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On new λ^2-convergent difference BK-spaces

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In this paper, we introduce the spaces $c(\lambda^2, \Delta)$ and $c_0(\lambda^2, \Delta)$, which are BK-spaces of non-absolute type and we prove that these spaces are linearly isomorphic to the spaces c and c_0, respectively. Moreover, we give some inclusion relations and compute the $\alpha-$, $\beta-$ and $\gamma-$duals of these spaces. We also determine the Schauder basis of the $c(\lambda^2, \Delta)$ and $c_0(\lambda^2, \Delta)$. Lastly we give some matrix transformations between of these spaces and others.

1 Introduction

A sequence space is defined to be a linear space of real or complex sequences. Let w denote the spaces of all complex sequences. If $x \in w$, then we simply write $x = (x_k)$ instead of $x = (x_k)_{k=0}^{\infty}$.

Let X be a sequence space. If X is a Banach space and

$$\tau_k : X \to C, \quad \tau_k(x) = x_k \quad (k = 1, 2, ...)$$

is a continuous for all k, X is called a $BK-$space.

We shall write ℓ_∞, c and c_0 for the sequence spaces of all bounded, convergent and null sequences, respectively, which are $BK-$spaces with the norm given by $\|x\|_\infty = \sup_k |x_k|$ for all $k \in \mathbb{N}$.

For a sequence space X, the matrix domain X_A of an infinite matrix A defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\} \tag{1}$$

which is a sequence space. We denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$.

M. Mursaleen and A. K. Noman [9] introduced the sequence spaces ℓ_∞^λ, c^λ and c_0^λ as the sets of all $\lambda - bounded, \lambda - convergent$ and $\lambda - null$ sequences as follows;

$$\ell_\infty^\lambda = \{x \in w : \sup_n |\Lambda_n(x)| < \infty\}$$
$$c^\lambda = \{x \in w : \lim_{n \to \infty} \Lambda_n(x) \ exists\}$$
$$c_0^\lambda = \{x \in w : \lim_{n \to \infty} \Lambda_n(x) = 0\}$$

where $\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k$, $k \in \mathbb{N}$. Also they generalized c^λ and c_0^λ spaces defining $c^\lambda(\Delta)$, $c_0^\lambda(\Delta)$ spaces using the difference operator. They studied some properties of these spaces in [8]. N. L. Braha and F. Başar introduced the infinite matrix $A(\lambda) = \{a_{nk}(\lambda)\}_{n,k=0}^{\infty}$ such as;

$$a_{nk}(\lambda) = \begin{cases} \frac{\Delta^2 \lambda_k}{\Delta \lambda_k}, & 0 \leq k \leq n; \\ 0, & k > n \end{cases}$$

for all $k, n \in \mathbb{N}$ and they defined $A_\lambda(\ell_\infty)$, $A_\lambda(c)$ and $A_\lambda(c_0)$ spaces in [11] as follows;

$$A_\lambda(\ell_\infty) = \left\{x \in w : \sup_n |(A_\lambda x)_n| < \infty\right\},$$
$$A_\lambda(c) = \left\{x \in w : \exists l \in \mathbb{C} \ni \lim_n (A_\lambda x)_n = l\right\},$$
$$A_\lambda(c_0) = \left\{x \in w : \lim_n (A_\lambda x)_n = 0\right\}$$

where $(A_\lambda x)_n = \frac{1}{\Delta \lambda_n} \sum_{k=0}^{n} (\Delta^2 \lambda_k) x_k$. They examined some properties of these spaces. In literature, some authors have constructed new sequence spaces by using matrix domain of infinite matrix and have introduced some topological properties. (see [2], [4], [12])

1

2 The sequence spaces $c(\lambda^2, \Delta)$ and $c_0(\lambda^2, \Delta)$

In this section, we define the sequence spaces $c(\lambda^2, \Delta)$ and $c_0(\lambda^2, \Delta)$ as follows;

$$c(\lambda^2, \Delta) = \left\{ x \in w : \lim_{n \to \infty} \Lambda_n^2(x) \ exists \right\}$$

$$c_0(\lambda^2, \Delta) = \left\{ x \in w : \lim_{n \to \infty} \Lambda_n^2(x) = 0 \right\}$$

where $\Lambda_n^2(x) = \frac{1}{\Delta \lambda_n} \sum_{k=0}^{n} \left(\Delta^2 \lambda_k \right) (x_k - x_{k-1})$ for all $k, n \in \mathbb{N}$. Δ denotes the difference operator. i.e., $\Delta^0 \lambda_n = \lambda_n$, $\Delta \lambda_n = \lambda_n - \lambda_{n-1}$, $\Delta^2 \lambda_n = \lambda_n - 2\lambda_{n-1} + \lambda_{n-2}$ and $\Delta x_k = x_k - x_{k-1}$. $\lambda = (\lambda_k)_{k=0}^{\infty}$ is a strictly increasing sequence of positive reals tending to infinity, that is $0 < \lambda_0 < \lambda_1 < ...$ and $\lambda_k \to \infty$ as $k \to \infty$ and $\lambda_{n+1} \geq 2\lambda_n$ for all $n \in \mathbb{N}$. Here and in sequel, we use the convention that any term with a negative subscript is equal to naught. e.g. $\lambda_{-1} = \lambda_{-2} = 0$ and $x_{-1} = 0$. On the other hand, we define the matrix $\Lambda^2 = \left(\lambda_{nk}^2 \right)$ for all $k, n \in \mathbb{N}$ by

$$\lambda_{nk}^2 = \begin{cases} \frac{\Delta^2 (\lambda_k - \lambda_{k+1})}{\Delta \lambda_k}; & k < n, \\ \frac{\Delta^2 \lambda_n}{\Delta \lambda_n}; & n = k, \\ 0; & n > k. \end{cases} \tag{2}$$

The equality can be eaisly seen from

$$\Lambda_n^2(x) = \frac{1}{\Delta \lambda_n} \sum_{k=0}^{n} \left(\Delta^2 \lambda_k \right) (x_k - x_{k-1}) \tag{3}$$

for all $m, n \in \mathbb{N}$ and every $x = (x_k) \in w$. Then it leads us together with (1) to the fact that

$$c_0 \left(\lambda^2, \Delta \right) = (c_0)_{\Lambda^2} \ \text{and} \ c \left(\lambda^2, \Delta \right) = (c)_{\Lambda^2}. \tag{4}$$

The matrix $\Lambda^2 = \lambda_{nk}^2$ is a triangle, i.e., $\lambda_{nn}^2 \neq 0$ and $\lambda_{nk}^2 = 0$ $(k > n)$ for all $n, k \in \mathbb{N}$. Further, for any sequence $x = (x_k)$ we define the sequence $y \left(\lambda^2 \right) = \left\{ y_k \left(\lambda^2 \right) \right\}$ as the Λ^2-transform of x, i.e., $y \left(\lambda^2 \right) = \Lambda^2(x)$ and so we have that

$$y_k \left(\lambda^2 \right) = \sum_{j=0}^{k-1} \frac{\Delta^2 (\lambda_j - \lambda_{j+1})}{\Delta \lambda_k} x_j + \frac{\Delta^2 \lambda_k}{\Delta \lambda_k} x_k \tag{5}$$

for $k \in \mathbb{N}$. Here and in what follows, the summation running from 0 to $k - 1$ is equal to zero when $k = 0$. Also it can be written from (3) with (5) for $k \in \mathbb{N}$ such as;

$$y_k \left(\lambda^2 \right) = \sum_{j=0}^{k} \frac{\Delta^2 \lambda_j}{\Delta \lambda_k} (x_j - x_{j-1}).$$

Theorem 1 $c_0(\lambda^2, \Delta)$ and $c(\lambda^2, \Delta)$ are BK-spaces with the norm

$$\|x\|_{(c_0)_{\Lambda^2}} = \|x\|_{(c)_{\Lambda^2}} = \sup_n \left| \Lambda_n^2(x) \right|.$$

Proof. We know that c and c_0 are $BK-$spaces with their natural norms from [6]. (4) holds and $\Lambda^2 = \lambda_{nk}^2$ is a triangle matrix and from Theorem 4.3.12 of Wilansky [1], we derive that $c_0(\lambda^2, \Delta)$ and $c(\lambda^2, \Delta)$ are $BK-$spaces. This completes the proof. ∎

Remark 2 The absolute property does not hold on the $c_0(\lambda^2, \Delta)$ and $c(\lambda^2, \Delta)$ spaces. For instance, if we take $|x| = (|x_k|)$ we hold $\|x\|_{(c)_{\Lambda^2}} \neq \||x|\|_{(c)_{\Lambda^2}}$. Thus, the space $c_0(\lambda^2, \Delta)$ and $c(\lambda^2, \Delta)$ are BK-space of non-absolute type.

Theorem 3 The sequence spaces $c_0(\lambda^2, \Delta)$ and $c(\lambda^2, \Delta)$ of non-absolute type are linearly isomorphic to the spaces c_0 and c, respectively, that is $c_0(\lambda^2, \Delta) \cong c_0$ and $c(\lambda^2, \Delta) \cong c$.

2

Proof: We only consider $c_0(\lambda^2, \Delta) \cong c_0$ and others will prove similarly. To prove the theorem we must show the existence of linear bijection operator between $c_0(\lambda^2, \Delta)$ and c_0. Hence, let define the linear operator with the notation (5), from $c_0(\lambda^2, \Delta)$ and c_0 by $x \to y\left(\lambda^2\right) = Tx$.

Then $Tx = y\left(\lambda^2\right) = \Lambda^2(x) \in c_0$ for every $x \in c_0(\lambda^2, \Delta)$. Also, the linearity of T is clear. Further, it is trivial that $x = 0$ whenever $Tx = 0$. Hence T is injective.

Let $y = (y_k) \in c_0$ and define the sequence $x = \left\{x\left(\lambda^2\right)\right\}$ by

$$x_k\left(\lambda^2\right) = \sum_{j=0}^{k} \sum_{i=j-1}^{j} (-1)^{j-i} \frac{\Delta \lambda_i}{\Delta^2 \lambda_j} y_i. \tag{6}$$

and we have

$$x_k\left(\lambda^2\right) - x_{k-1}\left(\lambda^2\right) = \sum_{i=k-1}^{k} (-1)^{k-i} \frac{\Delta \lambda_i}{\Delta^2 \lambda_k} y_i.$$

Thus, for every $k \in \mathbb{N}$, we have by (5) that

$$\Lambda_n^2(x) = \frac{1}{\Delta \lambda_n} \sum_{k=0}^{n} \left[\Delta\left(\lambda_k y_k - \lambda_{k-1} y_{k-1}\right)\right] = y_n$$

This shows that $\Lambda^2(x) = y$ and since $y \in c_0$, we obtain that $\Lambda^2(x) \in c_0$. Thus we deduce that $x \in c_0(\lambda^2, \Delta)$ and $Tx = y$. Hence T is surjective.

Further, we have for every $x \in c_0(\lambda^2, \Delta)$ that

$$\|Tx\|_{c_0} = \|Tx\|_{\ell_\infty} = \left\|y\left(\lambda^2\right)\right\|_{\ell_\infty} = \left\|\Lambda^2(x)\right\|_{\ell_\infty} = \|x\|_{(c_0)_{\Lambda^2}}$$

which means that $c_0(\lambda^2, \Delta)$ and c_0 are linearly isomorphic.

3   Some inclusion relations

Theorem 4 *The inclusion* $c_0\left(\lambda^2, \Delta\right) \subset c\left(\lambda^2, \Delta\right)$ *strictly holds.*

Proof. $c_0\left(\lambda^2, \Delta\right) \subset c\left(\lambda^2, \Delta\right)$ is clear. To show strict, consider the sequence $x = (x_k)$ defined by $x_k = k + 1$ for all $k \in \mathbb{N}$. Then we obtain that

$$\Lambda_n^2(x) = \frac{1}{\Delta \lambda_n} \sum_{k=0}^{n} \left(\Delta^2 \lambda_k\right) (x_k - x_{k-1}) = 1; \quad (n \in \mathbb{N})$$

for $n \in \mathbb{N}$ which shows that $\Lambda^2(x) \in c - c_0$. Thus, the sequence x is in $c\left(\lambda^2, \Delta\right)$ but not in $c_0\left(\lambda^2, \Delta\right)$. Hence the inclusion $c_0\left(\lambda^2, \Delta\right) \subset c\left(\lambda^2, \Delta\right)$ is strict and this completes the proof. ■

Theorem 5 *The inclusion* $c \subset c_0\left(\lambda^2, \Delta\right)$ *strictly holds.*

Proof. Let $x \in c$. Then, $\Lambda^2(x) \in c_0$. This shows that $x \in c_0\left(\lambda^2, \Delta\right)$. Hence, the inclusion $c \subset c_0\left(\lambda^2, \Delta\right)$ holds. Then, consider the sequence $y = (y_k)$ defined by $y_k = \sqrt{k+1}$ for $k \in \mathbb{N}$. It is trivial that $y \notin c$. On the other hand, it can easily be seen that $\Lambda^2(y) \in c_0$ and $y \in c_0\left(\lambda^2, \Delta\right)$. Consequently, the sequence y is in $c_0\left(\lambda^2, \Delta\right)$ but not in c. We therefore deduce that the inclusion $c \subset c_0\left(\lambda^2, \Delta\right)$ is strict. ■

Corollary 6 $c_0 \subset c_0\left(\lambda^2, \Delta\right)$ *and* $c \subset c\left(\lambda^2, \Delta\right)$ *strictly hold.*

Theorem 7 *Although the spaces* ℓ_∞ *and* $c_0\left(\lambda^2, \Delta\right)$ *overlap, the space* ℓ_∞ *does not include the space* $c_0\left(\lambda^2, \Delta\right)$.

Proof. It can be seen from the sequence y, which was defined in Theorem 5, is in $c_0\left(\lambda^2, \Delta\right)$ but not in ℓ_∞. ■

3

Lemma 8 $A \in (\ell_\infty : c_0)$ *if and only if* $\lim_n \sum_k |a_{nk}| = 0$.

Theorem 9 *The inclusion* $\ell_\infty \subset c_0 \left(\lambda^2, \Delta\right)$ *strictly holds if and only if* $z \in A_\lambda \left(c_0\right)$ *where the sequence* $z = (z_k)$ *is defined by*

$$z_k = \left| 1 - \frac{\Delta^2 \lambda_{k+1}}{\Delta^2 \lambda_{k-1}} \right| ; \quad (k \in \mathbb{N}).$$

Proof. Let $\ell_\infty \subset c_0 \left(\lambda^2, \Delta\right)$. Then, we obtain that $\Lambda^2 (x) \in c_0$ for every $x \in \ell_\infty$ and the matrix $\Lambda^2 = \left(\lambda_{nk}^2\right)$ is in the class $(\ell_\infty : c_0)$. It follows by Lemma 8

$$\lim_n \sum_k \left| \lambda_{nk}^2 \right| = 0. \tag{7}$$

From definition of $\Lambda^2 = \left(\lambda_{nk}^2\right)$ given in (2) we have

$$\sum_k \left| \lambda_{nk}^2 \right| = \frac{1}{\Delta \lambda_n} \sum_{k=0}^{n-1} \left| \left(\Delta^2 \lambda_k - \Delta^2 \lambda_{k-1}\right) \right| + \frac{\Delta^2 \lambda_n}{\Delta \lambda_n}. \tag{8}$$

From (7)

$$\lim_n \frac{\Delta^2 \lambda_n}{\Delta \lambda_n} = 0 \tag{9}$$

and

$$\lim_n \frac{1}{\Delta \lambda_n} \sum_{k=0}^{n-1} \left| \Delta^2 \left(\lambda_k - \lambda_{k+1}\right) \right| = 0. \tag{10}$$

We have

$$\frac{1}{\Delta \lambda_n} \sum_{k=0}^{n-1} \left| \Delta^2 \left(\lambda_k - \lambda_{k+1}\right) \right| = \frac{\Delta \lambda_{n-1}}{\Delta \lambda_n} \left[\frac{1}{\Delta \lambda_{n-1}} \sum_{k=0}^{n-1} \left(\Delta^2 \lambda_k\right) z_k \right]$$

and since $\lim_n \frac{\Delta \lambda_{n-1}}{\Delta \lambda_n} = 1$ by (9); we have from (10) that

$$\lim_n \frac{1}{\Delta \lambda_n} \sum_{k=0}^{n-1} \left(\Delta^2 \lambda_k\right) z_k = 0 \tag{11}$$

which shows that $z = (z_k) \in A_\lambda \left(c_0\right)$. ∎
　　Conversely, let $z = (z_k) \in A_\lambda \left(c_0\right)$. Then we have that (11) holds. Also we obtain that

$$\frac{1}{\Delta \lambda_n} \sum_{k=0}^{n} \left| \Delta^2 \left(\lambda_k - \lambda_{k+1}\right) \right| = \frac{1}{\Delta \lambda_n} \sum_{k=0}^{n-1} \Delta^2 \lambda_k z_k$$

$$\leq \frac{1}{\Delta \lambda_{n-1}} \sum_{k=0}^{n-1} \Delta^2 \lambda_k z_k.$$

This and (11) provides (10). On the other hand, we have that

$$\left| \frac{\Delta^2 \lambda_n - \lambda_0}{\Delta \lambda_n} \right| = \left| \frac{2\lambda_{n-1} - (\lambda_n + \lambda_{n-2} - \lambda_0)}{\Delta \lambda_n} \right|$$

$$= \left| \frac{1}{\Delta \lambda_n} \sum_{k=0}^{n-1} \Delta^2 \left(\lambda_k - \lambda_{k+1}\right) \right|$$

$$\leq \frac{1}{\Delta \lambda_n} \sum_{k=0}^{n-1} \left| \Delta^2 \left(\lambda_k - \lambda_{k+1}\right) \right|.$$

From (10), we derive that

$$\lim_n \frac{\Delta^2 \lambda_n}{\Delta \lambda_n} = \lim_n \frac{\Delta^2 \lambda_n - \lambda_0}{\Delta \lambda_n} = 0.$$

　　This provides (9). Hence, we obtain from (8) that (7) holds. From Lemma 8 $\Lambda^2 \in (\ell_\infty : c_0)$. Hence, the inclusion $\ell_\infty \subset c_0 \left(\lambda^2, \Delta\right)$ holds. This inclusion is strict from Theorem 7. The proof is completed.

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Corollary 10 *If* $\lim_n \frac{\Delta^2 \lambda_{n+1}}{\Delta^2 \lambda_n} = 1$, *then the inclusion* $\ell_\infty \subset c_0 \left(\lambda^2, \Delta \right)$ *is strict.*

4   The bases for the spaces $c \left(\lambda^2, \Delta \right)$ and $c_0 \left(\lambda^2, \Delta \right)$

If a normed sequence space X contains a sequence (b_n) with the property that for every $x \in X$ there is a unique sequence (α_n) of scalars such that

$$\lim_n \| x - (\alpha_0 b_0 + \alpha_1 b_1 + ... + \alpha_n b_n) \| = 0.$$

Then (b_n) is called a Schauder basis (or briefly basis) for X. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x = \sum_k \alpha_k b_k$.

Theorem 11 *Define the sequence* $b^{(k)} \left(\lambda^2 \right) \in c_0 \left(\lambda^2, \Delta \right)$ *for every fixed* $k \in \mathbb{N}$ *and by*

$$b_n^{(k)} \left(\lambda^2 \right) = \begin{cases} \frac{\Delta \lambda_k}{\Delta^2 \lambda_k} - \frac{\Delta \lambda_k}{\Delta^2 \lambda_{k+1}}; & n > k, \\ \frac{\Delta \lambda_k}{\Delta^2 \lambda_k}; & n = k, \\ 0; & n < k. \end{cases}$$

(i) The sequence $\left\{ b_n^{(k)} \left(\lambda^2 \right) \right\}_{k=0}^\infty$ is a Schauder basis for the space $c_0 \left(\lambda^2, \Delta \right)$ and every $x \in c_0 \left(\lambda^2, \Delta \right)$ has a unique representation of the form

$$x = \sum_k \alpha_k \left(\lambda^2 \right) b^{(k)} \left(\lambda^2 \right)$$

(ii) The sequence $\left\{ b, b_n^{(0)} \left(\lambda^2 \right), b_n^{(1)} \left(\lambda^2 \right), ... \right\}$ is a Schauder basis for the space $c \left(\lambda^2, \Delta \right)$ and every $x \in c \left(\lambda^2, \Delta \right)$ has a unique representation of the form

$$x = lb + \sum_k \left[\alpha_k \left(\lambda^2 \right) - l \right] b_n^{(k)} \left(\lambda^2 \right)$$

where $\alpha_k \left(\lambda^2 \right) = \Lambda^2 (x)$ for all $k \in \mathbb{N}$ and the sequence $b = (b_k)$ is defined by $b_k = k + 1$.

Corollary 12 *The difference sequence spaces* $c \left(\lambda^2, \Delta \right)$ *and* $c_0 \left(\lambda^2, \Delta \right)$ *are seperable.*

5   The $\alpha-, \beta-$ and $\gamma-$duals of the spaces $c \left(\lambda^2, \Delta \right)$ and $c_0 \left(\lambda^2, \Delta \right)$

In this section, we introduce and prove the theorems determining the $\alpha-, \beta-$ and $\gamma-$ duals of the difference sequence spaces $c \left(\lambda^2, \Delta \right)$ and $c_0 \left(\lambda^2, \Delta \right)$ of non-absolute type. For arbitrary sequence spaces X and Y, the set $M (X, Y)$ defined by

$$M (X, Y) = \{ a = (a_k) \in w : ax = (a_k x_k) \in Y \text{ for all } x = (x_k) \in X \} \qquad (12)$$

is called the multipier space of X and Y. With the notation of (12); the $\alpha-, \beta-$ and $\gamma-$duals of a sequence space X, which are respectively denoted by X^α, X^β and X^γ are defined by

$$X^\alpha = M (X, \ell_1), \ X^\beta = M (X, cs) \text{ and } X^\gamma = M (X, bs).$$

Now, we may begin with lemmas which are given in [10]. We are needed them in proving theorems.

Lemma 13 $A \in (c_0 : \ell_1) = (c : \ell_1)$ *if and only if*

$$\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} a_{nk} \right| < \infty.$$

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Lemma 14 $A \in (c_0 : c)$ *if and only if*

$$\lim_n a_{nk} \ \text{exists for each } k \in \mathbb{N}, \tag{13}$$

$$\sup_n \sum_k |a_{nk}| < \infty. \tag{14}$$

Lemma 15 $A \in (c : c)$ *if and only if (13) and (14) hold, and*

$$\lim_n \sum_k a_{nk} \ \text{exists.} \tag{15}$$

Lemma 16 $A \in (c_0 : \ell_\infty) = (c : \ell_\infty)$ *if and only if (14) holds.*

Lemma 17 $A \in (\ell_\infty : c)$ *if and only if (13) holds and*

$$\lim_{n \to \infty} \sum_k |a_{nk}| = \sum_k |\alpha_k| .$$

Theorem 18 *The $\alpha-$dual of the space $c \left(\lambda^2, \Delta \right)$ and $c_0 \left(\lambda^2, \Delta \right)$ is the set*

$$h_1 = \left\{ a = (a_k) \in w : \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} b_{nk} \left(\lambda^2 \right) \right| < \infty \right\};$$

where the matrix $B^{\lambda^2} = \left(b_{nk}^{\lambda^2} \right)$ is defined via the sequence $a = (a_k)$ by

$$b_{nk}^{\lambda^2} = \begin{cases} \left(\frac{\Delta \lambda_k}{\Delta^2 \lambda_k} - \frac{\Delta \lambda_k}{\Delta^2 \lambda_{k+1}} \right) a_n; & n > k, \\ \frac{\Delta \lambda_n}{\Delta^2 \lambda_n} a_n; & n = k, \\ 0; & n < k. \end{cases}$$

Proof. We prove the theorem for the space $c_0 \left(\lambda^2, \Delta \right)$. Let $a = (a_k) \in w$. Then, we obtain the equality

$$a_k x_k = \sum_{k=0}^n \sum_{j=k-1}^k (-1)^{k-j} \frac{\Delta \lambda_j}{\Delta^2 \lambda_k} y_j a_n = B_n^{\lambda^2} (y); \quad (n \in \mathbb{N}). \tag{16}$$

Thus, we observe by (16) that $ax = (a_k x_k) \in \ell_1$ whenever $x = (x_k) \in c_0 \left(\lambda^2, \Delta \right)$ or $c \left(\lambda^2, \Delta \right)$ if and only if $B^{\lambda^2} y \in \ell_1$ whenever $y = (y_k) \in c_0$ or c. This means that the sequence $a = (a_k)$ is in the $\alpha-$dual of the spaces $c_0 \left(\lambda^2, \Delta \right)$ or $c \left(\lambda^2, \Delta \right)$ if and only if $B^\lambda \in (c_0 : \ell_1) = (c : \ell_1)$. We therefore obtain by Lemma 13 with B^λ instead of A that $a \in \left\{ c_0 \left(\lambda^2, \Delta \right) \right\}^\alpha = \left\{ c \left(\lambda^2, \Delta \right) \right\}^\alpha$ if and only if

$$\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} b_{nk}^{\lambda^2} \right| < \infty.$$

Which leads us to the consequence that $\left\{ c_0 \left(\lambda^2, \Delta \right) \right\}^\alpha = \left\{ c \left(\lambda^2, \Delta \right) \right\}^\alpha = h_1$. This concludes proof. ■

Theorem 19 *Define the sets*

$$h_2 = \left\{ a = (a_k) \in w : \sum_{j=k}^\infty a_j \ \text{exists for each } k \in \mathbb{N}. \right\}$$

$$h_3 = \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n-1} |g_k (n)| < \infty. \right\}$$

$$h_4 = \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \left| \frac{\Delta \lambda_n}{\Delta^2 \lambda_n} a_n \right| < \infty. \right\}$$

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$$h_5 = \left\{ a = (a_k) \in w : \sum_k (k+1) \, a_k \text{ converges.} \right\}$$

where

$$g_k (n) = \Delta\lambda_k \left(\frac{a_k}{\Delta\lambda_k} + (\frac{1}{\Delta\lambda_k} - \frac{1}{\Delta\lambda_{k+1}}) \sum_{j=k+1}^n a_j \right)$$

for $k < n$. Then $\left\{ c\left(\lambda^2, \Delta\right) \right\}^\beta = h_3 \cap h_4 \cap h_5$ and $\left\{ c_0\left(\lambda^2, \Delta\right) \right\}^\beta = h_2 \cap h_3 \cap h_4$.

Proof. We have from (6) that

$$
\begin{aligned}
\sum_{k=0}^n a_k x_k &= \sum_{k=0}^n \left[\sum_{j=0}^k \left(\sum_{i=j-1}^j (-1)^{j-i} \frac{\Delta\lambda_i}{\Delta^2\lambda_j} y_i \right) \right] a_k \\
&= \sum_{k=0}^{n-1} \Delta\lambda_k \left[\frac{a_k}{\Delta^2\lambda_k} + \left(\frac{1}{\Delta^2\lambda_k} - \frac{1}{\Delta^2\lambda_{k+1}} \right) \sum_{j=k+1}^n a_j \right] y_k + \frac{\Delta\lambda_n}{\Delta^2\lambda_n} a_n y_n \qquad (17) \\
&= \sum_{k=0}^{n-1} g_k (n) \, y_k + \frac{\Delta\lambda_n}{\Delta^2\lambda_n} a_n y_n \\
&= T_n (y) ; \quad (n \in \mathbb{N})
\end{aligned}
$$

where the matrix $T = (t_{nk})$

$$t_{nk} = \begin{cases} g_k(n); & k < n, \\ \frac{\Delta\lambda_n}{\Delta^2\lambda_n} a_n; & k = n, \quad (k, n \in \mathbb{N}). \\ 0; & k > n. \end{cases}$$

Then we derive that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in c_0 \left(\lambda^2, \Delta\right)$ if and only if $Ty \in c$ whenever $y = (y_k) \in c_0$. This means that $a = (a_k) \in \left\{ c_0 \left(\lambda^2, \Delta\right) \right\}^\beta$ if and only if $T \in (c_0 : c)$. Therefore, by using Lemma 14, we obtain from (13) and (14) that

$$\sum_{j=k}^\infty a_j \quad \text{exists for each } k \in \mathbb{N}, \qquad (18)$$

$$\sup_n \sum_{k=0}^{n-1} |g_k (n)| < \infty, \qquad (19)$$

$$\sup_k \frac{\Delta\lambda_n}{\Delta^2\lambda_n} a_k < \infty. \qquad (20)$$

Hence we conclude that $\left\{ c_0 \left(\lambda^2, \Delta\right) \right\}^\beta = h_2 \cap h_3 \cap h_4$. We can derive from Lemma 15 and 16 that $a = (a_k) \in \left\{ c \left(\lambda^2, \Delta\right) \right\}^\beta$ if and only if $T \in (c : c)$. Therefore, we have from (13) and (14) that (18), (19) and (20) hold. It can be seen that the equality

$$\sum_{k=0}^n (k+1) \, a_k = \sum_{k=0}^{n-1} g_k (n) + \frac{\Delta\lambda_n}{\Delta^2\lambda_n} a_n; \; (n \in \mathbb{N})$$

holds, which can be written as follows;

$$\sum_{k=0}^n (k+1) \, a_k = \sum_k t_{nk}; \quad (n \in \mathbb{N}).$$

Consequently, we have from (15) that

$$\{ (k+1) \, a_k \} \in cs.$$

Hence (18) is redundant. We conclude that $\left\{ c \left(\lambda^2, \Delta\right) \right\}^\beta = h_3 \cap h_4 \cap h_5$. ∎

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Theorem 20 $\left\{ c_0 \left(\lambda^2, \Delta \right) \right\}^\gamma = \left\{ c \left(\lambda^2, \Delta \right) \right\}^\gamma = h_3 \cap h_4$.

Proof. It can be proved similarly as the proof of the Theorem 19 with Lemma 16 instead of Lemma 14. ∎

6 Some matrix transformations

In this section, we state some matrix classes of matrix mappings on the $c_0 \left(\lambda^2, \Delta \right)$ and $c \left(\lambda^2, \Delta \right)$. Let $x, y \in w$ be connected by the relation $y = \Lambda^2 (x)$ like given in (5). For an infinite matrix $A = (a_{nk})$, we have by (17)

$$\sum_{k=0}^m a_{nk} x_k = \sum_{k=0}^{m-1} g_{nk} (m) y_k + \frac{\Delta \lambda_m}{\Delta^2 \lambda_m} a_{nm} y_m \tag{21}$$

where

$$g_{nk} (m) = \Delta \lambda_k \left[\frac{a_{nk}}{\Delta \lambda_k} + \left(\frac{1}{\Delta \lambda_k} - \frac{1}{\Delta \lambda_{k+1}} \right) \sum_{j=k+1}^m a_{nj} \right].$$

Let $x \in c \left(\lambda^2, \Delta \right)$ and $A_n = (a_{nk})_{k=0}^\infty \in \left(c \left(\lambda^2, \Delta \right) \right)^\beta$ for all $n \in \mathbb{N}$. By passing limits in (21) as $m \to \infty$

$$\begin{aligned}
\sum_k a_{nk} x_k &= \sum_k g_{nk} y_k + l a_n \\
&= \sum_k g_{nk} (y_k - l) + l \left(\sum_k g_{nk} + a_n \right) \tag{22}
\end{aligned}$$

where $l = \lim_{k \to \infty} y_k$ and $a_n = \lim_{k \to \infty} \left(\frac{\Delta \lambda_k}{\Delta^2 \lambda_k} a_{nk} \right)$ for all $n \in \mathbb{N}$. Let consider following conditions;

$$\sup_{F \in \mathcal{F}} \sum_n \left| \sum_{k \in F} g_{nk} \right|^p < \infty, \tag{23}$$

$$\sup_m \sum_{k=0}^{m-1} |g_{nk} (m)| < \infty, \tag{24}$$

$$\left\{ (k+1) a_{nk} \right\}_{k=0}^\infty \in cs, \tag{25}$$

$$\lim_k \frac{\Delta \lambda_k}{\Delta^2 \lambda_k} a_{nk} = a_n, \tag{26}$$

$$\sum_n |a_n|^p < \infty, \tag{27}$$

$$\sup_n \sum_k |g_{nk}| < \infty, \tag{28}$$

$$\sup_n |a_n| < \infty, \tag{29}$$

$$\sum_{j=k}^\infty a_{nj} \; exists, \tag{30}$$

$$\left\{ \frac{\Delta \lambda_k}{\Delta^2 \lambda_k} a_{nk} \right\}_{k=0}^\infty \in \ell_\infty, \tag{31}$$

$$\lim_n a_n = a, \tag{32}$$

$$\lim_n g_{nk} = \alpha_k, \tag{33}$$

$$\lim_n \sum_k g_{nk} = \alpha, \tag{34}$$

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$$\lim_n a_n = 0, \tag{35}$$

$$\lim_n g_{nk} = 0, \tag{36}$$

$$\lim_n \sum_k g_{nk} = 0. \tag{37}$$

Using Theorem 19 and the results given in [10] with (21) and (22), we derive the following result:

Theorem 21

(a) Let $1 \le p < \infty$. Then $A \in \left(c\left(\lambda^2, \Delta\right) : \ell_p \right)$ if and only if (23), (24), (25), (26) and (27).

(b) $A \in \left(c\left(\lambda^2, \Delta\right) : \ell_p \right)$ if and only if (25), (26), (28), (29).

(c) Let $1 \le p < \infty$. Then $A \in \left(c_0\left(\lambda^2, \Delta\right) : \ell_p \right)$ if and only if (23), (24), (30) and (31).

(d) $A \in \left(c_0\left(\lambda^2, \Delta\right) : \ell_\infty \right)$ if and only if (28), (30) and (31).

(e) $A \in \left(c\left(\lambda^2, \Delta\right) : c \right)$ if and only if (25), (26), (28), (32), (33) and (34).

(f) $A \in \left(c\left(\lambda^2, \Delta\right) : c_0 \right)$ if and only if (25), (26), (28), (35), (36) and (37).

(g) $A \in \left(c_0\left(\lambda^2, \Delta\right) : c \right)$ if and only if (28), (30), (31) and (33).

(h) $A \in \left(c_0\left(\lambda^2, \Delta\right) : c_0 \right)$ if and only if (28), (30), (31) and (36).

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Stable cubic sets

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Abstract. The notions of (almost) stable cubic set, stable element, evaluative set and stable degree are introduced, and related properties are investigated. Regarding internal (external) cubic sets and the complement of cubic set, their (almost) stableness and unstableness are discussed. Regarding the P-union, R-union, P-intersection and R-intersection of cubic sets, their (almost) stableness and unstableness are investigated.

1. **Introduction**

Fuzzy sets are initiated by Zadeh [14]. In [15], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set, i.e., a fuzzy set with an interval-valued membership function. In traditional fuzzy logic, to represent, e.g., the expert's degree of certainty in different statements, numbers from the interval $[0, 1]$ are used. It is often difficult for an expert to exactly quantify his or her certainty; therefore, instead of a real number, it is more adequate to represent this degree of certainty by an interval or even by a fuzzy set. In the first case, we get an interval-valued fuzzy set. In the second case, we get a second-order fuzzy set. Interval-valued fuzzy sets have been actively used in real-life applications. For example, Sambuc [8] in Medical diagnosis in thyroidian pathology, Kohout [7] also in Medicine, in a system CLINAID, Gorzalczany [10] in Approximate reasoning, Turksen [10, 11] in Interval-valued logic, in preferences modelling [12], etc. These works and others show the importance of these sets. Using a fuzzy set and an interval-valued fuzzy set, Jun et al. [4] introduced a new notion, called a (internal, external) cubic set, and investigated several properties. They dealt with P-union, P-intersection, R-union and R-intersection of cubic sets, and investigated several related properties. Cubic set theory is applied to CI-algebras (see [1]), B-algebras (see [9]), BCK/BCI-algebras (see [5, 6]), KU-Algebras (see [2, 13]), and semigroups (see [3]).

In this paper, we introduce the notions of (almost) stable cubic set, stable element, evaluative set and stable degree. We investigate related properties. Regarding internal (external) cubic sets and the complement of cubic set, we investigate their (almost) stableness and unstableness.

Regarding the P-union, R-union, P-intersection and R-intersection of cubic sets, we deal with their (almost) stableness and unstableness.

2. Preliminaries

A *fuzzy set* in a set X is defined to be a function $\lambda : X \to [0,1]$. Denote by I^X the collection of all fuzzy sets in a set X. Define a relation $\leq$ on I^X as follows:

$$(\forall \lambda, \mu \in I^X)\, (\lambda \leq \mu \iff (\forall x \in X)(\lambda(x) \leq \mu(x))).$$

The join ($\vee$) and meet ($\wedge$) of λ and μ are defined by $(\lambda \vee \mu)(x) = \max\{\lambda(x), \mu(x)\}$, and $(\lambda \wedge \mu)(x) = \min\{\lambda(x), \mu(x)\}$, respectively, for all $x \in X$. The complement of λ, denoted by λ^c, is defined by $(\forall x \in X)\,(\lambda^c(x) = 1 - \lambda(x))$. For a family $\{\lambda_i \mid i \in \Lambda\}$ of fuzzy sets in X, we define the join ($\vee$) and meet ($\wedge$) operations as follows: $\left(\bigvee_{i \in \Lambda} \lambda_i\right)(x) = \sup\{\lambda_i(x) \mid i \in \Lambda\}$, $\left(\bigwedge_{i \in \Lambda} \lambda_i\right)(x) = \inf\{\lambda_i(x) \mid i \in \Lambda\}$, respectively, for all $x \in X$.

Let $D[0,1]$ be the set of all closed subintervals of the unit interval $[0,1]$. The elements of $D[0,1]$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M^-, M^+]$, where M^- and M^+ are the lower and the upper end points respectively. Especially, we denote $\mathbf{0} = [0,0]$, $\mathbf{1} = [1,1]$, and $\mathbf{a} = [a,a]$ for every $a \in (0,1)$. We also note that

  (i) $(\forall M, N \in D[0,1])\, (M = N \Leftrightarrow M^- = N^-,\ M^+ = N^+)$.
  (ii) $(\forall M, N \in D[0,1])\, (M \leq N \Leftrightarrow M^- \leq N^-,\ M^+ \leq N^+)$.

For every $M \in D[0,1]$, the *complement* of M, denoted by M^c, is defined by $M^c = 1 - M = [1 - M^+, 1 - M^-]$.

Let X be a nonempty set. A function $A : X \to D[0,1]$ is called an *interval-valued fuzzy set* (briefly, an *IVF set*) in X. For each $x \in X$, $A(x)$ is a closed interval whose lower and upper end points are denoted by $A(x)^-$ and $A(x)^+$, respectively. For any $[a,b] \in D[0,1]$, the IVF set whose value is the interval $[a,b]$ for all $x \in X$ is denoted by $\widetilde{[a,b]}$. Denote by D^X the collection of all interval-valued fuzzy sets in a set X. In particular, for any $a \in [0,1]$, the IVF set whose value is $\mathbf{a} = [a,a]$ for all $x \in X$ is denoted by simply $\tilde{a}$.

For every $A, B \in D^X$, we define

$$A = B \Leftrightarrow (\forall x \in X)\, (A(x)^- = B(x)^-,\ A(x)^+ = B(x)^+),$$

$$A \subseteq B \Leftrightarrow (\forall x \in X)\, (A(x)^- \leq B(x)^-,\ A(x)^+ \leq B(x)^+).$$

The *complement* A^c of A is defined by $(\forall x \in X)\, (A^c(x)^- = 1 - A(x)^+,\ A^c(x)^+ = 1 - A(x)^-)$. For a family $\{A_i \mid i \in \Lambda\}$ of IVF sets where Λ is an index set, the *union* $G = \bigcup_{i \in \Lambda} A_i$ and the

intersection $F = \bigcap\limits_{i \in \Lambda} A_i$ are defined by

$$(\forall x \in X)\left(G(x)^- = \sup\limits_{i \in \Lambda} A_i(x)^-,\ G(x)^+ = \sup\limits_{i \in \Lambda} A_i(x)^+\right),$$

$$(\forall x \in X)\left(F(x)^- = \inf\limits_{i \in \Lambda} A_i(x)^-,\ F(x)^+ = \inf\limits_{i \in \Lambda} A_i(x)^+\right),$$

respectively.

Definition 2.1 ([4]). Let X be a nonempty set. By a *cubic set* in X we mean a structure

$$\mathscr{A} = \{\langle x, A(x), \lambda(x)\rangle \mid x \in X\}$$

in which A is an IVF set in X and λ is a fuzzy set in X.

A cubic set $\mathscr{A} = \{\langle x, A(x), \lambda(x)\rangle \mid x \in X\}$ is simply denoted by $\mathscr{A} = \langle A, \lambda\rangle$. Note that a cubic set is a generalization of an intuitionistic fuzzy set.

Definition 2.2 ([4]). Let X be a nonempty set. A cubic set $\mathscr{A} = \langle A, \lambda\rangle$ in X is said to be an *internal cubic set* (briefly, ICS) if $A(x)^- \leq \lambda(x) \leq A(x)^+$ for all $x \in X$.

Definition 2.3 ([4]). Let X be a nonempty set. A cubic set $\mathscr{A} = \langle A, \lambda\rangle$ in X is said to be an *external cubic set* (briefly, ECS) if $\lambda(x) \notin (A(x)^-, A(x)^+)$ for all $x \in X$.

Theorem 2.4 ([4]). *Let $\mathscr{A} = \langle A, \lambda\rangle$ be a cubic set in X. If $\mathscr{A}$ is both an ICS and an ECS, then $(\forall x \in X)\,(\lambda(x) \in U(A) \cup L(A))$ where $U(A) = \{A(x)^+ \mid x \in X\}$ and $L(A) = \{A(x)^- \mid x \in X\}$.*

Definition 2.5 ([4]). Let $\mathscr{A} = \langle A, \lambda\rangle$ and $\mathscr{B} = \langle B, \mu\rangle$ be cubic sets in X. Then we define

   (a) (Equality) $\mathscr{A} = \mathscr{B} \Leftrightarrow A = B$ and $\lambda = \nu$.
   (b) (P-order) $\mathscr{A} \sqsubseteq \mathscr{B} \Leftrightarrow A \subseteq B$ and $\lambda \leq \nu$.
   (c) (R-order) $\mathscr{A} \Subset \mathscr{B} \Leftrightarrow A \subseteq B$ and $\lambda \geq \nu$.

Definition 2.6 ([4]). Let $\mathscr{A} = \langle A, \lambda\rangle$, $\mathscr{B} = \langle B, \mu\rangle$ and $\mathscr{A}_i = \{\langle x, A_i(x), \lambda_i(x)\rangle \mid x \in X\}, i \in \Lambda$, be cubic sets in X for $i \in \Lambda$. The *complement, P-union, P-intersection, R-union and R-intersection* are defined as follows;

   (a) (Complement) $\mathscr{A}^c = \{\langle x, A^c(x), 1 - \lambda(x)\rangle \mid x \in X\}$.
   (b) (P-union) $\mathscr{A} \sqcup \mathscr{B} = \{\langle x, (A \cup B)(x), (\lambda \vee \nu)(x)\rangle \mid x \in X\}$ and
       $\sqcup\mathscr{A}_i = \{\langle x, (\bigcup A_i)(x), (\bigvee \lambda_i)(x)\rangle \mid x \in X\}$ for $i \in \Lambda$.
   (c) (P-intersection) $\mathscr{A} \sqcap \mathscr{B} = \{\langle x, (A \cap B)(x), (\lambda \wedge \nu)(x)\rangle \mid x \in X\}$ and
       $\sqcap\mathscr{A}_i = \{\langle x, (\bigcap A_i)(x), (\bigwedge \lambda_i)(x)\rangle \mid x \in X\}$ for $i \in \Lambda$.
   (d) (R-union) $\mathscr{A} \Cup \mathscr{B} = \{\langle x, (A \cup B)(x), (\lambda \wedge \nu)(x)\rangle \mid x \in X\}$ and
       $\Cup\mathscr{A}_i = \{\langle x, (\bigcup A_i)(x), (\bigwedge \lambda_i)(x)\rangle \mid x \in X\}$ for $i \in \Lambda$.
   (e) (R-intersection) $\mathscr{A} \Cap \mathscr{B} = \{\langle x, (A \cap B)(x), (\lambda \vee \nu)(x)\rangle \mid x \in X\}$ and
       $\Cap\mathscr{A}_i = \{\langle x, (\bigcap A_i)(x), (\bigvee \lambda_i)(x)\rangle \mid x \in X\}$ for $i \in \Lambda$.

3. (Almost) stable cubic sets

In what follows, let X denote a nonempty set unless otherwise specified.

Definition 3.1. Let $\mathscr{A} = \langle A, \lambda \rangle$ be a cubic set in X. Then the *evaluative set* of $\mathscr{A} = \langle A, \lambda \rangle$ is defined to be a structure

$$\mathbf{E}_{\mathscr{A}} = \{(x, E_{\mathscr{A}}(x)) \mid x \in X\} \tag{3.1}$$

where $E_{\mathscr{A}}(x) = \langle l(E_{\mathscr{A}}(x)), r(E_{\mathscr{A}}(x)) \rangle$ with $l(E_{\mathscr{A}}(x)) = \lambda(x) - A(x)^-$ and $r(E_{\mathscr{A}}(x)) = A(x)^+ - \lambda(x)$ which are called the *left evaluative point* and the *right evaluative point*, respectively, of $\mathscr{A} = \langle A, \lambda \rangle$ at $x \in X$. We say that $E_{\mathscr{A}}(x)$ is the *evaluative point* of $\mathscr{A} = \langle A, \lambda \rangle$ at $x \in X$.

Example 3.2. Let $\mathscr{A} = \{\langle x, A(x), \lambda(x) \rangle \mid x \in I\}$ be a cubic set in $I = [0, 1]$.
  (1) If $A(x) = [0.3, 0.7]$ and $\lambda(x) = 0.4$ for all $x \in I$, then $\mathbf{E}_{\mathscr{A}} = \{(x, \langle 0.1, 0.3 \rangle) \mid x \in I\}$.
  (2) If $A(x) = [0.3, 0.7]$ and $\lambda(x) = 0.2$ for all $x \in I$, then $\mathbf{E}_{\mathscr{A}} = \{(x, \langle -0.1, 0.5 \rangle) \mid x \in I\}$.
  (3) If $A(x) = [0.3, 0.7]$ and $\lambda(x) = 0.8$ for all $x \in I$, then $\mathbf{E}_{\mathscr{A}} = \{(x, \langle 0.5, -0.1 \rangle) \mid x \in I\}$.

Example 3.3. Let $\mathscr{B} = \{\langle x, B(x), \mu(x) \rangle \mid x \in I\}$ be a cubic set in $I = [0, 1]$ with $B(x) = [\frac{x}{4}, 1 - \frac{x}{4}]$ and $\mu(x) = \frac{x}{3}$. Then $\mathbf{E}_{\mathscr{B}} = \{(x, \langle \frac{x}{12}, 1 - \frac{7x}{12} \rangle) \mid x \in I\}$, and so the evaluative point of $\mathscr{B}$ at $\frac{1}{2} \in I$ is $E_{\mathscr{B}}(\frac{1}{2}) = \langle \frac{1}{24}, \frac{17}{24} \rangle$.

Example 3.4. Let $\mathscr{A} = \{\langle x, A(x), \lambda(x) \rangle \mid x \in I\}$ be a cubic set in $X = \{0, a, b, c\}$ which is defined by Table 1.

TABLE 1. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
0	$[\frac{1}{8}, \frac{7}{8}]$	$\frac{7}{8} = 0.875$
a	$[\frac{1}{4}, \frac{3}{4}]$	$\frac{3}{8} = 0.375$
b	$[\frac{3}{8}, \frac{5}{8}]$	$\frac{1}{4} = 0.250$
c	$[\frac{1}{2}, \frac{1}{2}]$	$\frac{5}{8} = 0.625$

Then every evaluative point of $\mathscr{A}$ at each $x \in X$ is $E_{\mathscr{A}}(0) = \langle \frac{3}{4}, 0 \rangle$, $E_{\mathscr{A}}(a) = \langle \frac{1}{8}, \frac{3}{8} \rangle$, $E_{\mathscr{A}}(b) = \langle -\frac{1}{8}, \frac{3}{8} \rangle$, and $E_{\mathscr{A}}(c) = \langle \frac{1}{8}, -\frac{1}{8} \rangle$, respectively. Hence the evaluative set of $\mathscr{A}$ is

$$\mathbf{E}_{\mathscr{A}} = \{(0, \langle \tfrac{3}{4}, 0 \rangle), (a, \langle \tfrac{1}{8}, \tfrac{3}{8} \rangle), (b, \langle -\tfrac{1}{8}, \tfrac{3}{8} \rangle), (c, \langle \tfrac{1}{8}, -\tfrac{1}{8} \rangle)\}.$$

Definition 3.5. Let $\mathscr{A} = \langle A, \lambda \rangle$ be a cubic set in X with the evaluative set

$$\mathbf{E}_{\mathscr{A}} = \{(x, E_{\mathscr{A}}(x)) \mid x \in X\}.$$

An element $a \in X$ is called a *stable element* of $\mathscr{A} = \langle A, \lambda \rangle$ in X if it satisfies: $l(E_{\mathscr{A}}(a)) = \lambda(a) - A(a)^- \geq 0$, $r(E_{\mathscr{A}}(a)) = A(a)^+ - \lambda(a) \geq 0$. Otherwise, we say that a is an *unstable element* of $\mathscr{A} = \langle A, \lambda \rangle$ in X. The set of all stable elements of $\mathscr{A} = \langle A, \lambda \rangle$ in X is called the *stable cut* of

$\mathscr{A} = \langle A, \lambda \rangle$ in X and is denoted by $S_{\mathscr{A}}$. The set of all unstable elements of $\mathscr{A} = \langle A, \lambda \rangle$ in X is called the *unstable cut* of $\mathscr{A} = \langle A, \lambda \rangle$ in X and is denoted by $U_{\mathscr{A}}$. We say that $\mathscr{A} = \langle A, \lambda \rangle$ is a *stable* cubic set if $S_{\mathscr{A}} = X$. Otherwise, $\mathscr{A} = \langle A, \lambda \rangle$ is called an *unstable* cubic set.

It is clear that $X = S_{\mathscr{A}} \cup U_{\mathscr{A}}$, $S_{\mathscr{A}} = \{x \in X \mid l(E_{\mathscr{A}}(x)) \geq 0, \ r(E_{\mathscr{A}}(x)) \geq 0\}$ and $U_{\mathscr{A}} = \{x \in X \mid l(E_{\mathscr{A}}(x)) < 0\} \cup \{x \in X \mid r(E_{\mathscr{A}}(x)) < 0\}$.

Example 3.6. Let $\mathscr{A} = \langle A, \lambda \rangle$ be a cubic set in $X = \{0, a, b, c\}$ given by Table 2.

TABLE 2. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
0	$[0.2, 0.3]$	0.10
a	$[0.2, 0.3]$	0.25
b	$[0.7, 0.8]$	0.75
c	$[0.3, 0.7]$	0.80

Then a and b are stable elements of $\mathscr{A}$ in X, and 0 and c are unstable elements of $\mathscr{A}$ in X. Hence $S_{\mathscr{A}} = \{a, b\}$ and $U_{\mathscr{A}} = \{0, c\}$.

Example 3.7. (1) Let $\mathscr{A} = \langle A, \lambda \rangle$ be a cubic set in $X = \{a, b, c\}$ defined by Table 3.

TABLE 3. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
a	$[0.1, 0.6]$	0.5
b	$[0.6, 0.9]$	0.7
c	$[0.1, 0.9]$	0.6

It is routine to verify that $\mathscr{A} = \langle A, \lambda \rangle$ is a stable cubic set.
(2) Let $\mathscr{B} = \langle B, \mu \rangle$ be a cubic set in $X = \{a, b, c\}$ defined by Table 4.

TABLE 4. Tabular representation of the cubic set $\mathscr{B}$

X	$B(x)$	$\mu(x)$
a	$[0.1, 0.3]$	0.5
b	$[0.6, 0.9]$	0.7
c	$[0.1, 0.9]$	0.6

Then $\mathscr{B}$ is an unstable cubic set since $E_{\mathscr{B}}(a) = (0.5 - 0.1, 0.3 - 05) = (0.4, -0.2)$.

Theorem 3.8. *Every ICS is a stable cubic set.*

Proof. Straightforward. □

The following example shows that every ECS would be stable or unstable.

Example 3.9. (1) Let $\mathscr{A} = \langle A, \lambda \rangle$ be an ECS in $X = \{a, b, c\}$ given by Table 5.

TABLE 5. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
a	$[0.1, 0.6]$	0.6
b	$[0.6, 0.9]$	0.5
c	$[0.1, 0.9]$	0.1

Then $\mathscr{A}$ is unstable because $E_{\mathscr{A}}(b) = (0.5 - 0.6, 0.9 - 0.5) = (-0.1, 0.4)$.
(2) Let $\mathscr{B} = \langle B, \mu \rangle$ be an ECS in $X = \{a, b, c\}$ defined by Table 6.

TABLE 6. Tabular representation of the cubic set $\mathscr{B}$

X	$B(x)$	$\mu(x)$
a	$[0.1, 0.3]$	0.1
b	$[0.6, 0.9]$	0.9
c	$[0.1, 0.9]$	0.1

Then $\mathscr{B}$ is stable since $E_{\mathscr{B}}(a) = (0, 0.2)$, $E_{\mathscr{B}}(b) = (0.3, 0)$, and $E_{\mathscr{B}}(c) = (0, 0.8)$.

We provide a condition for an ECS to be a stable cubic set.

Theorem 3.10. *If an ECS $\mathscr{A} = \langle A, \lambda \rangle$ in X satisfies the following condition*

$$(\forall x \in X) \left(\mathscr{A}^-(x) = \lambda(x) \text{ or } \mathscr{A}^+(x) = \lambda(x) \right), \tag{3.2}$$

then $\mathscr{A} = \langle A, \lambda \rangle$ is a stable cubic set.

Proof. Straightforward. □

Corollary 3.11. *Let $\mathscr{A} = \langle A, \lambda \rangle$ be a cubic set in X. If $\mathscr{A}$ is both an ICS and an ECS, then $\mathscr{A}$ is stable.*

Proof. Straightforward. □

Theorem 3.12. *The complement of a stable cubic set is also stable.*

Proof. Let $\mathscr{A} = \langle A, \lambda \rangle$ be a stable cubic set in X. Then $X = S_{\mathscr{A}} = \{x \in X \mid l(E_{\mathscr{A}}(x)) \geq 0, \; r(E_{\mathscr{A}}(x)) \geq 0\}$. Hence $\lambda(x) - A(x)^- \geq 0$ and $A(x)^+ - \lambda(x) \geq 0$ for all $x \in X$. It follows that $l(E_{\mathscr{A}^c}(x)) = (1 - \lambda(x)) - (1 - A(x)^+) = A(x)^+ - \lambda(x) \geq 0$ and $r(E_{\mathscr{A}^c}(x)) = (1 - A(x)^-) - (1 - \lambda(x)) = \lambda(x) - A(x)^- \geq 0$. Therefore $\mathscr{A}^c = \langle A^c, \lambda^c \rangle$ is a stable cubic set. □

Theorem 3.13. *The complement of an unstable cubic set is also unstable.*

Proof. Let $\mathscr{A} = \langle A, \lambda \rangle$ be an unstable cubic set in X. Then $U_{\mathscr{A}} = \{x \in X \mid l(E_{\mathscr{A}}(x)) < 0\} \cup \{x \in X \mid r(E_{\mathscr{A}}(x)) < 0\} \neq \emptyset$, and so there exist $x \in X$ such that $\lambda(x) - A(x)^- < 0$ or $A(x)^+ - \lambda(x) < 0$. It follows that $l(E_{\mathscr{A}^c}(x)) = (1 - \lambda(x)) - (1 - A(x)^+) = A(x)^+ - \lambda(x)) < 0$ or $r(E_{\mathscr{A}^c}(x)) = (1 - A(x)^-) - (1 - \lambda(x)) = \lambda(x) - A(x)^- < 0$. Hence $U_{\mathscr{A}^c} \neq \emptyset$, and therefore $\mathscr{A}^c = \langle A^c, \lambda^c \rangle$ is an unstable cubic set in X. $\square$

The following example illustrates Theorem 3.13.

Example 3.14. Note that the cubic set $\mathscr{B} = \langle B, \mu \rangle$ in Example 3.7(2) is unstable, and its complement is represented by Table 7.

TABLE 7. Tabular representation of the cubic set $\mathscr{B}^c$

X	$B^c(x)$	$\mu^c(x)$
a	$[0.7, 0.9]$	0.5
b	$[0.1, 0.4]$	0.3
c	$[0.1, 0.9]$	0.4

Then $\mathscr{B}^c = \langle B^c, \mu^c \rangle$ is unstable since $a \in U_{\mathscr{B}^c}$.

Theorem 3.15. *The P-union and P-intersection of two stable cubic sets in X are stable cubic sets in X.*

Proof. Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be stable cubic sets in X. Then $S_{\mathscr{A}} = \{x \in X \mid l(E_{\mathscr{A}}(x)) \geq 0, \ r(E_{\mathscr{A}}(x)) \geq 0\} = X$ and $S_{\mathscr{B}} = \{x \in X \mid l(E_{\mathscr{B}}(x)) \geq 0, \ r(E_{\mathscr{B}}(x)) \geq 0\} = X$. It follows that $\lambda(x) - A(x)^- \geq 0, \ A(x)^+ - \lambda(x) \geq 0$ for all $x \in X$ and $\mu(x) - B(x)^- \geq 0, \ B(x)^+ - \mu(x) \geq 0$ for all $x \in X$. Assume that $\lambda(x) \geq \mu(x)$ and consider four cases:

   (i) $A(x)^- \geq B(x)^-$ and $A(x)^+ \geq B(x)^+$,
   (ii) $A(x)^- \geq B(x)^-$ and $A(x)^+ \leq B(x)^+$,
   (iii) $A(x)^- \leq B(x)^-$ and $A(x)^+ \geq B(x)^+$,
   (iv) $A(x)^- \leq B(x)^-$ and $A(x)^+ \leq B(x)^+$.

The first case implies that $\max\{\lambda(x), \mu(x)\} = \lambda(x) \geq A(x)^- = \max\{A(x)^-, B(x)^-\}$ and $\max\{\lambda(x), \mu(x)\} = \lambda(x) \leq A(x)^+ = \max\{A(x)^+, B(x)^+\}$. It follows that $\lambda(x) - A(x)^- \geq 0$ and $A(x)^+ - \lambda(x) \geq 0$. From the second case, we have $\max\{\lambda(x), \mu(x)\} = \lambda(x) \geq A(x)^- = \max\{A(x)^-, B(x)^-\}$ and $\max\{\lambda(x), \mu(x)\} = \lambda(x) \leq B(x)^+ = \max\{A(x)^+, B(x)^+\}$. Hence $\lambda(x) - A(x)^- \geq 0$ and $B(x)^+ - \lambda(x) \geq A(x)^+ - \lambda(x) \geq 0$. The third case induces $\max\{\lambda(x), \mu(x)\} = \lambda(x) \geq \mu(x) \geq B(x)^- = \max\{A(x)^-, B(x)^-\}$ and $\max\{\lambda(x), \mu(x)\} = \lambda(x) \leq A(x)^+ = \max\{A(x)^+, B(x)^+\}$, and so $\lambda(x) - B(x)^- \geq \mu(x) - B(x)^- \geq 0$ and $A(x)^+ - \lambda(x) \geq 0$. For the final case, we get $\max\{\lambda(x), \mu(x)\} = \lambda(x) \geq \mu(x) \geq B(x)^- = \max\{A(x)^-, B(x)^-\}$ and $\max\{\lambda(x), \mu(x)\} =$

$\lambda(x) \leq A(x)^+ \leq B(x) = \max\{A(x)^+, B(x)^+\}$. Thus $\lambda(x) - B(x)^- \geq \mu(x) - B(x)^- \geq 0$ and $B(x)^+ - \lambda(x) \geq 0$. In the case of $\mu(x) \geq \lambda(x)$, we can obtain the same results in a similar way. Therefore $\mathscr{A} \sqcup \mathscr{B}$ is a stable cubic set in X. By the similar method, we know that $\mathscr{A} \sqcap \mathscr{B}$ is a stable cubic set in X. $\qquad \square$

The following example shows that the R-union and the R-intersection of two stable cubic sets in X may not be stable in X.

Example 3.16. Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be cubic sets in $X = \{a, b, c\}$ defined by Tables 8 and 9, respectively.

TABLE 8. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
a	$[0.2, 0.3]$	0.20
b	$[0.7, 0.8]$	0.75
c	$[0.3, 0.7]$	0.60

TABLE 9. Tabular representation of the cubic set $\mathscr{B}$

X	$B(x)$	$\mu(x)$
a	$[0.1, 0.3]$	0.15
b	$[0.6, 0.9]$	0.70
c	$[0.1, 0.9]$	0.80

Then

$$\mathscr{A} \Cup \mathscr{B} = \{\langle a, [0.2, 0.3], 0.15 \rangle, \langle b, [0.7, 0.9], 0.7 \rangle, \langle c, [0.3, 0.9], 0.6 \rangle\}$$

and

$$\mathscr{A} \Cap \mathscr{B} = \{\langle a, [0.1, 0.3], 0.2 \rangle, \langle b, [0.6, 0.8], 0.75 \rangle, \langle c, [0.1, 0.7], 0.8 \rangle\}.$$

Hence we know that $E_{\mathscr{A} \Cup \mathscr{B}}(a) = \langle -0.05, 0.15 \rangle$ and $E_{\mathscr{A} \Cap \mathscr{B}}(c) = \langle 0.7, -0.1 \rangle$. Thus $\mathscr{A} \Cup \mathscr{B}$ and $\mathscr{A} \Cap \mathscr{B}$ are unstable.

Now, we provide conditions for the R-union (resp. R-intersection) of two ICSs to be stable.

Theorem 3.17. *Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be ICSs in X such that*

$$(\forall x \in X) \left(\max\{A(x)^-, B(x)^-\} \leq (\lambda \wedge \mu)(x) \right). \tag{3.3}$$

Then the R-union of $\mathscr{A}$ and $\mathscr{B}$ is a stable cubic set in X.

Proof. Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be ICSs in X. Then $A(x)^- \leq \lambda(x) \leq A(x)^+$ and $B(x)^- \leq \mu(x) \leq B(x)^+$ for all $x \in X$. It follows from (3.3) that $\max\{A(x)^-, B(x)^-\} \leq (\lambda \wedge \mu)(x) \leq \max\{A(x)^+, B(x)^+\}$ for all $x \in X$. Hence the R-union of $\mathscr{A}$ and $\mathscr{B}$ is an ICS, and so it is stable by Theorem 3.8. $\qquad\square$

Theorem 3.18. *Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be ICSs in X such that*

$$(\forall x \in X) \left(\max\{A(x)^+, B(x)^+\} \leq (\lambda \vee \mu)(x) \right). \tag{3.4}$$

Then the R-intersection of $\mathscr{A}$ and $\mathscr{B}$ is a stable cubic set in X.

Proof. The proof is by the similar method to Theorem 3.17. $\qquad\square$

Theorem 3.19. *Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be ECSs in X such that $\mathscr{A}^* = \langle A, \mu \rangle$ and $\mathscr{B}^* = \langle B, \lambda \rangle$ are ICSs in X. Then the P-union $\mathscr{A} \sqcup \mathscr{B}$ and the P-intersection $\mathscr{A} \sqcap \mathscr{B}$ of $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ are stable in X.*

Proof. It is straightforward by Theorems 3.20 and 3.21 in [4] and Theorem 3.8. $\qquad\square$

Definition 3.20. Let $\mathscr{A} = \langle A, \lambda \rangle$ be a cubic set with the evaluative set $\mathbf{E}_{\mathscr{A}} = \{(x, E_{\mathscr{A}}(x)) \mid x \in X\}$ in X. Then the *stable degree* of $\mathscr{A}$ in X is denoted by $SD_{\mathscr{A}}$ and is defined by

$$SD_{\mathscr{A}} = \left(\sum_{x \in X} l(E_{\mathscr{A}}(x)), \sum_{x \in X} r(E_{\mathscr{A}}(x)) \right). \tag{3.5}$$

Definition 3.21. A cubic set $\mathscr{A} = \langle A, \lambda \rangle$ with the evaluative set $\mathbf{E}_{\mathscr{A}} = \{(x, E_{\mathscr{A}}(x)) \mid x \in X\}$ in X is said to be *almost stable* if there exists the stable degree $SD_{\mathscr{A}}$ in which $\sum_{x \in X} l(E_{\mathscr{A}}(x)) \geq 0$ and $\sum_{x \in X} r(E_{\mathscr{A}}(x)) \geq 0$.

Example 3.22. Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be cubic sets in $X = \{a, b, c\}$ defined by Tables 10 and 11, respectively.

TABLE 10. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
a	$[0.2, 0.3]$	0.2
b	$[0.7, 0.8]$	0.9
c	$[0.3, 0.7]$	0.6

Then

$$\mathbf{E}_{\mathscr{A}} = \{(a, \langle 0, 0.1 \rangle), (b, \langle 0.2, -0.1 \rangle), (c, \langle 0.3, 0.1 \rangle)\}$$

and

TABLE 11. Tabular representation of the cubic set $\mathscr{B}$

X	$B(x)$	$\mu(x)$
a	$[0.2, 0.3]$	0.9
b	$[0.6, 0.9]$	0.7
c	$[0.1, 0.9]$	1

$$\mathbf{E}_{\mathscr{B}} = \{(a, \langle 0.7, -0.6 \rangle), (b, \langle 0.1, 0.2 \rangle), (c, \langle 0.9, -0.1 \rangle)\}.$$

Thus $SD_{\mathscr{A}} = (0 + 0.2 + 0.3, \, 0.1 - 0.1 + 0.1) = (0.5, 0.1)$ and so $\mathscr{A}$ is almost stable. But $\mathscr{B}$ is not almost stable since $SD_{\mathscr{B}} = (0.7 + 0.1 + 0.9, \, -0.6 + 0.2 - 0.1) = (1.7, -0.5)$.

Theorem 3.23. *Every stable cubic set* $\mathscr{A} = \langle A, \lambda \rangle$ *in* X *is almost stable.*

Proof. Straightforward. □

In Example 3.22, the almost stable cubic set $\mathscr{A} = \langle A, \lambda \rangle$ is not stable. This shows that the converse of Theorem 3.23 is not true in general.

Combining Theorems 3.8, 3.10, 3.15, 3.19 and 3.23, we know that

(1) Every ICS is almost stable.
(2) Every ESC satisfying the condition (3.2) is almost stable.
(3) The P-union and P-intersection of two stable cubic sets is almost stable.
(4) If $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ are ECSs in X such that $\mathscr{A}^* = \langle A, \mu \rangle$ and $\mathscr{B}^* = \langle B, \lambda \rangle$ are ICSs in X, then the P-union and the P-intersection of $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ are almost stable in X.

Proposition 3.24. *If* $\mathscr{A} = \langle A, \lambda \rangle$ *and* $\mathscr{B} = \langle B, \mu \rangle$ *are cubic sets in* X*, then either*

$$(\forall x \in X) \left(\max\{\lambda(x), \mu(x)\} - \max\{A(x)^-, B(x)^-\} \leq \lambda(x) - A(x)^- \right) \tag{3.6}$$

or

$$(\forall x \in X) \left(\max\{\lambda(x), \mu(x)\} - \max\{A(x)^-, B(x)^-\} \leq \mu(x) - B(x)^- \right). \tag{3.7}$$

Proof. For each $x \in X$, we consider the four cases as follows:

(1) $\max\{\lambda(x), \mu(x)\} = \lambda(x)$ and $\max\{A(x)^-, B(x)^-\} = A(x)^-$.
(2) $\max\{\lambda(x), \mu(x)\} = \lambda(x)$ and $\max\{A(x)^-, B(x)^-\} = B(x)^-$.
(3) $\max\{\lambda(x), \mu(x)\} = \mu(x)$ and $\max\{A(x)^-, B(x)^-\} = A(x)^-$.
(4) $\max\{\lambda(x), \mu(x)\} = \mu(x)$ and $\max\{A(x)^-, B(x)^-\} = B(x)^-$.

First two cases induce the inequality (3.6), and the inequality (3.7) is induced by the last two cases. □

Proposition 3.25. *If $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ are cubic sets in X, then either*

$$(\forall x \in X) \left(\max\{A(x)^+, B(x)^+\} - \max\{\lambda(x), \mu(x)\} \leq A(x)^+ - \lambda(x) \right) \quad (3.8)$$

or

$$(\forall x \in X) \left(\max\{A(x)^+, B(x)^+\} - \max\{\lambda(x), \mu(x)\} \leq B(x)^+ - \mu(x) \right). \quad (3.9)$$

Proof. It is similar to the proof of Proposition 3.24. □

In the following example, we know that the P-union and the R-union of almost stable cubic sets may not be almost stable.

Example 3.26. Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be cubic sets in $X = \{a, b, c\}$ defined by Tables 12 and 13, respectively.

TABLE 12. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
a	$[1.0, 1.0]$	0.7
b	$[0.5, 1.0]$	0.7
c	$[0.6, 1.0]$	0.7

TABLE 13. Tabular representation of the cubic set $\mathscr{B}$

X	$B(x)$	$\mu(x)$
a	$[0.5, 1.0]$	0.7
b	$[1.0, 1.0]$	0.7
c	$[0.6, 1.0]$	0.7

Then $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ are almost stable cubic sets in X because

$$\sum_{x \in X} l(E_{\mathscr{A}}(x)) = 0, \ \sum_{x \in X} r(E_{\mathscr{A}}(x)) = 0.9, \ \sum_{x \in X} l(E_{\mathscr{B}}(x)) = 0, \ \text{and} \ \sum_{x \in X} r(E_{\mathscr{B}}(x)) = 0.9.$$

But the P-union $\mathscr{A} \sqcup \mathscr{B}$ and the R-union $\mathscr{A} \uplus \mathscr{B}$ of $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ are not almost stable because $\sum_{x \in X} l(E_{\mathscr{A} \sqcup \mathscr{B}}(x)) = \sum_{x \in X} (\max\{\lambda(x), \mu(x)\} - \max\{A(x)^-, B(x)^-\}) = -0.5 \not\geq 0$ and $\sum_{x \in X} l(E_{\mathscr{A} \uplus \mathscr{B}}(x)) = \sum_{x \in X} (\min\{\lambda(x), \mu(x)\} - \max\{A(x)^-, B(x)^-\}) = -0.5 \not\geq 0.$

We now provide conditions for the P-union of almost stable cubic sets to be almost stable.

Theorem 3.27. *Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be almost stable cubic sets in X such that*

$$(\forall x \in X) \left(\sum_{x \in X} (|\lambda(x) - \mu(x)| - A(x)^-) \geq 0, \ \sum_{x \in X} (|A(x)^+ - B(x)^+| - \lambda(x)) \geq 0 \right). \quad (3.10)$$

Then the P-union $\mathscr{A} \sqcup \mathscr{B} = \langle A \cup B, \lambda \vee \mu \rangle$ of $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ is almost stable in X.

Proof. Assume that $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ are almost stable in X. Then there exist stable degrees $SD_{\mathscr{A}}$ and $SD_{\mathscr{B}}$, respectively, such that

$$\sum_{x \in X} l(E_{\mathscr{A}}(x)) = \sum_{x \in X} (\lambda(x) - A(x)^-) \geq 0, \sum_{x \in X} r(E_{\mathscr{A}}(x)) = \sum_{x \in X} (A(x)^+ - \lambda(x)) \geq 0,$$

$$\sum_{x \in X} l(E_{\mathscr{B}}(x)) = \sum_{x \in X} (\mu(x) - B(x)^-) \geq 0, \text{ and } \sum_{x \in X} r(E_{\mathscr{B}}(x)) = \sum_{x \in X} (B(x)^+ - \mu(x)) \geq 0.$$

Now, we have to show that $\sum_{x \in X} l(E_{\mathscr{A} \sqcup \mathscr{B}}(x)) \geq 0$ and $\sum_{x \in X} r(E_{\mathscr{A} \sqcup \mathscr{B}}(x)) \geq 0$ in the stable degree $SD_{\mathscr{A} \sqcup \mathscr{B}}$ of $\mathscr{A} \sqcup \mathscr{B}$. Using (3.10), we have

$$\sum_{x \in X} l(E_{\mathscr{A} \sqcup \mathscr{B}}(x)) = \sum_{x \in X} \left((\lambda \vee \mu)(x) - (A \cup B)(x)^- \right)$$

$$= \sum_{x \in X} \left(\max\{\lambda(x), \mu(x)\} - \max\{A(x)^-, B(x)^-\} \right)$$

$$= \sum_{x \in X} \left(\frac{|\lambda(x) - \mu(x)| + \lambda(x) + \mu(x)}{2} - \frac{|A(x)^- - B(x)^-| + A(x)^- + B(x)^-}{2} \right)$$

$$= \sum_{x \in X} \left(\frac{|\lambda(x) - \mu(x)| - |A(x)^- - B(x)^-| + \lambda(x) - A(x)^- + \mu(x) - B(x)^-}{2} \right)$$

$$= \tfrac{1}{2} \sum_{x \in X} \left(|\lambda(x) - \mu(x)| - |A(x)^- - B(x)^-| + \lambda(x) - A(x)^- + \mu(x) - B(x)^- \right)$$

$$= \tfrac{1}{2} \sum_{x \in X} \left(|\lambda(x) - \mu(x)| - |A(x)^- - B(x)^-| \right)$$

$$\qquad + \tfrac{1}{2} \sum_{x \in X} \left(\lambda(x) - A(x)^- \right) + \tfrac{1}{2} \sum_{x \in X} \left(\mu(x) - B(x)^- \right)$$

$$\geq \tfrac{1}{2} \sum_{x \in X} \left(|\lambda(x) - \mu(x)| - A(x)^- \right) + \tfrac{1}{2} \sum_{x \in X} \left(\lambda(x) - A(x)^- \right) + \tfrac{1}{2} \sum_{x \in X} \left(\mu(x) - B(x)^- \right)$$

$$\geq 0.$$

Similarly, we have $\sum_{x \in X} r(E_{\mathscr{A} \sqcup \mathscr{B}}(x)) \geq 0$. Therefore $\mathscr{A} \sqcup \mathscr{B} = \langle A \cup B, \lambda \vee \mu \rangle$ is almost stable in X. $\qquad \square$

Theorem 3.28. *The complement of an almost stable cubic set is also almost stable.*

Proof. Let $\mathscr{A} = \langle A, \lambda \rangle$ be an almost stable cubic set in X. Then there exists a stable degree $SD_{\mathscr{A}}$ such that

$$\sum_{x \in X} l(E_{\mathscr{A}}(x)) = \sum_{x \in X} (\lambda(x) - A(x)^-) \geq 0, \text{ and } \sum_{x \in X} r(E_{\mathscr{A}}(x)) = \sum_{x \in X} (A(x)^+ - \lambda(x)) \geq 0.$$

It follows that $\sum\limits_{x \in X} l(E_{\mathscr{A}^c}(x)) = \sum\limits_{x \in X} ((1 - \lambda(x)) - (1 - A(x)^+)) = \sum\limits_{x \in X} (A(x)^+ - \lambda(x)) \geq 0$ and $\sum\limits_{x \in X} r(E_{\mathscr{A}^c}(x)) = \sum\limits_{x \in X} ((1 - A(x)^-) - (1 - \lambda(x))) = \sum\limits_{x \in X} (\lambda(x) - A(x)^-) \geq 0$. Therefore $\mathscr{A}^c = \langle A^c, \lambda^c \rangle$ is almost stable. $\square$

We now provide conditions for the R-union of almost stable cubic sets to be almost stable.

Theorem 3.29. *Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be almost stable cubic sets in X such that*

$$\sum_{x \in X} (|\lambda(x) - \mu(x)| + |A(x)^- - B(x)^-|) \leq \left(\sum_{x \in X} \lambda(x) - A(x)^- \right) + \sum_{x \in X} (\mu(x) - B(x)^-) \quad (3.11)$$

and

$$\sum_{x \in X} (|\lambda(x) - \mu(x)| + |A(x)^+ - B(x)^+|) \geq \sum_{x \in X} (\lambda(x) - A(x)^+) + \sum_{x \in X} (\mu(x) - B(x)^+) \quad (3.12)$$

for all $x \in X$. Then the R-union $\mathscr{A} \Cup \mathscr{B} = \langle A \cup B, \lambda \wedge \mu \rangle$ is almost stable in X.

Proof. Assume that $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ are almost stable in X. Then there exist stable degrees $SD_{\mathscr{A}}$ and $SD_{\mathscr{B}}$, respectively, such that

$$\sum_{x \in X} l(E_{\mathscr{A}}(x)) = \sum_{x \in X} (\lambda(x) - A(x)^-) \geq 0, \sum_{x \in X} r(E_{\mathscr{A}}(x)) = \sum_{x \in X} (A(x)^+ - \lambda(x)) \geq 0,$$

$$\sum_{x \in X} l(E_{\mathscr{B}}(x)) = \sum_{x \in X} (\mu(x) - B(x)^-) \geq 0, \text{ and } \sum_{x \in X} r(E_{\mathscr{B}}(x)) = \sum_{x \in X} (B(x)^+ - \mu(x)) \geq 0.$$

It follows from (3.11) that

$$\sum_{x \in X} l(E_{\mathscr{A} \Cup \mathscr{B}}(x)) = \sum_{x \in X} \left((\lambda \wedge \mu)(x) - (A \cup B)(x)^- \right)$$

$$= \sum_{x \in X} \left(\min\{\lambda(x), \mu(x)\} - \max\{A(x)^-, B(x)^-\} \right)$$

$$= \sum_{x \in X} \left(\frac{-|\lambda(x) - \mu(x)| + \lambda(x) + \mu(x)}{2} - \frac{|A(x)^- - B(x)^-| + A(x)^- + B(x)^-}{2} \right)$$

$$= \sum_{x \in X} \left(\frac{-|\lambda(x) - \mu(x)| - |A(x)^- - B(x)^-| + \lambda(x) - A(x)^- + \mu(x) - B(x)^-}{2} \right)$$

$$= -\frac{1}{2} \sum_{x \in X} \left(|\lambda(x) - \mu(x)| + |A(x)^- - B(x)^-| \right)$$

$$+ \frac{1}{2} \sum_{x \in X} \left(\lambda(x) - A(x)^- \right) + \frac{1}{2} \sum_{x \in X} \left(\mu(x) - B(x)^- \right)$$

$$\geq -\frac{1}{2} \left(\sum_{x \in X} \left(\lambda(x) - A(x)^- \right) + \sum_{x \in X} \left(\mu(x) - B(x)^- \right) \right)$$

$$+ \frac{1}{2} \sum_{x \in X} \left(\lambda(x) - A(x)^- \right) + \frac{1}{2} \sum_{x \in X} \left(\mu(x) - B(x)^- \right) = 0.$$

Using (3.12), we have

$$\sum_{x \in X} r(E_{\mathscr{A} \Cup \mathscr{B}}(x)) = \sum_{x \in X} \left((A \cup B)(x)^+ - (\lambda \wedge \mu)(x) \right)$$

$$= \sum_{x \in X} \left(\max\{A(x)^-, B(x)^-\} - \min\{\lambda(x), \mu(x)\} \right)$$

$$= \sum_{x \in X} \left(\tfrac{|A(x)^+ - B(x)^+| + A(x)^+ + B(x)^+}{2} - \tfrac{-|\lambda(x) - \mu(x)| + \lambda(x) + \mu(x)}{2} \right)$$

$$= \tfrac{1}{2} \sum_{x \in X} \left(|\lambda(x) - \mu(x)| + |A(x)^+ - B(x)^+| \right)$$

$$- \tfrac{1}{2} \left(\sum_{x \in X} \left(\lambda(x) - A(x)^+ \right) + \sum_{x \in X} \left(\mu(x) - B(x)^+ \right) \right) \geq 0.$$

Hence $\mathscr{A} \Cup \mathscr{B} = \langle A \cup B, \lambda \wedge \mu \rangle$ is almost stable in X. $\square$

The following examples show that the P-intersection and the R-intersection of almost stable cubic sets may not be almost stable.

Example 3.30. (1) Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be cubic sets in $X = \{a, b, c\}$ defined by Tables 14 and 15, respectively.

TABLE 14. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
a	$[0.7, 1.0]$	0.4
b	$[0.5, 1.0]$	0.8
c	$[0.6, 1.0]$	0.7

TABLE 15. Tabular representation of the cubic set $\mathscr{B}$

X	$B(x)$	$\mu(x)$
a	$[0.5, 1.0]$	0.8
b	$[0.6, 1.0]$	0.7
c	$[0.7, 1.0]$	0.4

Then $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ are almost stable cubic sets in X because

$$\sum_{x \in X} l(E_{\mathscr{A}}(x)) = 0.1, \quad \sum_{x \in X} r(E_{\mathscr{A}}(x)) = 1.1, \quad \sum_{x \in X} l(E_{\mathscr{B}}(x)) = 0.1, \text{ and } \sum_{x \in X} r(E_{\mathscr{B}}(x)) = 1.1.$$

But the P-intersection $\mathscr{A} \sqcap \mathscr{B}$ of $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ is not almost stable because

$$\sum_{x \in X} l(E_{\mathscr{A} \sqcap \mathscr{B}}(x)) = \sum_{x \in X} \left(\min\{\lambda(x), \mu(x)\} - \min\{A(x)^-, B(x)^-\} \right) = -0.1 \not\geq 0.$$

(2) Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be cubic sets in $X = \{a, b, c\}$ defined by Tables 16 and 17, respectively.

TABLE 16. Tabular representation of the cubic set $\mathscr{A}$

X	$A(x)$	$\lambda(x)$
a	$[0.2, 0.7]$	0.8
b	$[0.3, 0.6]$	0.5
c	$[0.1, 0.5]$	0.5

TABLE 17. Tabular representation of the cubic set $\mathscr{B}$

X	$B(x)$	$\mu(x)$
a	$[0.2, 0.7]$	0.6
b	$[0.3, 0.6]$	0.7
c	$[0.1, 0.5]$	0.5

Then $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ are almost stable cubic sets in X because

$$\sum_{x \in X} l(E_{\mathscr{A}}(x)) = 1.2, \ \sum_{x \in X} r(E_{\mathscr{A}}(x)) = 0, \ \sum_{x \in X} l(E_{\mathscr{B}}(x)) = 1.2, \ \text{and} \ \sum_{x \in X} r(E_{\mathscr{B}}(x)) = 0.$$

But the R-intersection $\mathscr{A} \sqcap\!\!\!\!\!\cap \mathscr{B}$ of $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ is not almost stable since

$$\sum_{x \in X} r(E_{\mathscr{A} \sqcap\!\!\!\cap \mathscr{B}}(x)) = \sum_{x \in X} \left(\min\{A(x)^+, B(x)^+\} - \max\{\lambda(x), \mu(x)\} \right) = -0.2 \not\geq 0.$$

We now provide conditions for the P-intersection and the R-intersection of almost stable cubic sets to be almost stable.

Theorem 3.31. *Let $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ be almost stable cubic sets in X.*
(i) Assume that the following condition is valid.

$$(\forall x \in X) \left(\begin{array}{c} \sum_{x \in X} (|A(x)^- - B(x)^-| - |\lambda(x) - \mu(x)|) \geq 0, \\ \sum_{x \in X} (|\lambda(x) - \mu(x)| - |A(x)^+ - B(x)^+|) \geq 0 \end{array} \right). \tag{3.13}$$

Then the P-intersection $\mathscr{A} \sqcap \mathscr{B} = \langle A \cap B, \lambda \wedge \mu \rangle$ of $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ is almost stable in X.

(ii) *If $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ satisfy the following condition*

$$(\forall x \in X) \left(\sum_{x \in X} \left(|\lambda(x) - \mu(x)| + |A(x)^+ - B(x)^+| \right) = 0 \right), \tag{3.14}$$

then the R-intersection $\mathscr{A} \Cap \mathscr{B} = \langle A \cap B, \lambda \vee \mu \rangle$ of $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ is almost stable in X.

Proof. Since $\mathscr{A} = \langle A, \lambda \rangle$ and $\mathscr{B} = \langle B, \mu \rangle$ are almost stable in X, there exist stable degrees $SD_{\mathscr{A}}$ and $SD_{\mathscr{B}}$, respectively, such that

$$\sum_{x \in X} l(E_{\mathscr{A}}(x)) = \sum_{x \in X} (\lambda(x) - A(x)^-) \geq 0, \ \sum_{x \in X} r(E_{\mathscr{A}}(x)) = \sum_{x \in X} (A(x)^+ - \lambda(x)) \geq 0,$$
$$\sum_{x \in X} l(E_{\mathscr{B}}(x)) = \sum_{x \in X} (\mu(x) - B(x)^-) \geq 0, \text{and} \sum_{x \in X} r(E_{\mathscr{B}}(x)) = \sum_{x \in X} (B(x)^+ - \mu(x)) \geq 0.$$

(i) We have to show that $\sum_{x \in X} l(E_{\mathscr{A} \sqcap \mathscr{B}}(x)) \geq 0$ and $\sum_{x \in X} r(E_{\mathscr{A} \sqcap \mathscr{B}}(x)) \geq 0$ in the stable degree $SD_{\mathscr{A} \sqcap \mathscr{B}}$ of $\mathscr{A} \sqcap \mathscr{B}$. Using (3.13), we have

$$\sum_{x \in X} l(E_{\mathscr{A} \sqcap \mathscr{B}}(x)) = \sum_{x \in X} \left((\lambda \wedge \mu)(x) - (A \cap B)(x)^- \right)$$
$$= \sum_{x \in X} \left(\min\{\lambda(x), \mu(x)\} - \min\{A(x)^-, B(x)^-\} \right)$$
$$= \sum_{x \in X} \left(\frac{-|\lambda(x) - \mu(x)| + \lambda(x) + \mu(x)}{2} + \frac{|A(x)^- - B(x)^-| - A(x)^- - B(x)^-}{2} \right)$$
$$= \sum_{x \in X} \left(\frac{-|\lambda(x) - \mu(x)| + |A(x)^- - B(x)^-| + \lambda(x) - A(x)^- + \mu(x) - B(x)^-}{2} \right)$$
$$= \tfrac{1}{2} \sum_{x \in X} \left(-|\lambda(x) - \mu(x)| + |A(x)^- - B(x)^-| + \lambda(x) - A(x)^- + \mu(x) - B(x)^- \right)$$
$$= \tfrac{1}{2} \sum_{x \in X} \left(|A(x)^- - B(x)^-| - |\lambda(x) - \mu(x)| \right)$$
$$+ \tfrac{1}{2} \sum_{x \in X} \left((\lambda(x) - (A(x)^-) + \tfrac{1}{2} \sum_{x \in X} \left((\mu(x)) - B(x)^- \right) \right) \geq 0.$$

Similarly, we have $\sum_{x \in X} r(E_{\mathscr{A} \sqcap \mathscr{B}}(x)) \geq 0$. Therefore $\mathscr{A} \sqcap \mathscr{B} = \langle A \cap B, \lambda \wedge \mu \rangle$ is almost stable in X.

(ii) We have

$$\sum_{x \in X} l(E_{\mathscr{A} \Cap \mathscr{B}}(x)) = \sum_{x \in X} \left((\lambda \vee \mu)(x) - (A \cap B)(x)^- \right)$$

$$= \sum_{x \in X} \left(\max\{\lambda(x), \mu(x)\} - \min\{A(x)^-, B(x)^-\} \right)$$

$$= \sum_{x \in X} \left(\frac{|\lambda(x) - \mu(x)| + \lambda(x) + \mu(x)}{2} + \frac{|A(x)^- - B(x)^-| - A(x)^- - B(x)^-}{2} \right)$$

$$= \sum_{x \in X} \left(\frac{|\lambda(x) - \mu(x)| + |A(x)^- - B(x)^-| + \lambda(x) - A(x)^- + \mu(x) - B(x)^-}{2} \right)$$

$$= \tfrac{1}{2} \sum_{x \in X} \left(|\lambda(x) - \mu(x)| + |A(x)^- - B(x)^-| \right)$$

$$+ \tfrac{1}{2} \sum_{x \in X} \left(\lambda(x) - A(x)^- \right) + \tfrac{1}{2} \sum_{x \in X} \left(\mu(x) - B(x)^- \right)$$

$$\geq \tfrac{1}{2} \left(\sum_{x \in X} \left(\lambda(x) - A(x)^- \right) + \sum_{x \in X} \left(\mu(x) - B(x)^- \right) \right) \geq 0.$$

Using (3.14), we have

$$\sum_{x \in X} r(E_{\mathscr{A} \Cap \mathscr{B}}(x)) = \sum_{x \in X} \left((A \cap B)(x)^+ - (\lambda \vee \mu)(x) \right)$$

$$= \sum_{x \in X} \left(\min\{A(x)^+, B(x)^+\} - \max\{\lambda(x), \mu(x)\} \right)$$

$$= \sum_{x \in X} \left(\frac{-|A(x)^+ - B(x)^+| + A(x)^+ + B(x)^+}{2} - \frac{|\lambda(x) - \mu(x)| + \lambda(x) + \mu(x)}{2} \right)$$

$$= \tfrac{1}{2} \sum_{x \in X} \left(-|\lambda(x) - \mu(x)| - |A(x)^+ - B(x)^+| \right)$$

$$+ \tfrac{1}{2} \left(\sum_{x \in X} \left(A(x)^+ - \lambda(x) \right) + \sum_{x \in X} \left(B(x)^+ - \mu(x) \right) \right)$$

$$= \tfrac{1}{2} \left(\sum_{x \in X} \left(A(x)^+ - \lambda(x) \right) + \sum_{x \in X} \left(B(x)^+ - \mu(x) \right) \right) \geq 0.$$

Hence $\mathscr{A} \Cap \mathscr{B} = \langle A \cap B, \lambda \vee \mu \rangle$ is almost stable in X. $\qquad\square$

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SOME IDENTITIES OF CHEBYSHEV POLYNOMIALS ARISING FROM NON-LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we investigate some properties of Chebyshev polynomials arising from non-linear differential equations. From our investigation, we derive some new and interesting identities on Chebyshev polynomials.

1. INTRODUCTION

As is well known, the Chebyshev polynomials of the first kind, $T_n(x)$, $(n \geq 0)$, are defined by the generating function

$$(1.1) \qquad \frac{1 - t^2}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!}, \quad \text{(see [1, 3, 5, 8, 17, 21])}.$$

The higher-order Chebyshev polynomials are given by the generating function

$$(1.2) \qquad \left(\frac{1 - t^2}{1 - 2xt + t^2} \right)^{\alpha} = \sum_{n=0}^{\infty} T_n^{(\alpha)}(x) t^n,$$

and Chebyshev polynomials of the second kind are denoted by U_n and given by generating function

$$(1.3) \qquad \frac{1}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} U_n(x) t^n, \quad \text{(see [1, 7, 12, 17])}.$$

The higher-order Chebyshev polynomials of the second kind are also defined by

$$(1.4) \qquad \left(\frac{1}{1 - 2xt + t^2} \right)^{\alpha} = \sum_{n=0}^{\infty} U_n^{(\alpha)}(x) t^n.$$

The Chebyshev polynomials of the third kind are defined by the generating function

$$(1.5) \qquad \frac{1 - t}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} V_n(x) t^n, \quad \text{(see [1, 7, 8, 17])}.$$

and the higher-order Chebyshev polynomials of the third kind are also given by the generating function

$$(1.6) \qquad \left(\frac{1 - t}{1 - 2xt + t^2} \right)^{\alpha} = \sum_{n=0}^{\infty} V_n^{(\alpha)}(x) t^n.$$

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Finally, we introduce the Chebyshev polynomials of the fourth kind defined by the generating function

$$(1.7) \qquad \frac{1+t}{1-2xt+t^2} = \sum_{n=0}^{\infty} W_n(x) t^n.$$

The higher-order Chebyshev polynomials of the fourth kind are defined by

$$(1.8) \qquad \left(\frac{1+t}{1-2xt+t^2}\right)^{\alpha} = \sum_{n=0}^{\infty} W_n^{(\alpha)}(x) t^n.$$

It is well known that the Legendre polynomials are defined by the generating function

$$(1.9) \qquad \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} p_n(x) t^n, \quad (\text{see } [2, 20]).$$

Chebyshev polynomials are important in approximation theory because the roots of the Chebyshev polynomials of the first kind, which are also called Chebyshev nodes, are used as nodes in polynomial nodes (see [19]).

The Chebyshev polynomials of the first kind and of the second kind are solutions of the following Chebyshev differential equations

$$(1.10) \qquad (1-x^2) y'' - xy' + n^2 y = 0,$$

and

$$(1.11) \qquad (1-x^2) y'' - 3xy' + n(n+2) y = 0.$$

These equations are special cases of the Strum-Liouville differential equation (see [1–3]).

The Chebyshev polynomials of the first kind can be defined by the contour integral

$$(1.12) \qquad T_n(z) = \frac{1}{4\pi i} \oint \frac{(1-t^2)}{1-2tz+t^2} t^{-n-1} dt,$$

where the contour encloses the origin and is traversed in a counterclockwise direction (see [1, 19, 21]). The formula for $T_n(x)$ is given by

$$(1.13) \qquad T_n(x) = \sum_{m=0}^{\left[\frac{n}{2}\right]} \binom{n}{2m} x^{n-2m} (x^2-1)^m.$$

From (1.3), we note that

$$(1.14) \qquad 2(x-t)(1-2xt+t^2)^{-2} = \sum_{n=0}^{\infty} nU_n(x) t^{n-1}.$$

Thus, by (1.14), we get

$$(1.15) \qquad (2xt-2t^2)(1-2xt+t^2)^{-2} = \sum_{n=0}^{\infty} nU_n(x) t^n.$$

From (1.3) and (1.15), we can derive the following equation:

$$(1.16) \qquad \frac{(2xt-2t^2)+(1-2xt+t^2)}{(1-2xt+t^2)^2} = \frac{1-t^2}{(1-2xt+t^2)^2}$$

$$= \sum_{n=0}^{\infty} (n+1) U_n (x) t^n.$$

Note that

$$(1.17) \qquad \frac{1-t^2}{(1-2xt+t^2)^2}$$

$$= \left(\frac{1-t^2}{1-2xt+t^2} \right) \left(\frac{1}{1-2xt+t^2} \right)$$

$$= \left(\sum_{l=0}^{\infty} T_l (x) t^l \right) \left(\sum_{m=0}^{\infty} U_m (x) t^m \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} T_l (x) U_{n-l} (x) \right) t^n.$$

From (1.16) and (1.17), we have

$$U_n (x) = \frac{1}{n+1} \sum_{l=0}^{n} T_l (x) U_{n-l} (x).$$

The Chebyshev polynomials have been studied by many authors in the several areas (see [1–21]).

In [11], Kim-Kim studied non-linear differential equations arising from Changhee polynomials and numbers related to Chebyshev poynomials.

In this paper, we study non-linear differential equations arising from Chebyshev polynomials and give some new and explicit formulas for those polynomials.

2. Differential equations arising from Chebyshev polynomials and their applications

Let

$$(2.1) \qquad F = F (t, x) = \frac{1}{1-2tx+t^2}.$$

Then, by (1.1), we get

$$(2.2) \qquad F^{(1)} = \frac{d}{dt} F (t, x) = 2 (x - t) F^2.$$

From (2.2), we note that

$$(2.3) \qquad 2F^2 = (x - t)^{-1} F^{(1)}.$$

By using (2.3) and (2.2), we obtain the following equations:

$$(2.4) \qquad 2^2 \cdot 2F^3 = (x - t)^{-3} F^{(1)} + (x - t)^{-2} F^{(2)},$$

$$(2.5) \qquad 2^3 \cdot 2 \cdot 3F^4 = 3 (x - t)^{-5} F^{(1)} + 3 (x - t)^{-4} F^{(2)} + (x - t)^{-3} F^{(3)}$$

and

$$(2.6) \qquad 2^4 \cdot 2 \cdot 3 \cdot 4F^5 = 3 \cdot 5 (x - t)^{-6} F^{(1)} + 3 \cdot 5 (x - t)^{-6} F^{(2)}$$
$$+ (3 \cdot 2) (x - t)^{-5} F^{(3)} + (x - t)^{-4} F^{(4)},$$

where

$$F^N = \underbrace{F \times \cdots \times F}_{N-\text{times}} \quad \text{and} \quad F^{(N)} = \left(\frac{d}{dt}\right)^N F\,(t,x)\,.$$

Continuing this process, we set

$$(2.7) \qquad 2^N N! F^{N+1} = \sum_{i=1}^{N} a_i\,(N)\,(x-t)^{i-2N}\,F^{(i)},$$

where $N \in \mathbb{N}$.

From (2.7), we note that

$$(2.8) \qquad 2^N N! F^N\,(N+1)\,F^{(1)}$$
$$= \sum_{i=1}^{N} a_i\,(N)\,(2N-i)\,(x-t)^{i-2N-1}\,F^{(i)} + \sum_{i=1}^{N} a_i\,(N)\,(x-t)^{i-2N}\,F^{(i+1)}.$$

By (2.2) and (2.8), we get

$$(2.9) \qquad 2^N N!\,(N+1)\,F^N\left(2\,(x-t)\,F^2\right)$$
$$= \sum_{i=1}^{N} a_i\,(N)\,(2N-i)\,(x-t)^{i-2N-1}\,F^{(i)}$$
$$+ \sum_{i=1}^{N} a_i\,(N)\,(x-t)^{i-2N}\,F^{(i+1)}.$$

Thus, from (2.9), we have

$$(2.10) \qquad 2^{N+1}\,(N+1)! F^{N+2}$$
$$= \sum_{i=1}^{N} a_i\,(N)\,(2N-i)\,(x-t)^{i-2(N+1)}\,F^{(i)}$$
$$+ \sum_{i=2}^{N+1} a_{i-1}\,(N)\,(x-t)^{i-2(N+1)}\,F^{(i)}.$$

On the other hand, by replacing N by $N+1$, in (2.7), we get

$$(2.11) \qquad 2^{N+1}\,(N+1)! F^{N+2} = \sum_{i=1}^{N+1} a_i\,(N+1)\,(x-t)^{i-2(N+1)}\,F^{(i)}.$$

Comparing the coefficients on both sides of (2.10) and (2.11), we have

$$(2.12) \qquad a_1\,(N+1) = (2N-1)\,a_1\,(N)\,,$$
$$(2.13) \qquad a_{N+1}\,(N+1) = a_N\,(N)\,,$$

and

$$(2.14) \qquad a_i\,(N+1) = a_{i-1}\,(N) + (2N-i)\,a_i\,(N)\,, \quad (2 \le i \le N)\,.$$

Moreover, by (2.4) and (2.7), we get

$$(2.15) \qquad 2F^2 = (x-t)^{-1}\,F^{(1)} = a_1\,(1)\,(x-t)^{-1}\,F^{(1)}.$$

By comparing the coefficients on both sides of (2.15), we get

$$(2.16) \qquad a_1\,(1) = 1.$$

Now, by (2.12) and (2.16), we have

$$
\begin{aligned}
(2.17) \qquad a_1\,(N+1) &= (2N-1)\,a_1\,(N)\\
&= (2N-1)\,(2N-3)\,a_1\,(N-1)\\
&= (2N-1)\,(2N-3)\,(2N-5)\,a_1\,(N-2)\\
&\;\;\vdots\\
&= (2N-1)\,(2N-3)\,(2N-5)\cdots 1\cdot a_1\,(1)\\
&= (2N-1)!!,
\end{aligned}
$$

where $(2N-1)!!$ is Arfken's double factorial.

From (2.13), we easily note that

$$
(2.18) \qquad a_{N+1}\,(N+1) = a_N\,(N) = \cdots = a_1\,(1) = 1.
$$

For $2 \le i \le N$, from (2.14), we can derive the following equation:

(2.19)

$$
\begin{aligned}
a_i\,(N+1) &= a_{i-1}\,(N) + (2N-i)\,a_i\,(N)\\
&= a_{i-1}\,(N) + (2N-i)\,a_{i-1}\,(N-1) + (2N-i)\,(2N-2-i)\,a_i\,(N-1)\\
&\;\;\vdots\\
&= \sum_{k=0}^{N-i}\left(\prod_{l=0}^{k-1}(2\,(N-l)-i)\right) a_{i-1}\,(N-k) + \prod_{l=0}^{N-i}(2\,(N-l)-i)\,a_i\,(i)\\
&= \sum_{k=0}^{N-i} 2^k \left(N-\frac{i}{2}\right)_k a_{i-1}\,(N-k) + 2^{N-i+1}\left(N-\frac{i}{2}\right)_{N-i+1}\\
&= \sum_{k=0}^{N-i+1} 2^k \left(N-\frac{i}{2}\right)_k a_{i-1}\,(N-k),
\end{aligned}
$$

where $(x)_n = x\,(x-1)\cdots(x-n+1)$, $(n \ge 1)$ and $(x)_0 = 1$.

As the above is also valid for $i = N+1$, by (2.19), we get

$$
(2.20) \qquad a_i\,(N+1) = \sum_{k=0}^{N+1-i} 2^k \left(N-\frac{i}{2}\right)_k a_{i-1}\,(N-k),
$$

where $2 \le i \le N+1$.

Now, we give an explicit expression for $a_i\,(N+1)$.

From (2.17) and (2.20), we can derive the following equations:

$$
\begin{aligned}
(2.21) \qquad a_2\,(N+1) &= \sum_{k_1=0}^{N-1} 2^{k_1}\left(N-\frac{2}{2}\right)_{k_1} a_1\,(N-k_1)\\
&= \sum_{k_1=0}^{N-1} 2^{k_1}\left(N-\frac{2}{2}\right)_{k_1}(2\,(N-k_1-1)-1)!!,
\end{aligned}
$$

(2.22)

$$a_3(N+1) = \sum_{k_2=0}^{N-2} 2^{k_2} \left(N - \frac{3}{2}\right)_{k_2} a_2(N - k_2)$$

$$= \sum_{k_2=0}^{N-2} \sum_{k_1=0}^{N-2-k_2} 2^{k_1+k_2} \left(N - \frac{3}{2}\right)_{k_2} \left(N - k_2 - \frac{4}{2}\right)_{k_1} (2(N - 2 - k_1 - k_2) - 1)!!,$$

and
(2.23)

$$a_4(N+1) = \sum_{k_3=0}^{N-3} 2^{k_3} \left(N - \frac{4}{2}\right)_{k_3} a_3(N - k_3)$$

$$= \sum_{k_3=0}^{N-3} \sum_{k_2=0}^{N-3-k_3} \sum_{k_1=0}^{N-3-k_3-k_2} 2^{k_1+k_2+k_3} \left(N - \frac{4}{2}\right)_{k_3} \left(N - k_3 - \frac{5}{2}\right)_{k_2} \left(N - k_3 - k_2 - \frac{6}{2}\right)_{k_1}$$

$$\times (2(N - 3 - k_1 - k_2 - k_3) - 1)!!.$$

Thus, we see that, for $2 \le i \le N + 1$,

(2.24)

$$a_i(N+1) = \sum_{k_{i-1}=0}^{N-i+1} \sum_{k_{i-2}=0}^{N-i+1-k_{i-1}} \cdots \sum_{k_1=0}^{N-i+1-k_{i-1}-\cdots-k_2} 2^{\sum_{j=1}^{i-1} k_j}$$

$$\times \prod_{j=2}^{i} \left(N - \sum_{l=j}^{i-1} k_l - \frac{2i-j}{2}\right)_{k_{j-1}} \left(2\left(N - i + 1 - \sum_{j=1}^{i-1} k_j\right) - 1\right)!!.$$

Therefore, we obtain the following theorem.

Theorem 1. *The nonlinear differential equations*

$$2^N N! F^{N+1} = \sum_{i=1}^{N} a_i(N)(x - t)^{i-2N} F^{(i)}, \quad (N \in \mathbb{N})$$

has a solution $F = F(t, x) = \frac{1}{1 - 2tx + t^2}$, *where*

$$a_1(N) = (2N - 3)!!,$$

$$a_i(N) = \sum_{k_{i-1}=0}^{N-i} \sum_{k_{i-2}=0}^{N-i-k_{i-1}} \cdots \sum_{k_1=0}^{N-i-k_{i-1}-\cdots-k_2} 2^{\sum_{j=1}^{i-1} k_j}$$

$$\times \prod_{j=2}^{i} \left(N - \sum_{l=j}^{i-1} k_l - \frac{2i+2-j}{2}\right)_{k_{j-1}} \left(2\left(N - i - \sum_{j=1}^{i-1} k_j\right) - 1\right)!!$$

$(2 \le i \le N)$.

From (1.3) and (1.9), we note that

(2.25)

$$\sum_{n=0}^{\infty} U_n(x) t^n$$

$$= \frac{1}{1 - 2xt + t^2}$$

$$= \left(\frac{1}{\sqrt{1 - 2xt + t^2}} \right)^2$$

$$= \left(\sum_{l=0}^{\infty} p_l(x) t^l \right) \left(\sum_{m=0}^{\infty} p_m(x) t^m \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} p_l(x) p_{n-l}(x) \right) t^n.$$

Thus, from (2.25), we have

$$U_n(x) = \sum_{l=0}^{n} p_l(x) p_{n-l}(x).$$

From (1.4), we obtain

(2.26) $$2^N N! F^{N+1} = 2^N N! \sum_{n=0}^{\infty} U_n^{(N+1)}(x) t^n.$$

On the other hand, by Theorem 1, we get

(2.27)

$$2^N N! F^{N+1} = \sum_{i=1}^{N} a_i(N) (x - t)^{i-2N} F^{(i)}$$

$$= \sum_{i=1}^{N} a_i(N) \left(\sum_{m=0}^{\infty} \binom{2N + m - i - 1}{m} x^{i-2N-m} t^m \right) \left(\sum_{l=0}^{\infty} U_{i+l}(x) (l+i)_i t^l \right)$$

$$= \sum_{i=1}^{N} a_i(N) \sum_{n=0}^{\infty} \left\{ \sum_{l=0}^{n} \binom{2N + n - l - i - 1}{n - l} x^{i-2N-n+l} U_{l+i}(x) (l+i)_i \right\} t^n$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{i=1}^{N} a_i(N) \sum_{l=0}^{n} \binom{2N + n - l - i - 1}{n - l} x^{i+l-2N-n} U_{i+l}(x) (l+i)_i \right\} t^n.$$

Comparing the coefficients on the both sides of (2.26) and (2.27), we obtain the following theorem.

Theorem 2. *For $N \in \mathbb{N}$, and $n \in \mathbb{N} \cup \{0\}$, the following identity holds.*

$$U_n^{(N+1)}(x) = \frac{1}{2^N N!} \sum_{i=1}^{N} a_i(N) \sum_{l=0}^{n} \binom{2N + n - l - i - 1}{n - l} U_{l+i}(x) x^{i+l-2N-n} (l+i)_i.$$

The higher-order Legendre polynomials are given by the generating function

(2.28) $$\left(\frac{1}{\sqrt{1 - 2xt + t^2}} \right)^{\alpha} = \sum_{n=0}^{\infty} p_n^{(\alpha)}(x) t^n.$$

Thus, by 1.4 and (2.27), we get

(2.29) $$\sum_{n=0}^{\infty} U_n^{(\alpha)}(x) t^n$$

$$= \left(\frac{1}{1 - 2xt + t^2} \right)^{\alpha}$$

$$= \left(\frac{1}{\sqrt{1 - 2xt + t^2}} \right)^{2\alpha}$$

$$= \left(\sum_{l=0}^{\infty} p_l^{(\alpha)}(x) t^l \right) \left(\sum_{m=0}^{\infty} p_m^{(\alpha)}(x) t^m \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} p_l^{(\alpha)}(x) p_{n-l}^{(\alpha)}(x) \right) t^n.$$

From (2.29), we note that

$$(2.30) \qquad U_n^{(\alpha)}(x) = \sum_{l=0}^{n} p_l^{(\alpha)}(x) p_{n-l}^{(\alpha)}(x).$$

Therefore, we obtian the following corollaries.

Corollary 3. *For $N \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, we have*

$$\sum_{l=0}^{n} p_l^{(N+1)} p_{n-l}^{(N+1)}(x)$$

$$= \frac{1}{2^N N!} \sum_{i=1}^{N} a_i(N) \sum_{l=0}^{n} \binom{2N + n - l - i - 1}{n - l} U_{l+i}(x) (l + i)_i \, x^{i+l-2N-n}.$$

Corollary 4. *For $N \in \mathbb{N}$ and $n \in \mathbb{N}$, we have*

$$U_n^{(N+1)}(x)$$

$$= \frac{1}{2^N N!} \sum_{i=1}^{N} a_i(N) \sum_{l=0}^{n} \sum_{j=0}^{l+i} \binom{2N + n - l - i - 1}{n - l} x^{i+l-2N-n} (l + i)_i (x) \, p_{l+i-j}(x).$$

By (1.6), we get

$$(2.31) \qquad 2^N N! F^{N+1}$$

$$= 2^N N! (1 - t)^{-N-1} \left(\frac{1 - t}{1 - 2xt + t^2} \right)^{N+1}$$

$$= 2^N N! \left(\sum_{m=0}^{\infty} \binom{N + m}{m} t^m \right) \left(\sum_{l=0}^{\infty} V_l^{(N+1)}(x) t^l \right)$$

$$= 2^N N! \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{N + n - l}{n - l} V_l^{(N+1)}(x) \right) t^n.$$

On the other hand, by Theorem 1, we have

$$(2.32) \qquad 2^N N! F^{N+1} = \sum_{i=1}^{N} a_i(N) (x - t)^{i-2N} F^{(i)}$$

$$= \sum_{i=1}^{N} a_i(N) (x - t)^{i-2N} \left(\frac{d}{dt} \right)^i \left(\frac{1}{1 - t} \cdot \frac{1 - t}{1 - xt + t^2} \right).$$

From Leibniz formula, we note that

$$(2.33) \qquad \left(\frac{d}{dt} \right)^i \left(\frac{1 - t}{1 - 2xt + t^2} \cdot \frac{1}{1 - t} \right)$$

$$= \sum_{l=0}^{i} \binom{i}{l} \left(\left(\frac{d}{dt} \right)^{i-l} \frac{1}{1-t} \right) \left(\left(\frac{d}{dt} \right)^{l} \frac{1-t}{1-2xt+t^2} \right)$$

$$= \sum_{l=0}^{i} \binom{i}{l} (i-l)! (1-t)^{-i+l-1} \left(\frac{d}{dt} \right)^{l} \left(\frac{1-t}{1-2xt+t^2} \right)$$

$$= \sum_{l=0}^{i} \binom{i}{l} (i-l)! \sum_{s=0}^{\infty} \binom{i-l+s}{s} t^s \sum_{p=0}^{\infty} V_{p+l}(x) (p+l)_l t^p$$

$$= \sum_{l=0}^{i} \frac{i!}{l!} \sum_{s=0}^{\infty} \binom{i-l+s}{s} t^s \sum_{p=0}^{\infty} V_{p+l}(x) (p+l)_l t^p.$$

By (2.32) and (2.33), we get

(2.34) $\qquad 2^N N! F^{N+1}$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{i=1}^{N} \sum_{l=0}^{i} a_i(N) \frac{i!}{l!} \sum_{m+s+p=n} \binom{2N+m-i-1}{m} \binom{i-l+s}{s} \right.$$
$$\left. \times (p+l)_l \, x^{i-2N-m} V_{p+l}(x) \right\} t^n.$$

Therefore, by (2.31) and (2.34), we obtain the following theorem.

Theorem 5. *For $N \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, we have the following identity:*

$$\sum_{l=0}^{n} \binom{N+n-l}{n-l} V_l^{(N+1)}(x)$$

$$= \frac{1}{2^N N!} \sum_{i=1}^{N} \sum_{l=0}^{i} a_i(N) \frac{i!}{l!} \sum_{m+s+p=n} \binom{2N+m-i-1}{m} \binom{i-l+s}{s} (p+l)_l$$
$$\times x^{i-2N-m} V_{p+l}(x).$$

From (1.8), we note that

(2.35) $\qquad 2^N N! F^{N+1}$

$$= 2^N N! (1+t)^{-N-1} \left(\frac{1+t}{1-2xt+t^2} \right)^{N+1}$$

$$= 2^N N! \left(\sum_{m=0}^{\infty} \binom{N+m}{m} (-1)^m t^m \right) \left(\sum_{l=0}^{\infty} W_l^{(N+1)}(x) t^l \right)$$

$$= 2^N N! \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} (-1)^{n-l} \binom{N+n-l}{n-l} W_l^{(N+1)}(x) \right) t^n.$$

On the other hand, by Theorem 1, we get

(2.36) $\qquad 2^N N! F^{N+1} = \sum_{i=1}^{N} a_i(N) (x-t)^{i-2N} \left(\frac{d}{dt} \right)^i \left\{ \frac{1}{1+t} \cdot \frac{1+t}{1-2xt+t^2} \right\}.$

Now, we observe that

(2.37) $\qquad \left(\frac{d}{dt} \right)^i \left\{ \left(\frac{1}{1+t} \right) \left(\frac{1+t}{1-2xt+t^2} \right) \right\}$

$\qquad$

$$= \sum_{l=0}^{i} \binom{i}{l} (-1)^{i-l} (i-l)! \left(\frac{1}{1+t} \right)^{i-l+1} \left(\frac{d}{dt} \right)^{l} \left(\frac{1+t}{1-2xt+t^2} \right)$$

$$= \sum_{l=0}^{i} \binom{i}{l} (-1)^{i-l} (i-l)! \sum_{s=0}^{\infty} \binom{i-l+s}{s} (-1)^s t^s \sum_{p=0}^{\infty} W_{p+l}(x) (p+l)_l t^p.$$

From (2.36) and (2.37), we have

(2.38) $2^N N! F^{N+1}$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{i=1}^{N} a_i(N) \sum_{l=0}^{i} (-1)^{i-l} \frac{i!}{l!} \sum_{m+s+p=n} (-1)^s \binom{2N+m-i-1}{m} \right.$$
$$\left. \times \binom{i-l+s}{s} (p+l)_l x^{i-2N-m} W_{p+l}(x) \right\} t^n.$$

Therefore, by (2.35) and (2.38), we obtain the following theorem.

Theorem 6. *For $N \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, the following identity is valid:*

$$\sum_{l=0}^{n} (-1)^{n-l} \binom{N+n-l}{n-l} W_l^{(N+1)}(x)$$

$$= \frac{1}{2^N N!} \sum_{i=1}^{N} \sum_{l=0}^{i} (-1)^{i-l} a_i(N) \frac{i!}{l!} \sum_{m+s+p=n} (-1)^s \binom{2N+m-i-1}{m}$$

$$\times \binom{i-l+s}{s} (p+l)_l x^{i-2N-m} W_{p+l}(x).$$

From (1.1), we have

(2.39)

$$2^N N! F^{N+1}$$

$$= 2^N N! \left(\frac{1}{1-t^2} \cdot \frac{1-t^2}{1-2xt+t^2} \right)^{N+1}$$

$$= 2^N N! \left(\frac{1}{1-t} \right)^{N+1} \left(\frac{1}{1+t} \right)^{N+1} \left(\frac{1-t^2}{1-2xt+t^2} \right)^{N+1}$$

$$= 2^N N! \left(\sum_{l=0}^{\infty} \binom{N+l}{l} t^l \right) \left(\sum_{m=0}^{\infty} \binom{m+N}{m} (-1)^m t^m \right) \left(\sum_{p=0}^{\infty} T_p^{(N+1)}(x) t^p \right)$$

$$= 2^N N! \sum_{n=0}^{\infty} \left(\sum_{l+m+p=n} \binom{N+l}{l} \binom{m+N}{m} (-1)^m T_p^{(N+1)}(x) \right) t^n.$$

On the other hand, by Theorem 1, we get

(2.40) $2^N N! F^{N+1}$

$$= \sum_{i=1}^{N} a_i(N) (x-t)^{i-2N} F^{(i)}$$

$$= \frac{1}{2} \sum_{i=1}^{N} a_i(N)(x-t)^{i-2N} \left(\frac{d}{dt}\right)^i \left\{ \left(\frac{1}{1-t} + \frac{1}{1+t}\right) \frac{1-t^2}{1-2xt+t^2} \right\}.$$

From Leibniz formula, we note that the following equations:

$$(2.41) \qquad \left(\frac{d}{dt}\right)^i \left\{ \left(\frac{1}{1-t}\right) \cdot \left(\frac{1-t^2}{1-2xt+t^2}\right) \right\}$$

$$= \sum_{l=0}^{i} \binom{i}{l}(i-l)! \sum_{s=0}^{\infty} \binom{i+s-l}{s} t^s \sum_{p=0}^{\infty} T_{p+l}(x)(p+l)_l t^p,$$

and

$$(2.42) \quad \left(\frac{d}{dt}\right)^i \left\{ \left(\frac{1}{1+t}\right) \left(\frac{1-t^2}{1-2xt+t^2}\right) \right\}$$

$$= \sum_{l=0}^{i} \binom{i}{l}(i-l)!(-1)^{i-l} \sum_{s=0}^{\infty} \binom{i-l+s}{s} (-1)^s t^s \sum_{p=0}^{\infty} T_{p+l}(x)(p+l)_l t^p.$$

By (2.40), (2.41), and (2.42), we obtain

(2.43)

$$2^N N! F^{N+1}$$

$$= \frac{1}{2} \sum_{i=1}^{N} a_i(N)(x-t)^{i-2N} \sum_{l=0}^{i} \binom{i}{l}(i-l)! \sum_{s=0}^{\infty} \binom{i+s-l}{s} t^s \sum_{k=0}^{\infty} T_{p+l}(x)(p+l)_l t^p$$

$$+ \frac{1}{2} \sum_{i=1}^{N} a_i(N)(x-t)^{i-2N} \sum_{l=0}^{i} \binom{i}{l}(i-l)!(-1)^{i-l} \sum_{s=0}^{\infty} \binom{i-l+s}{s} (-1)^s t^s$$

$$\times \sum_{p=0}^{\infty} T_{p+l}(x)(p+l)_l t^p$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \sum_{i=1}^{N} \sum_{l=0}^{i} a_i(N) \frac{i!}{l!} \sum_{m+s+p=n} \binom{2N+m-i-1}{m} \binom{i+s-l}{s} (p+l)_l$$

$$\times x^{i-2N-m} T_{p+l}(x) t^n + \frac{1}{2} \sum_{n=0}^{\infty} \sum_{i=1}^{N} \sum_{l=0}^{i} a_i(N) \frac{i!}{l!} (-1)^{i-l}$$

$$\times \sum_{m+s+p=n} (-1)^s \binom{2N+m-i-1}{m} \binom{i+s-l}{s} (p+l)_l x^{i-2N-m} T_{p+l}(x) t^n.$$

Therefore, by (2.39) and (2.43), we obtain the following theorem.

Theorem 7. *For $n \in \mathbb{N} \cup \{0\}$ and $N \in \mathbb{N}$, we have the following identity*

$$2^{N+1} N! \sum_{s+m+p=n} \binom{N+s}{s} \binom{m+N}{m} (-1)^m T_p^{(N+1)}(x)$$

$$= \sum_{i=1}^{N} \sum_{l=0}^{i} a_i(N) \frac{i!}{l!} \sum_{m+s+p=n} \binom{2N+m-i-1}{m} \binom{i+s-l}{s} (p+l)_l$$

$$\times x^{i-2N-m}T_{p+l}(x) + \sum_{i=1}^{N}\sum_{l=0}^{i} a_i(N)\frac{i!}{l!}(-1)^{i-l}\sum_{m+s+p=n}(-1)^s\binom{2N+m-i-1}{m}$$

$$\times \binom{i+s-l}{s}(p+l)_l\, x^{i-2N-m}T_{p+l}(x).$$

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Blowup singularity for a degenerate and singular parabolic equation with nonlocal boundary *

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Abstract

In this paper, we are interested in the blowup behavior of the solution to a degenerate and singular parabolic equation

$$u_t = (x^\alpha u_x)_x + \int_0^l u^p dx - ku^q, \quad (x,t) \in (0,l) \times (0,+\infty)$$

with nonlocal boundary condition

$$u(0,t) = \int_0^l f(x) u(x,t)dx, \quad u(l,t) = \int_0^l g(x) u(x,t)dx, \quad t \in (0,+\infty),$$

where p, $q \in [1,\infty)$, $\alpha \in [0,1)$ and $k \in (0,\infty)$. In view of comparison principle, we investigate the conditions on the global existence and blowup of the solutions. Moreover, under some suitable hypotheses, we discuss the global blowup and the uniform blowup profile of the blowup solution.

Keywords: Degenerate and singular parabolic equation; Nonlocal boundary; Global existence; Blowup singularity

Mathematics Subject Classification(2000) : 35K50, 35K55, 35K65

1　Introduction

The main purpose of this paper is to deal with the blowup singularity of the following degenerate and singular parabolic equation with nonlocal source and nonlocal boundary condition

$$\begin{cases} u_t = (x^\alpha u_x)_x + \int_0^l u^p dx - ku^q, & (x,t) \in (0,l) \times (0,+\infty), \\ u(0,t) = \int_0^l f(x) u(x,t)dx, & t \in (0,+\infty), \\ u(l,t) = \int_0^l g(x) u(x,t)dx, & t \in (0,+\infty), \\ u(x,0) = u_0(x) \geq 0, & x \in [0,l], \end{cases} \tag{1.1}$$

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where $0 \leq \alpha < 1$, p, $q \geq 1$, $k > 0$, the weight functions $f(x)$ and $g(x)$ in the boundary condition are nonnegative continuous on $[0, l]$ and not identically zero, and the initial value $u_0(x) \in C^{2+\delta}(0, l) \cap C[0, l]$ with $0 < \delta < 1$, and satisfies the compatibility conditions. It is obvious that the equation in problem (1.1) is singular and degenerate because the coefficients of u_x and u_{xx} may tend to ∞ and 0 as $x \to 0$.

This type equation in problem (1.1) can be viewed as a model which describes the conduction of heat related to the geometric shape of the body (see [1] and the references therein for more details of the physical background). On the other hand, lots of physical phenomena were formulated into nonlocal mathematical models, for example, Day [4, 5] derived a heat equation with nonlocal boundary in the study of the heat conduction with thermoelastcity. From then on, a lot of mathematicians devoted to studying the blowup behavior of the solutions of various parabolic problems with nonlocal boundary conditions (see [6, 7, 8, 9, 10, 11, 13, 15, 16, 21]).

The blowup phenomenon related to problem (1.1) attracted extensive attention of mathematicians in the past several decades (see [2, 3, 12, 18, 20, 22, 23]), but most of them considered the problems with null Dirichlet boundary conditions. Inspired by the works mentioned above, we consider problem (1.1), and our main attention is focused on evaluating the effects of the weighted nonlocal boundary, the nonlocal source and absorption term on the asymptotic blowup behavior of the solution $u(x, t)$ of problem (1.1). Compared with [3] and [18], we need more skills to handle the difficulties, which are produced by the degeneration and singularity of problem (1.1), and the appearance of the nonlinear nonlocal boundary condition.

Before stating our main results, for the sake of convenience, we denote

$$\mathcal{N} = \max \left\{ \int_0^l f(x)dx, \int_0^l g(x)dx \right\},$$

and let λ_1 be the first eigenvalue and $\zeta_1(x)$ be the corresponding eigenfunction of the following eigenvalue problem

$$-(x^\alpha \zeta_x)_x = \lambda_1 \zeta, \quad 0 < x < l; \quad \zeta(0) = \zeta(l) = 0. \tag{1.2}$$

Indeed, from [3, 14], we know that the principle eigenvalue λ_1 of the eigenvalue problem (1.2) is the first zero of

$$J_{\frac{1-\alpha}{2-\alpha}}\left(\frac{2\sqrt{\lambda}}{2-\alpha} l^{\frac{2-\alpha}{2}} \right) = 0,$$

and $\zeta_1(x)$ can be expressed in an explicit form as follows

$$\zeta_1(x) = a x^{\frac{1-\alpha}{2}} J_{\frac{1-\alpha}{2-\alpha}}\left(\frac{2\sqrt{\lambda_1}}{2-\alpha} x^{\frac{2-\alpha}{2}} \right), \tag{1.3}$$

where $J_{\frac{1-\alpha}{2-\alpha}}$ is Bessel function of the first kind of order $\frac{1-\alpha}{2-\alpha}$, and a is an appropriate positive parameter such that $\|\zeta_1(x)\|_{L^1([0,l])} = 1$. Furthermore, we know easily that $\zeta_1(x)$ is a positive smooth function in $(0, l)$, and in light of

$$\frac{d}{d\tau} J_\vartheta(\tau) = \frac{\vartheta}{2} J_\vartheta(\tau) - J_{\vartheta+1}(\tau),$$

we can deduce that, for $x \in (0, l)$,

$$\frac{d}{dx}\zeta_1(x) = \frac{a(1-\alpha)}{2}\left(1 + \frac{\sqrt{\lambda_1}}{2-\alpha} x^{\frac{2-\alpha}{2}} \right) x^{-\frac{1+\alpha}{2}} J_{\frac{1-\alpha}{2-\alpha}}\left(\frac{2\sqrt{\lambda_1}}{2-\alpha} x^{\frac{2-\alpha}{2}} \right) - a\sqrt{\lambda_1} x^{\frac{1-2\alpha}{2}} J_{\frac{3-2\alpha}{2-\alpha}}\left(\frac{\sqrt{\lambda_1}}{2-\alpha} x^{\frac{2-\alpha}{2}} \right).$$

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And hence, by making use of

$$J_\vartheta(\tau) \to \frac{1}{\Gamma(\vartheta+1)}\left(\frac{\tau}{2}\right)^\vartheta \text{ as } \tau \to 0,$$

where $\Gamma(\cdot)$ is the Gamma function, we find that

$$\lim_{x\to 0^+} \zeta_1(x) = 0$$

and

$$\lim_{x\to 0^+} \frac{d}{dx}\zeta_1(x) = \frac{a(1-\alpha)}{2\Gamma\left(\frac{3-2\alpha}{2-\alpha}\right)}\left(\frac{2\sqrt{\lambda_1}}{2-\alpha}\right)^{\frac{1-\alpha}{2-\alpha}},$$

which imply that

$$\sup_{x\in[0,l]} \zeta_1(x) < \infty \text{ and } \sup_{x\in[0,l]} \frac{d}{dx}\zeta_1(x) < \infty. \tag{1.4}$$

The main results of this paper are stated as follows.

Theorem 1.1. *Assume that $q > p \geq 1$, then all the solutions of problem (1.1) exist globally.*

Theorem 1.2. *Assume that $p > q \geq 1$, then problem (1.1) admits blowup solutions as well as global solutions. More precisely,*

(i) *if $\mathcal{N} \leq 1$, then the solution exists globally provided that $u_0(x) \leq \left(\frac{k}{l}\right)^{\frac{1}{p-q}}$;*

(ii) *if $\mathcal{N} > 1$, then the solution of problem (1.1) blows up in finite time provided that $u_0(x) > \eta_1$, where $\eta_1 > 1$ is an appropriate constant;*

(iii) *there is a suitable positive small constant η_2 such that the solution $u(x,t)$ of problem (1.1) blows up in finite time for any $f(x)$ and $g(x)$ provided that*

$$u_0(x) > \eta_2^{-\xi}\left(\frac{l}{2-\alpha}x^{1-\alpha} - \frac{1}{2-\alpha}x^{2-\alpha}\right),$$

where $\xi > \frac{1}{p-1}$.

Theorem 1.3. *Assume that $p = q > 1$. The solution $u(x,t)$ of problem (1.1) exists globally provided that $\mathcal{N} < 1$ and $u_0(x) \leq \epsilon_1\mathcal{N}$, where ϵ_1 is given by (3.13). For any nonnegative weight functions $f(x)$ and $g(x)$, the solution $u(x,t)$ of problem (1.1) blows up in finite time provided that the initial value $u_0(x)$ is sufficiently large.*

Remark 1.1. *If $p = q = 1$, one can show that problem (1.1) has no blowup solution.*

The remaining part is devote to discussing the global blowup and the uniform blowup profile of the blowup solution, to this end, we assume that $p > q \geq 1$ (or $p = q > 1$), $\mathcal{N} \leq 1$ and $u_0(x)$ is large enough in some suitable sense. Moreover, we assume that $u_0(x)$ satisfies extra

$$(x^\alpha u_{0x})_x + \int_0^l u_0^p dx - ku_0^q \geq 0 \text{ for } x \in (0,l), \tag{1.5}$$

$$(x^\alpha u_{0x})_x \leq 0 \text{ in } (0,l), \tag{1.6}$$

and

$$\lim_{x\to 0^+}\left[(x^\alpha u_{0x})_x + \int_0^l u_0^p dx - ku_0^q\right] = \lim_{x\to l^-}\left[(x^\alpha u_{0x})_x + \int_0^l u_0^p dx - ku_0^q\right] = 0. \tag{1.7}$$

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Theorem 1.4. *Assume that $p > q \geq 1$ and $\mathcal{N} \leq 1$. Suppose that hypotheses (1.5), (1.6) and (1.7) hold. Then*

$$u(x,t) \sim [l(p-1)(T-t)]^{-\frac{1}{p-1}} \qquad a.e. \ in \ (0,l) \ as \ t \to T,$$

where T is the blowup time.

Corollary 1.1. *Under the assumptions of Theorem 1.4, we see that the blowup set of the blowup solution $u(x,t)$ of problem (1.1) is the whole interval $(0,l)$.*

Theorem 1.5. *Assume that $p = q > 1$, $\mathcal{N} \leq 1$ and $0 < k < l$. Suppose that hypotheses (1.5), (1.6) and (1.7) hold. Then*

$$u(x,t) \sim [l(p-1)(T-t)]^{-\frac{1}{p-1}} \qquad a.e. \ in \ (0,l) \ as \ t \to T,$$

where T is the blowup time.

Corollary 1.2. *Under the assumptions of Theorem 1.5, we know that the blowup set of the blowup solution $u(x,t)$ of problem (1.1) is the whole interval $(0,l)$.*

The rest of this paper is organized as follows. In Section 2, we shall state the comparison principle and local existence theorem for problem (1.1). In section 3, we shall concern with the conditions on the global existence of solution and prove Theorems 1.1, 1.2 and 1.3. In section 4, we shall estimate the uniform blowup profile and give the proofs of Theorems 1.4 and 1.5.

2 Comparison principle and local existence

In this section, we will establish a suitable comparison principle for problem (1.1) and state the existence and uniqueness result on the local solution. For the sake of simplify, we denote $I_T = (0,l) \times (0,T)$ and $\overline{I}_T = [0,l] \times [0,T]$. First, we give the definitions of the super-solution and sub-solution to problem (1.1).

Definition 2.1. *A nonnegative function $\overline{u}(x,t)$ is called a super-solution of problem (1.1) if $\overline{u}(x,t) \in C^{2,1}(I_T) \cap C(\overline{I}_T)$ satisfies*

$$\begin{cases} \overline{u}_t \geq (x^\alpha \overline{u}_x)_x + \int_0^l \overline{u}^p dx - k\overline{u}^q, & (x,t) \in I_T, \\ \overline{u}(0,t) \geq \int_0^l f(x) \overline{u}(x,t)dx, & t \in (0,T), \\ \overline{u}(l,t) \geq \int_0^l g(x) \overline{u}(x,t)dx, & t \in (0,T), \\ \overline{u}(x,0) \geq \overline{u}_0(x), & x \in [0,l]. \end{cases} \tag{2.1}$$

Similarly, $\underline{u}(x,t) \in C^{2,1}(I_T) \cap C(\overline{I}_T)$ is called a sub-solution of problem (1.1) if it satisfies all the reversed inequalities in (2.1). We say that $u(x,t)$ is a solution of problem (1.1) if it is both a sub-solution and a super-solution of problem (1.1).

Now, by using the similar arguments as those in [6] (or [10]), we give directly the following maximum principle.

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Lemma 2.1. *Let* $\omega(x,t) \in C^{2,1}(I_T) \cap C(\overline{I}_T)$ *satisfy*

$$\begin{cases} \omega_t - (x^\alpha \omega_x)_x \geq \theta_1(x,t)\omega + \int_0^l \theta_1(x,t)\omega(x,t)\,dx, & (x,t) \in I_T, \\ \omega(0,t) \geq \int_0^l \theta_3(x)\omega(x,t)dx, & t \in (0,T), \\ \omega(l,t) \geq \int_0^l \theta_4(x)\omega(x,t)dx, & t \in (0,T), \end{cases} \tag{2.2}$$

where $\theta_i(x,t)$, $i = 1,2,3,4$, *are bounded functions,* $\theta_2(x,t)$ *is nonnegative for* $(x,t) \in I_T$, $\theta_3(x)$ *and* $\theta_4(x)$ *are nonnegative, nontrivial in* $(0,l)$. *Then* $\omega(x,0) > 0$ *in* $[0,l]$ *implies that* $\omega(x,t) > 0$ *for* $(x,t) \in I_T$. *Moreover, if one of the following conditions holds, (i)* $\theta_3(x) = \theta_4(x) \equiv 0$ *for* $x \in (0,l)$; *(ii)* $\theta_3(x), \theta_4(x) \geq 0$ *for* $x \in (0,l)$ *and* $\max\left\{\int_0^l \theta_3(x)\,dx, \int_0^l \theta_4(x)\,dx\right\} \leq 1$, *then* $\omega(x,0) \geq 0$ *in* $[0,l]$ *leads to* $\omega(x,t) \geq 0$ *for* $(x,t) \in I_T$.

Based on the idea of [10], we can establish the comparison principle for problem (1.1) as follows, which is the main tool of establishing the conditions on the global existence and blowup of the solution.

Proposition 2.1 (Comparison principle). *Let* $\overline{u}(x,t)$ *and* $\underline{u}(x,t)$ *be a nonnegative super-solution and sub-solution of problem (1.1), respectively. Then* $\overline{u}(x,t) \geq \underline{u}(x,t)$ *holds in* $\overline{I}_T$ *if* $\overline{u}(x,0) \geq \underline{u}(x,0)$ *for* $x \in [0,l]$.

Next, we state the result on the existence and uniqueness of the local solution of problem (1.1) at the end of this section.

Theorem 2.1 (Local existence and uniqueness). *Assume that (1.5) holds, then there exists a small positive real number* T *such that problem (1.1) admits a unique nonnegative solution* $u(x,t) \in C(\overline{I}_T) \cap C^{2,1}(I_T)$.

Remark 2.1. *We can get the proof of Theorem 2.1 by using regularization method and Schauder's fixed point theorem. For more details, we refer the readers to [3, 23].*

3 Global existence of solution

The main goal of this section is to discuss the global existence and blowup property of the solution $u(x,t)$ to the problem (1.1). To this end, by Proposition 2.1, we only need to construct some suitable global super-solutions (or blowup sub-solutions).

Proof of Theorem 1.1. Let T be any positive number and $\overline{u}_1(x,t)$ be defined as

$$\overline{u}_1(x,t) = \frac{\chi_2}{\chi_1 \zeta_1(x) + 1} e^{\chi_3 t}$$

where χ_1 is large enough such that

$$\int_0^l \frac{1}{1 + \chi_1 \zeta_1(x)}\,dx \leq \max\left\{\max_{x \in [0,l]} f(x), \max_{x \in [0,l]} g(x)\right\},$$

and

$$\chi_2 = \max\left\{\max_{x \in [0,l]}(u_0(x) + 1)(\chi_1 \zeta_1(x) + 1), \max_{x \in [0,l]}\left[\frac{(\chi_1 \zeta_1(x) + 1)^q}{k}\int_0^l \frac{1}{(1 + \chi_1 \zeta_1(x))^p}\,dx\right]^{\frac{1}{q-p}}\right\},$$

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$$\chi_3 = \lambda_1 + \max_{x \in [0,l]} \frac{2l^\alpha \chi_1^2}{(\chi_1 \zeta_1(x) + 1)^2} \left| \frac{d\zeta_1(x)}{dx} \right|^2.$$

By the direct calculation, one has

$$
\begin{aligned}
P\overline{u}_1 := & \ \overline{u}_{1t} - (x^\alpha \overline{u}_{1x})_x - \int_0^l \overline{u}_1^p dx + k\overline{u}_1^q \\
= & \ \overline{u}_1 \left[\chi_3 - \left(\frac{\lambda_1 \chi_1 \zeta_1(x)}{1 + \chi_1 \zeta_1(x)} + \frac{2x^\alpha \chi_1^2}{(\chi_1 \zeta_1(x) + 1)^2} \left| \frac{d}{dx} \zeta_1(x) \right|^2 \right) \right] \\
& + k \left(\frac{\chi_2 e^{\chi_3 t}}{1 + \chi_1 \zeta_1(x)} \right)^q - (\chi_2 e^{\chi_3 t})^p \int_0^l \frac{1}{(1 + \chi_1 \zeta_1(x))^p} dx \\
\geq & \ 0,
\end{aligned}
$$

(3.1)

and

$$\overline{u}_1(x,0) = \frac{\chi_2}{1 + \chi_1 \zeta_1(x)} \geq \frac{\max_{x \in [0,l]} (u_0(x) + 1)(1 + \chi_1 \zeta_1(x))}{1 + \chi_1 \zeta_1(x)} > u_0(x).$$

(3.2)

On the other hand, we can verify that

$$\overline{u}_1(0,t) = \chi_2 e^{\chi_3 t} \geq \chi_2 e^{\chi_3 t} \max_{x \in [0,l]} f(x) \int_0^l \frac{1}{1 + \chi_1 \zeta_1(x)} dx \geq \int_0^l \frac{f(x) \chi_2 e^{\chi_3 t}}{1 + \chi_1 \zeta_1(x)} dx = \int_0^l f(x) \overline{u}_1(x,t) dx,$$

(3.3)

and

$$\overline{u}_1(l,t) \geq \int_0^l g(x) \overline{u}_1(x,t) dx.$$

(3.4)

Combining now from (3.1) to (3.4), we know that $\overline{u}_1(x,t)$ is a global super-solution of (1.1) in I_T and the solution $u(x,t)$ of (1.1) exists globally by Proposition 2.1. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. **(i)** If $p > q$ and $\mathcal{N} > 1$, then it is easy to check that $\overline{u}_2(x) = \left(\frac{k}{l} \right)^{\frac{1}{p-q}}$ is a global super-solution of problem (1.1) provided that $u_0(x) \leq \left(\frac{k}{l} \right)^{\frac{1}{p-q}}$.

  (ii) Consider the following ordinary differential equation

$$
\begin{cases}
\underline{v}_1'(t) = l\underline{v}_1^p - k\underline{v}_1^q, & t > 0, \\
\underline{v}_1(0) = \underline{v}_{10}.
\end{cases}
$$

(3.5)

From $p > q \geq 1$, it follows that $\underline{v}_1^q \leq \underline{v}_1^p + 1$, and hence, we have

$$l\underline{v}_1^p - k\underline{v}_1^q \geq (l - k)\underline{v}_1^p - k,$$

which tells us that the solution $\underline{v}_1(t)$ of (3.5) is a super-solution of the following problem

$$
\begin{cases}
\underline{v}_2'(t) = (l - k)\underline{v}_2^p - k, & t > 0, \\
\underline{v}_2(0) = \underline{v}_{10}
\end{cases}
$$

(3.6)

provided $l > k$. Noticing that $(l - k)\underline{v}_2^p$ is convex, then there exists $\eta_1 > 1$ such that $(l - k)\underline{v}_2^p \geq 2k$ holds for $\underline{v}_2 \geq \eta_1$. It follows easily that if $\underline{v}_2(0) = \underline{v}_{10} > \eta_1$, then $\underline{v}_2(t)$ is increasing on its interval of the existence and

$$\underline{v}_2'(t) \geq \frac{l - k}{2} \underline{v}_2^p.$$

(3.7)

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From the above inequality it follows that

$$\underline{v}_2(t) \to \infty \text{ as } t \to \frac{2}{(l-k)(p-1)\underline{v}_{10}^{p-1}}, \tag{3.8}$$

which leads to that $\underline{v}_1(t)$ will become infinite in a finite time. Recalling that $\mathcal{N} > 1$, then $\underline{v}_1(t)$ is a blowup sub-solution of problem (1.1) when $u_0(x) \geq \underline{v}_{10} > \eta$, so the solution $u(x,t)$ of problem (1.1) blows up in finite time for sufficiently large initial value.

(iii) Let $v(x,t)$ be the solution of the following auxiliary problem

$$\begin{cases} v_t = (x^\alpha v_x)_x + \int_0^l v^p(x,t)dx - kv^q, & 0 < x < l, t > 0, \\ v(0,t) = v(l,t) = 0, & t > 0, \\ v(x,0) = u_0(x), & 0 < x < l, \end{cases} \tag{3.9}$$

then $v(x,t)$ is a sub-solution of problem (1.1). Set

$$\underline{v}_3(x,t) = (\eta_2 - t)^{-\xi}\left(\frac{l}{2-\alpha}x^{1-\alpha} - \frac{1}{2-\alpha}x^{2-\alpha}\right) := (\eta_2 - t)^{-\xi}\mu(x),$$

where η_2 and $\xi > 0$ will be chosen later. Calculating directly, we have

$$P\underline{v}_3 := \underline{v}_{3t} - (x^\alpha \underline{v}_{3x})_x - \int_0^l \underline{v}_3^p(x,t)dx + k\underline{v}_3^q$$

$$= (\eta_2 - t)^{-\xi p}\left[\xi(\eta_2 - t)^{\xi p - \xi - 1}\mu(x) + (\eta_2 - t)^{\xi(p-1)} + k(\eta_2 - t)^{\xi(p-q)}\mu^q(x) - \int_0^l \mu^p(x)dx\right].$$

Since $p > q \geq 1$, we can take ξ large enough such that $\xi p - \xi - 1 > 0$, then we have $P\underline{v}_3 \leq 0$ with $\eta_2 - t$ small enough, which implies that $\underline{v}_3(x,t)$ is a blowup sub-solution to problem (3.9) provided that $v(x,0) = u_0(x) > \mu(x)\eta_2^{-\xi}$. And hence, Proposition 2.1 tells us that the solution $u(x,t)$ of problem (1.1) blows up in finite time for large initial value. The proof of Theorem 1.2 is completed. $\qquad\square$

Proof of Theorem 1.3. For any given constant

$$\epsilon_0 \in \left(0, \frac{(1-\mathcal{N})(2-\alpha)^{3-\alpha}}{l^{2-\alpha}(1-\alpha)^{1-\alpha}}\right), \tag{3.10}$$

let $\sigma(x)$ be the unique positive solution of the following ordinary differential equation

$$\begin{cases} -(x^\alpha \sigma_x)_x = \epsilon_0, & 0 < x < l, \\ \sigma(0) = \sigma(l) = \mathcal{N}. \end{cases} \tag{3.11}$$

In fact, we can solve the explicit expression of $\sigma(x)$ as follows

$$\sigma(x) = \frac{l\epsilon_0}{2-\alpha}x^{1-\alpha} - \frac{\epsilon_0}{2-\alpha}x^{2-\alpha} + \mathcal{N}, \quad x \in [0,l].$$

Moreover, according to $\mathcal{N} < 1$, we can verify that

$$0 < \min_{x \in [0,l]} \sigma(x) = \mathcal{N} < \max_{x \in [0,l]} \sigma(x) = \mathcal{N} + \frac{\epsilon_0 l^{2-\alpha}(1-\alpha)^{1-\alpha}}{(2-\alpha)^{3-\alpha}} < 1. \tag{3.12}$$

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Define
$$\overline{u}_3(x,t) = \epsilon_1 \sigma(x),$$

where

$$\epsilon_1 = \begin{cases} \left(\frac{\epsilon_0}{k\mathcal{N}^p - l}\right)^{\frac{1}{p-1}}, & \text{if } k\mathcal{N}^p - l > 0, \\ \text{any fixed positive constant}, & \text{if } k\mathcal{N}^p - l \le 0. \end{cases} \tag{3.13}$$

Calculating directly, one has

$$\begin{aligned} P\overline{u}_3 &:= \overline{u}_{3t} - (x^\alpha \overline{u}_{3x})_x - \int_0^l \overline{u}_3^p dx + k\overline{u}_3^p \\ &= \epsilon_0 \epsilon_1 - \epsilon_1^p \int_0^l \sigma^p dx + k\epsilon_1^p \sigma^p \\ &\ge \epsilon_0 \epsilon_1 - l\epsilon_1^p \left[\max_{x\in[0,l]} \sigma(x)\right]^p + k\epsilon_1^p \left[\min_{x\in[0,l]} \sigma(x)\right]^p \\ &> \epsilon_0 \epsilon_1 - \epsilon_1^p (k\mathcal{N}^p - l) \\ &\ge 0. \end{aligned} \tag{3.14}$$

Meanwhile, we can prove that

$$\overline{u}_3(0,t) = \epsilon_1 \mathcal{N} \ge \int_0^l \epsilon_1 f(x)\, dx > \int_0^l \epsilon_1 \sigma(x) f(x)\, dx = \int_0^l \overline{u}_3(x,t) f(x)\, dx \tag{3.15}$$

and

$$\overline{u}_3(l,t) > \int_0^l \overline{u}_3(x,t) f(x)\, dx. \tag{3.16}$$

Then $\overline{u}_3(x,t)$ is a global super-solution of problem (1.1) if $u_0(x) \le \epsilon_1 \mathcal{N}$, and hence, we obtain our global existence result by Proposition 2.1.

The proof of blowup conclusion in this case is similar to the arguments of (iii) in Theorem 1.2, we omit the details here. The proof of Theorem 1.3 is completed. $\qquad\square$

4   Global blowup set and uniform blowup profile

This section is mainly about the global blowup and the uniform blowup profile of the blowup solution for problem (1.1). Throughout this section, we assume that $p > q \ge 1$ (or $p = q > 1$), $\mathcal{N} \le 1$ and $u_0(x)$ is large enough in some suitable sense. From Theorems 1.2 and 1.3, it follows that the solution $u(x,t)$ of problem (1.1) blows up in finite. For convenience, we denote T the blowup time.

From the assumptions on the initial value $u_0(x)$ and (1.5), (1.6) and (1.7), we can find a sufficiently small positive constant ε_1 and a nonnegative function $w_{0\varepsilon}(x)$ such that

(1) $w_{0\varepsilon} \in C^{2+\delta}(\varepsilon, l-\varepsilon) \cap C[\varepsilon, l-\varepsilon]$ with $\delta \in (0,1)$ and $\varepsilon \in (0, \varepsilon_1]$.

(2) $w_{0\varepsilon}(\varepsilon) = \int_\varepsilon^{l-\varepsilon} f(x) w_{0\varepsilon}(x)\, dx$ and $w_{0\varepsilon}(l-\varepsilon) = \int_\varepsilon^{l-\varepsilon} g(x) w_{0\varepsilon}(x)\, dx$.

(3) $w_{0\varepsilon}(x) < u_0(x)$ for $x \in (\varepsilon, 2\varepsilon) \cup (l - 2\varepsilon, l - \varepsilon)$, and $w_{0\varepsilon}(x) = u_0(x)$ for $x \in [2\varepsilon, l - 2\varepsilon]$.

(4) $(x^\alpha w_{0\varepsilon x})_x \le 0$ for $x \in (\varepsilon, l - \varepsilon)$.

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(5) $(x^\alpha w_{0\varepsilon x})_x + \int_\varepsilon^{l-\varepsilon} w_{0\varepsilon}^p dx - k w_{0\varepsilon}^q \geq 0$ for $\varepsilon \in (0, \varepsilon_1]$ and $x \in (\varepsilon, l - \varepsilon)$.

(6) $w_{0\varepsilon}$ is non-increasing with respect to ε in $(0, \varepsilon_1]$. Moreover

$$\lim_{x \to \varepsilon^+} \left[(x^\alpha w_{0\varepsilon x})_x + \int_\varepsilon^{l-\varepsilon} w_{0\varepsilon}^p dx - k w_{0\varepsilon}^q \right] = \lim_{x \to (l-\varepsilon)^-} \left[(x^\alpha w_{0\varepsilon x})_x + \int_\varepsilon^{l-\varepsilon} w_{0\varepsilon}^p dx - k w_{0\varepsilon}^q \right] = 0.$$

It is obvious that

$$\lim_{\varepsilon \to 0^+} w_{0\varepsilon}(x) = u_0(x).$$

Now, we consider the following regularized problem

$$
\begin{cases}
w_{\varepsilon t} = (x^\alpha w_{\varepsilon x})_x + \int_\varepsilon^{l-\varepsilon} w_\varepsilon^p dx - k w_\varepsilon^q, & (x,t) \in (\varepsilon, l - \varepsilon) \times (0, +\infty), \\
w_\varepsilon(\varepsilon, t) = \int_\varepsilon^{l-\varepsilon} f(x) w_\varepsilon(x,t) dx, & t \in (0, +\infty), \\
w_\varepsilon(l - \varepsilon, t) = \int_\varepsilon^{l-\varepsilon} g(x) w_\varepsilon(x,t) dx, & t \in (0, +\infty), \\
w_\varepsilon(x, 0) = w_{0\varepsilon}(x), & x \in [0, l].
\end{cases}
\tag{4.1}
$$

Then it is not difficult to show that there exists a unique solution $w_\varepsilon(x,t)$ for problem (4.1). In addition, from the arguments of Section 2 in [23], it follows that

$$\lim_{\varepsilon \to 0^+} w_\varepsilon(x,t) = u(x,t),$$

where $u(x,t)$ is the solution of problem (1.1).

Lemma 4.1. *Suppose that hypotheses* (1.5), (1.6) *and* (1.7) *hold, and assume that* $p \geq q > 1$ *and* $\mathcal{N} \leq 1$. *Then* $(x^\alpha u_x)_x \leq 0$ *holds for* $(x,t) \in I_T$.

Proof. Taking $\eta = (x^\alpha w_{\varepsilon x})_x$, then from (4.1), we have

$$\eta_t = \left\{ x^\alpha \left[(x^\alpha w_{\varepsilon x})_x + \int_\varepsilon^{l-\varepsilon} w_\varepsilon^p dx - k w_\varepsilon^q \right]_x \right\}_x = (x^\alpha \eta_x)_x - kq w_\varepsilon^{q-1} \eta - kq(q-1) w_\varepsilon^{q-2} |w_{\varepsilon x}|^2 \tag{4.2}$$

holds for any $(x,t) \in (\varepsilon, l - \varepsilon) \times (0, T)$, which tells us that

$$\eta_t - (x^\alpha \eta_x)_x + kq w_\varepsilon^{q-1} \eta \leq 0. \tag{4.3}$$

On the other hand, for any $t \in (0, T)$, we have

$$
\begin{aligned}
\eta(\varepsilon, t) &= \int_\varepsilon^{l-\varepsilon} f(x) w_{\varepsilon t}(x,t) dx - \int_\varepsilon^{l-\varepsilon} w_\varepsilon^p(x,t) dx + k \left(\int_\varepsilon^{l-\varepsilon} f(x) w_\varepsilon(x,t) dx \right)^q \\
&= \int_\varepsilon^{l-\varepsilon} f(x) \left((x^\alpha w_{\varepsilon x})_x + \int_\varepsilon^{l-\varepsilon} w_\varepsilon^p(x,t) dx - k w_\varepsilon^q \right) dx \\
&\quad - \int_\varepsilon^{l-\varepsilon} w_\varepsilon^p(x,t) dx + k \left(\int_\varepsilon^{l-\varepsilon} f(x) w_\varepsilon(x,t) dx \right)^q \\
&= \int_\varepsilon^{l-\varepsilon} f(x) \eta(x,t) dx + \left(\int_\varepsilon^{l-\varepsilon} f(x) dx - 1 \right) \int_\varepsilon^{l-\varepsilon} w_\varepsilon^p(x,t) dx \\
&\quad - k \left[\int_\varepsilon^{l-\varepsilon} f(x) w_\varepsilon^q(x,t) dx - \left(\int_\varepsilon^{l-\varepsilon} f(x) w_\varepsilon(x,t) dx \right)^q \right].
\end{aligned}
\tag{4.4}
$$

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It follows from Jensen's inequality that

$$
\int_{\varepsilon}^{l-\varepsilon} f(x) w_{\varepsilon}^{q} dx - \left(\int_{\varepsilon}^{l-\varepsilon} f(x) w_{\varepsilon}(x,t) dx \right)^{q}
$$

$$
\geq \int_{\varepsilon}^{l-\varepsilon} f(x) dx \left(\frac{\int_{\varepsilon}^{l-\varepsilon} f(x) w_{\varepsilon}(x,t) dx}{\int_{\varepsilon}^{l-\varepsilon} f(x) dx} \right)^{q} - \left(\int_{\varepsilon}^{l-\varepsilon} f(x) w_{\varepsilon}(x,t) \right)^{q}
$$

$$
\geq 0.
$$

Exploiting the above inequality and the assumption $\mathcal{N} \leq 1$ to (4.4), we can claim that

$$
\eta(\varepsilon,t) \leq \int_{\varepsilon}^{l-\varepsilon} f(x) \eta(x,t) dx, \quad t \in (0,T). \tag{4.5}
$$

By the analogous arguments, one can also show that

$$
\eta(l-\varepsilon,t) \leq \int_{\varepsilon}^{l-\varepsilon} g(x) \eta(x,t) dx \tag{4.6}
$$

holds for all $t \in (0,T)$.

Moreover, noticing that $\eta(x,0) = (x^{\alpha} w_{0\varepsilon x})_x \leq 0$ holds for $x \in (\varepsilon, l-\varepsilon)$. Then, maximum principle tells us that $\eta(x,t) = (x^{\alpha} w_{\varepsilon x})_x \leq 0$ holds for all $(x,t) \in (\varepsilon, l-\varepsilon) \times (0,T)$. In addition, by the arbitrariness of ε, we know that $(x^{\alpha} u_x)_x \leq 0$ holds in I_T. The proof of Lemma 4.1 is complete. $\qquad\square$

In what follows, for the sake of simplicity, we denote

$$
\psi(t) = \int_0^l u^p(x,t) dx \text{ and } \Psi(t) = \int_0^t \psi(\tau) d\tau.
$$

Lemma 4.2. *Assume that (1.5), (1.6) and (1.7) hold, $p > q \geq 1$ and $\mathcal{N} \leq 1$, then there exists a positive constant C such that*

$$
\sup_{x \in K_d} (\Psi(t) - u(x,t)) \leq \frac{C}{d^2} \left(1 + Z(t) + \int_0^t \Psi(\tau) d\tau \right)
$$

in $[0,l] \times [\frac{T}{2}, T)$, where

$$
Z(t) = o(\Psi(t)) \ as \ t \to T,
$$

and

$$
K_d = \{ x \in (0,l) : dist(x,0) \geq d, dist(x,l) \geq d \} \subset (0,l).
$$

Proof. Put

$$
\mathfrak{F}(t) = \int_0^l (\Psi(t) - u(x,t)) \zeta_1(x) dx, \tag{4.7}
$$

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where $\zeta_1(x)$ is given by (1.3). Taking the derivative of $\mathfrak{F}(t)$ with respect to t, we arrive at

$$
\begin{aligned}
\mathfrak{F}'(t) &= \int_0^l (\psi(t) - u_t)\,\zeta_1(x)\,dx \\
&= \int_0^l \left(-(x^\alpha u_x)_x + ku^q\right)\zeta_1(x)\,dx \\
&= \lambda_1 \int_0^l u(x,t)\,\zeta_1(x)\,dx + k\int_0^l u^q(x,t)\,\zeta_1(x)\,dx \\
&\quad + l^\alpha \zeta_{1x}\mid_{x=l} \int_0^l g(x)\,u(x,t)\,dx \\
&\leq \lambda_1 \int_0^l u(x,t)\,\zeta_1(x)\,dx + k\int_0^l u^q(x,t)\,\zeta_1(x)\,dx \\
&= -\lambda_1 \mathfrak{F}(t) + \lambda_1 \Psi(t) + k\int_0^l u^q(x,t)\,\zeta_1(x)\,dx.
\end{aligned}
\tag{4.8}
$$

On the other hand, it follows from Lemma 4.1 that

$$
u_t \leq \psi(t) - ku^q,
$$

which implies that

$$
-\max_{x\in[0,l]} u_0(x) \leq \Psi(t) - u(x,t).
\tag{4.9}
$$

Then (4.9) and (4.8) lead to

$$
\mathfrak{F}'(t) \leq \lambda_1 \max_{x\in[0,l]} u_0(x) + \lambda_1 \Psi(t) + k\int_0^l u^q(x,t)\,\zeta_1(x)\,dx.
$$

Integrating above inequality over from 0 to t, one has

$$
\mathfrak{F}(t) \leq \max\left\{\lambda_1, k\max_{x\in[0,l]}\zeta_1(x), \mathfrak{F}(0) + \lambda_1 T \max_{x\in[0,l]} u_0(x)\right\}\left(1 + \int_0^t \Psi(\tau)\,d\tau + \int_0^t \int_0^l u^q(x,\tau)\,dxd\tau\right).
\tag{4.10}
$$

Further, since $p > q \geq 1$, Hölder's inequality implies that

$$
\int_0^t \int_0^l u^q(y,\tau)\,dyd\tau \leq (lT)^{\frac{p-q}{p}}\left(\int_0^t \int_0^l u^p(y,\tau)\,dyd\tau\right)^{\frac{q}{p}} := Z(t).
\tag{4.11}
$$

It is not difficult to verify that

$$
Z(t) = o(\Psi(t)) \text{ as } t \to T.
\tag{4.12}
$$

Combining (4.13), (4.11) with (4.12), we see that

$$
\mathfrak{F}(t) \leq \max\left\{\lambda_1, k\max_{x\in[0,l]}\zeta_1(x), \mathfrak{F}(0) + \lambda_1 T \max_{x\in[0,l]} u_0(x)\right\}\left(1 + Z(t) + \int_0^t \Psi(\tau)\,d\tau\right).
\tag{4.13}
$$

Now, by Lemma 4.5 in [17], we can claim that

$$
\sup_{x\in K_d}(\Psi(t) - u(x,t)) \leq \frac{C}{d^2}\left(1 + \int_0^t \Psi(\tau)\,d\tau + o(\Psi(t))\right)
$$

holds for $(x,t) \in [0,l] \times \left[\frac{T}{2}, T\right)$, where C is an appropriate positive constant. The proof of Lemma 4.2 is complete. $\quad\square$

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In view of Lemma 4.2, and by a slight variant of the proof of Lemma 4.5 in [17], we have the following Lemma.

Lemma 4.3. *Assume that (1.6) and (1.7) hold, $p > q \geq 1$ and $\mathcal{N} \leq 1$, then*

$$\lim_{t \to T} \sup_{[0,l]} |u(\cdot, t)| = +\infty \tag{4.14}$$

is equivalent to

$$\lim_{t \to T} \Psi(t) = +\infty \tag{4.15}$$

Moreover, if (4.14) or (4.15) is fulfilled, then

$$\lim_{t \to T} \frac{u(x,t)}{\Psi(t)} = \lim_{t \to T} \frac{|u(\cdot,t)|_\infty}{\Psi(t)} = 1 \tag{4.16}$$

uniformly on any compact subset of $(0, l)$.

Next, we give the proofs of Theorems 1.4 and Theorem 1.5, respectively.

Proof of Theorem 1.4. It follows from (4.16) that

$$u^p(x,t) \sim \Psi^p(t), \quad t \to T.$$

By the Lebesgue's dominated convergence theorem, we have

$$\Psi'(t) = \psi(t) = \int_0^l u^p(x,t)\, dx \sim l\Psi^p(t), \quad t \to T.$$

Therefore, by integrating the above equality, we can claim that

$$\Psi(t) \sim (l(p-1)(T-t))^{-\frac{1}{p-1}}. \tag{4.17}$$

Combining (4.16) with (4.17), we find that

$$u(x,t) \sim (l(p-1)(T-t))^{-\frac{1}{p-1}}, \quad t \to T, \tag{4.18}$$

which means that

$$\lim_{t \to T}(T-t)^{\frac{1}{p-1}}u(x,t) = \lim_{t \to T}(T-t)^{\frac{1}{p-1}}|u(\cdot,t)|_\infty = (l(p-1))^{-\frac{1}{p-1}}.$$

The proof of Theorem 1.4 is complete. □

Proof of Theorem 1.5. Denote

$$\varphi(t) = \int_0^l u^p(y,t)\, dy - k\left(\max_{x \in [0,l]} u(x,t)\right)^p \text{ and } \Phi(t) = \int_0^t g(\tau)\, d\tau.$$

Similar to Lemma 4.3, we can get

$$\lim_{t \to T} \frac{u(x,t)}{\Phi(t)} = \lim_{t \to T} \frac{|u(\cdot,t)|_\infty}{\Phi(t)} = 1, \tag{4.19}$$

uniformly on any compact subset of $(0, l)$.

Since, the remaining arguments are the same as those in the proof of Theorem 1.4, we omit it here. The proof of Theorem 1.5 is complete. □

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Competing interests

The authors declare that they have no competing interests.

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Approximation properties of Kantorovich-type q-Bernstein-Stancu-Schurer operators

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Abstract. In this paper, we introduce a Kantorovich-type Bernstein-Stancu-Schurer operators $K_{n,p,q}^{\alpha,\beta}$ based on the concept of q-integers. We investigate statistical approximation properties and establish a local approximation theorem, we also give a convergence theorem for the Lipschitz continuous functions. Finally, we give some graphics to illustrate the convergence properties of operators to some functions.

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1   Introduction

In 2013, Özarslan and Vedi [7] introduced the q-Bernstein-Schurer-Kantorovich operators as follows:

$$K_n^p(f;q;x) = \sum_{r=0}^{n+p} \begin{bmatrix} n+p \\ r \end{bmatrix}_q x^r \prod_{s=o}^{n+p-r-1} (1-q^s x) \int_0^1 f\left(\frac{[r]_q}{[n+1]_q} + \frac{1+(q-1)[r]_q}{[n+1]_q} t\right) d_q t$$

for any real number $0 < q < 1$, fixed $p \in \mathbb{N}_0$ and $f \in C[0, p+1]$. They gave the Korovkin-type approximation theorem, obtained the rate of convergence of the operators and so on. In 2014, Ren and Zeng [8] introduced two kinds of Kantorovich-type q-Bernstein-Stancu operators based on q-Jackson integral and Riemann-type q-integral respectively and got some approximation properties. In 2015, Acu [1] introduced and studied q analogue of Stancu-Schurer-Kantorovich operators. They proved a convergence theorem, established the rate of convergence, obtained a Voronovskaya type result and so on, they constructed

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the operators as follows:

$$K_{n,p}^{\alpha,\beta}(f;x) = \sum_{k=0}^{n+p} \begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k (1-x)_q^{n+p-k} \int_0^1 f\left(\frac{[k]_q + q^k t + \alpha}{[n+1]_q + \beta}\right) d_q t.$$

In 2015, Agrawal, Finta and Kumar [2] introduced a new Kantorovich-type generalization of the q-Bernstein-Schurer operators, they gave the basic convergence theorem, obtained the local direct results, estimated the rate of convergence and so on. The operators are defined as

$$K_{n,p}(f;q;x) = [n+1]_q \sum_{k=0}^{n+p} b_{n+p,k}(q;x) q^{-k} \int_{\frac{[k]_q}{[n+1]_q}}^{\frac{[k+1]_q}{[n+1]_q}} f(t) d_q^R t, \tag{1}$$

where $b_{n+p,k}(q;x)$ is defined by

$$b_{n+p,k}(q;x) = \begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k (1-x)_q^{n+p-k}. \tag{2}$$

Motivated by above investigations, it seems there have no papers mentioned about the Stancu-type of the operators defined in (1). In present paper, we will introduce the Kantorovich-type q-Bernstein-Stancu-Schurer operators $\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)$ which will be defined in (4). We will investigate statistical approximation properties, establish a local approximation theorem and give a convergence theorem for the Lipschitz continuous functions. Furthermore, we will give some graphics to illustrate the convergence properties of operators to some functions.

Before introducing the operators, we mention certain definitions based on q-integers, detail can be found in [5, 6]. For any fixed real number $0 < q \le 1$ and each nonnegative integer k, we denote q-integers by $[k]_q$, where

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q}, & q \ne 1; \\ k, & q = 1. \end{cases}$$

Also q-factorial and q-binomial coefficients are defined as follows:

$$[k]_q! = \begin{cases} [k]_q[k-1]_q...[1]_q, & k = 1, 2, ...; \\ 1, & k = 0, \end{cases} \quad , \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad (n \ge k \ge 0).$$

For $x \in [0, 1]$ and $n \in \mathbb{N}_0$, we recall that

$$(1-x)_q^n = \begin{cases} 1, & n = 0; \\ \prod_{j=0}^{n-1} (1-q^j x) = (1-x)(1-qx)...(1-q^{n-1}x), & n = 1, 2, ... \end{cases}.$$

The Riemann-type q-integral is defined by

$$\int_a^b f(t) d_q^R t = (1-q)(b-a) \sum_{j=0}^{\infty} f\left(a + (b-a)q^j\right) q^j, \tag{3}$$

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where the real numbers a, b and q satisfy that $0 \leq a < b$ and $0 < q < 1$.

For $f \in C(I)$, $I = [0, 1+p]$, $p \in \mathbb{N}_0$, $0 \leq \alpha \leq \beta$, $q \in (0,1)$ and $n \in \mathbb{N}$, we introduce the Kantorovich-type q-Bernstein-Stancu-Schurer operators as follows:

$$\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x) = ([n+1]_q + \beta) \sum_{k=0}^{n+p} b_{n+p,k}(q;x) q^{-k} \int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} f(t) d_q^R t, \tag{4}$$

where $b_{n+p,k}(q;x)$ is defined by (2).

2 Auxiliary Results

In order to obtain the approximation properties, We need the following lemmas:

Lemma 2.1. *Using the definition (3), we easily get*

$$\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} d_q^R t = \frac{q^k}{[n+1]_q + \beta}, \tag{5}$$

$$\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} t\, d_q^R t = \frac{([k]_q+\alpha)q^k}{([n+1]_q + \beta)^2} + \frac{q^{2k}}{[2]_q([n+1]_q + \beta)^2}, \tag{6}$$

$$\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} t^2\, d_q^R t = \frac{q^k([k]_q+\alpha)^2}{([n+1]_q + \beta)^3} + \frac{2q^{2k}([k]_q+\alpha)}{[2]_q([n+1]_q + \beta)^3} + \frac{q^{3k}}{[3]_q([n+1]_q + \beta)^3}. \tag{7}$$

Lemma 2.2. *(See [2], Lemma 2.1) The following equalities hold*

$$\sum_{k=0}^{n+p} b_{n+p,k}(q;x) q^k = 1 - (1-q)[n+p]_q x, \tag{8}$$

$$\sum_{k=0}^{n+p} b_{n+p,k}(q;x) q^{2k} = 1 - \left(1-q^2\right)[n+p]_q x + q(1-q)^2[n+p]_q[n+p-1]_q x^2. \tag{9}$$

Lemma 2.3. *For the Kantorovich-type q-Bernstein-Stancu-Schurer operators (4), we have*

$$\widetilde{K_{n,p,q}^{\alpha,\beta}}(1;x) = 1, \tag{10}$$

$$\widetilde{K_{n,p,q}^{\alpha,\beta}}(t;x) = \frac{2q[n+p]_q x + 1 + [2]_q \alpha}{[2]_q([n+1]_q + \beta)}, \tag{11}$$

$$\widetilde{K_{n,p,q}^{\alpha,\beta}}\left(t^2;x\right) = \frac{\left(q^2[3]_q + 3q^4\right)[n+p]_q[n+p-1]_q}{[2]_q[3]_q([n+1]_q + \beta)^2} x^2 + \frac{[2]_q[3]_q\alpha^2 + 2[3]_q\alpha + [2]_q}{[2]_q[3]_q([n+1]_q + \beta)^2}$$
$$+ \frac{\left(4q[3]_q\alpha + 3q + 5q^2 + 4q^3\right)[n+p]_q}{[2]_q[3]_q([n+1]_q + \beta)^2} x. \tag{12}$$

Proof. (10) is easily obtained from (4) and (5). Using (4), (6) and (8), we have

$$\widetilde{K_{n,p,q}^{\alpha,\beta}}(t;x)$$

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$$= \sum_{k=0}^{n+p} b_{n+p,q}(k;x) \left(\frac{[k]_q + \alpha}{[n+1]_q + \beta} + \frac{q^k}{[2]_q([n+1]_q + \beta)} \right)$$

$$= \frac{[n+p]_q}{[n+1]_q + \beta} \sum_{k=0}^{n+p} b_{n+p,q}(k;x) \frac{[k]_q}{[n+p]_q} + \frac{\alpha}{[n+1]_q + \beta} + \frac{1 - (1-q)[n+p]_q x}{[2]_q([n+1]_q + \beta)}$$

$$= \frac{[n+p]_q}{[n+1]_q + \beta} \sum_{k=0}^{n+p-1} \begin{bmatrix} n+p-1 \\ k \end{bmatrix}_q x^{k+1}(1-x)_q^{n+p-k-1} + \frac{1 - (1-q)[n+p]_q x}{[2]_q([n+1]_q + \beta)}$$

$$+ \frac{\alpha}{[n+1]_q + \beta}$$

$$= \frac{[n+p]_q}{[n+1]_q + \beta} x - \frac{(1-q)[n+p]_q}{[2]([n+1]_q + \beta)} x + \frac{1 + [2]_q \alpha}{[2]_q([n+1]_q + \beta)}.$$

Thus, (11) is proved. Finally, from (4) and (7), we have

$$\widetilde{K_{n,p,q}^{\alpha,\beta}}\left(t^2;x\right)$$

$$= \sum_{k=0}^{n+p} b_{n+p,q}(k;x) \left(\frac{[k]_q^2 + 2\alpha[k]_q + \alpha^2}{([n+1]_q + \beta)^2} + \frac{2q^k([k]_q + \alpha)}{[2]_q([n+1]_q + \beta)^2} + \frac{q^{2k}}{[3]_q([n+1]_q + \beta)^2} \right),$$

since $[k]_q^2 = [k]_q[k-1]_q + q^{k-1}[k]_q$, and from lemma 2.2, we have

$$\widetilde{K_{n,p,q}^{\alpha,\beta}}\left(t^2;x\right)$$

$$= \sum_{k=0}^{n+p} b_{n+p,k}(q;x) \frac{[k]_q[k-1]_q}{([n+1]_q + \beta)^2} + \sum_{k=0}^{n+p} b_{n+p,k}(q;x) \frac{2\alpha[k]_q}{([n+1]_q + \beta)^2}$$

$$+ \sum_{k=0}^{n+p} b_{n+p,k}(q;x) \frac{q^{k-1}[k]_q}{([n+1]_q + \beta)^2} + \frac{\alpha^2}{([n+1]_q + \beta)^2} + \sum_{k=0}^{n+p} b_{n+p,k}(q;x) \frac{2q^k[k]_q}{[2]_q([n+1]_q + \beta)^2}$$

$$+ \frac{2\alpha}{[2]_q([n+1]_q + \beta)^2} \sum_{k=0}^{n+p} b_{n+p,k}(q;x)q^k + \sum_{k=0}^{n+p} b_{n+p,k}(q;x) \frac{q^{2k}}{[3]([n+1]_q + \beta)^2}$$

$$= \frac{[n+p]_q[n+p-1]_q x^2}{([n+1]_q + \beta)^2} + \frac{2\alpha[n+p]_q x}{([n+1]_q + \beta)^2} + \frac{[n+p]_q x}{([n+1]_q + \beta)^2} - \frac{(1-q)[n+p]_q[n+p-1]_q x^2}{([n+1]_q + \beta)^2}$$

$$+ \frac{\alpha^2}{([n+1]_q + \beta)^2} + \frac{2q[n+p]_q x}{[2]_q([n+1]_q + \beta)^2} - \frac{2q(1-q)[n+p]_q[n+p-1]_q x^2}{[2]_q([n+1]_q + \beta)^2}$$

$$+ \frac{2\alpha\left(1 - (1-q)[n+p]_q x\right)}{[2]_q([n+1]_q + \beta)^2} + \frac{1 - \left(1-q^2\right)[n+p]_q x + q(1-q)^2[n+p]_q[n+p-1]_q x^2}{[3]_q([n+1]_q + \beta)^2}$$

$$= \frac{[n+p]_q[n+p-1]_q}{([n+1]_q + \beta)^2} x^2 + \frac{(2[2]_q\alpha + [2]_q + 2q)[n+p]_q}{[2]_q([n+1]_q + \beta)^2} x + \frac{[2]_q[3]_q\alpha^2 + 2[3]_q\alpha + [2]_q}{[2]_q[3]_q([n+1]_q + \beta)^2}$$

$$- \frac{(1-q)\left(1 - q + 4q[3]_q\right)[n+p]_q[n+p-1]_q}{[2]_q[3]_q([n+1]_q + \beta)^2} x^2 - \frac{(1-q)\left(2\alpha[3]_q + [2]_q^2\right)[n+p]_q}{[2]_q[3]_q([n+1]_q + \beta)^2} x$$

$$= \frac{\left(q^2[3]_q + 3q^4\right)[n+p]_q[n+p-1]_q}{[2]_q[3]_q([n+1]_q + \beta)^2} x^2 + \frac{\left(4q[3]_q\alpha + 3q + 5q^2 + 4q^3\right)[n+p]_q}{[2]_q[3]_q([n+1]_q + \beta)^2} x$$

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$$+\frac{[2]_q[3]_q\alpha^2 + 2[3]_q\alpha + [2]_q}{[2]_q[3]_q([n+1]_q + \beta)^2}.$$

Thus, (12) is proved. □

Remark 2.4. *From lemma 2.3, it is observed that for $\alpha = \beta = 0$, we get the moments for the operators defined in* (1), *which are the corresponding results of lemma 2.1 in [2].*

Lemma 2.5. *Using lemma 2.3 and easily computations, we have*

$$\widetilde{K_{n,p,q}^{\alpha,\beta}}(t-x;x) = \left[\frac{2q[n+p]_q}{[2]_q([n+1]_q + \beta)} - 1\right]x + \frac{1 + [2]_q\alpha}{[2]_q([n+1]_q + \beta)} \doteq A_{n,p,q}^{\alpha,\beta}(x), \qquad (13)$$

$$\widetilde{K_{n,p,q}^{\alpha,\beta}}\left((t-x)^2;x\right) \leq \left[\frac{\left(q^2[3]_q + 3q^4\right)[n+p]_q[n+p-1]_q}{[2]_q[3]_q([n+1]_q + \beta)^2} + 1 - \frac{4q[n+p]_q}{[2]_q([n+1]_q + \beta)}\right]x^2$$

$$+\frac{[2]_q[3]_q\alpha^2 + 2[3]_q\alpha + [2]_q}{[2]_q[3]_q([n+1]_q + \beta)^2} + \frac{\left(4q[3]_q\alpha + 3q + 5q^2 + 4q^3\right)[n+p]_q}{[2]_q[3]_q([n+1]_q + \beta)^2}x \doteq B_{n,p,q}^{\alpha,\beta}(x). \quad (14)$$

3   Statistical approximation properties

In this section, we present the statistical approximation properties of the operator $\widetilde{K_{n,p,q}^{\alpha,\beta}}$ by using the Korovkin-type statistical approximation theorem proved in [4].

Let K be a subset of $\mathbb{N}$, the set of all natural numbers. The density of K is defined by $\delta(K) := \lim_n \frac{1}{n}\sum_{k=1}^n \chi_K(k)$ provided the limit exists, where χ_K is the characteristic function of K. A sequence $x := \{x_n\}$ is called statistically convergent to a number L if, for every $\varepsilon > 0$, $\delta\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} = 0$. Let $A := (a_{jn}), j, n = 1, 2, ...$ be an infinite summability matrix. For a given sequence $x := \{x_n\}$, the A−transform of x, denoted by $Ax := ((Ax)_j)$, is given by $(Ax)_j = \sum_{k=1}^\infty a_{jn}x_n$ provided the series converges for each j. We say that A is regular if $\lim_n(Ax)_j = L$ whenever $\lim x = L$. Assume that A is a non-negative regular summability matrix. A sequence $x = \{x_n\}$ is called A-statistically convergent to L provided that for every $\varepsilon > 0$, $\lim_j \sum_{n:|x_n - L|\geq\varepsilon} a_{jn} = 0$. We denote this limit by $st_A - \lim_n x_n = L$. For $A = C_1$, the Cesàro matrix of order one, A-statistical convergence reduces to statistical convergence. It is easy to see that every convergent sequence is statistically convergent but not conversely.

We consider a sequence $q := \{q_n\}$ for $0 < q_n < 1$ satisfying

$$st_A - \lim_n q_n = 1, \qquad (15)$$

If $e_i = t^i$, $t \in \mathbb{R}^+$, $i = 0, 1, 2, ...$ stands for the ith monomial, then we have

Theorem 3.1. *Let $A = (a_{nk})$ be a non-negative regular summability matrix and $q := \{q_n\}$ be a sequence satisfying* (15), *then for all $f \in C(I)$, $x \in [0, 1]$, we have*

$$st_A - \lim_n \left\|\widetilde{K_{n,p,q}^{\alpha,\beta}}f - f\right\|_{C(I)} = 0. \qquad (16)$$

Proof. Obviously

$$st_A - \lim_n \left\| \widetilde{K_{n,p,q_n}^{\alpha,\beta}}(e_0) - e_0 \right\|_{C(I)} = 0. \tag{17}$$

By (13), we have

$$\left| \widetilde{K_{n,p,q_n}^{\alpha,\beta}}(e_1; x) - e_1(x) \right| \leq \left| \frac{2q_n[n+p]_{q_n}}{[2]_{q_n}([n+1]_{q_n} + \beta)} - 1 \right| + \frac{1 + [2]_{q_n}\alpha}{[2]_{q_n}([n+1]_{q_n} + \beta)}.$$

Now for a given $\varepsilon > 0$, let us define the following sets:

$$U := \left\{ k : \left\| \widetilde{K_{n,p,q_k}^{\alpha,\beta}}(e_1) - e_1 \right\|_{C(I)} \geq \varepsilon \right\}, \quad U_1 := \left\{ k : \left| \frac{2q_k[n+p]_{q_k}}{[2]_{q_k}([n+1]_{q_k} + \beta)} - 1 \right| \geq \frac{\varepsilon}{2} \right\},$$

$$U_2 := \left\{ k : \frac{1 + [2]_{q_k}\alpha}{[2]_{q_k}([n+1]_{q_k} + \beta)} \geq \frac{\varepsilon}{2} \right\}.$$

Then one can see that $U \subseteq U_1 \cup U_2$, so we have

$$\delta \left\{ k \leq n : \left\| \widetilde{K_{n,p,q_k}^{\alpha,\beta}}(e_1) - e_1 \right\|_{C(I)} \right\} \leq \delta \left\{ k \leq n : \left| \frac{2q_k[n+p]_{q_k}}{[2]_{q_k}([n+1]_{q_k} + \beta)} - 1 \right| \geq \frac{\varepsilon}{2} \right\}$$

$$+ \delta \left\{ k \leq n : \frac{1 + [2]_{q_k}\alpha}{[2]_{q_k}([n+1]_{q_k} + \beta)} \geq \frac{\varepsilon}{2} \right\},$$

since $st_A - \lim_n q_n = 1$, we have

$$st_A - \lim_n \left| \frac{[n+p]_{q_n}}{[n+1]_{q_n} + \beta} - 1 \right| = 0, \quad st_A - \lim_n \frac{1 + [2]_{q_n}\alpha}{[2]_{q_n}([n+1]_{q_n} + \beta)} = 0,$$

which implies that the right-hand side of the above inequality is zero, thus we have

$$st_A - \lim_n \left\| \widetilde{K_{n,p,q_n}^{\alpha,\beta}}(e_1) - e_1 \right\|_{C(I)} = 0. \tag{18}$$

Finally, by (10) and (12), we get

$$\left| \widetilde{K_{n,p,q_n}^{\alpha,\beta}}(e_2; x) - e_2(x) \right|$$

$$\leq \left| \frac{\left(q_n^2[3]_{q_n} + 3q_n^4\right)[n+p]_{q_n}[n+p-1]_{q_n}}{[2]_{q_n}[3]_{q_n}([n+1]_{q_n} + \beta)^2} - 1 \right| + \frac{\left(4q_n[3]_{q_n}\alpha + 3q_n + 5q_n^2 + 4q_n^3\right)[n+p]_{q_n}}{[2]_{q_n}[3]_{q_n}([n+1]_{q_n} + \beta)^2}$$

$$+ \frac{[2]_{q_n}[3]_{q_n}\alpha^2 + 2[3]_{q_n}\alpha + [2]_{q_n}}{[2]_{q_n}[3]_{q_n}([n+1]_{q_n} + \beta)^2} \doteq \alpha_n + \beta_n + \gamma_n.$$

Since $st_A - \lim_n q_n = 1$, one can see that

$$st_A - \lim_n \alpha_n = st_A - \lim_n \beta_n = st_A - \lim_n \gamma_n = 0. \tag{19}$$

For $\varepsilon > 0$, we define the following four sets

$$V := \left\{ k : \left\| \widetilde{K_{n,p,q_k}^{\alpha,\beta}}(e_2) - e_2 \right\|_{C(I)} \geq \varepsilon \right\}, \quad V_1 := \left\{ k : \alpha_k \geq \frac{\varepsilon}{3} \right\}, \quad V_2 := \left\{ k : \beta_k \geq \frac{\varepsilon}{3} \right\},$$

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$$V_3 := \left\{ k : \gamma_k \geq \frac{\varepsilon}{3} \right\}.$$

Hence, from (19) we obtain the right-hand side of the above inequality is zero, so we have

$$\delta \left\{ k \leq n : \left\| \widetilde{K_{n,p,q_k}^{\alpha,\beta}}(e_2) - e_2 \right\|_{C(I)} \geq \varepsilon \right\} = 0,$$

thus

$$st_A - \lim_n \left\| \widetilde{K_{n,p,q_n}^{\alpha,\beta}}(e_2) - e_2 \right\|_{C(I)} = 0. \tag{20}$$

Combining (17), (18) and (20), theorem 3.1 follows from the Korovkin-type statistical approximation theorem established in [4], the proof is completed. $\square$

4   Local approximation properties

Let $f \in C(I)$, endowed with the norm $\|f\| = \sup_{x \in I} |f(x)|$. The Peetre's $K-$functional is defined by

$$K_2(f;\delta) = \inf_{g \in C^2} \left\{ \|f - g\| + \delta \|g''\| \right\},$$

where $\delta > 0$ and $C^2 = \{ g \in C(I) : g', g'' \in C(I) \}$. By [3, p.177, Theorem 2.4], there exits an absolute constant $C > 0$ such that

$$K_2(f;\delta) \leq C\omega_2(f;\sqrt{\delta}), \tag{21}$$

where

$$\omega_2(f;\delta) = \sup_{0 < h \leq \delta} \sup_{x,x+h,x+2h \in I} |f(x+2h) - 2f(x+h) + f(x)|$$

is the second order modulus of smoothness of $f \in C(I)$. We denote the usual modulus of continuity of $f \in C(I)$ by

$$\omega(f;\delta) = \sup_{0 < h \leq \delta} \sup_{x,x+h \in I} |f(x+h) - f(x)|.$$

Now we give a direct local approximation theorem for the operators $\widetilde{K_{n,p,q}^{\alpha,\beta}}(f,x)$.

Theorem 4.1. *For $q \in (0,1)$, $x \in [0,1]$ and $f \in C(I)$, we have*

$$\left| \widetilde{K_{n,p,q}^{\alpha,\beta}}(f,x) - f(x) \right| \leq C\omega_2 \left(f; \sqrt{\left(A_{n,p,q}^{\alpha,\beta}(x) \right)^2 + B_{n,p,q}^{\alpha,\beta}(x)/2} \right) + \omega \left(f; \left| A_{n,p,q}^{\alpha,\beta}(x) \right| \right), \tag{22}$$

where C is a positive constant, $A_{n,p,q}^{\alpha,\beta}(x)$ and $B_{n,p,q}^{\alpha,\beta}(x)$ are defined in (13) and (14).

Proof. We define the auxiliary operators

$$\overline{K_{n,p,q}^{\alpha,\beta}}(f;x) = \widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x) - f\left(\frac{2q[n+p]_q x + 1 + [2]_q \alpha}{[2]_q([n+1]_q + \beta)} \right) + f(x), \tag{23}$$

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$x \in [0, 1]$. The operators $\overline{K_{n,p,q}^{\alpha,\beta}}(f; x)$ are linear and preserve the linear functions:

$$\overline{K_{n,p,q}^{\alpha,\beta}}(t - x; x) = 0 \tag{24}$$

(see Lemma 2.3).

Let $g \in C^2$. By Taylor's expansion

$$g(t) = g(x) + g'(x)(t - x) + \int_x^t (t - u)g''(u)du,$$

and (24), we get

$$\overline{K_{n,p,q}^{\alpha,\beta}}(g; x) = g(x) + \overline{K_{n,p,q}^{\alpha,\beta}}\left(\int_x^t (t - u)g''(u)du; x\right).$$

Hence, by (23), (13) and (14), we have

$$
\begin{aligned}
\left|\overline{K_{n,p,q}^{\alpha,\beta}}(g; x) - g(x)\right| &\leq \left|\widetilde{K_{n,p,q}^{\alpha,\beta}}\left(\int_x^t (t - u)g''(u)du; x\right)\right| \\
&\quad + \left|\int_x^{\frac{2q[n+p]_qx+1+[2]_q\alpha}{[2]_q([n+1]_q+\beta)}} \left(\frac{2q[n+p]_qx + 1 + [2]_q\alpha}{[2]_q([n+1]_q + \beta)} - u\right)g''(u)du\right| \\
&\leq \widetilde{K_{n,p,q}^{\alpha,\beta}}\left(\left|\int_x^t (t - u)|g''(u)|du\right|; x\right) \\
&\quad + \int_x^{\frac{2q[n+p]_qx+1+[2]_q\alpha}{[2]_q([n+1]_q+\beta)}} \left|\frac{2q[n+p]_qx + 1 + [2]_q\alpha}{[2]_q([n+1]_q + \beta)} - u\right| |g''(u)|du \\
&\leq \left\{\widetilde{K_{n,p,q}^{\alpha,\beta}}\left((t - x)^2; x\right) + \left[\frac{2q[n+p]_qx + 1 + [2]_q\alpha}{[2]_q([n+1]_q + \beta)} - x\right]^2\right\} ||g''|| \\
&\leq \left[\left(A_{n,p,q}^{\alpha,\beta}(x)\right)^2 + B_{n,p,q}^{\alpha,\beta}(x)\right] ||g''||,
\end{aligned}
$$

where $A_{n,p,q}^{\alpha,\beta}(x)$ and $B_{n,p,q}^{\alpha,\beta}(x)$ are defined in (13) and (14). On the other hand, by (23), (4) and lemma 2.3, we have

$$\left|\overline{K_{n,p,q}^{\alpha,\beta}}(f; x)\right| \leq \left|\widetilde{K_{n,p,q}^{\alpha,\beta}}(f; x)\right| + 2||f|| \leq ||f|| \widetilde{K_{n,p,q}^{\alpha,\beta}}(1; x) + 2||f|| \leq 3||f||. \tag{25}$$

Now (23) and (25) imply

$$
\begin{aligned}
\left|\widetilde{K_{n,p,q}^{\alpha,\beta}}(f; x) - f(x)\right| &\leq \left|\overline{K_{n,p,q}^{\alpha,\beta}}(f - g; x) - (f - g)(x)\right| + \left|\overline{K_{n,p,q}^{\alpha,\beta}}(g; x) - g(x)\right| \\
&\quad + \left|f\left(\frac{2q[n+p]_qx + 1 + [2]_q\alpha}{[2]_q([n+1]_q + \beta)}\right) - f(x)\right| \\
&\leq 4||f - g|| + \left[\left(A_{n,p,q}^{\alpha,\beta}(x)\right)^2 + B_{n,p,q}^{\alpha,\beta}(x)\right] ||g''|| + \omega\left(f; \left|A_{n,p,q}^{\alpha,\beta}(x)\right|\right).
\end{aligned}
$$

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Hence taking infimum on the right hand side over all $g \in C^2$, we get

$$\left| \widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x) - f(x) \right| \leq 4K_2 \left(f; \left[\left(A_{n,p,q}^{\alpha,\beta}(x) \right)^2 + B_{n,p,q}^{\alpha,\beta}(x) \right] /4 \right) + \omega \left(f; \left| A_{n,p,q}^{\alpha,\beta}(x) \right| \right).$$

By (21), for every $q \in (0,1)$, we have

$$\left| \widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x) - f(x) \right| \leq C\omega_2 \left(f; \sqrt{ \left(A_{n,p,q}^{\alpha,\beta}(x) \right)^2 + B_{n,p,q}^{\alpha,\beta}(x)}/2 \right) + \omega \left(f; \left| A_{n,p,q}^{\alpha,\beta}(x) \right| \right),$$

where $A_{n,p,q}^{\alpha,\beta}(x)$ and $B_{n,p,q}^{\alpha,\beta}(x)$ are defined in (13) and (14). This completes the proof of Theorem 4.1. □

Remark 4.2. *For any fixed $x \in [0,1]$, $0 \leq \alpha \leq \beta$, $p \in \mathbb{N}_0$ and $n \in \mathbb{N}$, let $q := \{q_n\}$ be a sequence satisfying $0 < q_n < 1$ and $\lim_n q_n = 1$, we have*

$$\lim_n A_{n,p,q}^{\alpha,\beta}(x) = 0 \quad and \quad \lim_n B_{n,p,q}^{\alpha,\beta}(x) = 0,$$

where $A_{n,p,q}^{\alpha,\beta}(x)$ and $B_{n,p,q}^{\alpha,\beta}(x)$ are defined in (13) and (14). These give us a rate of point-wise convergence of the operators $\widetilde{K_{n,p,q_n}^{\alpha,\beta}}(f;x)$ to $f(x)$.

Next we study the rate of convergence of the operators $K_{n,q}(f;x)$ with the help of functions of Lipschitz class $Lip_M(\xi)$, where $M > 0$ and $0 < \xi \leq 1$. A function f belongs to $Lip_M(\xi)$ if

$$|f(y) - f(x)| \leq M|y - x|^\xi \quad (y, x \in \mathbb{R}). \tag{26}$$

We have the following theorem.

Theorem 4.3. *Let $q := \{q_n\}$ be a sequence satisfying $0 < q_n < 1$, $\lim_n q_n = 1$ and $f \in Lip_M(\xi)$, $0 < \xi \leq 1$. Then we have*

$$\left| \widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x) - f(x) \right| \leq M \left(B_{n,p,q}^{\alpha,\beta}(x) \right)^{\frac{\xi}{2}}, \tag{27}$$

where $B_{n,p,q}^{\alpha,\beta}(x)$ is defined in (14).

Proof. Since $\widetilde{K_{n,p,q}^{\alpha,\beta}}$ is a linear positive operator and $f \in Lip_M(\xi)$ $(0 < \xi \leq 1)$, we have

$$|\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x) - f(x)|$$

$$\leq \widetilde{K_{n,p,q}^{\alpha,\beta}} \left(|f(t) - f(x)|; x \right)$$

$$= ([n+1]_q + \beta) \sum_{k=0}^{n+p} b_{n+p,k}(q;x) q^{-k} \int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} |f(t) - f(x)| d_q^R t$$

$$\leq M([n+1]_q + \beta) \sum_{k=0}^{n+p} b_{n+p,k}(q;x) q^{-k} \int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}} |t - x|^\xi d_q^R t$$

$$\leq \quad M([n+1]_q+\beta)\sum_{k=0}^{n+p}b_{n+p,k}(q;x)q^{-k}\left(\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}}\left[(t-x)^\xi\right]^{\frac{2}{\xi}}d_q^Rt\right)^{\frac{\xi}{2}}\left(\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}}d_q^Rt\right)^{\frac{2-\xi}{2}}$$

$$= \quad M([n+1]_q+\beta)\sum_{k=0}^{n+p}b_{n+p,k}(q;x)q^{-k}\left(\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}}(t-x)^2d_q^Rt\right)^{\frac{\xi}{2}}\left(\frac{q^k}{[n+1]_q+\beta}\right)^{\frac{2-\xi}{2}}$$

$$= \quad M\sum_{k=0}^{n+p}b_{n+p,k}(q;x)\left(\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}}(t-x)^2d_q^Rt\right)^{\frac{\xi}{2}}\left(\frac{[n+1]_q+\beta}{q^k}\right)^{\frac{\xi}{2}}$$

$$= \quad M\sum_{k=0}^{n+p}[b_{n+p,k}(q;x)]^{\frac{2-\xi}{2}}\left(([n+1]_q+\beta)b_{n+p,k}(q;x)q^{-k}\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}}(t-x)^2d_q^Rt\right)^{\frac{\xi}{2}}.$$

Applying Hölder's inequality for sums, we obtain

$$|\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)-f(x)|$$

$$\leq \quad M\left(\sum_{k=0}^{n+p}b_{n+p,k}(q;x)\right)^{\frac{2-\xi}{2}}\left(\sum_{k=0}^{n+p}([n+1]_q+\beta)b_{n+p,k}(q;x)q^{-k}\int_{\frac{[k]_q+\alpha}{[n+1]_q+\beta}}^{\frac{[k+1]_q+\alpha}{[n+1]_q+\beta}}(t-x)^2d_q^Rt\right)^{\frac{\xi}{2}}$$

$$= \quad M\left(\widetilde{K_{n,p,q}^{\alpha,\beta}}\left((t-x)^2;x\right)\right)^{\frac{\xi}{2}}=M\left(B_{n,p,q}^{\alpha,\beta}(x)\right)^{\frac{\xi}{2}}.$$

Thus, theorem 4.3 is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

5   Graphical analysis

In this section, we will illustrate two examples to state the convergence of operators $\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)$ to $f(x)$ by means of Graphs.

Example 1: From figure 1, we can observe that as q increases, $n=50$ be fixed, Kantorovich-type q-Bernstein-Stancu-Schurer operators given by (4) converge to the function $f(x)=\sin(2\pi x)$.

In comparison to figure 1, let $q=0.99$ be fixed, as n increases, operators given by (4) converge to the function as shown in figure 2.

Example2: Similarly for different values of parameters q and n, let $p=1$, $\alpha=2$ and $\beta=3$, convergence of operators to the function $f(x)=1-\cos(4e^x)$ is shown in figure 3 and 4, respectively.

Acknowledgement

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Figure 1: Convergence of $\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)$ for $n = 50$, $p = 1$, $\alpha = 2$, $\beta = 3$ and different values of q.

Figure 2: Convergence of $\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)$ for $q = 0.99$, $p = 1$, $\alpha = 2$, $\beta = 3$ and different values of n.
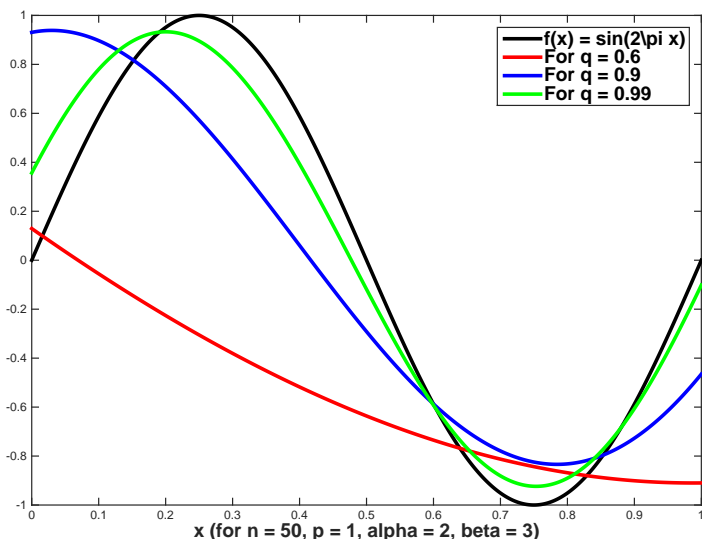
Q. -B. CAI
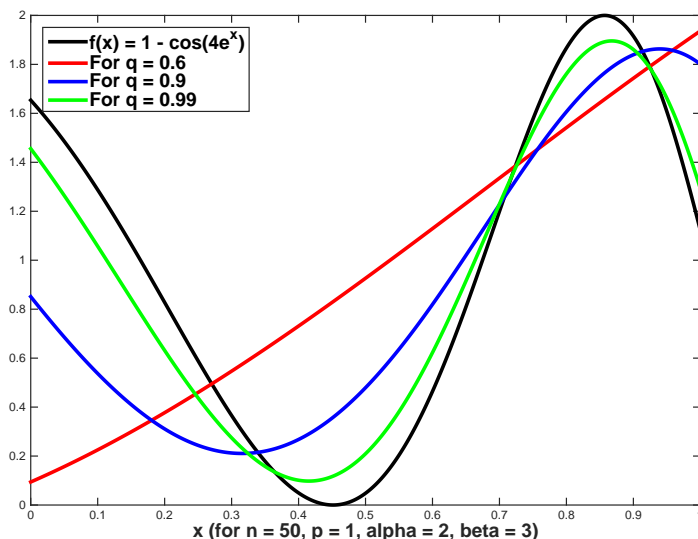
Figure 3: Convergence of $\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)$ for $n = 50$, $p = 1$, $\alpha = 2$, $\beta = 3$ and different values of q.
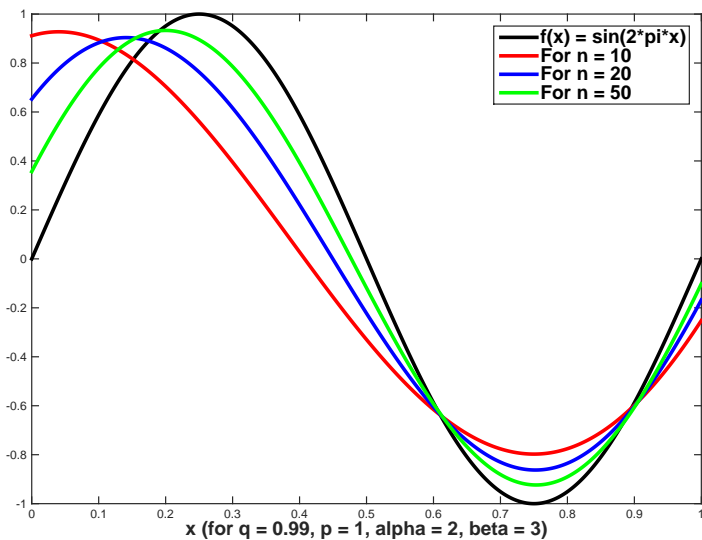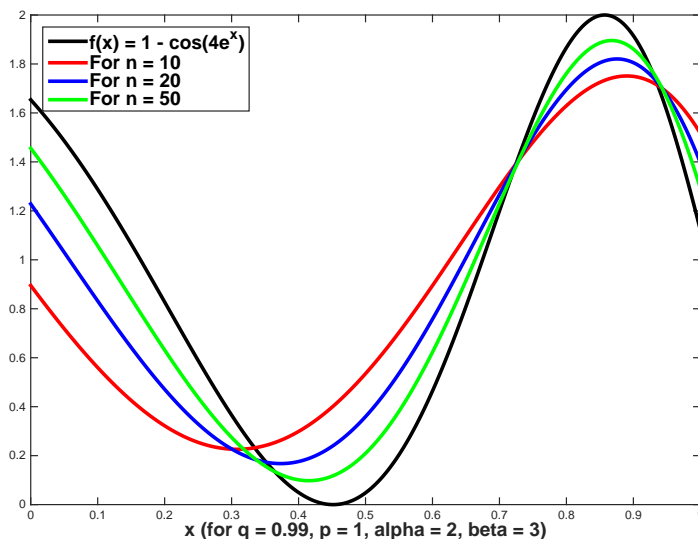

Figure 4: Convergence of $\widetilde{K_{n,p,q}^{\alpha,\beta}}(f;x)$ for $q = 0.99$, $p = 1$, $\alpha = 2$, $\beta = 3$ and different values of n.

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On the generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$

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Abstract: In this paper, we study the exact values of the generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$ for X being $l_\infty - l_1$ and $l_q - l_1$ spaces. Moreover, we shown that some new conditions for uniformly normal structure of a Banach space X.

Keywords: generalized von Neumann-Jordan constant; $l_\infty - l_1$ and $l_q - l_1$ space; uniformly normal structure

2000 Mathematics subject classification : 46B20.

1. Introduction

In order to study the geometric structure of a Banach space, many geometric constant have been investigated. In particular, the von Neuman-Jordan constant $C_{NJ}(X)$ is widely treated. In[1], as a generalization of the von Neuman-Jordan constant, a new geometric constant called the generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$ was introduced. It is proved that the $C_{NJ}^{(p)}(X)$ is strongly connected with geometric structure, such as uniformly non-square, uniformly normal structure. Hence it's necessary to compute the $C_{NJ}^{(p)}(X)$ for some concrete spaces.

Throughout this paper, let $X = (X, \|\cdot\|)$ be a real Banach spaces. We will use B_X, S_X and $ex(B_X)$ to denote unit ball, unit sphere of X and the set of extreme points of B_X, respectively.

Recall that the von Neumann-Jordan constant $C_{NJ}(X)$ of a Banach space X was introduced by Clarkson[3], as the smallest constant C for which,

$$\frac{1}{C} \le \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \le C,$$

holds for all $x, y \in X$.

An equivalent definition of the constant is

$$C_{NJ}(X) = \sup\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x \in S_X, y \in B_X\}.$$

The properties of $C_{NJ}(X)$ have been investigated in many papers(see for instances [2],[4],[8],[9],[10]). Recently, a generalized form of this constant was introduced as following

Definition 1.[1] The generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$ is defined by

$$C_{NJ}^{(p)}(X) := \sup\{\frac{\|x+y\|^p + \|x-y\|^p}{2^{p-1}(\|x\|^p + \|y\|^p)} : x, y \in X, (x, y) \ne (0, 0)\},$$

where $1 \le p < \infty$.
It's equivalent to

$$C_{NJ}^{(p)}(X) = \sup\{\frac{\|x+ty\|^p + \|x-ty\|^p}{2^{p-1}(1 + t^p)} : x, y \in S_X, 0 \le t \le 1\},$$

1

2

where $1 \le p < \infty$.

Now let us collect some properties of this constant (see [1]):
(i) $1 \le C_{NJ}^{(p)}(X) \le 2$;
(ii) X is uniformly non-square if and only if $C_{NJ}^{(p)}(X) < 2$;
(iii) Let $r \in (1, 2]$ and $\frac{1}{r} + \frac{1}{r'} = 1$. Then for $X = L_r[0, 1]$,
(1) if $1 < p \le r$ then $C_{NJ}^{(p)}(X) = 2^{2-p}$ and if $r < p \le r'$ then $C_{NJ}^{(p)}(X) = 2^{\frac{p}{r}-p+1}$,
(2) if $r' < p < \infty$ then $C_{NJ}^{(p)}(X) = 1$.

In this paper, we study the exact values of the generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$ for X being $l_\infty - l_1$ and $l_q - l_1$ space. Moreover, we shown that some new conditions for uniformly normal structure of a Banach space X.

2. Main Results

Firstly, we consider $l_\infty - l_1$ space. As $C_{NJ}^{(1)}(X) = 2$ for any Banach space X, we only consider the case $p > 1$.

Theorem 2.1. ($l_\infty - l_1$ spaces). Let $p > 1$ and $X = l_\infty - l_1$ which is $\mathbb{R}^2$ endowed with the norm

$$\|x\| = \begin{cases} \|x\|_\infty, & \text{if } x_1 x_2 \ge 0, \\ \|x\|_1, & \text{if } x_1 x_2 \le 0. \end{cases}$$

Then

$$C_{NJ}^{(p)}(l_\infty - l_1) = \frac{(1 + t_0)^p + 1}{2^{p-1}(1 + t_0^p)} = \frac{1}{2^{p-1}(1 - t_0^{p-1})}, \tag{2.1}$$

where $t_0 \in (0, 1)$ is the unique solution of the equation

$$(1 + t)^{p-1} - t^{p-1} - t^{p-1}(1 + t)^{p-1} = 0. \tag{2.2}$$

Proof. Firstly we shall show that $\|x + ty\|^p + \|x - ty\|^p \le 1 + (1 + t)^p$ for any $x, y \in S_X$ and every $t \in [0, 1]$.
By Minkowski inequality, for any $\alpha, \beta \in [0, 1]$ and any $x_1, x_2, y_1, y_2 \in B_X$ with $x = \alpha X_1 + (1 - \alpha)x_2, y = \beta y_2 + (1 - \beta)y_2$, we have

$$\begin{aligned} &\|x + ty\|^p + \|x - ty\|^p \\ &= \|\alpha(x_1 + ty) + (1 - \alpha)(x_2 + ty)\|^p + \|\alpha(x_1 - ty) + (1 - \alpha)(x_2 - ty)\|^p \\ &\le \alpha\|x_1 + ty\|^p + (1 - \alpha)\|x_2 + ty\|^p + \alpha\|x_1 - ty\|^p + (1 - \alpha)\|x_2 - ty\|^p \\ &= \alpha[\|\beta(x_1 + ty_1) + (1 - \beta)(x_1 + ty_2)\|^p + \|\beta(x_1 - ty_1) + (1 - \beta)(x_1 - ty_2)\|^p] \\ &\quad + (1 - \alpha)[\|\beta(x_2 + ty_1) + (1 - \beta)(x_2 + ty_2)\|^p + \|\beta(x_2 - ty_1) + (1 - \beta)(x_2 - ty_2)\|^p] \\ &\le \alpha\beta[\|x_1 + ty_1\|^p + \|x_1 - ty_1\|^p] + \alpha(1 - \beta)[\|x_1 + ty_2\|^p + \|x_1 - ty_2\|^p] \\ &\quad + (1 - \alpha)\beta[\|x_2 + ty_1\|^p + \|x_2 - ty_1\|^p] + (1 - \alpha)(1 - \beta)[\|x_2 + ty_2\|^p + \|x_2 - ty_2\|^p] \end{aligned}$$

Hence, we only need to prove $\|x + ty\|^p + \|x - ty\|^p \le 1 + (1 + t)^p$ for any $x, y \in ex(B_X)$ and every $t \in [0, 1]$.
Since $ex(B_X) = \{(1, 0), (0, 1), (1, 1), (-1, 0), (-1, -1), (0, -1)\}$ and we can change x into $-x$ or y into $-y$. So we may assume that $x, y = (0, 1), (1, 0)$ or $(1, 1)$. Obviously, for these x, y we easily have $\|x + ty\|^p + \|x - ty\|^p \le 1 + (1 + t)^p$ for every $t \in [0, 1]$. Therefore,

$$C_{NJ}^{(p)}(l_\infty - l_1) \le \sup_{t \in [0,1]} \{ \frac{(1+t)^p + 1}{2^{p-1}(1+t^p)} \}.$$

Let $f(t) = \frac{(1+t)^p + 1}{1+t^p}$, then

$$f'(t) = \frac{p(1+t)^{p-1}}{(1+t^p)^2}[1 - t^{p-1} - (\frac{t}{1+t})^{p-1}].$$

Defining $h(t) = 1 - t^{p-1} - (\frac{t}{1+t})^{p-1}$, we have $h(t)$ is decreasing from 1 to $-\frac{1}{2^{p-1}}$ on $[0,1]$. Whence there exists an unique $t_0 \in (0,1)$ such that $h(t_0) = 0$. Therefore,

$$C_{NJ}^{(p)}(l_\infty - l_1) \le \frac{(1+t_0)^p + 1}{2^{p-1}(1+t_0^p)}.$$

On the other hand, by taking $x_0 = (1,0), y_0 = (t_0, t_0)$, we have

$$C_{NJ}^{(p)}(l_\infty - l_1) \ge \frac{(1+t_0)^p + 1}{2^{p-1}(1+t_0^p)}.$$

Hence,

$$C_{NJ}^{(p)}(l_\infty - l_1) = \frac{(1+t_0)^p + 1}{2^{p-1}(1+t_0^p)},$$

where $t_0 \in (0,1)$ is the unique solution of $1 - t^{p-1} = (\frac{t}{1+t})^{p-1}$.
From (2.2), we also have

$$(1+t_0)^p + 1 = (1+t_0)\frac{t_0^{p-1}}{1-t_0^{p-1}} + 1 = \frac{1+t_0^p}{1-t_0^{p-1}}.$$

Therefore (2.1) is obtained.

Corollary 2.2. For $X = l_\infty - l_1$, we have

$$C_{NJ}^{(\frac{3}{2})}(X) = \frac{1}{\sqrt{2} - \sqrt{2\sqrt{2} + 1 - \sqrt{5 + 4\sqrt{2}}}} \approx 1.5077. \tag{2.3}$$

and

$$C_{NJ}^{(3)}(X) = \frac{3 + 2\sqrt{2} + \sqrt{5 + 4\sqrt{2}}}{8} \approx 1.1366. \tag{2.4}$$

Proof. (1) For $p = \frac{3}{2}$, (2.2) is equivalent to $t^4 + 1 - 2t^3 - 2t - 5t^2 = 0$.
that is

$$t^2 + \frac{1}{t^2} - 2(t + \frac{1}{t}) = 5.$$

Hence, we can get $t = \frac{2\sqrt{2} + 1 - \sqrt{5 + 4\sqrt{2}}}{2}$ and (2.3) is valid by (2.1).
(2) For $p = 3$, (2.2) is equivalent to $t^2 = (1+t)^2(1-t^2)$. Letting $t = u - 1$, we have

$$u^4 + 1 - 2u^3 - 2u + u^2 = 0.$$

that is

$$u^2 + \frac{1}{u^2} - 2(u + \frac{1}{u}) = -1.$$

Hence, $u = \frac{\sqrt{2} + 1 + \sqrt{2\sqrt{2} - 1}}{2}$ and $t = \frac{\sqrt{2} - 1 + \sqrt{2\sqrt{2} - 1}}{2}$. Therefore

$$C_{NJ}^{(3)}(X) = \frac{1}{4(1-t^2)} = \frac{1}{2 - 2(\sqrt{2} - 1)\sqrt{2\sqrt{2} - 1}} = \frac{3 + 2\sqrt{2} + \sqrt{5 + 4\sqrt{2}}}{8} \approx 1.1366.$$

4

Theorem 2.3. $(l_q - l_1$ spaces$)$. If $p \geq q > 1$. Let $X = \mathbb{R}^2$ endowed with the norm

$$\|x\| = \begin{cases} \|x\|_q, if \ x_1 x_2 \geq 0 \\ \|x\|_1, if \ x_1 x_2 \leq 0 \end{cases},$$

then

$$C_{NJ}^{(p)}(l_q - l_1) = 1 + 2^{\frac{p}{q}-p}.$$

In order to prove this theorem, firstly we give the following lemma.

Lemma2.4. Let $a, b, c, d \geq 0$ and $p \geq q > 1$ such that $a^q + b^q = 1$ and $c^q + d^q = 1$. If $0 \leq t \leq 1$, $a \geq ct$ and $b \leq dt$, then

$$[(a + ct)^q + (b + dt)^q]^{\frac{p}{q}} + (a - ct + dt - b)^p \leq (1 + t)^p + (1 + t^q)^{\frac{p}{q}}.$$

Proof. Clearly, $0 \leq a - ct + dt - b \leq 1 + t$. So we will consider the following two cases.

Case I. if $0 \leq a - ct + dt - b \leq (1 + t^q)^{\frac{1}{q}}$, then

$$[(a + ct)^q + (b + dt)^q]^{\frac{p}{q}} + (a - ct + dt - b)^p$$
$$\leq [(a^q + b^q)^{\frac{1}{q}} + t(c^q + d^q)^{\frac{1}{q}}]^p + (1 + t^q)^{\frac{p}{q}}$$
$$= (1 + t)^p + (1 + t^q)^{\frac{p}{q}}.$$

Case II. if $(1 + t^q)^{\frac{1}{q}} \leq a - ct + dt - b \leq 1 + t$, then

$$[(a + ct)^q + (b + dt)^q]^{\frac{1}{q}} + (a - ct + dt - b)$$
$$\leq (a^q + d^q t^q)^{\frac{1}{q}} + (c^q t^q + b^q)^{\frac{1}{q}} + a - ct + dt - b$$
$$\leq (1 + t^q)^{\frac{1}{q}} + ct + b + a - ct + dt - b$$
$$\leq (1 + t^q)^{\frac{1}{q}} + 1 + t.$$

So,

$$[(a + ct)^q + (b + dt)^q]^{\frac{1}{q}} \leq (1 + t^q)^{\frac{1}{q}} + 1 + t - (a - ct + dt - b).$$

Thus,

$$[(a + ct)^q + (b + dt)^q]^{\frac{p}{q}} + (a - ct + dt - b)^p$$
$$\leq [(1 + t^q)^{\frac{1}{q}} + 1 + t - (a - ct + dt - b)]^p + (a - ct + dt - b)^p$$
$$\leq \max_{u \in [(1+t^q)^{\frac{1}{q}}, 1+t]} [(1 + t^q)^{\frac{1}{q}} + 1 + t - u]^p + u^p$$
$$= (1 + t)^p + (1 + t^q)^{\frac{p}{q}}.$$

Proof of Theorem 2.3

Note that $ex(B_X) = \{(x_1, x_2) : x_1^q + x_2^q = 1, x_1 x_2 \geq 0\}$.

Now we prove that

$$\|x + ty\|^p + \|x - ty\|^p \leq (1 + t)^p + (1 + t^q)^{\frac{p}{q}},$$

holds for any $x, y \in ex(B_X)$ and any $t \in [0, 1]$.

Case I. If $(a - ct)(b - dt) \geq 0$. By Minkowski inequality, we have

$$
\begin{aligned}
&\|x + ty\|^p + \|x - ty\|^p \\
&= \|x + ty\|_q^p + \|x - ty\|_q^p \\
&= [(a + ct)^q + (b + dt)^q]^{\frac{p}{q}} + [|a - ct|^q + |b - dt|^q]^{\frac{p}{q}} \\
&\leq [(a^q + b^q)^{\frac{1}{q}} + (c^q t^q + d^q t^q)^{\frac{1}{q}}]^p + 1 \\
&\leq (1 + t)^p + 1 \\
&\leq (1 + t)^p + (1 + t^q)^{\frac{p}{q}}.
\end{aligned}
$$

Case II. If $(a - ct)(b - dt) \leq 0$. By Lemma2.4, we have that

$$
\begin{aligned}
&\|x + ty\|^p + \|x - ty\|^p \\
&= \|x + ty\|_q^p + \|x - ty\|_1^p \\
&= [(a + ct)^q + (b + dt)^q]^{\frac{p}{q}} + (a - ct + dt - b)^p \\
&\leq (1 + t)^p + (1 + t^q)^{\frac{p}{q}}.
\end{aligned}
$$

Therefore, $\|x + ty\|^p + \|x - ty\|^p \leq (1 + t)^p + (1 + t^q)^{\frac{p}{q}}$ is also valid for any $x, y \in S_X$. Hence ,

$$
C_{NJ}^{(p)}(l_q - l_1) \leq \frac{(1 + t)^p + (1 + t^q)^{\frac{p}{q}}}{2^{p-1}(1 + t^p)}.
$$

On the other hand, for every $t \in [0, 1]$, taking $x_0 = (1, 0), y_0 = (0, 1)$, we have

$$
\begin{aligned}
&C_{NJ}^{(p)}(l_q - l_1) \\
&\geq \frac{\|x_0 + ty_0\|^p + \|x_0 - ty_0\|^p}{2^{p-1}(1 + t^p)} \\
&= \frac{(1 + t)^p + (1 + t^q)^{\frac{p}{q}}}{2^{p-1}(1 + t^p)}.
\end{aligned}
$$

Hence,

$$
C_{NJ}^{(p)}(l_q - l_1) = \max_{t \in [0,1]} \frac{(1 + t)^p + (1 + t^q)^{\frac{p}{q}}}{2^{p-1}(1 + t^p)}.
$$

We let $f(t) = \frac{(1+t)^p + (1+t^q)^{\frac{p}{q}}}{1 + t^p}$, so

$$
f'(t) = \frac{p\{(1 + t^q)^{\frac{p}{q}-1}(t^{q-1} - t^{p-1}) + (1 + t)^{p-1}(1 - t^{p-1})\}}{(1 + t^p)^2} \geq 0.
$$

That imply $f(t)$ is not decreasing. Hence,

$$
\begin{aligned}
&C_{NJ}^{(p)}(l_q - l_1) \\
&= 2^{1-p} \max_{t \in [0,1]} f(t) \\
&= 2^{1-p} f(1) = 1 + 2^{\frac{p}{q}-p}.
\end{aligned}
$$

Lemma2.6. Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$
C_{NJ}^{(p)}(X) = 2^{1-\frac{p}{q}} C_{NJ}^{(q)}(X^*)^{\frac{p}{q}}
$$

and

$$
C_{NJ}^{(p)}(X) = C_{NJ}^{(p)}(X^{**}),
$$

where X^* is the dual of X.

Proof. Let $l_p(X) = \{(x_1, x_2) : \|(x_1, x_2)\| = (\|x_1\|^p + \|x_2\|^p)^{\frac{1}{p}}\}$ and define the operator $A : l_p(X) \to l_p(X)$ by $(x_1, x_2) \mapsto (x_1 + x_2, x_1 - x_2)$. Then we easily have $C_{NJ}^{(p)}(X) = \frac{\|A\|^p}{2^{p-1}}$. Similarly, $C_{NJ}^{(q)}(X^*) =$

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$\frac{\|A^*\|^q}{2^{q-1}}$. So $C_{NJ}^{(p)}(X) = 2^{1-\frac{p}{q}} C_{NJ}^{(q)}(X^*)^{\frac{p}{q}}$ by $\|A\| = \|A^*\|$, and hence $C_{NJ}^{(q)}(X^*) = 2^{1-\frac{q}{p}} C_{NJ}^{(p)}(X^{**})^{\frac{q}{p}}$. Therefore, we have $C_{NJ}^{(p)}(X) = C_{NJ}^{(p)}(X^{**})$.

The relationship between the constant $C_{NJ}^{(p)}(X)$ and the uniformly normal structure of X as follows:

Theorem 2.7. The Banach space X has uniformly normal structure if any one of the following conditions is valid

(i)$C_{NJ}^{(p)}(X) < \dfrac{\left(1+\sqrt{1+2^{\frac{2p-3}{p-1}}}\right)^{p-1}}{2^{2p-3}}$ for some $p \in \left(1, \frac{3-log_2 3}{2-log_2 3}\right)$;

(ii)$C_{NJ}^{(q)}(X^*) < \frac{1+(1+2^{3-q})^{\frac{1}{2}}}{2}$ for some $q > 1$,
where $p^{-1} + q^{-1} = 1$.

Proof. According to $C_{NJ}^{(p)}(X) < 2$, we have X is uniformly non-square, so we only need to prove X has weak normal structure.

Assume that X has no weak normal structure. Then it is well known (see[5])that for any $\varepsilon > 0$ there exists $z_1, z_2, z_3 \in S_X$ and $g_1, g_2, g_3 \in S_{X^*}$ satisfying the following statements:
(i) for all $i \neq j$, we have $|\|z_i - z_j\| - 1| < \varepsilon, |g_i(z_j)| < \varepsilon$,
(ii) $g_i(z_j) = 1$ for $i = 1, 2, 3$,
(iii) $\|z_3 - (z_2 + z_1)\| \geq \|z_2 + z_1\| - \varepsilon$.

Let us fix $\varepsilon > 0$ as small as needed. Then, we can find $z_1, z_2, z_3 \in S_X$ and $g_1, g_2, g_3 \in S_{X^*}$ satisfying the above properties.

(1)Taking $\alpha = \dfrac{\left(1+\sqrt{1+2^{\frac{2p-3}{p-1}}}\right)^{p-1}}{2^{2p-3}}$. We will consider the following two cases:

Case I. If $\|z_2 + z_1\| \leq \alpha$. Then,

$$\frac{\|g_1+g_2\|^q + \|g_2 - g_1\|^q}{2^{q-1}(\|g_2\|^q + \|g_1\|^q)}$$
$$\geq \frac{[(g_1+g_2)(\frac{z_2+z_1}{\alpha})]^q + [(g_2-g_1)(\frac{z_2-z_1}{\|z_2-z_1\|})]^q}{2^q}$$
$$\geq \frac{(\frac{2-2\varepsilon}{\alpha})^q + (\frac{2-2\varepsilon}{1+\varepsilon})^q}{2^q}$$
$$= \left(\frac{1-\varepsilon}{\alpha}\right)^q + \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^q.$$

Case II. If $\|z_2 + z_1\| > \alpha$. Then, the contains two sub-cases:

(i) If $\|z_3 - z_2 + z_1\| \leq \alpha$. Then,

$$\frac{\|g_1+g_3\|^q + \|g_3 - g_1\|^q}{2^{q-1}(\|g_3\|^q + \|g_1\|^q)}$$
$$\geq \frac{[(g_1+g_3)(\frac{z_3-z_2+z_1}{\alpha})]^q + [(g_3-g_1)(\frac{z_3-z_1}{\|z_3-z_1\|})]^q}{2^q}$$
$$\geq \frac{(\frac{2-4\varepsilon}{\alpha})^q + (\frac{2-2\varepsilon}{1+\varepsilon})^q}{2^q}$$
$$= \left(\frac{1-2\varepsilon}{\alpha}\right)^q + \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^q.$$

(ii) If $\|z_3 - z_2 + z_1\| > \alpha$. Then,

$$\frac{\|z_3-z_2+z_1\|^p + \|z_3-z_2-z_1\|^p}{2^{p-1}(\|z_3-z_2\|^p + \|z_1\|^p)}$$
$$\geq \frac{\alpha^p + (\|z_2+z_1\| - \varepsilon)^p}{2^{p-1}[(1+\varepsilon)^p + 1]}$$
$$\geq \frac{\alpha^p + (\alpha-\varepsilon)^p}{2^{p-1}[(1+\varepsilon)^p + 1]}.$$

Letting $\varepsilon \to 0$, and by lemma2.6 we have

$$C_{NJ}^{(p)}(X) \geq \min\{2^{1-\frac{p}{q}}(\frac{1}{\alpha^q}+1)^{\frac{p}{q}}, \frac{\alpha^p}{2^{p-1}}\} = \frac{\left(1+\sqrt{1+2^{\frac{2p-3}{p-1}}}\right)^{p-1}}{2^{2p-3}},$$

which contradicts to the hypothesis(i).

(2)Taking $\alpha = \frac{1+(1+2^{3-q})^{\frac{1}{2}}}{2}$. By the proof of (1), we have

$$C_{NJ}^{(q)}(X^*) \geq \min\{\frac{1}{\alpha^q}+1, 2^{1-\frac{q}{p}}(\frac{\alpha^p}{2^{p-1}})^{\frac{q}{p}}\} = \min\{\frac{1}{\alpha^q}+1, \frac{\alpha^q}{2^{q-1}}\} = \frac{1+(1+2^{3-q})^{\frac{1}{2}}}{2},$$

which contradicts to the hypothesis(ii).

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Discrete dynamical systems in soft topological spaces *

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Abstract   In this paper the iteration of soft continuous functions is investigated and their discrete dynamical systems in soft topological spaces are defined. Some basic concepts related to discrete dynamical systems (such as soft ω-limit set, soft invariant set, soft periodic point, soft nonwandering point, and soft recurrent point) are introduced into soft topological spaces. Soft topological mixing and soft topological transitivity are also studied. At last, soft topological entropy is defined and several properties of it are discussed.

Keywords   Soft point, Soft ω-limit set, Soft nonwandering point, Soft topological mixing, Soft topological transitivity, Soft topological entropy

1   Introduction and preliminaries

The real world is too complex for our immediate and direct understanding, so we create *models* which are simplifications of the real word. In 1999, Molodtsov [1] introduced the concept of soft set which gives a new approach to modeling uncertainties. And he also discussed the application of soft set theory in many fields, such as: operations analysis, game theory, the smoothness of function, and so on[2]. Maji et al.[3] and Ali et al.[4] defined some operators of soft sets. Beyond these theoretical works of soft set, research works on its applications in various fields are progressing rapidly, and great progress has been achieved, including soft set theory in abstract algebras[5−10], decision making, data analysis, information system, and so on[11−14]. The application of soft set theory in algebraic structures was introduced by Aktaş and Çağman[5], they defined the notion of soft groups and progressed some basic properties. Jun[6,7] investigated soft BCK/BCI-algebras and its application in ideal theory. Dudek et al.[8] discussed soft ideals in BCC-algebras. Zhang[9] studied intuitionistic fuzzy soft rings. Feng et al.[10] worked on soft semirings, soft ideals and idealistic soft semirings. Maji et al.[11] first applied soft sets to solve the decision making problem that is based on the concept of knowledge reduction in the theory of rough sets[12]. Based on the analysis of the rough set model on a tolerance relation and the fuzzy rough set, two types of fuzzy rough sets models on tolerance relations are constructed and researched by Xu et al.[13]. Chen et al.[14] presented a

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new definition of soft set parametrization reduction so as to improve the soft set based decision making in [11]. Yang[15] combined the multi-fuzzy set and soft set, from which they obtained a new soft set model named multi-fuzzy soft set, and applied it to decision making. Soft set theory is also be used in topology. Shabir and Naz's work[16] on soft topological spaces defined over an initial universe with a fixed set of parameters. The notions of soft open set, soft closed set, soft closure, soft interior point, soft neighborhood of a point, and soft separation axioms (such as soft T_i-space for $i = 1, 2, 3, 4$, soft normal space, and soft regular space) were also introduced and their basic properties were investigated. Min[17] pointed out some mistakes of [16] and investigated some properties of the soft separation axioms defined in [15]. Zorlutuna etc.[18] introduced some new concepts in soft topological spaces (such as soft point, interior point, interior, neighborhood, continuity, and compactness).

Motivated by Chen etc.[19] and Liu[20], this paper will investigate iteration of soft continuous functions and their discrete dynamical systems in soft topological spaces. Some basic concepts on dynamical systems (such as soft ω-limit set, soft invariant set, soft periodic point, soft nonwandering point, and soft recurrent point) are introduced in soft topological spaces, soft topological mixing, soft topological transitivity, soft topological entropy and its several properties are studied. As a result, some conclusions of discrete dynamical systems in ordinary topological spaces are generalized. Now we give some definitions and results to be used in this paper.

Definition 1[1]   A soft set on a set X is a triple (M, E, X), where $M : E \longrightarrow 2^X$ (the set of all subsets of X) is a mapping. The set of all soft sets on X is denoted by $\mathbb{S}(X, E)$.

Roughly speaking, a soft set on a set X is just a family $\{M_e\}_{e \in E}$ of subsets of X; it can be looked to be a subset of X if E is a singleton.

Let $(M, E, X), (N, E, X) \in \mathbb{S}(X, E)$. If $M(e) \subseteq N(e)$ ($\forall e \in E$), then (M, E, X) is called a soft subset of (N, E, X), denoted by $(M, E, X) \widetilde{\subseteq} (N, E, X)$. If $(M, E, X) \widetilde{\subseteq} (N, E, X)$ and $(M, E, X) \widetilde{\supseteq} (N, E, X)$, then (M, E, X) and (N, E, X) are said to be soft equal, denoted by $(M, E, X) = (N, E, X)$.

Remark 1[16] (1) Let X be a set, and $A \in 2^X$. Define $\widetilde{A} : E \longrightarrow 2^X$ as $\widetilde{A}(e) = A$ ($\forall e \in E$), then $(\widetilde{A}, E, X) \in \mathbb{S}(X, E)$; we use $\widetilde{A}$ to denote this soft set (particularly, we use $\widetilde{x}$ to denote the soft set $\widetilde{\{x\}}$).

(2) Let X be a set, and $(M, E, X) \in \mathbb{S}(X, E)$. Then $(M', E, X) \in \mathbb{S}(X, E)$, where $M' : E \longrightarrow 2^X$ is defined as

$$M'(e) = X - M(e) \ (\forall e \in E).$$

Sometimes we use $(M, E, X)'$ to replace (M', E, X).

(3) Let X be a set, $\{(H_j, E, X)\}_{j \in J} \subseteq \mathbb{S}(X, E)$. Then $(M, E, X), (N, E, X) \in \mathbb{S}(X, E)$, called the union (denoted as $\widetilde{\bigcup}_{j \in J}(H_j, E, X)$) and intersection (denoted as $\widetilde{\bigcap}_{j \in J}(H_j, E, X)$)

of the family $\{(H_j, E, X)\}_{j \in J}$ respectively, where

$$M(e) = \bigcup_{j \in J} H_j(e) \ (\forall e \in E)$$

and

$$N(e) = \bigcap_{j \in J} H_j(e) \ (\forall e \in E).$$

(4) Let X be a set, $(H, E, X) \in \mathbb{S}(X, E)$, and $x \in X$. Write $x \in (H, E, X)$ if $x \in H(e) \ (\forall e \in E)$, and $x \not\in (H, E, X)$ if $x \not\in H(e)$ for some $e \in E$.

(5) Let X be a set. The difference of the two soft sets (M, E, X) and (N, E, X) is a soft set (H, E, X) over X (usually, denoted by $(M, E, X) - (N, E, X)$) which is defined by $H(e) = M(e) - N(e) \ (\forall e \in E)$.

(6) Let X be a set, and $(M, E, X), (N, E, X) \in \mathbb{S}(X, E)$. Then

(i) $((M, E, X)\widetilde{\cup}(N, E, X))' = (M, E, X)'\widetilde{\cap}(N, E, X)'$;

(ii) $((M, E, X)\widetilde{\cap}(N, E, X))' = (M, E, X)'\widetilde{\cup}(N, E, X)'$.

Definition 2[18] (1) A soft set $(M, E, X) \in \mathbb{S}(X, E)$ is called elementary (or a soft point in $\widetilde{X}$, denoted by e_M) if $M(e) \neq \emptyset$ for some $e \in E$ and $M(e') = \emptyset$ for all $e' \in E - \{e\}$.

(2) Let e_M be a soft point in $\widetilde{X}$, and (N, E, X) is a soft set. If $M(e) \subseteq N(e)$, then e_M is said to be in (N, E, X), denoted by $e_M \widetilde{\in} (N, E, X)$.

Definition 3[17] Let X and Y be two sets, E and F be two nonempty parameter sets, and $f : E \longrightarrow F$ and $g : X \longrightarrow Y$ are mappings. For each $(M, E, X) \in \mathbb{S}(X, E)$, define

$$(f, g)(M, E, X) = (g^{\rightarrow}(M), f(E), Y),$$

where

$$g^{\rightarrow}(M)(\alpha) = \bigcup_{f(e) = \alpha} g(M(e)) \ (\forall \alpha \in F).$$

Then we obtain a mapping

$$(f, g) : \mathbb{S}(X, E) \longrightarrow \mathbb{S}(Y, F).$$

For each $(N, F, Y) \in \mathbb{S}(Y, F)$, define

$$(f, g)^{-1}(N, F, Y) = (g^{-1} \circ N \circ f, f^{-1}(F), X),$$

where

$$(g^{-1} \circ N \circ f)(e) = g^{-1}(N(f(e))) \ (\forall e \in f^{-1}(F)).$$

Then we obtain another mapping

$$(f, g)^{-1} : \mathbb{S}(Y, F) \longrightarrow \mathbb{S}(X, E).$$

Definition 4[16] (1) Let X be a set, and $\mathscr{T} \subseteq \mathbb{S}(X, E)$ satisfies

(1) $\emptyset$ and $X \in \mathscr{T}$;

(ii) $\mathscr{T}$ is closed under arbitrary unions;

(ii) $\mathscr{T}$ is closed under finite intersections.

Then $\mathscr{T}$ is called a soft topology on X, and $(X, \mathscr{T}, E)$ is called a soft topological space. The members of $\mathscr{T}$ are called soft open sets, members of $\mathscr{T}' = \{(M', E, X) \mid (M, E, X) \in \mathscr{T}\}$ are called soft closed sets.

(2) Let $(X, \mathscr{T}, E)$ be a soft topological space, and Y be a non-empty subset of X. Then

$$\mathscr{T}_Y = \{(M_Y, E, X) \mid (M, E, X) \in \mathscr{T}\}$$

is a soft topology on Y, it is called the soft relative topology on Y, and $(Y, \mathscr{T}_Y, E)$ is called a soft subspace of $(X, \mathscr{T}, E)$, where

$$(M_Y, E, X) = \widetilde{Y} \widetilde{\cap} (M, E, X) \ (\forall (M, E, X) \in \mathscr{T}).$$

Example 1 (1) Let $X = \{x_1, x_2, x_3\}$ be a 3-element set, $E = \{e_1, e_2\}$ be a 2-element set, and

$$\mathscr{T} = \{(M_i, E, X) \mid i = 1, 2, \cdots, 6\} \cup \{\widetilde{\emptyset}, \widetilde{X}\},$$

where $(M_i, E, X) \ (i = 1, 2, \cdots, 6)$ are defined as follows:

$$M_1(e) = \begin{cases} \{x_2\}, & \text{if } e = e_1; \\ \{x_1\}, & \text{if } e = e_2. \end{cases}$$

$$M_2(e) = \begin{cases} \{x_1\}, & \text{if } e = e_1; \\ \{x_3\}, & \text{if } e = e_2. \end{cases}$$

$$M_3(e) = \begin{cases} \{x_3\}, & \text{if } e = e_1; \\ \{x_2\}, & \text{if } e = e_2. \end{cases}$$

$$M_4(e) = \begin{cases} \{x_2, x_3\}, & \text{if } e = e_1; \\ \{x_1, x_2\}, & \text{if } e = e_2. \end{cases}$$

$$M_5(e) = \begin{cases} \{x_1, x_2\}, & \text{if } e = e_1; \\ \{x_1, x_3\}, & \text{if } e = e_2. \end{cases}$$

$$M_6(e) = \begin{cases} \{x_1, x_3\}, & \text{if } e = e_1; \\ \{x_2, x_3\}, & \text{if } e = e_2. \end{cases}$$

Then $\mathscr{T}$ is a soft topology on X and hence $(X, \mathscr{T}, E)$ is a soft topological space.

(2) Let $X = R$ (the set of all real numbers), $E = \{e_1, e_2\}$ be a 2-element set,

$$\mathcal{J} = \{A \subseteq X \mid X - A \text{ is a finite subset of } X\} \cup \{\emptyset, X\}$$

(i.e. the finite complement topology on X), and

$$\mathscr{T} = \{(M, E, X) \mid M(e_1), M(e_2) \in \mathcal{J}\}.$$

Then $\mathscr{T}$ is a soft topology on X and hence $(X, \mathscr{T}, E)$ is a soft topological space.

(3) Let $X = R$, $E = \{e_1, e_2\}$ be a 2-element set, $\mathcal{J}$ be the ordinary topology on X (i.e. $\mathcal{J}$ is the topology on X generated by the basis $\mathcal{B} = \{(a, b) \mid a, b \in R, a < b\}$), and

$$\mathcal{T} = \{(M, E, X) \mid M(e_1), M(e_2) \in \mathcal{J}\}.$$

Then $\mathcal{T}$ is a soft topology on X and hence $(X, \mathcal{T}, E)$ is a soft topological space.

(4) Let $X = [0, 1]$, $E = \{e_1, e_2\}$ be a 2-element set, $\mathcal{J}$ be the ordinary topology on X (i.e. $\mathcal{J}$ is the topology on $[0, 1]$ generated by the basis

$$\mathcal{B} = \{(a, b) \mid a \in [0, 1), b \in (0, 1], a < b\}),$$

and

$$\mathcal{T} = \{(M, E, X) \mid M(e_1), M(e_2) \in \mathcal{J}\}.$$

Then $\mathcal{T}$ is a soft topology on X and hence $(X, \mathcal{T}, E)$ is a soft topological space.

Remark 2 (1)[16] Let $(X, \mathcal{T}, E)$ be a soft topological space, e_M is a soft point in $\widetilde{X}$, $(N, E, X) \in \mathbb{S}(X, E)$. If there exists a $(A, E, X) \in \mathcal{T}$ such that

$$e_M \widetilde{\in} (A, E, X) \widetilde{\subseteq} (N, E, X),$$

then (N, E, X) is called a neighborhood of e_M.

(2) It can be easily seen that $\widetilde{\emptyset}, \widetilde{X} \in \mathcal{T}'$, and $\mathcal{T}'$ is closed under the operations of arbitrary intersections and finite unions. It can be also seen that $(N, E, X) \in \mathcal{T}'$ if and only if

$$((A, E, X) - e_M) \widetilde{\cap} (N, E, X) \neq \widetilde{\emptyset}$$

for any $e_M \in \widetilde{X}$ and any neighborhood (A, E, X) of e_M.

(3)[16] Let $(X, \mathcal{T}, E)$ be a soft topological space, and $(M, E, X) \in \mathbb{S}(X, E)$. Then

$$\overline{(M, E, X)} = \widetilde{\bigcap} \{(N, E, X) \mid (M, E, X) \widetilde{\subseteq} (N, E, X),$$

$$(N, E, X) \in \mathcal{T}'_X\}$$

is called the closure of (M, E, X). Clearly, $(M, E, X) \in \mathbb{S}(X, E)$ is a soft closed set of $(X, \mathcal{T}, E)$ if and only if $\overline{(M, E, X)} = (M, E, X)$.

(4)[16] Let $(X, \mathcal{T}, E)$ be a soft topological space over X, then $\mathcal{T}^e = \{M(e) \mid (M, E, X) \in \mathcal{T}\}$ is a topology on X ($e \in E$).

(5) If E is a single point set, then a soft topological space $(X, \mathcal{T}, E)$ can be seen as a common topological space.

Definition 5 Let $(X, \mathcal{T}_X, E)$ and $(Y, \mathcal{T}_Y, E)$ be soft topological spaces. A soft function

$$(f, g) : \mathbb{S}(X, E) \longrightarrow \mathbb{S}(Y, E)$$

is said to be a soft continuous function from $(X, \mathcal{T}_X, E)$ to $(Y, \mathcal{T}_Y, E)$ if

$$(f, g)^{-1}(N, E, Y) \in \mathcal{T}_X \quad (\forall (N, E, Y) \in \mathcal{T}_Y).$$

Remark 3 Let $(X, \mathscr{T}_X, E)$ and $(Y, \mathscr{T}_Y, E)$ be soft topological spaces, and

$$(id_E, g) : \mathbb{S}(X, E) \longrightarrow \mathbb{S}(Y, E)$$

be a soft continuous function from $(X, \mathscr{T}_X, E)$ to $(Y, \mathscr{T}_Y, E)$. Then $g : X \longrightarrow Y$ is a continuous function from $(X, \mathscr{T}_X^e)$ to $(Y, \mathscr{T}_Y^e)$ ($\forall e \in E$).

Definition 6[18] (1) Let $(X, \mathscr{T}, E)$ be a soft topological space, $(P, E, X) \in \mathbb{S}(X, E)$, and $\mathscr{A} \subseteq \mathscr{T}$. If

$$\widetilde{\bigcup} \mathscr{A} = (P, E, X),$$

then $\mathscr{A}$ is called an soft open cover of (P, E, X).

(2) Let $(X, \mathscr{T}, E)$ be a soft topological space, and $(P, E, X) \in \mathbb{S}(X, E)$. (P, E, X) is said to be soft compact if every open soft cover of it has a finite subcover. If $\widetilde{X}$ is compact, then $(X, \mathscr{T}, E)$ is called a soft compact topological space.

Theorem 1[18] Let $(X, \mathscr{T}, E)$ be a soft compact topological space, then each soft closed subset (P, E, X) is a soft compact subset of $\widetilde{X}$.

Theorem 2 Let $(X, \mathscr{T}_X, E)$ and $(Y, \mathscr{T}_Y, E)$ be soft topological spaces, and

$$(id_E, g) : \mathbb{S}(X, E) \longrightarrow \mathbb{S}(Y, E)$$

is a soft function. Then the following conditions are equivalent:

(1) (id_E, g) is a soft continuous function from $(X, \mathscr{T}_X, E)$ to $(Y, \mathscr{T}_Y, E)$.

(2) $(id_E, g)^{-1}(N, E, Y) \in \mathscr{T}'_X \quad (\forall \, (N, E, Y) \in \mathscr{T}'_Y)$.

(3) $(id_E, g)\overline{(M, E, X)} \widetilde{\subseteq} \overline{(id_E, g)(M, E, X)} \quad (\forall (M, E, X) \in \mathbb{S}(X, E))$.

(4) $(id_E, g)^{-1}\overline{(P, E, Y)} \widetilde{\supseteq} \overline{(id_E, g)^{-1}(P, E, Y)} \quad (\forall (P, E, Y) \in \mathbb{S}(Y, E))$.

Proof Straightforward. $\square$

2 Discrete dynamical systems in soft topological spaces

Let X be a topological space, and $g : X \longrightarrow X$ a continuous mapping, then the family $\{g^n\}_{n \in N}$ defines a (discrete) semi-dynamical system in topological space X, where N stands for the set of all nonnegative integers. In addition, if g is a homeomorphism (i.e. g is a one-to-one correspondence and both g and its inverse mapping g^{-1} are continuous), then we can define g^{-n} by $g^{-n} = (g^{-1})^n$ ($\forall n \in N$), then $\{g^n\}_{n \in Z}$ defines a discrete dynamical system in topological space X, where Z stands for the set of all integers.

Let $(X, \mathscr{T}, E)$ be a soft topological space and

$$(id_E, g) : \mathbb{S}(X, E) \longrightarrow \mathbb{S}(X, E)$$

be a soft continuous function from $(X, \mathscr{T}, E)$ to $(X, \mathscr{T}, E)$. It can be seen from definition 3 that

$$(g^n)^{\rightarrow} = (g^{\rightarrow})^n,$$

so we can define the n-th iterate of (id_E, g) $(n \in N)$ as follows:

$$(id_E, g)^n = (id_E, g) \circ (id_E, g)^{n-1} = (id_E \circ id_E, g \circ g^{n-1})$$
$$= (id_E, g^n),$$

$(id_E, g)^0 = (id_E, g^0) = (id_E, id_X)$,

where id_E (resp. id_X) denotes the identity mapping of E (resp., X) onto itself. Then the family $\{(id_E, g)^n\}_{n \in N}$ defines a (discrete) semi-dynamical system in soft topological space $(X, \mathcal{T}, E)$, where N stands for the set of all nonnegative integers. If g is a one-to-one correspondence and both (id_E, g) and its inverse mapping $(id_E, g)^{-1}$ are continuous, it can be seen from definition 3 that

$$(g^{\leftarrow})^n = (g^n)^{\leftarrow} \quad (\forall n \in N - \{0\})$$

and

$$((g^n)^{\leftarrow})^m = (g^{\leftarrow})^{nm} \quad (\forall n \in N - \{0\}, \forall m \in N).$$

Let

$$(id_E, g)^{-n} = (id_E, g^{-n}) = (id_E, (g^n)^{-1}) \quad (\forall n \in N),$$

then $\{(id_E, g)^n\}_{n \in Z}$ defines a discrete dynamical system in soft topological space, and it is denoted by $(X, (id_E, g))$. If $(X, \mathcal{T}, E)$ is a soft compact topological space, then $(X, (id_E, g))$ is called a soft compact discrete topological dynamical system. It is easy to show that $(id_E, g)^n(e_M)$

$(\forall n \in Z)$ is a soft point when e_M is a soft point.

Example 2 Let us consider the soft topological space in Example 1(1). Define $g : X \longrightarrow X$ as follows:

$$g(x_1) = x_2, \quad g(x_2) = x_3, \quad g(x_3) = x_1.$$

We will verify that both (id_E, g) and its inverse mapping $(id_E, g)^{-1}$ are continuous. In fact,

$$(id_E, g)^{-1}(M_1, E, X) = (g^{-1} \circ M_1 \circ id_E, E, X),$$

where

$$
\begin{aligned}
g^{-1} \circ M_1 \circ id_E(e) &= g^{-1}((M_1)(e)) \\
&= \begin{cases} g^{-1}(\{x_2\}), & \text{if } e = e_1; \\ g^{-1}(\{x_1\}), & \text{if } e = e_2. \end{cases} \\
&= \begin{cases} \{x_1\}, & \text{if } e = e_1; \\ \{x_3\}, & \text{if } e = e_2. \end{cases} \\
&= M_2(e)
\end{aligned}
$$

Thus $(id_E, g)^{-1}(M_1, E, X) = (M_2, E, X) \in \mathcal{T}$.

$$(id_E, g)^{-1}(M_2, E, X) = (g^{-1} \circ M_2 \circ id_E, E, X),$$

where

$$
\begin{aligned}
g^{-1} \circ M_2 \circ id_E(e) &= g^{-1}((M_2)(e)) \\
&= \begin{cases} g^{-1}(\{x_1\}), & \text{if } e = e_1; \\ g^{-1}(\{x_3\}), & \text{if } e = e_2. \end{cases} \\
&= \begin{cases} \{x_3\}, & \text{if } e = e_1; \\ \{x_2\}, & \text{if } e = e_2. \end{cases} \\
&= M_3(e)
\end{aligned}
$$

Thus $(id_E, g)^{-1}(M_2, E, X) = (M_3, E, X) \in \mathscr{T}$.

$$
(id_E, g)^{-1}(M_3, E, X) = (g^{-1} \circ M_3 \circ id_E, E, X),
$$

where

$$
\begin{aligned}
g^{-1} \circ M_3 \circ id_E(e) &= g^{-1}((M_3)(e)) \\
&= \begin{cases} g^{-1}(\{x_3\}), & \text{if } e = e_1; \\ g^{-1}(\{x_2\}), & \text{if } e = e_2. \end{cases} \\
&= \begin{cases} \{x_2\}, & \text{if } e = e_1; \\ \{x_1\}, & \text{if } e = e_2. \end{cases} \\
&= M_1(e)
\end{aligned}
$$

Thus $(id_E, g)^{-1}(M_3, E, X) = (M_1, E, X) \in \mathscr{T}$.

$$
(id_E, g)^{-1}(M_4, E, X) = (g^{-1} \circ M_4 \circ id_E, E, X),
$$

where

$$
\begin{aligned}
g^{-1} \circ M_4 \circ id_E(e) &= g^{-1}((M_4)(e)) \\
&= \begin{cases} g^{-1}(\{x_2, x_3\}), & \text{if } e = e_1; \\ g^{-1}(\{x_1, x_2\}), & \text{if } e = e_2. \end{cases} \\
&= \begin{cases} \{x_1, x_2\}, & \text{if } e = e_1; \\ \{x_3, x_1\}, & \text{if } e = e_2. \end{cases} \\
&= M_5(e)
\end{aligned}
$$

Thus $(id_E, g)^{-1}(M_4, E, X) = (M_5, E, X) \in \mathscr{T}$.

$$
(id_E, g)^{-1}(M_5, E, X) = (g^{-1} \circ M_5 \circ id_E, E, X),
$$

where

$$
\begin{aligned}
g^{-1} \circ M_5 \circ id_E(e) &= g^{-1}((M_5)(e)) \\
&= \begin{cases} g^{-1}(\{x_1, x_2\}), & \text{if } e = e_1; \\ g^{-1}(\{x_3, x_1\}), & \text{if } e = e_2. \end{cases} \\
&= \begin{cases} \{x_3, x_1\}, & \text{if } e = e_1; \\ \{x_2, x_3\}, & \text{if } e = e_2. \end{cases} \\
&= M_6(e)
\end{aligned}
$$

Thus $(id_E, g)^{-1}(M_5, E, X) = (M_6, E, X) \in \mathscr{T}$.

$$
(id_E, g)^{-1}(M_6, E, X) = (g^{-1} \circ M_6 \circ id_E, E, X),
$$

where

$$
\begin{aligned}
g^{-1} \circ M_6 \circ id_E(e) &= g^{-1}((M_6)(e)) \\
&= \begin{cases} g^{-1}(\{x_1, x_3\}), & \text{if } e = e_1; \\ g^{-1}(\{x_2, x_3\}), & \text{if } e = e_2. \end{cases} \\
&= \begin{cases} \{x_3, x_2\}, & \text{if } e = e_1; \\ \{x_1, x_2\}, & \text{if } e = e_2. \end{cases} \\
&= M_4(e)
\end{aligned}
$$

Thus $(id_E, g)^{-1}(M_6, E, X) = (M_4, E, X) \in \mathscr{T}$. It is easy to see that

$$(id_E, g)^{-1}(\widetilde{\emptyset}) = \widetilde{\emptyset} \in \mathscr{T}$$

and

$$(id_E, g)^{-1}(\widetilde{X}) = \widetilde{X} \in \mathscr{T}.$$

Therefore, (id_E, g) is continuous.

From the above, it is easy to see that

$$(id_E, g)^{-1} = (id_E, g^{-1}),$$

since for any $(M, E, X) \in \mathscr{T}$,

$$(id_E, g^{-1})(M, E, X) = ((g^{-1})^{\rightarrow}(M), E, X),$$

where

$$(g^{-1})^{\rightarrow}(M)(e) = g^{-1}(M)(e) = g^{-1} \circ M \circ id_E(e).$$

Thus for any $(M, E, X) \in \mathscr{T}$,

$$
\begin{aligned}
& ((id_E, g)^{-1})^{-1}(M, E, X) \\
=~ & (id_E, g^{-1})^{-1}(M, E, X) \\
=~ & ((g^{-1})^{-1} \circ M \circ id_E, E, X) \\
=~ & (g \circ M \circ id_E, E, X)
\end{aligned}
$$

Hence

$$((id_E, g)^{-1})^{-1}(M_1, E, X) = (g \circ M_1 \circ id_E, E, X),$$

where

$$
\begin{aligned}
g \circ M_1 \circ id_E(e) ~=~ & g((M_1)(e)) \\
=~ & \begin{cases} g(\{x_2\}), & \text{if } e = e_1; \\ g(\{x_1\}), & \text{if } e = e_2. \end{cases} \\
=~ & \begin{cases} \{x_3\}, & \text{if } e = e_1; \\ \{x_2\}, & \text{if } e = e_2. \end{cases} \\
=~ & M_3(e)
\end{aligned}
$$

Thus $((id_E, g)^{-1})^{-1}(M_1, E, X) = (M_3, E, X) \in \mathscr{T}$.

$$((id_E, g)^{-1})^{-1}(M_2, E, X) = (g \circ M_2 \circ id_E, E, X),$$

where

$$
\begin{aligned}
g \circ M_2 \circ id_E(e) ~=~ & g((M_2)(e)) \\
=~ & \begin{cases} g(\{x_1\}), & \text{if } e = e_1; \\ g(\{x_3\}), & \text{if } e = e_2. \end{cases} \\
=~ & \begin{cases} \{x_2\}, & \text{if } e = e_1; \\ \{x_1\}, & \text{if } e = e_2. \end{cases} \\
=~ & M_1(e)
\end{aligned}
$$

Thus $((id_E, g)^{-1})^{-1}(M_2, E, X) = (M_1, E, X) \in \mathscr{T}$.

$$((id_E, g)^{-1})^{-1}(M_3, E, X) = (g \circ M_3 \circ id_E, E, X),$$

where

$$
\begin{aligned}
g \circ M_3 \circ id_E(e) &= g((M_3)(e)) \\
&= \begin{cases} g(\{x_3\}), & \text{if } e = e_1; \\ g(\{x_2\}), & \text{if } e = e_2. \end{cases} \\
&= \begin{cases} \{x_1\}, & \text{if } e = e_1; \\ \{x_3\}, & \text{if } e = e_2. \end{cases} \\
&= M_2(e)
\end{aligned}
$$

Thus $((id_E, g)^{-1})^{-1}(M_3, E, X) = (M_2, E, X) \in \mathscr{T}$.

$$((id_E, g)^{-1})^{-1}(M_4, E, X) = (g \circ M_4 \circ id_E, E, X),$$

where

$$
\begin{aligned}
g \circ M_4 \circ id_E(e) &= g((M_4)(e)) \\
&= \begin{cases} g(\{x_2, x_3\}), & \text{if } e = e_1; \\ g(\{x_1, x_2\}), & \text{if } e = e_2. \end{cases} \\
&= \begin{cases} \{x_1, x_3\}, & \text{if } e = e_1; \\ \{x_2, x_3\}, & \text{if } e = e_2. \end{cases} \\
&= M_6(e)
\end{aligned}
$$

Thus $((id_E, g)^{-1})^{-1}(M_4, E, X) = (M_6, E, X) \in \mathscr{T}$.

$$((id_E, g)^{-1})^{-1}(M_5, E, X) = (g \circ M_5 \circ id_E, E, X),$$

where

$$
\begin{aligned}
g \circ M_5 \circ id_E(e) &= g((M_5)(e)) \\
&= \begin{cases} g(\{x_1, x_2\}), & \text{if } e = e_1; \\ g(\{x_3, x_1\}), & \text{if } e = e_2. \end{cases} \\
&= \begin{cases} \{x_2, x_3\}, & \text{if } e = e_1; \\ \{x_1, x_2\}, & \text{if } e = e_2. \end{cases} \\
&= M_4(e)
\end{aligned}
$$

Thus $((id_E, g)^{-1})^{-1}(M_5, E, X) = (M_4, E, X) \in \mathscr{T}$.

$$((id_E, g)^{-1})^{-1}(M_6, E, X) = (g \circ M_6 \circ id_E, E, X),$$

where

$$
\begin{aligned}
g \circ M_6 \circ id_E(e) &= g((M_6)(e)) \\
&= \begin{cases} g(\{x_1, x_3\}), & \text{if } e = e_1; \\ g(\{x_2, x_3\}), & \text{if } e = e_2. \end{cases} \\
&= \begin{cases} \{x_2, x_1\}, & \text{if } e = e_1; \\ \{x_3, x_1\}, & \text{if } e = e_2. \end{cases} \\
&= M_5(e)
\end{aligned}
$$

Thus $((id_E, g)^{-1})^{-1}(M_6, E, X) = (M_5, E, X) \in \mathscr{T}$. It is easy to see that

$$((id_E, g)^{-1})^{-1}(\widetilde{\emptyset}) = \widetilde{\emptyset} \in \mathscr{T}$$

and

$$((id_E, g)^{-1})^{-1}(\widetilde{X}) = \widetilde{X} \in \mathscr{T}.$$

Therefore, $(id_E, g)^{-1}$ is continuous. Hence, $(X, (id_E, g))$ is a soft topological dynamical system.

Example 3 Let us consider the soft topological space in Example 1(2). Let $g : X \longrightarrow X$ be an arbitrary one-to-one respondence on X. Then for any $(M, E, X) \in \mathscr{T}$,

$$(id_E, g)^{-1}(M, E, X) = (g^{-1} \circ M \circ id_E, E, X),$$

where

$$g^{-1} \circ M \circ id_E(e) = g^{-1}(M(e)) \quad (\forall e \in E),$$

the complement $X - g^{-1} \circ M \circ id_E(e)$ is still a finite subset of X since g is an one-to-one respondence, thus $(id_E, g)^{-1}(M, E, X) \in \mathscr{T}$. Therefore, (id_E, g) is continuous.

On the other hand, for any $(M, E, X) \in \mathscr{T}$,

$$
\begin{aligned}
& ((id_E, g)^{-1})^{-1}(M, E, X) \\
= & (id_E, g^{-1})^{-1}(M, E, X) \\
= & ((g^{-1})^{-1} \circ M \circ id_E, E, X) \\
= & (g \circ M \circ id_E, E, X)
\end{aligned}
$$

where

$$g \circ M \circ id_E(e) = g(M(e)) \quad (\forall e \in E),$$

the complement $X - g \circ M \circ id_E(e)$ is still a finite subset of X since g is an one-to-one respondence, thus

$$(id_E, g)(M, E, X) \in \mathscr{T}.$$

Therefore, $(id_E, g)^{-1}$ is continuous. Hence, $(X, (id_E, g))$ is a soft topological dynamical system.

Example 4 Let us consider the soft topological space in Example 1(3). Define $g : X \longrightarrow X$ as follows:

$$g(x) = x + 1 \quad (\forall x \in X).$$

Then for every $(a, b) \in \mathcal{B}$, $g(a, b) = (a + 1, b + 1)$, and $g^{-1}(a, b) = (a - 1, b - 1)$, thus $g(\mathcal{B}) = g^{-1}(\mathcal{B}) = \mathcal{B}$. Denote the topology on X generated by $g(\mathcal{B})$ and $g^{-1}(\mathcal{B})$ by $g(\mathcal{J})$ and $g^{-1}(\mathcal{J})$. Then $g(\mathcal{J}) = g^{-1}(\mathcal{J}) = \mathcal{J}$.

For any $(M, E, X) \in \mathscr{T}$,

$$(id_E, g)^{-1}(M, E, X) = (g^{-1} \circ M \circ id_E, E, X),$$

where

$$g^{-1} \circ M \circ id_E(e) = g^{-1}(M(e)) \quad (\forall e \in E),$$

since $M(e) \in \mathcal{J}$, we have $g^{-1}(M(e)) \in g^{-1}(\mathcal{J}) = \mathcal{J}$, thus $(id_E, g)^{-1}(M, E, X) \in \mathscr{T}$. Therefore, (id_E, g) is continuous.

On the other hand, for any $(M, E, X) \in \mathscr{T}$,

$$
\begin{aligned}
& ((id_E, g)^{-1})^{-1}(M, E, X) \\
= & (id_E, g^{-1})^{-1}(M, E, X) \\
= & ((g^{-1})^{-1} \circ M \circ id_E, E, X) \\
= & (g \circ M \circ id_E, E, X)
\end{aligned}
$$

where

$$g \circ M \circ id_E(e) = g(M(e)) \quad (\forall e \in E),$$

since $M(e) \in \mathcal{J}$, we have $g(M(e)) \in g(\mathcal{J}) = \mathcal{J}$, thus $(id_E, g)(M, E, X) \in \mathscr{T}$. Therefore, $(id_E, g)^{-1}$ is continuous. Hence, $(X, (id_E, g))$ is a soft topological dynamical system.

Example 5 Let us consider the soft topological space in Example 1(4). Define $g : X \longrightarrow X$ as follows:

$$g(x) = \begin{cases} 2x, & x \in [0, \frac{1}{2}]; \\ 2 - 2x, & x \in (\frac{1}{2}, 1]. \end{cases}$$

For every $(a, b) \in \mathcal{B}$,

$$g^{-1}(a, b) = \begin{cases} (\frac{a}{2}, \frac{b}{2}), & b \leq \frac{1}{2}; \\ (\frac{2-a}{2}, \frac{2-b}{2}), & a \geq \frac{1}{2}; \\ (\frac{a}{2}, \frac{2-b}{2}), & a < \frac{1}{2} < b. \end{cases}$$

Thus $g^{-1}(\mathcal{B}) \subseteq \mathcal{B}$. Let $g^{-1}(\mathcal{J})$ be the topology on X generated by $g^{-1}(\mathcal{B})$, then $g^{-1}(\mathcal{J}) \subseteq \mathcal{J}$.

For any $(M, E, X) \in \mathscr{T}$,

$$(id_E, g)^{-1}(M, E, X) = (g^{-1} \circ M \circ id_E, E, X),$$

where

$$g^{-1} \circ M \circ id_E(e) = g^{-1}(M(e)) \quad (\forall e \in E),$$

since $M(e) \in \mathcal{J}$, we have $g^{-1}(M(e)) \in g^{-1}(\mathcal{J}) \subseteq \mathcal{J}$, thus $(id_E, g)^{-1}(M, E, X) \in \mathscr{T}$. Therefore, (id_E, g) is continuous. Hence, $(X, (id_E, g))$ is a semi-soft topological dynamical system.

Definition 7 Let $(X, (id_E, g))$ be a soft discrete topological dynamical system and $e_M \in \widetilde{X}$ is a soft point. Define several soft sets as follows:

$$Orb_{(id_E, g)}(e_M) = \{(id_E, g)^n(e_M) \mid n \in Z\},$$

$$Orb^+_{(id_E, g)}(e_M) = \{(id_E, g)^n(e_M) \mid n \in N - \{0\}\}.$$

$$Orb^-_{(id_E, g)}(e_M) = \{(id_E, g)^{-n}(e_M) \mid n \in N - \{0\}\}.$$

Then we call $Orb_{(id_E, g)}(e_M)$ (resp., $Orb^+_{(id_E, g)}(e_M)$, $Orb^-_{(id_E, g)}(e_M)$) the soft orbit (resp., soft positive semi-orbit, soft negative semi-orbit) of the soft dynamical system of (id_E, g).

Let $e_M \in \widetilde{X}$, if $(id_E, g)^n(e_M) = e_M$ for some $n \in N - \{0\}$, then e_M is called a soft periodic point of (id_E, g), the smallest one of such integers is referred to as the soft period of e_M. In particular, if $(id_E, g)(e_M) = e_M$, then e_M is called a soft fixed point of (id_E, g). Let $Per(id_E, g)$ (resp. $Fix(id_E, g)$) be the set of all soft periodic points (resp. all soft fixed points) of (id_E, g). Then $Fix(id_E, g) \subseteq Per(id_E, g)$.

Definition 8 Let $e_M \in \widetilde{X}$ be a soft point, then the soft set

$$\omega(e_M) = \widetilde{\bigcap}_{n \in N - \{0\}} \widetilde{\overline{\bigcup\{(id_E, g)^k(e_M) \mid k \geq n\}}},$$

is called a soft ω-limit set.

Obviously $\omega(e_M)$ is a soft closed set of $(X, \mathscr{T}, E)$. If the soft topological space $(X, \mathscr{T}, E)$ is soft compact, then $\omega(e_M) \neq \widetilde{\emptyset}$ by Theorem 7.4 in [20].

Definition 9 Let $(X, (id_E, g))$ be a soft discrete topological dynamical system, and $e_M \in \widetilde{X}$ a soft point.

(1) If for each soft open neighborhood (N, E, X) of e_M, there exists an $n \in N - \{0\}$ such that $(id_E, g)^n(e_M)$

$\widetilde{\in}(N, E, X)$, then e_M is called a soft recurrent points of (id_E, g). The set of all soft recurrent points of (id_E, g) is denoted by $Rec(id_E, g)$. Clearly, $Per(id_E, g) \subseteq Rec(id_E, g)$.

(2) If for each soft open neighborhood (N, E, X) of e_M, there exists an $n \in N - \{0\}$ such that

$$(id_E, g)^{-n}(N, E, X) \widetilde{\cap} (N, E, X) \neq \widetilde{\emptyset}.$$

Then e_M is called a soft nonwandering point of (id_E, g). The set of all soft nonwandering points of (id_E, g) is denoted by $\Omega(id_E, g)$, i.e.,

$\Omega(id_E, g) = \{e_M \in \widetilde{X} \mid e_M$ be a soft nonwandering point of $(id_E, g)\}$.

Each soft point of $\widetilde{X} - \Omega(id_E, g)$ is called a soft wandering point.

Definition 10 Let (id_E, g) be a soft continuous function from $(X, \mathscr{T}, E)$ to $(X, \mathscr{T}, E)$.

(1) (id_E, g) is called soft topological mixing if, for any pair (M, E, X) and $(N, E, X) \in \mathscr{T}$ of nonempty soft open sets of $(X, \mathscr{T}, E)$, there exists an $n \in N - \{0\}$ such that $(id_E, g)^n(M, E, X) \widetilde{\cap} (N, E, X) \neq \widetilde{\emptyset}$.

(2) (id_E, g) is called soft topological transitivity if there exists a soft point $e_M \in \widetilde{X}$ such that $Orb_{(id_E, g)}(e_M)$ is dense in $\widetilde{X}$ (i.e. $\overline{Orb_{(id_E, g)}(e_M)} = \widetilde{X}$).

(3) A soft set (N, E, X) is said to be soft invariant of (id_E, g) if $(id_E, g)(N, E, X) \widetilde{\subseteq} (N, E, X)$ (i.e. $g(N(e)) \subseteq N(e)$ for each $e \in E$).

Theorem 3 Let $(X, \mathscr{T}, E)$ be a soft topological space, and $(id_E, g) : \mathbb{S}(X, E) \longrightarrow \mathbb{S}(\mathbb{X}, \mathbb{E})$ be a soft continuous function from $(X, \mathscr{T}, E)$ to $(X, \mathscr{T}, E)$. Then

(1) $\Omega(id_E, g)$ is a soft closed set of $\widetilde{X}$, and $Rec(id_E, g)$ $\widetilde{\subseteq} \Omega(id_E, g)$.

(2) $Orb_{(id_E, g)}(e_M)$, $\omega(e_M)$, $Per(id_E, g)$, $Fix(id_E, g)$ and $\Omega(id_E, g)$ are invariant of (id_E, g).

(3) $\Omega((id_E, g)^m)$ is an invariant and closed soft set, and

$$\Omega((id_E, g)^m) \widetilde{\subseteq} \Omega(id_E, g) \quad (m \in N - \{0\}).$$

(4) Each soft point $e_M \in \widetilde{X}$ is a soft nonwandering point if one of the following conditions is satisfied:

(i) (id_E, g) is soft topological mixing, g is a one-to-one correspondence, and both (id_E, g) and its inverse mapping $(id_E, g)^{-1}$ are continuous

(ii) $Per(id_E, g) = X$.

Proof (1) Suppose that a soft point e_M is not a soft wandering point of (id_E, g), then there exists some soft open neighborhood (N, E, X) and some $n \in N - \{0\}$ such that

$$(id_E, g)^{-n}(N, E, X) \widetilde{\cap} (N, E, X) = \widetilde{\emptyset}.$$

So all the soft points in (N, E, X) are not soft wandering points of (id_E, g), it follows that $\Omega(id_E, g)$ be a soft closed set of $\widetilde{X}$.

Now let soft point $e_M \in Rec(id_E, g)$, then for each soft open neighborhood (N, E, X) of e_M, there exists some $n \in N - \{0\}$ such that $(id_E, g)^n(e_M) \widetilde{\subseteq} (N, E, X)$, so for any $e \in E$, $g^n(M(e)) \subseteq N(e)$, thus $M(e) \subseteq g^{-n}(N(e))$, it implies that

$$e_M \widetilde{\in} (id_E, g^{-n})(N, E, X) = (id_E, g)^{-n}(N, E, X),$$

then

$$e_M \widetilde{\in} (id_E, g)^{-n}(N, E, X) \widetilde{\cap} (N, E, X),$$

hence

$$Rec(id_E, g) \widetilde{\subseteq} \Omega(id_E, g).$$

(2) We only show that $\omega(e_M)$ and $\Omega(id_E, g)$ are invariant sets of (id_E, g). Firstly, we have

$$(id_E, g)(\omega(e_M))$$

$$= (id_E, g)(\widetilde{\bigcap}_{n \in N - \{0\}} \overline{\widetilde{\bigcup}\{(id_E, g)^k(e_M) \mid k \geq n\}})$$

$$\widetilde{\subseteq} \widetilde{\bigcap}_{n \in N - \{0\}} (id_E, g) \overline{\widetilde{\bigcup}\{(id_E, g)^k(e_M) \mid k \geq n\}}$$

$$\widetilde{\subseteq} \widetilde{\bigcap}_{n \in N - \{0\}} \overline{\widetilde{\bigcup}\{(id_E, g)^{k+1}(e_M) \mid k \geq n\}}$$

$$\widetilde{\subseteq} \widetilde{\bigcap}_{n \in N - \{0\}} \overline{\widetilde{\bigcup}\{(id_E, g)^k(e_M) \mid k \geq n\}} = \omega(e_M)$$

Now let soft point $e_M \widetilde{\in} \Omega(id_E, g)$ and (N, E, X) a soft open neighborhood of soft point $(id_E, g)(e_M)$, we can obtain that $(id_E, g)^{-1}(N, E, X)$ is a soft open neighborhood of soft point e_M since (id_E, g) is a soft continuous function, then there exists some $n \in N - \{0\}$ such that

$$(id_E, g)^{-1}((id_E, g)^{-n}(N, E, X)) \widetilde{\cap} (N, E, X))$$

$$= (id_E, g)^{-n}((id_E, g)^{-1}(N, E, X)) \widetilde{\cap} (id_E, g)^{-1}(N, E, X)$$

$$\neq \widetilde{\emptyset}$$

So

$$(id_E, g)^{-n}(N, E, X)) \widetilde{\cap} (N, E, X) \neq \widetilde{\emptyset}.$$

Therefore

$$(id_E, g)(e_M) \widetilde{\in} \Omega(id_E, g),$$

Hence

$$(id_E, g)(\Omega(id_E, g)) \widetilde{\subseteq} \Omega(id_E, g).$$

(3) Straightforward.

(4) Let (i) hold, $e_M \widetilde{\in} \widetilde{X}$ be a soft point and $(N, E, X) \in \mathscr{T}$ be a soft open neighborhood of e_M. Because (id_E, g) is soft topological mixing, there exists some $n \in N - \{0\}$ such that

$$(id_E, g)^n(M, E, X) \widetilde{\cap} (M, E, X) \neq \widetilde{\emptyset}.$$

Then

$$(id_E, g)^{-n}(M, E, X) \widetilde{\cap} (M, E, X) \neq \widetilde{\emptyset}$$

since g is a one-to-one correspondence and both (id_E, g) and its inverse mapping $(id_E, g)^{-1}$ are continuous. Thus $e_M \in \Omega(id_E, g)$.

Let (ii) hold. Then

$$\widetilde{X} = \overline{Per(id_E, g)} \widetilde{\subseteq} \overline{Rec(id_E, g)} \widetilde{\subseteq} \overline{\Omega(id_E, g)} = \Omega(id_E, g) \widetilde{\subseteq} \widetilde{X}.$$

Therefore $\Omega(id_E, g) = \widetilde{X}$. □

Remark 4 If g is a one-to-one correspondence, both

$$(id_E, g) : \mathbb{S}(X, E) \longrightarrow \mathbb{S}(X, E)$$

and its inverse mapping

$$(id_E, g)^{-1} : \mathbb{S}(X, E) \longrightarrow \mathbb{S}(X, E)$$

are continuous, and $(M, E, X) \in \mathbb{S}(X, E)$. Then

$$(id_E, g)^n(M, E, X) \widetilde{\cap} (M, E, X) \neq \widetilde{\emptyset}$$

if and only if

$$(id_E, g)^{-n}(M, E, X) \widetilde{\cap} (M, E, X) \neq \widetilde{\emptyset} \ (\forall n \in N - \{0\}).$$

So $\Omega(id_E, g) = \Omega(id_E, g)^{-1}$.

Definition 11 Let $(X, \mathscr{T}_X, E)$ and $(Y, \mathscr{T}_Y, E)$ be soft topological spaces,

$$(id_E, g) : \mathbb{S}(X, E) \longrightarrow \mathbb{S}(X, E)$$

be a soft continuous function from $(X, \mathscr{T}_X, E)$ to $(X, \mathscr{T}_X, E)$,

$$(id_E, f) : \mathbb{S}(Y, E) \longrightarrow \mathbb{S}(Y, E))$$

be a soft continuous function from $(Y, \mathscr{T}_Y, E)$ to $(Y, \mathscr{T}_Y, E))$. If there exists a soft continuous function $(id_E, h) : \mathbb{S}(X, E) \longrightarrow \mathbb{S}(Y, E)$ from $(X, \mathscr{T}_X, E)$ to $(Y, \mathscr{T}_Y, E)$ such that

$$(id_E, h) \circ (id_E, f) = (id_E, g) \circ (id_E, h)$$

(i.e. $(id_E, h \circ f) = (id_E, g \circ h)$), then (id_E, h) is said to be soft topology semi-conjugate from (id_E, g) to (id_E, f). If g is a one-to-one correspondence and both (id_E, g) and its inverse

mapping (id_E, g) are continuous, then (id_E, h) is said to soft topological conjugate from (id_E, g) to (id_E, f). Here, we denote $(id_E, g) \cong (id_E, f)$.
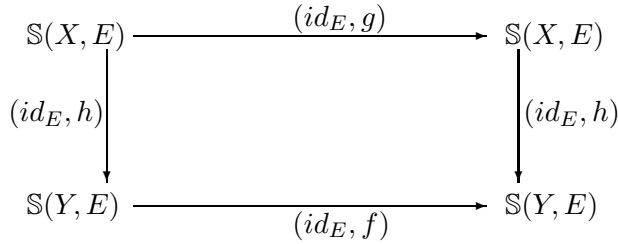

fig.1

Remark 5 (1) $\cong$ is an equivalence relation.

(2) If (id_E, h) is a soft topological conjugate mapping from (id_E, g) to (id_E, f), then for each soft point $e_M \in \widetilde{X}$ and $n \in N - \{0\}$, we have

$$(id_E, h)((id_E, f)^n(e_M)) = (id_E, g^n)((id_E, h)(e_M)),$$

it follows that

$$(id_E, h)(Orb_{(id_E, g)}(e_M)) = Orb_{(id_E, g)}((id_E, h)(e_M)),$$

and it is easy to show that

$$(id_E, h)(\omega(e_M)) = \omega((id_E, h)(e_M));$$

$$(id_E, h)(Per(id_E, g)) = Per(id_E, f);$$

$$(id_E, h)(Fix(id_E, g)) = Fix(id_E, f);$$

$$(id_E, h)(Rec(id_E, g)) = Rec(id_E, f);$$

$$(id_E, h)(\Omega(id_E, g)) = \Omega(id_E, f).$$

3 Soft topological entropy

In this section, the definition of soft topological entropy will be given and some fundamental properties of the soft topological entropy will be studied.

Definition 12 Let $(X, (id_E, g))$ be a soft compact discrete topological dynamical system, and α be a soft open cover of $\widetilde{X}$. Denote the smallest cardinality of all subcovers (for $\widetilde{X}$) of α by $N_{\widetilde{X}}(\alpha)$, i.e.,

$$N_{\widetilde{X}}(\alpha) = \min \left\{ |\beta| \mid \beta \subseteq \alpha \text{ and } \widetilde{X} = \widetilde{\bigcup}\beta \right\}.$$

Since $\widetilde{X}$ is compact soft set, $N_{\widetilde{X}}(\alpha)$ is a positive integer. Let $H_{\widetilde{X}}(\alpha) = \log N_{\widetilde{X}}(\alpha)$.

Let α and β be two soft open covers of $\widetilde{X}$. Define their join by

$$\alpha\widehat{\bigcup}\beta = \{(P, E, X)\widetilde{\cap}(Q, E, X) \mid (P, E, X) \in \alpha, (Q, E, X) \in \beta\}.$$

Clearly, the join $\alpha \cup \beta$ is a soft open cover of X. It is well known that β is called a refinement of α (denoted by $\alpha \prec \beta$) if for each $(Q, E, X) \in \beta$, there exists a $(P, E, X) \in \alpha$ such that $(Q, E, X) \widetilde{\subseteq} (P, E, X)$.

Theorem 4 Let $(X, (id_E, g))$ be a soft compact discrete topological dynamical system, α and β be two soft open covers of $\widetilde{X}$. Then the following hold.

(1) $H_{\widetilde{X}}(\alpha) \geq 0$.

(2) if $\beta \prec \alpha$, then $H_{\widetilde{X}}(\alpha) \leq H_{\widetilde{X}}(\beta)$.

(3) $H_{\widetilde{X}}(\alpha \widehat{\cup} \beta) \leq H_{\widetilde{X}}(\alpha) + H_{\widetilde{X}}(\beta)$.

(4) $H_{\widetilde{X}}((id_E, g)^{-1}(\alpha)) = H_{\widetilde{X}}(\alpha)$.

Proof we only prove (4). Let $N_{\widetilde{X}}(\alpha) = n$, then any subcover of α containing less than n elements of α would not cover $\widetilde{X}$. Let

$$\{(P_1, E, X), (P_2, E, X), \cdots, (P_n, E, X)\}$$

be a subcover (for $\widetilde{X}$) of α with a cardinality n, since (id_E, g) is continuous,

$$\{(id_E, g)^{-1}(P_1, E, X), (id_E, g)^{-1}(P_2, E, X),$$

$$\cdots, (id_E, g)^{-1}(P_n, E, X)\}$$

is a subcover (for $(id_E, g)^{-1}(\widetilde{X})$) of $(id_E, g)^{-1}(\alpha)$. By $(id_E, g)(\widetilde{X}) = \widetilde{X}$ we can know $\widetilde{X} = (id_E, g)^{-1}(\widetilde{X})$, so

$$\{(id_E, g)^{-1}(P_1, E, X), (id_E, g)^{-1}(P_2, E, X),$$

$$\cdots, (id_E, g)^{-1}(P_n, E, X)\}$$

is a finite open subcover (for $\widetilde{X}$) of $(id_E, g)^{-1}(\alpha)$. Therefore,

$$N_{\widetilde{X}}((id_E, g)^{-1}(\alpha)) \leq n = N_{\widetilde{X}}(\alpha)$$

which implies $H_{\widetilde{X}}((id_E, g)^{-1}(\alpha)) \leq H_{\widetilde{X}}(\alpha)$.

Now, suppose that $N_{\widetilde{X}}((id_E, g)^{-1}(\alpha)) = m$. Let

$$\{(id_E, g)^{-1}(Q_1, E, X), (id_E, g)^{-1}(Q_2, E, X),$$

$$\cdots, (id_E, g)^{-1}(Q_m, E, X)\}$$

be a finite open subcover (for $\widetilde{X}$) of $(id_E, g)^{-1}(\alpha)$. Therefore,

$$\widetilde{X} = \widetilde{\bigcup}_{i=1}^{m} \{(id_E, g)^{-1}(Q_i, E, X)\}.$$

Since $(id_E, g)(\widetilde{X}) = \widetilde{X}$, then

$$\begin{aligned} \widetilde{X} &= (id_E, g)(\widetilde{X}) = \widetilde{\bigcup}_{i=1}^{m} \{(id_E, g)^{-1}(Q_i, E, X)\} \\ &= \widetilde{\bigcup}_{i=1}^{m} \{(id_E, g)(id_E, g)^{-1}((Q_i, E, X))\} \\ &= \widetilde{\bigcup}_{i=1}^{n} \{(Q_i, E, X)\}. \end{aligned}$$

$$\{(Q_i, E, X) \mid i = 1, 2, \cdots, m\}$$

is a finite open subcover (for $\widetilde{X}$) of α, Hence, $m \geq N_{\widetilde{X}}(\alpha)$, i.e.,

$$N_{\widetilde{X}}((id_E, g)^{-1}(\alpha)) \geq N_{\widetilde{X}}(\alpha)$$

which implies

$$H_{\widetilde{X}}((id_E, g)^{-1}(\alpha)) \geq H_{\widetilde{X}}(\alpha).$$

By the above, we can get that

$$H_{\widetilde{X}}((id_E, g)^{-1}(\alpha)) = H_{\widetilde{X}}(\alpha). \quad \square$$

Theorem 5 Let $(X, (id_E, g))$ be a soft compact discrete topological dynamical system, α be a soft open cover of $\widetilde{X}$. Then the limit

$$\lim_{n \to \infty} \frac{1}{n} H_{\widetilde{X}}(\widehat{\bigcup}_{k=1}^{n-1} \{(id_E, g)^{-k}(\alpha)\})$$

exists.

Proof. Let

$$a_n = H_{\widetilde{X}}(\widehat{\bigcup}_{k=1}^{n-1} \{(id_E, g)^{-k}(\alpha)\}).$$

We only need to show that

$$a_{n+p} \leq a_n + a_p \quad (\forall n, p \in N - \{0\}).$$

From theorem 2.7(3) and (4), we have

$$
\begin{aligned}
a_{n+p} &= H_{\widetilde{X}}\left(\widehat{\bigcup}_{k=0}^{n+p-1} \{(id_E, g)^{-k}(\alpha)\}\right) \\
&= H_{\widetilde{X}}\left(\left(\widehat{\bigcup}_{k=0}^{n-1} \{(id_E, g)^{-k}(\alpha)\}\right) \right. \\
&\qquad \left. \widehat{\bigcup}\left(\widehat{\bigcup}_{k=n}^{n+p-1} \{(id_E, g)^{-k}(\alpha)\}\right)\right) \\
&= H_{\widetilde{X}}\left(\left(\widehat{\bigcup}_{k=0}^{n-1} \{(id_E, g)^{-k}(\alpha)\}\right) \right. \\
&\qquad \left. \widehat{\bigcup}\left((id_E, g)^{-n}\widehat{\bigcup}_{k=0}^{p-1} \{(id_E, g)^{-k}(\alpha)\}\right)\right) \\
&\leq H_{\widetilde{X}}\left(\widehat{\bigcup}_{k=0}^{n-1} \{(id_E, g)^{-k}(\alpha)\}\right) \\
&\qquad + H_{\widetilde{X}}\left(\widehat{\bigcup}_{k=0}^{p-1} \{(id_E, g)^{-k}(\alpha)\}\right).
\end{aligned}
$$

Thus $a_{n+p} \leq a_n + a_p$. $\quad \square$

Definition 13 Let $(X, (id_E, g))$ be a soft compact discrete topological dynamical system, let α be a soft open cover of $\widetilde{X}$. Then

$$Ent((id_E, g), \alpha, \widetilde{X}) = \lim_{n \to \infty} \frac{1}{n} H_{\widetilde{X}}\left(\widehat{\bigcup}_{k=1}^{n-1} \{(id_E, g)^{-k}(\alpha)\}\right)$$

is called the soft topological entropy of (id_E, g) on $\widetilde{X}$ relative to α, and

$$Ent(id_E, g) = \sup_{\alpha}\{Ent((id_E, g), \alpha, \widetilde{X}) \mid$$

$$\alpha \text{ is a soft open cover of } \widetilde{X}\}$$

is called the soft topological entropy of (id_E, g).

By Theorem 1, each soft closed subset of $\widetilde{X}$ is a soft compact subset of $\widetilde{X}$, then the following theorem holds.

Theorem 6 Let $(X, (id_E, g))$ be a soft compact discrete topological dynamical system, α be a soft open cover of $\widetilde{X}$, (A_1, E, X) and (A_2, E, X) be two closed soft sets, and $(A_1, E, X) \widetilde{\subseteq} (A_2, E, X)$, Then

(1)
$$Ent((id_E, g), \alpha, (A_1, E, X)) \leq Ent((id_E, g), \alpha, (A_2, E, X)).$$

(2)
$$Ent((id_E, g), (A_1, E, X)) \leq Ent((id_E, g), (A_2, E, X)).$$

Proof. (1) Let
$$N_{(A_2, E, X)}(\widehat{\bigcup}_{k=0}^{n-1}\{(id_E, g)^{-k}(\alpha)\}) = s.$$

Then there exists a soft open subcover

$$\{(P_1, E, X), (P_2, E, X), \cdots, (P_s, E, X)\}$$

of

$$\widehat{\bigcup}_{k=0}^{n-1}\{(id_E, g)^{-k}(\alpha)\}$$

for (A_2, E, X). Since $(A_1, E, X) \widetilde{\subseteq} (A_2, E, X)$, we have

$$\{(P_1, E, X), (P_2, E, X), \cdots, (P_s, E, X)\}$$

is also a subcover of

$$\widehat{\bigcup}_{k=0}^{n-1}\{(id_E, g)^{-k}(\alpha)\}$$

for (A_1, E, X), and hence

$$N_{(A_1, E, X)}(\widehat{\bigcup}_{k=0}^{n-1}\{(id_E, g)^{-k}(\alpha)\}) \leq s$$

$$= N_{(A_2, E, X)}(\widehat{\bigcup}_{k=0}^{n-1}\{(id_E, g)^{-k}(\alpha)\}).$$

So

$$H_{(A_1, E, X)}(\widehat{\bigcup}_{k=0}^{n-1}\{(id_E, g)^{-k}(\alpha)\})$$

$$\leq H_{(A_2, E, X)}(\widehat{\bigcup}_{k=0}^{n-1}\{(id_E, g)^{-k}(\alpha)\}).$$

$$Ent((id_E, g), \alpha, (A_1, E, X))$$

$$\leq Ent((id_E, g), \alpha, (A_2, E, X)).$$

(2)

$$Ent((id_E, g), (A_1, E, X))$$

$$= \sup_{\alpha}\{Ent((id_E, g), \alpha, (A_1, E, X)) \mid \alpha \text{ is a soft open}$$
$$\text{cover of } \widetilde{X}\}$$

$$\leq \sup_{\alpha}\{Ent((id_E, g), \alpha, (A_2, E, X)) \mid \alpha \text{ is a soft open}$$
$$\text{cover of } \widetilde{X}\}$$

$$= Ent((id_E, g), (A_2, E, X)). \ \square$$

Theorem 7 Let $(X, (id_E, g))$ be a soft compact discrete topological dynamical system, and α be a soft open cover of $\widetilde{X}$. Then $Ent(id_E, id_X) = 0$.

Proof Straightforward.

Theorem 8 $Ent(id_E, g^m) \geq m \cdot Ent(id_E, g)$ ($\forall m \in N - \{0\}$).

Proof As

$$((g^n)^{\leftarrow})^m = (g^{\leftarrow})^{nm} \quad (\forall n \in N - \{0\}, \forall m \in N),$$

we have

$$\widehat{\bigcup}_{t=0}^{n-1}\{(id_E, g^m)^{-s}\widehat{\bigcup}_{t=0}^{m-1}\{(id_E, g)^{-t}(\alpha)\}\}$$

$$= \widehat{\bigcup}_{s=0}^{mn-1}\{(id_E, g)^{-s}(\alpha)\}$$

Hence

$$H_{\widetilde{X}}(\widehat{\bigcup}_{t=0}^{n-1}\{(id_E, g^m)^{-s}\widehat{\bigcup}_{t=0}^{m-1}\{(id_E, g)^{-t}(\alpha))\}\}$$

$$= H_{\widetilde{X}}(\widehat{\bigcup}_{s=0}^{mn-1}\{(id_E, g)^{-s}(\alpha)\}).$$

Denote

$$\beta = \widehat{\bigcup}_{s=0}^{m-1}\{(id_E, g)^{-s}(\alpha)\}.$$

Then

$$Ent(id_E, g^m) = Ent(id_E, g)^m \geq Ent((id_E, g)^m, \beta, \widetilde{X})$$

$$= \lim_{n\to\infty}\frac{1}{n}H_{\widetilde{X}}\left(\widehat{\bigcup}_{t=0}^{n-1}\{(id_E, g^m)^{-s}\widehat{\bigcup}_{t=0}^{m-1}\{(id_E, g)^{-t}(\alpha))\}\}\right)$$

$$= \lim_{n\to\infty} m \cdot \frac{1}{mn}H_{\widetilde{X}}(\widehat{\bigcup}_{s=0}^{mn-1}\{(id_E, g)^{-s}(\alpha)\})$$

$$= m \cdot Ent((id_E, g), \alpha, \widetilde{X}).$$

Hence,

$$Ent(id_E, g^m) \geq m \cdot \sup_{\alpha} Ent((id_E, g), \alpha, \widetilde{X})$$

$$= m \cdot Ent(id_E, g). \ \square$$

4   Conclusion

In this paper, the discrete dynamical systems in soft topological spaces are defined, and simple examples are also given. Some basic concepts (such as soft ω-limit set, soft invariant set, soft periodic point, soft nonwandering point, and soft recurrent point) of the discrete dynamical system are introduced into soft topological spaces. Soft topological mixing and soft topological transitivity are also studied. At last, soft topological entropy is defined and several properties of it are discussed.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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FUNCTIONAL INEQUALITIES IN VECTOR BANACH SPACE

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ABSTRACT. In this paper, we prove that the generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(ax + by + cz) + f(bx + ay + bz) + f(cx + cy + az)\| \le \|(a + b + c)f(x + y + z)\|$$

in vector Banach space, where $a \ne b \ne c \in \mathbb{R}$ are fixed points with $3 > |a + b + c|$.

1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations originated from a question of Ulam [32] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [24] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias [24] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [9] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. The stability problems for several functional equations or inequations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2]–[8],[10], [12]–[16], [22]–[25],[26]-[31],[34]).

We recall some basic facts concerning generalized norm.

Definition 1.1 (see [15]). Let E be a real vector space. A generalized norm for E is a mapping $\| \cdot \|_G : E \to \mathbb{R}_+^k$ denoted by

$$\|x\|_G = (\alpha_1(x), \alpha_2(x), \alpha_3(x), \cdots, \alpha_k(x))$$

such that
(a) $\|x\|_G \ge 0$, that is, $\alpha_i(x) \ge 0$ for all $i = 1, 2, \cdots, k$;
(b) $\|x\|_G = 0$ if and only if $x = 0$, that is, $\alpha_i(x) = 0$ for all i, if and only if $x = 0$;
(c) $\|\lambda x\|_G = |\lambda| \|x\|_G$, that is, $\alpha_i(\lambda x) = |\lambda| \alpha_i(x)$;
(d) $\|x + y\|_G \le \|x\|_G + \|y\|_G$, which means, $\alpha_i(x + y) \le \alpha_i(x) + \alpha_i(y)$;

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Example 1.2. *In $\mathbb{R}^2$, $\|x\|_G = (|x_1|, |x_2|)$.*

Definition 1.3. Let $(X, \|\cdot\|_G)$ be a general normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} \alpha_i(x_n - x) = 0$ for all $i = 1, 2 \cdots, k$. In that case, x is called the limit of the sequence x_n and we denote it by $G\text{-}\lim_{n\to\infty} x_n = x$.

Definition 1.4. A sequence x_n in X is called *Cauchy* if for each $\epsilon > 0$ and each $a > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $\|x_{n+p} - x_n\|_G \leq \epsilon$, that is, $\alpha_i(x_{n+p} - x_n) \leq \epsilon$.

It is known that every convergent sequence in the general normed space is Cauchy. If each Cauchy sequence is convergent, then the the general normed space is said to be complete and the general normed space is called a *vector Banach space*.

2. HYers-Ulam Stability In vector Banach Space

From now on , Let $\mathcal{X}$ be a normed linear space and $\mathcal{Y}$ a vector Banach space.

This paper,we prove that the generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(ax + by + cz) + f(bx + cy + bz) + f(cx + ay + az)\|_G \leq \|(a + b + c)f(x + y + z)\|_G$$

in the vector Banach space, where $a \neq b \neq c \in \mathbb{R}$ are fixed points with $3 > |a + b + c|$.

Lemma 2.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping. If it satisfies*

$$\begin{aligned}
&\|f(ax + by + cz) + f(bx + cy + bz) + f(cx + ay + az)\|_G \\
&\leq \|(a + b + c)f(x + y + z)\|_G
\end{aligned} \tag{2.1}$$

for all $x, y, z \in \mathcal{X}$ and a, b, c are fixed real numbers with $3 > |a + b + c|$. Then f is additive.

Proof. Letting $x = y = z = 0$ in (2.1) for all $x, y, z \in \mathcal{X}$, we get

$$\|3f(0)\|_G \leq \|(a + b + c)f(0)\|_G \tag{2.2}$$

for $a, b, c \in \mathbb{R}$.

For any $i = 1, 2, \cdots, k$,

$$\alpha_i(3f(0)) \leq \alpha_i((a + b + c)f(0))$$

we get

$$3\alpha_i(f(0)) \leq |a + b + c|\alpha_i(f(0)),$$

Thus $f(0) = 0$.

Letting $x = 0$ and Replacing z by $-y$ in (2.1), we get

$$\|f((b - c)y) + f((c - b)y)\|_G \leq \|(a + b + c)f(0)\|_G = |a + b + c|\alpha_i(f(0)) = 0$$

and so $f(-x) = -f(x)$ for all $x \in \mathcal{X}$.

Replacing x by $-y - z$ in (2.1), we have

$$\|f((b-a)y + (c-a)z) + f((a-b)y) + f((a-c)z)\|_G \leq 0$$

for all $y, z \in \mathcal{X}$. Then we can obtain

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathcal{X}$. $\qquad\square$

Theorem 2.2. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$. If there is a function $\varphi : X^3 \to [0, \infty)$ such that*

$$\|f(ax + by + cz) + f(bx + cy + bz) + f(cx + ay + az)\|_G$$
$$\leq \|(a + b + c)f(x + y + z)\|_G + (\underbrace{\varphi(x,y,z), \varphi(x,y,z), \cdots, \varphi(x,y,z)}_{k}) \qquad (2.3)$$

and

$$\widetilde{\varphi}(x,y,z) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi\left((-2)^j x, (-2)^j x, (-2)^j x\right) < \infty \qquad (2.4)$$

for all $x, y, z \in \mathcal{X}$ and $a \neq b \neq c \in \mathbb{R}$ are fixed points with $3 > |a + b + c|$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\|_G$$

$$\leq \left(\underbrace{\widetilde{\varphi}\left(\frac{b+c-2a}{(a-b)(a-c)}x, \frac{1}{a-b}x, \frac{1}{a-c}x\right), \cdots, \widetilde{\varphi}\left(\frac{b+c-2a}{(a-b)(a-c)}x, \frac{1}{a-b}x, \frac{1}{a-c}x\right)}_{k}\right) \qquad (2.5)$$

for all $x \in \mathcal{X}$.

Proof. Letting $x = -y - z$ in (2.3), we get

$$\|f((b-a)y + (c-a)z) + f((a-b)y) + f((a-c)z)\|_G$$
$$\leq \underbrace{(\varphi(-y-z, y, z), \cdots, \varphi(-y-z, y, z))}_{k} \qquad (2.6)$$

for all $y, z \in \mathcal{X}$.

Letting $y = \frac{x}{b-a}, z = \frac{y}{c-a}$ in (2.6), we get

$$\|f(x + y) + f(-x) + f(-y)\|_G$$
$$\leq \left(\underbrace{\varphi\left(\frac{x}{a-b} + \frac{y}{a-c}, \frac{x}{b-a}, \frac{y}{c-a}\right), \cdots, \varphi\left(\frac{x}{a-b} + \frac{y}{a-c}, \frac{x}{b-a}, \frac{y}{c-a}\right)}_{k}\right) \qquad (2.7)$$

for all $x, z \in \mathcal{X}$.

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Letting $x = y$ in (2.7) we get

$$\|2f(-x) + f(2x)\|_G$$
$$\leq \left(\varphi \left(\frac{2a - b - c}{(a - b)(a - c)}x, \frac{1}{b - a}x, \frac{1}{c - a}x \right), \cdots, \right.$$
$$\left. \varphi \left(\frac{2a - b - c}{(a - b)(a - c)}x, \frac{1}{b - a}x, \frac{1}{c - a}x \right) \right)$$

for all $x \in \mathcal{X}$. Thus

$$\left\| f(x) - \frac{f(-2x)}{-2} \right\|_G$$
$$\leq \frac{1}{2} \left(\varphi \left(\frac{b + c - 2a}{(a - b)(a - c)}x, \frac{1}{a - b}x, \frac{1}{a - c}x \right), \cdots, \right.$$
$$\left. \varphi \left(\frac{b + c - 2a}{(a - b)(a - c)}x, \frac{1}{a - b}x, \frac{1}{a - c}x \right) \right)$$

for all $x \in \mathcal{X}$.

Hence one may have the following formula for positive integers m, l with $m > l$,

$$\left\| \frac{1}{(-2)^l} f\left((-2)^l x \right) - \frac{1}{(-2)^m} f\left((-2)^m x \right) \right\|_G$$
$$\leq \sum_{i=l}^{m-1} \frac{1}{2^i} \left(\varphi \left(\frac{(-2)^i (b + c - 2a)}{(a - b)(a - c)}x, \frac{(-2)^i}{a - b}x, \frac{(-2)^i}{a - c}x \right), \cdots, \right.$$
$$\left. \varphi \left(\frac{(-2)^i (b + c - 2a)}{(a - b)(a - c)}x, \frac{(-2)^i}{a - b}x, \frac{(-2)^i}{a - c}x \right) \right)$$

for all $x \in \mathcal{X}$. That is,

$$\alpha_i \left(\frac{1}{(-2)^l} f\left((-2)^l x \right) - \frac{1}{(-2)^m} f\left((-2)^m x \right) \right)$$
$$\leq \sum_{i=l}^{m-1} \frac{1}{2^i} \varphi \left(\frac{(-2)^i (b + c - 2a)}{(a - b)(a - c)}x, \frac{(-2)^i}{a - b}x, \frac{(-2)^i}{a - c}x \right) \tag{2.8}$$

for all $x \in mathcalX$. It follows from (2.8) that the sequence $\left\{ \frac{f((-2)^k x)}{(-2)^k} \right\}$ is a Cauchy sequence for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is a generalized norm space, the sequence $\left\{ \frac{f((-2)^k x)}{(-2)^k} \right\}$ converges. So one may define the mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := G - \lim_{k \to \infty} \left\{ \frac{f((-2)^k x)}{(-2)^k} \right\}, \quad \forall x \in \mathcal{X}.$$

Taking $m = 0$ and letting l tend to ∞ in (2.8), we have the inequality (2.5).

It follows from (2.3) that

$$\|A(ax + by + cz) + A(bx + ay + bz) + A(cx + cy + az)\|_G$$

$$= \lim_{k \to \infty} \left| \frac{1}{(-2)^k} \right| \left\| f((-2)^k(ax + by + cz)) + f((-2)^k(bx + ay + bz)) \right.$$

$$\left. + f((-2)^k(cx + cy + az)) \right\|_G$$

$$\leq \lim_{k \to \infty} \left| \frac{1}{(-2)^k} \right| \left\| (a + b + c)f((-2)^k(x + y + z)) \right\|_G \tag{2.9}$$

$$+ \lim_{k \to \infty} \left| \frac{1}{(-2)^k} \right| \left(\underbrace{\varphi((-2)^k x, (-2)^k y, (-2)^k z), \cdots, \varphi((-2)^k x, (-2)^k y, (-2)^k z)}_{k} \right)$$

$$\leq \|(a + b + c)A(x + y + z)\|_G$$

for all $x, y, z \in \mathcal{X}$. One see that A satisfies the inequality (2.1) and so it is additive by Lemma (2.1).

Now, we show that the uniqueness of A. Let $T : X \to Y$ be another additive mapping satisfying (2.5). Then one has

$$\|A(x) - T(x)\|_G = \left\| \frac{1}{(-2)^k} A\left((-2)^k x\right) - \frac{1}{(-2)^k} T\left((-2)^k x\right) \right\|_G$$

$$\leq \frac{1}{2^k} \left(\left\| A\left((-2)^k x\right) - f\left((-2)^k x\right) \right\|_G \right.$$

$$+ \left\| T\left((-2)^k x\right) - f\left((-2)^k x\right) \right\|_G \right)$$

$$\leq 2\frac{1}{2^k} \left(\underbrace{\widetilde{\varphi}\left(\frac{(b + c - 2a)(-2)^k}{(a - b)(a - c)} x, \frac{(-2)^k}{a - b} x, \frac{(-2)^k}{a - c} x \right), \cdots}_{k} \right)$$

which tends to zero as $k \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. $\qquad \square$

Theorem 2.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping with $f(0) = 0$. If there is a function $\varphi : X^3 \to [0, \infty)$ satisfying (2.3) such that*

$$\widetilde{\varphi}(x, y, z) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{(-2)^j}, \frac{y}{(-2)^j}, \frac{z}{(-2)^j} \right) < \infty \tag{2.10}$$

for all $x, y, z \in \mathcal{X}$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\|_G \leq \widetilde{\varphi}(x, x, -2x) \tag{2.11}$$

for all $x \in \mathcal{X}$.

Proof. The proof is similar with Theorem (2.2), we can get

$$\left\| f(x) - (-2)f\left(\frac{x}{-2}\right) \right\|_G$$

$$\leq \left(\underbrace{\varphi\left(\frac{(2a-b-c)x}{2(a-b)(a-c)}, \frac{x}{2(b-a)}, \frac{x}{2(c-a)}\right) \cdots \varphi\left(\frac{(2a-b-c)x}{2(a-b)(a-c)}, \frac{x}{2(b-a)}, \frac{x}{2(c-a)}\right)}_{k} \right)$$

for all $x \in \mathcal{X}$.

Next, we can prove that the sequence $\{(-2)^n f\left(\frac{x}{(-2)^n}\right)$ is a Cauchy sequence for all $x \in \mathcal{X}$,and define a mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := \lim_{n \to \infty} (-2)^n f\left(\frac{x}{(-2)^n}\right)$$

for all $x \in \mathcal{X}$ that is similar to the corresponding part of the proof of Theorem (2.2). $\square$

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Coupled fixed point theorems for generalized $(\psi, \phi)-$ weak contraction in partially ordered G-metric spaces

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In this manuscript, we give coupled fixed point results for generalized $(\psi, \phi)-$ weak contraction, satisfying rational type expression in the context of partially ordered G-metric spaces. The derived results generalize the result of K. Chakrabarti (K. Chakrabarti, Coupled fixed point theorems with rational type contractive condition in a partially ordered G-metric space, Journal of Mathematics, Volume 2014, Article ID 785357, 7 pages). To demonstrate our result and also to demonstrate the authenticity of our result from the previous one, we give suitable example.

Key Words: Coupled fixed point, Mixed monotone property, Partially ordered G-metric space, $(\psi, \phi)-$weak contraction.
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1 Introduction and preliminaries

Fixed point theory provide one of the most important and useful technique for the existence of fixed point, coincidence point, common fixed point and coupled fixed pint for self map under different condition. It is used for the existence and uniqueness of the solution of mathematical model which may be in the form of differential equations, matrix equations, integral equations, functional equations, linear inequalities or mixed see ([5], [17], [19], [30]). In this area the first well known result proved by Banach [8] known as Banach contraction principle. Many authors generalized this principle in various spaces by using different contractive conditions ([6], [13], [15], etc.).

In recent years, metric fixed point theory has been developed rapidly in partially ordered metric space. Ran and Reurings [30] extended the Banach contraction principle in partially ordered sets and also discuss some applications to linear and nonlinear matrix equations. Nieto and Rodriguez-Lopez [23]

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extended the result of Ran and Reurings and used their established result to obtain a unique solution for first order ODEs. Jaggi [15] construct rational type contraction in complete metric space. Harjani et al [13] extend the result of Jaggi to partial ordered complete metric space. For more details (see[13], [33]).

Alber and Gurre [6] gave the concept of weak contraction as a generalization of contraction and established the existence of fixed points for a self map in a Hilbert space. Rhoades [31] extended this concept to metric spaces and defined ϕ-weak contraction. Dutta and Choudhury [12] generalized ϕ-weak contraction to the concept of (ψ, ϕ) weak contraction and studies some fixed point results. Zhang and Song [34] extend weak contraction for the study of two self map. Furthermore Djorić [11] generalized the result of Zhang and Song and studied common fixed point for generalized (ψ, ϕ) weak contraction. For some other similar results see [22], [25], [29], [32].

The concept of mixed monotone mappings introduced by Bhaskar and Lakshmikantham [9] and derived some coupled fixed point results. Furthermore, they applied their results on a first order differential equation with periodic boundary conditions [14]. Lakshmikanthem and Ćirić [17] generalized the concept of mixed monotone mapping and established a coupled fixed point theorem for nonlinear contractions in partially ordered metric spaces. Recently Chakrabarti [10] investigated coupled fixed point theorems for map satisfying nonlinear rational type contraction and mixed monotone property in partially ordered G-metric space.

In this work, using the concept of generalized rational type $(\psi, \phi)-$weak contraction condition, coupled fixed point results in the framework of complete partially ordered generalized metric spaces are investigated. Through out the paper $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{N}_0$ will denote the set of all non-negative real numbers, the set of positive integer and the set of non-negative integer respectively.

Definition 1. *[20] Let $(X, \preceq)$ be a partially ordered set and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following conditions:*

1. *$G(u, v, w) = 0$ if $u = v = w$;*

2. *$0 < G(u, u, v)$ for all $u, v \in X$ with $u \neq v$;*

3. *$G(u, u, v) \leq G(u, v, w)$ for all $u, v, w \in X$ with $v \neq w$;*

4. *$G(u, v, w) = G(u, w, v) = G(v, w, u) = \ldots$ (symmetry in all three variables);*

5. *$G(u, v, w) \leq G(u, p, p) + G(p, v, w)$ for all $u, v, w, p \in X$ (rectangle inequality).*

Then it is called a G-metric on X and the triple $(X, G, \preceq)$ is called partially ordered G-metric space.

Definition 2. *[20] The pair (X, G) is said to be symmetric G-metric space if $G(u, v, v) = G(u, u, v)$ for all $u, v \in X$.*

Example 1. *(1) Let $X = \mathbb{R}^+$ and $G : X \times X \times X \to \mathbb{R}^+$ be the function defined is follows $G(u, v, w) = \max\{|u - v|, |v - w|, |w - u|\}$, for all $u, v, w \in X$. Then G is symmetric G-metric on X.*

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(2) Let $X = \{a, b\}$. Define $G(a, a, a) = G(b, b, b) = 0, G(a, a, b) = 1, G(a, b, b) = 2$, and extend G to X^3 by using the symmetry in the variables. Then it is clear that (X, G) is an asymmetric $G-$metric space.

(3) Also see examples of asymmetric $G-$metric spaces in ([2], Example 2.6; [3], Example 2.2; [18], Example 2.2; [22], Example 3.4.).

Definition 3. *[20] Let (X, G) be a G-metric space and let α_n be a sequence in X. A point $\alpha \in X$ is said to be the limit of the sequence α_n if*

$$\lim_{n,m \to \infty} G(\alpha_n, \alpha_m, \alpha) = 0$$

and the sequence α_n is said to be G-convergent in X.

Definition 4. *[20] A sequence α_n is called a G-Cauchy sequence if for every $\varepsilon > 0$, there is a positive integer $\mathbb{N}$ such that $G(\alpha_n, \alpha_m, \alpha_l) < \varepsilon$ for all $n, m, l > \mathbb{N}$.*

Definition 5. *[20] A metric space (X, G) is said to be G-complete (or a complete G-metric space) if every G-Cauchy sequence in (X, G) is G-convergent in X.*

Definition 6. *[9] Let $(X, \preceq)$ be a partially ordered set, $T : X \times X \to X$. Then T is said to have mixed-monotone property if $T(x, y)$ is monotone non-decreasing in x and monotone non-increasing in y. That is., for all $x, y \in X$*

Definition 7. *[17] Let $(X, \preceq)$ be a partially ordered set, $T : X \times X \to X$ and $g : X \to X$. We say T is the g-mixed monotone property if T is monotone g-nondecreasing in its first argument and monotone g-non-increasing in its second argument. That is., for all $x, y \in X$*

$$x_1, x_2 \in X, \ gx_1 \preceq gx_2 \Rightarrow T(x_1, y) \preceq T(x_2, y),$$
$$y_1, y_2 \in X, \ gy_1 \preceq gy_2 \Rightarrow T(x, y_1) \succeq T(x, y_2).$$

Definition 8. *[9] Let $T : X \times X \to X$ be a map such that $T(x, y) = x$ and $T(y, x) = y$ then the pair $(x, y) \in X \times X$ is called a coupled fixed point of T. It is clear that (x, y) is a coupled fixed point if and only if (y, x) is such.*

Definition 9. *[17] Let $T : X \times X \to X$ and $g : X \to X$ be two map such that $T(x, y) = gx$ and $T(y, x) = gy$ then the pair $(x, y) \in X \times X$ is called a coupled coincidence point of T and g.*

Definition 10. *[17] Two maps $T : X \times X \to X$ and $g : X \to X$ are said to be commutative if $g(T(x, y)) = T(gx, gy)$.*

Chakrababati [10] proved the following results.

Theorem 1. *[10]Let $(X, \preceq)$ be a partially ordered set and let (X, G) be a $G-$complete G-metric space. Suppose $T : X \times X \longrightarrow X$ be a continuous mapping on X having the mixed monotone property. Suppose for all $(x, y), (u, v), (w, z) \in X \times X$ with $(x, y) \preceq (u, v) \preceq (w, z)$ holds*

$$\begin{aligned}
&G(T(x, y), T(u, v), T(w, z)) \\
&\leq \alpha \frac{G(x, T(x, y), T(x, y)) \, G(u, T(u, v), T(u, v)) \, G(w, T(w, z), T(w, z))}{G^2(x, u, w)} \\
&\quad + \beta G(x, u, w),
\end{aligned}$$

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where $8\alpha + \beta < 1$. *If there exist* $x_0, y_0 \in X$ *such that* $x_0 \preceq T(x_0, y_0)$ *and* $y_0 \succeq T(y_0, x_0)$, *then* T *has a coupled fixed point* $(x_*, y_*) \in X$.

Theorem 2. *[10] Let* $(X, \preceq)$ *be a partially ordered set and let* (X, G) *be a* $G-$*complete* G-*metric space. Suppose* $T : X \times X \longrightarrow X$ *and* $g : X \longrightarrow X$ *be a continues mappings on* X *such that* T *has the mixed g-monotone property. Suppose that* $T(X \times X) \subseteq g(X)$, g *commute with* T *and for* $(x, y), (u, v), (w, z) \in X \times X$ *with* $(x, y) \preceq (u, v) \preceq (w, z)$ *and* $gx \preceq gu \preceq gw$ *or* $gy \succeq gu \succeq gz$ *holds*

$$G(T(x, y), T(u, v), T(w, z))$$
$$\leq \alpha \frac{G(gx, T(x, y), T(x, y)) \, G(gu, T(u, v), T(u, v)) \, G(gw, T(w, z), T(w, z))}{G^2(gx, gu, gw)}$$
$$+ \beta G(gx, gu, gw),$$

where $8\alpha + \beta < 1$. *If there exist* $x_0, y_0 \in X$ *such that* $gx_0 \preceq T(x_0, y_0)$ *and* $gy_0 \succeq T(y_0, x_0)$ *then* T *and* g *have a coupled coincidence point* $(x_*, y_*) \in X \times X$, *that is.,* (x_*, y_*) *satisfies* $gx_* = T(x_*, y_*)$, $gy_* = T(y_*, x_*)$.

2 Main Results

In our main results we used the following two classes.

$\psi \in \Psi$ if and only if $\psi : [0, \infty) \to [0, \infty)$, ψ is continuous and non-decreasing function such that $\psi(t) = 0$ if and only if $t = 0$.

$\phi \in \Phi$ if and only if $\phi : [0, \infty) \to [0, \infty)$, ψ is a lower semi continuous and non-decreasing function such that $\phi(t) = 0$ if and only if $t = 0$.

Also, for more details of G-metric spaces see ([1]-[4], [7], [16], [18], [21], [26]-[28]).

Remark 1. *It is worth to noticing that both results in [10] without the conditions* $G^2(x, u, w) \neq 0$ *that is.,* $G(gx, gu, gw) \neq 0$ *are not correct.*

Now, we announce the first our result.

Theorem 3. *Let* $(X, \preceq)$ *be a partially ordered set and let* (X, G) *be a* $G-$*complete symmetric* G-*metric space. Suppose* $T : X \times X \longrightarrow X$ *be a continuous mapping having the mixed monotone property and satisfying*

$$\psi(G(T(x, y), T(u, v), T(w, z))) \leq \psi(M(x, u, w, y, v, z)) - \phi(M(x, u, w, y, v, z)),$$
$$(2.1)$$

for all $x, y, z, u, v, w \in X$ *with* $G(x, u, w) \neq 0$ *and* $(x, y) \preceq (u, v) \preceq (w, z)$ *or* $(x, y) \succeq (u, v) \succeq (w, z)$, *where*

$$M(x, u, w, y, v, z)$$
$$= \max \left\{ \frac{[G(x, T(x, y), T(x, y)) G(u, T(u, v), T(u, v)) G(w, T(w, z), T(w, z))]}{G^2(x, u, w)}, \right.$$
$$\left. G(x, u, w) \right\},$$

$\psi \in \Psi$ *and* $\phi \in \Phi$. *If there exist* $x_0, y_0 \in X$ *such that* $x_0 \preceq T(x_0, y_0)$ *and* $y_0 \succeq T(y_0, x_0)$. *Then* T *has a coupled fixed point* $(x_*, y_*) \in X$.

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Proof. Suppose that there exist $x_0, y_0 \in X$ such that $x_0 \preceq T(x_0, y_0)$ and $y_0 \succeq T(y_0, x_0)$. Further, define $x_{n+1} = T(x_n, y_n)$ and $y_{n+1} = T(y_n, x_n)$. Using the mixed monotone property and the mathematical induction we obtain that $x_n \preceq x_{n+1}$ and $y_n \succeq y_{n+1}$ for all $n \in \mathbb{N}$ (very known method).

Consider now

$$\psi\left(G\left(x_{n+1}, x_n, x_n\right)\right) = \psi\left(G\left(T\left(x_n, y_n\right), T\left(x_{n-1}, y_{n-1}\right), T\left(x_{n-1}, y_{n-1}\right)\right)\right).$$

Using (2.1) we have that

$$\psi\left(G\left(x_{n+1}, x_n, x_n\right)\right) \leq \psi\left(M\left(x_n, x_{n-1}, x_{n-1}, y_n, y_{n-1}, y_{n-1}\right)\right) \qquad (2.2)$$
$$- \phi\left(M\left(x_n, x_{n-1}, x_{n-1}, y_n, y_{n-1}, y_{n-1}\right)\right)$$

where

$$M\left(x_n, x_{n-1}, x_{n-1}, y_n, y_{n-1}, y_{n-1}\right)$$
$$= \max\left\{ \frac{G\left(x_n, T\left(x_n, y_n\right), T\left(x_n, y_n\right)\right) G^2\left(x_{n-1}, T\left(x_{n-1}, y_{n-1}\right) T\left(x_{n-1}, y_{n-1}\right)\right)}{G^2\left(x_n, x_{n-1}, x_{n-1}\right)}, \right.$$
$$\left. G\left(x_n, x_{n-1}, x_{n-1}\right)\right\}.$$

Let $G_n = G(x_n, x_{n-1}, x_{n-1})$ then,

$$M(x_n, x_{n-1}, x_{n-1}, y_n, y_{n-1}, y_{n-1}) = \max\{G_{n+1}, G_n\}.$$

Further we show that G_n is non-increasing. Suppose their exist n_0 such that $G_{n_0+1} > G_{n_0}$ then from (2.2)

$$\psi(G_{n_0+1}) \leq \psi(G_{n_0+1}) - \phi(G_{n_0+1}).$$

Which implies that $\phi(G_{n_0+1}) \leq 0$. A contradiction. Hence $G_n \geq G_{n+1}$ for all $n \geq 1$. Since $\{G_n\}$ is a non-increasing sequence of positive real numbers there exists $r \geq 0$ such that

$$\lim_{n\to\infty} G_n = r. \qquad (2.3)$$

We shall show that $r = 0$. Suppose $r > 0$ then applying limit in (2.2) and using (2.3), we have

$$\psi(r) \leq \psi(r) - \phi(r) < \psi(r).$$

We obtain a contradiction. Therefore $r = 0$ that is.,

$$\lim_{n\to\infty} G_n = 0. \qquad (2.4)$$

Now, we show that $\{x_n\}$ is a G-Cauchy sequence. Suppose that, $\{x_n\}$ is not G-Cauchy. Then, there exist $\epsilon > 0$ and subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that,

$$G(x_{m(k)}, x_{m(k)}, x_{n(k)}) \geq \epsilon, \quad \forall\, k \in \mathbb{N}. \qquad (2.5)$$

Furthermore, corresponding to $m(k)$ one can choose $n(k)$ such that, it is the smallest integer with $n(k) > m(k)$ satisfying (2.5) then,

$$G(x_{m(k)}, x_{m(k)}, x_{n(k)-1}) < \epsilon, \quad \forall\, k \in \mathbb{N} \qquad (2.6)$$

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Now

$$\epsilon \le G(x_{m(k)}, x_{m(k)}, x_{n(k)}) = G(x_{n(k)}, x_{m(k)}, x_{m(k)})$$
$$\le G(x_{m(k)}, x_{m(k)}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{n(k)-1}, x_{n(k)}),$$

Taking limit $k \to \infty$ and using (2.4) we get

$$\lim_{k \to \infty} G(x_{m(k)}, x_{m(k)}, x_{n(k)}) = \epsilon. \tag{2.7}$$

Now

$$G(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}) = G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1})$$
$$\le G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{m(k)-1}, x_{m(k)-1})$$
$$\le G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{m(k)}, x_{m(k)})$$
$$+ G(x_{m(k)}, x_{m(k)-1}, x_{m(k)-1}), \tag{2.8}$$

and

$$G(x_{n(k)}, x_{m(k)}, x_{m(k)})$$
$$\le G(x_{n(k)}, x_{m(k)-1}, x_{m(k)-1}) + G(x_{m(k)-1}, x_{m(k)}, x_{m(k)})$$
$$\le G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{m(k)-1}, x_{m(k)-1})$$
$$+ G(x_{m(k)-1}, x_{m(k)}, x_{m(k)}), \tag{2.9}$$

Using limit $k \to \infty$ in (2.8) and (2.9) and using (2.4) and (2.7) we get

$$\lim_{k \to \infty} G(x_{m(k)-1}, x_{m(k)-1} x_{n(k)-1}) = \epsilon. \tag{2.10}$$

Consider

$$\psi\Big(G(x_{m(k)}, x_{m(k)}, x_{n(k)}\Big)$$
$$\le \psi\Big(M(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}, y_{n(k)-1})\Big)$$
$$- \phi\Big(M(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}, y_{n(k)-1})\Big), \tag{2.11}$$

where

$$M(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}, y_{n(k)-1})$$
$$= \max\left\{\frac{[[G(x_{m(k)-1}, x_{m(k)}, x_{m(k)})]^2 G(x_{n(k)-1}, x_{n(k)}, x_{n(k)})]}{G(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1})^2}, \tag{2.12}\right.$$

$$\left. G(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1})\right\}. \tag{2.13}$$

Applying limit $k \to \infty$ in (2.13), using (2.7), (2.10) and (2.4) we get

$$\lim_{k \to \infty} M(x_{m(k)-1}, x_{m(k)-1}, x_{n(k)-1}, y_{m(k)-1}, y_{m(k)-1}, y_{n(k)-1}) = \epsilon. \tag{2.14}$$

Taking limit of (2.11) using (2.7), (2.14) and lower semi continuity of ϕ we have

$$\psi(\epsilon) \le \psi(\epsilon) - \phi(\epsilon) < \psi(\epsilon),$$

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which is contradiction. So $\epsilon = 0$. Therefore x_n is a G-Cauchy sequence. Similarly by the same argument we can show that y_n is a G-Cauchy sequence. By completeness of X, there is $x_*, y_* \in X$ such that $x_n \to x_*$ and $y_n \to y_*$ as $n \to \infty$.

Now we have to show that (x_*, y_*) is a coupled fixed point of T. Since T is continuous on X and G is also continuous in each of its variable, so

$$G(T(x_*, y_*), x_*, x_*) = G(\lim_{n \to \infty} T(x_n, y_n), x_*, x_*) = G(x_*, x_*, x_*) = 0.$$

Hence, we proved that $T(x_*, y_*) = x_*$ Similarly by the same argument we obtain that $T(y_*, x_*) = y_*$. So (x_*, y_*) is a coupled fixed point of T. $\square$

Theorem 4. *Suppose that the conditions of Theorem 3 are valid. In addition suppose that for each $(x, y), (u, v) \in X \times X$ exists $(w, z) \in X \times X$ which is comparable to (x, y) and (u, v). Then coupled fixed point of T is unique.*

Proof. Suppose that $(x_*, y_*), (x', y') \in X \times X$ are two coupled fixed points.
   Case 1
   If (x_*, y_*), (x', y') are comparable then from (2.1)

$$\psi(G(T(x_*, y_*), T(x', y'), T(x', y') \leq \psi(M(x_*, x', x', y_*, y', y')$$
$$- \phi(M(x_*, x', x', y_*, y', y'), \qquad (2.15)$$

where

$$M(x_*, x', x', y_*, y', y')$$
$$= \max \left\{ \frac{G(x_*, T(x_*, y_*), T(x_*, y_*))[G(x', T(x', y'), T(x', y')]^2]}{G(x_*, x', x')}, G(x_*, x', x') \right\}$$
$$= \max \left\{ \frac{G(x_*, x_*, x_*)[G(x', x', x']^2}{G(x_*, x', x')}, G(x_*, x', x') \right\}.$$

Which implies that

$$M(x_*, x', x', y_*, y', y') = G(x_*, x', x').$$

From (2.15) we have

$$\psi(G(x_*, x', x') = \psi(G(T(x_*, y_*), T(x', y'), T(x', y')) < \phi(G(x_*, x', x'),$$

which is contradiction. Hence we must have $x_* = x'$. Similarly we can easily show that $y_* = y'$ so couple fixed point is unique.
   Case 2
   If (x_*, y_*), (x', y') are not comparable by Theorem 3 there is a $(u, v) \in X \times X$ comparable to (x_*, y_*) and (x', y') if there is $m_0 \in \mathbb{N}$ such that $T^{m_0}(u, v) = (x_*, y_*)$, then
$T^{m_0+1}(u, v) = T(x_*, y_*) = x_*$, in last we get $T^m(u, v) = x_*$ for $m \geq m_0$ this mean $T^m(u, v) \to x_*$ for $m \to \infty$
if there is no such m_0 then for any $m \geq 1$

$$\psi(G(T^m(u, v), x_*, x_*) = \psi(G(T^m(u, v), T^m(x_*, y_*), T^m(x_*, y_*)$$
$$\leq \psi(M(u, x_*, x_*, v, y_*, y_*) - \phi(M(u, x_*, x_*, v, y_*, y_*), \quad (2.16)$$

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where

$$
\begin{aligned}
&M(u, x_*, x_*, v, y_*, y_*) \\
&= \max \left\{ \frac{G(T^{m-1}(u,v), T^m(u,v), T^m(u,v))[G(T^{m-1}(x_*,y_*), T^m(x_*,y_*), T^m(x_*,y_*))]^2}{G(T^{m-1}(u,v), T^{m-1}(x_*,y_*), T^{m-1}(x_*,y_*))}, \right. \\
&\qquad \left. G(T^{m-1}(u,v), T^{m-1}(x_*,y_*), T^{m-1}(x_*,y_*)) \right\} \\
&= \max \left\{ \frac{G(T^{m-1}(u,v), x_*, x_*)[G(x_*,x_*,x_*)]^2}{G(T^{m-1}(u,v), x_*, x_*)}, G(T^{m-1}(u,v), x_*, x_*) \right\}.
\end{aligned}
$$

Which implies that

$$
M(u, x_*, x_*, v, y_*, y_*) = G(T^{m-1}(u,v), x_*, x_*).
$$

Putting M in (2.16), we have

$$
\psi(G(T^m(u,v), x_*, x_*) \leq \psi(G(T^{m-1}(u,v), x_*, x_*))
$$
$$
\phi(G(T^{m-1}(u,v), x_*, x_*)). \tag{2.17}
$$

This implies that

$$
\psi(G(T^m(u,v), x_*, x_*) < \psi(G(T^{m-1}(u,v), x_*, x_*),
$$

since ψ is non-decreasing therefore,

$$
G(T^m(u,v), x_*, x_*) < G(T^{m-1}(u,v), x_*, x_*)
$$

that is, $\{G(T^m(u,v), x_*, x_*)\}$ is a decreasing sequence of positive real numbers. Therefore, there is an α_1 such that $\{G(T^m(u,v), x_*, x_*)\} \to \alpha_1$. We shall show that $\alpha_1 = 0$. Suppose, to the contrary, that $\alpha_1 > 0$. Taking the limit in equation (2.17) we get contradiction. So $\alpha_1 = 0$. Implies $G(T^m(u,v), x_*, x_*) = 0$, that is., $T^m(u,v) = x_*$. Similarly we can show that $T^m(u,v) = y_*$, $(T^m(u,v) = x'$ and $(T^m(u,v) = y'$. Hence the coupled fixed point is unique. $\square$

The next result is the generalization of Theorem 3. Because the proof is similar, then it is omitted.

Theorem 5. *Let $(X, \preceq)$ be a partially ordered set and let (X, G) be a $G-$complete symmetric G-metric space. Suppose that $T : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ are a continues mappings such that T has the $g-$mixed monotone property. Suppose that $T(X \times X) \subseteq g(X)$, g commute with T and satisfying*

$$
\psi(G(T(x,y), T(u,v), T(w,z)) \leq \psi(M(x,u,w,y,v,z) - \phi(M(x,u,w,y,v,z), \tag{2.18}
$$

for all $x, y, z, u, v, w \in X$ with $G(gx, gu, gw) \neq 0$ and $(gx, gy) \preceq (gu, gv) \preceq (gw, gz)$ or $(gx, gy) \succeq (gu, gv) \succeq (gw, gz)$, where

$$
\begin{aligned}
&M(x, u, w, y, v, z) \\
&= \max \left\{ \frac{G(gx, T(x,y), T(x,y)) \, G(gu, T(u,v), T(u,v)) \, G(gw, T(w,z), T(w,z))}{G^2(gx, gu, gw)}, \right. \\
&\qquad \left. G(gx, gu, gw) \right\},
\end{aligned}
$$

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$\psi \in \Psi$ and $\phi \in \Phi$. If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq T(x_0, y_0)$ and $gy_0 \succeq T(y_0, x_0)$ then T and g have a coupled coincidence point $(x_*, y_*) \in X \times X$, that is., (x_*, y_*) satisfies $gx_* = T(x_*, y_*)$, $gy_* = T(y_*, x_*)$.

Corollary 1. *Let (X, G) be a partially ordered set and let (X, G) be a $G-$complete symmetric G-metric space. Suppose that $T : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ are a continues mappings such that T has the $g-$mixed monotone property. Suppose that $T(X \times X) \subseteq g(X)$, g commute with T and for $0 < k < 1$ satisfying*

$$G(T(x,y), T(u,v), T(w,z)) \leq k(M(x,u,w,y,v,z),$$

for all $x, y, z, u, v, w \in X$ with $G(gx, gu, gw) \neq 0$ and $(gx, gy) \preceq (gu, gv) \preceq (gw, gz)$ or $(gx, gy) \succeq (gu, gv) \succeq (gw, gz)$, where

$$
\begin{aligned}
&M(x, u, w, y, v, z) \\
&= \max \left\{ \frac{G(gx, T(x,y), T(x,y)) \, G(gu, T(u,v), T(u,v)) \, G(gw, T(w,z), T(w,z))}{G^2(gx, gu, gw)}, \right. \\
&\qquad \left. G(gx, gu, gw) \right\}.
\end{aligned}
$$

If there exist $x_0, y_0 \in X$ such that $gx_0 \preceq T(x_0, y_0)$ and $gy_0 \succeq T(y_0, x_0)$ then T and g have a coupled coincidence point $(x_*, y_*) \in X \times X$, that is., (x_*, y_*) satisfies $gx_* = T(x_*, y_*)$, $gy_* = T(y_*, x_*)$.

Proof. The proof follows by taking $\psi(t) = t$, $\phi(t) = (1-k)t$ where $0 < k < 1$ in Theorem 5. □

Remark 2. *For $0 < \alpha < \frac{1}{8}$, $0 < \beta < \frac{1}{16}$ and for all $x, y, z, u, v, w \in X$ with $G(gx, gu, gw) \neq 0$ and $(gx, gy) \preceq (gu, gv) \preceq (gw, gz)$ or $(gx, gy) \succeq (gu, gv) \succeq (gw, gz)$ we have*

$$
\begin{aligned}
&G(T(x,y), T(u,v), T(w,z)) \\
&\leq \alpha \frac{[G(gx, T(x,y), T(x,y)G(gu, T(u,v), T(u,v)G(gw, T(w,z), T(w,z)]}{G(gx, gu, gw)^2} \\
&\quad + \beta G(gx, gu, gw), \\
&\leq (\alpha + \beta) \max \left\{ \frac{G(gx, T(x,y), T(x,y)) \, G(gu, T(u,v), T(u,v)) \, G(gw, T(w,z), T(w,z))}{G^2(gx, gu, gw)}, \right. \\
&\qquad \left. G(gx, gu, gw) \right\}.
\end{aligned}
$$

where $k = \alpha + \beta < 1$. Clearly, the relation $0 < 8\alpha + \beta < 1$ implies that Corollary 1 is the generalization of Theorem 2. Therefore Theorem 5 is the generalization of Theorem 2.

Now we give example which satisfying Theorem 5 but does not Theorem 2.

Example 2. *Let $X = [0, 1]$ and consider the natural ordered relation in X, defined $G : X \times X \times X \to \mathbb{R}^+$ by*

$$
G(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x, y, z\}. \end{cases}
$$

Then (X, G) is $G-$complete symmetric G-metric space. Let $T : X \times X \to X$, $g : X \to X$, $\phi : [0, \infty) \to [0, \infty)$ and $\psi : [0, \infty) \to [0, \infty)$ define by,

$$T(x, y) = \begin{cases} \frac{x^3 - y^3}{4}, & \text{if } x \geq y, \\ 0, & \text{if } x < y, \end{cases}$$

$$g(x) = x^2, \quad \phi(t) = \frac{t}{2}, \quad \psi(t) = \frac{t}{4}.$$

We discuss the following cases.

Case 1. $(x, y) = (0, 0), (u, v) = (0, 0), (w, z) = (1, 0)$ *it is clear that* $(gx, gy) \preceq (gu, gv) \preceq (gw, gz)$ *or* $(gx, gy) \succeq (gu, gv) \succeq (gw, gz)$ *and*

$$\psi(G(T(0,0), T(0,0), T(1,0)) \leq \psi(M(0,0,1,0,0,0) - \phi(M(0,0,1,0,0,0)),$$

where $G(T(0,0), T(0,1), T(0,1)) = 1$ *and* $M(0,1,1,1,1,1) = 1$.

Case 2. $(x, y) = (0, 1), (u, v) = (1, 1), (w, z) = (1, 1)$ *it is clear that* $(gx, gy) \preceq (gu, gv) \preceq (gw, gz)$ *or* $(gx, gy) \succeq (gu, gv) \succeq (gw, gz)$ *and*

$$\psi(G(T(0,1), T(1,1), T(1,1)) \leq \psi(M(0,1,1,1,1,1) - \phi(M(0,1,1,1,1,1)),$$

where $G(T(0,1), T(1,1), T(1,1)) = 0$ *and* $M(0,1,1,1,1,1) = 1$.

Case 3. $(x, y) = (0, 0), (u, v) = (1, 0), (w, z) = (1, 0)$ *it is clear that* $(gx, gy) \preceq (gu, gv) \preceq (gw, gz)$ *or* $(gx, gy) \succeq (gu, gv) \succeq (gw, gz)$ *and*

$$\psi(G(T(0,0), T(1,0), T(1,0)) \leq \psi(M(0,1,1,0,0,0) - \phi(M(0,1,1,0,0,0)),$$

where $G(T(0,0), T(1,0), T(1,0)) = \frac{1}{4}$ *and* $M(0,1,1,0,0,0) = 1$.

Case 4. $(x, y) = (0, 1), (u, v) = (1, 1), (w, z) = (1, 1)$ *again it is clear that* $(gx, gy) \preceq (gu, gv) \preceq (gw, gz)$ *or* $(gx, gy) \succeq (gu, gv) \succeq (gw, gz)$ *and*

$$\psi(G(T(0,1), T(1,1), T(1,1)) \leq \psi(M(0,1,1,1,1,1) - \phi(M(0,1,1,1,1,1)),$$

where $G(T(0,1), T(1,1), T(1,1)) = 0$ *and* $M(0,1,1,1,1,1) = 1$.

Case 5. $(x, y) = (u, v) = (0, 1), (w, z) = (1, 1)$ *also it is clear that* $(gx, gy) \preceq (gu, gv) \preceq (gw, gz)$ *or* $(gx, gy) \succeq (gu, gv) \succeq (gw, gz)$ *and*

$$\psi(G(T(0,1), T(0,1), T(1,1)) \leq \psi(M(0,1,1,1,1,1) - \phi(M(0,1,1,1,1,1)),$$

where $G(T(0,1), T(0,1), T(1,1)) = 0$ *and* $M(0,0,1,1,1,1) = 1$.

Clearly for $(gx, gy) \preceq (gu, gv) \preceq (gw, gz)$ or $(gx, gy) \succeq (gu, gv) \succeq (gw, gz)$ all the conditions of Theorem 5 hold. So $(0, 0)$ is the unique common coupled fixed point of T and g. On the other side if we taking in the Case 3 $\alpha = \beta = \frac{1}{6}$ then Theorem 2 fail to satisfy.

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TRIANGULAR NORMS BASED ON INTUITIONISTIC FUZZY BCK-SUBMODULES

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Abstract: *We introduce the concept of intuitionistic fuzzy BCK-submodules of a BCK-module with respect to a t-norm and a s-norm and present some basic properties.*

Keywords : Intuitionistic fuzzy BCK-submodules, Triangular Norms, (Imaginable) Intuitionistic (T, S)-fuzzy BCK-submodules.

1. Introduction

The theory of fuzzy sets proposed by Zadeh [11] in 1965, and later on several researchers worked in this field. As a natural advancement of these research works we get one of the interesting generalizations of the theory of fuzzy sets that is the theory of intuitionstic fuzzy sets propounded by Atanassov [1, 2]. In 1966 Imai and Iseki [5] proposed the concept of BCK-algebra. Xi [10] applied the concept of fuzzy set to BCK-algebras. Also Bakhshi [3] in 2011 introduced the concept of fuzzy BCK-submodule of BCK-module and gave some related results. Recently, Badhurays and Bashammakh [4] considered the intuitionistic fuzzification of the concept of BCK-submodules in a BCK-module and investigated some properties of such BCK-modules. In this paper, we are going to introduce the notion of intuitionistic (T,S)-fuzzy BCK-submodules by using triangular norms, say T and S, and investigate several properties. We obtain some results on level sets of an intuitionistic (T,S)-fuzzy BCK-submodule by using the concept of level sets and triangular norms.

For the notations and terminology not given in this paper, the reader is referred to Atanassov [1, 2] (1986, 1994), Jun [8] (2001), Janiŝ [6] (2010), and Zadeh [11] (1965).

2. Preliminaries

First we present the fundamental definitions.

Definition 2.1. (Imai and Iseki [5]) a BCK-algebra is a set X with a binary operation $*$ and a constant 0 satisfying the following axioms :
(BCK1) $((x * y) \star (x * z)) \star (z * y) = 0$

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(BCK2) $(x * (x * y)) * y = 0$,
(BCK3) $x * x = 0$,
(BCK4) $0 * x = 0$,
(BCK5) $x * y = 0$ and $y * x = 0$ imply that $x = y$,
for all $x, y, z \in X$.
A partial ordering "$\leq$" is defined on X by $x \leq y$ iff $x * y = 0$.

Definition 2.2. (Zadeh [11]) By a fuzzy set μ in a nonempty set X we mean a function $\mu \colon X \longmapsto [0,1]$, and the complement of μ denoted by $\bar{\mu}$ is the fuzzy set in X given by $\bar{\mu}(x) = 1 - \mu(x)$ for all $x \in X$.

Definition 2.3. (Atanassov [1]) An intuitionistic fuzzy set *(IFS)* in a universe X is an object of the form
$$A = \{(x, \mu_A(x), \lambda_A(x)) | x \in X\},$$
where the functions $\mu \colon X \longmapsto [0,1]$ and $\lambda \colon X \longmapsto [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\lambda_A(x)$) of each element $x \in X$ to the set A respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$. For the sake of simplicity, we shall use the symbol $A = (\mu_A(x), \lambda_A(x))$ for the *IFS*
$$A = \{(x, \mu_A(x), \lambda_A(x)) | x \in X\}$$

Definition 2.4. (Atanassov [1]) Let X be a non-empty set and $A = (\mu_A(x), \lambda_A(x))$, $B = (\mu_B(x), \lambda_B(x))$ be *IFS*'s of X. Then
(1) $A \subset B$ iff $\mu_A(x) < \mu_B(x)$ and $\lambda_A(x) > \lambda_B(x)$ for all $x \in X$.
(2) $A = B$ iff $\mu_A(x) = \mu_B(x)$ and $\lambda_A(x) = \lambda_B(x)$ for all $x \in X$
(3) $A^C = (\lambda_A, \mu_A)$.
(4) $A \cap B = \{x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\lambda_A(x), \lambda_B(x)\} : x \in X\}$.
(5) $A \cup B = \{x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\lambda_A(x), \lambda_B(x)\} : x \in X\}$.
(6) $\square A = \{(x, \mu_A(x), \bar{\mu_A}(x)) | x \in X\}$.
(7) $\Diamond A = \{(x, \bar{\lambda_A}(x), \lambda_A(x)) | x \in X\}$.

Definition 2.5. (Atanassov [1]) Let $A = (\mu_A(x), \lambda_A(x))$ be an intuitionistic fuzzy set in M and let $\alpha \in [0,1]$. Then the sets
$$U(\mu_A, \alpha) = \{x \in M : \mu_A(x) \geq \alpha\},$$
$$L(\lambda_A, \alpha) = \{x \in M : \lambda_A(x) \leq \alpha\}$$
are called a μ-level α-cut and a λ-level α-cut of A, respectively.

Theorem 2.1. (Bakhshi [3]) *Let X be a bounded implicative BCK-algebra. Then $(X, +, 0)$ is an X-module where " + " is defined as $x + y = (x \star y) \vee (y \star x)$ and $xy = x \wedge y$.*

Theorem 2.2. (Bakhshi [3]) *A subset A of a BCK-module M is a BCK-submodule of M iff $a - b, xa \in A$, for every $a, b \in A$ and $x \in X$.*

Definition 2.6. (Bakhshi [3]) A fuzzy subset A of M is said to be a fuzzy *BCK*-submodule if for all $m, m_1, m_2 \in M$ and $x \in X$, the following axioms hold :
(1) $A(m_1 + m_2) \geq \min\{A(m_1), A(m_2)\}$

(2) $A(m) = A(-m)$

(3) $A(xm) \geq A(m)$

Definition 2.7. (Badhurays and Bashammakh [4]) An intuitionistic fuzzy subset $A = (\mu_A(x), \lambda_A(x))$ of M is said to be an intuitionistic fuzzy BCK-submodule of M if for all $m, m_1, m_2 \in M$ and $x \in X$, the following axioms hold :

(1) $\mu_A(m_1 + m_2) \geq \min\{\mu_A(m_1), \mu_A(m_2)\}$,
  $\lambda_A(m_1 + m_2) \leq \max\{\lambda_A(m_1), \lambda_A(m_2)\}$.

(2) $\mu_A(m) = \mu_A(-m), \lambda_A(m) = \lambda_A(-m)$,

(3) $\mu_A(xm) \geq \mu_A(m), \lambda_A(xm) \leq \lambda_A(m)$.

Definition 2.8 (Klir and Yuan [9]) a triangular norm (or t-norm) T is a mapping $T \colon [0,1] \times [0,1] \longmapsto [0,1]$, which satisfies the following axioms for every $x, y, z, \in [0,1]$:

(T1) $T(x, 1) = x$ (boundary condition);

(T2) $y \leq z$ implies $T(x, y) \leq T(x, z)$ (monotonicity);

(T3) $T(x, y) = T(y, x)$ (commutativity);

(T4) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity).

Definition 2.9. (Klir and Yuan [9]) a triangular conorm (or t-conorm) S is a mapping $S \colon [0,1] \times [0,1] \longmapsto [0,1]$, which satisfies the following axioms for every $x, y, z, \in [0,1]$:

(S1) $S(x, 0) = x$ (boundary condition);

(S2) $y \leq z$ implies $S(x, y) \leq S(x, z)$ (monotonicity);

(S3) $S(x, y) = S(y, x)$ (commutativity);

(S4) $S(x, S(y, z)) = S(S(x, y), z)$ (associativity).

Both t-norm and s-norm are called triangular norms. For all $\alpha, \beta \in [0,1]$, It is clear that

$$T(\alpha, \beta) \leq \min\{\alpha, \beta\} \leq \max\{\alpha, \beta\} \leq S(\alpha, \beta).$$

Definition 2.10. (Jun and Hong [7]) For a t-norm T and a s-norm S, we use the symbols Δ_T and Δ_S as the sets :

$$\Delta_T = \{a \in [0,1] | T(a, a) = a\},$$
$$\Delta_S = \{a \in [0,1] | S(a, a) = a\},$$

respectively.

Definition 2.11. (Jun and Hong [7]) We say that the intuitionistic fuzzy set $A = (\mu_A(x), \lambda_A(x))$ in M satisfies the imaginable property if

$$Im(\mu_A) \subseteq \Delta_T \text{ and } Im(\lambda_A) \subseteq \Delta_S.$$

Definition 2.12. (Klir and Yuan [9]) The norms T and S are called dual if and only if

D1) $\bar{T}(x, y) = S(\bar{x}, \bar{y})$,

D2) $\bar{S}(x, y) = T(\bar{x}, \bar{y})$ for all $x, y \in [0,1]$

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A few t-norms which are frequently encountered are T_l, T_m, and T_w defined by $T_l(a,b) = \max\{a+b-1, 0\}$ (Lukasiewicz), $T_m(a,b) = \min\{a,b\}$ (minimum) and

$$T_w(a,b) := \{ \begin{array}{ll} \min\{a,b\} & \text{if } a = 1 \text{ or } b = 1, \\ 0 & \text{otherwise (weak).} \end{array} \text{ ,}$$

A few s-norms which are frequently encountered are S_l, S_m, and S_w defined by $S_l(a,b) = \min\{a+b, 1\}$ (Lukasiewicz), $S_m(a,b) = \max\{a,b\}$ (maximum) andæ

$$S_w(a,b) := \{ \begin{array}{ll} \max\{a,b\} & \text{if } a = 0 \text{ or } b = 0, \\ 1 & \text{otherwise (strong).} \end{array} \text{ ,}$$

3. Intuitionistic (T, S)-fuzzy BCK-submodules

Throughout this paper, M is a BCK-module and T is a t-norm and S is a s-norm unless otherwise specified. we can extend the concept of the intuitionistic fuzzy BCK-submodules of M to the concept of intuitionistic (T, S)-fuzzy BCK-submodules in the following way:

Definition 3.1. Let T be a t-norm and S be a s-norm on $[0,1]$. An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in M is called an intuitionistic fuzzy BCK-submodule of M with respect to t-norm and s-norm (briefly, intuitionistic (T, S)-fuzzy BCK-submodule of M) if it satisfies the following conditions for all $m, m_1, m_2 \in M$:

(1) $\mu_A(m_1 + m_2) \geq T\{\mu_A(m_1), \mu_A(m_2)\}$,
   $\lambda_A(m_1 + m_2) \leq S\{\lambda_A(m_1), \lambda_A(m_2)\}$.
(2) $\mu_A(m) = \mu_A(-m), \lambda_A(m) = \lambda_A(-m)$,
(3) $\mu_A(xm) \geq \mu_A(m), \lambda_A(xm) \leq \lambda_A(m)$.

Example 3.2. Let $X = \{0, 1, 2, 3\}$ and consider the following table:

*	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

Then $(X, *)$ is a BCK-module over itself. Define a fuzzy set $\mu_A : M \longmapsto [0,1]$ by $\mu(0) = 0.5$, $\mu(m) = 0.3$, $m \in M$ and $\lambda_A : M \longmapsto [0,1]$ by $\lambda_A(0) = 0.3$, $\lambda_A(m) = 0.5$, $m \in M$. Let $T_l : [0,1] \times [0,1] \longmapsto [0,1]$ be a function defined by $T_l(a,b) = \max(a+b-1, 0)$ for all $a, b \in [0,1]$ and let $S_l : [0,1] \times [0,1] \longmapsto [0,1]$ be a function defined by $S_l(a,b) = \min(a+b, 1)$ for all $a, b \in [0,1]$. Then T_l is a t-norm and S_l is a s-norm. By routine calculations, we know that $A = (\mu_A(x), \lambda_A(x))$ is an intuitionistic (T_l, S_l)-fuzzy BCK-submodule of M.

Theorem 3.3. *An intuitionistic fuzzy subset A of M is an intuitionistic (T, S)-fuzzy BCK-submodule of M if and only if*

(1) $\mu_A(m_1 - m_2) \geq T\{\mu_A(m_1), \mu_A(m_2)\}$,
   $\lambda_A(m_1 - m_2) \leq S\{\lambda_A(m_1), \lambda_A(m_2)\}$.
(2) $\mu_A(xm) \geq \mu_A(m), \lambda_A(xm) \leq \lambda_A(m)$.

proof. Let A be an intuitionistic (T, S)-fuzzy BCK-submodule of M, then

$$\mu_A(m_1 - m_2) = \mu_A(m_1 + (-m_2))$$
$$\geq T(\mu_A(m_1), \mu_A(-m_2))$$
$$= T(\mu_A(m_1), \mu_A(m_2)),$$

Similarly, $\lambda_A(m_1 - m_2) \leq S(\lambda_A(m_1), \lambda_A(m_2))$. Condition 2 is hold by definition. Conversely suppose A satisfies 1 and 2. Then we have by 2

$$\mu_A(-m) = \mu_A((-1).m) \geq \mu_A(m),$$

and

$$\mu_A(m) = \mu_A((-1).(-1).m) \geq \mu_A(-m).$$

Thus $_A(m) = _A(-m)$. Similarly, $\lambda_A(m) = \lambda_A(-m)$.
Also we have

$$\mu_A(m_1 + m_2) = \mu_A(m_1 - (-m_2))$$
$$\geq T(\mu_A(m_1), \mu_A(-m_2))$$
$$\geq T(\mu_A(m_1), \mu_A(m_2))$$

Similarly,

$$\lambda_A(m_1 + m_2) \leq S(\lambda_A(m_1), \lambda_A(m_2)).$$

Thus A is an intuitionistic (T, S)-fuzzy BCK-submodule of M.

Proposition 3.4. *Let T and S be dual norms. If $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M, then so is $\Box A = (\mu_A, \overline{\mu}_A)$.*

Proof. For all $m_1, m_2 \in M$, we have

$$T(\mu_A(m_1), \mu_A(m_2)) \leq \mu_A(m_1 + m_2)$$

and so

$$T(1 - \overline{\mu}_A(m_1), 1 - \overline{\mu}_A(m_2)) \leq 1 - \overline{\mu}_A(m_1 + m_2)$$

hence

$$1 - T(1 - \overline{\mu}_A(m_1), 1 - \overline{\mu}_A(m)) \geq 1 - (1 - \overline{\mu}_A(m_1 + m_2))$$

which implies

$$\overline{T}(1 - \overline{\mu}_A(m_1), 1 - \overline{\mu}_A(m_2)) \geq \overline{\mu}_A(m_1 + m_2)$$

since T and S are dual, we get

$$S(\overline{\mu}_A(m_1), \overline{\mu}_A(m_2)) \geq \overline{\mu}_A(m_1 + m_2) ,$$

Moreover $\mu_A(m) = \mu_A(-m)$ imply that

$$1 - \mu_A(m) = 1 - \mu_A(-m),$$

Thus $\overline{\mu}_A(m) = \overline{\mu}_A(-m)$. Now, let $m \in M$ and $x \in X$, since μ_A is T-fuzzy BCK-submodule of M, we have $\mu_A(x.m) \geq \mu_A(m)$. Hence $1 - \mu_A(x.m) \leq 1 - \mu_A(m)$ which implies $\overline{\mu}_A(xm) \leq \overline{\mu}_A(m)$. Therefore $\Box A = (\mu_A, \overline{\mu}_A)$ is an intuitionistic (T, S) - fuzzy BCK-submodule of M.

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Proposition 3.5. *Let T and S be dual norms. If $A = (\mu_A, \lambda_A)$ is an intuitionistic (T,S)-fuzzy BCK- submodule of M, then so is $\diamond A = (\overline{\lambda}_A, \lambda_A)$.*

Proof. For all $m_1, m_2 \in M$, we have

$$S(\lambda_A(m_1), \lambda_A(m_2)) \geq \lambda_A(m_1 + m_2)$$

and so

$$S(1 - \overline{\lambda}_A(m_1), 1 - \overline{\lambda}_A(m_2)) \geq 1 - \overline{\lambda}_A(m_1 + m_2)$$

hence

$$1 - S(1 - \overline{\lambda}_A(m_1), 1 - \overline{\lambda}_A(m_2)) \leq 1 - (1 - \overline{\lambda}_A(m_1 + m_2))$$

which implies

$$1 - S(\overline{\overline{\lambda}}_A(m_1), \overline{\overline{\lambda}}_A(m_2)) \leq \overline{\lambda}_A(m_1 + m_2)$$

since T and S are dual

$$1 - \overline{T}(\overline{\lambda}_A(m_1), \overline{\lambda}_A(m_2)) \leq \overline{\lambda}_A(m_1 + m_2)$$

that is

$$T(\overline{\lambda}_A(m_1), \overline{\lambda}_A(m_2)) \leq \overline{\lambda}_A(m_1 + m_2).$$

Moreover

$$\overline{\lambda}_A(m) = \overline{\lambda}_A(-m)$$

imply that $1 - \lambda_A(m) = 1 - \lambda_A(-m)$, Thus $\lambda_A(m) = \lambda_A(-m)$. Now, let $m \in M$ and $x \in X$, since λ_A is T-fuzzy BCK-submodule of M we have $\lambda_A(x.m) \leq \lambda_A(m)$. Hence $1 - \lambda_A(x.m) \geq 1 - \lambda_A(m)$ which implies $\overline{\lambda}_A(xm) \geq \overline{\lambda}_A(m)$. Therefore $\diamond A = (\overline{\lambda}_A, \lambda_A)$ is an intuitionistic (T,S) - fuzzy BCK-submodule of M.

Combining the above two Propositions it is not difficult to verify that the following theorem is valid.

Theorem 3.6. *Let T and S be dual norms. Then $A = (\mu_A, \lambda_A)$ is an intuitionistic (T,S)-fuzzy BCK-submodule of M if and only if $\square A$ and $\diamond A$ are intuitionistic (T,S)-fuzzy BCK-submodule of M.*

Corollary 3.7. *Let T and S be dual norms. Then $A = (\mu_A, \lambda_A)$ is an intuitionistic (T,S)-fuzzy BCK-submodule of M if and only if μ_A and $\overline{\lambda_A}$ are T-fuzzy BCK-submodule of M.*

From corollary 3.7 we immediately obtain the following result.

Theorem 3.8. *An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ is an intuitionistic (T_m, S_m)- fuzzy BCK- submodule of M if and only if the fuzzy sets μ_A and $\overline{\lambda}_A$ are fuzzy BCK-submodule of M.*

Theorem 3.9. *An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ is an intuitionistic (T_m, S_m)-fuzzy BCK- submodule of M if and only if $\square A = (\mu_A, \overline{\mu}_A)$ and $\diamond A = (\overline{\lambda}_A, \lambda_A)$ are intuitionistic (T_m, S_m)-fuzzy BCK- submodule of M.*

Proof. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic (T_m, S_m)-fuzzy BCK-submodule of M. By Theorem 3.8, we get $\mu_A = \overline{\overline{\mu}}_A$ and $\overline{\lambda}_A$ are fuzzy BCK-submodule of M.

Therefore $\Box A = (\mu_A, \overline{\mu}_A)$ and $\Diamond A = (\overline{\lambda}_A, \lambda_A)$ are intuitionistic (T_m, S_m)-fuzzy BCK-submodule of M. Conversely, assume that $A = (\mu_A, \lambda_A)$ and $\Box A = (\mu_A, \overline{\mu}_A)$ and $\Diamond A = (\overline{\lambda}_A, \lambda_A)$ are intuitionistic (T_m, S_m)-fuzzy BCKsubmodule of M. Then the fuzzy sets μ_A and $\overline{\lambda}_A$ are fuzzy BCK-submodule of M. Therefore $A = (\mu_A, \lambda_A)$ is an intuitionistic (T_m, S_m)-fuzzy BCK- submodule of M.

Definition 3.10. An intuitionistic (T, S)-fuzzy BCK-submodule of M is called an imaginable intuitionistic (T, S)-fuzzy BCK-submodule of M if it satisfies the imaginable property.

Proposition 3.11. *Every imaginable intuitionistic (T, S)-fuzzy BCK-submodule of M is an intuitionistic fuzzy BCK-submodule of M.*

Proof. Let $A = (\mu_A, \lambda_A)$ be an imaginable intuitionistic (T, S)-fuzzy BCK-submodule of M. Then

$$\mu_A(m_1 + m_2) \geq T(\mu_A(m_1), \mu_A(m_2))$$

and

$$\lambda_A(m_1 + m_2) \leq S(\lambda_A(m_1), \lambda_A(m_2))$$

for all $m_1, m_2 \in M$.
Since $A = (\mu_A, \lambda_A)$ is imaginable, we have

$$\begin{aligned}
&\min\{\mu_A(m_1), \mu_A(m_2)\} \\
&= T(\min\{\mu_A(m_1), \mu_A(m_2)\}, \min\{\mu_A(m), \mu_A(m_2)\}) \\
&\leq T(\mu_A(m_1), \mu_A(m_2)) \\
&\leq \min\{\mu_A(m_1), \mu_A(m_2)\},
\end{aligned}$$

and

$$\begin{aligned}
&\max\{\lambda_A(m_1), \lambda_A(m_2)\} \\
&= S(\max\{\lambda_A(m_1), \lambda_A(m_2)\}, \max\{\lambda_A(m), \lambda_A(m_2)\}) \\
&\geq S(\lambda_A(m_1), \lambda_A(m_2)) \\
&\geq \max \lambda_A(m_1), \lambda_A(m_2).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mu_A(m_1 - m_2) \\
&\geq T(\mu_A(m_1), \mu_A(m_2)) \\
&= \min\{\mu_A(m_1), \mu_A(m_2)\},
\end{aligned}$$

and

$$\begin{aligned}
&\lambda_A(m_1 - m_2) \\
&\leq S(\lambda_A(m_1), \lambda_A(m_2)) \\
&= \max\{\lambda_A(m_1), \lambda_A(m_2)\}.
\end{aligned}$$

Now let $x \in X$ and $m \in M$. Since $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M, we have $\mu_A(xm) \geq \mu_A(m)$, $\lambda_A(xm) \leq \lambda_A(m)$. Therefore $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M.

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Note that every intuitionistic fuzzy BCK-submodule is an intuitionistic (T, S)-fuzzy BCK-submodule but the converse is not true as seen in the following Example.

Example 3.12. We consider the BCK-module M which is given in Example 3.2. Define an intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in M

$$\mu_A(m) = \begin{cases} 0.2 & \text{if} & m = 1 \\ 0.3 & \text{if} & m = 2, 3 \\ 0.5 & \text{if} & m = 0 \end{cases} \quad ; \quad \lambda_A(m) = \begin{cases} 0.5 & \text{if} & m = 1 \\ 0.3 & \text{if} & m = 2, 3 \\ 0.1 & \text{if} & m = 0 \end{cases}$$

Then $A = (\mu_A, \lambda_A)$ is an intuitionistic (T_w, S_w)-fuzzy BCK-submodule of M, but it is not an intuitionistic fuzzy BCK-submodule of M since

$$\mu_A(2 + 3) = \mu_A(1) = 0.2 < 0.3 = \min(\mu_A(2), \mu_A(3)).$$

Proposition 3.13. *If an intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in M is an imaginable intuitionistic (T, S)-fuzzy BCK-submodule of M, then for all $m \in M$, $\mu_A(0) \geq \mu_A(m)$ and $\lambda_A(0) \leq \lambda_A(m)$.*

Proof. From Definition 3.1 (3) it follows that

$$\mu_A(0) = \mu_A(0.m) \geq \mu_A(m)$$

and

$$\lambda_A(0) = \lambda_A(0.m) \leq \lambda_A(m)$$

for all $m \in M$.

Theorem 3.14. *If $A = (\mu_A, \lambda_A)$ is an imaginable intuitionistic (T, S)-fuzzy BCK-submodule of M, then the set $H = \{m \in M | \mu(m) = \mu(0)\}$ and $K = \{m \in M | \lambda_A(m) = \lambda_A(0)\}$ are BCK-submodule of M.*

Proof. Assume that $A = (\mu_A, \lambda_A)$ is an imaginable intuitionistic (T, S)-fuzzy BCK-submodule of M, and let $m_1, m_2 \in M$. Since $A = (\mu_A, \lambda_A)$ is an imaginable intuitionistic (T, S)-fuzzy BCK-submodule of M, we have

$$\mu_A(m_1 - m_2) \geq T(\mu_A(m), \mu_A(m))$$
$$= T(\mu_A(0), \mu_A(0))$$
$$= \mu_A(0)$$

for all $m_1, m_2 \in M$, Using Lemma Proposition 3.11., we get $\mu_A(m_1 - m_2) = \mu_A(0)$. Hence $m_1 - m_2 \in H$. Now let $x \in X$ and $m \in M$. Since $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M, we have $\mu_A(x.m) \geq \mu_A(m) = \mu_A(0)$. Using Lemma Proposition 3.11., we get $\mu_A(x.m) = \mu_A(0)$ and so $x.m \in H$. Therefore H is a BCK-submodule of M. By similar method, we get K is a BCK-submodule of M.

Definition 3.15. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set in BCK-submodule M and let $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$. Then the set

$$A_{(\alpha, \beta)} := \{m \in M | \mu_A(m) \geq \alpha, \lambda_A(m) \leq \beta\}$$

is called an (α, β)-level set of $A = (\mu_A, \lambda_A)$.

Theorem 3.16. *Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set in M such that*

$A_{(\alpha,\beta)}$ is a BCK-submodule of M , for all $(\alpha, \beta) \in [0,1]$ with $\alpha + \beta \leq 1$. Then $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M .

Proof. Let $m_1, m_2, m \in M$ and $x \in X$ be such that $A(m_1) = (\alpha_1, \beta_1)$, $A(m_2) = (\alpha_2, \beta_2)$ where $\alpha_i + \beta_i \leq 1$ for $i = 1, 2$. Then $m_1, m_2 \in A_{(\min(\alpha_1,\alpha_2),\max(\beta_1,\beta_2))}$, and so $m_1 - m_2 \in A_{\min(\alpha_1,\alpha_2),\max(\beta_1,\beta_2))}$.
Hence

$$\mu_A(m_1 - m_2) \geq \min(\alpha_1, \alpha_2) \geq T(\alpha_1, \alpha_2),$$

and

$$\lambda_A(m_1 - m_2)) \leq \max(\beta_1, \beta_2) \leq S(\beta_1, \beta_2).$$

Also, if we put $s' = {}_A(m), t' = {}_A(m)$ where $s' + t' \leq 1$. Then $m \in A_{(s',t')}$. Since $A_{(s',t')}$ is a BCK- submodule of M, we have $xm \in A_{(s',t')}$. It follows that

$$\mu_A(xm) \geq s' = \mu_A(m)$$

and

$$\lambda_A(xm) \leq t' = \lambda_A(m)$$

Therefore $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M.

The following Example shows that the converse of Theorem 3.16 is not true.

Example 3.17. We consider the intuitionistic (T_w, S_w)-fuzzy BCK-submodule A of M which is given in Example 3.2. Then $A_{(0.3,0.5)} = \{2, 3, 0\}$ is not BCK-submodule of M since $2 + 3 = 1 \notin A_{(0.3,0.5)}$

Theorem 3.18. If $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M, then $A_{(1,0)}$ is either empty or a BCK-submodule of M .

Proof. Let $m_1, m_2 \in A_{(1,0)}$. Then $\mu_A(m_1) \geq 1$, $\mu_A(m_2) \geq 1$, $\lambda_A(m_1) \leq 0$ and $\lambda_A(m_2) \leq 0$. It follows from Definitions 2.10 and Theorem 3.3 that

$$\mu_A(m_1 - m_2) \geq T(\mu_A(m_1), \mu_A(m_2)) \geq T(1, 1) = 1$$

and

$$\lambda_A(m_1 - m_2) \leq S(\lambda_A(m_1), \lambda_A(m_2)) \leq S(0, 0) = 0,$$

so $m_1 - m_2 \in A_{(1,0)}$. Let $m \in A_{(1,0)}$ and $x \in X$. Then

$$\mu_A(xm) \geq \mu_A(m) \geq 1$$

and

$$\lambda_A(xm) \leq \lambda_A(m) \leq 0,$$

so $xm \in A_{(1,0)}$.

As a generalization of Theorem 3.18, we get the following Theorem.

Theorem 3.19. If $A = (\mu_A, \lambda_A)$ is an imaginable intuitionistic (T, S)-fuzzy BCK-submodule of M, then $A_{(\alpha,\beta)}$ is either empty or a BCK-submodule of M for all $\alpha \in \Delta_T$ and $\beta \in \Delta_S$. with $\alpha + \beta \leq 1$.

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Proof. Let $m_1, m_2 \in A_{(\alpha,\beta)}$ where $\alpha \in \Delta_T$, $\beta \in \Delta_S$ and $\alpha + \beta \leq 1$. Then

$$\mu_A(m_1 - m_2)$$
$$\geq T(\mu_A(m_1), \mu_A(m_2))$$
$$\geq T(\alpha, \alpha) = \alpha$$

and

$$\lambda_A(m_1 - m_2)$$
$$\leq S(\lambda_A(m_1), \lambda_A(m_2))$$
$$\leq S(\beta, \beta) = \beta,$$

and so $m_1 - m_2 \in A_{(\alpha,\beta)}$. Let $m \in A_{(\alpha,\beta)}$ and $x \in X$. Then

$$\mu_A(xm) \geq \mu_A(m) \geq \alpha$$

and

$$\lambda_A(xm) \leq \lambda_A(m) \leq \beta,$$

so $xm \in A_{(\alpha,\beta)}$. Hence $A_{(\alpha,\beta)}$ is a BCK-submodule of M .

Proposition 3.20. (Bakhshi [3]) *A fuzzy set in M is a fuzzy BCK-submodule of M if and only if the non-empty $U(\mu, \alpha)$, $\alpha \in [0,1]$ is a BCK-submodule of M.*

By the above Proposition , we get the following result.

Corollary 3.21. *If $A = (\mu_A, \lambda_A)$ is an imaginable intuitionistic fuzzy set in M. Then $A = (\mu_A, \lambda_A)$ is an intuitionistic (T,S)-fuzzy BCK-submodule of M if and only if the non-empty sets $U(\mu, \alpha)$ and $L(\lambda, \alpha)$ are BCK-submodules of M, for every $(\alpha, \beta) \in [0,1]$.*

From corollary 3.21 we immediately obtain the following Theorem.

Theorem 3.22. *Let T be the minimum t-norm and let S the maximum s-norm dual of T . Then an intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ of M is is an intuitionistic (T,S)-fuzzy BCK-submodule of M if and only if*

$$A_{(\alpha,\beta)} := \{m \in M | \mu_A(m) \geq \alpha, \lambda_A(m) \leq \beta\}$$

is a BCK-submodule of M, where $(\alpha, \beta) \in [0,1]$.

Proposition 3.23. *Let S be a non-empty subset of a BCK-module M. Then an intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ defined by*

$$\mu_A(m) = \begin{cases} 1 & \text{if} \quad m \in S, \\ \alpha & \text{otherwise.} \end{cases}, \lambda_A(m) = \begin{cases} 0 & \text{if} \quad m \in S, \\ \beta & \text{otherwise.} \end{cases}$$

where $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$ and $\alpha + \beta \leq 1$ is an intuitionistic (T,S)-fuzzy BCK-submodule of M if and only if S is a BCK-submodule of M.

Proof. Let S be a BCK-submodule of M. Let $m_1, m \in M$. If $m_1, m_2 \in S$,

then $m_1 - m_2 \in S$, and so

$$\mu_A(m_1 - m_2) = 1 \geq 1$$
$$= T(1, 1)$$
$$= T(\mu_A(m_1), \mu_A(m_2))$$

and

$$\lambda_A(m_1 - m_2) = 0$$
$$= S(0, 0)$$
$$= S(\lambda_A(m_1), \lambda_A(m_2))$$

For $m_1 \in S$, $m_2 \notin S$, we have

$$\mu_A(m_1 - m_2) = \alpha \geq \alpha$$
$$= T(1, \alpha)$$
$$= T(\mu_A(m_1), \mu_A(m_2))$$

and

$$\lambda_A(m_1 - m_2) = \beta \leq \beta$$
$$= S(0, \beta)$$
$$= S(\lambda_A(m_1), \lambda_A(m_2))$$

Similarly, for the case $m_1 \notin S$, $m_2 \in S$, we have

$$\mu_A(m_1 - m_2) \geq T(\mu_A(m_1), \mu_A(m_2))$$

and

$$\lambda_A(m_1 - m_2) \leq S(\lambda_A(m_1), \lambda_A(m_2)).$$

For $m_1 \notin S$, $m_2 \notin S$,

$$\mu_A(m_1 - m_2) \geq \alpha$$
$$= T(1, \alpha)$$
$$\geq T(\alpha, \alpha)$$
$$= T(\mu_A(m_1), \mu_A(m_2)),$$

and

$$\lambda_A(m_1 - m_2) \leq \beta$$
$$= S(0, \beta)$$
$$\leq S(\beta, \beta)$$
$$= S(\lambda_A(m_1), \lambda_A(m_2)).$$

Thus for all cases,

$$\mu_A(m_1 - m_2) \geq T(\mu_A(m_1), \mu_A(m_2))$$

and

$$\lambda_A(m_1 - m_2) \leq S(\lambda_A(m_1), \lambda_A(m_2)).$$

Next, let $m \in M$ and $x \in X$, Then, if $m \in S$ then $xm \in S$ and so,

$$\mu_A(xm) = 1 \geq 1 = \mu_A(m)$$

and

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$$\lambda_A(xm) = 0 \leq 0 = \lambda_A(m).$$

If $m \notin S$, then

$$\mu_A(xm) \geq \alpha = {}_A(m)$$

and

$$\lambda_A(xm) \leq \beta = \lambda_A(m).$$

Therefore $\mu_A(xm) \geq \mu_A(m)$ and $\lambda_A(xm) \leq \lambda_A(m)$. Thus $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M .

Conversely, we assume $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M. Let $m_1, m_2 \in S$, $x \in X$. Then,

$$\mu_A(m_1 - m_2) \geq T(\mu_A(m_1), \mu_A(m_2)) = T(1, 1) = 1,$$

hence $\mu_A(m_1 - m_2) = 1$. Thus $m_1 - m_2 \in S$. Also, $\mu_A(xm) \geq \mu_A(m) = 1$ implies $\mu_A(xm) = 1$ implies $xm \in S$. Hence, S is a BCK-submodule of M .

Corollary 3.24. *Let S be a non-empy subset of a BCK-module M and let χ_s be the characteristic function of S. Then $A = (\chi_s, \chi_s^c)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M if and only if S is a BCK-submodule of M.*

Definition 3.25. (Janiŝ [6]) Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set of X and let T be a t-norm. Then $A_{T,\alpha}$ is a subset of X defined by

$$A_{T,\alpha} = \{x \in X | T(\mu_A(x), 1 - \lambda_A(x)) \geq \alpha\},$$

for every $\alpha \in [0, 1]$

Theorem 3.26. *Let T and S be dual norms. If $A = (\mu_A, \lambda_A)$ is an intuitionistic (T, S)-fuzzy BCK-submodule of M. Then*

$$A_{T,1} = \{m \in M | T(\mu_A(m), 1 - \lambda_A(m)) = 1\}$$

is a BCK-submodule of M.

Proof. Let $m_1, m_2 \in A_{T,1}$. Then,

$$T(\mu_A(m_1 - m_2), 1 - {}_A(m_1 - m_2))$$
$$\geq T(T(\mu_A(m_1), {}_A(m_2)), 1 - S({}_A(m_1), {}_A(m_2)))$$
$$= T(T(\mu_A(m_2), (\mu_A(m_1)), T(1 - \lambda_A(m_1), 1 - \lambda_A(m_2)))$$
$$= T(\mu_A(m_2), T(\mu_A(m_1), T(1 - \lambda_A(m_1), 1 - \lambda_A(m_2))))$$
$$= T(\mu_A(m_2), T(T(\mu_A(m_1), 1 - \lambda_A(m_1)), 1 - \lambda_A(m_2)))$$
$$= T(\mu_A(m_2), T(1 - \lambda_A(m_2), T(\mu_A(m_1), 1 - \lambda_A(m_1))))$$
$$= T(T(\mu_A(m_2), 1 - \lambda_A(m_2)), T(\mu_A(m_1), 1 - \lambda_A(m_1)))$$
$$= T(1, 1) = 1$$

Thus, we have $T(\mu_A(m_1 - m_2), 1 - \lambda_A(m_1 - m_2)) = 1$ Therefore $m_1 - m_2 \in A_{T,1}$. Also, let $x \in X$ and $m \in A_{T,1}$. Then $T(\mu_A(m), 1 - \lambda_A(m)) = 1$. Further, $T(\mu_A(xm), 1 - \lambda_A(xm)) \geq T(\mu_A(m), 1 - \lambda_A(m)) = 1$. Therefore $xm \in A_{T,1}$. Hence, $A_{T,1}$ is a is a BCK-submodule of M.

For any tringular norm T, the level set $A_{T,\alpha}$ of an intuitionistic (T, S)-fuzzy BCK-submodule of M is not necessarily to be a BCK-submodule of M. However, if T is the minimum tringular norm, then all level sets $A_{T,\alpha}$ of an intuitionistic (T, S)-fuzzy BCK-submodule of M are BCK-submodules of M.

Theorem 3.27. *Let $A = (\mu_A, \lambda_A)$ be an intuitionistic (T_m, S_m)-fuzzy BCK-submodule of M such that T_m, S_m are dual. Then for every $\alpha \in [0, 1]$,*
$$A_{T_m,\alpha} = \{m \in M | T(\mu_A(m), 1 - \lambda_A(m)) \geq \alpha\}$$
is a BCK-submodule of M.

Proof. Let $A = (\mu_A(x), \lambda_A(x))$ is an intuitionistic (T_m, S_m)-fuzzy BCK-submodule of M. Let $m_1, m_2 \in A_{T,}$. Then,

$$T_m(\mu_A(m_1 - m_2), 1 - {}_A(m_1 - m_2))$$
$$\geq T_m(T_m(\mu_A(m_1), \mu_A(m_2)), 1 - S_m(\lambda_A(m_1), \lambda_A(m_2)))$$
$$= T_m(T_m(\mu_A(m_2), (\mu_A(m_1)), T_m(1 - \lambda_A(m_1), 1 - \lambda_A(m_2)))$$
$$= T_m(\mu_A(m_2), T_m(\mu_A(m_1), T_m(1 - \lambda_A(m_1), 1 - \lambda_A(m_2))))$$
$$= T_m(\mu_A(m_2), T_m(T_m(\mu_A(m_1), 1 - \lambda_A(m_1)), 1 - \lambda_A(m_2)))$$
$$= T_m(\mu_A(m_2), T_m(1 - \lambda_A(m_2), T_m(\mu_A(m_1), 1 - \lambda_A(m_1))))$$
$$= T_m(T_m(\mu_A(m_2), 1 - \lambda_A(m_2)), T_m(\mu_A(m_1), 1 - \lambda_A(m_1)))$$
$$\geq T_m(\alpha, \alpha) = \alpha$$

Thus, we have

$$T_m(\mu_A(m_1 - m_2), 1 - {}_A(m_1 - m_2)) \geq \alpha$$

Therefore, $m_1 - m_2 \in A_{T_m,\alpha}$. Also, let $x \in X$ and $m \in A_{T_m,\alpha}$. Then
$$T_m(\mu_A(m), 1 - \lambda_A(m)) \geq \alpha$$

Further,

$$T_m(\mu_A(xm), 1 - \lambda_A(xm)) \geq T_m(\mu_A(m), 1 - \lambda_A(m)) \geq \alpha$$

Therefore we have $T_m(\mu_A(xm), 1 - \lambda_A(xm)) \geq \alpha$. Hence $xm \in A_{T_m,\alpha}$. Thus $A_{T_m,\alpha}$ is a is a BCK-submodule of M.

Definition 3.28. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy set of X, let T and S be dual norms. Then $A_{T,S,\alpha}$ is a subset of X defined by
$$A_{T,S,1} = \{x \in XT(\mu_A(x), S(\mu_A(x), \lambda_A(x))) \geq \alpha\}$$
for every $\alpha \in [0, 1]$.

Theorem 3.29. *Let $A = (\mu_A, \lambda_A)$ be an intuitionistic (T, S)-fuzzy BCK-submodule of M, then*
$$A_{T,S,1} = \{m \in M | T(\mu_A(m), S(\mu_A(m), \lambda_A(m))) = 1\}$$
is a BCK-submodule of M.

Proof. Let $A = (\mu_A, \lambda_A)$ be an intuitionistic (T, S)-fuzzy BCK-submodule of M. Let $m_1, m_2 \in A_{T,S,1}$, then
$$T(\mu_A(m_1), S(\mu_A(m_1), \lambda_A(m_1))) = 1$$

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and

$$T(\mu_A(m_2), S(\mu_A(m_2), \lambda_A(m_2))) = 1.$$

Therefore $\mu_A(m_1) \geq 1$ and $\mu_A(m_2) \geq 1$ which mean that $\mu_A(m_1) = 1$ and $\mu_A(m_2) = 1$. From monotonicity of T, we have,

$$
\begin{aligned}
& T(\mu_A(m_1 - m_2), S(\mu_A(m_1 - m_2), \lambda_A(m_1 - m_2))) \\
& \geq T(T(\mu_A(m_1 - m_2)), T(\mu_A(m_1 - m_2))) \\
& \geq T(T(\mu_A(m), \mu_A(m)), T(\mu_A(m), \mu_A(m))) \\
& = T(T(1,1), T(1,1)) \\
& = T(1,1) = 1
\end{aligned}
$$

Therefore, $T(\mu_A(m_1 - m_2), S(\mu_A(m_1 - m_2), \lambda_A(m_1 - m_2))) = 1$ implies $m_1, m_2 \in A_{T,S,1}$. Also, let $x \in X$ and $m \in A_{T,S,1}$. Then, $T(\mu_A(m), S(\mu_A(m), \lambda_A(m))) = 1$. which impliese $\mu_A(m) = 1$. Now,

$$
\begin{aligned}
& T(\mu_A(xm), S(\mu_A(xm), \lambda_A(xm))) \\
& \geq T(\mu_A(xm), \mu_A(xm)) \\
& \geq T(\mu_A(m), \mu_A(m)) \\
& = T(1,1) = 1
\end{aligned}
$$

Thus, we have, $T(\mu_A(xm), S(\mu_A(xm), \lambda_A(xm))) = 1$. Therefore, $xm \in A_{T,S,1}$. Hence, $A_{T,S,1}$ is a BCK-submodule of M.

Theorem 3.30. *Let $A = (\mu_A, \lambda_A)$ be an intuitionistic (T_m, S_m)-fuzzy BCK-submodule of M such that T_m, S_m are dual. Then for every $\alpha \in [0,1]$,*

$$A_{T,S,\alpha} = \{m \in M | T(\mu_A(m), S(\mu_A(m), \lambda_A(m))) \geq \alpha\}$$

is a BCK-submodule of M.

Proof. Let $A = (\mu_A, \lambda_A)$ is an intuitionistic (T_m, S_m)-fuzzy BCK-submodule of M. Let $m_1, m_2 \in A_{T,S,\alpha}$, then

$$T_m(\mu_A(m_1), S_m(\mu_A(m_1), \lambda_A(m_1))) \geq \alpha$$

and

$$T_m(\mu_A(m_2), S_m(\mu_A(m_2), \lambda_A(m_2))) \geq \alpha.$$

Therefore $\mu_A(m_1) \geq \alpha$ and $\mu_A(m_2 \geq \alpha$. Due monotonicity of T_m, we have,

$$
\begin{aligned}
& T_m(\mu_A(m_1 - m_2), S_m(\mu_A(m_1 - m_2), \lambda_A(m_1 - m_2))) \\
& \geq T_m(\mu_A(m_1 - m_2)), (\mu_A(m_1 - m_2))) \\
& = \mu_A(m_1 - m_2) \\
& \geq T_m(\mu_A(m_1), \mu_A(m_2)) \\
& \geq T_m(\alpha, \alpha) \\
& = \alpha
\end{aligned}
$$

Therefore, $T_m(\mu_A(m_1 - m_2), S_m(\mu_A(m_1 - m_2), \lambda_A(m_1 - m_2))) \geq \alpha$ and hence $m_1 - m_2 \in A_{T_m,S_m,\alpha}$. Also, let $m \in A_{T_m,S_m,\alpha}$ and $x \in X$. Then,

$$T_m(\mu_A(m), S_m(\mu_A(m), \lambda_A(m))) \geq \alpha.$$

which impliese $\mu_A(m) \geq \alpha$. From monotonicity of T_m, we have,

$$T_m(\mu_A(xm), S_m(\mu_A(xm), \lambda_A(xm))$$
$$\geq T_m(\mu_A(xm), \mu_A(xm))$$
$$= \mu_A(xm)$$
$$\geq \mu_A(m)$$
$$\geq \alpha$$

Thus $T_m(\mu_A(xm), S_m(\mu_A(xm), \lambda_A(xm)) \geq \alpha$. Therefore, $xm \in A_{T_m,S_m,\alpha}$. Hence, $A_{T_m,S_m,\alpha}$ is a BCK-submodule of M.

4. Conclusion

One of the generalizations of fuzzy BCK-submodules, namely, intuitionistic (T,S)-fuzzy BCK- submodules was defined and some properties of intuitionistic (T,S)-fuzzy BCK-submodules are investigated. Also, some related results on level sets of an intuitionistic (T,S)-fuzzy BCK-submodule are investigated. These investigations of generalized fuzzy on BCK-modules could be enable us to discuss further study in this field.

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On strongly almost generalized difference lacunary ideal convergent sequences of fuzzy numbers

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Abstract

The purpose of this paper is to introduce some new sequence spaces of fuzzy numbers defined by lacunary ideal convergence using generalized difference matrix and Orlicz functions. We also study some algebraic and topological properties of these classes of sequences. Moreover, some illustrative examples are given in support of our results.

Keywords and phrases: Ideal convergence; fuzzy number; difference sequence; Orlicz function; lacunary sequence.

AMS subject classification (2010): 40A05; 40C05; 40G15; 06B99.

1 Introduction and preliminaries

The concept of ideal convergence is the dual (equivelant) to the notion of filter convergence introduced by Cartan [4]. The filter convergence is a generalization of the classical notion of convergence of sequences of real or complex numbers and it has been an important tool in the study of functional analysis. Nowadays many authors studied this notion from various aspects and applied this notion to various problems arising in the convergence theory. Kostyrko et al. [13] and Nuray and Ruckle [23] independently studied in detalis about the notion of ideal convergence which is based upon the structure of the admissible ideal I of subsets $\mathbb{N}$ of natural numbers. Later on it was further investigated by many authors, e.g. Tripathy and Hazarika [26], Mursaleen and Mohiuddine [22] and references therein.

Let S be a non-empty set. Then a non empty class $I \subseteq P(S)$ is said to be an *ideal* on S if and only if (i) $\phi \in I$; (ii) I is additive; (iii) hereditary. An ideal $I \subseteq P(S)$ is said to be *non trivial* if $I \neq \phi$ and $S \notin I$. A non-empty family of sets $F \subseteq P(S)$ is said to be a *filter* on S if and only if (i) $\phi \notin F$ (ii) for each $A, B \in F$ we have $A \cap B \in F$; (iii) for each $A \in F$ and each $B \supset A$, we have $B \in F$. For each ideal I, there is a filter $F(I)$ corresponding to I i.e. $F(I) = \{K \subseteq S : K^c \in I\}$, where $K^c = S - K$. We say that a non-trivial ideal $I \subseteq P(S)$ is an *admissible ideal* on S if and only if it contains all singletons, i.e. if it contains $\{\{s\} : s \in S\}$. Recall that a sequence $x = (x_k)$ of points in $\mathbb{R}$ is said to be I-convergent to the number ℓ (denoted by $I\text{-}\lim x_k = \ell$) if for every $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\} \in I$.

We used the standard notation $\theta = (k_r)$ to denote the *lacunary sequence*, where θ is a sequence of positive integers such that $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r := k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $J_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ ($r \neq 1$) by q_r (see [8]).

The notion of lacunary ideal convergence for sequences of real numbers and fuzzy numbers, respectively, has been defined and studied in [27] and [9]. Let $I \subset 2^{\mathbb{N}}$ be a non-trivial ideal. A real sequence

$x = (x_k)$ is said to be *lacunary I-convergent* to $L \in \mathbb{R}$, in symbol we shall write I_θ-$\lim x = L$, if for every $\varepsilon > 0$, the set

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} |x_k - L| \geq \varepsilon \right\} \in I.$$

Throughout the paper we use w to denotes the set of all real sequences $x = (x_k)$. The difference sequence spaces have been introduced by Kızmaz [12] by using the difference operator Δ as follows:

$$Z(\Delta) = \{(x_k) \in w : \Delta x_k \in Z\},$$

for $Z = \ell_\infty, c, c_0$ and $\Delta x_k = \Delta^1 x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$, where the standard notations ℓ_∞, c and c_0 are used to denote the set of bounded, convergent and null sequences, respectively. Later this idea was generalized by Et and Çolak [6] by considering Δ^n instead of Δ, where $(\Delta^n x_k) = \Delta^1(\Delta^{n-1} x_k)$ for $n \geq 2$ and all $k \in \mathbb{N}$. In case of $n = 0$ we obtain x_k. Tripathy et al. [28] presented another generalization of difference sequence spaces by introducing the operator Δ_m^n and is given by $\Delta_m^n x = (\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m})$ so that $\Delta_m^n x_k$ has the following binomial representation:

$$\Delta_m^n x_k = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} x_{k+m\nu},$$

for all $k \in \mathbb{N}$. If we take $n = 1$, then $Z(\Delta_m^n)$ is reduced to $Z(\Delta_m)$ which was introduced by Tripathy and Esi [25], in this case the operator $\Delta_m x$ is given by $\Delta_m x = (\Delta_m x_k) = (x_k - x_{k+m})$ for all $k, m \in \mathbb{N}$. The choice of $m = 1$ in the definition of $Z(\Delta_m^n)$ gives us the difference sequence spaces introduced by Et and Colak [6]. Başar and Altay [1] introduced the generalized difference matrix $B(r,s) = (b_{nk}(r,s))$ by

$$b_{nk}(r,s) = \begin{cases} r, & \text{if } k = n; \\ s, & \text{if } k = n-1; \\ 0, & \text{if } 0 \leq k < n-1 \text{ or } k > n. \end{cases}$$

for all $k, n \in \mathbb{N}$ and all non-zero real numbers r, s. The generalized difference matrix B^n of order n has been recently defined by Başarir and Kayikçi [2] and its binomial representation is given by

$$B^n x_k = \sum_{\nu=0}^{n} \binom{n}{\nu} r^{n-\nu} s^\nu x_{k-\nu},$$

for all $n \in \mathbb{N}$ and $r, s \in \mathbb{R} - \{0\}$. Another generalization of above difference matrix was given by Başarir et al. [3] as $B_{(m)}^n$, where $B_{(m)}^n x = (B_{(m)}^n x_k) = (r B_{(m)}^{n-1} x_k + s B_{(m)}^{n-1} x_{k-m})$ and $B_{(m)}^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation:

$$B_{(m)}^n x_k = \sum_{\nu=0}^{n} \binom{n}{\nu} r^{n-\nu} s^\nu x_{k-m\nu}.$$

In [24], Orlicz introduced functions nowadays called Orlicz functions and constructed the sequence space (L^M). Krasnoselskii and Rutitsky further investigated the Orlicz space in [14]. Some recent related work we refer to Mohiuddine et al. [19, 20]. A function $M : [0, \infty) \to [0, \infty)$ is said to be an *Orlicz function* if it is non-decreasing, continuous, convex with $M(0) = 0$, $M(x) > 0$ as $x > 0$ and $M(x) \to \infty$ as $x \to \infty$ (see [24]). It is well known that if M is a convex function and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$ for all $\lambda \in (0, 1)$. An Orlicz function M is said to be satisfy Δ_2-*condition* for all values of u, if there exists a constant $K > 0$ such that $M(Lu) \leq KLM(u)$ for all values of $L > 1$ (see, Krasnoselskii and Rutitsky [14]).

Lindenstrauss and Tzafriri [16] introduced the sequence space ℓ_M by using the notion of Orlicz function by

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

and proved that this space is a Banach space with the norm

$$||x|| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

Every space ℓ_M contains a subspace isomorphic to the classical sequence space ℓ_p for some $1 \leq p < \infty$. The space ℓ_p, $1 \leq p < \infty$ is itself an Orlicz sequence space with $M(t) = |t|^p$.

A sequence space E is said to be (i) *normal* (or *solid*) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequence (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$, (ii) *symmetric* if $(x_{\pi(k)}) \in E$, whenever $(x_k) \in E$, where π is a permutation of $\mathbb{N}$.

Let E be a sequence space and $K = \{k_1 < k_2 < ...\} \subseteq \mathbb{N}$. A sequence space of the form $\lambda_K^E = \{(x_{k_n}) \in w : (k_n) \in E\}$ is called a *K-step space* of E. A canonical preimage of a sequence $(x_{k_n}) \in \lambda_K^E$ is a sequence $(y_k) \in w$ and is defined by

$$y_k = \begin{cases} x_k, & \text{if } k \in K \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of canonical pre-images of all elements in λ_K^E. We say that E is *monotone* if E contains the canonical pre-image of all its step spaces. Note that every normal space is monotone (see [11], pp. 53).

A sequence $x = (x_k) \in \ell_\infty$ (the space of bounded sequences) is said to be *almost convergent*, denoted by $\widehat{c}$, if all of its Banach limits coincide. Lorentz [17] introduced this sequence space as follows:

$$\widehat{c} = \left\{ x \in \ell_\infty : \lim_k t_{jk}(x) \text{ exists uniformly in } j \right\},$$

where

$$t_{jk}(x) = \frac{x_j + x_{j+1} + ... + x_{j+k}}{k+1}.$$

It is clear that

$$t_{jk}(x) = \begin{cases} \frac{1}{k}\sum_{i=1}^{k} x_{j+i} & \text{for } k \geq 1; \\ x_j & \text{for } k = 0. \end{cases}$$

Zadeh [29] introduced the concept of fuzzy set theory and its applications can be found in many branches of mathematical and engineering sciences including management science, control engineering, computer science, artificial intelligence. Matloka [18] introduced the bounded and convergent sequences of fuzzy numbers and proved that every convergent sequence of fuzzy numbers is bounded. Later, various classes of sequences of fuzzy numbers have been defined and studied by Colak et al. [5], Et et al. [7], Mursaleen and Başarir [21], Hazarika [10] and references therein.

Now recalling some notions of fuzzy numbers which we will used to prove our main results. Throughout the paper we used w^F, ℓ_∞^F, c^F and c_0^F to denote the set of all, bounded, convergent and null sequence spaces of fuzzy numbers, respectively. A fuzzy number X is a fuzzy subset of the real line $\mathbb{R}$ i.e., a mapping $X : \mathbb{R} \to J(= [0,1])$ associating each real number t with its grade of membership $X(t)$. A fuzzy number X is said to be (i) *upper-semi continuous* if for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon))$ for all $a \in [0,1]$ is

open in the usual topology of $\mathbb{R}$, (ii) *convex* if $X(t) \geq X(s) \wedge X(r) = \min\{X(s), X(r)\}$ for $s < t < r$ (iii) *normal* if there exists $t_0 \in \mathbb{R}$ such that $X(t_0) = 1$.

We used the notation X^α to denotes α-level set of a fuzzy number $X, 0 < \alpha \leq 1$ and is given by $X^\alpha = \{t \in \mathbb{R} : X(t) \geq \alpha\}$. The set of all normal, convex and upper semi-continuous fuzzy number with compact support will be denoted by $\mathbb{R}(J)$ and the fuzzy number we mean that the number belongs to $\mathbb{R}(J)$. We used the symbol D to denote the set of all closed and bounded intervals $X = [x_1, x_2]$ on $\mathbb{R}$. For any two sets $X, Y \in D$, we define $X \leq Y$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$. A metric d on D is given by $d(X, Y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$. It is easy to see that (D, d) is a complete metric space. Also, the relation $\leq$ is a partial order on D.

The absolute value $|X|$ of $X \in \mathbb{R}(J)$ is given by

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{if } t > 0, \\ 0, & \text{if } t < 0. \end{cases}$$

Suppose that $\bar{d} : \mathbb{R}(J) \times \mathbb{R}(J) \to \mathbb{R}$ is a mapping such that $\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha)$. Then $(\mathbb{R}(J), \bar{d})$ is a complete metric space.

We define $X \leq Y$ if and only if $X^\alpha \leq Y^\alpha$, for all $\alpha \in J$. By $\bar{0}$ and $\bar{1}$ we denotes the additive and multiplicative identities in $\mathbb{R}(J)$, respectively.

A sequence $u = (u_k)$ of fuzzy numbers is said to be (i) *bounded* if the set $\{u_k : k \in \mathbb{N}\}$ of fuzzy numbers is bounded, (ii) *convergent* to a fuzzy number u_0 if for every $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $\bar{d}(u_k, u_0) < \varepsilon$, for all $k \geq n_0$, (iii) *I-convergent* (see [15]) if there exists a fuzzy number u_0 such that for each $\varepsilon > 0$, the set $\{k \in \mathbb{N} : \bar{d}(u_k, u_0) \geq \varepsilon\} \in I$. We write I-lim $u_k = u_0$, (iv) *I-bounded* if there exists $K > 0$ such that the set $\{k \in \mathbb{N} : \bar{d}(u_k, \bar{0}) \geq K\} \in I$.

2 Main results

Throughout the article we assume that I is an admissible ideal of $\mathbb{N}$. In this section, we introduce the following definitions. We introduce some new strongly almost ideal convergent sequence spaces using the generalized difference matrix $B^n_{(m)}$ and Orlicz function M. Let us consider a sequence $p = (p_k)$ of positive real numbers and let m, n be any nonnegative integers. For some $\rho > 0$, we define the following sequence spaces.

$$[\widehat{w}_0^{IF}(M, \theta, B^n_{(m)}, p)] = \left\{(u_k) \in w^F : \left\{r \in \mathbb{N} : \frac{1}{h_r}\right.\right.$$

$$\left.\left. \times \sum_{k \in J_r} \left[M\left(\frac{\bar{d}(t_{jk}(B^n_{(m)} u_k), \bar{0})}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I, \text{uniformly in } j \in \mathbb{N}\right\}$$

$$[\widehat{w}^{IF}(M, \theta, B^n_{(m)}, p)] = \left\{(u_k) \in w^F : \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k \in J_r}\left[M\left(\frac{\bar{d}(t_{jk}(B^n_{(m)} u_k), u_0)}{\rho}\right)\right]^{p_k} \geq \varepsilon\right\} \in I,\right.$$

$$\left. \text{uniformly in } j \in \mathbb{N} \text{ and for some } u_0 \in \mathbb{R}(J)\right\}$$

$$[\widehat{w}_\infty^F(M, \theta, B^n_{(m)}, p)] = \left\{(u_k) \in w^F : \sup_r \frac{1}{h_r}\right.$$

$$\times \sum_{k \in J_r} \left[M \left(\frac{\overline{d}(t_{jk}(B^n_{(m)} u_k), \overline{0})}{\rho} \right) \right]^{p_k} < \infty, \text{ uniformly in } j \in \mathbb{N} \Bigg\}$$

$$[\widehat{w}^{IF}_\infty(M, \theta, B^n_{(m)}), p)] = \Bigg\{ (u_k) \in w^F : \exists K > 0 \ s.t. \Bigg\{ r \in \mathbb{N} : \frac{1}{h_r}$$

$$\times \sum_{k \in J_r} \left[M \left(\frac{\overline{d}(t_{jk}(B^n_{(m)} u_k), \overline{0})}{\rho} \right) \right]^{p_k} \geq K \Bigg\} \in I, \text{ uniformly in } j \in \mathbb{N} \Bigg\}.$$

Particular cases:

(i) If $p = (p_k) = 1$ for all $k \in \mathbb{N}$, we denote $[\widehat{w}^{IF}_0(M, \theta, B^n_{(m)}), p)] = [\widehat{w}^{IF}_0(M, \theta, B^n_{(m)})]$, $[\widehat{w}^{IF}(M, \theta, B^n_{(m)}), p)] = [\widehat{w}^{IF}(M, \theta, B^n_{(m)})]$, $[\widehat{w}^F_\infty(M, \theta, B^n_{(m)}), p)] = [\widehat{w}^F_\infty(M, \theta, B^n_{(m)})]$ and $[\widehat{w}^{IF}_\infty(M, \theta, B^n_{(m)}), p)] = [\widehat{w}^{IF}_\infty(M, \theta, B^n_{(m)})]$.

(ii) If $M(x) = x$, we denote $[\widehat{w}^{IF}_0(M, \theta, B^n_{(m)}), p)] = [\widehat{w}^{IF}_0(\theta, B^n_{(m)}), p)]$, $[\widehat{w}^{IF}(M, \theta, B^n_{(m)}), p)] = [\widehat{w}^{IF}(\theta, B^n_{(m)}), p)]$, $[\widehat{w}^F_\infty(M, \theta, B^n_{(m)}), p)] = [\widehat{w}^F_\infty(\theta, B^n_{(m)}), p)]$ and $[\widehat{w}^{IF}_\infty(M, \theta, B^n_{(m)}), p)] = [\widehat{w}^{IF}_\infty(\theta, B^n_{(m)}), p)]$.

(iii) If $\theta = (2^r)$, we denote $[\widehat{w}^{IF}_0(M, \theta, B^n_{(m)}), p)] = [\widehat{w}^{IF}_0(M, B^n_{(m)}), p)]$, $[\widehat{w}^{IF}(M, \theta, B^n_{(m)}), p)] = [\widehat{w}^{IF}(M, B^n_{(m)}), p)]$, $[\widehat{w}^F_\infty(M, \theta, B^n_{(m)}), p)] = [\widehat{w}^F_\infty(M, B^n_{(m)}), p)]$ and $[\widehat{w}^{IF}_\infty(M, \theta, B^n_{(m)}), p)] = [\widehat{w}^{IF}_\infty(M, B^n_{(m)}), p)]$.

Throughout the manuscript, we will used the following well-known inequality. Suppose that $p = (p_k)$ is a sequence of positive real numbers with $0 < p_k \leq \sup_k p_k = H$, $D = \max\{1, 2^{H-1}\}$. Then

$$|a_k + b_k|^{p_k} \leq D(|a_k|^{p_k} + |b_k|^{p_k}) \text{ for all } k \in \mathbb{N} \text{ and } a_k, b_k \in \mathbb{C}.$$

Also $|a|^{p_k} \leq \max\{1, |a|^H\}$ for all $a \in \mathbb{C}$.

Now we are ready to give our main results as follows.

Theorem 2.1. *Let* $p = (p_k)$ *be a bounded sequence of positive real numbers. The spaces* $[\widehat{w}^{IF}_0(M, \theta, B^n_{(m)}), p)]$, $[\widehat{w}^{IF}(M, \theta, B^n_{(m)}), p)]$, $[\widehat{w}^F_\infty(M, \theta, B^n_{(m)}), p)]$, *and* $[\widehat{w}^{IF}_\infty(M, \theta, B^n_{(m)}), p)]$ *are closed with respect to addition and scalar multiplication.*

Proof. We prove the result only for the space $[\widehat{w}^{IF}(M, \theta, B^n_{(m)}), p)]$. The others can be treated similarly. Let $u = (u_k)$ and $v = (v_k)$ be two elements of $[\widehat{w}^{IF}(M, \theta, B^n_{(m)}), p)]$ and α_1, α_2 be scalars. Let $\varepsilon > 0$ be given. Then there exist positive numbers ρ_1, ρ_2 such that

$$P = \Bigg\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M \left(\frac{\overline{d}(t_{jk}(B^n_{(m)} u_k), u_0)}{\rho_1} \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \Bigg\} \in I \qquad (\text{uniformly in } j \in \mathbb{N})$$

and

$$Q = \Bigg\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M \left(\frac{\overline{d}(t_{jk}(B^n_{(m)} v_k), v_0)}{\rho_2} \right) \right]^{p_k} \geq \frac{\varepsilon}{2} \Bigg\} \in I \qquad (\text{uniformly in } j \in \mathbb{N}).$$

Let $\rho_3 = \max(2|\alpha_1|\rho_1, 2|\alpha_2|\rho_2)$. Since M is non-decreasing and convex function, we have

$$\frac{1}{h_r} \sum_{k \in J_r} \left[M \left(\frac{\overline{d}(t_{jk}(B^n_{(m)}(\alpha_1 u_k + \alpha_2 v_k)), \alpha_1 u_0 + \alpha_2 v_0)}{\rho_3} \right) \right]^{p_k}$$

$$\leq \frac{1}{h_r} \sum_{k \in J_r} \left[M \left(\frac{\alpha_1 \overline{d}(t_{jk}(B^n_{(m)} u_k), u_0)}{\rho_3} \right) \right]^{p_k} + \frac{1}{h_r} \sum_{k \in J_r} \left[M \left(\frac{\alpha_2 \overline{d}(t_{jk}(B^n_{(m)} v_k), v_0)}{\rho_3} \right) \right]^{p_k}$$

$$\leq \frac{1}{h_r} \sum_{k \in J_r} \left[M \left(\frac{\overline{d}(t_{jk}(B^n_{(m)} u_k), u_0)}{\rho_1} \right) \right]^{p_k} + \frac{1}{h_r} \sum_{k \in J_r} \left[M \left(\frac{\overline{d}(t_{jk}(B^n_{(m)} v_k), v_0)}{\rho_2} \right) \right]^{p_k},$$

uniformly in j. Therefore, we have

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M \left(\frac{\overline{d}(t_{jk}(B^n_{(m)} (\alpha_1 u_k + \alpha_2 v_k)), \alpha_1 u_0 + \alpha_2 v_0)}{\rho_3} \right) \right]^{p_k} \geq \varepsilon \right\} \subseteq P \cup Q \in I.$$

uniformly in j. This yields $(\alpha_1 u + \alpha_2 v) \in [\widehat{w}^{IF}(M, \theta, B^n_{(m)}, p)]$. This completes the proof. $\square$

Theorem 2.2. *Let M_1 and M_2 be two Orlicz functions. Then*

(i) $[Z(M_2, \theta, B^n_{(m)}, p)] \subseteq [Z(M_1 M_2, \theta, B^n_{(m)}, p)]$.

(ii) $[Z(M_1, \theta, B^n_{(m)}, p)] \cap Z(M_2, \theta, B^n_{(m)}, p)] \subseteq [Z(M_1 + M_2, \theta, B^n_{(m)}, p)]$,

where $Z = \widehat{w}^{IF}_0, \widehat{w}^{IF}, \widehat{w}^{IF}_\infty, \widehat{w}^F_\infty$.

Proof. (i) Let $u = (u_k) \in [\widehat{w}^{IF}(M_2, \theta, B^n_{(m)}, p)]$ and let $\varepsilon > 0$ be given. For some $\rho > 0$, we have

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M_2 \left(\frac{\overline{d}(t_{jk}(B^n_{(m)} u_k), u_0)}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in I, \qquad (2.1)$$

uniformly in $j \in \mathbb{N}$. Choose λ with $0 < \lambda < 1$ such that $M_1(t) < \varepsilon$ for $0 \leq t \leq \lambda$. We define

$$v_k = \frac{\overline{d}(t_{jk}(B^n_{(m)} u_k), u_0)}{\rho}$$

and consider

$$\lim_{k \in \mathbb{N}; 0 \leq v_k \leq \lambda} [M_1(v_k)]^{p_k} = \lim_{k \in \mathbb{N}; v_k \leq \lambda} [M_1(v_k)]^{p_k} + \lim_{k \in \mathbb{N}; v_k > \lambda} [M_1(v_k)]^{p_k}.$$

Therefore, one obtains

$$\lim_{k \in \mathbb{N}; v_k \leq \lambda} [M_1(v_k)]^{p_k} \leq [M_1(2)]^H \lim_{k \in \mathbb{N}; v_k \leq \lambda} [v_k]^{p_k}, \qquad (H = \sup_k p_k). \qquad (2.2)$$

For the second summation (i.e. $v_k > \lambda$), we go through the following procedure. We have

$$v_k < \frac{v_k}{\lambda} < 1 + \frac{v_k}{\lambda}.$$

It follows from the fact that M_1 is convex and non-decreasing,

$$M_1(v_k) < M_1 \left(1 + \frac{v_k}{\lambda} \right) \leq \frac{1}{2} M_1(2) + \frac{1}{2} M_1 \left(\frac{2v_k}{\lambda} \right).$$

Since M_1 satisfies Δ_2-condition, we can write

$$M_1(v_k) < \frac{1}{2} K \frac{v_k}{\lambda} M_1(2) + \frac{1}{2} K \frac{v_k}{\lambda} M_1(2) = K \frac{v_k}{\lambda} M_1(2).$$

This yields the following estimates:

$$\lim_{k \in \mathbb{N}; v_k > \lambda} [M_1(v_k)]^{p_k} \leq \max \left\{ 1, (K \lambda^{-1} M_1(2))^H \right\} \lim_{k \in \mathbb{N}; v_k > \lambda} [v_k]^{p_k}. \qquad (2.3)$$

It follows from (2.1), (2.2) and (2.3) that

$$(u_k) \in [\widehat{w}^{IF}(M_1.M_2, \theta, B_{(m)}^n, p)].$$

Hence, $[\widehat{w}^{IF}(M_2, \theta, B_{(m)}^n, p)] \subseteq [\widehat{w}^{IF}(M_1.M_2, \theta, B_{(m)}^n, p)].$

(ii) Let $(u_k) \in [\widehat{w}^{IF}(M_1, \theta, B_{(m)}^n, p)] \cap [\widehat{w}^{IF}(M_2, \theta, B_{(m)}^n, p)].$ Let $\varepsilon > 0$ be given. Then there exists $\rho > 0$ such that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M_1 \left(\frac{\overline{d}(t_{jk}(B_{(m)}^n u_k), u_0)}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in I \qquad \text{(uniformly in } j \in \mathbb{N})$$

and

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M_2 \left(\frac{\overline{d}(t_{jk}(B_{(m)}^n u_k), u_0)}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in I \qquad \text{(uniformly in } j \in \mathbb{N}).$$

The rest of the proof follows from the following relation:

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[(M_1 + M_2) \left(\frac{\overline{d}(t_{jk}(B_{(m)}^n u_k), u_0)}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\}$$

$$\subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M_1 \left(\frac{\overline{d}(t_{jk}(B_{(m)}^n u_k), u_0)}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\}$$

$$\bigcup \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M_2 \left(\frac{\overline{d}(t_{jk}(B_{(m)}^n u_k), u_0)}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\}.$$

$\square$

Note that if we take $M_1(x) = M(x)$ and $M_2(x) = x$ for all $x \in [0, \infty)$ in the above theorem, then we obtain the following corollary:

Corollary 2.3. *One has* $[Z(\theta, B_{(m)}^n, p)] \subseteq [Z(M, \theta, B_{(m)}^n, p)],$ *where* $Z = \widehat{w}_0^{IF}, \widehat{w}^{IF}, \widehat{w}_\infty^{IF}, \widehat{w}_\infty^F.$

As in classical theory, the following is easy to prove.

Theorem 2.4. (a) *If* $M_1(x) \leq M_2(x)$ *for all* $x \in [0, \infty),$ *then* $[Z(M_1, \theta, B_{(m)}^n, p)] \subseteq [Z(M_2, \theta, B_{(m)}^n, p)]$ *for* $Z = \widehat{w}_0^{IF}, \widehat{w}^{IF}$ *and* $\widehat{w}_\infty^F.$

(b) *If* $n_1 < n_2$ *then* $[Z(\theta, B_{(m)}^{n_1}, p)] \subseteq [Z(\theta, B_{(m)}^{n_2}, p)]$ *for* $Z = \widehat{w}_0^{IF}, \widehat{w}^{IF}$ *and* $\widehat{w}_\infty^F.$

Theorem 2.5. *Let* M *be an Orlicz function. Then*

$$[\widehat{w}_0^{IF}(M, \theta, B_{(m)}^n, p)] \subset [\widehat{w}^{IF}(M, \theta, B_{(m)}^n, p)] \subset [\widehat{w}_\infty^F(M, \theta, B_{(m)}^n, p)]$$

and the inclusions are proper.

Proof. Suppose that $(u_k) \in [\widehat{w}^{IF}(M, \theta, B_{(m)}^n, p)].$ Let $\varepsilon > 0$ be given. Then there exists $\rho > 0$ such that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M \left(\frac{\overline{d}(t_{jk}(B_{(m)}^n u_k), u_0)}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in I.$$

Clearly,

$$M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k),\overline{0})}{\rho}\right) \le \frac{1}{2}M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k),u_0)}{\rho}\right) + \frac{1}{2}M\left(\frac{\overline{d}(u_0,\overline{0})}{\rho}\right).$$

Taking supremum over k on both sides of above inequalities implies that $(u_k) \in [\widehat{w}^F_\infty(M,\theta,B^n_{(m)},p)]$. Thus, we have $[\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)] \subset [\widehat{w}^F_\infty(M,\theta,B^n_{(m)},p)]$.

The inclusion $[\widehat{w}^{IF}_0(M,\theta,B^n_{(m)},p)] \subset [\widehat{w}^{IF}(M,\theta,B^n_{(m)},p)]$ is obvious.

We now show that the inclusion is strict in the above theorem by constructing the following illustrative example.

Example 2.1. Suppose that $\theta = (2^r)$ and $M(x) = x$ for all $x \in [0,\infty)$. Suppose also that $r = 1$, $s = -1$, $n = 1$, $m = 2$. Let us define the sequence (u_k) of fuzzy numbers by

$$u_k(t) = \begin{cases} \frac{6}{k}t + 1 & \text{if} \quad -\frac{k}{6} \le t \le 0; \\ -\frac{6}{k}t + 1 & \text{if} \quad 0 < t \le \frac{k}{6}; \\ 0 & , \quad \text{otherwise}, \end{cases}$$

where $k = 2^i$ $(i = 1,2,3,...)$, otherwise $u_k(t) = \overline{0}$. For $\alpha \in (0,1]$, the α-level sets of u_k and $B^1_{(m)}u_k$ are

$$[u_k]^\alpha = \begin{cases} [\frac{k}{6}(\alpha-1), \frac{k}{6}(1-\alpha)] & \text{if} \quad k = 2^i, i = 1,2,3,... \\ [0,0] & , \quad \text{otherwise} \end{cases}$$

and

$$[B^1_{(2)}u_k]^\alpha = \begin{cases} [\frac{1}{3}(\alpha-1), \frac{1}{3}(1-\alpha)] & \text{for} \quad k = 2^i \\ [0,0] & , \quad \text{otherwise}. \end{cases}$$

It is easy to prove that $-\frac{\overline{1}}{3} < [T_j]^\alpha < \frac{\overline{1}}{3}$ for $\alpha \in (0,1]$, where $[T_j]^\alpha = [t_{j,k}(B^1_{(2)}u_k)]^\alpha = [\frac{1}{j+1}\sum_{i=1}^j B^1_{(2)}u_k]^\alpha$. Because

$$[t_{j,k}(B^1_{(2)}u_k)]^\alpha = \begin{cases} \frac{1}{1+\frac{1}{j}}[\frac{1}{3}(\alpha-1), \frac{1}{3}(1-\alpha)] & \text{for} \quad k = 2^i; \ j \ge 1 \\ [0,0] & , \quad \text{otherwise} \end{cases}$$

and

$$[t_{j,k}(B^1_{(2)}u_k)]^\alpha = \begin{cases} [\frac{1}{3}(\alpha-1), \frac{1}{3}(1-\alpha)] & \text{if} \quad j = 0 \\ [0,0] & , \quad \text{otherwise}. \end{cases}$$

Thus (T_j) is I-bounded but not I-convergent. $\square$

Theorem 2.6. *The inclusions* $[Z(M,\theta,B^{n-1}_{(m)},p)] \subseteq [Z(M,\theta,B^n_{(m)},p)]$ *are strict for* $n \ge 1$. *In general* $[Z(M,\theta,B^i_{(m)},p)] \subseteq [Z(M,\theta,B^n_{(m)},p)]$ $(i = 1,2,...,n-1)$ *and the inclusion is strict, where* $Z = \widehat{w}^{IF}_0, \widehat{w}^{IF}, \widehat{w}^{IF}_\infty, \widehat{w}^F_\infty$.

Proof. Suppose that $u = (u_k) \in [\widehat{w}^{IF}_0(M,\theta,B^{n-1}_{(m)},p)]$. Let $\varepsilon > 0$ be given. Then there exists $\rho > 0$ such that

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k \in J_r}\left[M\left(\frac{\overline{d}(t_{jk}(B^{n-1}_{(m)}u_k),\overline{0})}{\rho}\right)\right]^{p_k} \ge \varepsilon\right\} \in I.$$

Since M is non-decreasing and convex it follows that

$$\left[M\left(\frac{\overline{d}(t_{jk}(B^n_{(m)}u_k),\overline{0})}{2\rho}\right)\right]^{p_k}$$

$$\leq \left[M \left(\frac{\overline{d}(t_{jk}(B^{n-1}_{(m)}u_k), t_{jk}(B^{n-1}_{(m)}u_{k+1}), \overline{0})}{2\rho} \right) \right]^{p_k}$$

$$\leq D \left[\frac{1}{2} M \left(\frac{\overline{d}(t_{jk}(B^{n-1}_{(m)}u_k), \overline{0})}{\rho} \right) \right]^{p_k} + D \left[\frac{1}{2} M \left(\frac{\overline{d}(t_{jk}(B^{n-1}_{(m)}u_{k+1}), \overline{0})}{\rho} \right) \right]^{p_k}$$

$$\leq DK \left[M \left(\frac{\overline{d}(t_{jk}(B^{n-1}_{(m)}u_k), \overline{0})}{\rho} \right) \right]^{p_k} + DK \left[M \left(\frac{\overline{d}(t_{jk}(B^{n-1}_{(m)}u_{k+1}), \overline{0})}{\rho} \right) \right]^{p_k},$$

where $K = \max\{1, \left(\frac{1}{2}\right)^H\}$. Therefore we have

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M \left(\frac{\overline{d}(t_{jk}(B^{n}_{(m)}u_k), \overline{0})}{2\rho} \right) \right]^{p_k} \geq \varepsilon \right\}$$

$$\subseteq \left\{ r \in \mathbb{N} : DK \frac{1}{h_r} \sum_{k \in J_r} \left[M \left(\frac{\overline{d}(t_{jk}(B^{n-1}_{(m)}u_k), \overline{0})}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\}$$

$$\bigcup \left\{ r \in \mathbb{N} : DK \frac{1}{h_r} \sum_{k \in J_r} \left[M \left(\frac{\overline{d}(t_{jk}(B^{n-1}_{(m)}u_{k+1}), \overline{0})}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\},$$

i.e.,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M \left(\frac{\overline{d}(t_{jk}(B^{n}_{(m)}u_k), \overline{0})}{2\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in I.$$

Hence, $(u_k) \in [\widehat{w}^{IF}_0(M, \theta, B^n_{(m)}, p)]$.

We now show that the inclusion is strict in the above theorem (Theorem 2.6) by constructing the following illustrative example.

Example 2.2. Let $\theta = (2^r)$ and $M(x) = x$ for all $x \in [0, \infty)$ Suppose also that $r = 1$, $s = -1$, $n = 2$, $m = 2$ and $p_k = 1$ for all $k \in \mathbb{N}$. We now define the sequence (u_k) of fuzzy numbers by

$$u_k(t) = \begin{cases} -\frac{t}{k^2-1} + 1 & , \quad \text{if } k^2 - 1 \leq t \leq 0; \\ -\frac{t}{k^2+1} + 1 & , \quad \text{if } 0 < t \leq k^2 + 1; \\ 0 & , \quad \text{otherwise.} \end{cases}$$

For $\alpha \in (0, 1]$, the α-level sets of u_k, $B^1_{(2)}u_k$ and $B^2_{(2)}u_k$ are as follow:

$$[u_k]^\alpha = [(1-\alpha)(k^2-1), (1-\alpha)(k^2+1)],$$

and

$$[B^1_{(2)}u_k]^\alpha = [(1-\alpha)(4k-6), (1-\alpha)(4k-2)],$$

$$[B^2_{(2)}u_k]^\alpha = [4(1-\alpha), 12(1-\alpha)].$$

It is easy to verified that the sequence $[B^1_{(2)}u_k]^\alpha$ is not I-convergent but $[B^2_{(2)}u_k]^\alpha$ is I-convergent. $\square$

Theorem 2.7. *Let* $0 < p_k \leq q_k < \infty$ *for each* k. *Then* $[Z(M, \theta, B^n_{(m)}, p)] \subseteq [Z(M, \theta, B^n_{(m)}, q)]$ *for* $Z = \widehat{w}^{IF}_0$ *and* $\widehat{w}^{IF}$.

Proof. Let $(u_k) \in [\widehat{w}_0^{IF}(M, \theta, B_{(m)}^n), p)$. Then there exists a number $\rho > 0$ such that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M\left(\frac{\overline{d}(t_{jk}(B_{(m)}^n u_k), \overline{0})}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in I \qquad \text{(uniformly in } j \in \mathbb{N}).$$

For sufficiently large k, since $p_k \leq q_k$ for each k, therefore we obtain

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M\left(\frac{\overline{d}(t_{jk}(B_{(m)}^n u_k), \overline{0})}{\rho} \right) \right]^{q_k} \geq \varepsilon \right\}$$

$$\subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \left[M\left(\frac{\overline{d}(t_{jk}(B_{(m)}^n u_k), \overline{0})}{\rho} \right) \right]^{p_k} \geq \varepsilon \right\} \in I,$$

uniformly in $j \in \mathbb{N}$, i.e. $(u_k) \in [\widehat{w}_0^{IF}(M, \theta, B_{(m)}^n), q)]$.

Similarly, we can show that $]\widehat{w}^{IF}(M, \theta, B_{(m)}^n), p)] \subseteq [\widehat{w}^{IF}(M, \theta, B_{(m)}^n), q)]$. □

Corollary 2.8. *(a) Let $0 < \inf_k p_k \leq p_k \leq 1$. Then $[Z(M, \theta, B_{(m)}^n), p)] \subseteq [Z(M, \theta, B_{(m)}^n)]$ for $Z = \widehat{w}_0^{IF}$ and $\widehat{w}^{IF}$.*

(b) Let $1 \leq p_k \leq \sup_k p_k < \infty$. Then $[Z(M, \theta, B_{(m)}^n)] \subseteq [Z(M, \theta, B_{(m)}^n), p)]$ for $Z = \widehat{w}_0^{IF}$ and $\widehat{w}^{IF}$.

Theorem 2.9. *If I is an admissible ideal and $I \neq I_f$, then the sequence spaces $[\widehat{w}_0^{IF}(M, \theta, B_{(m)}^n), p)]$ and $\widehat{w}^{IF}(M, \theta, B_{(m)}^n), p)]$ are neither normal nor monotone, where I_f denotes the class of all finite subsets of $\mathbb{N}$.*

Proof. To prove our result, we construct the following example.

Example 2.3. Suppose that $M(x) = x$ for all $x \in [0, \infty)$ and $r = 1$, $s = -1$, $n = 1$, $m = 1$. Consider that $I = I_\delta$, where $I_\delta = \{A \subset \mathbb{N} : \text{asymptotic density of } A \text{ (in symbol, } \delta(A)) = 0\}$ and note that I_δ is an ideal of $\mathbb{N}$, and $p_k = 1$ for all $k \in \mathbb{N}$. We now define the sequence (u_k) of fuzzy numbers by

$$u_k(t) = \begin{cases} 1 + t - k & , & \text{if } t \in [k-1, k]; \\ 1 - t + k & , & \text{if } t \in [k, k+1]; \\ 0 & , & \text{otherwise.} \end{cases}$$

Let us define

$$\alpha_k = \begin{cases} 1 & , & \text{if } k \text{ is odd}; \\ 0 & , & \text{if } k \text{ is even.} \end{cases}$$

Thus $(\alpha_k u_k) \notin [\widehat{w}_0^{IF}(M, \theta, B_{(m)}^n), p)]$ and $\widehat{w}^{IF}(M, \theta, B_{(m)}^n), p)]$. Therefore, we conclude that the spaces $[\widehat{w}_0^{IF}(M, \theta, B_{(m)}^n), p)]$ and $\widehat{w}^{IF}(M, \theta, B_{(m)}^n), p)]$ are not normal and hence these spaces are not monotone. □

Theorem 2.10. *If I is an admissible ideal and $I \neq I_f$, then the sequence space $[Z(M, \theta, B_{(m)}^n), p)]$ is not symmetric, where $Z = \widehat{w}_0^{IF}, \widehat{w}^{IF}$.*

Proof. We shall prove the result only for the space $[\widehat{w}^{IF}(M, \theta, B_{(m)}^n), p)]$ with the help of the following example. For other space, the proof is similar so we omitted.

Example 2.4. Suppose that $M(x) = x$ for all $x \in [0, \infty)$ and $r = 1$, $s = -1$, $n = 1$, $m = 1$. Let $I = I_\delta$ and $p_k = 1$ for all $k \in \mathbb{N}$. We now define the sequence (u_k) of fuzzy numbers by

$$u_k(t) = \begin{cases} t - 4k + 1 & , \quad \text{if } t \in [4k - 1, 4k]; \\ -t + 4k + 1 & , \quad \text{if } t \in [4k, 4k + 1]; \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Thus, we have $(u_k) \in [\widehat{w}^{IF}(M, \theta, B_{(m)}^n, p)]$. But the rearrangement (v_k) of (u_k) defined as

$$v_k = \{u_1, u_4, u_2, u_9, u_3, u_{16}, u_5, u_{25}, u_6, ...\}.$$

This implies that $(v_k) \notin [\widehat{w}^{IF}(M, \theta, B_{(m)}^n, p)]$. Hence $[\widehat{w}^{IF}(M, \theta, B_{(m)}^n, p)]$ is not symmetric. $\qquad \square$

3   Acknowledgement

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The Catalan Numbers: a Generalization, an Exponential Representation, and some Properties

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Abstract

In the paper, the authors establish an exponential representation for a function involving the gamma function and originating from investigation of the Catalan numbers in combinatorics, find necessary and sufficient conditions for the function to be logarithmically completely monotonic, introduce a generalization of the Catalan numbers, derive an exponential representation for the generalization, and present some properties of the generalization.

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1   Introduction

It is known [4, 21, 22] that, in combinatorics, the Catalan numbers C_n for $n \geq 0$ form a sequence of natural numbers that occur in tree enumeration problems such as "In how many ways can a regular n-gon be divided into $n-2$ triangles if different orientations are counted separately?" whose solution is the Catalan number C_{n-2}. Explicit formulas of C_n for $n \geq 0$ include

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{2^n(2n-1)!!}{(n+1)!} = \frac{1}{n}\binom{2n}{n-1} = {}_2F_1(1-n,-n;2;1) = \frac{4^n\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+2)}, \quad (1)$$

where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t$ for $\Re(z) > 0$ is the classical Euler gamma function and

$$\,_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) = \sum_{n=0}^\infty \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!} \qquad (2)$$

1

is the generalized hypergeometric series defined for $a_i \in \mathbb{C}$ and $b_i \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, for positive integers $p, q \in \mathbb{N}$, and in terms of the rising factorials $(x)_n = \prod_{k=0}^{n-1}(x+k)$. The asymptotic form for the Catalan function C_x is

$$C_x \sim \frac{4^x}{\sqrt{\pi}}\left(\frac{1}{x^{3/2}} - \frac{9}{8}\frac{1}{x^{5/2}} + \frac{145}{128}\frac{1}{x^{7/2}} + \cdots\right),$$

see [3, 4, 21, 22, 24]. Recently, among other things, the formula

$$C_n = (-1)^n \frac{2^n}{n!} \sum_{k=0}^{n} \frac{1}{2^k} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \prod_{m=0}^{n-1} (\ell - 2m) = \frac{2^n}{n!} \sum_{k=0}^{n} \frac{k!}{2^k} \binom{2n-k-1}{2(n-k)}[2(n-k)-1]!!$$

was found in [18, Theorem 3]. For more information on the Catalan numbers C_n, please refer to two monographs [2, 3] and references cited therein.

In the paper [20], motivated by the explicit expression (1), the authors established an integral representation of the Catalan function C_x for $x \geq 0$.

Theorem 1.1 ([20, Theorem 1]). *For $x \geq 0$, we have*

$$C_x = \frac{e^{3/2}4^x(x+1/2)^x}{\sqrt{\pi}(x+2)^{x+3/2}} \exp\left[\int_0^\infty \frac{1}{t}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)(e^{-t/2} - e^{-2t})e^{-xt}\,\mathrm{d}t\right]. \tag{3}$$

Recall from [8, Chapter XIII], [19, Chapter 1], and [25, Chapter IV] that an infinitely differentiable function f is said to be completely monotonic on an interval I if it satisfies $0 \leq (-1)^k f^{(k)}(x) < \infty$ on I for all $k \geq 0$. Recall from [11] that an infinitely differentiable and positive function f is said to be logarithmically completely monotonic on an interval I if $0 \leq (-1)^k[\ln f(x)]^{(k)} < \infty$ hold on I for all $k \in \mathbb{N}$. For more information on logarithmically completely monotonic functions, please refer to [14, 19].

The formula (3) can be rearranged as

$$\ln\left[\frac{\sqrt{\pi}(x+2)^{x+3/2}}{e^{3/2}4^x(x+1/2)^x}C_x\right] = \int_0^\infty \frac{1}{t}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)(e^{-t/2} - e^{-2t})e^{-xt}\,\mathrm{d}t. \tag{4}$$

Since the function $\frac{1}{t}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)$ is positive on $(0, \infty)$, the right-hand side of (4) is a completely monotonic function on $(0, \infty)$. This means that the function

$$\frac{(x+2)^{x+3/2}}{4^x(x+1/2)^x}C_x \tag{5}$$

is logarithmically completely monotonic on $(0, \infty)$. Because any logarithmically completely monotonic function must be completely monotonic, see [14, Eq. (1.4)] and references therein, the function (5) is also completely monotonic on $(0, \infty)$.

By virtue of (1), the function (5) can be rewritten as

$$\frac{(x+2)^{x+3/2}\Gamma(x+1/2)}{(x+1/2)^x\Gamma(x+2)}, \quad x > 0. \tag{6}$$

Hence, the logarithmically complete monotonicity of (5) implies the logarithmically complete monotonicity of (6). The function (6) is the special case $F_{1/2,2}(x)$ of the general function

$$F_{a,b}(x) = \frac{\Gamma(x+a)}{(x+a)^x}\frac{(x+b)^{x+b-a}}{\Gamma(x+b)}, \quad a, b \in \mathbb{R}, \quad a \neq b \quad x > -\min\{a, b\}. \tag{7}$$

2

We notice that the function $F_{a,b}(x)$ does not appear in the expository and survey articles [9, 14] and plenty of references therein. Therefore, it is significant to naturally pose an open problem below.

Open Problem 1.1 ([20, Open Problem 1]). *What are the necessary and sufficient conditions on $a, b \in \mathbb{R}$ such that the function $F_{a,b}(x)$ defined by (7) is (logarithmically) completely monotonic in $x \in (-\min\{a, b\}, \infty)$?*

This problem was answered in [6, Theorem 2] as follows.

Theorem 1.2 ([6, Theorem 2]). *The sufficient conditions on a, b such that the function $[F_{a,b}(x)]^{\pm 1}$ defined by (7) is logarithmically completely monotonic in $x \in (-\min\{a, b\}, \infty)$ are $(a, b) \in D_{\pm}(a, b)$, where*

$$D_{\pm}(a, b) = \{(a, b) : a \gtrless b, a \geq 1\} \cup \left\{(a, b) : a \lessgtr b, a \leq \frac{1}{2}\right\}.$$

The necessary conditions on a, b for the function $[F_{a,b}(x)]^{\pm 1}$ to be logarithmically completely monotonic in $x \in (-\min\{a, b\}, \infty)$ are $a(a - b) \gtrless \frac{a-b}{2}$.

The aims of this paper are to establish an exponential representation for the function $F_{a,b}(x)$, to find necessary and sufficient conditions on a, b for $[F_{a,b}(x)]^{\pm 1}$ to be logarithmically completely monotonic on $[0, \infty)$, to introduce a generalization of the Catalan numbers C_n, and to derive an exponential representation for the generalization of C_n.

The first main result in this paper can be stated as the following theorem.

Theorem 1.3. *For $a, b > 0$, the function $F_{a,b}(x)$ defined by (7) has the exponential representation*

$$F_{a,b}(x) = \exp\left[b - a + \int_0^\infty \frac{1}{t}\left(a + \frac{1}{t} - \frac{1}{1 - e^{-t}}\right)\left(e^{-bt} - e^{-at}\right)e^{-xt}\, \mathrm{d}\, t\right] \qquad (8)$$

on $[0, \infty)$ and the function $[F_{a,b}(x)]^{\pm 1}$ is logarithmically completely monotonic on $[0, \infty)$ if and only if $(a, b) \in D_{\pm}(a, b)$.

Comparing (3) with (8) hints and stimulates us to consider the three-variable function

$$C(a, b; z) = \frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^z \frac{\Gamma(z + a)}{\Gamma(z + b)}, \quad \Re(a), \Re(b) > 0, \quad \Re(z) \geq 0. \qquad (9)$$

Since $C\left(\frac{1}{2}, 2; n\right) = C_n$ for $n \geq 0$ is of the form (1), we can regard $C(a, b; x)$ as an analytical generalization of the Catalan numbers C_n. For uniqueness and convenience of referring to the quantity $C(a, b; x)$, we call $C(a, b; x)$ the Catalan–Qi function and, when taking $x = n \in \{0\} \cup \mathbb{N}$, call $C(a, b; n)$ the Catalan–Qi numbers.

By virtue of the integral representation (8) in Theorem 1.3, we immediately derive an integral representation for the Catalan–Qi function $C(a, b; x)$.

Theorem 1.4. *For $a, b > 0$ and $x \geq 0$, we have*

$$C(a, b; x) = \frac{\Gamma(b)}{\Gamma(a)}\left(\frac{b}{a}\right)^x \frac{(x + a)^x}{(x + b)^{x+b-a}}$$

$$\times \exp\left[b - a + \int_0^\infty \frac{1}{t}\left(\frac{1}{1 - e^{-t}} - \frac{1}{t} - a\right)\left(e^{-at} - e^{-bt}\right)e^{-xt}\, \mathrm{d}\, t\right]. \quad (10)$$

<center>3</center>

Remark 1.1. Can one give a combinatorial interpretation of the Catalan–Qi function $C(a, b; x)$ defined by (9) and its integral representation (10)?

In [22] and related references therein, the following simple properties of the Catalan numbers C_n are listed:

$$C_{n+1} = \frac{2(2n+1)}{n+2}C_n, \quad C_n = \frac{1}{(n+1)!}\prod_{k=1}^{n}(4k-2), \quad \sum_{n=1}^{\infty}\frac{C_n}{4^n} = 1, \tag{11}$$

$$\sum_{n=0}^{\infty}C_n\frac{x^{2n}}{(2n)!} = \frac{I_1(2x)}{x}, \quad e^{2x}\big[I_0(2x) - I_1(2x)\big] = \sum_{n=0}^{\infty}C_n\frac{x^n}{n!}, \tag{12}$$

where

$$I_\nu(z) = \sum_{k=0}^{\infty}\frac{1}{k!\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{2k+\nu}$$

for $\nu \in \mathbb{R}$ and $z \in \mathbb{C}$, see [1, p. 375, 9.6.10], is the modified Bessel function of the first kind. Corresponding to these properties, the following properties of the Catalan–Qi function $C(a, b; z)$ can be obtained.

Theorem 1.5. *For $n \geq 0$ and $\Re(z) \geq 0$, we have*

$$C(a, b; z+1) = \frac{b}{a}\frac{z+a}{z+b}C(a, b; z); \quad C(a, b; n) = \left(\frac{b}{a}\right)^n\prod_{k=0}^{n-1}\frac{a+k}{b+k};$$

$$\sum_{n=1}^{\infty}\left(\frac{a}{b}\right)^n C(a, b; n) = \frac{a}{b-a-1}, \quad b > a+1 > 1;$$

$$\sum_{n=0}^{\infty}C(a, b; n)\frac{x^{2n}}{(2n)!} = {}_1F_2\left(a; \frac{1}{2}, b; \frac{b}{4a}x^2\right); \quad \sum_{n=0}^{\infty}C(a, b; n)\frac{x^n}{n!} = {}_1F_1\left(a; b; \frac{b}{a}x\right).$$

Remark 1.2. When $a = \frac{1}{2}$ and $b = 2$, the formulas in Theorem 1.5 become those listed in (11) and (12).

Remark 1.3. The last two formulas in Theorem 1.5 show that the functions ${}_1F_2\left(a; \frac{1}{2}, b; \frac{b}{4a}x^2\right)$ and ${}_1F_1\left(a; b; \frac{b}{a}x\right)$ can be regarded as the generating functions of the Catalan–Qi numbers $C(a, b; n)$.

2 Proofs of Theorems 1.3 to 1.5

We are now start out to prove Theorem 1.3 by two approaches and to prove Theorems 1.4 and 1.5.

First proof of Theorem 1.3. Taking the logarithm of $F_{a,b}(x)$ gives

$$\ln F_{a,b}(x) = \ln\Gamma(x+a) - x\ln(x+a) - \ln\Gamma(x+b) + (x+b-a)\ln(x+b) \triangleq f_a(x) - f_a(x+b-a).$$

Differentiating twice with respect to the variable x of $f_a(x)$ yields

$$f_a'(x) = \psi(x+a) - \ln(x+a) + \frac{a}{x+a} - 1 \quad \text{and} \quad f_a''(x) = \psi'(x+a) - \frac{1}{x+a} - \frac{a}{(x+a)^2}.$$

4

By virtue of the formulas

$$\psi^{(n)}(z) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-zt}}{1 - e^{-t}} \, dt \quad \text{and} \quad \Gamma(z) = k^z \int_0^\infty t^{z-1} e^{-kt} \, dt$$

for $\Re(z) > 0$, $\Re(k) > 0$, and $n \in \mathbb{N}$ in [1, p. 260, 6.4.1] and [1, p. 255, 6.1.1], we obtain

$$f_a''(x - a) = \psi'(x) - \frac{1}{x} - \frac{a}{x^2} = \int_0^\infty \left(\frac{1}{1 - e^{-t}} - \frac{1}{t} - a \right) t e^{-xt} \, dt.$$

Accordingly, we have

$$
\begin{aligned}
[\ln F_{a,b}(x)]'' = f_a''(x) - f_a''(x + b - a) &= \int_0^\infty \left(\frac{1}{1 - e^{-t}} - \frac{1}{t} - a \right) t \left[e^{-(x+a)t} - e^{-(x+b)t} \right] dt \\
&= \int_0^\infty \left(\frac{1}{1 - e^{-t}} - \frac{1}{t} - a \right) t \left(e^{-at} - e^{-bt} \right) e^{-xt} \, dt.
\end{aligned}
\tag{13}
$$

The famous Bernstein-Widder theorem, [25, p. 161, Theorem 12b], states that a necessary and sufficient condition for $f(x)$ to be completely monotonic on $(0, \infty)$ is that $f(x) = \int_0^\infty e^{-xt} \, d\mu(t)$, where μ is a positive measure on $[0, \infty)$ such that the above integral converges on $(0, \infty)$. Hence, in order to find necessary and sufficient conditions on a, b such that the function $[\ln F_{a,b}(x)]''$ is completely monotonic on $(0, \infty)$, it is necessary and sufficient to discuss the positivity or negativity of the function

$$\left(\frac{1}{1 - e^{-t}} - \frac{1}{t} - a \right) t \left(e^{-at} - e^{-bt} \right) \tag{14}$$

on $(0, \infty)$.

It is clear that the factor $e^{-at} - e^{-bt}$ is positive (or negative, respectively) if and only if $b > a$ (or $b < a$, respectively). Since the function $\frac{1}{1-e^{-t}} - \frac{1}{t} = \frac{1}{e^t - 1} - \frac{1}{t} + 1$ is strictly increasing on $(0, \infty)$ and has the limits $\lim_{t \to 0^+} \left(\frac{1}{1-e^{-t}} - \frac{1}{t} \right) = \frac{1}{2}$ and $\lim_{t \to \infty} \left(\frac{1}{1-e^{-t}} - \frac{1}{t} \right) = 1$, see [5, 15] and references therein, the factor $\frac{1}{1-e^{-t}} - \frac{1}{t} - a$ is positive (or negative, respectively) on $(0, \infty)$ if and only if $a \leq \frac{1}{2}$ (or $a \geq 1$, respectively). Consequently, the function (14) is

1. positive if and only if either $b > a$ and $a \leq \frac{1}{2}$ or $b < a$ and $a \geq 1$,

2. negative if and only if either $b < a$ and $a \leq \frac{1}{2}$ or $b > a$ and $a \geq 1$.

As a result, the function $\pm [\ln F_{a,b}(x)]''$ is completely monotonic on $(0, \infty)$ if and only if $(a, b) \in D_\pm(a, b)$.

By a straightforward computation, we see that

$$\lim_{x \to \infty} [\ln F_{a,b}(x)]' = \lim_{x \to \infty} \left[\psi(x + a) - \psi(x + b) + \ln \frac{x + b}{x + a} + \frac{a(b - a)}{(x + a)(x + b)} \right] = 0 \tag{15}$$

for all $a, b \in \mathbb{R}$. This implies that, if and only if $(a, b) \in D_\pm(a, b)$, the first logarithmic derivative satisfies $[\ln F_{a,b}(x)]' \lessgtr 0$. By the definition of logarithmically completely monotonic functions, we conclude that, if and only if $(a, b) \in D_\pm(a, b)$, the function $[F_{a,b}(x)]^{\pm 1}$ is logarithmically completely monotonic on $(0, \infty)$.

5

Integrating from u to ∞ with respect to x on the very ends of (13) and considering the limit (15) give

$$-[\ln F_{a,b}(u)]' = \int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t} - a \right) \left(e^{-at} - e^{-bt} \right) e^{-ut} \, \mathrm{d}\, t.$$

Further integrating with respect to u from x to ∞ on both sides of the above equality and employing the limit $\lim_{x\to\infty} F_{a,b}(x) = e^{b-a}$ reveal that

$$\ln F_{a,b}(x) = b - a + \int_0^\infty \frac{1}{t} \left(\frac{1}{1-e^{-t}} - \frac{1}{t} - a \right) \left(e^{-at} - e^{-bt} \right) e^{-xt} \, \mathrm{d}\, t.$$

The first proof of Theorem 1.3 is thus complete. □

Second proof of Theorem 1.3. As did in the proof of [20, Theorem 1], employing the formula

$$\ln \Gamma(z) = \ln\left(\sqrt{2\pi}\, z^{z-1/2} e^{-z} \right) + \int_0^\infty \frac{1}{t} \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-zt} \, \mathrm{d}\, t$$

in [23, (3.22)] and utilizing $\ln \frac{b}{a} = \int_0^\infty \frac{e^{-au} - e^{-bu}}{u} \, \mathrm{d}\, u$ in [1, p. 230, 5.1.32] yield

$$\ln F_{a,b}(x) = b - a + \left(a - \frac{1}{2} \right) \ln \frac{x+a}{x+b} + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-xt}}{t} \left(e^{-at} - e^{-bt} \right) \mathrm{d}\, t$$

$$= b - a + \left(a - \frac{1}{2} \right) \int_0^\infty \frac{e^{-xt}}{t} \left(e^{-bt} - e^{-at} \right) \mathrm{d}\, t + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-xt}}{t} \left(e^{-at} - e^{-bt} \right) \mathrm{d}\, t$$

$$= b - a + \int_0^\infty \frac{1}{t} \left(a - \frac{1}{2} - \frac{1}{2} + \frac{1}{t} - \frac{1}{e^t - 1} \right) \left(e^{-bt} - e^{-at} \right) e^{-xt} \, \mathrm{d}\, t$$

$$= b - a + \int_0^\infty \frac{1}{t} \left(a + \frac{1}{t} - \frac{1}{1-e^{-t}} \right) \left(e^{-bt} - e^{-at} \right) e^{-xt} \, \mathrm{d}\, t.$$

The rest of the second proof is the same as in the first proof after the equation (13). The second proof of Theorem 1.3 is complete. □

Proof of Theorem 1.4. This follows from straightforwardly combining (7) and (8) with (9). □

Proof of Theorem 1.5. It is easy to see that

$$C(a, b; z+1) = \frac{\Gamma(b)}{\Gamma(a)} \left(\frac{b}{a} \right)^{z+1} \frac{\Gamma(z+a+1)}{\Gamma(z+b+1)} = \frac{b}{a} \frac{z+a}{z+b} \frac{\Gamma(b)}{\Gamma(a)} \left(\frac{b}{a} \right)^z \frac{\Gamma(z+a)}{\Gamma(z+b)} = \frac{b}{a} \frac{z+a}{z+b} C(a, b; z).$$

Consequently, when taking $z = n - 1$,

$$C(a, b; n) = \frac{b}{a} \frac{n+a-1}{n+b-1} C(a, b; n-1) = \left(\frac{b}{a} \right)^2 \frac{n+a-1}{n+b-1} \frac{n+a-2}{n+b-2} C(a, b; n-2)$$

$$= \cdots = \left(\frac{b}{a} \right)^n \frac{n+a-1}{n+b-1} \frac{n+a-2}{n+b-2} \cdots \frac{a+1}{b+1} \frac{a}{b} C(a, b; 0) = \left(\frac{b}{a} \right)^n \prod_{k=0}^{n-1} \frac{a+k}{b+k}.$$

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By (9), it follows that

$$\sum_{n=1}^{\infty} \left(\frac{a}{b}\right)^n C(a,b;n) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{n=1}^{\infty} \frac{\Gamma(n+a)}{\Gamma(n+b)} = \frac{\Gamma(b)}{\Gamma(a)} \frac{\Gamma(a+1)\Gamma(b-a-1)}{\Gamma(b)\Gamma(b-a)} = \frac{a}{b-a-1}.$$

The last two formulas in Theorem 1.5 can be straightforwardly derived from the definition (2) of the generalized hypergeometric series. The proof of Theorem 1.5 is complete. $\qquad\square$

Remark 2.1. This paper is a companion of the articles [6, 7, 12, 13, 16, 18, 20] and the preprints [10, 18] and is a revised version of the preprint [17].

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Semiring structures based on meet and plus ideals in lower BCK-semilattices

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Abstract. The notion of the meet set based on two subsets of a lower BCK-semilattice X is introduced, and related properties are investigated. Conditions for the meet set to be a (positive implicative, commutative, implicative) ideal are discussed. The meet ideal based on subsets, and the plus ideal of two subsets in a lower BCK-semilattice X are also introduced, and related properties are investigated. Using meet operation and addition, the semiring structure is induced.

1. **Introduction**

Ideal theory has an important role in the development BCK/BCI-algebras (see [1, 3, 4]). It was shown in [5] that if X is a BCK-algebra then $(X, \leq)$ is a poset, and moreover if X is a commutative BCK-algebra, i.e., $x * (x * y) = y * (y * x)$ holds in X, then $(X, \leq)$ is a lower semilattice. Pałasiński [7] discussed properties of certain ideals in BCK-algebras which are lower semilattices.

In this paper, we introduce the notion of the meet set based on two subsets of a lower BCK-semilattice X and we discuss conditions for the meet set to be a (positive implicative, commutative, implicative) ideal. We also introduced the meet ideal based on subsets, and the plus ideal of two subsets in a lower BCK-semilattice X. We investigate several related properties, and we induce the semiring structure by using meet operation and addition.

2. **Prliminaries**

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a BCI-algebra if it satisfies the following conditions

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(I) $(\forall x, y, z \in X) (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X) ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X) (x * x = 0)$,

(IV) $(\forall x, y \in X) (x * y = 0, \; y * x = 0 \; \Rightarrow \; x = y)$.

If a BCI-algebra X satisfies the following identity

(V) $(\forall x \in X) (0 * x = 0)$,

then X is called a BCK-*algebra*. Any BCK/BCI-algebra X satisfies the following conditions

(a1) $(\forall x \in X) (x * 0 = x)$,

(a2) $(\forall x, y, z \in X) (x \leq y \Rightarrow x * z \leq y * z, \; z * y \leq z * x)$,

(a3) $(\forall x, y, z \in X) ((x * y) * z = (x * z) * y)$,

(a4) $(\forall x, y, z \in X) ((x * z) * (y * z) \leq x * y)$

where $x \leq y$ if and only if $x * y = 0$. A BCK-algebra X is called a *lower BCK-semilattice* (see [6]) if X is a lower semilattice with respect to the BCK-order.

A subset A of a BCK/BCI-algebra X is called an *ideal* of X (see [6]) if it satisfies

$$0 \in A, \tag{2.1}$$

$$(\forall x \in X) (\forall y \in A) (x * y \in A \; \Rightarrow \; x \in A). \tag{2.2}$$

Note that every ideal A of a BCK/BCI-algebra X satisfies the following implication (see [6]).

$$(\forall x, y \in X) (x \leq y, \; y \in A \; \Rightarrow \; x \in A). \tag{2.3}$$

For any subset A of X, the ideal generated by A is defined to be the intersection of all ideals of X containing A, and it is denoted by $\langle A \rangle$. If A is finite, then we say that $\langle A \rangle$ is *finitely generated ideal* of X (see [6]).

A subset A of a BCK-algebra X is called a *commutative ideal* of X (see [6]) if it satisfies (2.1) and

$$(\forall x, y \in X)(\forall z \in A) ((x * y) * z \in A \; \Rightarrow \; x * (y * (y * x)) \in A). \tag{2.4}$$

A subset A of a BCK-algebra X is called a *positive implicative ideal* of X (see [6]) if it satisfies (2.1) and

$$(\forall x, y, z \in X) ((x * y) * z \in A, \; y * z \in A \; \Rightarrow \; x * z \in A). \tag{2.5}$$

A subset A of a BCK-algebra X is called an *implicative ideal* of X (see [6]) if it satisfies (2.1) and

$$(\forall x, y \in X)(\forall z \in A) ((x * (y * y)) * z \in A \; \Rightarrow \; x \in A). \tag{2.6}$$

A proper ideal P of a lower BCK-semilattice X is said to be *prime* if it satisfies

$$(\forall a, b \in X) (a \wedge b \in P \; \Rightarrow \; a \in P \text{ or } b \in P). \tag{2.7}$$

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We refer the reader to the books [2, 6] for further information regarding BCK/BCI-algebras.

3. **Meet and plus ideals**

In what follows, let X be a lower BCK-semilattice unless otherwise specified. For any nonempty subsets A and B of X, we consider the set
$$K := \{a \wedge b \mid a \in A,\ b \in B\}$$
where $a \wedge b$ is the greatest lower bound of a and b. We say that K is the *meet set* based on A and B. Note that $A \cap B \subseteq K$, but the reverse inclusion is not true as seen in the following example.

Example 3.1. (1) Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

For $A = \{2, 3\}$ and $B = \{1, 4\}$, we have
$$K := \{a \wedge b \mid a \in A,\ b \in B\} = \{0, 1, 2\} \nsubseteq A \cap B.$$

(2) Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	1	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

For subsets $A = \{1, 2, 3\}$ and $B = \{1, 3, 4\}$ of X, we have
$$K := \{a \wedge b \mid a \in A,\ b \in B\} = \{0, 1, 3\} \nsubseteq \{1, 3\} = A \cap B.$$

The following example shows that the set $K := \{a \wedge b \mid a \in A,\ b \in B\}$ may not be an ideal of X for some subsets A and B of X.

Example 3.2. Let $X = \{0, 1, 2, 3, 4\}$ be a lower BCK-semilattice in Example 3.1(1). For $A = \{2, 3\}$ and $B = \{1, 4\}$, we have
$$\{a \wedge b \mid a \in A,\ b \in B\} = \{0, 1, 2\},$$
which is not an ideal of X.

We provide conditions for the meet set $K := \{a \wedge b \mid a \in A,\ b \in B\}$ based on A and B to be an ideal.

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Theorem 3.3. *If A and B are ideals of X, then so is the meet set*

$$K := \{a \wedge b \mid a \in A, \ b \in B\}$$

based on A and B.

Proof. Obviously, $0 \in K$. Let $x \in K$ and $y * x \in K$ for $x, y \in X$. Then $x = a \wedge b$ and $y * x = a' \wedge b'$ where $a, a' \in A$ and $b, b' \in B$. Since $a \wedge b \leq a$ and A is an ideal, we have $x = a \wedge b \in A$. Similarly, we have

$$y * x = a' \wedge b' \leq a' \in A.$$

Since A is an ideal of X, it follows that $y \in A$. By the similar way, we get $y \in B$. Therefore,

$$y = y \wedge y \in \{a \wedge b \mid a \in A, \ b \in B\} = K$$

and K is an ideal of X. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\ \square$

Lemma 3.4 ([6])**.** *For an ideal A of a BCK-algebra X, the following are equivalent.*

    (i) *A is positive implicative.*
    (ii) *$(\forall x, y \in X)\,((x * y) * y \in A \ \Rightarrow \ x * y \in A)$.*

Lemma 3.5 ([6])**.** *For an ideal A of a BCK-algebra X, the following are equivalent.*

    (i) *A is commutative.*
    (ii) *$(\forall x, y \in X)\,(x * y \in A \ \Rightarrow \ x * (y * (y * x)) \in A)$.*

Lemma 3.6 ([6])**.** *Let A be an ideal of a BCK-algebra X. Then A is implicative if and only if A is both positive implicative and commutative.*

Theorem 3.7. *If A and B are positive implicative (resp., commutative, implicative) ideals of X, then so is the meet set*

$$K := \{a \wedge b \mid a \in A, \ b \in B\}$$

based on A and B.

Proof. Assume that A and B are positive implicative ideals of X. Then A and B are ideals of X, and so the set $K := \{a \wedge b \mid a \in A, \ b \in B\}$ is an ideal of X by Theorem 3.3. Let $(x * y) * y \in K$ for every $x, y \in X$. Then $(x * y) * y = a \wedge b$ for some $a \in A$ and $b \in B$. Since $a \wedge b \leq a$ and A is an ideal, we have $(x * y) * y \in A$. Similarly, $(x * y) * y \in B$. Since A and B are positive implicative ideals, it follows from Lemma 3.4 that $x * y \in A$ and $x * y \in B$. Therefore

$$x * y = (x * y) \wedge (x * y) \in \{a \wedge b \mid a \in A, \ b \in B\} = K,$$

and so K is a positive implicative ideal of X by Lemma 3.4.

    Now suppose that A and B are commutative ideals of X. Then A and B are ideals of X, and so the set $K := \{a \wedge b \mid a \in A, \ b \in B\}$ is an ideal of X by Theorem 3.3. Let $x * y \in K$ for every $x, y \in X$. Then $x * y = a \wedge b$ for some $a \in A$ and $b \in B$. Since $a \wedge b \leq a$ and $a \wedge b \leq b$, it follows

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that $x * y \in A \cap B$. Since A and B are commutative, we have $x * (y * (y * x)) \in A \cap B$ by Lemma 3.5. Hence

$$x * (y * (y * x)) = (x * (y * (y * x))) \wedge (x * (y * (y * x)))$$

$$\in \{a \wedge b \mid a \in A, \ b \in B\} = K,$$

and therefore K is a commutative ideal if X.

Now, if A and B are implicative ideals of X, then they are both positive implicative and commutative by Lemma 3.6. Thus K is both a positive implicative ideal and a commutative ideal of X, and so it is an implicative ideal of X. $\qquad\square$

Given two nonempty subsets A and B of X, we consider the ideal of X generated by the meet set $K := \{a \wedge b \mid a \in A, \ b \in B\}$ based on A and B.

Definition 3.8. For any nonempty subsets A and B of X, we denote

$$A \wedge B := \langle \{a \wedge b \mid a \in A, b \in B\} \rangle$$

which is called the *meet ideal* of X generated by A and B. In this case, we say that the operation "$\wedge$" is a *meet operation*. If $A = \{a\}$, then $\{a\} \wedge B$ is denoted by $a \wedge B$. Also, if $B = \{b\}$, then $A \wedge \{b\}$ is denoted by $A \wedge b$.

Obviously, $A \wedge B = B \wedge A$ for any nonempty subsets A and B of X. If A and B are ideals of X, then

$$A \wedge B = \{a \wedge b \mid a \in A, b \in B\}.$$

Example 3.9. For two subsets $A = \{2, 3\}$ and $B = \{1, 4\}$ of X in Example 3.1, the meet ideal of X generated by A and B is $A \wedge B = \langle \{0, 1, 2\} \rangle = \{0, 1, 2, 3\}$.

For any nonempty subsets A, B and C of X, we have

$$A \subseteq B, \ A \subseteq C \ \Rightarrow \ A \subseteq B \wedge C. \tag{3.1}$$

The following example shows that there are subsets A, B and C of X such that $A \subseteq B$ and $A \subseteq C$, but $B \wedge C \nsubseteq A$.

Example 3.10. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

For subsets $A = \{0, 1\}$, $B = \{0, 1, 2, 3\}$ and $C = \{0, 1, 2, 4\}$ of X, we have

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$$B \wedge C = \langle \{b \wedge c \mid b \in B, \ c \in C\} \rangle = \{0, 1, 2\} \nsubseteq \{0, 1\} = A.$$

Proposition 3.11. *If A, B and C are ideals of X, then*

$$A \wedge \{0\} = \{0\}. \tag{3.2}$$

$$A \wedge B = A \cap B. \tag{3.3}$$

$$(A \wedge B) \wedge C = A \wedge (B \wedge C) = \{a \wedge b \wedge c \mid a \in A, b \in B, c \in C\}. \tag{3.4}$$

Proof. It is clear that $A \wedge \{0\} = \{0\}$. Using (3.1), we have $A \cap B \subseteq A \wedge B$. Let $x \in A \wedge B$. Then there exist $a \in A$ and $b \in B$ such that $x = a \wedge b$. Since $a \wedge b \leq a$ and $a \wedge b \leq b$, we have $x \in A \cap B$ by (2.3). Hence $A \wedge B = A \cap B$. The result (3.4) is straightforward. $\quad\square$

Corollary 3.12. *If A, B and C are ideals of X, then the condition (3.1) is valid.*

By Proposition 3.11, we know that for ideals $A_1, A_2, \cdots, A_n$ of X

$$\bigwedge_{i=1}^{n} A_i := A_1 \wedge A_2 \wedge \cdots \wedge A_n$$

$$= \{a_1 \wedge a_2 \wedge \cdots \wedge a_n \mid a_1 \in A_1, a_2 \in A_2, \cdots, a_n \in A_n\} \tag{3.5}$$

$$= \bigcap_{i=1}^{n} A_i.$$

For any nonempty subsets A and B of X, denote by $A + B$ the ideal generated by $A \cup B$, and is called the *plus ideal* of A and B. The operation "+" is called the *addition*. Obviously, $A, B \subseteq A + B$, $A + \{0\} = A$ and $A + B = B + A$.

Example 3.13. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

For subsets $A = \{1, 3\}$ and $B = \{2\}$ of X, we have

$$A + B = \langle A \cup B \rangle = \{0, 1, 2, 3\},$$

which is a plus ideal of X.

Proposition 3.14. *For any nonempty subsets A and B of X, we have $A \wedge B \subseteq A + B$.*

Proof. If $x \in A \wedge B$, then there exists $z_1, z_2, \cdots, z_n \in \{a \wedge b \mid a \in A, \ b \in B\}$ such that

$$(\cdots((x * z_1) * z_2) * \cdots) * z_n = 0. \tag{3.6}$$

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For each $i \in \{1, 2, \cdots, n\}$, we have $z_i = a_i \wedge b_i$ where $a_i \in A$ and $b_i \in B$. Thus

$$a_i \wedge b_i \leq a_i \in A \subseteq A \cup B \subseteq A + B,$$

and so $z_i \in A + B$ for all $i \in \{1, 2, \cdots, n\}$. Since $0 \in A + B$, it follows from (3.6) and (2.2) that $x \in A + B$. Hence $A \wedge B \subseteq A + B$. □

Given two nonempty subsets A and B of X, we note that every ideal I of X is represented by the meet ideal based on some A and B, and every ideal J of X is represented by the plus ideal of A and B. But we know that they are different, that is, $I \neq J$ in general as seen in the following example.

Example 3.15. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

For two subsets $A = \{1\}$ and $B = \{2, 3\}$ of X, the ideal $I = \{0, 1\}$ is represented by the meet ideal based on A and B as follows

$$I = \langle A \wedge B \rangle = \langle \{0, 1\} \rangle = \{0, 1\}.$$

Also the ideal $J = \{0, 1, 2, 3\}$ is represented by the plus ideal of A and B as follows:

$$J = A + B = \langle A \cup B \rangle = \langle \{1, 2, 3\} \rangle = \{0, 1, 2, 3\}.$$

We know that $I \neq J$.

The following example shows that the reverse inclusion in Proposition 3.14 is not true in general.

Example 3.16. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ which is given in Example 3.13. For subsets $A = \{1, 2\}$ and $B = \{1, 3\}$ of X, we have

$$A \wedge B = \langle \{0, 1\} \rangle = \{0, 1\}$$

and

$$A + B = \langle \{1, 2, 3\} \rangle = \{0, 1, 2, 3\}.$$

Thus $A + B \nsubseteq A \wedge B$.

For any nonempty subsets A, B and C of X, consider the following condition.

$$A \subseteq C, \ B \subseteq C \ \Rightarrow A + B \subseteq C. \tag{3.7}$$

The following example shows that the condition (3.7) is not valid in general.

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Example 3.17. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

For subsets $A = \{1, 3\}$, $B = \{2, 3\}$ and $C = \{1, 2, 3\}$ of X, we have

$$A + B = \langle A \cup B \rangle = \{0, 1, 2, 3\} \nsubseteq C.$$

We provide conditions for the implication (3.7) to be hold.

Proposition 3.18. *If A and B are nonempty subsets of X and C is an ideal of X, then the implication (3.7) is valid.*

Proof. Let A and B be subsets of X and C be an ideal of X such that $A \subseteq C$ and $B \subseteq C$. If $x \in A + B$, then

$$(\cdots((x * z_1) * z_2) * \cdots) * z_n = 0 \tag{3.8}$$

for some $z_1, z_2, \cdots, z_n \in A \cup B$. It follows that $z_i \in C$ for all $i = 1, 2, \cdots, n$ and $0 \in C$. Since C is an ideal of X, it follows from (3.8) and (2.2) that $x \in C$. Therefore $A + B \subseteq C$. $\square$

Let A be an ideal of a BCI-algebra X and S be a subset of X with a nilpotent element. Then

$$x \in \langle A \cup S \rangle \text{ if and only if } (\cdots((x * s_1) * s_2) * \cdots) * s_n \in A$$

for some $s_1, s_2, \cdots, s_n \in S$ (see [2]). Since every element of a BCK-algebra is nilpotent, we can apply the result above to BCK-algebras as follows.

Lemma 3.19. *Let A an ideal of a BCK-algebra X. For any subset S of X, we have*

$$x \in \langle A \cup S \rangle \text{ if and only if } (\cdots((x * s_1) * s_2) * \cdots) * s_n \in A$$

for some $s_1, s_2, \cdots, s_n \in S$.

Lemma 3.20 ([2]). *Let X be a commutative BCK-algebra and $x, y, z \in X$. Then*

$$(x \wedge y) * (x \wedge z) = (x \wedge y) * z.$$

Theorem 3.21. *For any ideals A, B and C of a commutative BCK-algebra X, we have*

$$A \wedge (B + C) = (A \wedge B) + (A \wedge C) \text{ and } (B + C) \wedge A = (B \wedge A) + (C \wedge A).$$

Proof. Note that $A \wedge B \subseteq A$ and $A \wedge B \subseteq B \subseteq B + C$. It follows from (3.1) that

$$A \wedge B \subseteq A \wedge (B + C).$$

Similarly $A \wedge C \subseteq A \wedge (B + C)$, and thus

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$$(A \wedge B) + (A \wedge C) \subseteq A \wedge (B + C)$$

by Proposition 3.18. Now let $x \in A \wedge (B + C)$. Then $x = a \wedge z$ for some $a \in A$ and $z \in B + C = \langle B \cup C \rangle$. It follows from Lemma 3.19 that there exist $c_1, c_2, \cdots, c_n \in C$ such that

$$(\cdots ((z * c_1) * c_2) * \cdots) * c_n \in B. \tag{3.9}$$

Note that $a \wedge c_1, a \wedge c_2, \cdots, a \wedge c_n \in A \wedge C$. Using Lemma 3.20 and (a3) induces

$$\begin{aligned}
((a \wedge z) * (a \wedge c_1)) * (a \wedge c_2) &= ((a \wedge z) * c_1) * (a \wedge c_2) \\
&= ((a \wedge z) * (a \wedge c_2)) * c_1 \\
&= ((a \wedge z) * c_2) * c_1 \\
&= ((a \wedge z) * c_1) * c_2
\end{aligned}$$

which implies from Lemma 3.20 and (a3) again that

$$\begin{aligned}
(((a \wedge z) &* (a \wedge c_1)) * (a \wedge c_2)) * (a \wedge c_3) \\
&= (((a \wedge z) * c_1) * c_2) * (a \wedge c_3) \\
&= (((a \wedge z) * (a \wedge c_3)) * c_1) * c_2 \\
&= (((a \wedge z) * c_3) * c_1) * c_2 \\
&= (((a \wedge z) * c_1) * c_2) * c_3.
\end{aligned}$$

By the mathematical induction, we conclude that

$$\begin{aligned}
(\cdots (((a \wedge z) &* (a \wedge c_1)) * (a \wedge c_2)) * \cdots) * (a \wedge c_n) \\
&= (\cdots (((a \wedge z) * c_1) * c_2) * \cdots) * c_n.
\end{aligned} \tag{3.10}$$

The inequality $a \wedge z \leq z$ implies from (a2) that

$$(\cdots (((a \wedge z) * c_1) * c_2) * \cdots) * c_n \leq (\cdots ((z * c_1) * c_2) * \cdots) * c_n. \tag{3.11}$$

Since $(\cdots ((z * c_1)) * c_2) * \cdots) * c_n \in B$ and B is an ideal, it follows from (2.3) that

$$(\cdots (((a \wedge z) * c_1) * c_2) * \cdots) * c_n \in B. \tag{3.12}$$

Note that $(\cdots (((a \wedge z) * c_1) * c_2) * \cdots) * c_n \leq a \wedge z \leq a$ and $a \in A$, and so

$$(\cdots (((a \wedge z) * c_1) * c_2) * \cdots) * c_n \in A. \tag{3.13}$$

Combining (3.10), (3.12) and (3.13), we have

$$(\cdots (((a \wedge z) * (a \wedge c_1)) * (a \wedge c_2)) * \cdots) * (a \wedge c_n) \in A \wedge B. \tag{3.14}$$

Since $a \wedge c_1, a \wedge c_2, \cdots, a \wedge c_n \in A \wedge C$, it follows from Lemma 3.20 that

$$x = a \wedge z \in \langle (A \wedge B) \cup (A \wedge C) \rangle = (A \wedge B) + (A \wedge C). \tag{3.15}$$

Consequently $A \wedge (B + C) = (A \wedge B) + (A \wedge C)$. Similarly we have $(B + C) \wedge A = (B \wedge A) + (C \wedge A)$. $\square$

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Through our discussion above, we make a semiring as follows.

Theorem 3.22. *Let $\mathcal{I}(X)$ be the set of all ideals of a commutative BCK-algebra X. Then $(\mathcal{I}(X), +, \wedge)$ is a semiring, that is, two operations $+$ and $\wedge$ are associative on $\mathcal{I}(X)$ such that*

(i) *addition $+$ is a commutative operation,*

(ii) *there exist $\{0\} \in \mathcal{I}(X)$ such that $A + \{0\} = A$ and $A \wedge \{0\} = \{0\} \wedge A = \{0\}$ for each $A \in \mathcal{I}(X)$, and*

(iii) *the meet operation $\wedge$ distributes over addition $(+)$ both from the left and from the right.*

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The solutions of some types of q-shift difference differential equations [*]

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Abstract

In this paper, we investigate some properties of solutions of some types of q-shift difference differential equations. In addition, we also generalize the Rellich-Wittich-type theorem about differential equations to the case of q-shift difference differential equations. Moreover, we give some example to show the existence and growth of some q-shift difference differential equations.

Key words: q-shift; difference differential equation; zero order.

Mathematical Subject Classification (2010): 39A 50, 30D 35.

1　Introduction and Some Results

The main purpose of this paper is to investigate some properties of solutions of some q-shift difference differential equations by using Nevanlinna theory in the fields of complex analysis. Thus, we firstly assume that readers are familiar with the basic results and the notations of the Nevanlinna value distribution theory of meromorphic functions such as $m(r, f)$, $N(r, f)$, $T(r, f)$, $\cdots$, (see Hayman [15], Yang [33] and Yi and Yang [34]). For a meromorphic function f, we use $S(r, f)$ to denote any quantity satisfying $S(r, f) = o(T(r, f))$ for all r outside a possible exceptional set of finite logarithmic measure, $\mathbb{S}(f)$ denotes the family of all meromorphic function $a(z)$ such that $T(r, a) = S(r, f) = o(T(r, f))$, where $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. Besides, we use $S_1(r, f)$ to denote any quantity satisfying $S_1(r, f) = o(T(r, f))$ for all r on a set F of logarithmic density 1, the logarithmic density of a set F is defined by

$$\limsup_{r \to \infty} \frac{1}{\log r} \int_{[1,r] \cap F} \frac{1}{t} dt.$$

For convenience, we claim that the set F of logarithmic density can be not necessarily the same at each occurrence.

About forty years ago, F. Rellich, H. Wittich and I. Laine investigated the existence or growth of solutions of some differential equations (see [17, 18, 20, 22]) and obtained the following results.

Theorem 1.1 *(see [17, Rellich]). Let the differential equation be the following form*

$$w'(z) = f(w), \tag{1}$$

If $f(w)$ is transcendental meromorphic function of w, then equation (1) has no non-constant entire solution.

Theorem 1.2 *(see [26, Wittich]). Let*

$$\Phi(z, w) = \sum a_{(i)}(z) w^{i_0} (w')^{i_1} \cdots (w^{(n)})^{i_n}$$

be differential polynomial, with coefficients $a_{(i)}(z)$ are polynomials of z. If the right-hand side of the differential equation

$$\Phi(z, w) = f(w), \tag{2}$$

$f(w)$ is the transcendental meromorphic function of w, then equation (2) has no non-constant entire solution.

Remark 1.1 *H. Wittich [26] studied the more general differential equation than equation (1).*

Later, Yanagihara and Shimomura extended the above type theorem to the case of difference equations (see [25, 31, 32]), and obtained the following two results

Theorem 1.3 *(see [25, Shimomura]). For any non-constant polynomial $P(w)$, the difference equation*

$$w(z + 1) = P(w(z))$$

has a non-trivial entire solution.

Theorem 1.4 *(see [31, Yanagihara]). For any non-constant rational function $R(w)$, the difference equation*

$$w(z + 1) = R(w(z))$$

has a non-trivial meromorphic solution in the complex plane.

After theirs work, by using Nevanlinna theory in complex difference equations (see [1, 3, 7, 8, 11, 12, 14]), many mathematicians have done a lot of researches in difference equations, difference product and q-difference in the complex plane $\mathbb{C}$, there were a number of articles (including [5, 13, 16, 19, 24, 36]) focused on the existence and growth of solutions of difference equations. In addition, K. Liu, H.Y. Xu and X. G. Qi investigated some properties of complex q-shift difference equations [23, 24, 28]. Inspired by these papers, the purpose of this paper is to study the above Rellich-Wittich-type theorem of q-shift difference differential equation.

Definition 1.1 *We call the equation as q-shift difference differential equation if a equation contains the q-shift term $f(z+c)$, q-difference term $f(qz)$ and differential term $f'(z)$ of one function $f(z)$ at the same time.*

We consider the q-shift difference differential equation of the form

$$\Omega(z, w) := \sum_J a_J(z) \prod_{j=1}^n \left(w^{(j)}(q_j z + c_j)^{i_j} \right) = P_s[f(w)], \qquad (3)$$

where $a_J(z)$ are polynomials of z and $q_j, c_j \in \mathbb{C} \setminus \{0\}$, $P_m[f]$ is a polynomial of f of degree m,

$$P_m[f] = d_m(z)f^m + d_{m-1}(z)f^{m-1} + \cdots + d_0(z),$$

and $d_m(z), \ldots, d_0(z)$ are polynomials of z, and obtain the following results.

Theorem 1.5 *For equation (3), if $s \geq 1$ and f is a transcendental meromorphic function, then equation (3) has no non-constant transcendental entire solution with zero order.*

Theorem 1.6 *Under the assumptions of Theorem 1.5, the q-shift difference differential equation*

$$\sum_J a_J(z) \prod_{j=1}^n (w^{(j)}(q_j z + c_j))^{i_j} = \frac{P_s[f(w)]}{Q_t[f(w)]},$$

has no non-constant transcendental entire solution with zero order, where $s \geq 1$, and $P_s[f]$ and $Q_t[f]$ are irreducible polynomials in f.

In 2012, Beardon [4] studied entire solutions of the generalized functional equation

$$f(qz) = qf(z)f'(z), \qquad f(0) = 0, \qquad (4)$$

where q is a non-zero complex number. Beardon [4] obtained the main theorem as follows.

Theorem 1.7 *[4]. Any transcendental solution f of equation (4) is of the form*

$$f(z) = z + z(bz^p + \cdots),$$

where p is a positive integer, $b \neq 0$ and $q \in \mathcal{K}_p$. In particular, if $q \notin \mathcal{K}$, then the only formal solutions of (4) are $\mathcal{O}$ and $\mathcal{I}$, where $\mathcal{K}, \mathcal{K}_p, \mathcal{O}$ and $\mathcal{I}$ were stated as in [4].

In 2013, Zhang [35] further the growth of solutions of equation (4) and obtained the following theorem

Theorem 1.8 *[35, Theorem 1.1]. Suppose that f is a transcendental solution of (4) for $q \in \mathcal{K}$, then we have*

$$\rho(f) \leq \frac{\log 2}{\log |q|},$$

where

$$\rho(f) = \limsup_{r \to +\infty} \frac{\log T(r, f)}{\log r},$$

where $\mathcal{K}$ is stated as in Theorem 1.7.

Inspired by the ideas of Xu [27, 30] and Beardon [4], we investigate the growth of solutions of some q-shift difference differential equations and obtain the following results.

Theorem 1.9 *Suppose that f is a solution of*

$$f(qz + c) = \eta f(z) f'(z), \tag{5}$$

where $q, c, \eta \in \mathbb{C} \setminus \{0\}$ and $|q| > 1$. If f is a transcendental entire function, then we have

$$\rho(f) \leq \frac{\log 2}{\log |q|}.$$

Furthermore, if f is a polynomial, then f is a polynomial of degree 1, that is, $f(z) = a_1 z + a_0$, where

$$a_1 = \frac{q}{\eta}, \qquad a_0 = \frac{qc}{\eta(1 + q)}.$$

The following example shows that equation (5) had a transcendental entire solution.

Example 1.1 *Let $q = 2, c = 2\pi$ and $\eta = 2$. Then $f(z) = \sin z$ satisfies equation*

$$f(2z + 2\pi) = 2f(z)f'(z),$$

and

$$\rho(f) = 1 = \frac{\log 2}{\log 2}.$$

We also investigate the existence and growth of solutions of equation (5) when the constant η in equation (5) is replaced by a function, and obtain the following result.

Theorem 1.10 *Let f be a transcendental solution of equation*

$$f(qz + c)^n = R(z)f(z)[f^{(j)}(z)]^s, \tag{6}$$

where $q, c, \in \mathbb{C}$ and $|q| > 1$, n, j, s are positive integers and $R(z)$ is rational function in z. If f is an entire function, then $n \leq s + 1$ and

$$\rho(f) \leq \frac{\log(s + 1) - \log n}{\log |q|}.$$

Furthermore, if $n = 1$ and f is a meromorphic function with infinitely many poles, then we have

$$\frac{\log(s + 1)}{\log |q|} \leq \mu(f) \leq \rho(f) \leq \frac{\log(sj + s + 1)}{\log |q|}.$$

The following example shows that equation (6) has transcendental entire and meromorphic solutions.

Example 1.2 *Let $q = 2, c = 2\pi i$, $n = 1$ and $s = 1$, then $f(z) = ze^z$ satisfies system*

$$f(2z + 2\pi i) = \frac{2z + 2\pi i}{z(z + 1)} f(z)f'(z).$$

and

$$\rho(f) = 1 \leq \frac{\log 2}{\log 2}.$$

Example 1.3 *Let $q = 2, c = \pi i$, $n = 1$ and $s = 1$, then $f(z) = \frac{e^{2z}}{z^2}$ satisfies equation*

$$f(2z + 2\pi i) = \frac{z^5}{(2z - 2)(2z + 2\pi i)^2} f(z) f'(z),$$

and

$$\frac{\log 2}{\log 2} = 1 \leq \mu(f) = \rho(f) = 1 \leq \frac{\log 3}{\log 2}.$$

Theorem 1.11 *Let f be a transcendental solution of the equation*

$$f(qz + c)^n = \varphi(z) f(z) [f^{(j)}(z)]^s, \tag{7}$$

where $q, c, \in \mathbb{C}$ and $|q| > 1$, n, j, s are positive integers and $\varphi(z)$ is a small function with respect of f. If f is a meromorphic function with $\overline{N}(r, f) = S(r, f)$, then $n < s + 1$ and f satisfies

$$\rho(f) \leq \frac{\log(s + 1) - \log n}{\log |q|}.$$

Furthermore, if $n = 1$ and f has infinitely many poles, and the number of distinct common poles of f and $\frac{1}{\varphi}$ is finite, then we have

$$\rho(f) = \frac{\log(s + 1)}{\log |q|}.$$

The following example shows that equation (7) has transcendental meromorphic solution f with the order $\rho(f) = \frac{\log(s+1)}{\log |q|}$.

Example 1.4 *Let $n = j = s = 1$ and $q = \sqrt{2}, c = \frac{1}{2\sqrt{2}}$, then $f(z) = e^{z^2}$ satisfies equation*

$$f(2z + \frac{1}{2\sqrt{2}}) = \frac{1}{2z} e^{\frac{1}{8}} e^z f(z) f'(z).$$

Thus, $\varphi(z) = \frac{1}{2z} e^{\frac{1}{8}} e^z$ with $T(r, \varphi) = S(r, f)$ and the order of $f(z)$ satisfies

$$\rho(f) = 2 = \frac{\log 2 - \log 1}{\frac{1}{2} \log 2}.$$

2   Some Lemmas

Lemma 2.1 *(Valiron-Mohon'ko). [18] Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,*

$$R(z, f(z)) = \frac{\sum_{i=0}^{m} a_i(z) f(z)^i}{\sum_{j=0}^{n} b_j(z) f(z)^j},$$

with meromorphic coefficients $a_i(z), b_j(z)$, the characteristic function of $R(z, f(z))$ satisfies

$$T(r, R(z, f(z))) = dT(r, f) + O(\Psi(r)),$$

where $d = \max\{m, n\}$ and $\Psi(r) = \max_{i,j}\{T(r, a_i), T(r, b_j)\}$.

Lemma 2.2 *(see [23]). Let $f(z)$ be a nonconstant zero-order meromorphic function and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$m\left(r, \frac{f(qz + \eta)}{f(z)}\right) = S_1(r, f).$$

Lemma 2.3 *(see [28]). Let $f(z)$ be a transcendental meromorphic function of zero order and q, η be two nonzero complex constants. Then*

$$T(r, f(qz + \eta)) = T(r, f(z)) + S_1(r, f), \quad N(r, f(qz + \eta)) \le N(r, f) + S_1(r, f).$$

Lemma 2.4 *(see [34, p.37] or [33]). Let $f(z)$ be a nonconstant meromorphic function in the complex plane and l be a positive integer. Then*

$$N(r, f^{(l)}) = N(r, f) + l\overline{N}(r, f), \quad T(r, f^{(l)}) \le T(r, f) + l\overline{N}(r, f) + S(r, f).$$

Lemma 2.5 *Let $q, c \in \mathbb{C} \setminus \{0\}$ and $f(z)$ be a nonconstant meromorphic function with zero order. Then for any positive finite integer k, we have*

$$m\left(r, \frac{f^{(k)}(qz + c)}{f(z)}\right) = S_1(r, f),$$

and

$$m\left(r, f^{(k)}(qz + c)\right) \le m(r, f) + S_1(r, f).$$

Proof: It follows from Lemma 2.2 that

$$m\left(r, \frac{f^{(k)}(qz + c)}{f(z)}\right) \le m\left(r, \frac{f^{(k)}(qz + c)}{f(qz + c)}\right) + m\left(r, \frac{f(qz + c)}{f(z)}\right) = S_1(r, f).$$

Moreover, we have

$$m\left(r, f^{(k)}(qz + c)\right) = m\left(r, \frac{f^{(k)}(qz + c)}{f(z)} f(z)\right) \le m(r, f) + S_1(r, f).$$

This completes the proof of Lemma 2.5. $\qquad\square$

Lemma 2.6 *(see [11]). Let $\Phi : (1, \infty) \to (0, \infty)$ be a monotone increasing function, and let f be a nonconstant meromorphic function. If for some real constant $\alpha \in (0, 1)$, there exist real constants $K_1 > 0$ and $K_2 \ge 1$ such that*

$$T(r, f) \le K_1 \Phi(\alpha r) + K_2 T(\alpha r, f) + S(\alpha r, f),$$

then the order of growth of f satisfies

$$\rho(f) \le \frac{\log K_2}{-\log \alpha} + \limsup_{r \to +\infty} \frac{\log \Phi(r)}{\log r}.$$

Lemma 2.7 *(see [9]). Let $f(z)$ be a transcendental meromorphic function and $p(z) = p_k z^k + p_{k-1} z^{k-1} + \cdots + p_1 z + p_0$ be a complex polynomial of degree $k > 0$. For given $0 < \delta < |p_k|$, let $\lambda = |p_k| + \delta, \mu = |p_k| - \delta$, then for given $\varepsilon > 0$ and for r large enough,*

$$(1 - \varepsilon)T(\mu r^k, f) \le T(r, f \circ p) \le (1 + \varepsilon)T(\lambda r^k, f).$$

Lemma 2.8 *(see [2, 10] or [6]). Let $g : (0, +\infty) \to R, h : (0, +\infty) \to R$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E with finite linear measure, or $g(r) \leq h(r)$, $r \notin H \cup (0, 1]$, where $H \subset (1, \infty)$ is a set of finite logarithmic measure. Then, for any $\alpha > 1$, there exists r_0 such that $g(r) \leq h(\alpha r)$ for all $r \geq r_0$.*

3 Proofs of Theorems 1.5 and 1.6

3.1 The proof of Theorem 1.5

Suppose that w be non-constant entire solution of equation (3) with zero order. Let $E_1 = \{z : |w(z)| > 1\}$ and $E_2 = \{z : |w(z)| \leq 1\}$, then we have

$$|\Omega(z, w)| = \left| \sum_J a_J(z)(w(z))^{\lambda_i} \left(\frac{w'(q_1 z + c_1)}{w(z)} \right)^{i_1} \cdots \left(\frac{w'(q_{n_1} z + c_{n_1})}{w(z)} \right)^{i_{n_1}} \right|$$

$$\leq \begin{cases} |w(z)|^\lambda \sum_J |a_J(z)| \left| \frac{w'(q_1 z + c_1)}{w(z)} \right|^{i_1} \cdots \left| \frac{w'(q_{n_1} z + c_{n_1})}{w(z)} \right|^{i_{n_1}}, & if \ \ z \in E_1, \\ \sum_J |a_J(z)| \left| \frac{w'(q_1 z + c_1)}{w(z)} \right|^{i_1} \cdots \left| \frac{w'(q_{n_1} z + c_{n_1})}{w(z)} \right|^{i_{n_1}}, & if \ \ z \in E_2, \end{cases}$$

where $\lambda = \max\{\lambda_i\}, \lambda_i = i_1 + \cdots + i_{n_1}$. It follows from Lemma 2.2 and Lemma 2.5 that

$$m(r, \Omega(z, w)) = \frac{1}{2\pi} \left(\int_{E_1} + \int_{E_2} \right) \log^+ |\Omega(z, w)| d\theta \leq \lambda m(r, w) + S_1(r, w).$$

And since $w(z)$ is a non-constant entire function, we have $N(r, w) = 0$. Thus, we have $N(r, \Omega(z, w)) = 0$ and

$$T(r, \Omega) = m(r, \Omega) \leq \lambda m(r, w) + S_1(r, w) = \lambda T(r, w) + S_1(r, w). \tag{8}$$

Since $P_s[f(w)]$ is a polynomial of $f(w)$, we can take a complex constant α such that

$$P_s[f(w)] - \alpha = [f(w) - \alpha_1] \cdots [f(w) - \alpha_s],$$

where $\alpha_1, \ldots, \alpha_s$ are complex constants, and there at least exists a constant $\beta \in \{\alpha_1, \ldots, \alpha_s\}$, which is not a Picard exceptional value of $f(w)$. Let $\xi_j, j = 1, 2, \ldots, p$ be the zeros of $f(w) - \beta$, where p is an any positive integer with $p \geq 1$. Then it follows

$$\sum_{j=1}^p N(r, \frac{1}{w - \xi_j}) \leq N(r, \frac{1}{f(w) - \beta}) \leq N(r, \frac{1}{P_s[f(w)] - \alpha}). \tag{9}$$

Thus, by using the second main theorem and (8), (9), we can get that

$$
\begin{aligned}
(p-2)T(r,w) &\leq \sum_{j=1}^{p} N(r, \frac{1}{w-\xi_j}) + S(r,w) \\
&\leq N(r, \frac{1}{P_s[f(w)] - \alpha}) + S(r,w) \\
&\leq T(r, P_s[f(w)]) + S(r,w) \\
&\leq T(r, \Omega(z,w)) + S(r,w) \\
&\leq \lambda T(r,w) + S_1(r,w).
\end{aligned}
\tag{10}
$$

It follows from (8) and (10) that

$$
(p - 2 - \lambda)T(r,w) \leq S_1(r,w).
\tag{11}
$$

Since w is transcendental and p is arbitrary, we can get a contradiction with (11). Hence, we complete the proof of Theorem 1.5.

3.2   The proof of Theorem 1.6

By using the same argument as in Theorem 1.5, and applying Lemma 2.1, we can prove the conclusion of Theorem 1.6 easily.

4   The proof of Theorem 1.9

Suppose that f is a solution of (5). If f is a polynomial of degree $m \geq 1$, let

$$
f(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0,
$$

where $a_m, \ldots, a_0$ are complex constants. From (5), we have

$$
\begin{aligned}
&a_m(qz+c)^m + a_{m-1}(qz+c)^{m-1} + \cdots + a_0 \\
=&\eta(a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0)[m a_m z^{m-1} + (m-1)a_{m-1} z^{m-1} + \cdots + a_1].
\end{aligned}
\tag{12}
$$

By computing the degree of two sides in z in (12), we can get that $m = 2m - 1$, that is, $m = 1$. Thus, $f(z)$ can be rewritten as $f(z) = a_1 z + a_0$. It follows

$$
a_1(qz+c) + a_0 = \eta(a_1 z + a_0)a_1,
$$

that is,

$$
a_1 q = \eta a_1^2, \quad a_1 c + a_0 = \eta a_1 a_0.
$$

Thus, we have $a_1 = \frac{q}{\eta}$, $a_0 = \frac{qc}{\eta(1+q)}$.

If f is a transcendental entire function, from Lemma 2.4, we have

$$
T(r, f(qz+c)) \leq 2T(r,f) + S(r,f) \leq 2(1+\varepsilon)T(\beta r, f),
\tag{13}
$$

for sufficiently large r and any given $\beta > 1, \varepsilon > 0$. By Lemma 2.7 and (13), for $\theta = |q| - \delta (0 < \delta < |q|, 0 < \theta < 1), i = 1, 2$ and sufficiently larger r, we get

$$(1 - \varepsilon)T(\theta r, f) \leq 2(1 + \varepsilon)T(\beta r, f),$$

outside of a possible exceptional set E of finite linear measure. From Lemma 2.8, for any given $\gamma > 1$ and sufficiently large r, we obtain

$$(1 - \varepsilon)T(\theta r, f) \leq 2(1 + \varepsilon)T(\gamma \beta r, f). \tag{14}$$

that is,

$$\frac{(1 - \varepsilon)}{2(1 + \varepsilon)}T(r, f) \leq T\left(\frac{\beta \gamma}{\theta}r, f\right). \tag{15}$$

Since $|q| > 1$, we can choose $\delta > 0$ such that $\theta > 1$, and let $\varepsilon \to 0$, $\delta \to 0, \beta \to 1, \gamma \to 1$, and for sufficiently large r, by Lemma 2.6, we have

$$\rho(f) \leq \frac{\log 2}{\log |q|}.$$

Thus, this completes the proof of Theorem 1.9.

5   Proofs of Theorems 1.10 and 1.11

5.1   The Proof of Theorem 1.10

Since $R(z)$ is a rational function, then we have $T(r, R(z)) = O(\log r)$. If f is a transcendental entire function, similar to the argument as in Theorem 1.9, we can get $\rho(f) \leq \frac{\log(s+1) - \log n}{\log |q|}$ easily.

If f is a meromorphic function, by Lemma 2.1 and Lemma 2.4, it follows from (6) that

$$T(r, f(qz + c)) \leq \frac{sj + s + 1}{n}T(r, f(z)) + S(r, f).$$

Since $|q| > 1$, by Lemma 2.7 and using the same argument as in Theorem 1.9, we have $\rho(f) \leq \frac{\log(sj+s+1) - \log n}{\log |q|}$.

Suppose that $n = 1$. Since $R(z)$ is a rational function, we can choose a sufficiently large constant $R(> 0)$ such that $R(z)$ has no zeros or poles in $\{z \in \mathbb{C} : |z| > R\}$. Since f has infinitely many poles, we can choose a pole z_0 of f of multiplicity $\tau \geq 1$ satisfying $|z_0| > R$. Thus, it follows that the right side of the equation (6) has a pole of multiplicity $\tau_1 = (s + 1)\tau + sj$ at z_0, and f has a pole of multiplicity τ_1 at $qz_0 + c$. Replacing z by $qz_0 + c$ in equation (6), we have that f has a pole of multiplicity $\tau_2 = (s + 1)\tau_1 + sj$ at $q^2 z_0 + qc + c$. We proceed to follow the step above. Since $R(z)$ has no zeros or poles in $\{z \in \mathbb{C} : |z| > R\}$ and f has infinitely many poles again, we may construct poles $\zeta_k = q^k z_0 + q^{k-1}c + \cdots + c, k \in \mathbb{N}_+$ of f of multiplicity τ_k satisfying

$$\tau_k = (s + 1)\tau_{k-1} + sj = (s + 1)^k \tau + sj[(s + 1)^{k-1} + \cdots + 1],$$

as $k \to \infty, k \in \mathbb{N}$. Since $|q| > 1$, then $|\zeta_k| \to \infty$ as $k \to \infty$, for sufficiently large k, we have

$$\tau(s+1)^k \leq (\tau + j)(s+1)^k - j = \tau_k \leq \tau + \tau_1 + \cdots + \tau_k \leq n(|\zeta_k|, f) \quad (16)$$
$$\leq n(|q|^k|z_0| + |C|(|q|^{k-1} + \cdots + |q| + 1), f).$$

Thus, for each sufficiently large r, there exists a $k \in \mathbb{N}_+$ such that

$$r \in [|q|^k|z_0| + |C| \sum_{i=0}^{k-1} |q|^i, |q|^{(k+1)}|z_0| + |C| \sum_{i=0}^{k} |q|^i),$$

that is,

$$k > \frac{\log r - \log(|z_0| + \frac{|c|}{|q|-1}) - \log \frac{|c|}{|q|-1} - \log |q|}{\log |q|}. \quad (17)$$

Thus, it follows from (17) that

$$n(r, f) \geq \tau(s+1)^k \geq K_1(s+1)^{\frac{\log r}{\log |q|}}, \quad (18)$$

where

$$K_1 = \tau(s+1)^{\frac{-\log(|z_0| + \frac{|c|}{|q|-1}) - \log \frac{|c|}{|q|-1} - \log |q|}{\log |q|}}.$$

Since for all $r \geq r_0$,

$$K_1(s+1)^{\frac{\log r}{\log |q|}} \leq n(r, f) \leq \frac{1}{\log 2} N(2r, f) \leq \frac{1}{\log 2} T(2r, f),$$

it follows from (18) that

$$\rho(f) \geq \mu(f) \geq \frac{\log(s+1)}{\log |q|}.$$

Thus, this completes the proof of Theorem 1.10.

5.2 The proof of Theorem 1.11

By using the same argument as in Theorem 1.10, we can prove the conclusion of Theorem 1.11 easily.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

HW and HYX completed the main part of this article. All authors read and approved the final manuscript.

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Numerical method for solving inequality constrained matrix operator minimization problem[☆]

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Abstract

In this paper, we considered a matrix inequality constrained linear matrix operator minimization problems with a particular structure, some of whose reduced versions can be applicable to image restoration. We present an efficient iteration method to solve this problem. The approach belongs to the category of Powell-Hestense-Rockafellar augmented Lagrangian method, and combines a nonmonotone projected gradient type method to minimize the augmented Lagrangian function at each iteration. Several propositions and one theorem on the convergence of the proposed algorithm were established. Numerical experiments are performed to illustrate the feasibility and efficiency of the proposed algorithm, including when the algorithm is tested with randomly generated data and on image restoration problems with some special symmetry pattern images.

Key words:
matrix equation, matrix minimization problem, matrix inequality, augmented lagrangian method, image restoration.
2000 MSC: 65F30, 65H15, 15A24

1. Introduction

Let m, n, l_1, s_1, l_2, s_2 be positive integers. Let $\mathcal{A}(X; A_1, \cdots, A_p)$ be a linear mapping from $\mathbb{R}^{m \times n}$ onto $\mathbb{R}^{l_1 \times s_1}$ and $\mathcal{G}(X; E_1, \cdots, E_q)$ be a linear mapping from $\mathbb{R}^{m \times n}$ onto $\mathbb{R}^{l_2 \times s_2}$, where A_i ($i = 1, \ldots, p$) and E_j ($j = 1, \ldots, q$) with suitable sizes are the parameter matrices. In this paper we are interested in solving the following constrained matrix minimization problem

$$
\begin{aligned}
\text{minimize} \quad & \frac{1}{2}\left\|\mathcal{A}\big(X; A_1, \cdots, A_p\big) - C\right\|^2 \\
\text{subject to} \quad & X \in \mathcal{S} \\
& L \le \mathcal{G}\big(X; E_1, \cdots, E_q\big) \le U.
\end{aligned}
\tag{1.1}
$$

where $\|\cdot\|$ denotes the Frobenius norm, the symbol $\ge$ means nonnegative, the set $\mathcal{S} \subseteq \mathbb{R}^{m \times n}$ shows the constraint, $C \in \mathbb{R}^{l_1 \times s_1}$ and $L, U \in \mathbb{R}^{l_2 \times s_2}$ are given matrices. In general, $\mathcal{S} \subseteq \mathbb{R}^{m \times n}$ is a linear space

possessing special structures, such as symmetry/skew-symmetry, centrosymmetry/centro skew-symmetry, mirror-symmetry/mirror-skew-symmetry, P-commuting symmetry/ skew-symmetry with respect to a given symmetric matrix P, Toeplitz matrix and so on. It is obvious that the linear operator equation in (1.1) is quite general and includes several linear matrix equations such as the Lyapunov and Sylvester matrix equations which are shown in Table 1. For an instant, the Lyapunov matrix equation

$$A_1^T X A_2 + A_2^T X A_1 = -C$$

is equivalent to the linear operator equation in (1.1), if we define the operator $\mathcal{A}$ as:

$$\mathcal{A} : X \rightarrow A_1^T X A_2 + A_2^T X A_1.$$

Table 1: *One-sided and two-sided Lyapunov and Sylvester matrix equations.*

Name	Matrix equation
Continuous-time (CT) Lyapunov	$A_1 X + X A_1^T + B B^T = 0$
Generalized continuous-time (CT) Lyapunov	$A_1^T X A_2 + A_2^T X A_1 = -C$
Generalized discrete-time (CT) Lyapunov	$A_1^T X A_1 + A_2^T X A_2 = -C$
Continuous-time (CT) Sylvester	$A_1 X + X A_2 = C$
Discrete-time (DT) Sylvester	$A_1 X A_2^T + X = C$
Generalized Sylvester	$A_1 X A_2^T + A_3 X A_4^T = C$

Throughout we always assume that the matrix operator inequality in model (1.1) is consistent with these given matrices E_j, L, U and unknown $X \in \mathcal{S}$, then we known that the solution set of Problem (1.1) is nonempty.

The interest that we have in this problem stems from the following reasons. Firstly, by using the vec operator $vec(.)$ and the Kronecker produc $\otimes$, the model (1.1) can be equivalently rewritten as the convex linearly constrained quadratic programming(LCQP) in the vector-form

$$\begin{aligned} \text{minimize } f(x) &= \frac{1}{2} x^T Q x + g^T x + c \\ \text{subject to } & l \leq G x \leq u, \end{aligned} \tag{1.2}$$

where

$$Q = P^T M^T M P, \quad g = -P^T M^T vec(C), \quad c = \frac{1}{2} vec(C)^T vec(C) \tag{1.3}$$

and

$$P x = vec(X), \quad l = vec(L), \quad u = vec(U). \tag{1.4}$$

The matrices M and G are the Kronecker product of the parameter matrices $\{A_i\}_{i=1}^p$ and $\{E_j\}_{j=1}^q$ which satisfies $vec(\mathcal{A}(X; A_1, \ldots, A_p)) = M vec(X)$ and $vec(\mathcal{G}(X; E_1, \ldots, E_q)) = G vec(X)$, respectively. Specifically, in (1.3)-(1.4), P is the matrix that characterizes the elements $X \in \mathcal{S}$ by $vec(X) = P x$ in terms of its independent parameter vector x of X[18]. In theory, the model (1.2) can be solved by some classical optimization methods, such as interior point method, active set method, trust region method, Newton method, and other available methods. In particular, Delbos F. in [2] considered the vector LCQP(1.2) by using an augmented Lagrangian method and given a global linear convergence of the proposed algorithm. However, using this transformation will on the one hand destroy the original structure of the unknown matrix $X \in \mathcal{S}$ if the linear

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subspace $\mathcal{S}$ has some special symmetrical structure. On the other hand, using this transformation will result in a coefficient matrix in large scale, and then increase computational complexity and storage requirement. Indeed, taking $l = m = n = s = p = q = 200$ in (1.1), then the matrices Q and G in the transformed model (1.2) have sizes of about 40000×40000. For these reasons, it cannot be a practicable method for solving Problem (1.1) by the vec operator and the Kronecker produc if the system scale is large. In this paper we will consider directly from the perspective of matrices.

Secondly, various simplified versions of Problem (1.1) have been studied extensively. If we drop the matrix inequality constraint, then Problem (1.1) is reduced to the minimization problem with special structures. Methods proposed for solving such problems can be broadly classified into two classes, including factorization techniques for small size problems, based on the special structure of the linear subspace $\mathcal{S}$ that produce a low-dimensional problems that are then solved using direct methods[3, 4, 5, 6, 7, 8, 9, 10, 11], and iterative schemes, for large-scale problems, based on Krylov subspace-type methods, such as the well-known Jacobi and Gauss-Seidel iterations[12, 13], the conjugate gradient-type methods[14, 15] and the least squares QR(LSQR) methods[16, 17, 18] and so on. On the other hand, if we simplify the general matrix inequality constraint in (1.1) into the nonnegative constraint $X \geq \mathbf{0}$ or the bound constraint $L \leq X \leq U$, then the similar problem has been studied with Dykstra's alternating projection algorithm[19, 20] and spectral projection gradient method[21]. In particular, Problem (1.1) can be regarded as a natural generalization of the problems in [21, 22, 23]. The authors in [21] considered the following constrained minimization problem

$$\text{Minimize } \left\| \sum_{i=1}^{q} A_i X B_i - C \right\|^2 \quad \text{subject to } X \in \Omega = \{X \in \mathbb{R}^{m \times n} : L \leq X \leq U\}. \tag{1.5}$$

They propose a globalized variants projected gradient method and apply the left and right preconditioning strategies to solve (1.5). While the authors in [22, 23] devoted to solve the matrix equation $AX = B$ or minimize $\|AX - B\|$ with special structures under the constraint $CXD \geq E$, respectively. The problems considered in [22] and [23] can be transformed into least nonnegative correction problems based on the fact that close-form optimal solutions of $AX = B$ or minimizing $\|AX - B\|$ with special structures can be readily derived, and then some fixed point-like algorithms can be applied to solve these transformed problems. However, all these previous ideas show difficulties when dealing with the Problem (1.1), due to the generalization of the objection function and the matrix operator inequality, so that either the projection onto the set $\{X \in \mathbb{R}^{m \times n} | L \leq \mathcal{G}(X) \leq U\}$ is not available, or a close-form optimal solution of minimizing the objection function in (1.1) with $X \in \mathcal{S}$ is not tractable.

Thirdly, we consider the application of the model (1.1) in image restoration. In fact, the authors in [21, 24] consider the problem of image restoration, combined with a Tikhonov regularization term, as a convex constrained minimization problem by use a Kronecker decomposition of the blurring matrix and the Tikhonov regularization matrix. And then they propose and show the effectiveness of their approaches, a globalized variants projected gradient method [21] and a conditional gradient-type method[21], to restore some blurred and highly noisy images. However, in this paper, we are only concerned with the restoration problems with some special symmetric pattern images, which have not yet studied in [21, 24]. Moreover, to the best of our knowledge, this class of image restoration problems have received little attention in the other literature. The main difficult is due to the fact that the restore image should preserve the same special symmetric structure with the original images. In this paper we undertake some significant attempts in this field.

In this paper, we will propose and study an algorithm in the framework of the classic Powell-Hestenes-Rockafellar augmented Lagrangian method, first suggested by Hestenes [25] and Powell [26], and developed by E.G. Birgin [27, 28] for solving Problem (1.1). The classic PHR-AL method is a fundamental and

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effective approach in inequality-constrained optimization. The algorithm effectively combines a nonmonotone projected gradient type method to minimize the augmented Lagrangian function at each iteration. We will give several propositions and one theorem on the convergence of the proposed algorithm, and apply it to solving Problem (1.1) with randomly generated data and comparing it with existing methods. We also apply our approach, combined with a Tikhonov regularization term, to restore some blurred and highly noisy symmetric pattern images.

Throughout this paper, we use the following notations. Let e_i be the ith column of the identity matrix I_k and $S_k = (e_k, e_{k-1}, \ldots, e_1)$, i.e., the kth backward identity matrix. Let $\mathbf{0}$ be the zero matrix of suitable size and P_S be the Euclidean projection onto set S. We write $\varepsilon_k \downarrow 0$ to indicate that ε_k is a (not necessarily decreasing) sequence of non-negative numbers that tends to zero. We denote $\mathbb{N} = \{0, 1, 2, \ldots\}$. For $A = (a_{ij}) \in \mathbb{R}^{m \times n}$, A_+ (or A_-) be the matrix with the (i, j)-entry equals to $\max\{0, a_{ij}\}$ (or $\min\{0, a_{ij}\}$), respectively. For $A, B \in \mathbb{R}^{m \times n}$, $\{A, B\}_-$ denotes a matrix with the ijth entry being equal to $\min\{a_{ij}, b_{ij}\}$, $\langle A, B \rangle = trace(B^T A)$ denotes the inner product of matrices A and B. Then $\mathbb{R}^{m \times n}$ is a Hilbert inner product space and the norm generated is the Frobenius norm $\| \cdot \|$. For any linear operator $\mathcal{L}$ form $\mathbb{R}^{m \times n}$ onto $\mathbb{R}^{l_1 \times s_1}$, there is another operator called the adjoint of $\mathcal{L}$, written $\mathcal{L}^T \colon \mathbb{R}^{l_1 \times s_1} \to \mathbb{R}^{m \times n}$. What defines the adjoint is that for any two matrices $X \in \mathbb{R}^{m \times n}$ and $Y \in \mathbb{R}^{l_1 \times s_1}$,

$$\langle \mathcal{L}(X), Y \rangle = \langle X, \mathcal{L}^T(Y) \rangle.$$

The rest of this paper is organized as follows. In section 2, we will briefly characterize the application of model (1.1) in image restoration. Based on the classic augmented Lagrangian method, in section 3 we propose, analyze and test an algorithm for solving the inequality-constrained matrix minimization problem (1.1). Some numerical results are reported in section 4 to verify the efficiency of the proposed algorithm. Numerical tests on the proposed algorithm applied to some special image restoration problems are also reported in this section.

2. The application of model (1.1) in image restoration

For completeness, in this section we briefly characterize how to apply the model (1.1) into image restoration and we refer to [21, 24] for detailed description. Consider solving the following model in image restoration with Tikhonov regularization:

$$\min_{l \leq x \leq u} \frac{1}{2} \|Hx - g\|_2^2 + \frac{\lambda^2}{2} \|Tx\|_2^2, \tag{2.6}$$

where $\| \cdot \|_2$ is the 2-norm. In image restoration, H will be the blurring operator, g the observed image, T the regularization operator, λ the regularization parameter, and x the restored image to be sought. The constraints represent the dynamic range of the image.

The minimizer of (2.6) can be computed by the following linear system

$$H_\lambda x = H^T g, \quad \text{where } H_\lambda = H^T H + \lambda^2 T^T T. \tag{2.7}$$

In some practical problems in image restoration, often the system (2.7) may not be consistent due to measurement errors in the data matrices, and hence it is useful to consider the following minimization problem with constraints

$$\min_{l \leq x \leq u} \frac{1}{2} \left\| H_\lambda x - H^T g \right\|_2^2. \tag{2.8}$$

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Here we assume that the matrices H and T can be separated as Kronecker product of matrices with a smaller size, i.e., $H = H_1 \otimes H_2$ and $T = T_1 \otimes T_2$. In the case of nonseparable, one can still obtain an approximation solution of H_1 and H_2 by solving the Kronecker product approximation problem (KPA) of the form $(H_1, H_2) = \operatorname{argmin}_{\hat{H}_1, \hat{H}_2} \|H - \hat{H}_1 \otimes \hat{H}_2\|$[29]. Then, (2.8) can be written as

$$\min_{L \leq X \leq U} \frac{1}{2} \left\| \left\{ (H_1{}^T H_1) \otimes (H_2{}^T H_2) + \lambda^2 (T_1{}^T T_1) \otimes (T_2{}^T T_2) \right\} vec(X) - (H_1 \otimes H_2)^T vec(G) \right\|^2, \qquad (2.9)$$

where X, G, L and U are the matrices such that $vec(X) = x$, $vec(G) = g$, $vec(L) = l$ and $vec(U) = u$. If some special symmetry pattern images are considered, by using some properties of the Kronecker product, (2.9) is then written as

$$\begin{aligned} \min \quad & \frac{1}{2} \left\| A_1 X B_1 + \lambda^2 A_2 X B_2 - C \right\|^2 \\ \text{subject to} \quad & L \leq X \leq U, \quad X \in \mathcal{S}, \end{aligned} \qquad (2.10)$$

with $A_1 = H_2{}^T H_2$, $B_1 = H_1{}^T H_1$, $A_2 = T_2{}^T T_2$, $B_2 = T_1{}^T T_1$ and $C = H_2{}^T G H_1$ and $\mathcal{S}$ is the matrix set whose elements have the same symmetry structure with the original images. The parameter λ in (2.10) is a scalar need to be determined, and its optimal value can be obtained by the classical Generalized cross-validation (GCV) method[21, 24], which is chosen to minimize the GCV function defined by

$$GCV(\lambda) = \frac{\|H\hat{x}_\lambda - g\|_2^2}{\{trace(I - HH_\lambda^{-1}H^T)\}^2} = \frac{\|(I - HH_\lambda^{-1}H^T)g\|_2^2}{\{trace(I - HH_\lambda^{-1}H^T)\}^2},$$

where $H_\lambda = H^T H + \lambda^2 T^T T$. Then, the method proposed for solving Problem (1.1) could be applied directly to the model (2.10) by considering the linear matrix operators $\mathcal{A}(X) = A_1 X B_1 + \lambda^2 A_2 X B_2$ and $\mathcal{G}(X) = X$.

3. Augmented Lagrangian method for solving Problem (1.1)

In this section we propose a matrix-form iteration method, in the framework of the classic Powell-Hestense-Rockafellar augmented Lagrangian(PHR-AL) method, to compute the solution of Problem (1.1). We then prove some convergence results for the proposed algorithm at the end of this section. For convenience, the two linear matrix operators will be simply denote by $\mathcal{A}(X)$ and $\mathcal{G}(X)$ in the following discussion.

Lemma 1. *Assume x^* is a local minimizer of the quadratic program*

$$\min_{x \in R^s} f(x) = \frac{1}{2} x^T M x + g^T x + c \ \text{ subject to } \ Gx \geq b,$$

then there exists a vector y^* such that

$$M x^* + g - G^T y^* = 0, \ \ Gx^* \geq b, \ \ \langle y^*, Gx^* - b \rangle = 0, \ \ y^* \geq \mathbf{0}.$$

Theorem 1. *Matrix $X^* \in \mathbb{R}^{m \times n}$ is a solution of Problem (1.1) if and only if there exists nonnegative matrices $Y_1^*, Y_2^* \in \mathbb{R}^{l_2 \times s_2}$ such that the following conditions are satisfied:*

$$\begin{cases} P_{\mathcal{S}} \left\{ \mathcal{A}^T (\mathcal{A}(X^*) - C) - \mathcal{G}^T (Y_1^* - Y_2^*) \right\} = \mathbf{0} \\ \mathcal{G}(X^*) - L \geq \mathbf{0} \\ U - \mathcal{G}(X^*) \geq \mathbf{0} \\ \langle Y_1^*, \mathcal{G}(X^*) - L \rangle = 0 \\ \langle Y_2^*, U - \mathcal{G}(X^*) \rangle = 0. \end{cases} \qquad (3.11)$$

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Proof. Assume that there are nonnegative matrices $Y_1^*, Y_2^* \in \mathbb{R}^{l_2 \times s_2}$ such that the conditions (3.11) are satisfied. Let

$$f(X) = \frac{1}{2}\|\mathcal{A}(X) - C\|^2$$

and

$$\widetilde{f}(X) = f(X) + \langle Y_1^*, L - \mathcal{G}(X) \rangle + \langle Y_2^*, \mathcal{G}(X) - U \rangle.$$

Then for any $\widetilde{W} \in \mathcal{S}$, we have

$$
\begin{aligned}
&\widetilde{f}(X^* + \widetilde{W}) \\
&= \tfrac{1}{2}\|\mathcal{A}(X^* + \widetilde{W}) - C\|^2 + \left\langle Y_1^*, L - \mathcal{G}(X^* + \widetilde{W}) \right\rangle + \left\langle Y_2^*, \mathcal{G}(X^* + \widetilde{W}) - U \right\rangle \\
&= \widetilde{f}(X^*) + \tfrac{1}{2}\|\mathcal{A}(\widetilde{W})\|^2 + \left\langle \mathcal{A}(\widetilde{W}), \mathcal{A}(X^*) - C \right\rangle - \left\langle Y_1^* - Y_2^*, \mathcal{G}(\widetilde{W}) \right\rangle \\
&= \widetilde{f}(X^*) + \tfrac{1}{2}\|\mathcal{A}(\widetilde{W})\|^2 + \left\langle \widetilde{W}, \mathcal{A}^T(\mathcal{A}(X^*) - C) - \mathcal{G}^T(Y_1^* - Y_2^*) \right\rangle \\
&= \widetilde{f}(X^*) + \tfrac{1}{2}\|\mathcal{A}(\widetilde{W})\|^2 + \tfrac{1}{2}\left\langle \widetilde{W}, P_{\mathcal{S}}(\mathcal{A}^T(\mathcal{A}(X^*) - C) - \mathcal{G}^T(Y_1^* - Y_2^*)) \right\rangle \\
&= \widetilde{f}(X^*) + \tfrac{1}{2}\|\mathcal{A}(\widetilde{W})\|^2 \\
&\geq \widetilde{f}(X^*).
\end{aligned}
$$

This implies that X^* is a global minimizer of the function $\widetilde{f}(X)$. Since $\langle Y_1^*, \mathcal{G}(X^*) - L \rangle = 0$, $\langle Y_2^*, U - \mathcal{G}(X^*) \rangle = 0$ and $\widetilde{f}(X) \geq \widetilde{f}(X^*)$ for all $X \in \mathcal{S}$, we have

$$
\begin{aligned}
f(X) \;&\geq\; f(X^*) + \left\langle Y_1^*, L - \mathcal{G}(X^*) \right\rangle + \left\langle Y_2^*, \mathcal{G}(X^*) - U \right\rangle - \left\langle Y_1^*, L - \mathcal{G}(X) \right\rangle - \left\langle Y_2^*, \mathcal{G}(X) - U \right\rangle \\
&= f(X^*) - \left\langle Y_1^*, L - \mathcal{G}(X) \right\rangle - \left\langle Y_2^*, \mathcal{G}(X) - U \right\rangle.
\end{aligned}
$$

Hence, we have from $Y_1^* \geq \mathbf{0}$ and $Y_2^* \geq \mathbf{0}$ that $f(X) \geq f(X^*)$ for all $X \in \mathcal{S}$ with $\mathcal{G}(X) - L \geq \mathbf{0}$ and $U - \mathcal{G}(X) \geq \mathbf{0}$. Hence X^* is a solution to Problem (1.1).

Conversely, assuming that X^* is a solution to Problem (1.1), then X^* certainly satisfies the Karush-Kuhn-Tucker conditions of Problem (1.1). That is, there exists a nonnegative matrix Y^* such that satisfies conditions (3.11).

We now define the following Powell-Hestenes-Rockafellar(PHR) Augmented Lagrangian function

$$L_\rho(X, Z_1, Z_2) = \frac{1}{2}\|\mathcal{A}(X) - C\|^2 + \frac{\rho}{2}\left\|\left(L - \mathcal{G}(X) + \frac{Z_1}{\rho}\right)_+\right\|^2 + \frac{\rho}{2}\left\|\left(\mathcal{G}(X) - U + \frac{Z_2}{\rho}\right)_+\right\|^2, \tag{3.12}$$

where $Z_1 \geq \mathbf{0}$ and $Z_2 \geq \mathbf{0}$ are the Lagrangian multiplier matrices and $\rho > 0$ is the penalty parameter. Clearly, the partial derivative of function $L_\rho(X, Z_1, Z_2)$ with respect to X is given by

$$\nabla_X L_\rho(X, Z_1, Z_2) = \mathcal{A}^T(\mathcal{A}(X) - C) - \rho\mathcal{G}^T\left(\left(L - \mathcal{G}(X) + \frac{Z_1}{\rho}\right)_+ - \left(\mathcal{G}(X) - U + \frac{Z_2}{\rho}\right)_+\right).$$

The augmented Lagrangian method proposed by E.G. Birgin et al in in [27, 28] (with necessary modifications) to solve Problem (1.1) can be described as follows:

Algorithm PHR-AL. *(The PHR-AL method for solving Problem* (1.1)*.)*

1. Input coefficient matrices $A_i, B_i(i = 1, \ldots p)$ in the linear operator $\mathcal{A}$ and matrices $E_i, E_j(i = 1, \ldots q)$ in the linear operator $\mathcal{G}$. Input matrices C, L, U and a large parameter matrix $Z_{max} > \mathbf{0}$. Input $\gamma > 1$, $r \in (0, 1)$, $\rho_1 > 0$, a small tolerance $\varepsilon > 0$ and tolerance $\varepsilon_k \downarrow 0$. Choose initial matrices $\overline{Z}_1^1$ and $\overline{Z}_2^1$ with $\mathbf{0} \leq \overline{Z}_1^1, \overline{Z}_2^1 \leq Z_{max}$. Set $k \leftarrow 1$.

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2. Compute X^k as an approximate stationary point of

$$\text{minimize} \quad L_{\rho_k}(X, \overline{Z}_1^k, \overline{Z}_2^k) \quad \text{subject to } X \in \mathcal{S}. \tag{3.13}$$

That is, compute X^k such that $\left\| P_{\mathcal{S}}\{\nabla L_{\rho_k}(X^k, \overline{Z}_1^k, \overline{Z}_2^k)\} \right\| < \varepsilon_k$.

3. Define

$$Z_1^k = \left(\overline{Z}_1^k + \rho_k(L - \mathcal{G}(X^k))\right)_+, \quad Z_2^k = \left(\overline{Z}_2^k + \rho_k(\mathcal{G}(X^k) - U)\right)_+.$$

4. If $k = 1$ or

$$\left(\left\|\{\mathcal{G}(X^k) - L, Z_1^k\}_-\right\|^2 + \left\|\{U - \mathcal{G}(X^k), Z_2^k\}_-\right\|^2\right)^{1/2}$$
$$\leq r\left(\left\|\{\mathcal{G}(X^{k-1}) - L, Z_1^{k-1}\}_-\right\|^2 + \left\|\{U - \mathcal{G}(X^{k-1}), Z_2^{k-1}\}_-\right\|^2\right)^{1/2}, \tag{3.14}$$

define $\rho_{k+1} = \rho_k$. Else, define $\rho_{k+1} = \gamma \rho_k$.

5. If

$$\left(\left\|P_{\mathcal{S}}\{\nabla L_{\rho_k}(X^k, \overline{Z}_1^k, \overline{Z}_2^k)\}\right\|^2 + \left\|\{\mathcal{G}(X^k) - L, Z_1^k\}_-\right\|^2 + \left\|\{U - \mathcal{G}(X^k), Z_2^k\}_-\right\|^2\right)^{1/2} < \varepsilon,$$

then stop.

6. Update $\overline{Z}_1^{k+1}$ and $\overline{Z}_2^{k+1}$ with $\mathbf{0} \leq \overline{Z}_1^{k+1}, \overline{Z}_2^{k+1} \leq Z_{max}$ in such a way that $(\overline{Z}_1^{k+1})_{ij} = (Z_1^k)^{ij}$ and $(\overline{Z}_2^{k+1})_{ij} = (Z_2^k)^{ij}$ if $0 \leq (Z_1^k)_{ij}, (Z_2^k)_{ij} \leq (Z_{max})_{ij}, i = 1, 2, \ldots, p, j = 1, 2, \ldots, q$.

7. Set $k \leftarrow k + 1$ and go to step 2.

Problem (3.13) in Algorithm PHR-AL is a linear constrained matrix minimization problem. It is certainly solvable for all the known matrices and the scalar ρ_k. Here we will use the spectral projected gradient (SPG) method to compute the approximation stationary point X^k of problem (3.13). The SPG method is a nonmonotone projected gradient type method for minimizing general smooth functions on convex sets[27]. The SPG method is simple, easy to code, and does not require matrix factorizations. Moreover, it overcomes the traditional slowness of the gradient method by incorporating a spectral step length and a nonmonotone globalization strategy. The main steps of SPG algorithm (with necessary modifications) to compute an approximate stationary point of problem (3.13) can be described as follows:

Algorithm SPG. *(Compute an approximate stationary point of problem* (3.13)*)*

1. Input matrices $\overline{Z}_1^k$ and $\overline{Z}_2^k$; an integer $M > 1$, parameters $\alpha_{\min} > 0$, $\alpha_{\max} > \alpha_{\min}$, $\widetilde{\gamma} \in (0, 1)$, $0 < \sigma_1 < \sigma_2 < 1$ and $\alpha_1 \in [\alpha_{\min}, \alpha_{\max}]$. Choose an initial matrix $X_1 \in \mathcal{S}$ and let $i \leftarrow 1$.

2. If $\left\| P_{\mathcal{S}}\{\nabla L_{\rho_k}(X_i, \overline{Z}_1^k, \overline{Z}_2^k)\} \right\| < \varepsilon_k$, stop. (In this case, X_i is an approximate stationary point of problem (3.13).)

3. Compute $dX_i = -\alpha_i P_{\mathcal{S}}\{\nabla L_{\rho_k}(X_i, \overline{Z}_1^k, \overline{Z}_2^k)\}$. Let $\lambda = 1$.

4. Compute $\check{X} = X_i + \lambda dX_i$.

5. If

$$L_{\rho_k}(\check{X}, \overline{Z}_1^k, \overline{Z}_2^k) \leq \max_{1 \leq j \leq \min\{i, M\}} L_{\rho_k}(X_{i-j}, \overline{Z}_1^k, \overline{Z}_2^k) + \widetilde{\gamma}\lambda\left\langle dX_i, \nabla L_{\rho_k}(X_i, \overline{Z}_1^k, \overline{Z}_2^k)\right\rangle, \tag{3.15}$$

define $\lambda_i = \lambda$, $X_{i+1} = \check{X}$, $s_i = X_{i+1} - X_i$, $y_i = \nabla L_{\rho_k}(X_{i+1}, \overline{Z}_1^k, \overline{Z}_2^k) - \nabla L_{\rho_k}(X_i, \overline{Z}_1^k, \overline{Z}_2^k)$. Then goto step 6. If (3.15) does not hold, define $\lambda_{new} \in [\sigma_1\lambda, \sigma_2\lambda]$, Let $\lambda = \lambda_{new}$ and goto step 4.

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6. Compute $b_i = \langle s_i, y_i \rangle$. If $b_i \leq 0$, let $\alpha_i = \alpha_{\max}$, otherwise, compute

$$a_i = \langle s_i, s_i \rangle, \quad \alpha_i = \min\{\alpha_{\max}, \max\{\alpha_{\min}, a_i/b_i\}\}.$$

7. Let $i \leftarrow i + 1$ and goto step2.

In the practical implementation of Algorithm PHR-AL, similarly to [27], we take the parameters $\gamma = 5$, $r = 0.5$, $\rho_1 = 1$, and the large matrix $Z_{\max}$ with all elements equal to 10^{10}. The initial matrices $\overline{Z}_1^1$ and $\overline{Z}_2^1$ are chosen as $\overline{Z}_1^1 = \overline{Z}_2^1 = \mathbf{0}$. For the implementation of Algorithm SPG, similarly to [30], we take the parameters $M = 10$, $\gamma = 10^{-4}$, $\alpha_{min} = 10^{-30}$, $\alpha_{max} = 10^{30}$, $\sigma_1 = 0.1$, $\sigma_2 = 0.9$, $\lambda_{new} = (\sigma_1\lambda + \sigma_2\lambda)/2$ and $\alpha_0 = 1$. The initial matrix X_1 is chosen as the $(k-1)$th approximate solution of Algorithm PHR-AL.

Lemma 2. *Assume that X^* is limit point of a sequence generated by Algorithm PHR-AL and the sequence ρ_k is bounded, then we have*

$$L \leq \mathcal{G}(X^*) \leq U.$$

Proof. Let $\mathbb{K}$ be an infinite subset of $\mathbb{N}$ such that $\lim_{k \in \mathbb{K}} X^k = X^*$. Since $\lim_{k \to \infty} \rho_k = \infty$ when (3.14) does not hold, the boundedness of ρ_k implies that there exists $k_0 \in \mathbb{N}$ such that (3.14) takes place for all $k \geq k_0$. Therefore,

$$\lim_{k \in \mathbb{K}} \left\| \{\mathcal{G}(X^k) - L, Z_1^k\}_- \right\| = 0 \quad \text{and} \quad \lim_{k \in \mathbb{K}} \left\| \{U - \mathcal{G}(X^k), Z_2^k\}_- \right\| = 0.$$

Note that $Z_1^k \geq \mathbf{0}$ and $Z_2^k \geq \mathbf{0}$ for all $k \in \mathbb{N}$, we have

$$\lim_{k \in \mathbb{K}} (L - \mathcal{G}(X^k))_+ = \mathbf{0} \quad \text{and} \quad \lim_{k \in \mathbb{K}} (\mathcal{G}(X^k) - U)_+ = \mathbf{0},$$

that is, $\mathcal{G}(X^*) - L \geq \mathbf{0}$ and $U - \mathcal{G}(X^*) \geq \mathbf{0}$.

Lemma 3. *Assume that X^* is limit point of a sequence generated by Algorithm PHR-AL, then X^* is a first-order stationary point of the problem*

$$\text{minimize} \quad \frac{1}{2}\left\{ \left\|(L - \mathcal{G}(X^*))_+\right\|^2 + \left\|(\mathcal{G}(X^*) - U)_+\right\|^2 \right\} \quad \text{subject to} \ X \in \mathcal{S}. \qquad (3.16)$$

In other words, $X^* \in \mathcal{S}$ satisfies

$$P_{\mathcal{S}}\left\{ \mathcal{G}^T\left((L - \mathcal{G}(X^*))_+ - (\mathcal{G}(X^*) - U)_+\right)\right\} = \mathbf{0}.$$

Proof. Let $\mathbb{K}$ be an infinite subset of $\mathbb{N}$ such that $\lim_{k \in \mathbb{K}} X^k = X^*$. Consider first the case in which the sequence ρ_k is bounded. By the proof of Lemma 2, we have that

$$\lim_{k \in \mathbb{K}} \left\|(L - \mathcal{G}(X^k))_+\right\| = 0 \quad \text{and} \quad \lim_{k \in \mathbb{K}} \left\|(\mathcal{G}(X^k) - U)_+\right\| = 0.$$

Note that

$$\left\| \mathcal{G}^T\left((L - \mathcal{G}(X^*))_+\right)\right\| \leq \left\|(L - \mathcal{G}(X^*))_+\right\| \quad \text{and} \quad \left\| \mathcal{G}^T\left((\mathcal{G}(X^*) - U)_+\right)\right\| \leq \left\|(\mathcal{G}(X^*) - U)_+\right\|,$$

we have that

$$\lim_{k \in \mathbb{K}} \left\| \mathcal{G}^T\left((L - \mathcal{G}(X^k))_+ - (\mathcal{G}(X^k) - U)_+\right)\right\| = 0.$$

Since $X^k \in \mathcal{S}$ for all k, this implies the desired result in the case that $\{\rho_k\}$ is bounded.

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Assume now that $\{\rho_k\}$ is not bounded. Therefore there exists an infinite sequence of indices $\mathbb{K}' \subset \mathbb{K}$ such that $\lim\limits_{k \in \mathbb{K}'} \rho_k = \infty$. Note that $\varepsilon_k \downarrow 0$ and $\left\| P_{\mathcal{S}}\{\nabla L_{\rho_k}(X^k, \overline{Z}_1^k, \overline{Z}_2^k)\} \right\| < \varepsilon_k$, we have

$$\lim_{k \in \mathbb{K}} \left\| P_{\mathcal{S}}\left\{ \mathcal{A}^T\left(\mathcal{A}(X^k) - C\right) - \rho_k \mathcal{G}^T\left(\left(L - \mathcal{G}(X^k) + \frac{\overline{Z}_1^k}{\rho_k}\right)_+ - \left(\mathcal{G}(X^k) - U + \frac{\overline{Z}_2^k}{\rho_k}\right)_+\right)\right\}\right\| = 0.$$

Therefore we have

$$\lim_{k \in \mathbb{K}} \left\| P_{\mathcal{S}}\left\{ \mathcal{A}^T\left(\mathcal{A}(X^k) - C\right)\big/\rho_k - \mathcal{G}^T\left(\left(L - \mathcal{G}(X^k) + \frac{\overline{Z}_1^k}{\rho_k}\right)_+ - \left(\mathcal{G}(X^k) - U + \frac{\overline{Z}_2^k}{\rho_k}\right)_+\right)\right\}\right\| = 0.$$

Since $\{X^k\}$, $\{\overline{Z}_1^k\}$ and $\{\overline{Z}_2^k\}$ are bounded, we obtain

$$\left\| P_{\mathcal{S}}\{\mathcal{G}^T\left((L - \mathcal{G}(X^*))_+ - (\mathcal{G}(X^*) - U)_+\right)\}\right\| = 0.$$

This implies that X^* is a stationary point of (3.16).

Theorem 2. *Assume that X^* is limit point of a sequence generated by Algorithm PHR-AL and the sequence $\{\rho_k\}$ is bounded, then X^* is a solution to Problem* (1.1).

Proof. Let $\mathbb{K}$ be an infinite subset of $\mathbb{N}$ such that

$$\lim_{k \in \mathbb{K}} X^k = X^*, \quad \lim_{k \in \mathbb{K}} \rho_k = \rho^*, \quad \lim_{k \in \mathbb{K}} \overline{Z}_1^k = \overline{Z}_1^* \quad \text{and} \quad \lim_{k \in \mathbb{K}} \overline{Z}_2^k = \overline{Z}_2^*.$$

By Lemma 2, we have $L \leq \mathcal{G}(X^*) \leq U$. Since

$$\left\| P_{\mathcal{S}}\{\nabla L_{\rho_k}(X^k, \overline{Z}_1^k, \overline{Z}_2^k)\}\right\| < \varepsilon_k$$

holds for all $\varepsilon_k \downarrow 0$, we have

$$\left\| P_{\mathcal{S}}\{\nabla L_{\rho^*}(X^*, \overline{Z}_1^*, \overline{Z}_2^*)\}\right\| = 0. \tag{3.17}$$

Let

$$Y_1^* = \rho^*\left(L - \mathcal{G}(X^*) + \overline{Z}_1^*/\rho^*\right)_+ \quad \text{and} \quad Y_2^* = \rho^*\left(\mathcal{G}(X^*) - U + \overline{Z}_2^*/\rho^*\right)_+,$$

then $Y_1^* \geq \mathbf{0}$ and $Y_2^* \geq \mathbf{0}$, and, from (3.17), we have

$$P_{\mathcal{S}}\{\mathcal{A}^T\left(\mathcal{A}(X^*) - C\right) - \mathcal{G}^T(Y^* - Z^*)\} = \mathbf{0}.$$

Since $\{\rho_k\}$ is bounded, then there exists $k_0 \in \mathbb{N}$ such that (3.14) takes place for all $k \geq k_0$. Hence, we have

$$\lim_{k \to \infty}\{\mathcal{G}(X^k) - L, \ Z_1^k\}_- = \{\mathcal{G}(X^*) - L, \ Z_1^*\}_- = \mathbf{0}$$

and

$$\lim_{k \to \infty}\{U - \mathcal{G}(X^k), \ Z_2^k\}_- = \{U - \mathcal{G}(X^*), \ Z_2^*\}_- = \mathbf{0},$$

which imply that $\langle \mathcal{G}(X^*) - L, \ Z_1^* \rangle = 0$ and $\langle U - \mathcal{G}(X^*), \ Z_2^* \rangle = 0$. By the definition of Z_1^k, Z_2^k and Y_1^*, Y_2^* we know that $(Z_1^*)_{ij} > 0$ if and only if $(Y_1^*)_{ij} > 0$ and $(Z_2^*)_{ij} > 0$ if and only if $(Y_2^*)_{ij} > 0$ $(i = 1, 2, \ldots, l_2, j = 1, 2, \ldots, s_2)$. So we have $\langle \mathcal{G}(X^*) - L, Y_1^* \rangle = 0$ and $\langle U - \mathcal{G}(X^*), Y_2^* \rangle = 0$. Hence X^* satisfies conditions (3.11). By Theorem 1, we know that X^* is a solution to Problem (1.1).

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4. Numerical examples

In this section, we first report some numerical results when Algorithm PHR-AL is implemented to solve Problem (1.1) with random data, and then we illustrate the applicability when the algorithm is applied to solve the model (2.10) in image restoration. All the tested algorithms were coded by MATLAB 7.8 (R2009a) and all our computational experiments were run on a personal computer with an Intel(R) Core i3 processor at 2.13 GHz with 2.00 GB of memory.

4.1. Tested with random data

In this example, we test the two linear operators as $\mathcal{A}(X) = A_1 X B_1 + A_2 X B_2$ and $\mathcal{G}(X) = E_1 X F_1$, and $\mathcal{S}$ as the set of all real $m \times n$ rectangular centrosymmetric matrices[31].

Example 1. Given the matrices $A_1, B_1, A_2, B_2, E_1, F_1, C, L$ and U in Matlab style as follows:

$$A_1 = randn(l_1, m), \quad B_1 = randn(n, s_1), \quad A_2 = randn(l_1, m), \quad B_2 = randn(n, s_1),$$
$$E_1 = rand(l_2, m), \quad F_1 = rand(n, s_2), \quad C = A_1 \overline{X} B_1 + A_2 \overline{X} B_2,$$
$$L = E_1 \overline{X} F_1 - 10 * ones(l_2, s_2), \quad U = E_1 \overline{X} F_1 + 10 * ones(l_2, s_2),$$

where $\overline{X} = Z + S_m Z S_n$ with $Z = rand(m, n)$. Matrices L, U and C are chosen in this way to guarantee that Problem (1.1) is solvable.

Note that the Algorithm PHR-AL involve an outer iteration and an inner iteration, the convergence stopping criterion of the outer iterations are all set to be $\varepsilon = 10^{-8}$, and the small tolerance ε_k in the inner iterations is set to

$$\varepsilon_0 = 10^0 \quad \text{and} \quad \varepsilon_k = \begin{cases} 0.1\varepsilon_{k-1} & \text{if } \varepsilon_{k-1} > \varepsilon, \\ \varepsilon_{k-1} & \text{if } \varepsilon_{k-1} < \varepsilon. \end{cases} \tag{4.18}$$

The largest number of the inner iteration is set to be 200. We consider the following two cases to be tested: (a) $l_1 \geq m$ and $s_1 \geq n$ and (b) $l_1 < m$ and $s_1 < n$.

Table 2: Numerical results for the case (a) $l_1 \geq m$ and $s_1 \geq n$ in Example 1.

l_1, m, n, s_1, l_2, s_2	CPU	$\frac{\|X^* - \overline{X}\|}{\|\overline{X}\|}$
10,10,10,10,10,10	0.1248	5.1294×10^{-11}
30,18,20,30,25,30	0.3588	3.4006×10^{-13}
50,50,50,50,50,50	3.4476	1.2540×10^{-12}
80,60,70,100,80,80	4.0404	6.7827×10^{-14}
100,100,100,100,100,100	13.3537	6.7580×10^{-14}
150,100,100,150,120,120	10.1401	4.8226×10^{-15}
150,150,150,150,150,150	44.2263	4.7307×10^{-14}
200,180,180,200,150,150	53.3367	1.2976×10^{-14}
250,250,250,250,200,200	161.7106	1.1052×10^{-13}

For case $l_1 \geq m$ and $s_1 \geq n$, Problem (1.1) has unique solution and the true solution is $\overline{X}$. Therefore in Table 2, we report the mean computing time in seconds and the mean relative error based on their average values of 10 repeated tests with randomly generated matrices A_1, B_1, A_2, B_2, E_1 and F_1 for each problem size. Here the relative error is defined as $Re = \frac{\|X^* - \overline{X}\|}{\|\overline{X}\|}$, where X^* is the estimated solution.

For case $l < n$ and $s < n$, as Problem(1.1) has multiple solutions, the algorithm is not guaranteed to converge to the solution $\overline{X}$, it is not meaningful to record the relative errors. In this case, we report the mean

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Table 3: Numerical results for the case (b)$l_1 < m$ and $s_1 < n$ in Example 1.

l_1, m, n, s_1, l_2, s_2	CPU	$\|A_1XB_1 + A_2XB_2 - C\|$
6,10,10,6,10,10	0.1560	9.3373×10^{-9}
15,30,25,15,20,20	0.7644	1.4437×10^{-9}
30,60,75,35,50,50	4.2432	3.2598×10^{-10}
50,120,125,65,80,80	17.6749	2.6637×10^{-10}
50,200,200,50,100,100	43.9299	7.1933×10^{-11}
70,150,150,70,120,120	44.9595	1.7993×10^{-10}
100,200,200,100,150,150	132.8817	1.7718×10^{-10}
100,300,300,100,180,180	348.3970	5.1966×10^{-11}

computing time in seconds and the mean residual $\|A_1XB_1 + A_2XB_2 - C\|$ (see Table 3) based on 10 repeated tests with randomly generated matrices A, B, E and F for each problem size in each test.

4.2. Application to image restoration with some special symmetry pattern images

In this subsection, we test the efficiency when Algorithm PHR-AL is applied to solve the model (2.10) in image restoration. We only focus on some special symmetry pattern images. The original image is denoted by $\widehat{X}$ in each example and it consists of $m \times n$ grayscale pixel values in the range $[0, d]$ with $d = 255$ is the maximum possible pixel value of the image. Let $\hat{x} = vec(\widehat{X})$ denotes the vector obtained by stacking the columns of $\widehat{X}$ and H represents the blurring matrix. The vector $\hat{g} = H\hat{x}$ represents the associated blurred and noise-free image. In our tests, similarly to [24], we generated a blurred and noisy image g by

$$g = \hat{g} + \mathbf{n}_0 \times \sigma_{\hat{x}} \times 10^{-\frac{SNR}{20}},$$

where $\mathbf{n}_0$ is a random vector noise with a zero mean and a variance equal to one, and SNR is the signal to noise ratio defined by

$$SNR = 10 \log_{10} \Big(\frac{\sigma_{\hat{x}}^2}{\sigma_{\mathbf{n}}^2} \Big),$$

where $\sigma_{\hat{x}}^2$ and $\sigma_{\mathbf{n}}^2$ are the variance of the noise and the original image, respectively. The performance of the Algorithm PHR-AL and its comparison are evaluated by the peak signal-to-noise ratio (PSNR) in decible (dB):

$$PSNR(X) = 10 \log_{10} \Big(\frac{d^2 mn}{\|\hat{x} - x\|_2^2} \Big) = 10 \log_{10} \Big(\frac{d^2 mn}{\|\widehat{X} - X\|^2} \Big).$$

In all the tests, the largest number of the involved inner iteration(Algorithm SPG) in the Algorithm PHR-AL is set to be 20. The algorithm started with the degraded images and terminated when the relative difference between the successive iterates of the restored image satisfy

$$R_{error} = \frac{\|X^{k+1} - X^k\|}{\|X^k\|} \leq 0.5 \times 10^{-2}.$$

Example 2. In the first example, we consider the *"butterfly"* original image of size 192×254 and is shown on the left side of Figure 1. The original image has perfectly mirror-symmetry[32], that is, the pixel value
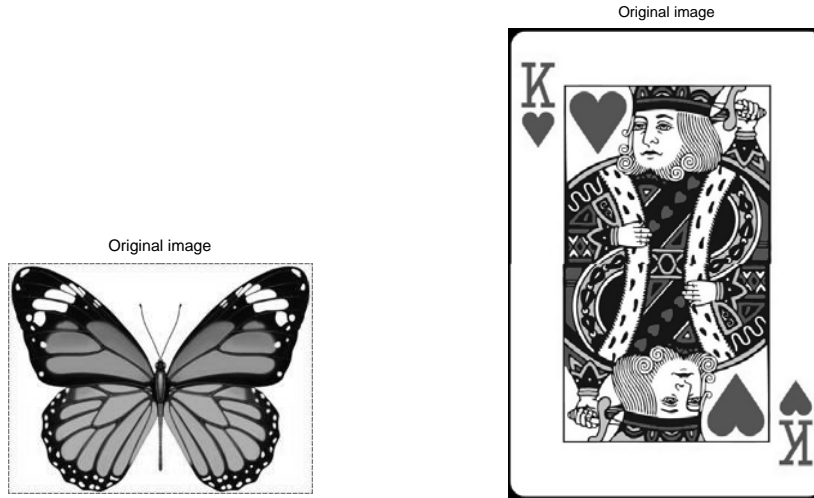
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Original image

Original image

Figure 1: Original images. Left: "*Butterfly*"(mirror-symmetric). Right: "*PlayCard-K-Heart*" (centro-symmetric).

matrix $\widehat{X}$ can be expressed as $\widehat{X} = (X_L, X_L S_n)$, where X_L is the left half of the matrix $\widehat{X}$. Actually, we have $\|\widehat{X} - \mathcal{P}_S(\widehat{X})\| = 0$, where $\mathcal{S}$ is the set of all real 192×254 column mirror-symmetry matrices and

$$\mathcal{P}_S(X) = \left(\frac{X_L + X_R S_n}{2}, \frac{X_L S_n + X_R}{2} \right), \quad \forall X \in \mathbb{R}^{192 \times 254}$$

where X_R is the left half and the right half of X. The blurring matrix H is chosen to be $H = H_1 \otimes H_2 \in \mathbb{R}^{192^2 \times 254^2}$, where $H_1 = [h_{ij}^{(1)}] \in \mathbb{R}^{192 \times 192}$ and $H_2 = [h_{ij}^{(2)}] \in \mathbb{R}^{254 \times 254}$ are the Toeplitz matrices whose entries are given by

$$h_{ij}^{(1)} = \begin{cases} \frac{1}{\sigma \sqrt{2\pi}} exp\left(-\frac{(i-j)^2}{2\sigma^2} \right), & |i-j| \le r, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad h_{ij}^{(2)} = \begin{cases} \frac{1}{2r-1}, & |i-j| \le r, \\ 0, & \text{otherwise}. \end{cases}$$

In this example we choose the band $r = 3$ and the variance $\sigma = 0.4$. A random Gaussian noise, with $SNR = 15dB$, was added to produce a blurred and noisy image G with $PSNR(G) = 8.1411$. The blurred and noisy image is shown on the left side of Figure 4. The restoration of the image from the degraded image is obtained by solving the minimization problem (2.10) using the PHR-AL algorithm. The regularization matrix T is chosen to be $T = T_1 \otimes T_2 \in \mathbb{R}^{192^2 \times 254^2}$, where $T_1 = I_{192}$ and T_2 is the tridiagonal matrix, of size 254×254, generated by vector $(1, 2, 1)$. The optimal value of the parameter $\lambda = 0.015$ was obtained by using the GCV method. The corresponding GCV curve is plotted on the right side of Figure 2.

The restored image obtained by using Algorithm PHR-AL is given on the left of Figure 4, the relative error was $Re(X) = 1.2521 \times 10^{-1}$ with $PSNR(X) = 21.0231$, and the iterations are terminated after 3 iterations with a cpu time of 13.9309 s. Table 1 reports on more results for three levels of noise corresponding to different $SNR = 5, 10, 15$ and to different values of $\sigma = 0.35, 0.55, 0.85$ given in the definition of the blurring matrices H_1 and H_2 in Example 2.

Example 3. In the second example, the original image is the "*PlayCard-K-Heart*" image of size 628×423 and is shown on the right side of Figure 1. The original image is centrosymmetric, that is, the pixel value

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Table 4: *Results for Example 3*.

σ	$SNR(dB)$	λ_{opt}	$PSNR(G)(dB)$	$PSNR(X)(dB)$	$Re(X)$	CPU-times(s)
	5	0.036	5.3075	19.6357	1.4690×10^{-1}	23.4002
	10	0.025	6.0042	20.9344	1.2650×10^{-1}	17.8621
0.35	15	0.017	6.4097	21.3394	1.2073×10^{-1}	18.0337
	20	0.011	6.6397	21.6077	1.1706×10^{-1}	18.3145
	25	0.007	6.7709	21.9395	1.1267×10^{-1}	19.5781
	5	0.036	8.3142	18.7410	1.6284×10^{-1}	29.6090
	10	0.025	9.3290	21.1153	1.2389×10^{-1}	40.3419
0.55	15	0.018	9.9286	21.8547	1.1378×10^{-1}	38.4386
	20	0.012	10.2655	21.9397	1.1267×10^{-1}	28.2830
	25	0.008	10.4569	21.1417	1.2351×10^{-1}	18.8137
	5	0.035	8.4387	18.5387	1.6667×10^{-1}	38.4542
	10	0.026	9.4712	20.7428	1.2932×10^{-1}	39.1875
0.85	15	0.019	10.0763	20.9952	1.2561×10^{-1}	27.9086
	20	0.014	10.4170	20.5296	1.3253×10^{-1}	12.9949
	25	0.010	10.6154	20.7946	1.2855×10^{-1}	18.8137
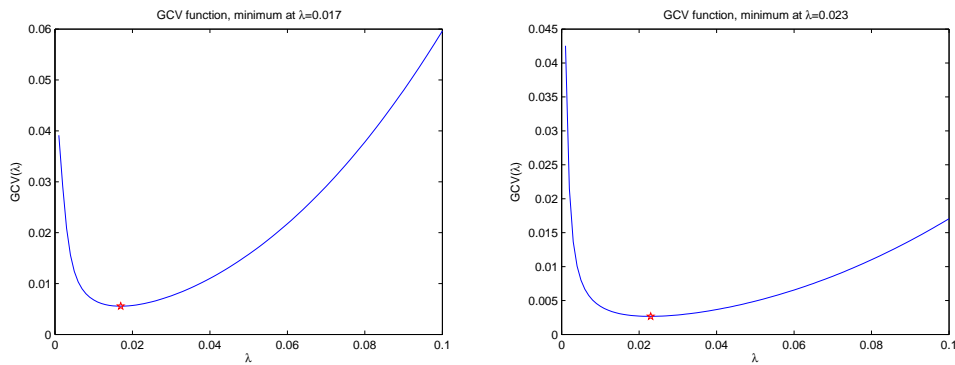
Figure 2: The GCV curve for the Example 2 with the optimal value of $\lambda = 0.017$ (left) and the GCV curve for the Example 3 with the optimal value of $\lambda = 0.023$.
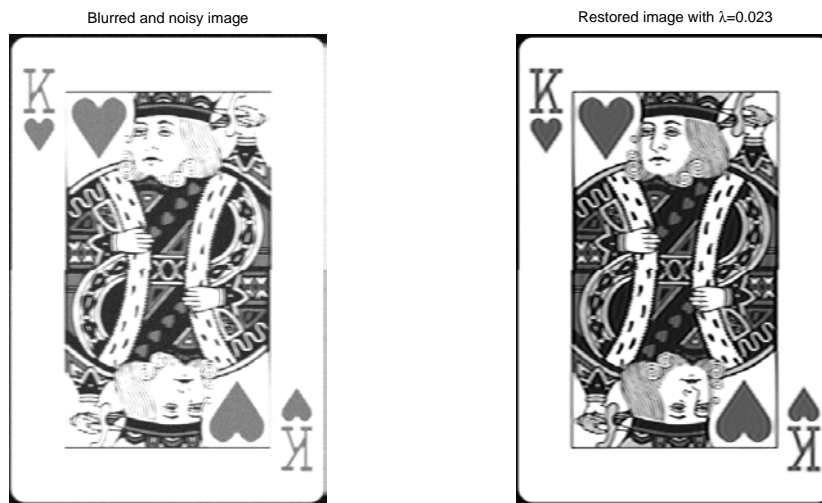

Figure 3: The blurred and noisy image (left) with $PSNR(G) = 8.1411$, $r = 3$ and $\sigma = 0.45$ and the restored image (right) with $PSNR(X) = 21.0231$ and $Re(X) = 1.2521 \times 10^{-1}$.

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matrix $\widehat{X}$ satisfies $\widehat{X} = S_{628}\widehat{X}S_{423}$. Actually, we have $\|\widehat{X} - P_{\mathcal{S}}(\widehat{X})\| = 0$, where $\mathcal{S}$ is the set of all real 628×423 rectangle centrosymmetry matrices and $P_{\mathcal{S}} = \frac{1}{2}(X + S_{628}XS_{423})$ for any $X \in \mathbb{R}^{628\times423}$. The blurring matrix H is chosen to be $H = H_1 \otimes H_2 \in \mathbb{R}^{256^2\times256^2}$, where $H_1 = I_{628}$ is the identity matrix and $H_2 = [h_{ij}^{(2)}]$ is the Toeplitz matrices of dimension 423×423 given by

$$h_{ij}^{(2)} = \begin{cases} \frac{1}{2r-1}, & |i-j| \le r, \\ 0, & \text{otherwise.} \end{cases}$$

The blurring matrix H models a uniform blur. The regularization matrix T is chosen to be $T = T_1 \otimes T_2 \in$

Figure 4: The blurred and noisy image (left) with $PSNR(G) = 8.0481$, $r = 3$ and $\sigma = 0.45$ and the restored image (right) with $PSNR(X) = 20.1459$ and $Re(X) = 1.5784 \times 10^{-1}$.

$\mathbb{R}^{256^2\times256^2}$, where T_1 and T_2 are similar to the ones given in Example 2. In this example we set $r = 3$ and a random Gaussian noise, with $SNR = 15dB$, was added to produce a blurred and noisy image G with $PSNR(G) = 8.0481$. The obtained image is shown on the middle of Figure 2. The optimal value of the parameter $\lambda = 0.023$ was obtained by using the GCV method. The corresponding GCV curve is plotted on the right side of Figure 2.

The restored image obtained by using our proposed Algorithm PHR-AL is also denoted by X and it is given on the right side of Figure 4. The relative error was $Re(X) = 1.5784 \times 10^{-1}$ with the $PSNR(X) = 20.1459$. The iterations are terminated after 5 iterations with a cpu time of 86.9699s.

5. Conclusion

In this paper, we consider solving a class of inequality constrained matrix-form minimization problems, whose various simplified versions have been studied extensively. These matrix-form minimization problems problem can be transformed into the convex linearly constrained quadratic programming in the vector-form by using the vec operator $vec(.)$ and the Kronecker produc $\otimes$. However, using this transformation will destroy the preindicated linear structure of the unknown matrix and will increase computational complexity and storage requirement. In this paper we will consider the problem from a general point of view and

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directly from the perspective of matrices. We propose, analyze and test a matrix-form iteration algorithm framework with the augmented Lagrangian method for solving this problem and its reduced versions which are applicable in image restoration. The numerical results, including when the algorithm is tested with some randomly generated data and on some image restoration problems with special symmetry pattern images, illustrate the effectiveness of the proposed algorithm.

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