap h_X(y) \subseteq h_X(x * y)$. Hence H_X is a hesitant fuzzy subalgebra of X. □

The converse of Proposition 3.3 may not be true in general (see Example 3.4).

Example 3.4. Let $X = \{0, 1, a, b, c\}$ be a BE-algebra ([4]) with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \{(1, [0, 1]), (a, (0, \tfrac{1}{8})), (b, (\tfrac{1}{4}, \tfrac{3}{4}])), (c, (0, \tfrac{1}{2}))\}$$

Then H_X is a hesitant fuzzy subalgebra of X, but not a hesitant fuzzy filter of X since $h_X(b * a) \cap h_X(b) = h_X(1) \cap h_X(b) = [0, 1] \cap (\frac{1}{4}, \frac{3}{4}] \not\subseteq h_X(a) = (0, \frac{1}{8})$.

Definition 3.5. Given a non-empty subset (subalgebra as much as possible) A of a BE-algebra X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X. Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is called a *hesitant fuzzy mighty filter of X related to A* (briefly, *A-hesitant fuzzy mighty filter* of X) if it satisfies (3.1) and

(3.3) $(\forall x, y, z \in A)(h_A(z * (y * x)) \cap h_A(z) \subseteq h_A(((x * y) * y) * x)$.

An A-hesitant fuzzy mighty filter of X with $A = X$ is called a *hesitant fuzzy mighty filter* of X.

Example 3.6. Let $X = \{1, a, b, c, d, 0\}$ be a BE-algebra ([6]) with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	b	c	d	c
b	1	a	1	b	a	d
c	1	a	1	1	a	a
d	1	1	1	b	1	b
0	1	1	1	1	1	1

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \{(1, [0, 1]), (a, [\tfrac{3}{4}, 1]), (b, [\tfrac{1}{2}, 1]), (c, [\tfrac{1}{2}, 1]), (d, \{\tfrac{3}{4}, 1\}), (0, \{\tfrac{1}{2}, 1\})\}$$

It is easy to check that H_X is a hesitant fuzzy fuzzy mighty filter of X.

Proposition 3.7. *Every hesitant fuzzy mighty filter of a BE-algebra X is a hesitant fuzzy filter of X.*

Proof. Let $H_X = \{(x, h_X(x)) | x \in X\}$ be a hesitant fuzzy mighty filter of X. Putting $y := 1$ in (3.3), we have $h_X(z * (1 * x)) \cap h_X(z) = h_X(z * x) \cap h_X(z) \subseteq h_X(((x * 1) * 1) * x) = h_X(x)$. Hence H_X is a hesitant fuzzy filter of X. □

The converse of Proposition 3.7 may not be true in general (see Example 3.8).

Example 3.8. Let $X = \{1, a, b, c, d\}$ be a BE-algebra ([5]) with the following Cayley table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \{(1, [0, 1]), (a, [\tfrac{1}{2}, 1]), (b, [\tfrac{1}{4}, 1]), (c, [\tfrac{1}{5}, 1]), (d, \{\tfrac{3}{4}, 1\})\}.$$

It is easy to check that H_X is a hesitant fuzzy fuzzy filter of X, but not a hesitant fuzzy mighty filter of X since $h_X(1 * (c * a)) \cap h_X(1) = h_X(1) = [0, 1] \not\subseteq h_X(((a * c) * c) * a) = h_X(a) = [\frac{1}{2}, 1]$.

Theorem 3.9. *Any hesitant fuzzy filter $H_X = \{(x, h_X(x))|x \in X\}$ of a BE-algebra X is mighty if and only if it satisfies*

(3.4) $(\forall x, y \in X)(h_X(y * x) \subseteq h_X(((x * y) * y) * x))$.

Proof. Assume that a hesitant fuzzy filter H_X is mighty. Setting $z := 1$ in (3.3), we have $h_X(1 * (y * x)) \cap h_X(1) = h_X(y * x) \subseteq h_X(((x * y) * y) * x)$. Hence (3.4) holds.

Conversely, suppose that the hesitant fuzzy filter $H_X = \{(x, h_X(x))|x \in X\}$ satisfies the condition (3.4). Using (3.2) and (3.4), we have $h_X(z * (y * x)) \cap h_X(z) \subseteq h_X(y * x) \subseteq h_X(((x * y) * y) * x)$, for any $x, y \in X$. Hence H_X is a hesitant fuzzy mighty filter of X. □

Proposition 3.10. *Let $H_X = \{(x, h_X(x))|x \in X\}$ be a hesitant fuzzy mighty filter of a BE-algebra X. Then $X_{H_X} := \{x \in X|h_X(x) = h_X(1)\}$ is a mighty filter of X.*

Proof. Clearly, $1 \in X_{H_X}$. Let $z*(y*x), z \in X_{H_X}$. Then $h_X(z*(y*x)) = h_X(1)$ and $h_X(z) = h_X(1)$. It follows from (3.3) that $h_X(z * (y * x)) \cap h_X(z) = h_X(1) \subseteq h_X(((x * y) * y) * x)$. By (3.1), we get $h_X(((x * y) * y) * x) = h_X(1)$. Hence $((x * y) * y) * x \in X_{H_X}$. Therefore X_{H_X} is a mighty filter of X. □

Theorem 3.11. *Let $H_X = \{(x, h_X(x))|x \in X\}$ and $G_X = \{(x, g_X(x))|x \in X\}$ be hesitant fuzzy filters of a transitive BE-algebra such that $H_X \widetilde{\subseteq} G_X$ and $h_X(1) = g_X(1)$. If H_X is mighty, then so is G_X.*

Proof. Let $x, y \in X$. Note that $y * ((y * x) * x) = (y * x) * (y * x) = 1$. Since H_X is a hesitant fuzzy mighty filter of a BE-algebra X, by (3.4) and $H_X \widetilde{\subseteq} G_X$ we have $h_X(1) = h_X(y * ((y * x) * x)) \subseteq h_X(((((y*x)*x)*y)*y)*((y*x)*x)) \subseteq g_X(((((y*x)*x)*y)*y)*((y*x)*x))$. Since $h_X(1) = g_X(1)$, we get $g_X((y * x) * ((((y * x) * x) * y) * y) * x)) = g_X(((((y * x) * x) * y) * y) * ((y * x) * x)) = g_X(1)$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} g_X(y*x) =& g(1)\cap g_X(y*x) \\ =& g_X((y*x)*(((((y*x)*x)*y)*y)*x))\cap g_X(y*x) \\ \subseteq& g_X(((((y*x)*x)*y)*y)*x). \end{aligned} \tag{3.5}$$

Since X is transitive, we get

$$\begin{aligned} [((((y*x)*x)*y)*y)*x]*&[((x*y)*y)*x] \\ &\geq ((x*y)*y)*((((y*x)*x)*y)*y) \\ &\geq (((y*x)*x)*y)*(x*y) \\ &\geq x*((y*x)*x) \\ &= (y*x)*(x*x) \\ &= (y*x)*1 = 1. \end{aligned}$$

It follows from Proposition 3.2 that $g_X(((((y*x)*x)*y)*y)*x)\cap g_X(1) = g_X(((((y*x)*x)*y)*y)*x) \subseteq g_X(((x*y)*y)*x)$. Using (3.5), we have $g_X(y*x) \subseteq g_X(((((y*x)*x)*y)*y)*x) \subseteq g_X(((x*y)*y)*x)$. Therefore $g_X(y*x) \subseteq g_X(((x*y)*y)*x)$. By Theorem 3.9, G_X is a hesitant fuzzy mighty filter of X. □

Corollary 3.12. *Every hesitant fuzzy filter H_X of a transitive BE-algebra X is mighty if and only if the hesitant fuzzy filter $H_{\{1\}}$ is mighty.*

Proof. Straightforward. □

Let $H_X := \{(x, h_X(x))|x\in X\}$ be a hesitant fuzzy filter of a transitive BE-algebra X. Define a binary relation " $\sim_{h_X}$ " on X by putting $x\sim_{h_X} y$ if and only if $h_X(x*y) = h_X(y*x) = h_X(1)$ for any $x, y\in X$.

Lemma 3.13. *The relation " $\sim_{h_X}$ " is an equivalence relation on a transitive BE-algebra X.*

Proof. For any $x\in X$, $x*x = 1$ by (2.1). So $h_X(x*x) = h_X(1)$, hence $x\sim_{h_X} x$, which $\sim_{h_X}$ is reflexive. Suppose that $x\sim_{h_X} y$ for any $x, y\in X$. Then $h_X(x*y) = h_X(y*x) = h_X(1)$. Hence $\sim_{h_X}$ is symmetric. Assume that $x\sim_{h_X} y$ and $y\sim_{h_X} z$ for any $x, y, z\in X$. Then $h_X(x*y) = h_X(y*x) = h_X(1)$ and $h_X(y*z) = h_X(z*y) = h_X(1)$. By transitivity, $(x*y)*[(y*z)*(x*z)] = 1$ and $(z*y)*[(y*x)*(z*x)] = 1$. By Proposition 3.2, we have $h_X(x*y)\cap h_X(y*z) = h_X(1) \subseteq h_X(x*z)$ and $h_X(z*y)\cap h_X(y*x) = h_X(1) \subseteq h_X(z*x)$. Hence $h_X(z*x) = h_X(z*x) = h_X(1)$, i.e., $x\sim_{h_X} z$. Thus $\sim_{h_X}$ is an equivalence relation on X. □

Lemma 3.14. *The relation " $\sim_{h_X}$ " is a congruence relation on a transitive BE-algebra X.*

Proof. If $x\sim_{h_X} y$ and $u\sim_{h_X} v$ for any $x, y, u, v\in X$, then $h_X(x*y) = h_X(y*x) = h_X(1)$ and $h_X(u*v) = h_X(v*u) = h_X(1)$. By transitivity, $(u*v)*[(x*u)*(x*v)] = 1$ and

$(v * u) * [(x * v) * (x * u)] = 1$, it follows from Proposition 3.2 that $h_X(1) = h_X(u * v) \subseteq h_X((x*u)*((x*v))$ and $h_X(1) = h_X(v*u) \subseteq h_X((x*v)*(x*u))$. Hence $h_X((x*u)*(x*v)) = h_X(1)$ and $h_X((x*v)*(x*u)) = h_X(1)$. Therefore $x*u \sim_{h_X} x*v$. By a similar way, we can prove that $x*v \sim_{h_X} y*v$. Therefore $\sim_{h_X}$ is a congruence relation on X. □

X is decomposed by the congruence relation $\sim_{h_X}$. The class containing x is denoted by $[x]_{h_X}$. Denote $X/h_X := \{[x]_{h_X} | x \in X\}$. We define a binary relation $*'$ on X/h_X by $[x]_{h_X} *' [y]_{h_X} := [x*y]_{h_X}$. This definition is well defined since $\sim_{h_X}$ is a congruence relation on X.

Lemma 3.15. $[1]_{h_X} = X_{H_X}$.

Proof. $[1]_{h_X} = \{x \in X | 1 \sim_{h_X} x\} = \{x \in X | h_X(1*x) = h_X(x*1) = h_X(1)\} = \{x \in X | h_X(x) = h_X(1)\} = X_{H_X}$. □

Theorem 3.16. *Let X be a transitive BE-algebra X. Then $(X/h_X; *', [1]_{h_X})$ is a transitive BE-algebra.*

Proof. Straightforward. □

Theorem 3.17. *A hesitant fuzzy filter of a transitive BE-algebra X is mighty if and only if every filter of the quotient algebra X/h_X is mighty.*

Proof. Assume that a hesitant fuzzy filter H_X is mighty and let $x, y \in X$ be such that $[y]_{h_X} *' [x]_{h_X} \in [1]_{h_X}$. Then $h_X(y*x) = h_X(1)$. It follows from (2.3) and (3.3) that $h_X(1*(y*x)) \cap h_X(1) = h_X(y*x) = h_X(1) \subseteq h_X(((x*y)*y)*x)$. Hence $h_X(((x*y)*y)*x) = h_X(1)$. So $(((([x]_{h_X} *' [y]_{h_X}) *' [y]_{h_X})) *' [x]_{h_X} = [((x*y)*y)*x]_{h_X} \in [1]_{h_X}$ which proves that $\{[1]_{h_X}\}$ is a mighty filter of X/h_X. By Corollary 3.13, every filter of X/h_X is mighty.

Conversely, suppose that every filter of the quotient algebra X/h_X is mighty and let $x, y \in X$ be such that $y * x \in [1]_{h_X}$. Then $h_X(y*x) = h_X(1)$ and so $[y]_{h_X} *' [x]_{h_X} \in [1]_{h_X}$. Since $\{[1]_{h_X}\}$ is a mighty filter of X/h_X, it follows from Theorem 2.2 that $[((x*y)*y)*x]_{h_X} = (([x]_{h_X} *' [y]_{h_X}) *' [y]_{h_X}) *' [x]_{h_X} \in [1]_{h_X}$. Hence $h_X((((x*y)*y)*)*x) = h_X(1)$. Therefore $h_X(y*x) = h_X(((x*y)*y))*x)$. Thus H_X is a hesitant fuzzy filter of Theorem 3.9. □

Theorem 3.18. *A hesitant fuzzy set $H_X := \{(x, h_X(x) | x \in X\}$ of a BE-algebra X is a hesitant fuzzy mighty filter of X if and only if the set $H_X(\gamma) := \{x \in X | \gamma \subseteq h_X(x)\}$ is a mighty filter of X for all $\gamma \in \mathscr{P}([0,1])$ whenever it is nonempty.*

Proof. Suppose that H_X is a hesitant fuzzy mighty filter of X. Let $x, y, z \in X$ and $\gamma \in \mathscr{P}([0,1])$ be such that $z*(y*x) \in H_X(\gamma)$ and $z \in H_X(\gamma)$. Then $h_X(z*(y*x)) \supseteq \gamma$ and $h_X(z) \supseteq \gamma$. It follows from (3.1) and (3.3) that $h_X(1) \supseteq h_X(((x*y)*y)*x) \supseteq h_X(z*(y*x)) \cap h_X(z) \supseteq \gamma$. Hence $1 \in H_X(\gamma)$ and $((x*y)*y)*x \in H_X(\gamma)$, and therefore $H_X(\gamma)$ is a mighty filter of X.

Conversely, assume that $H_X(\gamma)$ is a mighty filter of X for all $\gamma \in \mathscr{P}([0,1])$ with $H_X(\gamma) \neq \emptyset$. For any $x \in X$, let $h_X(x) = \gamma$. Then $x \in H_X(\gamma)$. Since $H_X(\gamma)$ is a mighty filter of X, we have

$1 \in h_X(\gamma)$ and so $h_X(x) = \gamma \subseteq h_X(1)$. For any $x, y, z \in X$, let $h_X(z*(y*x)) = \gamma_{z*(y*x)}$ and $h_X(z) = \gamma_z$. Let $\gamma := \gamma_{z*(y*x)} \cap \gamma_z$. Then $z*(y*x) \in H_X(\gamma)$ and $z \in H_X(\gamma)$ which imply that $((x*y)*y)*x \in H_X(\gamma)$. Hence $h_X(((x*y)*y)*x) \supseteq \gamma = \gamma_{z*(y*x)} \cap \gamma_z = h_X(z*(y*x)) \cap h_X(z)$. Thus H_X is a hesitant fuzzy mighty filter of X. □

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A Class of New General Iteration Approximation of Common Fixed Points for Total Asymptotically Nonexpansive Mappings in Hyperbolic Spaces

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Abstract. In this paper, we introduce and study a class of new general iteration processes for two finite families of total asymptotically nonexpansive mappings in hyperbolic spaces, which includes asymptotically nonexpansive mapping, (generalized) nonexpansive mapping of all normed linear spaces, Hadamard manifolds and CAT(0) spaces as special cases. Some important related properties to the new general iterative processes are also given and analyzed, and Δ-convergence and strong convergence of the iteration in hyperbolic spaces are proved. Furthermore, some meaningful illustrations for clarifying our results and two open questions are proposed. The results presented in this paper extend and improve the corresponding results announced in the current literature.

Key Words and Phrases: common fixed point, new general iterative approximation, Δ-convergence and strong convergence, total asymptotically nonexpansive mapping, hyperbolic space.

AMS Subject Classification: 47H09, 47H10, 54E70.

1 Introduction and preliminaries

Let $(\mathcal{H}, d)$ be a metric space, $\{T_i\}_{i=1}^r$ and $\{S_i\}_{i=1}^r$ be two finite families of nonlinear mappings on nonempty set $K \subset \mathcal{H}$. Suppose that $\{\alpha_{in}\}$ and $\{\beta_{in}\}$ are two real sequences in $[a, b]$ for some $a, b \in (0, 1)$ and $\theta_{in} := \frac{\beta_{in}}{1-\alpha_{in}}$. For $r \geq 2$ and $n \geq 1$, in this paper, we consider the following general iterative sequence $\{x_n\}$:

$$\begin{aligned}
x_{n+1} &= W(T_1^n y_{n+r-2}, W(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}), \alpha_{1n}),\\
y_{n+r-2} &= W(T_2^n y_{n+r-3}, W(y_{n+r-3}, S_2^n y_{n+r-3}, \theta_{2n}), \alpha_{2n}),\\
y_{n+r-3} &= W(T_3^n y_{n+r-4}, W(y_{n+r-4}, S_3^n y_{n+r-4}, \theta_{3n}), \alpha_{3n}),\\
&\ \vdots\\
y_{n+1} &= W(T_{r-1}^n y_n, W(y_n, S_{r-1}^n y_n, \theta_{(r-1)n}), \alpha_{(r-1)n}),\\
y_n &= W(T_r^n x_n, W(x_n, S_r^n x_n, \theta_{rn}), \alpha_{rn}).
\end{aligned} \tag{1.1}$$

Remark 1.1 For appropriate and suitable choices of the nonlinear mappings $\{T_i\}_{i=1}^r$ and $\{S_i\}_{i=1}^r$, the positive integer r and the underlying spaces, the iteration (1.1) includes a number of known iterative processes, which were studied previously by many authors. For more details, see [1–20] and the references therein, and the following examples:

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Example 1.1 If $\beta_{in} = 0$ for $i = 1, 2, 3, \cdots, r$ and all $n \geq 1$, and $\{\alpha_{in}\}$ is a real sequence in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$, then the sequence $\{x_n\}$ in (1.1) reduces to

$$\begin{aligned} x_{n+1} &= \alpha_{1n} y_{n+r-2} + (1-\alpha_{1n}) T_1^n y_{n+r-2}, \\ y_{n+r-2} &= \alpha_{2n} y_{n+r-3} + (1-\alpha_{2n}) T_2^n y_{n+r-3}, \\ y_{n+r-3} &= \alpha_{3n} y_{n+r-4} + (1-\alpha_{3n}) T_3^n y_{n+r-4}, \\ &\vdots \\ y_{n+1} &= \alpha_{(r-1)n} y_n + (1-\alpha_{(r-1)n}) T_{r-1}^n y_n, \\ y_n &= \alpha_{rn} x_n + (1-\alpha_{rn}) T_r^n x_n, \end{aligned} \tag{1.2}$$

which was considered by Yildirim and Ozdemir [1] when $\{T_i\}_{i=1}^r$ is a family of asymptotically quasi-nonexpansive self-mappings on $K \subset \mathcal{H}$ and $\mathcal{H}$ is a Banach space. Further, the iteration process (1.2) was introduced and studied by Basarir and Sahin [2] for a generalized nonexpansive mapping of the CAT(0) spaces.

Example 1.2 For $r = 3$ and $\alpha_{in} = 0$, then (1.1) changes into the iterative process introduced by Noor [3], which was dealt for variational inequalities of the Hilbert spaces. Moreover, a unified treatment regarding the iterative process for nonexpansive mapping in hyperbolic spaces was considered by Akbulut and Gündüz [4]. For many more, see, for example, the research works of Sahin and Basarir [5], Suantai [6] and many others in the literature.

Example 1.3 Let $r = 2$, and $\alpha_{1n} = 1$, and $\alpha_{2n} = 0$, and $T_2 = S_2$, then (1.1) becomes to the following iteration:

$$x_{n+1} = T_1^n y_n, y_n = W(x_n, T_2^n x_n, \theta_{2n}). \tag{1.3}$$

The iteration (1.3) is called a modified hybrid Picard-Mann iteration process, which was introduced and studied by Thakur et al. [7] in CAT(0) space. This process (1.3) is independent of Picard and Mann iterative process and the convergence process is faster than Picard and Mann iteration process. For more on (hybrid) Picard-Mann iteration process and a comparison between different process of modified hybrid Picard-Mann iteration process, see, for example, [7, 8] and the references therein.

Example 1.4 Let $r = 2$, and $\alpha_{1n} = 0$, and $\beta_{1n} = 1$, $\alpha_{2n} = 1$, then (1.1) is equivalent to

$$x_{n+1} = W(x_n, S^n x_n, \theta_n),$$

which is well-known modified Mann iteration process, and was studied by Schu [9] in Banach spaces.

In 2013, Fukhar-ud-din and Khan [21] pointed out "structural properties of the space under consideration are very important in establishing the fixed point property of the space, for example, strict convexity, uniform convexity and uniform smoothness etc". In fact, in recent decades, motivated and governed by questions in most of science problems about hyperbolic groups, the study on hyperbolic spaces has been developed unremittingly in geometric group theory and metric fixed point theory in normed linear spaces or Banach spaces. Especially, the concept of hyperbolic spaces introduced by Kohlenbach [22] and defined below, is more restrictive and more general than that of being considered in [23] and in [24], respectively (see also [25]). Furthermore, all normed linear spaces, convex subsets wherein Hadamard manifolds and CAT(0) spaces are the special cases of the class of hyperbolic spaces due to Kohlenbach [22].

Definition 1.1 A hyperbolic spaces is a metric space $(\mathcal{H}, d)$ together with a mapping $W : \mathcal{H}^2 \times [0,1] \to \mathcal{H}$ satisfying

(i) $d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1-\alpha) d(u, y)$,

(ii) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$,

(iii) $W(x, y, \alpha) = W(y, x, (1-\alpha))$,

(iv) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq \alpha d(x, y) + (1-\alpha) d(z, w)$ for all $u, x, y, z, w \in \mathcal{H}$ and $\alpha, \beta \in [0,1]$.

Remark 1.1 (1) The class of hyperbolic spaces is general in nature and its important example is the open unit ball B in a complex domain C with respect to the Poincare metric (also called

"Poincare distance")

$$d_B(x,y) := \arg\tanh\left|\frac{x-y}{1-x\overline{y}}\right| = \arg\tanh(1-\sigma(x,y))^{\frac{1}{2}},$$

where $\sigma(x,y) := \frac{(1-|x|^2)(1-|y|^2)}{|1-x\overline{y}|^2}$ for all $x,y \in B$. Further, the above example can be extended from C to general complex Hilbert spaces $(H, \langle\cdot\rangle)$ (see [21, 22]).

(2) A metric space $(\mathcal{H}, d)$ satisfying only (i) in Definition 1.1 is a convex metric space introduced by Takahashi [26]. A nonempty subset K of a hyperbolic space $\mathcal{H}$ is convex if $W(x,y,\alpha) \in K$ for all $x,y \in K$ and $\alpha \in [0,1]$. For more on hyperbolic spaces and a comparison between different notions of hyperbolic space, see, for example, [27] and the references therein.

(3) A hyperbolic space is uniformly convex if for any $r > 0$ and $\epsilon \in (0,2]$, and all $u,x,y \in \mathcal{H}$, there exists $\delta \in (0,1]$ such that

$$d(W(x,y,\frac{1}{2}),u) \le (1-\delta)r,$$

provided $\max\{d(x,u), d(y,u)\} \le r$ and $d(x,y) \ge r\epsilon$ (see [28, 29]). A map $\eta : (0,+\infty)\times(0,2] \to (0,1]$, which provides such $\delta = \eta(r,\epsilon)$ for given $r > 0$ and $\epsilon \in (0,2]$, is known as a modulus of uniform convexity of $\mathcal{H}$. We call η monotone if it decreases with r (for fixed ϵ), i.e., for all $\epsilon > 0$, $r_2 \ge r_1 > 0(\eta(r_2,\epsilon) \le \eta(r_1,\epsilon))$. CAT(0) spaces are uniformly convex hyperbolic spaces with modulus of uniform convexity $\eta(r,\varepsilon) = \frac{\varepsilon^2}{8}$ (see [28, 30]). Thus, the class of uniformly convex hyperbolic spaces includes both uniformly convex normed spaces and CAT(0) spaces as special cases.

In the sequel, let $(\mathcal{H}, d)$ be a metric space, and let K be a nonempty subset of $\mathcal{H}$. We shall denote the fixed point set of a self-mapping on K of T by $F(T) = \{x \in K : Tx = x\}$.

Definition 1.2 A mapping $T : K \to K$ is said to be

(i) semi-compact if every bounded sequence $\{x_n\} \subset K$, satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$, has a convergent subsequence;

(ii) nonexpansive if $d(Tx,Ty) \le d(x,y)$ for any $x,y \in K$;

(iii) quasi-nonexpansive if $d(Tx,p) \le d(x,p)$ for all $x \in K$ and $p \in F(T) \neq \emptyset$;

(iv) asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [0,+\infty)$ and $\lim_{n\to\infty} k_n = 0$ such that

$$d(T^n x, T^n y) \le (1+k_n)d(x,y), \quad \forall x,y \in K, n \ge 1;$$

(v) asymptotically quasi-nonexpansive if there exists a sequence $\{k_n\} \subset [0,+\infty)$ and $\lim_{n\to\infty} k_n = 0$ such that

$$d(T^n x, p) \le (1+k_n)d(x,p), \quad \forall x \in K, p \in F(T), n \ge 1;$$

(vi) $(\{\mu_n\},\{\xi_n\},\rho)$-total asymptotically nonexpansive, if there exist nonnegative sequences $\{\mu_n\}$, $\{\xi_n\}$ with $\mu_n \to 0$, $\xi_n \to 0$ and a strictly increasing continuous function $\rho : [0,+\infty) \to [0,+\infty)$ with $\rho(0) = 0$ such that

$$d(T^n x, T^n y) \le d(x,y) + \mu_n \rho\big(d(x,y)\big) + \xi_n, \quad \forall x,y \in K, n \ge 1;$$

(vii)$(\{\mu_n\},\{\xi_n\},\rho)$-total asymptotically quasi-nonexpansive, if there exist nonnegative sequences $\{\mu_n\}$, $\{\xi_n\}$ with $\mu_n \to 0$, $\xi_n \to 0$ and a strictly increasing continuous function $\rho : [0,+\infty) \to [0,+\infty)$ with $\rho(0) = 0$ such that

$$d(T^n x, p) \le d(x,p) + \mu_n \rho\big(d(x,p)\big) + \xi_n, \quad \forall x \in K, p \in F, n \ge 1;$$

(viii) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$d(T^n x, T^n y) \le L d(x,y), \quad \forall x,y \in K, n \ge 1.$$

Remark 1.2 From Definition 1.2, it follows that a (quasi-)nonexpansive mapping is an asymptotically (quasi-)nonexpansive mapping with $k_n \equiv 0$ for $n \ge 1$, and each asymptotically (quasi-)nonexpansive mapping is a $(\{\mu_n\},\{\xi_n\},\rho)$-total asymptotically (quasi-)nonexpansive mapping with $\xi_n = 0$, and $\rho(t) = t \ge 0$. However, in general, the converse of these statement is not true.

As all we know, the study of such types of problems on the iterative approximation of (common) fixed points for generalizations of nonexpansive mappings in hyperbolic spaces, is motivated by an increasing interest in the problems of finding a common fixed point of some nonlinear mappings, which is the only main tool for analysis of generalized nonexpansive mappings and provides us a general and unified framework for studying the existence of fixed points of various nonlinear mappings arising in many branches of nonlinear analysis, topology and applied mathematics, etc.

Inspired and motivated and by the above recent works, in this paper, we shall study some important related properties to the new general iterative process (1.1) for two finite families of total asymptotically nonexpansive mappings as well as two finite families of total asymptotically quasi-nonexpansive mappings in hyperbolic spaces. Results concerning Δ-convergence as well as strong convergence of this iteration are proved. The results presented in the paper extend and improve some recent results given in [1, 2, 4–7, 9, 21].

In order to define the concept of Δ-convergence in the general setup of hyperbolic spaces, in the next moment, we first give some basic concepts.

In 1976, Lim [31] introduced the notion of asymptotic center and, consequently, coined the concept of Δ-convergence in a general setting of a metric space. Kirk and Panyanak [32] proposed an analogous version of convergence in geodesic spaces, namely Δ-convergence, which was originally introduced by Lim [31]. Further, Kirk and Panyanak [32] showed that Δ-convergence coincides with the usual weak convergence in Banach spaces and both concepts share many useful properties.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space $\mathcal{H}$. For $x \in \mathcal{H}$, we define a continuous functional $r(\cdot, \{x_n\}) : \mathcal{H} \to [0, +\infty)$ by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $\hat{r}(\{x_n\})$ of $\{x_n\}$ is given by

$$\hat{r}(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in \mathcal{H}\}.$$

The asymptotic center of a bounded sequence $\{x_n\}$ with respect to $K \subset \mathcal{H}$ is defined as follows:

$$A_K(\{x_n\}) = \{x \in \mathcal{H} : r(x, \{x_n\}) \leq r(y, \{x_n\}), \forall y \in K\},$$

which is the set of minimizers for $r(\cdot, \{x_n\})$. Further, it is simply denoted by $A(\{x_n\})$ when the asymptotic center is taken with respect to $\mathcal{H}$, and a sequence $\{x_n\}$ in $\mathcal{H}$ is said to Δ-converge to $x \in \mathcal{H}$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

It is well known that uniformly convex Banach spaces and even CAT(0) spaces enjoy the property that "bounded sequences have unique asymptotic centers with respect to closed convex subsets". The following lemma ensures that this property also holds in a complete uniformly convex hyperbolic space.

Lemma 1.1 ([30]) Let $(\mathcal{H}, d, W)$ be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity. Then every bounded sequence $\{x_n\}$ in $\mathcal{H}$ has a unique asymptotic center with respect to any nonempty closed convex subset K of $\mathcal{H}$.

In the sequel, we need the following lemmas.

Lemma 1.2 ([10]) Let $(\mathcal{H}, d, W)$ be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in \mathcal{H}$ and $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in $\mathcal{H}$ such that for some $c \geq 0$,

$$\limsup_{n\to\infty} d(x_n, x) \leq c, \quad \limsup_{n\to\infty} d(y_n, x) \leq c, \quad \lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = c,$$

Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Lemma 1.3 ([10]) Let K be a nonempty closed convex subset of uniformly convex hyperbolic space, and let $\{x_n\}$ be a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \zeta$. If $\{y_m\}$ is another sequence in K such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \zeta$, then $\lim_{m\to\infty} y_m = y$.

Lemma 1.4 ([33]) Let $\{a_n\}$, $\{b_n\}$ and $\{\omega_n\}$ be nonnegative real sequences satisfying

$$a_{n+1} \le (1+\omega_n)a_n + b_n, \quad \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} \omega_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then the limit $\lim_{n\to\infty} a_n$ exist. If there exists a subseqence $\{a_{n_i}\} \subset \{a_n\}$ such that $a_{n_i} \to 0$, then $\lim_{n\to\infty} a_n = 0$.

2 Some important related properties

Throughout in this paper, we assume that $I = \{1, 2, \cdots, r\}$, $\{T_i\}_{i=1}^r$ and $\{S_i\}_{i=1}^r$ are two finite families of total asymptotically nonexpansive mappings on a nonempty subset K of the hyperbolic space $\mathcal{H}$ defined by Definition 1.2, for each $i \in I$ and all $n \ge 1$, $\{\alpha_{in}\}$, $\{\beta_{in}\}$ and $\{\theta_{in}\}$ are the same as in (1.1). We start with the following important related property of the general iterative process (1.1) for two finite families of total asymptotically nonexpansive mappings in a hyperbolic space.

Theorem 2.1 Let K be a nonempty closed and convex subset of a hyperbolic space $\mathcal{H}$. For $i \in I$, let $T_i : K \to K$ be a $(\{\mu_n^i\}, \{\xi_n^i\}, \rho^i)$-total asymptotically nonexpansive mapping with $\lim_{n\to\infty} \mu_n^i = 0$ and $\lim_{n\to\infty} \xi_n^i = 0$, and a strictly increasing continuous function $\rho^i : [0, +\infty) \to [0, +\infty)$ satisfying $\rho^i(0) = 0$, and let $S_i : K \to K$, be a $(\{\hat{\mu}_n^i\}, \{\hat{\xi}_n^i\}, \hat{\rho}^i)$-total asymptotically nonexpansive mapping with $\lim_{n\to\infty} \hat{\mu}_n^i = 0$ and $\lim_{n\to\infty} \hat{\xi}_n^i = 0$, and a strictly increasing continuous function $\hat{\rho}^i : [0, +\infty) \to [0, +\infty)$ satisfying $\hat{\rho}^i(0) = 0$. Assume that $F = \bigcap_{i=1}^r (F(T_i) \cap F(S_i)) \ne \emptyset$, and for each $i \in I$, the following conditions hold:

(i) $\sum_{n=1}^{\infty} \mu_n^i < +\infty$, $\sum_{n=1}^{\infty} \hat{\mu}_n^i < +\infty$, $\sum_{n=1}^{\infty} \xi_n^i < +\infty$, $\sum_{n=1}^{\infty} \hat{\xi}_n^i < +\infty$;

(ii) there exists a constant $M^* > 0$ such that

$$\rho^i(r) \le M^* r, \quad \hat{\rho}^i(r) \le M^* r, \quad \forall r > 0.$$

Then, for the sequence $\{x_n\}$ in (1.1), $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F$.

Proof. Set $\mu_n = \max_{i\in I}\{\mu_n^i, \hat{\mu}_n^i\}$, and $\xi_n = \max_{i\in I}\{\xi_n^i, \hat{\xi}_n^i\}$, $\rho = \max_{i\in I}\{\rho^i, \hat{\rho}^i\}$. By condition (i), we know that $\sum_{n=1}^{\infty} \mu_n < +\infty$, $\sum_{n=1}^{\infty} \xi_n < +\infty$. For any $p \in F$ and all $n \ge 1$, it follows from (1.1) that

$$\begin{aligned}
d(y_n, p) &\le \alpha_{rn} d(T_r^n x_n, p) + (1-\alpha_{rn}) d(W(x_n, S_r^n x_n, \theta_{rn}), p)\\
&\le \alpha_{rn} d(T_r^n x_n, p) + \beta_{rn} d(x_n, p) + (1-\alpha_{rn}-\beta_{rn}) d(S_r^n x_n, p)\\
&\le \alpha_{rn}[d(x_n, p) + \mu_n^r \rho^r\big(d(x_n, p)\big) + \xi_n^r] + \beta_{rn} d(x_n, p)\\
&\quad +(1-\alpha_{rn}-\beta_{rn})[d(x_n, p) + \hat{\mu}_n^r \hat{\rho}^r\big(d(x_n, p)\big) + \hat{\xi}_n^r]\\
&\le \alpha_{rn}[d(x_n, p) + \mu_n \rho\big(d(x_n, p)\big) + \xi_n] + \beta_{rn} d(x_n, p)\\
&\quad +(1-\alpha_{rn}-\beta_{rn})[d(x_n, p) + \mu_n \rho\big(d(x_n, p)\big) + \xi_n]\\
&\le \alpha_{rn}[(1+\mu_n M^*) d(x_n, p) + \xi_n] + \beta_{rn} d(x_n, p)\\
&\quad +(1-\alpha_{rn}-\beta_{rn})[(1+\mu_n M^*) d(x_n, p) + \xi_n]\\
&\le (1+\mu_n M^*) d(x_n, p) + \xi_n
\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}
d(y_{n+1}, p) &\le \alpha_{(r-1)n} d(T_{r-1}^n y_n, p) + (1-\alpha_{(r-1)n}) d(W(y_n, S_{r-1}^n y_n, \theta_{(r-1)n}), p)\\
&\le \alpha_{(r-1)n} d(T_{r-1}^n y_n, p) + \beta_{(r-1)n} d(y_n, p)\\
&\quad +(1-\alpha_{(r-1)n}-\beta_{(r-1)n}) d(S_{r-1}^n y_n, p)\\
&\le \alpha_{(r-1)n}[d(y_n, p) + \mu_n \rho\big(d(y_n, p)\big) + \xi_n] + \beta_{(r-1)n} d(y_n, p)\\
&\quad +(1-\alpha_{(r-1)n}-\beta_{(r-1)n})[d(y_n, p) + \mu_n \rho\big(d(y_n, p)\big) + \xi_n]\\
&\le \alpha_{(r-1)n}[(1+\mu_n M^*) d(y_n, p) + \xi_n] + \beta_{(r-1)n} d(y_n, p)\\
&\quad +(1-\alpha_{(r-1)n}-\beta_{(r-1)n})[(1+\mu_n M^*) d(y_n, p) + \xi_n]\\
&\le (1+\mu_n M^*) d(y_n, p) + \xi_n.
\end{aligned} \tag{2.2}$$

Similarly, we have

$$d(y_{n+r-2}, p) \le (1 + \mu_n M^*) d(y_{n+r-3}, p) + \xi_n,$$
$$d(x_{n+1}, p) \le (1 + \mu_n M^*) d(y_{n+r-2}, p) + \xi_n.$$

Thus,

$$\begin{aligned} d(x_{n+1}, p) &\le (1 + \mu_n M^*)^r d(x_n, p) + \sum_{j=1}^{r-1} (1 + \mu_n M^*)^j \xi_n \\ &\le d(x_n, p)\Big[1 + \tbinom{r}{1}\mu_n M^* + \tbinom{r}{2}(\mu_n M^*)^2 + \tbinom{r}{3}(\mu_n M^*)^3 \\ &\quad + \cdots + \tbinom{r}{r}(\mu_n M^*)^r\Big] + \sum_{j=1}^{r-1} (1 + \mu_n M^*)^j \xi_n \\ &\le (1 + a_n^r \mu_n) d(x_n, p) + \sum_{j=1}^{r-1} (1 + \mu_n M^*)^j \xi_n \\ &\le (1 + M_1 \mu_n) d(x_n, p) + M_2 \xi_n, \end{aligned}$$

where $a_n^r = \binom{r}{1}M^* + \binom{r}{2}(M^*)^2\mu_n + \binom{r}{3}(M^*)^3(\mu_n)^2 + \cdots + \binom{r}{r}(M^*)^r(\mu_n)^{r-1}$, and by virtue of condition(i), there exist positive constants M_1 and M_2 such that $a_n^r \le M_1$, $\sum_{j=1}^{r-1}(1+\mu_n M^*)^j \le M_2$ for each $n \ge 1$. Applying Lemma 1.4 to the above inequality, we have $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. □

In 1993, Bruck et al. [34] introduced a notion of asymptotically nonexpansive mapping in the intermediate sense. More accurately, a mapping $T : K \to K$ is said to be asymptotically nonexpansive mapping in the intermediate sense, provided that T is uniformly continuous and $\limsup_{n\to\infty} \sup_{x,y\in K}\{d(T^n x, T^n y) - d(x, y)\} \le 0$. Put $\xi_n = \max\{0, \sup_{x,y\in K}\{d(T^n x, T^n y) - d(x, y)\}\}$ and $\sum_{n=1}^{\infty} \xi_n^i < +\infty$, then $d(T^n x, T^n y) \le d(x, y) + \xi_n$ for any $n \ge 1$ and $x, y \in K$. In more detail, see, for example, [20] and the references therein.

The following result can be obtained from Theorem 2.1 immediately.

Corollary 2.1 Let K be a nonempty closed and convex subset of a hyperbolic space $\mathcal{H}$. For $i \in I$, let $T_i : K \to K$ be a $\{\xi_n^i\}$-asymptotically nonexpansive mapping in the intermediate sense and let $S_i : K \to K$ be a $\{\hat{\xi}_n^i\}$-asymptotically nonexpansive mapping in the intermediate sense. If $\sum_{n=1}^{\infty} \xi_n^i < +\infty$, $\sum_{n=1}^{\infty} \hat{\xi}_n^i < +\infty$ for $i \in I$ and $F = \bigcap_{i=1}^{r}(F(T_i) \cap F(S_i)) \neq \emptyset$, then, for the sequence $\{x_n\}$ in (1.1), $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F$.

Proof. Let $\xi_n = \max_{i\in I}\{\xi_n^i, \hat{\xi}_n^i\}$, then $\sum_{n=1}^{\infty} \xi_n < +\infty$. The rest of the proof is trivial. □

Corollary 2.2 Let K be a nonempty closed and convex subset of a hyperbolic space $\mathcal{H}$. Let $T_i : K \to K$ be a $\{k_n^i\}$-asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty} k_n^i < +\infty$ and $S_i : K \to K$ be a $\{\hat{k}_n^i\}$-asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty} \hat{k}_n^i < +\infty$ for $i \in I$. Assume that $F = \bigcap_{i=1}^{r}(F(T_i) \cap F(S_i)) \neq \emptyset$. Then, for the sequence $\{x_n\}$ in (1.1), $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F$.

Proof. Taking $k_n = \max_{i\in I}\{k_n^i, \hat{k}_n^i\}$, then $\sum_{n=1}^{\infty} k_n < +\infty$. Let $\rho^i(t) = \hat{\rho}^i(t) = t$, $\xi_n^i = \hat{\xi}_n^i = 0$, $\mu_n^i = k_n^i$ in Theorem 2.1 for $i \in I$. Then all the conditions in Theorem 2.1 are satisfied and so the result holds. □

Theorem 2.2 Let K be a nonempty closed and convex subset of a uniformly convex hyperbolic space $\mathcal{H}$ with monotone modulus of uniform convexity η. For $i \in I$, let $T_i : K \to K$ be a uniformly L_i-Lipschitzian and $(\{\mu_n^i\}, \{\xi_n^i\}, \rho^i)$-total asymptotically nonexpansive mapping with $\lim_{n\to\infty} \mu_n^i = 0$ and $\lim_{n\to\infty} \xi_n^i = 0$, and a strictly increasing continuous function $\rho^i : [0, +\infty) \to [0, +\infty)$ satisfying $\rho^i(0) = 0$, and let $S_i : K \to K$ be a uniformly $\hat{L}_i$-Lipschitzian and $(\{\hat{\mu}_n^i\}, \{\hat{\xi}_n^i\}, \hat{\rho}^i)$-total asymptotically nonexpansive mapping with $\lim_{n\to\infty} \hat{\mu}_n^i = 0$ and $\lim_{n\to\infty} \hat{\xi}_n^i = 0$, and a strictly increasing continuous function $\hat{\rho}^i : [0, +\infty) \to [0, +\infty)$ satisfying $\hat{\rho}^i(0) = 0$. Suppose that $F = \bigcap_{i=1}^{r}(F(T_i) \cap F(S_i)) \neq \emptyset$ and the conditions (i) and (ii) in Theorem 2.1 hold. Then, for $i \in I$ and the sequence $\{x_n\}$ generated by (1.1), we have

$$\lim_{n\to\infty} d(x_n, T_i x_n) = \lim_{n\to\infty} d(x_n, S_i x_n) = 0.$$

Proof. It follows from Theorem 2.1 that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. Assume that $\lim_{n\to\infty} d(x_n, p) = c > 0$. Otherwise the proof is trivial.

Take lim sup on both sides of inequalities (2.1) and (2.2). Since $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$, we have $\limsup_{n\to\infty} d(y_n, p) \le c$ and $\limsup_{n\to\infty} d(y_{n+1}, p) \le c$. Similarly, we get $\limsup_{n\to\infty} d(y_{n+r-2}, p) \le c$, and so in total

$$\limsup_{n\to\infty} d(y_{n+k-1}, p) \le c, \quad \forall k = 1, 2, \cdots, r-1. \tag{2.3}$$

Carry lim inf on both side of (2.4). Since

$$d(x_{n+1}, p) \le (1 + \mu_n M^*)^{r-1} d(y_n, p) + \sum_{j=1}^{r-2} (1 + \mu_n M^*)^j \xi_n \tag{2.4}$$

we have

$$\liminf_{n\to\infty} d(y_n, p) \ge c,$$

$$d(x_{n+1}, p) \le (1 + \mu_n M^*)^{r-k} d(y_{n+k-1}, p) + \sum_{j=1}^{r-k-1} (1 + \mu_n M^*)^j \xi_n, \quad \forall k = 2, 3, \cdots, r-1.$$

Also taking lim inf on both side of the above estimate, then we get

$$\liminf_{n\to\infty} d(y_{n+k-1}, p) \ge c, \quad \forall k = 2, 3, \cdots, r-1.$$

Thus, in total,

$$\liminf_{n\to\infty} d(y_{n+k-1}, p) \ge c, \quad \forall k = 1, 2, \cdots, r-1. \tag{2.5}$$

Combining (2.3) and (2.5), we have

$$\lim_{n\to\infty} d(y_{n+k-1}, p) = c, \quad \forall k = 1, 2, \cdots, r-1. \tag{2.6}$$

For $k = 1$ in (2.6), we get

$$\lim_{n\to\infty} d(W(T_r^n x_n, W(x_n, S_r^n x_n, \theta_{rn}), \alpha_{rn}), p) = c. \tag{2.7}$$

Moreover,

$$\begin{aligned} d(W(x_n, S_r^n x_n, \theta_{rn}), p) \quad & \le \theta_{rn} d(x_n, p) + (1 - \theta_{rn}) d(S_r^n x_n, p) \\ & \le \theta_{rn} d(x_n, p) + (1 - \theta_{rn})[(1 + \mu_n M^*) d(x_n, p) + \xi_n] \\ & \le (1 + \mu_n M^*) d(x_n, p) + \xi_n \end{aligned}$$

implies that

$$\limsup_{n\to\infty} d(W(x_n, S_r^n x_n, \theta_{rn}), p) \le c. \tag{2.8}$$

Obviously,

$$\limsup_{n\to\infty} d(T_r^n x_n, p) \le c. \tag{2.9}$$

It follows from (2.7)-(2.9) and Lemma 1.2 that

$$\lim_{n\to\infty} d(T_r^n x_n, W(x_n, S_r^n x_n, \theta_{rn})) = 0. \tag{2.10}$$

Again, for $k = 2, 3, \cdots, r-1$, (2.6) can be expressed as

$$\lim_{n\to\infty} d(W(T^n_{r-(k-1)}y_{n+k-2}, W(y_{n+k-2}, S^n_{r-(k-1)}y_{n+k-2}, \theta_{(r-k+1)n}), \alpha_{(r-k+1)n}), p) = c. \quad (2.11)$$

By (2.3) and the inequality

$$\begin{aligned}
&d(W(y_{n+k-2}, S^n_{r-(k-1)}y_{n+k-2}, \theta_{(r-k+1)n}), p)\\
&\leq \theta_{(r-k+1)n}d(y_{n+k-2}, p) + (1-\theta_{(r-k+1)n})d(S^n_{r-(k-1)}y_{n+k-2}, p)\\
&\leq \theta_{(r-k+1)n}d(y_{n+k-2}, p) + (1-\theta_{(r-k+1)n})[(1+\mu_n M^*)d(y_{n+k-2}, p) + \xi_n]\\
&\leq (1+\mu_n M^*)d(y_{n+k-2}, p) + \xi_n,
\end{aligned}$$

now we know that

$$\limsup_{n\to\infty} d(W(y_{n+k-2}, S^n_{r-(k-1)}y_{n+k-2}, \theta_{(r-k+1)n}), p) \leq c. \quad (2.12)$$

Further,

$$\limsup_{n\to\infty} d(T^n_{r-(k-1)}y_{n+k-2}, p) \leq c, \quad \forall k = 2, 3, \cdots, r-1. \quad (2.13)$$

From (2.11)-(2.13) and Lemma 1.2, it follows that

$$\lim_{n\to\infty} d(T^n_{r-(k-1)}y_{n+k-2}, W(y_{n+k-2}, S^n_{r-(k-1)}y_{n+k-2}, \theta_{(r-k+1)n})) = 0 \quad (2.14)$$

for $k = 2, 3, \cdots, r-1$ and for $k = r$, we have

$$\lim_{n\to\infty} d(x_{n+1}, p) = \lim_{n\to\infty} d(W(T^n_1 y_{n+r-2}, W(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n}), \alpha_{1n}), p) = c. \quad (2.15)$$

Applying (2.3), the following estimate

$$\begin{aligned}
&d(W(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n}), p)\\
&\leq \theta_{1n}d(y_{n+r-2}, p) + (1-\theta_{1n})d(S^n_1 y_{n+r-2}, p)\\
&\leq \theta_{1n}d(y_{n+r-2}, p) + (1-\theta_{1n})[(1+\mu_n M^*)d(y_{n+r-2}, p) + \xi_n]\\
&\leq (1+\mu_n M^*)d(y_{n+r-2}, p) + \xi_n
\end{aligned}$$

implies that

$$\limsup_{n\to\infty} d(W(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n}), p) \leq c. \quad (2.16)$$

Also,

$$\limsup_{n\to\infty} d(T^n_1 y_{n+r-2}, p) \leq c. \quad (2.17)$$

Hence, (2.15)-(2.17) and Lemma 1.2 imply that

$$\lim_{n\to\infty} d(T^n_1 y_{n+r-2}, W(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n})) = 0. \quad (2.18)$$

Observe that

$$\begin{aligned}
d(x_{n+1}, T^n_1 y_{n+r-2}) &= d(W(T^n_1 y_{n+r-2}, W(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n}), \alpha_{1n}), T^n_1 y_{n+r-2})\\
&\leq (1-\alpha_{1n})d(W(y_{n+r-2}, S^n_1 y_{n+r-2}, \theta_{1n}), T^n_1 y_{n+r-2})\\
&\quad +\alpha_{1n}d(T^n_1 y_{n+r-2}, T^n_1 y_{n+r-2}).
\end{aligned}$$

Based on (2.18), this implies

$$\lim_{n\to\infty} d(x_{n+1}, T^n_1 y_{n+r-2}) = 0. \quad (2.19)$$

Similarly, since $a \leq \alpha_{in}, \beta_{in} \leq b$ for all $i \in I$, we have

$$\begin{aligned} d(x_{n+1}, p) &\leq \alpha_{1n} d(T_1^n y_{n+r-2}, p) + (1-\alpha_{1n}) d(W(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}), p) \\ &\leq \alpha_{1n} d(x_{n+1}, p) + \alpha_{1n} d(x_{n+1}, T_1^n y_{n+r-2}) \\ &\quad +(1-\alpha_{1n}) d(W(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}), p) \\ &\leq \frac{\alpha_{1n}}{1-\alpha_{1n}} d(x_{n+1}, T_1^n y_{n+r-2}) + d(W(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}), p) \\ &\leq \frac{b}{1-b} d(x_{n+1}, T_1^n y_{n+r-2}) + d(W(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}), p). \end{aligned} \tag{2.20}$$

Taking lim inf on both side of the estimate (2.20) and using (2.19), we have

$$\liminf_{n\to\infty} d(W(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}), p) \geq c. \tag{2.21}$$

Combining (2.16) and (2.21), we get

$$\lim_{n\to\infty} d(W(y_{n+r-2},, S_1^n y_{n+r-2}, \theta_{1n}), p) = c. \tag{2.22}$$

By Lemma 1.2 and (2.22), we have

$$\lim_{n\to\infty} d(y_{n+r-2}, S_1^n y_{n+r-2}) = 0.$$

In a similar way, for $k = 2, 3, \cdots, r-1$, we compute

$$\begin{aligned} & d(y_{n+k-1}, T_{r-(k-1)}^n y_{n+k-2}) \\ & = d(W(T_{r-(k-1)}^n y_{n+k-2}, W(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}, \theta_{(r-k+1)n}), \alpha_{(r-k+1)n}), \\ & T_{r-(k-1)}^n y_{n+k-2}) \\ & \leq (1-\alpha_{(r-k+1)n}) d(W(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}, \theta_{(r-k+1)n}), T_{r-(k-1)}^n y_{n+k-2}) \\ & \quad +\alpha_{(r-k+1)n} d(T_{r-(k-1)}^n y_{n+k-2}, T_{r-(k-1)}^n y_{n+k-2}). \end{aligned}$$

Utilizing (2.14), we have

$$\lim_{n\to\infty} d(y_{n+k-1}, T_{r-(k-1)}^n y_{n+k-2}) = 0, \quad \forall k = 2, 3, \cdots, r-1. \tag{2.23}$$

For $k = 1$, we calculate

$$\begin{aligned} d(y_n, T_r^n x_n) &= d(W(T_r^n x_n, W(x_n, S_r^n x_n, \theta_{rn}), \alpha_{rn}), T_r^n x_n) \\ &\leq \alpha_{rn} d(T_r^n x_n, T_r^n x_n) + (1-\alpha_{rn}) d(W(x_n, S_r^n x_n, \theta_{rn}), T_r^n x_n). \end{aligned}$$

Now, using (2.10), we have

$$\lim_{n\to\infty} d(y_n, T_r^n x_n) = 0. \tag{2.24}$$

Reasoning as above, we get that

$$d(y_n, p) \leq \frac{b}{1-b} d(T_r^n x_n, y_n) + d(W(x_n, S_r^n x_n, \theta_{rn}), p). \tag{2.25}$$

Setting lim inf on both sides of the estimate (2.25) and utilizing (2.6) and (2.24), we know

$$\liminf_{n\to\infty} d(W(x_n, S_r^n x_n, \theta_{rn}), p) \geq c. \tag{2.26}$$

Inequalities (2.8) and (2.26) collectively imply that

$$\lim_{n\to\infty} d(W(x_n, S_r^n x_n, \theta_{rn}), p) = c. \tag{2.27}$$

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Consequently, Lemma 1.2 and (2.27) imply that

$$\lim_{n\to\infty} d(x_n, S_r^n x_n) = 0. \tag{2.28}$$

Note that

$$\begin{aligned} d(x_n, T_r^n x_n) &\le d(x_n, y_n) + d(y_n, T_r^n x_n) \\ &\le \alpha_{rn} d(x_n, T_r^n x_n) + (1-\alpha_{rn}) d(W(x_n, S_r^n x_n, \theta_{rn}), x_n) + d(y_n, T_r^n x_n) \\ &\le (1-\theta_{rn}) d(x_n, S_r^n x_n) + \frac{1}{1-\alpha_{rn}} d(y_n, T_r^n x_n) \\ &\le \frac{1-2a}{1-b} d(x_n, S_r^n x_n) + \frac{1}{1-b} d(y_n, T_r^n x_n). \end{aligned}$$

From (2.24) and (2.28), we have

$$\lim_{n\to\infty} d(x_n, T_r^n x_n) = 0. \tag{2.29}$$

Moreover

$$\begin{aligned} d(x_n, y_n) &\le \alpha_{rn} d(x_n, T_r^n x_n) + (1-\alpha_{rn}) d(x_n, W(x_n, S_r^n x_n, \theta_{rn})) \\ &\le \alpha_{rn} d(x_n, T_r^n x_n) + (1-\alpha_{rn}-\beta_{rn}) d(x_n, S_r^n x_n) \\ &\le b d(x_n, T_r^n x_n) + (1-2a) d(x_n, S_r^n x_n). \end{aligned}$$

By (2.28) and (2.29), we have

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{2.30}$$

Again, reasoning as above, we have

$$\begin{aligned} d(y_{n+k-1}, p) &\le d(W(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}, \theta_{(r-k+1)n}), p) \\ &\quad + \frac{b}{1-b} d(T_{r-(k-1)}^n y_{n+k-2}, y_{n+k-1}). \end{aligned}$$

Now, Utilizing (2.6) and (2.23), we get

$$\liminf_{n\to\infty} d(W(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}, \theta_{(r-k+1)n}), p) \ge c. \tag{2.31}$$

Thus, (2.12) and (2.31) imply in total

$$\lim_{n\to\infty} d(W(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}, \theta_{(r-k+1)n}), p) = c,$$

and by Lemma 1.2, we conclude that

$$\lim_{n\to\infty} d(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}) = 0, \quad \forall k = 2, 3, \cdots, r-1. \tag{2.32}$$

Also,

$$\begin{aligned} & d(y_{n+k-2}, T_{r-(k-1)}^n y_{n+k-2}) \\ & \le d(y_{n+k-2}, y_{n+k-1}) + d(y_{n+k-1}, T_{r-(k-1)}^n y_{n+k-2}), \\ & S_{r-(k-1)}^n y_{n+k-2}, \theta_{(r-k+1)n}), \alpha_{(r-k+1)n})) + d(y_{n+k-1}, T_{r-(k-1)}^n y_{n+k-2}) \\ & \le d(y_{n+k-1}, T_{r-(k-1)}^n y_{n+k-2}) + \alpha_{(r-k+1)n} d(y_{n+k-2}, T_{r-(k-1)}^n y_{n+k-2}) \\ & \quad + (1-\alpha_{(r-k+1)n}) d(y_{n+k-2}, W(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}, \theta_{(r-k+1)n})) \\ & \le d(y_{n+k-1}, T_{r-(k-1)}^n y_{n+k-2}) + \alpha_{(r-k+1)n} d(y_{n+k-2}, T_{r-(k-1)}^n y_{n+k-2}) \\ & \quad + (1-\alpha_{(r-k+1)n} - \beta_{(r-k+1)n}) d(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}) \\ & \le \frac{1}{1-b} d(y_{n+k-1}, T_{r-(k-1)}^n y_{n+k-2}) + \frac{1-2a}{1-b} d(y_{n+k-2}, S_{r-(k-1)}^n y_{n+k-2}). \end{aligned}$$

Now, it follows from (2.23) and (2.32) that

$$\lim_{n\to\infty} d(y_{n+k-2}, T^n_{r-(k-1)} y_{n+k-2}) = 0, \quad \forall k = 2, 3, \cdots, r-1. \tag{2.33}$$

For $k = 2, 3, \cdots, r-1$, we have

$$d(y_{n+k-2}, y_{n+k-1}) \le d(y_{n+k-2}, T^n_{r-(k-1)} y_{n+k-2}) + d(T^n_{r-(k-1)} y_{n+k-2}, y_{n+k-1}).$$

Hence, (2.23) and (2.33) imply that

$$\lim_{n\to\infty} d(y_{n+k-2}, y_{n+k-1}) = 0. \tag{2.34}$$

Additionally,

$$d(x_n, y_{n+k-1}) \le d(x_n, y_n) + d(y_n, y_{n+1}) + \cdots + d(y_{n+k-2}, y_{n+k-1}).$$

By (2.30) and (2.34), we have

$$\lim_{n\to\infty} d(x_n, y_{n+k-1}) = 0, \quad \forall k = 1, 2, \cdots, r-1. \tag{2.35}$$

Let $L = \max_{i\in I}\{L_i, \hat{L}_i\}$, where L_i and $\hat{L}_i$ are Lipschitz constants for T_i and S_i for $i \in I$, respectively. Since each T_i is uniformly L-Lipschitzian for $i \in I$, we have

$$\begin{aligned} d(x_n, T_i^n x_n) &\le d(x_n, y_{n+r-i-1}) + d(y_{n+r-i-1}, T_i^n x_n) \\ &\le d(x_n, y_{n+r-i-1}) + d(y_{n+r-i-1}, T_i^n y_{n+r-i-1}) + d(T_i^n y_{n+r-i-1}, T_i^n x_n) \\ &\le (1+L) d(x_n, y_{n+r-i-1}) + d(y_{n+r-i-1}, T_i^n y_{n+r-i-1}) \end{aligned}$$

for $1 \le i \le r-1$.

It follows from (2.33) and (2.35) that

$$\lim_{n\to\infty} d(x_n, T_i^n x_n) = 0, \quad \forall 1 \le i \le r-1. \tag{2.36}$$

Moreover,

$$\begin{aligned} d(x_n, T_i x_n) &\le d(x_n, T_i^n x_n) + d(T_i^n x_n, T_i^n y_{n+r-i-1}) + d(T_i^n y_{n+r-i-1}, T_i x_n) \\ &\le d(x_n, T_i^n x_n) + L d(x_n, y_{n+r-i-1}) + L d(T_i^{n-1} y_{n+r-i-1}, x_n) \\ &\le d(x_n, T_i^n x_n) + 2L d(x_n, y_{n+r-i-1}) + L d(T_i^{n-1} y_{n+r-i-1}, y_{n+r-i-1}). \end{aligned}$$

Thus, (2.33), (2.35) and (2.36) (or (2.29)) imply that $d(x_n, T_i x_n) \to 0$ as $n \to \infty$ and so

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0, \quad \forall 1 \le i \le r.$$

Similarly, we have

$$\lim_{n\to\infty} d(x_n, S_i x_n) = 0, \quad \forall 1 \le i \le r.$$

This completes the proof. □

The following results can be obtained from Theorem 2.2 immediately. The proof is similar to Corollaries 2.1 and 2.2, respectively, and so they are omitted.

Corollary 2.3 Assume that K and F are the same as in Theorem 2.2. For $i \in I$, let $T_i : K \to K$ be a uniformly L_i-Lipschitzian and $\{\xi_n^i\}$-asymptotically nonexpansive mapping in the intermediate sense and $S_i : K \to K$ be a uniformly $\hat{L}_i$-Lipschitzian and $\{\hat{\xi}_n^i\}$-asymptotically nonexpansive mapping in the intermediate sense. If $\sum_{n=1}^{\infty} \xi_n^i < +\infty$ and $\sum_{n=1}^{\infty} \hat{\xi}_n^i < +\infty$ for $i \in I$, then, for the sequence $\{x_n\}$ in (1.1),

$$\lim_{n\to\infty} d(x_n, T_i x_n) = \lim_{n\to\infty} d(x_n, S_i x_n) = 0, \quad \forall i \in I.$$

Corollary 2.4 Suppose that K and F are the same as in Theorem 2.2. For $i \in I$, let $T_i : K \to K$ be a uniformly L_i-Lipschitzian and $\{k_n^i\}$-asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty} k_n^i < +\infty$, and $S_i : K \to K$ be a uniformly $\hat{L}_i$-Lipschitzian and $\{\hat{k}_n^i\}$-asymptotically nonexpansive mapping with $\sum_{n=1}^{\infty} \hat{k}_n^i < +\infty$. Then,

$$\lim_{n\to\infty} d(x_n, T_i x_n) = \lim_{n\to\infty} d(x_n, S_i x_n) = 0, \quad i \in I,$$

where $\{x_n\}$ is the sequence defined by (1.1).

Remark 2.1 (1) It is worth mentioning that Theorems 2.1-2.2 can easily be extended to a more general class of total asymptotically quasi-nonexpansive mappings for the iteration process (1.1). And the proofs of Theorems 2.1-2.2 are greatly differ from those of Lemmas 2.1 and 2.2 in [21]. Further, Corollaries 2.1 and 2.3 (Corollaries 2.2 and 2.4, respectively) are so.

(2) Moreover, conclusion of the Theorem 2.2 (Corollaries 2.3 and 2.4, respectively) can be extended to a more general class of weakly total-asymptotically quasi-nonexpansive mappings (weakly asymptotically quasi-nonexpansive mappings asymptoticallyin in the intermediate sense and weakly quasi-nonexpansive mappings). For concepts of the weakly properly, see, for example, Fukhar-ud-din and Khan [21].

3 Approximation of common fixed points

In this section, we approximate common fixed points of two finite families of total asymptotically nonexpansive mappings in a hyperbolic space. More briefly, we establish Δ-convergence and strong convergence of the iteration process (1.1) for two finite families of total asymptotically nonexpansive mappings in a hyperbolic space.

Theorem 3.1 Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space $\mathcal{H}$ with monotone modulus of uniform convexity η. For $i \in I$, let $T_i : K \to K$, $i \in I = \{1, 2, 3, \cdots, r\}$ be a uniformly L_i-Lipschitzian and $(\{\mu_n^i\}, \{\xi_n^i\}, \rho^i)$-total asymptotically nonexpansive mapping with $\lim_{n\to\infty} \mu_n^i = 0$ and $\lim_{n\to\infty} \xi_n^i = 0$, and a strictly increasing continuous function $\rho^i : [0, +\infty) \to [0, +\infty)$ satisfying $\rho^i(0) = 0$, and let $S_i : K \to K$ be a uniformly $\hat{L}_i$-Lipschitzian and $(\{\hat{\mu}_n^i\}, \{\hat{\xi}_n^i\}, \hat{\rho}^i)$-total asymptotically nonexpansive mapping with $\lim_{n\to\infty} \hat{\mu}_n^i = 0$ and $\lim_{n\to\infty} \hat{\xi}_n^i = 0$, and with a strictly increasing continuous function $\hat{\rho}^i : [0, +\infty) \to [0, +\infty)$ satisfying $\hat{\rho}^i(0) = 0$. Assume that $F = \bigcap_{i=1}^{r}(F(T_i) \cap F(S_i)) \neq \emptyset$ and for $i \in I$, the following conditions hold:

(i) $\sum_{n=1}^{\infty} \mu_n^i < +\infty$, $\sum_{n=1}^{\infty} \hat{\mu}_n^i < +\infty$, $\sum_{n=1}^{\infty} \xi_n^i < +\infty$, $\sum_{n=1}^{\infty} \hat{\xi}_n^i < +\infty$.

(ii)There exists a constant $M^* > 0$ such that $\rho^i(r) \le M^* r$ and $\hat{\rho}^i(r) \le M^* r$ for all $r > 0$.

Then the sequence $\{x_n\}$ defined in (1.1) Δ-converges to a common fixed point $p \in F$.

Proof. Since the sequence $\{x_n\}$ is bounded (by Theorem 2.1), therefore Lemma 1.1 asserts that $\{x_n\}$ has a unique asymptotic center in K. That is, $A(\{x_n\}) = \{x\}$. Let $\{v_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{v_n\}) = \{v\}$. Then, by Theorem 2.2, we have

$$\lim_{n\to\infty} d(v_n, T_i v_n) = \lim_{n\to\infty} d(v_n, S_i v_n) = 0, \quad \forall i \in I. \tag{3.1}$$

We claim that v is the common fixed point of $\{T_i\}_{i=1}^r$ and $\{S_i\}_{i=1}^r$.

For each $i \in I$, define a sequence $\{z_m\}$ in K by $z_m = T_i^m v$. Then, we calculate

$$\begin{aligned} d(z_m, v_n) &\le d(T_i^m v, T_i^m v_n) + d(T_i^m v_n, T_i^{m-1} v_n) + \cdots + d(T_i v_n, v_n) \\ &\le [d(v, v_n) + \mu_m^i \rho^i(d(v, v_n)) + \xi_m^i] + \sum_{j=0}^{m-1} d(T_i^{j+1} v_n, T_i^j v_n). \end{aligned}$$

Since each T_i is uniformly L_i-Lipschitzian with the Lipschitz constant L_i for $i \in I$, the above estimate yields

$$d(z_m, v_n) \le [(1 + \mu_m M^*) d(v, v_n) + \xi_m] + mLd(T_i v_n, v_n), \tag{3.2}$$

where $L = \max_{i\in I}\{L_i, \hat{L}_i\}$.

Taking lim sup on both sides of (3.2) and using (3.1), we have

$$r(z_m, \{v_n\}) = \limsup_{n\to\infty} d(z_m, v_n) \leq \limsup_{n\to\infty} d(v, v_n) = r(v, \{v_n\}),$$

which implies that $|r(z_m, \{v_n\}) - r(v, \{v_n\})| \to 0$ as $m \to \infty$. It follows Lemma 1.3 that $\lim_{m\to\infty} T_i^m v = v$. by the uniform continuity of T_i, we know that

$$T_i(v) = T(\lim_{m\to\infty} T_i^m v) = \lim m \to \infty T_i^{m+1} v = v.$$

From the arbitrariness of $i \in I$, we conclude that v is the common fixed point of $\{T_i\}_{i=1}^r$. Similarly, we can show that v is the common fixed point of $\{S_i\}_{i=1}^r$. Hence, $v \in F$.

Next, we claim that the common fixed point v is the unique asymptotic center for each subsequence $\{v_n\}$ of $\{x_n\}$.

Contrarily, $v \neq x$. It follows Theorem 2.1 that $\lim_{n\to\infty} d(x_n, v)$ exists, and by the uniqueness of asymptotic centers, we have

$$\begin{aligned} \limsup_{n\to\infty} d(v_n, v) \quad & < \limsup_{n\to\infty} d(v_n, x) \leq \limsup_{n\to\infty} d(x_n, x) \\ & < \limsup_{n\to\infty} d(x_n, v) = \limsup_{n\to\infty} d(v_n, v), \end{aligned}$$

a contradiction. Therefore $v = x$. Since $\{v_n\}$ is an arbitrary subsequence of $\{x_n\}$, $A(\{v_n\}) = \{x\}$ for all subsequence $\{v_n\}$ of $\{x_n\}$, this proves that $\{x_n\}$ Δ-converges to a common fixed point x of $\{T_i\}_{i=1}^r$ and $\{S_i\}_{i=1}^r$. □

From Theorem 3.1, we have the following result.

Corollary 3.1 Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space $\mathcal{H}$ with monotone modulus of uniform convexity η. For $i \in I$, let $T_i : K \to K$ be a uniformly L_i-Lipschitzian and $\{\xi_n^i\}$-asymptotically nonexpansive mapping in the intermediate sense and $S_i : K \to K$ be a uniformly $\hat{L}_i$-Lipschitzian and $\{\hat{\xi}_n^i\}$-asymptotically nonexpansive mapping in the intermediate sense. If for all $i \in I$, $\sum_{n=1}^\infty \xi_n^i < +\infty$ and $\sum_{n=1}^\infty \hat{\xi}_n^i < +\infty$, and $F = \bigcap_{i=1}^r (F(T_i) \cap F(S_i)) \neq \emptyset$, then the sequence $\{x_n\}$ defined in (1.1) Δ-converges to a common fixed point $p \in F$.

Corollary 3.2 Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space $\mathcal{H}$ with monotone modulus of uniform convexity η. For $i \in I$, let $T_i : K \to K$ be a uniformly L_i-Lipschitzian and $\{k_n^i\}$-asymptotically nonexpansive mapping with $\sum_{n=1}^\infty k_n^i < +\infty$, and $S_i : K \to K$ be a uniformly $\hat{L}_i$-Lipschitzian and $\{\hat{k}_n^i\}$-asymptotically nonexpansive mapping with $\sum_{n=1}^\infty \hat{k}_n^i < +\infty$. Assume that $F = \bigcap_{i=1}^r (F(T_i) \cap F(S_i)) \neq \emptyset$. Then the sequence $\{x_n\}$ defined in (1.1) Δ-converges to a common fixed point $p \in F$.

Proof. Based on Corollaries 2.2 and 2.4, and the proof of Theorem 3.1 in [21], the result holds. □

In order to prove strong convergence of the iteration (1.1) for two finite families of total asymptotically nonexpansive mappings in a hyperbolic space, we first give the following conditions:

- (**H**) There exists a nondecreasing self-mapping on $[0, +\infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, +\infty)$ such that $d(x, Tx) \geq f(d(x, F(T)))$ for all $x \in K$, where $T : K \to K$ is a nonlinear mapping with $F(T) \neq \emptyset$ and $d(x, F(T)) = \inf\{d(x, y) : y \in F(T)\}$.

The condition (**H**) was introduced by Senter and Dotson [35]. Further, based on works of [21, 36, 37], for two finite families of total asymptotically nonexpansive mappings $\{T_i, i \in I\}_{i=1}^r$ and $\{S_i, i \in I\}_{i=1}^r$ on $K \subset \mathcal{H}$ with $F = \bigcap_{i=1}^n (F(T_i) \cap F(S_i)) \neq \emptyset$, condition (**H**) becomes as follows:

- (**A**) $d(x, Tx) \geq f(d(x, F))$ or $d(x, Sx) \geq f(d(x, F))$ holds for $x \in K$ and for at least one $T \in \{T_i\}_{i=1}^r$ or $S \in \{S_i\}_{i=1}^r$, where $d(x, F) = \inf\{d(x, y) : y \in F\}$.
- (**B**) $d(x, T_i x) + d(x, S_i x) \geq f(d(x, F))$ for $x \in K$ and $i \in I$.
- ($\mathbf{C_1}$) $\frac{1}{2r}\left(\sum_{i=1}^r d(x, T_i x) + \sum_{i=1}^r d(x, S_i x)\right) \geq f(d(x, F))$ for $x \in K$.

(C_2) $\frac{1}{2}\big(\max_{1\le i\le r} d(x,T_ix)+\max_{1\le i\le r} d(x,S_ix)\big)\ge f(d(x,F))$ for $x\in K$.

(C_3) $\max\big\{\max_{1\le i\le r} d(x,T_ix),\max_{1\le i\le r} d(x,S_ix)\big\}\ge f(d(x,F))$ for $x\in K$.

Note that the conditions (**A**), (**B**) and (**C_1**)-(**C_3**) are equivalent to the condition (**H**), if $T_i=S_i$ for $i\in I$. We shall use condition (**C_1**) or (**C_2**) or (**C_3**) to study strong convergence of the iteration (1.1).

Now we give the following lemma for proving the strong convergence.

Lemma 3.1 Let K, $\mathcal{H}$, $\{T_i\}_{i=1}^r$, $\{S_i\}_{i=1}^r$ and $\{x_n\}$ be as in Theorem 3.1. Then $\{x_n\}$ converges strongly to some $p\in F$ if and only if

$$\liminf_{n\to\infty} d(x_n,F)=0.$$

Proof. If $\{x_n\}$ converges strongly to $p\in F$, then $\lim_{n\to\infty} d(x_n,p)=0$. Since $0\le d(x_n,F)\le d(x_n,p)$, we have $\liminf_{n\to\infty} d(x_n,F)=0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n,F)=0$. It follows from Theorem 2.1 that $\lim_{n\to\infty} d(x_n,F)$ exists. Now $\liminf_{n\to\infty} d(x_n,F)=0$ reveals that $\lim_{n\to\infty} d(x_n,F)=0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. By last inequalities in the proof of Theorem 2.1

$$d(x_{n+1},p)\le(1+M_1\mu_n)d(x_n,p)+M_2\xi_n,$$

taking infimum on $p\in F$ on both sides in the above inequality, we have

$$d(x_{n+1},F)\le(1+M_1\mu_n)d(x_n,F)+M_2\xi_n.$$

On account of $\sum_{n=1}^{\infty}\mu_n<\infty$, $\sum_{n=1}^{\infty}\xi_n<\infty$, set $e^{M_1\sum_{n=1}^{\infty}\mu_n}=M$. Let $\forall\varepsilon>0$. Since $\lim_{n\to\infty} d(x_n,F)=0$, for any given $\varepsilon>0$, there exists a positive integer n_0 such that

$$d(x_{n_0},F)<\frac{\varepsilon}{4(M+1)}\quad\text{and}\quad\sum_{n=n_0}^{\infty}\xi_n<\frac{\varepsilon}{2MM_2}\tag{3.3}$$

The first inequality in (3.3) implies that there exists $p_0\in F$ such that $d(x_{n_0},p_0)<\frac{\varepsilon}{2(M+1)}$. Hence, for any $n\ge n_0$ and $m\ge 1$, we have

$$\begin{aligned}
d(x_{n_0+m},x_{n_0}) &\le d(x_{n_0+m},p_0)+d(x_{n_0},p_0)\\
&\le [e^{M_1\sum_{k=n_0}^{n_0+m-1}\mu_k}+1]d(x_{n_0},p_0)+M_2[\xi_{n_0+m-1}\\
&\quad+\xi_{n_0+m-2}e^{M_1\mu_{n_0+m-1}}+\xi_{n_0+m-3}e^{M_1\sum_{k=n_0+m-2}^{n_0+m-1}\mu_k}\\
&\quad+\cdots+\xi_{n_0}e^{M_1\sum_{k=n_0+1}^{n_0+m-1}\mu_k}]\\
&\le (M+1)d(x_{n_0},p_0)+MM_2\sum_{n=n_0}^{\infty}\xi_n\\
&<(M+1)\frac{\varepsilon}{2(M+1)}+MM_2\frac{\varepsilon}{2MM_2}=\varepsilon.
\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in $\mathcal{H}$. Sine K is a closed subset of a complete hyperbolic space $\mathcal{H}$, it is complete. We can assume that $\lim_{n\to\infty} x_n=q$, and $q\in K$. It is easy to see that $F(T)$ is a close subset in K, so is $F(T)$. Since $\lim_{n\to\infty} d(x_n,F)=0$, we obtain $q\in F(T)$. This completes the proof. □

We now establish strong convergence of the iteration process (1.1) based on Theorem 2.2.

Theorem 3.2 Suppose that K, $\mathcal{H}$, $\{T_i\}_{i=1}^r$, $\{S_i\}_{i=1}^r$ and F be the same as in Theorem 3.1, and $\{T_i\}_{i=1}^r$, and $\{S_i\}_{i=1}^r$, satisfies condition (**C_1**) (or (**C_2**), or (**C_3**)). Then the sequence $\{x_n\}$ defined in (1.1) converges strongly to some $p\in F$.

Proof. It follows from Theorem 2.1 that $\lim_{n\to\infty} d(x_n,F)$ exists. Moreover, Theorem 2.2 implies that $\lim_{n\to\infty} d(x_n,T_ix_n)=\lim_{n\to\infty} d(x_n,S_ix_n)=0$ for each $i\in I$. Thus, the condition (**C_1**) (or (**C_2**), or (**C_3**)) guarantees that $\lim_{n\to\infty} f(d(x_n,F))=0$. Since f is nondecreasing with $f(0)=0$, it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. Then, Lemma 3.1 implies that $\{x_n\}$ converges strongly to a common fixed point $p \in F$. □

From Theorem 3.2, we have the following results.

Corollary 3.3 Let K, $\mathcal{H}$, $\{T_i\}_{i=1}^r$, $\{S_i\}_{i=1}^r$ and F be the same as in Corollary 3.1. Suppose that $\{T_i\}_{i=1}^r$, and $\{S_i\}_{i=1}^r$, satisfies condition $(\mathbf{C_1})$ (or $(\mathbf{C_2})$, or $(\mathbf{C_3})$). Then the sequence $\{x_n\}$ defined in (1.1) converges strongly to some $p \in F$.

Corollary 3.4 Assume that K, $\mathcal{H}$, $\{T_i\}_{i=1}^r$, $\{S_i\}_{i=1}^r$ and F are the same as in Corollary 3.2, and $\{T_i\}_{i=1}^r$, and $\{S_i\}_{i=1}^r$, satisfies condition $(\mathbf{C_1})$ (or $(\mathbf{C_2})$, or $(\mathbf{C_3})$). Then the sequence $\{x_n\}$ defined in (1.1) converges strongly to some $p \in F$.

Theorem 3.3 Let K, $\mathcal{H}$, $\{T_i\}_{i=1}^r$, $\{S_i\}_{i=1}^r$ and F be the same as in Theorem 3.1. Suppose that either $T_l \in \{T_i\}_{i=1}^r$ or $S_l \in \{S_i\}_{i=1}^r$ is semi-compact. Then the sequence $\{x_n\}$ defined in (1.1) converges strongly to $p \in F$.

Proof. Let $T_l \in \{T_i\}_{i=1}^r$ is semi-compact. By Theorem 2.2, we know that $\lim_{n\to\infty} d(T_i x_n, x_n) = 0$ for all $i \in I$. By Theorem 2.1, $\{x_n\}$ is bounded and T_l is semi-compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to q$ as $j \to \infty$. By continuity of T_i and Theorem 2.2, we obtain

$$d(q, T_i q) = \lim_{j\to\infty} d(x_{n_j}, T_i x_{n_j}) = 0, \quad i \in I.$$

This implies that q is the common fixed point of $\{T_i\}_{i=1}^r$. Similarly, we can show that q is the common fixed point of $\{S_i\}_{i=1}^r$. Hence, $q \in F$. Again, by Theorem 2.1, $\lim_{n\to\infty} d(x_n, q)$ exists. Therefore, q is the strong limit of the sequence $\{x_n\}$. As a result, $\{x_n\}$ converges strongly to a point q. □

From Theorem 3.3, we have the following results.

Corollary 3.5 Let K, $\mathcal{H}$, $\{T_i\}_{i=1}^r$, $\{S_i\}_{i=1}^r$ and F be the same as in Corollary 3.1. Suppose that either $T_l \in \{T_i\}_{i=1}^r$ or $S_l \in \{S_i\}_{i=1}^r$ is semi-compact. Then the sequence $\{x_n\}$ defined in (1.1) converges strongly to $p \in F$.

Corollary 3.6 Suppose that K, $\mathcal{H}$, $\{T_i\}_{i=1}^r$, $\{S_i\}_{i=1}^r$ and $\{x_n\}$ be the same as in Corollary 3.2, and either $T_l \in \{T_i\}_{i=1}^r$ or $S_l \in \{S_i\}_{i=1}^r$ is semi-compact. Then the sequence $\{x_n\}$ defined in (1.1) converges strongly to $p \in F$.

Remark 3.1 (1) If the uniformly convex hyperbolic spaces with modulus of uniform convexity reduce to CAT(0) spaces, and iterative process (1.1) reduce to iterative process (1.3), Theorem 3.1, Lemma 3.1, Theorem 3.2 reduce to Theorems 3.1-3.3 proved by Thakur et al. [7], respectively.

(2) If $r = 3$ and $\alpha_{in} = 0$ and $S_1 = S_2 = \cdots = S_r = T$, Theorem 3.1, Lemma 3.1, Theorem 3.2 and Theorem 3.3 become to Theorems 1-4 in [5], respectively.

(3) If the uniformly convex hyperbolic spaces with modulus of uniform convexity reduce to CAT(0) spaces, and $r = 3$ and $\alpha_{in} = 0$ and $S_1^n = S_2^n = \cdots = S_r^n = T$, where T is a nonexpansive mappings on $K \subset \mathcal{H}$, Theorem 3.1, Lemma 3.1, Theorem 3.2 are equivalent to Theorems 1-3 of [6], respectively.

4 Concluding remarks

In this paper, we introduced and studied the following new general iteration for two finite families of total asymptotically nonexpansive mappings in hyperbolic spaces $\mathcal{H}$:

$$\begin{aligned}
x_{n+1} &= W(T_1^n y_{n+r-2}, W(y_{n+r-2}, S_1^n y_{n+r-2}, \theta_{1n}), \alpha_{1n}),\\
y_{n+r-2} &= W(T_2^n y_{n+r-3}, W(y_{n+r-3}, S_2^n y_{n+r-3}, \theta_{2n}), \alpha_{2n}),\\
y_{n+r-3} &= W(T_3^n y_{n+r-4}, W(y_{n+r-4}, S_3^n y_{n+r-4}, \theta_{3n}), \alpha_{3n}),\\
&\vdots\\
y_{n+1} &= W(T_{r-1}^n y_n, W(y_n, S_{r-1}^n y_n, \theta_{(r-1)n}), \alpha_{(r-1)n}),\\
y_n &= W(T_r^n x_n, W(x_n, S_r^n x_n, \theta_{rn}), \alpha_{rn}),
\end{aligned} \tag{4.1}$$

where $\{T_i\}_{i=1}^r$ and $\{S_i\}_{i=1}^r$ be two finite families of total asymptotically nonexpansive mappings on nonempty closed and convex subset $K \subset \mathcal{H}$, $\{\alpha_{in}\}$ and $\{\beta_{in}\}$ are two double real sequences in $[0, 1]$, and for each $i \in I = \{1, 2, \cdots, r\}$, $r \geq 2$ and $n \geq 1$, $\theta_{in} := \frac{\beta_{in}}{1-\alpha_{in}}$.

In order to prove Δ-convergence and strong convergence of the iteration (4.1) in hyperbolic spaces, we gave and analyzed some important related properties to the new general iterative processes (4.1), and proposed some meaningful illustrations for clarifying the results presented in this paper, which show that our results extend and improve the corresponding results of iterative approximation for asymptotically (quasi-)nonexpansive mapping, (generalized) (quasi-)nonexpansive mapping of all normed linear spaces, Hadamard manifolds and CAT(0) spaces as special cases. Our results extended and improved the corresponding results of [1, 2, 4–7, 9, 21].

It is well known that iterative processes as ubiquitous in the area of abstract nonlinear analysis and still remain as a main tool for approximation of fixed points of generalizations of nonexpansive maps. Furthermore, the analysis of general iterative processes, in a more general setup, is a problem of interest in theoretical numerical analysis. Therefore, on two finite families of total asymptotically nonexpansive mappings in the setting of the general iteration (4.1), the following two **open questions** will be worth further studying:.

(1) If some errors are added in the iteration (4.1), such as the iterative approximating scheme (3.1) in [11], can the Δ-convergence and strong convergence presented in this paper be proved?

(2) When T_i and S_i $(i \in I)$ in (4.1) become total asymptotically quasi-nonexpansive mappings, whether do the results of Theorems 3.1-3.3 hold?

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On Simpson's type inequalities utilizing fractional integrals

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Abstract

In the present article, we establish an integral identity for Riemann-Liouville fractional integrals. Some Simpson type integral inequalities utilizing this integral identity are obtained. It is worth mentioning that the presented results have close connection with those in [M. Z Sarikaya, E. Set, M. E Ozdemir, On new inequalities of Simpson's type for s-convex functions, Computers and Mathematics with Applications, 60 (2010), 2191–2199)].
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1. Introduction

The following definition for convex functions is well known in the mathematical literature:

A function $f : \Phi \neq I \subseteq R \to R$. is said to be convex on I, if inequality

$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$, for all $x, y \in I, t \in [0,1]$

Many inequalities have been established for convex functions but the most famous is the Simpson's inequality, due to its rich geometrical significance and applications, which is stated as [9]:

Theorem 1 *Let $f : [a,b] \to R$ be a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$, then we have the following inequality:*

$$\left|\left[\frac{1}{6}f(a)+\frac{2}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4. \tag{1}$$

For recent refinements, counterparts, generalizations and new Simpson's type inequalities, see [[9]-[11]].

In [10], Dragomir et. al proved the following recent developments on Simpson's inequality for which the remainder is expressed in terms of derivatives lower than the fourth.

Theorem 2 *Let $f : [a,b] \to \mathbb{R}$ is a differentiable mapping whose derivative is continuous on (a,b) and $f' \in L[a,b]$. Then we have the following inequality:*

$$\left|\left[\frac{1}{6}f(a)+\frac{2}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{3}\left\|f'\right\|_1, \tag{2}$$

where $\left\|f'\right\|_1 = \int_a^b |f'(x)\,dx|$.

The bound of (2) for L-Lipschitzian mapping was given in [8] by $\frac{5}{36}(b-a)$.

In [8], Sarikaya et. al presented inequalities for differentiable convex functions which are linked with Simpson's inequality, and the main inequality in [8], pointed out, is as follows.

Theorem 3 *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I^0 (interior of I) such that $f' \in L_1[a,b]$ where $a,b \in I$ with $a < b$. If $|f'|$ is $s-$convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds:*

$$\left|\left[\frac{1}{6}f(a)+\frac{2}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\le (b-a)\frac{(s-4)6^{s+1}+2\times 5^{s+2}-2\times 3^{s+2}+2}{6^{s+1}(s+1)(s+2)}(\ |f'(a)|+|f'(b)|\ ). \tag{3}$$

Proposition 1 *Under the assumptions of Theorem 3 with $s = 1$, we have the following inequality,*

$$\left|\left[\frac{1}{6}f(a)+\frac{2}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{5(b-a)}{72}(\ |f'(a)|+|f'(b)|\ ). \tag{4}$$

Proposition 2 *Under the assumptions of Theorem 3 with $f(a) = f\left(\frac{a+b}{2}\right) = f(b)$, we have the following inequality,*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{5\,(b-a)}{72}(\ |f'(a)|+|f'(b)|\ ). \tag{5}$$

Theorem 4 *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I^0 such that $f' \in L_1[a,b]$ where $a,b \in I$ with $a < b$. If $|f'|^q$ is $s-$convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q \geq 1$, then the following inequality holds:*

$$\left|\left[\frac{1}{6}f(a)+\frac{2}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{2}\left(\frac{5}{36}\right)^{1-1/q}\times$$
$$\left\{\left(\left[\frac{2\times 5^{s+2}+(s-4)6^{s+1}-(2s+7)3^{s+1}}{3\times 6^{s+1}(s+1)(s+2)}\right]|f'(b)|^q+\left[\frac{(2s+1)3^{s+1}+2}{3\times 6^{s+1}(s+1)(s+2)}\right]|f'(a)|^q\right)^{1/q}\right.$$
$$\left.+\left(\left[\frac{(2s+1)3^{s+1}+2}{3\times 6^{s+1}(s+1)(s+2)}\right]|f'(b)|^q+\left[\frac{2\times 5^{s+2}+(s-4)6^{s+1}-(2s+7)3^{s+1}}{3\times 6^{s+1}(s+1)(s+2)}\right]|f'(a)|^q\right)^{1/q}\right\}.$$

Proposition 3 *Under the assumptions of Theorem 4 with $s = 1$, we have the following inequality,*

$$\left|\left[\frac{1}{6}f(a)+\frac{2}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{2}\left(\frac{5}{36}\right)^{1-1/q}\times$$
$$\left\{\left(\frac{61}{648}|f'(b)|^q+\frac{29}{648}|f'(a)|^q\right)^{1/q}+\left(\frac{61}{648}|f'(b)|^q+\frac{29}{648}|f'(a)|^q\right)^{1/q}\right\}.$$

Proposition 4 *Under the assumptions of Theorem 4 with $f(a) = f\left(\frac{a+b}{2}\right) = f(b)$, we have the following inequality,*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{72}(5)^{1-1/q}\times$$
$$\left\{\left(\frac{61}{648}|f'(b)|^q+\frac{29}{648}|f'(a)|^q\right)^{1/q}+\left(\frac{61}{648}|f'(b)|^q+\frac{29}{648}|f'(a)|^q\right)^{1/q}\right\}$$

Definition 1 *Let $f \in L^1[a,b]$. The left$-$sided and right$-$sided Riemann$-$Liouville fractional integrals of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \qquad a < x$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \qquad x < b$$

respectively, where $\Gamma(.)$ is Gamma function and its definition is $\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}du$. It is to be noted that $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Properties relating to this operator can be found in [5] and for useful details on Simpson's type inequalities connected with fractional integral inequalities, the interested readers are directed to [1]

The main aim of this paper is to establish new Simpson's type inequalities for Riemann−Liouville fractional integral using the convexity as well as concavity, for the class of functions whose derivatives in absolute value at certain powers are convex functions. we will derive a general integral identity for convex functions.

2. Main Results

In order to prove our main results we need the following integral identity:

Lemma 1 *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function such that f' is integrable and $0 < \alpha \leq 1$ on (a, b) with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following identity for Riemann−Liouville fractional integrals holds:*

$$\left[\frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right] - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right] = \frac{b-a}{2^{\alpha+1}}\left[I_1 + I_2 + (2^{\alpha}-1)(I_3 + I_4)\right],$$

where

$I_1 = \int_0^1 \left(\frac{1}{6} - \frac{1}{2}(1-t)^{\alpha}\right) f'(tb + (1-t)\frac{a+b}{2})dt,$

$I_2 = \int_0^1 \left(\frac{1}{2}(1-t)^{\alpha} - \frac{1}{6}\right) f'(ta + (1-t)\frac{a+b}{2})dt,$

$I_3 = \int_0^1 \left(\frac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha} - \frac{1}{2(2^{\alpha}-1)} - \frac{1}{3}\right) f'(tb + (1-t)\frac{a+b}{2})dt,$

$I_4 = \int_0^1 \left(\frac{1}{2(2^{\alpha}-1)} - \frac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha} + \frac{1}{3}\right) f'(ta + (1-t)\frac{a+b}{2})dt.$

Proof. Integrating by parts, we have

$$\begin{aligned} I_1 &= \int_0^1 \left(\frac{1}{6} - \frac{1}{2}(1-t)^{\alpha}\right) f'(tb + (1-t)\frac{a+b}{2})dt \\ &= \left.\frac{2\left(\frac{1}{6} - \frac{1}{2}(1-t)^{\alpha}\right) f'(tb + (1-t)\frac{a+b}{2})dt}{b-a}\right|_0^1 \\ &\qquad - \frac{2\alpha}{b-a}\int_0^1 (1-t)^{\alpha+1} f(tb + (1-t)\frac{a+b}{2})dt \\ &= \frac{2}{b-a}\left[\frac{1}{6}f(b) + \frac{1}{3}f\left(\frac{a+b}{2}\right)\right] - \frac{2\alpha}{b-a}\int_0^1 (1-t)^{\alpha+1} f(tb + (1-t)\frac{a+b}{2})dt \\ &= \frac{2}{b-a}\left[\frac{1}{6}f(b) + \frac{1}{3}f\left(\frac{a+b}{2}\right)\right] - \frac{2^{\alpha}\alpha}{(b-a)^{\alpha}}J_3. \end{aligned}$$

$$I_3 = \int_0^1 (\frac{1}{2\left(2^{\alpha}-1\right)}\left(1+t\right)^{\alpha} - \frac{1}{2\left(2^{\alpha}-1\right)} - \frac{1}{3})f'(tb+(1-t)\frac{a+b}{2})dt$$

$$= \frac{2\left[\frac{1}{2(2^{\alpha}-1)}\left(1+t\right)^{\alpha} - \frac{1}{2(2^{\alpha}-1)} - \frac{1}{3}\right] f(tb+(1-t)\frac{a+b}{2})dt)}{b-a}\Bigg|_0^1$$

$$- \frac{2\alpha}{(b-a)\left(2^{\alpha}-1\right)}\int_0^1 \left(1+t\right)^{\alpha+1} f(tb+(1-t)\frac{a+b}{2})dt)$$

$$\left(2^{\alpha}-1\right) I_3 = \frac{2}{b-a}\left[\frac{1}{6}f\left(b\right) + \frac{1}{3}f\left(\frac{a+b}{2}\right)\right] + \frac{2\left(\alpha+1\right)}{b-a}\int_0^1 \left(1+t\right)^{\alpha+1} f(tb+(1-t)\frac{a+b}{2})dt)$$

$$= \frac{2}{b-a}\left[\frac{1}{6}f\left(b\right) + \frac{1}{3}f\left(\frac{a+b}{2}\right)\right] - \frac{2^{\alpha}\alpha}{\left(b-a\right)^{\alpha+1}} J_2.$$

Analogously:

$I_2 = \frac{2}{b-a}\left[\frac{1}{6}f\left(b\right) + \frac{1}{3}f\left(\frac{a+b}{2}\right)\right] - \frac{2^{\alpha}\alpha}{(b-a)^{\alpha}} J_1.$

$\left(2^{\alpha}-1\right) I_4 = \frac{2}{b-a}\left[\frac{1}{6}f\left(b\right) + \frac{1}{3}f\left(\frac{a+b}{2}\right)\right] - \frac{2^{\alpha}\alpha}{(b-a)^{\alpha+1}} J_4.$

Adding above equalities, we get

$$\frac{2}{b-a}\left[\frac{1}{6}f\left(a\right) + \frac{1}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f\left(b\right)\right] - \frac{\alpha}{2\left(b-a\right)^{\alpha}}\left[J_1+J_2+J_3+J_4\right]$$
$$= I_1 + I_2 + \left(2^{\alpha}-1\right)\left(I_3+I_4\right).$$

Now making suitable substitutions, we have

$J_1 = \int_0^1 \left(1-t\right)^{\alpha+1} f'(ta+(1-t)\frac{a+b}{2})dt = \frac{2^{\alpha}}{(b-a)^{\alpha}}\int_a^{a+b/2} \left(u-a\right)^{\alpha-1} f(u)du$

$J_2 = \int_0^1 \left(1+t\right)^{\alpha+1} f'(tb+(1-t)\frac{a+b}{2})dt = \frac{2^{\alpha}}{(b-a)^{\alpha}}\int_{a+b/2}^{b} \left(u-a\right)^{\alpha-1} f(u)du$

$J_1 + J_2 = \frac{2^{\alpha}}{(b-a)^{\alpha}}\int_a^b (u-a)^{\alpha-1} f(u)du = \frac{2^{\alpha}\Gamma(\alpha)}{(b-a)^{\alpha}} J_{b^-}^{\alpha} f(a),$

likewise

$J_3 = \int_0^1 \left(1-t\right)^{\alpha+1} f'(tb+(1-t)\frac{a+b}{2})dt = \frac{2^{\alpha}}{(b-a)^{\alpha}}\int_{a+b/2}^{b} \left(b-u\right)^{\alpha-1} f(u)du$

$J_4 = \int_0^1 \left(1+t\right)^{\alpha+1} f'(ta+(1-t)\frac{a+b}{2})dt = \frac{2^{\alpha}}{(b-a)^{\alpha}}\int_a^{a+b/2} \left(b-u\right)^{\alpha-1} f(u)du$

$J_3 + J_4 = \frac{2^{\alpha}}{(b-a)^{\alpha}}\int_a^b (b-u)^{\alpha-1} f(u)du = \frac{2^{\alpha}\Gamma(\alpha)}{(b-a)^{\alpha}} J_{a^+}^{\alpha} f(b),$

which completes our proof. ∎

Theorem 5 *Let f and f' be defined as in Theorem 4 and if $|f'|$ is convex on $[a,b]$, then the following identity for Riemann–Liouville fractional integrals holds:*

$$\left[\frac{1}{6}f\left(a\right) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f\left(b\right)\right] - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]$$
$$\leq \frac{(b-a)}{2^{\alpha}}(\psi_1+\psi_2)(\ |f'(a)| + |f'(b)|\ ). \quad (6)$$

where $\psi_1 = K_1 + K_2,\quad \psi_2 = K_3 + K_4$

Proof. By using the properties of modulus on Lemma 1, we have

$$\left|\left[\frac{1}{6}f\left(a\right) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f\left(b\right)\right] - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]\right| \leq \frac{b-a}{2^{\alpha+1}}\times$$
$$\left[\frac{2c-\alpha+2}{6\left(\alpha+1\right)} + \left\{\left(\frac{2^{\alpha}-1}{3} + \frac{1}{2}\right)(2d-3) - \frac{1}{\alpha+1}\left(\frac{5d}{3} - \frac{2^{\alpha+1}+1}{2}\right)\right\}\right](\ |f'(a)|+|f'(b)|,$$

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where $c = \left(\frac{1}{3}\right)^{\frac{1}{\alpha}}$ and $d^{\alpha} = \frac{2(2^{\alpha}-1)}{3} + 1$.

Using convexity of $|f'|$, we have

$$\begin{aligned}
|I_1| &\leq \int_0^1 \left(\frac{1}{6} - \frac{1}{2}(1-t)^{\alpha}\right) |f'(tb + (1-t)\frac{a+b}{2})| dt \\
&= \int_0^1 \left(\frac{1}{6} - \frac{1}{2}(1-t)^{\alpha}\right) |f'(\frac{1+t}{2}b + \frac{1-t}{2}a)| dt \\
&\leq \int_0^1 \left(\frac{1}{6} - \frac{1}{2}(1-t)^{\alpha}\right) \left\{\left(\frac{1+t}{2}\right)|f'(b)| + \left(\frac{1-t}{2}\right)|f'(a)|\right\} dt \\
&= \frac{K_1}{2}|f'(b)| + \frac{K_2}{2}|f'(a)|.
\end{aligned}$$

Analogously:

$$|I_2| \leq \frac{K_1}{2}|f'(a)| + \frac{K_2}{2}|f'(b)|.$$

Using the convexity on $|f'|$ and the fact that for $\alpha \in (0,1]$ and $\forall\ t \in [0,1]$,

$$\begin{aligned}
|I_3| &\leq \int_0^1 \left(\frac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha} - \frac{1}{2(2^{\alpha}-1)} - \frac{1}{3}\right) |f'(ta + (1-t)\frac{a+b}{2})| dt \\
&= \int_0^1 \left(\frac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha} - \frac{1}{2(2^{\alpha}-1)} - \frac{1}{3}\right) |f'(\frac{1+t}{2}a + \frac{1-t}{2}b)| dt \\
&\leq \int_0^1 \left(\frac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha} - \frac{1}{2(2^{\alpha}-1)} - \frac{1}{3}\right) \left\{\left(\frac{1+t}{2}\right)|f'(a)| + \left(\frac{1-t}{2}\right)|f'(b)|\right\} dt \\
&= \frac{K_3}{2}|f'(a)| + \frac{K_4}{2}|f'(b)|.
\end{aligned}$$

Similarly

$$|I_4| \leq \frac{K_3}{2}|f'(b)| + \frac{K_4}{2}|f'(a)|.$$

To get desired result, adding above inequalities and it is very easy to check

$$\begin{aligned}
K_1 &= \int_0^{1-c} \left(\frac{1}{2}(1-t)^{\alpha} - \frac{1}{6}\right) dt = -\frac{1}{6}(1-c) - \frac{1}{2(\alpha+1)}c^{\alpha+1} + \frac{1}{2(\alpha+1)}, \\
K_2 &= \int_{1-c}^1 \left(\frac{1}{6} - \frac{1}{2}(1-t)^{\alpha}\right) dt = \frac{1}{6} - \frac{1}{6}(1-c) - \frac{1}{2(\alpha+1)}c^{\alpha+1},
\end{aligned}$$

$$\begin{aligned}
K_3 &= \int_0^{d-1} \left(\frac{1}{2(2^{\alpha}-1)} - \frac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha} + \frac{1}{3}\right) dt \\
&= \left[\frac{1}{3} + \frac{1}{2(2^{\alpha}-1)}\right](d-1) - \frac{d^{\alpha+1}}{2(2^{\alpha}-1)(\alpha+1)} + \frac{1}{2(2^{\alpha}-1)(\alpha+1)}, \\
K_4 &= \int_{d-1}^1 \left(\frac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha} - \frac{1}{2(2^{\alpha}-1)} - \frac{1}{3}\right) dt \\
&= \frac{2^{\alpha+1}}{2(2^{\alpha}-1)(\alpha+1)} - \left[\frac{1}{3} + \frac{1}{2(2^{\alpha}-1)}\right] - \frac{d^{\alpha+1}}{2(2^{\alpha}-1)(\alpha+1)} + \left[\frac{1}{3} + \frac{1}{2(2^{\alpha}-1)}\right](d-1).
\end{aligned}$$

This completes the proof. ∎

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Remark 1 *If we take* $\alpha = 1$ *in Theorem 5 then inequality* (6) *reduces to inequality* (4).

The corresponding version for powers of the absolute value of the derivative is incorporated in the following theorem.

Theorem 6 *Let* f *and* f' *be defined as in Theorem 4 and if* $|f'|^q$ *is a convex on* $[a,b]$, *with* $q \geq 1$, *then the following inequality holds*:

$$\left|\left[\frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right] - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\right]\right| \leq \frac{(b-a)}{2^\alpha}$$
$$\left[{\psi_1}^{1-1/q}\left\{\left(\frac{K_5|f'(a)|^q + K_6|f'(b)|^q}{2}\right)^{1/q} + \left(\frac{K_5|f'(a)|^q + K_6|f'(b)|^q}{2}\right)^{1/q}\right\} + \right.$$
$$\left.{\psi_2}^{1-1/q}\left\{\left(\frac{K_7|f'(a)|^q + K_8|f'(b)|^q}{2}\right)^{1/q} + \left(\frac{K_7|f'(a)|^q + K_8|f'(b)|^q}{2}\right)^{1/q}\right\}\right]. \quad (7)$$

Proof. Using the well-known power-mean integral inequality for $q > 1$, we have

$$|I_1| \leq \left(\int_0^1 \left|\left(\frac{1}{6} - \frac{1}{2}(1-t)^\alpha\right)\right| dt\right)^{1-1/q} \left(\int_0^1 \left|\left(\frac{1}{6} - \frac{1}{2}(1-t)^\alpha\right)\right| \left|f'\left(ta + (1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{1/q}$$

Using the convexity of $|f'|^q$, we have

$$|I_1| \leq {\psi_1}^{1-1/q}\left(K_5\frac{|f'(a)|^q}{2} + K_6\frac{|f'(b)|^q}{2}\right)^{1/q}.$$

Analogously:

$$|I_2| \leq {\psi_1}^{1-1/q}\left(K_5\frac{|f'(b)|^q}{2} + K_6\frac{|f'(a)|^q}{2}\right)^{1/q}.$$

$$|I_2| \leq {\psi_2}^{1-1/q}\left(\int_0^1((1+t)^{\alpha+1} - 2^\alpha(1+t) + \alpha 2^\alpha(1-t))\left|f'\left(tb + (1-t)\tfrac{a+b}{2}\right)\right|^q dt\right)^{1/q}.$$

By the convexity of $|f'|^q$, we have

$$|I_3| \leq {\psi_2}^{1-1/q}\left(K_7\frac{|f'(a)|^q}{2} + K_8\frac{|f'(b)|^q}{2}\right)^{1/q}.$$

Analogously:

$$|I_4| \leq {\psi_2}^{1-1/q}\left(K_7\frac{|f'(b)|^q}{2} + K_8\frac{|f'(a)|^q}{2}\right)^{1/q}.$$

It is very easy to check that

$$K_5 = \int_0^1\left|\left(\tfrac{1}{6} - \tfrac{1}{2}(1-t)^\alpha\right)\right|(1+t)\,dt = \tfrac{3(\alpha+1)+4\alpha(\alpha+2)c-\alpha(\alpha+1)c^2}{12(\alpha+1)(\alpha+2)} - \tfrac{1}{8},$$
$$K_6 = \int_0^1\left|\left(\tfrac{1}{6} - \tfrac{1}{2}(1-t)^\alpha\right)\right|(1-t)\,dt = \tfrac{2\alpha c^2-\alpha+4}{24(\alpha+2)},$$
$$K_7 = \int_0^1\left|\tfrac{1}{2(2^\alpha-1)} - \tfrac{1}{2(2^\alpha-1)}(1+t)^\alpha + \tfrac{1}{3}\right|(1+t)\,dt,$$
$$= \tfrac{1}{2(2^\alpha-1)}\left[\left(d^2 - \tfrac{5}{2}\right)\left(\tfrac{2^\alpha-1}{3} + \tfrac{1}{2}\right) - \tfrac{1}{(\alpha+2)}\left(\tfrac{5}{3}d^2 - \tfrac{2^{\alpha+1}+1}{2}\right)\tfrac{1}{3} + \tfrac{1}{2(2^\alpha-1)}\right]$$
$$K_8 = \int_0^1\left|\tfrac{1}{2(2^\alpha-1)} - \tfrac{1}{2(2^\alpha-1)}(1+t)^\alpha + \tfrac{1}{3}\right|(1-t))dt$$
$$= \tfrac{1}{2(2^\alpha-1)}\left[\left(\tfrac{1}{2} - (2-d)^2\right)\left(\tfrac{2^\alpha-1}{3} + \tfrac{1}{2}\right) + \tfrac{1}{(\alpha+1)}\left(\tfrac{1}{2} - \tfrac{5d}{3}(2-d)\right) + \right.$$
$$\left.\tfrac{1}{(\alpha+1)(\alpha+2)}\left(\tfrac{2^{\alpha+2}+1}{2} - \tfrac{5}{3}d^2\right)\right].$$

This completes the proof. ⊓⊔

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Remark 2 *If we take $\alpha = 1$ in Theorem 6, then inequality* (7) *reduces to inequality as obtained in Proposition 3.*

In the following theorem, we obtain estimate of Simpson's inequality (1) for concave functions.

Theorem 7 *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) such that $f' \in L^1[a,b]$. If $|f'|^q$ is concave on $[a,b]$, for some fixed $p > 1$ with $q = \frac{p}{p-1}$, then the following inequality for fractional integrals holds for $\alpha > 0$:*

$$\left|\left[\frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right] - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)\right]\right| \le \frac{(b-a)}{2^{\alpha+1}} \times \left[\psi_1\left\{\left|f'\left(\frac{K_5 b + K_6 a}{\psi_1}\right)\right| + \left|f'\left(\frac{K_5 a + K_6 b}{\psi_1}\right)\right|\right\} + \psi_2\left(2^\alpha - 1\right)\left|f'\left(\frac{K_7 b + K_8 a}{\psi_2}\right)\right| + \left|f'\left(\frac{K_7 a + K_8 b}{\psi_2}\right)\right|\right]. \quad (8)$$

Proof. Using the concavity of $|f'|^q$ and the power-mean inequality, we obtain

$$\begin{aligned} |f'|^q &> t|f'|^q + (1-t)|f'|^q \\ &\ge t|f'|^q + (1-t)|f'|^q. \end{aligned}$$

Hence

$$|f'(tx + (1-t)y)| \ge t|f'(x)| + (1-t)|f'(y)|.$$

So $|f'|$ is also concave. By the Jensen integral inequality, we have

$$\begin{aligned} |I_1| &\le \left(\int_0^1 \left|\left(\frac{1}{6} - \frac{1}{2}(1-t)^\alpha\right)\right| dt\right)\left|f''\left(\frac{\int_0^1 \left|\left(\frac{1}{6} - \frac{1}{2}(1-t)^\alpha\right)\right|\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt}{\int_0^1 \left|\left(\frac{1}{6} - \frac{1}{2}(1-t)^\alpha\right)\right| dt}\right)\right| \\ &= \psi_1\left|f'\left(\frac{K_5 b + K_6 a}{\psi_1}\right)\right|. \end{aligned}$$

Analogously:

$$\begin{aligned} |I_2| &\le \psi_1\left|f'\left(\frac{K_5 a + K_6 b}{\psi_1}\right)\right|, \\ |I_3| &\le \psi_2\left|f'\left(\frac{K_7 b + K_8 a}{\psi_2}\right)\right|, \\ |I_4| &\le \psi_2\left|f'\left(\frac{K_7 a + K_8 b}{\psi_2}\right)\right|. \end{aligned}$$

This completes the proof. ◻

Corollary 1 *If we take $\alpha = 1$ in Theorem 7, then inequality* (8) *becomes as:*

$$\left|\left[\frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{5(b-a)}{72}\left[\left|f'\left(\frac{29a + 61b}{90}\right)\right| + \left|f'\left(\frac{61a + 29b}{90}\right)\right|\right]. \quad (9)$$

Remark 3 *Inequality* (9) *is an generalization of obtained inequality as in [9, Theorem 8]*

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The permanence and global attractivity in a nonautonomous Gilpin-Ayala competition system with several delayed negative feedbacks

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Abstract: In this paper, a nonautonomous delayed Gilpin-Ayala competition system without instantaneous negative feedbacks (i.e., pure-delay-type system) is investigated. By the techniques of comparison arguments and constructing Lyapunov functionals something different to usual case, several results to guarantee the permanence of the system are derived by means of Ahmad and Lazer's definitions of lower and upper averages of a function. Moreover, the sufficient conditions for the global attractivity of the positive solution are also obtained, in which it is not necessarily to require the exponent of nonlinear intraspecific interference to exceed that of nonlinear interspecific interactions. These results are more general and practical, and possess a wide range of applications. Obviously, they are basically an extension of many existing conclusions for nonlinear competitive systems.

Keywords: Permanence; Global attractivity; Nonlinear competition; Lyapunov functionals; Pure-delays

1 Introduction

The permanence and global stability of ecological systems are always the most important and ubiquitous problems in mathematical biology. As pointed out by Li and Kuang [1], more realistic and interesting models of single or multiple species growth should take into account both the seasonality of the changing environment and the effects of time delays. Moreover, in view of the fact that in real-life species interactions, instantaneous responses are rare or weak relatively to delayed responses, more realistic models should consist of delay differential systems instead of the ones with instantaneous feedbacks. Recently, some model with discrete delay and distributed delay was studied [2–5]. In the meantime, some scholars [6,7] argue that continuously distributed delays as ecologically and biologically are more realistic than discrete delays to species interactions, which is proved true by Caperon [8]. Therefore, a reasonable alternative way is to study the pure-delay-type systems with both discrete delays and continuously distributed delays.

One the other hand, it is well know that for Lotka-Volterra model with delays, the stability is ordinarily delineated in two ways: the one that contain delay independent terms which dominate other intra-specific and inter-specific interaction effects with and without delays, called a "no-pure-delay-type", and the other with only delay feedbacks, is named as "pure-delay-type". For no-pure-delay-type system, one can use the no-delay terms to control the delay terms. Various results have been obtained recently under so-called diagonally dominant conditions and the conditions are often independent of delays (see [9–13]). However, for the pure-delay-type

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systems, the analysis of the permanence and the global asymptotic stability of the system is very difficult, let along the nonlinear type system.

Motivated by the works on Gilpin-Ayala competition systems with delays (see [12, 14–16]), in particular, strongly stimulated by the works [1, 17–19], which all contain several time delay, we consider the following Gilpin-Ayala competitive system with several discrete arguments and continuous time delays

$$\dot{x}_i(t) = x_i(t)\Big[r_i(t) - \sum_{j=1}^{n}\sum_{k=1}^{k_{ij}} a_{ijk}(t)x_j^{\alpha_{ijk}}(t-\tau_{ijk}(t)) - \sum_{j=1}^{n}\sum_{l=1}^{l_{ij}} \int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)x_j^{\beta_{ijl}}(t+s)ds\Big]. \quad (1.1)$$

The aim of this paper is, by developing the analytic technique the analytic technique of the literatures [10, 11, 14–16, 21, 22], to obtain conditions which guarantee the permanence of the system (1.1); after that, by constructing a suitable Lyapunov functional, sufficient conditions about the global attractivity of the positive solution of system (1.1) are gained.

For convenience, we will use following notations in the rest of this paper, let $\tau_{ijk} = \sup\{\tau_{ijk}(t) \mid t \in R\}$ and $\tau = \max\{\tau_{ijk},\ \sigma_{ijl}\}$, then we have $0 < \tau_{ijk},\ \sigma_{ijl} \le \tau$. Denote by $\Psi_{ijk}(t) = t - \tau_{ijk}(t)$, and the functions $\Psi_{ijk}^{-1}(t)$ is the inverse functions of $\Psi_{ijk}(t)$, respectively. In this paper, for system (1.1) we always assume that

(H_1) $\alpha_{ijk} > 0,\ \beta_{ijl} > 0$.

(H_2) $r_i(t),\ a_{ijk}(t),\ \tau_{ijk}(t)$, are positively continuous and bounded functions on $[c, +\infty)$.

(H_3) Functions $b_{ijl}(t, s)$ are defined on $[c, +\infty) \times [-\tau, 0]$ such that they are integrable with respect to s, and $\int_{-\sigma_{ijl}}^{0} b_{ijl}(t, s)ds$ are positive, continuous and bounded above with respect to t on $[c, +\infty)$.

(H_4) $\tau_{ijk}(t)$ are nonnegative, continuous and bounded, $\Psi_{ijk}(t) = t - \tau_{ijk}(t)$ are all invertible. Furthermore, it is differentiable and satisfy $1 - \tau'_{ijk}(t) > 0$ $(t \ge c)$.

Stimulated by the application of system (1.1) to population dynamics, we assume that solutions of system (1.1) satisfy the following initial condition

$$x_i(\theta) = \phi_i(\theta) \ge 0,\ \theta \in [-\tau, 0],\ \ \phi_i(0) > 0,\ \sup_{\theta\in[-\tau,0]} \phi_i(\theta) < +\infty. \quad (1.2)$$

2 Basic results

Let $g(t)$ be a continuous function define on $[c, +\infty)$. Denote

$$g^u = \sup\{g(t) \mid c \le t < +\infty\},\ \ g^l = \sup\{g(t) \mid c \le t < +\infty\}.$$

According to Ahmad and Lazer [10], we define the lower and upper averages of a function $g(t)$. If $c \le t_1 < t_2$, set

$$A[g,\ t_1,\ t_2] = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} g(s)ds.$$

The lower and upper averages of $g(t)$ denoted by $m[g]$ and $M[f]$ are follows

$$m[g] = \lim_{s\to+\infty} \inf\{A[g,\ t_1,\ t_2] \mid t_2 - t_1 \ge s\},$$

and

$$M[g] = \lim_{s\to+\infty} \sup\{A[g,\ t_1,\ t_2] \mid t_2 - t_1 \ge s\}.$$

Since the set $\{A[g,\ t_1,\ t_2] \mid t_2 - t_1 \ge s\}$ decreases as s increases, the limits exist; and since $g^l \le A[g,\ t_1,\ t_2] \le g^u$, it follows that $g^l \le m[g] \le A[g,\ t_1,\ t_2] \le M[g] \le g^u$.

Definition 2.1. The system of differential equation

$$\dot{x}(t) = F(t,\ x(t)),\quad x \in R^n$$

is said to be permanent if there exits a compact set D in $R^n_+ = \{(x_1,\ x_2,\ ...,\ x_n) \in R^n \mid x_i > 0\ (i = 1, 2, ..., n)\}$, such that all solutions starting in the interior of R^n_+ ultimately enter D.

Now we consider following single species Logistic type equation

$$\dot{x}(t) = x(t)\Big[r(t) - \sum_{k=1}^{n} a_k(t)x^{\alpha_k}(t)\Big]. \tag{2.1}$$

Where $r(t)$ and $a_k(t)$ $(k = 1, 2, ..., n)$ are all continuous functions on $[0, +\infty)$, $r(t)$ may be negative, $a_k(t)$ $(k = 1, 2, ..., n)$ are nonnegative and there exists at least one $k \in 1,\ 2,\ ...,\ n$ such that $m[a_k] > 0$, and α_k $(k = 1, 2, ..., n)$ are positive constants.

From the Lemma of [11], we have

Lemma 2.1. Suppose that $m[r] > 0$, $a_k(t)$ $(k = 1, 2, ..., n)$ are nonnegative and there exists at least one $k \in \{1,\ 2,\ ...,\ n\}$ such that $m[a_k] > 0$, then any solution $x(t)$ of (2.1) with initial value $x(t_0) > 0$ is bounded above and below on $[t_0, +\infty)$ and globally attractive. Specially, if $r(t),\ a_k(t)$ $(k = 1, 2, ..., n)$ are continuous T-periodic functions, then (2.1) has a unique positive, global attractive T-periodic solution $x^*(t)$.

As a matter of fact, according to Lemma 2.2 of [11], if $r(t)$ may be negative but $M[r] > 0$, $a_k(t)$ $(k = 1, 2, ..., n)$ are nonnegative and there exists at least one $k \in \{1,\ 2,\ ...,\ n\}$ such that $m[a_k] > 0$, then we have Lemma 2.2 below corresponding to Lemma 2.1:

Lemma 2.2. Assume that $M[r] > 0$ and $a_k(t)$ $(k = 1, 2, ..., n)$ are nonnegative and there exists at least one $k \in \{1,\ 2,\ ...,\ n\}$ such that $m[a_k] > 0$, then any solution $x(t)$ of (2.1) with initial value $x(t_0) > 0$ is bounded above and below by strictly positive real numbers on $[t_0, +\infty)$ and globally attractive. Specially, if $r(t),\ a_k(t)$ $(k = 1, 2, ..., n)$ are all continuous T-periodic functions, then system (2.1) has a unique positive, globally asymptotically stable T-periodic solution $x^*(t)$.

By developing the analytic technique of [11, 16], it is not difficult to verify the following results

Lemma 2.3. If $(H_2) - (H_4)$ are hold, then we have

$$M\Big[a_{ijk}(t)X_j^{\alpha_{ijk}}(t - \tau_{ijk}(t))\Big] = M\Big[\frac{a_{ijk}\big(\Psi^{-1}_{ijk}(t)\big)}{1 - \tau'_{ijk}\big(\Phi^{-1}_{ijk}(t)\big)}X_j^{\alpha_{ijk}}(t)\Big].$$

$$m\Big[a_{ijk}(t)X_j^{\alpha_{ijk}}(t - \tau_{ijk}(t))\Big] = m\Big[\frac{a_{ijk}\big(\Phi^{-1}_{ijk}(t)\big)}{1 - \tau'_{ijk}\big(\Phi^{-1}_{ijk}(t)\big)}X_j^{\alpha_{ijk}}(t)\Big].$$

where $X_i(t)$ is the unique solution of the Logistic system corresponding to Eqs. (1.1) with initial condition $X_i(t_0) > 0$.

Proof. From $(H_2) - (H_4)$ and Lemma 2.1, 2.2, we infer that $\tau_{ijk}(t)$, $\frac{a_{ijk}(\Phi^{-1}_{ijk}(t))}{1-\tau'_{ijk}(\Phi^{-1}_{ijk}(t))}$ and $X_j^{\alpha_{ijk}}(t)$ are all bounded, we claim that

$$\int_{t_1-\tau_{ijk}(t_1)}^{t_1} \frac{a_{ijk}\big(\Phi^{-1}_{ijk}(s)\big)}{1 - \tau'_{ijk}\big(\Phi^{-1}_{ijk}(s)\big)}X_j^{\alpha_{ijk}}(s)ds,\ \int_{t_2}^{t_2-\tau_{ijk}(t_2)} \frac{a_{ijk}\big(\Phi^{-1}_{ijk}(s)\big)}{1 - \tau'_{ijk}\big(\Phi^{-1}_{ijk}(s)\big)}X_j^{\alpha_{ijk}}(s)ds$$

are all bounded above and below. Then from the definition of lower and upper averages of a function, we obtain that for $t_2 > t_1 \geq t_0$

$$M\Big[a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t))\Big] = \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t))ds \mid t_2 - t_1 \geq s\Big\}$$

$$= \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2 - t_1}\int_{t_1-\tau_{ijk}(t_1)}^{t_2-\tau_{ijk}(t_2)} \frac{a_{ijk}\big(\Phi^{-1}_{ijk}(t)\big)}{1 - \tau'_{ijk}\big(\Phi^{-1}_{ijk}(t)\big)}X_j^{\alpha_{ijk}}(t)ds \mid t_2 - t_1 \geq s\Big\}$$

$$= \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2 - t_1}\Big(\int_{t_1-\tau_{ijk}(t_1)}^{t_1} + \int_{t_1}^{t_2} + \int_{t_2}^{t_2-\tau_{ijk}(t_2)}\Big)\frac{a_{ijk}\big(\Phi^{-1}_{ijk}(t)\big)X_j^{\alpha_{ijk}}(t)}{1 - \tau'_{ijk}\big(\Psi^{-1}_{ijk}(t)\big)}dt \mid t_2 - t_1 \geq s\Big\}$$

$$= \lim_{s\to+\infty} \sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)X_j^{\alpha_{ijk}}(t)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}dt \mid t_2-t_1\ge s\Big\} = M\Big[\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)X_j^{\alpha_{ijk}}(t)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}\Big].$$

Similarly, we can testify that the equality for the case of $m\big[a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t))\big]$ is also true.

Lemma 2.4. If $(H_2)-(H_4)$ hold, then

$$M\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\Big] = M\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big],$$
$$m\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\Big] = m\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big].$$

where $X_i(t)$ is the unique solution of the Logistic system corresponding to Eqs. (1.1) with initial condition $X_i(t_0) > 0$.

Proof. From $(H_2)-(H_4)$ and Lemma 2.1, 2.2, it follows that $b_{ijl}(t,.)$ and $\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)ds$, $X_j^{\beta_{ijl}}(t)$ are all bounded functions, we conclude that

$$\int_{-\sigma_{ijl}}^{0}\int_{t_1+s}^{t_1} b_{ijl}(t-s,s)X_j^{\beta_{ijl}}(s)ds,\ \int_{-\sigma_{ijl}}^{0}\int_{t_2}^{t_2+s} b_{ijl}(t-s,s)X_j^{\beta_{ijl}}(s)ds$$

are all bounded. Therefore, according to the definition of lower and upper averages of a function, we find that for $t_2 > t_1 \ge t_0$

$$M\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\Big]$$
$$= \lim_{s\to+\infty} \sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\Big(\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ij}}(t+s)ds\Big)dt \mid t_2-t_1\ge s\Big\}$$
$$= \lim_{s\to+\infty} \sup\Big\{\frac{1}{t_2-t_1}\int_{-\sigma_{ijl}}^{0}\int_{t_1+s}^{t_2+s} b_{ijl}(t-s,s)X_j^{\beta_{ijl}}(t)dt \mid t_2-t_1\ge s\Big\}ds$$
$$= \lim_{s\to+\infty} \sup\Big\{\frac{1}{t_2-t_1}\int_{-\sigma_{ijl}}^{0}\Big(\int_{t_1+s}^{t_1}+\int_{t_1}^{t_2}+\int_{t_2}^{t_2+s}\Big)b_{ijl}(t-s,s)X_j^{\beta_{ijl}}(t)dt \mid t_2-t_1\ge s\Big\}ds$$
$$= \lim_{s\to+\infty} \sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\Big(\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big)dt \mid t_2-t_1\ge s\Big\}$$
$$= M\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big].$$

In a similar way, we can show that the equality for the case of $m\big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\big]$ is also hold.

3 Permanence

In this section, we are mainly concerned with the permanence of the system (1.1)-(1.2). Firstly, for the sake of the permanence with regarding to the system (1.1), we introduce the following notations

$$a^*_{ijk}(t) = a_{ijk}(t)\exp\Big\{\alpha_{ijk}\int_t^{t-\tau_{ijk}(t)} r_i(s)ds\Big\},$$
$$b^*_{ijl}(t) = \int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)\exp\Big\{\beta_{ijl}\int_t^{t+s} r_i(u)du\Big\}ds.$$

Then, let us consider the following logistic type equation corresponding to Eqs. (1.1)

$$\dot{x}_i(t) = x_i(t)\Big[r_i(t) - \sum_{k=1}^{k_{ii}} a^*_{iik}(t)x_i^{\alpha_{iik}}(t) - \sum_{l=1}^{l_{ii}}\int_{-\sigma_{iil}}^{0} b^*_{iil}(t,s)dsx_i^{\beta_{iil}}(t)\Big]. \tag{3.1}$$

Theorem 3.1. In addition to $(H_1)-(H_4)$, assume further that

(H_5) $M\Big[r_i(t) - \sum_{j=1,\ j\neq i}^{n}\Big(\sum_{k=1}^{k_{ij}} a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t)) + \sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\Big)\Big] > 0.$

Where $X_i(t)$ is the unique globally attractive positive solution of the (3.1) with initial condition $X_i(t_0) > 0$. Then Eqs. (1.1)-(1.2) is permanent.

Proof. Firstly, we show that any positive solution of system (1.1) is ultimately bounded above by some positive constant. Let $x(t) = \big(x_1(t),\ ...,\ x_n(t)\big)$ be any positive solution of system (1.1), then it follows from (1.1) that for all $t \geq 0$

$$\dot{x}_i(t) \leq r_i(t)x_i(t). \tag{3.2}$$

Thus for any $t \geq 0,\ s \leq 0$ and $t+s \geq 0$, by integrating (2.11) over interval $[t+s, t]$ we derive

$$x_i(t+s) \geq x_i(t)\exp\Big\{\int_t^{t+s} r_i(s)ds\Big\} \quad \text{for, } t \geq \tau. \tag{3.3}$$

Integrate with (3.3), we obtain directly from the system (1.3) that

$$\begin{aligned}\dot{x}_i(t) &= x_i(t)\Big[r_i(t) - \sum_{j=1}^{n}\sum_{k=1}^{k_{ij}} a_{ijk}(t)x_j^{\alpha_{ijk}}(t-\tau_{ijk}(t)) - \sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)x_j^{\beta_{ijl}}(t+s)ds\Big] \\ &\leq x_i(t)\Big[r_i(t) - \sum_{k=1}^{k_{ii}} a_{iik}(t)x_i^{\alpha_{iik}}(t-\tau_{iik}(t)) - \sum_{l=1}^{l_{ii}}\int_{-\sigma_{iil}}^{0} b_{iil}(t,s)x_i^{\beta_{iil}}(t+s)ds\Big] \\ &\leq x_i(t)\Big[r_i(t) - \sum_{k=1}^{k_{ii}} a^*_{iik}(t)x_i^{\alpha_{iik}}(t) - \sum_{l=1}^{l_{ii}}\int_{-\sigma_{iil}}^{0} b^*_{iil}(t,s)dsx_i^{\beta_{iil}}(t)\Big].\end{aligned} \tag{3.4}$$

By using the comparison theorem, we find

$$x_i(t) \leq X_i(t), \ \text{ for all } \ t \geq t_0. \tag{3.5}$$

Where $X_i(t)$ is the positive solution of system (3.1) with initial condition $X_i(0)$ which satisfies $x_i(0) \leq X_i(0)$. From Lemma 2.1, Lemma 2.2 and (3.5), it is not difficult to obtain that

$$\limsup_{t\to+\infty} x_i(t) \leq X_i(t), \ \text{ for all } \ t \geq t_0.$$

Hence, for a sufficiently small $\varepsilon > 0$, there exists a $T_{i1}(\varepsilon) > 0$ such that for $t \geq T_{i1}(\varepsilon)$

$$x_i(t) \leq X_i(t) \leq X_i(t) + \varepsilon. \tag{3.6}$$

Now choose $M_0 = \sup\{X_i(t) + \varepsilon \mid t \geq 0,\ i = 1,\ 2,\ ...,\ n\}$, then M_0 does not depend on any solution of system (3.1), also $x_i(t) \leq M_0$, for all $t \geq T_1$, where $T_1 = \max_{1\leq i\leq n}\{T_{i1}\}$.

Secondly, we shall show that any positive solution of system (1.1) is ultimately bounded below by some positive constant. To this end, we proceed with following two steps.

Step 1: We show that there exists $\epsilon_0 > 0$ such that $\limsup_{t\to+\infty} x_i(t) \geq \epsilon_0$, for all $i = 1,\ 2,\ ...,\ n$. For the convenience of the following discuss, for any constant $\varepsilon > 0$, we denote by

$$\begin{aligned}R_i(t,\varepsilon) = r_i(t) &- \sum_{j=1,\ j\neq i}^{n}\sum_{k=1}^{k_{ij}} a_{ijk}(t)\Big(X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t)) + \varepsilon\Big) \\ &- \sum_{j=1,\ j\neq i}^{n}\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)\Big(X_j^{\beta_{ijl}}(t+s) + \varepsilon\Big)ds\Big]\end{aligned}$$

On the one hand, according to (H_5) in Theorem 3.1, one finds that for any given small number $\varepsilon > 0$, there is $M[R_i(t,\varepsilon)] > 0$ $(i = 1, 2, ..., n)$. Therefore, we can choose a sufficiently small number $\epsilon_0 > 0,\ \delta > 0$ such that

$$M\Big[R_i(t,\varepsilon) - \sum_{k=1}^{k_{ii}} a_{iik}(t)\epsilon_0^{\alpha_{iik}} - \sum_{l=1}^{l_{ii}}\int_{-\sigma_{iil}}^{0} b_{iil}(t,s)ds\epsilon_0^{\beta_{iil}}\Big] \geq \delta,$$

for all $i = 1,\ 2,\ ...,\ n$, i.e.,

$$\lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\Big[R_i(t,\varepsilon) - \sum_{k=1}^{k_{ii}} a_{iik}(t)\epsilon_0^{\alpha_{iik}} - \sum_{l=1}^{l_{ii}}\int_{-\sigma_{iil}}^{0} b_{iil}(t,s)ds\epsilon_0^{\beta_{iil}}\Big]dt \mid t_2 - t_1 \geq s\Big\} \geq \delta.$$

Which implies that

$$\lim_{s\to+\infty}\sup\Big\{\int_{t_1}^{t_2}\Big[R_i(t,\varepsilon)-\sum_{k=1}^{k_{ii}}a_{iik}(t)\epsilon_0^{\alpha_{iik}}\\ -\sum_{l=1}^{l_{ii}}\int_{-\sigma_{iil}}^{0}b_{iil}(t,s)\epsilon_0^{\beta_{iil}}ds\Big]dt\mid t_2-t_1\ge s\Big\}=+\infty.$$

Therefore, there must exist $\lambda>0$ and a positive number $\gamma_0>0$ such that

$$\int_t^{t+\lambda}\Big[R_i(t,\varepsilon)-\sum_{k=1}^{k_{ii}}a_{iik}(t)\epsilon_0^{\alpha_{iik}}-\sum_{l=1}^{l_{ii}}\int_{-\sigma_{iil}}^{0}b_{iil}(t,s)ds\epsilon_0^{\beta_{iil}}\Big]dt\ge\gamma_0,\ \text{for all}\ t\ge T_2. \tag{3.7}$$

Now we claim that the following inequality holds

$$\limsup_{t\to+\infty}x_i(t)\ge\epsilon_0,\ \text{for all}\ i=1,\ 2,\ ...,\ n. \tag{3.8}$$

By way of contradiction, suppose that $\limsup_{t\to+\infty}x_i(t)<\epsilon_0$ for a certain $p\in\{1,\ 2,\ ...,\ n\}$, then there exists $T_2>T_1$ such that $x_p(t)<\delta$, for all $t\ge T_2$. This, together with the (3.6), gives out that for all $t\ge T_2$

$$\begin{aligned}\dot{x}_p(t)&=x_p(t)\Big[r_p(t)-\sum_{j=1}^{n}\Big(\sum_{k=1}^{k_{pj}}a_{pjk}(t)x_j^{\alpha_{pjk}}(t-\tau_{pjk}(t))+\sum_{l=1}^{l_{pj}}\int_{-\sigma_{pjl}}^{0}b_{pjl}(t,s)x_j^{\beta_{pjl}}(t+s)ds\Big)\Big]\\ &\ge x_p(t)\Big[r_p(t)-\sum_{j=1,\ j\ne p}^{n}\sum_{k=1}^{k_{pj}}a_{pjk}(t)\Big(X_j^{\alpha_{pjk}}(t-\tau_{pjk}(t))+\varepsilon\Big)\\ &\quad-\sum_{j=1,\ j\ne p}^{n}\sum_{l=1}^{l_{pj}}\int_{-\sigma_{pjl}}^{0}b_{pjl}(t,s)\Big(X_j^{\beta_{pjl}}(t+s)+\varepsilon\Big)ds\Big]\\ &\quad-\sum_{k=1}^{k_{pp}}a_{ppk}(t)\epsilon_0^{\alpha_{ppk}}-\sum_{l=1}^{l_{pp}}\int_{-\sigma_{ppl}}^{0}b_{ppl}(t,s)ds\epsilon_0^{\beta_{ppl}}\Big]\\ &\ge x_p(t)\Big[R_p(t,\varepsilon)-\sum_{k=1}^{k_{pp}}a_{ppk}(t)\epsilon_0^{\alpha_{ppk}}-\sum_{l=1}^{l_{pp}}\int_{-\sigma_{ppl}}^{0}b_{ppl}(t,s)ds\epsilon_0^{\beta_{ppl}}\Big].\end{aligned} \tag{3.9}$$

An integration of (3.9) over time interval $[T_2,t]$ leads to

$$x_p(t)\ge x_p(T_2)\exp\Big\{\int_{T_2}^{t}\Big[R_p(t,\varepsilon)-\sum_{k=1}^{k_{pp}}a_{ppk}(t)\epsilon_0^{\alpha_{ppk}}-\sum_{l=1}^{l_{pp}}\int_{-\sigma_{ppl}}^{0}b_{ppl}(t,s)ds\epsilon_0^{\beta_{ppl}}\Big]\Big\}. \tag{3.10}$$

Obviously, which, together with (3.7) result into the conclusion that $x_p(t)\to+\infty$ as $t\to+\infty$, which contradicts to the boundedness of $x_i(t)$, for all $t\ge T_{i1}$ in (3.6). Hence, the inequality (3.8) is true.

Step 2: We shall prove that there exists a constant $m_0>0$, m_0 is independent of any solution of system (1.1), i.e., there is a positive constant $m_0>0$ such that for any solution $x(t)=(x_1(t),\ ...,\ x_n(t))$, one has

$$\liminf_{t\to+\infty}x_i(t)\ge m_0,\ \text{for all}\ i=1,\ 2,\ ...,\ n. \tag{3.11}$$

Assume that it is not true, then there exist a certain integer $q\in\{1,\ 2,\ ...,\ n\}$ and a sequence of initial functions $\{\phi_q^{(k)}(t)\}_{k=1}^{+\infty}$ for system (1.1) such that $x_q^{(k)}(t)=x_q(t,\phi_q^{(k)}),\ k=1,\ 2,\ ...$ satisfy

$$\liminf_{t\to+\infty}x_q^{(k)}(t)\le\frac{\epsilon_0}{(k+1)^2},\ \text{for all}\ k=1,\ 2,\ ... \tag{3.12}$$

For each $k=1,\ 2,\ ...$, from (3.8) we claim that $\limsup_{t\to+\infty}x_q^{(k)}(t)\ge\frac{1}{(k+1)}\epsilon_0$. Hence, by (3.12) one can infer that there exists two time sequences $\{s_n^{(k)}\}$ and $\{t_n^{(k)}\}$ such that for each $k=1,\ 2,\ ...$

$$0<s_1^{(k)}<t_1^{(k)}<s_2^{(k)}<t_2^{(k)}<\ ...\ <s_n^{(k)}<t_n^{(k)}<\ ...\ ,\ \text{for all}\ n=1,\ 2,\ ...,$$

$$s_n^{(k)} \to +\infty, \quad t_n^{(k)} \to +\infty, \text{ as } n \to +\infty, \; x_q^{(k)}(t_n^{(k)}) = \frac{\epsilon_0}{(k+1)^2}, \; x_q^{(k)}(s_n^{(k)}) = \frac{\epsilon_0}{(k+1)}. \tag{3.13}$$

$$\frac{\epsilon_0}{(k+1)^2} < x_q^{(k)}(t) < \frac{\epsilon_0}{(k+1)}, \text{ for all } t \in (s_n^{(k)}, \; t_n^{(k)}). \tag{3.14}$$

It follows from (3.6) that for a given small number ϵ_0, there exists $T_2^{(k)} > T_1$ such that $x_i^{(k)}(t) \leq X_i(t) + \epsilon_0, \;\; t \geq T_2^{(k)}$.

Obviously, by (3.13) there exists a large enough integer $N_1^{(k)} > 0$ such that $s_n^{(k)} > T_2^{(k)} + \tau$ for all $n \geq N_1^{(k)}$ for each $k = 1, \; 2, \; \ldots$. Hence, for any $t \in [s_n^{(k)}, \; t_n^{(k)}]$ and $n \geq N_1^{(k)}$, we have

$$\begin{aligned}
\dot{x}_q^{(k)}(t) &= x_q^{(k)}(t)\Big[r_q(t) - \sum_{j=1}^{n}\sum_{\nu=1}^{\nu_{qj}} a_{qj\nu}(t)\Big(x_j^{(k)}(t-\tau_{qj\nu}(t))\Big)^{\alpha_{qj\nu}} \\
&\qquad - \sum_{j=1}^{n}\sum_{l=1}^{l_{qj}} \int_{-\sigma_{qjl}}^{0} b_{qjl}(t,s)\Big(x_j^{(k)}(t+s)\Big)^{\beta_{qjl}} ds\Big] \\
&\geq x_q^{(k)}(t)\Big[r_q(t) - \sum_{j=1}^{n}\sum_{\nu=1}^{\nu_{qj}} a_{qj\nu}(t)\Big(X_j^{(k)}(t-\tau_{qj\nu}(t)) + \varepsilon\Big)^{\alpha_{qj\nu}} \\
&\qquad - \sum_{j=1}^{n}\sum_{l=1}^{l_{qj}} \int_{-\sigma_{qjl}}^{0} b_{qjl}(t,s)\Big(X_j^{(k)}(t+s) + \varepsilon\Big)^{\beta_{qjl}} ds\Big] \geq -\gamma x_q^{(k)}(t).
\end{aligned} \tag{3.15}$$

Where

$$\gamma = \sup_{t \in R}\Big\{\sum_{j=1}^{n}\Big[\sum_{\nu=1}^{\nu_{qj}} a_{qj\nu}(t)\Big(X_j^{(k)}(t-\tau_{qj\nu}(t))+\varepsilon\Big)^{\alpha_{qj\nu}} + \sum_{l=1}^{l_{qj}} \int_{-\sigma_{qjl}}^{0} b_{qjl}(t,s)\Big(X_j^{(k)}(t+s)+\varepsilon\Big)^{\beta_{qjl}} ds\Big]\Big\}.$$

Therefore, for any $n \geq N_1^{(k)}$ and $k = 1, \; 2, \; \ldots$, an integration of (3.15) over $[s_n^{(k)}, \; t_n^{(k)}]$ makes one lead to

$$\begin{aligned}
\frac{\epsilon_0}{(k+1)^2} &= x_q^{(k)}(t_n^{(k)}) \geq x_q^{(k)}(s_n^{(k)}) \exp\big\{-\gamma(t_n^{(k)} - s_n^{(k)})\big\} \\
&= \frac{\epsilon_0}{(k+1)} \exp\big\{-\gamma(t_n^{(k)} - s_n^{(k)})\big\}.
\end{aligned}$$

Which means

$$t_n^{(k)} - s_n^{(k)} \geq \frac{\ln(k+1)}{\gamma}, \text{ for all } n \geq N_1^{(k)}, \; k = 1, \; 2, \; \ldots \tag{3.16}$$

It follows from (3.16) that there exists a sufficient large integer K_0 such that

$$t_n^{(k)} - s_n^{(k)} \geq \lambda, \text{ for all } k \geq K_0, \;\; n \geq N_1^{(k)}. \tag{3.17}$$

Hence, for any $k \geq K_0, \;\; n \geq N_1^{(k)}$ and $t \in [s_n^{(k)}, \; t_n^{(k)}]$, it follows from (3.13) and (3.14) that

$$\begin{aligned}
\dot{x}_q^{(k)}(t) &= x_q^{(k)}(t)\big[r_q(t) - \sum_{j=1}^{n}\sum_{\nu=1}^{\nu_{qj}} a_{qj\nu}(t)\big(x_j^{(k)}(t-\tau_{qj\nu}(t))\big)^{\alpha_{qj\nu}} \\
&\qquad - \sum_{j=1}^{n}\sum_{l=1}^{l_{qj}} \int_{-\sigma_{qjl}}^{0} b_{qjl}(t,s)\big(x_j^{(k)}(t+s)\big)^{\beta_{qjl}} ds\big] \\
&\geq x_q^{(k)}(t)\big[r_q(t) - \sum_{\nu=1}^{\nu_{qq}} a_{qq\nu}(t)\big(\frac{\epsilon_0}{k+1}\big)^{\alpha_{qq\nu}} - \sum_{l=1}^{l_{qq}} \int_{-\sigma_{qql}}^{0} b_{qql}(t,s)ds\big(\frac{\epsilon_0}{k+1}\big)^{\beta_{qql}} \\
&\qquad - \sum_{j=1, \; j\neq q}^{n} \sum_{\nu=1}^{\nu_{qj}} a_{qj\nu}(t)\big(X_j^{(k)}(t-\tau_{qj\nu}(t)) + \varepsilon\big)^{\alpha_{qj\nu}} \\
&\qquad - \sum_{j=1, \; j\neq p}^{n} \sum_{l=1}^{l_{qj}} \int_{-\sigma_{qjl}}^{0} b_{qjl}(t,s)\big(X_j^{(k)}(t+s) + \varepsilon\big)^{\beta_{qjl}} ds\big] \\
&\geq x_q^{(k)}(t)\big[r_q(t) - \sum_{\nu=1}^{\nu_{qq}} a_{qq\nu}(t)\epsilon_0^{\alpha_{qq\nu}} - \sum_{l=1}^{l_{qq}} \int_{-\sigma_{qql}}^{0} b_{qql}(t,s)ds\epsilon_0^{\beta_{qql}}
\end{aligned}$$

$$-\sum_{j=1,\ j\neq q}^{n}\sum_{\nu=1}^{\nu_{qj}}a_{qj\nu}(t)\big(X_j^{(k)}(t-\tau_{qj\nu}(t))+\varepsilon\big)^{\alpha_{qj\nu}}$$
$$-\sum_{j=1,\ j\neq p}^{n}\sum_{l=1}^{l_{qj}}\int_{-\sigma_{qjl}}^{0}b_{qjl}(t,s)\big(X_j^{(k)}(t+s)+\varepsilon\big)^{\beta_{qjl}}ds\big]. \tag{3.18}$$

According to (3.7), (3.13) and (3.14), an integration of (3.18) over time interval $[t_n^{(k)}-\lambda,\ t_n^{(k)}]$ makes it reach

$$\frac{\epsilon_0}{(k+1)^2}=x_q^{(k)}(t_n^{(k)})\geq x_q^{(k)}(t_n^{(k)}-\lambda)\exp\Big\{\int_{t_n^{(k)}-\lambda}^{t_n^{(k)}}\big[B_q(t,\epsilon_0)-\sum_{j=1,\ j\neq q}^{n}\big(\sum_{\nu=1}^{\nu_{qj}}a_{qj\nu}(t)$$
$$\times\big(X_j^{(k)}(t-\tau_{qj\nu}(t))+\varepsilon\big)^{\alpha_{qj\nu}}+\sum_{l=1}^{l_{qj}}\int_{-\sigma_{qjl}}^{0}b_{qjl}(t,s)\big(X_j^{(k)}(t+s)+\varepsilon\big)^{\beta_{qjl}}ds\big)\big]dt\Big\}$$
$$>\frac{\epsilon_0}{(k+1)^2}\exp\epsilon_0>\frac{\epsilon_0}{(k+1)^2}. \tag{3.19}$$

Where

$$B_q(t,\epsilon_0)=r_q(t)-\sum_{\nu=1}^{\nu_{qq}}a_{qq\nu}(t)\epsilon_0^{\alpha_{qq\nu}}-\sum_{l=1}^{l_{qq}}\int_{-\sigma_{qql}}^{0}b_{qql}(t,s)ds\epsilon_0^{\beta_{qql}}.$$

Which is contradiction. This shows that there exists a constant $m_0>0$ ($m_0>0$ is independent of any initial function) such that the inequality (2.15) is correct. That is to say, any positive solution $x(t)$ of the initial value problem (1.1)-(1.2) is ultimately bounded below by a positive constant $m_0>0$. From Definition 2.1, the proof of Theorem 3.1 is complete.

Theorem 3.2. In addition to $(H_1)-(H_4)$, assume further that

$$(H_5)'\ M\big[r_i(t)\big]-\sum_{j=1,\ j\neq i}^{n}\sum_{k=1}^{k_{ij}}m\Big[\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}X_j^{\alpha_{ijk}}(t)\Big]$$
$$-\sum_{j=1,\ j\neq i}^{n}\sum_{l=1}^{l_{ij}}m\Big[\int_{-\sigma_{ijl}}^{0}b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big]>0.$$

Where $X_i(t)$ is the unique globally attractive positive solution of the (3.1) with initial condition $X_i(t_0)>0$. Then the system (1.1)-(1.2) is permanent.

Proof. In order to prove the correct of Theorem 3.2, We only need to show that $(H_5)'$ implies the assumption (H_5). Actually, if take into account the fact that

$$M[X_{i0}(t)+c]=M[X_{i0}(t)]+c,\qquad m[f_i(t)]\leq A[f_i(t),\ t_1,\ t_2].$$

Then we may obtain that

$$M\big[r_i(t)\big]-\sum_{j=1,\ j\neq i}^{n}\Big(\sum_{k=1}^{k_{ij}}m\Big[\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)X_j^{\alpha_{ijk}}(t)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}\Big]+\sum_{l=1}^{l_{ij}}m\Big[\int_{-\sigma_{ijl}}^{0}b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big]\Big)$$
$$-M\big[r_i(t)-\sum_{j=1,\ j\neq i}^{n}\big(\sum_{k=1}^{k_{ij}}a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t))+\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0}b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\big)\big]$$
$$=\lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\Big[r_i(t)-\sum_{j=1,\ j\neq i}^{n}\big(\sum_{k=1}^{k_{ij}}m\Big[\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)X_j^{\alpha_{ijk}}(t)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}\Big]$$
$$+\sum_{l=1}^{l_{ij}}m\Big[\int_{-\sigma_{ijl}}^{0}b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big]\big)\Big]dt\mid t_2-t_1\geq s\Big\}-\lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\big[r_i(t)$$
$$-\sum_{j=1,\ j\neq i}^{n}\big(\sum_{k=1}^{k_{ij}}a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t))+\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0}b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\big)\big]dt\mid t_2-t_1\geq s\Big\}$$

$$= \lim_{s\to+\infty} \sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\Big[r_i(t)-\sum_{j=1,\ j\neq i}^{n}\big(\sum_{k=1}^{k_{ij}} m\big[a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t))\big]$$

$$+\sum_{l=1}^{l_{ij}} m\big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\big]\big)\Big]dt \mid t_2-t_1\geq s\Big\} - \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\big[r_i(t)$$

$$-\sum_{j=1,\ j\neq i}^{n}\big(\sum_{k=1}^{k_{ij}} a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t))+\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\big)\big]dt \mid t_2-t_1\geq s\Big\}$$

$$\geq \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2} r_i(t)dt-\sum_{j=1,\ j\neq i}^{n}\big(\sum_{k=1}^{k_{ij}}\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\big[a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t))\big]dt$$

$$+\sum_{l=1}^{l_{ij}}\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\big]\big)dt \mid t_2-t_1\geq s\Big\} - \lim_{s\to+\infty}\sup\Big\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\big[r_i(t)$$

$$-\sum_{j=1,\ j\neq i}^{n}\big(\sum_{k=1}^{k_{ij}} a_{ijk}(t)X_j^{\alpha_{ijk}}(t-\tau_{ijk}(t))+\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)X_j^{\beta_{ijl}}(t+s)ds\big)\big]dt \mid t_2-t_1\geq s\Big\}=0.$$

Therefore, we claim from Theorem 3.1 that Theorem 3.2 is correct. The proof is complete.

Theorem 3.3. In addition to $(H_1)-(H_4)$, assume further that

$$(H_5)''\ M\big[r_i(t)\big]-\sum_{j=1,\ j\neq i}^{n}\sum_{k=1}^{k_{ij}} M\big[\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}X_j^{\alpha_{ijk}}(t)\big]$$

$$-\sum_{j=1,\ j\neq i}^{n}\sum_{l=1}^{l_{ij}} M\big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\big]>0.$$

Where $X_i(t)$ is the unique globally attractive positive solution of the (3.1) with initial condition $X_i(t_0)>0$. Then Eqs. (1.1)-(1.2) is permanent.

Proof. Noticing the following facts that

$$M[X_{i0}(t)+c]=M[X_{i0}(t)]+c,\quad m[f_i(t)]\leq M[f_i(t)]\ \text{ and }\ \sum_{i=1}^{n} m[f_i(t)]\leq\sum_{i=1}^{n} M[f_i(t)].$$

We find that the condition $(H_5)''$ means the hypothesis $(H_5)'$, and so it does the assumption (H_5). Hence, one can confirm that the result of Theorem 3.3 is also true.

Theorem 3.4. In addition to $(H_1)-(H_4)$, assume further that

$$(H_5)'''\ m\big[r_i(t)\big]-\sum_{j=1,\ j\neq i}^{n}\sum_{k=1}^{k_{ij}} M\big[\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}X_j^{\alpha_{ijk}}(t)\big]$$

$$-\sum_{j=1,\ j\neq i}^{n}\sum_{l=1}^{l_{ij}} M\big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\big]>0.$$

Where $X_i(t)$ is the unique globally attractive positive solution of the (3.1) with initial condition $X_i(t_0)>0$. Then Eqs. (1.1)-(1.2) is permanent.

Proof. Taking into account the facts that

$$M[X_{i0}(t)+c]=M[X_{i0}(t)]+c,\quad m[f_i(t)]\leq M[f_i(t)].$$

We declare that the assumption $(H_5)''$ can be deduced from the hypothesis $(H_5)'''$, so it is evident that Theorem 3.3 implies the Theorem 3.4.

Theorem 3.5. In addition to $(H_1)-(H_4)$, assume further that

$$(H_5)''''\ m\big[r_i(t)\big]-\sum_{j=1,\ j\neq i}^{n}\sum_{k=1}^{k_{ij}} m\big[\frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}X_j^{\alpha_{ijk}}(t)\big]$$

$$-\sum_{j=1,\ j\neq i}^{n}\sum_{l=1}^{l_{ij}} m\Big[\int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)dsX_j^{\beta_{ijl}}(t)\Big] > 0.$$

Where $X_i(t)$ is the unique globally attractive positive solution of the (3.1) with initial condition $X_i(t_0) > 0$. Then Eqs. (1.1)-(1.2) is permanent.

Proof. As a matter of fact, $m[f_i(t)] \le M[f_i(t)]$ and assumption $(H_5)''''$ means that the hypothesis (H_5) is true, so it follows from Theorem 3.1 that the conclusion of Theorem 3.5 is right.

Remark. 3.1 It is easy to verify that $M[g] = m[g] = \frac{1}{T}\int_0^T g(t)dt$ for a T-periodic function $g(t)$. So if system (1.1) is a periodic system, i.e., $r_i(t)$, $a_{ijk}(t)$, $b_{ijl}(t,\cdot)$ are the continuous T-periodic functions, then $X_i(t)$ in above mentioned Theorems can be replaced by the unique positive T-periodic solution $X_i^*(t)$ of (3.1), and the assumptions of Theorem 3.1-Theorem 3.5 are equivalent to each other.

Remark. 3.2 Theorems 3.1-3.5 generalize the main results of Zhao et al. [11], Chen et al. [14,15] and Xia et al. [16]. We mention here that for general nonautonomous Lotka-Volterra system (1.1), Teng et al. [21, 22] also obtained some similar results as that of Zhao [11]. It is in this sense, our results can also be seen as the generalization of Theorems of [21, 22].

4 Global attractivity

A very basic and important problem accompanying with the ecological dynamics systems is the global stability of the positive solution for the system. In this section, we will devote ourselves to give some new criteria to guarantee global attractivity of the positive solution.

Definition 4.1. The bounded solution $X^*(t) = (x_1^*(t), x_2^*(t), ..., x_n^*(t))$ of system (1.1) with $X^*(t_0) > 0$ is said to be globally attractive, if for any other solution $X(t) = (x_1(t), x_2(t), ..., x_n(t))^T$ with $X(0) > 0$, there is

$$\lim_{t\to+\infty} \mid x_i(t) - x_i^*(t) \mid = 0, \quad i = 1, 2, ..., n.$$

Before we state the main result of this section, we first introduce some notations which will be used in the following discussion. Let $\Phi_{ijk}^{-1}(t)$ be the inverse function of $\Phi_{ijk}(t) = t - \tau_{ijk}(t)$, and

$$A_{ijk}^{(1)}(t) = \frac{a_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)}{1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)},\ A_{ijk}^{(2)}(t) = \frac{a_{ijk}\Big(\Phi_{ijk}^{-1}\big(\Phi_{ijk}^{-1}(t)\big)\Big)}{\Big(1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}\big(\Phi_{ijk}^{-1}(t)\big)\big)\Big)\Big(1-\tau'_{ijk}\big(\Phi_{ijk}^{-1}(t)\big)\Big)},$$

$$B_{ijl}^{(1)}(t) = \int_{-\sigma_{ijl}}^{0} b_{ijl}(t-s,s)ds, \quad B_{ijl}^{(2)}(t) = \int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t} b_{ijl}(\theta-s,s)d\theta ds,$$

$$\big(B_{ijl}^{(2)}\cdot A_{i\tilde{j}\tilde{k}}^{(1)}\big)(t) = \int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t} A_{i\tilde{j}\tilde{k}}^{(1)}(\theta-s)b_{ijl}(t-s,s)d\theta ds,$$

$$\big(B_{ijl}^{(2)}\cdot B_{i\tilde{j}\tilde{l}}^{(1)}\big)(t) = \int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t} B_{i\tilde{j}\tilde{l}}^{(1)}(\theta-s)b_{ijl}(t-s,s)d\theta ds.$$

Let $u_i(t) = \ln x_i(t)$, then Eqs. (1.1) can be reformulated as

$$\begin{aligned}\dot{u}_i(t) = r_i(t) - \sum_{j=1}^{n}\sum_{k=1}^{k_{ij}} a_{ijk}(t)\exp\Big\{\alpha_{ijk}u_j\big(t-\tau_{ijk}(t)\big)\Big\} \\ -\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0} b_{ijl}(t,s)\exp\big\{\beta_{ijl}u_j(t+s)\big\}ds. \qquad (4.1)\end{aligned}$$

Now we are in the position of stating the sufficient conditions which guarantee the global attractivity of system (1.1).

Theorem 4.1. In addition to $(H_1)-(H_5)$, we assume further that

(H_6) There exist positive constants $\lambda_i > 0$ $(i = 1, 2, ..., n)$, $\zeta > 0$ such that

$$\liminf_{t\to+\infty}\{\Lambda_i(t)\}>\zeta,\quad \liminf_{t\to+\infty}\{\Delta_i(t)\}>\zeta.$$

Where
$$\Lambda_i(t)=2\sum_{k=1}^{k_{ii}}\lambda_iA_{iik}^{(1)}(t)-\sum_{j=1,j\neq i}^{n}\Big[\sum_{k=1}^{k_{ji}}\frac{\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}}{\alpha_{iik}}A_{jik}^{(1)}(t)+\sum_{k=1}^{k_{ij}}\frac{\lambda_i}{\alpha_{iik}m_{i0}^{\alpha_{iik}}}A_{ijk}^{(1)}(t)\Big]$$
$$-\sum_{j=1}^{n}\sum_{\tilde{j}=1}^{n}\Big[\sum_{k=1}^{k_{ji}}\frac{\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}}{\alpha_{iik}}A_{jik}^{(1)}(t)\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\int_{t-\tau_{i\tilde{j}\tilde{k}}(t)}^{t}A_{i\tilde{j}\tilde{k}}^{(1)}(s)ds+\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\int_{t-\tau_{jik}(t)}^{t}A_{i\tilde{j}\tilde{k}}^{(2)}(s)ds\Big)$$
$$+\sum_{k=1}^{k_{ji}}\frac{\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}}{\alpha_{iik}}A_{jik}^{(1)}(t)\Big(\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\int_{t-\tau_{jik}(t)}^{t}B_{i\tilde{j}\tilde{l}}^{(1)}(\Phi_{jik}^{-1}(\theta))d\theta+\sum_{l=1}^{l_{i\tilde{j}}}B_{i\tilde{j}l}^{(2)}(t)\Big)\Big],$$
$$\Delta_i(t)=2\sum_{l=1}^{l_{ii}}\lambda_iB_{iil}^{(1)}(t)-\sum_{j=1,j\neq i}^{n}\Big[\sum_{l=1}^{l_{ji}}\frac{\lambda_j\beta_{jil}^2M_{i0}^{2\beta_{jil}}}{\beta_{iil}}B_{jil}^{(1)}(t)+\sum_{l=1}^{l_{ij}}\frac{\lambda_i}{\beta_{iil}m_{i0}^{\beta_{iil}}}B_{ijl}^{(1)}(t)\Big]$$
$$-\sum_{j=1}^{n}\sum_{\tilde{j}=1}^{n}\Big[\sum_{l=1}^{l_{ji}}\frac{\lambda_j\beta_{jil}^2M_{i0}^{2\beta_{jil}}}{\beta_{iil}}B_{jil}^{(1)}(t)\Big(\sum_{k=1}^{k_{i\tilde{j}}}\int_{t-\tau_{i\tilde{j}k}(t)}^{t}A_{i\tilde{j}k}^{(1)}(s)ds+\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}B_{jil}^{(1)}(t)B_{i\tilde{j}\tilde{l}}^{(2)}(t)\Big)$$
$$+\sum_{l=1}^{l_{ji}}\frac{\lambda_j\beta_{jil}^2M_{i0}^{2\beta_{jil}}}{\beta_{iil}}\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\big(B_{jil}^{(2)}\cdot A_{i\tilde{j}\tilde{k}}^{(1)}\big)(t)+\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\big(B_{jil}^{(2)}\cdot B_{i\tilde{j}\tilde{l}}^{(1)}\big)(t)\Big)\Big].$$

Then the solution $X^*(t)=(x_1^*(t),x_2^*(t),...,x_n^*(t))$ of $(1.1)-(1.2)$ is globally attractive.

Proof. Let $X^*(t)=\big(x_1^*(t),...,x_n^*(t)\big)$ with $x_i^*(t_0)>0$ be a positive solution of (1.1), and $X(t)=\big(x_1(t),...,x_n(t)\big)$ with $x_i(t_0)>0$ be an any given solution of system (1.1). In order to show the global attractivity of the bounded solution $X^*(t)$ of system (1.1), we shall show that the solution $U^*(t)=(u_1^*(t),\ ...,\ u_n^*(t)$ of system (4.1) is globally attractive. Let $U(t)=(u_1(t)\ ,...,\ u_n(t))$ be any other positive solution of system (4.1). According to Theorem 3.1, there exist positive constants $m_{i0},\ M_{i0}\ (i=1,2,...,n)$ and enough large $T>0$ such that for all $t\geq T$, there are

$$m_{i0}\leq u_i(t),\quad u_i^*(t)\leq M_{i0}\ (i=1,2,...,n). \tag{4.2}$$

Obviously, So to prove the global attractivty of the system (1.1), it is suffices to verify that system (4.1) is globally attractive. Firstly, construct a Lyapunov functional as follows

$$V_1(t)=\sum_{i=1}^{n}\lambda_i\Big[\big(u_i(t)-u_i^*(t)\big)-\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(t)\Big(\exp\{\alpha_{ijk}u_j(s)\}-\exp\{\alpha_{ijk}u_j^*(s)\}\Big)ds$$
$$-\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(\theta-s,s)\Big(\exp\{\beta_{ijl}u_j(\theta)\}-\exp\{\beta_{ijl}u_j^*(\theta)\}\Big)d\theta ds\Big]^2.$$

By calculating the right upper derivative of $V_1(t)$, we find

$$\dot{V}_1(t)=-2\sum_{i=1}^{n}\lambda_i\Big[\big(u_i(t)-u_i^*(t)\big)-\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)\Big(\exp\{\alpha_{ijk}u_j(s)\}-\exp\{\alpha_{ijk}u_j^*(s)\}\Big)ds$$
$$-\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(\theta-s,s)\Big(\exp\{\beta_{ijl}u_j(\theta)\}-\exp\{\beta_{ijl}u_j^*(\theta)\}\Big)d\theta ds\Big]$$
$$\times\Big[\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}A_{ijk}^{(1)}(t)\Big(\exp\{\alpha_{ijk}u_j(t)\}-\exp\{\alpha_{ijk}u_j^*(t)\}\Big)$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}B_{ijl}^{(1)}(t)\Big(\exp\{\beta_{ijl}u_j(t)\}-\exp\{\beta_{ijl}u_j^*(t)\}\Big)\Big]$$
$$\leq-2\sum_{i=1}^{n}\sum_{k=1}^{k_{ii}}\lambda_iA_{iik}^{(1)}(t)\Big(\exp\{\alpha_{iik}u_i(t)\}-\exp\{\alpha_{iik}u_i^*(t)\}\Big)\big(u_i(t)-u_i^*(t)\big)$$

$$
\begin{aligned}
&-2\sum_{i=1}^{n}\sum_{l=1}^{l_{ii}}\lambda_i B_{iil}^{(1)}(t)\Big(\exp\{\beta_{iil}u_i(t)\}-\exp\{\beta_{iil}u_i^*(t)\}\Big)\big(u_i(t)-u_i^*(t)\big)\\
&+2\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1}^{k_{ji}}\lambda_j A_{jik}^{(1)}(t)\Big(\exp\{\alpha_{jik}u_i(t)\}-\exp\{\alpha_{jik}u_i^*(t)\}\Big)\big(u_j(t)-u_j^*(t)\big)\\
&+2\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{l=1}^{l_{ji}}\lambda_j B_{jil}^{(1)}(t)\Big(\exp\{\beta_{jil}u_i(t)\}-\exp\{\beta_{jil}u_i^*(t)\}\Big)\big(u_j(t)-u_j^*(t)\big)\\
&+2\sum_{i=1}^{n}\lambda_i\Big[\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}A_{i\tilde{j}\tilde{k}}^{(1)}(t)\Big(\exp\{\alpha_{i\tilde{j}\tilde{k}}u_j(t)\}-\exp\{\alpha_{i\tilde{j}\tilde{k}}u_j^*(t)\}\Big)\Big]\\
&\quad\times\Big[\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)\Big(\exp\{\alpha_{ijk}u_j(s)\}-\exp\{\alpha_{ijk}u_j^*(s)\}\Big)ds\Big]\\
&+2\sum_{i=1}^{n}\lambda_i\Big[\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}A_{i\tilde{j}\tilde{k}}^{(1)}(t)\Big(\exp\{\alpha_{i\tilde{j}\tilde{k}}u_j(t)\}-\exp\{\alpha_{i\tilde{j}\tilde{k}}u_j^*(t)\}\Big)\Big]\\
&\quad\times\Big[\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(\theta-s,s)\Big(\exp\{\beta_{ijl}u_j(s)\}-\exp\{\beta_{ijl}u_j^*(s)\}\Big)d\theta ds\Big]\\
&+2\sum_{i=1}^{n}\lambda_i\Big[\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}B_{i\tilde{j}\tilde{l}}^{(1)}(t)\Big(\exp\{\beta_{i\tilde{j}\tilde{l}}u_j(t)\}-\exp\{\beta_{i\tilde{j}\tilde{l}}u_j^*(t)\}\Big)\Big]\\
&\quad\times\Big[\sum_{j=1,j\neq\tilde{j}}^{n}\sum_{k=1}^{k_{ij}}\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)\Big(\exp\{\alpha_{ijk}u_j(s)\}-\exp\{\alpha_{ijk}u_j^*(s)\}\Big)ds\Big]\\
&+2\sum_{i=1}^{n}\lambda_i\Big[\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}B_{i\tilde{j}\tilde{l}}^{(1)}(t)\Big(\exp\{\beta_{i\tilde{j}\tilde{l}}u_j(t)\}-\exp\{\beta_{i\tilde{j}\tilde{l}}u_j^*(t)\}\Big)\Big]\\
&\quad\times\Big[\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(\theta-s,s)\Big(\exp\{\beta_{ijl}u_j(s)\}-\exp\{\beta_{ijl}u_j^*(s)\}\Big)d\theta ds\Big]. \qquad (4.3)
\end{aligned}
$$

That is

$$
\begin{aligned}
\dot{V}_1(t)\le&-2\sum_{i=1}^{n}\sum_{k=1}^{k_{ii}}\lambda_i A_{iik}^{(1)}(t)\Big(\exp\{\alpha_{iik}u_i(t)\}-\exp\{\alpha_{iik}u_i^*(t)\}\Big)\big(u_i(t)-u_i^*(t)\big)\\
&-2\sum_{i=1}^{n}\sum_{l=1}^{l_{ii}}\lambda_i B_{iil}^{(1)}(t)\Big(\exp\{\beta_{iil}u_i(t)\}-\exp\{\beta_{iil}u_i^*(t)\}\Big)\big(u_i(t)-u_i^*(t)\big)\\
&+2\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1}^{k_{ji}}\lambda_j A_{jik}^{(1)}(t)\Big(\exp\{\alpha_{jik}u_i(t)\}-\exp\{\alpha_{jik}u_i^*(t)\}\Big)\big(u_j(t)-u_j^*(t)\big)\\
&+2\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{l=1}^{l_{ji}}\lambda_j B_{jil}^{(1)}(t)\Big(\exp\{\beta_{jil}u_i(t)\}-\exp\{\beta_{jil}u_i^*(t)\}\Big)\big(u_j(t)-u_j^*(t)\big)\\
&+2\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_i A_{i\tilde{j}\tilde{k}}^{(1)}(t)\Big(\exp\{\alpha_{i\tilde{j}\tilde{k}}u_{\tilde{j}}(t)\}-\exp\{\alpha_{i\tilde{j}\tilde{k}}u_j^*(t)\}\Big)\\
&\qquad\times\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)\Big(\exp\{\alpha_{ijk}u_j(s)\}-\exp\{\alpha_{ijk}u_j^*(s)\}\Big)ds\\
&+2\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_i A_{i\tilde{j}\tilde{k}}^{(1)}(t)\Big(\exp\{\alpha_{i\tilde{j}\tilde{k}}u_{\tilde{j}}(t)\}-\exp\{\alpha_{i\tilde{j}\tilde{k}}u_j^*(t)\}\Big)\Big]\\
&\qquad\times\Big[\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(\theta-s,s)\Big(\exp\{\beta_{ijl}u_j(s)\}-\exp\{\beta_{ijl}u_j^*(s)\}\Big)d\theta ds
\end{aligned}
$$

$$+2\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_i B_{i\tilde{j}\tilde{l}}^{(1)}(t)\Big(\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_{\tilde{j}}(t)\big\}-\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_{\tilde{j}}^*(t)\big\}\Big)$$
$$\times\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)\Big(\exp\big\{\alpha_{ijk}u_j(s)\big\}-\exp\big\{\alpha_{ijk}u_j^*(s)\big\}\Big)ds$$
$$+2\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_i B_{i\tilde{j}\tilde{l}}^{(1)}(t)\Big(\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_{\tilde{j}}(t)\big\}-\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_{\tilde{j}}^*(t)\big\}\Big)$$
$$\times\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(\theta-s,s)\Big(\exp\big\{\beta_{ijl}u_j(s)\big\}-\exp\big\{\beta_{ijl}u_j^*(s)\big\}\Big)d\theta ds. \tag{4.4}$$

By further using the inequality $a^2+b^2\geq 2ab$, it follows from (4.4) that

$$\dot{V}_1(t)\leq -2\sum_{i=1}^{n}\sum_{k=1}^{k_{ii}}\lambda_i A_{iik}^{(1)}(t)\Big(\exp\big\{\alpha_{iik}u_i(t)\big\}-\exp\big\{\alpha_{iik}u_i^*(t)\big\}\Big)\big(u_i(t)-u_i^*(t)\big)$$
$$-2\sum_{i=1}^{n}\sum_{l=1}^{l_{ii}}\lambda_i B_{iil}^{(1)}(t)\Big(\exp\big\{\beta_{iil}u_i(t)\big\}-\exp\big\{\beta_{iil}u_i^*(t)\big\}\Big)\big(u_i(t)-u_i^*(t)\big)$$
$$+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1}^{k_{ji}}\lambda_j A_{jik}^{(1)}(t)\Big[\Big(\exp\big\{\alpha_{jik}u_i(t)\big\}-\exp\big\{\alpha_{jik}u_i^*(t)\big\}\Big)^2+\big(u_j(t)-u_j^*(t)\big)^2\Big]$$
$$+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{l=1}^{l_{ji}}\lambda_j B_{jil}^{(1)}(t)\Big[\Big(\exp\big\{\beta_{jil}u_i(t)\big\}-\exp\big\{\beta_{jil}u_i^*(t)\big\}\Big)^2+\big(u_j(t)-u_j^*(t)\big)^2\Big]$$
$$+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_i A_{i\tilde{j}\tilde{k}}^{(1)}(t)\Big[\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)ds\Big(\exp\big\{\alpha_{i\tilde{j}\tilde{k}}u_{\tilde{j}}(t)\big\}-\exp\big\{\alpha_{i\tilde{j}\tilde{k}}u_{\tilde{j}}^*(t)\big\}\Big)^2$$
$$+\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)\Big(\exp\big\{\alpha_{ijk}u_j(s)\big\}-\exp\big\{\alpha_{ijk}u_j^*(s)\big\}\Big)^2ds\Big]$$
$$+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_i A_{i\tilde{j}\tilde{k}}^{(1)}(t)\Big[B_{ijl}^{(2)}(t)\Big(\exp\big\{\alpha_{i\tilde{j}\tilde{k}}u_{\tilde{j}}(t)\big\}-\exp\big\{\alpha_{i\tilde{j}\tilde{k}}u_{\tilde{j}}^*(t)\big\}\Big)^2$$
$$+\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(\theta-s,s)\Big(\exp\big\{\beta_{ijl}u_j(s)\big\}-\exp\big\{\beta_{ijl}u_j^*(s)\big\}\Big)^2d\theta ds\Big]$$
$$+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_i B_{i\tilde{j}\tilde{l}}^{(1)}(t)\Big[\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)ds\Big(\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_{\tilde{j}}(t)\big\}-\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_{\tilde{j}}^*(t)\big\}\Big)^2$$
$$+\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)\Big(\exp\big\{\alpha_{ijk}u_j(s)\big\}-\exp\big\{\alpha_{ijk}u_j^*(s)\big\}\Big)^2ds\Big]$$
$$+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_i B_{i\tilde{j}\tilde{l}}^{(1)}(t)\Big[B_{ijl}^{(2)}(t)\Big(\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_{\tilde{j}}(t)\big\}-\exp\big\{\beta_{i\tilde{j}\tilde{l}}u_{\tilde{j}}^*(t)\big\}\Big)^2$$
$$+\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(\theta-s,s)\Big(\exp\big\{\beta_{ijl}u_j(s)\big\}-\exp\big\{\beta_{ijl}u_j^*(s)\big\}\Big)^2d\theta ds\Big]$$

Now let us define the Lyapunov functional $V_2(t)$ as follows

$$V_2(t)=\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_i\int_{t-\tau_{ijk}(t)}^{t}A_{i\tilde{j}\tilde{k}}^{(2)}(s)\big(\Phi_{ijk}^{-1}\big)\int_s^t A_{ijk}^{(1)}(r)$$
$$\times\Big(\exp\big\{\alpha_{ijk}u_j(r)\big\}-\exp\big\{\alpha_{ijk}u_j^*(r)\big\}\Big)^2drds$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_i\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}A_{i\tilde{j}\tilde{k}}^{(1)}(\theta-s)\int_{\theta}^{t}b_{ijl}(r-s,s)\\
&\qquad\times\Big(\exp\big\{\beta_{ijl}u_j(r)\big\}-\exp\big\{\beta_{ijl}u_j^*(r)\big\}\Big)^2drd\theta ds\\
&+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_i\int_{t-\tau_{ijk}(t)}^{t}B_{i\tilde{j}\tilde{l}}^{(1)}\big(\Phi_{ijk}^{-1}(\theta)\big)\int_{\theta}^{t}A_{ijk}^{(1)}(r)\\
&\qquad\times\Big(\exp\big\{\alpha_{ijk}u_j(r)\big\}-\exp\big\{\alpha_{ijk}u_j^*(r)\big\}\Big)^2drd\theta\\
&+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_i\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}B_{i\tilde{j}\tilde{l}}^{(1)}(\theta-s)\int_{\theta}^{t}b_{ijl}(r-s,s)\\
&\qquad\times\Big(\exp\big\{\beta_{ijl}u_j(r)\big\}-\exp\big\{\beta_{ijl}u_j^*(r)\big\}\Big)^2drd\theta ds.
\end{aligned}
$$

Calculating the derivative of $V_2(t)$ along the positive solution of system (1.1), it follows:

$$
\begin{aligned}
\dot{V}_2(t)=&\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_i\int_{t-\tau_{ijk}(t)}^{t}A_{i\tilde{j}\tilde{k}}^{(2)}(s)dsA_{ijk}^{(1)}(t)\\
&\qquad\times\Big(\exp\big\{\alpha_{ijk}u_j(t)\big\}-\exp\big\{\alpha_{ijk}u_j^*(t)\big\}\Big)^2\\
&-\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_iA_{i\tilde{j}\tilde{k}}^{(1)}(t)\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)\\
&\qquad\times\Big(\exp\big\{\alpha_{ijk}u_j(s)\big\}-\exp\big\{\alpha_{ijk}u_j^*(s)\big\}\Big)^2ds\\
&+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_i\big(B_{ijl}^{(2)}\cdot A_{i\tilde{j}\tilde{k}}^{(1)}\big)(t)\Big(\exp\big\{\beta_{ijl}u_j(t)\big\}-\exp\big\{\beta_{ijl}u_j^*(t)\big\}\Big)^2\\
&-\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_iA_{i\tilde{j}\tilde{k}}^{(1)}(t)\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(r-s,s)\\
&\qquad\times\Big(\exp\big\{\beta_{ijl}u_j(r)\big\}-\exp\big\{\beta_{ijl}u_j^*(r)\big\}\Big)^2drds\\
&+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_i\int_{t-\tau_{ijk}(t)}^{t}B_{i\tilde{j}\tilde{l}}^{(1)}\big(\Phi_{ijk}^{-1}(\theta)\big)d\theta A_{ijk}^{(1)}(t)\\
&\qquad\times\Big(\exp\big\{\alpha_{ijk}u_j(t)\big\}-\exp\big\{\alpha_{ijk}u_j^*(t)\big\}\Big)^2\\
&-\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_iB_{i\tilde{j}\tilde{l}}^{(1)}(t)\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(r)\\
&\qquad\times\Big(\exp\big\{\alpha_{ijk}u_j(r)\big\}-\exp\big\{\alpha_{ijk}u_j^*(r)\big\}\Big)^2dr\\
&+\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_i\big(B_{ijl}^{(2)}\cdot B_{i\tilde{j}\tilde{l}}^{(1)}\big)(t)\Big(\exp\big\{\beta_{ijl}u_j(t)\big\}-\exp\big\{\beta_{ijl}u_j^*(t)\big\}\Big)^2\\
&-\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_iB_{i\tilde{j}\tilde{l}}^{(1)}(t)\int_{-\sigma_{ijl}}^{0}\int_{t+s}^{t}b_{ijl}(r-s,s)\\
&\qquad\times\Big(\exp\big\{\beta_{ijl}u_j(r)\big\}-\exp\big\{\beta_{ijl}u_j^*(r)\big\}\Big)^2drds.
\end{aligned}
\tag{4.5}
$$

Finally, we consider the following Lyapunov functional $V(t)$

$$V(t) = V_1(t) + V_2(t). \tag{4.6}$$

Calculating the upper right derivative of $V(t)$ along the solution of system (1.2), and integrating with the above-mentioned analysis, one claims that

$$\begin{aligned}
D^+V(t) \le & -2\sum_{i=1}^{n}\sum_{k=1}^{k_{ii}}\lambda_i A_{iik}^{(1)}(t)\Big(\exp\big\{\alpha_{iik}u_i(t)\big\} - \exp\big\{\alpha_{iik}u_i^*(t)\big\}\Big)\big(u_i(t)-u_i^*(t)\big) \\
& -2\sum_{i=1}^{n}\sum_{l=1}^{l_{ii}}\lambda_i B_{iil}^{(1)}(t)\Big(\exp\big\{\beta_{iil}u_i(t)\big\} - \exp\big\{\beta_{iil}u_i^*(t)\big\}\Big)\big(u_i(t)-u_i^*(t)\big) \\
& +\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{k=1}^{k_{ji}}\lambda_j A_{jik}^{(1)}(t)\Big(\exp\big\{\alpha_{jik}u_i(t)\big\} - \exp\big\{\alpha_{jik}u_i^*(t)\big\}\Big)^2 \\
& +\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{l=1}^{l_{ji}}\lambda_j B_{jil}^{(1)}(t)\Big(\exp\big\{\beta_{jil}u_i(t)\big\} - \exp\big\{\beta_{jil}u_i^*(t)\big\}\Big)^2 \\
& +\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\Big[\sum_{k=1}^{k_{ij}}\lambda_i A_{ijk}^{(1)}(t)\big(u_i(t)-u_i^*(t)\big)^2 + \sum_{l=1}^{l_{ij}}\lambda_i B_{ijl}^{(1)}(t)\big(u_i(t)-u_i^*(t)\big)^2\Big] \\
& +\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{k_{\tilde{j}i}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_{\tilde{j}} A_{\tilde{j}i\tilde{k}}^{(1)}(t)\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)ds\Big(\exp\big\{\alpha_{\tilde{j}i\tilde{k}}u_i(t)\big\} - \exp\big\{\alpha_{\tilde{j}i\tilde{k}}u_i^*(t)\big\}\Big)^2 \\
& +\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{\tilde{j}i}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_{\tilde{j}} A_{\tilde{j}i\tilde{k}}^{(1)}(t)B_{ijl}^{(2)}(t)\Big(\exp\big\{\alpha_{\tilde{j}i\tilde{k}}u_i(t)\big\} - \exp\big\{\alpha_{\tilde{j}i\tilde{k}}u_i^*(t)\big\}\Big)^2 \\
& +\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{\tilde{j}i}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ij}}\lambda_{\tilde{j}} B_{\tilde{j}i\tilde{l}}^{(1)}(t)\int_{t-\tau_{ijk}(t)}^{t}A_{ijk}^{(1)}(s)ds\Big(\exp\big\{\beta_{\tilde{j}i\tilde{l}}u_i(t)\big\} - \exp\big\{\beta_{\tilde{j}i\tilde{l}}u_i^*(t)\big\}\Big)^2 \\
& +\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{\tilde{j}i}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ij}}\lambda_{\tilde{j}} B_{\tilde{j}i\tilde{l}}^{(1)}(t)B_{ijl}^{(2)}(t)\Big(\exp\big\{\beta_{\tilde{j}i\tilde{l}}u_i(t)\big\} - \exp\big\{\beta_{\tilde{j}i\tilde{l}}u_i^*(t)\big\}\Big)^2 \\
& +\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ji}}\lambda_j\int_{t-\tau_{jik}(t)}^{t}A_{i\tilde{j}\tilde{k}}^{(2)}(s)dsA_{jik}^{(1)}(t)\Big(\exp\big\{\alpha_{jik}u_i(t)\big\} - \exp\big\{\alpha_{jik}u_i^*(t)\big\}\Big)^2 \\
& +\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ji}}\lambda_j\big(B_{jil}^{(2)}\cdot A_{i\tilde{j}\tilde{k}}^{(1)}\big)(t)\Big(\exp\big\{\beta_{jil}u_i(t)\big\} - \exp\big\{\beta_{jil}u_i^*(t)\big\}\Big)^2 \\
& +\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{k=1}^{k_{ji}}\lambda_j\int_{t-\tau_{jik}(t)}^{t}B_{i\tilde{j}\tilde{l}}^{(1)}\big(\Phi_{jik}^{-1}(\theta)\big)d\theta A_{jik}^{(1)}(t)\Big(\exp\big\{\alpha_{jik}u_i(t)\big\} - \exp\big\{\alpha_{jik}u_i^*(t)\big\}\Big)^2 \\
& +\sum_{i=1}^{n}\sum_{\tilde{j}=1}^{n}\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\sum_{j=1}^{n}\sum_{l=1}^{l_{ji}}\lambda_j\big(B_{jil}^{(2)}\cdot B_{i\tilde{j}\tilde{l}}^{(1)}\big)(t)\Big(\exp\big\{\beta_{jil}u_i(t)\big\} - \exp\big\{\beta_{jil}u_i^*(t)\big\}\Big)^2.
\end{aligned} \tag{4.7}$$

Meanwhile, by making use of mean value theorem, we can obtain that for any given positive number $\epsilon > 0$, there are

$$\begin{aligned}
&\exp\big\{\epsilon u_i(t)\big\} - \exp\big\{\epsilon u_i^*(t)(t)\big\} = \epsilon\exp\{\epsilon\vartheta_i^{(1)}(t)\}\big(u_i(t)-u_i^*(t)\big), \\
&\exp\big\{\epsilon u_i(t)\big\} - \exp\big\{\epsilon u_i^*(t)\big\} = \frac{\epsilon}{\alpha_{iik}}\exp\{\epsilon\vartheta_i^{(2)}(t)\} \\
&\qquad\qquad\times\big(\exp\big\{\alpha_{iik}u_i(t)\big\} - \exp\big\{\alpha_{iik}u_i^*(t)\big\}\big), \\
&\exp\big\{\epsilon u_i(t)\big\} - \exp\big\{\epsilon u_i^*(t)\big\} = \frac{\epsilon}{\beta_{iil}}\exp\{\epsilon\vartheta_i^{(3)}(t)\}
\end{aligned}$$

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$$\times\big(\exp\{\beta_{iil}u_i(t)\}-\exp\{\beta_{iil}u_i^*(t)\}\big). \tag{4.8}$$

Where $\vartheta_i^{(1)}(t),\ \vartheta_i^{(2)}(t),\ \vartheta_i^{(3)}(t)$ are all lie between $u_i(t)$ and $u_i^*(t)$. Thus, it follows from (4.2) and (4.8) that for any given positive number $\epsilon>0$, we have

$$\begin{aligned}
&\exp\{\epsilon u_i(t)\}-\exp\{\epsilon u_i^*(t)(t)\}\ge \epsilon m_{i0}^{\epsilon}\big(u_i(t)-u_i^*(t)\big),\\
&\exp\{\epsilon u_i(t)\}-\exp\{\epsilon u_i^*(t)(t)\}\le \epsilon M_{i0}^{\epsilon}\big(u_i(t)-u_i^*(t)\big).
\end{aligned}\tag{4.9}$$

$$\begin{aligned}
\exp\{\epsilon u_i(t)\}-\exp\{\epsilon u_i^*(t)\}\ge{}& \frac{\epsilon}{\alpha_{iik}}m_{i0}^{\epsilon}\\
&\times\big(\exp\{\alpha_{iik}u_i(t)\}-\exp\{\alpha_{iik}u_i^*(t)\}\big),\\
\exp\{\epsilon u_i(t)\}-\exp\{\epsilon u_i^*(t)\}\le{}& \frac{\epsilon}{\alpha_{iik}}M_{i0}^{\epsilon}\\
&\times\Big(\exp\{\alpha_{iik}u_i(t)\}-\exp\{\alpha_{iik}u_i^*(t)\}\Big).
\end{aligned}\tag{4.10}$$

and

$$\begin{aligned}
\exp\{\epsilon u_i(t)\}-\exp\{\epsilon u_i^*(t)\}\ge{}& \frac{\epsilon}{\beta_{iil}}m_{i0}^{\epsilon}\\
&\times\big(\exp\{\beta_{iil}u_i(t)\}-\exp\{\beta_{iil}u_i^*(t)\}\big),\\
\exp\{\epsilon u_i(t)\}-\exp\{\epsilon u_i^*(t)\}\le{}& \frac{\epsilon}{\beta_{iil}}M_{i0}^{\epsilon}\\
&\times\Big(\exp\{\beta_{iil}u_i(t)\}-\exp\{\beta_{iil}u_i^*(t)\}\Big).
\end{aligned}\tag{4.11}$$

Inequality (4.7), (4.9), (4.10) and (4.11) implies that for $t\ge T_1$

$$\begin{aligned}
D^+V(t)\le{}&\sum_{i=1}^{n}\Big\{\sum_{k=1}^{k_{ii}}-2\lambda_iA_{iik}^{(1)}(t)+\sum_{j=1,j\neq i}^{n}\Big[\sum_{k=1}^{k_{ji}}\frac{\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}}{\alpha_{iik}}A_{jik}^{(1)}(t)+\sum_{k=1}^{k_{ij}}\frac{\lambda_i}{\alpha_{iik}m_{i0}^{\alpha_{iik}}}A_{ijk}^{(1)}(t)\Big]\\
&+\sum_{j=1}^{n}\sum_{\tilde j=1}^{n}\Big[\sum_{k=1}^{k_{ji}}\frac{\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}}{\alpha_{iik}}A_{jik}^{(1)}(t)\Big(\sum_{\tilde k=1}^{\tilde k_{i\tilde j}}\int_{t-\tau_{i\tilde j\tilde k}(t)}^{t}A_{i\tilde j\tilde k}^{(1)}(s)ds+\sum_{\tilde k=1}^{\tilde k_{i\tilde j}}\int_{t-\tau_{jik}(t)}^{t}A_{i\tilde j\tilde k}^{(2)}(s)ds\Big)\\
&+\sum_{k=1}^{k_{ji}}\frac{\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}}{\alpha_{iik}}A_{jik}^{(1)}(t)\Big(\sum_{\tilde l=1}^{\tilde l_{i\tilde j}}\int_{t-\tau_{jik}(t)}^{t}B_{i\tilde j\tilde l}^{(1)}\big(\Phi_{jik}^{-1}(\theta)\big)d\theta+\sum_{l=1}^{l_{i\tilde j}}B_{i\tilde j\tilde l}^{(2)}(t)\Big)\Big]\Big\}\\
&\times\big(\exp\{\alpha_{iik}u_i(t)\}-\exp\{\alpha_{iik}u_i^*(t)\}\big)\big(u_i(t)-u_i^*(t)\big)\\
&+\sum_{i=1}^{n}\Big\{-2\sum_{l=1}^{l_{ii}}\lambda_iB_{iil}^{(1)}(t)+\sum_{j=1,j\neq i}^{n}\Big[\sum_{l=1}^{l_{ji}}\frac{\lambda_j\beta_{jil}^2M_{i0}^{2\beta_{jil}}}{\beta_{iil}}B_{jil}^{(1)}(t)+\sum_{l=1}^{l_{ij}}\frac{\lambda_i}{\beta_{iil}m_{i0}^{\beta_{iil}}}B_{ijl}^{(1)}(t)\Big]\\
&+\sum_{j=1}^{n}\sum_{\tilde j=1}^{n}\Big[\sum_{l=1}^{l_{ji}}\frac{\lambda_j\beta_{jil}^2M_{i0}^{2\beta_{jil}}}{\beta_{iil}}B_{jil}^{(1)}(t)\Big(\sum_{k=1}^{k_{i\tilde j}}\int_{t-\tau_{i\tilde j k}(t)}^{t}A_{i\tilde j k}^{(1)}(s)ds+\sum_{\tilde l=1}^{\tilde l_{i\tilde j}}B_{jil}^{(1)}(t)B_{i\tilde j\tilde l}^{(2)}(t)\Big)\\
&+\sum_{l=1}^{l_{ji}}\frac{\lambda_j\beta_{jil}^2M_{i0}^{2\beta_{jil}}}{\beta_{iil}}\Big(\sum_{\tilde k=1}^{\tilde k_{i\tilde j}}\big(B_{jil}^{(2)}\cdot A_{i\tilde j\tilde k}^{(1)}\big)(t)+\sum_{\tilde l=1}^{\tilde l_{i\tilde j}}\big(B_{jil}^{(2)}\cdot B_{i\tilde j\tilde l}^{(1)}\big)(t)\Big)\Big]\Big\}\\
&\times\big(\exp\{\beta_{iil}u_i(t)\}-\exp\{\beta_{iil}u_i^*(t)\}\big)\big(u_i(t)-u_i^*(t)\big).\\
=:{}&-\sum_{i=1}^{n}\Lambda_i(t)\mid\big(\exp\{\alpha_{iik}u_i(t)\}-\exp\{\alpha_{iik}u_i^*(t)\}\big)\big(u_i(t)-u_i^*(t)\big)\mid\\
&-\sum_{i=1}^{n}\Delta_i(t)\mid\big(\exp\{\beta_{iil}u_i(t)\}-\exp\{\beta_{iil}u_i^*(t)\}\big)\big(u_i(t)-u_i^*(t)\big)\mid.
\end{aligned}\tag{4.12}$$

At the same time, according to hypotheses (H_6) of Theorem 4.1, we declare that there exists a constant $\zeta>0$ such that $\Lambda_i(t),\ \Delta_i(t)>\zeta$, so it follows from (4.12) that $V(t)$ is nonincreasing, and it not difficult to see that $\dot u_i(t)$ are bounded for $t\ge T_1$. Hence, one can further infer that $\mid u_i(t)-u_i^*(t)\mid,\ \mid\exp\{\alpha_{iik}u_i(t)\}-\exp\{\alpha_{iik}u_i^*(t)\}\mid,\ \mid\exp\{\beta_{iil}u_i(t)\}-\exp\{\beta_{iil}u_i^*(t)\}\mid$ are uniformly continuous on $[T_1, +\infty)$. An integration on both sides of (4.10) over time interval $[T_1, t)$ leads to

$$V(t) + \zeta \sum_{i=1}^{n} \int_{T_1}^{t} \Big[\mid \big(\exp\big\{\alpha_{iik} u_i(s)\big\} - \exp\big\{\alpha_{iik} u_i^*(s)\big\}\big)\big(u_i(s) - u_i^*(s)\big) \mid + \mid \big(\exp\big\{\beta_{iil} u_i(s)\big\} - \exp\big\{\beta_{iil} u_i^*(s)\big\}\big)\big(u_i(s) - u_i^*(s)\big) \mid \Big] ds \le V(T_1) < +\infty.$$

Thus

$$\limsup_{t\to+\infty} \sum_{i=1}^{n} \int_{T_1}^{t} \Big[\mid \big(\exp\big\{\alpha_{iik} u_i(s)\big\} - \exp\big\{\alpha_{iik} u_i^*(s)\big\}\big)\big(u_i(s) - u_i^*(s)\big) \mid + \mid \big(\exp\big\{\beta_{iil} u_i(s)\big\} - \exp\big\{\beta_{iil} u_i^*(s)\big\}\big)\big(u_i(s) - u_i^*(s)\big) \mid \Big] ds \le \frac{V(T_1)}{\zeta} < +\infty. \tag{4.13}$$

It follows from (4.13) that

$$\mid \big(\exp\big\{\alpha_{iik} u_i(s)\big\} - \exp\big\{\alpha_{iik} u_i^*(s)\big\}\big)\big(u_i(s) - u_i^*(s)\big) \mid \in L[T_1, +\infty),$$

$$\mid \big(\exp\big\{\beta_{iil} u_i(s)\big\} - \exp\big\{\beta_{iil} u_i^*(s)\big\}\big)\big(u_i(s) - u_i^*(s)\big) \mid \in L[T_1, +\infty).$$

According to Barbalat's lemma, we conclude that

$$\lim_{t\to+\infty} \mid \big(\exp\big\{\alpha_{iik} u_i(t)\big\} - \exp\big\{\alpha_{iik} u_i^*(t)\big\}\big)\big(u_i(t) - u_i^*(t)\big) \mid = 0. \tag{4.14}$$

$$\lim_{t\to+\infty} \mid \big(\exp\big\{\beta_{iil} u_i(t)\big\} - \exp\big\{\beta_{iil} u_i^*(t)\big\}\big)\big(u_i(t) - u_i^*(t)\big) \mid = 0. \tag{4.15}$$

By way of contradiction, it easy to obtain from (4.14) and (4.15) that

$$\lim_{t\to+\infty} \mid u_i(t) - u_i^*(t) \mid = 0. \tag{4.16}$$

Therefore, the positive solution $X^*(t)$ of the system (1.1) is also globally attractive. This completes the proof.

Theorem 4.2. In addition to $(H_1) - (H_5)$, we assume further that
$(H_6)'$ There exist positive constants $\lambda_i > 0$ $(i = 1, 2, ..., n)$, $\zeta > 0$ such that

$$\liminf_{t\to+\infty} \big\{\Lambda_i(t)\big\} > \zeta.$$

Where

$$\begin{aligned}
\Lambda_i(t) = {} & 2\sum_{k=1}^{k_{ii}} \lambda_i \alpha_{iik} m_{i0}^{\alpha_{iik}} A_{iik}^{(1)}(t) - \sum_{j=1, j\neq i}^{n} \Big[\sum_{k=1}^{k_{ji}} \lambda_j \alpha_{jik}^2 M_{i0}^{2\alpha_{jik}} A_{jik}^{(1)}(t) + \sum_{k=1}^{k_{ij}} \lambda_i A_{ijk}^{(1)}(t) \Big] \\
& + 2\sum_{l=1}^{l_{ii}} \lambda_i \beta_{iil} m_{i0}^{\beta_{iil}} B_{iil}^{(1)}(t) - \sum_{j=1, j\neq i}^{n} \Big[\sum_{l=1}^{l_{ji}} \lambda_j B_{jil}^{(1)}(t) \beta_{jil}^2 M_{i0}^{2\beta_{jil}} + \sum_{l=1}^{l_{ij}} \lambda_i B_{ijl}^{(1)}(t) \Big] \\
& - \sum_{j=1}^{n} \sum_{\tilde{j}=1}^{n} \Big[\sum_{k=1}^{k_{ji}} \lambda_j \alpha_{jik}^2 M_{i0}^{2\alpha_{jik}} A_{jik}^{(1)}(t) \Big(\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}} \int_{t-\tau_{i\tilde{j}\tilde{k}}(t)}^{t} A_{i\tilde{j}\tilde{k}}^{(1)}(s) ds + \sum_{l=1}^{l_{i\tilde{j}}} B_{i\tilde{j}l}^{(2)}(t) \Big) \\
& + \sum_{k=1}^{k_{ji}} \lambda_j \alpha_{jik}^2 M_{i0}^{2\alpha_{jik}} A_{jik}^{(1)}(t) \Big(\sum_{\tilde{k}=1}^{\tilde{k}_{\tilde{j}i}} \int_{t-\tau_{jik}(t)}^{t} A_{i\tilde{j}\tilde{k}}^{(2)}(s) ds + \sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}} \int_{t-\tau_{jik}(t)}^{t} B_{i\tilde{j}\tilde{l}}^{(1)}\big(\Phi_{jik}^{-1}(\theta)\big) d\theta \Big) \\
& + \sum_{l=1}^{l_{ji}} \lambda_j \beta_{jil}^2 M_{i0}^{2\beta_{jil}} B_{jil}^{(1)}(t) \Big(\sum_{k=1}^{k_{i\tilde{j}}} \int_{t-\tau_{i\tilde{j}k}(t)}^{t} A_{i\tilde{j}k}^{(1)}(s) ds + \sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}} B_{i\tilde{j}\tilde{l}}^{(2)}(t) \Big) \\
& + \sum_{l=1}^{l_{ji}} \lambda_j \beta_{jil}^2 M_{i0}^{2\beta_{jil}} \Big(\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}} (B_{jil}^{(2)} \cdot A_{i\tilde{j}\tilde{k}}^{(1)})(t) + \sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}} (B_{jil}^{(2)} \cdot B_{i\tilde{j}\tilde{l}}^{(1)})(t) \Big) \Big].
\end{aligned}$$

Then the solution $X^*(t) = (x_1^*(t), x_2^*(t), ..., x_n^*(t))$ of $(1.1) - (1.2)$ is globally attractive.

Proof. Let $U^*(t) = (u_1^*(t),\ ...,\ u_n^*(t)$ be the solution of system (4.1), and $U(t) = (u_1(t)\ , ...,\ u_n(t))$

be any other positive solution of system (4.1). Then for the Lyapunov functional $V(t)$ as defined in (4.6), similarly to the discuss of Theorem 4.1, one can obtain that the inequality (4.7) is true. By further making use of (4.9), (4.10) and (4.11), it follows that (4.7) implies

$$\begin{aligned}
D^+V(t) \le & \sum_{i=1}^{n}\Big\{-2\sum_{k=1}^{k_{ii}}\lambda_i\alpha_{iik}m_{i0}^{\alpha_{iik}}A_{iik}^{(1)}(t)+\sum_{j=1,j\neq i}^{n}\Big[\sum_{k=1}^{k_{ji}}\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}A_{jik}^{(1)}(t)+\sum_{k=1}^{k_{ij}}\lambda_iA_{ijk}^{(1)}(t)\Big] \\
& -2\sum_{l=1}^{l_{ii}}\lambda_i\beta_{iil}m_{i0}^{\beta_{iil}}B_{iil}^{(1)}(t)+\sum_{j=1,j\neq i}^{n}\Big[\sum_{l=1}^{l_{ji}}\lambda_jB_{jil}^{(1)}(t)\beta_{jil}^2M_{i0}^{2\beta_{jil}}+\sum_{l=1}^{l_{ij}}\lambda_iB_{ijl}^{(1)}(t)\Big] \\
& +\sum_{j=1}^{n}\sum_{\tilde{j}=1}^{n}\Big[\sum_{k=1}^{k_{ji}}\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}A_{jik}^{(1)}(t)\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}\int_{t-\tau_{i\tilde{j}\tilde{k}}(t)}^{t}A_{i\tilde{j}\tilde{k}}^{(1)}(s)ds+\sum_{l=1}^{l_{i\tilde{j}}}B_{i\tilde{j}l}^{(2)}(t)\Big) \\
& +\sum_{k=1}^{k_{ji}}\lambda_j\alpha_{jik}^2M_{i0}^{2\alpha_{jik}}A_{jik}^{(1)}(t)\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{\tilde{j}i}}\int_{t-\tau_{jik}(t)}^{t}A_{i\tilde{j}\tilde{k}}^{(2)}(s)ds+\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}\int_{t-\tau_{jik}(t)}^{t}B_{i\tilde{j}\tilde{l}}^{(1)}\big(\Phi_{jik}^{-1}(\theta)\big)d\theta\Big) \\
& +\sum_{l=1}^{l_{ji}}\lambda_j\beta_{jil}^2M_{i0}^{2\beta_{jil}}B_{jil}^{(1)}(t)\Big(\sum_{k=1}^{k_{i\tilde{j}}}\int_{t-\tau_{i\tilde{j}k}(t)}^{t}A_{i\tilde{j}k}^{(1)}(s)ds+\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}B_{i\tilde{j}\tilde{l}}^{(2)}(t)\Big) \\
& +\sum_{l=1}^{l_{ji}}\lambda_j\beta_{jil}^2M_{i0}^{2\beta_{jil}}\Big(\sum_{\tilde{k}=1}^{\tilde{k}_{i\tilde{j}}}(B_{jil}^{(2)}\cdot A_{i\tilde{j}\tilde{k}}^{(1)})(t)+\sum_{\tilde{l}=1}^{\tilde{l}_{i\tilde{j}}}(B_{jil}^{(2)}\cdot B_{i\tilde{j}\tilde{l}}^{(1)})(t)\Big)\Big]\Big\}\big(u_i(t)-u_i^*(t)\big)^2. \\
& =: -\sum_{i=1}^{n}\Lambda_i(t)\big(u_i(t)-u_i^*(t)\big)^2
\end{aligned} \tag{4.17}$$

An integration on both sides of (4.17) over time interval $[T_1, t)$ leads to

$$V(t)+\zeta\sum_{i=1}^{n}\int_{T_1}^{t}\big(u_i(s)-u_i^*(s)\big)^| 2ds \le V(T_1) < +\infty.$$

Thus

$$\limsup_{t\to+\infty}\sum_{i=1}^{n}\int_{T_1}^{t}\big(u_i(s)-u_i^*(s)\big)^| 2ds \le \frac{V(T_1)}{\zeta} < +\infty. \tag{4.18}$$

It follows from (4.18) that

$$\big(u_i(s)-u_i^*(s)\big)^2 \in L[T_1,+\infty),$$

According to Barbalat's lemma, we conclude that

$$\lim_{t\to+\infty}\big(u_i(t)-u_i^*(t)\big)^2=0. \tag{4.19}$$

Taking into account the fact that for $t \ge T_1$

$$\big(x_i(t)-x_i^*(t)\big)=\exp\big\{u_i(t)\big\}-\exp\big\{u_i^*(t)\big\}$$

One infers that

$$(m_{i0}) \mid u_i(t)-u_i^*(t) \mid \le \mid x_i(t)-x_i^*(t) \mid \le (M_{i0}) \mid u_i(t)-u_i^*(t) \mid$$

So it follows that

$$\lim_{t\to+\infty} \mid x_i(t)-x_i^*(t) \mid = 0. \tag{4.20}$$

Thus, we have verified that the positive solution $X^*(t)$ of the system (1.1) is globally attractive.

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Some approximations of the Bateman's $G-$function

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Abstract

In the paper, we presented a family $M(\mu,x)$ of approximations of the Bateman function $G(x)$. The family $M(\mu,x)=G(x)$ for a certain μ whenever x is fixed and it presented asymptotical approximation of the Bateman's $G-$function as $x\to\infty$. We studied the order of convergence of the approximations $M(\mu,x)$ of the function $G(x)$. Some properties and bounds of the error are deduced. We presented new sharp double inequality of $G(x)$ with the upper and lower bounds $M(1,x)$ and $M(\frac{4}{e^2-4},x)$ (resp.). Also, we show that the approximations $M(\mu,x)$ are better than the approximation $\frac{1}{x}+\frac{1}{2x^2}$ for any μ in an open subinterval of $\left[1,\frac{4}{e^2-4}\right]$.

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Key Words: Bateman function, digamma function, monotonicity, sharp inequality, approximation, error.

1 Introduction.

In 1953, Erdélyi [6] defined the Bateman's $G-$function as

$$G(x)=\psi\left(\frac{x+1}{2}\right)-\psi\left(\frac{x}{2}\right),\qquad x\neq 0,-1,-2,... \tag{1}$$

where the digamma function $\psi(x)$ is given by

$$\psi(x)=\frac{d}{dx}\log\Gamma(x)$$

and $\Gamma(x)$ is the ordinary gamma function defined by [3]

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \qquad x > 0.$$

The function $G(x)$ is very useful in estimating and summing certain numerical and algebraic series [18]. For more details on bounding the function $\Gamma(x)$ and its logarithmic derivatives $\psi^{(n)}(x)$, please refer to the papers [2]-[5], [7]-[23] and plenty of references therein.

The function $G(x)$ can be also defined by

$$G(x) = \frac{2}{x} \, {}_2F_1(1, x; 1 + x; -1),$$

where

$${}_rF_s(a_1, ..., a_r; b_1, ..., b_s; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k ... (a_r)_k}{(b_1)_k ... (b_s)_k} \frac{x^k}{k!}$$

is the generalized hypergeometric series [1] defined for $r, s \in \mathbb{N}$, $a_j \in \mathbb{C}$, $b_j \in \mathbb{C} - \{0, -1, -2, ...\}$ and the Pochhammer symbol $(a)_n$ is defined by

$$(a)_0 = 1 \quad \text{and} \quad (a)_n = \prod_{i=0}^{n-1} (a + i) = \frac{\Gamma(a + n)}{\Gamma(a)}, \qquad n \geq 1.$$

The function $G(x)$ satisfies the functional equation [6]:

$$G(1 + x) = -G(x) + \frac{2}{x} \tag{2}$$

and it has the integral representation

$$G(x) = 2 \int_0^\infty \frac{e^{-xt}}{1 + e^{-t}} dt, \qquad x > 0 \tag{3}$$

which can be deduced from the following known integral representation of the digamma [3]

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}} \right) dt, \qquad x > 0.$$

Qiu and Vuorinen [24] deduced the inequality

$$\frac{1}{x} + \frac{4(1.5 - \log 4)}{x^2} < G(x) < \frac{1}{x} + \frac{1}{2x^2}, \qquad x > 1/2. \tag{4}$$

Mahmoud and Agarwal [9] presented the following asymptotic formula for Bateman's G-function

$$G(x) \sim \frac{1}{x} + \sum_{k=1}^{\infty} \frac{(2^{2k} - 1) B_{2k}}{k x^{2k}}, \qquad x \to \infty \tag{5}$$

and they deduced the double inequality

$$\frac{1}{2x^2+1.5} < G(x) - \frac{1}{x} < \frac{1}{2x^2}, \qquad x > 0 \tag{6}$$

which improve the lower bound of the inequality (4). Also, Mahmoud and Almuashi [11] proved that the Bateman's $G-$function satisfies the double inequality

$$\sum_{n=1}^{2m} \frac{(2^n-1)B_{2n}}{nx^{2n}} < G(x) - \frac{1}{x} < \sum_{n=1}^{2m-1} \frac{(2^n-1)B_{2n}}{nx^{2n}}, \qquad m \in \mathbb{N} \tag{7}$$

with best bounds, where $B_r{}'s$ are the Bernoulli numbers and they presented some estimates for the error term of a class of the alternating series, which improve and generalize some recent results. Mortici [13] established the inequality

$$0 < \psi(x+v) - \psi(x) \leq \psi(v) + \gamma + \frac{1}{v} - v \qquad x \geq 1;\ 0 < v < 1, \tag{8}$$

where γ is the Euler constant, which also improves the inequality (4) of Qiu and Vuorinen. Also, Alzer presented the double inequality [2]

$$\frac{1}{x} - T_n(v;x) - \rho_n(v;x) < \psi(x+v) - \psi(x) < \frac{1}{x} - T_n(v;x),$$

where $n \geq 0$ be an integer, $x > 0$, $0 < v < 1$,

$$T_n(v;x) = (1-v)\left[\frac{1}{v+n+1} + \sum_{i=0}^{n-1} \frac{1}{(x+i+1)(x+i+v)}\right]$$

and

$$\rho_n(v;x) = \frac{1}{x+n+v} \log \frac{(x+n)^{(x+n)(1-v)}(x+n+1)^{(x+n+1)v}}{(x+n+v)^{x+n+v}}.$$

In 2006, Muqattash and Yahdi [17] presented an infinite family of functions $I_a(x) = \psi(x)$ for a certain a when x is fixed. Local and global bounding error functions are found and new inequalities for the Digamma function are introduced. These functions are shown to approximate ψ locally and asymptotically. The approximations are compared to another approximations of the Digamma function. The technique of construct of Muqattash and Yahdi is very useful and can be updated to another functions as we will see in this paper.

In 2014, Guo and Qi improved the results of [8] and presented the two sharp inequalities

$$\ln\left(x + \frac{1}{2}\right) < \psi(x) + \frac{1}{x} < \ln\left(x + e^{-\gamma}\right), \qquad x > 0$$

where the constants $\frac{1}{2}$ and $e^{-\gamma}$ are the best possible, and

$$\ln\left(n + \frac{1}{2}\right) + \gamma < H_n(n) < \ln\left(n + e^{1-\gamma} - 1\right) + \gamma, \qquad n \in \mathbb{N}$$

where the n-th harmonic numbers are defined by

$$H_n = \sum_{i=1}^{n} \frac{1}{i}, \qquad n \in \mathbb{N}$$

and is related to the Psi function by the relation

$$H_n = \gamma + \psi(n+1).$$

In this paper, we presented a family of functions $M(\mu, x)$ satisfies that for all $x > 0$ there exists $\mu \in [1, 2]$ such that $M(\mu, x) = G(x)$ and is asymptotically equivalent to $G(x)$ as $x \to \infty$. We proved that the approximations $M(\mu, x)$ of the function $G(x)$ are of an order of convergence of $O\left(\ln \frac{(x+2)[(e^2-4)x+4]}{(x+1)[(e^2-4)x+e^2]}\right)$ for $x > 2$ and $\mu \in \left(1, \frac{4}{e^2-4}\right)$. Some properties and bounds of the error are deduced. Also, we presented a new sharp double inequality of the function $G(x)$ between the lower bound $M(\frac{4}{e^2-4}, x)$ and the upper bound $M(1, x)$. We proved that the approximations $M(\mu, x)$ are better than the approximation $\frac{1}{x} + \frac{1}{2x^2}$ for any μ in an open subinterval of $\left[1, \frac{4}{e^2-4}\right]$.

2 Main Results

Lemma 2.1. *For $x > 0$, we have*

$$\ln\left(1 + \frac{1}{x+2}\right) + \frac{2}{x(x+1)} \le G(x) \le \ln\left(1 + \frac{1}{x+1}\right) + \frac{2}{x(x+1)}. \tag{9}$$

Proof. Consider the function

$$H_\mu(x) = \ln\left(1 + \frac{1}{x+\mu}\right) + \frac{2}{x(x+1)} - G(x), \qquad x > 0;\ \mu > 0$$

which can be represented using (3) by the integral formula

$$H_\mu(x) = \int_0^\infty \frac{e^{-(\mu+1)t}[e^{2t} - 1 - 2te^{\mu t}]}{t(1+e^t)} e^{-xt} dt.$$

The function $m_1(t) = e^{2t} - 1 - 2te^t$ is strictly increasing pass through the origin, then $H_1(x) > 0$, that is

$$\ln\left(1 + \frac{1}{x+1}\right) + \frac{2}{x(x+1)} > G(x).$$

Also, $m_2(t) = e^{2t} - 1 - 2te^{2t}$ is strictly decreasing function pass through the origin, then $H_2(x) < 0$, that is

$$\ln\left(1 + \frac{1}{x+2}\right) + \frac{2}{x(x+1)} < G(x).$$

□

The double inequality (9) show that the function $G(x)$ lies between two functions of the following family of functions

$$M(\mu, x) = \ln\left(1 + \frac{1}{x+\mu}\right) + \frac{2}{x(x+1)} \qquad x > 0;\ \mu > 0. \tag{10}$$

and hence we can conclude the following result:

Theorem 1. *For every $x > 0$, there exists $\mu \in [1, 2]$ such that*

$$M(\mu, x) = G(x).$$

Proof. For a positive fixed x, consider the function $M_2(\mu) = M(\mu, x)$ with $1 \le \mu \le 2$ and $G(x) = \lambda$. $M_2(\mu)$ is a continuous on $[1, 2]$ and using the inequality (9), we obtain

$$M_2(2) \le \lambda \le M_2(1).$$

Then by the Intermediate Value Theorem, there exists $\mu \in [1, 2]$ such that $M_2(\mu) = \lambda$. □

Also, by using the relations

$$\frac{\partial M(\mu, x)}{\partial x} = -\frac{2\mu + 2\mu^2 + 2x + 8\mu x + 4\mu^2 x + 7x^2 + 8\mu x^2 + 6x^3 + x^4}{x^2(1+x)^2(\mu + \mu^2 + x + 2\mu x + x^2)} < 0$$

and

$$\frac{\partial M(\mu, x)}{\partial \mu} = \frac{-1}{(x+\mu+1)(x+\mu)} < 0,$$

we obtain the following properties of the family $M(\mu, x)$.

Lemma 2.2.

1. *$M_1(x) = M(\mu, x)$ is a positive and strictly decreasing as a function of x, $x > 0$.*
2. *$M_2(\mu) = M(\mu, x)$ is strictly decreasing as a function of μ, $1 \le \mu \le 2$*

and hence

$$0 < M(2, x) \le M(\mu, x) \le M(1, x), \qquad x > 0;\ \mu \in [1, 2]. \tag{11}$$

Now, we will show that the family $M(\mu, x)$ presented asymptotical approximation of the Bateman's $G-$function for all $\mu \in [1, 2]$.

Theorem 2. *For all $\mu \in [1, 2]$, the Bateman's $G-$function and the family $M(\mu, x)$ are asymptotically equivalent as $x \to \infty$, that is*

$$\lim_{x\to\infty} \frac{G(x)}{M(\mu, x)} = 1$$

and this is written symbolically as $G(x) \sim M(\mu, x)$.

Proof. Using the inequality (9), we get

$$M(2,x) \le G(x) \le M(1,x) \tag{12}$$

and hence

$$\frac{M(2,x)}{M(1,x)} \le \frac{G(x)}{M(1,x)} \le 1.$$

But

$$\lim_{x\to\infty} \frac{M(2,x)}{M(1,x)} = \frac{12+34x+23x^2+6x^3+x^4}{(3+x)(4+10x+5x^2+x^3)} = 1$$

and then

$$\lim_{x\to\infty} \frac{G(x)}{M(1,x)} = 1. \tag{13}$$

Similarly, we have

$$\lim_{x\to\infty} \frac{G(x)}{M(2,x)} = 1. \tag{14}$$

Using the inequality (11), we obtain

$$\frac{G(x)}{M(1,x)} \le \frac{G(x)}{M(\mu,x)} \le \frac{G(x)}{M(2,x)}. \tag{15}$$

From (13), (14) and (15), we get

$$1 \le \lim_{x\to\infty} \frac{G(x)}{M(\mu,x)} \le 1.$$

□

Now, we will study the error of the approximation $M(\mu,x)$ of the function $G(x)$.

Theorem 3. *For any $\mu \in [1,2]$, the error*

$$e_\mu(x) = G(x) - M(\mu,x)$$

approaches zero as $x \to \infty$ and

$$G(x) = \ln\left(1+\frac{1}{x+\mu}\right) + \frac{2}{x(x+1)} + \mathrm{O}\left(\ln\left(1+\frac{1}{(x+1)(x+3)}\right)\right). \tag{16}$$

Proof. From inequality (12), we have

$$M(2,x) - M(\mu,x) \le G(x) - M(\mu,x) \le M(1,x) - M(\mu,x)$$

and using (11), we get

$$M(2,x) - M(1,x) \le M(2,x) - M(\mu,x).$$

Hence

$$0 \le |G(x) - M(\mu,x)| \le M(1,x) - M(2,x) \tag{17}$$

or

$$0 \le |e_\mu(x)| \le \ln\left(1+\frac{1}{(x+1)(x+3)}\right). \tag{18}$$

Then

$$G(x) = M(\mu, x) + \mathrm{O}\left(\ln\left(1+\frac{1}{(x+1)(x+3)}\right)\right)$$

and

$$\lim_{x\to\infty} e_\mu(x) = 0.$$

□

As a consequence of the above result, we obtain some bounds of the error $e_\mu(x)$.

Corollary 2.3. *The error $e_\mu(x)$ is uniformly bounded by $\pm\ln\left(1+\frac{1}{(\varepsilon+1)(\varepsilon+3)}\right)$ $\forall\ x>\varepsilon>0$ and $\forall\ \mu\in[1,2]$.*

Proof. Using the inequality (18), we obtain

$$\sup_{0<x<\infty} |e_\mu(x)| \le \ln\left(1+\frac{1}{(x+1)(x+3)}\right).$$

Also, the function $g(x)=\ln\left(1+\frac{1}{(x+1)(x+3)}\right)$ for $x>0$ is decreasing. Then the errors $e_\mu(x)$ are uniformly bounded between $-\ln\left(1+\frac{1}{(\varepsilon+1)(\varepsilon+3)}\right)$ and $\ln\left(1+\frac{1}{(\varepsilon+1)(\varepsilon+3)}\right)$. □

3 The best bounds of the double inequality (9).

Firstly, we will prove the following auxiliary results:

Lemma 3.1.

$$\lim_{x\to\infty}\left(\frac{1}{e^{G(x+2)}-1}-x\right)=1 \tag{19}$$

and

$$\lim_{x\to\infty}\frac{G'(x+2)e^{G(x+2)}}{(e^{G(x+2)}-1)^2}=-1. \tag{20}$$

Proof. Using the double inequality (6) with

$$\beta(x)=\frac{1}{x}+\frac{1}{2x^2+3/2} \quad \text{and} \quad \alpha(x)=\frac{1}{x}+\frac{1}{2x^2},$$

we get

$$\lim_{x\to\infty}\left(\frac{1}{e^{\alpha(x+2)}-1}-x\right) \le \lim_{x\to\infty}\left(\frac{1}{e^{G(x+2)}-1}-x\right) \le \lim_{x\to\infty}\left(\frac{1}{e^{\beta(x+2)}-1}-x\right).$$

But

$$\lim_{x\to\infty}\left(\frac{1}{e^{\alpha(x+2)}-1}-x\right)=\lim_{x\to\infty}\left(\frac{1}{\left[1+\frac{1}{x}-\frac{1}{x^2}+\frac{2}{3x^3}+\frac{5}{12x^4}-O\left(\frac{1}{x^5}\right)\right]-1}-x\right)=1$$

and

$$\lim_{x\to\infty}\left(\frac{1}{e^{\beta(x+2)}-1}-x\right)=\lim_{x\to\infty}\left(\frac{1}{\left[1+\frac{1}{x}-\frac{1}{x^2}+\frac{2}{3x^3}+\frac{1}{24x^4}-O\left(\frac{1}{x^5}\right)\right]-1}-x\right)=1.$$

Also, using the double inequality (6), we have

$$\lim_{x\to\infty}\frac{G'(x+2)e^{\alpha(x+2)}}{(e^{\beta(x+2)}-1)^2}\le\lim_{x\to\infty}\frac{G'(x+2)e^{G(x+2)}}{(e^{G(x+2)}-1)^2}\le\lim_{x\to\infty}\frac{G'(x+2)e^{\beta(x+2)}}{(e^{\alpha(x+2)}-1)^2}.$$

Now, using the asymptotic formula for Bateman's G-function (5), we obtain

$$G'(x)=\frac{-1}{x^2}-O\left(\frac{1}{x^3}\right).$$

Then

$$\lim_{x\to\infty}\frac{G'(x+2)e^{\alpha(x+2)}}{(e^{\beta(x+2)}-1)^2}=\lim_{x\to\infty}\frac{\left[\frac{-1}{(x+2)^2}-O\left(\frac{1}{x^3}\right)\right]\left[1+\frac{1}{x}-\frac{1}{x^2}+\frac{2}{3x^3}+\frac{5}{12x^4}-O\left(\frac{1}{x^5}\right)\right]}{\left(\left[1+\frac{1}{x}-\frac{1}{x^2}+\frac{2}{3x^3}+\frac{1}{24x^4}-O\left(\frac{1}{x^5}\right)\right]-1\right)^2}=-1$$

and

$$\lim_{x\to\infty}\frac{G'(x+2)e^{\beta(x+2)}}{(e^{\alpha(x+2)}-1)^2}=\lim_{x\to\infty}\frac{\left[\frac{-1}{(x+2)^2}-O\left(\frac{1}{x^3}\right)\right]\left[1+\frac{1}{x}-\frac{1}{x^2}+\frac{2}{3x^3}+\frac{1}{24x^4}-O\left(\frac{1}{x^5}\right)\right]}{\left(\left[1+\frac{1}{x}-\frac{1}{x^2}+\frac{2}{3x^3}+\frac{5}{12x^4}-O\left(\frac{1}{x^5}\right)\right]-1\right)^2}=-1$$

□

Now, we will present the sharp bounds of the double inequality (9).

Theorem 4. *For all $x\in(0,\infty)$*

$$\ln\left(1+\frac{1}{x+\frac{4}{e^2-4}}\right)+\frac{2}{x(x+1)}<G(x)<\ln\left(1+\frac{1}{x+1}\right)+\frac{2}{x(x+1)},\tag{21}$$

where the constants 1 *and* $\frac{4}{e^2-4}$ *are the best possible.*

Proof. Using the inequality (9) and functional equation (2), we get

$$0<\frac{1}{e^{G(x+2)}-1}-x<2.$$

Now consider the two functions

$$f(x)=e^{G(x+2)}-1,\qquad x>0$$

and

$$q(x)=\frac{1}{f(x)}-x,\qquad x>0.$$

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Then $f'(x) = G'(x+2)e^{G(x+2)} < 0$ and $f(x)$ is strictly decreasing function. Hence $\frac{1}{f(x)}$ is strictly increasing function. Since $\frac{d}{dx}\frac{1}{f(x)}|_{x=0} \simeq 0.91$, and $\frac{d}{dx}\frac{1}{f(x)}|_{x=1} \simeq 0.96$. Then the function $\frac{1}{f(x)}$ is convex and $\frac{d}{dx}\frac{1}{f(x)}$ is increasing function. Thus we get

$$\frac{d}{dx}\frac{1}{f(x)} < \lim_{x\to\infty}\frac{d}{dx}\frac{1}{f(x)} = -\lim_{x\to\infty}\frac{G'(x+2)e^{G(x+2)}}{(e^{G(x+2)}-1)^2}.$$

Using the limit (20), we obtain

$$\frac{d}{dx}\frac{1}{f(x)} < 1, \qquad x > 0.$$

Then $q(x)$ is strictly decreasing function for all $x > 0$, where $\frac{dq(x)}{dx} = \frac{d}{dx}\frac{1}{f(x)} - 1 < 0$. Hence

$$\lim_{x\to\infty} q(x) < q(x) < \lim_{x\to 0^+} q(x)$$

and using the limit (19) and $G(2) = 2 - \ln 4$, we have

$$1 < q(x) < \frac{4}{e^2-4}. \tag{22}$$

with best bounds. □

In the proof of theorem (4), we proved that the function $\frac{1}{f(x)}$ is convex. Also, the second derivatives of the functions $q(x)$ and $\frac{1}{f(x)}$ have the same sign, then we get the following resuts:

Corollary 3.2. *The function $q(x)$ is strictly decreasing and convex for all $x > 0$.*

Corollary 3.3. *For every $x > 0$ there exists a unique number $\mu \in \left(1, \frac{4}{e^2-4}\right)$ such that $G(x) = M(\mu, x)$. Conversely for every $\mu \in \left(1, \frac{4}{e^2-4}\right)$ there exists a unique number $x > 0$ such that $M(\mu, x) = G(x)$.*

Proof. The function $q(x)$ is strictly decreasing from $(0, \infty)$ onto $\left(1, \frac{4}{e^2-4}\right)$ then the mapping $q(x) : (0, \infty) \to \left(1, \frac{4}{e^2-4}\right)$ is bijective and the proof is easy consequence of this result. □

Corollary 3.4. *For $x > 2$ and $\mu \in \left(1, \frac{4}{e^2-4}\right)$ we have*

1) the errors $e_\mu(x)$ are uniformly bounded by $\pm \ln\left(\frac{4(2e^2-4)}{3(3e^2-8)}\right)$.

2) $G(x) = M(\mu, x) + O\left(\ln \frac{(x+2)[(e^2-4)x+4]}{(x+1)[(e^2-4)x+e^2]}\right)$.

Proof. Analogues to inequality (17), we can deduce for all $x > 2$ and $\mu \in \left(1, \frac{4}{e^2-4}\right)$ that

$$0 \le |G(x) - M(\mu, x)| \le \left| M(1, x) - M\left(\frac{4}{e^2-4}, x\right)\right|$$

which is equivalent to

$$0 \le |e_\mu(x)| \le \left|\ln \frac{(x+2)[(e^2-4)x+4]}{(x+1)[(e^2-4)x+e^2]}\right| \le \left|\ln\left(\frac{4(2e^2-4)}{3(3e^2-8)}\right)\right|.$$

□

4 Comparing approximations

Firstly, we will prove the following one side inequality the function $G(x)$ which proves a special case of a conjecture posed in [9] and proved in [11] about the best bounds of the Bateman's function but with different proof.

Lemma 4.1. *For all $x > 0$, we have*

$$G(x) - \frac{1}{x} > \frac{1}{2x^2} - \frac{1}{4x^4}. \tag{23}$$

Proof. Consider the function

$$K(x) = G(x) - \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{4x^4}, \qquad x > 0.$$

Using the integral representation (3) of $G(x)$ and the formula

$$\frac{1}{x^r} = \frac{1}{(r-1)!}\int_0^\infty t^{r-1}e^{-xt}dt, \qquad r \in N$$

we get

$$K(x) = \int_0^\infty \varphi(t)\frac{e^{-xt}}{1+e^t}dt,$$

where

$$\varphi(t) = e^t - 1 - \frac{1}{2}t(1+e^t) + \frac{1}{24}t^3(1+e^t).$$

But

$$\begin{aligned}
\varphi(t) &= \sum_{k=4}^{\infty}\frac{t^k}{k!} - \frac{1}{2}\sum_{k=3}^{\infty}\frac{t^{k+1}}{k!} + \frac{1}{24}\sum_{k=1}^{\infty}\frac{t^{k+3}}{k!} \\
&= \sum_{k=0}^{\infty}\frac{t^{(k+4)}}{(k+4)!}(1 + \frac{1}{24}(k+4)[(k+3)(k+2) - 12]) \\
&= \sum_{k=0}^{\infty}\frac{t^{(k+5)}}{(k+5)!}(1 + \frac{1}{24}k(k+5)(k+7)) > 0.
\end{aligned}$$

Hence $\varphi(x) > 0$ and then $K(x) > 0$. □

As by-product of the the inequalities (6) and (23), we obtain the following double inequality.

Corollary 4.2. *For all $x > 1$, we have*

$$0 < \frac{(2x+1)(x-1)(x^2+1)}{2x^4(x+1)} < 2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2} < \frac{2x^2-x+1}{2x^2(x+1)}. \tag{24}$$

Now, we will prove the following auxiliary results:

Lemma 4.3. *For all $x > x_0 \approx 2.5315129$, we have*

$$\frac{1}{e^{2G(x)-\frac{2}{x(x+1)}-\frac{1}{x}-\frac{1}{2x^2}} - 1} - x > \frac{1}{e^{\frac{2x^2-x+1}{2x^2(x+1)}} - 1} - x > 1. \tag{25}$$

Proof. Using the inequality (24), we have

$$2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2} - \ln\left(\frac{x+2}{x+1}\right) < u(x)$$

where

$$u(x) = \frac{2x^2 - x + 1}{2x^2(x+1)} - \ln\left(\frac{x+2}{x+1}\right), \qquad x > 0.$$

Then

$$u'(x) = \frac{(x - \frac{3+\sqrt{17}}{2})(x - \frac{3-\sqrt{17}}{2})}{x^3(x+1)^2},$$

and the function $u(x)$ has only one positive critical point at $x_m = \frac{3+\sqrt{17}}{2}$. Now,

$$u(x_m) = \frac{10}{(3+\sqrt{17})^2} - \ln\frac{7+\sqrt{17}}{5+\sqrt{17}} \approx -0.00113 < 0,$$

$$\lim_{x\to\infty} u(x) = 0$$

and

$$\lim_{x\to 0^-} u(x) = \infty.$$

Hence $u(x)$ has only one positive root $x_0 \approx 2.5315129$ and

$$u(x) < 0, \qquad \forall x > x_0.$$

Then

$$2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2} < \ln\left(\frac{x+2}{x+1}\right), \qquad \forall x > x_0.$$

□

Lemma 4.4. *For all $x > x_1 \approx 2.6925094$, we have*

$$\frac{1}{e^{2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2}} - 1} - x < \frac{4}{e^2 - 4}. \tag{26}$$

Proof. Using the inequality (24), we have

$$2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2} - \ln\left(\frac{e^2 + (e^2-4)x)}{4 + (e^2-4)x}\right) > v(x),$$

where

$$v(x) = \frac{(2x+1)(x-1)(x^2+1)}{2x^4(x+1)} - \ln\left(\frac{e^2 + (e^2-4)x)}{4 + (e^2-4)x}\right), \qquad x > 1.$$

Hence

$$v'(x) = \frac{L(x)}{S(x)},$$

where

$$L(x) = 8e^2 + (-32 + 16e^2 + 2e^4)x + (-32 - 12e^2 + 6e^4)x^2 + (48 - 36e^2 + 5e^4)x^3 + (32 - 4e^2)x^4 + (-16 - 4e^2 + e^4)x^5 + (64 - 24e^2 + 2e^4)x^6$$

and

$$S(x) = x^5(x+1)^2(4e^2 + (e^4 - 16)x + (16 - 8e^2 + e^4)x^2 > 0, \qquad x > 0.$$

The function $L''(x)$ is a polynomial of fourth degree has one positive root at $x_I \approx 2.31866$ with $L''(3) < 0$, then $L(x)$ is concave function on (x_I, ∞). Also, $L(x_I) > 0$ and $\lim_{x\to\infty} L(x) = -\infty$. Hence, the function $L(x)$ has only one root on (x_I, ∞) at $x_3 \approx 4.0635204$, where $L(4.063) > 0$ and $L(4.064) < 0$. Then $L(x) > 0$ on $[x_I, x_3)$ and $L(x) < 0$ for all $x > x_3$. Hence $v(x)$ is increasing on (x_I, x_3) and decreasing function on (x_3, ∞) and it has a maximum point at x_3. But $v(2.69) < 0$ and $v(2.7) > 0$ and then $v(x)$ has a root $x_1 \approx 2.6925094 \in (x_I, x_3)$. Also, $\lim_{x\to\infty} v(x) = 0$, then we have

$$v(x) > 0, \qquad x > x_1$$

and hence

$$2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2} - \ln\left(\frac{e^2 + (e^2-4)x)}{4 + (e^2-4)x}\right) > 0, \qquad x > x_1.$$

□

Theorem 5. *For a fixed $x > x_1$, consider I_x be the nonempty open interval of $\left[1, \frac{4}{e^2-4}\right]$ defined by*

$$I_x = \left(\frac{1}{e^{-\frac{2}{x(x+1)} + \frac{1}{x} + \frac{1}{2x^2}} - 1} - x, \frac{1}{e^{2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2}} - 1} - x\right).$$

For any $\mu \in I_x$, we have

$$|e_\mu(x)| < \left|G(x) - \left(\frac{1}{x} + \frac{1}{2x^2}\right)\right|.$$

Proof. Using the inequalities (25) and (26), we obtain

$$I_x \subset \left[1, \frac{4}{e^2-4}\right].$$

For any positive real number μ,

$$\frac{1}{e^{-\frac{2}{x(x+1)} + \frac{1}{x} + \frac{1}{2x^2}} - 1} - x < \mu \ \text{ iff } \ -M(\mu, x) > -\frac{1}{x} - \frac{1}{2x^2}$$

and hence

$$\frac{1}{e^{-\frac{2}{x(x+1)} + \frac{1}{x} + \frac{1}{2x^2}} - 1} - x < \mu \ \text{ iff } \ G(x) - M(\mu, x) > G(x) - \frac{1}{x} - \frac{1}{2x^2}. \tag{27}$$

Also,

$$\frac{1}{e^{2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2}} - 1} - x > \mu \ \text{ iff } \ 2G(x) - \frac{2}{x(x+1)} - \frac{1}{x} - \frac{1}{2x^2} < \ln\left(1 + \frac{1}{x+\mu}\right)$$

and hence

$$\frac{1}{e^{2G(x)-\frac{2}{x(x+1)}-\frac{1}{x}-\frac{1}{2x^2}}-1}-x>\mu \text{ iff } G(x)-M(\mu,x)<-G(x)+\frac{1}{x}+\frac{1}{2x^2}. \tag{28}$$

From the inequalities (27) and (28) we have

$$G(x)-\frac{1}{x}-\frac{1}{2x^2}<G(x)-M(\mu,x)<-G(x)+\frac{1}{x}+\frac{1}{2x^2}, \quad \forall\mu\in I_x.$$

Thus

$$|G(x)-M(\mu,x)|<\left|G(x)-\left(\frac{1}{x}+\frac{1}{2x^2}\right)\right|, \quad \forall\mu\in I_x. \tag{29}$$

□

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