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Implicit Midpoint Type Picard Iterations for Strongly Accretive and Strongly Pseudocontractive Mappings

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Abstract

We study the convergence of implicit midpoint type Picard sequence for strongly accretive and strongly pseudocontractive mappings. We have also improved the results of some authors.

2010 Mathematics Subject Classification: 47H06, 47J05, 47J25

Key words and phrases: Banach space, Lipschitzian mapping, strongly pseudocontractive mapping, strongly accretive mapping, implicit midpoint type Picard iteration

1 Introduction and Preliminaries

Let E be a real Banach space with dual E^*. A mapping T with domain $D(T)$ and range $R(T)$ in E is called *strongly pseudocontractive* if and only if for all $x, y \in D(T)$, the following inequality is satisfied:

$$\|x - y\| \leq \|(1 + r)(x - y) - rt(Tx - Ty)\| \tag{1.1}$$

for $r > 0$ and some $t > 1$. If $t = 1$ in inequality (1.1), then T is called *pseudocontractive*.

*Corresponding author

For E, we will denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. As a consequence of a result of Kato [15], it follows from inequality (1.1) that T is strongly pseudocontractive if and only if

$$\langle (I-T)x - (I-T)y, j(x-y) \rangle \geq k\|x-y\|^2 \tag{1.2}$$

holds for all $x, y \in D(T)$ and for some $j(x-y) \in J(x-y)$, where $k = \frac{t-1}{t} \in (0,1)$.

Consequently, it follows easily (again from Kato [15] and inequality (1.2) that T is strongly pseudocontractive if and only if the following inequality holds:

$$\|x-y\| \leq \|x-y+s[(I-T-kI)x-(I-T-kI)y\| \tag{1.3}$$

for all $x, y \in D(T)$ and $s > 0$.

Closely related to the class of pseudocontractive mappings is the class of *accretive operators*. A mapping A with domain $D(A)$ and range $R(A)$ in E is called *accretive* if the following inequality holds:

$$\|x-y\| \leq \|x-y+s(Ax-Ay)\|$$

for all $x, y \in D(A)$ and $s > 0$. Also, as a consequence of Kato [15], this accretive condition can be expressed in terms of the duality map as follows: For each $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \geq 0. \tag{1.4}$$

Consequently, inequality (1.1) with $t = 1$ yields that A is accretive if and only if $T := (I-A)$ is pseudocontractive. Furthermore, setting $A := (I-T)$, it follows from inequality (1.3) that T is *strongly pseudocontractive* if and only if $(A-kI)$ is *accretive*, and using (1.4), this implies that $T(=I-A)$ is *strongly pseudocontractive* if and only if the following inequality holds

$$\langle Ax - Ay, j(x-y) \rangle \geq k\|x-y\|^2 \tag{1.5}$$

for all $x, y \in D(A)$ and some $k \in (0,1)$. Operators A satisfying inequality (1.5) for all $x, y \in D(A)$ and some $k \in (0,1)$ are called *strongly accretive*. It is then clear that A is strongly accretive if and only if $T := (I-A)$ is strongly pseudocontractive. Thus, the mapping theory for strongly accretive operators is closely related to the fixed point theory of strongly pseudocontractive maps. We shall exploit this connection in the sequel.

The notion of accretive operators was introduced independently in 1967 by Browder [2] and Kato [15]. An early fundamental result in the theory of accretive operators, due to Browder, states that the initial value problem

$$\frac{du}{dt} + Au = 0, \quad u(0) = u_0 \tag{1.6}$$

is solvable if A is locally Lipschitzian and accretive on E. If u is independent of t, then $Au = 0$ and the solution of this equation corresponds to the equilibrium points of the system (1.6). Consequently, considerable research efforts have been devoted, especially within the past 15 years or so, to developing constructive techniques for the determination of the kernels of accretive operators in Banach spaces (see [3,4,8–12,14,16,17,19,20,22]). Two well known iterative schemes, the *Mann iterative method* (see [18]) and the *Ishikawa iteration scheme* (see [13]) have successfully been employed.

In [16], Liu obtained a fixed point of the strictly pseudocontractive mapping as the limit of an iteratively constructed sequence with error estimation in general Banach spaces.

Theorem 1.1. *Let E be a Banach space, and let K be a nonempty closed convex and bounded subset of E. Let $T : K \to K$ be a Lipschitzian strictly pseudocontractive mapping. If $Fix(T) \neq \emptyset$, where $Fix(T)$ is the fixed point set of T, then $\{x_n\}$ is a sequence in K generated by $x_1 \in K$,*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n,$$

where $\{\alpha_n\}$ is a seqnence in $(0, 1]$ satisfying

$$\sum_{n=1}^{\infty} \alpha_n = \infty, \quad \alpha_n \to 0 \quad n \to \infty$$

strongly converges to $q \in Fix(T)$ and $Fix(T)$ is a single set.

In [21], Sastry and Babu showed that any fixed point of a Lipschitzian, strictly pseudocontractive mapping T on a closed convex subset K of a Banach space E is necessarily unique, and may be norm approximated by an iterative procedure. They also provided a convergence rate estimate and removes the boundedness assumption on K, generalizing Theorems of Liu.

Theorem 1.2. *Let $(E, \|\cdot\|), K, T, L$ and k be as described above. Let $q \in K$ be a fixed point of T. Suppose that $\{\alpha_n\}$ is a sequence in $(0, 1]$ such that for some $\eta \in (0, k)$, for all $n \in \mathbb{N}$,*

$$\alpha_n \leq \frac{k - \eta}{(L + 1)(L + 2 - k)}, \quad \textit{while} \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Fix $x_1 \in K$. Define for all $n \in \mathbb{N}$,

$$x_{n+1} := (1 - \alpha_n)x_n + \alpha_n T x_n.$$

Then there exists a sequence $\{\beta_n\}$ in $(0, 1)$ with each $\beta_n \geq \frac{\eta}{1+k}\alpha_n$ such that for all $n \in \mathbb{N}$,

$$\|x_{n+1} - q\| \leq \prod_{j=1}^{n} (1 - \beta_j)\|x_1 - q\|.$$

In particular, $\{x_n\}$ converges strongly to q, and q is the unique fixed point of T.

The Mann and Ishikawa iteration schemes are global and their rate of convergence is generally of the order $O(n^{-\frac{1}{2}})$. It is clear that if, for an operator U, *the classical iteration sequence* of the form, $x_{n+1} = Ux_n$, $x_0 \in D(U)$ (the so-called *Picard sequence*) converges, then it is certainly superior and preferred to either the Mann or the Ishikawa sequence since it requires less computations and moreover, its rate of convergence is always at least as fast as that of a geometric progression.

In [5, 6], Chidume proved the following results.

Theorem 1.3. *Let E be an arbitrary real Banach space and $A : E \to E$ be a Lipschitz (with constant $L > 0$) and strongly accretive mapping with strong accretivity constant $k \in (0,1)$. Let x^* denote a solution of the equation $Ax = 0$. Set $\epsilon := \frac{1}{2}\big(\frac{k}{1+L(3+L-k)}\big)$ and define $A_\epsilon : E \to E$ by $A_\epsilon x := x - \epsilon Ax$ for each $x \in E$. For arbitrary $x_0 \in E$, define the sequence $\{x_n\}$ in E by*

$$x_{n+1} = A_\epsilon x_n, \quad n \geq 0. \tag{1.7}$$

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to x^* with

$$\|x_{n+1} - x^*\| \leq \delta^n \|x_0 - x^*\|,$$

where $\delta = 1 - \frac{1}{2}k\epsilon \in (0,1)$ is the Lipschitz constant of the operator A. Moreover, x^* is unique.

Corollary 1.4. *Let E be an arbitrary real Banach space and K be nonempty convex subset of E. Let $T : K \to K$ be Lipschitz (with constant $L > 0$) and strongly pseudocontractive (i.e., T satisfies inequality (1.3) for all $x, y \in K$). Assume that T has a fixed point $x^* \in K$. Set $\epsilon_0 := \frac{1}{2}\big(\frac{k}{1+L(3+L-k)}\big)$ and define $T_{\epsilon_0} : K \to K$ by $T_{\epsilon_0}x = (1-\epsilon_0)x + \epsilon_0 Tx$ for each $x \in K$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}_{n=0}^{\infty}$ in K by*

$$x_{n+1} = T_{\epsilon_0} x_n, \quad n \geq 0. \tag{1.8}$$

Then $\{x_n\}$ converges strongly to x^* with

$$\|x_{n+1} - x^*\| \leq \delta^n \|x_0 - x^*\|,$$

where $\delta := 1 - \frac{1}{2}k\epsilon_0 \in (0,1)$. Moreover, x^* is unique.

Recently Ćirić et al. [7] presented the following results.

Theorem 1.5. *Let E be an arbitrary real Banach space, $A : E \to E$ be a Lipschitz (with constant $L > 0$) and strongly accretive mapping with strong accretivity constant $k \in (0,1)$. Let x^* denote a solution of the equation $Ax = 0$. Set $\epsilon := \frac{k-\eta}{L(2+L)}$, $\eta \in (0,k)$ and define $A_\epsilon : E \to E$ by $A_\epsilon x := x - \epsilon Ax$ for each $x \in E$. For arbitrary $x_0 \in E$, define the sequence $\{x_n\}$ in E by*

$$x_{n+1} = A_\epsilon x_n, \quad n \geq 0.$$

Then $\{x_n\}$ converges strongly to x^* with

$$\|x_{n+1} - x^*\| \leq \theta^n \|x_0 - x^*\|,$$

where $\theta = 1 - \frac{k-\eta}{k-\eta+L(2+L)}\eta \in (0,1)$. Thus the choice $\eta = \frac{k}{2}$ yields $\theta = 1 - \frac{k^2}{2[k+2L(2+L)]}$. Moreover, x^* is unique.

Corollary 1.6. *Let E be an arbitrary real Banach space, K be a nonempty convex subset of E. Let $T : K \to K$ be Lipschitz (with constant $L > 0$) and strongly pseudocontractive (i.e., T satisfies inequality (1.3) for all $x, y \in K$). Assume that T has a fixed point $x^* \in K$. Set $\epsilon_0 := \frac{k-\eta}{L(2+L)}$, $\eta \in (0,k)$ and define $T_{\epsilon_0} : K \to K$ by $T_{\epsilon_0}x = (1-\epsilon_0)x + \epsilon_0 Tx$ for each $x \in K$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ in K by*

$$x_{n+1} = T_{\epsilon_0} x_n, n \geq 0.$$

Then $\{x_n\}$ *converges strongly to* x^* *with*

$$\|x_{n+1} - x^*\| \le \theta^n \|x_0 - x^*\|,$$

where $\theta := 1 - \frac{k-\eta}{k-\eta+L(2+L)}\eta \in (0,1)$. *Moreover,* x^* *is unique.*

However Kang et al. [14] established the following results.

Theorem 1.7. *Let* E *be an arbitrary real Banach space,* $A : E \to E$ *be a Lipschitz (with constant* $L > 1$*) and strongly accretive mapping with strong accretivity constant* $k \in (0,1)$. *Let* x^* *denote a solution of the equation* $Ax = 0$. *Set* $\epsilon := \frac{k-\eta}{L+(1+L)(k-\eta)}$, $\eta \in (0,k)$ *and define* $A_\epsilon : E \to E$ *by* $A_\epsilon x_n := (1-\epsilon)\, x_{n-1} + \epsilon x_n - \epsilon A x_n$ *for each* $x_n \in E$. *For arbitrary* $x_0 \in E$, *define the sequence* $\{x_n\}$ *in* E *by*

$$x_n = A_\epsilon x_n, \quad n \ge 1.$$

Then $\{x_n\}$ *converges strongly to* x^* *with*

$$\|x_{n+1} - x^*\| \le \lambda^n \|x_0 - x^*\|,$$

where $\lambda = 1 - \frac{k-\eta}{L+(k-\eta)(1+L+k)}\eta \in (0,1)$. *Thus the choice* $\eta = \frac{k}{2}$ *yields* $\lambda = 1 - \frac{k^2}{2[2L+k(1+L+k)]}$. *Moreover,* x^* *is unique.*

Corollary 1.8. *Let* E *be an arbitrary real Banach space ans* K *be a nonempty closed convex subset of* E. *Let* $T : K \to K$ *be Lipschitz (with constant* $L > 0$*) and strongly pseudocontractive (i.e.,* T *satisfies inequality (1.3) for all* $x, y \in K$*). Assume that* T *has a fixed point* $x^* \in K$. *Set* $\epsilon_0 := \frac{k-\eta}{L+(1+L)(k-\eta)}$, $\eta \in (0,k)$ *and define* $A_{\epsilon_0} : K \to K$ *by* $A_{\epsilon_0} x_n := (1-\epsilon_0)\, x_{n-1} + \epsilon_0 x_n - \epsilon_0 A x_n$ *for each* $x_n \in K$. *For arbitrary* $x_0 \in K$, *define the sequence* $\{x_n\}$ *in* K *by*

$$x_n = A_{\epsilon_0} x_n, \quad n \ge 1.$$

Then $\{x_n\}$ *converges strongly to* x^* *with*

$$\|x_{n+1} - x^*\| \le \lambda_0^n \|x_0 - x^*\|,$$

where $\lambda_0 = 1 - \frac{k-\eta}{L+(k-\eta)(1+L+k)}\eta \in (0,1)$. *Thus the choice* $\eta = \frac{k}{2}$ *yields* $\lambda_0 = 1 - \frac{k^2}{2[2L+k(1+L+k)]}$. *Moreover,* x^* *is unique.*

Let H be the Hilbert space. Recently Alghamdi et al. [1] defined the following algorithm.

Algorithm 1.9. *Initialize* $x_n \in H$ *arbitrarily and iterate*

$$x_{n+1} = (1-t_n)x_n + t_n T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \ge 0,$$

where $t_n \in (0,1)$ *for all* n.

For the approximation of fixed points of nonexpansive mappings under the setting of Hilbert spaces, they provide the following results.

Lemma 1.10. *Let* $\{x_n\}$ *be the sequence generated by Algorithm 1.9. Then*

(i) $\|x_{n+1} - p\| \leq \|x_n - p\|$ *for all* $n \geq 0$ *and* $p \in Fix(T)$,

(ii) $\sum_{n=1}^{\infty} t_n \|x_n - x_{n+1}\|^2 < \infty$,

(iii) $\sum_{n=1}^{\infty} t_n(1 - t_n)\|x_n - T(\frac{x_n+x_{n+1}}{2})\|^2 < \infty$.

Lemma 1.11. *Let* $\{x_n\}$ *be the sequence generated by Algorithm 1.9. Suppose that* $t_{n+1}^2 \leq at_n$ *for all* $n \geq 0$ *and* $a > 0$. *Then*

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Lemma 1.12. *Assume that*

(i) $t_{n+1}^2 \leq at_n$ *for all* $n \geq 0$ *and* $a > 0$,

(ii) $\limsup_{n\to\infty} t_n > 0$.

Then the sequence $\{x_n\}$ *generated by Algorithm 1.9 satisfies the property*

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Theorem 1.13. *Let* H *be a Hilbert space and* $T : H \to H$ *be a nonexpansive mapping with* $Fix(T) \neq \emptyset$. *Assume that* $\{x_n\}$ *is generated by Algorithm 1.9, where the sequence* $\{t_n\}$ *of parameters satisfies the conditions:*

(i) $t_{n+1}^2 \leq at_n$ *for all* $n \geq 0$ *and* $a > 0$,

(ii) $\limsup_{n\to\infty} t_n > 0$.

Then $\{x_n\}$ *converges weakly to a fixed point of* T.

In this paper, we study the convergence of implicit Picard sequence for strongly accretive and strongly pseudocontractive mappings. We have also improved the results of [5–7, 14, 16, 19–21].

2 Main results

In the following theorems, $L > 1$ will denote the Lipschitz constant of the operator A and $k > 0$ will denote the strong accretivity constant of A (as in inequality (1.5)). Furthermore, $\epsilon > 0$ is defined by

$$\epsilon := \frac{k - \eta}{L + \frac{1}{2}(1 + L)(k - \eta)}, \quad \eta \in (0, k).$$

With these notations, we prove the following theorem.

Theorem 2.1. *Let* E *be an arbitrary real Banach space,* $A : E \to E$ *be a Lipschitz and strongly accretive mapping with strong accretivity constant* $k \in (0, 1)$. *Let* x^* *denote a solution of the equation* $Ax = 0$. *Define* $A_\epsilon : E \to E$ *by* $A_\epsilon x_n := (1 - \epsilon)\, x_{n-1} + \epsilon \frac{x_{n-1}+x_n}{2} - \epsilon A \frac{x_{n-1}+x_n}{2}$ *for each* $x_n \in E$. *For arbitrary* $x_0 \in E$, *define the sequence* $\{x_n\}_{n=0}^{\infty}$ *in* E *by*

$$x_n = A_\epsilon x_n, \quad n \geq 1.$$

Then $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* x^* *with*

$$\|x_{n+1} - x^*\| \leq \rho^n \|x_0 - x^*\|,$$

where $\rho = 1 - \frac{2(k-\eta)}{2L+(k-\eta)(2+L+k)}\eta \in (0, 1)$. *Thus the choice* $\eta = \frac{k}{2}$ *yields* $\rho = 1 - \frac{k^2}{4L+k(2+L+k)}$. *Moreover,* x^* *is unique.*

Proof. Existence of x^* follows from Theorem 13.1 of [8]. Define $T := (I - A)$ where I denotes the identity mapping on E. Observe that $Ax^* = 0$ if and only if x^* is a fixed point of T. Moreover, T is strongly pseudocontractive (satisfies inequality (1.2) since A satisfies (1.5), and so T also satisfies inequality (1.3) for all $x, y \in E$ and all $s > 0$. Furthermore, the recursion formula $x_n = A_\epsilon x_n$ becomes

$$x_n = (1-\epsilon)x_{n-1} + \epsilon T\left(\frac{x_{n-1}+x_n}{2}\right), \quad n \geq 1. \tag{2.1}$$

Observe that

$$x^* = (1+\epsilon)x^* + \epsilon(I - T - kI)x^* - (1-k)\epsilon x^*,$$

and from the recursion formula (2.1) that

$$\begin{aligned} x_{n-1} &= (1+\epsilon)x_n + \epsilon(I - T - kI)\frac{x_{n-1}+x_n}{2} - (1-k)\epsilon\frac{x_{n-1}+x_n}{2} \\ &\quad + \epsilon^2\left(x_{n-1} - T\left(\frac{x_{n-1}+x_n}{2}\right)\right), \end{aligned} \tag{2.2}$$

so that

$$\begin{aligned} x_{n-1} - x^* &= (1+\epsilon)(x_n - x^*) + \epsilon\left[(I - T - kI)\frac{x_{n-1}+x_n}{2} - (I - T - kI)x^*\right] \\ &\quad - (1-k)\epsilon\left(\frac{x_{n-1}+x_n}{2} - x^*\right) + \epsilon^2\left(x_{n-1} - T\left(\frac{x_{n-1}+x_n}{2}\right)\right). \end{aligned}$$

Assume that $x_n \simeq x_{n-1}$, which yields that $\frac{x_{n-1}+x_n}{2} \simeq x_n$. Replace $\frac{x_{n-1}+x_n}{2}$ by x_n in the second term of right hand side, we get

$$\begin{aligned} x_{n-1} - x^* &= (1+\epsilon)(x_n - x^*) + \epsilon\left[(I - T - kI)x_n - (I - T - kI)x^*\right] \\ &\quad - (1-k)\epsilon\left(\frac{x_{n-1}+x_n}{2} - x^*\right) + \epsilon^2\left(x_{n-1} - T\left(\frac{x_{n-1}+x_n}{2}\right)\right). \end{aligned}$$

This implies, using inequality (1.3) with $s = \frac{\epsilon}{1+\epsilon}$ and $y = x^*$ that

$$\begin{aligned} &\|x_{n-1} - x^*\| \\ &\geq (1+\epsilon)\left[\left\|(x_n - x^*) + \frac{\epsilon}{1+\epsilon}\left[(I - T - kI)x_n - (I - T - kI)x^*\right]\right\|\right] \\ &\quad - (1-k)\epsilon\left\|\frac{x_{n-1}+x_n}{2} - x^*\right\| - \epsilon^2\left\|x_{n-1} - T\left(\frac{x_{n-1}+x_n}{2}\right)\right\| \\ &\geq (1+\epsilon)\|x_n - x^*\| - (1-k)\frac{\epsilon}{2}\|x_{n-1} - x^*\| \\ &\quad - (1-k)\frac{\epsilon}{2}\|x_n - x^*\| - \epsilon^2\left\|x_{n-1} - T\left(\frac{x_{n-1}+x_n}{2}\right)\right\| \\ &= -(1-k)\frac{\epsilon}{2}\|x_{n-1} - x^*\| + \left(1 + (1+k)\frac{\epsilon}{2}\right)\|x_n - x^*\| \\ &\quad - \epsilon^2\left\|x_{n-1} - T\left(\frac{x_{n-1}+x_n}{2}\right)\right\|. \end{aligned} \tag{2.3}$$

Observe that

$$\begin{aligned}
&\left\|x_{n-1} - T\left(\frac{x_{n-1}+x_n}{2}\right)\right\| \\
&\le \|x_{n-1} + Tx_{n-1}\| + \left\|Tx_{n-1} + T\left(\frac{x_{n-1}+x_n}{2}\right)\right\| \\
&\le \|Ax_{n-1}\| + \frac{1}{2}\|x_{n-1} - x_n\| + \left\|Ax_{n-1} - A\left(\frac{x_{n-1}+x_n}{2}\right)\right\| \\
&\le L\|x_{n-1} - x^*\| + \frac{1}{2}\|x_{n-1} - x_n\| + L\left\|x_{n-1} - \frac{x_{n-1}+x_n}{2}\right\| \\
&= L\|x_{n-1} - x^*\| + \frac{1}{2}(1+L)\|x_{n-1} - x_n\| \\
&= L\|x_{n-1} - x^*\| + \frac{1}{2}(1+L)\epsilon\left\|x_{n-1} - T\left(\frac{x_{n-1}+x_n}{2}\right)\right\|,
\end{aligned}$$

and so

$$\|x_{n-1} - T\left(\frac{x_{n-1}+x_n}{2}\right)\| \le \frac{L}{1-\frac{1}{2}(1+L)\epsilon}\|x_{n-1} - x^*\|, \tag{2.4}$$

so that from (2.3) we obtain

$$\begin{aligned}
&\left(1 + (1-k)\frac{\epsilon}{2}\right)\|x_{n-1} - x^*\| \\
&\ge \left(1 + (1+k)\frac{\epsilon}{2}\right)\|x_n - x^*\| - \frac{L\epsilon^2}{1-\frac{1}{2}(1+L)\epsilon}\|x_{n-1} - x^*\|.
\end{aligned}$$

Therefore

$$\|x_n - x^*\| \le \frac{1 + (1-k)\frac{\epsilon}{2} + \frac{L\epsilon^2}{1-\frac{1}{2}(1+L)\epsilon}}{1+(1+k)\frac{\epsilon}{2}}\|x_{n-1} - x^*\|, \tag{2.5}$$

and consider

$$\begin{aligned}
\rho &= \frac{1 + (1-k)\frac{\epsilon}{2} + \frac{L\epsilon^2}{1-\frac{1}{2}(1+L)\epsilon}}{1+(1+k)\frac{\epsilon}{2}} \\
&= 1 - \frac{\epsilon}{1+(1+k)\frac{\epsilon}{2}}\left[k - \frac{L\epsilon}{1-\frac{1}{2}(1+L)\epsilon}\right] \\
&= 1 - \frac{\epsilon}{1+(1+k)\frac{\epsilon}{2}}\eta \\
&= 1 - \frac{2(k-\eta)}{2L + (k-\eta)(2+L+k)}\eta.
\end{aligned} \tag{2.6}$$

From (2.5) and (2.6), we get

$$\begin{aligned}
\|x_n - x^*\| &\le \rho\|x_{n-1} - x^*\| \\
&\le \cdots \le \rho^n\|x_0 - x^*\| \\
&\to 0
\end{aligned}$$

as $n \to \infty$. Hence $x_n \to x^*$ as $n \to \infty$. Uniqueness follows from the strong accretivity property of A. This completes the proof. □

The following is an immediate corollary of Theorem 2.1.

Corollary 2.2. *Let E be an arbitrary real Banach space and K be anonempty closed convex subset of E. Let $T : K \to K$ be Lipschitz (with constant $L > 1$) and strongly pseudocontractive (i.e., T satisfies inequality (1.3) for all $x, y \in K$). Assume that T has a fixed point $x^* \in K$. Set $\epsilon_0 := \frac{k-\eta}{L+\frac{1}{2}(1+L)(k-\eta)}$; $\eta \in (0, k)$ and Define $A_{\epsilon_0} : K \to K$ by $A_{\epsilon_0}x_n := (1-\epsilon_0)\, x_{n-1} + \epsilon_0\frac{x_{n-1}+x_n}{2} - \epsilon_0 A\frac{x_{n-1}+x_n}{2}$ for each $x_n \in K$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ in K by*

$$x_n = A_{\epsilon_0}x_n, \quad n \geq 1. \tag{2.7}$$

Then $\{x_n\}$ converges strongly to x^* with

$$\|x_{n+1} - x^*\| \leq \rho_0^n\|x_0 - x^*\|,$$

where $\rho_0 = 1 - \frac{2(k-\eta)}{2L+(k-\eta)(2+L+k)}\eta \in (0, 1)$. Thus the choice $\eta = \frac{k}{2}$ yields $\rho_0 = 1 - \frac{k^2}{4L+k(2+L+k)}$. Moreover, x^* is unique.

Proof. Observe that x^* is a fixed point of T if and only if it is a fixed point of T_{ϵ_0}. Furthermore, the recursion formula (2.7) simplifies to the formula

$$x_n = (1-\epsilon_0)x_{n-1} + \epsilon_0 Tx_n,$$

which is similar to (2.1). Following the method of computations as in the proof of the Theorem 2.1, we obtain

$$\begin{aligned}\|x_n - x^*\| &\leq \frac{1+(1-k)\frac{\epsilon_0}{2} + \frac{L\epsilon_0^2}{1-\frac{1}{2}(1+L)\epsilon_0}}{1+(1+k)\frac{\epsilon_0}{2}}\|x_{n-1} - x^*\| \\ &= \left(1 - \frac{2(k-\eta)}{2L+(k-\eta)(2+L+k)}\eta\right)\|x_{n-1} - x^*\|.\end{aligned} \tag{2.8}$$

Set $\rho_0 = 1 - \frac{2(k-\eta)}{2L+(k-\eta)(2+L+k)}\eta$. Then from (2.8) we obtain

$$\begin{aligned}\|x_n - x^*\| &\leq \rho_0\|x_{n-1} - x^*\| \\ &\leq \cdots \leq \rho_0^n\|x_0 - x^*\| \\ &\to 0\end{aligned}$$

as $n \to \infty$. This completes the proof. □

Remark 2.3. Since $L > 1$, consider

$$\begin{aligned}\rho &= 1 - \frac{2(k-\eta)}{2L+(k-\eta)(2+L+k)}\eta \\ &= 1 - \frac{k-\eta}{L+(k-\eta)(1+L+k)} \\ &\quad - \left(\frac{2}{2L+(k-\eta)(2+L+k)} - \frac{1}{L+(k-\eta)(1+L+k)}\right)(k-\eta)\eta \\ &< 1 - \frac{k-\eta}{L+(k-\eta)(1+L+k)} \\ &\quad - \frac{(k-\eta)^2(L+k)\eta}{(2L+(k-\eta)(2+L+k))(L+(k-\eta)(1+L+k))}.\end{aligned}$$

Thus the relation between Kang et al. [14] and our parameter of convergence, that is, λ and ρ, respectively, is the following:

$$\rho < \lambda.$$

Our convergence parameter ρ shows the overall improvement for λ, and consequently the results of Chidume [5, 6], Ćirić et al. [7] and Kang et al. [14] are improved.

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JORDAN HOMOMORPHISMS IN C^*-TERNARY ALGEBRAS AND JB^*-TRIPLES

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Abstract. In this paper, we investigate Jordan homomorphisms between C^*-ternary algebras and Jordan derivations on C^*-ternary algebras, and Jordan homomorphisms between JB^*-triples and Jordan derivations on JB^*-triples, associated with the following Apollonius type additive functional equation

$$f(z-x)+f(z-y)=-\frac{1}{2}f(x+y)+2f\Big(z-\frac{x+y}{4}\Big).$$

1. Introduction and preliminaries

We say that a functional equation (Q) is stable if any function g satisfying the equation (Q) approximately is near to true solution of (Q).

Also, we say that a functional equation is superstable if every approximately solution is an exact solution of it.

Ternary algebraic operations were considered in the 19 th century by several mathematicians such as A.

Cayley [3] who introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii in 1990 [17]. As an application in physics, the quark model inspired a particular brand of ternary algebraic systems. The so-called Nambu mechanics which has been proposed by Y. Nambu [19] in 1973, is based on such structures.
There are also some applications, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics (the anyons), supersymmetric theories, Yang-Baxter equation, etc, (cf. [1, 40]).
The comments on physical applications of ternary structures can be found in [2, 5, 7, 18].

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x,y,z)\mapsto [x,y,z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x,y,[z,w,v]]=[x,[w,z,y],v]=[[x,y,z],w,v]$, and satisfies $\|[x,y,z]\|\le\|x\|\cdot\|y\|\cdot\|z\|$ and $\|[x,x,x]\|=\|x\|^3$ (see [41]).

If a C^*-ternary algebra $(A,[\cdot,\cdot,\cdot])$ has an identity, i.e., an element $e\in A$ such that $x=[x,e,e]=[e,e,x]$ for all $x\in A$, then it is routine to verify that A, endowed with $x\circ y:=[x,e,y]$ and $x^*:=[e,x,e]$, is a unital C^*-algebra. Conversely, if $(A,\circ)$ is a unital C^*-algebra, then

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$[x, y, z] := x \circ y^* \circ z$ makes A into a C^*-ternary algebra.
A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *C^*-ternary algebra homomorphism* if

$$H\Big([x, y, z]\Big) = [H(x), H(y), H(z)]$$

for all $x, y, z \in A$.
A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *C^*-ternary derivation* if

$$\delta\Big([x, y, z]\Big) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)]$$

for all $x, y, z \in A$ (see [22]).
A $\mathbb{C}$-linear mapping $H : A \to B$ is called a ternary Jordan homomorphism if

$$H\Big([x, x, x]\Big) = [H(x), H(x), H(x)]$$

for all $x \in A$.
A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a ternary Jordan derivation if

$$\delta\Big([xxx]\Big) = [\delta(x)xx] + [x\delta(x)x] + [xx\delta(x)]$$

for all $x \in A$. Suppose that $\mathcal{J}$ is a complex vector space endowed with a real trilinear composition $\mathcal{J} \times \mathcal{J} \times \mathcal{J} \ni (x, y, z) \mapsto \{xy^*z\} \in \mathcal{J}$ which is complex bilinear in (x, z) and conjugate linear in y. Then $\mathcal{J}$ is called a *Jordan triple system* if $\{xy^*z\} = \{zy^*x\}$ and

$$\{\{xy^*z\}u^*v\} + \{\{xy^*v\}u^*z\} - \{xy^*\{zu^*v\}\} = \{z\{yx^*u\}^*v\}$$

hold.

We are interested in Jordan triple systems having a Banach space structure. A complex Jordan triple system $\mathcal{J}$ with a Banach space norm $\|\cdot\|$ is called a *J^*-triple* if, for every $x \in \mathcal{J}$, the operator $x\Box x^*$ is hermitian in the sense of Banach algebra theory. Here the operator $x\Box x^*$ on $\mathcal{J}$ is defined by $(x\Box x^*)y := \{xx^*y\}$. This implies that $x\Box x^*$ has real spectrum $\sigma(x\Box x^*) \subset \mathbb{R}$. A J^*-triple $\mathcal{J}$ is called a JB^*-triple if every $x \in \mathcal{J}$ satisfies $\sigma(x\Box x^*) \geq 0$ and $\|x\Box x^*\| = \|x\|^2$.
A $\mathbb{C}$-linear mapping $H : \mathcal{J} \to \mathcal{L}$ is called a *JB^*-triple homomorphism* if

$$H\Big(\{xyz\}\Big) = \{H(x)H(y)H(z)\}$$

for all $x, y, z \in \mathcal{J}$.
A $\mathbb{C}$-linear mapping $\delta : \mathcal{J} \to \mathcal{J}$ is called a *JB^*-triple derivation* if

$$\delta\Big(\{xyz\}\Big) = \{\delta(x)yz\} + \{x\delta(y)z\} + \{xy\delta(z)\}$$

for all $x, y, z \in \mathcal{J}$ (see [20]).
A $\mathbb{C}$-linear mapping $H : \mathcal{J} \to \mathcal{L}$ is called a *JB^*-triple Jordan homomorphism* if

$$H\Big(\{xxx\}\Big) = \{H(x)H(x)H(x)\}$$

for all $x \in \mathcal{J}$.
A $\mathbb{C}$-linear mapping $\delta : \mathcal{J} \to \mathcal{J}$ is called a *JB^*-triple Jordan derivation* if

$$\delta\Big(\{xxx\}\Big) = \{\delta(x)xx\} + \{x\delta(x)x\} + \{xx\delta(x)\}$$

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for all $x \in \mathcal{J}$

The study of stability problems originated from a famous talk given by Ulam [39] in 1940: "Under what condition does there exist a homomorphism near an approximate homomorphism?" In the next year 1941, Hyers [11] answered affirmatively the question of Ulam for additive mappings between Banach spaces. Th.M. Rassias [25] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

Theorem 1.1. (Th.M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p$$

for all $x \in E$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.

Th.M. Rassias [26] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Gajda [9] following the same approach as in Th.M. Rassias [25], gave an affirmative solution to this question for $p > 1$. For further research developments in stability of functional equations the readers are referred to the works of Găvruta [10], Jung [16], Park [23], Th.M. Rassias [27]–[30], Th.M. Rassias and Šemrl [31], F. Skof [38] and the references cited therein. See also [32, 33, 34, 35, 36, 37] for functional equations.

In an inner product space, the equality

$$\|z-x\|^2 + \|z-y\|^2 = \frac{1}{2}\|x-y\|^2 + 2\left\|z - \frac{x+y}{2}\right\|^2$$

holds, and is called the *Apollonius' identity*. The following functional equation, which was motivated by this equation,

$$Q(z-x) + Q(z-y) = \frac{1}{2}Q(x-y) + 2Q\Big(z - \frac{x+y}{2}\Big), \tag{1.1}$$

is quadratic. For this reason, the function equation (1.1) is called a *quadratic functional equation of Apollonius type*, and each solution of the functional equation (1.1) is said to be a *quadratic mapping of Apollonius type*. Jun and Kim [15] investigated the quadratic functional equation of Apollonius type.

In this paper, employing the above equality (1.1), we introduce a new functional equation, which is called the *Apollonius type additive functional equation* and whose solution of the functional equation is said to be the *Apollonius type additive mapping*:

$$L(z-x) + L(z-y) = -\frac{1}{2}L(x+y) + 2L\Big(z - \frac{x+y}{4}\Big).$$

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In this paper, we investigate Jordan homomorphisms and Jordan derivations in C^*-ternary algebras, and Jordan homomorphisms and Jordan derivations in JB^*-triples.

2. Jordan homomorphisms between C^*-ternary algebras

Throughout this section, assume that A is a C^*-ternary algebra with norm $\|\cdot\|_A$ and that B is a C^*-ternary algebra with norm $\|\cdot\|_B$.

In this section, we investigate Jordan homomorphisms between C^*-ternary algebras.
The following lemma was proved in [24].

Lemma 2.1. *Let $f : A \to B$ be a mapping such that*

$$\Big\|f(z-x)+f(z-y)+\frac{1}{2}f(x+y)\Big\|_B \le \Big\|2f(z-\frac{x+y}{4})\Big\|_B$$

for all $x, y, z \in A$. Then f is additive.

The following lemma was proved in [8].

Lemma 2.2. *Let $f : A \to B$ be an additive mapping. Then the following assertions are equivalent*

$$f\Big([x,x,x]\Big) = [f(x), f(x), f(x)]$$

for all $x \in A$, and

$$f\Big([x,y,z]+[y,z,x]+[z,x,y]\Big) = [f(x),f(y),f(z)]+[f(y),f(z),f(x)]+[f(z),f(x),f(y)]$$

for all $x, y, z \in A$.

The following lemma was proved in [6].

Lemma 2.3. *Let $f : A \to A$ be an additive mapping. Then the following assertions are equivalent.*

$$f\Big([x,x,x]\Big) = [f(x),x,x]+[x,f(x),x]+[x,x,f(x)]$$

for all $x \in A$, and

$$\begin{aligned}&f\Big([xyz]+[yzx]+[zxy]\Big)\\&=[f(x),b,c]+[x,f(y),z]+[x,y,f(z)]+[f(y),z,x]+[y,f(z),x]\\&+[y,z,f(x)]+[f(z),x,y]+[z,f(x),y]+[z,x,f(y)],\end{aligned}$$

for all $x, y, z \in A$.

Theorem 2.4. *Let $r \neq 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that*

$$\Big\|f(z-\mu x)+\mu f(z-y)+\frac{1}{2}f(x+y)\Big\|_B \le \Big\|2f(z-\frac{x+y}{4})\Big\|_B, \tag{2.1}$$

$$\begin{aligned}&\Big\|f\Big([x,y,z]+[y,z,x]+[z,x,y]\Big)-[f(x),f(y),f(z)]-[f(y),f(z),f(x)]-[f(z),f(x),f(y)]\Big\|_B\\&\le \theta\Big(\|x\|_A^{3r}+\|y\|_A^{3r}+\|z\|_A^{3r}\Big)\end{aligned} \tag{2.2}$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ *and all* $x, y, z \in A$. *Then the mapping* $f : A \to B$ *is a* C^**-ternary algebra Jordan homomorphism.*

Proof. Assume $r > 1$.

Let $\mu = 1$ in (2.1). By Lemma 2.1, the mapping $f : A \to B$ is additive.

Letting $y = -x$ and $z = 0$, we get

$$\|f(-\mu x) + \mu f(x)\|_B \leq \|2f(0)\|_B = 0$$

for all $x \in A$ and all $\mu \in \mathbb{T}^1$. So

$$-f(\mu x) + \mu f(x) = f(-\mu x) + \mu f(x) = 0$$

for all $x \in A$ and all $\mu \in \mathbb{T}^1$. Hence $f(\mu x) = \mu f(x)$ for all $x \in A$ and all $\mu \in \mathbb{T}^1$. By the same reasoning as in the proof of [21, Theorem 2.1], the mapping $f : A \to B$ is $\mathbb{C}$-linear.

It follows from (2.2) that

$$\begin{aligned}
&\left\|f\Big([x,y,z]+[y,z,x]+[z,x,y]\Big) - [f(x),f(y),f(z)] - [f(y),f(z),f(x)] - [f(z),f(x),f(y)]\right\|_B \\
&\quad = \lim_{n\to\infty} 8^n \left\|f\Big(\frac{[x,y,z]}{8^n} + \frac{[y,z,x]}{8^n} + \frac{[z,x,y]}{8^n}\Big)\right. \\
&-\Big[f(\frac{x}{2^n}), f(\frac{y}{2^n}), f(\frac{z}{2^n})\Big] - \Big[f(\frac{y}{2^n}), f(\frac{z}{2^n}), f(\frac{x}{2^n})\Big] - \left.\Big[f(\frac{z}{2^n}), f(\frac{x}{2^n}), f(\frac{y}{2^n})\Big]\right\|_B \\
&\quad \leq \lim_{n\to\infty} \frac{8^n\theta}{8^{nr}}\Big(\|x\|_A^{3r} + \|y\|_A^{3r} + \|z\|_A^{3r}\Big) = 0
\end{aligned}$$

for all $x, y, z \in A$. Thus

$$f\Big([x,y,z]+[y,z,x]+[z,x,y]\Big) = [f(x),f(y),f(z)] + [f(y),f(z),f(x)] + [f(z),f(x),f(y)]$$

for all $x, y, z \in A$. Hence the mapping $f : A \to B$ is a C^*-ternary algebra Jordan homomorphism.

Similarly, one obtains the result for the case $r < 1$. ☐

3. Jordan derivations on C^*-ternary algebras

Throughout this section, assume that A is a C^*-ternary algebra with norm $\|\cdot\|_A$.

In this section, we investigate Jordan derivations on C^*-ternary algebras.

Theorem 3.1. *Let* $r \neq 1$ *and* θ *be nonnegative real numbers, and let* $f : A \to A$ *be a mapping satisfying* (2.1) *such that*

$$\begin{aligned}
&\Big\|f\Big([x,y,z]+[y,z,x]+[z,x,y]\Big) - [f(x),b,c] - [x,f(y),z] - [x,y,f(z)] - [f(y),z,x] \\
&- [y,f(z),x] - [y,z,f(x)] - [f(z),x,y] - [z,f(x),y] - [z,x,f(y)]\Big\|_A \\
&\leq \theta\Big(\|x\|_A^{3r} + \|y\|_A^{3r} + \|z\|_A^{3r}\Big)
\end{aligned} \tag{3.1}$$

for all $x, y, z \in A$. *Then the mapping* $f : A \to A$ *is a* C^**-ternary Jordan derivation.*

Proof. Assume $r > 1$.

By the same reasoning as in the proof of Theorem 2.4, the mapping $f : A \to A$ is $\mathbb{C}$-linear.

It follows from (3.1) that

$$\begin{aligned}
&\Big\|f\Big([x,y,z]+[y,z,x]+[z,x,y]\Big)-[f(x),y,z]-[x,f(y),z]-[x,y,f(z)]-[f(y),z,x]\\
&-[y,f(z),x]-[y,z,f(x)]-[f(z),x,y]-[z,f(x),y]-[z,x,f(y)]\|_A\\
&=\lim_{n\to\infty}8^n\Big\|f\Big([\frac{x,y,z}{8^n}]+[\frac{y,z,x}{8^n}]+[\frac{z,x,y}{8^n}]\Big)-\Big[f(\frac{x}{2^n}),\frac{y}{2^n},\frac{z}{2^n}\Big]\\
&-\Big[\frac{x}{2^n},f(\frac{y}{2^n}),\frac{z}{2^n}\Big]-\Big[\frac{x}{2^n},\frac{y}{2^n},f(\frac{z}{2^n})\Big]-\Big[f(\frac{y}{2^n}),\frac{z}{2^n},\frac{x}{2^n}\Big]-\Big[\frac{y}{2^n},f(\frac{z}{2^n}),\frac{x}{2^n}\Big]\\
&-\Big[\frac{y}{2^n},\frac{z}{2^n},f(\frac{x}{2^n})\Big]-\Big[f(\frac{z}{2^n}),\frac{x}{2^n},\frac{y}{2^n}\Big]-\Big[\frac{z}{2^n},f(\frac{x}{2^n}),\frac{y}{2^n}\Big]-\Big[\frac{z}{2^n},\frac{x}{2^n},f(\frac{y}{2^n})\Big]\Big\|_A\\
&\le\lim_{n\to\infty}\frac{8^n\theta}{8^{nr}}\Big(\|x\|_A^{3r}+\|y\|_A^{3r}+\|z\|_A^{3r}\Big)=0
\end{aligned}$$

for all $x, y, z \in A$. So

$$\begin{aligned}
f\Big([x,y,z]+[y,z,x]+[z,x,y]\Big)&=[f(x),y,z]+[x,f(y),z]+[x,y,f(z)]+[f(y),z,x]\\
&+[y,f(z),x]+[y,z,f(x)]+[f(z),x,y]+[z,f(x),y]+[z,x,f(y)]
\end{aligned}$$

for all $x, y, z \in A$.

Thus the mapping $f : A \to A$ is a C^*-ternary Jordan derivation.

Similarly, one obtains the result for the case $r < 1$. □

4. Jordan homomorphisms between JB^*-triples

Throughout this paper, assume that $\mathcal{J}$ is a JB^*-triple with norm $\|\cdot\|_{\mathcal{J}}$ and that $\mathcal{L}$ is a JB^*-triple with norm $\|\cdot\|_{\mathcal{L}}$.

In this section, we investigate Jordan homomorphisms between JB^*-triples.

Theorem 4.1. *Let $r \neq 1$ and θ be nonnegative real numbers, and let $f : \mathcal{J} \to \mathcal{L}$ be a mapping such that*

$$\|f(z-\mu x)+\mu f(z-y)+\frac{1}{2}f(x+y)\|_{\mathcal{L}} \quad\le\quad \|2f(z-\frac{x+y}{4})\|_{\mathcal{L}}, \tag{4.1}$$

$$\begin{aligned}
&\Big\|f\Big(\{xyz\}+\{yzx\}+\{zxy\}\Big)-\{f(x)f(y)f(z)\}-\{f(y)f(z)f(x)\}-\{f(z)f(x)f(y)\}\Big\|_{\mathcal{L}}\\
&\le\theta\Big(\|x\|_{\mathcal{J}}^{3r}+\|y\|_{\mathcal{J}}^{3r}+\|z\|_{\mathcal{J}}^{3r}\Big)
\end{aligned} \tag{4.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in \mathcal{J}$. Then the mapping $f : \mathcal{J} \to \mathcal{L}$ is a JB^*-triple Jordan homomorphism.

Proof. Assume $r > 1$.

By the same reasoning as in the proof of Theorem 2.4, the mapping $f : \mathcal{J} \to \mathcal{L}$ is $\mathbb{C}$-linear.

It follows from (4.2) that

$$\begin{aligned}
&\Big\|f\Big(\{xyz\}+\{yzx\}+\{zxy\}\Big)-\{f(x)f(y)f(z)\}-\{f(y)f(z)f(x)\}-\{f(z)f(x)f(y)\}\Big\|_{\mathcal{L}}\\
&\quad=\lim_{n\to\infty}8^n\Big\|f\Big(\frac{\{xyz\}}{2^n.2^n.2^n}+\frac{\{yzx\}}{2^n.2^n.2^n}+\frac{\{zxy\}}{2^n.2^n.2^n}\Big)-\Big\{f(\frac{x}{2^n})f(\frac{y}{2^n})f(\frac{z}{2^n})\Big\}\\
&\quad-\Big\{f(\frac{y}{2^n})f(\frac{z}{2^n})f(\frac{x}{2^n})\Big\}-\Big\{f(\frac{z}{2^n})f(\frac{x}{2^n})f(\frac{y}{2^n})\Big\}\Big\|_{\mathcal{L}}\\
&\quad\le\lim_{n\to\infty}\frac{8^n\theta}{8^{nr}}\Big(\|x\|_{\mathcal{J}}^{3r}+\|y\|_{\mathcal{J}}^{3r}+\|z\|_{\mathcal{J}}^{3r}\Big)=0
\end{aligned}$$

for all $x,y,z\in\mathcal{J}$. Thus

$$f\Big(\{xyz\}+\{yzx\}+\{zxy\}\Big)=\{f(x)f(y)f(z)\}+\{f(y)f(z)f(x)\}+\{f(z)f(x)f(y)\}$$

for all $x,y,z\in\mathcal{J}$. Hence the mapping $f:\mathcal{J}\to\mathcal{L}$ is a JB^*-triple Jordan homomorphism.

Similarly, one obtains the result for the case $r<1$. □

5. Jordan derivations on JB^*-triples

Throughout this paper, assume that $\mathcal{J}$ is a JB^*-triple with norm $\|\cdot\|_{\mathcal{J}}$.

In this section, we investigate Jordan derivations on JB^*-triples.

Theorem 5.1. *Let $r\neq 1$ and θ be nonnegative real numbers, and let $f:\mathcal{J}\to\mathcal{J}$ be a mapping satisfying* (4.1) *such that*

$$\begin{aligned}
&\Big\|f(\{xyz\}+\{yzx\}+\{zxy\})-\{f(x)yz\}-\{xf(y)z\}-\{xyf(z)\}-\{f(y)zx\}\\
&-\{yf(z)x\}-\{yzf(x)\}-\{f(z)xy\}-\{zf(x)y\}-\{zxf(y)\}\Big\|_A \qquad (5.1)\\
&\le\theta\Big(\|x\|_A^{3r}+\|y\|_A^{3r}+\|z\|_A^{3r}\Big)
\end{aligned}$$

for all $x,y,z\in\mathcal{J}$. Then the mapping $f:\mathcal{J}\to\mathcal{J}$ is a JB^*-triple Jordan derivation.

Proof. Assume $r>1$.

By the same reasoning as in the proof of Theorem 2.4, the mapping $f:\mathcal{J}\to\mathcal{J}$ is $\mathbb{C}$-linear.

It follows from (5.1) that

$$\begin{aligned}
&\Big\|f\Big(\{xyz\}+\{yzx\}+\{zxy\}\Big)-\{f(x)yz\}-\{xf(y)z\}-\{xyf(z)\}-\{f(y)zx\}\\
&-\{yf(z)x\}-\{yzf(x)\}-\{f(z)xy\}-\{zf(x)y\}-\{zxf(y)\}\Big\|_{\mathcal{J}}\\
&=\lim_{n\to\infty}8^n\Big\|f\Big(\frac{\{xyz\}}{2^n.2^n.2^n}+\frac{\{yzx\}}{2^n.2^n.2^n}+\frac{\{zxy\}}{2^n.2^n.2^n}\Big)-\Big\{f(\frac{x}{2^n})\frac{y}{2^n}\frac{z}{2^n}\Big\}\\
&-\Big\{\frac{x}{2^n}f(\frac{y}{2^n})\frac{z}{2^n}\Big\}-\Big\{\frac{x}{2^n}\frac{y}{2^n}f(\frac{z}{2^n})\Big\}-\Big\{f(\frac{y}{2^n})\frac{z}{2^n}\frac{x}{2^n}\Big\}-\Big\{\frac{y}{2^n}f(\frac{z}{2^n})\frac{x}{2^n}\Big\}\\
&-\Big\{\frac{y}{2^n}\frac{z}{2^n}f(\frac{x}{2^n})\Big\}-\Big\{f(\frac{z}{2^n})\frac{x}{2^n}\frac{y}{2^n}\Big\}-\Big\{\frac{z}{2^n}f(\frac{x}{2^n})\frac{y}{2^n}\Big\}-\Big\{\frac{z}{2^n}\frac{x}{2^n}f(\frac{y}{2^n})\Big\}\Big\|_{\mathcal{J}}\\
&\le\lim_{n\to\infty}\frac{8^n\theta}{8^{nr}}\Big(\|x\|_{\mathcal{J}}^{3r}+\|y\|_{\mathcal{J}}^{3r}+\|z\|_{\mathcal{J}}^{3r}\Big)=0
\end{aligned}$$

for all $x, y, z \in \mathcal{J}$. So $f(\{xyz\}+\{yzx\}+\{zxy\}) = \{f(x)yz\}+\{xf(y)z\}+\{xyf(z)\}+\{f(y)zx\}+\{yf(z)x\}+\{yzf(x)\}+\{f(z)xy\}+\{zf(x)y\}+\{zxf(y)\}$
for all $x, y, z \in \mathcal{J}$.

Thus the mapping $f : \mathcal{J} \to \mathcal{J}$ is a JB^*-triple Jordan derivation.

Similarly, one obtains the result for the case $r < 1$. □

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THE GENERIC STABILITY OF KKM POINTS IN PMT SPACES

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Abstract. In this paper, we consider the generic stability of generalized KKM points and present a result concerning the generic continuity of set-valued mappings in PMT spaces. Then we prove that almost all of generalized KKM points of probabilistic upper semicontinuous set-valued mappings defined on compact subsets of such spaces are stable in the sense of Baire category theory. Also, we discuss on existence of the essential component of generalized KKM points.

Keywords: KKM point, PMT space, Hausdorff distance, Generic stability, Essential component, Generic continuity

1. Introduction

In 2003, Yu et.al. [1], introduced the concept of KKM points of a KKM mapping $G : X \to K(X)$, from a bounded complete convex subset X of a normed linear space E into nonempty compact subsets of X. By Fort theorem, they prove that if M be the collection of all KKM mappings G, then there exists a dense residual subset Q of M such that for each $G \in Q$, G is essential. They also proved there exists at least one essential component of KKM points for each $G \in M$; (see also [2, 3]). In this paper, we present a result concerning generic continuity of set-valued mappings based upon extensions of Fort's theorems in probabilistic metric type spaces.

2. Preliminaries

First, let us give the background and auxiliary results which will be needed. For more details see [4, 5, 6, 7, 8].

Definition 2.1. ([9, 10]) mapping $F : (-\infty, \infty) \to [0, 1]$ is called a distribution function if it is non-decreasing and left-continuous with $\inf_{x\in R} F(x) = 0$ and $\sup_{x\in R} F(x) = 1$. If in addition $F(0) = 0$, then F is called a distance distribution function. The set of all distance distribution functions ($d.d.f$)is denoted by Δ^+. The maximal element for Δ^+ in this order is the $d.d.f$, ϵ_0, given by

$$\epsilon_0(t) = \begin{cases} 0 & \text{if } t \le 0\,, \\ 1 & \text{if } t > 0\,. \end{cases}$$

Definition 2.2. ([9]) A triangular norm (shorter t-norm) is a binary operation T on $[0, 1]$, i.e., a function $T : [0, 1]^2 \to [0, 1]$ which satisfies the following conditions:

(1) T is associative and commutative;
(2) $T(a, 1) = a$ for all $a \in [0, 1]$;
(3) $T(a, b) \le T(c, d)$ whenever $a \le c$ and $b \le d$, for each $a, b, c, d \in [0, 1]$.

In particular, a $t-$norm T is said to be continuous if it is a continuous function in $[0, 1]^2$. A $t-$norm is called sup-continuous if $\sup_{\lambda\in\Lambda} T(a_\lambda, b) = T(\sup_{\lambda\in\Lambda} a_\lambda, b)$ for any family $\{a_\lambda : \lambda \in \Lambda\} \subset [0, 1]$ and $b \in [0, 1]$. The operatins $T_L(a, b) = \max(a + b - 1, 0)$, $T_M(a, b) = \min\{a, b\}$ and $T_p(a, b) = ab$ on $[0, 1]$ are T norms.

Lemma 2.3. ([11]) *Let T be a $t-$norm.*

(1) *If T is left-continuous, then T satisfies $\sup_{0<a<1} T(a, b) = b$ for all $b \in [0, 1]$;*
(2) *If $\sup_{0<a<1} T(a, b) = b$ for all $b \in [0, 1]$, then T satisfies $\sup_{0<a<1} T(a, a) = 1$.*

*The corresponding author.

Definition 2.4. A probabilistic metric type space (PMT space) is a triple (X, F, T), where X is a nonempty set, T is a continuous t-norm and F is a mapping from $X \times X$ into Δ^+ such that, if $F_{x,y}$ denote the value of F at the pair (x, y), the following conditions hold:

(PMT1) $F_{x,y}(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = y$;
(PMT2) $F_{x,y}(t) = F_{y,x}(t)$;
(PMT3) $F_{x,y}(K(s + t)) \geq T(F_{x,z}(s), F_{z,y}(t))$ for any $x, y, z \in M$, $t, s \geq 0$ for some constant $K \geq 1$;

Observe that if $K = 1$, then the PMT space is a probabilistic metric space, however it does not hold true when $K > 1$.

Example 2.5. ([12]) Let X be the set of Lebesgue measurable functions on $[0, 1]$ such that $\int_0^1 |f(x)|^p dx < \infty$, where $p > 0$ is a real number. Define

$$F_{x,y}(t) = \begin{cases} 0 & \text{if } t \leq 0\,, \\ \frac{t}{t+\int_0^1 |f(x)-g(x)|^p dx)^{\frac{1}{p}}} & \text{if } t > 0\,. \end{cases}$$

Then (X, F, T_p) is a PMT space with $K = 2^{\frac{1}{p}}$.

In [12], the authors proved that every PMT space (X, F, T), generated a topology τ on X which has as a base the family of sets of the form $\{B_x(r, t) \,:\, x \in X\,, 0 < r < 1\,, t > 0\}$, where $B_x(r, t) = \{y \in X \,:\, F_{x,y}(t) > 1 - r\}$ for all $r \in (0, 1)$ and $t > 0$, and (X, F, T) is a Hausdorff topological space. In virtue of this topology τ, a sequence $\{x_n\}$ in (X, F, T) is said to be convergent to x (we write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$) if $\lim_{n\to\infty} F_{x_n,x}(t) = 1$ for all $t > 0$; $\{x_n\}$ is called a Cauchy sequence in (X, F, T) if for any given $t > 0$ and $r \in (0, 1)$, there exists $N = N(\varepsilon, \lambda) \in Z^+$ such that $F_{x_n,x_m}(t) > 1 - r$, whenever $n, m \geq N$. Let $t > 0$ and $r \in (0, 1]$, A is said to have a finite (r, t)-net if there exists a finite set $S \subset A$ such that $A \subset \bigcup_{x\in S} B_x(r, t)$, i.e. , for each $y \in A$ there is $x \in S$ such that $F_{x,y}(t) > 1 - r$. A is said to be totally bounded if for each $t > 0$ and $r \in (0, 1]$, A has a finite (r, t)-net. A is said to be probabilistically bounded($P-$bounded) if $\sup_{t>0} \inf_{x,y\in A} F_{x,y}(t) = 1$. Let $P(X)$ denote the class of all nonempty subsets of X. We use the notions:

1. $P_{cl}(X) = \{Y \in P(X) :\ Y \textit{ is closed}\}$;
2. $P_{bd}(X) = \{Y \in P(X) :\ Y \textit{ is probabilistic bounded}\}$;
3. $P_{cp}(X) = \{Y \in P(X) :\ Y \textit{ is compact}\}$;
4. $P_{cl,bd}(X) = P_{cl}(X) \bigcap P_{bd}(X)$.

Let $\Psi : X \to X$ be a mapping. Ψ is said to be closed if $\Psi A \in P_{cl}(X)$ for each $A \in P_{cl}(X)$. It is said to be bounded if $\Psi A \in P_{bd}(X)$ for each $A \in P_{bd}(X)$.

Lemma 2.6. ([12]) *Let (X, F, T) be a PMT space. Let $A \subset X$.*

1. *A is compact if and only if A is sequentially compact;*
2. *If A is compact, then A is closed and totally bounded;*
3. *If A is totally bounded, then $A \in P_{bd}(X)$ and $\bar{A}$ is also totally bounded.*

3. Probabilistic Hausdorff distance type

Given $x \in X$, $B \in P(X)$, the "probabilistic distance type" from x to B is defined as

$$F_{x,B}(t) = F_{B,x}(t) = \begin{cases} 0 & \text{if } t = 0\,, \\ \sup_{s<t} \sup_{y\in B} F_{x,y}(s) & \text{if } t \in (0, \infty)\,, \end{cases}$$

with the convention $F_{x,\emptyset} = 1 - \epsilon_0$.

Given $A, B \in P(X)$, the "probabilistic type distance" from A to B is defined as

$$F_{A,B}(t) = F_{B,A}(t) = \begin{cases} 0 & \text{if } t = 0\,, \\ \sup_{s<t} \inf_{x\in A} \sup_{y\in B} F_{x,y}(s) & \text{if } t \in (0, \infty)\,. \end{cases}$$

For convenience, we write $\tilde{F}_{A,B}(s) = \inf_{x\in A} \sup_{y\in B} F_{x,y}(s)$. Then $F_{A,B}(t) = \sup_{s<t} \tilde{F}_{A,B}(s)$. Based on the above formulas, we can obtain the following definition.

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Definition 3.1. ([13, 14]) Let (X, F, T) be a PMT space and let $A, B \in P(X)$. The probabilistic Hausdorff distance type between A and B is a mapping $H_{A,B} : [0, \infty) \to [0, 1]$ defined by

$$\begin{aligned} H_{A,B}(t) &= \sup_{s<t} \min\{\tilde{F}_{A,B}(s), \tilde{F}_{B,A}(s)\} \\ &= \sup_{s<t} \min\{\inf_{x\in A} \sup_{y\in B} F_{x,y}(s), \inf_{y\in B} \sup_{x\in A} F_{x,y}(s)\}, \quad t \in (0, \infty), \end{aligned}$$

where $H_{A,B}(0) = 0$.

According to the above definition, some results related to the probabilistic Hausdorff distance type can be obtained.

Lemma 3.2. ([13]) *Let (X, F, T) be a PMT space, Let $A, B, C \in P(X), x, y \in X$ and $s, t \in \Re^+$. Then*
(1) $F_{x,B}(K(s + t)) \geq T(F_{x,y}(s), F_{y,B}(t))$;
(2) $H_{x,B}(K(s + t)) \geq T(F_{x,A}(s), H_{A,B}(t))$;
(3) $F_{A,B}(K(s + t)) \geq T(F_{A,C}(s), F_{C,B}(t))$.

Theorem 3.3. *Let (X, F, T) be a PMT space. Then $(P_{cl,bd}(X), H, T)$ is also a PMT space.*

Proof. The conditions (PMT1) and (PMT2) are obvious. Now, we have to show that the condition (PMT3) is satisfied.

Let $A, B, C \in P_{cl,bd}(X)$. If at least one of these three sets is empty, by Definition (3.1) it can easily be verified that the inequality is true. Moreover, if $s = 0$ or $t = 0$, the inequality is also obvious. Thus, we assume that these three sets are non-empty and $t > 0$, $s > 0$.

Set $u < s$, $v < t$. For every $x \in A$ we may assume that $\sup_{y\in B} F_{x,y}(u) > 0$. Then for each $\epsilon \in (0, \sup_{y\in B} F_{x,y}(u))$ there exists $y_x \in B$ such that

$$\sup_{y\in B} F_{x,y}(u) - \epsilon \leq F_{x,y_x}(u) .$$

Moreover, since F is a probabilistic metric type, it follows that

$$T(F_{x,y}(u), \sup_{z\in C} F_{x,z}(v)) \leq \sup_{z\in C} F_{x,z}(K(u + v)) ,$$

for every $y \in B$ and constant $K > 0$. Thus we can obtain

$$T((\sup_{y\in B} F_{x,y}(u) - \epsilon), \inf_{y\in B} \sup_{z\in C} F_{y,z}(v)) \leq T(F_{x,y_x}(u), \sup_{z\in C} F_{y_x,z}(v)) \leq \sup_{z\in C} F_{x,z}(K(u + v))$$

By the arbitrariness of ϵ and the continuity of T we have

$$T(\sup_{y\in B} F_{x,y}(u), \inf_{y\in B} \sup_{z\in C} F_{y,z}(v)) \leq \sup_{z\in C} F_{x,z}(K(u + v)). \tag{3.1}$$

Then we have that

$$T(\inf_{x\in A} \sup_{y\in B} F_{x,y}(u), \inf_{y\in B} \sup_{z\in C} F_{y,z}(v)) \leq \inf_{x\in A} \sup_{z\in C} F_{x,z}(K(u + v)) ,$$

which implies that

$$T(\tilde{F}_{A,B}(u), \tilde{F}_{B,C}(v)) \leq \tilde{F}_{A,C}(K(u + v)). \tag{3.2}$$

In addition, if $\sup_{y\in B} F_{y,y}(u) = 0$ for every $x \in A$, then the inequality (1) still holds. So the inequality (2) is also true in this case. Analogously, we also obtain that

$$T(\tilde{F}_{B,A}(u), \tilde{F}_{C,B}(v)) \leq \tilde{F}_{C,A}(K(u + v)).$$

Therefore, we have

$$\begin{aligned} T &\quad (\min\{\tilde{F}_{A,B}(u), \tilde{F}_{B,A}(u)\}, \min\{\tilde{F}_{B,C}(v), \tilde{F}_{C,B}(v)\}) \\ &\leq \min\{T(\tilde{F}_{A,B}(u), \tilde{F}_{B,C}(v)), T(\tilde{F}_{B,A}(u), \tilde{F}_{C,B}(v))\} \\ &\leq \min\{\tilde{F}_{A,C}(K(u + v)), \tilde{F}_{C,A}(K(u + v))\}. \end{aligned}$$

Furthermore, we can get that

$$\begin{aligned} T \quad (& \sup_{u<s} \min\{\tilde{F}_{A,B}(u), \tilde{F}_{B,A}(u)\}, \sup_{v<t} \min\{\tilde{F}_{B,C}(v), \tilde{F}_{C,B}(v)\}) \\ \leq & \sup_{u<s,\, v<t} \min\{\tilde{F}_{A,C}(K(u+v)), \tilde{F}_{C,A}(K(u+v))\} \\ \leq & \sup_{u+v<s+t} \min\{\tilde{F}_{A,C}(K(u+v)), \tilde{F}_{C,A}(K(u+v))\}. \end{aligned}$$

Then, it follows from the above inequality that

$$T(H_{A,B}(s), H_{B,C}(t)) \leq H_{A,C}(K(s+t)) \; .$$

Hence, we conclude that $(P_{cl,bd}(X), H, T)$ is a PMT space. This completes the proof. □

As a consequence of Theorem (3.3) and Lemma (2.6) we have the following result.

Corollary 3.4. *Let (X, F, T) be a PMT space. Then (P_{cp}, H, T) is also a PMT space.*

Theorem 3.5. *Let (X, F, T) be a complete PMT space. Then $(P_{cl,bd}(X), H, T)$ is a complete PMT space.*

Proof. By Theorem (3.3), $(P_{cl,bd}(X), H, T)$ is a PMT space. Now let $\{A_n\}_{n=1}^{\infty} \subset P_{cl,bd}(X)$, $A_n \to A$ with respect to H. We shall prove that $A \in P_{cl,bd}(X)$. Take an arbitrary number $\lambda \in (0, 1]$. By continuity of T, we have $\sup_{0<a<1} T(a, a) = 1$, applying the Lemma (2.3), we have $\mu \in (0, \lambda]$ and $\nu \in (0, \lambda]$ such that

$$T(1-\mu, 1-\mu) > 1-\lambda \quad and \quad T(1-\nu, 1-\nu) > 1-\mu. \tag{3.3}$$

The convergence of $\{A_n\}$ implies that there exists $N \in Z^+$ such that

$$H_{A,A_n}(1) > 1-\nu \quad for\, all\, n \geq N. \tag{3.4}$$

Since A_N is probabilistically bounded, we have $\sup_{t>0} inf_{x,y\in A_N} F_{x,y}(t) = 1$. Thus, there exists $M = M(\nu) > 0$ such that $F_{x,y}(M/K) > 1-\nu$ for all $x, y \in A_N$. Suppose that $u, w \in A$ are two arbitrary points. From (4) it follows that there exists $x, y \in A_N$ such that $F_{u,x}(1/K) > 1-\nu$ and $F_{w,y}(1) > 1-\nu$. Thus, from(3) we have

$$F_{u,y}(M+1) \geq T(F_{u,x}(1/K), F_{x,y}(M/K)) \geq T(1-\nu, 1-\nu) > 1-\mu,$$

and moreover,

$$F_{u,w}(M+2) \geq T(F_{u,y}((M+1)/K), F_{y,w}(1/K)) \geq T(1-\mu, 1-\nu) \geq T(1-\mu, 1-\mu) > 1-\lambda.$$

Hence $\sup_{t>0} \inf_{u,w\in A} F_{u,w}(t) \geq \inf_{u,w\in A} F_{u,w}(M+2) \geq 1-\lambda$. By the arbitrariness of λ, we have $\sup_{t>0} \inf_{u,w\in A} F_{u,w}(t) = 1$, i.e. , A is probabilistically bounded. Also similar to proof of Theorem 2.2, [13] we can prove that A is closed. From Theorem (3.3) we see that A is closed. Therefore $A \in P_{cl,bd}(X)$, and so the proof is complete. □

Theorem 3.6. *Let (X, F, T) be a complete PMT space. Then $(P_{cp}(X), H, T)$ is complete PMT space.*

Proof. By Theorem (3.3), $(P_{cl,bd}(X), H, T)$ is a complete PMT space. By Corollary (3.4) we see that $P_{cp}(X)$ is PMT space. Since $P_{cp}(X) \subset P_{cl,bd}(X)$, it is enough that $P_{cp}(X)$ is closed with respect to H.

Suppose that $\{A_n\}_{n=1}^{\infty} \subset P_{cp}(X)\,, A_n \to A$ with respect to H. We shall prove that $A \in P_{cp}(X)$. Choose any $\varepsilon > 0$ and $\lambda \in (0, 1]$. By the left-continuity of T and lemma (2.3), we have $\sup_{0<a<1} T(a, a) = 1$, and then there exists $\mu \in (0, \lambda]$ such that

$$T(1-\mu, 1-\mu) > 1-\lambda. \tag{3.5}$$

By the convergence of $\{A_n\}$, there exists $N \in Z^+$ such that

$$H_{A,A_n}(\frac{\varepsilon}{2}) > 1-\mu\,, for\ all\ n \geq N.$$

Since by Lemma (2.6), A_N is compact, A_N is also totally bounded. Thus, A_N has a finite $(\frac{\varepsilon}{2}, \mu)$-net S_N. From this we infer that S_N is a finite (ϵ, λ)-net of A. In fact, for each $x \in A$, it follows the existence of

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$y \in A_N$ such that $F_{x,y}(\frac{\varepsilon}{2K}) > 1-\mu$. For such y we can select $z \in S_N$ with $F_{y,z}(\frac{\varepsilon}{2K}) > 1-\mu$. Hence, from (5) we have

$$F_{x,z}(\varepsilon) \geq T(F_{x,y}(\frac{\varepsilon}{2K}), F_{y,z}(\frac{\varepsilon}{2K})) \geq T(1-\mu, 1-\mu) > 1-\lambda.$$

This shows that S_N is a finite (ε, λ)-net of A, and so A is totally bounded. From the completeness of (X, F, T) it follows that A is compact, i.e. , $A \in P_{cp}(X)$. □

4. Stability of KKM points

Stability of solution maps has been intensively investigated recently [15, 16, 17]. In this section, we first give some Lemmas and concepts, then we investigate on exictence of essential components and the stability of the set of KKM points in PMT space.

For a set A, we denote the set of all nonempty finite subsets of A by $\langle A\rangle$. Let A be a nonempty p-bounded subset of PMT space (X, F, T). Then:

1. $co\,(A) = \bigcap\{B \subset X,\ B \text{ is a closed ball in } X \text{ such that } A \subset B\}$;
2. $\mathbb{A}(X) = \{A \subset X,\ A = co\,(A)\}$, i.e. $A \in \mathbb{A}(X)$ if and only if A is an intersection of all closed balls containing A. In this case, we say that A is an admissible set in X;
3. A is called subadmissible, if for each $D \subset < A >$, $co(D) \subset A$. Obviously, if A is an admissible subset of X, then A must be subadmissible.

Recall that closed and open balls of X are defined as

$$B_x[r,t] = \{y \in X,\ F_{x,y}(t) \geq 1-r\}, \quad B_x(r,t) = \{y \in X,\ F_{x,y}(t) > 1-r\},$$

for any $x \in X$ and $0 < r < 1$ and $t > 0$. Let (X, F, T) be a PMT space and A a subadmissible subset of X and $P_{cp}(X)$ the set of all nonempty compact subsets of X. $G : X \to P_{cp}(X)$ is called a KKM mapping, if for each $A \in < X >$, we have $co(A) \subset G(A)$. More generally, if $G : X \to P_{cp}(X)$, $S : X \to P_{cp}(X)$ are two set-valued functions such that for any $A \in < X >$, $S(co(A)) \subseteq G(A)$, then G is called a generalized KKM mapping with respect to S. If the set-valued function $S : X \to P_{cp}(X)$ satisfies the requirement that for any generalized KKM mapping $G : X \to P_{cp}(X)$ with respect to S the family $\{\overline{G(x)} :\ x \in X\}$ has the finite intersection property, then S is said to have the KKM property. We define

$$KKM(X, P_{cp}(X)) := \{S : X \to P_{cp}(X) \,:\, S \textit{ has the KKM property}\}.$$

Thus if $S \in KKM(X, P_{cp}(X))$, then for any generalized KKM mapping $G : X \to P_{cp}(X)$ with respect to S we have $\bigcap_{x\in X} G(x) \neq \emptyset$. then such a point $x^* \in \bigcap_{x\in X} G(x)$, is called the KKM point of G and denote by $\mathcal{K}(G)$ the set of all generalized KKM points of G. Let M be the collection of all KKM mappings $G : X \to P_{cp}(X)$ with respect to S. For each $G_1, G_2 \in M$ define

$$\tilde{H}_{G_1,G_2}(t) = \inf_{x\in X} H_{G_1(x),G_2(x)}(t),$$

where H is the probabilistic Hausdorff distance type defined on all compact subsets of X. Clearly $(M, \tilde{H}, T)$ is a PMT space.

Lemma 4.1. *$(M, \tilde{H}, T)$ is a complete PMT space.*

Proof. Let $\{G_n\}_{n=1}^{\infty}$ be any Cauchy sequence in M, then for any $t > 0$ and $r \in (0, 1)$, there exists a positive integer k such that $\tilde{H}_{G_n,G_m}(t) > 1-r$ whenever $n, m \geq k$, i.e.

$$\inf_{x\in X} H_{G_n(x),G_m(x)}(t) > 1-r \quad ,$$

for any $n, m \geq k$. It follows that for each $x \in X$, $\{G_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $(P_{cp}(X), H, T)$. By Theorem (3.6), there is $G : X \to P_{cp}(X)$ such that $H_{G_n(x),G(x)}(t) > 1-r$ for each $x \in X$. And it is easy to prove that $\inf_{x\in X} H_{G_n(x),G(x)}(t) > 1-r$. Suppose that G were not generalized KKM mapping with respect to S, then there exist $\{x_1, ..., x_m\} \subset X$ and $x' \in S(co\{x_1, ..., x_m\})$ such that $x' \in \bigcup_{i=1}^m G(x_i)$. Since $\sup_{x\in X} H_{G_n(x),G(x)}(t) > 1-r$, there is n_2 such that $\bigcup_{i=1}^m G_n(x_i) \subset \bigcup_{i=1}^m B_{G(x_i)}(r, t)$ for any $n \geq n_2$. Thus $x' \notin \bigcup_{i=1}^m G_n(x_i)$ for any $n \geq n_2$ which contradicts the assumption that G_n is generalized KKM mapping with respect to S for all $n = 1, 2, \ldots$. Hence G must be generalized KKM mapping with respect to S, and $(M, \tilde{H}, T)$ is complete. □

Now we state some definitions. A set-valued mapping S from PMT space (X, F, T), into nonempty subsets of a PMT space (Y, F^*, T^*) is said to be probabilistic upper (lower) semicontinuous at $x_0 \in X$, if for any $0 < r < 1$, there exists $0 < r' < 1$ such that $S(x') \subset B_{S(x_0)}(r, t)$ $(S(x_0) \subset B_{S(x')}(r, t))$ for each $x' \in X$ with $F_{x_0,x'}(t) > 1 - r'$, for $t > 0$. S is probabilistic continuous at $x_0 \in X$ if S is both probabilistic upper semicontinuous and probabilistic lower semicontinuous at x_0. Also S is said probabilistic metric upper semicontinuous at $x_0 \in X$ if, for any $0 < r < 1$, there exists a neighborhood U of x_0 such that $S(U) \subset B_{S(x_0)}(r, t)$ for $t > 0$. It is easily verified that if $S(x_0)$ is compact, then S is probabilistic metric upper semicontinuous at x_0 if and only if S is probabilistic upper semicontinuous at x_0. In general probabilistic metric upper semicontinuity is a weaker notion than probabilistic upper semicontinuity. On the other hand, the set-valued mapping S is said to be probabilistic metric lower semicontinuous at x_0 if for any $0 < r < 1$ there exists a neighborhood U of x_0 such that $S(x_0) \subset B_{S(x)}(r, t)$ for every $x \in U$ and $t > 0$. It is easy to see that if $S(x_0)$ is totally bounded, then S is probabilistic lower semicontinuous at x_0 if and only if S is probabilistic metric lower semicontinuous at x_0. However, we can also show that in general probabilistic lower semicontinuity is a weaker notion than probabilistic metric lower semicontinuity.

Also a subset Q in X is called a residual set if it contains a countable intersection of open dense subsets of X. A set Q is called nowhere dense in X if $int(\overline{Q}) = \emptyset$. If there exists a dense residual set Q of X such that S is continuous at each point of Q then we say that S is continuous at most point of X. In this case we shall also say that S is generically continuous on X. Result concerning generic continuity of set-valued mappings were first considered by Fort in [18]. After Fort's theorems were published there have been several extensions of his original results; see [19, 20]. In the following we will extend Fort's theorem in PMT space.

Theorem 4.2. *Let (X, F, T) be a complete PMT space, (Y, F^*, T^*) be a PMT space and $S : X \to 2^Y$ be a probabilistic metric upper semicontinuous. Then there exists a dense residual set $Q \subset X$ such that S is probabilistic metric lower semicontinuous at each $x \in Q$.*

Proof. For each $0 < r < 1$ let

$$C(r) = \{x \in X \,:\, \forall\, 0 < r_0 < r \, and \, 0 < r' < 1,\, \exists y \in B_x(r', t),\, such\, that\, B_{S(y)}(r_0, t) \not\supseteq S(x)\, for\, t > 0\}$$

First, we prove that $C(r)$ is a closed set. For any $0 < k < r$ and $0 < r' < 1$, let $r_0 < r_{00} < r$ and $r_{00} - r_0 = \eta$. Due to the probabilistic metric upper semicontinuity of S, for each $z \in \overline{C(r)}$, there exists $0 < r'' < r'$ such that $S(x) \subset B_{S(z)}(\eta, t)$ for all $x \in B_z(r'', t)$. Then there exists $x \in C(r) \cap B_z(r'', t)$ such that $S(x) \subset B_{S(z)}(\eta, t)$. From $x \in B_z(r'', t)$, Choose $0 < r''' < r''$ such that $B_x(r''', t) \subset B_z(r'', t)$. As $x \in C(r)$, it is easy to see that there exists $y \in B_x(r''', t) \subset B_z(r'', t) \subset B_z(r', t)$ such that $B_{S(y)}(r_{00}, t) \not\supseteq S(x)$. Thus, it follows that $B_{S(y)}(r_0, t) \supset S(z)$. In fact, if $B_{S(y)}(r_0, t) \supset S(z)$, then $B_{B_{S(y)}(r_0,t)}(\eta, t) \supset B_{S(y)}(\eta, t)$, so that $B_{S(y)}(r_{00}, t) \not\supseteq S(x)$. Thus, it follows that $B_{S(y)}(r_0, t) \not\supseteq S(z)$. In fact, if $B_{S(q)}(r_0, t) \supset S(z)$, then $B_{B_{S(y)}(r_0,t)}(\eta, t) \supset B_{S(z)}(\eta, t)$, so that $B_{S(y)}(r_{00}, t) \supset S(x)$ which contradicts $x \in C(r)$. Thus, it is proved that $z \in C(r)$ and $C(r)$ is a closed set. Next, we will prove that $C(r)$ is a nowhere dense set, that is, to prove that $C(r)$ contains no interior point. If not, let $x_1 \in C(r)$ be a interior point of $C(r)$. For any $0 < r_0 < r$, choose $r_0 < r_{00} < r$ and set $r_{00} - r_0 = \eta$. Then there exists $0 < r_1 < 1$ such that $B_{x_1}(r_1, t) \subset C(r)$ and $S(x') \subset B_{S(x_1)}(\eta, t)$, $\forall x' \in B_{x_1}(r_1, t)$. From $x_1 \in C(r)$, it is known that there exists $x_2 \in B_{x_1}(r_1, t)$ such that $B_{S(x_2)}(r_{00}, t) \not\supseteq S(x_1)$. For $x_2 \in B_{x_1}(r_1, t) \subset C(r)$, choose $0 < r_2 < \frac{r_1}{2}$ such that $B_{x_2}(r_2, t) \subset B_{x_1}(r_1, t) \subset C(r)$ and $S(x') \subset B_{S(x_2)}(\eta, t)$ for all $x' \in B_{x_2}(r_2, t)$. From $x_2 \in C(r)$, there exists $x_3 \in B_{x_2}(r_2, t)$ such that $B_{S(x_3)}(r_{00}, t) \not\supseteq S(x_2)$. The rest may be deduced by analogy, thus there exists $r_1, r_2, \ldots, r_{n-1}, r_n, \ldots$ such that $0 < r_n < \dfrac{r_{n-1}}{2}$,

$$B_{x_n}(r_n, t) \subset B_{x_{n-1}}(r_{n-1}, t) \subset \ldots \subset B_{x_2}(r_2, t) \subset B_{x_1}(r_1, t) \subset C(\varepsilon)$$

and

$$S(x') \subset B_{S(x_n)}(\eta, t)\,,\ \forall x' \in B_{x_n}(r_n, t)\,.$$

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We also have

$$B_{S(x_{n+1})}(r_{00},t) \not\supseteq S(x_n)\,,\; n = 1, 2, \ldots .$$

From the completeness of (X, F, T) and the closedness of $C(r)$, it is known that there exists $x^* \in C(r)$ and $x_n \to x^*$. As $x^* \in B_{x_n}(r_n, t)$ for each $n = 1, 2, ...,$ we have $S(x^*) \subset B_{S(x_n)}(\eta, t)$. Therefore,$B_{S(p^*)}(r_0, t) \subset B_{B_{S(x_n)}(\eta,t)}(r_0, t) = B_{S(x_n)}(r_{00}, t)$. It follows from $B_{S(x_n)}(r_{00}, t) \not\supseteq S(x_{n-1})$ that $B_{S(x^*)}(r_0, t) \not\supseteq S(x_{n-1})$. In addition, from $x_n \to x^*$ and the upper semicontinuity of S at x^*, for given $r_0 > 0$, it is known that $S(x_{n-1}) \subset B_{S(x^*)}(r_0, t)$ when n is sufficiently large, which is a contradiction. Thus, we can prove that $C(r)$ is nowhere dense.

Let $(0, 1)_R$ be the rational number set in $[0, 1]$, $C = \bigcup_{r\in(0,1)_R} C(r)$ and $Q = X \setminus C$. From the completeness of X and nowhere density of $C(r)$, it is easy to see that C is of first category. Hence, Q is a dense residual set and of second category for any $0 < r < 1$, choose $r^{'} \in (0, 1)_R$ such that $0 < r_0 < r$. For each $x \in Q$, by the definition of Q we have $x \notin C(r_0)$. Also by the definition of $C(r_0)$, there exists $0 < r^* < r_0$ and $r^{'} > 0$ such that $B_{S(y)}(r^*, t) \supset S(x)$ for all $y \in B_x(r^{'}, t)$, and hence $B_{S(y)}(r, t) \supset S(x)$. From the arbitrariness of $0 < r < 1$, it is known that S is probabilistic metric lower semicotinuous at x. Therefore, S is probabilistic metric lower semicontinuous at each $p \in Q$.

□

Because in general probabilistic metric upper semi-continuity is a weaker notion than probabilistic upper semicontinuity, the following corollary is obvious.

Corollary 4.3. *Let (X, F, T) be a complete PMT space, (Y, F^*, T^*) be a PMT space and $S : X \to 2^Y$ be probabilistic upper semicontinuous. Then there exists a dense residual set $Q \subset X$ such that S is probabilistic metric lower semicontinuous at each $x \in Q$, and hence S is also probabilistic lower semi-continuity at each $x \in Q$.*

For each $G \in M$, $\mathcal{K}(G)$ is the set of all KKM points of G, $G \to \mathcal{K}(G)$ indeed defines a set-valued mapping $\mathcal{K} : M \to 2^X$

Lemma 4.4. *$\mathcal{K} : M \to 2^X$ is a probabilistic upper semicontinuous and compact-valued (pusco) mapping.*

Proof. For any $G \in M$, for any sequence $\{x_n\}_{m=1}^{\infty}$ in $\mathcal{K}(G)$ with $x_n \to x^*$, then $x_n \in G(x)$ for each $x \in X$. Since $G(x)$ is compact, then $x^* \in G(x)$ for each $x \in X$ and $x^* \in \bigcap_{x\in X} G(x)$, $x^* \in \mathcal{K}(G)$. Hence $\mathcal{K}(G)$ is closed, $\mathcal{K}(G) \subseteq G(x)$ must be compact. Fix $t > 0$, suppose that $\mathcal{K}$ were not probabilitic upper semicontinuous at $G \in M$, then there exist $0 < r_0 < 1$ and a sequence $\{G_n\}_{n=1}^{\infty}$ in M with $G_n \to G$ such that for each $n = 1, 2, ...,$ there is $x_n \in \mathcal{K}(G_n)$ with $x_n \notin B_{\mathcal{K}(G)}(r_0, t)$. Since $x_n \in \mathcal{K}(G_n)$, we have $x_n \in \bigcap_{x\in X} G_n(x)$. For any $x \in X$, since $G_n(x) \to G(x)$, $G_n(x)\,(n = 1, 2, ...)$ *and* $G(x)$ is compact, thus $\bigcap_{n=1}^{\infty} G_n(x) \bigcup G(x)$ is compact. $x_n \in G_n(x)$, we may assume that $x_n \to x^*$, we obtain $x^* \in G(x)$. Thus $x^* \in \bigcap_{x\in X} G_n(x)$ and $x^* \in \mathcal{K}(G) \subset B_{K(G)}(r_0, t)$ which contradicts the assumption that $x_n \to x^*$ and $x_n \notin B_{\mathcal{K}(G)}(r_0, t)$ for each $n = 1, 2, \ldots$. Therefore, $\mathcal{K}$ must be probabilistic upper semicontinuous on M. □

Definition 4.5. $G \in M$, (1) $x \in \mathcal{K}(G)$ is essential if for any $0 < r < 1$, there exists $0 < r^{'} < 1$ such that for each $G^{'} \in M$ with $\tilde{H}_{G,G'}(t) > 1 - r^{'}$, there exists $x^{'} \in \mathcal{K}(G^{'})$, with $F_{x,x'}(t) > 1 - r$, (2) G is weakly essential if there exists $x \in \mathcal{K}(G)$ which is essential and (3) G is essential if every $x \in \mathcal{K}(G)$ is essential.

Theorem 4.6. *$\mathcal{K} : M \to P_{cp}(X)$ is probabilistic lower semicontinuous at $G \in M$ if and only if G is essential.*

Proof. If $\mathcal{K}$ is probabilistic lower semicontinuous at $G \in M$, then for any $0 < r < 1$, there exists $0 < r^{'} < 1$ such that $\mathcal{K}(G) \subset B_{\mathcal{K}(G')}(r, t)$ for each $G^{'} \in M$ with $\tilde{H}_{G,G'}(t) > 1 - \delta$. For each $x \in \mathcal{K}(G)$ there exists $x^{'} \in \mathcal{K}(G^{'})$ with $F_{x,x'}(t) > 1 - r$, x is essential and G is essential. Conversely, suppose that G is essential. If $\mathcal{K}$ were not probabilistic lower semicontinuous at G, then there exist $0 < r_0 < 1$ and a sequence $\{G_n\}_{n=1}^{\infty}$ in M with $G_n \to G$ such that for each $n = 1, 2, ...,$ there is an $x_n \in \mathcal{K}(G)$ with $x_n \in B_{\mathcal{K}(G_n)}(r_0, t)$. Since $\mathcal{K}(G)$ is compact, we map assume that $x_n \to x \in \mathcal{K}(G)$. Since x is essential, $G_n \to G$, $x_n \to x$, there is an N such that $F_{x_n,x}(t) > 1 - \frac{r_1}{2}$ and $B_{\mathcal{K}(G_n)}(\frac{r_1}{2}, t)$ for all $n \geq N$. Hence $x_n \in B_{\mathcal{K}(G_n)}(r_1, t)$ for all $n \geq N$ which contracts the assumption that $x_n \notin B_{\mathcal{K}(G_n)}(r_0, t)$ for all $n = 1, 2, \ldots$. Hence $\mathcal{K}$ must be probabilistic lower semicontinuous at G. ☐

Theorem 4.7. *There exists a dense residual subset Q of M such that for each $G \in Q$, G is essential.*

Proof. by Lemma (4.4), $\mathcal{K} : M \to P_{cp}(X)$ is an pusco mapping. By Corollary (4.3), there exists a dense residual subset Q of M such that for each $G \in Q$, $\mathcal{K}$ is probabilistic lower semicontinuous at G. By Theorem (4.6), for each $G \in Q$, G is essential. ☐

Remark 4.8. *If $G \in Q$, by Lemma (4.4) and Theorem (4.6), $\mathcal{K}$ is probabilistic continuous, then for any $0 < r < 1$, there exists $0 < r^{'} < 1$ such that for any $G^{'} \in M$ with $\tilde{H}_{G,G'}(t) > 1 - r^{'}$, $H_{\mathcal{K}(G),\mathcal{K}(G')}(t) > 1 - r$, G is stable.*

Now we shall introduce some definitions. For each $G \in M$, the component of a point $x \in \mathcal{K}(G)$ is the union of all connected subsets of $\mathcal{K}(G)$ which contain the point x. Note that components are connected closed subsets of $\mathcal{K}(G)$ and are also connected compact. It is easy to see that the components of two distinct points $\mathcal{K}(G)$ either coincide or are disjoint, so that all components constitute a decomposition of $\mathcal{K}(G)$ into connected pairwise disjoint compact subsets, i.e. ,

$$\mathcal{K}(G) = \bigcup_{\alpha \in \Lambda} C_\alpha(G)$$

where Λ is an index set; for any $\alpha \in \Lambda$, $C_\alpha(G)$ is a nonempty connected compact and for any $\alpha, \beta \in \Lambda(\alpha \neq \beta)$, $C_\alpha(G) \bigcap C_\beta(G) = \emptyset$.

Definition 4.9. For each $G \in M$, let $e(G)$ be a nonempty closed subset of $\mathcal{K}(G)$. Fix $t > 0$, $e(G)$ is called an essential set of $\mathcal{K}(G)$ if for eny $0 < r < 1$, there exists $0 < r^{'} < 1$ such that for any $G^{'} \in M$ with $\tilde{H}_{G,G'}(t) > 1 - r^{'}$, $\mathcal{K}(G^{'}) \bigcap B_{e(G)}(r, t) \neq \emptyset$. If $C_\alpha(G)$, the component of $\mathcal{K}(G)$ is essential, then $C_\alpha(G)$ is called an essential component of $\mathcal{K}(G)$.

Following theorem is the main result .

Theorem 4.10. *For each $G \in M$, there exists at least one essential component of $\mathcal{K}(G)$.*

Proof. By Lemma (4.4), $\mathcal{K} : M \to P_{cp}(X)$ is probabilistic upper semicontinuous, that is for any $0 < r < 1$, there exists $0 < r^{'} < 1$ such that for any $G^{'} \in M$ with $\tilde{H}_{G,G'}(t) > 1 - r^{'}$, $\mathcal{K}(G^{'}) \subset B_{\mathcal{K}(G)}(r, t)$. Hence $\mathcal{K}(G^{'}) \bigcap B_{\mathcal{K}(G)}(r, t) \neq \emptyset$, $\mathcal{K}(G)$ is essential set of itself. Let Φ denote the family of all essential sets of $\mathcal{K}(G)$ ordered by set inclusion. Thus Φ is nonempty and every decreasing chain of elements in Φ has a lower bound (because by the compactness the intersection is in Φ); therefore by Zorn's Lemma, Φ has a minimal element $m(G)$ and $m(G)$ is essential. Suppose that $m(G)$ were not connected. Then there exist two nonempty closed sets $c_1(G)$, $c_2(G)$ and two open sets V_1 and V_2 such that $m(G) = c_1(G) \bigcup c_2(G)$, $c_1(G) \subset V_1$, $c_2(G) \subset V_2$, $V_1 \bigcap V_2 = \emptyset$. Since $m(G)$ is minimal, neither $c_1(G)$ nor $c_2(G)$ is essential, there exist $0 < r_1 < 1$, $0 < r_2 < 1$ such that for any $0 < r^{'} < 1$, there exists $G_1, G_2 \in M$ such that $\tilde{H}_{G_1,G_2}(t) > 1 - r^{'}$, $\tilde{H}_{G,G_2}(t) > 1 - r^{'}$ with $\mathcal{K}(G_1) \bigcap [B_{c_1(G)}(r_1, t)] = \emptyset$, $\mathcal{K}(G_2) \bigcap [B_{c_2(G)}(r_2, t)] = \emptyset$. Denote $W_1 = V_1 \bigcap [B_{c_1(G)}(r_1, t)]$, $W_2 = V_2 \bigcap [B_{c_2(G)}(r_2, t)]$, then W_1, W_2 are open. Since $c_1(G) \subset W_1$, $c_2(G) \subset W_2$, there exists $0 < r_0 < 1$ such that $[B_{c_1(G)}(r_1, t)] \subset W_1$, $[B_{c_2(G)}(r_2, t)] \subset W_2$. Denote $W_1^{'} = B_{c_1(G)}(r^{'}, t)$, $W_2^{'} = B_{c_2(G)}(r^{'}, t)$, we know that $W_1^{'}$, $W_2^{'}$ are open. Since $W_1^{'} \bigcup W_2^{'} = B_{c_1(G) \cup c_2(G)}(r^{'}, t) = B_{m(G)}(r^{'}, t) \supset m(G)$, then there exists $0 < r^* < 1$ such that for any $G^{'} \in M$ with $\tilde{H}_{G,G'}(t) > 1 - r^*$, $K(G^{'}) \bigcap (W_1 \bigcup W_2) \neq \emptyset$. We may suppose that $r^* > a$, where $a = F_{W_1^{'}, W_2^{'}}(t) > 0$. For this r^* we can find an $r^\dagger > r^*$ such that $T(1 - r^\dagger, 1 - r^\dagger) \geq 1 - r^*$. Thus for $0 < r^\dagger < 1$ there exist $G_1^{'}, G_2^{'} \in M$ such that $\tilde{H}_{G,G_1^{'}}(\frac{t}{2K}) > 1 - r^\dagger$, $\tilde{H}_{G,G_2^{'}}(\frac{t}{2K}) > 1 - r^\dagger$, $\mathcal{K}(G_1^{'}) \bigcap W_1^{'} = \emptyset$, $\mathcal{K}(G_2^{'}) \bigcap W_2^{'} = \emptyset$. Note that

$$\tilde{H}_{G_1^{'}, G_2^{'}}(t) > T(\tilde{H}_{G,G_1^{'}}(\frac{t}{2K}))\ \tilde{H}_{G,G_1^{'}}(\frac{t}{2K}) > T(1 - r^\dagger, 1 - r^\dagger) \geq 1 - r^* > a\,.$$

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Now define $G^{'} : X \to P_{cp}(X)$ as follows:

$$G^{'}(x) = [G_1^{'} \setminus W_2^{'}] \bigcap [G_2^{'} \setminus W_1^{'}], \quad x \in X$$

Suppose that $G^{'}$ were not a generalized KKM mapping with respect to S, then there exist $\{x_1, ..., x_m\} \subset X$ and $x^{'} \in S(co\{x_1, ..., x_m\})$ such that $x^{'} \notin \bigcup_{i=1}^m G^{'}(x_i)$. Since $W_1^{'} \bigcap W_2^{'} = \emptyset$, then $x^{'} \notin W_1^{'}$ or $x^{'} \notin W_2^{'}$. Without loss of generality, we may assume that $x^{'} \notin W_1^{'}$. Since $x^{'} \notin G_2(x_i) \setminus W_1^{'}$, then $x^{'} \notin G_2(x_i)$ for each $i = 1, ..., m$, $x^{'} \notin \bigcup_{i=1}^m G_2(x_i)$ which contradicts that $G_2 \in M$. Thus $G^{'} \in M$.

Next we are going to prove that $\tilde{H}_{G^{'},G_1^{'}}(t) \geq \tilde{H}_{G_1^{'},G_2^{'}}(t)$. Note that

$$\tilde{H}_{G^{'},G_1^{'}}(t) = \inf_{x \in X} H_{G^{'}(x),G_1^{'}(x)}(t) = \inf_{x \in X} \{\sup_{s<t} \min\{ \inf_{y \in G^{'}(x)} F_{y,G_1^{'}(x)}(s), \inf_{y \in G_1^{'}(x)} F_{y,G^{'}(x)}(s)\}\}$$

and

$$\tilde{H}_{G_1^{'},G_2^{'}}(t) = \inf_{x \in X} H_{G_1^{'}(x),G_2^{'}(x)}(t) = \inf_{x \in X} \{\sup_{s<t} \min\{ \inf_{y \in G_1^{'}(x)} F_{y,G_2^{'}(x)}(s), \inf_{y \in G_2^{'}(x)} F_{y,G_1^{'}(x)}(s)\}\} .$$

For any $y \in G_1^{'}(x)$, if $y \in G_1^{'}(x) \setminus W_2^{'}$, then $F_{y,G_1^{'}}(s) = 1$; if $y \in G_1^{'}(x) \setminus W_1^{'}$, then

$$F_{y,G_1^{'}(x)}(s) \geq \inf_{y \in G_2^{'}(x)} F_{y,G_1^{'}(x)}(s) ,$$

and

$$\inf_{y \in G^{'}(x)} F_{y,G_1^{'}(x)}(s) \geq \inf_{y \in G_2^{'}(x)} F_{y,G_1^{'}(x)}(s) \geq H_{G_1^{'}(x),G_2^{'}(x)}(t).$$

For any $y \in G_1^{'}(x)$, since $W_1^{'} \bigcap W_2^{'} = \emptyset$, then $y \notin W_1^{'}$ or $y \notin W_2^{'}$. If $y \notin W_2^{'}$, then $y \in G_1^{'}(x) \setminus W_2^{'}$, $F_{y,G^{'}(x)}(s) = 1$; if $y \in W_2^{'}$, then $y \notin W_1^{'}$,

$$F_{y,G^{'}(x)}(s) \geq F_{y,G_2^{'}(x)}(s) \setminus W_1^{'} = \sup_{z \in G_2^{'}(x) \setminus W_1^{'}} F_{y,z}(s).$$

Since $F_{y,G_2^{'}(x)}(s) \geq F_{G_1^{'}(x),G_2^{'}(x)}(s) \geq \tilde{H}_{G_1^{'},G_2^{'}}(t) > 1 - r^*$, and when $y \in W_2^{'}$, $z \in W_1^{'}$, $F_{y,z}(s) \leq FW_1^{'}, W_2^{'}(s) \leq a \leq r^*$, thus $F_{y,G^{'}(x)}(s) = \sup_{z \in G_2^{'}(x)} F_{y,z}(s)$, $F_{y,G^{'}(x)}(s) \geq F_{y,G_2^{'}(x)}(s) \geq \inf_{y \in G_1^{'}(x)} F_{y,G_2^{'}(x)}(s)$, $\inf_{y \in G_1^{'}(x)} F_{y,G^{'}(x)}(s) \geq \inf_{y \in G_1^{'}(x)} F_{y,G_2^{'}(x)}(s) \geq H_{G_1^{'}(x),G_2^{'}(x)}(s)$.

Thus $H_{G_1^{'}(x),G_2^{'}(x)}(s) \leq H_{G^{'}(x),G_1^{'}(x)}(t)$, for any $x \in X$, $\tilde{H}_{G_1^{'},G_2^{'}}(t) \leq \tilde{H}_{G^{'},G_1^{'}}(t)$. Since

$$\begin{aligned} \tilde{H}_{G^{'},G}(2Kt) &\geq T(\tilde{H}_{G^{'},G_1^{'}}(t), \tilde{H}_{G_1^{'},G}(t)) \\ &\geq T(\tilde{H}_{G_1^{'},G_2^{'}}(t), \tilde{H}_{G_1^{'},G}(t)) \\ &\geq T(1 - r^*, 1 - r^{\dagger}) \geq T(1 - r^{\dagger}, 1 - r^{\dagger}) \geq 1 - r^*, \end{aligned}$$

$\mathcal{K}(G^{'}) \bigcap [W_1^{'} \bigcup W_2^{'}] \neq \emptyset$. Take $x^* \in \mathcal{K}(G^{'}) \bigcap [W_1^{'} \bigcup W_2^{'}]$, without losing the generality, we may assume that $x^* \in \mathcal{K}(G^{'}) \bigcap W_1^{'}$, then $x^* \notin G_2^{'}(x) \setminus W_1^{'}$ for any $x \in X$. Since $\mathcal{K}(G_1^{'}) \bigcap W_1^{'} = \emptyset$, then $x^* \notin \mathcal{K}(G_1^{'})$, that is $x^* \notin \bigcap_{x \in X} G_1^{'}(x)$. There is $\tilde{x} \in X$ such that $x^* \notin G_1^{'}(\tilde{x})$, so $x^* \notin G_1^{'}(\tilde{x}) \setminus W_2^{'}$, $x^* \notin G^{'}(\tilde{x})$, $x^* \notin \mathcal{K}(G^{'})$ which is a contradiction. Thus, $m(G)$ must be connected. Thus there is a component $c_{\alpha_0}(G)$ of $\mathcal{K}(G)$ such that $m(G) \subset c_{\alpha_0}(G)$. It is obvious that $c_{\alpha_0}(G)$ is essential which complete the proof. □

5. Conclusions

In this paper, at first we studied probabilistic metric type spaces. Next, we presented a result concerning generic continuity of set-valued mappings based upon extensions of Fort's theorems in these spaces.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally and significantly in writing this paper.

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The essential norm of the generalized integration operator *

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Abstract: We provide a function-theoretic estimate for the essential norm of the generalized integration operator $I_{g,\varphi}^{(n)}$ from mixed-norm spaces $H(p,\, q,\, \phi)$ to Zygmund-type spaces $\mathcal{Z}_\mu$ by means of the definition of the essential norm of an operator and the properties of the analytic function.

Keywords Essential norm; generalized integration operator; mixed-norm spaces; Zygmund-type space; mean value theorem

MR(2000) Subject Classification 47B38, 47G10, 47A30, 30D45.

1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the linear space of all analytic functions on $\mathbb{D}$ and $S(\mathbb{D})$ the set of all analytic self-maps on $\mathbb{D}$. Let $g \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. The following generalized integration operator:

$$I_{g,\varphi}^{(n)} f(z) = \int_0^z f^{(n)}(\varphi(\xi))g(\xi)d\xi,\ z \in \mathbb{D},$$

has been recently introduced in [23] and considerably studied (see, e.g., [10, 11, 21, 26, 36]). One of the reasons is that it provides connections between operator theorem and complex analysis and helps us to gain a deep understanding of both areas. Recently, there has a lot of work on some related operators on different spaces of analytic functions on the unit disc, see also [1, 6, 20, 28, 29, 30, 31, 32] and the related references therein.

Recall that, for $0 < p,\, q < \infty$ and ϕ normal, let $H(p, q, \phi)$ denote the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{p,q,\phi} = \left(\int_0^1 M_q^p(f,r)\frac{\phi^p(r)}{1-r}dr\right)^{1/p} < \infty,$$

where the integral means $M_p(f, r)$ are defined by

$$M_q(f,r) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^q d\theta\right)^{1/q}, \quad 0 \le r < 1.$$

For $1 \le p < \infty$, the mixed-norm space $H(p, q, \phi)$ equipped with the norm $\|\cdot\|_{p,q,\phi}$ is a Banach space (see [13, 34]). For related results in the setting of the unit ball, see, for example, [12, 24] and the

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references therein. Given a bounded, continuous and strictly positive function μ on $\mathbb{D}$. Let $\mathcal{Z}_\mu$ denote the space of all $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ such that

$$\sup_{z\in\mathbb{D}} \mu(z)|f''(z)| < \infty.$$

Under the norm

$$\|f\|_{\mathcal{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \mu(z)|f''(z)|, \tag{1}$$

it is easy to see that $\mathcal{Z}_\mu$ is a Banach space.

Let X and Y be two Banach spaces. T is a bounded linear operator from X to Y. The essential norm of T is defined by

$$\|T\|_{e,X\to Y} = \inf_S \{\|T-S\|_{X\to Y}\},$$

where $\|\cdot\|_{X\to Y}$ denote the operator norm and the infimum is taken over all admissible compact operator S from X to Y. For results on the essential norms of composition, weighted composition and some integral operators, see, for instance, [2, 3, 4, 5, 7, 8, 9, 14, 15, 16, 17, 18, 19, 22, 27, 33, 35, 37, 38], and the references therein.

Inspired by the above results, for $1 \le p < \infty$ and $0 < q < \infty$, we provide a function-theoretic estimate for the essential norms of the generalized integration operator $I_{g,\varphi}^{(n)}$: $H(p, q, \phi) \to \mathcal{Z}_\mu$ on the basis of the characterizations of the boundedness and compactness of the generalized integration operator ([10]). The main result is Theorem 3.1.

Throughout the paper, the letter C denotes a positive constant which may vary at each occurrence but it is independent of the essential variables. For two functions A and B, we use the abbreviation $A \preceq B$ if there is a positive constant C independent of A and B such that $A \le CB$. We write $A \asymp B$ if $A \preceq B \preceq A$.

2 Some Lemmas

To prove our main results, we need several lemmas. The next Schwartz-type lemma is proved in a standard way, hence we omit its proof.

Lemma 2.1 *Assume that $0 < p, q < \infty$. Then $I_{g,\varphi}^{(n)}$: $H(p, q, \phi) \to \mathcal{Z}_\mu$ is compact if and only $I_{g,\varphi}^{(n)}$: $H(p, q, \phi) \to \mathcal{Z}_\mu$ is bounded and for every bounded sequence $\{f_m\}$ in $H(p, q, \phi)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $m \to \infty$, we have $\|I_{g,\varphi}^{(n)} f_m\|_{\mathcal{Z}_\mu} \to 0$ as $m \to \infty$.*

The following characterizations of boundedness and compactness of the operator $I_{g,\varphi}^{(n)}$: $H(p, q, \phi) \to \mathcal{Z}_\mu$ have been essentially proved in [10]. The parts of this work which are relevant here are given in the following two lemmas.

Lemma 2.2 ([10]) *Assume that $0 < p, q < \infty$. Then $I_{g,\varphi}^{(n)}$: $H(p, q, \phi) \to \mathcal{Z}_\mu$ is bounded if and only if*

$$M_1 = \sup_{z\in\mathbb{D}} \frac{\mu(z)|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+n+1}} < \infty, \tag{2}$$

and

$$M_2 = \sup_{z\in\mathbb{D}} \frac{\mu(z)|g'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+n}} < \infty. \tag{3}$$

Lemma 2.3 *([10]) Assume that $0 < p,\, q < \infty$. Then $I^{(n)}_{g,\varphi}$: $H(p,\, q,\, \phi) \to \mathcal{Z}_\mu$ is compact if and only if $I^{(n)}_{g,\varphi}$: $H(p,\, q,\, \phi) \to \mathcal{Z}_\mu$ is bounded,*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+n+1}} = 0, \tag{4}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|g'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{1/q+n}} = 0. \tag{5}$$

On the basis of the compactness criterion, the following lemma provides a sufficient condition of the compactness of the operator $I^{(n)}_{g,\varphi}$: $H(p,\, q,\, \phi) \to \mathcal{Z}_\mu$.

Lemma 2.4 *Assume that $1 \le p < \infty$, $0 < q < \infty$. Then $I^{(n)}_{g,\varphi}$: $H(p,\, q,\, \phi) \to \mathcal{Z}_\mu$ is compact, if $I^{(n)}_{g,\varphi}$: $H(p,\, q,\, \phi) \to \mathcal{Z}_\mu$ is bounded and $\|\varphi\|_\infty < 1$.*

Proof. Since $I^{(n)}_{g,\varphi}$: $H(p,\, q,\, \phi) \to \mathcal{Z}_\mu$ is bounded and $\|\varphi\|_\infty < 1$, the conditions (4) and (5) hold, Lemma 2.3 implies that the operator $I^{(n)}_{g,\varphi}$: $H(p,\, q,\, \phi) \to \mathcal{Z}_\mu$ is compact.

The last auxiliary results needed are the following two lemmas.

Lemma 2.5 *([25]) Let $0 < p,\, q < \infty$ and $f \in H(p,\, q,\, \phi)$. Then for every $m \in \mathbb{N}_0$, there is a positive constant C independent of f such that*

$$|f^{(m)}(z)| \le C \frac{\|f\|_{p,\,q,\,\phi}}{\phi(|z|)\,(1-|z|^2)^{m+1/q}},\ z \in \mathbb{D}.$$

Lemma 2.6 *([10, P. 384]) Let $0 < p,\, q < \infty$, $\omega \in \mathbb{D}$ and*

$$f_{j,\omega}(z) = \frac{(1-|\omega|^2)^{j+t}}{\phi(|\omega|)(1-\overline{\omega}z)^{j+t+1/q}},\ z \in \mathbb{D}.$$

Then $f_{j,\omega} \in H(p,\, q,\, \phi)\,(j \in \mathbb{N})$, where the constant t is from the definition of the normality of the function ϕ.

3 The essential norm of the operator $I^{(n)}_{g,\varphi} : H(p,\, q,\, \phi) \to \mathcal{Z}_\mu$

The following theorem is the main result of the paper.

Theorem 3.1 *Assume that $1 \le p < \infty$, $0 < q < \infty$, $I^{(n)}_{g,\varphi}$: $H(p,\, q,\, \phi) \to \mathcal{Z}_\mu$ is bounded. Then*

$$\|I^{(n)}_{g,\varphi}\|_{e,H(p,\,q,\,\phi)\to\mathcal{Z}_\mu} \asymp \max\left\{ \lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1+1/q}},\ \lim_{|\varphi(z)|\to 1} \frac{\mu(z)|g'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1/q}} \right\}.$$

Proof. If $\|\varphi\|_\infty < 1$, by Lemma 2.4, $I^{(n)}_{g,\varphi}$: $H(p,\, q,\, \phi) \to \mathcal{Z}_\mu$ is compact, so $\|I^{(n)}_{g,\varphi}\|_{e,H(p,\,q,\,\phi)\to\mathcal{Z}_\mu} = 0$. The result holds.

Now assume that $\|\varphi\|_\infty = 1$. We first give the lower estimate of $\|I_{g,\varphi}^{(n)}\|_{e,H(p,\,q,\,\phi)\to\mathcal{Z}_\mu}$. Let $C_j(q,t) = 1/q + j + t,\ j = 1, 2, \ldots, n+2$. For a fixed $\omega \in \mathbb{D}$, set the test function

$$\begin{aligned} f_\omega(z) &= C_{n+1}(q,t) f_{1,\omega}(z) - C_1(q,t) f_{2,\omega}(z) \\ &= C_{n+1}(q,t)\frac{(1-|\omega|^2)^{1+t}}{\phi(|\omega|)(1-\overline{\omega}z)^{1+t+1/q}} - C_1(q,t)\frac{(1-|\omega|^2)^{2+t}}{\phi(|\omega|)(1-\overline{\omega}z)^{2+t+1/q}},\ z \in \mathbb{D}. \end{aligned}$$

By Lemma 2.6, we have $f_\omega \in H(p,\,q,\,\phi)$ and $\sup_{\omega\in\mathbb{D}} \|f_\omega\|_{p,\,q,\,\phi} \le C$. By an easy calculation,

$$\begin{aligned} f_\omega^{(n)}(z) &= \frac{C_{n+1}(q,t)C_1(q,t)C_2(q,t)\cdots C_n(q,t)(1-|\omega|^2)^{1+t}\,(\overline{\omega})^n}{\phi(|\omega|)(1-\overline{\omega}z)^{n+1+t+1/q}} \\ &\quad - \frac{C_1(q,t)C_2(q,t)C_3(q,t)\cdots C_{n+1}(q,t)(1-|\omega|^2)^{2+t}\,(\overline{\omega})^n}{\phi(|\omega|)(1-\overline{\omega}z)^{n+2+t+1/q}}, \end{aligned} \tag{6}$$

$$\begin{aligned} f_\omega^{(n+1)}(z) &= \frac{C_{n+1}^2(q,t)C_1(q,t)C_2(q,t)\cdots C_n(q,t)(1-|\omega|^2)^{1+t}\,(\overline{\omega})^{n+1}}{\phi(|\omega|)(1-\overline{\omega}z)^{n+2+t+1/q}} \\ &\quad - \frac{C_1(q,t)C_2(q,t)\cdots C_{n+2}(q,t)(1-|\omega|^2)^{2+t}\,(\overline{\omega})^{n+1}}{\phi(|\omega|)(1-\overline{\omega}z)^{n+3+t+1/q}}, \end{aligned} \tag{7}$$

Let $\{z_k\}_{k\in\mathbb{N}} \subset \mathbb{D}$, satisfying $\lim_{k\to\infty} |\varphi(z_k)| = 1$. Let

$$\begin{aligned} & f_k(z) = f_{\varphi(z_k)}(z) \\ &= \frac{C_{n+1}(q,t)(1-|\varphi(z_k)|^2)^{1+t}}{\phi(|\varphi(z_k)|)(1-\overline{\varphi(z_k)}z)^{1+t+1/q}} - \frac{C_1(q,t)(1-|\varphi(z_k)|^2)^{2+t}}{\phi(|\varphi(z_k)|)(1-\overline{\varphi(z_k)}z)^{2+t+1/q}},\ k \in \mathbb{N}. \end{aligned}$$

Then $f_k \in H(p,\,q,\,\phi)$, $\sup_{k\in\mathbb{N}} \|f_k\|_{p,\,q,\,\phi} \le C$ and $\{f_k\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$, so for every compact operator $K\colon\ H(p,\,q,\,\phi) \to \mathcal{Z}_\mu$, by Lemma 2.1 , one has $\lim_{k\to\infty} \|Kf_k\|_{\mathcal{Z}_\mu} = 0$. Using (6) and (7), we have

$$f_k^{(n)}(\varphi(z_k)) = 0,\ f_k^{(n+1)}(\varphi(z_k)) = \frac{-C_1(q,t)C_2(q,t)\cdots C_{n+1}(q,t)\left(\overline{\varphi(z_k)}\right)^{n+1}}{\phi(|\varphi(z_k)|)\left(1-|\varphi(z_k)|^2\right)^{n+1+1/q}}.$$

Therefore

$$\begin{aligned} & C\|I_{g,\varphi}^{(n)} - K\|_{H(p,\,q,\,\phi)\to\mathcal{Z}_\mu} \ge \limsup_{k\to\infty} \|\left(I_{g,\varphi}^{(n)} - K\right) f_k\|_{\mathcal{Z}_\mu} \\ & \ge \limsup_{k\to\infty} \left(\|I_{g,\varphi}^{(n)} f_k\|_{\mathcal{Z}_\mu} - \|Kf_k\|_{\mathcal{Z}_\mu}\right) \\ & \ge \limsup_{k\to\infty} \|I_{g,\varphi}^{(n)} f_k\|_{\mathcal{Z}_\mu} \\ & \ge C_1(q,t)C_2(q,t)\cdots C_{n+1}(q,t) \limsup_{k\to\infty} \frac{\mu(z_k)|\varphi(z_k)|^{n+1}\cdot|\varphi'(z_k)g(z_k)|}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{n+1+1/q}}, \end{aligned}$$

thus

$$\|I_{g,\varphi}^{(n)}\|_{e,H(p,\,q,\,\phi)\to\mathcal{Z}_\mu} \geq C \lim_{|\varphi(z)|\to 1} \frac{\mu(z)\,|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)\,(1-|\varphi(z)|^2)^{n+1+1/q}}. \tag{8}$$

Further, let

$$h_k(z) = \frac{C_{n+2}(q,t)(1-|\varphi(z_k)|^2)^{1+t}}{\phi(|\varphi(z_k)|)(1-\overline{\varphi(z_k)}z)^{1+t+1/q}} - \frac{C_1(q,t)(1-|\varphi(z_k)|^2)^{2+t}}{\phi(|\varphi(z_k)|)(1-\overline{\varphi(z_k)}z)^{2+t+1/q}},\ k\in\mathbb{N}.$$

Similarly $h_k \in H(p,\,q,\,\phi)$, $\sup_{k\in\mathbb{N}} \|h_k\|_{p,\,q,\,\phi} \leq C$, and

$$h_k^{(n)}(\varphi(z_k)) = \frac{C_1(q,t)C_2(q,t)\cdots C_n(q,t)\left(\overline{\varphi(z_k)}\right)^n}{\phi(|\varphi(z_k)|)\,(1-|\varphi(z_k)|^2)^{n+1/q}},\ h_k^{(n+1)}(\varphi(z_k)) = 0.$$

It is clear that $\{h_k\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$, Lemma 2.1 implies that $\lim_{k\to\infty} \|Kh_k\|_{\mathcal{Z}_\mu} = 0$ and consequently

$$\begin{aligned}
&C\|I_{g,\varphi}^{(n)} - K\|_{H(p,\,q,\,\phi)\to\mathcal{Z}_\mu} \geq \limsup_{k\to\infty} \|\left(I_{g,\varphi}^{(n)} - K\right)h_k\|_{\mathcal{Z}_\mu}\\
&\geq \limsup_{k\to\infty}\left(\|I_{g,\varphi}^{(n)}h_k\|_{\mathcal{Z}} - \|Kh_k\|_{\mathcal{Z}_\mu}\right)\\
&\geq \limsup_{k\to\infty} \|I_{g,\varphi}^{(n)}h_k\|_{\mathcal{Z}_\mu}\\
&\geq C_1(q,t)C_2(q,t)\cdots C_n(q,t)\limsup_{k\to\infty}\frac{\mu(z_k)|\varphi(z_k)|^n\cdot|g'(z_k)|}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{n+1/q}}\\
&= C_1(q,t)C_2(q,t)\cdots C_n(q,t)\limsup_{k\to\infty}\frac{\mu(z_k)\cdot|g'(z_k)|}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{n+1/q}},
\end{aligned}$$

thus

$$\|I_{g,\varphi}^{(n)}\|_{e,H(p,\,q,\,\phi)\to\mathcal{Z}_\mu} \geq C \lim_{|\varphi(z)|\to 1} \frac{\mu(z)\,|g'(z)|}{\phi(|\varphi(z)|)\,(1-|\varphi(z)|^2)^{n+1/q}}. \tag{9}$$

By (8) and (9), we get

$$\begin{aligned}
&\max\left\{\lim_{|\varphi(z)|\to 1}\frac{\mu(z)|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)\,(1-|\varphi(z)|^2)^{n+1+1/q}},\ \lim_{|\varphi(z)|\to 1}\frac{\mu(z)\,|g'(z)|}{\phi(|\varphi(z)|)\,(1-|\varphi(z)|^2)^{n+1/q}}\right\}\\
&\preceq \|I_{g,\varphi}^{(n)}\|_{e,H(p,\,q,\,\phi)\to\mathcal{Z}_\mu},
\end{aligned}$$

as desired.

Next, we intend to give the upper estimate of $\|I_{g,\varphi}^{(n)}\|_{e,H(p,\,q,\,\phi)\to\mathcal{Z}_\mu}$. Since $I_{g,\varphi}^{(n)}$: $H(p,\,q,\,\phi)\to\mathcal{Z}_\mu$ is bounded, then for any $f\in H(p,\,q,\,\phi)$, there is a constant C such that $\|I_{g,\varphi}^{(n)}f\|_{\mathcal{Z}_\mu} \leq C\|f\|_{p,\,q,\,\phi}$. Taking test function $f(z) = \frac{z^n}{n} \in H(p,\,q,\,\phi)$, we obtain

$$K_1 := \sup_{z\in\mathbb{D}} \mu(z)\,|g'(z)| < \infty. \tag{10}$$

If take $f(z)=\frac{z^{n+1}}{n+1}\in H(p,\,q,\,\phi)$, we have

$$K_2:=\sup_{z\in\mathbb{D}}\mu(z)|\varphi'(z)g(z)|<\infty. \tag{11}$$

For a fixed $\rho\in(0,1)$, using the normality of ϕ we have

$$\sup_{z\in\mathbb{D}}\frac{\mu(z)\,|\rho\varphi'(z)g(z)|}{\phi(\rho|\varphi(z)|)\,(1-|\rho\varphi(z)|^2)^{n+1+1/q}}\le\sup_{z\in\mathbb{D}}\frac{\mu(z)\,|\rho\varphi'(z)g(z)|}{\phi(\rho)\,(1-|\rho|^2)^{n+1+1/q}}<\infty,$$

and

$$\sup_{z\in\mathbb{D}}\frac{\mu(z)|g'(z)|}{\phi(\rho|\varphi(z)|)\,(1-|\rho\varphi(z)|^2)^{n+1/q}}\le\sup_{z\in\mathbb{D}}\frac{\mu(z)\,|g'(z)|}{\phi(\rho)\,(1-|\rho|^2)^{n+1/q}}<\infty.$$

By Lemma 2.2, we get $I_{g,\rho\varphi}^{(n)}$: $H(p,\,q,\,\phi)\to\mathcal{Z}_\mu$ is bounded. Since $|\rho\varphi(z)|\le\rho<1$, by Lemma 2.6, $I_{g,\rho\varphi}^{(n)}$: $H(p,\,q,\,\phi)\to\mathcal{Z}_\mu$ is compact, thus

$$\begin{aligned}
\|I_{g,\varphi}^{(n)}\|_{e,H(p,\,q,\,\phi)\to\mathcal{Z}_\mu} &\le \left\|I_{g,\varphi}^{(n)}-I_{g,\rho\varphi}^{(n)}\right\|_{H(p,\,q,\,\phi)\to\mathcal{Z}_\mu}\\
&=\sup_{\|f\|_{H(p,\,q,\,\phi)}\le1}\left\|\left(I_{g,\varphi}^{(n)}-I_{g,\rho\varphi}^{(n)}\right)f\right\|_{\mathcal{Z}_\mu}\\
&=\sup_{\|f\|_{H(p,\,q,\,\phi)}\le1}\sup_{z\in\mathbb{D}}\mu(z)\left|\left(I_{g,\varphi}^{(n)}f-I_{g,\rho\varphi}^{(n)}f\right)''(z)\right|\\
&\le\sup_{\|f\|_{H(p,\,q,\,\phi)}\le1}\sup_{z\in\mathbb{D}}\mu(z)\left|f^{(n+1)}(\varphi(z))-\rho f^{(n+1)}(\rho\varphi(z))\right||\varphi'(z)g(z)|\\
&+\sup_{\|f\|_{H(p,\,q,\,\phi)}\le1}\sup_{z\in\mathbb{D}}\mu(z)|f^{(n)}(\varphi(z))-f^{(n)}(\rho\varphi(z))||g'(z)|\\
&\le\sup_{\|f\|_{H(p,\,q,\,\phi)}\le1}\sup_{|\varphi(z)|\le r}\mu(z)\left|f^{(n+1)}(\varphi(z))-\rho f^{(n+1)}(\rho\varphi(z))\right||\varphi'(z)g(z)|\\
&+\sup_{\|f\|_{H(p,\,q,\,\phi)}\le1}\sup_{r<|\varphi(z)|<1}\mu(z)\left|f^{(n+1)}(\varphi(z))-\rho f^{(n+1)}(\rho\varphi(z))\right||\varphi'(z)g(z)|\\
&+\sup_{\|f\|_{H(p,\,q,\,\phi)}\le1}\sup_{|\varphi(z)|\le r}\mu(z)|f^{(n)}(\varphi(z))-f^{(n)}(\rho\varphi(z))||g'(z)|\\
&+\sup_{\|f\|_{H(p,\,q,\,\phi)}\le1}\sup_{r<|\varphi(z)|<1}\mu(z)|f^{(n)}(\varphi(z))-f^{(n)}(\rho\varphi(z))||g'(z)|\\
&:=I_{1,\rho,r}^{(n)}+I_{2,\rho,r}^{(n)}+I_{3,\rho,r}^{(n)}+I_{4,\rho,r}^{(n)},
\end{aligned}\tag{12}$$

Using the mean value theorem, the triangle inequality, (10), (11) and Lemma 2.5 we have

$$\begin{aligned}
I_{1,\rho,r}^{(n)}&=\sup_{\|f\|_{H(p,\,q,\,\phi)}\le1}\sup_{|\varphi(z)|\le r}\mu(z)\left|f^{(n+1)}(\varphi(z))-\rho f^{(n+1)}(\rho\varphi(z))\right||\varphi'(z)g(z)|\\
&=\sup_{\|f\|_{H(p,\,q,\,\phi)}\le1}\sup_{|\varphi(z)|\le r}|\mu(z)|\varphi'(z)g(z)|\left|f^{(n+1)}(\varphi(z))-f^{(n+1)}(\rho\varphi(z))+(1-\rho)f^{(n+1)}(\rho\varphi(z))\right|\\
&\le\sup_{\|f\|_{H(p,\,q,\,\phi)}\le1}\sup_{|\varphi(z)|\le r}(1-\rho)\mu(z)|\varphi'(z)g(z)|\left(\sup_{|w|\le r}|f^{(n+2)}(w)|+\sup_{|w|\le r}|f^{(n+1)}(w)|\right)\\
&\le C(1-\rho)\left(\frac{1}{\phi(r)(1-r^2)^{n+2+1/q}}+\frac{1}{\phi(r)(1-r^2)^{n+1+1/q}}\right)\to0\ (\rho\to1),
\end{aligned}\tag{13}$$

$$\begin{aligned}
I_{3,\rho,r}^{(n)} &= \sup_{\|f\|_{H(p,\,q,\,\phi)}\le 1}\ \sup_{|\varphi(z)|\le r} \mu(z)|f^{(n)}(\varphi(z)) - f^{(n)}(\rho\varphi(z))||g'(z)| \\
&\le \sup_{\|f\|_{H(p,\,q,\,\phi)}\le 1}\ \sup_{|\varphi(z)|\le r} (1-\rho)\mu(z)|g'(z)| \sup_{|w|\le r}|f^{(n+1)}(w)| \\
&\le C(1-\rho)\frac{1}{\phi(r)(1-r^2)^{n+1+1/q}} \to 0\ (\rho\to 1), \qquad (14)
\end{aligned}$$

$$\begin{aligned}
&I_{2,\rho,r}^{(n)} = \\
&\sup_{\|f\|_{H(p,\,q,\,\phi)}\le 1}\ \sup_{r<|\varphi(z)|<1} \mu(z)\left|f^{(n+1)}(\varphi(z)) - \rho f^{(n+1)}(\rho\varphi(z))\right||\varphi'(z)g(z)| \\
&\le \sup_{\|f\|_{H(p,\,q,\,\phi)}\le 1}\ \sup_{r<|\varphi(z)|<1} \mu(z)|\varphi'(z)g(z)|\left(\left|f^{(n+1)}(\varphi(z))\right| + \left|\rho f^{(n+1)}(\rho\varphi(z))\right|\right) \\
&\le \sup_{\|f\|_{H(p,\,q,\,\phi)}\le 1}\ \sup_{r<|\varphi(z)|<1} \mu(z)|\varphi'(z)g(z)|\frac{C\|f\|_{p,\,q,\,\phi}}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1+1/q}} \\
&+ \sup_{\|f\|_{H(p,\,q,\,\phi)}\le 1}\ \sup_{r<|\varphi(z)|<1} \mu(z)|\varphi'(z)g(z)|\frac{C\|f\|_{p,\,q,\,\phi}}{\phi(\rho|\varphi(z)|)(1-|\rho\varphi(z)|^2)^{n+1+1/q}} \\
&\le \sup_{r<|\varphi(z)|<1} \frac{C\mu(z)|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1+1/q}} + \sup_{r<|\varphi(z)|<1} \frac{C\mu(z)|\varphi'(z)g(z)|}{\phi(\rho|\varphi(z)|)(1-|\rho\varphi(z)|^2)^{n+1+1/q}} \\
&\to 2C \sup_{r<|\varphi(z)|<1} \frac{\mu(z)|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1+1/q}}\ (\rho\to 1), \qquad (15)
\end{aligned}$$

and

$$\begin{aligned}
I_{4,\rho,r}^{(n)} &= \sup_{\|f\|_{H(p,\,q,\,\phi)}\le 1}\ \sup_{r<|\varphi(z)|<1} \mu(z)|f^{(n)}(\varphi(z)) - f^{(n)}(\rho\varphi(z))||g'(z)| \\
&\le \sup_{\|f\|_{H(p,\,q,\,\phi)}\le 1}\ \sup_{r<|\varphi(z)|<1} \frac{C\mu(z)|g'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1/q}}\|f\|_{p,\,q,\,\phi} \\
&+ \sup_{\|f\|_{H(p,\,q,\,\phi)}\le 1}\ \sup_{r<|\varphi(z)|<1} \frac{C\mu(z)|g'(z)|}{\phi(\rho|\varphi(z)|)(1-|\rho\varphi(z)|^2)^{n+1/q}}\|f\|_{p,\,q,\,\phi} \\
&\le C \sup_{r<|\varphi(z)|<1} \frac{\mu(z)|g'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1/q}} + C \sup_{r<|\varphi(z)|<1} \frac{\mu(z)|g'(z)|}{\phi(\rho|\varphi(z)|)(1-|\rho\varphi(z)|^2)^{n+1/q}} \\
&\to 2C \sup_{r<|\varphi(z)|<1} \frac{\mu(z)|g'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1/q}}\ (\rho\to 1). \qquad (16)
\end{aligned}$$

Using the conditions (13-16), let $\rho\to 1$ in (12), then let $r\to 1$, we have

$$\begin{aligned}
&\|I_{g,\varphi}^{(n)}\|_{e,H(p,\,q,\,\phi)\to\mathcal{Z}_\mu} \\
&\le 2C \lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1+1/q}} + 2C \lim_{|\varphi(z)|\to 1} \frac{\mu(z)|g'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1/q}} \\
&\preceq \max\left\{\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\varphi'(z)g(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1+1/q}},\ \lim_{|\varphi(z)|\to 1} \frac{\mu(z)|g'(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{n+1/q}}\right\},
\end{aligned}$$

we get the desired result, so we are done.

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Solution of a third order fractional system of difference equations

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ABSTRACT

This paper is devoted to study the form of the solutions and the periodicity of the following third order systems of rational difference equations

$$x_{n+1} = \frac{y_{n-2}}{1 - y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{x_{n-2}}{\pm 1 \pm x_{n-2}y_{n-1}x_n},$$

with initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1}$ and y_0 are real numbers.

Keywords: System of difference equations, Periodic solutions.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

Recently, many researchers have studied systems of difference equations. This concern is due to the importance of some systems that can be used in some mathematical models in biology, economics, genetics, psychology, populations etc [1-37]. Many articles discuss difference equations systems, for example, Cinar [1] investigated the following system

$$x_{n+1} = \frac{m}{y_n}, \ \ y_{n+1} = \frac{py_n}{x_{n-1}y_{n-1}}.$$

Elsayed et al. [2] dealt with the solutions of the system of the difference equations

$$x_{n+1} = \frac{1}{x_{n-p}y_{n-p}}, \ \ y_{n+1} = \frac{x_{n-p}y_{n-p}}{x_{n-q}y_{n-q}},$$

and

$$x_{n+1} = \frac{1}{x_{n-p}y_{n-p}z_{n-p}}, \ \ y_{n+1} = \frac{x_{n-p}y_{n-p}z_{n-p}}{x_{n-q}y_{n-q}z_{n-q}}, \ z_{n+1} = \frac{x_{n-q}y_{n-q}z_{n-q}}{x_{n-r}y_{n-r}z_{n-r}}.$$

The behavior of positive solutions of the following system

$$x_{n+1} = \frac{x_{n-1}}{1 + x_{n-1}y_n}, \ y_{n+1} = \frac{y_{n-1}}{1 + y_{n-1}x_n}.$$

has been discussed by Kurbanli et al. [3].

In [7] Yalçınkaya studied the sufficient condition for the global asymptotic stability of the following system

$$z_{n+1} = \frac{t_n z_{n-1} + a}{t_n + z_{n-1}}, \ t_{n+1} = \frac{z_n t_{n-1} + a}{z_n + t_{n-1}}.$$

In [5] El-Dessoky considered the solutions of the following system

$$x_{n+1}=\frac{y_{n-1}y_{n-2}}{x_n\left(\pm1\pm y_{n-1}y_{n-2}\right)},\ y_{n+1}=\frac{x_{n-1}x_{n-2}}{y_n\left(\pm1\pm x_{n-1}x_{n-2}\right)}.$$

Touafek et al. [6] examined the periodic nature and found the form of the solutions of the following system

$$x_{n+1}=\frac{x_{n-3}}{\pm1\pm x_{n-3}y_{n-1}},\ y_{n+1}=\frac{y_{n-3}}{\pm1\pm y_{n-3}x_{n-1}}.$$

Kurbanli [7] investigated solutions' behavior for the following

$$x_{n+1}=\frac{x_{n-1}}{x_{n-1}y_n-1},\ y_{n+1}=\frac{y_{n-1}}{y_{n-1}x_n-1},\ z_{n+1}=\frac{x_n}{z_{n-1}y_n}.$$

El-Dessoky et al. [8] obtained the form of the solutions of the following system

$$x_{n+1}=\frac{x_{n-1}}{1+y_nx_{n-1}},\ y_{n+1}=\frac{y_{n-1}}{1+x_ny_{n-1}},\ z_{n+1}=\frac{z_{n-m}}{x_ny_n}.$$

We consider in this paper, the solution of the systems of difference equations

$$x_{n+1}=\frac{y_{n-2}}{1-y_{n-2}x_{n-1}y_n},\quad y_{n+1}=\frac{x_{n-2}}{\pm1\pm x_{n-2}y_{n-1}x_n},$$

with initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1}$ and y_0 are real numbers.

2. ON THE SYSTEM: $X_{N+1}=\frac{Y_{N-2}}{1-Y_{N-2}X_{N-1}Y_N},\ Y_{N+1}=\frac{X_{N-2}}{1+X_{N-2}Y_{N-1}X_N}$

In this section, we investigate the solutions of the system of two difference equations

$$x_{n+1}=\frac{y_{n-2}}{1-y_{n-2}x_{n-1}y_n},\quad y_{n+1}=\frac{x_{n-2}}{1+x_{n-2}y_{n-1}x_n},\tag{1}$$

where $n\in\mathbb{N}_0$ and the initial conditions are real numbers with $y_{-2}x_{-1}y_0\neq1$ and $x_{-2}y_{-1}x_0\neq-1$.

The following theorem is concerned with the form of the solutions of system (1).

Theorem 1. Suppose that $\{x_n,y_n\}$ are solutions of system (1). Then every solution of system (1) are periodic with period six and for $n=0,1,2,...,$

$$\begin{aligned}
x_{6n-2} &= x_{-2},\quad x_{6n-1}=x_{-1},\quad x_{6n}=x_0,\quad x_{6n+1}=\frac{y_{-2}}{(1-y_{-2}x_{-1}y_0)},\\
x_{6n+2} &= y_{-1}(1+x_{-2}y_{-1}x_0),\quad x_{6n+3}=\frac{y_0}{(1-y_{-2}x_{-1}y_0)},\\
\text{and}\ y_{6n-2} &= y_{-2},\quad y_{6n-1}=y_{-1},\quad y_{6n}=y_0,\quad y_{6n+1}=\frac{x_{-2}}{(1+x_{-2}y_{-1}x_0)},\\
y_{6n+2} &= x_{-1}(1-y_{-2}x_{-1}y_0),\quad y_{6n+3}=\frac{x_0}{(1+x_{-2}y_{-1}x_0)}.
\end{aligned}$$

Proof: If $n=0$, the result holds. We now assume that $n>0$ and that our assumption holds for $n-1$. That is,

$$\begin{aligned}
x_{6n-8} &= x_{-2},\quad x_{6n-7}=x_{-1},\quad x_{6n-6}=x_0,\quad x_{6n-5}=\frac{y_{-2}}{(1-y_{-2}x_{-1}y_0)},\\
x_{6n-4} &= y_{-1}(1+x_{-2}y_{-1}x_0),\quad x_{6n-3}=\frac{y_0}{(1-y_{-2}x_{-1}y_0)},\\
\text{and}\ y_{6n-8} &= y_{-2},\quad y_{6n-7}=y_{-1},\quad y_{6n-6}=y_0,\quad y_{6n-5}=\frac{x_{-2}}{(1+x_{-2}y_{-1}x_0)},\\
y_{6n-4} &= x_{-1}(1-y_{-2}x_{-1}y_0),\quad y_{6n-3}=\frac{x_0}{(1+x_{-2}y_{-1}x_0)}.
\end{aligned}$$

Now it follows from Eq.(1) that

$$
\begin{aligned}
x_{6n-2} &= \frac{y_{6n-5}}{1-y_{6n-5}x_{6n-4}y_{6n-3}} \\
&= \frac{\frac{x_{-2}}{(1+x_{-2}y_{-1}x_0)}}{1-\left(\frac{x_{-2}}{(1+x_{-2}y_{-1}x_0)}y_{-1}(1+x_{-2}y_{-1}x_0)\frac{x_0}{(1+x_{-2}y_{-1}x_0)}\right)} \\
&= \frac{x_{-2}}{(1+x_{-2}y_{-1}x_0)\left(1-\frac{x_{-2}y_{-1}x_0}{(1+x_{-2}y_{-1}x_0)}\right)} = x_{-2}, \\
y_{6n-1} &= \frac{x_{6n-4}}{1+x_{6n-4}y_{6n-3}x_{6n-2}} \\
&= \frac{y_{-1}(1+x_{-2}y_{-1}x_0)}{1+\left(y_{-1}(1+x_{-2}y_{-1}x_0)\frac{x_0}{(1+x_{-2}y_{-1}x_0)}x_{-2}\right)} \\
&= \frac{y_{-1}(1+x_{-2}y_{-1}x_0)}{1+y_{-2}y_{-1}x_0} = y_{-1}, \\
x_{6n} &= \frac{y_{6n-3}}{1-y_{6n-3}x_{6n-2}y_{6n-1}} = \frac{\frac{x_0}{(1+x_{-2}y_{-1}x_0)}}{1-\left(\frac{x_0}{(1+x_{-2}y_{-1}x_0)}x_{-2}y_{-1}\right)} = x_0, \\
y_{6n+1} &= \frac{x_{6n-2}}{1+x_{6n-2}y_{6n-1}x_{6n}} = \frac{x_{-2}}{1+x_{-2}y_{-1}x_0}.
\end{aligned}
$$

Also, we can prove the other relations and so the proof is completed.

Example 1. We look at interesting numerical example for the system (1) with the initial conditions $x_{-2} = 0.5,\ x_{-1} = 0.9,\ x_0 = 0.2,\ y_{-2} = 1.1,\ y_{-1} = 0.7$ and $y_0 = 0.3$. (See Fig. 1).

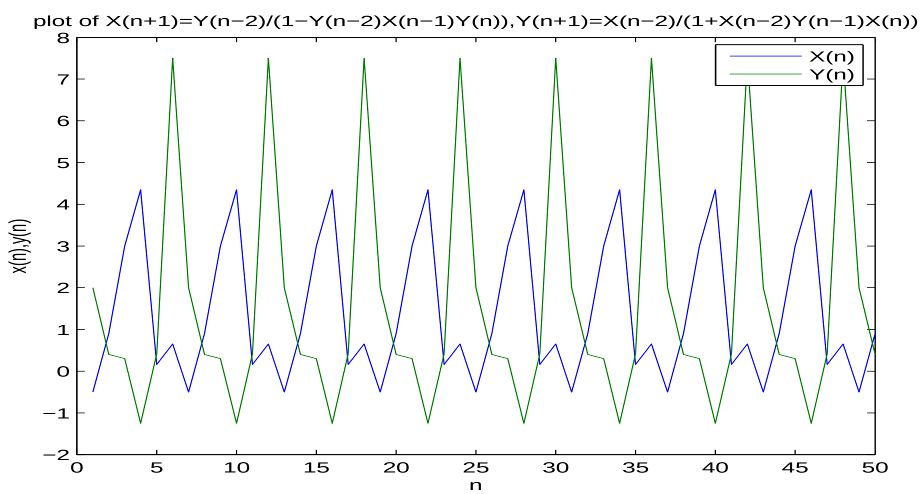

Figure 1.

3. ON THE SYSTEM: $X_{N+1} = \frac{Y_{N-2}}{1-Y_{N-2}X_{N-1}Y_N}$, $Y_{N+1} = \frac{X_{N-2}}{1-X_{N-2}Y_{N-1}X_N}$

In this section, we get the form of the solutions of the system of two difference equations

$$x_{n+1} = \frac{y_{n-2}}{1-y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{x_{n-2}}{1-x_{n-2}y_{n-1}x_n}, \tag{2}$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary real numbers.

Theorem 2. Suppose that $\{x_n, y_n\}$ are solutions of system (2). Then for $n = 0, 1, 2, ...,$

$$
\begin{aligned}
x_{6n-2} &= x_{-2}\prod_{i=0}^{n-1}\frac{(1-(6i)x_{-2}y_{-1}x_0)(1-(6i+3)x_{-2}y_{-1}x_0)}{(1-(6i+1)x_{-2}y_{-1}x_0)(1-(6i+4)x_{-2}y_{-1}x_0)},\\
x_{6n-1} &= x_{-1}\prod_{i=0}^{n-1}\frac{(1-(6i+1)y_{-2}x_{-1}y_0)(1-(6i+4)y_{-2}x_{-1}y_0)}{(1-(6i+2)y_{-2}x_{-1}y_0)(1-(6i+5)y_{-2}x_{-1}y_0)},\\
x_{6n} &= x_0\prod_{i=0}^{n-1}\frac{(1-(6i+2)x_{-2}y_{-1}x_0)(1-(6i+5)x_{-2}y_{-1}x_0)}{(1-(6i+3)x_{-2}y_{-1}x_0)(1-(6i+6)x_{-2}y_{-1}x_0)},\\
x_{6n+1} &= \frac{y_{-2}}{(1-y_{-2}x_{-1}y_0)}\prod_{i=0}^{n-1}\frac{(1-(6i+3)y_{-2}x_{-1}y_0)(1-(6i+6)y_{-2}x_{-1}y_0)}{(1-(6i+4)y_{-2}x_{-1}y_0)(1-(6i+7)y_{-2}x_{-1}y_0)},\\
x_{6n+2} &= \frac{y_{-1}(1-x_{-2}y_{-1}x_0)}{(1-2x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-1}\frac{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)},\\
x_{6n+3} &= \frac{y_0(1-2y_{-2}x_{-1}y_0)}{(1-3y_{-2}x_{-1}y_0)}\prod_{i=0}^{n-1}\frac{(1-(6i+5)y_{-2}x_{-1}y_0)(1-(6i+8)y_{-2}x_{-1}y_0)}{(1-(6i+6)y_{-2}x_{-1}y_0)(1-(6i+9)y_{-2}x_{-1}y_0)},\\
y_{6n-2} &= y_{-2}\prod_{i=0}^{n-1}\frac{(1-(6i)y_{-2}x_{-1}y_0)(1-(6i+3)y_{-2}x_{-1}y_0)}{(1-(6i+1)y_{-2}x_{-1}y_0)(1-(6i+4)y_{-2}x_{-1}y_0)},\\
y_{6n-1} &= y_{-1}\prod_{i=0}^{n-1}\frac{(1-(6i+1)x_{-2}y_{-1}x_0)(1-(6i+4)x_{-2}y_{-1}x_0)}{(1-(6i+2)x_{-2}y_{-1}x_0)(1-(6i+5)x_{-2}y_{-1}x_0)},\\
y_{6n} &= y_0\prod_{i=0}^{n-1}\frac{(1-(6i+2)y_{-2}x_{-1}y_0)(1-(6i+5)y_{-2}x_{-1}y_0)}{(1-(6i+3)y_{-2}x_{-1}y_0)(1-(6i+6)y_{-2}x_{-1}y_0)},\\
y_{6n+1} &= \frac{x_{-2}}{(1-x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-1}\frac{(1-(6i+3)x_{-2}y_{-1}x_0)(1-(6i+6)x_{-2}y_{-1}x_0)}{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)},\\
y_{6n+2} &= \frac{x_{-1}(1-y_{-2}x_{-1}y_0)}{(1-2y_{-2}x_{-1}y_0)}\prod_{i=0}^{n-1}\frac{(1-(6i+4)y_{-2}x_{-1}y_0)(1-(6i+7)y_{-2}x_{-1}y_0)}{(1-(6i+5)y_{-2}x_{-1}y_0)(1-(6i+8)y_{-2}x_{-1}y_0)},\\
y_{6n+3} &= \frac{x_0(1-2x_{-2}y_{-1}x_0)}{(1-3x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-1}\frac{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)}{(1-(6i+6)x_{-2}y_{-1}x_0)(1-(6i+9)x_{-2}y_{-1}x_0)}.
\end{aligned}
$$

where $\prod_{i=0}^{-1} C_i = 1$

Proof: The result holds for $n = 0$. Now suppose that $n > 1$ and that our assumption holds for $n - 1$. that is,

$$
\begin{aligned}
x_{6n-5} &= \frac{y_{-2}}{(1-y_{-2}x_{-1}y_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+3)y_{-2}x_{-1}y_0)(1-(6i+6)y_{-2}x_{-1}y_0)}{(1-(6i+4)y_{-2}x_{-1}y_0)(1-(6i+7)y_{-2}x_{-1}y_0)},\\
x_{6n-4} &= \frac{y_{-1}(1-x_{-2}y_{-1}x_0)}{(1-2x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)},\\
x_{6n-3} &= \frac{y_0(1-2y_{-2}x_{-1}y_0)}{(1-3y_{-2}x_{-1}y_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+5)y_{-2}x_{-1}y_0)(1-(6i+8)y_{-2}x_{-1}y_0)}{(1-(6i+6)y_{-2}x_{-1}y_0)(1-(6i+9)y_{-2}x_{-1}y_0)},\\
y_{6n-5} &= \frac{x_{-2}}{(1-x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+3)x_{-2}y_{-1}x_0)(1-(6i+6)x_{-2}y_{-1}x_0)}{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)},
\end{aligned}
$$

$$y_{6n-4} = \frac{x_{-1}(1-y_{-2}x_{-1}y_0)}{(1-2y_{-2}x_{-1}y_0)} \prod_{i=0}^{n-2} \frac{(1-(6i+4)y_{-2}x_{-1}y_0)(1-(6i+7)y_{-2}x_{-1}y_0)}{(1-(6i+5)y_{-2}x_{-1}y_0)(1-(6i+8)y_{-2}x_{-1}y_0)},$$

$$y_{6n-3} = \frac{x_0(1-2x_{-2}y_{-1}x_0)}{(1-3x_{-2}y_{-1}x_0)} \prod_{i=0}^{n-2} \frac{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)}{(1-(6i+6)x_{-2}y_{-1}x_0)(1-(6i+9)x_{-2}y_{-1}x_0)}.$$

It follows from Eq.(2) that

$$\begin{aligned}
x_{6n-2} &= \frac{y_{6n-5}}{1-y_{6n-5}x_{6n-4}y_{6n-3}} \\
&= \frac{\frac{x_{-2}}{(1-x_{-2}y_{-1}x_0)} \prod_{i=0}^{n-2} \frac{(1-(6i+3)x_{-2}y_{-1}x_0)(1-(6i+6)x_{-2}y_{-1}x_0)}{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}}{\left(\begin{array}{c} 1-\frac{x_{-2}}{(1-x_{-2}y_{-1}x_0)} \prod_{i=0}^{n-2} \frac{(1-(6i+3)x_{-2}y_{-1}x_0)(1-(6i+6)x_{-2}y_{-1}x_0)}{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)} \\ \frac{y_{-1}(1-x_{-2}y_{-1}x_0)}{(1-2x_{-2}y_{-1}x_0)} \prod_{i=0}^{n-2} \frac{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)} \\ \frac{x_0(1-2x_{-2}y_{-1}x_0)}{(1-3x_{-2}y_{-1}x_0)} \prod_{i=0}^{n-2} \frac{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)}{(1-(6i+6)x_{-2}y_{-1}x_0)(1-(6i+9)x_{-2}y_{-1}x_0)} \end{array}\right)} \\
&= \frac{\frac{x_{-2}}{(1-x_{-2}y_{-1}x_0)} \prod_{i=0}^{n-2} \frac{(1-(6i+3)x_{-2}y_{-1}x_0)(1-(6i+6)x_{-2}y_{-1}x_0)}{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}}{\left(1-\frac{x_0y_{-1}x_{-2}}{(1-3x_{-2}y_{-1}x_0)} \prod_{i=0}^{n-2} \frac{(1-(6i+3)x_{-2}y_{-1}x_0)}{(1-(6i+9)x_{-2}y_{-1}x_0)}\right)} \\
&= \frac{\frac{x_{-2}}{(1-x_{-2}y_{-1}x_0)} \prod_{i=0}^{n-2} \frac{(1-(6i+3)x_{-2}y_{-1}x_0)(1-(6i+6)x_{-2}y_{-1}x_0)}{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}}{\left(1-\frac{x_{-2}y_{-1}x_0}{(1-(6n-3)x_{-2}y_{-1}x_0)}\right)} \\
&= \frac{\frac{x_{-2}}{(1-x_{-2}y_{-1}x_0)} \prod_{i=0}^{n-2} \frac{(1-(6i+3)x_{-2}y_{-1}x_0)(1-(6i+6)x_{-2}y_{-1}x_0)}{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}}{\left(\frac{1-(6n-2)x_{-2}y_{-1}x_0}{1-(6n-3)x_{-2}y_{-1}x_0}\right)} \\
&= \frac{\frac{x_{-2}}{(1-x_{-2}y_{-1}x_0)} \prod_{i=0}^{n-2} \frac{(1-(6i+3)x_{-2}y_{-1}x_0)(1-(6i+6)x_{-2}y_{-1}x_0)}{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}}{\left(\frac{1-(6n-2)x_{-2}y_{-1}x_0}{1-(6n-3)x_{-2}y_{-1}x_0}\right)}.
\end{aligned}$$

Then we see that

$$x_{6n-2} = x_{-2} \prod_{i=0}^{n-1} \frac{(1-(6i)x_{-2}y_{-1}x_0)(1-(6i+3)x_{-2}y_{-1}x_0)}{(1-(6i+1)x_{-2}y_{-1}x_0)(1-(6i+4)x_{-2}y_{-1}x_0)}.$$

Also, we see from Eq.(2) that

$$\begin{aligned}
y_{6n-1} &= \frac{x_{6n-4}}{1-x_{6n-4}y_{6n-3}x_{6n-2}} \\
&= \frac{\frac{y_{-1}(1-x_{-2}y_{-1}x_0)}{(1-2x_{-2}y_{-1}x_0)} \prod_{i=0}^{n-2} \frac{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)}}{1-\left(\begin{array}{c} \frac{y_{-1}(1-x_{-2}y_{-1}x_0)}{(1-2x_{-2}y_{-1}x_0)} \prod_{i=0}^{n-2} \frac{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)} \\ \frac{x_0(1-2x_{-2}y_{-1}x_0)}{(1-3x_{-2}y_{-1}x_0)} \prod_{i=0}^{n-2} \frac{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)}{(1-(6i+6)x_{-2}y_{-1}x_0)(1-(6i+9)x_{-2}y_{-1}x_0)} \\ x_{-2} \prod_{i=0}^{n-1} \frac{(1-(6i)x_{-2}y_{-1}x_0)(1-(6i+3)x_{-2}y_{-1}x_0)}{(1-(6i+1)x_{-2}y_{-1}x_0)(1-(6i+4)x_{-2}y_{-1}x_0)} \end{array}\right)}
\end{aligned}$$

$$y_{6n-1} = \frac{\frac{y_{-1}(1-x_{-2}y_{-1}x_0)}{(1-2x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)}}{1-\left(\begin{array}{c}\frac{x_{-2}x_0y_{-1}(1-x_{-2}y_{-1}x_0)}{(1-3x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}{(1-(6i+6)x_{-2}y_{-1}x_0)(1-(6i+9)x_{-2}y_{-1}x_0)} \\ \prod_{i=0}^{n-1}\frac{(1-(6i)x_{-2}y_{-1}x_0)(1-(6i+3)x_{-2}y_{-1}x_0)}{(1-(6i+1)x_{-2}y_{-1}x_0)(1-(6i+4)x_{-2}y_{-1}x_0)}\end{array}\right)}$$

$$= \frac{\frac{y_{-1}(1-x_{-2}y_{-1}x_0)}{(1-2x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)}}{1-\left(x_{-2}x_0y_{-1}\prod_{i=0}^{n-2}(1-(6i+4)x_{-2}y_{-1}x_0)\prod_{i=0}^{n-1}\frac{1}{(1-(6i+4)x_{-2}y_{-1}x_0)}\right)}$$

$$= \frac{\frac{y_{-1}(1-x_{-2}y_{-1}x_0)}{(1-2x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)}}{1-\left(\frac{x_{-2}x_0y_{-1}}{1-(6n-2)x_{-2}y_{-1}x_0}\right)}$$

$$= \frac{\frac{y_{-1}(1-x_{-2}y_{-1}x_0)}{(1-2x_{-2}y_{-1}x_0)}\prod_{i=0}^{n-2}\frac{(1-(6i+4)x_{-2}y_{-1}x_0)(1-(6i+7)x_{-2}y_{-1}x_0)}{(1-(6i+5)x_{-2}y_{-1}x_0)(1-(6i+8)x_{-2}y_{-1}x_0)}}{\left(\frac{1-(6n-1)x_{-2}y_{-1}x_0}{1-(6n-2)x_{-2}y_{-1}x_0}\right)}.$$

Then

$$y_{6n-1} = y_{-1}\prod_{i=0}^{n-1}\frac{(1-(6i+1)x_{-2}y_{-1}x_0)(1-(6i+4)x_{-2}y_{-1}x_0)}{(1-(6i+2)x_{-2}y_{-1}x_0)(1-(6i+5)x_{-2}y_{-1}x_0)}.$$

Similarly we can prove the other relations. Hence, the proof is completed.

Lemma 1. If $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}$ and y_0 are arbitrary real numbers and $\{x_n, y_n\}$ are solutions of system (2), then the following statements are true:

(i) If $x_{-2} = 0,\ y_{-1} \neq 0,\ x_0 \neq 0$, then we have $x_{6n-2} = y_{6n+1} = 0$ *and* $x_{6n} = y_{6n+3} = x_0,\ x_{6n+2} = y_{6n-1} = y_{-1}$.

(ii) If $x_{-1} = 0,\ y_{-2} \neq 0,\ y_0 \neq 0$, then we have $x_{6n-1} = y_{6n+2} = 0$ *and* $x_{6n+1} = y_{6n-2} = y_{-2},\ x_{6n+3} = y_{6n} = y_0$.

(iii) If $x_0 = 0,\ y_{-1} \neq 0,\ x_{-2} \neq 0$, then we have $x_{6n} = y_{6n+3} = 0$ *and* $x_{6n-2} = y_{6n+1} = x_{-2},\ x_{6n+2} = y_{6n-1} = y_{-1}$.

(iv) If $y_{-2} = 0,\ x_{-1} \neq 0,\ y_0 \neq 0$, then we have $y_{6n-2} = x_{6n+1} = 0$ *and* $x_{6n-1} = y_{6n+2} = x_{-1},\ x_{6n+3} = y_{6n} = y_0$.

(v) If $y_{-1} = 0,\ x_0 \neq 0,\ x_{-2} \neq 0$, then we have $y_{6n-1} = x_{6n+2} = 0$ *and* $x_{6n-2} = y_{6n+1} = x_{-2},\ x_{6n} = y_{6n+3} = x_0$.

(vi) If $y_0 = 0,\ y_{-2} \neq 0,\ x_{-1} \neq 0$, then we have $y_{6n} = x_{6n+3} = 0$ *and* $x_{6n-1} = y_{6n+2} = x_{-1},\ x_{6n+1} = y_{6n-2} = y_{-2}$.

Proof: The proof follows from the form of the solutions of system (2).

Example 2. We assume the initial conditions $x_{-2} = -0.5,\ x_{-1} = 0.9,\ x_0 = 3,\ y_{-2} = 2,\ y_{-1} = 0.4$ *and* $y_0 = 0.3$, for the difference system (2), see Fig. 2.

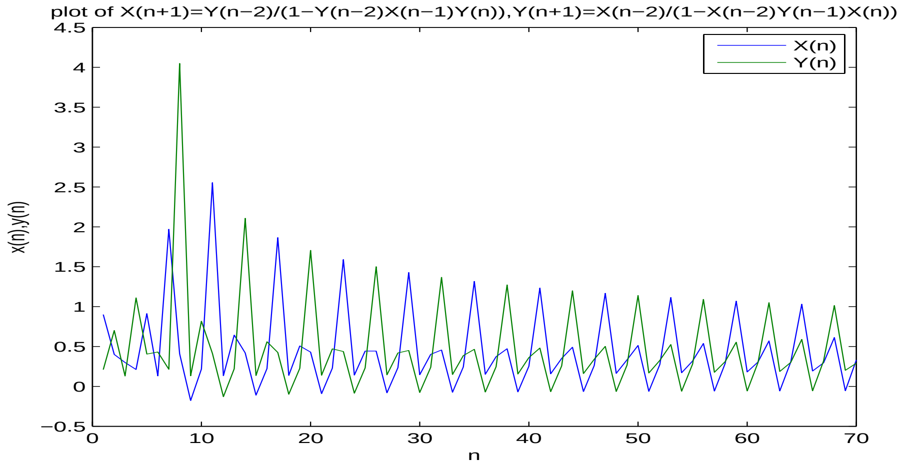

Figure 2.

The following cases can be proved similarly.

4. ON THE SYSTEM: $X_{N+1} = \frac{Y_{N-2}}{1-Y_{N-2}X_{N-1}Y_N}$, $Y_{N+1} = \frac{X_{N-2}}{-1+X_{N-2}Y_{N-1}X_N}$

In this section, we get the solutions of the system of the difference equations

$$x_{n+1} = \frac{y_{n-2}}{1 - y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{x_{n-2}}{-1 + x_{n-2}y_{n-1}x_n}, \tag{3}$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary real numbers such that $x_{-2}y_{-1}x_0 \neq \pm 1$ and $y_{-2}x_{-1}y_0 \neq 1, \neq \frac{1}{2}$.

Theorem 3. If $\{x_n, y_n\}$ are solutions of difference equation system (3). Then every solution of system (3) are periodic with period twelve and for $n = 0, 1, 2, ...,$

$$\begin{aligned}
x_{12n-2} &= x_{-2}, \quad x_{12n-1} = x_{-1}, \quad x_{12n} = x_0, \quad x_{12n+1} = \frac{y_{-2}}{(1 - y_{-2}x_{-1}y_0)}, \\
x_{12n+2} &= -y_{-1}(-1 + x_{-2}y_{-1}x_0), \quad x_{12n+3} = \frac{y_0(-1 + 2y_{-2}x_{-1}y_0)}{(-1 + y_{-2}x_{-1}y_0)}, \\
x_{12n+4} &= \frac{x_{-2}(1 + x_{-2}y_{-1}x_0)}{(-1 + x_{-2}y_{-1}x_0)}, \; x_{12n+5} = \frac{x_{-1}}{(-1 + 2y_{-2}x_{-1}y_0)}, \\
x_{12n+6} &= \frac{x_0(-1 + x_{-2}y_{-1}x_0)}{(1 + x_{-2}y_{-1}x_0)}, \quad x_{12n+7} = \frac{-y_{-2}(-1 + 2y_{-2}x_{-1}y_0)}{(-1 + y_{-2}x_{-1}y_0)}, \\
x_{12n+8} &= -y_{-1}(1 + x_{-2}y_{-1}x_0), \; x_{12n+9} = \frac{-y_0}{(1 - y_{-2}x_{-1}y_0)}, \\
y_{12n-2} &= y_{-2}, \; y_{12n-1} = y_{-1}, \quad y_{12n} = y_0, \quad y_{12n+1} = \frac{x_{-2}}{(-1 + x_{-2}y_{-1}x_0)}, \\
y_{12n+2} &= \frac{-x_{-1}(-1 + y_{-2}x_{-1}y_0)}{(-1 + 2y_{-2}x_{-1}y_0)}, \; y_{12n+3} = \frac{-x_0}{(1 + x_{-2}y_{-1}x_0)}, \\
y_{12n+4} &= -y_{-2}, \; y_{12n+5} = -y_{-1}, \; y_{12n+6} = -y_0, \; y_{12n+7} = \frac{-x_{-2}}{-1 + x_{-2}y_{-1}x_0}, \\
y_{12n+8} &= \frac{x_{-1}(-1 + y_{-2}x_{-1}y_0)}{(-1 + 2y_{-2}x_{-1}y_0)}, \; y_{12n+9} = \frac{x_0}{(1 + x_{-2}y_{-1}x_0)}.
\end{aligned}$$

Example 3. Figure (3) shows the behavior of the solution of the difference system (3) with the initial conditions $x_{-2} = -0.7,\ x_{-1} = 1.6,\ x_0 = -0.3,\ y_{-2} = -0.2,\ y_{-1} = -1.9$ and $y_0 = 1.1$.

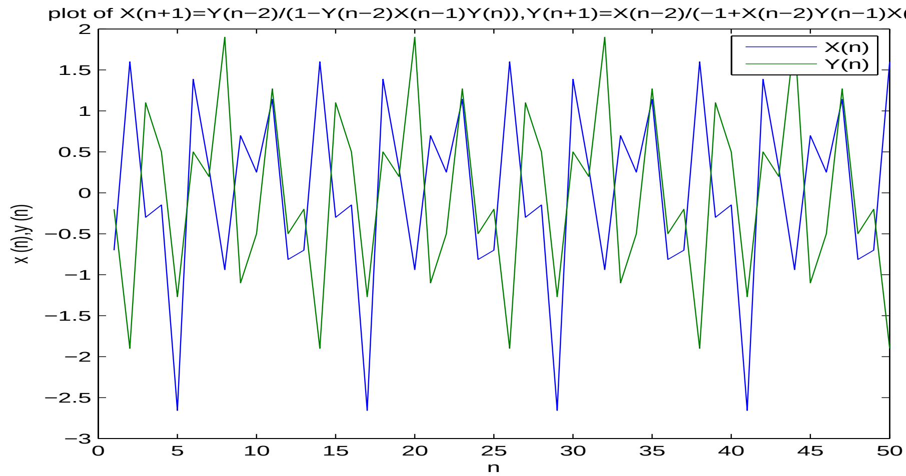

Figure 3.

5. ON THE SYSTEM $X_{N+1} = \frac{Y_{N-2}}{1-Y_{N-2}X_{N-1}Y_N}$, $Y_{N+1} = \frac{X_{N-2}}{-1-X_{N-2}Y_{N-1}X_N}$

In this section, we discuss the solutions of the following system of the difference equations

$$x_{n+1} = \frac{y_{n-2}}{1 - y_{n-2}x_{n-1}y_n}, \quad y_{n+1} = \frac{x_{n-2}}{-1 - x_{n-2}y_{n-1}x_n}, \tag{4}$$

where $n \in \mathbb{N}_0$ and the initial conditions are real numbers with $x_{-2}y_{-1}x_0 \neq -1, \neq -\frac{1}{2}$ and $y_{-2}x_{-1}y_0 \neq \pm 1$.

Theorem 4. Suppose that $\{x_n, y_n\}$ are solutions of system (4). Then every solution of system (4) are periodic with period twelve and for $n = 0, 1, 2, ...,$

$$\begin{aligned}
x_{12n-2} &= x_{-2}, \quad x_{12n-1} = x_{-1}, \quad x_{12n} = x_0, \quad x_{12n+1} = \frac{y_{-2}}{(1 - y_{-2}x_{-1}y_0)}, \\
x_{12n+2} &= \frac{y_{-1}(1 + x_{-2}y_{-1}x_0)}{(1 + 2x_{-2}y_{-1}x_0)}, \ x_{12n+3} = \frac{y_0}{(1 + y_{-2}x_{-1}y_0)}, \ x_{12n+4} = -x_{-2}, \\
x_{12n+5} &= -x_{-1}, \quad x_{12n+6} = -x_0, \quad x_{12n+7} = \frac{-y_{-2}}{(1 - y_{-2}x_{-1}y_0)}, \\
x_{12n+8} &= \frac{-y_{-1}(1 + x_{-2}y_{-1}x_0)}{(1 + 2x_{-2}y_{-1}x_0)}, \quad x_{12n+9} = \frac{-y_0}{(1 + y_{-2}x_{-1}y_0)}, \\
y_{12n-2} &= y_{-2}, \quad y_{12n-1} = y_{-1}, \quad y_{12n} = y_0, \quad y_{12n+1} = \frac{-x_{-2}}{(1 + x_{-2}y_{-1}x_0)}, \\
y_{12n+2} &= -x_{-1}(1 - y_{-2}x_{-1}y_0), \ y_{12n+3} = \frac{-x_0(1 + 2x_{-2}y_{-1}x_0)}{(1 + x_{-2}y_{-1}x_0)}, \\
y_{12n+4} &= \frac{-y_{-2}(1 + y_{-2}x_{-1}y_0)}{(1 - y_{-2}x_{-1}y_0)}, \ y_{12n+5} = \frac{-y_{-1}}{(1 + 2x_{-2}y_{-1}x_0)}, \\
y_{12n+6} &= \frac{-y_0(1 - y_{-2}x_{-1}y_0)}{(1 + y_{-2}x_{-1}y_0)}, \quad y_{12n+7} = \frac{x_{-2}(1 + 2x_{-2}y_{-1}x_0)}{(1 + x_{-2}y_{-1}x_0)}, \\
y_{12n+8} &= x_{-1}(1 + y_{-2}x_{-1}y_0), \ y_{12n+9} = \frac{x_0}{(1 + x_{-2}y_{-1}x_0)}.
\end{aligned}$$

Example 4. Figure (4) shows the periodicity of the solution of the difference system (4) when we put the initial conditions $x_{-2} = -0.7,\ x_{-1} = .6,\ x_0 = 5,\ y_{-2} = -.5,\ y_{-1} = .18$ and $y_0 = 6$.

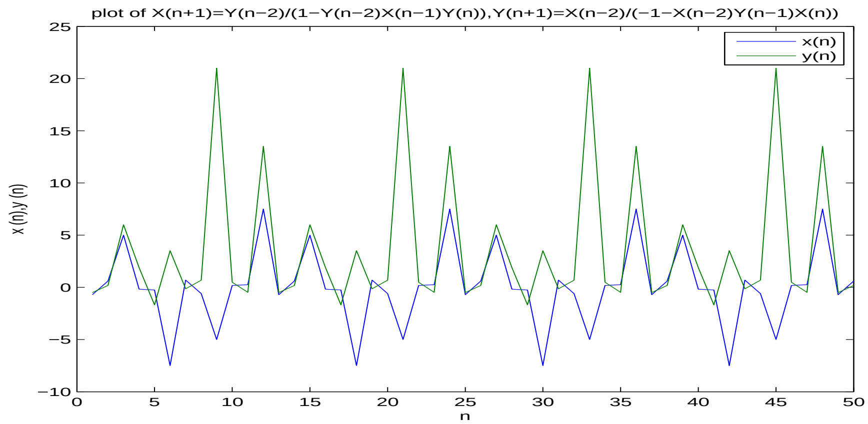

Figure 4.

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Inequalities For Orlicz mixed Harmonic Quermassintegrals *

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Abstract: In this paper, we introduce the notion of Orlicz mixed harmonic quermassintegrals. The variational formula of harmonic quermassintegrals with respect to the Orlicz combination is proved. The Minkowski type inequality and the Brunn-Minkowski type inequality are established for Orlicz mixed harmonic quermassintegrals.
Keywords: Orlicz mixed harmonic quermassintegrals, variational formula, harmonic quermassintegrals.
2000 Mathematics Subject Classification: 52A20 52A40.

1. Introduction

Associated with a body $K \in \mathcal{K}_0^n$ are its harmonic quermassintegrals, $\widehat{W}_0, \widehat{W}_1, \cdots, \widehat{W}_n$. These quermassintegrals were introduced by Hadwiger [6,section 6.4.8], and can be defined by letting $\widehat{W}_0 = V(K), \widehat{W}_n = \omega_n$, and for $j = 1, 2, \cdots, n-1$,

$$\widehat{W}_{n-j} = \frac{\omega_n}{\omega_j}[\int_{Gr(n,j)} V^{\{j\}}(K|\xi)^{-1} d\mu_j(\xi)]^{-1}. \tag{1.1}$$

Where $K|\xi$ denote the image of the orthogonal projection of K onto ξ. From the Schwarz or Hӧder inequality, we have

$$\widehat{W}_i(K) \leq W_i(K).$$

Here $0 < i < n$, with equality hold if and only if K has constant outer $n-i$-measure. For more details, please see [8,10].

Recently, the Orlicz-Brunn Minkowski theory originated with the work of Lutwak, Yang and Zhang [11,12]and Haberl [5]. This theory is much more general than the L_p Brunn-Minkowski theory(see [2,9,13]), for the development of the Orlicz Brunn Minkowski theory, see[3-5,7,11-12,15-16]. Gardner [3] extended the L_p Brunn-Minkowski theory to a Orlicz Brunn-Minkowski theory. As the same time, Xi, jin and leng[15] defined the Orlicz addition and given the Orlicz Brunn-Minkowski inequality. Note that xi use a completely different approach technique of Steiner symmetrization, although these results coincide with Gardner.

Following the spirit of Hadwiger, we introduce the Orlicz mixed harmonic quermassintegrals as follow: Let $K, L \in \mathcal{K}_0^n, \phi \in \mathcal{C}^+$, we have, for $j = 1, 2, \cdots, n-1$,

$$\widehat{W}_{\phi,n-j}(K, L) = \frac{\omega_n}{\omega_j}[\int_{Gr(n,j)} \frac{V_\phi^{\{j\}}(K|\xi, L|\xi)}{V^{\{j\}}(K|\xi)} V^{\{j\}}(K|\xi)^{-1} d\mu_j(\xi)]^{-1}. \tag{1.2}$$

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Note that, taking $K = L, \phi(1) = 1$, we have $\widehat{W}_{\phi,n-j}(K,K) = \widehat{W}_{n-j}(K)$. where $V^{\{j\}}(K|\xi)$ denotes the $j-$dimensional volume of intersection of K with an $j-$ dimensional subspace $\xi \subset R^n$. The Grassmann manifold $Gr(n,j)$ is endowed with the normalized Haar measure.

In Section 3. Using the Orlicz combination, we give the variational formula of harmonic quermassintegrals. That is, if $K, L \in \mathcal{K}_0^n, \phi \in \mathcal{C}^+$, then for each $j = 1, 2, \cdots, n-1$, $\xi \in Gr(n,j)$,

$$\lim_{\varepsilon \to 0^+} \frac{\widehat{W}_{n-j}(K +_\phi \varepsilon \cdot L) - \widehat{W}_{n-j}(K)}{\varepsilon} = \frac{j}{\phi_l'(1)} \widehat{W}_{\phi,n-j}(K,L)^{-1} \widehat{W}_{n-j}(K)^2.$$

Orlicz mixing homogeneous quermassintegrals as a generalization of the harmonic homogeneous integration. A nature question is whether there is a Minkowski type isoperimetric inequality for Orlicz mixing homogeneous quermassintegrals, we give a definite answer.

In the section 4, we prove the following Minkowski type inequality: Let $K, L \in \mathcal{K}_0^n, 0 < j < n, \phi \in \mathcal{C}^+$. Then

$$\widehat{W}_{\phi,n-j}(K,L)^{-1} \geq \widehat{W}_{n-j}(K)^{-1} \phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K)})^{1/j}).$$

If ϕ is strictly convex, with equality holds if and only if K and L are dilations.

We also establish the Brunn-Minkowski type inequality for convex bodies. The following Brunn-Minkowski type inequality: Let $K, L \in \mathcal{K}_0^n, 0 < j < n, \varepsilon > 0$. Then for any $\phi \in \mathcal{C}^+$,

$$\phi(1) \geq \phi((\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K +_\phi \varepsilon \cdot L)})^{1/j}) + \varepsilon \phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K +_\phi \varepsilon \cdot L)})^{1/j}).$$

If ϕ is strictly convex, with equality holds if and only if K and L are dilations.

In the section 5, We obtain the Brunn-Minkowski type inequality for Orlicz sum $+_\phi(K_1, K_2, \cdots, K_n)$. Let $K_1, K_2, \cdots, K_n \in \mathcal{K}_0^n \in \mathcal{K}_0^n$, ϕ is $n \geq 2$ variate functions, $j = 1, 2, \cdots, n-1$. Then following inequality hold,

$$1 \geq \phi((\frac{\widehat{W}_{n-j}(K_1)}{\widehat{W}_{n-j}(+_\phi(K_1, K_2, \cdots, K_n))})^{1/j}, \cdots, (\frac{\widehat{W}_{n-j}(K_n)}{\widehat{W}_{n-j}(+_\phi(K_1, K_2, \cdots, K_n))})^{1/j}).$$

If ϕ is strictly convex, with equality holds if and only if $K_1, K_2, \cdots, K_n$ are dilations.

2. Notation and background material

We collect some basic facts about convex bodies that are needed in our paper.

2.1. Mixed volumes

The unit ball in R^n and unit sphere denoted by B and S^{n-1}, respectively. Let $Gr(n,j)$ denote the Grassmann manifold of $j-$dimensional subspaces ξ through the origin in R^n. $d\mu_j(\xi)$ is the normalized rotation invariant measure on $Gr(n,j)$ and to emphasize the dependence of j.

The Minkowski addition and scalar product of sets K and L in R^n is defined by(see[1])

$$aK + bL = \{ax + by : x \in K, y \in L\}, \text{for all } \ a, b \in R. \tag{2.1}$$

If $K, L \in \mathcal{K}_0^n$ can be defined as a convex body such that

$$h_{aK+bL}(u) = ah_K(u) + bh_L(u), \text{for all } \ u \in S^{n-1}.$$

The volume of a Minkowski combination $\lambda_1 L_1 + \cdots + \lambda_m L_m$ of convex bodies $L_1, \ldots, L_m$ can be expressed as a homogeneous polynomial of degree n (see [1]):

$$V(\lambda_1 L_1 + \cdots + \lambda_m L_m) = \sum_{i_1,\ldots,i_n} V(L_{i_1}, \ldots, L_{i_n})\lambda_{i_1} \cdots \lambda_{i_n}.$$

The coefficients $V(L_{i_1}, \ldots, L_{i_n})$ are called mixed volumes of $L_{i_1}, \ldots, L_{i_n}$.

2.2. Orlicz mixed volumes

Consider convex function $\phi : (-\infty, 0) \cup (0, \infty) \to (0, \infty)$ such that $\lim_{t\to\infty} \phi(t) = \infty, \lim_{t\to 0} \phi(t) = 0$. we assume that $\mathcal{C}$ be the class of convex function $\phi : (0, \infty) \to (0, \infty)$. Let $\mathcal{C}^+$ be the class of convex and strictly increasing functions $\phi : [0, \infty) \to (0, \infty)$ such that $\lim_{t\to\infty} \phi(t) = \infty, \lim_{t\to 0} \phi(t) = 0$. It is easy to conclude from [14, pp. 23-24] that $\phi \in \mathcal{C}$ is continuous on $[0, \infty)$, and the left derivative ϕ'_l and right derivative ϕ'_r exists and ϕ'_l are positive on $(0, \infty)$.

The Orlicz combination were defined by Gardner [3] and xi [15]. Let $\alpha, \beta > 0$(not both zero) and $\phi \in \mathcal{C}^+$, the Orlicz combination $\alpha \cdot K +_\phi \beta \cdot L$ given by, for all $u \in S^{n-1}$

$$h_{\alpha\cdot K+_\phi\beta\cdot L}(u) = \inf\{t > 0 : \alpha\phi\Big(\frac{h_K(u)}{t}\Big) + \beta\phi\Big(\frac{h_L(u)}{t}\Big) \le \phi(1)\}. \tag{2.2}$$

By the definition of Orlicz combination and Orlicz mixed volume, we derive

$$\phi(1) = \alpha\phi(\frac{h_K(u)}{h_{\alpha\cdot K+_\phi\beta\cdot L}(u)}) + \beta\phi(\frac{h_K(u)}{h_{\alpha\cdot L+_\phi\beta\cdot L}(u)}). \tag{2.3}$$

$$V_\phi(K, K) = \phi(1)V(K)$$

$$V(K +_\phi \varepsilon \cdot L)\phi(1) = V_\phi(K +_\phi \varepsilon \cdot L, K) + \varepsilon V_\phi(K +_\phi \varepsilon \cdot L, L) \quad \text{for all } \varepsilon > 0. \tag{2.4}$$

If $\phi(t) = t^p$, $p \ge 1$, then the Orlicz combination reduces to the L_p combination. Xi[15] defined the Orlicz mixed volume $V_\phi(K, L)$ of $K, L \in \mathcal{K}^n_0$ by

$$V_\phi(K, L) = \frac{\phi'_l(1)}{n} \lim_{\varepsilon\to 0^+} \frac{V(K +_\phi \varepsilon \cdot L) - V(K)}{\varepsilon}$$

and obtain the following integral formula of the Orlicz mixed volume:

$$V_\phi(K, L) = \frac{1}{n}\int_{S^{n-1}} \phi(\frac{h_L(u)}{h_K(u)})h_K(u)dS(K, u).$$

Jensen' inequality: Let μ be a probability measure in a space X, let U be an open convex set in R^n, and let φ be a convex real-valued function on U. Assume that $g : X \to U$ is measurable and component-wise μ-integrable, and that $\varphi \circ g$ is μ-integrable. Let $z_0 = \int_X g(x)d\mu(x)$. Then $z_0 \in U$ and

$$\int_X \varphi(g(x))d\mu(x) \ge \varphi(\int_X g(x)d\mu(x)).$$

If ϕ is strictly convex, equality holds if and only if $g(x)$ is constant for $\mu-$almost all $x \in X.(see[4])$

3. Orlicz mixed harmonic quermassintegrals

In this section, We are ready to derive the variational formula of harmonic quermassintegrals with respect to Orlicz combination. For this aim, the following Lemma is needed.

According to the proof of lemma 5.2 and theorem 2(see[15]) , we give the following Lemma :

Lemma 3.1 Let $K, L \in \mathcal{K}_0^n, \phi \in \mathcal{C}^+, 0 < \varepsilon_0 < 1$ and $K_\varepsilon = K +_\phi \varepsilon \cdot L$. Then for $\varepsilon \in (0, \varepsilon_0]$, the family of functions $\{\frac{V^{\{j\}}(K_\varepsilon|\cdot) - V^{\{j\}}(K|\cdot)}{\varepsilon}\}$ uniformly bounded on $Gr(n, j)$.

Lemma 3.2[7] Let $K, L \in \mathcal{K}_0^n, \phi \in \mathcal{C}^+$ and $j = 1, 2, \cdots, n-1$. Then for each $\xi \in Gr(n, j)$ and $\varepsilon > 0$, we have

$$(K +_\phi \varepsilon \cdot L)|\xi = (K|\xi) +_\phi \varepsilon \cdot (L|\xi).$$

Theorem 3.3 If $K, L \in \mathcal{K}_0^n, \phi \in \mathcal{C}^+$, then for each $j = 1, 2, \cdots, n-1$,

$$\lim_{\varepsilon \to 0^+} \frac{\widehat{W}_{n-j}(K +_\phi \varepsilon \cdot L) - \widehat{W}_{n-j}(K)}{\varepsilon} = \frac{j}{\phi_l'(1)} \widehat{W}_{\phi, n-j}(K, L)^{-1} \widehat{W}_{n-j}(K)^2.$$

Proof. By the Lemma 3.1, there exist a positive constant c, such that for all $\xi \in Gr(n, j)$ and $0 < \varepsilon \leq \varepsilon_0$,

$$\frac{V^{\{j\}}(K_\varepsilon|\xi) - V^{\{j\}}(K|\xi)}{\varepsilon} < c.$$

Therefore,

$$\left| \frac{V^{\{j\}}(K_\varepsilon|\xi)^{-1} - V^{\{j\}}(K|\xi)^{-1}}{\varepsilon} \right|$$

$$= \frac{V^{\{j\}}(K_\varepsilon|\xi) - V^{\{j\}}(K|\xi)}{\varepsilon V^{\{j\}}(K_\varepsilon|\xi) V^{\{j\}}(K|\xi)} \leq \frac{c}{\min_{\xi \in Gr(n,j)} V^{\{j\}}(K|\xi)^2}.$$

Thus, we know that the family of functions $\{\frac{V^{\{j\}}(K_\varepsilon|\cdot)^{-1} - V^{\{j\}}(K|\cdot)^{-1}}{\varepsilon}\}_{0<\varepsilon\leq\varepsilon_0}$ uniformly bounded on $Gr(n, j)$ and calculating the limitation

$$\lim_{\varepsilon \to 0^+} \frac{V^{\{j\}}(K_\varepsilon|\xi)^{-1} - V^{\{j\}}(K|\xi)^{-1}}{\varepsilon} = -\frac{j}{\phi_l'(1)} V_\phi^{\{j\}}(K|\xi, L|\xi) V^{\{j\}}(K|\xi)^{-2},$$

pointwise on $Gr(n, j)$. It follows from Lebesgue dominated convergence theorem, Lemma 3.2 that

$$\begin{aligned}
&\lim_{\varepsilon \to 0^+} \frac{\widehat{W}_{n-j}(K +_\phi \varepsilon \cdot L) - \widehat{W}_{n-j}(K)}{\varepsilon} \\
&= -\frac{\omega_n}{\omega_j} [\int_{Gr(n,j)} V^{\{j\}}(K|\xi)^{-1} d\mu_j(\xi)]^{-2} \lim_{\varepsilon \to 0^+} \int_{Gr(n,j)} \frac{V^{\{j\}}((K +_\phi \varepsilon \cdot L)|\xi)^{-1} - V^{\{j\}}(K|\xi)^{-1}}{\varepsilon} d\mu_j(\xi) \\
&= \frac{\omega_n}{\omega_j} [\int_{Gr(n,j)} V^{\{j\}}(K|\xi)^{-1} d\mu_j(\xi)]^{-2} \frac{j}{\phi_l'(1)} \int_{Gr(n,j)} V_\phi^{\{j\}}(K|\xi, L|\xi) V^{\{j\}}(K|\xi)^{-2} d\mu_j(\xi) \\
&= \frac{j}{\phi_l'(1)} \widehat{W}_{n-j}(K)^2 \widehat{W}_{\phi, n-j}(K, L)^{-1},
\end{aligned}$$

as desired. □

4. Brunn-Minkowski type inequalities

In the section. We establish the Minkowski type isoperimetric inequality for Orlicz mixed affine quermassintegrals, on this basis, we obtain the Brunn-Minkowski type inequality with respect to Orlicz combination for Orlicz mixed harmonic quermassintegrals.
Theorem 4.1 Let $K, L \in \mathcal{K}_0^n, 0 < j < n, \phi \in \mathcal{C}^+$. Then

$$\widehat{W}_{\phi,n-j}(K,L)^{-1} \geq \widehat{W}_{n-j}(K)^{-1}\phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K)})^{1/j}). \tag{4.1}$$

If K and L are dilations, then equality holds in (4.1). Conversely, if ϕ is strictly convex, then K and L are dilations.

Proof. The condition on K guarantees that $\widehat{W}_{n-j}(K) > 0$. Since

$$\widehat{W}_{n-j}(K) = \frac{\omega_n}{\omega_j}[\int_{Gr(n,j)} V^{\{j\}}(K|\xi)^{-1}d\mu_j(\xi)]^{-1}.$$

$\frac{(\frac{\omega_n}{\omega_j})^{-1}V^{\{j\}}(K|\xi)^{-1}d\mu_j(\xi)}{\widehat{W}_{n-j}^{-1}(K)}$ is a probability measure on $Gr(n,j)$. It follows from Jensen's inequality and Hölder inequality that

$$\begin{aligned}
(\frac{\widehat{W}_{\phi,n-j}(K,L)}{\widehat{W}_{n-j}(K)})^{-1} &= \frac{(\frac{\omega_n}{\omega_j})^{-1}\int_{Gr(n,j)} V_\phi^{\{j\}}(K|\xi, L|\xi)V^{\{j\}}(K|\xi)^{-2}d\mu_j(\xi)}{\widehat{W}_{n-j}(K)^{-1}} \\
&\geq \phi(\frac{(\frac{\omega_n}{\omega_j})^{-1}\int_{Gr(n,j)} V^{\{j\}}(K|\xi)^{-1-1/j}V^{\{j\}}(L|\xi)^{1/j}d\mu_j(\xi)}{\widehat{W}_{n-j}(K)^{-1}}) \\
&\geq \phi(\frac{\widehat{W}_{n-j}(K)^{-(1+1/j)}\widehat{W}_{n-j}(L)^{1/j}}{\widehat{W}_{n-j}(K)^{-1}}) = \phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K)})^{1/j}).
\end{aligned}$$

If K and L are dilations, taking $K = aL, a > 0$, then by Orlicz mixed harmonic quermassintegrals and harmonic quermassintegrals, we have following equality holds,

$$[\frac{\widehat{W}_{\phi,n-j}(K,L)}{\widehat{W}_{n-j}(K)}]^{-1} = \phi(a) = \phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K)})^{1/j})$$

conversely, we will show K and L are dilations when ϕ is strictly convex. We are divided into two cases to prove.

Firstly, we suppose $j = 1$. By the define of Orlicz mixed volume, we derive

$$\frac{V_\phi^{\{1\}}(K|u, L|u)}{V^{\{1\}}(K|u)} = \phi(\frac{h_L(u)}{h_K(u)})\frac{h_K(u)}{h_K(u)+h_K(-u)} + \phi(\frac{h_L(-u)}{h_K(-u)})\frac{h_K(-u)}{h_K(u)+h_K(-u)}.$$

Note that,

$$\frac{h_L(u)}{h_K(u)}\frac{h_K(u)}{h_K(u)+h_K(-u)} + \frac{h_L(-u)}{h_K(-u)}\frac{h_K(-u)}{h_K(u)+h_K(-u)} = \frac{V^{\{1\}}(L|u)}{V^{\{1\}}(K|u)}.$$

Since equality hold in inequality (4.1), we know that $\frac{V_\phi^{\{1\}}(K|u,L|u)}{V^{\{1\}}(K|u)} = \frac{V^{\{1\}}(L|u)}{V^{\{1\}}(K|u)}$. It follows from ϕ is strictly convex that

$$\frac{h_L(u)}{h_K(u)} = \frac{h_L(-u)}{h_K(-u)}.$$

And by Hölder equality condition, we conclude, exist a positive constant $b > 0$, for any $u \in S^{n-1}$, such that $V^{\{1\}}(K|u) = bV^{\{1\}}(L|u)$. Consequently, we obtain K and L are dilations.

Now, we suppose $2 \leq j \leq n-1$. For any $\xi \in Gr(n,j)$, we have $\frac{V_\phi^{\{j\}}(K|\xi, L|\xi)}{V^{\{j\}}(K|\xi)} = \phi((\frac{V^{\{j\}}(L|\xi)}{V^{\{j\}}(K|\xi)})^{1/j})$ according to equality hold in inequality (4.1). By the Mimkowski isoperimetric inequality of Orlicz mixed volume, we have, for $\xi \in Gr(n,j)$, $K|\xi$ and $L|\xi$ are dilations of each. By Lemma 3.12(see[7]), we derive K and L are dilations for each $\xi \in Gr(n,j)$. □

In Theorem 4.1, we derive the following corollary:

Corollary 4.2 Let $K, L \in \mathcal{K}_0^n, 0 < j < n, p \geq 1$. Then

$$\widehat{W}_{p,n-j}(K,L)^{-1} \geq \widehat{W}_{n-j}(K)^{-1}(\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K)})^{p/j}. \tag{4.2}$$

When $p = 1$, the equality holds for $2 \leq j \leq n-1$ if and only if K and L are homothetic; the equality holds for $j = 1$ if and only if exist a positive constant a such that for $u \in S^{n-1}$, $V^{\{1\}}(K|u) = aV^{\{1\}}(L|u)$. When $p > 1$, the equality holds if and only if K and L are dilations.

Now, we obtain the following Brunn-Minkowski type inequality for Orlicz mixed harmonic quermassintegrals as follows:

Theorem 4.3 Let $K, L \in \mathcal{K}_0^n, 0 < j < n, \varepsilon > 0$. Then for any $\phi \in \mathcal{C}^+$,

$$\phi(1) \geq \phi((\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K +_\phi \varepsilon \cdot L)})^{1/j}) + \varepsilon\phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K +_\phi \varepsilon \cdot L)})^{1/j}). \tag{4.3}$$

If ϕ is strictly convex, equality holds in (4.3) if and only if K and L are dilations.

Proof. Let $K_\phi = K +_\phi \varepsilon \cdot L$. From Orlicz mixed harmonic quermassintegrals, Orlicz combination, the Theorem 4.1 and (2.4), it follows that

$$\begin{aligned}\phi(1) &= \frac{(\frac{\omega_n}{\omega_j})^{-1}\int_{Gr(n,j)}\phi(1)V^{\{j\}}(K_\phi|\xi)^{-1}d\mu_j(\xi)}{\widehat{W}_{n-j}^{-1}(K_\phi)} \\ &= \frac{\widehat{W}_{n-j}^{-1}(K_\phi, K) + \varepsilon\widehat{W}_{n-j}(K_\phi, L)^{-1}}{\widehat{W}_{n-j}(K_\phi)^{-1}} \geq \phi((\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}) + \varepsilon\phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}).\end{aligned}$$

The equality holds in theorem 4.1, we know that, if K and L are dilations, then equality holds in (4.3). Conversely, if ϕ is strictly convex, then K and L are dilations.

□

Using Theorem 4.3, we give the following Corollary:

Corollary 4.4 Let $K, L \in \mathcal{K}_0^n, 0 < j < n, \varepsilon > 0$. Then for any $p \geq 1$,

$$\widehat{W}_{n-j}(K +_\phi \varepsilon \cdot L)^{p/j} \geq (\widehat{W}_{n-j}(K))^{p/j} + \varepsilon(\widehat{W}_{n-j}(L))^{p/j}. \tag{4.4}$$

When $p = 1$, the equality holds for $2 \leq j \leq n-1$ if and only if K and L are homothetic; the equality holds for $j = 1$ if and only if exist a positive constant a such that for $u \in S^{n-1}$, $V^{\{1\}}(K|u) = aV^{\{1\}}(L|u)$. When $p > 1$, the equality holds if and only if K and L are dilations.

We now derive the equivalence between the inequality (4.1) and the inequality (4.3). We have proved Theorem 4.3 by Theorem 4.1. Thus, we only need to prove the inequality (4.1) by the inequality (4.3).

Proof. Let $K_\phi = K +_\phi \varepsilon \cdot L$, by (4.3), the following function

$$f(\varepsilon) = \phi((\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}) + \varepsilon\phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}) - \phi(1)$$

is non-positive, then

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \frac{f(\varepsilon)-f(0)}{\varepsilon} &= \lim_{\varepsilon\to 0^+} \frac{\phi((\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}) + \varepsilon\phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}) - \phi(1)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0^+} \frac{\phi((\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}) - \phi(1)}{\varepsilon} + \phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}) \\
&= \lim_{\varepsilon\to 0^+} \frac{\phi((\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}) - \phi(1)}{(\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K_\phi)})^{1/j} - 1} \lim_{\varepsilon\to 0^+} \frac{(\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K_\phi)})^{1/j} - 1}{\varepsilon} \\
&\quad + \phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}).
\end{aligned}$$

Let $a = (\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}$ and $a \to 1^-$ as $\varepsilon \to 0^+$, consequently,

$$\lim_{\varepsilon\to 0^+} \frac{\phi((\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}) - \phi(1)}{(\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K_\phi)})^{1/j} - 1} = \lim_{a\to 1^+} \frac{\phi(a)-\phi(1)}{a-1} = \phi_l'(1)$$

and by theorem 3.3, we have

$$\lim_{\varepsilon\to 0^+} \frac{(\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K_\phi)})^{1/j} - 1}{\varepsilon} = -\frac{1}{\phi_l'(1)}(\frac{\widehat{W}_{\phi,n-j}(K,L)}{\widehat{W}_{n-j}(K)})^{-1}.$$

Thus,

$$\lim_{\varepsilon\to 0^+} \frac{f(\varepsilon)-f(0)}{\varepsilon} = -(\frac{\widehat{W}_{\phi,n-j}(K,L)}{\widehat{W}_{n-j}(K)})^{-1} + \phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K_\phi)})^{1/j}) \le 0,$$

which finishes the proof. □

Obviously, We derive the equivalence between the inequality (4.2) and the inequality (4.4).

5. Brunn-Minkowski inequality with respect to n variables

Gardner in [3] introduced the notion of Orlicz sum: consider a young function $\phi : [0,\infty)^n \to [0,\infty)$, that is for every $u \in S^{n-1} \cap [0,\infty)^n, \phi(tu)$ is convex and strictly increasing with $\phi(0) = 0$. For $K_1, K_2, \cdots, K_n \in \mathcal{K}_0^n$, exist a unique convex body $+_\phi(K_1, K_2, \cdots, K_n) \in \mathcal{K}_0^n$, the support function defined by

$$h_{+_\phi(K_1,K_2,\cdots,K_n)}(u) = \inf\{\lambda > 0 : \phi(\frac{h_{K_1}(u)}{\lambda}, \frac{h_{K_2}(u)}{\lambda}, \cdots, \frac{h_{K_n}(u)}{\lambda}) \le 1\}, u \in R^n.$$

The convex body $+_\phi(K_1, K_2, \cdots, K_n)$ is called Orlicz sum of $K_1, K_2, \cdots, K_n$.

We obtain the Brunn-Minkowski type inequality on above Orlicz sum.

Theorem 5.1 Let $K_1, K_2, \cdots, K_n \in \mathcal{K}_0^n \in \mathcal{K}_0^n$, ϕ is $n \geq 2$ variate young functions, $j = 1, 2, \cdots, n-1$. Then following inequality hold,

$$1 \geq \phi((\frac{\widehat{W}_{n-j}(K_1)}{\widehat{W}_{n-j} +_\phi (K_1, K_2, \cdots, K_n)})^{1/j}, \cdots, (\frac{\widehat{W}_{n-j}(K_n)}{\widehat{W}_{n-j} +_\phi (K_1, K_2, \cdots, K_n)})^{1/j}).$$

If $K_1, K_2, \cdots, K_n$ are dilations, then the above equality holds . Conversely, if ϕ is strictly convex, then $K_1, K_2, \cdots, K_n$ are dilations.

Proof. Considering a probability measure ν on $Gr(n, j)$, is given by

$$d\nu(\xi) = \frac{V^{\{j\}}(K|\xi)^{-1}}{\int_{Gr(n,j) V^{\{j\}}(K|\xi)^{-1} d\mu_j(\xi)}} d\mu_j(\xi).$$

Using the same method of theory 4.1 . We have , for all $\xi \in Gr(n, j)$

$$1 \geq \phi((\frac{V^{\{j\}}(K_1|\xi)}{V^{\{j\}}(K|\xi)})^{1/j}, \cdots, (\frac{V^{\{j\}}(K_n|\xi)}{V^{\{j\}}(K|\xi)})^{1/j}),$$

here $K = +_\phi(K_1, K_2, \cdots, K_n)$.

It follows from Jensen's inequality that

$$\begin{aligned}
1 &\geq \int_{Gr(n,j)} \phi((\frac{V^{\{j\}}(K_1|\xi)}{V^{\{j\}}(K|\xi)})^{1/j}, \cdots, (\frac{V^{\{j\}}(K_n|\xi)}{V^{\{j\}}(K|\xi)})^{1/j}) d\nu(\xi) \\
&\geq \phi(\int_{Gr(n,j)} ((\frac{V^{\{j\}}(K_1|\xi)}{V^{\{j\}}(K|\xi)})^{1/j}, \cdots, (\frac{V^{\{j\}}(K_n|\xi)}{V^{\{j\}}(K|\xi)})^{1/j}) d\nu(\xi)) \\
&= \phi(\frac{\int_{Gr(n,j)} V^{\{j\}}(K_1|\xi)^{1/j} V^{\{j\}}(K|\xi)^{-1/j+1} d\mu_j(\xi)}{\int_{Gr(n,j)} V^{\{j\}}(K|\xi)^{-1} d\mu_j(\xi)}, \\
&\cdots, \frac{\int_{Gr(n,j)} V^{\{j\}}(K_n|\xi)^{1/j} V^{\{j\}}(K|\xi)^{-1/j+1} d\mu_j(\xi)}{\int_{Gr(n,j)} V^{\{j\}}(K|\xi)^{-1} d\mu_j(\xi)}) \\
&\geq \phi((\frac{\widehat{W}_{n-j}(K_1)}{\widehat{W}_{n-j}(K)})^{1/j}, \cdots, (\frac{\widehat{W}_{n-j}(K_n)}{\widehat{W}_{n-j}(K)})^{1/j}).
\end{aligned}$$

Here equality holds according to the equality holds of theorem 4.1, we obtain, if $K_1, K_2, \cdots, K_n$ are dilations, then the above equality holds . Conversely, if ϕ is strictly convex, then $K_1, K_2, \cdots, K_n$ are dilations. □

Let $\phi_1, \phi_2 \in \mathcal{C}^+$ and $\phi(t_1, t_2) = \phi_1(t_1) + \phi_2(t_2)$, Then for every $K, L \in \mathcal{K}_o^n$, we have

$$1 \geq \phi_1(\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(+_\phi(K, L)})^{1/j}) + \phi_2(\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(+_\phi(K, L)})^{1/j}).$$

More generally, taking $\phi(t_1, t_2) = \frac{\phi_1(t_1) + \varepsilon \phi_2(t_2)}{\phi(1)}$, then, for $+_\phi(K, L) = K +_\phi \varepsilon \cdot L$, we have,

$$\phi(1) \geq \phi((\frac{\widehat{W}_{n-j}(K)}{\widehat{W}_{n-j}(K +_\phi \varepsilon \cdot L)})^{1/j}) + \varepsilon \phi((\frac{\widehat{W}_{n-j}(L)}{\widehat{W}_{n-j}(K +_\phi \varepsilon \cdot L)})^{1/j}).$$

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ROUGHNESS IN $(\in_\gamma, \in_\gamma \vee q_\delta)$-FUZZY SUBSTRUCTURES OF SEMIGROUPS BASED ON SET VALUED MAPPING

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Abstract. Study of generalized roughness for fuzzy algebraic substrucures of semigroups has been initiated. Many different kinds of set valued maps are needed to preserve an algebraic substrucure while considering its lower and upper approximations. In the present paper generalized lower and upper approximations in $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals of semigroups have been investigated. An $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subset of a subsemigroup have two parts viz. lower and upper parts. Several properties of lower and upper approximations have been given for these. To conclud this paper, lower and upper approximations for $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals have been discussed in semigroups.

1. INTRODUCTION

A semigroup is an algebraic structure consisting of a nonempty set S together with an associative binary operation. Semigroups are important in many areas of mathematics, for example coding and language theory, automata theory, combinatorics and mathematical analysis. Zadeh introduced the notion of fuzzy subset in [24].

The idea of quasi coincidence of a fuzzy point with a fuzzy set which is mentioned in [17], played a vital role to generate some different types of fuzzy subgroups, called (α, β)-fuzzy subgroups, introduced by Bakat and Das (see [2]-[4]) . Fuzzy point play a vital role in the study of (α, β)-fuzzy subgroups initiated by Bhakat and Das [3], using the combined notions of *" belongingness"* and *"quasi-coincidence"* of a fuzzy point with a fuzzy set. Shabir et al. have applied this concept in semigroup [20]. Rehman and Shabir initiated the study of (α, β)-fuzzy substructures in ternary semigroups [18]. Jun introduced $(\in, \in \vee q_k)$-fuzzy subalgebras in BCK /BCI algebras [11]. Shabir et al. introduced $(\in, \in \vee q_k)$-fuzzy ideals in semigroup [21]. Rehman and Shabir initiated the study of $(\in, \in \vee q_k)$-fuzzy substructures in ternary semigroups [22]. Shabir and Ali introduced $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal in semigroup [23]. Rehman and Shabir initiated the study of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy substructures in ternary semigroups [19].

Pawlak was the first to discuss rough set with the help of equivalence relation among the elements of a set which is a key point to discuss the uncertainty [16]. There are at least two methods for the development of rough set theory, the constructive and axiomatic approaches. In rough sets, equivalence classes play an important role in the construction of both lower and upper approximations. But some times in algebraic structures, it is difficult to find equivalence relations. Many researchers have worked on this to initiate rough set without equivalence relations.

Key words and phrases. Semigroup, Generalized fuzzy roughness, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals
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Couso and Dubois in [7] initiated generalized rough set or "T-rough set" with the help of a set valued mapping, which is a more generalized rough form of the Pawlak rough set. The notion of roughness in fuzzy set introduced by Dubois and Prade in [9]. Some researchers applied this concept in [1] and [6].

Many researchers have taken interest to apply the concept of roughness in different algebraic structures (see [5],[8],[12],[15]). Hossini has applied generalized rough set in fuzzy algebraic structures (See [13],[14]). However, in the case of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy algebraic structures much attention has not been paid. Therefore it is important to study the roughness in generalized fuzzy algebraic structures such as in $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy structures.

2. Preliminaries

In this section some basic concepts of fuzzy set, fuzzy point, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy substructures of semigroup, different types of set valued homomorphism are given. Throughout this paper S will denote a semigroup unless specified otherwise

A fuzzy subset μ of S is a function $\mu : S \rightarrow [0,1]$. A fuzzy subset μ of S is called fuzzy left (right) ideal of S if $\mu(ab) \geq \mu(b)\,(\mu(ab) \geq \mu(a))$ for all $a, b \in S$. μ is fuzzy ideal of S if it is both fuzzy left and fuzzy right ideal of S.

Definition 1. [17] *A fuzzy subset μ of S of the form*

$$\mu(x) = \begin{cases} t \in (0,1] & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

Definition 2. [23] *Let μ be a fuzzy subset of S. Then μ is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S if the following condition hold:*

$$(\forall x, y \in S)\,(\forall t_1, t_2 \in (\gamma, 1])\left(x_{t_1}, y_{t_2} \in_\gamma \mu \rightarrow (xy)_{\min\{t_1,t_2\}} \in_\gamma \vee q_\delta \mu\right)$$

Theorem 1. [23] *Let μ be a fuzzy subset of S. Then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S if and only if $\mu(xy) \vee \gamma \geq \min\{\mu(x), \mu(y), \delta\}$ for all $x, y \in S$.*

Definition 3. [23] *Let μ be a fuzzy subset of S. Then μ is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S if the following condition holds:*

$$(\forall x, y \in S)\,(\forall t \in (\gamma, 1])\,(y_t \in \mu \rightarrow (xy)_t \in_\gamma \vee q_\delta \mu)$$

Theorem 2. [23] *Let μ be a fuzzy subset of a semigroup S. Then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S if and only if $\mu(ab) \vee \gamma \geq \min\{\mu(b), \delta\}$ for all $a, b \in S$.*

Definition 4. [23] *Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S. Then μ is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S if the following condition holds:*

$$(\forall x, a, y \in S)\,(\forall t \in (0,1])\,(a_t \in_\gamma \mu \rightarrow (xay)_t \in_\gamma \vee q_\delta \mu)$$

Theorem 3. [23] *Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S. Then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S if and only if $\mu(xay) \vee \gamma \geq \min\{\mu(a), \delta\}$ for all x, y and $a \in S$.*

Definition 5. [23] *Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S. Then μ is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S if the following condition holds:*

$$(\forall x, a, y \in S)\,(\forall t_1, t_2 \in (\gamma, 1])\left(x_{t_1}, y_{t_2} \in_\gamma \mu \rightarrow (xay)_{\min\{t_1,t_2\}} \in_\gamma \vee q_\delta \mu\right)$$

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Theorem 4. [23] *Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S. Then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S if and only if $\mu(xay) \vee \gamma \geq \min\{\mu(x), \mu(y), \delta\}$ for all x, y and $a \in S$.*

Definition 6. [23] *Let λ, μ be fuzzy subsets of a semigroup S, we define the fuzzy subsets $\lambda^*, (\lambda \wedge \mu)^*, (\lambda \vee \mu)^*$ of S as follows:*

$$\lambda^*(x) = (\lambda(x) \vee \gamma) \wedge \delta, (\lambda \wedge \mu)^*(x) = ((\lambda \wedge \mu)(x) \vee \gamma) \wedge \delta$$

$$\text{and } (\lambda \vee \mu)^*(x) = ((\lambda \vee \mu)(x) \vee \gamma) \wedge \delta.$$

Definition 7. *Let λ, μ be fuzzy subsets of S, we define the fuzzy subsets $\lambda^\diamond, (\lambda \wedge \mu)^\diamond$, $(\lambda \vee \mu)^\diamond$ of S as follows:*

$$\lambda^\diamond(x) = \lambda(x) \vee \delta, (\lambda \wedge \mu)^\diamond(x) = ((\lambda \wedge \mu)(x) \vee \delta)$$

$$\text{and } (\lambda \vee \mu)^\diamond(x) = ((\lambda \vee \mu)(x) \vee \delta).$$

Definition 8. *Let $T : S \to \mathcal{P}(S)$ be a set valued (SV) mapping. Then T is called an SV-homomorphism, if $T(a)T(b) \subseteq T(ab)$ for all $a, b \in S$.*

Definition 9. *Let $T : S \to \mathcal{P}(S)$ be an SV-homomorphism. Then T is called reflexive if $a \in T(a)$ for all $a \in S$. In this paper reflexive set valued homomorphism will be denoted by RSV-homomorphism.*

Definition 10. *Let $T : S \to \mathcal{P}(S)$ be a set valued (SV) mapping. Then T is called a strong set valued (SSV) homomorphism, if $T(a)T(b) = T(ab)$ for all $a, b \in S$.*

3. Generalized roughness in $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroups

This section deals with generalized roughness in fuzzy sets and the approximation of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroups. We provide an example to show that the lower approximation of an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of a semigroup S is not an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of a semigroup S under an SV-homomorphism.

Definition 11. *Let S be a semigroup and $T : S \to P(S)$ be an SV-mapping. Let μ be a fuzzy subset of S. For every $x \in S$, we define T-rough lower and T-rough upper fuzzy subsets of S by*

$$\underline{T}(\mu)(x) = \bigwedge_{a \in T(x)} \mu(a) \text{ and } \overline{T}(\mu)(x) = \bigvee_{a \in T(x)} \mu(a).$$

Proposition 1. *Let $T : S \to P(S)$ be an SV-homomorphism and μ be a fuzzy subset of S. If μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S, then $\overline{T}(\mu)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S.*

Proof. Let $a, b \in S$ and μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup of S. Let $a_{t_1}, b_{t_2} \in_\gamma \overline{T}(\mu)$. Then $\overline{T}(\mu)(a) \geq t_1 > \gamma$ and $\overline{T}(\mu)(b) \geq t_2 > \gamma$, where $t_1, t_2 \in (\gamma, 1]$. It follows that

$$\begin{aligned}
\min\{t_1,t_2,\delta\} &\leq \overline{T}(\mu)(a)\wedge\overline{T}(\mu)(b)\wedge\delta \\
&= \bigvee_{x\in T(a)}(\mu(x))\wedge\bigvee_{y\in T(b)}(\mu(y))\wedge\delta \\
&= \bigvee_{x\in T(a),y\in T(b)}(\mu(x)\wedge\mu(y)\wedge\delta) \\
&\leq \bigvee_{x\in T(a),y\in T(b)}\mu(xy)\vee\gamma=\left(\bigvee_{xy\in T(a)T(b)}\mu(xy)\right)\vee\gamma \\
&= \left(\bigvee_{z\in T(a)T(b)}\mu(z)\right)\vee\gamma \qquad (\text{where } z=xy) \\
&\leq \left(\bigvee_{z\in T(ab)}\mu(z)\right)\vee\gamma=\overline{T}(\mu)(ab)\vee\gamma.
\end{aligned}$$

This implies that $\min\{t_1,t_2,\delta\}\leq\overline{T}(\mu)(ab)\vee\gamma$.

If $\min\{t_1,t_2\}>\delta$, then $\overline{T}(\mu)(ab)>\delta$. This implies that $\overline{T}(\mu)(ab)+\min\{t_1,t_2\}>\delta+\delta=2\delta$. Which implies $(ab)_{\min\{t_1,t_2\}}\,q_\delta\overline{T}(\mu)$. If $\min\{t_1,t_2\}\leq\delta$, then $\overline{T}(\mu)(ab)\geq\min\{t_1,t_2\}>\gamma$. This implies that $(ab)_{\min\{t_1,t_2\}}\in_\gamma\overline{T}(\mu)$. Hence $(ab)_{\min\{t_1,t_2\}}\in_\gamma\vee q_\delta\overline{T}(\mu)$. Therefore $\overline{T}(\mu)$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy subsemigroup of S. □

Proposition 2. *Let $T:S\to P(S)$ be an SSV-homomorphism and μ be a fuzzy subset of S. If μ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy subsemigroup of S, then $\underline{T}(\mu)$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy subsemigroup of S.*

Proof. Let $a,b\in S$ and let μ be an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy subsemigroup of S. Let $a_{t_1},b_{t_2}\in\underline{T}(\mu)$. Then $\underline{T}(\mu)(a)\geq t_1>\gamma$ and $\underline{T}(\mu)(b)\geq t_2>\gamma$, where $t_1,t_2\in(\gamma,1]$. It follows that

$$\begin{aligned}
\{\underline{T}(\mu)(ab)\vee\gamma\} &= \bigwedge_{z\in T(ab)}\mu(z)\vee\gamma=\bigwedge_{z\in T(a)T(b)}\mu(z)\quad\vee\gamma \\
&= \bigwedge_{xy\in T(a)T(b)}\mu(xy)\vee\gamma \qquad (\text{where } z=xy) \\
&\geq \bigwedge_{x\in T(a),y\in T(b)}(\mu(x)\wedge\mu(y)\wedge\delta) \\
&= \left(\bigwedge_{x\in T(a)}(\mu(x))\right)\wedge\left(\bigwedge_{y\in T(b)}(\mu(y))\right)\wedge\delta \\
&= \underline{T}(\mu)(a)\wedge\underline{T}(\mu)(b)\wedge\delta\geq\min\{t_1,t_2,\delta\}.
\end{aligned}$$

This implies that $\{\underline{T}(\mu)(ab)\vee\gamma\}\geq\min\{t_1,t_2,\delta\}$.

If $\min\{t_1,t_2\}>\delta$, then$\underline{T}(\mu)(ab)>\delta$. This implies that $\underline{T}(\mu)(ab)+\min\{t_1,t_2\}>\delta+\delta=2\delta$. Which implies $(ab)_{\min\{t_1,t_2\}}\,q_\delta\underline{T}(\mu)$. If $\min\{t_1,t_2\}\leq\delta$, then $\underline{T}(\mu)(ab)\geq\min\{t_1,t_2\}>\gamma$. This implies that $(ab)_{\min\{t_1,t_2\}}\in_\gamma\underline{T}(\mu)$. Hence $(ab)_{\min\{t_1,t_2\}}\in_\gamma\vee q_\delta\underline{T}(\mu)$. Therefore $\underline{T}(\mu)$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy subsemigroup of S. □

Remark 1. *If T is an SV-homomorphism, then $\underline{T}(\mu)$ may not be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemigroup as seen in the following example.*

Example 1. *Consider the semigroup S with the following table:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Define an SV-mapping $T : S \to P(S)$ by $T(a) = \{a, b, c, d\}, T(b) = \{a, b, c\}$, and $T(c) = T(d) = \{a, b\}$. Then T is an SV homomorphism. Let μ be a fuzzy subset of S define by $\mu(a) = 0.5, \mu(b) = 0.6, \mu(c) = 0.6, \mu(d) = 0.2$. Also $\underline{T}(\mu)(d) = \underline{T}(\mu)(b) = \underline{T}(\mu)(c) = 0.5$ and $\underline{T}(\mu)(a) = 0.2$. Let $a_{0.2}, b_{0.3}, c_{0.36}, d_{0.15} \in \underline{T}(\mu)$. Then μ is an $(\in_{0.1}, \in_{0.1} \vee q_{0.3})$-fuzzy subsemigroup of S. But $(bb)_{\min\{0.3,0.3\}} = (a)_{0.3}$. Since $\underline{T}(\mu)(a) = 0.2 \ngeq 0.3$. This implies that $(a)_{0.3}\, \overline{\in}_{0.1} \underline{T}(\mu)$. Also $\underline{T}(\mu)(bb) + \min\{0.3, 0.3\} \ngtr 2\delta$. This implies that $(a)_{0.3}\, \overline{q}_{0.3} \underline{T}(\mu)$. Hence $\underline{T}(\mu)$ is not an $(\in_{0.1}, \in_{0.1} \vee q_{0.3})$-fuzzy subsemigroup of S.

4. Approximation of some types of fuzzy subsets

The notion of different types of fuzzy subsets related to $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsets of S is given in [23]. In this section we discuss the lower and upper approximations of these subsets.

Proposition 3. *Let $T : S \to P(S)$ be an SV-homomorphism and let λ, μ be fuzzy subsets of S. Then the following hold:*

(i) $\overline{T}(\lambda \wedge \mu)^\diamond = \overline{T}(\lambda)^\diamond \wedge \overline{T}(\mu)^\diamond$
(ii) $\overline{T}(\lambda \vee \mu)^\diamond = \overline{T}(\lambda)^\diamond \vee \overline{T}(\mu)^\diamond$.

Proof. (i) Let $x \in S$. Then

$$\begin{aligned}
\overline{T}(\lambda \wedge \mu)(x)^\diamond &= \bigvee_{z \in T(x)} (\lambda \wedge \mu)^\diamond(z) = \bigvee_{z \in T(x)} ((\lambda \wedge \mu)(z) \vee \delta) \\
&= \bigvee_{z \in T(x)} ((\lambda)(z) \vee \delta) \wedge ((\mu)(z) \vee \delta) \\
&= \bigvee_{z \in T(x)} ((\lambda)(z) \vee \delta) \wedge ((\mu)(z) \vee \delta) \\
&= \bigvee_{z \in T(x)} ((\lambda)(z) \vee \delta) \wedge \bigvee_{z \in T(x)} ((\mu)(z) \vee \delta) \\
&= \bigvee_{z \in T(x)} \lambda^\diamond(z) \wedge \bigvee_{z \in T(x)} \mu^\diamond(z) = \overline{T}(\lambda)^\diamond(x) \wedge \overline{T}(\mu)^\diamond(x).
\end{aligned}$$

(ii) The proof is similar to (i) using Definition 7. □

Proposition 4. *Let $T : S \to P(S)$ be an SV-homomorphism and let λ,μ be fuzzy subsets of S. Then the following hold:*

(i) $\underline{T}(\lambda \wedge \mu)^\diamond = \underline{T}(\lambda)^\diamond \wedge \underline{T}(\mu)^\diamond$
(ii) $\underline{T}(\lambda \vee \mu)^\diamond = \underline{T}(\lambda)^\diamond \vee \underline{T}(\mu)^\diamond$.

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Proof. (i) Let $x \in S$. Then

$$\begin{aligned}
\underline{T}(\lambda \wedge \mu)^{\diamond}(x) &= \bigwedge_{z \in T(x)} (\lambda \wedge \mu)^{\diamond}(z) = \bigwedge_{z \in T(x)} ((\lambda \wedge \mu)(z) \vee \delta) \\
&= \bigwedge_{z \in T(x)} ((\lambda)(z) \vee \delta) \wedge ((\mu)(z) \vee \delta) \\
&= \bigwedge_{z \in T(x)} ((\lambda)(z) \vee \delta) \wedge ((\mu)(z) \vee \delta) \\
&= \bigwedge_{z \in T(x)} ((\lambda)(z) \vee \delta) \wedge \bigwedge_{z \in T(x)} ((\mu)(z) \vee \delta) \\
&= \bigwedge_{z \in T(x)} \lambda^{\diamond}(z) \wedge \bigwedge_{z \in T(x)} \mu^{\diamond}(z) = \underline{T}(\lambda)^{\diamond}(x) \wedge \underline{T}(\mu)^{\diamond}(x).
\end{aligned}$$

(ii) The proof is similar to (i) using Definition 7. □

Proposition 5. *Let $T : S \rightarrow P(S)$ be an SV homomorphism and let λ, μ be fuzzy subsets of S. Then the following hold:*

(i) $\overline{T}(\lambda \wedge \mu)^{*} = \overline{T}(\lambda)^{*} \wedge \overline{T}(\mu)^{*}$
(ii) $\overline{T}(\lambda \vee \mu)^{*} = \overline{T}(\lambda)^{*} \vee \overline{T}(\mu)^{*}$.

Proof. (i) Let $x \in S$. Then

$$\begin{aligned}
\overline{T}(\lambda \wedge \mu)^{*}(x) &= \bigvee_{z \in T(x)} (\lambda \wedge \mu)^{*}(z) = \bigvee_{z \in T(x)} ((\lambda \wedge \mu)(z) \vee \gamma) \wedge \delta \\
&= \bigvee_{z \in T(x)} (((\lambda)(z) \vee \gamma) \wedge ((\mu)(z) \vee \gamma)) \wedge \delta \\
&= \bigvee_{z \in T(x)} (((\lambda)(z) \vee \gamma) \wedge \delta) \wedge (((\mu)(z) \vee \gamma) \wedge \delta) \\
&= \bigvee_{z \in T(x)} (((\lambda)(z) \vee \gamma) \wedge \delta) \wedge \bigvee_{z \in T(x)} (((\mu)(z) \vee \gamma) \wedge \delta) \\
&= \bigvee_{z \in T(x)} \lambda^{*}(z) \wedge \bigvee_{z \in T(x)} \mu^{*}(z) = \overline{T}(\lambda)^{*}(x) \wedge \overline{T}(\mu)^{*}(x).
\end{aligned}$$

(ii) The proof is similar to (i) using Definition 6. □

Proposition 6. *Let S be a semigroup and $T : S \rightarrow \mathcal{P}(S)$ be an SV-homomorphism. If λ and μ are fuzzy ideals of S, then*

$$\overline{T}(\mu \circ \lambda)^{*}(y) \leq \overline{T}(\mu)^{*}(y) \wedge \overline{T}(\lambda)^{*}(y).$$

Proof. Let μ and λ be fuzzy ideals of S. Then

$$\begin{aligned}
(\mu \circ \lambda)(y) &\leq (\mu \circ S)(y) = \bigvee_{y=ab} \{\mu(a) \wedge S(b)\} \\
&= \bigvee_{y=ab} \{\mu(a) \wedge 1\} = \mu(a) \leq \mu(ab) = \mu(y).
\end{aligned}$$

That is $(\mu \circ \delta)(y) \leq \mu(y)$. Now $((\mu \circ \delta)(y) \vee \gamma) \wedge \delta \leq (\mu(y) \vee \gamma) \wedge \delta$. This implies that $(\mu \circ \lambda)^{*}(y) \leq (\mu)^{*}(y)$. Similarly $(\mu \circ \lambda)^{*}(y) \leq (\lambda)^{*}(y)$. Hence $\overline{T}(\mu \circ \lambda)^{*}(y) \leq \overline{T}(\mu)^{*}(y)$ and $\overline{T}(\mu \circ \lambda)^{*}(y) \leq \overline{T}(\lambda)^{*}(y)$.

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Therefore $\overline{T}(\mu\circ\lambda)^*(y) \leq \left\{\overline{T}(\mu)^*(y) \wedge \overline{T}(\lambda)^*(y)\right\}.$ □

In general equality does not hold in above proposition. Following example makes the situation clear.

Example 2. *Consider the semigroup S of Example 1. Define an SV-mapping $T : S \to \mathcal{P}(S)$ by $T(a) = \{a,b,c\}, T(b) = \{b,c\}, T(c) = \{c\}$ and $T(d) = \{d\}$. Then T is an SV-homomorphism. Let λ, μ be fuzzy subsets of S defined by $\mu(a) = 0.4, \mu(b) = 0.35, \mu(c) = 0.2$ and $\mu(d) = 0.2$ also $\lambda(a) = 0.38, \lambda(b) = 0.3, \lambda(c) = 0.1 = \lambda(d)$. Then clearly λ and μ are fuzzy ideals of S. Also $(\mu\circ\lambda)^*(a) = 0.36$, $(\mu\circ\lambda)^*(b) = (\mu\circ\lambda)^*(c) = (\mu\circ\lambda)^*(d) = 0.1$. This implies that $\overline{T}(\mu\circ\lambda)^*(a) = 0.36, \overline{T}(\mu\circ\lambda)^*(b) = \overline{T}(\mu\circ\lambda)^*(c) = \overline{T}(\mu\circ\lambda)^*(d) = 0.1$. Also $\overline{T}(\mu)^*(b) = 0.35$ and $\overline{T}(\lambda)^*(b) = 0.3$. Therefore $\overline{T}(\mu\circ\lambda)^*(b) \ngeq \left\{\overline{T}(\mu)^*(b) \wedge \overline{T}(\lambda)^*(b)\right\}$, where $\gamma = 0.1$ and $\delta = 0.36$.*

However in case of an idempotent semigroup equality can be shown.

Proposition 7. *Let S be an idempotent semigroup and $T : S \to \mathcal{P}(S)$ be an SV-homomorphism. If μ and λ are fuzzy ideals of S, then*

$$\overline{T}(\mu\circ\lambda)^*(y) = \overline{T}(\mu)^*(y) \wedge \overline{T}(\lambda)^*(y).$$

Proof. From Proposition 6, it is obvious that $\overline{T}(\mu\circ\lambda)^*(y) \leq \overline{T}(\mu)^*(y) \wedge \overline{T}(\lambda)^*(y)$. For the reverse inequality, let $y \in S$. It follows that

$$\begin{aligned}
\overline{T}(\mu)^*(y) \wedge \overline{T}(\lambda)^*(y) &= \bigvee_{a\in T(y)} (\mu)^*(a) \wedge \bigvee_{b\in T(x)} (\lambda)^*(b) \\
&= \bigvee_{a\in T(y)} (((\mu)(y) \vee \gamma) \wedge \delta) \wedge \bigvee_{b\in T(y)} (((\lambda)(b) \vee \gamma) \wedge \delta) \\
&= \bigvee_{a\in T(y), b\in T(y)} (((\mu)(a) \vee \gamma) \wedge \delta) \wedge (((\lambda)(b) \vee \gamma) \wedge \delta) \\
&= \bigvee_{ab\in T(y)T(y)} (((\mu)(a) \vee \gamma) \wedge ((\lambda)(b) \vee \gamma)) \wedge \delta \\
&= \bigvee_{ab\in T(y)T(y)} (((\mu)(a) \wedge (\lambda)(b)) \vee \gamma) \wedge \delta \\
&\leq \bigvee_{z=ab\in T(yy)} (((\mu)(a) \wedge (\lambda)(b)) \vee \gamma) \wedge \delta \\
&= \bigvee_{z\in T(y)} \left(\bigvee_{z=ab} ((\mu)(a) \wedge (\lambda)(b)) \vee \gamma \right) \wedge \delta \\
&= \bigvee_{z\in T(y)} ((\mu\circ\lambda)(z) \vee \gamma) \wedge \delta = \\
&= \bigvee_{z\in T(y)} (\mu\circ\lambda)^*(z) = \overline{T}(\mu\circ\lambda)^*(y).
\end{aligned}$$

This implies that $\overline{T}(\mu)^*(y) \wedge \overline{T}(\lambda)^*(y) \leq \overline{T}(\mu\circ\lambda)^*(y)$. Hence $\overline{T}(\mu\circ\lambda)^*(y) = \overline{T}(\mu)^*(y) \wedge \overline{T}(\lambda)^*(y)$. □

Proposition 8. *Let $T : S \to P(S)$ be an SV-homomorphism and let λ,μ be fuzzy subsets of S. Then the following hold:*

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(i) $\underline{T}(\lambda \wedge \mu)^* = \underline{T}(\lambda)^* \wedge \underline{T}(\mu)^*$
(ii) $\underline{T}(\lambda \vee \mu)^* = \underline{T}(\lambda)^* \vee \underline{T}(\mu)^*$.

Proof. (i) Let $x \in S$. Then

$$\begin{aligned}\underline{T}(\lambda \wedge \mu)^*(x) &= \bigwedge_{z \in T(x)} (\lambda \wedge \mu)^*(z) = \bigwedge_{z \in T(x)} ((\lambda \wedge \mu)(z) \vee \gamma) \wedge \delta \\ &= \bigwedge_{z \in T(x)} (((\lambda)(z) \vee \gamma) \wedge ((\mu)(z) \vee \gamma)) \wedge \delta \\ &= \bigwedge_{z \in T(x)} (((\lambda)(z) \vee \gamma) \wedge \delta) \wedge (((\mu)(z) \vee \gamma) \wedge \delta) \\ &= \bigwedge_{z \in T(x)} (((\lambda)(z) \vee \gamma) \wedge \delta) \wedge \bigwedge_{z \in T(x)} (((\mu)(z) \vee \gamma) \wedge \delta) \\ &= \bigwedge_{z \in T(x)} \lambda^*(z) \wedge \bigwedge_{z \in T(x)} \mu^*(z) = \underline{T}(\lambda)^*(x) \wedge \underline{T}(\mu)^*(x).\end{aligned}$$

(ii) The proof is similar to (i) using Definition 6. □

Proposition 9. *Let S be an idempotent semigroup and $T : S \to \mathcal{P}(S)$ be an SV-homomorphism. If μ and λ are fuzzy ideals of S, then*

$$\underline{T}(\mu \circ \lambda)^*(y) = \left\{\underline{T}(\mu)^*(y) \wedge \underline{T}(\lambda)^*(y)\right\}.$$

Proof. The proof is similar to the proof of Proposition 7. □

5. Upper and lower approximations of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals of semigroups

In this section some properties of upper an lower approximation for fuzzy ideals of semigroups are studied.

Proposition 10. *Let $T : S \to P(S)$ be an RSV-homomorphism. If μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S, then $\overline{T}(\mu)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S.*

Proof. Let $a, b \in S$ and let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S. Let $b_t \in_\gamma \overline{T}(\mu)$. Then $\overline{T}(\mu)(b) \geq t > \gamma$, where $t \in (\gamma, 1]$. It follows that

$$\begin{aligned}\min\{t, \delta\} &\leq \overline{T}(\mu)(b) \wedge \delta = \bigvee_{y \in T(b)} (\mu(y)) \wedge \delta \\ &= \bigvee_{y \in T(b)} (\mu(y) \wedge \delta) \leq \bigvee_{a \in T(a), y \in T(b)} \mu(ay) \vee \gamma \\ &= \left(\bigvee_{ay \in T(a)T(b)} \mu(ay)\right) \vee \gamma = \left(\bigvee_{z \in T(a)T(b)} \mu(z)\right) \vee \gamma \quad (z = ay) \\ &\leq \left(\bigvee_{z \in T(ab)} \mu(z)\right) \vee \gamma = \overline{T}(\mu)(ab) \vee \gamma.\end{aligned}$$

This implies that $\min\{t, \delta\} \leq \left\{\overline{T}(\mu)(ab) \vee \gamma\right\}$.

Hence by Theorem 2, $\overline{T}(\mu)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S. □

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Proposition 11. *Let $T : S \to P(S)$ be an SSV-homomorphism and and μ be a fuzzy subset of S. If μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S, then $\underline{T}(\mu)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S.*

Proof. Let $a, b \in S$ and let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S. Let $b_t \in \underline{T}(\mu)$. Then $\underline{T}(\mu)(b) \geq t > \gamma$, where $t \in (\gamma, 1]$. It follows that

$$\begin{aligned}
\underline{T}(\mu)(ab) \vee \gamma &= \bigwedge_{z \in T(ab)} \mu(z) \vee \gamma = \bigwedge_{z \in T(a)T(b)} \mu(z) \vee \gamma \\
&= \bigwedge_{xy \in T(a)T(b)} \mu(xy) \vee \gamma \quad \text{(where } z = xy\text{)} \\
&\geq \bigwedge_{y \in T(b)} (\mu(y) \wedge \delta) = \left(\bigwedge_{y \in T(b)} (\mu(y)) \right) \wedge \delta \\
&= \underline{T}(\mu)(b) \wedge \delta \geq \min\{t, \delta\}.
\end{aligned}$$

This implies that $\underline{T}(\mu)(ab) \vee \gamma \geq \min\{t, \delta\}$. Hence by Theorem 2, $\underline{T}(\mu)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S. □

Proposition 12. *Let $T : S \to P(S)$ be an RSV-homomorphism and μ be a fuzzy subset of S. If μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S, then $\overline{T}(\mu)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S.*

Proof. It is straightforward. □

Proposition 13. *Let $T : S \to P(S)$ be an SSV-homomorphism and μ be a fuzzy subset of S. If μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S, then $\underline{T}(\mu)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideal of S.*

Proof. It is straightforward. □

Proposition 14. *Let $T : S \to P(S)$ be an RSV-homomorphism and μ be a fuzzy subset of S. If μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S, then $\overline{T}(\mu)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S.*

Proof. Let $a, x, y \in S$ and let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S. Let $x_{t_s}, y_{t_p} \in_\gamma \overline{T}(\mu)$. Then $\overline{T}(\mu)(x) \geq t_s > \gamma$ and $\overline{T}(\mu)(y) \geq t_p > \gamma$, where $t_s, t_p \in (\gamma, 1]$. It follows that

$$\begin{aligned}
\min\{t_s, t_p, \delta\} &\leq \left\{ \overline{T}(\mu)(x) \wedge \overline{T}(\mu)(y) \wedge \delta \right\} = \bigvee_{b \in T(x)} \mu(b) \wedge \bigvee_{d \in T(y)} \mu(d) \wedge \delta \\
&= \bigvee_{b \in T(x), d \in T(y)} (\mu(b) \wedge \mu(d) \wedge \delta) \leq \bigvee_{b \in T(x), a \in T(a), d \in T(y)} \mu(bad) \vee \gamma \\
&= \bigvee_{ba \in T(x)T(a), d \in T(y)} \mu(bad) \vee \gamma \leq \bigvee_{bc \in T(xa), d \in T(y)} \mu(bad) \vee \gamma \\
&= \bigvee_{(ba)d \in T(xa)T(y)} \mu(bad) \vee \gamma \leq \bigvee_{(ba)d \in T((xa)y)} \mu(bad) \vee \gamma \\
&= \bigvee_{z \in T((xa)y)} \mu(z) \vee \gamma \quad \text{(where } z = bad\text{)} \\
&= \left\{ \overline{T}(\mu)(xay) \vee \gamma \right\}.
\end{aligned}$$

This implies that $\min\{t_s, t_p, \delta\} \leq \overline{T}(\mu)((xa)y) \vee \gamma$.

Hence by Theorem 4, $\overline{T}(\mu)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-deal of S. □

Proposition 15. *Let $T : S \to P(S)$ be an SSV-homomorphism and μ be a fuzzy subset of S. If μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S, then $\underline{T}(\mu)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S.*

Proof. Let $a, x, y \in S$ and let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S. Let $x_{t_s}, y_{t_p} \in_\gamma \underline{T}(\mu)$. Then $\underline{T}(\mu)(x) \geq t_s > \gamma$ and $\underline{T}(\mu)(y) \geq t_p > \gamma$, where $t_s, t_p \in (\gamma, 1]$. It follows that

$$\begin{aligned}
\underline{T}(\mu)((xa)y) \vee \gamma &= \bigwedge_{z \in T((xa)y)} \mu(z) \vee \gamma = \bigwedge_{z \in T(xa)T(y)} \mu(z) \vee \gamma \\
&= \bigwedge_{(bc)d \in T((xa)y)} \mu((bc)d) \vee \gamma \quad (\text{where } \ z = (bc)d) \\
&= \bigwedge_{bc \in T(xa), d \in T(y)} \mu((bc)d) \vee \gamma = \bigwedge_{bc \in T(x)T(a), d \in T(y)} \mu((bc)d) \vee \gamma \\
&= \bigwedge_{b \in T(x), c \in T(a), d \in T(y)} \mu((bc)d) \vee \gamma \\
&\geq \bigwedge_{b \in T(x), d \in T(y)} (\mu(b) \wedge \mu(d) \wedge \delta) \\
&= \bigwedge_{b \in T(x)} \mu(b) \wedge \bigwedge_{d \in T(y)} \mu(d) \wedge \delta = \underline{T}(\mu)(x) \wedge \underline{T}(\mu)(y) \wedge \delta.
\end{aligned}$$

This implies that $\underline{T}(\mu)((xa)y) \vee \gamma \geq \min\{t_1, t_2, \delta\}$.

Hence by Theorem 4, $\underline{T}(\mu)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S. □

Conclusion; Associative algebras are being studied all over the globe, in particular semigroups have attracted many authors and researchers. The $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy algebraic substructures are generalizations of fuzzy algebraic substructures and $(\in, \in \vee q_k)$-fuzzy algebraic substructures. In this paper, generalized roughness have been studied for $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy algebraic substructures of semigroups. It is seen that, in order to preserve a particular algebraic substructure in case of its approximations, many types of set valued homomorphisms are required. This aspect of roughness study in semigroups makes this study more interesting.

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Equivalence between some iterations in $CAT(0)$ spaces

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Abstract. We obtain some equivalence conditions for the convergence of iterative sequences for set-valued contraction mapping in $CAT(0)$ spaces.

1. Introduction

Let (X, d) be a metric space. One of the most interesting aspects of metric fixed point theory is to extend a linear version of known result to the nonlinear case in metric spaces. To achieve this, Takahashi [32] introduced a convex structure in a metric space (X, d). A mapping $W : X \times X \times [0, 1] \to X$ is a *convex structure* in X if

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

for all $x, y \in X$ and $\lambda \in [0, 1]$. A metric space together with a convex structure W is known as a convex metric space. A nonempty subset K of a convex metric space is said to be *convex* if

$$W(x, y, \lambda) \in K$$

for all $x, y \in K$ and $\lambda \in [0, 1]$. In fact, every normed space and its convex subsets are convex metric spaces but the converse is not true, in general (see, [32]).

Example 1.1. ([15, 16]) Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\ ,\ x_2 > 0\}$. For all $x = (x_1, x_2), y = (y_1, y_2) \in X$ and $\lambda \in [0, 1]$. We define a mapping $W : X \times X \times [0, 1] \to X$ by

$$W(x, y, \lambda) = \left(\lambda x_1 + (1 - \lambda)y_1, \frac{\lambda x_1 x_2 + (1 - \lambda)y_1 y_2}{\lambda x_1 + (1 - \lambda)y_1}\right)$$

and define a metric $d : X \times X \to [0, \infty)$ by

$$d(x, y) = |x_1 - y_1| + |x_1 x_2 - y_1 y_2|.$$

Then we can show that (X, d, W) is a convex metric space but not a normed linear space.

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A metric space X is a $CAT(0)$ space. This term is due to M. Gromov [10] and it is an acronym for E. Cartan, A.D. Aleksandrov and V.A. Toponogov. If it is geodesically connected, and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane(see, *e.g.*, [2], p.159). It is well known that any complete, simply connected Riemannian manifold nonpositive sectional curvature is a $CAT(0)$ space. The precise definition is given below. For a thorough discussion of these spaces and of the fundamental role they play in various branches of mathematics, see Bridson and Haefliger [2] or Burago *et al.* [1].

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a mapping c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic* (or, *metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ is a geodesic metric space (X, d) consists of three points $x_1, x_2, x_3 \in X$ (the *vertices* of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\triangle}(x_1, x_2, x_3) = \triangle(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists(see, [2]).

A geodesic metric space is said to be a $CAT(0)$ *space* if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

> Let $\triangle$ be a geodesic triangle in X and let $\bar{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ *inequality* if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$,
>
> $$d(x, y) \leq d(\bar{x}, \bar{y}).$$

Complete $CAT(0)$ spaces are often called *Hadamard spaces*(see [22]). If x, y_1, y_2 are points of a $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, which we will denote by $\frac{y_1 \oplus y_2}{2}$, then the $CAT(0)$ inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2).$$

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This inequality is the (CN) inequality of Bruhat and Tits [3]. In fact, a geodesic space is a $CAT(0)$ space if and only if satisfies the (CN) inequality (*cf.* [2], p.163). The above inequality has been extended by Khamsi and Kirk [12] as

$$\begin{aligned} &d^2(z, \alpha x \oplus (1-\alpha)y) \\ &\le \alpha d^2(z,x) + (1-\alpha)d^2(z,y) - \alpha(1-\alpha)d^2(x,y), \end{aligned} \tag{CN*}$$

for any $\alpha \in [0,1]$ and $x, y, z \in X$. The inequality (CN*) also appeared in [5].

Let us recall that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality(see, [2], p.163). Moreover, if X is a $CAT(0)$ metric space and $x, y \in X$, then for any $\alpha \in [0,1]$, there exists a unique point $\alpha x \oplus (1-\alpha)y \in [x,y]$ such that

$$d(z, \alpha x \oplus (1-\alpha)y) \le \alpha d(z,x) + (1-\alpha)d(z,y)$$

for any $z \in X$ and $[x,y] = \{\alpha x \oplus (1-\alpha)y : \alpha \in [0,1]\}$. In view of the above inequality, $CAT(0)$ space have Takahashi's convex structure $W(x,y,\alpha) = \alpha x \oplus (1-\alpha)y$. It is easy to see that for any $x, y \in X$ and $\lambda \in [0,1]$,

$$\begin{aligned} d(x, (1-\lambda)x \oplus \lambda y) &= \lambda d(x,y), \\ d(y, (1-\lambda)x \oplus \lambda y) &= (1-\lambda)d(x,y). \end{aligned}$$

As a consequence,

$$\begin{aligned} 1 \cdot x \oplus 0 \cdot y &= x, \\ (1-\lambda)x \oplus \lambda x &= \lambda x \oplus (1-\lambda)x = x. \end{aligned} \tag{1.1}$$

Moreover, a subset K of $CAT(0)$ space X is convex if for any $x, y \in K$, we have $[x,y] \subset K$.

2. Preliminaries

Let D be a nonempty subset of a $CAT(0)$ space X. We shall denote by $CB(D)$ the family of nonempty bounded closed subset of D. Let $H(\cdot,\cdot)$ be the Hausdorff metric on $CB(D)$, *i.e.*,

$$H(A,B) = \max\left\{\sup_{a\in A} dist(a,B), \sup_{b\in B} dist(b,A)\right\}, \quad A, B \in CB(D),$$

where $dist(a,B) = \inf\{d(a,b) : b \in B\}$ is the distance from the point a to the set B.

A multivalued mapping $T : D \to CB(D)$ is said to be a *contraction* if there exists a constant $k \in (0,1)$ such that

$$H(Tx, Ty) \le k \cdot d(x,y), \quad \forall\, x, y \in D.$$

A point x is called a *fixed point* of any mapping T if $x \in Tx$. We denote by $F(T)$ the set of all fixed points of T.

Let X be a $CAT(0)$ space, and let $\{x_n\}$ be a bounded sequence in X, for $x \in X$ we let

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known that in a $CAT(0)$ space asymptotic center $A(\{x_n\})$ consists of exactly one point(see, *e.g.*, [6], Proposition 7).

Definition 2.1. ([23]) A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to $\triangle$-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case one must write

$$x_n \xrightarrow{\triangle} x \quad \text{or} \quad \triangle - \lim_{n\to\infty} x_n = x$$

and call x the $\triangle$-limit of $\{x_n\}$.

Remark 2.1. In a $CAT(0)$ space X, strong convergence implies $\triangle$-convergence.

Lemma 2.1. ([28]) *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multivalued mapping. Then for any given $\varepsilon > 0$ and for any given $x, y \in X$, $u \in Tx$, there exists $v \in Ty$ such that*

$$d(u, v) \leq (1 + \varepsilon)H(Tx, Ty)$$

where $H(\cdot, \cdot)$ is the Hausdorff metric on $CB(X)$.

Definition 2.2. Let D be a nonempty convex subset of a $CAT(0)$ space X, $T : D \to CB(D)$ be a multivalued mapping. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying some conditions.
(1) The sequence of Picard iterates (*cf.*, [30]) is defined by $w_0 \in D$,

$$w_{n+1} = \nu_n, \tag{$\mathbb{P}$}$$

where $\nu_n \in Tw_n$ such that

$$d(\nu_{n+1}, \nu_n) \leq (1 + \varepsilon)H(Tw_{n+1}, Tw_n).$$

(2) The sequence of Mann iterates (*cf.*, [27]) is defined by $u_0 \in D$,

$$u_{n+1} = (1 - \alpha_n)u_n \oplus \alpha_n\theta_n, \tag{$\mathbb{M}$}$$

where $\theta_n \in Tu_n$ such that

$$d(\theta_{n+1}, \theta_n) \leq (1 + \varepsilon)H(Tu_{n+1}, Tu_n).$$

(3) The sequence of Ishikawa iterates (*cf.*, [11]) is defined by $r_0 \in D$,

$$\begin{aligned} s_n &= (1-\beta_n)r_n \oplus \beta_n\delta_n, \\ r_{n+1} &= (1-\alpha_n)r_n \oplus \alpha_n\sigma_n, \end{aligned} \tag{$\mathbb{I}$}$$

where $\delta_n \in Tr_n$ and $\sigma_n \in Ts_n$ such that

$$\begin{aligned} d(\delta_{n+1}, \delta_n) &\le (1+\varepsilon)H(Tr_{n+1}, Tr_n), \\ d(\sigma_{n+1}, \sigma_n) &\le (1+\varepsilon)H(Ts_{n+1}, Ts_n). \end{aligned}$$

(4) The sequence of three-step iterates (*cf.*, [13, 14]) is defined by $x_0 \in D$,

$$\begin{aligned} z_n &= (1-\gamma_n)x_n \oplus \gamma_n\mu_n, \\ y_n &= (1-\beta_n)x_n \oplus \beta_n\xi_n, \\ x_{n+1} &= (1-\alpha_n)x_n \oplus \alpha_n\eta_n, \end{aligned} \tag{$\mathbb{TH}$}$$

where $\mu_n \in Tx_n$, $\xi_n \in Tz_n$ and $\eta_n \in Ty_n$ such that

$$\begin{aligned} d(\mu_{n+1}, \mu_n) &\le (1+\varepsilon)H(Tx_{n+1}, Tx_n), \\ d(\xi_{n+1}, \xi_n) &\le (1+\varepsilon)H(Tz_{n+1}, Tz_n), \\ d(\eta_{n+1}, \eta_n) &\le (1+\varepsilon)H(Ty_{n+1}, Ty_n). \end{aligned}$$

Another iteration processes and other some results in $CAT(0)$ space have been studied extensively by various authors(see *e.g.* [4, 9, 17, 24, 26, 31]).

Lemma 2.2. ([7]) *Let $\{a_n\}$ be recursively generated by*

$$a_{n+1} = (1-t_n)a_n + b_n^2$$

with $n \ge 1$, $a_n \ge 0$, $\{t_n\} \subseteq [0,1]$ and

$$\sum_{n=1}^{\infty} b_n^2 < \infty, \quad \sum_{n=1}^{\infty} t_n = \infty.$$

Then

$$\lim_{n\to\infty} a_n = 0.$$

3. Main theorems

Theorem 3.1. *Let (X,d) be a $CAT(0)$ space and D be a nonempty convex subset of X. Let $T : D \to CB(D)$ be a multivalued contraction mapping with $k < \frac{1}{1+\varepsilon}$ and $F(T) \ne \emptyset$ satisfying $Tp = \{p\}$ for any fixed point $p \in F(T)$. Let a constant L satisfying $\sup_{w\in Tx, x\in D} d(p,w) \le L$, for all $x \in D$. Let $\{w_n\}$ and $\{x_n\}$ be the Picard and three step iterative sequence defined by $(\mathbb{P})$ and $(\mathbb{TH})$ respectively and satisfying the following conditions:*

(i) $\alpha_n, \beta_n, \gamma_n \in [0,1], \quad \forall\, n \ge 0$;

(ii) $\lim_{n\to\infty} \beta_n = 0$;
(iii) $\sum_{n=0}^{\infty} \alpha_n\beta_n < \infty, \quad \sum_{n=0}^{\infty}(1-\alpha_n) = \infty$.

If $w_0 = x_0$, then the following statements are equivalent:
(1) *the Picard iterative sequence $\{w_n\}$ $\triangle$-converegs to $x^* \in F(T)$;*
(2) *the three step iterative sequence $\{x_n\}$ $\triangle$-converegs to $x^* \in F(T)$.*

Furthermore, x^* is the unique fixed point of T.

Proof. From Nadler [28], there exists a fixed point $x^* \in F(T)$. Put

$$M' = L + d(p, x_0).$$

From the contractive of T, we have

$$\begin{aligned}
d^2(x_{n+1}, p) &= d^2((1-\alpha_n)x_n \oplus \alpha_n\eta_n, p) \\
&\le (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(\eta_n, p) - (1-\alpha_n)\alpha_n d^2(_n, \eta_n) \\
&\le (1-\alpha_n)d^2(x_n, p) + \alpha_n(H(Ty_n, Tp))^2 \\
&\le (1-\alpha_n)d^2(x_n, p) + \alpha_n k^2 d^2(y_n, p) \\
&\le (1-\alpha_n)d^2(x_n, p) + \alpha_n k^2(d^2((1-\beta_n)x_n \oplus \beta_n\xi_n, p)) \\
&\le (1-\alpha_n)d^2(x_n, p) \\
&\quad + \alpha_n k^2((1-\beta_n)d^2(x_n, p) + \beta_n d^2(\xi_n, p) - \beta_n(1-\beta_n)d^2(x_n, \xi_n)) \\
&\le (1-\alpha_n)d^2(x_n, p) + \alpha_n \cdot k^2(1-\beta_n)d^2(x_n, p) \\
&\quad + \alpha_n\beta_n \cdot k^4 \cdot d^2(z_n, p) - \alpha_n\beta_n(1-\beta_n)k^2 \cdot d^2(x_n, \xi_n) \\
&\le (1-\alpha_n)d^2(x_n, p) + \alpha_n(1-\beta_n) \cdot k^2 d^2(x_n, p) \\
&\quad + \alpha_n\beta_n \cdot k^4((1-\gamma_n)d^2(x_n, p) + \gamma_n d^2(\mu_n, p) - (1-\gamma_n)\gamma_n d^2(x_n, \mu_n)) \\
&\quad - \alpha_n\beta_n(1-\beta_n)k^2 \cdot d^2(x_n, \xi_n) \\
&\le d^2(x_n, p) - \alpha_n\beta_n\gamma_n(1-\gamma_n) \cdot k^4 \cdot d^2(x_n, \mu_n),
\end{aligned}$$

for $p \in F(T)$. This implies

$$0 \le \alpha_n\beta_n\gamma_n(1-\gamma_n)k^4 d^2(x_n, \mu_n) \le d^2(x_n, p) - d^2(x_{n+1}, p).$$

Therefore, we have

$$d(x_{n+1}, p) \le d(x_n, p).$$

By induction, it is easy to see that

$$\sup_{n\ge 0}\{d(p, \mu_n), d(p, \eta_n), d(p, \xi_n), d(p, x_n), d(p, y_n), d(p, z_n)\} \le M',$$

for $\mu_n \in Tx_n$, $\eta_n \in Ty_n$ and $\xi_n \in Tz_n$, $n \ge 0$. By hypothesis, let

$$M'' = d(p, w_0) + d(p, w_1) < \infty, \quad \forall\, p \in F(T).$$

Put

$$M = \max\{M', M''\}.$$

From $\{w_n\}$ be the Picard iterative sequence defined by ($\mathbb{P}$), we have

$$\begin{aligned} d(\nu_n, \nu_{n+1}) &\le (1+\varepsilon)H(Tw_n, Tw_{n-1}) \\ &\le (1+\varepsilon)k \cdot d(w_n, w_{n-1}) \\ &= (1+\varepsilon)k \cdot d(\nu_{n-1}, \nu_{n-2}) \\ &\le (1+\varepsilon)k(1+\varepsilon)H(Tw_{n-1}, Tw_{n-2}) \\ &\le ((1+\varepsilon)k)^2 d(w_{n-1}, w_{n-2}) \\ &\vdots \\ &\le ((1+\varepsilon)k)^n d(w_1, w_0) \\ &\le ((1+\varepsilon)k)^n M \end{aligned} \tag{3.1}$$

for any given $\varepsilon > 0$. From $\{x_n\}$ be the three step iterative sequence defined by ($\mathbb{TH}$) and (3.1), for each $n \ge 0$

$$\begin{aligned} d(x_{n+1}, w_{n+1}) &= d((1-\alpha_n)x_n \oplus \alpha_n \eta_n, \nu_n) \\ &\le (1-\alpha_n)d(x_n, \nu_n) + \alpha_n \cdot d(\eta_n, \nu_n) \\ &\le (1-\alpha_n)\{d(x_n, w_n) + d(w_n, \nu_n)\} \\ &\quad + \alpha_n k \cdot d(y_n, w_n) \\ &\le (1-\alpha_n)\{d(x_n, w_n) + ((1+\varepsilon)k)^n M\} \\ &\quad + \alpha_n k \cdot d(y_n, w_n), \end{aligned} \tag{3.2}$$

$$\begin{aligned} d(y_n, w_n) &= d((1-\beta_n)x_n \oplus \beta_n \xi_n, w_n) \\ &\le (1-\beta_n)d(x_n, w_n) + \beta_n \cdot d(\xi_n, \nu_{n-1}) \\ &\le (1-\beta_n)d(x_n, w_n) \\ &\quad + \beta_n(d(\xi_n, , \nu_n) + d(\nu_n, \nu_{n-1})) \\ &\le (1-\beta_n)d(x_n, w_n) \\ &\quad + \beta_n k \cdot d(z_n, w_n) + \beta_n((1+\varepsilon)k)^n M \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} d(z_n, w_n) &= d((1-\gamma_n)x_n \oplus \gamma_n \mu_n, w_n) \\ &\le (1-\gamma_n)d(x_n, w_n) + \gamma_n \cdot d(\mu_n, \nu_{n-1}) \\ &\le (1-\gamma_n)d(x_n, w_n) \\ &\quad + \gamma_n k\{d(x_n, w_n) + d(w_n, w_{n-1})\} \\ &\le (1-\gamma_n)d(x_n, w_n) \\ &\quad + \gamma_n k\{d(x_n, w_n) + ((1+\varepsilon)k)^{n-1} M\}. \end{aligned} \tag{3.4}$$

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Substituting (3.4) into (3.3), we get

$$\begin{aligned}
&d(y_n, w_n)\\
&\le (1-\beta_n)d(x_n, w_n)\\
&\quad + \beta_n k\Big[(1-\gamma_n)d(x_n, w_n) + \gamma_n k\{d(x_n, w_n) + ((1+\varepsilon)k)^{n-1}M\}\Big]\\
&\quad + \beta_n((1+\varepsilon)k)^n M\\
&\le (1-\beta_n)d(x_n, w_n) + \beta_n((1+\varepsilon)k)^n M\\
&\quad + \beta_n k\{(1-\gamma_n(1-k))d(x_n, w_n) + \gamma_n((1+\varepsilon)k)^n M\}.
\end{aligned} \tag{3.5}$$

Combining (3.5) and (3.2), we can obtain

$$\begin{aligned}
&d(x_{n+1}, w_{n+1})\\
&\le (1-\alpha_n)\{d(x_n, w_n) + ((1+\varepsilon)k)^n M\}\\
&\quad + \alpha_n k\Big[(1-\beta_n)d(x_n, w_n) + \beta_n((1+\varepsilon)k)^n M\\
&\quad + \beta_n k\Big\{(1-\gamma_n(1-k))d(x_n, w_n) + \gamma_n((1+\varepsilon)k)^n M\Big\}\Big]\\
&= (1-\alpha_n)d(x_n, w_n) + (1-\alpha_n)((1+\varepsilon)k)^n M\\
&\quad + \alpha_n k(1-\beta_n)d(x_n, w_n) + \alpha_n\beta_n k((1+\varepsilon)k)^n M\\
&\quad + \alpha_n\beta_n k^2\Big\{(1-\gamma_n(1-k))d(x_n, w_n) + \gamma_n((1+\varepsilon)k)^n M\Big\}\\
&= \Big[1-\alpha_n + \alpha_n(1-\beta_n)k + \alpha_n\beta_n k^2(1-\gamma_n(1-k))\Big]d(x_n, w_n)\\
&\quad + (1-\alpha_n)((1+\varepsilon)k)^n M + \alpha_n\beta_n k(1+k\gamma_n)((1+\varepsilon)k)^n M\\
&\le (1-\alpha_n(1-k))d(x_n, w_n)\\
&\quad + \{(1-\alpha_n) + \alpha_n\beta_n(1+k\gamma_n)\}((1+\varepsilon)k)^n M.
\end{aligned} \tag{3.6}$$

Take

$$a_n = d(x_n, w_n), \quad t_n = \alpha_n(1-k)$$

and

$$b_n^2 = \{(1-\alpha_n) + \alpha_n\beta_n(1+k\gamma_n)\}((1+\varepsilon)k)^n M$$

in (3.6). Since $(1+\varepsilon)k < 1$, $\sum_{n=0}^{\infty}\alpha_n\beta_n < \infty$ and $\sum_{n=0}^{\infty}(1-\alpha_n) < \infty$, we have

$$\sum_{n=0}^{\infty} t_n = \infty, \quad \sum_{n=0}^{\infty} b_n^2 < \infty.$$

By Lemma 2.2, we know that

$$\triangle - \lim_{n\to\infty} d(x_n, w_n) = 0.$$

If $w_n \overset{\triangle}{\longrightarrow} x^* \in F(T)$ as $n \to \infty$, by Definition 2.1, we have

$$d(x_{n_k}, x^*) \leq d(x_{n_k}, w_{n_k}) + d(w_{n_k}, x^*) \overset{\triangle}{\longrightarrow} 0$$

as $n \to \infty$. If $x_n \overset{\triangle}{\longrightarrow} x^* \in F(T)$ as $n \to \infty$, we have

$$d(w_{n_k}, x^*) \leq d(w_{n_k}, x_{n_k}) + d(x_{n_k}, x^*) \overset{\triangle}{\longrightarrow} 0$$

as $n \to \infty$. Therefore, the equivalence between the statement (1) and (2) was proved. Finally, we prove that $x^* \in X$ is the unique fixed point of T. In fact, let $x^*, y^* \in X$ be two fixed points of T. Since T is a multivalued contraction with constant $0 < k < 1$, we have

$$\begin{aligned} d(x^*, y^*) &\leq (1+\varepsilon)H(Tx^*, Ty^*) \\ &\leq (1+\varepsilon)k \cdot d(x^*, y^*). \end{aligned}$$

Since ε is arbitrary, this implies that

$$d(x^*, y^*) = 0,$$

i.e.,

$$x^* = y^*.$$

This completes the proof. □

If $\gamma_n = 0$ in $(\mathbb{TH})$, then it reduces to $(\mathbb{I})$. So we can easily prove the following corollary.

Corollary 3.1. *Let (X, d) be a $CAT(0)$ space and D be a nonempty convex subset of X. Let $T : D \to CB(D)$ be a multivalued contraction mapping with $k < \frac{1}{1+\varepsilon}$ and $F(T) \neq \emptyset$ satisfying $Tp = \{p\}$ for any fixed point $p \in F(T)$. Let a constant L satisfying $\sup_{w \in Tx, x \in D} d(p, w) \leq L$, for all $x \in D$. Let $\{w_n\}$ and $\{r_n\}$ be the Picard and Ishikawa iterative sequence defined by $(\mathbb{P})$ and $(\mathbb{I})$ respectively and satisfying the following conditions:*

(i) $\alpha_n, \beta_n \in [0, 1], \quad \forall\, n \geq 0$;
(ii) $\lim_{n\to\infty} \beta_n = 0$;
(iii) $\sum_{n=0}^{\infty} \alpha_n \beta_n < \infty, \quad \sum_{n=0}^{\infty} (1 - \alpha_n) = \infty$.

If $w_0 = r_0$, then the following statements are equivalent:

(1) *the Picard iterative sequence $\{w_n\}$ $\triangle$-converegs to $x^* \in F(T)$;*
(2) *the Ishikawa iterative sequence $\{x_n\}$ $\triangle$-converegs to $x^* \in F(T)$.*

Furthermore, x^* is the unique fixed point of T.

If $\beta_n = 0$ in $(\mathbb{I})$, then it reduces to $(\mathbb{M})$. So we can easily prove the following corollary.

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Corollary 3.2. *Let (X, d) be a $CAT(0)$ space and D be a nonempty convex subset of X. Let $T : D \to CB(D)$ be a multivalued contraction mapping with $k < \frac{1}{1+\varepsilon}$ and $F(T) \neq \emptyset$ satisfying $Tp = \{p\}$ for any fixed point $p \in F(T)$. Let a constant L satisfying $\sup_{w \in Tx, x \in D} d(p, w) \leq L$, for all $x \in D$. Let $\{w_n\}$ and $\{r_n\}$ be the Picard and Mann iterative sequence defined by $(\mathbb{P})$ and $(\mathbb{M})$ respectively and satisfying the following conditions:*

(i) $\alpha_n \in [0, 1], \quad \forall\, n \geq 0$;
(ii) $\sum_{n=0}^{\infty}(1 - \alpha_n) = \infty$.

If $w_0 = u_0$, then the following statements are equivalent:

(1) *the Picard iterative sequence $\{w_n\}$ $\triangle$-converegs to $x^* \in F(T)$;*
(2) *the Mann iterative sequence $\{x_n\}$ $\triangle$-converegs to $x^* \in F(T)$.*

Furthermore, x^* is the unique fixed point of T.

4. Some remarks and open problem

For a real number κ, a $CAT(\kappa)$ space is defined by a geodesic metric space whose geodesic triangle is sufficiently thinner than the corresponding triangle in a model space with curvature κ.

For $\kappa = 0$, the 2-dimensional model space $M_\kappa^2 = M_0^2$ is the Euclidean space $\mathbb{R}^2$ with the metric induced from the Euclidean norm. For $\kappa > 0$, M_κ^2 is the 2-dimensional sphere $\frac{1}{\sqrt{\kappa}}\mathbb{S}^2$ whose metric is length of a minimal great arc joining each two points. For $\kappa < 0$, M_κ^2 is the 2-dimensional hyperbolic space $\frac{1}{\sqrt{-\kappa}}\mathbb{H}^2$ with the metric defined by a usual hyperbolic distance. For more details about the properties of $CAT(\kappa)$ spaces, see [2], [8], [20], [21], [29].

Open Problem 1. It will be interesting to obtain a generalization of both Theorem 3.1 and Theorem 3.2 to $CAT(\kappa)$ space.

Open Problem 2. Can Theorem 3.1 be generalized to more than one contractive, or a commutative or left amenable semigroup S of mappings for which the sequence is defined by a strongly left invariant sequence (or net) of finite means on S(see [18], [19], [25])?

Competing interests

The author declares to have no competing interests.

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Strong convergence theorems for the generalized viscosity implicit rules of asymptotically nonexpansive mappings in Hilbert spaces

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Abstract

The purpose of this paper is to introduce the generalized viscosity implicit rules of one asymptotically nonexpansive mapping in Hilbert spaces. We obtain some strong convergence theorems under certain assumptions imposed on the parameters. We also apply our main results to solve mixed equilibrium problem in Hilbert spaces. A numerical example is also given to support our main results. The results obtained in this paper improve and extend many recent ones in this field.

Keywords:
Fixed point; Generalized implicit rule; Asymptotically nonexpansive mapping; Hilbert spaces.
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1. Introduction

Let C be a subset of real Hilbert space H. Let $F(T)$ be the set of fixed points of mapping T. We recall some basic definitions.

A mapping $f : C \to C$ is called a strict contraction, if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \ \forall\, x, y \in C. \tag{1.1}$$

A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \forall\, x, y \in C. \tag{1.2}$$

A mapping $T : C \to C$ is called asymptotically nonexpansive if there exists a sequence $\{\theta_n\} \subset [0, +\infty)$ with $\lim_{n\to\infty} \theta_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + \theta_n) \|x - y\|, \forall\, n \geq 0,\ x, y \in C. \tag{1.3}$$

It is easy to see that asymptotically nonexpansive mapping contains strict contraction, nonexpansive mapping as a special case.

A mapping $A : C \to H$ is called monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0,\ \forall x, y \in C. \tag{1.4}$$

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A mapping $A : C \to H$ is called α-inverse strongly monotone if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \ \forall x, y \in C. \tag{1.5}$$

Recently, viscosity iterative algorithms for finding a common element of the set of fixed point of nonexpansive mappings, the set of solution of variational inequality problem and mixed equilibrium problems have been investigated extensively by many authors, see [1-15] and the references therein. For instance, Moudafi[1] introduced the viscosity technique for nonexpansive mappings in Hilbert spaces. Xu [2] refined the main results of [1] in Hilbert spaces and extended them to more general uniformly smooth spaces. Precisely, he proved that the suggested viscosity iterative sequence converges strongly to a fixed point of one nonexpansive mapping, which also solves some variational inequality.

Very recently, the implicit midpoint rule has become a powerful methods for solving ordinary differential equations; see [16-22] and the references therein. Xu et al. [20] considered the following viscosity implicit midpoint rule:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(\frac{x_n + x_{n+1}}{2}), \ n \geq 0. \tag{1.6}$$

By using contractions to regularize the implicit midpoint rule for nonexpansive mappings, they proved that the iterative sequence defined by (1.6) converges in norm to a fixed point of T, which also solves the variational inequality:

$$\langle (I - f)q, x - q\rangle \geq 0, \ x \in F(T). \tag{1.7}$$

On the other hand, many authors studied the Mann and Ishikawa iterations processes for asymptotically nonexpansive mapping in Hilbert spaces or Banach spaces, see [23-30] and the references therein. For example, Lou et al.[24] investigated some iterative algorithms for asymptotically nonexpansive mapping on a uniformly convex Banach space with uniformly Gâteaux differentiable norm.

In this paper, we introduce a viscosity implicit rules for an asymptotically nonexpansive mapping in Hilbert spaces. Under suitable assumptions imposed on the parameters, we obtain some strong convergence theorems for finding a fixed point of an asymptotically nonexpansive mapping. We also apply our main results to solve mixed equilibrium problem in Hilbert spaces.

2. Preliminaries

Let C be a nonempty closed convex subset of H. For all $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \ \text{ for all } y \in C. \tag{2.1}$$

In this case, P is called a metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \forall x, y \in H. \tag{2.2}$$

Furthermore, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.3}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \forall x \in H, \ y \in C. \tag{2.4}$$

We need the following lemmas for proving our main results.

Lemma 2.1 ([2]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \ n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) either $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 ([21]). *Let T be an asymptotically nonexpansive mapping defined on a nonempty bounded closed convex subset C of a Hilbert space H. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup z$ and $Tx_n - x_n \to 0$, then $z \in F(T)$.*

3. Main results

Theorem 3.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H. Let $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{\theta_n\}$ such that $F(T) \neq \emptyset$ and $f : C \to C$ a strict contraction with coefficient $\alpha \in [0, 1)$. Pick any $x_0 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T^n(\frac{x_n + x_{n+1}}{2}), \tag{3.1}$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$ satisfying the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$;
(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(iv) $\sum_{n=0}^{\infty} \sup_{x\in C'} \|T^{n+1}x - T^n x\| < \infty$, where $C^{'}$ is a closed convex subset of C that contains sequence $\{x_n\}$.
Then $\{x_n\}$ converges strongly to a fixed point q of the asymptotically nonexpansive mapping T, which is also the solution of the variational inequality

$$\langle (I - f)q, y - q\rangle \geq 0, \text{ for all } y \in F(T).$$

Proof. First, we show that $\{x_n\}$ is bounded. Indeed, take $p \in F(T)$ arbitrarily, since $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\frac{\theta_n}{\alpha_n} \leq \frac{1-\alpha}{2}$. Choose a constant $M_1 > 0$ sufficiently large such that

$$\|x_N - p\| \leq M_1, \|f(p) - p\| \leq \frac{1-\alpha}{2} M_1.$$

We proceed by induction to show that $\|x_n - p\| \leq M_1, \forall n \geq 1$. Assume $\|x_n - p\| \leq M_1$, for some $n \geq N$. We show that $\|x_{n+1} - p\| \leq M_1$. We observe

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&= \|\alpha_n f(x_n) + (1-\alpha_n)T^n(\frac{x_n + x_{n+1}}{2}) - p\| \\
&= \|\alpha_n(f(x_n) - p) + (1-\alpha_n)[T^n(\frac{x_n + x_{n+1}}{2}) - p]\| \\
&\leq \alpha_n\|f(x_n) - f(p)\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\|T^n(\frac{x_n + x_{n+1}}{2}) - p\| \\
&\leq \alpha\alpha_n\|x_n - p\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)(\theta_n + 1)\|\frac{x_n + x_{n+1}}{2} - p\| \\
&\leq \alpha\alpha_n\|x_n - p\| + \alpha_n\|f(p) - p\| + \frac{(1-\alpha_n)(\theta_n + 1)}{2}\|x_n - p\| + \frac{(1-\alpha_n)(\theta_n + 1)}{2}\|x_{n+1} - p\|
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \frac{1 - \alpha_n + 2\alpha_n\alpha + (1-\alpha_n)\theta_n}{1 + \alpha_n - (1-\alpha_n)\theta_n}\|x_n - p\| + \frac{\alpha_n}{1 + \alpha_n - (1-\alpha_n)\theta_n}\|f(p) - p\| \\
&= [1 - \frac{2\alpha_n(1-\alpha) - 2(1-\alpha_n)\theta_n}{1 + \alpha_n - (1-\alpha_n)\theta_n}]\|x_n - p\| + \frac{\alpha_n}{1 + \alpha_n - (1-\alpha_n)\theta_n}\|f(p) - p\| \\
&\leq [1 - \frac{\alpha_n(1-\alpha)}{1 + \alpha_n - (1-\alpha_n)\theta_n}]\|x_n - p\| + \frac{\alpha_n(1-\alpha)}{1 + \alpha_n - (1-\alpha_n)\theta_n}\frac{\|f(p) - p\|}{1-\alpha} \\
&\leq \max\{\|x_n - p\|, \frac{\|f(p) - p\|}{1-\alpha}\} \\
&\leq M_1.
\end{aligned} \tag{3.2}$$

This implies that $\{x_n\}$ is bounded.

Next, we prove that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. It follows from (3.1) that

$$\begin{aligned}
&\|x_{n+1}-x_n\|\\
&=\|\alpha_n f(x_n)+(1-\alpha_n)T^n(\frac{x_n+x_{n+1}}{2})-\alpha_{n-1}f(x_{n-1})-(1-\alpha_{n-1})T^{n-1}(\frac{x_{n-1}+x_n}{2})\|\\
&=\|\alpha_n(f(x_n)-f(x_{n-1}))+(\alpha_n-\alpha_{n-1})f(x_{n-1})+(1-\alpha_n)[T^n(\frac{x_n+x_{n+1}}{2})-T^n(\frac{x_{n-1}+x_n}{2})]\\
&\quad+(1-\alpha_n)T^n(\frac{x_{n-1}+x_n}{2})-(1-\alpha_{n-1})T^n(\frac{x_{n-1}+x_n}{2})\\
&\quad+(1-\alpha_{n-1})[T^n(\frac{x_{n-1}+x_n}{2})-T^{n-1}(\frac{x_{n-1}+x_n}{2})]\|\\
&=\|\alpha_n(f(x_n)-f(x_{n-1}))+(1-\alpha_n)[T^n(\frac{x_n+x_{n+1}}{2})-T^n(\frac{x_{n-1}+x_n}{2})]+(\alpha_n-\alpha_{n-1})\\
&\quad\cdot[f(x_{n-1})-T^n(\frac{x_{n-1}+x_n}{2})]+(1-\alpha_{n-1})[T^n(\frac{x_{n-1}+x_n}{2})-T^{n-1}(\frac{x_{n-1}+x_n}{2})]\|\\
&\le\alpha\alpha_n\|x_n-x_{n-1}\|+(1-\alpha_n)(\theta_n+1)(\frac{\|x_{n+1}-x_n\|}{2}+\frac{\|x_n-x_{n-1}\|}{2})\\
&\quad+|\alpha_n-\alpha_{n-1}|\,\|f(x_{n-1})-T^n(\frac{x_{n-1}+x_n}{2})\|+\sup_{x\in C'}\|T^nx-T^{n-1}x\|\\
&=\frac{2\alpha\alpha_n+(1-\alpha_n)(\theta_n+1)}{2}\|x_n-x_{n-1}\|+\frac{(1-\alpha_n)(\theta_n+1)}{2}\|x_{n+1}-x_n\|\\
&\quad+|\alpha_n-\alpha_{n-1}|\,M_2+\sup_{x\in C'}\|T^nx-T^{n-1}x\|,
\end{aligned}$$

where M_2 is a constant such that

$$M_2=\sup_{n\ge0}\|f(x_{n-1})-T^n(\frac{x_{n-1}+x_n}{2})\|.$$

It follows that

$$\begin{aligned}
\frac{2-(1-\alpha_n)(\theta_n+1)}{2}\|x_{n+1}-x_n\|\le{}&\frac{2\alpha\alpha_n+(1-\alpha_n)(\theta_n+1)}{2}\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\,M_2\\
&+\sup_{x\in C'}\|T^nx-T^{n-1}x\|.
\end{aligned}$$

This implies

$$\begin{aligned}
&\|x_{n+1}-x_n\|\\
&\le\frac{2\alpha\alpha_n+(1-\alpha_n)(\theta_n+1)}{2-(1-\alpha_n)(\theta_n+1)}\|x_n-x_{n-1}\|+\frac{2M_2}{2-(1-\alpha_n)(\theta_n+1)}\cdot\\
&\quad|\alpha_n-\alpha_{n-1}|+\frac{2}{2-(1-\alpha_n)(\theta_n+1)}\sup_{x\in C'}\|T^nx-T^{n-1}x\|\\
&=\left(1-\frac{2[1-\alpha\alpha_n-(1-\alpha_n)(\theta_n+1)]}{2-(1-\alpha_n)(\theta_n+1)}\right)\|x_n-x_{n-1}\|+\frac{2M_1}{2-(1-\alpha_n)(\theta_n+1)}\cdot\\
&\quad|\alpha_n-\alpha_{n-1}|+\frac{2}{2-(1-\alpha_n)(\theta_n+1)}\sup_{x\in C'}\|T^nx-T^{n-1}x\|.
\end{aligned}\tag{3.3}$$

Let

$$\gamma_n=\frac{2[1-\alpha\alpha_n-(1-\alpha_n)(\theta_n+1)]}{2-(1-\alpha_n)(\theta_n+1)}.$$

We note

$$\begin{aligned}\gamma_n &= \frac{2[\alpha_n(1-\alpha)+\theta_n(\alpha_n-1)]}{1-\theta_n+\alpha_n(\theta_n+1)}\\ &\geq \frac{2[\alpha_n(1-\alpha)+\theta_n(\alpha_n-1)]}{1-\theta_n+(\theta_n+1)}\\ &= \alpha_n(1-\alpha)+\theta_n(\alpha_n-1)\\ &\geq \alpha_n(1-\alpha)-\theta_n \geq \frac{1-\alpha}{2}\alpha_n.\end{aligned}$$

By condition (i), we have $\sum_{n=0}^{\infty}\gamma_n=\infty$. Apply Lemma 2.1 to (3.3), we get

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.4}$$

Next, we prove that $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. In fact, we have

$$\begin{aligned}\|x_{n+1}-T^n(\frac{x_n+x_{n+1}}{2})\| &= \alpha_n\|f(x_n)-T^n(\frac{x_n+x_{n+1}}{2})\|\\ &\to 0 \text{ as } n\to\infty.\end{aligned} \tag{3.5}$$

Moreover, we get

$$\begin{aligned}&\|x_n-T^nx_n\|\\ &= \|x_n-x_{n+1}+x_{n+1}-T^n(\frac{x_n+x_{n+1}}{2})+T^n(\frac{x_n+x_{n+1}}{2})-T^nx_n\|\\ &\leq \|x_{n+1}-x_n\|+\|x_{n+1}-T^n(\frac{x_n+x_{n+1}}{2})\|+\|T^n(\frac{x_n+x_{n+1}}{2})-T^nx_n\|\\ &\leq \|x_{n+1}-x_n\|+\|x_{n+1}-T^n(\frac{x_n+x_{n+1}}{2})\|+\frac{\theta_n+1}{2}\|x_{n+1}-x_n\|\\ &= \frac{\theta_n+3}{2}\|x_{n+1}-x_n\|+\|x_{n+1}-T^n(\frac{x_n+x_{n+1}}{2})\|.\end{aligned}$$

Combining (3.4) and (3.5), we can obtain

$$\lim_{n\to\infty}\|x_n-T^nx_n\|=0. \tag{3.6}$$

We notice

$$\begin{aligned}&\|x_n-Tx_n\|\\ &= \|x_n-T^nx_n+T^nx_n-T^{n+1}x_n+T^{n+1}x_n-Tx_n\|\\ &\leq \|x_n-T^nx_n\|+\|T^nx_n-T^{n+1}x_n\|+(1+\theta_1)\|T^nx_n-x_n\|\\ &\leq \|x_n-T^nx_n\|+\sup_{x\in C'}\|T^nx-T^{n+1}x\|+(1+\theta_1)\|T^nx_n-x_n\|.\end{aligned}$$

By condition (iv) and (3.6), we have

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0. \tag{3.7}$$

Next, we claim that

$$\limsup_{n\to\infty}\langle q-f(q),q-x_n\rangle\leq 0, \tag{3.8}$$

where $q=P_{F(T)}f(q)$. Indeed, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle q-f(q),q-x_n\rangle=\lim_{i\to\infty}\langle q-f(q),q-x_{n_i}\rangle.$$

Since $\{x_n\}$ is bounded, there exists a subsequence of $\{x_n\}$ which converges weakly to p. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup p$. From (3.7) and Lemma 2.2, we have $p \in F(T)$. This together with the property of the metric projection implies that

$$\limsup_{n\to\infty} \langle q - f(q), q - x_n\rangle = \lim_{i\to\infty} \langle q - f(q), q - x_{n_i}\rangle = \langle q - f(q), q - p\rangle \le 0.$$

Then (3.8) holds. Finally, we show that $x_n \to q$ as $n \to \infty$. In fact, we have

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&= \langle \alpha_n f(x_n) + (1-\alpha_n)T^n(\frac{x_n + x_{n+1}}{2}) - q, x_{n+1} - q\rangle \\
&= \langle \alpha_n (f(x_n) - q) + (1-\alpha_n)(T^n(\frac{x_n + x_{n+1}}{2}) - q), x_{n+1} - q\rangle \\
&= \alpha_n \langle f(x_n) - f(q), x_{n+1} - q\rangle + \alpha_n \langle f(q) - q, x_{n+1} - q\rangle + (1-\alpha_n)\langle T^n(\frac{x_n + x_{n+1}}{2}) - q, x_{n+1} - q\rangle \\
&\le \alpha\alpha_n \|x_n - q\| \cdot \|x_{n+1} - q\| + (1-\alpha_n)(\theta_n + 1)\|\frac{x_n - q}{2} + \frac{x_{n+1} - q}{2}\| \cdot \|x_{n+1} - q\| \\
&\quad + \alpha_n \langle f(q) - q, x_{n+1} - q\rangle \\
&\le \frac{\alpha\alpha_n}{2}\|x_n - q\|^2 + \frac{\alpha\alpha_n}{2}\|x_{n+1} - q\|^2 + \frac{(1-\alpha_n)(\theta_n+1)}{4}\|x_n - q\|^2 \\
&\quad + \frac{(1-\alpha_n)(\theta_n+1)}{4}\|x_{n+1} - q\|^2 + \frac{(1-\alpha_n)(\theta_n+1)}{2}\|x_{n+1} - q\|^2 + \alpha_n\langle f(q) - q, x_{n+1} - q\rangle,
\end{aligned}$$

which implies

$$\begin{aligned}
&\frac{4 - 2\alpha\alpha_n - 3(1-\alpha_n)(\theta_n+1)}{4}\|x_{n+1} - q\|^2 \\
&\le \frac{2\alpha\alpha_n + (1-\alpha_n)(\theta_n+1)}{4}\|x_n - q\|^2 + \alpha_n\langle f(q) - q, x_{n+1} - q\rangle.
\end{aligned}$$

That is

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\le \frac{2\alpha\alpha_n + (1-\alpha_n)(\theta_n+1)}{4 - 2\alpha\alpha_n - 3(1-\alpha_n)(\theta_n+1)}\|x_n - q\|^2 + \frac{4\alpha_n}{4 - 2\alpha\alpha_n - 3(1-\alpha_n)(\theta_n+1)}\langle f(q) - q, x_{n+1} - q\rangle \\
&= [1 - \frac{4(\alpha_n\theta_n + \alpha_n - \alpha\alpha_n - \theta_n)}{4 - 2\alpha\alpha_n - 3(1-\alpha_n)(\theta_n+1)}]\|x_n - q\|^2 + \frac{4\alpha_n}{4 - 2\alpha\alpha_n - 3(1-\alpha_n)(\theta_n+1)} \cdot \\
&\quad \langle f(q) - q, x_{n+1} - q\rangle. \qquad (3.9)
\end{aligned}$$

Put

$$\gamma_n = \frac{4(\alpha_n\theta_n + \alpha_n - \alpha\alpha_n - \theta_n)}{4 - 2\alpha\alpha_n - 3(1-\alpha_n)(\theta_n+1)}.$$

We have

$$\begin{aligned}
\gamma_n &= \frac{4[\theta_n(\alpha_n - 1) + \alpha_n(1-\alpha)]}{1 - 2\alpha\alpha_n + 3\theta_n(\alpha_n - 1) + 3\alpha_n} \\
&\ge \frac{4[\theta_n(\alpha_n - 1) + \alpha_n(1-\alpha)]}{1 + 3\alpha_n} \\
&\ge \theta_n(\alpha_n - 1) + \alpha_n(1-\alpha) \\
&\ge \alpha_n(1-\alpha) - \theta_n \ge \frac{1-\alpha}{2}\alpha_n.
\end{aligned}$$

Apply Lemma 2.1 to (3.9), we obtain $x_n \to q$ as $n \to \infty$. This completes the proof. □

Since nonexpansive mapping is asymptotically nonexpansive, so we obtain the main results of [20].

Theorem 3.2. *Let H be a Hilbert space, C a nonempty closed convex subset of H. Let $T : C \to C$ be a nonexpansive mapping with such that $F(T) \neq \emptyset$ and $f : C \to C$ a strict contraction with coefficient $\alpha \in [0, 1)$. Pick any $x_0 \in C$. Let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(\frac{x_n + x_{n+1}}{2}), \tag{3.10}$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$ satisfying the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n-1}} = 0$.
Then $\{x_n\}$ converges strongly to a fixed point q of nonexpansive mapping T, which is also the solution of the variational inequality

$$\langle (I - f)q, y - q\rangle \geq 0, \text{ for all } y \in F(T).$$

Now we give an example that one mapping satisfies condition (iv) in Theorem 3.1.

Example 3.3. *Let $T : C \to C$ be a strict contraction with a constant $\beta \in (0, 1)$ and let C' be a bounded subset of C. Then*

$$\left\|T^{n+1}x - T^n x\right\| \leq \beta^n \|Tx - x\| \leq \beta^n K_1, \forall\, x \in C',$$

where K_1 is a constant such that $K_1 = \sup_{x\in C'}(\|x\| + \|Tx\|)$. It follows that

$$\sum_{n=1}^{\infty} \sup_{x\in C'} \left\|T^{n+1}x - T^n x\right\| \leq \sum_{n=1}^{\infty} \beta^n K_1 = \frac{\beta K_1}{1-\beta} < \infty.$$

Example 3.4. *Let C be a nonempty closed convex subset of a Banach space. Define mapping $T : C \to C$ as $T^n x = (1 + \frac{1}{n})x$ for any $x \in C$. It is easy to see that T is asymptotically nonexpansive mapping in the intermediate sense. Let $\{x_n\}$ be a bounded sequence in C, we observe*

$$\left\|T^{n+1}x_n - T^n x_n\right\| = \frac{1}{n(n+1)} \|x_n\| \leq \frac{1}{n^2} \|x_n\| \leq \frac{1}{n^2} K_2,$$

where K_2 is a constant such that $K_2 = \sup_{n\geq 1} \|x_n\|$. Hence we obtain

$$\sum_{n=1}^{\infty} \left\|T^{n+1}x_n - T^n x_n\right\| \leq \sum_{n=1}^{\infty} \frac{1}{n^2} K_2 < \infty.$$

4. Applications

In this section, we apply our main results to solve mixed equilibrium problems.

Let $\varphi : C \to \mathbb{R}$ be a real-valued function and $F : C \times C \to \mathbb{R}$ be a bifunction. The mixed equilibrium problem is to find $x \in C$ such that

$$F(x, y) + \varphi(y) - \varphi(x) \geq 0, \forall\, y \in C. \tag{4.1}$$

The set of solutions of (1.1) is denoted by $MEP(F, \varphi)$. If $\varphi = 0$, then problem (4.1) reduces to equilibrium problem which is to find $x \in C$ such that

$$F(x, y) \geq 0, \forall\, y \in C. \tag{4.2}$$

We denote the set of solutions of (4.2) by $EP(F)$.

For solving the mixed equilibrium problem, let us give the following assumptions for the bifunction F, φ and the set C([13]):

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y)+F(y,x)\le 0$ for any $x,y\in C$;
(A3) for each $y\in C$, $x\mapsto F(x,y)$ is weakly upper semicontinuous;
(A4) for each $x\in C$, $y\mapsto F(x,y)$ is convex;
(A5) for each $x\in C$, $y\mapsto F(x,y)$ is lower semicontinuous;
(B1) for each $x\in H$ and $r>0$, there exists a bounded subset $D_x\subseteq C$ and $y_x\in C$ such that for any $z\in C\backslash D_x$,

$$F(z,y_x)+\varphi(y_x)+\frac{1}{r}\langle y_x-z,z-x\rangle<\varphi(z);$$

(B2) C is bounded set.

Lemma 4.1 ([13])**.** *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)-(A5) and let $\varphi: C\to\mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r>0$ and $x\in H$, define a mapping $T_r^{(F,\varphi)}: H\to C$ as follows.*

$$T_r^{(F,\varphi)}(x)=\left\{z\in C: F(z,y)+\varphi(y)+\frac{1}{r}\langle y-z,z-x\rangle\ge\varphi(z),\ \forall\, y\in C\right\}$$

for all $x\in H$. Then the following conclusions hold:
(1) for each $x\in H$, $T_r^{(F,\varphi)}(x)\neq\emptyset$;
(2) $T_r^{(F,\varphi)}$ is single-valued;
(3) $T_r^{(F,\varphi)}$ is firmly nonexpansive, i.e., for any $x,y\in H$,

$$\left\|T_r^{(F,\varphi)}(x)-T_r^{F,\varphi}(y)\right\|^2\le\left\langle T_r^{(F,\varphi)}(x)-T_r^{(F,\varphi)}(y),x-y\right\rangle;$$

(4) $F(T_r^{(F,\varphi)})=MEP(F,\varphi)$;
(5) $MEP(F,\varphi)$ is closed and convex.

Theorem 4.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H. Let F be a bifunction from $C\times C$ to R satisfying (A1)-(A5), $\varphi: C\to R\cup\{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. Let $f: C\to C$ be a strict contraction with coefficient $\alpha\in[0,1)$. Pick any $x_0\in C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)z_n,\\ z_n\in C \text{ such that } F(z_n,y)+\varphi(y)+\frac{1}{r}\langle y-z_n,z_n-u_n\rangle\ge\varphi(z_n),\forall\, r>0,y\in C,\\ u_n=\frac{x_n+x_{n+1}}{2},\end{cases}\tag{4.3}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$ satisfying the following conditions:
(i) $\lim_{n\to\infty}\alpha_n=0,\sum_{n=0}^{\infty}\alpha_n=\infty$;
(iii) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ or $\lim_{n\to\infty}\frac{\alpha_n}{\alpha_{n-1}}=0$.
Then $\{x_n\}$ converges strongly to an element of mixed equilibrium problem (4.1), which is also the solution of the variational inequality

$$\langle(I-f)q,y-q\rangle\ge 0,\ \text{for all } y\in MEP(F,\varphi).$$

Proof. We can rewrite (4.3) as follows:

$$x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)T_r^{(F,\varphi)}(\frac{x_n+x_{n+1}}{2}).$$

Then we obtain the desired results by Theorem 3.2 easily. □

5. Numerical Examples

Example 5.1. *Let inner product* $<\cdot,\cdot>: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ *be defined by*

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x} \cdot \boldsymbol{y} = x_1 \cdot y_1 + x_2 \cdot y_2 + x_3 \cdot y_3$$

and the usual norm $\|\cdot\| : \mathbb{R}^3 \to \mathbb{R}$ *be defined by*

$$\|\boldsymbol{x}\| = \sqrt{x_1^2 + y_1^2 + z_1^2},\ \forall\ \boldsymbol{x} = (x_1, x_2, x_3), \boldsymbol{y} = (y_1, y_2, y_3) \in \mathbb{R}^3.$$

Let $T, f : \mathbb{R}^3 \to \mathbb{R}^3$ *be defined by*

$$T\boldsymbol{x} = \frac{1}{3}(\boldsymbol{x}),\ f(\boldsymbol{x}) = \frac{1}{4}\boldsymbol{x},\ \forall\ x \in \mathbb{R}.$$

Let $\alpha_n = \frac{1}{5n}, \forall\ n \in \mathbb{N}$ *and let* $\{x_n\}$ *be a sequence generated by (3.10). It is easy to see that* $F(T) = \{0\}$. *Then* $\{\boldsymbol{x}_n\}$ *converges strongly to* 0 *by Theorem 3.2.*

We can rewrite (3.10) as follows:

$$\boldsymbol{x}_{n+1} = \frac{10n+1}{50n+2}\boldsymbol{x}_n. \tag{5.1}$$

Choosing $\boldsymbol{x}_1 = (1, 3, 5)$ *in (5.1), we have the following numerical results in Figure* 1 *and Figure* 2.

Figure 1:

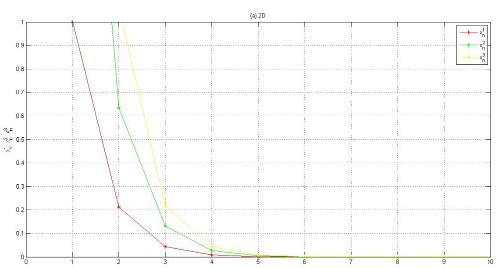
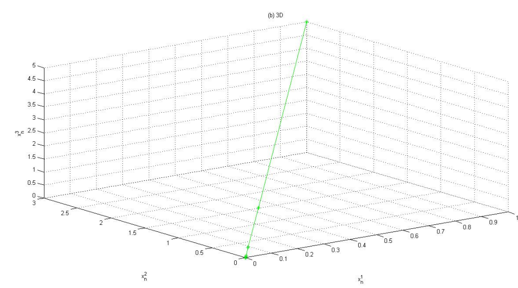


Figure 2:



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HERMITE HADAMARD TYPE INEQUALITIES FOR m-CONVEX AND (α, m)-CONVEX FUNCTIONS FOR FUZZY INTEGRALS

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Abstract. In this paper we prove Hermite–Hadamard type inequalities for m-convex and (α, m)-convex functions for fuzzy integrals. Some examples are also given to illustrate the results.

1. Main Results

Let $[0, b]$, where $b > 0$, be an interval of the real line $\mathbb{R}$. A function f is said to be convex on $[0, b]$ if

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y),$$

holds for all $x, y \in [0, b]$ and $t \in [0, 1]$ and
a function f is starshaped with respect to the origin on $[0, b]$ if

$$f(tx) \leq tf(x),$$

holds for all $x \in [0, b]$ and $t \in [0, 1]$.

In [26] G. Toader, (see also [1, 2, 4]) defined m-convexity: another intermediate between the usual convexity and starshaped convexity as follow:

Definition 1. *The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be m-convex, where $m \in [0, 1]$, if we have*

$$f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$. We say that f is m-concave if $-f$ is m-convex.

The class of all m-convex functions on $[0, b]$ for which $f(0) \leq 0$ is denoted by $K_m(b)$. Obviously, for $m = 1$, m-convexity is the standard convexity of functions on $[0, b]$, and for $m = 0$ the concept of starshaped functions.

The following lemmas hold (see [26] see also [1, Lemma A & Lemma B, Page 2]).

Lemma 1. [1, Lemma A, Page 2] *If f is in the class $K_m(b)$, then it is starshaped.*

Lemma 2. [1, Lemma B, Page 2] *If f is in the class $K_m(b)$ and $0 < n < m \leq 1$, then f is in the class $K_n(b)$.*

From Lemma 2 and Lemma 3 it follows that

$$K_1(b) \subset K_m(b) \subset K_0(b)$$

whenever $m \in (0, 1)$. Note that in the class $K_1(b)$ are only convex functions $f : [0, b] \to \mathbb{R}$ for which $f(0) \leq 0$, that is, $K_1(b)$ is a proper subclass of the class of

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convex functions on $[0,b]$. It is interesting to point out that for any $m \in (0,1)$ there are continuous and differentiable functions which are m-convex, but which are not convex in the standard sense (see [27]).

The notion of m-convexity was further generalized in the following definition (see also [1, 2]).

Definition 2. [11] *The function $f:[0,b]\to\mathbb{R}$, $b>0$, is said to be (α,m)-convex, where $(\alpha,m)\in[0,1]^2$, if we have*

$$f(tx+m(1-t)y)\le t^{\alpha}f(x)+m(1-t^{\alpha})f(y)$$

for all $x,y\in[0,b]$ and $t\in[0,1]$.

The class of all (α,m)-convex functions on $[0,b]$ for which $f(0)\le 0$ is denoted by $K_m^{\alpha}(b)$.

If we take $(\alpha,m)=(1,m)$, it can be easily seen that (α,m)-convexity reduces to m-convexity and for $(\alpha,m)=(1,1)$, (α,m)-convexity reduces to the concept of usual convexity defined on $[0,b], b>0$.

For further results on inequalities related to m-convex and (α,m)-convex functions we refer the readers to [1, 2, 4].

In [4], S.S. Dragomir and G. Toader proved the following Hadamard type inequality for m-convex functions:

Theorem 1. *Let $f:[0,\infty)\to\mathbb{R}$ be an m-convex function with $m\in(0,1]$. If $0\le a<b<\infty$ and $f\in L^1([a,b])$ then*

$$\frac{1}{b-a}\int_a^b f(x)dx\le\min\left\{\frac{f(a)+mf\left(\frac{b}{m}\right)}{2},\frac{f(b)+mf\left(\frac{a}{m}\right)}{2}\right\} \tag{2.1}$$

We will see that this inequalitiy does not valid for fuzzy integrals in general.

To prove our assertion we consider the function $f:[0,\infty)\to[0,\infty)$, $f(x)=ax^n$, $n\in\mathbb{N}$, $n\ge 2$, $a\ge 0$, then f is m-convex on $[0,\infty)$, $m\in(0,1]$.

Example 1. *Take $X=[0,1]$ and let μ be the usual Lebesgue measure on X. Let $f:[0,\infty)\to[0,\infty)$ be defined as $f(x)=\frac{x^2}{3}$ with $m=\frac{9}{10}$. Now to calculate the Sugeno integral $\int_0^1\frac{x^2}{3}d\mu$, consider the distribution function F associated to f on $[0,1]$ then*

$$\begin{aligned}F(\alpha)&=\mu([0,1]\cap\{f\ge\alpha\})=\mu\left([0,1]\cap\left\{\frac{x^2}{3}\ge\alpha\right\}\right)\\&=\mu\left([0,1]\cap\left\{x\ge\sqrt{3\alpha}\right\}\right)=1-\sqrt{3\alpha}\end{aligned}$$

and we solve the equation $1-\sqrt{3\alpha}=\alpha$. It can be easily seen that the solution of this equation is $\frac{5}{2}-\frac{1}{2}\sqrt{21}$, therefore by Remark 1, we have that

$$\int_0^1\frac{x^2}{3}d\mu=\frac{5}{2}-\frac{1}{2}\sqrt{21}\approx 0.208\,71.$$

Now

$$\frac{f(a)+mf\left(\frac{b}{m}\right)}{2}=\frac{5}{27}\approx 0.1851852$$

and on the other hand

$$\frac{f(b)+mf\left(\frac{a}{m}\right)}{2}=\frac{1}{6}\approx 0.16666667.$$

Therefore

$$\min\left\{\frac{f(a)+mf\left(\frac{b}{m}\right)}{2},\frac{f(b)+mf\left(\frac{a}{m}\right)}{2}\right\}=\frac{1}{6}\approx 0.16666667.$$

Which follows that (2.1) is not satisfied in the fuzzy context.

Now we prove Hdadamard type inequalites like (2.1) but for Sugeno Integral or fuzzy integral.

Theorem 2. *Let $g:[0,\infty)\to[0,\infty)$ be an m-convex function with $m\in(0,1]$ such that $mg(0)<g(1)$ and $g(0)<mg\left(\frac{1}{m}\right)$. Let μ be the Lebesgue measure on $[0,1]\subset[0,\infty)$, then*

$$\int_0^1 gd\mu\le\min\left\{1,\frac{mg\left(\frac{1}{m}\right)}{1+mg\left(\frac{1}{m}\right)-g(0)},\frac{g\left(1\right)}{1+g\left(1\right)-mg(0)}\right\}. \tag{2.2}$$

Proof. Since g is an m-convex function, therefore for $x\in[0,1]$ and $m\in(0,1]$, we have

$$g(x)=g\left(\left(1-x\right)0+1\cdot x\right)\le\left(1-x\right)g(0)+mxg\left(\frac{1}{m}\right)=h(x)$$

and hence by (3) of Proposition 1,

$$\int_0^1 gd\mu\le\int_0^1\left(\left(1-x\right)g(0)+mxg\left(\frac{1}{m}\right)\right)d\mu=\int_0^1 h(x)d\mu.$$

Let F be the distribution function associated to h on $[0,1]$, then

$$\begin{aligned}F(\alpha)&=\mu\left([0,1]\cap\{h\ge\alpha\}\right)=\mu\left([0,1]\cap\left\{\left(1-x\right)g(0)+mxg\left(\frac{1}{m}\right)\ge\alpha\right\}\right)\\&=\mu\left([0,1]\cap\left\{x\ge\frac{\alpha-g(0)}{mg\left(\frac{1}{m}\right)-g(0)}\right\}\right)\\&=1-\frac{\alpha-g(0)}{mg\left(\frac{1}{m}\right)-g(0)}\end{aligned}$$

and as a solution of the equation $\alpha=1-\frac{\alpha-g(0)}{mg\left(\frac{1}{m}\right)-g(0)}$, we get

$$\alpha=\frac{mg\left(\frac{1}{m}\right)}{1+mg\left(\frac{1}{m}\right)-g(0)}. \tag{2.3}$$

Analogously by the m-convexity of g, we also have

$$g(x)=g\left(\left(1-x\right)0+1\cdot x\right)\le m\left(1-x\right)g(0)+xg(1)=h_1(x).$$

Arguing smilarly, let F_1 be the distribution function accociated to h_1 on $[0,1]$, then

$$\alpha=\frac{g\left(1\right)}{1+g\left(1\right)-mg(0)}. \tag{2.4}$$

By (1) of Proposition 1, we have that

$$\int_0^1 h(x)d\mu=\int_0^1 h_1(x)d\mu\le\mu\left([0,1]\right)=1. \tag{2.5}$$

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The equations (2.3), (2.4), (2.5) and the definition of Sugeno integral give us the desired inequality. □

A similar results may be stated as follow, however, we leave the details for the intrested readers.

Proposition 1. *Let $g : [0, \infty) \to [0, \infty)$ be an m-convex function with $m \in (0, 1]$ such that $mg(0) > g(1)$ and $g(0) > mg\left(\frac{1}{m}\right)$. Let μ be the Lebesgue measure on $[0, 1] \subset [0, \infty)$, then*

$$\int_0^1 g d\mu \leq \min\left\{1, \frac{g(0)}{1 - mg\left(\frac{1}{m}\right) + g(0)}, \frac{mg(0)}{1 - g(1) + mg(0)}\right\}. \tag{2.6}$$

Remark 1. *If $m = 1$, then the inequalities (2.2) and (2.6) become those inequalities proved in Theroem 1 and Theorem 2 from* [3, p. 3].

Now we give general cases of Theroem 2 and Theorem 3.

Theorem 3. *Let $f : [0, \infty) \to [0, \infty)$ be an m-convex function with $m \in (0, 1]$ such that $mg(\frac{a}{m}) < g(b)$ and $g(a) < mg\left(\frac{b}{m}\right)$. Let μ be the Lebesgue measure on $[a, b]$ and $0 \leq a < b < \infty$. Then*

$$\int_a^b g d\mu \leq \min\left\{1, \frac{mg\left(\frac{b}{m}\right)(b-a)}{b - a + mg\left(\frac{b}{m}\right) - g(a)}, \frac{(b-a)\, g(b)}{b - a + g(b) - mg\left(\frac{a}{m}\right)}\right\}. \tag{2.7}$$

Proof. Since g is an m-convex function $m \in (0, 1]$, therefore for $x \in [a, b]$, $0 \leq a < b < \infty$, we have

$$\begin{aligned} g(x) &= g\left(\left(1 - \frac{x-a}{b-a}\right) a + \frac{x-a}{b-a} \cdot b\right) \\ &\leq \left(\frac{b-x}{b-a}\right) g(a) + m\left(\frac{x-a}{b-a}\right) g\left(\frac{b}{m}\right) = h(x) \end{aligned}$$

By (3) of Proposition 1, we have

$$\int_a^b g d\mu \leq \int_a^b \left[\left(\frac{b-x}{b-a}\right) g(a) + m\left(\frac{x-a}{b-a}\right) g\left(\frac{b}{m}\right)\right] d\mu = \int_a^b h(x) d\mu.$$

Let us consider the distribution function F given by

$$\begin{aligned} F(\alpha) &= \mu\left([a, b] \cap \{h \geq \alpha\}\right) \\ &= \mu\left([a, b] \cap \left\{\left(\frac{b-x}{b-a}\right) g(a) + m\left(\frac{x-a}{b-a}\right) g\left(\frac{b}{m}\right) \geq \alpha\right\}\right) \\ &= \mu\left([a, b] \cap \left\{x \geq \frac{\alpha(b-a) + mag\left(\frac{b}{m}\right) - bg(a)}{mg\left(\frac{b}{m}\right) - g(a)}\right\}\right) \\ &= b - \frac{\alpha(b-a) + mag\left(\frac{b}{m}\right) - bg(a)}{mg\left(\frac{b}{m}\right) - g(a)} \end{aligned}$$

and as solution of the equation $b - \frac{\alpha(b-a)+mag\left(\frac{b}{m}\right)-bg(a)}{mg\left(\frac{b}{m}\right)-g(a)} = \alpha$, we get that

$$\alpha = \frac{mg\left(\frac{b}{m}\right)(b-a)}{b - a + mg\left(\frac{b}{m}\right) - g(a)}. \tag{2.8}$$

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Analogously by the m-convexity of g, we also have

$$\begin{aligned} g(x) &= g\left(\left(1-\frac{x-a}{b-a}\right)a+\frac{x-a}{b-a}\cdot b\right) \\ &\le m\left(\frac{b-x}{b-a}\right)g(\frac{a}{m})+\left(\frac{x-a}{b-a}\right)g\left(b\right)=h_1(x). \end{aligned}$$

Arguing smilarly, let F_1 be the distribution function accociated to h_1 on $[a,b]$, then

$$\alpha=\frac{(b-a)\,g\left(b\right)}{b-a+g(b)-mg\left(\frac{a}{m}\right)}. \tag{2.9}$$

Moreover by (1) of Proposition 1, we have

$$\int_a^b h(x)d\mu=\int_a^b h_1(x)d\mu\le\mu\left([a,b]\right)=b-a. \tag{2.10}$$

From (2.8), (2.9), (2.10) and by the definition of fuzzy integral, we obtain (2.7). This completes the proof of the Teorem. □

Again, we state smilar results like the one proved in Theorem 4, however, the details are left to the intrested readers.

Proposition 2. *Let $f:[0,\infty)\to[0,\infty)$ be an m-convex function with $m\in(0,1]$ such that $mg(\frac{a}{m})>g(b)$ and $g(a)>mg\left(\frac{b}{m}\right)$. Let μ be the Lebesgue measure on $[a,b]$ and $0\le a<b<\infty$. Then*

$$\int_a^b gd\mu\le\min\left\{1,\frac{(b-a)\,g\left(a\right)}{b-a+g(a)-mg\left(\frac{b}{m}\right)},\frac{m\left(b-a\right)g\left(\frac{a}{m}\right)}{b-a+mg\left(\frac{a}{m}\right)-g(b)}\right\}. \tag{2.11}$$

Remark 2. *If $m=1$, then the inequalities (2.7) and (2.11) become those inequalities proved in Theorem 3 from* [3, p. 4].

Example 2. *Take $X=[0,1]$ and let μ be the usual Lebesgue measure on X. Let $f:[0,\infty)\to[0,\infty)$ be defined as $f(x)=x^2$, then f is an m-convex function on $[0,1]$ with $mg(0)<g(1)$ and $g(0)<mg\left(\frac{1}{m}\right)$, $m\in(0,1]$. Now*

$$\frac{mg\left(\frac{1}{m}\right)}{1+mg\left(\frac{1}{m}\right)-g(0)}=\frac{1}{m+1}$$

and

$$\frac{g\left(1\right)}{1+g\left(1\right)-mg(0)}=\frac{1}{2}$$

Therefore by Theorem 2, we have

$$\int_0^1 x^2d\mu\le\frac{1}{2}.$$

Now we give our results for (α,m)-convex functions

Theorem 4. *Let $g:[0,\infty)\to[0,\infty)$ be an (α,m)-convex function with α, $m\in(0,1]^2$ such that $mg(0)<g(1)$ and $g(0)<mg\left(\frac{1}{m}\right)$. Let μ be the Lebesgue measure on $[0,1]$, then*

$$\int_0^1 gd\mu\le\min\left\{1,\alpha_1,\alpha_2\right\}, \tag{2.12}$$

where α_1 and α_2 are positive real solutions of the equations $\alpha' = 1-\left(\frac{\alpha'-g(0)}{mg\left(\frac{1}{m}\right)-g(0)}\right)^{\frac{1}{\alpha}}$ and $\alpha' = 1-\left(\frac{\alpha'-g(0)}{g(1)-mg(0)}\right)^{\frac{1}{\alpha}}$ respectively.

Proof. Since g is an (α, m)-convex function, therefore for $x \in [a,b]$ and α, $m \in (0,1]^2$, we have

$$g(x) = g\left((1-x)\,0 + 1\cdot x\right) \leq \left(1-x^{\alpha}\right)g(0) + mx^{\alpha}g\left(\frac{1}{m}\right) = h(x)$$

and hence by (3) of Proposition 1,

$$\int_0^1 g d\mu \leq \int_0^1 \left(\left(1-x^{\alpha}\right)g(0) + mx^{\alpha}g\left(\frac{1}{m}\right)\right)d\mu = \int_0^1 h(x)d\mu.$$

Let F be the distribution function associated to h on $[0,1]$, then

$$\begin{aligned} F(\alpha') &= \mu\left([0,1]\cap\left\{h \geq \alpha'\right\}\right) = \mu\left([0,1]\cap\left\{\left(1-x^{\alpha}\right)g(0) + mx^{\alpha}g\left(\frac{1}{m}\right) \geq \alpha'\right\}\right) \\ &= \mu\left([0,1]\cap\left\{x \geq \left(\frac{\alpha'-g(0)}{mg\left(\frac{1}{m}\right)-g(0)}\right)^{\frac{1}{\alpha}}\right\}\right) \\ &= 1-\left(\frac{\alpha'-g(0)}{mg\left(\frac{1}{m}\right)-g(0)}\right)^{\frac{1}{\alpha}} \end{aligned}$$

and hence we get the equation

$$\alpha' = 1-\left(\frac{\alpha'-g(0)}{mg\left(\frac{1}{m}\right)-g(0)}\right)^{\frac{1}{\alpha}}. \tag{2.13}$$

Analogously by the (α, m)-convexity of g, we also have

$$g(x) = g\left((1-x)\,0 + 1\cdot x\right) \leq m\left(1-x^{\alpha}\right)g(0) + x^{\alpha}g(1) = h_1(x).$$

Arguing smilarly, let F_1 be the distribution function accociated to h_1 on $[0,1]$, then we have that the following equation:

$$\alpha' = 1-\left(\frac{\alpha'-g(0)}{g\left(1\right)-mg(0)}\right)^{\frac{1}{\alpha}}. \tag{2.14}$$

By (1) of Proposition 1, we have that

$$\int_0^1 h(x)d\mu = \int_0^1 h_1(x)d\mu \leq \mu\left([0,1]\right) = 1. \tag{2.15}$$

The equations (2.13), (2.14), (2.15) and the definition of Sugeno integral give us the required inequality. □

A similar result can be stated as follow, however, the details are left to the intrested reasers:

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Proposition 3. *Let $g : [0,\infty) \to [0,\infty)$ be an (α, m)-convex function with α, $m \in (0,1]^2$ such that $mg(0) > g(1)$ and $g(0) > mg\left(\frac{1}{m}\right)$. Let μ be the Lebesgue measure on $[0,1] \subset [0,\infty)$, then*

$$\int_0^1 g d\mu \leq \min\{1, \alpha_1, \alpha_2\}, \tag{2.16}$$

where α_1 and α_2 are positive real solutions of the equations $\alpha' = \left(\frac{g(0)-\alpha'}{g(0)-mg\left(\frac{1}{m}\right)}\right)^{\frac{1}{\alpha}}$ and $\alpha' = \left(\frac{g(0)-\alpha'}{mg(0)-g(1)}\right)^{\frac{1}{\alpha}}$ respectively.

Remark 3. *If $(\alpha, m) = (1,1)$, then the inequalities (2.12) and (2.16) become those inequalities proved in Theroem 1 and Theorem 2 from* [3, p.4].

Now in following results we give the general case of the last two results.

Theorem 5. *Let $g : [0,\infty) \to [0,\infty)$ be an (α, m)-convex function with α, $m \in (0,1]^2$ such that $mg(\frac{a}{m}) < g(b)$ and $g(a) < mg\left(\frac{b}{m}\right)$. Let μ be the Lebesgue measure on $[a,b]$, $0 \leq a < b < \infty$, then*

$$\int_0^1 g d\mu \leq \min\{1, \alpha_1, \alpha_2\}, \tag{2.17}$$

where α_1 and α_2 are positive real solutions of the equations $\alpha' = (b-a)\left[1 - \left(\frac{\alpha' - g(a)}{mg\left(\frac{b}{m}\right)-g(a)}\right)^{\frac{1}{\alpha}}\right]$ and $\alpha' = (b-a)\left[1 - \left(\frac{\alpha' - g\left(\frac{a}{m}\right)}{g(b)-mg\left(\frac{a}{m}\right)}\right)^{\frac{1}{\alpha}}\right]$ respectively.

Proof. Since g is an (α, m)-convex function, therefore for $x \in [0,1]$ and α, $m \in (0,1]^2$, we have

$$\begin{aligned} g(x) &= g\left(\left(1 - \frac{x-a}{b-a}\right)a + \frac{x-a}{b-a} \cdot b\right) \\ &\leq \left(1 - \left(\frac{x-a}{b-a}\right)^{\alpha}\right) g(a) + m\left(\frac{x-a}{b-a}\right)^{\alpha} g\left(\frac{b}{m}\right) = h(x) \end{aligned}$$

and hence by (3) of Proposition 1,

$$\int_0^1 g d\mu \leq \int_0^1 \left[\left(1 - \left(\frac{x-a}{b-a}\right)^{\alpha}\right) g(a) + m\left(\frac{x-a}{b-a}\right)^{\alpha} g\left(\frac{b}{m}\right)\right] d\mu = \int_0^1 h(x) d\mu.$$

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Let F be the distribution function associated to h on $[0,1]$, then

$$\begin{aligned}F(\alpha^{'}) &= \mu\left([a,b]\cap\left\{h\geq\alpha^{'}\right\}\right)\\ &= \mu\left([a,b]\cap\left\{\left(1-\left(\frac{x-a}{b-a}\right)^{\alpha}\right)g(a)+m\left(\frac{x-a}{b-a}\right)^{\alpha}g\left(\frac{b}{m}\right)\geq\alpha^{'}\right\}\right)\\ &= \mu\left([a,b]\cap\left\{x\geq a+(b-a)\left(\frac{\alpha^{'}-g(a)}{mg\left(\frac{b}{m}\right)-g(a)}\right)^{\frac{1}{\alpha}}\right\}\right)\\ &= (b-a)\left[1-\left(\frac{\alpha^{'}-g(a)}{mg\left(\frac{b}{m}\right)-g(a)}\right)^{\frac{1}{\alpha}}\right]\end{aligned}$$

and hence we get the equation

$$\alpha^{'}=(b-a)\left[1-\left(\frac{\alpha^{'}-g(a)}{mg\left(\frac{b}{m}\right)-g(a)}\right)^{\frac{1}{\alpha}}\right]. \tag{2.18}$$

Analogously by the (α,m)-convexity of g, we also have

$$\begin{aligned}g(x) &= g\left(\left(1-\frac{x-a}{b-a}\right)a+\frac{x-a}{b-a}\cdot b\right)\\ &\leq m\left(1-\left(\frac{x-a}{b-a}\right)^{\alpha}\right)g\left(\frac{a}{m}\right)+\left(\frac{x-a}{b-a}\right)^{\alpha}g(b)=h_1(x).\end{aligned}$$

Arguing smilarly, let F_1 be the distribution function accociated to h_1 on $[0,1]$, then we have that the following equation:

$$\alpha^{'}=(b-a)\left[1-\left(\frac{\alpha^{'}-g\left(\frac{a}{m}\right)}{g\left(b\right)-mg\left(\frac{a}{m}\right)}\right)^{\frac{1}{\alpha}}\right]. \tag{2.19}$$

By (1) of Proposition 1, we have that

$$\int_0^1 h(x)d\mu=\int_0^1 h_1(x)d\mu\leq\mu\left([a,b]\right)=b-a. \tag{2.20}$$

The equations (2.18), (2.19), (2.20) and the definition of Sugeno integral give us the required inequality. □

A similar resutl is stated below, however, the details are left:

Theorem 6. *Let $g:[0,\infty)\to[0,\infty)$ be an (α,m)-convex function with α, $m\in(0,1]^2$ such that $mg(\frac{a}{m})>g(b)$ and $g(a)>mg\left(\frac{b}{m}\right)$. Let μ be the Lebesgue measure on $[a,b]$, $0\leq a<b<\infty$, then*

$$\int_0^1 gd\mu\leq\min\left\{1,\alpha_1,\alpha_2\right\}, \tag{2.21}$$

where α_1 and α_2 are positive real solutions of the equations $\alpha^{'}=(b-a)\left[\left(\frac{g(a)-\alpha^{'}}{g(a)-mg\left(\frac{b}{m}\right)}\right)^{\frac{1}{\alpha}}\right]$ and $\alpha^{'}=(b-a)\left[\left(\frac{g\left(\frac{a}{m}\right)-\alpha^{'}}{mg\left(\frac{a}{m}\right)-g(b)}\right)^{\frac{1}{\alpha}}\right]$ respectively.

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Remark 4. *If $(\alpha, m) = (1, 1)$, then the inequalities (2.17) and (2.21) become those inequalities proved in Theroem 3 from* [3, p.4].

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Very true operators on equality algebras

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Abstract

In this paper, we introduce the concept of very true operators on equality algebras and investigate some related properties of such operators. As an application of properties of very true operators on equality algebras, we give a characterization of prelinear equality algebras, and discuss the relation between very true operator and internal state operator on equality algebras. Moreover, we put forth the notion of very true filter on equality algebras and obtain some important results. In particular, using very true filter, we characterize the simple very true equality algebras and establish the uniform structures on very true equality algebras.

Keywords: fuzzy higher logic; equality algebras; very true operators; very true filters; uniform structures.

1. Introduction

Fuzzy type theory [13, 14, 15, 16], whose basic connective is a fuzzy equality $\sim$, was developed as a fuzzy counterpart of the classical higher-order logic (type theory in which identity is a basic connective, see [9]). Since the truth values for algebra of fuzzy type theory is no longer a residuated lattice, a specific algebra called an EQ-algebra [17] was proposed. Viewing the axioms of EQ-algebras with a purely algebraic eye it appears that unlike in the case of residuated lattices where the adjointness condition ties product with implication. By contrast, the product in EQ-algebras is quite loosely related to the other connectives, which lead to the product in EQ-algebra may be replaced by other similar connectives. Furthermore, the freedom in choosing the product might prohibit to find deep related algebraic results. For this reason, a new algebraic structure was introduced in [11], called equality algebra, which consisting of two binary operations-meet and equivalence, and constant 1. It was proved in [12] that any equality algebra has a corresponding BCK-meet-semilattice and any BCK-meet-semilattice with (D) has a corresponding equality algebras. Apart from their algebraic interest, the general motivation for equality algebras from the side of logic was to define an algebraic structure which (with appropriate extensions) is suitable to axiomatize a large class of substructural logics based on an equivalence connective rather than implication. The very first step toward this aim has been done in [11]. Indeed, the equality algebras could also be candidates for a possible algebraic semantics for fuzzy type theory, which lead to the study of equality algebra is highly motivated.

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In the sense of Zadeh [5](1975), fuzzy logic distinguish in broad and narrow sense. In narrow sense, fuzzy logic deals with some many valued logic but asks questions different from those asked by logicians, who devote to completeness and soundness, etc. Compare to narrow sense, Zadeh stress the importance of fuzzy truth values as very true, quite true and so on, that themselves are fuzzy subsets of all truth degrees. He always gives some examples of handling these fuzzy truth values but seems uninterested in any sort of axiomatization. In order to answer the question " if any natural axiomatization is possible and how far can even this sort of fuzzy logic be captured by standard methods of mathematical logic", Hájek[8] introduced the concept of very true operator on BL-algebras as a tool for reducing the number of possible logical values in many valued fuzzy logic. Consequently, the notion of very true operator has been extended to other logical algebras such as MV-algebra [2], Rℓ-monoid[3], residuated lattices[10, 20] and provided an algebraic foundation for reasoning about fuzzy truth valued of events in many valued logic system, which belong to a subclass of substructural logic based on an fuzzy implication.

As for BL-algebras, MV-algebras Rℓ-monoid and residuated lattices, we observed that although they are different algebras they all are essentially particular types of equality algebras. Thus, it is natural to generalized the concept of very true operators to equality algebras for studying the most general results regarding very true operators in the above-mentioned algebras. On the other hand, BL, Łukasiewicz, ML, they are all many valued logic system belong to a subclass of substructural logic based on an fuzzy implication. However, the logic system corresponding to equality algebra is a fuzzy higher logic, which belong to subclass of substructural logic based on fuzzy equality and is different from above common many valued logic system that based on fuzzy implication. Moreover, all results of this paper may be considered providing an algebraic foundation for reasoning about probabilities of fuzzy events for higher fuzzy logic. This is the motivation for us to investigate very true operators on equality algebras.

Based on above consideration, we enrich the language of equality algebras by adding a very true operator to get algebras named very true equality algebras. This paper is structured in five sections. In order to make the paper as self-contained as possible, we recapitulate in Section 2 the definition of equality algebras, and review their basic properties that will be used in the remainder of the paper. In Section 3, we introduce very true operator on equality algebras and study some properties of them. Also, we give a characterizations of a prelinear equality algebras and discuss the relation between very true operator and internal state on equality algebras. In Section 4, we investigate very true filter of very true equality algebras. In particular, by using very true filter, simple very true equality algebras are characterized and the uniform structures on very true equality algebras are established.

2. Preliminaries

In this section, we summarize some definitions and results about equality algebras which will be used in the following and we shall not cite them every time they are used.

Definition 2.1. [11] An algebra $(\mathcal{E}, \sim, \wedge, 1)$ of type $(2, 2, 0)$ is called an *equality algebra* if it satisfies the following conditions:

(E1) $(\mathcal{E}, \wedge, 1)$ is a commutative idempotent integral monoid (i.e., $\wedge$-semilattice with top element 1),

(E2) $x \sim y = y \sim x$,
(E3) $x \sim x = 1$,
(E4) $x \sim 1 = x$,
(E5) $x \leq y \leq z$ implies $x \sim z \leq y \sim z$ and $x \sim z \leq x \sim y$,
(E6) $x \sim y \leq (x \wedge z) \sim (y \wedge z)$,
(E7) $x \sim y \leq (x \sim z) \sim (y \sim z)$,

for all $x, y, z \in \mathcal{E}$.

In what follows, by $\mathcal{E}$ we denote the universe of $(\mathcal{E}, \sim, \wedge, 1)$. For any $x, y \in \mathcal{E}$, we define fuzzy implication as $x \to y = x \sim (x \wedge y)$ and agree that $\sim$ and $\to$ have higher priority than $\wedge$.

On an equality algebra $(\mathcal{E}, \sim, \wedge, 1)$ we define $x \leq y$ iff $x \wedge y = x$. It is easy to check that $\leq$ is a partial order relation on $\mathcal{E}$ and for all $x \in \mathcal{E}$, $x \leq 1$.

Definition 2.2. [6, 7] Let $(\mathcal{E}, \sim, \wedge, 1)$ be an equality algebra. Then $\mathcal{E}$ is called:

(1) *bounded* if there exists an element $0 \in \mathcal{E}$ such that $0 \leq x$ for all $x \in \mathcal{E}$,
(2) *prelinear* if for all $x, y \in \mathcal{E}$, 1 is the unique upper bounded in $\mathcal{E}$ of the set $\{(y \to x), (x \to y)\}$.

Proposition 2.3. [6, 7] If $(\mathcal{E}, \sim, \wedge, 1)$ is a prelinear equality algebra, then $(\mathcal{E}, \leq)$ is a lattice, where the join operation is given by $x \vee y = ((x \to y) \to y) \wedge (y \to x) \to x)$, for any $x, y \in \mathcal{E}$.

Proposition 2.4. [6, 7] An equality algebra $(\mathcal{E}, \sim, \wedge, 1)$ is prelinear if and only if $(x \to y) \to z \leq ((y \to x) \to z) \to z$, for all $x, y, z \in \mathcal{E}$.

Proposition 2.5. [11, 12] In every equality algebra $(\mathcal{E}, \sim, \wedge, 1)$ the following properties hold for all $x, y, z \in \mathcal{E}$:

(1) $x \sim y \leq x \leftrightarrow y \leq x \to y$,
(2) $x \sim y = 1$ iff $x = y$,
(3) $x \to y = 1$ iff $x \leq y$,
(4) $x \to y = 1$ and $y \to x = 1$ imply $x = y$,
(5) $1 \to x = x$, $x \to 1 = 1$, $x \to x = 1$,
(6) $x \leq y \to x$,
(7) $x \leq (x \to y) \to y$,
(8) $x \to y \leq (y \to z) \to (x \to z)$,
(9) $x \leq y \to z$ iff $y \leq x \to z$,
(10) if $x \leq y$, then $x \leq x \sim y$,
(11) $x \leq y$ imply $y \to x = y \sim x$,
(12) $x \leq y$ imply $y \to z \leq x \to z$, $z \to x \leq z \to y$,
(13) If $\mathcal{E}$ is a prelinear equality algebra, then $\bigwedge_{i \in I}(x_i \to y) = \bigvee_{i \in I} x_i \to y$, provided that both infimum as well as supremum exist.

Definition 2.6. [11] Let $(\mathcal{E}, \sim, \wedge, 1)$ and $(\mathcal{E}', \rightsquigarrow, \sqcap, 1')$ be two equality algebras and $f : \mathcal{E} \longrightarrow \mathcal{E}'$ be a mapping. We call f a *homomorphism* if the following conditions hold for all $x, y \in \mathcal{E}$:

(1) $f(x \wedge y) = f(x) \sqcap f(y)$,
(2) $f(x \sim y) = f(x) \rightsquigarrow f(y)$.

The following theorem provides a connection of equality algebras with a special class of BCK-algebras with meet.

Theorem 2.7. [4, 11, 12] The following two statements hold:

(1) For any equality algebra $(\mathcal{E}, \sim, \wedge, 1)$, the structure $\Psi(\mathcal{E}) = \{\mathcal{E}, \wedge, \rightarrow, 1\}$ is a BCK-meet-semilattice, where $\rightarrow$ denotes the implication of $\mathcal{E}$,
(2) For any BCK(D)-meet-semilattice $(B, \wedge, \rightarrow, 1)$, the structure $\Phi(B) = \{B, \wedge, \leftrightarrow, 1)$ is an equality algebra, where $\leftrightarrow$ denotes the equivalence operation of B. Moreover, the implication of $\Phi(B)$ coincide with $\rightarrow$, that is, $x \rightarrow y = x \leftrightarrow (x \wedge y)$.

Let $(\mathcal{E}, \sim, \wedge, 1)$ be an equality algebra. A nonempty set F is called a *filter* of $\mathcal{E}$ if it satisfies: (1) $x \in F$, $x \leq y$ implies $y \in F$, (2) $x \in F$, $x \sim y \in F$ implies $y \in F$. One can prove that the set of filters of an equality algebra coincide with the set of filter of its underlying BCK-algebra. A filter F of an equality algebra $\mathcal{E}$ is *proper* if $F \neq \mathcal{E}$. A proper filter is called *maximal* if it is not strictly contained in any other proper filter of $\mathcal{E}$. We will denote by $F(\mathcal{E})$ the set of all filter of $\mathcal{E}$. Clearly, $\{1\}$, $\mathcal{E} \subseteq F(\mathcal{E})$ and $F(\mathcal{E})$ is closed under arbitrary intersections. As a consequence, $(F(\mathcal{E}), \subseteq)$ forms a complete lattice. An equality algebra $(\mathcal{E}, \sim, \wedge, 1)$ is calle a *simple* if $F(\mathcal{E}) = \{\{1\}, \mathcal{E}\}$. (see [4, 12])

Definition 2.8. [4, 11, 12] Let $(\mathcal{E}, \sim, \wedge, 1)$ be an equality algebra. A subset $\theta \subseteq \mathcal{E} \times \mathcal{E}$ is called a *congruence* of $\mathcal{E}$ if it is an equivalence relation on $\mathcal{E}$ and for all $x_1, y_1, x_2, y_2 \in \mathcal{E}$ such that $(x_1, y_1), (x_2, y_2) \in \theta$ the following hold:

(1) $(x_1 \wedge x_2, y_1 \wedge y_2) \in \theta$,
(2) $(x_1 \sim x_2, y_1 \sim y_2) \in \theta$.

Let F be a filter of $\mathcal{E}$. Define the congruence relation $\equiv_F$ on $\mathcal{E}$ by $x \equiv_F y$ if and only if $x \sim y \in F$. The set of all congruence class is denote by $\mathcal{E}/F$, i.e. $\mathcal{E}/F = \{[x] | x \in \mathcal{E}\}$, where $[x] = \{x \in \mathcal{E} | x \equiv_F y\}$. Define $\bullet$, $\rightharpoonup$ on $\mathcal{E}/F$ as follows: $[x] \bullet [y] = [x \wedge y]$, $[x] \rightharpoonup [y] = [x \sim y]$. Therefore, $(\mathcal{E}/F, \bullet, \rightharpoonup, [1])$ is an equality algebra which is called a *quotient equality algebra* of $\mathcal{E}$ with respect to F. (see [4])

In what follows, we review some notions about uniformity which will be necessary in the following section.

Let X be a nonempty set and U, V be any subset of $X \times X$. Defined $U \circ V = \{(x, y) \in X \times X |$ for some $z \in X, (z, y) \in U$ and $(x, z) \in V\}$, $U^{-1} = \{(x, y) \in X \times X | (y, x) \in U\}$, $\triangle = \{(x, x) \in X \times X | x \in X\}$.

Definition 2.9. [1, 18, 19] By an *uniformity* on X we shall mean a nonempty collection K of subsets of $X \times X$ which satisfies the following conditions:

(U1) $\triangle \subseteq U$ for any $U \in K$,
(U2) If $U \in K$, then $U^{-1} \in K$,
(U3) If $U \in K$, then there exists a $V \in K$ such that $V \circ V \subseteq U$,
(U4) If $U, V \in K$, then $U \cap V \in K$,
(U5) If $U \in K$ and $U \subseteq V \subseteq X \times X$ then $V \in K$,

The pair (X, K) is called an *uniform structure* (uniform space).

3. Very true operators on equality algebras

In this section, we introduce the notion of very true operators in an equality algebras and investigate some related properties of such operators. Also, we give characterizations of prelinea equality algebras and discuss relations between very true operators and internal state operators on equality algebras.

Definition 3.1. Let $\mathcal{E}$ be an equality algebra. The mapping $\tau : \mathcal{E} \longrightarrow \mathcal{E}$ is called a *very true operator* if it satisfies the following conditions:

(VE1) $\tau(1) = 1$,
(VE2) $\tau(x) \leq x$,
(VE3) $\tau(x) \leq \tau\tau(x)$,
(VE4) $\tau(x \sim y) \leq \tau(x) \sim \tau(y)$,
(VE5) $\tau(x \wedge y) = \tau(x) \wedge \tau(y)$.

The pair $(\mathcal{E}, \tau)$ is said to be *very true equality algebra.*

Note. (1) Such a proliferation of logics deserves some explanation. Since 1 is considered as the logical value absolutely true. First note that (VE1) means that absolutely true is very true, which is the standard axiom obtain the classical logic. (VE2) means that if φ is very true then it is true. (VE3) says that very true of very true is very true, which is a kind of necessitation with respect to very true connective. (VE4) means that if both φ and $\varphi \sim \psi$ are very true then so is ψ, that means the connective τ preserve modus ponens based on fuzzy equality $\sim$. (VE5) says that if both φ and ψ are very true then so is conjunction $\varphi \wedge \psi$, one can easily to check that (VE5) is sound for each natural interpretation in equality algebra. Indeed, the order in equality algebra is lattice order, that is the conjunction $\wedge$ is interpreted as the lattice meet $\wedge$.

(2) Although equality algebras belong to some subclasses of substructural logics based on fuzzy equality rather than on fuzzy implication, we define very true operators on them in the way which is in accordance with traditional definitions in residuated lattices. In fact, a very true operator on residuated lattice was introduced by Vychodil [10] in 2005 as a mapping $\tau : \mathcal{E} \to \mathcal{E}$ satisfying conditions (VE1)-(VE3) in Definition 3.1 and (VE4') $\tau(x \to y) \leq \tau(x) \to \tau(y)$. We know that residuated lattices are special cases of equality algebras satisfying the residuated law. In residuated lattice, fuzzy equality $\sim$ can be defined by $x \sim y = (x \to y) \wedge (y \to x)$. From (VE4) and (VE5), one can obtain that the connective τ always preserve modus ponens based on fuzzy implication. Based on (VE4') and the isotonicity of very true connective τ, one can easily obtain $\tau(x \sim y) = \tau((x \to y) \wedge (y \to x)) \leq \tau(x \to y) \wedge \tau(y \to x) \leq (\tau(x) \to \tau(y)) \wedge (\tau(y) \to \tau(x)) = \tau(x) \sim \tau(y)$, thus the (VE5) hold. From this point of view, the very true equality algebra essentially generalize residuated lattice with very true operator. Thus, it is the most general logic algebras with very true in the existing ones founded in the literature, at least to the authors knowledge.

Now, we will give some important examples to illustrate above definition is existing and meaningful.

Example 3.2. For every equality algebra $\mathcal{E}$ there exist at least two very true operator. One is the identical mapping $\tau_0(x) = x$ for any $x \in \mathcal{E}$, and the other is defined by $\tau_1(1) = 1$ and $\tau_1(x) = 0$ for any $x < 1$. It is evident that if τ is a very true operator on $\mathcal{E}$, we have $\tau_1(x) \leq \tau(x) \leq \tau_0(x)$. Thus these τ_0 ,τ_1 are extremal.

Example 3.3. Let $\mathcal{E} = \{0, a, b, 1\}$ with $0 \leq a \leq b \leq 1$. Consider the operation $\sim$ and $\rightarrow$ given by the following tables:

$\sim$	0	a	b	1
0	1	a	0	0
a	a	1	a	a
b	0	a	1	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(\mathcal{E}, \sim, \wedge, 1)$ is an equality algebra in [4]. Now,we define $\tau(0) = 0$, $\tau(a) = a$, $\tau(b) = a$, $\tau(1) = 1$. One can easily check that τ is a very true operator on $\mathcal{E}$. However, τ is not a endhomomorphism on $\mathcal{E}$ since $\tau(a \sim b) = a \neq 1 = \tau(a) \sim \tau(b)$.

Next, we present some useful properties of very true operator on equality algebras.

Proposition 3.4. Let $(\mathcal{E}, \tau)$ be a very true equality algebra, then for any $x, y, z \in \mathcal{E}$ we have,

(1) If $\mathcal{E}$ is a bounded equality algebra, then $\tau(0) = 0$,
(2) $\tau(x) = 1$ if and only if $x = 1$,
(3) $\tau\tau(x) = \tau(x)$,
(4) $\tau(x \rightarrow y) \leq \tau(x) \rightarrow \tau(y)$,
(5) $x \leq y$ implies $\tau(x) \leq \tau(y)$,
(6) $\tau(x) \leq y$ if and only if $\tau(x) \leq \tau(y)$,
(7) $\tau(\mathcal{E}) = Fix_\tau$,where $Fix_\tau = \{x \in \mathcal{E} | \tau(x) = x\}$,
(8) Fix_τ is closed under $\wedge$,
(9) If $y \leq x$, then $\tau(x) \rightarrow \tau(y) = \tau(x) \sim \tau(y)$,
(10) $\tau(x \sim y) \leq \tau(x) \rightarrow \tau(y)$,
(11) $\tau(x \sim y) \leq (x \wedge z) \sim (y \wedge z)$,
(12) If $\tau(\mathcal{E}) = \mathcal{E}$, then $\tau = id_{\mathcal{E}}$,
(13) $\mathrm{Ker}(\tau) = \{1\}$, where $\mathrm{Ker}(\tau) = \{x \in \mathcal{E} | \tau(x) = 1\}$,
(14) $\mathrm{Ker}(\tau)$ is a filter of $\mathcal{E}$,
(15) $\tau(x) = x$ or $\tau(x)$ and x are not comparable,
(16) If $\mathcal{E}$ is linearly order, then $\tau = id_{\mathcal{E}}$.

Proof. (1) Applying (VE2), we have $\tau(0) \leq 0$ and hence $\tau(0) = 0$.

(2) If $\tau(x) = 1$ for some $x \in \mathcal{E}$ then by (VE2), $1 = \tau(x) \leq x$ giving $x = 1$. The converse follows by (VE1).

(3) Applying (VE2) and (VE3), we have $\tau\tau(x) = \tau(x)$.

(4) By (VE4) and (VE5) we have $\tau(x \rightarrow y) = \tau(x \wedge y \sim x) = \tau(x \wedge y) \sim \tau(x) \leq \tau(x) \wedge \tau(y) \sim \tau(x) = \tau(x) \rightarrow \tau(y)$.

(5) If $x \leq y$, then $x \rightarrow y = 1$. It follows from (VE1) and (4) that $\tau(x) \rightarrow \tau(y) = 1$. Therefore, $\tau(x) \leq \tau(y)$.

(6) Assume that $\tau(x) \leq y$, we have $\tau\tau(x) \leq \tau(y)$. By (3), we get $\tau\tau(x) = \tau(x)$. Thus $\tau(x) \leq \tau(y)$. Conversely, suppose that $\tau(x) \leq \tau(y)$, we have $\tau(x) \leq \tau(y) \leq y$.

(7) Let $y \in \tau(\mathcal{E})$, so there exists $x \in \mathcal{E}$ such that $y = \tau(x)$. Hence $\tau(y) = \tau\tau(x) = \tau(x) = y$. It follows that $y \in Fix_\tau$. Conversely, if $y \in Fix_\tau$, we have $y \in \tau(\mathcal{E})$. Therefore, $\tau(\mathcal{E}) = Fix_\tau$.

(8) By (VE5), we obtain that Fix_τ is closed under $\wedge$.

(9) Since $y \leq x$, we have $\tau(y) \leq \tau(x)$ and $\tau(x) \to \tau(y) = \tau(x) \sim \tau(x) \wedge \tau(y) = \tau(x) \sim \tau(y)$.

(10) By Proposition 2.5(1) and (4),(5), we have $\tau(x \sim y) \leq \tau(x \to y) \leq \tau(x) \to \tau(y)$.

(11) By (E6) and (VE2), one can obtain it very easy and hence we omit the process of this proof.

(12) For any $x \in \mathcal{E}$, we have $x = \tau(x_0)$ for some $x_0 \in \mathcal{E}$. By (3), we have $\tau(x) = \tau(\tau(x_0)) = \tau(x_0) = x$. Therefore, $\tau = id_{\mathcal{E}}$.

(13) Assume that $x \in \mathcal{E}$ but $x \neq 1$ such that $\tau(x) = 1$. Applying (VE2), we have $1 = \tau(x) \leq x$ and hence $x = 1$, which is a contradiction. Therefore, $\mathrm{Ker}(\tau) = \{1\}$.

(14) It is easy to check it and hence we omit the process.

(15) Assume $x \in \mathcal{E}$ such that $\tau(x) \neq x$ and $\tau(x)$ and x are comparable. Then $\tau(x) < x$ or $x < \tau(x)$, from (3),(5), we have $\tau(x) < \tau(x)$, which is a contradiction.

(16) It follows from (15) directly.

From the above Proposition 3.4, one can see that $\tau(\mathcal{E})$ is closed under the operation $\wedge$. However, the following example shows that $\tau(\mathcal{E})$ is not a subalgebra of $\mathcal{E}$ since it is not closed under $\sim$ in general.

Example 3.5. Let $\mathcal{E} = \{0, a, b, c, 1\}$ with $0 \leq a \leq b, c \leq 1$. Consider the operation $\sim$ and $\to$ given by the following tables:

$\sim$	0	a	b	c	1
0	1	0	0	0	0
a	0	1	c	b	a
b	0	c	1	0	b
c	0	b	0	1	c
1	0	a	b	1	b

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

Then $(\mathcal{E}, \sim, \wedge, 1)$ is an equality algebra. Now,we define $\tau(0) = 0$, $\tau(a) = a$, $\tau(b) = a$, $\tau(c) = c$, $\tau(1) = 1$. One can easily check that τ is a very true operator on $\mathcal{E}$. However, $\tau(\mathcal{E})$ is not a subalgebra of $\mathcal{E}$ since $a \sim c = b \notin \tau(\mathcal{E})$.

Although the $\tau(\mathcal{E})$ is not necessary a subalgebra of an equality algebra in general, while it forms an equality algebra after redefined its fuzzy equality, which reveals the essence of the fixed point set.

Theorem 3.6. Let $(\mathcal{E}, \tau)$ be a very true equality algebra. Then $(\tau(\mathcal{E}), \wedge, \rightsquigarrow, 1)$ is also an equality algebra, where $x \rightsquigarrow y = \tau(x \sim y)$ for all $x, y \in \tau(\mathcal{E})$.

Proof. Now, we will show that $(\tau(\mathcal{E}), \wedge, \rightsquigarrow, 1)$ is an equality algebra.

For (E1), we show that $(\tau(\mathcal{E}), \wedge, 1)$ is a semilattice with 1 as the greatest element. From Definition 3.1(5), we have that $\tau(\mathcal{E})$ is closed under $\wedge$. Therefore $(\tau(\mathcal{E}), \wedge)$ is a semilattice. For all $x \in \tau(\mathcal{E})$, one can easily check that $x \wedge 1 = x$. Thus, 1 is the greatest element in $\tau(\mathcal{E})$.

For (E2), we will show that $x \rightsquigarrow y = y \rightsquigarrow x$. It is easy to prove.

For (E3), we will show that $x \rightsquigarrow x = 1$. Applying (VE1), we have $\tau(1) = 1$ and hence $x \rightsquigarrow x = \tau(x \sim x) = \tau(1) = 1$.

For (E4), we will show that $x \rightsquigarrow 1 = x$. Since $x \in \tau(\mathcal{E})$, we have $\tau(x) = x$ and hence $x \rightsquigarrow 1 = \tau(x \sim 1) = \tau(x) = x$.

In the similarly way, we can show that (E5)-(E7) hold.

Combine them, we obtain that $(\tau(\mathcal{E}), \wedge, \rightsquigarrow, 1)$ is an equality algebra.

Note. In fact, it is easily checked that $(\tau(\mathcal{E}), \wedge, \rightsquigarrow, 1)$ is an equality algebra, where $\tau(x) \rightsquigarrow \tau(y) = \tau(x \sim y)$. Furthermore, we also obtain that if $\tau(x) \rightsquigarrow \tau(y) = 1$, then $\tau(x) = \tau(y)$. Indeed, if $\tau(x), \tau(y) \in Fix_\tau$, by above Theorem 3.6, $\tau(x) \rightsquigarrow \tau(y) = \tau(\tau(x) \sim \tau(y)) = \tau(x) \sim \tau(y)$. Thus $\tau(x) \sim \tau(y) = 1$, and we have $\tau(x) = \tau(y)$ by Proposition 2.5(3).

In what following, we will give an analogy first isomorphism theorem related to very true operator on equality algebras, which will be used in the next section.

Theorem 3.7. Let $(\mathcal{E}, \tau)$ be a very true equality algebra and $(\tau(\mathcal{E}), \wedge, \rightsquigarrow, 1)$ be an equality algebra. Then the following properties hold:

(1) $\tau : \mathcal{E} \longrightarrow \tau(\mathcal{E})$ is a homomorphism,
(2) The mapping $\tau_0 : \mathcal{E}/\mathrm{Ker}(\tau) \longrightarrow \tau(\mathcal{E})$ defined by $\tau_0([x]) = \tau(x)$ is a isomorphism.

Proof. (1) It follows from (VE1) and (VE5) in Definition 3.1 that $\tau(1) = 1$ and $\tau(x \wedge y) = \tau(x) \wedge \tau(y)$. Moreover, from the above Note, we have $\tau(x) \rightsquigarrow \tau(y) = \tau(x \sim y)$. Therefore $\tau : \mathcal{E} \longrightarrow \tau(\mathcal{E})$ is a homomorphism.

(2) Assume that $[x] = [y]$ and hence $(x, y) \in \mathrm{Ker}(\tau)$. Then $x \sim y \in \mathrm{Ker}(\tau)$, that is, $\tau(x \sim y) = 1$. From the above Note, we have $\tau(x) \rightsquigarrow \tau(y) = 1$ and $(\tau(\mathcal{E}), \wedge, \rightsquigarrow, 1)$ is an equality algebras and so $\tau(x) = \tau(y)$. Therefore, τ_0 is well defined. Now, we will show that τ_0 is a isomorphism. First, we will show that τ_0 is a homomorphism. From (1), we have $\tau_0([x] \rightharpoonup [y]) = \tau_0([x \sim y]) = \tau(x \sim y) = \tau(x) \rightsquigarrow \tau(y) = \tau_0([x]) \rightsquigarrow \tau_0([y])$. Moreover, we have $\tau_0([x] \bullet [y]) = \tau_0([x \wedge y]) = \tau(x \wedge y) = \tau(x) \wedge \tau(y) = \tau_0([x]) \wedge \tau_0([y])$. Clearly, $\tau_0([1]) = 1$. Hence τ_0 is a homomorphism. Next, we will show that τ_0 is one to one. From the above Note, if $\tau_0([x]) = \tau_0([y])$ then $\tau(x) = \tau(y)$ and hence $\tau(x) \rightsquigarrow \tau(y) = \tau(x \sim y) = 1$, that is means $(x, y) \in \mathrm{Ker}(\tau)$ and hence $[x] = [y]$ and so τ_0 is one to one. Furthermore, since τ is onto, τ_0 is onto. Combine them, we obtain that τ_0 is a isomorphism.

As an application of the properties respect to very true operators, we give a characterization of prelinear equality algebras. For obtaining this important result, we need the following theorem.

Theorem 3.8. The following conditions are equivalent in each very true equality algebra $(\mathcal{E}, \tau)$:

(1) $\tau(x \to y) \leq (x \wedge z) \to (y \wedge z)$,
(2) $\tau(y) \leq z \to (y \wedge z)$,
(3) $\tau(y) \leq u \to (u \wedge (z \to (y \wedge z))$.

Proof. (1) $\Rightarrow$ (2) Assume (1) holds. From Proposition 2.5 (1) and Proposition 3.5 (11), we have $\tau(x \to y) \leq ((1 \wedge z) \sim (y \wedge z)) \leq ((1 \wedge z) \to (y \wedge z)) = z \to (y \wedge z)$.

(2) $\Rightarrow$ (3) Assume (2) holds. By (VE3), we get $\tau(y) \leq \tau\tau(y) \leq \tau(z \to (y \wedge z))$, which implies, by (2) again, $\tau(y) \leq u \to (u \wedge (z \to (z \wedge y)))$.

(3) $\Rightarrow$ (2) Taking $u = 1$, we obtain that (2) holds.

(2) $\Rightarrow$ (1) Assume (2) holds. By (3), we have $\tau(x \to y) \leq (x \wedge z) \to ((x \to y) \wedge (x \wedge z))$. Thus, $\tau(x \to y) \leq (x \wedge z) \to ((x \to y) \wedge x \wedge z)$. Furthermore, from Proposition 2.5(12), we have $\tau(x \to y) \leq (x \wedge z) \to ((x \to y) \wedge x \wedge z) \leq (x \wedge z) \to (y \wedge z)$.

Conditions for an equality algebra to be a prelinear equality algebra is gave via very true operator on equality algebra.

Theorem 3.9. The following conditions are equivalent in each very true equality algebras:

(1) $\mathcal{E}$ is prelinear,
(2) $\tau(x \vee y) = \tau(x) \vee \tau(y)$,
(3) $\tau(x \to y) \vee \tau(y \to x) = 1$,
(4) $\tau((x \to y) \to z) \leq \tau(y \to x) \to z) \to z$.

Proof. (1) $\Rightarrow$ (2) Assume that $\mathcal{E}$ is prelinear equality algebra. By(VE5) and Proposition 2.3, we obtain, for all $x, y \in L$, $\tau(x \vee y) = \tau((x \to y) \to y) \wedge \tau((y \to x) \to x)$. By Proposition 3.4(5) and (VE3), we get $\tau(x \vee y) \leq \tau((x \to y) \to (\tau(x) \vee \tau(y))) \wedge (\tau(y \to x) \to (\tau(x) \vee \tau(y)))$. Hence by Proposition 2.5(13), we obtain $\tau(x \vee y) \leq (\tau(x \to y)) \vee (\tau(y \to x)) \to (\tau(x) \vee \tau(y))$ and hence $\tau(x \vee y) \leq \tau(x) \vee \tau(y)$. The other inequality easily by Proposition 3.4(5).

(2) $\Rightarrow$ (3) straightforward.

(3) $\Rightarrow$ (4) This follows exactly by a proof similar to the proof of the equivalence between prelinearity and $(x \to y) \to z \leq ((y \to x) \to z) \to z)$ in Proposition 2.4.

(4) $\Rightarrow$ (1) Taking very true operator $\tau = id_{\mathcal{E}}$ and direct from Proposition 2.4.

Note. We know a BL-algebras is an prelinear quality algebra satisfies the divisibility. From the above theorem, one can check that the very true prelinear equality algebra essentially generalize BL-algebras with very true operator, which introduced by Hájek [8] in 2001 as a mapping $\tau : \mathcal{E} \to \mathcal{E}$ satisfying conditions (VE1)-(VE3), (4) in Proposition 3.5 and (2) in Theorem 3.9.

In what follows, we will give a relationship between very true equality algebras and sate (morphism) equality algebras, which was introduced by L.C. Ciungu [4] to providing an algebraic foundation for reasoning about probabilities of fuzzy events of a large class of substructural logics based on an fuzzy equality.

Definition 3.10. [4] An equality algebra with internal state or *state equality algebra* is a structure $(\mathcal{E}, \sigma) = (\mathcal{E}, \wedge, \sim, \sigma, 1)$, where $(\mathcal{E}, \wedge, \sim, 1)$ is an equality algebra and $\sigma : \mathcal{E} \to \mathcal{E}$ is a unary operator on $\mathcal{E}$, called internal state or state operator, such that for all $x, y \in \mathcal{E}$ the following conditions are satisfied:

(1) $\sigma(x) \leq \sigma(y)$, whenever $x \leq y$,
(2) $\sigma(x \sim x \wedge y) = \sigma((x \sim x \wedge y) \sim y) \sim \sigma(y)$,
(3) $\sigma(\sigma(x) \sim \sigma(y)) = \sigma(x) \sim \sigma(y)$,
(4) $\sigma(\sigma(x) \wedge \sigma(y)) = \sigma(x) \wedge \sigma(y)$.

Definition 3.11. [4] Let $(\mathcal{E}, \wedge \sim, 1)$ be an equality algebra. A *state morphism operator* on $\mathcal{E}$ is a map $\sigma : \mathcal{E} \to \mathcal{E}$ satisfying the following condition for all $x, y \in \mathcal{E}$:

(1) $\sigma(x \sim y) = \sigma(x) \sim \sigma(y)$,
(2) $\sigma(x \wedge y) = \sigma(x) \wedge \sigma(y)$,
(3) $\sigma(\sigma(x)) = \sigma(x)$.

The pair $(\mathcal{E}, \sigma)$ is called a *state morphism equality algebra.*

Theorem 3.12. [4] Any state morphism on a linearly ordered equality algebra $\mathcal{E}$ is an internal state on $\mathcal{E}$.

Theorem 3.13. Let $(\mathcal{E}, \tau)$ be a very true equality algebra. Then the following conditions are equivalent:

(1) $(\mathcal{E}, \tau)$ is a state morphism equality algebras,
(2) $\tau = id_{\mathcal{E}}$.

Proof. (1) $\Rightarrow$ (2) We note that $(\mathcal{E}, \tau)$ is not only a state morphism equality algebras but also a very true equality algebras. In order to prove this important result, we need some useful result. First, we will prove that $x < y$ implies $\tau(x) < \tau(y)$. Assume that $x < y$, then $\tau(x) \leq \tau(y)$. If $\tau(x) = \tau(y)$, then $\tau(x \to y) = \tau(x) \to \tau(y) = 1$. By Proposition 3.4(13), we have $y \to x = 1$ and hence $y \leq x$, which is a contraction to $x < y$. Now, we will show that $\tau(x) = x$ for all $x \in \mathcal{E}$. By (VE2), we have $\tau(x) \leq x$. Assume that $\tau(x) \neq x$, then $\tau(x) < x$ and hence $\tau\tau(x) < \tau(x)$. Thus $\tau(x) < \tau(x)$, which is a contradiction. Therefore, $\tau = id_{\mathcal{E}}$.

(2) $\Rightarrow$ (1) straightforward.

Theorem 3.14. Let $(\mathcal{E}, \tau)$ be a very true linearly order equality algebra. Then the following conditions are equivalent:
(1) $(\mathcal{E}, \tau)$ is a state equality algebras,
(2) $\tau = id_{\mathcal{E}}$.

Proof. It follows from Proposition 3.4(16), we can easily to obtain this result.

4. Very true filters in very true equality algebras

In this section, we introduce very true filters of very true equality algebras and obtain some important result of them. In particular, using very true filter, we give a characterization of simple very true equality algebras and construct the uniform structures on very true equality algebras.

Definition 4.1. Let $(\mathcal{E}, \tau)$ be a very true equality algebra. A nonempty subset F of $\mathcal{E}$ is called a *very true filter* of $(\mathcal{E}, \tau)$ if F is a filter of $\mathcal{E}$ such that if $x \in F$, then $\tau(x) \in F$ for all $x \in \mathcal{E}$.

Example 4.2. Let $(\mathcal{E}, \tau)$ be a very true equality algebra. Then $\text{Ker}(\tau) = \{x \in X | \tau(x) = 1\}$ is a very true filter of $(\mathcal{E}, \tau)$.

Example 4.3. Consider the Example 3.3, one can easily check that the very true filter of $(\mathcal{E}, \tau)$ are $\{1\}$, $\{a, b, 1\}$ and $\mathcal{E}$. On the other hand, one can see that $\{b, 1\}$ is not a very true filter of $(\mathcal{E}, \tau)$, but it is a filter of $\mathcal{E}$, that is to say, not every filter is very true filter.

As an application of very true filters, we give a characterizations of simple very true equality algebras. For obtaining this important result, we need the following proposition.

Proposition 4.4. Let $(\mathcal{E}, \tau)$ be a very true equality algebra.

(1) If F is a filter of $\tau(\mathcal{E})$, then $\tau^{-1}(F)$ is a very true filter of $(\mathcal{E}, \tau)$,

(2) If F is a very true filter of $(\mathcal{E}, \tau)$, then $\tau(F)$ is a filter of $\tau(\mathcal{E})$.

Proof. (1) Suppose that F is a filter of $\tau(\mathcal{E})$. If $x, x \sim y \in \tau^{-1}(F)$. Then $\tau(x), \tau(x \sim y) \in F$. Since $\tau(x \sim y) \leq \tau(x) \sim \tau(y)$ and $\tau(x) \sim \tau(y) \in F$, thus $\tau(y) \in F$, that is $y \in \tau^{-1}(F)$. Let $x, y \in \mathcal{E}$ such that $x \in \tau^{-1}(F)$ and $x \leq y$. Then $\tau(x) \leq \tau(y)$. Since $\tau(x) \in F$ and $\tau(y) \in \tau(\mathcal{E})$, we can obtain that $\tau(y) \in F$, that is $y \in \tau^{-1}(F)$. Obviously, $1 \in \tau^{-}(F)$. Therefore $\tau^{-1}(F)$ is a filter of $\mathcal{E}$.

If $x \in \tau^{-1}(F)$, then $\tau(x) \in F$, so $\tau\tau(x) = \tau(x) \in F$, that is, $\tau(x) \in \tau^{-1}(F)$. Therefore, $\tau^{-1}(F)$ is a very true filter of $(\mathcal{E}, \tau)$.

(2) First, we have $\tau(F) = F \cap \tau(\mathcal{E})$. Indeed, if $x \in F \cap \tau(\mathcal{E})$, then $x = \tau(x)$ as $x \in \tau(\mathcal{E})$, and $\tau(x) \in \tau(F)$ as $x \in F$. Thus, we have $x \in \tau(F)$. It follows that $F \cap \tau(\mathcal{E}) \subseteq \tau(F)$. Conversely, if $y \in \tau(F)$, then there exists $x \in F$ such that $y = \tau(x)$. Since F is an very true filter of $(\mathcal{E}, \tau)$, we obtain $y = \tau(x) \in F$. Therefore, $\tau(F) = F \cap \tau(\mathcal{E})$.

If $x, x \rightsquigarrow y \in \tau(F) = F \cap \tau(\mathcal{E})$, then $\tau(y) \in F \cap \tau(\mathcal{E}) = \tau(F)$ and hence $y \in \tau(F)$. On the other hand, if $x \in \tau(F) = F \cap \tau(\mathcal{E})$, $y \in \tau(\mathcal{E})$ such that $x \leq y$, then $y \in F \cap \tau(\mathcal{E}) = \tau(F)$. Therefore, $\tau(F)$ is a filter of $\tau(\mathcal{E})$.

Now, we introduce simple very true equality algebras and give a characterizations of it via very true filter.

Definition 4.5. A very true equality algebra $(\mathcal{E}, \tau)$ is called *simple very true* if it has exactly two very true filter: $\{1\}$ and $\mathcal{E}$.

Example 4.6. For each equality algebra $\mathcal{E}$, $(\mathcal{E}, \tau_1)$ is a simple very true equality algebra, where $\tau_1(1) = 1$ and $\tau_1(0) = 0$ if $x < 1$.

Theorem 4.7. Let $(\mathcal{E}, \tau)$ be a very true equality algebra. Then the following are equivalent:

(1) $(\mathcal{E}, \tau)$ is a simple very true equality algebra,
(2) $\tau(\mathcal{E})$ is a simple equality algebra.

Proof. (1) $\Rightarrow$ (2) Let F be a filter of $\tau(\mathcal{E})$ and $F \neq \{1\}$. It follows from Proposition 4.4(1) that $\tau^{-1}(F)$ is a very true filter of $(\mathcal{E}, \tau)$. Since $(\mathcal{E}, \tau)$ is very true simple, we have that $\tau^{-1}(F) = \{1\}$ or $\tau^{-1}(F) = \mathcal{E}$. Notice that $F \subseteq \tau^{-1}(F)$ (if $x \in F$, then $\tau(x) = x$, that is, $x \in \tau^{-1}(F)$, we obtain that $\tau^{-1}(F) \neq \{1\}$. Thus, $\tau^{-1}(F) = \mathcal{E}$. Then $0 \in \tau^{-1}(F)$, that is, $0 = \tau(0) \in F$. So we obtain that $F = \tau(\mathcal{E})$. Therefore, $\tau(\mathcal{E})$ is simple equality algebra.

(2) $\Rightarrow$ (1) Let F be a very true filter of $(\mathcal{E}, \tau)$ and $F \neq \{1\}$. By Proposition 4.4(2), we obtain that $\tau(F)$ is a filter of $\tau(\mathcal{E})$. Since $\tau(\mathcal{E})$ is simple equality algebra, we obtain that $\tau(F) = \{1\}$ or $\tau(F) = \mathcal{E}$. Since $\mathrm{Ker}(\tau) = \{1\}$, we have $F \neq \{1\}$. Thus, $\tau(F) = \mathcal{E}$. Then $0 \in \tau(F)$, that is, $0 \in F$. It follows that $F = \mathcal{E}$. Therefore $(\mathcal{E}, \tau)$ is a simple very true equality algebra.

Note. The above theorem brings a method of how to check a very true equality algebra is simple very true. As an application of above theorem, one can easily check that the very true equality algebra in example 4.6 is simple very true since $\tau(\mathcal{E}) = \{0, 1\}$ is a simple equality algebra.

For any very true filter F of $(\mathcal{E}, \tau)$. Defined by $\tau_F : \mathcal{E}/F \longrightarrow \mathcal{E}/F$ as a mapping $\tau_F([x]) = [\tau(x)]$.

Proposition 4.8. Let $(\mathcal{E}, \tau)$ be a very true equality algebra and F be a very true filter of very true equality algebra $(\mathcal{E}, \tau)$. Then τ_F is a very true operator on $\mathcal{E}/F$.

Proof. First, we will prove that τ_F is well defined. Indeed, assume that $[x] = [y]$ for $x, y \in \mathcal{E}$. Then $(x, y) \in \equiv_F$, i.e., $x \sim y \in F$. Since F is a very true filter and hence $\tau(x \sim y) \in F$. Now, applying (VE4) of Definition 3.1, $\tau(x \sim y) \leq \tau(x) \sim \tau(y) \in F$. Thus $(\tau(x), \tau(y)) \in F$ and $[\tau(x)] = [\tau(y)]$. The rest of the proof is easy:

(VE1) $\tau_F([1]) = [\tau(1)] = [1]$,
(VE2) $\tau_F([x]) = [\tau(x)] \leq [x]$,
(VE3) $\tau_F([x]) = [\tau(x)] \leq [\tau\tau(x)] \leq \tau_F\tau_F([x])$,
(VE4) $\tau_F([x] \sim [y]) = [\tau(x \sim y)] \leq [\tau(x) \sim \tau(y)] = \tau_F([x]) \sim \tau_F([y])$,
(VE5) $\tau_F([x] \wedge [y]) = [\tau(x \wedge y)] = [\tau(x) \wedge \tau(y)] = \tau_F([x]) \wedge \tau_F([y])$.

Combine them, one can obtain that $(\mathcal{E}/F, \tau_F)$ is a very true equality algebra.

Proposition 4.9. In the very true equality algebra $(\mathcal{E}/Ker(\tau), \tau_{Ker(\tau)})$ we have:

(1) $[x] \leq [y]$ iff $\tau(x \sim x \wedge y) = 1$ iff $\tau(x \to y) = 1$,
(2) $[x] = [y]$ iff $\tau(x \sim y) = 1$.

Proof. (1) Applying the definition of $\mathcal{E}/Ker(\tau)$ we get: $[x] \leq [y]$ iff $[x] \bullet [y] = [x]$ iff $[x] = [x \wedge y]$ iff $[x] \rightharpoonup [x \wedge y] = [1]$ iff $[x \sim x \wedge y] = [1]$ iff $x \sim x \wedge y \in Ker(\tau)$ iff $\tau(x \sim x \wedge y) = 1$ iff $\tau(x \to y) = 1$.

(2) We have $[x] = [y]$ iff $[x] \rightharpoonup [y] = [1]$ iff $[x \sim y] = [1]$ iff $x \sim y \in Ker(\tau)$ iff $\tau(x \sim y) = 1$.

Definition 4.10. Let $(\mathcal{E}, \tau)$ be a very true equality algebra and θ be a congruence on $\mathcal{E}$. Then θ is called a *very true congruence* on $(\mathcal{E}, \tau)$ if $(x, y) \in \theta$ implies $(\tau(x), \tau(y)) \in \theta$ for each $x, y \in \mathcal{E}$.

Example 4.11. Consider the Example 3.4, one can see that $R = \{\{0, 0\}, \{a, a\}, \{b, b\}, \{1, 1\}, \{a, b\}, \{b, a\}, \{a, 1\}, \{1, a\}, \{1, b\}, \{b, 1\}\}$ is a very true congruence on a very true equality algebra $(\mathcal{E}, \tau)$.

Theorem 4.12. Let $(\mathcal{E}, \tau)$ be a very true equality algebra. Then there is a one to one correspondence between its very true filters and its very true congruences.

Proof. Suppose that θ is a very true congruence relation on $(\mathcal{E}, \tau)$. Clearly $F_\theta = \{x \in \mathcal{E} | (x, 1) \in \theta\}$ is a very true filter of $(\mathcal{E}, \tau)$. Now given $x \in F_\theta$, we have $(x, 1) \in \eta$ and hence $(\tau(x), 1) = (\tau(x), \tau(1)) \in \theta$ and therefore $\tau(x) \in F_\theta$. This proves that F_θ is a very true filter on $(\mathcal{E}, \tau)$. Conversely, let F be a very true filter. Then θ_F is a very true congruence on very true equality algebra, since for each $(x, y) \in \theta_F$, we have $x \sim y \in F$. Since F is a very true filter and hence $\tau(x \sim y) \in F$. By (VE4), we have $\tau(x) \sim \tau(y) \in \theta_F$, thus θ_F is an very true congruence of $(\mathcal{E}, \tau)$. It can be easily shown that $gh(\theta) = \theta$ and $hg(F) = F$, for all very true congruence θ and very true filter F of $(\mathcal{E}, \tau)$.

As another applications of very true filter on very true equality algebra, we consider the uniformity structure on a very true equality algebra.

Theorem 4.13. Let $(\mathcal{E}, \tau)$ be a very true equality algebra and F be a very true filter of $(\mathcal{E}, \tau)$. Define $U_F = \{(x, y) \in \mathcal{E} \times \mathcal{E} | \tau(x) \sim \tau(y) \in F\}$ and $K^* = \{U_F | F$ is a very true filter of $(\mathcal{E}, \tau)\}$. Then K^* satisfies the conditions (U_1)-(U_4).

Proof. Now, we will show that K^* satisfies the conditions (U_1)-(U_4).

(U_1) Let $U_F \in K^*$ and $(x, x) \in \triangle$. Since $x \sim x = 1 \in F$, we have $(x, x) \in U_F$. Therefore (U_1) holds.

(U_2) Note that $(x, y) \in U_F$ if and only if $\tau(x) \sim \tau(y) \in F$ if and only if $\tau(y) \sim \tau(x) \in F$ if and only if $(y, x) \in U_F$ if and only if $(x, y) \in U_F^{-1}$. Thus $U_F^{-1} = U_F \in K^*$. Therefore (U_2) is true.

(U_3) Let $\Sigma(F) = \{F_a | F_a \subseteq F\}$ be the collection of very true filters contained in F. Clearly, $\Sigma(F)$ is not empty. Let G be the very true filter generated by $\cup_a F_a$. Then $U_G \subseteq K^*$. It is sufficient to show that $U_G \circ U_G \subseteq U_F$. If $(x, y) \in U_G \circ U_G$, then there exists $z \in \mathcal{E}$ such that $(x, z) \in U_G$ and $(z, y) \in U_G$. It follows from $(E7)$ in Definition 2.1 that $(x, y) \in U_G$, that is, $\tau(x) \sim \tau(y) \in G$. Since G is the minimal very true filter containing $\cup_a F_a$ and $\cup_a F_a \subseteq F$, it follows that $G \subseteq F$. Thus $\tau(x) \sim \tau(y) \in F$ and hence $(x, y) \in U_F$. Therefore $U_G \circ U_G \subseteq U_F$ is true.

(U_4) This will from the observation that $U_G \cap U_F = U_{G \cap F}$ for all U_G, $U_F \in K^*$. Indeed, if $(x, y) \in U_G \cap U_F$. Then $(x, y) \in U_G$ and $(x, y) \in U_F$, which implies $\tau(x) \sim \tau(y) \in G$ and $\tau(x) \sim \tau(y) \in F$. Thus $\tau(x) \sim \tau(y) \in G \cap F$ and hence $(x, y) \in U_{G \cap F}$. Similary, we can show that $U_{F \cap G} \subseteq U_F \cap U_G$, whence $U_{F \cap G} = U_F \cap U_G$. This completeness the proof.

Theorem 4.14. Let $(\mathcal{E}, \tau)$ be a very true equality algebra. Define $K = \{U \in \mathcal{E} \times \mathcal{E} | U_F \subseteq U$, for some $U_F \in K^*\}$. Then K satisfies a uniformity on $(\mathcal{E}, \tau)$ and hence the pair $((\mathcal{E}, \tau), K)$ is a uniform space.

Proof. Using Theorem 4.13, we can show that K satisfies the conditions $(U_1) - (U_4)$. Now, we will show that it satisfies (U_5). If $U \in K$ and $U \subseteq V \subseteq \mathcal{E} \times \mathcal{E}$. Then there exists a $U_F \in K^*$ such that $U_F \subseteq U \subseteq V$, which implies that $V \in K$. Therefore,$((\mathcal{E}, \tau), K)$ is a uniform space.

For $x \in \mathcal{E}$ and $U \in K$, we define

$$U[x] = \{y \in \mathcal{E} | (x, y) \in U\}.$$

Theorem 4.15. Let $(\mathcal{E}, \tau)$ be a very true equality algebra. Define $T = \{G \subseteq \mathcal{E} | \tau x \in G, \exists U \in K, U[x] \subseteq G\}$. Then T is a topology on $(\mathcal{E}, \tau)$.

Proof. It is clear $\emptyset \in T$ and $\mathcal{E} \in T$. Also from the nature of that definition, it is clear that T is closed under arbitrary union. Finally to show that T is closed under finite intersection, let $G, W \in T$ and suppose $x \in G \cap W$. Then there exist U and $V \in K$ such that $U[x] \subseteq G$ and $V[x] \subseteq W$. Let $N = U \cap V$, then $N \in K$. Also $N[x] \subseteq U[x] \cap V[x]$ and hence $N[x] \subseteq G \cap W$, thus $G \cap W \in T$. Therefore T is a topology on $(\mathcal{E}, \tau)$.

Note We denote the uniform topology obtained by an arbitrary family Λ, by T_Λ and If $\Lambda = \{F\}$, we denote it by T_F.

Theorem 4.16. Let $(\mathcal{E}, \tau)$ be a very true equality algebra. For each $x \in \mathcal{E}$, the collection $U_x = \{U[x] | U \in K\}$ forms a neighborhood base at x, making $(\mathcal{E}, \tau)$ a topological space.

Proof. We note that $x \in U[x]$ for each x. Since $U_1[x] \cap U_2[x] = (U_1 \cap U_2)[x]$, which means that the intersection of neighborhoods is a neighborhood. Finally, if $U[x] \in U_x$ then there exists a $W \in K$ such that $W \circ W \subseteq U$ by (U_3). Then for any $y \in W[x], W[y]$, so this property of neighborhoods is satisfied.

Theorem 4.17. Let $(\mathcal{E}, \tau)$ be a very true equality algebra and $(\mathcal{E}, T_{Ker(\tau)})$ be a compact topological space. Then $\tau(\mathcal{E})$ is finite.

Proof. Since $\mathcal{E} = \cup\{U_F[x] | x \in \mathcal{E}\}$ and $(\mathcal{E}, T_F)$ is a compact topological space. Thus there exist $x_1, x_2 \cdots, x_n \in \mathcal{E}$ such that $\mathcal{E} = \cup U_F[x_i]$. Now, let x be a arbitrary element of $\mathcal{E}$. By Theorem 3.7, $\tau : \mathcal{E} \longrightarrow \tau(\mathcal{E})$ is a homomorphism and by the Note after the Theorem 3.6, $(\tau(\mathcal{E}), \rightsquigarrow, 1)$ is an equality algebra, hence we have $\tau(U_F[x]) = \{\tau(y) | y \in U_F[x]\} = \{\tau(y) | (\tau(x), \tau(y) \in \theta_F\} = \{\tau(y) | \tau(x) \sim \tau(y) \in F\} = \{\tau(y) | \tau(x) = \tau(y)\} = \{\tau(x)\}$. Thus $\tau(U_F[x]) = \{\tau(x)\}$ and so $Fix_\tau = \cup U_F[x_i] = \{\tau(x_1), \cdots, \tau(x_n)\}$, that is, $\tau(\mathcal{E})$ is finite.

Corollary 4.18. Let $(\mathcal{E}, \tau)$ be a very true equality algebra and $(\mathcal{E}, T_{Ker(\tau)})$ be a compact topological space. Then $\tau(\mathcal{E})$ is a compact subset of $\mathcal{E}$.

Proof. It follows from Theorem 4.17 directly and hence we omit it.

5. Conclusion

As we mentioned in the introduction, the study of equality algebras is motivated by the goal to develop appropriate algebraic semantics for fuzzy type theory, so a concept of fuzzy type theory should be introduced based on these algebras. In this paper, motivating by the previous research on very true residuated lattice, very true MV-algebras and very true BL-algebras, we extended the concept of very true operators to the more general fuzzy structures, namely equality algebras. We introduce and study very true equality algebras and prove some new results regarding these structures. The aim of this paper is to provide an algebraic foundation for fuzzy type theory. Since the above topics are of current interest we suggest further directions of research:

1. Characterize very true filter generated by a subset of an very true equality algebra in terms of fuzzy equality operation.

2. Define and characterize subdirectly irreducible very true equality algebras.

3. Establish the logic system corresponding to very true equality algebra and prove the soundness and completeness theorem of this logic.

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A Splitting Iterative Method for a System of Accretive Inclusions in Banach Spaces

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Abstract

In this paper, a system of accretive inclusions is proposed and a splitting iterative method is investigated for solutions of proposed system of operator inclusion problems. Under suitable conditions on the parameters, strong convergence of our splitting iterative method is established in a reflexive Banach space.

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1 Introduction

In the area of nonlinear analysis, the theory of accretive operators is an important and developing field [3, 4]. The class of accretive operators is firmly connected with equations of evolutions found in the heat, wave, Schrödinger and similar other equations [5]. Many problems in operations research and mathematical physics can be written as variational inequalities, equilibrium problems or operator inclusions with accretive operators [2, 10, 17].

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. One popular method for solving the following inclusion problem:

$$\text{find } z \in H \text{ such that } 0 \in Az, \tag{1.1}$$

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where $A : H \to 2^H$ is an m-accretive operator, is the proximal point algorithm, which was proposed by Martinet [15, 16] and generalized by Rockafellar [20, 21]. Rockafellar [20] proved the weak convergence of the sequence $\{x_n\}$ defined by

$$x_{n+1} = J_{r_n}^A x_n, \quad n \in \mathbb{N} \tag{1.2}$$

to an element of solution set of problem (1.1). The weak and strong convergence of the sequence $\{x_n\}$ defined by (1.2) have been extensively discussed in Hilbert and Banach spaces (see [7, 28, 29, 30, 31] and the references therein). The proximal point like methods for finding solutions of problem (1.1) have been studied by Lehdili and Moudafi [12] and Tossings [26] in Hilbert spaces and by Sahu and Yao [23] in Banach spaces.

Many nonlinear problems arising in applied areas such as image recovery, signal processing, and machine learning can be mathematically modeled in form of inclusion problem:

to find $z \in C$ such that

$$0 \in (A + B)z, \tag{1.3}$$

where C is a nonempty closed convex subset of H, $A : H \to 2^H$ and $B : C \to H$ are monotone operators. For instance, a stationary solution to the initial value problem of the evolution equation

$$0 \in \frac{\partial u}{\partial t} + Fu, \quad u_0 = u(0)$$

can be recast as (1.3) when the governing maximal monotone F is of the form $F = A + B$, see, for example, [13].

The central problem is to iteratively find the solution of the inclusion problem (1.3) when A and B are two monotone operators in a Hilbert space H. One method for finding solutions of problem (1.3) is *splitting method*. A splitting method for (1.3) means an iterative method for which each iteration involves only with the individual operators A and B, but not the sum $A + B$. Splitting methods for linear equations were introduced by Douglas and Rachford [8] and Peaceman and Rachford [18]. Extensions to nonlinear equations in Hilbert spaces were carried out by Kellogg [9] and Lions and Mercier [13] (see also [22, 27]).

In this paper, we are interested in the following system of operator inclusion problems:

$$\text{find } z \in C \text{ such that } 0 \in (A_i + B_i)z, \quad i \in \Delta_N := \{1, 2, \cdots, N\}, \tag{1.4}$$

in the framework of a Banach space X, where $N \geq 1$ is a positive integer, C is a nonempty closed convex subset of X, $A_i : X \to 2^X$ is an m-accretive operator such that $\bigcap_{i \in \Delta_N} \overline{D(A_i)} \subseteq C$ and $B_i : C \to X$ an operator. The inclusion problem (1.4) is more general in nature. For instance, if B_i is the operator constantly zero for all $i \in \Delta_N$, the problem (1.4) reduces

$$\text{find } z \in C \text{ such that } 0 \in A_i z, \quad i \in \Delta_N. \tag{1.5}$$

The purpose of this paper is to introduce a forward-backward splitting method to solve the system of operator inclusion problem (1.4) in a Banach space. We prove strong convergence of iterative sequences generated by our algorithm. In Section 2, we give duality mappings, nonexpansive mappings and their properties and accretive operators and their properties. In Section 3, we introduce a forward-backward splitting method and state main theoretical result of the paper. Our iterative method improves and generalizes the corresponding results of inclusion problem (1.5).

2 Preliminaries

2.1 Duality mappings

A continuous strictly increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a *gauge* if $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. The mapping $J_\varphi : X \to 2^{X^*}$ defined by

$$J_\varphi(x) = \{j \in X^* : \langle x, j\rangle = \|x\|\|j\|_*,\ \|j\|_* = \varphi(\|x\|)\}, \quad x \in X,$$

is called the duality mapping with gauge φ. In the special case where $\varphi(t) = t$, the duality mapping $J_\varphi =: J$ is the classical normalized duality mapping. In the case $\varphi(t) = t^{p-1}$, $p > 1$, the duality mapping $J_\varphi =: J_p$ is called the generalized duality mapping and it is given by

$$J_p(x) := \{j \in X^* : \langle x, j\rangle = \|x\|\|j\|_*, \|j\|_* = \|x\|^{p-1}\}, \quad x \in X.$$

Note that if $p = 2$, then $J_2 = J$ is the normalized duality mapping. By φ we always mean a gauge and by Φ the corresponding function defined by

$$\Phi(t) = \int_0^t \varphi(s)ds.$$

For a smooth Banach space X, we have

$$\Phi(\|x + y\|) \le \Phi(\|x\|) + \langle y, J_\varphi(x + y)\rangle \quad \text{for all } x, y \in X; \tag{2.1}$$

or considering the normalized duality mapping J, we have

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, J(x + y)\rangle \quad \text{for all } x, y \in X.$$

2.2 Nonexpansive mappings

Let C be a nonempty subset of a Banach space X and $T : C \to X$ a mapping. T is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\| \quad \text{for all } x, y \in C.$$

The set of fixed point of T is denoted by $Fix(T)$.

The following result was proved by Kirk [11].

Lemma 2.1. (Kirk [11]) Let X be a reflexive Banach space and let C be a nonempty closed convex bounded subset of X which has normal structure. Let T be a nonexpansive mapping of C into itself. Then T has a fixed point.

A subset C of a Banach space X is called a retract of X if there exists a continuous mapping P from X onto C such that $Px = x$ for all x in C. We call such P a *retraction* of X onto C. It follows that if a mapping P is a retraction, then $Py = y$ for all y in the range of P. A retraction P is said to be *sunny* if $P(Px + t(x - Px)) = Px$ for each x in X and $t \ge 0$. If a sunny retraction P is also nonexpansive, then C is said to be a *sunny nonexpansive* retract of X.

The following lemmas will be useful for our main result.

Lemma 2.2. ([1, Corollary 5.6.4]) Let X be a Banach space with a weakly continuous duality mapping $J_\varphi : X \to X^*$ with gauge function φ. Let C be a nonempty closed convex subset of X and $T : C \to C$ an nonexpansive mapping. Then $I - T$ is demiclosed at zero, that is, if $\{x_n\}$ is a sequence in C which converges weakly to x and if the sequence $\{x_n - Tx_n\}$ converges strongly to zero, then $x - Tx = 0$.

Lemma 2.3. ([6]) Let X be a strictly convex Banach space. Let C be a nonempty closed convex subset of X and let $N \geq 1$ be a positive integer. For each $i \in \Delta_N$, let $T_i : C \to C$ be a nonexpansive mapping such that $\bigcap_{i\in\Delta_N} Fix(T_i) \neq \emptyset$. Let $\{\delta_i\}_{i\in\Delta_N} \subset (0, 1)$ such that $\sum_{i=1}^N \delta_i = 1$. Then the mapping $\sum_{i=1}^N \delta_i T_i$ is nonexpansive with $Fix(\sum_{i=1}^N \delta_i T_i) = \bigcap_{i\in\Delta_N} Fix(T_i)$.

Lemma 2.4. *Let C be a convex subset of a smooth Banach space X, D a nonempty subset of C and P a retraction from C onto D. Then the following are equivalent:*
(a) *P is a sunny and nonexpansive.*
(b) *$\langle x - Px, J(z - Px)\rangle \leq 0$ for all $x \in C, z \in D$.*
(c) *$\langle x - y, J(Px - Py)\rangle \geq \|Px - Py\|^2$ for all $x, y \in C$.*

Lemma 2.5. *Let X be a reflexive Banach space which has a weakly continuous duality map J_φ. Let C be a nonempty closed convex subset of X and $T : C \to C$ a nonexpansive mapping such that $Fix(T) \neq \emptyset$. Then $Fix(T)$ is the sunny nonexpansive retract of C.*

The property $(\mathcal{N})$ alludes to the fact that in order to solve the system of operator inclusions (1.4).

Definition 2.6. ([22]) Let C be a nonempty closed convex subset of a Banach space X. An operator $B : C \to X$ is said to satisfy the *property* $(\mathcal{N})$ on $(0, \gamma_{X,B})$ if there exists $\gamma_{X,B} \in (0, \infty]$, depends on X and B, such that $I - \xi B : C \to C$ is nonexpansive for each $\xi \in (0, \gamma_{X,B})$.

Remark 2.7. For a nonexpansive mapping $T : C \to C$ with $B = I - T$, the average mapping $T_w = I - wB$ is always nonexpansive for each $w \in (0, \gamma_{X,B})$, where $\gamma_{X,B} = 1$.

2.3 Accretive operators

Let X be a real Banach space. For an operator $A : X \to 2^X$, we define its domain, range and graph as follows:

$$D(A) = \{x \in X : Ax \neq \emptyset\},$$

$$R(A) = \bigcup\{Az : z \in D(A)\},$$

and

$$G(A) = \{(x, y) \in X \times X : x \in D(A), y \in Ax\},$$

respectively. Thus, we write $A : X \to 2^X$ as follows: $A \subseteq X \times X$. The inverse A^{-1} of A is defined by

$$x \in A^{-1}y \text{ if } \text{ and only if } y \in Ax.$$

The operator A is said to be *accretive* if for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j\rangle \geq 0$. An accretive operator A is said to be *maximal accretive* if there is no proper accretive extension of A and *m-accretive* if $R(I+A) = X$, where I stands for the identity operator on X (It follows that $R(I+rA) = X$ for all $r > 0$). If A is m-accretive, then it is maximal accretive, but the converse is not true in general. If A is accretive, then we can define, for each $\lambda > 0$, a nonexpansive single-valued mapping $J_\lambda^A : R(1 + \lambda A) \to D(A)$ by

$$J_\lambda^A = (I + \lambda A)^{-1}.$$

It is called the *resolvent* of A. It is well known that if A is an m-accretive operator on a Banach space X, then for each $\lambda > 0$, the resolvent $J_\lambda^A = (I + \lambda A)^{-1}$ is a single-valued nonexpansive mapping whose domain is entire space X. An accretive operator A defined on a Banach space X is said to satisfy the *range condition* if $\overline{D(A)} \subset R(1+\lambda A)$ for all $\lambda > 0$, where $\overline{D(A)}$ denotes the closure of the domain of A. It is well known that for an accretive operator A which satisfies the range condition, $A^{-1}(0) = Fix(J_\lambda^A)$ for all $\lambda > 0$. We also define the *Yosida approximation* A_r by $A_r = (I - J_r^A)/r$. We know that $A_r x \in AJ_r^A x$ for all $x \in R(I + rA)$ and $\|A_r x\| \leq |Ax| = \inf\{\|y\| : y \in Ax\}$ for all $x \in D(A) \cap R(I + rA)$. We also know the following [25]: For each $\lambda, \mu > 0$ and $x \in R(I + \lambda A) \cap R(I + \mu A)$, it holds that

$$\left\|J_\lambda^A x - J_\mu^A x\right\| \leq \frac{|\lambda - \mu|}{\lambda}\|x - J_\lambda^A x\|.$$

Lemma 2.8. ([22]) *Let C be a nonempty closed convex subset of a Banach space X, $A \subseteq X \times X$ an accretive operator such that $\overline{D(A)} \subseteq C \subseteq \bigcap_{t>0} R(I + tA)$ and $B : C \to X$ an operator such that $Zer(A+B) \neq \emptyset$ and B has the property $(\mathcal{N})$ on $(0, \gamma_{X,B})$, where $\gamma_{X,B}$ is a constant depends on X and B. For $r \in (0, \gamma_{X,B})$, define an operator $J_r^{A,B} : C \to C$ by*

$$J_r^{A,B} x = J_r^A (I - rB)x, \quad x \in C.$$

Then the following statements hold.
(a) *$J_r^{A,B}$ is nonexpansive.*
(b) *$Fix(J_r^{A,B}) = Zer(A + B)$.*

Lemma 2.9. ([24]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that*

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n y_n \quad \text{for all } n \in \mathbb{N}$$

and

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.10. ([14]) *Let $\{a_n\}$ and $\{c_n\}$ be two sequences of nonnegative real numbers and let $\{b_n\}$ be a sequence in $\mathbb{R}$ satisfying the following condition:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + b_n + c_n \quad \text{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0, 1]$. Assume that $\sum_{n=1}^{\infty} c_n < \infty$. Then the following statements hold:

(*a*) *If* $b_n \leq K\alpha_n$ *for all* $n \in \mathbb{N}$ *and for some* $K \geq 0$, *then*

$$a_{n+1} \leq \delta_n a_1 + (1-\delta_n)K + \sum_{j=1}^{n} c_j \quad \text{for all } n \in \mathbb{N},$$

where $\delta_n = \prod_{j=1}^{n}(1-\alpha_j)$ *and hence* $\{a_n\}$ *is bounded.*

(*b*) *If* $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\limsup_{n\to\infty} \frac{b_n}{\alpha_n} \leq 0$, *then* $\{a_n\}_{n=1}^{\infty}$ *converges to zero.*

3 Main results

Now we are ready to prove our main result for solving the system of operator inclusions (1.4) in the framework of Banach space.

Theorem 3.1. *Let* X *be a strictly convex and reflexive Banach space which has a weakly continuous duality map* J_φ. *Let* C *be a nonempty closed convex subset of* X *and let* $N \geq 1$ *be a positive integer. Let* $f : C \to C$ *be a contraction mapping with Lipschitz constant* k_f. *For each* $i \in \Delta_N$, *let* $A_i : X \to 2^X$ *be an* m*-accretive operator such that* $\bigcap_{i\in\Delta_N} \overline{D(A_i)} \subseteq C$ *and* $B_i : C \to X$ *an operator such that* $\mathbb{S} := \bigcap_{i\in\Delta_N} Zer(A_i + B_i) \neq \emptyset$. *For each* $i \in \Delta_N$, *let* B_i *has the property* ($\mathcal{N}$) *on* $(0, \gamma_{X,B_i})$, *where* γ_{X,B_i} *is a constant depends on* X *and* B_i.

Let $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *be real number sequences in* $(0, 1)$ *and let* $\{\delta_{n,i}\}$ *be a real number sequence in* $(0, 1)$ *for each* $i \in \Delta_N$ *satisfying the following conditions*:

($C1$) $\alpha_n + \beta_n + \gamma_n = \sum_{i=1}^{N} \delta_{n,i} = 1$ *for all* $n \in \mathbb{N}$,

($C2$) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

($C3$) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

($C4$) $\lim_{n\to\infty} \delta_{n,i} = \delta_i \in (0, 1)$ *for all* $i \in \Delta_N$.

Let $\{x_n\}$ *be a sequence in* C *generated by the following splitting iterative method:*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{n=1}^{N} \delta_{n,i} J_{r_i}^{A_i}(I - r_i B)x_n \quad \text{for all } n \in \mathbb{N}, \tag{3.1}$$

where $\{r_i\}_{i\in\Delta_N}$ *is a set of positive real numbers. Then* $\{x_n\}$ *converges strongly to* $x^* \in C$, *which is the unique solution to the following variational inequality:*

$$\text{to find } z \in \bigcap_{i\in\Delta_N} Zer(A_i + B_i) \text{ such that } \langle (I-f)z, x - z\rangle \geq 0 \tag{3.2}$$

for all $x \in \bigcap_{i\in\Delta_N} Zer(A_i + B_i)$.

Proof. (a) Define $T = \sum_{i=1}^{N} \delta_i J_{r_i}^{A_i}(I - r_i B)$. From Lemmas 2.3 and 2.8, we see that T is nonexpansive with $Fix(T) = \bigcap_{i\in\Delta_N} Zer(A_i + B_i)$. From Lemma 2.5 shows that $\mathbb{S}$ is the Sunny nonexpansive retract of C. Let $Q_{\mathbb{S}}$ be the sunny nonexpansive retraction of C onto $\mathbb{S}$. It follows that $Q_{\mathbb{S}} f$ is a contraction. Hence there exists a unique fixed point $x^* \in C$ of $Q_{\mathbb{S}} f$. From Lemma 2.4 that the variational inequality problem (3.2) has a unique solution $x^* \in C$.

(b) We proceed with the following steps:

Step I: $\{x_n\}$ *is bounded.*

From (3.1), we have

$$\begin{aligned}
\|x_{n+1}-x^*\| &\le \alpha_n\|f(x_n)-x^*\| + \beta_n\|x_n-x^*\| + \gamma_n\left\|\sum_{i=1}^N \delta_{n,i}J_{r_i}^{A_i}(I-r_iB)x_n - x^*\right\| \\
&\le k_f\alpha_n\|x_n-x^*\| + \alpha_n\|f(x^*)-x^*\| + \beta_n\|x_n-x^*\| \\
&\quad + \gamma_n\sum_{i=1}^N \delta_{n,i}\|J_{r_i}^{A_i}(I-r_iB)x_n - x^*\| \\
&\le k_f\alpha_n\|x_n-x^*\| + \alpha_n\|f(x^*)-x^*\| + (1-\alpha_n)\|x_n-x^*\| \\
&= (1-(1-k_f)\alpha_n)\|x_n-x^*\| + \alpha_n\|f(x^*)-x^*\| \\
&\le \max\left\{\|x_n-x^*\|, \frac{\|f(x^*)-x^*\|}{1-k_f}\right\} \\
&\le \max\left\{\|x_1-x^*\|, \frac{\|f(x^*)-x^*\|}{1-k_f}\right\}.
\end{aligned}$$

Thus, $\{x_n\}$ is bounded.

Step II: $\|x_{n+1}-x_n\| \to 0$ *and* $\|x_n - Tx_n\| \to 0$ *as* $n \to \infty$.

Set $y_n = \sum_{n=1}^N \delta_{n,i}J_{r_i}^{A_i}(I-r_iB)x_n$. Note

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \left\|\sum_{i=1}^N \delta_{n+1,i}J_{r_i}^{A_i}(I-r_iB)x_{n+1} - \sum_{i=1}^N \delta_{n,i}J_{r_i}^{A_i}(I-r_iB)x_n\right\| \\
&\le \left\|\sum_{i=1}^N \delta_{n+1,i}J_{r_i}^{A_i}(I-r_iB)x_{n+1} - \sum_{i=1}^N \delta_{n+1,i}J_{r_i}^{A_i}(I-r_iB)x_n\right\| \\
&\quad + \left\|\sum_{i=1}^N \delta_{n+1,i}J_{r_i}^{A_i}(I-r_iB)x_n - \sum_{i=1}^N \delta_{n,i}J_{r_i}^{A_i}(I-r_iB)x_n\right\| \\
&\le \sum_{i=1}^N \delta_{n+1,i}\|J_{r_i}^{A_i}(I-r_iB)x_{n+1} - J_{r_i}^{A_i}(I-r_iB)x_n\| \\
&\quad + \left\|\sum_{i=1}^N (\delta_{n+1,i}-\delta_{n,i})J_{r_i}^{A_i}(I-r_iB)x_n\right\| \\
&\le \|x_{n+1}-x_n\| + \sum_{i=1}^N |\delta_{n+1,i}-\delta_{n,i}|\ \|J_{r_i}^{A_i}(I-r_iB)x_n\|.
\end{aligned}$$

From (3.1), we have

$$\begin{aligned}
x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n \\
&= \beta_n x_n + (1-\beta_n)z_n,
\end{aligned}$$

where

$$z_n = \frac{1}{1-\beta_n}\left[\alpha_n f(x_n) + \gamma_n y_n\right].$$

Hence

$$\begin{aligned}
z_{n+1} - z_n &= \frac{1}{1-\beta_{n+1}}\left[\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} y_{n+1}\right] - \frac{1}{1-\beta_n}\left[\alpha_n f(x_n) + \gamma_n y_n\right] \\
&= \frac{1}{1-\beta_{n+1}}\left[\alpha_{n+1} f(x_{n+1}) + (1-\alpha_{n+1}-\beta_{n+1}) y_{n+1}\right] \\
&\quad - \frac{1}{1-\beta_n}\left[\alpha_n f(x_n) + (1-\alpha_n-\beta_n) y_n\right] \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left[f(x_{n+1}) - y_{n+1}\right] - \frac{\alpha_n}{1-\beta_n}\left[f(x_n) - y_n\right] + y_{n+1} - y_n.
\end{aligned}$$

Note

$$\begin{aligned}
\|z_{n+1} - z_n\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1}) - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|f(x_n) - y_n\| + \|y_{n+1} - \gamma_n\| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1}) - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|f(x_n) - y_n\| \\
&\quad + \|x_{n+1} - x_n\| + \sum_{i=1}^{N} |\delta_{n+1,i} - \delta_{n,i}|\ \|J_{r_i}^{A_i}(I - r_i B)x_n\|.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1}) - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|f(x_n) - y_n\| \\
&\quad + \sum_{i=1}^{N} |\delta_{n+1,i} - \delta_{n,i}|\|J_{r_i}^{A_i}(I - r_i B)x_n\|.
\end{aligned}$$

From the conditions $(C1)$-$(C4)$, we get

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.$$

From Lemma 2.9, we obtain that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0.$$

Since $x_{n+1} - x_n = (1-\beta_n)(z_n - x_n)$, we have

$$\|x_{n+1} - x_n\| = (1-\beta_n)\|z_n - x_n\| \le \|z_n - x_n\| \to 0 \quad \text{as } n \to \infty.$$

Observe that

$$\begin{aligned}
\|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - Tx_n\| + \beta_n\|x_n - Tx_n\| \\
&\quad + \gamma_n\left\|\sum_{i=1}^{N}(\delta_{n,i} - \delta_i)J_{r_i}^{A_i}(I - r_i B)x_n\right\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - Tx_n\| + \beta_n\|x_n - Tx_n\| \\
&\quad + \gamma_n\sum_{i=1}^{N}|\delta_{n,i} - \delta_i|\ \|J_{r_i}^{A_i}(I - r_i B)x_n\|,
\end{aligned}$$

which implies that

$$(1-\beta_n)\|x_n - Tx_n\| \le \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - Tx_n\| + \gamma_n \sum_{i=1}^{N} |\delta_{n,i} - \delta_i| \; \|J_{r_i}^{A_i}(I - r_i B)x_n\|.$$

Thus, we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Step III: $\{x_n\}$ *converges strongly to* x^*.

Take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, J_\varphi(x_n - x^*)\rangle = \lim_{i\to\infty}\langle f(x^*) - x^*, J_\varphi(x_{n_i} - x^*)\rangle.$$

Since X is reflexive, we may further assume that $x_{n_i} \rightharpoonup z$ for some $z \in C$. It follows from Lemma 2.2 that $z \in Fix(T)$. From the weak continuity of the duality mapping J_φ and (3.2) we obtain that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(x^*) - x^*, J_\varphi(x_n - x^*)\rangle &= \lim_{i\to\infty}\langle f(x^*) - x^*, J_\varphi(x_{n_i} - x^*)\rangle \\ &= \langle f(x^*) - x^*, J_\varphi(z - x^*)\rangle \\ &\le 0.\end{aligned}$$

From (2.1) and (3.1), we have

$$\begin{aligned}&\Phi(\|x_{n+1} - x^*\|) \\ &= \Phi(\|\alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - x^*\|) \\ &\le \Phi(\|\alpha_n(f(x_n) - f(x^*) + \beta_n(x_n - x^*) + \gamma_n(y_n - x^*) + \alpha_n(f(x^*) - x^*)\|) \\ &\le \Phi(\|\alpha_n(f(x_n) - f(x^*) + \beta_n(x_n - x^*) + \gamma_n(y_n - x^*)\|) \\ &\quad + \alpha_n\langle f(x^*) - x^*, J_\varphi(x_{n+1} - x^*)\rangle \\ &\le \Phi(\alpha_n\|f(x_n) - f(x^*)\| + \beta_n\|x_n - x^*\| + \gamma_n\|y_n - x^*\|) \\ &\quad + \alpha_n\langle f(x^*) - x^*, J_\varphi(x_{n+1} - x^*)\rangle \\ &\le \Phi((1 - (1 - k_f)\alpha_n)\|x_n - x^*\|) + \alpha_n\langle f(x^*) - x^*, J_\varphi(x_{n+1} - x^*)\rangle \\ &\le (1 - (1 - k_f)\alpha_n)\Phi(\|x_n - x^*\|) + \alpha_n\langle f(x^*) - x^*, J_\varphi(x_{n+1} - x^*)\rangle.\end{aligned}$$

Noticing that $\limsup_{n\to\infty}\langle f(x^*) - x^*, J_\varphi(x_{n+1} - x^*)\rangle \le 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Therefore, we conclude from Lemma 2.10 that $\Phi(\|x_n - x^*\|) \to 0$, that is, $\{x_n\}$ converges strongly to x^*. □

Theorem 3.1 is more general in nature due to the property $(\mathcal{N})$ of operators B_i, therefore, we are able to derive the some new and known results from it. To demonstrate the wide range of applicability of our convergence theory, a few examples are detailed below. In particular for $B_i = 0$, we immediately obtain an improvement upon Qing and Lv [19, Theorem 2.1] as follows:

Corollary 3.2. *Let X be a strictly convex and reflexive Banach space which has a weakly continuous duality map J_φ. Let C be a nonempty closed convex subset of X and let $N \geq 1$ be a positive integer. Let $f : C \to C$ be a contraction mapping with Lipschitz constant k_f. For each $i \in \Delta_N$, let $A_i : X \to 2^X$ be an m-accretive operator such that $\bigcap_{i\in\Delta_N} \overline{D(A_i)} \subseteq C$ such that $\bigcap_{i\in\Delta_N} A_i^{-1}(0) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0, 1)$ and let $\{\delta_{n,i}\}$ be a real number sequence in $(0, 1)$ for each $i \in \Delta_N$ satisfying the conditions $(C1)$-$(C4)$.*

Let $\{x_n\}$ be a sequence in C generated in the following iterative process:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{n=1}^{N} \delta_{n,i} J_{r_i}^{A_i} x_n \quad \text{for all } n \in \mathbb{N},$$

where $\{r_i\}_{i\in\Delta_N}$ is a set of positive real numbers. Then $\{x_n\}$ converges strongly to $x^* \in C$, which is the unique solution to the following variational inequality:

$$\text{to find } z \in \bigcap_{i\in\Delta_N} A_i^{-1}(0) \text{ such that } \langle (I - f)z, x - z\rangle \geq 0$$

for all $x \in \bigcap_{i\in\Delta_N} A_i^{-1}(0)$.

Theorem 3.3. *Let X be a strictly convex and reflexive Banach space which has a weakly continuous duality map J_φ. Let C be a nonempty closed convex subset of X and let $N \geq 1$ be a positive integer. Let $f : C \to C$ be a contraction mapping with Lipschitz constant k_f. For each $i \in \Delta_N$, let $A_i : X \to 2^X$ be an m-accretive operator such that $\bigcap_{i\in\Delta_N} \overline{D(A_i)} \subseteq C$ and $T_i : C \to C$ a nonexpansive with $B_i = I - T_i$ such that $\mathbb{S} := \bigcap_{i\in\Delta_N} Zer(A_i + B_i) \neq \emptyset$.*

Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0, 1)$ and let $\{\delta_{n,i}\}$ be a real number sequence in $(0, 1)$ for each $i \in \Delta_N$ satisfying the conditions $(C1)$-$(C4)$.

Let $\{x_n\}$ be a sequence in C generated in the following iterative process:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{n=1}^{N} \delta_{n,i} J_{r_i}^{A_i}(I - r_i B_i)x_n \quad \text{for all } n \in \mathbb{N},$$

where $\{r_i\}_{i\in\Delta_N}$ is a set of positive real numbers. Then $\{x_n\}$ converges strongly to $x^* \in C$, which is the unique solution to the following variational inequality:

$$\text{to find } z \in \bigcap_{i\in\Delta_N} Zer(A_i + B_i) \text{ such that } \langle (I - f)z, x - z\rangle \geq 0$$

for all $x \in \bigcap_{i\in\Delta_N} Zer(A_i + B_i)$.

Proof. Note each T_i is nonexpansive with $B_i = I - T_i$. It follows from Remark 2.7 that the average mapping $T_w^{(i)} = I - wB_i$ is always nonexpansive for each $w \in (0, \gamma_{X,B_i})$, where $\gamma_{X,B_i} = 1$. Therefore, Theorem 3.3 follows from Theorem 3.1. □

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Sidi-Israeli Quadrature Method for Steady-State Anisotropic Field Problems by Direct Domain Mapping*

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Abstract

In this paper, the two-dimensional steady-state anisotropic field problems are transformed into the Laplace equation by direct domain mapping, and then the Sidi-Israeli quadrature method is applied to solve the weakly singular boundary integral equation of the Laplace equation. Especially, the kress's variable transformation is used for the polygon case in order to improve the accuracy by smoothing the singularities of the exact solution at the corner points of the boundary. The convergence and error analysis of numerical solutions are given by use of collective compact theory. At last, numerical examples are tested and results verify the theoretical analysis.

Keyword: Boundary integral equation, singularity, variable transformation, convergence

1 Introduction

Consider an anisotropic medium in domain $\Omega \in R^2$ bounded by its boundary $\Gamma = \cup_{j=1}^{m}\Gamma_j$ $(m \geq 1)$ which may consist of m segments each being sufficiently smooth (in the sense of Liapunov). In the absence of heat sources, the equation governing steady-state heat conduction with Dirichlet condition can be described as (see as [1, 2, 5])

$$\begin{cases} \kappa_{ij}\dfrac{\partial^2 u(x)}{\partial x_i \partial x_j} = 0, \quad (i,j=1,2), \\ u(x) = g, \quad x \in \Gamma, \end{cases} \tag{1.1}$$

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where u represents the temperature, and $\kappa = \{\kappa_{ij}\}_{1\le i,j\le 2}$ denotes the thermal conductivity matrix which satisfies the symmetry ($\kappa_{12} = \kappa_{21}$) and positive-definite ($|\kappa| = \kappa_{11}\kappa_{22} - \kappa_{12}^2 > 0$) conditions.

Using the following coordinate transformation [2]

$$\hat{x}_1 = \frac{\sqrt{|\kappa|}}{\kappa_{11}} x_1, \quad \hat{x}_2 = x_2 - \frac{\kappa_{12}}{\kappa_{11}} x_1, \tag{1.2}$$

Eq. (1.1) can be written as the 'isotropic' Laplacian form in a mapped plane in the transformed $\hat{x}_i-$system

$$\frac{\partial^2 u(\hat{x})}{\partial \hat{x}_1^2} + \frac{\partial^2 u(\hat{x})}{\partial \hat{x}_2^2} = 0, \quad \hat{x} = (\hat{x}_1, \hat{x}_2) \in \Omega'. \tag{1.3}$$

Then by single-layer potential theory [8], Eq. (1.3) can be converted into the following weakly singular boundary integral equation (BIE):

$$-\frac{1}{2\pi} \int_{\Gamma'} v(\hat{x}) \ln |\hat{x} - \hat{y}| \mathrm{d}s_{\hat{x}} = g(\hat{y}), \quad \hat{y} \in \Gamma', \tag{1.4}$$

where Γ' is the boundary of the transformed domain Ω'. The solution of Eq. (1.4) exists and is unique as long as $C_{\Gamma'} \neq 1$, where $C_{\Gamma'}$ is the logarithmic capacity [11, 12]. As soon as $v(\hat{x})$ is solved from (1.4), the solution $u(\hat{x})$ of the problem (1.3) can be calculated by the following

$$u(\hat{y}) = -\frac{1}{2\pi} \int_{\Gamma'} v(\hat{x}) \ln |\hat{x} - \hat{y}| \mathrm{d}s_{\hat{x}}, \quad \hat{y} \in \Omega'. \tag{1.5}$$

Finally, using the inverse transformation of (1.2)

$$x_1 = \frac{\kappa_{11}}{\sqrt{|\kappa|}} \hat{x}_1, \quad x_2 = \hat{x}_2 + \frac{\kappa_{12}}{\sqrt{|\kappa|}} \hat{x}_1, \tag{1.6}$$

we can obtain the solution $u(x)$ of the problem (1.1).

As far as we know, the most popular numerical methods for engineering problems include, for example, the finite element method (FEM) [17], the finite difference method (FDM) [15], and the boundary element method (BEM) [11, 12]. The former two, used frequently in numerical modeling, are referred to as domain solution techniques and require full discretization of the whole domain and are often computationally costly and mathematically tricky in the volume mesh generation [1]. The BEM has been recognized as an efficient computational method that only the boundary needs to be modeled and owing the high approximation. For the application of BEM for the steady-state anisotropic heat conduction problems (1.1), various types of elements, namely constant, continuous and discontinuous linear elements and continuous and discontinuous quadratic elements has been investigated in the literature [5]. In this article, the Sidi-Israeli quadrature formula [3] is applied to calculate weakly singular integrals. Especially for the case of closed curved polygons Γ', we use the

Kress's variable transformation [4] to overcome the corner singularities and improve the accuracy at the boundary corners.

This paper is organized as follows: in Section 2, the convergence and error analysis are carried out based on the theory of collectively compact operators [6, 7] for closed smooth boundaries. The Kress's variable transformation is introduced to overcome the corner singularities for curved polygons in Section 3. Numerical examples are provided to verify the theoretical results in Section 4, and some useful conclusions are made in Section 5.

2 Sidi-Israeli quadrature method for boundary integral equation on smooth domain

Suppose that the boundary Γ' $(=\partial\Omega')$ to be a smooth closed curve and assume that the curve Γ' can be parameterized by $\hat{y}(t)=\varphi(t)=(\varphi_1(t),\varphi_2(t)):[0,2\pi)\to\partial\Omega'$. Then Eq. (1.4) can be written as

$$g(s)=-\frac{1}{2\pi}\int_0^{2\pi}\ln\left|\varphi(s)-\varphi(t)\right|v(t)\mathrm{d}t, \tag{2.1}$$

where $v(t)=|\varphi'(t)|v(\varphi(t))$ and $g(t)=g(\varphi(t))$ are periodic functions with period 2π.

In order to achieve high accuracy for numerical computation of finite-range integrals with weakly singular kernels, the following lemma about Sidi's quadrature formula is introduced.

Lemma 2.1. [3] Assume that the functions $H_1(t,\tau)$ and $H_2(t,\tau)$ are 2ℓ times differentiable on $[0,2\pi]$. Assume also that the functions $H(t,\tau)$ are periodic with period $T=2\pi$, and that they are 2ℓ times differentiable on $\widetilde{R}=(-\infty,\infty)\backslash\{\tau+kT\}_{k=-\infty}^{\infty}$. If $a(t,\tau)=H_1(t,\tau)\ln|t-\tau|+H_2(t,\tau)$, then

$$Q_n[a(t,\tau)]=h\sum_{\substack{j=0\\t_j\neq\tau}}^{n}a(t_j,\tau)+H_2(t_j,\tau)h+\ln(\frac{h}{2\pi})H_1(t_j,\tau)h,\ h=\frac{2\pi}{n},\ t_j=jh,$$

and

$$E_n[a(t,\tau)]=2\sum_{\mu=1}^{\ell-1}\frac{\zeta'(-2\mu)}{(2\mu)!}\frac{\partial^{2\mu}}{\partial\tau^{2\mu}}\big(H_1(t_j,\tau)\big)h^{2\mu+1}+O\left(h^{2\ell}\right),\quad\text{as}\quad h\to0,$$

where $\zeta(z)$ is a Riemann function [9, 13] and $E_n[a(t,\tau)]=\int_0^{2\pi}a(t,\tau)dt-Q_n[a(t,\tau)]$.

Define the integral operator

$$(Lv)(s)=\int_0^{2\pi}l(s,t)v(t)\mathrm{d}t, \tag{2.2}$$

with the kernel

$$(l(s,t)=-\frac{1}{2\pi}\ln\left|\varphi(s)-\varphi(t)\right|,$$

and construct the Nyström operator by the Sidi's quadrature formula

$$(L^h v)(s) = -\frac{h}{2\pi}\big[\ln\big(\frac{h}{2\pi}\big) + \ln|\varphi'(s)|\big]v(s) + h\sum_{\substack{j=0\\ t_j\neq s}}^{n-1} l(s,t_j)v(t_j),\ t_j = jh,\ h = \frac{2\pi}{n}.$$

Let $V^h = \mathrm{span}\{e_0(s), e_1(s), \cdots, e_n(s)\} \subset C[0, 2\pi)$ be a piecewise linear function subspace with nodes $\{s_j\}_{j=0}^n$, where $e_i(s)$ is the basis function satisfying $e_i(s_j) = \delta_{ij}$. Define a prolongation operator $P^h : \mathfrak{R}^n \to V^h$ and a restricted operator Q^h satisfying

$$\begin{cases} P^h v = v\cdot e,\ v = (v_0, \cdots, v_n),\ e = (e_0, \cdots, e_n) \in \mathfrak{R}^n, \\ Q^h v = (v(s_0), \cdots, v(s_n)) \in \mathfrak{R}^n,\ \ v \in C[0.2\pi). \end{cases}$$

Then Eq. (2.1) and its approximation equation are

$$\begin{cases} Lv = g \\ L^h v^h = Q^h g \end{cases}$$

where $L^h = [l_{ij}]_{i,j=0}^{n-1}$ and the entries are

$$l_{ij} = \begin{cases} hl(s_i, t_j), & i \neq j, \\ -\frac{h}{2\pi}\ln\big(\frac{h|\varphi'(t_i)|}{2\pi}\big), & i = j. \end{cases}$$

Define the following integral operator

$$(A_0 v)(s) = \int_0^{2\pi} a_0(s,t)v(t)\mathrm{d}t, \tag{2.3}$$

with the kernel

$$a_0(s,t) = -\frac{1}{2\pi}\ln\big|2e^{-1/2}\sin\frac{s-t}{2}\big|.$$

Let $L - A_0 = A_1$, then the integral equation (1.4) can be split into a singularity part and a compact perturbation part

$$A_0 v + A_1 v = g, \tag{2.4}$$

where $(A_1 v)(s) = (v(.), a_1(s, .))_{L^2}$ with the kernel

$$a_1(s,t) = \begin{cases} -\frac{1}{2\pi}\ln\big|e^{1/2}\frac{\varphi(s)-\varphi(t)}{2\sin\frac{s-t}{2}}\big|, & s-t \neq 2\pi Z, \\ \\ -\frac{1}{2\pi}\ln\big(e^{1/2}|\varphi'(s)|\big), & s-t = 2\pi Z. \end{cases}$$

Now we construct the approximations of A_0 and A_1. For the logarithmically singular operators A_0, by the Sidi's quadrature formula [3], we can construct the Fredholm approximation

$$(A_0^h v)(s) = h\sum_{j=0}^{n-1} a_0(s,t_j)v(t_j), \tag{2.5}$$

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where

$$a_0(s,t_j)=\begin{cases} -\dfrac{1}{2\pi}\ln\left|2e^{-1/2}\sin\left(\dfrac{s-t_j}{2}\right)\right| & s\neq t_j,\\ -\dfrac{1}{2\pi}\ln\left(\dfrac{e^{-1/2}h}{2\pi}\right) & s=t_j,\end{cases} \tag{2.6}$$

which has the following error bounds:

$$(A_0^h v)(s_i)-(A_0 v)(s_i)=\frac{-2}{\pi}\sum_{\mu=1}^{2l-1}\frac{\zeta'(-2\mu)}{(2\mu)!}[v(s)]^{(2\mu)}\Big|_{s=s_i}h^{2\mu+1}+O(h^{2l}). \tag{2.7}$$

For the integral operators A_1 with periodic kernels, we can construct the Nyström approximation by the trapezoidal rule [8],

$$(A_1^h v)(s)=h\sum_{i=0}^{n-1}a_1(s,t_j)v(t_j) \quad j=0,1,\cdots,n-1,$$

which has the error bounds $O\left(h^{2l}\right),\ l\in N$.

Consider the discrete approximation of (2.4)

$$(A_0^h+A_1^h)v^h=g^h, \tag{2.8}$$

where $v^h=(v_0^h,v_1^h,\cdots,v_{n-1}^h)^T$, $A_0^h=[a_0(s_i,t_j)]_{i,j=0}^{n-1}$, $A_1^h=[a_1(s_i,t_j]_{i,j=0}^{n-1}$, and $g^h=(g(\varphi(s_0)),\cdots,g(\varphi(s_{n-1})))^T$. Obviously, (2.8) is a linear equation system with n unknowns. Once v^h is solved from (2.8), the solution of (1.5) $u(\hat{y})$ $(\hat{y}\in\Omega')$ can be computed by

$$u^h(\hat{y})=-\frac{h}{2\pi}\sum_{i=0}^{n}v^h(s_i)\left|\hat{x}'(s_i)\right|\ln\left|\hat{x}(s_i)-\hat{y}\right|. \tag{2.9}$$

From (2.6), we have

$$A_0^h=-\frac{h}{2\pi}\begin{bmatrix} \ln\left(e^{-\frac{1}{2}}\frac{h}{2\pi}\right) & \ln\left|2e^{-\frac{1}{2}}\sin\frac{h}{2}\right| & \cdots & \ln\left|2e^{-\frac{1}{2}}\sin\frac{(n-1)h}{2}\right| \\ \ln\left|2e^{-\frac{1}{2}}\sin\frac{h}{2}\right| & \ln\left(e^{-\frac{1}{2}}\frac{h}{2\pi}\right) & \cdots & \ln\left|2e^{-\frac{1}{2}}\sin\frac{(n-2)h}{2}\right| \\ \vdots & \vdots & \vdots & \vdots \\ \ln\left|2e^{-\frac{1}{2}}\sin\frac{(n-1)h}{2}\right| & \ln\left|2e^{-\frac{1}{2}}\sin\frac{(n-2)h}{2}\right| & \cdots & \ln\left(e^{-\frac{1}{2}}\frac{h}{2\pi}\right) \end{bmatrix}.$$

By [10], we know that $\|(A_0^h)^{-1}\|\leq cn$. Hence the Eq. (2.8) is equivalent to

$$(E^h+P^h(A_0^h)^{-1}Q^hA_1^h)v^h=P^h(A_0^h)^{-1}Q^hg^h, \tag{2.10}$$

where E^h is the identity operator.

Lemma 2.2. The operator sequence $\left\{P^h(A_0^h)^{-1}Q^hA_0^h : C^3[0,2\pi)\to C[0,2\pi)\right\}$ is uniformly bounded and

$$P^h(A_0^h)^{-1}Q^hA_0\xrightarrow{p}I.$$

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where $\xrightarrow{p}$ denotes the pointwisely convergence and I is the embedding operator .

Proof. Let $v \in C^3[0,2\pi)$ and v^h be the solutions of the auxiliary equations $A_0 v = g$ and $A_0^h v^h = Q^h g$ respectively, where

$$A_0^h Q^h v = A_0^h v(t_i) = h \sum_{\substack{j=0 \\ j \neq i}}^{n-1} a_0(s_i, t_j) v(t_j) - \frac{1}{2\pi} \ln\big(\frac{h e^{-1/2}}{2\pi}\big) v(t_i),$$

and

$$Q^h g = g(s_i) = \int_0^{2\pi} a_0(s_i, t) v(t) \mathrm{d}t.$$

From (2.7), we obtain that

$$Q^h g - A_0^h Q^h v = O\ (h^3),$$

and

$$\|Q^h g - A_0^h Q^h v\|_2 = (n (O\ (h^3))^2)^{\frac{1}{2}} = O\ (h^{\frac{5}{2}}). \tag{2.11}$$

By (2.11), the following holds

$$\begin{aligned} \|Q^h v - (A_0^h)^{-1} Q^h A_0 v\|_2 &= \|Q^h v - (A_0^h)^{-1} Q^h g\|_2 \\ &= \|Q^h v - v^h\|_2 \\ &= \|(A_0^h)^{-1} A_0^h (Q^h v - v^h)\|_2 \\ &\leq \frac{1}{h} \|A_0^h (Q^h v - v^h)\|_2 \\ &= \frac{1}{h} \|A_0^h Q^h v - A_0^h v^h\|_2 \\ &= \frac{1}{h} \|Q^h g - A_0^h Q^h v\|_2 = O\ (h^{\frac{3}{2}}), \end{aligned}$$

the proof of Lemma 2.2 is completed. □

Theorem 2.3. Assume that $\partial\Omega'$ is a simply smooth and closed curve, Q^h is a restricted operator and P^h is a prolongation operator with nodes $\{s_i\}_{i=0}^n$, then the operator sequence $\{P^h (A_0^h)^{-1} Q^h A_1^h\}$ is collectively compactly convergent to $A_0^{-1} A_1$ in $C[0,2\pi)$, that is,

$$P^h (A_0^h)^{-1} Q^h A_1^h \xrightarrow{c.c} A_0^{-1} A_1.$$

Proof. Since $(P^h (A_0^h)^{-1} Q^h)(P^h A_1^h Q^h) = (P^h (A_0^h)^{-1} Q^h A_0)((A_0)^{-1} P^h A_1^h Q^h)$, we get

$$\|(P^h (A_0^h)^{-1} Q^h)(P^h A_1^h Q^h)\| \leq \|P^h (A_0^h)^{-1} Q^h A_0\|_{0,3} \|(A_0)^{-1} P^h A_1^h Q^h\|_{3,0}.$$

From [10] and by Lemma 2.2, we have $(A_0)^{-1} P^h A_1^h Q^h \xrightarrow{c.c} A_0^{-1} A_1$ and $P^h (A_0^h)^{-1} Q^h A_0 \xrightarrow{p} I$. The proof of Theorem 2.3 is completed. □

Replacing $(A_0^h)^{-1}$, A_0^h, A_1^h, v^h and g^h by $(\hat{A_0}^h)^{-1} = P^h(A_0^h)^{-1}Q^h$, $\hat{A_0}^h = P^h A_0^h Q^h$, $\hat{A_1}^h = P^h A_1^h Q^h$, $\hat{v}^h = P^h v^h$ and $\hat{g}^h = P^h(A_0^h)^{-1}Q^h g^h$, respectively. Then (2.10) can be written as

$$(E^h + (\hat{A_0}^h)^{-1}\hat{A_1}^h)\hat{v}^h = \hat{g}^h. \tag{2.12}$$

Theorem 2.4. Assume that $\partial\Omega'$ is a simply smooth and closed curve, v and $\hat{v}^h$ are the solutions of (2.4) and (2.12), respectively, $x_i \in C^6[0, 2\pi)$ and $g(s) \in C^5[0, 2\pi)$, then the following holds

$$(\hat{v}^h - v)\big|_{s=s_i} = O\ (h^3). \tag{2.13}$$

Proof. By the trapezoidal rule, the asymptotic expansion holds [10]

$$(g - g^h)\big|_{s=s_i} = h^3 P^h Q^h \varphi_1\big|_{s=s_i} + O\ (h^5), \tag{2.14}$$

with $\varphi_1(s) = -\zeta'(-2)g''(s)/\pi$. Using (2.7) and (2.14), we can obtain

$$\begin{aligned}
&P^h(A_0^h + A_1^h)Q^h(v^h - v)\big|_{s=s_i} \\
&= g^h - P^h(A_0^h + A_1^h)Q^h v\big|_{s=s_i} \\
&= g^h - [((A_0 + A_1)v - h^3 P^h Q^h \varphi_2)\big|_{s=s_i} + O\ (h^5)] \\
&= (g^h - g)\big|_{s=s_i} + h^3 P^h Q^h \varphi_2\big|_{s=s_i} + O\ (h^5) \\
&= h^3 P^h Q^h \varphi\big|_{s=s_i} + O\ (h^5),
\end{aligned}$$

where $\varphi_2(s) = -\xi'(-2)v''(s)/\pi$, and $\varphi(s) = \varphi_1(s) + \varphi_2(s)$. From Theorem 2.3, we have

$$(E^h + (\hat{A_0}^h)^{-1}A_1^h)(v - \hat{v}^h)\big|_{s=s_i} = h^3(\hat{A_0}^h)^{-1}P^h Q^h \varphi(s)\big|_{s=s_i} + O\ (h^5). \tag{2.15}$$

Since $(E^h + (\hat{A_0}^h)^{-1}A_1^h)^{-1}$ is uniformly bounded, we immediately get

$$(\hat{v}^h - v)\big|_{s=s_i} = O\ (h^3). \tag{2.16}$$

□

3 Corner singularity and convergence analysis

Definition 3.1. [14] A real-valued function γ is said to be a sigmoidal transformation if the following conditions are satisfied:

(i) $\gamma \in C^1[0, 1] \cup C^\infty(0, 1)$ with $\gamma(0) = 0$;

(ii) $\gamma(x) + \gamma(1 - x) = 1,\ 0 \le x \le 1$;

(iii) γ is strictly increasing on $[0, 1]$ and its derivative γ' is strictly increasing on $[0, 1/2]$ with $\gamma'(0) = 0$.

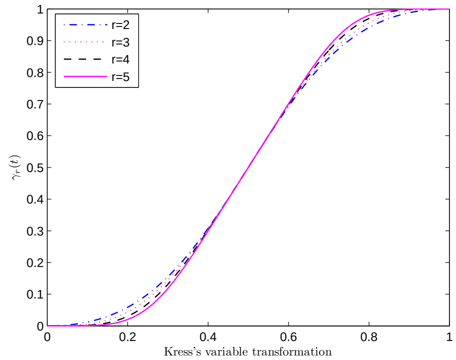

Figure 1: γ_r *transformation*

The Kress's variable transformation is an example of "algebraic" sigmoidal transformations, which was first proposed by Kress [4]. The function γ, defined on $[0, 1]$ by

$$\gamma(x) = f(x)/(f(x) + f(1 - x)),$$

where

$$f(x) = (x + cB_3(x))^r,$$

here $r > 1$, c is a constant to be determined and B_3 is the Bernoulli polynomial of degree 3 defined by

$$B_3(x) := x(x - 1/2)(x - 1).$$

If we choose $c = -8(1/r - 1/2)$, then we have

$$\gamma_r(t) = \frac{\Theta_1(t)}{\Theta_2(t)} = \frac{(\theta(t))^r}{(\theta(t))^r + (\theta(2\pi - 2\pi t))^r} : [0, 1] \to [0, 1], \quad r \geq 1, \tag{3.1}$$

where $\theta(t) = (\frac{1}{r} - \frac{1}{2})(1 - 2t)^3 + \frac{1}{r}(2t - 1) + \frac{1}{2}$. The plots for γ_r is shown in Figure 1.

Assume that $\Gamma' = \cup_{q=1}^{m} \Gamma'_q$ $(q > 1)$ be the boundary of a polygonal domain Ω' in $\mathbb{R}^2$, $\Gamma'_q \in C^{2\ell+1}(q = 1, ..., m,\ \ell \in N)$, and let $g_q = g\big|_{\Gamma'_q}$. Define the boundary integral operators on Γ'_q,

$$(L_{pq}v_q)(\hat{y}) = -\frac{1}{2\pi} \int_{\Gamma'_q} v_q(\hat{x}) \ln\left|\hat{x} - \hat{y}\right| \mathrm{d}s_{\hat{x}},\ \hat{y} = (\hat{y}_1, \hat{y}_2) \in \Gamma'_p \ \ (p, q = 1, ..., m). \tag{3.2}$$

Then Eq. (1.4) can be converted into a matrix operator equation

$$Lv = G, \tag{3.3}$$

where $L = [l_{pq}]_{p,q=1}^{m}$, $G = (g_1(\hat{y}),\ ...,\ g_m(\hat{y}))^T$ and $v = (v_1(\hat{x}),\ ...,\ v_m(\hat{x}))^T$. Assume that Γ'_q can be described by the parameter mapping: $\hat{x}_q(s) = \varphi(s) = (\varphi_{q1}(s),\ \varphi_{q2}(s))$

: $[0,1] \to \Gamma'_q$ with $|\varphi'_q(s)| = [|\varphi'_{q1}(s)|^2 + |\varphi'_{q2}(s)|^2]^{1/2} > 0$. In order to degrade the singularities at corners, we apply the Kress's variable transformation to (3.3) and give the following decomposition of L_{pq},

$$L_{pq} = A_{0(pp)} + A_{1(pq)}.$$

The operators $A_{0(pp)}$ and $A_{1(pq)}$ are singular and compact operators respectively, and their Nyström approximation are

$$\begin{cases} (A_{0(pp)}w_p)(t) = \int_0^1 a_{0(pq)}(t,\tau)w_p(\tau)\mathrm{d}\tau, \quad t \in [0,1], \\ (A^{h_p}_{0(pp)}w_p)(t) = h_p \sum_{\substack{j=1 \\ t\neq\tau_j}}^{n_m} a_{0(pp)}(t,\tau_j)w_p(\tau_j) - h_p \ln\left|e^{-1/2}h_p\right|w_p(t), \; h_p = 1/n_p, \end{cases}$$

and

$$\begin{cases} A_{1(pq)}(w_q)(t) = \int_0^1 a_{1(pq)}(t,\tau)w_q(\tau)\mathrm{d}\tau, \quad t \in [0,1], \\ (A^{h_q}_{1(pq)}w_q)(t) = h_q \sum\limits_{j=1}^{n_q} a_{1(pq)}(t,\tau_j)w_q(\tau_j), \quad t \in [0,1], \; \tau_j = jh \;\; p,q = 1,...,m, \end{cases}$$

where

$$a_{0(pp)}(t,\tau) = -\frac{1}{2\pi}\ln\left|2e^{-1/2}\sin\pi(t-\tau)\right|,$$

and

$$a_{1(pq)}(t,\tau) = \begin{cases} -\dfrac{1}{2\pi}\ln\dfrac{|\hat{x}_p(t) - \hat{y}_q(\tau)|}{|2e^{-1/2}\sin\pi(t-\tau)|} & \text{as} \;\; p = q, \\ -\dfrac{1}{2\pi}\ln|\hat{x}_p(t) - \hat{y}_q(\tau)| & \text{as} \;\; p \neq q, \end{cases}$$

here

$$\hat{x}_q(t) = (\varphi_{q1}(\gamma_r(t)), \varphi_{q2}(\gamma_r(t))),$$
$$w_q(t) = v_q(\varphi_q(\gamma_r(t)))|\varphi'_q(\gamma_r(t))|\gamma'_r(t).$$

Then Eq. (3.3) and its discrete equations are

$$\begin{cases} (A_0 + A_1)W = G, \\ (A^h_0 + A^h_1)W^h = G^h. \end{cases} \tag{3.4}$$

where

$$\begin{aligned} A_0 &= \mathrm{diag}(A_{0(11)}, \cdots, A_{0(mm)}), \quad A_1 = [A_{1(pq)}]^m_{p,q=1}, \\ W &= (w_1, \cdots, w_m)^T, \quad G = (g_1, \cdots, g_m)^T, \quad g_q(t) = g_q(\varphi_q(t)), \\ W^h &= (w^{h_1}_1(t_1), ..., w^{h_1}_1(t_{n_1}), ..., w^{h_m}_m(t_1), ..., w^{h_m}_m(t_{n_m}))^T, \\ A^h_0 &= \mathrm{diag}(A^{h_1}_{0(11)}, ..., A^{h_m}_{0(mm)}), \quad A^h_{0(pp)} = [a_{0(pp)}(t_j, \tau_i)]^{n_p}_{j,i=1}, \\ A^h_1 &= [A^{h_q}_{1(pq)}]^m_{p,q=1}, \quad A^h_{1(pq)} = [a_{1(pq)}(t_j, \tau_i)]^{n_p,n_q}_{j,i=1}, \\ G^h &= (g_1(t_1), ..., g_1(t_{n_1}), ..., g_m(t_1), ..., g_m(t_{n_m}))^T. \end{aligned}$$

The operator $A_{0(pp)}$ is an isometry operator from $(H^r[0,1])^m$ to $(H^{r+1}[0,1])^m$ [11, 12]. In addition, from [10], we know that $A^{h_q}_{0(qq)}$ and A^h_0 are invertible, and $||(A^{h_q}_{0(qq)})^{-1}|| = O(n_q)$ and $||(A^h_0)^{-1}|| = O(h_q^{-1})$, where $\|\cdot\|$ denotes the spectral norm. Hence Eq. (3.4) is equivalent to

$$\begin{cases} (E + A_0^{-1}A_1)W = A_0^{-1}G = \tilde{G} \\ (E^h + (A_0^h)^{-1}A_1^h)W^h = \tilde{G}^h. \end{cases} \tag{3.5}$$

Obviously, the second equation of (3.5) is a system of linear equations with n $(= \Sigma_{j=1}^m n_j)$ unknowns. Once W^h is solved by (3.5), the solution $u(\hat{y})$ $(\hat{y} \in \Omega')$ can be computed by

$$u^h(\hat{y}) = -\frac{1}{2\pi}\sum_{p=1}^{m}\sum_{q=1}^{n_p} h_p \ln|\hat{x}_{pq}(\tau_q) - \hat{y}||\hat{x}'_p(\tau_q)|w_q^h(\tau_q).$$

Let the function $v_q(t) = t^{\alpha_q}\phi_q(t)$ $(0 > \alpha_q \geq -1/2)$, where $\phi_q(t)$ is differentiable enough on $[0,1]$ with $\phi_q(0) \neq 0$. From Taylor's formula we have

$$v_q(t) = \sum_{j=0}^{l}\frac{\phi_q^{(j)}(0)}{j!}t^{j+\alpha_q} + O(t^{l+\alpha_q+1}) \quad \text{as} \quad t \to 0^+ \tag{3.6}$$

and

$$\gamma'_r(t) \sim \sum_{j=0}^{\infty}\delta_j t^{r-1+j} \quad \text{as} \quad t \to 0^+, \quad \text{and} \quad \delta_0 > 0. \tag{3.7}$$

By substituting (3.6) and (3.7) into the expression of $w_q(t)$, then the function $w_q(t)$ can be expressed by

$$w_q(t) = c_1\phi_q(0)t^{r(\alpha_q+1)-1}(1 + O(t)) \quad \text{as} \quad t \to 0^+, \tag{3.8}$$

where c_1 is a constant independent of t.

Similarly, let the function $v_q(t) = (1-t)^{\alpha_q}\tilde{\phi}_q(t)$ $(0 > \alpha_q \geq -1/2)$, where $\tilde{\phi}_q(t)$ is differentiable enough on $[0,1]$ with $\tilde{\phi}_q(1) \neq 0$. Then the function $w_q(t)$ can be expressed by

$$w_q(t) = c_2\tilde{\phi}_q(1)(1-t)^{r(\alpha_q+1)-1}(1 + O(1-t)) \quad \text{as} \quad t \to 1^-, \tag{3.9}$$

where c_2 is a constant independent of t.

Remark 1 The function $v_q(t)$ has singularities at endpoints $t = 0$ and $t = 1$ [16], but $w_q(t)$ has no singularities by Kress transformation at $t = 0$ and $t = 1$.

Lemma 3.2. Let

$$\tilde{a}_{1(pq)}(t,\tau) = a_{1(pq)}(t,\tau)\gamma'_r(t), \quad \gamma \geq 1, \quad \Gamma_p \cap \Gamma_q \neq \emptyset, \tag{3.10}$$

then $\tilde{a}_{1(pq)}(t,\tau)$ is smooth on $[0,1]^2$.

Proof. By using the continuity of $\tilde{a}_{1(pp)}(t,\tau)$ and the boundness of $\gamma'_r(t)$, we can

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immediately complete the proof for the case $p = q$. Let $\Gamma_{p-1} \cap \Gamma_p = P_p = (0,0)$ and $\beta_p \in (0, 2\pi)$ be the corresponding interior angle. Since $\varphi_{p-1}(1) = \varphi_{p-1}(0)$, the kernel $a_{1(p-1,p)}(t,\tau)$ have singularities at the points $(t,\tau) = (0,1)$ and $(t,\tau) = (1,0)$. For convenience of analysis, we only discuss the case in that $(t,\tau) = (1,0)$. If $(t,\tau) \neq (1,0)$, we write

$$\tilde{a}_{1(p-1,p)}(t,\tau) = -\frac{1}{4\pi}(S_1(t,\tau) + S_2(t,\tau)), \tag{3.11}$$

where

$$S_1(t,\tau) = \gamma'(t)\ln(|\varphi_{p-1}(t)|^2 + |\varphi_p(\tau)|^2)$$

and

$$S_2(t,\tau) = \gamma'(t)\ln[1 - 2|\varphi_{p-1}(t)||\varphi_p(\tau)|\cos\beta_{p-1}/(|\varphi_{p-1}(t)|^2 + |\varphi_p(\tau)|^2)].$$

Since

$$\left|2|\varphi_{p-1}(t)||\varphi_p(\tau)|\cos\beta_{p-1}/(|\varphi_{p-1}(t)|^2 + |\varphi_p(\tau)|^2)\right| < 1,$$

the function $S_2(t,\tau)$ and its first derivative are bounded. Noting that

$$\gamma_r^{(k)}(0) = \gamma_r^{(k)}(1) = 0, \quad k = 0, \cdots, \gamma,$$

we have

$$|\varphi_{\bar{p}}^{(k)}(0)| = |\varphi_{\bar{p}}^{(k)}(1)| = 0, \quad \bar{p} = p-1 \text{ or } p, \quad k = 1, \cdots, \gamma.$$

Let $(t,\tau) \in [\varepsilon/2, \varepsilon] \times [1-\varepsilon, 1-\varepsilon/2]$ for all $\varepsilon > 0$, we have

$$|S_1(t,\tau)| = O(\varepsilon^{r-1}|\ln\varepsilon|),$$

so $S_1(t,\tau)$ is also bounded. In addition, from

$$\begin{aligned}\left|\frac{\partial}{\partial\tau}S_1(t,\tau)\right| &\leq O(t^{r-1})\frac{2|\varphi_p(\tau)||\hat{x}'_{p-1}(\gamma_r(\tau))||\gamma_r'(\tau)|}{|\varphi_{p-1}(t)|^2 + |\varphi_p(\tau)|^2} \\ &= O(\varepsilon^{r-1})O(\varepsilon^{2r-2})/O(\varepsilon^{2r-2}) = O(\varepsilon^{r-1}),\end{aligned}$$

we know $\frac{\partial\tilde{a}_{1(pq)}(t,\tau)}{\partial\tau}$ is also continuous in $(C[0,1])^2$. The proof of Lemma 3.2 is completed. □

Suppose that

$$t_\nu = (\nu+1)/2 \quad \text{for } -1 < \nu \leq 1,$$

so that $-1 < \nu \leq 1$ with $t_0 = 1/2$ and $t_1 = 1$. The offset trapezoidal rule $Q_n^{[\nu,r]}f$ with transformation γ_r, $n \in N$, $-1 < \nu \le 1$ is defined by

$$Q_n^{[\nu,r]} f = \begin{cases} \frac{1}{n}\sum_{j=0}^{n-1} f(\gamma_r((j+t_\nu)/n))\gamma_r'((j+t_\nu)/n), & -1 < \nu < 1, \\ \frac{1}{n}\sum_{j=1}^{n-1} f(\gamma_r(j/n))\gamma_r'(j/n), & \nu = 1. \end{cases}$$

At the same time, we define the truncation error $E_n^{[\nu,r]} f$ by

$$E_n^{[\nu,r]} f := \mathrm{I}f - Q_n^{[\nu,r]} f. \tag{3.12}$$

Let us assume that near $x = 0$ we can write

$$\gamma_r(x) = c_0(r)x^r(1 + \sum_{k=1}^{\infty} d_k(r)x^k). \tag{3.13}$$

Theorem 3.3. [14] Assume that f is holomorphic at both 0 and 1, and $f_r(\tau) := f(\gamma_r(\tau))\gamma_r'(\tau)$, $0 \le \tau \le 1$ can be continuous into the strip S such that

(i) $f_r(\tau)$ is continuous in S and holomorphic in $\mathrm{int}(S)$;

(ii) $f_r(\tau) = o(exp(2\pi n|Rz|))$ as $Rz \to \infty$ in S, uniformly with respect to Rz.

For any sigmoidal transformation γ_r of order $r > 1$ and for $-1 < \nu \le 1$ then for $n >> 1$

$$\begin{aligned}(2\pi n)^r E_n^{[0,r]} f \sim\ & 2c_0(r)\Gamma(r+1)(f(0)+f(1)) \times \{\cos(r\pi/2)(1-2^{1-r})\zeta(r) \\ & - \sin(r\pi/2)(r+1)d_1(r)(1-2^{-r})\zeta(r+1)/(2\pi n)\} + O(1/n^{\min(2,r)}),\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}(2\pi n)^r E_n^{[1,r]} f \sim\ & 2c_0(r)\Gamma(r+1)(f(0)+f(1)) \times \{\cos(r\pi/2)\zeta(r) \\ & - \sin(r\pi/2)(r+1)d_1(r)\zeta(r+1)/(2\pi n)\} + O(1/n^{\min(2,r)}).\end{aligned} \tag{3.15}$$

Theorem 3.4. [14] Suppose f is defined on S by

$$f(z) = z^\alpha(1-z)^\beta g(z) \quad \text{for } \alpha,\ \beta > 1, \tag{3.16}$$

where g is holomorphic on S, real on $[0,1]$ and such that $g(0) \ne 0$, $g(1) \ne 0$. Let γ_r be a sigmoidal transformation of order r, $r > 1$. Then for $n >> 1$

$$n^r E_n^{[\nu,r]} f \sim J_\nu(\alpha, r, n)g(0) + J_{-\nu}(\beta, r, n)g(1),$$

where the strip S of the complex $z-$ plane defined by

$$S := \{z : 0 \le x = \mathcal{R}z \le 1,\ -\infty < y = \mathcal{I}z < \infty\},$$

and

$$J_\nu(\alpha,r,n) := -r(c_0(r))^{1+\alpha}\{\zeta(1-r(\alpha+1),t_\nu) + (\alpha+1+1/r)d_1(r)\zeta(-r(\alpha+1),t_\nu)/n\}/n^{r\alpha}. \tag{3.17}$$

and

$$t_{-\nu} = 1-t_\nu, \quad \text{for } -1<\nu\leq 1.$$

Remark 2 If we choose $r=3$ for γ_r, then $\gamma_3 \sim O(t^3)$ and $d_1(r)=0$. In addition, if $\beta=0$, we have

$$E_n^{[\nu,r]}f \sim n^{-\omega}, \quad \omega=\min\{r,(\alpha+1)r\}. \tag{3.18}$$

For the Nyström approximation operator $A_{1(pq)}^{h_q}$ of the integral operator $A_{1(pq)}$, we have the error bounds [8]

$$(A_{1(pq)}w_q)(t)-(A_{1(pq)}^{h_q}w_q)(t)=O(h^r), \quad \text{for } \Gamma_p=\Gamma_q \text{ or } \Gamma_p\cap\Gamma_q=\emptyset, \quad r\in N, \tag{3.19}$$

and

$$A_{1(pq)}(w_q)(t)-(A_{1(pq)}^{h_q}w_q)(t)=O(h^\omega), \quad \text{for } \Gamma_p\cap\Gamma_q\in\{P_q\}, \tag{3.20}$$

where $\omega=\min\{r,(\alpha+1)r\}$.

For the approximate operator $A_{0(pp)}^{h_p}$ of the logarithmically singular operator $A_{0(pp)}$,

$$(A_{0(pp)}^{h_p}w_p)(t) = -\frac{h_l}{2\pi}\left\{\sum_{\substack{j=1\\ t\neq\tau_j}}^{n_q}\ln\left|2e^{-1/2}\sin\pi(t-\tau_j)\right|w_p(\tau_j)\right\} - \frac{h_p}{2\pi}\left\{\ln\left|2e^{-1/2}h_p\right|w_p(t)\right\} \quad (i=1,...,n_p), \tag{3.21}$$

which have the error bounds [3]

$$(A_{0(pq)}^{h_p}w_p)(t)-(A_{0(pp)}w_p)(t) = -\frac{2}{\pi}\sum_{\mu=1}^{2\ell-1}\frac{\zeta'(-2\mu)}{(2\mu)!}[w_p(t)]^{(2\mu)}h_p^{2\mu+1}+O(h_p^{2\ell}), \quad t\in\{t_i\},$$

where $\zeta'(t)$ is the derivative of the Riemann zeta function.

From (3.21), we can obtain

$$A_{0(pp)}^{h_p} = -\frac{h_p}{2\pi}\begin{bmatrix} \ln(\frac{h_p}{e^{1/2}}) & \ln(\frac{\sin(\pi h_p)}{e^{1/2}/2}) & \cdots & \ln(\frac{\sin((n_p-1)\pi h_p)}{e^{1/2}/2}) \\ \ln(\frac{\sin(\pi h_m)}{e^{-1/2}/2}) & \ln(\frac{h_p}{e^{1/2}}) & \cdots & \ln(\frac{\sin((n_p-2)\pi h_p)}{e^{1/2}/2}) \\ \vdots & \vdots & \vdots & \vdots \\ \ln(\frac{\sin((n_p-1)\pi h_p)}{e^{1/2}/2}) & \ln(\frac{\sin((n_p-2)\pi h_p)}{e^{1/2}/2}) & \cdots & \ln(\frac{h_p}{e^{1/2}}) \end{bmatrix},$$

Define the subspace $C_0[0,1]=\{v(t)\in C[0,1]: v(t)/\gamma_3(\pi t)\in C[0,1]\}$ of the space $C[0,1]$ with the norm $||v||^*=\max_{0\leq t\leq 1}|v(t)/\gamma_3(\pi t)|$. Let $S^{h_p}=\text{span}\{e_j(t), j=1,...,n_p\}\subset C_0[0,1]$ be a piecewise linear function subspace with the basis nodes

$\{t_i\}_{i=1}^{n_p}$, where $e_j(t)$ are the basis functions satisfying $e_{pj}(t_{pi}) = \delta_{ji}$. Also define a prolongation operator $I^{h_p} : \Re^{n_p} \to S^{h_p}$ and a restricted operator $R^{h_p} : C_0[0,1] \to \Re^{n_p}$ satisfying

$$\begin{cases} P^{h_p} v = \sum_{j=1}^{n_p} v_{pj} e_{pj}(\tau), & v = (v_{p1}, \cdots, v_{pn_p}) \in \mathfrak{R}^n, \\ Q^{h_p} v = (v(\tau_{p1}), \cdots, v(\tau_{pn_p})) \in \mathfrak{R}^n, & v \in C[0, 2\pi). \end{cases}$$

For the properties of $A_{0(pp)}^{h_p}$, we have the following lemma from [10].

Lemma 3.5. The operator sequence $\{P^{h_p}(A_{0(pp)}^{h_p})^{-1}Q^{h_p}A_{0(pp)} : C^2[0,1] \to C[0,1]\}$ is uniformly bounded and

$$P^{h_p}(A_{0(pp)}^{h_p})^{-1}Q^{h_p}A_{0(pp)} \xrightarrow{p} I. \tag{3.22}$$

If $|p-q| \neq 1$ or $m-1$, by the definition of $A_{1(pq)}$ we know the kernel $a_{1(pq)}(t,\tau)$ of the operator $A_{1(pq)}$ and its derivatives of higher order are continuous .

Lemma 3.6. Let $\Gamma' = \cup_{q=1}^m \Gamma'_q$ satisfy $C_{\Gamma'} \neq 1$, and also let

$$\bar{A}_{1(pq)}^{h_q} = \begin{cases} A_{1(pq)}^{h_q}, & \Gamma'_p = \Gamma'_q \text{ or } \Gamma'_p \cap \Gamma'_q = \emptyset, \\ \tilde{A}_{1(pq)}^{h_q}, & \Gamma'_p \cap \Gamma'_q \in \{P_q\}, \end{cases}$$

where the kernel $\tilde{a}_{1(pq)}(t,\tau)$ of $\tilde{A}_{1(pq)}$ is defined by (3.10). Then under the transformation (3.1), we have

$$\|(A_{0(pp)})^{-1}\bar{A}_{1(pq)}^{h_q}\|_{2,0} \leq M \tag{3.23a}$$

and

$$P^{h_p}(A_{0(pp)}^{h_p})^{-1}Q^{h_p}\bar{A}_{1(pq)}^{h_q} \xrightarrow{c.c} (A_{0(pp)})^{-1}A_{1(pq)}, \quad \text{in } C[0,1] \to C[0,1], \tag{3.23b}$$

where M is a constant.

proof. From [10] and by Lemma 3.2, $a_{1(pq)}(t,\tau)$ and $\tilde{a}_{1(pq)}(t,\tau)$ are continuous on $(C^2[0,1])^2$, and then we have (3.23a). Using the results of Lemma 3.5, and by

$$\begin{aligned} \|P^{h_p}(A_{0(pp)}^{h_p})^{-1}Q^{h_p}\bar{A}_{1(pq)}^{h_q}\|_{0,0} &= \|(P^{h_p}(A_{0(pp)}^{h_p})^{-1}Q^{h_p}A_{0(pp)})((A_{0(pp)})^{-1}\bar{A}_{1(pq)}^{h_q})\|_{0,0} \\ &\leq \|P^{h_p}(A_{0(pp)}^{h_p})^{-1}Q^{h_p}A_{0(pp)}\|_{0,2}\|(A_{0(pp)})^{-1}\bar{A}_{1(pq)}^{h_q}\|_{2,0} \\ &\leq C, \end{aligned}$$

where C is a constant. Thus, we complete the proof of Lemma 3.6. □

Consider the following discrete equation

$$(E^h + P^h(A_0^h)^{-1}Q^hA_1^h)W^h = P^h(A_0^h)^{-1}Q^hG^h, \tag{3.24}$$

where P^h =diag$(P^{h_1}, ..., P^{h_m})$ and Q^h =diag$(Q^{h_1}, ..., Q^{h_m})$.

Theorem 3.7. Assume $\Gamma' = \cup_{q=1}^m \Gamma'_q$ satisfy $C_{\Gamma'} \neq 1$, and Γ'_q $(q = 1, ..., m)$ are smooth curves. Then we have

$$P^h(A_0^h)^{-1}Q^hA_1^h \xrightarrow{c.c} A_0^{-1}A_1. \tag{3.25}$$

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Proof. Let $\mathcal{B} = \{z : ||z|| \leq 1, z \in (C[0,1])^m\}$ be the unit ball in the space $V = (C[0,1])^m$, and

$$H = \{H^{(1)}, H^{(2)}, \cdots\}, \quad H^{(n)} = \{h_1^{(n)}, \cdots, h_m^{(n)}\}$$

are multi-parameter sequences. Also let $\max_{1\leq q\leq m} h_q^{(n)} \to 0$ as $n_q \to \infty$. Choosing the sequence $\{Z_h, h \in H\} \subset \Theta$ and

$$Z_h = (Z_{1h}, \cdots, Z_{mh}), \quad Z_{qh} = (z_{q1}, \cdots, z_{qn_q}),\ q = 1, \cdots, m,$$

satisfying

$$\max_{1\leq p\leq m} \max_{0\leq q\leq n_p} \max_{0\leq t\leq 1} |z_{pq}^h(t)/\gamma_3(\pi t)| \leq 1. \tag{3.26}$$

From

$$P^h(A_0^h)^{-1}Q^h = \begin{bmatrix} P^{h_1}(A_{0(11)}^{h_1})^{-1}Q^{h_1} & & & \\ & P^{h_2}(A_{0(22)}^{h_2})^{-1}Q^{h_2} & & \\ & & \ddots & \\ & & & P^{h_m}(A_{0(mm)}^{h_m})^{-1}Q^{h_m} \end{bmatrix}$$

and

$$P^hA_1^hQ^h = \begin{bmatrix} P^{h_1}A_{1(11)}^{h_1}Q^{h_1} & P^{h_2}A_{1(12)}^{h_2}Q^{h_2} & \dots & P^{h_m}A_{1(1m)}^{h_m}Q^{h_m} \\ P^{h_1}A_{1(21)}^{h_1}Q^{h_1} & P^{h_2}A_{1(22)}^{h_2}Q^{h_2} & \dots & P^{h_m}A_{1(2m)}^{h_m}Q^{h_m} \\ \vdots & \vdots & & \vdots \\ P^{h_1}A_{1(m1)}^{h_1}Q^{h_1} & P^{h_2}A_{1(m2)}^{h_2}Q^{h_2} & \dots & P^{h_m}A_{1(mm)}^{h_m}Q^{h_m} \end{bmatrix},$$

we have

$$P^h(A_0^h)^{-1}Q^hA_1^hQ^hZ_h = \begin{bmatrix} \sum_{q=1}^m P^{h_1}(A_{0(11)}^{h_1})^{-1}Q^{h_1}A_{1(1q)}^{h_q}Q^{h_q}Z_{qh} \\ \sum_{q=1}^m P^{h_2}(A_{0(22)}^{h_2})^{-1}Q^{h_2}A_{1(2q)}^{h_q}Q^{h_q}Z_{qh} \\ \vdots \\ \sum_{q=1}^m P^{h_m}(A_{0(mm)}^{h_m})^{-1}Q^{h_m}A_{1(mq)}^{h_q}Q^{h_q}Z_{qh} \end{bmatrix}.$$

If $\Gamma_p' \cap \Gamma_q' = \emptyset$, from Lemma 3.6 we obtain

$$P^{h_p}(A_{0(pp)}^{h_p})^{-1}Q^{h_p}A_{1(pq)}^{h_q} \xrightarrow{cc} A_{0(pp)}^{-1}A_{1(pq)}. \tag{3.27}$$

If $\Gamma_p' \cap \Gamma_q' \neq \emptyset$, using Lemma 3.5 and Lemma 3.6, and by

$$\begin{aligned} ||P^{h_p}(A_{0(pp)}^h)^{-1}Q^{h_p}A_{1(pq)}^{h_q}Q^{h_q}Z_{qh}||_0 &= ||P^{h_p}(A_{0(pp)}^h)^{-1}Q^{h_p}\tilde{A}_{1(pq)}^{h_q}Q^{h_q}Z_{qh}/\gamma_3(\pi t)||_0 \\ &\leq \left\|P^{h_{pp}}(A_{0(pp)}^h)^{-1}Q^{h_p}A_{0(pp)}\right\|_{0,3} ||A_{0(pp)}^{-1}\tilde{A}_{1(pq)}^{h_q}||_{3,0} \left\|||Q^{h_q}Z_{qh}\right\|^*, \end{aligned} \tag{3.28}$$

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we can find an infinite subsequence in $\{P^{h_p}(A^{h_p}_{0(pp)})^{-1}Q^{h_p}A^{h_q}_{1(pq)}Q^{h_q}Z_{qh}\}$ which converges as $h \to 0$. Hence, there exists an infinite subsequence $H_l \subset H$ such that (3.28) converges, As above, there exists an infinite subsequence $\{H_l,\ l = 1, \cdots, m\}$ such that $\{P^h(A^h_0)^{-1}Q^hA^h_1,\ h \in H_m\}$ is a convergent sequence in the space $V = (C_0[0,1])^m$. This shows that $\{P^h(A^h_0)^{-1}Q^hA^h_1\}$ is a collectively compact sequence, and $P^h(A^h_0)^{-1}Q^hA^h_1$ is pointwisely convergent to $A_0^{-1}A_1$. The proof of Theorem 3.7 is completed. □

Similar to Theorem 2.4, we have the following theorem.

Theorem 3.8. Assume $\Gamma' = \cup_{q=1}^m \Gamma'_q$ satisfy $C_{\Gamma'} \neq 1$, $g_q = g|_{\Gamma'_q} \in C^6(\Gamma'_q)$, then when we choose an appropriate number r in (3.18) such that $\omega > 3$, there exists a vector function $\Phi = (\Phi_1, ..., \Phi_m)^T \in (C_0[0,1])^m$ independent of $h = (h_1, ..., h_m)^T$ such that the following multi-parameter asymptotic expansions hold at nodes

$$w - \hat{w}^h = \mathrm{diag}(h_1^3, ..., h_m^3)\Phi + O(h^5_{\max})e, \tag{3.29}$$

where $h_{\max} = \max_{1\leq q\leq m} h_q$, and $e = (1, 1, \cdots, 1)^T$ is a m dimensional vector.

4 Numerical experiments

In this section, two numerical examples are presented to verify the efficiency of the Sidi-Israeli quadrature method for anisotropic heat conduction problems.

Suppose that $e_n = |u - u_n|$ be the errors by Sidi-Israeli quadrature method using n boundary nodes, and let $r_n = \log_2(e_n/e_{n/2})$ be the error ratio.

Example 1. [5] Consider the steady state heat conduction in an anisotropic material in the two dimensional disc Ω of radius unity. The thermal conductivity tensor is chosen to be $\kappa_{11} = 5.0$, $\kappa_{12} = \kappa_{21} = 2.0$, and $\kappa_{22} = 1.0$. Dirichlet boundary conditions corresponding to the analytical solution $u(x_1, x_2) = x_1^3/5 - x_1^2x_2 + x_1x_2^2 + x_2^3/3$ are applied to the whole boundary $\Gamma = \{(x_1, x_2)|x_1^2 + x_2^2 = 1\}$. Under the transformation (1.2), the physical domain is distorted into an oblique ellipse Ω' with the boundary $\Gamma' = \{(\hat{x}_1, \hat{x}_2)|(5\hat{x}_1)^2 + (\hat{x}_2 + 2\hat{x}_1)^2 = 1\}$ on the mapped plane, as shown in Fig. 2. The computed values at the interior points $P_1 = (0.2, 0.2)$, $P_2 = (0.4, 0.4)$ and $P_3 = (0.6, 0.6)$ using different boundary nodes are listed in Table 1, from the numerical results we can see that $r_n \approx 3$.

In addition, the numerical solution u of the interior points along the line $x_2 = x_1$ are computed, where $x_1 = \sqrt{2}/2\rho$ and $\rho = -0.9 : 0.05 : 0.9$. The plots of computed errors are shown in Figure 3 to Figure 5.

Example 2. Consider a square domain Ω with the boundary $\Gamma = \sum_{q=1}^4 \Gamma_q$, where $\Gamma_1 = \{(x_1, 0) :\ 0 \leq x_1 \leq 1\}$, $\Gamma_2 = \{(1, x_2) :\ 0 \leq x_2 \leq 1\}$, $\Gamma_3 = \{(x_1, 1) :\ 0 \leq x_1 \leq 1\}$, and $\Gamma_4 = \{(0, x_2) :\ 0 \leq x_2 \leq 1\}$. The invariant coefficients are chosen to be $\kappa_{11} = 1$, $\kappa_{12} = 0.5$, and $\kappa_{22} = 1$. The Dirichlet condition are applied to the Γ is $u(x_1, x_2) = \frac{1}{2}x_1^2 + x_1x_2 - x_2^2 + 2$. Let each boundary Γ_q $(q = 1, \cdots, 4)$ be divided into 2^k $(k = 4, \cdots, 10)$ segments. The physical domain and the 'isotropic' mapped domain (parallelogram) are shown in Fig. 6. In order to overcome the singularities at the corners, we use Sidi-Israeli quadrature method with the Kress's

Table 1: The errors for u at the points $P_1 = (0.2, 0.2)$, $P_2 = (0.4, 0.4)$ and $P_3 = (0.6, 0.6)$

n	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}
$e_n(P_1)$	8.633E-4	2.420E-5	3.672E-6	4.589E-7	5.735E-8	7.169E-9	8.961E-10
$r_n(P_1)$	-	5.517	2.720	3.000	3.000	3.000	3.000
$e_n(P_2)$	1.533E-3	9.791E-6	6.180E-6	7.621E-7	9.525E-8	1.191E-8	1.488E-9
$r_n(P_2)$	-	7.291	0.644	3.020	3.000	3.000	3.000
$e_n(P_3)$	2.377E-3	1.532E-3	7.495E-5	3.283E-7	9.447E-8	1.178E-8	1.472E-9
$r_n(P_3)$	-	0.6324	4.355	7.835	1.797	3.004	3.000

Figure 2: *Left: The physical domain* Ω*; Right: The mapped domain* Ω'*.*

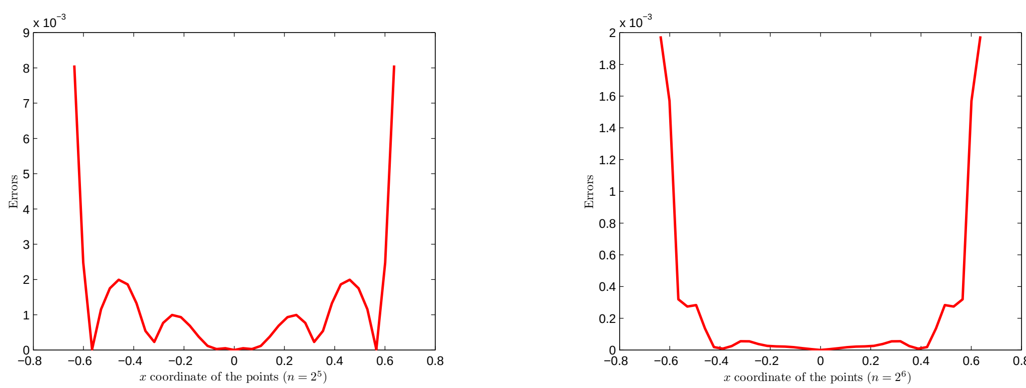

Figure 3: *Left: Errors for* u *by* 2^5 *boundary nodes; Right: Errors for* u *by* 2^6 *boundary nodes.*

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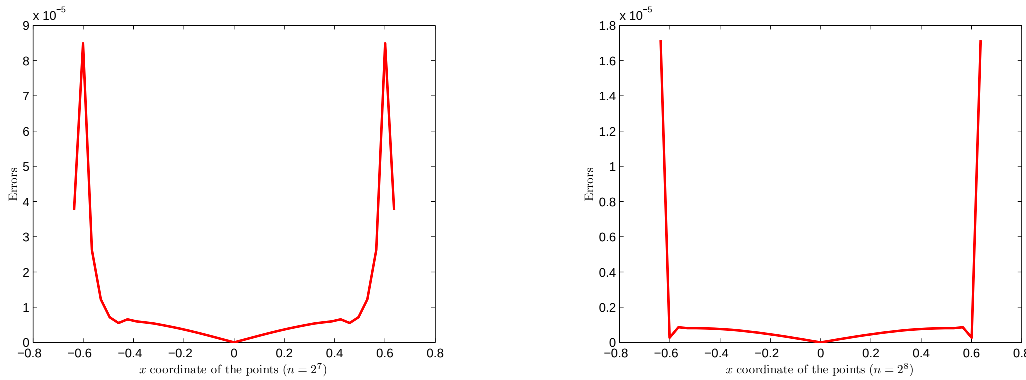

Figure 4: *Left: Errors for* u *by* 2^7 *boundary nodes; Right: Errors for* u *by* 2^8 *boundary nodes.*

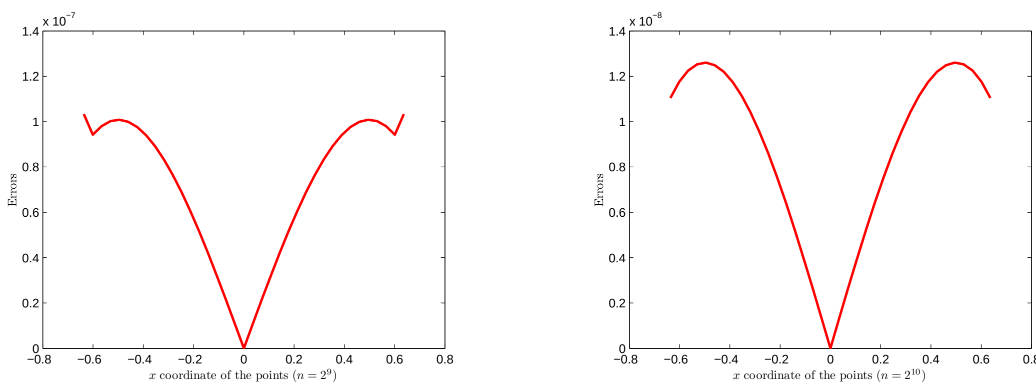

Figure 5: *Left: Errors for* u *by* 2^9 *boundary nodes; Right: Errors for* u *by* 2^{10} *boundary nodes.*

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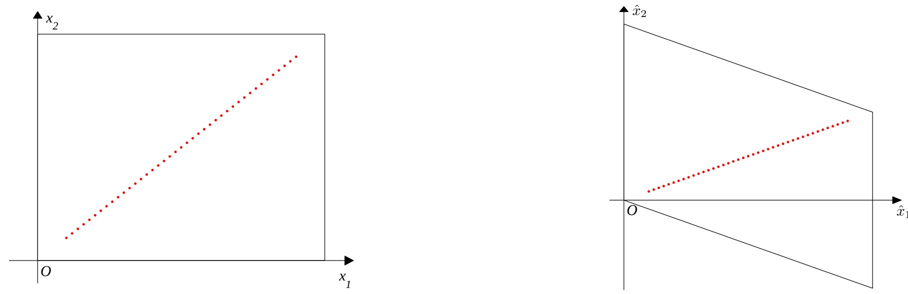

Figure 6: *Left: The physical domain* Ω*; Right: The mapped domain* Ω'*.*

variable transformation γ_3 for this problem. The computed values at the interior points $P_1 = (0.1, 0.1)$, $P_2 = (0.3, 0.3)$ and $P_3 = (0.5, 0.5)$ using n $(= 4 \times 2^k, k = 4, \cdots, 10$ nodes are listed in Table 2, from the numerical results we can also see that $r_n \approx 3$.

In addition, the numerical solution u of the interior points along the line $x_2 = x_1$ are computed, where $x_1 = \rho$ and $\rho = 0.1 : 0.02 : 0.9$. The plots of computed errors are shown in Figure 7 to Figure 9.

Table 2: The errors for u at the points $P_1 = (0.1, 0.1)$, $P_2 = (0.3, 0.3)$ and $P_3 = (0.5, 0.5)$

n	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
$e_n(P_1)$	1.154E-4	1.254E-5	1.565E-6	1.956E-7	2.444E-8	3.055E-9	3.819E-10
$r_n(P_1)$	-	3.202	3.002	3.000	3.000	3.000	3.000
$e_n(P_2)$	1.158E-4	1.445E-5	1.805E-6	2.256E-7	2.820E-8	3.525E-9	4.406E-10
$r_n(P_2)$	-	3.003	3.001	3.000	3.000	3.000	3.000
$e_n(P_3)$	1.198E-4	1.495E-5	1.869E-6	2.336E-7	2.919E-8	3.649E-9	4.561E-10
$r_n(P_3)$	-	3.002	3.001	3.000	3.000	3.000	3.000

5 Conclusions

In this paper, the Sidi-Israeli quadrature method is used to solve the boundary integral equations of steady state anisotropic heat conduction problems on the two-dimensional domain with smooth boundaries and polygons respectively. Especially, in order to provide a good accuracy in the solution near the singular points, the Kress's variable transformation is used for the weakly singular integral equations of

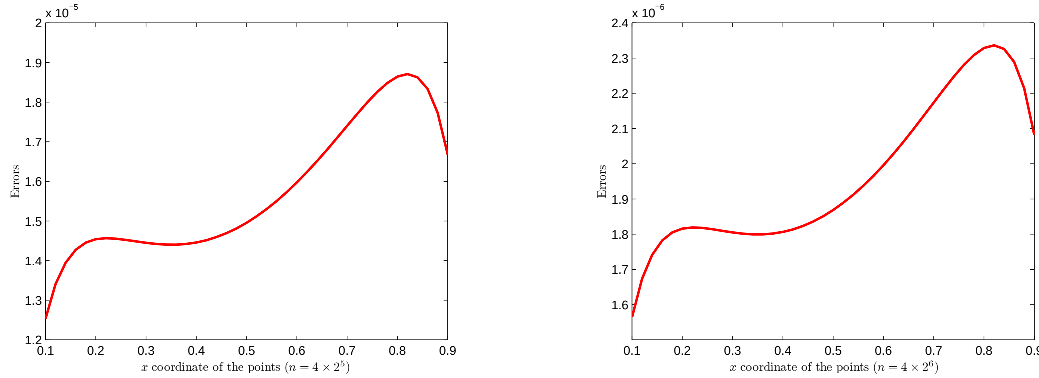

Figure 7: *Left: Errors for u by 4×2^5 boundary nodes; Right: Errors for u by 4×2^6 boundary nodes.*

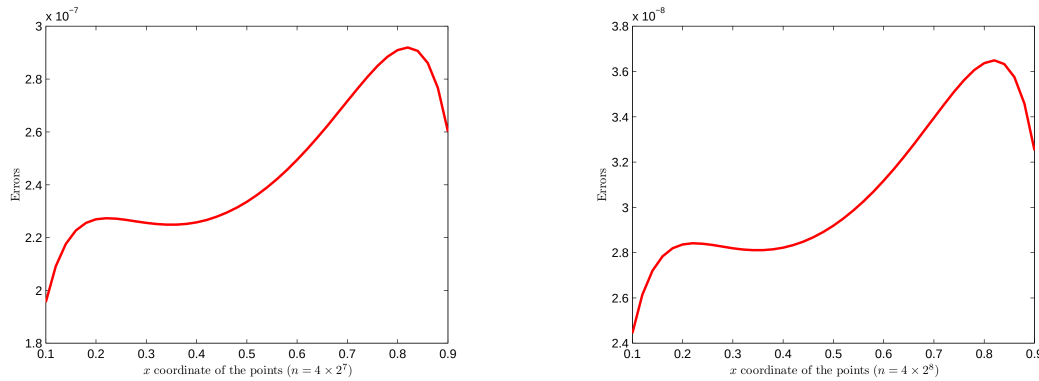

Figure 8: *Left: Errors for u by 4×2^7 boundary nodes; Right: Errors for u by 4×2^8 boundary nodes.*

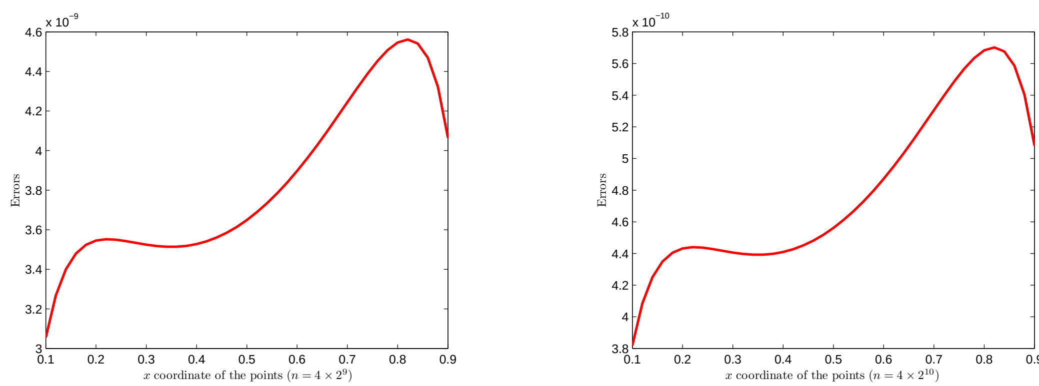

Figure 9: *Left: Errors for u by 4×2^9 boundary nodes; Right: Errors for u by 4×2^{10} boundary nodes.*

problems (1.1). The numerical results show that the presented algorithm has a high accuracy of $O(n^{-3})$, which coincides with our theoretical analysis.

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Hyers-Ulam stability of an additive set-valued functional equation

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Abstract. In this paper, we define the following additive set-valued functional equation

$$\begin{aligned} &f(2x+3y-z)+f(2y+3z-x)+f(3x+2z-y) \\ &= f(x+y)+f(y+z)+f(x+z)+f(2x)+f(2y)+f(2z) \end{aligned} \tag{1}$$

and prove the Hyers-Ulam stability of the above additive set-valued functional equation.

1. Introduction and preliminaries

The stability problem of functional equations was originated from a question of Ulam [28] concerning the stability of group homomorphisms. Hyers [6] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [19] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach (see [1, 4, 5, 7, 8, 14, 15, 20, 21, 22, 23, 24, 25, 26, 27]).

It is easy to show that if $f:\mathbb{R}\to\mathbb{R}$ is a solution of the inequality

$$|f(x+y)-f(x)-f(y)|<\varepsilon \tag{1.1}$$

for some $\varepsilon>0$ then there exists a linear function $g(x)=mx, m\in\mathbb{R}$, such that $|f(x)-g(x)|<\varepsilon$ for all $x\in\mathbb{R}$.

The inequality (1.1) can be written as the form

$$f(x+y)-f(x)-f(y)\in B(0,\varepsilon),$$

where $B(0,\varepsilon):=(-\varepsilon,\varepsilon)$. Hence we have

$$f(x+y)+B(0,\varepsilon)\subseteq f(x)+B(0,\varepsilon)+f(y)+B(0,\varepsilon)$$

and denoting by $F(x)=f(x)+B(0,\varepsilon), x\in\mathbb{R}$, we get

$$F(x+y)\subseteq F(x)+F(y), x,y\in\mathbb{R}$$

and

$$g(x)\in F(x).$$

Let Y be a real normed space. The family of all closed and convex subsets, containing 0, of Y will be denoted by $ccz(Y)$.

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Let A, B be nonempty subsets of a real vector space X and λ a real number. We define

$$\begin{aligned} A+B &= \{x \in X : x = a+b, \quad a \in A, b \in B\}, \\ \lambda A &= \{x \in X : x = \lambda a, \quad a \in A\}. \end{aligned}$$

Lemma 1.1. ([13]) *Let λ and μ be real numbers. If A and B are nonempty subsets of a real vector space X, then*

$$\lambda(A+B) = \lambda A + \lambda B,$$
$$(\lambda+\mu)A \subseteq \lambda A + \mu B.$$

Moreover, if A is a convex set and $\lambda\mu \geq 0$, then we have

$$(\lambda+\mu)A = \lambda A + \mu A.$$

A subset $A \subseteq X$ is said to be a *cone* if $A + A \subseteq A$ and $\lambda A \subseteq A$ for all $\lambda > 0$. If the zero vector in X belongs to A, then we say that A is a *cone with zero.*

Set-valued functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [3, 10, 11, 12]).

2. Stability of the set-valued functional equation (1)

In this section, let X be a real vector space, $A \subseteq X$ a cone with zero and Y a Banach space.

The following theorem is similar to the results of [16] and [18]

Theorem 2.1. *If $F : A \to ccz(Y)$ is a set-valued map satisfying*

$$\begin{aligned} &F(2x+3y-z) + F(2y+3z-x) + F(3x+2z-y) \\ &\subseteq F(x+y) + F(y+z) + F(x+z) + F(2x) + F(2y) + F(2z) \end{aligned} \tag{2.1}$$

and

$$\sup\{diam(F(x)) : x \in A\} < +\infty$$

for all $x, y, z \in A$, then there exists a unique additive mapping $g : A \to Y$ such that $g(x) \in F(x)$.

Proof. Take an element $x \in A$. Letting $y = z = x$ in (2.1) and using Lemma 1.1, we get

$$3F(4x) \quad \subseteq \quad 6F(2x). \tag{2.2}$$

Replacing $2x$ by $2^n x$ in (2.2), we obtain

$$F\left(2^{n+1}x\right) \subseteq 2F\left(2^n x\right)$$

and

$$\frac{F(2^{n+1}x)}{2^{n+1}} \subseteq \frac{F(2^n x)}{2^n}.$$

Denoting by $F_n(x) = \frac{F(2^n x)}{2^n}, x \in A, n \in \mathbb{N}$, we obtain that $(F_n(x))_{n\geq 0}$ is a decreasing sequence of closed subsets of the Banach space Y. We have also

$$diam(F_n(x)) = \frac{1}{2^n} diam\left(F\left(2^n x\right)\right).$$

Taking account of $\sup\{diam(F(x)) : x \in A\} < +\infty$, we get

$$\lim_{n\to\infty} diam(F_n(x)) = 0.$$

Using the Cantor theorem for the sequence $(F_n(x))_{n\geq 0}$, we obtain that the intersection $\cap_{n\geq 0}F_n(x)$ is a singleton set and we denote this intersection by $g(x)$ for all $x \in A$. Thus we get a mapping $g : A \to Y$ and $g(x) \in F_0(x) = F(x)$ for all $x \in A$.

We now show that g is additive. For all $x_i \in A, i = 1, 2, \cdots, N$ and $n \in \mathbb{N}$,

$$\begin{aligned} & F_n(2x+3y-z) + F_n(2y+3z-x) + F_n(3x+2z-y) \\ = & \frac{F(2^n(2x+3y-z))}{2^n} + \frac{F(2^n(2y+3z-x))}{2^n} + \frac{F(2^n(3x+2z-y))}{2^n} \\ \subseteq & \frac{F(2^n(x+y)) + F(2^n(y+z)) + F(2^n(x+z)) + F(2^n(2x)) + F(2^n(2y)) + F(2^n(2z))}{2^n}. \\ = & F_n(x+y) + F_n(y+z) + F_n(x+z) + F_n(2x) + F_n(2y) + F_n(2z). \end{aligned}$$

By the definition of g, we obtain

$$\begin{aligned} & g(2x+3y-z) + g(2y+3z-x) + g(3x+2z-y) \\ & = \bigcap_{n=0}^{\infty} F_n(2x+3y-z) + \bigcap_{n=0}^{\infty} F_n(2y+3z-x) + \bigcap_{n=0}^{\infty} F_n(3x+2z-y) \\ & \subseteq \bigcap_{n=0}^{\infty}\{F_n(2x+3y-z) + F_n(2y+3z-x) + F_n(3x+2z-y)\} \\ & \subseteq \bigcap_{n=0}^{\infty}\{F_n(x+y) + F_n(y+z) + F_n(x+z) + F_n(2x) + F_n(2y) + F_n(2z)\} \end{aligned}$$

and $g(x_i) \in F_n(x_i)$. Thus we get

$$\begin{aligned} & \|g(2x+3y-z) + g(2y+3z-x) + g(3x+2z-y) \\ & -g(x+y) - g(y+z) - g(z+x) - g(2x) - g(2y) - g(2z)\| \\ & \leq diam\,(F_n\,(x+y)) + diam\,(F_n\,(y+z)) + diam\,(F_n\,(x+z)) + diam\,(F_n\,(2x)) \\ & +diam\,(F_n\,(2y)) + diam\,(F_n\,(2z)) \end{aligned}$$

which tends to zero as n tends to ∞. Thus

$$\begin{aligned} & g(2x+3y-z) + g(2y+3z-x) + g(3x+2z-y) \\ & = g(x+y) + g(y+z) + g(x+z) + g(2x) + g(2y) + g(2z) \end{aligned} \tag{2.3}$$

for all $x, y, z \in A$.

Letting $x = y = z = 0$ in (2.3), we have $3g(0) = 6g(0)$. Thus $g(0) = 0$. Letting $x = y = z$ in (2.3), we get $g(2x) = 2g(x)$ for all $x \in A$. And letting $y = z = 0$ in (2.3), we have

$$g(-x) + g(3x) = 2g(x) = g(2x) \tag{2.4}$$

for all $x \in A$. Letting $z = -x, y = 2x$ in (2.4), we get

$$g(z) + g(y-z) = g(y) \tag{2.5}$$

for all $y, z \in A$. Letting $y = 0$ in (2.5), we have $g(-z) = -g(z)$ for all $z \in A$. Hence

$$g(y-z) = g(y) + g(-z)$$

for all $y, z \in A$. That is, g is an additive mapping. □

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Topological spaces induced by fuzzy prime ideals in BCC-algebras

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Abstract. In this paper, we construct a topological space on the set of all fuzzy prime ideals of a commutative BCC-algebra X, and we discuss fuzzy prime ideals in commutative BCC-algebras.

1. Introduction

In 1966, Imai and Iséki ([7]) defined a class of algebras of type (2,0), called *BCK-algebras* which generalized the notion of an algebra of sets with the set subtraction as the only fundamental non-nullary operation, and also the notion of implication algebras ([8]). The class of all BCK-algebras is a quasi-variety. Iséki posed an interesting problem whether the class of BCK-algebras is a variety. That problem was solved by Wroński ([11]), who proved that BCK-algebra do not form a variety. In connection with this problem, Komori ([9]) introduced the notion of BCC-algebras, and Dudeck ([1, 2]) redefined the notion of BCC-algebras by using a dual form of the original definition in the sense of Komori. In [5], Dudek and Zhang introduced a new notion of ideals in BCC-algebras and described some connections between such ideals and congruences. Dudek and Jun ([3]) considered the fuzzification of ideals in BCC-algebras. Dudek, Jun and Stojakovic ([4]) described fuzzy BCC-ideals and its image. In this paper, we define a topology on the set of all fuzzy prime ideals of a commutative BCC-algebra X and the resulting space, denoted by F-spec(X), and obtain some related properties.

2. Preliminaries

By a *BCC-algebra* ([6]) we mean a nonempty set X with a constant 0 and a binary operation " $*$ " satisfying axioms: for all $x, y, z \in X$,

(I) $((x*y)*(z*y))*(x*z) = 0$,
(II) $0*x = 0$,
(III) $x*0 = x$,
(IV) $x*y = 0$ and $y*x = 0$ imply $x = y$.

For brevity, we also call X a BCC-algebra. In X we can define a binary relation " $\leq$ " by $x \leq y$ if and only if $x*y = 0$. Then $\leq$ is a partial ordering on X. The relation " $\leq$ " is called a *BCC-order* on X. A non-empty subset S of a BCC-algebra X is called a *subalgebra* of X if $x*y \in S$ for all $x, y \in S$.

In a BCC-algebra X, the following hold: for any $x, y, z \in X$,

(2.1) $x*x = 0$,
(2.2) $(x*y)*x = 0$,
(2.3) $x \leq y \Rightarrow x*z \leq y*z$,

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(2.4) $x \leq y \Rightarrow z * y \leq z * x$.

Any BCK-algebra is a BCC-algebra, but there exist BCC-algebras which are not BCK-algebras (cf. [2]). Note that a BCC-algebra is a BCK-algebra if and only if it satisfies:

(2.5) $(x * y) * z = (x * z) * y$, for all $x, y, z \in X$.

Definition 2.1 ([5])**.** Let X be a BCC-algebra and $\emptyset \neq I \subseteq X$. I is called an *ideal* (or a BCK*-ideal*) of X if it satisfies the following conditions:

(i) $0 \in I$,
(ii) $x * y, y \in I$ imply $x \in I$ for all $x, y \in X$.

Theorem 2.2 ([5])**.** *In a BCC-algebra X, every ideal of X is a subalgebra of X.*

Definition 2.3 ([5]). Let X be a BCC-algebra and $\emptyset \neq I \subseteq X$. I is called a BCC*-ideal* of X if it satisfies the following conditions:

(i) $0 \in I$,
(ii) $(x * y) * z \in I$ and $y \in I$ imply $x * z \in I$, for all $x, y, z \in X$.

Lemma 2.4 ([5]). *In a BCC-algebra X, any BCC-ideal of X is an ideal of X.*

Corollary 2.5 ([5]). *Any BCC-ideal X of a BCC-algebra X is a subalgebra of X.*

Remark. *In a BCC-algebra, a subalgebra need not be an ideal, and an ideal need not be a BCC-ideal in general (see* [2, 4]).

We now review some fuzzy logic concept. Let X be a BCC-algebra. A fuzzy set μ in X is a function $\mu : X \to [0,1]$. The set $\mu_t := \{x \in X | \mu(x) \geq t\}$, where $t \in [0,1]$ is fixed, is called a *level set* of X. By $Im(\mu)$ we denote the image set of μ. A fuzzy set $\mu : X \to [0,1]$ is called a *fuzzy subalgebra* ([3]) of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

Definition 2.6. For $t \in [0,1]$, *fuzzy point* x_t is a fuzzy subset of X such that

$$x_t(y) := \begin{cases} t & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

Definition 2.7 ([3])**.** A fuzzy set μ in a BCC-algebra X is called a *fuzzy* BCK*-ideal* if

(i) $\mu(0) \geq \mu(x)$ for all $x \in X$,
(ii) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$.

Lemma 2.8 ([3])**.** *Let X be a BCC-algebra and μ be a fuzzy BCK-ideal of X.*

(i) *if $x * y = 0$, then $\mu(x) \geq \mu(y)$ for any $x, y \in X$,*
(ii) *$\mu(x * y) \geq \min\{\mu(x * z), \mu(z * y)\}$ for all $x, y, z \in X$.*

Definition 2.9 ([3])**.** A fuzzy set μ in a BCC-algebra X is called a *fuzzy* BCC*-ideal* if

(i) $\mu(0) \geq \mu(x)$ for all $x \in X$,
(ii) $\mu(x * z) \geq \min\{\mu((x * y) * z), \mu(y)\}$ for all $x, y, z \in X$.

Any fuzzy BCC-ideal is a fuzzy BCK-ideal in BCC-algebras.

Lemma 2.10 ([4]). *If μ is a fuzzy BCC-ideal of a BCC-algebra X, then, for any $x, y, z \in X$,*

(i) $x \leq y$ *implies* $\mu(y) \leq \mu(x)$,
(ii) $\mu(x * y) = \mu(0)$, *then* $\mu(x) \geq \mu(y)$,
(iii) $\mu(x * y) \geq \mu(x)$,
(iv) $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$,
(v) $\mu(x * (y * z)) \geq \min\{\mu(x), \mu(y), \mu(z)\}$,
(i) $\mu((x * y) * (x * z)) \geq \mu(z * y)$.

Proposition 2.11 ([3]). *A fuzzy set μ in a BCC-algebra X is a fuzzy $BCK(BCC$, resp.)-ideal (subalgebra, resp.) if and only if for every $t \in [0, 1]$, the level subset μ_t is either empty or a $BCK(BCC$, resp.)-ideal (subalgebra, resp.) of X.*

Theorem 2.12. If I is an ideal of a BCC-algebra, then the characteristic function $\chi_I : X \to [0, 1]$ of I, is a fuzzy ideal of X with $I = X_{\chi_I}$, where $X_{\chi_I} =: \{x \in X | \chi_I(x) = \chi_I(0)$.

Proof. It is easily checked that χ_I is a fuzzy ideal of X. Given an ideal I, we have

$$\begin{aligned} X_{\chi_I} &= \{x \in X | \chi_I(x) = \chi_I(0)\} \\ &= \{x \in X | \chi_I(x) = 1\} \\ &= I. \end{aligned}$$

□

3. Toplogical Spaces by fuzzy prime ideals

Theorem 3.1. *Let X be a BCC-algebra and let $\{\eta_i\}_{\{i \in \Lambda\}}$ be a family of fuzzy $BCK(BCC$, resp.)-ideals of X. Then $\cap_{i \in \Lambda} \eta_i$ is a fuzzy $BCK(BCC$, resp.)-ideal of X.*

Proof. Straighforward. □

If μ is a fuzzy subset of a BCC-algebra X, then the ideal generated by μ which is denoted by $\langle \mu \rangle$ is defined as follows:

$$\langle \mu \rangle = \cap \{\eta | \mu \subseteq \eta, \eta \text{ is a fuzzy } BCC(BCK, \text{ resp.})\text{-ideal of } X\}.$$

For all x, y in a BCC-algebra X, $y * (y * x)$ is denoted by $x \wedge y$. A BCC-algebra X is said to be *commutative* ([2]) if $x * (x * y) = y * (y * x)$, for all $x, y \in X$, i.e., $x \wedge y = y \wedge x$. If X is a commutative BCC-algebra, then it is easy to check that

$$x \wedge y \leq x \text{ and } x \wedge y \leq y. \qquad (*)$$

A proper ideal P of a BCC-algebra X is said to be *prime* if for all ideals A, B of X such that $AB \subseteq P$, either $A \subseteq P$ or $B \subseteq P$, where

$$AB = \{a \wedge b | a \in A, b \in B\}.$$

In what follows, let X be a BCC-algebra, unless otherwise specified.

Definition 3.2. Let μ and η be two fuzzy subsets of X. Then the fuzzy set $\mu\eta$ is defined by

$$\mu\eta(x) = \sup\{\min\{\mu(y), \eta(z)\} | x = y \wedge z\}.$$

Clearly, since $x \wedge x = x$, each $x \in X$ is expressible as $x = y \wedge z$ for some $y, z \in X$.

Definition 3.3. A non-constant fuzzy ideal μ of X is said to be *fuzzy prime* if for all fuzzy ideals θ, σ such that $\theta\sigma \subseteq \mu$, either $\theta \subseteq \mu$ or $\sigma \subseteq \mu$.

Lemma 3.4. *Let μ and η be two fuzzy $BCK(BCC$, resp.)-ideals of X. Then $\mu\eta \subseteq \mu \cap \eta$.*

Proof. Let $x \in X$ such that $x = a \wedge b$ for some $a, b \in X$ and let μ, η be fuzzy $BCK(BCC$, resp.)-ideals of X. Then $\mu(a) \leq \mu(a \wedge b) = \mu(x)$ and $\eta(b) \leq \eta(a \wedge b) = \eta(x)$ from Lemma 2.10-(i). Hence $\min\{\mu(a), \eta(b)\} \leq (\mu \cap \eta)(x)$. Thus $\mu\eta(x) \leq (\mu \cap \eta)(x)$. □

Let Y be the set of all fuzzy prime ideals of X. Let $V(\theta) := \{\mu \in Y | \theta \subseteq \mu\}$, where θ is any fuzzy subset of X. Put $Y(\theta) = Y \setminus V(\theta)$, the complement of $V(\theta)$ in Y.

Lemma 3.5. *If θ is a fuzzy subset of X, then $V(\langle\sigma\rangle) = V(\sigma)$. In particular, $V(\langle x_\beta\rangle) = V(x_\beta)$ for any fuzzy point x_β of X.*

Proof. Clearly, $V(\sigma) \subseteq V(\langle\sigma\rangle)$. Now, let $\mu \in V(\langle\sigma\rangle) = \{\mu \in Y | \langle\sigma\rangle \subseteq \mu\}$. Then we have $\langle\sigma\rangle \subseteq \mu$. Since $\sigma \subseteq \langle\sigma\rangle$, we have $\sigma \subseteq \mu$ which implies that $\mu \in V(\sigma)$. Thus $V(\langle\sigma\rangle) \subseteq V(\sigma)$. □

Theorem 3.6. *Let $\tau = \{Y(\theta) | \theta$ is a fuzzy $BCK(BCC)$-ideal of $X\}$. Then the pair (Y, τ) is a topological space.*

Proof. Consider the fuzzy ideals θ and σ of Y defined by $\theta(x) := 0$ and $\sigma(x) := 1$ for all $x \in X$. Then $V(\theta) = Y$ and $V(\sigma) = \emptyset$ so that $\emptyset, Y \in \tau$. Now let θ_1 and θ_2 be two fuzzy $BCK(BCC)$-ideals of X. We show that $V(\theta_1) \cup V(\theta_2) = V(\theta_1 \cap \theta_2)$. To do this, if $\mu \in V(\theta_1) \cup V(\theta_2)$, then $\mu \in V(\theta_1)$ or $\mu \in V(\theta_2)$, i.e., $\theta_1 \subseteq \mu$ or $\theta_2 \subseteq \mu$ and hence $\theta_1 \cap \theta_2 \subseteq \mu$. Therefore $\mu \in V(\theta_1 \cap \theta_2)$. Thus $V(\theta_1) \cup V(\theta_2) \subseteq V(\theta_1 \cap \theta_2)$. On the other hand, if $\mu \in V(\theta_1 \cap \theta_2)$, then $\theta_1 \cap \theta_2 \subseteq \mu$. By Lemma 3.4, $\theta_1\theta_2 \subseteq \theta_1 \cap \theta_2 \subseteq \mu$ and $\theta_1\theta_2 \subseteq \mu$. Since μ is a fuzzy prime ideal of X, $\theta_1 \subseteq \mu$ or $\theta_2 \subseteq \mu$. Hence $\mu \in V(\theta_1) \cup V(\theta_2)$. Therefore $V(\theta_1 \cap \theta_2) \subseteq V(\theta_1) \cup V(\theta_2)$ and hence $V(\theta_1) \cup V(\theta_2) = V(\theta_1 \cap \theta_2)$. Thus $Y(\theta_1) \cap Y(\theta_2) = Y(\theta_1 \cap \theta_2)$, i.e., τ is closed under finite intersection.

Now we will prove that if $\{\theta_i\}_{i\in\Lambda}$ is a family of fuzzy $BCK(BCC)$-ideal of X, then

$$\cap_{i\in\Lambda} V(\theta_i) = V(\langle\cup_{i\in\Lambda}\theta_i\rangle). \tag{$**$}$$

Let $\mu \in Y$. Then we have

$$\begin{aligned}\mu \in V(\theta_i), \forall i \in \Lambda &\Leftrightarrow \theta_i \subseteq \mu, \forall i \in \Lambda\\ &\Leftrightarrow \cup_{i\in\Lambda}\, \theta_i \subseteq \mu\\ &\Leftrightarrow \langle\cup_{i\in\Lambda}\theta_i\rangle \subseteq \mu\\ &\Leftrightarrow \mu \in V(\langle\cup_{i\in\Lambda}\theta_i\rangle).\end{aligned}$$

Therefore $(**)$ holds. Thus $\cap_{i\in\Lambda} Y(\theta_i) = Y(\langle\cup_{i\in\Lambda}\theta_i\rangle)$. This proves that τ is closed under arbitrary union. Thus (Y, τ) is a topological space. □

The topological space (Y, τ) described in Theorem 3.6 is called a *fuzzy spectrum* of Y or *F-spectrum* of Y and is denoted by F-spec (Y).

Theorem 3.7. *Let (Y, τ) be a topological space. Then the subfamily $\mathcal{B} = \{Y(x_\beta) | x \in X, \beta \in (0,1]\}$ of τ is a base for τ.*

Proof. It is enough to show that for all $Y(\theta) \in \tau$ and $\mu \in Y(\theta)$ there exists $Y(x_\beta) \in \mathcal{B}$ such that $\mu \in Y(x_\beta)$ and $Y(x_\beta) \subseteq Y(\theta)$. To do this, if $Y(\theta) \in \tau$ and $\mu \in Y(\theta)$, then $\theta \nsubseteq \mu$. Hence there exists $x \in X$ such that $\theta(x) > \mu(x)$. If $\theta(x) = \beta$, then

$$\mu \in Y(x_\beta). \tag{1}$$

If $\sigma \in V(\theta)$ is an arbitrary element, then $\sigma(x) \geq \theta(x) = \beta = x_\beta(x)$, which implies that $x_\beta \subseteq \sigma$. Therefore $\sigma \in V(x_\beta)$ and hence $V(\theta) \subseteq V(x_\beta)$. Thus we have

$$Y(x_\beta) \subseteq Y(\theta). \tag{2}$$

By (1) and (2), the proof is complete. □

4. Fuzzy prime ideals of commutative BCC-algebras

Proposition 4.1. *Let μ be a fuzzy $BCK(BCC)$-ideal of a BCC-algebra X. Then $X_\mu := \{x \in X | \mu(x) = \mu(0)\}$ is a $BCK(BCC)$-ideal of X.*

Proof. Straightforward. □

If μ is not a fuzzy $BCK(BCC)$-ideal of a BCC-algebra X, then Proposition 4.1 need not be true as shown in the following example.

Example 4.2. Let X be a BCC-algebra with the following table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Then $(X; *, 0)$ is not a BCK-algebra, since $(2 * 1) * 4 = 1 \neq 0 = (2 * 4) * 1$. Let $S := \{0, 1, 2, 3, 4\}$ and $T := \{0, 1, 2\}$. Then S is a BCC-ideal of X and T is a BCC-subalgebra of X, but not a BCK-ideal of X, since $3 * 2 = 0 \in T$ and $3 \notin T$ ([5]). Let $\mu : X \to [0, 1]$ be a map defined by $\mu(0) = \mu(1) = \mu(2) = 1$ and $\mu(3) = \mu(4) = \mu(5) = \frac{1}{2}$. Then μ is not a BCK-ideal, since $\frac{1}{2} = \mu(3) < \min\{\mu(3 * 2), \mu(2)\} = 1$. $X_\mu = \{0, 1, 2\}$ is not a BCK-ideal of X, since $1 = 3 * 2 \in X_\mu$ and $2 \in X_\mu$, but $3 \notin X_\mu$. Define a fuzzy subset ν in X by $\nu(0) = \nu(1) = \nu(5) = 0.9$ and $\nu(2) = \nu(3) = \nu(4) = 0.3$. Then ν is not a fuzzy BCC-ideal of X, since $\nu(4 * 2) = \nu(4) = 0.3 < \min\{\nu((4 * 5) * 2) = \nu(1 * 2) = \nu(0) = 0.9, \nu(5) = 0.9\} = 0.9$. But $X_\nu = \{0, 1, 5\}$ is not a BCC-ideal of X, since $(4 * 5) * 0 = 1, 5 \in X_\nu$ but $4 * 0 = 4 \notin X_\nu$.

A proper ideal P of a BCC-algebra X is said to be *s-prime* if

$$x \wedge y \in P \Leftrightarrow x \in P \text{ or } y \in P, \text{ for all } x, y \in X.$$

Definition 4.3. Let X be a commutative BCC-algebra. A non-constant fuzzy BCK-ideal (or fuzzy ideal) μ of X is said to be *s-prime* if for all $x, y \in X$, either $\mu(x \wedge y) = \mu(x)$ or $\mu(x \wedge y) = \mu(y)$.

Lemma 4.4. *A non-constant fuzzy set μ of a commutative X is a fuzzy s-prime ideal of X if and only if for each $t \in [0, 1]$, μ_t is either empty or an s-prime ideal of X if it is proper.*

Proof. Suppose that μ is a fuzzy s-prime ideal of X. For each $t \in [0, 1]$, assume that $\mu_t \neq \emptyset$ and $x \wedge y \in \mu_t$, where $x, y \in X$. Then $\mu(x \wedge y) \geq t$. Since μ is a fuzzy s-prime ideal of X, we obtain $\mu(x \wedge y) = \mu(x) \geq t$ or $\mu(x \wedge y) = \mu(y) \geq t$. Hence either $x \in \mu_t$ or $y \in \mu_t$. Thus μ_t is an s-prime ideal of X.

Conversely, if μ is not an s-prime ideal of X, then $\mu(x \wedge y) \neq \mu(x)$ and $\mu(x \wedge y) \neq \mu(y)$ for all $x, y \in X$. Let $x \wedge y \in \mu_t$ for all $x, y \in X$. Then $\mu(x \wedge y) \geq t$. Since μ is not an s-prime ideal of X, we obtain $\mu(x) < t$ and $\mu(y) < t$. Hence $x \notin \mu_t$ and $y \notin \mu_t$, which is a contradiction. □

Lemma 4.5. *Let μ be a fuzzy prime ideal of a BCC-algebra X. Then for any $t \in [0, 1]$, μ_t is either empty or a prime ideal of X if it is proper.*

Proof. Let $t \in [0, 1]$ and $\mu_t \neq \emptyset$. By Proposition 2.11, μ_t is a BCK-ideal of X. Now let A, B be two ideals of X such that

$$AB \subseteq \mu_t = \{x \in X | \mu(x) \geq t\}.$$

If we define the fuzzy subsets $\theta := \chi_A$ and $\sigma := \chi_B$, then it is easy to show that $\theta\sigma \subseteq \mu$, which implies $\theta \subseteq \mu$ or $\sigma \subseteq \mu$, since μ is a fuzzy prime ideal of X. It follows that $A \subseteq \mu_t$ or $B \subseteq \mu_t$. □

Lemma 4.6. *Let X be a commutative BCC-algebra X. If $z \leq x$ and $z \leq y$ for all $x, y, z \in X$, then $z \leq x \wedge y$.*

Proof. Since $z \leq x$ and $z \leq y$, we have $z * x = 0$ and $z * y = 0$. Then $z = z * 0 = z * (z * x) = x * (x * z)$ and $z = z * 0 = z * (z * y) = y * (y * z)$, since X is commutative. Hence $z = x * (x * z) = x * (x * (y * (y * z))) \leq x * (x * y) = y \wedge x = x \wedge y$. This competes the proof. □

A BCC-algebra X is said to be *positive implicative* ([2]) if for any $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

Lemma 4.7. *Let X be a positive implicative BCC-algebra which is commutative. Then a proper ideal P of X is an s-prime ideal of X if and only if P is a prime ideal of X.*

Proof. Suppose that P is an s-prime ideal such that $AB \subseteq P$ for some ideals A, B of X. In order to prove that $A \subseteq P$ or $B \subseteq P$, let us assume that neither $A \subseteq P$ nor $B \subseteq P$. Then there exist $a \in A$, $b \in B$ such that $a \notin P$ and $b \notin P$. Since $a \wedge b \in AB$ and $AB \subseteq P$, we have $a \wedge b \in P$. Since P is an s-prime ideal of X, $a \in P$ or $b \in P$, which is a contradiction. Thus $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

Conversely, suppose that for any ideals A, B of X, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. We prove that P is an s-prime ideal of X. Let $a \wedge b \in P$, where $a, b \in X$. Put $A(a) := \{x \in X | x \leq a\}$ and $A(b) := \{y \in X | y \leq b\}$. Clearly, $0, a \in A(a)$. Let $x * y, y \in A(a)$. Then $x * y \leq a$ and $y \leq a$. Since X is positive implicative, we have $(x * y) * a = (x * a) * (y * a) = (x * a) * 0 = x * a = 0$. Hence $x \in A(a)$. Therefore $A(a)$ is a BCK-ideal of X. Similarly, $A(b)$ is a BCK-ideal of X. We claim that $A(a)A(b) \subseteq P$. Let $x \in A(a)$ and $y \in A(b)$. Then $x \leq a$ and $y \leq b$. Since X is commutative, we obtain $x \wedge y \leq x$. Since $(X, \leq)$ is a partially ordered set, we have $x \wedge y \leq a$. Similarly, $x \wedge y \leq b$. By Lemma 4.6, we obtain $x \wedge y \leq a \wedge b$ and $a \wedge b \in P$. Since P is a BCK-ideal, $x \wedge y \in P$. Hence $A(a)B(b) \subseteq P$. By hypothesis, $A(a) \subseteq P$ or $A(b) \subseteq P$. Since $a \in A(a), b \in A(b)$, we have $a \in P$ or $b \in P$. Thus P is an s-prime ideal of X. □

Theorem 4.8. *Let μ be a fuzzy prime ideal of a BCC-algebra X. Then X_μ is a prime ideal of X.*

Proof. Clearly, $X_\mu = \mu_{\mu(0)}$. By Lemma 4.5, X_μ is a prime ideal of X. □

Theorem 4.9. *Let μ be a fuzzy prime ideal of a positive implicative BCC-algebra X which is commutative. Then μ is a fuzzy s-prime ideal of X.*

Proof. Let μ be a fuzzy prime ideal of X. By Lemma 4.5, μ_t is a prime ideal of X. Using Lemma 4.7, μ_t is an s-prime ideal of X. It follows from Lemma 4.4 that μ is a fuzzy s-prime ideal of X. □

The following example shows that the converse of Theorem 4.9 does not hold.

Example 4.10. Let $X := \{0, e\}$ be a set with the following table:

$*$	0	e
0	0	0
e	e	0

Then $(X; *, 0)$ is a positive implicative BCC-algebra which is commutative. Define the fuzzy subset μ of X by $\mu(0) = 0.7, \mu(e) = 0$. Clearly μ is a fuzzy s-prime ideal of X. Now consider the fuzzy ideals σ and θ of X which are defined by $\sigma(x) = \frac{1}{2}$ for all $x \in X$ and $\theta(e) = 0, \theta(0) = 1$. Then we have $\sigma\theta \subseteq \mu$ but $\sigma \nsubseteq \mu$ and $\theta \nsubseteq \mu$. Thus μ is not a fuzzy prime ideal of X.

Theorem 4.11. *Let X be a commutative BCC-algebra. Then I is an s-prime ideal of X if and only if χ_I is a fuzzy s-prime ideal of X.*

Proof. Suppose that I is an s-prime ideal of X. By Theorem 2.12, χ_I is a fuzzy ideal of X. Since I is proper, χ_I is a non-constant function. Let $x, y \in X$. If $x \in I$ or $y \in I$, then $x \wedge y \in I$. Hence $\chi_I(x \wedge y) = 1 = \chi_I(x) \vee \chi_I(y)$. If $x \notin I$ and $y \notin I$, then $x \wedge y \notin I$. Hence $\chi_I(x \wedge y) = 0 = \chi(x) \vee \chi_I(y)$. Thus χ_I is a fuzzy s-prime ideal of X.

Conversely, since $I = X_{\chi_I}$, if χ_I is a fuzzy s-prime ideal of X, it follows by Lemma 4.4 that I is an s-prime ideal of X. □

Corollary 4.12. *Let X be a positive implicative BCC-algebra which is commutative. Then P is a prime ideal of X if and only if χ_P is a fuzzy prime ideal of X.*

Proof. Let P be a prime ideal of X. Then χ_P is a fuzzy ideal of X. Now let θ, σ be two fuzzy ideals such that $\theta\sigma \subseteq \chi_P$. We shall show that

$$\theta \subseteq \chi_P \text{ or } \sigma \subseteq \chi_P. \qquad (***)$$

If $(***)$ does not hold, then there exist $x, y \in X \setminus P$ such that $\theta(x) > 0$ and $\sigma(y) > 0$. By Lemma 4.7, we have $x \wedge y \notin P$. Since $\theta\sigma \subseteq \chi_P$, we have

$$0 < \min\{\theta(x), \sigma(y)\} \leq \theta\sigma(x \wedge y) \leq \chi_P(x \wedge y).$$

In other words, $x \wedge y \in P$, which is a contradiction. Hence $(***)$ holds.

Conversely, let χ_P be a fuzzy prime ideal. By Theorem 4.9, χ_P is a fuzzy s-prime ideal. By Theorem 4.11, P is an s-prime ideal of X. By Lemma 4.7, P is a prime ideal of X. □

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RIESZ FUZZY NORMED SPACES AND STABILITY OF A LATTICE PRESERVING FUNCTIONAL EQUATION

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ABSTRACT. The main objective of this paper is to introduce and to study fuzzy normed Riesz spaces. By the direct method, we prove the Hyers-Ulam stability of the following lattice preserving functional equation in fuzzy Banach Riesz space

$$\mathcal{N}_2\left(f(\tau x \vee \eta y) - \tau f(x) \vee \eta f(y), t\right) \geq \mathcal{N}_1(\varphi(\tau x \vee \eta y, \tau x \wedge \eta y), t)$$

where $(\mathcal{X}, \mathcal{N}_1), (\mathcal{Y}, \mathcal{N}_2)$ are fuzzy normed Riesz space and fuzzy Banach Riesz space, respectively; and $\varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is a mapping such that

$$\varphi(x, y) \leq (\tau\eta)^{\frac{\alpha}{2}} \varphi\left(\frac{x}{\tau}, \frac{y}{\eta}\right)$$

for all $\tau, \eta \geq 1$ and $\alpha \in [0, 1)$.

1. Introduction

Riesz spaces are named after Frigyes Riesz who first defined them in [1] . Riesz spaces are real vector spaces equipped with a partial order. Under this partial order the Riesz space must satisfy some axioms, including the axiom that it is a lattice.

For the basic theory of vector lattices (Riesz spaces) and Banach lattices and for unexplained terminology we refer to [2, 3, 4].

In 1984, Katrasas [5] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space. Later, some mathematicians have defined fuzzy norms on a linear space from various points of view [6, 7]. In particular, Bag and Samanta [8], following Cheng and Mordeson [9], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [10]. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces.

A classical question in the theory of functional equations is the following: When is it true that a function which approximately satisfies a functional equation $\mathcal{D}$ must be close to an exact solution of $\mathcal{D}$? If the problem accepts a solution, we say that the equation $\mathcal{D}$ is stable. The first stability problem concerning group homomorphisms was raised by Ulam [11] in 1940. In 1941, Hyers [12] solved this stability problem for additive mappings subject to the Hyers condition on approximately additive mappings. The result of Hyers was generalized by Rassias [13] for linear mapping by considering an unbounded Cauchy difference. The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem ([14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]). Recently, considerable attention has been increasing to the problem of fuzzy stability of functional equations. Several fuzzy stability results concerning Cauchy, Jensen, simple quadratic, and cubic functional equations have been investigated [26]–[31].

In this paper, Riesz fuzzy normed spaces are defined and the stability condition are verified.

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2. Preliminary estimates

A non empty set $\mathcal{M}$ with a relation "$\leq$" is said to be an ordered set whenever the following conditions are satisfied :

1. $x \leq x$ for every $x \in \mathcal{M}$.
2. $x \leq y$ and $y \leq x$ implies that $x = y$.
3. $x \leq y$ and $y \leq z$ implies that $x \leq z$.

If, in addition, for all $x, y \in \mathcal{M}$ either $x \leq y$ or $y \leq x$, then $\mathcal{M}$ is called a totally ordered set. Let $\mathcal{A}$ be subset of an ordered set $\mathcal{M}$. $x \in \mathcal{M}$ is called an upper bound of $\mathcal{A}$ if $y \leq x$ for every $y \in \mathcal{A}$. $z \in \mathcal{M}$ is called a lower bound of $\mathcal{A}$ if $y \geq z$ for all $y \in \mathcal{A}$. Moreover, if there is an upper bound of $\mathcal{A}$, then $\mathcal{A}$ is said to be bounded from above. If there is a lower bound of $\mathcal{A}$, then $\mathcal{A}$ is said to be bounded from below. If $\mathcal{A}$ is bounded from above and from below, then we will briefly say that $\mathcal{A}$ is order bounded.

An order set $(\mathcal{M}, \leq)$ is called a *lattice* if any two elements $x, y \in \mathcal{M}$ have a least upper bound denoted by $x \vee y = \sup(x, y)$ and a greatest lower bound denoted by $x \wedge y = \inf(x, y)$.

A real vector space $\mathcal{E}$ which is also an ordered set is an ordered vector space if the order and the vector space structure are compatible in the following sense:

1. If $x, y \in \mathcal{E}$ such that $x \leq y$, then $x + z \leq y + z$ for all $z \in \mathcal{E}$.
2. If $x, y \in \mathcal{E}$ such that $x \leq y$, then $\alpha x \leq \alpha y$ for all $\alpha \geq 0$.

$(\mathcal{E}, \leq)$ is called a Riesz space if $(\mathcal{E}, \leq)$ is a lattice and ordered vector space.
A norm $\|\cdot\|$ on Riesz space $\mathcal{E}$, is called a lattice norm if $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$. In the latter case $(\mathcal{E}, \|\cdot\|)$ is called a normed Riesz space.
$(\mathcal{E}, \|\cdot\|)$ is called a Banach lattice if for all $x, y \in \mathcal{E}$

1. $(\mathcal{E}, \|\cdot\|)$ is a Banach space.
2. $\mathcal{E}$ is a Riesz space.
3. $\|\cdot\|$ is a lattice norm.

Example 2.1. *Suppose that $\mathcal{X}$ is compact Hausdorff space. We denote by $\mathcal{C}(\mathcal{K})$ the Banach space of all real continuous functions on $\mathcal{X}$. Let "$\leq$" be a point-wise order on $\mathcal{C}(\mathcal{K})$, $f \leq g$ if and only if $f(t) \leq g(t)$ for all $t \in K$. It is easy to see that $(\mathcal{C}(\mathcal{K}), \leq)$ is a Banach lattice.*

Let $\mathcal{E}$ be a Riesz space and let the positive cone $\mathcal{E}^+$ of $\mathcal{E}$ consist of all $x \in \mathcal{E}$ such that $x \geq 0$. For every $x \in \mathcal{E}$ let

$$x^+ = x \vee 0 \quad x^- = -x \vee 0 \quad |x| = x \vee -x.$$

Let $\mathcal{E}$ be a Riesz space. For all $x, y, z \in \mathcal{E}$ and $a \in \mathcal{R}$ the following assertions hold

1. $x + y = x \vee y + x \wedge y$, $-(x \vee y) = -x \wedge y$.
2. $x + (y \vee z) = (x + y) \vee (x + z)$, $x + (y \wedge z) = (x + y) \wedge (x + z)$.
3. $|x| = x^+ + x^-$, $|x + y| \leq |x| + |y|$.
4. $x \leq y$ is equivalent to $x^+ \leq y^+$ and $y^- \leq x^-$.
5. $(x \vee y) \wedge z = (x \wedge y) \vee (y \wedge z)$, $(x \wedge y) \vee z = (x \vee y) \wedge (y \vee z)$.

A Riesz space $\mathcal{E}$ is **Archimedean** if $x \leq 0$ holds whenever the set $\{nx : n \in \mathbf{N}\}$ is bounded from above.

Definition 2.1. [2] *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach lattices. A mapping $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$ is called positive if $\mathcal{T}(\mathcal{X}^+) = \{\mathcal{T}(|x|) : x \in \mathcal{X}\} \subset \mathcal{Y}^+$.*

Theorem 2.1. [3] *For an operator $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$ between two Riesz spaces the following statements are equivalent:*

1. *$\mathcal{T}$ is a lattice homomorphism.*
2. *$\mathcal{T}(x^+) = \mathcal{T}(x)^+$ for all $x \in \mathcal{X}$.*
3. *$\mathcal{T}(x \wedge y) = \mathcal{T}(x) \wedge \mathcal{T}(y)$.*
4. *If $x \wedge y = 0$ in $\mathcal{X}$, then $\mathcal{T}(x) \wedge \mathcal{T}(y) = 0$ holds in $\mathcal{Y}$.*

5. $\mathcal{T}(|x|) = |\mathcal{T}(x)|$.

Definition 2.2. [4] *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach lattices and let $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$ be a positive mapping. We define*

($\mathcal{P}_1$) *lattice homomorphism functional equation:*

$$\mathcal{T}\left(|x| \vee |y|\right) = \mathcal{T}(|x|) \vee \mathcal{T}(|y|);$$

($\mathcal{P}_2$) *semi-homogeneity: for all $x \in \mathcal{X}$ and every number $\alpha \in \mathcal{R}^+$*

$$\mathcal{T}(\alpha|x|) = \alpha\mathcal{T}(|x|).$$

Remark 2.1. [4] *Given two Banach lattices $\mathcal{X}$ and $\mathcal{Y}$, let a positive mapping $f : \mathcal{X} \to \mathcal{Y}$ satisfy the property ($\mathcal{P}_1$). Then the following statements are valid.*

1. *$f(|x \vee y|) \leq f(|x|) \vee f(|y|)$ for all $x, y \in \mathcal{X}$.*
2. *The semi-homogeneity implies that $f(0) = 0$.*
3. *f is an increasing operator, in the sense that if $x, y \in \mathcal{X}$ are such that $|x| \leq |y|$, then $f(|x|) \leq f(|y|)$.*

3. Main results

Definition 3.1. *Let $(\mathcal{X}, \leq)$ be a Riesz space. A function $\mathcal{N} : \mathcal{X} \times \mathcal{R} \to [0, 1]$ is called a Riesz fuzzy norm on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$ and all $s, t \in \mathcal{R}$*

($\mathcal{N}_1$) $\mathcal{N}(x, t) = 0$ for $t \leq 0$;
($\mathcal{N}_2$) $x = 0$ if and only if $\mathcal{N}(x, t) = 1$ for all t> 0 ;
($\mathcal{N}_3$) $\mathcal{N}(cx, t) = \mathcal{N}(x, t/|c|)$ if c$\neq$ 0 ;
($\mathcal{N}_4$) $\mathcal{N}(x + y, t + s) \geq \min\,\{\mathcal{N}(x, t), \mathcal{N}(y, s)\}$;
($\mathcal{N}_5$) $\mathcal{N}(x, .)$ is a non-decreasing function of R and

$$lim_{t\to\infty}\, \mathcal{N}(x, t) = 1\ ;$$

($\mathcal{N}_6$) for $x \neq 0, \mathcal{N}(x, .)$ is continuous on R ;
($\mathcal{N}_7$) $\mathcal{N}(x, t) \geq \mathcal{N}(y, t)$ whenever $|x| \leq |y|$.
Then $(\mathcal{X}, \leq, \mathcal{N})$ is called a Riesz fuzzy normed space.

Example 3.1. *Let $(\mathcal{X}, \leq, \|.\|)$ be a normed Riesz space. One can easily verify that for each $k > 0$,*

$$\mathcal{N}_k(x, t) = \begin{cases} \dfrac{t}{t + k\|x\|} & if\ t > 0 \\ \\ 0 & if\ t \leq 0 \end{cases}$$

defines a Riesz fuzzy norm on $\mathcal{X}$.

Note that $(\mathcal{N}_1) - (\mathcal{N}_6)$ have been checked in [8]. We show that $(\mathcal{N}_7)$ is satisfied. Suppose that $|x| \leq |y|$ for all $x, y \in \mathcal{X}$. Then $||x|| \leq ||y||$ since $(\mathcal{X}, \leq, \|\cdot\|)$ is a normed Riesz space. So

$$\frac{t}{t + k\|x\|} \geq \frac{t}{t + k\|y\|}$$

and so

$$\mathcal{N}(x, t) \geq \mathcal{N}(y, t)$$

for all $t > 0$ and $k > 0$. Therefore, $(\mathcal{X}, \leq, \mathcal{N})$ is a Riesz fuzzy normed space.

Example 3.2. *Let $(\mathcal{X}, \leq, \|.\|$) be a normed Riesz space. We define*

$$\mathcal{N}(x, t) = \begin{cases} 0 & if\ t \leq \|x\| \\ \\ 1 & if\ t > \|x\|. \end{cases}$$

It is very easy to show that $(\mathcal{X}, \leq, \mathcal{N})$ is a Riesz fuzzy normed space.

Remark 3.1. *Convergent and Cauchy sequences in Riesz fuzzy normed space are same as in fuzzy normed space.*

Definition 3.2. *Let $(\mathcal{X}, \leq, \mathcal{N})$ be a fuzzy normed Riesz space. A sequence $\{x_n\}$ in $\mathcal{X}$ is said to be convergent if there exists an $x \in \mathcal{X}$ such that $lim_{n\to\infty}\mathcal{N}(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ and we denote it by*

$$\mathcal{N} - lim_{n\to\infty}\mathcal{N}(x_n - x, t) = x.$$

Definition 3.3. *Let $(\mathcal{X}, \leq, \mathcal{N})$ be a fuzzy normed Riesz space. A sequence $\{x_n\}$ in $\mathcal{X}$ is said to be Cauchy if for each $\epsilon > 0$ and each $\delta > 0$ there exists an $n_0 \in \mathbf{N}$ such that*

$$\mathcal{N}(x_m - x_n, \delta) > 1 - \epsilon \ (m, n \geq n_0).$$

Definition 3.4. *Let $(\mathcal{X}, \leq, \mathcal{N})$ be a fuzzy normed Riesz space. A sequence y_n in $\mathcal{X}$ is called order fuzzy convergent to y as $n \to \infty$ if there exists a sequence $x_n \downarrow 0$ in $\mathcal{X}$ as $n \to \infty$ and*

$$\mathcal{N}(y_n - y, t) \geq \mathcal{N}(x_n, t)$$

for all $n \in \mathbf{N}$. We write $y = of - \lim_{n\to\infty} y_n$.

It is well-known that every convergent sequence in a fuzzy normed Riesz space is Cauchy. If each Cauchy sequence is convergent, then the Riesz fuzzy norm is said to be complete and the fuzzy normed Riesz space is called a fuzzy Banach Riesz space.

Theorem 3.1. *Let $(\mathcal{X}, \leq, \mathcal{N})$ be a fuzzy normed Riesz space and let $\{x_n\}, \{y_n\}$ be sequences in $\mathcal{X}$ such that*

$$x = of - \lim_{n\to\infty} x_n \quad and \quad y = of - \lim_{n\to\infty} y_n.$$

Then

$$\begin{aligned} x_n + y_n &= of - \lim_{n\to\infty} x + y, \\ x_n \vee y_n &= of - \lim_{n\to\infty} x \vee y, \\ x_n \wedge y_n &= of - \lim_{n\to\infty} x \wedge y. \end{aligned}$$

Theorem 3.2. *Let $(\mathcal{X}, \leq, \mathcal{N})$ be a fuzzy normed Riesz space. Then lattice operators are continuous.*

Proof. Assume that

$$\lim_{n\to\infty} \mathcal{N}(x_n - x, s) = 1 \quad \lim_{n\to\infty} \mathcal{N}(y_n - y, t) = 1 \tag{3.1}$$

for all $t, s > 0$. Therefore,

$$\begin{aligned} \mathcal{N}(x_n \wedge y_n - x \wedge y, t + s) &= \mathcal{N}(x_n \wedge y_n - x_n \wedge y + x_n \wedge y - x \wedge y, t + s) \\ &\geq \min\{\mathcal{N}(x_n \wedge y_n - x_n \wedge y, t), \mathcal{N}(x_n \wedge y - x \wedge y, s)\} \\ &\geq \min\{\mathcal{N}(y_n - y, t), \mathcal{N}(x_n - x, s)\}. \end{aligned}$$

So

$$\lim_{n\to\infty} \mathcal{N}(x_n \wedge y_n - x \wedge y, t + s) = 1.$$

It is easy to see that the other lattice operations are continuous. □

Theorem 3.3. *Every fuzzy normed Riesz space is Archimedean.*

Proof. Let $(\mathcal{X}, \leq, \mathcal{N})$ be fuzzy normed Riesz space. We show that $\mathcal{X}$ has Archimedean properties. Suppose that $x, y \in \mathcal{X}_+$ and $nx \leq y$ for all $n \in \mathbf{N}$.

$$\mathcal{N}(nx, t) \geq \mathcal{N}(y, t) \qquad t > 0$$

and so

$$\mathcal{N}\left(x, \frac{t}{n}\right) \geq \mathcal{N}(y, t) \qquad t > 0.$$

Therefore,

$$\mathcal{N}(x, t) \geq \mathcal{N}(y, nt) \qquad t > 0$$

and

$$\mathcal{N}(x, t) \geq \mathcal{N}(n^{-1}y, t) \qquad t > 0.$$

Since $n \in \mathbf{N}$ is arbitrary as $n \to \infty$, we have $x = 0$. Hence $\mathcal{X}$ has Archimedean properties. □

Theorem 3.4. *Let $(\mathcal{E}, \leq, \mathcal{N})$ be a fuzzy normed Riesz space. Then the positive cone $\mathcal{E}_+$ is closed.*

Proof. Assume that $x_n \in \mathcal{E}_+$

$$\lim_{n\to\infty} \mathcal{N}\left(x_n - x, \frac{t}{2}\right) = 1 \quad \text{for all } t > 0, x \in \mathcal{E}.$$

By Theorem 3.2, we have

$$\lim_{n\to\infty} \mathcal{N}\left(x_n \vee 0 - x \vee 0, \frac{t}{2}\right) = 1 \quad \text{for all } t > 0, x \in \mathcal{E}.$$

So $x_n \vee 0 = x_n$ since $x_n \in \mathcal{E}_+$. Therefore,

$$\lim_{n\to\infty} \mathcal{N}\left(x_n - x \vee 0, \frac{t}{2}\right) = 1 \quad \text{for all } t > 0, x \in \mathcal{E}$$

and hence by $(\mathcal{N}_4)$, we get

$$\mathcal{N}\left(x - x \vee 0, t\right) \geq \min\left\{\mathcal{N}\left(x_n - x \vee 0, \frac{t}{2}\right), \mathcal{N}\left(x_n - x, \frac{t}{2}\right)\right\}$$

for all $t > 0$ and $x \in \mathcal{E}$. Two terms on the right hand side of the above inequality tend to 1 as $n \to \infty$, and so $x = x \vee 0$. Hence $x \in \mathcal{E}_+$. Thus the proof is complete. □

Theorem 3.5. *Let $(\mathcal{E}, \leq, \mathcal{N})$ be a fuzzy normed Riesz space. For every increasing convergent sequence $\{x_n\} \subset \mathcal{E}$*

$$\lim_{n\to\infty} \mathcal{N}(x_n - u, t) = 1 \quad \textit{for all} \quad t > 0,$$

where $u = \sup\{x_n \ : \ n \in \mathbf{N}\}$.

Proof. Suppose that $\{x_n\}$ is an increasing convergent sequence and

$$\lim_{n\to\infty} \mathcal{N}(x_n - x, t) = 1 \quad \text{for all} \quad t > 0 \text{ and all } n \in \mathbf{N}. \tag{3.2}$$

For every $m \geq n$, we have

$$x_m - x_n \in \mathcal{E}_+.$$

It follows from Theorem 3.4 that $x - x_n \geq 0$ and $x_n \leq u \leq x$ for all $n \in \mathbf{N}$. So by $(\mathbf{N}_7)$

$$\mathcal{N}(x_n - x, t) \leq \mathcal{N}(x_n - u, t) \text{ for all } t > 0.$$

Therefore, as $n \to \infty$, we have $\lim_{n\to\infty} \mathcal{N}(x_n - u, t) = 1$ and hence $u = x$. This completes the proof. □

Definition 3.5. *The sequence $\{x_n\}$ is called uniformly bounded if there exist $e \in \mathcal{E}_+$ and $a_n \in l^1$ such that $x_n \leq a_n \cdot e$.*

Definition 3.6. *Let $(\mathcal{E}, \leq, \mathcal{N})$ be a fuzzy normed Riesz space. Then $\mathcal{E}$ is called uniformly complete if* $\sup\{\sum_{i=1}^{n} x_i \ : \ n \in \mathbf{N}\}$ *exists for every uniformly bounded sequence $x_n \subset \mathcal{E}_+$.*

Theorem 3.6. *Every fuzzy Banach Riesz space is uniformly complete.*

Proof. Let $(\mathcal{E}, \leq, \mathcal{N})$ be a fuzzy Banach Riesz space and $\{x_n\} \subset \mathcal{E}_+$ be a sequence such that $x_n \leq a_n e$ for a suitable sequence $\{a_n\} \in l^1$ and some $e \in \mathcal{E}_+$. We show that $\sup\{\sum_{i=1}^{n} x_i \ : \ n \in \mathbf{N}\}$ exists. We set

$$y_n = x_1 + x_2 + ... + x_n \quad \text{and} \quad b_n = \sum_{j=n+1}^{\infty} a_j. \tag{3.3}$$

By (3.3) and $(\mathcal{N}_7)$ we have

$$\begin{aligned} \mathcal{N}(y_{n+p} - y_n, t) &= \mathcal{N}(x_{n+1} + \cdots + x_{n+p}, t) \\ &\geq \mathcal{N}(\sum_{j=1}^{\infty} a_{n+j} \cdot e, t) \\ &= \mathcal{N}(b_n \cdot e, t) \end{aligned}$$

for all $t > 0$. As $n \to \infty$, we get

$$\lim_{n\to\infty} \mathcal{N}(y_{n+p} - y_n, t) = 1.$$

So (y_n) is a Cauchy sequence in fuzzy Banach Riesz space and therefore there exists $y \in \mathcal{E}$ such that $y_n \longrightarrow^{\mathcal{N}} y$. Since y_n is increasing and convergence sequence, by Theorem 3.5, we have

$$\lim_{n\to\infty} \mathcal{N}(y_n - \vee y_n, t) = 1$$

that is, $y_n \longrightarrow^{\mathcal{N}} \sup\{\sum_{i=1}^{\infty} x_i \ : \ n \in \mathbf{N}\}$. Using a unique limit we have

$$y = \sup\{\sum_{i=1}^{\infty} x_i \ : \ n \in \mathbf{N}\}.$$

Thus the proof is complete. □

Definition 3.7. *Let $(\mathcal{E}, \leq, \mathcal{N})$ be a fuzzy normed Riesz space. $\mathcal{A} \subset \mathcal{E}$ is solid if*

(1) *$f \in \mathcal{A}$ if and only if $|f| \in \mathcal{A}$;*
(2) *if $0 \leq f \in \mathcal{A}$ and $g \in \mathcal{E}_+$ then $f \wedge g \in \mathcal{A}$.*

Definition 3.8. *Every solid subset I of $\mathcal{E}$ is called an ideal in $\mathcal{E}$.*

Definition 3.9. *An ordered closed ideal is referred to as a band.*

Theorem 3.7. *Let $(\mathcal{E}, \leq, \mathcal{N})$ be a fuzzy normed Riesz space. Then the closure of every solid subset of $\mathcal{E}$ is solid.*

Proof. Let $\mathcal{A} \subset \mathcal{E}$ be solid and $f \in \overline{\mathcal{A}}$. We show that $|f| \in \overline{\mathcal{A}}$. There exists $\{f_n\} \in \mathcal{A}$ such that $f_n \longrightarrow^{\mathcal{N}} f$. It follows from $||f_n| - |f|| \leq |f_n - f|$ and $(\mathcal{N}_7)$ that

$$\begin{aligned} \mathcal{N}(|f_n| - |f|, t) &\geq \mathcal{N}(|f_n - f|, t) \\ &= \mathcal{N}(f_n - f, t) \end{aligned}$$

for all $n \in \mathbf{N}$ and $t > 0$. Therefore, $|f_n| \to^{\mathcal{N}} |f|$ as n tends to infinity. Hence $|f| \in \overline{\mathcal{A}}$, since $\mathcal{A}$ is solid and $f_n \in \mathcal{A}$. Conversely, assume that $|f| \in \overline{\mathcal{A}}$. Then there exists $f_n \in \mathcal{A}_+$ such that $f_n \longrightarrow^{\mathcal{N}} |f|$. By (3.2) we have

$$f_n \wedge f \longrightarrow^{\mathcal{N}} f \wedge |f| = f.$$

Therefore, $f \in \overline{\mathcal{A}}$, since $f_n \wedge f \in \mathcal{A}$.
Now, assume that $0 \leq f \in \overline{\mathcal{A}}$ and $g \in \mathcal{E}_+$. There exists a sequence $\{f_n\} \in \mathcal{A}_+$ such that $f_n \overset{\mathcal{N}}{\longrightarrow} f$. Hence by (3.2)

$$f_n \wedge g \to^{\mathcal{N}} f \wedge g.$$

So $f_n \wedge g \in \mathcal{A}$. Since $f_n \in \mathcal{A}$ and $\mathcal{A}$ is solid, $f \wedge g \in \overline{\mathcal{A}}$. □

Theorem 3.8. *Let $(\mathcal{E}, \leq, \mathcal{N})$ be a fuzzy normed Riesz space. Then every band in $\mathcal{E}$ is closed.*

Proof. Suppose that $\mathcal{B}$ is a band and assume that $\{f_n\} \subset \mathcal{B}$ is a sequence such that $f_n \overset{\mathcal{N}}{\longrightarrow} f$ for some $f \in \mathcal{E}$. It follows from (3.2) that

$$|f_n| \wedge |f| \to^{\mathcal{N}} |f|$$

as $n \to \infty$. For every $n \in \mathbf{N}$, let

$$g_n = (|f_n| \vee \cdots \vee |f_1|) \wedge |f|.$$

Then $\{g_n\}$ is an increasing sequence and

$$g_n = (|f_n| \wedge |f|) \vee ... \vee (|f_1| \wedge |f|).$$

So $|f_n| \wedge |f| \leq g_n \leq |f|$. Therefore, by $(\mathcal{N}_7)$ we have

$$\mathcal{N}(|f| - g_n, t) \geq \mathcal{N}(|f| - |f_n| \wedge |f|, t)$$

for all $t > 0$. Hence $g_n \overset{\mathcal{N}}{\longrightarrow} |f|$ as $n \to \infty$. □

4. Hyers-Ulam stability of lattice homomorphisms in fuzzy normed Riesz spaces

Using the direct method, we prove the Hyers-Ulam stability of lattice homomorphisms in fuzzy Banach Riesz space as below.

Theorem 4.1. *Let f be a positive operator from a fuzzy normed Riesz space $(\mathcal{X}, \mathcal{N}_1)$ to a fuzzy Banach Riesz space $(\mathcal{Y}, \mathcal{N}_2)$ such that*

$$\mathcal{N}_2\left(f(\tau x \vee \eta y) - \tau f(x) \vee \eta f(y), t\right) \geq \mathcal{N}_1(\varphi(\tau x \vee \eta y, \tau x \wedge \eta y), t) \tag{4.4}$$

for all $x, y \in \mathbf{X}$ and $t > 0$. Here $\varphi : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is a mapping such that

$$\varphi(x, y) \leq (\tau\eta)^{\frac{\alpha}{2}} \varphi\left(\frac{x}{\tau}, \frac{y}{\eta}\right)$$

for all $\tau, \eta \geq 1$ and for which there are a number $\alpha \in [0, 1)$ and a unique positive operator $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$ satisfying the properties $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ for $x \in \mathcal{X}_+$ and the inequality

$$\mathcal{N}_2(\mathcal{T}(x) - f(x), t) \geq \mathcal{N}_1(\varphi(x, x), \frac{\tau - \tau^\alpha}{\tau^\alpha}, t).$$

Proof. Putting $y = x$ and $\tau = \eta$ in (**??**), we have

$$\begin{aligned}
\mathcal{N}_2(f(\tau x) - \tau f(x), t) &\geq \mathcal{N}_1(\varphi(\tau x, \tau x), t) \\
&\geq \mathcal{N}_1\left(\tau^\alpha \varphi(x, x), t\right) \\
&= \mathcal{N}_1\left(\varphi(x, x), \frac{t}{\tau^\alpha}\right).
\end{aligned}$$

Therefore,

$$\mathcal{N}_2\left(\frac{1}{\tau} f(\tau x) - f(x), \tau^{\alpha - 1} t\right) \geq \mathcal{N}_1(\varphi(x, x), t). \tag{4.5}$$

Now, replacing x by τx in (4.5) and using the assumption that $\varphi(\tau x, \tau x) \leq \tau^\alpha \varphi(x,x)$ and the property $(\mathcal{N}_3)$ and $(\mathcal{N}_7)$ of Definition 3.1, we obtain

$$\begin{aligned} \mathcal{N}_2\left(\frac{1}{\tau}f(\tau^2x) - f(\tau x), \tau^{\alpha-1}t\right) &\geq \mathcal{N}_1(\varphi(\tau x, \tau x), t) \\ &\geq \mathcal{N}_1\left(\tau^\alpha \varphi(x,x), t\right) \\ &= \mathcal{N}_1\left(\varphi(x,x), \frac{t}{\tau^\alpha}\right). \end{aligned}$$

Hence

$$\mathcal{N}_2\left(\frac{1}{\tau^2}f(\tau^2x) - \frac{1}{\tau}f(\tau x), \tau^{2\alpha-2}t\right) \geq \mathcal{N}_1(\varphi(x,x), t). \tag{4.6}$$

By comparing (4.5) and (4.6) and using property $(\mathcal{N}_4)$, we obtain

$$\mathcal{N}_2\left(\frac{1}{\tau^2}f(\tau^2x) - f(x), \left(\tau^{\alpha-1} + \tau^{2(\alpha-1)}\right)t\right) \geq \mathcal{N}_1(\varphi(x,x), t). \tag{4.7}$$

Again, replacing x by τx in (4.7), we have

$$\mathcal{N}_2\left(\frac{1}{\tau^3}f(\tau^3x) - \frac{1}{\tau}f(\tau x), \left(\tau^{2(\alpha-1)} + \tau^{3(\alpha-1)}\right)t\right) \geq \mathcal{N}_1(\varphi(x,x), t). \tag{4.8}$$

By comparing (4.5) and (4.8) and the property $(\mathcal{N}_4)$, we obtain

$$\mathcal{N}_2\left(\frac{1}{\tau^3}f(\tau^3x) - f(x), \left(\tau^{(\alpha-1)} + \tau^{2(\alpha-1)} + \tau^{3(\alpha-1)}\right)t\right) \geq \mathcal{N}_1(\varphi(x,x), t).$$

With this process, we obtain

$$\mathcal{N}_2\left(\frac{1}{\tau^n}f(\tau^nx) - f(x), \sum_{k=1}^{n} \tau^{k(\alpha-1)}t\right) \geq \mathcal{N}_1(\varphi(x,x), t) \tag{4.9}$$

for all $n \in \mathbf{N}$. If $m \in \mathbf{N}$ and $n > m > 0$, then $n - m \in \mathbf{N}$. Replacing n by $n - m$ in (4.9), we get

$$\mathcal{N}_2\left(\frac{1}{\tau^{n-m}}f(\tau^{n-m}x) - f(x), \sum_{k=1}^{n-m} \tau^{k(\alpha-1)}t\right) \geq \mathcal{N}_1(\varphi(x,x), t). \tag{4.10}$$

By replacing x by $\tau^m x$ and using $(\mathcal{N}_7)$, we obtain

$$\begin{aligned} \mathcal{N}_2\left(\frac{1}{\tau^n}f(\tau^nx) - \frac{1}{\tau^m}f(\tau^mx), \frac{1}{\tau^m}\sum_{k=1}^{n-m} \tau^{k(\alpha-1)}t\right) &\geq \mathcal{N}_1(\varphi(\tau^m x, \tau^m x), t) \\ &\geq \mathcal{N}_1\left(\tau^{m\alpha}\varphi(x,x), t\right) \\ &= \mathcal{N}_1\left(\varphi(x,x), \frac{t}{\tau^{m\alpha}}\right). \end{aligned}$$

It follows that

$$\mathcal{N}_2\left(\frac{1}{\tau^n}f(\tau^nx) - \frac{1}{\tau^m}f(\tau^mx), \frac{1}{\tau^m}\sum_{k=m+1}^{n} \tau^{k(\alpha-1)}t\right) \geq \mathcal{N}_1\left(\varphi(x,x), t\right). \tag{4.11}$$

Let $c > 0$, and let ϵ be given. Since $\lim_{t\to\infty} \mathcal{N}_1(\varphi(x,x), t) = 1$, there is some $t_0 > 0$ such that

$$\mathcal{N}_1(\varphi(x,x), t) \geq 1 - \epsilon.$$

Fix $t > t_0$. The convergence of series $\sum_{k=1}^{\infty} \tau^{k(\alpha-1)}$ guarantees that there exists some $n_0 \geq 0$ such that, for each $n > m > n_0$, the inequality

$$\sum_{k=m+1}^{n} \tau^{k(\alpha-1)} < c$$

holds. It follows that

$$\begin{aligned}\mathcal{N}_2\left(\frac{1}{\tau^n}f(\tau^n x)-\frac{1}{\tau^m}f(\tau^m x),c\right) &\geq \mathcal{N}_2\left(\frac{1}{\tau^n}f(\tau^n x)-\frac{1}{\tau^m}f(\tau^m x),\sum_{k=m+1}^{n}\tau^{k(\alpha-1)}t\right)\\ &\geq \mathcal{N}_1(\varphi(x,x),t)\geq 1-\epsilon.\end{aligned}$$

Hence $\left\{\frac{f(\tau^n x)}{\tau^n}\right\}$ is a Cauchy sequence in fuzzy Banach Riesz space $(\mathcal{Y},\mathcal{N}_2)$ and thus this sequence converges to some $\mathcal{T}(x)\in\mathcal{Y}$. It means that

$$\mathcal{T}(x)=\mathcal{N}_2-\lim_{n\to\infty}\frac{f(\tau^n x)}{\tau^n}$$

Furthermore, by putting $m=0$ in (4.11), we have

$$\mathcal{N}_2\left(\frac{1}{\tau^n}f(\tau^n x)-f(x),\sum_{k=1}^{n}\tau^{k(\alpha-1)}t\right) \geq \mathcal{N}_1(\varphi(x,x),t).$$

So

$$\mathcal{N}_2\left(\frac{1}{\tau^n}f(\tau^n x)-f(x),t\right) \geq \mathcal{N}_1\left(\varphi(x,x),\frac{t}{\sum_{k=1}^{n}\tau^{k(\alpha-1)}}\right).$$

As $n\to\infty$, we have

$$\mathcal{N}_2\left(\mathcal{T}(x)-f(x),t\right) \geq \mathcal{N}_1\left(\varphi(x,x),\frac{\tau-\tau^\alpha}{\tau^\alpha}t\right).$$

Next, we show that $\mathcal{T}$ satisfies $(\mathcal{P}_1)$. Putting $\tau=\eta=\tau^n$ in (4.4), we get

$$\begin{aligned}\mathcal{N}_2\left(f\left(\tau^n x\vee\tau^n y\right)-\tau^n f(x)\vee\tau^n f(y),t\right) &\geq \mathcal{N}_1\left(\varphi(\tau^n x\vee\tau^n y,\tau^n x\wedge\tau^n y,t\right)\\ &\geq \mathcal{N}_1\left(\varphi(x\vee y,x\wedge y),\frac{t}{\tau^{n\alpha}}\right).\end{aligned}$$

Substituting x with $\tau^n x$ and y with $\tau^n y$ in this last inequality, one can get

$$\begin{aligned}&\mathcal{N}_2\left(f(\tau^n\left(\tau^n x\vee\tau^n y\right))-\tau^n f(\tau^n x)\vee\tau^n f(\tau^n y),t\right)\\ &\geq \mathcal{N}_1\left(\varphi(\tau^n x\vee\tau^n y,\tau^n x\wedge\tau^n y,\frac{t}{\tau^{n\alpha}}\right)\\ &\geq \mathcal{N}_1\left(\varphi(x\vee y,x\wedge y),\frac{t}{\tau^{2n\alpha}}\right),\end{aligned}$$

which yields

$$\mathcal{N}_2\left(\frac{f\left(\tau^{2n}(x\vee y)\right)}{\tau^{2n}}-\frac{f(\tau^n x)}{\tau^n}\vee\frac{f(\tau^n y)}{\tau^n},\frac{t}{\tau^{2n}}\right) \geq \mathcal{N}_1\left(\varphi(x\vee y,x\wedge y),\tau^{2(1-\alpha)}t\right).$$

The term on the right-hand side of the above inequality tends to 1 as $n\to\infty$. By Theorem 3.2, we obtain

$$\mathcal{N}_2(\mathcal{T}(x\vee y)-\mathcal{T}x\vee\mathcal{T}y,t) \geq 1.$$

This means that

$$\mathcal{T}(x\vee y) = \mathcal{T}x\vee\mathcal{T}y,$$

consequently, the property $(\mathcal{P}_1)$ holds.

Next, we show that $\mathcal{T}(\tau x)=\tau\mathcal{T}(x)$ for all $x\in\mathbf{X}_+$ and $\tau\geq 1$. In fact, in the inequality (4.4), choose $\eta=\tau$, $y=0$ and substitute $2^n\tau$ for τ and consider Remark 2.1.

$$\mathcal{N}_2\left(f\left(2^n\tau x\vee 0\right)-2^n\tau f(x)\vee 0,t\right) \geq \mathcal{N}_1\left(\varphi\left(2^n\tau x,0\right),t\right)$$

for all $x \in \mathbf{X}$. Now we replace x with $2^n x$. Consequently, by (2.1)

$$\begin{aligned}\mathcal{N}_2\left(\frac{f(4^n\tau x)}{4^n}-\frac{\tau f(2^n x)}{2^n},\frac{t}{4^n}\right) &\geq \mathcal{N}_1\left(\varphi\left(4^n\tau x,0\right),t\right)\\ &\geq \mathcal{N}_1\left(4^{n\alpha}\tau^{\alpha}\varphi\left(x,0\right),t\right).\end{aligned}$$

Therefore,

$$\mathcal{N}_2\left(\frac{f(4^n\tau x)}{4^n}-\frac{\tau f(2^n x)}{2^n},t\right) \geq \mathcal{N}_1\left(4^{n(\alpha-1)}\tau^{\alpha}\varphi\left(x,0\right),t\right).$$

The term on the right-hand side of the above inequality tends to 1 as $n \to \infty$. Thus

$$\mathcal{T}(\tau x) = \tau(\mathcal{T}x)$$

for all $x \in \mathbf{X}_+$. □

Theorem 4.2. *Let $\mathcal{X},\mathcal{Y}$ be Banach lattices and $p:[0,\infty)\to[0,\infty)$ be a continuous function. Consider a positive map $f:\mathcal{X}\to\mathcal{Y}$ for which there are numbers $\nu\in\mathbf{R}$ and $0\leq r<1$ such that*

$$\mathcal{N}_2\left(f(\alpha|x|\vee\beta|y|)-\frac{\alpha p(\alpha)f(|x|)\vee\beta p(\beta)f(|y|)}{p(\alpha)\vee p(\beta)},t\right)\geq\mathcal{N}_1\left(\nu\left(x^r\vee y^r\right),t\right) \tag{4.12}$$

for all $x,y\in\mathcal{X}$ and $\alpha,\beta\in\mathbf{R}^+$. Then there exists a unique positive mapping $T:\mathcal{X}\to\mathcal{Y}$ which satisfies the properties $(\mathcal{P}_1),(\mathcal{P}_2)$ and the inequality

$$\mathcal{N}_2\left(F(|x|)-T(|x|),t\right) \geq \mathcal{N}_1\left(\frac{2\nu x}{2-2^r},t\right).$$

Proof. Putting $\alpha=\beta=2$ and $x=y$ in (4.12), we get

$$\mathcal{N}_2\left(f(2|x|)-\frac{2p(2)f(|x|)\vee 2p(2)f(|x|)}{p(2)\vee p(2)},t\right)\geq\mathcal{N}_1\left(\nu x^r,t\right)$$

for all $x\in\mathcal{X}$ and $r\in[0,1)$. Therefore,

$$\begin{aligned}\mathcal{N}_2\left(f(2|x|)-2f(|x|),t\right) &\geq \mathcal{N}_1\left(\nu x^r,t\right),\\ \mathcal{N}_2\left(\frac{1}{2}f(2|x|)-f(|x|),t\right) &\geq \mathcal{N}_1\left(\nu x^r,2t\right).\end{aligned}$$

The rest of the proof is similar to the previous one. □

5. Conclusion

In the classical Riesz space theory, Banach lattice requires more attention. In the present research work, we briefly introduced and studied the fuzzy normed Riesz spaces. Thus we think that there are many open problems and applications in this new research area. For example we will introduce M-space, L-space and order unit in fuzzy normed Riesz spaces in our future research work.

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BIPOLAR FUZZY SETS OF BCK-MODULES

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Abstract. The notion of bipolar fuzzy BCK-submodules are introduced, and some characterizations of bipolar fuzzy BCK-submodules are given. The conceopt of homomorphic images and preimages of bipolar fuzzy BCK-submodules are investigated. Normality and completely normality of bipolar fuzzy BCK-submodules are discussed.

keywords: bipolar fuzzy BCK-submodule, normal (completely normal) bipolar fuzzy BCK-submodule, maximal bipolar fuzzy BCK-submodule.

1. Introduction

In the traditional fuzzy sets, which presented by Zadeh [8] in 1965, the membership of elements are expressed in degrees ranging from 0 to 1. The membership degree 0 is assigned to elements which do not satisfy a corresponding property to the concerned fuzzy set. It is of interest to know whether these elements are satisfying a counter-property of our fuzzy set, but the restriction of the membership degrees to the interval [0,1] led to a great difficulty in doing this. For this reason, Lee [7] introduced the concept of the bipolar-valued fuzzy sets as an extension of the fuzzy sets. In the case of bipolar-valued fuzzy sets, the membership degrees range is increased from the interval [0,1] to the interval [-1,1]. The representation of bipolar-valued fuzzy sets express the difference of contrary elements from irrelevant elements.

The notion of bipolar-valued fuzzy subalgebra and bipolar-valued fuzzy ideal was introduced by Lee [6].

H. A. S. Abujabal, M. Aslam and A. B. Thaheem [1], introduced the notion of BCK-modules as an action of BCK-algebra over a commutative group. The concept of fuzzy BCK-submodules was introduced by M. Bakhshi [2], where he characterized the fuzzy BCK-submodules and provided some operations of it.

In this paper, we apply the notion of bipolar-valued fuzzy set on BCK-modules and introduce the notion of bipolar-valued fuzzy BCK-submodules. Then we present some characterization of bipolar-valued fuzzy BCK-submodules by means of positive t-level cut and negative s-level cut. Moreover, a certain form of bipolar fuzzy BCK-submodules is derived from a given BCK-submodule. We investigate the homomorphic image and preimage of the bipolar-valued fuzzy BCK-submodules under some conditions. The later work is devoted to discuss the normality and completely normality of bipolar fuzzy BCK-submodules. A maximal bipolar fuzzy BCK-submodule is defined and its range is specified quit so. Many examples are given to illustrate our concepts and results.

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2. **Preliminaries**

In this section we review some definitions and results regarding BCK-algebras, BCK-modules and bipolar fuzzy sets.

By a BCK-algebra, we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms:

(I) $((x * y) * (x * z))(z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $0 * x = 0$,
(V) $x * y = 0$ and $y * x = 0$ implies $x = y$, for all $x, y, z \in X$.

Let $(X, *, 0)$ be a BCK-algebra. Then X is a partially ordered set with the partial ordering $\leq$ defined on X by: $x \leq y$ if and only if $x * y = 0$. X is said to be bounded if there is an element $1 \in X$ such that $x \leq 1$ for all $x \in X$. X is said to be commutative (implicative) if $x \wedge y = y \wedge x$ $(x * (y * x) = x)$ for all $x, y \in X$ where $x \wedge y = y * (y * x)$.

Definition 2.1[1] Let X be a BCK-algebra. Then by a left X-module (abbreviated X-module), we mean an abelian group M with an operation $X \times M \longrightarrow M$ with $(x, m) \mapsto xm$ satisfies the following axioms for all $x, y \in X$ and $m, n \in M$,

(1) $(x \wedge y)m = x(ym)$,
(2) $x(m + n) = xm + xn$,
(3) $0m = 0$.

Moreover, if X is bounded and M satisfies $1m = m$, for all $m \in M$, then M is said to be unitary.

Example 2.2 Any bounded implicative BCK-algebra X forms an X-module, where "+ "is defined as $x + y = (x * y) \vee (y * x)$ and $xy = x \wedge y$.

A subgroup N of an X-module M is called submodule of M if N is also an X-module.

Theorem 2.3 [2] A subset N of a BCK-module M is a BCK-submodule of M if and only if $n_1 - n_2, xn \in N$ for all $n_1, n_2, n \in N$ and $x \in X$.

Definition 2.4 [1] Let M, N be modules over a BCK-algebra X. A mapping $f : M \longrightarrow N$ is called an X-homomorphism if

(1) $f(m_1 + m_2) = f(m_1) + f(m_2)$
(2) $f(xm) = xf(m)$ for all $m_1, m_2, m \in M$, $x \in X$.

A BCK-module homomorphism is said to be monomorphism (epimorphism) if it is one to one (onto). If it is both one to one and onto, then we say that it is an isomorphism.

Let X be the universe of discourse. A bipolar valued fuzzy set Φ of X is an object having the form

$$\Phi = \{(x; \Phi^+(x), \Phi^-(x)) \mid x \in X\}$$

where $\Phi^+ : X \longrightarrow [0, 1]$ and $\Phi^- : X \longrightarrow [-1, 0]$ are mappings. The positive membership degree $\Phi^+(x)$ denotes the satisfaction degree of an element x to the

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property corresponding to a bipolar -valued fuzzy set $\Phi = \{(x; \Phi^+(x), \Phi^-(x)) \mid x \in X\}$, and the negative membership degree $\Phi^-(x)$ denotes the satisfaction degree of x to some implicit counter-property of $\Phi = \{(x; \Phi^+(x), \Phi^-(x)) \mid x \in X\}$.

For the sake of simplicity, we shall use the symbol $\Phi = (X; \Phi^+, \Phi^-)$ for the bipolar-valued fuzzy set $\Phi = \{(x; \Phi^+(x), \Phi^-(x)) \mid x \in X\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

For a bipolar fuzzy set $\Phi = (X; \Phi^+, \Phi^-)$ and $(t, s) \in [0,1] \times [-1, 0]$, we define

$$\begin{aligned} P(\Phi; t) &= \{x \in X \mid \Phi^+(x) \geq t\}, \\ N(\Phi; s) &= \{x \in X \mid \Phi^-(x) \leq s\} \end{aligned}$$

which are called the positive t-level cut of $\Phi = (X; \Phi^+, \Phi^-)$ and the negative s-level cut of $\Phi = (X, \Phi^+, \Phi^-)$, respectively. For $k \in [0,1]$, the set

$$L(\Phi; k) = P(\Phi; k) \cap N(\Phi; -k)$$

is called the k-level cut of $\Phi = (X; \Phi^+, \Phi^-)$ (see [6]).

If $\Phi = (X; \Phi^+, \Phi^-)$ and $\Psi = (X; \Psi^+, \Psi^-)$ are bipolar fuzzy sets defined on X, then the union and the intersection of Φ and Ψ are bipolar fuzzy sets of X defined as follows:

$\Phi \cup \Psi = (X; (\Phi \cup \Psi)^+, (\Phi \cup \Psi)^-)$ and $\Phi \cap \Psi = (X; (\Phi \cap \Psi)^+, (\Phi \cap \Psi)^-)$, respectively, where

$$(\Phi \cup \Psi)^+(x) = \max\left\{\Phi^+(x), \Psi^+(x)\right\},\ (\Phi \cup \Psi)^-(x) = \min\left\{\Phi^-(x), \Psi^-(x)\right\},$$

and

$$(\Phi \cap \Psi)^+(x) = \min\left\{\Phi^+(x), \Psi^+(x)\right\},\ (\Phi \cap \Psi)^-(x) = \max\left\{\Phi^-(x), \Psi^-(x)\right\},$$

for all $x \in X$.

Definition 2.5 [3] Let $\Phi = (X; \Phi^+, \Phi^-)$ and $\Psi = (X; \Psi^+, \Psi^-)$ be bipolar fuzzy sets of X. If $\Psi^+(x) \geq \Phi^+(x)$ and $\Psi^-(x) \leq \Phi^-(x)$ for all $x \in X$, then we say that $\Psi = (X; \Psi^+, \Psi^-)$ is a bipolar fuzzy extension of $\Phi = (X; \Phi^+, \Phi^-)$ (simply Φ is subset of Ψ) and we write $\Phi \subseteq \Psi$.

In what follows, X will denote a bounded BCK-algebra and M, N are X-modules unless otherwise specified.

3. Bipolar Fuzzy BCK-Submodules

In this section applying bipolar fuzzy sets theory to BCK-modules, we introduce the notion of bipolar fuzzy BCK-submodules and discuss their properties.

Definition 3.1 A bipolar fuzzy set $\Phi = (M; \Phi^+, \Phi^-)$ of a BCK-module M is said to be a bipolar fuzzy BCK-submodule if it satisfies:

(BFS1) $\Phi^+(m_1 + m_2) \geq \min\{\Phi^+(m_1), \Phi^+(m_2)\}$ and
$\Phi^-(m_1 + m_2) \leq \max\{\Phi^-(m_1), \Phi^-(m_2)\}$,

(BFS2) $\Phi^+(-m) = \Phi^+(m)$ and $\Phi^-(-m) = \Phi^-(m)$,

(BFS3) $\Phi^+(xm) \geq \Phi^+(m)$ and $\Phi^-(xm) \leq \Phi^-(m)$.

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For the sake of simplicity, we shall use the symbols $\mathcal{BF}(M)$ and $\mathcal{BFS}(M)$ for the set of all bipolar fuzzy sets of M, and the set of all bipolar fuzzy BCK-submodules of M, respectively.

Example 3.2 Let $X = P(\mathbb{Z})$ with $*$ defined by $A * B = A \cap B^c$ and 0 is the empty set $\varnothing$, then X is a BCK-algebra. $M = \mathbb{Z}^{\mathbb{Z}} = \{f \mid f : \mathbb{Z} \longrightarrow \mathbb{Z}\}$, considered with the traditional addition of maps and 0 is the zero map, is an abelian group. If we define an action of X on M by $Af = \chi_A f$, then M forms an X-module. Define a bipolar fuzzy set $\Phi = (M; \Phi^+, \Phi^-)$ on M by

$$\Phi^+(f) = \begin{cases} 1 & \text{if } f = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\Phi^-(f) = \begin{cases} -1 & \text{if } f = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then Φ is a bipolar fuzzy BCK-submodule of M.

Example 3.3 Let $X = \{0, a, b, c, d\}$ and consider the following table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	b
c	c	b	a	0	d
d	d	d	d	d	0

Tab. 3.1

Then $(X, *, 0)$ is a commutative BCK-algebra which is not bounded. The subset $M = \{0, a, b, c\}$ of X along with the operation $+$ defined by Table 3.2 is an abelian group. Table 3.3 shows the action of X on M ($xm = x \wedge m$). Consequently, M forms an X-module.

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Tab. 3.2

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	b	b	b
c	0	a	b	c
d	0	0	0	0

Tab. 3.3

Now let $\Phi = (M; \Phi^+, \Phi^-)$ be a bipolar fuzzy set on M defined as follows:

M	0	a	b	c
Φ^+	1	0.7	0.7	0.7
Φ^-	-0.8	-0.6	-0.6	-0.6

Tab. 3.4

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Then $\Phi = (M; \Phi^+, \Phi^-)$ is a bipolar fuzzy BCK-submodule of M.

Theorem 3.4 A bipolar fuzzy set $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$ if and only if
(i) $\Phi^+(xm) \geq \Phi^+(m)$ and $\Phi^-(xm) \leq \Phi^-(m)$,
(ii) $\Phi^+(m_1 - m_2) \geq \min\{\Phi^+(m_1), \Phi^+(m_2)\}$ and
$\Phi^-(m_1 - m_2) \leq \max\{\Phi^-(m_1), \Phi^-(m_2)\}$ for all $m, m_1, m_2 \in M$ and $x \in X$.
Proof Let $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$, then

$$\begin{aligned}
\Phi^+(m_1 - m_2) &\geq \min\{\Phi^+(m_1), \Phi^+(-m_2)\} \\
&= \min\{\Phi^+(m_1), \Phi^+(m_2)\}, \\
\Phi^-(m_1 - m_2) &\leq \max\{\Phi^-(m_1), \Phi^-(-m_2)\} \\
&= \max\{\Phi^-(m_1), \Phi^-(m_2)\}.
\end{aligned}$$

Conversely, assume that (i) and (ii) are satisfied. Put $x = 0$ in (i), then $\Phi^+(0) \geq \Phi^+(m)$, and $\Phi^-(0) \leq \Phi^-(m)$ for all $m \in M$. So using (ii), we have

$$\begin{aligned}
\Phi^+(-m) &= \Phi^+(0 - m) \geq \min\{\Phi^+(0), \Phi^+(m)\} \geq \Phi^+(m), \\
\Phi^+(m) &= \Phi^+(0 - (-m)) \geq \min\{\Phi^+(0), \Phi^+(-m)\} \geq \Phi^+(-m),
\end{aligned}$$

which implies that

$$\Phi^+(-m) = \Phi^+(m).$$

Moreover,

$$\begin{aligned}
\Phi^+(m_1 + m_2) &= \Phi^+(m_1 - (-m_2)) \\
&\geq \min\{\Phi^+(m_1), \Phi^+(-m_2)\} \\
&= \min\{\Phi^+(m_1), \Phi^+(m_2)\}.
\end{aligned}$$

We can verify that $\Phi^-(-m) = \Phi^-(m)$ and $\Phi^-(m_1+m_2) \leq \max\{\Phi^-(m_1), \Phi^-(m_2)\}$ by similar argument.

Theorem 3.5 A bipolar fuzzy set $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$ if and only if
(i) $\Phi^+(0) \geq \Phi^+(m)$ and $\Phi^-(0) \leq \Phi^-(m)$,
(ii) $\Phi^+(x_1m_1 - x_2m_2) \geq \min\{\Phi^+(m_1), \Phi^+(m_2)\}$ and $\Phi^-(x_1m_1 - x_2m_2) \leq \max\{\Phi^-(m_1), \Phi^-(m_2)\}$
for all $m, m_1, m_2 \in M$ and $x_1, x_2 \in X$.
Proof Let $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$ and let $m, m_1, m_2 \in M$ and $x_1, x_2 \in X$. (i) is already shown in the proof of Theorem 3.4. Moreover,

$$\begin{aligned}
\Phi^+(x_1m_1 - x_2m_2) &\geq \min\{\Phi^+(x_1m_1), \Phi^+(x_2m_2)\} \\
&\geq \min\{\Phi^+(m_1), \Phi^+(m_2)\},
\end{aligned}$$

and

$$\begin{aligned}
\Phi^-(x_1m_1 - x_2m_2) &\leq \max\{\Phi^-(x_1m_1), \Phi^-(x_2m_2)\} \\
&\leq \max\{\Phi^-(m_1), \Phi^-(m_2)\}.
\end{aligned}$$

Now, let $\Phi = (M; \Phi^+, \Phi^-)$ be a bipolar fuzzy set of M. Assume that (i) and (ii) hold. Let $m, m_1, m_2 \in M$ and $x \in X$, then $\Phi^+(xm) = \Phi^+(xm - 0) = \Phi^+(xm - 0m) \geq \min\{\Phi^+(m)), \Phi^+(m)\} = \Phi^+(m)$, and $\Phi^-(xm) = \Phi^-(xm - 0) = \Phi^-(xm - 0m) \leq \max\{\Phi^-(m)), \Phi^-(m)\} = \Phi^-(m)$. Now

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$\Phi^+(m_1 - m_2) = \Phi^+(1m_1 - 1m_2) \geq \min\{\Phi^+(m_1), \Phi^+(m_2)\}$, and $\Phi^-(m_1 - m_2) = \Phi^-(1m_1 - 1m_2) \leq \max\{\Phi^-(m_1), \Phi^-(m_2)\}$. Using Theorem 3.4, $\Phi = (M; \Phi^+, \Phi^-)$ is a bipolar fuzzy BCK-submodule of M.

Theorem 3.6 Let $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BF}(M)$. Then $\Phi \in BFS(M)$ if and only if $\varnothing \neq P(\Phi; t)$ and $\varnothing \neq N(\Phi; s)$ are submodules of M for all $(t, s) \in [0, 1] \times [-1, 0]$.

Proof Assume that $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$ and $(t, s) \in [0, 1] \times [-1, 0]$ be such that $\varnothing \neq P(\Phi; t), \varnothing \neq N(\Phi; s)$. Let $m,\ m_1, m_2 \in P(\Phi; t)$ and $m', m_1', m_2' \in N(\Phi; s)$ and $x \in X$. Then

$$\begin{aligned} \Phi^+(m_1 - m_2) &\geq \min\{\Phi^+(m_1), \Phi^+(m_2)\} \geq t, \\ \Phi^-(m_1' - m_2') &\leq \max\{\Phi^-(m_1'), \Phi^-(m_2')\} \leq s. \end{aligned}$$

i.e. $m_1 - m_2 \in P(\Phi; t)$ and $m_1' - m_2' \in N(\Phi; s)$.

Further,

$$\begin{aligned} \Phi^+(xm) &\geq \Phi^+(m) \geq t, \\ \Phi^-(xm') &\leq \Phi^-(m') \leq s. \end{aligned}$$

Thus we have $xm \in P(\Phi; t)$ and $xm' \in N(\Phi; s)$. Hence $P(\Phi; t)$ and $N(\Phi; s)$ are submodules of M.

Conversely, assume that $\varnothing \neq P(\Phi; t)$ and $\varnothing \neq N(\Phi; s)$ are submodules of M for all $(t, s) \in [0, 1] \times [-1, 0]$. For $m, m' \in M$, let $t_0 = \min\{\Phi^+(m), \Phi^+(m')\}$ and $s_0 = \max\{\Phi^-(m), \Phi^-(m')\}$. Then $m, m\prime \in P(\Phi; t_0)$ and $m, m\prime \in N(\Phi; s_0)$. Since $P(\Phi; t_0)$ and $N(\Phi; s_0)$ are submodules of M, then $m - m\prime \in P(\Phi; t_0)$ and $m - m\prime \in N(\Phi; s_0)$, which means that

$$\Phi^+(m - m') \geq t_0 = \min\{\Phi^+(m), \Phi^+(m')\},$$

and

$$\Phi^-(m - m') \leq s_0 = \max\{\Phi^-(m), \Phi^-(m')\}.$$

Now, let $\Phi^+(m) = t_1$, $\Phi^-(m') = s_1$ and $x \in X$. Then $m \in P(\Phi, t_1)$ and $m' \in N(\Phi; s_1)$ which implies that $xm \in P(\Phi; t_1)$ and $xm' \in N(\Phi; s_1)$. i.e. $\Phi^+(xm) \geq t_1 = \Phi^+(m)$, and $\Phi^-(xm') \leq s_1 = \Phi^-(m')$. Thus by Theorem 3.4, $\Phi \in \mathcal{BFS}(M)$.

Corollary 3.7 If $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$, then the intersection of a non-empty positive t-level cut and a non-empty negative s-level cut of $\Phi = (M; \Phi^+, \Phi^-)$ is a submodule of M for all $(t, s) \in [0, 1] \times [-1, 0]$. In particular, the non-empty k-level cut of $\Phi = (M; \Phi^+, \Phi^-)$ is a submodule of M for all $k \in [0, 1]$.

The union of a non-empty positive t-level cut and a non-empty negative s-level cut of $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$ is not a submodule of M in general as seen in the following example.

Example 3.8 Let $X = \{0, a, b, c, d, 1\}$ with a binary operation $*$ defined on Table 3.5. For the subset $M = \{0, a, c, d\}$ of X, define an operation $+$ as $x + y = (x * y) \vee (y * x)$. It follows by Table 3.6 that $(M, +)$ is an abelian group. M is an X-module according to Table 3.7.

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$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	a	0	0	a	0	0
b	b	a	0	b	a	0
c	c	c	c	0	0	0
d	d	c	c	a	0	0
1	1	d	c	b	a	0

Tab. 3.5

$+$	0	a	c	d
0	0	a	c	d
a	a	0	d	c
c	c	d	0	a
d	d	c	a	0

Tab. 3.6

$\wedge$	0	a	c	d
0	0	0	0	0
a	0	a	0	a
b	0	a	0	a
c	0	0	c	c
d	0	a	c	d
1	0	a	c	d

Tab. 3.7

Define a bipolar fuzzy set on M by the following table:

M	0	a	c	d
Φ^+	0.9	0.6	0.4	0.4
Φ^-	-1	-0.5	-0.7	-0.5

Tab. 3.8

We can easily check that $\Phi = (X; \Phi^+, \Phi^-)$ is a bipolar fuzzy BCK-submodule of M. The positive 0.5-level cut is $P(\Phi; 0.5) = \{0, a\}$, and the negative -0.7-level cut is $N(\Phi; -0.7) = \{0, c\}$. It is clear that $P(\Phi; 0.5) \cup N(\Phi; -0.7) = \{0, a, c\}$ is not a submodule of M. Furthermore, $P(\Phi; 0.6) \cup N(\Phi; -0.6) = \{0, a, c\}$ is not a submodule of M.

A sufficient condition for $P(\Phi; k) \cup N(\Phi; -k)$ to be a submodule of M is given in the next theorem. without this condition, $P(\Phi; k) \cup N(\Phi; -k)$ need not be a submodule of M as we have already seen.

Theorem 3.9 If $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$ such that

(*) $$\Phi^+(m) + \Phi^-(m) \geq 0,$$

or

(**) $$\Phi^+(m) + \Phi^-(m) \leq 0,$$

for all $m \in M$, then the union of a non-empty positive k-level cut and a non-empty negative $-k$-level cut of $\Phi = (M; \Phi^+, \Phi^-)$ is a submodule of M for all $k \in [0, 1]$.

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Proof Let $k \in [0,1]$ such that $P(\Phi;k) \neq \varnothing$ and $N(\Phi;-k) \neq \varnothing$. Then they are submodules of M by Theorem 3.6. Let $x \in X$ and $m, m_1, m_2 \in P(\Phi;k) \cup N(\Phi;-k)$. If $m \in P(\Phi;k)$, then

$$xm \in P(\Phi;k) \subseteq P(\Phi;k) \cup N(\Phi;-k).$$

If $m \in N(\Phi^-, -k)$, then

$$xm \in N(\Phi;-k) \subseteq P(\Phi;k) \cup N(\Phi;-k).$$

Now we shall prove that $m_1 - m_2 \in P(\Phi^+;k) \cup N(\Phi^-;-k)$. We have the following three cases:

i. $m_1, m_2 \in P(\Phi;k)$,
ii. $m_1, m_2 \in N(\Phi;-k)$,
iii. $m_1 \in P(\Phi;k)$, $m_2 \in N(\Phi;-k)$.

Case i. implies that

$$m_1 - m_2 \in P(\Phi;k) \subseteq P(\Phi;k) \cup N(\Phi;-k).$$

Case ii. gives

$$m_1 - m_2 \in N(\Phi;-k) \subseteq P(\Phi;k) \cup N(\Phi;-k).$$

In case iii. $\Phi^+(m_1) \geq k$ and $\Phi^-(m_2) \leq -k$. If we consider (*), then $\Phi^+(m_2) + \Phi^-(m_2) \geq 0$ which means that $\Phi^+(m_2) \geq k$ and so

$$m_1 - m_2 \in P(\Phi;k) \subseteq P(\Phi;k) \cup N(\Phi;-k).$$

If we consider (**), then $\Phi^+(m_1) + \Phi^-(m_1) \leq 0$ implies that $\Phi^-(m_1) \leq -k$. Hence

$$m_1 - m_2 \in N(\Phi;-k) \subseteq P(\Phi;k) \cup N(\Phi;-k).$$

Therefore $P(\Phi;k) \cup N(\Phi;-k)$ is a submodule of M.

For a bipolar fuzzy set $\Phi = (M;\Phi^+,\Phi^-)$ and an element $m \in M$, we shall write $\Phi(m) = (\alpha, \beta)$ in the meaning of $\Phi^+(m) = \alpha$ and $\Phi^-(m) = \beta$.

Theorem 3.10 Let M be a module over a BCK-algebra X and $\varnothing \neq N \subseteq M$. Suppose that $\Phi = (M;\Phi^+,\Phi^-)$ is a bipolar fuzzy set on M defined as follows:

$$\Phi(m) = \begin{cases} (\alpha,\gamma) & \text{if } m \in N, \\ (\beta,\delta) & \text{otherwise}, \end{cases}$$

where $(\alpha,\gamma), (\beta,\delta) \in [0,1] \times [-1,0]$ with $\alpha > \beta$ and $\gamma < \delta$. Then $\Phi = (M;\Phi^+,\Phi^-)$ is a bipolar fuzzy BCK-submodule of M if and only if N is a submodule of M.

Proof Assume that $\Phi = (M;\Phi^+,\Phi^-) \in \mathcal{BFS}(M)$ and we shall prove that N is a submodule of M. Let $n \in N$ and $x \in X$. Then $\Phi^+(xn) \geq \Phi^+(n) = \alpha > \beta$ which implies that $\Phi^+(xn) = \alpha$ i.e. $xn \in N$.

Now let $n_1, n_2 \in N$, then $\Phi^+(n_1) = \Phi^+(n_2) = \alpha$ and $\Phi^+(n_1 - n_2) \geq \min\{\Phi^+(n_1), \Phi^+(n_2)\} = \alpha > \beta$ and this gives $\Phi^+(n_1 - n_2) = \alpha$ i.e. $n_1 - n_2 \in N$. Hence N is a submodule of M.

Conversely, let N be a submodule of M and let $m, m_1, m_2 \in M$ and $x \in X$. We shall prove that $\Phi^+(xm) \geq \Phi^+(m)$ and $\Phi^-(xm) \leq \Phi^-(m)$. If $m \in N$,then $xm \in N$ and we obtain

$$\Phi^+(xm) = \alpha = \Phi^+(m),$$

and

$$\Phi^-(xm) = \gamma = \Phi^-(m).$$

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If $m \notin N$, then $\Phi^+(m) = \beta$ and $\Phi^-(m) = \delta$. So we have

$$\Phi^+(xm) \geq \beta = \Phi^+(m)$$

and

$$\Phi^-(xm) \leq \delta = \Phi^-(m).$$

To show that

$\Phi^+(m_1 - m_2) \geq \min\{\Phi^+(m_1), \Phi^+(m_2)\}$ and $\Phi^-(m_1 - m_2) \leq \max\{\Phi^-(m_1), \Phi^-(m_2)\}$,

we consider the following cases:

i. $m_1, m_2 \in N$,

ii. $m_1, m_2 \notin N$,

iii. $m_1 \in N, m_2 \notin N$.

For case i. we have $m_1 - m_2 \in N$ and so

$$\Phi^+(m_1 - m_2) = \alpha = \min\{\Phi^+(m_1), \Phi^+(m_2)\}$$

and

$$\Phi^-(m_1 - m_2) = \gamma = \max\{\Phi^-(m_1), \Phi^-(m_2)\}.$$

For case ii. and iii. we have

$$\Phi^+(m_1 - m_2) \geq \beta = \min\{\Phi^+(m_1), \Phi^+(m_2)\}$$

and

$$\Phi^-(m_1 - m_2) \leq \delta = \max\{\Phi^-(m_1), \Phi^-(m_2)\}.$$

Therefore $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$.

For a submodule N of M, denote by $\Phi_N = (M, \Phi_N^+, \Phi_N^-)$ the bipolar fuzzy set defined in the theorem above with $(\alpha, \gamma) = (1, -1)$ and $(\beta, \delta) = (0, 0)$.

Example 3.11 Let $f : M \longrightarrow N$ be a homomorphism of BCK-modules. We know that $\ker f$ and $\operatorname{Im} f$ are submodules of M and N respectively. So $\Phi_{\ker f} = (M; \Phi^+_{\ker f}, \Phi^-_{\ker f})$ and $\Psi_{\operatorname{Im} f} = (N; \Psi^+_{\operatorname{Im} f}, \Psi^-_{\operatorname{Im} f})$ are bipolar fuzzy BCK-submodules.

Definition 3.12 Let $f : M \longrightarrow N$ be a BCK-module homomorphism and let $\Phi \in \mathcal{BFS}(M)$. Then the homomorphic image $f(\Phi) = (N; f(\Phi^+), f(\Phi^-))$ of Φ under f defined as follows:

$$f(\Phi^+)(n) = \begin{cases} \sup\limits_{m \in f^{-1}(n)} \Phi^+(m) & \text{if } f^{-1}(n) \neq \varnothing, \\ 0 & \text{if } f^{-1}(n) = \varnothing, \end{cases}$$

and

$$f(\Phi^-)(n) = \begin{cases} \inf\limits_{m \in f^{-1}(n)} \Phi^-(m) & \text{if } f^{-1}(n) \neq \varnothing, \\ 0 & \text{if } f^{-1}(n) = \varnothing. \end{cases}$$

Theorem 3.13 Let $f : M \longrightarrow N$ be a BCK-module epimorphism. If $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$, then the homomorphic image $f(\Phi) \in \mathcal{BFS}(N)$.

Proof According to Theorem 3.6, it is sufficient to prove that $P(f(\Phi); t)$ and $N(f(\Phi); s)$ are submodules of N for all $(t, s) \in [0, 1] \times [-1, 0]$ satisfying $P(f(\Phi); t) \neq$

$\varnothing$, $N(f(\Phi);s) \neq \varnothing$. Let $n_0, n_1, n_2 \in P(f(\Phi);t)$ and $x \in X$. Since f is an epimorphism, then there exist $m_0 \in f^{-1}(n_0), m_1 \in f^{-1}(n_1), m_2 \in f^{-1}(n_2)$ such that $\Phi^+(m_i) \geq t$, $i = 0, 1, 2$. Now

$$f(\Phi^+)(xn_0) = \sup_{m \in f^{-1}(xn)} \Phi^+(m) \geq \Phi^+(xm_0) \geq \Phi^+(m_0) \geq t,$$

and

$$f(\Phi^+)(n_1 - n_2) = \sup_{m \in f^{-1}(n_1 - n_2)} \Phi^+(m) \geq \Phi^+(m_1 - m_2) \geq \min\{\Phi^+(m_1), \Phi^+(m_2)\} \geq t.$$

Which implies that $xn_0, n_1 - n_2 \in P(f(\Phi);t)$. Therefore $P(f(\Phi);t)$ is a submodule of N for all $t \in [0,1]$. Analogously, we can verify that $N(f(\Phi);s)$ is a submodules of N for all $s \in [-1,0]$. This completes the proof.

Definition 3.14 Let $f : M \longrightarrow N$ be a homomorphism of BCK-modules, and $\Psi = (N; \Psi^+, \Psi^-)$ be a bipolar fuzzy set of N. Then the preimage of Ψ, $f^{-1}(\Psi) = (M; f^{-1}(\Psi^+), f^{-1}(\Psi^-))$, is the bipolar fuzzy set on M given by $f^{-1}(\Psi^+)(m) = \Psi^+(f(m))$, $f^{-1}(\Psi^-)(m) = \Psi^-(f(m))$ for all $m \in M$.

Theorem 3.15 Let $f : M \longrightarrow N$ be a homomorphism of BCK-modules, and $\Psi = (N; \Psi^+, \Psi^-) \in \mathcal{BFS}(N)$, then the preimage $f^{-1}(\Psi) = (M; f^{-1}(\Psi^+), f^{-1}(\Psi^-)) \in \mathcal{BFS}(M)$.

Proof Suppose that $\Psi = (N; \Psi^+, \Psi^-) \in \mathcal{BFS}(N)$ and f is a homomorphism of BCK-modules from M to N. Then for all $m_1, m_2 \in M$, we have

$$\begin{aligned} f^{-1}(\Psi^+)(m_1 - m_2) &= \Psi^+(f(m_1 - m_2)) = \Psi^+(f(m_1) - f(m_2)) \\ &\geq \min\{\Psi^+(f(m_1)), \Psi^+(f(m_2))\} \\ &= \min\{f^{-1}(\Psi^+)(m_1), f^{-1}(\Psi^+)(m_2)\} \end{aligned}$$

Moreover, let $x \in X$ and $m \in M$. then

$$\begin{aligned} f^{-1}(\Psi^+)(xm) &= \Psi^+(f(xm)) = \Psi^+(xf(m)) \\ &\geq \Psi^+(f(m)) = f^{-1}(\Psi^+)(m). \end{aligned}$$

Analogously, we have

$$f^{-1}(\Psi^-)(m_1 - m_2) \leq \max\{f^{-1}(\Psi^-)(m_1), f^{-1}(\Psi^-)(m_2)\}$$

and

$$f^{-1}(\Psi^-)(xm) \leq \max f^{-1}(\Psi^-)(m).$$

Hence, $f^{-1}(\Psi) = (M; f^{-1}(\Psi^+), f^{-1}(\Psi^-))$ is a bipolar fuzzy BCK-submodule of M.

Theorem 3.16 Let $f : M \longrightarrow N$ be an epimorphism of BCK-modules. If $\Psi = (N; \Psi^+, \Psi^-)$ is a bipolar fuzzy set on N such that the preimage $f^{-1}(\Psi) = (M; f^{-1}(\Psi^+), f^{-1}(\Psi^-)) \in \mathcal{BFS}(M)$, then $\Psi \in \mathcal{BFS}(N)$.

Proof Let $f^{-1}(\Psi) = (M; f^{-1}(\Psi^+), f^{-1}(\Psi^-))$ be a bipolar fuzzy BCK-submodule on M and let $f : M \longrightarrow N$ be an epimorphism. For $n_0, n_1, n_2 \in N$, there exist $m_0, m_1, m_2 \in M$ such that $f(m_0) = n_0$, $f(m_1) = n_1$, and $f(m_2) = n_2$.

Now

$$\begin{aligned}\Psi^+(n_1 - n_2) &= \Psi^+(f(m_1 - m_2)) = f^{-1}(\Psi^+)(m_1 - m_2)\\ &\geq \min\{f^{-1}(\Psi^+)(m_1), f^{-1}(\Psi^+)(m_2)\}\\ &= \min\{\Psi^+(f(m_1), \Psi^+(f(m_2))\}\\ &\quad \min\{\Psi^+(n_1), \Psi^+(n_2)\}\end{aligned}$$

and

$$\Psi^+(xn_0) = \Psi^+(f(xm_0)) = f^{-1}(\Psi^+)(xm_0) \geq f^{-1}(\Psi^+)(m_0) = \Psi^+(f(m_0)) = \Psi^+(n_0),$$

for all $x \in X$. By similar argument, we have

$$\Psi^-(n_1 - n_2) \leq \max\{\Psi^-(n_1), \Psi^-(n_2)\},$$

and

$$\Psi^-(xn_0) \leq \Psi^-(n_0).$$

This finishes the proof.

For a bipolar fuzzy set $\Phi = (M; \Phi^+, \Phi^-)$, we define M_Φ to be the set of all elements $m \in M$ with $\Phi(m) = \Phi(0)$.

Proposition 3.17 If $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$, then M_Φ is a submodule of M.

Proof Clearly $M_\Phi \neq \varnothing$, since $0 \in M_\Phi$. Let $m, m_1, m_2 \in M_\Phi$ and $x \in X$. Then

$$\Phi^+(0) \geq \Phi^+(xm) \geq \Phi^+(m) = \Phi^+(0)$$

i.e. $\Phi^+(xm) = \Phi^+(0)$. Similarly, $\Phi^-(xm) = \Phi^-(0)$ and so $xm \in M_\Phi$. Furthermore,

$$\Phi^+(0) \geq \Phi^+(m_1 - m_2) \geq \min\left\{\Phi^+(m_1), \Phi^+(m_2)\right\} = \Phi^+(0),$$

which means $\Phi^+(m_1 - m_2) = \Phi^+(0)$. Analogously, $\Phi^-(m_1 - m_2) = \Phi^-(0)$. Hence $m_1 - m_2 \in M_\Phi$. Therefore M_Φ is a submodule of M.

Definition 3.18 If $\Phi(0) = (1, -1)$ for $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$, then Φ is said to be normal.

Theorem 3.19 Let $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$. The normalization $\bar{\Phi} = (M; \bar{\Phi}^+, \bar{\Phi}^-)$ of Φ defined by $\bar{\Phi}^+(m) = \Phi^+(m) + 1 - \Phi^+(0)$ and $\bar{\Phi}^-(m) = \Phi^-(m) - 1 - \Phi^-(0)$, for all $m \in M$, is normal bipolar fuzzy BCK-submodule of M containing Φ.

Proof Clearly, $\bar{\Phi}^+(m) \geq \Phi^+(m)$ and $\bar{\Phi}^-(m) \leq \Phi^-(m)$ for all $m \in M$. i.e.$\Phi \subseteq \bar{\Phi}$. Now let $m, m_1, m_2 \in M$ and $x \in X$. Then

$$\bar{\Phi}^+(xm) = \Phi^+(xm) + 1 - \Phi^+(0) \geq \Phi^+(m) + 1 - \Phi^+(0) = \bar{\Phi}^+(m),$$

and

$$\begin{aligned}\bar{\Phi}^+(m_1 - m_2) &= \Phi^+(m_1 - m_2) + 1 - \Phi^+(0)\\ &\geq \min\left\{\Phi^+(m_1), \Phi^+(m_2)\right\} + 1 - \Phi^+(0)\\ &= \min\left\{\Phi^+(m_1) + 1 - \Phi^+(0), \Phi^+(m_2) + 1 - \Phi^+(0)\right\}\\ &= \min\left\{\bar{\Phi}^+(m_1), \bar{\Phi}^+(m_2)\right\}.\end{aligned}$$

Analogously, $\bar{\Phi}^-(xm) \leq \bar{\Phi}^-(m)$ and $\bar{\Phi}^-(m_1 - m_2) \leq \max\left\{\bar{\Phi}^-(m_1), \bar{\Phi}^-(m_2)\right\}$. Hence $\bar{\Phi} \in \mathcal{BFS}(M)$. Moreover,

$$\bar{\Phi}^+(0) = \Phi^+(0) + 1 - \Phi^+(0) = 1,$$

and

$$\bar{\Phi}^-(0) = \Phi^-(0) - 1 - \Phi^-(0) = -1.$$

Which means that $\bar{\Phi}$ is normal.

Let $\mathcal{S}(M)$ (respectively $\mathcal{N}(M)$) denote the set of all submodules (respectively, normal bipolar fuzzy BCK-submodules) of M. We define functions $F : \mathcal{S}(M) \longrightarrow \mathcal{N}(M)$ and $G : \mathcal{N}(M) \longrightarrow \mathcal{S}(M)$ by $F(N) = \Phi_N$ and $G(\Phi) = M_\Phi$. Then $GF = 1_{\mathcal{S}(M)}$ and $FG(\Phi) = F(M_\Phi) = \Phi_{M_\Phi} \subseteq \Phi$.

Note that $\mathcal{S}(M)$ (respectively $\mathcal{N}(M)$) is a poset under the set inclusion (respectively, bipolar fuzzy set inclusion).

Theorem 3.20 If $N, K \in \mathcal{S}(M)$, then $\Phi_{N\cap K} = \Phi_N \cap \Phi_K$, that is, $F(N \cap K) = F(N) \cap F(K)$. If $\Phi, \Psi \in \mathcal{N}(M)$, then $M_{\Phi\cap\Psi} = M_\Phi \cap M_\Psi$, that is, $G(\Phi \cap \Psi) = G(\Phi) \cap G(\Psi)$.

Proof Let $m \in M$. If $m \in N \cap K$, then $\Phi_{N\cap K}(m) = (1, -1)$. From $m \in N$ and $m \in K$ it follows that $\Phi_N(m) = \Phi_K(m) = (1, -1)$. Hence

$$(\Phi_N \cap \Phi_K)^+(m) = \min\{\Phi_N^+(m), \Phi_K^+(m)\} = 1 = \Phi_{N\cap K}^+(m),$$

and

$$(\Phi_N \cap \Phi_K)^-(m) = \max\{\Phi_N^-(m), \Phi_K^-(m)\} = -1 = \Phi_{N\cap K}^-(m).$$

If $m \notin N \cap K$, then $m \notin N$ or $m \notin K$. Thus

$$(\Phi_N \cap \Phi_K)^+(m) = \min\{\Phi_N^+(m), \Phi_K^+(m)\} = 0 = \Phi_{N\cap K}^+(m),$$

and

$$(\Phi_N \cap \Phi_K)^-(m) = \max\{\Phi_N^-(m), \Phi_K^-(m)\} = 0 = \Phi_{N\cap K}^-(m).$$

Therefore $\Phi_{N\cap K} = \Phi_N \cap \Phi_K$, and so $F(N \cap K) = F(N) \cap F(K)$. Now let $\Phi, \Psi \in \mathcal{N}(M)$. Then

$$\begin{aligned}
M_{\Phi\cap\Psi} &= \left\{m \in M \mid (\Phi \cap \Psi)^+(m) = 1, (\Phi \cap \Psi)^-(m) = -1\right\} \\
&= \left\{m \in M \mid \min\{\Phi^+(m), \Psi^+(m)\} = 1, \max\left\{\Phi^-(m), \Psi^-(m)\right\} = -1\right\} \\
&= \left\{m \in M \mid \Phi^+(m) = \Psi^+(m) = 1, \Phi^-(m) = \Psi^-(m) = -1\right\} \\
&= \left\{m \in M \mid \Phi^+(m) = 1, \Phi^-(m) = -1\right\} \cap \\
&\quad\ \left\{m \in M \mid \Psi^+(m) = 1, \Psi^-(m) = -1\right\} \\
&= \{m \in M \mid \Phi(m) = \Phi(0)\} \cap \{m \in M \mid \Psi(m) = \Psi(0)\} \\
&= M_\Phi \cap M_\Psi,
\end{aligned}$$

that is, $G(\Phi \cap \Psi) = G(\Phi) \cap G(\Psi)$. This completes the proof.

Proposition 3.21 Let $\Phi = (M; \Phi^+, \Phi^-)$ be a non-constant normal bipolar fuzzy BCK-submodule which is maximal in $(\mathcal{N}(M), \subseteq)$, then Φ takes only a value among $(0, 0), (1, 0), (0, -1)$ and $(1, -1)$.

Proof Let $\Phi = (M; \Phi^+, \Phi^-)$ be a non-constant maximal element in $(\mathcal{N}(M), \subseteq)$. Since Φ is normal, then $\Phi(0) = (1, -1)$. Let $m_0 \in M$ be such that $\Phi^+(m_0) \neq 1$.

Then $\Phi^+(m_0) = 0$. Otherwise, $0 < \Phi^+(m_0) < 1$. Consider $\Psi = (M, \Psi^+, \Psi^-)$ defined by

$$\Psi^+(m) = \frac{1}{2}(\Phi^+(m) + \Phi^+(m_0)),\ \Psi^-(m) = \frac{1}{2}(\Phi^-(m) + \Phi^-(m_0)),$$

for all $m \in M$. Clearly Ψ is well defined. Let $m, m_1, m_2 \in M$, then

$$\begin{aligned}
\Psi^+(m_1 - m_2) &= \frac{1}{2}(\Phi^+(m_1 - m_2) + \Phi^+(m_0)) \\
&\geq \frac{1}{2}(\min\{\Phi^+\big(m_1), \Phi^+(m_2)\big\} + \Phi^+(m_0)) \\
&= \min\{\frac{1}{2}(\Phi^+(m_1) + \Phi^+(m_0)), \frac{1}{2}(\Phi^+(m_2) + \Phi^+(m_0))\} \\
&= \min\left\{\Psi^+(m_1), \Psi^+(m_2)\right\},
\end{aligned}$$

and

$$\begin{aligned}
\Psi^+(xm) &= \frac{1}{2}(\Phi^+(xm) + \Phi^+(m_0)) \\
&\geq \frac{1}{2}(\Phi^+(m) + \Phi^+(m_0)) \\
&= \Psi^+(m).
\end{aligned}$$

By similar argument, we can show

$$\Psi^-(m_1 - m_2) \leq \max\{\Psi^-(m_1), \Psi^-(m_2)\},$$

and

$$\Psi^-(xm) \leq \Psi^-(m).$$

Hence, $\Psi \in \mathcal{BFS}(M)$. Clearly, $\bar{\Psi}$ is non-constant and by Theorem 3.19, $\bar{\Psi} = \left(M, \bar{\Psi}^+, \bar{\Psi}^-\right) \in \mathcal{N}(M)$. Now for all $m \in M$, we have

$$\begin{aligned}
\bar{\Psi}^+(m) &= \Psi^+(m) + 1 - \Psi^+(0) \\
&= \frac{1}{2}(\Phi^+(m) + \Phi^+(m_0)) + 1 - \frac{1}{2}(\Phi^+(0) + \Phi^+(m_0)) \\
&= \frac{1}{2}(1 + \Phi^+(m)) \geq \Phi^+(m),
\end{aligned}$$

and

$$\begin{aligned}
\bar{\Psi}^-(m) &= \Psi^-(m) - 1 - \Psi^-(0) \\
&= \frac{1}{2}(\Phi^-(m) + \Phi^-(m_0)) - 1 - \frac{1}{2}(\Phi^-(0) + \Phi^-(m_0)) \\
&= \frac{1}{2}(\Phi^-(m) - 1) \\
&\leq \Phi^-(m).
\end{aligned}$$

Furthermore,

$$\bar{\Psi}^+(m_0) = \frac{1}{2}(1 + \Phi^+(m_0)) > \Phi^+(m_0).$$

This means that Φ is a proper subset of $\bar{\Psi}$, which contradicts the maximality of Φ in $\mathcal{N}(M)$. Thus the possible values of Φ^+ are only 0 and 1. Likewise, we can show that 0 and -1 are the only possible values of Φ^-. Therefore, Φ takes only a value among $(0,0), (1,0), (0,-1)$ and $(1,-1)$. This finishes the proof.

For $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$, consider the following sets:

$$M_\Phi^{(0,0)} = \{m \in M \mid \Phi^+(m) \geq 0, \Phi^-(m) \leq 0\} = M,$$

$$M_\Phi^{(1,0)} = \{m \in M \mid \Phi^+(m) \geq 1, \Phi^-(m) \leq 0\} = \{m \in M \mid \Phi^+(m) = 1\},$$

$$M_\Phi^{(0,-1)} = \{m \in M \mid \Phi^+(m) \geq 0, \Phi^-(m) \leq -1\} = \{m \in M \mid \Phi^-(m) = -1\},$$

$$\begin{aligned} M_\Phi^{(1,-1)} &= \{m \in M \mid \Phi^+(m) \geq 1, \Phi^-(m) \leq -1\} \\ &= \{m \in M \mid \Phi^+(m) = 1, \Phi^-(m) = -1\}. \end{aligned}$$

Clearly, we have the relations

$M_\Phi^{(1,-1)} \subseteq M_\Phi^{(1,0)} \subseteq M_\Phi^{(0,0)}$, $M_\Phi^{(1,-1)} \subseteq M_\Phi^{(0,-1)} \subseteq M_\Phi^{(0,0)}$, $M_\Phi^{(1,0)} \cap M_\Phi^{(0,-1)} = M_\Phi^{(1,-1)}$.

Definition 3.22 A normal bipolar fuzzy BCK-submodule $\Phi = (M; \Phi^+, \Phi^-)$ is said to be completely normal if there exists $m \in M$ such that $\Phi(m) = (0, 0)$.

Example 3.23 Let N be a proper submodule of M. Then $\Phi_N = (M; \Phi_N^+, \Phi_N^-)$

is completely normal bipolar fuzzy BCK-submodule.

Denote by $\mathcal{C}(M)$ the set of all completely normal bipolar fuzzy BCK-submodules of M. Note that $\mathcal{C}(M) \subseteq \mathcal{N}(M)$ and the restriction of the partial ordering " $\subseteq$ " of $\mathcal{N}(M)$ gives a partial ordering of $\mathcal{C}(M)$.

Theorem 3.24 A non-constant maximal element of $(\mathcal{N}(M), \subseteq)$ is also a maximal element of ($\mathcal{C}(M), \subseteq$).

Proof First, we show that if $\Phi = (M; \Phi^+, \Phi^-)$ is a non-constant maximal element of $(\mathcal{N}(M), \subseteq)$,then $\Phi \in \mathcal{C}(M)$. Suppose that there is no $m \in M$ with $\Phi(m) = (0, 0)$ i.e. $M_\Phi^{(0,0)} - \left(M_\Phi^{(1,0)} \cup M_\Phi^{(0,-1)}\right) = \varnothing$. Since Φ is non-constant normal, then Φ assumes the value $(1, 0)$ or (and) $(0, -1)$ at some points in M and so we have the following cases:

i. $M_\Phi^{(1,0)} - M_\Phi^{(1,-1)} \neq \varnothing$, $M_\Phi^{(0,-1)} - M_\Phi^{(1,-1)} = \varnothing$.
ii. $M_\Phi^{(0,-1)} - M_\Phi^{(1,-1)} \neq \varnothing$, $M_\Phi^{(1,0)} - M_\Phi^{(1,-1)} = \varnothing$.
iii. $M_\Phi^{(1,0)} - M_\Phi^{(1,-1)} \neq \varnothing$, $M_\Phi^{(0,-1)} - M_\Phi^{(1,-1)} \neq \varnothing$.

For case i. let

$$\Psi(m) = \begin{cases} (1, -1) & \text{if } m \in M_\Phi^{(1,-1)}, \\ \left(1, -\frac{1}{2}\right) & \text{if } m \in M_\Phi^{(1,0)} - M_\Phi^{(1,-1)}. \end{cases}$$

For case ii. let

$$\Psi(m) = \begin{cases} (1, -1) & \text{if } m \in M_\Phi^{(1,-1)}, \\ \left(\frac{1}{2}, -1\right) & \text{if } m \in M_\Phi^{(0,-1)} - M_\Phi^{(1,-1)}. \end{cases}$$

For case iii. let

$$\Psi(m) = \begin{cases} (1, -1) & \text{if } m \in M_\Phi^{(1,-1)}, \\ \left(1, -\frac{1}{2}\right) & \text{if } m \in M_\Phi^{(1,0)} - M_\Phi^{(1,-1)}, \\ \left(\frac{1}{2}, -1\right) & \text{if } m \in M_\Phi^{(0,-1)} - M_\Phi^{(1,-1)}. \end{cases}$$

Noting that $M_\Phi^{(1,-1)}$, $M_\Phi^{(1,0)}$, and $M_\Phi^{(0,-1)}$ are submodules of M, it is not difficult to show that $\Psi = (M; \Psi^+, \Psi^-) \in \mathcal{BFS}(M)$ in each case. obviously, Ψ is non-constant

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normal and $\Phi \subset \Psi$. But this contradicts the fact that Φ is non-constant maximal in $(\mathcal{N}(M), \subseteq)$. Thus Φ should has the value $(0,0)$ at some points $m \in M$ and so $\Phi \in \mathcal{C}(M)$. Now let $\Phi' \in \mathcal{C}(M)$ such that $\Phi \subseteq \Phi'$. It follows that $\Phi \subseteq \Phi'$ in $\mathcal{N}(M)$. Since Φ is maximal in $(\mathcal{N}(M), \subseteq)$ and since Φ' is non-constant, then $\Phi = \Phi'$. Therefore Φ is maximal in $(\mathcal{C}(M), \subseteq)$.

Definition 3.25 A bipolar fuzzy set $\Phi = (M; \Phi^+, \Phi^-) \in \mathcal{BFS}(M)$ is said to be maximal if it satisfies:
(i) Φ is non-constant
(ii) $\bar{\Phi}$ is maximal element in $(\mathcal{N}(M), \subseteq)$.

Theorem 3.26 A maximal bipolar fuzzy BCK-submodule is completely normal and equivalent to its normalization.

Proof If $\Phi = (M; \Phi^+, \Phi^-)$ is a maximal bipolar fuzzy BCK-submodule of M, then $\bar{\Phi}$ is non-constant maximal in $(N(M), \subseteq)$ and so it is maximal in $(\mathcal{C}(M), \subseteq)$. So for some $m \in M$,

$$\begin{aligned} 0 &= \bar{\Phi}^+(m) = \Phi^+(m) + 1 - \Phi^+(0), \\ 0 &= \bar{\Phi}^-(m) = \Phi^-(m) - 1 - \Phi^-(0). \end{aligned}$$

Which implies that

$$\begin{aligned} \Phi^+(m) &= \Phi^+(0) - 1 \leq 0, \\ \Phi^-(m) &= \Phi^-(0) + 1 \geq 0. \end{aligned}$$

Since $\Phi^+(m) \geq 0$ and $\Phi^-(m) \leq 0$, then $\Phi^+(0) = 1$ and $\Phi^-(0) = -1$. Therefore $\Phi = \bar{\Phi}$ and so Φ is completely normal.

Now we arrive at one of our main theorems

Theorem 3.27 A maximal bipolar fuzzy BCK-submodule takes exactly the values $(1,-1)$, $(0,0)$.

Proof Assume that $\Phi = (M; \Phi^+, \Phi^-)$ is a maximal bipolar fuzzy BCK-submodule. Then Φ takes a value among $(0,0)$, $(1,0)$, $(0,-1)$ and $(1,-1)$ and it is completely normal. So $M_\Phi^{(0,0)} - \left(M_\Phi^{(1,0)} \cup M_\Phi^{(0,-1)}\right) \neq \varnothing$. The subsets $M_\Phi^{(1,0)} - M_\Phi^{(1,-1)}$ and $M_\Phi^{(0,-1)} - M_\Phi^{(1,-1)}$ of M are empty. If not, then we have the following cases:
i. $M_\Phi^{(1,0)} - M_\Phi^{(1,-1)} \neq \varnothing$, $M_\Phi^{(0,-1)} - M_\Phi^{(1,-1)} = \varnothing$
ii. $M_\Phi^{(0,-1)} - M_\Phi^{(1,-1)} \neq \varnothing$, $M_\Phi^{(1,0)} - M_\Phi^{(1,-1)} = \varnothing$
iii. $M_\Phi^{(1,0)} - M_\Phi^{(1,-1)} \neq \varnothing$, $M_\Phi^{(0,-1)} - M_\Phi^{(1,-1)} \neq \varnothing$
For case i. let

$$\Psi(m) = \begin{cases} (1,-1) & \text{if } m \in M_\Phi^{(1,-1)}, \\ \left(1, -\frac{1}{2}\right) & \text{if } m \in M_\Phi^{(1,0)} - M_\Phi^{(1,-1)}, \\ (0,0) & \text{if } m \in M_\Phi^{(0,0)} - M_\Phi^{(1,0)}. \end{cases}$$

For case ii. let

$$\Psi(m) = \begin{cases} (1,-1) & \text{if } m \in M_\Phi^{(1,-1)}, \\ \left(\frac{1}{2}, -1\right) & \text{if } m \in M_\Phi^{(0,-1)} - M_\Phi^{(1,-1)}, \\ (0,0) & \text{if } m \in M_\Phi^{(0,0)} - M_\Phi^{(0,-1)}. \end{cases} .$$

For case iii. let

$$\Psi(m) = \begin{cases} (1,-1) & \text{if } m \in M_\Phi^{(1,-1)}, \\ \left(1,-\frac{1}{2}\right) & \text{if } m \in M_\Phi^{(1,0)} - M_\Phi^{(1,-1)}, \\ \left(\frac{1}{2},-1\right) & \text{if } m \in M_\Phi^{(0,-1)} - M_\Phi^{(1,-1)}, \\ (0,0) & \text{if } m \in M_\Phi^{(0,0)} - \left(M_\Phi^{(1,0)} \cup M_\Phi^{(0,-1)}\right). \end{cases}$$

In each case, Ψ is non-constant completely normal bipolar fuzzy BCK-submodules containing Φ. A contradiction. Hence $M_\Phi^{(1,0)} - M_\Phi^{(1,-1)} = \varnothing$ and $M_\Phi^{(0,-1)} - M_\Phi^{(1,-1)} = \varnothing$, i.e. Φ does not assume the values $(1,0)$ and $(0,-1)$ at any point in M. Therefore Φ takes exactly the values $(0,0)$ and $(1,-1)$.

Clearly $M_\Phi^{(1,-1)} = M_\Phi$, for $\Phi \in \mathcal{N}(M)$. We are thus led to the following result.

Corollary 3.28 If $\Phi = (M; \Phi^+, \Phi^-)$ is a maximal bipolar fuzzy BCK-submodule, then $\Phi_{M_\Phi} = \Phi$.

Theorem 3.29 For a maximal bipolar fuzzy BCK-submodule $\Phi = (M; \Phi^+, \Phi^-)$, M_Φ is a maximal submodule of M.

Proof Let $\Phi = (M; \Phi^+, \Phi^-)$ be a maximal bipolar fuzzy BCK-submodule and suppose that N is a proper submodule of M such that $M_\Phi \subset N$. Consider the normal bipolar fuzzy BCK-submodule $\Psi_N = (M; \Psi_N^+, \Psi_N^-)$. If M_Φ is a proper submodule of N then Φ is proper subset of Ψ_N. A contradiction. Hence M_Φ is a maximal submodule of M.

4. **Conclusion**

The notion of bipolar-valued fuzzy set was introduced by K. M. Lee in 2000. Since then, bipolarity has been applied to various algebraic structures by many researchers. In this paper, we have applied the notion of bipolar-valued fuzzy sets to BCK-modules. We have characterized our new concept "bipolar fuzzy BCK-submodule " in several ways. Next, the homomorphic images and pre images of bipolar fuzzy BCK-submodules were discussed. The remainder of the paper was focused on the normal and completely normal bipolar fuzzy BCK-submodules which guided finally to the concept of maximality.

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