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OSCILLATION CRITERIA FOR FORCED AND DAMPED NABLA FRACTIONAL DIFFERENCE EQUATIONS

JEHAD ALZABUT, THABET ABDELJAWAD, HUSSAM ALRABAIAH

ABSTRACT. Based on the properties of Riemann–Liouville difference and sum operators, sufficient conditions are established to guarantee the oscillation of solutions for forced and damped nabla fractional difference equations. Numerical examples are presented to show the applicability of the proposed results. We finish the paper by a concluding remark.

1. INTRODUCTION AND PRELIMINARIES

The study of oscillation of solutions for various type of equations including differential, difference and dynamic equations on time scales has been the object of many researchers in the last five decades. Indeed, great efforts have been put in the direction of establishing new oscillation criteria for these type of equations; see the monographs [1, 2, 3, 4].

Due to their widespread applications in science and technology, the fractional differential equations have started to attract more attention among physicists and mathematicians. Indeed, it was found that various interdisciplinary applications can be elegantly modeled by the help of these equations. For instance, the nonlinear oscillation of earthquake and heart beats and the stability of brain tumor growth can be modeled by using fractional derivatives; the reader can consult the paper [5, 6] for more details. In parallel to the recent developments in research, the investigation of oscillation property for fractional differential equations has continued and thus many significant results have lately appeared [7, 8, 9, 10, 11, 12, 13]. On the other hand, difference equations have shown tremendous applications in numerical and computational simulations [14, 15]. However, the oscillation of damped or forced difference equations have comparably gained less attention among researchers [16, 17, 18, 19]. For fractional difference equations, fairly few papers have recently published on the oscillation of their solutions [20, 21, 22].

Following this trend in this paper, we consider the following forced and damped nabla fractional difference equation

$$(1 - p(n))\nabla\nabla_0^\alpha y(n) + p(n)\nabla_0^\alpha y(n) + q(n)f(y(n)) = g(n), \quad n \in \mathbb{N}_1,$$

(1.1)

$$\nabla_0^{-(1-\alpha)} y(1) = y(1) = c,$$

where $\nabla_0^\alpha y$ and $\nabla_0^{-\alpha} y$ are the Riemann–Liouville fractional difference and sum operators of y of order α, respectively, $\alpha \in (0, 1)$ is a real number, c is a constant, $\mathbb{N}_1 = \{1, 2, \ldots\}$ and

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(H) p, g are real sequences from $\mathbb{N}_1 \to \mathbb{R}$, $p(n) < 1$, q is a positive real sequence from $\mathbb{N}_1 \to \mathbb{R}^+$ and $f : \mathbb{R} \to \mathbb{R}$ such that $\frac{f(s)}{s} > 0$ for all $s \neq 0$.

To the best of their observations, the authors claim that there is no paper in the literature concerning with the oscillation of solutions of nabla fractional difference equation involving forcing and damping terms. Therefore, the equation under consideration and the obtained results are essentially new and have their own merits.

Definition 1. *A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise, it is called non–oscillatory.*

Throughout this paper, we will make use of the following notations, definitions and some known results of ∇–fractional operators [23]. For any $\alpha, t \in \mathbb{R}$, the α rising function is defined by

$$(1.2) \qquad t^{\overline{\alpha}} = \frac{\Gamma(t + \alpha)}{\Gamma(t)}, \ t \in \mathbb{R} \backslash \{\ldots, -2, -1, 0\},$$

with the convention that $0^{\overline{\alpha}} = 0$. Γ is the well known Gamma function satisfying the property $\Gamma(\alpha + 1) = \alpha \Gamma(\alpha)$.

Definition 2. [23] *For $\alpha \in (0, 1)$ and $\rho(s) = s - 1$, we have*

    I. *The nabla operator is defined by $\nabla y(n) = y(n) - y(n-1)$, $n \in \mathbb{N}_1$.*
    II. *The Riemann–Liouville fractional sum $\nabla_0^{-\alpha}$ of order α is defined by*

$$(1.3) \qquad \nabla_0^{-\alpha} y(n) = \frac{1}{\Gamma(\alpha)} \sum_{s=1}^{n} (n - \rho(s))^{\overline{\alpha-1}} y(s), \ n \in \mathbb{N}_1.$$

    III. *The Riemann–Liouville fractional difference ∇_0^{α} of order α is defined by*

$$(1.4) \qquad \nabla_0^{\alpha} y(n) = \nabla \nabla_0^{-(1-\alpha)} y(n), \ n \in \mathbb{N}_1$$

    and hence

$$(1.5) \qquad \nabla_0^{\alpha} y(n) = \frac{\nabla}{\Gamma(1-\alpha)} \sum_{s=1}^{n} (n - \rho(s))^{-\overline{\alpha}} y(s), \ n \in \mathbb{N}_1.$$

Lemma 1. [23] *For $\alpha \in (0, 1)$ and $n \in \mathbb{N}_0$, we have*

    1. *The relation*

$$(1.6) \qquad \nabla_1^{-\alpha} \nabla_0^{\alpha} y(n) = y(n) - \frac{n^{\overline{\alpha-1}}}{\Gamma(\alpha)} y(1).$$

    2. *The power rule is defined by*

$$(1.7) \qquad \nabla_0^{-\alpha} n^{\overline{\mu}} = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + \alpha + 1)} n^{\overline{\alpha+\mu}}, \ \mu > -1.$$

Lemma 2. [24] *For $\varepsilon > 0$, we have*

$$(1.8) \qquad \lim_{n \to \infty} \frac{\Gamma(n) n^{\varepsilon}}{\Gamma(n + \varepsilon)} = 1.$$

2. The main results

In this section, we establish the main results of this paper. Indeed, sufficient conditions are established for the oscillation of the solutions of equation (1.1).

Theorem 1. *Let the assumption (H) and the following conditions hold*

$$(2.1) \qquad \liminf_{n \to \infty} \sum_{s=1}^{n} \frac{(n - \rho(s))^{\overline{\alpha-1}}}{P(s)} \Big[M + \sum_{r=n_0+1}^{s} g(r)P(r) \Big] < 0,$$

and

$$(2.2) \qquad \limsup_{n \to \infty} \sum_{s=1}^{n} \frac{(n - \rho(s))^{\overline{\alpha-1}}}{P(s)} \Big[M + \sum_{r=n_0+1}^{s} g(r)P(r) \Big] > 0,$$

where M is a constant and $P(n) = \prod_{s=n_0}^{n} \left(\frac{1}{1-p(s)} \right)$ for $n_0 \in \mathbb{N}_1$. Then every solution of (1.1) is oscillatory.

Proof. For the sake of contradiction, assume that $y(n)$ is a non–oscillatory solution of (1.1). Then, the solution y is either $y(n) > 0$ or $y(n) < 0$ for $n \in \mathbb{N}_1$. Let $y(n) > 0, \ n \in \mathbb{N}_1$. From equation (1.1), we obtain

$$\big(1 - p(n)\big)\nabla\nabla_0^\alpha y(n) + p(n)(\nabla_0^\alpha y)(n) = -q(n)f(y(n)) + g(n) < g(n)$$

Multiplying both sides of the above inequality by $P(n)$, we get

$$(2.3) \qquad \nabla\nabla_0^\alpha y(n)P(n-1) + \nabla_0^\alpha y(n)\nabla P(n) < g(n)P(n),$$

where $P(n-1) = \big(1-p(n)\big)P(n)$ and $\nabla P(n) = p(n)P(n)$ have been used. However, the left side of (2.3) can be written in the form

$$(2.4) \qquad \nabla\Big(\nabla_0^\alpha y(n)P(n) \Big) < g(n)P(n).$$

Taking the sum of both sides from $n_0 + 1$ to n, we have

$$\nabla_0^\alpha y(n)P(n) < \nabla_0^\alpha y(n_0)P(n_0) + \sum_{s=n_0+1}^{n} g(s)P(s),$$

or

$$(2.5) \qquad \nabla_0^\alpha y(n) < \frac{M}{P(n)} + \frac{1}{P(n)} \sum_{s=n_0+1}^{n} g(s)P(s),$$

where $M = \nabla_0^\alpha y(n_0)P(n_0)$. Applying the operator $\nabla_1^{-\alpha}$ on both sides of the above inequality, we get

$$(2.6) \qquad \nabla_1^{-\alpha}\nabla_0^\alpha y(n) < \nabla_1^{-\alpha}\Big[\frac{M}{P(n)} + \frac{1}{P(n)} \sum_{s=n_0+1}^{n} g(s)P(s) \Big].$$

In view of (1.3) and (1.6), we observe that

$$(2.7) \qquad \nabla_1^{-\alpha}\nabla_0^\alpha y(n) = y(n) - \frac{n^{\overline{\alpha-1}}}{\Gamma(\alpha)}y(1).$$

and

$$\nabla^{-\alpha}\Big[\frac{M}{P(n)} + \frac{1}{P(n)}\sum_{s=n_0+1}^{n} g(s)P(s)\Big]$$

$$(2.8)\qquad = \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{n}(n-\rho(s))^{\overline{\alpha-1}}\Big[\frac{M}{P(s)} + \frac{1}{P(s)}\sum_{r=n_0+1}^{s} g(r)P(r)\Big].$$

Combining (2.7) and (2.8), the inequality (2.6) becomes

$$y(n) < \frac{n^{\overline{\alpha-1}}}{\Gamma(\alpha)}y(1) + \frac{1}{\Gamma(\alpha)}\sum_{s=1}^{n}\frac{(n-\rho(s))^{\overline{\alpha-1}}}{P(s)}\Big[M + \sum_{r=n_0+1}^{s} g(r)P(r)\Big].$$

Multiplying the above inequality by $n^{1-\alpha}$, we get

$$n^{1-\alpha}y(n) < n^{1-\alpha}n^{\overline{\alpha-1}}\frac{y(1)}{\Gamma(\alpha)} + \frac{n^{1-\alpha}}{\Gamma(\alpha)}\sum_{s=1}^{n}\frac{(n-\rho(s))^{\overline{\alpha-1}}}{P(s)}\Big[M + \sum_{r=n_0+1}^{s} g(r)P(r)\Big].$$

However, we note that $n^{1-\alpha}n^{\overline{\alpha-1}} = \frac{n\Gamma(n+\alpha-1)}{n^\alpha\Gamma(n)}$. By the help of Stirling formula (1.8), we observe that

$$\begin{aligned}
\lim_{n\to\infty} n^{1-\alpha}n^{\overline{\alpha-1}} &= \lim_{n\to\infty}\frac{n\Gamma(n+\alpha-1)}{n^\alpha\Gamma(n)}\frac{\Gamma(n+\alpha)}{\Gamma(n+\alpha)} \\
&= \lim_{n\to\infty}\frac{n\Gamma(n+\alpha-1)}{\Gamma(n+\alpha)} = \lim_{n\to\infty}\frac{n\Gamma(n+\alpha-1)}{\Gamma(n+\alpha-1+1)} \\
(2.9)\qquad &= \lim_{n\to\infty}\frac{n\Gamma(n+\alpha-1)}{(n+\alpha-1)\Gamma(n+\alpha-1)} = 1.
\end{aligned}$$

By virtue of (2.9) and condition (2.1), we conclude that

$$\liminf_{n\to\infty} n^{1-\alpha}y(n) \le -\infty,$$

which contradicts the assumption that $y(n) > 0$.

In case $y(n) < 0$, $n \in \mathbb{N}_1$, nevertheless, we get

$$\nabla\Big(\nabla_0^\alpha y(n)P(n)\Big) > g(n)P(n)$$

or

$$\nabla_0^\alpha y(n) > \frac{M}{P(n)} + \frac{1}{P(n)}\sum_{s=n_0+1}^{n} g(s)P(s),$$

where $M = \nabla_0^\alpha y(n_0)P(n_0)$. Following similar steps, we end up with

$$n^{1-\alpha}y(n) > n^{1-\alpha}n^{\overline{\alpha-1}}\frac{y(1)}{\Gamma(\alpha)} + \frac{n^{1-\alpha}}{\Gamma(\alpha)}\sum_{s=1}^{n}\frac{(n-\rho(s))^{\overline{\alpha-1}}}{P(s)}\Big[M + \sum_{r=n_0+1}^{s} g(r)P(r)\Big].$$

The analysis in (2.9) and condition (2.2) implies that $\limsup_{n\to\infty} n^{1-\alpha}y(n) \ge \infty$, which contradicts the assumption that $y(n) < 0$. The proof is finished. □

Let $z(n) = \sum_{s=1}^{n}(n-\rho(s))^{-\overline{\alpha}}y(s)$ where $y(t)$ is a solution of equation (1.1). Then, by relations (1.3) and (1.4) we get

$$(2.10)\qquad \nabla z(n) = \Gamma(1-\alpha)\nabla_0^\alpha y(n).$$

It is clear that z and y have the same dynamical character. We will use relation (2.10) to prove the following theorem.

Theorem 2. *Let the assumption (H) and the following conditions hold*

(2.11)
$$\liminf_{n\to\infty} \sum_{s=1}^{n} \frac{1}{P(s)} \Big[M + \sum_{r=n_0+1}^{s} g(r)P(r) \Big] = -\infty,$$

and

(2.12)
$$\limsup_{n\to\infty} \sum_{s=1}^{n} \frac{1}{P(s)} \Big[M + \sum_{r=n_0+1}^{s} g(r)P(r) \Big] = \infty,$$

where M is a constant and $P(n) = \prod_{s=n_0}^{n} \left(\frac{1}{1-p(s)} \right)$ for $n_0 \in \mathbb{N}_0$. Then every solution of (1.1) is oscillatory.

Proof. For the sake of contradiction, assume that $y(n)$ is a non–oscillatory solution of (1.1). Then, the solution y is either $y(n) > 0$ or $y(n) < 0$ for $n \in \mathbb{N}_1$. Let $y(n) > 0$, $n \in \mathbb{N}_1$. Following similar steps as in the proof of Theorem 1, we reach to

$$\nabla_0^\alpha y(n) < \frac{M}{P(n)} + \frac{1}{P(n)} \sum_{s=n_0+1}^{n} g(s)P(s),$$

It follows from (2.10) that

$$\nabla z(n) < \frac{\Gamma(1-\alpha)}{P(n)} \Big[M + \sum_{s=n_0+1}^{n} g(s)P(s) \Big].$$

Taking the sum of both sides from $n_0 + 1$ to n, we have

$$z(n) < z(n_0) + \Gamma(1-\alpha) \sum_{s=n_0+1}^{n} \frac{1}{P(s)} \Big[M + \sum_{r=n_0+1}^{s} g(r)P(r) \Big].$$

In view of condition (2.11), one can easily see that the right hand side of the above inequality tends to $-\infty$ as $n \to \infty$ which contradicts with the fact that $z(n) > 0$.

In case $y(n) < 0$, $n \in \mathbb{N}_1$, then in the same manner we obtain

$$z(n) > z(n_0) + \Gamma(1-\alpha) \sum_{s=n_0+1}^{n} \frac{1}{P(s)} \Big[M + \sum_{r=n_0+1}^{s} g(r)P(r) \Big].$$

In view of condition (2.12), the right hand side of the above inequality tends to ∞ as $n \to \infty$ which contradicts the fact that $z(n) < 0$. $\qquad\square$

3. Examples

In this section, we present two examples to demonstrate the validity of the results in Theorem 1 and Theorem 2. By finding a non–oscillatory solution, the first example shows that the assumptions of Theorem 1 cannot be ignored whereas in the second example we verify that the assumptions of Theorem 2 are satisfied thus we conclude that all solutions are oscillatory.

Example 1. *Consider the following forced and damped nabla fractional difference equation*

$$\frac{3}{2}\nabla\nabla_0^{\frac{3}{4}}y(n) - \frac{1}{2}\nabla_0^{\frac{3}{4}}y(n) + \frac{3\Gamma(n)}{4\Gamma(n+\frac{3}{4})}y(n) = \frac{6 - 3\Gamma(\frac{3}{4})}{8}, \quad n \in \mathbb{N}_1,$$

(3.1)

$$\nabla_0^{-\frac{1}{4}}y(1) = y(1) = \frac{3}{4}\Gamma\left(\frac{3}{4}\right),$$

where $\alpha = \frac{3}{4}$, $p(n) = -\frac{1}{2}$, $q(n) = \frac{3\Gamma(n)}{4\Gamma(n+\frac{3}{4})}$, $f(y) = y$ *and* $g(n) = \frac{6-3\Gamma(\frac{3}{4})}{8}$. *Therefore,* $P(n) = \prod_{s=1}^{n}\left(\frac{2}{3}\right) = (\frac{2}{3})^n$. *By simple calculations, we get*

$$\sum_{s=1}^{n}(n-s+1)^{-\frac{1}{4}}\left(\frac{2}{3}\right)^s\left[M + \sum_{r=2}^{s}\frac{6 - 3\Gamma(\frac{3}{4})}{8}\left(\frac{3}{2}\right)^r\right] > 0.$$

(3.2)

Inequality (3.2) implies that condition (2.1) of Theorem 1 is not satisfied. On the other hand, one can easily figure out that $y(n) = n^{\frac{3}{4}}$ *is a non–oscillatory solution for equation (3.1).*

Example 2. *Consider the following forced and damped nabla fractional difference equation*

$$\frac{2}{3}\nabla\nabla_0^{\frac{1}{4}}y(n) - \frac{1}{3}\nabla_0^{\frac{1}{4}}y(n) + q(n)f(y(n)) = n^2\Gamma(n), \quad n \in \mathbb{N}_1,$$

(3.3)

$$\nabla_0^{-\frac{3}{4}}y(1) = y(1) = c,$$

where $\alpha = \frac{1}{4}$, $p(n) = -\frac{1}{3}$ *and* $g(n) = n^2\Gamma(n)$. *Therefore,* $P(n) = \prod_{s=1}^{n}\left(\frac{3}{4}\right) = \left(\frac{3}{4}\right)^n$. *Furthermore, we get*

$$\limsup_{n\to\infty}\sum_{s=1}^{n}\left(\frac{4}{3}\right)^s\left[M + \sum_{r=2}^{s}r^2\Gamma(r)\left(\frac{3}{4}\right)^r\right] = \infty,$$

which implies that condition (2.11) of Theorem 2 holds. Therefore, every solution of (3.3) is oscillatory.

4. A CONCLUDING REMARKS

The oscillation of solutions of fractional differential equations is considered to be one of the hottest topics among researchers nowdays. For fractional difference equations, however, the oscillation of solutions is still at its first stages of progress. We consider here this problem for a general form of equations involving forcing and damping terms which make the equation under consideration adequately represents real phenomena. Two main results are established to guarantee the oscillation of solutions. For numerical treatment, we examined our results by presenting two examples which show that the proposed assumptions are sufficient.

There are some points that feature the results of this paper and make them innovative. Equation (1.1) is considered in general form which covers many particular cases. No restriction is imposed on the forced term g in equation (1.1). The results of this paper can be extended to equation involving delay argument

$$\bigl(1 - p(n)\bigr)\nabla\nabla_0^{\alpha}y(n) + p(n)(\nabla_0^{\alpha}y)(n) + q(n)f(y(n-\tau)) = g(n), \quad n \in \mathbb{N}_1,$$

where $\tau > 0$. It is worthy to mention that every solution of the above equation is also oscillatory under assumption (H). That is, the delay argument has no influence on the oscillatory behavior. The results of this paper can be also carried out upon

employing the right fractional difference operator, we recommend the reader to consult the paper [23] for more details. The above equation with delay argument is then converted to an equation with advanced argument. Unlike most of the papers appeared in the literature which studied oscillation property for difference equations involving delta operator, we consider here an equation involving nabla operator which is more appropriate upon calculations.

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A type of C^2 smooth surfaces generated by bivariate rational spline interpolation

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Abstract

In this paper, we present a novel C^2 surface modeling method only based on the values of the original function. There are two schemes to generate a family of the interpolation surfaces. Identified uniquely by the values of the tension parameters $\alpha_{i,j}$ and $\beta_{i,j}$, each interpolant of the family is C^2 continuous in the whole interpolating region, and which can be represented by using basis functions clearly. More important, since there are free positive parameters in the interpolants, the shape of the interpolating surfaces can be modified by selecting suitable parameters for the unchanged interpolating data. Also, the interpolants are stable for any positive parameters, and the error estimate formula of the interpolators are derived. Numerical examples show that the interpolators give a good approximation to the interpolated function and the shape of the interpolating surfaces can be modified.

Keywords: C^2-spline; bivariate rational interpolation; surface modeling; shape control

1   Introduction

In various applications such as industrial design and manufacture, atmospheric analysis, geology and medical imaging, etc., it is often necessary to generate a smooth surface that interpolates a prescribed set of data. For most applications, C^1 smoothness is generally sufficient. However, curvature continuity sometimes is needed and this leads to the need for C^2 smoothness.

For generating a C^1 smooth surface, there are many ways to tackle this problem [3, 9, 10, 16, 22]. But, generating a C^2 bivariate interpolation is a more difficult task. In [9], a bicubic spline interpolation scheme was proposed as a extension of the theory of cubic splines to two dimension, this type of interpolation scheme has become the standard scheme for rectangular regions, and which has studied in many literatures [4, 7, 17]. In recent years, some of the literatures have contributed to the C^2 bivariate interpolation also. For example, in [5], Brou and Méhauté proposed a construction of C^r bivariate rational splines over a triangulation, via a finite element approach; In [6], a novel surface modeling scheme was presented based on an envelope template, and G^2 or C^2 composite surfaces can be obtained utilizing the envelope template sweeping over the data points; In [8], a rationally corrected quintic Bézier triangular patch scheme of degree 9 over degree 4 and controlled by 27 Bézier points was used to define a smooth surface through scattered data, and a convex combination technique was employed to enable C^2 continuity conditions on the boundaries of the

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triangle to be satisfied; In [11], a refinable function vector of C^2-quartic splines was introduced for generating approximation quadrilateral subdivisions, and that of C^2-quintic splines was constructed for generating a second order Hermite interpolatory quadrilateral subdivision; In [12], two families of solutions provided by two Hermite subdivision schemes HD^2 and HR^2 were investigate, and a C^2 interpolant on any semiregular rectangular mesh was generated with Hermite data of degree 2; In [18], two C^2 shape-preserving bivariate interpolant on rectangular grids were developed by using polynomial splines; In [20], C^1- and C^2-continuous spline-interpolation surfaces were constructed in a regular triangular net with the help of polynomial basic functions; In [21], author proved that there exists a C^3 piecewise polynomial of degree 7 on the twice CT type split of a triangle, which interpolate arbitrarily given values and derivatives of orders up to 3 at the vertices and on the edges of the triangle.

Most of above C^2 bivariate spline interpolations are in fact polynomial interpolations. However, one of the disadvantages of the polynomial spline method is that the local shape can not be modified for the interpolating surfaces while interpolating data is unchanged. Further, generating a C^2 bivariate interpolation usually requires up to second-order partial derivative values of the interpolated function, or to solve a system of consistency equations for second-order partial derivatives at the data sites. Unfortunately, in many practical problems, the partial derivatives are difficult to get. Thus, in order to generate the C^2 surfaces required for CAGD, the following conditions must be satisfied: (a) the surfaces are constructed based only on function values; (b) the parameters of constructed surfaces can be modified without changing the given data.

In fact, in recent years, motivated by the univariate rational spline interpolation, the C^1 bivariate rational spline, which has a simple and explicit mathematical representation with parameters, has been studied [1, 13, 14, 15]. Since the parameters in the interpolation function are selective according to the control constrains, the constrained control of the shape becomes possible.

In this paper, motivated by C^2 rational spline curve [2], a C^2 piecewise bivariate rational spline interpolation which can be modified by using new parameters, will be concerned based only on function values. To solve the problem, a new approach is proposed by using a constructed interpolation function comprising a simple and explicit mathematical representation with the new parameters $\alpha_{i,j}$ and $\beta_{i,j}$. This paper is arranged as follows. In Section 2, a piecewise bivariate rational spline interpolation with parameters is constructed over rectangular mesh. Section 3 discusses the C^2 continuity of the interpolant. In Section 4, the basis of this interpolator is derived, and the bounded property is obtained. Sections 5 deals with the error estimates of the interpolator. Some numerical examples are given in Section 6, which show that this interpolator gives a good approximation to the interpolated function and the shape of the interpolating surfaces can be modified by selecting suitable parameters.

2   Construction of bivariate rational spline interpolant

Let $\Omega : [a, b; c, d]$ be the plane region, and $\{(x_i, y_i, f_{i,j}), i = 1, 2, \cdots, n; j = 1, 2, \cdots, m\}$ be a given set of data points, where $a = x_1 < x_2 < \cdots < x_n = b$ and $c = y_1 < y_2 < \cdots < y_m = d$ are the knot spacings, $f_{i,j} = f(x_i, y_j)$. $d_{i,j}$ and $e_{i,j}$ are chosen partial derivative values $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$ at the knots (x_i, y_j), respectively. Let $h_i = x_{i+1} - x_i$, and $l_j = y_{j+1} - y_j$, and for any point $(x, y) \in [x_i, x_{i+1}; y_j, y_{j+1}]$ in the xy-plane, let $\theta = (x - x_i)/h_i$ and $\eta = (y - y_j)/l_j$. Denoting

$$\Delta_{i,j}^{(x)} = \frac{f_{i+1,j} - f_{i,j}}{h_i}, \ \Delta_{i,j}^{(y)} = \frac{f_{i,j+1} - f_{i,j}}{l_j}.$$

First, for each $y = y_j, j = 1, 2, \cdots, m$, construct the x-direction interpolating curve, this is given

by

$$P_{i,j}^*(x) = \frac{p_{i,j}^*(x)}{q_{i,j}^*(x)}, \quad i = 1, 2, \cdots, n-1, \tag{1}$$

where

$$p_{i,j}^*(x) = (1-\theta)^3 f_{i,j} + \theta(1-\theta)^2 V_{i,j}^*(x) + \theta^2(1-\theta) W_{i,j}^*(x) + \theta^3 f_{i+1,j},$$
$$q_{i,j}^*(x) = (1-\theta)^2 + \theta(1-\theta)\alpha_{i,j} + \theta^2,$$

with

$$V_{i,j}^*(x) = (\alpha_{i,j} + 1)f_{i,j} + h_i d_{i,j} + \theta(1 - (1-\theta)\alpha_{i,j})(f_{i+1,j} - f_{i,j} - h_i d_{i,j}),$$
$$W_{i,j}^*(x) = (\alpha_{i,j} + 1)f_{i+1,j} - h_i d_{i+1,j} - (1-\theta)(1 - \theta\alpha_{i,j})(f_{i+1,j} - f_{i,j} - h_i d_{i+1,j}),$$

and $\alpha_{i,j} > 0$. This interpolation $P_{i,j}^*(x)$ defined by (1) is called the rational quintic interpolator which satisfies

$$P_{i,j}^*(x_i) = f_{i,j}, \; P_{i,j}^*(x_{i+1}) = f_{i+1,j}, \; P_{i,j}^{*\,\prime}(x_i) = d_{i,j}, \; P_{i,j}^{*\,\prime}(x_{i+1}) = d_{i+1,j}.$$

Further, when $\alpha_{i,j} \to +\infty$, the interpolant is the well-known standard cubic Hermite interpolation. That is to say, in this special case, the interpolant $P_{i,j}^*(x)$ defined by (1) will give approximately the Hermite interpolation.

If the partial derivative values $d_{i,j}$ at the data sites are estimated using the arithmetic mean method:

$$d_{i,j} = \frac{h_{i-1}\Delta_{i,j}^{(x)} + h_i \Delta_{i-1,j}^{(x)}}{h_{i-1} + h_i}, \; i = 2, 3, \cdots, n-1, \tag{2}$$

then the interpolation function $P_{i,j}^*(x)$ defined by (1) is C^2 continuous in $[a, b]$, and which satisfies

$$P''(x_i) = \frac{2}{h_{i-1} + h_i}(\Delta_{i,j}^{(x)} - \Delta_{i-1,j}^{(x)}), \; i = 2, 3, \cdots, n-1.$$

Remark 1. At the end knots x_1, x_n, the derivative values are given as

$$d_{1,j} = \Delta_{1,j}^{(x)} - \frac{h_1}{h_1 + h_2}(\Delta_{2,j}^{(x)} - \Delta_{1,j}^{(x)}),$$
$$d_{n,j} = \Delta_{n-1,j}^{(x)} + \frac{h_{n-1}}{h_{n-1} + h_{n-2}}(\Delta_{n-1,j}^{(x)} - \Delta_{n-2,j}^{(x)}), \tag{3}$$

For each pair of $(i, j), i = 1, 2, \cdots, n-1$ and $j = 1, 2, \cdots, m-1$, using the x-direction interpolation $P_{i,j}^*(x)$, define the interpolation function $P_{i,j}(x, y)$ on $[x_i, x_{i+1}; y_j, y_{j+1}]$ as follows:

$$P_{i,j}(x, y) = \frac{p_{i,j}(x, y)}{q_{i,j}(y)}, \quad i = 1, 2, \cdots, n-1; j = 1, 2, \cdots, m-1, \tag{4}$$

where

$$p_{i,j}(x, y) = (1-\eta)^3 P_{i,j}^*(x) + \eta(1-\eta)^2 V_{i,j} + \eta^2(1-\eta)W_{i,j} + \eta^3 P_{i,j+1}^*(x),$$
$$q_{i,j}(y) = (1-\eta)^2 + \eta(1-\eta)\beta_{i,j} + \eta^2,$$

with

$$V_{i,j} = (\beta_{i,j} + 1)P_{i,j}^*(x) + l_j\phi_{i,j}(x) + \varphi_{i,j}(x,y),$$
$$W_{i,j} = (\beta_{i,j} + 1)P_{i,j+1}^*(x) - l_j\phi_{i,j+1}(x) + \psi_{i,j}(x,y),$$

and

$$\phi_{i,s}(x) = (1-\theta)^3(1 + 4\theta + 9\theta^2)e_{i,s} + \theta^3(6 - 8\theta + 3\theta^2)e_{i+1,s}, \ s = j, j+1,$$
$$\varphi_{i,j}(x,y) = (\eta - \eta(1-\eta)(\beta_{i,j}+1))(P_{i,j+1}^*(x) - P_{i,j}^*(x) - l_j\phi_{i,j}(x)),$$
$$\psi_{i,j}(x,y) = (1 - \eta - \eta(1-\eta)(\beta_{i,j}+1))(P_{i,j}^*(x) - P_{i,j+1}^*(x) + l_j\phi_{i,j+1}(x)),$$

and $\beta_{i,j} > 0$. The interpolation function $P_{i,j}(x,y)$ defined by (4) is called a bivariate piecewise rational interpolator, which satisfies

$$P_{i,j}(x_r, y_s) = f(x_r, y_s), \ \frac{\partial P_{i,j}(x_r, y_s)}{\partial x} = d_{r,s}, \ \frac{\partial P_{i,j}(x_r, y_s)}{\partial y} = e_{r,s}, \ \ r = i, i+1, \ s = j, j+1.$$

The interpolating scheme above begins in x-direction first. Now, let the interpolation begins with y-direction first. For each $x = x_i, i = 1, 2, \cdots, n$, denote the y-direction interpolation in $[y_j, y_{j+1}]$ by

$$Q_{i,j}^*(y) = \frac{(1-\eta)^3 f_{i,j} + \eta(1-\eta)^2\overline{V}_{i,j}^*(y) + \eta^2(1-\eta)\overline{W}_{i,j}^*(y) + \eta^3 f_{i,j+1}}{(1-\eta)^2 + \eta(1-\eta)\beta_{i,j} + \eta^2}, \ \ j = 1, 2, \cdots, m-1, \quad (5)$$

where

$$\overline{V}_{i,j}^*(y) = (\beta_{i,j} + 1)f_{i,j} + l_je_{i,j} + \eta(1 - (1-\eta)\beta_{i,j})(f_{i,j+1} - f_{i,j} - l_je_{i,j}),$$
$$\overline{W}_{i,j}^*(y) = (\beta_{i,j} + 1)f_{i,j+1} - l_jd_{i,j+1} - (1-\eta)(1 - \eta\beta_{i,j})(f_{i,j+1} - f_{i,j} - l_je_{i,j+1}),$$

with $\beta_{i,j} > 0$. This interpolant $Q_{i,j}^*(y)$ defined by (5) is C^1-continuous in $[c, d]$, and which satisfies

$$Q_{i,j}^*(y_j) = f_{i,j}, \ Q_{i,j}^*(y_{j+1}) = f_{i,j+1}, \ Q_{i,j}^{*\prime}(y_j) = e_{i,j}, \ Q_{i,j}^{*\prime}(y_{j+1}) = e_{i,j+1}.$$

If the partial derivative values $e_{i,j}$ at the data sites are estimated using the arithmetic mean method:

$$e_{i,j} = \frac{l_{j-1}\Delta_{i,j}^{(y)} + l_j\Delta_{i,j-1}^{(y)}}{l_{j-1} + l_j}, j = 2, 3, \cdots, m-1,$$
$$e_{i,1} = \Delta_{i,1}^{(y)} - \frac{l_1}{l_1 + l_2}(\Delta_{i,2}^{(y)} - \Delta_{i,1}^{(y)}), \quad (6)$$
$$e_{i,m} = \Delta_{i,m-1}^{(y)} + \frac{l_{m-1}}{l_{m-1} + l_{m-2}}(\Delta_{i,m-1}^{(y)} - \Delta_{i,m-2}^{(y)}),$$

then the interpolation function $Q_{i,j}^*(y)$ defined by (5) is C^2-continuous in $[c, d]$, and which satisfies

$$Q''(y_j) = \frac{2}{l_{j-1} + l_j}(\Delta_{i,j}^{(y)} - \Delta_{i,j-1}^{(y)}), \ j = 2, 3, \cdots, m-1.$$

For each pair (i, j), $i = 1, 2, \cdots, n-1$ and $j = 1, 2, \cdots, m-1$, using the y-direction interpolation function $Q_{i,j}^*(y)$, define the bivariate rational Hermite interpolating function $Q_{i,j}(x,y)$ on $[x_i, x_{i+1}; y_j, y_{j+1}]$ as follows:

$$Q_{i,j}(x,y) = \frac{(1-\theta)^3 Q_{i,j}^*(y) + \theta(1-\theta)^2\overline{V}_{i,j} + \theta^2(1-\theta)\overline{W}_{i,j} + \theta^3 Q_{i+1,j}^*(y)}{(1-\theta)^2 + \theta(1-\theta)\alpha_{i,j} + \theta^2}, \quad (7)$$

where

$$\overline{V}_{i,j} = (\alpha_{i,j} + 1)Q^*_{i,j}(y) + h_i\overline{\phi}_{i,j}(y) + \overline{\varphi}_{i,j}(x,y),$$
$$\overline{W}_{i,j} = (\alpha_{i,j} + 1)Q^*_{i+1,j}(y) - h_i\overline{\phi}_{i+1,j}(y) + \overline{\psi}_{i,j}(x,y),$$

with

$$\overline{\phi}_{i,s}(y) = (1 - \eta)^3(1 + 4\eta + 9\eta^2)d_{r,j} + \eta^3(6 - 8\eta + 3\eta^2)d_{r,j+1}, \ r = i, i+1,$$
$$\overline{\varphi}_{i,j}(x,y) = (\theta - \theta(1 - \theta)(\alpha_{i,j} + 1))(Q^*_{i+1,j}(y) - Q^*_{i,j}(y) - h_i\overline{\phi}_{i,j}(y)),$$
$$\overline{\psi}_{i,j}(x,y) = (1 - \theta - \theta(1 - \theta)(\alpha_{i,j} + 1))(Q^*_{i,j}(y) - Q^*_{i+1,j}(y) + h_i\overline{\phi}_{i+1,j}(y)),$$

and $\alpha_{i,j} > 0$. The interpolation function $Q_{i,j}(x,y)$ defined by (7) satisfies

$$Q_{i,j}(x_r, y_s) = f(x_r, y_s), \ \frac{\partial Q_{i,j}(x_r, y_s)}{\partial x} = d_{r,s}, \ \frac{\partial Q_{i,j}(x_r, y_s)}{\partial y} = e_{r,s}, \ r = i, i+1, \ s = j, j+1.$$

The interpolating functions $P_{i,j}(x,y)$ defined by (4) and $Q_{i,j}(x,y)$ defined by (7) satisfy the same interpolating data, but they are not the same interpolation functions. In the following, unless pointed out specifically, the bivariate rational interpolation based on function values means they are defined by (4).

3 C^2 continuity of the interpolant

For the C^2 continuity of the interpolation function $P_{i,j}(x,y)$ defined by (4), we have the following theorem.

Theorem 1. *If the knots are equally spaced for variable x, namely, $h_i = (b - a)/n$, a sufficient condition for the interpolation function $P_{i,j}(x,y), i = 1, 2, \cdots, n - 1; j = 1, 2, \cdots, m - 1$, to be C^2 in the whole interpolating region $[x_1, x_n; y_1, y_m]$ is that the parameters $\beta_{i,j} =$constant for each $j \in \{1, 2, \cdots, m - 1\}$ and all $i = 1, 2, \cdots, n - 1$, no matter what the parameters $\alpha_{i,j}$ might be.*

Proof. When the conditions of the theorem are satisfied, we can easily obtain that the interpolation function $P_{i,j}(x,y)$ is C^1 continuous in the interpolating region $[x_1, x_n; y_1, y_m]$ (see [13]).

Furthermore, since the rational interpolation function $P^*_{i,j}(x)$ defined by (1) is C^2 continuous in $[x_1, x_n]$, it is easy to show that the bivariate interpolation function $P_{i,j}(x,y)$ has continuous second-order partial derivatives $\frac{\partial P^2_{i,j}(x,y)}{\partial x^2}$, $\frac{\partial P^2_{i,j}(x,y)}{\partial x \partial y}$ and $\frac{\partial P^2_{i,j}(x,y)}{\partial y^2}$ in the interpolating region $[x_1, x_n; y_1, y_m]$ except $\frac{\partial P^2_{i,j}(x,y)}{\partial x^2}$ for every $y \in [y_j, y_{j+1}], j = 1, 2, \cdots, m - 1$, at the points $(x_i, y), i = 2, 3, \cdots, n - 1$, and $\frac{\partial P^2_{i,j}(x,y)}{\partial y^2}$ for every $x \in [x_i, x_{i+1}], i = 1, 2, \cdots, n - 1$, at the points $(x, y_j), j = 2, 3, \cdots, m - 1$. Thus, it is sufficient for $P_{i,j}(x,y) \in C^2$ in the whole interpolating region $[x_1, x_n; y_1, y_m]$ if $\frac{\partial P^2_{i,j}(x_i^+, y)}{\partial x^2} = \frac{\partial P^2_{i,j}(x_i^-, y)}{\partial x^2}$ and $\frac{\partial P^2_{i,j}(x, y_j^+)}{\partial y^2} = \frac{\partial P^2_{i,j}(x, y_j^-)}{\partial y^2}$ hold.

From (4), it can be derived

$$\begin{aligned}
\frac{\partial P^2_{i,j}(x,y)}{\partial x^2} = \frac{1}{q_{i,j}(y)}&[(1 - \eta)^3(1 + \eta + 2\eta^2 + \eta(1 + 2\eta)\beta_{i,j})\frac{d^2 P^*_{i,j}(x)}{dx^2} \\
&+ \eta^3(4 - 5\eta + 2\eta^2 + (3 - 5\eta + 2\eta^2)\beta_{i,j})\frac{d^2 P^*_{i,j+1}(x)}{dx^2} \\
&+ l_j\eta(1 - \eta)^3(1 + \eta + \eta\beta_{i,j})\frac{d^2\phi_{i,j}(x)}{dx^2} \\
&- l_j\eta^3(1 - \eta)(2 - \eta + (1 - \eta)\beta_{i,j})\frac{d^2\phi_{i,j+1}(x)}{dx^2}].
\end{aligned} \quad (8)$$

Since

$$\frac{d^2\phi_{i,s}(x_i^+)}{dx^2} = 0, \ \frac{d^2\phi_{i,s}(x_{i+1}^-)}{dx^2} = 0, \ s = j, j+1,$$

and the interpolation function $P_{i,j}^*(x)$ is C^2 continuous, it is easy to see from (8) that $\frac{\partial P_{i,j}^2(x,y)}{\partial x^2}$ is continuous at the points $(x_i, y), i = 2, 3, \cdots, n-1$ for every $y \in [y_j, y_{j+1}], j = 1, 2, \cdots, m-1$, when $\beta_{i-1,j} = \beta_{i,j}$ and $h_{i-1} = h_i$. The proof of the case which $\frac{\partial P_{i,j}^2(x,y)}{\partial y^2}$ is continuous at the points (x, y_j) is similar. This completes the proof. $\square$

Similarly, for the interpolation function $Q_{i,j}(x, y)$ defined by (7), the following theorem can be derived.

Theorem 2. *If the knots are equally spaced for variable y, namely, $l_j = (b-a)/m$, a sufficient condition for the interpolating function $Q_{i,j}(x, y), i = 1, 2, \cdots, n-1; j = 1, 2, \cdots, m-1$, to be C^2 in the whole interpolating region $[x_1, x_n; y_1, y_m]$ is that the parameters $\alpha_{i,j} =$constant for each $i \in \{1, 2, \cdots, n-1\}$ and all $j = 1, 2, \cdots, m-1$, no matter what the parameters $\beta_{i,j}$ might be.*

4 Basis of the interpolant

From (1) and (4), the interpolation function $P_{i,j}(x, y)$ can be written as follows:

$$P_{i,j}(x,y) = \sum_{r=i}^{i+1} \sum_{s=j}^{j+1} [a_{r,s}(\theta, \eta) f_{r,s} + b_{r,s}(\theta, \eta) h_i d_{r,s} + c_{r,s}(\theta, \eta) l_j e_{r,s}], \tag{9}$$

where

$$a_{i,j}(\theta, \eta) = \frac{(1-\theta)^2(1-\eta)^3(1+\theta(1+\theta-2\theta^2)\alpha_{i,j})(1+\eta+2\eta^2+\eta(1+2\eta)\beta_{i,j})}{((1-\theta)^2+\theta(1-\theta)\alpha_{i,j}+\theta^2)((1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2)},$$

$$a_{i,j+1}(\theta, \eta) = \frac{(1-\theta)^2\eta^3(1+\theta(1+\theta-2\theta^2)\alpha_{i,j+1})(4-5\eta+2\eta^2+(3-5\eta+2\eta^2)\beta_{i,j})}{((1-\theta)^2+\theta(1-\theta)\alpha_{i,j+1}+\theta^2)((1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2)},$$

$$a_{i+1,j}(\theta, \eta) = \frac{\theta^2(1-\eta)^3(1+\theta(3-5\theta+2\theta^2)\alpha_{i,j})(1+\eta+2\eta^2+\eta(1+2\eta)\beta_{i,j})}{((1-\theta)^2+\theta(1-\theta)\alpha_{i,j}+\theta^2)((1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2)},$$

$$a_{i+1,j+1}(\theta, \eta) = \frac{\theta^2\eta^3(1+\theta(3-5\theta+2\theta^2)\alpha_{i,j+1})(4-5\eta+2\eta^2+(3-5\eta+2\eta^2)\beta_{i,j})}{((1-\theta)^2+\theta(1-\theta)\alpha_{i,j+1}+\theta^2)((1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2)},$$

$$b_{i,j}(\theta, \eta) = \frac{\theta(1-\theta)^3(1-\eta)^3(1+\theta\alpha_{i,j})(1+\eta+2\eta^2+\eta(1+2\eta)\beta_{i,j})}{((1-\theta)^2+\theta(1-\theta)\alpha_{i,j}+\theta^2)((1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2)},$$

$$b_{i,j+1}(\theta, \eta) = \frac{\theta(1-\theta)^3\eta^3(1+\theta\alpha_{i,j+1})(4-5\eta+2\eta^2+(3-5\eta+2\eta^2)\beta_{i,j})}{((1-\theta)^2+\theta(1-\theta)\alpha_{i,j+1}+\theta^2)((1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2)},$$

$$b_{i+1,j}(\theta, \eta) = -\frac{\theta^3(1-\theta)(1-\eta)^3(1+(1-\theta)\alpha_{i,j})(1+\eta+2\eta^2+\eta(1+2\eta)\beta_{i,j})}{((1-\theta)^2+\theta(1-\theta)\alpha_{i,j}+\theta^2)((1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2)},$$

$$b_{i+1,j+1}(\theta, \eta) = -\frac{\theta^3(1-\theta)\eta^3(1+(1-\theta)\alpha_{i,j+1})(4-5\eta+2\eta^2+(3-5\eta+2\eta^2)\beta_{i,j})}{((1-\theta)^2+\theta(1-\theta)\alpha_{i,j+1}+\theta^2)((1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2)},$$

$$c_{i,j}(\theta, \eta) = \frac{(1-\theta)^3\eta(1-\eta)^3(1+4\theta+9\theta^2)(1+\eta+\eta\beta_{i,j})}{(1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2},$$

$$c_{i,j+1}(\theta, \eta) = -\frac{(1-\theta)^3\eta^3(1-\eta)(1+4\theta+9\theta^2)(2-\eta+(1-\eta)\beta_{i,j})}{(1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2},$$

$$c_{i+1,j}(\theta, \eta) = \frac{\theta^3 \eta(1-\eta)^3(6-8\theta+3\theta^2)(1+\eta+\eta\beta_{i,j})}{(1-\eta)^2 + \eta(1-\eta)\beta_{i,j} + \eta^2},$$

$$c_{i+1,j+1}(\theta, \eta) = -\frac{\theta^3 \eta^3(1-\eta)(6-8\theta+3\theta^2)(2-\eta+(1-\eta)\beta_{i,j})}{(1-\eta)^2 + \eta(1-\eta)\beta_{i,j} + \eta^2}.$$

The teams $a_{r,s}(\theta, \eta), b_{r,s}(\theta, \eta), c_{r,s}(\theta, \eta), r = i, i+1, s = j, j+1$ are called the basis of the interpolant defined by (4), which satisfy

$$
\begin{aligned}
&a_{i,j}(\theta, \eta) + a_{i,j+1}(\theta, \eta) + a_{i+1,j}(\theta, \eta) + a_{i+1,j+1s}(\theta, \eta) = 1, \\
&b_{i,j}(\theta, \eta) + b_{i,j+1}(\theta, \eta) - b_{i+1,j}(\theta, \eta) - b_{i+1,j+1s}(\theta, \eta) = \theta(1-\theta), \\
&c_{i,j}(\theta, \eta) - c_{i,j+1}(\theta, \eta) + c_{i+1,j}(\theta, \eta) - c_{i+1,j+1s}(\theta, \eta) \\
&\quad = \frac{\eta(1-\eta)(1-\eta+\eta^2+\eta(1-\eta)\beta_{i,j})(1+\theta-10\theta^3+15\theta^4-6\theta^5)}{(1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2}.
\end{aligned}
\tag{10}
$$

Denote

$$
\begin{aligned}
M &= \max\{|f_{r,s}|, r = i, i+1; s = j, j+1\}, \\
Q_1 &= \max\{h_i|d_{r,s}|, r = i, i+1; s = j, j+1\}, \\
Q_2 &= \max\{l_j|e_{r,s}|, r = i, i+1; s = j, j+1\}.
\end{aligned}
$$

For the given data, the values of the piecewise bivariate interpolation function $P_{i,j}(x, y)$ defined by (4) are bounded in the interpolating interval as described by the following theorem.

Theorem 3. *Let $P_{i,j}(x, y)$ is the interpolation function over $[x_i, x_{i+1}; y_j, y_{j+1}]$ defined by (4). No matter what positive number the parameters $\alpha_{i,s}$ $(s = j, j+1)$ and $\beta_{i,j}$ take, the values of $P_{i,j}(x, y)$ in $[x_i, x_{i+1}; y_j, y_{j+1}]$ satisfy*

$$|P_{i,j}(x, y)| \le M + \frac{1}{4}Q_1 + 0.430029Q_2.$$

Proof. From (9) and (10), it is easy to derive that

$$
|P_{i,j}(x, y)| \le M \sum_{r=i}^{i+1}\sum_{s=j}^{j+1} |a_{r,s}(\theta, \eta)| + Q_1 \sum_{r=i}^{i+1}\sum_{s=j}^{j+1} |b_{r,s}(\theta, \eta)| + Q_2 \sum_{r=i}^{i+1}\sum_{s=j}^{j+1} |c_{r,s}(\theta, \eta)|
$$

$$
\le M + \theta(1-\theta)Q_1 + Q_2 \sum_{r=i}^{i+1}\sum_{s=j}^{j+1} |c_{r,s}(\theta, \eta)| \le M + \frac{1}{4}Q_1 + Q_2 \sum_{r=i}^{i+1}\sum_{s=j}^{j+1} |c_{r,s}(\theta, \eta)|.
$$

Since

$$
\sum_{r=i}^{i+1}\sum_{s=j}^{j+1} |c_{r,s}(\theta, \eta)| = \frac{\eta(1-\eta)(1-\eta+\eta^2+\eta(1-\eta)\beta_{i,j})(1+\theta-10\theta^3+15\theta^4-6\theta^5)}{(1-\eta)^2+\eta(1-\eta)\beta_{i,j}+\eta^2}
$$

$$
\le (1+\theta-10\theta^3+15\theta^4-6\theta^5)\frac{\eta(1-\eta)(1-\eta+\eta^2)}{1-2\eta+2\eta^2},
$$

and

$$
\max_{\theta \in [0,1]} (1+\theta-10\theta^3+15\theta^4-6\theta^5) = 1.14675,
$$

$$
\max_{\eta \in [0,1]} \frac{\eta(1-\eta)(1-\eta+\eta^2)}{1-2\eta+2\eta^2} = \frac{3}{8},
$$

thus, the proof is completed. □

5 Error estimates of the interpolant

Note that the interpolant defined by (4) is local, without loss of generality, it is only necessary to consider the interpolating region $[x_i, x_{i+1}; y_j, y_{j+1}]$ in order to process its error estimates. Let $f(x,y) \in C^2$ be the interpolated function, and $P_{i,j}(x,y)$ be the interpolation function defined by (4) over $[x_i, x_{i+1}; y_j, y_{j+1}]$.

Denoting

$$\|\frac{\partial f}{\partial y}\| = \max_{(x,y) \in D} |\frac{\partial f(x,y)}{\partial y}|, \quad \|\frac{\partial P}{\partial y}\| = \max_{(x,y) \in D} |\frac{\partial P_{i,j}(x,y)}{\partial y}|,$$

where $D = [x_i, x_{i+1}; y_j, y_{j+1}]$. By the Taylor expansion and the Peano-Kernel Theorem [19] gives the following:

$$|f(x,y) - P_{i,j}(x,y)| \le |f(x,y) - f(x,y_j)| + |P_{i,j}(x,y_j) - P_{i,j}(x,y)| + |f(x,y_j) - P_{i,j}(x,y_j)|$$

$$\le l_j(\|\frac{\partial f}{\partial y}\| + \|\frac{\partial P}{\partial y}\|) + |\int_{x_i}^{x_{i+1}} \frac{\partial^2 f(\tau, y_j)}{\partial x^2} R_x[(x-\tau)_+] d\tau| \tag{11}$$

$$\le l_j(\|\frac{\partial f}{\partial y}\| + \|\frac{\partial P}{\partial y}\|) + \|\frac{\partial^2 f(x, y_j)}{\partial x^2}\| \int_{x_i}^{x_{i+1}} |R_x[(x-\tau)_+]| d\tau,$$

where $\|\frac{\partial^2 f(x,y_j)}{\partial x^2}\| = \max_{x \in [x_i, x_{i+1}]} |\frac{\partial^2 f(x,y_j)}{\partial x^2}|$, and

$$R_x[(x-\tau)_+] = \begin{cases} (x-\tau) - a_{i+1,j}(\theta, 0)(x_{i+1} - \tau) - b_{i+1,j}(\theta, 0)h_i, x_i < \tau < x; \\ -a_{i+1,j}(\theta, 0)(x_{i+1} - \tau) - b_{i+1,j}(\theta, 0)h_i, x < \tau < x_{i+1}, \end{cases}$$

$$= \begin{cases} r(\tau), & x_i < \tau < x; \\ s(\tau), & x < \tau < x_{i+1}. \end{cases}$$

Thus, by simple integral calculation, it can be derived that

$$\int_{x_i}^{x_{i+1}} |R_x[(x-\tau)_+]| d\tau == h_i^2 B(\theta, \alpha_{i,j}), \tag{12}$$

where

$$B(\theta, \alpha_{i,j}) = \frac{\theta^2(1-\theta)^2(1 + 2\theta(1-\theta)\alpha_{i,j})^2}{(1 + \theta(3 - 5\theta + 2\theta^2)\alpha_{i,j})(1 + \theta(1 + \theta - 2\theta^2)\alpha_{i,j})}. \tag{13}$$

For the fixed $\alpha_{i,j}$, let

$$B_{i,j}^{(x)} = \max_{\theta \in [0,1]} B(\theta, \alpha_{i,j}). \tag{14}$$

This leads to the following theorem.

Theorem 4. *Let $f(x,y) \in C^2$ be the interpolated function, and $P_{i,j}(x,y)$ be its interpolator defined by (4) in $[x_i, x_{i+1}; y_j, y_{j+1}]$. Whatever the positive values of the parameters $\alpha_{i,s}, \beta_{r,j}$ might be, the error of the interpolation satisfies*

$$|f(x,y) - P_{i,j}(x,y)| \le l_j(\|\frac{\partial f}{\partial y}\| + \|\frac{\partial P}{\partial y}\|) + h_i^2 \|\frac{\partial^2 f(x, y_j)}{\partial x^2}\| B_{i,j}^{(x)},$$

where $B_{i,j}^{(x)}$ defined by (14).

Similarly, denoting $\|\frac{\partial^2 f(x,y_{j+1})}{\partial x^2}\| = \max_{x\in[x_i,x_{i+1}]}|\frac{\partial^2 f(x,y_{j+1})}{\partial x^2}|$, then the following theorem holds.

Theorem 5. *Let $f(x,y) \in C^2$ be the interpolated function, and $P_{i,j}(x,y)$ be its interpolation function defined by (4) in $[x_i, x_{i+1}; y_j, y_{j+1}]$. Whatever the positive values of the parameters $\alpha_{i,s}, \beta_{r,j}$ might be, the error of the interpolation satisfies*

$$|f(x,y) - P_{i,j}(x,y)| \le l_j(\|\frac{\partial f}{\partial y}\| + \|\frac{\partial P}{\partial y}\|) + h_i^2\|\frac{\partial^2 f(x,y_{j+1})}{\partial x^2}\|B_{i,j+1}^{(x)},$$

where $B_{i,j+1}^{(x)} = \max_{\theta\in[0,1]} B(\theta, \alpha_{i,j+1})$, and $B(\theta, \alpha_{i,j})$ defined by (13).

Furthermore, for $B_{i,s}^{(x)}$, we can conclude the following theorem.

Theorem 6. *For any positive parameters $\alpha_{i,s}, s = j, j+1$, $B_{i,s}^{(x)}$ are bounded, and*

$$\frac{1}{16} \le B_{i,s}^{(x)} \le \frac{3}{16}.$$

6   Numerical examples

For the bivariate rational spline interpolant defined by (4) , since there are three shape parameters in the interpolation function, when the parameters vary, the interpolation function can be changed for the unchanged interpolating data. Thus, the shape of the interpolating surface can be modified by selecting suitable shape parameters according to the control need. Also, the interpolator can give a good approximation to the interpolated function. In this section, in order to show the effectiveness which the interpolator defined by (4) approximate a function, and to describe that the shape of the interpolating surface can be modified by free shape parameters, some examples will be given.

Example 1. Let the interpolated function be $f(x,y) = \sin(x^2 + y)$, $(x,y) \in [0, 0.8; 0, 0.8]$, and let $h_i = l_j = 0.2$, then $x_i = 0.2(i-1)$, $y_j = 0.2(j-1)$, $i, j = 1, 2, 3, 4, 5$. Also let $\alpha_{i,j} = 0.3 + 0.2i + 0.1j$, $\beta_{i,j} = 0.6 + 0.1j$. The partial derivative values $d_{i,j}$ at the knots (x_i, y_j) $(i, j = 1, 2, 3, 4, 5)$ are conducted by using (2) and (3). The partial derivative values $e_{i,j}$ at the knots (x_i, y_j) $(i, j = 1, 2, 3, 4, 5)$ are given by (6).

Figure 1 shows the graph of the interpolated function $f(x,y)$. Figure 2 shows the graph of the interpolation function $P(x,y)$ defined by (4). Figure 3 shows the surface of the error $f(x,y) - P(x,y)$. From Figure 3, it is evident that the error of the interpolation is smaller than $\pm 8 \times 10^{-3}$, this means the interpolator defined by (4) gives a good approximation to the interpolated function.

Example 2. Let $\Omega : [0, 1.5; 0, 1.5]$ be the plane region, and the interpolation data are given in Table 1. The interpolation function $P_{i,j}(x,y)$ defined by (4) can be constructed in $[0, 1.5; 0, 1.5]$ for the given positive parameters $\alpha_{i,j}, \alpha_{i,j+1}$ and $\beta_{i,j}$. In order to show that the shape of the interpolating surface can be modified by selecting suitable parameters according to control need, we consider the value control of the interpolating surface. Assume $\alpha_{i,j} = \alpha_{i,j+1}$ and $\beta_{i,j} =$ constant for each $j \in \{1, 2, 3\}$ and all $i = 1, 2, 3$, then the interpolant $P_{i,j}(x,y)$ defined by (4) is C^2 in interpolating region $[0, 1.5; 0, 1.5]$. Without loss of generality, we only consider a subinterval $[0.5, 1; 0.5, 1]$.

Let $\alpha_{i,j} = 0.5, \beta_{i,j} = 0.8$. The partial derivative values $d_{i,j}$ and $e_{i,j}$ at the knots (x_i, y_j) are conducted by using (2) and (6), respectively. For the given interpolation data, denote the interpolation function by $P_1(x,y)$ which is defined over $[0.5, 1; 0.5, 1]$. Figure 4 shows the graph of the bivariate rational interpolating surface $P_1(x,y)$ with the parameters $\alpha_{i,j} = 0.5, \beta_{i,j} = 0.8$. It is
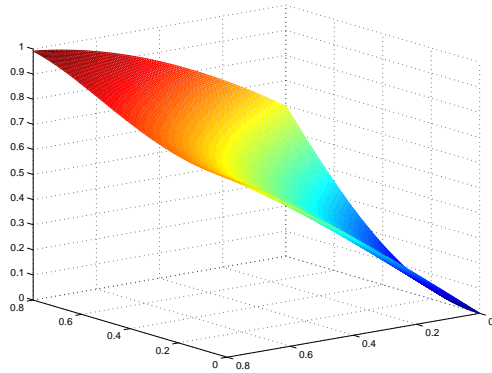
Figure 1: Graph of surface $f(x,y)$.
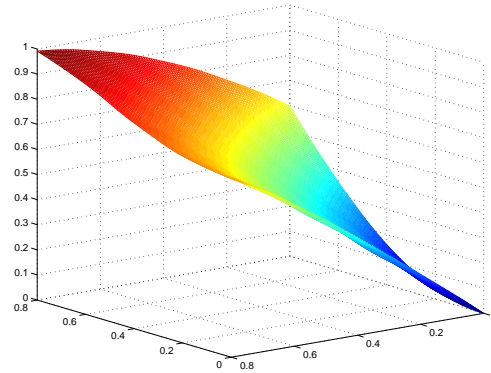
Figure 2: Graph of surface $P(x,y)$.
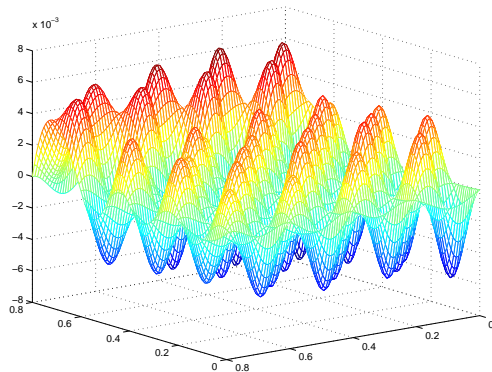
Figure 3: Graph of surface $f(x,y) - P(x,y)$.
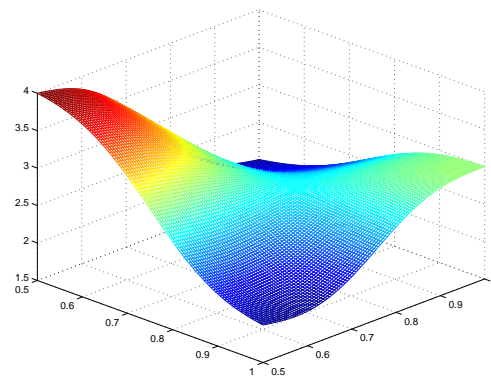
Figure 4: Graph of surface $P_1(x,y)$ with $\alpha_{i,j} = 0.5$, $\beta_{i,j} = 0.8$.

easy to compute that $P_1(0.875, 0.875) = 2.84818$. If the practical design requires $P(0.875, 0.875) = 2.8$, then $\alpha_{i,j} = 5$ and $\beta_{i,j} = 11.2921$ can be obtained. Denote the interpolation by $P_2(x,y)$. Figure 5 shows the graph of the surface $P_2(x,y)$ with the parameters $\alpha_{i,j} = 5, \beta_{i,j} = 11.2921$, and in this case $P_2(0.875, 0.875) = 2.8$.

Furthermore, if the practical design requires $P(0.875, 0.875) = 2.86$, then $\alpha_{i,j} = 0.112391$ and $\beta_{i,j} = 0.2$ can be derived. Denote the interpolation by $P_3(x,y)$. Figure 6 shows the graph of the surface $P_3(x,y)$ with the parameters $\alpha_{i,j} = 0.112391, \beta_{i,j} = 0.2$, and in this case $P_3(0.875, 0.875) = 2.86$.

Remark 2. Each interpolant of the family of the C^2 bivariate rational spline interpolation defined by (4) is identified uniquely by the values of the shape parameters $\alpha_{i,j}$ and $\beta_{i,j}$. For different shape

Table 1: Set of the interpolating data.

(x_i, y_j)	(0,0)	(0,0.5)	(0,1)	(0,1.5)	(0.5,0)	(0.5,0.5)	(0.5,1)	(0.5,1.5)
$f_{i,j}$	3	3	2	4	3	4	2	3
(x_i, y_j)	(1,0)	(1,0.5)	(1,1)	(1,1.5)	(1.5,0)	(1.5,0.5)	(1.5,1)	(1.5,1.5)
$f_{i,j}$	4	2	3	2	3	3	2	4
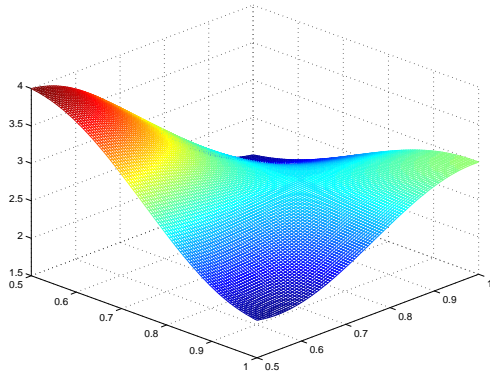
Figure 5: Graph of surface $P_2(x, y)$ with $\alpha_{i,j} = 5$, $\beta_{i,j} = 11.2921$.
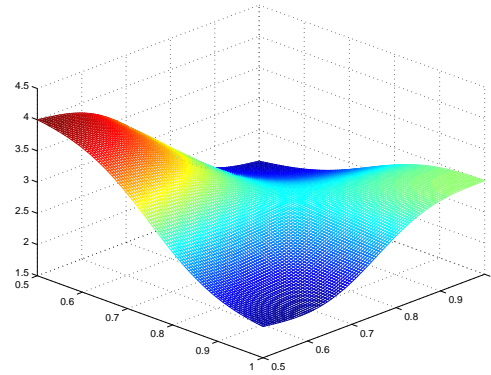
Figure 6: Graph of surface $P_3(x, y)$ with $\alpha_{i,j} = 0.112391$, $\beta_{i,j} = 0.2$.

parameters, from Figures 4 to 6, we can catch sight of some minor changes of the surfaces in shape. It means that the shape modification of interpolating surface can be achieved by selecting suitable shape parameters according to needs of practical design.

7   Concluding remarks

Generally speaking, generating a C^2 bivariate interpolation is a very difficult task, it requires up to second-order partial derivatives values of the interpolated function. Usual NURBS method is the most popular technology in modern surface modeling, however, preset weights are needed to generate a C^2 rational surface, the given points play the role of the control points.

This paper develops a new interpolating approach for construction of C^2 bivariate rational spline interpolants only based on the values of a function. There are two schemes to generate this type of interpolation function, one is interpolating from the x-direction first, another is from the y-direction first. In the interpolant beginning from the x-direction first, there are three positive shape parameters: $\alpha_{i,j}, \alpha_{i,j+1}$ and $\beta_{i,j}$; In the interpolant beginning from the y-direction first, there are also three positive shape parameters: $\beta_{i,j}, \beta_{i+1,j}$ and $\alpha_{i,j}$. Generally, when the interpolating data are given, because of the uniqueness of the interpolation function, the shape of interpolating surface is fixed. However, note that there are some free shape parameters in (4) and (7), the shape of the interpolating surfaces can be modified by selecting suitable shape parameters for the unchanged interpolating data according to the control need, and numerical examples illustrate this case. It means that the uniqueness of the interpolating surfaces for the given interpolating data becomes that of for the given interpolating data and the shape parameters.

For each pitch of the interpolating surface, the value of the interpolation function depends on the interpolating data. Theorem 3 shows that the values of the interpolant is bounded in whole interpolating region, it means that the interpolant is stable for the positive shape parameters. Also, numerical example shows that the interpolator can give a good approximation to the original function.

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Brunn-Minkowski type inequalities for width-integrals of index i

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Abstract

In this paper, we investigate Brunn-Minkowski type inequalities for width-integrals of index i related to the Blaschke Minkowski homomorphism. Some inequalities similar to Lutwak's inequality are established.

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1 Introduction and Main Results

Let $\mathcal{K}^n$ be the set of convex bodies, which is a compact, convex subsets with nonempty interiors in Euclidean space $\mathbb{R}^n$. S^{n-1} denotes the unit sphere in $\mathbb{R}^n$. We denote by $V(K)$ the n-dimensional volume of a body K. For the standard unit ball B in $\mathbb{R}^n$, we denote its volume by $\omega_n = V(B)$.

The support function, $h_K = h(K, \cdot) : \mathbb{R}^n \to (-\infty, \infty)$, of a convex body $K \in \mathcal{K}^n$ is defined by (see [5, 14])

$$h(K, x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y.

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The study of width-integral has a long history, and has received considerable attention. It was first considered by Blaschke [2], and a book by Hadwiger [6] detailed this problem. In 1975, width-integral was extended to width-integrals of index i by Lutwak in the reference [10]. For the more results associated with width-integrals of index i, we refer the interested reader to [8, 11-13].

Define by $\overline{B}_i(K)$, $i \in \mathbb{R}$, the width-integrals of index i of $K \in \mathcal{K}^n$ (see [10])

$$\overline{B}_i(K) = \frac{1}{n} \int_{S^{n-1}} \bar{b}(K,u)^{n-i} dS(u), \qquad (1.1)$$

where $dS(u)$ and $\bar{b}(K,u)$ denote the $(n-1)$-dimensional volume element on S^{n-1} and the half width, $\bar{b}(K,u) = \frac{1}{2}(h(K,u) + h(K,-u))$, of K in the direction u, respectively. If we take $i = 0$ in (1.1), $\overline{B}_i(K)$ is just width-integral $\overline{B}(K)$.

The map, $\overline{B}_i : \mathcal{K}^n \to \mathbb{R}$, is continuous, positive, invariant under motion and homogeneous of degree $n-i$. If there exists a constant $\lambda > 0$ such that $\bar{b}(K,u) = \lambda \bar{b}(L,u)$ for all $u \in S^{n-1}$, then we call K and L with similar width.

Lutwak [10] showed that if $K, L \in \mathcal{K}^n$, then for $u \in S^{n-1}$

$$\bar{b}(K+L,u) = \bar{b}(K,u) + \bar{b}(L,u). \qquad (1.2)$$

From the above formula (1.2), Lutwak [10] established the following Brunn-Minkowski inequality for width-integrals of index i.

Theorem 1.A *If $K, L \in \mathcal{K}^n$ and $i \leq n-1$, then*

$$\overline{B}_i(K+L)^{\frac{1}{n-i}} \leq \overline{B}_i(K)^{\frac{1}{n-i}} + \overline{B}_i(L)^{\frac{1}{n-i}},$$

with equality if and only if K and L have similar width.

The main results of the present paper are the following:

We first establish a Brunn-Minkowski type inequality for width-integrals of index i similar to Theorem 1.A.

Theorem 1.1 *If $K, L \in \mathcal{K}^n$, then for $i \leq n-1$*

$$\overline{B}_i(\Phi(K\hat{+}L))^{\frac{1}{n-i}} \leq \overline{B}_i(\Phi K)^{\frac{1}{n-i}} + \overline{B}_i(\Phi L)^{\frac{1}{n-i}}, \qquad (1.3)$$

with equality if and only if ΦK and ΦL are homothetic.

Here $K\hat{+}L$ is the Blaschke sum of $K, L \in \mathcal{K}^n$ (see (2.7)) and Φ denotes Blaschke Minkowski homomorphism (see Section 2.2). In fact, the more general result than Theorem 1.1 will be obtained at the beginning of Section 3.

Moreover, we get the following Brunn-Minkowski type inequality of width-integrals of index i based on the Blaschke Minkowski homomorphism.

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Theorem 1.2 *If $K_1, K_2, \cdots, K_{n-2}, L_1, L_2 \in \mathcal{K}^n$, and let $C = (K_1, K_2, \cdots, K_{n-2})$, then for $i \leq n - 1$*

$$\overline{B}_i(\Phi(C, L_1 + L_2))^{\frac{1}{n-i}} \leq \overline{B}_i(\Phi(C, L_1))^{\frac{1}{n-i}} + \overline{B}_i(\Phi(C, L_2))^{\frac{1}{n-i}}; \qquad (1.4)$$

for $i > n$

$$\overline{B}_i(\Phi(C, L_1 + L_2))^{\frac{1}{n-i}} \geq \overline{B}_i(\Phi(C, L_1))^{\frac{1}{n-i}} + \overline{B}_i(\Phi(C, L_2))^{\frac{1}{n-i}}, \qquad (1.5)$$

with equality in every inequality if and only if $\Phi(C, L_1)$ and $\Phi(C, L_2)$ are homothetic.

Here, we should note that Φ denotes mixed Blaschke-Minkowski homomorphism (see Section 2.2 for precise definition), and $L_1 + L_2$ is the Minkowski sum of $L_1, L_2 \in \mathcal{K}^n$ (see (2.5)).

Finally, we show the following result which is the more general form of the Brunn-Minkowski type inequality of width-integrals of index i.

Theorem 1.3 *If $K_1, K_2, \cdots, K_{n-2}, K, L \in \mathcal{K}^n$, let $C = (K_1, K_2, \cdots, K_{n-2})$ and $i, j \in \mathbb{R}$, then for $i \leq n - 1 \leq j \leq n$ and $i \neq j$*

$$\left(\frac{\overline{B}_i(\Phi(C, (K +_{-1} L)^*))}{\overline{B}_j(\Phi(C, (K +_{-1} L)^*))} \right)^{\frac{1}{j-i}} \leq \left(\frac{\overline{B}_i(\Phi(C, K^*))}{\overline{B}_j(\Phi(C, K^*))} \right)^{\frac{1}{j-i}} + \left(\frac{\overline{B}_i(\Phi(C, L^*))}{\overline{B}_j(\Phi(C, L^*))} \right)^{\frac{1}{j-i}}; \qquad (1.6)$$

for $n - 1 \leq i \leq n \leq j$ and $i \neq j$

$$\left(\frac{\overline{B}_i(\Phi(C, (K +_{-1} L)^*))}{\overline{B}_j(\Phi(C, (K +_{-1} L)^*))} \right)^{\frac{1}{j-i}} \geq \left(\frac{\overline{B}_i(\Phi(C, K^*))}{\overline{B}_j(\Phi(C, K^*))} \right)^{\frac{1}{j-i}} + \left(\frac{\overline{B}_i(\Phi(C, L^*))}{\overline{B}_j(\Phi(C, L^*))} \right)^{\frac{1}{j-i}}, \qquad (1.7)$$

with equality in every inequality if and only if $\Phi(C, K^*)$ and $\Phi(C, L^*)$ are homothetic.

Here $K +_{-1} L$ denotes the harmonic radial sum of $K, L \in \mathcal{S}^n$ (see (2.4)) and K^* denotes the polar of K (see (2.2)).

2 Preliminaries

2.1 Radial function and polar of convex bodies

For a compact star-shaped set K in $\mathbb{R}^n$, let $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$ denote the radial function of K (see[5, 14]), that is

$$\rho(K, u) = \max\{\lambda \geq 0 : \lambda \cdot u \in K\}, \quad u \in S^{n-1}. \qquad (2.1)$$

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If ρ_K is positive and continuous, then K is said to be a star body, and $\mathcal{S}^n$ denotes the set of star bodies in Euclidean space $\mathbb{R}^n$.

For $K \in \mathcal{K}^n$, its polar body is defined by (see[5, 14])

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in K\}. \tag{2.2}$$

Obviously, it follows from (2.2) that $(K^*)^* = K$ and

$$h_{K^*} = \rho_K^{-1}, \quad \rho_{K^*} = h_K^{-1}. \tag{2.3}$$

For $\lambda, \mu \geq 0$ (not both zero), define by $\lambda \star K +_{-1} \mu \star L$ the harmonic radial combination of $K, L \in \mathcal{S}^n$ (see [4]). Namely,

$$\rho^{-1}(\lambda \star K +_{-1} \mu \star L, \cdot) = \lambda \rho^{-1}(K, \cdot) + \mu \rho^{-1}(L, \cdot). \tag{2.4}$$

For $\lambda, \mu \geq 0$ (not both zero), define by $\lambda \cdot K + \mu \cdot L$ the Minkowski linear combination of $K, L \in \mathcal{K}^n$ (see [5, 14]), Namely,

$$h(\lambda \cdot K + \mu \cdot L, \cdot) = \lambda h(K, \cdot) + \mu h(L, \cdot). \tag{2.5}$$

Combining (2.3), (2.4) with (2.5), we obtain that for $K, L \in \mathcal{K}^n$ and $\lambda, \mu \geq 0$ (not both zero)

$$(\lambda \star K +_{-1} \mu \star L)^* = \lambda \cdot K^* + \mu \cdot L^*. \tag{2.6}$$

2.2 Blaschke Minkowski homomorphism

For $\lambda, \mu \geq 0$ (not both zero), define by $\lambda \circ K \hat{+} \mu \circ L$ the Blaschke addition of $K, L \in \mathcal{K}^n$ such that (see [5, 14])

$$S(\lambda \circ K \hat{+} \mu \circ L, \cdot) = \lambda S(K, \cdot) + \mu S(L, \cdot), \tag{2.7}$$

where $S(K, \cdot)$ denotes the surface area measure of K.

Schuster [15] introduced the definition of Blaschke Minkowski homomorphism as follows: A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is called Blaschke Minkowski homomorphism if it satisfies the following conditions

(a) Φ is continuous.

(b) Φ is Blaschke Minkowski additive, i.e., for all $K, L \in \mathcal{K}^n$

$$\Phi(K \hat{+} L) = \Phi K + \Phi L. \tag{2.8}$$

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(c) Φ intertwines rotation, i.e., for all $K \in \mathcal{K}^n$ and $\vartheta \in SO(n)$

$$\Phi(\vartheta K) = \vartheta \Phi K.$$

Here $SO(n)$ is the group of rotation in n dimensions.

The following result is a direct extension for the Blaschke Minkowski homomorphism which is said to be the mixed Blaschke Minkowski homomorphism.

Theorem 2.A(*[15]*) *There is a continuous operator*

$$\Phi : \underbrace{\mathcal{K}^n \times \cdots \times \mathcal{K}^n}_{n-1} \to \mathcal{K}^n,$$

symmetric in its arguments such that for $K_1, \cdots, K_m \in \mathcal{K}^n$ and $\lambda_1, \cdots, \lambda_m \geq 0$,

$$\Phi(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1, \cdots, i_{n-1}} \lambda_{i_1} \cdots \lambda_{i_{n-1}} \Phi(K_{i_1}, \cdots, K_{i_{n-1}}),$$

where the operator $\Phi : \underbrace{\mathcal{K}^n \times \cdots \times \mathcal{K}^n}_{n-1} \to \mathcal{K}^n$ is called mixed Blaschke Minkowski homomorphism.

Further, the author of [15] established the following properties for the mixed Blaschke Minkowski homomorphism.

(i) $\Phi : \mathcal{K}^n \times \cdots \times \mathcal{K}^n \to \mathcal{K}^n$ is continuous and symmetric with respect to origin.

(ii) If $K, L, K_1, K_2, \cdots, K_{n-2} \in \mathcal{K}^n$, $\lambda, \mu \geq 0$, and let $C = (K_1, K_2, \cdots, K_{n-2})$, then

$$\Phi(C, \lambda \cdot K + \mu \cdot L) = \lambda \Phi(C, K) + \mu \Phi(C, L). \tag{2.9}$$

3 The Proofs of Main Results

Here, we first establish Theorem 3.1 which is the more general version of Theorem 1.1. Next, we will prove Theorems 1.2 and 1.3.

Theorem 3.1 *If $K, L \in \mathcal{K}^n$ and $i, j \in \mathbb{R}$, then for $i \leq n-1 \leq j \leq n$ and $i \neq j$*

$$\left(\frac{\overline{B}_i(\Phi(K \hat{+} L))}{\overline{B}_j(\Phi(K \hat{+} L))} \right)^{\frac{1}{j-i}} \leq \left(\frac{\overline{B}_i(\Phi K)}{\overline{B}_j(\Phi K)} \right)^{\frac{1}{j-i}} + \left(\frac{\overline{B}_i(\Phi L)}{\overline{B}_j(\Phi L)} \right)^{\frac{1}{j-i}}; \tag{3.1}$$

for $n-1 \leq i \leq n \leq j$ and $i \neq j$

$$\left(\frac{\overline{B}_i(\Phi(K \hat{+} L))}{\overline{B}_j(\Phi(K \hat{+} L))} \right)^{\frac{1}{j-i}} \geq \left(\frac{\overline{B}_i(\Phi K)}{\overline{B}_j(\Phi K)} \right)^{\frac{1}{j-i}} + \left(\frac{\overline{B}_i(\Phi L)}{\overline{B}_j(\Phi L)} \right)^{\frac{1}{j-i}}, \tag{3.2}$$

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with equality in every inequality if and only if ΦK and ΦL are homothetic.

The proof of Theorem 3.1 requires the following lemmas.

Lemma 3.1 *([10]) If K is a convex body in $\mathbb{R}^n$, then*

$$\overline{B}_{2n}(K) \leq V(K^*), \tag{3.3}$$

with equality if and only if K is origin-symmetric.

Beckenbach-Dresher inequality [3] is an extension of Beckenbach's inequality [1] which is proved by Dresher through the method of moment-space techniques.

Lemma 3.2 *(The Beckenbach-Dresher inequality) If $p \geq 1 \geq r \geq 0$, $p \neq r$, $f, g \geq 0$, and ϕ is a distribution function, then*

$$\left(\frac{\int_{\mathbb{E}}(f+g)^p d\phi}{\int_{\mathbb{E}}(f+g)^r d\phi}\right)^{\frac{1}{p-r}} \leq \left(\frac{\int_{\mathbb{E}} f^p d\phi}{\int_{\mathbb{E}} f^r d\phi}\right)^{\frac{1}{p-r}} + \left(\frac{\int_{\mathbb{E}} g^p d\phi}{\int_{\mathbb{E}} g^r d\phi}\right)^{\frac{1}{p-r}}, \tag{3.4}$$

with equality if and only if the functions f and g are positively proportional. Here $\mathbb{E}$ is a bounded measurable subset in $\mathbb{R}^n$.

The inverse Beckenbach-Dresher inequality was established in [9].

Lemma 3.3 *(The inverse Beckenbach-Dresher inequality) If $1 \geq p \geq 0 \geq r$, $p \neq r$, $f, g \geq 0$, and ϕ is a distribution function, then*

$$\left(\frac{\int_{\mathbb{E}}(f+g)^p d\phi}{\int_{\mathbb{E}}(f+g)^r d\phi}\right)^{\frac{1}{p-r}} \geq \left(\frac{\int_{\mathbb{E}} f^p d\phi}{\int_{\mathbb{E}} f^r d\phi}\right)^{\frac{1}{p-r}} + \left(\frac{\int_{\mathbb{E}} g^p d\phi}{\int_{\mathbb{E}} g^r d\phi}\right)^{\frac{1}{p-r}}, \tag{3.5}$$

with equality if and only if the functions f and g are positively proportional.

Proof of Theorem 3.1. According to (1.1) and (2.8), we obtain that for $p \geq 1 \geq r \geq 0$

$$\overline{B}_{n-p}(\Phi(K\hat{+}L)) = \frac{1}{n}\int_{S^{n-1}} \bar{b}(\Phi(K\hat{+}L), u)^p dS(u)$$

$$= \frac{1}{n}\int_{S^{n-1}} (\bar{b}(\Phi K, u) + \bar{b}(\Phi L, u))^p dS(u). \tag{3.6}$$

Similarly,

$$\overline{B}_{n-r}(\Phi(K\hat{+}L)) = \frac{1}{n}\int_{S^{n-1}} (\bar{b}(\Phi K, u) + \bar{b}(\Phi L, u))^r dS(u). \tag{3.7}$$

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From Lemma 3.2, (3.6) and (3.7), this implies

$$\left(\frac{\overline{B}_{n-p}(\Phi(K\hat{+}L))}{\overline{B}_{n-r}(\Phi(K\hat{+}L))}\right)^{\frac{1}{p-r}}$$

$$= \left(\frac{\int_{S^{n-1}}(\bar{b}(\Phi K,u)+\bar{b}(\Phi L,u))^p dS(u)}{\int_{S^{n-1}}(\bar{b}(\Phi K,u)+\bar{b}(\Phi L,u))^r dS(u)}\right)^{\frac{1}{p-r}}$$

$$\leq \left(\frac{\int_{S^{n-1}}\bar{b}(\Phi K,u)^p dS(u)}{\int_{S^{n-1}}\bar{b}(\Phi K,u)^r dS(u)}\right)^{\frac{1}{p-r}} + \left(\frac{\int_{S^{n-1}}\bar{b}(\Phi L,u)^p dS(u)}{\int_{S^{n-1}}\bar{b}(\Phi L,u)^r dS(u)}\right)^{\frac{1}{p-r}}$$

$$= \left(\frac{\overline{B}_{n-p}(\Phi K)}{\overline{B}_{n-r}(\Phi K)}\right)^{\frac{1}{p-r}} + \left(\frac{\overline{B}_{n-p}(\Phi L)}{\overline{B}_{n-r}(\Phi L)}\right)^{\frac{1}{p-r}}. \tag{3.8}$$

Let $p = n - i$ and $r = n - j$. By $0 \leq r \leq 1 \leq p$ and $p \neq r$, we obtain that $i \leq n - 1 \leq j \leq n$ and $i \neq j$. Suppose $p = n - i$ and $r = n - j$ in (3.8), this gives inequality (3.1). Similar to the above method, it follows from Lemma 3.3 that inequality (3.2).

The equality conditions of Lemmas 3.2 and 3.3 imply that equality holds in inequalities (3.1) and (3.2) if and only if $\bar{b}(\Phi K, u)$ and $\bar{b}(\Phi L, u)$ are positively proportional, namely, ΦK and ΦL have similar width. Since ΦK and ΦL are origin-symmetric, we have ΦK and ΦL are homothetic. Therefore, equality holds in every inequality if and only if ΦK and ΦL are homothetic.□

If $j = n$ in (3.1), then $\overline{B}_n(K) = \frac{1}{n}\int_{S^{n-1}}dS(u) = \omega_n$ is a constant. thus we get Theorem 1.1.

Let $i = 2n$ and $j = n$ in (3.2). Note that ΦK, ΦL and $\Phi(K\hat{+}L)$ are origin-symmetric. Thus Lemma 3.1 implies that inequality (3.2) has the following result.

Corollary 3.1 *If $K, L \in \mathcal{K}^n$, then*

$$V(\Phi^*(K\hat{+}L))^{-\frac{1}{n}} \geq V(\Phi^*K)^{-\frac{1}{n}} + V(\Phi^*L)^{-\frac{1}{n}},$$

with equality if and only if ΦK and ΦL are homothetic.

Proof of Theorem 1.2. We first prove inequality (1.4). From (1.1), (1.2), (2.9) and the Minkowski's integral inequality (see [7]), we obtain that for $i \leq n - 1$

$$\overline{B}_i(\Phi(C, L_1 + L_2))^{\frac{1}{n-i}} = \left(\frac{1}{n}\int_{S^{n-1}}\bar{b}(\Phi(C, L_1 + L_2), u)^{n-i}dS(u)\right)^{\frac{1}{n-i}}$$

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$$= \left(\frac{1}{n} \int_{S^{n-1}} (\bar{b}(\Phi(C, L_1), u) + \bar{b}(\Phi(C, L_2), u))^{n-i} dS(u) \right)^{\frac{1}{n-i}}$$

$$\leq \left(\frac{1}{n} \int_{S^{n-1}} \bar{b}(\Phi(C, L_1), u)^{n-i} dS(u) \right)^{\frac{1}{n-i}}$$

$$+ \left(\frac{1}{n} \int_{S^{n-1}} \bar{b}(\Phi(C, L_2), u)^{n-i} dS(u) \right)^{\frac{1}{n-i}}$$

$$= \overline{B}_i(\Phi(C, L_1))^{\frac{1}{n-i}} + \overline{B}_i(\Phi(C, L_2))^{\frac{1}{n-i}}.$$

This implies inequality (1.4). Similarly, it follows from the inverse Minkowski's integral inequality that inequality (1.5).

From the equality condition of Minkowski's integrals inequalities, it follows that equality holds in inequalities (1.4) and (1.5) if and only if $\Phi(C, L_1)$ and $\Phi(C, L_2)$ have similar width. Since $\Phi(C, L_1)$ and $\Phi(C, L_2)$ are origin-symmetric, we obtain that equality holds in every inequality if and only if $\Phi(C, L_1)$ and $\Phi(C, L_2)$ are homothetic.$\square$

If $i = 0$ in (1.4), then inequality (1.4) implies the following result.

Corollary 3.2 If $K_1, K_2, \cdots, K_{n-2}, L_1, L_2 \in \mathcal{K}^n$, and let $C = (K_1, K_2, \cdots, K_{n-2})$, then

$$\overline{B}(\Phi(C, L_1 + L_2))^{\frac{1}{n}} \leq \overline{B}(\Phi(C, L_1))^{\frac{1}{n}} + \overline{B}(\Phi(C, L_2))^{\frac{1}{n}},$$

with equality if and only if $\Phi(C, L_1)$ and $\Phi(C, L_2)$ are homothetic.

If we let $i = 2n$ in (1.5), and note that $\Phi(C, L_1)$, $\Phi(C, L_2)$ and $\Phi(C, L_1 + L_2)$ are origin-symmetric, then Lemma 3.1 implies that inequality (1.5) has the following result.

Corollary 3.3 If $K_1, K_2, \cdots, K_{n-2}, L_1, L_2 \in \mathcal{K}^n$, and let $C = (K_1, K_2, \cdots, K_{n-2})$, then

$$V(\Phi^*(C, L_1 + L_2))^{-\frac{1}{n}} \geq V(\Phi^*(C, L_1))^{-\frac{1}{n}} + V(\Phi^*(C, L_2))^{-\frac{1}{n}},$$

with equality if and only if $\Phi(C, L_1)$ and $\Phi(C, L_2)$ are homothetic.

Proof of Theorem 1.3. Combining (1.1) with (2.6), we obtain that for $p \geq 1 \geq r \geq 0$

$$\overline{B}_{n-p}(\Phi(C, (K +_{-1} L)^*))$$

$$= \frac{1}{n} \int_{S^{n-1}} \bar{b}(\Phi(C, (K +_{-1} L)^*), u)^p dS(u)$$

$$= \frac{1}{n} \int_{S^{n-1}} \bar{b}(\Phi(C, K^* + L^*), u)^p dS(u)$$

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$$= \frac{1}{n} \int_{S^{n-1}} (\bar{b}(\Phi(C, K^*), u) + \bar{b}(\Phi(C, L^*), u))^p dS(u). \qquad (3.9)$$

Similarly,

$$\overline{B}_{n-r}(\Phi(C, (K +_{-1} L)^*))$$

$$= \frac{1}{n} \int_{S^{n-1}} (\bar{b}(\Phi(C, K^*), u) + \bar{b}(\Phi(C, L^*), u))^r dS(u). \qquad (3.10)$$

By Lemma 3.2, (3.9) and (3.10), it follows that

$$\left(\frac{\overline{B}_{n-p}(\Phi(C, (K +_{-1} L)^*))}{\overline{B}_{n-r}(\Phi(C, (K +_{-1} L)^*))} \right)^{\frac{1}{p-r}}$$

$$= \left(\frac{\int_{S^{n-1}} (\bar{b}(\Phi(C, K^*), u) + \bar{b}(\Phi(C, L^*), u))^p dS(u)}{\int_{S^{n-1}} (\bar{b}(\Phi(C, K^*), u) + \bar{b}(\Phi(C, L^*), u))^r dS(u)} \right)^{\frac{1}{p-r}}$$

$$\leq \left(\frac{\int_{S^{n-1}} \bar{b}(\Phi(C, K^*), u)^p dS(u)}{\int_{S^{n-1}} \bar{b}(\Phi(C, K^*), u)^r dS(u)} \right)^{\frac{1}{p-r}} + \left(\frac{\int_{S^{n-1}} \bar{b}(\Phi(C, L^*), u)^p dS(u)}{\int_{S^{n-1}} \bar{b}(\Phi(C, L^*), u)^r dS(u)} \right)^{\frac{1}{p-r}}$$

$$= \left(\frac{\overline{B}_{n-p}(\Phi(C, K^*))}{\overline{B}_{n-r}(\Phi(C, K^*))} \right)^{\frac{1}{p-r}} + \left(\frac{\overline{B}_{n-p}(\Phi(C, L^*))}{\overline{B}_{n-r}(\Phi(C, L^*))} \right)^{\frac{1}{p-r}}. \qquad (3.11)$$

If $p = n - i$ and $r = n - j$ in (3.11), then $0 \leq r \leq 1 \leq p$ and $p \neq r \Rightarrow i \leq n-1 \leq j \leq n$ and $i \neq j$.

This implies that inequality (1.6) is given. Similar to the above method, Lemma 3.3 implies that inequality (1.7) holds.

The equality conditions of Lemmas 3.2 and 3.3 see that with equality in inequalities (1.6) and (1.7) if and only if $\bar{b}(\Phi(C, K^*), u)$ and $\bar{b}(\Phi(C, L^*), u)$ are positively proportional. Since $\Phi(C, K^*)$ and $\Phi(C, L^*)$ are origin-symmetric, $\Phi(C, K^*)$ and $\Phi(C, L^*)$ are homothetic. Therefore, equality holds in every inequality if and only if $\Phi(C, K^*)$ and $\Phi(C, L^*)$ are homothetic. $\square$

Analogue to the proofs of Theorem 1.1 and Corollary 3.1, Theorem 1.3 has the following facts.

Corollary 3.4 *If $K_1, K_2, \cdots, K_{n-2}, K, L \in \mathcal{K}^n$, and let $C = (K_1, K_2, \cdots, K_{n-2})$, then for $i \leq n - 1$*

$$\overline{B}_i(\Phi(C, (K +_{-1} L)^*))^{\frac{1}{n-i}} \leq \overline{B}_i(\Phi(C, K^*))^{\frac{1}{n-i}} + \overline{B}_i(\Phi(C, L^*))^{\frac{1}{n-i}},$$

with equality if and only if $\Phi(C, K^*)$ and $\Phi(C, L^*)$ are homothetic.

Corollary 3.5 *If $K_1, K_2, \cdots, K_{n-2}, K, L \in \mathcal{K}^n$, and let $C = (K_1, K_2, \cdots, K_{n-2})$,*

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then

$$V(\Phi^*(C,(K +_{-1} L)^*))^{-\frac{1}{n}} \geq V(\Phi^*(C,K^*))^{-\frac{1}{n}} + V(\Phi^*(C,L^*))^{-\frac{1}{n}},$$

with equality if and only if $\Phi(C,K^*)$ *and* $\Phi(C,L^*)$ *are homothetic.*

4 Conclusions

We first establish some Brunn-Minkowski type inequalities of width-integrals of index i which are related to the Blaschke Minkowski homomorphism. Then together with the Blaschke Minkowski homomorphism, we use the Beckenbach-Dresher inequalities to give more general Brunn-Minkowski type inequalities of width-integrals of index i, in which some inequalities similar to Lutwak's inequality are established.

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this article.

Acknowledgments

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On fuzzy mighty filters in BE-algebras

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Abstract. In this paper, we study several degrees in defining a fuzzy mighty filter, which is a generalization of a fuzzy filter in BE-algebras.

1. Introduction

In [6], H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra. S. S. Ahn and K. S. So [3, 4] introduced the notion of ideals in BE-algebras. S. S. Ahn et al. [2] fuzzified the concept of BE-algebras, and investigated some of their properties. H. R. Lee and S. S. Ahn [7] defined the notions of a mighty filter and an n-fold mighty filter, and considered some related properties of them.

In this paper, we study several degrees in defining a fuzzy mighty filter, which is a generalization of a fuzzy filter in BE-algebras.

2. Prelimiaries

We recall some definitions and results discussed in [3, 6].

An algebra $(X; *, 1)$ of type (2, 0) is called a *BE-algebra* if

(BE1) $x * x = 1$ for all $x \in X$,
(BE2) $x * 1 = 1$ for all $x \in X$,
(BE3) $1 * x = x$ for all $x \in X$,
(BE4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$ (*exchange*).

We introduce a relation "$\leq$" on a BE-algebra X by $x \leq y$ if and only if $x * y = 1$. A non-empty subset S of a BE-algebra X is said to be a *subalgebra* of X if it is closed under the operation "$*$". Noticing that $x * x = 1$ for all $x \in X$, it is clear that $1 \in S$. A BE-algebra $(X; *, 1)$ is said to be *self distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$.

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Let X be a BE-algebra and n denote a positive integer. For any elements $x, y \in X$, let $x^n * y$ denote $x * (\cdots (x * (x * y)) \cdots)$, in which x occurs n times, and $x^0 * y = y$.

Definition 2.1. Let $(X; *, 1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is called a *filter* ([6]) of X if

(F1) $1 \in F$,

(F2) $x * y \in F$ and $x \in F$ imply $y \in F$

for all $x, y \in X$. A non-empty subset F of a BE-algebra X is called a *mighty filter* ([7]) of X if it satisfies (F1) and

(F3) $x * (y * z) \in F$ and $x \in F$ imply $((z * y) * y) * z \in F$

for all $x, y, z \in X$. A non-empty subset F of a BE-algebra X is called an *n-fold mighty filter* ([7]) of X if it satisfies (F1) and

(F4) $x * (y * z) \in F$ and $x \in F$ imply $((z^n * y) * y) * z \in F$

for all $x, y, z \in X$.

Note that every mighty filter of a BE-algebra X is a filter of X.

Proposition 2.2. *Let* $(X; *, 1)$ *be a* BE-*algebra and let* F *be a filter of* X. *If* $x \leq y$ *and* $x \in F$ *for any* $y \in X$, *then* $y \in F$.

Proposition 2.3 *Let* $(X; *, 1)$ *be a self distributive* BE-*algebra. Then the following hold, for any* $x, y, z \in X$:

(i) if $x \leq y$, then $z * x \leq z * y$ and $y * z \leq x * z$,

(ii) $y * z \leq (z * x) * (y * x)$,

(iii) $y * z \leq (x * y) * (x * z)$.

A BE-algebra $(X; *, 1)$ is said to be *transitive* if it satisfies Proposition 2.3 (iii). If a BE-algebra X is transitive, then $y \leq z$ imply $x * y \leq x * z$ and $z * x \leq y * x$ for all $x, y, z \in X$.

Definition 2.4. ([5]) A fuzzy subset μ of a BE-algebra X is called a *fuzzy filter* of X if it satisfies for all $x, y \in X$

(d1) $\mu(1) \geq \mu(x)$,

(d2) $\mu(x) \geq \min\{\mu(y * x), \mu(y)\}$.

Proposition 2.5. *Every fuzzy filter of a* BE-*algebra* X *satisfies the following assertions:*

(i) $(\forall x, y \in X)(y \leq x \Rightarrow \mu(y) \leq \mu(x))$,

(ii) $(\forall x, y, z \in X)(x \leq y * z \Rightarrow \mu(z) \geq \min\{\mu(y), \mu(x)\})$.

Definition 2.6. ([5]) Let F be a non-empty subset of a BE-algebra X which is not necessary a filter of X. One says that a subset G of X is an *enlarged filter* of X related to F if it satisfies:

(1) F is a subset of G,

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(2) $1 \in G$,

(3) $(\forall x, y \in X)(\forall x \in F)(x * y \in F \Rightarrow y \in G)$.

Definition 2.7. ([5]) A fuzzy subset μ of a BE-algebra X is called a *fuzzy filter* of X with degree (λ, κ) if it satisfies:

(e1) $(\forall x \in X)(\mu(1) \geq \lambda \mu(x))$,

(e2) $(\forall x, y \in X)(\mu(x) \geq \kappa \min\{\mu(y * x), \mu(y)\})$.

Proposition 2.8. ([5]) *Every fuzzy filter of a BE-algebra X with degree (λ, κ) satisfies the following assertions:*

(i) $(\forall x, y \in X)(\mu(x * y) \geq \lambda \kappa \mu(y))$,

(ii) $(\forall x, y \in X)(y \leq x \Rightarrow \mu(x) \geq \lambda \kappa \mu(y))$,

(iii) $(\forall x, y, z \in X)(x \leq y * z \Rightarrow \mu(z) \geq \min\{\kappa \mu(y), \lambda \kappa^2 \mu(x)\})$.

3. Fuzzy mighty filters of BE-algebras

In what follows, let X denote a BE-algebra unless specified otherwise.

Definition 3.1. A fuzzy subset μ of a BE-algebra X is called a *fuzzy mighty filter* of X if it satisfies (d1) and

(d3) $\mu(((x * y) * y) * x) \geq \min\{\mu(z * (y * x)), \mu(z)\}$,

for all $x, y, z \in X$.

Example 3.2. Let $X := \{1, a, b, c, d, 0\}$ be a BE-algebra ([7]) with the following table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	b	c	b	c
b	1	a	1	b	a	d
c	1	a	1	1	a	a
d	1	1	1	b	1	b
0	1	1	1	1	1	1

Define a fuzzy subset $\mu : X \to [0, 1]$ by

$$\mu = \begin{pmatrix} 1 & a & b & c & d & 0 \\ 0.7 & 0.4 & 0.7 & 0.7 & 0.4 & 0.4 \end{pmatrix}$$

Then μ is a fuzzy mighty filter of X.

Proposition 3.3. *Every fuzzy mighty filter of a BE-algebra X is a fuzzy filter of X.*

Proof. Let $y := 1$ in (d3). Then $\mu(x) = \mu(((x * 1) * 1) * x) \geq \min\{\mu(z * (1 * x)), \mu(z)\} = \min\{\mu(z * x), \mu(z)\}$. Hence (d2) holds. $\qquad \square$

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The converse of Proposition 3.3 may not be true in general (see Example 3.4).

Example 3.4. Let $X := \{1, a, b, c, d\}$ be a BE-algebra ([6]) with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Define a fuzzy subset $\nu : X \to [0, 1]$ by

$$\nu = \begin{pmatrix} 1 & a & b & c & d \\ 0.7 & 0.3 & 0.3 & 0.3 & 0.3 \end{pmatrix}$$

Then ν is a fuzzy filter of X, but not a fuzzy mighty filter of X, since $\mu(((a*c)*c)*a) = \mu(a) = 0.3 \ngeq 0.7 = \mu(1) = \min\{\mu(1*(c*a)), \mu(1)\}$.

Theorem 3.5. *A fuzzy filter μ of a BE-algebra X is mighty if and only if it satisfies the following inequality:*

(3.1) $\mu(((x*y)*y)*x) \geq \mu(y*x)$ *for all $x, y \in X$.*

Proof. Suppose that a fuzzy filter μ of a BE-algebra X is mighty. Putting $z := 1$ in (d3), we have $\mu(((x*y)*y)*x) \geq \min\{\mu(1*(y*x)), \mu(1)\} = \mu(y*x)$ for all $x, y \in X$.

Conversely, assume that μ is a fuzzy filter of X satisfying (3.1). It follows from (d2) and (3.1) that $\mu(((x*y)*y)*x) \geq \mu(y*x) \geq \min\{\mu(z*(y*x)), \mu(z)\}$. Hence μ is mighty. $\square$

Theorem 3.6. *Let μ, ν be fuzzy filters of a transitive BE-algebra X such that $\mu \subseteq \nu$ and $\mu(1) = \nu(1)$. If μ is mighty, then so is ν.*

Proof. Let $x, y \in X$. Since μ is a fuzzy mighty filter of a BE-algebra X, by (3.1) and $\mu \subseteq \nu$ we have $\mu(1) = \mu(y*((y*x)*x)) \leq \mu(((((y*x)*x)*y)*y)*((y*x)*x)) \leq \nu(((((y*x)*x)*y)*y)*((y*x)*x))$. Since $\mu(1) = \nu(1)$, we get $\nu((y*x)*(((((y*x)*x)*y)*y)*x)) = \nu(((((y*x)*x)*y)*y)*((y*x)*x)) = \nu(1)$. It follows from (d1) and (d2) that

$$\begin{aligned} \nu(y*x) &= \min\{\nu(1), \nu(y*x)\} \\ &= \min\{\nu((y*x)*(((((y*x)*x)*y)*y)*x)), \nu(y*x)\} \\ &\leq \nu(((((y*x)*x)*y)*y)*x). \end{aligned} \qquad (3.2)$$

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Since X is transitive, we get

$$[((((y * x) * x) * y) * y) * x] * [((x * y) * y) * x]$$
$$\geq ((x * y) * y) * (((((y * x) * x) * y) * y)$$
$$\geq (((y * x) * x) * y) * (x * y)$$
$$\geq x * ((y * x) * x)$$
$$= (y * x) * (x * x)$$
$$= (y * x) * 1 = 1.$$

It follows from Proposition 2.5 that $\min\{\nu((((((y*x)*x)*y)*y)*x), \nu(1)\} = \nu((((((y*x)*x)*y)*y)*x) \leq \nu(((x*y)*y)*x)$. Using (3.2), we have $\nu(y*x) \leq \nu((((((y*x)*x)*y)*y)*x) \leq \nu(((x*y)*y)*x)$. Therefore $\nu(y * x) \leq \nu(((x * y) * y) * x)$. By Theorem 3.5, ν is a fuzzy mighty filter of X.

Proposition 3.7. *Let μ be a fuzzy mighty filter of a BE-algebra. Then $X_\mu := \{x \in X | \mu(x) = \mu(1)\}$ is a mighty filter.*

Proof. Clearly, $1 \in X_\mu$. Let $x * (y * z), x \in X_\mu$. Then $\mu(x * (y * z)) = \mu(1)$ and $\mu(x) = \mu(1)$. It follows from (d3) that $\mu(1) = \min\{\mu(x * (y * z)), \mu(x)\} \leq \mu((((z * y) * y) * z)$. By (d1), $\mu((((z * y) * y) * z) = \mu(1)$. Hence $((z * y) * y) * z \in X_\mu$, Therefore X_μ is a mighty filter of X. $\square$

Definition 3.8. A fuzzy subset μ of a BE-algebra X is called a *fuzzy n-fold mighty filter* of X if it satisfies (d1) and

(d4) $\mu(((x^n * y) * y) * x) \geq \min\{\mu(z * (y * x)), \mu(z)\}$,

for all $x, y, z \in X$.

Putting $n := 1$ in (d4), every fuzzy 1-fold mighty filter of a BE-algebra X is a fuzzy mighty filter of X.

Theorem 3.9. *A fuzzy filter μ of a BE-algebra X is a fuzzy n-fold mighty if and only if it satisfies the following inequality*

(3.3) $\mu(((x^n * y) * y) * x) \geq \mu(y * x)$ *for all $x, y \in X$.*

Proof. Suppose that a fuzzy filter μ is a fuzzy n-fold mighty. Letting $z := 1$ in (d4), we have $\mu(((x^n * y) * y) * x) \geq \min\{\mu(1 * (y * x)), \mu(1)\} = \mu(y * x)$ for all $x, y \in X$.

Conversely, assume that μ is a fuzzy filter of X satisfying (3.3). It follows from (d2) and (3.3) that $\mu(((x^n * y) * y) * x) \geq \mu(y * x) \geq \min\{\mu(z * (y * x)), \mu(z)\}$. Hence μ is a fuzzy n-fold mighty filter of X. $\square$

Theorem 3.10. *Let μ and ν be fuzzy filters of a transitive BE-algebra X with $\mu \subseteq \nu$ and $\mu(1) = \nu(1)$. If μ is a fuzzy n-fold mighty filter of X, then so is ν.*

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Proof. Let $x, y \in X$. Setting $w := (y * x) * x$, we have $y * w = y * ((y * x) * x) = (y * x) * (y * x) = 1$. Since μ is a fuzzy n-fold mighty filter of a BE-algebra X, by (3.3) and $\mu \subseteq \nu$ we have $\mu(1) = \mu(y * w) \leq \mu(((w^n * y) * y) * w) \leq \nu(((w^n * y) * y) * w)$. Since $\mu(1) = \nu(1)$, we get $\nu(((w^n * y) * y) * w) = \nu(1)$. It follows from (d1) and (d2) that

$$
\begin{aligned}
\nu(y * x) &= \min\{\nu(1), \nu(y * x)\} \\
&= \min\{\nu(((w^n * y) * y) * w), \nu(y * x)\} \\
&= \min\{\nu(((w^n * y) * y) * ((y * x) * x)), \nu(y * x)\} \qquad (3.4) \\
&= \min\{\nu((y * x) * (((w^n * y) * y) * *x)), \nu(y * x)\} \\
&\leq \nu(((w^n * y) * y) * x).
\end{aligned}
$$

Since $y \leq w$, we have $w^n * y \leq x^n * y$, and so $((w^n * y) * y) * x \leq ((x^n * y) * y) * x$. Using (3.4) and Proposition 2.5(i), we have $\nu(y * x) \leq \nu((w^n * y) * y) * x) \leq \nu(((x^n * y) * y) * x)$. Therefore $\nu(y * x) \leq \nu(((x^n * y) * y) * x)$. By Theorem 3.9, ν is a fuzzy n-fold mighty filter of X. $\qquad \square$

4. **Fuzzy mighty filters of BE-algebras with degrees in the interval** $(0, 1]$

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Definition 4.1. Let F be a non-empty subset of a BE-algebra X, which is not necessary a mighty filter of X. One says that a subset G of X is an *enlarged mighty filter* of X related to F if it satisfies:

(1) F is a subset of G,
(2) $1 \in G$,
(3) $(\forall x, y \in X)(\forall z \in F)(z * (y * x) \in F \Rightarrow ((x * y) * y) * x \in G)$.

Obviously, every mighty filter is an enlarged mighty filter of X related to itself. Note that there exists an enlarged mighty filter of X related to any non-empty subset F of X.

Example 4.2. Consider a BE-algebra $X = \{1, a, b, c, d\}$ as in Example 3.4. Then $F := \{1\}$ is not a mighty filter of X, since $1 * (c * a) = 1 \in F$ and $((a * c) * c) * a = a \notin F$. But $G := \{1, a, b, c\}$ is an enlarged mighty filter of X related to F and G is not a mighty filter of X, since $c * (1 * d) = b \in G, c \in G$ and $((d * 1) * 1) * d = d \notin G$.

Proposition 4.3. *Let F be a non-empty subset of a BE-algebra X. Every enlarged mighty filter of X related to F is an enlarged filter of X related to F.*

Proof. Let G be an enlarged mighty filter of X related to F. Putting $y := 1$ in Definition 4.1 (3), we have for all $x \in X$, all $z \in F$ $z * (1 * x) = z * x \in F$ imply $((x * 1) * 1) * x = x \in G$. Hence G is an enlarged filter of X related to F. $\qquad \square$

The converse of Proposition 4.3 is not true in general as seen in the following example.

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Example 4.4. Consider a BE-algebra $X = \{1, a, b, c, d\}$ as in Example 3.4. Let $H := \{1, b\}$. Then H is an enlarged filter of X related to $F := \{1\}$, but it is not an enlarged mighty filter of X related to F since $1 * (c * a) = 1 \in F$ and $((a * c) * c) * a = a \notin H$.

In what follows let λ and κ be members of $(0, 1]$, and let n and k denote a natural number and a real number, respectively, such that $k < n$ unless otherwise specified.

Definition 4.5. A fuzzy subset μ of a BE-algebra X is called a *fuzzy mighty filter* of X with degree (λ, κ) if it satisfies (e1) and

(e3) $(\forall x, y, z \in X)(\mu(((x * y) * y) * x) \geq \kappa \min\{\mu(z * (y * x)), \mu(z)\})$.

Example 4.6. Consider a BE-algebra $X = \{1, a, b, c, d, 0\}$ which is given in Example 3.2. Define a fuzzy subset $\nu : X \to [0, 1]$ by

$$\nu = \begin{pmatrix} 1 & a & b & c & d & 0 \\ 0.6 & 0.4 & 0.7 & 0.7 & 0.4 & 0.4 \end{pmatrix}$$

Then ν is a fuzzy filter of X with degree $(\frac{3}{5}, \frac{2}{5})$, but it not a fuzzy mighty filter of X, since $\mu(1) = 0.6 \not\geq \mu(b) = 0.7$.

Note that a fuzzy mighty filter with degree (λ, κ) is a fuzzy mighty filter if and only if $(\lambda, \kappa) = (1, 1)$.

Proposition 4.7. *If μ is a fuzzy mighty filter of a BE-algebra X with degree (λ, κ), then μ is a fuzzy filter of X with degree (λ, κ).*

Proof. Putting $y := 1$ in (e3), we have

$$\mu(x) = \mu(((x * 1) * 1) * x) \geq \kappa \min\{\mu(z * (1 * x)), \mu(z)\}$$
$$= \kappa \min\{\mu(z * x), \mu(z)\}$$

for any $x, y \in X$. Thus μ is a fuzzy filter of X with degree (λ, κ). $\square$

The converse of Proposition 4.7 is not true in general (see Example 4.8).

Example 4.8. Consider a BE-algebra $X = \{1, a, b, c, d\}$ which is given in Example 3.4. Define a fuzzy subset $\nu : X \to [0, 1]$ by

$$\nu = \begin{pmatrix} 1 & a & b & c & d \\ 0.6 & 0.4 & 0.7 & 0.4 & 0.4 \end{pmatrix}$$

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Then ν is a fuzzy filter of X with degree $(\frac{4}{5}, \frac{4}{5})$, but it is not a fuzzy mighty filter of X with degree $(\frac{4}{5}, \frac{4}{5})$, since

$$
\begin{aligned}
\nu(((a * c) * c) * a) = \nu(a) = 0.4 &\not\geq 0.48 = \frac{4}{5} \times 0.6 \\
&= \frac{4}{5} \times \nu(1) \\
&= \frac{4}{5} \times \min\{\nu(1 * (c * a)) = \nu(1), \nu(1)\}.
\end{aligned}
$$

Proposition 4.9. *Let μ be a fuzzy mighty filter of a BE-algebra X with degree (λ, κ). Then the following holds:*

$$(\forall x, y \in X)(\mu(((x * y) * y) * x) \geq \kappa\lambda\mu(y * x)).$$

Proof. Assume that μ is a fuzzy mighty filter of a BE-algebra X with degree (λ, κ) and let $x, y \in X$. Let $z := 1$ in (e3). Then we have

$$
\begin{aligned}
\mu(((x * y) * y) * x) &\geq \kappa\min\{\mu(1 * (y * x)), \mu(1)\} \\
&\geq \kappa\min\{\mu(y * x), \lambda\mu(y * x)\} \\
&= \kappa\lambda\mu(y * x).
\end{aligned}
$$

This completes the proof. $\square$

Proposition 4.10. *Let μ be a fuzzy filter of a BE-algebra X with degree (λ, κ) satisfying*

$$(\forall x, y \in X)(\mu(((x * y) * y) * x) \geq \mu(y * x)).$$

Then μ is a fuzzy mighty filter of X with degree (λ, κ).

Proof. Let $x, y, z \in X$. Using (e2), we have

$$
\begin{aligned}
\mu(((x * y) * y) * x) &\geq \mu(y * x) \\
&\geq \kappa\min\{\mu(z * (y * x)), \mu(z)\}.
\end{aligned}
$$

Thus μ is a fuzzy mighty filter of a BE-algebra X with degree (λ, κ). $\square$

Corollary 4.11. *Let μ be a fuzzy filter of X. Then μ is a fuzzy mighty filter of X if and only if*

$$(\forall x, y \in X)(\mu(((x * y) * y) * x) \geq \mu(y * x)).$$

Proof. It follows from Proposition 4.9 and Proposition 4.10. $\square$

Definition 4.12. A fuzzy subset μ of a BE-algebra X is called a *fuzzy n-fold mighty filter* of X with degree (λ, κ) if it satisfies (e1) and

(e4) $(\forall x, y, z \in X)(\mu(((x^n * y) * y) * x) \geq \kappa\min\{\mu(z * (y * x)), \mu(z)\}).$

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A fuzzy subset μ of a BE-algebra X is called a *fuzzy weak n-fold mighty filter* of X with degree (λ, κ) if it satisfies (e1) and

(e5) $(\forall x, y, z \in X)(\mu((x * y) * y) \geq \kappa \min\{\mu(z * ((y^n * x) * x)), \mu(z)\})$.

Putting $y := 1$ and $x := y$ in (e4), and using (BE1), (BE2) and (BE3), we know that every fuzzy n-fold mighty filter with degree (λ, κ) is a fuzzy mighty filter of X with degree (λ, κ). Setting $x := y$ in (e5) and using (BE1), (BE2) and (BE3), we know that every fuzzy weak n-fold mighty filter with degree (λ, κ) is a fuzzy mighty filter of X with degree (λ, κ). Hence every fuzzy (weak) n-fold mighty filter of X with degree (λ, κ) is a fuzzy filter of X with degree (λ, κ).

Example 4.13. Let $X := \{1, a, b, c, d, 0\}$ be a BE-algebra ([6]) with the following table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Define a fuzzy subset $\mu : X \to [0, 1]$ by

$$\mu = \begin{pmatrix} 1 & a & b & c & d & 0 \\ 0.6 & 0.7 & 0.3 & 0.3 & 0.3 & 0.3 \end{pmatrix}$$

Then μ is a fuzzy $n(\geq 2)$-fold mighty filter of X with degree $(\frac{5}{7}, \frac{4}{7})$, but it is not a fuzzy mighty filter of X with degree $(\frac{5}{7}, \frac{4}{7})$, since $\mu(((b * c) * c) * b) = \mu(b) = 0.3 \not\geq 0.399 = \frac{4}{7} \times 0.7 = \mu(a) = \min\{\mu(a * (c * b)), \mu(a)\}$.

Proposition 4.14. *Let μ be a fuzzy n-fold mighty filter of a BE-algebra X with degree (λ, κ). Then the following holds:*

$$(\forall x, y \in X)(\mu(((x^n * y) * y) * x) \geq \kappa \lambda \mu(y * x)).$$

Proof. Assume that μ is a fuzzy n-fold mighty filter of a BE-algebra X with degree (λ, κ) and let $x, y \in X$. Let $z := 1$ in (e4). Then we have

$$\begin{aligned} \mu(((x^n * y) * y) * x) &\geq \kappa \min\{\mu(1 * (y * x)), \mu(1)\} \\ &\geq \kappa \min\{\mu(y * x), \lambda \mu(y * x)\} \\ &= \kappa \lambda \mu(y * x). \end{aligned}$$

This completes the proof. □

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Proposition 4.15. *Let μ be a fuzzy filter of a BE-algebra X with degree (λ, κ) satisfying*

$$(\forall x, y \in X)(\mu(((x^n * y) * y) * x) \geq \mu(y * x)).$$

Then μ is a fuzzy n-fold mighty filter of X with degree (λ, κ).

Proof. Let $x, y, z \in X$. Using (e2), we have

$$\mu(((x^n * y) * y) * x) \geq \mu(y * x)$$
$$\geq \kappa \min\{\mu(z * (y * x)), \mu(z)\}.$$

Thus μ is a fuzzy n-fold mighty filter of a BE-algebra X with degree (λ, κ). $\square$

Proposition 4.16. *If μ is a fuzzy weak n-fold filter of X, then the following holds:*

(4.1) $(\forall x, y \in X)(\mu((x * y) * y) \geq \kappa \lambda \mu((y^n * x) * x)).$

Proof. Let $z := 1$ in (e5). Then $\mu((x * y) * y) \geq \kappa \min\{\mu(1 * ((y^n * x) * x)), \mu(1)\} \geq \kappa \min\{\mu((y^n * x) * x), \lambda \mu((y^n * x) * x)\} = \kappa \lambda \mu((y^n * x) * x)$. This completes the proof. $\square$

Proposition 4.17. *Let μ be a fuzzy filter of a BE-algebra X with degree (λ, κ) satisfying*

$$(\forall x, y \in X)(\mu((y^n * x) * x) \leq \mu((x * y) * y)).$$

Then μ is a fuzzy weak n-fold mighty filter of X with degree (λ, κ).

Proof. Let $x, y, z \in X$. Using (e2), we have

$$\mu((x * y) * y) \geq \mu((y^n * x) * x)$$
$$\geq \kappa \min\{\mu(z * ((y^n * x) * x)), \mu(z)\}.$$

Thus μ is a fuzzy weak n-fold mighty filter of a BE-algebra X with degree (λ, κ). $\square$

A BE-algebra X is said to be *n-fold mighty* if $((x^n * y) * y) * x = y * x$ for all $x, y \in X$.

Lemma 4.18. *Let X be an n-fold mighty BE-algebra. If μ is a fuzzy filter of X with degree (λ, κ), then the following holds:*

(4.2) $(\forall x, y \in X)(\mu((y * x) * x) \geq \lambda \kappa \mu((x^n * y) * y)).$

Proof. Since X is an n-fold mighty BE-algebra, $((x^n * y) * y) * ((y * x) * x) = (y * x) * (((x^n * y) * y) * x) = (y * x) * (y * x) = 1$, for all $x, y \in X$. Hence $(x^n * y) * y \leq ((y * x) * x)$. Using Proposition 2.8(ii), we have $\mu((y * x) * x) \geq \lambda \kappa \mu((x^n * y) * y))$. This completes the proof. $\square$

Proposition 4.19. *Let X be an n-fold mighty transitive BE-algebra. If μ is a fuzzy filter of X with degree (λ, κ), then it is a fuzzy weak n-fold mighty filter of X with degree $(\lambda, \lambda \kappa^2)$.*

Proof. Let μ be a fuzzy filter of X with degree (λ, κ). Using Lemma 4.18 and (e2), we have $\mu((y * x) * x) \geq \lambda \kappa \mu((x^n * y) * y)) \geq \lambda \kappa^2 \min\{\mu(z * ((x^n * y) * y), \mu(z)\}$. Hence μ is a fuzzy weak n-fold mighty filter of X with degree $(\lambda, \lambda \kappa^2)$.

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Proposition 4.20. *Let μ be a fuzzy filter with degree (λ, κ) of an n-fold mighty BE-algebra X. Then μ is a fuzzy n-fold mighty filter of X with degree (λ, κ).*

Proof. Let μ be a fuzzy filter with degree (λ, κ) of an n-fold mighty BE-algebra X. Since X is an n-fold mighty BE-algebra, we have $\mu(((x^n * y) * y) * x) = \mu(y * x) \geq \mu(y * x)$. By Proposition 4.15, μ is a fuzzy n-fold mighty filter of X with degree (λ, κ). $\qquad\square$

Denote by $\mathcal{F}_M(X)$ the set of all mighty filters of a BE-algebra X. Note that a fuzzy subset μ of a BE-algebra X is a fuzzy mighty filter of X if and only if

$$(\forall t \in [0,1])(U(\mu;t) := \{x \in X | \mu(x) \geq t\} \in \mathcal{F}_M(X) \cup \{\emptyset\}).$$

But we know that for any fuzzy subset μ of a BE-algebra X there exist $\lambda, \kappa \in (0,1)$ and $t \in [0,1]$ such that

(1) μ is a fuzzy mighty filter of X with degree (λ, κ),
(2) $U(\mu;t) \notin \mathcal{F}_M(X) \cup \{\emptyset\}$.

Example 4.21. Consider a BE-algebra $X = \{1, a, b, c, d, 0\}$ which is given Example 3.2. Define a fuzzy subset $\mu : X \to [0,1]$ by

$$\mu = \begin{pmatrix} 1 & a & b & c & d & 0 \\ 0.8 & 0.3 & 0.9 & 0.3 & 0.3 & 0.3 \end{pmatrix}$$

If $t \in (0.8, 1]$, then $U(\mu;t) = \{1, b\}$ is not a mighty filter of X, since $b * (a * c) = b \in U(\mu;t)$ and $((c * a) * a) * c = c \notin U(\mu;t)$. But μ is a fuzzy mighty filter of X with degree $(0.6, 0.3)$.

Theorem 4.22. *Let μ be a fuzzy subset of a BE-algebra X. For any $t \in [0,1]$ with $t \leq \max\{\lambda, \kappa\}$, if $U(\mu;t)$ is an enlarged mighty filter of X related to $U(\mu; \frac{t}{\max\{\lambda, \kappa\}})$, then μ is a fuzzy mighty filter of X with degree (λ, κ).*

Proof. Assume that $\mu(1) < t \leq \lambda\mu(x)$ for some $x \in X$ and $t \in (0, \lambda]$. Then $\mu(x) \geq \frac{t}{\lambda} \geq \frac{t}{\max\{\lambda, \kappa\}}$ and so $x \in U(\mu; \frac{t}{\max\{\lambda, \kappa\}})$, i.e., $U(\mu; \frac{t}{\max\{\lambda, \kappa\}}) \neq \emptyset$. Since $U(\mu;t)$ is an enlarged filter of X related to $U(\mu; \frac{t}{\max\{\lambda, \kappa\}})$, we have $1 \in U(\mu;t)$, i.e., $\mu(1) \geq t$. This is a contradiction, and thus $\mu(1) \geq \lambda\mu(x)$ for all $x \in X$.

Now suppose that there exist $a, b, c \in X$ such that $\mu(((a * b) * b) * a) < \kappa\min\{\mu(c * (b * a)), \mu(c)\}$. If we take $t := \kappa\min\{\mu(c * (b * a)), \mu(c)\}$, then $t \in (0, \kappa] \subseteq (0, \max\{\lambda, \kappa\}]$. Hence $c * (b * a) \in U(\mu; \frac{t}{\kappa}) \subseteq U(\mu; \frac{t}{\max\{\lambda, \kappa\}})$ and $c \in U(\mu; \frac{t}{\kappa}) \subseteq U(\mu; \frac{t}{\max\{\lambda, \kappa\}})$. It follows from an enlarged mighty filter that $((a * b) * b) * a \in U(\mu;t)$ so that $\mu(((a * b) * b) * a) \geq t$, which is impossible. Therefore

$$\mu(((x * y) * y) * x) \geq \kappa\min\{\mu(z * (y * x)), \mu(z)\}$$

for all $x, y, z \in X$. Thus μ is a fuzzy mighty filter of X with degree (λ, κ). $\qquad\square$

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Corollary 4.23. *Let μ be a fuzzy subset of a BE-algebra X. For any $t \in [0,1]$ with $t \leq \frac{k}{n}$, if $U(\mu;t)$ is an enlarged mighty filter of X related to $U(\mu;\frac{n}{k}t)$, then μ is a fuzzy mighty filter of X with degree $(\frac{k}{n}, \frac{k}{n})$.*

Theorem 4.24. *Let $t \in [0,1]$ be such that $U(\mu;t)(\neq \emptyset)$ is not necessary a mighty filter of a BE-algebra X. If μ is a fuzzy mighty filter of X with degree (λ, κ), then $U(\mu; t\min\{\lambda, \kappa\})$ is an enlarged mighty filter of X related to $U(\mu;t)$.*

Proof. Since $t\min\{\lambda, \kappa\} \leq t$, we have $U(\mu;t) \subseteq U(\mu; t\min\{\lambda, \kappa\})$. Since $U(\mu;t) \neq \emptyset$, there exists $x \in U(\mu;t)$ and so $\mu(x) \geq t$. By (e1), we obtain $\mu(1) \geq \lambda\mu(x) \geq \lambda t \geq t\min\{\lambda, \kappa\}$. Therefore $1 \in U(\mu; t\min\{\lambda, \kappa\})$.

Let $x, y, z \in X$ be such that $x * (y * z) \in U(\mu;t)$ and $x \in U(\mu;t)$. Then $\mu(x * (y * z)) \geq t$ and $\mu(x) \geq t$. It follows from (e3) that

$$\mu(((z * y) * y) * z) \geq \kappa\min\{\mu(x * (y * z)), \mu(x)\}$$
$$\geq \kappa t \geq t\min\{\lambda, \kappa\}.$$

so that $((z * y) * y) * z \in U(\mu; t\min\{\lambda, \kappa\})$. Thus $U(\mu; t\min\{\lambda, \kappa\})$ is an enlarged mighty filter of X related to $U(\mu;t)$. $\square$

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SYMMETRIC IDENTITIES FOR (h,q)-EXTENSIONS OF THE GENERALIZED HIGHER ORDER MODIFIED q-EULER POLYNOMIALS

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ABSTRACT. In this paper, we introduce the (h,q)-extensions of the generalized modified q-Euler polynomials. The main objective of this paper is to consider symmetric identities of the (h,q)-extensions of the generalized modified q-Euler polynomials attached to χ which are derived from the p-adic fermionic integral on $\mathbb{Z}_p$.

1. INTRODUCTION

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$. Let q be an indeterminate in $\mathbb{C}_p$ such that $|q-1|_p < p^{-\frac{1}{p-1}}$. The q-analogue of number x is defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$.

Let $f(x)$ be a continuous function on $\mathbb{Z}_p$. Then the p-adic fermionic q-integral on $\mathbb{Z}_p$ is defined by

$$
\begin{aligned}
\int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_{-q}\left(x+p^N\mathbb{Z}_p\right) \\
&= \lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \\
&= \frac{[2]_q}{2} \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x q^x, \quad (\text{see } [1, 3, 4, 6, 7, 9, 11, 12, 14]).
\end{aligned}
$$

(1.1)

Thus, by (1.1), we get

$$
q\int_{\mathbb{Z}_p} f(x+1)d\mu_{-q}(x) + \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = [2]_q f(0),
$$

(1.2)

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and

$$q^n \int_{\mathbb{Z}_p} f(x+n)d\mu_{-q}(x) + (-1)^{n-1} \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = [2]_q \sum_{l=0}^{n-1} f(l)q^l(-1)^{n-1-l}, \quad (1.3)$$

where $n \in \mathbb{N}$ (see [5, 10-13]).

The Euler polynomials are defined by the generating function to be

$$\frac{2}{e^t + 1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad (\text{see } [2, 8]), \quad (1.4)$$

when $x = 0$, $E_n = E_n(0), (n \geq 0)$, are called Euler numbers.

It is known that the *q-Euler polynomials* are given by the generating function as follows:

$$\int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_{-q}(y) = \frac{[2]_q}{qe^t + 1}e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}, \quad (\text{see } [1, 3 - 7, 9 - 14]), \quad (1.5)$$

when $x = 0$, $E_{n,q} = E_{n,q}(0)$ are called *q-Euler numbers*.

It is known that

$$\int_X f(x)d\mu_{-1}(x) = \int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_{-1}(x). \quad (1.6)$$

Let χ be a primitive Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. Then, by (1.3), we get

$$\int_X \chi(y)e^{(x+y)t}d\mu_{-1}(y) = \left(\frac{2}{e^{dt} + 1}\sum_{a=0}^{d-1}\chi(a)(-1)^a e^{at}\right)e^{xt}$$
$$= \sum_{n=0}^{\infty} E_{n,\chi}(x)\frac{t^n}{n!}, \quad (\text{see } [4, 11]) \quad (1.7)$$

where $E_{n,\chi}(x)$ are called the generalized Euler polynomials attached to χ and $x = 0$, $E_{n,\chi} = E_{n,\chi}(0)$ is called the n-th generalized Euler number attached to χ.

From (1.7), we can derive the generalized higher order Euler polynomials attached to χ as follows:

$$\int_X \cdots \int_X \left(\prod_{l=1}^{r}\chi(y_l)\right)e^{(y_1+y_2+\cdots+y_r+x)t}d\mu_{-1}(y_1)d\mu_{-1}(y_2)\cdots d\mu_{-1}(y_r)$$
$$= \left(\frac{2}{e^{dt} + 1}\sum_{a=0}^{d-1}\chi(a)(-1)^a e^{at}\right)^r e^{xt} \quad (1.8)$$
$$= \sum_{n=0}^{\infty} E_{n,\chi}^{(r)}(x)\frac{t^n}{n!}, \quad (\text{see } [9, 11]).$$

From (1.8), we consider the (h,q)-extensions of higher order modified Euler polynomials attached to χ as follows:

$$\int_X \cdots \int_X q^{\sum_{l=1}^r (h-l)y_l} (\prod_{l=1}^r \chi(y_l)) e^{[y_1+y_2+\cdots+y_r+x]_q t} d\mu_{-1}(y_1) d\mu_{-1}(y_2) \cdots d\mu_{-1}(y_r)$$

$$= \sum_{n=0}^\infty E_{n,\chi,q}^{(h,r)}(x) \frac{t^n}{n!}, \quad (\text{see } [9, 11]),$$

$$(1.9)$$

where $r \in \mathbb{N}, h \in \mathbb{Z}$.

Recently, T. Kim and D.Kim studied (h,q)-extensions of the generalized higher order Euler polynomials (see [11]). So, we introduce (h,q)-extensions of the generalized higher order modified q-Euler polynomials. The purpose of this paper is to investigate symmetric identities of (h,q)-extensions of the generalized higher order modified q-Euler polynomials.

2. Symmetric identities for (h,q)-extensions of the generalized higher order modified q-Euler polynomials

We introduce the (h,q)-extensions of the generalized higher order modified q-Euler polynomials attached to χ as follows:

$$\int_X \cdots \int_X q^{\sum_{l=1}^r (h-l)y_l} (\prod_{l=1}^r \chi(y_l)) e^{[y_1+y_2+\cdots+y_r+x]_q t} d\mu_{-q}(y_1) d\mu_{-q}(y_2) \cdots d\mu_{-q}(y_r)$$

$$= \sum_{n=0}^\infty \mathcal{E}_{n,\chi,q}^{(h,r)}(x) \frac{t^n}{n!}.$$

$$(2.1)$$

Now, we consider the symmetric identities of (h,q)-extensions of the generalized higher order modified q-Euler polynomials.

Let $w_1, w_2 \in \mathbb{N}$ such that $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. Then, we consider the following identity.

$$\int_X \cdots \int_X q^{\sum_{l=1}^r (h-l)y_l} (\prod_{l=1}^r \chi(y_l))$$

$$\times e^{[w_1 w_2 x + w_2 \sum_{l=1}^r j_l + w_1 \sum_{l=1}^r y_l]_q t} d\mu_{-q}(y_1) d\mu_{-q}(y_2) \cdots d\mu_{-q}(y_r)$$

$$= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{i_1,i_2,\cdots,i_r=0}^{dw_2-1} \sum_{y_1,y_2,\cdots,y_r=0}^{p^N-1} q^{w_1 \sum_{l=1}^r (h-l)y_l} (\prod_{l=1}^r \chi(y_l))$$

$$\times e^{[w_1 w_2 x + w_2 \sum_{l=1}^r j_l + w_1 \sum_{l=1}^r (i_l + dw_2 y_l)]_q t} (-q)^{\sum_{l=1}^r (i_l + dw_2 y_l)}.$$

$$(2.2)$$

From (1.9), we have

$$\sum_{j_1,j_2,\cdots,j_r=0}^{dw_1-1} (-q)^{\sum_{l=1}^{r}(i_l+dw_2 y_l)} q^{w_2 \sum_{l=1}^{r}(h-l)j_l} (\prod_{l=1}^{r}\chi(j_l)) \int_X \cdots \int_X q^{w_1 \sum_{l=1}^{r}(h-l)y_l}$$

$$\times (\prod_{l=1}^{r}\chi(y_l)) e^{[w_1 w_2 x+w_2 \sum_{l=1}^{r}j_l+w_1 \sum_{l=1}^{r}y_l]_q t} d\mu_{-q}(y_1)d\mu_{-q}(y_2)\cdots d\mu_{-q}(y_r)$$

$$=\lim_{N\to\infty}\sum_{j_1,j_2,\cdots,j_r=0}^{dw_1-1}\sum_{i_1,i_2,\cdots,i_r=0}^{dw_2-1}\sum_{y_1,y_2,\cdots,y_r=0}^{p^N-1}(-q)^{\sum_{l=1}^{r}(i_l+j_l+y_l)}q^{\sum_{l=1}^{r}(h-l)(w_2 j_l+w_1 i_l+w_1 w_2 y_l)}$$

$$\times e^{[w_1 w_2(x+d\sum_{l=1}^{r}y_l)+\sum_{l=1}^{r}(w_2 j_l+w_1 i_l)]_q t}(\prod_{l=1}^{r}\chi(i_l j_l)).$$

(2.3)

By the similar method as (2.3), we get

$$\sum_{j_1,j_2,\cdots,j_r=0}^{dw_2-1} (-q)^{\sum_{l=1}^{r}(i_l+dw_2 y_l)} q^{w_1 \sum_{l=1}^{r}(h-l)j_l} (\prod_{l=1}^{r}\chi(j_l)) \int_X \cdots \int_X q^{w_2 \sum_{l=1}^{r}(h-l)y_l}$$

$$\times (\prod_{l=1}^{r}\chi(y_l)) e^{[w_1 w_2 x+w_1 \sum_{l=1}^{r}j_l+w_2 \sum_{l=1}^{r}y_l]_q t} d\mu_{-q}(y_1)d\mu_{-q}(y_2)\cdots d\mu_{-q}(y_r)$$

$$=\lim_{N\to\infty}\sum_{j_1,j_2,\cdots,j_r=0}^{dw_2-1}\sum_{i_1,i_2,\cdots,i_r=0}^{dw_1-1}\sum_{y_1,y_2,\cdots,y_r=0}^{p^N-1}(-q)^{\sum_{l=1}^{r}(i_l+j_l+y_l)}q^{\sum_{l=1}^{r}(h-l)(w_1 j_l+w_2 i_l+w_1 w_2 y_l)}$$

$$\times e^{[w_1 w_2(x+d\sum_{l=1}^{r}y_l)+\sum_{l=1}^{r}(w_1 j_l+w_2 i_l)]_q t}(\prod_{l=1}^{r}\chi(i_l j_l)).$$

(2.4)

Therefore, by (2.3) and (2.4), we obtain the following theorem.

Theorem 2.1. *Let* $w_1, w_2 \in \mathbb{N}$ *such that* $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. *Then, we have*

$$\sum_{j_1,j_2,\cdots,j_r=0}^{dw_1-1} (-q)^{\sum_{l=1}^{r}(i_l+dw_2 y_l)} q^{w_2 \sum_{l=1}^{r}(h-l)j_l} (\prod_{l=1}^{r}\chi(j_l)) \int_X \cdots \int_X q^{w_1 \sum_{l=1}^{r}(h-l)y_l}$$

$$\times (\prod_{l=1}^{r}\chi(y_l)) e^{[w_1 w_2 x+w_2 \sum_{l=1}^{r}j_l+w_1 \sum_{l=1}^{r}y_l]_q t} d\mu_{-q}(y_1)d\mu_{-q}(y_2)\cdots d\mu_{-q}(y_r)$$

$$=\sum_{j_1,j_2,\cdots,j_r=0}^{dw_2-1} (-q)^{\sum_{l=1}^{r}(i_l+dw_2 y_l)} q^{w_1 \sum_{l=1}^{r}(h-l)j_l} (\prod_{l=1}^{r}\chi(j_l)) \int_X \cdots \int_X q^{w_2 \sum_{l=1}^{r}(h-l)y_l}$$

$$\times (\prod_{l=1}^{r}\chi(y_l)) e^{[w_1 w_2 x+w_1 \sum_{l=1}^{r}j_l+w_2 \sum_{l=1}^{r}y_l]_q t} d\mu_{-q}(y_1)d\mu_{-q}(y_2)\cdots d\mu_{-q}(y_r).$$

(2.5)

Now, we consider that

$$\left[w_1 w_2 x + w_2 \sum_{l=1}^{r}j_l + w_1 \sum_{l=1}^{r}y_l\right]_q = [w_1]_q \left[w_2 x + \frac{w_2}{w_1}\sum_{l=1}^{r}j_l + \sum_{l=1}^{r}y_l\right]_q^{w_1},$$
(2.6)

and

$$\left[w_1 w_2 x + w_1 \sum_{l=1}^{r} j_l + w_2 \sum_{l=1}^{r} y_l\right]_q = [w_2]_q \left[w_1 x + \frac{w_1}{w_2} \sum_{l=1}^{r} j_l + \sum_{l=1}^{r} y_l\right]_q^{w_2}, \quad (2.7)$$

Therefore, by Theorem 2.1 , (2.6) and (2.7), we obtain the following theorem.

Theorem 2.2. *Let $w_1, w_2 \in \mathbb{N}$ such that $w_1 \equiv 1 \ (mod \ 2)$, $w_2 \equiv 1 \ (mod \ 2)$ and $n \geq 0$. Then, we have*

$$[w_1]_q^n \sum_{j_1,j_2,\cdots,j_r=0}^{dw_1-1} (-q)^{\sum_{l=1}^{r} j_l} q^{w_2 \sum_{l=1}^{r}(h-l)j_l} (\prod_{l=1}^{r} \chi(j_l)) \int_X \cdots \int_X q^{w_1 \sum_{l=1}^{r}(h-l)y_l}$$

$$\times (\prod_{l=1}^{r} \chi(y_l)) \left[w_2 x + \frac{w_2}{w_1}(j_1 + j_2 + \cdots + j_r) \sum_{l=1}^{r} y_l\right]_{q^{w_1}}^{n} d\mu_{-q}(y_1)d\mu_{-q}(y_2)\cdots d\mu_{-q}(y_r)$$

$$= [w_2]_q^n \sum_{j_1,j_2,\cdots,j_r=0}^{dw_2-1} (-q)^{\sum_{l=1}^{r} j_l} q^{w_1 \sum_{l=1}^{r}(h-l)j_l} (\prod_{l=1}^{r} \chi(j_l)) \int_X \cdots \int_X q^{w_2 \sum_{l=1}^{r}(h-l)y_l}$$

$$\times (\prod_{l=1}^{r} \chi(y_l)) \left[w_1 x + \frac{w_1}{w_2}(j_1 + j_2 + \cdots + j_r) \sum_{l=1}^{r} y_l\right]_{q^{w_2}}^{n} d\mu_{-q}(y_1)d\mu_{-q}(y_2)\cdots d\mu_{-q}(y_r).$$

$$(2.8)$$

Thus, by (1.7) and Theorem 2.2, we obtain the following theorem.

Theorem 2.3. *Let $w_1, w_2 \in \mathbb{N}$ such that $w_1 \equiv 1 \ (mod \ 2)$, $w_2 \equiv 1 \ (mod \ 2)$ and $n \geq 0$. Then, we have*

$$[w_1]_q^n \sum_{j_1,j_2,\cdots,j_r=0}^{dw_1-1} (-q)^{\sum_{l=1}^{r} j_l} q^{w_2 \sum_{l=1}^{r}(h-l)j_l} (\prod_{l=1}^{r} \chi(j_l)) E_{n,\chi,q^{w_1}}^{(h,r)} (w_2 x + \frac{w_2}{w_1} \sum_{l=1}^{r} j_l)$$

$$= [w_2]_q^n \sum_{j_1,j_2,\cdots,j_r=0}^{dw_2-1} (-q)^{\sum_{l=1}^{r} j_l} q^{w_2 \sum_{l=1}^{r}(h-l)j_l} (\prod_{l=1}^{r} \chi(j_l)) E_{n,\chi,q^{w_2}}^{(h,r)} (w_1 x + \frac{w_1}{w_2} \sum_{l=1}^{r} j_l).$$

$$(2.9)$$

Now, we consider that

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$$\int_X \cdots \int_X q^{w_1 \sum_{l=1}^{r}(h-l)y_l}(\prod_{l=1}^{r}\chi(y_l))$$

$$\times \left[w_2 x + \frac{w_2}{w_1}\sum_{l=1}^{r} j_l + \sum_{l=1}^{r} y_l \right]_{q_1^w}^{n} d\mu_{-q}(y_1)d\mu_{-q}(y_2)\cdots d\mu_{-q}(y_r)$$

$$=\sum_{i=0}^{n}\binom{n}{i}\left(\frac{[w_2]_q}{[w_1]_q}\right)^{i}[\sum_{l=1}^{r}j_l]_{q^{w_2}}^{i}q^{w_2(n-i)\sum_{l=1}^{r}j_l}\int_X\cdots\int_X\left[w_2x+\sum_{l=1}^{r}y_l\right]_{q_1^w}^{n-i}$$

$$\times (q)^{w_1\sum_{l=1}^{r}(h-l)y_l}(\prod_{l=1}^{r}\chi(y_l))d\mu_{-q}(y_1)d\mu_{-q}(y_2)\cdots d\mu_{-q}(y_r) \qquad (2.10)$$

$$=\sum_{i=0}^{n}\binom{n}{i}\left(\frac{[w_2]_q}{[w_1]_q}\right)^{i}\left[\sum_{l=1}^{r}j_l\right]_{q^{w_2}}^{i}q^{w_2(n-i)\sum_{l=1}^{r}j_l}E_{n-i,\chi,q^{w_1}}^{(h,r)}(w_2x)$$

$$=\sum_{i=0}^{n}\binom{n}{i}\left(\frac{[w_2]_q}{[w_1]_q}\right)^{n-i}\left[\sum_{l=1}^{r}j_l\right]_{q^{w_2}}^{n-i}q^{w_2 i\sum_{l=1}^{r}j_l}E_{i,\chi,q^{w_1}}^{(h,r)}(w_2x).$$

From (2.6), we have

$$[w_1]_q^n\sum_{j_1,j_2,\cdots,j_r=0}^{dw_1-1}(-q)^{\sum_{l=1}^{r}j_l}q^{w_2\sum_{l=1}^{r}(h-l)j_l}(\prod_{l=1}^{r}\chi(j_l))\int_X\cdots\int_X q^{w_1\sum_{l=1}^{r}(h-l)y_l}$$

$$\times(\prod_{l=1}^{r}\chi(y_l))\left[w_2x+\frac{w_2}{w_1}(j_1+j_2+\cdots+j_r)+\sum_{l=1}^{r}y_l\right]_{q^{w_1}}^{n}d\mu_{-q}(y_1)d\mu_{-q}(y_2)\cdots d\mu_{-q}(y_r)$$

$$=\sum_{j_1,j_2,\cdots,j_r=0}^{dw_1-1}(-q)^{\sum_{l=1}^{r}j_l}q^{w_2\sum_{l=1}^{r}(h-l)j_l}(\prod_{l=1}^{r}\chi(j_l))$$

$$\times\sum_{i=0}^{n}\binom{n}{i}[w_1]_q^{i}[w_2]_q^{n-i}\left[\sum_{l=1}^{r}j_l\right]_{q^{w_2}}^{n-i}q^{w_2\sum_{l=1}^{r}j_l}E_{i,\chi,q^{w_1}}^{(h,r)}(w_2x)$$

$$=\binom{n}{i}[w_1]_q^{i}[w_2]_q^{n-i}E_{i,\chi,q^{w_1}}^{(h,r)}(w_2x)\sum_{j_1,j_2,\cdots,j_r=0}^{dw_1-1}(-q)^{\sum_{l=1}^{r}j_l}q^{w_2\sum_{l=1}^{r}(h-l)j_l}(\prod_{l=1}^{r}\chi(j_l))$$

$$\times\left[\sum_{l=1}^{r}j_l\right]_{q^{w_2}}^{n-i}q^{w_2\sum_{l=1}^{r}(h-l-i)j_l}$$

$$=\binom{n}{i}[w_1]_q^{i}[w_2]_q^{n-i}E_{i,\chi,q^{w_1}}^{(h,r)}(w_2x)\sum_{j_1,j_2,\cdots,j_r=0}^{w_1-1}(-q)^{\sum_{l=1}^{r}j_l}q^{\sum_{l=1}^{r}(h-l)j_l}(\prod_{l=1}^{r}\chi(j_l))$$

$$\times\left[\sum_{l=1}^{r}j_l\right]_{q^{w_2}}^{n-i}.$$

$$(2.11)$$

By the similar method as (2.6), we get

$$
[w_2]_q^n \sum_{j_1,j_2,\cdots,j_r=0}^{dw_2-1} (-q)^{\sum_{l=1}^r j_l} q^{w_1 \sum_{l=1}^r (h-l)j_l} (\prod_{l=1}^r \chi(j_l)) \int_X \cdots \int_X q^{w_2 \sum_{l=1}^r (h-l)y_l}
$$

$$
\times (\prod_{l=1}^r \chi(y_l)) \left[w_1 x + \frac{w_1}{w_2}(j_1 + j_2 + \cdots + j_r) + \sum_{l=1}^r y_l \right]_{q^{w_2}}^n d\mu_{-q}(y_1) d\mu_{-q}(y_2) \cdots d\mu_{-q}(y_r)
$$

$$
= \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} E_{i,\chi,q^{w_2}}^{(h,r)}(w_1 x) \sum_{j_1,j_2,\cdots,j_r=0}^{w_2-1} (-q)^{\sum_{l=1}^r j_l} q^{\sum_{l=1}^r (h-l)j_l} (\prod_{l=1}^r \chi(j_l))
$$

$$
\times \left[\sum_{l=1}^r j_l \right]_{q^{w_1}}^{n-i}.
$$

$$(2.12)$$

Therefore, by (2.6) and (2.7), we obtain the following theorem.

Theorem 2.4. *Let* $w_1, w_2 \in \mathbb{N}$ *such that* $w_1 \equiv 1 \ (mod \ 2)$ *,* $w_2 \equiv 1 \ (mod \ 2)$ *and* $n \geq 0$*. Then, we have*

$$
\binom{n}{i} [w_1]_q^i [w_2]_q^{n-i} E_{i,\chi,q^{w_1}}^{(h,r)}(w_2 x) \sum_{j_1,j_2,\cdots,j_r=0}^{w_1-1} (-q)^{\sum_{l=1}^r j_l} q^{\sum_{l=1}^r (h-l)j_l}
$$

$$
\times (\prod_{l=1}^r \chi(j_l)) \left[\sum_{l=1}^r j_l \right]_{q^{w_2}}^{n-i}
$$

$$
= \binom{n}{i} [w_2]_q^i [w_1]_q^{n-i} E_{i,\chi,q^{w_2}}^{(h,r)}(w_1 x) \sum_{j_1,j_2,\cdots,j_r=0}^{w_2-1} (-q)^{\sum_{l=1}^r j_l} q^{\sum_{l=1}^r (h-l)j_l}
$$

$$
\times (\prod_{l=1}^r \chi(j_l)) \left[\sum_{l=1}^r j_l \right]_{q^{w_1}}^{n-i}.
$$

$$(2.13)$$

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Some properties of (p, q)-tangent polynomials

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Abstract We introduce (p, q)-tangent polynomials and their basic properties including (p, q)-derivative and (p, q)-integral. By using Mathematica, we find roots of (p, q)-tangent polynomials. We also investigate relations of zeros between (p, q)-tangent polynomials and classical tangent polynomials.

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1. Introduction

In [1, 2, 3], R. Chakrabarti and R. Jagannathan, G. Brodimas, et al. and M. Arik, et al. introduced the (p, q)-number in order to unify various forms of q-oscillator algebras.

For any $n \in \mathbb{C}$, the (p, q)-number is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

It is clear that (p, q)-number contains symmetric property, and this number reduces to q-number when $p = 1$. In particular, we can see that $\lim_{q \to 1}[n]_{p,q} = n$ with $p = 1$(see [9]). By using the above numbers, many researchers have studied (p, q)-calculus(see [4, 5, 8]).

Definition 1.1. We define the (p, q)-derivative operator

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p - q)x}, \quad x \neq 0,$$

and $D_{p,q}f(0) = f'(0)$. The following properties of (p, q)-derivative operator are immediate.

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Theorem 1.2. For the operator $D_{p,q}$ the following hold

(i) Derivative of a product
$$D_{p,q}(f(x)g(x)) = f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x)$$
$$= g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x).$$

(ii) Derivative of a ratio
$$D_{p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{g(qx)D_{p,q}f(x) - f(qx)D_{p,q}g(x)}{g(px)g(qx)}$$
$$= \frac{g(px)D_{p,q}f(x) - f(px)D_{p,q}g(x)}{g(px)g(qx)}.$$

In [4], R. B. Corcino found (p,q)-extension of binomials coefficients and used it to establish various properties related to horizontal function, the triangular function, and the vertical function.

Definition 1.3. The (p,q)-analogue of $(x+a)^n$ is defined by

(i) $(x+a)^n_{p,q} = \begin{cases} 1 & \text{if } n = 0 \\ (x+a)(px+aq)\cdots(p^{n-2}x+aq^{n-2})(p^{n-1}x+aq^{n-1}) & \text{if } n \neq 0 \end{cases}$,

(ii) $(x+a)^n_{p,q} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} q^{\binom{n-k}{2}} x^k a^{n-k}$,

where $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$ is (p,q)-Gauss-Binomial coefficients.

Definition 1.4. Let z be any complex numbers with $|z| < 1$. Two forms of (p,q)-exponential functions are defined by

$$e_{p,q}(z) = \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{z^n}{[n]_{p,q}!},$$

$$\mathcal{E}_{p,q}(z) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]_{p,q}!}.$$

These forms are connected by the following interesting relations

$$e_{p,q}(z)\mathcal{E}_{p,q}(-z) = 1, \qquad \mathcal{E}_{p,q}(z) = e_{p^{-1},q^{-1}}(z).$$

Bernoulli, Euler, and Genocchi polynomials have been studied extensively by many mathematicians(see [5-7, 10-13]). In 2013, C. S. Ryoo introduced tangent polynomials and he developed several properties of these polynomials(see [10]). The tangent numbers are closely related to Euler numbers.

Definition 1.5. Tangent numbers T_n and tangent polynomials $T_n(x)$ are defined by

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means of the generating functions

$$\sum_{n=0}^{\infty} T_n \frac{t^n}{n!} = \frac{2}{e^{2t}+1} = 2\sum_{m=0}^{\infty}(-1)^m e^{2mt},$$

$$\sum_{n=0}^{\infty} T_n \frac{t^n}{n!} = \frac{2}{e^{2t}+1} e^{tx} 2 \sum_{m=0}^{\infty}(-1)^m e^{(2m+x)t}.$$

Theorem 1.6. For any positive integer n, we have

$$T_n(x) = (-1)^n T_n(2-x).$$

Theorem 1.7. For any positive integer $m(= odd)$, we have

$$T_n(x) = m^n \sum_{i=0}^{m-1} (-1)^i T_n\left(\frac{2i+x}{m}\right), \quad n \in \mathbb{Z}_+.$$

Theorem 1.8. For $n \in \mathbb{Z}_+$, we have

$$T_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} T_k(x) y^{n-k}.$$

The main aim of this paper is to extend tangent numbers and polynomials, and study some of their properties. Our paper is organised as follows: In Section 2, we define (p,q)-tangent polynomials and find some properties of these polynomials. In Section 3, we consider (p,q)-tangent polynomials in two parameters and establish some relations between (p,q)-tangent polynomials and (p,q)-Euler or Bernoulli polynomials. In Section 4, we observe roots distributions of (p,q)-tangent polynomials and demonstrate interesting phenomenon.

2. Some properties of the (p,q)-tangent polynomials

In this section we define the (p,q)-tangent numbers and polynomials and establish some of their basic properties. We also define (p,q)-derivative and (p,q)-integral of (p,q)-tangent polynomials.

Definition 2.1. For $x, p, q \in \mathbb{C}$, we define (p,q)-tangent polynomials as

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q}}{e_{p,q}(2t)+1} e_{p,q}(tx), \quad |t| < \frac{\pi}{2}.$$

From Definition 2.1, it follows that

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(0) \frac{t^n}{[n]_{p,q}!} = \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q} \frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q}}{e_{p,q}(2t)+1},$$

where $\mathcal{T}_{n,p,q}$ is (p,q)-tangent number. If $p = 1$, $q \to 1$, then it reduces to the classical tangent polynomial(see [10]).

Theorem 2.2. Let $x, p, q \in \mathbb{C}$. Then, the following hold

$$\text{(i)} \quad \mathcal{T}_{n,p,q} + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} 2^{n-k} p^{\binom{n-k}{2}} \mathcal{T}_{k,p,q} = \begin{cases} [2]_{p,q} & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases},$$

$$\text{(ii)} \quad \mathcal{T}_{n,p,q}(x) + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} 2^{n-k} p^{\binom{n-k}{2}} \mathcal{T}_{k,p,q}(x) = [2]_{p,q} p^{\binom{n}{2}} x^n.$$

Proof. From the Definition 2.1, we have

$$[2]_{p,q} = (1 + e_{p,q}(2t)) \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q} \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\mathcal{T}_{n,p,q} + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} 2^{n-k} p^{\binom{n-k}{2}} \mathcal{T}_{k,p,q} \right) \frac{t^n}{n!}.$$

Now comparing the coefficients of t^n we find (i). For (ii) we use the relation

$$[2]_{p,q} e_{p,q}(tx) = (1 + e_{p,q}(2t)) \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!}$$

$$= \sum_{n=0}^{\infty} \left(\mathcal{T}_{n,p,q}(x) + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} 2^{n-k} p^{\binom{n-k}{2}} \mathcal{T}_{k,p,q}(x) \right) \frac{t^n}{n!},$$

and again compare the coefficients of t^n.

$\square$

Theorem 2.3. Let n be a non-negative integer. Then, the following holds

$$\mathcal{T}_{n,p,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} \mathcal{T}_{n-k,p,q} x^k.$$

Proof. From the definition of the (p,q)-exponential function, we have

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q}}{e_{p,q}(2t) + 1} e_{p,q}(tx) = \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q} \frac{t^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} p^{\binom{n}{2}} x^n \frac{t^n}{[n]_{p,q}!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} \mathcal{T}_{n-k,p,q}(x) x^k \right) \frac{t^n}{[n]_{p,q}!}.$$

The required relation now follows on comparing the coefficients of t^n on both sides.

$\square$

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Theorem 2.4. Let n be a non-negative integer. Then, the following holds

$$\mathcal{T}_{n,p,q} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (-1)^{n-k} q^{\binom{n-k}{2}} \mathcal{T}_{k,p,q}(x) x^{n-k}.$$

Proof. From the property of (p, q)-exponential function, it follows that

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q} \frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q}}{e_{p,q}(2t)+1} e_{p,q}(tx) e_{p^{-1},q^1}(-tx)$$

$$= \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} q^{\binom{n}{2}} (-1)^n x^n \frac{t^n}{[n]_{p,q}!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (-1)^{n-k} q^{\binom{n-k}{2}} \mathcal{T}_{k,p,q}(x) x^{n-k} \right) \frac{t^n}{[n]_{p,q}!}.$$

The required relation now follows immediately.

$\square$

In what follows, we consider (p, q)-derivative of $e_{p,q}(tx)$. Using the Mathematical Induction, we find

(i) $k = 1:$ $D_{p,q}^{(1)} e_{p,q}(tx) = \sum_{n=1}^{\infty} p^{\binom{n}{2}} x^{n-1} \frac{t^n}{[n-1]_{p,q}!}.$

(ii) $k = i:$ $D_{p,q}^{(i)} e_{p,q}(tx) = \sum_{n=i}^{\infty} p^{\binom{n}{2}} x^{n-i} \frac{t^n}{[n-i]_{p,q}!}.$

If (ii) is true, then it follows that

(iii) $k = i + 1:$ $D_{p,q}^{(i+1)} e_{p,q}(tx) = D_{p,q;x}^{(1)} \left(\sum_{n=i}^{\infty} p^{\binom{n}{2}} x^{n-i} \frac{t^n}{[n-i]_{p,q}!} \right)$

$$= \sum_{n=i+1}^{\infty} p^{\binom{n}{2}} x^{n-(i+1)} \frac{t^n}{[n-(i+1)]_{p,q}!}$$

$$= t^{i+1} p^{\binom{i+1}{2}} e_{p,q}(p^{i+1} tx).$$

We are now in the position to prove the following theorem.

Theorem 2.5. For $k \in \mathbb{N}$, the following holds

$$D_{p,q}^{(k)} \mathcal{T}_{n,p,q}(x) = \frac{[n]_{p,q}!}{[n-k]_{p,q}!} p^{\binom{k}{2}} \mathcal{T}_{n-k,p,q}(p^k x).$$

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Proof. Considering (p, q)-derivative of $e_{p,q}(tx)$, we find

$$
\begin{aligned}
D_{p,q}^{(i+1)} \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} &= \sum_{n=0}^{\infty} D_{p,q}^{(i+1)} \mathcal{T}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} \\
&= \frac{[2]_{p,q}}{e_{p,q}(2t)+1} D_{p,q}^{(i+1)} e_{p,q}(tx) \\
&= t^{i+1} p^{\binom{i+1}{2}} \frac{[2]_{p,q}}{e_{p,q}(2t)+1} e_{p,q}(p^{i+1}tx) \\
&= p^{\binom{i+1}{2}} \sum_{n=0}^{\infty} [n+(i+1)]_{p,q} \cdots [n+2]_{p,q}[n+1]_{p,q} \\
&\quad \times \mathcal{T}_{n,p,q}(p^{i+1}x) \frac{t^{n+i+1}}{[n+(i+1)]_{p,q}!} \\
&= p^{\binom{i+1}{2}} \sum_{n=0}^{\infty} \frac{[n]_{p,q}}{[n+(i+1)]_{p,q}!} \mathcal{T}_{n-(i+1),p,q}(p^{i+1}x) \frac{t^n}{[n]_{p,q}!},
\end{aligned}
$$

which immediately gives the required result.

$\square$

Theorem 2.6. Let a, b be any real numbers. Then, we have

$$
\int_a^b \mathcal{T}_{n,p,q}(x) d_{p,q}x = \sum_{k=0}^{n+1} \frac{p^k}{p^n [n+1]_{p,q}} \left(\mathcal{T}_{n+1,p,q}(b) - \mathcal{T}_{n+1,p,q}(a) \right).
$$

Proof. From Theorem 2.3, we find

$$
\begin{aligned}
\int_a^b \mathcal{T}_{n,p,q}(x) d_{p,q}x &= \int_a^b \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} \mathcal{T}_{k,p,q} x^{n-k} d_{p,q}x \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} \mathcal{T}_{k,p,q} \frac{1}{[n-k+1]_{p,q}} x^{n-k+1} \Big|_a^b \\
&= \sum_{k=0}^{n+1} \frac{p^k (\mathcal{T}_{n+1,p,q}(b) - \mathcal{T}_{n+1,p,q}(a))}{p^n [n+1]_{p,q}}.
\end{aligned}
$$

$\square$

3. Some properties of the (p, q)-tangent polynomial in two parameters

In this section, we shall study the (p, q)-tangent polynomials involving two parameters. We shall also find some important relations between these polynomials and other polynomials.

Definition 3.1. For $x, y \in \mathbb{C}$, we define (p, q)-tangent polynomial with two parameters as

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x,y) \frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q}}{e_{p,q}(2t)+1} e_{p,q}(tx) e_{p,q}(ty), \quad |t| < \frac{\pi}{2}.$$

From the Definition 3.1, it is clear that

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x,0) \frac{t^n}{[n]_{p,q}!} = \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q}}{e_{p,q}(2t)+1} e_{p,q}(tx),$$

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(0,0) \frac{t^n}{[n]_{p,q}!} = \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q} \frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q}}{e_{p,q}(2t)+1},$$

where $\mathcal{T}_{n,p,q}$ is (p,q)-tangent number. We also note that the original tangent number, $\mathcal{T}_n$,

$$\lim_{q \to 1} \sum_{n=0}^{\infty} \mathcal{T}_{n,1,q} \frac{t^n}{[n]_{1,q}!} = \sum_{n=0}^{\infty} T_n \frac{t^n}{n!} = \frac{2}{e^{2t}+1},$$

where $p = 1$ and $q \to 1$.

Theorem 3.2. Let x, y be any complex numbers. Then, the following hold

(i) $\quad \mathcal{T}_{n,p,q}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} \mathcal{T}_{n-k,p,q}(x) y^k,$

(ii) $\quad \mathcal{T}_{n,p,q}(x,y) = \sum_{l=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \mathcal{T}_{n-l,p,q} \sum_{k=0}^{l} \begin{bmatrix} l \\ k \end{bmatrix}_{p,q} p^{\frac{(l+k)^2-l}{2}} x^{l-k} y^k.$

Proof. From the Definition 3.1, we have

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x,y) \frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q}}{e_{p,q}(2t)+1} e_{p,q}(tx) e_{p,q}(ty)$$

$$= \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} p^{\binom{n}{2}} y^n \frac{t^n}{[n]_{p,q}!}.$$

Using Cauchy's product and the method of coefficient comparison in the above relation, we find (i). Next, we transform (p, q)-tangent polynomials in two parameters as

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x,y) \frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q}}{e_{p,q}(2t)+1} e_{p,q}(tx) e_{p,q}(ty)$$

$$= \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q} \frac{t^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} p^{\binom{n}{2}} x^n \frac{t^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} p^{\binom{n}{2}} y^n \frac{t^n}{[n]_{p,q}!}.$$

Now following same procedure as in (i), we obtain (ii).

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$\square$

Theorem 3.3. Setting $y = 2$ in (p, q)-tangent polynomials with two parameters, the following relation holds

$$[2]_{p,q} p^{\binom{n}{2}} x^n = \mathcal{T}_{n,p,q}(x, 2) + \mathcal{T}_{n,p,q}(x).$$

Proof. Using (p, q)-tangent polynomials and its polynomials with two parameters, we have

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x, 2) \frac{t^n}{[n]_{p,q}!} + \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q} e_{p,q}(2t)}{e_{p,q}(2t) + 1} e_{p,q}(tx) + \frac{[2]_{p,q}}{e_{p,q}(2t) + 1} e_{p,q}(tx)$$

$$= [2]_{p,q} e_{p,q}(tx)$$

Now from the definition of (p, q)-exponential function, the required relation follows.

$\square$

Theorem 3.3 is interesting as it leads to the relation

$$x^n = \frac{\mathcal{T}_{n,p,q}(x, 2) + \mathcal{T}_{n,p,q}(x)}{[2]_{p,q} p^{\binom{n}{2}}}.$$

Theorem 3.4. Let $\left| \frac{q}{p} \right| < 1$. Then, the following holds

$$\mathcal{T}_{n,p,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (-1)^k p^{\binom{k}{2}} \mathcal{T}_{k,1,\frac{p}{q}}(2) x^{n-k}.$$

Proof. To prove the relation, we note that

$$e_{1,\frac{p}{q}}(-2t) = \mathcal{E}_{p,q}(-2t),$$

where $\mathcal{E}_{p,q}(t) = e_{p^{-1},q^{-1}}(t)$. Using the above equation we can represent the (p, q)-tangent polynomials as

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q}}{e_{p,q}(2t) + 1} e_{p,q}(tx)$$

$$= \frac{[2]_{p,q}}{1 + \mathcal{E}_{p,q}(-2t)} \mathcal{E}_{p,q}(-2t) e_{p,q}(tx)$$

$$= \frac{[2]_{p,q}}{e_{1,\frac{p}{q}}(-2t) + 1} e_{1,\frac{p}{q}}(-2t) e_{p,q}(tx)$$

$$= \sum_{n=0}^{\infty} \mathcal{T}_{n,1,\frac{p}{q}}(2) \frac{(-t)^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} p^{\binom{n}{2}} x^n \frac{t^n}{[n]_{p,q}!}$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} (-1)^k p^{\binom{k}{2}} T_{k,1,\frac{p}{q}}(2) x^{n-k} \right\} \frac{t^n}{[n]_{p,q}!},$$

which leads to the required relation immediately.

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$\square$

Now we shall find relations between (p,q)-tangent polynomials and others polynomials. For this, first we introduce well known polynomials by using (p,q)-numbers.

Definition 3.5. We define (p,q)-Euler polynomials, $E_{n,p,q}(x)$, and (p,q)-Bernoulli polynomials, $B_{n,p,q}(x)$, as

$$\sum_{n=0}^{\infty} E_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q}}{e_{p,q}(t)+1}e_{p,q}(tx), \quad |t| < \pi,$$

$$\sum_{n=0}^{\infty} B_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!} = \frac{t}{e_{p,q}(t)-1}e_{p,q}(tx), \quad |t| < 2\pi.$$

Theorem 3.6. For $x,y \in \mathbb{C}$, the following relation holds

$$\mathcal{T}_{n,p,q}(x,y)$$
$$= \frac{1}{[2]_{p,q}}\sum_{l=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\left(\frac{\mathcal{T}_{n-l,p,q}(x)}{m^l} + \sum_{k=0}^{n-l}\begin{bmatrix}n-l\\k\end{bmatrix}_{p,q}\frac{p^{\binom{n-l-k}{2}}\mathcal{T}_{k,p,q}(x)}{m^{n-k}}\right)E_{l,p,q}(my).$$

Proof. Transforming (p,q)-tangent polynomials containing two parameters, we find

$$\frac{[2]_{p,q}}{e_{p,q}(2t)+1}e_{p,q}(tx)e_{p,q}(ty)$$
$$= \left(\frac{[2]_{p,q}}{e_{p,q}(\frac{t}{m})+1}e_{p,q}(ty)\right)\left(\frac{e_{p,q}(\frac{t}{m})+1}{[2]_{p,q}}\right)\left(\frac{[2]_{p,q}}{e_{p,q}(2t)+1}e_{p,q}(tx)\right).$$

Thus, for the relation between (p,q)-tangent polynomials of two parameters and (p,q)-Euler polynomials, we have

$$\sum_{n=0}^{\infty}\mathcal{T}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}$$
$$= \sum_{n=0}^{\infty}E_{n,p,q}(my)\frac{t^n}{m^n[n]_{p,q}!}\sum_{n=0}^{\infty}\mathcal{T}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}\left(\sum_{n=0}^{\infty}\frac{1}{[2]_{p,q}}p^{\binom{n}{2}}\frac{t^n}{m^n[n]_{p,q}!} + \frac{1}{[2]_{p,q}}\right)$$
$$= \frac{1}{[2]_{p,q}}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_{p,q}E_{l,p,q}(my)\sum_{k=0}^{n-l}\begin{bmatrix}n-l\\k\end{bmatrix}_{p,q}p^{\binom{n-l-k}{2}}\frac{\mathcal{T}_{k,p,q}(x)}{m^{n-k}}\frac{t^n}{[n]_{p,q}!}$$
$$+ \frac{1}{[2]_{p,q}}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_{p,q}E_{l,p,q}(my)\frac{\mathcal{T}_{n-l,p,q}(x)}{m^l}\frac{t^n}{[n]_{p,q}!},$$

which on comparing the coefficients immediately gives the required relation.

$\square$

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Corollary 3.7. From Theorem 3.6, the following hold

(i) $\quad \mathcal{T}_{n,q}(x,y) = \dfrac{1}{[2]_q} \sum\limits_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\dfrac{\mathcal{T}_{n-l,q}(x)}{m^l} + \sum\limits_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_q \dfrac{\mathcal{T}_{k,q}(x)}{m^{n-k}} \right) E_{l,q}(my).$

(ii) $\quad \mathcal{T}_{n}(x,y) = \dfrac{1}{2} \sum\limits_{l=0}^{n} \binom{n}{l} \left(\dfrac{\mathcal{T}_{n-l}(x)}{m^l} + \sum\limits_{k=0}^{n-l} \binom{n-l}{k} \dfrac{\mathcal{T}_{k}(x)}{m^{n-k}} \right) E_{l}(my).$

Theorem 3.8. For $x, y \in \mathbb{C}$, the following relation holds

$$\mathcal{T}_{n-1,p,q}(x,y)$$
$$= \frac{1}{[n]_{p,q}} \sum_{l=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \left(\sum_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_{p,q} \frac{p^{\binom{n-l-k}{2}} \mathcal{T}_{k,p,q}(x)}{m^{n-k}} - \frac{\mathcal{T}_{n-l,p,q}(x)}{m^l} \right) B_{l,p,q}(my).$$

Proof. We note that

$$\frac{[2]_{p,q}}{e_{p,q}(2t)+1} e_{p,q}(tx) e_{p,q}(ty)$$
$$= \left(\frac{t}{e_{p,q}(\frac{t}{m})-1} e_{p,q}(ty) \right) \left(\frac{e_{p,q}(\frac{t}{m})-1}{t} \right) \left(\frac{[2]_{p,q}}{e_{p,q}(2t)+1} e_{p,q}(tx) \right).$$

Thus as in Theorem 3.6, we have

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x,y) \frac{t^n}{[n]_{p,q}!}$$
$$= \left(\sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{t^{n-1}}{m^n [n]_{p,q}!} - \frac{1}{t} \right) \sum_{n=0}^{\infty} B_{n,p,q}(my) \frac{t^n}{m^n [n]_{p,q}!} \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} \sum_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_{p,q} p^{\binom{n-l-k}{2}} \frac{\mathcal{T}_{k,p,q}(x)}{m^{n-k}} B_{l,p,q}(my) \right) \frac{t^{n-1}}{[n]_{p,q}!}$$
$$- \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} \frac{\mathcal{T}_{n-l,p,q}(x)}{m^l} B_{l,p,q}(my) \right) \frac{t^{n-1}}{[n]_{p,q}!}.$$

The required relation now follows on comparing the coefficients.

$\square$

Corollary 3.9. From the Theorem 3.8, the following relations hold

(i) $\quad \mathcal{T}_{n-1,q}(x,y) = \dfrac{1}{[n]_q} \sum\limits_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\sum\limits_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_q \dfrac{\mathcal{T}_{k,q}(x)}{m^{n-k}} - \dfrac{\mathcal{T}_{n-l,q}(x)}{m^l} \right) B_{l,q}(my).$

(ii) $\quad \mathcal{T}_{n-1}(x,y) = \dfrac{1}{n} \sum\limits_{l=0}^{n} \binom{n}{l} \left(\sum\limits_{k=0}^{n-l} \binom{n-l}{k} \dfrac{\mathcal{T}_{k}(x)}{m^{n-k}} - \dfrac{\mathcal{T}_{n-l}(x)}{m^l} \right) B_{l}(my).$

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4. **The observation of scattering zeros of the (p,q)-tangent polynomials**

In this section, our aim is to find zeros of the (p,q)-tangent polynomials. From this work, we can investigate relations between (p,q)-tangent polynomials and classical tangent polynomials. In addition, we shall observe scattering of zeros of the (p,q)-tangent polynomials in three dimension. For this, we use Theorem 2.2 to calculate some elements of (p,q)-tangent numbers and polynomials. The first few (p,q)-tangent numbers are

$$\mathcal{T}_{0,p,q} = \frac{[2]_{p,q}}{3},$$

$$\mathcal{T}_{1,p,q} = -\frac{2[2]_{p,q}}{9},$$

$$\mathcal{T}_{2,p,q} = -\frac{4[2]_{p,q}}{9}p + \frac{4[2]_{p,q}^2}{27},$$

$$\mathcal{T}_{3,p,q} = -\frac{72[2]_{p,q}}{81}p^3 + \left(\frac{24[2]_{p,q}^2}{81} - \frac{8[2]_{p,q}[3]_{p,q}}{81}\right)p - \frac{8[2]_{p,q}[3]_{p,q}}{81},$$

$\cdots$.

To compute (p,q)-tangent polynomials we employ Mathematica. The first few (p,q)-tangent polynomials are

$$\mathcal{T}_{0,p,q}(x) = \frac{p+q}{2},$$

$$\mathcal{T}_{1,p,q}(x) = \frac{1}{2}(p+q)(-1+x),$$

$$\mathcal{T}_{2,p,q}(x) = \frac{1}{2}(-q^2(-1+x) + pq(-2+x)x + p^2(-1-x+x^2)),$$

$$\mathcal{T}_{3,p,q}(x) = \frac{1}{2}(p+q)(q^3(-1+x) - p^2q(-2+x^2) - pq^2(-2+x^2) + p^3(-1-x-x^2+x^3)),$$
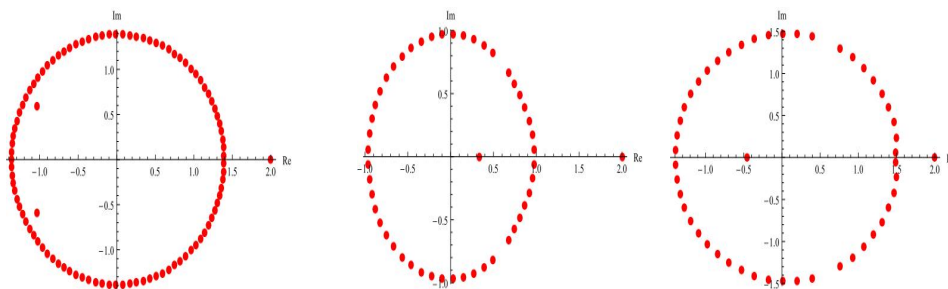
$\cdots$.

Figure 1: Zeros of $\mathcal{T}_{n,p,q}(x)$ for $q = 0.19, 0.39, 0.59$

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Using computer we can investigate the zeros of $\mathcal{T}_{n,p,q}(x)$. Here, our expectation is that a plot of $\mathcal{T}_{n,p,q}(x)$ will approach to a plot of $T_n(x)$ when $p = 1$ and $q \to 1$. We let $n = 100$. In Figure 1, for $q = 0.19, 0.39, 0.59$, we observe from left to the right that the middle shape is similar to a sphere, but it seems like an ellipse because one of real roots is exists near to 2.
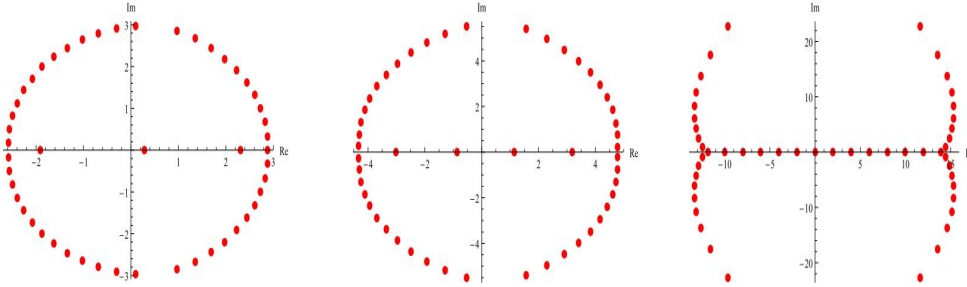

Figure 2: Zeros of $\mathcal{T}_{n,p,q}(x)$ for $q = 0.79, 0.89, 0.99$

A similar pattern we see in Figure 2 for $q = 0.79, 0.89, 0.99$. From Figures 1 and 2, it is clear that zeros of $T_{n,p,q}(x)$ are very similar to zeros of $T_n(x)$ given in([11]) for $n = 100, p = 1, q = 0.99$. We can also expect that $Im(x) = 0$ of (p, q)-tangent polynomials have reflective and symmetric properties. An interesting point is the location of roots. We can see the empty spot in the middle of Figure 1 and observe spread of empty space in Figure 2.

For $n = 100, p = 1$, and $q = 0.19$, the approximate zeros of (p, q)-tangent polynomials are listed in the following table.

degree	Approximate roots of (p, q)-tangent polynomials for $p = 1, q = 0.19$
100	-1.36843, -1.36556-0.087566i, -1.36556+0.087566i, -1.35692-0.174899i, -1.35692+0.174899i, -1.34243-0.261765i, -1.34243+0.261765i, -1.32191-0.347931i, -1.32191+0.347931i, -1.29512-0.433163i, -1.29512+0.433163i, -1.26178-0.517337i, -1.26178+0.517337i, -1.2219-0.600756i, -1.2219+0.600756i, -1.17691-0.683948i, -1.17691+0.683948i, -1.12913-0.764992i, -1.12913+0.764992, -1.07803-0.84129i, -1.07803+0.84129i, -1.03576-0.587811i, -1.03576+0.587811i, -1.02268-0.912499i-1.02268+0.912499i, -0.963125-0.978877i, -0.963125+0.978877i, -0.899648-1.0407i, -0.899648+1.0407i, -0.832427-1.09814i, -0.832427+1.09814i, -0.761516-1.15116i,-0.761516+1.15116i, -0.68703-1.19947i, -0.68703+1.19947i, -0.609252-1.24274i, -0.609252+1.24274i, -0.528594-1.28064i, -0.528594+1.28064, -0.445526-1.31298i, -0.445526+1.31298i, -0.36053-1.33962i, -0.36053+1.33962i,

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-0.274073-1.36053i, -0.274073+1.36053i, -0.186589-1.37571i, -0.186589+1.37571i, -0.0984547-1.38522i-0.0984547+1.38522i, -0.0099887-1.38914i, -0.0099887+1.38914i, 0.0785479-1.38753i, 0.0785479+1.38753i, 0.166936-1.38039i, 0.166936+1.38039i, 0.254936-1.36763i, 0.254936+1.36763i, 0.342202-1.34913i, 0.342202+1.34913i, 0.428262-1.32482i, 0.428262+1.32482i, 0.512581-1.29476i, 0.512581+1.29476i, 0.594658-1.25916i, 0.594658+1.25916i, 0.674095-1.21831i, 0.674095+1.21831i, 0.750625-1.17257i, 0.750625+1.17257i, 0.8241-1.1222i, 0.8241+1.1222i, 0.894413-1.06734i, 0.894413+1.06734i, 0.961386-1.00806i, 0.961386+1.00806i, 1.02471-0.944367i, 1.02471+0.944367i, 1.08399-0.876392i, 1.08399+0.876392i, 1.13879-0.804367i, 1.13879+0.804367i, 1.18869-0.728642i, 1.18869+0.728642i, 1.23339-0.649655i, 1.23339+0.649655i, 1.27264-0.567879i, 1.27264+0.567879i, 1.30629-0.48378i, 1.30629+0.48378i, 1.33425-0.397798i, 1.33425+0.397798i, 1.35643-0.310367i, 1.35643+0.310367i, 1.37275-0.221972i, 1.37275+0.221972i, 1.38331-0.133155i, 1.38331+0.133155i, 1.3884-0.0443556i, 1.3884+0.0443556i, 2

The above table for $\mathcal{T}_{100,1,0.19}(x)$ shows that there are only two real roots, and the shape is similar to a sphere.
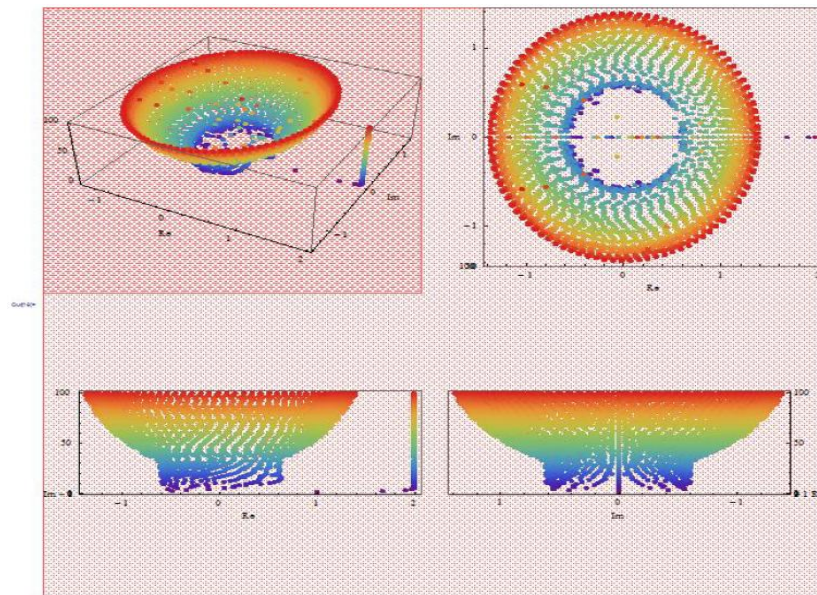
Figure 3: Zeros of $\mathcal{T}_{n,p,q}(x)$ for $p = 1, q = 0.19, 1 < n < 100$

From Figure 3 and the following table, we observe that (p, q)-tangent polynomials for different values of n do not have a regular pattern of the number of zeros.

degree	Approximate roots of (p, q)-tangent polynomials for $p = 1, q = 0.19$
100	-1.36843, 2
99	2
98	-1.37033, 0.141925, 1.39158, 2
97	-1.37233, 1.38851, 2
96	1.37639, 2
95	-1.16958, 1.36103, 2
94	1.32543, 2
93	1.32543, 2
92	0.362552, 1.33302, 2

In Figure 4, we stack the zeros of (p, q)-tangent polynomials for $1 \leq n \leq 100$. We put $q = 0.19$(the top-left), 0.39(the top-middle), 0.59(the top-right),0.79(the bottom-left), 0.89(the bottom-middle), 0.99(the bottom-right) and $p = 1$. We observe scattering of zeros of (p, q)-tangent polynomials including a stick when $q \leq 0.59$. We expect that this stick exists for all values of q smaller than 0.59. From Figures 1-3, it appears that the shape of zeros of (p, q)-tangent polynomials is similar to a sphere for all large values of n and small values of q.
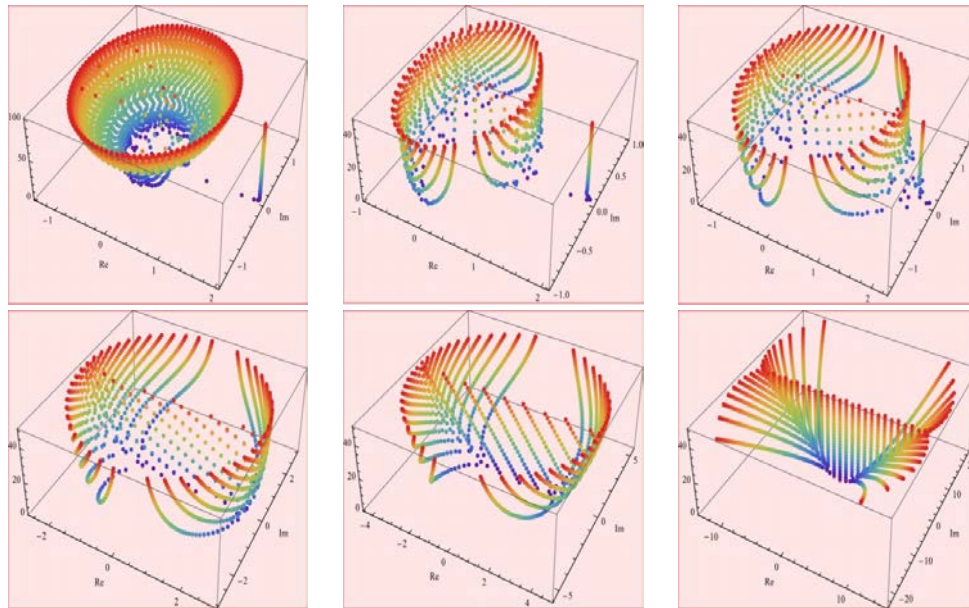
Figure 4: Stacks of zeros of $\mathcal{T}_{n,p,q}(x)$ for $q = 0.19, 0.39, 0.59, 0.79, 0.89, 0.99$

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Figure 5 shows the front view of Figure 4.
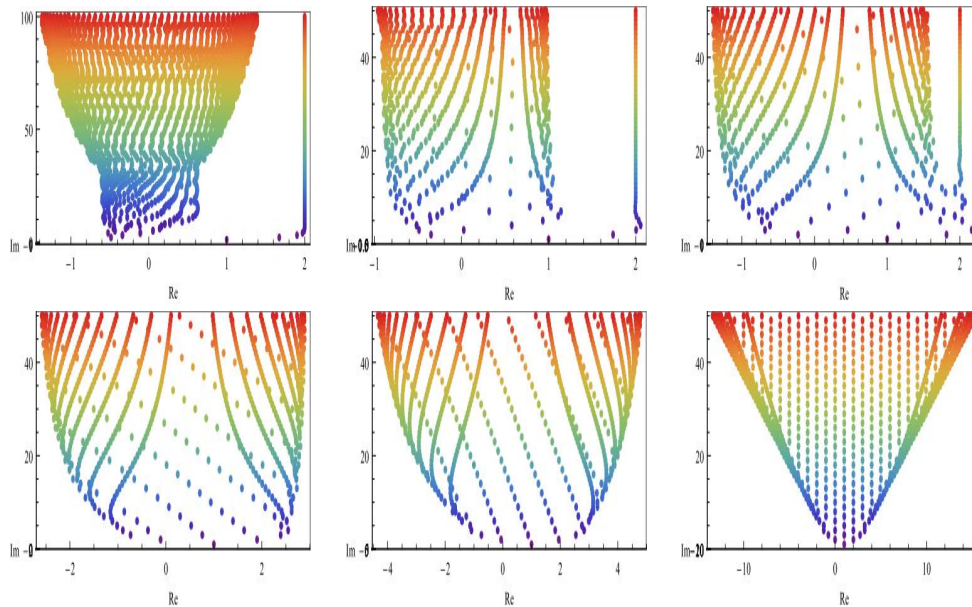
Figure 5: Zeros of $\mathcal{T}_{n,p,q}(x)$ for $1 \leq n \leq 100$

Conclusion

Our first main contribution in the work is to show that the zeros of (p,q)-tangent polynomials approach to zeros of classical tangent polynomials as $q \to 1$. To support our claim we obtain some specific results. Next, we show that the shape of the roots in (p,q)-tangent polynomials is almost identical to a circle when q is small. We also observe that as q is large the zeros are separated into two parts maintaining the symmetry.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Strong Convergence Theorems for a Non-convex Hybrid Method for Quasi-Lipschitz Mappings and Applications

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Abstract

In this note, the strong convergence theorems of a non-convex hybrid iteration algorithm corresponding to Noor iterative scheme about common fixed points for a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in a Hilbert space has been proved. Moreover some applications of developed algorithm is presented.

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1 Introduction

Fixed points of special mappings like nonexpansive, asymptotically nonexpansive, contractive and other mappings has become a field of interest on its on and has a various applications in related fields like image recovery, signal processing and geometry of objects [13]. Almost in all branches of mathematics we see some versions of theorems relating to fixed points of functions of special nature. As a result we apply them in industry, toy making, finance, aircrafts and manufacturing of new model cars. A fixed-point iteration

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scheme has been applied in intensity modulated radiation therapy optimization optimization to pre-compute dose-deposition coefficient matrix, see [12]. Because of its vast range of applications almost in all directions, the research in it is moving rapidly and an immense literature is present now. Constructive fixed point theorems (for example, Banach fixed point theorem) which not only claims the existence of a fixed point but yields an algorithm, too (in the Banach case fixed point iteration $x_{n+1} = f(x_n)$). Any equation that can be written as $x = f(x)$ for some mapping f that is contracting with respect to some (complete) metric will provide such a fixed point iteration. Mann's iteration method was the stepping stone in this regard and is invariably used in most of the occasions see [6]. But it only ensures weak convergence, see [2] but more often then not, we require strong convergence in many real world problems relating to Hilbert spaces, see [1]. So mathematician are in search for the modifications of the Mann's process to control and guarantee the strong convergence (see [2–5, 7–9, 11] and references therein).

Most probably the first noticeable modification of Mann's Iteration process was proposed by Nakajo and Takahashi [9] in 2003. They introduced this modification for only one nonexpansive mapping, where Kim and Xu [4] introduced a variant for asymptotically nonexpansive mappings in a Hilbert space in 2006. In the same year Martinez-Yanes and Xu [7] introduced a variation of the Ishikawa iteration process for a nonexpansive mapping in a Hilbert space. They also gave a variant of the Halpern method. Su and Qin [11] gave a hybrid iteration process for nonexpansive mappings which is monotone. Liu et al. [5] gave a novel method for a finite family of quasi-asymptotically pseudo-contractive mappings.

Let H be a Hilbert space and C be a nonempty, closed and convex subset of H. Let $P_c(\cdot)$ be the metric projection onto C. A mapping $T : C \to C$ is said to be *nonexpensive* if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in C$. Denote by $F(T)$ the set of fixed points of T. It is well known that $F(T)$ is closed and convex. A mapping $T : C \to C$ is said to be *quasi-Lipschitz* if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq L\|x - p\|$ for all $x \in C$, $p \in F(T)$, where $1 \leq L < \infty$ is a constant. If $L = 1$, then T is known as *quasi-nonexpansive*. It is well-known that T is said to be *closed* if $x_n \to x$ and $\|Tx_n - x_n\| \to 0$ as $n \to \infty$ implies $Tx = x$. T is said to be *weak closed* if $x_n \rightharpoonup x$ and $\|Tx_n - x_n\| \to 0$ as $n \to \infty$ implies $Tx = x$. It is obvious that a weak closed mapping should be closed, but converse is no longer true.

Let $\{T_n\}$ be a sequence of mappings from C into itself with a nonempty common fixed points set F. Then $\{T_n\}$ is said to be *uniformly closed* if for any convergent sequences $\{z_n\} \subset C$ with conditions $\|T_n z_n - z_n\| \to 0$ as $n \to \infty$, the limit of $\{z_n\}$ belongs to F.

In 1953 Mann [6] proposed an iterative scheme given as

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(x_n), \quad n = 0, 1, 2, \ldots.$$

Guan et al. [3] established the following non-convex hybrid iteration algorithm corresponding to Mann iterative scheme:

$$
\begin{cases}
x_0 \in C = Q_0, \quad \text{choosen arbitrarily,} \\
y_n = (1 - \alpha_n)x_n + \alpha_n T_n x_n, \quad n \geq 0, \\
C_n = \{z \in C : \|y_n - z\| \leq (1 + (L_n - 1)\alpha_n)\|x_n - z\| \cap A, \quad n \geq 0, \\
Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \quad n \geq 1, \\
x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0
\end{cases}
$$

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and proved some strong convergence results about common fixed points relating to a family of countable uniformly closed asymptotically quasi-Lipschitz mappings in H. They applied their results for the finite case to obtain fixed points.

The Noor iterative scheme [10] was defined in 2000 as

$$\begin{cases} x_{n+1} = (1 - a_n)x_n + a_n T(y_n), \\ y_n = (1 - b_n)x_n + b_n T(z_n), \\ z_n = (1 - c_n)x_n + c_n T(x_n), \quad n \geq 0. \end{cases}$$

In this article, we established a non-convex hybrid algorithm corresponding to Noor iterative scheme. Then we also establish strong convergence theorems with proofs about common fixed points related to a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in a Hilbert space. Some applications of presented algorithm were also given.

2 Main results

In this section we formulate our main results.

Definition 2.1. Let C be a closed convex subset of a Hilbert space H, and let $\{T_n\}$ be a family of countable quasi-L_n-Lipschitz mappings from C into itself. $\{T_n\}$ is said to be *asymptotically* if $\lim_{n\to\infty} L_n = 1$.

The following lemmas are well known and useful for our conclusions

Proposition 2.2. *Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if we have the relation $\langle x - z, z - y \rangle \geq 1$ for all $y \in C$.*

Proposition 2.3. ([3]) *Let C be a closed convex subset of a Hilbert space H and let $\{T_n\}$ be a uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings from C into itself. Then the common fixed point set F is closed and convex.*

Proposition 2.4. *Let C be a closed convex subset of a Hilbert space H. For any given $x_0 \in H$, we have $p = P_C x_0$ if and only if $\langle p - z, x_0 - p \rangle \geq 0$ for all $z \in C$.*

Theorem 2.5. *Let C be a closed convex subset of a Hilbert space H, and let $\{T_n\} : C \to C$ be a uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings from C into itself. Assume that $a_n, b_n, c_n \in (a, 1]$ holds for some $a \in (0, 1)$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \text{chosen arbitrarily}, \\ y_n = (1 - a_n)x_n + a_n T_n z_n, \quad n \geq 0, \\ z_n = (1 - b_n)x_n + b_n T_n t_n, \quad n \geq 0, \\ t_n = (1 - c_n)x_n + c_n T_n x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1 + (L_n(1 - b_n) + L_n^2(1 - c_n)b_n \\ \qquad + b_n c_n L_n^3 - 1)a_n]\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0, \end{cases}$$

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converges to $P_F x_0$, where $\overline{co}C_n$ denotes the closed convex closure of C_n for all $n \geq 1$ and $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof. We divide our proof in following seven steps.

Step 1. It is obvious that $\overline{co}C_n$ and Q_n are closed and convex for all $n \geq 0$. Next, we show that $F \cap A \subset \overline{co}C_n$ for all $n \geq 0$. Indeed, for each $p \in F \cap A$, we have

$$
\begin{aligned}
\|y_n - p\| &= \|(1 - a_n)x_n + a_n T_n z_n - p\| \\
&= \|(1 - a_n)x_n + a_n T_n[(a - b_n)x_n + b_n T_n t_n] - p\| \\
&= \|(1 - a_n)x_n + a_n T_n[(a - b_n)x_n + b_n T_n((1 - c_n)x_n + c_n T_n x_n)] - p\| \\
&= \|(1 - a_n)(x_n - p) + (a_n - a_n b_n)(T_n x_n - p) \\
&\quad + (a_n b_n - a_n b_n c_n)(T_n^2 x_n - p) + a_n b_n c_n(T_n^3 x_n - p)\| \\
&\leq (1 - a_n)\|x_n - p\| + (a_n - a_n b_n)L_n\|x_n - p\| \\
&\quad + (a_n b_n - a_n b_n c_n)L_n^2\|x_n - p\| + a_n b_n c_n L_n^3\|x_n - p\| \\
&= [1 + (L_n(1 - b_n) + L_n^2(1 - c_n)b_n + b_n c_n L_n^3 - 1)a_n]\|x_n - p\|,
\end{aligned}
$$

and $p \in A$, so $p \in C_n$ which implies that $F \cap A \subset C_n$ for all $n \geq 0$. therefore, $F \cap A \subset \overline{co}C_n$ for all $n \geq 0$.

Step 2. We show that $F \cap A \subset \overline{co}C_n \cap Q_n$ for all $n \geq 0$. it suffices to show that $F \cap A \subset Q_n$, for all $n \geq 0$. We prove this by mathematical induction. For $n = 0$ we have $F \cap A \subset C = Q_0$. Assume that $F \cap A \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $\overline{co}C_n \cap Q_n$, from Proposition 2.2, we have

$$
\langle x_{n+1} - z, x_{n+1} - x_0 \rangle \leq 0, \quad \forall z \in \overline{co}C_n \cap Q_n
$$

as $F \cap A \subset \overline{co}C_n \cap Q_n$, the last inequality holds, in particular, for all $z \in F \cap A$. This together with the definition of Q_{n+1} implies that $F \cap A \subset Q_{n+1}$. Hence the $F \cap A \subset \overline{co}C_n \cap Q_n$ holds for all $n \geq 0$.

Step 3. We prove that $\{x_n\}$ is bounded. Using closeness and convexity properties of F, we can say that we have a unique element $z_0 \in F$ such that $z_0 = P_F x_0$. From $x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0$, we have

$$
\|x_{n+1} - x_0\| \leq \|z - x_0\|
$$

for every $z \in \overline{co}C_n \cap Q_n$. As $z_0 \in F \cap A \subset \overline{co}C_n \cap Q_n$, we get

$$
\|x_{n+1} - x_0\| \leq \|z_0 - x_0\|
$$

for each $n \geq 0$. This shows that $\{x_n\}$ is bounded.

Step 4. We show that $\{x_n\}$ converges strongly to a point of C by showing that $\{x_n\}$ is a Cauchy sequenc. As $x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0 \subset Q_n$ and $x_n = P_{Q_n} x_0$ (Proposition 2.4), we have

$$
\|x_{n+1} - x_0\| \geq \|x_n - x_0\|
$$

for every $n \geq 0$, which together with the boundedness of $\|x_n - x_0\|$ implies that there exsists the limit of $\|x_n - x_0\|$. On the other hand, from $x_{n+m} \in Q_n$, we have $\langle x_n - x_{n+m}, x_n - x_0 \rangle \leq$

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0 and hence

$$\|x_{n+m} - x_n\|^2 = \|(x_{n+m} - x_0) - (x_n - x_0)\|^2$$
$$\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+m} - x_n, x_n - x_0\rangle$$
$$\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 \to 0, \ n \to \infty$$

for any $m \geq 1$. Therefore $\{x_n\}$ is a Cauchy sequence in C, then there exists a point $q \in C$ such that $\lim_{n \to \infty} x_n = q$.

Step 5. We show that $y_n \to q$, as $n \to \infty$. Let

$$D_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (L_n^3 - 1)(L_n^3 + 1)\}.$$

From the definition of D_n, we have

$$D_n = \{z \in C : \langle y_n - z, y_n - z\rangle \leq \langle x_n - z, x_n - z\rangle + (L_n^3 - 1)(L_n^3 + 1)\}$$
$$= \{z \in C : \|y_n\|^2 - 2\langle y_n, z\rangle + \|z\|^2 \leq \|x_n\|^2 - 2\langle x_n, z\rangle + \|z\|^2 + (L_n^3 - 1)(L_n^3 + 1)\}$$
$$= \{z \in C : 2\langle x_n - y_n, z\rangle \leq \|x_n\|^2 - \|y_n\|^2 + (L_n^3 - 1)(L_n^3 + 1)\}.$$

This shows that D_n is convex and closed, $n \in \mathbb{Z}^+ \cup \{0\}$. Next, we want to prove that $C_n \subset D_n$, $n \geq 0$.

In fact, for any $z \in C_n$, we have

$$\|y_n - z\|^2 \leq [1 + (L_n(1 - b_n) + L_n^2(1 - c_n)b_n + b_n c_n L_n^3 - 1)a_n]^2 \|x_n - z\|^2$$
$$= \|x_n - z\|^2 + [2(L_n(1 - b_n) + L_n^2(1 - c_n)b_n + b_n c_n L_n^3 - 1)a_n$$
$$+ (L_n(1 - b_n) + L_n^2(1 - c_n)b_n + b_n c_n L_n^3 - 1)^2 a_n^2]\|x_n - z\|^2$$
$$\leq \|x_n - z\|^2 + [2(L_n^3 - 1) + (L_n^3 - 1)^2]\|x_n - z\|^2$$
$$= \|x_n - z\|^2 + (L_n^3 - 1)(L_n^3 + 1)\|x_n - z\|^2.$$

From

$$C_n = \{z \in C : \|y_n - z\| \leq [1 + (L_n(1 - b_n) + L_n^2(1 - c_n)b_n$$
$$+ b_n c_n L_n^3 - 1)a_n]\|x_n - z\|\} \cap A, \quad n \geq 0,$$

we have $C_n \subset A$, $n \geq 0$. Using convexity of A, we have $\overline{co}C_n \subset A$, $n \geq 0$. Consider $x_n \in \overline{co}C_{n-1}$, we know that

$$\|y_n - z\| \leq \|x_n - z\|^2 + (L_n^3 - 1)(L_n^3 + 1)\|x_n - z\|^2$$
$$\leq \|x_n - z\|^2 + (L_n^3 - 1)(L_n^3 + 1).$$

This implies that $z \in D_n$ and hence $C_n \subset D_n$, $n \geq 0$. Since D_n is convex, we have $\overline{co}(C_n) \subset D_n$, $n \geq 0$. Therefore

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + (L_n^3 - 1)(L_n^3 + 1) \to 0$$

as $n \to \infty$. That is, $y_n \to q$ as $n \to \infty$.

Step 6. To prove that $q \in F$, we use definition of y_n. So we have

$$(a_n + a_n b_n T_n + a_n b_n c_n T_n^2)\|T_n x_n - x_n\| = \|y_n - x_n\| \to 0$$

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as $n \to \infty$. Since $a_n \in (a, 1] \subset [0, 1]$, from the above limit we have

$$\lim_{n \to \infty} \|T_n x_n - x_n\| = 0.$$

Since $\{T_n\}$ is uniformly closed and $x_n \to q$, we have $q \in F$.

Step 7. We claim that $q = z_0 = P_F x_0$, if not, we have that $\|x_0 - p\| > \|x_0 - z_0\|$. There must exist a positive integer N, if $n > N$, then $\|x_0 - x_n\| > \|x_0 - z_0\|$, which leads to

$$\|z_0 - x_n\|^2 = \|z_0 - x_n + x_n - x_0\|^2$$
$$= \|z_0 - x_n\|^2 + \|x_n - x_0\|^2 + 2\langle z_0 - x_n, x_n - x_0 \rangle.$$

It follows that $\langle z_0 - x_n, x_n - x_0 \rangle < 0$, which implies that $z_0 \overline{\in} Q_n$, so that $z_0 \overline{\in} F$, which is a contradiction. This completes the proof. $\qquad\square$

In [3], we show an example of C_n which does not involve a convex subset.

Corollary 2.6. *Let C be a closed convex subset of a Hilbert space H, and let $T : C \to C$ be a closed quasi-nonexpansive mapping from C into itself. Assume that $a_n, b_n, c_n \in (a, 1]$ holds for some $a \in (0, 1)$. Then $\{x_n\}$ generated by*

$$\begin{cases}
x_0 \in C = Q_0, \quad \text{chosen arbitrarily}, \\
y_n = (1 - a_n)x_n + a_n T z_n, \quad \geq 0, \\
z_n = (1 - b_n)x_n + b_n T t_n, \quad n \geq 0, \\
t_n = (1 - c_n)x_n + c_n T x_n, \quad n \geq 0, \\
C_n = \{z \in C : \|y_n - z\| \leq [1 + (L_n(1 - b_n) + L_n^2(1 - c_n)b_n \\
\qquad + b_n c_n L_n^3 - 1)a_n]\|x_n - z\|\} \cap A, \quad n \geq 0, \\
Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \quad n \geq 1, \\
x_{n+1} = P_{C_n \cap Q_n} x_0,
\end{cases}$$

converges strongly to $P_{F(T)} x_0$, where $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof. Take $T_n \equiv T$, $L_n \equiv 1$ in Theorem 2.5, in this case, C_n is convex and closed for all $n \geq 0$, by using Theorem 2.5, we obtain Corollary 2.6. $\qquad\square$

Corollary 2.7. *Let C be a closed convex subset of a Hilbert space H, and let T be a nonexpansive mapping from C into itself. Assume that $a_n, b_n, c_n \in (a, 1]$ holds for some $a \in (0, 1)$. Then $\{x_n\}$ generated by*

$$\begin{cases}
x_0 \in C = Q_0, \quad \text{chosen arbitrarily}, \\
y_n = (1 - a_n)x_n + a_n T z_n, \quad n \geq 0, \\
z_n = (1 - b_n)x_n + b_n T t_n, \quad n \geq 0, \\
t_n = (1 - c_n)x_n + c_n T x_n, \quad n \geq 0, \\
C_n = \{z \in C : \|y_n - z\| \leq [1 + (L_n(1 - b_n) + L_n^2(1 - c_n)b_n \\
\qquad + b_n c_n L_n^3 - 1)a_n]\|x_n - z\|\} \cap A, \quad n \geq 0, \\
Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \quad n \geq 1, \\
x_{n+1} = P_{C_n \cap Q_n} x_0,
\end{cases}$$

converges strongly to $P_{F(T)} x_0$, where $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

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3 Applications

Here, we give an application of our result for the following case of finite family of asymptotically quasi-nonexpansive mappings $\{T_n\}_{n=0}^{N-1}$. Let

$$\|T_i^j x - p\| \leq k_{i,j}\|x - p\|, \quad \forall x \in C, \, p \in F,$$

where F is the common fixed point set of $\{T_n\}_{n=0}^{N-1}$ and $\lim_{j\to\infty} k_{i,j} = 1$ for all $0 \leq i \leq N - 1$. The finite family of asymptotically quasi-nonexpansive mappings $\{T_n\}_{n=0}^{N-1}$ is said to be *uniformly L-Lipschitz* if

$$\|T_i^j x - T_i^j y\| \leq L_{i,j}\|x - y\|, \quad \forall x, y \in C$$

for all $i \in \{0, 1, 2, ..., N - 1\}$, $j \geq 1$, where $L \geq 1$.

Theorem 3.1. *Let C be a closed convex subset of a Hilbert space H, and let $\{T_n\}_{n=0}^{N-1} : C \to C$ be a uniformly L-Lipschitz finite family of asymptotically quasi-nonexpansive mappings with nonempty common fixed point set F. Assume that $a_n, b_n, c_n \in (a, 1]$ holds for some $a \in (0, 1)$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \text{chosen arbitrarily,} \\ y_n = (1 - a_n)x_n + a_n T_{i(n)}^{j(n)} z_n, \quad n \geq 0, \\ z_n = (1 - b_n)x_n + b_n T_{i(n)}^{j(n)} t_n, \quad n \geq 0, \\ t_n = (1 - c_n)x_n + c_n T_{i(n)}^{j(n)} x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1 + (k_{i(n),j(n)}(1 - b_n) + k_{i(n),j(n)}^2(1 - c_n)b_n \\ \quad\quad + b_n c_n k_{i(n),j(n)}^3 - 1)a_n]\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_F x_0$, where $\overline{co}C_n$ denotes the closed convex closure of C_n for all $n \geq 1$, $n = (j(n) - 1)N + i(n)$ for all $n \geq 0$ and $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof. We can acquire the prove from the conclusions:
Conclusion 1. $\{T_{n=0}^{N-1}\}_{n=0}^{\infty}$ is asymptotically family of uniformly closed countable quasi-L_n-Lipschitz mappings from C into itself.
Conclusion 2. $F = \bigcap_{n=0}^{N} F(T_n) = \bigcap_{n=0}^{\infty} F(T_{i(n)}^{j(n)})$, where $F(T_n)$ is the fixed point set of T_n. $\square$

Corollary 3.2. *Let C be a closed convex subset of a Hilbert space H, and let $T : C \to C$ be a L-Lipschitz asymptotically quasi-nonexpansive mappings with nonempty common fixed point set F. Assume that $a_n, b_n, c_n \in (a, 1]$ holds for some $a \in (0, 1)$. Then $\{x_n\}$ generated*

by

$$\begin{cases} x_0 \in C = Q_0, \quad chosen\ arbitrarily, \\ y_n = (1 - a_n)x_n + a_n T^n z_n, \quad n \geq 0, \\ z_n = (1 - b_n)x_n + b_n T^n t_n, \quad n \geq 0, \\ t_n = (1 - c_n)x_n + c_n T^n x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1 + (k_n(1 - b_n) + k_n^2(1 - c_n)b_n \\ \quad\quad + b_n c_n k_n^3 - 1)a_n]\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0, \end{cases}$$

converges strongly to $P_F x_0$, where $\overline{co}C_n$ denotes the closed convex closure of C_n for all $n \geq 1$ and $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$.

Proof. Take $T_n \equiv T$ in 3.1, we get the desired result. $\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\square$

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9

Certain subclasses of k-uniformly starlike functions associated with symmetric q-derivative operator

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Abstract

In this paper, we have established the inclusion relations for k-uniformly starlike functions under the $(\widetilde{D}_q f)(z)$ operator. We define two new subclass of k-uniformly starlike functions of order α. Moreover, for functions belonging to these function classes, we investigate necessary and sufficient coefficient conditions, distortion bounds, extreme points.

Keywords: Analytic functions, k-uniformly starlike functions, symmetric q-derivative operator.

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1   Introduction, Definitions and Notations

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and S be the subclass of A consisting of the form (1) which are also univalent in U. A function $f \in A$ is called starlike of order α, $0 \leq \alpha < 1$, if and only if $\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha$ $(z \in U)$. We denote by $S(\alpha)$ the subset of A consisting of all

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starlike functions of order α. For $\alpha = 0$ we get the class S of functions f that maps U onto a starlike domain with respect to the origin.

Also represent by T the subclass of A consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0). \tag{2}$$

A function $f \in A$ is said to be in $US(k, \alpha)$, the class of k-uniformly starlike functions of order α, $0 \leq \alpha < 1$, if f satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)} - \alpha\right) > k\left|\frac{zf'(z)}{f(z)} - 1\right| \quad (k \geq 0),$$

(for details see [5]).

We remark here that the class of k-uniformly starlike functions is an extension of the relatively more familiar class of uniformly starlike functions investigated earlier by (for example) Goodman [9], Rønning [18], (see also the more recent contributions on this function class by Srivastava and Mishra [21] and others [11, 12, 19]).

In the field of Geometric Function Theory, various subclasses of analytic functions have been studied from different viewpoints. The fractional q-calculus is the important tools that are used to investigate subclasses of analytic functions. Historically speaking, a firm footing of the usage of the the q-calculus in the context of Geometric Function Theory was actually provided and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details, [20]). In fact, the theory of univalent functions can be described by using the theory of the q-calculus. Moreover, in recent years, such q-calculus operators as the fractional q-integral and fractional q-derivative operators were used to construct several subclasses of analytic functions (see, for example, [1, 2, 4, 14] and [16]). In particular, Purohit and Raina [17] investigated applications of fractional q-calculus operators to define several classes of functions which are analytic in the open unit disk $\mathbb{U}$. On the other hand, Mohammed and Darus [13] studied approximation and geometric properties of these q-operators in regard to some subclasses of analytic functions in a compact disk.

For the convenience, we provide some basic definitions and concept details of q-calculus which are used in this paper. We suppose throughout the paper that $0 < q < 1$. We shall follow the notation and terminology as in [8]. We recall the definitions of fractional q-calculus operators of complex valued function $f(z)$.

Definition 1 *Let* $q \in (0, 1)$ *and define the* q*-number* $[\lambda]_q$ *by*

$$[\lambda]_q = \begin{cases} \dfrac{1 - q^{\lambda}}{1 - q} & (\lambda \in \mathbb{C}) \\[2em] \displaystyle\sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \cdots + q^{n-1} & (\lambda = n \in \mathbb{N} = \{1, 2, \ldots\}). \end{cases} \tag{3}$$

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Definition 2 *Let $q \in (0,1)$ and define the q-fractional $[n]_q!$ by*

$$[n]_q! = \begin{cases} \prod_{k=1}^{n} [k]_q, & n \in \mathbb{N} \\ \\ 1, & n = 0 \end{cases}.$$

Definition 3 *For $\alpha \in \mathbb{C}$, the q-shifted factorial is defined as a product of $n \in \mathbb{N}$ factors by*

$$(\alpha;q)_0 = 1, \quad (\alpha;q)_n = \prod_{i=0}^{n-1}(1 - \alpha q^i), \quad (\alpha;q)_\infty = \prod_{i=0}^{\infty}(1 - \alpha q^i).$$

Definition 4 *(see [10]) The q-derivative of a function f is defined on a subset of $\mathbb{C}$ is given by*

$$(D_q f)(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad if \quad z \neq 0, \tag{4}$$

and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists.

Note that

$$\lim_{q \to 1^-} (D_q f)(z) = \lim_{q \to 1^-} \frac{f(z) - f(qz)}{(1-q)z} = \frac{df(z)}{dz}$$

if f is differentiable. From (4), we deduce that

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}. \tag{5}$$

Definition 5 *(see [6]) The symmetric q-derivative $\widetilde{D}_q f$ of a function f given by (1) is defined as follows:*

$$(\widetilde{D}_q f)(z) = \frac{f(qz) - f(q^{-1}z)}{(q - q^{-1})z}, \quad if \quad z \neq 0, \tag{6}$$

and $(\widetilde{D}_q f)(0) = f'(0)$ provided $f'(0)$ exists.

From (6), we deduce that

$$(\widetilde{D}_q f)(z) = 1 + \sum_{n=2}^{\infty} \widetilde{[n]}_q a_n z^{n-1},$$

where the symbol $\widetilde{[n]}_q$ denotes the number

$$\widetilde{[n]}_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

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frequently occurring in the study of q-deformed quantum mechanical simple harmonic oscillator (see [7]).

The following properties hold

$$\widetilde{D}_q(f(z) + g(z)) = (\widetilde{D}_q f)(z) + (\widetilde{D}_q g)(z)$$

$$\widetilde{D}_q(f(z)g(z)) = g(q^{-1}z)(\widetilde{D}_q f)(z) + f(qz)(\widetilde{D}_q g)(z)$$

$$= g(qz)(\widetilde{D}_q f)(z) + f(q^{-1}z)(\widetilde{D}_q g)(z)$$

$$\widetilde{D}_q z^n = \widetilde{[n]}_q z^{n-1}.$$

Finally, we have the following relation

$$\widetilde{D}_q f(z) = D_{q^2} f(q^{-1}z).$$

Definition 6 *Let $0 \le k < \infty$ and $0 \le \alpha < 1$. By $US(q; k, \lambda, \alpha)$ we denote the class of functions $f \in A$ satisfying the condition*

$$\Re\left(\frac{z(\widetilde{D}_q f)(z)}{(1-\lambda)f(z) + \lambda z(\widetilde{D}_q f)(z)} - \alpha\right) > k\left|\frac{z(\widetilde{D}_q f)(z)}{(1-\lambda)f(z) + \lambda z(\widetilde{D}_q f)(z)} - 1\right|, \ (z \in U).$$

We also let $UTS(q; k, \lambda, \alpha) = US(q; k, \lambda, \alpha) \cap T$.

We note that
$$\lim_{q \to 1^-} US(q; k, \lambda, \alpha) = US(k, \lambda, \alpha)$$

where $US(k, \lambda, \alpha)$ is the class of defined by Murugusundaramoorthy and Magesh [15].

Murugusundaramoorthy and Magesh [15] and Srivastava and Mishra [21] defined the new subclasses of the families UCV and US making use of hypergeometric functions and fractional calculus respectively and obtained various interesting properties. In light of this in this paper, we study the classes $US(q; k, \lambda, \alpha)$ and $UTS(q; k, \lambda, \alpha)$ defining by symmetric q-derivative operator. We provide necessary and suffcient coefficient conditions, distortion bounds, extreme points for functions in $UTS(q; k, \lambda, \alpha)$.

2 Main Results

Theorem 7 *Let $f \in A$ be given by (1). If the inequality*

$$\sum_{n=2}^{\infty} \left[\widetilde{[n]}_q(k+1) - (k+\alpha)(1 - \lambda + \lambda\widetilde{[n]}_q)\right]|a_n| \le 1 - \alpha \tag{7}$$

holds true for some k $(0 \le k < \infty)$ and α $(0 \le \alpha < 1)$, then $f \in US(q; k, \lambda, \alpha)$.

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From Definition 6, it suffices to prove that

$$k\left|\frac{z(\widetilde{D}_q f)(z)}{(1-\lambda)f(z)+\lambda z(\widetilde{D}_q f)(z)}-1\right|-\Re\left(\frac{z(\widetilde{D}_q f)(z)}{(1-\lambda)f(z)+\lambda z(\widetilde{D}_q f)(z)}-1\right)\leq 1-\alpha.$$

Observe that

$$k\left|\frac{z(\widetilde{D}_q f)(z)}{(1-\lambda)f(z)+\lambda z(\widetilde{D}_q f)(z)}-1\right|-\Re\left(\frac{z(\widetilde{D}_q f)(z)}{(1-\lambda)f(z)+\lambda z(\widetilde{D}_q f)(z)}-1\right)$$

$$\leq (k+1)\left|\frac{z(\widetilde{D}_q f)(z)}{(1-\lambda)f(z)+\lambda z(\widetilde{D}_q f)(z)}-1\right|$$

$$= (k+1)\left|\frac{\sum\limits_{n=2}^{\infty}\left(\widetilde{[n]}_q-\lambda\widetilde{[n]}_q-1+\lambda\right)a_n z^{n-1}}{1+\sum\limits_{n=2}^{\infty}\left(1-\lambda+\lambda\widetilde{[n]}_q\right)a_n z^{n-1}}\right|; \quad (|z|<1)$$

$$\leq (k+1)\frac{\sum\limits_{n=2}^{\infty}\left(\widetilde{[n]}_q-\lambda\widetilde{[n]}_q-1+\lambda\right)|a_n|}{1-\sum\limits_{n=2}^{\infty}\left(1-\lambda+\lambda\widetilde{[n]}_q\right)|a_n|}.$$

The last expression is bounded by $1-\alpha$ if the inequality (7) holds.

The next Corollary can be easily obtained from Theorem 7.

Corollary 8 *Let* $f(z)=z+a_n z^n$. *If*

$$|a_n|\leq \frac{1-\alpha}{\widetilde{[n]}_q(k+1)-(k+\alpha)(1-\lambda+\lambda\widetilde{[n]}_q)}, \quad (n\geq 2)$$

then $f\in US(q;k,\lambda,\alpha)$.

Theorem 9 *A necessary and sufficient condition for* $f(z)$ *of the form (2) to be in the class* $UTS(q;k,\lambda,\alpha)$, *is that*

$$\sum_{n=2}^{\infty}\left[\widetilde{[n]}_q(k+1)-(k+\alpha)(1-\lambda+\lambda\widetilde{[n]}_q)\right]a_n\leq 1-\alpha,$$

$$\tag{8}$$

$$(0\leq k<\infty;\ 0\leq\alpha<1;\ z\in U).$$

The result is sharp for the function $f(z)$ *given by*

$$f(z)=z-\frac{1-\alpha}{\widetilde{[n]}_q(k+1)-(k+\alpha)(1-\lambda+\lambda\widetilde{[n]}_q)}z^n.$$

Proof. In view of Theorem 7, we need only to prove the necessity. If $f\in UTS(q;k,\lambda,\alpha)$, using the fact that $|\Re(z)|\leq|z|$ for any z, then

$$\left|\frac{1-\sum\limits_{n=2}^{\infty}\widetilde{[n]}_q a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}(1-\lambda+\lambda\widetilde{[n]}_q)a_n z^{n-1}}-\alpha\right|\geq k\left|\frac{\sum\limits_{n=2}^{\infty}\left(\widetilde{[n]}_q-\lambda\widetilde{[n]}_q-1+\lambda\right)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}(1-\lambda+\lambda\widetilde{[n]}_q)a_n z^{n-1}}\right|.$$

$$\tag{9}$$

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Choose values of z on the real axis so that $\widetilde{D}_q f(z)$ is real. Upon clearing the dominator in (9) and letting $z \to 1^-$ through the real values, we obtain (8). ∎

This completes the proof.

Letting $q \to 1^-$ we get desired Corollary.

Corollary 10 *As special cases of Theorem 9, for $k = 0$ and $\lambda = 0$, see [3].*

3 Distortion Theorems

Theorem 11 *Let the function f defined by (2) in the class $UTS(q; k, \lambda, \alpha)$. Then*

$$r - \frac{q(1-\alpha)}{(q^2+1)(k+1)-(k+\alpha)((1-\lambda)q+\lambda(q^2+1)} r^2 \quad \leq \quad |f(z)| \leq r + \frac{q(1-\alpha)}{(q^2+1)(k+1)-(k+\alpha)((1-\lambda)q+\lambda(q^2+1)} r^2 \tag{10}$$

$$(|z| \quad = \quad r < 1)$$

for $z \in U$.

Equality in (10) holds true for the function $f(z)$ given by

$$|f(z)| = |z| + \frac{q(1-\alpha)}{(q^2+1)(k+1)-(k+\alpha)((1-\lambda)q+\lambda(q^2+1)} |z|^2. \tag{11}$$

Proof. Since $f \in UTS(q; k, \lambda, \alpha)$, in view of Theorem 9, we have

$$\left[\widetilde{[2]}_q(k+1) - (k+\alpha)(1-\lambda+\lambda\widetilde{[2]}_q) \right] \sum_{n=2}^{\infty} |a_n| \quad \leq \quad \sum_{n=2}^{\infty} \left[\widetilde{[n]}_q(k+1) - (k+\alpha)(1-\lambda+\lambda\widetilde{[n]}_q) \right] a_n$$

$$\leq \quad 1 - \alpha,$$

which gives

$$\sum_{n=2}^{\infty} a_n \leq \frac{1-\alpha}{\widetilde{[2]}_q(k+1) - (k+\alpha)(1-\lambda+\lambda\widetilde{[2]}_q)}. \tag{12}$$

Therefore

$$|f(z)| \quad \leq \quad |z| + \sum_{n=2}^{\infty} a_n |z|^n$$

$$\leq \quad r + \frac{q(1-\alpha)}{(q^2+1)(k+1)-(k+\alpha)((1-\lambda)q+\lambda(q^2+1)} r^2.$$

On the other hand,

$$|f(z)| \quad \geq \quad |z| - \sum_{n=2}^{\infty} a_n |z|^n$$

$$\geq \quad r - \frac{q(1-\alpha)}{(q^2+1)(k+1)-(k+\alpha)((1-\lambda)q+\lambda(q^2+1)} r^2.$$

∎

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Theorem 12 *Let the function f defined by (2) in the class $UTS(q; k, \lambda, \alpha)$. Then*

$$1 - \frac{2q(1-\alpha)}{(q^2+1)(k+1)-(k+\alpha)((1-\lambda)q+\lambda(q^2+1)}r \quad \leq \quad |f'(z)| \leq 1 + \frac{2q(1-\alpha)}{(q^2+1)(k+1)-(k+\alpha)((1-\lambda)q+\lambda(q^2+1)}r \tag{13}$$

$$|z| \quad = \quad r < 1$$

for $z \in U$.

From (2),

$$|f'(z)| \leq 1 + \sum_{n=2}^{\infty} na_n |z|^{n-1} \leq 1 + r\sum_{n=2}^{\infty} na_n, \tag{14}$$

and

$$|f'(z)| \geq 1 - \sum_{n=2}^{\infty} na_n |z|^{n-1} \geq 1 - r\sum_{n=2}^{\infty} na_n. \tag{15}$$

The assertion (13) of Theorem 12 would now follow from (14) and (15) by means of a rather simple consequence of (12) given by

$$\sum_{n=2}^{\infty} na_n \leq \frac{2(1-\alpha)}{\widetilde{[2]}_q(k+1) - (k+\alpha)(1-\lambda+\lambda\widetilde{[2]}_q)}.$$

This completes the proof of Theorem 12.

4   Extreme Points of the Function Class $UTS(q; k, \lambda, \alpha)$

Theorem 13 *Let*

$$f_1(z) = z$$

and

$$f_n(z) = z - \frac{1-\alpha}{\widetilde{[n]}_q(k+1) - (k+\alpha)(1-\lambda+\lambda\widetilde{[n]}_q)}z^n \quad (n = 2, 3, \ldots).$$

Then $f \in UTS(q; k, \lambda, \alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z),$$

where $\lambda_n > 0$ and $\sum\limits_{n=1}^{\infty} \lambda_n = 1$.

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Proof. Suppose that

$$
\begin{aligned}
f(z) &= \sum_{n=1}^{\infty} \lambda_n f_n(z) = \lambda_1 f_1(z) + \sum_{n=2}^{\infty} \lambda_n f_n(z) \\
&= \lambda_1 f_1(z) + \sum_{n=2}^{\infty} \lambda_n \left[z - \frac{1-\alpha}{\widetilde{[n]}_q(k+1) - (k+\alpha)(1 - \lambda + \lambda \widetilde{[n]}_q)} z^n \right] \\
&= \lambda_1 z + \sum_{n=2}^{\infty} \lambda_n z - \sum_{n=2}^{\infty} \lambda_n \frac{1-\alpha}{\widetilde{[n]}_q(k+1) - (k+\alpha)(1 - \lambda + \lambda \widetilde{[n]}_q)} z^n \\
&= \left(\sum_{n=1}^{\infty} \lambda_n \right) z - \sum_{n=2}^{\infty} \lambda_n \frac{1-\alpha}{\widetilde{[n]}_q(k+1) - (k+\alpha)(1 - \lambda + \lambda \widetilde{[n]}_q)} z^n \\
&= z - \sum_{n=2}^{\infty} \lambda_n \frac{1-\alpha}{\widetilde{[n]}_q(k+1) - (k+\alpha)(1 - \lambda + \lambda \widetilde{[n]}_q)} z^n \,.
\end{aligned}
$$

Then

$$
\begin{aligned}
\sum_{n=2}^{\infty} \lambda_n \frac{1-\alpha}{\widetilde{[n]}_q(k+1) - (k+\alpha)} \cdot \frac{\widetilde{[n]}_q(k+1) - (k+\alpha)}{1-\alpha} &= \sum_{n=2}^{\infty} \lambda_n \\
&= \sum_{n=2}^{\infty} \lambda_n - \lambda_1 \\
&= 1 - \lambda_1 \leq 1.
\end{aligned}
$$

Thus we have $f \in UTS(q; k, \lambda, \alpha)$. ∎

Conversely, suppose that $f \in UTS(q; k, \lambda, \alpha)$. Since

$$
|a_n| \leq \frac{1-\alpha}{\widetilde{[n]}_q(k+1) - (k+\alpha)(1 - \lambda + \lambda \widetilde{[n]}_q)},
$$

we may set

$$
\lambda_n = \frac{\widetilde{[n]}_q(k+1) - (k+\alpha)(1 - \lambda + \lambda \widetilde{[n]}_q)}{1-\alpha} |a_n| \quad and \quad \lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n.
$$

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Then

$$
\begin{aligned}
f(z) &= z + \sum_{n=2}^{\infty} a_n z^n = z + \sum_{n=2}^{\infty} \lambda_n \frac{1-\alpha}{\widetilde{[n]_q}(k+1) - (k+\alpha)(1-\lambda+\lambda\widetilde{[n]_q})} z^n \\
&= z + \sum_{n=2}^{\infty} \lambda_n (z + f_n(z)) = z + \sum_{n=2}^{\infty} \lambda_n z + \sum_{n=2}^{\infty} \lambda_n f_n(z) \\
&= \left(1 - \sum_{n=2}^{\infty} \lambda_n\right) z + \sum_{n=2}^{\infty} \lambda_n f_n(z) = \lambda_1 z + \sum_{n=2}^{\infty} \lambda_n f_n(z) \\
&= \lambda_1 f_1(z) + \sum_{n=2}^{\infty} \lambda_n f_n(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)
\end{aligned}
$$

This completes the proof.

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10

ON SOLUTION OF SYSTEM OF INTEGRAL EQUATIONS VIA FIXED POINT METHOD

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ABSTRACT. In this article, following the approach of F-contractions, we establish a common fixed point theorem for a pair of self-mappings satisfying F-contraction of rational type in complete metric spaces. An example is constructed to illustrate this result. An application to system of integral equations is presented.

Keywords: metric space; fixed point; rational type F-contraction; integral equation.
AMS 2010 Subject Classification: 47H09; 47H10; 54H25.
[1]

1. INTRODUCTION AND PRELIMINARIES

Banach Contraction Principle has been extended and generalized in many directions (see [3, 7, 8, 9, 10]). One of the most interesting generalization of it was given by Wardowski [22]. Later on, Abbas et al.[1] further generalized the concept of F-contraction and proved certain fixed point results. Hussain and Salimi [13] introduced α-GF contraction with respect to a general family of functions G and established Wardowski type fixed point results in ordered metric spaces. Batra et al. [5, 6] extended the concept of F-contraction on graphs and altered distances. Batra also proved some fixed point and coincidence point results. Recently, Cosentino and Vetro [11] followed the approach of F-contraction and obtained some fixed point theorems for Hardy-Rogers-type self-mappings in complete metric spaces and complete ordered metric spaces.

In this article, following Cosentino and Vetro [11], we prove common fixed point theorems for a pair of self-mappings satisfying F-contraction of rational type in complete metric spaces. An example is constructed to illustrate this result. An application to system of Volterra type integral equations is presented.

Throughout this paper, we denote $(0, \infty)$ by $\mathbb{R}^+$, $[0, \infty)$ by $\mathbb{R}_0^+$, $(-\infty, +\infty)$ by $\mathbb{R}$ and the set of natural numbers by $\mathbb{N}$. The following concepts and results will be required for the proofs of main results.

Definition 1. A mapping $T : X \to X$ is said to be an F-contraction if it satisfies the following condition:

$$d(T(x), T(y)) > 0 \Rightarrow t + F(d(T(x), T(y))) \leq F(d(x, y)) \qquad (1.1)$$

for all $x, y \in X$ and some $t > 0$. Here $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying the following properties:

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(F_1) F is strictly increasing.

(F_2) For each sequence $\{a_n\}$ of positive numbers $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$.

(F_3) There exists $\theta \in (0,1)$ such that $\lim_{\alpha\to 0^+} (\alpha)^\theta F(\alpha) = 0$.

Wardowski [22] established the following result using F-contraction:

Theorem 1. [22] Let (X, d) be a complete metric space and $T : X \to X$ be an F-contraction. Then T has a unique fixed point $v \in X$ and for every $X_0 \in X$ a sequence $\{T^n(X_0)\}$ is convergent to v.

We denote by Δ_F the set of all functions satisfying the conditions $(F_1) - (F_3)$.

Example 1. [22] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be given by the formula $F(\alpha) = \ln \alpha$. It is clear that F satisfies $(F_1) - (F_3)$ for any $\kappa \in (0,1)$. Each mapping $T : X \to X$ satisfying (1.1) is an F-contraction such that

$$d(T(x), T(y)) \le e^{-\tau} d(x, y) \text{ for all } x, y \in X \text{ with } T(x) \ne T(y).$$

Obviously, for all $x, y \in X$ such that $T(x) = T(y)$, the inequality $d(T(x), T(y)) \le e^{-\tau} d(x, y)$ holds, that is, T is a Banach contraction.

Remark 1. From (F_1) and (1.1) it is easy to conclude that every F-contraction is necessarily continuous.

2. MAIN RESULT

We begin with the following definitions.

Definition 2. Let (X, d) be a metric space. A mapping $T : X \to X$ is called a rational type F-contraction if, for all $x, y \in X$, we have

$$\tau + F(d(T(x), T(y))) \le F(N(x, y)), \tag{2.1}$$

where $F \in \Delta_F$ and $\tau > 0$, and

$$N(x, y) = \max \left\{ d(x, y), \frac{d(x, T(x))d(y, T(y))}{1 + d(x, y)}, \frac{d(x, T(x))d(y, T(y))}{1 + d(T(x), T(y))} \right\}.$$

Definition 3. Let (X, d) be a metric space. Mappings $S, T : X \to X$ are called a pair of rational type F-contractions if for all $x, y \in X$, we have

$$\tau + F(d(S(x), T(y))) \le F(M(x, y)), \tag{2.2}$$

where $F \in \Delta_F$ and $\tau > 0$, and

$$M(x, y) = \max \left\{ d(x, y), \frac{d(x, S(x))d(y, T(y))}{1 + d(x, y)}, \frac{d(x, S(x))d(y, T(y))}{1 + d(S(x), T(y))} \right\}.$$

The following theorem is one of our main results.

Theorem 2. *Let (X, d) be a complete metric space and $S, T : X \to X$ be a pair of mappings such that*

(1) (S, T) *is a pair of continuous mappings,*

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(2) (S, T) *is a pair of rational type F-contractions.*
Then there exists a common fixed point v of a pair (S, T) in X.

Proof. We begin with the following observation:

$$M(x, y) = 0 \text{ if and only if } x = y \text{ is a common fixed point of } (S, T).$$

Indeed, if $x = y$ is a common fixed point of (S, T), then $T(y) = T(x) = x = y = S(y) = S(x)$ and

$$M(x, y) = \max\left\{d(x, x), \frac{d(x, S(x))d(y, T(y))}{1 + d(x, y)}, \frac{d(x, S(x))d(y, T(y))}{1 + d(S(x), T(y))}\right\} = 0.$$

Conversely, if $M(x, y) = 0$, it is easy to check that $x = y$ is a fixed point of S and T. In order to find common fixed points of S and T for the situation when $M(x, y) > 0$ for all $x, y \in X$ with $x \neq y$, we construct an iterative sequence $\{x_n\}$ of points in X such a way that $x_{2i+1} = S(x_{2i})$ and $x_{2i+2} = T(x_{2i+1})$ where $i = 0, 1, 2, \dots$. If $x_n \neq x_{n+1}$ for all $n \geq 0$, then from contractive condition (2.2), we get

$$F\left(d(x_{2i+1}, x_{2i+2})\right) = F\left(d(S(x_{2i}), T(x_{2i+1}))\right) \leq F\left(M(x_{2i}, x_{2i+1})\right) - \tau$$

for all $i \in \mathbb{N} \cup \{0\}$, where

$$M(x_{2i}, x_{2i+1}) = \max\left\{ \begin{array}{c} d(x_{2i}, x_{2i+1}), \dfrac{d(x_{2i}, S(x_{2i}))d(x_{2i+1}, T(x_{2i+1}))}{1 + d(x_{2i}, x_{2i+1})}, \\ \dfrac{d(x_{2i}, S(x_{2i}))d(x_{2i+1}, T(x_{2i+1}))}{1 + d(S(x_{2i}), T(x_{2i+1}))} \end{array} \right\}$$

$$= \max\left\{ d(x_{2i}, x_{2i+1}), \dfrac{d(x_{2i}, x_{2i+1})d(x_{2i+1}, x_{2i+2})}{1 + d(x_{2i}, x_{2i+1})}, \dfrac{d(x_{2i}, x_{2i+1})d(x_{2i+1}, x_{2i+2})}{1 + d(x_{2i+1}, x_{2i+2})} \right\}$$

$$\leq \max\left\{ d(x_{2i}, x_{2i+1}), d(x_{2i+1}, x_{2i+2}) \right\}.$$

If $M(x_{2i}, x_{2i+1}) = d(x_{2i+1}, x_{2i+2})$, then

$$F\left(d(x_{2i+1}, x_{2i+2})\right) \leq F\left(d(x_{2i+1}, x_{2i+2})\right) - \tau,$$

which is a contradiction due to F_1. Therefore,

$$F\left(d(x_{2i+1}, x_{2i+2})\right) \leq F\left(d(x_{2i}, x_{2i+1})\right) - \tau,$$

for all $i \in \mathbb{N} \cup \{0\}$. Hence

$$F\left(d(x_{n+1}, x_{n+2})\right) \leq F\left(d(x_n, x_{n+1})\right) - \tau, \tag{2.3}$$

for all $n \in \mathbb{N} \cup \{0\}$. By (2.3), we obtain

$$F\left(d(x_n, x_{n+1})\right) \leq F\left(d(x_{n-2}, x_{n-1})\right) - 2\tau.$$

Repeating these steps, we get

$$F\left(d(x_n, x_{n+1})\right) \leq F\left(d(x_0, x_1)\right) - n\tau. \tag{2.4}$$

From (2.4), we obtain $\lim_{n\to\infty} F\left(d(x_n, x_{n+1})\right) = -\infty$. Since $F \in \Delta_F$,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.5}$$

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From the property (F_3) of F-contraction, there exists $\kappa \in (0,1)$ such that

$$\lim_{n\to\infty} ((d(x_n, x_{n+1}))^\kappa F(d(x_n, x_{n+1}))) = 0. \tag{2.6}$$

By (2.4), for all $n \in \mathbb{N}$, we obtain

$$(d(x_n, x_{n+1}))^\kappa (F(d(x_n, x_{n+1})) - F(d(x_0, x_1))) \leq -(d(x_n, x_{n+1}))^\kappa n\tau \leq 0. \tag{2.7}$$

Considering (2.5), (2.6) and letting $n \to \infty$ in (2.7), we have

$$\lim_{n\to\infty} (n(d(x_n, x_{n+1}))^\kappa) = 0. \tag{2.8}$$

Since (2.8) holds, there exists $n_1 \in \mathbb{N}$, such that $n(d(x_n, x_{n+1}))^\kappa \leq 1$ for all $n \geq n_1$ or,

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{\kappa}}} \text{ for all } n \geq n_1. \tag{2.9}$$

Using (2.9), we get for $m > n \geq n_1$,

$$\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \cdots + d(x_{m-1}, x_m); \\
&= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \\
&\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}.
\end{aligned}$$

The convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{\kappa}}}$ entails $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, there exists $\upsilon \in X$ such that $x_n \to \upsilon$ as $n \to \infty$, moreover, $x_{2n+1} \to \upsilon$ and $x_{2n+2} \to \upsilon$.

Now the continuity of T implies

$$\upsilon = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} T(x_{2n+1}) = T(\lim_{n\to\infty} x_{2n+1}) = T(\upsilon).$$

Analogously, $\upsilon = S(\upsilon)$. Thus we have $S(\upsilon) = T(\upsilon) = \upsilon$. Hence (S, T) has a common fixed point. Now we show that υ is the unique common fixed point of S and T. Assume the contrary, that is, there exists $\omega \in X$ such that $\omega = T(\omega)$. From the contractive condition (2.2), we have

$$\tau + F(d(S(\upsilon), T(\omega))) \leq F(M(\upsilon, \omega)), \tag{2.10}$$

where

$$M(\upsilon, \omega) = \max\left\{ d(\upsilon, \omega), \frac{d(\upsilon, S(\upsilon))d(\omega, T(\omega))}{1 + d(\upsilon, y)}, \frac{d(\upsilon, S(\upsilon))d(\omega, T(\omega))}{1 + d(S(\upsilon), T(\omega))} \right\}.$$

From (2.10), we have

$$\tau + F(d(\upsilon, \omega)) \leq F(d(\upsilon, \omega)),, \tag{2.11}$$

which implies

$$d(\upsilon, \omega) < d(\upsilon, \omega),$$

which is a contradiction. Hence $\upsilon = \omega$ and υ is a unique common fixed point of a pair (S, T). $\qquad\square$

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Let us consider an example to illustrate Theorem 2.

Example 2. Let $X = [1, \infty]$ and $d(x, y) = |x - y|$. Then (X, d) is a complete metric space. Define the mappings $S, T : X \to X$ as follows:

$$S(x) = x^2 \text{ and } T(x) = x + 3 \text{ for all } x \in X.$$

Define the function $F : R^+ \to R$ by $F(x) = \ln(x)$ for all $x \in R^+ > 0$ and $\tau > 0$. Then the contractive condition (2.2) is satisfied. Indeed, for all $x, y \in X$, the following inequality

$$\tau + \ln(d(S(x), T(y))) \leq \ln(M(x, y))$$

holds. Particularly, for $x = 2$ and $y = 3$, we have

$$
\begin{aligned}
M(2, 3) &= \max \left\{ d(2, 3), \frac{d(2, S(2))d(3, T(3))}{1 + d(2, 3)}, \frac{d(2, S(2))d(3, T(3))}{1 + d(S2, T3)} \right\} \\
&= \max \{1, 3, 2\} = 3
\end{aligned}
$$

and

$$d(S2, T3) = d(4, 6) = 2.$$

Thus

$$\tau + \ln(d(S(2), T(3))) = \tau + \ln 2 \leq \ln(M(2, 3)) = \ln 3,$$

which implies

$$\tau + F(d(S(x), T(y))) \leq F(M(x, y)).$$

Hence all the hypotheses of Theorem 2 are satisfied and so (S, T) have a common fixed point.

By setting $S = T$, we obtain the following result.

Corollary 1. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that*

(1) *T is a continuous mapping,*
(2) *T is a rational type F-contraction.*

Then T has a unique fixed point v in X.

Remark 2. If we set $N(x, y) = \max \{d(x, y), d(x, T(x)), d(y, T(y))\}$ in (2.1), then Corollary 1 remains true. Similarly, if we set $M(x, y) = \max \{d(x, y), d(x, T(x)), d(y, S(y))\}$ in (2.2), then Theorem 2 remains true.

3. APPLICATION TO SYSTEM OF INTEGRAL EQUATIONS

Now we discuss an application of fixed point theorem, proved in the previous section, in solving the system of Volterra type integral equations. Such a system is given by the

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following equations

$$u(t) \;=\; f(t) + \int_0^t K_1(t, s, u(s))ds. \tag{3.1}$$

$$w(t) \;=\; f(t) + \int_0^t K_2(t, s, w(s))ds. \tag{3.2}$$

for all $t \in [0, a]$, and $a > 0$. We shall show, by using Theorem 2, that the solution of integral equations (3.1) and (3.2) exists. Let $C([0, a], \mathbb{R})$ be the space of all continuous functions defined on $[0, a]$. For $u \in C([0, a], \mathbb{R})$, define supremum norm as: $\|u\|_\tau = \sup_{t \in [0,a]} \{u(t)e^{-\tau t}\}$, where $\tau > 0$. Let $C([0, a], \mathbb{R})$ be endowed with the metric

$$d_\tau(u, v) = \sup_{t \in [0,a]} \| \, |u(t) - v(t)| \, e^{-\tau t}\|_\tau \tag{3.3}$$

for all $u, v \in C([0, a], \mathbb{R})$. Obviously, $C([0, a], \mathbb{R}, \|\cdot\|_\tau)$ is a Banach space.

Now we prove the following theorem to ensure the existence of solution of system of integral equations.

Theorem 3. *Assume the following conditions are satisfied:*
(i) $K_1, K_2 : [0, a] \times [0, a] \times \mathbb{R} \to \mathbb{R}$ and $f, g : [0, a] \to \mathbb{R}$ are continuous;
(ii) Define the operators

$$Su(t) \;=\; f(t) + \int_0^t K_1(t, s, u(s))ds,$$

$$Tu(t) \;=\; f(t) + \int_0^t K_2(t, s, u(s))ds,$$

and there exists $\tau \geq 1$ such that

$$|K_1(t, s, u) - K_2(t, s, v)| \leq \tau e^{-\tau}[M(u, v)]$$

for all $t, s \in [0, a]$ and $u, v \in C([0, a], \mathbb{R})$, where

$$M(u, v) = \max\{|u(t) - v(t)|, \frac{|u(t) - Su(t)| \, |v(t) - Tv(t)|}{1 + |u(t) - v(t)|}, \frac{|u(t) - Su(t)| \, |v(t) - Tv(t)|}{1 + |Su(t) - Tv(t)|}\}.$$

Then the system of integral equations given in (3.1) and (3.2) has a unique solution.

Proof. By assumption (ii), we have

$$
\begin{aligned}
|Tu(t) - Sv(t)| &= \int_0^t |K_1(t, s, u(s) - K_2(t, s, v(s)))| \, ds \\
&\leq \int_0^t \tau e^{-\tau}([M(u,v)]e^{-\tau s})e^{\tau s} ds \\
&\leq \int_0^t \tau e^{-\tau}\|M(u,v)\|_\tau e^{\tau s} ds \\
&\leq \tau e^{-\tau}\|M(u,v)\|_\tau \int_0^t e^{\tau s} ds \\
&\leq \tau e^{-2\tau}\|M(u,v)\|_\tau \frac{1}{\tau}e^{\tau t} \\
&\leq e^{-\tau}\|M(u,v)\|_\tau e^{\tau t}.
\end{aligned}
$$

This implies

$$
|Tu(t) - Sv(t)|\, e^{-\tau t} \leq e^{-\tau}\|M(u,v)\|_\tau,
$$

that is,

$$
\|Tu(t) - Sv(t)\|_\tau \leq e^{-\tau}\|M(u,v)\|_\tau.
$$

So we have

$$
\tau + \ln \|Tu(t) - Sv(t)\|_\tau \leq \ln \|M(u,v)\|_\tau.
$$

Thus all the conditions of Theorem 2 are satisfied. Hence the system of integral equations given in (3.1) and (3.2) has a unique common solution. $\qquad\square$

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On continuous Fibonacci functions

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Abstract. In this paper, we define and study a function $F : [0, \infty) \to \mathbf{R}$ and extensions $F : \mathbf{R} \to \mathbf{C}$, $\widetilde{F} : \mathbf{C} \to \mathbf{C}$ which are continuous and such that if $n \in \mathbf{Z}$, the set of all integers, then $F(n) = F_n$, the nth Fibonacci number based on $F_0 = F_1 = 1$. If x is not an integer and $x < 0$, then $F(x)$ may be a complex number, e.g., $F(-1.5) = \frac{1}{2} + i$. If $z = a + bi$, then $\widetilde{F}(z) = F(a) + iF(b-1)$ defines complex Fibonacci numbers. In connection with this function (and in general) we define a Fibonacci derivative of $f : \mathbf{R} \to \mathbf{R}$ as $(\triangle f)(x) = f(x+2) - f(x+1) - f(x)$ so that if $(\triangle f)(x) \equiv 0$ for all $x \in \mathbf{R}$, then f is a (real) Fibonacci function. A complex Fibonacci derivative $\widetilde{\triangle}$ is given as $\widetilde{\triangle} f(a + bi) = \triangle f(a) + i \triangle f(b-1)$ and its properties are discussed in same detail.

1. Introduction

Fibonacci-numbers have been studied in many different forms for centuries and the literature on the subject is consequently incredibly vast. One of the amazing qualities of these numbers is the variety of mathematical models where they play some sort of role and where their properties are of importance in elucidating the ability of the model under discussion to explain whatever implications are inherent in it. The fact that the ratio of successive Fibonacci numbers approaches the Golden ratio (section) rather quickly as they go to infinity probably has a good deal to do with the observation made in the previous sentence. Surveys and connections of the type just mentioned are provided in [1] and [2] for a very minimal set of examples of such texts, while in [7] Kim and Neggers showed that there is a mapping $D : M \to DM$ on means such that if M is a Fibonacci mean so is DM, that if M is the harmonic mean, then DM is the arithmetic mean, and if M is a Fibonacci mean, then $\lim_{n \to \infty} D^n M$ is the golden section mean. Hyers-Ulam stability of Fibonacci functional equation was studied in [6]. Surprisingly novel perspectives are still available and will presumably continue to be so for the future as long as mathematical investigations continue to be made. In the following the authors of the present paper are making another small offering at the same spot many previous contributors have visited in both recent and more distant pasts.

Han et al. [4] considered several properties of Fibonacci sequences in arbitrary groupoids. They discussed Fibonacci sequences in both several groupoids and groups. The present authors [8] introduced the notion of generalized Fibonacci sequences over a groupoid and discussed these in particular for the case where the groupoid contains idempotents and pre-idempotents. Using the notion of Smarandache-type P-algebras they obtained several relations on groupoids which are derived from generalized Fibonacci sequences.

In [5] Han et al. discussed Fibonacci functions on the real numbers $\mathbf{R}$, i.e., functions $f : \mathbf{R} \to \mathbf{R}$ such that for all $x \in \mathbf{R}$, $f(x+2) = f(x+1) + f(x)$, and developed the notion of Fibonacci functions using the concept of f-even and f-odd functions. Moreover, they showed that if f is a Fibonacci function then $\lim_{x \to \infty} \frac{f(x+1)}{f(x)} = \frac{1+\sqrt{5}}{2}$. KNS[4445] discussed Fibonacci functions using the (ultimately) periodicity and we also discuss the exponential

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Fibonacci functions. Especially, given a non-negative real-valued function, we obtain several exponential Fibonacci functions.

In this paper we are interested in describing properties of a function $F : \mathbf{R} \to \mathbf{C}$, the complex numbers, where $F(x) := (F_{\lfloor x-1 \rfloor})^{x - \lfloor x \rfloor} + F_{\lfloor x \rfloor} + (x - \lfloor x \rfloor - 1)$ and $\lfloor x \rfloor$ is the greatest integer function. It follows that if $x = n \in \mathbf{Z}$, then $\lfloor n \rfloor = n$ and $F(n) = (F_{n-1})^0 + F_n + (n - n - 1)$, where $(F_{n-1})^0 = 1$ implies $F(n) = F_n$ with $F_0 = F_1 = 1$ and $F_{-n} = (-1)^n F_{n-2}$ so that $F_{-1} = -F_{-1} = 0$ for example. If one computes F_1 directly, then $F_1 = F_0 + F_{-1}$ and $1 = 1 + 0$ yields $F_0 = 1$ as well.

It also follows that $F(x)$ is not itself a Fibonacci function in the sense that $F(x + 2) \neq F(x + 1) + F(x)$ if $x - \lfloor x \rfloor \neq 0$, i.e., if $x \notin \mathbf{Z}$. Nevertheless it is a very interesting function which allows one to define continuous Fibonacci numbers in an interesting manner. If one computes $F(-1.5)$ for example, then one finds that $F(-1.5) = \frac{1}{2} + i$, which suggests that the function $F(x)$ may deserve looking at in the context of the study of the zeta-function.

Given the fact that $F : \mathbf{R} \to \mathbf{C}$ and that $F(\mathbf{R}) \cap \mathbf{C} \neq \emptyset$, it also becomes a question of interest to study possible complex extensions of F to $\widetilde{F} : \mathbf{C} \to \mathbf{C}$ where $z = a + bi$ means $\widetilde{F}(z) = F(a) + iF(b - 1)$, so that if $z = a$, then $\widetilde{F}(z) = F(a) + iF(-1)$, where $F(-1) = 0$ implies $\widetilde{F}(a) = F(a)$, i.e., $\widetilde{F}$ is an extension of F. According to this construction we find that $\widetilde{F}(1 + i) = F(1) + iF(0) = 1 + i$ as one would hope.

A second component of the paper is a study of properties of the Fibonacci derivative $\triangle f$ of a function $f : \mathbf{R} \to \mathbf{R}$, given by the formula $(\triangle f)(x) = f(x + 2) - f(x + 1) - f(x)$, so that $(\triangle f)(x) \equiv 0$ means that f is then a Fibonacci function. If one notes that $(\triangle f)$ exists for any function $f : \mathbf{R} \to \mathbf{R}$, then a variety of questions may be asked about properties of this operator. For example $(\triangle f)(x) \equiv f(x)$ is a simple type of Fibonacci derivative equation with many types of solutions. Other analogs of standard differential equations may also be addressed.

Given that $F : \mathbf{R} \to \mathbf{C}$ is itself a function of interest in this context, $\triangle F : \mathbf{R} \to \mathbf{C}$ is looked at below. Finally, a complex version $\widetilde{\triangle} F : \mathbf{C} \to \mathbf{C}$ defined by $\widetilde{\triangle} F(z) = \widetilde{\triangle} F(a + bi) = \triangle F(a) + i \triangle F(b - 1)$, reduces for $b = 0$ to $\widetilde{\triangle} F(a) = \triangle F(a)$, i.e., $\widetilde{\triangle}$ extends the operator $\triangle$, and thus again it is a matter of interest to study the behavior of the function $\widetilde{\triangle} \widetilde{F}(z)$ for complex numbers.

Note that because of the very rich structure of relations among the coefficients F_n, we may expect there to eventually be development of an equally rich structure of relations among the various values of $F(x)$ (and $\widetilde{F}(z)$) extending the ones already known.

2. **Preliminaries**

A function f defined on the real numbers is said to be a *Fibonacci function* ([5]) if it satisfies the formula

$$f(x + 2) = f(x + 1) + f(x)$$

for any $x \in \mathbf{R}$, where $\mathbf{R}$ (as usual) is the set of real numbers.

Example 2.1. ([5]) Let $f(x) := a^x$ be a Fibonacci function on $\mathbf{R}$ where $a > 0$. Then $a^x a^2 = f(x + 2) = f(x + 1) + f(x) = a^x(a + 1)$. Since $a > 0$, we have $a^2 = a + 1$ and $a = \frac{1 + \sqrt{5}}{2}$. Hence $f(x) = (\frac{1 + \sqrt{5}}{2})^x$ is a Fibonacci function, and the unique Fibonacci function of this type on $\mathbf{R}$.

If we let $u_0 = 0, u_1 = 1$, then we consider the *full Fibonacci sequence*: $\cdots, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, \cdots$, i.e., $u_{-n} = (-1)^n u_n$ for $n > 0$, and $u_n = F_n$, the nth Fibonacci number.

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Example 2.2. ([5]) Let $\{u_n\}_{n=-\infty}^{\infty}$ and $\{v_n\}_{n=-\infty}^{\infty}$ be full Fibonacci sequences. We define a function $f(x)$ by $f(x) := u_{\lfloor x \rfloor} + v_{\lfloor x \rfloor}t$, where $t = x - \lfloor x \rfloor \in (0,1)$. Then $f(x+2) = u_{\lfloor x+2 \rfloor} + v_{\lfloor x+2 \rfloor}t = u_{(\lfloor x \rfloor+2)} + v_{(\lfloor x \rfloor+2)}t = (u_{(\lfloor x \rfloor+1)} + u_{\lfloor x \rfloor}) + (v_{(\lfloor x \rfloor+1)} + v_{\lfloor x \rfloor})t = f(x+1) + f(x)$ for any $x \in \mathbf{R}$. This proves that f is a Fibonacci function.

Note that if a Fibonacci function is differentiable on $\mathbf{R}$, then its derivative is also a Fibonacci function.

Proposition 2.3. ([5]) *Let f be a Fibonacci function. If we define $g(x) := f(x+t)$ where $t \in \mathbf{R}$ for any $x \in \mathbf{R}$, then g is also a Fibonacci function.*

For example, since $f(x) = (\frac{1+\sqrt{5}}{2})^x$ is a Fibonacci function, $g(x) = (\frac{1+\sqrt{5}}{2})^{x+t} = (\frac{1+\sqrt{5}}{2})^t f(x)$ is also a Fibonacci function where $t \in \mathbf{R}$.

Theorem 2.4. ([5]) *If $f(x)$ is a Fibonacci function, then the limit of the quotient $\frac{f(x+1)}{f(x)}$ exists.*

Corollary 2.5. ([5]) *If $f(x)$ is a Fibonacci function, then*

$$\lim_{x \to \infty} \frac{f(x+1)}{f(x)} = \frac{1+\sqrt{5}}{2}$$

3. Continuous Fibonacci functions

Given a real number $x \notin \mathbf{Z}$, we define a map $F(x)$ by

(1) $$F(x) := (F_{\lfloor x-1 \rfloor})^{x - \lfloor x \rfloor} + F_{\lfloor x \rfloor} + (x - \lfloor x \rfloor - 1)$$

where $\{F_n\}$ is the sequence of Fibonacci numbers with $F_0 = F_1 = 1$.

Example 3.1. We compute some $F(x)$ as follows: $F(1.5) = (F_{\lfloor 1.5-1 \rfloor})^{1.5 - \lfloor 1.5 \rfloor} + F_{\lfloor 1.5 \rfloor} + (1.5 - \lfloor 1.5 \rfloor - 1) = (F_0)^{0.5} + F_1 + (1.5 - 1 - 1) = 1.5$ and $F(1.75) = (F_0)^{0.75} + F_1 + (1.75 - \lfloor 1.75 \rfloor - 1) = 1.75$. Moreover, $F(3.25) = (F_{\lfloor 3.25-1 \rfloor})^{0.25} + F_{\lfloor 3.25 \rfloor} + (3.25 - \lfloor 3.25 \rfloor - 1) = (F_2)^{0.25} + F_3 + (3.25 - 3 - 1) = 4\sqrt{2} + 2.25$.

Theorem 3.2. *If we define $F(n) := F_n$, the nth Fibonacci function, then $F(x)$ is continuous for all $x \in \mathbf{R}$.*

Proof. Let $x := n + \epsilon$ where $n \in \mathbf{Z}$ and $0 < \epsilon < 1$. Then $F(x) = F_{n-1}^{\epsilon} + F_n + (\epsilon - 1)$. It follows that $\lim_{\epsilon \to 0^+} F(x) = \lim_{\epsilon \to 0^+}(F_{n-1}^{\epsilon} + F_n + (\epsilon - 1)) = F_n$. Let $x := n - \epsilon$ where $n \in \mathbf{Z}$ and $0 < \epsilon < 1$. Then

$$
\begin{aligned}
F(x) &= (F_{\lfloor n-\epsilon-1 \rfloor})^{n-\epsilon - \lfloor n-\epsilon \rfloor} + F_{\lfloor n-\epsilon \rfloor} + (n - \epsilon - \lfloor n - \epsilon \rfloor - 1) \\
&= (F_{n-2})^{n-\epsilon-(n-1)} + F_{n-1} + (n - \epsilon - (n-1) - 1) \\
&= (F_{n-2})^{1-\epsilon} + F_{n-1} - \epsilon
\end{aligned}
$$

It follows that $\lim_{\epsilon \to 0^+} F(x) = \lim_{\epsilon \to 0^+}[(F_{n-2})^{1-\epsilon} + F_{n-1} - \epsilon] = F_{n-2} + F_{n-1} = F_n$. $\square$

In Theorem 3.2, we call the real number $F(x)$ the (*continuous*) *Fibonacci function* at x.

Let $f : \mathbf{R} \to \mathbf{R}$ be a real-valued function. We shall consider the expression

$$(\triangle f)(x) := f(x+2) - f(x+1) - f(x)$$

to be the *Fibonacci derivative* of $f(x)$. For example, if $\Phi := \frac{1+\sqrt{5}}{2}$, then $f(x) = \Phi^x$ yields $(\triangle f)(x) = \Phi^{x+2} - \Phi^{x+1} - \Phi^x = \Phi^x(\Phi^2 - \Phi - 1) = 0$ and similarly, if f is any Fibonacci function, then $(\triangle f)(x) = 0$ for all $x \in \mathbf{R}$.

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We are next concerned with determining the Fibonacci derivative of $F(x)$ as we have defined above.

Theorem 3.3. *If $F(x)$ is a continuous Fibonacci function, then its Fibonacci derivative is*

(2)
$$(\triangle F)(x) = (F_{n+1})^\epsilon - (F_n)^\epsilon - (F_{n-1})^\epsilon - (\epsilon - 1)$$

where $x = n + \epsilon$, $n \in \mathbf{Z}, 0 < \epsilon < 1$

Proof. Given $x = n + \epsilon$, $n \in \mathbf{Z}, 0 < \epsilon < 1$, by using the formula (1), we obtain

$$
\begin{aligned}
(\triangle F)(x) &= F(n+2+\epsilon) - F(n+1+\epsilon) - F(n+\epsilon) \\
&= (F_{\lfloor n+1+\epsilon \rfloor})^\epsilon + F_{n+2} + (\epsilon - 1) \\
&\quad -(F_{\lfloor n+\epsilon \rfloor})^\epsilon - F_{n+1} - (\epsilon - 1) \\
&\quad -(F_{\lfloor n-1+\epsilon \rfloor})^\epsilon - F_n - (\epsilon - 1) \\
&= (F_{n+1})^\epsilon - (F_n)^\epsilon - (F_{n-1})^\epsilon - (\epsilon - 1)
\end{aligned}
$$

$\square$

Note that the map $F(x)$ in Theorem 3.3 is not necessarily a Fibonacci function.

The formula (2) is a function depending on ϵ, and so we need to know the value of $\frac{d}{d\epsilon}[\triangle F(x)]$.

$$
\begin{aligned}
\frac{d}{d\epsilon}[(\triangle F)(x)] &= \frac{d}{d\epsilon}[(F_{n+1})^\epsilon - (F_n)^\epsilon - (F_{n-1})^\epsilon - (\epsilon - 1)] \\
&= \ln(F_{n+1})(F_{n+1})^\epsilon - \ln(F_n)(F_n)^\epsilon - \ln(F_{n-1})(F_{n-1})^\epsilon - 1
\end{aligned}
$$

We denote $\frac{d}{d\epsilon}[\triangle F(x)]$ by $(\triangle F)'(x)$.

Proposition 3.4. *If $F(x)$ is a continuous Fibonacci function, then*

(3)
$$(\triangle(\triangle F))(x) = (F_{n+3})^\epsilon - 2(F_{n+2})^\epsilon - (F_{n+1})^\epsilon + 2(F_n)^\epsilon + (F_{n-1})^\epsilon + (\epsilon - 1)$$

Proof. It follows from the formula (2) that

$$
\begin{aligned}
(\triangle(\triangle F))(x) &= \triangle F(x+2) - \triangle F(x+1) - \triangle F(x) \\
&= (F_{n+3})^\epsilon - (F_{n+2})^\epsilon - (F_{n+1})^\epsilon - (\epsilon - 1) \\
&\quad -(F_{n+2})^\epsilon + (F_{n+1})^\epsilon - (F_n)^\epsilon + (\epsilon - 1) \\
&\quad -(F_{n+1})^\epsilon + (F_n)^\epsilon - (F_{n-1})^\epsilon + (\epsilon - 1) \\
&= (F_{n+3})^\epsilon - 2(F_{n+2})^\epsilon - (F_{n+1})^\epsilon + 2(F_n)^\epsilon + (F_{n-1})^\epsilon + (\epsilon - 1),
\end{aligned}
$$

proving the proposition. $\square$

Proposition 3.5. *$(\triangle F)(x)$ is a continuous function and $(\triangle F)(n) = 0$ for all $n \in \mathbf{Z}$.*

Proof. It follows from Theorem 3.3 that $(\triangle F)(x)$ is a continuous function. Since $\lim_{\epsilon \to 0}(\triangle F)(x) = \lim_{\epsilon \to 0}[(F_{n+1})^\epsilon - (F_n)^\epsilon - (F_{n-1})^\epsilon - (\epsilon - 1)] = 0$ and $\lim_{\epsilon \to 1}(\triangle F)(x) = \lim_{\epsilon \to 1}[(F_{n+1})^\epsilon - (F_n)^\epsilon - (F_{n-1})^\epsilon - (\epsilon - 1)] = F_{n+1} - F_n - F_{n-1} = 0$ for any $n \in \mathbf{Z}$. $\square$

Theorem 3.6. *If $F(x)$ is a continuous Fibonacci function, then there exists a $\gamma_n \in (n, n+1)$ such that $(\triangle F)'(\gamma_n) = 0$ for all $n \in \mathbf{Z}$.*

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Proof. Since $(\triangle F)(x)$ is a continuous function and $(\triangle F)(n) = (\triangle F)(n+1) = 0$, by Rolle's Theorem, there exists a $\gamma_n \in (n, n+1)$ such that $(\triangle F)'(\gamma_n) = \frac{d}{d\epsilon}(\triangle F)(\gamma_n) = 0$. $\square$

Theorem 3.7. *If $F(x)$ is a continuous Fibonacci function, then $(\triangle F)(x)$ is concave down.*

Proof. If we let $T(x) := \frac{d^2}{d\epsilon^2}[(\triangle F)(x)]$, then

$$
\begin{aligned}
T(x) &= \frac{d}{d\epsilon}[\frac{d}{d\epsilon}((\triangle F)(x))] \\
&= \frac{d}{d\epsilon}[\ln(F_{n+1})(F_{n+1})^\epsilon - \ln(F_n)(F_n)^\epsilon - \ln(F_{n-1})(F_{n-1})^\epsilon - 1] \\
&= \{\ln(F_{n+1})\}^2(F_{n+1})^\epsilon - \{\ln(F_n)\}^2(F_n)^\epsilon - \{\ln(F_{n-1})\}^2(F_{n-1})^\epsilon.
\end{aligned}
$$

Let n be very large so that $\frac{F_{k+1}}{F_k} = \Phi = \frac{1+\sqrt{5}}{2}$. It follows that

$$
\begin{aligned}
\frac{T(x)}{\{\ln(F_{n-1})\}^2(F_{n-1})^\epsilon} &= \{\frac{\ln(F_{n+1})}{\ln(F_{n-1})}\}^2[\frac{F_{n+1}}{F_{n-1}}]^\epsilon - \{\frac{\ln(F_n)}{\ln(F_{n-1})}\}^2[\frac{F_n}{F_{n-1}}]^\epsilon - 1 \\
&= \{\frac{2\ln\Phi + \ln(F_{n-1})}{\ln(F_{n-1})}\}^2(\Phi)^{2\epsilon} \\
&\quad - \{\frac{\ln\Phi + \ln(F_{n-1})}{\ln(F_{n-1})}\}^2(\Phi)^\epsilon - 1.
\end{aligned}
$$

If we let $n \to \infty$, then

(4)
$$
\lim_{n\to\infty} \frac{T(x)}{\{\ln(F_{n-1})\}^2(F_{n-1})^\epsilon} = \Phi^{2\epsilon} - \Phi^\epsilon - 1
$$

If we let $\epsilon := \frac{1}{2}$, then $\Phi^{2\epsilon} - \Phi^\epsilon - 1 = \Phi - \sqrt{\Phi} - 1$ and $\Phi - 1 = \frac{\sqrt{5}-1}{2}$, $\sqrt{\Phi} = \sqrt{\frac{1+\sqrt{5}}{2}}$, so that $(\Phi - 1)^2 - (\sqrt{\Phi})^2 = \frac{4-4\sqrt{5}}{4} < 0$, proving that $T(x) < 0$. This shows that $(\triangle F)(x)$ is concave down. $\square$

We discuss a Fibonacci derivative of a function which is not a Fibonacci function as below.

Proposition 3.8. *Let $f(x) := ax + b$ for some $a, b \in \mathbf{R}$. Then*

$$
\triangle^{k+1}(f)(x) = (-1)^{k+1}ax + (-1)^k[(k+1)a - b]
$$

Proof. The Fibonacci derivative $(\triangle f)(x)$ of $f(x) = ax + b$ is $f(x+2) - f(x+1) - f(x) = [a(x+2)+b] - [a(x+1)+b] - [ax+b] = -ax + a - b$. Similarly, we obtain $[\triangle^2(f)](x) = ax - 2a + b$, $[\triangle^3(f)](x) = -ax + 3a - b$ and $[\triangle^4(f)](x) = ax - 4a + b$. Assume $\triangle^k(f)(x) = (-1)^kax + (-1)^{k-1}(ka - b)$. Then

$$
\begin{aligned}
\triangle^{k+1}(f)(x) &= \triangle[\triangle^k(f)(x)] \\
&= \triangle^k(f)(x+2) - \triangle^k(f)(x+1) - \triangle^k(f)(x) \\
&= (-1)^ka(x+2) + (-1)^{k-1}(ka - b) \\
&\quad - [(-1)^ka(x+1) + (-1)^{k-1}(ka - b)] \\
&\quad - [(-1)^kax + (-1)^{k-1}(ka - b)] \\
&= (-1)^{k+1}ax + (-1)^k[(k+1)a - b].
\end{aligned}
$$

$\square$

Note that $[(\triangle^{k+3} + \triangle^{k+2}) - (\triangle^{k+1} + \triangle^k)](f)(x) = (-1)^{k+2}2a - (-1)^k2a = 0$.

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We need to find some conditions for a map $f : \mathbf{R} \to \mathbf{R}$ satisfying $(\triangle f)(x) = f(x)$.

Proposition 3.9. *Let* $f : \mathbf{R} \to \mathbf{R}$ *be a map. If it satisfies the condition either* (i) $2f(x) = f(x+2) - f(x+1)$ *or* (ii) $f(x+1) = -f(x)$ *for all* $x \in \mathbf{R}$, *then* $(\triangle f)(x) = f(x)$.

Proof. Straightforward. □

Example 3.10. If $f(x) := 2^x$, then $\frac{1}{2}(2^{x+2} - 2^{x+1}) = 2^x$ and hence $\triangle(2^x)(x) = 2^x$. If we let $f(x) := \sin(\pi x)$, then $f(x+1) = \sin \pi(x+1) = -\sin \pi x = -f(x)$ and hence $\triangle(\sin \pi x)(x) = \sin \pi x$.

Now, we define a function $f : \mathbf{R} \to \mathbf{R}$ satisfying the condition: $2f(x) = f(x+2) - f(x+1)$. If we make such a function, then it satisfies the condition $(\triangle f)(x) = f(x)$.

Suppose that one defines $f(x)$ for $0 \le x < 2$ at will. Then for $x \in [2,3)$ one defines $f(2+\theta) := 2f(\theta) + f(1+\theta)$ where $0 \le \theta < 1$. If $f(x)$ has been defined for $x \in [m-1, m)$, then we define $f(m+\theta) := 2f((m-2)+\theta) + f(m-1+\theta)$, where $0 \le \theta < 1$. Then $f(x)$ is uniquely determined on $[0, \infty)$. To define $f(x)$ for $[-1, 0)$, we have $f(1+\theta) = 2f(-1+\theta) + f(\theta)$ or $f(-1+\theta) = (f(1+\theta) - f(\theta))/2$, and thus $f(-m+\theta) = [f(-m+2+\theta) - f(-m+1+\theta)]/2$ inductively as well to obtain $f(x)$ defined on the entire real line.

Example 3.11. if $f(x) := 1$ on $[0, 2)$, then $f(2+\theta) = 2f(\theta) + f(1+\theta) = 2+1 = 3$, $f(3+\theta) = 2f(1+\theta) + f(2+\theta) = 5$ and $f(4+\theta) = 2f(2+\theta) + f(3+\theta) = 2 \cdot 3 + 5 = 11$. If we take $F_1^* = 1, F_2^* = 3, F_3^* = 5, F_4^* = 11, \cdots$, then $\{F_n^*\}$ is a Fibonacci sequence type satisfying $F_{n+1}^* = 2F_{n-1}^* + F_n^*$. We have $f(4+\theta) = 2f(2+\theta) + f(3+\theta) = 5f(1+\theta) + 6f(\theta) = F_3^* f(1+\theta) + 2F_2^* f(\theta)$. Assume that $f(n+\theta) = F_{n-1}^* f(1+\theta) + 2F_{n-2}^* f(\theta)$. Then

$$
\begin{aligned}
f(n+1+\theta) &= 2f(n-1+\theta) + 2f(n+\theta) \\
&= 2[F_{n-2}^* f(1+\theta) + 2F_{n-3}^* f(\theta)] + [F_{n-1}^* f(1+\theta) + 2F_{n-2}^* f(\theta)] \\
&= (2F_{n-2}^* + F_{n-1}^*)f(1+\theta) + 2(2F_{n-3}^* + F_{n-2}^*)f(\theta) \\
&= F_n^* f(1+\theta) + 2F_{n-1}^* f(\theta)
\end{aligned}
$$

4. Complex Fibonacci functions

Given the Fibonacci sequence $F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3, \cdots$, we may compute F_{-n}, $n = 1, 2, \cdots$ via the equation

$$F_{-n+2} = F_{-n+1} + F_{-n} \tag{5}$$

so that $F_{-1} = F_1 - F_0 = 0$, $F_{-2} = F_0 - F_{-1} = 1 - 0 = 1$, $F_{-3} = F_{-1} - F_{-2} = 0 - 1 = -1$, $F_4 = F_{-2} - F_{-3} = 1 - (-1) = 2$, $F_{-5} = F_{-3} - F_{-4} = -1 - 2 = (-1)^5 F_3$ and $F_{-6} = F_{-4} - F_{-5} = (-1)^6 F_4$. Assume $F_{-n} = (-1)^n F_{n-2}$ $(n \ge 5)$. Then $F_{-(n+1)} = F_{-(n+1)+2} - F_{-(n+1)+1} = (-1)^{n-1} F_{n-1} - (-1)^n F_n = (-1)^{n+1}(F_{n-3} + F_{n-2}) = (-1)^n F_{n-1}$ so that $F_{-2} = (-1)^2 F_0 = 1$, $F_{-3} = (-1)^1 F_1 = -1$. For F_{-1}, the formula would yield $F_{-1} = (-1)^1 F_{-1}$, i.e., $F_{-1} = -F_{-1}$ which would imply $F_{-1} = 0$ as well. Hence we have the result: for $n \ge 1$,

$$F_{-n} = (-1)^n F_{n-2} \tag{6}$$

Thus, we may apply the formula (1) for $x < 0$ as well. For example, $F(-1.5) = (F_{\lfloor -1.5-1 \rfloor})^{(-1.5-\lfloor -1.5 \rfloor)} + F_{\lfloor -1.5 \rfloor} + (-1.5 - \lfloor -1.5 \rfloor - 1) = (F_{-3})^{0.5} + F_{-2} + (-1.5 - (-2) - 1) = \sqrt{-1} + 1 - \frac{1}{2} = \frac{1}{2} + i$, i.e., $F(-1.5) = \frac{1}{2} + i$, the complex number.

On continuous Fibonacci functions

Example 4.1. We compute $F(-4 + \theta)$, $(0 \le \theta < 1)$ as follows:

$$
\begin{aligned}
F(-4 + \theta) &= (F_{\lfloor -4+\theta-1 \rfloor})^{-4+\theta-\lfloor -4+\theta \rfloor} + F_{\lfloor -4+\theta \rfloor} + (-4 + \theta - \lfloor -4 + \theta \rfloor - 1) \\
&= (F_{-5})^{\theta} + F_{-4} + (\theta - 1) \\
&= (-3)^{\theta} + 2 + (\theta - 1) \\
&= 3^{\theta}(-1)^{\theta} + \theta + 1
\end{aligned}
$$

where $(-1)^{\theta} = \exp^{\theta \ln(-1)}$, so that $\ln(-1) = Log(-1)$, where $Log(-1)$ is a "suitable branch of the Log-function". Note that $i^2 = -1$, so $Log(-1) = Log(i^2) = 2Log(i) = 2\ln(i)$. If we set $\ln i := a + bi$, then $\exp^{\ln i} = i = \exp^a \exp^b i$ and $a = 0$, $b = \frac{\pi}{2}$ yields $\ln i = \frac{\pi}{2}i$. Thus $2\ln i = \pi i = Log(-1)$ and $(-1)^{\theta} = \exp^{\pi \theta i}$, e.g., $\theta = 1$ yields $(-1)^1 = \exp^{\pi i} = -1$ as required. Thus we set $(-3)^{\theta} = 3^{\theta} \exp^{\pi \theta i} = 3^{\theta}(\cos \pi \theta + i \sin \pi \theta)$. Hence

$$
F(-4 + \theta) = (3^{\theta} \cos \pi \theta + 1 + \theta) + (3^{\theta} \sin \pi \theta)i
$$

Hence the evaluation of $F(x)$ for $x < 0$ may involve complex numbers.

Definition 4.2. Given a complex number $z := a + bi \in \mathbf{C}$, we define a map $\widetilde{F} : \mathbf{C} \to \mathbf{C}$ by

$$
\widetilde{F}(z) := F(a) + iF(b - 1)
$$

where $F(x)$ is the continuous Fibonacci function on $\mathbf{R}$. We call such a map $\widetilde{F}$ a *complex Fibonacci function*.

Given a real number $a \in \mathbf{R}$, we have $\widetilde{F}(a) = \widetilde{F}(a + i0) = F(a) + iF(-1) = F(a)$, so that $\widetilde{F}$ extends the function F already defined on $\mathbf{R}$ to the complex numbers $\mathbf{C}$.

Proposition 4.3. *Given a Gaussian integer $z = m + in$ $(m, n \in \mathbf{Z})$, we have*

$$
\widetilde{F}((m + 2) + (n + 3)i) = \widetilde{F}((m + 1) + (n + 2)i) + \widetilde{F}(m + (n + 1)i))
$$

Proof. Since $\widetilde{F}(m + ni) = F(m) + F(n - 1)i = F_m + F_{n-1}i$, we obtain $\widetilde{F}((m+2) + (n+3)i) = F_{m+2} + F_{n+2}i = (F_{m+1} + F_{n+1}i) + (F_m + F_n i) = \widetilde{F}((m + 1) + (n + 2)i) + \widetilde{F}(m + (n + 1)i))$. $\square$

Using the fact that the Fibonacci derivative is a linear mapping, we define the Fibonacci derivative for complex Fibonacci numbers as follows: Given $z = a + bi$ $(a, b \in \mathbf{R})$,

$$
\widetilde{\triangle}F(z) := \triangle F(a) + i \triangle F(b - 1)
$$

Proposition 4.4. *Given a complex Fibonacci function $\widetilde{F}(z)$, we have*

$$
\widetilde{\triangle}\widetilde{F}(z) = \widetilde{F}(z + 2(1 + i)) - \widetilde{F}(z + 1 + i) - \widetilde{F}(z)
$$

Proof. Given $z := a + bi \in \mathbf{R}$, we have

$$
\begin{aligned}
\widetilde{\triangle}\widetilde{F}(z) &= \triangle F(a) + i \triangle F(b - 1) \\
&= [F(a + 2) - F(a + 1) - F(a)] + i[F(b + 1) - F(b) - F(b - 1)] \\
&= \widetilde{F}((a + 2) + i(b + 2)) - \widetilde{F}((a + 1) + i(b + 1)) - \widetilde{F}(a + ib) \\
&= \widetilde{F}(z + 2(1 + i)) - \widetilde{F}(z + 1 + i) - \widetilde{F}(z),
\end{aligned}
$$

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i.e., $\widetilde{\triangle}$ is the complex Fibonacci derivative. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

5. **Concluding remark**

As was already been mentioned in the introduction and has been demonstrated in the paper, the extensions of $\{F_n\}_{n\in\mathbf{Z}}$ to $F(x)$ and $\widetilde{F}(z)$ show themselves to be rather remarkable functions. We should note that $F(0) = F(1) = 1$, only selects one among a family of functions of this type. Considering the usual property $\lim_{x\to\infty} \frac{f(x+1)}{f(x)} = \frac{1+\sqrt5}{2}$ for Fibonacci function $f : \mathbf{R} \to \mathbf{R}$, it is naturally of interest to check on a variety of limit problems of this type and discover properties and solutions to these problems among others.

6. **Future works**

One area which needs further investigation is the adapting of the theory developed above to general groupoids. In order to do this we will need to reduce formulas such as given above, which involve two (closely related) binary operations to those using only one such binary operation. Thus, consider an arbitrary groupoid $(X, *)$ and an element $a \in X$. We consider functions $f : X \to X$ such that $(\bigtriangledown_a^c f)(x) = (f(x) * f(x*a)) * f((x*a)*a) \equiv c$ and let these be (a, c)-$(X, *)$-Fibonacci-functions. That this is a true generalization can be seen as follows. Let $(X, *) = (\mathbf{R}, -)$ and let $a = 1, c = 0$. Then $(\bigtriangledown_1^0 f)(x) = f(x) - f(x-1) - f(x-2) = 0$ means $f(x) = f(x-1) + f(x-2)$, i.e., an $(1, 0)$-$(\mathbf{R}, -)$-Fibonacci-function as defined above. At the same time, if $(\triangle f)(x) = f(x+2) - f(x+1) - f(x)$, then $(\triangle f)(x-2) = f(x) - f(x-1) - f(x-2) = (\bigtriangledown_1^0 f)(x)$, so that $\triangle f$ is a translation of $\bigtriangledown_1^0 f$ on $(\mathbf{R}, -)$. Using these approach one hopes to develop very general Fibonacci properties which may be directly applied to a great variety of situations and thus also with an improved chance for possible applications, due to a much larger range of possible models which may be available. Therefore it is among our plans to follow through with this approach as well as what has been mentioned in the concluding remark section also.

Acknowledgement

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Decomposition and improved hyperbolic cross approximation of bivariate functions on $[0,1]^2$ *

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Abstract. For a bivariate function on the unit square, if we extend it robustly into a periodic function on the plane, then its Fourier coefficients decay very slowly due to the discontinuity on the boundary of the unit square, therefore, we need a lot of Fourier coefficients to reconstruct this bivariate function. In order to solve this problem, for any bivariate smooth function on the unit square, we introduce a Fourier expansion with a polynomial term and several polynomial factors such that the corresponding Fourier coefficients decay fast. Using this expansion, we can construct a good approximation tool for any bivariate function on the unit square.

1. Introduction

It is well-known that smooth period functions can be approximated well by Fourier series. But, for a bivariate function f on the unit square $[0,1]^2$, if we extend it into a periodic function on the plane, then its Fourier coefficients decay very slowly due to the discontinuity on the boundary of $[0,1]^2$. So we need a lot of Fourier coefficients to reconstruct this bivariate function. In order to reconstruct f by fewest Fourier coefficients, we will develop a new approximation tool in this paper. We first construct four simple univariate polynomial φ_i $(i = 1, ..., 4)$ of degree 3 which is independent of f. With the help of these polynomials, we express f into a sum:

$$f = f_1 + f_2 + f_3.$$

In this decomposition formula, f_1 is a linear combination of $\varphi_i(x)\varphi_j(y)$ $(i, j = 1, ..., 4)$, f_2 is a sum of products of a polynomial φ_i and a univariate function, and f_3 is a bivariate function whoes partial derivatives vanish on the boundary of $[0,1]^2$. Then we expand these univariate functions and this bivariate function into Fourier series, where the corresponding Fourier coefficients will decay fast. We call this process a Fourier expansion of f with a polynomial term and several polynomial factors. Based on this expansion, we can develop a good approximation tool of f by using the partial sums of these univariate Fourier series and the hyperbolic cross truncation of bivariate Fourier series. Precisely say, for a function f satisfying $\frac{\partial^4 f}{\partial x^2 \partial y^2} \in C([0,1]^2)$, if we use our

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approximation tool, then the approximation error is equivalent to $\frac{\log^4 N_d}{N_d^3}$, however, if we directly expand f into Fourier series, the approximation errors of its partial sums and hyperbolic cross truncation are equivalent to $\frac{1}{\sqrt{N_d}}$, $\frac{\log^2 N_d}{N_d}$, respectively, where N_d is the number of Fourier coefficients used. It is clear that our approximation tool is much better than traditional Fourier approximation. At the end of this paper, we will extend these results to the case of random processes.

Throughout this paper we always assume bivariate functions on $[0,1]^2$ are real-valued. Denote by $\{0,1\}^2$ vertexes of the unit square $[0,1]^2$ and by $\partial([0,1]^2)$ the boundary of $[0,1]^2$. We say $f \in C^{(2,2)}([0,1]^2)$ if $\frac{\partial^4 f}{\partial x^2 \partial y^2}$ is a continuous function on $[0,1]^2$. Denote by $s_N(f;x,y)$ the Fourier series partial sum of f on $[0,1]^2$, i.e.,

$$s_N(f;x,y) = \sum_{|m| \leq N} \sum_{|n| \leq N} c_{mn}(f) \, e^{2\pi imx} \, e^{2\pi iny}.$$

Denote by $s_n^{(h)}(f;x,y)$ the Fourier series hyperbolic cross truncation of f on $[0,1]^2$, i.e.,

$$s_N^{(h)}(f;x,y) = \sum_{|m|=0}^{N} c_{m0}(f) \, e^{2\pi imx}$$

$$+ \sum_{|n|=1}^{N} c_{0n} \, e^{2\pi iny} + \sum_{1 \leq |mn| \leq N} c_{mn}(f) \, e^{2\pi i(mx+ny)}.$$

$(1,1)$

For a random variable ξ, denote by $E[\xi]$ and $\mathrm{Var}(\xi)$ its expectation and variance, respectively. For two random variables ξ, η, denote by $\mathrm{Cov}(\xi, \eta)$ their covariance. We also always assume that ξ is real-valued. The concept of the random calculus may refer the reference [2,6].

This paper is organized as follows: In section 2 we give a decomposition formula of a bivariate function on $[0,1]^2$. In section 3 we discuss Fourier expansions with polynomial term and polynomial factors and estimate Fourier coefficients. In Section 4 we present a new approximation tool and estimate the corresponding approximation error. In Section 5 we generalize these results to random processes on $[0,1]^2$.

2. Decomposition of bivariate functions on the unit square

Suppose that $f(x,y)$ is a real-valued function on $[0,1]^2$ and $f \in C^{(2,2)}([0,1]^2)$. First we introduce four fundamental polynomials:

$$\varphi_1(x) = (1+2x)(x-1)^2 = -(3-2x)x^2 + 1,$$

$$\varphi_2(x) = (3-2x)x^2 = -\varphi_1(x) + 1,$$

$$\varphi_3(x) = x(x-1)^2,$$

$$\varphi_4(x) = x^2(x-1)$$

(2.1)

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satisfying the following conditions

$$
\begin{aligned}
\varphi_1(0) &= 1, & \varphi_1(1) &= \varphi_1'(0) = \varphi_1'(1) = 0, \\
\varphi_2(1) &= 1, & \varphi_2(0) &= \varphi_2'(0) = \varphi_2'(1) = 0, \\
\varphi_3'(0) &= 1, & \varphi_3(0) &= \varphi_3(1) = \varphi_3'(1) = 0, \\
\varphi_4'(1) &= 1, & \varphi_4(0) &= \varphi_4(1) = \varphi_4'(0) = 0.
\end{aligned}
\tag{2.2}
$$

Define a bivariate polynomial:

$$
\begin{aligned}
f_1(x,y) \;=\;& \sum_{\nu=0}^{1} \left(f(0,\nu)\varphi_1(x) + f(1,\nu)\varphi_2(x) \right) \varphi_{1+\nu}(y) \\
&+ \sum_{\nu=0}^{1} \left(\tfrac{\partial f}{\partial x}(0,\nu)\varphi_3(x) + \tfrac{\partial f}{\partial x}(1,\nu)\varphi_4(x) \right) \varphi_{1+\nu}(y) \\
&+ \sum_{\nu=0}^{1} \left(\tfrac{\partial f}{\partial y}(0,\nu)\varphi_1(x) + \tfrac{\partial f}{\partial y}(1,\nu)\varphi_2(x) \right) \varphi_{3+\nu}(y).
\end{aligned}
\tag{2.3}
$$

This is a linear combination of $\{\varphi_i(x)\varphi_j(y)\}_{i,j=1,\ldots,4}$ whose coefficients depend only on values of f and partial derivatives of f at vertexes $\{0,1\}^2 = \{(0,0),\,(0,1),\,(1,0),\,(1,1)\}$. Denote

$$
g(x,y) = f(x,y) - f_1(x,y).
$$

We construct a bivariate function f_2 which depends only on values of g and partial derivatives of g at the boundary of $[0,1]^2$. Define

$$
\begin{aligned}
f_2(x,y) =\;& g(x,0)\varphi_1(y) + g(x,1)\varphi_2(y) + g(0,y)\varphi_1(x) + g(1,y)\varphi_2(x) \\
&+ \tfrac{\partial g}{\partial x}(0,y)\varphi_3(x) + \tfrac{\partial g}{\partial x}(1,y)\varphi_4(x) + \tfrac{\partial g}{\partial y}(x,0)\varphi_3(y) + \tfrac{\partial g}{\partial y}(x,1)\varphi_4(y).
\end{aligned}
\tag{2.4}
$$

This is a sum of products of univariate functions and fundamental polynomials φ_i. Finally, we let

$$
f_3(x,y) = f(x,y) - f_1(x,y) - f_2(x,y).
$$

Then the following decomposition formula holds.

Theorem 2.1. Let $f \in C^{(2,2)}([0,1]^2)$. Then

$$
f(x,y) = f_1(x,y) + f_2(x,y) + f_3(x,y),
\tag{2.5}
$$

where f_1, f_2, and f_3 are stated as above and satisfy

(i) $f_1(x,y)$ is a bivariate polynomial and for $(x,y) \in \{0,1\}^2$,

$$
\begin{aligned}
f_1(x,y) &= f(x,y), & \tfrac{\partial f_1}{\partial x}(x,y) &= \tfrac{\partial f}{\partial x}(x,y), \\
\tfrac{\partial f_1}{\partial y}(x,y) &= \tfrac{\partial f}{\partial y}(x,y), & \tfrac{\partial^2 f_1}{\partial x \partial y}(x,y) &= \tfrac{\partial^2 f}{\partial x \partial y}(x,y),
\end{aligned}
$$

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i.e., $g = f - f_1$ satisfies

$$g(x, y) = \frac{\partial g}{\partial x}(x, y) = \frac{\partial g}{\partial y}(x, y) = \frac{\partial^2 g}{\partial x \partial y}(x, y) = 0, \qquad (x, y) \in \{0, 1\}^2;$$

(ii) the remainder $f_3 \in C^{(2,2)}([0, 1]^2)$ and for $(x, y) \in \partial([0, 1]^2)$,

$$f_3(x, y) = \frac{\partial f_3}{\partial x}(x, y) = \frac{\partial f_3}{\partial y}(x, y) = \frac{\partial^2 f_3}{\partial x \partial y}(x, y) = 0.$$

From the definitions of f_1, f_2, f_3, and (2.2), we can directly check (i) and (ii).

With the help of this decomposition formula, we give a Fourier expansion with polynomial factors and a new approximation tool such that we can reconstruct functions on $[0, 1]^2$ by fewest Fourier coefficients.

3. A kind of new Fourier expansions

In this section we give a Fourier expansion of the function on $[0, 1]^2$ with a polynomial term and several polynomial factors.

Suppose that $f \in C^{(2,2)}([0, 1]^2)$. By Theorem 2.1,

$$f(x, y) = f_1(x, y) + f_2(x, y) + f_3(x, y),$$

where f_1 is a polynomial which is stated in (2.3) and f_2 is stated in (2.4). We expand the first factor of each term in (2.4) into univariate Fourier series, such as we expand $g(x, \nu)$ ($\nu = 0, 1$) into the Fourier series:

$$g(x, \nu) = \sum_m a_m^{(\nu)} e^{2\pi i m x},$$

where $a_m^{(\nu)} = \int_0^1 g(x, \nu) e^{-2\pi i m x} dx$ and $\sum_m = \sum_{m=-\infty}^{\infty}$. Using the integration by parts, by Theorem 2.1 (i) and Riemann-Lebesgue lemma, the Fourier coefficients satisfy

$$a_m^{(\nu)} = \int_0^1 g(x, \nu) e^{-2\pi i m x} dx = \frac{1}{2\pi i m} \int_0^1 \frac{\partial g}{\partial x}(x, \nu) e^{-2\pi i m x} dx$$

$$= -\frac{1}{4\pi^2 m^2} \int_0^1 \frac{\partial^2 g}{\partial x^2}(x, \nu) e^{-2\pi i m x} dx = o\left(\frac{1}{m^2}\right).$$

Similarly, we expand $g(\nu, y)$, $\frac{\partial g}{\partial x}(\nu, y)$, $\frac{\partial g}{\partial y}(x, \nu)$ ($\nu = 0, 1$) into Fourier series:

$$g(\nu, y) = \sum_n b_n^{(\nu)} e^{2\pi i n y},$$

$$\frac{\partial g}{\partial x}(\nu, y) = \sum_n \alpha_n^{(\nu)} e^{2\pi i n y},$$

$$\frac{\partial g}{\partial y}(x, \nu) = \sum_m \beta_m^{(\nu)} e^{2\pi i m x}.$$

4

From this, we see that $f_2(x, y)$ can be expanded into the following Fourier series with polynomial factors φ_i:

$$
\begin{aligned}
f_2(x, y) = {} & \varphi_1(y) \sum_m a_m^{(0)} e^{2\pi i m x} + \varphi_2(y) \sum_m a_m^{(1)} e^{2\pi i m x} \\[4pt]
& + \varphi_1(x) \sum_n b_n^{(0)} e^{2\pi i n y} + \varphi_2(x) \sum_n b_n^{(1)} e^{2\pi i n y} \\[4pt]
& + \varphi_3(x) \sum_n \alpha_n^{(0)} e^{2\pi i n y} + \varphi_4(x) \sum_n \alpha_n^{(1)} e^{2\pi i n y} \\[4pt]
& + \varphi_3(y) \sum_m \beta_m^{(0)} e^{2\pi i m x} + \varphi_4(y) \sum_m \beta_m^{(1)} e^{2\pi i m x}
\end{aligned} \tag{3.1}
$$

and

$$
a_m^{(\nu)} = o\left(\tfrac{1}{m^2}\right), \qquad b_n^{(\nu)} = o\left(\tfrac{1}{n^2}\right),
$$

$$
\alpha_n^{(\nu)} = o\left(\tfrac{1}{n^2}\right), \qquad \beta_m^{(\nu)} = o\left(\tfrac{1}{m^2}\right).
$$

Finally, expand f_3 into a bivariate Fourier series:

$$
f_3(x, y) = \sum_{m,n} c_{mn}(f_3)\, e^{2\pi i (m x + n y)},
$$

where $c_{mn}(f_3) = \int_0^1 \int_0^1 f_3(x, y)\, e^{-2\pi i (m x + n y)}\, \mathrm{d}x\mathrm{d}y$ and $\sum_{m,n} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}$. By Theorem 2.1 (ii), the interior integral is equal to

$$
\int_0^1 f_3(x, y)\, e^{-2\pi i (m x + n y)}\, \mathrm{d}x = \frac{1}{(2\pi i m)^2} \int_0^1 \frac{\partial^2 f_3}{\partial x^2}(x, y)\, e^{-2\pi i m x}\, \mathrm{d}x.
$$

So the Fourier coefficients:

$$
c_{mn}(f_3) = \frac{1}{(2\pi i m)^2} \int_0^1 e^{-2\pi i m x} \left(\int_0^1 \frac{\partial^2 f_3}{\partial x^2}(x, y)\, e^{-2\pi i n y} \mathrm{d}y \right) \mathrm{d}x.
$$

Again, by Theorem 2.1 (ii),

$$
\begin{aligned}
c_{mn}(f_3) = {} & \frac{1}{(2\pi i m)^2} \int_0^1 e^{-2\pi i m x} \left(\frac{1}{2\pi i n} \int_0^1 \frac{\partial^3 f_3}{\partial x^2 \partial y}(x, y)\, e^{-2\pi i n y} \mathrm{d}y \right) \mathrm{d}x \\[4pt]
= {} & \frac{1}{16\pi^4 m^2 n^2} \int_0^1 \int_0^1 \frac{\partial^4 f_3}{\partial x^2 \partial y^2}(x, y)\, e^{-2\pi i (m x + n y)} \mathrm{d}x\mathrm{d}y.
\end{aligned} \tag{3.2}
$$

Summarizing up the above results, we get a Fourier expansion with polynomial term and polynomial factors, where Fourier coefficients decay fast.

Theorem 3.1. Let $f \in C^{(2,2)}([0,1]^2)$ and f_1, f_2, f_3 be stated as in (2.3)-(2.5), and $\varphi_i (i = 1, ..., 4)$ be stated as in (2.1). Then f can be expanded into Fourier series with a polynomial term and several polynomial factors as follows:

$$
\begin{aligned}
f(x, y) = {} & f_1(x, y) + \sum_{\nu=0}^{1} \left(\varphi_{1+\nu}(y) \sum_m a_m^{(\nu)} e^{2\pi i m x} + \varphi_{1+\nu}(x) \sum_n b_n^{(\nu)} e^{2\pi i n y} \right) \\[4pt]
& + \sum_{\nu=0}^{1} \left(\varphi_{3+\nu}(x) \sum_n \alpha_n^{(\nu)} e^{2\pi i n y} + \varphi_{3+\nu}(y) \sum_m \beta_m^{(\nu)} e^{2\pi i m x} \right) \\[4pt]
& + \sum_{m,n} c_{mn}(f_3)\, e^{2\pi i (m x + n y)},
\end{aligned} \tag{3.3}
$$

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where $f_1(x, y)$ is a polynomial which is stated in (2.3), both the second term and the third term are a combination of four univariate Fourier expansions and four fundamental polynomials, and for $\nu = 0, 1$,

$$a_m^{(\nu)} = \int_0^1 g(x, \nu) \, e^{-2\pi i m x} \mathrm{d}x = o\left(\tfrac{1}{m^2}\right),$$

$$b_n^{(\nu)} = \int_0^1 g(\nu, y) \, e^{-2\pi i n y} \mathrm{d}x = o\left(\tfrac{1}{n^2}\right),$$

$$\alpha_n^{(\nu)} = \int_0^1 \tfrac{\partial g}{\partial x}(\nu, y) \, e^{-2\pi i n y} \mathrm{d}y = o\left(\tfrac{1}{n^2}\right),$$

$$\beta_m^{(\nu)} = \int_0^1 \tfrac{\partial g}{\partial y}(x, \nu) \, e^{-2\pi i m x} \mathrm{d}x = o\left(\tfrac{1}{m^2}\right)$$

(3.4)

and $g(x, y) = f(x, y) - f_1(x, y)$, and the last term is a bivariate Fourier series of f_3 whose coefficients satisfy

$$c_{mn}(f_3) = o\left(\tfrac{1}{m^2 n^2}\right) \qquad (m \to \infty \text{ or } n \to \infty),$$

$$c_{0n}(f_3) = o\left(\tfrac{1}{n^2}\right) \qquad (n \to \infty),$$

$$c_{m0}(f_3) = o\left(\tfrac{1}{m^2}\right) \qquad (m \to \infty).$$

(3.5)

4. A new approximation tool

We want to reconstruct the bivariate function $f(x, y)$ by the fewest Fourier coefficients. For this purpose, we take partial sums of univariate Fourier series and hyperbolic cross truncation of the bivariate Fourier series of $\sum\limits_{m,n} c_{mn}(f_3) \, e^{2\pi i(mx+ny)}$ in (3.3), we get a hyperbolic cross truncation of Fourier expansion of f with a polynomial term and several polynomial factors. For an appropriate $N \in \mathbb{Z}_+$, we define such a combination of polynomials and trigonometric polynomials:

$$T_N^{(h)}(x, y) = f_1(x, y) + \sum_{\nu=0}^{1} \left(\varphi_{1+\nu}(y) \sum_{|n| \le N} a_n^{(\nu)} e^{2\pi i n x} + \varphi_{1+\nu}(x) \sum_{|n| \le N} b_n^{(\nu)} e^{2\pi i n y} \right)$$

$$+ \sum_{\nu=0}^{1} \left(\varphi_{3+\nu}(x) \sum_{|n| \le N} \alpha_n^{(\nu)} e^{2\pi i n y} + \varphi_{3+\nu}(y) \sum_{|n| \le N} \beta_n^{(\nu)} e^{2\pi i n x} \right) + s_N^{(h)}(f_3; x, y),$$

(4.1)

where the last term is the hyperbolic cross truncation of f_3 which is stated in (1.1).

From this and (3.3), it follows that

$$f(x, y) - T_N^{(h)}(x, y) = s_N^{(1)}(x, y) + s_N^{(2)}(x, y),$$

(4.2)

where

$$s_N^{(1)}(x, y) = \sum_{\nu=0}^{1} \left(\varphi_{1+\nu}(y) \sum_{|n|>N} a_n^{(\nu)} e^{2\pi i n x} + \varphi_{1+\nu}(x) \sum_{|n|>N} b_n^{(\nu)} e^{2\pi i n y} \right)$$

$$+ \sum_{\nu=0}^{1} \left(\varphi_{3+\nu}(x) \sum_{|n|>N} \alpha_n^{(\nu)} e^{2\pi i n y} + \varphi_{3+\nu}(y) \sum_{|n|>N} \beta_n^{(\nu)} e^{2\pi i n x} \right)$$

(4.3)

$$s_N^{(2)}(x, y) = f_3(x, y) - s_N^{(h)}(f_3, x, y).$$

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Consider the square error:

$$\| f - T_N^{(h)} \|_2^2 = \int_0^1 \int_0^1 |f(x,y) - T_N^{(h)}(f;x,y)|^2 \mathrm{d}x \mathrm{d}y.$$

By (4.2), we have

$$\| f - T_N^{(h)} \|_2^2 \leq 2 \| s_N^{(1)} \|_2^2 + 2 \| s_N^{(2)} \|_2^2 . \tag{4.4}$$

By (4.3) and $\| h_1(x) h_2(y) \|_2^2 = \| h_1(x) \|_2^2 \| h_2(y) \|_2^2$, where $\| h(t) \|_2^2 = \int_0^1 |h(t)|^2 \mathrm{d}t$, we have

$$
\begin{aligned}
\| s_N^{(1)} \|_2^2 \;\leq\; & 64 \sum_{\nu=0}^1 \left(\| \varphi_{1+\nu} \|_2^2 \| \sum_{|n|>N} a_n^{(\nu)} e^{2\pi i n x} \|_2^2 + \| \varphi_{1+\nu} \|_2^2 \| \sum_{|n|>N} b_n^{(\nu)} e^{2\pi i n y} \|_2^2 \right) \\
& + 64 \sum_{\nu=0}^1 \left(\| \varphi_{3+\nu} \|_2^2 \| \sum_{|n|>N} \alpha_n^{(\nu)} e^{2\pi i n y} \|_2^2 + \| \varphi_{3+\nu} \|_2^2 \| \sum_{|n|>N} \beta_n^{(\nu)} e^{2\pi i n x} \|_2^2 \right) .
\end{aligned}
$$

By the Parseval identity of the univariate Fourier series, we get

$$\| s_N^{(1)} \|_2^2 \leq 64 \sum_{\nu=0}^1 \left(\| \varphi_{1+\nu} \|_2^2 \sum_{|n|>N} (|a_n^{(\nu)}|^2 + |b_n^{(\nu)}|^2) + \| \varphi_{3+\nu} \|_2^2 \sum_{|n>N|} (|\alpha_n^{(\nu)}| + |\beta_n^{(\nu)}|^2) \right) .$$

Again, by (3.4),

$$\| s_N^{(1)} \|_2^2 = o \left(\sum_{|n|>N} \frac{1}{n^4} \right) = o \left(\frac{1}{N^3} \right) . \tag{4.5}$$

For $s_N^{(2)}$, by (1.1), we have

$$
s_N^{(2)}(f_3; x, y) = \sum_{|m|>N} c_{m0}(f_3) e^{2\pi i m x} + \sum_{|n|>N} \sum_{|m|=0}^{\infty} c_{mn}(f_3) e^{2\pi i (mx+ny)}
$$
$$
+ \sum_{|n|=1}^{N} \sum_{|m|>\frac{N}{|n|}} c_{mn}(f_3) e^{2\pi i (mx+ny)} .
$$

By the Parseval identity and (3.5), we have $\| s_N^{(2)} \|_2^2 = J_N^{(1)} + J_N^{(2)} + J_N^{(3)}$, where

$$J_N^{(1)} = \sum_{|m|>N} |c_{m0}(f_3)|^2 = o \left(\frac{1}{m^4} \right) = o \left(\frac{1}{N^3} \right),$$

$$J_N^{(2)} = \sum_{|n|>N} \sum_{|m|=1}^{\infty} |c_{mn}(f_3)|^2 = o \left(\sum_{|n|>N} \frac{1}{n^4} \right) \left(\sum_{|m|=1}^{\infty} \frac{1}{m^4} \right) = o \left(\frac{1}{N^3} \right),$$

$$J_N^{(3)} = \sum_{|n|=1}^{N} \sum_{|m|>\left[\frac{N}{|n|}\right]} |c_{mn}(f_3)|^2 = O \left(\sum_{|n|=1}^{N} \sum_{|m|>\frac{N}{|n|}} \frac{1}{m^4 n^4} \right)$$

$$= O \left(\sum_{|n|=1}^{N} \frac{1}{n^4} \sum_{|m|>\frac{N}{|n|}} \frac{1}{m^4} \right) = O \left(\sum_{|n|>N} \frac{1}{n^4} \frac{|n|^3}{N^3} \right) = O \left(\frac{1}{N^3} \log N \right).$$

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Therefore,

$$\| s_N^{(2)} \|_2^2 = O\left(\frac{1}{N^3} \log N \right).$$

From this and (4.4), and (4.5), it follows that

$$\| f - T_N^{(h)} \|_2^2 = O\left(\frac{\log N}{N^3} \right).$$

Since $\sum_{1 \le |mn| \le N} 1 \sim N \log N$, by (4.1), we see that the number N_d of Fourier coefficients in the approximation tool $T_N^{(h)}(x, y)$ is equivalent to $N \log N$, i.e.,

$$N_d \sim N \log N.$$

This implies the following:

Theorem 4.1. Let $f \in C^{(2,2)}([0,1]^2)$ and $T_N^{(h)}$ be the hyperbolic cross truncation of its Fourier expansion with a polynomial term and several polynomial factors which are stated in (4.1). Then

$$\| f - T_N^{(h)} \|_2^2 = O\left(\frac{\log^4 N_d}{N_d^3} \right), \tag{4.6}$$

where N_d is the number of Fourier coefficients in $T_N^{(h)}$.

For $f \in C^{(2,2)}([0,1]^2)$, consider the partial sums of the Fourier series of f:

$$s_N(x, y) = \sum_{|m| \le N} \sum_{|n| \le N} c_{mn}(f) \, e^{2\pi i (mx + ny)}.$$

By the Parseval identity,

$$\| f - s_N \|_2^2 = \sum_{|n| > N} \sum_{|m|=0}^{\infty} |c_{mn}(f)|^2 + \sum_{|n|=0}^{N} \sum_{|m| > N} |c_{mn}(f)|^2.$$

From this and $c_{m0}(f) = O\left(\frac{1}{m}\right)$, $c_{0n}(f) = O\left(\frac{1}{n}\right)$, $c_{mn}(f) = O\left(\frac{1}{mn}\right)$, it follows that

$$\| f - s_N \|_2^2 = O\left(\frac{1}{N} \right).$$

Note that the number N_d of Fourier coefficients in the partial sum is equivalent to N^2, i.e., $N_d \sim N^2$. So

$$\| f - s_N \|_2^2 = O\left(\frac{1}{\sqrt{N_d}} \right). \tag{4.7}$$

Consider the hyperbolic cross truncation of the Fourier series of f. By (1.1) and the Parseval identity,

$$\| f - s_N^{(h)} \|_2^2 = \sum_{|m|>N} |c_{m0}(f)|^2 + \sum_{|n|>N} \sum_{|m|=1}^{\infty} |c_{mn}(f)|^2 + \sum_{|n|=1}^{N} \sum_{|m| > \frac{N}{|n|}} |c_{mn}(f)|^2 = o\left(\frac{\log N}{N} \right). \tag{4.8}$$

From this and $N_d \sim N \log N$,

$$\| f - s_N^{(h)} \|_2^2 = o\left(\frac{\log^2 N_d}{N_d} \right).$$

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Comparing (4.7), (4.8) with (4.6), we see that the approximation tool $T_N^{(h)}$ is such that we reconstruct f by the fewest Fourier coefficients.

5. Uncertainty analysis

Now we extend the above results to the case of random processes. Suppose that f is a real-valued random process on $[0,1]^2$ and $f \in C^{(2,2)}([0,1]^2)$ (Refer to [6] for random calculus). Then the decomposition formula (2.5) is still valid:

$$f = f_1 + f_2 + f_3,$$

where f_1 and f_2 are stated as in (2.3) and (2.4), respectively, and f_3 is the residual. However, now f_1 is a random polynomial, f_2 is a sum of products of univariate random processes and fundamental polynomials φ_i. Theorem 2.1 and the expansion (3.3) are still valid. However, Fourier coefficients in (3.3): $a_m^{(\nu)}, b_n^{(\nu)}, \alpha_n^{(\nu)}, \beta_m^{(\nu)}$ $(\nu = 0, 1)$, and $c_{mn}(f_3)$ are now all random variables. Consider their expectations and variances. Note that

$$a_m^{(\nu)} = \int_0^1 g(x, \nu) e^{-2\pi i m x} \mathrm{d}x \qquad (\nu = 0, 1). \tag{5.1}$$

Since the expectation and the integral can be exchanged,

$$E[a_m^{(\nu)}] = \int_0^1 E[g(x, \nu)] e^{-2\pi i m x} \mathrm{d}x \qquad (\nu = 0, 1). \tag{5.2}$$

Since the random process $g = f - f_1$ belongs to $C^{(2,2)}([0,1]^2)$ and the expectation and the partial derivatives can be exchanged, the deterministic function $E[g(x, y)] \in C^{(2,2)}([0,1]^2)$. Noticing that Theorem 2.1 is still valid, for $(x, y) \in \{0, 1\}^2$,

$$E[g(x, y)] = E\left[\frac{\partial g}{\partial x}(x, y)\right] = E\left[\frac{\partial g}{\partial y}(x, y)\right] = E\left[\frac{\partial^2 g}{\partial x \partial y}(x, y)\right] = 0.$$

Exchanging the expectation and the partial derivatives, for $(x, y) \in \{0, 1\}^2$, we get

$$E[g(x, y)] = \frac{\partial}{\partial x} E[g(x, y)] = \frac{\partial}{\partial y} E[g(x, y)] = \frac{\partial^2}{\partial x \partial y} E[g(x, y)] = 0.$$

Therefore,

$$
\begin{aligned}
E[a_m^{(\nu)}] &= -\frac{1}{2\pi i m} E[g(x, \nu)]\big|_{x=0}^1 + \frac{1}{2\pi i m} \int_0^1 \frac{\partial}{\partial x}(E[g(x, \nu)]) e^{-2\pi i m x} \mathrm{d}x \\
&= -\frac{1}{(2\pi i m)^2} \frac{\partial}{\partial x} E[g(x, \nu)]\big|_{x=0}^1 + \frac{1}{(2\pi i m)^2} \int_0^1 \frac{\partial^2}{\partial x^2}(E[g(x, \nu)]) e^{-2\pi i m x} \mathrm{d}x \\
&= \frac{1}{(2\pi i m)^2} \int_0^1 \frac{\partial^2}{\partial x^2}(E[g(x, \nu)]) e^{-2\pi i m x} \mathrm{d}x = o\left(\frac{1}{m^2}\right) \qquad (\nu = 0, 1).
\end{aligned}
$$

This implies that

$$E[a_m^{(\nu)}] \leq \frac{1}{4\pi^2 m^2} \max_{0 \leq x \leq 1} |\frac{\partial^2}{\partial x^2} E[g(x, \nu)]| \qquad (\nu = 0, 1).$$

Similarly, we compute $E[b_n^{(\nu)}]$, $E[\alpha_n^{(\nu)}]$, $E[\beta_m^{(\nu)}]$, and $E[c_{mn}(f_3)]$. So we have the following:

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Theorem 5.1. Let f be a random process on $[0,1]^2$ and $f \in C^{(2,2)}([0,1]^2)$. Then, in the Fourier expansion (3.3) with a random polynomial and several random polynomial factors, Fourier coefficients satisfy

$$E[a_m^{(\nu)}] = o\left(\tfrac{1}{m^2}\right), \qquad E[a_m^{(\nu)}] \le \tfrac{1}{4\pi^2 m^2} \max_{0 \le x \le 1} |\tfrac{\partial^2}{\partial x^2} E[g(x,\nu)]|,$$

$$E[b_n^{(\nu)}] = o\left(\tfrac{1}{n^2}\right), \qquad E[b_n^{(\nu)}] \le \tfrac{1}{4\pi^2 n^2} \max_{0 \le y \le 1} |\tfrac{\partial^2}{\partial y^2} E[g(\nu,y)]|,$$

$$E[\alpha_n^{(\nu)}] = o\left(\tfrac{1}{n^2}\right), \qquad E[\alpha_n^{(\nu)}] \le \tfrac{1}{4\pi^2 n^2} \max_{0 \le y \le 1} |\tfrac{\partial^3}{\partial x \partial y^2} E[g(\nu,y)]|,$$

$$E[\beta_m^{(\nu)}] = o\left(\tfrac{1}{m^2}\right), \qquad E[\beta_m^{(\nu)}] \le \tfrac{1}{4\pi^2 m^2} \max_{0 \le x \le 1} |\tfrac{\partial^3}{\partial x^2 \partial y} E[g(x,\nu)]| \qquad (\nu = 0, 1),$$

$$E[c_{mn}(f_3)] = o\left(\tfrac{1}{m^2 n^2}\right), \qquad E[c_{mn}(f_3)] \le \tfrac{1}{16\pi^4 m^2 n^2} \max_{0 \le x,y \le 1} |\tfrac{\partial^4}{\partial x^2 \partial y^2} E[f_3(x,y)]|,$$

where $g = f - f_1$ and f_1, f_2 are stated as above.

Now we consider the variances of Fourier coefficients in (3.3).

Since g is a real-valued, by (5.1), we deduce that for $\nu = 0, 1$,

$$|a_m^{(\nu)}|^2 = \int_0^1 \int_0^1 g(x,\nu) g(t,\nu) \, e^{-2\pi i m(x-t)} \mathrm{d}x \mathrm{d}t,$$

$$E[\,|a_m^{(\nu)}|^2\,] = \int_0^1 \int_0^1 E[\,g(x,\nu) g(t,\nu)\,] \, e^{-2\pi i m(x-t)} \mathrm{d}x \mathrm{d}t,$$

Since $E[g(x,\nu)g(t,\nu)] \in C^{(2,2)}([0,1]^2)$, by Theorem 2.1 (i) and using integration by parts, it follows that

$$E[\,|a_m^{(\nu)}|^2\,] = \frac{1}{16\pi^4 m^4} \int_0^1 \int_0^1 \frac{\partial^4}{\partial x^2 \partial t^2} E[g(x,\nu) g(t,\nu)] \, e^{-2\pi i m(x-t)} \mathrm{d}x \mathrm{d}t \qquad (\nu = 0, 1), \tag{5.3}$$

and so

$$E[\,|a_m^{(\nu)}|^2\,] = o\left(\frac{1}{m^4}\right) \qquad (\nu = 0, 1).$$

Noticing that $\mathrm{Var}(a_m^{(\nu)}) = E[\,|a_m^{(\nu)}|^2\,] - (\,E[a_m^{(\nu)}]\,)^2$, we get

$$\mathrm{Var}(a_m^{(\nu)}) \le E[(a_m^{(\nu)})^2] = o\left(\frac{1}{m^4}\right) \qquad (\nu = 0, 1). \tag{5.4}$$

From (5.2), it follows that

$$(E[\,a_m^{(\nu)}\,])^2 = \frac{1}{16\pi^4 m^4} \int_0^1 \int_0^1 \frac{\partial^4}{\partial x^2 \partial t^2} (E[g(x,\nu)] E[g(t,\nu)]) \, e^{-2\pi i m(x-t)} \mathrm{d}x \mathrm{d}t \qquad (\nu = 0, 1). \tag{5.5}$$

Again, by the covariance formula: $\mathrm{Cov}(g(x,\nu), \, g(t,\nu)) = E[g(x,\nu)g(t,\nu)] - E[g(x,\nu)]E[g(t,\nu)]$, we have

$$\mathrm{Var}(a_m^{(\nu)}) = \frac{1}{16\pi^4 m^4} \int_0^1 \int_0^1 \frac{\partial^4}{\partial x^2 \partial t^2} \mathrm{Cov}(g(x,\nu), \, g(t,\nu)) \, e^{-2\pi i m(x-t)} \mathrm{d}x \mathrm{d}t \qquad (\nu = 0, 1).$$

Similarly, we compute $\mathrm{Var}(b_n^{(\nu)}), \mathrm{Var}(\alpha_n^{(\nu)}), \mathrm{Var}(\beta_m^{(\nu)})$, and $\mathrm{Var}(c_{mn}(f_3))$. So we have

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Theorem 5.2. Under conditions of Theorem 5.1, for $\nu = 0, 1$, we have

$$\mathrm{Var}(a_m^{(\nu)}) \leq \frac{1}{16\pi^4 m^4} \max_{0 \leq x, t \leq 1} |\frac{\partial^4}{\partial x^2 \partial t^2} \mathrm{Cov}(g(x, \nu),\ g(t, \nu))|,$$

$$\mathrm{Var}(b_n^{(\nu)}) \leq \frac{1}{16\pi^4 n^4} \max_{0 \leq y, t \leq 1} |\frac{\partial^4}{\partial y^2 \partial t^2} \mathrm{Cov}(g(\nu, y),\ g(\nu, t))|,$$

$$\mathrm{Var}(\alpha_n^{(\nu)}) \leq \frac{1}{16\pi^4 n^4} \max_{0 \leq y, t \leq 1} |\frac{\partial^4}{\partial y^2 \partial t^2} \mathrm{Cov}\left(\frac{\partial g}{\partial x}(\nu, y),\ \frac{\partial g}{\partial x}(\nu, t)\right)|,$$

$$\mathrm{Var}(\beta_m^{(\nu)}) \leq \frac{1}{16\pi^4 m^4} \max_{0 \leq x, t \leq 1} |\frac{\partial^4}{\partial x^2 \partial t^2} \mathrm{Cov}\left(\frac{\partial g}{\partial y}(x, \nu),\ \frac{\partial g}{\partial y}(t, \nu)\right)|,$$

$$\mathrm{Var}(c_{mn}(f_3)) \leq \frac{1}{256\pi^8 m^4 n^4} \max_{0 \leq x, y, t, s \leq 1} |\mathrm{Cov}\left(\frac{\partial^4 f_3}{\partial x^2 \partial y^2}(x, y),\ \frac{\partial^4 f_3}{\partial t^2 \partial s^2}(t, s)\right)|,$$

Similar to (5.5), we can obtain that the second-order moments are as follows. For $\nu = 0, 1$,

$$E[|a_m^{(\nu)}|^2] = o\left(\frac{1}{m^4}\right), \qquad E[|b_n^{(\nu)}|^2] = o\left(\frac{1}{n^4}\right),$$

$$E[|\alpha_n^{(\nu)}|^2] = o\left(\frac{1}{n^4}\right), \qquad E[|\beta_m^{(\nu)}|^2] = o\left(\frac{1}{m^4}\right), \tag{5.6}$$

$$E[|c_{mn}(f_3)|^2] = o\left(\frac{1}{m^4 n^4}\right).$$

Finally, for a random process, we still define an approximation tool $T_N^{(h)}(x, y)$ as in (4.1). Now $T_N^{(h)}(x, y)$ is a combination of random polynomials of degree 3 and random trigonometric polynomials of degree N. Using the Parseval identity, by (5.6), we have the following:

Theorem 5.3. Let f be a random process on $[0, 1]^2$ and $f \in C^{(2,2)}([0, 1]^2)$, and $T_N^{(h)}$ be stated as above. Then the mean square error of approximation by $T_N^{(h)}$ satisfies

$$E[\parallel f - T_N^{(h)} \parallel_2^2] = o\left(\frac{\log^4 N_d}{N_d^3}\right),$$

where N_d is the number of Fourier coefficients in $T_N^{(h)}$.

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Gronwall-Bellman type inequalities for the distributional Henstock-Kurzweil integral and applications *

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Abstract: This paper is devoted to studying the Gronwall-Bellman type inequalities involving the distributional Henstock-Kurzweil integral. Moreover, an linear differential equation with distributional coefficients is considered as an application.

Keywords: distributional derivative, distributional Henstock-Kurzweil integral, Gronwall-Bellman inequality.

MSC 2010: 26D15, 26A39, 46G12.

1 Introduction

It is well-known that the Gronwall-Bellman inequality (also called the Gronwall's lemma or the Gronwall's inequality) has played a fundamental role in the study of the qualitative behaviour of solutions of differential and integral equations.

In 1919, T. H. Gronwall [1] firstly established the following integral inequality.

Lemma 1.1. *Let $f(t)$ be a continuous function defined on $[a, a + \delta] \subset \mathbb{R}$, for which the inequality*

$$0 \leq f(t) \leq \int_a^t (\alpha f(s) + \beta)ds, \quad t \in [a, a + \delta] \tag{1.1}$$

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holds, where α, β, δ are nonnegative constants. Then

$$0 \le f(t) \le \beta\delta \exp(\alpha\delta), \quad t \in [a, a+\delta]. \tag{1.2}$$

In 1943, R. Bellman [2] generalized Lemma 1.1 to the following result.

Lemma 1.2. *Let $f(t)$ and $g(t)$ be nonnegative, continuous functions on $[a,b] \subset \mathbb{R}$, for which the inequality*

$$0 \le f(t) \le \eta + \int_a^t g(s)f(s)ds, \quad t \in [a,b] \tag{1.3}$$

holds, where η is a nonnegative constant. Then

$$0 \le f(t) \le \eta \exp\left(\int_a^t g(s)ds \right), \quad t \in [a,b]. \tag{1.4}$$

Because of the importance of this inequality, over the years investigators have discovered many useful generalizations in order to achieve a diversity of desired goals in various branches of differential and integral equations [3–18].

However, almost all the generalizations are based on continuous functions under Riemann and Lebesgue integrals, they are not applicable to generalized ordinary differential equations [19, 20]. This did not really change until Š. Schwabik [11] presented the Gronwall-Bellman inequality for the Henstock-Kurzweil integral in 1985, while K. Ostaszewski and J. Sochacki [12] gave a simpler and significant proof in 1987. As for the Gronwall-Bellman type inequalities for the Stieltjes integrals, we refer the reader to [14–18].

In this paper, we study the Gronwall-Bellman type inequalities for the distributional Henstock-Kurzweil integral, which defined by using Schwartz distributional derivative. It is a very wide integral form including the Riemann integral, the Lebesgue integral, and the Henstock-Kurzweil integral (see [19–22, 27–30] for details). The space of such integrable distributions, denoted by D_{HK}, is a completion of the space of Henstock-Kurzweil integrable functions (shortly, HK).

This paper is organized as follows. Section 2 is devoted to the basic notations of the distributional Henstock-Kurzweil integral. Section 3 contains our main results on the Gronwall-Bellman type inequalities involving the distributional Henstock-Kurzweil integral, while Section 4 sets forth an application to an linear differential equation with distributional coefficients.

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2   The Distributional Henstock-Kurzweil Integral

Let (a, b) be an open interval in $\mathbb{R}$, we define

$$\mathcal{D}((a,b)) = \{\phi : (a,b) \to \mathbb{R} \mid \phi \in C_c^\infty \text{ and } \phi \text{ has a compact support in } (a,b)\}.$$

The distributions on (a, b) are defined to be the continuous linear functionals on $\mathcal{D}((a,b))$. The dual space of $\mathcal{D}((a,b))$ is denoted by $\mathcal{D}'((a,b))$.

For all $f \in \mathcal{D}'((a,b))$, we define the distributional derivative f' of f to be a distribution satisfying $\langle f', \phi \rangle = -\langle f, \phi' \rangle$, where $\phi \in \mathcal{D}((a,b))$ is a test function. Further, we write distributional derivative as f' and its pointwise derivative as $f'(t)$ where $t \in \mathbb{R}$. From now on, all derivative in this paper will be distributional derivatives unless stated otherwise.

Denote the space of continuous functions on $[a, b]$ by $C[a, b]$. Let

$$C_0 = \{F \in C[a, b] : F(a) = 0\}. \tag{2.1}$$

Then C_0 is an Banach space under the norm

$$\|F\|_\infty = \sup_{t \in [a,b]} |F(t)| = \max_{t \in [a,b]} |F(t)|.$$

Definition 2.1 ([29, Definition 1]). A distribution $f \in \mathcal{D}'((a,b))$ is said to be Henstock–Kurzweil integrable (shortly D_{HK}) on an interval $[a, b]$ if there exists a continuous function $F \in C_0$ such that $F' = f$, i.e., the distributional derivative of F is f. The distributional Henstock–Kurzweil integral of f on $[a, b]$ is denoted by $\int_a^b f(t)dt = F(b) - F(a)$. The function F is called the primitive of f.

For every $f \in D_{HK}$, $\phi \in \mathcal{D}((a,b))$, we write

$$\langle f, \phi \rangle = \int_a^b f(t)\phi(t)dt = -\int_a^b F(t)\phi'(t)dt.$$

The distributional Henstock–Kurzweil integral is very wide and it includes Riemann integral, Lebesgue integral, Henstock–Kurzweil integral, restricted and wide Denjoy integral (see [21, 22, 27–29]). From now on, we write " $\int_a^b f(t)dt$ " as " $\int_a^b f$ " for short.

For $f \in D_{HK}$, define the Alexiewicz norm in D_{HK} as

$$\|f\| = \|F\|_\infty = \sup_{t \in [a,b]} |F(t)| = \max_{t \in [a,b]} |F(t)|.$$

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Under the Alexiewicz norm, D_{HK} is a Banach space, see [28, Theorem 2].

For $F \in C_0$, the positive part $F^+ = \max_{t \in [a,b]}\{F(t), 0\}$, the negative part $F^- = \max_{t \in [a,b]}\{-F(t), 0\}$, and hence $F = F^+ - F^-$ and the absolute value $|F| = F^+ + F^-$. Moreover, F^+, F^-, $|F|$ all belong to C_0. Let $f \in D_{HK}$ with the primitive $F \in C_0$, as in [28], define

$$f^+ = (F^+)', \quad f^- = (F^-)', \quad |f| = |F|'. \tag{2.2}$$

Then,

$$f = f^+ - f^-, \quad |f| = f^+ + f^-. \tag{2.3}$$

In C_0 there exists a pointwise order: for $F, G \in C_0$, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in [a,b]$. For $f, g \in D_{HK}$ with primitives $F, G \in C_0$, respectively, we say

$$f \overset{(p)}{\preceq} g \ (\text{or } g \overset{(p)}{\succeq} f) \quad \text{if and only if} \quad F(t) \leq G(t), \ \forall t \in [a,b], \tag{2.4}$$

and

$$f \overset{(m)}{\preceq} g \ (\text{or } g \overset{(m)}{\succeq} f) \quad \text{if and only if} \quad \int_I f \leq \int_I g, \tag{2.5}$$

where I is arbitrary subinterval of $[a,b]$. Obviously,

$$f \overset{(m)}{\preceq} g \quad \Rightarrow \quad f \overset{(p)}{\preceq} g, \tag{2.6}$$

but the converse is not true. Particularly, if f, g are functions, then

$$f(t) \leq g(t) \ (\forall t \in [a,b]) \quad \Leftrightarrow \quad f \overset{(m)}{\preceq} g \quad \Rightarrow \quad f \overset{(p)}{\preceq} g. \tag{2.7}$$

Lemma 2.2. *Let $f, g \in D_{HK}$. Then*

(I) $|f| \in D_{HK}$ and $|\int_a^t f| \leq \int_a^t |f|$ for all $t \in [a,b]$;

(II) $\| \, |f| \, \| = \| \, |F'| \, \| = \| \, |F| \, \|_\infty = \|f\|$;

(III) $|f+g| \overset{(p)}{\preceq} |f| + |g|$.

Proof. (I) and (II) see [28, Theorem 24].

Since $|F+G| \leq |F| + |G|$ in C_0, (III) follows immediately from (2.3) and (2.4). $\square$

If $g : [a,b] \to \mathbb{R}$, its variation is $Vg = \sup \sum_n |g(t_n) - g(s_n)|$, where the supremum is taken over every sequence $\{(t_n, s_n)\}$ of disjoint intervals in $[a,b]$. If $Vg < \infty$ then g is called a function with bounded variation. Denote the set of functions with bounded variation by $\mathcal{BV}$ (see [21–23]).

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Lemma 2.3 ([23, Theorem 2.2]). *Let g, $h \in \mathcal{BV}$. Then*

(i) $g \pm h \in \mathcal{BV}$ and $V(g \pm h) \leq Vg + Vh$;

(ii) $gh \in \mathcal{BV}$;

(iii) $gh^{-1} = \frac{g}{h} \in \mathcal{BV}$ if there exists constant $\delta > 0$ such that $|h| \geq \delta$.

Moreover, we have the following result.

Lemma 2.4 ([25, Lemma 1.5]). *Let $F \in C[a, b]$ and $g \in \mathcal{BV}$. Then*

$$F'g = \left(\int_a^t g\,dF \right)', \tag{2.8}$$

and

$$Fg' = \left(\int_a^t F\,dg \right)'. \tag{2.9}$$

Lemma 2.5 ([29, Lemma 2, Integration by parts]). *Let $f \in D_{HK}$, and $g \in \mathcal{BV}$. Then $fg \in D_{HK}$ and*

$$\int_a^b fg = F(b)g(b) - \int_a^b F\,dg.$$

By Lemmas 2.4 and 2.5, it is easy to see that

Lemma 2.6. *Let $f \in D_{HK}$ be the distributional derivative of $F \in C[a, b]$, and $g \in \mathcal{BV}$. Then*

$$(Fg)' = fg + Fg'.$$

From (2.5) and Lemma 2.5, the following lemma holds.

Lemma 2.7. *Let $f \in D_{HK}$ and let g be a nonnegative function on $[a, b]$.*

(I) If $f \overset{(m)}{\succeq} 0$ and g is monotone on $[a, b]$, then

$$fg \overset{(m)}{\succeq} 0.$$

(II) If $f \overset{(p)}{\succeq} 0$ and g is nonincreasing on $[a, b]$, then

$$fg \overset{(p)}{\succeq} 0.$$

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Proof. (I) Let $F(t) = \int_a^t f$, $t \in [a,b]$. Then, $F \in C[a,b]$, and $F \geq 0$ on $[a,b]$, because $f \overset{(m)}{\succeq} 0$. Since $g \geq 0$ is monotone, then $g \in \mathcal{BV}$. By the first mean value theorem for Riemann integrals, there exists $\xi \in [c,d] \subset [a,b]$ such that

$$\int_c^d F dg = F(\xi)(g(d) - g(c)), \quad \xi \in [c,d].$$

In view of Lemma 2.5 and (2.5), one has $fg \in D_{HK}$, and

$$\begin{aligned}
\int_c^d fg &= Fg \mid_c^d - \int_c^d F dg \\
&= F(d)g(d) - F(c)g(c) - F(\xi)(g(d) - g(c)) \\
&= g(d) \int_\xi^d f + g(c) \int_c^\xi f \geq 0, \quad \forall \, [c,d] \subset [a,b].
\end{aligned}$$

Hence, by (2.5), (I) follows.

(II) Let $F(t) = \int_a^t f$, $t \in [a,b]$. Then, $F \in C[a,b]$, and $F \geq 0$ on $[a,b]$, because $f \overset{(p)}{\succeq} 0$. Since $g \geq 0$ is nonincreasing on $[a,b]$, then $g \in \mathcal{BV}$ and $fg \in D_{HK}$. Moreover,

$$\begin{aligned}
\int_a^t fg &= Fg|_a^t + \int_a^t F d(-g) \\
&\geq F(t)g(t) - F(a)g(a) + F(\eta)(g(a) - g(t)) \geq 0, \quad t \in [a,b],
\end{aligned}$$

where $F(\eta) = \min_{s \in [a,t]} F(s)$. Thus, by(2.4), (II) holds. The proof is therefore complete. □

Remark 2.8. In Lemma 2.7, (II) is not true if g is nondecreasing on $[a,b]$. For example, let

$$f = \sin t, \ t \in \left[0, \frac{5\pi}{4}\right], \text{ and } g(t) = \begin{cases} 0, & t \in [0, \pi), \\ 1, & t \in \left[\pi, \frac{5\pi}{4}\right]. \end{cases}$$

It is easy to see that $f \overset{(p)}{\succeq} 0$ and g is nonnegative and nondecreasing on $\left[0, \frac{5\pi}{4}\right]$. However, $\int_0^{\frac{5\pi}{4}} fg = \int_\pi^{\frac{5\pi}{4}} \sin t = \frac{\sqrt{2}}{2} - 1 < 0$. This implies by (2.4) that (II) is not true.

3   Main Results

In this section, we shall prove that the Gronwall-Bellman type inequalities involving the distributional Henstock-Kurzweil integral remain valid.

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Theorem 3.1. *Let* $f \in D_{HK}$, $g : [a, b] \to \mathbb{R}$ *be a nonnegative nonincreasing function. If there is a constant* η *such that*

$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} \eta + \int_a^t fg, \quad t \in [a, b]. \tag{3.1}$$

Then

$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} \eta \exp\left(\int_a^t g\right), \quad t \in [a, b]. \tag{3.2}$$

Proof. Since g is a nonnegative nonincreasing function on $[a, b]$, then $g \exp\left(-\int_a^t g\right)$ is also nonnegative and nonincreasing on $[a, b]$. This together with Lemma 2.5 implies that

$$fg \in D_{HK}, \quad fg \exp\left(-\int_a^t g\right) \in D_{HK}.$$

Let $x(t) = \int_a^t fg$, then $x(t) \in C_0$, $x' = fg$, and (3.1) can be transformed into

$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} \eta + x, \quad \text{on } [a, b]. \tag{3.3}$$

Furthermore, by Lemma 2.7, $0 \leq x(t)$, $t \in [a, b]$. Multiplying by $g \exp\left(-\int_a^t g\right)$ on both sides of (3.3), one has

$$(fg - xg) \exp\left(-\int_a^t g\right) \overset{(p)}{\preceq} \eta g \exp\left(-\int_a^t g\right), \quad t \in [a, b]. \tag{3.4}$$

By Lemma 2.6,

$$\left(x \exp\left(-\int_a^t g\right)\right)' \overset{(p)}{\preceq} \left(\eta - \eta \exp\left(-\int_a^t g\right)\right)', \quad t \in [a, b]. \tag{3.5}$$

Taking in account (2.4), we get

$$0 \leq x(t) \leq \eta \left(\exp\left(\int_a^t g\right) - 1\right), \quad \forall t \in [a, b]. \tag{3.6}$$

It follows from (2.7), (3.3) and (3.6) that

$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} \eta + \eta \left(\exp\left(\int_a^t g\right) - 1\right) = \eta \exp\left(\int_a^t g\right), \quad t \in [a, b].$$

This completes the proof. $\qquad\square$

As Lemma 1.1, we have the following consequence.

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Corollary 3.2. *Let $f \in D_{HK}$. If there exist positive constants K and η such that*

$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} \int_a^t (Kf(s) + \eta)ds, \quad t \in [a, b].$$

Then

$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} \eta(b - a)\exp\left(K(b - a)\right).$$

Remark 3.3. If $f(t)$ is nonnegative and Henstock-Kurzweil integrable on $[a, b]$, then Theorem 3.1 and Corollary 3.2 are still valid. Therefore, Lemma 1.1 and the corresponding result in [12] are only special cases of our results.

For the ordering (2.5), we have the following result.

Theorem 3.4. *Let $f \in D_{HK}$, $g : [a, b] \to \mathbb{R}$ be a positive monotone function. If there is a constant η such that*

$$0 \overset{(m)}{\preceq} f \overset{(m)}{\preceq} \eta + \int_a^t fg, \quad t \in [a, b]. \tag{3.7}$$

Then

$$0 \overset{(m)}{\preceq} f \overset{(m)}{\preceq} \eta \exp\left(\int_a^t g\right), \quad t \in [a, b]. \tag{3.8}$$

Proof. The proof is similar to Theorem 3.1, so we omit it. $\square$

Remark 3.5. Assume that f, g are nonnegative continuous functions. Since the ordering (2.5) equals to the pointwise ordering (see (2.7)), it is easy to see that Theorem 3.4 is a generalization of Lemma 1.2.

Next we give a more general version of the Gronwall-Bellman type inequality due to H. E. Gollwitzer [13].

Theorem 3.6. *Let $f \in D_{HK}$, $g : [a, b] \to \mathbb{R}$ be a nonnegative nonincreasing function. If there exist $l \in D_{HK}$, $l \overset{(p)}{\succeq} 0$ on $[a, b]$, and $h \in HK$, $h \geq 0$ on $[a, b]$, such that*

$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} l + h\int_a^t fg, \quad t \in [a, b]. \tag{3.9}$$

Then

$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} l + h\int_a^t lg \exp\left(\int_a^t gh\right), \quad t \in [a, b]. \tag{3.10}$$

Proof. Since $g(t) \in \mathcal{BV}$ and $f, l \in D_{HK}$. Then $\exp\left(-\int_a^t g\right) \in \mathcal{BV}$, $fg \in D_{HK}$, and $lg \in D_{HK}$. Suppose that $x(t) = \int_a^t fg$, one has $x(a) = 0$ and

$$x' = fg, \quad \text{on } [a, b]. \tag{3.11}$$

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According to (3.9),
$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} l + hx, \quad \text{on } [a,b].\tag{3.12}$$

It turns out from (3.11), (3.12) and Lemma 2.7 that
$$x' - ghx \overset{(p)}{\preceq} lg, \quad \text{on } [a,b].\tag{3.13}$$

Multiplying $\exp\left(-\int_a^t gh\right)$ on both sides of (3.13), we have
$$\left(\exp\left(-\int_a^t gh\right)x\right)' \overset{(p)}{\preceq} lg \exp\left(-\int_a^t gh\right) \le lg, \quad t \in [a,b].\tag{3.14}$$

Applying (2.5) yields that
$$0 \le x(t) \le \exp\left(\int_a^t gh\right)\int_a^t gl, \quad t \in [a,b],\tag{3.15}$$

and hence, by (2.7) and (3.12),
$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} l + h\exp\left(\int_a^t gh\right)\int_a^t gl, \quad t \in [a,b],$$

which completes the proof. □

Corollary 3.7. *Let $f \in D_{HK}$, K be a positive constant. If there exist $l \in D_{HK}$, $l \overset{(p)}{\succeq} 0$ on $[a,b]$, and $h \in HK$, $h \ge 0$ on $[a,b]$, such that*
$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} l + h\int_a^t fK, \quad t \in [a,b].\tag{3.16}$$

Then
$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} \|l\|Kh\exp\left(\int_a^t Kh\right), \quad t \in [a,b].\tag{3.17}$$

Proof. Let $F(t) = \int_a^t f$ and $L(t) = \int_a^t l$. By Theorem 3.6 and Lemma 2.7, we get
$$0 \overset{(p)}{\preceq} f \overset{(p)}{\preceq} l + Kh\exp\left(\int_a^t Kh\right)\int_a^t l, \quad t \in [a,b].\tag{3.18}$$

Moreover, in view of (2.4),
$$
\begin{aligned}
0 \le F(t) &\le L(t) + \int_a^t\left(Kh\exp\left(\int_a^s Kh\right)\int_a^s l\right)\\
&\le \|l\|\left(1 + \int_a^t\left(Kh\exp\left(\int_a^s Kh\right)\right)\right)\\
&= \|l\|\exp\left(\int_a^t Kh\right), \quad t \in [a,b].
\end{aligned}\tag{3.19}
$$

Therefore, by (2.4) and Lemma 2.4, the assertion follows. □

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Remark 3.8. In Corollary 3.7, without loss of generality, let $h \in C[a, b]$. Obviously, the inequality (3.16) implies by (2.5) that

$$0 \leq F(t) \leq L(t) + \int_a^t KhF \leq \|l\| + \int_a^t KhF, \quad t \in [a, b]. \qquad (3.20)$$

Since F, $h \in C[a, b]$ are nonnegative, K, $\|l\|$ are positive constants, then by Lemma 1.2,

$$0 \leq F(t) \leq \|l\| \exp\left(\int_a^t Kh\right), \quad t \in [a, b],$$

which is the same result as in (3.19). Hence, our results are extensions of Lemma 1.2.

Furthermore, we have another result for the ordering (2.5).

Theorem 3.9. *Let $f \in D_{HK}$, $g : [a, b] \to \mathbb{R}$ be a nonnegative nonincreasing function. If there exist $l \in D_{HK}$, $l \overset{(m)}{\succeq} 0$ on $[a, b]$, and $h \in HK$, $h \geq 0$ on $[a, b]$, such that*

$$0 \overset{(m)}{\preceq} f \overset{(m)}{\preceq} l + h \int_a^t fg, \quad t \in [a, b]. \qquad (3.21)$$

Then

$$0 \overset{(m)}{\preceq} f \overset{(m)}{\preceq} l + h \int_a^t lg \exp\left(\int_a^t gh\right), \quad t \in [a, b]. \qquad (3.22)$$

Remark 3.10. If f, g, h, l are nonnegative continuous functions, the inequalities in Theorem 3.6 also hold for the pointwise order in $C[a, b]$, because of (2.7). Therefore, Theorem 3.6 extends the corresponding result in [13].

4   Application

In this section, we will give an application concerned about the Gronwall-Bellman type inequalities.

We consider the system

$$A_1(A_0 x)' - A_2' x = F', \qquad (4.1)$$

where the derivatives, products and equality are understood in the sense of distributions, see [26].

Assumptions 4.1. The function $A_0 \in C[0, T]$, $A_0 \neq 0$ on $[0, T]$, $A_1^{-1} \in \mathcal{BV}$ with $|A_1^{-1}| \geq \delta_1 > 0$, and $A_2 \in C[0, T] \cap \mathcal{BV}$. Furthermore, $F \in C[0, T]$.

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Let us notice that under Assumptions 4.1 the products $A_1(A_0x)'$ and $A_2'x$, by Lemma 2.4, are well defined for any $x \in C[0,T]$.

Definition 4.2. A function $x(t)$ is called a solution to the equation (4.1) on the interval $[0,T]$ if $x \in C[0,T]$ and $A_1(A_0x)' - A_2'x - F'$ is the zero distribution.

Firstly, we show the estimate of solutions to the equation (4.1).

Theorem 4.3. *Let the assumptions 4.1 be satisfied. If $x(t) \in C[0,T]$ is a solution to the equation (4.1) on $[0,T]$, then*

$$|x(t)| \le |A_0^{-1}(t)|\|l\| \exp\left(\int_0^t h\right), \quad t \in [0,T], \tag{4.2}$$

where

$$l = |A_1^{-1}F'|, \qquad h = |A_1^{-1}A_2'A_0^{-1}|. \tag{4.3}$$

Proof. By Assumptions 4.1 and (4.1),

$$(A_0x)' = A_1^{-1}F' + A_1^{-1}A_2'A_0^{-1}(A_0x). \tag{4.4}$$

Hence, by Lemma 2.2,

$$|(A_0x)'| \overset{(p)}{\le} |A_1^{-1}F'| + |A_1^{-1}A_2'A_0^{-1}| \int_0^t |(A_0x)'|. \tag{4.5}$$

Let

$$l = |A_1^{-1}F'|, \quad g = 1, \quad h = |A_1^{-1}A_2'A_0^{-1}|. \tag{4.6}$$

It is easy to see that $l \in D_{HK}$, $h \in HK$. Therefore, by Corollary 3.7,

$$|(A_0x)'| \overset{(p)}{\le} \|l\|h \exp\left(\int_0^t h\right), \quad t \in [0,T], \tag{4.7}$$

which yields by (2.4) and Lemma 2.2 that

$$|x(t)| \le |A_0^{-1}(t)|\|l\| \exp\left(\int_0^t h\right), \quad t \in [0,T]. \tag{4.8}$$

The proof is therefore complete. $\qquad\square$

Finally, we give an existence and uniqueness result.

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Theorem 4.4. *Let the Assumptions 4.1 be satisfied. Moreover, either $A_0^{-1} \in \mathcal{BV}$ with $|A_0^{-1}| \geq \delta_0 > 0$ or $A_2 \in AC$ holds. Then (4.1) has a unique solution $x(t) \in C[0,T]$ satisfying*

$$x(t) = A_0^{-1}(t)k^{-1}(t) \int_0^t kA_1^{-1}dF, \quad t \in [0,T], \tag{4.9}$$

where

$$k(t) = \exp\left(-\int_0^t A_0^{-1}A_1^{-1}dA_2\right), \quad t \in [0,T]. \tag{4.10}$$

Proof. We only prove the necessity, the sufficiency is easy to prove.

By Assumptions 4.1 and (4.1),

$$(A_0x)' - A_1^{-1}A_2'A_0^{-1}(A_0x) = A_1^{-1}F'. \tag{4.11}$$

Let

$$k(t) = \exp\left(-\int_0^t A_0^{-1}A_1^{-1}dA_2\right), \quad t \in [0,T]. \tag{4.12}$$

It is easy to see that $k(t) \in C[0,T] \cap \mathcal{BV}$. Multiplying both side of (4.11) by $k(t)$, we get

$$k(A_0x)' - kA_1^{-1}A_2'A_0^{-1}(A_0x) = kA_1^{-1}F'. \tag{4.13}$$

By Lemma 2.6,

$$(kA_0x)' = kA_1^{-1}F'. \tag{4.14}$$

Therefore,

$$x(t) = A_0^{-1}(t)k^{-1}(t)\int_0^t kA_1^{-1}dF, \quad t \in [0,T].$$

Let $y(t)$ be another solution to (4.1). Then,

$$x(t) - y(t) = 0, \quad t \in [0,T].$$

Therefore, $x(t)$ satisfying (4.9) is a unique solution of (4.1). $\square$

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Formulas and properties of some class of nonlinear difference equations

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ABSTRACT

We obtain the formulas of the solutions of the recursive sequences

$$x_{n+1} = \frac{x_n x_{n-5}}{x_{n-4}(\pm 1 \pm x_n x_{n-5})}, \quad n = 0, 1, ...,$$

where the initial conditions are arbitrary non zero real numbers. Also, we discuss and illustrate the stability of the solutions in the neighborhood of the critical points and the periodicity of the considered equations.

Keywords: equilibrium point, recursive sequences, periodicity.

Mathematics Subject Classification: 39A10.

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1. INTRODUCTION

In recent years, the qualitative study of difference equations has become an active research area among a considerable number of mathematicians. Some economical and biological examples can be seen in [9,36,40,47,48,54]. It is commonly known that nonlinear difference equations are able to produce and present sophisticated behaviors regardless their orders.

Some articles show that a great effort has been done to demonstrate and explore the dynamics of nonlinear difference equations (see [40]-[61]). In fact, investigating these equations is a challenge and still new in the mathematical world. It is strongly believed that the rational difference equations are significant in their own right.

Abo-Zeid and Cinar [1] illustrated the global stability, cyclical behavior, oscillation of all acceptable solutions of the equation

$$x_{n+1} = \frac{Ax_{n-1}}{B - Cx_n x_{n-2}}.$$

In [7], [8] Cinar considered the solutions of the equations

$$y_{n+1} = \frac{y_{n-1}}{1 + ay_n y_{n-1}}, \qquad y_{n+1} = \frac{y_{n-1}}{-1 + ay_n y_{n-1}}.$$

A. El-Moneam, and Alamoudy [16] examined the positive solutions of the equation in terms of its periodicity, boundedness and the global stability. The considered difference equation is given by

$$x_{n+1} = ax_n + \frac{bx_{n-1} + cx_{n-2} + fx_{n-3} + rx_{n-4}}{dx_{n-1} + ex_{n-2} + gx_{n-3} + sx_{n-4}}.$$

Khatibzadeh and Ibrahim [42] studied the boundedness, asymptotic stability, oscillatory behavior and discovered the closed form of solutions of the equation

$$x_{n+1} = ax_n + \frac{bx_n x_{n-1}}{cx_n + dx_{n-1}}.$$

Simsek et al. [49] has found and explored solutions for the recursive formula

$$y_{n+1} = \frac{y_{n-3}}{1 + y_{n-1}}.$$

For other related papers, see [25–46].

We analyze and explore the solutions of the following nonlinear recursive equation

$$x_{n+1} = \frac{x_n x_{n-5}}{x_{n-4}(\pm 1 \pm x_n x_{n-5})}, \qquad n = 0, 1, ..., \tag{1}$$

with conditions posed on the initial values are arbitrary non zero real numbers. Also, we will survey some dynamic behaviors of its solutions.

The linearized equation of equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{2}$$

about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}.$$

Theorem A [43]: Assume that $p_i \in R, \ i = 1, 2, ..., k$ and $k \in \{0, 1, 2, ...\}$. Then $\sum_{i=1}^{k} |p_i| < 1$, is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, ...$$

2. THE FIRST EQUATION $X_{N+1} = \frac{X_N X_{N-5}}{X_{N-4}(1 + X_N X_{N-5})}$

This section is devoted to give a specific solution of the first difference equation which is

$$x_{n+1} = \frac{x_n x_{n-5}}{x_{n-4}(1 + x_n x_{n-5})}. \tag{3}$$

THEOREM 2.1. *Let* $\{x_n\}_{n=-5}^{\infty}$ *be a solution of Eq.(3). Then*

$$x_{10n-5} = f \prod_{i=0}^{n-1} \left(\frac{1 + (10i)af}{1 + (10i+5)af} \right), \qquad x_{10n-4} = e \prod_{i=0}^{n-1} \left(\frac{1 + (10i+1)af}{1 + (10i+6)af} \right),$$

$$x_{10n-3} = d \prod_{i=0}^{n-1} \left(\frac{1 + (10i+2)af}{1 + (10i+7)af} \right), \qquad x_{10n-2} = c \prod_{i=0}^{n-1} \left(\frac{1 + (10i+3)af}{1 + (10i+8)af} \right),$$

$$x_{10n-1} = b \prod_{i=0}^{n-1} \left(\frac{1 + (10i+4)af}{1 + (10i+9)af} \right), \qquad x_{10n} = a \prod_{i=0}^{n-1} \left(\frac{1 + (10i+5)af}{1 + (10i+10)af} \right),$$

$$x_{10n+1} = \frac{af}{e(1+af)} \prod_{i=0}^{n-1} \left(\frac{1 + (10i+6)af}{1 + (10i+11)af} \right), \qquad x_{10n+2} = \frac{af}{d(1+2af)} \prod_{i=0}^{n-1} \left(\frac{1 + (10i+7)af}{1 + (10i+12)af} \right),$$

$$x_{10n+3} = \frac{af}{c(1+3af)} \prod_{i=0}^{n-1} \left(\frac{1 + (10i+8)af}{1 + (10i+13)af} \right), \qquad x_{10n+4} = \frac{af}{b(1+4af)} \prod_{i=0}^{n-1} \left(\frac{1 + (10i+9)af}{1 + (10i+14)af} \right),$$

where we put $x_{-5} = f, \ x_{-4} = e, \ x_{-3} = d, \ x_{-2} = c, \ x_{-1} = b, \ x_0 = a.$

Proof: The result holds for $n = 0$. Assume that $n > 0$ and our assumption true for $n - 1$. Then;

$$x_{10n-15} = f \prod_{i=0}^{n-2} \left(\frac{1 + (10i)af}{1 + (10i+5)af} \right), \qquad x_{10n-14} = e \prod_{i=0}^{n-2} \left(\frac{1 + (10i+1)af}{1 + (10i+6)af} \right),$$

$$x_{10n-13} = d \prod_{i=0}^{n-2} \left(\frac{1 + (10i+2)af}{1 + (10i+7)af} \right), \qquad x_{10n-12} = c \prod_{i=0}^{n-2} \left(\frac{1 + (10i+3)af}{1 + (10i+8)af} \right),$$

$$x_{10n-11} = b \prod_{i=0}^{n-2} \left(\frac{1 + (10i+4)af}{1 + (10i+9)af} \right), \qquad x_{10n-10} = a \prod_{i=0}^{n-2} \left(\frac{1 + (10i+5)af}{1 + (10i+10)af} \right),$$

$$x_{10n-9} = \frac{af}{e(1+af)} \prod_{i=0}^{n-2} \left(\frac{1 + (10i+6)af}{1 + (10i+11)af} \right), \quad x_{10n-8} = \frac{af}{d(1+2af)} \prod_{i=0}^{n-2} \left(\frac{1 + (10i+7)af}{1 + (10i+12)af} \right),$$

$$x_{10n-7} = \frac{af}{c(1+3af)} \prod_{i=0}^{n-2} \left(\frac{1 + (10i+8)af}{1 + (10i+13)af} \right), \quad x_{10n-6} = \frac{af}{b(1+4af)} \prod_{i=0}^{n-2} \left(\frac{1 + (10i+9)af}{1 + (10i+14)af} \right).$$

Now, it follows from Eq.(3) that

$$x_{10n-5} = \frac{x_{10n-6}x_{10n-11}}{x_{10n-10}(1 + x_{10n-6}x_{10n-11})}$$

$$= \frac{\frac{af}{b(1+4af)} \prod_{i=0}^{n-2} \left(\frac{1+(10i+9)af}{1+(10i+14)af} \right) b \prod_{i=0}^{n-2} \left(\frac{1+(10i+4)af}{1+(10i+9)af} \right)}{\left[a \prod_{i=0}^{n-2} \left(\frac{1+(10i+5)af}{1+(10i+10)af} \right) \right] \left[1 + \frac{af}{b(1+4af)} \prod_{i=0}^{n-2} \left(\frac{1+(10i+9)af}{1+(10i+14)af} \right) b \prod_{i=0}^{n-2} \left(\frac{1+(10i+4)af}{1+(10i+9)af} \right) \right]}$$

$$= \frac{\frac{f}{(1+4af)} \prod_{i=0}^{n-2} \left(\frac{1 + (10i+4)af}{1 + (10i+14)af} \right)}{\left[\prod_{i=0}^{n-2} \left(\frac{1 + (10i+5)af}{1 + (10i+10)af} \right) \right] \left[1 + \frac{af}{(1+4af)} \prod_{i=0}^{n-2} \left(\frac{1 + (10i+4)af}{1 + (10i+14)af} \right) \right]}$$

$$= \frac{\left(\frac{f}{1+(10n-6)af} \right)}{\left[\prod_{i=0}^{n-2} \left(\frac{1 + (10i+5)af}{1 + (10i+10)af} \right) \right] \left[1 + \frac{af}{1+(10n-6)af} \right]}$$

$$= \frac{f}{\left[\prod_{i=0}^{n-2} \left(\frac{1 + (10i+5)af}{1 + (10i+10)af} \right) \right] [1 + (10n-6)af + af]}$$

$$= \frac{f}{[1+(10n-5)af]} \prod_{i=0}^{n-2} \left(\frac{1 + (10i+10)af}{1 + (10i+5)af} \right) = f \prod_{i=0}^{n-1} \left(\frac{1 + (10i)af}{1 + (10i+5)f} \right).$$

Also, we have

$$x_{10n-4} = \frac{x_{10n-5}x_{10n-10}}{x_{10n-9}(1 + x_{10n-5}x_{10n-10})}$$

$$= \frac{f \prod_{i=0}^{n-1}\left(\frac{1+(10i)af}{1+(10i+5)af}\right) a \prod_{i=0}^{n-2}\left(\frac{1+(10i+5)af}{1+(10i+10)af}\right)}{\left[\frac{af}{e(1+af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+6)af}{1+(10i+11)af}\right)\right]\left[1+f\prod_{i=0}^{n-1}\left(\frac{1+(10i)af}{1+(10i+5)af}\right)a\prod_{i=0}^{n-2}\left(\frac{1+(10i+5)af}{1+(10i+10)af}\right)\right]}$$

$$= \frac{af\left(\frac{1}{1+(10n-5)af}\right)}{\left[\frac{af}{e(1+af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+6)af}{1+(10i+11)af}\right)\right]\left[1+\frac{af}{1+(10n-5)af}\right]}$$

$$= \frac{af}{\left[\frac{af}{e(1+af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+6)af}{1+(10i+11)af}\right)\right][1+(10n-5)af+af]}$$

$$= \frac{e(1+af)}{[1+(10n-4)af]}\prod_{i=0}^{n-2}\left(\frac{1+(10i+11)af}{1+(10i+6)af}\right) = e\prod_{i=0}^{n-1}\left(\frac{1+(10i+1)af}{1+(10i+6)af}\right).$$

Similarly

$$x_{10n-3} = \frac{x_{10n-4}x_{10n-9}}{x_{10n-8}(1+x_{10n-4}x_{10n-9})}$$

$$= \frac{e\prod_{i=0}^{n-1}\left(\frac{1+(10i+1)af}{1+(10i+6)af}\right)\frac{af}{e(1+af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+6)af}{1+(10i+11)af}\right)}{\left[\frac{af}{d(1+2af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+7)af}{1+(10i+12)af}\right)\right]\left[1+e\prod_{i=0}^{n-1}\left(\frac{1+(10i+1)af}{1+(10i+6)af}\right)\frac{af}{e(1+af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+6)af}{1+(10i+11)af}\right)\right]}$$

$$= \frac{\prod_{i=0}^{n-1}\left(\frac{af}{1+(10i+6)af}\right)\prod_{i=0}^{n-2}(1+(10i+6)af)}{\left[\frac{af}{d(1+2af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+7)af}{1+(10i+12)af}\right)\right]\left[1+\prod_{i=0}^{n-1}\left(\frac{af}{1+(10i+6)af}\right)\prod_{i=0}^{n-2}(1+(10i+6)af)\right]}$$

$$= \frac{\left(\frac{af}{1+(10n-4)af}\right)}{\left[\frac{af}{d(1+2af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+7)af}{1+(10i+12)af}\right)\right]\left[1+\left(\frac{af}{1+(10n-4)af}\right)\right]}$$

$$= \frac{d}{\left[\frac{1}{(1+2af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+7)af}{1+(10i+12)af}\right)\right][1+(10n-4)af+af]}$$

$$= \frac{d}{\left[\frac{1}{(1+2af)}\prod_{i=0}^{n-2}\left(\frac{1+(10i+7)af}{1+(10i+12)af}\right)\right][1+(10n-3)af]} = d\prod_{i=0}^{n-1}\left(\frac{1+(10i+2)af}{1+(10i+7)af}\right).$$

Similarly, one can simply find the other relations. Thus, the proof is done.

THEOREM 2.2. *The unique equilibrium point of Eq.(3) is the number zero which is not locally asymptotically stable.*

Proof: The equilibrium points of Eq.(3) obtained by

$$\overline{x} = \frac{\overline{x}^2}{\overline{x}\left(1 + \overline{x}^3\right)}.$$

Arranging the previous equation gives $\overline{x}^4 = 0$. Thus $\overline{x} = 0$.

Let $f : (0, \infty)^3 \longrightarrow (0, \infty)$ be a function takes the form

$$f(u, v, w) = \frac{uw}{v\left(1 + uw\right)}.$$

Therefore

$$f_u(u, v, w) = \frac{w}{v\left(1 + uw\right)^2}, \ \ f_v(u, v, w) = -\frac{uw}{v^2\left(1 + uw\right)}, \ \ f_w(u, v, w) = \frac{u}{v\left(1 + uw\right)^2}.$$

So

$$f_u(\overline{x}, \overline{x}, \overline{x}) = 1, \quad f_v(\overline{x}, \overline{x}, x) = 1, \quad f_w(\overline{x}, \overline{x}, \overline{x}) = 1.$$

Then by using Theorem A the proof follows.

Example 1. We assume $x_{-5} = 6, \ x_{-4} = 11, \ x_{-3} = 3, \ x_{-2} = 2, \ x_{-1} = 1.8, \ x_0 = -7.$ See Fig. 1.
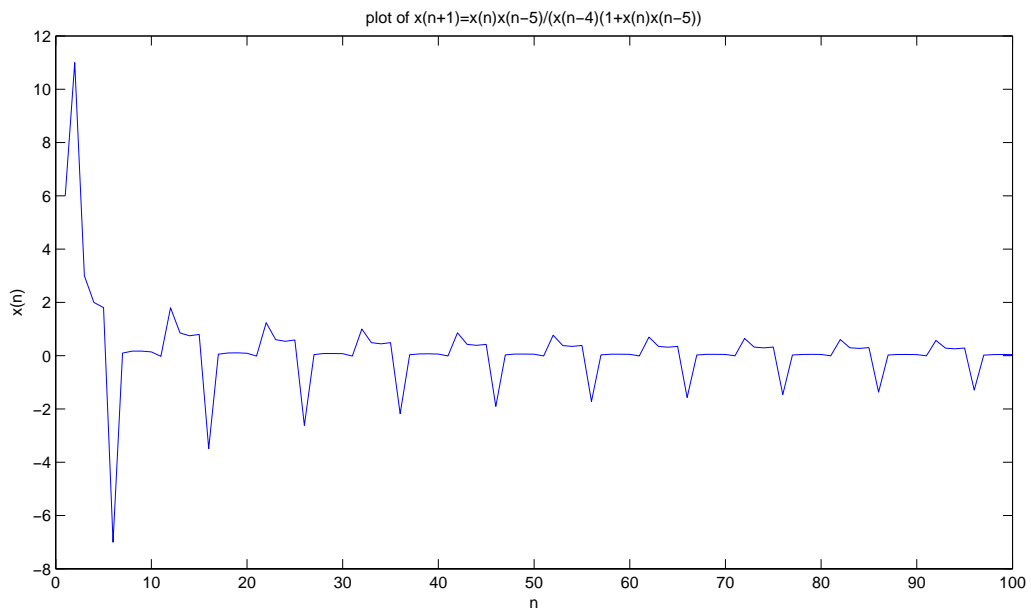
Figure 1.

Example 2. See Fig. 2, since $x_{-5} = 1.6, \ x_{-4} = 1.2, \ x_{-3} = -3, \ x_{-2} = .7, \ x_{-1} = 1.8, \ x_0 = 3$.
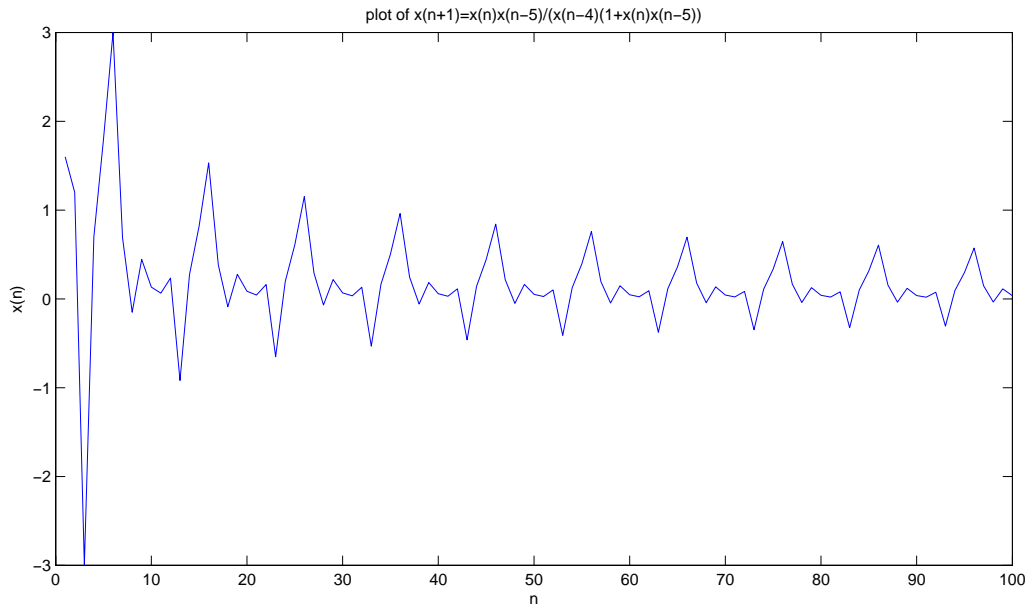
Figure 2.

3. THE SECOND EQUATION $X_{N+1} = \dfrac{X_N X_{N-5}}{X_{N-4}(-1+X_N X_{N-5})}$

This section is devoted to obtain the solution of the difference equation which is

$$x_{n+1} = \frac{x_n x_{n-5}}{x_{n-4}(-1 + x_n x_{n-5})}, \tag{4}$$

where $x_0 x_{-5} \neq 1$.

THEOREM 3.1. *Let $\{x_n\}_{n=-5}^{\infty}$ be a solution of Eq.(4). Then for*

$$
\begin{aligned}
x_{10n-5} &= \frac{f}{(-1+af)^n}, & x_{10n-4} &= e(-1+af)^n, \\
x_{10n-3} &= \frac{d}{(-1+af)^n}, & x_{10n-2} &= c(-1+af)^n, \\
x_{10-1} &= \frac{b}{(-1+af)^n}, & x_{10n} &= a(-1+af)^n, \\
x_{10n+1} &= \frac{af}{e(-1+af)^{n+1}}, & x_{10n+2} &= \frac{af(-1+af)^n}{d}, \\
x_{10n+3} &= \frac{af}{c(-1+af)^{n+1}}, & x_{10n+4} &= \frac{af(-1+af)^n}{b}.
\end{aligned}
$$

Proof: The result holds for $n = 0$. Assume that $n > 0$ and that our assumption true for $n - 1$. Then;

$$x_{10n-15} = \frac{f}{(-1+af)^{n-1}}, \qquad x_{10n-14} = e(-1+af)^{n-1},$$

$$x_{10n-13} = \frac{d}{(-1+af)^{n-1}}, \qquad x_{10n-12} = c(-1+af)^{n-1},$$

$$x_{10-11} = \frac{b}{(-1+af)^{n-1}}, \qquad x_{10n-10} = a(-1+af)^{n-1},$$

$$x_{10n-9} = \frac{af}{e(-1+af)^{n}}, \qquad x_{10n-8} = \frac{af(-1+af)^{n-1}}{d},$$

$$x_{10n-7} = \frac{af}{c(-1+af)^{n}}, \qquad x_{10n-6} = \frac{af(-1+af)^{n-1}}{b}.$$

It follows from (4) that

$$x_{10n-5} = \frac{x_{10n-6}x_{10n-11}}{x_{10n-10}(-1+x_{10n-6}x_{10n-11})} = \frac{af}{a(-1+af)^{n-1}[-1+af]} = \frac{af}{a(-1+af)^{n}},$$

$$
\begin{aligned}
x_{10n-4} &= \frac{x_{10n-5}x_{10n-10}}{x_{10n-9}(-1+x_{10n-5}x_{10n-10})} \\
&= \frac{\frac{af}{a(-1+af)^{n}}a(-1+af)^{n-1}}{\left[\frac{af}{e(-1+af)^{n}}\right]\left[-1+\frac{af}{a(-1+af)^{n}}a(-1+af)^{n-1}\right]} \\
&= \frac{e(-1+af)^{n}}{(-1+af)\left[-1+\frac{af}{(-1+af)}\right]} = \frac{e(-1+af)^{n}}{[1-af+af]} = e(-1+af)^{n}.
\end{aligned}
$$

Similarly one can simply prove the other relations.

THEOREM 3.2. *Eq.(4) has a period ten solution iff $af = 2$ and will be in the following form*

$$\left\{f,e,d,c,b,a,\frac{af}{e},\frac{af}{d},\frac{af}{c},\frac{af}{b},f,e,d,...\right\}.$$

Proof: Firstly, assume that there exists a period ten solution

$$\left\{f,e,d,c,b,a,\frac{af}{e},\frac{af}{d},\frac{af}{c},\frac{af}{b},f,e,d,...\right\},$$

of Eq.(4). Then, we can notice from the solution of Eq.(4) that

$$
\begin{aligned}
f &= \frac{f}{(-1+af)^{n}}, \quad e = e(-1+af)^{n}, \quad d = \frac{d}{(-1+af)^{n}}, \quad c = c(-1+af)^{n}, \\
b &= \frac{b}{(-1+af)^{n}}, \quad a = a(-1+af)^{n}, \quad \frac{af}{e} = \frac{af}{e(-1+af)^{n+1}}, \\
\frac{af}{d} &= \frac{af(-1+af)^{n}}{d}, \quad \frac{af}{c} = \frac{af}{c(-1+af)^{n+1}}, \quad \frac{af}{b} = \frac{af(-1+af)^{n}}{b}.
\end{aligned}
$$

or,

$$(-1+af)^{n} = 1.$$

Then
$$af = 2.$$

Secondly, suppose that $af = 2$. Then, it is easily seen from the solution of Eq.(4) that

$$
\begin{aligned}
x_{10n-5} &= f, \quad x_{10n-4} = e, \quad x_{10n-3} = d, \quad x_{10n-2} = c, \quad x_{10-1} = b, \\
x_{10n} &= a, \quad x_{10n+1} = \frac{af}{e}, \qquad x_{10n+2} = \frac{af}{d}, \qquad x_{10n+3} = \frac{af}{c}, \qquad x_{10n+4} = \frac{af}{b}.
\end{aligned}
$$

Thus, the periodic solution of period ten is obtained and this proves the theorem.

THEOREM 3.3. *Eq.(4) has two equilibrium points which are $0, \sqrt[3]{2}$ and these equilibrium points are not locally asymptotically stable.*

Proof: The equilibrium points of Eq.(4) can be written in the following form

$$\overline{x} = \frac{\overline{x}^2}{\overline{x}\left(-1 + \overline{x}^2\right)}.$$

Arranging this gives

$$
\begin{aligned}
\overline{x}^2\left(-1 + \overline{x}^2\right) &= \overline{x}^2 \\
\Rightarrow \quad \overline{x}^2\left(\overline{x}^2 - 2\right) &= 0.
\end{aligned}
$$

Therefore, the fixed points are $0, \pm\sqrt{2}$.

Let $f : (0, \infty)^3 \longrightarrow (0, \infty)$ be a function defined by

$$f(u, v, w) = \frac{uw}{v\left(-1 + uw\right)}.$$

Then it follows that

$$
\begin{aligned}
f_u(u, v, w) &= -\frac{w}{v\left(-1 + uw\right)^2}, \\
f_v(u, v, w) &= -\frac{uw}{v^2\left(-1 + uw\right)}, \\
f_w(u, v, w) &= -\frac{u}{v\left(-1 + uw\right)^2},
\end{aligned}
$$

It can be seen that
$$f_u(\overline{x}, \overline{x}, \overline{x}) = -1, \quad f_v(\overline{x}, \overline{x}, x) = \pm 1, \quad f_w(\overline{x}, \overline{x}, \overline{x}) = -1.$$

Then by using Theorem A the proof follows.

Example 3. We consider $x_{-5} = .8$, $x_{-4} = 1.7$, $x_{-3} = .3$, $x_{-2} = 2$, $x_{-1} = 1.8$, $x_0 = .7$. See Fig. 3.
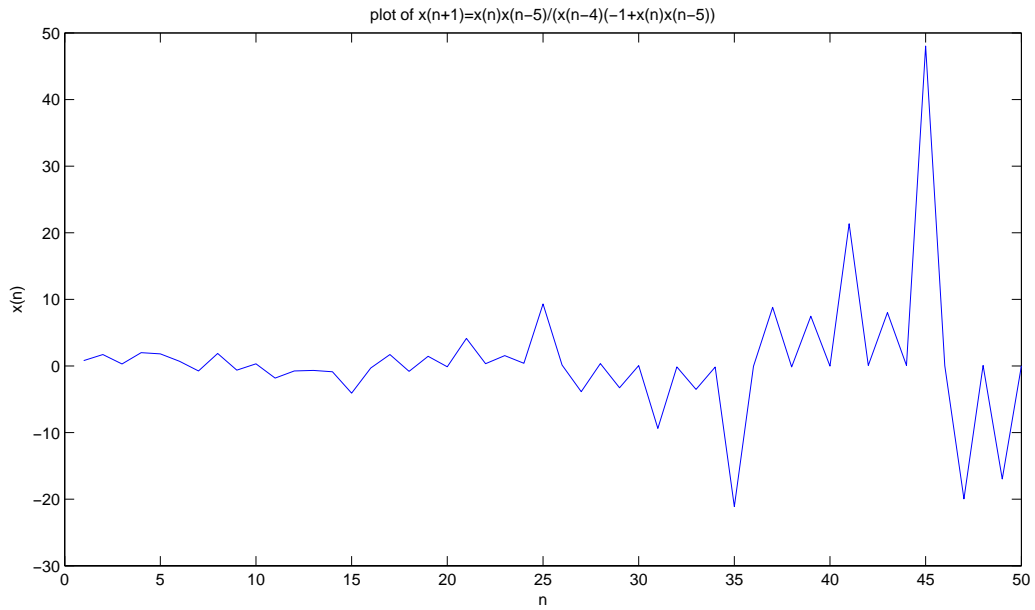
Figure 3.

Example 4. See Fig. 4, since $x_{-5} = 8$, $x_{-4} = 1.7$, $x_{-3} = .3$, $x_{-2} = 2$, $x_{-1} = 1.8$, $x_0 = 1/4$.
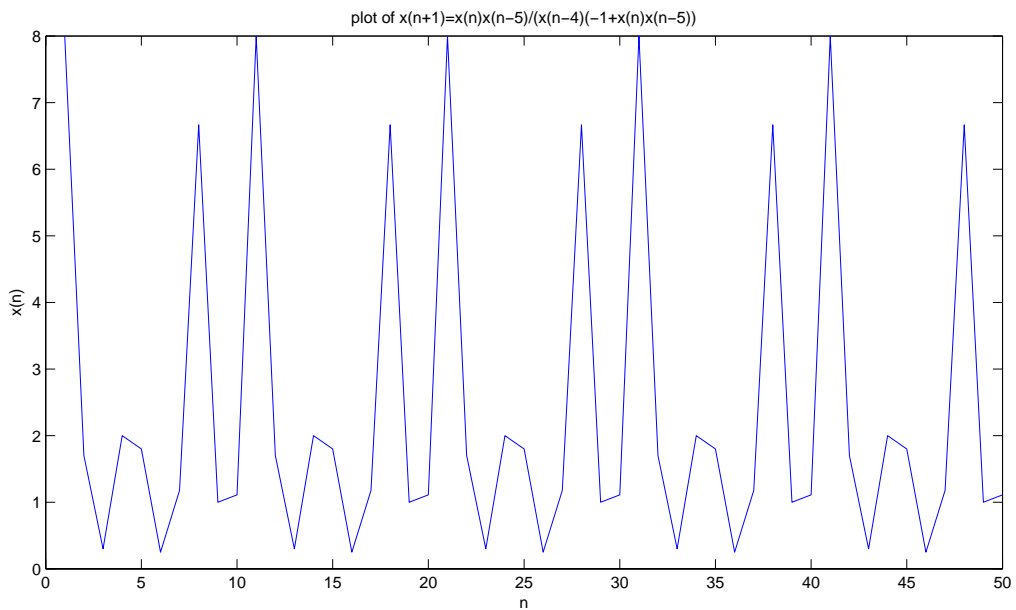
Figure 4.

4. THE THIRD EQUATION $X_{N+1} = \dfrac{X_N X_{N-5}}{X_{N-4}(1 - X_N X_{N-5})}$

In this section we will obtain and present the solution of the third difference equation which is

$$x_{n+1} = \frac{x_n x_{n-5}}{x_{n-4}(1 - x_n x_{n-5})}. \tag{5}$$

THEOREM 4.1. *Let $\{x_n\}_{n=-5}^{\infty}$ be a solution of Eq.(5). Then for $n = 0, 1, ...$*

$$x_{10n-5} = f \prod_{i=0}^{n-1} \left(\frac{1 - (10i)af}{1 - (10i+5)af} \right), \qquad x_{10n-4} = e \prod_{i=0}^{n-1} \left(\frac{1 - (10i+1)af}{1 - (10i+6)af} \right),$$

$$x_{10n-3} = d \prod_{i=0}^{n-1} \left(\frac{1 - (10i+2)af}{1 - (10i+7)af} \right), \qquad x_{10n-2} = c \prod_{i=0}^{n-1} \left(\frac{1 - (10i+3)af}{1 - (10i+8)af} \right),$$

$$x_{10n-1} = b \prod_{i=0}^{n-1} \left(\frac{1 - (10i+4)af}{1 - (10i+9)af} \right), \qquad x_{10n} = a \prod_{i=0}^{n-1} \left(\frac{1 - (10i+5)af}{1 - (10i+10)af} \right),$$

$$x_{10n+1} = \frac{af}{e(1-af)} \prod_{i=0}^{n-1} \left(\frac{1 - (10i+6)af}{1 - (10i+11)af} \right),$$

$$x_{10n+2} = \frac{af}{d(1-2af)} \prod_{i=0}^{n-1} \left(\frac{1 - (10i+7)af}{1 - (10i+12)af} \right),$$

$$x_{10n+3} = \frac{af}{c(1-3af)} \prod_{i=0}^{n-1} \left(\frac{1 - (10i+8)af}{1 - (10i+13)af} \right),$$

$$x_{10n+4} = \frac{af}{b(1-4af)} \prod_{i=0}^{n-1} \left(\frac{1 - (10i+9)af}{1 - (10i+14)af} \right).$$

THEOREM 4.2. *The unique critical point of Eq.(5) is the number zero which is not locally asymptotically stable.*

Example 5. Suppose that $x_{-5} = 8$, $x_{-4} = 1.7$, $x_{-3} = .3$, $x_{-2} = 2$, $x_{-1} = 1.8$, $x_0 = 1/4$ see Fig. 5.
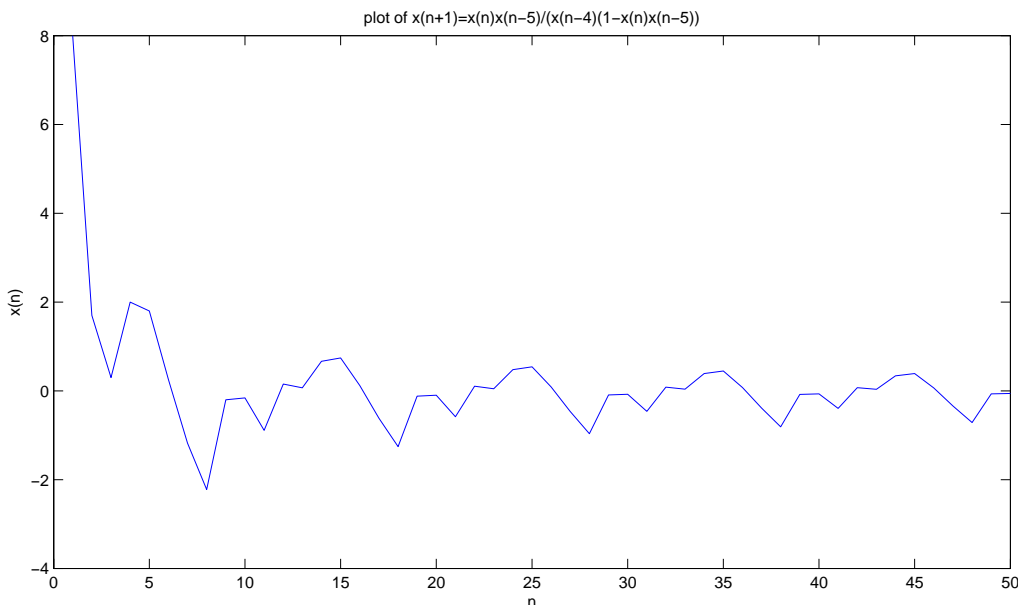
Figure 5.

Example 6. See Fig. 6 since $x_{-5} = -7$, $x_{-4} = 1.5$, $x_{-3} = -3$, $x_{-2} = 2$, $x_{-1} = 12$, $x_0 = 4$.
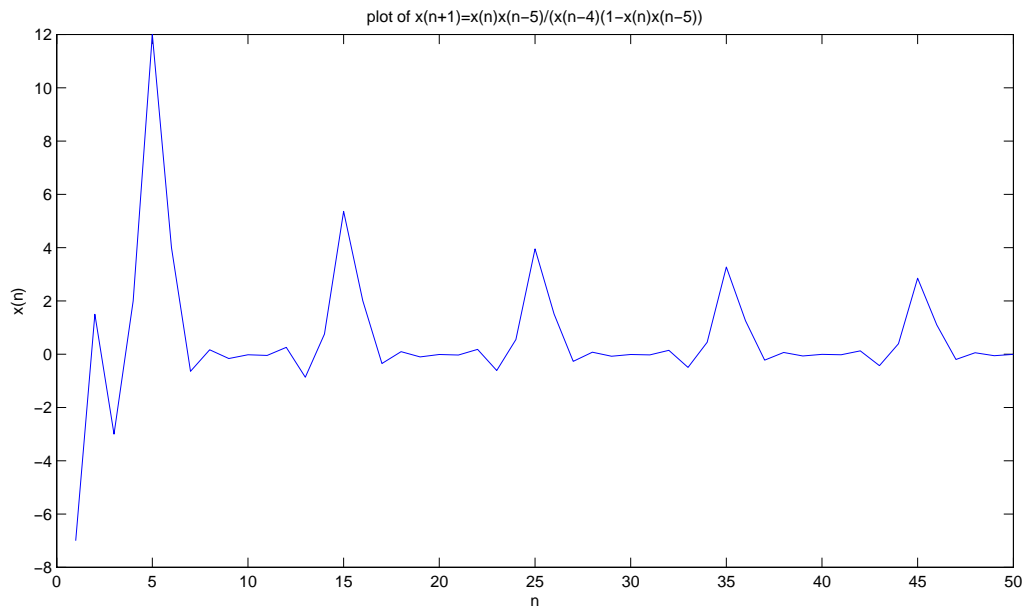
Figure 6.

5. THE FOURTH EQUATION $X_{N+1} = \dfrac{X_N X_{N-5}}{X_{N-4}(-1-X_N X_{N-5})}$

Now, we will explore and discover the solution of the following difference equation

$$x_{n+1} = \frac{x_n x_{n-5}}{x_{n-4}(-1 - x_n x_{n-5})}, \quad n = 0, 1, ..., \tag{6}$$

where $x_{-5}x_0 \neq -1$.

THEOREM 5.1. *Let $\{x_n\}_{n=-5}^{\infty}$ be a solution of Eq.(6). Then Eq.(6) has unboundedness solution (except in the case if $af = -2$) and for $n = 0, 1, ...$*

$$
\begin{aligned}
x_{10n-5} &= \frac{f}{(-1-af)^n}, & x_{10n-4} &= e(-1-af)^n, \\
x_{10n-3} &= \frac{d}{(-1-af)^n}, & x_{10n-2} &= c(-1-af)^n, \\
x_{10-1} &= \frac{b}{(-1-af)^n}, & x_{10n} &= a(-1-af)^n, \\
x_{10n+1} &= \frac{af}{e(-1-af)^{n+1}}, & x_{10n+2} &= \frac{af(-1-af)^n}{d}, \\
x_{10n+3} &= \frac{af}{c(-1-af)^{n+1}}, & x_{10n+4} &= \frac{af(-1-af)^n}{b}.
\end{aligned}
$$

THEOREM 5.2. *Eq.(6) has a periodic solution of period ten iff $af = -2$ and written in the following form $\left\{ f, e, d, c, b, a, \dfrac{af}{e}, \dfrac{af}{d}, \dfrac{af}{c}, f, e, d, c, ... \right\}.$*

THEOREM 5.3. *The unique equilibrium of Eq.(6) is the number zero which is not locally asymptotically stable.*

Example 7. Consider $x_{-5} = -7$, $x_{-4} = 1.5$, $x_{-3} = -3$, $x_{-2} = 2$, $x_{-1} = 12$, $x_0 = 4$ see Fig. 7.
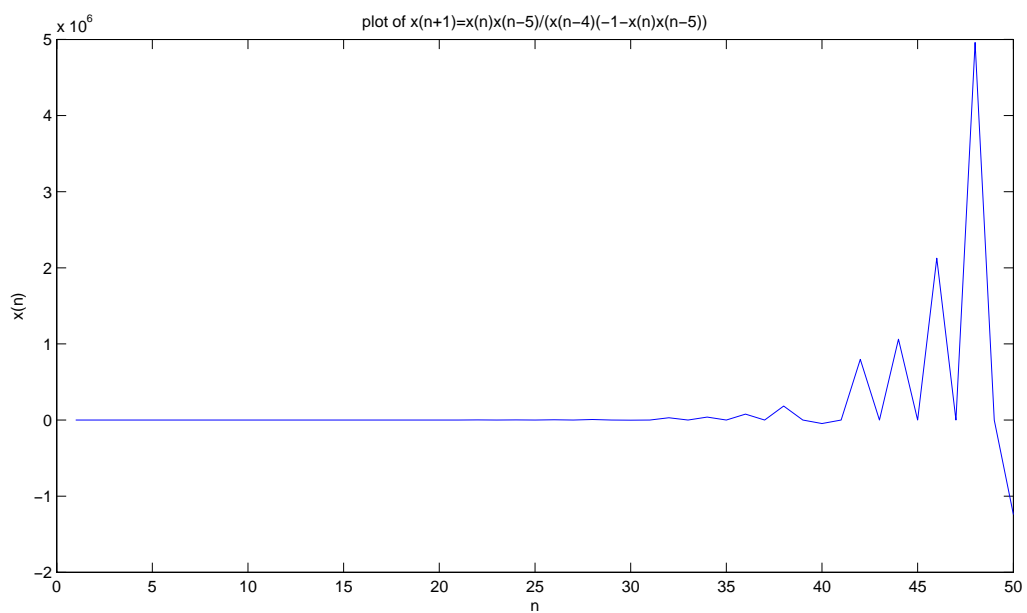
Figure 7.

Example 8. Fig. 8 illustrates the solutions when $x_{-5} = -7$, $x_{-4} = 1.5$, $x_{-3} = -3$, $x_{-2} = 2$, $x_{-1} = 12$, $x_0 = 2/7$.
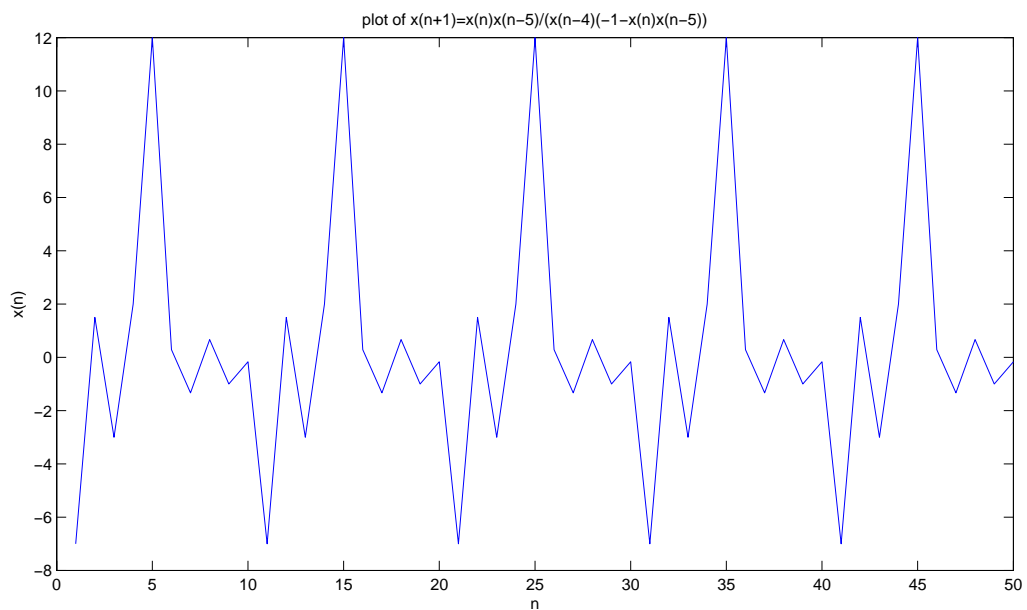
Figure 8.

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A rational bicubic spline for visualization of shaped data

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Abstract

The shaped data usually needs to be represented in such a way that its visual display looks smooth and pleasant, its shape is preserved everywhere and the computation cost is economical. This work contributes to the graphical display of positive or monotone data. For this purpose, a new bicubic rational interpolating spline with biquadratic denominator is developed based on function values and partial derivatives, and simple sufficient conditions are derived on the shape parameters in the description of the rational function to visualize the positive or monotone data in the view of positive or monotone surfaces.

Keywords: Rational bicubic spline, shape parameter, positivity, monotonicity.

1 Introduction

The construction method of curve and surface and the mathematical description of them is a important issue in Computer-Aided Geometry Design (CAGD). Generally speaking, the interpolating data are often given as a set of values, in order to display these data, it is first necessary to construct an interpolant through those data; and then this interpolant is used in the subsequent contouring or curve and surface drawing. Thus, for the data obtained from some complex function or from some scientific phenomena, smooth curve or surface expression becomes crucial to incorporate the inherited features of the data. In many problems of industrial design and manufacturing, the given data often have some special shape properties, such as positivity, monotonicity and convexity, it is usually needed to generate a smooth function, which passes through the given set of data and preserves those certain shape properties of the data.

In recent years, a good amount of work has been published that focuses on shape preserving curves and surfaces. Goodman and Ong [7] presented a local convexity preserving interpolation scheme using parametric C^2 cubic splines with uniform knots produced by a vector subdivision scheme. In [2, 4, 8, 9, 13, 14, 16, 18], several shape-preserving rational curves were shown for shaped data, such as positive data, monotonic data and convex data. Beatson and Ziegler [1] interpolated monotone data, given on a rectangular grid, with a C^1 monotone quadratic spline, and derived necessary and sufficient conditions to visualize monotone data. Floater and Pena [6] defined three kinds of monotonicity preservation of systems of bivariate functions on a triangle, and investigated some geometric applications. In [10], authors proposed a kind of monotonicity-preserving interpolating schemes for 2D/3D monotone data by constraints on shape parameters in the description of rational spline interpolants. In [12], Hussain et al. presented the C^1 rational bi-cubic local interpolation schemes for the shape preservation of convex, monotone and positive surface data. In [11], A bi-quadratic trigonometric interpolation scheme with four free parameters is developed for the positive and monotone 3D data. Piah et al. [15] discussed the problem of positivity preserving for scattered data interpolation.

In [5], Duan et al. developed a C^1 bivariate rational spline interpolant based on function values and partial derivatives under some suitable hypotheses. In [17], Sun et al. proposed a surface modeling method by using C^2 piecewise rational spline interpolation. This paper is concerned with the preservation of 3D positive data and monotone data. To solve the problem, motivated by [5, 17], we will first construct a new bicubic rational interpolating spline with biquadratic denominator based function values and derivative values of an original function. Further more, a positivity-preserving scheme and a monotonicity-preserving scheme are developed to visualize 3D positive data and monotone data in the view of positive surfaces and monotone surfaces, respectively.

This paper is arranged as follows. Section 2 describes about the C^1 bicubic rational spline interpolant to be used in the surface schemes. In Section 3, the positivity problem is discussed for the generation of a smooth surface which can preserve the shape of positive data. In Section 4, a method is developed to preserve the shape of monotone data in the view of monotone surfaces by making constraints on shape parameters in the description of bicubic rational interpolant. Finally, Numerical examples are presented to discuss and demonstrate the performance of the method in Section 5.

2　Bicubic rational interpolant

Let $\Omega = [a, b; c, d]$ be the plane region, and $\{(x_i, y_j, f_{i,j}) : i = 1, 2, \cdots, n; j = 1, 2, \cdots, m\}$ be a given set of data points, where $a = x_1 < x_2 < ... < x_n = b$, $c = y_1 < y_2 < \cdots < y_m = d$ are the knot spacings, $f_{i,j}$ represents $f_{i,j}(x, y)$ at the point (x_i, y_j). Let $d^*_{i,j}$ and $d_{i,j}$ be chosen partial derivative values $\frac{\partial f(x,y)}{\partial x}$ and $\frac{\partial f(x,y)}{\partial y}$ at the knots (x_i, y_j), respectively. Denote $h_i = x_{i+1} - x_i$, $l_j = y_{j+1} - y_j$, $I = \{1, 2, \cdots, n\}$, $J = \{1, 2, \cdots, m\}$, and for any point $(x, y) \in [x_i, x_{i+1}; y_j, y_{j+1}]$, $\theta := \frac{x-x_i}{h_i}$, $\eta := \frac{y-y_j}{l_j}$. Let $\alpha^*_{i,j}, \beta^*_{i,j}$ and $\gamma^*_{i,j}$ be the positive parameters. First, we construct the x-direction interpolating curve $P^*_{i,j}(x)$ in $[x_i, x_{i+1}]$, this is given by

$$P^*_{i,j}(x) = \frac{(1-\theta)^3 \alpha^*_{i,j} f_{i,j} + \theta(1-\theta)^2 V^*_{i,j} + \theta^2(1-\theta) W^*_{i,j} + \theta^3 \beta^*_{i,j} f_{i+1,j}}{(1-\theta)^2 \alpha^*_{i,j} + \theta(1-\theta)\gamma^*_{i,j} + \theta^2 \beta^*_{i,j}}, \tag{1}$$

where

$$V^*_{i,j} = (\alpha^*_{i,j} + \gamma^*_{i,j}) f_{i,j} + h_i \alpha^*_{i,j} d^*_{i,j},$$
$$W^*_{i,j} = (\beta^*_{i,j} + \gamma^*_{i,j}) f_{i+1,j} - h_i \beta^*_{i,j} d^*_{i+1,j},$$

The interpolant $P^*_{i,j}(x)$ is called a rational cubic spline interpolation in $[x_1, x_n]$, and which satisfies:

$$P^*_{i,j}(x_i) = f_{i,j}, \ P^*_{i,j}(x_{i+1}) = f_{i+1,j}, \ P^{*}_{i,j}{}'(x_i) = d^*_{i,j}, \ P^{*}_{i,j}{}'(x_{i+1}) = d^*_{i+1,j}.$$

Using the x-direction $P^*_{i,j}(x)$ defines the bivariate function on $[x_i, x_{i+1}; y_j, y_{j+1}]$ as follow:

$$P(x, y) \equiv P_{i,j}(x, y) = \frac{p_{i,j}(x, y)}{q_{i,j}(y)}, \tag{2}$$

where

$$p_{i,j}(x, y) = (1-\eta)^3 \alpha_{i,j} P^*_{i,j}(x) + \eta(1-\eta)^2 V_{i,j} + \eta^2(1-\eta) W_{i,j} + \eta^3 \beta_{i,j} P^*_{i,j+1}(x),$$
$$q_{i,j}(y) = (1-\eta)^2 \alpha_{i,j} + \eta(1-\eta)\gamma_{i,j} + \eta^3 \beta_{i,j},$$

with

$$V_{i,j} = (\alpha_{i,j} + \gamma_{i,j}) P^*_{i,j}(x) + l_j \alpha_{i,j} D_{i,j}(x),$$
$$W_{i,j} = (\beta_{i,j} + \gamma_{i,j}) P^*_{i,j+1}(x) - l_j \beta_{i,j} D_{i,j+1}(x),$$

and

$$D_{i,j}(x) = \frac{(1-\theta)^2(\alpha_{i,j}^* + \theta\gamma_{i,j}^*)d_{i,j}}{(1-\theta)^2\alpha_{i,j}^* + \theta(1-\theta)\gamma_{i,j}^* + \theta^2\beta_{i,j}^*} + \frac{\theta^2((1-\theta)\gamma_{i,j}^* + \beta_{i,j}^*)d_{i+1,j}}{(1-\theta)^2\alpha_{i,j}^* + \theta(1-\theta)\gamma_{i,j}^* + \theta^2\beta_{i,j}^*}, \tag{3}$$

and $\alpha_{i,j} > 0, \beta_{i,j} > 0, \gamma_{i,j} > 0$. The interpolant $P(x,y)$ defined by (2) is called a bicubic rational interpolation in $[x_1, x_n; y_1, y_m]$, and which satisfies

$$P(x_r, y_s) = f(x_r, y_s), \quad \frac{\partial P(x_r, y_s)}{\partial x} = d_{r,s}^*,$$
$$\frac{\partial P(x_r, y_s)}{\partial y} = d_{r,s}, \quad r = i, i+1; s = j, j+1.$$

It is easy to test that the interpolant $P(x,y)$ is C^1 in the interpolating region $[a, b; c, d]$ if the shape parameters satisfy $\alpha_{i,j}$ =constant, $\beta_{i,j}$ =constant and $\gamma_{i,j}$ =constant, for each $j \in J$ and all $i \in I$, no matter what the shape parameters $a_{i,j}^*, \beta_{i,j}^*$ and $\gamma_{i,j}^*$ might be.

3 Positivity-preserving surface interpolating scheme

In engineering, industrial, and scientific problems, the construction of shape preserving interpolants is an everlasting demand and one of the major research areas of computer aided design. In this section, we identify suitable values for the shape parameters involved in $P(x,y)$ defined by (2), which make the interpolating surface to preserve positive property of given data.

Let $\{(x_i, y_j, f_{i,j}) : i = 1, 2, \cdots, n; j = 1, 2, \cdots, m\}$ be a monotone data set defined over the rectangular grid $[x_i, x_{i+1}; y_j, y_{j+1}]$ such that $f_{i,j} > 0$ for all i, j. Here, the aim is to construct a piecewise rational bivariate function $P(x,y)$ on $\Omega = [x_1, x_n; y_1, y_m]$ such that

$$P(x_i, y_j) = f_{i,j}, \quad i = 1, 2, \cdots, n; j = 1, 2, \cdots, m,$$

and $P(x,y) > 0$ for $(x, y) \in \Omega$. From (2), $q_{i,j}(y) > 0$ for the positive shape parameters, therefore, $P(x,y)$ is positive if the following constraints hold:

$$P_{i,j}^*(x) > 0, \quad V_{i,j} > 0, \quad W_{i,j} > 0, \quad P_{i,j+1}^*(x) > 0.$$

From (1), it is easy to see that $P_{i,j}^*(x) > 0$ holds if the following equalities are satisfied:

$$V_{i,j}^* = (\alpha_{i,j}^* + \gamma_{i,j}^*)f_{i,j} + h_i\alpha_{i,j}^*d_{i,j}^* > 0,$$
$$W_{i,j}^* = (\beta_{i,j}^* + \gamma_{i,j}^*)f_{i+1,j} - h_i\beta_{i,j}^*d_{i+1,j}^* > 0.$$

Thus, $P_{i,j}^*(x) > 0$ if

$$\frac{\gamma_{i,j}^*}{\alpha_{i,j}^*} > -\frac{h_i d_{i,j}^*}{f_{i,j}} - 1, \quad \frac{\gamma_{i,j}^*}{\beta_{i,j}^*} > \frac{h_i d_{i+1,j}^*}{f_{i+1,j}} - 1. \tag{4}$$

Similarly, $P_{i,j+1}^*(x) > 0$ if

$$\frac{\gamma_{i,j+1}^*}{\alpha_{i,j+1}^*} > -\frac{h_i d_{i,j+1}^*}{f_{i,j+1}} - 1, \quad \frac{\gamma_{i,j+1}^*}{\beta_{i,j+1}^*} > \frac{h_i d_{i+1,j+1}^*}{f_{i+1,j+1}} - 1. \tag{5}$$

Consider $V_{i,j}$ which, after simplification, leads to

$$V_{i,j} = \frac{(1-\theta)^3\kappa_1 + \theta(1-\theta)^2\kappa_2 + \theta^2(1-\theta)\kappa_3 + \theta^3\kappa_4}{(1-\theta)^2\alpha_{i,j}^* + \theta(1-\theta)\gamma_{i,j}^* + \theta\beta_{i,j}^*},$$

where

$$\kappa_1 = \alpha_{i,j}^*((\alpha_{i,j} + \gamma_{i,j})f_{i,j} + l_j d_{i,j}\alpha_{i,j}),$$
$$\kappa_2 = (\alpha_{i,j} + \gamma_{i,j})((\alpha_{i,j}^* + \gamma_{i,j}^*)f_{i,j} + h_i d_{i,j}^*\alpha_{i,j}^*) + l_j d_{i,j}\alpha_{i,j}(\alpha_{i,j}^* + \gamma_{i,j}^*),$$
$$\kappa_3 = (\alpha_{i,j} + \gamma_{i,j})((\beta_{i,j}^* + \gamma_{i,j}^*)f_{i+1,j} - h_i d_{i+1,j}^*\beta_{i,j}^*) + l_j d_{i+1,j}\alpha_{i,j}(\beta_{i,j}^* + \gamma_{i,j}^*),$$
$$\kappa_4 = \beta_{i,j}^*((\alpha_{i,j} + \gamma_{i,j})f_{i+1,j} + l_j d_{i+1,j}\alpha_{i,j}).$$

Note that $V_{i,j} > 0$ if

$$\begin{cases} \dfrac{\gamma_{i,j}^*}{\alpha_{i,j}^*} > -\dfrac{2h_i d_{i,j}^*}{f_{i,j}} - 1, \quad \dfrac{\gamma_{i,j}^*}{\beta_{i,j}^*} > \dfrac{2h_i d_{i+1,j}^*}{f_{i+1,j}} - 1, \\ \dfrac{\gamma_{i,j}}{\alpha_{i,j}} > \max\{0, -\dfrac{2h_i d_{i,j}}{f_{i,j}} - 1, -\dfrac{2h_i d_{i+1,j}}{f_{i+1,j}} - 1\}. \end{cases} \tag{6}$$

Moreover, $W_{i,j}$ can be rewritten as

$$W_{i,j} = \frac{(1-\theta)^3\tau_1 + \theta(1-\theta)^2\tau_2 + \theta^2(1-\theta)\tau_3 + \theta^3\tau_4}{(1-\theta)^2\alpha_{i,j}^* + \theta(1-\theta)\gamma_{i,j}^* + \theta\beta_{i,j}^*},$$

where

$$\tau_1 = \alpha_{i,j+1}^*((\beta_{i,j} + \gamma_{i,j})f_{i,j+1} - l_j d_{i,j+1}\beta_{i,j}),$$
$$\tau_2 = (\beta_{i,j} + \gamma_{i,j})((\alpha_{i,j+1}^* + \gamma_{i,j+1}^*)f_{i,j+1} + h_i d_{i,j+1}^*\alpha_{i,j+1}^*) - l_j d_{i,j+1}\beta_{i,j}(\alpha_{i,j+1}^* + \gamma_{i,j+1}^*),$$
$$\tau_3 = (\beta_{i,j} + \gamma_{i,j})((\beta_{i,j+1}^* + \gamma_{i,j+1}^*)f_{i+1,j+1} - h_i d_{i+1,j+1}^*\beta_{i,j+1}^*) - l_j d_{i+1,j+1}\beta_{i,j}(\beta_{i,j+1}^* + \gamma_{i,j+1}^*),$$
$$\tau_4 = \beta_{i,j+1}^*((\beta_{i,j} + \gamma_{i,j})f_{i+1,j+1} - l_j d_{i+1,j+1}\beta_{i,j}).$$

Thus, it is easy to derive that $V_{i,j} > 0$ if

$$\begin{cases} \dfrac{\gamma_{i,j+1}^*}{\alpha_{i,j+1}^*} > -\dfrac{2h_i d_{i,j+1}^*}{f_{i,j+1}} - 1, \quad \dfrac{\gamma_{i,j+1}^*}{\beta_{i,j+1}^*} > \dfrac{2h_i d_{i+1,j+1}^*}{f_{i+1,j+1}} - 1, \\ \dfrac{\gamma_{i,j}}{\beta_{i,j}} > \max\{0, \dfrac{2l_j d_{i,j+1}}{f_{i,j+1}} - 1, \dfrac{2l_j d_{i+1,j+1}}{f_{i+1,j+1}} - 1\}. \end{cases} \tag{7}$$

Based on the analysis above, from (4)-(7), the following theorem can be obtained.

Theorem 1. *Let $\{(x_i, y_j, f_{i,j}) : i = 1, 2, \cdots, n; j = 1, 2, \cdots, m\}$ be a positive data set defined over the plane region $[x_1, x_n; y_1, y_m]$ such that $f_{i,j} > 0$ for all i and j, where $f_{i,j}$ represents $f_{i,j}(x, y)$ at the point (x_i, y_j). Let $d_{i,j}^*$ and $d_{i,j}$ be the chosen partial derivatives. Then the bivariate rational spline interpolant $P(x, y)$ defined in (2) visualize positive data in the view of positive surface if the positive shape parameters satisfy the following constraints:*

$$\frac{\gamma_{i,j}^*}{\alpha_{i,j}^*} > \max\{0, -\frac{2h_i d_{i,j}^*}{f_{i,j}} - 1\},$$
$$\frac{\gamma_{i,j}^*}{\beta_{i,j}^*} > \max\{0, \frac{2h_i d_{i+1,j}^*}{f_{i+1,j}} - 1\},$$
$$\frac{\gamma_{i,j}}{\alpha_{i,j}} > \max\{0, -\frac{2h_i d_{i,j}}{f_{i,j}} - 1, -\frac{2h_i d_{i+1,j}}{f_{i+1,j}} - 1\},$$
$$\frac{\gamma_{i,j}}{\beta_{i,j}} > \max\{0, \frac{2l_j d_{i,j+1}}{f_{i,j+1}} - 1, \frac{2l_j d_{i+1,j+1}}{f_{i+1,j+1}} - 1\}.$$

4   Monotonicity-preserving surface interpolating scheme

In this section, we will develop a monotonicity preserving surface interpolating scheme for the given monotone interpolating data.

Denote $\Delta_{i,j}^{(x)} = (f_{i+1,j} - f_{i,j})/h_i$, $\Delta_{i,j}^{(y)} = (f_{i,j+1} - f_{i,j})/l_j$. Let $\{(x_i, y_j, f_{i,j}) : i = 1, 2, \cdots, n; j = 1, 2, \cdots, m\}$ be a monotone data set defined over the rectangular grid $[x_i, x_{i+1}; y_j, y_{j+1}]$ such that $f_{i+1,j} > f_{i,j}$, $f_{i,j+1} > f_{i,j}$ for all i, j, or equivalently $\Delta_{i,j}^{(x)} > 0$, $\Delta_{i,j}^{(y)} > 0$. For a monotone surface $P(x, y)$, it is necessary that the corresponding first partial derivatives $d_{i,j}^*$ and $d_{i,j}$ should meet:

$$d_{i,j}^* > 0, \quad d_{i,j} > 0, \quad \text{for all} \quad i = 1, 2, \cdots, n; j = 1, 2, \cdots, m.$$

Using the result developed in [3]: *Bicubic partially blended surface patch inherits all the properties of network of boundary curves*, we just need to consider the monotonicity of the boundary curves in the interpolating surface.

The function $P(x, y_j)$ is monotonic increasing if and only if $P'(x, y_j) > 0$. From (2), we can derive that

$$P'(x, y_j) = \frac{(1-\theta)^4 C_1 + \theta(1-\theta)^3 C_2 + \theta^2(1-\theta)^2 C_3 + \theta^3(1-\theta)C_4 + \theta^4 C_5}{((1-\theta)^2 \alpha_{i,j}^* + \theta(1-\theta)\gamma_{i,j}^* + \theta^2 \beta_{i,j}^*)^2}, \tag{8}$$

with

$$C_1 = \alpha_{i,j}^{*2} d_{i,j},$$
$$C_2 = 2\alpha_{i,j}^*[(\beta_{i,j}^* + \gamma_{i,j}^*)\Delta_{i,j}^{(x)} - \beta_{i,j}^* d_{i+1,j}],$$
$$C_3 = (\gamma_{i,j}^{*2} + \gamma_{i,j}^*(\alpha_{i,j}^* + \beta_{i,j}^*) + 4\alpha_{i,j}^*\beta_{i,j}^*)\Delta_{i,j}^{(x)} - \alpha_{i,j}^*(\beta_{i,j}^* + \gamma_{i,j}^*)d_{i,j} - \beta_{i,j}^*(\alpha_{i,j}^* + \gamma_{i,j}^*)d_{i+1,j},$$
$$C_4 = 2\beta_{i,j}^*[(\alpha_{i,j}^* + \gamma_{i,j}^*)\Delta_{i,j}^{(x)} - \alpha_{i,j}^* d_{i,j}],$$
$$C_5 = \beta_{i,j}^{*2} d_{i+1,j}.$$

It is evident that C_1 and C_5 are positive, and $C_2 > 0$ if $\frac{\gamma_{i,j}^*}{\alpha_{i,j}^*} > \frac{d_{i,j}^*}{\Delta_{i,j}^{(x)}} - 1$, $C_4 > 0$ if $\frac{\gamma_{i,j}^*}{\beta_{i,j}^*} > \frac{d_{i+1,j}^*}{\Delta_{i,j}^{(x)}} - 1$. Further, since

$$C_3 = \frac{1}{2}(C_2 + C_4) + (\gamma_{i,j}^{*2} + 2\alpha_{i,j}^*\beta_{i,j}^*)\Delta_{i,j}^{(x)} - \alpha_{i,j}^*\gamma_{i,j}^* d_{i,j} - \beta_{i,j}^*\gamma_{i,j}^* d_{i+1,j}$$
$$\geq \frac{1}{2}(C_2 + C_4) + \gamma_{i,j}^*(\frac{1}{2}\gamma_{i,j}^*\Delta_{i,j}^{(x)} - \alpha_{i,j}^* d_{i,j}) + \gamma_{i,j}^*(\frac{1}{2}\gamma_{i,j}^*\Delta_{i,j}^{(x)} - \beta_{i,j}^* d_{i+1,j}),$$

it is easy to see that $C_3 > 0$ if $\frac{\gamma_{i,j}^*}{\alpha_{i,j}^*} > \frac{2d_{i,j}^*}{\Delta_{i,j}^{(x)}}$ and $\frac{\gamma_{i,j}^*}{\beta_{i,j}^*} > \frac{2d_{i+1,j}^*}{\Delta_{i,j}^{(x)}}$.

Similarly, we have

$$P'(x_i, y) = \frac{(1-\eta)^4 K_1 + 2\eta(1-\eta)^3 K_2 + \eta^2(1-\eta)^2 K_3 + \eta^3(1-\eta)K_4 + \eta^4 K_5}{((1-\eta)^2 \alpha_{i,j} + \eta(1-\eta)\gamma_{i,j} + \eta^2 \beta_{i,j})^2}, \tag{9}$$

where

$$K_1 = \alpha_{i,j}^2 d_{i,j},$$
$$K_2 = 2\alpha_{i,j}[(\beta_{i,j} + \gamma_{i,j})\Delta_{i,j}^{(y)} - \beta_{i,j} d_{i,j+1}],$$
$$K_3 = (\gamma_{i,j}^2 + \gamma_{i,j}(\alpha_{i,j} + \beta_{i,j}) + 4\alpha_{i,j}\beta_{i,j})\Delta_{i,j}^{(y)} - \beta_{i,j}(\alpha_{i,j} + \gamma_{i,j})d_{i,j+1} - \alpha_{i,j}(\beta_{i,j} + \gamma_{i,j})d_{i,j},$$
$$K_4 = 2\beta_{i,j}[(\alpha_{i,j} + \gamma_{i,j})\Delta_{i,j}^{(y)} - \alpha_{i,j} d_{i,j}],$$
$$K_5 = \beta_{i,j}^2 d_{i,j+1}.$$

Hence, $K_i > 0$ $(k = 1, 2, \cdots, 5)$ hold if $\frac{\gamma_{i,j}}{\alpha_{i,j}} > \frac{2d_{i,j}}{\Delta_{i,j}^{(y)}}$ and $\frac{\gamma_{i,j}}{\beta_{i,j}} > \frac{2d_{i,j+1}}{\Delta_{i,j}^{(y)}}$.

Thus, the above discussion is epitomized in the form of following theorem.

Theorem 2. *Let* $\{(x_i, y_j, f_{i,j}) : i = 1, 2, \cdots, n; j = 1, 2, \cdots, m\}$ *be a monotone data set defined over the plane region* $[x_1, x_n; y_1, y_m]$ *such that* $f_{i+1,j} > f_{i,j}$, $f_{i,j+1} > f_{i,j}$ *for all* i *and* j, *where* $a = x_1 < x_2 < ... < x_n = b$, $c = y_1 < y_2 < \cdots < y_m = d$ *are the knot spacings,* $f_{i,j}$ *represents* $f_{i,j}(x, y)$ *at the point* (x_i, y_j). *Let* $d_{i,j}^*$ *and* $d_{i,j}$ *be the chosen partial derivatives, so that* $d_{i,j}^* > 0$ *and* $d_{i,j} > 0$. *Then the bivariate rational spline interpolant* $P(x, y)$ *defined in (2) visualize monotonic data in the view of monotone surface if the positive shape parameters satisfy the following constraints:*

$$\frac{\gamma_{i,j}^*}{\alpha_{i,j}^*} > \frac{2d_{i,j}^*}{\Delta_{i,j}^{(x)}}, \quad \frac{\gamma_{i,j}^*}{\beta_{i,j}^*} > \frac{2d_{i+1,j}^*}{\Delta_{i,j}^{(x)}}, \quad \frac{\gamma_{i,j}}{\alpha_{i,j}} > \frac{2d_{i,j}}{\Delta_{i,j}^{(y)}}, \quad \frac{\gamma_{i,j}}{\beta_{i,j}} > \frac{2d_{i,j+1}}{\Delta_{i,j}^{(y)}}.$$

5 Demonstration

In this section, we shall illustrate the positivity preserving scheme and the monotonicity preserving scheme developed in Sections 3 and 4 with some examples, respectively.

Example 1. First of all, let us take the example of a positive data in Table 1. This data is generated approximately from the following smooth positive function by taking the values truncated to four decimal places:

$$f(x, y) = \exp(-x^2 - 2y) + 0.01, \quad 0 \le x, y \le 8. \tag{10}$$

The interpolant $P(x, y)$ defined by (2) is identified uniquely by the given interpolating data and

Table 1: A positive data set taken from function (10).

y/x	0	2	4	6	8
0	1.0100	0.0283	0.0103	0.0100	0.0100
2	0.0283	0.0103	0.0100	0.0100	0.0100
4	0.0100	0.0100	0.0100	0.0100	0.0100
6	0.0100	0.0100	0.0100	0.0100	0.0100
8	0.0100	0.0100	0.0100	0.0100	0.0100

the values of shape parameters. We take $\alpha_{i,j}^* = \beta_{i,j}^* = \alpha_{i,j} = \beta_{i,j} = 1$ and $\gamma_{i,j}^* = \gamma_{i,j} = 2$, the interpolant coincides with the bicubic Hermite interpolant. Figure 1 shows the graph of corresponding interpolating surface $P_1(x, y)$, which loses the positivity.

For the same data set in table 1, we employ Theorem 1 to compute the values of shape parameters, which are taken as: $\alpha_{i,j}^* = \beta_{i,j}^* = \alpha_{i,j} = \beta_{i,j} = 0.4$ and $\gamma_{i,j}^* = \gamma_{i,j} = 3$. Figure 2 provides the graph of the corresponding bicubic rational interpolant $P_2(x, y)$. It is obvious to see from Figure 2 that the shape of the data has been preserved by the surface representation.

Example 2. Here, we consider the example of a monotonic data in Table 2. This data has been generated approximately from the following smooth function:

$$f(x, y) = \sin(x^2 + y^2 + xy), \quad 0 \le x, y \le 0.6. \tag{11}$$

This example will illustrate how visualization of 3D monotone data can be achieved in the view of monotone surfaces only by selecting suitable shape parameters for unchanged interpolating data in Table 2. For any values of the shape parameters, it cannot be guaranteed that the bicubic rational surface generated by (2) is monotone. For example, we take $\alpha_{i,j}^* = \alpha_{i,j} = 1$, $\beta_{i,j}^* = \beta_{i,j} = 8$ and
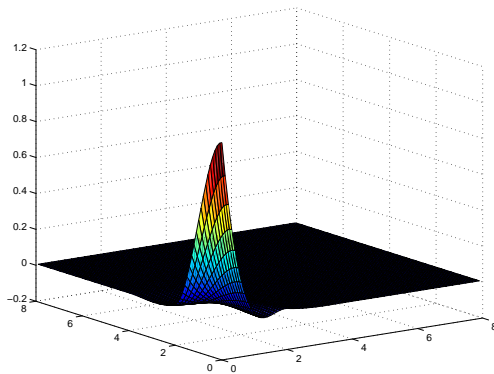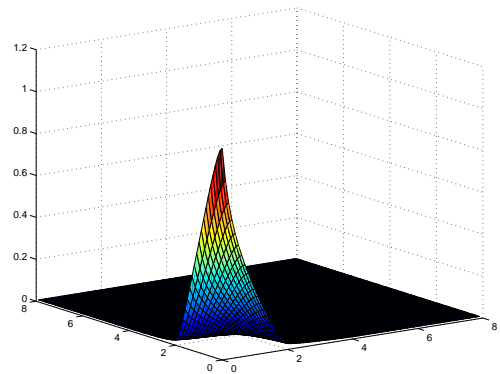
Figure 1: Non-positive Hermite surface P_1.

Figure 2: Positive rational surface P_2.

Table 2: A monotone data set taken from function (11).

y/x	0	0.2	0.4	0.6
0	0	0.0400	0.1593	0.3523
0.2	0.0400	0.1197	0.2764	0.4969
0.4	0.1593	0.2764	0.4618	0.6889
0.6	0.3523	0.4969	0.6889	0.8820

$\gamma_{i,j}^* = \gamma_{i,j} = 0.1$. Figure 3 shows the graph of the corresponding rational surface $P_3(x, y)$, which loses the monotonicity in its display. Figure 4 is a different view of Figure 3 obtained after making a rotation, it confirms quite clearly that the surface is not preserving monotonicity feature.
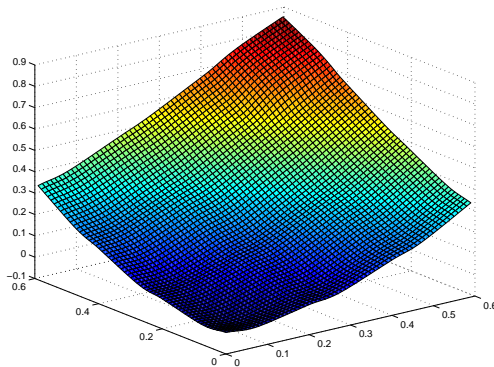
Figure 3: Non-monotonic rational surface P_3.
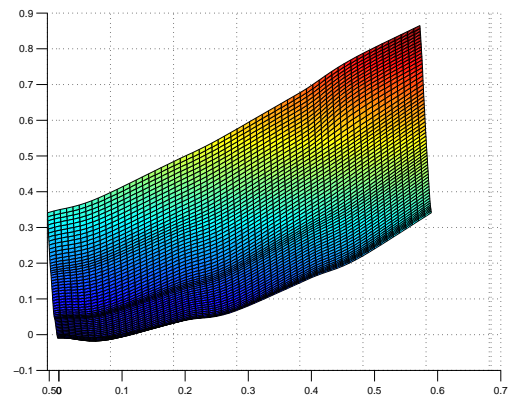
Figure 4: A different view of surface P_3.

Now, we employ Theorem 2 to compute the values of shape parameters, which are taken as: $\alpha_{i,j}^* = \beta_{i,j}^* = \alpha_{i,j} = \beta_{i,j} = 0.6$ and $\gamma_{i,j}^* = \gamma_{i,j} = 1.2$. Figure 5 provides the graph of the corresponding bicubic rational interpolant $P_4(x, y)$. It is obvious to see from Figure 5 that the shape of the monotone data in Table 2 has been preserved by the surface representation. Figure 6 is produced from this data set using a bicubic Hermite interpolant.
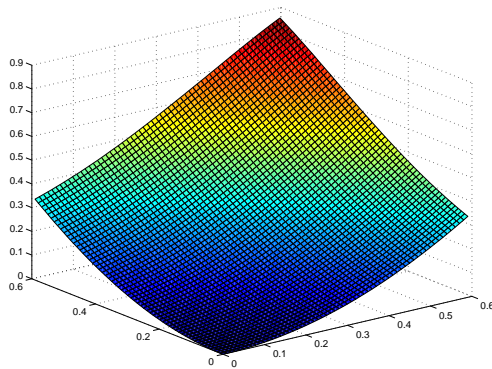
Figure 5: Monotonic rational surface P_4.
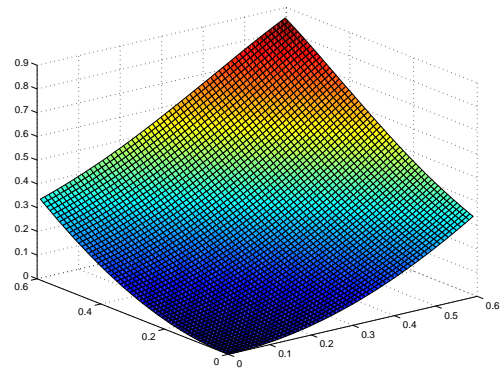
Figure 6: Monotonic Hermite surface P_5.

6 Concluding remarks

In engineering, industrial and scientific problems, the construction of shape preserving interpolants is an everlasting demand and one of the major research areas of computer aided design. This work is a contribution towards the graphical display of data when it is positive or monotone. To overcome the problem, we present a new bicubic rational interpolating spline with the shape parameters based on function values and partial derivatives. Further, a positivity-preserving scheme and a monotonicity-preserving scheme are developed to visualize positive data and monotone data in the view of positive surface and monotone surface, respectively, and the simple sufficient conditions are derived on the shape parameters in the description of the rational function.

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Explicit Viscosity Rules and Applications of Nonexpansive Mappings

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Abstract

This article presents a new explicit viscosity rule for nonexpansive mappings in Hilbert spaces. The strong convergence theorems of the rule is proved under certain assumptions imposed on the sequence of parameters. Moreover, we give applications to a more general system of variational inequalities, the constrained convex minimization problem and K-mapping.

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1 Introduction

In this paper, we will take H as a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$, and C as a nonempty closed subset of the Hilbert space H. A mapping $T : H \to H$ is called *nonexpansive* if

$$\|T(x) - T(y)\| \leq \|x - y\|, \quad \forall x, y \in H.$$

A mapping $f : H \to H$ is called a *contraction* if there exists $\theta \in [0, 1)$ such that

$$\|f(x) - f(y)\| \leq \theta \|x - y\|, \quad \forall x, y \in H.$$

Note that $F(T)$ is the set of fixed points of T. The following strong convergence theorem for nonexpansive mappings in real Hilbert spaces is given by Moudafi [8] in 2000.

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Theorem 1.1. *Let C be a noneempty closed convex subset of the a Hilbert space H. Let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. Let f be a contraction of C into itself with coefficient $\theta \in [0,1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \frac{\epsilon_n}{1+\epsilon_n}f(x_n) + \frac{1}{1+\epsilon_n}T(x_n), \quad n \geq 0,$$

where the sequence $\{\epsilon_n\}$ in $(0,1)$ satisfies
  (i) $\lim_{n\to\infty} \epsilon_n = 0$,
  (2) $\sum_{n=0}^{\infty} \epsilon_n = \infty$,
  (3) $\lim_{n\to\infty} |\frac{1}{\epsilon_{n+1}} - \frac{1}{\epsilon_n}| = 0$.
  Then $\{x_n\}$ converges strongly to a fixed point x^* of the mapping T, which is also the unique solution of the variational inequality

$$\langle (I-f)x, y-x \rangle \leq 0, \quad \forall \in F(T).$$

In other words, x^* is the unique fixed point of the contraction $P_{F(T)}f$, that is, $P_{F(T)}f(x^*) = x^*$.

This type of method for approximation of fixed points is called the *viscosity approximation method.*

In 2015, Xu et al. [11] applied the viscosity method on the midpoint rule for nonexpansive mappings and give the following generalized viscosity implicit rule:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\left(\frac{x_n + x_{n+1}}{2}\right), \quad \forall n \geq 0.$$

This use contraction to regularize the implicit midpoint rule for nonexpansive mappings. They also proved that the sequence generated by the generalized viscosity implicit rule converges strongly to a fixed point of T, which can also solved variational inequality.

Ke and Ma [6] motivated and inspired by the idea of Xu et al. [11] and they proposed two generalized viscosity implicit rules:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T\left(s_n x_n + (1-s_n)x_{n+1}\right)$$

and

$$x_{n+1} = \alpha_n x_n + \beta f(x_n) + \gamma_n T(s_n x_n + (1-s_n)x_{n+1})$$

for $n \geq 0$.

In $[3, 7]$, new viscosity rules and applications are developed. But they correspond to one step viscosity rule.

In this paper, we give the following new two step explicit viscosity rule:

$$\begin{cases} x_{n+1} = (1-\alpha_n)f(x_n) + \alpha_n T(y_n), \\ y_n = (1-\beta_n)x_n + \beta_n T(x_n). \end{cases}$$

We also give many applications of above rule.

2

2   Preliminaries

Now, we recall the properties of the metric projection.

Definition 2.1. $P_C : H \to C$ is called a *metric projection* if for every point $x \in H$, there exist a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

The following lemma gives the condition for a projection mapping to be nonexpansive.

Lemma 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and $P_C : H \to H$ be a metric projection. Then*
(1) *$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle$ for all $x, y \in H$.*
(2) *P_C is a nonexpansive mapping, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$.*
(3) *$\langle x - P_C x, y - P_C x \rangle \leq 0$ for all $x \in H$ and $y \in C$.*

In order to verify the weak convergence of an algorithm to a fixed point of a nonexpansive mapping we need the demiclosedness principle:

Lemma 2.3. (The demiclosedness principle) ([2]) *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ such that*

$$x_n \rightharpoonup x^* \in C \quad and \quad (I - T)x_n \to 0.$$

Then $x^* = Tx^*$, where $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively.

In addition, we also need the following convergence lemma.

Lemma 2.4. ([11]) *Assume that $\{x_n\}$ is a sequence of non-negative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \forall n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and$\{\delta_n\}$ is a sequence such that
(1) *$\sum_{n=0}^{\infty} \gamma_n = \infty$,*
(2) *$\lim_{n \to \infty} \sup \frac{\delta_n}{\gamma_n} \leq 0 or \sum_{n=0}^{\infty} |\delta_n| < \infty$.*
Then $\lim_{n \to \infty} a_n = 0$ as $n \to \infty$.

3   Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_{n+1} = (1 - \alpha_n)f(x_n) + \alpha_n T(y_n), \\ y_n = (1 - \beta_n)x_n + \beta_n T(x_n), \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

3

(1) $\lim_{n\to\infty} \alpha_n = 1$ and $\lim_{n\to\infty} \beta_n = 1$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\sum_{n=0}^{\infty} \beta_n = \infty$,

(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ and $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\forall n \geq 0$,

(iv) $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$.

Then $\{x_n\}$ converges strongly to a fixed point x^* of the mapping T which is also the unique solution of the variational inequality

$$\langle (I - f)x, y - x \rangle \geq 0, \quad \forall y \in F(T).$$

In other words, x^* is the unique fixed point of the contraction $P_{F(T)}f$, that is, $P_{F(T)}f(x^*) = x^*$.

Proof. We divide the proof into the following five steps.

Step 1. Firstly, we show that x_n is bounded.

Indeed, take $p \in F(T)$ arbitrarily, we have

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|(1 - \alpha_n)f(x_n) + \alpha_n T(y_n) - p\| \\
&= \|(1 - \alpha_n)f(x_n) - (1 - \alpha_n)p + \alpha_n T(y_n) - \alpha_n p\| \\
&\leq (1 - \alpha_n)\|f(x_n) - p\| + \alpha_n\|T(y_n) - p\| \\
&\leq (1 - \alpha_n)\|f(x_n) - f(p)\| + (1 - \alpha_n)\|f(p) - p\| \qquad (3.2) \\
&\quad + \alpha_n\|y_n - p\| \\
&\leq (1 - \alpha_n)\theta\|x_n - p\| + (1 - \alpha_n)\|f(p) - p\| \\
&\quad + \alpha_n\|y_n - p\|.
\end{aligned}
$$

Now, consider

$$
\begin{aligned}
\|y_n - p\| &= \|(1 - \beta_n)x_n + \beta_n T(x_n) - p\| \\
&= \|(1 - \beta_n)x_n - (1 - \beta_n)p + \beta_n T(x_n) - \beta_n p\| \\
&\leq (1 - \beta_n)\|x_n - p\| + \beta_n\|T(x_n) - p\| \\
&\leq (1 - \beta_n)\|x_n - p\| + \beta_n\|x_n - p\| \\
&\leq \|x_n - p\|.
\end{aligned}
$$

Using this in (3.2) we have

$$
\begin{aligned}
\|x_{n+1} - p\| &\leq (1 - \alpha_n)\theta\|x_n - p\| + (1 - \alpha_n)\|f(p) - p\| + \alpha_n\|x_n - p\| \\
&= [(1 - \alpha_n)\theta + \alpha_n]\|x_n - p\| + (1 - \alpha_n)\|f(p) - p\| \\
&= [1 - 1 + \alpha + (1 - \alpha_n)\theta]\|x_n - p\| + (1 - \alpha_n)\|f(p) - p\| \\
&= [1 - (1 - \alpha) + (1 - \alpha_n)\theta]\|x_n - p\| + (1 - \alpha_n)\|f(p) - p\| \\
&= [1 - (1 - \alpha_n)(1 - \theta)]\|x_n - p\| \\
&\quad + (1 - \alpha_n)(1 - \theta)\left(\frac{1}{1 - \theta}\|f(p) - p\|\right).
\end{aligned}
$$

Thus, we have

$$\|x_{n+1} - p\| \leq \max\left\{\|x_n - p\|, \left(\frac{1}{1 - \theta}\|f(p) - p\|\right)\right\}.$$

4

Similarly

$$\|x_n - p\| \leq \max\left\{\|x_{n-1} - p\|, \left(\frac{1}{1-\theta}\|f(p) - p\|\right)\right\}.$$

From this

$$\|x_{n+1} - p\| \leq \max\left\{\|x_n - p\|, \left(\frac{1}{1-\theta}\|f(p) - p\|\right)\right\}$$

$$\leq \max\left\{\|x_{n-1} - p\|, \left(\frac{1}{1-\theta}\|f(p) - p\|\right)\right\}$$

$$\vdots$$

$$\leq \max\left\{\|x_0 - p\|, \left(\frac{1}{1-\theta}\|f(p) - p\|\right)\right\},$$

which shows that $\{x_n\}$ is bounded. From this we deduce immediately that $\{f(x_n)\}$, $\{T(x_n)\}$ are bounded.

Step 2. Next, we want to prove that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

For this consider

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&= \|(1-\alpha_n)f(x_n) + \alpha_n T(y_n) - (1-\alpha_{n-1})f(x_{n-1}) - \alpha_{n-1}T(y_{n-1})\| \\
&= \|(1-\alpha_n)(f(x_n) - f(x_{n-1})) - (\alpha_n - \alpha_{n-1})f(x_{n-1}) \\
&\quad + \alpha_n(T(y_n) - T(y_{n-1})) + (\alpha_n - \alpha_{n-1})T(y_{n-1})\| \\
&\leq (1-\alpha_n)\theta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|T(y_{n-1}) - f(x_{n-1})\| \\
&\quad + \alpha_n\|y_n - y_{n-1}\|.
\end{aligned} \tag{3.3}$$

Now, consider

$$\begin{aligned}
&\|y_n - y_{n-1}\| \\
&= \|(1-\beta_n)x_n + \beta_n T(x_n) - (1-\beta_{n-1})x_{n-1} - \beta_{n-1}T(x_{n-1})\| \\
&= \|(1-\beta_n)(x_n - x_{n-1}) - (\beta_n - \beta_{n-1})x_{n-1} + \beta_n(T(x_n) - T(x_{n-1})) \\
&\quad + (\beta_n - \beta_{n-1})T(x_{n-1})\| \\
&\leq (1-\beta_n)\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|T(x_{n-1}) - x_{n-1}\| + \beta_n\|x_n - x_{n-1}\| \\
&\leq \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|T(x_{n-1}) - x_{n-1}\|.
\end{aligned}$$

Using this in (3.3) we get

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\leq (1-\alpha_n)\theta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|T(y_{n-1}) - f(x_{n-1})\| \\
&\quad + \alpha_n\|x_n - x_{n-1}\| + \alpha_n|\beta_n - \beta_{n-1}|\|T(x_{n-1}) - x_{n-1}\| \\
&= [(1-\alpha_n)\theta + \alpha_n]\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|T(y_{n-1}) - f(x_{n-1})\| \\
&\quad + \alpha_n|\beta_n - \beta_{n-1}|\|T(x_{n-1}) - x_{n-1}\|.
\end{aligned}$$

Let $\lambda_n = (1-\alpha_n)$ so $\lambda_n \in (0,1)$ since $\alpha_n \in (0,1)$ $\sum_{n=0}^{\infty}\lambda_n = \infty$, $\sum_{n=0}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$ and $\sum_{n=0}^{\infty}|\beta_n - \beta_{n-1}| < \infty$. By using Lemma 2.4, we get $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Step 3. Now we want to prove that $\lim_{n\to\infty}\|x_n - T(y_n)\| = 0$.

5

Now, consider

$$\begin{aligned}
\|x_n - T(y_n)\| &\leq \|x_n - T(x_n)\| + \|T(x_n) - T(y_n)\| \\
&\leq \|x_n - T(x_n)\| + \|x_n - y_n\| \\
&= \|x_n - T(x_n)\| + \|x_n - (1 - \beta_n)x_n - \beta_n T(x_n)\| \\
&\leq \|x_n - T(x_n)\| + \beta_n \|x_n - T(x_n)\| \\
&\leq (1 + \beta_n)\|x_n - T(x_n)\| \\
&\rightarrow 0 \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

Step 4. In this step, we claim that $\limsup_{n\to\infty}\langle x^* - f(x^*), x^* - x_n\rangle \leq 0$, where $x^* = P_{F(T)}f(x^*)$.

Indeed, we take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to a fixed point p of T. Without loss of generality, we may assume that $\{x_{n_i}\} \rightharpoonup p$. From $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and Lemma 2.3 we have $p = Tp$. This together with the property of the metric projection implies that

$$\begin{aligned}
\limsup_{n\to\infty}\langle x^* - f(x^*), x^* - x_n\rangle &= \limsup_{n\to\infty}\langle x^* - f(x^*), x^* - x_{n_i}\rangle \\
&= \langle x^* - f(x^*), x^* - p\rangle \\
&\leq 0.
\end{aligned}$$

Step 5. Finally, we show that $x_n \to x^*$ as $n \to \infty$.

Here again $x^* \in F(T)$ is the unique fixed point of the contraction $P_{F(T)}f$. Consider

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(1 - \alpha_n)f(x_n) + \alpha_n T(y_n) - x^*\|^2 \\
&= \|(1 - \alpha_n)[f(x_n) - x^*] + \alpha_n[T(y_n) - x^*]\|^2 \\
&= (1 - \alpha_n)_n^2\|f(x_n) - x^*\|^2 + (1 - \alpha_n)^2\|T(x_n) - x^*\|^2 \\
&\quad + 2\alpha_n(1 - \alpha_n)\langle f(x_n) - x^*, T(y_n) - x^*\rangle \\
&\leq \alpha_n^2\|y_n - x^*\|^2 + (1 - \alpha_n)^2\|f(x_n) - x^*\|^2 \\
&\quad + 2\alpha_n(1 - \alpha_n)\langle f(x_n) - f(x^*), T(y_n) - x^*\rangle \\
&\quad + 2\alpha_n(1 - \alpha_n)\langle f(x^*) - x^*, T(y_n) - x^*\rangle \\
&\leq \alpha_n^2\|y_n - x^*\|^2 + (1 - \alpha_n)^2\|f(x_n) - x^*\|^2 \\
&\quad + 2\alpha_n(1 - \alpha_n)\|f(x_n) - f(x^*)\|\|T(y_n) - x^*\| \\
&\quad + 2\alpha_n(1 - \alpha_n)\langle f(x^*) - x^*, T(y_n) - x^*\rangle \\
&\leq \alpha_n^2\|y_n - x^*\|^2 + 2\alpha_n(1 - \alpha_n)\theta\|x_n - x^*\|\|y_n - x^*\| \\
&\quad + (1 - \alpha_n)^2\|f(x_n) - x^*\|^2 \\
&\quad + 2\alpha_n(1 - \alpha_n)\langle f(x^*) - x^*, T(y_n) - x^*\rangle.
\end{aligned}$$
(3.4)

Now, consider

$$\begin{aligned}
\|y_n - x^*\| &= \|(1 - \beta_n)x_n + \beta_n T(x_n) - x^*\| \\
&= \|(1 - \beta_n)x_n - (1 - \beta_n)x^* + \beta_n T(x_n) - \beta_n x^*\| \\
&\leq (1 - \beta_n)\|x_n - x^*\| + \beta_n\|T(x_n) - x^*\| \\
&\leq (1 - \beta_n)\|x_n - x^*\| + \beta_n\|x_n - x^*\| \\
&\leq \|x_n - x^*\|.
\end{aligned}$$

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Using (3.5) in (3.4) we get

$$
\begin{aligned}
\|x_{n+1} &- x^*\|^2 \\
&\leq \alpha_n^2 \|x_n - x^*\|^2 + 2\alpha_n(1-\alpha_n)\theta\|x_n - x^*\|\|x_n - x^*\| \\
&\quad + (1-\alpha_n)^2\|f(x_n) - x^*\|^2 + 2\alpha_n(1-\alpha_n)\langle f(x^*) - x^*, T(y_n) - x^*\rangle \\
&\leq \alpha_n^2\|x_n - x^*\|^2 + 2\alpha_n(1-\alpha_n)\theta\|x_n - x^*\|\|x_n - x^*\| \\
&\quad + (1-\alpha_n)^2\|f(x_n) - x^*\|^2 + 2\alpha_n(1-\alpha_n)\langle f(x^*) - x^*, T(y_n) - x^*\rangle \\
&\leq [\alpha_n^2 + 2\alpha_n(1-\alpha_n)\theta]\|x_n - x^*\|^2 + (1-\alpha_n)^2\|f(x_n) - x^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x^*) - x^*, T(y_n) - x^*\rangle.
\end{aligned}
$$

Note that $\alpha_n\theta < \alpha_n$ since $\alpha_n \in (0,1)$ and $\theta \in [0,1)$.

$$2\alpha_n\theta < 2\alpha_n$$

implies

$$2\alpha_n\theta(1-\alpha_n) < 2\alpha_n(1-\alpha_n)$$

implies

$$\alpha_n^2 + 2\alpha_n\theta(1-\alpha_n) < \alpha_n^2 + 2\alpha_n(1-\alpha_n).$$

So we have

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq [\alpha_n^2 + 2\alpha_n(1-\alpha_n)]\|x_n - x^*\|^2 + (1-\alpha_n)^2\|f(x_n) - x^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x^*) - x^*, T(y_n) - x^*\rangle \\
&\leq [2\alpha_n - \alpha_n^2]\|x_n - x^*\|^2 + (1-\alpha_n)^2\|f(x_n) - x^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x^*) - x^*, T(y_n) - x^*\rangle \\
&\leq 2\alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)^2\|f(x_n) - x^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x^*) - x^*, T(y_n) - x^*\rangle \\
&\leq 2[1-(1-\alpha_n)]\|x_n - x^*\|^2 + (1-\alpha_n)^2\|f(x_n) - x^*\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle f(x^*) - x^*, T(y_n) - x^*\rangle.
\end{aligned}
\tag{3.6}
$$

By $\lim_{n\to\infty}\alpha_n = 1$ we have

$$
\begin{aligned}
\limsup_{n\to\infty} &\frac{(1-\alpha_n)^2\|f(x_n) - x^*\|^2 + 2\alpha_n(1-\alpha_n)\langle f(x^*) - x^*, T(y_n) - x^*\rangle}{(1-\alpha_n)} \\
&= \limsup_{n\to\infty}[(1-\alpha_n)\|f(x_n) - x^*\|^2 + 2\alpha_n\langle f(x^*) - x^*, T(y_n) - x^*\rangle] \leq 0.
\end{aligned}
\tag{3.7}
$$

From (3.6), (3.7) and Lemma 2.4 we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\|^2 = 0,$$

which implies that $x_n \to x^*$ as $n \to \infty$. This completes the proof. $\square$

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4   Applications

4.1   A more general system of variational inequalities

Let C be a nonempty closed convex subset of a real Hilbert space H and $\{A_i\}_{i=1}^N$: $C \to H$ be a family of mappings. In [1], Cai and Bu considered the problem of finding $x_1^*, x_2^*, ..., x_N^* \in C \times C \times \cdots \times C$ such that

$$\begin{cases} \langle \lambda_N A_N x_N^* + x_1^* - x_N^*, x - x_1^* \rangle \geq 0, \\ \langle \lambda_{N-1} A_{N-1} x_{N-1}^* + x_N^* - x_{N-1}^*, x - x_N^* \rangle \geq 0, \\ \quad\vdots \\ \langle \lambda_2 A_2 x_2^* + x_3^* - x_2^*, x - x_3^* \rangle \geq 0, \\ \langle \lambda_1 A_1 x_1^* + x_2^* - x_1^*, x - x_2^* \rangle \geq 0, \quad \forall x \in C. \end{cases} \tag{4.1}$$

The equation (4.1) can be written as

$$\begin{cases} \langle x_1^* - (I - \lambda_N A_N) x_N^*, x - x_1^* \rangle \geq 0, \\ \langle x_N^* - (I - \lambda_{N-1} A_{N-1}) x_{N-1}^*, x - x_N^* \rangle \geq 0, \\ \quad\vdots \\ \langle x_3^* - (I - \lambda_2 A_2) x_2^*, x - x_3^* \rangle \geq 0, \\ \langle x_2^* - (I - \lambda_1 A_1) x_1^*, x - x_2^* \rangle \geq 0, \quad \forall x \in C, \end{cases}$$

which is a more general system of variational inequalities in a Hilbert space, where $\lambda_i > 0$ for all $i \in \{1, 2, 3, ..., N\}$. We also have following lemmas.

Lemma 4.1. ([6]) *Let C be a nonempty closed convex subset of a real Hilbert spaces H. For $i \in \{1, 2, 3, ..., N\}$, let $A_i : C \to H$ be δ_i-inverse-strongly monotone for some positive real number δ_i, namely,*

$$\langle A_i x - A_i y, x - y \rangle \geq \delta_i \|A_i x - A_i y\|^2, \quad \forall x, y \in C$$

Let $G : C \to C$ be a mapping defined by

$$\begin{aligned} G(x) = P_C(I - \lambda_N A_N) P_C(I - \lambda_{N-1} A_{N-1}) \cdots \\ P_C(I - \lambda_2 A_2) P_C(I - \lambda_1 A_1) x, \quad \forall x \in C. \end{aligned} \tag{4.2}$$

If $0 < \lambda_i \leq 2\delta_i$ for all $i \in \{1, 2, ..., N\}$, then G is nonexpansive.

Lemma 4.2. ([5]) *Let C be a nonempty closed convex subject of a real Hilbert space H. Let $A_i : C \to H$ be a nonlinear mapping, where $i \in \{1, 2, 3, ..., N\}$. For given $x_i^* \in C$, $i \in \{1, 2, 3, ..., N\}$, $(x_1^*, x_2^*, x_3^*, ..., x_N^*)$ is a solution of the problem (4.1) if and onli if*

$$\begin{aligned} x_1^* = P_C(I - \lambda_N A_N) x_N^*, \\ x_i^* = P_C(I - \lambda_{i-1} A_{i-1}) x_{i-1}^*, \quad i = 2, 3, 4, ..., N, \end{aligned} \tag{4.3}$$

that is,

$$\begin{aligned} x_1^* = P_C(I - \lambda_N A_N) P_C(I - \lambda_{N-1} A_{N-1}) \cdots \\ P_C(I - \lambda_2 A_2) P_C(I - \lambda_1 A_1) x_1^*, \quad \forall x \in C. \end{aligned}$$

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From Lemma 4.2, we know that $x_1^* = G(x_1^*)$, that is, x_1^* is a fixed point of the mapping G, where G is defined by (4.2). Moreover, if we find the fixed point x_1^*, it is easy to get the other points by (4.3). Applying Theorem 3.1 we get the result

Theorem 4.3. *Let C be a nonempty closed convex subject of a real Hilbert space H. For $i \in \{1, 2, 3, ..., N\}$, let $A_i : C \to H$ be δ_i-inverse-strongly monotone for some positive real number δ_i with $F(G) \neq \emptyset$, where $G : C \to C$ is defined by*

$$G(x) = P_C(I - \lambda_N A_N)P_C(I - \lambda_{N-1}A_{N-1})\cdots$$
$$P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x, \quad \forall x \in C.$$

Let $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_{n+1} = (1 - \alpha_n)f(x_n) + \alpha_n G(y_n), \\ y_n = (1 - \beta_n)x_n + \beta_n G(x_n), \end{cases}$$

where $\lambda_i \in (0, 2\delta_i)$, $i = 1, 2, 3, ..., N$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying the conditions (i)-(iv).

Then $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping G, which is also the unique solution of the variational inequality

$$\langle (I - f)x, y - x \rangle \geq 0, \quad \forall y \in F(T).$$

In other words, x^* is the unique fixed point of the contraction $P_{F(G)}f$, that is, $P_{F(G)}f(x^*) = x^*$.

4.2   The constrained convex minimization problem

Now, we consider the following constrained convex minimization problem:

$$\min_{x \in C} \phi(x), \tag{4.4}$$

where $\phi : C \to R$ is a real-valued convex function and assumes that the problem (4.4) is consistent. Let Ω denote its solution set.

For the minimization problem (4.4), if ϕ is (Fréchet)differentiable, then we have the following lemma.

Lemma 4.4. (Optimality Condition) ([5]) *A necessary condition of optimality for a point $x^* \in C$ to be a solution of the minimization problem (4.4) is that x^* solves the variational inequality*

$$\langle \nabla \phi(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{4.4}$$

Equivalently, $x^* \in C$ solves the fixed point equation

$$x^* = P_C\left(x^* - \lambda \nabla \phi(x^*)\right)$$

for every constant $\lambda > 0$. If, in a addition ϕ is convex, then the optimality condition (4.5) is also sufficient.

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It is well known that the mapping $P_C(I - \lambda A)$ is nonexpansive when the mapping A is δ-inverse-strongly monotone and $0 < \lambda < 2\delta$. We therefore have the following result.

Theorem 4.5. *Let C be a nonempty closed convex subset of the real Hilbert Space H. For the minimization problem (4.4), assume that ϕ is (Fréchet) differentiable and the gradient $\nabla\phi$ is a δ-inverse-strongly monotone mapping for some positive real number δ. Let $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_{n+1} = (1 - \alpha_n)f(x_n) + \alpha_n P_C(1 - \lambda\nabla\phi)(y_n), \\ y_n = (1 - \beta_n)x_n + \beta_n P_C(1 - \lambda\nabla\phi)(x_n), \end{cases}$$

where $\lambda_i \in (0, 2\delta)$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying the conditions (i)-(iv).

Then $\{x_n\}$ converges strongly to a solution x^* of the minimization problem (4.4), which is also the unique solution of the variational inequality

$$\langle (I - f)x, y - x \rangle \geq 0, \quad \forall y \in \Omega.$$

In other words, x^* is the unique fixed point of the contraction $P_\Omega f$, that is, $P_\Omega f(x^*) = x^*$.

4.3 The K-mapping

In 2009, Kangtunyakarn and Suantai [4] gave a K-mapping generated by $T_1, T_2, T_3, ..., T_N$ and $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$ as follows.

Definition 4.6. Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a family of mappings of C into itself and let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, 2, 3, ..., N$. We define a mapping $K : C \to C$ as follows:

$$\begin{cases} U_1 = \lambda_1 T_1 + (1 - \lambda_1)I, \\ U_2 = \lambda_2 T_2 U_1 + (1 - \lambda_2)U_1, \\ \quad \vdots \\ U_{N-1} = \lambda_{N-1}T_{N-1}U_{N-2} + (1 - \lambda_{N-1})U_{N-2}, \\ K = U_N = \lambda_N T_N U_{N-1} + (1 - \lambda_N)U_{N-1}. \end{cases}$$

Such a mapping is called a K-*mapping* generated by $T_1, T_2, T_3, ..., T_N$ and $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$.

In 2014, Suwannaut and Kangtunyakarn [10] established the following result for K-mapping generated by $T_1, T_2, T_3, ..., T_N$ and $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$.

Lemma 4.7. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, 3, ..., N$, let $\{T_i\}_{i=1}^N$ be a finite family of k_i-strictly pseudo-contractive mapping of C into itself with $k_i \leq \omega_1$ and $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, namely, there exist constants $k_i \in [0, 1)$ such that*

$$\|T_i x - T_i y\|^2 \leq \|x - y\|^2 + k_i\|(I - T_i)x - (I - T_i)y\|^2, \quad \forall x, y \in C.$$

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Let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$ be real numbers with $0 < \lambda_i < \omega_2$ for all $i = 1, 2, 3, ..., N$ and $\omega_1 + \omega_2 < 1$. Let K be the K-mapping generated by $T_1, T_2, T_3, ..., T_N$ and $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$. Then the following properties hold:

(a) $F(K) = \bigcap_{i=1}^{N} F(T_i)$,

(b) K is a nonexpansive mapping.

On the bases of above lemma, we have the following results.

Theorem 4.8. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1, 2, 3, ..., N$, let $\{T_i\}_{i=1}^{N}$ be a finite family of k_i-strictly pseudo-contractive mapping of C into itself with $k_i \leq \omega_1$ and $\bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$ be real numbers with $0 < \lambda_i < \omega_2$ for all $i = 1, 2, 3, ..., N$ and $\omega_1 + \omega_2 < 1$. Let K be the K-mapping generated by $T_1, T_2, T_3, ..., T_N$ and $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_N$. Let $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be sequence generated by*

$$\begin{cases} x_{n+1} = (1 - \alpha_n)f(x_n) + \alpha_n K(y_n), \\ y_n = (1 - \beta_n)x_n + \beta_n K(x_n). \end{cases},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying the conditions (i)-(iv).

Then $\{x_n\}$ converges strongly to a fixed point x^* of the mappings $\{T_i\}_{i=1}^{N}$, which is also the unique solution of the variational inequality

$$\langle (I - f)x, y - x \rangle, \quad \forall y \in F(K) = \bigcap_{i=1}^{N} F(T_i).$$

In other words, x^* is the unique fixed point of the contraction $P_{\bigcap_{i=1}^{N} F(T_i)} f$, that is, $P_{\bigcap_{i=1}^{N} F(T_i)} f(x^*) = x^*$.

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Applications and Strong Convergence Theorems of Asymptotically Nonexpansive Non-self Mappings

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Abstract

We give some strong convergence theorems about asymptotically nonexpansive non-self mappings. In the end we give some applications of our results in the form of examples. Our results extend the results already proved.

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1 Introduction

Fixed points of special mappings like asymptotically nonexpansive, nonexpansive, contractive and other mappings has become a field of interest on its on and has a variety of applications in related fields like image recovery, signal processing and geometry of objects. Almost in all branches of mathematics we see some versions of theorems relating to fixed points of functions of special nature. As a result we apply them in industry, toy making, finance, aircrafts and manufacturing of new model cars. Because of its vast range of applications almost in all directions, the research in it is moving rapidly and an immense literature is present now.

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Let C be a nonempty subset of a real normed space E of A mapping $T : C \to C$ is *asymptotically nonexpansive* if there exist a sequence $\{k_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \le k_n \|x - y\|$ for all $x, y \in C$ and $n \ge 1$. This class was introduced by Goebel and Kirk in 1972, see [4]. Ever since it has occupied a central place in fixed point theory and other related fields concerning about special mappings.

Theorem 1.1. *If C is a nonempty closed convex subset of a real uniformly convex Banach space E and $T : C \to C$ is an asymptotically nonexpansive mapping, then T has a unique fixed point in C.*

However, in 1991, Schu [9] developed the modified Mann process for the approximation of fixed points of an asymptotically nonexpansive self mapping which is defined on a nonempty closed convex bounded subset of a Hilbert space given as follows:

Theorem 1.2. *Let C be a nonempty closed convex bounded subset of a Hilbert space H and $T : C \to C$ be a completely continuous and asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty]$ having $\lim_{n\to\infty} k_n = 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) \le \infty$. Let $\{\alpha_n\}$ be a real sequence in $[0, 1]$ with condition $\epsilon \le \alpha_n \le 1 - \epsilon$ for all $n > 1$ and for some $\epsilon > 0$. Then the sequence $\{x_n\}$ defined recursively by*

$$\begin{cases} x_1 \in C \quad (arbitrarily), \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \ge 1 \end{cases} \tag{1.1}$$

converges strongly to some fixed point of T.

In the above theorem, T is a self mapping of C, where C is a nonempty closed convex subset of H. If, however, the domain $D(T)$ of T is a proper subset of H and $T : D(T) \to H$ is a mapping the modified iteration $\{x_n\}$ may fail to be well defined.

To overcome this, in 2003, Chidume et al. [3] introduce the concept of asymptotically nonexpansive non-self mappings.

Let E be a real Banach space. A subset C of E is called a *retract* of E if there exist a continuous mapping $P : E \to E$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a uniform convex Banach space is a retract. A mapping $P : E \to E$ is called a *retraction* if $P^2 = P$. It is clear that if P is a retraction, then $Py = y$ for all y in the the range of P (see [3]).

Definition 1.3. Let C be a nonempty subset of a normed linear space E. Let $P : E \to C$ be a nonexpansive retraction of E onto C. A non-self mapping $T : C \to E$ is said to be *asymptotically nonexpansive* if there exist a sequence $k_n \subset [0, 1)$ with $\lim_{n\infty} k_n = 1$ such that

$$\|T(PT)^{n-1}(x) - T(PT)^{n-1}(y)\| \le k_n \|x - y\| \tag{1.2}$$

for all $x, y \in C$ and $n \ge 1$.

Chidume et al. [3] introduced the following iterative scheme:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n) \end{cases} \tag{1.3}$$

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for all $n \geq 1$, where $\{\alpha_n\} \subset (0, 1)$ and proved some results of strong convergence and weak convergence for asymptotic nonexpansive non-self mapping.

Throughout this paper, we denote $Fix(T) = \{x \in C : Tx = x\}$.

Remark 1.4. If $T : C \to C$ is a self mapping, then P becomes identity and we have

1. The non-self mapping with (1.2) coincides with an asymptotically nonexpansive self mapping in (1.1).

2. Both iterations (1.3) and (1.1) coincide.

After Chidume et al. many authors proved weak and strong convergence theorems for asymptotically nonexpansive non-self mapping in Banach spaces [6–8, 10]. Guo and Guo [7] introduced following new iterative scheme which is given as:

Let E be a real Banach space, C be a nonempty closed convex subset of E and $P : C \to E$ be a nonexpansive retraction of E onto C. Let $T : C \to E$ be an asymptotically nonexpansive non-self mapping defined by

$$\begin{cases} x_1 \in C, \\ y_n = P((1 - \beta_n)x_n + \beta_n T(PT)^{n-1}x_n), \\ x_{n+1} = P((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}y_n) \end{cases} \tag{1.4}$$

for all $n \geq 1$, where $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \subset [0, 1]$. They proved some theorems of strong convergence and weak convergence of the above iteration for an asymptotically nonexpansive non-self mapping $T : C \to E$.

In this paper, we first introduce a new iterative scheme $\{x_n\}$ defined as follows:

$$\begin{cases} x_1 \in C, \\ y_n = P((1 - \beta_n)x_n + \beta_n T(PT)^{n-1}x_n), \\ z_n = P((1 - \gamma_n)y_n + \gamma_n T(PT)^{n-1}y_n), \\ x_{n+1} = P\left((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}\left(\frac{y_n + z_n}{2}\right)\right) \end{cases} \tag{1.5}$$

for all $n \geq 1$, where $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\}, \{\gamma_n\} \subset [0, 1]$. We first use the condition which is weaker than the completely continuous mappings, given in [5] named as the condition (BP). Secondly, we prove some strong convergence theorems for our iteration scheme for an asymptotically nonexpansive non-self mapping. It is important to remark that our results extend the results in [3, 5]. Finally, we give examples to explain the main results of this paper.

2 Some lemmas

In this section we give some key lemmas which will be used to prove the main results of this paper.

Lemma 2.1. ([11]) *Let $p > 1$ and $R > 0$ and E be a Banach space. Then E is uniformly convex if and only if there exist a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda)y\|^p \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - w_p(\lambda)g(\|x - y\|)$$

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for all $x, y \in B(0, R) = \{x \in E : |x| \leq R\}$ with $\lambda \in [0, 1]$, where $w_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda)$.

Lemma 2.2. *Let E be a real uniformly convex Banach space and C be a nonempty convex subset of E. Let $T : C \to E$ be an asymptotically nonexpansive non-self mapping with $\{h_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(h_n - 1) < \infty$ and $Fix(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (1.5), where $\{\alpha_n\} \subset [0, 1)$ and $\{\beta_n\}, \{\gamma_n\} \subset [0, 1]$. Then*
(a) $\|x_{n+1} - p\| \leq h_n^2 \|x_n - p\|$ for all $p \in Fix(T)$.
(b) $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in Fix(T)$.

Proof. Take $T_2 = T_1 = T$ and $S_2 = S_1 = I$ in [6], we obtain the required result (see [5]). $\square$

Lemma 2.3. *Let E be a real uniform convex Banach space and C be a nonempty closed convex subset of E. Let $T : C \to E$ be an asmyptotically nonexpansive non-self mapping with $\{h_n\}, \{h'_n\} \subset [1, \infty)$ and $h'_n \leq h_n$ such that $\sum_{n=1}^{\infty}(h_n - 1) < \infty$, $\sum_{n=1}^{\infty}(h'_n - 1) < \infty$ and $Fix(T) \neq \emptyset$. Let $\{x_n\}$ a sequence defined in (1.5), where*

$$0 < \liminf_{n\to\infty} \alpha_n, \quad \limsup_{n\to\infty} \alpha_n < 1, \limsup_{n\to\infty} \beta_n < 1, \quad \limsup_{n\to\infty} \gamma_n < 1.$$

Then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof. By Lemma 2.2, we know that $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in Fix(T)$. So $\{x_n - p\}, \{y_n - p\}, \{z_n - p\}, \{T(PT)^{n-1}x_n - p\}, \{T(PT)^{n-1}y_n - p\}$ and $\{T(PT)^{n-1}z_n - p\}$ are all bounded so we have a real number $R > 0$ such that

$$\{x_n - p, y_n - p, z_n - p, T(PT)^{n-1}x_n - p, T(PT)^{n-1}y_n - p \, T(PT)^{n-1}z_n - p\} \subset B(0, R),$$

for all $n \geq 1$. It follows from (1.5) and Lemma 2.1 that

$$\begin{aligned}
\|y_n - p\|^2 &\leq \|(1 - \beta_n)(x_n - p) + \beta_n(T(PT)^{n-1}x_n - p)\|^2 \\
&\leq (1 - \beta_n)\|(x_n - p)\|^2 + \beta_n\|(T(PT)^{n-1}x_n - p)\|^2 \\
&\quad - \beta_n(1 - \beta_n)g(\|x_n - T(PT)^{n-1}x_n\|) \\
&\leq (1 - \beta_n)h_n^2\|(x_n - p)\|^2 + \beta_n h_n^2\|x_n - p\|^2 \\
&= h_n^2\|(x_n - p)\|^2,
\end{aligned}$$

$$\begin{aligned}
\|z_n - p\|^2 &\leq \|(1 - \gamma_n)(y_n - p) + \gamma_n(T(PT)^{n-1}y_n - p)\|^2 \\
&\leq (1 - \gamma_n)\|(y_n - p)\|^2 + \gamma_n\|(T(PT)^{n-1}y_n - p)\|^2 \\
&\quad - \gamma_n(1 - \gamma_n)g(\|y_n - T(PT)^{n-1}y_n\|) \\
&\leq (1 - \gamma_n)h_n'^2\|(y_n - p)\|^2 + \gamma_n h_n'^2\|y_n - p\|^2 \\
&= h_n'^2\|(y_n - p)\|^2 \\
&\leq h_n'^2 h_n^2\|(x_n - p)\|^2 \\
&\leq h_n^4\|(x_n - p)\|^2.
\end{aligned}$$

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Also

$$\|x_{n+1} - p\|^2 \le \left\|(1 - \alpha_n)(x_n - p) + \alpha_n\left(T(PT)^{n-1}\left(\frac{y_n + z_n}{2}\right) - p\right)\right\|^2$$

$$\le (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\left\|\left(T(PT)^{n-1}\left(\frac{y_n + z_n}{2}\right) - p\right)\right\|^2$$
$$- \alpha_n(1 - \alpha_n)g\left(\left\|x_n - T(PT)^{n-1}\left(\frac{y_n + z_n}{2}\right)\right\|\right)$$

$$\le (1 - \alpha_n)h_n^8\|x_n - p\|^2 + \alpha_n h_n'^2 h_n^2\left\|\frac{y_n + z_n}{2} - p\right\|^2$$
$$- \alpha_n(1 - \alpha_n)g\left(\left\|x_n - T(PT)^{n-1}\left(\frac{y_n + z_n}{2}\right)\right\|\right)$$

$$\le (1 - \alpha_n)h_n^8\|x_n - p\|^2 + \frac{1}{2}\alpha_n h_n'^2 h_n^2(\|y_n - p\|^2 + \|z_n - p\|^2)$$
$$- \alpha_n(1 - \alpha_n)g\left(\left\|x_n - T(PT)^{n-1}\left(\frac{y_n + z_n}{2}\right)\right\|\right)$$

$$\le (1 - \alpha_n)h_n^8\|x_n - p\|^2 + \alpha_n h_n^8\|x_n - p\|^2$$
$$- \alpha_n(1 - \alpha_n)g\left(\left\|x_n - T(PT)^{n-1}\left(\frac{y_n + z_n}{2}\right)\right\|\right)$$

$$= h_n^8\|x_n - p\|^2 - \alpha_n(1 - \alpha_n)g\left(\left\|x_n - T(PT)^{n-1}\left(\frac{y_n + z_n}{2}\right)\right\|\right),$$

where $g : [0, \infty) \to [0, \infty)$ is a continuous, strictly increasing and convex function with the condition $g(0) = 0$. Since we have condition $0 < \liminf_{n\to\infty} \alpha_n$ and $\limsup_{n\to\infty} \alpha_n < 1$. So there exist two real numbers $a, b \in (0, 1)$ and a positive integer n_0 such that $a \le \alpha_n \le b$ for all $n \ge n_0$, so

$$a(1 - b)g(\| x_n - T(PT)^{n-1}y_n \|) \le h^4 \| x_n - p \|^2 - \| x_{n+1} - p \|^2$$

and

$$a(1 - b)g(\| x_n - T(PT)^{n-1}z_n \|) \le h'^4 \| x_n - p \|^2 - \| x_{n+1} - p \|^2 .$$

Since $\lim_{n\to\infty} h_n = 1$ and $\lim_{n\to\infty} h'_n = 1$, there exist positive integer m_0, to and real number $s, s' \in (0, 1)$ such that $\beta_n h_n \le s$ and $\gamma_n h'_n \le s'$ for all $n > m_0$ and

$$\|y_n - x_n\| \le \beta_n\|T(PT)^{n-1}x_n - x_n\|$$
$$\le \beta_n\|T(PT)^{n-1}x_n - T(PT)^{n-1}y_n\| + \beta_n\|x_n - T(PT)^{n-1}y_n\|$$
$$\le \beta_n h_n\|x_n - y_n\| + \|x_n - T(PT)^{n-1}y_n\|.$$

Similarly,

$$\|z_n - x_n\| \le \gamma_n h'_n\|x_n - z_n\| + \|x_n - T(PT)^{n-1}z_n\|.$$

Hence

$$(1 - s)\|y_n - x_n\| \le (1 - \beta_n h_n)\|y_n - x_n\| \le \|x_n - T(PT)^{n-1}y_n\|$$

and

$$(1 - s')\|z_n - x_n\| \le (1 - \gamma_n h'_n)\|z_n - x_n\| \le \|x_n - T(PT)^{n-1}z_n\|.$$

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So we have
$$\lim_{n \to \infty} \|y_n - x_n\| = 0 \quad \text{and} \quad \lim_{n \to \infty} \|z_n - x_n\| = 0.$$

Furthermore from
$$\|x_n - T(PT)^{n-1} x_n\| \leq \|x_n - T(PT)^{n-1} y_n\| + \|T(PT)^{n-1} y_n - T(PT)^{n-1} x_n\|$$
$$+ \|x_n - T(PT)^{n-1} y_n\| + h_n \|y_n - x_n\|,$$

it follows
$$\lim_{n \to \infty} \|x_n - T(PT)^{n-1} x_n\| = 0.$$

Using equations, we get
$$\lim_{n \to \infty} \|x_{n+1} - T(PT)^{n-1} y_n\| = 0.$$

$$\lim_{n \to \infty} \|x_{n+1} - T(PT)^{n-1} z_n\| = 0.$$

$$\lim_{n \to \infty} \|x_{n+1} - y_n\| = 0.$$

$$\lim_{n \to \infty} \|x_{n+1} - z_n\| = 0.$$

Now
$$\|x_n - T x_n\| \leq \|x_n - T(PT)^{n-1} x_n\| + \|T(PT)^{n-1} x_n - T x_n\|$$
$$\leq \|x_n - T(PT)^{n-1} x_n\| + \|T(PT)^{n-1} x_n - T(PT)^{n-1} y_{n-1}\|$$
$$+ \|T(PT)^{n-1} y_{n-1} - T x_n\| + \|T(PT)^{n-1} x_n - T(PT)^{n-1} z_{n-1}\|$$
$$+ \|T(PT)^{n-1} z_{n-1} - T x_n\|,$$

and using above condition we get required result $\lim_{n \to \infty} \|x_n - T x_n\| = 0$. The completes the proof. $\square$

3   The conditions (BP)

Here we recall the condition introduced in [5]. Let E be a Banach space and $T : E \to E$ be a bounded linear operator. In 1966, Browdin and Petryshyn [2] considered the existence of a solution of the equation $f = u - Tu$ by iteration of Picard-Poincoré-Newmann,

$$\begin{cases} x_0 \in E \\ x_{n+1} = T x_n + f, \end{cases} \tag{3.1}$$

or

$$\begin{cases} x_0 \in E \\ x_{n+1} = T^n x_0 + (f + Tf + \cdots + T^{n-1} f), \quad \forall n \geq 0, f \in E. \end{cases} \tag{3.2}$$

In fact, in 1958, Browder [1] proved the following.

Theorem 3.1. *Let E be a reflexive Banach space. Then a solution u of the equation $u - Tu = f$ exists for a given point $f \in E$ and an operator T which is asmyptotic bounded if and only if the sequence $\{x_n\}$ defined by (3.1) is bounded for any fixed $x_0 \in E$.*

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Relaxing the assumption of reflexivity on E, under a slight sharp condition on T, Browden and Petryshyn proved the following:

Theorem 3.2. *Let E be a Banach space, $T : E \to E$ be a bounded linear operator which is asymptotically convergent, that is, $\{T^n x\}$ converges in E for all $x \in E$. Then we have the following:*

(1) If $f \in R(1 - T)$, the sequence $\{x_n\}$ converges to a solution u of the equation $u + Tu = f$.

(2) If any subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ converges to an element $y \in E$, then y is a solution of the equation $y - Ty = f$.

(3) If E is a reflexive Banach space and the sequence $\{x_n\}$ is bounded, then the sequence $\{x_n\}$ converges to a solution of the equation $u + Tu = f$.

Motivated by the above theorem, we have the concept of the condition (BP) as in [5] given by

Let E be a real normed linear space, C be a nonempty subset of E and $T : C \to E$ be a mapping

Definition 3.3. (Condition) The pair (T, C) is said to satisfy the *condition* (BP) if for any bounded closed subset G of C, $\{z : z = x - Tx, x \in G\}$ is a closed subset of E.

Let E and F be Banach spaces. Recall that a mapping $T : E \to F$ is completely continuous if it is continuous and compact (that is, C is bounded implies that $\overline{T(C)}$ is compact) or a weakly convergent sequence ($x_n \to x$ weakly) implies a strong convergent sequence ($Tx_n \to Tx$).

Next we establish a relation between the condition (BP) and completely continuous mapping given as:

Proposition 3.4. *Let E be a real normed linear space, C be a nonempty subset of E and $T : C \to E$ be a completely continuous mapping. Then the pair (T, C) satisfy the condition (BP).*

Proof. It is similar as in [5]. $\qquad \qquad \square$

Remark 3.5. ([5]) The converse of the above proposition does not holds in general.

4   Strong convergence theorems

Now we turn to strong convergence theorems for the asymptotically nonexpansive non-self mappings with condition in the real uniformly convex Banach spaces. First two results correspond to our new scheme where as remaining results are the same results in [3] and [5], which are the simple derivations of our result.

Theorem 4.1. *Let E be a real uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $T : C \to E$ be an asymptotically nonexpansive non-self mapping*

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with $\{h_n\}, \{h'_n\} \subset [1, \infty)$ and $h'_n \leq h_n$ such that $\sum_{n=1}^{\infty}(h_n - 1) < \infty$, $\sum_{n=1}^{\infty}(h'_n - 1) < \infty$ and $Fix(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by

$$\begin{cases} x_1 \in C, \\ y_n = P((1 - \beta_n)x_n + \beta_n T(PT)^{n-1}x_n), \\ z_n = P((1 - \gamma_n)y_n + \gamma_n T(PT)^{n-1}y_n), \\ x_{n+1} = P\left((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}\left(\frac{y_n + z_n}{2}\right)\right) \end{cases} \quad (4.1)$$

for all $n \geq 1$, where

$$0 \leq \liminf_{n \to \infty} \alpha_n, \quad \limsup_{n \to \infty} \alpha_n < 1, \quad \limsup_{n \to \infty} \beta_n < 1, \quad \limsup_{n \to \infty} \gamma_n < 1.$$

If the pair (T, C) satisfy the condition (BP), then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Let $G = \{\overline{x}_n\}$, where $\overline{A}$ denotes the closure of A. Since the sequence $\{x_n\}$ is bounded in C by Lemma 2.2 and so G is a bounded closed subset of C. As pair (T, C) satisfy (BP) conditions, it follows that

$$N = \{z = x - Tx : x \in G\}$$

is closed. Lemma 2.3 guarantees $\{x_n - Tx_n\} \subset N$ and $x_n - Tx_n \to 0$, $\{y_n - Ty_n\} \subset N$ and $y_n - Ty_n \to 0$ as $n \to \infty$. Clearly the zero vector $0 \in N$ so there exist a $q \in G$ such that $q = Tq$ so $q \in Fix(T)$. Since $q \in G$ so there exists a positive integer n_0 such that $x_{n_0} = q$ or there exists a subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ such that $x_{n_k} \to q$ as $k \to \infty$.

If $x_{n_k} = q$, then it follows from Lemma 2.2 that $x_n = q$ for all $n \geq n_0$ and so $x_n \to q$ as $n \to \infty$.

If $x_{n_k} \to q$, then, since $\lim_{n \to \infty} \|x_n - q\|$ exists, we have $x_n \to q$ as $n \to \infty$. This completes the proof. $\square$

Using Theorem 4.1 and Proposition 3.4, we have

Corollary 4.2. *Let E be a real uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $T : C \to E$ be an asymptotically nonexpansive non-self mapping with $\{h_n\}, \{h'_n\} \subset [1, \infty)$ and $h'_n \leq h_n$ such that $\sum_{n=1}^{\infty}(h_n - 1) < \infty$, $\sum_{n=1}^{\infty}(h'_n - 1) < \infty$ and $Fix(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (4.1), where*

$$0 \leq \liminf_{n \to \infty} \alpha_n, \quad \limsup_{n \to \infty} \alpha_n < 1, \quad \limsup_{n \to \infty} \beta_n < 1, \quad \limsup_{n \to \infty} \gamma_n < 1.$$

If T is completely continuous, then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 4.3. *Let E be a real uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $T : C \to E$ be an asymptotically nonexpansive non-self mapping with $\{h_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(h_n - 1) < \infty$ and $Fix(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (1.3), where*

$$0 \leq \liminf_{n \to \infty} \alpha_n \quad and \quad \limsup_{n \to \infty} \alpha_n < 1.$$

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If the pair (T, C) satisfy the condition (BP), then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. In Theorem 4.1, take $\beta_n = 0$ and $\gamma_n = 0$ for all $n \geq 1$, we arrive at the result. $\square$

Using Theorem 4.3 and Proposition 3.4, we have

Corollary 4.4. *Let E be a real uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $T : C \to E$ be an asymptotically nonexpansive non-self mapping with $\{h_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(h_n - 1) < \infty$ and $Fix(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined by (1.3), where*

$$0 \leq \liminf_{n\to\infty} \alpha_n \quad and \quad \limsup_{n\to\infty} \alpha_n < 1.$$

If T is completely continuous, then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Remark 4.5. The results proved in [3] can also be obtained from our Theorem 4.1 under special assumptions of sequences on α_n, β_n and γ_n.

5   Examples

Here we focus on the families of examples to apply on our results. First example also extends the example presented in [5].

Example 5.1. Let X be a Hilbert space and

$$C = \{x \in X : \|x\| \leq r, \forall r > 0\}.$$

Let $P : C \to C$ by

$$Px = \begin{cases} x, & \text{if } x \in C, \\ \frac{rx}{\|x\|}, & \text{if } x \in X - C. \end{cases}$$

Then P is a nonexpansive retraction of X onto C (see [5]).
Take $X = R^n$ with

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

and

$$\|x\| = \left(\sum_{i=1}^{n} (x_i)^2 \right)^{\frac{1}{2}}.$$

Then X is a Hilbert space.
Let $C = \{x \in X : \|x\| \leq 1, \}$.
Take $P : C \to C$ by

$$Px = \begin{cases} x, & \text{if } x \in C, \\ \frac{x}{\|x\|}, & \text{if } x \in X - C. \end{cases}$$

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Then P is a nonexpansive retraction of X onto C by [8], taking $r = 1$.

Define $T_i : C \to X$ by $T_i = (0, 0, ..., 1 - x_i, 0, 0, ..., 0)$ for all $X = (x_1, x_2, x_3, ..., x_n) \in C$. Then

$$\|Tx_i - Ty_i\| = \|(0, 0, ...1 - x_i, 0, 0, ..., 0) - (0, 0, ..., 1 - y_i, 0, 0, ..., 0)\|$$
$$= \|(0, 0, ..., x_i - y_i, 0, 0, ..., 0)\|$$
$$\leq \|x - y\|.$$

So $T_i x$ are a family of nonexpansive non-self mappings, so

$$\|T_i(PT_i)^{2-1}x - T_i(PT_i)^{2-1}y\| \leq \|P(T_i x) - P(T_i y)\|$$
$$\leq \|T_i x - T_i y\|$$
$$\leq \|x - y\|.$$

Suppose that

$$\|T_i(PT_i)^{k-1}x - T_i(PT_i)^{k-1}y\| \leq \|x - y\|, \quad \forall n = k.$$

Taking $n = k + 1$, we have

$$\|T_i(PT_i)^{(k+1)-1}x - T_i(PT_i)^{(k+1)-1}y\| \leq \|P(T_i x)^{(k+1)-1} - P(T_i y)^{(k+1)-1}\|$$
$$= \|P(T_i)P(T_i x)^{k-1} - P(T_i)P(T_i y)^{k-1}\|$$
$$\leq \|T_i P(T_i x)^{k-1} - T_i P(T_i y)^{k-1}\|$$
$$\leq \|x - y\|.$$

It follws that from Mathematical Induction, T_i is a family of an asymptotically non expensive non-self mapping with sequence $\{h_{i,n}\}$ defined by $h_{i,n} = 1$ for all $n \geq 1$. Put

$$Fix(T_i) = \left\{(0, 0, ..., \frac{1}{2}, 0, 0, ..., 0)\right\}.$$

Now, we prove that the pairs (T_i, C) for each i satisfy the condition. For any closed subset G of C, we denoted $N_i = \{z = x - T_i x : x \in G\}$ Then N_i are closed. Reality is that, for any $z_n \in N_i$ with $z_n \to z$, there exist $x_n \in G$ such that $z_n = x_n - T_i x_n$. As G is bounded and closed in C so is compact. Therefore, there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Letting $x_{n_k} \to x_0$ as $k \to \infty$, we have $x_0 \in G$. Also as T_i are continuous so

$$z = \lim_{k \to \infty} z_{n_k} = \lim_{k \to \infty} (x_{n_k} - T_i x_{n_k}) = x_0 - T_i x_0 \in N_i.$$

For any given $x_1 \in C$, take a sequence $\{x_n\}$

$$\begin{cases} y_n = P((1 - \beta_1)x_n + \beta_n T_i(PT_i)^{n-1}x_n), \\ z_n = P((1 - \gamma_1)y_n + \gamma_n T_i(PT_i)^{n-1}y_n), \\ x_{n+1} = P\left((1 - \alpha_1)x_n + \alpha_n T_i(PT_i)^{n-1}\left(\frac{y_n + z_n}{2}\right)\right) \end{cases}$$

for any $n \geq 1$, where

$$\alpha_n = \frac{4}{5} + \frac{1}{6n}, \quad n = 0 \pmod 2,$$
$$\alpha_n = \frac{1}{10} + \frac{1}{2n}, \quad n = 1 \pmod 2,$$
$$\beta_n = \gamma_n = \frac{2n}{3n + 2}, \quad n \geq 1.$$

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Clearly

$$\liminf_{n\to\infty} \alpha_n = \frac{1}{10}, \quad \limsup_{n\to\infty} \alpha_n = \frac{4}{5},$$

$$\limsup_{n\to\infty} \beta_n = \limsup_{n\to\infty} \gamma_n = \frac{2}{3}.$$

So all condition of Theorem 4.1 are satisfied and so $\{x_n\}$ converge strongly to a fixed point $(0,0,0,...,\frac{1}{2},0,0,...,0)$ of T_i.

Remark 5.2. If we take $T_1(x_1,x_2) = (1 - x_1, 0)$ with $x_1 = (0,0)$, then $\{x_n\}$ converges strongly to $Fix(T_i) = (\frac{1}{2}, 0)$ after single iteration.

Example 5.3. Let $X = l^2$ with $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$ and $\|x\| = \left(\sum_{i=1}^{\infty} x_i^2\right)^{\frac{1}{2}}$. Then X is a real infinite dimensional Hilbert space. Let $C = \{x \in X : \|x\| \leq 1\}$. Define a mapping $P : X \to C$ by

$$Px = \begin{cases} x, & \text{if } x \in X, \\ \frac{x}{\|x\|}, & \text{if } x \in X - C. \end{cases}$$

Then P is nonexpansive retraction of X onto C. Define $T : C \to X$ by

$$Tx = (-x_1, -x_2, ..., -x_i, ...), \quad \forall x \in C.$$

Then we have

$$\|Tx - Ty\| = \|(y_1 - x_1, y_2 - x_2, ..., y_i - x_i, ...)\|$$

$$= \left(\sum_{i=1}^{\infty} (y_i - x_i)^2\right)^{\frac{1}{2}} = \|x - y\|$$

for all $x = (x_1, x_2, ..., x_i, ...)$, $y = (y_1, y_2, ..., y_i, ...) \in C$ so T is an asymptotically non-expansive non-self mapping with sequence $\{h_n\}$ defined by $h_n = 1$ for all $n \geq 1$ and $Fix(T) = \{(0,0,...,0)\}$.

Now we prove that the pair (T, C) satisfy our condition and T is not completely continuous. In fact for any closed subset G of C, we denote $N = \{z = x - Tx : c \in G\}$. For any $z_n \in N$ with $z_n \to z$ as $n \to \infty$, there exists $z_n = x_n - Tx_n = 2x_n$. It follows from $z_n \to z$ that $x_n \to \frac{1}{2}z$ as $n \to \infty$. Since G is closed in C, it follows that $\frac{1}{2}z \in G$. Since T is continuous, it follows that

$$z = \lim_{n\to\infty} z_n = \lim_{n\to\infty} (x_n - Tx_n) = \frac{1}{2}z - T\left(\frac{1}{2}z\right) \in N.$$

This shows that the pair (T, C) satisfy the condition (BP). Since T is surjective, and unit ball $C = \{x \in X : \|x\| \leq 1\}$, is not sequentially compact, so T is not completely continuous (see [5]).

For $x_1 \in C$, define sequence $\{x_n\}$ by

$$\begin{cases} y_n = P((1 - \beta_n)x_n + \beta_n T(PT)^{n-1}x_n), \\ z_n = P((1 - \gamma_n)y_n + \gamma_n T(PT)^{n-1}y_n), \\ x_{n+1} = P\left((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}\left(\frac{y_n + z_n}{2}\right)\right), \end{cases}$$

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where $\alpha_n = \frac{3}{5} + \frac{1}{4n}$, $n \equiv 1 \pmod 2$, and $\alpha_n = \frac{1}{7} + \frac{1}{2n}$, $n \equiv 0 \pmod 2$ for all $c \geq 1$, $\beta_n = \frac{3n}{4n+2}$, $\gamma_n = \frac{3n}{4n+2}$ for all $n \geq 1$. Clearly $\liminf_{n\to\infty} \alpha_n = \frac{1}{7}$ and $\limsup_{n\to\infty} \alpha_n = \frac{3}{5}$ and $\limsup_{n\to\infty} \beta_n = \limsup_{n\to\infty} \gamma_n = \frac{3}{4}$. So all conditions of Theorem 4.1 are satisfied. Hence the sequence $\{x_n\}$ converges strongly to fixed point $(0, 0, ..., 0)$ of T.

Acknowledgement

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On existence of nondecreasing solutions of q-quadratic integral equations

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Abstract

We investigate a q-fractional integral equation with supremum and prove an existence theorem for it. We will prove that our q-integral equation has a solution in $C[0, 1]$ which is monotonic on $[0, 1]$. The monotonicity measure of noncompactness due to Banaś and Olszowy and Darbo's theorem are the main tools used in the proof our main result.

1 Introduction

Jackson in [20, 21] introduced the concept of quantum calculus (q-calculus). This area of research has rich history and several applications, see [1, 3, 22, 23] and references therein. There are several developments and applications of the q-calculus in mathematical physics,

especially concerning quantum mechanics, the theory of relativity and special functions [1, 3, 17, 23, 24]. Recently, several researchers attracted their attention by the concept of q-calculus, and we can find several new results in [2, 3, 18, 25] and the references therein.

In several papers among them [4, 19], differential equations with supremum as well as integral equations with supremum have been studied. In [10, ?, 12, 13, 14, 15] Darwish *et al.* studied differential and integral equations of arbitrary orders with supremum. Also, Caballero *et al.* [8, 9] introduced and studied the quadratic Volterra equations with supremum. They showed that these equations have monotonic solutions in the space $C[0, 1]$. In [10], Darwish generalized and extend Caballero *et al.* [8] results to the quadratic integral equations of arbitrary orders with supremum.

In this paper we will study the q-quadratic integral equation with supremum

$$y(t) = f(t) + \frac{t^{\beta-1}(Ty)(t)}{\Gamma_q(\beta)} \int_0^t (qs/t; q)_{\beta-1}\, \kappa(t, s) \max_{[0, \sigma(s)]} |y(\tau)|\, d_q s,\ t \in J = [0, 1], \qquad (1.1)$$

where $0 < \beta$, $q \in (0, 1)$, $f : J \to \mathbb{R}$, $T : C(J) \to C(J)$, $\kappa : J \times J \to \mathbb{R}_+$ and $\sigma : J \to J$.

By using Darbo fixed point theorem and the monotonicity measure of noncompactness due to Banaś and Olszowy [6] we prove the existence of monotonic solution to Eq.(1.1) in $C[0, 1]$.

2 Fractional q-calculus

We collect basic definitions and results of the q-fractional integrals and q-derivatives, for more details, see [2, 3, 7, 17, 18, 24, 25] and references therein.

First, for a real parameter $q \in (0, 1)$, we define a q-real number $[a]_q$ by

$$[a]_q = \frac{1 - q^a}{1 - q},\ a \in \mathbb{R},$$

and a q-analog of the Pochhammer symbol (q-shifted factorial) is defined by

$$(a; q)_n = \begin{cases} 1, & n = 0, \\ \prod\limits_{k=0}^{n-1}(1 - aq^k), & n \in \mathbb{N}. \end{cases}$$

Also, the q-analog of the power $(a - b)^n$ is given by

$$(a - b)^{(n)} = \begin{cases} 1, & n = 0, \\ \prod\limits_{k=0}^{n-1}(a - bq^k), & n \in \mathbb{N}; a, b \in \mathbb{R}. \end{cases}$$

Moreover,

$$(a - b)^{(n)} = a^n (b/a; q)_n,\ a \neq 0.$$

Notice that, $\lim_{n \to \infty} (a; q)_n$ exists and we will denote it by $(a; q)_\infty$.

More generally, for $\beta \in \mathbb{R}$, $aq^\beta \neq q^{-n}$ $(n \in \mathbb{N})$, we define

$$(a; q)_\beta = \frac{(a; q)_\infty}{(aq^\beta; q)_\infty}$$

and

$$(a - b)^{(\beta)} = a^\beta \frac{(b/a; q)_\infty}{(q^\beta b/a; q)_\infty}.$$

Notice that $(a - b)^{(\beta)} = a^\beta (b/a; q)_\beta$. Therefore, if $b = 0$, then $a^{(\beta)} = a^\beta$.

Now, the q-gamma function is given by

$$\Gamma_q(x) = \frac{G(q^x)}{(1 - q)^{x-1} G(q)}, \quad x \in \mathbb{R} \backslash \{0, -1, -2, \cdots\},$$

where $G(q^x) = \frac{1}{(q^x; q)_\infty}$. Or, equivalently, $\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}$ and satisfies $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

Next, the q-derivative of a function f is given by

$$(D_q f)(t) = \frac{f(t) - f(qt)}{t - qt}, \quad (D_q f)(0) = \lim_{t \to 0} (D_q f)(t),$$

and the q-derivative of higher order of a function f is defined by

$$(D_q^n f)(t) = \begin{cases} f(t), & n = 0, \\ D_q(D_q^{n-1} f)(t), & n \in \mathbb{N}. \end{cases}$$

Let f be a function defined on $[0, b]$. The q-integral of f is defined as follows

$$(I_q f)(t) = \int_0^t f(s) \, d_q s = t(1 - q) \sum_{n=0}^\infty q^n f(tq^n), \ t \in [0, b]. \tag{2.2}$$

If f is given on the interval $[0, b]$ and $a \in [0, b]$, then

$$\int_a^b f(s) \, d_q s = \int_0^b f(s) \, d_q s - \int_0^a f(s) \, d_q s.$$

The operator I_q^n is defined by

$$(I_q^n f)(t) = \begin{cases} f(t), & n = 0, \\ I_q(I_q^{n-1} f)(t), & n \in \mathbb{N}. \end{cases}$$

The fundamental theorem of calculus satisfies for D_q and I_q, i.e., $(D_q I_q f)(t) = f(t)$, and if f is continuous at $t = 0$, then $(I_q D_q f)(t) = f(t) - f(0)$.

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The following four formulas will be used later in this paper

$$[a(t-s)]^{(\beta)} = a^\beta (t-s)^{(\beta)},$$

$$_t D_q (t-s)^{(\beta)} = [\beta]_q (t-s)^{(\beta-1)},$$

$$_s D_q (t-s)^{(\beta)} = -[\beta]_q (t-qs)^{(\beta-1)}$$

and

$$_t D_q \int_0^t f(t,s) \, d_q s = \int_0^t {_t D_q} f(t,s) \, d_q s + f(qt,t),$$

where $_t D_q$ denotes the derivative with respect to variable t.

Notice that, if $\beta > 0$ and $a \le b \le t$, then $(t-b)^{(\beta)} \le (t-a)^{(\beta)}$.

Definition 1. [1] Let f be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type of order $\beta \ge 0$ is given by

$$(I_q^\beta f)(t) = \begin{cases} f(t), & \beta = 0, \\ \frac{1}{\Gamma_q(\beta)} \int_0^t (t-qs)^{(\beta-1)} f(s) \, d_q s = t^\beta (1-q)^\beta \sum_{n=0}^{\infty} q^n \frac{(q^\beta;q)_n}{(q;q)_n} f(tq^n), & \beta > 0, \ t \in [0,1]. \end{cases}$$

Notice that, for $\beta = 1$, the above q-integral reduces to (2.2).

Definition 2. [1] The fractional q-derivative of the Riemann-Liouville type of order $\beta \ge 0$ is given by

$$(D_q^\beta f)(t) = \begin{cases} f(t), & \beta = 0, \\ (D_q^{[\beta]} I_q^{[\beta]-\beta} f)(t), & \beta > 0, \end{cases}$$

where $[\beta]$ stands for the smallest integer equal or greater than β.

In $q-$calculus, the derivative rule for the product of two functions and integration by parts formulas are

$$(D_q fg)(t) = (D_q f)(t) g(t) + f(qt)(D_q g)(t),$$

$$\int_0^t f(s)(D_q g)(s) \, d_q s = [f(s)g(s)]_0^t - \int_0^t (D_q f)(s) g(qs) \, d_q s.$$

Lemma 1. *Let $\beta, \gamma \ge 0$. Then the following are verified for a function f defined on $[0,1]$:*
(1) $(I_q^\gamma I_q^\beta f)(t) = (I_q^{\beta+\gamma} f)(t),$
(2) $(D_q^\beta I_q^\beta f)(t) = f(t).$

Lemma 2. *[24] For $\beta > 0$. Then q-integration by parts allows us to have*

$$(I_q^\beta 1)(t) = \frac{t^{(\beta)}}{\Gamma_q(\beta+1)}$$

or

$$\int_0^t (t-qs)^{(\beta-1)} \, d_q s = \frac{t^{(\beta)}}{[\beta]_q}.$$

3   Measure of noncompactness

We assume that $(E, \|.\|)$ is a real Banach space with zero element θ and we denote by $B(x, r)$ the closed ball with radius r and centre x, where $B_r \equiv B(\theta, r)$.

Now, let $X \subset E$ and denote by $\overline{X}$ and $\mathrm{Conv}X$ the closure and convex closure of X, respectively. Also, the symbols $X + Y$ and λY stands for the usual algebraic operators on sets. Moreover, the families $\mathfrak{M}_E$ and $\mathfrak{N}_E$ are defined by $\mathfrak{M}_E = \{A \subset E : A \neq \emptyset, \ A \text{ is bounded}\}$ and $\mathfrak{N}_E = \{B \subset \mathfrak{M}_E : B \text{ is relatively compact}\}$, respectively.

Definition 3. [5] Let $\mu : \mathfrak{M}_E \to [0, +\infty)$. If the following conditions

1° $\emptyset \neq \{X \in \mathfrak{M}_E : \mu(X) = 0\} = \ker\mu \subset \mathfrak{N}_E$,

2° if $X \subset Y$, then $\mu(X) \leq \mu(Y)$,

3° $\mu(X) = \mu(\overline{X}) = \mu(\mathrm{Conv}X)$,

4° $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$, $0 \leq \lambda \leq 1$ and

5° if (X_n) is a sequence of closed subsets of $\mathfrak{M}_E$ with $X_{n+1} \subset X_n$, $n = 1, 2, 3, ...,$ and $\lim_{n \to \infty} \mu(X_n) = 0$ then $X_\infty = \cap_{n=1}^{\infty} X_n \neq \emptyset$

hold. Then, the mapping μ is said to be a measure of noncompactness in E.

Here, $\ker\mu$ is the kernel of the measure of noncompactness μ.

Our result will establish in $C(J)$ the Banach space of all defined, continuous and real functions on $J \equiv [0, 1]$ with $\|y\| = \max\{|y(\tau)| : \tau \in J\}$.

Next, we defined the measure of noncompactness related to monotonicity in $C(J)$, see [5, 6].

We fix a bounded subset $Y \neq \emptyset$ of $C(J)$. For $\varepsilon \geq 0$ and $y \in Y$, $\omega(y, \varepsilon)$ denotes the modulus of continuity of the function y given by

$$\omega(y, \varepsilon) = \sup\{|y(t) - y(s)| : t, s \in J, \ |t - s| \leq \varepsilon\}.$$

Moreover, we let

$$\omega(Y, \varepsilon) = \sup\{\omega(y, \varepsilon) : y \in Y\}$$

and

$$\omega_0(Y) = \lim_{\varepsilon \to 0} \omega(Y, \varepsilon).$$

Define

$$d(y) = \sup_{t, s \in J, \ s \leq t} (|y(t) - y(s)| - [y(t) - y(s)])$$

and

$$d(Y) = \sup_{y \in Y} d(y).$$

Notice that all functions in Y are nondecreasing on J if and only if $d(Y) = 0$.

6

Now, we define the map μ on $\mathfrak{M}_{C(J)}$ as

$$\mu(Y) = d(Y) + \omega_0(Y).$$

Clearly, μ verifies all conditions in Definition 3 and, therefore it is a measure of noncompactness in $C(J)$ [6].

Definition 4. Let $\emptyset \neq M \subset E$. Let $\mathcal{P} : M \to E$ be a continuous operator. Suppose that $\mathcal{P}$ maps bounded sets onto bounded ones. If there exists a bounded $Y \subset M$ with $\mu(\mathcal{P}Y) \leq \alpha\mu(Y)$, $\alpha \geq 0$, then $\mathcal{P}$ is said to be satisfies the Darbo condition with respect to a measure of noncompactness μ.

If $\alpha < 1$, then $\mathcal{P}$ is called a contraction operator with respect to μ.

Theorem 1. *[16] Let $\Omega \neq \emptyset$ be a bounded, convex and closed subset of E. If $\mathcal{P} : \Omega \to \Omega$ is a contraction operator with respect to μ. Then $\mathcal{P}$ has at least one fixed point belongs to Ω.*

We will need the following two lemmas throughout our proof [8].

Lemma 3. *Let $r : J \to J$ be a continuous function and $y \in C(J)$. If, for $t \in J$,*

$$(Fy)(t) = \max_{[0,\sigma(t)]} |y(\tau)|,$$

then $Fy \in C(J)$.

Lemma 4. *Let (y_n) be a sequence in $C(J)$ and $y \in C(J)$. If (y_n) converges to $y \in C(J)$, then (Fy_n) converges uniformly to Fy uniformly J.*

4   Main Theorem

Let us consider the following hypotheses:

(h_1) $f \in C(J)$. Moreover, f is nondecreasing and nonnegative on J.

(h_2) The operator $T : C(J) \to C(J)$ is continuous and satisfies the Darbo condition with a constant c for the measure of noncompactness μ . Moreover, $Tx \geq 0$ if $x \geq 0$.

(h_3) $\exists\, a, b \geq 0$ s.t. $|(Tx)(t)| \leq a + b\|x\|$ $\forall x \in C(J)$, $t \in J$.

(h_4) The function $\kappa : J \times J \to \mathbb{R}_+$ is continuous on $J \times J$ and nondecreasing $\forall t$ and s separately. Moreover, $\kappa^* = \sup\limits_{(t,s)\in J\times J} \kappa(t,s)$.

(h_5) The function $\sigma : J \to J$ is nondecreasing and continuous on J.

(h_6) $\exists\, r_0 > 0$ such that

$$\|f\| + \frac{\kappa^* r_0(a + br_0)}{\Gamma_q(\beta + 1)} \leq r_0 \tag{4.3}$$

and $\frac{ck^* r_0}{\Gamma_q(\beta+1)} < 1$.

Now, we rewrite Eq.(1.1) as

$$x(t) = f(t) + \frac{(Tx)(t)}{\Gamma_q(\beta)} \int_0^t \kappa(t,s)\,(t-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s,\ 0 < \beta \le 1,\ t \in J, \qquad (4.4)$$

and define the two operators $\mathcal{K}$ and $\mathcal{F}$ on $C(J)$ as follows

$$(\mathcal{K}y)(t) = \frac{1}{\Gamma_q(\beta)} \int_0^t \kappa(t,s)(t-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |y(\tau)|\, d_q s \qquad (4.5)$$

and

$$(\mathcal{F}y)(t) = f(t) + (Ty)(t) \cdot (\mathcal{K}y)(t), \qquad (4.6)$$

respectively. Finding a fixed point of the operator $\mathcal{F}$ is equivalent to solving Eq.(4.4).

Under the above hypotheses, we state and prove our main theorem.

Theorem 2. *Assume the hypotheses $(h_1) - (h_6)$ be verified. Then Eq.(4.4) has at least one solution $x \in C(J)$ which is nondecreasing on J.*

Proof. First, we will show that the operator $\mathcal{F}$ maps $C(J)$ into itself. For this, it is sufficient to show that $\mathcal{K}x \in C(J)$ if $x \in C(J)$. Fix $\varepsilon > 0$ and let $x \in C(J)$ and $t_1,\ t_2 \in J\ (t_1 \le t_2)$ with $|t_2 - t_1| \le \varepsilon$. We have

$$
\begin{aligned}
|(\mathcal{K}x)(t_2) - (\mathcal{K}x)(t_1)| &= \left| \frac{1}{\Gamma_q(\beta)} \int_0^{t_2} \kappa(t_2,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \right. \\
&\quad \left. - \frac{1}{\Gamma_q(\beta)} \int_0^{t_1} \kappa(t_1,s)(t_1-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \right| \\
&\le \left| \frac{1}{\Gamma_q(\beta)} \int_0^{t_2} \kappa(t_2,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \right. \\
&\quad \left. - \frac{1}{\Gamma_q(\beta)} \int_0^{t_2} \kappa(t_1,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \right| \\
&\quad + \left| \frac{1}{\Gamma_q(\beta)} \int_0^{t_2} \kappa(t_1,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \right. \\
&\quad \left. - \frac{1}{\Gamma_q(\beta)} \int_0^{t_1} \kappa(t_1,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \right| \\
&\quad + \left| \frac{1}{\Gamma_q(\beta)} \int_0^{t_1} \kappa(t_1,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \right. \\
&\quad \left. - \frac{1}{\Gamma_q(\beta)} \int_0^{t_1} \kappa(t_1,s)(t_1-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \right| \\
&\le \frac{1}{\Gamma_q(\beta)} \int_0^{t_2} |\kappa(t_2,s) - \kappa(t_1,s)|(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s \\
&\quad + \frac{1}{\Gamma_q(\beta)} \int_{t_1}^{t_2} |\kappa(t_1,s)|(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_q s
\end{aligned}
$$

$$+\frac{1}{\Gamma_q(\beta)}\int_0^{t_1}|\kappa(t_1,s)|\,|(t_2-qs)^{(\beta-1)}-(t_1-qs)^{(\beta-1)}|\max_{[0,\sigma(s)]}|x(\tau)|\,d_qs$$

$$\leq \frac{\|x\|}{\Gamma_q(\beta)}\,\omega_\kappa(\varepsilon,.)\int_0^{t_2}(t_2-qs)^{(\beta-1)}\,d_qs$$

$$+\frac{\kappa^*\|x\|}{\Gamma_q(\beta)}\left\{\int_0^{t_1}[(t_1-qs)^{(\beta-1)}-(t_2-qs)^{(\beta-1)}]d_qs+\int_{t_1}^{t_2}(t_2-qs)^{(\beta-1)}d_qs\right\}$$

$$=\frac{\|x\|}{\Gamma_q(\beta+1)}\,\omega_\kappa(\varepsilon,.)\,t_2^{(\beta)}+\frac{\kappa^*\,\|x\|}{\Gamma_q(\beta+1)}[t_1^{(\beta)}-t_2^{(\beta)}+2(t_2-t_1)^{(\beta)}]$$

$$\leq\frac{\|x\|}{\Gamma_q(\beta+1)}\,\omega_\kappa(\varepsilon,.)\,t_2^{(\beta)}+\frac{2\kappa^*\,\|x\|}{\Gamma_q(\beta+1)}(t_2-t_1)^{(\beta)}$$

$$\leq\frac{\|x\|}{\Gamma_q(\beta+1)}\,\omega_\kappa(\varepsilon,.)\,t_2^{\beta}+\frac{\kappa^*\,\|x\|}{\Gamma(\beta+1)}\,\varepsilon^\beta,\tag{4.7}$$

where we used

$$\omega_\kappa(\varepsilon,.)=\sup_{t,\,\tau\in J,\,|t-\tau|\leq\varepsilon}|\kappa(t,s)-\kappa(\tau,s)|.$$

Notice that, since the function κ is uniformly continuous on $J\times J$, then when $\varepsilon\to 0$ we have that $\omega_\kappa(\varepsilon,.)\to 0$.

Therefore, $\mathcal{K}x\in C(J)$ and consequently, $\mathcal{F}x\in C(J)$.

Now, $\forall t\in J$, we have

$$|(\mathcal{F}x)(t)|\leq\left|f(t)+\frac{(Tx)(t)}{\Gamma_q(\beta)}\int_0^t\kappa(t,s)(t-qs)^{(\beta-1)}\max_{[0,\sigma(s)]}|x(\tau)|\,d_qs\right|$$

$$\leq\|f\|+\frac{a+b\|x\|}{\Gamma_q(\beta)}\int_0^t\kappa(t,s)(t-qs)^{(\beta-1)}\max_{[0,\sigma(s)]}|x(\tau)|\,d_qs$$

$$\leq\|f\|+\frac{a+b\|x\|}{\Gamma_q(\beta+1)}\kappa^*\|x\|.$$

Hence

$$\|\mathcal{F}x\|\leq\|f\|+\frac{a+b\|x\|}{\Gamma_q(\beta+1)}\kappa^*\|x\|.$$

From hypothesis (h_6), if $\|x\|\leq r_0$, we get

$$\|\mathcal{F}x\|\leq\|f\|+\frac{a+br_0}{\Gamma_q(\beta+1)}\kappa^*r_0$$

$$\leq r_0.$$

Therefore, $\mathcal{F}$ maps B_{r_0} into itself.

Next, we consider the operator $\mathcal{F}$ on the set $B_{r_0}^+=\{x\in B_{r_0}:x(t)\geq 0,\ \forall t\in J\}$. It is clear that $B_{r_0}^+\neq\emptyset$ is closed, convex and bounded. By this facts and hypotheses (h_1), (h_3) and (h_5), we obtain $\mathcal{F}$ transforms $B_{r_0}^+$ into itself.

In what follows, we will show that $\mathcal{F}$ is continuous on $B_{r_0}^+$. For, let (x_n) be us a sequence in $B_{r_0}^+$ such that $x_n\to x$ and we will show that $\mathcal{F}x_n\to\mathcal{F}x$. We have, $\forall t\in J$,

$$|(\mathcal{F}x_n)(t)-(\mathcal{F}x)(t)|=\left|\frac{(Tx_n)(t)}{\Gamma_q(\beta)}\int_0^t\kappa(t,s)(t-qs)^{(\beta-1)}\max_{[0,\sigma(s)]}|x_n(\tau)|d_qs\right.$$

$$- \frac{(Tx)(t)}{\Gamma_q(\beta)} \int_0^t \kappa(t,s)(t-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| d_q s \Bigg|$$

$$\leq \Bigg| \frac{(Tx_n)(t)}{\Gamma_q(\beta)} \int_0^t \kappa(t,s)(t-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x_n(\tau)| \, d_q s$$

$$- \frac{(Tx)(t)}{\Gamma_q(\beta)} \int_0^t \kappa(t,s)(t-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x_n(\tau)| \, d_q s \Bigg|$$

$$+ \Bigg| \frac{(Tx)(t)}{\Gamma_q(\beta)} \int_0^t \kappa(t,s)(t-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x_n(\tau)| \, d_q s$$

$$- \frac{(Tx)(t)}{\Gamma_q(\beta)} \int_0^t \kappa(t,s)(t-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_q s \Bigg|$$

$$\leq \frac{|(Tx_n)(t) - (Tx)(t)|}{\Gamma_q(\beta)} \int_0^t |\kappa(t,s)|(t-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x_n(\tau)| \, d_q s$$

$$+ \frac{|(Tx)(t)|}{\Gamma_q(\beta)} \int_0^t |\kappa(t,s)|(t-qs)^{(\beta-1)} \left| \max_{[0,\sigma(s)]} |x_n(\tau)| - \max_{[0,\sigma(s)]} |x(\tau)| \right| \, d_q s.$$

Applying Lemma 4, we obtain

$$\|\mathcal{F}x_n - \mathcal{F}x\| \leq \frac{\kappa^* r_0}{\Gamma_q(\beta+1)} \|Tx_n - Tx\| + \frac{\kappa^*(a+b\,r_0)}{\Gamma_q(\beta+1)} \|x_n - x\|. \tag{4.8}$$

By the continuity of T, $\exists n_1 \in \mathbb{N}$ such that

$$\|Tx_n - Tx\| \leq \frac{\varepsilon \Gamma_q(\beta+1)}{2\kappa^* r_0} \, \forall n \geq n_1.$$

Also, $\exists n_2 \in \mathbb{N}$ such that

$$\|x_n - x\| \leq \frac{\varepsilon \Gamma_q(\beta+1)}{2\kappa^*(a+br_0)} \, \forall n \geq n_2.$$

Now, take $\max\{n_1, n_2\} \leq n$, then (4.8) gives us that

$$\|\mathcal{F}x_n - \mathcal{F}x\| \leq \varepsilon.$$

This shows that $\mathcal{F}$ is continuous in $B_{r_0}^+$.

Now, we take $\emptyset \neq X \subset B_{r_0}^+$. Let us fix an arbitrarily number $\varepsilon > 0$ and choose $x \in X$ and $t_1, t_2 \in J$ with $|t_2 - t_1| \leq \varepsilon$. We will assume that $t_1 \leq t_2$ because no generality will be loss. Then, by using our hypotheses and inequality (4.7), we get

$$
\begin{aligned}
|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| &\leq |f(t_2) - f(t_1)| + |(Tx)(t_2)\,(\mathcal{K}x)(t_2) - (Tx)(t_2)\,(\mathcal{K}x)(t_1)| \\
&\quad + |(Tx)(t_2)\,(\mathcal{K}x)(t_1) - (Tx)(t_1)\,(\mathcal{K}x)(t_1)| \\
&\leq \omega(f,\varepsilon) + |(Tx)(t_2)|\,|(\mathcal{K}x)(t_2) - (\mathcal{K}x)(t_1)| \\
&\quad + |(Tx)(t_2) - (Tx)(t_1)|\,|(\mathcal{K}x)(t_1)| \\
&\leq \omega(f,\varepsilon) + \frac{(a+b\|x\|)}{\Gamma_q(\beta+1)} \left[\|x\|\omega_\kappa(\varepsilon,.) + 2\kappa^*\|x\|\varepsilon^\beta \right] + \frac{\omega(Tx,\varepsilon)}{\Gamma_q(\beta+1)} \|x\|\kappa^*
\end{aligned}
$$

$$\leq \quad \omega(f,\varepsilon) + \frac{r_0(a+br_0)}{\Gamma_q(\beta+1)}[\omega_\kappa(\varepsilon,.) + 2\,\kappa^*\varepsilon^\beta] + \frac{\kappa^*r_0}{\Gamma_q(\beta+1)}\omega(Tx,\varepsilon).$$

Hence,

$$\omega(\mathcal{F}x,\varepsilon) \leq \omega(f,\varepsilon) + \frac{r_0(a+br_0)}{\Gamma_q(\beta+1)}[\omega_\kappa(\varepsilon,.) + 2\kappa^*\varepsilon^\beta] + \frac{\kappa^*\,r_0}{\Gamma_q(\beta+1)}\omega(Tx,\varepsilon).$$

Consequently,

$$\omega(\mathcal{F}X,\varepsilon) \leq \omega(f,\varepsilon) + \frac{r_0(a+br_0)}{\Gamma_q(\beta+1)}[\omega_\kappa(\varepsilon,.) + 2\kappa^*\varepsilon^\beta] + \frac{\kappa^*r_0}{\Gamma_q(\beta+1)}\omega(TX,\varepsilon).$$

Since the function κ is uniformly continuous on $J \times J$ and the function f is continuous on J, then the last inequality gives us that

$$\omega_0(\mathcal{F}X) \leq \frac{\kappa^*r_0}{\Gamma_q(\beta+1)}\omega_0(TX). \tag{4.9}$$

Further, fix arbitrary $x \in X$ and $t_1, t_2 \in J$ with $t_2 > t_1$. Then, by our hypotheses, we have

$$|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| - [(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)]$$

$$= \left| f(t_2) + \frac{(Tx)(t_2)}{\Gamma_q(\beta)} \int_0^{t_2} \kappa(t_2,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_qs \right.$$

$$\left. -f(t_1) - \frac{(Tx)(t_1)}{\Gamma_q(\beta)} \int_0^{t_1} \kappa(t_1,s)(t_1-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_qs \right|$$

$$- \left[f(t_2) + \frac{(Tx)(t_2)}{\Gamma_{(q\beta)}} \int_0^{t_2} \kappa(t_2,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_qs \right.$$

$$\left. -f(t_1) - \frac{(Tx)(t_1)}{\Gamma_q(\beta)} \int_0^{t_1} \kappa(t_1,s)(t_1-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_qs \right]$$

$$\leq \{|f(t_2) - f(t_1)| - [f(t_2) - f(t_1)]\} + \left| \frac{(Tx)(t_2)}{\Gamma_q(\beta)} \int_0^{t_2} \kappa(t_2,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_qs \right.$$

$$\left. - \frac{(Tx)(t_1)}{\Gamma_q(\beta)} \int_0^{t_2} \kappa(t_2,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_qs \right|$$

$$+ \left| \frac{(Tx)(t_1)}{\Gamma_q(\beta)} \int_0^{t_2} \kappa(t_2,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_qs \right.$$

$$\left. - \frac{(Tx)(t_1)}{\Gamma_q(\beta)} \int_0^{t_1} \kappa(t_1,s)(t_1-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_qs \right|$$

$$- \left\{ \left[\frac{(Tx)(t_2)}{\Gamma_q(\beta)} \int_0^{t_2} \kappa(t_2,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_qs \right. \right.$$

$$\left. \left. - \frac{(Tx)(t_1)}{\Gamma_q(\beta)} \int_0^{t_2} \kappa(t_2,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_qs \right] \right.$$

$$+ \left[\frac{(Tx)(t_1)}{\Gamma_q(\beta)} \int_0^{t_2} \kappa(t_2,s)(t_2-qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)|\, d_qs \right.$$

$$- \frac{(Tx)(t_1)}{\Gamma_q(\beta)} \int_0^{t_1} \kappa(t_1, s)(t_1 - qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_qs \Bigg] \Bigg\}$$

$$\leq \{|(Tx)(t_2) - (Tx)(t_1)| - [(Tx)(t_2) - (Tx)(t_1)]\}$$

$$\times \frac{1}{\Gamma_q(\beta)} \int_0^{t_2} \kappa(t_2, s)(t_2 - qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_qs$$

$$+ \frac{(Tx)(t_1)}{\Gamma_q(\beta)} \left\{ \left| \int_0^{t_2} \kappa(t_2, s)(t_2 - qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_qs \right. \right.$$

$$- \int_0^{t_1} \kappa(t_1, s)(t_1 - qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_qs \Bigg|$$

$$- \left[\int_0^{t_2} \kappa(t_2, s)(t_2 - qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_qs \right.$$

$$\left. \left. - \int_0^{t_1} \kappa(t_1, s)(t_1 - qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_qs \right] \right\}. \tag{4.10}$$

Now, we will prove that

$$\int_0^{t_2} \kappa(t_2, s)(t_2 - qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_qs - \int_0^{t_1} \kappa(t_1, s)(t_1 - qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_qs \geq 0.$$

In fact, we have

$$\int_0^{t_2} \kappa(t_2, s)(t_2 - qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_qs - \int_0^{t_1} \kappa(t_1, s)(t_1 - qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_qs$$

$$= \int_0^{t_2} \kappa(t_2, s)(t_2 - qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_qs - \int_0^{t_2} \kappa(t_1, s)(t_2 - qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_qs$$

$$+ \int_0^{t_2} \kappa(t_1, s)(t_2 - qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_qs - \int_0^{t_1} \kappa(t_1, s)(t_2 - qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_qs$$

$$+ \int_0^{t_1} \kappa(t_1, s)(t_2 - qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_qs - \int_0^{t_1} \kappa(t_1, s)(t_1 - qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_qs$$

$$= \int_0^{t_2} (\kappa(t_2, s) - \kappa(t_1, s))(t_2 - qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_qs$$

$$+ + \int_{t_1}^{t_2} \kappa(t_1, s)(t_2 - qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_qs$$

$$+ \int_0^{t_1} \kappa(t_1, s)[(t_2 - qs)^{(\beta-1)} - (t_1 - qs)^{(\beta-1)}] \max_{[0,\sigma(s)]} |x(\tau)| \, d_qs.$$

But, $\kappa(t_1, s) \leq \kappa(t_2, s)$ because $\kappa(t, s)$ is increasing with respect to t, then

$$\int_0^{t_2} (\kappa(t_2, s) - \kappa(t_1, s))(t_2 - qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_qs \geq 0 \tag{4.11}$$

and, since $(t_2 - qs)^{(\beta-1)} - (t_1 - qs)^{(\beta-1)} \geq 0$ for $s \in [0, t_1)$ then

$$\int_0^{t_1} \kappa(t_1, s)[(t_2 - qs)^{(\beta-1)} - (t_1 - qs)^{(\beta-1)}] \max_{[0,\sigma(s)]} |x(\tau)| \, d_qs$$

$$+ \int_{t_1}^{t_2} \kappa(t_1, s)(t_2 - qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_q s$$

$$\geq \int_0^{t_1} \kappa(t_1, t_1)[(t_2 - qs)^{(\beta-1)} - (t_1 - qs)^{(\beta-1)}] \max_{[0,\sigma(t_1)]} |x(\tau)| \, d_q s$$

$$+ \int_{t_1}^{t_2} \kappa(t_1, t_1)(t_2 - qs)^{(\beta-1)} \max_{[0,\sigma(t_1)]} |x(\tau)| \, d_q s$$

$$= \kappa(t_1, t_1) \max_{[0,\sigma(t_1)]} |x(\tau)| \left[\int_0^{t_2} (t_2 - qs)^{(\beta-1)} \, d_q s - \int_0^{t_1} (t_1 - qs)^{(\beta-1)} \, d_q s \right]$$

$$= \kappa(t_1, t_1) \frac{t_2^\beta - t_1^\beta}{[\beta]_q} \max_{[0,\sigma(t_1)]} |x(\tau)|$$

$$\geq 0. \tag{4.12}$$

Finally, (4.11) and (4.12) imply that

$$\int_0^{t_2} \kappa(t_2, s)(t_2 - qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_q s - \int_0^{t_1} \kappa(t_1, s)(t_1 - qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_q s \geq 0.$$

The above inequality and (4.10) leads us to

$$|(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)| - [(\mathcal{F}x)(t_2) - (\mathcal{F}x)(t_1)]$$
$$= \{|(Tx)(t_2) - (Tx)(t_1)| - [(Tx)(t_2) - (Tx)(t_1)]\}$$
$$\times \frac{1}{\Gamma_q(\beta)} \int_0^{t_2} \kappa(t_2, s)(t_2 - qs)^{(\beta-1)} \max_{[0,\sigma(s)]} |x(\tau)| \, d_q s$$
$$\leq \frac{\kappa^* r_0}{\Gamma_q(\beta + 1)} d(Tx).$$

Thus,

$$d(\mathcal{F}x)\Gamma_q(\beta + 1) \leq \kappa^* r_0 d(Tx)$$

and therefore,

$$d(\mathcal{F}X)\Gamma_q(\beta + 1) \leq \kappa^* r_0 d(TX). \tag{4.13}$$

Finally, (4.9) and (4.13) gives us that

$$\omega_0(\mathcal{F}X) + d(\mathcal{F}X) \leq \frac{\kappa^* r_0}{\Gamma_q(\beta + 1)} (\omega_0(\mathcal{F}X) + d(TX))$$

or

$$\mu(\mathcal{F}X) \leq \frac{r_0 \kappa^*}{\Gamma_q(\beta + 1)} \mu(TX)$$
$$\leq \frac{\kappa^* c r_0}{\Gamma_q(\beta + 1)} \mu(X).$$

But $\frac{\kappa^* r_0 c}{\Gamma_q(\beta+1)} < 1$, then

$$\mu(\mathcal{F}X) \leq \mu(X). \tag{4.14}$$

Inequality (4.14) enables us to use Theorem 1, then there are solutions to Eq.(1.1) in $C(J)$. This finishes our proof. $\qquad\square$

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