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Dynamics of a difference equation with maximum

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Abstract The purpose of this work is to investigate the convergence of the solutions of the following max-type difference equation

$$z_n = \max\{\frac{1}{z_{n-s}}, \frac{P_n}{z_{n-t}^{\alpha_n}}\}, \quad n = 0, 1, 2, \cdots,$$

where $s, t \in \{1, 2, 3, \cdots\}$ with $s \neq t$, $\alpha_n \in (0, 1)$ is an s-periodic sequence, $\{P_n\}_{n=0}^{+\infty}$ is a constant sequence satisfying $P_n \in (0, 1]$ for every $n \geq 0$. We show that if $\{z_n\}_{n=-r}^{+\infty}$ ($r = \max\{s, t\}$) is a positive solution of the above equation with the initial conditions $z_{-r}, z_{-r+1}, \cdots, z_{-1} \in (0, +\infty)$, then $\lim_{n\longrightarrow\infty} z_n = 1$ or $\{z_{2sn+k}\}_{n=0}^{+\infty}$ is eventually monotone for every $0 \leq k \leq 2s-1$. Further, we show that if P_n is a periodic sequence, $s = 1$ and t is even, then $\lim_{n\longrightarrow\infty} z_n = 1$ or $\{z_n\}_{n=-t}^{+\infty}$ is eventually periodic with period 2.

AMS Subject Classification: 39A10; 39A11.

Keywords: max-type equation, positive solution, eventual periodicity, monotonicity, periodic sequence.

1. Introduction

The max operator arises naturally in certain models in automatic control theory (see [6,7]). In the recent years, there has been a lot of interest in studying the convergence and boundedness of max-type difference equations (see [1,3,5,8-11]). In [2], Chen studied the second order max-type difference equation

$$z_{n+1} = \max\{\frac{1}{z_n}, \frac{A_n}{z_{n-1}}\}, \quad n = 0, 1, 2, \cdots, \tag{1.1}$$

and showed that every positive solution of (1.1) is eventually periodic with period 2 when $\{A_n\}_{n=0}^{+\infty}$ is a periodic sequence with period $k \geq 2$ and $A_n \in (0, 1)$ for all $n \geq 0$.

In [4], the authors studied the following non-autonomous max-type difference equation with two delays

$$z_n = \max\{\frac{f_n}{z_{n-m}^{\alpha}}, \frac{A}{z_{n-r}^{\beta}}\}, \quad n = 0, 1, 2, \cdots,$$

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where $\alpha, \beta \in \mathbb{R}$, $\{A_n\}_{n=0}^{+\infty}$ is a sequence of positive real numbers with a finite limit and $m, r \in \mathbb{N} \equiv \{1, 2, 3, \cdots\}$ with $m \neq r$.

In this paper, we study the periodicity, the boundedness and the convergence of the following max-type difference equation

$$z_n = \max\{\frac{1}{z_{n-s}}, \frac{P_n}{z_{n-t}^{\alpha_n}}\}, \quad n = 0, 1, 2, \cdots, \tag{1.2}$$

where $s, t \in \mathbb{N}$ with $s \neq t$, $\alpha_n \in (0, 1)$ is an s-periodic sequence, $\{P_n\}_{n=0}^{+\infty}$ is a constant sequence satisfying $P_n \in (0, 1]$ for every $n \geq 0$.

2. Some Propositions

In the following, suppose that $\{z_n\}_{n=-r}^{+\infty}$ is a positive solution of (1.2). To obtain the main results of this paper, we need the following propositions.

Proposition 2.1 (i) $z_n z_{n-s} \geq 1$ for all $n \geq 0$.

(ii) For any $n \geq r$, $z_n \leq \max\{z_{n-2s}, P_n z_{n-s-t}^{\alpha_{n-s}}\}$.

(iii) If $z_n = P_n / z_{n-t}^{\alpha_n} > 1/z_{n-s}$ for some $n \geq r$, then $z_n > z_{n-2s}$. If $z_n = 1/z_{n-s}$ for some $n \geq s$, then $z_n \leq z_{n-2s}$.

Proof (i) Since $z_n \geq 1/z_{n-s}$ for any $n \geq 0$, we have $z_n z_{n-s} \geq 1$.

(ii) According to (i), we get that for every $n \geq r$,

$$z_n = \max\Big\{\frac{z_{n-2s}}{z_{n-s}z_{n-2s}}, \frac{P_n z_{n-s-t}^{\alpha_n}}{z_{n-s-t}^{\alpha_n} z_{n-t}^{\alpha_n}}\Big\} \leq \max\{z_{n-2s}, P_n z_{n-s-t}^{\alpha_n}\}.$$

(iii) If $z_n = P_n / z_{n-t}^{\alpha_n} > 1/z_{n-s}$ for some $n \geq r$, then by (i) we obtain that

$$\begin{aligned} 1 \quad < \quad & z_n z_{n-s} = \max\{\frac{z_n}{z_{n-2s}}, \frac{z_n z_{n-t}^{\alpha_n} P_{n-s}}{z_{n-t-s}^{\alpha_{n-s}} z_{n-t}^{\alpha_n}}\} \\ \leq \quad & \max\{\frac{z_n}{z_{n-2s}}, P_n P_{n-s}\} = \frac{z_n}{z_{n-2s}}. \end{aligned}$$

Which implies $z_n > z_{n-2s}$. If $z_n = 1/z_{n-s}$ for some $n \geq s$, then by (i) we obtain that

$$z_n = \frac{z_{n-2s}}{z_{n-s}z_{n-2s}} \leq z_{n-2s}.$$

The proof is complete.

Define

$$U_n = \max\{z_{n-1}, z_{n-2}, \cdots, z_{n-s-r}\} \quad (n \geq r). \tag{2.1}$$

According to Proposition 2.1 (i), we get $\max\{z_{n-1}, z_{n-s-1}\} \geq 1$, from which it follows $U_n \geq 1$ for any $n \geq r$.

Proposition 2.2 (i) Let U_n be as in (2.1). Then $z_n \leq U_n$ for any $n \geq r$ and $\{U_n\}_{n=r}^{+\infty}$ is a decreasing sequence.

(ii) There exist constants $R \geq R' > 0$ such that $R' \leq z_n \leq R$ for any $n \geq -r$.

Proof (i) If $z_{n-s-t} \leq 1$, then $z_{n-s-t}^{\alpha_n} \leq 1$. If $z_{n-s-t} \geq 1$, then $z_{n-s-t}^{\alpha_n} \leq z_{n-s-t}$. According to Proposition 2.1 (ii), we have that for any $n \geq r$,

$$z_n \leq \max\{z_{n-2s}, z_{n-s-t}^{\alpha_n}\} \leq \max\{z_{n-1}, z_{n-2}, \cdots, z_{n-s-r}\} = U_n.$$

Further, we get

$$U_{n+1} = \max\{z_n, z_{n-1}, \cdots, z_{n-s-r+1}\} \leq U_n.$$

(ii) Let $R = \max\{U_r, z_{r-1}, \cdots, z_{-r}\}$ and $R' = \min\{1/U_r, z_{r-1}, \cdots, z_{-r}\}$. Then $R' \leq z_n \leq R$ for any $n \geq -r$. The proof is complete.

Now we assume $\lim_{n\longrightarrow\infty} U_n = U$ and $\liminf_{n\longrightarrow\infty} U_n = u$. According to Proposition 2.2 (i), we obtain the following corollary.

Corollary 2.3 There exists a sequence $1 < n_1 < n_2 < \cdots < n_k < \cdots$ such that $z_{n_k} \geq U$ and $n_{k+1} - n_k \leq s + r$.

Proposition 2.4 The following statements hold:

(i) $U = \limsup_{n\longrightarrow\infty} z_n$.

(ii) Assume that $U > 1$. Then $\{n : U \leq z_n = P_n/z_{n-t}^{\alpha_n}\}$ is a finite set. Further, there exists $N \in \mathbb{N}$ such that:

i) $z_{N+2ks} \geq U$ and $z_{N+2ks} = 1/z_{N+(2k-1)s}$ for any $k \geq 0$, and z_{N+2ks} is decreasing.

ii) $\lim_{k\longrightarrow\infty} z_{N+(2k-1)s} = u = 1/U$.

Proof (i) According to (2.1), we see that U_n is a subsequence of z_n. Thus $U \leq \limsup_{n\longrightarrow\infty} z_n$. Further, since $z_n \leq U_n$ for all $n \geq r$, we obtain

$$\limsup_{n\longrightarrow\infty} z_n \leq \limsup_{n\longrightarrow\infty} U_n = U.$$

(ii) If $\{n : U \leq z_n = P_n/z_{n-t}^{\alpha_n}\}$ is an infinite set, then there exists a sequence $t < n_1 < n_2 < \cdots < n_k < \cdots$ such that

$$U \leq z_{n_k} = \frac{P_{n_k}}{z_{n_k-t}^{\alpha_{n_k}}} \leq P_{n_k} z_{n_k-t-s}^{\alpha_{n_k}} \leq z_{n_k-t-s}^{\alpha_{n_k}}.$$

Without loss of generality, suppose that $\lim_{k\longrightarrow\infty} z_{n_k-t-s} = u_1$ and $\lim_{k\longrightarrow\infty} \alpha_{n_k} = \alpha < 1$. Thus we obtain $U = \lim_{k\longrightarrow\infty} z_{n_k} \leq u_1^\alpha \leq U^\alpha < U$ since $U > 1$. A contradiction.

It follows from the above that there exists $M \in \mathbb{N}$ such that if $n \geq M$ and $z_n \geq U$, then $z_n = 1/z_{n-s}$. By Corollary 2.3 we see that there exists a sequence $1 < n_1 < n_2 < \cdots < n_k < \cdots$ such that $z_{n_k} \geq U$ and $\lim_{k\longrightarrow\infty} z_{n_k} = U$. Without loss of generality, suppose that $n_k = 2sr_k + \tau > M$ with $0 \leq \tau < 2s$ for all $k \in \mathbb{N}$. Then $z_{n_k} = 1/z_{n_k-s}$. Write $N = 2sr_1 + \tau$. By Proposition 2.1 (iii), we see that for any $k \geq 0$,

$$z_{N+2ks} \geq U \quad \text{and} \quad \frac{1}{z_{N+2ks-s}} = z_{N+2ks} \geq z_{N+2(k+1)s} = \frac{1}{z_{N+2(k+1)s-s}}.$$

Let $i_k \longrightarrow +\infty$ such that $z_{i_k} \longrightarrow u$ and $z_{i_k-s} \longrightarrow u_1$. Then

$$\frac{1}{U} = \lim_{k\longrightarrow\infty} \frac{1}{z_{N+2ks}} = \lim_{k\longrightarrow\infty} z_{N+(2k-1)s} \geq u = \lim_{k\longrightarrow\infty} z_{i_k} \geq \lim_{k\longrightarrow\infty} \frac{1}{z_{i_k-s}} = \frac{1}{u_1} \geq \frac{1}{U},$$

this implies $\lim_{k\longrightarrow\infty} z_{N+(2k-1)s} = u = 1/U$. The proof is complete.

Proposition 2.5 Let $N, p, q \in \mathbb{N}$ with $q \geq 2$ such that

(i) $\{z_{N+2ks}\}_{k=0}^{+\infty}$ is monotone.

(ii) $z_{N+2s(p+\lambda)+t} = P_{N+2s(p+\lambda)+t}/z_{N+2s(p+\lambda)}^{\alpha_{N+2s(p+\lambda)+t}} > 1/z_{N+2s(p+\lambda)+t-s}$ for every $\lambda \in \{0, q\}$.

(iii) $z_{N+2s(p+\lambda)+t} = 1/z_{N+2s(p+\lambda)+t-s}$ for every $1 \leq \lambda \leq q-1$.
Then $z_{N+2s(p+\lambda)+t} = z_{N+2s(p+\lambda+1)+t}$ for every $0 \leq \lambda \leq q-2$.

Proof There are two cases to be considered.

Case 1 $\{z_{N+2sk}\}_{k=0}^{+\infty}$ is decreasing. In this case, we claim that $z_{N+2s(p+\lambda)+t-s} = 1/z_{N+2s(p+\lambda-1)+t}$ for any $1 \leq \lambda \leq q-1$. Since, otherwise, if for some $1 \leq \lambda \leq q-1$,

$$z_{N+2s(p+\lambda)+t-s} = \frac{P_{N+2s(p+\lambda)+t-s}}{z_{N+2s(p+\lambda)-s}^{\alpha_{N+2s(p+\lambda)+t-s}}} > 1/z_{N+2s(p+\lambda-1)+t},$$

then by Proposition 2.1 (iii) it follows that

$$\begin{aligned}\frac{P_{N+2sp+t}}{z_{N+2s(p+\lambda)}^{\alpha_{N+2s(p+\lambda)+t}}} &\geq \frac{P_{N+2sp+t}}{z_{N+2sp}^{\alpha_{N+2sp+t}}} = z_{N+2sp+t} \geq z_{N+2s(p+\lambda-1)+t} \\ &> \frac{1}{z_{N+2s(p+\lambda)+t-s}} = \frac{z_{N+2s(p+\lambda)-s}^{\alpha_{N+2s(p+\lambda)+t-s}}}{P_{N+2s(p+\lambda)+t-s}}.\end{aligned}$$

This implies

$$1 \geq P_{N+2sp+t}P_{N+2s(p+\lambda)+t-s} > z_{N+2s(p+\lambda)}^{\alpha_{N+2s(p+\lambda)+t}} z_{N+2s(p+\lambda)-s}^{\alpha_{N+2s(p+\lambda)+t-s}} \geq 1.$$

A contradiction. From the above claim it follows that

$$z_{N+2s(p+\lambda)+t} = \frac{1}{z_{N+2s(p+\lambda)+t-s}} = z_{N+2s(p+\lambda-1)+t} \geq z_{N+2s(p+\lambda)+t}.$$

Thus $z_{N+2s(p+\lambda-1)+t} = z_{N+2s(p+\lambda)+t}$ for every $1 \leq \lambda \leq q-1$.

Case 2 $\{z_{N+2ks}\}_{k=0}^{+\infty}$ is increasing. In this case, it follows from Proposition 2.1 (iii) that

$$\begin{aligned}\frac{P_{N+2s(p+q)+t}}{z_{N+2s(p+q-1)}^{\alpha_{N+2s(p+q-1)+t}}} &\geq \frac{P_{N+2s(p+q)+t}}{z_{N+2s(p+q)}^{\alpha_{N+2s(p+q)+t}}} = z_{N+2s(p+q)+t} > z_{N+2s(p+q-1)+t} \\ &= \frac{1}{z_{N+2s(p+q-1)+t-s}} = \min\{z_{N+2s(p+q-2)+t}, \frac{z_{N+2s(p+q-1)-s}^{\alpha_{N+2s(p+q-1)+t-s}}}{P_{N+2s(p+q-1)+t-s}}\} \\ &= z_{N+2s(p+q-2)+t} \geq z_{N+2s(p+q-1)+t}\end{aligned}$$

since

$$P_{N+2s(p+q)+t}P_{N+2s(p+q-1)+r-s} \leq 1 \quad \text{and} \quad z_{N+2s(p+q-1)}^{\alpha_{N+2s(p+q-1)+t}} z_{N+2s(p+q-1)-s}^{\alpha_{N+2s(p+q-1)+t-s}} \geq 1,$$

we have

$$z_{N+2s(p+q-1)+t} = z_{N+2s(p+q-2)+t}.$$

In a similar fashion, we may obtain that $z_{N+2s(p+q-1)+t} = z_{N+2s(p+\lambda)+t}$ for any $0 \leq \lambda \leq q-2$. The proof is complete.

Proposition 2.6 If there exists $N \in \mathbb{N}$ such that $\{z_{N+2ks}\}_{k=0}^{+\infty}$ is monotone, then $\{z_{N+t+2ks}\}_{k=0}^{+\infty}$ is eventually monotone.

Proof If there exists $K \in \mathbb{N}$ such that

$$z_{N+2ks+t} = 1/z_{N+2sk+t-s} \quad \text{for all} \quad k \geq K$$

or

$$z_{N+2ks+t} = P_{N+2ks+t}/z_{N+2ks}^{\alpha_{N+2ks+t}} > 1/z_{N+2ks+t-s} \quad \text{for all} \quad k \geq K,$$

then by Proposition 2.1 (iii) we obtain that $z_{N+2ks+t} \leq z_{N+2(k-1)s+t}$ for all $k \geq K$ (or $z_{N+2ks+t} > z_{N+2(k-1)s+t}$ for all $k \geq K$). Thus $\{z_{N+t+2ks}\}_{k=K}^{+\infty}$ is monotone.

If there exists a sequence $1 < p_1 < q_1 < p_2 < q_2 < \cdots < p_k < q_k < \cdots$ such that

$$z_{N+2rs+t} = \frac{P_{N+2rs+t}}{z_{N+2rs}^{\alpha_{N+2rs+t}}} > \frac{1}{z_{N+2rs+t-s}} \quad \text{for every} \quad p_i \leq r < q_i$$

and

$$z_{N+2rs+t} = \frac{1}{z_{N+2rs+t-s}} \quad \text{for every} \quad q_i \leq r < p_{i+1},$$

then by Proposition 2.1 (iii) and Proposition 2.5 it follows that $z_{N+2(r-1)s+t} < z_{N+2rs+t}$ for every $p_i \leq r < q_i$ and $z_{N+2(r-1)s+t} = z_{N+2rs+t}$ for every $q_i \leq r < p_{i+1}$, this follows that $\{z_{N+t+2rs}\}_{r=p_1}^{+\infty}$ is increasing. The proof is complete.

3. Main Results

In section, we state the main results of this paper.

Theorem 3.1 Let $\{z_n\}_{n=-r}^{+\infty}$ be a positive solution of (1.2). Then $\lim_{n\longrightarrow\infty} z_n = 1$ or $\{z_{2ns+k}\}_{n=0}^{+\infty}$ is eventually monotone for every $0 \leq k \leq 2s-1$.

Proof If $U = \limsup_{n\longrightarrow\infty} z_n = 1$, then let $i_k \longrightarrow +\infty$ such that $z_{i_k} \longrightarrow u = \liminf_{n\longrightarrow\infty} z_n$ and $z_{i_k-s} \longrightarrow u_1$. Thus

$$1 \geq u = \lim_{k\longrightarrow\infty} z_{i_k} \geq \lim_{k\longrightarrow\infty} \frac{1}{z_{i_k-s}} = \frac{1}{u_1} \geq \frac{1}{U} = 1.$$

Which implies $\lim_{n\longrightarrow\infty} z_n = 1$. Now assume that $U = \limsup_{n\longrightarrow\infty} z_n > 1$.

First we suppose that $\gcd(s,t) = 1$. Then by Proposition 2.4 (iii) we see that there exists $N \in \mathbb{N}$ such that the following statements hold:

(1) $z_{N+2ns}z_{N+(2n-1)s} = 1$ for any $n \geq 0$.

(2) z_{N+2ns} is decreasing ($n \geq 0$) and $\lim_{n\longrightarrow\infty} z_{N+2ns} = U$. $x_{N+(2n-1)s}$ is increasing ($n \geq 0$) and $\lim_{n\longrightarrow\infty} z_{N+(2n-1)s} = u = 1/U$.

Using Proposition 2.6 repeatedly, it follows that for every $1 \leq i \leq s-1$, $\{z_{N+2ns+it}\}_{n=0}^{+\infty}$ and $\{z_{N+(2n-1)s+it}\}_{n=0}^{+\infty}$ are eventually monotone. Since $\gcd(s,t) = 1$, it follows that for every $j \in \{0,1,2,\cdots,s-1\}$ there exist some $0 \leq i_j \leq s-1$ and integer λ_j such that $i_jt = \lambda_j s + j$ and $i_jt - s = (\lambda_j - 1)s + j$. Thus $\{z_{N+2ns+\lambda_j s+j}\}_{n=0}^{+\infty}$ and $\{z_{N+2ns+(\lambda_j-1)s+j}\}_{n=0}^{+\infty}$ are eventually monotone for every $j \in \{0,1,2,\cdots,s-1\}$, which implies that $\{z_{2ns+k}\}_{n=0}^{+\infty}$ is eventually monotone for every $0 \leq k \leq 2s-1$.

If $\gcd(s,t) = d > 1$, then we consider the max-type equation

$$z_n = \max\{\frac{1}{z_{n-ds_1}}, \frac{P_n}{z_{n-dt_1}^{\alpha_n}}\}, \quad n = 0,1,2,\cdots\cdots, \tag{3.1}$$

where $s = ds_1$ and $t = dt_1$ with $\gcd(s_1,t_1) = 1$. Write $y_n^i = z_{nd+i}$ for every $0 \leq i \leq d-1$ and $n = 0,1,2,\cdots$. Then (3.1) reduces to the equations

$$y_n^i = \max\{\frac{1}{y_{n-s_1}^i}, \frac{P_{nd+i}}{(y_{n-t_1}^i)^{\alpha_{nd+i}}}\}, \quad 0 \leq i \leq d-1, \quad n = 0,1,2,\cdots. \tag{3.2}$$

By an analogous way as in the above, we obtain that for every $0 \leq i \leq d-1$, y_n^i is a solution of equation

$$y_n^i = \max\{1/y_{n-s_1}^i, \frac{P_{nd+i}}{(y_{n-t_1}^i)^{\alpha_{nd+i}}}\}.$$

Then $\{y_{2s_1n+k}^i\}_{n=0}^{+\infty}$ is eventually monotone for every $0 \leq k \leq 2s_1-1$. Thus for every $0 \leq k \leq 2s-1$, $\{z_{2ns+k}\}_{n=0}^{+\infty}$ is eventually monotone. The proof is complete.

Theorem 3.2 Assume that $s=1$, and t is even, and P_n is a periodic sequence. Let $\{z_n\}_{n=-t}^{+\infty}$ be a positive solution of (1.2). Then $\lim_{n\longrightarrow\infty} z_n = 1$ or $\{z_n\}_{n=-t}^{+\infty}$ is eventually periodic with period 2.

Proof If $U = \limsup_{n\longrightarrow\infty} z_n = 1$, then using arguments similar to ones developed in the proof of Theorem 3.1 we can obtain $\lim_{n\longrightarrow\infty} z_n = 1$. Now assume that $U = \limsup_{n\longrightarrow\infty} z_n > 1$.

According to Proposition 2.4 (iii) and Theorem 3.1, we see that there exists $N \in \mathbb{N}$ such that the following statements hold:

(1) $z_{N+2n}z_{N+2n-1} = 1$ for any $n \geq 0$.

(2) z_{N+2n} is decreasing $(n \geq 0)$ and $\lim_{n\longrightarrow\infty} z_{N+2n} = U$. z_{N+2n-1} is increasing $(n \geq 0)$ and $\lim_{n\longrightarrow\infty} z_{N+2n-1} = u = 1/U$.

We claim that $z_{N+2n+1} = 1/z_{N+2n}$ eventually. In fact, if there exist $1 \leq k_1 < k_2 < \cdots < k_i < \cdots$ such that

$$z_{N+2k_i+1} = \frac{P_{N+2k_i+1}}{z_{N+2k_i+1-t}^{\alpha_{N+2k_i+1}}},$$

then by taking a subsequence we may assume that P_{N+2k_i+1} and α_{N+2k_i+1} are constant sequences since P_n and α_n are periodic sequences. Thus z_{N+2k_i+1} is decreasing since $z_{N+2k_i+1-t}^{\alpha_{N+2k_i+1}}$ is increasing. A contradiction. Which implies that $\{z_n\}_{n=-t}^{+\infty}$ is eventually periodic with period 2. The proof is complete.

Example 3.3 Assume that $s=1$ and t is odd. Let $P_n = P \in (0,1)$ and $\alpha_n = \alpha \in (0,1)$ for any $n \geq 0$. Then there exists a positive solution $\{z_n\}_{n=-t}^{\infty}$ of (1.2) which is not eventually periodic such that $\lim_{n\longrightarrow\infty} z_n \neq 1$.

Proof Choose the initial values $z_{-t}, z_{1-t}, \cdots, z_{-1} \in (0,+\infty)$ satisfying

$$z_{-t} < z_{2-t} < \cdots < z_{-1} < z_{-t}/P, \quad z_{-t} < P^{2/(1-\alpha)}, \quad z_{k-t} = 1/z_{k-t-1} \quad k \in \{1,3,\cdots,t-2\}.$$

Now we show that $z_{2k-1} < z_{2k+1}$ and $z_{2k} < z_{2k-2}$ for any $k \in \mathbb{N}$.

By $z_{-1} < z_{-t}/P$ and $z_{-t} < P^{2/(1-\alpha)}$, we have $z_{-1} < z_{-t}/P < Pz_{-t}^{\alpha}$. Which implies

$$\begin{aligned}
z_0 &= \max\{\frac{1}{z_{-1}}, \frac{P}{z_{-t}^{\alpha}}\} = \frac{1}{z_{-1}} < \frac{1}{z_{-3}} = z_{-2}.\\
z_1 &= \max\{\frac{1}{z_0}, \frac{P}{z_{1-t}^{\alpha}}\} = \max\{z_{-1}, Pz_{-t}^{\alpha}\} = Pz_{-t}^{\alpha} > z_{-1}.\\
z_2 &= \max\{\frac{1}{z_1}, \frac{P}{z_{2-t}^{\alpha}}\} = \max\{\frac{1}{Pz_{-t}^{\alpha}}, \frac{P}{z_{2-t}^{\alpha}}\} = \frac{1}{Pz_{-t}^{\alpha}} = \frac{1}{z_1} < \frac{1}{z_{-1}} = z_0.\\
z_3 &= \max\{\frac{1}{z_2}, \frac{P}{z_{3-t}^{\alpha}}\} = \max\{z_1, Pz_{2-t}^{\alpha}\} = \max\{Pz_{-t}^{\alpha}, Pz_{2-t}^{\alpha}\} = Pz_{2-t}^{\alpha} > \frac{1}{z_2} = z_1.\\
z_4 &= \max\{\frac{1}{z_3}, \frac{P}{z_{4-t}^{\alpha}}\} = \max\{\frac{1}{Pz_{2-t}^{\alpha}}, \frac{P}{z_{4-t}^{\alpha}}\} = \frac{1}{Pz_{2-t}^{\alpha}} = \frac{1}{z_3} < \frac{1}{z_1} = z_2.
\end{aligned}$$

Assume that there exists some $m \in \mathbb{N}$ such that

(1) $z_{2k-1} < z_{2k+1}$ and $z_{2k+2} < z_{2k}$ for any $(-t+1)/2 \leq k \leq m$.

(2) $z_{2k+1} = Pz^{\alpha}_{2k-t}$ for any $0 \leq k \leq m$ and $z_{2k+2}z_{2k+1} = 1$ for any $(-t+1)/2 \leq k \leq m$.

Then

$$\begin{aligned}
z_{2m+3} &= \max\{\frac{1}{z_{2m+2}}, \frac{P}{z^{\alpha}_{2m+3-t}}\} = \max\{z_{2m+1}, Pz^{\alpha}_{2m+2-t}\} \\
&= \max\{Pz^{\alpha}_{2m-t}, Pz^{\alpha}_{2m+2-t}\} = Pz^{\alpha}_{2m+2-t} > Pz^{\alpha}_{2m-t} = z_{2m+1}. \\
z_{2m+4} &= \max\{\frac{1}{z_{2m+3}}, \frac{P}{z^{\alpha}_{2m+4-t}}\} = \max\{\frac{1}{Pz^{\alpha}_{2m+2-t}}, \frac{P}{z^{\alpha}_{2m+4-t}}\} \\
&= \frac{1}{Pz^{\alpha}_{2m+2-t}} = \frac{1}{z_{2m+3}} < \frac{1}{z_{2m+1}} = z_{2m+2}.
\end{aligned}$$

Therefore $z_{2k-1} < z_{2k+1}$ and $z_{2k+2} < z_{2k}$ for any $k \geq (-t+1)/2$, which implies that $\{z_n\}^{\infty}_{n=-t}$ is not eventually periodic. Since $z_{2n+1} = Pz^{\alpha}_{2n-t}$ $(n \in \mathbb{N})$, we obtain $\lim_{n\longrightarrow\infty} z_n \neq 1$. The proof is complete.

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General properties of concave functions defined by the generalized Srivastava-Attiya operator

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Abstract

In this paper we introduce a class $\Im_{\mu,b}^{m,k} C_0(\alpha)$ of concave functions by using the generalized Srivastava-Attiya operator. Also, we get distortion bounds for this class.

Keywords: Hadamard product, concave functions, linear operator, distortion theorem, Hurwitz-Lerch Zeta functions, Srivastava-Attiya operator.

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1 Introduction

Let A denote the class of analytic functions in the unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Further, by S we shall denote the class of all functions in A which are univalent in U.

The study of operators plays an important role in Geometric Function Theory in Complex Analysis and its related fields. Many derivative and integral operators can be written in terms of convolution of certain analytic functions. For functions

$$f_j(z) = \sum_{n=0}^{\infty} a_{n,j} z^n \quad (j = 1, 2)$$

analytic in U, we define the Hadamard product of f_1 and f_2 as

$$(f_1 * f_2)(z) = \sum_{n=0}^{\infty} a_{n,1}, a_{n,2} z^n = (f_2 * f_1)(z) \quad (z \in U). \tag{2}$$

In terms of the Hadamard product (or convolution), the Dziok-Srivastava linear convolution operator involving the generalized hypergeometric function was introduced and studied systematically by Dziok and Srivastava [9], [10] and (subsequently) by many other authors (see, for details, [11] and [20]).

We recall here a general Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ defined in [19] by

$$\Phi(z,s,a) := \sum_{n=2}^{\infty} \frac{z^n}{(n+a)^s}$$

$\big(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C},\ \text{when } |z|<1;\ Re(s)>1 \text{ when } |z|=1\big)$ where, as usual, $\mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N},$ and $\mathbb{N} := \{1,2,3,\ldots\}$). Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ can be found in [8], and the references stated there in (see also [16], [21], [22]). Srivastava and Attiya [21] (also see [4], [12]) introduced and investigated the linear operator.

$$\Im_b^{\mu} : A \to A$$

defined in terms of the Hadamard product by

$$\Im_b^{\mu} f\ (z) = (G_b^{\mu} * f)\ (z), \qquad \big(z \in U;\ b \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \mu \in \mathbb{C};\ f \in A\big) \tag{3}$$

where, for convenience,

$$G_b^{\mu}(z) := (1+b)^{\mu}[\Phi(z,\mu,b) - b^{-\mu}] \quad (z \in U). \tag{4}$$

We recall here the following relationships which follow easily by using (1), (3) and (4)

$$\Im_b^{\mu} f\ (z) = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b}\right)^{\mu} a_n z^n. \tag{5}$$

Motivated essentially by the Srivastava-Attiya operator, Murugusundaramoorthy [17] introduced the generalized integral operator $\Im_{\mu,b}^{m,k}$ given by

$$\Im_{\mu,b}^{m,k}\ f(z) = z + \sum_{n=2}^{\infty} C_n^m(b,\mu,k) a_n z^n \tag{6}$$

where

$$\Psi_n = C_n^m(b,\mu,k) = \left| \left(\frac{1+b}{n+b}\right)^{\mu} \right| \frac{m!(n+k-2)!}{(k-2)!(n+m-1)!} \tag{7}$$

and (throughout this paper unless otherwise mentioned) the parameters μ, b are constrained as $b \in \mathbb{C} \setminus \mathbb{Z}_0^-; \mu \in \mathbb{C}, k \geq 2$ and $m > -1$. It is of interest to note that $\Im_{\mu,b}^{1,2}$ is the Srivastava-Attiya operator and $\Im_{0,b}^{m,k}$ is the well-known Choi-Saigo-Srivastava operator (see [15]). Suitably specializing the parameters m, k, μ and b in $\Im_{\mu,b}^{m,k}\ f(z)$ we can get various integral operators introduced by Alexander [1] and Bernardi [5], Libera and Livingston [13], [14].

2 Preliminaries

Conformal maps of the unit disk onto convex domains are a classical topic. Recently Avkhadiev and Wirths [2] discovered that conformal maps onto concave domains (the complements of convex closed sets) have some novel properties.

A function $f : U \to \mathbb{C}$ is said to belong to the family $C_0(\alpha)$ if f satisfies the following conditions:

- f is analytic in U with the standard normalization $f(0) = f'(0) - 1 = 0$. In addition it satisfies $f(1) = \infty$.
- f maps U conformally onto a set whose complement with respect to $\mathbb{C}$ is convex.
- The opening angle of $f(U)$ at ∞ is less than or equal to $\pi\alpha$, $\alpha \in (1, 2]$.

The class $C_0(\alpha)$ is referred to as the class of concave univalent functions and for a detailed discussion about concave functions, we refer to Avkhadiev et al. [3], Cruz and Pommerenke [7] and references there in.

In particular, the inequality

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < 0 \qquad (z \in U)$$

is used - sometimes also as a definition - for concave functions $f \in C_{0_O}$ (see e.g. [18] and others).

Bhowmik et al. [6] showed that an analytic function f maps U onto a concave domain of angle $\pi\alpha$, if and only if $ReP_f(z) > 0$, where

$$P_f(z) = \frac{2}{\alpha - 1}\left[\frac{\alpha + 1}{2}\frac{1 + z}{1 - z} - 1 - z\frac{f''(z)}{f'(z)}\right].$$

Definition 1 *Let $f(z) \in A$ and $\alpha \in (1, 2]$. Then $f(z) \in \Im_{\mu,b}^{m,k} C_0(\alpha)$ if and only if*

$$Re\frac{2}{\alpha - 1}\left[\frac{\alpha + 1}{2}\frac{1 + z}{1 - z} - 1 - z\frac{\left[\Im_{\mu,b}^{m,k} f(z)\right]''}{\left[\Im_{\mu,b}^{m,k} f(z)\right]'}\right] > 0 \qquad (z \in U).$$

3 Main results

Theorem 2 *If $f(z) \in A$ satisfies the inequality*

$$\sum_{n=2}^{\infty}\left[(\alpha - 1)n + 2n^2\right]|C_n^m(b, \mu, k)||a_n| < 3 - \alpha,$$

for some $\alpha \in (1, 2]$, $n \in \mathbb{N}$, then $f(z) \in \Im_{\mu,b}^{m,k} C_0(\alpha)$.

Proof. We want to prove that

$$Re\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z}-z\frac{\left[\Im_{\mu,b}^{m,k}f(z)\right]''}{\left[\Im_{\mu,b}^{m,k}f(z)\right]'}\right]>0.$$

By using the fact that

$$Re\frac{1}{w}>\frac{1}{2}\Leftrightarrow|w-1|<1,$$

it is enough to show that $|w|<1$.

$$\frac{1}{w}=\frac{2}{\alpha-1}\left[\frac{\alpha+1}{2}\frac{1+z}{1-z}-z\frac{g'(z)}{g(z)}\right] \tag{8}$$

where

$$g(z)=z\left(\Im_{\mu,b}^{m,k}f(z)\right)'=z\left\{1+\sum_{n=2}^{\infty}C_n^m(b,\mu,k)na_nz^{n-1}\right\} \tag{9}$$

and

$$g'(z)=1+\sum_{n=2}^{\infty}C_n^m(b,\mu,k)n^2a_nz^{n-1}. \tag{10}$$

Using (9) and (10), in (8) we obtain

$$|w|\le\frac{\alpha-1}{2}\left|\frac{2(1-z)z\left[1+\sum_{n=2}^{\infty}C_n^m(b,\mu,k)na_nz^{n-1}\right]}{(\alpha+1)(1+z)z\left(1+\sum_{n=2}^{\infty}C_n^m(b,\mu,k)na_nz^{n-1}\right)-2(1-z)z\left(1+\sum_{n=2}^{\infty}C_n^m(b,\mu,k)n^2a_nz^{n-1}\right)}\right|.$$

Using triangle inequality and letting $z\to-1$, then

$$|w|<\frac{\alpha-1}{2}\left(\frac{1+\sum_{n=2}^{\infty}C_n^m(b,\mu,k)|a_n|n}{1-\sum_{n=2}^{\infty}C_n^m(b,\mu,k)|a_n|n^2}\right).$$

The last expression is bounded by 1, if

$$\frac{1+\sum_{n=2}^{\infty}C_n^m(b,\mu,k)|a_n|n}{1-\sum_{n=2}^{\infty}C_n^m(b,\mu,k)|a_n|n^2}<\frac{2}{\alpha-1}.$$

Finally, we can easily see that

$$\sum_{n=2}^{\infty}\left[(\alpha-1)n+2n^2\right]C_n^m(b,\mu,k)|a_n|<3-\alpha. \tag{11}$$

■

4

4 Distortion Bounds

Theorem 3 *If* $f(z) \in \Im_{\mu,b}^{m,k} C_0(\alpha)$, *then*

$$|z| - \frac{3-\alpha}{2(3+\alpha)}|z|^2 \leq \left|\Im_{\mu,b}^{m,k} f(z)\right| \leq |z| + \frac{3-\alpha}{2(3+\alpha)}|z|^2.$$

Proof. From the Theorem 2, we have

$$2(3+\alpha)\sum_{n=2}^{\infty} C_n^m(b,\mu,k)|a_n| \leq \sum_{n=2}^{\infty}\left[(\alpha-1)n+2n^2\right] C_n^m(b,\mu,k)|a_n| < 3-\alpha,$$

That is

$$\sum_{n=2}^{\infty} C_n^m(b,\mu,k)|a_n| \leq \frac{3-\alpha}{2(3+\alpha)}.$$

According to (11) we obtain

$$\begin{aligned} |\Im_{\mu,b}^{m,k} f(z)| &\leq |z| + \sum_{n=2}^{\infty} C_n^m(b,\mu,k)|a_n||z|^n \\ &\leq |z| + \sum_{n=2}^{\infty} C_n^m(b,\mu,k)|a_n||z|^2 \\ &\leq |z| + \frac{3-\alpha}{2(3+\alpha)}|z|^2. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |\Im_{\mu,b}^{m,k} f(z)| &\geq |z| - \sum_{n=2}^{\infty} C_n^m(b,\mu,k)|a_n||z|^n \\ &\geq |z| - \sum_{n=2}^{\infty} C_n^m(b,\mu,k)|a_n||z|^2 \\ &\geq |z| - \frac{3-\alpha}{2(3+\alpha)}|z|^2. \end{aligned}$$

This completes the proof. ■

Theorem 4 *If* $f(z) \in \Im_{\mu,b}^{m,k} C_0(\alpha)$, *then*

$$|z| - \frac{(3-\alpha)}{2(3+\alpha)}\frac{m+1}{k(k-1)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right||z|^2 \leq |f(z)| \leq |z| + \frac{(3-\alpha)}{2(3+\alpha)}\frac{m+1}{k(k-1)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right||z|^2.$$

Proof. According to the Theorem 2 we get that

$$2(3+\alpha)\left|\left(\frac{1+b}{2+b}\right)^{\mu}\right|\frac{k(k-1)}{m+1}\sum_{n=2}^{\infty}|a_n| \leq \sum_{n=2}^{\infty}\left[(\alpha-1)n+2n^2\right] C_n^m(b,\mu,k)|a_n| < 3-\alpha.$$

Thus we get

$$\sum_{n=2}^{\infty}|a_n| \leq \frac{(3-\alpha)}{2(3+\alpha)}\frac{m+1}{k(k-1)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right|.$$

Next from (1), we have

$$\begin{aligned} |f(z)| &\leq |z| + \sum_{n=2}^{\infty}|a_n||z|^n \\ &\leq |z| + \sum_{n=2}^{\infty}|a_n||z|^2 \\ &\leq |z| + \frac{(3-\alpha)}{2(3+\alpha)}\frac{m+1}{k(k-1)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right||z|^2. \end{aligned}$$

The other assertion can be proved as follows

$$\begin{aligned} |f(z)| &\geq |z| - \sum_{n=2}^{\infty}|a_n||z|^n \\ &\geq |z| - \sum_{n=2}^{\infty}|a_n||z|^2 \\ &\geq |z| - \frac{(3-\alpha)}{2(3+\alpha)}\frac{m+1}{k(k-1)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right||z|^2 \end{aligned}$$

This completes the proof. ■

Theorem 5 *If $f(z) \in \Im_{\mu,b}^{1,2}C_0(\alpha)$, then*

$$|z| - \frac{(3-\alpha)}{2(3+\alpha)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right||z|^2 \leq |f(z)| \leq |z| + \frac{(3-\alpha)}{2(3+\alpha)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right||z|^2.$$

Proof. According to the Theorem 2 we get that

$$2(3+\alpha)\sum_{n=2}^{\infty}C_n^1(b,\mu,2)|a_n| \leq \sum_{n=2}^{\infty}\left[(\alpha-1)n+2n^2\right]C_n^m(b,\mu,k)|a_n| < 3-\alpha,$$

or, equivalently

$$2(3+\alpha)\left|\left(\frac{1+b}{2+b}\right)^{\mu}\right|\sum_{n=2}^{\infty}|a_n| \leq \sum_{n=2}^{\infty}\left[(\alpha-1)n+2n^2\right]C_n^m(b,\mu,k)|a_n| < 3-\alpha.$$

Thus we get

$$\sum_{n=2}^{\infty}|a_n| \leq \frac{(3-\alpha)}{2(3+\alpha)}\left|\left(\frac{2+b}{1+b}\right)^{\mu}\right|.$$

Next from (1), we have

$$\begin{aligned} |f(z)| &\leq |z| + \sum_{n=2}^{\infty} |a_n||z|^n \\ &\leq |z| + \sum_{n=2}^{\infty} |a_n||z|^2 \\ &\leq |z| + \frac{(3-\alpha)}{2(3+\alpha)} \left| \left(\frac{2+b}{1+b} \right)^{\mu} \right| |z|^2. \end{aligned}$$

The other assertion can be proved as follows

$$\begin{aligned} |f(z)| &\geq |z| - \sum_{n=2}^{\infty} |a_n||z|^n \\ &\geq |z| - \sum_{n=2}^{\infty} |a_n||z|^2 \\ &\geq |z| - \frac{(3-\alpha)}{2(3+\alpha)} \left| \left(\frac{2+b}{1+b} \right)^{\mu} \right| |z|^2. \end{aligned}$$

■

Theorem 6 *If* $f(z) \in \Im_{0,b}^{m,k} C_0(\alpha)$, *then*

$$|z| - \frac{(3-\alpha)}{2(3+\alpha)} \left| \frac{(k-2)!(n+m-1)!}{m!(n+k-2)!} \right| |z|^2 \leq |f(z)| \leq |z| + \frac{(3-\alpha)}{2(3+\alpha)} \left| \frac{(k-2)!(n+m-1)!}{m!(n+k-2)!} \right| |z|^2.$$

Proof. According to the Theorem 2 we get that

$$2(3+\alpha) \left| \frac{m!(n+k-2)!}{(k-2)!(n+m-1)!} \right| \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} \left[(\alpha-1)n + 2n^2 \right] C_n^m(b,\mu,k) |a_n| \leq 3-\alpha.$$

Thus we get

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{(3-\alpha)}{2(3+\alpha)} \left| \frac{(k-2)!(n+m-1)!}{m!(n+k-2)!} \right|.$$

Next from (1), we have

$$\begin{aligned} |f(z)| &\leq |z| + \sum_{n=2}^{\infty} |a_n||z|^n \\ &\leq |z| + \sum_{n=2}^{\infty} |a_n||z|^2 \\ &\leq |z| + \frac{(3-\alpha)}{2(3+\alpha)} \left| \frac{(k-2)!(n+m-1)!}{m!(n+k-2)!} \right| |z|^2. \end{aligned}$$

The other assertion can be proved as follows

$$\begin{aligned}|f(z)| &\geq |z| - \sum_{n=2}^{\infty}|a_n||z|^n \\ &\geq |z| - \sum_{n=2}^{\infty}|a_n||z|^2 \\ &\geq |z| - \frac{(3-\alpha)}{2(3+\alpha)}\left|\frac{(k-2)!(n+m-1)!}{m!(n+k-2)!}\right||z|^2.\end{aligned}$$

■

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On the zeros of eigenfunctions of discontinuous Sturm-Liouville problems

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Abstract : In this paper, we prove analogues of the classical Sturm comparison and oscillation theorems for Sturm-Liouville problem together with boundary -transmission conditions on two disjoint intervals. We present a new version for Sturm's comparison and oscillation theorems. The obtained results generalizes the recently obtained oscillation and comparison theorems for regular Sturm-Liouville problem which contained transmission conditions.

Keywords : Sturm-Liouville problems, transmission conditions, Sturm comparison and oscillation theorems.

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1 Introduction

The oscillation theory for the solutions of differential equations is one of the traditional trends in the qualitative theory of differential equations. Its essence is to establish conditions for the existence of oscillating (nonoscillating) solutions, to study the laws of distribution of the zeros, to obtain estimates of the distance between the consecutive zeros and of the number of zeros in a given interval. The relationship between the oscillatory and other fundamental properties of the solutions of Sturm-Liouville type differential equations are of central importance in the theory of boundary value problems There are substantial literature on this subject. Many authors have expounded on various aspects of this theory, see [1, 9, 10] and the references cited therein. A considerable number of studies have been made on the oscillation and nonoscillation for a long time. Those results can be found in [14, 15] and the references contained therein. While the extensions and generalizations have much intrinsic interest, we believe their continued relevance is due in no small part to their important connection with problems of physical origin. Particularly the connections with the minimization problems of the calculus of variations and optimal control as well as the spectral theory of differential operators are important. Since the second order equations have applications in various problems in physics, biology, and economics (see for example [1, 5, 13], and the references cited therein) there is a permanent interest in obtaining new sufficient conditions for the oscillation or nonoscillation of solutions of various types of second order equations. In this study we investigated same aspects of comparison and oscillation properties for one discontinuous eigenvalue problem which consists of Sturm-Liouville equation,

$$Ly := -y''(x) + q(x)y(x) = \lambda y(x) \tag{1.1}$$

to hold on two disjoint intervals $(-1, 0)$ and $(0, 1)$, where discontinuity in y and y' at the interior singular point $x = 0$ are prescribed by transmission conditions

$$y(0-) = \delta y(0+), \quad y'(0-) = \frac{1}{\delta} y'(0+), \tag{1.2}$$

together with the boundary conditions

$$y(-1) = y(1) = 0 \tag{1.3}$$

where the potential $q(x)$ is real-valued, continuous on $[-1, 0) \cup \in (0, 1]$ and has a finite limits $q(c\mp) = \lim_{x\to 0\mp} q(x)$; λ is a complex eigenparameter; $\delta \neq 0$ any real number. Since various type transmission problems appear frequently in various fields of physics and technics, Sturm-Liouville problems with transmission conditions have been an important research topic in mathematical physics [2, 8, 11]. For the earlier developments about Sturm comparison and oscillation theory, we refer to [4, 5, 6, 9, 14, 15] and for recent developments, we refer to [1, 3, 7, 13, 16, 17].

2 Comparison Theorem for discontinuous Sturm-Liouville problems

At first we shall extend and generalize the classical Sturm-liouville comparison theorem.

Theorem 2.1. *Let $y = y_1(x)$ be solution of the equation*

$$L_1 y := -y'' + q_1(x)y = 0 \tag{2.1}$$

satisfying transmission conditions at the point of interaction $x = 0$ given by

$$y(0-) = \delta y(0+), \ y'(0-) = \frac{1}{\delta} y'(0+) \tag{2.2}$$

and let $y = y_2(x)$ be the solution of the equation

$$L_2 y := -y'' + q_2(x)y = 0 \tag{2.3}$$

satisfying the same transmission conditions (2.2) where $\delta \neq 0$ any real number if $q_1(x) > q_2(x)$ on $[-1, 0) \cup \in (0, 1]$, then between any two consecutive zeros of $y_1(x)$ there is at least one zero of $y_2(x)$.

Proof. Let x_1 and x_2 with $x_1 < x_2$ be consecutive zeroes of y_1. Suppose, it possible,that y_2 does not have a zero on (x_1, x_2). Lagrange's identity (see, [12]) gives

$$y_2 L_1 y_1 - y_1 L_2 y_2 = \frac{d}{dx}\{y_2' y_1 - y_1' y_2\} + \{q_1(x) - q_2(x)\} y_1 y_2 \tag{2.4}$$

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Hence

$$\frac{d}{dx}\{y_1'y_2 - y_2'y_1\} = \{q_1(x) - q_2(x)\}y_1y_2 \tag{2.5}$$

Case 1. Let $x_1 \in [-1, 0)$, $x_2 \in (0, 1]$ and $\delta > 0$. Integrating on both sides of the equation (2.9) over $[x_1, 0)$ and $(0, x_2]$ and then adding we get

$$\lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_1 > 0}} (y_1'y_2 - y_2'y_1)|_{x_1}^{0-\epsilon_1} + \lim_{\substack{\epsilon_2 \to 0 \\ \epsilon_2 > 0}} (y_1'y_2 - y_2'y_1)|_{0+\epsilon_2}^{x_2}$$

$$= \lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_1 > 0}} \int_{x_1}^{0-\epsilon_1} \{q_1(x) - q_2(x)\}y_1y_2dx + \lim_{\substack{\epsilon_2 \to 0 \\ \epsilon_2 > 0}} \int_{0+\epsilon_2}^{x_2} \{q_1(x) - q_2(x)\}y_1y_2dx \tag{2.6}$$

Since $y_1(x_1) = y_1(x_2) = 0$ we get

$$\lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_1 > 0}} W(y_1, y_2; 0 - \epsilon_1) - \lim_{\substack{\epsilon_2 \to 0 \\ \epsilon_2 > 0}} W(y_1, y_2; 0 + \epsilon_2) - y_1'(x_1)y_2(x_1) + y_1'(x_2)y_2(x_2)$$

$$= \lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_1 > 0}} \int_{x_1}^{0-\epsilon_1} \{q_1(x) - q_2(x)\}y_1y_2dx + \lim_{\substack{\epsilon_2 \to 0 \\ \epsilon_2 > 0}} \int_{0+\epsilon_2}^{x_2} \{q_1(x) - q_2(x)\}y_1y_2dx \tag{2.7}$$

Using the transmission conditions we obtain

$$-y_1'(x_1)y_2(x_1) + y_1'(x_2)y_2(x_2) = \lim_{\substack{\epsilon_1 \to 0 \\ \epsilon_1 > 0}} \int_{x_1}^{0-\epsilon_1} \{q_1(x) - q_2(x)\}y_1y_2dx$$

$$+ \lim_{\substack{\epsilon_2 \to 0 \\ \epsilon_2 > 0}} \int_{0+\epsilon_2}^{x_2} \{q_1(x) - q_2(x)\}y_1y_2dx \tag{2.8}$$

In this case with no restriction we can assume that $y_1(x) > 0$ and $y_2(x) > 0$ over $(x_1, 0) \cup (0, x_2)$. These conditions ensure that the integral on the right in (2.8) is positive. On the left, since $y_1(x) > 0$ by assumption, the function is increasing at the point x_1. Hence $y_1'(x_1) > 0$(it cannot vanish, because then it would follow from the uniqueness theorem for the solutions of (2.1) that $y_1(x) \equiv 0$, which is impossible). Similarly, $y_1'(x_2) < 0$. Thus, the left-hand side of the equation (2.8) is less or equal to zero, which is a contradiction.

Case 2. Let $x_1 \in [-1, 0)$, $x_2 \in (0, 1]$ and $\delta < 0$. In this case with no restriction it can be assumed that, $y_1(x) > 0$ over $(x_1, 0)$, $y_1(x) < 0$ over $(0, x_2)$, $y_2(x) > 0$ over $(x_1, 0)$ and $y_2(x) < 0$ over $(0, x_2)$. Since $y_1(x_1) = 0$ and $y_1(x_1) > 0$ over $(x_1, 0)$ $y_1'(x_1) > 0$. Further, since $y_2(x_2) = 0$ and $y_2(x_2) < 0$ immediately to left of x_2, $y_2'(x) < 0$. Hence, the left-hand side of (2.8) is is less

or equal zero, but the right-hand side is positive which shows that (2.8) is impossible.
Case 3. Let $(x_1, x_2) \subset [-1, 0)$. Integrating on both sides of the equation (2.5) from x_1 to x_2, we get

$$(y_1' y_2 - y_2' y_1)|_{x_1}^{x_2} = \int_{x_1}^{x_2} \{q_1(x) - q_2(x)\} y_1 y_2 dx \tag{2.9}$$

Then with no restriction it can be assumed that $y_1(x) > 0$ and $y_2(x) > 0$ over (x_1, x_2). These conditions ensure that the integral on the right in (2.9) is positive. However, on the left, we have $y_1(x_1) = y_1(x_2) = 0$ with $y_1'(x_1) > 0$ and $y_1'(x_2) < 0$. The left-hand side therefore becomes

$$y_1'(x_2) y_2(x_2) - y_1'(x_1) y_2(x_1) \leq 0$$

which presents us with a contradiction: right-hand side > 0 and left-hand side < 0. Thus $y_2(x) = 0$ (at least once) between the zeros of $y_1(x)$. Since the conditions describing $y_1(x)$ are given, we conclude that $y_2(x)$ must change sign between $x = x_1$ and $x = x_2$.
Case 4. Let $(x_1, x_2) \subset (0, 1]$. This case is totaly similar to the previous case. □

3 On the zeros of eigenfunctions

In this section we examine the number of zeros of eigenfunctions.

Lemma 3.1. *There is an unique solution $y(x, \lambda)$ of the equation (1.1) satisfying the initial conditions*

$$y(x_0, \lambda) = \alpha(\lambda), \; y'(x_0, \lambda) = \beta(\lambda) \tag{3.1}$$

and the transmission conditions (1.2) where $\alpha(\lambda), \beta(\lambda)$ are given entire functions of $\lambda \in \mathbb{C}$ and $x_0 \in [-1, 0) \cup (0, 1]$. Moreover, $y(x, \lambda)$ is entire function of $\lambda \in \mathbb{C}$ for each fixed $x \in [-1, 0) \cup (0, 1]$.

Proof. The proof is totally similar to [**?**] and therefore is omitted. □

Theorem 3.2. *Let $\phi(x, \lambda_1) = \begin{cases} \phi_1(x, \lambda_1), & x \in [-1, 0) \\ \phi_2(x, \lambda_1), & x \in (0, 1] \end{cases}$ be solution of the equation (1.1), for $\lambda = \lambda_1$ satisfying the initial conditions*

$$\phi_1(-1, \lambda_1) = \alpha, \quad \phi_1'(-1, \lambda_1) = \beta \tag{3.2}$$

and the transmission conditions

$$\phi_2(0^+, \lambda_1) = \frac{1}{\delta} \phi_1(0^-, \lambda_1), \phi_2'(0^+, \lambda_1) = \delta \phi_1'(0^-, \lambda_1) \tag{3.3}$$

and $\varphi(x, \lambda_2) = \begin{cases} \varphi_1(x, \lambda_2), & x \in [-1, 0) \\ \varphi_2(x, \lambda_2), & x \in (0, 1] \end{cases}$ be solution of the equation (1.1), for $\lambda = \lambda_2$ satisfying the initial conditions

$$\varphi_1(-1, \lambda_2) = \alpha, \quad \varphi_1'(-1, \lambda_2) = \beta \tag{3.4}$$

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and the transmission conditions

$$\varphi_2(0^+,\lambda_2)=\frac{1}{\delta}\phi_2(0^-,\lambda_2),\varphi_2'(0^+,\lambda_2 1)=\delta\phi_1'(0^-,\lambda_1). \tag{3.5}$$

where δ,β,δ any real numbers with $\alpha^2+\beta^2\neq 0,\delta\neq 0$. Suppose that $\phi(x,\lambda_1)$ has a zeros in $[-1,0)\cup(0,1]$ and let $x_1(x_1\neq -1)$ be zero of the function $\phi(x,\lambda_1)$, nearest to $x=-1$. If $\lambda_2>\lambda_1$ then $\varphi(x_2,\lambda_2)$ has at least one zero in $[-1,x_1)$.

Proof. From the well-known Lagrange's identity (see,for example, [12]) we have

$$\frac{d}{dx}\{\phi_1'\varphi_1-\varphi_1'\phi_1\}=\{\lambda_2-\lambda_1\}\phi_1\varphi_1 \tag{3.6}$$

in the interval $(0,1)$.

$$\frac{d}{dx}\{\phi_2'\varphi_2-\varphi_2'\phi_2\}=\{\lambda_2-\lambda_1\}\phi_2\varphi_2 \tag{3.7}$$

Case 1. Let $x_1>0$ and $\delta>0$. Integrating on both sides of the equation (3.11) from -1 to x_1, we get

$$\begin{aligned}&\lim_{\substack{\epsilon_1\to 0\\ \epsilon_1>0}}(\phi_1'\varphi_1-\varphi_1'\phi_1)|_{-1}^{0-\epsilon_1}+\lim_{\substack{\epsilon_2\to 0\\ \epsilon_2>0}}(\phi_2'\varphi_2-\varphi_2'\phi_2)|_{0+\epsilon_2}^{x_1}\\ &=\lim_{\substack{\epsilon_1\to 0\\ \epsilon_1>0}}\{\lambda_2-\lambda_1\}\int_{-1}^{0-\epsilon_1}\phi_1\varphi_1dx+\lim_{\substack{\epsilon_2\to 0\\ \epsilon_2>0}}\{\lambda_2-\lambda_1\}\int_{0+\epsilon_2}^{x_1}\phi_2\varphi_2dx\end{aligned} \tag{3.8}$$

Since $W(\phi_1,\varphi_1;-1)=0$ by (3.2) and (3.4) we get

$$\begin{aligned}&\lim_{\substack{\epsilon_1\to 0\\ \epsilon_1>0}}W(\phi_1,\varphi_1;0-\epsilon_1)-\lim_{\substack{\epsilon_2\to 0\\ \epsilon_2>0}}W(\phi_2,\varphi_2;0+\epsilon_2)+\phi_2'(x_1,\lambda_1)\varphi_2(x_1,\lambda_2)\\ &=\lim_{\substack{\epsilon_1\to 0\\ \epsilon_1>0}}\{\lambda_2-\lambda_1\}\int_{-1}^{0-\epsilon_1}\phi_1\varphi_1dx+\lim_{\substack{\epsilon_2\to 0\\ \epsilon_2>0}}\{\lambda_2-\lambda_1\}\int_{0+\epsilon_2}^{x_1}\phi_2\varphi_2dx\end{aligned} \tag{3.9}$$

Using the transmission conditions we obtain

$$\begin{aligned}\phi_2'(x_1,\lambda_1)\varphi_2(x_1,\lambda_2)\;&=\;\lim_{\substack{\epsilon_1\to 0\\ \epsilon_1>0}}\{\lambda_2-\lambda_1\}\int_{-1}^{0-\epsilon_1}\phi_1\varphi_1dx\\ &+\;\lim_{\substack{\epsilon_2\to 0\\ \epsilon_2>0}}\{\lambda_2-\lambda_1\}\int_{0+\epsilon_2}^{x_1}\phi_2\varphi_2dx\end{aligned} \tag{3.10}$$

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With no restriction it can be assumed that $\phi(x,\lambda_1) < 0$ and $\varphi(x,\lambda_2) < 0$ in $[-1,x_1)$. These conditions ensure that the integral on the right in (3.10) is positive. Since $\phi_2(x_1,\lambda_1) = 0$ and $\phi_2(x,\lambda_1) > 0$ immediately to the left of x_1 by assumption, the function is increasing at the point x_1. Hence $\phi_2'(x_1,\lambda_1) > 0$(it cannot vanish, because then it would follow from the uniqueness theorem for the solutions of (2.1) that $\phi_2(x,\lambda_1) \equiv 0$, which is impossible). Thus, the left-hand side of the equation (3.10) is less or equal to zero, but the right-hand side is positive, which is a contradiction.

Case 2. Let $x_1 > 0$ and $\delta < 0$. In this case with no restriction it can be assumed that $\phi(x,\lambda_1) > 0$ and $\varphi(x,\lambda_2) < 0$ in $[-1,0)$ but $\phi(x,\lambda_1) < 0$ and $\varphi(x,\lambda_2) > 0$ in $(0,x_1]$. As in the previous case, these conditions ensure that the integral on the right of (3.10) is negative, but left hand side of (3.10) is positive or is equal to zero, i.e. the equality (3.10)is impossible.

Case 3. Let $x_1 \in [-1,0)$. Integrating on both sides of the equation (2.5) from a to x_1, we get

$$(\phi_1'\varphi_1 - \varphi_1'\phi_1)|_{-1}^{x_1} = \int_{-1}^{x_1} \{\lambda_2 - \lambda_1\}\phi_1\varphi_1 dx \tag{3.11}$$

Since $\phi_1(x,\lambda_1) = 0$ by using the initial conditions $\phi_1(-1,\lambda_1) = 0, \phi_1'(-1,\lambda_1) = 0$ we get

$$\phi_1'(x_1)\varphi_1(x_1) = \int_{-1}^{x_1} \{\lambda_2 - \lambda_1\}\phi_1\varphi_1 dx \tag{3.12}$$

Let $x_1 < 0$. Without loss of generality, we can put $\phi(x,\lambda_1) > 0$ and $\varphi(x,\lambda_2) > 0$ in $[-1,x_1)$. Since, by assumption, $\phi_1(x,\lambda_1) > 0$ and $\varphi_1(x,\lambda_2) > 0$ in $[-1,x_1)$ and $\lambda_2 > \lambda_1$, the right-hand side of the equality (3.12) is positive. However, on the left-hand side, since $\phi_1(x_1,\lambda_1) = 0$ and $\phi_1(x,\lambda_1) > 0$ immediately to the left of x_1, the function $\phi_1(x,\lambda_1)$ is decreasing in the vicinity of the point x_1. Therefore, $\phi_1'(x_1,\lambda_1) \leq 0$(it cannot vanish, because then it would follow from the uniqueness theorem for the solutions of (1.1) that $\phi_1(x,\lambda_1) \equiv 0$, which is impossible). The left-hand side therefore becomes

$$\phi_1'(x_1,\lambda_1)\varphi_1(x_1,\lambda_1) \leq 0$$

which presents us with a contradiction: right-hand side > 0 and left-hand side ≤ 0. The proof is complete. □

Now we are ready to establish the main result.

Theorem 3.3. *Let $\psi_1(x)$ and $\psi_2(x)$ be two eigenfunction corresponding to the eigenvalues λ_1 and λ_2 of the problem (1.1)-(1.3) and let $\lambda_2 > \lambda_1$. Then if $\psi_1(x)$ has m zeros in $[-1,0) \cup (0,1]$, $\psi_2(x)$ has not fewer than m zeros in the same two-interval $[-1,0) \cup (0,1]$. Moreover, $n-th$ zero of $\psi_2(x)$ is less than the $n-th$ zero of $\psi_1(x)$.*

Proof. Let $x_1', x_2', ..., x_m'$ with $x_1' < x_2 <, ... < x_m'$ be zeros of the eigenfunctions $\psi_1(x)$. By virtue of the Theorem 3.2 $\psi_2(x)$ has at least one zero in $[-1,x_1')$. Moreover, by applying the Theorem 2.1 to the solutions ψ_1 and ψ_2 we see that $\psi_2(x)$ has at least one zero in each of the intervals $(x_1',x_2'), (x_2',x_3'), ..., (x_{m-1}',x_m')$. Consequently the number of zeros of $\psi_2(x)$ is not fewer than the number of zeros $\psi_1(x)$ and $n-th$ zero of $\psi_2(x)$ is less than $n-th$ zero of $\psi_1(x)$. The proof is complete. □

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Fuzzy stability of an additive-quadratic functional equation in matrix fuzzy normed spaces

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Abstract. A mapping $f : X \times X \to Y$ is called additive-quadratic if f satisfies the system of equations

$$\begin{cases} f(x+y,z) = f(x,z) + f(y,z), \\ f(x,y+z) + f(x,y-z) = 2f(x,y) + 2f(x,z). \end{cases}$$

In this paper, using the fixed point method, we prove the Hyers-Ulam stability in matrix fuzzy normed spaces associated to the following additive-quadratic functional equation

$$f(x+y,z+w) + f(x+y,z-w) = 2f(x,z) + 2f(x,w) + 2f(y,z) + 2f(y,w)$$

for all $x, y, z, w \in X$.

1. Introduction and preliminaries

A definition of fuzzy norm on a vector space, to construct a fuzzy vector topological structure, introduced by Katsaras [15]. During the last four decades some mathematicians have defined fuzzy norms on a vector space from various points of view [13, 16, 32]. In particular, Bag and Samanta [1], following Cheng and Mordeson [6], presented an idea of a fuzzy norm in such a manner the corresponding fuzzy metric is of Kramosil and Michalek type [6]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [2].

We use the definition of fuzzy normed spaces given in [1, 19, 21] to investigate a fuzzy version of the Hyers-Ulam stability of an additive-quadratic additive functional equation in the fuzzy normed vector space setting.

Definition 1.1. *Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a* fuzzy norm *on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*

(N_1) $N(x,t) = 0$ *for* $t \leqslant 0$*;*
(N_2) $x = 0$ *if and only if* $N(x,t) = 1$ *for all* $t > 0$*;*
(N_3) $N(cx,t) = N(x, \frac{t}{|c|})$ *if* $c \neq 0$
(N_4) $N(x+y, s+t \geqslant \min\{N(x,s), N(y,t)\}$*;*

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(N_5) *$N(x,.)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;*
(N_6) *for $x \neq 0, N(x,.)$ is continuous on $\mathbb{R}$.*

The pair (X, N) is called a *fuzzy normed vector space*. To see more properties and examples of fuzzy normed vector spaces, we refer to [19, 20].

Definition 1.2. *Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be* convergent *or* converge *if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the* limit *of the sequence $\{x_n\}$ and we denote it by $N - \lim_{n\to\infty} x_n = x$.*

Definition 1.3. *Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called a* Cauchy sequence *if for each $\varepsilon > 0$ and each $t > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.*

It is well known that every convergent sequence in a fuzzy normed vector space is a Cauchy sequence. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, the sequence $\{f(x_n)\}$ converges $f(x_0)$. If $f : X \to Y$ continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [2]).

We will use the following notations:
$M_n(X)$ is the set of all $n \times n$-matrices in X;
$e_j \in M_{1,n}(\mathbb{C})$ is that the jth component is 1 and the other components are zero;
$E_{ij} \in M_n(\mathbb{C})$ is that the (i, j)-component is 1 and the other components are zero;
$E_{ij} \otimes x \in M_n(X)$ is that the (i, j)-component is x and the other components are zero.
For $x \in M_n(X), y \in M_k(X)$,

$$x \otimes y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Let $(X, \|\cdot\|)$ be a normed space. Note that $(X, \{\|\cdot\|_n\})$ is a matrix normed space if and only if $(M_n(X), \|\cdot\|_n)$ is a normed space for each positive integer n and $\|AxB\|_k \leqslant \|A\|\|B\|\|x\|_n$ holds for $A \in M_{k,n}(\mathbb{C}), x = (x_{ij}) \in M_n(X)$ and $B \in M_{n,k}(\mathbb{C})$, and that $(X, \{\|.\|_n\})$ is a matrix Banach space if and only if X is a Banach space and $(X, \{\|\cdot\|_n\})$ is a matrix normed space.

A matrix normed space $(X, \{\|\cdot\|_n\})$ is called an *L^∞-matrix normed space* if $\|x \oplus y\|_{n+k} = \max\{\|x\|_n, \|y\|_k\}$ holds for all $x \in M_n(X)$ and all $y \in M_k(X)$.

Let E, F be vector spaces. For a given mapping $h : E \to F$ and a given positive integer n, define $h_n : M_n(E) \to M_n(F)$ by

$$h_n([x_{ij}]) = [h(x_{ij})]$$

for all $[x_{ij}] \in M_n(E)$.

Throughout this paper, let $(X, \{\|\cdot\|_n\})$ be a matrix normed space and $(Y, \{\|\cdot\|_n\})$ be a matrix Banach space.

We introduce the concept of a matrix fuzzy normed space.

Definition 1.4. *Let (X, N) be a fuzzy normed space.*

(1) (X, N) *is called a* matrix fuzzy normed space *if for each positive integer* n, $(M_n(X), N_n)$ *is a fuzzy normed space and* $N_k(AxB, t) \geqslant N_n\left(x, \frac{t}{\|A\|\cdot\|B\|}\right)$ *for all* $t > 0$, $A \in M_{k,n}(\mathbb{R})$, $x = [x_{ij}] \in M_n(X)$ *and* $B \in M_{n,k}(\mathbb{R})$ *with* $\|A\| \neq 0.\|B\| \neq 0$.

(2) $(X, \{N_n\})$ *is called a* matrix fuzzy Banach space *if* (X, N) *is a fuzzy Banach space and* $(X, \{N_n\})$ *is a matrix fuzzy normed space.*

Example 1.5. *Let $(X, \{\|\cdot\|_n\})$ be a matrix normed space. Let $N_n(x, t) := \frac{t}{t+\|x\|_n}$ for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$. Then*

$$N_k(AxB, t) = \frac{t}{t + \|AxB\|_k} \geqslant \frac{t}{t + \|A\|.\|x\|_n.\|B\|} = \frac{\frac{t}{\|A\|.\|B\|}}{\frac{t}{\|A\|.\|B\|} + \|x\|_n}$$

for all $t > 0, A \in M_{k,n}(\mathbb{R}), x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}(\mathbb{R})$ with $\|A\|.\|B\| \neq 0$. So, $(X, \{N_n\})$ is a matrix fuzzy normed space.

The abstract characterization given for linear spaces of bounded Hilbert space operators in terms of *matricially normed spaces* [29] implies that quotients, mapping spaces, and various tensor product of operator spaces may again be regarded as operator spaces. Owing in part to this result, the theory of operator spaces have an increasingly significant effect on operator algebra theory(see [10]).

The proof given in [29] appealed to the theory of ordered operator spaces [7]. Effros and Ruan [11] showed that one can give a purely metric proof of this important theorem by using a technique of Pisier [26] and Effors [9].

The study of stability problems have been formulated by Ulam [31] in 1940: Under what condition does there exist a homomorphism near an approximate homomorphism? In the following year, Hyers [14] answered affirmatively the question of Ulam for Banach spaces, which was stated that if $\varepsilon > 0$ and $f : X \to Y$ is a mapping with X a normed space and Y is a Banach space such that

$$\|f(x + y) - f(x) - f(y)\| \leqslant \varepsilon \tag{1.1}$$

for all $x, y \in X$, then there exists a unique additive map $T : X \to Y$ such that

$$\|f(x + y) - f(x) - f(y)\| \leqslant \varepsilon$$

for all $x \in X$. A generalized version of the theorem of Hyers for approximately linear mappings presented by Rassias [27] in 1978 by considering the case when (1.1) is unbounded.

In 2003, Cădariu and Radu applied the fixed point method to the investigation of the Jensen functional equation [3]. They could present a short and a simple proof (different of the "direct method", initiated by Hyers in 1941) for the Hyers-Ulam stability of the Jensen functional equation [3] and forthe quadratic functional equation [4]. See [12, 22, 23, 24, 28, 30] for more information on functional equations.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x,y)=0$ if and only if $x=y$;
(2) $d(x,y)=d(y,x)$ for all $x,y\in X$;
(3) $d(x,y)\leqslant d(x,z)+d(z,y)$ for all $x,y,z\in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.6. [8] *Let (Ω,d) be a complete generalized metric space and $J:\Omega\to\Omega$ be a strictly contractive mapping with Lipschitz constant $0<L<1$. Then for each given $x\in\Omega$, either*

$$d(J^n x, J^{n+1}x)=\infty$$

for all nonnegative n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x)<\infty, \quad \forall n\geqslant n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $\Lambda=\{y\in\Omega:\ d(J^{n_0}x,y)<\infty\}$;*
(4) *$d(y,y^*)\leqslant\frac{1}{1-L}d(y,Jy)$ for all $y\in\Lambda$.*

Definition 1.7. *A mapping $f:X\times X\to Y$ is called additive-quadratic if f satisfies the system of equations*

$$\begin{cases} f(x+y,z)=f(x,z)+f(y,z), \\ f(x,y+z)+f(x,y-z)=2f(x,y)+2f(x,z). \end{cases} \tag{1.2}$$

When $X=Y=\mathbb{R}$, the function $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ given by $f(x,y):=cxy^2$ is a solution of (1.2). In particular, letting $x=y$, we get a cubic function $g:\mathbb{R}\to\mathbb{R}$ given by $g(x):=f(x,x)=cx^3$.

For a mapping $f:X\times X\to Y$, consider the functional equation:

$$f(x+y,z+w)+f(x+y,z-w)=2f(x,z)+2f(x,w)+2f(y,z)+2f(y,w). \tag{1.3}$$

for all $x,y,z,w\in X$. The solution of (1.3) was discussed in [25].

In this paper, by using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic functional equation (1.3) in matrix fuzzy normed spaces.

2. Fuzzy stability of the additive-quadratic functional equation (1.3)

In this section, using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic functional equation (1.3) in matrix fuzzy normed space.

We need the following lemma.

Lemma 2.1. [17, Lemma 2.1] *Let $(X,\{N_n\})$ be a matrix fuzzy normed space.*

(1) $N_n(E_{kl}\otimes x,t)=N(x,t)$ *for all $t>0$ and $x\in X$.*
(2) *for all $[x_{ij}]\in M_n(X)$ and $t=\sum_{i,j=1}^n t_{ij}$,*

$$N(x_{kl},t)\geqslant N([x_{ij}],t)\geqslant\min\{N(x_{ij},t_{ij}):i,j=1,2,\cdots,n\},$$

$$N(x_{kl},t)\geqslant N([x_{ij}],t)\geqslant\min\left\{N\Big(x_{ij},\frac{t}{n^2}\Big):i,j=1,2,\cdots,n\right\}$$

(3) $\lim_{n\to\infty}x_n=x$ *if and only if* $\lim_{n\to\infty}x_{ijn}=x_{ij}$ *for* $x_n=[x_{ijn}],x=[x_{ij}]\in M_k(X)$

Proof. (1) Since $E_{kl} \otimes x = e_k^* x e_l$ and $\|e_k^*\| = \|e_l\| = 1, N_n(E_{kl} \otimes x, t) \geqslant N(x, t)$. Since $e_k(E_{kl} \otimes x)e_l^* = x, N_n(E_{kl} \otimes x, t) \leqslant N(x, t)$. So $N(E_{kl} \otimes x, t)$=N(x,t).

(2) $N(x_{kl}, t) = N(e_k[x_{ij}]e_l^*, t) \geqslant N_n\left([x_{ij}], \frac{t}{\|e_k\| \cdot \|e_l\|}\right) = N_n([x_{ij}], t)$.

$$\begin{aligned} N_n([x_{ij}], t) &= N_n\Big(\sum_{i,j=1}^{n} E_{ij} \otimes x_{ij}, t\Big) \geqslant \min\{N_n(E_{ij} \otimes x_{ij}, t_{ij}) : i, j = 1, 2, \cdots, n\} \\ &= \min\{N(x_{ij}, t_{ij}) : i, j = 1, 2, \cdots, n\}, \end{aligned}$$

where $t = \sum_{i,j=1}^{n} t_{ij}$. So, $N_n([x_{ij}], t) \geqslant \min\{N(x_{ij}, \frac{t}{n^2}) : i, j = 1, 2, \cdots, n\}$.

(3) By $N(x_{kl}, t) \geqslant N_n([x_{ij}], t) \geqslant \min\{N(x_{ij}, \frac{t}{n^2}) : i, j = 1, 2, \cdots, n\}$, we obtain the result. This completes the proof. □

For a mapping $f : X \to Y$, define $Df : X^m \to Y$ and $Df_n : M_n(X^4) \to M_n(Y)$ by

$$\begin{aligned} Df(a, b, c, d) &:= f(a + b, c + d) + f(a + b, c - d) \\ &\quad - 2f(a, c) - 2f(a, d) - 2f(b, c) - 2f(b, d), \end{aligned}$$

$$\begin{aligned} Df_n\Big([x_{ij}], [y_{ij}], [z_{ij}], [w_{ij}]\Big) &:= f_n\Big([x_{ij}] + [y_{ij}], [z_{ij}] + [w_{ij}]\Big) + f_n\Big([x_{ij}] + [y_{ij}], [z_{ij}] - [w_{ij}]\Big) \\ &\quad - 2f_n\Big([x_{ij}], [z_{ij}]\Big) - 2f_n\Big([x_{ij}], [w_{ij}]\Big) - 2f_n\Big([y_{ij}], [z_{ij}]\Big) - 2f_n\Big([y_{ij}], [w_{ij}]\Big) \end{aligned}$$

for all $a, b, c, d \in X$ and all $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}], w = [w_{ij}] \in M_n(X)$.

Theorem 2.2. *Let $f : X \to Y$, with $f(x, 0) = 0$, be a mapping for which there exists a function $\varphi : X^4 \to [0, \infty)$ such that*

$$N_n\Big(f_n([x_{ij}], [y_{ij}], [z_{ij}], [w_{ij}]), t\Big) \geqslant \frac{t}{t + \sum_{i,j=1}^{n} \varphi(x_{ij}, y_{ij}, z_{ij}, w_{ij})} \tag{2.1}$$

for all $t > 0$ and all $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}], w = [w_{ij}] \in M_n(X)$. If there exists an $\alpha < 1$ such that

$$\varphi(a, b, c, d) \leqslant 8\alpha\varphi\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}, \frac{d}{2}\right) \tag{2.2}$$

for all $a, b, c, d \in X$, then there exists a unique additive-quadratic mapping $T : X \times X \to Y$ such that

$$N_n\left(f_n\left([x_{ij}], [y_{ij}]\right) - T_n\left([x_{ij}], [y_{ij}]\right), t\right) \geqslant \frac{8(1-\alpha)t}{8(1-\alpha)t + n^2 \sum_{i,j=1}^{n} \varphi(x_{ij}, x_{ij}, y_{ij}, y_{ij})} \tag{2.3}$$

for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Proof. Putting $n = 1$ in (2.1), we have

$$N(Df(x, y, z, w), t) \geqslant \frac{t}{t + \varphi(x, y, z, w)} \tag{2.4}$$

for all $t > 0$ and $x, y, z, w \in X$.

Letting $x = y$ and $z = w$ in (2.4), we obtain

$$N(f(2x, 2z) - 8f(x, z), t) \geqslant \frac{t}{t + \varphi(x, x, z, z)} \tag{2.5}$$

and also

$$N\left(\frac{1}{8}f(2x, 2z) - f(x, z), \frac{t}{8}\right) \geqslant \frac{t}{t + \varphi(x, x, z, z)}$$

for all $t > 0$ and $x, z \in X$. Also it can be written as

$$N\left(\frac{1}{8}f(2x, 2y) - f(x, y), \frac{t}{8}\right) \geqslant \frac{t}{t + \varphi(x, x, y, y)} \tag{2.6}$$

for all $t > 0$ and $x, y \in X$.

By considering the set of

$$\Omega := \{g : X \to Y\},$$

we introduce the generalized metric on Ω as following:

$$d(g, h) = \inf\left\{k \in \mathbb{R}^+ : N(g(x, y) - h(x, y), kt) \geqslant \frac{t}{t + \varphi(x, x, y, y)}, \forall x, y \in X, \forall t > 0\right\}$$

where, as usual $\inf \emptyset = +\infty$. It is easy to show that (Ω, d) is complete (see [5, 18]).

Now we define $J : \Omega \to \Omega$ by

$$Jg(x, y) := \frac{1}{8}h(2x, 2y)$$

for all $x, y \in X$.

Let $g, h \in \Omega$ be given such that $d(g, h) = c$. Then

$$\begin{aligned}
&N(g(x, y) - h(x, y), ct) \geqslant \frac{t}{t + \varphi(2x, 2x, 2y, 2y)} \\
&\Rightarrow N\left(\frac{1}{8}g(2x, 2y) - \frac{1}{8}h(2x, 2y), \frac{c}{8}t\right) \geqslant \frac{t}{t + \varphi(2x, 2x, 2y, 2y)} \\
&\Rightarrow N\left(\frac{1}{8}g(2x, 2y) - \frac{1}{8}h(2x, 2y), \frac{c}{8}t\right) \geqslant \frac{t}{t + 8\alpha\varphi(x, x, y, y)} \\
&\Rightarrow N\left(\frac{1}{8}g(2x, 2y) - \frac{1}{8}h(2x, 2y), \alpha ct\right) \geqslant \frac{t}{t + \varphi(x, x, y, y)} \\
&\Rightarrow d(Jg, Jh) \leqslant \alpha c
\end{aligned}$$

for all $x, y \in X$. Hence we get that

$$d(Jg, Jh) \leqslant \alpha d(g, h)$$

for all $g, h \in \Omega$. It follows from (2.6) that $d(f, Jf) \leqslant \frac{1}{8}$.

By Theorem 1.6, there exists a mapping $T : X \to Y$ satisfying the following:

(1) T is a fixed point of J, i.e., $T(2x, 2y) = 8T(x, y)$ for all $x \in X$. The mapping T is a unique fixed point of J in the set $X = \{g \in \Omega : d(f, g) < \infty\}$.
(2) $d(J^k f, T) \to 0$ as $k \to \infty$. This implies the inequality $N - \lim_{k\to\infty} \frac{1}{8^k} f(2^k x, 2^k y) = T(x, y)$ for all $x, y \in X$.

(3) $d(f,T) \leqslant \frac{1}{1-\alpha} d(f,Jf)$, which implies the inequality

$$(f,T) \leqslant \frac{1}{8(1-\alpha)}. \tag{2.7}$$

By (2.2) and (2.4),

$$\begin{aligned} N\left(\frac{1}{8^k} Df(2^k x, 2^k y, 2^k z, 2^k w)\right) &\geqslant \frac{t}{t+\varphi(2^k x, 2^k y, 2^k z, 2^k w)} \\ &\geqslant \frac{8^k t}{8^k t + 8^k \alpha^k \varphi(x,y,z,w)} \end{aligned}$$

for all $x,y,z,w \in X$ and $t>0$. Since $\lim_{k\to\infty} \frac{8^k t}{8^k t+8^k\alpha^k\varphi(x,y,z,w)} = 1$ for all $x,y,z,w \in X$ and $t>0$,

$$N(DT(x,y,z,w),t) = 1$$

for all $x,y,z,w \in X$ and $t>0$. Therefore

$$T(x+y,z+w)+T(x+y,z-w) = 2T(x,z)+2T(x,w)+2T(y,z)+2T(y,w).$$

for all $x,y,z,w \in X$. Then, the mapping $T : X \times X \to Y$ is additive-quadratic.

It follows from Lemma 2.1 and (2.7) that

$$\begin{aligned} N_n\Big(f_n([x_{ij}],[y_{ij}]) - T_n([x_{ij}],[y_{ij}]), t\Big) &\geqslant \left\{ N\left(f(x_{ij},y_{ij}) - T(x_{ij},y_{ij}), \frac{t}{n^2}\right) : i,j = 1,2,\cdots,n \right\} \\ &\geqslant \min\left\{ \frac{8(1-\alpha)t}{8(1-\alpha)t + n^2\varphi(x_{ij},x_{ij},y_{ij},y_{ij})} : i,j=1,2,\cdots,n \right\} \\ &\geqslant \frac{8(1-\alpha)t}{8(1-\alpha)t + n^2 \sum_{i,j=1}^n \varphi(x_{ij},x_{ij},y_{ij},y_{ij})} \end{aligned}$$

for all $x = [x_{ij}] \in M_n(X)$. Therefore, we conclude that $T : X \times X \to Y$ is the unique mapping satisfying (2.3). $\square$

Corollary 2.3. *Let p,θ be positive real numbers $p<1$. Let $f : X \times X \to Y$, with $f(x,0)=0$, be a mapping satisfying*

$$N_n(Df_n([x_{ij}],[y_{ij}],[z_{ij}],[w_{ij}]),t) \geqslant \frac{t}{t+\sum_{i,j=1}^n \theta(\|x_{ij}\|^p + \|y_{ij}\|^p + \|z_{ij}\|^p + \|w_{ij}\|^p)} \tag{2.8}$$

for all $x=[x_{ij}], y=[y_{ij}], z=[z_{ij}], w=[w_{ij}] \in M_n(X)$ and $t>0$. Then $T(x,y) := N-\lim_{k\to\infty} \frac{1}{8^k} f(2^k x, 2^k y)$ exists for each $x,y \in X$ and defines an additive-quadratic mapping $T : X \times X \to Y$ such that

$$N_n\Big(f_n([x_{ij}],[y_{ij}]) - T_n([x_{ij}],[y_{ij}]), t\Big) \geqslant \frac{2(2-2^p)t}{2(2-2^p)t + n^2\sum_{i,j=1}^n \theta(\|x_{ij}\|^p + \|y_{ij}\|^p)}$$

for all $x=[x_{ij}], y=[y_{ij}] \in M_n(X)$ and $t>0$.

Proof. Putting $\varphi(a,b,c,z) := \theta\sum_{i=1}^m (\|a\|^p + \|b\|^p + \|c\|^p + \|d\|^p)$ for all $a,b,c,d \in X$ and letting $\alpha = 2^{p-1}$ in Theorem 2.2, we obtain the desired result. $\square$

Theorem 2.4. *Let $f : X \times X \to Y$, with $f(x, 0) = 0$, be a mapping for which there exists a function $\varphi : X^4 \to [0, \infty)$ satisfying (2.1). If there exists an $\alpha < 1$ such that*

$$\varphi\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}, \frac{d}{2}\right) \leqslant \frac{\alpha}{8}\varphi(a, b, c, d)$$

for all $a, b, c, d \in X$, then there exists a unique additive-quadratic mapping $T : X \times X \to Y$ such that

$$N\Big(f_n\left([x_{ij}], [y_{ij}]\right) - T_n\left([x_{ij}], [y_{ij}]\right), t\Big) \geqslant \frac{8(1-\alpha)t}{8(1-\alpha)t + n^2\alpha\sum_{i,j=1}^{n}\varphi(x_{ij}, x_{ij}, y_{ij}, y_{ij})}$$

for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Proof. Let (Ω, d) be the generalized metric space defined in the proof of Theorem 2.2. Here, we define the linear mapping $J : \Omega \to \Omega$ such that

$$Jg(x, y) := 8g(\frac{x}{2}, \frac{y}{2})$$

for all $x, y \in X$.

It follows from (2.5) that $d(f, Jf) \leqslant \frac{\alpha}{8}$. Thus

$$d(f, T) \leqslant \frac{\alpha}{8(1-\alpha)}.$$

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let p, θ be positive real numbers with $p > 1$. Let $f : X \times X \to Y$, with $f(x, 0) = 0$, be a mapping satisfying (2.8). Then $T(x, y) := N - \lim_{k\to\infty} 8^k f(\frac{x}{2^k}, \frac{y}{2^k})$ exists for all $x \in X$ and defines an additive-quadratic mapping $T : X \times X \to Y$ such that*

$$N_n\Big(f_n\left([x_{ij}], [y_{ij}]\right) - T_n\left([x_{ij}], [y_{ij}]\right), t\Big) \geqslant \frac{4(2^p-2)t}{4(2^p-2)t + n^2 \cdot 2^p\sum_{i,j=1}^{n}\theta\left(\|x_{ij}\|^p + \|y_{ij}\|^p\right)}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$ and $t > 0$.

Proof. Putting $\varphi(a, b, c, d) := \theta(\|a\|^p + \|b\|^p + \|c\|^p + \|d\|^p)$ for all $a, b, c, d \in X$ and letting $\alpha = 2^{1-p}$ in Theorem 2.4, we get the desired result. □

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Closed Form Expressions of some systems of Nonlinear Partial Difference Equations

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Abstract

In this paper we give the closed form expressions of some two dimensional systems of nonlinear rational partial difference equations of second order.We shall use a new method to prove the results by using (odd-even) double mathematical induction. As a direct consequences , we investigate and drive the explicit solutions of some partial difference equations and some (systems of) ordinary difference equations .

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Key Words and Phrases: (partial)difference equations, solutions , double mathematical induction.

1 Introduction

While the study of (ordinary)difference equations has been widely treated in the past , partial difference equations (PΔEs) have not received the same full attention .Both of ordinary and partial difference equations may be found in the study of probability ,dynamics and other branches of mathematical physics .Moreover,partial difference equations arise in applications involving population dynamics with spatial migrations , chemical reactions and finite difference schemes . Indeed Laplace and Lagrange considered the solution of partial difference equations in their studies of dynamics and probability.
An example of a partial difference equation is the following well known relation

$$C_m^{(n)} = C_{m-1}^{(n-1)} + C_m^{(n-1)} \quad ,1 \leq m < n.$$

The solution of this equation is the celebrated binomial coefficient function $C_m^{(n)}$ defined by

$$C_m^{(n)} = \frac{n!}{m!(n-m)!} \quad ,0 \leq m < n.$$

An another example , the following PΔEs :

$$s_k^{(n+1)} = s_{k-1}^{(n)} - n s_k^{(n)} \quad ,1 \le k < n.$$

$$S_k^{(n+1)} = S_{k-1}^{(n)} + k S_k^{(n)} \quad ,1 \le k < n.$$

The solutions of these PΔEs are the stirling numbers of the first kind $s_k^{(n)}$ and the stirling numbers of the second kind $S_k^{(n)}$ respectively .

Some authors investigate the closed form solutions for certain Partial difference equations .
For instance , Heins [[9]] considered the solution of the partial difference equation

$$X_{n+1,m} + X_{n-1,m} = 2X_{n,m+1}$$

under some conditions .
In [[3]] Carlitz has studied a solution of the partial difference equation

$$X_{n,m} - X_{n,m-1} - X_{n-1,m} - X_{n,m-2} + 3X_{n-1,m-1} - X_{n-2,m} = 0$$

He used a power series expansion related to the Fibonacci numbers .

For more results about partial difference equations we refer to ([1],[2], [4]-[8],[10],[11]-[15]).

In this paper , we studied the closed form solutions of the following systems of partial difference equations

$$\alpha X_{n,m} + \beta X_{n,m} X_{n-2,m-2} Y_{n-1,m-1} - X_{n-2,m-2} = 0 \tag{1}$$

$$\gamma Y_{n,m} + \delta Y_{n,m} Y_{n-2,m-2} X_{n-1,m-1} - Y_{n-2,m-2} = 0 \tag{2}$$

where $n, m \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$,$\alpha, \beta, \gamma, \delta \in \{1, -1\}$ and the initial values $X_{n,0}$,$X_{n,-1}$,$X_{0,m}$,$X_{-1,m}$, $Y_{n,0}$,$Y_{n,-1}$,$Y_{0,m}$,and $Y_{-1,m}$ are real numbers .

As a direct consequence , we can drive the explicit solutions of a family of partial difference equations in the following form

$$\alpha X_{n,m} + \beta X_{n,m} X_{n-2,m-2} X_{n-1,m-1} - X_{n-2,m-2} = 0$$

where $n, m \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$,$\alpha, \beta \in \{1, -1\}$ and the initial values $X_{n,0}$,$X_{n,-1}$,$X_{0,m}$, ,and $X_{-1,m}$ are real numbers .
Moreover , we can derive the exact solution for the following systems of ordinary difference equations

$$\alpha X_n + \beta X_n X_{n-2} Y_{n-1} - X_{n-2} = 0$$

$$\gamma Y_n + \delta Y_n Y_{n-2} X_{n-1} - Y_{n-2} = 0$$

where $n \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$,$\alpha, \beta, \gamma, \delta \in \{1, -1\}$ and the initial values X_0,X_{-1},Y_0,and Y_{-1} are real numbers .

2 Forms of Solutions

In this section we shall give explicit forms of solutions of the system (1)-(2) for particular values of $\alpha, \beta, \gamma, \delta \in \{1, -1\}$. We can rewrite system (1)-(2) in the following form

$$X_{n,m} = \frac{X_{n-2,m-2}}{\alpha + \beta X_{n-2,m-2} Y_{n-1,m-1}} \quad , \quad Y_{n,m} = \frac{Y_{n-2,m-2}}{\gamma + \delta Y_{n-2,m-2} X_{n-1,m-1}} \tag{3}$$

2.1 Form of Solutions when $(\alpha,\beta)=(\gamma,\delta)=(1,-1)$

In this case we have the system

$$X_{n,m}=\frac{X_{n-2,m-2}}{1-X_{n-2,m-2}Y_{n-1,m-1}} \quad , \quad Y_{n,m}=\frac{Y_{n-2,m-2}}{1-Y_{n-2,m-2}X_{n-1,m-1}} \tag{4}$$

Theorem 1. *Let $\{X_{n,m},Y_{n,m}\}_{n,m=-k}^{\infty}$ be a solution of system (4) with initial conditions*

$$X_{n,0},X_{n,-1},X_{0,m},X_{-1,m},Y_{n,0},Y_{n,-1},Y_{0,m},Y_{-1,m}$$

where $n,m\in\mathbb{N}_0$, $\mathbb{N}_0=\mathbb{N}\bigcup\{0\}$. Suppose $X_{-1,m-2}Y_{0,m-1}\neq 1$,$X_{n-2,-1}Y_{n-1,0}\neq 1$,$Y_{-1,m-2}X_{0,m-1}\neq 1$,$Y_{n-2,-1}X_{n-1,0}\neq 1$. Then, the form of solutions of system (4) ,for $n,m\geq 1$ and $n\geq m$, are as follows:

$$X_{n,m}=\begin{cases} X_{n-m,0}\prod\limits_{k=0}^{\frac{m-2}{2}}\frac{-1+(2k+1)X_{n-m,0}Y_{n-m-1,-1}}{-1+(2k+2)X_{n-m,0}Y_{n-m-1,-1}}, & m \quad even;\\ X_{n-m-1,-1}\prod\limits_{k=0}^{\frac{m-1}{2}}\frac{1-(2k)X_{n-m-1,-1}Y_{n-m,0}}{1-(2k+1)X_{n-m-1,-1}Y_{n-m,0}}, & m \quad odd;\end{cases} \tag{5}$$

$$Y_{n,m}=\begin{cases} Y_{n-m,0}\prod\limits_{k=0}^{\frac{m-2}{2}}\frac{-1+(2k+1)Y_{n-m,0}X_{n-m-1,-1}}{-1+(2k+2)Y_{n-m,0}X_{n-m-1,-1}}, & m \quad even;\\ Y_{n-m-1,-1}\prod\limits_{k=0}^{\frac{m-1}{2}}\frac{1-(2k)Y_{n-m-1,-1}X_{n-m,0}}{1-(2k+1)Y_{n-m-1,-1}X_{n-m,0}}, & m \quad odd;\end{cases} \tag{6}$$

$$X_{m,n}=\begin{cases} X_{-1,n-m-1}\prod\limits_{k=0}^{\frac{m-1}{2}}\frac{1-(2k)X_{-1,n-m-1}Y_{0,n-m}}{1-(2k+1)X_{-1,n-m-1}Y_{0,n-m}}, & m \quad odd;\\ X_{0,n-m}\prod\limits_{k=0}^{\frac{m-2}{2}}\frac{-1+(2k+1)X_{0,n-m}Y_{-1,n-m-1}}{-1+(2k+2)X_{0,n-m}Y_{-1,n-m-1}}, & m \quad even;\end{cases} \tag{7}$$

$$Y_{m,n}=\begin{cases} Y_{-1,n-m-1}\prod\limits_{k=0}^{\frac{m-1}{2}}\frac{1-(2k)Y_{-1,n-m-1}X_{0,n-m}}{1-(2k+1)Y_{-1,n-m-1}X_{0,n-m}}, & m \quad odd;\\ Y_{0,n-m}\prod\limits_{k=0}^{\frac{m-2}{2}}\frac{-1+(2k+1)Y_{0,n-m}X_{-1,n-m-1}}{-1+(2k+2)Y_{0,n-m}X_{-1,n-m-1}}, & m \quad even;\end{cases} \tag{8}$$

Proof. We shall use the principle of (odd-even)double mathematical induction . Firstly , we shall prove that the relations (5)-(8) hold for $(n,m)=(1,1)$. From equations in system (4)we can see

$$X_{1,1}=\frac{X_{-1,-1}}{1-X_{-1,-1}Y_{0,0}}=X_{-1,-1}\prod_{k=0}^{\frac{1-1}{2}}\frac{1-(2k)X_{-1,-1}Y_{0,0}}{1-(2k+1)X_{-1,-1}Y_{0,0}}$$

$$Y_{1,1}=\frac{Y_{-1,-1}}{1-Y_{-1,-1}X_{0,0}}=Y_{-1,-1}\prod_{k=0}^{\frac{1-1}{2}}\frac{1-(2k)Y_{-1,-1}X_{0,0}}{1-(2k+1)Y_{-1,-1}X_{0,0}}$$

Now , we shall prove that the relations (5)-(8)hold for $(n, m) = (2, 2)$.

$$X_{2,2} = \frac{X_{0,0}}{1 - X_{0,0}Y_{1,1}} = \frac{X_{0,0}}{1 - X_{0,0}(\frac{Y_{-1,-1}}{1-Y_{-1,-1}X_{0,0}})} = X_{0,0}(\frac{1 - X_{0,0}Y_{-1,-1}}{1 - 2X_{0,0}Y_{-1,-1}})$$

$$= X_{0,0}\prod_{k=0}^{\frac{2-2}{2}} \frac{-1 + (2k+1)X_{0,0}Y_{-1,-1}}{-1 + (2k+2)X_{0,0}Y_{-1,-1}}$$

$$Y_{2,2} = \frac{Y_{0,0}}{1 - Y_{0,0}X_{1,1}} = \frac{Y_{0,0}}{1 - Y_{0,0}(\frac{X_{-1,-1}}{1-X_{-1,-1}Y_{0,0}})} = Y_{0,0}(\frac{1 - Y_{0,0}X_{-1,-1}}{1 - 2Y_{0,0}X_{-1,-1}})$$

$$= Y_{0,0}\prod_{k=0}^{\frac{2-2}{2}} \frac{-1 + (2k+1)Y_{0,0}X_{-1,-1}}{-1 + (2k+2)Y_{0,0}X_{-1,-1}}$$

Moreover ,We shall prove that the relations (5)-(8) hold for $(n, m) = (1, 2)$ and $(n, m) = (2, 1)$.

$$X_{1,2} = \frac{X_{-1,0}}{1 - X_{-1,0}Y_{0,1}} = X_{-1,0}\prod_{k=0}^{\frac{1-1}{2}} \frac{1 - (2k)X_{-1,0}Y_{0,1}}{1 - (2k+1)X_{-1,0}Y_{0,1}}$$

$$Y_{1,2} = \frac{Y_{-1,0}}{1 - Y_{-1,0}X_{0,1}} = Y_{-1,0}\prod_{k=0}^{\frac{1-1}{2}} \frac{1 - (2k)Y_{-1,0}X_{0,1}}{1 - (2k+1)Y_{-1,0}X_{0,1}}$$

$$X_{2,1} = \frac{X_{0,-1}}{1 - X_{0,-1}Y_{1,0}} = X_{0,-1}\prod_{k=0}^{\frac{1-1}{2}} \frac{1 - (2k)X_{0,-1}Y_{1,0}}{1 - (2k+1)X_{0,-1}Y_{1,0}}$$

$$Y_{2,1} = \frac{Y_{0,-1}}{1 - Y_{0,-1}X_{1,0}}$$

Now suppose that the relations (5)-(8) hold for $m = 1$ and $m = 2$ with $n \in \mathbb{N}$. So we have ,

$$X_{n,1} = X_{n-2,-1}\prod_{k=0}^{0} \frac{1 - (2k)X_{n-2,-1}Y_{n-1,0}}{1 - (2k+1)X_{n-2,-1}Y_{n-1,0}} = \frac{X_{n-2,-1}}{1 - X_{n-2,-1}Y_{n-1,0}}$$

$$Y_{n,1} = \frac{Y_{n-2,-1}}{1 - Y_{n-2,-1}X_{n-1,0}}$$

$$X_{n,2} = X_{n-2,0}(\frac{1 - X_{n-2,0}Y_{n-3,-1}}{1 - 2X_{n-2,0}Y_{n-3,-1}})$$

$$Y_{n,2} = Y_{n-2,0}(\frac{1 - Y_{n-2,0}X_{n-3,-1}}{1 - 2Y_{n-2,0}X_{n-3,-1}})$$

Now we try to prove that relations (5)-(8) hold for $m=1$ with $n+2$.

$$X_{n+2,1}=\frac{X_{n,-1}}{1-X_{n,-1}Y_{n+1,0}}=X_{n,-1}\prod_{k=0}^{\frac{0}{2}}\frac{1-(2k)X_{n,-1}Y_{n+1,0}}{1-(2k+1)X_{n,-1}Y_{n+1,0}}$$

$$Y_{n+2,1}=\frac{Y_{n,-1}}{1-Y_{n,-1}X_{n+1,0}}=Y_{n,-1}\prod_{k=0}^{\frac{1-1}{2}}\frac{1-(2k)Y_{n,-1}X_{n+1,0}}{1-(2k+1)Y_{n,-1}X_{n+1,0}}$$

Now we try to prove that relations (5)-(8) hold for $m=2$ with $n+2$.

$$X_{n+2,2}=\frac{X_{n,0}}{1-X_{n,0}Y_{n+1,1}}=\frac{X_{n,0}}{1-X_{n,0}(\frac{Y_{n-1,-1}}{1-Y_{n-1,-1}X_{n,0}})}$$

$$=\frac{X_{n,0}(1-Y_{n-1,-1}X_{n,0})}{1-2Y_{n-1,-1}X_{n,0}}=X_{n,0}\prod_{k=0}^{\frac{2-2}{2}}\frac{1-(2k+1)X_{n,0}Y_{n-1,-1}}{1-(2k+2)X_{n,0}Y_{n-1,-1}}$$

$$Y_{n+2,2}=\frac{Y_{n,0}}{1-Y_{n,0}X_{n+1,1}}=Y_{n,0}\prod_{k=0}^{\frac{2-2}{2}}\frac{1-(2k+1)Y_{n,0}X_{n-1,-1}}{1-(2k+2)Y_{n,0}X_{n-1,-1}}$$

Finally , we suppose that relations (5)-(8) hold for $n,m\in\mathbb{N}$. We shall prove that relations (5)-(8) hold for $n,m+2\in\mathbb{N}$.
From (4)we have

$$X_{n,m+2}=\frac{X_{n-2,m}}{1-X_{n-2,m}Y_{n-1,m+1}} \tag{9}$$

There are four cases :

(1) If $n>m+2$ and m even .

$$X_{n,m+2}=\frac{X_{n-2,m}}{1-X_{n-2,m}Y_{n-1,m+1}}$$

$$=\frac{X_{n-m-2,0}\prod_{k=0}^{\frac{m-2}{2}}\frac{1-(2k+1)X_{n-m-2,0}Y_{n-m-3,-1}}{1-(2k+2)X_{n-m-2,0}Y_{n-m-3,-1}}}{1-(X_{n-m-2,0}\prod_{k=0}^{\frac{m-2}{2}}\frac{1-(2k+1)X_{n-m-2,0}Y_{n-m-3,-1}}{1-(2k+2)X_{n-m-2,0}Y_{n-m-3,-1}})(Y_{n-m-3,-1}\prod_{k=0}^{\frac{m}{2}}\frac{1-(2k)Y_{n-m-3,-1}X_{n-m-2,0}}{1-(2k+1)Y_{n-m-3,-1}X_{n-m-2,0}})}$$

$$=\frac{X_{n-m-2,0}\prod_{k=0}^{\frac{m-2}{2}}\frac{1-(2k+1)X_{n-m-2,0}Y_{n-m-3,-1}}{1-(2k+2)X_{n-m-2,0}Y_{n-m-3,-1}}}{1-\frac{X_{n-m-2,0}Y_{n-m-3,-1}}{1-(m+1)X_{n-m-2,0}Y_{n-m-3,-1}}}$$

$$=X_{n-m-2,0}\prod_{k=0}^{\frac{m}{2}}\frac{1-(2k+1)X_{n-m-2,0}Y_{n-m-3,-1}}{1-(2k+2)X_{n-m-2,0}Y_{n-m-3,-1}}$$

(2) If $n > m+2$ and m odd

$$X_{n,m+2} = \frac{X_{n-2,m}}{1 - X_{n-2,m}Y_{n-1,m+1}}$$

$$= \frac{X_{n-m-3,-1} \prod_{k=0}^{\frac{m-1}{2}} \frac{1-(2k)X_{n-m-3,-1}Y_{n-m-2,0}}{1-(2k+1)X_{n-m-3,-1}Y_{n-m-2,0}}}{1 - (X_{n-m-3,-1} \prod_{k=0}^{\frac{m-1}{2}} \frac{1-(2k)X_{n-m-3,-1}Y_{n-m-2,0}}{1-(2k+1)X_{n-m-3,-1}Y_{n-m-2,0}})(Y_{n-m-2,0} \prod_{k=0}^{\frac{m-1}{2}} \frac{1-(2k+1)Y_{n-m-2,0}X_{n-m-3,-1}}{1-(2k+2)Y_{n-m-2,0}X_{n-m-3,-1}})}$$

$$= \frac{X_{n-m-3,-1} \prod_{k=0}^{\frac{m-1}{2}} \frac{1-(2k)X_{n-m-3,-1}Y_{n-m-2,0}}{1-(2k+1)X_{n-m-3,-1}Y_{n-m-2,0}}}{1 - \frac{X_{n-m-3,-1}Y_{n-m-2,0}}{1-(m+1)X_{n-m-3,-1}Y_{n-m-2,0}}}$$

$$= X_{n-m-3,-1} \prod_{k=0}^{\frac{m+1}{2}} \frac{1-(2k)X_{n-m-3,-1}Y_{n-m-2,0}}{1-(2k+1)X_{n-m-3,-1}Y_{n-m-2,0}}$$

(3) If $n < m+2$ and m even
By symmetry ,using (7) and (8), we can prove it like part (1) .

(4) If $n < m+2$ and m odd
By symmetry ,using (7) and (8), we can prove it like part (2) ..

$$Y_{n,m+2} = \frac{Y_{n-2,m}}{1 - Y_{n-2,m}X_{n-1,m+1}}$$

We can do that by the same way in proving equation (9)

□

Proposition 1. *We have the following properties for the solutions of system (4) :*

(1) If m even and $X_{n-m,0} = 0$, then $X_{n,m} = 0$.

(2) If m odd and $X_{n-m,0} = 0$, then $Y_{n,m} = Y_{n-m-1,-1}$.

(3) If m even and $Y_{n-m,0} = 0$, then $Y_{n,m} = 0$.

(4) If m odd and $Y_{n-m,0} = 0$, then $X_{n,m} = X_{n-m-1,-1}$.

(5) If m even and $X_{n-m-1,-1} = 0$, then $Y_{n,m} = Y_{n-m,0}$.

(6) If m odd and $X_{n-m-1,-1} = 0$, then $X_{n,m} = 0$.

(7) If m even and $Y_{n-m-1,-1} = 0$, then $X_{n,m} = X_{n-m,0}$.

(8) If m odd and $Y_{n-m-1,-1} = 0$, then $Y_{n,m} = 0$.

Proposition 2. *We have the following properties for the solutions of system (4) :*

(1) If m even and $X_{0,n-m} = 0$, then $X_{m,n} = 0$.

(2) If m odd and $X_{0,n-m} = 0$, then $Y_{m,n} = Y_{-1,n-m-1}$.

(3) If m even and $Y_{0,n-m} = 0$, then $Y_{m,n} = 0$.

(4) If m odd and $Y_{0,n-m} = 0$, then $X_{m,n} = X_{-1,n-m-1}$.

(5) If m even and $X_{-1,n-m-1} = 0$, then $Y_{m,n} = Y_{0,n-m}$.

(6) If m odd and $X_{-1,n-m-1} = 0$, then $X_{m,n} = 0$.

(7) If m even and $Y_{-1,n-m-1} = 0$, then $X_{m,n} = X_{0,n-m}$.

(8) If m odd and $Y_{-1,n-m-1} = 0$, then $Y_{m,n} = 0$.

Remark 1. *If we take into account the one dimensional case of system (4) we have a partial difference equation in the form*

$$X_{n,m} = \frac{X_{n-2,m-2}}{1 - X_{n-2,m-2}X_{n-1,m-1}} \tag{10}$$

We can see that the closed form solution of equation(10) is given ,from theorem(1) , by the following corollary .

Corollary 2. *Let $\{X_{n,m}\}_{n,m=-k}^{\infty}$ be a solution of equation (10) with initial conditions $X_{n,0}, X_{n,-1}, X_{0,m}, X_{-1,m}$, where $n, m \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$. Suppose $X_{-1,m-2}X_{0,m-1} \neq 1$,$X_{n-2,-1}X_{n-1,0} \neq 1$. Then, the form of solutions of equation (10) ,for $n, m \geq 1$ and $n \geq m$, are as follows:*

$$X_{n,m} = \begin{cases} X_{n-m,0} \prod\limits_{k=0}^{\frac{m-2}{2}} \frac{-1+(2k+1)X_{n-m,0}X_{n-m-1,-1}}{-1+(2k+2)X_{n-m,0}X_{n-m-1,-1}}, & m \quad even; \\ X_{n-m-1,-1} \prod\limits_{k=0}^{\frac{m-1}{2}} \frac{1-(2k)X_{n-m-1,-1}X_{n-m,0}}{1-(2k+1)X_{n-m-1,-1}X_{n-m,0}}, & m \quad odd; \end{cases}$$

$$X_{m,n} = \begin{cases} X_{-1,n-m-1} \prod\limits_{k=0}^{\frac{m-1}{2}} \frac{1-(2k)X_{-1,n-m-1}X_{0,n-m}}{1-(2k+1)X_{-1,n-m-1}X_{0,n-m}}, & m \quad odd; \\ X_{0,n-m} \prod\limits_{k=0}^{\frac{m-2}{2}} \frac{-1+(2k+1)X_{0,n-m}X_{-1,n-m-1}}{-1+(2k+2)X_{0,n-m}X_{-1,n-m-1}}, & m \quad even; \end{cases}$$

Proposition 3. *We have the following properties for the solutions of equation (4):*

(1) If m even and $X_{n-m,0} = 0$, then $X_{n,m} = 0$.

(2) If m odd and $X_{n-m,0} = 0$, then $X_{n,m} = X_{n-m-1,-1}$.

(3) If m even and $X_{n-m-1,-1} = 0$, then $X_{n,m} = X_{n-m,0}$.

(4) If m odd and $X_{n-m-1,-1} = 0$, then $X_{n,m} = 0$.

(5) If m even and $X_{0,n-m} = 0$, then $X_{m,n} = 0$.

(6) If m odd and $X_{0,n-m} = 0$, then $X_{m,n} = X_{-1,n-m-1}$.

(7) If m even and $X_{-1,n-m-1} = 0$, then $X_{m,n} = X_{0,n-m}$.

(8) If m odd and $X_{-1,n-m-1} = 0$, then $X_{m,n} = 0$.

Remark 2. *If we put $n = m$ in system (4) we have a system of ordinary difference equations in the following form*

$$X_n = \frac{X_{n-2}}{1 - X_{n-2}Y_{n-1}}, \qquad Y_n = \frac{Y_{n-2}}{1 - Y_{n-2}X_{n-1}} \tag{11}$$

Corollary 3. *Let $\{X_n, Y_n\}_{n=-k}^{\infty}$ be a solution of system (11) with initial conditions X_0, X_{-1}, Y_0, Y_{-1} . Suppose $X_{-1}Y_0 \neq 1$,and $Y_{-1}X_0 \neq 1$,. Then, the form of solutions of system (11) ,for $n \geq 1$ are as follows:*

$$X_n = \begin{cases} X_0 \prod\limits_{k=0}^{\frac{n-2}{2}} \frac{-1+(2k+1)X_0Y_{-1}}{-1+(2k+2)X_0Y_{-1}}, n, even \\ X_{-1} \prod\limits_{k=0}^{\frac{n-1}{2}} \frac{1-(2k)X_{-1}Y_0}{1-(2k+1)X_{-1}Y_0}, n, odd \end{cases} \qquad Y_n = \begin{cases} Y_0 \prod\limits_{k=0}^{\frac{n-2}{2}} \frac{-1+(2k+1)Y_0X_{-1}}{-1+(2k+2)Y_0X_{-1}}, n, even \\ Y_{-1} \prod\limits_{k=0}^{\frac{n-1}{2}} \frac{1-(2k)Y_{-1}X_0}{1-(2k+1)Y_{-1}X_0}, n, odd \end{cases}$$

Remark 3. *If we put $X = Y$ in system(11) we get an ordinary difference equation in the form*

$$X_n = \frac{X_{n-2}}{1 - X_{n-2}X_{n-1}} \tag{12}$$

We can see that the closed form solution of equation(12) is given ,from corollary(3) , by the following

$$X_n = \begin{cases} X_0 \prod\limits_{k=0}^{\frac{n-2}{2}} \frac{-1+(2k+1)X_0X_{-1}}{-1+(2k+2)X_0X_{-1}}, & n \quad even; \\ X_{-1} \prod\limits_{k=0}^{\frac{n-1}{2}} \frac{1-(2k)X_{-1}X_0}{1-(2k+1)X_{-1}X_0}, & n \quad odd; \end{cases}$$

where $n \in \mathbb{N}$, and $X_{-1}X_0 \neq -1$ .We can easy see that if n even (or odd) and $X_0 = 0$ then $X_n = 0 (X_n = X_{-1})$. Also if n even (or odd) and $X_{-1} = 0$ then $X_n = X_0 (X_n = 0)$.

2.2 Form of Solutions when $(\alpha,\beta)=(1,1)\&(\gamma,\delta)=(1,-1)$

In this case we have the system

$$X_{n,m}=\frac{X_{n-2,m-2}}{1+X_{n-2,m-2}Y_{n-1,m-1}} \quad , \quad Y_{n,m}=\frac{Y_{n-2,m-2}}{1-Y_{n-2,m-2}X_{n-1,m-1}} \tag{13}$$

Theorem 4. *Let $\{X_{n,m},Y_{n,m}\}_{n,m=-k}^{\infty}$ be a solution of system (13) with initial conditions $X_{n,0},X_{n,-1},X_{0,m},X_{-1,m},Y_{n,0},Y_{n,-1},Y_{0,m},Y_{-1,m}$ where $n,m\in\mathbb{N}_0$, $\mathbb{N}_0=\mathbb{N}\bigcup\{0\}$. Suppose $X_{-1,m-2}Y_{0,m-1}\neq-1$,$X_{n-2,-1}Y_{n-1,0}\neq-1$, $Y_{-1,m-2}X_{0,m-1}\neq1$,$Y_{n-2,-1}X_{n-1,0}\neq1$. Then, the form of solutions of system (13) ,for $n,m\geq1$ and $n\geq m$, are as follows:*

$$X_{n,m}=\begin{cases}\frac{X_{n-m-1,-1}}{(1+X_{n-m-1,-1}Y_{n-m,0})^{\frac{m+1}{2}}}, & m \quad odd;\\ (-1)^{\frac{m}{2}}X_{n-m,0}(-1+X_{n-m,0}Y_{n-m-1,-1})^{\frac{m}{2}}, & m \quad even;\end{cases}$$

$$Y_{n,m}=\begin{cases}\frac{(-1)^{\frac{m+1}{2}}Y_{n-m-1,-1}}{(-1+Y_{n-m-1,-1}X_{n-m,0})^{\frac{m+1}{2}}}, & m \quad odd;\\ Y_{n-m,0}(1+Y_{n-m,0}X_{n-m-1,-1})^{\frac{m}{2}}, & m \quad even;\end{cases}$$

$$X_{m,n}=\begin{cases}\frac{X_{-1,n-m-1}}{(1+X_{-1,n-m-1}Y_{0,n-m})^{\frac{m+1}{2}}}, & m \quad odd;\\ (-1)^{\frac{m}{2}}X_{0,n-m}(-1+X_{0,n-m}Y_{-1,n-m-1})^{\frac{m}{2}}, & m \quad even;\end{cases}$$

$$Y_{m,n}=\begin{cases}\frac{(-1)^{\frac{m+1}{2}}Y_{-1,n-m-1}}{(-1+Y_{-1,n-m-1}X_{0,n-m})^{\frac{m+1}{2}}}, & m \quad odd;\\ Y_{0,n-m}(1+Y_{0,n-m}X_{-1,n-m-1})^{\frac{m}{2}}, & m \quad even;\end{cases}$$

Proof. We can prove the theorem by odd-even double mathematical induction as in theorem (1). □

Remark 4. *We can see that both of proposition (1) and proposition (2) hold for the solutions of system (13) included in theorem(4) .*

Remark 5. *If we put $n=m$ in system (13) we have a system of ordinary difference equations in the following form*

$$X_n=\frac{X_{n-2}}{1+X_{n-2}Y_{n-1}}, \qquad Y_n=\frac{Y_{n-2}}{1-Y_{n-2}X_{n-1}} \tag{14}$$

We can drive the formulas for solutions from theorem(4) in the following corollary .

Corollary 5. *Let $\{X_n,Y_n\}_{n=-k}^{\infty}$ be a solution of system (14) with initial conditions X_0,X_{-1},Y_0,Y_{-1} . Suppose $X_{-1}Y_0\neq-1$,and $Y_{-1}X_0\neq1$,. Then, the form of solutions of system (14) ,for $n\geq1$ are as follows:*

$$X_n=\begin{cases}\frac{X_{-1}}{(1+X_{-1}Y_0)^{\frac{n+1}{2}}}; n, odd\\ (-1)^{\frac{n}{2}}X_0(-1+X_0Y_{-1})^{\frac{n}{2}}; n, even\end{cases} \qquad Y_n=\begin{cases}\frac{(-1)^{\frac{n+1}{2}}Y_{-1}}{(-1+Y_{-1}X_0)^{\frac{n+1}{2}}}; n, odd\\ Y_0(1+Y_0X_{-1})^{\frac{n}{2}}; n, even\end{cases}$$

.

2.3 Form of Solutions when $(\alpha,\beta)=(1,1)\&(\gamma,\delta)=(-1,1)$

In this case we have the system

$$X_{n,m}=\frac{X_{n-2,m-2}}{1+X_{n-2,m-2}Y_{n-1,m-1}}\quad,\quad Y_{n,m}=\frac{Y_{n-2,m-2}}{-1+Y_{n-2,m-2}X_{n-1,m-1}}\qquad(15)$$

Theorem 6. *Let $\{X_{n,m},Y_{n,m}\}_{n,m=-k}^{\infty}$ be a solution of system (15) with initial conditions $X_{n,0},X_{n,-1},X_{0,m},X_{-1,m},Y_{n,0},Y_{n,-1},Y_{0,m},Y_{-1,m}$ where $n,m\in\mathbb{N}_0$, $\mathbb{N}_0=\mathbb{N}\bigcup\{0\}$. Suppose $X_{-1,m-2}Y_{0,m-1}\neq-1$,$X_{n-2,-1}Y_{n-1,0}\neq-1$, $Y_{-1,m-2}X_{0,m-1}\neq 1$,$Y_{n-2,-1}X_{n-1,0}\neq 1$. Then, the form of solutions of system (15) ,for $n,m\geq 1$ and $n\geq m$, are as follows:*

$$X_{n,m}=\begin{cases}\frac{(-1)^{\frac{m-1}{4}}X_{n-m-1,-1}}{(1+X_{n-m-1,-1}Y_{n-m,0})^{\frac{m+3}{4}}(-1+X_{n-m-1,-1}Y_{n-m,0})^{\frac{m-1}{4}}}, & m=4K+1;\\ \frac{(-1)^{\frac{m-2}{4}}X_{n-m,0}(-1+X_{n-m,0}Y_{n-m-1,-1})^{\frac{m}{2}}}{(-1+2X_{n-m,0}Y_{n-m-1,-1})^{\frac{m+2}{4}}}, & m=4K+2;\\ \frac{(-1)^{\frac{m+1}{4}}X_{n-m-1,-1}}{(-1+X_{n-m-1,-1}Y_{n-m,0})^{\frac{m+1}{4}}(1+X_{n-m-1,-1}Y_{n-m,0})^{\frac{m+1}{4}}}, & m=4K+3;\\ \frac{(-1)^{\frac{m}{4}}X_{n-m,0}(-1+X_{n-m,0}Y_{n-m-1,-1})^{\frac{m}{2}}}{(-1+2X_{n-m,0}Y_{n-m-1,-1})^{\frac{m}{4}}}, & m=4K+4;\end{cases}$$

$$Y_{n,m}=\begin{cases}\frac{(-1)^{\frac{m-1}{4}}Y_{n-m-1,-1}(-1+2Y_{n-m-1,-1}X_{n-m,0})^{\frac{m-1}{4}}}{(-1+Y_{n-m-1,-1}X_{n-m,0})^{\frac{m+1}{2}}}, & m=4K+1;\\ (-1)^{\frac{m+2}{4}}Y_{n-m,0}(-1+Y_{n-m,0}X_{n-m-1,-1})^{\frac{m-2}{4}} & \\ \quad.(1+Y_{n-m,0}X_{n-m-1,-1})^{\frac{m+2}{4}}, & m=4K+2;\\ \frac{(-1)^{\frac{m+1}{4}}Y_{n-m-1,-1}(-1+2Y_{n-m-1,-1}X_{n-m,0})^{\frac{m+1}{4}}}{(-1+Y_{n-m-1,-1}X_{n-m,0})^{\frac{m+1}{2}}}, & m=4K+3;\\ (-1)^{\frac{m}{4}}Y_{n-m,0}(-1+Y_{n-m,0}X_{n-m-1,-1})^{\frac{m}{4}} & \\ \quad.(1+Y_{n-m,0}X_{n-m-1,-1})^{\frac{m}{4}}, & m=4K+4;\end{cases}$$

$$X_{m,n}=\begin{cases}\frac{(-1)^{\frac{m-1}{4}}X_{-1,n-m-1}}{(1+X_{-1,n-m-1}Y_{0,n-m})^{\frac{m+3}{4}}(-1+X_{-1,n-m-1}Y_{0,n-m})^{\frac{m-1}{4}}}, & m=4K+1;\\ \frac{(-1)^{\frac{m-2}{4}}X_{0,n-m}(-1+X_{0,n-m}Y_{-1,n-m-1})^{\frac{m}{2}}}{(-1+2X_{0,n-m}Y_{-1,n-m-1})^{\frac{m+2}{4}}}, & m=4K+2;\\ \frac{(-1)^{\frac{m+1}{4}}X_{-1,n-m-1}}{(-1+X_{-1,n-m-1}Y_{0,n-m})^{\frac{m+1}{4}}(1+X_{-1,n-m-1}Y_{0,n-m})^{\frac{m+1}{4}}}, & m=4K+3;\\ \frac{(-1)^{\frac{m}{4}}X_{0,n-m}(-1+X_{0,n-m}Y_{-1,n-m-1})^{\frac{m}{2}}}{(-1+2X_{0,n-m}Y_{-1,n-m-1})^{\frac{m}{4}}}, & m=4K+4;\end{cases}$$

$$Y_{m,n}=\begin{cases}\frac{(-1)^{\frac{m-1}{4}}Y_{-1,n-m-1}(-1+2Y_{-1,n-m-1}X_{0,n-m})^{\frac{m-1}{4}}}{(-1+Y_{-1,n-m-1}X_{0,n-m})^{\frac{m+1}{2}}}, & m=4K+1;\\ (-1)^{\frac{m+2}{4}}Y_{0,n-m}(-1+Y_{0,n-m}X_{-1,n-m-1})^{\frac{m-2}{4}} & \\ \quad.(1+Y_{0,n-m}X_{-1,n-m-1})^{\frac{m+2}{4}}, & m=4K+2;\\ \frac{(-1)^{\frac{m+1}{4}}Y_{-1,n-m-1}(-1+2Y_{-1,n-m-1}X_{0,n-m})^{\frac{m+1}{4}}}{(-1+Y_{-1,n-m-1}X_{0,n-m})^{\frac{m+1}{2}}}, & m=4K+3;\\ (-1)^{\frac{m}{4}}Y_{0,n-m}(-1+Y_{0,n-m}X_{-1,n-m-1})^{\frac{m}{4}} & \\ \quad.(1+Y_{0,n-m}X_{-1,n-m-1})^{\frac{m}{4}}, & m=4K+4;\end{cases}$$

where $k = 0, 1, 2, 3......$.

Proof. We can prove the theorem by piecewise double mathematical induction as in theorem (1). □

Proposition 4. *We have the following properties for the solutions of system* (15) *:*

(1) *If* m *even and* $X_{n-m,0} = 0$, *then* $X_{n,m} = 0$.

(2) *If* m *odd and* $X_{n-m,0} = 0$, *then* $Y_{n,m} = \pm Y_{n-m-1,-1}$.

(3) *If* m *even and* $Y_{n-m,0} = 0$, *then* $Y_{n,m} = 0$.

(4) *If* m *odd and* $Y_{n-m,0} = 0$, *then* $X_{n,m} = X_{n-m-1,-1}$.

(5) *If* m *even and* $X_{n-m-1,-1} = 0$, *then* $Y_{n,m} = \pm Y_{n-m,0}$.

(6) *If* m *odd and* $X_{n-m-1,-1} = 0$, *then* $X_{n,m} = 0$.

(7) *If* m *even and* $Y_{n-m-1,-1} = 0$, *then* $X_{n,m} = \pm X_{n-m,0}$.

(8) *If* m *odd and* $Y_{n-m-1,-1} = 0$, *then* $Y_{n,m} = 0$.

Proposition 5. *We have the following properties for the solutions of system* (15) *:*

(1) *If* m *even and* $X_{0,n-m} = 0$, *then* $X_{m,n} = 0$.

(2) *If* m *odd and* $X_{0,n-m} = 0$, *then* $Y_{m,n} = \pm Y_{-1,n-m-1}$.

(3) *If* m *even and* $Y_{0,n-m} = 0$, *then* $Y_{m,n} = 0$.

(4) *If* m *odd and* $Y_{0,n-m} = 0$, *then* $X_{m,n} = X_{-1,n-m-1}$.

(5) *If* m *even and* $X_{-1,n-m-1} = 0$, *then* $Y_{m,n} = \pm Y_{0,n-m}$.

(6) *If* m *odd and* $X_{-1,n-m-1} = 0$, *then* $X_{m,n} = 0$.

(7) *If* m *even and* $Y_{-1,n-m-1} = 0$, *then* $X_{m,n} = \pm X_{0,n-m}$.

(8) *If* m *odd and* $Y_{-1,n-m-1} = 0$, *then* $Y_{m,n} = 0$.

Remark 6. *If we put* $n = m$ *in system* (15) *we have a system of ordinary difference equations in the following form*

$$X_n = \frac{X_{n-2}}{1 + X_{n-2}Y_{n-1}}, \qquad Y_n = \frac{Y_{n-2}}{-1 + Y_{n-2}X_{n-1}} \tag{16}$$

We can drive the formulas for solutions from theorem(6) *in the following corollary .*

Corollary 7. *Let* $\{X_n, Y_n\}_{n=-k}^{\infty}$ *be a solution of system* (16) *with initial conditions* X_0, X_{-1}, Y_0, Y_{-1} . *Suppose* $X_{-1}Y_0 \neq -1$ *,and* $Y_{-1}X_0 \neq 1$ *,. Then, the form of solutions of system* (16) *,for* $n \geq 1$ *are as follows:*

$$X_n = \begin{cases} \frac{(-1)^{\frac{n-1}{4}} X_{-1}}{(1+X_{-1}Y_0)^{\frac{n+3}{4}}(-1+X_{-1}Y_0)^{\frac{n-1}{4}}}, & n = 4K+1; \\ \frac{(-1)^{\frac{n-2}{4}} X_0(-1+X_0Y_{-1})^{\frac{n}{2}}}{(-1+2X_0Y_{-1})^{\frac{n+2}{4}}}, & n = 4K+2; \\ \frac{(-1)^{\frac{n+1}{4}} X_{-1}}{(-1+X_{-1}Y_0)^{\frac{n+1}{4}}(1+X_{-1}Y_0)^{\frac{n+1}{4}}}, & n = 4K+3; \\ \frac{(-1)^{\frac{n}{4}} X_0(-1+X_0Y_{-1})^{\frac{n}{2}}}{(-1+2X_0Y_{-1})^{\frac{n}{4}}}, & n = 4K+4; \end{cases}$$

$$Y_n = \begin{cases} \frac{(-1)^{\frac{n-1}{4}} Y_{-1}(-1+2Y_{-1}X_0)^{\frac{n-1}{4}}}{(-1+Y_{-1}X_0)^{\frac{n+1}{2}}}, & n = 4K+1; \\ (-1)^{\frac{n+2}{4}} Y_0(-1+Y_0X_{-1})^{\frac{n-2}{4}}(1+Y_0X_{-1})^{\frac{n+2}{4}}, & n = 4K+2; \\ \frac{(-1)^{\frac{n+1}{4}} Y_{-1}(-1+2Y_{-1}X_0)^{\frac{n+1}{4}}}{(-1+Y_{-1}X_0)^{\frac{n+1}{2}}}, & n = 4K+3; \\ (-1)^{\frac{n}{4}} Y_0(-1+Y_0X_{-1})^{\frac{n}{4}}(1+Y_0X_{-1})^{\frac{n}{4}}, & n = 4K+4; \end{cases}$$

where $k = 0, 1, 2, 3......$.

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TWO-DIMENSIONAL CHLODOWSKY VARIANT OF q-BERNSTEIN-SCHURER-STANCU OPERATORS

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ABSTRACT. In this paper, two-dimensional Chlodowsky variant q-based Bernstein-Schurer-Stancu operators are introduced. Korovkin-type approximation theorems in different function spaces are studied. The error of approximation by using full modulus of continuity and partial modulus of continuities are given. Moreover, we introduce a generalization of our operators and investigate its approximation in more general weighted space.

1. INTRODUCTION

It was Chlodowsky [3] who introduced the classical Bernstein-Chlodowsky operators as

$$C_n(f;x) = \sum_{r=0}^{n} f\left(\frac{r}{n}b_n\right)\binom{n}{r}\left(\frac{x}{b_n}\right)^r\left(1-\frac{x}{b_n}\right)^{n-r},$$

where the function f is defined on $[0,\infty)$ and $\{b_n\}$ is a positive increasing sequence with $b_n \to \infty$ and $\dfrac{b_n}{n} \to 0$ as $n \to \infty$.

In 2008, the q-analogue of Chlodowsky operators were introduced and investigated by Karslı and Gupta [8] as

$$C_n(f;q;x) = \sum_{k=0}^{n+p} f\left(\frac{[k]}{[n]}b_n\right)\begin{bmatrix} n+p \\ k \end{bmatrix}\left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right), \qquad 0 \le x \le b_n$$

where $\{b_n\}$ has the same property of Bernstein-Chlodowsky operators.

On the other hand, the q-Bernstein-Schurer operators were defined by Muraru [9], for fixed $p \in \mathbb{N}_0$ and for all $x \in [0,1]$, by

$$B_n^p(f;q;x) = \sum_{k=0}^{n+p} f\left(\frac{[k]}{[n]}\right)\begin{bmatrix} n+p \\ k \end{bmatrix} x^k \prod_{s=0}^{n+p-k-1}(1-q^s x). \tag{1.1}$$

Note that the case $q \to 1^-$ in (1.1) reduces to the operators considered by Schurer [12]. Then, some properties of the q-Bernstein-Schurer operators were given in [13]. In 2013, the q-analogue of Bernstein-Schurer-Stancu operators $S_{n,p}^{\alpha,\beta} : C[0,1+p] \to C[0,1]$ were introduced by Agrawal, et al in [4] by

$$S_{n,p}^{(\alpha,\beta)}(f;q;x) = \sum_{k=0}^{n+p} f\left(\frac{[k]+\alpha}{[n]+\beta}\right)\begin{bmatrix} n+p \\ k \end{bmatrix} x^k \prod_{s=0}^{n+p-k-1}(1-q^s x), \tag{1.2}$$

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where α and β are non-negative numbers which satisfy $0 \leq \alpha \leq \beta$ and also p is a non-negative integer. Notice that, if we choose $\alpha = \beta = 0$ in (1.2), $S_{n,p}^{(\alpha,\beta)}(f;q;x)$ reduces to the classical q-Bernstein operator [10].

Recently, Chlodowsky variant of q-Bernstein-Schurer-Stancu operators were introduced by the authors in [14] as

$$(1.3)\quad C_{n,p}^{(\alpha,\beta)}(f;q;x) := \sum_{k=0}^{n+p} f\left(\frac{[k]+\alpha}{[n]+\beta} b_n\right) \begin{bmatrix} n+p \\ k \end{bmatrix} \left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n+p-k-1} \left(1 - q^s \frac{x}{b_n}\right),$$

where $n \in \mathbb{N}$, $p \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$, $0 \leq \alpha \leq \beta$, $0 \leq x \leq b_n$ and $0 < q < 1$. If $\alpha = \beta = p = 0$ in (1.3), we get the operators $C_n(f;q;x)$ and if $q \to 1^-$ and $\alpha = \beta = p = 0$ in (1.3), we get the operators $C_n(f;x)$.

In 2009, Büyükyazıcı [1] defined the two-dimensional q-Bernstein-Chlodowsky polynomials as

$$\widetilde{B}_{n,m}^{q_n,q_m}(f;x,y) = \sum_{k=0}^{n}\sum_{j=0}^{m} f\left(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n, \frac{[j]_{q_m}}{[m]_{q_m}}\beta_m\right) \Omega_{k,n,q_n}\left(\frac{x}{\alpha_n}\right) \Omega_{k,m,q_m}\left(\frac{y}{\beta_m}\right)$$

where $\Omega_{k,n,q_n}(u) = \begin{bmatrix} n \\ k \end{bmatrix} u^k \prod_{s=0}^{n-k-1}(1-q_n^s)$ and investigated its approximation properties on the rectangular unbounded domain.

On the other hand, Büyükyazıcı and Sharma [2] defined the two-dimensional q-Bernstein-Chlodowsky-Durrmeyer operators on the rectangular unbounded domain and derived the Korovkin type approximation properties. They also computed the order of convergence by means of the modulus of continuity and then examined the weighted approximation properties for these operators.

In the present paper we consider the two dimensional Chlodowsky variant of q-Bernstein-Schurer-Stancu operators. Some of the results about the operators $C_{n,p}^{(\alpha,\beta)}(f;q;x)$ defined in (1.3) will be useful in our investigations. For instance, the first three moments first three moments of the operator $C_{n,p}^{(\alpha,\beta)}(f;q;x)$ are as follows [14]:

Lemma 1.1. *Let $C_{n,p}^{(\alpha,\beta)}(f;q;x)$ defined. Then the first few moments of the operators are,*

(i) $C_{n,p}^{(\alpha,\beta)}(1;q;x) = 1$,

(ii) $C_{n,p}^{(\alpha,\beta)}(t;q;x) = \dfrac{[n+p]\,x + \alpha b_n}{[n]+\beta}$,

(iii) $C_{n,p}^{(\alpha,\beta)}(\mathrm{t}^2;q;x) = \dfrac{1}{([n]+\beta)^2}\left\{[n+p-1]\,[n+p]\,qx^2\right.$

$\left. + (2\alpha+1)\,[n+p]\,b_n x + \alpha^2 b_n^2\right\}$.

Before proceeding further let us recall that the some basic definitions of q-calculus. The q-integer of $k \in \mathbb{R}$ is [7]

$$[k]_q = \begin{cases} \left(1-q^k\right)/(1-q), & q \neq 1 \\ k, & q = 1, \end{cases}$$

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the q-factorial is defined by

$$[k]_q! = \begin{cases} [k]_q [k-1]_q \dots [1]_q, & k = 1, 2, 3, \dots, \\ 1, & k = 0 \end{cases}$$

and q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q! \, [k]_q!}$$

The organization of the paper as follows:
In section two, the two dimensional Chlodowsky variant of q-Bernstein-Schurer-Stancu operators is established and the first few moments of the operator is given. In section three, some Korovkin-type theorems in different function spaces are studied. In section four, we obtain the order of convergence of the Chlodowsky variant of q-Bernstein-Schurer-Stancu operators by means of the first modulus of continuity and partial modulus of continuity. In section five, we study the generalization of the two-dimensional Chlodowsky variant of q-Bernstein-Schurer-Stancu operators and seek its approximation properties in more general weighted space.

2. Construction of the operators

Let $\{a_n\}$ and $\{b_m\}$ be increasing sequences of real numbers satisfying

$$\lim_{n\to\infty} a_n = \lim_{m\to\infty} b_m = \infty.$$

Let, D_{a_n,b_m} denotes

$$D_{a_n,b_m} = \{(x,y) : 0 \le x \le a_n, \, 0 \le y \le b_m\}. \tag{2.1}$$

For $(x,y) \in D_{a_n,b_m}$, we construct the two dimensional Chlodowsky variant of q-Bernstein-Schurer-Stancu operators as

$$\begin{aligned} & C_{n,m,p}^{(\alpha,\beta)}(f; q_n, q_m; x, y) \\ & := \sum_{k=0}^{n+p} \sum_{j=0}^{m+p} f\left(\frac{[k]_{q_n} + \alpha}{[n]_{q_n} + \beta} a_n, \frac{[j]_{q_m} + \alpha}{[m]_{q_m} + \beta} b_m \right) \Phi_{k,n,q_n}\left(\frac{x}{a_n}\right) \Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \end{aligned} \tag{2.2}$$

where $n \in \mathbb{N}$, $p \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$, $0 \le \alpha \le \beta$. $\Phi_{k,n,q_n}(z) = \begin{bmatrix} n+p \\ k \end{bmatrix}_{q_n} z^k \prod_{s=0}^{n+p-k-1} (1 - q_n^s z)$.
We also let $0 < q_n < 1$ $(n \in \mathbb{N})$ for the positivity of the operators. It is easy to show that $C_{n,p}^{(\alpha,\beta)}(f; q_n, q_m; x, y)$ is a linear and positive operator.
Now, we start by giving the following lemma which will be used throughout the paper.

Lemma 2.1. *Let $C_{n,m,p}^{(\alpha,\beta)}(f; q_n, q_m; x, y)$ be given in (2.2). Then the first few moments of the operators are,*

(i) $C_{n,m,p}^{(\alpha,\beta)}(1; q_n, q_m; x, y) = 1$,

(ii) $C_{n,m,p}^{(\alpha,\beta)}(t_1; q_n, q_m; x, y) = \dfrac{[n+p]_{q_n} x + \alpha a_n}{[n]_{q_n} + \beta}$,

(iii) $C_{n,m,p}^{(\alpha,\beta)}\left(t_2;q_n,q_m;x,y\right)=\dfrac{[m+p]_{q_m}y+\alpha b_m}{[m]_{q_m}+\beta}$

(iv) $C_{n,m,p}^{(\alpha,\beta)}\left(t_1^2+t_2^2;q_n,q_m;x,y\right)$

$$=\frac{1}{\left([n]_{q_n}+\beta\right)^2}\left\{[n+p-1]_{q_n}[n+p]_{q_n}q_nx^2+(2\alpha+1)[n+p]_{q_n}a_nx+\alpha^2a_n^2\right\}$$

$$+\frac{1}{\left([m]_{q_m}+\beta\right)^2}\left\{[m+p-1]_{q_m}[m+p]_{q_m}q_my^2+(2\alpha+1)[m+p]_{q_m}b_my+\alpha^2b_m^2\right\}.$$

Proof. Using Lemma 1.1 and the linearity of the operators, the proof is easily obtained. □

3. Korovkin-type approximation theorems

In this section, Korovkin-type approximation theorems are given for the two dimensional Chlodowsky variant of q-Bernstein-Schurer-Stancu operators. For fixed $\nu\geq 0$ consider the space C_{ρ^ν} which consists of all continuous functions f, satisfying the condition

$$|f(x,y)|\leq M_f\rho^\nu(x,y),\quad (x,y)\in[0,\infty)\times[0,\infty):=\mathbb{R}_+^2 \text{ and } \rho(x,y)=1+x^2+y^2.$$

Clearly, C_{ρ^ν} is a linear normed space with the following norm

$$\|f\|_{\rho^\nu}=\sup_{0\leq x,y<\infty}\frac{|f(x,y)|}{\rho^\nu(x,y)}.$$

The following theorem will be used in the investigation of approximation properties of $C_{n,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)$ in the weighted spaces.

Theorem 3.1. *Let the numbers A and B be any fixed positive real numbers. Let $D_{A,B}=\{(x,y):0\leq x\leq A,\ 0\leq y\leq B\}$, $q:=\{q_n\}$ with $0<q_n<1$, $\lim_{n\to\infty}q_n=1$ and $\{a_n\}$ and $\{b_m\}$ be increasing sequences of positive real numbers that satisfy the following properties:*

$$\lim_{n\to\infty}a_n=\lim_{m\to\infty}b_m=\infty \text{ and } \lim_{n\to\infty}\frac{a_n}{[n]_{q_n}}=\lim_{m\to\infty}\frac{b_m}{[m]_{q_m}}=0.$$

For all $f\in C(D_{A,B})$, we have

$$\lim_{n,m\to\infty}\max_{(x,y)\in D_{A,B}}\left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right|=0.$$

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Proof. Using Lemma 2.1, we get

$$\left\| C_{n,m,p}^{(\alpha,\beta)}(1;q_n,q_m;\cdot,\cdot)-1\right\|_{C(D_{A,B})}=0$$

$$\left\| C_{n,m,p}^{(\alpha,\beta)}(t_1;q_n,q_m;\cdot,\cdot)-x\right\|_{C(D_{A,B})}\leq A\left|\frac{[n+p]_{q_n}}{[n]_{q_n}+\beta}-1\right|+\frac{\alpha a_n}{[n]_{q_n}+\beta}$$

$$\left\| C_{n,m,p}^{(\alpha,\beta)}(t_2;q_n,q_m;\cdot,\cdot)-y\right\|_{C(D_{A,B})}\leq B\left|\frac{[m+p]_{q_m}}{[m]_{q_m}+\beta}-1\right|+\frac{\alpha b_m}{[m]_{q_m}+\beta}.$$

And again using Lemma 2.1 we have

$$C_{n,m,p}^{(\alpha,\beta)}\left(t_1^2+t_2^2;q_n,q_m;\cdot,\cdot\right)-\left(x^2+y^2\right)=\frac{1}{\left([n]_{q_n}+\beta\right)^2}$$
$$\times\left\{\left([n+p+1]_{q_n}[n+p]_{q_n}q_n-\left([n]_{q_n}+\beta\right)^2\right)x^2+(2\alpha+1)[n+p]_{q_n}a_nx+\alpha^2a_n^2\right\}$$
$$+\frac{1}{\left([m]_{q_m}+\beta\right)^2}$$
$$\times\left\{\left([m+p+1]_{q_m}[m+p]_{q_m}q_m-\left([m]_{q_m}+\beta\right)^2\right)y^2+(2\alpha+1)[m+p]_{q_m}b_my+\alpha^2b_m^2\right\}.$$

Finally, from the above equality we obtain

$$\left\| C_{n,m,p}^{(\alpha,\beta)}\left(t_1^2+t_2^2;q_n,q_m;\cdot,\cdot\right)-\left(x^2+y^2\right)\right\|_{C(D_{A,B})}$$
$$\leq\frac{1}{\left([n]_{q_n}+\beta\right)^2}$$
$$\times\left\{\left|[n+p+1]_{q_n}[n+p]_{q_n}q_n-\left([n]_{q_n}+\beta\right)^2\right|A^2+(2\alpha+1)[n+p]_{q_n}a_nA+\alpha^2a_n^2\right\}$$
$$+\frac{1}{\left([m]_{q_m}+\beta\right)^2}$$
$$\times\left\{\left|[m+p+1]_{q_m}[m+p]_{q_m}q_m-\left([m]_{q_m}+\beta\right)^2\right|B^2+(2\alpha+1)[m+p]_{q_m}b_mB+\alpha^2b_m^2\right\}.$$

Therefore, from the hypothesis of the theorem, we have

$$\begin{aligned}\left\| C_{n,m,p}^{(\alpha,\beta)}(t_1;q_n,q_m;\cdot,\cdot)-x\right\|_{C(D_{A,B})}&\to 0\\ \left\| C_{n,m,p}^{(\alpha,\beta)}(t_2;q_n,q_m;\cdot,\cdot)-y\right\|_{C(D_{A,B})}&\to 0\\ \left\| C_{n,m,p}^{(\alpha,\beta)}\left(t_1^2+t_2^2;q_n,q_m;\cdot,\cdot\right)-\left(x^2+y^2\right)\right\|_{C(D_{A,B})}&\to 0\end{aligned}$$

when n and $m\to\infty$.

Hence, the proof is completed by the two dimensional Korovkin theorem. □

In studying Korovkin-type weighted approximation, the following theorem plays an important role.

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Theorem 3.2. *(See [6]) There exists a sequence of positive operators $T_{n,m}$, acting from $C_\rho\left(\mathbb{R}_+^2\right)$ to $C_\rho\left(\mathbb{R}_+^2\right)$, satisfying the conditions*

$$\lim_{n,m\to\infty}\left\|T_{n,m}\left(1;\cdot,\cdot\right)-1\right\|_\rho=0$$
$$\lim_{n,m\to\infty}\left\|T_{n,m}\left(t_1;\cdot,\cdot\right)-x\right\|_\rho=0$$
$$\lim_{n,m\to\infty}\left\|T_{n,m}\left(t_2;\cdot,\cdot\right)-y\right\|_\rho=0$$
$$\lim_{n,m\to\infty}\left\|T_{n,m}\left(t_1^2+t_2^2;\cdot,\cdot\right)-\left(x^2+y^2\right)\right\|_\rho=0$$

and there exists a function $f^*\in C_\rho$ for which

$$\lim_{n,m\to\infty}\left\|T_{n,m}f^*-f^*\right\|_\rho\geq\frac{1}{4}$$

where $\rho=1+x^2+y^2$.

Now, consider the following operator

$$T_{n,m}\left(f;q_n,q_m;x,y\right)=\begin{cases} C_{n,m,p}^{(\alpha,\beta)}\left(f;q_n,q_m;x,y\right), & (x,y)\in D_{a_n,b_n} \\ f\left(x,y\right), & \mathbb{R}_+^2\backslash D_{a_n,b_n}\end{cases}.$$

Theorem 3.3. *Let $f\in C_\rho\left(\mathbb{R}_+^2\right)$. Then for any $\gamma>0$*

$$\lim_{n,m\to\infty}\left\|T_{n,m}\left(f;q_n,q_m;\cdot,\cdot\right)-f\left(\cdot\right)\right\|_{C_{\rho^{1+\gamma}}}=0$$

where $\{a_n\}$, $\{b_m\}$, $\{q_n\}$ and $\{q_m\}$have the same conditions as in Theorem 3.1.

Proof. For all $\varepsilon>0$, there exist sufficiently large positive real numbers A and B such that

$$\left(1+x^2+y^2\right)^{-\gamma}<\varepsilon \tag{3.1}$$

when $x>A$ and $y>B$.
Let n, m be sufficiently large so that $D_{A,B}\subset D_{a_n,b_m}$

$$\begin{aligned}&\left\|T_{n,m}\left(f;q_n,q_m;\cdot,\cdot\right)-f\left(\cdot\right)\right\|_{C_{\rho^{1+\gamma}}}\\ &\leq\sup_{(x,y)\in D_{A,B}}\frac{\left|C_{n,m,p}^{(\alpha,\beta)}\left(f;q_n,q_m;x,y\right)-f(x,y)\right|}{\left(1+x^2+y^2\right)^{1+\gamma}}\\ &+\sup_{(x,y)\in D_{a_n,b_n}\backslash D_{A,B}}\frac{\left|C_{n,m,p}^{(\alpha,\beta)}\left(f;q_n,q_m;x,y\right)-f(x,y)\right|}{\left(1+x^2+y^2\right)^{1+\gamma}}\\ &=y_{n,m}^{'}+y_{n,m}^{''}.\end{aligned}$$

By Theorem 3.1, $\lim_{n,m\to\infty}y_{n,m}^{'}=0$ and for the proof of the second term we have

$$y_{n,m}^{''}\leq\left(1+x^2+y^2\right)^{-\gamma}\left(\frac{\left|C_{n,m,p}^{(\alpha,\beta)}\left(f;q_n,q_m;x,y\right)\right|}{1+x^2+y^2}+\frac{|f(x,y)|}{1+x^2+y^2}\right).$$

Finally, since $f\in C_\rho\left(\mathbb{R}_+^2\right)$, the term $\frac{|f(x,y)|}{1+x^2+y^2}$ is bounded. Furthermore, because of the fact that

$$\left|C_{n,m,p}^{(\alpha,\beta)}\left(f;q_n,q_m;x,y\right)\right|\leq\left|C_{n,m,p}^{(\alpha,\beta)}\left(1+t_1^2+t_2^2;q_n,q_m;x,y\right)\right|,$$

using Lemma 2.1, the term $\frac{\left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)\right|}{1+x^2+y^2}$ is bounded for sufficiently large n and m. Hence, we get by (3.1) that

$$y''_{n,m} \leq \varepsilon(1+M)$$

Since $\varepsilon > 0$ is arbitrary, then $\lim\limits_{n,m\to\infty} y''_{n,m} = 0$. This completes the proof. □

Now, consider the subspace C_ρ^0 of C_ρ which is defined by

$$C_\rho^0 := \left\{ f \in C_\rho : \lim_{x,y\to 0} \frac{|f(x,y)|}{1+x^2+y^2} = 0 \right\}.$$

Theorem 3.4. *Let the sequences $\{q_n\}$, $\{a_n\}$ and $\{b_m\}$ satisfy the same properties as in Theorem 3.1. Then for all $f \in C_\rho^0\left(\mathbb{R}_+^2\right)$, we obtain*

$$\lim_{n,m\to\infty} \left\| T_{n,m}\left(f;q_n,q_m;\cdot,\cdot\right) - f\left(\cdot\right) \right\|_{C_\rho} = 0.$$

Proof. For all $f \in C_\rho^0\left(\mathbb{R}_+^2\right)$, observe that

$$\lim_{x,y\to\infty} \frac{|f(x,y)|}{1+x^2+y^2} = 0, \qquad \lim_{n,m\to\infty} \frac{\left| f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, \frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m \right) \right|}{1+\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n\right)^2+\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right)^2} = 0.$$

Therefore, for all $\varepsilon > 0$, we can find sufficiently large numbers A and B such that

$$|f(x,y)| < \varepsilon\left(1+x^2+y^2\right) \tag{3.2}$$

for $x > A$ and $y > B$ and there exists natural numbers n_0 and m_0 such that

$$\left| f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, \frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m \right) \right| < \varepsilon\left(1+\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n\right)^2+\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right)^2 \right) \tag{3.3}$$

for all $n > n_0$ and $m > m_0$.
Hence, for large n and m, we have

$$\begin{aligned}
&\left\| T_{n,m}\left(f;q_n,q_m;\cdot,\cdot\right) - f\left(\cdot\right) \right\|_{C_\rho} \\
&\leq \sup_{(x,y)\in D_{A,B}} \frac{\left|C_{n,m}^{\alpha,\beta}\left(f;q_n,q_m;x,y\right) - f(x,y)\right|}{1+x^2+y^2} \\
&+ \sup_{(x,y)\in D_{a_n,b_m}\setminus D_{A,B}} \frac{\left|C_{n,m}^{\alpha,\beta}\left(f;q_n,q_m;x,y\right) - f(x,y)\right|}{1+x^2+y^2} = z'_{n,m} + z''_{n,m}.
\end{aligned}$$

By Theorem 3.1 it is sufficient to show that $z''_{n,m} \to 0$ as $n \to \infty$.
Using (3.2) and (3.3), we get

$$\begin{aligned}
z''_{n,m} &\leq \varepsilon + \sup_{(x,y)\in D_{a_n,b_m}\setminus D_{A,B}} \frac{\left|C_{n,m}^{\alpha,\beta}\left(f;q_n,q_m;x,y)\right)\right|}{1+x^2+y^2} \\
&\leq \varepsilon + \varepsilon \sup_{(x,y)\in D_{a_n,b_m}\setminus D_{A,B}} t_{n,m}(q_n,q_m;x;y) \\
&= \varepsilon\left(1+\sup_{(x,y)\in D_{a_n,b_m}/D_{A,B}} t_{n,m}(q_n,q_m;x;y)\right)
\end{aligned}$$

where $t_{n,m}(q_n,q_m;x;y) := \frac{C_{n,m}^{\alpha,\beta}(1;q_n,q_m;x,y))+C_{n,m}^{\alpha,\beta}(t_1^2;q_n,q_m;x,y))+C_{n,m}^{\alpha,\beta}(t_2^2;q_n,q_m;x,y))}{1+x^2+y^2}$.
By Lemma 2.1, it is clear that there exist K independent of n and m such that

$$\sup_{(x,y)\in D_{a_n,b_m}/D_{A,B}} t_{n,m}(q_n,q_m;x;y) \leq K.$$

Therefore, for $n > n_0$ and $m > m_0$ we have

$$z''_{n,m} < (1+K)\varepsilon.$$

This completes the proof. □

4. Order of convergence

In this section, we compute the rate of convergence of the operators in terms of the the full modulus of continuity and partial modulus of continuities.

Let $f \in D_{A,B}$ and $x \geq 0$. Then the definition of the modulus of continuity of f is given by

$$\omega(f;\delta) = \max_{\substack{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\leq\delta \\ x,y\in C(D_{A,B})}} |f(x_1,y_1)-f(x_2,y_2)|. \tag{4.1}$$

It is known that for any $\delta > 0$ we know that

$$|f(x_1,y_1)-f(x_2,y_2)| \leq \omega(f,\delta)\left(\frac{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}}{\delta}+1\right) \tag{4.2}$$

and its partial modulus of continuies are defined by

$$\begin{aligned}
\omega^{(1)}(f;\delta) &= \max_{0\leq y\leq A}\max_{|x_1-x_2|\leq\delta} |f(x_1,y)-f(x_2,y)| \\
\omega^{(2)}(f;\delta) &= \max_{0\leq x\leq B}\max_{|y_1-y_2|\leq\delta} |f(x,y_1)-f(x,y_2)|.
\end{aligned}$$

Also, for any $\delta > 0$ we have

$$\begin{aligned}
|f(x_1,y_1)-f(x_2,y_2)| &\leq \omega^{(1)}(f,\delta)\left(\frac{|x_1-x_2|}{\delta}+1\right), \\
|f(x_1,y_1)-f(x_2,y_2)| &\leq \omega^{(2)}(f,\delta)\left(\frac{|y_1-y_2|}{\delta}+1\right).
\end{aligned}$$

Theorem 4.1. *For any $f \in C(D_{A,B})$, the following inequalities*

$$\left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right| \leq 2\left[\omega^{(1)}(f;\delta_m)+\omega^{(2)}(f;\delta_n)\right] \tag{4.3}$$

$$\left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right| \leq 2\omega\left(f;\sqrt{\delta_m^2+\delta_n^2}\right) \tag{4.4}$$

are satisfied where

(4.5)

$$\delta_n^2 := \frac{1}{\left([n]_{q_n}+\beta\right)^2} \times\left\{\left|[n+p+1]_{q_n}[n+p]_{q_n}q_n-\left([n]_{q_n}+\beta\right)^2\right|A^2+(2\alpha+1)[n+p]_{q_n}a_nA+\alpha^2a_n^2\right\}$$

and

$$\delta_m^2 := \frac{1}{\left([m]_{q_m}+\beta\right)^2} \times \left\{\left|[m+p+1]_{q_m}[m+p]_{q_m}q_m - \left([m]_{q_m}+\beta\right)^2\right| B^2 + (2\alpha+1)[m+p]_{q_m} b_m B + \alpha^2 b_m^2\right\}. \tag{4.6}$$

Proof. We directly have,

$$\begin{aligned}
&C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y) - f(x,y) \\
&= \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left[f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, \frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right) - f(x,y)\right] \\
&\times \Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \\
&= \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left[f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, \frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right) - f(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, y)\right. \\
&\left. + f(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, y) - f(x,y)\right]\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right).
\end{aligned}$$

By linearity and positivity of the operators, we get

$$\begin{aligned}
&\left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y) - f(x,y)\right| \\
&\leq \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, \frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right) - f(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, y)\right| \\
&\times \Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \\
&+ \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|f(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n, y) - f(x,y)\right|\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\omega^{(2)}\left(f;\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m - y\right|\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \\
&+ \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\omega^{(1)}\left(f;\left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n - x\right|\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \\
&= \Omega_1(x,y) + \Omega_2(x,y).
\end{aligned}$$

Using Lemma 1.1 and Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&\Omega_1(x,y)\\
=\;& \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\omega^{(2)}\left(f;\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
=\;& \sum_{j=0}^{m+p}\omega^{(2)}\left(f;\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
\leq\;& \omega^{(2)}(f;\delta_m)\left\{1+\frac{1}{\delta_m}\left[\sum_{j=0}^{m+p}\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)^2\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\right]^{1/2}\right\}.
\end{aligned}$$

Finally, using Lemma 2.1, we get

$$\Omega_1(x,y)\leq 2\omega^{(2)}(f;\delta_m) \tag{4.7}$$

where we choose δ_m as in (4.6).
In the same way, we obtain

$$\Omega_2(x,y)\leq 2\omega^{(1)}(f;\delta_n) \tag{4.8}$$

where δ_n is given in (4.5). Combining (4.7) and (4.8), we obtain (4.3) .
Now, by using linearity and the monotonicity of the operators, and taking into account (4.1), we have

$$\begin{aligned}
&\left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right|\\
&\leq \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\omega\left(f;\sqrt{\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)^2+\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)^2}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&\leq \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right)-f(x,y)\right|\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&\leq 1+\frac{1}{\delta}\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\omega\left(f;\sqrt{\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)^2+\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)^2}\right)
\end{aligned}$$

$$\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \tag{4.9}$$

Using (4.2) and the Cauchy-Schwartz inequality, we get (4.4). □

Theorem 4.2. *Let $f(x,y)$ have continuous partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$, let $\omega^1(f_x;.)$ and $\omega^2(f_y;.)$ denote the partial moduli of $\partial f/\partial x$ and $\partial f/\partial y$, respectively* *on $D_{A,B}$. Then the inequality*

$$\begin{aligned}
&\left|C_{n,m,p}^{(\alpha,\beta)}\left(f;q_n,q_m;x,y\right)-f(x,y)\right| \\
&\leq N\left(\left|\frac{[n+p]_{q_n}}{[n]_{q_n}+\beta}-1\right|A+\frac{\alpha a_n}{[n]_{q_n}+\beta}\right)+2\left[\delta_n\omega^{(1)}\left(\frac{\partial f}{\partial x};\delta_n\right)\right] \\
&+M\left(\left|\frac{[m+p]_{q_m}}{[m]_{q_m}+\beta}-1\right|B+\frac{\alpha b_m}{[m]_{q_n}+\beta}\right)+2\left[\delta_m\omega^{(2)}\left(\frac{\partial f}{\partial y};\delta_m\right)\right].
\end{aligned}$$

where δ_n and δ_m are the same as in Theorem 4.1 and $\left|\frac{\partial f}{\partial x}\right|\leq N$, $\left|\frac{\partial f}{\partial y}\right|\leq M$ on $D_{A,B}$.

Proof. By the mean value theorem, we can write

$$\begin{aligned}
&f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right)-f(x,y) \\
&=f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,y\right)-f(x,y)+f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,\frac{[j]+\alpha}{[m]+\beta}b_m\right) \\
&-f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,y\right)
\end{aligned}$$

$$\begin{aligned}
&=\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)\frac{\partial f(x,y)}{\partial x}+\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)\left[\frac{\partial f(\psi_1,y)}{\partial x}-\frac{\partial f(x,y)}{\partial x}\right] \\
&+\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)\frac{\partial f(x,y)}{\partial y}+\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right) \\
&\times\left[\frac{\partial f(x,\psi_2)}{\partial y}-\frac{\partial f(x,y)}{\partial y}\right]
\end{aligned} \tag{4.10}$$

for any fixed $y\in[0,B]$ and $x\in[0,A]$, where

$$x<\psi_1<\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n$$

and

$$y<\psi_2<\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m.$$

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Applying the operator $C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)$ to (4.10)

$$\begin{aligned}
&C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\\
&=\frac{\partial f(x,y)}{\partial x}\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right)\left[\frac{\partial f(\psi_1,y)}{\partial x}-\frac{\partial f(x,y)}{\partial x}\right]\\
&\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&+\frac{\partial f(x,y)}{\partial y}\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)\left[\frac{\partial f(x,\psi_2)}{\partial y}-\frac{\partial f(x,y)}{\partial y}\right]\\
&\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right).
\end{aligned}$$

Hence, taking $\left|\frac{\partial f}{\partial x}\right|\leq N$ and $\left|\frac{\partial f}{\partial y}\right|\leq M$, we get

$$\begin{aligned}
&\left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right|\\
&\leq\left|\frac{\partial f(x,y)}{\partial x}\right|\left|C_{n,m,p}^{(\alpha,\beta)}(t_1-x;q_n,q_m;x,y)\right|\\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n-x\right|\left|\frac{\partial f(\psi_1,y)}{\partial x}-\frac{\partial f(x,y)}{\partial x}\right|
\end{aligned}$$

$$\begin{aligned}
&\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&+\left|\frac{\partial f(x,y)}{\partial y}\right|\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\\
&+\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right|\left|\frac{\partial f(x,\psi_2)}{\partial y}-\frac{\partial f(x,y)}{\partial y}\right|\\
&\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq N \left| C_{n,m,p}^{(\alpha,\beta)}\left(t_1 - x; q_n, q_m; x, y\right)\right| \\
&+ \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n - x\right|\left|\frac{\partial f(\psi_1, y)}{\partial x} - \frac{\partial f(x,y)}{\partial x}\right| \\
&\times \Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \\
&+ M \left| C_{n,m,p}^{(\alpha,\beta)}\left(t_2 - x; q_n, q_m; x, y\right)\right| \\
&+ \sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m - y\right|\left|\frac{\partial f(x, \psi_2)}{\partial y} - \frac{\partial f(x,y)}{\partial y}\right| \\
&\times \Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right).
\end{aligned}
$$

Then using the properties of partial modulus of continuities, we have

$$
\begin{aligned}
&\left| C_{n,m,p}^{(\alpha,\beta)}\left(f; q_n, q_m; x, y\right) - f(x,y)\right| \\
&\leq N \left(\left|\frac{[n+p]_{q_n}}{[n]_{q_n}+\beta} - 1\right| A + \frac{\alpha a_n}{[n]_{q_n}+\beta}\right) \\
&+ \omega^{(1)}\left(f_x; \delta_n\right)\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n - x\right|\left(\frac{\left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n - x\right|}{\delta_n} + 1\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ M \left(\left|\frac{[m+p]_{q_m}}{[m]_{q_m}+\beta} - 1\right| B + \frac{\alpha b_m}{[m]_{q_n}+\beta}\right) \\
&+ \omega^{(2)}\left(f_y; \delta_m\right)\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m - y\right|\left(\frac{\left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m - y\right|}{\delta_n} + 1\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)
\end{aligned}
$$

since

$$
|\psi_1 - x| \leq \left|\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n - x\right|, \quad |\psi_2 - y| \leq \left|\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m - y\right|.
$$

Applying the Cauchy-Schwarz inequality we have

$$
\begin{aligned}
&\left| C_{n,m,p}^{(\alpha,\beta)}\left(f; q_n, q_m; x, y\right) - f(x,y)\right| \\
&\leq N \left(\left|\frac{[n+p]_{q_n}}{[n]_{q_n}+\beta} - 1\right| A + \frac{\alpha a_n}{[n]_{q_n}+\beta}\right) \\
&+ \omega^{(1)}\left(f_x; \delta_n\right)\left(\sum_{k=0}^{n+p}\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n - x\right)^2\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\right)^{1/2} \\
&+ \frac{\omega^{(1)}(f_x; \delta_n)}{\delta_n}\sum_{k=0}^{n+p}\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n - x\right)^2\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)
\end{aligned}
$$

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$$+M\left(\left|\frac{[m+p]_{q_m}}{[m]_{q_m}+\beta}-1\right|B+\frac{\alpha b_m}{[m]_{q_n}+\beta}\right)$$
$$+\omega^{(2)}(f_y;\delta_m)\left(\sum_{j=0}^{m+p}\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)^2\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)\right)^{1/2}$$
$$+\frac{\omega^{(2)}(f_y;\delta_m)}{\delta_m}\sum_{j=0}^{m+p}\left(\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m-y\right)^2\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right).$$

Therefore

$$\left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right|\leq N\left(\left|\frac{[n+p]_{q_n}}{[n]_{q_n}+\beta}-1\right|A+\frac{\alpha a_n}{[n]_{q_n}+\beta}\right)$$
$$+\omega^{(1)}(f_x;\delta_n)\left(\left(\sqrt{C_{n,m,p}^{(\alpha,\beta)}\left((t_1-x)^2;q_n,q_m;x,y\right)}\right)\right.$$
$$\left.+\frac{\omega^{(1)}(f_x;\delta_n)}{\delta_n}\left(C_{n,m,p}^{(\alpha,\beta)}\left((t_1-x)^2;q_n,q_m;x,y\right)\right)\right)$$
$$+M\left(\left|\frac{[m+p]_{q_m}}{[m]_{q_m}+\beta}-1\right|B+\frac{\alpha b_m}{[m]_{q_n}+\beta}\right)$$
$$+\omega^{(2)}(f_y;\delta_m)\sqrt{C_{n,m,p}^{(\alpha,\beta)}\left((t_2-y)^2;q_n,q_m;x,y\right)}$$
$$+\frac{\omega^{(2)}(f_y;\delta_m)}{\delta_m}C_{n,m,p}^{(\alpha,\beta)}\left(C_{n,m,p}^{(\alpha,\beta)}\left((t_2-y)^2;q_n,q_m;x,y\right)\right).$$

Now using Lemma 2.1 and choosing δ_n and δ_m as in (4.5) and (4.6), respectively, we get

$$\left|C_{n,m,p}^{(\alpha,\beta)}(f;q_n,q_m;x,y)-f(x,y)\right|$$
$$\leq N\left(\left|\frac{[n+p]_{q_n}}{[n]_{q_n}+\beta}-1\right|A+\frac{\alpha a_n}{[n]_{q_n}+\beta}\right)+2\left[\delta_n\omega^{(1)}\left(\frac{\partial f}{\partial x};\delta_n\right)\right]$$
$$+M\left(\left|\frac{[m+p]_{q_m}}{[m]_{q_m}+\beta}-1\right|B+\frac{\alpha b_m}{[m]_{q_n}+\beta}\right)+2\left[\delta_m\omega^{(2)}\left(\frac{\partial f}{\partial y};\delta_m\right)\right].$$

Whence the result. □

5. Generalization of the Two Dimensional of Chlodowsky Variant of q-Bernstein-Schurer-Stancu Operators

In this section, we introduce generalization of Chlodowsky variant of q-Bernstein-Schurer-Stancu operators. The generalized operators help us to approximate continuous functions defined on more general weighted spaces. Note that this kind of generalization was considered earlier for the Chlodowsky-Bernstein polynomials [5]. For $x\geq 0$, consider any continuous function $\omega(x,y)\geq 1$ and define

$$G_f(t,s)=f(t,s)\frac{1+t^2+s^2}{w(t,s)}.$$

Let us consider the generalization of the $C_{n,p}^{\alpha,\beta}(f;q_n,q_m;x,y)$ as follows
(5.1)

$$L_{n,p}^{\alpha,\beta}(f;q_n,q_m;x,y)=\begin{cases}\frac{w(x,y)}{1+x^2+y^2}\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}G_f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right) \\ \times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right), & (x,y)\in D_{a_n,b_n} \\ f(x,y), & \mathbb{R}_+^2\backslash D_{a_n,b_n}\end{cases}$$

where $(x,y)\in D_{a_nb_m}$ and $\{a_n\}$ and $\{b_m\}$ have the same properties of two dimensional of Chlodowsky variant of q-Bernstein-Schurer-Stancu operators.

Theorem 5.1. *For all continuous functions f satisfying $|f(x,y)|\leq M_f w(x,y)$, $x,y\geq 0$, and $\lim\limits_{x,y\to\infty}\frac{f(x,y)}{w(x,y)}=0$, we have*

$$\lim_{n,m\to\infty}\left\|L_{n,p}^{\alpha,\beta}(f;q_n,q_m;\cdot,\cdot)-f(\cdot,\cdot)\right\|_w=0$$

where $\rho(x,y)=1+x^2+y^2$.

Proof. Clearly,

$$\begin{aligned}&\left|L_{n,p}^{\alpha,\beta}(f;q_n,q_m;x,y)-f(x,y)\right| \\ =\ &\frac{w(x,y)}{1+x^2+y^2}\left|\sum_{k=0}^{n+p}\sum_{j=0}^{m+p}G_f\left(\frac{[k]_{q_n}+\alpha}{[n]_{q_n}+\beta}a_n,\frac{[j]_{q_m}+\alpha}{[m]_{q_m}+\beta}b_m\right)\right. \\ &\left.\times\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{k,n,q_n}\left(\frac{x}{a_n}\right)\Phi_{j,m,q_m}\left(\frac{y}{b_m}\right)-G_f(x,y)\right|,\end{aligned}$$

thus

$$\begin{aligned}&\left\|L_{n,p}^{\alpha,\beta}(f;q_n,q_m;\cdot,\cdot)-f(\cdot,\cdot)\right\|_w \\ =\ &\sup_{x,y\in\mathbb{R}_+^2}\frac{\left|L_{n,p}^{\alpha,\beta}(f;q_n,q_m;x,y)-f(x,y)\right|}{w(x,y)}=\sup_{x,y\in\mathbb{R}_+^2}\frac{\left|T_{n,p}(G_f;q_n,q_m;x,y)-G_f(x,y)\right|}{1+x^2+y^2}.\end{aligned}$$

Since $|f(x,y)|\leq M_f w(x,y)$, then $|G_f(x,y)|\leq M_f\rho(x,y)$ for $x,y\geq 0$ and $G_f(x,y)$ is continuous function on R_+^2. Furthermore, from $\lim\limits_{x,y\to\infty}\frac{f(x,y)}{w(x,y)}=0$, we have

$$\lim_{x,y\to\infty}\frac{G_f(x,y)}{\rho(x,y)}=0.$$

Thus, from Theorem 3.4 we get the result. □

Finally, note that, taking $w(x,y)=1+x^2+y^2$, then the operators $L_{n,p}^{\alpha,\beta}(f;q_n,q_m;x,y)$ reduces $T_{n,p}^{\alpha,\beta}(G_f;q_n,q_m;x,y)$.

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Global stability in stochastic difference equations for predator-prey models

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Abstract

There are many publications on theoretical analysis of deterministic difference equations and stochastic differential equations. However, relatively few theoretical papers are published to consider the positivity of solutions of discrete-time stochastic difference equations (DSDEs), and no theoretical papers investigate the global stability of nontrivial solutions of DSDEs with nonlinear terms. In this paper, we consider a DSDE model that is a generalization of two-dimensional nonlinear models of stochastic predator-prey interactions, and show the positivity and global stability of the nontrivial solutions by using our new discretized version of the Itô formula. In addition, our results are compared with those of continuous-time stochastic differential equations and discrete-time deterministic difference equations. Numerical simulations are introduced to support the results.

Key words: Discrete-time stochastic difference equations, Positivity, Global stability.

1. Introduction

Many predator-prey models have been studied to describe the dynamics of biological systems in which two species interact, one as a predator and the other as a prey. A classic predator-prey model is given by

$$\frac{dx}{dt} = x(r_1 - a_{11}x - a_{12}y), \ \frac{dy}{dt} = y(r_2 + a_{21}x - a_{22}y), \tag{1}$$

where $x(t)$ and $y(t)$ denote the population density of the prey and predator at time t, respectively. In the model (1), r_1 is the intrinsic growth rate of the prey in the absence of the predator, $-r_2$ is the death rate of the predator in the absence of the prey, the coefficients $a_{ij}(i \neq j)$ give the strength of the interaction between the two species, and $a_{ii}(i = 1, 2)$ measure the inhibiting effect of environment on the two species.

In the model (1), the predator consumes the prey with functional response of type $a_{12}x(t)y(t)$. However the rate of prey capture is saturated when the population of the prey is relatively large. Such phenomena are described by nonlinear functions including Holling types [1–5], Beddington-DeAngelis type [6–8], Crowley-Martin type [9–11], and

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Ivlev-type of functional responses [12–14]. Other types of nonlinear functions have been applied to express the Allee effect [15–19], which describes a positive relation between the population density and the per capita growth rate of a species. There have been also models to take into account of diffusion of species ([15] and [20–22]).

On the other hand, the population is inevitably affected by environmental noise in nature, so that the reproduction rates can change randomly. In order to be more realistic, stochastic models should be considered. Stochastic differential equation (SDE) models have been increasingly used in a range of application areas, including biology, chemistry, mechanics, economics, and finance. The SDE models have been studied to understand extinction, stochastic permanence and stationary distributions of the stochastic systems. In particular, many authors have taken stochastic perturbation into deterministic predator prey models with Beddington-DeAngelis and Holling types of functional responses [23–33]. For example, putting noise into the deterministic model (1) gives the SDE model

$$\begin{aligned} dx(t) &= x(t)\{r_1 - a_{11}x(t) - a_{12}y(t)\}dt + \sigma_1 x(t)dW_1(t), \\ dy(t) &= y(t)\{r_2 + a_{21}x(t) - a_{22}y(t)\}dt + \sigma_2 y(t)dW_2(t), \end{aligned} \tag{2}$$

which is a special model studied in [25] with zero-time delays. Here the positive coefficients σ_1 and σ_2 measure the intensity of environmental perturbations on the underlying growth rate of the prey and the death rate of the predator, respectively. The processes W_i are independent and real valued Wiener processes on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

In general, the exact solutions of SDEs are not known, so one has to numerically solve these SDEs. This leads us to consider and analyze discrete-time stochastic difference equations (DSDEs), which can be also viewed as stochastically perturbed versions of deterministic difference equations (DDEs) (see [34], [35] and references therein). There are many publications on estimations of the difference between solutions of SDEs and DSDEs. The global asymptotic stability of the trivial solution of DSDEs has been also widely addressed (see [36], [37], [38] and references therein). However, relatively few theoretical studies consider the positivity of solutions of DSDEs that are scalar equations on a finite time interval (see [39] references therein). In particular, to the best of our knowledge, there is no paper that theoretically deals with the global stability of nontrivial solutions of DSDEs. Therefore, to investigate the positivity and global stability, we consider the DSDE model for (2)

$$x^i_{k+1} = x^i_k \left\{1 + h\left(r_i + \sum\nolimits_{j=1}^{i-1} a_{ij}x^j_k - \sum\nolimits_{j=i}^{2} a_{ij}x^j_k\right) + h^{0.5}\sigma_i\xi^i_{k+1}\right\}, \tag{3}$$

where $1 \le i \le 2$, $k \ge 0$, $x^i_0 > 0$ and $0 < h < 1$. Although $r_1 > 0$, $r_2 < 0$ and $a_{ij} > 0$ in the SDE model (2) and the DDE model (3) with $\sigma_i = 0$ (see [34] and [35]), we weaken the conditions on the parameters and use the following conditions in the DSDE model (3): for $1 \le i, j \le 2$ and $i \ne j$

$$r_i \in \mathbb{R}, a_{ii} > 0, a_{ij} \ge 0, \sigma_i > 0. \tag{4}$$

The discrete Wiener processes $W_i(t_{k+1}) - W_i(t_k)$ are $h^{0.5}\xi^i_{k+1}$ with a mutually independent and identically distributed sequence $(\xi^1_k, \xi^2_k)^\infty_{k=1}$ of the standard normal random variables. The solutions of (3) are defined with respect to a complete, filtered probability space $(\Omega_h, \mathcal{F}_h, \{\mathcal{F}_k\}^\infty_{k=1}, \mathbb{P}_h)$, where $\{\mathcal{F}_k\}^\infty_{k=1}$ is the natural filtration generated by the stochastic sequence $(\xi^1_k, \xi^2_k)^\infty_{k=1}$, i.e., $\mathcal{F}_k = \sigma(\xi^1_1, \xi^2_1, \cdots, \xi^1_k, \xi^2_k)$ for $k \ge 1$. Therefore $(x^1_k, x^2_k)^\infty_{k=1}$ is adapted to the filtration for any initial vector (x_0^1, x_0^2), which is supposed to be non-random.

The positivity of solutions of the SDEs (2) is obtained in the infinite time interval $[0, \infty)$ without boundedness of the noises $W_i(t)$ by using the concept of explosion time (see [25] and [40]). However, to the best of our knowledge, there is no method for applying the concept of explosion time to DSDEs. Then for obtaining the positivity of solutions of the DSDE model (3) in the infinite time interval, we restrict the noises to bounded noises, which means that $\xi_k^i (1 \leq i \leq 2, k \geq 1)$ are assumed to be doubly truncated standard normal random variables with support $[-\varsigma, \varsigma]$ for a positive constant ς

$$-\varsigma \leq \xi_k^i \leq \varsigma \tag{5}$$

and the probability density function

$$\psi(x) = \begin{cases} q(x) \left\{ \Phi(\varsigma) - \Phi(-\varsigma) \right\}^{-1} & \text{if } x \in [-\varsigma, \varsigma], \\ 0 & \text{if } x \notin [-\varsigma, \varsigma], \end{cases} \tag{6}$$

where q and Φ are the probability density and cumulative distribution functions of the standard normal random variable, respectively. Denoting $\eta_\varsigma = 2\varsigma q(\varsigma) \left\{ \Phi(\varsigma) - \Phi(-\varsigma) \right\}^{-1}$ gives that for $1 \leq i \leq 2$ and $k \geq 1$

$$E(\xi_k^i) = 0, \ E\left((\xi_k^i)^2\right) = 1 - \eta_\varsigma, \tag{7}$$

in which the positive value η_ς can be assumed to be sufficiently close to 0. For example, when $\varsigma = 20$, we have $0 < \eta_\varsigma < 10^{-85}$. The truncation constant ς will be first used in (12) for the positivity of the solutions x_k^i of the DSDE model (3).

The paper is organized as follows. Section 2 gives the positivity and boundedness of solutions of the model (3). In Section 3, we develop a new discrete Itô formula for (3) by using a known discrete Itô formula for DSDEs (see [41], [42] and [43]). The new discrete Itô formula is the main tool for finding conditions for the global stability of solutions of (3). Section 4 introduces auxiliary equations, the solutions of which are used for the upper bounds of solutions of (3). In Section 5, we present sufficient conditions for extinction and non-extinction of solutions of (3). Our results are compared with those for the DDEs in [35] and the SDEs in [25]. Section 6 gives simulation results to confirm the theoretical analysis obtained in this paper.

2. Positivity and boundedness of solutions of DSDEs

In this section, we show the positivity and boundedness of solutions of the DSDE model (3) by applying the approach used in the DDE model (3) with $\sigma_1 = \sigma_2 = 0$ (see [34] and [35]).

Notation 1. For simplicity, we use the symbols $\tilde{a}$ and $\hat{a}$ for every constant a to denote

$$\tilde{a} = a \cdot h^{0.5}, \ \hat{a} = a \cdot h$$

and the symbols $\varkappa_k^1$ and $\varkappa_k^2$ for a vector $\varkappa_k = (x_k^1, x_k^2)$ to denote

$$\varkappa_k^1 = x_k^2, \ \varkappa_k^2 = x_k^1.$$

Write the model (3) as

$$x_{k+1}^i = F_{k,\varkappa_k^i}^i(x_k^i),$$

where

$$\begin{aligned} F_{k,y}^1(x) = &\ x\left(1+\hat{r}_1 - \hat{a}_{11}x - \hat{a}_{12}y + \tilde{\sigma}_1\xi_{k+1}^1\right), \\ F_{k,x}^2(y) = &\ y\left(1+\hat{r}_2 + \hat{a}_{21}x - \hat{a}_{22}y + \tilde{\sigma}_2\xi_{k+1}^2\right). \end{aligned} \tag{8}$$

Note that for a vector $\boldsymbol{\zeta}_k = (\zeta_k^1, \zeta_k^2)$ of real numbers ζ_k^1 and ζ_k^2,

$$F_{k,\zeta_k^i}^i(\tau) \text{ is strictly increasing on } 0 \leq \tau < V_k^i(\boldsymbol{\zeta}_k), \tag{9}$$

in which

$$V_k^i(\boldsymbol{\zeta}_k) = (2\hat{a}_{ii})^{-1}\left(1+\hat{r}_i + \sum\nolimits_{j=1}^{i-1}\hat{a}_{ij}\zeta_k^j - \sum\nolimits_{j=i+1}^{2}\hat{a}_{ij}\zeta_k^j + \tilde{\sigma}_i\xi_{k+1}^i\right). \tag{10}$$

Denote that for $1 \leq i \leq 2$

$$\chi_i = \hat{a}_{ii}^{-1}\left(\hat{r}_i + \sum\nolimits_{j=1}^{i-1}\hat{a}_{ij}\chi_j + \tilde{\sigma}_i\varsigma_*\right), \tag{11}$$

where ς_* is a constant satisfying

$$\varsigma_* > \varsigma, \tag{12}$$

$$\chi_i \leq (2\hat{a}_{ii})^{-1}\left(1+\hat{r}_i - \sum\nolimits_{j=i+1}^{2}\hat{a}_{ij}\chi_j - \tilde{\sigma}_i\varsigma_*\right), \tag{13}$$

$$\hat{r}_i + \sum\nolimits_{j=1}^{i-1}\hat{a}_{ij}\chi_j + \tilde{\sigma}_i\varsigma_* < 1. \tag{14}$$

The relation (12) will be first used in (69) to find upper solutions of the model (3). The initial condition of the model (3) is assumed to satisfy

$$(x_0^1, x_0^2) \in (0, \chi_1) \times (0, \chi_2). \tag{15}$$

Remark 1. The definition (11) gives that $\chi_1 = \frac{\hat{r}_1 + \tilde{\sigma}_1\varsigma_*}{\hat{a}_{11}}$ and $\chi_2 = \hat{a}_{22}^{-1}(\hat{r}_2 + \hat{a}_{21}\chi_1 + \tilde{\sigma}_2\varsigma_*)$. Letting h in (3) be small, we can choose ς_* satisfying the two conditions (13) and (14). For example, let $h = 0.0001$, $\varsigma_* = 20$, $r_1 = 2$, $r_2 = a_{ij} = 1$ and $\sigma_i = 0.1$ $(1 \leq i, j \leq 2)$. Denoting by R_i and L_i the right and left-hand sides of (13) and (14), respectively, gives

$$(\chi_1, R_1, L_1) = (202, 4699.5, 0.3848), (\chi_2, R_2, L_2) = (403, 4900.5, 0.3518),$$

which show that the conditions (13) and (14) are satisfied.

Theorem 1. *Let x_k^i be the solutions of (3) and χ_i be defined in (11). Assume that (5), (12), (13), (14) and (15) hold. Then*

$$(x_k^1, x_k^2) \in (0, \chi_1) \times (0, \chi_2), \ \ k \geq 0.$$

Proof. The proof is divided into the following three steps.
Step 1. We prove the positivity: $x_1^i > 0$ for $1 \leq i \leq 2$.
Note that for $\varkappa_0 = (x_0^1, x_0^2)$

$$0 < x_0^i < \chi_i \leq (2\hat{a}_{ii})^{-1}\left(1 + \hat{r}_i - \sum_{j=i+1}^{2} \hat{a}_{ij}\chi_j - \tilde{\sigma}_i\varsigma_*\right) < V_0^i(\varkappa_0),$$

where the first two inequalities are obtained from (15), the third from (13) and the last from (10), (15), (5) and (12). Then using (9) with $\boldsymbol{\zeta}_0 = \varkappa_0$ and (15), we have the positivity

$$x_1^i = F^i_{0,\varkappa_0^i}(x_0^i) > F^i_{0,\varkappa_0^i}(0) = 0.$$

Step 2. We prove the upper-bound property: $x_1^i < \chi_i$ for $1 \leq i \leq 2$.
Let $\omega \in \Omega_h$. If $\hat{r}_i + \sum_{j=1}^{i-1} \hat{a}_{ij}x_0^j - \sum_{j=i+1}^{2} \hat{a}_{ij}x_0^j + \tilde{\sigma}_i\xi_1^i(\omega) \leq 0$, then

$$x_1^i(\omega) = F^i_{0,\varkappa_0^i}(x_0^i)(\omega) \leq x_0^i < \chi_i.$$

Otherwise, we have $0 < x_0^i < f_{0,i}(\varkappa_0^i)(\omega)$ with

$$f_{0,i}(\varkappa_0^i) = \hat{a}_{ii}^{-1}\left(\hat{r}_i + \sum_{j=1}^{i-1} \hat{a}_{ij}x_0^j - \sum_{j=i+1}^{2} \hat{a}_{ij}x_0^j + \tilde{\sigma}_i\xi_1^i\right).$$

Since $0 < f_{0,i}(\varkappa_0^i) < V_0^i(\varkappa_0)$ by (14), we get

$$0 < x_0^i < f_{0,i}(\varkappa_0^i)(\omega) < V_0^i(\varkappa_0)(\omega)$$

and further

$$x_1^i(\omega) = F^i_{0,\varkappa_0^i}(x_0^i)(\omega) < F^i_{0,\varkappa_0^i}(f_{0,i}(\varkappa_0^i))(\omega) = f_{0,i}(\varkappa_0^i)(\omega) < \chi_i,$$

where the first inequality is obtained from (9) with $\boldsymbol{\zeta}_0 = \varkappa_0$ and the last inequality from (11) and (15).
Step 3. We prove the boundedness: $(x_k^1, x_k^2) \in (0, \chi_1) \times (0, \chi_2)$ for $k \geq 0$.
Since Step1 and 2 give that

$$\text{if } (x_0^1, x_0^2) \in (0, \chi_1) \times (0, \chi_2), \text{ then } (x_1^1, x_1^2) \in (0, \chi_1) \times (0, \chi_2),$$

we can obtain the desired result by both applying mathematical induction and replacing $\left(x_0, \xi_1^i, \varkappa_0, \boldsymbol{\zeta}_0, V_0^i, F^i_{0,\varkappa_0^i}, f_{0,i}\right)$ with $\left(x_k, \xi_{k+1}^i, \varkappa_k, \boldsymbol{\zeta}_k, V_k^i, F^i_{k,\varkappa_k^i}, f_{k,i}\right)$ in Step 1 and 2. Here the function $f_{k,i}$ is defined as $f_{k,i}(\varkappa_k^i) = \hat{a}_{ii}^{-1}\left(\hat{r}_i + \sum_{j=1}^{i-1} \hat{a}_{ij}x_k^j - \sum_{j=i+1}^{2} \hat{a}_{ij}x_k^j + \tilde{\sigma}_i\xi_{k+1}^i\right)$. □

Remark 2. For simplicity, from now on we assume that the conditions (5), (12), (13), (14) and (15) used in Theorem 1 hold. Then we will not write the conditions explicitly in later sections when we need the positivity and boundedness of the solutions x_k^i.

3. A new discretized version of the Itô formula

In order to find conditions for the stability of (3), we need a discretized form of the Itô formula. Although there are discretized versions of the Itô formula (see [41], [42] and [43]), we need to formulate a variant which is suitable for our model (3). The proof of our new discrete Itô formula is almost the same as that of the discrete Itô formula in [42] and [43]. For the completeness of this paper, we reproduce the proof in the Appendix.

We write $q_1(h) = O(q_2(h))$ (or $q_1(h) = O(q_2(h))$ for $h \to 0$ to be more precise) if there exist positive constants C and h_0 such that $|q_1(h)| \leq C|q_2(h)|$ for all h with $0 < h \leq h_0$.

We make the two assumptions about the noise ξ: First, the noise ξ satisfies that for some constants M_1 and μ with $0<\mu<1$

$$E(\xi)=0,\ E\left(\xi^2\right)=1-\mu,\ E\left(|\xi|^\ell\right)\le M_1\ (\ell=1,3). \tag{16}$$

Second, the probability density function p of the noise ξ exists with the property that for some constant M_2 and all sufficiently large $|x|$

$$|x|^3p(x)\le M_2|x|^{-1}. \tag{17}$$

Using $\mu=\eta_\varsigma$ in (7) and the probability density function $p(x)=\psi(x)$ in (6), one can obtain that the truncated standard normal random variables ξ_k^i satisfy the two assumptions (16) and (17). Let the symbol $\mathbb{R}$ denote the set of all real numbers and $C^3(\mathbb{R})$ denote the set of all functions defined on $\mathbb{R}$ that are continuously differentiable up to the order 3.

Lemma 1. *Let $\mathcal{G}$ be a sub σ-algebra of $\mathcal{F}_h$. Consider functions $\phi,\ \varphi:\mathbb{R}\to\mathbb{R}$ satisfying that for some $\delta>0$,*

(i) $\varphi=\phi$ *on* $[1-\delta,1+\delta]$

(ii) $\varphi\in C^3(\mathbb{R})$ *and* $|\varphi'''(x)|\le M_3$ *for some constant* M_3 *and all* $x\in\mathbb{R}$

(iii) $\int_{\mathbb{R}}|\varphi(x)-\phi(x)|dx<M_4$ *for some constant* M_4

and ϕ is almost everywhere continuous. Let f and g be $\mathcal{G}$-measurable random variables satisfying that for some positive constants ε and M_5,

$$\max\{h|f|,h^{0.5}|g|\}\le M_5h^\varepsilon. \tag{18}$$

Let ξ be a $\mathcal{G}$-independent random variable satisfying (16) and (17). Then the conditional expectation of the random variable $\phi\left(1+hf+h^{0.5}g\xi\right)$ with respect to the σ-algebra $\mathcal{G}$ becomes

$$\begin{aligned}&E\left[\phi\left(1+hf+h^{0.5}g\xi\right)|\mathcal{G}\right]\\&\quad=\phi(1)+\phi'(1)hf+2^{-1}\phi''(1)hg^2\cdot(1-\mu)+hfO\left(h^\varepsilon\right)+hg^2O\left(h^\varepsilon\right),\end{aligned}$$

where the first big O denotes

$$2^{-1}\varphi''(1)M_5h^\varepsilon+6^{-1}M_3\left(M_5h^\varepsilon\right)^2\{1+3(1-\mu)\}$$

and the last denotes

$$\left(M_1M_5+M_4M_2M_5\delta_1\right)h^\varepsilon$$

for some positive constant δ_1 less than δ. Here M_1 and M_2 are defined in (16) and (17).

Proof. See the Appendix. □

Remark 3. Differently from the discretized Itô formulas in [43], [41] and [42], our discretized Itô formula in Lemma 1 does not require that the upper bounds of f and g are independent of h. Let $\mathcal{G}=\mathcal{F}_k$ and

$$f=r_i+\sum\nolimits_{j=1}^{i-1}a_{ij}x_k^j-\sum\nolimits_{j=i}^{2}a_{ij}x_k^j,\ g=\sigma_i,\ \xi=\xi_{k+1}^i \tag{19}$$

for the solutions x_k^i of (3) with $1\le i\le 2$. Then f and g are $\mathcal{F}_k$-measurable and satisfy (18) with $\varepsilon=0.5$ by applying the upper bound $\chi_i=O(h^{-0.5})$ of x_k^i to the definition of f. In addition, $\xi=\xi_{k+1}^i$ is an $\mathcal{F}_k$-independent random variable satisfying (16) and (17).

Remark 4. In order to construct φ in Lemma 1 corresponding to the function

$$\phi(x) = \begin{cases} \ln|x| & (|x| > 0), \\ 0 & (x = 0), \end{cases}$$

we modify the function φ used in [37]. Define the function φ as follows.

$$\varphi(x) = \begin{cases} \ln|x| & (|x| \geq e^{-1}), \\ -4^{-1}e^4x^4 + e^2x^2 - 4^{-1}7 + 6^{-1}e^6(x - e^{-1})^3(x + e^{-1})^3 & (|x| \leq e^{-1}). \end{cases}$$

Then ϕ and φ satisfy all the conditions in Lemma 1 with $\delta = 1 - e^{-1}$.

Notation 2. For simplicity, we use the notations

$$\overline{E}(x_k^i) = k^{-1}\sum_{s=0}^{k-1} E\left(x_s^i\right) \tag{20}$$

and

$$\mathring{a} = a \cdot \left\{1 + O(h^{0.5})\right\}, \ a_\eta = a \cdot (1 - \eta_\varsigma), \ r_{i\sigma} = r_i - 0.5\sigma_{i\eta}^2$$

for $k > 0$, $1 \leq i \leq 2$, constants a and η_ς in (7). Here $\sigma_{i\eta}^2$ is equal to $\{\sigma_i \cdot (1 - \eta_\varsigma)\}^2$.

Remark 5. Since the solutions x_k^i of (3) are positive by Theorem 1, we can take logarithm of (3), which gives

$$E\left[\ln x_{k+1}^i \middle| \mathcal{F}_k\right] = E\left[\ln x_k^i \middle| \mathcal{F}_k\right] + E\left[\ln\left(1 + hf + h^{0.5}g\xi_{k+1}^i\right) \middle| \mathcal{F}_k\right], \tag{21}$$

where f and g are defined in (19). In order to simplify the equation (21), applying $\mathcal{F}_k$-independence of ξ_{k+1}, $\mathcal{F}_k$-measurability of x_k^i and Lemma 1 with Remarks 3 and 4 to the three expectation terms in (21), respectively, we have

$$\begin{aligned} E(\ln x_{k+1}^i) &= \ln x_k^i + hf - \frac{1}{2}hg^2 \cdot (1 - \eta_\varsigma) + hfO\left(h^{0.5}\right) + hg^2O\left(h^{0.5}\right) \\ &= \ln x_k^i + \mathring{h}\left(r_i - \frac{1}{2}\sigma_{i\eta}^2 + \sum_{j=1}^{i-1} a_{ij}x_k^j - \sum_{j=i}^{2} a_{ij}x_k^j\right). \end{aligned} \tag{22}$$

Taking expectation of (22) and adding the result, we obtain

$$E(\ln x_k^i) = E(\ln x_0^i) + k\mathring{h}\left\{r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\overline{E}(x_k^j) - \sum_{j=i}^{2} a_{ij}\overline{E}(x_k^j)\right\}. \tag{23}$$

4. Auxiliary equations

In order to find upper bounds of x_k^i, we consider the auxiliary equations

$$z_{k+1}^i = z_k^i\left(1 + \hat{r}_i + \sum_{j=1}^{i-1} \hat{a}_{ij}z_k^j - \hat{a}_{ii}z_k^i + \tilde{\sigma}_i\xi_{k+1}^i\right), \ z_0^i = x_0^i \tag{24}$$

for $1 \leq i \leq 2$ and $k \geq 0$. Since (24) is the system (3) with $a_{12} = 0$, Theorem 1 with (4) gives that for $k \geq 0$

$$(z_k^1, z_k^2) \in (0, \chi_1) \times (0, \chi_2). \tag{25}$$

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Let β_i be the solutions of the equations

$$r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\beta_j - a_{ii}\beta_i = 0 \tag{26}$$

for $1 \le i \le 2$. Note that (22) and (23) with $a_{12} = 0$ become

$$E\left(\ln z_{k+1}^1\right) = \ln z_k^1 + \mathring{h}\left(r_{1\sigma} - a_{11} z_k^1\right), \tag{27}$$

$$\begin{aligned} E\left(\ln z_k^1\right) &= E\left(\ln z_0^1\right) + k\mathring{h}\left\{r_{1\sigma} - a_{11}\overline{E}\left(z_k^1\right)\right\} \\ &= E\left(\ln z_0^1\right) + k\mathring{h}a_{11}\left\{\beta_1 - k^{-1}\sum_{s=0}^{k-1} E\left(z_s^1\right)\right\} \end{aligned} \tag{28}$$

due to (20) and $\beta_1 = a_{11}^{-1} r_{1\sigma}$ in (26). Similarly, we have

$$E\left(\ln z_{k+1}^2\right) = \ln z_k^2 + \mathring{h}\left(r_{2\sigma} + a_{21} z_k^1 - a_{22} z_k^2\right), \tag{29}$$

$$\begin{aligned} E\left(\ln z_k^2\right) &= E\left(\ln z_0^2\right) + k\mathring{h}\left\{r_{2\sigma} + a_{21}\overline{E}(z_k^1) - a_{22}\overline{E}(z_k^2)\right\} \\ &= E\left(\ln z_0^2\right) + k\mathring{h}a_{22}\left\{\frac{r_{2\sigma}}{a_{22}} + \frac{a_{21}}{a_{22}}\overline{E}(z_k^1) - k^{-1}\sum_{s=0}^{k-1} E\left(z_s^2\right)\right\}. \end{aligned} \tag{30}$$

Lemma 2. *Let z_k^1 and β_1 be the solutions of (24) and (26), respectively. If $\beta_1 \ge 0$, then for $\epsilon > 0$ and all sufficiently large k*

$$k^{-1}\sum_{s=0}^{k-1} E\left(z_s^1\right) \le \beta_1 + \epsilon.$$

Proof. Suppose, on the contrary, that the theorem is false, which means that there exist a constant $\varepsilon_0 > 0$ and an infinite increasing sequence $\{k_m\}$ satisfying both for all k_m

$$k_m^{-1}\sum_{s=0}^{k_m-1} E\left(z_s^1\right) > \beta_1 + \varepsilon_0 \tag{31}$$

and for all k with $k \ne k_m$

$$k^{-1}\sum_{s=0}^{k-1} E\left(z_s^1\right) \le \beta_1 + \varepsilon_0. \tag{32}$$

Combining (31) and (28), we have

$$\lim_{m\to\infty} E\left(\ln z_{k_m}^1\right) = -\infty. \tag{33}$$

Substituting (33) and the boundedness of z_k^1 into (27) gives

$$\lim_{m\to\infty} \ln z_{k_m-1}^1 = -\infty \quad a.s.$$

and then

$$\lim_{m\to\infty} z_{k_m-1}^1 = 0 \quad a.s. \tag{34}$$

Thus the dominated convergence theorem with (25) leads to

$$\lim_{m\to\infty} E(z_{k_m-1}^1) = 0. \tag{35}$$

In order to obtain a contraction we follow the two steps:

Step 1. If there exists $k = k_m - 1$ satisfying (32), then the system of (31) and (32) becomes

$$\begin{aligned} \sum\nolimits_{s=0}^{k_m-1} E\left(z_s^1\right) &> k_m\left(\beta_1+\varepsilon_0\right), \\ \sum\nolimits_{s=0}^{k_m-2} E\left(z_s^1\right) &\leq \left(k_m-1\right)\left(\beta_1+\varepsilon_0\right), \end{aligned}$$

which gives

$$E(z_{k_m-1}^1) > \beta_1 + \varepsilon_0, \tag{36}$$

and hence there exist finitely many k satisfying (32) due to (35) and (36). Therefore for all sufficiently large k

$$k^{-1}\sum\nolimits_{s=0}^{k-1} E\left(z_s^1\right) > \beta_1 + \varepsilon_0. \tag{37}$$

Step 2. As (31) implies (35), the equation (37) implies

$$\lim\nolimits_{k\to\infty} E(z_k^1) = 0,$$

which is contradictory to (37) due to $\beta_1 + \varepsilon_0 > 0$ and so the proof is completed. □

Lemma 3. *Let (z_k^1, z_k^2) and (β_1, β_2) be the solutions of (24) and (26), respectively.*

(a) *Assume $r_{1\sigma} < 0$. Then $\lim_{k\to\infty} z_k^1 = 0$ a.s.*
 (i) *If $r_{1\sigma} < 0$ and $r_{2\sigma} < 0$, then $\lim_{k\to\infty} z_k^2 = 0$ a.s.*
 (ii) *If $r_{1\sigma} < 0$ and $r_{2\sigma} \geq 0$, then $\lim_{k\to\infty} k^{-1}\sum_{s=0}^{k-1} E\left(z_s^2\right) = a_{22}^{-1} r_{2\sigma}$.*

(b) *Assume $r_{1\sigma} \geq 0$. Then $\lim_{k\to\infty} k^{-1}\sum_{s=0}^{k-1} E\left(z_s^1\right) = \beta_1$.*
 (i) *If $r_{1\sigma} \geq 0$ and $r_{2\sigma} + a_{21}\beta_1 < 0$, then $\lim_{k\to\infty} z_k^2 = 0$ a.s.*
 (ii) *If $r_{1\sigma} \geq 0$ and $r_{2\sigma} + a_{21}\beta_1 \geq 0$, then $\lim_{k\to\infty} k^{-1}\sum_{s=0}^{k-1} E\left(z_s^2\right) = \beta_2$.*

Proof. (a) Since $r_{1\sigma} < 0$ is equivalent to $\beta_1 = a_{11}^{-1} r_{1\sigma} < 0$, it follows from (28) and the positivity of z_k^1 in (25) that if $r_{1\sigma} < 0$, then $\lim_{k\to\infty} E\left(\ln z_k^1\right) = -\infty$, and further

$$\lim\nolimits_{k\to\infty} z_k^1 = 0 \ \ a.s. \tag{38}$$

as (33) implies (34).
(a)-(i) Assume that $r_{1\sigma} < 0$ and $r_{2\sigma} < 0$.
As (34) implies (35), the equation (38) yields $\lim_{m\to\infty} E(z_k^1) = 0$ and then

$$\lim\nolimits_{k\to\infty} \overline{E}(z_k^1) = 0. \tag{39}$$

Combining (39) and (30) with $r_{2\sigma} < 0$ and using $z_k^2 > 0$, we have from (30) that

$$\lim\nolimits_{k\to\infty} E\left(\ln z_k^2\right) = -\infty. \tag{40}$$

Therefore, as (33) implies (34), the equation (40) gives

$$\lim\nolimits_{k\to\infty} z_k^2 = 0 \ \ a.s.$$

(a)-(ii) Assume that $r_{1\sigma} < 0$ and $r_{2\sigma} \geq 0$.
Using $(z_k^2, a_{22}^{-1} r_{2\sigma})$, (29) and (30) instead of (z_k^1, β_1), (27) and (28) in the proof of Lemma 2, respectively, and applying (39) to (30), we can obtain that for $\epsilon > 0$ and all sufficiently large k

$$k^{-1}\sum\nolimits_{s=0}^{k-1} E\left(z_s^2\right) \leq a_{22}^{-1} r_{2\sigma} + \epsilon. \tag{41}$$

In order to show $\lim_{k\to\infty} k^{-1}\sum_{s=0}^{k-1} E\left(z_s^2\right) = a_{22}^{-1} r_{2\sigma}$, it is enough to prove that for $\epsilon > 0$ and all sufficiently large k

$$a_{22}^{-1} r_{2\sigma} - \epsilon \leq k^{-1}\sum_{s=0}^{k-1} E\left(z_s^2\right). \tag{42}$$

Suppose that (42) is false, which means that there exist a constant $\varepsilon_0 > 0$ and an infinite increasing sequence $\{k_m\}$ satisfying

$$a_{22}^{-1} r_{2\sigma} - \varepsilon_0 > k_m^{-1}\sum_{s=0}^{k_m-1} E\left(z_s^2\right). \tag{43}$$

Then the boundedness of z_k^2 and (30) imply that for all k_m

$$\infty > E\left(\ln z_{k_m}^2\right) > E\left(\ln z_0^2\right) + k_m \mathring{h} a_{22}\varepsilon_0, \tag{44}$$

which is a contradiction. Therefore (42) is true and so the proof is completed due to (41) and (42).
(b) Assume $r_{1\sigma} \geq 0$, which means $\beta_1 = a_{11}^{-1} r_{1\sigma} \geq 0$.
In order to show $\lim_{k\to\infty} k^{-1}\sum_{s=0}^{k-1} E\left(z_s^1\right) = \beta_1$, it is enough to prove that for $\epsilon > 0$ and all sufficiently large k

$$\beta_1 - \epsilon \leq k^{-1}\sum_{s=0}^{k-1} E\left(z_s^1\right) \tag{45}$$

due to Lemma 2. Suppose that (45) is false, so that there exist a constant $\varepsilon_0 > 0$ and an infinite increasing sequence $\{k_m\}$ such that

$$\beta_1 - \varepsilon_0 > k_m^{-1}\sum_{s=0}^{k_m-1} E\left(z_s^1\right). \tag{46}$$

Then the boundedness of z_k^1 and (28) imply that for all k_m

$$\infty > E\left(\ln z_{k_m}^1\right) > E\left(\ln z_0^1\right) + k_m \mathring{h} a_{11}\varepsilon_0, \tag{47}$$

which is a contradiction. Hence (45) is true and, therefore, Lemma 2 with (45) gives

$$\lim_{k\to\infty} \overline{E}(z_k^1) = \beta_1. \tag{48}$$

(b)-(i) Assume that $r_{1\sigma} \geq 0$ and $r_{2\sigma} + a_{21}\beta_1 < 0$.
Applying (48) to (30) with both $r_{2\sigma} + a_{21}\beta_1 < 0$ and $z_k^2 > 0$, we have

$$\lim_{k\to\infty} E(\ln z_k^2) = -\infty.$$

Therefore, as (33) implies (34), we can obtain $\lim_{k\to\infty} z_k^2 = 0 \ \ a.s.$
(b)-(ii) Assume that $r_{1\sigma} \geq 0$ and $r_{2\sigma} + a_{21}\beta_1 \geq 0$.
Following the proof of Lemma 2, we can obtain that

$$k^{-1}\sum_{s=0}^{k-1} E\left(z_s^2\right) \leq \beta_2 + \epsilon \tag{49}$$

for $\epsilon > 0$ and all sufficiently large k by using (z_k^2, β_2), (29) and (30) instead of (z_k^1, β_1), (27) and (28), respectively, and applying (48) and $\beta_2 = a_{22}^{-1}\left(r_{2\sigma} + a_{21}\beta_1\right) \geq 0$ to (30).
Similarly, following the proof of (45), we can obtain that

$$\beta_2 - \epsilon \leq k^{-1}\sum_{s=0}^{k-1} E\left(z_s^2\right) \tag{50}$$

for $\epsilon > 0$ and all sufficiently large k by replacing (z_k^1, β_1) and (28) with (z_k^2, β_2) and (30), respectively, and applying (48) to (30). Therefore (49) and (50) give the desired result. □

Remark 6. The equations (28) and (30) can be written as

$$E(\ln z_k^i) = E(\ln z_0^i) + k\mathring{h}\left\{r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\overline{E}(z_k^j) - a_{ii}\overline{E}(z_k^i)\right\}. \tag{51}$$

Substituting (26) to (51) yields

$$E(\ln z_k^i) = E(\ln z_0^i) + k\mathring{h}\left[\sum_{j=1}^{i-1} a_{ij}\left\{\overline{E}(z_k^j) - \beta_j\right\} - a_{ii}\left\{\overline{E}(z_k^i) - \beta_i\right\}\right]. \tag{52}$$

Applying Lemma 3-(b) and (b)-(ii) to (52) with the notation (20), we have

$$\lim_{k\to\infty} k^{-1}E(\ln z_k^i) = 0 \tag{53}$$

under the condition that $\min\{r_{1\sigma}, r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\beta_j\} \geq 0$ for $1 \leq i \leq 2$.

Lemma 4. *Let x_k^i and z_k^i be the solutions of (3) and (24), respectively for $i = 1, 2$. Then for $k \geq 0$*

$$0 < x_k^i \leq z_k^i.$$

Proof. Theorem 1 with Remark 2 gives

$$0 < x_k^i. \tag{54}$$

Note that

$$F_{k,y}^1(x) \text{ is nonincreasing in } y \text{ for } x \geq 0 \text{ and } k \geq 0 \tag{55}$$

and

$$F_{k,x}^2(y) \text{ is nondecreasing in } x \text{ for } y \geq 0 \text{ and } k \geq 0 \tag{56}$$

by the definition (8). The proof of this lemma is divided into the following two cases.
Case 1. Let $i = 1$.
Using $\not{x}_0^1 = x_0^2 > 0$ and (55), we have

$$x_1^1 = F_{0,\not{x}_0^1}^1(x_0^1) \leq F_{0,0}^1(x_0^1). \tag{57}$$

It follows from Remark 2, (24), (25), (10) and (13) that

$$0 < x_0^1 \leq z_0^1 < \chi_1 < V_0^1(0,0),$$

with which (9) yields

$$F_{0,0}^1(x_0^1) \leq F_{0,0}^1(z_0^1) = z_1^1. \tag{58}$$

Hence combining (54), (57) and (58) gives

$$0 < x_1^1 \leq z_1^1. \tag{59}$$

Assume that for some positive integer k

$$0 < x_k^1 \leq z_k^1. \tag{60}$$

Using (54), (60), (25), (10) and (13), we have

$$\not{x}_k^1 > 0,\ 0 < x_k^1 \leq z_k^1 < \chi_1 < V_k^1(0,0)$$

and so

$$x^1_{k+1} = F^1_{k,\varkappa^1_k}(x^1_k) \le F^1_{k,0}(x^1_k) \le F^1_{k,0}(z^1_k) = z^1_{k+1},$$

where the first inequality is obtained from (55) and the second inequality from (9).
Case 2. Let $i = 2$.
Using $\varkappa^2_0 = x^1_0 \le z^1_0$ and $0 < x^2_0 \le z^2_0 < \chi_2 < V^2_0(0,0)$, we have

$$x^2_1 = F^2_{0,\varkappa^2_0}(x^2_0) \le F^2_{0,z^1_0}(x^2_0) \le F^2_{0,z^1_0}(z^2_0) = z^2_1 \tag{61}$$

due to (56) and (9). Similarly as in Case 1, using mathematical induction and $z^2_k \le \chi_2 < V^2_k(0,0)$ instead of $z^1_k < \chi_1 < V^1_k(0,0)$ in Case 1, we can obtain the desired result. □

Remark 7. If $\min\{r_{1\sigma}, r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\beta_j\} \ge 0$ for $1 \le i \le 2$, then Lemma 4 and (53) imply that for $\epsilon > 0$ and all sufficiently large k

$$k^{-1}E(\ln x^i_k) \le \epsilon, \tag{62}$$

which will be first used in Theorem 4.

5. Extinction and persistence of the discrete solutions

In this section, we present several conditions sufficient for the extinction and persistence (non-extinction) of the solutions x^i_k of (3).

Theorem 2. *Let x^i_k and β_i be the solutions of (3) and (26), respectively for $i = 1, 2$.*

(a) *If $r_{1\sigma} < 0$, then $\lim_{k\to\infty} x^1_k = 0$ a.s.*
(b) *If $r_{1\sigma} < 0$ and $r_{2\sigma} < 0$, then $\lim_{k\to\infty} x^2_k = 0$ a.s.*

Proof. The proof is followed by combining Lemma 3-(a) and (a)-(i) with Lemma 4. □

Remark 8. Since $r_{1\sigma} = 0$ gives $\beta_1 = a^{-1}_{11} r_{1\sigma} = 0$, we obtain that

$$\text{if } r_{1\sigma} = 0, \text{ then } \lim_{k\to\infty} k^{-1} \textstyle\sum_{s=0}^{k-1} E\left(x^1_s\right) = 0$$

by combining Lemma 3-(b) with Lemma 4. Similarly, Lemma 3-(b)-(ii) gives

$$\text{if } r_{1\sigma} = r_{2\sigma} = 0, \text{ then } \lim_{k\to\infty} k^{-1} \textstyle\sum_{s=0}^{k-1} E\left(x^2_s\right) = 0$$

since $\beta_2 = a^{-1}_{22}\left(r_{2\sigma} + a_{21}\beta_1\right) = 0$.

Remark 9. By Theorem 2-(a), we find that if $r_1 < \frac{1}{2}\sigma^2_{1\eta}$, then the prey population will be extinct in the future, no matter whether the predator exists. It implies that environmental noise plays a very important role in the biological system.

In order to establish the sufficient condition for the extinction of the predator and the persistence of the prey, we will use the following Lemma 5 as well as Lemma 3-(b).
Using Lemmas 4 and 3-(b) with $\beta_1 = a^{-1}_{11} r_{1\sigma}$ we obtain that

$$\text{if } r_{1\sigma} > 0, \text{ then } \lim_{k\to\infty} k^{-1} \sum_{s=0}^{k-1} E\left(x^1_s\right) \le a^{-1}_{11} r_{1\sigma}. \tag{63}$$

For finding a lower function of x_k^1, we consider the solution $u_{k,\epsilon}$ of the equation

$$u_{k+1,\epsilon} = u_{k,\epsilon}(1 + \hat{r}_1 - \hat{a}_{11}u_{k,\epsilon} - \hat{a}_{12}\epsilon + \tilde{\sigma}_1\xi^1_{N_\epsilon+k+1}),\ u_{0,\epsilon} = x^1_{N_\epsilon}, \tag{64}$$

in which ϵ satisfies that for some positive integer N_ϵ and all $k \geq N_\epsilon$

$$0 < x_k^2 \leq \epsilon, \tag{65}$$

$$\hat{r}_1 - \hat{a}_{12}\epsilon + \tilde{\sigma}_1\varsigma_* < 1, \tag{66}$$

$$\hat{a}_{12}\epsilon + \tilde{\sigma}_1\varsigma < \tilde{\sigma}_1\varsigma_*, \tag{67}$$

where (65) is possible under the conditions $r_{1\sigma} > 0$ and $r_{2\sigma} + a_{21}\beta_1 < 0$ due to Lemmas 4 and 3-(b)-(i). The inequalities (66) and (67) are possible by (14) and (12), respectively.

Lemma 5. *Assume that $r_{1\sigma} > 0$ and $r_{2\sigma} + a_{21}\beta_1 < 0$. Let ϵ and N_ϵ satisfy (65)–(67). Let x_k^1 and $u_{k,\epsilon}$ be the solutions of (3) and (64), respectively. Then*

(a) $0 < u_{k,\epsilon} < \chi_1$ *for* $k \geq 0$.
(b) $u_{k,\epsilon} \leq x^1_{N_\epsilon+k}$ *for* $k \geq 0$.
(c) *If* $r_{1\sigma} - a_{12}\epsilon > 0$, *then* $\lim_{k\to\infty} k^{-1}\sum_{s=0}^{k-1} E\left(u_{s,\epsilon}\right) = a_{11}^{-1}\left(r_{1\sigma} - a_{12}\epsilon\right)$.

Proof. (a)We proceed by induction on k.
Since (64) and Theorem 1 with Remark 2 give

$$u_{0,\epsilon} = x^1_{N_\epsilon},\ 0 < x^1_{N_\epsilon} < \chi_1,$$

the statement (a) is true for $k = 0$.
Assume that for a nonnegative integer k

$$0 < u_{k,\epsilon} < \chi_1. \tag{68}$$

Now, in the case of $k+1$, the proof of (a) is divided into the following two steps.
Step 1. We prove the positivity of $u_{k+1,\epsilon}$.
Denoting

$$\mathcal{U}_k = (2\hat{a}_{11})^{-1}\left(1 + \hat{r}_1 - \hat{a}_{12}\epsilon + \tilde{\sigma}_1\xi^1_{N_\epsilon+k+1}\right)$$

gives that for $k \geq 0$

$$0 < \chi_1 < (2\hat{a}_{11})^{-1}\left(1 + \hat{r}_1 - \tilde{\sigma}_1\varsigma_*\right) < \mathcal{U}_k, \tag{69}$$

where the second inequality is obtained from (13) and the last from (67), (12) and (5).
Letting

$$G_k(x) = x\left(1 + \hat{r}_1 - \hat{a}_{11}x - \hat{a}_{12}\epsilon + \tilde{\sigma}_1\xi^1_{N_\epsilon+k+1}\right),$$

we have

$$G_k(x) \text{ is strictly increasing on } 0 \leq x < \mathcal{U}_k. \tag{70}$$

Applying (68) and (69) to (70), we have the desired positivity.
Step 2. We prove that χ_1 is an upper bound of $u_{k+1,\epsilon}$.
Let $\omega \in \Omega_h$. If $\hat{r}_1 - \hat{a}_{11}u_{k,\epsilon}(\omega) - \hat{a}_{12}\epsilon + \tilde{\sigma}_1\xi^1_{N_\epsilon+k+1}(\omega) \leq 0$, then

$$u_{k+1,\epsilon}(\omega) = G_k(u_{k,\epsilon})(\omega) \leq u_{k,\epsilon}(\omega) < \chi_1,$$

in which (68) gives the last inequality. Otherwise, we have $0 < u_{k,\epsilon}(\omega) < \Delta_k(\omega)$ with

$$\Delta_k = \hat{a}_{11}^{-1}\left(\hat{r}_1 - \hat{a}_{12}\epsilon + \tilde{\sigma}_1\xi^1_{N_\epsilon+k+1}\right).$$

Since $\Delta_k < \mathcal{U}_k$ by (66), we have $0 < u_{k,\epsilon}(\omega) < \Delta_k(\omega) < \mathcal{U}_k(\omega)$ and then (70) gives

$$u_{k+1,\epsilon}(\omega) = G_k(u_{k,\epsilon})(\omega) < G_k(\Delta_k)(\omega) = \Delta_k(\omega) < \chi_1,$$

where the last inequality is obtained from (11), (12) and (5).
(b)We proceed by induction on k.
The statement (b) is true for $k = 0$ due to (64).
Assume that for a nonnegative integer k

$$u_{k,\epsilon} \le x^1_{N_\epsilon+k}. \tag{71}$$

It follows from (a) in this theorem, (71), Theorem 1, Remark 2 and (69) that

$$0 < u_{k,\epsilon} \le x^1_{N_\epsilon+k} < \chi_1 < \mathcal{U}_k$$

and then

$$u_{k+1,\epsilon} = G_k(u_{k,\epsilon}) \le G_k(x^1_{N_\epsilon+k}) = F^1_{N_\epsilon+k,\epsilon}(x^1_{N_\epsilon+k}) \tag{72}$$

due to (70). Combining (55) and (65) also gives

$$F^1_{N_\epsilon+k,\epsilon}(x^1_{N_\epsilon+k}) \le F^1_{N_\epsilon+k,x^2_{N_\epsilon+k}}(x^1_{N_\epsilon+k}) = x^1_{N_\epsilon+k+1}. \tag{73}$$

Therefore, (72) and (73) give the desired result.
(c) Let $\gamma_1 = a_{11}^{-1}(r_{1\sigma} - a_{12}\epsilon)$. Note that

$$\begin{aligned} E(\ln u_{k+1,\epsilon}) &= \ln u_{k,\epsilon} + \mathring{h}\left(r_{1\sigma} - a_{11}u_{k,\epsilon} - a_{12}\epsilon\right), & (74)\\ E(\ln u_{k,\epsilon}) &= E(\ln u_{0,\epsilon}) + k\mathring{h}\left\{r_{1\sigma} - a_{12}\epsilon - a_{11}\overline{E}(u_{k,\epsilon})\right\} \\ &= E(\ln u_{0,\epsilon}) + k\mathring{h}a_{11}\left\{\gamma_1 - k^{-1}\sum_{s=0}^{k-1} E(u_{s,\epsilon})\right\} & (75)\end{aligned}$$

as in (27) and (28). Following the proof of Lemma 2, we can obtain that

$$k^{-1}\sum_{s=0}^{k-1} E(u_{s,\epsilon}) \le \gamma_1 + \epsilon' \tag{76}$$

for $\epsilon' > 0$ and all sufficiently large k by replacing (27), (28) and $(z^1_k, r_{1\sigma}, \beta_1)$ with (74), (75) and $(u_{k,\epsilon}, r_{1\sigma} - a_{12}\epsilon, \gamma_1)$, respectively.
Similarly, replacing (28) and (z^1_k, β_1) in (45)–(47) with (75) and $(u_{k,\epsilon}, \gamma_1)$, respectively, we can obtain that for $\epsilon' > 0$ and all sufficiently large k

$$\gamma_1 - \epsilon' \le k^{-1}\sum_{s=0}^{k-1} E(u_{s,\epsilon}),$$

with which (76) gives the desired result. □

Theorem 3. *Let x^i_k and β_1 be the solutions of (3) and (26), respectively for $i = 1, 2$.*

If $r_{1\sigma} \ge 0$ and $r_{2\sigma} + a_{21}\beta_1 < 0$, then $\lim_{k\to\infty} \overline{E}(x^1_k) = \beta_1$ and $\lim_{k\to\infty} x^2_k = 0$ a.s.

Proof. It follows from Lemma 3-(b)-(i), Lemma 4, Theorem 1 and Remark 2 that

$$\lim_{k\to\infty} x_k^2 = 0 \quad a.s.$$

Using Lemma 5-(a) and Lemma 4, we obtain that for $\epsilon > 0$ and all sufficiently large k

$$0 < u_{k,\epsilon} \le x^1_{N_\epsilon+k} \le z^1_{N_\epsilon+k}. \tag{77}$$

Lemma 5-(c) and Lemma 3-(b) give

$$\lim_{k\to\infty} \overline{E}\left(u_{k,\epsilon}\right) = a_{11}^{-1}\left(r_{1\sigma} - a_{12}\epsilon\right), \ \lim_{k\to\infty} \overline{E}\left(z_k^1\right) = a_{11}^{-1} r_{1\sigma}, \tag{78}$$

where the first and second equalities are valid under the conditions $r_{1\sigma} - a_{12}\epsilon > 0$ and $r_{1\sigma} \ge 0$, respectively. Therefore using (77), (78) and Remark 8, we obtain the desired result. □

Remark 10. By Theorems 2 and 3, we find that the value $r_{1\sigma}$ is the threshold between the extinction and persistence for the prey population. In addition, although the prey population converges to a non-extinction state in the mean when $r_{1\sigma} > 0$ and $r_{2\sigma} + a_{21}\beta_1 < 0$, the predators dies out when the diffusion coefficient σ_2 is large enough and then

$$-r_{2\sigma} = -r_2 + 0.5\left\{\sigma_2 \cdot (1 - \eta_\varsigma)\right\}^2$$

becomes too large.

Remark 11. We can establish one condition for the extinction of the prey and the persistence of the predator as follows. Lemmas 4 and 3-(a)-(ii) yield

$$\text{if } r_{1\sigma} < 0 \text{ and } r_{2\sigma} \ge 0, \text{ then } \lim_{k\to\infty} k^{-1} \sum\nolimits_{s=0}^{k-1} E\left(x_s^2\right) \le a_{22}^{-1} r_{2\sigma}. \tag{79}$$

For finding a lower function of x_k^2, we consider the solution $v_{k,\epsilon}$ of the equation

$$v_{k+1,\epsilon} = v_{k,\epsilon}(1 + \hat{r}_2 - \hat{a}_{21}\epsilon - \hat{a}_{22} v_{k,\epsilon} + \tilde{\sigma}_2 \xi^2_{N_\epsilon+k+1}), \ v_{0,\epsilon} = x^2_{N_\epsilon}, \tag{80}$$

in which ϵ satisfies that for some positive integer N_ϵ and all $k \ge N_\epsilon$

$$0 < x_k^1 \le \epsilon, \tag{81}$$
$$\hat{r}_2 - \hat{a}_{21}\epsilon + \tilde{\sigma}_2\varsigma_* < 1, \tag{82}$$
$$\hat{a}_{21}\epsilon + \tilde{\sigma}_2\varsigma < \tilde{\sigma}_2\varsigma_*. \tag{83}$$

The inequality (81) is possible under the condition $r_{1\sigma} < 0$ due to Lemma 3-(a).
Replacing (64)–(67), $r_{1\sigma} > 0$, $r_{2\sigma} + a_{21}\beta_1 < 0$ and $(u_{k,\epsilon}, r_1, a_{11}, a_{12}, \xi^1)$ in the proof of Lemma 5 with (80)–(83), $r_{1\sigma} < 0$, $r_{2\sigma} > 0$ and $(v_{k,\epsilon}, r_2, a_{22}, a_{21}, \xi^2)$, we can obtain that

$$v_{k,\epsilon} \le x^2_{N_\epsilon+k}, \ \lim_{k\to\infty} k^{-1} \sum\nolimits_{s=0}^{k-1} E\left(v_{s,\epsilon}\right) = a_{22}^{-1}\left(r_{2\sigma} - a_{21}\epsilon\right), \tag{84}$$

if $r_{2\sigma} - a_{21}\epsilon > 0$. Therefore (79) and (84) give the desired result:

$$\text{if } r_{1\sigma} < 0 \text{ and } r_{2\sigma} > 0, \text{ then } \lim_{k\to\infty}\left(x_k^1, \overline{E}(x_k^2)\right) = \left(0, a_{22}^{-1} r_{2\sigma}\right) \quad a.s.$$

Now, it remains to establish one condition for persistence of the prey and the predator. Define the matrix A and the constants D_i as

$$A = \begin{pmatrix} a_{11} & a_{12} \\ -a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} r_{1\sigma} \\ r_{2\sigma} \end{pmatrix} = A \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}, \tag{85}$$

which give

$$|A| = a_{11}a_{22} + a_{12}a_{21} > 0, \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = A^{-1} \begin{pmatrix} r_{1\sigma} \\ r_{2\sigma} \end{pmatrix} = |A|^{-1} \begin{pmatrix} a_{22}r_{1\sigma} - a_{12}r_{2\sigma} \\ a_{11}\left(r_{2\sigma} + a_{21}\beta_1\right) \end{pmatrix} \geq 0 \tag{86}$$

under the conditions $r_{1\sigma} \geq a_{22}^{-1}a_{12}r_{2\sigma}$ and $r_{2\sigma} + a_{21}\beta_1 \geq 0$.
Using (85), the system (23) can be written as the matrix equation

$$\begin{pmatrix} E\left(\ln x_k^1\right) \\ E\left(\ln x_k^2\right) \end{pmatrix} = \begin{pmatrix} E\left(\ln x_0^1\right) \\ E\left(\ln x_0^2\right) \end{pmatrix} + k\mathring{h}A \begin{pmatrix} D_1 - \overline{E}\left(x_k^1\right) \\ D_2 - \overline{E}\left(x_k^2\right) \end{pmatrix} \tag{87}$$

and multiplying the matrix $|A|A^{-1}$ to (87), we have

$$\begin{aligned} a_{22}E(\ln x_k^1) - a_{12}E(\ln x_k^2) &= C_1 + k\mathring{h}|A|\left\{D_1 - \overline{E}\left(x_k^1\right)\right\}, &(88)\\ a_{21}E(\ln x_k^1) + a_{11}E(\ln x_k^2) &= C_2 + k\mathring{h}|A|\left\{D_2 - \overline{E}\left(x_k^2\right)\right\}, &(89)\end{aligned}$$

where $C_1 = a_{22}E(\ln x_0^1) - a_{12}E(\ln x_0^2)$ and $C_2 = a_{21}E(\ln x_0^1) + a_{11}E(\ln x_0^2)$.

Lemma 6. *Let x_k^1 and β_1 be the solutions of (3) and (26), respectively. If $r_{1\sigma} \geq a_{22}^{-1}a_{12}r_{2\sigma}$ and $r_{2\sigma} + a_{21}\beta_1 \geq 0$, then for $\epsilon > 0$ and all sufficiently large k*

$$\overline{E}(x_k^1) \leq D_1 + \epsilon, \tag{90}$$

where D_1 is defined in (85).

Proof. Suppose that (90) is false, which means that there exist a constant $\epsilon_0 > 0$ and an infinite increasing sequence $\{k_m\}$ satisfying both for all k_m

$$k_m^{-1} \sum_{s=0}^{k_m-1} E\left(x_s^1\right) > D_1 + \epsilon_0, \tag{91}$$

and for all k with $k \neq k_m$

$$k^{-1} \sum_{s=0}^{k-1} E\left(x_s^1\right) \leq D_1 + \epsilon_0. \tag{92}$$

Replace (z_k^1, β_1), (31), (32), (28) and (27) in the proof of Lemma 2 with (x_k^1, D_1), (91), (92), (88) and (22), respectively, where we apply (22) with $i = 1$. Then using the boundedness of x_k^1 and following the proof for (37), we can obtain that for all sufficiently large k

$$k^{-1} \sum_{s=0}^{k-1} E\left(x_s^1\right) > D_1 + \epsilon_0. \tag{93}$$

Combining (93) and (88) gives

$$a_{22}E(\ln x_k^1) - a_{12}E(\ln x_k^2) < C_1 + k\mathring{h}|A|(-\epsilon_0). \tag{94}$$

Applying Theorem 1 to (22) with $i = 2$, we obtain

$$\sup_{k\geq 0} E(\ln x_k^2) < \infty$$

and then (94) yields

$$\lim_{k\to\infty} E\left(\ln x_k^1\right) = -\infty. \tag{95}$$

Substituting (95) into (22) with $i = 1$ and using the boundedness of x_k^1, we obtain

$$\lim_{k\to\infty} \ln x_k^1 = -\infty \quad a.s.,$$

which implies

$$\lim_{k\to\infty} x_k^1 = 0 \quad a.s.$$

Hence the dominated convergence theorem with Theorem 1 leads to

$$\lim_{k\to\infty} E(x_k^1) = 0,$$

which is contradictory to (93) due to $D_1 + \varepsilon_0 > 0$. This completes the proof. □

Remark 12. The equation (90) with (87) gives that for $\epsilon > 0$ and all sufficiently large k

$$E(\ln x_k^2) \leq E(\ln x_0^2) + k\mathring{h}a_{22}\left\{a_{22}^{-1}a_{21}\epsilon + D_2 - \overline{E}(x_k^2)\right\}. \tag{96}$$

Following the proof of Lemma 6 with (96), we can obtain that

$$\text{if } r_{1\sigma} \geq a_{22}^{-1}a_{12}r_{2\sigma} \text{ and } r_{2\sigma} + a_{21}\beta_1 \geq 0, \text{ then } \overline{E}(x_k^2) \leq a_{22}^{-1}a_{21}\epsilon + D_2 + \epsilon' \tag{97}$$

for $\epsilon' > 0$ and all sufficiently large k by replacing (x_k^1, D_1) and (88) in the proof of Lemma 6 with $(x_k^2, a_{22}^{-1}a_{21}\epsilon + D_2)$ and (96), respectively.

Theorem 4. *Let x_k^i and β_i be the solutions of (3) and (26), respectively for $i = 1, 2$.*

$$\text{If } r_{1\sigma} \geq a_{22}^{-1}a_{12}r_{2\sigma} \text{ and } r_{2\sigma} + a_{21}\beta_1 \geq 0, \text{ then } \lim_{k\to\infty} \overline{E}(x_k^i) = D_i,$$

where D_i are defined in (85).

Proof. Substituting (62) into (89) gives that for $\epsilon' > 0$ and all sufficiently large k

$$\epsilon' \geq D_2 - \overline{E}(x_k^2). \tag{98}$$

Combining (98) and (97), we have

$$\lim_{k\to\infty} \overline{E}(x_k^2) = D_2. \tag{99}$$

Applying (99) to (89) with (62) yields

$$\lim_{k\to\infty} k^{-1}E(\ln x_k^1) = \lim_{k\to\infty} k^{-1}E(\ln x_k^2) = 0,$$

with which (88) gives the desired result $\lim\limits_{k\to\infty} \overline{E}(x_k^1) = D_1$. □

Remark 13. Let (x_k, y_k) be the solutions of DDEs (3) with $\sigma_1 = \sigma_2 = 0$ in [35].

(i) If $r_1 > 0$, $r_2 < 0$ and $r_2 + a_{21}a_{11}^{-1}r_1 \leq 0$, then $\lim_{k\to\infty}(x_k, y_k) = (a_{11}^{-1}r_1, 0)$.

(ii) If $r_1 > 0$, $r_2 < 0$ and $r_2 + a_{21}a_{11}^{-1}r_1 > 0$, then $\lim_{k\to\infty}(x_k, y_k) = (D_x, D_y)$, where (D_x, D_y) is equal to (D_1, D_2) with $\sigma_1 = \sigma_2 = 0$.

Note that the sign of r_2 in the DDE model is fixed to $r_2 < 0$. Adding the noise to the DDEs, we have from Theorems 3 and 4 that

(i)′ If $r_{1\sigma} \geq 0$ and $r_{2\sigma} + a_{21}a_{11}^{-1}r_{1\sigma} < 0$, then $\lim_{k\to\infty}\left(\overline{E}(x_k^1), x_k^2\right) = \left(a_{11}^{-1}r_{1\sigma}, 0\right)$ *a.s.*

(ii)′ If $r_{1\sigma} \geq a_{22}^{-1}a_{12}r_{2\sigma}$ and $r_{2\sigma}+a_{21}a_{11}^{-1}r_{1\sigma} \geq 0$, then $\lim_{k\to\infty}\left(\overline{E}(x_k^1), \overline{E}(x_k^2)\right) = (D_1, D_2)$.

Hence we demonstrate that the solutions of the DDEs and the DSDEs with small noise have similar asymptotic behavior by comparing (i), (ii) and (i)′, (ii)′, respectively. In addition, when comparing $r_2 + a_{21}a_{11}^{-1}r_1 > 0$ in (ii) and $r_{2\sigma} + a_{21}a_{11}^{-1}r_{1\sigma} < 0$ in (i)′, we understand the effect of strong noise, which changes the behavior of the predator population from non-extinction into extinction. Therefore the main difference between the deterministic and stochastic models is that large stochastic perturbation may result in the extinction of the predator population.

Remark 14. Let (x, y) be the solutions of the SDE model (2), which is a special model in [25] with zero time delays. Note that the sign of r_2 in the SDE model is also negative.

(i) If $r_1 - 0.5\sigma_1^2 < 0$ and $r_2 - 0.5\sigma_2^2 < 0$, then $\lim_{t\to\infty}(x(t), y(t)) = (0, 0)$ *a.s.*

(ii) If $r_1 - 0.5\sigma_1^2 > 0$, $r_2 - 0.5\sigma_2^2 < 0$ and $(r_2 - 0.5\sigma_2^2) + a_{21}a_{11}^{-1}(r_1 - 0.5\sigma_1^2) < 0$, then x is stable in the mean and y goes to extinction:

$$\lim_{t\to\infty} t^{-1}\int_0^t x(s)ds = a_{11}^{-1}r_{1\sigma},\ \ \lim_{t\to\infty} y(t) = 0 \ \ a.s.$$

(iii) If $r_2 - 0.5\sigma_2^2 < 0$ and $(r_2 - 0.5\sigma_2^2) + a_{21}a_{11}^{-1}(r_1 - 0.5\sigma_1^2) > 0$, then both x and y are stable in the mean:

$$\lim_{t\to\infty}\left(t^{-1}\int_0^t x(s)ds,\ t^{-1}\int_0^t y(s)ds\right) = (D_1, D_2) \ \ a.s.$$

Since $r_2 < 0$ in the SDE model (2), the sign of $r_2 - 0.5\sigma_2^2$ in (2) is also negative, which is the reason why the condition $r_2 - 0.5\sigma_2^2 < 0$ is assumed in (i)–(iii). The three results, (i), (ii) and (iii) in this remark, are corresponding to Theorem 2-(b), (i)′ and (ii)′ in Remark 13, respectively. Hence, when replacing the stability of $(x(t), y(t))$ in the mean with the stability of $\left(\overline{E}(x_k^1), \overline{E}(x_k^2)\right)$, we demonstrate that the sufficient conditions for the almost sure global stability of the SDE model (2) also suffice to give the same global stability of the DSDE model (3). In this case, note that there is no constraint on the sign of r_2 in the DSDE model. Therefore we show that the DSDE model (3) is a good discrete model for the corresponding SDE model (2).

6. Numerical examples

In this section, we provide some simulations that illustrate the results in Theorems 1, 2, 3 and 4 with truncation constants $(\varsigma, \varsigma_*) = (19.9, 20)$ in (5) and (12). In this case, we have $0 < \eta_\varsigma < 10^{-85}$, so that we can ignore the effect of the term η_ς when using the values of parameters in the following three examples, where the conditions (12)–(14) are satisfied. In Figures 1, 2 and 3, the DSDE model (3) is simulated 1000 times at each time kh for calculating the expectation values $E(x_k)$ and $E(y_k)$, where x_k and y_k denote the solutions x_k^1 and x_k^2, respectively. We compare our results for the DSDE model (3) with the results for the DDE model in [35], which is the model (3) with $\sigma_1 = \sigma_2 = 0$.

Example 1. Let $h = 0.0001, r_1 = 0.8, r_2 = -0.1, a_{11} = 0.4, a_{12} = 0.001, a_{21} = 0.1, a_{22} = 0.3, \sigma_1^2 = 2.5$ and $\sigma_2^2 = 0.1$. Since $r_1 > 0$, $r_2 < 0$ and $r_2 + a_{21}a_{11}^{-1}r_1 > 0$, the solutions x_k and y_k of the DDE model converge to the positive numbers D_x and D_y in Remark 13-(ii), respectively, as displayed in Figure 1-(a). However, since $r_{i\sigma} < 0$ $(i = 1, 2)$, the noises have a large effect on the convergence and, as a result, the solutions of the stochastically perturbed model (3) go to extinction, which are shown in Figures 1-(b) and (c), as in Theorem 2-(a) and (b), respectively. Therefore Figures 1 demonstrates the important role of noise.

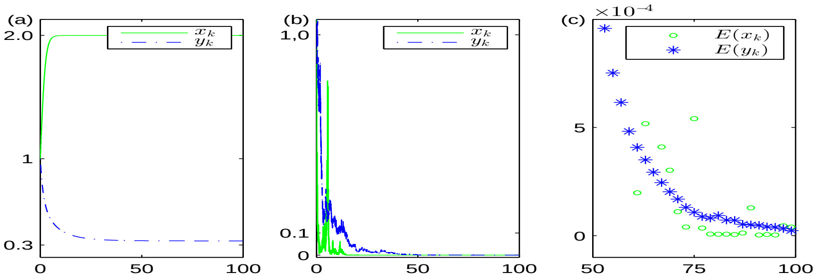

Figure 1: All the x-axes denote time kh. (a) Curves of the solutions of the DDE model. (b) Two realizations of the solutions x_k and y_k of the DSDE model, which converge to zero. (c) Expectation values of the solutions x_k and y_k of the DSDE model, which converge to zero in the mean.

Example 2. Let $h = 0.001, r_1 = 2, r_2 = -2, a_{11} = 1.0, a_{12} = 0.4, a_{21} = a_{22} = 0.3, \sigma_1^2 = 0.2$ and $\sigma_2^2 = 4$. Figure 2-(a) shows that the solutions x_k and y_k of the DDE model converge to $a_{11}^{-1}r_1$ and 0, respectively, as in Remark 13-(i) when $r_1 > 0$, $r_2 < 0$ and $r_2 + a_{21}a_{11}^{-1}r_1 \leq 0$. The noises satisfy both $r_{1\sigma} > 0$ and $r_{2\sigma} + a_{21}a_{11}^{-1}r_{1\sigma} < 0$, which are the conditions in Theorem 3. Then Figures 2-(b), (c) and (d) show that the stochastically perturbed model (3) behaves similarly to the DDE model in the sense that $k^{-1}\sum_{i=0}^{k-1} E(x_i)$ and y_k converge to $a_{11}^{-1}r_{1\sigma}$ and 0, respectively, which confirms Theorem 3.

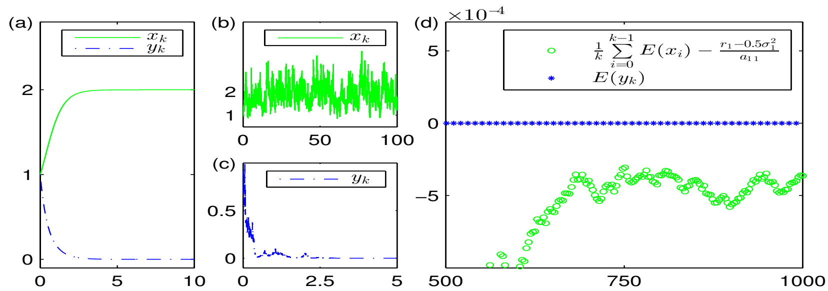

Figure 2: All the x-axes denote time kh. (a) Curves of the solutions of the DDE model. Curves in (b) and (c) are realizations of the solutions x_k and y_k of the DSDE model, respectively. (d) Convergence of average of expectation values of x_k to non-zero and convergence of y_k to zero in the mean.

Example 3. Let $h = 0.001, r_1 = 2.0, r_2 = -0.1, a_{11} = a_{12} = 0.4, a_{21} = 1, a_{22} = 0.3$ and $\sigma_1^2 = \sigma_2^2 = 0.02$, which give that $r_1 > 0$, $r_2 < 0$ and $r_2 + a_{21}a_{11}^{-1}r_1 > 0$. Thus Figure 3-(a) shows that the solutions x_k and y_k of the DDE model converge to D_x and D_y in Remark 13-(ii), respectively, as displayed in Figure 1-(a) in Example 1. However, the condition $r_{1\sigma} > 0$ is different from that in Example 1. Realizations of the solutions of the DSDE model are given in Figures 3-(b) and (c). Since $r_{1\sigma} > a_{22}^{-1}a_{12}r_{2\sigma}$ and $r_{2\sigma} + a_{21}a_{11}^{-1}r_{1\sigma} > 0$, Figure 3-(d) shows that the DSDE model behaves similarly to the DDE model in the sense that $k^{-1}\sum_{i=0}^{k-1} E(x_i)$ and $k^{-1}\sum_{i=0}^{k-1} E(y_i)$ converge to positive D_1 and D_2, respectively, which demonstrate Theorem 4.

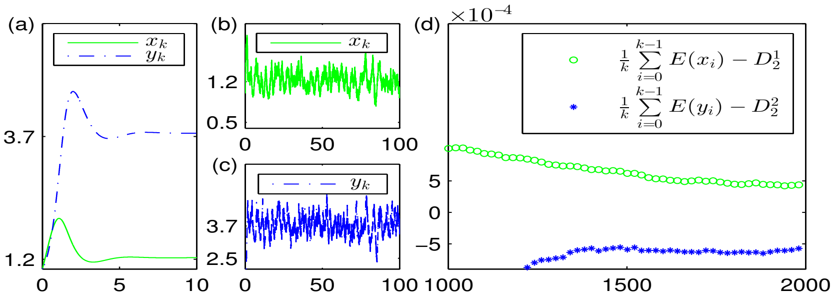

Figure 3: All the x-axes denote time kh. (a) Curves of the solutions of the DDE model. Curves in (b) and (c) are realizations of the solutions x_k and y_k of the DSDE model, respectively. The symbols D_2^1 and D_2^2 in (d) denote D_1 and D_2 defined in (85).

7. Conclusion

In this paper, we have considered a system of discrete-time stochastic difference equations for predator-prey interactions and established sufficient conditions for extinction and non-extinction of the two species. Our results show that if the positive equilibrium point of the deterministic difference system is globally stable, then the stochastic difference model will preserve the nice property in mean provided that the noise is sufficiently small. It is shown, however, that large noise can change the behavior of the predator population from non-extinction into extinction.

Our new discrete Itô formula has played an important role in the two-dimensional DSDE model. In addition we can apply the new formula for the n-dimensional DSDE model

$$x_{k+1}^i = x_k^i\left\{1 + h\left(r_i + \sum\nolimits_{j=1}^{i-1} a_{ij}x_k^j - \sum\nolimits_{j=i}^{n} a_{ij}x_k^j\right) + h^{0.5}\sigma_i\xi_{k+1}^i\right\}$$

for $1 \le i \le n$ and $k \ge 0$. Therefore it is a further study to establish sufficient conditions for the extinction and non-extinction of the n species.

Appendix

A.1. The proof of Lemma 1

By Taylor expansion,

$$\varphi(1+x) = \varphi(1) + \varphi'(1)x + 2^{-1}\varphi''(1)x^2 + 6^{-1}\varphi'''(\theta)x^3 \tag{100}$$

with θ lying between 1 and x. Let $x = hf + h^{0.5}g\xi$. Since f, g are $\mathcal{G}$-measurable and ξ is $\mathcal{G}$-independent with $E(\xi) = 0$, we have

$$E\left(x \middle| \mathcal{G}\right) = E\left(hf \middle| \mathcal{G}\right) + E\left(h^{0.5}g\xi \middle| \mathcal{G}\right) = hf + h^{0.5}gE(\xi) = hf \tag{101}$$

and further

$$\begin{aligned} E\left(x^2 \middle| \mathcal{G}\right) &= E\left((hf)^2 \middle| \mathcal{G}\right) + E\left(2hfh^{0.5}g\xi \middle| \mathcal{G}\right) + E\left(hg^2\xi^2 \middle| \mathcal{G}\right) \\ &= (hf)^2 + hg^2 \cdot (1-\mu) \\ &\leq hfM_5h^{\varepsilon} + hg^2 \cdot (1-\mu) \end{aligned} \tag{102}$$

due to $E(\xi^2) = 1 - \mu$ and (18). Using Lemma 1-(ii) gives

$$\left|E\left(6^{-1}\varphi'''(\theta)x^3 \,\middle|\, \mathcal{G}\right)\right| \leq 6^{-1}M_3E\left(\left|x^3\right| \middle| \mathcal{G}\right) \tag{103}$$

and expanding $x^3 = (hf + h^{0.5}g\xi)^3$ yields

$$\begin{aligned} E\left(\left|x^3\right| \middle| \mathcal{G}\right) &\leq hf\left\{(hf)^2 + 3hg^2 \cdot (1-\mu)\right\} + hg^2M_1h^{0.5}g \\ &\leq hf\left(M_5h^{\varepsilon}\right)^2\left\{1 + 3(1-\mu)\right\} + hg^2M_1M_4h^{\varepsilon} \end{aligned} \tag{104}$$

because of (18) and (16). Inserting (101)–(104) into (100), we have

$$E\left(\varphi(1+x) \middle| \mathcal{G}\right) \tag{105}$$
$$= \varphi(1) + \varphi'(1)hf + 2^{-1}\varphi''(1)hg^2 \cdot (1-\mu) + hfO_1\left(h^{\varepsilon}\right) + hg^2O_2\left(h^{\varepsilon}\right), \tag{106}$$

in which the two big O notations denote

$$\begin{aligned} O_1\left(h^{\varepsilon}\right) &= 2^{-1}\varphi''(1)M_5h^{\varepsilon} + 6^{-1}M_3\left(M_5h^{\varepsilon}\right)^2\left\{1 + 3(1-\mu)\right\}, \\ O_2\left(h^{\varepsilon}\right) &= M_1M_5h^{\varepsilon}. \end{aligned}$$

Now it remains to show

$$E\left(\phi\left(1 + hf + h^{0.5}g\xi\right) - \varphi\left(1 + hf + h^{0.5}g\xi\right) \middle| \mathcal{G}\right) = hg^2O\left(h^{\varepsilon}\right).$$

Let $c_1 = 1 + hf$ and $c_2 = h^{0.5}g$. Then the disintegration formula for conditional expectations with respect to $\mathcal{G}$ gives

$$\begin{aligned} &E\left(\phi\left(1 + hf + \sqrt{h}g\xi\right) - \varphi\left(1 + hf + \sqrt{h}g\xi\right) \middle| \mathcal{G}\right) \\ &\quad = \int_{\mathbb{R}} \left\{\phi\left(c_1 + c_2x\right) - \varphi\left(c_1 + c_2x\right)\right\} p(x)\, dx \end{aligned} \tag{107}$$

due to Lemma 1-(iii) and the fact that f, g are $\mathcal{G}$-measurable, ξ is $\mathcal{G}$ -independent, ϕ is almost everywhere continuous and φ is also continuous (see Theorem 5.4 in [44] for the disintegration formula). Let $U_\delta = [1-\delta, 1+\delta]$ and $s = c_1 + c_2x$. Then (107) becomes

$$\int_{\mathbb{R}-U_\delta} \left\{\phi\left(s\right) - \varphi\left(s\right)\right\} p\left(\frac{s - c_1}{c_2}\right) \frac{ds}{|c_2|} \tag{108}$$

because of Lemma 1-(i). Here p is the probabilty density function of ξ.

Lemma 1-(iii) gives that

$$
\begin{aligned}
&\left|\int_{\mathbb{R}-U_\delta} \{\phi(s)-\varphi(s)\}\, p\left(\frac{s-c_1}{c_2}\right)\,\frac{ds}{|c_2|}\right| \\
&\quad\le \left\{\int_{\mathbb{R}-U_\delta} |\phi(s)-\varphi(s)|\,\frac{ds}{|c_2|}\right\} \sup_{s\notin U_\delta}\left\{p\left(\frac{s-c_1}{c_2}\right)\,\frac{1}{|c_2|}\right\} \\
&\quad\le M_4|c_2|^2 \sup_{s\notin U_\delta}\left\{p\left(\frac{s-c_1}{c_2}\right)\,\frac{1}{|c_2|^3}\right\} \\
&\quad= M_4 h g^2 \sup_{s\notin U_\delta}\left\{p\left(\frac{s-1-hf}{h^{0.5}g}\right)\,\frac{1}{|h^{0.5}g|^3}\right\}.
\end{aligned}
$$

Since there exists some δ_0 such that for $s \notin U_\delta$ and all sufficiently small $h > 0$

$$
|s-1-hf| > |s-1| - h|f| > \delta - M_5 h^\epsilon > \delta_0 > 0, \tag{109}
$$

letting $y = (s-1-hf)/(h^{0.5}g)$ yields

$$
|y| = \frac{|s-1-hf|}{h^{0.5}\,|g|} > \frac{\delta_0}{M_5 h^\varepsilon} \tag{110}
$$

and further

$$
\sup_{s\notin U_\delta}\left\{p\left(\frac{s-1-hf}{h^{0.5}g}\right)\,\frac{1}{|h^{0.5}g|^3}\right\} = \sup_{s\notin U_\delta}\,\frac{p(y)\,|y|^3}{|s-1-hf|^3}.
$$

Hence it follows from (17), (109) and (110) that

$$
\sup_{s\notin U_\delta}\,\frac{p(y)\,|y|^3}{|s-1-hf|^3} < M_2 \sup_{s\notin U_\delta}\,\frac{|y|^{-1}}{|s-1-hf|^3} < M_2\frac{M_5}{\delta_0^2}h^\varepsilon,
$$

which gives

$$
\left|\int_{\mathbb{R}-U_\delta} \{\phi(s)-\varphi(s)\}\, p\left(\frac{s-c_1}{c_2}\right)\,\frac{ds}{|c_2|}\right| < hg^2\cdot M_4 M_2 \frac{M_5}{\delta_0^2}h^\varepsilon. \tag{111}
$$

Therefore using (105), (108) and (111), we obtain the desired result.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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WEIGHTED SUPERPOSITION OPERATORS FROM ZYGMUND SPACES TO μ-BLOCH SPACES

ZHI JIE JIANG, TING WANG, JUAN LIU, TING LUO, TING SONG

ABSTRACT. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. Let φ be an entire function on $\mathbb{C}$ and $u \in H(\mathbb{D})$. The boundedness and compactness of the operators $S_{u,\varphi} : f \mapsto u \cdot \varphi \circ f$ from Zygmund spaces to μ-Bloch spaces are characterized.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$ and $H^\infty(\mathbb{D})$ the space of bounded analytic functions. Let φ be a complex-valued function on $\mathbb{C}$ and $u \in H(\mathbb{D})$. We introduce a class of nonlinear operators by

$$S_{u,\varphi} f = u \cdot \varphi \circ f, \quad f \in H(\mathbb{D}).$$

This operator can be regarded as a generalization of the superposition operator $S_\varphi f = \varphi \circ f$ and the multiplication operator $M_u f = u \cdot f$.

Suppose that X and Y are two metric spaces of analytic functions on $\mathbb{D}$. Note that if X contains the linear functions and S_φ maps X into Y, then φ must be an entire function. In recent years, the following natural questions of the superposition operators are considered.

(a) When does φ induce a superposition operator from X into Y?

(b) When is a superposition operator from X into Y bounded?

(c) When is a superposition operator from X into Y compact?

Although analogous concepts also make sense in the context of real-valued functions and their theory has a long history (see [2]), the study of such natural questions on analytic function spaces has only begun fairly recently. The operators S_φ that map Bergman spaces into area Nevanlinna classes were characterized in [6], which have been extended by other authors to some other analytic function spaces, where it is remarkable the works of Vukotić et. al. in [1], [4] and [5]. It must be mentioned that the authors of [4] gave a very interesting geometric construction of simple connected domain in several analytic function spaces. This technique has been used by many authors; in particular, Xu used it to study the superposition operators from α-Bloch spaces into β-Bloch spaces in [20] and Xiong used it to characterize the superposition operators from Q_p spaces into α-Bloch spaces with $0 < \alpha < 1$ in [18]. It should be noted that quite recently, Castillo et.al. and Ramos Fernández have studied the superposition operators from Bloch-Orlicz spaces into α-Bloch spaces and between weighted Banach spaces of analytic functions in [7] and [14], respectively. In this paper we characterize the boundedness and compactness of the operators $S_{u,\varphi}$ from weighted Zygmund spaces to μ-Bloch spaces. We also consider the superposition operators from weighted Zygmund spaces to weighted Bloch spaces.

Now we present the needed spaces and some facts. The Zygmund space $\mathcal{Z}$ consists of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}} (1 - |z|^2)|f''(z)| < \infty.$$

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With the norm
$$\|f\|_{\mathcal{Z}} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f''(z)|,$$
it is a Banach space. By Zygmund's theorem (see Theorem 5.3 in [9]), we know that $f \in \mathcal{Z}$ if and only if f is continuous on $\overline{D}$ and
$$\sup_{h>0,\theta\in\mathbb{R}} \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty.$$
In closed subspaces of $\mathcal{Z}$, the little Zygmund space $\mathcal{Z}_0$ is usually considered, which is defined by
$$\mathcal{Z}_0 = \big\{f \in \mathcal{Z} : \lim_{|z|\to 1}(1-|z|^2)|f''(z)| = 0\big\}.$$

Let $\alpha \in (0,\infty)$. The weighted Zygmund space $\mathcal{Z}_\alpha$ consists of all $f \in H(\mathbb{D})$ such that
$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f''(z)| < +\infty.$$
With the norm
$$\|f\|_{\mathcal{Z}_\alpha} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f''(z)|,$$
$\mathcal{Z}_\alpha$ is also a Banach space. For the weighted Zygmund spaces and the operators from them into some other spaces, see, e.g., [10], [12] and [15].

Suppose that μ is a positive continuous radial function on $\mathbb{D}$ (that is, $\mu(z) = \mu(|z|)$) and decreasing on $[0,1)$ with $\lim_{r\to 1}\mu(r) = 0$. Let μ be a weight. The μ-Bloch space $\mathcal{B}_\mu$ consists of all $f \in H(\mathbb{D})$ such that $\sup_{z\in\mathbb{D}}\mu(z)|f'(z)| < \infty$. With
$$\|f\|_{\mathcal{B}_\mu} = |f(0)| + \sup_{z\in\mathbb{D}}\mu(z)|f'(z)|,$$
$\mathcal{B}_\mu$ is a Banach space. When $\mu(z) = 1-|z|^2$, the space $\mathcal{B}_\mu$ is just Bloch space and denoted by $\mathcal{B}$; while when $\mu(z) = (1-|z|^2)^\alpha$ with $\alpha > 0$, the space $\mathcal{B}_\mu$ becomes the weighted Bloch space $\mathcal{B}_\alpha$. The μ-Bloch spaces appear in the literature in a natural way when one considers properties of some operators in certain spaces of analytic functions; for example, if $\mu(z) = (1-|z|)\log\frac{2}{1-|z|}$, Attele in [3] proved that the Hankel operator on Bergman spaces induced by a function f is bounded if and only if $f \in \mathcal{B}_\mu$. The logarithmic Bloch type space has been defined and studied in [16]. Recently, the Bloch-Orlicz spaces have been introduced by Ramos-Fernandez in [13].

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \simeq b$ means that there is a positive constant C such that $a/C \le b \le Ca$.

2. The operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$

First we enumerate several useful lemmas. The first one below is well-known.

Lemma 2.1 *There is a positive constant C_α depending only on α such that for any $z \in \mathbb{D}$ and $f \in \mathcal{Z}_\alpha$*

(i)
$$\big|f(z)\big| \le \begin{cases} C_\alpha\|f\|_{\mathcal{Z}_\alpha}, & 0<\alpha<2,\\ C_\alpha\|f\|_{\mathcal{Z}_\alpha}\log\frac{2}{1-|z|^2}, & \alpha = 2,\\ C_\alpha\|f\|_{\mathcal{Z}_\alpha}(1-|z|^2)^{2-\alpha}, & \alpha>2.\end{cases}$$

(ii)
$$\big|f'(z)\big| \le \begin{cases} C_\alpha\|f\|_{\mathcal{Z}_\alpha}, & 0<\alpha<1,\\ C_\alpha\|f\|_{\mathcal{Z}_\alpha}\log\frac{2}{1-|z|^2}, & \alpha = 1,\\ C_\alpha\|f\|_{\mathcal{Z}_\alpha}(1-|z|^2)^{1-\alpha}, & \alpha>1.\end{cases}$$

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Let $a \in \mathbb{D}$ and $1/\sqrt{2} < |a| < 1$, define

$$f(z) = (z-1)\Big(\Big(1 + \log \frac{1}{1-z}\Big)^2 + 1\Big)$$

and

$$g_a(z) = \frac{f(\bar{a}z)}{\bar{a}}\Big(\log \frac{1}{1-|a|^2}\Big)^{-1}.$$

The function g_a is called the test function with the following property (see [11]).

Lemma 2.2 *The function g_a belongs to $\mathcal{Z}$ and $\|g_a\|_{\mathcal{Z}} \simeq 1$.*

The following result can be found in [17].

Lemma 2.3 *Let $\alpha \in (0,1]$. Then for every bounded sequence $\{f_n\}$ in $\mathcal{Z}_\alpha$ and $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n \to \infty$, we have*

(i) if $\alpha = 1$, then $\lim_{n\to\infty} \sup_{z\in\mathbb{D}} \big|f_n(z)\big| = 0$.

(ii) if $0 < \alpha < 1$, then $\lim_{n\to\infty} \sup_{z\in\mathbb{D}} \big|f_n'(z)\big| = 0$.

The next result is often used in dealing with the compactness of operators on analytic function spaces. Since the proof is standard (see Proposition 3.11 in [8]), it is omitted .

Lemma 2.4 *Let $u \in H(\mathbb{D})$ and φ an entire function. Then the bounded operator $S_{u,\varphi} : \mathcal{Z}_\alpha \to \mathcal{B}_\mu$ is compact if and only if for any bounded sequence $\{f_n\}$ in $\mathcal{Z}_\alpha$ such that $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n \to \infty$, it follows that $\lim_{n\to\infty} \|S_{u,\varphi} f_n\|_{\mathcal{B}_\mu} = 0$.*

Now we characterize the boundedness of the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$.

Theorem 2.1 *Let $u \in H(\mathbb{D})$ and φ an entire function with $\varphi'(0) \neq 0$. Then the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is bounded if and only if $u \in \mathcal{B}_\mu$ and*

$$L := \sup_{z\in\mathbb{D}} \mu(z)|u(z)| \log \frac{2}{1-|z|^2} < \infty.$$

Proof. Suppose that the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is bounded. By taking f_1 the constant function, we obtain $u \in \mathcal{B}_\mu$. Since operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is bounded, for the function $f_2 = g_a$ there exists a positive constant C such that

$$\begin{aligned}
\infty > C\|S_{u,\varphi}\| \geq \|S_{u,\varphi} f_2\|_{\mathcal{B}_\mu} &\geq \mu(a)\big|(S_{u,\varphi} f_2)'(a)\big| \\
&= \mu(a)\big|u'(a)\varphi(f_2(a)) + u(a)\varphi'(f_2(a))f_2'(a)\big| \\
&\geq \mu(a)\big(\big|u(a)\big|\big|\varphi'(f_2(a))\big|\big|f_2'(a)\big| - \big|u'(a)\big|\big|\varphi(f_2(a))\big|\big).
\end{aligned}$$

From this, we get

$$\mu(a)\big|u'(a)\big|\big|\varphi(f_2(a))\big| + C\|S_{u,\varphi}\| \geq \mu(a)\big|u(a)\big|\big|\varphi'(f_2(a))\big|\big|f_2'(a)\big|.$$

Set $M = C_\alpha \|f_2\|_{\mathcal{Z}}$ and $M_1 = \max_{|z|=M} |\varphi(z)|$. By Lemma 2.1 (i), we have

$$\begin{aligned}
M_1\|u\|_{\mathcal{B}_\mu} + C\|S_{u,\varphi}\| &\geq \mu(a)\big|u'(a)\big|\big|\varphi(f_2(a))\big| + C\|S_{u,\varphi}\| \\
&\geq \mu(a)\big|u(a)\big|\big|\varphi'(f_2(a))\big|\big|f_2'(a)\big| \\
&= \mu(a)\big|u(a)\big|\big|\varphi'(g_a(a))\big| \log \frac{1}{1-|a|^2} \\
&\geq \frac{1}{2}\mu(a)\big|u(a)\big|\big|\varphi'(g_a(a))\big| \log \frac{2}{1-|a|^2},
\end{aligned}$$

where we have used that when $|a| > 1/\sqrt{2}$,

$$\log \frac{1}{1-|a|^2} \geq \frac{1}{2} \log \frac{2}{1-|a|^2}.$$

It is easy to see that $g_a(a) \to 0$ as $|a| \to 1$. Therefore from this and the fact that

$$\lim_{|a|\to 1} \big|\varphi'(g_a(a))\big| = |\varphi'(0)| \neq 0,$$

we obtain

$$\sup_{1/2<|z|<1} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} < \infty.$$

It is clear that

$$\sup_{|z|\leq 1/2} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} < \infty.$$

Consequently, we obtain $L < \infty$.

Now let $u \in \mathcal{B}_\mu$ and $L < \infty$. Let $f \in \mathcal{Z}$ and $\|f\|_{\mathcal{Z}} \leq M$. Set $M_1 = \max_{|z|=C_\alpha M} |\varphi(z)|$ and $M_2 = \max_{|z|=C_\alpha M} |\varphi'(z)|$. Then by Lemma 2.1, we have

$$\begin{aligned}
\|S_{u,\varphi} f\|_{\mathcal{B}_\mu} &= \big|u(0)\varphi(f(0))\big| + \sup_{z\in\mathbb{D}} \mu(z)\big|(S_{u,\varphi} f)'(z)\big| \\
&= \big|u(0)\varphi(f(0))\big| + \sup_{z\in\mathbb{D}} \mu(z)\big|u'(z)\varphi(f(z)) + u(z)\varphi'(f(z))f'(z)\big| \\
&\leq C_\alpha M\|u\|_{\mathcal{B}_\mu} + \sup_{z\in\mathbb{D}} \mu(z)\big|u'(z)\big|\big|\varphi(f(z))\big| + \sup_{z\in\mathbb{D}} \mu(z)\big|u(z)\big|\big|\varphi'(f(z))\big|\big|f'(z)\big| \\
&\leq C_\alpha M\|u\|_{\mathcal{B}_\mu} + M_1\|u\|_{\mathcal{B}_\mu} + C_\alpha M M_2 \sup_{z\in\mathbb{D}} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} \\
&\leq (C_\alpha M + M_1)\|u\|_{\mathcal{B}_\mu} + C_\alpha L M M_2 \\
&< \infty.
\end{aligned}$$

This shows that the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is bounded. □

There are a lot of examples satisfying the conditions of Theorem 2.1. Here we take the following two examples. Since the first is clear, its proof is omitted.

Example 2.1 *Let $u(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ and $\varphi(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_m z^m$, where $b_1 \neq 0$. Then the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is bounded.*

Example 2.2 *Let $u(z) = \lambda \frac{a-z}{1-\bar{a}z}$ be the automorphism of $\mathbb{D}$ and $\varphi(z) = e^z$. Then $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is bounded.*

Proof. Since $\|u\|_\infty \leq 1$ and it is easy to see that

$$|u'(z)| = \frac{1-|a|^2}{|1-\bar{a}z|^2} \leq \frac{2}{1-|a|},$$

we get $u \in \mathcal{B}_\mu$ and $L < \infty$. By Theorem 2.1, the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is bounded. □

From the proof of Theorem 2.1, we can obtain the following sufficient condition of boundedness for the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$.

Theorem 2.2 *Let $u \in H(\mathbb{D})$ and φ an entire function. If $u \in \mathcal{B}_\mu$ and $L < \infty$, then $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is bounded.*

We begin to study when the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is compact.

Theorem 2.3 *Let $u \in H(\mathbb{D})$ and φ an entire function with $\varphi(0) = 0$ and $\varphi'(0) \neq 0$. Then the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is compact if and only if $u \in \mathcal{B}_\mu$ and*

$$\lim_{|z|\to 1} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} = 0.$$

Proof. Suppose that the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is compact. Of course, it is bounded, and then $u \in \mathcal{B}_\mu$. Now let us suppose, by the way of contradiction, that

$$\lim_{|z|\to 1} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} \neq 0.$$

Then there exists some $\varepsilon_0 > 0$ and a sequence $\{z_n\} \subseteq \mathbb{D}$ such that $|z_n| \to 1$ as $n \to \infty$ and

$$\mu(z_n)\big|u(z_n)\big| \log \frac{2}{1-|z_n|^2} \geq \varepsilon_0.$$

For each $n \in \mathbb{N}$, take the function $f_n = g_{z_n}$. From Lemma 2.2 it follows that $\|f_n\|_{\mathcal{Z}} \leq C$. One can easily check that $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n \to \infty$. Thus it follows from Lemma 2.4 that

$$\begin{aligned}
\|S_{u,\varphi}f_n\|_{\mathcal{B}_\mu} &\geq \mu(z_n)\big|(S_{u,\varphi}f_n)'(z_n)\big| \\
&= \mu(z_n)\big|u'(z_n)\varphi(f_n(z_n)) + u(z_n)\varphi'(f_n(z_n))f_n'(z_n)\big| \\
&\geq \mu(z_n)\big(\big|u(z_n)\varphi'(f_n(z_n))f_n'(z_n)\big| - \big|u'(z_n)\varphi(f_n(z_n))\big|\big) \\
&= \mu(z_n)\big|u(z_n)\big|\big|\varphi'(f_n(z_n))\big|\big|f_n'(z_n)\big| - \mu(z_n)\big|u'(z_n)\big|\big|\varphi(f_n(z_n))\big| \\
&\geq \mu(z_n)\big|u(z_n)\big|\big|\varphi'(f_n(z_n))\big| \log\frac{1}{1-|z_n|^2} - \|u\|_{\mathcal{B}_\mu}\big|\varphi(f_n(z_n))\big| \\
&\geq \frac{1}{2}\mu(z_n)\big|u(z_n)\big|\big|\varphi'(f_n(z_n))\big| \log\frac{2}{1-|z_n|^2} - \|u\|_{\mathcal{B}_\mu}\big|\varphi(f_n(z_n))\big| \\
&\geq \frac{1}{2}\big|\varphi'(f_n(z_n))\big|\varepsilon_0 - \|u\|_{\mathcal{B}_\mu}\big|\varphi(f_n(z_n))\big|.
\end{aligned}$$

From this and since Lemma 2.3 (i) implies that $|\varphi(f_n(z_n))| = 0$ as $n \to \infty$, we get

$$0 = \lim_{n\to\infty} \|S_{u,\varphi}f_n\|_{\mathcal{B}_\mu} \geq \frac{1}{2}\big|\varphi'(0)\big|\varepsilon_0,$$

which arrives at a contradiction.

Conversely, by the definition of limit we have that for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\mu(z)\big|u(z)\big| \log\frac{2}{1-|z|^2} < \varepsilon$$

for all $z \in \{z \in \mathbb{D} : \delta < |z| < 1\}$. Let $M_0 > 0$ and $\|f_n\|_{\mathcal{Z}} \leq M_0$ and $f_n \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $n \to \infty$. By the Cauchy integral formula and an easy calculation, it is clear that $\{f_n'\}$ also uniformly converges to zero on every compact subset of $\mathbb{D}$ as $n \to \infty$. Let $M = \max\limits_{|z|=C_\alpha M_0} |\varphi'(z)|$. By Lemma 2.1 and Lemma 2.3 (i), we have

$$\begin{aligned}
\|S_{u,\varphi}f_n\|_{\mathcal{B}_\mu} &= \big|u(0)\varphi(f_n(0))\big| + \sup_{z\in\mathbb{D}}\mu(z)\big|(S_{u,\varphi}f_n)'(z)\big| \\
&= \big|u(0)\varphi(f_n(0))\big| + \sup_{z\in\mathbb{D}}\mu(z)\big|u'(z)\varphi(f_n(z)) + u(z)\varphi'(f_n(z))f_n'(z)\big| \\
&\leq \big|u(0)\varphi(f_n(0))\big| + \sup_{z\in\mathbb{D}}\mu(z)\big|u'(z)\big|\big|\varphi(f_n(z))\big| + \sup_{z\in\mathbb{D}}\mu(z)\big|u(z)\big|\big|\varphi'(f_n(z))\big|\big|f_n'(z)\big| \\
&\leq \big|u(0)\varphi(f_n(0))\big| + \|u\|_{\mathcal{B}_\mu}\sup_{z\in\mathbb{D}}\big|\varphi(f_n(z))\big| + \sup_{|z|\leq\delta}\mu(z)\big|u(z)\big|\big|\varphi'(f_n(z))\big|\big|f_n'(z)\big| \\
&\quad + \sup_{\delta<|z|<1}\mu(z)\big|u(z)\big|\big|\varphi'(f_n(z))\big|\big|f_n'(z)\big| \\
&\leq \big|u(0)\varphi(f_n(0))\big| + \|u\|_{\mathcal{B}_\mu}\sup_{z\in\mathbb{D}}\big|\varphi(f_n(z))\big| + M\max_{|z|\leq\delta}\mu(z)\big|u(z)\big|\max_{|z|\leq\delta}\big|f_n'(z)\big| \\
&\quad + C_\alpha M_0 M\sup_{\delta<|z|<1}\mu(z)\big|u(z)\big|\log\frac{2}{1-|z|^2}.
\end{aligned}$$

Taking the limit as $n \to \infty$ in this inequality, we obtain $\lim_{n\to\infty}\|S_{u,\varphi}f_n\|_{\mathcal{B}_\mu} = 0$. By Lemma 2.4, the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ is compact. $\square$

Remark 2.1 Considering Theorem 2.3, we have a reason to regard as the limit

$$\lim_{|z|\to 1^-}\mu(z)\log\frac{2}{1-|z|^2}$$

as an important factor for the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_\mu$ to be compact.

Theorem 2.4 *Let $u \in H(\mathbb{D})$ and φ an entire function with $\varphi(0) = 0$ and $\varphi'(0) \neq 0$. Then $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_{\mu,0}$ is bounded if and only if $u \in \mathcal{B}_{\mu,0}$ and*

$$\lim_{|z|\to 1} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} = 0.$$

Proof. Suppose that the operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_{\mu,0}$ is bounded, then by taking f the constant function we have $u \in \mathcal{B}_{\mu,0}$. Now let us suppose, by the way of contradiction, that

$$\lim_{|z|\to 1} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} \neq 0.$$

Then there exist some $\varepsilon_0 > 0$ and a sequence $\{z_n\} \subseteq \mathbb{D}$ with $|z_n| \to 1$ such that

$$\mu(z_n)\big|u(z_n)\big| \log \frac{2}{1-|z_n|^2} \geq \frac{2}{|\varphi'(0)|}\varepsilon_0.$$

Take the function $f = g_{z_n}$. Since $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_{\mu,0}$ is bounded, $S_{u,\varphi} f \in \mathcal{B}_{\mu,0}$, that is,

$$\lim_{|z|\to 1} \mu(z)\big|(S_{u,\varphi} f)'(z)\big| = 0;$$

in particular,

$$\lim_{n\to\infty} \mu(z_n)\big|(S_{u,\varphi} f)'(z_n)\big| = 0.$$

Letting $n \to \infty$ in

$$\begin{aligned}
\mu(z_n)\big|(S_{u,\varphi} f)'(z_n)\big| &= \mu(z_n)\big|u'(z_n)\varphi(f(z_n)) + u(z_n)\varphi'(f(z_n))f'(z_n)\big| \\
&\geq \mu(z_n)\big|u(z_n)\big|\big|\varphi'(f(z_n))\big|\big|f'(z_n)\big| - \mu(z_n)\big|u'(z_n)\big|\big|\varphi(f(z_n))\big| \\
&\geq \frac{1}{2}\mu(z_n)\big|u(z_n)\big| \log \frac{2}{1-|z_n|^2}\big|\varphi'(f(z_n)\big| - \mu(z_n)\big|u'(z_n)\big|\big|\varphi(f(z_n))\big| \\
&\geq \frac{\big|\varphi'(f(z_n))\big|}{|\varphi'(0)|}\varepsilon_0 - \mu(z_n)\big|u'(z_n)\big|\big|\varphi(f(z_n))\big|
\end{aligned}$$

arrives at a contradiction.

Conversely, by Theorem 2.1, we know that $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_{\mu}$ is bounded. It is enough to prove that for any $f \in \mathcal{Z}$, it holds $S_{u,\varphi} f \in \mathcal{B}_{\mu,0}$. Let $f \in \mathcal{Z}$, $M_1 = \max\limits_{|z|=\|f\|_{\mathcal{Z}}} |\varphi(z)|$ and $M_2 = \max\limits_{|z|=\|f\|_{\mathcal{Z}}} |\varphi'(z)|$. Then for any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\mu(z)|u'(z)| < \frac{\varepsilon}{2M_1}$$

and

$$\mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} < \frac{\varepsilon}{2M_2\|f\|_{\mathcal{Z}}}$$

for all $z \in \{z \in \mathbb{D} : \delta < |z| < 1\}$. So for $z \in \{z \in \mathbb{D} : \delta < |z| < 1\}$, it follows that

$$\begin{aligned}
\mu(z)\big|(S_{u,\varphi} f)'(z)\big| = \mu(z)\big|(S_{u,\varphi} f)'(z)\big| &= \mu(z)\big|u'(z)\varphi(f(z)) + u(z)\varphi'(f(z))f'(z)\big| \\
&\leq M_1\mu(z)\big|u'(z)\big| + M_2\|f\|_{\mathcal{Z}}\mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} \\
&< \varepsilon.
\end{aligned}$$

This shows that $S_{u,\varphi} f \in \mathcal{B}_{\mu,0}$. □

Theorem 2.5 *Let $u \in H(\mathbb{D})$ and φ an entire function with $\varphi(0) = 0$ and $\varphi'(0) \neq 0$. Then the bounded operator $S_{u,\varphi} : \mathcal{Z} \to \mathcal{B}_{\mu,0}$ is compact if and only if $u \in \mathcal{B}_{\mu,0}$ and*

$$\lim_{|z|\to 1} \mu(z)\big|u(z)\big| \log \frac{2}{1-|z|^2} = 0.$$

Proof. Similarly as in the proof of Theorem 2.3, this result is true. □

3. The operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$

Although we can obtain some results of the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ from the preceding discussions, we still will individually consider this operator.

Theorem 3.1 *Let $\alpha \in (0,1)$ and φ an entire function. Then the following assertions hold:*

(i) The operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded .

(ii) If $\varphi(0) = 0$, then the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is compact.

Proof. We first prove (i). Let $M > 0$, $f \in \mathcal{Z}_\alpha$ and $\|f\|_{\mathcal{Z}_\alpha} \le M$. Set $M_1 = \max_{|z|=C_\alpha M} |\varphi'(z)|$. Then we have

$$(1-|z|^2)^\beta \big|(S_\varphi f)'(z)\big| = (1-|z|^2)^\beta \big|\varphi'(f(z))\big|\big|f'(z)\big| \le C_\alpha M M_1 (1-|z|^2)^\beta < \infty.$$

This means that the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded.

Now we prove (ii). Suppose that $\|f_n\|_{\mathcal{Z}_\alpha} \le M$ and $\{f_n\}$ uniformly converges to zero on every compact subset of $\mathbb{D}$ as $n \to \infty$, then

$$\begin{aligned}\|S_\varphi f_n\|_{\mathcal{B}_\beta} &= |\varphi(f_n(0))| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta \big|(S_\varphi f_n)'(z)\big| \\ &= |\varphi(f_n(0))| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta \big|\varphi'(f_n(z))\big|\big|f_n'(z)\big| \\ &\le |\varphi(f_n(0))| + M_1 \sup_{z\in\mathbb{D}} |f_n'(z)|,\end{aligned}$$

where $M_1 = \max_{|z|=C_\alpha M} |\varphi'(z)|$. By $\varphi(0) = 0$ and Lemma 2.3 (ii), we know that $\lim_{n\to\infty} \|S_\varphi f_n\|_{\mathcal{B}_\beta} = 0$. By Lemma 2.4, the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is compact. $\square$

When $\alpha = 1$, from Theorem 2.1 and Theorem 2.2 we can obtain characterizations of the boundedness and compactness of the operator $S_\varphi : \mathcal{Z} \to \mathcal{B}_\beta$. It is unnecessary to go into details here.

Theorem 3.2 *Let $\alpha \in (1,2)$ and φ an entire function. We have the following assertions:*

(1) If $\alpha \le 1+\beta$, then (i) the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded, and
(ii) when $\varphi(0) = 0$, the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is compact.
(2) If $\alpha > 1+\beta$, then the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded if and only if φ is a constant function.

Proof. We first prove the assertion (i) of (1). Let $M > 0$, $f \in \mathcal{Z}_\alpha$ and $\|f\|_{\mathcal{Z}_\alpha} \le M$. Set $M_1 = \max_{|z|=C_\alpha M} |\varphi'(z)|$. Then we have

$$(1-|z|^2)^\beta \big|(S_\varphi f)'(z)\big| = (1-|z|^2)^\beta \big|\varphi'(f(z))\big|\big|f'(z)\big| \le CMM_1(1-|z|^2)^{1-\alpha+\beta} < \infty.$$

This shows that the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded. As the proof of Theorem 3.1 (ii), the assertion (ii) follows.

Note that we have the relation $\mathcal{Z}_\alpha = \mathcal{B}_{\alpha-1}$. By this and Theorem 4 in [5], the assertion (2) is true. $\square$

Theorem 3.3 *Let $\alpha = 2$ and φ an entire function.*

(1) When $\beta > 1$, (i) the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded if and only if φ is a polynomial of degree $s \le 1$, and
(ii) the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is compact.
(2) When $\beta = 1$, (i) the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded if and only if φ is a linear function, and
(ii) the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is compact.
(3) When $0 < \beta < 1$, the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded if and only φ is a constant function.

Proof. By Theorem 7 of [5], the assertions (i) of (1) and (i) of (2) hold. Also from Theorem 4 of [5], the assertion (3) follows. Now we want to prove the assertion (ii) of (1). Let the operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ be compact. From the assertion (i) of (1), we know that, if φ is not a constant function, then $\varphi(z) = az + b$ with $a \neq 0$. Therefore, it is enough to show that $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is compact when $\varphi(z) = az$. At this time, S_φ is just the multiplication operator M_a defined by $M_a f = a \cdot f$. Thus, by Theorem 3.1 of [19], we know that $M_a : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is compact. Similar to the proof of the assertion (ii) of (1), the assertion (ii) of (2) is right. □

Theorem 3.4 *Let $\alpha > 2$, $\beta > 1$ and φ an entire function.*

(1) The operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is bounded if and only if
(i) when $\alpha > \beta$, φ is a constant.
(ii) when $\alpha = \beta$, φ is a linear function.
(iii) when $\alpha < \beta$, φ is a polynomial of degree $s \leq \frac{\beta-1}{\alpha-2}$.

(2) The operator $S_\varphi : \mathcal{Z}_\alpha \to \mathcal{B}_\beta$ is compact if and only if φ is a polynomial of degree $s < \frac{\beta-1}{\alpha-2}$.

Proof. Note that when $\alpha > 2$, it follows that $\mathcal{Z}_\alpha = \mathcal{B}_{\alpha-1} = H_{\alpha-2}$, where $H_{\alpha-2}$ is called the weighted Banach space of analytic functions defined by

$$H_{\alpha-2} = \{f \in H(\mathbb{D}) : (1 - |z|^2)^{\alpha-2}|f(z)| < \infty\}.$$

Then (1) and (2) follow from Theorem 4.2 of [14] and Proposition 3.1 of [4]. □

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Dynamical Analysis Of The Rational Difference Equation $x_{n+1} = \frac{\alpha x_{n-3}}{A + B x_{n-1} x_{n-3}}$

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ABSTRACT

This article is concerned with the following rational difference equation $x_{n+1} = \frac{\alpha x_{n-3}}{A + B x_{n-1} x_{n-3}}$ with the initial conditions, $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, and $x_0 = a$ are arbitrary real numbers, α, A and B are arbitrary constants. A detailed analytical study of the convergence of the solutions including their dependence on parameters and initial conditions is investigated. The local stability and global attractivity of the difference equation's equilibrium points are discussed. The existence of periodic solutions in the proposed difference equation is also verified analytically. Moreover, numerical simulations are carried out to verify the correctness of the analytical results.

Keywords: Difference equations, Recursive sequences, Analytical study, Infinite products, Convergence, Periodic solution.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

Difference equations arise from the study of the evolution of natural phenomena. The applications of difference equations are rapidly increasing to various fields such as economics [1], [12]-[14], mathematical, biology [15]-[16] physics and engineering [7]. Indeed, difference equations represent chief tools of investigating the qualitative behaviors of dynamical systems [33]. Consequently, studying the solutions of difference equations and its qualitative behaviors have become focal topics for research [1]-[36].

In recent years, difference equations have been investigated by many authors. For some results: In [3], Aloqeili found the solution of the difference equation $x_{n+1} = \frac{d x_{n-1} x_{n-k}}{b - c x_{n-s}}$. Cinar [5] obtained the solution of the difference equation $x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}$. In [9], Elabbasy *et al.* discussed the solution and the periodicity character of the difference equations $x_{n+1} = a x_n - \frac{b x_n}{c x_n - d x_{n-1}}$.
In this paper, we study to the following sequence defined recursively by

$$x_{n+1} = \frac{\alpha x_{n-3}}{A + B x_{n-1} x_{n-3}}, \tag{1}$$

with the initial data: $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, and $x_0 = a$.
Note first that, if $\alpha = 0$, then for all $n \in \mathbb{N}$, $x_n = 0$. Then we will consider that $\alpha \neq 0$. Although we can (by dividing the numerator and denominator by α) obtain a more simply form of such sequences, we will keep them in order to study of the behaviors with respect to α.
Note also that, if one or more of the initial data a, b, c and d is zero, then it will be seen that one or more of the subsequences of $(x_n)_n$ modulo 4 vanish, so that we will suppose that $abcd \neq 0$.
The cases $A = 0$ and $B = 0$ are a trivial, therefore we will assume that $A \neq 0$ and $B \neq 0$. Finally, we will consider the convention: if $(a_p)_p$ is a sequence of complex numbers, and $n > m$, in $\mathbb{Z}$, then $\prod_{p=n}^{m} a_p = 1$.

2. DEFINITIONS AND PRELIMINARIES.

A difference equation of order k is an equation of the form

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-(k-1)}), n = 0, 1, ..., \tag{2}$$

where F is a function that maps on some set I^k into I. A solution of Eq. (2) is a sequence x_n that satisfies Eq. (2) for all $n \geq 0$. With each solution x_n of the Eq. (1), we associate the vector of initial conditions $v_0(x) = (x_0, x_{-1}, ..., x_{-k+1}) \in I^k$.
The norm of the vector $u \in I^k$ will be defined as $\|u\| = \sum_{i=-k+1}^{0} |u_i|$.

Definition 1. (Equilibrium point)
A point $\bar{x} \in \mathbb{R}$ is called an equilibrium point of Eq. (2), if

$$\bar{x} = F(\bar{x}, \bar{x}, ..., \bar{x}).$$

Let $\bar{x} \in \mathbb{R}$ be an equilibrium point of Eq. (2), and denote by $v(\bar{x}) \in I^k$ the vector $v(\bar{x}) = (\bar{x}, \bar{x}, ..., \bar{x})$.
Suppose that the function F is continuously differentiable in some open neighborhood of an equilibrium point $\bar{x}$. Consider the linearized equation of Eq. (2) about the equilibrium point $\bar{x}$:

$$y_{n+1} = q_0 y_n + q_1 y_{n-1} + ... + q_{k-1} y_{n-(k-1)}, \tag{3}$$

where $q_i = \frac{\partial F}{\partial x_i}(\bar{x}, \bar{x}, ..., \bar{x})$, $i = 0, 1, ..., k-1$, and the characteristic equation of Eq. (3) about $\bar{x}$:

$$\lambda^k - q_0 \lambda^{k-1} - ... - q_{k-2}\lambda - q_{k-1} = 0. \tag{4}$$

Definition 2.

1. When all the roots of Eq. (4) have absolute value less than one, then the equilibrium point of Eq. (2) is locally asymptotically stable.
2. If at least a root of Eq. (4) have absolute value greater than one, then the equilibrium point of Eq. (2) is unstable.

Definition 3.

1. An equilibrium point $\bar{x}$ of Eq. (2) is called hyperbolic if no root of Eq. (4) has absolute value equal one.
2. If there exists a root of Eq. (4) with absolute value equal to one, then the equilibrium point $\bar{x}$ is called nonhyperbolic.
3. An equilibrium point $\bar{x}$ of Eq. (2) is called saddle if there exists a root of Eq. (4) has absolute value less than one. and another root of Eq. (4) greater than one.
4. An equilibrium point $\bar{x}$ of Eq. (2) is called a repeller if all roots of Eq. (4) has absolute value greater than one.
5. A solution x_n of Eq. (2) is called nonoscillatory about $\bar{x}$ or simply nonoscillatory if there exists $N \geq -k$ such that either $x_n \geq \bar{x}$, $\forall n \geq N$ or $x_n \leq \bar{x}$, $\forall n \geq N$. Otherwise, the solution x_n is called oscillatory about $\bar{x}$, or simply oscillatory.
6. A solution x_n of Eq. (2) is called periodic with period p if there exists an integer p, such that

   $$x_{n+p} = x_n, \qquad \forall n \geq -k. \tag{5}$$

   A solution is called periodic with prime period p if p is the smallest positive integer for which Eq. (5) holds.

3. ANALYTICAL EXPRESSIONS OF $(X_N)_N$

The following Theorem gives an analytical expression of the sequence $(x_n)_n$.

Theorem 1. *Let $(x_n)_n$ be the sequence given by (1) and the initial data that follow, then For all $n \geq 2$*

$$x_{4n-3} = \frac{d\alpha^n \prod_{p=0}^{n-2} \left(A^{2p+2} + Bbd \sum_{i=0}^{2p+1} A^i \alpha^{2p+1-i} \right)}{\prod_{p=0}^{n-1} \left(A^{2p+1} + Bbd \sum_{i=0}^{2p} A^i \alpha^{2p-i} \right)}, \quad x_{4n-2} = \frac{c\alpha^n \prod_{p=0}^{n-2} \left(A^{2p+2} + Bac \sum_{i=0}^{2p+1} A^i \alpha^{2p+1-i} \right)}{\prod_{p=0}^{n-1} \left(A^{2p+1} + Bac \sum_{i=0}^{2p} A^i \alpha^{2p-i} \right)}. \quad (6)$$

$$x_{4n-1} = \frac{b\alpha^n \prod_{p=0}^{n-1} \left(A^{2p+1} + Bbd \sum_{i=0}^{2p} A^i \alpha^{2p-i} \right)}{\prod_{p=0}^{n-1} \left(A^{2p+2} + Bbd \sum_{i=0}^{2p+1} A^i \alpha^{2p+1-i} \right)}, \quad x_{4n} = \frac{a\alpha^n \prod_{p=0}^{n-1} \left(A^{2p+1} + Bac \sum_{i=0}^{2p} A^i \alpha^{2p-i} \right)}{\prod_{p=0}^{n-1} \left(A^{2p+2} + Bac \sum_{i=0}^{2p+1} A^i \alpha^{2p+1-i} \right)}. \quad (7)$$

Proof. By induction, we prove the result for x_{4n-3}. Take $n \geq 2$, and assume that the results hold for the step n, then prove the result for the step $n+1$, we get:

$$\begin{aligned}
x_{4(n+1)-3} &= \frac{\alpha x_{4n-3}}{A + Bx_{4n-1}x_{4n-3}} \\
&= \frac{d\alpha^{n+1} \prod_{p=0}^{n-1} \left(A^{2p+2} + Bbd \sum_{i=0}^{2p+1} A^i \alpha^{2p+1-i} \right)}{\prod_{p=0}^{n-1} \left(A^{2p+1} + Bbd \sum_{i=0}^{2p} A^i \alpha^{2p-i} \right) \left[A \left(A^{2n} + Bbd \sum_{i=0}^{2n-1} A^i \alpha^{2n-1-i} \right) + Bbd\alpha^{2n} \right]} \\
&= \frac{d\alpha^{n+1} \prod_{p=0}^{n-1} \left(A^{2p+2} + Bbd \sum_{i=0}^{2p+1} A^i \alpha^{2p+1-i} \right)}{\prod_{p=0}^{n-1} \left(A^{2p+1} + Bbd \sum_{i=0}^{2p} A^i \alpha^{2p-i} \right) \left(A^{2n+1} + Bbd \left(\sum_{i=1}^{2n} A^i \alpha^{2n-i} + \alpha^{2n} \right) \right)}.
\end{aligned}$$

Hence, we obtain

$$x_{4(n+1)-3} = \frac{d\alpha^{n+1} \prod_{p=0}^{n-1} \left(A^{2p+2} + Bbd \sum_{i=0}^{2p+1} A^i \alpha^{2p+1-i} \right)}{\prod_{p=0}^{n} \left(A^{2p+1} + Bbd \sum_{i=0}^{2p} A^i \alpha^{2p-i} \right)}.$$

Similarly, the expression for x_{4n-2}, x_{4n-1}, x_{4n} can be easily proved.

Notation. If we denote by $(P_n)_n$ the sequence of two variables polynomials defined for every $n \in \mathbb{N}$, x and y as,

$$P_n(x, y) = (A - \alpha + Bxy)A^n - Bxy\alpha^n.$$

The following Corollary gives a simplified analytic expression when $A \neq \alpha$.

Corollary 1. *Consider the sequence $(x_n)_n$ defined by the Eq. (1) for $A \neq \alpha$, the subsequences can be written as:*

$$x_{4n-3} = \frac{d\alpha^n (A - \alpha) \prod_{p=0}^{n-2} P_{2p+2}(b, d)}{\prod_{p=0}^{n-1} P_{2p+1}(b, d)}, \quad x_{4n-2} = \frac{c\alpha^n (A - \alpha) \prod_{p=0}^{n-2} P_{2p+2}(a, c)}{\prod_{p=0}^{n-1} P_{2p+1}(a, c)},$$

$$x_{4n-1} = \frac{b\alpha^n \prod_{p=0}^{n-1} P_{2p+1}(b,d)}{\prod_{p=0}^{n-1} P_{2p+2}(b,d)}, \qquad and \qquad x_{4n} = \frac{a\alpha^n \prod_{p=0}^{n-1} P_{2p+1}(a,c)}{\prod_{p=0}^{n-1} P_{2p+2}(a,c)}.$$

Proof. It is sufficient to use the binomial identity $x^{p+1} - y^{p+1} = (x-y)\sum_{k=0}^{p} x^k y^{p-k}$ in the analytical expression of the subsequences defined by Eq. (6) and (7).

Corollary 2. *Consider the sequence $(x_n)_n$ defined by the Eq. (1). For $A = \alpha \neq 0$, the sequence can be expressed in Gamma form as*

$$x_{4n-3} = \frac{A2^{2n-2}\Gamma^2(\frac{A}{2Bbd}+n)\Gamma(\frac{A}{Bbd}+1)}{Bb\Gamma^2(\frac{A}{2Bbd}+1)\Gamma(\frac{A}{Bbd}+2n)}, \qquad x_{4n-2} = \frac{A2^{2n-2}\Gamma^2(\frac{A}{2Bac}+n)\Gamma(\frac{A}{Bac})}{Ba\Gamma^2(\frac{A}{2Bac}+1)\Gamma(\frac{A}{Bac}+2n)},$$

$$x_{4n-1} = \frac{b\Gamma(\frac{A}{Bbd}+2n+1)\Gamma^2(\frac{A}{2Bbd}+1)}{2^{2n}\Gamma(\frac{A}{Bbd}+1)\Gamma^2(\frac{A}{2Bbd}+n+1)}, \qquad x_{4n} = \frac{a\Gamma(\frac{A}{Bac}+2n+1)\Gamma^2(\frac{A}{2Bac}+1)}{2^{2n}\Gamma(\frac{A}{Bac}+1)\Gamma^2(\frac{A}{2Bac}+n+1)},$$

where Γ is the Euler's Gamma function.

Proof. Using Eq. (6) we have:

$$\begin{aligned}
x_{4n-3} &= \frac{dA^n \prod_{p=0}^{n-2}\left(A^{2p+2} + Bbd\sum_{i=0}^{2p+1} A^{2p+1}\right)}{\prod_{p=0}^{n-1}\left(A^{2p+1} + Bbd\sum_{i=0}^{2p} A^{2p}\right)}, \\
&= \frac{dA\prod_{p=0}^{n-2} Bbd\Big(\frac{A}{Bbd}+2p+2\Big)}{\prod_{p=0}^{n-1} Bbd\Big(\frac{A}{Bbd}+2p+1\Big)} = \frac{A\Big[\prod_{p=1}^{n-1} 2\Big(\frac{A}{2Bbd}+p\Big)\Big]^2}{Bb\prod_{p=1}^{2n-1}\Big(\frac{A}{Bbd}+p\Big)} \\
&= \frac{A2^{2n-2}\Gamma^2\Big(\frac{A}{2Bbd}+n\Big)\Gamma(\frac{A}{Bbd}+1)}{Bb\,\Gamma\Big(\frac{A}{Bbd}+2n\Big)\Gamma^2\Big(\frac{A}{2Bbd}+1\Big)}.
\end{aligned}$$

Similarly, one can prove the other relations. This ended the proof.

Remark 1.

1. A common hypothesis in the study of rational difference equations is the choice of positive coefficients and initial data. Therefore, all the solutions will be automatically well defined. It is, in general a problem of great difficulty to determine the good set of initial conditions without finding the analytical expression of the considered sequence.

2. According to the Corollaries 1 and 2, the good set G of the sequence $(x_n)_n$ is given as

   (a) When $A \neq \alpha$,

$$G = \left\{(a,b,c,d) \in \mathbb{R}^4 \text{ such that } bd,\ ac \in \mathbb{R} - \left\{\frac{-(A-\alpha)A^n}{B(A^n - \alpha^n)}, \quad n \in \mathbb{N}\right\}\right\}.$$

(b) When $A=\alpha$, $G=\{(a,b,c,d)\in\mathbb{R}^4$ such that $\frac{A}{Bbd}, \frac{A}{Bac}\notin 2\mathbb{Z}_-\}$.

3. If we choose for example $\alpha=A=B$, we obtain the expression of the general term which can be written and in gamma form as

$$x_{4n-3}=\frac{2^{2n-2}\Gamma^2(\frac{1}{2bd}+n)\Gamma(\frac{1}{bd})}{b\Gamma^2(\frac{1}{2bd}+1)\Gamma(\frac{1}{bd}+2n)},\qquad x_{4n-2}=\frac{2^{2n-2}\Gamma^2(\frac{1}{2ac}+n)\Gamma(\frac{1}{ac})}{a\Gamma^2(\frac{1}{2ac}+1)\Gamma(\frac{1}{ac}+2n)},$$

$$x_{4n-1}=\frac{b\Gamma(\frac{1}{bd}+2n+1)\Gamma^2(\frac{1}{2bd}+1)}{2^{2n}\Gamma(\frac{1}{bd}+1)\Gamma^2(\frac{1}{2bd}+n+1)},\qquad x_{4n}=\frac{a\Gamma(\frac{1}{ac}+2n+1)\Gamma^2(\frac{1}{2ac}+1)}{2^{2n}\Gamma(\frac{1}{ac}+1)\Gamma^2(\frac{1}{2ac}+n+1)}.$$

In the following section we will study the convergence of sequence $(x_n)_n$. This will depend evidently on the parameters α, A, B and the initial data.

4. CONVERGENCE OF SOLUTIONS OF *EQ. (1)*

Consider the function F defined on $\mathbb{R}^4$ as: $F(u_0,u_1,u_2,u_3)=\frac{\alpha u_3}{A+Bu_1u_3}$. Using the function F, Eq. (1) can be written as $x_{n+1}=F(x_n,x_{n-1},x_{n-2},x_{n-3})$.

Theorem 2. The following statements are true:

(1) For $B(A-\alpha)\geq 0$, Eq.(1) has a unique equilibrium point $\overline{x}=0$, then

(a) If $A=\alpha$, the equilibrium point is nonhyperbolic.

(b) If $\frac{A}{\alpha}>1$, the equilibrium point is locally asymptotically stable.

(2) For $B(A-\alpha)<0$, then

(a) The Eq. (1) has exactly three equilibrium points which are

$$\overline{x}_1=0,\ \overline{x}_2=\sqrt{\tfrac{\alpha-A}{B}},\ \overline{x}_3=-\sqrt{\tfrac{\alpha-A}{B}}. \tag{8}$$

(b) If $0<A<\alpha$, then

(i) The equilibrium point $\overline{x}_1=0$ is a repeller.

(ii) The equilibrium points $\overline{x}_2$, $\overline{x}_3$ are hyperbolic.

Proof. (1) For $B(A-\alpha)\geq 0$, $\bar{x}$ is an equilibrium point is equivalent to

$$\bar{x}=\frac{\alpha\bar{x}}{A+B\bar{x}^2}\quad\Rightarrow\ B\bar{x}^3+(A-\alpha)\bar{x}=0\quad\Rightarrow\ \bar{x}(B\bar{x}^2+A-\alpha)=0.$$

This shows clearly that if $B(A-\alpha)\geq 0$, $\overline{x}=0$ is the unique equilibrium point of Eq. (1).
$q_i=\frac{\partial F}{\partial u_i}(0,0,0,0)$, then $q_0=q_1=q_2=0$ and $q_3=-\frac{\alpha}{A}$, the characteristic equation of the linearized equation associated with Eq. (1) is then all real roots have absolute value equal to one, so the equilibrium points is nonhyperbolic.$\bar{x}$ is an equilibrium point is equivalent to

$$\lambda^4-\frac{\alpha}{A}=0. \tag{9}$$

(a) Suppose that $A=\alpha$, then all real roots have absolute value equal to one, so the equilibrium points is nonhyperbolic.

(b) Suppose that $\frac{A}{\alpha}>1$, so all the roots of Eq. (9) have absolute value less than one, according the linearized stability Theorem, the equilibrium point $\overline{x}=0$ is locally asymptotically stable.

(2) For $B(A-\alpha)<0$, the equation $\bar{x}(B\bar{x}^2+A-\alpha)=0$ has exactly three solutions which are the equilibrium points in Eqs. (8).

(a) The characteristic equation about $\overline{x}_1 = 0$ is $\lambda^4 - \frac{\alpha}{A} = 0$, since $0 < A < \alpha$ then all roots has absolute value greater than one and $\overline{x}_1 = 0$ is repeller.

(b) The characteristic equation about $\overline{x}_2$ is $\lambda^4 + \frac{\alpha - A}{A}\lambda^2 - \frac{A}{\alpha} = 0$. The real roots of this equation are $\sqrt{\frac{A}{\alpha}}$ and $-\sqrt{\frac{A}{\alpha}}$, they are less than one, so the equilibrium point $\overline{x}_2$ is hyperbolic. The proof for $\overline{x}_3$ can be similarly obtained.

As it is expected, the convergence of $(x_n)_n$ depends on the parameters α, A, B, and the initial data. We will distinguish the following cases:

(i) Case $|\frac{A}{\alpha}| > 1$.

Theorem 3. *Assume that* $|\frac{A}{\alpha}| > 1$

(1) If $(A - \alpha + Bbd)(A - \alpha + Bac) \neq 0$, then every solution of Eq. (1) converges toward zero.

(2) If $A - \alpha + Bbd = A - \alpha + Bac = 0$, then the solution of Eq. (1) converges iff $a = b = c = d = \pm\sqrt{\frac{\alpha - A}{B}}$.

(3) If $(A - \alpha + Bbd)(A - \alpha + Bac) = 0$ but not both terms of the product are zero, then every solution of Eq. (1).

Proof. (1) Suppose that $(A - \alpha + Bbd)(A - \alpha + Bac) \neq 0$, then Corollary 1 implies that

$$\begin{aligned} x_{4n-3} &= \frac{d\alpha^n (A-\alpha) \prod_{p=0}^{n-2} \left(A^{2p+2}(A-\alpha+Bbd) - Bbd\alpha^{2p+2} \right)}{\prod_{p=0}^{n-1} \left(A^{2p+1}(A-\alpha+Bbd) - Bbd\alpha^{2p+1} \right)} \\ &= \frac{d\alpha^n (A-\alpha) A^{n-1} \prod_{p=0}^{n-2} \left(1 - \frac{Bbd}{A-\alpha+Bbd} (\frac{\alpha}{A})^{2p+2} \right)}{(A-\alpha+Bbd) A^{2n-1} \prod_{p=0}^{n-2} \left(1 - \frac{Bbd}{A-\alpha+Bbd} (\frac{\alpha}{A})^{2p+1} \right)}. \end{aligned}$$

Denote by $\beta = \frac{Bbd}{A-\alpha+Bbd}$ and by $(U_p)_p$ the sequence defined as $U_p = \frac{1-\beta(\frac{\alpha}{A})^{2p+2}}{1-\beta(\frac{\alpha}{A})^{2p+1}}$, we get

$$x_{4n-3} = \frac{d(\frac{\alpha}{A})^n (A-\alpha)}{(A-\alpha+Bbd)\left(1-\beta(\frac{\alpha}{A})^{2n-1}\right)} \prod_{p=0}^{n-2} U_p.$$

We have either: for $p \in \mathbb{N}$ big enough, $U_p > 1$ or for $p \in \mathbb{N}$ big enough, $0 < U_p < 1$.
Using Taylor expansion of the U_p, we obtain

$$U_p = (1 - \beta(\frac{\alpha}{A})^{2p+2})(1 + \beta(\frac{\alpha}{A})^{2p+1} + o(\frac{\alpha}{A})^{2p+1}) = 1 + \beta(\frac{\alpha}{A})^{2p+1} + o(\frac{\alpha}{A})^{2p+1},$$

then U_p is equivalent to $1 + \beta(\frac{\alpha}{A})^{2p+1}$ which is the general term of a convergent infinite product.
We can easily deduce that $(x_{4n-3})_n$ converges toward zero. same discussion can be obtained for the other subsequences.

(2) If $A - \alpha + Bbd = A - \alpha + Bac = 0$, then by the proof of (1), the subsequences $(x_{4n-3})_n$ and $(x_{4n-1})_n$ are constants: $x_{4n-3} = d$ and $x_{4n-1} = b$, also the subsequences $x_{4n-2} = c$ and $x_{4n} = a$. Thus every solution of Eq. (1) converges to a real number l if and only if $a = b = c = d = l$.

(3) Consider for instance the case $A - \alpha + Bbd = 0$ and $A - \alpha + Bac \neq 0$, by (2), the subsequences $(x_{4n-3})_n$ and $(x_{4n-1})_n$ are constants $x_{4n-3} = d$ and $x_{4n-1} = b$, in other hand and also by the proof of case (1), the subsequences $(x_{4n-2})_n$ and $(x_{4n})_n$ converge to zero, then the sequence $(x_n)_n$ diverges. The proof is completed.

(ii) Case $|\frac{A}{\alpha}| = 1$.

Theorem 4. *Assume that* $|\frac{A}{\alpha}| = 1$. We distinguish two subcases, $A = \alpha$ and $A = -\alpha$.

(1) If $A = \alpha$, and let sequence $(x_n)_n$ be the sequence given by the formula (1), then the sequence $(x_n)_n$ converges toward zero.

(2) If $A = -\alpha$, and let sequence $(x_n)_n$ be the sequence given by the formula (1), then we have $x_{4n-1} = \frac{b}{dx_{4n-3}}$, $x_{4n-2} = \frac{c}{ax_{4n}}$ and the sequence $(x_n)_n$ is divergent.

Proof. (1) For $A = \alpha$, let δ the parameter $\delta = \frac{A}{Bbd}$. In the proof of Corollary 2, we find that

$$x_{4n-3} = \tfrac{A}{Bb(\delta+1)} \prod_{p=1}^{n-1} \Big(\frac{\frac{\delta}{2p}+1}{\frac{\delta+1}{2p}+1} \Big).$$

Denote by $(W_p)_p$ the sequence defined as $W_p = \frac{\frac{\delta}{2p}+1}{\frac{\delta+1}{2p}+1}$, then we get:
For p big enough, we have $0 < W_p < 1$. The Taylor expansion for W_p gives:
$W_p = (1 + \frac{\delta}{2p})(1 - \frac{\delta+1}{2p} + o(\frac{1}{p})) = 1 - \frac{1}{2p} + o(\frac{1}{p})$,
which is a general term of divergent infinite product. Since for p big enough, $0 < W_p < 1$, then $\lim_{n\to\infty} \Pi_{p=1}^{n-1} W_p = 0$. So, we get $\lim_{n\to\infty} x_{4n-3} = 0$. Similarly, one can easily prove that the other subsequences converge to zero, therefore the sequence $(x_n)_n$ converges to zero.

(2) To prove the second part, we replace α by $(-A)$ in the expression of x_{4n-3} of Eq. (6), we obtain

$$x_{4n-3} \quad = \quad \frac{d(-A)^n \prod_{p=0}^{n-2} A^{2p+1}\Big(A+Bbd\sum_{k=0}^{2p+1}(-1)^k\Big)}{\prod_{p=0}^{n-1} A^{2p}\Big(A+Bbd\sum_{k=0}^{2p}(-1)^k\Big)} = \frac{d}{(-1-\delta^{-1})^n}, \ .$$

In other hand, If we replace α by $(-A)$ in the first term of Eq. (7), we obtain

$$x_{4n-1} \quad = \quad b(-A)^n \prod_{p=0}^{n-1} \Big(\frac{A^{2p}(A+Bbd)}{A^{2p+2}} \Big) = b(-1-\delta^{-1})^n.$$

Thus $x_{4n-1} = \frac{b}{dx_{4n-3}}$, hence

(a) If $|1 + \delta^{-1}| > 1$, then the subsequence $(x_{4n-3})_n$ converges to zero, so $(|x_{4n-1}|)_n$ goes to infinity.

(b) If $|1 + \delta^{-1}| < 1$, then the subsequence $(|x_{4n-3}|)_n$ goes to infinity.

This completed the proof.

(iii) Case $|\frac{A}{\alpha}| < 1$.

Theorem 5. *Let $(x_n)_n$ be the sequence given by the formula (1), then*
For $|\frac{A}{\alpha}| < 1$, then the subsequences $(x_{4n-3})_n$, $(x_{4n-1})_n$, $(x_{4n-2})_n$ and $(x_{4n})_n$ converge.

Proof. We need to prove that $(x_{4n-3})_n$ converges. Using Corollary (1), we obtain

$$x_{4n-3} \quad = \quad \frac{d\alpha^n(A-\alpha)\prod_{p=0}^{n-2}\Big(A^{2p+2}(A-\alpha+Bbd)-Bbd\alpha^{2p+2}\Big)}{\prod_{p=0}^{n-1}\Big(A^{2p+1}(A-\alpha+Bbd)-Bbd\alpha^{2p+1}\Big)} = \frac{\alpha-A}{Bb(1-\gamma\lambda^{2n-1})}\prod_{p=0}^{n-2} V_p,$$

where $\gamma = \frac{A-\alpha+Bbd}{Bbd}$, $\lambda = \frac{A}{\alpha}$ and $(V_p)_p$ is the sequence defined by $V_p = \frac{1-\gamma\lambda^{2p+2}}{1-\gamma\lambda^{2p+1}}$. For $p \in \mathbb{N}$ big enough, we have two cases; either $V_p > 1$ or $0 < V_p < 1$. Applying the transformation of infinite product of positive terms to infinite series, and assuming p_0 to be big enough, we get

$$x_{4n-3} = \frac{\alpha - A}{Bb(1-\gamma\lambda^{2n-1})}\Big(\prod_{p=0}^{p_0} V_p\Big)\exp\Big(\sum_{p=p_0+1}^{n-2} \ln(V_p)\Big).$$

It is clear that the sequence $\Big(\frac{\alpha-A}{Bb(1-\gamma\lambda^{2n-1})}\Big)_n$ converges toward $\frac{\alpha-A}{Bb}$. The Taylor expansion of V_p to the first order gives

$$V_p = \tfrac{1-\gamma\lambda^{2p+2}}{1-\gamma\lambda^{2p+1}} = 1 + \gamma(1-\lambda)\lambda^{2p+1} + o(\lambda^{2p+1}).$$

So $\ln(V_p)$ is equivalent to $\gamma(1-\lambda)\lambda^{2p+1}$, which is the general term of a convergent infinite series, then the sequence $(x_{4n-3})_n$ is convergent. Similarly, one can prove that the other subsequences are convergent.

Remark 2. (Commentary on the convergence of $(x_n)_n$ in the case $|\frac{A}{\alpha}| < 1$).
Suppose that $|\frac{A}{\alpha}| < 1$, according to Theorem 5, the subsequences $(x_{4n-3})_n$, $(x_{4n-1})_n$, $(x_{4n-2})_n$ and $(x_{4n})_n$ converge, denote by: l_3, l_2, l_1 and l_0 their limits respectively.
The subsequences $(x_{4n-3})_n$ and $(x_{4n-1})_n$ are related by the equations:

$$x_{4(n+1)-3} = \frac{\alpha x_{4n-3}}{A+Bx_{4n-1}x_{4n-3}}, \tag{10}$$

$$x_{4(n+1)-1} = \frac{\alpha x_{4n-1}}{A+Bx_{4(n+1)-3}x_{4n-1}}. \tag{11}$$

Passing to the limit as n goes to infinity in Eq. (10), we obtain $l_3 = \frac{\alpha l_3}{A+Bl_3 l_1}$, then $(S_1): \begin{cases} l_3 = 0, \\ or \\ l_3 \neq 0 \text{ and } l_1 = \frac{\alpha - A}{Bl_3}. \end{cases}$

Passing to the limit as n goes to infinity in Eq. (11), we obtain $l_1 = \frac{\alpha l_1}{A+Bl_3 l_1}$, then $(S_2): \begin{cases} l_1 = 0, \\ or \\ l_1 \neq 0 \text{ and } l_3 = \frac{\alpha - A}{Bl_1}. \end{cases}$

Combining systems (S_1) and (S_2), since $\alpha \neq A$, we obtain

$$\begin{cases} l_3 = l_1 = 0 \\ or \\ l_1 \neq 0, \quad l_3 \neq 0 \text{ and } (S): \begin{cases} l_3 = \frac{\alpha - A}{Bl_1}, \\ and \\ l_1 = \frac{\alpha - A}{Bl_3}. \end{cases} \end{cases}$$

The proposition $l_3 = l_1 = 0$ contradicts the fact that the infinite product $\prod_{p\geq 0} V_p$ converges, in fact if $\lim_{n\to\infty} \prod_{p=0}^{n} V_p = 0$, then $\lim_{n\to\infty} \sum_{p=p_0}^{n} \ln(V_p) = -\infty$, and this is absurd. Hence the only possibility is that

$$l_1 \neq 0, \quad l_3 \neq 0 \text{ and } (S): \begin{cases} l_3 = \frac{\alpha - A}{Bl_1}, \\ and \\ l_1 = \frac{\alpha - A}{Bl_3}. \end{cases}$$

One can easily see that (S) is equivalent to $l_3 = \frac{\alpha - A}{Bl_1}$. Let f be the function defined on $\mathbb{R}^*$ as $f(x) = \frac{\alpha - A}{Bx}$, we have $fof = Id$ and, l_1 and l_3 are related by $f(l_1) = l_3$.

$$f(x) = x \Leftrightarrow \frac{\alpha - A}{Bx} = x \Leftrightarrow x = \mp\sqrt{\frac{\alpha - A}{B}}.$$

Hence: f has fixed points if and only if $\frac{\alpha - A}{B} > 0$.
The numerical example (Figure 4) given in the end of this paper confirm that even we chose $\frac{\alpha - A}{B} > 0$ and $|\frac{A}{\alpha}| < 1$, l_1 and l_3 may be different, which implies the sequence $(x_n)_n$ may converge or diverge.
Finally based on the preview discussion of all preview cases, The following Theorem is now proved.
Theorem 6. (Boundedness of $(x_n)_n$). *The Eq. (1) has an unbounded solutions if and only if $A = -\alpha$.*

5. PERIODICITY CHARACTER OF SOLUTIONS OF *EQ. (1)*

In the sequel, we need the following lemma, which describes sufficient conditions for Eq. (1) to have a periodic solution.
Lemma 1. *Let $(x_n)_{n\geq -3}$ be a solution of Eq. (1) and the initial data that follow. Suppose that there are real numbers l_3, l_2, l_1, l_0 such that $\lim_{n\to\infty} x_{4n-j} = l_j$ for $j = 0, ..., 3$.*

Let $(y_n)_{n\geq -3}$ be the period-4 sequence such that $y_{-j} = l_j$, for all $j = 0, ..., 3$, then the sequence $(y_n)_{n\geq -3}$ is a period-4 solution of Eq. (1). The periodicity results are given by the following Theorem

Theorem 7. *Let $(x_n)_{n\geq -3}$ be a solution of Eq. (1) and the initial data that follow, then*

(1) For $|\frac{A}{\alpha}| > 1$,

(a) If $(A - \alpha + Bbd)(A - \alpha + Bac) \neq 0$, then then Eq. (1) has no periodic solutions.

(b) If $A - \alpha + Bbd = A - \alpha + Bac = 0$, then the solution of Eq. (1) is a periodic-4 solution.

(c) If either $A - \alpha + Bbd$ or $A - \alpha + Bac$ equals zero but not both of them, then Eq. (1) has a periodic-4 solution.

(2) For $|\frac{A}{\alpha}| = 1$, Eq. (1) has no periodic solutions.

(3) For $|\frac{A}{\alpha}| < 1$, Eq. (1) has periodic-4 solutions.

Proof. (1) Suppose that $|\frac{A}{\alpha}| > 1$,

(a) If $(A - \alpha + Bbd)(A - \alpha + Bac) \neq 0$, then by Theorem 3, every solution of Eq. (1) converges to zero, hence, the solutions are not allowed to be periodic (since the solutions are not identically zero).

(b) If $A - \alpha + Bbd = A - \alpha + Bac = 0$, then by Theorem 3, the subsequences of $(x_n)_n$ $(x_{4n-j})_n$, $j = 0, .., 3$ are constants: $x_{4n-3} = d$, $x_{4n-2} = c$, $x_{4n-1} = b$ and $x_{4n} = a$, and the sequence $d, c, b, a, d, c, b, a...$ is a periodic-4 solution of Eq. (1).

(c) Consider for instance the case $A - \alpha + Bbd = 0$ and $A - \alpha + Bac \neq 0$, by the proof of Theorem 3, the subsequences $(x_{4n-3})_n$ and $(x_{4n-1})_n$ are constants and equal d and b respectively. Also according to the proof of Theorem 3, the subsequences $(x_{4n-2})_n$ and $(x_{4n})_n$ converge to zero. Applying Lemma 1, the sequence $d, 0, b, 0, d, 0, b, 0, ...$ is a periodic-4 solution of Eq. (1).

(2) The case $A = \alpha$ is similar to (1) (a).
If $A = -\alpha$, then every solution of Eq. (1) is unbounded, so Eq. (1) has no periodic solutions.

(3) If $|\frac{A}{\alpha}| < 1$, then by Theorem 5, there are real numbers l_3, l_2, l_1 and l_0, such that $\lim_{n\to\infty} x_{4n-j} = l_j$ for all $j = 0, .., 3$.
Applying Lemma 1, the sequence $l_3, l_2, l_1, l_0, l_3, l_2, l_1, l_0...$ is a periodic-4 solution of Eq. (1).
This completes the proof.

Remark 3.

(1) Note that if $|\frac{A}{\alpha}| > 1$, $A - \alpha + Bbd = A - \alpha + Bac = 0$, $a = c$, $b = d$, then Eq. (1) has periodic-2 solution $a, b, a, b, ...$

(2) If $|\frac{A}{\alpha}| < 1$, $A - \alpha + Bbd = A - \alpha + Bac = 0$, then, by the proof of Theorem 7, we deduce that the values of the limits of the subsequences are $l_3 = d$, $l_2 = c$, $l_1 = b$ and $l_0 = a$.

6. NUMERICAL SIMULATION

Example 1. Figure (1) illustrates the case $|\frac{A}{\alpha}| > 1$, $(A - \alpha + Bbd)(A - \alpha + Bac) \neq 0$, we choose $a = 2$, $b = -3$, $c = 2$, $d = -2$, $B = 2$, $A = 1.1$ and $\alpha = 1$. We notice that the solution is oscillating about zero with a decreasing amplitude. In fact, according to Theorem 3, the solution has to converge to zero.

Example 2. In order illustrate the case $|\frac{A}{\alpha}| > 1$, $A - \alpha + Bbd = A - \alpha + Bac = 0$, we choose $a = c = 2$, $b = d = -2$, $B = -3$, $A = 13$ and $\alpha = 1$. Figure (2) depicts that the obtained solution is a 2-prime periodic solution. This is coherent with Remark 3.

Example 3. The case $|\frac{A}{\alpha}| > 1$, $A - \alpha + Bbd = 0$ and $A - \alpha + Bac \neq 0$ is illustrated in figure (3), in which we set $a = c = 1$, $b = d = -2$, $B = -2$, $A = 9$ and $\alpha = 1$. The subsequences $(x_{4n-3})_n$ and $(x_{4n-1})_n$ are constants $(x_{4n-3})_n = d$ and $(x_{4n-1})_n = b$, and the subsequences $(x_{4n-2})_n$ and $(x_{4n})_n$ converge to zero. by Lemma 1, the sequence $d, 0, b, 0, d, 0, b, 0, ...$ is a periodic-4 solution of Eq. (1).

Example 4. Figure (4) illustrates the case $|\frac{A}{\alpha}| < 1$, we choose $a = -1$, $b = 0.5$, $c = -0.2$, $d = 0.8$, $B = 1$, $A = 0.5$ and $\alpha = 1$. the subsequences $(x_{4n-3})_n$, $(x_{4n-1})_n$, $(x_{4n-2})_n$ and $(x_{4n})_n$ converge.

Example 5. To illustrate the case $A = \alpha$, we choose $a = 0.1$, $b = 0.2$, $c = 0.3$, $d = -0.4$, $B = 1$, $\alpha = 0.5$ and $A = 0.5$. We notice in the figure (5), that the solution converges to zero (which is coherent to Theorem 4 part (1)), and the Eq. (1) has no periodic solutions (which is coherent to Theorem 7 part (2)).

Example 6. In figure (6) (case $A = -\alpha$), we choose $a = 0.2$, $b = 0.3$, $c = 0.1$, $d = -0.3$, $B = 2$, $\alpha = -0.4$ and $A = 0.4$. We notice that the solution is oscillating about zero with an increasing amplitude and the solution is unbounded, which is coherent to Theorem 4 part (2).

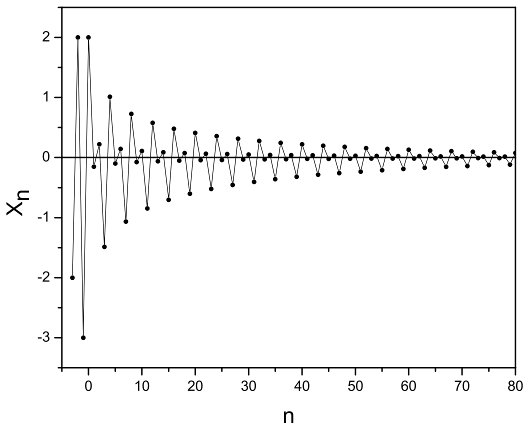

Figure 1.

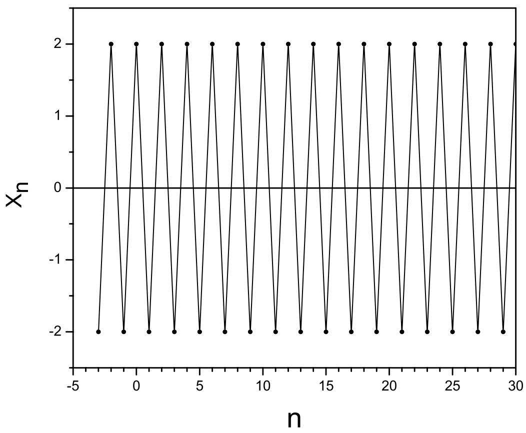

Figure 2.

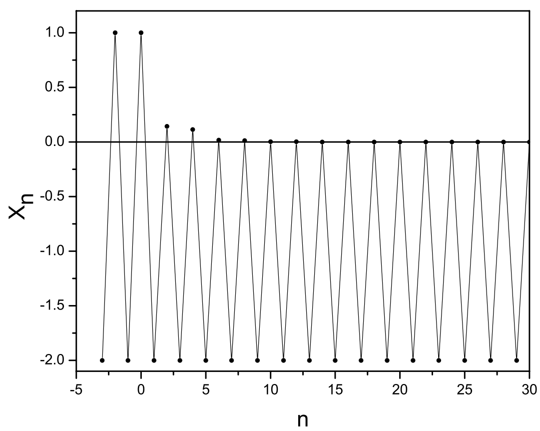

Figure 3.

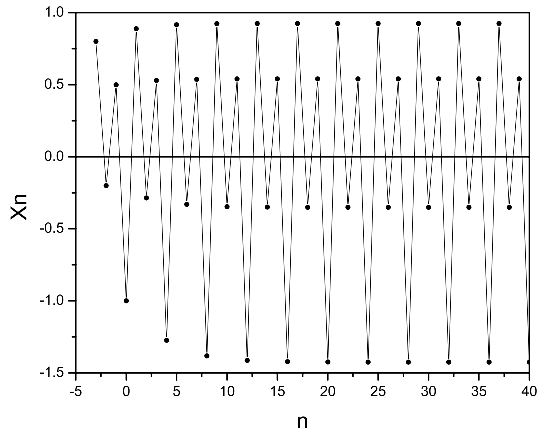

Figure 4.

Conclusion

In this work, some dynamical behaviors of the rational difference equation $x_{n+1} = \frac{\alpha x_{n-3}}{A+Bx_{n-1}x_{n-3}}$ with the initial conditions, $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, and $x_0 = a$ are arbitrary real numbers, A and B are arbitrary constants, have been investigated. A detailed analytical study of the convergence of the solutions including their dependence on parameters and initial conditions has been illustrated. The local stability and global attractivity of the difference equation's equilibrium points have been demonstrated. The existence of periodic solutions in the proposed difference equation has also been shown analytically. Finally, numerical simulations have been carried out to match the analytical results.

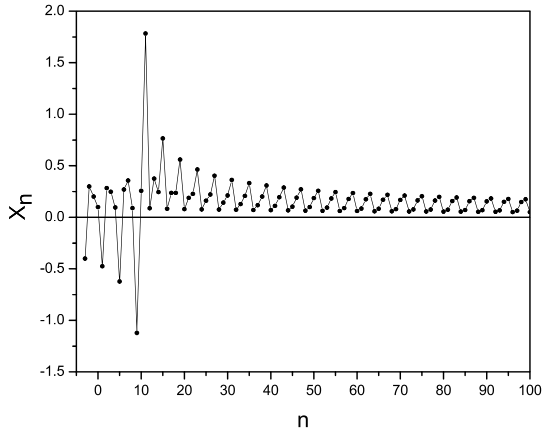

Figure 5.

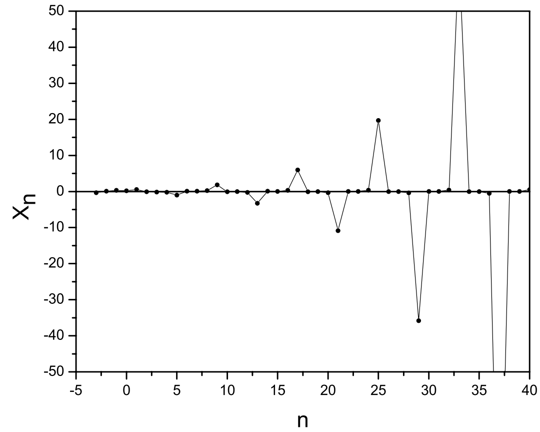

Figure 6.

Acknowledgements

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QUADRATIC ρ-FUNCTIONAL EQUATIONS

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Abstract. In this paper, we solve the quadratic ρ-functional equations

$$\begin{aligned} & f(x+y)+f(x-y)-2f(x)-2f(y) \\ & \quad = \rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right), \end{aligned} \tag{0.1}$$

where ρ is a fixed non-Archimedean number or a fixed real or complex number with $\rho \neq -1, 2$, and

$$\begin{aligned} & 2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y) \\ & \quad = \rho(f(x+y)+f(x-y)-2f(x)-2f(y)), \end{aligned} \tag{0.2}$$

whereρ is a fixed non-Archimedean number or a fixed real or complex number with $\rho \neq -1, \frac{1}{2}$.

Using the direct method, we prove the Hyers-Ulam stability of the quadratic ρ-functional equations (0.1) and (0.2) in non-Archimedean Banach spaces and in Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [25] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [13] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [22] for linear mappings by considering an unbounded Cauchy difference. Gajda [11] following the same approach as in Rassias [22], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [11], as well as by Rassias and Šemrl [21] that one cannot prove a Rassias' type theorem when $p = 1$. The counterexamples of Gajda [11], as well as of Rassias and Šemrl [21] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, cf. Găvruta [12], who among others studied the Hyers-Ulam stability of functional equations (cf. the books of Czerwik [8, 9], Hyers, Isac and Th.M. Rassias [14]). The hyperstability of the Cauchy equation was proved by Brzdek [4].

The functional equation

$$f(x+y)+f(x-y) = 2f(x)+2f(y)$$

is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [24] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [7] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. See [1, 5, 6, 10, 16, 17, 18, 19, 20, 23] for more

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functional equations. The survey on the Hyers-Ulam stability of functional equations was given by Brillouet-Bulluot, Brzdek and Cieplinski [3].

The functional equation

$$2f\left(\frac{x+y}{2}\right)+2\left(\frac{x-y}{2}\right)=f(x)+f(y)$$

is called a *Jensen type quadratic equation.*

A *valuation* is a function $|\cdot|$ from a field K into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs|=|r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s|\le|r|+|s|, \qquad \forall r,s\in K.$$

A field K is called a *valued field* if K carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s|\le\max\{|r|,|s|\}, \qquad \forall r,s\in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1|=|-1|=1$ and $|n|\le 1$ for all $n\in\mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0|=0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

Definition 1.1. ([15]) Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\|:X\to[0,\infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\|=0$ if and only if $x=0$;
(ii) $\|rx\|=|r|\|x\| \qquad (r\in K, x\in X)$;
(iii) the strong triangle inequality

$$\|x+y\|\le\max\{\|x\|,\|y\|\}, \qquad \forall x,y\in X$$

holds. Then $(X,\|\cdot\|)$ is called a *non-Archimedean normed space.*

In Section 2, we solve the quadratic functional equation (0.1) in vector spaces and prove the Hyers-Ulam stability of the quadratic functional equation (0.1) in non-Archimedean Banach spaces.

In Section 3, we solve the quadratic functional equation (0.2) in vector spaces and prove the Hyers-Ulam stability of the quadratic functional equation (0.2) in non-Archimedean Banach spaces.

In Section 4, we prove the Hyers-Ulam stability of the quadratic functional equation (0.1) in Banach spaces.

In Section 5, we prove the Hyers-Ulam stability of the quadratic functional equation (0.2) in Banach spaces.

2. Quadratic ρ-functional equation (0.1) in non-Archimedean Banach spaces

Throughout Sections 2 and 3, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let $|2|\ne 1$ and let ρ be a fixed non-Archimedean number with $\rho\ne -1,2$.

Lemma 2.1. *Let X and Y be vector spaces. A mapping $f:X\to Y$ satisfies*

$$f(x+y)+f(x-y)-2f(x)-2f(y)=0 \tag{2.1}$$

for all $x, y \in X$ if and only if the mapping $f : X \to Y$ satisfies

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) = 0 \tag{2.2}$$

for all $x, y \in X$.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $f(0) = 0$.

Letting $y = x$ in (2.1), we get $f(2x) - 4f(x) = 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$ for all $x \in X$. So $f : X \to Y$ satisfies (2.2).

Assume that $f : X \to Y$ satisfies (2.2).

Letting $x = y = 0$ in (2.2), we get $f(0) = 0$.

Letting $y = 0$ in (2.2), we get $4f\left(\frac{x}{2}\right) = f(x)$ for all $x \in X$. and so $f(2x) = 4f(x)$ for all $x \in X$. So $f : X \to Y$ satisfies (2.1). □

We solve the quadratic ρ-functional equation (0.1) in vector spaces.

Lemma 2.2. *Let X and Y be vector spaces. If a mapping $f : X \to Y$ satisfies*

$$\begin{aligned} f(x+y) + f(x-y) - 2f(x) - 2f(y) & \\ = \rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right) & \end{aligned} \tag{2.3}$$

for all $x, y \in X$, then $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (2.3).

Letting $x = y = 0$ in (2.3), we get $-2f(0) = 2\rho f(0)$. So $f(0) = 0$.

Letting $y = x$ in (2.3), we get

$$f(2x) - 4f(x) = 0$$

and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{2.4}$$

for all $x \in X$.

It follows from (2.3) and (2.4) that

$$\begin{aligned} f(x+y) + f(x-y) - 2f(x) - 2f(y) & \\ = \rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right) & \\ = \frac{\rho}{2}(f(x+y) + f(x-y) - 2f(x) - 2f(y)) & \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the quadratic ρ-functional equation (2.3) in non-Archimedean Banach spaces.

Theorem 2.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\lim_{j\to\infty} |4|^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) = 0, \tag{2.5}$$

$$\begin{aligned} &\|f(x+y)+f(x-y)-2f(x)-2f(y) \\ &-\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\| \leq \varphi(x,y) \end{aligned} \tag{2.6}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x)-h(x)\| \leq \sup_{j\in\mathbb{N}}\left\{|4|^{j-1}\varphi\left(\frac{x}{2^j},\frac{x}{2^j}\right)\right\} \tag{2.7}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.6), we get

$$\|f(2x)-4f(x)\| \leq \varphi(x,x) \tag{2.8}$$

for all $x \in X$. So

$$\left\|f(x)-4f\left(\frac{x}{2}\right)\right\| \leq \varphi\left(\frac{x}{2},\frac{x}{2}\right)$$

for all $x \in X$. Hence

$$\begin{aligned} &\left\|4^l f\left(\frac{x}{2^l}\right)-4^m f\left(\frac{x}{2^m}\right)\right\| \\ &\leq \max\left\{\left\|4^l f\left(\frac{x}{2^l}\right)-4^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|,\cdots,\left\|4^{m-1} f\left(\frac{x}{2^{m-1}}\right)-4^m f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &\leq \max\left\{|4|^l\left\|f\left(\frac{x}{2^l}\right)-4f\left(\frac{x}{2^{l+1}}\right)\right\|,\cdots,|4|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right)-4f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &\leq \sup_{j\in\{l,l+1,\cdots\}}\left\{|4|^j\varphi\left(\frac{x}{2^{j+1}},\frac{x}{2^{j+1}}\right)\right\} \end{aligned} \tag{2.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.9) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.9), we get (2.7).

It follows from (2.5) and (2.6) that

$$\begin{aligned} &\|h(x+y)+h(x-y)-2h(x)-2h(y) \\ &-\rho\left(2h\left(\frac{x+y}{2}\right)+2h\left(\frac{x-y}{2}\right)-h(x)-h(y)\right)\| \\ &= \lim_{n\to\infty}|4|^n\left\|f\left(\frac{x+y}{2^n}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right. \\ &\qquad \left.-\rho\left(2f\left(\frac{x+y}{2^{n+1}}\right)+2f\left(\frac{x-y}{2^{n+1}}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right)\right\| \\ &\leq \lim_{n\to\infty}|4|^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right)=0 \end{aligned}$$

for all $x, y \in X$. So

$$h(x+y)+h(x-y)-2h(x)-2h(y)=\rho\left(2h\left(\frac{x+y}{2}\right)+2h\left(\frac{x-y}{2}\right)-h(x)-h(y)\right)$$

for all $x, y \in X$. By Lemma 2.2, the mapping $h : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.7). Then we have

$$\begin{aligned}
\|h(x) - T(x)\| &= \left\| 4^q h\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right) \right\| \\
&\le \max\left\{ \left\| 4^q h\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\|, \left\| 4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\| \right\} \\
&\le \sup_{j\in\mathbb{N}} \left\{ |4|^{q+j-1} \varphi\left(\frac{x}{2^{q+j}}, \frac{x}{2^{q+j}}\right) \right\},
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $h : X \to Y$ is a unique quadratic mapping satisfying (2.7). □

Corollary 2.4. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned}
&\|f(x+y) + f(x-y) - 2f(x) - 2f(y) \\
&-\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right) \| \le \theta(\|x\|^r + \|y\|^r)
\end{aligned} \tag{2.10}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le \frac{2\theta}{|2|^r}\|x\|^r$$

for all $x \in X$.

Theorem 2.5. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (2.6) and*

$$\lim_{j\to\infty} \left\{ \frac{1}{|4|^j} \varphi(2^{j-1}x, 2^{j-1}y) \right\} = 0$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le \sup_{j\in\mathbb{N}} \left\{ \frac{1}{|4|^j} \varphi(2^{j-1}x, 2^{j-1}x) \right\} \tag{2.11}$$

for all $x \in X$.

Proof. It follows from (2.8) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \le \frac{1}{|4|}\varphi(x, x)$$

for all $x \in X$. Hence

$$\begin{aligned}
&\left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x) \right\| \\
&\le \max\left\{ \left\| \frac{1}{4^l} f\left(2^l x\right) - \frac{1}{4^{l+1}} f\left(2^{l+1}x\right) \right\|, \cdots, \left\| \frac{1}{4^{m-1}} f\left(2^{m-1}x\right) - \frac{1}{4^m} f\left(2^m x\right) \right\| \right\} \\
&\le \max\left\{ \frac{1}{|4|^l} \left\| f\left(2^l x\right) - \frac{1}{4} f\left(2^{l+1}x\right) \right\|, \cdots, \frac{1}{|4|^{m-1}} \left\| f\left(2^{m-1}x\right) - \frac{1}{4} f\left(2^m x\right) \right\| \right\} \\
&\le \sup_{j\in\{l, l+1, \cdots\}} \left\{ \frac{1}{|4|^{j+1}} \varphi(2^j x, 2^j x) \right\}
\end{aligned} \tag{2.12}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.12) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.12), we get (2.11).

The rest of the proof is similar to the proof of Theorem 2.3. □

Corollary 2.6. *Let $r > 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.10). *Then there exists a unique quadratic mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \le \frac{2\theta}{|4|}\|x\|^r$$

for all $x \in X$.

3. Quadratic ρ-functional equation (0.2) in non-Archimedean Banach spaces

Let $|2| \ne 1$ and let ρ be a fixed non-Archimedean number with $\rho \ne -1, \frac{1}{2}$.

We solve the quadratic ρ-functional equation (0.2) in vector spaces.

Lemma 3.1. *Let X and Y be vector spaces. If a mapping $f : X \to Y$ satisfies*

$$\begin{aligned} 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \\ = \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned} \tag{3.1}$$

for all $x, y \in X$, then $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $2f(0) = -2\rho f(0)$. So $f(0) = 0$.

Letting $y = 0$ in (3.1), we get

$$4f\left(\frac{x}{2}\right) - f(x) = 0 \tag{3.2}$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{4} f(x)$ for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} &\frac{1}{2}(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \\ &= 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \\ &= \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the quadratic ρ-functional equation (3.1) in non-Archimedean Banach spaces.

Theorem 3.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &\lim_{j\to\infty} \left\{ |4|^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right)\right\} = 0, \\ &\|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \\ &-\rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right)\| \le \varphi(x,y) \end{aligned} \tag{3.3}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le \sup_{j\in\mathbb{N}} \left\{ |4|^{j-1} \varphi\left(\frac{x}{2^{j-1}}, 0\right)\right\} \tag{3.4}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.3), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \le \varphi(x,0) \tag{3.5}$$

for all $x \in X$. So

$$\begin{aligned} &\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \\ &\le \max\left\{\left\|4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &\le \max\left\{|4|^l \left\|f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |4|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &\le \sup_{j\in\{l,l+1,\cdots\}} \left\{|4|^j \varphi\left(\frac{x}{2^j}, 0\right)\right\} \end{aligned} \tag{3.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.6) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.4).

The rest of the proof is similar to the proof of Theorem 2.3. $\square$

Corollary 3.3. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} &\|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \\ &-\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| \le \theta(\|x\|^r + \|y\|^r) \end{aligned} \tag{3.7}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le \theta\|x\|^r$$

for all $x \in X$.

Theorem 3.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (3.3) and*

$$\lim_{j\to\infty} \left\{\frac{1}{|4|^j} \varphi(2^j x, 2^j y)\right\} = 0$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le \sup_{j\in\mathbb{N}} \left\{ \frac{1}{|4|^j} \varphi(2^j x, 0) \right\} \tag{3.8}$$

for all $x \in X$.

Proof. It follows from (3.5) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \le \frac{1}{|4|} \varphi(2x, 0)$$

for all $x \in X$. Hence

$$\begin{aligned} &\left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x) \right\| \\ &\le \max\left\{ \left\| \frac{1}{4^l} f\left(2^l x\right) - \frac{1}{4^{l+1}} f\left(2^{l+1} x\right) \right\|, \cdots, \left\| \frac{1}{4^{m-1}} f\left(2^{m-1} x\right) - \frac{1}{4^m} f\left(2^m x\right) \right\| \right\} \\ &\le \max\left\{ \frac{1}{|4|^l} \left\| f\left(2^l x\right) - \frac{1}{4} f\left(2^{l+1} x\right) \right\|, \cdots, \frac{1}{|4|^{m-1}} \left\| f\left(2^{m-1} x\right) - \frac{1}{4} f\left(2^m x\right) \right\| \right\} \\ &\le \sup_{j\in\{l+1, l+2, \cdots\}} \left\{ \frac{1}{|4|^j} \varphi(2^j x, 0) \right\} \end{aligned} \tag{3.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.9) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.9), we get (3.8).

The rest of the proof is similar to the proof of Theorems 2.3. □

Corollary 3.5. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying* (3.7)*. Then there exists a unique quadratic mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \le \frac{|2|^r \theta}{|4|} \|x\|^r$$

for all $x \in X$.

4. Quadratic ρ-functional equation (0.1) in Banach spaces

Throughout Sections 4 and 5, assume that X is a normed space and that Y is a Banach space. Let ρ be a fixed real or complex number with $\rho \ne -1, 2$.

We prove the Hyers-Ulam stability of the quadratic ρ-functional equation (2.3) in Banach spaces.

Theorem 4.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\Psi(x, y) := \sum_{j=1}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty, \tag{4.1}$$

$$\begin{aligned} &\|f(x + y) + f(x - y) - 2f(x) - 2f(y) \\ &-\rho\left(2f\left(\frac{x + y}{2}\right) + 2f\left(\frac{x - y}{2}\right) - f(x) - f(y)\right)\| \le \varphi(x, y) \end{aligned} \tag{4.2}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{4}\Psi(x, x) \tag{4.3}$$

for all $x \in X$.

Proof. Letting $y = x$ in (4.2), we get

$$\|f(2x) - 4f(x)\| \le \varphi(x, x) \tag{4.4}$$

for all $x \in X$. So

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \le \varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1}\left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \sum_{j=l}^{m-1} 4^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \end{aligned} \tag{4.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (4.5) that the sequence $\{4^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{4^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} 4^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.5), we get (4.3).

Now, let $T : X \to Y$ be another quadratic mapping satisfying (4.3). Then we have

$$\begin{aligned}\|Q(x) - T(x)\| &= \left\|4^q Q\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right)\right\| \\ &\le \left\|4^q Q\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\| + \left\|4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\| \\ &\le \frac{4^q}{2}\Psi\left(\frac{x}{2^q}, \frac{x}{2^q}\right),\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q.

It follows from (4.1) and (4.2) that

$$\begin{aligned}&\|Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y) \\ &-\rho\left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - Q(x) - Q(y)\right)\| \\ &= \lim_{n\to\infty} 4^n \left\|f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right. \\ &\qquad \left. -\rho\left(2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right\| \\ &\le \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0\end{aligned}$$

for all $x, y \in X$. So

$$Q(x+y)+Q(x-y)-2Q(x)-2Q(y)=\rho\left(2Q\left(\frac{x+y}{2}\right)+2Q\left(\frac{x-y}{2}\right)-Q(x)-Q(y)\right)$$

for all $x, y \in X$. By Lemma 2.2, the mapping $Q : X \to Y$ is quadratic. □

Corollary 4.2. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} &\|f(x+y)+f(x-y)-2f(x)-2f(y) \\ &-\rho\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\| \leq \theta(\|x\|^r+\|y\|^r) \end{aligned} \tag{4.6}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x)\| \leq \frac{2\theta}{2^r-4}\|x\|^r$$

for all $x \in X$.

Theorem 4.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (4.2) and*

$$\Psi(x,y) := \sum_{j=0}^{\infty} \frac{1}{4^j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x)\| \leq \frac{1}{4}\Psi(x,x) \tag{4.7}$$

for all $x \in X$.

Proof. It follows from (4.4) that

$$\left\|f(x)-\frac{1}{4}f(2x)\right\| \leq \frac{1}{4}\varphi(x,x)$$

for all $x \in X$. Hence

$$\begin{aligned} \left\|\frac{1}{4^l}f(2^l x)-\frac{1}{4^m}f(2^m x)\right\| &\leq \sum_{j=l}^{m-1}\left\|\frac{1}{4^j}f\left(2^j x\right)-\frac{1}{4^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\leq \sum_{j=l}^{m-1}\frac{1}{4^{j+1}}\varphi(2^j x, 2^j x) \end{aligned} \tag{4.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (4.8) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.8), we get (4.7).

The rest of the proof is similar to the proof of Theorem 4.1. □

Corollary 4.4. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying* (4.6). *Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{2\theta}{4 - 2^r}\|x\|^r$$

for all $x \in X$.

5. Quadratic ρ-functional equation (0.2) in Banach spaces

Let ρ be a fixed real or complex number with $\rho \ne -1, \frac{1}{2}$.

In this section, we prove the Hyers-Ulam stability of the quadratic ρ-functional equation (3.1) in Banach spaces.

Theorem 5.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\Psi(x, y) := \sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty,$$

$$\begin{aligned} &\|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \\ &-\rho\left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right)\| \le \varphi(x, y) \end{aligned} \tag{5.1}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \Psi(x, 0) \tag{5.2}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (5.1), we get

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| = \left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \le \varphi(x, 0) \tag{5.3}$$

for all $x \in X$. So

$$\begin{aligned} \left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| &\le \sum_{j=l}^{m-1} \left\|4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right)\right\| \\ &\le \sum_{j=l}^{m-1} 4^j \varphi\left(\frac{x}{2^j}, 0\right) \end{aligned} \tag{5.4}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (5.4) that the sequence $\{4^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{4^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{k\to\infty} 4^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (5.4), we get (5.2).

The rest of the proof is similar to the proof of Theorem 4.1. □

Corollary 5.2. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &\|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \\ &-\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| \le \theta(\|x\|^r + \|y\|^r) \end{aligned} \tag{5.5}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{2^r\theta}{2^r - 4}\|x\|^r$$

for all $x \in X$.

Theorem 5.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$, (5.1) and*

$$\Psi(x, y) := \sum_{j=1}^{\infty} \frac{1}{4^j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \Psi(x, 0) \tag{5.6}$$

for all $x \in X$.

Proof. It follows from (5.3) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{1}{4}\varphi(2x, 0)$$

for all $x \in X$. Hence

$$\begin{aligned}\left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\right\| &\leq \sum_{j=l+1}^{m}\left\|\frac{1}{4^j}f\left(2^j x\right) - \frac{1}{4^{j+1}}f\left(2^{j+1}x\right)\right\| \\ &\leq \sum_{j=l+1}^{m}\frac{1}{4^j}\varphi(2^j x, 0) \qquad (5.7)\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (5.7) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty}\frac{1}{4^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (5.7), we get (5.6).

The rest of the proof is similar to the proof of Theorem 4.1. □

Corollary 5.4. *Let $r < 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (5.5). Then there exists a unique quadartic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{2^r\theta}{4 - 2^r}\|x\|^r$$

for all $x \in X$.

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ON MODIFIED DEGENERATE GENOCCHI POLYNOMIALS AND NUMBERS

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Abstract. In this paper, we consider the modified partially degenerate Genocchi polynomials and investigate some properties of these polynomials. From these properties, we give some new and interesting identities of them.

1. Introduction

The Genocchi polynomials are defined by the generating function

$$\frac{2t}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}\qquad \text{(see [2, 3, 7, 9, 12, 14, 17, 19, 27, 28])}. \tag{1}$$

When $x=0$, $G_n=G_n(0)$ are called the Genocchi numbers. From (1), we see that

$$\begin{aligned}\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!} &= \left(\frac{2t}{e^t+1}\right)e^{xt}\\ &= \left(\sum_{l=0}^{\infty}G_l\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}x^m\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}G_lx^{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{2}$$

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Thus, by comparing the coefficients on both sides of (2), we get

$$G_n(x) = \sum_{l=0}^{n} \binom{n}{l} G_l x^{n-l}. \tag{3}$$

From (1), we can derive the following equation:

$$\begin{aligned} \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} &= -\frac{-2t}{e^{-t}+1} e^{-(1-x)t} \\ &= \sum_{n=0}^{\infty} (-1)^{n-1} G_n(1-x) \frac{t^n}{n!}. \end{aligned} \tag{4}$$

By comparing the coefficients on both sides of (4), we get

$$G_n(x) = (-1)^{n-1} G_n(1-x). \tag{5}$$

The gamma and beta function are defined by the following definite integrals: for $(\alpha > 0, \beta > 0)$,

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt \tag{6}$$

and

$$\begin{aligned} B(\alpha, \beta) &= \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt \\ &= \int_0^{\infty} \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}} dt \end{aligned} \tag{7}$$

(see [15,23,24]).Thus by (6) and (7), we get

$$\Gamma(\alpha+1) = \alpha \Gamma(\alpha), \qquad B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{8}$$

The classical Genocchi numbers, a sequence of integers introduced by Angelo Genocchi (1817-1889), have been studied in various context in such diverse areas of mathematics and physics as number theory, combinatorics, complex analysis, topology, and quantum physics. In recent years, Genocchi polynomials and numbers have received considerable attention and many researchers have worked on them, their extensions and their connections with some combinatorial counting.

The degenerate Bernoulli polynomials, the rst degenerate version of well-known families of polynomials, were introduced by Carlitz and rediscovered by Ustinov under the name of Korobov polynomials of the second kind. On the other hand, Korobov polynomials (of the rst kind) are the degenerate version of the Bernoulli polynomials of the second kind. Recently, many researchers began to study various kinds of degenerate versions of the familiar polynomials like Bernoulli, Euler, Genocchi, falling factorial and Bell polynomials by using generating functions, umbral calculus, and p-adic integrals.

The goal of this paper is to introduce the modified degenerate Genocchi polynomials and numbers, a degenerate version of the classical Genocchi polynomials and numbers, in order to study their properties and obtain several new and interesting identities involving them. More precisely, we give some properties, explicit formulas, several identities, a connection with Genocchi polynomials, and some integral formulas. Here they were named as the modified degenerate Genocchi polynomials, since there existed what are called the degenerate Genocchi polynomials whose definition is slightly different from ours (see [1, 4-6, 8, 11-16, 18, 20, 21, 22-26, 28]).

2. Modified degenerate Genocchi polynomials

First, we note that

$$e^t = \lim_{\lambda\to 0}(1+\lambda)^{\frac{t}{\lambda}}, \qquad t = loge^t = \lim_{\lambda\to 0} log(1+\lambda)^{\frac{t}{\lambda}} = \lim_{\lambda\to 0}\frac{t}{\lambda}log(1+\lambda). \tag{9}$$

From (1) and (9), we define the modified degenerate Genocchi polynomials as

$$\frac{2t}{(1+\lambda)^{\frac{t}{\lambda}}+1}(1+\lambda)^{\frac{tx}{\lambda}} = \sum_{n=0}^{\infty} g_{n,\lambda}(x)\frac{t^n}{n!} \tag{10}$$

When $x = 0$, $g_{n,\lambda} = g_{n,\lambda}(0)$ are called the modified degenerate Genocchi numbers. From (10), we get

$$\begin{aligned} 2t &= \left((1+\lambda)^{\frac{t}{\lambda}}+1\right)\left(\frac{2t}{(1+\lambda)^{\frac{t}{\lambda}}+1}\right) \\ &= \frac{2t}{(1+\lambda)^{\frac{t}{\lambda}}+1}(1+\lambda)^{\frac{t}{\lambda}} + \frac{2t}{(1+\lambda)^{\frac{t}{\lambda}}+1} \\ &= \sum_{n=0}^{\infty} g_{n,\lambda}(1)\frac{t^n}{n!} + \sum_{n=0}^{\infty} g_{n,\lambda}\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}(g_{n,\lambda}(1)+g_{n,\lambda})\frac{t^n}{n!}. \end{aligned} \tag{11}$$

By comparing the coefficients on both sides of (11), we get

$$\begin{cases} g_{0,\lambda} = 0 \\ g_{n,\lambda}(1) + g_{n,\lambda} = 2\delta_{1,n}, \end{cases} \tag{12}$$

where $\delta_{1,n}$ is the Kronecker delta. From (10), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} g_{n,\lambda}(x)\frac{t^n}{n!} &= \left(\sum_{m=0}^{\infty} g_{m,\lambda}\frac{t^m}{m!}\right)\left(\sum_{m=0}^{\infty}\left(\frac{log(1+\lambda)}{\lambda}\right)^m x^m\frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} g_{n-m,\lambda}\left(\frac{log(1+\lambda)}{\lambda}\right)^m x^m\right)\frac{t^n}{n!}. \end{aligned} \tag{13}$$

Thus, by comparing the coefficients on both sides of (13), we obtain the following theorem.

Theorem 2.1. *Let $n \in \mathbb{N}\cup\{0\}$. Then we have*

$$g_{n,\lambda}(x) = \sum_{m=0}^{n}\binom{n}{m} g_{n-m,\lambda}\left(\frac{log(1+\lambda)}{\lambda}\right)^m x^m. \tag{14}$$

From (10), we derive the following equation:

$$\begin{aligned} \sum_{n=0}^{\infty} g_{n,\lambda}(x)\frac{t^n}{n!} &= -\frac{-2t}{(1+\lambda)^{\frac{-t}{\lambda}}+1}(1+\lambda)^{\frac{-(1-x)t}{\lambda}} \\ &= \sum_{n=0}^{\infty}(-1)^{n-1} g_{n,\lambda}(1-x)\frac{t^n}{n!}. \end{aligned} \tag{15}$$

By comparing the coefficients on both sides of (15),

$$g_{n,\lambda}(x) = (-1)^{n-1} g_{n,\lambda}(1-x) \quad (n \geq 0). \tag{16}$$

By (10), we see that

$$\frac{d}{dx}g_{n,\lambda}(x) = g_{n-1,\lambda}(x)\left(\frac{log(1+\lambda)}{\lambda}\right)n \quad (n \geq 1). \tag{17}$$

From (17), we get

$$\begin{aligned}\frac{g_{n+1,\lambda}(1) - g_{n+1,\lambda}}{n+1} &= \int_0^1 \frac{d}{dx}\frac{g_{n+1,\lambda}(x)}{n+1}dx \\ &= \int_0^1 g_{n,\lambda}(x)\left(\frac{log(1+\lambda)}{\lambda}\right)dx. \quad (n \geq 1).\end{aligned} \tag{18}$$

By (18), we obtain the following theorem.

Theorem 2.2. *Let $n \in \mathbb{N} \cup \{0\}$. Then we have*

$$\frac{g_{n+1,\lambda}(1) - g_{n+1,\lambda}}{n+1} = \int_0^1 g_{n,\lambda}(x)\left(\frac{log(1+\lambda)}{\lambda}\right)dx. \tag{19}$$

We note that the Stirling numbers of the first kind are defined as

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \qquad (n \geq 0), \qquad (\text{see } [1,4-6,8,11-16,18,20,21,22-26,28]). \tag{20}$$

where $(x)_n = x(x-1)\cdots(x-n+1)(n \geq 1)$, and $(x)_0 = 1$. By (10), we see that

$$\begin{aligned}&\frac{2t}{(1+\lambda)^{\frac{t}{\lambda}}+1}(1+\lambda)^{\frac{tx}{\lambda}} \\ =& \left(\sum_{k=0}^{\infty} g_{k,\lambda}\frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{tx}{\lambda}\right)_m\lambda^m\right) \\ =& \left(\sum_{k=0}^{\infty} g_{k,\lambda}\frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty}\sum_{l=0}^{m} S_1(m,l)\left(\frac{tx}{\lambda}\right)^l\right)\frac{\lambda^m}{m!} \\ =& \left(\sum_{k=0}^{\infty} g_{k,\lambda}\frac{t^k}{k!}\right)\left(\sum_{l=0}^{\infty}\left(\sum_{m=l}^{\infty} S_1(m,l)\left(\frac{x}{\lambda}\right)^l\frac{\lambda^m}{m!}l!\right)\frac{t^l}{l!}\right) \\ =& \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=l}^{\infty}\binom{n}{l}g_{n-l,\lambda}S_1(m,l)\left(\frac{x}{\lambda}\right)^l\frac{\lambda^m}{m!}l!\right)\frac{t^n}{n!}\end{aligned} \tag{21}$$

From (21), we obtain the following theorem.

Theorem 2.3. *Let $n \in \mathbb{N} \cup \{0\}$. Then we have*

$$g_{n,\lambda}(x) = \sum_{l=0}^{n}\sum_{m=l}^{\infty}\binom{n}{l}g_{n-l,\lambda}S_1(m,l)\left(\frac{x}{\lambda}\right)^l\frac{\lambda^m}{m!}l!. \tag{22}$$

Let d be an odd integer. Then we see that

$$\begin{aligned}&2t\sum_{l=0}^{d-1}(-1)^l(1+\lambda)^{\frac{lt}{\lambda}} \\ =& \frac{2t}{1+(1+\lambda)^{\frac{t}{\lambda}}}\left(1-\left(-(1+\lambda)^{\frac{t}{\lambda}}\right)^d\right)\end{aligned}$$

$$\begin{aligned}
&= \frac{2t}{1+(1+\lambda)^{\frac{t}{\lambda}}}\left(1+(1+\lambda)^{\frac{dt}{\lambda}}\right)\\
&= \frac{2t}{1+(1+\lambda)^{\frac{t}{\lambda}}}+\frac{2t}{1+(1+\lambda)^{\frac{t}{\lambda}}}(1+\lambda)^{\frac{dt}{\lambda}}\\
&= \sum_{n=1}^{\infty} g_{n,\lambda}\frac{t^n}{n!}+\sum_{n=1}^{\infty} g_{n,\lambda}(d)\frac{t^n}{n!}\\
&= \sum_{n=1}^{\infty}(g_{n,\lambda}+g_{n,\lambda}(d))\frac{t^n}{n!}\\
&= t\sum_{n=0}^{\infty}\left(\frac{g_{n+1,\lambda}+g_{n+1,\lambda}(d)}{n+1}\right)\frac{t^n}{n!}. \qquad (23)
\end{aligned}$$

Also, we see that

$$\begin{aligned}
&2\sum_{l=0}^{d-1}(-1)^l(1+\lambda)^{\frac{lt}{\lambda}}\\
&= 2\sum_{l=0}^{d-1}\left(\sum_{n=0}^{\infty}(-1)^l\left(\frac{log(1+\lambda)}{\lambda}\right)^n l^n\right)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(2\sum_{l=0}^{d-1}(-1)^l\left(\frac{log(1+\lambda)}{\lambda}\right)^n l^n\right)\frac{t^n}{n!}. \qquad (24)
\end{aligned}$$

From (23) and (24), we obtain the following theorem.

Theorem 2.4. *Let $n \in \mathbb{N}\cup\{0\}$. Then we have*

$$2\sum_{l=0}^{d-1}(-1)^l\left(\frac{log(1+\lambda)}{\lambda}\right)^n l^n = \frac{g_{+1n,\lambda}+g_{n+1,\lambda}(d)}{n+1}. \qquad (25)$$

From (10) and (14), we note that

$$\begin{aligned}
\int_0^1 y^n g_{n,\lambda}(x+y)dy &= \sum_{m=0}^{n}\binom{n}{m}g_{n-m,\lambda}\left(\frac{log(1+\lambda)}{\lambda}\right)^m\int_0^1 y^{n+m}dy\\
&= \sum_{m=0}^{n}\binom{n}{m}\frac{g_{n-m,\lambda}(x)}{n+m+1}\left(\frac{log(1+\lambda)}{\lambda}\right)^m
\end{aligned} \qquad (26)$$

By (16), we get

$$\begin{aligned}
&\int_0^1 y^n g_{n,\lambda}(x+y)dy = (-1)^{n-1}\int_0^1 y^n g_{n,\lambda}(1-(x+y))dy\\
&= (-1)^{n-1}\sum_{m=0}^{n}\binom{n}{m}g_{n-m}(-x)\left(\frac{log(1+\lambda)}{\lambda}\right)^m\int_0^1 y^n(1-y)^m dy\\
&= \sum_{m=0}^{n}\binom{n}{m}(-1)^m g_{n-m,\lambda}(1+x)\left(\frac{log(1+\lambda)}{\lambda}\right)^m B(n+1,m+1)\\
&= \sum_{m=0}^{n}\binom{n}{m}(-1)^m\frac{g_{n-m,\lambda}(1+x)}{n+m+1}\left(\frac{log(1+\lambda)}{\lambda}\right)^m\binom{n+m}{m}^{-1} \qquad (27)
\end{aligned}$$

By (26) and (27), we obtain the following theorem.

Theorem 2.5. *For $n \in \mathbb{N}$, we have*

$$\begin{aligned}
&\sum_{m=0}^{n} \binom{n}{m} \frac{g_{n-m,\lambda}(x)}{n+m+1} \left(\frac{log(1+\lambda)}{\lambda} \right)^m \\
= &\sum_{m=0}^{n} \binom{n}{m} (-1)^m \frac{g_{n-m,\lambda}(1+x)}{n+m+1} \left(\frac{log(1+\lambda)}{\lambda} \right)^m \binom{n+m}{m}^{-1} \qquad (28)
\end{aligned}$$

From (17), we note that

$$\begin{aligned}
&\int_0^1 y^n g_{n,\lambda}(x+y)dy \\
= &\frac{g_{n,\lambda}(x+1)}{n+1} - \frac{n}{n+1} \frac{log(1+\lambda)}{\lambda} \int_0^1 y^{n+1} g_{n-1,\lambda}(x+y)dy \\
= &\frac{g_{n,\lambda}(x+1)}{n+1} - \frac{g_{n-1,\lambda}(x+1)}{n+1} \frac{n}{n+2} \frac{log(1+\lambda)}{\lambda} \\
&+(-1)^2 \frac{n(n-1)}{(n+1)(n+2)} \left(\frac{log(1+\lambda)}{\lambda} \right)^2 \int_0^1 y^{n+2} g_{n-2,\lambda}(x+y)dy \\
= &\frac{g_{n,\lambda}(x+1)}{n+1} - \frac{g_{n-1,\lambda}(x+1)}{n+1} \frac{n}{n+2} \frac{log(1+\lambda)}{\lambda} \\
&+(-1)^2 \frac{g_{n-2,\lambda}(x+1)}{n+1} \frac{n(n-1)}{(n+2)(n+3)} \left(\frac{(log(1+\lambda)}{\lambda} \right)^2 \\
&+(-1)^3 \frac{n(n-1)(n-2)}{(n+1)(n+2)(n+3)} \left(\frac{(log(1+\lambda)}{\lambda} \right)^3 \int_0^1 y^{n+3} g_{n-3,\lambda}(x+y)dy \qquad (29)
\end{aligned}$$

By continuing this process, we have

$$\begin{aligned}
\int_0^1 y^n g_{n,\lambda}(x+y)dy = &\frac{g_{n,\lambda}(x+1)}{n+1} \\
&+ \sum_{m=1}^{n-1} (-1)^m \frac{n(n-1)\cdots(n-m+1)}{(n+1)(n+2)\cdots(n+m+1)} \left(\frac{(log(1+\lambda)}{\lambda} \right)^m g_{n-m,\lambda}(x+1) \qquad (30)
\end{aligned}$$

Therefore by (26) and (30), we obtain the following theorem.

Theorem 2.6. *For $n \in \mathbb{N}$, we have*

$$\sum_{m=0}^{n} \binom{n}{m} \frac{g_{n-m,\lambda}(x)}{n+m+1} = \sum_{m=0}^{n-1} (-1)^m \frac{\binom{n}{m}}{\binom{n+m}{m}} \frac{g_{n-m,\lambda}(x+1)}{n+m+1} \left(\frac{(log(1+\lambda)}{\lambda} \right)^m \qquad (31)$$

Taking $x = 0$, From (16) and (31), we obtain the following corollary.

Corollary 2.7. *For $n \in \mathbb{N}$, we have*

$$\sum_{m=0}^{n} \binom{n}{m} \frac{g_{n-m,\lambda}}{n+m+1} = \sum_{m=0}^{n-1} (-1)^m \frac{\binom{n}{m}}{\binom{n+m}{m}} \frac{g_{n-m,\lambda}(1)}{n+m+1} \left(\frac{(log(1+\lambda)}{\lambda} \right)^m \qquad (32)$$

For $n \in \mathbb{N}$, we observe that

$$\int_0^1 y^n g_{n,\lambda}(x+y)dy$$

$$
\begin{aligned}
&= \frac{\lambda}{log(1+\lambda)}\frac{g_{n+1,\lambda}(x+1)}{n+1} - \frac{\lambda}{log(1+\lambda)}\frac{n}{n+1}\int_0^1 y^{n-1}g_{n+1,\lambda}(x+y)dy \\
&= \frac{\lambda}{log(1+\lambda)}\left(\frac{g_{n+1,\lambda}(x+1)}{n+1} - \frac{n}{n+1}\int_0^1 y^{n-1}(-1)^n g_{n+1,\lambda}(1-(x+y))dy\right) \\
&= \frac{\lambda}{log(1+\lambda)}\left(\frac{g_{n+1,\lambda}(x+1)}{n+1} - \frac{n}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}g_{n+1-l,\lambda}(-x)(-1)^n\int_0^1 y^{n-1}(1-y)^l dy\right) \\
&= \frac{\lambda}{log(1+\lambda)}\left(\frac{g_{n+1,\lambda}(x+1)}{n+1} - \frac{n}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}g_{n+1-l,\lambda}(-x)(-1)^n B(n,l+1)\right) \\
&= \frac{\lambda}{log(1+\lambda)}\left(\frac{g_{n+1,\lambda}(x+1)}{n+1} - \frac{1}{n+1}\sum_{l=0}^{n+1}\frac{\binom{n+1}{l}}{\binom{n+l}{l}}(-1)^n g_{n+1-l,\lambda}(-x)\right) \\
&= \frac{\lambda}{log(1+\lambda)}\left(\frac{g_{n+1,\lambda}(x+1)}{n+1} - \frac{1}{n+1}\sum_{l=0}^{n+1}\frac{\binom{n+1}{l}}{\binom{n+l}{l}}(-1)^l g_{n+1-l,\lambda}(1+x)\right)
\end{aligned}
$$
(33)

Therefore, by (30) and (33), we obtain the following theorem.

Theorem 2.8. *For $n \in \mathbb{N}$, we have*

$$
\begin{aligned}
&\sum_{l=0}^{n-1}(-1)^l\frac{\binom{n}{l}}{\binom{n+l}{l}}\frac{g_{n-l,\lambda}(1+x)}{n+l+1}\left(\frac{log(1+\lambda)}{\lambda}\right)^{l+1} \\
&= \frac{g_{n+1,\lambda}(x+1)}{n+1} - \frac{1}{n+1}\sum_{l=0}^{n+1}(-1)^l\frac{\binom{n+1}{l}}{\binom{n+l}{l}}g_{n+1-l,\lambda}(1+x) \qquad (34)
\end{aligned}
$$

Replacing λ by $e-1$ and t by $(e-1)t$ in (10), we get

$$
\begin{aligned}
\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!} &= \frac{2t}{e^t+1}e^{xt} \\
&= \sum_{n=0}^{\infty}g_{n,e-1}(x)(e-1)^{n-1}\frac{t^n}{n!}, \qquad (35)
\end{aligned}
$$

where $G_n(x)$ are the Genocchi polynomials. By comparing both sides of (35), we obtain the following theorem.

Theorem 2.9. *For $n \in \mathbb{N}\cup\{0\}$, we have*

$$G_n(x) = g_{n,e-1}(x)(e-1)^{n-1}. \qquad (36)$$

By (12) and (18), we get

$$
\begin{aligned}
\int_0^1 g_{n,\lambda}(x)dx &= \frac{\lambda}{log(1+\lambda)}(n+1)^{-1}\int_0^1\frac{d}{dx}g_{n+1,\lambda}(x)dx \\
&= \frac{\lambda}{log(1+\lambda)}(n+1)^{-1}(g_{n+1,\lambda}(1)-g_{n+1,\lambda}(0)) \\
&= \frac{(-2)\lambda}{log(1+\lambda)}(n+1)^{-1}g_{n+1,\lambda} \qquad (37)
\end{aligned}
$$

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where $n \in \mathbb{N}$. Also, we have

$$\begin{aligned}
&\int_0^1 g_{n,\lambda}(x)g_{m,\lambda}(x)dx \\
=\ & \frac{\lambda}{log(1+\lambda)}\frac{1}{n+1}g_{n+1,\lambda}(x)g_{m,\lambda}(x)\ |_0^1 - \frac{\lambda}{log(1+\lambda)}\frac{1}{n+1}\int_0^1 g_{n+1,\lambda}(x)\frac{d}{dx}g_{m,\lambda}(x)dx \\
=\ & \frac{\lambda}{log(1+\lambda)}\frac{1}{n+1}(g_{n+1,\lambda}(1)g_{m,\lambda}(1) - g_{n+1,\lambda}(0)g_{m,\lambda}(0) \\
& -\frac{\lambda}{log(1+\lambda)}\frac{1}{n+1}\frac{log(1+\lambda)}{\lambda}m\int_0^1 g_{n+1,\lambda}(x)g_{m-1,\lambda}(x)dx \\
=\ & -\frac{m}{n+1}\int_0^1 g_{n+1,\lambda}(x)g_{m-1,\lambda}(x)dx \\
=\ & (-1)^2\frac{m(m-1)}{(n+1)(n+2)}\int_0^1 g_{n+2,\lambda}(x)g_{m-2,\lambda}(x)dx \qquad (38)
\end{aligned}$$

By continuing this process, we obtain the following theorem.

Theorem 2.10. *For $m, n \in \mathbb{N}$, we have*

$$\begin{aligned}
&\int_0^1 g_{n,\lambda}(x)g_{m,\lambda}(x)dx \\
=\ & (-1)^{m-2}\frac{m(m-1)\cdots 3}{(n+1)(n+2)\cdots(n+m-2)}\int_0^1 g_{n+m-2,\lambda}(x)g_{2,\lambda}(x)dx. \qquad (39)
\end{aligned}$$

Now, we have

$$\begin{aligned}
&\int_0^1 g_{n+m-2,\lambda}(x)g_{2,\lambda}(x)dx \\
=\ & -\frac{2}{n+m-1}\int_0^1 g_{n+m-1,\lambda}(x)g_{1,\lambda}(x)dx \\
=\ & -\frac{2}{n+m-1}\frac{g_{n+m,\lambda}(x)}{n+m}\frac{\lambda}{log(1+\lambda)}|_0^1 \\
=\ & -\frac{2}{n+m-1}\frac{\lambda}{log(1+\lambda)}\frac{-2g_{n+m,\lambda}}{n+m}. \qquad (40)
\end{aligned}$$

By (41), we obtain the following theorem.

Theorem 2.11. *For $m, n \in \mathbb{N}$, we have*

$$\begin{aligned}
&\int_0^1 g_{n,\lambda}(x)g_{m,\lambda}(x)dx \\
=\ & (-1)^m 2\binom{n+m}{m}^{-1}\frac{\lambda}{log(1+\lambda)}g_{n+m,\lambda}. \qquad (41)
\end{aligned}$$

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Hesitant fuzzy implicative filters in BE-algebras

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Abstract. The notion of hesitant fuzzy implicative filter of a BE-algebra is introduced and related properties are investigated. We provide conditions for a hesitant fuzzy filter to be a hesitant fuzzy implicative filter. Also, as a generalization of hesitant fuzzy implicative filter, we consider the hesitant fuzzy n-fold implicative filter. Characterizations of hesitant fuzzy n-fold implicative filter are discussed.

1. **Introduction**

In 2007, Kim and Kim [5] introduced the notion of a BE-algebra, and investigated several properties. In [1], Ahn and So introduced the notion of ideals in BE-algebras. They gave several descriptions of ideals in BE-algebras. Song et al. [8] considered the fuzzification of ideals in BE-algebras. They introduced the notion of fuzzy ideals in BE-algebras, and investigated related properties. They gave characterizations of a fuzzy ideal in BE-algebras.

The notions of Atanassov's intuitionistic fuzzy sets, type 2 fuzzy sets and fuzzy multisets etc. are a generalization of fuzzy sets. As another generalization of fuzzy sets, Torra [9] introduced the notion of hesitant fuzzy sets which are a very useful to express peoples hesitancy in daily life. The hesitant fuzzy set is a very useful tool to deal with uncertainty, which can be accurately and perfectly described in terms of the opinions of decision makers. Also, hesitant fuzzy set theory is used in decision making problem etc. (see [11, 12, 13, 14, 15]), and is applied to residuated lattices and MTL-algebras (see [4, 6]).

In this paper, we introduce the notion of hesitant fuzzy implicative filter of a BE-algebra, and investigate some properties of it. We consider characterizations of hesitant fuzzy implicative filter of a BE-algebra. We provide conditions for a hesitant fuzzy filter to be a hesitant fuzzy implicative filter. Also, as a generalization of hesitant fuzzy implicative filter, we consider the hesitant fuzzy n-fold implicative filter. We discuss characterizations of hesitant fuzzy n-fold implicative filter.

2. **Preliminaries**

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[0]**Keywords**: filter; implicative filter; hesitant fuzzy (implicative) filter.

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By a *BE-algebra* ([5]) we mean a system $(X;*,1)$ of type $(2,0)$ which the following axioms hold:

(2.1) $(\forall x \in X)\,(x * x = 1)$,
(2.2) $(\forall x \in X)\,(x * 1 = 1)$,
(2.3) $(\forall x \in X)\,(1 * x = x)$,
(2.4) $(\forall x, y, z \in X)\,(x * (y * z) = y * (x * z)$ (exchange).

We introduce a relation " $\leq$ " on X by $x \leq y$ if and only if $x * y = 1$.

A BE-algebra $(X;*,1)$ is said to be *transitive* ([5]) if it satisfies: for any $x, y, z \in X$, $y * z \leq (x * y) * (x * z)$. A BE-algebra $(X;*,1)$ is said to be *self distributive* ([5]) if it satisfies: for any $x, yz \in X$, $x * (y * z) = (x * y) * (x * z)$. Note that every self distributive BE-algebra is transitive, but the converse is not true in general ([5]).

Every self distributive BE-algebra $(X;*,1)$ satisfies the following properties:

(2.5) $(\forall x, y, z \in X)\,(x \leq y \Rightarrow z * x \leq z * y$ and $y * z \leq x * z)$,
(2.6) $(\forall x, y \in X)\,(x * (x * y) = x * y)$,
(2.7) $(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))$,

Definition 2.1.([5]) Let $(X;*,1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is a *filter* of X if

(F1) $1 \in F$;
(F2) $(\forall x, y \in X)(x * y, x \in F \Rightarrow y \in F)$.

F is an *implicative filter* of X if it satisfies (F1) and

(F3) $(\forall x, y, z \in X)(x * (y * z), x * y \in F \Rightarrow x * z \in F)$.

Definition 2.2.([9]) Let E be a reference set. A *hesitant fuzzy set* on E is defined in terms of a function that when applied to E returns a subset of $[0,1]$, which can be viewed as the following mathematical representation:

$$H_E := \{(e, h_E(e)) | e \in E\}$$

where $h_E : E \to \mathscr{P}([0,1])$.

Definition 2.3. Given a non-empty subset A of X, a hesitant fuzzy set

$$H_X := \{(x, h_X(x)) | x \in X\}$$

on satisfying the following condition:

$$h_X(x) = \emptyset \text{ for all } x \notin A \tag{2.8}$$

is called a *hesitant fuzzy set related to* A (briefly, A-*hesitant fuzzy set*) on X, and is represented by $H_A := \{(x, h_A(x)) \mid x \in X\}$, where h_A is a mapping from X to $\mathscr{P}([0,1])$ with $h_A(x) = \emptyset$ for all $x \notin A$.

For a hesitant set $H_X := \{(x, h_X(x)) \mid x \in X\}$ of a BE-algebra X and a subset γ of $[0, 1]$, the hesitant fuzzy γ-inclusive set of H_X, denoted by $H_X(\gamma)$, is defined to be the set

$$H_X(\gamma) := \{x \in X | \gamma \subseteq h_X(x)\}.$$

For any hesitant fuzzy set $H_X = \{(x, h_X(x)|x \in X\}$ and $G_X = \{(x, g_X(x))|x \in X\}$, we call H_X a *hesitant fuzzy subset* of G_X, denoted by $H_X \widetilde{\subseteq} G_X$, if $h_X(x) \subseteq g_X(x)$ for all $x \in X$. The *hesitant fuzzy union* of H_X and G_X, denoted by $H_X \widetilde{\cup} G_X$, is defined to be the hesitant fuzzy set $(h_X \widetilde{\cup} g_X)(x) = h_X(x) \cup g_X(x)$ for all $x \in X$. The *hesitant fuzzy intersection* of H_X and G_X, denoted by $H_X \widetilde{\cap} G_X$, is defined to be the hesitant fuzzy set $(h_X \widetilde{\cap} g_X)(x) = h_X(x) \cap g_X(x)$ for all $x \in X$.

3. **Hesitant fuzzy implicative filters**

Definition 3.1.([3]) Given a non-empty subset (subalgebra as much as possible) A of X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X. Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is called a *hesitant fuzzy filter of X related to A* (briefly, *A-hesitant fuzzy filter* of X) if it satisfies the following condition:

$$(\forall x \in A)\left(h_A(x) \subseteq h_A(1)\right), \tag{3.1}$$

$$(\forall x, y \in A)\left(h_A(x * y) \cap h_A(x) \subseteq h_A(y)\right). \tag{3.2}$$

An A-hesitant fuzzy filter of X with $A = X$ is called a *hesitant fuzzy filter* of X.

Proposition 3.2.([3]) *Let $H_A := \{(x, h_A(x))|x \in X\}$ be an A-hesitant fuzzy filter of X where A is a subalgebra of X. Then the following assertions are valid.*

(i) $(\forall x, y \in A)(x \leq y \Rightarrow h_A(x) \subseteq h_A(y))$,

(ii) $(\forall x, y, z \in A)(h_A(x * (y * z)) \cap h_A(y) \subseteq h_A(x * z))$,

(iii) $(\forall a, x \in A)(h_A(a) \subseteq h_A((a * x) * x)$.

Definition 3.3. Given a non-empty subset (subalgebra as much as possible) A of X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X. Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is called a *hesitant fuzzy implicative filter of X related to A* (briefly, *A-hesitant fuzzy implicative filter* of X) if it satisfies (3.1) and

$$(\forall x, y, z \in A)\left(h_A(x * (y * z)) \cap h_A(x * y) \subseteq h_A(x * z)\right). \tag{3.3}$$

An A-hesitant fuzzy implicative filter of X with $A = X$ is called a *hesitant fuzzy implicative filter* of X.

Example 3.4. Let $X = \{1, a, b, c, d, 0\}$ be a BE-algebra with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	1	a
d	1	1	a	1	1	a
0	1	1	1	1	1	1

For a subalgebra $A = \{1, a, b\}$ of X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X defined by

$$H_A = \left\{(1, [0, 1]), (a, (0, \tfrac{1}{2}]), (b, (0, \tfrac{1}{2})), (c, (0, \tfrac{1}{4})), (d, \emptyset), (0, \emptyset)\right\}$$

It is easy to check that H_A is an A-hesitant fuzzy implicative filter of X.

Proposition 3.5. *Every A-hesitant fuzzy implicative filter of a BE-algebra X is an A-hesitant fuzzy filter of X.*

Proof. Let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy implicative filter of X. It follows from (2.4) and (3.3) that

$$\begin{aligned} h_A(y * (x * z)) \cap h_A(x * y) =& h_A(x * (y * z)) \cap h_A(x * y) \\ \subseteq& h_A(x * z) \end{aligned} \tag{3.4}$$

for any $x, y, z \in X$. Setting $x := 1$ in (3.4), we have $h_A(y * z) \cap h_A(y) \subseteq h_A(z)$. Therefore $H_A := \{(x, h_A(x)) \mid x \in X\}$ is an A-hesitant fuzzy filter of X. □

The converse of Proposition may not be true in general (see Example 3.6).

Example 3.6. Let $X = \{1, a, b, c, d, 0\}$ be a BE-algebra as in Example 3.4. Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined as follows:

$$h_X : X \to \mathscr{P}([0, 1]),\ x \mapsto \begin{cases} \gamma_2 & \text{if } x = 1 \\ \gamma_1 & \text{if } x \in \{a, b, c, d, 0\}, \end{cases}$$

where γ_1 and γ_2 are subsets of $[0, 1]$ with $\gamma_1 \subsetneq \gamma_2$. It is easy to check that H_X is a hesitant fuzzy filter of X. But it is not a hesitant fuzzy implicative filter of X, since $h_X(d*(a*0)) \cap h_X(d*a) = \gamma_2 \nsubseteq \gamma_1 = h_X(d * 0)$.

We provide conditions for a hesitant fuzzy filter to be a hesitant fuzzy implicative filter.

Proposition 3.7. *Let X be a self distributive BE-algebra. Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy filter of X satisfying*

$$(\forall x, y, z \in X)(h_X(x * (y * (y * z))) \cap h_X(y * x)) \subseteq h_X(y * z). \tag{3.5}$$

Then $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy implicative filter of X.

Proof. Since $x*(y*z) = y*(x*z) \leq (x*y)*(x*(x*z)) = x*(y*(x*z)) = y*(x*(x*z))$ for all $x, y \in X$, we have $h_X(x*(y*z)) \subseteq h_X(y*(x*(x*z)))$ by Proposition 3.2(i). It follows from (3.5) that $h_X(x*z) \supseteq h_X(y*(x*(x*z)) \cap h_X(x*y) \supseteq h_X(x*(y*z)) \cap h_X(x*y)$. Thus $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy implicative filter of X. □

Theorem 3.8. *Let X be a transitive BE-algebra. For any hesitant fuzzy filter $H_X := \{(x, h_X(x)) \mid x \in X\}$ of X, the following are equivalent:*

(i) $H_X := \{(x, h_X(x)) \mid x \in X\}$ *is a hesitant fuzzy implicative filter,*
(ii) $(\forall x, y \in X)\,(h_X(x*(x*y)) \subseteq h_X(x*y))$,
(iii) $(\forall x, y, z \in X)\,(h_X(x*(y*z)) \subseteq h_X((x*y)*(x*z)))$.

Proof. (i)⇒(ii) Assume that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy implicative filter of X. Setting $z := y, y := x$ in (3.3), we get

$$\begin{aligned} h_X(x*y) \supseteq & h_X(x*(x*y)) \cap h_X(x*x) \\ = & h_X(x*(x*y)) \cap h_X(1) \\ = & h_X(x*(x*y)). \end{aligned}$$

Hence (ii) holds.
(ii)⇒(iii) Suppose that (ii) holds. Since $x*(y*z) \leq x*((x*y)*(x*z)) = x*(x*((x*y)*z))$, by Proposition 3.2(i) we have $h_X(x*((x*y)*(x*z))) = h_X(x*(x*((x*y)*z))) \supseteq h_X(x*(y*z))$. It follows from (ii) that

$$\begin{aligned} h_X((x*y)*(x*z)) = & h_X(x*((x*y)*z)) \\ \supseteq & h_X(x*(x*((x*y)*z))) \\ \supseteq & h_X(x*(y*z)). \end{aligned}$$

Thus (iii) holds.
(iii)⇒(ii) Assume that (iii) holds. By (3.2) and (iii), we have

$$\begin{aligned} h_X(x*z) \supseteq & h_X((x*y)*(x*z)) \cap h_X(x*y) \\ \supseteq & h_X(x*(y*z)) \cap h_X(x*y). \end{aligned}$$

Therefore $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy implicative filter of X. □

Theorem 3.9. *Let X be a self distributive BE-algebra. Then the hesitant fuzzy set $H_X := \{(x, h_X(x)) \mid x \in X\}$ of X is a hesitant fuzzy implicative filter of X if and only if it is a hesitant fuzzy filter of X.*

Proof. By Proposition 3.5, every hesitant fuzzy implicative filter of X is a hesitant fuzzy filter of X.

Conversely, assume that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy filter of X. For any $x, y, z \in X$, by (3.2) we have

$$\begin{aligned} h_X(x * z) \supseteq & h_X((x * y) * (x * z)) \cap h_X(x * y) \\ = & h_X(x * (y * z)) \cap h_X(x * y). \end{aligned}$$

Hence $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy implicative filter of X. □

For any element x and y of a BE-algebra X and positive integer n, let $x^n * y$ denote $x * (\cdots * (x * (x * y)) \cdots)$ in which x occurs n times, and $x^0 * y = 1$.

Definition 3.10. Let X be a BE-algebra and let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X. Then $H_X := \{(x, h_X(x)) \mid x \in X\}$ is called a *hesitant fuzzy n-fold implicative filter* of X if it satisfies (3.1) and

(3.6) $(\forall x, y, z \in X)\, (h_X(x^n * (y * z)) \cap h_X(x^n * y)) \subseteq h_X(x^n * z))$.

Note that a hesitant fuzzy 1-fold implicative filter of X is a hesitant fuzzy implicative filter of X.

Example 3.11. Let $X := \{1, a, b, c, d, 0\}$ is a transitive BE-algebra ([11]) with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	b	c	b	c
b	1	a	1	b	a	d
c	1	a	1	1	a	a
d	1	1	1	b	1	b
0	1	1	1	1	1	1

Let $H_X := \{(x, h_X(x)) \mid x \in X\}$ be a hesitant fuzzy set on X defined as follows:

$$h_X : X \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \gamma_2 & \text{if } x \in \{1, b, c\} \\ \gamma_1 & \text{if } x \in \{a, d, 0\}, \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$. It is easy to check that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy n-fold implicative filter of X.

Theorem 3.12. *Every hesitant n-fold fuzzy implicative filter of X is a hesitant fuzzy filter of X.*

Proof. Taking $x := 1$ in (3.6) and (2.3), we have $h_X(z) \supseteq h_X(y * z) \cap h_X(y)$. Hence $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy filter of X. □

The converse of Theorem 3.12 may not be not true in general (see Example 3.13).

Example 3.13. Let $X := \{1, a, b, c, d, 0\}$ be a BE-algebra as in Example 3.11. Let H_X be a hesitant fuzzy set on X defined as follows:

$$h_X : X \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \gamma_2 & \text{if } x = 1 \\ \gamma_1 & \text{if } x \in \{a, b, c, d, 0\}, \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$. It is easy to check that H_X is a hesitant fuzzy filter of X. But it is not a hesitant fuzzy 1-fold implicative filter of X, since $h_X(d*c) = h_X(b) = \gamma_1 \not\supseteq \gamma_2 = h_X(1) = h_X(d*(b*c)) \cap h_X(d*b)$.

Theorem 3.14. *Let X be a transitive BE-algebra. For any hesitant fuzzy filter $H_X := \{(x, h_X(x)) \mid x \in X\}$ of X, the following are equivalent:*

(i) $H_X := \{(x, h_X(x)) \mid x \in X\}$ *is a hesitant fuzzy n-fold implicative filter,*

(ii) $(\forall x, y \in X)\,(h_X(x^{n+1} * y) \subseteq h_X(x^n * y))$,

(iii) $(\forall x, y, z \in X)\,(h_X(x^n * (y * z)) \subseteq h_X((x^n * y) * (x^n * z)))$.

Proof. (i)⇒(ii) Assume that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy n-fold implicative filter of X. Setting $z := y, y := x$ in (3.6), we have

$$\begin{aligned} h_X(x^n * y) &\supseteq h_X(x^n * (x * y)) \cap h_X(x^n * x) \\ &= h_X(x^{n+1} * y) \cap h_X(1) \\ &= h_X(x^{n+1} * y). \end{aligned}$$

Hence (ii) holds.

(ii)⇒(iii) Suppose that (ii) holds. Since $x^n * (y * z) \leq x^n * ((x^n * y) * (x^n * z))$, we have $h_X(x^n * ((x^n * y) * (x^n * z))) \supseteq h_X(x^n * (y * z))$. Since $x^{n+1} * (x^{n-1} * ((x^n * y) * z)) = x^n * (x^n * ((x^n * y) * z))) = x^n * ((x^n * y)) * (x^n * z))$ and using (ii), we have

$$\begin{aligned} h_X(x^{n+1} * (x^{n-2} * ((x^n * y) * z)) &= h_X(x^n * (x^{n-1} * ((x^n * y) * z)) \\ &\supseteq h_X(x^{n+1} * (x^{n-1} * ((x^n * y) * z))) \\ &= h_X(x^n * ((x^n * y) * (x^n * z))) \\ &\supseteq h_X(x^n * (y * z)). \end{aligned} \tag{3.7}$$

By (ii) and (3.7), we have

$$\begin{aligned} h_X(x^{n+1} * (x^{n-3} * ((x^n * y) * z))) &= h_X(x^n * (x^{n-2} * ((x^n * y) * z))) \\ &\supseteq h_X(x^{n+1} * (x^{n-2} * ((x^n * y) * z))) \\ &\supseteq h_X(x^n * (y * z)). \end{aligned}$$

Continuing this process, we conclude that

$$\begin{aligned} h_X((x^n * y) * (x^n * z)) &= h_X(x^n * ((x^n * y) * z)) \\ &\supseteq h_X(x^n * (y * z)). \end{aligned}$$

(iii)⇒(i) Let $x, y, z \in X$. It follows from (iii) that

$$\begin{aligned} h_X(x^n * z) \supseteq & h_X((x^n * y) * (x^n * z)) \cap h_X(x^n * y) \\ \supseteq & h_X((x^n * (y * z)) \cap h_X(x^n * y). \end{aligned}$$

Hence $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant n-fold fuzzy implicative filter □

Definition 3.15. Let n be a positive integer. A BE-algebra X is said to be *n-fold implicative* if it satisfies the equality $x^{n+1} * y = x^n * y$ for all $x, y \in X$.

Corollary 3.16. *In an n-fold implicative BE-algebra, the notion of hesitant fuzzy filters and hesitant fuzzy n-fold implicative filters coincide.*

Proof. Straightforward. □

Theorem 3.17. *A hesitant fuzzy set $H_X := \{(x, h_X(x)) \mid x \in X\}$ of a BE-algebra X is a hesitant fuzzy implicative filter of X if and only if the hesitant fuzzy γ-inclusive set $H_X(\gamma)$ is an implicative filter of X for all $\gamma \in \mathscr{P}([0,1])$ with $H_X(\gamma) \neq \emptyset$.*

The filter $H_X(\gamma)$ in Theorem 3.17 is called the *γ-inclusive filter* of X.

Proof. Assume that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy implicative filter of X. Let $x, y, z \in X$ and $\gamma \in \mathscr{P}([0,1])$ be such that $x * (y * z) \in H_X(\gamma)$ and $x * y \in H_X(\gamma)$. Then $\gamma \subseteq h_X(x * (y * z))$ and $\gamma \subseteq h_X(x * y)$. Using (3.1) and (3.3), we have $\gamma \subseteq h_X(1)$ and $\gamma \subseteq h_X(x * (y * z) \cap h_X(x * y) \subseteq h_X(x * z)$ for $x, y, z \in X$. Hence $1 \in H_X(\gamma)$ and $x * z \in H_X(\gamma)$. Thus $H_X(\gamma)$ is an implicative filter of X.

Conversely, suppose that $H_X(\gamma)$ is an implicative filter of X for all $\gamma \in \mathscr{P}([0,1])$ with $H_X(\gamma) \neq \emptyset$. For any $x \in X$, let $h_X(x) = \gamma$. Since $H_X(\gamma)$ is an implicative filter of X, we have $1 \in H_X(\gamma)$ and so $h_X(x) = \gamma \subseteq h_X(1)$. For any $x, y, z \in X$, let $h_X(x*(y*z)) = \gamma_{x*(y*z)}$ and $h_X(x*y) = \gamma_{x*y}$. Take $\gamma = \gamma_{x*(y*z)} \cap \gamma_{x*y}$. Then $x * (y * z) \in H_X(\gamma)$ and $x * y \in H_X(\gamma)$ which imply that $x * z \in H_X(\gamma)$. Hence

$$h_X(x * z) \supseteq \gamma = \gamma_{x*(y*z)} \cap \gamma_{x*y} = h_X(x * (y * z)) \cap h_X(x * y).$$

Thus $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy implicative filter of X. □

Theorem 3.18. *Every hesitant fuzzy implicative filter of a BE-algebra can be represented as a hesitant fuzzy γ-inclusive set of a hesitant fuzzy implicative filter.*

Proof. Let F be an implicative filter of a BE-algebra X. For a subset γ of $[0,1]$, define a hesitant fuzzy set $H_X := \{(x, h_X(x)) \mid x \in X\}$ of X by

$$h_X : X \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \gamma & \text{if } x \in F, \\ \emptyset & \text{if } x \notin F. \end{cases}$$

Obviously, $F = H_X(\gamma)$. We now prove that H_X is a hesitant fuzzy implicative filter of X. Since $1 \in F = H_X(\gamma)$, we have $h_X(1) = \gamma \supseteq h_X(x)$ for all $x \in X$. Let $x, y, z \in X$. If $x*(y*z), x*y \in F$, then

$x*z \in F$ because F is an implicative filter of X. Hence $h_X(x*(y*z)) = h_X(x*y) = h_X(x*z) = \gamma$, and so $h_X(x*(y*z)) \cap h_X(x*y) \subseteq h_X(x*z)$. If $x*(y*z) \in F$ and $x*y \notin F$, then $h_X(x*(y*z)) = \gamma$ and $h_X(x*y) = \emptyset$ which imply that

$$h_X(x*(y*z)) \cap h_X(x*y) = \gamma \cap \emptyset = \emptyset \subseteq h_X(x*z).$$

Similarly, if $x*(y*z) \notin F$ and $x*y \in F$, then $h_X(x*(y*z)) \cap h_X(x*y) \subseteq h_X(x*z)$. Obviously, if $x*(y*z) \notin F$ and $x*y \notin F$, then $h_X(x*(y*z)) \cap h_X(x*y) \subseteq h_X(x*z)$. Therefore $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy implicative filter of X. $\square$

For two elements a and b of X, consider a hesitant fuzzy set $H_X^{a,b} := \{(x, h_X^{a,b} x)) \mid x \in X\}$ where

$$h_X^{a,b} : X \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \gamma_1 & \text{if } a*(b*x) = 1, \\ \gamma_2 & \text{otherwise,} \end{cases}$$

where γ_1 and γ_2 are subsets of X with $\gamma_2 \subsetneq \gamma_1$.

There exist $a, b \in X$ such that $H_X^{a,b}$ is not a hesitant fuzzy implicative filter of X (see Example 3.19).

Example 3.19. Consider the BE-algebra $X = \{1, a, b, c, d, 0\}$ which is given in Example 3.4. Then $H_X^{1,a}$ is not a hesitant fuzzy implicative filter of X since

$$h_X^{1,a}(1*(a*b)) \cap h_X^{1,a}(1*a) = \gamma_1 \nsubseteq h_X^{1,a}(1*b) = \gamma_2.$$

Now we provide a condition for the hesitant fuzzy set $H_X^{a,b}$ to be a hesitant fuzzy implicative filter of X for all $a, b \in X$.

Theorem 3.20. *If X is a self distributive BE-algebra, then the hesitant fuzzy set $H_X^{a,b}$ is a hesitant fuzzy implicative filter of X for all $a, b \in X$.*

Proof. Let $a, b \in X$. Obviously, $h_X^{a,b}(1) \supseteq h_X^{a,b}(x)$ for all $x \in X$. Let $x, y, z \in X$ be such that $a*(b*(x*(y*z))) \neq 1$ or $a*(b*(x*y)) \neq 1$. Then $h_X^{a,b}(x*(y*z)) = \gamma_2$ or $h_X^{a,b}(x*y) = \gamma_2$. Hence

$$h_X^{a,b}(x*(y*z)) \cap h_X^{a,b}(x*y) = \gamma_2 \subseteq h_X^{a,b}(x*z).$$

Assume that $a*(b*(x*(y*z)))=1$ and $a*(b*(x*y))=1$. Then

$$\begin{aligned}1&=a*(b*(x*(y*z)))\\&=a*(b*((x*y)*(x*z)))\\&=a*((b*(x*y))*(b*(x*z)))\\&=(a*(b*(x*y)))*(a*(b*(x*z)))\\&=1*(a*(b*(x*z)))\\&=a*(b*(x*z)),\end{aligned}$$

and so $h_X^{a,b}(x*(y*z))\cap h_X^{a,b}(x*y)=\gamma_1=h_X^{a,b}(x*z)$. Therefore $H_X^{a,b}$ is a hesitant fuzzy implicative filter of X for all $a,b\in X$. □

Theorem 3.21. *If H_X and G_X are hesitant fuzzy implicative filters of a BE-algebra X, then the hesitant fuzzy intersection $H_X\tilde{\cap}G_X$ of H_X and G_X is a hesitant fuzzy implicative filter of X.*

Proof. For any $x\in X$, we have

$$(h_X\tilde{\cap}g_X)(1)=h_X(1)\cap g_X(1)\supseteq h_X(x)\cap g_X(x)=(h_X\tilde{\cap}g_X)(x).$$

Let $x,y,z\in X$. Then

$$\begin{aligned}(h_X\tilde{\cap}g_X)(x*z)&=h_X(x*z)\cap g_X(x*z)\\&\supseteq(h_X(x*(y*z))\cap h_X(x*y))\cap(g_X(x*(y*z))\cap g_X(x*y))\\&=(h_X(x*(y*z))\cap g_X(x*(y*z)))\cap(h_X(x*y)\cap g_X(x*y))\\&=(h_X\tilde{\cap}g_X)(x*(y*z))\cap(h_X\tilde{\cap}g_X)(x*y).\end{aligned}$$

Hence $H_X\tilde{\cap}G_X$ is a hesitant fuzzy implicative filter of X. □

The hesitant fuzzy union of hesitant fuzzy implicative filters of a BE-algebra X may not be a hesitant fuzzy implicative filter of X as the following example.

Example 3.22. Let $X=\{1,a,b,c,d\}$ is a BE-algebra with the following Cayley table ([5]):

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Let H_X and G_X be hesitant fuzzy sets of X defined, respectively, as follows:

$$h_X:X\to\mathscr{P}([0,1]),\ x\mapsto\begin{cases}\gamma_3 & \text{if } x\in\{1,b\}\\ \gamma_1 & \text{if } x\in\{a,c,d\}\end{cases}$$

and

$$g_X : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_4 & \text{if } x \in \{1, a, c\} \\ \gamma_2 & \text{if } x \in \{b, d\} \end{cases}$$

where γ_1, γ_2, γ_3, and γ_4 are subsets of $[0, 1]$ with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3 \subsetneq \gamma_4$. It is easy to check that H_X and G_X are hesitant fuzzy implicative filters of X. But $H_X \tilde{\cup} G_X$ is not a hesitant fuzzy implicative filter of X, since

$$\begin{aligned}(h_X \tilde{\cup} g_X)(1 * (c * d)) \cap (h_X \tilde{\cup} g_X)(1 * c) &= (h_X \tilde{\cup} g_X)(b) \cap (h_X \tilde{\cup} g_X)(c) \\ &= (h_X(b) \cup g_X(b)) \cap (h_X(c) \cup g_X(c)) \\ &= \gamma_3 \cap \gamma_4 = \gamma_3 \nsubseteq \gamma_2 = \gamma_1 \cup \gamma_2 \\ &= h_X(1 * d) \cup g_X(1 * d) = (h_X \tilde{\cup} g_X)(1 * d).\end{aligned}$$

Let H_X be a hesitant fuzzy set set of a BE-algebra X. For any $a, b \in X$ and $k \in \mathbb{N}$, consider the set

$$h_X[a^k; b] := \left\{x \in X \mid h_X\left(a^k * (b * x)\right) = h_X(1)\right\}$$

where $h_X(a^k * x) = h_X(a * (a * (\cdots * (a * (a * x)) \cdots)))$ in which a appears k-times. Note that $a, b, 1 \in h_X[a^k; b]$ for all $a, b \in X$ and $k \in \mathbb{N}$.

Proposition 3.23. *Let H_X be a hesitant fuzzy set of a BE-algebra X such that the condition (3.1) and $h_X(x*y) = h_X(x) \cup h_X(y)$ for all $x, y \in X$. For any $a, b \in X$ and $k \in \mathbb{N}$, if $x \in h_X[a^k; b]$, then $y * x \in h_X[a^k; b]$ for all $y \in X$.*

Proof. Assume that $x \in h_X[a^k; b]$. Then $h_X(a^k * (b * x)) = h_X(1)$, and so

$$\begin{aligned}h_X(a^k * (b * (y * x))) &= h_X(a^k * (y * (b * x))) \\ &= h_X(y * (a^k * (b * x))) \\ &= h_X(y) \cup h_X(a^k * (b * x)) \\ &= h_X(y) \cup h_X(1) = h_X(1)\end{aligned}$$

for all $y \in X$ by the exchange property of the operation $*$. Hence $y*x \in h_X[a^k; b]$ for all $y \in X$. □

Proposition 3.24. *For any hesitant fuzzy set H_X of a BE-algebra X, let $a \in X$ satisfy the following condition $a * x = 1$ for all $x \in X$. Then $h_X[a^k; b] = X = h_X[b^k; a]$ for all $b \in X$ and $k \in \mathbb{N}$.*

Proof. For any $x \in X$, we have

$$\begin{aligned} h_X(a^k * (b * x)) &= h_X(a^{k-1} * (a * (b * x))) \\ &= h_X(a^{k-1} * (b * (a * x))) \\ &= h_X(a^{k-1} * (b * 1)) \\ &= h_X(1), \end{aligned}$$

and so $x \in h_X[a^k; b]$. Similarly, $x \in h_X[b^k; a]$. □

Proposition 3.25. *Let X be a self distributive BE-algebra and let H_X be an order-preserving soft set of X with the property (3.1). If $b \le c$ in X, then $h_X[a^k; c] \subseteq h_X[a^k; b]$ for all $a \in X$ and $k \in \mathbb{N}$.*

Proof. Let $a, b, c, \in X$ be such that $b \le c$. For any $k \in \mathbb{N}$, if $x \in h_X[a^k; c]$, then

$$\begin{aligned} h_X(1) = h_X(a^k * (c * x)) &= h_X(c * (a^k * x)) \\ &\subseteq h_X(b * (a^k * x)) = h_X(a^k * (b * x)) \end{aligned}$$

by (2.5), Proposition 3.2(i) and (2.4), and so $h_X(a^k * (b * x)) = h_X(1)$. Thus $x \in h_X[a^k; b]$, which completes the proof. □

The following example shows that there exists a hesitant fuzzy set H_X of X, $a, b \in X$ and $k \in \mathbb{N}$ such that $h_X[a^k; b]$ is not a filter of X.

Example 3.26. Let $X = \{1, a, b, c\}$ is a BE-algebra with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	a	a	1

Let H_X be a hesitant fuzzy set of X U defined as follows:

$$h_X : X \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \gamma_2 & \text{if } x = 1 \\ \gamma_1 & \text{if } x \in \{a, b, c\}, \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$. Then it is a hesitant fuzzy set of X. But $h_X[c; b] = \{x \in X | h_X(c * (b * x)) = h_X(1)\} = \{1, a, b\}$ is not an implicative filter, since $1 * (a * c) = a \in h_X[c; b]$, $1 * a = a \in h_X[c; b]$ and $1 * c = c \notin h_X[c; b]$.

We provide conditions for a set $h_X[a^k; b]$ to be an implicative filter.

Theorem 3.27. *Let H_X be a hesitant fuzzy set of a self distributive BE-algebra X. If h_X is injective, then $h_X[a^k; b]$ is an implicative filter of X for all $a, b \in X$ and $k \in \mathbb{N}$.*

Proof. Assume that X is a self distributive BE-algebra and h_X is injective. Obviously, $1 \in h_X[a^k; b]$. Let $a, b, x, y, z \in X$ and $k \in \mathbb{N}$ be such that $x * (y * z) \in h_X[a^k; b]$ and $x * y \in h_X[a^k; b]$. Then $h_X(a^k * (b * (x * (y * z)))) = h_X(1)$ which implies that $a^k * (b * (x * (y * z))) = 1$ since h_X is injective. Since X is a self distributive BE-algebra, we have

$$\begin{aligned}
h_X(1) &= h_X(a^k * (b * (x * (y * z)))) \\
&= h_X(a^{k-1} * (a * (b * (x * (y * z))))) \\
&= h_X(a^{k-1} * (a * ((b * (x * y)) * (b * (x * z))))) \\
&= \cdots \\
&= h_X((a^k * (b * (x * y))) * (a^k * (b * (x * z)))) \\
&= h_X(1 * (a^k * (b * (x * z)))) \\
&= h_X(a^k * (b * (x * z))),
\end{aligned}$$

which implies that $x * z \in h_X[a^k; b]$. Therefore $h_X[a^k; b]$ is an implicative filter of X for all $a, b \in X$ and $k \in \mathbb{N}$. □

Theorem 3.28. *Let H_X be a hesitant fuzzy set of a self distributive B-algebra X satisfying the condition (3.1) and*

$$(\forall x, y \in X)\,(h_X(x * y) = h_X(x) \cap h_X(y)). \tag{3.8}$$

Then $h_X[a^k; b]$ is an implicative filter of X for all $a, b \in X$ and $k \in \mathbb{N}$.

Proof. Let $a, b \in X$ and $k \in \mathbb{N}$. Obviously, $1 \in h_X[a^k; b]$. Let $x, y, z \in X$ be such that $x * (y * z) \in h_X[a^k; b]$ and $x * y \in h_X[a^k; b]$. Then $h_X\left(a^k * (b * (x * (y * z)))\right) = h_X(1)$, which implies from (3.8) and (3.1) that

$$\begin{aligned}
h_X(1) &= h_X(a^k * (b * (x * (y * z)))) \\
&= h_X(a^{k-1} * (a * (b * (x * (y * z))))) \\
&= h_X(a^{k-1} * (a * ((b * (x * y)) * (b * (x * z))))) \\
&= \cdots \\
&= h_X((a^k * (b * (x * y))) * (a^k * (b * (x * z)))) \\
&= h_X(a^k * (b * (x * y))) \cap h_X(a^k * (b * (x * z))) \\
&= h_X(1) \cap h_X(a^k * (b * (x * z))) \\
&= h_X(a^k * (b * (x * z))).
\end{aligned}$$

Hence $x * z \in h_X[a^k; b]$ and therefore $h_X[a^k; b]$ is an implicative filter of X for all $a, b \in X$ and $k \in \mathbb{N}$. □

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A new quadratic functional equation version and its stability and superstability

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Abstract. Let $\mathcal{X}$ and $\mathcal{Y}$ be vector spaces. It is shown that a mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfies the functional equation

$$f\left(\frac{x+y+z}{2}\right)+f\left(\frac{x-y-z}{2}\right)+f\left(\frac{y-x-z}{2}\right)+f\left(\frac{z-x-y}{2}\right) = f(x)+f(y)+f(z) \tag{0.1}$$

if and only if $f : \mathcal{X} \to \mathcal{Y}$ is a quadratic mapping.

Furthermore, we prove the superstability and the Hyers-Ulam stability for the quadratic functional equation (0.1) by using a direct method.

Keywords: Hyers-Ulam stability; quadratic functional equation; fixed point method; quadratic functional inequality; orthogonality space.

1. Introduction and preliminaries

Studying functional equations focusing on their approximate and exact solutions, conduces to one of the most substantial significant study brunches in functional equations, what we would call *"the theory of stability of functional equations"*. This theory specifically analyzes relationships between approximate and exact solutions of functional equations. Actually a functional equation is considered to be *stable*, if one can find an exact solution for any approximate solution of that certain functional equation. Another related and close term in this area is *superstability*, which has a similar nature and concept to the stability problem. As a matter of fact, superstability for a given functional equation occurs when any approximate solution is an exact solution too. In such this situation the functional equation is called *superstable*.

In 1940, the most preliminary form of stability problems was proposed by Ulam [40]. He gave a talk and asked the following: "when and under what conditions does an exact solution of a functional equation near an approximately solution of that exist?"

In 1941, this question that today is considered as the source of the stability theory, was formulated and solved by Hyers [14] for the Cauchy's functional equation in Banach spaces. Then the result of Hyers was generalized by Aoki [1] for additive mappings and by Rassias [32] for linear mappings by considering an unbounded Cauchy difference. In 1994, Găvruţa [13] provided a further generalization of Rassias' theorem in which he replaced the unbounded Cauchy difference by a general control function for the existence of a unique linear mapping. For more epochal information and various aspects about the stability of functional equations theory, we refer the

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reader to the monographs [15, 28, 33, 35], which also include many interesting results concerning the stability of different functional equations in many various spaces.

Consider the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y). \tag{1.1}$$

The function $f(x)=cx^2$ is a solution for the quadratic functional equation and obviously every satisfied function in this equation is said to be a quadratic function. A stability problem for this equation was first proved by Skof [39] and then was generalized by Cholewa [9], Czerwik [7, 8] and others [2, 4, 30, 31, 33, 34]. Moreover, there are some other different types of quadratic functional equations that their stability problems have been investigated by many authors. We refer the readers to the papers [3, 5, 6, 10, 11, 12, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 36, 37, 38, 41].

This paper is organized as follows: In Section 2, we consider the superstability of the quadratic functional equation (0.1) and in Sections 3 and 4, we prove two types of stability problems for the quadratic functional equation (0.1).

2. **Superstability of the functional equation (0.1)**

To commence proving the superstability of the quadratic functional equation (0.1), we first solve it and then will give a superstability theorem.

Proposition 2.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be vector spaces. A mapping $f:\mathcal{X}\to\mathcal{Y}$ satisfies* (0.1) *if and only if the mapping $f:\mathcal{X}\to\mathcal{Y}$ is a quadratic mapping.*

Proof. Sufficiency. Assume that $f:\mathcal{X}\to\mathcal{Y}$ satisfies (0.1).

Letting $x=y=z=0$ in (0.1), we have $4f(0)=3f(0)$. So $f(0)=0$.

Letting $y=z=0$ in (0.1), we get

$$\begin{aligned} 2f\left(\frac{x}{2}\right)+2f\left(-\frac{x}{2}\right)&=f(x), \\ 2f\left(-\frac{x}{2}\right)+2f\left(\frac{x}{2}\right)&=f(-x) \end{aligned} \tag{2.1}$$

for all $x\in\mathcal{X}$, which imply that $f(x)=f(-x)$ for all $x\in\mathcal{X}$.

It follows from (2.1) that $4f\left(\frac{x}{2}\right)=f(x)$ and so $f(2x)=4f(x)$ for all $x\in\mathcal{X}$.

Putting $z=0$ in (0.1), we see that

$$\frac{1}{2}f(x+y)+\frac{1}{2}f(x-y)=f(x)+f(y)$$

for all $x,y\in\mathcal{X}$, which means that $f:\mathcal{X}\to\mathcal{Y}$ is a quadratic mapping.

Necessity. Assume that $f:\mathcal{X}\to\mathcal{Y}$ is quadratic.

By (1.1), one can easily get $f(0)=0$, $f(x)=f(-x)$ and $f(2x)=4f(x)$ for all $x\in\mathcal{X}$. So by applying (1.1), we obtain

$$\begin{aligned} f\Big(\frac{x+y+z}{2}\Big)&+f\Big(\frac{x-y-z}{2}\Big)+f\Big(\frac{y-x-z}{2}\Big)+f\Big(\frac{z-x-y}{2}\Big) \\ &=\Big[2f\Big(\frac{x}{2}\Big)+2f\Big(\frac{y+z}{2}\Big)\Big]+\Big[2f\Big(-\frac{x}{2}\Big)+2f\Big(\frac{y-z}{2}\Big)\Big] \\ &=4f\left(\frac{x}{2}\right)+f\Big(\frac{y+z+y-z}{2}\Big)+f\Big(\frac{y+z-y+z}{2}\Big) \\ &=f(x)+f(y)+f(z) \end{aligned}$$

for all $x,y,z\in\mathcal{X}$, which is the functional equation (0.1) and the proof is complete. $\square$

Theorem 2.2. *Let $\mathcal{X}, \mathcal{Y}$ be normed spaces with norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$, respectively. Let δ be a nonnegative real number and $\varphi : \mathcal{X}^3 \to [0, \infty)$ be a function with*

$$\varphi(0, 0, 0) = 0, \qquad \varphi(x, y, 3x + y) = 0$$

for all $x, y \in \mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a mapping such that

$$\begin{aligned}
&\left\| f\Big(\frac{x+y+z}{2}\Big) + f\Big(\frac{x-y-z}{2}\Big) + f\Big(\frac{y-x-z}{2}\Big) - f(y) - f(z) \right\|_{\mathcal{Y}} \\
&\quad \le \left\| f(x) - f\Big(\frac{z-x-y}{2}\Big) \right\|_{\mathcal{Y}} + \delta \cdot \varphi(x, y, z)
\end{aligned} \tag{2.2}$$

for all $x, y, z \in \mathcal{X}$. Then f is a quadratic mapping.

Proof. Putting $x = y = z = 0$ in (2.2), we get

$$\big\| f(0) \big\|_{\mathcal{Y}} \le \big\| 0 \big\|_{\mathcal{Y}} + \delta \cdot \varphi(0, 0, 0) = 0.$$

So $f(0) = 0$.

Replacing x, y, z by $0, x, x$ in (2.2), respectively, we obtain

$$\big\| f(-x) - f(x) \big\|_{\mathcal{Y}} \le \big\| 0 \big\|_{\mathcal{Y}} + \delta \cdot \varphi(0, x, x) = 0.$$

So $f(x) = f(-x)$ for all $x \in \mathcal{X}$.

Replacing x, y and z by $x, -3x$ and 0, and then by $2x, -3x$ and $3x$ in (2.2), respectively, we have

$$\begin{aligned}
[f(x) - f(3x)] + 2f(2x) &= 0, \\
2[f(x) - f(3x)] + f(4x) &= 0,
\end{aligned}$$

which result that $f(2x) = 4f(x)$ and $f(3x) = 9f(x)$ for all $x \in \mathcal{X}$.

Letting $x = v - u, y = 2u - v$ and $z = 2v - u$ and then $x = u + v, y = -3v$ and $z = 3u$ in (2.2), respectively, we get the equalities

$$\begin{aligned}
f(2u - v) + f(2v - u) &= f(u) + f(v) + f(2u - 2v), \\
f(2u - v) + f(2v - u) &= f(3u) + f(3v) - f(2u + 2v).
\end{aligned}$$

Thus

$$f(u) + f(v) + 4f(u - v) = 9f(u) + 9f(v) - 4f(u + v),$$

which is simplified to

$$f(u + v) + f(u - v) = 2f(u) + 2f(v)$$

for all $u, v \in \mathcal{X}$. So f is quadratic. □

Theorem 2.2 covers several other cases for $\varphi : \mathcal{X}^3 \to [0, \infty)$. For example, we can define φ satisfying the mentioned conditions with $\varphi(x, y, z) := \|y\|_{\mathcal{X}} - \|3x - z\|_{\mathcal{X}}$ or $\varphi(x, y, z) := \|3x + y - z\|_{\mathcal{X}}$. In addition, to make a simpler result, one can put $\delta = 0$.

3. Hyers-Ulam stability of the functional equation (0.1): Type A

In this section, we prove the Hyers-Ulam stability of the quadratic functional equation (0.1). We will suppose that $\mathcal{X}$ is a normed space and $\mathcal{Y}$ is a complete normed space with norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$, respectively.

Theorem 3.1. *Let $\varphi : \mathcal{X}^3 \to [0,\infty)$ be a function with $\varphi(0,0,0)=0$ and the following condition holds:*

$$\text{if} \quad \left\{ \begin{array}{ll} \|x\|_{\mathcal{X}} \le \|x'\|_{\mathcal{X}}, & \text{or} \\ \|y\|_{\mathcal{X}} \le \|y'\|_{\mathcal{X}}, & \text{or} \\ \|z\|_{\mathcal{X}} \le \|z'\|_{\mathcal{X}}, & \end{array} \right. \qquad \Longrightarrow \qquad \varphi(x,y,z) \le \varphi(x',y',z') \tag{3.1}$$

for all $x,y,z,x',y',z' \in \mathcal{X}$. Denote by ϕ a function such that

$$\phi(x,y,z) := \sum_{n=0}^{\infty} 2^{2n} \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) < \infty \tag{3.2}$$

for all $x,y,z \in \mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is an even mapping satisfying

$$\begin{aligned} \left\| f\Big(\frac{x+y+z}{2}\Big) + f\Big(\frac{x-y-z}{2}\Big) + f\Big(\frac{y-x-z}{2}\Big) - f(y) - f(z) \right\|_{\mathcal{Y}} &\le \left\| f(x) - f\Big(\frac{z-x-y}{2}\Big) \right\|_{\mathcal{Y}} \\ &+ \varphi(x,y,z) \end{aligned} \tag{3.3}$$

for all $x,y,z \in \mathcal{X}$. Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\left\| f(x) - \mathcal{Q}(x) \right\|_{\mathcal{Y}} \le 2\phi(x,x,x) \tag{3.4}$$

for all $x \in \mathcal{X}$.

Proof. Letting $x=y=z=0$ in (3.3), we get

$$\left\| f(0) \right\|_{\mathcal{Y}} \le \left\| 0 \right\|_{\mathcal{Y}} + \varphi(0,0,0) = 0.$$

So $f(0)=0$.

Replacing x,y,z by $x,x,4x$ and $x,0,3x$ in (3.3), respectively, and then using (3.1), we obtain

$$\begin{aligned} \left\| f(3x) + 2f(2x) - f(x) - f(4x) \right\|_{\mathcal{Y}} &\le \varphi(x,x,4x) \le \varphi(4x,4x,4x), \\ \left\| 2f(2x) + f(x) - f(3x) \right\|_{\mathcal{Y}} &\le \varphi(x,0,3x) \le \varphi(4x,4x,4x) \end{aligned}$$

for all $x \in \mathcal{X}$. These inequalities give

$$\left\| 4f(2x) - f(4x) \right\|_{\mathcal{Y}} \le \left\| f(3x) + 2f(2x) - f(x) - f(4x) \right\|_{\mathcal{Y}} + \left\| 2f(2x) + f(x) - f(3x) \right\|_{\mathcal{Y}} \le 2\varphi(4x,4x,4x).$$

Thus

$$\left\| 4f\left(\frac{x}{2}\right) - f(x) \right\|_{\mathcal{Y}} \le 2\varphi(x,x,x) \tag{3.5}$$

for all $x \in \mathcal{X}$. Using the induction method, we show that

$$\left\| 4^n f\left(\frac{x}{2^n}\right) - f(x) \right\|_{\mathcal{Y}} \le \sum_{s=0}^{n-1} 2^{2s+1} \varphi\left(\frac{x}{2^s}, \frac{x}{2^s}, \frac{x}{2^s}\right) \tag{3.6}$$

for all $n \ge 1$ and all $x \in \mathcal{X}$. The case $n=1$ is the inequality (3.5). For the case $n+1$, by (3.5) and (3.6), we have

$$\begin{aligned} \left\| 4^{n+1} f\left(\frac{x}{2^{n+1}}\right) - f(x) \right\|_{\mathcal{Y}} &\le 4^n \left\| 4f\left(\frac{1}{2}\left(\frac{x}{2^n}\right)\right) - f\left(\frac{x}{2^n}\right) \right\|_{\mathcal{Y}} + \left\| 4^n f\left(\frac{x}{2^n}\right) - f(x) \right\|_{\mathcal{Y}} \\ &\le 4^n \cdot 2\varphi\left(\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n}\right) + \sum_{s=0}^{n-1} 2^{2s+1} \varphi\left(\frac{x}{2^s}, \frac{x}{2^s}, \frac{x}{2^s}\right) = \sum_{s=0}^{n} 2^{2s+1} \varphi\left(\frac{x}{2^s}, \frac{x}{2^s}, \frac{x}{2^s}\right) \end{aligned}$$

for all $x \in \mathcal{X}$, which ends the induction method.

Assume that m,l are positive integers with $m > l$. From (3.6), it follows that

$$\left\| 4^m f\left(\frac{x}{2^m}\right) - 4^l f\left(\frac{x}{2^l}\right) \right\|_{\mathcal{Y}} = 4^l \left\| 4^{m-l} f\left(\frac{1}{2^{m-l}}\left(\frac{x}{2^l}\right)\right) - f\left(\frac{x}{2^l}\right) \right\|_{\mathcal{Y}} \le \sum_{s=l}^{m-1} 2^{2s+1} \varphi\left(\frac{x}{2^s}, \frac{x}{2^s}, \frac{x}{2^s}\right)$$

for all $x \in \mathcal{X}$, in which by (3.2) the right-hand side tends to zero as $m, l \to \infty$. This clarifies that the sequence $\left\{4^n f\left(\frac{x}{2^n}\right)\right\}$ is Cauchy in the complete space $\mathcal{Y}$ and therefore convergent in it. So we can define for all $x \in \mathcal{X}$, the mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ by

$$\mathcal{Q}(x) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right).$$

Now passing the limit $n \to \infty$ in (3.6) and then using (3.2), we obtain (3.4).

To end the proof, we show that $\mathcal{Q}$ is a unique quadratic mapping. It follows from (3.3) that

$$\begin{aligned}
&\left\|\mathcal{Q}\Big(\frac{x+y+z}{2}\Big) + \mathcal{Q}\Big(\frac{x-y-z}{2}\Big) + \mathcal{Q}\Big(\frac{y-x-z}{2}\Big) - \mathcal{Q}(y) - \mathcal{Q}(z)\right\|_{\mathcal{Y}} \\
&= \lim_{n\to\infty} 4^n \left\| f\Big(\frac{x+y+z}{2^{n+1}}\Big) + f\Big(\frac{x-y-z}{2^{n+1}}\Big) + f\Big(\frac{y-x-z}{2^{n+1}}\Big) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right)\right\|_{\mathcal{Y}} \\
&\leq \lim_{n\to\infty} \left\| 4^n f\left(\frac{x}{2^n}\right) - 4^n f\Big(\frac{z-x-y}{2^{n+1}}\Big)\right\|_{\mathcal{Y}} + \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right)
\end{aligned}$$

for all $x, y, z \in \mathcal{X}$, in which by (3.2), the second term of the right-hand side tends to zero as $n \to \infty$, and therefore we obtain

$$\left\|\mathcal{Q}\Big(\frac{x+y+z}{2}\Big) + \mathcal{Q}\Big(\frac{x-y-z}{2}\Big) + \mathcal{Q}\Big(\frac{y-x-z}{2}\Big) - \mathcal{Q}(y) - \mathcal{Q}(z)\right\|_{\mathcal{Y}} \leq \left\|\mathcal{Q}(x) - \mathcal{Q}\Big(\frac{z-x-y}{2}\Big)\right\|_{\mathcal{Y}}$$

for all $x, y, z \in \mathcal{X}$. Now by applying Theorem 2.2 (with $\delta = 0$), we conclude that $\mathcal{Q}$ is a quadratic mapping.

Let $\mathcal{Q}' : \mathcal{X} \to \mathcal{Y}$ be another quadratic mapping satisfying (3.4). Then we have

$$\begin{aligned}
\left\|\mathcal{Q}(x) - \mathcal{Q}'(x)\right\|_{\mathcal{Y}} &\leq 4^n \left\|\mathcal{Q}\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_{\mathcal{Y}} + 4^n \left\|\mathcal{Q}'\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)\right\|_{\mathcal{Y}} \\
&\leq 2 \cdot 4^n \cdot 2\phi\left(\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n}\right) = 4\sum_{s=n}^{\infty} 2^{2s+1}\varphi\left(\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n}\right)
\end{aligned}$$

for all $x \in \mathcal{X}$. By (3.2), the right-hand side tends to zero as $n \to \infty$, and thus $\mathcal{Q}(x) = \mathcal{Q}'(x)$ for all $x \in \mathcal{X}$. This means the uniqueness of $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ and so the proof is complete. □

Theorem 3.2. *Let $\varphi : \mathcal{X}^3 \to [0, \infty)$ be a function satisfying $\varphi(0,0,0) = 0$ and* (3.1). *Denote by ϕ a function such that*

$$\phi(x, y, z) := \sum_{n=1}^{\infty} \frac{1}{2^{2n}} \varphi\big(2^n x, 2^n y, 2^n z\big) < \infty \tag{3.7}$$

for all $x, y, z \in \mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is an even mapping satisfying (3.3). *Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ satisfying* (3.4).

Proof. As in the proof of Theorem 3.1, we can first get the inequality (3.5), and then by replacing x by $2x$ in (3.5), we obtain

$$\left\|\frac{1}{4} f(2x) - f(x)\right\|_{\mathcal{Y}} \leq \frac{1}{2}\varphi(2x, 2x, 2x)$$

for all $x \in \mathcal{X}$.

Using the induction method, we get

$$\left\|\frac{1}{4^n} f(2^n x) - f(x)\right\|_{\mathcal{Y}} \leq \sum_{s=1}^{n} \frac{1}{2^{2s-1}} \varphi(2^s x, 2^s x, 2^s x) \tag{3.8}$$

for all $n \geq 1$ and all $x \in \mathcal{X}$.

Now by the same method which was done in the proof of Theorem 3.1, we have the Cauchy sequence $\left\{\frac{1}{4^n} f(2^n x)\right\}$, and then the mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ defined by

$$\mathcal{Q}(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in \mathcal{X}$.

And finally we can conclude the inequality (3.4) by (3.7) and (3.8).

The rest of the proof is similar to the proof of Theorem 3.1. ☐

Corollary 3.3. *Let δ be a nonnegative real number and p_1, p_2, p_3 be positive real numbers such that $p_1, p_2, p_3 > 2$ or $p_1, p_2, p_3 < 2$. Let $f : \mathcal{X} \to \mathcal{Y}$ be an even mapping satisfying*

$$\begin{aligned}&\Big\|f\Big(\frac{x+y+z}{2}\Big) + f\Big(\frac{x-y-z}{2}\Big) + f\Big(\frac{y-x-z}{2}\Big) - f(y) - f(z)\Big\|_{\mathcal{Y}} \\ &\leq \Big\|f(x) - f\Big(\frac{z-x-y}{2}\Big)\Big\|_{\mathcal{Y}} + \delta\big(\|x\|_{\mathcal{X}}^{p_1} + \|y\|_{\mathcal{X}}^{p_2} + \|z\|_{\mathcal{X}}^{p_3}\big)\end{aligned}$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - \mathcal{Q}(x)\big\|_{\mathcal{Y}} \leq \sum_{i=1}^{3} \frac{2^{p_i+1}}{\left|2^{p_i} - 4\right|}\delta\|x\|_{\mathcal{X}}^{p_i}$$

for all $x \in \mathcal{X}$.

Proof. Defining $\varphi(x, y, z) = \delta\big(\|x\|_{\mathcal{X}}^{p_1} + \|y\|_{\mathcal{X}}^{p_2} + \|z\|_{\mathcal{X}}^{p_3}\big)$ and applying Theorem 3.1 for the case $p_1, p_2, p_3 > 2$, and Theorem 3.2 for the case $p_1, p_2, p_3 < 2$, we get the desired results. ☐

Corollary 3.4. *Let δ be a nonnegative real number and p_1, p_2, p_3 be positive real numbers such that $p_1+p_2+p_3 \neq 2$. Let $f : \mathcal{X} \to \mathcal{Y}$ be an even mapping satisfying*

$$\begin{aligned}&\Big\|f\Big(\frac{x+y+z}{2}\Big) + f\Big(\frac{x-y-z}{2}\Big) + f\Big(\frac{y-x-z}{2}\Big) - f(y) - f(z)\Big\|_{\mathcal{Y}} \\ &\leq \Big\|f(x) - f\Big(\frac{z-x-y}{2}\Big)\Big\|_{\mathcal{Y}} + \delta\big(\|x\|_{\mathcal{X}}^{p_1} \cdot \|y\|_{\mathcal{X}}^{p_2} \cdot \|z\|_{\mathcal{X}}^{p_3}\big)\end{aligned}$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - \mathcal{Q}(x)\big\|_{\mathcal{Y}} \leq \frac{2^{p_1+p_2+p_3+1}}{\left|2^{p_1+p_2+p_3} - 4\right|}\delta\|x\|_{\mathcal{X}}^{p_1+p_2+p_3}$$

for all $x \in \mathcal{X}$.

Proof. Defining $\varphi(x, y, z) = \delta\big(\|x\|_{\mathcal{X}}^{p_1} \cdot \|y\|_{\mathcal{X}}^{p_2} \cdot \|z\|_{\mathcal{X}}^{p_3}\big)$ and applying Theorem 3.1 for the case $p_1 + p_2 + p_3 > 2$, and Theorem 3.2 for the case $p_1 + p_2 + p_3 < 2$, we get the desired results. ☐

4. Hyers-Ulam stability of the functional equation (0.1): Type B

In this section, we bring another type of stability theorems for the quadratic functional equation (0.1) which is more prevalent in considering stability problems rather than the given type in the previous section.

First of all, for convenience, we define for a given mapping $f : \mathcal{X} \to \mathcal{Y}$, the difference operator:

$$\begin{aligned}\mathcal{D}f(x, y, z) =: {}& f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \\ &- f(x) - f(y) - f(z)\end{aligned}$$

for all $x, y, z \in \mathcal{X}$.

Theorem 4.1. *Let $\varphi : \mathcal{X}^3 \to [0, \infty)$ be a function satisfying $\varphi(0, 0, 0) = 0$ and (3.1). Denote by ϕ a function such that*

$$\phi(x, y, z) := \sum_{n=0}^{\infty} \frac{9^n}{4^n}\varphi\left(\frac{2^n}{3^n}x, \frac{2^n}{3^n}y, \frac{2^n}{3^n}z\right) < \infty \tag{4.1}$$

for all $x, y, z \in \mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is an even mapping satisfying

$$\big\|\mathcal{D}f(x, y, z)\big\|_{\mathcal{Y}} \leq \varphi(x, y, z) \tag{4.2}$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - \mathcal{Q}(x)\big\|_{\mathcal{Y}} \leq \phi(x, x, x) \tag{4.3}$$

for all $x \in \mathcal{X}$.

Proof. Letting $x = y = z = 0$ in (4.2), we get $f(0) = 0$.

Replacing x, y, z by $0, x, 3x$ and then by $2x, 2x, 2x$ in (4.2), respectively, we obtain

$$\begin{aligned}\big\|2f(2x) + f(x) - f(3x)\big\|_{\mathcal{Y}} &\leq \varphi(0, x, 3x) \leq \varphi(3x, 3x, 3x),\\ \Big\|f(2x) - f(x) - \frac{1}{3}f(3x)\Big\|_{\mathcal{Y}} &\leq \frac{1}{3}\varphi(2x, 2x, 2x) \leq \frac{1}{3}\varphi(3x, 3x, 3x).\end{aligned}$$

Adding the above inequalities, we conclude that $\big\|3f(2x) - \frac{4}{3}f(3x)\big\|_{\mathcal{Y}} \leq \frac{4}{3}\varphi(3x, 3x, 3x)$ and therefore

$$\Big\|\frac{9}{4}f\left(\frac{2}{3}x\right) - f(x)\Big\|_{\mathcal{Y}} \leq \varphi(x, x, x)$$

for all $x \in \mathcal{X}$.

By the induction method, we can show that

$$\Big\|\frac{9^n}{4^n}f\left(\frac{2^n}{3^n}x\right) - f(x)\Big\|_{\mathcal{Y}} \leq \sum_{s=0}^{n-1}\frac{9^s}{4^s}\varphi\left(\frac{2^s}{3^s}x, \frac{2^s}{3^s}x, \frac{2^s}{3^s}x\right) \tag{4.4}$$

for all $x \in \mathcal{X}$.

Now similar to the method in the proof of Theorem 3.1, we have the Cauchy sequence $\left\{\frac{9^n}{4^n}f\left(\frac{2^n}{3^n}x\right)\right\}$, and then the mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ defined by

$$\mathcal{Q}(x) := \lim_{n\to\infty}\frac{9^n}{4^n}f\left(\frac{2^n}{3^n}x\right)$$

for all $x \in \mathcal{X}$. This definition and the inequality (4.4) lead us to the inequality (4.3).

It follows from (4.1) and (4.2) that

$$\big\|\mathcal{D}\mathcal{Q}(x, y, z)\big\|_{\mathcal{Y}} \leq \lim_{n\to\infty}\frac{9^n}{4^n}\Big\|\mathcal{D}f\left(\frac{2^n}{3^n}x, \frac{2^n}{3^n}y, \frac{2^n}{3^n}z\right)\Big\|_{\mathcal{Y}} \leq \lim_{n\to\infty}\frac{9^n}{4^n}\varphi\left(\frac{2^n}{3^n}x, \frac{2^n}{3^n}y, \frac{2^n}{3^n}z\right) = 0.$$

Hence $\mathcal{D}\mathcal{Q}(x, y, z) = 0$ for all $x, y, z \in \mathcal{X}$. Now Proposition 2.1 signifies that $\mathcal{Q}$ is a quadratic mapping.

The proof of the uniqueness of $\mathcal{Q}$ is similar to the proof of Theorem 3.1. □

Theorem 4.2. *Let $\varphi : \mathcal{X}^3 \to [0, \infty)$ be a function satisfying $\varphi(0, 0, 0) = 0$ and (3.1). Denote by ϕ a function such that*

$$\phi(x, y, z) := \sum_{n=0}^{\infty}\frac{4^n}{9^n}\varphi\left(\frac{3^n}{2^n}x, \frac{3^n}{2^n}y, \frac{3^n}{2^n}z\right) < \infty$$

for all $x, y, z \in \mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is an even mapping satisfying (4.2). Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ satisfying (4.3).

Proof. The proof is similar to the proof of the previous theorem and thus we omit it. □

Corollary 4.3. *Let δ be a nonnegative real number and p_1, p_2, p_3 be positive real numbers such that $p_1, p_2, p_3 > 2$ or $p_1, p_2, p_3 < 2$. Let $f : \mathcal{X} \to \mathcal{Y}$ be an even mapping satisfying*

$$\big\|\mathcal{D}f(x, y, z)\big\|_{\mathcal{Y}} \leq \delta(\|x\|_{\mathcal{X}}^{p_1} + \|y\|_{\mathcal{X}}^{p_2} + \|z\|_{\mathcal{X}}^{p_3})$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - \mathcal{Q}(x)\big\|_{\mathcal{Y}} \leq \sum_{i=1}^{3}\frac{2^{p_i - 2}}{\left|\frac{2^{p_i}}{9} - \frac{3^{p_i}}{4}\right|}\delta\|x\|_{\mathcal{X}}^{p_i}$$

for all $x \in \mathcal{X}$.

Proof. Defining $\varphi(x,y,z) = \delta\left(\|x\|_{\mathcal{X}}^{p_1} + \|y\|_{\mathcal{X}}^{p_2} + \|z\|_{\mathcal{X}}^{p_3}\right)$ and applying Theorem 4.1 for the case $p_1, p_2, p_3 > 2$, and Theorem 4.2 for the case $p_1, p_2, p_3 < 2$, we get the desired results. □

Corollary 4.4. *Let δ be a nonnegative real number and p_1, p_2, p_3 be positive real numbers such that $p_1+p_2+p_3 \neq 2$. Let $f : \mathcal{X} \to \mathcal{Y}$ be an even mapping satisfying*

$$\left\|\mathcal{D}f(x,y,z)\right\|_{\mathcal{Y}} \leq \delta(\|x\|_{\mathcal{X}}^{p_1} \cdot \|y\|_{\mathcal{X}}^{p_2} \cdot \|z\|_{\mathcal{X}}^{p_3})$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\left\|f(x) - \mathcal{Q}(x)\right\|_{\mathcal{Y}} \leq \frac{2^{p_1+p_2+p_3-2}}{\left|\frac{2^{p_1+p_2+p_3}}{9} - \frac{3^{p_1+p_2+p_3}}{4}\right|}\theta\|x\|_{\mathcal{X}}^{p_1+p_2+p_3}$$

for all $x \in \mathcal{X}$.

Proof. Defining $\varphi(x,y,z) = \delta\left(\|x\|_{\mathcal{X}}^{p_1} \cdot \|y\|_{\mathcal{X}}^{p_2} \cdot \|z\|_{\mathcal{X}}^{p_3}\right)$ and applying Theorem 4.1 for the case $p_1 + p_2 + p_3 > 2$, and Theorem 4.2 for the case $p_1 + p_2 + p_3 < 2$, we get the desired results. □

This paper is just a start for the quadratic functional equation (0.1). Actually this functional equation and its stability problems can be studied more in various mathematical structures and spaces. Such this studied approach can cause to have a deeper description of this equation's unknown properties which will probably be more interesting and remarkable.

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Some New Results on Preconditioned Generalized Mixed-Type Splitting Iterative Methods

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Abstract

In this paper, we present three preconditioned generalized mixed-type splitting (GMTS) methods for solving the weighted linear least square problem. We compare the spectral radii of the iteration matrices of the preconditioned and the original methods. The comparison results show that the preconditioned GMTS methods converge faster than the GMTS method whenever the GMTS method is convergent. Finally, we give two numerical examples to confirm our theoretical results.

Keywords: Preconditioning, GMTS method, linear system, convergence, comparison.

2000 AMS Classification: 65F10.

1. Introduction

We consider the following weighted least squares problem

$$(1.1) \qquad \min_{x\in R^n} (Ax-b)^T W^{-1} (Ax-b),$$

where $A \in R^{n\times n}$ is nonsingular, $b \in R^n$, $W \in R^{n\times n}$ is a symmetric positive definite matrix, see [1,4,9].

In order to solve it, one has to solve a nonsingular linear system as

$$(1.2) \qquad Hy = f,$$

where

$$(1.3) \qquad H = A^T W^{-1} A = \begin{pmatrix} I-B & U \\ L & I-C \end{pmatrix} \in R^{n\times n}$$

is an invertible matrix with

$$B = (b_{ij})_{p\times p}\,,\ C = (c_{ij})_{q\times q}\,,\ L = (l_{ij})_{q\times p}\,,\ U = (u_{ij})_{p\times q}\,,$$

$p+q=n$ and $f = A^T W^{-1} b \in R^n$, see [1,4].

Throughout the paper, we consider the following decomposition for the matrix H,

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$H = \hat{D} - \hat{L} - \hat{U}$, in which

$$\hat{D} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \hat{L} = \begin{pmatrix} 0 & 0 \\ -L & 0 \end{pmatrix}, \quad \hat{U} = \begin{pmatrix} B & -U \\ 0 & C \end{pmatrix}. \tag{1.4}$$

In [1], authors established a generalized AOR(GAOR) method to solve systems of linear equations (1.2). In paper [2, 3] , authors studied the preconditioned GAOR methods. In [4], authors presented a generalized mixed-type splitting (GMTS) iterative method which is generalized GAOR method. And they studied the preconditioned generalized mixed-type splitting iterative methods to solve (1.2). They showed that the preconditioned GMTS methods converge faster than the GMTS method, whenever the GMTS method is convergent.

In this paper, we propose three new preconditioners and give the comparison theorems between the preconditioned and original methods. These results show that the preconditioned GMTS methods converge faster than the GMTS method whenever the GMTS method is convergent. And we prove that in the case that the GMTS method is convergent, using the third preconditioned GMTS method leads to the better convergence rate than the first and the second preconditioned GMTS methods. In Section 4, we give two examples to confirm our theoretical results. And we know that the preconditioned GMTS methods with preconditioners in this paper have the better converge rate than the preconditioned GMTS method with preconditioner P^*.

2. Preliminaries

2.1 Definition [5] $A \in R^{n\times n}$ is called a Z-matrix if $a_{ij} \le 0$ for $i,j = 1,2,...,n$ $(i \neq j)$.

2.2 Definition [5] Let A be a Z-matrix with positive diagonal elements. Then the matrix A is called an M-matrix if A is nonsingular and $A^{-1} \ge 0$.

2.3 Definition [6] The splitting $A = M - N$ is called

(1) a regular splitting of A if $M^{-1} \ge 0$ and $N \ge 0$;
(2) a nonnegative splitting of A if $M^{-1} \ge 0$, $M^{-1}N \ge 0$ and $NM^{-1} \ge 0$;
(3) a weak nonnegative splitting of A if $M^{-1} \ge 0$ and either $M^{-1}N \ge 0$ (the first type) or $NM^{-1} \ge 0$ (the second type);
(4) a convergent splitting of A if $\rho(M^{-1}N) < 1$.

2.1. Lemma. [4] *Let A be a Z-matrix. Moreover, suppose that $A = M - N$ is a weak nonnegative splitting of the first type. Then $\rho(M^{-1}N) < 1$ if and only if A is an M-matrix.*

2.2. Lemma. [7] *Let $A = M - N$ be a regular splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if A is nonsingular and A^{-1} is nonnegative.*

2.3. Lemma. [8] *Let matrix $A = (a_{ij})_{n\times n}$ be given such that*

(1) $a_{ij} \le 0$ *for* $i,j = 1,2,...,n$ $(i \neq j)$,
(2) A *is nonsingular,*
(3) $A^{-1} \ge 0$.

Then,

(1) $a_{ii} > 0$ *for* $i = 1,2,...,n$, *i.e.,* A *is an M-matrix,*
(2) $\rho(B) < 1$ *where* $B = I - D^{-1}A$, *where* $D = diag\{a_{11},...,a_{nn}\}$.

2.4. Lemma. [6] *Let $A = M_1 - N_1 = M_2 - N_2$ be two convergent weak nonnegative splittings of A, where $A^{-1} \geq 0$, if $M_1^{-1} \geq M_2^{-1}$ then $\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2)$.*

3. Comparison results

Consider the linear system (1.2), the generalized mixed-type splitting (GMTS) iterative method is given as follows:

$$(\hat{D} + D_1 + L_1 - \hat{L})y^{(k+1)} = (D_1 + L_1 + \hat{U})y^{(k)} + f \tag{3.1}$$

where $\hat{D}$, $\hat{L}$ and $\hat{U}$ are defined by (1.4), and D_1 is an auxiliary nonnegative block diagonal matrix, L_1 is an auxiliary strictly nonnegative block lower triangular matrix such that $0 \leq D_1 \leq \hat{D}$ and $0 \leq L_1 \leq \hat{L}$. Evidently, the iteration matrix of the GMTS iterative method is given as follow:

$$T = (\hat{D} + D_1 + L_1 - \hat{L})^{-1}(D_1 + L_1 + \hat{U}).$$

In this paper, we propose the new preconditioners as follows,

$$P_i^* = \begin{pmatrix} I + S_i & 0 \\ 0 & I + V_i \end{pmatrix}, \quad i = 1, 2, 3 \tag{3.2}$$

where

$$S_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ b_{21} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{p-1,1} & 0 & \cdots & 0 & 0 \\ b_{p1} & 0 & \cdots & 0 & 0 \end{pmatrix}, \; S_2 = \begin{pmatrix} 0 & b_{12} & \cdots & b_{1,p-1} & b_{1p} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} 0 & b_{12} & \cdots & b_{1,p-1} & b_{1p} \\ b_{21} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{p-1,1} & 0 & \cdots & 0 & 0 \\ b_{p1} & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$V_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ c_{21} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{q-1,1} & 0 & \cdots & 0 & 0 \\ c_{q1} & 0 & \cdots & 0 & 0 \end{pmatrix}, \; V_2 = \begin{pmatrix} 0 & c_{12} & \cdots & c_{1,q-1} & c_{1q} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$V_3 = \begin{pmatrix} 0 & c_{12} & \cdots & c_{1,q-1} & c_{1q} \\ c_{21} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{q-1,1} & 0 & \cdots & 0 & 0 \\ c_{q1} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Then $P_i^* H$ can be expressed by

$$P_i^* H = \begin{pmatrix} I - B_i^* & U_i^* \\ L_i^* & I - C_i^* \end{pmatrix},$$

where $B_i^* = B - S_i(I - B)$, $C_i^* = C - V_i(I - C)$, $L_i^* = (I + V_i)L$, $U_i^* = (I + S_i)U$.

Let us consider the corresponding splitting for the preconditioned GMTS method, that is the generalized mixed-type splitting for the $\bar{H}_i = P_i^* H = \bar{M}_i - \bar{N}_i$, where

$$\bar{M}_i = \hat{D}_i^* + \bar{D}_1 + \bar{L}_1 - \hat{L}_i^*, \quad \bar{N}_i = \bar{D}_1 + \bar{L}_1 + \hat{U}_i^*$$

and

$$\hat{D}_i^* = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},\ \hat{L}_i^* = \begin{pmatrix} 0 & 0 \\ -L_i^* & 0 \end{pmatrix},\ \hat{U}_i^* = \begin{pmatrix} B_i^* & -U_i^* \\ 0 & C_i^* \end{pmatrix},\ i = 1, 2, 3,$$

$\bar{D}_1$ is an auxiliary nonnegative block diagonal matrix with $0 \le \bar{D}_1 \le \hat{D}_i^*$, $\bar{L}_1$ is an auxiliary strictly nonnegative block lower triangular matrix with $0 \le \bar{L}_1 \le \hat{L}_i^*$.

The iteration matrix of the preconditioned GMTS method is

$$T_i^* = (\hat{D}_i^* + \bar{D}_1 + \bar{L}_1 - \hat{L}_i^*)^{-1}(\bar{D}_1 + \bar{L}_1 + \hat{U}_i^*).$$

3.1. Lemma. [4] *Assume that $L \le 0, U \le 0, B \ge 0, C \ge 0$ and H in (1.2) is irreducible. If D_1 is nonsingular, then the iteration matrix of the GMTS method is irreducible.*

3.2. Lemma. [4] *Assume that $L \le 0, U \le 0, B \ge 0, C \ge 0$, then the corresponding splitting of GMTS method is a regular splitting for the matrix H.*

Similar to the proof of Lemma 3.2, we can prove the following lemma.

3.3. Lemma. *Assume that $L \le 0, U \le 0, B \ge 0, C \ge 0$, then the corresponding splitting of PGMTS method is a regular splitting for the matrix $P_i^* H$ $(i = 1, 2, 3)$.*

3.4. Theorem. *Let H be an M-matrix, then $P_i^* H$ $(i = 1, 2, 3)$ is an M-matrix.*

Proof. Consider the following splitting for H, $H = M_1 - N_1$,
where $M_1 = (P_1^*)^{-1}$, $N_1 = (P_1^*)^{-1}(\hat{L}^* + \hat{U}^*)$,
and $\hat{L}^* = \begin{pmatrix} 0 & 0 \\ -L_1^* & 0 \end{pmatrix}, \hat{U}^* = \begin{pmatrix} B_1^* & -U_1^* \\ 0 & C_1^* \end{pmatrix}$.

We can see that $M_1^{-1} N_1 = \hat{L}^* + \hat{U}^*$ and $M_1^{-1} \ge 0$. Then$H = M_1 - N_1$ is a weak nonnegative splitting of the first type. By the assumption H is an M-matrix, hence Lemma 2.1 implies that $\rho(M_1^{-1} N_1) < 1$. Let us assume that $P_1^* H = I - \hat{L}^* - \hat{U}^*$, using the fact that $\rho(\hat{L}^* + \hat{U}^*) = \rho(M_1^{-1} N_1) < 1$, by Lemma 2.2 and Lemma 2.3, it is easy to know that $P_1^* H$ is an M-matrix. The similar results can be gotten when $i = 2, 3$. □

Now, we will show that in the case that the GMTS method converges, the preconditioned GMTS methods converge faster.

3.5. Theorem. *Let T and T_1^* be the iteration matrices of the GMTS and the preconditioned GMTS methods, respectively, assume that the matrix H is irreducible, $L \le 0, U \le 0, B \ge 0, C \ge 0, 0 \le D_1 \le \hat{D}$, $0 \le \bar{D}_1 \le \hat{D}_1^*, 0 \le L_1 \le \hat{L}$, $0 \le \bar{L}_1 \le \hat{L}_1^*, b_{i,1} > 0, c_{j,1} > 0$, for some $i \in \{2, 3, ..., p\}, j \in \{2, 3, ..., q\}$. If $\rho(T) < 1, \bar{D}_1 \le D_1$ and $\bar{L}_1 \le L_1$, then $\rho(T_1^*) \le \rho(T)$.*

Proof. As the matrix H is irreducible, so the $P_1^* H$ is irreducible. And by Lemma 3.1, we know that T and T_1^* are irreducible. Consider the GMTS splitting for the matrix $H = M - N$, where $M = \hat{D} + D_1 + L_1 - \hat{L}$, $N = D_1 + L_1 + \hat{U}$.

Obviously, $H = M - N$ is a regular splitting, and by the assumption $\rho(M^{-1}N) < 1$, we can get that H is an M-matrix. From Theorem 3.4, we know that $P_1^* H$ is also an M-matrix. Thus, from Lemma 3.3, we know that $\bar{H}_1 = \bar{M}_1 - \bar{N}_1$ is a regular splitting. Therefore, as H is an M-matrix, we can get $\rho(T_1^*) = \rho(\bar{M}_1^{-1}\bar{N}_1) < 1$.

Now, we define the following splitting for the matrix H, $H = M_1^* - N_1^*$, in which $M_1^* = (I + \bar{S}_1)^{-1}\bar{M}_1$, $N_1^* = (I + \bar{S}_1)^{-1}\bar{N}_1$ and

$$\bar{S}_1 = \begin{pmatrix} S_1 & 0 \\ 0 & V_1 \end{pmatrix}.$$

Consider the iteration matrix of the GMTS method $T = M^{-1}N$, it is easy to see that

$$M - \bar{M}_1 = \begin{pmatrix} D_{11} - D_{11}^* & 0 \\ L_{21} + L - L_{21}^* - L_1^* & D_{22} - D_{22}^* \end{pmatrix},$$

where

$$D_1 = \begin{pmatrix} D_{11} & 0 \\ 0 & D_{22} \end{pmatrix} \le \hat{D},\ \bar{D}_1 = \begin{pmatrix} D_{11}^* & 0 \\ 0 & D_{22}^* \end{pmatrix} \le \hat{D}_1^*,$$

$L_1 = \begin{pmatrix} 0 & 0 \\ L_{21} & 0 \end{pmatrix} \le \hat{L}$ and $\bar{L}_1 = \begin{pmatrix} 0 & 0 \\ L_{21}^* & 0 \end{pmatrix} \le \hat{L}_1$.

It is known that $L_1^* = (I + V_1)L$, hence $L_1^* - L = V_1 L \le 0$.

By computations, we know that $\bar{M}_1 \le M$, so $\bar{M}_1^{-1} \ge M^{-1}$. Consequently,

$$M^{-1} \le \bar{M}_1^{-1} \le \bar{M}_1^{-1}(I + \bar{S}_1) = (M_1^*)^{-1}.$$

From Lemma 2.4, we deduce that

$$\rho(\bar{M}_1^{-1}\bar{N}_1) = \rho((M_1^*)^{-1}N_1^*) \le \rho(M^{-1}N),$$

so $\rho(T_1^*) \le \rho(T)$. □

Similar to the proof of Theorem 3.5, we can get the following two theorems.

3.6. Theorem. *Let T and T_2^* be the iteration matrices of the GMTS and the preconditioned GMTS methods, respectively. Assume that the matrix H is irreducible, $L \le 0, U \le 0, B \ge 0, C \ge 0, 0 \le D_2 \le \hat{D}$, $0 \le \bar{D}_2 \le \hat{D}_2^*, 0 \le L_2 \le \hat{L}$, $0 \le \bar{L}_2 \le \hat{L}_2^*, b_{1,i} > 0, c_{1,j} > 0$, for some $i \in \{2, 3, ..., p\}, j \in \{2, 3, ..., q\}$. If $\rho(T) < 1, \bar{D}_2 \le D_2$ and $\bar{L}_2 \le L_2$, then $\rho(T_2^*) \le \rho(T)$.*

3.7. Theorem. *Let T and T_3^* be the iteration matrices of the GMTS and the preconditioned GMTS methods, respectively. Assume that the matrix H is irreducible, $L \le 0, U \le 0, B \ge 0, C \ge 0$, $0 \le D_3 \le \hat{D}$, $0 \le \bar{D}_3 \le \hat{D}_3^*$, $0 \le L_3 \le \hat{L}$, $0 \le \bar{L}_3 \le \hat{L}_3^*$, $b_{i,1} > 0, c_{j,1} > 0, b_{1,i} > 0, c_{1,j} > 0$, for some $i \in \{2, 3, ..., p\}, j \in \{2, 3, ..., q\}$. If $\rho(T) < 1, \bar{D}_3 \le D_3$ and $\bar{L}_3 \le L_3$, then $\rho(T_3^*) \le \rho(T)$.*

Now, we prove that in the case that the GMTS method is convergent, using the third preconditioned GMTS method leads to the better convergence rate than the first and the second preconditioned GMTS methods.

3.8. Theorem. *Suppose that the matrix H is irreducible, $L \le 0, U \le 0, B \ge 0, C \ge 0$, $b_{i,1} > 0, c_{j,1} > 0, b_{1,i} > 0, c_{1,j} > 0$, for some $i \in \{2, 3, ..., p\}, j \in$*

$\{2, 3, ..., q\}$, *the auxiliary block diagonal matrices are chosen as* $\alpha_i I$ *and the auxiliary block lower triangular matrices as* $\beta_i L_i^*$ *for* $i = 1, 3,\ 0 \le \alpha_3 \le \alpha_1 \le 1, 0 \le \beta_1 \le \beta_3 \le 1$. *Then* $\rho(T_3^*) \le \rho(T_1^*)$ *if* $\rho(T) < 1$.

Proof. By the assumption $\rho(T) < 1$, and according the Lemma 2.1, H is an M-matrix. Assume that $P_i^* H = \widetilde{M}_i - \widetilde{N}_i, i = 1, 3$ where

$$\widetilde{M}_i = \begin{pmatrix} I + D_{11}^i & 0 \\ L_{21}^i + L_i^* & I + D_{22}^i \end{pmatrix}, \qquad \widetilde{N}_i = \begin{pmatrix} B_i^* + D_{11}^i & -U_i^* \\ L_{21}^i & C_i^* + D_{22}^i \end{pmatrix},$$

and $L_{21}^i = -\beta_i L_i^*, D_{11}^i = \alpha_i I_p, D_{22}^i = \alpha_i I_q$ for $i = 1, 3$.

Now, we define the following splitting for the matrix H, i.e. $H = M_i - N_i (i = 1, 3)$ such that $M_i = (I + \widetilde{S}_i)^{-1} \widetilde{M}_i$ and $N_i = (I + \widetilde{S}_i)^{-1} \widetilde{N}_i$,

where $\widetilde{S}_i = \begin{pmatrix} S_i & 0 \\ 0 & V_i \end{pmatrix}$.

Since

$$L_{21}^1 - L_{21}^3 = -\beta_1 L_1^* + \beta_3 L_3^* \ge \beta_1 L_3^* - \beta_1 L_1^* = -\beta_1 (L_1^* - L_3^*),$$

so

$$L_{21}^1 - L_{21}^3 + L_1^* - L_3^* \ge (1 - \beta_1)(L_1^* - L_3^*),$$

then

$$\begin{aligned} \widetilde{M}_1 - \widetilde{M}_3 &= \begin{pmatrix} D_{11}^1 - D_{11}^3 & 0 \\ L_{21}^1 - L_{21}^3 + L_1^* - L_3^* & D_{22}^1 - D_{22}^3 \end{pmatrix} \\ &\ge \begin{pmatrix} (\alpha_1 - \alpha_3) I_p & 0 \\ (1 - \beta_1)(L_1^* - L_3^*) & (\alpha_1 - \alpha_3) I_q \end{pmatrix}, \end{aligned}$$

as $L_1^* - L_3^* = (V_1 - V_3) L \ge 0$, then $\widetilde{M}_1 \ge \widetilde{M}_3$.

Notice that $\widetilde{M}_1^{-1} \ge 0, \widetilde{M}_3^{-1} \ge 0$, hence $\widetilde{M}_1^{-1} \le \widetilde{M}_3^{-1}$ and

$$\begin{aligned} M_1^{-1} &= \widetilde{M}_1^{-1}(I + \widetilde{S}_1) \\ &= \widetilde{M}_1^{-1} + \widetilde{M}_1^{-1} \widetilde{S}_1 \\ &\le \widetilde{M}_3^{-1} + \widetilde{M}_1^{-1}(\widetilde{S}_1 - \widetilde{S}_3) + \widetilde{M}_1^{-1} \widetilde{S}_3 \\ &\le \widetilde{M}_3^{-1} + \widetilde{M}_3^{-1} \widetilde{S}_3 \\ &= \widetilde{M}_3^{-1}(I + \widetilde{S}_3) = M_3^{-1}. \end{aligned}$$

Since H is an M-matrix, Lemma 2.4 implies that

$$\rho(M_3^{-1} N_3) \le \rho(M_1^{-1} N_1).$$

According $M_i^{-1} N_i = \widetilde{M}_i^{-1} \widetilde{N}_i$ for $i = 1, 3$, we can conclude that

$$\rho(T_3^*) \le \rho(T_1^*).$$

□

Similar to the proof of Theorem 3.8, we can get the following theorem.

3.9. Theorem. *Suppose that the matrix* H *is irreducible,* $L \le 0, U \le 0, B \ge 0, C \ge 0$, $b_{i,1} > 0, c_{j,1} > 0, b_{1,i} > 0, c_{1,j} > 0$, *for some* $i \in \{2, 3, ..., p\}, j \in \{2, 3, ..., q\}$, *the auxiliary block diagonal matrices are chosen as* $\alpha_i I$ *and the auxiliary block lower triangular matrices as* $\beta_i L_i^*$ *for* $i = 2, 3,\ 0 \le \alpha_3 \le \alpha_2 \le 1, 0 \le \beta_2 \le \beta_3 \le 1$. *Then* $\rho(T_3^*) \le \rho(T_2^*)$ *if* $\rho(T) < 1$.

4. Examples

4.1 Example Consider

$$H = \begin{pmatrix} I-B & U \\ L & I-C \end{pmatrix},$$

where $B = (b_{ij})_{p\times p}$, $C = (c_{ij})_{(n-p)\times(n-p)}$, $L = (l_{ij})_{(n-p)\times p}$ and $U = (u_{ij})_{p\times(n-p)}$ with

$$b_{ii} = \frac{1}{10\times(i+1)}, \quad i = 1,2,\cdots,p,$$

$$b_{ij} = \frac{1}{30} - \frac{1}{30\times j+i}, \quad i<j, \quad i=1,2,\cdots,p-1, \quad j=2,\cdots,p,$$

$$b_{ij} = \frac{1}{30} - \frac{1}{30\times(i-j+1)+i}, \quad i>j, \quad i=2,\cdots,p,\ j=1,2,\cdots,p-1,$$

$$c_{ii} = \frac{1}{10\times(p+i+1)}, \quad i=1,2,\cdots,n-p,$$

$$c_{ij} = \frac{1}{30} - \frac{1}{30\times(p+j)+p+i}, \ i<j,\ i=1,2,\cdots,n-p-1,\ j=2,\cdots,n-p,$$

$$c_{ij} = \frac{1}{30} - \frac{1}{30\times(i-j+1)+p+i}, \ i>j,\ i=2,\cdots,n-p,\ j=1,2,\cdots,n-p-1,$$

$$l_{ij} = \frac{1}{30\times(p+i-j+1)+p+i} - \frac{1}{30}, \ i=1,2,\cdots,n-p,\ j=1,2,\cdots,p,$$

$$u_{ij} = \frac{1}{30\times(p+j)+i} - \frac{1}{30}, \ i=1,2,\cdots,p,\ j=1,2,\cdots,n-p.$$

In the experiments, the auxiliary matrices are chosen such that

$$D_1 = 0.5(\frac{1}{\omega}-1)I,\ \overline{D}_1 = 0.5(\frac{1}{\omega}-1)I,\ L_1 = 0.5(1-\frac{\gamma}{\omega})\widehat{L_i},\ \overline{L}_1 = 0.5(1-\frac{\gamma}{\omega})\widehat{L}_i^*.$$

From Table 1, we see that these results accord with Theorems 3.5 - 3.9.

Table 1. The spectral radii of the GMTS and preconditioned GMTS iteration matrices

n	ω	r	p	$\rho(T)$	$\rho(T_1^*)$	$\rho(T_2^*)$	$\rho(T_3^*)$
10	0.9	0.8	5	0.2352	0.2156	0.2140	0.2048
20	0.8	0.6	5	0.5736	0.5609	0.5605	0.5568
20	0.8	0.6	10	0.5551	0.5413	0.5404	0.5334
25	0.8	0.6	8	0.7164	0.7074	0.7070	0.7033
30	0.9	0.7	10	0.8680	0.8635	0.8633	0.8613
30	0.9	0.7	20	0.8676	0.8630	0.8627	0.8605

In [4], the authors considered the following preconditioner

$$P^* = \begin{pmatrix} I+S & 0 \\ 0 & I+V \end{pmatrix}, \tag{4.1}$$

where

$$S = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ \frac{b_{p1}}{\alpha} & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$V = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ \frac{c_{q1}}{\beta} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Table 2. The spectral radii of the preconditioned GMTS iteration matrices

n	ω	r	p	$\alpha=\beta$	$\rho(T^*)$	$\rho(T_1^*)$	$\rho(T_2^*)$	$\rho(T_3^*)$
10	0.9	0.8	5	3	0.2335	0.2156	0.2140	0.2048
20	0.8	0.6	5	2	0.5729	0.5609	0.5605	0.5568
20	0.8	0.6	10	2	0.5542	0.5413	0.5404	0.5334
25	0.8	0.6	8	3	0.7161	0.7074	0.7070	0.7033
30	0.9	0.7	10	2	0.8678	0.8635	0.8633	0.8613
30	0.9	0.7	20	2	0.8673	0.8630	0.8627	0.8605

Here, T^* is the GMTS iteration matrix for solving $P^*Hy = P^*f$.

From Table 2, we see that the preconditioned GMTS methods with preconditioners in this paper have better converge rates than the preconditioned GMTS method with preconditioner P^*.

4.2 Example The coefficient matrix H in Equation (1.2) is given by

$$H = \begin{pmatrix} I-B & U \\ L & I-C \end{pmatrix},$$

where

$$B = \begin{pmatrix} b_{11} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 \end{pmatrix}, \quad C = \begin{pmatrix} c_{11} & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix},$$

$$L = \begin{pmatrix} -\frac{1}{4} & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} & 0 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} -\frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} \\ -\frac{1}{4} & 0 & 0 \end{pmatrix}.$$

Table 3 displays the spectral radii of the corresponding iteration matrices with $\omega = 0.9, \gamma = 0.8$ and different values of b_{11} and c_{11}.

From Table 3, we can see that $\rho(T_i^*) \le \rho(T)$ for $i = 1,2,3$ and $\rho(T_3^*) \le \rho(T_i^*)$ for $i = 1,2$ when $\rho(T) < 1$. These numerical results are in accordance with the theoretical results given in Theorems 3.5- 3.9.

Table 3. The spectral radii of the GMTS and preconditioned GMTS iteration matrices

b_{11}	c_{11}	$\rho(T)$	$\rho(T_1^*)$	$\rho(T_2^*)$	$\rho(T_3^*)$
0	0	0.6804	0.6303	0.6381	0.6140
0	0.3	0.7657	0.7253	0.7323	0.7071
0.2	0.2	0.7614	0.7186	0.7265	0.6987
0.2	0.5	0.8860	0.8677	0.8713	0.8596
0.5	0.5	0.9553	0.9483	0.9499	0.9453

5. Conclusion

In this paper, we propose three new preconditioners and give comparison theorems between the preconditioned and original methods. These results show that the preconditioned GMTS methods converge faster than the GMTS method whenever the GMTS method is convergent. Finally, we give two examples to confirm our theoretical results.

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A Linear Adaptive time-stepping Method for Solving Vibration Problems with Damping Terms

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Abstract

A linear adaptive time-stepping method is devised for linear or nonlinear damping vibration analysis, which has wide applications in civil engineering. In the time direction, the underlying problem is discretized by a linear C^0-continuous discontinuous Galerkin method combined with the technique of linearization. By means of the energy method, some optimal a posteriori error estimates are established for linear vibration problems. Motivated by these estimates, we design an adaptive time-stepping strategy for actual computation. Numerical results are performed to illustrate the efficiency of the adaptive method.

Keywords. Time-stepping method, Vibration, Damping, A posteriori error analysis, Adaptive algorithm

1 Introduction

This paper aims to design and analyze an adaptive time-stepping method for solving the following problem:

For any real number $T > 0$, find $\mathbf{u} : [0, T] \to \mathbb{R}^d$ (with d the spatial dimension) such that

$$\begin{cases} \mathbf{M}\mathbf{u}''(t) + \mathbf{F}\big(t, \mathbf{u}(t), \mathbf{u}'(t)\big) = 0, & 0 < t < T, \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}'(0) = \mathbf{v}_0, \end{cases} \tag{1.1}$$

where $(\cdot)'$ and $(\cdot)''$ denote respectively the first and second order derivatives in time; $\mathbf{M}$ is a given $(d \times d)$ matrix and $\mathbf{F}$ is a given vector-valued function from $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ into $\mathbb{R}^d$; $\mathbf{u}_0$ and $\mathbf{v}_0$ are two given vectors in $\mathbb{R}^d$.

The above problem is frequently encountered in structure analysis of dynamical transient response (cf. [5]). Concretely speaking, the mathematical models for structure analysis are described by a system of second-order linear/nonlinear evolution equations, which give rise to the problem (1.1), after spatial discretization by finite element methods, finite difference methods or spectral methods (cf. [2, 9, 11, 16, 17, 21, 22]).

When the vector-valued function $\mathbf{F}$ is linear with respect to $\mathbf{u}$ and $\mathbf{u}'$, there are various numerical methods for solving the problem (1.1). The most widely used may be classified as modal superposition (cf. [6, 14]) and direct-time integration methods including the Runge-Kutta, central difference, Houbolt, Newmark-β and Wilson-θ methods (see [11] and the references therein for details). The space-time finite element method (cf. [7, 12, 13]) is another widely developed approach for solving second order time evolution equations. One typical way is using the time-discontinuous Galerkin (TDG) method (cf. [7, 15]) in the time direction for the displacement and velocity fields together, but it has the disadvantage that an ill-conditioned (4×4) block system must be solved at each time step, which is time consuming. To overcome this difficulty, some linear C^0-continuous time-stepping methods were used in [18], where only the primal variables are involved and only a (1×1) block system

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should be solved at each time step. Moreover, an adaptive method was proposed in [18] for solving second order abstract evolution equations, where the optimal a posteriori error estimates are established, which, in conjunction with the error equidistribution strategy and some ideas implied in the Runge-Kutta-Felberg method, leads to an adaptive time-stepping method.

In this paper, we intend to use some ideas in [18] to develop an adaptive time-stepping method for solving the problem (1.1). In the time direction, the problem (1.1) is discretized by a linear C^0-continuous discontinuous Galerkin method combined with the technique of linearization (including three linearization methods). Then, by means of the energy method, some optimal a posteriori error estimates are established for linear vibration problems via some ideas in [18]. It deserves to emphasize that the mathematical argument developed here is greatly simplified by using the Lagrange basis functions instead of the Legendre polynomials. Motivated by these estimates, we construct a posteriori error estimates for nonlinear problems, based on which we design an adaptive time-stepping strategy for actual computation. Numerical results are performed to illustrate the efficiency of the adaptive method.

The rest of this paper is organized as follows. In Section 2, we present a time-stepping finite element method for the problem (1.1), and the detailed implementation of the previous method is also developed for actual computation. In Section 3, a posteriori error analysis is established in detail for linear vibration problems. In Section 4, we propose an adaptive algorithm based on some a posteriori error estimates. A series of numerical results are performed in the final section.

2 A linear time-stepping finite element method

2.1 The formulation of a linear time-stepping finite element method

Throughout this paper, we assume that Problem (1.1) has a unique solution and the matrix $\mathbf{M}$ is symmetric positive definite. We use a standard time-stepping method to discretize Problem (1.1) (cf. [10, 18, 19]). To this end, we first partition the time interval $I := (0, T)$ with the nodes

$$0 = t_0 < t_1 < \cdots < t_N = T,$$

to get the following subintervals:

$$J_n = (t_{n-1}, t_n], \quad k_n = t_n - t_{n-1}, \quad 1 \le n \le N.$$

Define

$$\mathcal{V}_1 = \left\{ \mathbf{v} : \bar{I} \to \mathbb{R}^d;\ \mathbf{v} \in C(\bar{I}),\ \mathbf{v}|_{J_n}(t) = \sum_{j=0}^{1} t^j \mathbf{w}_j,\ \mathbf{w}_j \in \mathbb{R}^d,\ 1 \le n \le N \right\},$$

$$\mathcal{W}_2 = \left\{ \mathbf{v} : \bar{I} \to \mathbb{R}^d;\ \mathbf{v} \in C^1(\bar{I}),\ \mathbf{v}|_{J_n}(t) = \sum_{j=0}^{2} t^j \mathbf{w}_j,\ \mathbf{w}_j \in \mathbb{R}^d,\ 1 \le n \le N \right\},$$

$$\mathcal{H}_q = \left\{ \mathbf{v} : \bar{I} \to L^2(I);\ \mathbf{v}|_{J_n}(t) = \sum_{j=0}^{q} t^j \mathbf{w}_j,\ \mathbf{w}_j \in \mathbb{R}^d,\ 1 \le n \le N \right\}, \quad q = 0, 1.$$

Let $\mathcal{V}_1(J_n)$ and $\mathcal{W}_2(J_n)$ be the restrictions of $\mathcal{V}_1$ and $\mathcal{W}_2$ to J_n, respectively. Similarly, denote by $\mathcal{H}_q(J_n)$ the restriction of $\mathcal{H}_q$ to J_n. Thus, our time-stepping method for (1.1) is to find $\mathbf{U} \in \mathcal{V}_1$ such that

$$\begin{cases} \displaystyle\int_{J_n} \left(\langle \mathbf{U}'', \ \mathbf{w}' \rangle_{\mathbf{M}} + \langle \mathbf{F}(t, \mathbf{U}, \mathbf{U}'), \ \mathbf{w}' \rangle\right) \mathrm{d}t + \left\langle \dot{\mathbf{U}}_+^{n-1} - \dot{\mathbf{U}}_-^{n-1}, \ \dot{\mathbf{w}}_+^{n-1} \right\rangle_{\mathbf{M}} = 0, \\ \mathbf{U}^0 = \mathbf{u}_0, \quad \dot{\mathbf{U}}_-^0 = \mathbf{v}_0, \qquad \mathbf{w} \in \mathcal{V}_1(J_n), \quad 1 \le n \le N, \end{cases} \tag{2.1}$$

where

$$\langle \mathbf{a}, \mathbf{b} \rangle := \mathbf{b}^{\mathrm{T}} \mathbf{a}, \quad \langle \mathbf{a}, \mathbf{b} \rangle_{\mathbf{A}} := \mathbf{b}^{\mathrm{T}} \mathbf{A} \mathbf{a}, \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^d, \ \mathbf{A} \in \mathbb{R}^{d \times d}, \tag{2.2}$$

$$\dot{\mathbf{w}}_\pm^{n-1} := \lim_{s \to 0^+} \mathbf{w}'(t_{n-1} \pm s), \quad \mathbf{w}^{n-1} := \mathbf{w}(t_{n-1}).$$

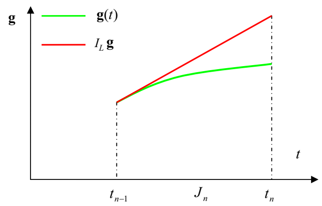
2.2 Implementation of the time-stepping method

Since $\mathbf{U} \in \mathcal{V}_1$, we have by a direct manipulation that, for any $t \in J_n$,

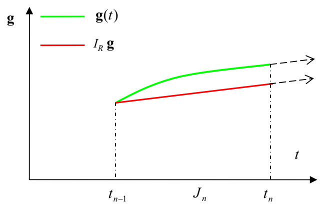
$$\mathbf{U}(t) = \mathbf{U}^{n-1} + (t - t_{n-1}) \dot{\mathbf{U}}_-^n, \qquad \mathbf{U}'(t) = \dot{\mathbf{U}}_-^n, \qquad \mathbf{U}''(t) = \mathbf{0}. \tag{2.3}$$

To implement the method (2.1) in actual computation, we require to linearize the nonlinear function $\mathbf{F}(\mathbf{t}, \mathbf{U}, \mathbf{U}')$ with respect to $\mathbf{U}$. As shown in Figure 1, for a given function $g(t)$, its linearization over J_n are usually the interpolants given by

$$\mathcal{I}_L \mathbf{g}(t) = \mathbf{g}(t_{n-1}) + (t - t_{n-1}) \mathbf{g}'(t_{n-1}) \quad \text{or} \quad \mathcal{I}_R \mathbf{g}(t) = \mathbf{g}(t_{n-1}) + (t - t_{n-1}) \mathbf{g}'(t_n), \qquad t \in J_n.$$



(a) $\mathcal{I}_L$



(b) $\mathcal{I}_R$

Figure 1: Diagrams of the (local) interpolate operators $\mathcal{I}_L$ and $\mathcal{I}_R$.

Note that the function $\mathbf{F} = \mathbf{F}(\mathbf{t}, \mathbf{U}, \mathbf{U}')$ is discontinuous at the interior node t_n. Recalling the expression (2.3), we have by the direct computation that the right limit of $\mathbf{F}$ at $t = t_{n-1}$ can be expressed as

$$\mathbf{F}_+^{n-1} = \mathbf{F}(t_{n-1}, \mathbf{U}^{n-1}, \dot{\mathbf{U}}_+^{n-1}) = \mathbf{F}(t_{n-1}, \mathbf{U}^{n-1}, \dot{\mathbf{U}}_-^n). \tag{2.4}$$

Using the chain rule for differentiation and (2.3), we find that, at $t = t_n$, the left limit of the full derivative of $\mathbf{F}(t, \mathbf{U}, \mathbf{U}')$ with respect to t is given as follows:

$$\begin{aligned} \dot{\mathbf{F}}_-^n &= \frac{\partial \mathbf{F}}{\partial t}(t_n, \mathbf{U}^n, \dot{\mathbf{U}}_-^n) + \frac{\partial \mathbf{F}}{\partial \mathbf{U}}(t_n, \mathbf{U}^n, \dot{\mathbf{U}}_-^n) \dot{\mathbf{U}}_-^n + \frac{\partial \mathbf{F}}{\partial \mathbf{U}'}(t_n, \mathbf{U}^n, \dot{\mathbf{U}}_-^n) \mathbf{0} \\ &= \frac{\partial \mathbf{F}}{\partial t}(t_n, \mathbf{U}^n, \dot{\mathbf{U}}_-^n) + \frac{\partial \mathbf{F}}{\partial \mathbf{U}}(t_n, \mathbf{U}^n, \dot{\mathbf{U}}_-^n) \dot{\mathbf{U}}_-^n \\ &=: \left. \frac{\partial \mathbf{F}}{\partial t} \right|_{t_-^n} + \left. \frac{\partial \mathbf{F}}{\partial \mathbf{U}} \right|_{t_-^n} \dot{\mathbf{U}}_-^n. \end{aligned}$$

Similarly, we have

$$\dot{\mathbf{F}}_+^{n-1} := \left. \frac{\partial \mathbf{F}}{\partial t} \right|_{t_+^{n-1}} + \left. \frac{\partial \mathbf{F}}{\partial \mathbf{U}} \right|_{t_+^{n-1}} \dot{\mathbf{U}}_-^n.$$

With these results in mind, we have by the definitions of the interpolation operators $\mathcal{I}_L$ and $\mathcal{I}_R$ that

$$\text{Left side Scheme}: \ \mathbf{F}\big(t, \mathbf{U}(t), \mathbf{U}'(t)\big) \ \approx \mathcal{I}_L \mathbf{F} = \mathbf{F}_+^{n-1} + (t - t_{n-1})\dot{\mathbf{F}}_+^{n-1}, \tag{2.5}$$

$$\text{Right side Scheme}: \ \mathbf{F}\big(t, \mathbf{U}(t), \mathbf{U}'(t)\big) \ \approx \mathcal{I}_R \mathbf{F} = \mathbf{F}_+^{n-1} + (t - t_{n-1})\dot{\mathbf{F}}_-^{n}. \tag{2.6}$$

Now, inserting (2.3) and (2.6) into the first equation of (2.1) and taking $\dot{\mathbf{w}}$ to be $\mathbf{w}^*$ or $(t - t_{n-1})\mathbf{w}^*$, where $\mathbf{w}^*$ is any constant vector in $\mathbb{R}^d$, we find that the method (2.1) is equivalent to finding $\{\dot{\mathbf{U}}_-^n\}_{n=0}^N$ such that

$$\Big(\mathbf{M} + \frac{k_n^2}{2}\frac{\partial \mathbf{F}}{\partial \mathbf{U}}\Big|_{t_-^n}\Big)\dot{\mathbf{U}}_-^n + \frac{1}{2}k_n^2 \frac{\partial \mathbf{F}}{\partial t}\Big|_{t_-^n} + k_n \mathbf{F}_+^{n-1} = \mathbf{M}\dot{\mathbf{U}}_-^{n-1}, \quad 1 \le n \le N. \tag{2.7}$$

Note that the quantities $\frac{\partial \mathbf{F}}{\partial t}\big|_{t_-^n}$, $\frac{\partial \mathbf{F}}{\partial \mathbf{U}}\big|_{t_-^n}$ and $\mathbf{F}_+^{n-1}$ are all the functions of the unknown vector $\dot{\mathbf{U}}_-^n$, so the above scheme is implicit. However, if we use the linearization formulation (2.5) instead of (2.6), then the system (2.1) reduces to

$$\Big(\mathbf{M} + \frac{k_n^2}{2}\frac{\partial \mathbf{F}}{\partial \mathbf{U}}\Big|_{t_+^{n-1}}\Big)\dot{\mathbf{U}}_-^n + \frac{1}{2}k_n^2 \frac{\partial \mathbf{F}}{\partial t}\Big|_{t_+^{n-1}} + k_n \mathbf{F}_+^{n-1} = \mathbf{M}\dot{\mathbf{U}}_-^{n-1}, \quad 1 \le n \le N. \tag{2.8}$$

It is noted that in most vibration problems, it suffices for us to deal with the linear damping case, indicating that the function $\mathbf{F}$ is linear with respect to the independent variable $\mathbf{u}'$. In this case, since the quantities $\frac{\partial \mathbf{F}}{\partial t}\big|_{t_+^{n-1}}$ and $\frac{\partial \mathbf{F}}{\partial \mathbf{U}}\big|_{t_+^{n-1}}$ in (2.8) do not depend on $\dot{\mathbf{U}}_-^n$, the system (2.8) is essentially a linear system of the unknown vector $\dot{\mathbf{U}}_-^n$. Hence, we can work out $\dot{\mathbf{U}}_-^n$ with much less computational cost, compared to the method (2.7).

In order to balance the efficiency and stability of the time-stepping method, it is very natural to split the nonlinear term $\mathbf{F}$ into two parts $\mathbf{F}_L$ and $\mathbf{F}_R$, which correspond to the non-stiff and the stiff terms of the original system (1.1), respectively. Then, it is better for us to use $\mathcal{I}_L\mathbf{F}_L + \mathcal{I}_R\mathbf{F}_R$ to approximate $\mathbf{F}$ in (2.1). In other words, we have

$$\text{Semi} - \text{side Scheme}: \ \mathbf{F} \ \approx \mathcal{I}_L\mathbf{F}_L + \mathcal{I}_R\mathbf{F}_R = \mathbf{F}_+^{n-1} + (t - t_{n-1})\big(\dot{\mathbf{F}}_{L+}^{n-1} + \dot{\mathbf{F}}_{R-}^{n}\big). \tag{2.9}$$

It is noted that for the linear damping system, the semi-side scheme also yields a linear system for getting the unknown vector $\dot{\mathbf{U}}_-^n$.

Now, let us present the solution process of the method (1.1) in detail. Once we obtain $\mathbf{U}$ in J_{n-1}, we can get $\dot{\mathbf{U}}_-^n$ by solving the system (2.7) or (2.8). Then the function $\mathbf{U}$ over J_n is completely determined using the formulation $\mathbf{U}(t) = \mathbf{U}^{n-1} + (t - t_{n-1})\dot{\mathbf{U}}_-^n$ for all $t \in J_n$. On implementing this computation recursively, we can thereby determine the function $\mathbf{U}$ completely.

In the last part of this subsection, we give the solution process explicitly for the vibration analysis related to linear transient dynamic response. At this moment, we can reformulate the problem (1.1) as follows.

For any real number $T > 0$, find $\mathbf{u} : [0, T] \to \mathbb{R}^d$ such that

$$\begin{cases} \mathbf{M}\mathbf{u}'' + \mathbf{C}\mathbf{u}' + \mathbf{K}\mathbf{u} = \mathbf{f}, & 0 < t < T, \\ \mathbf{u}(0) = \mathbf{u}_0, \\ \mathbf{u}'(0) = \mathbf{v}_0, \end{cases} \tag{2.10}$$

where $\mathbf{C}$ and $\mathbf{K}$ are the $(d \times d)$ damping and stiffness matrices of the dynamic system, respectively. We assume that $\mathbf{C}$ and $\mathbf{K}$ are symmetric and semi-definite. Observing that

$$\mathbf{F}\big(t, \mathbf{u}(t), \mathbf{u}'(t)\big) = \mathbf{C}\mathbf{u}' + \mathbf{K}\mathbf{u} - \mathbf{f},$$

we have from the variational formulation (2.1) that

$$\left(\frac{k_n^2}{2}\mathbf{K}+k_n\mathbf{C}+\mathbf{M}\right)\dot{\mathbf{U}}_-^n=\mathbf{M}\dot{\mathbf{U}}_-^{n-1}-k_n\mathbf{K}\mathbf{U}^{n-1}+\mathbf{f}^n,\quad 1\le n\le N, \tag{2.11}$$

where $\mathbf{f}^n := \int_{J_n}\mathbf{f}\mathrm{d}t$.

3 A posteriori error analysis for linear problems

For the numerical method (2.1) for the linear vibration problem (2.10), following the similar arguments leading to Theorem 2.5 in [18], we can derive some stability estimates to the numerical solution $\mathbf{U}$ and then establish the required a priori error estimates. Another way to derive such estimates is to use the mathematical argument due to [24]. Since the objective of this article is to develop efficient adaptive time stepping method for the linear vibration problem (2.10) and the generalized problem (1.1), we will focus on in this section a posteriori error analysis for the problem (2.10) discretized by the method (2.1). Motivated by such an analysis, we will heuristically mention in the next section some error estimators for the nonlinear problem (1.1) and then devise the corresponding adaptive time stepping method.

3.1 Reconstruction

As shown in [18], in order to get efficient a posteriori error estimates for the method (2.1), we require to construct a higher order reconstruction $\widetilde{\mathbf{U}}$ from the approximate solution $\mathbf{U}$. So let us first recall such a reconstruction given in [18]. Introduce an invertible linear operator $\widetilde{I}_2:\ \mathcal{V}_1\to\mathcal{W}_2$ as follows. With any $\mathbf{w}\in\mathcal{V}_1$ we associate an element $\widetilde{\mathbf{w}}:=\ \widetilde{I}_2\mathbf{w}\in\mathcal{W}_2$ defined by locally interpolating $\mathbf{w}$ in each subinterval $J_n(1\le n\le N)$, i.e., $\widetilde{\mathbf{w}}|_{J_n}\in\mathcal{W}_2(J_n)$ is uniquely determined by

$$\widetilde{\mathbf{w}}(t)=\widetilde{\mathbf{w}}(t_{n-1})+k_n\dot{\mathbf{w}}_-^{n-1}\Phi_0(\frac{t-t_{n-1}}{k_n})+k_n\dot{\mathbf{w}}_-^n\Phi_1(\frac{t-t_{n-1}}{k_n}),\qquad 1\le n\le N, \tag{3.1}$$

and the initial values $\widetilde{\mathbf{w}}(0)=\mathbf{w}(0)$, $\widetilde{\mathbf{w}}'(0)=\mathbf{w}'(0)$. In (3.1), the definition of Φ_0, Φ_1 are given as

$$\Phi_0(\xi)=-\frac{1}{2}\xi^2+\xi,\qquad\qquad \Phi_1(\xi)=\frac{1}{2}\xi^2. \tag{3.2}$$

We call $\widetilde{\mathbf{w}}$ a time reconstruction of $\mathbf{w}$, as shown in Figure 2. It is easy to check by the

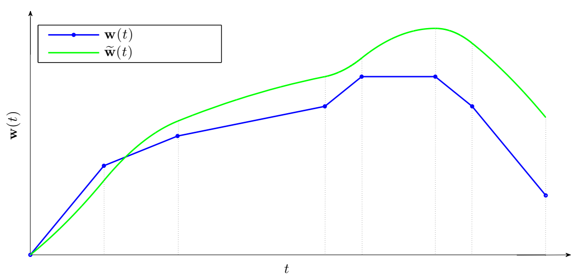


Figure 2: Diagram of $\widetilde{I}_2 w$.

above construction that

$$\widetilde{\mathbf{w}}'(t_n)=\dot{\mathbf{w}}_-^n,\quad 1\le n\le N. \tag{3.3}$$

Thus, for an approximate solution U, the reconstructed function we hope to find is $\widetilde{\mathbf{U}} \in \mathcal{W}_2$, defined by

$$\widetilde{\mathbf{U}}(t) = \widetilde{\mathbf{U}}(t_{n-1}) + k_n \dot{\mathbf{U}}_-^{n-1} \Phi_0(\frac{t - t_{n-1}}{k_n}) + k_n \dot{\mathbf{U}}_-^n \Phi_1(\frac{t - t_{n-1}}{k_n}), \qquad 1 \le n \le N. \tag{3.4}$$

By a direct computation we have

$$\widetilde{\mathbf{U}}''(t) = \frac{1}{k_n}(\dot{\mathbf{U}}_-^n - \dot{\mathbf{U}}_-^{n-1}), \qquad 1 \le n \le N. \tag{3.5}$$

Observing that the function $\mathbf{U}(t)$ can be rewritten as

$$\mathbf{U}(t) = \mathbf{U}(t_{n-1}) + k_n \dot{\mathbf{U}}_+^{n-1} \Phi_0(\frac{t - t_{n-1}}{k_n}) + k_n \dot{\mathbf{U}}_-^n \Phi_1(\frac{t - t_{n-1}}{k_n}), \quad t \in J_n,$$

subtracting which from (3.4) we know

$$\mathbf{U}(t) - \widetilde{\mathbf{U}}(t) = \mathbf{U}^{n-1} - \widetilde{\mathbf{U}}^{n-1} + k_n(\dot{\mathbf{U}}_+^{n-1} - \dot{\mathbf{U}}_-^{n-1}) \Phi_0(\frac{t - t_{n-1}}{k_n}), \quad t \in J_n. \tag{3.6}$$

Hence,

$$\mathbf{U}^n - \widetilde{\mathbf{U}}^n = \mathbf{U}^{n-1} - \widetilde{\mathbf{U}}^{n-1} + \frac{1}{2} k_n^2 \widetilde{\mathbf{U}}'', \quad t \in J_n,$$

i.e.,

$$\mathbf{U}^n - \widetilde{\mathbf{U}}^n = \frac{1}{2} \sum_{m=1}^{n} k_m^2 \widetilde{\mathbf{U}}''|_{J_m}, \quad t \in J_n. \tag{3.7}$$

Moreover, by integration by parts and (3.3), it follows that

$$\int_{J_n} \langle \widetilde{\mathbf{U}}'',\ \mathbf{w}' \rangle_{\mathbf{M}} \mathrm{d}t = \int_{J_n} \langle \mathbf{U}'',\ \mathbf{w}' \rangle_{\mathbf{M}} \mathrm{d}t + \langle \dot{\mathbf{U}}_+^{n-1} - \dot{\mathbf{U}}_-^{n-1}, \dot{\mathbf{w}}_+^{n-1} \rangle_{\mathbf{M}}, \quad \mathbf{w} \in \mathcal{V}_1(J_n),$$

and use the variational equation in (2.1) we further have

$$\int_{J_n} (\langle \widetilde{\mathbf{U}}'',\ \mathbf{w}' \rangle_{\mathbf{M}} + \langle \mathbf{C}\mathbf{U}' + \mathbf{K}\mathbf{U} - \mathbf{f},\ \mathbf{w}' \rangle)\ \mathrm{d}t = 0, \quad \mathbf{w} \in \mathcal{V}_1 \quad 1 \le n \le N,$$

i.e.,

$$\mathbf{M}\widetilde{\mathbf{U}}'' + P_0(\mathbf{C}\mathbf{U}' + \mathbf{K}\mathbf{U} - \mathbf{f}) = 0, \qquad t \in J_n, \tag{3.8}$$

where P_q $(q = 0, 1)$ stands for the (local) L^2 orthogonal projection operator on to $\mathcal{H}_q(J_n)$ (cf. [1]), defined by

$$\int_{J_n} \langle P_q \mathbf{v} - \mathbf{v},\ \mathbf{w} \rangle \mathrm{d}t = 0, \qquad \mathbf{w} \in \mathcal{H}_q(J_n). \tag{3.9}$$

3.2 Error estimates

Let $\|\cdot\|$, $\|\cdot\|_{\mathbf{M}}$, $\|\cdot\|_{\mathbf{C}}$ and $\|\cdot\|_{\mathbf{K}}$ be the norms (or seminorms) over $\mathbb{R}^d$, defined by the inner products (2.2), respectively. We further define

$$\|\mathbf{v}\|_{L^\infty_{\mathbf{M}}(G)} = \operatorname*{ess\,sup}_{t \in G} \|\mathbf{v}(t)\|_{\mathbf{M}}, \qquad \|\mathbf{v}\|_{L^\infty_{\mathbf{M}^{-1}}(G)} = \operatorname*{ess\,sup}_{t \in G} \|\mathbf{v}(t)\|_{\mathbf{M}^{-1}}, \tag{3.10}$$

where $\mathbf{M}^{-1}$ is the inverse of the matrix $\mathbf{M}$. We assume that for the given function f, the linear problem (2.10) has a unique solution satisfying that

$$\mathbf{u} \in C([0, T];\ \mathbb{R}^d) \cap C^1([0, T];\ \mathbb{R}^d).$$

Let $\widetilde{\mathbf{e}} := \mathbf{u} - \widetilde{\mathbf{U}}$ and $\widetilde{\mathbf{R}}$ be the residual of $\widetilde{\mathbf{U}}$ given by

$$\widetilde{\mathbf{R}}(t) := \mathbf{M}^{-1}(\mathbf{M}\widetilde{\mathbf{U}}''(t) + \mathbf{C}\widetilde{\mathbf{U}}'(t) + \mathbf{K}\widetilde{\mathbf{U}}(t) - \mathbf{f}(t)), \qquad t \in J_n,\ 1 \le n \le N. \tag{3.11}$$

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Theorem 3.1 *Let* $\mathbf{u}$ *and* $\mathbf{U}$ *be the solution of* (2.10) *and* (2.1), *respectively. Let* $\widetilde{\mathbf{U}}$ *be the reconstruction of* $\mathbf{U}$ *by* (3.1). *Then for any* $t \in [0, T]$, *there holds*

$$\max_{0\le\tau\le t} \|(\mathbf{u}-\widetilde{\mathbf{U}})'(\tau)\|_{\mathbf{M}} \le 2\int_0^t \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\, \mathrm{d}s, \tag{3.12}$$

where $\widetilde{\mathbf{R}}$ *is given by* (3.11).

Proof. Subtracting (3.11) from (2.10) gives

$$\mathbf{M}\widetilde{\mathbf{e}}''(t) + \mathbf{C}\widetilde{\mathbf{e}}'(t) + \mathbf{K}\widetilde{\mathbf{e}}(t) = -\mathbf{M}\widetilde{\mathbf{R}}(t). \tag{3.13}$$

Then, we test (3.13) by $\widetilde{\mathbf{e}}'$ and integrate over $t \in [0,\ \tau]$ to get

$$\begin{aligned} &\int_0^\tau \left(\langle \widetilde{\mathbf{e}}''(s),\ \widetilde{\mathbf{e}}'(s)\rangle_{\mathbf{M}} + \langle \widetilde{\mathbf{e}}'(s),\ \widetilde{\mathbf{e}}'(s)\rangle_{\mathbf{C}} + \langle \widetilde{\mathbf{e}}(s),\ \widetilde{\mathbf{e}}'(s)\rangle_{\mathbf{K}}\right) \mathrm{d}s \\ &\quad = \int_0^\tau \langle -\widetilde{\mathbf{R}}(s),\ \widetilde{\mathbf{e}}'(s)\rangle_{\mathbf{M}}\, \mathrm{d}s. \end{aligned} \tag{3.14}$$

Moreover, using integration by parts and noting that $\widetilde{\mathbf{e}}(0) = \widetilde{\mathbf{e}}'(0) = 0$, we arrive at

$$\frac{1}{2}\|\widetilde{\mathbf{e}}'(\tau)\|^2_{\mathbf{M}} + \int_0^\tau \|\widetilde{\mathbf{e}}'(s)\|^2_{\mathbf{C}}\, \mathrm{d}s + \frac{1}{2}\|\widetilde{\mathbf{e}}(\tau)\|^2_{\mathbf{K}} = \int_0^\tau \langle -\widetilde{\mathbf{R}}(s),\ \widetilde{\mathbf{e}}'(s)\rangle_{\mathbf{M}}\, \mathrm{d}s,\ \tau \in [0, t]. \tag{3.15}$$

Hence, it follows from (3.15) and the Cauchy-Schwarz inequality that

$$\begin{aligned} \frac{1}{2}\Big(\max_{0\le\tau\le t}\|\widetilde{\mathbf{e}}'(\tau)\|_{\mathbf{M}}\Big)^2 &\le \max_{0\le\tau\le t}\int_0^\tau |\langle \widetilde{\mathbf{R}}(s),\ \widetilde{\mathbf{e}}'(s)\rangle_{\mathbf{M}}|\, \mathrm{d}s \\ &\le \int_0^t |\langle \widetilde{\mathbf{R}}(s),\ \widetilde{\mathbf{e}}'(s)\rangle_{\mathbf{M}}|\, \mathrm{d}s \ \le\ \max_{0\le\tau\le t}\|\widetilde{\mathbf{e}}'(\tau)\|_{\mathbf{M}} \int_0^t \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\, \mathrm{d}s, \end{aligned}$$

which readily yields

$$\max_{0\le\tau\le t}\|\widetilde{\mathbf{e}}'(\tau)\|_{\mathbf{M}} \le 2\int_0^t \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\, \mathrm{d}s, \tag{3.16}$$

as required. ■

Now, we proceed with the efficiency of the above a posteriori error estimates.

Lemma 3.1 *For* $t \in J_n$, $1 \le n \le N$,

$$\mathbf{U}(t) - P_0\mathbf{U}(t) = (t - t_{n-1} - \frac{1}{2}k_n)\dot{\mathbf{U}}^{\mathbf{n}}_{-}. \tag{3.17}$$

Moreover, for $1 \le n \le N$,

$$\|(\mathbf{U}-\widetilde{\mathbf{U}})'\|_{L^\infty_{\mathbf{M}}(J_n)} = k_n\|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_n)}. \tag{3.18}$$

Furthermore, there holds

$$\begin{aligned} 2\int_0^t \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\, \mathrm{d}s \le \sum_{m-1}^{n} \bigg(&\frac{2}{3}k_m^3\|\mathbf{K}\widetilde{\mathbf{U}}^{(3)}\|_{L^\infty_{\mathbf{M}^{-1}}(J_m)} + tk_m^2\|\mathbf{K}\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}^{-1}}(J_m)} \\ &+ \frac{1}{2}k_m^2\|\mathbf{K}\mathbf{U}'\|_{L^\infty_{\mathbf{M}^{-1}}(J_m)} + k_m^2\|\mathbf{C}\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}^{-1}}(J_m)} \\ &+ 2\int_{J_m} \|\mathbf{f}(s) - P_0\mathbf{f}(s)\|_{\mathbf{M}^{-1}}\, \mathrm{d}s\bigg). \end{aligned} \tag{3.19}$$

Proof. First of all, recalling the definition of (Local) L^2 projection (3.9), we can deduce that

$$\begin{aligned} P_0\mathbf{U}(t) &= \frac{1}{k_n}\int_{J_n}\mathbf{U}(s)\mathrm{d}s = \frac{1}{k_n}\int_{J_n}\big(\mathbf{U}^{n-1}+(s-t_{n-1})\dot{\mathbf{U}}_-^n\big)\mathrm{d}s \\ &= \mathbf{U}^{n-1}+\frac{1}{2}k_n\dot{\mathbf{U}}_-^n, \qquad\qquad t\in J_n, \end{aligned}$$

so

$$\mathbf{U}(t)-P_0\mathbf{U}(t) = (t-t_{n-1}-\frac{1}{2}k_n)\dot{\mathbf{U}}_-^n, \qquad t\in J_n. \tag{3.20}$$

On the other hand, differentiating (3.6) with respect to the variable t directly yields

$$(\mathbf{U}-\widetilde{\mathbf{U}})'(t) = -(t-t_{n-1})\widetilde{\mathbf{U}}'', \qquad t\in J_n, \tag{3.21}$$

which implies (3.18).

Moreover, we have by (3.8) and (3.11) that

$$\mathbf{M}\widetilde{\mathbf{R}} = \mathbf{K}(\widetilde{\mathbf{U}}-P_0\mathbf{U}) + \mathbf{C}\big(\widetilde{\mathbf{U}}'-P_0(\mathbf{U}')\big) - (\mathbf{f}-P_0\mathbf{f}). \tag{3.22}$$

Write

$$\mathbf{K}(\widetilde{\mathbf{U}}-P_0\mathbf{U}) = \mathbf{K}(\widetilde{\mathbf{U}}-\mathbf{U}) + \mathbf{K}(\mathbf{U}-P_0\mathbf{U}),$$

and owing to the fact that$P_0(\mathbf{U}')=\mathbf{U}'$ we know

$$\mathbf{C}\big(\widetilde{\mathbf{U}}'-P_0(\mathbf{U}')\big) = \mathbf{C}(\widetilde{\mathbf{U}}-\mathbf{U})'.$$

Hence, the equation (3.22) can be reformulated as

$$\mathbf{M}\widetilde{\mathbf{R}}(s) = \mathbf{K}(\widetilde{\mathbf{U}}-\mathbf{U})(s) + \mathbf{K}(\mathbf{U}-P_0\mathbf{U})(s) + \mathbf{C}(\widetilde{\mathbf{U}}-\mathbf{U})'(s) - (\mathbf{f}-P_0\mathbf{f})(s),$$

which, in conjunction with (3.6), (3.20) and (3.21), yields the estimate (3.19). ■

Now, let us continue to discuss the lower and upper a posteriori error bound for the method (2.1).

Theorem 3.2 (lower and upper bounds) *Let* $\mathbf{u}$ *and* $\mathbf{U}$ *be the solution of* (2.10) *and* (2.1)*, respectively. Let* $\widetilde{\mathbf{U}}$ *be the reconstruction of* $\mathbf{U}$ *by* (3.1)*. Then for* $t\in[0,T]$, $1\le n\le N$*,*

$$\begin{aligned} \max_{1\le m\le n} k_m^2\|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_m)} &\le \|(\mathbf{u}-\mathbf{U})'\|_{L^\infty_{\mathbf{M}}(0,t)} + \max_{0\le\tau\le t}\|(\mathbf{u}-\widetilde{\mathbf{U}})'(\tau)\|_{\mathbf{M}} \\ &\le \max_{1\le m\le n} k_m\|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_m)} + 4\int_0^t \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\,\mathrm{d}s, \end{aligned} \tag{3.23}$$

where the a posteriori term $\widetilde{\mathbf{R}}$ *is given by* (3.11)*.*

Proof. Using the triangle inequality and (3.18), we obtain

$$\begin{aligned} \max_{1\le m\le n} k_m^2\|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_m)} &= \|(\mathbf{U}-\widetilde{\mathbf{U}})'\|_{L^\infty_{\mathbf{M}}(0,t)} \\ &\le \|(\mathbf{u}-\mathbf{U})'\|_{L^\infty_{\mathbf{M}}(0,t)} + \max_{0\le\tau\le t}\|(\mathbf{u}-\widetilde{\mathbf{U}})'(\tau)\|_{\mathbf{M}}, \end{aligned} \tag{3.24}$$

which implies the left side estimate of (3.23). Again, by the triangle inequality, (3.18) and (3.16), we have

$$\begin{aligned}\|(\mathbf{u}-\mathbf{U})'\|_{L^\infty_{\mathbf{M}}(0,t)} &\le \max_{0\le\tau\le t}\|(\mathbf{u}-\widetilde{\mathbf{U}})'(\tau)\|_{\mathbf{M}} + \|(\mathbf{U}-\widetilde{\mathbf{U}})'(\tau)\|_{L^\infty_{\mathbf{M}}(0,t)}\\ &\le \max_{1\le m\le n} k_m\|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_m)} + 2\int_0^t \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\,\mathrm{d}s. \end{aligned} \tag{3.25}$$

This together with (3.16) and (3.24) yields

$$\begin{aligned}&\max_{0\le\tau\le t}\|(\mathbf{u}-\widetilde{\mathbf{U}})'(\tau)\|_{\mathbf{M}} + \|(\mathbf{u}-\mathbf{U})'(\tau)\|_{L^\infty_{\mathbf{M}}(0,t)}\\ &\le \max_{1\le m\le n} k_m\|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_m)} + 4\int_0^t \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\,\mathrm{d}s,\end{aligned}$$

which leads to the right side estimate of (3.23). ■

4 An adaptive algorithm

Motivated by Theorem 3.2 (cf. the estimate (3.25)), we are tempted to introduce a posteriori error estimator of the time-stepping method (2.1) for solving even a nonlinear problem (1.1) heuristically. That means, let

$$\eta := \max_{1\le n\le N} k_n\|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_n)} + 2\int_0^T \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\mathrm{d}s, \tag{4.1}$$

where $\widetilde{\mathbf{R}}$ is the residual of a nonlinear problem, defined by

$$\widetilde{\mathbf{R}}(t) = \mathbf{M}^{-1}\Big(\mathbf{M}\widetilde{\mathbf{U}}''(t) + \mathbf{F}\big(t,\widetilde{\mathbf{U}}(t),\widetilde{\mathbf{U}}'(t)\big)\Big), \qquad t\in J_n,\ 1\le n\le N.$$

Then the quantity η may be viewed as a posteriori error estimator for the method (2.1). Until now, it is beyond our power to develop reliability and efficiency estimates for such an estimator.

Based on the above error estimator, using the error equidistribution strategy as used in [4, 20], we can construct the error indicator corresponding to the subinterval J_n as

$$\Theta := 2\ \max\{\ \Theta_1,\quad \Theta_2\ \}, \tag{4.2}$$

where

$$\Theta_1 := k_n\|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_n)}, \qquad \Theta_2 := 2\frac{T}{k_n}\int_{J_n}\|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\mathrm{d}s.$$

The magnitude of Θ affects the choice of k_n, the length of the subinterval J_n.

Next, let us study how to compute the quantities Θ_1 and Θ_2 after we get $\dot{\mathbf{U}}^n_-$ at each time step by (2.1). First of all, from (3.5) and the definition of Θ_1, we have

$$\Theta_1 = \|\dot{\mathbf{U}}^n_- - \dot{\mathbf{U}}^{n-1}_-\|_{\mathbf{M}}\ . \tag{4.3}$$

For deriving Θ_2, we should obtain $\widetilde{\mathbf{R}}(t)$ in advance. It follows from (3.4) that

$$\begin{aligned}&\widetilde{\mathbf{U}}(t) = \widetilde{\mathbf{U}}(t_{n-1}) + k_n\dot{\mathbf{U}}^{n-1}_-\Phi_0(\xi) + k_n\dot{\mathbf{U}}^n_-\Phi_1(\xi),\\ &\widetilde{\mathbf{U}}'(t) = \dot{\mathbf{U}}^{n-1}_-(1-\xi) + \dot{\mathbf{U}}^n_-\xi, \qquad \widetilde{\mathbf{U}}''(t) = \frac{1}{k_n}(-\dot{\mathbf{U}}^{n-1}_- + \dot{\mathbf{U}}^n_-),\end{aligned} \tag{4.4}$$

where $\xi = (t - t_{n-1})/k_n$ and Φ_0, Φ_1 are defined as in (3.2).

Furthermore, in actual computation, we will use the Gaussian quadrature formula (cf. [23]) to evaluate Θ_2 numerically. In other words, for $t \in J_n$, $1 \le n \le N$,

$$\int_{J_n} \|\widetilde{\mathbf{R}}(t)\|_{\mathbf{M}} \mathrm{d}t \ \approx \ \sum_{j=1}^{N_g} k_n \omega_j \|\widetilde{\mathbf{R}}(t_{n-1} + k_n \zeta_j)\|_{\mathbf{M}}, \tag{4.5}$$

where ζ_j and ω_j $(1 \le j \le N_g)$ are the Gaussian quadrature points and weights on reference interval $[0, 1]$, respectively.

Remark 4.1 *Let us discuss the cost of computing Θ_2 briefly. It is evident that the cost is taken in numerical integration by Gaussian quadrature formula* (4.5). *Since the quadrature method is highly accurate, very few nodes are enough for actual computation (with the number ≤ 10). Next, we have to evaluate $\|\widetilde{\mathbf{R}}(\cdot)\|_{\mathbf{M}}$ at the quadrature nodes, the main cost of which corresponds to numerical solution of a linear system with $\mathbf{M}$ as a coefficient matrix. Generally speaking, the mass matrix $\mathbf{M}$ is a well-conditioned symmetric positive definite matrix, so the linear system can be solved by the conjugate gradient method very efficiently. According to the above analysis, we find that the cost for computing Θ_2 is inexpensive.*

With the help of the previous preparations and using some ideas implied in the Runge-Kutta-Felberg method (cf. [23]), we are ready to present the following Algorithm 1 to compute the numerical solution of the problem (1.1) by using the adaptive time-stepping strategy.

Algorithm 1 Adaptive Time Stepping Method

Given a tolerance ϵ, a parameter $\delta \in (0, 1)$, and the max (min) time step size $k_{\max}$ ($k_{\min}$) by user

· **Step 0:** Initialize $n = 1$, $t_0 = 0$, $k_1 = k_{\max}$, $\mathbf{U}^0 = \mathbf{u}_0$, $\dot{\mathbf{U}}^0_- = \mathbf{v}_0$
· **WHILE** $t_{n-1} < T$
· **Step 1:** Given t_{n-1}, k_n, $\mathbf{U}^{n-1}$, $\dot{\mathbf{U}}^{n-1}_-$
· **1(a):** Get the numerical solution $\mathbf{U}^n$, $\dot{\mathbf{U}}^n_-$ by (2.7)
· **1(b):** Get the approximation $\widetilde{\mathbf{U}}^n$ by (3.4)
· **1(c):** Evaluate Θ_1 by (4.3)
· **1(d):** Get $\widetilde{\mathbf{R}}(t)$ at Gaussian quadrature points by (4.4) and (3.11)
· **1(e):** Summation to get the value of Θ_2 by (4.5)
· **1(f):** Get Θ by (4.2)
· **Step 2:** If $\delta\epsilon \le \Theta \le \epsilon$, $k_{n+1} = k_n$, go to **Step 5**
· **Step 3:** If $\Theta < \delta\epsilon$, $k_{n+1} = \min\{2k_n,\ k_{\max}\}$, go to **Step 5**
· **Step 4:** If $\Theta > \epsilon$, $k_n = \max\{k_n/2,\ k_{\min}\}$, go to **Step 1**
· **Step 5:** Let $t_n = t_{n-1} + k_n$, $n = n + 1$, go to loop condition judgment
· **END WHILE**

Remark 4.2 *Similar to the Runge-Kutta-Felberg method (cf. [23]), the parameter $\delta \in (0, 1)$ in Algorithm 1 is used to determine how to enlarge the step size during the computation process (see Step 3 in Algorithm 1). The choice of δ is very technical. If δ is chosen too small, the over-refined meshes would be used in time, deteriorating the efficiency of Algorithm 1. If it is chosen too large, the algorithm would enlarge the step size more frequently, increasing the extra computational cost remarkably. From our numerical experience, it's better to choose δ such that $1/32 \le \delta \le 1/2$.*

5 Numerical experiments

5.1 Efficiency of the estimators

Example 5.1 (Nonlinear lumped mass system) *For illustrating the effectiveness of the a posteriori error estimates developed in the previous sections, we first study the vibration of a multi-structure model, a similar one as given in [3]. As shown in Figure 3, the structure consists of two rigid elements (vehicles) with lumped masses equal to* m_1 *and* m_2*, respectively; these elements are connected with each other by soften, classical and harden springs with linear damping. And the restoring force of these springs are given as follows:*

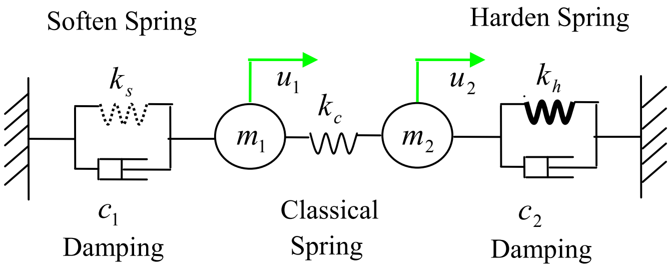


Figure 3: Example: 5.1: The nonlinear dynamic system.

$$\begin{aligned} &\text{Classical Spring } (k_c): && f_c = -\kappa_1 u, && (5.1)\\ &\text{Softening Spring } (k_s): && f_s = -\kappa_2 \tanh(u), && (5.2)\\ &\text{Hardening Spring } (k_h): && f_h = -\kappa_3 u(1+\kappa_4 u^2). && (5.3)\end{aligned}$$

In our actual computation, we choose $m_1 = m_2 = 1$*, and choose the spring stiffness as* $\kappa_1 = \kappa_2 = \kappa_3 = 1$*. The damping coefficients are taken as* $c_1 = c_2 = 1$*. Hence by d Alembert's principle, we can get the following system of nonlinear dynamic equations,*

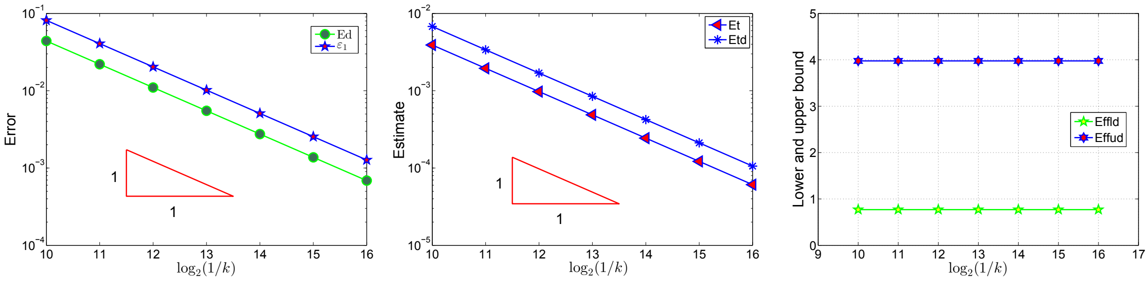
$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}'' = \begin{pmatrix} c_1u_1'(t) + f_s\big(u_1(t)\big) + f_c\big(u_1(t)\big) - f_c\big(u_2(t)\big) - f_1(t) \\ c_2u_2'(t) + f_h\big(u_2(t)\big) + f_c\big(u_2(t)\big) - f_c\big(u_1(t)\big) - f_2(t) \end{pmatrix}, \tag{5.4}$$

where f_1 *and* f_2 *are the external forces. We choose* $T = 1$ *and the exact solution to be* $\mathbf{u}(t) = (u_1, u_2)^{\mathrm{T}} = (\sin(\pi t), \sin(2\pi t))^{\mathrm{T}}$ *, so the force term* $\mathbf{f}$ *can be computed by the equations* (5.4)*. We solve the solution of the dynamical system by the method* (2.1) *combined with the right side scheme* (2.7)*.*

In our numerical computation, for a given natural number N*, we adopt the uniform partition in time with the mesh size* $k = T/N$*,* $1 \le n \le N$*. To show the computational performance of our method, define*

$$\begin{aligned} &\mathrm{Ed} = \max_{0\le\tau\le T} \|(\mathbf{u}-\mathbf{U})'(\tau)\|_{\mathbf{M}}\ , && \mathrm{Et} = \max_{0\le\tau\le T} \|(\mathbf{u}-\widetilde{\mathbf{U}})(\tau)\|_{\mathbf{M}}\ ,\\ &\mathrm{Etd} = \max_{0\le\tau\le T} \|(\mathbf{u}-\widetilde{\mathbf{U}})'(\tau)\|_{\mathbf{M}}\ , && \varepsilon_1 = 2\int_0^T \|\widetilde{\mathbf{R}}(s)\|_{\mathbf{M}}\ \mathrm{d}s\ ,\\ &\varepsilon_2 = \max_{0\le n\le N} k_n \|\widetilde{\mathbf{U}}''\|_{L^\infty_{\mathbf{M}}(J_n)}\ , && \varepsilon_3 = \eta = 2\varepsilon_1 + \varepsilon_2\ .\\ &\mathrm{Effld} = \frac{\varepsilon_2}{\mathrm{Ed}+\mathrm{Etd}}, && \mathrm{Effud} = \frac{\varepsilon_3}{\mathrm{Ed}+\mathrm{Etd}}.\end{aligned}$$

In Figure 4(a) we present the values of Et *and* ε_1 *as well as their orders (which are* 1*). In Figure 4(b) we give the estimates of the reconstruction solution Et and Etd as well as*



(a) Order of Ed and ε_1 (b) Order of Et and Etd (c) Lower and Upper bound

Figure 4: Example 5.1. Numerical results corresponding to estimators in Theorem 3.12 and Theorem 3.2.

their orders. Moreover, we present the values of these effectivity indices in Figure 4(c), from which we can observe that $0.77 \approx \mathrm{Effld} < 1 < \mathrm{Effud} \approx 3.98$*. Therefore, our a posteriori error estimator* (4.1) *is rather efficient.*

5.2 Efficiency of the adaptive algorithm

Example 5.2 (Nonlinear Klein-Gordon equation) *In order to test the effectiveness of our adaptive Algorithm 1, we consider the nonlinear Klein-Gordon equations (cf. [8]),*

$$u_{tt}(\mathbf{x},t) - \Delta u(\mathbf{x},t) + \beta u_t(\mathbf{x},t) + u^2(\mathbf{x},t) = f(\mathbf{x},t),$$

equipped with the homogeneous Dirichlet boundary condition and the initial conditions. After the discretization by P_1 *conforming element in the space direction, we obtain the following system of nonlinear ODEs,*

$$\begin{cases} \mathbf{M}\mathbf{u}''(t) + \mathbf{C}\mathbf{u}'(t) + \mathbf{K}\mathbf{u}(t) + \mathbf{M}\mathbf{u}^2(t) = \mathbf{f}(t), & 0 < t < T, \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}'(0) = \mathbf{v}_0, \end{cases} \tag{5.5}$$

where $\mathbf{u}$ *is the vector representation of the finite element solution* u_h *in terms of the shape basis functions* $\{\varphi_i\}$*, i.e.,* $u_h(\mathbf{x},t) = \sum_{i=1}^M \{\mathbf{u}(t)\}_i \varphi_i(\mathbf{x})$*. The mass matrix* $\mathbf{M}$*, the stiff matrix* $\mathbf{K}$*, the damping matrix* $\mathbf{C}$ *and the force* $\mathbf{F}$ *are defined respectively by* $[\mathbf{M}]_{ij} = \int_\Omega \varphi_j \varphi_i \mathrm{d}\Omega$*,* $[\mathbf{K}]_{ij} = \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \mathrm{d}\Omega$*,* $[\mathbf{C}]_{ij} = \beta \int_\Omega \varphi_j \varphi_i \mathrm{d}\Omega$ *and* $\{\mathbf{f}\}_i = \int_\Omega f(t)\varphi_i \mathrm{d}\Omega$*. In the numerical computation, we choose the damping coefficient* $\beta = 0.05$ *and the terminal time* $T = 1.0$*. Consider the* 1*-dim case of the above problem with the force* f *given such that the exact solution is*

$$u(x,t) = \mathrm{e}^{-t/2} x(1-x)\sin((1.5\pi + \mathrm{actan}(500(2t-1))x), \qquad 0 < x < 1,$$

which varies rapidly around $t = 0.5$*. After the discretization in space direction with a fine uniform mesh* $h = 1/5000$*, we solve the semi-discrete problem by using Algorithm 1 combined with the semi-side scheme* (2.9) *with* $\mathbf{F}$ *split into* $\mathbf{F}^R := \mathbf{C}\mathbf{u}' + \mathbf{K}\mathbf{u} - \mathbf{f}$ *and* $\mathbf{F}^L := \mathbf{M}\mathbf{u}^2$*, so that we only require to solve a linear system at each time subinterval. When implementing Algorithm 1 in this example, we set the related parameters by* $\epsilon = 2.5\mathrm{e}-1$*,* $\delta = 1/2$*,* $k_{\max} = 1\mathrm{e}-1$ *and* $k_{\min} = 2\mathrm{e}-4$*.*

To show the efficiency of Algorithm 1, we also carry out the numerical simulation using the uniform time stepping method with the same number of subintervals as for the adaptive method. The numerical solution obtained by the uniform time stepping method with $k = k_{\min}/100$ *is used as a reference solution.*

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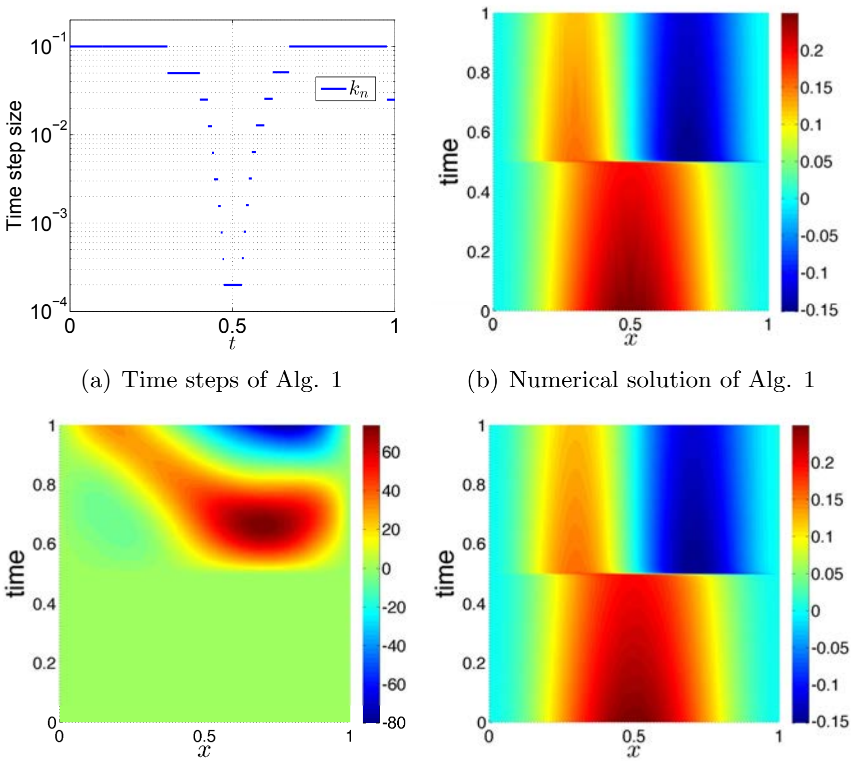

(a) Time steps of Alg. 1

(b) Numerical solution of Alg. 1

(c) Numerical solution with uniform stepsize

(d) Reference solution

Figure 5: Example 5.2. Comparison of numerical results.

From Figure 5(a) we can see the time step size becomes extremely small around $t = 0.5$ *in order to capture the rapid change of the solution, and the step size will become large automatically when the solution varies slowly, which illustrates the efficiency of Algorithm 1. The numerical results with Algorithm 1 and the uniform time stepping method, and the reference solution are shown in Figures 5(b), 5(c) and 5(d), respectively, from which we may find that the adaptive method can approximate the exact solution very well even if it varies rapidly, but the uniform time stepping method fails. We mention further that for the adaptive method in this example, the total CPU time used is approximately* 147.1 s*, while the one for computing* Θ *is only* 7.4 s*, only covers a very small amount of the total time.*

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A fractional Means inequality

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Abstract

Here we produce an interesting fractional means scalar inequality.

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Key Words and Phrases: Means inequality, fractional derivative.

We make

Remark 1 *Let $\nu > 0$, $n := \lceil \nu \rceil$ ($\lceil \cdot \rceil$ ceiling of the number), $f(\cdot, y) \in AC^n([a,b])$, $\forall\ y \in [c,d]$ (it means $\frac{\partial^{n-1} f(\cdot,y)}{\partial x^{n-1}} \in AC([a,b])$, $\forall\ y \in [c,d]$). Then the left Caputo partial fractional derivative with respect to x, is given by (see [1], p. 270)*

$$\frac{\partial_{*a}^{\nu} f(x,y)}{\partial x^{\nu}} = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \frac{\partial^n f(t,y)}{\partial x^n} dt, \tag{1}$$

$\forall\ y \in [c,d]$, and it exists almost everywhere for x in $[a,b]$, Γ denotes the gamma function.

Then, we get the left Caputo fractional Taylor formula ([2], p. 54)

$$f(x,y) = \sum_{k=0}^{n-1} \frac{\partial^k f(a,y)}{\partial x^k} (x-a)^k + \frac{1}{\Gamma(\nu)} \int_a^x (x-t)^{\nu-1} \frac{\partial_{*a}^{\nu} f(t,y)}{\partial x^{\nu}} dt, \tag{2}$$

$\forall\ x \in [a,b]$, for each $y \in [c,d]$.

Above $\left(\int_a^x (x-t)^{\nu-1} \frac{\partial_{*a}^{\nu} f(t,y)}{\partial x^{\nu}} dt \right) \in AC^n([a,b])$, $\forall\ y \in [c,d]$.

Let now $f(x,\cdot) \in AC^n([c,d])$, $\forall\ x \in [a,b]$ (it means $\frac{\partial^{n-1} f(x,\cdot)}{\partial y^{n-1}} \in AC([c,d])$, $\forall\ x \in [a,b]$). Then the left Caputo partial fractional derivative with respect to y, is given by

$$\frac{\partial_{*c}^{\nu} f(x,y)}{\partial y^{\nu}} = \frac{1}{\Gamma(n-\nu)} \int_c^y (y-s)^{n-\nu-1} \frac{\partial^n f(x,s)}{\partial y^n} ds, \tag{3}$$

$\forall\ x \in [a,b]$, and it exists almost everywhere for y in $[c,d]$.

Then, we get the left Caputo fractional Taylor formula

$$f(x,y)=\sum_{k=0}^{n-1}\frac{\partial^k f(x,c)}{\partial y^k}(y-c)^k+\frac{1}{\Gamma(\nu)}\int_c^y (y-s)^{\nu-1}\frac{\partial_{*c}^{\nu} f(x,s)}{\partial y^{\nu}}ds, \tag{4}$$

$\forall\ y\in[c,d]$, *for each* $x\in[a,b]$.

Above $\left(\int_c^y (y-s)^{\nu-1}\frac{\partial_{*c}^{\nu} f(x,s)}{\partial y^{\nu}}ds\right)\in AC^n([c,d])$, $\forall\ x\in[a,b]$.

Assume

$$\frac{\partial^k f(a,y)}{\partial x^k}=0,\quad \text{for } k=1,...,n-1,\ \forall\ y\in[c,d], \tag{5}$$

we get

$$f(x,y)-f(a,y)=\frac{1}{\Gamma(\nu)}\int_a^x (x-t)^{\nu-1}\frac{\partial_{*a}^{\nu} f(t,y)}{\partial x^{\nu}}dt. \tag{6}$$

Additionally assume $f(a,y)=0$, $\forall\ y\in[c,d]$, *then*

$$f(x,y)=\frac{1}{\Gamma(\nu)}\int_a^x (x-t)^{\nu-1}\frac{\partial_{*a}^{\nu} f(t,y)}{\partial x^{\nu}}dt, \tag{7}$$

$\forall\ y\in[c,d]$, $\forall\ x\in[a,b]$.

Assume

$$\frac{\partial^k f(x,c)}{\partial y^k}=0,\quad \text{for } k=1,...,n-1,\ \forall\ x\in[a,b], \tag{8}$$

we get

$$f(x,y)-f(x,c)=\frac{1}{\Gamma(\nu)}\int_c^y (y-s)^{\nu-1}\frac{\partial_{*c}^{\nu} f(x,s)}{\partial y^{\nu}}ds, \tag{9}$$

$\forall\ y\in[c,d]$, $\forall\ x\in[a,b]$.

Additionally assume that $f(x,c)=0$, $\forall\ x\in[a,b]$, *then*

$$f(x,y)=\frac{1}{\Gamma(\nu)}\int_c^y (y-s)^{\nu-1}\frac{\partial_{*c}^{\nu} f(x,s)}{\partial y^{\nu}}ds, \tag{10}$$

$\forall\ y\in[c,d]$, $\forall\ x\in[a,b]$.

Assuming (5) and (8), we get

$$2f(x,y)-f(a,y)-f(x,c)=$$
$$\frac{1}{\Gamma(\nu)}\left\{\int_a^x (x-t)^{\nu-1}\frac{\partial_{*a}^{\nu} f(t,y)}{\partial x^{\nu}}dt+\int_c^y (y-s)^{\nu-1}\frac{\partial_{*c}^{\nu} f(x,s)}{\partial y^{\nu}}ds\right\}, \tag{11}$$

$\forall\ x\in[a,b]$, $\forall\ y\in[c,d]$.

Additionally assume that $f(a,y)=0$, $\forall\ y\in[c,d]$, *and* $f(x,c)=0$, $\forall\ x\in[a,b]$, *we obtain*

$$f(x,y)=\frac{1}{2\Gamma(\nu)}\left\{\int_a^x (x-t)^{\nu-1}\frac{\partial_{*a}^{\nu} f(t,y)}{\partial x^{\nu}}dt+\int_c^y (y-s)^{\nu-1}\frac{\partial_{*c}^{\nu} f(x,s)}{\partial y^{\nu}}ds\right\}, \tag{12}$$

$\forall\ x \in [a,b]\,,\ \forall\ y \in [c,d]\,.$

We can rewrite (11) as follows:

$$f(x,y) - \left(\frac{f(a,y) + f(x,c)}{2}\right) =$$

$$\frac{1}{2\Gamma(\nu)} \left\{ \int_a^x (x-t)^{\nu-1} \frac{\partial_{*a}^{\nu} f(t,y)}{\partial x^{\nu}} dt + \int_c^y (y-s)^{\nu-1} \frac{\partial_{*c}^{\nu} f(x,s)}{\partial y^{\nu}} ds \right\}, \tag{13}$$

$\forall\ x \in [a,b]\,,\ \forall\ y \in [c,d]\,.$

If $0 < \nu < 1$, then $n = 1$, and (13) is valid without (5) and (8), which in this case are void conditions.

Call

$$\Delta f(x,y) := f(x,y) - \left(\frac{f(a,y) + f(x,c)}{2}\right). \tag{14}$$

Assume $f \in C([a,b] \times [c,d])$, then

$$\int_a^b \int_c^d \Delta f(x,y)\, dxdy = \int_a^b \int_c^d f(x,y)\, dxdy -$$

$$\left(\frac{(b-a)\int_c^d f(a,y)\,dy + (d-c)\int_a^b f(x,c)\,dx}{2}\right). \tag{15}$$

Hence it holds

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \Delta f(x,y)\, dxdy = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dxdy -$$

$$\left(\frac{\frac{1}{(d-c)}\int_c^d f(a,y)\,dy + \frac{1}{(b-a)}\int_a^b f(x,c)\,dx}{2}\right). \tag{16}$$

Assume now that

$$\frac{\partial_{*a}^{\nu} f(x,y)}{\partial x^{\nu}}, \frac{\partial_{*c}^{\nu} f(x,y)}{\partial y^{\nu}} \in C([a,b] \times [c,d]) \tag{17}$$

Clearly, it holds

$$|\Delta f(x,y)| \leq$$

$$\frac{1}{2\Gamma(\nu)} \left\{ \int_a^x (x-t)^{\nu-1} \left|\frac{\partial_{*a}^{\nu} f(t,y)}{\partial x^{\nu}}\right| dt + \int_c^y (y-s)^{\nu-1} \left|\frac{\partial_{*c}^{\nu} f(x,s)}{\partial y^{\nu}}\right| ds \right\} \leq$$

$$\frac{1}{2\Gamma(\nu)} \left\{ \frac{(x-a)^{\nu}}{\nu} \left\|\frac{\partial_{*a}^{\nu} f}{\partial x^{\nu}}\right\|_{\infty} + \frac{(y-c)^{\nu}}{\nu} \left\|\frac{\partial_{*c}^{\nu} f}{\partial y^{\nu}}\right\|_{\infty} \right\} \leq \tag{18}$$

$$\frac{1}{2\Gamma(\nu+1)} \left\{ (b-a)^{\nu} \left\|\frac{\partial_{*a}^{\nu} f}{\partial x^{\nu}}\right\|_{\infty} + (d-c)^{\nu} \left\|\frac{\partial_{*c}^{\nu} f}{\partial y^{\nu}}\right\|_{\infty} \right\}.$$

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That is

$$|\Delta f(x,y)| \leq \frac{1}{2\Gamma(\nu+1)} \left\{ (b-a)^{\nu} \left\| \frac{\partial_{*a}^{\nu} f}{\partial x^{\nu}} \right\|_{\infty} + (d-c)^{\nu} \left\| \frac{\partial_{*c}^{\nu} f}{\partial y^{\nu}} \right\|_{\infty} \right\} =: \lambda. \quad (19)$$

Hence

$$\frac{1}{(b-a)(d-c)} \left| \int_a^b \int_c^d \Delta f(x,y)\, dxdy \right| \leq$$

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d |\Delta f(x,y)|\, dxdy \leq \lambda.$$

We have derived:

Theorem 2 *Let $\nu > 0$, $n := \lceil \nu \rceil$, $f(\cdot, y) \in AC^n([a,b])$, $\forall\ y \in [c,d]$; and $f(x,\cdot) \in AC^n([c,d])$, $\forall\ x \in [a,b]$. Assume $\frac{\partial^k f(a,y)}{\partial x^k} = 0$, for $k = 1, ..., n-1$, $\forall\ y \in [c,d]$; and $\frac{\partial^k f(x,c)}{\partial y^k} = 0$, for $k = 1, ..., n-1$, $\forall\ x \in [a,b]$. Furthermore, assume $f \in C([a,b] \times [c,d])$ and $\frac{\partial_{*a}^{\nu} f(x,y)}{\partial x^{\nu}}, \frac{\partial_{*c}^{\nu} f(x,y)}{\partial y^{\nu}} \in C([a,b] \times [c,d])$. Then*

$$\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dxdy - \left(\frac{\frac{1}{(b-a)} \int_a^b f(x,c)\, dx + \frac{1}{(d-c)} \int_c^d f(a,y)\, dy}{2} \right) \right|$$

$$\leq \frac{1}{2\Gamma(\nu+1)} \left\{ (b-a)^{\nu} \left\| \frac{\partial_{*a}^{\nu} f}{\partial x^{\nu}} \right\|_{\infty} + (d-c)^{\nu} \left\| \frac{\partial_{*c}^{\nu} f}{\partial y^{\nu}} \right\|_{\infty} \right\}. \quad (20)$$

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