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THE EXTENSION OF A MODIFIED INTEGRAL OPERATOR TO A CLASS OF GENERALIZED FUNCTIONS

S. K. Q. Al-Omari[1] and Dumitru Baleanu[2]

Abstract. In this paper, we investigate a class of modified G-transforms having G-functions as kernels on a generalized space of sequences. We derive certain spaces of generalized functions named as Boehmians to legitimate the existence of the described integral. The modified G-transform is partially sharing the classical transform with some general properties. An inversion formula is also discussed on the generalized sense.

1. Introduction

H-functions being related to most of known special functions are defined by integrals of the Mellin-Barnes type with integrands involving products of Euler gamma functions. Being an intemperate generalization of the generalized hypergeometric functions ${}_pF_q$, H-functions are utilized for applications in a large variety of problems connected with statistical distributions, versatile integrals, reaction, diffusion, reaction diffusion, engineering, communications, fractional differential and integral equations and many areas of theoretical physics and statistical distribution theory as well.

Through a special case of H-integral transforms, the G-integral transform enfolds various integrals related to Laplace, Hankel, Hilbert and Riemann-Liouville fractional integral transforms and, that integrals of Gauss hypergeometric function kernel type. However, despite a variety of integral transforms may not be reduced to G-transform integral type they are indeed given in the form of H-transform integral type.

With the interest to study integral, dual and tripple equations, integral transforms of special kernel functions were motivated to include many mathematical problems and engineering applications.

Integral transforms having kernels of H-function type were frequently presented as [4]

$$(H\varphi)(\eta) = \int_0^\infty H_{p,q}^{m,n}\left[\eta\zeta \,\middle|\, \begin{matrix} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix}\right]\varphi(\zeta)\,\mathrm{d}\zeta,\ \eta>0, \quad (1)$$

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where $H_{p,q}^{m,n}$ are functions given in terms of the Mellin-Barnes type contour integral [6]

$$H_{p,q}^{m,n}(w)=\frac{1}{2\pi i}\int_{\mathcal{L}}X(\eta)\,w^{\eta}\mathtt{d}\eta, \tag{2}$$

where

$$X(\eta)=\frac{\prod_{1}^{m}\Gamma\left(b_j,\beta_j\eta\right)\prod_{1}^{n}\Gamma\left(1-a_j,\alpha_j\eta\right)}{\prod_{m+1}^{q}\Gamma\left(1-b_j,\beta_j\eta\right)\prod_{n+1}^{p}\Gamma\left(a_j,\alpha_j\eta\right)}. \tag{3}$$

The particular case of H-transforms where $\alpha_1=...=\alpha_p=1$ and $\beta_1=...=\beta_q=1$, gives the G-transform integral

$$(G\varphi)(\eta)=\int_0^{\infty}G_{p,q}^{m,n}\left[\eta\zeta\left|\begin{array}{l}(a_i)_{1,p}\\(b_j)_{1,q}\end{array}\right.\right]\varphi(\zeta)\,\mathtt{d}\zeta, \tag{4}$$

and that the amendment

$$G_{p,q}^{m,n}\left[\eta\left|\begin{array}{l}(a_i)_{1,p}\\(b_j)_{1,q}\end{array}\right.\right] \tag{5}$$

of the H-function is the so-called G-function.
For a somehow much more detailed account of G and H-functions we refer to $[1,8]$.
The numbers $a^*,\Delta^*,a_1^*,a_2^*,\alpha$ and β when they appear are given as follows $[4,(6.1.5)-(6.1.11)]$

$$\left.\begin{array}{l}a^*=2m+2n-p-q,\\ \Delta^*=q-p,\\ a_1^*=(m+n)-p,\\ a_2^*=(m+n)-q,\\ \alpha=\begin{cases}-\min\limits_{1\le j\le m}\operatorname{Re}(b_j) & ,m>0\\ -\infty & ,m=0\end{cases}\\ \beta=\begin{cases}1-\max\limits_{1\le i\le n}\operatorname{Re}(a_i) & ,n>0\\ \infty & ,n=0\end{cases}\\ \mu=\sum_{j=1}^{q}b_j-\sum_{i=1}^{p}a_i+\dfrac{p-q}{2}\end{array}\right\}. \tag{6}$$

By $\boldsymbol{l}_{v,r},v\in\mathbb{R},1\le r<\infty$, we denote the summable space of those Lebesgue measurable complex valued functions such that

$$\left.\begin{array}{c}\|\varphi\|_{v,r}=\left(\displaystyle\int_0^{\infty}\left|\zeta^{v}\varphi(\zeta)\right|^{r}\frac{\mathtt{d}\zeta}{\zeta}\right)^{\frac{1}{r}}<\infty\\ \text{and}\\ \|\varphi\|_{v,\infty}=\operatorname*{ess\,sup}_{\zeta>0}\left(\zeta^{v}\left|\varphi(\zeta)\right|\right),v\in\mathbb{R}\end{array}\right\}. \tag{7}$$

The modified G-transform we consider in this note is given by the integral equation $[4,(6.2.4)]$

$$\left(G_{\sigma,\kappa}^{1}\varphi\right)(\eta)=\eta^{\sigma}\int_0^{\infty}G_{p,q}^{m,n}\left[\frac{\zeta}{\eta}\left|\begin{array}{l}(a_i)_{1,p}\\(b_j)_{1,q}\end{array}\right.\right]\zeta^{\kappa}\varphi(\zeta)\,\frac{\mathtt{d}\zeta}{\zeta}. \tag{8}$$

It is associated with the the radical integral transform (4) by the equation

$$\left(G_{\sigma,\kappa}^{1}\varphi\right)(\eta)=M_{\sigma}\left(G\left(RM_{\kappa}\varphi\right)\right)(\eta),$$

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where R and M_κ are operators defined , respectively, by $[4,(3.3.13)$ and $(3.3.11)]$

$$(R\varphi)(\zeta)=\frac{1}{\zeta}\varphi\left(\frac{1}{\zeta}\right) \text{ and } (M_\kappa\varphi)(\zeta)=\zeta^\kappa\varphi(\zeta), \kappa\in\mathbb{C}.$$

Parseval Formula 1 The Parseval's formula for the modified G-transform is derived as

$$\int_0^\infty \varphi(\eta)\left(G^1_{\sigma,\kappa}g\right)(\eta)\,\mathrm{d}\eta=\int_0^\infty\left(G^2_{\kappa,\sigma}\varphi\right)(\eta)\,g(\eta)\,\mathrm{d}\eta,$$

where $G^2_{\kappa,\sigma}\varphi$ is the modified G-transform

$$\left(G^2_{\kappa,\sigma}\varphi\right)(\eta)=\eta^\sigma\int_0^\infty G^{m,n}_{p,q}\left[\frac{\zeta}{\eta}\left|\begin{array}{c}(a_i)_{1,p}\\(b_j)_{1,q}\end{array}\right.\right]\zeta^\kappa\varphi(\zeta)\frac{\mathrm{d}\zeta}{\eta}.$$

The following remark is of great importance to our investigation .

Remark 2 ([4, Theorem 6.50 (i)]) Let κ and σ be real numbers and that numbers a^*, Δ, μ, be defined as in (6). Suppose the following are satisfied :

(i) $\alpha<v-\kappa<\beta$;

(ii) Either of (a) $a^*>0$ or (b) $a^*=0$ and $\Delta^*[v-\kappa]+\operatorname{Re}\mu\leqq 0$ holds. Then the transform $G^1_{\sigma,\kappa}\varphi$ is a one-one mapping from $\boldsymbol{l}_{v,2}$ into $\boldsymbol{l}_{v-\kappa-\sigma,2}$.

For a somehow much more detailed account of several significant results on the modified G-transforms, we refer the reader to [4].

Boehmians are motivations of regular operators with algebraic character of Mikusinski operators and do not have restriction on the support. With different function spaces various spaces of Boehmians can be obtained. Distributions, ultradistributions, regular operators are indeed contained in some well established spaces of Boehmians.

In a Boehmian context, various generalizations of various integral transforms were given once the topic was started. A complete account of the theory of Boehmian spaces was given in [2, 3, 5, 7] , [10]-[17].

However, the existed results in this theory are classical and none were discussed in the space of Boehmians. In this article, we develope the classical theory of the modified $G^1_{\sigma,\kappa}$ transform to the theory of Boehmians. In the following section we discuss the construction of the spaces of Boehmians. In Section 3, we give the representative of the modified $G^1_{\sigma,\kappa}$ transform and its inverse in the defined spaces of Boehmians. We further discuss certain results related to the proposed integrals.

2. Construction of Spaces of Boehmians

Let us first agree for the products we demand for our investigation.

The first product we should use here is the so-called Mellin type convolution product of first kind defined as [9]

$$(\varphi\curlyvee g)(\xi)=\int_0^\infty y^{-1}\varphi\left(\xi y^{-1}\right)g(y)\,\mathrm{d}y, \tag{9}$$

provided the integral exists.

A number of the properties of this integral that we find it worthwhile to be described here :

(i) $g_1\curlyvee g_2=g_2\curlyvee g_1$;

(ii) $(g_1\curlyvee g_2)\curlyvee g_3=g_1\curlyvee(g_2\curlyvee g_3)$;

(iii) $(\alpha g_1)\curlyvee g_2=\alpha(g_1\curlyvee g_2)$;

(iv) $g_1\curlyvee(g_2+g_3)=g_1\curlyvee g_2+g_1\curlyvee g_3$.

It is of great importance to introduce the following convolution product that will be worthy of attention

$$(\varphi \bullet g)(\xi) = \int_0^\infty \varphi\left(\xi y^{-1}\right) y^{\kappa-1-\sigma} g(y)\, \mathtt{d}y, \tag{10}$$

where κ and σ are real numbers.

Properties of this integral are to be provided in the text of the paper.

The relation between the convolution products are given by the following theorem.

Theorem 3 Let φ and g be integrable functions on $(0,\infty)$. Then, we have

$$G^1_{\sigma,\kappa}(\varphi \curlyvee g)(\eta) = \left(G^1_{\sigma,\kappa}\varphi \bullet g\right)(\eta).$$

Proof Under the hypothesis of the theorem and by using (8) for (9) we get

$$\begin{aligned}
G^1_{\sigma,\kappa}(\varphi \curlyvee g)(\eta) &= \eta^\sigma \int_0^\infty G^{m,n}_{p,q}\left[\frac{\zeta}{\eta}\,\middle|\, \begin{matrix}(a_i)_{1,p}\\ (b_j)_{1,q}\end{matrix}\right] \zeta^\kappa \int_0^\infty y^{-1}\varphi\left(\zeta y^{-1}\right) g(y) \\
&\quad \times \mathtt{d}y \frac{\mathtt{d}\zeta}{\zeta} \\
&= \int_0^\infty g(y)\, y^{-1}\eta^\sigma \int_0^\infty G^{m,n}_{p,q}\left[\frac{\zeta}{\eta}\,\middle|\, \begin{matrix}(a_i)_{1,p}\\ (b_j)_{1,q}\end{matrix}\right] \zeta^\kappa \varphi\left(\zeta y^{-1}\right) \\
&\quad \times \frac{\mathtt{d}\zeta}{\zeta}\mathtt{d}y. \qquad (11)
\end{aligned}$$

On setting variables and using Fubini's theorem, (11) produce

$$\begin{aligned}
G^1_{\sigma,\kappa}(\varphi \curlyvee g)(\eta) &= \int_0^\infty g(y)\, y^{-1}\eta^\sigma \int_0^\infty G^{m,n}_{p,q}\left[\frac{yw}{\sigma}\,\middle|\, \begin{matrix}(a_i)_{1,p}\\ (b_j)_{1,q}\end{matrix}\right] (yw)^\kappa \varphi(w) \\
&\quad \times \frac{y\mathtt{d}w}{yw}\mathtt{d}y \\
&= \int_0^\infty g(y)\, y^{-1}\eta^\sigma \int_0^\infty G^{m,n}_{p,q}\left[\frac{w}{\eta y^{-1}}\,\middle|\, \begin{matrix}(a_i)_{1,p}\\ (b_j)_{1,q}\end{matrix}\right] y^\kappa w^\kappa \varphi(w) \\
&\quad \times \frac{\mathtt{d}w}{w}\mathtt{d}y \\
&= \int_0^\infty g(y)\, y^{\kappa-1-\sigma}\left(\eta y^{-1}\right)^\sigma \int_0^\infty G^{m,n}_{p,q}\left[\frac{w}{\eta y^{-1}}\,\middle|\, \begin{matrix}(a_i)_{1,p}\\ (b_j)_{1,q}\end{matrix}\right] \\
&\quad \times w^\kappa \varphi(w)\frac{\mathtt{d}w}{w}\mathtt{d}y \\
&= \int_0^\infty G^1_{\sigma,\kappa}\varphi\left(\eta y^{-1}\right) y^{\kappa-1-\sigma} g(y)\,\mathtt{d}y
\end{aligned}$$

The proof of the theorem is completely finished.

Lemma 4 Let φ, g and ψ be integrable functions on the open interval $(0,\infty)$. We get

$$\varphi \bullet (g \curlyvee \psi) = (\varphi \bullet g) \bullet \psi\ .$$

Proof On account of (10) we write

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$$\begin{aligned}\varphi \bullet (g \curlyvee \psi)(\xi) &= \int_0^\infty \varphi\left(\xi y^{-1}\right) y^{\kappa-1-\sigma}\left(\int_0^\infty \zeta^{-1} g\left(y\zeta^{-1}\right)\psi(\zeta)\,\mathtt{d}\zeta\right)\mathtt{d}y \\ &= \int_0^\infty \psi(\zeta)\zeta^{-1}\left(\int_0^\infty \varphi\left(\xi y^{-1}\right) y^{\kappa-1-\sigma} g\left(y\zeta^{-1}\right)\mathtt{d}y\right)\mathtt{d}\zeta. \quad (12)\end{aligned}$$

By change of variables, (12) yields

$$\begin{aligned}\varphi \bullet (g \curlyvee \psi)(\xi) &= \int_0^\infty \psi(\zeta)\int_0^\infty \varphi\left(\xi(\zeta w)^{-1}\right)(\zeta w)^{\kappa-1-\sigma} g(w)\,\zeta\,\mathtt{d}w\mathtt{d}\zeta \\ &= \int_0^\infty \psi(\zeta)\zeta^{\kappa-1-\sigma}\int_0^\infty \varphi\left(\xi\zeta^{-1}w^{-1}\right) \\ &\quad \times w^{\kappa-1-\sigma} g(w)\,\mathtt{d}w\mathtt{d}\zeta \\ &= \int_0^\infty \psi(\zeta)\zeta^{\kappa-1-\sigma}(\varphi \bullet g)\left(\xi\zeta^{-1}\right)\mathtt{d}\zeta.\end{aligned}$$

The proof is completely finished.

Let $\mathcal{D}$ denote the standard notation of the space of test functions of compact supports in $(0,\infty)$. Then we have the following results.

Theorem 5 Let $\varphi \in \boldsymbol{l}_{v-\kappa-\sigma,2}$ and $g \in \mathcal{D}$ be given. Then, we have

$$\varphi \bullet g \in \boldsymbol{l}_{v-\kappa-\sigma,2}.$$

Proof By appealing to (7) and the integral equation (10), we get

$$\|\varphi \bullet g\|^2_{v-\kappa-\sigma,2} = \int_0^\infty \left|\zeta^{v-\kappa-\sigma}\int_0^\infty \varphi\left(\zeta y^{-1}\right)^{\kappa-1-\sigma} g(y)\,\mathtt{d}y\right|^2 \frac{\mathtt{d}\zeta}{\zeta}.$$

Applying Jensen's inquality yields

$$\|\varphi \bullet g\|^2_{v-\kappa-\sigma,2} \le \int_0^\infty \left|\zeta^{v-\kappa-\sigma}\right|\int_0^\infty \left|\varphi\left(\zeta y^{-1}\right)\right|^2 \left|y^{\kappa-1-\sigma} g(y)\right|\mathtt{d}y\frac{\mathtt{d}\zeta}{\zeta}.$$

Using the Fubini's theorem implies

$$\|\varphi \bullet g\|^2_{v-\kappa-\sigma,2} \le \int_0^\infty \left|g(y)\,y^{\kappa-1-\sigma}\right|\left(\int_0^\infty \left|\zeta^{v-\kappa-\sigma}\varphi\left(\zeta y^{-1}\right)\right|^2\frac{\mathtt{d}\zeta}{\zeta}\right)\mathtt{d}y.$$

Now, let $[a,b]$, $0<a<b$, be an interval containing the support of g. Then, the hypothesis of the theorem $\varphi \in \boldsymbol{l}_{v-\kappa-\sigma,2}$, reveals

$$\|\varphi \bullet g\|^2_{v-\kappa-\sigma,2} \le M\|\varphi\|^2_{v-\kappa-\sigma,2}\int_a^b \left|y^{\kappa-1-\sigma} g(y)\right|\mathtt{d}y,$$

where $M = \int_a^b \left|y^{\kappa-1-\sigma} g(y)\right|\mathtt{d}y$.
Thus, the above equation further reveals

$$\|\varphi \bullet g\|_{v-\kappa-\sigma,2} < \infty.$$

The proof is completely finished.

Theorem 6 There hold the following identities.

(i) Let $\{\varphi_n\}$, $\varphi \in \boldsymbol{l}_{v-\kappa-\sigma,2}$ be such that $\varphi_n \to \varphi$ as $n\to\infty$. We have

$$\varphi_n \bullet g \to \varphi \bullet g \text{ as } n \to \infty$$

for every $g \in \mathcal{D}$.

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(ii) Let $\varphi_1, \varphi_2 \in \boldsymbol{l}_{v-\kappa-\sigma,2}$ and $g \in \mathcal{D}$. Then we have the following identities satisfy

$$(\varphi_1 + \varphi_2) \bullet g = \varphi_1 \bullet g + \varphi_2 \bullet g \text{ and } \rho (\varphi_1 \bullet g) = (\rho\varphi_1) \bullet g,$$

for arbitrary complex number ρ.

The proof of this theorem can be followed by using simple integral calculus. Hence, we avoid adding more details.

Definition 7 Let $\{\delta_n\} \in \mathcal{D}$ such that

(i) $\int_0^\infty \delta_n(\xi)\, \mathrm{d}\xi = 1 \ (n \in \mathbb{N})$.

(ii) $\int_0^\infty |\delta_n(\xi)|\, \mathrm{d}\xi < A \ (A \in \mathbb{R} \text{ being positive})$.

(iii) $\operatorname{supp} \delta_n = \{\xi : \delta_n(\xi) \neq 0\} \to 0$ as $n \to \infty$.

The set of all sequences $\{\delta_n\}$ are denoted by Δ. Every $\{\delta_n\}$ in Δ is said to be a delta sequence which corresponds to the delta distribution.

Theorem 8 Let $\{\delta_n\} \in \Delta$ and $\varphi \in \boldsymbol{l}_{v-\kappa-\sigma,2}$. Then, we have

$$\varphi \bullet \delta_n \to \varphi \text{ in } \boldsymbol{l}_{v-\kappa-\sigma,2} \text{ as } n \to \infty. \tag{14}$$

Proof By the first part of Definition 7 and Jensen's inequality we have

$$\|(\varphi \bullet \delta_n)(\zeta) - \varphi(\zeta)\|^2_{v-\kappa-\sigma,2} \le \int_0^\infty \left|\zeta^{v-\kappa-\sigma}\right|^2 \int_0^\infty \left|y^{\kappa-1-\sigma}\varphi\left(\zeta y^{-1}\right) - \varphi(\zeta)\right|^2 \times |\delta_n(y)|\, \mathrm{d}y \frac{\mathrm{d}\zeta}{\zeta}. \tag{15}$$

Therefore, by making use of Fubini's theorem, (15) gives

$$\|(\varphi \bullet \delta_n)(\zeta) - \varphi(\zeta)\|^2_{v-\kappa-\sigma,2} \le \int_{a_n}^{b_n} |\delta_n(y)| \times \int_0^\infty \left|\zeta^{v-\kappa-\sigma}\left(\psi_y(\zeta) - \varphi(\zeta)\right)\right|^2 \frac{\mathrm{d}\zeta}{\zeta} \mathrm{d}y, \tag{16}$$

where $\operatorname{supp} \delta_n(y) \subseteq [a_n, b_n], 0 < a_n < b_n, \forall n \in \mathbb{N}$.

Taking into account the fact that $\varphi(\zeta), \psi_y(\zeta) = \varphi\left(\zeta y^{-1}\right) y^{\kappa-1-\sigma} \in \boldsymbol{l}_{v-\kappa-\sigma,2}$, it follows from (16) that

$$\|(\varphi \bullet \delta_n)(\zeta) - \varphi(\zeta)\|^2_{v-\kappa-\sigma,2} \le M^* \int_{a_n}^{b_n} |\delta_n(y)|\, \mathrm{d}y.$$

for some positive constant M^*.

Therefore,

$$\|(\varphi \bullet \delta_n)(\zeta) - \varphi(\zeta)\|^2_{v-\kappa-\sigma,2} \le M^* M_1(a_n, b_n),$$

$M_1 > 0$.

The last inequality follows from the identity (iii) of Definition 7.

The proof of the theorem is completely finished.

The space $\mathcal{B}(\boldsymbol{l}_{v-\kappa-\sigma,2}, (\mathcal{D}, \curlyvee), \bullet, \Delta)$ is therefore generated and regarded as a space of Boehmians.

Construction of the space $\mathcal{B}(\boldsymbol{l}_{v,2}, (\mathcal{D}, \curlyvee), \curlyvee, \Delta)$ can be obtained by that technique similar to that of $\mathcal{B}(\boldsymbol{l}_{v-\kappa-\sigma,2}, (\mathcal{D}, \curlyvee), \bullet, \Delta)$ and the properties of $\curlyvee$ we have already cited above.

Sum of two Boehmians $\left[\frac{\varphi_n}{\delta_n}\right]$ and $\left[\frac{g_n}{\varepsilon_n}\right]$ in $\mathcal{B}\left(\boldsymbol{l}_{v,2},\left(\mathcal{D},\curlyvee\right),\curlyvee,\Delta\right)$ can be expressed as

$$\left[\frac{\varphi_n}{\delta_n}\right]+\left[\frac{g_n}{\varepsilon_n}\right]=\left[\frac{\varphi_n\curlyvee\delta_n+g_n\curlyvee\delta_n}{\delta_n\curlyvee\varepsilon_n}\right].$$

Multiplication in $\mathcal{B}\left(\boldsymbol{l}_{v,2},\left(\mathcal{D},\curlyvee\right),\curlyvee,\Delta\right)$ by $\alpha\in\mathbb{C}$ is defined as $\gamma\left[\frac{\varphi_n}{\delta_n}\right]=\left[\gamma\frac{\varphi_n}{\delta_n}\right]=\left[\frac{\gamma\varphi_n}{\delta_n}\right]$.

The extensions of $\curlyvee$ and $\mathcal{D}^\alpha$ to $\mathcal{B}\left(\boldsymbol{l}_{v,2},\left(\mathcal{D},\curlyvee\right),\curlyvee,\Delta\right)$ are introduced as

$$\left[\frac{\varphi_n}{\delta_n}\right]\curlyvee\left[\frac{g_n}{\varepsilon_n}\right]=\left[\frac{\varphi_n\curlyvee g_n}{\delta_n\curlyvee\varepsilon_n}\right]\text{ and }\mathcal{D}^\alpha\left[\frac{\varphi_n}{\delta_n}\right]=\left[\frac{\mathcal{D}^\alpha\varphi_n}{\delta_n}\right],\ \alpha\in\mathbb{R}.$$

Let $\left[\frac{\varphi_n}{\delta_n}\right]$ belong to $\mathcal{B}\left(\boldsymbol{l}_{v,2},\left(\mathcal{D},\curlyvee\right),\curlyvee,\Delta\right)$ and ω be in $\boldsymbol{l}_{v,2}$. The operation $\curlyvee$ can be extended to $\mathcal{B}\left(\boldsymbol{l}_{v,2},\left(\mathcal{D},\curlyvee\right),\curlyvee,\Delta\right)\times\boldsymbol{l}_{v,2}$ by

$$\left[\frac{\varphi_n}{\delta_n}\right]\curlyvee\omega=\left[\frac{\varphi_n\curlyvee\omega}{\delta_n}\right].$$

Let the sequence $\{\beta_n\}$ be in $\mathcal{B}\left(\boldsymbol{l}_{v,2},\left(\mathcal{D},\curlyvee\right),\curlyvee,\Delta\right)$. Then $\beta_n\xrightarrow{\delta}\beta$ in $\mathcal{B}\left(\boldsymbol{l}_{v,2},\left(\mathcal{D},\curlyvee\right),\curlyvee,\Delta\right)$, if there can be found a delta sequence $\{\delta_n\}$ such that for $(\beta_n\curlyvee\delta_k)$ and $(\beta\curlyvee\delta_k)\in\boldsymbol{l}_{v,2}$, $n,k\in\mathbb{N}$, we have

$$\lim_{n\to\infty}\beta_n\curlyvee\delta_k\to\beta\curlyvee\delta_k\text{ in }\boldsymbol{l}_{v,2}\text{ for every }k\in\mathbb{N}.$$

This can be expressed in $\mathcal{B}\left(\boldsymbol{l}_{v,2},\left(\mathcal{D},\curlyvee\right),\curlyvee,\Delta\right)$ as :

$\beta_n\xrightarrow{\delta}\beta$ (as $n\to\infty$) if and only if there are $\varphi_{n,k},\ \varphi_k\in\boldsymbol{l}_{v,2}$ and $\{\delta_k\}\in\Delta$, $\beta_n=\left[\frac{\varphi_{n,k}}{\delta_k}\right]$, $\beta=\left[\frac{\varphi_k}{\delta_k}\right]$ and to every $k\in\mathbb{N}$ we have $\lim_{n\to\infty}\varphi_{n,k}=\varphi_k$ in $\boldsymbol{l}_{v,2}$.

$\beta_n\xrightarrow{\Delta}\beta$ (as $n\to\infty$) if there can be found a $\{\delta_n\}\in\Delta$ such that $(\beta_n-\beta)\curlyvee\delta_n\in\boldsymbol{l}_{v,2}$ $(\forall n\in\mathbb{N})$ and $\lim_{n\to\infty}(\beta_n-\beta)\curlyvee\delta_n=0$ in $\boldsymbol{l}_{v,2}$.

On the other hand, addition of two Boehmians in $\mathcal{B}\left(\boldsymbol{l}_{v-\kappa-\sigma,2},\left(\mathcal{D},\curlyvee\right),\bullet,\Delta\right)$ is defined as

$$\left[\frac{\varphi_n}{\delta_n}\right]+\left[\frac{g_n}{\varepsilon_n}\right]=\left[\frac{\varphi_n\bullet\delta_n+g_n\bullet\delta_n}{\delta_n\curlyvee\varepsilon_n}\right].$$

Multiplication and convergence in $\mathcal{B}\left(\boldsymbol{l}_{v-\kappa-\sigma,2},\left(\mathcal{D},\curlyvee\right),\bullet,\Delta\right)$ can be defined similarly as in $\mathcal{B}\left(\boldsymbol{l}_{v-\kappa-\sigma,2},\left(\mathcal{D},\curlyvee\right),\bullet,\Delta\right)$.

3. $G^1_{\sigma,\kappa}$ of Boehmians

In view of Remark 2 and Theorem 3, we extend the transform $G^1_{\sigma,\kappa}$ to the space $\mathcal{B}\left(\boldsymbol{l}_{v,2},\left(\mathcal{D},\curlyvee\right),\curlyvee,\Delta\right)$ as

$$\overbrace{G^1_{\sigma,\kappa}}\left[\frac{\varphi_n}{\delta_n}\right]=\left[\frac{G^1_{\sigma,\kappa}\varphi_n}{\delta_n}\right]\tag{17}$$

in $\mathcal{B}\left(\boldsymbol{l}_{v-\kappa-\sigma,2},\left(\mathcal{D},\curlyvee\right),\bullet,\Delta\right)$.

We recite some properties of the transform $\overbrace{G^1_{\sigma,\kappa}}$ in the course of the following theorems.

Theorem 9 (i) The operator $\widehat{G^1_{\sigma,\kappa}}$ is well - defined and linear .

(ii) The operator $\widehat{G^1_{\sigma,\kappa}}$ is an isomorphism from $\mathcal{B}(\boldsymbol{l}_{v,2},(\mathcal{D},\curlyvee),\curlyvee,\Delta)$ onto the space $\mathcal{B}(\boldsymbol{l}_{v-\kappa-\sigma,2},(\mathcal{D},\curlyvee),\bullet,\Delta)$.

(iii) The operator $\widehat{G^1_{\sigma,\kappa}}$ is continuous with respect to δ and Δ - convergence.

(iv) The operator $\widehat{G^1_{\sigma,\kappa}} : \mathcal{B}(\boldsymbol{l}_{v,2},(\mathcal{D},\curlyvee),\curlyvee,\Delta) \rightarrow \mathcal{B}(\boldsymbol{l}_{v-\kappa-\sigma,2},(\mathcal{D},\curlyvee),\bullet,\Delta)$ transform is compatible with $G^1_{\sigma,\kappa} : \boldsymbol{l}_{v,2} \rightarrow \boldsymbol{l}_{v-\kappa-\sigma,2}$.

Proof We prove Part (iv) since similar proofs for Part(i) - Part(iii) are available in many cited papers of the same author and of Roopkumar in [20].

To prove the last part of the theorem, let $\sigma \in \boldsymbol{l}_{v,2}$ and $\left[\frac{\sigma \curlyvee \delta_n}{\delta_n}\right]$ be its representative in $\mathcal{B}(\boldsymbol{l}_{v,2},(\mathcal{D},\curlyvee),\curlyvee,\Delta)$ where $\{\delta_n\} \in \Delta$ $(\forall n \in \mathbb{N})$. Clearly, for all $n \in \mathbb{N}$, $\{\delta_n\}$ is independent from the representative. Hence, from (17) and Theorem 3, we get

$$\widehat{G^1_{\sigma,\kappa}}\left(\left[\frac{\sigma \curlyvee \delta_n}{\delta_n}\right]\right) = \widehat{G^1_{\sigma,\kappa}}\left(\left[\frac{\sigma \curlyvee \delta_n}{\delta_n}\right]\right) = \left[\frac{G^1_{\sigma,\kappa}(\sigma \curlyvee \delta_n)}{\delta_n}\right] = \left[\frac{G^1_{\sigma,\kappa}\sigma \bullet \delta_n}{\delta_n}\right].$$

Thus $\left[\frac{G^1_{\sigma,\kappa}\sigma \bullet \delta_n}{\delta_n}\right]$ is the representative of $G^1_{\sigma,\kappa}\sigma$ in the space $\boldsymbol{l}_{v-\kappa-\sigma,2}$.

The proof is therefore finished.

In view of Theorem 9, we introduce the inverse transform of $\widehat{G^1_{\sigma,\kappa}}$ as follows.

Definition 10 Let $\left[\frac{G^1_{\sigma,\kappa}\varphi_n}{\delta_n}\right] \in \mathcal{B}(\boldsymbol{l}_{v,2},(\mathcal{D},\curlyvee),\curlyvee,\Delta)$. We define the inverse $\widehat{G^1_{\sigma,\kappa}}$ integral operator of $\left[\frac{G^1_{\sigma,\kappa}\varphi_n}{\delta_n}\right]$ as

$$\left(\widehat{G^1_{\sigma,\kappa}}\right)^{-1}\left(\left[\frac{G^1_{\sigma,\kappa}\varphi_n}{\delta_n}\right]\right) = \left[\frac{\varphi_n}{\delta_n}\right],$$

for each $\{\delta_n\} \in \Delta$.

Theorem 11 Let $\left[\frac{G^1_{\sigma,\kappa}\varphi_n}{\delta_n}\right] \in \mathcal{B}(\boldsymbol{l}_{v-\kappa-\sigma,2},(\mathcal{D},\curlyvee),\bullet,\Delta)$ and $\varphi \in \mathcal{D}$. We have

$$\left(\widehat{G^1_{\sigma,\kappa}}\right)^{-1}\left(\left[\frac{G^1_{\sigma,\kappa}\varphi_n}{\delta_n}\right] \bullet \varphi\right) = \left[\frac{\varphi_n}{\delta_n}\right] \curlyvee \varphi$$

and

$$\widehat{G^1_{\sigma,\kappa}}\left(\left[\frac{\varphi_n}{\delta_n}\right] \curlyvee \varphi\right) = \left[\frac{G^1_{\sigma,\kappa}\varphi_n}{\delta_n}\right] \bullet \varphi.$$

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Proof Assume $\left[\frac{G^1_{\sigma,\kappa}\varphi_n}{\delta_n}\right] \in \mathcal{B}\left(\boldsymbol{l}_{v-\kappa-\sigma,2}, (\mathcal{D}, \curlyvee), \bullet, \Delta\right)$. For every $\phi \in \mathcal{D}$, we have

$$\begin{aligned} \left(\widetilde{G^1_{\sigma,\kappa}}\right)^{-1}\left(\left[\frac{G^1_{\sigma,\kappa}\varphi_n}{\delta_n}\right]\bullet\varphi\right) &= \left(\widetilde{G^1_{\sigma,\kappa}}\right)^{-1}\left(\left[\frac{G^1_{\sigma,\kappa}\varphi_n\bullet\varphi}{\delta_n}\right]\right) \\ \text{i.e.} \qquad &= \left[\frac{\left(G^1_{\sigma,\kappa}\right)^{-1}\left(G^1_{\sigma,\kappa}\varphi_n\bullet\varphi\right)}{\delta_n}\right]. \end{aligned}$$

Using Theorem 3 and Definition 10 we obtain

$$\begin{aligned} \left(\widetilde{G^1_{\sigma,\kappa}}\right)^{-1}\left(\left[\frac{G^1_{\sigma,\kappa}\varphi_n}{\delta_n}\right]\bullet\varphi\right) &= \left[\frac{\varphi_n\curlyvee\varphi}{\delta_n}\right] \\ &= \left[\frac{\varphi_n}{\delta_n}\right]\curlyvee\varphi. \end{aligned}$$

Proof of the part $\widetilde{G^1_{\sigma,\kappa}}\left(\left[\frac{\varphi_n}{\delta_n}\right]\curlyvee\varphi\right) = \left[\frac{G^1_{\sigma,\kappa}\varphi_n}{\delta_n}\right]\bullet\varphi$ is almost similar. We prefer to omit the details of the proof.
This completely finishes the proof of the theorem.

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HARMONIC QUASICONFORMAL MAPPINGS OF THE UNIT DISK ONTO THE HORIZONTAL STRIP AND HALF PLANE

JIAN-FENG ZHU

ABSTRACT. In this paper we consider two types of harmonic mappings $w(z) \in S_H(\mathbb{D}, \Omega_1)$ and $w(z) \in S_H(\mathbb{D}, R)$, where $\mathbb{D}$ is the unit disk and Ω_1, R are the domain defined by (2) and (3). Using the representation of harmonic mappings, we find the sufficient and necessary conditions to make $w(z)$ be a harmonic quasiconformal mapping. Furthermore, we obtain some estimates of $w(z)$.

1. INTRODUCTION

A real-valued function $u(x, y)$ on an open set $D \subseteq \mathbb{C}$ is harmonic if it is C^2 on D and satisfies Laplace's equation: $\triangle u = u_{xx} + u_{yy} = 0$. Assume that $z = x + iy$, $w(z) = u(x, y) + iv(x, y)$. Then a complex-valued function $w(z)$ is harmonic if and only if $u(x, y)$ and $v(x, y)$ are both harmonic. This has an equivalent form $w_{z\bar{z}} = 0$.

Let $w(z)$ be a harmonic mapping defined in the unit disk $\mathbb{D} = \{z : |z| < 1\}$. Then there exist two analytic functions $h(z)$ and $g(z)$ such that $w(z) = h(z) + \overline{g(z)}$.

For $z = re^{i\varphi} \in \mathbb{D}$, denote by

$$P(r, t - \varphi) = \frac{1}{2\pi} \frac{1 - r^2}{(1 - 2r\cos(t - \varphi) + r^2)}$$

the Poisson kernel. Then every bounded harmonic mapping $w(z)$ defined on $\mathbb{D}$ has the following representation

$$w(z) = P[f](z) = \int_0^{2\pi} P(r, t - \varphi) f(e^{it})\, dt, \tag{1}$$

where $z = re^{i\varphi} \in \mathbb{D}$ and f is a bounded integrable function defined on the unit circle $S^1 = \partial\mathbb{D}$.

For $z \in \mathbb{D}$, let

$$\Lambda_w(z) = \max_{0 \le \alpha \le 2\pi} |w_z(z) + e^{-2i\alpha} w_{\bar{z}}(z)| = |w_z(z)| + |w_{\bar{z}}(z)|$$

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and

$$\lambda_w(z) = \min_{0\leq\alpha\leq 2\pi} |w_z(z) + e^{-2i\alpha} w_{\bar{z}}(z)| = ||w_z(z)| - |w_{\bar{z}}(z)||.$$

According to Lewy's Theorem we know that $w(z)$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if its Jacobian satisfies

$$J_w(z) = |w_z(z)|^2 - |w_{\bar{z}}(z)|^2 > 0 \quad \text{for every} \quad z \in \mathbb{D}.$$

Suppose that $w(z)$ is a sense-preserving univalent harmonic mapping of $\mathbb{D}$ onto a domain $\Omega \subseteq \mathbb{C}$. Then $w(z)$ is a harmonic K-quasiconformal mapping if and only if

$$K(w) := \sup_{z\in\mathbf{U}} \frac{|w_z(z)| + |w_{\bar{z}}(z)|}{|w_z(z)| - |w_{\bar{z}}(z)|} \leq K.$$

It is interesting to consider such a question: under what conditions on f is $w = P[f](z)$ a harmonic quasiconformal mapping? Several authors have studied such a problem (see [5], [6], [8],[10], [11], [12], [13]). However, a univalent sense-preserving harmonic mapping defined on $\mathbb{D}$ doesn't determined by its image domain. In this paper we consider two types of harmonic mappings of $\mathbb{D}$ onto a unbounded convex domain. One maps the unit disk onto the horizontal strip and the other maps the unit disk onto the half plane.

Let S_H denote the class of all complex valued, sense-preserving univalent harmonic mappings $w(z)$ in $\mathbb{D}$, with the normalization $w(0) = w_z(0) - 1 = 0$. Let S_H^0 be the subclass of S_H with $w_{\bar{z}}(0) = 0$ and $w_z(0) > 0$. For a domain $\Omega \subseteq \mathbb{C}$ containing the origin, $S_H(\mathbb{D}, \Omega)$ will denote the class of all sense-preserving univalent harmonic mappings of $\mathbb{D}$ onto Ω normalized by $w(0) = w_{\bar{z}}(0) = 0$ and $w_z(0) > 0$.

Considering the following domains: the horizontal strip

$$\Omega_1 = \{w : -1 < \mathbf{Im} w < 1\}, \tag{2}$$

and the right half plane

$$R = \{w : \mathbf{Re} w > \frac{-1}{2}\}. \tag{3}$$

A conformal mapping φ from $\mathbb{D}$ onto Ω_1 normalized by $\varphi(0) = 0 < \varphi'(0)$ has the form

$$\varphi(z) = \frac{2}{\pi} \ln \frac{1+z}{1-z}. \tag{4}$$

For $z \in \mathbb{D}$ and $|\eta| = 1$, define the kernel

$$k(z,\eta) = \int_0^z \varphi'(\zeta) \frac{1+\bar{\eta}\zeta}{1-\bar{\eta}\zeta} \, d\zeta = \frac{4}{\pi} \int_0^z \frac{1+\bar{\eta}\zeta}{(1-\bar{\eta}\zeta)(1-\zeta^2)} \, d\zeta, \tag{5}$$

and the family

$$\mathcal{F} = \{w : w(z) = \mathbf{Re} \int_{|\eta|=1} k(z,\eta) d\mu(\eta) + i\mathbf{Im}\varphi(z), \mu \in \mathbb{P}\}, \tag{6}$$

where $\mathbb{P}$ is the set of probability measures on the Borel sets of the unit circle S^1.

According to [4] we have the following theorem.

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Theorem A. $\overline{S_H(\mathbb{D},\Omega_1)} = F$. *Here* $\overline{S_H(\mathbb{D},\Omega_1)}$ *is the closure of* $S_H(\mathbb{D},\Omega_1)$.

Similarly, define the kernel

$$F(z,\eta)=\begin{cases}\mathbf{Re}\frac{z}{1-z}+i\mathbf{Im}\frac{z}{(1-z)^2} & if\ \eta=1\\ \mathbf{Re}\frac{z}{1-z}+i\mathbf{Im}\left(\frac{2\eta}{(1-\eta)^2}\ln\frac{1-z}{1-\eta z}+\frac{1+\eta}{1-\eta}\frac{z}{1-z}\right) & if\ \eta\neq 1.\end{cases}$$

According to [1], we have the following theorem.

Theorem B. *Each harmonic mapping* $w(z)\in\overline{S_H(\mathbb{D},R)}$ *if and only if there is a probability measure* μ *on the unit circle such that*

$$w(z)=\int_{|\eta|=1}F(z,\eta)d\mu(\eta). \tag{7}$$

In this paper, we find the sufficient and necessary conditions on the kernel $k(z,\eta)$ and $F(z,\eta)$ which make harmonic mappings $w(z)$ of $S_H(\mathbb{D},\Omega_1)$ and $S_H(\mathbb{D},R)$ to be quasiconformal.

2. Necessary and sufficient conditions

Theorem 1. *Suppose that* $w\in S_H(\mathbb{D},\Omega_1)$, *which has the representation* (6). *Then* w *is a harmonic quasiconformal mapping if and only if it's kernel satisfies the following conditions:*

(i) $c:=\operatorname*{ess\,inf}_{z\in\mathbb{D}}\mathbf{Re}\int_{|\eta|=1}\frac{\eta+z}{\eta-z}d\mu(\eta)>0$,

(ii) $M_1:=\operatorname*{ess\,sup}_{z\in\mathbb{D}}\left|\int_{|\eta|=1}\frac{\eta+z}{\eta-z}d\mu(\eta)\right|<\infty$,

where c *and* M_1 *are positive constant.*

Proof. Let $w(z)=h(z)+\overline{g(z)}$ be a sense-preserving univalent harmonic mapping of $\mathbb{D}$ onto Ω_1. According to Theorem A we have

$$w(z)=\mathbf{Re}\int_{|\eta|=1}k(z,\eta)d\mu(\eta)+i\mathbf{Im}\varphi(z),$$

where $k(z,\eta)$ and μ are defined by (5) and (6). This implies that

$$w(z)=h(z)+\overline{g(z)}=\frac{1}{2}\left(\int_{|\eta|=1}k(z,\eta)d\mu(\eta)+\varphi(z)\right)+\frac{1}{2}\overline{\left(\int_{|\eta|=1}k(z,\eta)d\mu(\eta)-\varphi(z)\right)}.$$

Then

$$h'(z)=\frac{\varphi'(z)}{2}\left(\int_{|\eta|=1}\frac{\eta+z}{\eta-z}d\mu(\eta)+1\right) \tag{8}$$

and

$$g'(z)=\frac{\varphi'(z)}{2}\left(\int_{|\eta|=1}\frac{\eta+z}{\eta-z}d\mu(\eta)-1\right). \tag{9}$$

It follows from (8) and (9) that

$$\left|\frac{g'(z)}{h'(z)}\right| = \left|\frac{\int_{|\eta|=1} \frac{\eta+z}{\eta-z} d\mu(\eta) - 1}{\int_{|\eta|=1} \frac{\eta+z}{\eta-z} d\mu(\eta) + 1}\right|.$$

Let $A_1 = \mathbf{Re} \int_{|\eta|=1} \frac{\eta+z}{\eta-z} d\mu(\eta)$ and $A_2 = \mathbf{Im} \int_{|\eta|=1} \frac{\eta+z}{\eta-z} d\mu(\eta)$.

The proof of 'if' part: Since $A_1 \geq c > 0$ applying condition (ii) we see that

$$\operatorname*{ess\,sup}_{z\in\mathbb{D}} \left|\frac{g'(z)}{h'(z)}\right|^2 = \operatorname*{ess\,sup}_{z\in\mathbb{D}} \frac{(A_1-1)^2 + A_2^2}{(A_1+1)^2 + A_2^2} = \operatorname*{ess\,sup}_{z\in\mathbb{D}} \left(1 - \frac{4A_1}{(A_1+1)^2 + A_2^2}\right) < 1.$$

This shows that $w(z)$ is a harmonic quasiconformal mapping.

The proof of 'only if' part: Assume that $w(z) \in S_H(\mathbb{D}, \Omega_1)$ is a harmonic quasiconformal mapping. Then the following inequality

$$\operatorname*{ess\,sup}_{z\in\mathbb{D}} \left|\frac{g'(z)}{h'(z)}\right| = \operatorname*{ess\,sup}_{z\in\mathbb{D}} \left|\frac{\int_{|\eta|=1} \frac{\eta+z}{\eta-z} d\mu(\eta) - 1}{\int_{|\eta|=1} \frac{\eta+z}{\eta-z} d\mu(\eta) + 1}\right| \leq k$$

holds for some constant $k < 1$. Hence $\left|\int\limits_{|\eta|=1} \frac{\eta+z}{\eta-z} d\mu(\eta)\right| < \infty$ and $(A_1-1)^2 + A_2^2 \leq k^2(A_1+1)^2 + k^2 A_2^2$. This implies that $2A_1(1+k^2) \geq 1-k^2$. Thus $A_1 \geq \frac{1-k^2}{2(1+k^2)} > 0$.

The proof is completed. $\square$

Remark: For any $z = re^{i\theta} \in \mathbb{D}$ and $\eta = e^{it} \in \partial\mathbb{D}$, we have $\mathbf{Re}\frac{\eta+z}{\eta-z} = \frac{1-r^2}{1-2r\cos(\theta-t)+r^2} > 0$. If we additional assume that $\mu'(e^{it}) > 0$, then $\mathbf{Re} \int_{|\eta|=1} \frac{\eta+z}{\eta-z} d\mu(\eta) > 0$. According to (8) and (9) we see that

$$\operatorname*{ess\,sup}_{z\in\mathbb{D}} \left|\frac{g'(z)}{h'(z)}\right| < 1,$$

which implies that $w(z)$ is a harmonic quasiconformal mapping.

Theorem 2. *Let $w \in S_H(\mathbb{D}, R)$ be a sense-preserving harmonic mapping which has the representation (7). Then w is a quasiconformal mapping if and only if its kernel satisfies the following conditions:*

(i) $d := \operatorname*{ess\,inf}_{z\in\mathbb{D}} \left(1 + 2\mathbf{Re} \int_{|\eta|=1} \frac{\eta z}{1-\eta z} d\mu(\eta)\right) > 0,$

(ii) $M_2 := \operatorname*{ess\,sup}_{z\in\mathbb{D}} \left|\int_{|\eta|=1} \frac{\eta z}{1-\eta z} d\mu(\eta)\right| < \infty,$

where d and M_2 are positive constant.

Proof. According to [4] we know that $w(z)$ has the following representation:

$$\begin{aligned} w(z) &= \int_{|\eta|=1} F(z,\eta)d\mu(\eta) \\ &= \frac{1}{2}\int_{|\eta|=1}\left(\frac{z}{1-z}+\frac{2\eta}{(1-\eta)^2}\ln\frac{1-z}{1-\eta z}+\frac{1+\eta}{1-\eta}\frac{z}{1-z}\right)d\mu(\eta) \\ &+ \overline{\frac{1}{2}\int_{|\eta|=1}\left(\frac{z}{1-z}-\frac{2\eta}{(1-\eta)^2}\ln\frac{1-z}{1-\eta z}-\frac{1+\eta}{1-\eta}\frac{z}{1-z}\right)d\mu(\eta)} \\ &= h(z)+\overline{g(z)}. \end{aligned}$$

Then

$$h'(z)=\frac{1}{(1-z)^2}\int_{|\eta|=1}\frac{1}{1-\eta z}d\mu(\eta) \tag{10}$$

and

$$g'(z)=\frac{1}{(1-z)^2}\int_{|\eta|=1}\frac{-\eta z}{1-\eta z}d\mu(\eta). \tag{11}$$

This implies that

$$\left|\frac{g'(z)}{h'(z)}\right| = \frac{\left|\int_{|\eta|=1}\frac{\eta z}{1-\eta z}d\mu(\eta)\right|}{\left|\int_{|\eta|=1}\left(1+\frac{\eta z}{1-\eta z}\right)d\mu(\eta)\right|} = \frac{\left|\int_{|\eta|=1}\frac{\eta z}{1-\eta z}d\mu(\eta)\right|}{\left|1+\int_{|\eta|=1}\frac{\eta z}{1-\eta z}d\mu(\eta)\right|}.$$

The proof of 'if' part: Let

$$B=\int_{|\eta|=1}\frac{\eta z}{1-\eta z}d\mu(\eta). \tag{12}$$

Then

$$\left|\frac{g'(z)}{h'(z)}\right|^2=\frac{|B|^2}{|1+B|^2}=\frac{|B|^2}{1+|B|^2+2ReB}.$$

Applying condition(i) and (ii) we have

$$\operatorname*{ess\,sup}_{z\in\mathbb{D}}\left|\frac{g'(z)}{h'(z)}\right|^2\le\frac{|B|^2}{|B|^2+d}<1.$$

The proof of 'only if' part: Assume that $w(z)$ is a harmonic quasiconformal mapping of $\mathbb{D}$ onto R. Then the following inequality

$$\left|\frac{g'(z)}{h'(z)}\right|^2\le k^2$$

holds for some constant $k<1$. This is equivalent to $|B|<\infty$ and $|B|^2\le k^2(|B|^2+1+2\mathbf{Re}B)$. Hence

$$\left|\int_{|\eta|=1}\frac{\eta z}{1-\eta z}d\mu(\eta)\right|<\infty,$$

and

$$1+2\mathbf{Re}B\ge\frac{(1-k^2)|B|^2}{k^2},$$

where B is defined by (12).

This completes the proof. □

Remark: According to (6) we know that $\mu \in \mathbb{P}$ is a probability measure on the Borel set of $\mathbb{S}^1$. This implies that $\int_{|\eta|=1} d\mu(\eta) = 1$.

Theorem 3. *Suppose that $w(z) = h(z) + \overline{g(z)} \in S_H(\mathbb{D}, \Omega_1)$ is a harmonic K-quasiconformal mapping. Then its Jacobian satisfies*

$$J_w \geq \frac{c}{\pi^2},$$

where c is a positive constant depends on K.

Proof. According to (8) and (9) we have $h'(z) = \frac{\varphi'(z)}{2}\left(\int_{|\eta|=1} \frac{\eta+z}{\eta-z} d\mu(\eta) + 1\right)$ and $g'(z) = \frac{\varphi'(z)}{2}\left(\int_{|\eta|=1} \frac{\eta+z}{\eta-z} d\mu(\eta) - 1\right)$, where $\varphi(z) = \frac{2}{\pi} \ln \frac{1+z}{1-z}$. Let $A_1 = \mathbf{Re} \int_{|\eta|=1} \frac{\eta+z}{\eta-z} d\mu(\eta)$ and $A_2 = \mathbf{Im} \int_{|\eta|=1} \frac{\eta+z}{\eta-z} d\mu(\eta)$. Since $w(z)$ is a harmonic quasiconformal mapping, by using condition (i) in Theorem 1 we see that

$$J_w(z) = |h'(z)|^2 - |g'(z)|^2 = A_1 |\varphi'(z)|^2 \geq \frac{16c}{\pi^2(1+|z|)^4} \geq \frac{c}{\pi^2},$$

where c is a positive constant depends on K.

This completes the proof. □

Theorem 4. *Suppose that $w(z) = h(z) + \overline{g(z)} \in S_H(\mathbb{D}, R)$ is a harmonic K-quasiconformal mapping. Then its Jacobian satisfies*

$$J_w(z) \geq \frac{d}{16},$$

where d is is a positive constant depends on K.

Proof. According to (10) and (11) we have

$$h'(z) = \frac{1}{(1-z)^2} \int_{|\eta|=1} \frac{1}{1-\eta z} d\mu(\eta) = \frac{1}{(1-z)^2}\left(1 + \int_{|\eta|=1} \frac{\eta z}{1-\eta z} d\mu(\eta)\right)$$

and $g'(z) = \frac{1}{(1-z)^2} \int_{|\eta|=1} \frac{-\eta z}{1-\eta z} d\mu(\eta)$. Using (12) and condition (i) of Theorem2 we obtain that

$$J_w = |h'|^2 - |g'|^2 = \frac{1}{|1-z|^4}(1 + 2\mathbf{Re}B) \geq \frac{d}{(1+|z|)^4} \geq \frac{d}{16},$$

where d is a positive constant depends on K.

This completes the proof. □

3. co-Lipschitz condition of $w(z)$

A complex-valued mapping $w(z)$ in $\mathbb{D}$ is said to be a co-Lipschitz (Lipschitz) mapping if there exists a constant $L > 0$ such that the following inequality

$$|w(z_1) - w(z_2)| \geq \frac{|z_1 - z_2|}{L}$$

$$(|w(z_1) - w(z_2)| \leq L|z_1 - z_2|)$$

holds for any $z_1, z_2 \in \mathbb{D}$. Suppose that $w(z)$ is a harmonic quasiconformal mapping of $\mathbb{D}$ onto a bounded convex domain. Many mathematicians have discussed about the bi-Lipschitz property of $w(z)$ (cf.[6],[9] and [11]). We point out that if $w(z)$ is a harmonic quasiconformal mapping defined by (6) and (7), then it would be a co-Lipschitz mapping.

Theorem 5. *Given $K \geq 1$. Suppose that $w(z) = h(z) + \overline{g(z)} \in S_H(\mathbb{D}, \Omega_1)$ is a harmonic K-quasiconformal mapping. Then the following inequalities*

$$\frac{1+c}{2\pi} \leq |h'(z)| \leq \frac{2(M_1+1)}{\pi(1-|z|)^2}$$

hold for every $z \in \mathbb{D}$, where c and M_1 are positive constant depend on K. Furthermore, the inequality

$$|w(z_1) - w(z_2)| \geq \frac{(1-k)(1+c)}{2\pi}|z_1 - z_2|,$$

holds for any $z_1, z_2 \in \mathbb{D}$, where $k = \frac{K-1}{K+1}$.

Proof. According to (8), we have

$$|h'(z)| = \left|\frac{\varphi'(z)}{2}\right| \left|\int_{|\eta|=1} \frac{\eta+z}{\eta-z} d\mu(\eta) + 1\right| \leq \frac{2(1+M_1)}{\pi(1-|z|^2)}$$

and

$$|h'(z)| = \left|\frac{\varphi'(z)}{2}\right| \left|\int_{|\eta|=1} \frac{\eta+z}{\eta-z} d\mu(\eta) + 1\right| \geq \frac{2(1+A_1)}{\pi(1+|z|)^2} \geq \frac{(1+c)}{2\pi},$$

where c and M_1 are positive constant depend on K. Since $w(z) = h(z) + \overline{g(z)}$ is a harmonic K-quasiconformal mapping, we see that $\operatorname{ess\,sup}_{z\in\mathbb{D}} \left|\frac{g'(z)}{h'(z)}\right| \leq k < 1$, where $k = \frac{K-1}{K+1}$. Then

$$\Lambda_w(z) = |h'(z)| + |g'(z)| \leq |h'(z)|(1+k) \leq \frac{2(1+k)(1+M_1)}{\pi(1-|z|^2)}$$

and

$$\lambda_w(z) = ||h'(z)| - |g'(z)|| \geq \frac{2(1-k)(1+c)}{\pi(1+|z|)^2} \geq \frac{(1-k)(1+c)}{2\pi}.$$

Take $\zeta_1, \zeta_2 \in \Omega_1$ satisfying $z_1 = w^{-1}(\zeta_1)$, $z_2 = w^{-1}(\zeta_2)$. Let $\varphi(t) = w^{-1}(\zeta_1 + t(\zeta_2 - \zeta_1))$, $t \in [0,1]$. Then $\frac{d}{dt}w(\varphi(t)) = \zeta_2 - \zeta_1$. Since Ω_1 is a convex domain, we see that

$$\begin{aligned} |\zeta_1 - \zeta_2| &= \int_0^1 \left|\frac{d}{dt}w(\varphi(t))\right| dt = \int_0^1 \left|w_z(\varphi(t))\varphi'(t) + w_{\bar{z}}(\varphi(t))\overline{\varphi'(t)}\right| dt \\ &\geq \int_0^1 \left(|w_z(\varphi(t))\varphi'(t)| - |w_{\bar{z}}(\varphi(t))\overline{\varphi'(t)}|\right) dt \\ &\geq \inf_{u\in\mathbb{D}}(|w_z(u)| - |w_{\bar{z}}(u)|)\int_0^1 |\varphi'(t)|\, dt \qquad (13) \\ &\geq \frac{(1-k)(1+c)}{2\pi}|z_1 - z_2|. \end{aligned}$$

This completes the proof. □

Theorem 6. *Given $K \geq 1$. Suppose that $w(z) = h(z) + \overline{g(z)} \in S_H(\mathbb{D}, R)$ is a harmonic K-quasiconformal mapping. Then the following inequalities*

$$\frac{d}{4} \leq |h'(z)| \leq \frac{(M_2+1)}{(1-|z|)^2},$$

hold for any $z \in \mathbb{D}$, where d and M_2 are positive constant depend on K. Furthermore, the following inequality

$$|w(z_1) - w(z_2)| \geq \frac{(1-k)d}{4}|z_1 - z_2|,$$

holds for any $z_1, z_2 \in \mathbb{D}$, where $k = \frac{K-1}{K+1}$.

Proof. According to (10) and (12), we have

$$|h'(z)| = \left|\frac{1}{(1-z)^2}\right|\left|1 + \int_{|\eta|=1}\frac{\eta z}{1-\eta z}d\mu(\eta)\right| = \left|\frac{1}{(1-z)^2}\right|\sqrt{1 + 2ReB + B^2} \geq \frac{d}{4},$$

and

$$|h'(z)| = \left|\frac{1}{(1-z)^2}\right|\left|1 + \int_{|\eta|=1}\frac{\eta z}{1-\eta z}d\mu(\eta)\right| \leq \frac{1}{(1-|z|)^2}(M_2+1),$$

where d and M_2 are positive constant depend on K. Since $w(z) = h(z) + \overline{g(z)}$ is a harmonic K-quasiconformal mapping, we see that $k = \operatorname*{ess\,sup}_{z\in\mathbb{D}}\left|\frac{g'(z)}{h'(z)}\right| < 1$, where $k = \frac{K-1}{K+1}$. Hence

$$\Lambda_w(z) = |h'(z)| + |g'(z)| \leq |h'(z)|(1+k) \leq \frac{(1+k)(1+M_2)}{(1-|z|)^2},$$

and

$$\lambda_w(z) = ||h'(z)| - |g'(z)|| \geq |h'(z)|(1-k) \geq \frac{(1-k)d}{4}.$$

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For each $z_1, z_2 \in \mathbb{D}$, using (13) we have

$$|w(z_1) - w(z_2)| = \left| \int\limits_{[z_1,z_2]} w_z dz + w_{\bar{z}} d\bar{z} \right| \geq \inf_{z \in \mathbb{D}} \lambda_w(z)|z_1 - z_2| \geq \frac{(1-k)d}{4}|z_1 - z_2|.$$

This completes the proof. □

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Langevin fractional differential inclusions with nonlocal Katugampola fractional integral boundary conditions

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Abstract

In this paper, we study a boundary value problem consisting from a fractional differential inclusion of Riemann-Liouville Langevin type subject to Katugampola fractional integral conditions. Some new existence results for convex as well as non-convex multivalued maps are obtained by using standard fixed point theorems. Enlighten examples illustrating the obtained results are also presented.

Key words and phrases: Fractional differential inclusions; generalized fractional integral; Katugampola fractional integral; nonlocal boundary conditions; fixed point theorems
AMS (MOS) Subject Classifications: 34A60; 26A33; 34A08

1 Introduction

In this manuscript, we investigate the sufficient conditions of existence of solutions for the following fractional Langevin inclusion subject to the generalized nonlocal fractional integral conditions of the form

$$\begin{cases} D^{p_1}(D^{p_2}+\lambda)x(t) \in F(t,x(t)), \qquad 0<t<T, \\ x(0)=0, \\ x(\eta) = \displaystyle\sum_{i=1}^{n} \alpha_i \frac{\rho_i^{1-q_i}}{\Gamma(q_i)} \int_0^{\xi_i} \frac{s^{\rho_i-1}x(s)}{(t^{\rho_i}-s^{\rho_i})^{1-q_i}}ds := \sum_{i=1}^{n} \alpha_i \, {}^{\rho_i}I^{q_i} x(\xi_i), \end{cases} \tag{1}$$

where D^{p_i} denote the Riemann-Liouville fractional derivative of order p_i, $i=1,2$, $0<p_1,p_2\le 1$, $1<p_1+p_2\le 2$, λ is a given constant, ${}^{\rho_i}I^{q_i}$ are the generalized fractional integral of orders $q_i>0$, $\rho_i>0$, η,ξ_i arbitrary, with $\eta,\xi_i\in(0,T)$, $\alpha_i\in\mathbb{R}$ for all $i=1,2,\ldots,n$ and $F:[0,T]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

The search for the existence of solutions to nonlinear fractional boundary value problems has expanded greatly over the past years. For examples and recent development of the topic, see [1]-[11] and the references cited therein. In fractional calculus, the fractional derivatives are defined via fractional integrals. There are several known forms of the fractional integrals which have been studied extensively for their applications. The most known fractional integrals are the Caputo, Riemann-Liouville and the Hadamard fractional integral.

A new fractional integral, called *generalized Riemann-Liouville fractional integral*, which generalizes the Riemann-Liouville and the Hadamard integrals into a single form, was introduced in [12], [13]. See Definition 2.5 below. This integral is now known as *"Katugampola fractional integral"* see for example

[14, pp 15, 123]. The existence and uniqueness results for the Caputo-Katugampola derivative is given in [15]. For some recent work with this new operator and similar operators, for example, see [16]-[18] and the references cited therein.

The Langevin equation (first formulated by Langevin in 1908 to give an elaborate description of Brownian motion) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [19]. For some new developments on the fractional Langevin equation, see, for example, [20]-[24].

The present paper is motivated by a recent paper [25], where it is considered problem (1) with F single valued. Existence and uniqueness results were proved in [25] by using a variety of fixed point theorems, such as Banach contraction principle, Krasnoselskii fixed point theorem, Leray-Schauder nonlinear alternative and Leray-Schauder degree theory. Here, we cover the multi-valued case. We establish some existence results for the problem (1), when the right hand side is convex as well as non-convex valued. In the first result, we use the nonlinear alternative of Leray-Schauder type while in the second result, we shall combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values. The third result relies on the fixed point theorem for contraction multivalued maps due to Covitz and Nadler. Examples illustrating the obtained results are also presented. The methods used are well known, however their exposition in the framework of problem (1) is new.

2 Preliminaries

2.1 Basic material of fractional calculus

In this section, we introduce some notations and definitions of fractional calculus [1, 2] and present preliminary results needed in our proofs later.

Definition 2.1 *[2] The Riemann-Liouville fractional integral of order $p > 0$ of a continuous function $f : (0,\infty) \to \mathbb{R}$ is defined by*

$$\mathcal{I}^p f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} v(s) ds,$$

provided the right-hand side is point-wise defined on $(0,\infty)$, where Γ is the gamma function defined by $\Gamma(p) = \int_0^\infty e^{-s} s^{p-1} ds$.

Definition 2.2 *[2] The Riemann-Liouville fractional derivative of order $p > 0$ of a continuous function $f : (0,\infty) \to \mathbb{R}$ is defined by*

$$D^p f(t) = \frac{1}{\Gamma(n-p)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-p-1} v(s) ds, \quad n-1 \le p < n,$$

where $n = [p]+1$, $[p]$ denotes the integer part of a real number p, provided the right-hand side is point-wise defined on $(0,\infty)$.

Lemma 2.3 *[2] Let $p > 0$ and $x \in C(0,T) \cap L(0,T)$. Then the fractional differential equation $D^p x(t) = 0$ has a unique solution $x(t) = \sum_{i=1}^n c_i t^{p-i}$, and the following formula holds: $\mathcal{I}^p D^p x(t) = x(t) + \sum_{i=1}^n c_i t^{p-i}$, where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, and $n-1 \le p < n$.*

Lemma 2.4 *([2], page 71) Let $\alpha > 0$ and $\beta > 0$. Then the following properties hold:*

$$\mathcal{I}^\alpha (x-a)^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (t-a)^{\beta+\alpha-1}.$$

Definition 2.5 *([13]) The Katugampola fractional integral of order $q > 0$ and $\rho > 0$, of a function $f(t)$, for all $0 < t < \infty$, is defined as*

$$ {}^{\rho}I^{q}f(t) = \frac{\rho^{1-q}}{\Gamma(q)} \int_0^t \frac{s^{\rho-1}v(s)}{(t^{\rho}-s^{\rho})^{1-q}}\,ds, $$

provided the right-hand side is point-wise defined on $(0,\infty)$.

Lemma 2.6 *([25]) Let constants $q > 0$ and $p > 0$. Then the following formula holds*

$$ {}^{\rho}I^{q}t^{p} = \frac{\Gamma\left(\frac{p+\rho}{\rho}\right)}{\Gamma\left(\frac{p+\rho q+\rho}{\rho}\right)} \frac{t^{p+\rho q}}{\rho^{q}}. \tag{2} $$

Lemma 2.7 *([25]) Let $0 < p_1, p_2 \le 1$, $1 < p_1 + p_2 \le 2$, $q_i, \rho_i > 0$, η, $\xi_i \in (0,T)$, $\alpha_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, n$ and $h \in C([0,T],\mathbb{R})$. Then x is a solution of the problem*

$$ D^{p_1}(D^{p_2}+\lambda)x(t) = h(t), \qquad 0 < t < T, \tag{3} $$

$$ x(0) = 0, \qquad x(\eta) = \sum_{i=1}^{n} \alpha_i\, {}^{\rho_i}I^{q_i}x(\xi_i), \tag{4} $$

if and only if

$$ \begin{aligned} x(t) &= \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \Bigg[\mathcal{I}^{p_1+p_2}h(\eta) - \lambda\, \mathcal{I}^{p_2}x(\eta) \\ &\quad - \sum_{i=1}^{n} \alpha_i\, {}^{\rho_i}I^{q_i}\left(\mathcal{I}^{p_1+p_2}h(s) - \lambda\, \mathcal{I}^{p_2}x(s)\right)(\xi_i) \Bigg] + \mathcal{I}^{p_1+p_2}h(t) - \lambda\, \mathcal{I}^{p_2}x(t), \end{aligned} \tag{5} $$

where

$$ \Omega = \sum_{i=1}^{n} \frac{\alpha_i \Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{\Gamma\left(\dfrac{p_1+p_2+\rho_i-1}{\rho_i}\right)}{\Gamma\left(\dfrac{p_1+p_2+\rho_i q_i+\rho_i-1}{\rho_i}\right)} \frac{\xi_i^{p_1+p_2+\rho_i q_i+\rho_i-1}}{\rho_i^{q_i}} - \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \eta^{p_1+p_2-1} \neq 0. \tag{6} $$

2.2 Basic material for multivalued maps

Here, we outline some basic concepts of multivalued analysis [26, 27].

Let $C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ with the norm $\|x\| = \sup\{|x(t)|,\ t \in [0,T]\}$. Also by $L^1([0,T],\mathbb{R})$, we denote the space of functions $x : [0,T] \to \mathbb{R}$ such that $\|x\|_{L^1} = \int_0^T |x(t)|dt$.

For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cl,b}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed and bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

A multi-valued map $G : X \to \mathcal{P}(X)$:

(i) is *convex (closed) valued* if $G(x)$ is convex (closed) for all $x \in X$.

(ii) is *bounded* on bounded sets if $G(Y) = \cup_{x\in Y} G(x)$ is bounded in X for all $Y \in \mathcal{P}_b(X)$ (i.e. $\sup_{x\in Y}\{\sup\{|y| : y \in G(x)\}\} < \infty$).

(iii) is called *upper semi-continuous (u.s.c.)* on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$.

(iv) G is *lower semi-continuous (l.s.c.)* if the set $\{y \in X : G(y) \cap Y \neq \emptyset\}$ is open for any open set Y in X.

(v) is said to be *completely continuous* if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$; If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*,\ y_n \to y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$.

(vi) is said to be *measurable* if for every $y \in X$, the function $t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$ is measurable.

(vii) *has a fixed point* if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$.

3 Existence results

Let $\mathcal{C} = C([0,T], \mathbb{R})$ denotes the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup_{t\in[0,T]} |x(t)|$. Throughout of this paper, for convenience of proving, we let the notations $\mathcal{I}^z v(s)(y)$ and ${}^\rho I^z v(s)(y)$ defined by

$$\mathcal{I}^z v(s)(y) = \frac{1}{\Gamma(z)} \int_0^y (y-s)^{z-1} v(s) ds \quad \text{and} \quad {}^\rho I^z v(s)(y) = \frac{\rho^{1-z}}{\Gamma(z)} \int_0^y \frac{s^{\rho-1} v(s)}{(y^\rho - s^\rho)^{1-z}} ds,$$

where $z > 0$ and $y \in [0,T]$.

To simplify the notations, we use the following constants:

$$\Lambda_1 = \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{T^{p_1+p_2-1}}{|\Omega|}, \tag{7}$$

$$\Lambda_2 = \frac{T^{p_2}}{\Gamma(1+p_2)} + \Lambda_1 \left(\frac{\eta^{p_2}}{\Gamma(1+p_2)} + \sum_{i=1}^n |\alpha_i| \left[\frac{1}{\Gamma(1+p_2)} \frac{\xi_i^{p_2+\rho_i q_i}}{\rho_i^{q_i}} \frac{\Gamma\left(\frac{p_2+\rho_i}{\rho_i}\right)}{\Gamma\left(\frac{p_2+\rho_i q_i+\rho_i}{\rho_i}\right)} \right] \right). \tag{8}$$

Definition 3.1 *A function $x \in AC^2([0,T], \mathbb{R})$ is a solution of the problem (1) if $x(0) = 0,\ x(\eta) = \sum_{i=1}^n \alpha_i\, {}^{\rho_i} I^{q_i} x(\xi_i)$, and there exists a function $v \in L^1([0,T], \mathbb{R})$ such that $f(t) \in F(t, x(t))$ a.e. on $[0,T]$ and*

$$\begin{aligned} x(t) &= \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \left[\mathcal{I}^{p_1+p_2} v(\eta) - \lambda\, \mathcal{I}^{p_2} x(\eta) - \sum_{i=1}^n \alpha_i\, {}^{\rho_i} I^{q_i} \left(\mathcal{I}^{p_1+p_2} v(s) - \lambda\, \mathcal{I}^{p_2} x(s) \right)(\xi_i) \right] \\ &\quad + \mathcal{I}^{p_1+p_2} v(t) - \lambda\, \mathcal{I}^{p_2} x(t). \end{aligned}$$

3.1 The Carathéodory case

In this subsection, we consider the case when F has convex values and prove an existence result based on nonlinear alternative of Leray-Schauder type, assuming that F is Carathéodory.

Definition 3.2 *A multivalued map $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if*

(i) $t \longmapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;

(ii) $x \longmapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,T]$;

Further a Carathéodory function F is called L^1-Carathéodory if

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(iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1([0,T],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_\rho(t)$$

for all $\|x\| \leq \rho$ and for a.e. $t \in [0,T]$.

For each $y \in \mathcal{C}$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,T],\mathbb{R}) : v(t) \in F(t,y(t)) \text{ on } [0,T]\}.$$

We define the graph of G to be the set $Gr(G) = \{(x,y) \in X \times Y, y \in G(x)\}$ and recall a result for closed graphs and upper-semicontinuity.

Lemma 3.3 *([26, Proposition 1.2]) If $G : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

The following lemma will be used in the sequel.

Lemma 3.4 *([28]) Let X be a Banach space. Let $F : J \times \mathbb{R} \to \mathcal{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$. Then the operator*

$$\Theta \circ S_F : C(J,X) \to \mathcal{P}_{cp,c}(C(J,X)), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

We recall the well-known nonlinear alternative of Leray-Schauder for multivalued maps.

Lemma 3.5 *(Nonlinear alternative for Kakutani maps)[29]. Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{cp,c}(C)$ is a upper semicontinuous compact map. Then either*

(i) F has a fixed point in $\overline{U}$, or

(ii) there is a $u \in \partial U$ and $\nu \in (0,1)$ with $u \in \nu F(u)$.

Theorem 3.6 *Assume that:*

(H_1) $F : [0,T] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory;

(H_2) there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in L^1([0,T],\mathbb{R}^+)$ such that

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leq p(t)\psi(\|x\|) \ \ \text{for each} \ \ (t,x) \in [0,T] \times \mathbb{R};$$

(H_3) there exists a constant $M > 0$ such that

$$\begin{aligned} \frac{M}{\psi(M)} \quad &> \quad \frac{\Lambda_1}{(1-|\lambda|\Lambda_2)}\Bigg\{\Bigg[\mathcal{I}^{p_1+p_2}p(s)(\eta) + \sum_{i=1}^{n}|\alpha_i| \ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}p(s)(\tau)\Big)(\xi_i)\Bigg] \\ &\qquad + \mathcal{I}^{p_1+p_2}p(s)(T)\Bigg\} := \Omega_1, \quad |\lambda|\Lambda_2 < 1, \end{aligned}$$

where Λ_1 and Λ_2 are defined by (7) and (8) respectively.

Then the boundary value problem (1) has at least one solution on $[0,T]$.

Proof. Define the operator $\mathcal{F}:\mathcal{C}\to\mathcal{P}(\mathcal{C})$ by

$$\mathcal{F}(x) = \left\{\begin{array}{l} h\in\mathcal{C}: \\ h(t)=\left\{\begin{array}{l} \dfrac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\dfrac{t^{p_1+p_2-1}}{\Omega}\Big[\mathcal{I}^{p_1+p_2}v(s)(\eta)-\lambda\,\mathcal{I}^{p_2}x(s)(\eta) \\ -\displaystyle\sum_{i=1}^{n}\alpha_i\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}v(s)(\tau)-\lambda\,\mathcal{I}^{p_2}x(s)(\tau)\Big)(\xi_i)\Big] \\ +\mathcal{I}^{p_1+p_2}v(s)(t)-\lambda\,\mathcal{I}^{p_2}x(s)(t) \end{array}\right. \end{array}\right\} \tag{9}$$

for $v\in S_{F,x}$. It is obvious that the fixed points of $\mathcal{F}$ are solutions of the boundary value problem (1).

We will show that $\mathcal{F}$ satisfies the assumptions of Leray-Schauder Nonlinear alternative (Lemma 3.5). The proof consists of several steps.

Step 1. $\mathcal{F}(x)$ is convex for each $x\in\mathcal{C}$.

This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore, we omit the proof.

Step 2. $\mathcal{F}$ maps bounded sets (balls) into bounded sets in $\mathcal{C}$.

For a positive number ρ, let $B_\rho=\{x\in\mathcal{C}:\|x\|\le\rho\}$ be a bounded ball in $\mathcal{C}$. Then, for each $h\in\mathcal{F}(x), x\in B_\rho$, there exists $v\in S_{F,x}$ such that

$$\begin{aligned} h(t) &= \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{\Omega}\Bigg[\mathcal{I}^{p_1+p_2}v(s)(\eta)-\lambda\,\mathcal{I}^{p_2}x(s)(\eta) \\ &\quad -\sum_{i=1}^{n}\alpha_i\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}v(s)(\tau)-\lambda\,\mathcal{I}^{p_2}x(s)(\tau)\Big)(\xi_i)\Bigg]+\mathcal{I}^{p_1+p_2}v(s)(t)-\lambda\,\mathcal{I}^{p_2}x(s)(t). \end{aligned}$$

Then, we have

$$\begin{aligned} |h(x)| &\le \Bigg|\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{\Omega}\Bigg[\mathcal{I}^{p_1+p_2}v(s)(\eta)-\lambda\,\mathcal{I}^{p_2}x(s)(\eta) \\ &\quad -\sum_{i=1}^{n}\alpha_i\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}v(s)(\tau)-\lambda\,\mathcal{I}^{p_2}x(s)(\tau)\Big)(\xi_i)\Bigg]+\mathcal{I}^{p_1+p_2}v(s)(t)-\lambda\,\mathcal{I}^{p_2}x(s)(t)\Bigg| \\ &\le \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{|\Omega|}\Bigg[\mathcal{I}^{p_1+p_2}p(s)\psi(\|x\|)(\eta)+|\lambda|\ \mathcal{I}^{p_2}\|x\|(\eta) \\ &\quad +\sum_{i=1}^{n}|\alpha_i|\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}p(s)\psi(\|x\|)(\tau)+|\lambda|\ \mathcal{I}^{p_2}\|x\|(\tau)\Big)(\xi_i)\Bigg] \\ &\quad +\mathcal{I}^{p_1+p_2}p(s)\psi(\|x\|)(t)+|\lambda|\ \mathcal{I}^{p_2}\|x\|(t) \\ &\le \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{T^{p_1+p_2-1}}{|\Omega|}\psi(\|x\|)\Bigg[\mathcal{I}^{p_1+p_2}p(s)(\eta) \\ &\quad +\sum_{i=1}^{n}|\alpha_i|\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}p(s)(\tau)\Big)(\xi_i)\Bigg]+\psi(\|x\|)\mathcal{I}^{p_1+p_2}p(s)(T) \\ &\quad +\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{T^{p_1+p_2-1}}{|\Omega|}|\lambda|\|x\|\Bigg[\mathcal{I}^{p_2}(\eta)+\sum_{i=1}^{n}|\alpha_i|\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_2}\|x\|(\tau)\Big)(\xi_i)\Bigg]+|\lambda|\|x\|\mathcal{I}^{p_2}(T) \\ &\le \psi(\|x\|)\Bigg\{\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{T^{p_1+p_2-1}}{|\Omega|}\Bigg[\mathcal{I}^{p_1+p_2}p(s)(\eta) \\ &\quad +\sum_{i=1}^{n}|\alpha_i|\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}p(s)(\tau)\Big)(\xi_i)\Bigg]+\mathcal{I}^{p_1+p_2}p(s)(T)\Bigg\} \end{aligned}$$

$$+|\lambda|\|x\|\Bigg\{\frac{T^{p_2}}{\Gamma(1+p_2)}+\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{T^{p_1+p_2-1}}{|\Omega|}\Bigg(\frac{\eta^{p_2}}{\Gamma(1+p_2)}$$
$$+\sum_{i=1}^{n}|\alpha_i|\Bigg[\frac{1}{\Gamma(1+p_2)}\frac{\xi_i^{p_2+\rho_i q_i}}{\rho_i^{q_i}}\frac{\Gamma\left(\frac{p_2+\rho_i}{\rho_i}\right)}{\Gamma\left(\frac{p_2+\rho_i q_i+\rho_i}{\rho_i}\right)}\Bigg]\Bigg)\Bigg\}$$
$$= \psi(\|x\|)\Lambda_1\Bigg[\mathcal{I}^{p_1+p_2}p(s)(\eta)+\sum_{i=1}^{n}|\alpha_i|\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}p(s)(\tau)\Big)(\xi_i)\Bigg]+|\lambda|\|x\|\Lambda_2,$$

and consequently,

$$\|h\| \le \psi(\rho)\Lambda_1\Bigg[\mathcal{I}^{p_1+p_2}p(s)(\eta)+\sum_{i=1}^{n}|\alpha_i|\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}p(s)(\tau)\Big)(\xi_i)\Bigg]+|\lambda|\rho\Lambda_2.$$

Step 3. $\mathcal{F}$ maps bounded sets into equicontinuous sets of $\mathcal{C}$.

Let $\tau_1,\tau_2\in[0,T]$ with $\tau_1<\tau_2$ and $x\in B_\rho$. For each $h\in\mathcal{F}(x)$, we obtain

$$\begin{aligned}
|h(\tau_2)-h(\tau_1)| \le\ & \Bigg|\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t_2^{p_1+p_2-1}-t_1^{p_1+p_2-1}}{\Omega}\Bigg[\mathcal{I}^{p_1+p_2}v(s)(\eta)-\lambda\mathcal{I}^{p_2}x(s)(\eta)\\
&-\sum_{i=1}^{n}\alpha_i\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}v(s)(\tau)-\lambda\mathcal{I}^{p_2}x(s)(\tau)\Big)(\xi_i)\Bigg]\Bigg|\\
&+\big|\mathcal{I}^{p_1+p_2}v(s)(t_2)-\mathcal{I}^{p_1+p_2}v(s)(t_1)\big|+\big|\lambda\mathcal{I}^{p_2}x(s)(t_2)-\lambda\mathcal{I}^{p_2}x(s)(t_1)\big|\\
\le\ & \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{|t_2^{p_1+p_2-1}-t_1^{p_1+p_2-1}|}{|\Omega|}\Bigg[\mathcal{I}^{p_1+p_2}p(s)\psi(\|x\|)(\eta)+|\lambda|\,\mathcal{I}^{p_2}\|x\|(\eta)\\
&+\sum_{i=1}^{n}\alpha_i\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}p(s)\psi(\|x\|)(\tau)+|\lambda|\,\mathcal{I}^{p_2}\|x\|(\tau)\Big)(\xi_i)\Bigg]\\
&+\big|\mathcal{I}^{p_1+p_2}p(s)\psi(\|x\|)(t_2)-\mathcal{I}^{p_1+p_2}p(s)\psi(\|x\|)(t_1)\big|\\
&+\big|\lambda\,\mathcal{I}^{p_2}x(s)(t_2)-\lambda\,\mathcal{I}^{p_2}x(s)(t_1)\big|\\
\le\ & \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{|t_2^{p_1+p_2-1}-t_1^{p_1+p_2-1}|}{|\Omega|}\Bigg[\mathcal{I}^{p_1+p_2}p(s)\psi(\rho)(\eta)+|\lambda|\,\rho\mathcal{I}^{p_2}(\eta)\\
&+\sum_{i=1}^{n}\alpha_i\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}p(s)\psi(\rho)(\tau)+|\lambda|\,\rho\mathcal{I}^{p_2}(\tau)\Big)(\xi_i)\Bigg]\\
&+\big|\left(\mathcal{I}^{p_1+p_2}p(s)\psi(\rho)\right)(t_2)-\left(\mathcal{I}^{p_1+p_2}p(s)\psi(\rho)\right)(t_1)\big|\\
&+|\lambda|\rho\big|\left(\mathcal{I}^{p_2}\right)(t_2)-\left(\mathcal{I}^{p_2}\right)(t_1)\big|.
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x\in B_\rho$ as $\tau_2-\tau_1\to 0$. As $\mathcal{F}$ satisfies the above three assumptions, therefore it follows by the Ascoli-Arzelá theorem that $\mathcal{F}:\mathcal{C}\to\mathcal{P}(\mathcal{C})$ is completely continuous.

Since $\mathcal{F}$ is completely continuous, in order to prove that it is u.s.c. it is enough to prove that it has a closed graph (Lemma 3.3). Thus, in our next step, we show that

Step 4. $\mathcal{F}$ has a closed graph.

Let $x_n\to x_*, h_n\in\mathcal{F}(x_n)$ and $h_n\to h_*$. Then, we need to show that $h_*\in\mathcal{F}(x_*)$. Associated with $h_n\in\mathcal{F}(x_n)$, there exists $v_n\in S_{F,x_n}$ such that for each $t\in[0,T]$,

$$h_n(t) = \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{\Omega}\Bigg[\mathcal{I}^{p_1+p_2}v_n(s)(\eta)-\lambda\,\mathcal{I}^{p_2}x(s)(\eta)$$

$$-\sum_{i=1}^{n}\alpha_i\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}v_n(s)(\tau)-\lambda\,\mathcal{I}^{p_2}x(s)(\tau)\Big)(\xi_i)\Bigg]+\mathcal{I}^{p_1+p_2}v_n(s)(t)-\lambda\,\mathcal{I}^{p_2}x(s)(t).$$

Thus it suffices to show that there exists $v_*\in S_{F,x_*}$ such that for each $t\in[0,T]$,

$$\begin{aligned} h_*(t) &= \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{\Omega}\Bigg[\mathcal{I}^{p_1+p_2}v_*(s)(\eta)-\lambda\,\mathcal{I}^{p_2}x(s)(\eta) \\ &\quad -\sum_{i=1}^{n}\alpha_i\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}v_*(s)(\tau)-\lambda\,\mathcal{I}^{p_2}x(s)(\tau)\Big)(\xi_i)\Bigg]+\mathcal{I}^{p_1+p_2}v_*(s)(t)-\lambda\,\mathcal{I}^{p_2}x(s)(t). \end{aligned}$$

Let us consider the linear operator $\Theta: L^1([0,T],\mathbb{R})\to\mathcal{C}$ given by

$$\begin{aligned} v\mapsto\Theta(v)(t) &= \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{\Omega}\Bigg[\mathcal{I}^{p_1+p_2}v(s)(\eta)-\lambda\,\mathcal{I}^{p_2}x(s)(\eta) \\ &\quad -\sum_{i=1}^{n}\alpha_i\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}v(s)(\tau)-\lambda\,\mathcal{I}^{p_2}x(s)(\tau)\Big)(\xi_i)\Bigg]+\mathcal{I}^{p_1+p_2}v(s)(t)-\lambda\,\mathcal{I}^{p_2}x(s)(t). \end{aligned}$$

Observe that

$$\begin{aligned} \|h_n(t)-h_*(t)\| &= \Bigg\|\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{\Omega}\Bigg[\mathcal{I}^{p_1+p_2}(v_n(s)-v_*(s))(\eta) \\ &\quad -\sum_{i=1}^{n}\alpha_i\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}(v_n(s)-v_*(s))(\tau)\Big)(\xi_i)\Bigg]+\mathcal{I}^{p_1+p_2}(v_n(s)-v_*(s))(t)\Bigg\|\to 0, \end{aligned}$$

as $n\to\infty$. Thus, it follows by Lemma 3.4 that $\Theta\circ S_F$ is a closed graph operator. Further, we have $h_n(t)\in\Theta(S_{F,x_n})$. Since $x_n\to x_*$, therefore, we have

$$\begin{aligned} h_*(t) &= \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{\Omega}\Bigg[\mathcal{I}^{p_1+p_2}v_*(s)(\eta)-\lambda\,\mathcal{I}^{p_2}x(s)(\eta) \\ &\quad -\sum_{i=1}^{n}\alpha_i\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}v_*(s)(\tau)-\lambda\,\mathcal{I}^{p_2}x(s)(\tau)\Big)(\xi_i)\Bigg]+\mathcal{I}^{p_1+p_2}v_*(s)(t)-\lambda\,\mathcal{I}^{p_2}x(s)(t) \end{aligned}$$

for some $v_*\in S_{F,x_*}$.

Step 5. We show there exists an open set $U\subseteq\mathcal{C}$ with $x\notin\theta\mathcal{F}(x)$ for any $\theta\in(0,1)$ and all $x\in\partial U$.

Let $\theta\in(0,1)$ and $x\in\theta\mathcal{F}(x)$. Then there exists $v\in L^1([0,T],\mathbb{R})$ with $v\in S_{F,x}$ such that, for $t\in[0,T]$, we have

$$\begin{aligned} x(t) &= \theta\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{\Omega}\Bigg[\mathcal{I}^{p_1+p_2}v(s)(\eta)-\lambda\,\mathcal{I}^{p_2}x(s)(\eta) \\ &\quad -\sum_{i=1}^{n}\alpha_i\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}v(s)(\tau)-\lambda\,\mathcal{I}^{p_2}x(s)(\tau)\Big)(\xi_i)\Bigg]+\theta\mathcal{I}^{p_1+p_2}v(s)(t)-\theta\lambda\,\mathcal{I}^{p_2}x(s)(t). \end{aligned}$$

Using the computations of the second step above, we have

$$\|x\| \le \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{t^{p_1+p_2-1}}{|\Omega|}\Bigg[\mathcal{I}^{p_1+p_2}p(s)\psi(\|x\|)(\eta)+|\lambda|\,\mathcal{I}^{p_2}\|x\|(\eta)$$

$$
\begin{aligned}
& +\sum_{i=1}^{n}|\alpha_i|\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}p(s)\psi(\|x\|)(\tau)+|\lambda|\ \mathcal{I}^{p_2}\|x\|(\tau)\Big)(\xi_i)\Bigg] \\
& +\mathcal{I}^{p_1+p_2}p(s)\psi(\|x\|)(T)+|\lambda|\ \mathcal{I}^{p_2}\|x\|(T) \\
\leq\ & \psi(\|x\|)\Bigg\{\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{T^{p_1+p_2-1}}{|\Omega|}\Bigg[\mathcal{I}^{p_1+p_2}p(s)(\eta) \\
& +\sum_{i=1}^{n}|\alpha_i|\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}p(s)(\tau)\Big)(\xi_i)\Bigg]+\mathcal{I}^{p_1+p_2}p(s)(T)\Bigg\} \\
& +|\lambda|\|x\|\Bigg\{\frac{T^{p_2}}{\Gamma(1+p_2)}+\frac{\Gamma(p_1)}{\Gamma(p_1+p_2)}\frac{T^{p_1+p_2-1}}{|\Omega|}\Bigg(\frac{\eta^{p_2}}{\Gamma(1+p_2)} \\
& +\sum_{i=1}^{n}|\alpha_i|\Bigg[\frac{1}{\Gamma(1+p_2)}\frac{\xi_i^{p_2+\rho_i q_i}}{\rho_i^{q_i}}\frac{\Gamma\left(\frac{p_2+\rho_i}{\rho_i}\right)}{\Gamma\left(\frac{p_2+\rho_i q_i+\rho_i}{\rho_i}\right)}\Bigg]\Bigg)\Bigg\} \\
=\ & \psi(\|x\|)\Bigg\{\Lambda_1\Bigg[\mathcal{I}^{p_1+p_2}p(s)(\eta)+\sum_{i=1}^{n}|\alpha_i|\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}p(s)(\tau)\Big)(\xi_i)\Bigg] \\
& +\mathcal{I}^{p_1+p_2}p(s)(T)\Bigg\}+|\lambda|\|x\|\Lambda_2,
\end{aligned}
$$

which implies that

$$
\frac{\|x\|}{\psi(\|x\|)}\ \leq\ \frac{\Lambda_1}{(1-|\lambda|\Lambda_2)}\Bigg\{\Bigg[\mathcal{I}^{p_1+p_2}p(s)(\eta)+\sum_{i=1}^{n}|\alpha_i|\ {}^{\rho_i}I^{q_i}\Big(\mathcal{I}^{p_1+p_2}p(s)(\tau)\Big)(\xi_i)\Bigg]+\mathcal{I}^{p_1+p_2}p(s)(T)\Bigg\}.
$$

In view of (H_3), there exists M such that $\|x\| \neq M$. Let us set

$$U=\{x\in\mathcal{C}:\|x\|<M\}.$$

Note that the operator $\mathcal{F}:\overline{U}\to\mathcal{P}(\mathcal{C})$ is a compact multi-valued map, u.s.c. with convex closed values. From the choice of U, there is no $x\in\partial U$ such that $x\in\theta\mathcal{F}(x)$ for some $\theta\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.5), we deduce that $\mathcal{F}$ has a fixed point $x\in\overline{U}$ which is a solution of the problem (1). This completes the proof. □

3.2 The lower semicontinuous case

In the next result, F is not necessarily convex valued. Our strategy to deal with this problem is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo [30] for lower semi-continuous maps with decomposable values.

Let X be a nonempty closed subset of a Banach space E and $G:X\to\mathcal{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y\in X: G(y)\cap B\neq\emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,T]\times\mathbb{R}$. A is $\mathcal{L}\otimes\mathcal{B}$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{J}\times\mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,T]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([0,T],\mathbb{R})$ is decomposable if for all $u,v\in\mathcal{A}$ and measurable $\mathcal{J}\subset[0,T]=J$, the function $u\chi_{\mathcal{J}}+v\chi_{J-\mathcal{J}}\in\mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 3.7 *Let Y be a separable metric space and let $N:Y\to\mathcal{P}(L^1([0,T],\mathbb{R}))$ be a multivalued operator. We say N has a property (BC) if N is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.*

Let $F:[0,T]\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator $\mathcal{F}:C([0,T]\times\mathbb{R})\to\mathcal{P}(L^1([0,T],\mathbb{R}))$ associated with F as

$$\mathcal{F}(x)=\{w\in L^1([0,T],\mathbb{R}):w(t)\in F(t,x(t))\text{ for a.e. }t\in[0,T]\},$$

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which is called the Nemytskii operator associated with F.

Definition 3.8 *Let $F : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.*

Lemma 3.9 *([31]) Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0, T], \mathbb{R}))$ be a multivalued operator satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0, T], \mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Theorem 3.10 *Assume that (H_2), (H_3) and the following condition holds:*

(H_4) *$F : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a nonempty compact-valued multivalued map such that*

- *(a) $(t, x) \longmapsto F(t, x)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,*
- *(b) $x \longmapsto F(t, x)$ is lower semicontinuous for each $t \in [0, T]$;*

Then the boundary value problem (1) has at least one solution on $[0, T]$.

Proof. It follows from (H_2) and (H_4) that F is of l.s.c. type. Then from Lemma 3.9, there exists a continuous function $f : AC^2([0, T], \mathbb{R}) \to L^1([0, T], \mathbb{R})$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in C([0, T], \mathbb{R})$.

Consider the problem

$$\begin{cases} {}_{RL}D^q x(t) = f(x(t)), \quad 0 < t < T, \quad 1 < \alpha \leq 2, \\ x(0) = 0, \quad x(\eta) = \displaystyle\sum_{i=1}^{n} \alpha_i \, {}^{\rho_i}I^{q_i} x(\xi_i). \end{cases} \tag{10}$$

Observe that if $x \in AC^2([0, T], \mathbb{R})$ is a solution of (10), then x is a solution to the problem (1). In order to transform the problem (10) into a fixed point problem, we define the operator $\overline{\mathcal{F}}$ as

$$\begin{aligned} \overline{\mathcal{F}}x(t) \;&=\; \frac{\Gamma(p_1)}{\Gamma(p_1 + p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \Bigg[\mathcal{I}^{p_1+p_2} f(x(s))(\eta) - \lambda \, \mathcal{I}^{p_2} x(s)(\eta) \\ &\quad - \sum_{i=1}^{n} \alpha_i \, {}^{\rho_i}I^{q_i} \Big(\mathcal{I}^{p_1+p_2} f(x(s))(\tau) - \lambda \, \mathcal{I}^{p_2} x(s)(\tau) \Big)(\xi_i) \Bigg] + \mathcal{I}^{p_1+p_2} f(x(s))(t) - \lambda \, \mathcal{I}^{p_2} x(s)(t). \end{aligned}$$

It can easily be shown that $\overline{\mathcal{F}}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.6. So, we omit it. This completes the proof. □

3.3 The Lipschitz case

In this subsection, we prove the existence of solutions for the problem (1) with a not necessary nonconvex valued right hand side, by applying a fixed point theorem for multivalued maps due to Covitz and Nadler [32].

Let (X, d) be a metric space induced from the normed space $(X; \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\},$$

where $d(A, b) = \inf_{a \in A} d(a; b)$ and $d(a, B) = \inf_{b \in B} d(a; b)$. Then $(\mathcal{P}_{cl,b}(X), H_d)$ is a metric space (see [33]).

Definition 3.11 *A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called*

(a) *$\gamma-$Lipschitz if and only if there exists $\gamma > 0$ such that*

$$H_d(N(x), N(y)) \leq \gamma d(x, y) \text{ for each } x, y \in X;$$

(**b**) *a contraction if and only if it is $\gamma-$Lipschitz with $\gamma < 1$.*

Lemma 3.12 *([32]) Let (X, d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Theorem 3.13 *Assume that:*

(A_1) *$F : [0, T] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot, x) : [0, T] \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$.*

(A_2) *$H_d(F(t, x), F(t, \bar{x})) \leq m(t)|x - \bar{x}|$ for almost all $t \in [0, T]$ and $x, \bar{x} \in \mathbb{R}$ with $m \in L^1([0, T], \mathbb{R}^+)$ and $d(0, F(t, 0)) \leq m(t)$ for almost all $t \in [0, T]$.*

Then the boundary value problem (1) has at least one solution on $[0, T]$ if

$$\Omega_2 := \Lambda_1 \left[\mathcal{I}^{p_1+p_2} m(s)(\eta) + \sum_{i=1}^{n} |\alpha_i| \ {}^{\rho_i}I^{q_i} \Big(\mathcal{I}^{p_1+p_2} m(s)(\tau) \Big) \right] + \mathcal{I}^{p_1+p_2} m(s)(T) < 1.$$

Proof. Consider the operator $\mathcal{F}$ defined by (9). Observe that the set $S_{F,x}$ is nonempty for each $x \in \mathcal{C}$ by the assumption (A_1), so F has a measurable selection (see Theorem III.6 [34]). Now, we show that the operator $\mathcal{F}$ satisfies the assumptions of Lemma 3.12. We show that $\mathcal{F}(x) \in \mathcal{P}_{cl}(\mathcal{C})$ for each $x \in \mathcal{C}$. Let $\{u_n\}_{n\geq 0} \in \mathcal{F}(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $\mathcal{C}$. Then $u \in \mathcal{C}$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [0, T]$,

$$\begin{aligned} u_n(t) &= \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \Bigg[\mathcal{I}^{p_1+p_2} v_n(s)(\eta) - \lambda \, \mathcal{I}^{p_2} x(s)(\eta) \\ &\quad - \sum_{i=1}^{n} \alpha_i \ {}^{\rho_i}I^{q_i} \Big(\mathcal{I}^{p_1+p_2} v_n(s)(\tau) - \lambda \, \mathcal{I}^{p_2} x(s)(\tau) \Big)(\xi_i) \Bigg] + \mathcal{I}^{p_1+p_2} v_n(s)(t) - \lambda \, \mathcal{I}^{p_2} x(s)(t). \end{aligned}$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([0, T], \mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in [0, T]$, we have

$$\begin{aligned} u_n(t) \to v(t) &= \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \Bigg[\mathcal{I}^{p_1+p_2} v(s)(\eta) - \lambda \, \mathcal{I}^{p_2} x(s)(\eta) \\ &\quad - \sum_{i=1}^{n} \alpha_i \ {}^{\rho_i}I^{q_i} \Big(\mathcal{I}^{p_1+p_2} v(s)(\tau) - \lambda \, \mathcal{I}^{p_2} x(s)(\tau) \Big)(\xi_i) \Bigg] + \mathcal{I}^{p_1+p_2} v(s)(t) - \lambda \, \mathcal{I}^{p_2} x(s)(t). \end{aligned}$$

Hence, $u \in \mathcal{F}(x)$.

Next, we show that there exists $\delta < 1$ such that

$$H_d(\mathcal{F}(x), \mathcal{F}(\bar{x})) \leq \delta \|x - \bar{x}\| \quad \text{for each} \quad x, \bar{x} \in AC^2([0, T], \mathbb{R}).$$

Let $x, \bar{x} \in AC^2([0, T], \mathbb{R})$ and $h_1 \in \mathcal{F}(x)$. Then there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in [0, T]$,

$$\begin{aligned} h_1(t) &= \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \Bigg[\mathcal{I}^{p_1+p_2} v_1(s)(\eta) - \lambda \, \mathcal{I}^{p_2} x(s)(\eta) \\ &\quad - \sum_{i=1}^{n} \alpha_i \ {}^{\rho_i}I^{q_i} \Big(\mathcal{I}^{p_1+p_2} v_1(s)(\tau) - \lambda \, \mathcal{I}^{p_2} x(s)(\tau) \Big)(\xi_i) \Bigg] + \mathcal{I}^{p_1+p_2} v_1(s)(t) - \lambda \, \mathcal{I}^{p_2} x(s)(t). \end{aligned}$$

By (A_2), we have

$$H_d(F(t, x), F(t, \bar{x})) \leq m(t)|x(t) - \bar{x}(t)|.$$

So, there exists $w \in F(t, \bar{x}(t))$ such that

$$|v_1(t) - w| \leq m(t)|x(t) - \bar{x}(t)|, \quad t \in [0, T].$$

Define $U : [0,T] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \le m(t)|x(t) - \bar{x}(t)|\}.$$

Since the multivalued operator $U(t) \cap F(t, \bar{x}(t))$ is measurable (Proposition III.4 [34]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in [0,T]$, we have $|v_1(t) - v_2(t)| \le m(t)|x(t) - \bar{x}(t)|$.

For each $t \in [0,T]$, let us define

$$\begin{aligned} h_2(t) &= \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{t^{p_1+p_2-1}}{\Omega} \Bigg[\mathcal{I}^{p_1+p_2} v_2(s)(\eta) - \lambda\, \mathcal{I}^{p_2} x(s)(\eta) \\ &\quad - \sum_{i=1}^{n} \alpha_i \, {}^{\rho_i} I^{q_i} \Big(\mathcal{I}^{p_1+p_2} v_2(s)(\tau) - \lambda\, \mathcal{I}^{p_2} x(s)(\tau) \Big)(\xi_i) \Bigg] + \mathcal{I}^{p_1+p_2} v_2(s)(t) - \lambda\, \mathcal{I}^{p_2} x(s)(t). \end{aligned}$$

Thus,

$$\begin{aligned} |h_1(t) - h_2(t)| &\le \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{T^{p_1+p_2-1}}{|\Omega|} \Bigg[\mathcal{I}^{p_1+p_2} |v_1(s) - v_2(s)|(\eta) \\ &\quad + \sum_{i=1}^{n} |\alpha_i| \, {}^{\rho_i} I^{q_i} \Big(\mathcal{I}^{p_1+p_2} |v_1(s) - v_2(s)|(\tau) \Bigg] + \mathcal{I}^{p_1+p_2} |v_1(s) - v_2(s)|(t) \\ &\le \frac{\Gamma(p_1)}{\Gamma(p_1+p_2)} \frac{T^{p_1+p_2-1}}{|\Omega|} \Bigg[\mathcal{I}^{p_1+p_2} m(s) \|x - \bar{x}\|(\eta) \\ &\quad + \sum_{i=1}^{n} |\alpha_i| \, {}^{\rho_i} I^{q_i} \Big(\mathcal{I}^{p_1+p_2} m(s) \|x - \bar{x}\|(\tau) \Big) \Bigg] + \mathcal{I}^{p_1+p_2} m(s) \|x - \bar{x}\|(T) \\ &= \Bigg\{ \Lambda_1 \Bigg[\mathcal{I}^{p_1+p_2} m(s)(\eta) + \sum_{i=1}^{n} |\alpha_i| \, {}^{\rho_i} I^{q_i} \Big(\mathcal{I}^{p_1+p_2} m(s)(\tau) \Big) \Bigg] \\ &\quad + \mathcal{I}^{p_1+p_2} m(s)(T) \Bigg\} \|x - \bar{x}\|. \end{aligned}$$

Hence,

$$\|h_1 - h_2\| \le \Bigg\{ \Lambda_1 \Bigg[\mathcal{I}^{p_1+p_2} m(s)(\eta) + \sum_{i=1}^{n} |\alpha_i| \, {}^{\rho_i} I^{q_i} \Big(\mathcal{I}^{p_1+p_2} m(s)(\tau) \Big) \Bigg] + \mathcal{I}^{p_1+p_2} m(s)(T) \Bigg\} \|x - \bar{x}\|.$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$\begin{aligned} H_d(\mathcal{F}(x), \mathcal{F}(\bar{x})) &\le \Bigg\{ \Lambda_1 \Bigg[\mathcal{I}^{p_1+p_2} m(s)(\eta) + \sum_{i=1}^{n} |\alpha_i| \, {}^{\rho_i} I^{q_i} \Big(\mathcal{I}^{p_1+p_2} m(s)(\tau) \Big) \Bigg] \\ &\quad + \mathcal{I}^{p_1+p_2} m(s)(T) \Bigg\} \|x - \bar{x}\|. \end{aligned}$$

So $\mathcal{F}$ is a contraction. Therefore, it follows by Lemma 3.12 that $\mathcal{F}$ has a fixed point x which is a solution of (1). This completes the proof. □

3.4 Examples

In this section, we will illustrate our main results with the help of some examples.

Example 3.14 *Let us consider the following Langevin fractional differential inclusions with nonlocal Katugampola fractional integral boundary conditions*

$$\begin{cases} D^{2/3}\left(D^{4/5}+\frac{1}{8}\right)x(t)\in F_1(t,x(t)), \quad t\in\left(0,\frac{2}{3}\right), \\ x(0)=0, \\ x\left(\frac{2}{9}\right)=\frac{3}{4}\,{}^{\frac{\sqrt{2}}{5}}I^{\frac{1}{3}}x\left(\frac{1}{9}\right)+\frac{1}{\sqrt{7}}\,{}^{\frac{\sqrt{3}}{8}}I^{\frac{1}{\pi}}x\left(\frac{1}{3}\right)+\frac{\sqrt{2}}{5}\,{}^{\frac{4}{e^2}}I^{\frac{2}{3}}x\left(\frac{4}{9}\right) \\ \qquad +\frac{11}{15}\,{}^{\frac{2}{\sqrt{5}}}I^{\frac{6}{13}}x\left(\frac{5}{9}\right), \end{cases} \tag{11}$$

where

$$F_1(t,x)=\left[\frac{(t^{1/2}+1)}{20}\left(\frac{|x|\sin|x|}{5(3+2|x|)}+\frac{e^{-2t}}{3}\right),\frac{(t^{1/2}+1)}{15}\left(\frac{x^2}{3(1+|x|)}+\frac{e^{-t}}{2}\right)\right]. \tag{12}$$

Here $p_1=2/3$, $p_2=4/5$, $\lambda=1/8$, $T=2/3$, $\eta=2/9$, $n=4$, $\alpha_1=3/4$, $\rho_1=\sqrt{2}/5$, $q_1=1/3$, $\xi_1=1/9$, $\alpha_2=1/\sqrt{7}$, $\rho_2=\sqrt{3}/8$, $q_2=1/\pi$, $\xi_2=1/3$, $\alpha_3=\sqrt{2}/5$, $\rho_3=4/e^2$, $q_3=2/3$, $\xi_3=4/9$, $\alpha_4=11/15$, $\rho_4=2/\sqrt{5}$, $q_4=6/13$, $\xi_4=5/9$. From these constants, we can find that $\Omega=0.2660602470$, $\Lambda_1=4.756155970$, $\Lambda_2=5.624515148$ and also $|\lambda|\Lambda_2=0.7030643935<1$. It is obvious that the condition (H_1) is satisfied.

For $f\in F_1$, we have

$$\begin{aligned} |f| &\leq \max\left(\frac{(t^{1/2}+1)}{20}\left(\frac{|x|\sin|x|}{5(3+2|x|)}+\frac{e^{-2t}}{3}\right),\frac{(t^{\frac{1}{2}}+1)}{15}\left(\frac{x^2}{3(1+|x|)}+\frac{e^{-t}}{2}\right)\right) \\ &\leq \frac{(t^{1/2}+1)}{15}\left(\frac{1}{3}|x|+\frac{1}{2}\right), \quad t\in(0,2/3),\ x\in\mathbb{R}. \end{aligned}$$

Therefore, we have

$$\|F_1(t,x)\|_{\mathcal{P}}=\sup\{|y|\,:\,y\in F_1(t,x)\}\leq p(t)\psi(|x|), \quad t\in(0,2/3)\ x\in\mathbb{R},$$

where $p(t)=(t^{1/2}+1)/15$, $\psi(|x|)=(1/3)|x|+(1/2)$. This means that the condition (H_2) is fulfilled. By direct computation, we have $\Omega_1=1.123809144$ and also there exists a constant $M>0.8984766718$ satisfying condition (H_3).

Therefore, all the conditions of Theorem 3.6 are satisfied. So, the problem (11) with $F_1(t,x)$ given by (12) has at least one solution on $[0,2/3]$.

Example 3.15 *Let us consider the following Langevin fractional differential inclusions with nonlocal Katugampola fractional integral boundary conditions*

$$\begin{cases} D^{6/7}\left(D^{8/9}+\frac{1}{12}\right)x(t)\in F_2(t,x(t)), \quad t\in\left(0,\frac{3}{2}\right), \\ x(0)=0, \\ x\left(\frac{2}{3}\right)=\frac{2}{\sqrt{11}}\,{}^{\frac{3}{4}}I^{\frac{5}{8}}x\left(\frac{1}{3}\right)+\frac{4}{7}\,{}^{\frac{\sqrt{3}}{5}}I^{\frac{1}{\sqrt{6}}}x\left(\frac{1}{2}\right)+\frac{\pi}{e^2}\,{}^{\frac{2}{13}}I^{\frac{3}{\sqrt{8}}}x\left(\frac{5}{6}\right) \\ \qquad +\frac{5}{4}\,{}^{\frac{2}{9}}I^{\frac{3}{\pi^2}}x\left(\frac{7}{6}\right)+\frac{\sqrt{3}}{9}\,{}^{\frac{\sqrt{11}}{15}}I^{\frac{13}{17}}x\left(\frac{4}{3}\right), \end{cases} \tag{13}$$

where

$$F_2(t,x)=\left[0,\frac{(t^{1/3}+1)}{24}\left(\frac{x^2+2|x|}{(1+|x|)}\right)+\frac{t}{2}\right]. \tag{14}$$

Here $p_1 = 6/7$, $p_2 = 8/9$, $\lambda = 1/12$, $T = 3/2$, $\eta = 2/3$, $n = 5$, $\alpha_1 = 2/\sqrt{11}$, $\rho_1 = 3/4$, $q_1 = 5/8$, $\xi_1 = 1/3$, $\alpha_2 = 4/7$, $\rho_2 = \sqrt{3}/5$, $q_2 = 1/\sqrt{6}$, $\xi_2 = 1/2$, $\alpha_3 = \pi/e^2$, $\rho_3 = 2/13$, $q_3 = 3/\sqrt{8}$, $\xi_3 = 5/6$, $\alpha_4 = 5/4$, $\rho_4 = 2/9$, $q_4 = 3/\pi^2$, $\xi_4 = 7/6$, $\alpha_5 = \sqrt{3}/9$, $\rho_5 = \sqrt{11}/15$, $q_5 = 13/17$, $\xi_5 = 4/3$. From given constants, we can find that $\Omega = 2.105868955$, $\Lambda_1 = 0.7738927855$. In addition, we have

$$\sup\{|x| : x \in F_2(t,x)\} \le \frac{(t^{1/3}+1)}{24}\left(\frac{x^2+2|x|}{(1+|x|)}\right) + \frac{t}{2},$$

which yields

$$H_d(F_2(t,x), F_2(t,y)) \le \frac{(t^{1/3}+1)}{12}|x-y|.$$

Choosing $m(t) = (t^{1/3}+1)/12$, we obtain $H_d(F_2(t,x), F_2(t,y)) \le m(t)|x-y|$ such that $d(0, F_2(t,0)) \le m(t)$. By the previous setting, we find that $\Omega_2 = 0.3397697571 < 1$.

Thus all assumptions of Theorem 3.13 are fulfilled. Therefore, by the conclusion of Theorem 3.13, we deduce that the problem (13) with $F_2(t,x)$ given by (14) has at least one solution on $[0, 3/2]$.

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On the Riemann-Liouville fractional Hermite-Hadamard-type inequalities for differentiable α-preinvex mappings

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Abstract

In the paper, by discovering a Riemann-Liouville fractional integral identity involving twice differentiable preinvex mappings, the authors establish the right-sided new Hermite-Hadamard type inequalities via Riemann-Liouville fractional integrals for α-preinvex functions. The new fractional integral inequalities are then applied to some special means.

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1 Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

referred to as Hermite-Hadamard inequality, is one of the most famous results for convex functions. A number of papers have been written on this inequality providing new proofs, noteworthy extensions, generalizations, refinements and

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new inequalities connected with the Hermite-Hadamard inequality. The reader may refer to [7, 12, 14, 16, 20, 21, 26, 27, 34] and the references therein.

Let us recall some necessary definitions and preliminary results which are used for further discussion.

Definition 1.1 *([3, 32]) A set $S \subseteq \mathbb{R}^n$ is said to be invex set with respect to the mapping $\eta : S \times S \to \mathbb{R}^n$ if $x + t\eta(y, x) \in S$ for every $x, y \in S$ and $t \in [0, 1]$. The invex set S is also called an η-connected set.*

Notice that every convex set is invex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not necessarily true. See [3, 33], for example.

Definition 1.2 *([3]) Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}^n$. For every $x, y \in S$, the η-path P_{xv} joining the points x and $v = x + \eta(y, x)$ is defined by*

$$P_{xv} = \{z | z = x + t\eta(y, x), t \in [0, 1]\}.$$

A significant generalization of convex mappings is that of preinvex mappings introduced by Weir and Mond in [32].

Definition 1.3 *([32]) The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect to η if for every $x, y \in K$ and $t \in [0, 1]$ we have*

$$f\big(x + t\eta(y, x)\big) \le (1 - t)f(x) + tf(y).$$

The function f is said to be preincave if and only if $-f$ is preinvex.

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y, x) = y - x$. Further, there exist preinvex functions which are not convex.

Moreover, Wang et al. gave the so-called α-preinvex function in [29] as follows.

Definition 1.4 *([29]) Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}^n$. A function $f : S \to \mathbb{R}$ is said to be α-preinvex with respect to η for $\alpha \in (0, 1]$, if every $x, y \in S$ and $t \in [0, 1]$,*

$$f\big(x + t\eta(y, x)\big) \le (1 - t^\alpha)f(x) + t^\alpha f(y).$$

Certainly, α-preinvex mapping means just preinvex mapping when $\alpha = 1$.

For recent results on some new generalizations, refinements of integral inequalities involved with the preinvex functions, one can see [4, 13, 17–19, 23] and the references therein.

We also need the following fractional calculus background.

Definition 1.5 *([25]) Let $f \in L^1[a, b]$. The left-sided and right-sided Riemann-Liouville fractional integrals of order $\alpha > 0$ with $a \ge 0$ are defined by*

$$J^\alpha_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t)\mathrm{d}t, \quad a < x$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t, \quad x < b$$

respectively, where $\Gamma(.)$ *is Gamma function and its definition is* $\Gamma(\alpha) = \int_0^{\infty} e^{-u}u^{\alpha-1}\mathrm{d}u$*. It is to be noted that* $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$.

In the case $\alpha = 1$, the Riemann-Liouville fractional integral reduces to the classical and usual integral.

In [28], Sarikaya et al. established the following interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.1 *Let* $f : [a,b] \to \mathbb{R}$ *be a positive function with* $0 \le a < b$ *and* $f \in L^1[a,b]$*. If* f *is convex function on* $[a,b]$*, then the following inequalities for fractional integrals hold:*

$$f\Big(\frac{a+b}{2}\Big) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)] \le \frac{f(a)+f(b)}{2} \tag{1.2}$$

Observe that for $\alpha = 1$, the inequalities (1.2) reduces to the classical Hermite-Hadamard inequality (1.1).

For some recent results associated with the fractional integral inequalities, one can consult [1, 2, 5, 8–10, 15, 28, 30].

In the recently published article [25] by Qaisar et al., they obtained Riemann-Liouville fractional Hadamard-type integral inequalities for mappings whose absolute value of first derivatives are preinvex, and in the paper [11] Dragomir et al. also found some Hadamard-type fractional integral inequalities for differentiable mappings whose absolute value of second derivatives are convex. Motivated and inspired by this idea, in the present paper, by discovering a Riemann-Liouville fractional integral identity involving twice differentiable preinvex mappings, the authors establish the right-sided new Hermite-Hadamard type inequalities via Riemann-Liouville fractional integrals for α-preinvex functions. The new fractional integral inequalities are then applied to some special means.

2 Main Results

To derive main results in this section, we prove the following Lemma.

Lemma 2.1 *Let* $A \subseteq \mathbb{R}$ *be an open invex subset with respect to* $\eta : A \times A \to \mathbb{R}$ *and Let* $a, b \in A$ *with* $a < a + \eta(b,a)$*. Assume that* $f : A \to \mathbb{R}$ *be a twice differentiable mapping. If* f'' *is preinvex on* A *and* f'' *is integrable on the* η*-path* $P_{ac} : c = a + \eta(b,a)$*, then the following identity for Riemann-Liouville fractional integral with* $\alpha > 0$ *holds:*

$$\begin{aligned} &\frac{f(a) + f\big(a+\eta(b,a)\big)}{2} - \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\big[J_{a^+}^{\alpha} f\big(a+\eta(b,a)\big) + J_{(a+\eta(b,a))^-}^{\alpha} f(a)\big] \\ &= \frac{\eta^2(b,a)}{2} \int_0^1 \frac{1 - t^{\alpha+1} - (1-t)^{\alpha+1}}{\alpha+1} f''\big(a + t\eta(b,a)\big)\mathrm{d}t, \end{aligned} \tag{2.1}$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} u^{\alpha-1} \mathrm{d}u$.

Proof. Set

$$I = \frac{\eta^2(b,a)}{2} \int_0^1 \frac{1 - t^{\alpha+1} - (1-t)^{\alpha+1}}{\alpha+1} f''\big(a + t\eta(b,a)\big)\mathrm{d}t.$$

Since $a, b \in A$ and A is an invex set with respect to η, for every $t \in [0,1]$, we have $a + t\eta(b,a) \in A$. Integrating by part yields that

$$\begin{aligned} I =& \frac{\eta^2(b,a)}{2}\left[\frac{1 - t^{\alpha+1} - (1-t)^{\alpha+1}}{(\alpha+1)\eta(b,a)} f'\big(a + t\eta(b,a)\big)\Big|_0^1\right. \\ &\left. - \int_0^1 \frac{-(\alpha+1)t^{\alpha} + (\alpha+1)(1-t)^{\alpha}}{(\alpha+1)\eta(b,a)} f'\big(a + t\eta(b,a)\big)\mathrm{d}t\right] \\ =& \frac{\eta^2(b,a)}{2}\left[\frac{t^{\alpha} - (1-t)^{\alpha}}{\eta^2(b,a)} f\big(a + t\eta(b,a)\big)\Big|_0^1\right. \\ &\left. - \int_0^1 \frac{\alpha t^{\alpha-1} + \alpha(1-t)^{\alpha-1}}{\eta^2(b,a)} f\big(a + t\eta(b,a)\big)\mathrm{d}t\right] \\ =& \frac{f(a) + f\big(a + \eta(b,a)\big)}{2} - \frac{\alpha}{2}\left[\int_0^1 \Big(t^{\alpha-1} + (1-t)^{\alpha-1}\Big) f\big(a + t\eta(b,a)\big)\mathrm{d}t\right]. \end{aligned}$$

Let $u = a + t\eta(b,a)$, then $\mathrm{d}u = \eta(b,a)\mathrm{d}t$, and using the reduction formula $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)(\alpha > 0)$ for Euler gamma function, we have

$$\frac{\alpha}{2}\int_0^1 t^{\alpha-1} f\big(a + t\eta(b,a)\big)\mathrm{d}t = \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)} J^{\alpha}_{(a+\eta(b,a))^-} f(a)$$

and similarly we get

$$\frac{\alpha}{2}\int_0^1 (1-t)^{\alpha-1} f\big(a + t\eta(b,a)\big)\mathrm{d}t = \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)} J^{\alpha}_{a^+} f\big(a + \eta(b,a)\big).$$

Thus, we have conclusion (2.1).

Remark 2.1 *Applying Lemma 2.1 for* $\eta(b,a) = b - a$*, we can obtain the Lemma 2.1 in [31], which may be discovered also in [22]. Furthermore, let* $\alpha = 1$*, we can get lemma 1 in [24].*

With the help of Lemma 2.1, new upper bound for the right-hand side of (1.2) for α-preinvex functions via the Riemann-Liouville fractional integral is presented in the following theorem.

Theorem 2.1 *Let* $A \subseteq \mathbb{R}$ *be an open invex subset with respect to* $\eta : A \times A \to \mathbb{R}$ *and* $a, b \in A$ *with* $a < a + \eta(b,a)$*. Suppose that* $f : A \to \mathbb{R}$ *be a twice differentiable mapping and* f'' *is integrable on the* η*-path* $P_{ac} : c = a + \eta(b,a)$*.*

If $|f''|$ is α-preinvex on A then the following inequality for fractional integrals with $0 < \alpha \leq 1$ holds:

$$\begin{aligned}&\Big|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\big[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\big]\Big|\\ \leq&\frac{\eta^2(b,a)}{2(\alpha+1)}\Big[\Big(\frac{2\alpha^2+\alpha-2}{(\alpha+2)(2\alpha+2)}+\beta(\alpha+1,\alpha+2)\Big)|f''(a)|\\ &+\Big(\frac{1}{2\alpha+2}-\beta(\alpha+1,\alpha+2)\Big)|f''(b)|\Big].\end{aligned} \tag{2.2}$$

Proof. Since $a + t\eta(b, a) \in A$ for every $t \in [0, 1]$, by using the properties of modulus on Lemma 2.1, we can obtain that

$$\begin{aligned}&\Big|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\big[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\big]\Big|\\ \leq&\ \frac{\eta^2(b,a)}{2}\int_0^1\Big|\frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{\alpha+1}\Big||f''\big(a+t\eta(b,a)\big)|\mathrm{d}t.\end{aligned}$$

Using the α-preinvexity of $|f''|$, we have

$$\begin{aligned}&\int_0^1\Big|\frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{\alpha+1}\Big||f''\big(a+t\eta(b,a)\big)|\mathrm{d}t\\ \leq&\frac{1}{\alpha+1}\int_0^1\Big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\Big)\Big((1-t^{\alpha})|f''(a)|+t^{\alpha}|f''(b)|\Big)\mathrm{d}t\\ \leq&\frac{1}{\alpha+1}\Big[\Big(\frac{2\alpha^2+\alpha-2}{(\alpha+2)(2\alpha+2)}+\beta(\alpha+1,\alpha+2)\Big)|f''(a)|\\ &+\Big(\frac{1}{2\alpha+2}-\beta(\alpha+1,\alpha+2)\Big)|f''(b)|\Big].\end{aligned}$$

To prove the second inequality, we used the following fact that

$$\begin{aligned}&\int_0^1\big(1-t^{\alpha+1}-(1-t)^{\alpha+1}-t^{\alpha}+t^{2\alpha+1}\big)\mathrm{d}t=\frac{2\alpha^2+\alpha-2}{(\alpha+2)(2\alpha+2)},\\ &\int_0^1(t^{\alpha}-t^{2\alpha+1})\mathrm{d}t=\frac{1}{2\alpha+2},\end{aligned}$$

and

$$\int_0^1 t^{\alpha}(1-t)^{\alpha+1}\mathrm{d}t=\beta(\alpha+1,\alpha+2),$$

where the Beta function,

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t,\quad \forall x,y>0,$$

which completes the proof.

Another Riemann-Liouville fractional Hermit-Hadamard-type inequality for powers in terms of the second derivatives is obtained below.

Theorem 2.2 *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $a < a+\eta(b,a)$. Suppose that $f : A \to \mathbb{R}$ be a twice differentiable mapping and f'' is integrable on the η-path $P_{ac} : c = a + \eta(b,a)$. Assume that $q \in \mathbb{R}$, $q \geq 1$ such that $|f''|^q$ is α-preinvex on A, then the following inequality for fractional integrals with $0 < \alpha \leq 1$ holds:*

$$\begin{aligned}&\left|\frac{f(a) + f\big(a + \eta(b,a)\big)}{2} - \frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\left[J^\alpha_{a+} f\big(a+\eta(b,a)\big) + J^\alpha_{(a+\eta(b,a))-} f(a)\right]\right| \\ &\leq \frac{\eta^2(b,a)}{2(\alpha+1)}\Big(1 - \frac{1}{2^{q\alpha}}\Big)^{\frac{1}{q}}\Big(\frac{\alpha}{\alpha+1}|f''(a)|^q + \frac{1}{\alpha+1}|f''(b)|^q\Big)^{\frac{1}{q}}.\end{aligned} \tag{2.3}$$

Proof. Since $a + t\eta(b,a) \in A$ for every $t \in [0,1]$, by using the properties of modulus on Lemma 2.1 and using the well-known power-mean integral inequality for $q \geq 1$, we can obtain that

$$\begin{aligned}&\left|\frac{f(a) + f\big(a + \eta(b,a)\big)}{2} - \frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\left[J^\alpha_{a+} f\big(a+\eta(b,a)\big) + J^\alpha_{(a+\eta(b,a))-} f(a)\right]\right| \\ &\leq \frac{\eta^2(b,a)}{2}\int_0^1 \left|\frac{1 - t^{\alpha+1} - (1-t)^{\alpha+1}}{\alpha+1}\right| |f''\big(a + t\eta(b,a)\big)| \mathrm{d}t \\ &\leq \frac{\eta^2(b,a)}{2(\alpha+1)}\Big(\int_0^1 1 \mathrm{d}t\Big)^{1-\frac{1}{q}}\left[\int_0^1 \big(1 - t^{\alpha+1} - (1-t)^{\alpha+1}\big)^q \big|f''\big(a+t\eta(b,a)\big)\big|^q \mathrm{d}t\right]^{\frac{1}{q}} \\ &\leq \frac{\eta^2(b,a)}{2(\alpha+1)}\left[\int_0^1 \Big(1 - \frac{1}{2^{q\alpha}}\Big)\big|f''\big(a+t\eta(b,a)\big)\big|^q \mathrm{d}t\right]^{\frac{1}{q}}.\end{aligned}$$

To prove the third inequality above, we used the following inequality

$$\begin{aligned}\big(1 - (1-t)^{\alpha+1} - t^{\alpha+1}\big)^q &\leq 1 - [(1-t)^{\alpha+1} + t^{\alpha+1}]^q \\ &\leq 1 - (2^{-\alpha})^q \\ &\leq 1 - \frac{1}{2^{q\alpha}}\end{aligned}$$

for any $t \in [0,1]$ with $q \geq 1$, and also using the α-preinvexity of $|f''|^q$, that is

$$\begin{aligned}\int_0^1 \big|f''\big(a+t\eta(b,a)\big)\big|^q \mathrm{d}t &\leq \int_0^1 \big[(1-t^\alpha)|f''(a)|^q + t^\alpha |f''(b)|^q\big]\mathrm{d}t \\ &= \frac{\alpha}{\alpha+1}|f''(a)|^q + \frac{1}{\alpha+1}|f''(b)|^q.\end{aligned} \tag{2.4}$$

Therefore, we can get the required results (2.3).

Corollary 2.1 *With the same assumptions given in Theorem 2.2, if* $|f''(x)| \leq M$ *on* $[a, a+\eta(b,a)]$*, we can deduce that*

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\right]\right| \leq \frac{M\eta^2(b,a)}{2(\alpha+1)}\Big(1-\frac{1}{2^{q\alpha}}\Big)^{\frac{1}{q}}.$$

Another similar result may be presented in the following theorem.

Theorem 2.3 *Let* $A \subseteq \mathbb{R}$ *be an open invex subset with respect to* $\eta : A\times A \to \mathbb{R}$ *and* $a, b \in A$ *with* $a < a+\eta(b,a)$*. Suppose that* $f : A \to \mathbb{R}$ *be a twice differentiable mapping and* f'' *is integrable on the* η*-path* $P_{ac} : c = a + \eta(b,a)$*. Assume that* $p \in \mathbb{R}$*,* $p > 1$ *with* $q = \frac{p}{p-1}$ *such that* $|f''|^q$ *is* α*-preinvex on* A*, then the following inequality for fractional integrals with* $0 < \alpha \leq 1$ *holds:*

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\right]\right| \leq \frac{\eta^2(b,a)}{2(\alpha+1)}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\Big(\frac{\alpha}{\alpha+1}|f''(a)|^q+\frac{1}{\alpha+1}|f''(b)|^q\Big)^{\frac{1}{q}}. \tag{2.5}$$

Proof. Since $a + t\eta(b,a) \in A$ for every $t \in [0,1]$, by using the properties of modulus on Lemma 2.1 and making use of the well-known Hölder's integral inequality for $q > 1$, we can obtain that

$$\begin{aligned}&\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\right]\right|\\
&\leq \frac{\eta^2(b,a)}{2}\int_0^1\left|\frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{\alpha+1}\right||f''\big(a+t\eta(b,a)\big)|\mathrm{d}t\\
&\leq \frac{\eta^2(b,a)}{2(\alpha+1)}\Big(\int_0^1|1-t^{\alpha+1}-(1-t)^{\alpha+1}|^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1|f''\big(a+t\eta(b,a)\big)|^q\mathrm{d}t\Big)^{\frac{1}{q}}\\
&\leq \frac{\eta^2(b,a)}{2(\alpha+1)}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\Big(\int_0^1|f''\big(a+t\eta(b,a)\big)|^q\mathrm{d}t\Big)^{\frac{1}{q}},\end{aligned}$$

where we use the following inequality

$$\big(1-(1-t)^{\alpha+1}-t^{\alpha+1}\big)^p \leq 1-(1-t)^{p(\alpha+1)}-t^{p(\alpha+1)} \tag{2.6}$$

for any $t \in [0,1]$, which follows from

$$(A-B)^p \leq A^p - B^p \tag{2.7}$$

for any $A > B \geq 0$ and $p > 1$.
By applying (2.4) and (2.6), we can get (2.5). Hence the proof is completed.

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Corollary 2.2 *With the same assumptions given in Theorem 2.3, if* $|f''(x)| \leq M$ *on* $[a, a+\eta(b,a)]$*, we obtain that*

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big) + J^{\alpha}_{(a+\eta(b,a))^-}f(a)\right]\right| \leq \frac{M\eta^2(b,a)}{2(\alpha+1)}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}.$$

A different approach leads to the following result.

Theorem 2.4 *Let* $A \subseteq \mathbb{R}$ *be an open invex subset with respect to* $\eta : A\times A \to \mathbb{R}$ *and* $a, b \in A$ *with* $a < a+\eta(b,a)$*. Suppose that* $f : A \to \mathbb{R}$ *be a twice differentiable mapping and* f'' *is integrable on the* η*-path* $P_{ac} : c = a+\eta(b,a)$*. Assume that* $p \in \mathbb{R}$*,* $p > 1$ *with* $q = \frac{p}{p-1}$ *such that* $|f''|^q$ *is* α*-preinvex on* A*, then the following inequality for fractional integrals with* $0 < \alpha \leq 1$ *holds:*

$$\begin{aligned}&\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big) + J^{\alpha}_{(a+\eta(b,a))^-}f(a)\right]\right|\\ \leq&\frac{\eta^2(b,a)}{2(\alpha+1)}\Big(\frac{\alpha}{\alpha+2}\Big)^{1-\frac{1}{q}}\Big[\Big(\frac{2\alpha^2+\alpha-2}{(\alpha+2)(2\alpha+2)}+\beta(\alpha+1,\alpha+2)\Big)|f''(a)|^q\\ &+\Big(\frac{1}{2\alpha+2}-\beta(\alpha+1,\alpha+2)\Big)|f''(b)|^q\Big]^{\frac{1}{q}}.\end{aligned} \tag{2.8}$$

Proof. Since $a + t\eta(b,a) \in A$ for every $t \in [0,1]$, by utilizing the properties of modulus on Lemma 2.1 and using the Hölder's integral inequality for $q > 1$, we can obtain that

$$\begin{aligned}&\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big) + J^{\alpha}_{(a+\eta(b,a))^-}f(a)\right]\right|\\ \leq&\frac{\eta^2(b,a)}{2}\int_0^1\Big|\frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{\alpha+1}\Big||f''\big(a+t\eta(b,a)\big)|\mathrm{d}t\\ \leq&\frac{\eta^2(b,a)}{2(\alpha+1)}\Big[\int_0^1\big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\big)\mathrm{d}t\Big]^{1-\frac{1}{q}}\\ &\times\Big[\int_0^1\big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\big)|f''\big(a+t\eta(b,a)\big)|^q\mathrm{d}t\Big]^{\frac{1}{q}}\\ =&\frac{\eta^2(b,a)}{2(\alpha+1)}\Big(\frac{\alpha}{\alpha+2}\Big)^{1-\frac{1}{q}}\Big[\int_0^1\big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\big)|f''\big(a+t\eta(b,a)\big)|^q\mathrm{d}t\Big]^{\frac{1}{q}}.\end{aligned}$$

Using the α-preinvexity of $|f''|^q$, i.e. the inequality (2.4), we have

$$\begin{aligned}
&\int_0^1 \big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\big)|f''\big(a+t\eta(b,a)\big)|^q\mathrm{d}t\\
\le&\int_0^1 \big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\big)\Big((1-t^\alpha)|f''(a)|^q+t^\alpha|f''(b)|^q\Big)\mathrm{d}t\\
=&\Big(\frac{2\alpha^2+\alpha-2}{(\alpha+2)(2\alpha+2)}+\beta(\alpha+1,\alpha+2)\Big)|f''(a)|^q\\
&+\Big(\frac{1}{2\alpha+2}-\beta(\alpha+1,\alpha+2)\Big)|f''(b)|^q.
\end{aligned}$$

Thus, we get the desired inequality (2.8).

Corollary 2.3 *With the same assumptions given in Theorem 2.4, if $|f''(x)|\le M$ on $[a,a+\eta(b,a)]$, we obtain that*

$$\begin{aligned}
&\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\big[J^\alpha_{a^+}f\big(a+\eta(b,a)\big)+J^\alpha_{(a+\eta(b,a))^-}f(a)\big]\right|\\
&\le\frac{M\alpha\eta^2(b,a)}{2(\alpha+1)(\alpha+2)}.
\end{aligned}$$

Finally we shall prove the following result.

Theorem 2.5 *Suppose that all the assumptions of Theorem 2.4 are satisfied. Then the following inequalities hold:*

$$\begin{aligned}
&\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\big[J^\alpha_{a^+}f\big(a+\eta(b,a)\big)+J^\alpha_{(a+\eta(b,a))^-}f(a)\big]\right|\\
\le&\frac{\eta^2(b,a)}{2(\alpha+1)}\Big[\frac{(q-p)\alpha-p+1}{(q-p)\alpha+2q-p-1}\Big]^{\frac{q-1}{q}}\\
&\times\Bigg\{\Big[\frac{\alpha p+\alpha+1}{(\alpha+1)(p+1)}-\frac{2}{p(\alpha+1)+1}+\beta(\alpha+1,p(\alpha+1)+1)\Big]|f''(a)|^q\\
&+\Big[\frac{p}{(\alpha+1)(p+1)}-\beta(\alpha+1,p(\alpha+1)+1)\Big]|f''(b)|^q\Bigg\}.
\end{aligned}\tag{2.9}$$

Proof. Since $a+t\eta(b,a)\in A$ for every $t\in[0,1]$, by using the properties of modulus on Lemma 2.1 and making use of the well-known Hölder's integral inequality for $q>1$, we can obtain that

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$$\begin{aligned}
&\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\big[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\big]\right|\\
\leq&\frac{\eta^2(b,a)}{2}\int_0^1\Big|\frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{\alpha+1}\Big||f''\big(a+t\eta(b,a)\big)|\mathrm{d}t\\
\leq&\frac{\eta^2(b,a)}{2(\alpha+1)}\Big[\int_0^1\Big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\Big)^{\frac{q-p}{q-1}}\mathrm{d}t\Big]^{\frac{q-1}{q}}\\
&\times\Big[\int_0^1\Big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\Big)^{p}|f''\big(a+t\eta(b,a)\big)|^q\mathrm{d}t\Big]^{\frac{1}{q}}\\
\leq&\frac{\eta^2(b,a)}{2(\alpha+1)}\Big[\frac{(q-p)\alpha-p+1}{(q-p)\alpha+2q-p-1}\Big]^{\frac{q-1}{q}}\\
&\times\Big[\int_0^1\Big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\Big)^{p}|f''\big(a+t\eta(b,a)\big)|^q\mathrm{d}t\Big]^{\frac{1}{q}}
\end{aligned}$$

Using the α-preinvexity of $|f''|^q$, we have

$$\begin{aligned}
&\int_0^1\Big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\Big)^{p}|f''\big(a+t\eta(b,a)\big)|^q\mathrm{d}t\\
\leq&\int_0^1\Big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\Big)^{p}\Big((1-t^{\alpha})|f''(a)|^q+t^{\alpha}|f''(b)|^q\Big)\mathrm{d}t\\
\leq&\int_0^1\Big(1-t^{(\alpha+1)p}-(1-t)^{(\alpha+1)p}\Big)\Big((1-t^{\alpha})|f''(a)|^q+t^{\alpha}|f''(b)|^q\Big)\mathrm{d}t\\
\leq&\int_0^1\Big(1-(1-t)^{(\alpha+1)p}-t^{(\alpha+1)p}-t^{\alpha}+t^{\alpha}(1-t)^{(\alpha+1)p}+t^{\alpha+(\alpha+1)p}\Big)|f''(a)|^q\mathrm{d}t\\
&+\int_0^1\Big(t^{\alpha}-t^{\alpha}(1-t)^{(\alpha+1)p}-t^{\alpha+(\alpha+1)p}\Big)|f''(b)|^q\mathrm{d}t\\
=&\Big[\frac{\alpha p+\alpha+1}{(\alpha+1)(p+1)}-\frac{2}{p(\alpha+1)+1}+\beta(\alpha+1,p(\alpha+1)+1)\Big]|f''(a)|^q\\
&+\Big[\frac{p}{(\alpha+1)(p+1)}-\beta(\alpha+1,p(\alpha+1)+1)\Big]|f''(b)|^q.
\end{aligned}$$

Here, we use

$$(1-(1-t)^{\alpha+1}-t^{\alpha+1})^q\leq 1-(1-t)^{q(\alpha+1)}-t^{q(\alpha+1)} \tag{2.10}$$

for any $t\in[0,1]$, which follows from

$$(A-B)^q\leq A^q-B^q \tag{2.11}$$

for any $A>B\geq 0$ and $q\geq 1$.
Thus, we get the desired inequality (2.9).

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Corollary 2.4 *From Theorems (2.2),(2.3),(2.4) and (2.5), we have*

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))-}f(a)\right]\right|$$
$$\leq min\{K_1,K_2,K_3,K_4\},$$

where

$$K_1=\frac{\eta^2(b,a)}{2(\alpha+1)}\Big(1-\frac{1}{2^{q\alpha}}\Big)^{\frac{1}{q}}\Big(\frac{\alpha}{\alpha+1}|f''(a)|^q+\frac{1}{\alpha+1}|f''(b)|^q\Big)^{\frac{1}{q}},$$
$$K_2=\frac{\eta^2(b,a)}{2(\alpha+1)}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\Big(\frac{\alpha}{\alpha+1}|f''(a)|^q+\frac{1}{\alpha+1}|f''(b)|^q\Big)^{\frac{1}{q}},$$
$$K_3=\frac{\eta^2(b,a)}{2(\alpha+1)}\Big(\frac{\alpha}{\alpha+2}\Big)^{1-\frac{1}{q}}\Bigg[\Big(\frac{2\alpha^2+\alpha-2}{(\alpha+2)(2\alpha+2)}+\beta(\alpha+1,\alpha+2)\Big)|f''(a)|^q$$
$$+\Big(\frac{1}{2\alpha+2}-\beta(\alpha+1,\alpha+2)\Big)|f''(b)|^q\Bigg]^{\frac{1}{q}},$$

$$K_4=\frac{\eta^2(b,a)}{2(\alpha+1)}\Big[\frac{(q-p)\alpha-p+1}{(q-p)\alpha+2q-p-1}\Big]^{\frac{q-1}{q}}$$
$$\times\Bigg\{\Bigg[\frac{\alpha p+\alpha+1}{(\alpha+1)(p+1)}-\frac{2}{p(\alpha+1)+1}+\beta(\alpha+1,p(\alpha+1)+1)\Bigg]|f''(a)|^q$$
$$+\Bigg[\frac{p}{(\alpha+1)(p+1)}-\beta(\alpha+1,p(\alpha+1)+1)\Bigg]|f''(b)|^q\Bigg\}.$$

3 Applications to special means

In the following we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 3.1 *[6] A function $M:\mathbb{R}^2_+\to\mathbb{R}_+$, is called a Mean function if it has the following properties:*
(1) Homogeneity: $M(ax,ay)=aM(x,y)$, for all $a>0$,
(2) Symmetry: $M(x,y)=M(y,x)$,
(3) Reflexivity: $M(x,x)=x$,
(4) Monotonicity: If $x\leq x'$ and $y\leq y'$, then $M(x,y)\leq M(x',y')$,
(5) Internality: $min\{x,y\}\leq M(x,y)\leq max\{x,y\}$.

We consider some means for arbitrary positive real numbers $a>0$ and $b>0$, define $A:=A(a,b)=\frac{a+b}{2}$, $G:=G(a,b)=\sqrt{ab}$, $H:=H(a,b)=\frac{2ab}{a+b}$, $P_r:=P_r(a,b)=\left(\frac{a^r+b^r}{2}\right)^{\frac{1}{r}}, r\geq 1$

$$I:=I(a,b)=\begin{cases}\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & a\neq b,\\ a, & a=b,\end{cases}\qquad L:=L(a,b)=\begin{cases}\frac{b-a}{\ln b-\ln a}, & a\neq b,\\ a, & a=b,\end{cases}$$

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and

$$L_p := L_p(a,b) = \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & p \neq 0, -1, \text{ and } a \neq b, \\ L(a,b), & p = -1 \text{ and } a \neq b, \\ I(a,b), & p = 0 \text{ and } a \neq b, \\ a, & a = b. \end{cases}$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$, with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$.

Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(a,b) : [a + \eta(b,a)] \times [a, a + \eta(b,a)] \to \mathbb{R}^+$, which is one of the above mentioned means, therefore one can obtain various inequalities for these means below:

Setting $\eta(b,a) = M(b,a)$ in (2.2), (2.3), (2.5), (2.8) and (2.9), one can derive the following interesting inequalities concerning means:

$$(1)\left|\frac{f(a) + f\big(a + M(b,a)\big)}{2} - \frac{\Gamma(\alpha+1)}{2M^\alpha(b,a)}\left[J^\alpha_{a^+} f\big(a + M(b,a)\big) + J^\alpha_{(a+M(b,a))^-} f(a)\right]\right|$$
$$\leq \frac{M^2(b,a)}{2(\alpha+1)}\left[\Big(\frac{2\alpha^2+\alpha-2}{(\alpha+2)(2\alpha+2)} + \beta(\alpha+1,\alpha+2)\Big)|f''(a)| + \Big(\frac{1}{2\alpha+2} - \beta(\alpha+1,\alpha+2)\Big)|f''(b)|\right],$$

$$(2)\left|\frac{f(a) + f\big(a + M(b,a)\big)}{2} - \frac{\Gamma(\alpha+1)}{2M^\alpha(b,a)}\left[J^\alpha_{a^+} f\big(a + M(b,a)\big) + J^\alpha_{(a+M(b,a))^-} f(a)\right]\right|$$
$$\leq \frac{M^2(b,a)}{2(\alpha+1)}\Big(1 - \frac{1}{2^{q\alpha}}\Big)^{\frac{1}{q}}\Big(\frac{\alpha}{\alpha+1}|f''(a)|^q + \frac{1}{\alpha+1}|f''(b)|^q\Big)^{\frac{1}{q}},$$

$$(3)\left|\frac{f(a) + f\big(a + M(b,a)\big)}{2} - \frac{\Gamma(\alpha+1)}{2M^\alpha(b,a)}\left[J^\alpha_{a^+} f\big(a + M(b,a)\big) + J^\alpha_{(a+M(b,a))^-} f(a)\right]\right|$$
$$\leq \frac{M^2(b,a)}{2(\alpha+1)}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\left[\frac{\alpha}{\alpha+1}|f''(a)|^q + \frac{1}{\alpha+1}|f''(b)|^q \mathrm{d}t\right]^{\frac{1}{q}},$$

$$(4)\left|\frac{f(a) + f\big(a + M(b,a)\big)}{2} - \frac{\Gamma(\alpha+1)}{2M^\alpha(b,a)}\left[J^\alpha_{a^+} f\big(a + M(b,a)\big) + J^\alpha_{(a+M(b,a))^-} f(a)\right]\right|$$
$$\leq \frac{M^2(b,a)}{2(\alpha+1)}\Big(\frac{\alpha}{\alpha+2}\Big)^{1-\frac{1}{q}}\left[\Big(\frac{2\alpha^2+\alpha-2}{(\alpha+2)(2\alpha+2)} + \beta(\alpha+1,\alpha+2)\Big)|f''(a)|^q + \Big(\frac{1}{2\alpha+2} - \beta(\alpha+1,\alpha+2)\Big)|f''(b)|^q\right]^{\frac{1}{q}},$$

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and

$$(5)\left|\frac{f(a)+f\big(a+M(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2M^{\alpha}(b,a)}\left[J^{\alpha}_{a^+}f\big(a+M(b,a)\big)+J^{\alpha}_{(a+M(b,a))^-}f(a)\right]\right|$$
$$\leq\frac{M^2(b,a)}{2(\alpha+1)}\left[\frac{(q-p)\alpha-p+1}{(q-p)\alpha+2q-p-1}\right]^{\frac{q-1}{q}}$$
$$\times\left\{\left[\frac{\alpha p+\alpha+1}{(\alpha+1)(p+1)}-\frac{2}{p(\alpha+1)+1}+\beta(\alpha+1,p(\alpha+1)+1)\right]|f''(a)|^q\right.$$
$$\left.+\left[\frac{p}{(\alpha+1)(p+1)}-\beta(\alpha+1,p(\alpha+1)+1)\right]|f''(b)|^q\right\}.$$

Letting $M = A$, G, H, P_r, I, L, L_p in (1), (2), (3), (4) and (5), we get the inequalities involving means for a particular choice of a twice differentiable α-preinvex function f, and the details are left to the interested reader.

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CONVOLUTION PROPERTIES FOR CERTAIN SUBCLASSES OF MEROMORPHIC BOUNDED FUNCTIONS

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ABSTRACT. By making use of the Hadamard product, we derive necessary and sufficient conditions for certain meromorphic function to be in the class $\mathcal{S}^*(\lambda, \gamma, M)(\gamma \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}, M \geq 1, \lambda \in \mathbb{C})$ which unifies the classes of bounded starlike and convex functions of complex order. By using Al-Oboudi operator a more general class $\mathcal{S}^*(n, \lambda, \gamma, M)$ related to $\mathcal{S}^*(\lambda, \gamma, M)$ is also considered. Several properties of the class $\mathcal{S}^*(n, \lambda, \gamma, M)$ are also obtained.

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Key Words. Univalent meromorphic functions, bounded starlike functions of complex order, bounded convex functions of complex order, λ-starlike functions, Hadamard product, subordination.

1. Introduction

Let $\mathbb{C}$ be the complex plane and let Σ denote the class of all meromorphic functions having the form:

$$f(z) = z^{-1} + \sum_{k=0}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the punctured unit disc

$$E^* = \{z : z \in \mathbb{C}, \quad 0 < |z| < 1\} =: E\backslash\{0\}.$$

The familiar Hadamard product (or convolution) of two functions $f(z)$ given by (1.1 and $g(z)$ is given by

$$g(z) = z^{-1} + \sum_{k=0}^{\infty} b_k z^k, \tag{1.2}$$

is defined by

$$(f * g)(z) = z^{-1} + \sum_{k=0}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

An analytic function f is said to be subordinate to another analytic function g, written symbolically as follows:

$$f(z) \prec g(z) \ (z \in \mathbb{E}),$$

if there exists a function $\omega(z)$, analytic in $\mathbb{E}$ with

$$\omega(0) = 0 \ \ and \ \ |\omega(z)| < 1 \ \ (z \in \mathbb{E}),$$

such that

$$f(z) = g(w(z)) \ (z \in \mathbb{E}).$$

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Furthermore, if the function $g(z)$ is univalent in $\mathbb{E}$, then we have the following equivalence, (cf., e.g., [5], [9], [10]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{E}) \subset g(\mathbb{E}).$$

Making use of the principal of subordination between analytic functions, Aouf [2] defined the subclasses $\mathcal{S}^*(\gamma, M)$ and $\mathcal{C}(\gamma, M)$ of the class Σ as follows:

(1.4)

$$\mathcal{S}^*(\gamma, M) = \left\{ f \in \Sigma : -\frac{zf'(z)}{f(z)} \prec \frac{[\gamma(1+m) - m]\, z + 1}{1 - mz} \quad \left(\gamma \in \mathbb{C}^*,\ m = 1 - \frac{1}{M};\ M \geq 1;\ z \in \mathbb{E} \right) \right\}$$

or, equivalently,

$$\left| \frac{\gamma - 1 - \frac{zf'(z)}{f(z)}}{\gamma} - M \right| < M \quad \left(m = 1 - \frac{1}{M};\ M \geq 1;\ z \in \mathbb{E} \right), \tag{1.5}$$

and

(1.6)

$$\mathcal{C}(\gamma, M) = \left\{ f \in \Sigma : -\frac{zf''(z)}{f'(z)} \prec 2 + \frac{\gamma(1+m)z}{1 - mz}\ , \quad \left(\gamma \in \mathbb{C}^*,\ m = 1 - \frac{1}{M};\ M \geq 1;\ z \in \mathbb{E} \right) \right\}$$

or, equivalently,

$$\left| \frac{\gamma - 2 - \frac{zf''(z)}{f'(z)}}{\gamma} - M \right| < M \quad \left(m = 1 - \frac{1}{M};\ M \geq 1;\ z \in \mathbb{E} \right). \tag{1.7}$$

From inequalities (1.4) and (1.6), we get

$$f(z) \in \mathcal{C}(\gamma, M) \Leftrightarrow -zf'(z) \in \mathcal{S}^*(\gamma, M). \tag{1.8}$$

First let us define the class $S^*(\lambda, \gamma, M)$ which unifies the classes of bounded meromorphic starlike and convex functions of complex order.

Definition 1. *A function $f \in \Sigma$ is said to be in the class $\mathcal{S}^*(\lambda, \gamma, M)(\lambda \in \mathbb{C}, \gamma \in \mathbb{C}^*, M \geq 1)$ of bounded meromorphic $\lambda-$starlike functions of complex order, if and only if for fixed M, $\frac{f(z)f'(z)}{z} \neq 0$ and*

$$\left| 1 - \frac{1}{\gamma} \left(1 + \frac{\lambda z \left(zf'(z) \right)' + (1+\lambda) zf'(z)}{\lambda zf'(z) + (1+\lambda) f(z)} \right) - M \right| < M \quad (z \in \mathbb{E})\,, \tag{1.9}$$

or, equivalently,

(1.10)

$$\mathcal{S}^*(\lambda, \gamma, M) = \left\{ f \in \Sigma : -\frac{\lambda z \left(zf'(z) \right)' + (1+\lambda) zf'(z)}{\lambda zf'(z) + (1+\lambda) f(z)} \prec \frac{[\gamma(1+m) - m]\, z + 1}{1 - mz} \right\}$$

$$\left(\lambda \in \mathbb{C},\ \gamma \in \mathbb{C}^*,\ m = 1 - \frac{1}{M};\ M \geq 1;\ z \in \mathbb{E} \right)$$

One can easily show that $f \in \mathcal{S}^*(\lambda,\gamma,M)$ if and only if there is a function $g \in \mathcal{S}^*(1,M)$ such that

$$\lambda z f^{'}(z) + (1+\lambda)f(z) = \frac{(zg(z))^{\gamma}}{z} \tag{1.11}$$

It was shown in [8] $g \in \mathcal{S}^*(1,M)$ if and only if for $z \in \mathbb{E}$

$$-\frac{zg^{'}(z)}{g(z)} = \frac{1+\omega(z)}{1-m\omega(z)}, \quad \omega(0)=0,\ |\omega(z)|<1 \ \textit{and} \ \ m = 1-\frac{1}{M}. \tag{1.12}$$

Thus from (1.11) and (1.12) follows that $f \in \mathcal{S}^*(\lambda,\gamma,M)$ if and only if for $M \geq 1,\ \lambda \in \mathbb{C}$ and $z \in \mathbb{E}$

$$\frac{\lambda z\left(zf^{'}(z)\right)^{'} + (1+\lambda)zf^{'}(z)}{\lambda zf'(z) + (1+\lambda)f(z)} = -\frac{[\gamma(1+m)-m]\,\omega(z)+1}{1-m\omega(z)}. \tag{1.13}$$

By specializing λ,γ and M, we get the following subclasses studied by earlier authors:

Remark 1.

(i) $\mathcal{S}^*(0,\gamma,\infty) =: \mathcal{S}^*(\gamma)$, with $\gamma \in \mathbb{C}^*$, (see Aouf [2]);
(ii) $\mathcal{S}^*(-1,\gamma,\infty) =: \mathcal{C}(\gamma)$, with $\gamma \in \mathbb{C}^*$, (see Aouf [2]);
(iii) $\mathcal{S}^*(0,1-a,M) =: \mathcal{S}^*_M(a)$, with $0 \leq a < 1$, (see Kaczmarski [8]);
(iv)$\mathcal{S}^*(-1,1-a,M) =: \mathcal{C}_M(a)$, with $0 \leq a < 1$, (see Aouf [2]);
(v) $\mathcal{S}^*(0,1,\infty) =: \mathcal{S}^*(1)$, with $0 \leq a < 1$, (see Clunie [7]);
(vi) $\mathcal{S}^*(-1,1,\infty) =: \mathcal{C}(1)$, with $0 \leq a < 1$, (see Aouf [2]);
(vii) $\mathcal{S}^*(0,1-a,\infty) =: \mathcal{S}^*(1-a)$, with $0 \leq a < 1$, (see Kaczmarski [8] and Pommerenke [11]);
(viii) $\mathcal{S}^*(-1,1-a,\infty) =: \mathcal{C}(1-a)$,with $0 \leq a < 1$, (see Aouf [2]);
(ix) $\mathcal{S}^*(0,(1-a)e^{-i\beta}\cos\beta,M) =: \mathcal{S}^*_M(a,\beta)$, with $0 \leq a < 1$, $|\beta| \leq \frac{\pi}{2}$), (see Kaczmarski [8]);
(x) $\mathcal{S}^*(-1,(1-a)e^{-i\beta}\cos\beta,M) =: \mathcal{C}_M(a,\beta)$,with $0 \leq a < 1$, $|\beta| \leq \frac{\pi}{2}$), (see Aouf [2]);
(xi) $\mathcal{S}^*(0,(1-a)e^{-i\beta}\cos\beta,\infty) =: \mathcal{S}^*(a,\beta)$, with $0 \leq a < 1$, $|\beta| \leq \frac{\pi}{2}$), (see Kaczmarski [8]);
(xii) $\mathcal{S}^*(-1,(1-a)e^{-i\beta}\cos\beta,\infty) =: \mathcal{C}(a,\beta)$,with $0 \leq a < 1$, $|\beta| \leq \frac{\pi}{2}$), (see Aouf [2]).

For $f(z) \in \Sigma$, Al-Oboudi and Al-Zkeri [1] defined the following operator $D^n f(n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\} = \{0,1,2,3,\ldots\})$ which is called the Al-Oboudi operator:

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= (1-\mu)f(z) + \mu\frac{(z^2 f(z))'}{z}, \quad \mu \geq 0 \\ &= (1+\mu)f(z) + \mu z f'(z) = D_\mu f(z), \\ D^2 f(z) &= D_\mu D^1 f(z). \\ D^n f(z) &= D_\mu\left(D^{n-1} f(z)\right),\ n \in \mathbb{N} \qquad (1.14) \end{aligned}$$

From (1.1) and (1.14) we get

$$D^n f(z) = z^{-1} + \sum_{k=0}^{\infty} [\mu(k+1)+1]^n a_k z^k \quad (z \in \mathbb{E}^*)\,. \tag{1.15}$$

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With the aid of Al-Oboudi operator, we introduce the class $\mathcal{S}^*(n,\lambda,\gamma,M)$ as follows:

Definition 2. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{S}^*(n,\lambda,\gamma,M)$ if and only if for fixed M,*

$$\left|1-\frac{1}{\gamma}\left(1+\frac{\lambda z\left[z\left(D^n f(z)\right)'\right]'+(1+\lambda)z\left(D^n f(z)\right)'}{\lambda z\left(D^n f(z)\right)'+(1+\lambda)D^n f(z)}\right)-M\right|<M \quad (z\in\mathbb{E})$$

where, $M\geq 1,\gamma\in C^*,\lambda\in\mathbb{C}$ and $n\in\mathbb{N}_0$.

We note that $\mathcal{S}^*(0,\lambda,\gamma,M)\equiv\mathcal{S}^*(\lambda,\gamma,M)$.

In this paper we will investigate some convolution properties of the class $\mathcal{S}^*(\lambda,\gamma,M)$. Using these properties, we find the necessary and sufficient condition, and containment property for the subclass $\mathcal{S}^*(n,\lambda,\gamma,M)$. The results obtained here extend some known results in [3], [4] and [6].

2. Convolution properties

Unless otherwise mentioned, we assume throughout this article that $\gamma\in\mathbb{C}^*, M\geq 1,\lambda\in\mathbb{C}$ and $n\in\mathbb{N}_0$.

Theorem 1. *The function $f(z)$ defined by (1.1) be in the class $\mathcal{S}^*(\lambda,\gamma,M)$ if and only if*

$$z\left[f(z)*\left\{(1+\lambda)\frac{(C-1)\,z+1}{z(1-z)^2}+\lambda\frac{2(C-1)z^2+3z-1}{z(1-z)^3}\right\}\right]\neq 0 \ (z\in\mathbb{E}) \tag{2.1}$$

where $C=C_\theta=\frac{e^{-i\theta}-m}{\gamma(1+m)}$, $\theta\in[0,2\pi)$.

Proof. First suppose $f(z)$ defined by (1.1) is in the class $\mathcal{S}^*(\lambda,\gamma,M)$, we have

$$-\frac{\lambda z\left(zf'(z)\right)'+(1+\lambda)zf'(z)}{\lambda zf'(z)+(1+\lambda)f(z)}\prec\frac{\left[\gamma(1+m)-m\right]z+1}{1-mz} \ (z\in\mathbb{E}), \tag{2.2}$$

since the left-hand side of (2.2) is analytic in $\mathbb{E}$, it follows $\lambda zf'(z)+(1+\lambda)f(z)\neq 0$ for all $z\in\mathbb{E}^*$, i.e. $\lambda z^2f'(z)+(1+\lambda)zf(z)\neq 0$, $z\in\mathbb{E}$, so (2.1) holds for $C=0$. By using the principle of subordination, we can write (2.2) as

$$-\frac{\lambda z\left(zf'(z)\right)'+(1+\lambda)zf'(z)}{\lambda zf'(z)+(1+\lambda)f(z)}=\frac{[\gamma(1+m)-m]\omega(z)+1}{1-m\omega(z)}(z\in\mathbb{E}),$$

which is equivalent to

$$-\frac{z\left[\lambda\left(zf'(z)\right)+(1+\lambda)f(z)\right]'}{\lambda zf'(z)+(1+\lambda)f(z)}\neq\frac{[\gamma(1+m)-m]e^{i\theta}+1}{1-me^{i\theta}}, \quad (\ z\in\mathbb{E},\ \theta\in[0,2\pi)).$$

or

$$-z\left[\lambda\left(zf'(z)\right)+(1+\lambda)f(z)\right]'\left(1-me^{i\theta}\right)-\left[\lambda zf'(z)+(1+\lambda)f(z)\right]\left[\left[\gamma(1+m)-m\right]e^{i\theta}+1\right]\neq 0. \tag{2.3}$$

Since

$$f(z)=f(z)*\frac{1}{z(1-z)} \text{ and } -zf'(z)=f(z)*\left[\frac{1}{z(1-z)^2}-\frac{2}{(1-z)^2}\right]. \tag{2.4}$$

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Applying (2.4) it is not difficult to verify that

$$\lambda z f^{'}(z)+(1+\lambda)f(z)=f(z)*\frac{(\lambda-1)z+1}{z(1-z)^2}. \tag{2.5}$$

Since $z(f*g)^{'}=f*zg^{'}$, we can write

$$z\left[\lambda\left(zf^{'}(z)\right)+(1+\lambda)f(z)\right]^{'}=f(z)*\frac{2(\lambda-1)z^2+3z-1}{z(1-z)^3}. \tag{2.6}$$

Using (2.5) and (2.6) in (2.3), we get

$$z[f(z)*\{-(1+\lambda)(1-z)[\gamma(1+m)e^{i\theta}+\left([\gamma(1+m)-m]e^{i\theta}+1\right)z] \tag{2.7}$$

$$+\lambda(1-z)\gamma(1+m)e^{i\theta}-2\lambda(1-me^{i\theta})z^2+2(1-z)\gamma(1+m)e^{i\theta}z\}/z(1-z)^3]\neq 0.$$

The left hand side of (2.7) may be written as

$$z[f(z)*\{-(1+\lambda)(1-z)[\gamma(1+m)e^{i\theta}+\left(1-me^{i\theta}-\gamma(1+m)e^{i\theta}\right)z]$$
$$+\lambda\gamma(1+m)e^{i\theta}-3\lambda\gamma(1+m)e^{i\theta}z-2\lambda\left(1-me^{i\theta}\right)z^2+2\lambda\gamma(1+m)e^{i\theta}z^2]\}/z(1-z)^3\ ].$$

Equation (2.7) can be rewritten in the form

$$z\left[f(z)*\left\{(1+\lambda)\frac{\left(\frac{e^{-i\theta}-m}{\gamma(1+m)}-1\right)z+1}{z(1-z)^2}+\lambda\frac{2(\frac{e^{-i\theta}-m}{\gamma(1+m)}-1)z^2+3z-1}{z(1-z)^3}\right\}\right]\neq 0$$

where $z\in\mathbb{E},\ \theta\in[0,2\pi)$. Thus we have the first part of the proof.

(ii) Conversely, since (2.1) holds for $C=0$, then $\lambda z^2 f^{'}(z)+(1+\lambda)zf(z)\neq 0$ for all $z\in\mathbb{E}$, hence the function $\varphi(z)=-\frac{z\left[\lambda z f^{'}(z)+(1+\lambda)f(z)\right]^{'}}{\lambda z f^{'}(z)+(1+\lambda)f(z)}$ is analytic in $\mathbb{E}$ (i.e. it is regular at $z_0=0$, with $\varphi(0)=1$). Since (2.7) is equivalent to (2.1), we have

$$-\frac{z\left[\lambda z f^{'}(z)+(1+\lambda)f(z)\right]^{'}}{\lambda z f^{'}(z)+(1+\lambda)f(z)}\neq\frac{[\gamma(1+m)-m]e^{i\theta}+1}{1-me^{i\theta}}\quad (z\in\mathbb{E},\ \theta\in[0,2\pi)). \tag{2.8}$$

Assume that

$$\varphi(z)=-\frac{z\left[\lambda z f^{'}(z)+(1+\lambda)f(z)\right]^{'}}{\lambda z f^{'}(z)+(1+\lambda)f(z)},\ \psi(z)=\frac{[\gamma(1+m)-m]e^{i\theta}+1}{1-me^{i\theta}}.$$

The relation (2.8) means that $\varphi(\mathbb{E})\cap\psi(\partial\mathbb{E})=\emptyset$. Thus, the simply connected domain $\varphi(\mathbb{E})$ is included in a connected component of $\mathbb{C}\backslash\psi(\partial\mathbb{E})$. From this, using the fact that $\varphi(0)=\psi(0)$ and the univalence of the function ψ, it follows that $\varphi(z)\prec\psi(z)$, this implies that $f(z)\in\mathcal{S}^*(\lambda,\gamma,M)$. Thus the proof of Theorem 1 is completed. □

Remark 2.

(i) Taking $\lambda=0$ in Theorem 1, we obtain the result obtained by Aouf [4, Theorem 2.1].

(ii) Taking $\lambda=-1$ in Theorem 1, we obtain the result obtained by Aouf [4, Theorem 2.3].

(iii) Taking $\lambda=0$ and $m=1$ in Theorem 1, we obtain the result obtained by Bulboacă et al. [6, Theorem 1, with $A=1$ and $B=-1$] and Aouf et al. [3, Theorem 4, with $\lambda=0,\ A=1$ and $B=-1$].

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(iv) Taking $\lambda = 0,\ \gamma = m = 1$ and $e^{i\theta} = x$ in Theorem 1, we obtain the result obtained by Ponnusamy [12, Theorem 4, with $\lambda = 0,\ A = 1$ and $B = -1$].
(iiv) Taking $\lambda = 0,\ m = 1,\ \gamma = (1-\alpha)e^{-i\mu}\cos\mu$ ($\mu \in \mathbb{R},\ |\mu| \leq \frac{\pi}{2},\ 0 \leq \alpha < 1$) and $e^{i\theta} = x$ in Theorem 1, we obtain the result obtained by Ravichandran et al. [13, Theorem 1.2 with $p = 1$].

Theorem 2. *A necessary and sufficient condition for the function $f(z)$ defined by (1.1) to be in the class $\mathcal{S}^*(n, \lambda, \gamma, M)$ is that*

$$1 + \sum_{k=0}^{\infty} [\mu(k+1)+1]^n \, [\lambda(k+1)+1] a_k z^{k+1} \neq 0 \tag{2.9}$$

and

$$1 + \sum_{k=0}^{\infty} [\mu(k+1)+1]^n \left[\tfrac{(k+1)[e^{-i\theta}-m]+\gamma(1+m)}{\gamma(1+m)}\right] [(1+\lambda+\lambda k] a_k z^{k+1} \neq 0 \tag{2.10}$$

for all $\theta \in [0, 2\pi)$ and $z \in \mathbb{E}$.

Proof. From Theorem 1, we have $f(z) \in \mathcal{S}^*(n, \lambda, \gamma, M)$ if and only if
(2.11)

$$z\left[D^n f(z) * \left\{(1+\lambda)\frac{(C-1)\,z+1}{z(1-z)^2} + \lambda\frac{2(C-1)z^2+3z-1}{z(1-z)^3}\right\}\right] \neq 0 \ (z \in \mathbb{E})$$

for all $C = C_\theta = \frac{e^{-i\theta}-m}{\gamma(1+m)},\quad (0 \leq \theta < 2\pi)$, and also for $C = 0$. From (1.15) and the equations

$$\frac{1}{z\,(1-z)} = z^{-1} + \sum_{k=0}^{\infty} z^k, \quad \frac{1}{z(1-z)^2} = z^{-1} + \sum_{k=0}^{\infty} (k+2)\, z^k, \tag{2.12}$$

it is not difficult to show that (2.10) holds for $C = 0$ iff (2.9) satisfied. The left hand side of (2.11) may be written as
(2.13)

$$z\left[D^n f(z) * \left\{(1+\lambda)\left[\frac{1-C}{z\,(1-z)} + \frac{C}{z(1-z)^2}\right] + \lambda\left[\frac{2C}{z(1-z)^3} - \frac{4C-1}{z(1-z)^2} + \frac{2(C-1)}{z\,(1-z)}\right]\right\}\right].$$

Using (1.15), (2.12) and the formula

$$\frac{1}{z(1-z)^3} = z^{-1} + \sum_{k=0}^{\infty} \frac{(k+2)\,(k+3)}{2} z^k$$

Equation (2.13) can be written as

$$1 + \sum_{k=0}^{\infty} [\mu(k+1)+1]^n \left[\tfrac{(k+1)[e^{-i\theta}-m]+\gamma(1+m)}{\gamma(1+m)}\right] [1+\lambda+\lambda k] a_k z^{k+1}.$$

Thus, the proof of Theorem 2 is completed. □

Theorem 3. *If the function $f(z)$ given by (1.1) and satisfy the inequality*

$$\sum_{k=0}^{\infty} (k+1+|\gamma|)[\lambda(k+1)+1][\mu(k+1)+1]^n \, |a_k| \leq |\gamma| \tag{2.14}$$

then $f(z) \in \mathcal{S}^*(n, \lambda, \gamma, M)$.

Proof. Since

$$\left|\frac{(k+1)[e^{-i\theta}-m]+\gamma(1+m)}{\gamma(1+m)}\right| \leq \frac{(k+1+|\gamma|)}{|\gamma|}$$

then

$$\begin{aligned} &\left|1+\sum_{k=0}^{\infty}[\mu(k+1)+1]^n\left[\frac{(k+1)[e^{-i\theta}-m]+\gamma(1+m)}{\gamma(1+m)}\right][\lambda(k+1)+1]a_k z^{k+1}\right| \\ \geq\ & 1-\sum_{k=0}^{\infty}[\mu(k+1)+1]^n\left|\frac{(k+1)[e^{-i\theta}-m]+\gamma(1+m)}{\gamma(1+m)}\right|[\lambda(k+1)+1]\,|a_k|\,|z|^{k+1} \\ \geq\ & 1-\sum_{k=0}^{\infty}\frac{(k+1+|\gamma|)}{|\gamma|}[\lambda(k+1)+1]\,[\mu(k+1)+1]^n\,|a_k|>0 \qquad (z\in\mathbb{E})\,. \end{aligned}$$

Which implies that inequality (2.14). Thus this completes the proof of Theorem 3. □

Theorem 4. *For $\lambda \in \mathbb{C}$, we have $\mathcal{S}^*(n+1,\lambda,\gamma,M) \subset \mathcal{S}^*(n,\lambda,\gamma,M)$.*

Proof. If $f(z) \in \mathcal{S}^*(n+1,\lambda,\gamma,M)$, then Theorem 2 gives

$$1+\sum_{k=0}^{\infty}[\mu(k+1)+1]^{n+1}[\lambda(k+1)+1]a_k z^{k+1} \neq 0$$

and

$$1+\sum_{k=0}^{\infty}[\mu(k+1)+1]^{n+1}\left[\frac{(k+1)[e^{-i\theta}-m]+\gamma(1+m)}{\gamma(1+m)}\right][\lambda(k+1)+1]\,a_k z^{k+1} \neq 0 \tag{2.15}$$

we can write (2.15) as

$$\left[1+\sum_{k=0}^{\infty}[\mu(k+1)+1]\,z^{k+1}\right]*\left[1+\sum_{k=0}^{\infty}\frac{(k+1)[e^{-i\theta}-m]+\gamma(1+m)}{\gamma(1+m)}\,[\mu(k+1)+1]^n\,[\lambda(k+1)+1]\,a_k z^{k+1}\right] \neq 0. \tag{2.16}$$

But

$$\left[1+\sum_{k=0}^{\infty}[\mu(k+1)+1]\,z^{k+1}\right]*\left[1+\sum_{k=0}^{\infty}\frac{1}{[\mu(k+1)+1]}z^{k+1}\right]=1+\sum_{k=0}^{\infty}z^{k+1}. \tag{2.17}$$

By using the property, if $f \neq 0$ and $g*h \neq 0$, then $f*(g*h) \neq 0$, (2.16) can be written as

$$1+\sum_{k=0}^{\infty}[\mu(k+1)+1]^n\left[\frac{(k+1)[e^{-i\theta}-m]+\gamma(1+m)}{\gamma(1+m)}\right][\lambda(k+1)+1]\,a_k z^{k+1} \neq 0. \tag{2.18}$$

In view of Theorem 2, we conclude that $f(z) \in \mathcal{S}^*(n,\lambda,\gamma,M)$. □

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ON A GENERALIZED DEGENERATE λ-q-DAEHEE NUMBERS AND POLYNOMIALS

JIN-WOO PARK

ABSTRACT. In [4], Daehee numbers and polynomials are introduced by T. Kim et al. In this paper, we consider the generalized λ-q-Daehee polynomials by using the bosonic p-adic q-integral and give some relations between the generalized λ-q-Daehee polynomials and special polynomials.

1. Introduction

Let d be fixed positive integer and let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completions of algebraic closure of $\mathbb{Q}_p$. The p-adic norm is defined $|p|_p = \frac{1}{p}$.

We set

$$X = X_d = \varprojlim_N \mathbb{Z}/dp^N\mathbb{Z}, \ X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p),$$

$$a + dp^N\mathbb{Z}_p = \left\{x \in X | x \equiv a \pmod{dp^N}\right\},$$

where $a \in \mathbb{Z}$ and $0 \le a < dp^n$.

When one talks of q-extension, q is various considered as an indeterminate, a complex $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the *p-adic bosonic integral on* $\mathbb{Z}_p$ is defined by Kim to be

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \ \text{(see [8, 9, 10])}. \tag{1.1}$$

If we put $f_n(x) = f(x+n)$, then, by (1.1), we can derive the following very useful integral identity;

$$q^n I_q(f_n) - I_q(f) = (q-1)\sum_{j=0}^{n-1} q^j f(j) + \frac{q-1}{\log q} \sum_{j=0}^{n-1} f'(j) q^j, \tag{1.2}$$

where $f'(0) = \left.\frac{df(x)}{dx}\right|_{x=0}$.

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As is well known, the *generalized q-Bernoulli numbers* $B_{n,\chi,q}$ *attached to* χ are defined by the generating function to be

$$\frac{q-1+\frac{q-1}{\log q}t}{q^d e^{dt}-1}\sum_{a=0}^{d-1}\chi(a)q^a e^{at}=\sum_{n=0}^{\infty}B_{n,\chi,q}\frac{t^n}{n!},\ \text{(see [3, 7, 11, 12, 17, 18])}.$$

The Stirling numbers of the first kind is given by

$$(x)_n=x(x-1)\cdots(x-n+1)=\sum_{l=0}^{n}S_1(n,l)x^l\ (x\geq 0),$$

and the Stirling numbers of the second kind is defined by the generating function to be

$$(e^t-1)^n=n!\sum_{l=n}^{\infty}S_2(l,n)\frac{t^l}{l!}$$

(see [18, 16]). Note that

$$(\log(x+1))^n=n!\sum_{l=n}^{\infty}S_1(l,n)\frac{x^l}{l!},\ (n\geq 0),$$

(see [18, 16]).

Recently, q-Daehee numbers and polynomials are introduced by Kim et. al. in [8], and have been studied by many mathematicians, and possess many interesting properties (see [1, 4-6, 13-15]). In this paper, we consider the generalized λ-q-Daehee polynomials and numbers by using the bosonic p-adic q-integral, and give some relations between the generalized λ-q-Daehee numbers and polynomials and special numbers and polynomials.

2. The generalized degenerate λ-q-Daehee polynomials attached to χ

From now on, we assume that $t\in\mathbb{C}$ with $|t|_p<p^{-\frac{1}{p-1}}$ and $u,\ \lambda\in\mathbb{Z}_p$. Let χ be the Dirichlet character with conductor $d\in\mathbb{N}=\{1,2,\ldots\}$ with $d\equiv 1\pmod 2$.

The *generalized degenerate* λ*-q-Daehee polynomials* $D_{n,\chi,\lambda,q}(x)$ *attached to* χ are defined by the generating function to be

$$\begin{aligned}&\frac{q-1+\frac{q-1}{\log q}\lambda\log\left(1+\frac{1}{u}\log(1+ut)\right)}{q^d\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda d}-1}\sum_{j=0}^{d-1}\chi(j)q^j\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda j+x}\\&=\sum_{n=0}^{\infty}D_{n,\chi,\lambda,q}(x|u)\frac{t^n}{n!},\end{aligned}\tag{2.1}$$

where $t\in\mathbb{C}_p$ and $|ut|_p<p^{-\frac{1}{p-1}}$. In the special case $x=0$, $D_{n,\chi,\lambda,q}(0|u)=D_{n,\chi,\lambda,q}(u)$ are called *generalized degenerate* λ*-q-Daehee numbers attached to* χ.

By replacing t by $\frac{1}{u}\left(e^{u(e^t-1)}-1\right)$ in (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}D_{n,\chi,\lambda,q}(u)\frac{\left(\frac{1}{u}\left(e^{u(e^t-1)}-1\right)\right)^n}{n!}&=\frac{q-1+\frac{q-1}{\log q}\lambda t}{q^d e^{\lambda dt}-1}\sum_{j=0}^{d-1}\chi(j)q^j e^{\lambda jt}\\&=\sum_{m=0}^{\infty}\lambda^m B_{m,\chi,q}^{(r)}\frac{t^m}{m!},\end{aligned}\tag{2.2}$$

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and

$$\begin{aligned}
&\sum_{n=0}^{\infty} D_{n,\chi,\lambda,q}(u)\frac{\left(\frac{1}{u}\left(e^{u(e^t-1)}-1\right)\right)^n}{n!} \\
=&\sum_{n=0}^{\infty} \frac{D_{n,\chi,\lambda,q}(u)}{n!}n!\sum_{m=n}^{\infty} S_2(m,n)\frac{1}{m!}\left(\frac{1}{u}\left(e^{u(e^t-1)}-1\right)\right)^m \\
=&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{k}\sum_{k=0}^{m}\sum_{n=0}^{m} D_{k,\chi,\lambda,q}(u)u^{n-l-k}S_2(m,k)S_2(n,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.3}$$

Therefore, by (2.2) and (2.3), we obtain the following theorem.

Theorem 2.1. *For $m \geq 0$, we have*

$$\lambda^n B_{n,\chi,q} = \sum_{m=0}^{k}\sum_{k=0}^{m}\sum_{n=0}^{m} D_{k,\chi,\lambda,q}(u)u^{n-l-k}S_2(m,k)S_2(n,m).$$

If taking $f(x) = \chi(x)\left(1+\frac{1}{u}\log(1+ut)\right)^x$ in (1.2), we can have

$$\begin{aligned}
&q^d\int_X \chi(x)\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda(x+d)}d\mu_q(x) + \int_X \chi(x)\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda x}d\mu_q(x) \\
=&\left(q-1+\frac{q-1}{\log q}\lambda\log\left(1+\frac{1}{u}\log(1+ut)\right)\right)\sum_{j=0}^{d-1}\chi(j)q^j\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda j}.
\end{aligned} \tag{2.4}$$

By (2.4), we can easily have

$$\begin{aligned}
&\int_X \chi(x)\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda x}d\mu_q(x) \\
=&\frac{q-1+\frac{q-1}{\log q}\lambda\log\left(1+\frac{1}{u}\log(1+ut)\right)}{q^d\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda d}-1}\sum_{j=0}^{d-1}\chi(j)q^j\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda j} \\
=&\sum_{n=0}^{\infty} D_{n,\chi,\lambda,q}(u)\frac{t^n}{n!},
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
&\int_X \chi(x)\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda x}d\mu_q(x) \\
=&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} u^{n-m}S_1(n,m)\int_X \chi(x)(\lambda x)_m d\mu_q(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.6}$$

Therefore, by (2.5) and (2.6), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$D_{n,\chi,\lambda,q}(u) = \sum_{m=0}^{n} u^{n-m}S_1(n,m)\int_X \chi(x)(\lambda x)_m d\mu_q(x).$$

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By (2.1), we note that

$$\begin{aligned}
&\sum_{n=0}^{\infty} D_{n,\chi,\lambda,q}(x|u)\frac{t^n}{n!}\\
=&\frac{q-1+\frac{q-1}{\log q}\lambda\log\left(1+\frac{1}{u}\log(1+ut)\right)}{q^d\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda d}-1}\sum_{j=0}^{d-1}\chi(j)q^j\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda j+x}\\
=&\left(\sum_{m=0}^{\infty} D_{m,\chi,\lambda,q}(u)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{x}{l}u^{n-l}l!S_1(n,l)\frac{t^n}{n!}\right)\\
=&\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{l=0}^{n-m}\binom{x}{l}\binom{n}{m}u^{n-m-l}l!S_1(n-m,l)D_{m,\chi,\lambda,q}(u)\frac{t^n}{n!}.
\end{aligned}\tag{2.7}$$

So, by (2.7), we can have

$$D_{n,\chi,\lambda,q}(x|u)=\sum_{m=0}^{n}\sum_{l=0}^{n-m}\binom{x}{l}\binom{n}{m}u^{n-m-l}l!S_1(n-m,l)D_{m,\chi,\lambda,q}(u).\tag{2.8}$$

For $r\in\mathbb{N}$, let us consider the *generalized degenerate λ-q-Daehee numbers of order r attached to χ* as follows:

$$\begin{aligned}
&\left(\frac{q-1+\frac{q-1}{\log q}\lambda\log\left(1+\frac{1}{u}\log(1+ut)\right)}{q^d\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda d}-1}\sum_{j=0}^{d-1}\chi(j)q^j\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda j}\right)^r\\
=&\left(\frac{q-1+\frac{q-1}{\log q}\lambda\log\left(1+\frac{1}{u}\log(1+ut)\right)}{q^d\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda d}-1}\right)^r\\
&\times\sum_{a_1,\dots,a_r=0}^{d-1}\chi(a_1)\cdots\chi(a_r)q^{a_1+\cdots+a_r}\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda(a_1+\cdots+a_r)}\\
=&\sum_{n=0}^{\infty} D^{(r)}_{n,\chi,\lambda,q}(u)\frac{t^n}{n!}.
\end{aligned}\tag{2.9}$$

By (2.5), we can see that

$$\begin{aligned}
&\int_X\cdots\int_X\chi(x_1)\cdots\chi(x_r)\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda(x_1+\cdots+x_r)}d\mu_q(x_1)\cdots d\mu_q(x_r)\\
=&\left(\frac{q-1+\frac{q-1}{\log q}\lambda\log\left(1+\frac{1}{u}\log(1+ut)\right)}{q^d\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda d}-1}\right)^r\\
&\times\sum_{a_1,\dots,a_r=0}^{d-1}\chi(a_1)\cdots\chi(a_r)q^{a_1+\cdots+a_r}\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda(a_1+\cdots+a_r)}.
\end{aligned}\tag{2.10}$$

Thus, by (2.9) and (2.10), we get

$$\begin{aligned}
&D^{(r)}_{n,\chi,\lambda,q}(u)\\
&=\sum_{m=0}^{n} u^{n-m}S_1(n,m)\int_X\cdots\int_X \chi(x_1)\cdots\chi(x_r)(\lambda x_1+\cdots+\lambda x_r)_m d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&=\sum_{m=0}^{n} u^{n-m}S_1(n,m)\int_X\cdots\int_X \chi(x_1)\cdots\chi(x_r)\sum_{l=0}^{m}S_1(l,m)\lambda^l(x_1+\cdots+x_r)^l d\mu_q(x_1)\cdots d\mu_q(x_r)\\
&=\sum_{m=0}^{n}\sum_{l=0}^{m} u^{n-m}\lambda^l S_1(n,m)S_1(l,m)B^{(r)}_{l,\chi,q},
\end{aligned} \tag{2.11}$$

where $B^{(r)}_{l,\chi,q}$ are the l-th generalized q-Bernoulli numbers of order r attached to χ, given by

$$\left(\sum_{a=0}^{d-1}\frac{1-q+\frac{1-q}{\log q}t}{1-q^d e^{td}}\chi(a)q^a e^{at}\right)^r=\sum_{n=0}^{\infty}B^{(r)}_{n,\chi,q}\frac{t^n}{n!} \text{ (see [11])}.$$

Therefore, by (2.11), we obtain the following theorem.

Theorem 2.3. *For $n\geq 0$, we have*

$$D^{(r)}_{n,\chi,\lambda,q}(u)=\sum_{m=0}^{n}\sum_{l=0}^{m} u^{n-m}\lambda^l S_1(n,m)S_1(l,m)B^{(r)}_{l,\chi,q}.$$

By replacing t by $\frac{1}{u}\left(e^{u(e^t-1)}-1\right)$ in (2.9), we can get

$$\begin{aligned}
&\sum_{n=0}^{\infty}D^{(r)}_{n,\chi,\lambda,q}(u)\frac{\left(\frac{1}{u}\left(e^{u(e^t-1)}-1\right)\right)^n}{n!}\\
&=\sum_{a_1,\dots,a_r=0}^{d-1}\left(\frac{q-1+\frac{q-1}{\log q}\lambda t}{q^d e^{\lambda dt}-1}\right)^r\chi(a_1)\cdots\chi(a_r)q^{a_1+\cdots+a_r}e^{\lambda(a_1+\cdots+a_r)t} \qquad (2.12)\\
&=\sum_{m=0}^{\infty}\lambda^m B^{(r)}_{m,\chi,q}\frac{t^m}{m!},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=0}^{\infty}D^{(r)}_{n,\chi,\lambda,q}(u)\frac{\left(\frac{1}{u}\left(e^{u(e^t-1)}-1\right)\right)^n}{n!}\\
&=\sum_{n=0}^{\infty}\frac{D^{(r)}_{n,\chi,\lambda,q}(u)}{n!}u^{-n}n!\sum_{m=n}^{\infty}S_2(m,n)\frac{1}{m!}\left(u(e^t-1)\right)^m \qquad (2.13)\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{s=0}^{m}D^{(r)}_{n,\chi,\lambda,q}(u)u^{m-s}S_2(m,s)S_2(n,m)\right)\frac{t^n}{n!}.
\end{aligned}$$

Therefore, by (2.12) and (2.13), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$\lambda^m B_{m,\chi,q}^{(r)} = \sum_{m=0}^{n}\sum_{s=0}^{m} D_{n,\chi,\lambda,q}^{(r)}(u)u^{m-s}S_2(m,s)S_2(n,m).$$

From (2.9), we can consider the *generalized λ-q-Daehee polynomials of order r attached to χ* as follows:

$$\begin{aligned}
&\left(\frac{q-1+\frac{q-1}{\log q}\lambda\log\left(1+\frac{1}{u}\log(1+ut)\right)}{q^d\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda d}-1}\sum_{j=0}^{d-1}\chi(j)q^j\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda j}\right)^r \\
&\times\left(1+\frac{1}{u}\log(1+ut)\right)^x \\
=&\left(\frac{q-1+\frac{q-1}{\log q}\lambda\log\left(1+\frac{1}{u}\log(1+ut)\right)}{q^d\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda d}-1}\right)^r \\
&\times\sum_{a_1,\dots,a_r=0}^{d-1}\chi(a_1)\cdots\chi(a_r)q^{a_1+\cdots+a_r}\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda(a_1+\cdots+a_r)+x} \\
=&\sum_{n=0}^{\infty}D_{n,\chi,\lambda,q}^{(r)}(x|u)\frac{t^n}{n!}
\end{aligned} \tag{2.14}$$

By (2.14),

$$\begin{aligned}
&\int_X\cdots\int_X\chi(x_1)\cdots\chi(x_r)\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda x_1+\cdots+\lambda x_r+x}d\mu_q(x_1)\cdots d\mu_q(x_r) \\
=&\sum_{a_1,\dots,a_r=0}^{d-1}\left(\frac{q-1+\frac{q-1}{\log q}\lambda\log\left(1+\frac{1}{u}\log(1+ut)\right)}{q^d\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda d}-1}\right)^r \\
&\times\chi(a_1)\cdots\chi(a_r)q^{a_1+\cdots+a_r}\left(1+\frac{1}{u}\log(1+ut)\right)^{\lambda(a_1+\cdots+a_r)+x},
\end{aligned} \tag{2.15}$$

and so, from (2.14) and (2.15)

$$\begin{aligned}
&D_{n,\chi,\lambda,q}^{(r)}(x|u) \\
=&\sum_{m=0}^{n}u^{n-m}S_1(n,m)\int_X\cdots\int_X\chi(x_1)\cdots\chi(x_r)(\lambda x_1+\cdots+\lambda x_r+x)_m d\mu_q(x_1)\cdots d\mu_q(x_r) \\
=&\sum_{m=0}^{n}u^{n-m}S_1(n,m) \\
&\times\int_X\cdots\int_X\chi(x_1)\cdots\chi(x_r)\sum_{l=0}^{m}S_1(l,m)\lambda^l(\lambda x_1+\cdots+\lambda x_r+x)^l d\mu_q(x_1)\cdots d\mu_q(x_r) \\
=&\sum_{m=0}^{n}\sum_{l=0}^{m}u^{n-m}\lambda^l S_1(n,m)S_1(l,m)B_{l,\chi,q}^{(r)}\left(\frac{x}{\lambda}\right).
\end{aligned} \tag{2.16}$$

Therefore, by (2.16), we obtain the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$D^{(r)}_{n,\chi,\lambda,q}(x|u) = \sum_{m=0}^{n}\sum_{l=0}^{m} u^{n-m}\lambda^l S_1(n,m)S_1(l,m)B^{(r)}_{l,\chi,q}\left(\frac{x}{\lambda}\right).$$

In (2.14), by replacing t by $\frac{1}{u}\left(e^{u(e^t-1)}-1\right)$, we can get

$$\begin{aligned}&\sum_{n=0}^{\infty} D^{(r)}_{n,\chi,\lambda,q}(x|u)\frac{\left(\frac{1}{u}\left(e^{u(e^t-1)}-1\right)\right)^n}{n!}\\ &= \sum_{a_1,\ldots,a_r=0}^{d-1}\left(\frac{q-1+\frac{q-1}{\log q}\lambda t}{q^d e^{\lambda dt}-1}\right)^r \chi(a_1)\cdots\chi(a_r)q^{a_1+\cdots+a_r}e^{\lambda(a_1+\cdots+a_r)t+xt} \qquad (2.17)\\ &= \sum_{m=0}^{\infty}\lambda^m B^{(r)}_{m,\chi,q}\left(\frac{x}{\lambda}\right)\frac{t^m}{m!},\end{aligned}$$

and

$$\begin{aligned}&\sum_{n=0}^{\infty} D^{(r)}_{n,\chi,\lambda,q}(x|u)\frac{\left(\frac{1}{u}\left(e^{u(e^t-1)}-1\right)\right)^n}{n!}\\ &=\sum_{n=0}^{\infty}\frac{D^{(r)}_{n,\chi,\lambda,q}(x|u)}{n!}u^{-n}n!\sum_{m=n}^{\infty}S_2(m,n)\frac{1}{m!}\left(u(e^t-1)\right)^m \qquad (2.18)\\ &=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{s=0}^{m}D^{(r)}_{n,\chi,\lambda,q}(x|u)u^{m-s}S_2(m,s)S_2(n,m)\right)\frac{t^n}{n!}.\end{aligned}$$

Therefore, by (2.17) and (2.18), we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, we have*

$$\lambda^m B^{(r)}_{m,\chi,q}\left(\frac{x}{\lambda}\right) = \sum_{m=0}^{n}\sum_{s=0}^{m}D^{(r)}_{n,\chi,\lambda,q}(x|u)u^{m-s}S_2(m,s)S_2(n,m).$$

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On the $m-$extension of Fibonacci $p-$functions with period k

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Abstract

Let $f_{p,m}$ be a real valued function on $\mathbb{R}$, p be nonnegative integer, k be a positive integer and m be a nonnegative real number. For all $x \in \mathbb{R}$, $f_{p,m}(x + (p + 1)k) = mf_{p,m}(x + pk) + f_{p,m}(x)$, we call this function $m-$extension of Fibonacci $p-$function with period k. In this paper, we present basic properties of $m-$extension of Fibonacci $p-$functions with period k. Specifying p and m, we obtain Fibonacci ($p = 1$, $m = 1$) and Pell ($p = 1$, $m = 2$) functions. Furthermore, we define $m-$extension of odd Fibonacci $p-$functions with period k. Moreover, we analyze some properties by using notion of $f-$even and $f-$odd functions with period k. We also demonstrate the products and quotients of these functions and provide new results in the development of Fibonacci functions with period k.

Keywords: $m-$extension of Fibonacci $p-$function with period k, $m-$extension of odd Fibonacci $p-$function with period k, $f-$even function with period k, $f-$odd functions with period k.
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1. Introduction

Fibonacci numbers is one of the most popular and fascinating linear sequences in mathematics and related fields. The classical Fibonacci sequence is defined by $F_{n+2} = F_{n+1} + F_n$, for $n \in \mathbb{N}$, with initial conditions $F_0 = 0, \quad F_1 = 1$. Up until now, many authors have studied the sums, representations, properties, relations with another mathematical topics, applications and generalizations of the Fibonacci sequence extensively (see [1–15]). Falcon introduced kth Fibonacci numbers $\{F_{k,n}\}_{n=0}^{\infty}$ that arises in the study of the recursive application of two geometrical transformations used in the well known four triangle longest edge (4TLE) partition[2]. In [7], Yazlik and Taskara defined generalized $k-$Horadam sequence and proved the properties of this sequence by means of determinant. Stakhov and Rozin presented, one of the important mathematical discoveries of the modern Golden Section and Fibonacci numbers theory, Fibonacci $p-$numbers and some properties of this sequence, $F_p(n) = F_p(n-1) + F_p(n-p-1)$, in [10]. Later on, the authors defined the

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$m-$extension of the Fibonacci $p-$numbers as

$$F_{p,m}(n+p+1) = mF_{p,m}(n+p) + F_{p,m}(n) \tag{1}$$

with initial conditions $F_{p,m}(0) = 0, F_{p,m}(1) = 1, F_{p,m}(2) = m, F_{p,m}(3) = m^2, \ldots, F_{p,m}(p+1) = m^p$, where $p, n \in \mathbb{N}$ and m is positive real number. For different values of p and m in equation (1), it can be reduced into different numerical sequences. For example, if $(p,m) = (1,1)$, the Fibonacci sequence is obtained as $F_{n+2} = F_{n+1} + F_n$. If $(p,m) = (1,2)$, the Pell sequence is obtained as $P_{n+2} = 2P_{n+1} + P_n$. If $p = 1$ and $m = k$, the k-Fibonacci sequence is obtained as $F_{k,n+2} = kF_{k,n+1} + F_{k,n}$ [9]. Recently, one of the important application of these integer sequences is continuous functions. Han et al.,[16], considered Fibonacci functions on the real numbers $\mathbb{R}$, i.e., functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$, $\quad f(x+2) = f(x+1)+f(x)$. Also they presented some properties of these functions by using the concept of $f-$even and $f-$odd functions. Moreover, they showed that if f is Fibonacci function then $\lim\limits_{x\to\infty} \frac{f(x+1)}{f(x)} = \frac{1+\sqrt{5}}{2}$. Afterwards, Sroysang extended Fibonacci functions to Fibonacci functions with period k as $f(x+2k) = f(x+k)+f(x)$ for all $x \in \mathbb{R}$ in [17]. In [18], Rabago defined the second order linear recurrent function with period k, $w(x+2k) = rw(x+k)+sw(x)$, where r, s are nonnegative real numbers, which is generalization of the Fibonacci function with period k.

Up until now, authors investigated some properties of the continuous functions of the second order linear recursive integer sequences. In this paper, we extend these properties to the continuous function in terms of $m-$extension of Fibonacci $p-$numbers which is defined by the $(p+1)th$ order linear recursive relation. We present some properties of the $m-$extension of Fibonacci $p-$functions with period k using the concept of $f-$even and $f-$odd functions with period k. We also define $m-$extension of odd Fibonacci $p-$functions with period k, investigate the product and the limit of $m-$extension of Fibonacci $p-$functions with period k.

2. $m-$extension of Fibonacci $p-$functions with period k

In this section we define $m-$extension of Fibonacci $p-$functions with period k and present some properties of these functions.

Definition 2.1. *Let k be a positive integer, p be nonnegative integer and m be a nonnegative real number. A function $f_{p,m} : \mathbb{R} \to \mathbb{R}$ is called an $m-$extension of Fibonacci $p-$function with period k if it satisfies the equation*

$$f_{p,m}(x+(p+1)k) = mf_{p,m}(x+pk) + f_{p,m}(x), \quad \forall x \in \mathbb{R}. \tag{2}$$

Taking $(p,m) = (1,1)$ and $(p,m) = (1,2)$ in (2), we obtain Fibonacci and Pell function with period k, respectively (see [17, 18]).

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Example 2.1. Let α be the positive real number that satisfies the equation $\alpha^{p+1} = m\alpha^p + 1$, k be a positive integer, p be a nonnegative integer. Then, $f_{p,m}(x) = \alpha^{\frac{x}{k}}$ is an $m-$extension of Fibonacci $p-$function with period k.

The following are special cases of the previous example:

1. If $(p, m) = (1, 1)$ then the function $f_{1,1}(x) = \phi^{\frac{x}{k}}$, where $\phi = \frac{1+\sqrt{5}}{2}$ is known as golden ratio, is an example of $m-$extension of Fibonacci $p-$function with period k in [17].
2. If $(p, m) = (1, 2)$ then the function $f_{1,2}(x) = \sigma^{\frac{x}{k}}$, where $\sigma = 1 + \sqrt{2}$ is known as silver ratio, is an example of $m-$extension of Pell $p-$function with period k in [18].

Proposition 2.1. *Let p be a nonnegative integer, k be positive integer and $f_{p,m} : \mathbb{R} \to \mathbb{R}$ be an $m-$extension of Fibonacci $p-$function with period k. Assume that $f_{p,m}$ is s times differentiable. Then $\{f'_{p,m}, f''_{p,m}, \ldots, f^{(s)}_{p,m}\}$ are also $m-$extension of odd Fibonacci $p-$functions with period k.*

Proposition 2.2. *Let p be a nonnegative integer, k be positive integer and $f_{p,m} : \mathbb{R} \to \mathbb{R}$ be an $m-$extension of Fibonacci $p-$function with period k. Define $g_t(x) = f_{p,m}(x + t)$, for all $x \in \mathbb{R}$, where $t \in \mathbb{R}$. Then, $g_t(x)$ is also an $m-$extension of Fibonacci $p-$function with period k.*

Proof. Let $x \in \mathbb{R}$. Then,

$$\begin{aligned} g_t(x + (p+1)k) &= f_{p,m}(x + (p+1)k + t) \\ &= m f_{p,m}(x + pk + t) + f_{p,m}(x + t) \\ &= m g_t(x + pk) + g_t(x) \end{aligned}$$

is an $m-$extension of Fibonacci $p-$function with period k. □

Example 2.2. Let p be a nonnegative integer, k be positive integer and $t \in \mathbb{R}$. Define $g_t : \mathbb{R} \to \mathbb{R}$ by

$$g_t(x) = \alpha^{\frac{x+t}{k}}, \qquad \forall x \in \mathbb{R}, \tag{3}$$

then $g_t(x)$ is an $m-$extension of Fibonacci $p-$function with period k.

As special cases of the previous example, we have

1. If $(p, m) = (1, 1)$, then the function $g_t(x) = f_{1,1}(x + t) = \phi^{\frac{x+t}{k}}$ is an example of $m-$extension of Fibonacci $p-$function with period k in [17].
2. If $(p, m) = (1, 2)$, then the function $g_t(x) = f_{1,2}(x + t) = \sigma^{\frac{x}{k}}$ is an example of $m-$extension of Pell $p-$function with period k in [18].

Theorem 2.1. *Let $f_{p,m}$ be an $m-$extension of Fibonacci $p-$function with period k and $F_{p,m}$ be an $m-$extension of Fibonacci $p-$sequence with the initial conditions $F_{p,m}(0) = 0, F_{p,m}(1) = 1, F_{p,m}(2) = m, \dots, F_{p,m}(p) = m^{p-1}$. Then, for $n \geq 2p$ and $\forall x \in \mathbb{R}$,*

$$f_{p,m}(x + nk) = F_{p,m}(n - p + 1)f(x + pk) + \sum_{i=0}^{p-1} F_{p,m}(n - p - i)f(x + ik). \tag{4}$$

Proof. We prove the theorem by induction on n. For $n = 2p$, we get

$$\begin{aligned} f_{p,m}(x + 2pk) &= mf_{p,m}(x + (2p - 1)k) + f_{p,m}(x + (p - 1)k) \\ &= m\Big[mf_{p,m}(x + (2p - 2)k) + f_{p,m}(x + (p - 2)k)\Big] \\ &\quad +f_{p,m}(x + (p - 1)k) \\ &= m^2 f_{p,m}(x + (2p - 2)k) + f_{p,m}(x + (p - 1)k) \\ &\quad +mf_{p,m}(x + (p - 2)k) \\ &= m^3 f_{p,m}(x + (2p - 3)k) + f_{p,m}(x + (p - 1)k) \\ &\quad +mf_{p,m}(x + (p - 2)k) + m^2 f_{p,m}(x + (p - 3)k). \end{aligned}$$

Continuing this process $(p - 3)$ times, we have

$$\begin{aligned} f_{p,m}(x + 2pk) &= m^p f_{p,m}(x + pk) + f_{p,m}(x + (p - 1)k) \\ &\quad +mf_{p,m}(x + (p - 2)k) + \cdots + m^{p-1} f_{p,m}(x). \end{aligned}$$

By considering the initial conditions of the $m-$extension of Fibonacci $p-$sequence, we obtain

$$\begin{aligned} f_{p,m}(x + 2pk) &= F_{p,m}(p + 1)f_{p,m}(x + pk) + F_{p,m}(1)f_{p,m}(x + (p - 1)k) \\ &\quad +F_{p,m}(2)f_{p,m}(x + (p - 2)k) + \cdots + F_{p,m}(p - 1)f_{p,m}(x + k) \\ &\quad +F_{p,m}(p)f_{p,m}(x). \end{aligned}$$

Assume that equation (4) is true for $n \geq 2p+1$. Then we write

$$\begin{aligned}
f_{p,m}(x+(n+1)k) &= mf_{p,m}(x+nk)+f_{p,m}(x+(n-p)k) \\
&= m\Big[F_{p,m}(n-p+1)f_{p,m}(x+pk) \\
&\quad +F_{p,m}(n-2p+1)f_{p,m}(x+(p-1)k) \\
&\quad +\cdots+F_{p,m}(n-p)f_{p,m}(x)\Big] \\
&\quad +F_{p,m}(n-2p+1)f_{p,m}(x+pk) \\
&\quad +F_{p,m}(n-3p+1)f_{p,m}(x+(p-1)k) \\
&\quad +\cdots+F_{p,m}(n-2p)f_{p,m}(x) \\
&= (mF_{p,m}(n-p+1)+F_{p,m}(n-2p+1))f_{p,m}(x+pk) \\
&\quad +(mF_{p,m}(n-2p+1)+F_{p,m}(n-3p+1))f_{p,m}(x+(p-1)k) \\
&\quad +\cdots+(mF_{p,m}(n-p)+F_{p,m}(n-2p))f_{p,m}(x) \\
&= F_{p,m}(n-p+2)f(x+pk)+\sum_{i=0}^{p-1}F_{p,m}(n+1-p-i)f(x+ik),
\end{aligned}$$

which completes the proof. □

Corollary 2.1. *Let $f_{p,m}$ be an $m-$extension of Fibonacci $p-$function with period k and $F_{p,m}$ be the sequence of $m-$extension of Fibonacci $p-$numbers. Then, for any $x \in \mathbb{R}$ and $n \geq 2p$,*

$$\alpha^n = F_{p,m}(n-p+1)\alpha^p + \sum_{i=0}^{p-1}F_{p,m}(n-p-i)\alpha^i. \tag{5}$$

Proof. From example (2.1), we say that $f_{p,m}(x)=\alpha^{\frac{x}{k}}$, k is a positive integer, is an $m-$extension of Fibonacci $p-$function with period k, so it satisfies the Equation(2), for all $x \in \mathbb{R}$, i.e.

$$\begin{aligned}
\alpha^{\frac{x+nk}{k}} &= f_{p,m}(x+nk) \\
&= F_{p,m}(n-p+1)f(x+pk)+\sum_{i=0}^{p-1}F_{p,m}(n-p-i)f(x+ik) \\
&= \alpha^{\frac{x}{k}+p}F_{p,m}(n-p+1)+\alpha^{\frac{x}{k}}F_{p,m}(n-p)+\alpha^{\frac{x}{k}+1}F_{p,m}(n-p-1) \\
&+ \alpha^{\frac{x}{k}+2}F_{p,m}(n-p-2)+\cdots+\alpha^{\frac{x}{k}+p-1}F_{p,m}(n-2p+1).
\end{aligned}$$

Upon simplifying, we get

$$\alpha^n = F_{p,m}(n-p+1)\alpha^p + \sum_{i=0}^{p-1}F_{p,m}(n-p-i)\alpha^i, \tag{6}$$

which is desired. □

3. $m-$extension of odd Fibonacci $p-$functions with period k

In this section, we present the $m-$extension of odd Fibonacci $p-$function with period k and analyze some properties of these functions.

Definition 3.1. *Let p be a nonnegative integer, m be a nonnegative real number and k be a positive integer. A function $f_{p,m} : \mathbb{R} \to \mathbb{R}$ is said to be $m-$extension of odd Fibonacci $p-$function with period k, if $f_{p,m}$ satisfies*

$$f_{p,m}(x + (p + 1)k) = -mf_{p,m}(x + pk) + f_{p,m}(x), \quad \forall x \in \mathbb{R}. \tag{7}$$

Example 3.1. Let α be the positive real number that satisfies the equation $\alpha^{p+1} = m\alpha^p + 1$, k be a positive integer, p be a nonnegative integer. Therefore $f_{p,m}(x) = \alpha^{\frac{x}{k}}$, for all $x \in \mathbb{R}$, is an $m-$extension of odd Fibonacci $p-$function with period k.

Proposition 3.1. *Let p be a nonnegative integer, k be positive integer and $f_{p,m} : \mathbb{R} \to \mathbb{R}$ be an $m-$extension of odd Fibonacci $p-$function with period k. Assume that $f_{p,m}$ is s times differentiable. Then $\{f'_{p,m}, f''_{p,m}, \dots, f^{(s)}_{p,m}\}$ are also $m-$extension of odd Fibonacci $p-$functions with period k.*

Proposition 3.2. *Let p be a nonnegative integer, k be positive integer and $f_{p,m} : \mathbb{R} \to \mathbb{R}$ be an $m-$extension of odd Fibonacci $p-$function with period k. Define $g_t(x) = f_{p,m}(x + t)$, for all $x \in \mathbb{R}$, where $t \in \mathbb{R}$. Then, g_t is also an $m-$extension of odd Fibonacci $p-$function with period k.*

Proof. Let $x \in \mathbb{R}$. Then,

$$\begin{aligned} g_t(x + (p + 1)k) &= f_{p,m}(x + (p + 1)k + t) \\ &= -mf_{p,m}(x + pk + t) + f_{p,m}(x + t) \\ &= -mg_t(x + pk) + g_t(x). \end{aligned}$$

Therefore, $g_t(x)$ is an $m-$extension of odd Fibonacci $p-$function with period k. □

4. Products of $m-$extension of Fibonacci $p-$functions with period k

In this section, we present the product of $m-$extension of Fibonacci $p-$functions with period k by using the concept of $f-$even and $f-$odd functions with period k which are defined in [16].

Definition 4.1 ([16]). *Let $k \in \mathbb{N}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be such that if $\varphi h \equiv 0$ where $h : \mathbb{R} \to \mathbb{R}$ is continuous, then $h \equiv 0$. The map φ is said to be an $f-$even and $f-$odd function with period k if $\varphi(x + k) = \varphi(x)$ and if $\varphi(x + k) = -\varphi(x)$, respectively, for any $x \in \mathbb{R}$.*

Theorem 4.1. *Let k be a positive integer, $\varphi : \mathbb{R} \to \mathbb{R}$ be an $f-$even function with period k and $f_{p,m} : \mathbb{R} \to \mathbb{R}$ be a continuous function.Then, $f_{p,m}$ is an $m-$extension of Fibonacci $p-$function with period k if and only if $(\varphi f_{p,m})$ is an $m-$extension of Fibonacci $p-$function with period k.*

Proof. First, we assume that $f_{p,m}$ is an $m-$extension of Fibonacci $p-$function with period k. For any $x \in \mathbb{R}$,

$$\begin{aligned} (\varphi f_{p,m})(x+(p+1)k) &= \varphi(x+(p+1)k)f_{p,m}(x+(p+1)k) \\ &= \varphi(x+(p+1)k)\Big[mf_{p,m}(x+pk)+f_{p,m}(x)\Big] \\ &= m\varphi(x+pk)f_{p,m}(x+pk)+\varphi(x)f_{p,m}(x) \\ &= m(\varphi f_{p,m})(x+pk)+(\varphi f_{p,m})(x). \end{aligned}$$

Therefore, $(\varphi f_{p,m})$ is an $m-$extension of Fibonacci $p-$function with period k. Next, assume that $(\varphi f_{p,m})$ is an $m-$extension of Fibonacci $p-$function with period k, then

$$\begin{aligned} \varphi(x+k)f_{p,m}(x+(p+1)k) &= \varphi(x+(p+1)k)f_{p,m}(x+(p+1)k) \\ &= (\varphi f_{p,m})(x+(p+1)k) \\ &= m(\varphi f_{p,m})(x+pk)+(\varphi f_{p,m})(x) \\ &= m\varphi(x+pk)f_{p,m}(x+pk)+\varphi(x)f_{p,m}(x) \\ &= \varphi(x+k)\Big[mf_{p,m}(x+pk)+f_{p,m}(x)\Big]. \end{aligned}$$

Thus, $f_{p,m}$ is an $m-$extension of Fibonacci $p-$function with period k. This completes the proof. □

Example 4.1. Let k be a positive integer and define $\gamma(x) = x - \lfloor x \rfloor$ which is an example of $f-$even function. Moreover, recall that the function $f_{p,m}(x) = \alpha^{\frac{x}{k}}$, where α is positive real root of the characteristic equation $\alpha^{p+1} - m\alpha^p - 1 = 0$, is an $m-$extension of Fibonacci $p-$function with period k. By using Theorem 4.1, for all $x \in \mathbb{R}$

$$(\gamma f_{p,m})(x) = (x - \lfloor x \rfloor)\alpha^{\frac{x}{k}} \tag{8}$$

is an example of an $m-$extension of Fibonacci $p-$function with period k.

Theorem 4.2. *Let k be a positive integer, $\varphi : \mathbb{R} \to \mathbb{R}$ be an $f-$even function with period k and $f_{p,m} : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then, $f_{p,m}$ is an $m-$extension of odd Fibonacci $p-$function with period k if and only if $(\varphi f_{p,m})$ is an $m-$extension of odd Fibonacci $p-$function with period k.*

Proof. First, assume that $f_{p,m}$ is an $m-$extension of odd Fibonacci $p-$function with period k, for any $x \in \mathbb{R}$

$$\begin{aligned} (\varphi f_{p,m})(x+(p+1)k) &= \varphi(x+(p+1)k)f_{p,m}(x+(p+1)k) \\ &= \varphi(x+(p+1)k)\left[-mf_{p,m}(x+pk)+f_{p,m}(x)\right] \\ &= -m\varphi(x+pk)f_{p,m}(x+pk)+\varphi(x)f_{p,m}(x) \\ &= -m(\varphi f_{p,m})(x+pk)+(\varphi f_{p,m})(x). \end{aligned}$$

Therefore, $(\varphi f_{p,m})$ is an $m-$extension of odd Fibonacci $p-$function with period k. Next, assume that $(\varphi f_{p,m})$ is an $m-$extension of odd Fibonacci $p-$function with period k, for any $x \in \mathbb{R}$, then

$$\begin{aligned}
\varphi(x+k)f_{p,m}(x+(p+1)k) &= \varphi(x+(p+1)k)f_{p,m}(x+(p+1)k) \\
&= (\varphi f_{p,m})(x+(p+1)k) \\
&= -m(\varphi f_{p,m})(x+pk) + (\varphi f_{p,m})(x) \\
&= -m\varphi(x+pk)f_{p,m}(x+pk) + \varphi(x)f_{p,m}(x) \\
&= \varphi(x+k)\left[-mf_{p,m}(x+pk) + f_{p,m}(x)\right].
\end{aligned}$$

Thus, $f_{p,m}$ is an $m-$extension of odd Fibonacci $p-$function with period k. This completes the proof. □

Example 4.2. Let k be a positive integer and define $\gamma(x) = x - \lfloor x \rfloor$ which is an example of $f-$even function [16]. Moreover, recall that the function $f_{p,m}(x) = \alpha^{\frac{x}{k}}$, where α is positive real root of the characteristic equation $\alpha^{p+1} + m\alpha^p - 1 = 0$, is an $m-$extension of odd Fibonacci $p-$function with period k. By using Theorem (4.2), for all $x \in \mathbb{R}$

$$(\gamma f_{p,m})(x) = (x - \lfloor x \rfloor)\alpha^{\frac{x}{k}} \tag{9}$$

is an example of an $m-$extension of odd Fibonacci $p-$function with period k.

Theorem 4.3. *Let k be a positive integer, f_{p,m_1} and f_{p,m_2} be two $m-$extension of Fibonacci $p-$functions with period k satisfying*

$$f_{p,m_1}(x+(p+1)k) = m_1 f_{p,m_1}(x+pk) + f_{p,m_1}(x), \quad \forall x \in \mathbb{R}$$
$$f_{p,m_2}(x+(p+1)k) = m_2 f_{p,m_2}(x+pk) + f_{p,m_2}(x), \quad \forall x \in \mathbb{R},$$

where m_1, m_2 are nonnegative real numbers. Suppose that the following conditions are satisfied:

(C1) f_{p,m_1} is an $f-$even function,

(C2) f_{p,m_2} is an $f-$odd function,

(C3) if p is odd then $m_1 = m_2$,

(C4) if p is even then $m_1 = -m_2$,

(C5) $\mu = m_1.m_2$.

Then $(f_{p,m_1} f_{p,m_2})(x)$ is also an $m-$extension of Fibonacci $p-$function with period k.

Proof. Assume that f_{p,m_1} and f_{p,m_2} be two $m-$extension of Fibonacci $p-$functions with period k and the conditions *(C1)*,*(C2)*,*(C3)*,*(C4)* and *(C5)* are satisfied. Then,

$$\begin{aligned}
(f_{p,m_1}f_{p,m_2})(x+(p+1)k) &= f_{p,m_1}(x+(p+1)k)f_{p,m_2}(x+(p+1)k) \\
&= [m_1 f_{p,m_1}(x+pk)+f_{p,m_1}(x)] \\
&\quad [m_2 f_{p,m_2}(x+pk)+f_{p,m_2}(x)] \\
&= m_1 m_2 f_{p,m_1}(x+pk)f_{p,m_2}(x+pk) \\
&\quad +f_{p,m_1}(x)f_{p,m_2}(x)+m_1 f_{p,m_1}(x+pk)f_{p,m_2}(x) \\
&\quad +m_2 f_{p,m_2}(x+pk)f_{p,m_1}(x) \\
&= m_1 m_2 f_{p,m_1}(x+pk)f_{p,m_2}(x+pk) \\
&\quad +f_{p,m_1}(x)f_{p,m_2}(x) \\
&= \mu(f_{p,m_1}f_{p,m_2})(x+pk)+(f_{p,m_1}f_{p,m_2})(x), \quad \forall x \in \mathbb{R}.
\end{aligned}$$

Thus, $(f_{p,m_1}f_{p,m_2})$ is an $m-$extension of Fibonacci $p-$function with period k. □

Theorem 4.4. *Let k be a positive integer, f_{p,m_1} be an $m-$extension of Fibonacci $p-$function with period k and f_{p,m_2} be an $m-$extension of odd Fibonacci $p-$function with period k satisfying*

$$f_{p,m_1}(x+(p+1)k) = m_1 f_{p,m_1}(x+pk)+f_{p,m_1}(x), \quad \forall x \in \mathbb{R}$$
$$f_{p,m_2}(x+(p+1)k) = -m_2 f_{p,m_2}(x+pk)+f_{p,m_2}(x), \quad \forall x \in \mathbb{R},$$

where m_1, m_2 are nonnegative real numbers. Suppose that (C6), (C9) and one the following conditions (C7) and (C8) are satisfied:

(C6) if p is odd or even then $m_1 = m_2$,

(C7) f_{p,m_1} and f_{p,m_2} are both $f-$even functions,

(C8) f_{p,m_1} and f_{p,m_2} are both $f-$odd functions,

(C9) $\mu = m_1 m_2$

Then, $(f_{p,m_1}f_{p,m_2})$ is also an $m-$extension of odd Fibonacci $p-$function with period k.

Proof. First assume that f_{p,m_1} is an $m-$extension of Fibonacci $p-$function with period k and f_{p,m_2} is an $m-$extension of odd Fibonacci $p-$function with period k and the conditions *(C6)*, *(C9)* and *(C7)* are satisfied. Then,

$$\begin{aligned}
(f_{p,m_1}f_{p,m_2})(x+(p+1)k) &= f_{p,m_1}(x+(p+1)k)f_{p,m_2}(x+(p+1)k)\\
&= \Big[m_1 f_{p,m_1}(x+pk)+f_{p,m_1}(x)\Big]\Big[-m_2 f_{p,m_2}(x+pk)+f_{p,m_2}(x)\Big]\\
&= -m_1m_2 f_{p,m_1}(x+pk)f_{p,m_2}(x+pk)\\
&\quad +f_{p,m_1}(x)f_{p,m_2}(x)+m_1 f_{p,m_1}(x+pk)f_{p,m_2}(x)\\
&\quad -m_2 f_{p,m_2}(x+pk)f_{p,m_1}(x)\\
&= -m_1m_2 f_{p,m_1}(x+pk)f_{p,m_2}(x+pk)\\
&\quad +f_{p,m_1}(x)f_{p,m_2}(x)\\
&= -\mu(f_{p,m_1}f_{p,m_2})(x+pk)+(f_{p,m_1}f_{p,m_2})(x),
\end{aligned}$$

$\forall x \in \mathbb{R}$. Therefore, $(f_{p,m_1}f_{p,m_2})$ is an $m-$extension of odd Fibonacci $p-$function with period k. Next, assume that f_{p,m_1} is an $m-$extension of Fibonacci $p-$function with period k and f_{p,m_2} is an $m-$extension of odd Fibonacci $p-$function with period k and the conditions *(C6)*, *(C9)* and *(C8)* are satisfied. Then,

$$\begin{aligned}
(f_{p,m_1}f_{p,m_2})(x+(p+1)k) &= f_{p,m_1}(x+(p+1)k)f_{p,m_2}(x+(p+1)k)\\
&= \Big[m_1 f_{p,m_1}(x+pk)+f_{p,m_1}(x)\Big]\Big[-m_2 f_{p,m_2}(x+pk)+f_{p,m_2}(x)\Big]\\
&= -m_1m_2 f_{p,m_1}(x+pk)f_{p,m_2}(x+pk)\\
&\quad +f_{p,m_1}(x)f_{p,m_2}(x)-m_1 f_{p,m_1}(x+pk)f_{p,m_2}(x)\\
&\quad +m_2 f_{p,m_2}(x+pk)f_{p,m_1}(x)\\
&= -m_1m_2 f_{p,m_1}(x+pk)f_{p,m_2}(x+pk)\\
&\quad +f_{p,m_1}(x)f_{p,m_2}(x)\\
&= -\mu(f_{p,m_1}f_{p,m_2})(x+pk)+(f_{p,m_1}f_{p,m_2})(x),
\end{aligned}$$

$\forall x \in \mathbb{R}$. Thus, $(f_{p,m_1}f_{p,m_2})$ is an $m-$extension of odd Fibonacci $p-$function with period k. This proves the theorem. □

Theorem 4.5. *Let k be a positive integer, f_{p,m_1} and f_{p,m_2} be two $m-$extension of Fibonacci $p-$functions with period k satisfying*

$$f_{p,m_1}(x+(p+1)k) = m_1 f_{p,m_1}(x+pk)+f_{p,m_1}(x), \quad \forall x \in \mathbb{R}$$
$$f_{p,m_2}(x+(p+1)k) = m_2 f_{p,m_2}(x+pk)+f_{p,m_2}(x), \quad \forall x \in \mathbb{R},$$

where m_1, m_2 are nonnegative real numbers. Suppose that (C10),(C11) and one the conditions (C7) and (C8) are satisfied:

(C10) if p is odd or even then $m_1 = -m_2$,

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(C11) $\mu = -m_1.m_2$.

Then, $(f_{p,m_1} f_{p,m_2})$ *is an* $m-$*extension of odd Fibonacci* $p-$*function with period* k.

Proof. First assume that f_{p,m_1} and f_{p,m_2} are $m-$extension of Fibonacci $p-$functions with period k and the conditions *(C7)*, *(C10)* and *(C11)* are satisfied. Then,

$$\begin{aligned}
(f_{p,m_1} f_{p,m_2})(x+(p+1)k) &= f_{p,m_1}(x+(p+1)k) f_{p,m_2}(x+(p+1)k) \\
&= \Big[m_1 f_{p,m_1}(x+pk) + f_{p,m_1}(x)\Big]\Big[m_2 f_{p,m_2}(x+pk) + f_{p,m_2}(x)\Big] \\
&= m_1 m_2 f_{p,m_1}(x+pk) f_{p,m_2}(x+pk) \\
&\quad + f_{p,m_1}(x) f_{p,m_2}(x) + m_1 f_{p,m_1}(x+pk) f_{p,m_2}(x) \\
&\quad + m_2 f_{p,m_2}(x+pk) f_{p,m_1}(x) \\
&= m_1 m_2 f_{p,m_1}(x+pk) f_{p,m_2}(x+pk) \\
&\quad + f_{p,m_1}(x) f_{p,m_2}(x) \\
&= -\mu (f_{p,m_1} f_{p,m_2})(x+pk) + (f_{p,m_1} f_{p,m_2})(x),
\end{aligned}$$

$\forall x \in \mathbb{R}$. Therefore, $(f_{p,m_1} f_{p,m_2})$ is an $m-$extension of odd Fibonacci $p-$function with period k. Next, assume that f_{p,m_1} and f_{p,m_2} are $m-$extension of Fibonacci $p-$functions with period k and the conditions *(C8)*, *(C10)* and *(C11)* are satisfied. Then the same result can be obtained. Therefore, $(f_{p,m_1} f_{p,m_2})$ is an $m-$extension of odd Fibonacci $p-$function with period k. □

Theorem 4.6. *Let* k *be a positive integer,* f_{p,m_1} *and* f_{p,m_2} *be two* $m-$*extension of odd Fibonacci* $p-$*functions with period* k *satisfying*

$$f_{p,m_1}(x+(p+1)k) = -m_1 f_{p,m_1}(x+pk) + f_{p,m_1}(x), \quad \forall x \in \mathbb{R}$$
$$f_{p,m_2}(x+(p+1)k) = -m_2 f_{p,m_2}(x+pk) + f_{p,m_2}(x), \quad \forall x \in \mathbb{R},$$

where m_1, m_2 *are nonnegative real numbers. Suppose that the conditions (C1),(C2),(C3),(C4) and (C5) are satisfied. Then* $(f_{p,m_1} f_{p,m_2})(x)$ *is an* $m-$*extension of Fibonacci* $p-$*function with period* k.

Proof. Assume that f_{p,m_1} and f_{p,m_2} be two $m-$extension of odd Fibonacci $p-$functions with period k and the conditions *(C1),(C2),(C3),(C4)* and *(C5)* are satisfied. Then,

$$\begin{aligned}
(f_{p,m_1}f_{p,m_2})(x+(p+1)k) &= f_{p,m_1}(x+(p+1)k)f_{p,m_2}(x+(p+1)k) \\
&= \Big[-m_1f_{p,m_1}(x+pk)+f_{p,m_1}(x)\Big]\Big[-m_2f_{p,m_2}(x+pk)+f_{p,m_2}(x)\Big] \\
&= m_1m_2f_{p,m_1}(x+pk)f_{p,m_2}(x+pk) \\
&\quad +f_{p,m_1}(x)f_{p,m_2}(x)-m_1f_{p,m_1}(x+pk)f_{p,m_2}(x) \\
&\quad -m_2f_{p,m_2}(x+pk)f_{p,m_1}(x) \\
&= m_1m_2f_{p,m_1}(x+pk)f_{p,m_2}(x+pk) \\
&\quad +f_{p,m_1}(x)f_{p,m_2}(x) \\
&= \mu(f_{p,m_1}f_{p,m_2})(x+pk)+(f_{p,m_1}f_{p,m_2})(x), \quad \forall x\in\mathbb{R}.
\end{aligned}$$

Thus, $(f_{p,m_1}f_{p,m_2})$ is an $m-$extension of Fibonacci $p-$function with period k. □

Theorem 4.7. *Let k be a positive integer, f_{p,m_1} and f_{p,m_2} be two $m-$extension of odd Fibonacci $p-$functions with period k satisfying*

$$f_{p,m_1}(x+(p+1)k) = -m_1f_{p,m_1}(x+pk)+f_{p,m_1}(x), \quad \forall x\in\mathbb{R}$$
$$f_{p,m_2}(x+(p+1)k) = -m_2f_{p,m_2}(x+pk)+f_{p,m_2}(x), \quad \forall x\in\mathbb{R},$$

where m_1, m_2 are nonnegative real numbers. Suppose that (C10),(C11) and one the conditions (C7) and (C8) are satisfied. Then, $(f_{p,m_1}f_{p,m_2})$ is an $m-$extension of odd Fibonacci $p-$function with period k.

Proof. First assume that f_{p,m_1} and f_{p,m_2} are $m-$extension of odd Fibonacci $p-$functions with period k and the conditions *(C7)*, *(C10)* and *(C11)* are satisfied. Then,

$$\begin{aligned}
(f_{p,m_1}f_{p,m_2})(x+(p+1)k) &= f_{p,m_1}(x+(p+1)k)f_{p,m_2}(x+(p+1)k) \\
&= \Big[-m_1f_{p,m_1}(x+pk)+f_{p,m_1}(x)\Big]\Big[-m_2f_{p,m_2}(x+pk)+f_{p,m_2}(x)\Big] \\
&= m_1m_2f_{p,m_1}(x+pk)f_{p,m_2}(x+pk) \\
&\quad +f_{p,m_1}(x)f_{p,m_2}(x)-m_1f_{p,m_1}(x+pk)f_{p,m_2}(x) \\
&\quad -m_2f_{p,m_2}(x+pk)f_{p,m_1}(x) \\
&= m_1m_2f_{p,m_1}(x+pk)f_{p,m_2}(x+pk) \\
&\quad +f_{p,m_1}(x)f_{p,m_2}(x) \\
&= -\mu(f_{p,m_1}f_{p,m_2})(x+pk)+(f_{p,m_1}f_{p,m_2})(x),
\end{aligned}$$

$\forall x\in\mathbb{R}$. Therefore, $(f_{p,m_1}f_{p,m_2})$ is an $m-$extension of odd Fibonacci $p-$function with period k. Next, assume that f_{p,m_1} and f_{p,m_2} are $m-$extension of odd Fibonacci $p-$functions with period k and the conditions *(C8)*, *(C10)* and *(C11)* are satisfied. Then the same result can be obtained. Therefore, $(f_{p,m_1}f_{p,m_2})$ is an $m-$extension of odd Fibonacci $p-$function with period k. □

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5. Quotients of $m-$extension of Fibonacci $p-$functions with period k

In this section, we discuss the limit of quotients of $m-$extension of Fibonacci $p-$functions with period k.

Theorem 5.1. *If $f_{p,m}$ is an $m-$extension of Fibonacci $p-$functions with period k, then the limit of the quotient $\frac{f_{p,m}(x+k)}{f_{p,m}(x)}$ exists.*

Proof. Let $k \in \mathbb{N}$, $m \in \mathbb{R}^+$, p be a nonnegative integer and $n \geq 2p$. Consider the quotient $Q(x) = \frac{f_{p,m}(x+k)}{f_{p,m}(x)}$, where $f_{p,m}$ is an $m-$extension of Fibonacci $p-$function with period k. We have two possibilities such that either $Q(x) < 0$ or $Q(x) > 0$. First, suppose that $Q(x) < 0$ then without loss of generality, $f_{p,m}(x) > 0$ and $f_{p,m}(x+k) < 0$. Therefore,

$$\begin{aligned}
f_{p,m}(x+2pk) &= mf_{p,m}(x+(2p-1)k) + f_{p,m}(x+(p-1)k) \\
&= m^2 f_{p,m}(x+(2p-2)k) + f_{p,m}(x+(p-1)k) \\
&\quad + mf_{p,m}(x+(p-2)k) \\
&= m^3 f_{p,m}(x+(2p-3)k) + f_{p,m}(x+(p-1)k) \\
&\quad + mf_{p,m}(x+(p-2)k) + m^2 f_{p,m}(x+(p-3)k) \\
&\vdots \\
&= m^p f_{p,m}(x+pk) + f_{p,m}(x+(p-1)k) \\
&\quad + \cdots - m^{p-2} f_{p,m}(x+k) + m^{p-1} f_{p,m}(x) \\
&= F_{p,m}(p+1) f_{p,m}(x+pk) + F_{p,m}(1) f_{p,m}(x+(p-1)k) \\
&\quad + \cdots - F_{p,m}(p-1) f_{p,m}(x+k) + F_{p,m}(p) f_{p,m}(x),
\end{aligned}$$

$$\begin{aligned}
f_{p,m}(x+(2p+1)k) &= mf_{p,m}(x+2pk) + f_{p,m}(x+pk) \\
&= m\Big[m^p f_{p,m}(x+pk) + f_{p,m}(x+(p-1)k) \\
&\quad + \cdots - m^{p-2} f_{p,m}(x+k) + m^{p-1} f_{p,m}(x)\Big] + f(x+pk) \\
&= (m^{p+1}+1) f_{p,m}(x+pk) + mf_{p,m}(x+(p-1)k) \\
&\quad + \cdots - m^{p-1} f_{p,m}(x+k) + m^p f_{p,m}(x) \\
&= F_{p,m}(p+2) f_{p,m}(x+pk) + F_{p,m}(2) f_{p,m}(x+(p-1)k) \\
&\quad + \cdots - F_{p,m}(p) f_{p,m}(x+k) + F_{p,m}(p+1) f_{p,m}(x).
\end{aligned}$$

$$\begin{aligned} f_{p,m}(x+(2p+2)k) &= mf_{p,m}(x+(2p+1)k)+f_{p,m}(x+(p+1)k) \\ &= m\Big[(m^{p+1}+1)f_{p,m}(x+pk)+mf_{p,m}(x+(p-1)k) \\ &\quad +\cdots-m^{p-1}f_{p,m}(x+k)+m^{p}f_{p,m}(x)\Big] \\ &\quad +mf(x+pk)+f(x) \\ &= (m^{p+2}+2m)f_{p,m}(x+pk)+m^{2}f_{p,m}(x+(p-1)k) \\ &\quad +\cdots-m^{p}f_{p,m}(x+k)+(m^{p+1}+1)f_{p,m}(x) \\ &= F_{p,m}(p+3)f_{p,m}(x+pk)+F_{p,m}(3)f_{p,m}(x+(p-1)k) \\ &\quad +\cdots-F_{p,m}(p+1)f_{p,m}(x+k)+F_{p,m}(p+2)f_{p,m}(x). \end{aligned}$$

Continuing this process, we have

$$\begin{aligned} f_{p,m}(x+nk) &= F_{p,m}(n-p+1)f_{p,m}(x+pk) \\ &= +F_{p,m}(n-2p+1)f_{p,m}(x+(p-1)k) \\ &= +\cdots-F_{p,m}(n-p-1)f_{p,m}(x+k)+F_{p,m}(n-p)f_{p,m}(x) \end{aligned}$$

and

$$\begin{aligned} f_{p,m}(x+(n+1)k) &= F_{p,m}(n-p+2)f_{p,m}(x+pk) \\ &= +F_{p,m}(n-2p+2)f_{p,m}(x+(p-1)k) \\ &= +\cdots-F_{p,m}(n-p)f_{p,m}(x+k)+F_{p,m}(n-p+1)f_{p,m}(x), \end{aligned}$$

where $F_{p,m}$ is an $m-$extension of Fibonacci $p-$sequence with the initial conditions, $F_{p,m}(0)=0$, $F_{p,m}(1)=1$, $F_{p,m}(2)=m$, …, $F_{p,m}(p)=m^{p-1}$. Given $x'\in\mathbb{R}$, there exists $x\in\mathbb{R}$ such that $x'=x+nk$. Therefore,

$$\begin{aligned} \frac{f_{p,m}(x'+k)}{f_{p,m}(x')} &= \frac{f_{p,m}(x+(n+1)k)}{f_{p,m}(x+nk)} \\ &= \left(\frac{F_{p,m}(n-p+2)f_{p,m}(x+pk)+\cdots-F_{p,m}(n-p)f_{p,m}(x+k)+F_{p,m}(n-p+1)f_{p,m}(x)}{F_{p,m}(n-p+1)f_{p,m}(x+pk)+\cdots-F_{p,m}(n-p-1)f_{p,m}(x+k)+F_{p,m}(n-p)f_{p,m}(x)}\right) \\ &= \left(\frac{F_{p,m}(n-p+2)\left[f_{p,m}(x+pk)+\cdots-\frac{F_{p,m}(n-p)}{F_{p,m}(n-p+2)}f_{p,m}(x+k)+\frac{F_{p,m}(n-p+1)}{F_{p,m}(n-p+2)}f_{p,m}(x)\right]}{F_{p,m}(n-p+1)\left[f_{p,m}(x+pk)+\cdots-\frac{F_{p,m}(n-p-1)}{F_{p,m}(n-p+1)}f_{p,m}(x+k)+\frac{F_{p,m}(n-p)}{F_{p,m}(n-p+1)}f_{p,m}(x)\right]}\right). \end{aligned}$$

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$$\lim_{n\to\infty}\left(\frac{f_{p,m}(x'+k)}{f_{p,m}(x')}\right) = \lim_{n\to\infty}\left(\frac{F_{p,m}(n-p+2)}{F_{p,m}(n-p+1)}\right)\left(\frac{f_{p,m}(x+pk)+\cdots-\frac{F_{p,m}(n-p)}{F_{p,m}(n-p+2)}f_{p,m}(x+k)+\frac{F_{p,m}(n-p+1)}{F_{p,m}(n-p+2)}f_{p,m}(x)}{f_{p,m}(x+pk)+\cdots-\frac{F_{p,m}(n-p-1)}{F_{p,m}(n-p+1)}f_{p,m}(x+k)+\frac{F_{p,m}(n-p)}{F_{p,m}(n-p+1)}f_{p,m}(x)}\right)$$

$$= \left(\lim_{n\to\infty}\frac{F_{p,m}(n-p+2)}{F_{p,m}(n-p+1)}\right)\left(\frac{f_{p,m}(x+pk)+\cdots-\lim_{n\to\infty}\frac{F_{p,m}(n-p)}{F_{p,m}(n-p+2)}f_{p,m}(x+k)+\lim_{n\to\infty}\frac{F_{p,m}(n-p+1)}{F_{p,m}(n-p+2)}f_{p,m}(x)}{f_{p,m}(x+pk)+\cdots-\frac{F_{p,m}(n-p-1)}{F_{p,m}(n-p+1)}f_{p,m}(x+k)+\frac{F_{p,m}(n-p)}{F_{p,m}(n-p+1)}f_{p,m}(x)}\right).$$

Let $N=n+1$. If $n\to\infty$ then $N\to\infty$. So, we can write the above expression as

$$\left(\lim_{n\to\infty}\frac{f_{p,m}(x'+k)}{f_{p,m}(x')}\right) = \lim_{n\to\infty}\left(\frac{F_{p,m}(n-p+2)}{F_{p,m}(n-p+1)}\right)\left(\frac{f_{p,m}(x+pk)+\cdots-\lim_{N\to\infty}\frac{F_{p,m}(N-p-1)}{F_{p,m}(N-p+1)}f_{p,m}(x+k)+\lim_{N\to\infty}\frac{F_{p,m}(N-p)}{F_{p,m}(N-p+1)}f_{p,m}(x)}{f_{p,m}(x+pk)+\cdots-\lim_{n\to\infty}\frac{F_{p,m}(n-p-1)}{F_{p,m}(n-p+1)}f_{p,m}(x+k)+\lim_{n\to\infty}\frac{F_{p,m}(n-p)}{F_{p,m}(n-p+1)}f_{p,m}(x)}\right)$$

$$= \alpha_m\left(\frac{f_{p,m}(x+pk)+\cdots-\lim_{N\to\infty}\frac{F_{p,m}(N-p-1)}{F_{p,m}(N-p+1)}f_{p,m}(x+k)+\lim_{N\to\infty}\frac{F_{p,m}(N-p)}{F_{p,m}(N-p+1)}f_{p,m}(x)}{f_{p,m}(x+pk)+\cdots-\lim_{n\to\infty}\frac{F_{p,m}(n-p-1)}{F_{p,m}(n-p+1)}f_{p,m}(x+k)+\lim_{n\to\infty}\frac{F_{p,m}(n-p)}{F_{p,m}(n-p+1)}f_{p,m}(x)}\right)=\alpha_m.$$

Here α_m is the unique positive real root of the characteristic equation of $m-$extension of Fibonacci $p-$sequence. Next, suppose that $Q(x)>0$, without loss of generality we assume $f_{p,m}(x)>0$, $f_{p,m}(x+k)>0$. Identically, we can easily obtain that $\lim_{n\to\infty}\left(\frac{f_{p,m}(x+(n+1)k)}{f_{p,m}(x+nk)}\right)=\alpha_m$. Hence we omit the proof. □

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FOURIER SPECTRAL METHODS FOR STOCHASTIC SPACE FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS DRIVEN BY SPECIAL ADDITIVE NOISES

FANG LIU, MONZORUL KHAN AND YUBIN YAN *

Abstract. Fourier spectral methods for solving stochastic space fractional partial differential equations driven by special additive noises in one-dimensional case are introduced and analyzed. The space fractional derivative is defined by using the eigenvalues and eigenfunctions of Laplacian subject to some boundary conditions. The space-time noise is approximated by the piecewise constant functions in the time direction and by some appropriate approximations in the space direction. The approximated stochastic space fractional partial differential equations are then solved by using Fourier spectral methods. For the linear problem, we obtain the precise error estimates in the L_2 norm and find the relations between the error bounds and the fractional powers. For the nonlinear problem, we introduce the numerical algorithms and MATLAB codes based on the FFT transforms. Our numerical algorithms can be adapted easily to solve other stochastic space fractional partial differential equations with multiplicative noises. Numerical examples for the semilinear stochastic space fractional partial differential equations are given.

Key words. Space fractional partial differential equations, stochastic partial differential equations, Fourier spectral method, error estimates

AMS subject classifications. 65M12; 65M06; Secondary 65M70;35S10

1. Introduction. Fourier spectral methods for solving the following stochastic space fractional partial differential equation are considered in this work, with $1/2 < \alpha \leq 1$,

$$\frac{du(t)}{dt} + A^{\alpha}u(t) = f(u(t)) + \frac{dW(t)}{dt}, \quad 0 < t < T, \tag{1.1}$$

$$u(0) = u_0. \tag{1.2}$$

Here A is an unbounded positive self-adjoint operator, u_0 is an initial value and $f(u)$ is a nonlinear term. The space-time white noise $W(t)$ will be defined below.

Let H be a separable Hilbert space and $\|\cdot\|$, $(\cdot,\cdot)$ denote the norm and inner product in H. Let $A : \mathcal{D}(A) \subset H \to H$ be a positive definite self-adjoint operator such that A^{-1} is compact on H. From this we infer the existence of a complete orthonormal basis $\{e_k\}_{k\geq 0}$ for H of eigenfunctions of A such that the associated sequence of eigenvalues $\{\lambda_k\}$ form an increasing unbounded sequence.

Using the basis $\{e_k\}$ we may also define the fractional powers of A. Given $1/2 < \alpha \leq 1$ define

$$H^{2\alpha} := \mathcal{D}(A^{\alpha}) = \{v \in H : \sum_k \lambda_k^{2\alpha} |(v, e_k)|^2 < \infty\},$$

and

$$A^{\alpha}v := \sum_k \lambda_k^{\alpha}(v, e_k)e_k, \; v \in \mathcal{D}(A^{\alpha}), \tag{1.3}$$

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with the associated Hilbert norm defined by

$$\|A^\alpha v\|^2 = \sum_k \lambda_k^{2\alpha} |(v, e_k)|^2.$$

The special space-time noise considered in this work is

$$\frac{dW(t)}{dt} = \sum_{k=1}^{\infty} \sigma_k(t) \dot{\beta}_k(t) e_k, \tag{1.4}$$

where $\dot{\beta}_k(t) = \frac{d\beta_k(t)}{dt}, k = 1, 2, \dots$ is the derivative of the standard Brownian motions $\beta_k(t)$, $k = 1, 2, \dots$ and $\sigma_k(t), k = 1, 2, \dots$ are some appropriate functions of t. In particular, when $\sigma_k(t) = \bar{\gamma}_k^{1/2}, \bar{\gamma}_k > 0$, the noise (1.4) reduces to

$$\frac{dW(t)}{dt} = \sum_{k=1}^{\infty} \bar{\gamma}_k^{1/2} \dot{\beta}_k(t) e_k,$$

which is a so-called H-valued Wiener process with the covariance operator Q and the linear operator $Q : H \to H$ is a trace class operator, that is $\mathrm{Tr}(Q) = \sum_{k=1}^{\infty} \bar{\gamma}_k < \infty$ where $Qe_k = \bar{\gamma}_k e_k, k = 1, 2, \dots$.

Let us here give two possible operators in (1.1)-(1.2). One is $A = -\Delta$ with the homogeneous Dirichlet boundary condition, $\mathcal{D}(A) = H_0^1(0,1) \cap H^2(0,1)$, where $\Delta = \partial^2/\partial x^2$ denotes the Laplacian. In this case, A has the eigenvalues $\lambda_k = k^2\pi^2$ and eigenfunctions $e_k = \sqrt{2} \sin k\pi x, k = 1, 2, \dots$. Our error estimates in this work are based on these eigenvalues and eigenfunctions. Another one is $A = I - \Delta$ with periodic boundary conditions, $\mathcal{D}(A) = H^2_{per}(-\pi, \pi)$. Here $H^2_{per}(-\pi, \pi)$ denotes the completion with respect to the $H^2(-\pi, \pi)$ norm of the set of $u \in C^\infty([-\pi, \pi])$ such that the pth derivative $u^{(p)}(-\pi) = u^{(p)}(\pi)$ for $p = 0, 1, \dots$. It is a Hilbert space with the $H^2(-\pi, \pi)$ inner product, see [24, Definition 1.47]. In this case, A has the eigenvalues $\lambda_1 = 1, \lambda_{2k} = 1 + k^2, \lambda_{2k+1} = 1 + k^2$ and eigenfunctions $e_1(x) = \frac{1}{\sqrt{2\pi}}, e_{2k}(x) = \frac{1}{\sqrt{\pi}} \sin kx, e_{2k+1}(x) = \frac{1}{\sqrt{\pi}} \cos kx, k = 1, 2, \dots$, see [24, Example 1.84].

We obtain the detailed error estimates, i.e., Theorems 2.1, 3.1, 3.3 below for the linear stochastic space fractional partial differential equation subject to the Dirichlet boundary conditions. More precisely, we shall consider the error estimates for the following linear problem, with $1/2 < \alpha \leq 1$,

$$\frac{\partial u(t,x)}{\partial t} + (-\Delta)^\alpha u(t,x) = \frac{\partial^2 W(t,x)}{\partial t \partial x}, \quad 0 < t < T, \ 0 < x < 1, \tag{1.5}$$

$$u(t,0) = u(t,1) = 0, \quad 0 < t < T, \tag{1.6}$$

$$u(0,x) = u_0(x), \quad 0 < x < 1. \tag{1.7}$$

Here the space-time noise $\frac{\partial^2 W(t,x)}{\partial t \partial x} = \frac{dW(t)}{dt}$ is define by (1.4).

For the linear stochastic space fractional partial differential equation subject to the periodic boundary conditions, we may obtain the similar error estimates as in Theorems 2.1, 3.1, 3.3. For the length of the paper, we will not give the detailed proofs for the error estimates in this case. However, in the numerical examples in Section 4, we shall consider the spectral method for the semilinear stochastic space fractional partial differential equations subject to the periodic boundary conditions to illustrate the experimentally determined convergence orders.

The stochastic partial differential equations driven by the white noise (the covariance operator $Q = I$) often have poor regularity estimates. In the physical world, to take into account the short and long range correlations of the stochastic effects, both white noise and colored noises may be considered. There are many situations where colored noises model the reality more closely, and there are also instances where the important stochastic effects are the noises acting on a few selected frequencies. For example one may choose $\sigma_k(t) = \frac{\cos t}{k^3}$. [12]

Space-fractional partial differential equations are widely used to model complex phenomena, for example, quasi-geostrophic flows, fast rotating fluids, dynamic of the frontogenesis in meteorology, diffusion in fractal or disordered medium, pollution problems, mathematical finance and the transport problems, see, e.g., [3], [7], [21], [36], [2].

Let us here consider two examples which apply the fractional Laplacian in the physical models. The first example is about the surface quasi-geostrophic (SQG) equation,

$$\partial_t \theta + \vec{u} \cdot \nabla \theta + \kappa(-\Delta)^{\alpha} \theta = 0,$$

where $\kappa \geq 0$ and $\alpha > 0$, $\theta = \theta(x_1, x_2, t)$ denotes the potential temperature, $\vec{u} = (u_1, u_2)$ is the velocity field determined by θ. When $\kappa > 0$, the SQG equation takes into account the dissipation generated by a fractional Laplacian. The SQG equation with $\kappa > 0$ and $\alpha = 1/2$ arises in geophysical studies of strongly rotating fluids. For the dissipative SQG equation, $\alpha = 1/2$ appears to be a critical index. In the subcritical case when $\alpha > 1/2$, the dissipation is sufficient to control the nonlinearity and the global regularity is a consequence of global a priori bound. In the critical case when $\alpha = 1/2$, the global regularity issue is more delicate. The mystery in the supercritical case $\alpha < 1/2$ is only partially uncovered at the moment. [9]

The second example is about the wave propagation in complex solids, especially viscoelastic materials (for example Polymers).[4]. In this case, the relaxation function has the form $k(t) = ct^{-\nu}, 0 < \nu < 1,\ c \in \mathbb{R}$, instead of the exponential form known in the standard models. This polynomial relaxation is due to the non uniformity of the material. The far field is then described by a Burgers equation with the leading operator $(-\Delta)^{\frac{1+\nu}{2}}$ instead of the Laplacian

$$\partial_t u = -(-\Delta)^{\frac{1+\nu}{2}} u + \partial_x(u^2).$$

This equation also describes the far-field evolution of acoustic waves propagating in a gas-filled tube with a boundary layer.

Frequently, the initial value or the coefficients of the equation are random, therefore it is natural to consider the stochastic space-fractional partial differential equations. The existence, uniqueness and regularities of the solutions of stochastic space-fractional partial differential equations have been extensively studied, see, e.g., [3], [7], [10], [28]. In this work, we will focus on the case $1/2 < \alpha \leq 1$ since the existence and uniqueness and regularity of the solution in this case is well understood in literature, see [11, Theorem 1.3]. However the numerical methods for solving space-fractional stochastic partial differential equations are quite restricted even for the case $1/2 < \alpha \leq 1$. Debbi and Dozzi [11] introduced a discretization of the fractional Laplacian and used it to elaborate an approximation scheme for fractional heat equation perturbed by a multiplicative cylindrical white noise. As far as we know [11] is the only existing paper in the literature of dealing with the numerical approach of this kind of problems. In this work, we will use the ideas developed in [1] and [12], to consider the numerical methods for solving stochastic space fractional partial differential equations, see also [19], [8], [20]. We first approximate the noise by using piecewise constant functions and then obtain the approximate solution $\hat{u}(t)$ of the exact solution $u(t)$. Finally we provide error estimates in L^2-norm for $u(t) - \hat{u}(t)$.

For the deterministic space fractional partial differential equations, many numerical methods are available in literature. There are two approaches to define the fractional Laplacian. One approach is by using the eigenvalues and eigenfunctions of the Laplacian $-\Delta$ subject to the boundary conditions as in (1.3). Another approach is by using the left-handed and right-handed Riemann-Liouville fractional derivatives. For the deterministic space fractional partial differential equations defined by the Riemann-Liouville fractional derivatives, many numerical methods are available, e.g., finite difference methods [14]-[15], [26], [31]-[32], finite element methods [13], [18] and the spectral methods [22]-[23]. For the deterministic space fractional partial differential equations defined by (1.3), some numerical methods are also available, see, e.g., matric transfer technique (MTT) [14], [15], [6], Fourier spectral method [5]. In this work, we will use Fourier spectral method to solve the stochastic space fractional partial differential equations. The main advantage of this approach is that it gives a full diagonal representation of the fractional operator, being able to achieve spectral convergence regardless of the fractional power in the problem.

Let $N_t \in \mathbb{N}$ and let $0 = t_0 < t_1 < t_2 < \cdots < t_{N_t} = T$ be the time partition of $[0, T]$ and Δt the time step size. To find the approximate solution of (1.5)-(1.7), we approximate the noise $\frac{\partial^2 W(t,x)}{\partial t \partial x}$ by the piecewise constant functions in the time direction defined by, with $l = 1, 2, ..., N_t$, [12]

$$\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x} = \sum_{k=1}^{\infty} \sigma_k^M(t) e_k(x) \Big(\sum_{l=1}^{N_t} \frac{1}{\sqrt{\Delta t}} \eta_{k,l} \chi_l(t) \Big), \tag{1.8}$$

where

$$\eta_{k,l} = \frac{1}{\sqrt{\Delta t}} \int_{t_{l-1}}^{t_l} d\,\beta_k(t) = \frac{1}{\sqrt{\Delta t}} \Big(\beta_k(t_l) - \beta_k(t_{l-1}) \Big) \in \mathcal{N}(0,1), \tag{1.9}$$

and

$$\chi_l(t) = \begin{cases} 1, & t \in [t_{l-1}, t_l], \ l = 1, 2, \dots, N_t, \\ 0, & \text{otherwise}. \end{cases}$$

Here $\sigma_k^M(t)$ is the approximation of $\sigma_k(t)$ in the space direction. For example, we can choose, with some positive integer $M > 0$,

$$\sigma_k(t) = \frac{\cos t}{k^3}, \qquad \sigma_k^M(t) = \begin{cases} \sigma_k(t), & k \le M, \\ 0, & k > M. \end{cases}$$

More precisely, replacing $\sigma_k(t)$ by $\sigma_k^M(t)$, we get the noise approximation in space, and replacing $\dot{\beta}_k(t)$ by $\sum_{j=1}^{N_t} \frac{1}{\sqrt{\Delta t}} \eta_{k,j} \chi_j(t)$, we get the noise approximation in time.

Substituting $\frac{\partial^2 W(t,x)}{\partial t \partial x}$ with $\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}$ in (1.5)-(1.7), we get

$$\frac{\partial \hat{u}(t,x)}{\partial t} + (-\Delta)^\alpha \hat{u}(t,x) = \frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}, \quad 0 < t < T,\ 0 < x < 1, \tag{1.10}$$

$$\hat{u}(t,0) = \hat{u}(t,1) = 0, \quad 0 < t < T, \tag{1.11}$$

$$\hat{u}(0,x) = u_0(x), \quad 0 < x < 1. \tag{1.12}$$

Note that $\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}$ now is a function in $L^2((0,T)\times(0,1))$ and therefore we can solve (1.10)-(1.12) by using any numerical methods for deterministic space fractional partial differential equations. Assume that $\{\sigma_k(t)\}$ and its derivative are uniformly bounded, [12]

$$|\sigma_k(t)| \leq \beta_k, \quad |\sigma_k'(t)| \leq \gamma_k, \quad \forall\, t \in [0,T], \tag{1.13}$$

and the coefficients $\{\sigma_k^M\}$ are constructed such that

$$|\sigma_k(t) - \sigma_k^M(t)| \leq \alpha_k^M, \quad |\sigma_k^M(t)| \leq \beta_k^M, \quad |(\sigma_k^M)'(t)| \leq \gamma_k^M, \quad \forall\, t \in [0,T], \tag{1.14}$$

with positive sequences $\{\alpha_k^M\}$ being arbitrarily chosen, $\{\beta_k^M\}$ and $\{\gamma_k^M\}$ being related to $\{\beta_k\}$ and $\{\gamma_k\}$. Further we assume that

$$\beta_k^M \leq k^{-\tilde{\alpha}}, \quad \text{for some } 0 \leq \tilde{\alpha} < 1/2. \tag{1.15}$$

Let $\mathbb{E}$ denote the expectation, in Theorem 2.1, we prove that, with $1/2 < \alpha \leq 1$ and $0 \leq \tilde{\alpha} < 1/2$,

$$\begin{aligned} &\mathbb{E} \int_0^T \int_0^1 \big(u(t,x) - \hat{u}(t,x)\big)^2 \, dx dt \\ &\leq C\Big(\sum_{k=1}^\infty \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} + \Delta t^2 \sum_{k=1}^\infty \Big(\lambda_k^\alpha \beta_k^M + \gamma_k^M\Big)^2 + \Delta t^{1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}} \Big). \end{aligned} \tag{1.16}$$

Let $J \in \mathbb{N}$, we denote

$$S_J = \text{span}\{e_1, e_2, \dots, e_J\},$$

and define by $P_J : H \to S_J$ the projection from H to S_J,

$$P_J v = \sum_{j=1}^J (v, e_j) e_j. \tag{1.17}$$

The Fourier spectral method of (1.10)-(1.12) is to find $\hat{u}_J(t) \in S_J$ such that, with $\hat{g}(t,x) := \frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}$.

$$\frac{\partial \hat{u}_J(t,x)}{\partial t} + (-\Delta)^\alpha \hat{u}_J(t,x) = P_J \hat{g}(t,x), \quad 0 < t < T,\ 0 < x < 1, \tag{1.18}$$

$$\hat{u}_J(t,0) = \hat{u}_J(t,1) = 0, \quad 0 < t < T, \tag{1.19}$$

$$\hat{u}_J(0,x) = P_J u_0(x), \quad 0 < x < 1, \tag{1.20}$$

In Theorem 3.1, we prove that, with $1/2 < \alpha \leq 1$ and $0 \leq \tilde{\alpha} < 1/2$,

$$\|\hat{u}(t) - \hat{u}_J(t)\|^2 \leq C\|u_0 - P_J u_0\|^2 + C \frac{1}{(J+1)^{2\alpha}} \int_0^t \|\hat{g}(s)\|^2 \, ds. \tag{1.21}$$

Combining Theorem 2.1 with Theorem 3.1, we have, with $u_0 \in \mathcal{D}(A^\alpha), 1/2 < \alpha \leq 1$,

$$\begin{aligned} &\mathbb{E}\int_0^T \int_0^1 \big(u(t,x) - \hat{u}_J(t,x)\big)^2\, dxdt \\ &\leq C\Big(\sum_{k=1}^{\infty} \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} + \Delta t^2 \sum_{k=1}^{\infty} \Big(\lambda_k^\alpha \beta_k^M + \gamma_k^M\Big)^2 + \Delta t^{1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}\Big) \\ &+ C\mathbb{E}\|u_0 - P_J u_0\|^2 + C\frac{1}{(J+1)^{2\alpha}}\Big(\Delta t \mathbb{E}\|A^\alpha u_0\|^2 + \Delta t \sum_{k=1}^{\infty} \lambda_k^\alpha (\beta_k^M)^2 + \sum_{k=1}^{\infty} (\beta_k^M)^2\Big). \end{aligned}$$

The paper is organized as follows. In Section 2, we consider the approximation of noise. In Section 3, we introduce the Fourier spectral methods for solving the approximated space fractional partial differential equations and the error estimates for the linear stochastic space fractional partial differential equations are proved. In Section 4, we consider the numerical examples for solving the semilinear stochastic space fractional partial differential equations subject to the periodic boundary conditions. From now on we denote by C a generic constant, which may not be the same at different occurrences.

2. Approximate the noise and regularity. It is well known that the mild solution of (1.5)-(1.7) has the following form

$$u(t,x) = \int_0^1 G_\alpha(t,x,y)\, u_0(y)\, dy + \int_0^t \int_0^1 G_\alpha(t-s,x,y)\, dW(s,y), \tag{2.1}$$

where

$$G_\alpha(t,x,y) = \sum_{j=1}^{\infty} e^{-\lambda_j^\alpha t} e_j(x) e_j(y),$$

and the stochastic integral $\int_0^t \int_0^1 G_\alpha(t-s,x,y)\, dW(s,y)$ is well-defined. The existence and uniqueness of the solutions of (1.5)-(1.7) are discussed in, e.g., [10], [11], [28] and the references cited therein.

Similarly the mild solution of (1.10)-(1.12) has the form of, see, e.g., [12]

$$\hat{u}(t,x) = \int_0^1 G_\alpha(t,x,y)\, u_0(y)\, dy + \int_0^t \int_0^1 G_\alpha(t-s,x,y)\, d\widehat{W}(s,y), \tag{2.2}$$

THEOREM 2.1. *Let u and $\hat{u}$ be the solutions of* (1.5)-(1.7) *and* (1.10)-(1.12), *respectively. Assume that the assumptions* (1.13)-(1.15) *hold. Then we have*

$$\begin{aligned} &\mathbb{E}\int_0^T \int_0^1 \big(u(t,x) - \hat{u}(t,x)\big)^2\, dxdt \\ &\leq C\Big(\sum_{k=1}^{\infty} \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} + \Delta t^2 \sum_{k=1}^{\infty} \Big(\lambda_k^\alpha \beta_k^M + \gamma_k^M\Big)^2 + \Delta t^{1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}\Big). \end{aligned} \tag{2.3}$$

Proof. See the Appendix. □

REMARK 2.2. *When $\alpha = 1$, Theorem 2.1 should reduce to the Theorem 3.3 in [12]. However one term $\Delta t^{1/2+\tilde{\alpha}}, 0 \leq \tilde{\alpha} < 1/2$ of the bounds in (3.20) in Theorem 3.3 [12] is missing. The term $\Delta t^{1/2+\tilde{\alpha}}, 0 \leq \tilde{\alpha} < 1/2$ comes from the estimates II_1 and*

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II_3 of the estimate for $II = \mathbb{E}\int_0^T \int_0^1 F_2(t,x)\,dxdt$ in (4.12). The authors in [12] only considered the estimate II_2 and neglected the terms II_1 and II_3 which would produce the term $\Delta t^{1/2+\tilde{\alpha}}, 0 \leq \tilde{\alpha} < 1/2$. (See the estimates for the term II in [12, p.1441]). In Theorem 2.1, we include the terms $O(\Delta t^{1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}})$.

THEOREM 2.3. *Let $\hat{u}$ be the solution of (1.10)-(1.12). Assume that the assumptions (1.13)-(1.15) hold. Further assume that $u_0 \in \mathcal{D}(A^\alpha), 1/2 < \alpha \leq 1$ and $\mathbb{E}\|A^\alpha u_0\|^2 < \infty$. Then*

$$\mathbb{E}\int_{t_j}^{t_{j+1}} \int_0^1 \Big|\frac{\partial \hat{u}(t,x)}{\partial t}\Big|^2 dxdt \leq C\Big(\Delta t \mathbb{E}\|A^\alpha u_0\|^2 + \Delta t \sum_{k=1}^\infty \lambda_k^\alpha (\beta_k^M)^2 + \sum_{k=1}^\infty (\beta_k^M)^2\Big), \tag{2.4}$$

and

$$\mathbb{E}\int_{t_j}^{t_{j+1}} \int_0^1 \big|A^\alpha \hat{u}(t,x)\big|^2 dxdt \leq C\Big(\Delta t \mathbb{E}\|A^\alpha u_0\|^2 + \Delta t \sum_{k=1}^\infty \lambda_k^\alpha (\beta_k^M)^2\Big). \tag{2.5}$$

Proof. Assume that, with $0 < t \leq t_{j+1}$,

$$\hat{u}(t,x) = \sum_{k=1}^\infty \hat{u}_k(t) e_k(x), \tag{2.6}$$

and, with $\hat{u}_k(0) = (u_0, e_k)$, $k = 1, 2, \dots,$

$$\hat{u}(0,x) = u_0(x) = \sum_{k=1}^\infty \hat{u}_k(0) e_k(x).$$

Substituting (2.6) into (1.10), we get, with $0 < t \leq t_{j+1}$,

$$\frac{d\hat{u}_k(t)}{dt} + \lambda_k^\alpha \hat{u}_k(t) = \sigma_k^M(t)\Big(\sum_{l=1}^{j+1} \frac{1}{\sqrt{\Delta t}} \eta_{k,l}\chi_l(t)\Big), \tag{2.7}$$

which implies that, with $0 < t \leq t_{j+1}$,

$$\hat{u}_k(t) = e^{-\lambda_k^\alpha t}\hat{u}_k(0) + \int_0^t e^{-\lambda_k^\alpha (t-s)} \sigma_k^M(s)\Big(\sum_{l=1}^{j+1} \frac{1}{\sqrt{\Delta t}} \eta_{k,l}\chi_l(s)\Big)\, ds. \tag{2.8}$$

Let us first show (2.4). Note that $\{e_k\}$ is an orthonormal basis in $H = L^2(0,1)$, we have, by (2.7),

$$\begin{aligned}
&\mathbb{E}\int_{t_j}^{t_{j+1}} \int_0^1 \Big|\frac{\partial \hat{u}(t,x)}{\partial t}\Big|^2 dxdt = \mathbb{E}\sum_{k=1}^\infty \int_{t_j}^{t_{j+1}} \Big|\frac{d\hat{u}_k(t)}{dt}\Big|^2 dt \\
&\leq 2\mathbb{E}\sum_{k=1}^\infty \Big(\int_{t_j}^{t_{j+1}} |\lambda_k^\alpha \hat{u}_k(t)|^2\, dt + \int_{t_j}^{t_{j+1}} \Big|\frac{\sigma_k^M(t)}{\sqrt{\Delta t}} \sum_{l=1}^{j+1} \eta_{k,l}\chi_l(t)\Big|^2 dt\Big) \\
&= 2\mathbb{E}\sum_{k=1}^\infty \lambda_k^{2\alpha} \int_{t_j}^{t_{j+1}} |\hat{u}_k(t)|^2\, dt + 2\mathbb{E}\sum_{k=1}^\infty \int_{t_j}^{t_{j+1}} \Big|\frac{\sigma_k^M(t)}{\sqrt{\Delta t}} \eta_{k,j+1}\chi_{j+1}(t)\Big|^2 dt \\
&= 2(I + II).
\end{aligned}$$

For I, we have, by (2.8), with $t_l^* = t_l, 1 \le l \le j$ and $t_l^* = t,\ l = j+1$,

(2.9)
$$\begin{aligned}
I &\le 2\mathbb{E}\sum_{k=1}^{\infty} \lambda_k^{2\alpha} \int_{t_j}^{t_{j+1}} \Big| e^{-\lambda_k^\alpha t} \hat{u}_k(0)\Big|^2 dt + 2\mathbb{E}\sum_{k=1}^{\infty} \lambda_k^{2\alpha} \int_{t_j}^{t_{j+1}} \Big| \sum_{l=1}^{j+1} \frac{\eta_{k,l}}{\sqrt{\Delta t}} \int_{t_{l-1}}^{t_l^*} e^{-\lambda_k^\alpha (t-s)} \sigma_k^M(s)\, ds\Big|^2 dt \\
&= 2\mathbb{E}\sum_{k=1}^{\infty} \int_{t_j}^{t_{j+1}} e^{-2\lambda_k^\alpha t} \big(A^\alpha u_0, e_k\big)^2 dt + 2\sum_{k=1}^{\infty} \lambda_k^{2\alpha} \int_{t_j}^{t_{j+1}} \sum_{l=1}^{j+1} \frac{1}{\Delta t} \Big(\int_{t_{l-1}}^{t_l^*} e^{-\lambda_k^\alpha (t-s)} \sigma_k^M(s)\, ds\Big)^2 dt \\
&\le 2\mathbb{E}\sum_{k=1}^{\infty} \big(A^\alpha u_0, e_k\big)^2 \Delta t + 2\sum_{k=1}^{\infty} \lambda_k^{2\alpha} \int_{t_j}^{t_{j+1}} \sum_{l=1}^{j+1} \frac{1}{\Delta t} \Big(\int_{t_{l-1}}^{t_l^*} e^{-2\lambda_k^\alpha (t-s)} \big(\sigma_k^M(s)\big)^2 ds\Big) \Big(\int_{t_{l-1}}^{t_l^*} 1^2\, ds\Big) dt \\
&\le 2\mathbb{E}\sum_{k=1}^{\infty} \big(A^\alpha u_0, e_k\big)^2 \Delta t + 2\sum_{k=1}^{\infty} \lambda_k^{2\alpha} \int_{t_j}^{t_{j+1}} \Big(\int_0^t e^{-2\lambda_k^\alpha (t-s)} \big(\sigma_k^M(s)\big)^2 ds\Big) dt \\
&\le 2\mathbb{E}\sum_{k=1}^{\infty} \big(A^\alpha u_0, e_k\big)^2 \Delta t + 2\sum_{k=1}^{\infty} \lambda_k^{2\alpha} \big(\beta_k^M\big)^2 \int_{t_j}^{t_{j+1}} \frac{1 - e^{-2\lambda_k^\alpha t}}{2\lambda_k^\alpha}\, dt \\
&\le 2\mathbb{E}\|A^\alpha u_0\|^2 \Delta t + \Delta t \sum_{k=1}^{\infty} \lambda_k^\alpha \big(\beta_k^M\big)^2,
\end{aligned}$$

where in the last inequality, we use the fact $1 - e^{-2\lambda_k^\alpha t} \le 1$.

For II, we have

$$II = \mathbb{E}\sum_{k=1}^{\infty} \int_{t_j}^{t_{j+1}} \Big| \frac{\sigma_k^M(t)}{\sqrt{\Delta t}} \eta_{k,j+1} \chi_{j+1}(t) \Big|^2 dt = \sum_{k=1}^{\infty} \int_{t_j}^{t_{j+1}} \Big(\frac{\sigma_k^M(t)}{\sqrt{\Delta t}}\Big)^2 dt \le \sum_{k=1}^{\infty} \big(\beta_k^M\big)^2.$$

Combining I with II we get (2.4). Similarly we have,

$$\begin{aligned}
&\mathbb{E}\int_{t_j}^{t_{j+1}} \int_0^1 |A^\alpha \hat{u}(t)|^2\, dx dt = \mathbb{E}\int_{t_j}^{t_{j+1}} \|A^\alpha \hat{u}(t,x)\|^2\, dt \\
&= \mathbb{E}\int_{t_j}^{t_{j+1}} \Big(\sum_{k=1}^{\infty} \lambda_k^{2\alpha} \hat{u}_k^2(t)\Big) dt = \mathbb{E}\sum_{k=1}^{\infty} \lambda_k^{2\alpha} \int_{t_j}^{t_{j+1}} |\hat{u}_k(t)|^2\, dt = I,
\end{aligned}$$

which implies (2.5) also holds. Together these estimates complete the proof of Theorem 2.3.
□

3. Fourier spectral method. Denote $E_\alpha(t) = e^{-tA^\alpha}, 1/2 < \alpha \le 1$, where A^α is defined by (1.3). The mild solution of (1.10)-(1.12) has the form of, with $\hat{g}(t) = \frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}$,

$$\hat{u}(t) = E_\alpha(t)\hat{u}_0 + \int_0^t E_\alpha(t-s)\hat{g}(s)\, ds, \quad \hat{u}(0) = u_0. \tag{3.1}$$

Similarly the solution of (1.18)-(1.20) has the form of

$$\hat{u}_J(t) = E_\alpha(t)P_J\hat{u}_0 + \int_0^t E_\alpha(t-s)P_J\hat{g}(s)\, ds, \quad \hat{u}(0) = P_J u_0. \tag{3.2}$$

THEOREM 3.1. *Assume that* $\hat{u}$ *and* $\hat{u}_J$ *are the solutions of* (1.10)-(1.12) *and* (1.18)-(1.20), *respectively. Let* $0 \leq r < 1/2$ *and let* $u_0 \in H$. *Then we have*

$$\big\|A^{r/2}\big(\hat{u}(t) - \hat{u}_J(t)\big)\big\|^2 \leq C\big\|u_0 - P_J u_0\big\|^2 + C\frac{1}{(J+1)^{2\alpha(1-r/\alpha)}} \int_0^t \|\hat{g}(s)\|^2 \, ds. \tag{3.3}$$

In particular, with $r = 0$,

$$\|\hat{u}(t) - \hat{u}_J(t)\|^2 \leq C\|u_0 - P_J u_0\|^2 + C\frac{1}{(J+1)^{2\alpha}} \int_0^t \|\hat{g}(s)\|^2 \, ds. \tag{3.4}$$

To prove Theorem 3.1, we need the following smoothing property for the solution operator $E_\alpha(t)$.

LEMMA 3.2.

1. *Let* $s > 0$. *We have, with* $1/2 < \alpha \leq 1$,
$$\|A^s E_\alpha(t)\| \leq C t^{-\frac{s}{\alpha}} e^{-\delta t}, \; t > 0,$$
for some constants $C = C(s, \alpha) > 0$ *and* $\delta = \delta(\alpha) > 0$.
2. *Let* $P_J : H \to S_J$ *be defined by* (1.17). *We have*
$$\|E_\alpha(t)(I - P_J)v\| \leq e^{-t\lambda_{J+1}^\alpha}\|v\|, \, t > 0.$$

Proof. Recall that A is positive definite and A has the eigenvalues $0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots$. For any function $h(\cdot)$, we have

$$\|h(A)\| = \sup_{\lambda \in \sigma(A)} |h(\lambda)|,$$

where $\sigma(A)$ denotes the set of eigenvalues of A. Thus, with $\delta = \frac{1}{2}\lambda_1^\alpha$,

$$\begin{aligned}
&\|A^s E_\alpha(t)\| = \|A^s E_\alpha(t/2) E_\alpha(t/2)\| \leq \|A^s E_\alpha(t/2)\| \|E_\alpha(t/2)\| \\
&= \sup_{\lambda \in \sigma(A)} \big(\lambda^s e^{-\frac{t}{2}\lambda^\alpha}\big) \cdot \sup_{\lambda \in \sigma(A)} \big(e^{-\frac{t}{2}\lambda^\alpha}\big) = \sup_{\lambda \in \sigma(A)} \Big(\frac{\big(\frac{t}{2}\lambda^\alpha\big)^{s/\alpha}}{e^{\frac{t}{2}\lambda^\alpha}} \Big(\frac{t}{2}\Big)^{-s/\alpha}\Big) e^{-\frac{t}{2}\lambda_1^\alpha} \\
&\leq C(t/2)^{-s/\alpha} e^{-\delta t} \leq C t^{-s/\alpha} e^{-\delta t},
\end{aligned}$$

which shows (1). Further (2) follows from

$$\|E_\alpha(t)(I - P_J)v\| = \Big(\sum_{j=J+1}^\infty e^{-2t\lambda_j^\alpha}(v, e_j)^2\Big)^{1/2} \leq e^{-t\lambda_{J+1}^\alpha}\|v\|.$$

Together these estimates complete the proof of Lemma 3.2.
□

Proof. [Proof of Theorem 3.1] Subtracting (3.2) from (3.1), we get

$$\hat{u}(t) - \hat{u}_J(t) = E_\alpha(t)(u_0 - P_J u_0) + \int_0^t E_\alpha(t-s)\big(\hat{g}(s) - P_J\hat{g}(s)\big) \, ds = I + II. \tag{3.5}$$

For I, we have, with $0 \leq r < 1/2$,

$$\begin{aligned}
\|A^{r/2} I\| &= \|A^{\frac{r}{2}} E_\alpha(t)\big(u_0 - P_J u_0\big)\| \\
&= \Big(\sum_{j=J+1}^\infty e^{-2t\lambda_j^\alpha} \lambda_j^r (u_0, e_j)^2\Big)^{1/2} \leq e^{-t\lambda_{J+1}^\alpha}\|A^{r/2}(u_0 - P_J u_0)\|.
\end{aligned}$$

For II, we have, by Lemma 3.2, for some $\gamma \in (0,1)$,

$$\begin{aligned}\|A^{r/2} II\| &= \Big\| \int_0^t A^{r/2} E_\alpha(t-s)\big(I-P_J\big)\hat{g}(s)\,ds \Big\| \\ &= \Big\| \int_0^t \Big[A^{r/2} E_\alpha\big((1-\gamma)(t-s)\big)\Big]\Big[E_\alpha\big(\gamma(t-s)\big)(I-P_J)\Big]\hat{g}(s)\,ds \Big\| \\ &\le C\int_0^t (t-s)^{-\frac{r}{2\alpha}} e^{-\kappa_\alpha(t-s)} \|\hat{g}(s)\|\,ds,\end{aligned}$$

where $\kappa_\alpha = \delta(1-\gamma) + \lambda_{J+1}^\alpha \gamma$.

By Cauchy- Schwarz inequality, we have

$$\|A^{r/2} II\| \le C\Big(\int_0^t \big((t-s)^{-\frac{r}{2\alpha}} e^{-\kappa_\alpha(t-s)}\big)^2\,ds\Big)^{1/2}\Big(\int_0^t \|\hat{g}(s)\|^2\,ds\Big)^{1/2}.$$

Note that $r < \alpha$, we have, with $\lambda_{J+1} = (J+1)^2\pi^2$,

$$\begin{aligned}\int_0^t \frac{e^{-2\kappa_\alpha s}}{s^{r/\alpha}}\,ds \le \int_0^\infty \frac{e^{-2\kappa_\alpha s}}{s^{r/\alpha}}\,ds \le \frac{\int_0^\infty s^{-r/\alpha}e^{-2s}\,ds}{\kappa_\alpha^{1-r/\alpha}} \le C\frac{1}{\kappa_\alpha^{1-r/\alpha}} \\ \le C\frac{1}{(\lambda_{J+1}^\alpha)^{1-r/\alpha}} \le C\frac{1}{(J+1)^{2\alpha(1-r/\alpha)}}.\end{aligned}$$

Thus

$$\|A^{r/2} II\| \le C\frac{1}{(J+1)^{2\alpha(1-r/\alpha)}}\Big(\int_0^t \|\hat{g}(s)\|^2\,ds\Big)^{1/2}.$$

Together these estimates complete the proof of Theorem 3.1. □

Combining Theorem 2.1 with Theorem 3.1, we have

THEOREM 3.3. *Let u and $\hat{u}_J$ be the solutions of* (1.5)-(1.7) *and* (1.18)-(1.20), *respectively. Assume that the assumptions* (1.13)-(1.15) *hold. Further assume that $u_0 \in \mathcal{D}(A^\alpha), 1/2 < \alpha \le 1$ and $\mathbb{E}\|A^\alpha u_0\|^2 < \infty$. Then we have*

$$\begin{aligned}&\mathbb{E}\int_0^T\int_0^1 \big(u(t,x) - \hat{u}_J(t,x)\big)^2\,dxdt \\ &\le C\Big(\sum_{k=1}^\infty \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} + \Delta t^2 \sum_{k=1}^\infty \Big(\lambda_k^\alpha \beta_k^M + \gamma_k^M\Big)^2 + \Delta t^{1+\frac{\tilde{\alpha}}{\alpha} - \frac{1}{2\alpha}}\Big) \\ &+ C\mathbb{E}\|u_0 - P_J u_0\|^2 + C\frac{1}{(J+1)^{2\alpha}}\Big(\Delta t \mathbb{E}\|A^\alpha u_0\|^2 + \Delta t\sum_{k=1}^\infty \lambda_k^\alpha(\beta_k^M)^2 + \sum_{k=1}^\infty (\beta_k^M)^2\Big).\end{aligned}$$

Proof. Note that

$$\begin{aligned}&\mathbb{E}\int_0^T\int_0^1 \big(u(t,x) - \hat{u}_J(t,x)\big)^2\,dxdt \\ &\le 2\mathbb{E}\int_0^T\int_0^1 \big(u(t,x) - \hat{u}(t,x)\big)^2\,dxdt + 2\mathbb{E}\int_0^T\int_0^1 \big(\hat{u}(t,x) - \hat{u}_J(t,x)\big)^2\,dxdt \\ &= 2I + 2II.\end{aligned}$$

For I, we have, by Theorem 2.1,

$$I \leq C\Big(\sum_{k=1}^{\infty} \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} + \Delta t^2 \sum_{k=1}^{\infty} \Big(\lambda_k^\alpha \beta_k^M + \gamma_k^M \Big)^2 + \Delta t^{1+\frac{\tilde{\alpha}}{\alpha} - \frac{1}{2\alpha}} \Big).$$

For II, we have

$$II = \mathbb{E} \int_0^T \|\hat{u}(t) - \hat{u}_J(t)\|^2 \, dt \leq C\mathbb{E}\|u_0 - P_J u_0\|^2 + C\frac{1}{(J+1)^{2\alpha}} \mathbb{E} \int_0^T \int_0^t \|\hat{g}(s)\|^2 \, dsdt.$$

Note that $\hat{g}(s) = \frac{d\hat{u}(s)}{ds} + (-\Delta)^\alpha \hat{u}(s)$, we have, by Theorem 2.3,

$$\begin{aligned} &\mathbb{E} \int_0^T \int_0^t \|\hat{g}(s)\|^2 \, dsdt \leq \mathbb{E} \int_0^T \int_0^t \Big\| \frac{d\hat{u}(s)}{ds} + (-\Delta)^\alpha \hat{u}(s) \Big\|^2 \, dsdt \\ &\leq C\mathbb{E} \int_0^T \int_0^T \int_0^1 \Big(\Big| \frac{\partial \hat{u}(s,x)}{\partial s} \Big|^2 + |(-\Delta)^\alpha \hat{u}(s,x)|^2 \Big) \, dxdsdt \\ &\leq C\Big(\Delta t \mathbb{E}\|A^\alpha u_0\|^2 + \Delta t \sum_{k=1}^{\infty} \lambda_k^\alpha (\beta_k^M)^2 + \sum_{k=1}^{\infty} (\beta_k^M)^2 \Big). \end{aligned}$$

Together these estimates complete the proof of Theorem 3.3.
□

4. Numerical simulations. In this section, we will consider the numerical simulation of the Fourier spectral methods for solving the following semilinear stochastic space fractional partial differential equations subject to the periodic boundary conditions, with $1/2 < \alpha \leq 1,\ 0 < x < 1,\ 0 < t \leq T$,

$$\frac{\partial u(t,x)}{\partial t} + \epsilon(-\Delta)^\alpha u(t,x) = f(u(t,x)) + \frac{\partial^2 W(t,x)}{\partial t \partial x}, \tag{4.1}$$

$$u(t,0) = u(t,1), \quad u_x'(t,0) = u_x'(t,1), \tag{4.2}$$

$$u(0,x) = u_0(x), \tag{4.3}$$

where $(-\Delta)^\alpha$ is the fractional Laplacian defined by using the eigenvalues and eigenfunctions of the Laplacian $-\Delta$ subject to the periodic boundary conditions. Here $f : \mathbb{R} \to \mathbb{R}$ is a smooth function and $\epsilon > 0$ denotes the diffusion coefficient. Here we consider the problems with the periodic boundary conditions because we want to compare our numerical results with the results in [24, Example 10.39] where the algorithms of the spectral methods for stochastic semilinear parabolic equation subject to the periodic boundary conditions are given and discussed. One may also consider the algorithms and MATLAB codes for stochastic space fractional partial differential equations with the homogeneous boundary conditions following the approaches in, e.g., [16], [17]. Although the Laplacian is singular in (4.1)-(4.2) due to the periodic boundary conditions, we expect the errors to behave as in Theorem 3.3, see the comments in [24, Corollary 10.38].

Denote $A = -\frac{\partial^2}{\partial x^2}$ with $\mathcal{D}(A) = H^2_{per}(0,1)$, where $\mathcal{D}(A) = H^2_{per}(0,1)$ is defined in the Introduction section. Then the eigenvalues and eigenfunctions of A can also be expressed by

$$\lambda_k = (2\pi k)^2, \quad e_k = e^{i2\pi kx}, \ k \in \mathbb{Z}.$$

The noise has the form of

$$\frac{\partial^2 W(t,x)}{\partial t \partial x} = \sum_{k \in \mathbb{Z}} \sigma_k(t) \dot{\beta}_k(t) e_k(x), \tag{4.4}$$

where $\dot{\beta}_k(t) = \frac{d\beta_k(t)}{dt}, k \in \mathbb{Z}$ are the derivatives of the standard Brownian motions $\beta_k(t)$, $k \in \mathbb{Z}$ and $\sigma_k(t), k \in \mathbb{Z}$ are some appropriate functions of t. Here $k \in \mathbb{Z}$ since we consider the periodic boundary conditions. When $\sigma_k(t) = \bar{\gamma}_k^{1/2}, \bar{\gamma}_k > 0, k \in \mathbb{Z}$, the noise (4.4) reduces to

$$\frac{\partial^2 W(t,x)}{\partial t \partial x} = \sum_{k \in \mathbb{Z}} \bar{\gamma}_k^{1/2} \dot{\beta}_k(t) e_k(x). \tag{4.5}$$

The approximate noise $\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x}$ is, with some positive integer $M > 0$,

$$\frac{\partial^2 \widehat{W}(t,x)}{\partial t \partial x} = \sum_{k \in \mathbb{Z}, |k| \le M} \bar{\gamma}_k^{1/2} e_k(x) \sum_{l=1}^{N_t} \frac{\eta_{k,l}}{\Delta t} \chi_l(t). \tag{4.6}$$

In our numerical example below, we assume that, [24, Example 10.8],

$$\bar{\gamma}_0 = 0, \quad \bar{\gamma}_k = |k|^{-(2r_1 + 1 + \tilde{\epsilon})}, \ k \in \mathbb{Z}, \ k \neq 0. \tag{4.7}$$

where $\tilde{\epsilon} > 0$ is a very small positive number. When $r_1 = -1/2$, we obtain so-called space-time white noise. When $r_1 = 1$, we obtain the smooth noise.

Let $S_J := \text{span}\{e_0, e_1, \dots, e_{J/2}, e_{-J/2+1} \dots, e_{-1}\}$. We assume $J \le M$ where M is determined in (4.5). Here the ordering $0, 1, 2, \dots, J/2, -J/2+1, \dots, -1$ is consistent with the ordering in the MATLAB functions **fft** and **ifft** [33]. Let $0 = t_0 < t_1 < t_2 < \dots < t_{N_t} = T$, $N_t \in \mathbb{N}$ be the time partition of $[0, T]$ and Δt the time step size with $T = N_t \Delta t$. We use the semi-implicit Euler method to consider the time discretization.

We will consider the convergence rate against the different time steps. Choose $J = 64$. The reference solution is obtained by using the time step size $\Delta tref = T/Nref$ with $Nref = 10^4$. Let $\mathbf{kappa} = [5, 10, 20, 50, 100, 200, 500]$, we will consider the approximate solutions with the different time step sizes $\Delta t_i = \Delta tref * \mathbf{kappa}(i), i = 1, 2, \dots, 7$. By Theorem 2.1, we have

$$\begin{aligned} &\mathbb{E} \int_0^T \int_0^1 \Big(u(t,x) - \hat{u}(t,x) \Big)^2 dx dt \\ &\le C \Big(\sum_{k \in \mathbb{Z}} \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} + \Delta t^2 \sum_{k \in \mathbb{Z}} \Big(\lambda_k^\alpha \beta_k^M + \gamma_k^M \Big)^2 + \Delta t^{1 + \frac{\tilde{\alpha}}{\alpha} - \frac{1}{2\alpha}} \Big). \end{aligned} \tag{4.8}$$

We remark that here we choose $k \in \mathbb{Z}$ since we consider the periodic boundary conditions. In our numerical example, we will choose, with $\bar{\gamma}_k$ given by (4.7),

$$\begin{aligned} &\sigma_k(t) = \bar{\gamma}_k^{1/2}, \bar{\gamma}_k > 0, k \in \mathbb{Z}, \\ &\sigma_k^M(t) = \begin{cases} \sigma_k(t) = \bar{\gamma}_k^{1/2}, \ |k| \le M, \\ 0, \quad |k| > M, \end{cases} \end{aligned}$$

which implies that

$$|\sigma_k^M(t)| \le \beta_k^M, \text{ where } \beta_k^M = \bar{\gamma}_k^{1/2}, \ |k| \le M,$$

and

$$|\sigma_k(t) - \sigma_k^M(t)| \leq \alpha_k^M, \text{ where } \alpha_k^M = \bar{\gamma}_k^{1/2}, \ |k| > M.$$

We first observe that for sufficiently large M the convergence order of the L^2 norm of the error in (4.8) is dominated by $O\big(\Delta t^{\frac{1}{2}(1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha})}\big)$. In fact, we will choose $M = J$ where J is sufficiently large. Then the first term of the right side of (4.8) satisfies, with $\lambda_k = (2\pi k)^2, k \in \mathbb{Z}$,

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} = \sum_{|k|>M} \frac{(\alpha_k^M)^2}{2\lambda_k^\alpha} &\leq C\Big(\frac{1}{\lambda_{M+1}^\alpha} + \frac{1}{\lambda_{M+2}^\alpha} + \dots\Big) \\ &\leq C\Big(\frac{1}{(M+1)^{2\alpha}} + \frac{1}{(M+2)^{2\alpha}} + \dots\Big) \\ &= C\Big(\frac{1}{(J+1)^{2\alpha}} + \frac{1}{(J+2)^{2\alpha}} + \dots\Big).\end{aligned}$$

The second term of the right side of the error in (4.8) is $O(\Delta t^2)$. Hence for sufficiently large J, the convergence order of the L^2 norm of the error in (4.8) is $O\big(\Delta t^{\frac{1}{2}(1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha})}\big)$.

We now consider two cases $r_1 = -1/2$ and $r_1 = 1$ in (4.7). For $r_1 = -1/2$, we may choose $\tilde{\alpha} = 0$ which implies that the convergence order of the L^2 norm in (4.8) is $O\big(\Delta t^{\frac{1}{2}(1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha})}\big) = O\big(\Delta t^{\frac{1}{2}(1-\frac{1}{2\alpha})}\big)$. Indeed, $\tilde{\alpha} = 0$ satisfies (1.15), that is,

$$\beta_k^M = \bar{\gamma}_k^{1/2} = |k|^{-\frac{2r_1+1+\bar{\epsilon}}{2}} = |k|^{-\bar{\epsilon}/2} \leq |k|^{-\tilde{\alpha}}.$$

For $r_1 = 1$, we may choose $\tilde{\alpha} = 1/2 - \bar{\epsilon}$ (since $0 \leq \tilde{\alpha} < 1/2$) with arbitrarily small positive number $\bar{\epsilon}$ which implies that the convergence order of the L^2 norm in (4.8) is $O\big(\Delta t^{\frac{1}{2}(1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha})}\big) = O\big(\Delta t^{\frac{1}{2}(1-\frac{\bar{\epsilon}}{\alpha})}\big) \approx O(\Delta t^{1/2})$. Indeed, in this case, $\tilde{\alpha} = 1/2 - \bar{\epsilon}$ satisfies (1.15), that is,

$$\beta_k^M = \bar{\gamma}_k^{1/2} = |k|^{-\frac{2r_1+1+\bar{\epsilon}}{2}} = |k|^{-\frac{3+\bar{\epsilon}}{2}} \leq |k|^{-\tilde{\alpha}}.$$

Thus we have, by Theorem 2.1, the following error estimates, with $1/2 < \alpha \leq 1$ and $r_1 = -1/2$,

$$\|\hat{u} - u\|_{L^2(\Omega, L^2((0,T),H))} \leq C(\Delta t^{\frac{1}{2}(1-\frac{1}{2\alpha})}), \tag{4.9}$$

and, with $1/2 < \alpha \leq 1$ and $r_1 = 1$

$$\|\hat{u} - u\|_{L^2(\Omega, L^2((0,T),H))} \leq C(\Delta t^{1/2}), \tag{4.10}$$

where the norm is measured in L^2 both for time and space. In particular, when $\alpha = 1, r_1 = -1/2$, we have

$$\|\hat{u} - u\|_{L^2(\Omega, L^2((0,T),H))} \leq C(\Delta t^{1/4}),$$

which is consistent with the standard time discretization error for the stochastic heat equation driven by space-time white noise, see, e.g., [35].

In our numerical experiment below, we choose $f(u) = u - u^3$, $u_0(x) = \sin(2\pi x)$, and $\epsilon = 1$. See the simulation of this problem for $\alpha = 1$ in [30]. We will consider the error estimates $\|\hat{u}(t_n) - u(t_n)\|_{L^2(\Omega,H)}$ at time t_n. We hope to observe the same convergence order as in (4.9) and (4.10).

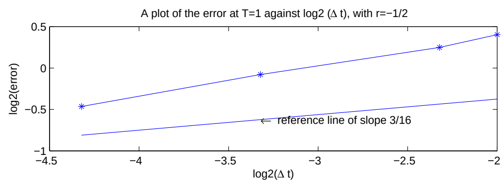

Fig. 1. *A plot of the error at $T = 1$ against log2(Δt) with $\alpha = 0.8, r_1 = -1/2$*

To do this, we consider $\overline{M} = 100$ simulations. For each simulation $\omega_m, m = 1, 2, \dots, \overline{M}$, we generate J independent Brownian motions $\beta_l, l = 0, 1, \dots, J/2, -J/2+1, \dots, -1$ and compute $\hat{u}_J(t_n) \approx \hat{u}(t_n)$ at time $t_n = 1$ by using the different time step sizes. We then compute the following L^2 norm of the error at $t_n = 1$ for the simulation $\omega_m, m = 1, 2, \dots, \overline{M}$,

$$\epsilon(\Delta t_i, \omega_m) = \epsilon(\Delta t_i, \omega_m, t_n) = \|\hat{u}_J(t_n, \omega_m) - \text{uref}(t_n, \omega_m)\|^2,$$

where the reference ("true ") solution $\text{uref}(t_n, \omega_m)$ is approximated by using the time step $\Delta tref = T/Nref$ and $Jref = J$. We then average $\epsilon(\Delta t_i, \omega_m)$ with respect to ω_m to obtain the following approximation of $\|\hat{u}_J(t_n) - \text{uref}(t_n)\|_{L^2(\Omega, H)}$ for the different time step size Δt_i,

$$S(\Delta t_i) = \Big(\frac{1}{\overline{M}} \sum_{m=1}^{\overline{M}} \epsilon(\Delta t_i, \omega_m) \Big)^{1/2} = \Big(\frac{1}{\overline{M}} \sum_{m=1}^{\overline{M}} \|\hat{u}_J(t_n, \omega_m) - \text{uref}(t_n, \omega_m)\|^2 \Big)^{1/2}.$$

For example, in the case $\alpha = 0.8, r_1 = -1/2$, the convergence rate against the time step size is $O(\Delta t^{\frac{1}{2}(1-\frac{1}{2\alpha})}) = O(\Delta t^{3/16})$, i.e., with some positive constant C,

$$S(\Delta t_i) \approx C \Delta t_i^{3/16},$$

which implies that

$$\log(S(\Delta t_i)) \approx \log(C) + \frac{3}{16} \log(\Delta t_i), i = 1, 2, \dots, 7.$$

In Figure 1, we consider the case $\alpha = 0.8, r_1 = -1/2$ and plot the points $\big(\log(\Delta t_i), \log(S(\Delta t_i))\big), i = 1, 2, \dots, 7$ and we observe that the experimentally determined convergence order is higher than the theoretical order in this case. Here the reference line has the slope $\frac{3}{16}$.

In Figure 2, we consider the case $\alpha = 0.8, r_1 = 1$ and in this case the theoretical convergence order with respect to the time step size is $O(\Delta t^{1/2})$. We plot the points $\big(\log(\Delta t_i), \log(S(\Delta t_i))\big), i = 1, 2, \dots, 7$ and we observe that the experimentally determined convergence order is also higher than the theoretical order in this case. Here the reference line has the slope $1/2$.

In Figure 3, we consider the convergence rate against the different J. Choose fixed time step $\Delta t = T/N_t$ with $N_t = 10^4$. We then consider the different $J = Jref * (\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \dots, \frac{1}{2^8})$ where $Jref = 2^{10}$.

We will first generate the reference Brownian motions

$$\beta_j(t), j = 0, 1, 2, \dots, Jref/2, -Jref/2 + 1, \dots - 1 \tag{4.11}$$

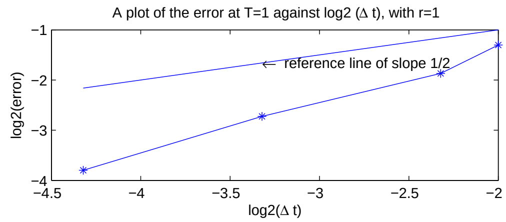

FIG. 2. *A plot of the error at $T = 1$ against log2(Δt) with $\alpha = 0.8, r_1 = 1$*

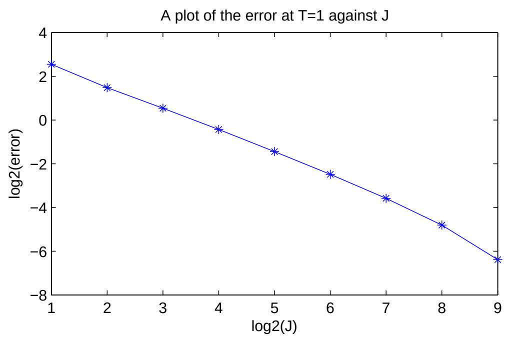

FIG. 3. *A plot of the error at $T = 1$ against the J with $\alpha = 0.8, r_1 = 1$*

for computing the reference ("true") solution uref. When we consider the approximate solution u with J truncated terms, we will use the Brownian motions $\beta_j(t), j = 0, 1, 2, \dots, J/2, -J/2+1, \dots -1$ from (4.11).

In Figure 3, we consider the case $\alpha = 0.8, r_1 = 1$ and plot the L^2 norm error against the different J where the L^2 norm error are approximated by using $\overline{M} = 100$ simulations. We indeed observe the spectral convergence with respect to the different J.

Appendix In the Appendix, we shall provide the proof of Theorem 2.1. To do this, we need the following lemma.

LEMMA 4.1. *Let $1/2 < \alpha \leq 1$ and $0 \leq \tilde{\alpha} < 1/2$. We have*

$$\int_0^\infty x^{-2(\tilde{\alpha}+\alpha)}\big(1 - e^{-x^{2\alpha}\Delta t}\big)\, dx \leq C\Delta t^{1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}.$$

Proof. With the variable change $y = x^{2\alpha}\Delta t$, we have

$$\int_0^\infty x^{-2(\tilde{\alpha}+\alpha)}\big(1 - e^{-x^{2\alpha}\Delta t}\big)\, dx = C\Delta t^{1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}\Big(\int_0^1 + \int_1^\infty\Big)\frac{1-e^{-y}}{y^{2+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}}\, dy$$

It is easy to see that, with $\alpha \in (1/2, 1]$,

$$\int_1^\infty \frac{1-e^{-y}}{y^{2+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}}\, dy \leq C.$$

Further, we have

$$\Big|\int_0^1 \frac{1-e^{-y}}{y^{2+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}}\, dy\Big| \leq C\int_0^1 \frac{y}{y^{2+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}}\, dy \leq C\int_0^1 \frac{1}{y^{1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}}\, dy < \infty.$$

if $1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}<1$, i.e., $0\le\tilde{\alpha}<1/2$.

Together these estimates complete the proof of Lemma 4.1.

□

Proof. [Proof of Theorem 2.1]

Subtracting (2.2) from (2.1), we have

$$\begin{aligned}
&u(t,x)-\hat{u}(t,x)\\
&=\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,dW(s,y)-\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,d\widehat{W}(s,y)\\
&=\Big[\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,dW(s,y)-\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,d\overline{W}(s,y)\Big]\\
&+\Big[\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,d\overline{W}(s,y)-\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,d\widehat{W}(s,y)\Big]\\
&=F_1(t,x)+F_2(t,x),
\end{aligned}$$

where, with $\eta_{k,l}$ and $\chi_l(t)$ defined as in (1.9),

$$\begin{aligned}
dW(s,y)&=\frac{\partial^2 W(s,y)}{\partial s\partial y}\,dsdy=\Big[\sum_{k=1}^\infty \sigma_k(s)e_k(y)\Big]\,d\beta_k(s)dy,\\
d\overline{W}(s,y)&=\frac{\partial^2 \overline{W}(s,y)}{\partial s\partial y}\,dsdy=\Big[\sum_{k=1}^\infty \sigma_k^M(s)e_k(y)\Big]\,d\beta_k(s)dy,\\
d\widehat{W}(s,y)&=\frac{\partial^2 \widehat{W}(s,y)}{\partial s\partial y}\,dsdy=\Big[\sum_{k=1}^\infty \sigma_k^M(s)\Big(\sum_{l=1}^{N_t}\frac{\eta_{k,l}}{\sqrt{\Delta t}}\chi_l(s)\Big)e_k(y)\Big]\,dsdy.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{E}\int_0^T\int_0^1 |u(t,x)-\hat{u}(t,x)|^2\,dxdt &\le C\mathbb{E}\int_0^T\int_0^1 F_1^2(t,x)\,dxdt\\
&+C\mathbb{E}\int_0^T\int_0^1 F_2^2(t,x)\,dxdt=C(I+II).
\end{aligned}$$

For I, we have, by using isometry property and (1.14), with $G_\alpha(t-s,x,y)=\sum_{j=1}^\infty e^{-(t-s)\lambda_j^\alpha}e_j(x)e_j(y)$,

$$\begin{aligned}
I&=\mathbb{E}\int_0^T\int_0^1\Big[\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,dW(s,y)-\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,d\overline{W}(s,y)\Big]^2\,dxdt\\
&=\int_0^T\int_0^1\int_0^t\Big[\int_0^1 G_\alpha(t-s,x,y)\Big(\sum_{k=1}^\infty\big(\sigma_k(s)-\sigma_k^M(s)\big)e_k(y)\Big)\,dy\Big]^2\,dsdxdt.\\
&=\int_0^T\int_0^t\sum_{k=1}^\infty e^{-2(t-s)\lambda_k^\alpha}\big(\alpha_k^M\big)^2\,dsdt=\int_0^T\sum_{k=1}^\infty\frac{1-e^{-2t\lambda_k^\alpha}}{2\lambda_k^\alpha}\big(\alpha_k^M\big)^2\,dt\le C\sum_{k=1}^\infty\frac{1}{2\lambda_k^\alpha}\big(\alpha_k^M\big)^2.
\end{aligned}$$

For II, we have

$$
\begin{aligned}
II &= \mathbb{E}\int_0^T\int_0^1 \Big\{\Big[\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,d\overline{W}(s,y) - \int_0^t\int_0^1 G_\alpha(t-s,x,y)\,d\widehat{W}(s,y)\Big]^2\Big\}dxdt \\
&\le 3\mathbb{E}\sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\int_0^1 \Big\{\Big[\int_0^t\int_0^1 G_\alpha(t-s,x,y)\,d\overline{W}(s,y) - \int_0^{t_j}\int_0^1 G_\alpha(t_j-s,x,y)\,d\overline{W}(s,y)\Big]^2 \\
&+ \Big[\int_0^{t_j}\int_0^1 G_\alpha(t_j-s,x,y)\,d\overline{W}(s,y) - \int_0^{t_j}\int_0^1 G_\alpha(t_j-s,x,y)\,d\widehat{W}(s,y)\Big]^2 \\
&+ \Big[\int_0^{t_j}\int_0^1 G_\alpha(t_j-s,x,y)\,d\widehat{W}(s,y) - \int_0^{t}\int_0^1 G_\alpha(t_j-s,x,y)\,d\widehat{W}(s,y)\Big]^2\Big\}dxdt \\
&\le 3\big(II_1+II_2+II_3\big).
\end{aligned} \tag{4.12}
$$

For II_2, we have, by isometry property,

$$
\begin{aligned}
II_2 &= \mathbb{E}\sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\int_0^1 \Big[\sum_{l=0}^{j-1}\int_{t_l}^{t_{l+1}}\int_0^1 G_\alpha(t_j-s,x,y)\Big(\sum_{k=1}^\infty \sigma_k^M(s)e_k(y)\,dy\Big)\,d\beta_k(s) \\
&- \sum_{l=0}^{j-1}\int_{t_l}^{t_{l+1}}\int_0^1 G_\alpha(t_j-\tilde{s},x,y)\sum_{k=1}^\infty \sigma_k^M(\tilde{s})e_k(y)\,dyd\tilde{s}\Big(\frac{1}{\Delta t}\int_{t_l}^{t_{l+1}} d\beta_k(s)\Big)\Big]^2 dxdt \\
&= \sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\int_0^1\sum_{l=0}^{j-1}\int_{t_l}^{t_{l+1}}\Big\{\int_0^1 G_\alpha(t_j-s,x,y)\Big(\sum_{k=1}^\infty\sigma_k^M(s)e_k(y)\Big)\,dy \\
&- \frac{1}{\Delta t}\int_{t_l}^{t_{l+1}}\int_0^1 G_\alpha(t_j-\tilde{s},x,y)\Big(\sum_{k=1}^\infty\sigma_k^M(\tilde{s})e_k(y)\Big)\,dyd\tilde{s}\Big\}^2 dsdxdt \\
&= \sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\int_0^1\sum_{l=0}^{j-1}\int_{t_l}^{t_{l+1}}\Big\{\frac{1}{\Delta t}\int_{t_l}^{t_{l+1}}\Big[\int_0^1 G_\alpha(t_j-s,x,y)\Big(\sum_{k=1}^\infty\sigma_k^M(s)e_k(y)\Big)\,dy \\
&- \int_0^1 G_\alpha(t_j-\tilde{s},x,y)\Big(\sum_{k=1}^\infty\sigma_k^M(\tilde{s})e_k(y)\Big)\,dy\Big]\,d\tilde{s}\Big\}^2 dsdxdt \\
&= \sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\sum_{l=0}^{j-1}\int_{t_l}^{t_{l+1}}\sum_{k=1}^\infty\Big\{\frac{1}{\Delta t}\int_{t_l}^{t_{l+1}}\Big[e^{-\lambda_k^\alpha(t_j-s)}\sigma_k^M(s) - e^{-\lambda_k^\alpha(t_j-\tilde{s})}\sigma_k^M(\tilde{s})\Big]\,d\tilde{s}\Big\}^2 dsdt \\
&= \sum_{j=0}^{N_t-1}\int_{t_j}^{t_{j+1}}\sum_{l=0}^{j-1}\int_{t_l}^{t_{l+1}}\sum_{k=1}^\infty\frac{e^{-2\lambda_k^\alpha t_j}}{\Delta t^2}\Big\{\int_{t_l}^{t_{l+1}}\Big[e^{\lambda_k^\alpha s}\sigma_k^M(s) - e^{\lambda_k^\alpha\tilde{s}}\sigma_k^M(\tilde{s})\Big]\,d\tilde{s}\Big\}^2 dsdt.
\end{aligned}
$$

By (1.14), we have, with some ξ_l^1, ξ_l^2 which lie between s and $\tilde{s}$,

$$
\begin{aligned}
&\Big|e^{\lambda_k^\alpha s}\sigma_k^M(s) - e^{\lambda_k^\alpha\tilde{s}}\sigma_k^M(\tilde{s})\Big| = \Big|\Big(e^{\lambda_k^\alpha s} - e^{\lambda_k^\alpha\tilde{s}}\Big)\sigma_k^M(s) + e^{\lambda_k^\alpha\tilde{s}}(\sigma_k^M(s)-\sigma_k^M(\tilde{s})\Big| \\
&\le \Big|\Big(\lambda_k^\alpha e^{\lambda_k^\alpha\xi_l^1}\Delta t\Big)\sigma_k^M(s) + e^{\lambda_k^\alpha\tilde{s}}\Big((\sigma_k^M)'(\xi_l^2)\Big)\Delta t\Big| \\
&\le \Big|\lambda_k^\alpha e^{\lambda_k^\alpha t_{l+1}}\beta_k^M\Delta t + e^{\lambda_k^\alpha t_{l+1}}\gamma_k^M\Delta t\Big| \le e^{\lambda_k^\alpha t_{l+1}}\Big(\lambda_k^\alpha\beta_k^M + \gamma_k^M\Big)\Delta t.
\end{aligned}
$$

Hence

$$\begin{aligned} II_2 &\leq \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \sum_{l=0}^{j-1} \int_{t_l}^{t_{l+1}} \sum_{k=1}^{\infty} \frac{e^{-2\lambda_k^\alpha t_j}}{\Delta t^2} \Big[e^{2\lambda_k^\alpha t_{l+1}} \Big(\lambda_k^\alpha \beta_k^M + \gamma_k^M \Big)^2 \Delta t^4 \Big] \, dsdt \\ &\leq \Delta t^2 \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_0^{t_j} \sum_{k=1}^{\infty} \Big(\lambda_k^\alpha \beta_k^M + \gamma_k^M \Big)^2 \, dsdt \leq C \Delta t^2 \sum_{k=1}^{\infty} \Big(\lambda_k^\alpha \beta_k^M + \gamma_k^M \Big)^2, \end{aligned}$$

where we use the inequality $e^{-2\lambda_k^\alpha (t_j - t_{l+1})} \leq 1$ for $l = 0, 1, 2, \ldots, j-1$.

For II_1, we have

$$\begin{aligned} II_1 &= \mathbb{E} \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_0^1 \Big[\int_0^t \int_0^1 G_\alpha(t-s,x,y) \, d\overline{W}(s,y) - \int_0^{t_j} \int_0^1 G_\alpha(t-s,x,y) \, d\overline{W}(s,y) \Big]^2 \, dxdt \\ &\leq 2\mathbb{E} \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_0^1 \Big[\int_0^{t_j} \int_0^1 \Big(G_\alpha(t-s,x,y) - G_\alpha(t_j-s,x,y) \Big) \, d\overline{W}(s,y) \Big]^2 \, dxdt \\ &+ 2\mathbb{E} \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_0^1 \Big[\int_{t_j}^t \int_0^1 G_\alpha(t-s,x,y) \, d\overline{W}(s,y) \Big]^2 \, dxdt = 2(II_1^1 + II_1^2). \end{aligned}$$

For II_1^1, we have, by the isometry property and (1.14),

$$\begin{aligned} II_1^1 &= \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_0^{t_j} \sum_{k=1}^{\infty} \Big(e^{-\lambda_k^\alpha (t-s)} - e^{-\lambda_k^\alpha (t_j - s)} \Big)^2 (\sigma_k^M(s))^2 \, dsdt \\ &\leq \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \sum_{k=1}^{\infty} (\beta_k^M)^2 \int_0^{t_j} \Big(e^{-\lambda_k^\alpha (t-s)} - e^{-\lambda_k^\alpha (t_j - s)} \Big)^2 \, dsdt \end{aligned}$$

Note that

$$\begin{aligned} &\int_0^{t_j} \Big(e^{-\lambda_k^\alpha (t-s)} - e^{-\lambda_k^\alpha (t_j - s)} \Big)^2 ds = \int_0^{t_j} e^{-2\lambda_k^\alpha (t-s)} \Big(1 - e^{-\lambda_k^\alpha (t_j - t)} \Big)^2 ds \\ &= \Big(1 - e^{-\lambda_k^\alpha (t_j - t)} \Big)^2 \frac{e^{-2\lambda_k^\alpha (t - t_j)} - e^{-2\lambda_k^\alpha t}}{2\lambda_k^\alpha} \leq \frac{\Big(1 - e^{-\lambda_k^\alpha (t - t_j)} \Big)^2}{2\lambda_k^\alpha}. \end{aligned}$$

Hence, we have

$$II_1^1 \leq \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \Big(\sum_{k=1}^{\infty} (\beta_k^M)^2 \Big) \frac{\Big(1 - e^{-\lambda_k^\alpha (t - t_j)} \Big)^2}{2\lambda_k^\alpha} \, dt \leq C \sum_{k=1}^{\infty} (\beta_k^M)^2 \frac{\Big(1 - e^{-\lambda_k^\alpha \Delta t} \Big)^2}{2\lambda_k^\alpha}.$$

By (1.15) and Lemma 4.1, we have

$$II_1^1 \leq C \sum_{k=1}^{\infty} k^{-2\tilde{\alpha}} \frac{\Big(1 - e^{-\lambda_k^\alpha \Delta t} \Big)^2}{2\lambda_k^\alpha} \leq C \int_1^\infty x^{-2(\tilde{\alpha} + \alpha)} \big(1 - e^{-x^{2\alpha} \Delta t} \big) \, dx \leq C \Delta t^{1 + \frac{\tilde{\alpha}}{\alpha} - \frac{1}{2\alpha}}.$$

For II_1^2, we have, by isometry property and (1.14) and (1.15),

$$
\begin{aligned}
II_1^2 &= \mathbb{E} \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_0^1 \Big[\int_{t_j}^{t} \int_0^1 G_\alpha(t-s,x,y)\, d\overline{W}(s,y) \Big]^2 dxdt \\
&= \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_{t_j}^{t} \sum_{k=1}^{\infty} e^{-2\lambda_k^\alpha (t-s)} \big(\sigma_k^M(s)\big)^2 dsdt \le \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \int_{t_j}^{t} \sum_{k=1}^{\infty} \Big(k^{-2\tilde{\alpha}} e^{-2\lambda_k^\alpha (t-s)} \Big) dsdt \\
&\le C \sum_{j=0}^{N_t-1} \int_{t_j}^{t_{j+1}} \sum_{k=1}^{\infty} \Big[k^{-2\tilde{\alpha}} \Big(\frac{1-e^{-2\lambda_k^\alpha \Delta t}}{\lambda_k^\alpha} \Big) \Big] dt = C \sum_{k=1}^{\infty} \Big[k^{-2\tilde{\alpha}} \Big(\frac{1-e^{-2\lambda_k^\alpha \Delta t}}{\lambda_k^\alpha} \Big) \Big] \\
&\le C \int_0^\infty \frac{1-e^{-2x^{2\alpha}\Delta t}}{x^{2\alpha+2\tilde{\alpha}}} dx \le C \int_0^\infty x^{-2(\tilde{\alpha}+\alpha)} \big(1-e^{-x^{2\alpha}\Delta t}\big) dx.
\end{aligned}
$$

By Lemma 4.1, we have

$$ II_1^2 \le C \Delta t^{1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}. \tag{4.13} $$

Similarly we may show, with $0 \le \tilde{\alpha} < 1/2$,

$$ II_3 \le C \Delta t^{1+\frac{\tilde{\alpha}}{\alpha}-\frac{1}{2\alpha}}. $$

Together these estimates complete the proof of Theorem 2.1.

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On anti-periodic type boundary value problems of sequential fractional differential equations of order $q \in (2,3]$

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Abstract

We investigate a new kind of anti-periodic type boundary value problems of sequential fractional differential equation of order $q \in (2,3]$. We make use of Banach's contraction mapping principle to obtain the uniqueness result while the existence of solutions is established via Krasnoselskii's fixed point theorem and Leray-Schauder nonlinear alternative. The paper concludes with some illustrative examples.

Key words and phrases: Fractional differential equations; sequential; antiperiodic; existence; fixed point
AMS (MOS) Subject Classifications: 34A08; 34A12; 34A37

1 Introduction

Boundary value problems constitute an important field of research and arise in several disciplines such as applied mathematics, control theory, mechanical structures and physics. The literature on the topic ranges from theoretical aspects of existence and uniqueness of solutions to analytic and numerical methods for finding solutions of the problems. Linear and nonlinear, singular and nonsingular, well-posed and ill-posed, local and nonlocal, free and fixed problems are well known types of boundary value problems. In relation to the boundary conditions, considerable attention has been given to two-point, multi-point, periodic/anti-periodic and integral boundary value problems. In particular, anti-periodic boundary conditions are found to be quite significant and important in the mathematical modeling of certain physical processes and phenomena, for example, wavelets, physics, trigonometric polynomials in the study of interpolation problems, etc., for example, see [1] and the references cited therein.

Differential and integral operators of fractional-order appear in the mathematical modelling of several phenomena occurring in engineering and scientific disciplines such as biological sciences, ecology, control theory, aerodynamics, fluid dynamics, polymer rheology, regular variation in thermodynamics, etc. For more details and explanation, for instance, see [2, 3, 4]. The interest in the study of fractional-order operators is mainly due to nonlocal nature of such operators which takes into account memory and hereditary properties of some important and useful materials and processes.

In recent years, fractional-order boundary value problems involving a variety of boundary conditions have been studied by several researchers. For details and examples, we refer the reader to a series of papers ([5]-[10]). For some works on sequential fractional differential equations, for example, see ([11]-[15]).

Anti-periodic boundary value problems of fractional-order have also been investigated in the literature ([16]-[19]). However, the study of sequential fractional differential equations equipped with anti-periodic boundary conditions has not been investigated yet.

In this paper, we consider a nonlinear anti-periodic boundary value problem of sequential fractional differential equations given by

$$\begin{cases} ({}^cD^q + k\ {}^cD^{q-1})u(t) = f(t, u(t)),\ 2 < q \leq 3,\ 0 < t < T, \\ \alpha_1 u(0) + \gamma_1 u(T) = a, \quad \alpha_2 u'(0) + \gamma_2 u'(T) = b, \quad \alpha_3 u''(0) + \gamma_3 u''(T) = c, \end{cases} \tag{1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, $\alpha_i, \gamma_i, (i = 1, 2, 3), a, b, c \in \mathbb{R}$, $k \in \mathbb{R}^+$ and f is a given continuous function.

2 Preliminaries and an auxiliary lemma

First of all, let us recall some basic definitions [2, 3].

Definition 2.1 *The fractional integral of order r with the lower limit zero for a function f is defined as*

$$I^r f(t) = \frac{1}{\Gamma(r)} \int_0^t \frac{f(s)}{(t-s)^{1-r}} ds, \quad t > 0, \quad r > 0,$$

provided the right hand-side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function, which is defined by $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt$.

Definition 2.2 *The Riemann-Liouville fractional derivative of order $r > 0$, $n - 1 < r < n$, $n \in N$, is defined as*

$$D_{0+}^r f(t) = \frac{1}{\Gamma(n-r)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-r-1} f(s) ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 2.3 *The Caputo derivative of order r for a function $f : [0, \infty) \to R$ can be written as*

$${}^cD^r f(t) = D_{0+}^r \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, \quad n - 1 < r < n.$$

Remark 2.4 *If $f(t) \in C^n[0, \infty)$, then*

$${}^cD^r f(t) = \frac{1}{\Gamma(n-r)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{r+1-n}} ds = I^{n-r} f^{(n)}(t),\ t > 0,\ n - 1 < q < n.$$

The following lemma plays a pivotal role in defining the solution for problem (1).

Lemma 2.5 *Let $h \in AC([0, T]), \mathbb{R})$. Then the following linear boundary value problem*

$$\begin{cases} ({}^cD^q + k\ {}^cD^{q-1})u(t) = h(t),\ 2 < q \leq 3,\ 0 < t < T, \\ \alpha_1 u(0) + \gamma_1 u(T) = a, \quad \alpha_2 u'(0) + \gamma_2 u'(T) = b, \quad \alpha_3 u''(0) + \gamma_3 u''(T) = c \end{cases} \tag{2}$$

is equivalent to the fractional integral equation

$$\begin{aligned} u(t) &= \nu_1(t) + \int_0^t e^{-k(t-s)} \Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} h(x) dx \Big) ds + \nu_2(t) \int_0^T e^{-k(T-s)} \Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} h(x) dx \Big) ds \\ &+ \nu_3(t) \int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)} h(s) ds + \nu_4(t) \int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)} h(s) ds, \end{aligned} \tag{3}$$

where

$$\begin{aligned}
\nu_1(t) &= \frac{a}{\lambda_1} - \frac{\gamma_1 T b}{\lambda_1\lambda_2} - \frac{c}{k^2\lambda_1\lambda_2\delta_3}\Big(\lambda_2\delta_1 + k\delta_2\gamma_1 T\Big) + \frac{ce^{-kt}}{k^2\delta_3} + \frac{t}{\lambda_2}\Big(b + \frac{c\delta_2}{k\delta_3}\Big),\\
\nu_2(t) &= \frac{\gamma_3}{\delta_3\lambda_1}\Big(\delta_1 - \frac{\delta_3\gamma_1}{\gamma_3}\Big) - \frac{k\gamma_1 T}{\lambda_1\lambda_2}\Big(\gamma_2 - \frac{\delta_2\gamma_3}{\delta_3}\Big) - \frac{\gamma_3 e^{-kt}}{\delta_3} + \frac{kt}{\lambda_2}\Big(\gamma_2 - \frac{\delta_2\gamma_3}{\delta_3}\Big),\\
\nu_3(t) &= \frac{-\delta_1\gamma_3}{k\lambda_1\delta_3} - \frac{\gamma_1 T}{\lambda_1\lambda_2}\Big(\frac{\delta_2\gamma_3}{\delta_3} - \gamma_2\Big) + \frac{\gamma_3 e^{-kt}}{k\delta_3} + \frac{t}{\lambda_2}\Big(\frac{\delta_2\gamma_3}{\delta_3} - \gamma_2\Big)\\
\nu_4(t) &= \frac{\gamma_3}{k^2\delta_3\lambda_1\lambda_2}\Big(\lambda_2\delta_1 + \delta_2 k\gamma_1 T\Big) - \frac{\gamma_3 e^{-kt}}{k^2\delta_3} - \frac{\gamma_3\delta_2 t}{k\delta_3\lambda_2},
\end{aligned} \tag{4}$$

$$\delta_i = \alpha_i + \gamma_i e^{-kT},\ i = 1,2,3,\ \delta_3 \neq 0,\ \lambda_1 = \alpha_1 + \gamma_1 \neq 0,\ \lambda_2 = \alpha_2 + \gamma_2 \neq 0. \tag{5}$$

Proof. Rewrite the equation $({}^cD^q + k\,{}^cD^{q-1})u(t) = h(t)$ as

$${}^cD^{q-1}(D + k)u(t) = h(t). \tag{6}$$

Applying the operator I^{q-1} on both sides of (6), and solving the resulting equation, we get

$$u(t) = A_0 e^{-kt} + A_1 + A_2 t + \int_0^t e^{-k(t-s)} I^{q-1}h(s)ds, \tag{7}$$

where A_0, A_1 and A_2 are arbitrary constants and

$$I^{q-1}h(t) = \int_0^t \frac{(t-x)^{q-2}}{\Gamma(q-1)} h(x)dx.$$

Differentiating (7) with respect to t, we obtain

$$u'(t) = -kA_0 e^{-kt} + A_2 - k\int_0^t e^{-k(t-s)} I^{q-1}h(s)ds + I^{q-1}h(t), \tag{8}$$

$$u''(t) = k^2 A_0 e^{-kt} + k^2\int_0^t e^{-k(t-s)} I^{q-1}h(s)ds - kI^{q-1}h(t) + I^{q-2}h(t). \tag{9}$$

Using the boundary conditions of (2) in (7)-(9), we get

$$\delta_1 A_0 + \lambda_1 A_1 + A_2\gamma_1 T + \gamma_1\int_0^T e^{-k(T-s)} I^{q-1}h(s)ds = a, \tag{10}$$

$$-k\delta_2 A_0 + \lambda_2 A_2 + \gamma_2\Big(-k\int_0^T e^{-k(T-s)} I^{q-1}h(s)ds + I^{q-1}h(T)\Big) = b, \tag{11}$$

$$A_0 k^2\delta_3 + \gamma_3\Big(k^2\int_0^T e^{-k(T-s)} I^{q-1}h(s)ds - kI^{q-1}h(T) + I^{q-2}h(T)\Big) = c. \tag{12}$$

Solving the system (10)-(12) for A_0, A_1 and A_2, we find that

$$A_0 = \frac{1}{k^2\delta_3}\Big\{c - \gamma_3\Big(k^2\int_0^T e^{-k(T-s)} I^{q-1}h(s)ds - kI^{q-1}h(T) + I^{q-2}h(T)\Big)\Big\},$$

$$\begin{aligned}
A_1 &= \frac{a}{\lambda_1} - \Big(\frac{\gamma_1 T b}{\lambda_1\lambda_2}\Big) - \Big(\frac{\delta_1}{k^2\delta_3\lambda_1} + \frac{\delta_2\gamma_1 T)}{k\delta_3\lambda_1\lambda_2}\Big)c\\
&+\Big(\frac{\delta_1\gamma_3}{\delta_3\lambda_1} - \frac{\gamma_1 T}{\lambda_1}\Big(\frac{\gamma_2 k}{\lambda_2} - \frac{k\delta_2\gamma_3}{\delta_3\lambda_2}\Big) - \frac{\gamma_1}{\lambda_1}\Big)\int_0^T e^{-k(T-s)} I^{q-1}h(s)ds\\
&-\Big(\frac{\delta_1\gamma_3}{k\delta_3\lambda_1} + \frac{\gamma_1 T}{\lambda_1}\Big(\frac{\delta_2\gamma_3}{\delta_3\lambda_2} - \frac{\gamma_2}{\lambda_2}\Big)\Big)I^{q-1}h(T) + \Big(\frac{\delta_1\gamma_3}{k^2\delta_3\lambda_1} + \frac{\delta_2\gamma_3\gamma_1 T}{k\delta_3\lambda_1\lambda_2}\Big)I^{q-2}h(T),
\end{aligned}$$

$$A_2 = \frac{b}{\lambda_2} + \frac{\delta_2 c}{k\delta_3\lambda_2} + \Big(\frac{\gamma_2 k}{\lambda_2} - \frac{k\delta_2\gamma_3}{\delta_3\lambda_2}\Big)\int_0^T e^{-k(T-s)} I^{q-1}h(s)ds$$

$$+\Big(\frac{\delta_2\gamma_3}{\delta_3\lambda_2} - \frac{\gamma_2}{\lambda_2}\Big)I^{q-1}h(T) - \frac{\delta_2\gamma_3}{k\delta_3\lambda_2}I^{q-2}h(T),$$

where we have used (5). Substituting the values of A_0, A_1 and A_2 in (7) and using the notations (4), we obtain the solution (3). By direct computation, it is easy to show that (3) satisfies the problem (2). This completes the proof. □

3 Main Result

Let $\mathcal{C} = C([0,T],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,T] \to \mathbb{R}$ endowed with the norm defined by $\|u\| = \sup\{|u(t)|, t \in [0,T]\}$.
Via Lemma 2.5, we transform the problem (1) to an equivalent fixed point problem as

$$u = \mathcal{H}u, \tag{13}$$

where $\mathcal{H} : \mathcal{C} \to \mathcal{C}$ is defined by

$$\begin{aligned}(\mathcal{H}u)(t) &= \nu_1(t) + \int_0^t e^{-k(t-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}f(x,u(x))dx\Big)ds \\ &+\nu_2(t)\int_0^T e^{-k(T-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}f(x,u(x))dx\Big)ds \\ &+\nu_3(t)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}f(s,u(s))ds + \nu_4(t)\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}f(s,u(s))ds.\end{aligned} \tag{14}$$

Notice that the problem (1) has solutions if the operator equation (13) has fixed points.
For computational convenience, we set

$$Q = \sup_{t\in[0,T]}\Big\{\frac{t^{q-1}(1-e^{-kt})}{k\Gamma(q)} + \frac{|\nu_2(t)|T^{q-1}(1-e^{-kT})}{k\Gamma(q)} + \frac{|\nu_3(t)|T^{q-1}}{\Gamma(q)} + \frac{|\nu_4(t)|T^{q-2}}{\Gamma(q-1)}\Big\}. \tag{15}$$

Now we are in a position to present our first result which deals with the existence of a unique solution of the problem (1) and is based on Banach's contraction mapping principle.

Theorem 3.1 *Assume that $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ is a continuous functions satisfying the Lipschitz condition:*

(A_1) *there exists a positive number ℓ such that $|f(t,u) - f(t,v)| \le \ell|u-v|$, $\forall t \in [0,T]$, $u,v \in \mathbb{R}$.*

Then the problem (1) has a unique solution on $[0,T]$ if $\ell < 1/Q$, where Q is given by (15).

Proof. Let us fix $r \ge \frac{QM + \|\nu_1\|}{1-\ell Q}$, where $\sup_{t\in[0,T]}|f(t,0)| = M$ and Q is given by (15), and define a set $B_r = \{u \in \mathcal{C} : \|u\| \le r\}$. In the first step, we show that $\mathcal{H}B_r \subset B_r$, where the operator $\mathcal{H}$ is defined by (14). For any $u \in B_r, t \in [0,T]$, we have

$$\begin{aligned}|f(t,u(t))| &= |f(t,u(t)) - f(t,0) + f(t,0)| \le |f(t,u(t)) - f(t,0)| + |f(t,0)| \\ &\le \ell\|u\| + M \le \ell r + M.\end{aligned}$$

Then, for $u \in B_r$, we obtain

$$\|(\mathcal{H}u)\| \le \sup_{t\in[0,T]}\Big\{|\nu_1(t)| + \int_0^t e^{-k(t-s)}\Big(\int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)}|f(x,u(x))|dx\Big)ds$$

$$
\begin{aligned}
&+ \quad |\nu_2(t)|\int_0^T e^{-k(T-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}|f(x,u(x))|dx\Big)ds \\
&+ \quad |\nu_3(t)|\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|ds + |\nu_4(t)|\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}|f(s,u(s))|ds.\Big\} \\
&\leq \quad (\ell r + M)\sup_{t\in[0,T]}\Big\{\int_0^t e^{-k(t-s)}\Big(\int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)}dx\Big)ds \\
&+ \quad |\nu_2(t)|\int_0^T e^{-k(T-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}dx\Big)ds \\
&+ \quad |\nu_3(t)|\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds + |\nu_4(t)|\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}ds.\Big\} + \|\nu_1\| \\
&\leq \quad (\ell r + M)Q + \|\nu_1\| \leq r.
\end{aligned}
$$

This shows that $\mathcal{H}B_r \subset B_r$. Next we show that the operator $\mathcal{H}$ is a contraction. Let $u, v \in \mathcal{C}$. Then

$$
\begin{aligned}
\|\mathcal{H}u - \mathcal{H}v\| \quad &\leq \quad \sup_{t\in[0,T]}\Big\{\int_0^t e^{-k(t-s)}\Big(\int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)}|f(x,u(x)) - f(x,v(x))|dx\Big)ds \\
&+ \quad |\nu_2(t)|\int_0^T e^{-k(T-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}\Big|f(x,u(x)) - f(x,v(x))\Big|dx\Big)ds \\
&+ \quad |\nu_3(t)|\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}\Big|f(s,u(s)) - f(s,v(s))\Big|ds \\
&+ \quad |\nu_4(t)|\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}\Big|f(s,u(s)) - f(s,v(s))\Big|ds\Big\} \\
&\leq \quad \ell Q \parallel u - v \parallel,
\end{aligned}
$$

where we have used (15). By the given assumption: $\ell < 1/Q$, it follows that the operator $\mathcal{H}$ is a contraction. Thus, by Banach's contraction mapping principle, we deduce that the operator $\mathcal{H}$ has a fixed point, which equivalently means that the problem (1) has a unique solution on $[0, T]$. □

Now we show the existence of solutions for the problem (1) by means of Krasnoselskiis fixed point theorem, which is stated below for the reader's convenience.

Lemma 3.2 *(Krasnoselskii's fixed point theorem [20]) Let $\mathcal{Y}$ be a closed bounded, convex and nonempty subset of a Banach space $\mathcal{X}$. Let φ_1, φ_2 be the operators such that (i) $\varphi_1 y_1 + \varphi_2 y_2 \in Y$ whenever $y_1, y_2 \in Y$; (ii) φ_1 is compact and continuous and (iii) φ_2 is a contraction mapping. Then there exists $y \in \mathcal{Y}$ such that $y = \varphi_1 y + \varphi_2 y$.*

Theorem 3.3 *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $|f(t, x)| \leq g(t)$, $\forall (t, x) \in [0, T] \times \mathbb{R}$, where $g \in C([0, T], \mathbb{R}^+)$, with $\sup_{t\in[0,T]} |g(t)| = \|g\|$. In addition, it is assumed that $\ell Q_1 < 1$, where*

$$
Q_1 \quad = \sup_{t\in[0,T]}\Big\{\frac{|\nu_2(t)|T^{q-1}(1-e^{-kT})}{k\Gamma(q)} + \frac{|\nu_3(t)|T^{q-1}}{\Gamma(q)} + \frac{|\nu_4(t)|T^{q-2}}{\Gamma(q-1)}\Big\}. \tag{16}
$$

Then the problem (1) has at least one solution on $[0, T]$.

Proof. With $\bar{r} \geq Q\|g\| + \|\nu_1\|$ (Q is given by (15)), we define operators $\mathcal{H}_1$ and $\mathcal{H}_2$ on $B_{\bar{r}} = \{u \in \mathcal{C} : \|u\| \leq \bar{r}\}$ as follows

$$
(\mathcal{H}_1 u)(t) = \int_0^t e^{-k(t-s)}\Big(\int_0^s \frac{(s-x)^{\alpha-2}}{\Gamma(\alpha-1)}f(x,u(x))dx\Big)ds,
$$

$$\begin{aligned}(\mathcal{H}_2u)(t) &= \nu_1(t)+\nu_2(t)\int_0^T e^{-k(T-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}f(x,u(x))dx\Big)ds\\ &+\nu_3(t)\int_0^T \frac{(T-s)^{q-2}}{\Gamma(q-1)}f(s,u(s))ds+\nu_4(t)\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}f(s,u(s))ds.\end{aligned}$$

For $u,v\in B_{\bar r}$, it is easy to verify that $\|\mathcal{H}_1u+\mathcal{H}_2v\|\le Q\|g\|+\|\nu_1\|$. Thus, $\mathcal{H}_1u+\mathcal{H}_2v\in B_{\bar r}$. Using the assumption (A_1) and (3.3),one can get $\|\mathcal{H}_2u-\mathcal{H}_2v\|\le \ell Q_1\parallel u-v\parallel$, which implies that the operator $\mathcal{H}_2$ is a contraction in view of the given condition: $\ell Q_1<1$.
Continuity of f implies that the operator $\mathcal{H}_1$ is continuous. Also, $\mathcal{H}_1$ is uniformly bounded on $B_{\bar r}$ as

$$\|\mathcal{H}_1u\|\le \frac{(1-e^{-kT})T^{q-1}\|g\|}{k\Gamma(q)}.$$

Finally, we establish that the operator $\mathcal{H}_1$ is compact. Letting $\sup_{(t,u)\in[0,T]\times B_{\bar r}}|f(t,u)|=f_{\bar r}$, for $t_1,t_2\in[0,T]$, we have

$$\begin{aligned}&\|(\mathcal{H}_1u)(t_2)-(\mathcal{H}_1u)(t_1)\|\\ \le\ & f_{\bar r}\left(|e^{-kt_2}-e^{-kt_1}|\int_0^{t_1} e^{ks}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}dx\Big)ds+\int_{t_1}^{t_2} e^{-k(t_2-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}dx\Big)ds\right)\\ &\to 0 \text{ as } t_2-t_1\to 0,\end{aligned}$$

independent of u. Thus the operator $\mathcal{H}_1$ is relatively compact on $B_{\bar r}$. Hence, by the Arzelá-Ascoli Theorem, the operator $\mathcal{H}_1$ is compact on $B_{\bar r}$. Thus all the assumptions of Lemma 3.2 are satisfied. In consequence, by the conclusion of Lemma 3.2, the problem (1) has at least one solution on $[0,T]$. □

In our last result, we prove the existence of solutions the problem(1) by applying Leray-Schauder nonlinear alternative.

Lemma 3.4 *(Nonlinear alternative for single valued maps [20]). Let $\mathcal{S}$ be a closed, convex subset of a Banach space $\mathcal{E}$, and $\mathcal{V}$ be an open subset of $\mathcal{S}$ with $0\in\mathcal{V}$. Suppose that $\mathcal{A}:\overline{\mathcal{V}}\to\mathcal{S}$ is continuous and compact (that is, $\mathcal{A}(\overline{\mathcal{V}})$ is a relatively compact subset of $\mathcal{S}$) map. Then either*

(i) $\mathcal{A}$ has a fixed point in $\overline{\mathcal{V}}$, or

(ii) there is a $v\in\partial\mathcal{V}$ (the boundary of $\mathcal{V}$ in $\mathcal{S}$) and $\lambda\in(0,1)$ with $v=\lambda\mathcal{A}(v)$.

Theorem 3.5 *Let $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Assume that*

(A_3) there exist a function $p\in C([0,T],\mathbb{R}^+)$, and a nondecreasing function $\psi:\mathbb{R}^+\to\mathbb{R}^+$ such that $|f(t,x)|\le p(t)\psi(\|u\|)$, $\forall(t,u)\in[0,T]\times\mathbb{R}$;

(A_4) there exists a constant $\overline{M}>0$ such that $\overline{M}/\overline{Q}>1$, where

$$\overline{Q}=\|\nu_1\|+\|p\|\psi(\|\overline{M}\|)Q. \tag{17}$$

Then the boundary value problem (1) has at least one solution on $[0,T]$.

Proof. We complete the proof in several steps. Firstly we show that the operator $\mathcal{H}:\mathcal{C}\to\mathcal{C}$ defined by (14) maps bounded sets into bounded sets in $\mathcal{C}$. For the positive number r, let $B_r=\{u\in\mathcal{C}:\|u\|\le r\}$ be a bounded set in $\mathcal{C}$. Then, for $u\in B_r$, we have

$$\begin{aligned}|(\mathcal{H}u)(t)| &\le |\nu_1(t)|+\int_0^t e^{-k(t-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}|f(x,u(x))|dx\Big)ds\\ &+ |\nu_2(t)|\int_0^T e^{-k(T-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}|f(x,u(x))|dx\Big)ds\\ &+ |\nu_3(t)|\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|ds+|\nu_4(t)|\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}|f(s,u(s))|ds.\\ &\le |\nu_1(t)|+\|p\|\psi(\|u\|)\Big\{\frac{t^{q-1}(1-e^{-kt})}{k\Gamma(q)}+|\nu_2(t)|\frac{T^{q-1}(1-e^{-kT})}{k\Gamma(q)}+|\nu_3(t)|\frac{T^{q-1}}{k\Gamma(q)}+|\nu_4(t)|\frac{T^{q-2}}{k\Gamma(q-1)}\Big\},\end{aligned}$$

which implies that $\|(\mathcal{H}u)\| \le \|\nu_1(t)\| + \|p\|\psi(r)Q$, where Q is given by (15).
Next we show that $\mathcal{H}$ maps bounded sets into equicontinuous sets of $\mathcal{C}$. Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and $u \in B_r$, where B_r is a bounded set in $\mathcal{C}$. Then we obtain

$$
\begin{aligned}
&|(\mathcal{H}u)(t_2) - (\mathcal{H}u)(t_1)| \le |\nu_1(t_2) - \nu_1(t_1)| \\
&+\Big|\int_0^{t_2} e^{-k(t_2-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} f(x,u(x))dx\Big)ds - \int_0^{t_1} e^{-k(t_1-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)} f(x,u(x))dx\Big)ds\Big| \\
&+|\nu_2(t_2) - \nu_2(t_1)|\int_0^T e^{-k(T-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}|f(x,u(x))|dx\Big)ds \\
&+|\nu_3(t_2) - \nu_3(t_1))|\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|ds + |\nu_4(t_2) - \nu_4(t_1)|\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}|f(s,u(s))|ds \\
&\le \Big|(\frac{b}{\lambda_2} + \frac{\delta_2 c}{k\lambda_2\delta_3})(t_2 - t_1) + \frac{c}{k^2\delta_2}(e^{-kt_2} - e^{-kt_1})\Big| + \|p\|\psi(r)\Big[\frac{(1-e^{-k(t_2-t_1)})}{k\Gamma(q)}\Big(t_1^{q-1}(1-e^{-kt_1}) + t_2^{q-1}\Big) \\
&+\frac{T^{q-2}e^{-kt_1}}{\Gamma(q)}\Big|\frac{\gamma_3}{\delta_3}\Big|\Big\{\frac{T}{k}\Big(2 - e^{-kT}\Big) + \frac{q-1}{k^2}\Big\}\Big(1 - e^{-k(t_2-t_1)}\Big) \\
&+\frac{T^{q-2}}{\Gamma(q)}\Big\{\Big|\frac{(\gamma_2\delta_3 - \gamma_3\delta_2)T}{\lambda_2\delta_3}\Big|\Big(2 - e^{-kT}\Big) + \frac{q-1}{k}\Big|\frac{\delta_2\gamma_3}{\delta_3\gamma_2}\Big|\Big\}(t_2 - t_1)\Big].
\end{aligned}
$$

Obviously the right hand side of the above inequality tends to zero independently of $u \in B_r$ as $t_2 - t_1 \to 0$. As $\mathcal{H}$ satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\mathcal{H} : \mathcal{C} \to \mathcal{C}$ is completely continuous. The conclusion will follow form the Leray-Schauder nonlinear alternative (Lemma 3.4) once we have proved the boundedness of the set of all solutions to equations $u = \lambda\mathcal{H}u$ for $\lambda \in [0,1]$. Let u be a solution. Then, for $t \in [0,T]$, and using the computations in proving that $\mathcal{H}$ is bounded, we have

$$
\begin{aligned}
|u(t)| \;&=\; |\lambda(\mathcal{H}u)(t)| \le |\nu_1(t)| + \int_0^t e^{-k(t-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}|f(x,u(x))|dx\Big)ds \\
&+\; |\nu_2(t)|\int_0^T e^{-k(T-s)}\Big(\int_0^s \frac{(s-x)^{q-2}}{\Gamma(q-1)}|f(x,u(x))|dx\Big)ds \\
&+\; |\nu_3(t)|\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s))|ds + |\nu_4(t)|\int_0^T \frac{(T-s)^{q-3}}{\Gamma(q-2)}|f(s,u(s))|ds \\
&\le\; |\nu_1(t)| + \|p\|\psi(\|u\|)\Big\{\frac{t^{q-1}(1-e^{-kt})}{k\Gamma(q)} + |\nu_2(t)|\frac{T^{q-1}(1-e^{-kT})}{k\Gamma(q)} + |\nu_3(t)|\frac{T^{q-1}}{k\Gamma(q)} + |\nu_4(t)|\frac{T^{q-2}}{k\Gamma(q-1)}\Big\}.
\end{aligned}
$$

In consequence, we get

$$\|u\|/\Big[\|\nu_1\| + \|p\|\psi(\|u\|)Q\Big] \le 1.$$

In view of (A_4), there exists $\overline{M}$ such that $\|u\| \ne \overline{M}$. Let us set $U = \{u \in \mathcal{C} : \|u\| < \overline{M}\}$. Note that the operator $\mathcal{H} : \overline{U} \to C([0,T],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u = \lambda\mathcal{H}(u)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.4), we deduce that $\mathcal{H}$ has a fixed point $u \in \overline{U}$ which is a solution of the problem (1). □

Example 3.6 *Consider the following anti-periodic fractional boundary value problem:*

$$
\begin{cases}
({}^cD^{5/2} + 2\,{}^cD^{3/2})u(t) = \dfrac{\sin u}{25} + e^{-t}\cos t,\; t \in [0,2], \\
u(0) + u(2) = 1,\;\; u'(0) - (1/2)u'(2) = 2,\;\; u''(0) + (1/4)u''(2) = 1,
\end{cases}
\tag{18}
$$

where $f(t,u(t)) = \frac{\sin u}{25} + e^{-t}\cos t$, $T = 2$, $k = 2$, $\alpha_1 = 1$, $\gamma_1 = 1$, $\alpha_2 = 1$, $\gamma_2 = -1/2$, $\alpha_3 = 1$, $\gamma_3 = 1/4$, $a = 1$, $b = 2$, $c = 1$. *With the given data, we find that the values of* Q *and* Q_1 *respectively given by (15) and* (16) *are* $Q \simeq 7.557935$, $Q_1 \simeq 6.513574$.

(a) *For the applicability of Theorem 3.1, we have that* $\ell and = 1/25$ *as* $|f(t,u) - f(t,v)| \leq \frac{1}{25}|u - v|$ *and* $\ell Q \simeq 0.302317 < 1$. *Thus all the conditions of Theorem 3.1 are satisfied. Hence the conclusion of Theorem 3.1 implies that there exists a unique solution for problem (18)on* $[0, 2]$.

(b) *Observe that* $|f(t,u)| \leq g(t) = \frac{1}{25} + e^{-t}\cos t$ *with* $\|g\| = \frac{26}{25}$ *and* $\ell Q_1 \simeq 0.260543 < 1$. *Thus all the conditions of Theorem (3.3) are satisfied. Hence, by the conclusion of Theorem (3.3), the problem (18) has at least one solution on* $[0, 2]$.

(c) *Obviously* $|f(t,u)| \leq 1/25 + e^{-t}\cos t$. *Taking* $\psi(\|u\|) = 1$, $p(t) = 1/25 + e^{-t}\cos t$, *we have* $\overline{Q} = \|\nu_1\| + \|p\|\psi(\|\overline{M}\|)Q \simeq 13.224428$ *($\overline{Q}$ is given by (17)) so that* $\overline{M} > 13.224428$. *Thus all the conditions of Theorem 3.5 are satisfied. Hence it follows by the conclusion of Theorem 3.5 that the problem (18) has at least one solution on* $[0, 2]$.

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ADDITIVE-QUADRATIC ρ-FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES

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Abstract. Let

$$\begin{aligned} M_1 f(x,y) : &= \frac{3}{4}f(x+y) - \frac{1}{4}f(-x-y) + \frac{1}{4}f(x-y) + \frac{1}{4}f(y-x) - f(x) - f(y), \\ M_2 f(x,y) : &= 2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) - f(x) - f(y). \end{aligned}$$

Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequalities

$$N(M_1 f(x,y), t) \geq N(\rho M_2 f(x,y), t) \tag{0.1}$$

where ρ is a fixed real number with $|\rho| < 1$, and

$$N(M_2 f(x,y), t) \geq N(\rho M_1 f(x,y), t) \tag{0.2}$$

where ρ is a fixed real number with $|\rho| < \frac{1}{2}$.

1. Introduction and preliminaries

Katsaras [19] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [15, 21, 48]. In particular, Bag and Samanta [3], following Cheng and Mordeson [11], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [20]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [4].

We use the definition of fuzzy normed spaces given in [3, 25, 26] to investigate the Hyers-Ulam stability of additive ρ-functional inequalities in fuzzy Banach spaces.

Definition 1.1. [3, 25, 26, 27] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x,t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N_3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N_4) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
(N_5) $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.
(N_6) for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed vector space.*

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [25, 28].

Definition 1.2. [3, 28, 26, 27] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.

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Definition 1.3. [3, 28, 26, 27] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [4]).

The stability problem of functional equations originated from a question of Ulam [47] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [17] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [39] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [16] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [46] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [12] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 5, 9, 10, 14, 22, 24, 29, 34, 35, 36, 40, 41, 42, 43, 44, 45, 49, 50]).

We recall a fundamental result in fixed point theory.

Theorem 1.4. [6, 13] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [18] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [7, 8, 30, 31, 38]).

Park [32, 33] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

In Section 2, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.1) in fuzzy Banach spaces by using the fixed point method.

In Section 3, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.2) in fuzzy Banach spaces by using the fixed point method.

2. Additive-quadratic ρ-functional inequality (0.1)

In this section, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.1) in fuzzy Banach spaces. Let ρ be a real number with $|\rho| < 1$.

We need the following lemma to prove the main results.

Lemma 2.1.
(i) *If an odd mapping* $f : X \to Y$ *satisfies*

$$N(M_1 f(x,y), t) \geq N(\rho M_2 f(x,y), t) \tag{2.1}$$

for all $x, y \in X$ *and all* $t > 0$*, then* f *is the Cauchy additive mapping.*
(ii) *If an even mapping* $f : X \to Y$ *satisfies* $f(0) = 0$ *and* (2.1)*, then* f *is the quadratic mapping.*

Proof. (i) Letting $y = x$ in (2.1), we get $N(f(2x) - 2f(x), t) = 1$ for all $t > 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} N(f(x+y) - f(x) - f(y), t) &= N(\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right), t) \\ &= N(\rho(f(x+y) - f(x) - f(y)), t) \end{aligned}$$

for all $t > 0$ and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$ by (N_5).

(ii) Letting $y = x$ in (2.1), we get $N\left(\frac{1}{2}f(2x) - 2f(x), t\right) = 1$ for all $t > 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4} f(x) \tag{2.3}$$

for all $x \in X$.

It follows from (2.1) and (2.3) that

$$\begin{aligned} & N\left(\frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) - f(x) - f(y), t\right) \\ &= N\left(\rho\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right), t\right) \\ &= N\left(\rho\left(\frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) - f(x) - f(y)\right), t\right) \end{aligned}$$

for all $t > 0$ and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$ by (N_5). □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.1) in fuzzy Banach spaces.

Theorem 2.2. *Let* $\varphi : X^2 \to [0, \infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x,y) \leq \frac{L}{4}\varphi(2x, 2y) \leq \frac{L}{2}\varphi(2x, 2y) \tag{2.4}$$

for all $x, y \in X$.
(i) *Let* $f : X \to Y$ *be an odd mapping satisfying*

$$N(M_1 f(x,y), t) \geq \min\left\{N(\rho M_2 f(x,y), t), \frac{t}{t + \varphi(x,y)}\right\} \tag{2.5}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{(2-2L)t}{(2-2L)t + L\varphi(x,x)} \tag{2.6}$$

for all $x \in X$ and all $t > 0$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.5). Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(2-2L)t}{(2-2L)t + L\varphi(x,x)} \tag{2.7}$$

for all $x \in X$ and all $t > 0$.

Proof. (i) Letting $y = x$ in (2.5), we get

$$N\left(f(2x) - 2f(x), t\right) \geq \frac{t}{t + \varphi(x,x)} \tag{2.8}$$

and so

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \tag{2.9}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(x,x)},\ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [23, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{2}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\varphi(x,x)} = \frac{t}{t + \varphi(x,x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.9) that $N\left(f(x) - 2f\left(\frac{x}{2}\right), \frac{L}{2}t\right) \geq \frac{t}{t+\varphi(x,x)}$ for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.4, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{2.10}$$

for all $x \in X$. Since $f : X \to Y$ is odd, $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.10) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - A(x), \mu t) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, A) \leq \frac{L}{2 - 2L}.$$

This implies that the inequality (2.6) holds.

By (2.5),

$$N\left(2^n M_1 f\left(\frac{x}{2^n}, \frac{y}{2^n}\right), 2^n t\right) \geq \min\left\{N\left(2^n M_2 f\left(\frac{x}{2^n}, \frac{y}{2^n}\right), 2^n t\right), \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}\right\}$$

and so

$$N\left(2^n M_1 f\left(\frac{x}{2^n}, \frac{y}{2^n}\right), t\right) \geq \min\left\{N\left(2^n M_2 f\left(\frac{x}{2^n}, \frac{y}{2^n}\right), t\right), \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n}\varphi(x, y)}\right\}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$N(M_1 A(x, y), t) \geq N(\rho M_2 A(x, y), t)$$

for all $x, y \in X$ and all $t > 0$. By Lemma 2.1, the mapping $A : X \to Y$ is Cauchy additive.

(ii) Letting $y = x$ in (2.5), we get

$$N\left(\frac{1}{2}f(2x) - 2f(x), t\right) \geq \frac{t}{t + \varphi(x, x)} \tag{2.11}$$

and so

$$N\left(f(x) - 4f\left(\frac{x}{2}\right), t\right) \geq \frac{\frac{t}{2}}{\frac{t}{2} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} = \frac{t}{t + 2\varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \tag{2.12}$$

for all $x \in X$.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{4}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \frac{L}{4}\varphi(x,x)} = \frac{t}{t + \varphi(x,x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.12) that $N\left(f(x) - 4f\left(\frac{x}{2}\right), \frac{L}{2}t\right) \geq \frac{t}{t+\varphi(x,x)}$ for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{2.13}$$

for all $x \in X$. Since $f : X \to Y$ is even, $Q : X \to Y$ is a even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.13) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality

$$d(f,Q) \leq \frac{L}{2 - 2L}.$$

This implies that the inequality (2.7) holds.

By (2.5),

$$N\left(4^n M_1 f\left(\frac{x}{2^n}, \frac{y}{2^n}\right), 4^n t\right) \geq \min\left\{N\left(4^n M_2 f\left(\frac{x}{2^n}, \frac{y}{2^n}\right), 4^n t\right), \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}\right\}$$

and so

$$N\left(4^n M_1 f\left(\frac{x}{2^n}, \frac{y}{2^n}\right), t\right) \geq \min\left\{N\left(4^n M_2 f\left(\frac{x}{2^n}, \frac{y}{2^n}\right), t\right), \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x,y)}\right\}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{4^n}}{\frac{t}{4^n} + \frac{L^n}{4^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$N(M_1 Q(x,y), t) \geq N(\rho M_2 Q(x,y), t)$$

for all $x, y \in X$ and all $t > 0$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic. □

Corollary 2.3. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$.*

(i) *Let $f : X \to Y$ be an odd mapping satisfying*

$$N(M_1 f(x,y), t) \geq \min\left\{N\left(\rho M_2 f(x,y), t\right), \frac{t}{t+\theta(\|x\|^p + \|y\|^p)}\right\} \tag{2.14}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.14). Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 4\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Choosing $L = 2^{1-p}$ for an odd mapping case and $L = 2^{2-p}$ for an even mapping case, then we obtain the desired results. □

Theorem 2.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{2.15}$$

for all $x, y \in X$..
(i) *Let $f : X \to Y$ be an odd mapping satisfying (2.5). Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N\left(f(x) - A(x), t\right) \geq \frac{(2-2L)t}{(2-2L)t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.5). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(2-2L)t}{(2-2L)t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.
(i) It follows from (2.8) that

$$N\left(f(x) - \frac{1}{2}f(2x), \frac{1}{2}t\right) \geq \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.
(ii) It follows from (2.11) that

$$N\left(f(x) - \frac{1}{4}f(2x), \frac{1}{2}t\right) \geq \frac{t}{t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.
The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\|\cdot\|$.*

(i) *Let $f : X \to Y$ be an odd mapping satisfying* (2.14). *Then $A(x) := N\text{-}\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N(f(x) - A(x), t) \ge \frac{(2-2^p)t}{(2-2^p)t + 2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and* (2.14). *Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \ge \frac{(4-2^p)t}{(4-2^p)t + 4\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Choosing $L = 2^{p-1}$ for an odd mapping case and $L = 2^{p-2}$ for an even mapping case, then we obtain the desired results. □

3. Additive-quadratic ρ-functional inequality (0.2)

In this section, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.2) in fuzzy Banach spaces. Let ρ be a real number with $|\rho| < \frac{1}{2}$.

Lemma 3.1.
(i) *If an odd mapping $f : X \to Y$ satisfies*

$$N(M_2 f(x, y), t) \ge N(\rho M_1 f(x, y), t) \tag{3.1}$$

for all $x, y \in X$ and all $t > 0$, then f is the Cauchy additive mapping.
(ii) *If an even mapping $f : X \to Y$ satisfies $f(0) = 0$ and* (3.1), *then f is the quadratic mapping.*

Proof. (i) Letting $y = 0$ in (3.1), we get $N\left(2f\left(\frac{x}{2}\right) - f(x), t\right) = 1$ for all $t > 0$. So

$$f\left(\frac{x}{2}\right) = \frac{1}{2} f(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} N(f(x+y) - f(x) - f(y), t) &\ge N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t\right) \\ &= N(\rho(f(x+y) - f(x) - f(y)), t) \end{aligned}$$

for all $t > 0$ and so

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$ by (N_5).

(ii) Letting $y = 0$ in (3.1), we get $N\left(4f\left(\frac{x}{2}\right) - f(x), t\right)$ for all $t > 0$. So

$$f\left(\frac{x}{2}\right) = \frac{1}{4} f(x) \tag{3.3}$$

for all $x \in X$.

It follows from (3.1) and (3.3) that

$$\begin{aligned} & N\left(\frac{1}{2} f(x+y) + \frac{1}{2} f(x-y) - f(x) - f(y), t\right) \\ & \ge N\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y), t\right) \\ & = N\left(\rho\left(\frac{1}{2} f(x+y) + \frac{1}{2} f(x-y) - f(x) - f(y)\right), t\right) \end{aligned}$$

for all $t > 0$ and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$ by (N_5). □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive-quadratic ρ-functional inequality (0.2) in fuzzy Banach spaces.

Theorem 3.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function satisfying* (2.4).
(i) *Let $f : X \to Y$ be an odd mapping satisfying*

$$N\left(M_2 f(x,y), t\right) \geq \min\left\{N\left(\rho M_1 f(x,y), t\right), \frac{t}{t+\varphi(x,y)}\right\} \tag{3.4}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{(1-L)t}{(1-L)t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and* (3.4)*. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(1-L)t}{(1-L)t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.

Proof. (i) Letting $y = 0$ in (3.4), we get

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) = N\left(2f\left(\frac{x}{2}\right) - f(x), t\right) \geq \frac{t}{t+\varphi(x,0)} \tag{3.5}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t+\varphi(x,0)},\ \forall x \in X, \forall t > 0\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [23, Lemma 2.1]).

The rest of the proof is similar to the proof of Theorem 2.2 (i).

(ii) Letting $y = 0$ in (3.4), we get

$$N\left(f(x) - 4f\left(\frac{x}{2}\right), t\right) = N\left(4f\left(\frac{x}{2}\right) - f(x), t\right) \geq \frac{t}{t+\varphi(x,0)} \tag{3.6}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2 (ii). □

Corollary 3.3. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$.*
(i) *Let $f : X \to Y$ be an odd mapping satisfying*

$$N\left(M_2 f(x,y) - \rho M_1 f(x,y), t\right) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{3.7}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that

$$N\left(f(x) - A(x), t\right) \geq \frac{(2^p - 2)t}{(2^p-2)t + 2^p\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.7). Then $Q(x) := N$-$\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \geq \frac{(2^p - 4)t}{(2^p - 4)t + 2^p\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Choosing $L = 2^{1-p}$ for an odd mapping case and $L = 2^{2-p}$ for an even mapping case, then we obtain the desired results. □

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function satisfying (2.15).*
(i) *Let $f : X \to Y$ be an odd mapping satisfying (3.4). Then $A(x) := N$-$\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N(f(x) - A(x), t) \geq \frac{(1 - L)t}{(1 - L)t + L\varphi(x, x)}$$

for all $x \in X$ and all $t > 0$.
(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.4). Then $Q(x) := N$-$\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \geq \frac{(1 - L)t}{(1 - L)t + L\varphi(x, x)}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.2.
(i) It follows from (3.5) that

$$N\left(f(x) - \frac{1}{2}f(2x), \frac{t}{2}\right) \geq \frac{t}{t + \varphi(2x, 0)}$$

and so

$$N\left(f(x) - \frac{1}{2}f(2x), Lt\right) \geq \frac{2Lt}{2Lt + \varphi(2x, 0)} = \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$.
(ii) It follows from (3.6) that

$$N\left(f(x) - \frac{1}{4}f(2x), \frac{t}{4}\right) \geq \frac{t}{t + \varphi(2x, 0)}$$

and so

$$N\left(f(x) - \frac{1}{4}f(2x), Lt\right) \geq \frac{4Lt}{4Lt + \varphi(2x, 0)} = \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$.
The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\|\cdot\|$.*
(i) *Let $f : X \to Y$ be an odd mapping satisfying (3.7). Then $A(x) := N$-$\lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$N(f(x) - A(x), t) \geq \frac{(2 - 2^p)t}{(2 - 2^p)t + 2^p\theta\|x\|^p}$$

for all $x \in X$.

(ii) *Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.7). Then $Q(x) := N$-$\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(4-2^p)t}{(4-2^p)t + 2^p\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Choosing $L = 2^{p-1}$ for an odd mapping case and $L = 2^{p-2}$ for an even mapping case, then we obtain the desired results. □

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Hyers-Ulam stability of set-valued functional equations: a fixed point approach

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Abstract. In [36], Park proved the Hyers-Ulam stability of set-valued functional equations by using the direct method.

In this paper, we prove the Hyers-Ulam stability of set-valued functional equations by using the fixed point method.

1. Introduction and preliminaries

Set-valued functions in Banach spaces have been developed in the last decades. The pioneering paper by Aumann [5] and Debreu [14] were inspired by problems arising in Control Theory and Mathematical Economics. We can refer to the papers by Arrow and Debreu [3], McKenzie [29], the momographs by Hindenbrand [20], Aubin and Frankowska [4], Castaing and Valadier [8], Klein and Thompson [26] and the survey by Hess [19].

The stability problem of functional equations originated from a question of Ulam [50] concerning the stability of group homomorphisms. Hyers [21] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [40] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [18] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach

The functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [49] for mappings $f: X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [12] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [13] proved the Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 17, 18, 22, 23], [41]–[48]).

In [25], Jun and Kim considered the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{1.1}$$

It is easy to show that the function $f(x)=x^3$ satisfies the functional equation (1.1), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*.

In [28], Lee et al. considered the following quartic functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \tag{1.2}$$

It is easy to show that the function $f(x)=x^4$ satisfies the functional equation (1.2), which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*.

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Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies
(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Let (X, d) be a generalized metric space. An operator $T : X \to X$ satisfies a Lipschitz condition with Lipschitz constant L if there exists a constant $L \ge 0$ such that $d(Tx, Ty) \le Ld(x, y)$ for all $x, y \in X$. If the Lipschitz constant L is less than 1, then the operator T is called a strictly contractive operator. Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity. We recall the following theorem by Margolis and Diaz.

Theorem 1.1. [9, 15] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that
(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \ge n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

In 1996, Isac and Rassias [24] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [10, 11, 31, 39]).

Let Y be a Banach space. We define the following:
2^Y : the set of all subsets of Y;
$C_b(Y)$: the set of all closed bounded subsets of Y;
$C_c(Y)$: the set of all closed convex subsets of Y;
$C_{cb}(Y)$: the set of all closed convex bounded subsets of Y.

On 2^Y we consider the addition and the scalar multiplication as follows:

$$C + C' = \{x + x' : x \in C, x' \in C'\}, \qquad \lambda C = \{\lambda x : x \in C\},$$

where $C, C' \in 2^Y$ and $\lambda \in \mathbb{R}$. Further, if $C, C' \in C_c(Y)$, then we denote by $C \oplus C' = \overline{C + C'}$.

It is easy to check that

$$\lambda C + \lambda C' = \lambda(C + C'), \qquad (\lambda + \mu)C \subseteq \lambda C + \mu C.$$

Furthermore, when C is convex, we obtain $(\lambda + \mu)C = \lambda C + \mu C$ for all $\lambda, \mu \in \mathbb{R}^+$.

For a given set $C \in 2^Y$, the distance function $d(\cdot, C)$ and the support function $s(\cdot, C)$ are respectively defined by

$$\begin{aligned} d(x, C) &= \inf\{\|x - y\| : y \in C\}, \qquad x \in Y, \\ s(x^*, C) &= \sup\{\langle x^*, x\rangle : x \in C\}, \qquad x^* \in Y^*. \end{aligned}$$

For every pair $C, C' \in C_b(Y)$, we define the Hausdorff distance between C and C' by

$$h(C, C') = \inf\{\lambda > 0 : C \subseteq C' + \lambda B_Y, \qquad C' \subseteq C + \lambda B_Y\},$$

where B_Y is the closed unit ball in Y.

The following proposition reveals some properties of the Hausdorff distance.

Proposition 1.2. *For every $C, C', K, K' \in C_{cb}(Y)$ and $\lambda > 0$, the following properties hold*

(a) $h(C \oplus C', K \oplus K') \leq h(C, K) + h(C', K')$;

(b) $h(\lambda C, \lambda K) = \lambda h(C, K)$.

Let $(C_{cb}(Y), \oplus, h)$ be endowed with the Hausdorff distance h. Since Y is a Banach space, $(C_{cb}(Y), \oplus, h)$ is a complete metric semigroup (see [8]). Debreu [14] proved that $(C_{cb}(Y), \oplus, h)$ is isometrically embedded in a Banach space as follows.

Lemma 1.3. [14] *Let $C(B_{Y^*})$ be the Banach space of continuous real-valued functions on B_{Y^*} endowed with the uniform norm $\|\cdot\|_u$. Then the mapping $j : (C_{cb}(Y), \oplus, h) \to C(B_{Y^*})$, given by $j(A) = s(\cdot, A)$, satisfies the following properties:*

(a) $j(A \oplus B) = j(A) + j(B)$;

(b) $j(\lambda A) = \lambda j(A)$;

(c) $h(A, B) = \|j(A) - j(B)\|_u$;

(d) $j(C_{cb}(Y))$ *is closed in* $C(B_{Y^*})$

for all $A, B \in C_{cb}(Y)$ and all $\lambda \geq 0$.

Let $f : \Omega \to (C_{cb}(Y), h)$ be a set-valued function from a complete finite measure space (Ω, Σ, ν) into $C_{cb}(Y)$. Then f is *Debreu integrable* if the composition $j \circ f$ is Bochner integrable (see [7]). In this case, the Debreu integral of f in Ω is the unique element $(D) \int_\Omega f d\nu \in C_{cb}(Y)$ such tha $j((D) \int_\Omega f d\nu)$ is the Bochner integral of $j \circ f$. The set of Debreu integrable functions from Ω to $C_{cb}(Y)$ will be denoted by $D(\Omega, C_{cb}(Y))$. Furthermore, on $D(\Omega, C_{cb}(Y))$, we define $(f + g)(\omega) = f(\omega) \oplus g(\omega)$ for all $f, g \in D(\Omega, C_{cb}(Y))$. Then we obtain that $((\Omega, C_{cb}(Y)), +)$ is an abelian semigroup.

Set-valued functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [6], [32]–[35], [37, 38]).

Using the fixed point method, we prove the additive set-valued functional equation, the quadratic set-valued functional equation, the cubic set-valued functional equation and the quartic set-valued functional equation.

Throughout this paper, let X be a real vector space and Y a Banach space.

2. Stability of the additive set-valued functional equation

Using the fixed point method, we prove the Hyers-Ulam stability of the additive set-valued functional equation.

Definition 2.1. [27] Let $f : X \to C_{cb}(Y)$. The additive set-valued functional equation is defined by

$$f(x + y) = f(x) \oplus f(y)$$

for all $x, y \in X$. Every solution of the additive set-valued functional equation is called an *additive set-valued mapping*.

Note that there are some examples in [27].

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{2} \varphi(2x, 2y)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying

$$h(f(x + y), f(x) \oplus f(y)) \leq \varphi(x, y) \tag{2.1}$$

for all $x, y \in X$. Then there exists a unique additive set-valued mapping $A : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), A(x)) \le \frac{L}{2-2L}\varphi(x,x) \tag{2.2}$$

for all $x \in X$.

Proof. Let $y = x$ in (2.1). Since $f(x)$ is convex, we get

$$h(f(2x), 2f(x)) \le \varphi(x,x) \tag{2.3}$$

and so

$$h\left(f(x), 2f\left(\frac{x}{2}\right)\right) \le \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \le \frac{L}{2}\varphi(x,x) \tag{2.4}$$

for all $x \in X$.

Consider

$$S := \{g : \ g : X \to C_{cb}(Y), \ g(0) = \{0\}\}$$

and introduce the generalized metric on X,

$$d(g,f) = inf\{\mu \in (0,\infty) : \ h(g(x), f(x)) \le \mu\varphi(x,x), \ x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [16, Theorem 2.4] and [30, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, f \in S$ be given such that $d(g,f) = \varepsilon$. Then

$$h(g(x), f(x)) \le \varepsilon\varphi(x,x)$$

for all $x \in X$. Hence

$$h(Jg(x), Jf(x)) \quad = \quad h\left(2g\left(\frac{x}{2}\right), 2f\left(\frac{x}{2}\right)\right) = 2h\left(g\left(\frac{x}{2}\right), f\left(\frac{x}{2}\right)\right) \le L\varphi(x,x)$$

for all $x \in X$. So $d(g,f) = \varepsilon$ implies that $d(Jg, Jf) \le L\varepsilon$. This means that

$$d(Jg, Jf) \le Ld(g,f)$$

for all $g, f \in S$.

It follows from (2.4) that $d(f, Jf) \le \frac{L}{2}$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{2.5}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.5) such that there exists a $\mu \in (0,\infty)$ satisfying

$$h(f(x), A(x)) \le \mu\varphi(x,x)$$

for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \le \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, A) \le \frac{L}{2-2L}.$$

This implies that the inequality (2.2) holds.

By (2.1),

$$h\left(2^n f\left(\frac{x+y}{2^n}\right), 2^n f\left(\frac{x}{2^n}\right) \oplus 2^n f\left(\frac{y}{2^n}\right)\right) \le 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \le L^n \varphi(x, y),$$

which tends to zero as $n \to \infty$ for all $x, y \in X$. Thus $A(x+y) = A(x) \oplus A(y)$, as desired. □

Corollary 2.3. *Let $p > 1$ and $\theta \ge 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying*

$$h(f(x+y), f(x) \oplus f(y)) \le \theta(||x||^p + ||y||^p) \tag{2.6}$$

for all $x, y \in X$. Then there exists a unique additive set-valued mapping $A : X \to Y$ satisfying

$$h(f(x), A(x)) \le \frac{2\theta}{2^p - 2} ||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{1-p}$ and we get the desired result. □

Theorem 2.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying (2.1). Then there exists a unique additive set-valued mapping $A : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), A(x)) \le \frac{1}{2-2L} \varphi(x, x)$$

for all $x \in X$.

Proof. It follows from (2.3) that

$$h\left(f(x), \frac{1}{2} f(2x)\right) \le \frac{1}{2} \varphi(x, x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $1 > p > 0$ and $\theta \ge 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying (2.6). Then there exists a unique additive set-valued mapping $A : X \to Y$ satisfying*

$$h(f(x), A(x)) \le \frac{2\theta}{2 - 2^p} ||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-1}$ and we get the desired result. □

3. Stability of the quadratic set-valued functional equation

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic set-valued functional equation.

Definition 3.1. [27] Let $f : X \to C_{cb}(Y)$. The quadratic set-valued functional equation is defined by

$$2f(x+y) \oplus 2f(x-y) = f(2x) \oplus f(2y)$$

for all $x, y \in X$. Every solution of the quadratic set-valued functional equation is called a *quadratic set-valued mapping.*

Note that there are some examples in [27].

Theorem 3.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le \frac{L}{4}\varphi(2x,2y)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and

$$h(2f(x+y) \oplus 2f(x-y), f(2x) \oplus f(2y)) \le \varphi(x,y) \tag{3.1}$$

for all $x, y \in X$. Then there exists a unique quadratic set-valued mapping $Q : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), Q(x)) \le \frac{L}{4-4L}\varphi(x,0)$$

for all $x \in X$.

Proof. Let $y = 0$ in (3.1). Since $f(x)$ is convex, we get

$$h(f(2x), 4f(x)) \le \varphi(x,0) \tag{3.2}$$

and

$$h\left(f(x), 4f\left(\frac{x}{2}\right)\right) \le \varphi\left(\frac{x}{2},0\right) \le \frac{L}{4}\varphi(x,0) \tag{3.3}$$

for all $x \in X$.

Consider

$$S := \{g : \ g : X \to C_{cb}(Y), \ g(0) = \{0\}\}$$

and introduce the generalized metric on X,

$$d(g,f) = inf\{\mu \in (0,\infty) : \ h(g(x), f(x)) \le \mu\varphi(x,0), \ x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [16, Theorem 2.4] and [30, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

By the same reasoning as in the proof of Theorem 2.2, one can show that

$$d(Jg, Jf) \le Ld(g,f)$$

for all $g, f \in S$.

It follows from (3.3) that $d(f, Jf) \leq \frac{L}{4}$.

The rest of the proof is similar to the proof of Theorem 2.2. $\square$

Corollary 3.3. *Let $p > 2$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and*

$$h(2f(x+y) \oplus 2f(x-y), f(2x) \oplus f(2y)) \leq \theta(||x||^p + ||y||^p) \tag{3.4}$$

for all $x, y \in X$. Then there exists a unique quadratic set-valued mapping $Q : X \to Y$ satisfying

$$h(f(x), Q(x)) \leq \frac{\theta}{2^p - 4}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{2-p}$ and we get the desired result. $\square$

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and (3.1). Then there exists a unique quadratic set-valued mapping $Q : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), Q(x)) \leq \frac{1}{4 - 4L}\varphi(x, 0)$$

for all $x \in X$.

Proof. It follows from (3.2) that

$$h\left(f(x), \frac{1}{4}f(2x)\right) \leq \frac{1}{4}\varphi(x, 0)$$

for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.2. $\square$

Corollary 3.5. *Let $0 < p < 2$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and (3.4). Then there exists a unique quadratic set-valued mapping $Q : X \to Y$ satisfying*

$$h(f(x), Q(x)) \leq \frac{\theta}{4 - 2^p}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-2}$ and we get the desired result. $\square$

4. Stability of the cubic set-valued functional equation

Using the fixed point method, we define a cubic set-valued functional equation and prove the Hyers-Ulam stability of the cubic set-valued functional equation.

Definition 4.1. [36] Let $f : X \to C_{cb}(Y)$. The cubic set-valued functional equation is defined by

$$f(2x+y) \oplus f(2x-y) = 2f(x+y) \oplus 2f(x-y) \oplus 12f(x)$$

for all $x, y \in X$. Every solution of the cubic set-valued functional equation is called a *cubic set-valued mapping.*

Theorem 4.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le \frac{L}{8}\varphi(2x,2y)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying $f(0) = \{0\}$ and

$$h(f(2x+y) \oplus f(2x-y), 2f(x+y) \oplus 2f(x-y) \oplus 12f(x)) \le \varphi(x,y) \tag{4.1}$$

for all $x, y \in X$. Then there exists a unique cubic set-valued mapping $C : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), C(x)) \le \frac{L}{16-16L}\varphi(x,0)$$

for all $x \in X$.

Proof. Let $y = 0$ in (4.1). Since $f(x)$ is convex, we get

$$h(2f(2x), 16f(x)) \le \varphi(x,0) \tag{4.2}$$

and

$$h\left(f(x), 8f\left(\frac{x}{2}\right)\right) \le \frac{1}{2}\varphi\left(\frac{x}{2},0\right) \le \frac{L}{16}\varphi(x,0) \tag{4.3}$$

for all $x \in X$.

Consider

$$S := \{g : \ g : X \to C_{cb}(Y), \ g(0) = \{0\}\}$$

and introduce the generalized metric on X,

$$d(g,f) = inf\{\mu \in (0,\infty) : \ h(g(x), f(x)) \le \mu\varphi(x,0), \ x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [16, Theorem 2.4] and [30, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 8g\left(\frac{x}{2}\right)$$

for all $x \in X$.

By the same reasoning as in the proof of Theorem 2.2, one can show that

$$d(Jg, Jf) \le Ld(g,f)$$

for all $g, f \in S$.

It follows from (4.3) that $d(f, Jf) \le \frac{L}{16}$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 4.3. *Let $p > 3$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying*

$$h(f(2x+y) \oplus f(2x-y), 2f(x+y) \oplus 2f(x-y) \oplus 12f(x)) \leq \theta(||x||^p + ||y||^p) \tag{4.4}$$

for all $x, y \in X$. Then there exists a unique cubic set-valued mapping $C : X \to Y$ satisfying

$$h(f(x), C(x)) \leq \frac{\theta}{2(2^p - 8)}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.2 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{3-p}$ and we get the desired result. □

Theorem 4.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 8L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying (4.1). Then there exists a unique cubic set-valued mapping $C : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), C(x)) \leq \frac{1}{16 - 16L}\varphi(x, 0)$$

for all $x \in X$.

Proof. It follows from (4.2) that

$$h\left(f(x), \frac{1}{8}f(2x)\right) \leq \frac{1}{16}\varphi(x, 0)$$

for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 2.2 and 4.2. □

Corollary 4.5. *Let $3 > p > 0$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying (4.4). Then there exists a unique cubic set-valued mapping $C : X \to Y$ satisfying*

$$h(f(x), C(x)) \leq \frac{\theta}{2(8 - 2^p)}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.4 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-3}$ and we get the desired result. □

5. Stability of the quartic set-valued functional equation

Using the fixed point method, we define a quartic set-valued functional equation and prove the Hyers-Ulam stability of the quartic set-valued functional equation.

Definition 5.1. [36] Let $f : X \to C_{cb}(Y)$. The quartic set-valued functional equation is defined by

$$f(2x+y) \oplus f(2x-y) \oplus 6f(y) = 4f(x+y) \oplus 4f(x-y) \oplus 24f(x)$$

for all $x, y \in X$. Every solution of the quartic set-valued functional equation is called a *quartic set-valued mapping*.

Theorem 5.2. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le \frac{L}{16}\varphi(2x,2y)$$

for all $x,y \in X$. Suppose that $f : X \to (C_{cb}(Y),h)$ is a mapping satisfying $f(0) = \{0\}$ and

$$h(f(2x+y) \oplus f(2x-y) \oplus 6f(y), 4f(x+y) \oplus 4f(x-y) \oplus 24f(x)) \le \varphi(x,y) \tag{5.1}$$

for all $x,y \in X$. Then there exists a unique quartic set-valued mapping $T : X \to (C_{cb}(Y),h)$ such that

$$h(f(x),T(x)) \le \frac{L}{32-32L}\varphi(x,0)$$

for all $x \in X$.

Proof. Let $y = 0$ in (5.1). Since $f(x)$ is convex, we get

$$h(2f(2x), 32f(x)) \le \varphi(x,0) \tag{5.2}$$

and

$$h\left(f(x), 16f\left(\frac{x}{2}\right)\right) \le \frac{1}{2}\varphi\left(\frac{x}{2},0\right) \le \frac{L}{32}\varphi(x,0) \tag{5.3}$$

for all $x \in X$.

Consider

$$S := \{g : \ g : X \to C_{cb}(Y), \ g(0) = \{0\}\}$$

and introduce the generalized metric on X,

$$d(g,f) = inf\{\mu \in (0,\infty) : \ h(g(x),f(x)) \le \mu\varphi(x,0), \ x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [16, Theorem 2.4] and [30, Lemma 2.1]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 16g\left(\frac{x}{2}\right)$$

for all $x \in X$.

By the same reasoning as in the proof of Theorem 2.2, one can show that

$$d(Jg,Jf) \le Ld(g,f)$$

for all $g,f \in S$.

It follows from (5.3) that $d(f,Jf) \le \frac{L}{32}$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 5.3. *Let $p > 4$ and $\theta \ge 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y),h)$ is a mapping satisfying*

$$h(f(2x+y) \oplus f(2x-y) \oplus 6f(y), 4f(x+y) \oplus 4f(x-y) \oplus 24f(x)) \le \theta(||x||^p + ||y||^p) \tag{5.4}$$

for all $x,y \in X$. Then there exists a unique quartic set-valued mapping $T : X \to Y$ satisfying

$$h(f(x),T(x)) \le \frac{\theta}{2(2^p-16)}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 5.2 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x,y \in X$. Then we can choose $L = 2^{4-p}$ and we get the desired result. □

Theorem 5.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 16L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying (5.1). Then there exists a unique quartic set-valued mapping $T : X \to (C_{cb}(Y), h)$ such that

$$h(f(x), T(x)) \leq \frac{1}{32 - 32L}\varphi(x, 0)$$

for all $x \in X$.

Proof. It follows from (5.2) that

$$h\left(f(x), \frac{1}{16}f(2x)\right) \leq \frac{1}{32}\varphi(x, 0)$$

for all $x \in X$.

The rest of the proof is similar to the proofs of Theorems 2.2 and 5.2. □

Corollary 5.5. *Let $4 > p > 0$ and $\theta \geq 0$ be real numbers, and let X be a real normed space. Suppose that $f : X \to (C_{cb}(Y), h)$ is a mapping satisfying (5.4). Then there exists a unique quartic set-valued mapping $T : X \to Y$ satisfying*

$$h(f(x), T(x)) \leq \frac{\theta}{2(2^p - 16)}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 5.4 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-4}$ and we get the desired result. □

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On precompactness of the Hausdorff fuzzy metric on closed sets

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In the paper, we construct a Hausdorff fuzzy metric on the family of nonempty closed subsets of a stationary and F-bounded fuzzy metric space. Using the construction of the Hausdorff fuzzy metric, we prove three equivalent characterizations for the given fuzzy metric space to be precompact. Furthermore, several examples are given.

Keywords: Fuzzy metric, Continuous t-norm, The Hausdorff fuzzy metric, Closed subset, Precompact.

AMS Subject Classifications: 54A40, 54B20, 54E35

1 Introduction

Fuzzy metric is an important notion in Fuzzy Topology. Many authors have introduced the concept of fuzzy metric from different points of view [2, 3, 12, 13]. In particular, George and Veeramani [3] obtained the concept of fuzzy metric with the help of continuous t-norms in 1994. Later, it was proved that the topological space induced by the fuzzy metric space is metrizable in [8]. This version of fuzzy metric determines the class of spaces that are tightly connected with the class of metrizable topological spaces. Hence it is interesting to study the version of fuzzy metric. Some contributions to the study of fuzzy metric spaces can be found in [4, 5, 6, 14, 15, 16, 19, 20].

In order to study the hyperspaces in a fuzzy metric space, Rodríguez-López and Romaguera [17] gave a definition of Hausdorff fuzzy metric on the family

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of nonempty compact sets. Unfortunately, the Hausdorff fuzzy metric defined by the authors does not provide a fuzzy metric when one consider the family of nonempty closed and F-bounded subsets of a given fuzzy metric space. In [17], Rodríguez-López and Romaguera illustrated the result above with the help of a example. It is a nature problem to explore under what condition the Hausdorff fuzzy metric defined by Rodríguez-López and Romaguera on the family of nonempty closed and F-bounded subsets of a given fuzzy metric space can provide a fuzzy metric. This is done in the present paper.

We construct a Hausdorff fuzzy metric on the family of nonempty closed subsets of a stationary and F-bounded fuzzy metric space. Also, we prove three necessary and sufficient conditions for the given fuzzy metric space to be precompact. Moreover, we give some illustrative examples.

2 Preliminaries

Throughout the paper the letter **N** shall denote the set of all nature numbers. Our basic reference for general topology is [1].

Definition 2.1 [3] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a *continuous t-norm* if it satisfies the following conditions:

(i) $*$ is associative and commutative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a \in [0,1]$;
(iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$, and $a, b, c, d \in [0,1]$.

The following are examples of t-norms: $a * b = \min\{a, b\}$; $a * b = a \cdot b$; $a * b = \max\{a + b - 1, 0\}$.

Definition 2.2 [3] A 3-tuple $(X, M, *)$ is said to be a *fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0,\infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t \in (0,\infty)$:

(i) $M(x, y, t) > 0$;
(ii) $M(x, y, t) = 1$ if and only if $x = y$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$;
(v) the function $M(x, y, \cdot) : (0,\infty) \to [0,1]$ is continuous.

If $(X, M, *)$ is a fuzzy metric space, we will say that $(M, *)$ is a fuzzy metric on X.

Definition 2.3 [3] Let $(X, M, *)$ be a fuzzy metric space and let $r \in (0,1), t > 0$ and $x \in X$. The set

$$B_M(x, r, t) = \{y \in X | M(x, y, t) > 1 - r\}$$

is called the *open ball with center x and radius r with respect to t.*

Obviously, $\{B_M(x, r, t) | x \in X, t > 0, r \in (0,1)\}$ forms a base of a topology in X. The topology is denoted by τ_M and is known to be metrizable (see [8]).

Lemma 2.4 [3] *Let $(X, M, *)$ be a fuzzy metric space. Then, for each $x \in X$, $\{B_M(x, \frac{1}{n}, \frac{1}{n}) | n \in \mathbf{N}\}$ is a neighborhood base at x for the topology τ_M.*

Definition 2.5 [3] Let (X, d) be a metric space. Define $a * b = ab$ for all $a, b \in [0, 1]$, and let M_d be the function on $X \times X \times (0, \infty)$ defined by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M_d, \cdot)$ is a fuzzy metric space and $(M_d, \cdot)$ is called *the standard fuzzy metric induced by d.*

Definition 2.6 [3] Let $(X, M, *)$ be a fuzzy metric space. A subset A of X is said to be *F-bounded* if there exist $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

We call $(X, M, *)$ a *F-bounded fuzzy metric space* provided that X is F-bounded. Clearly, a subset of an F-bounded fuzzy metric space is F-bounded.

Definition 2.7 [10] A fuzzy metric space $(X, M, *)$ is said to be *stationary* if M does not depend on t, i.e. if for each $x, y \in X$, the function $M(x, y, \cdot)$ is constant. In the case we write $M(x, y)$ and $B_M(x, r)$ instead of $M(x, y, t)$ and $B_M(x, r, t)$, respectively.

Lemma 2.8 [17] *Let $(X, M, *)$ be a fuzzy metric space. Then M is a continuous function on $X \times X \times (0, \infty)$.*

3 The Hausdorff fuzzy metric on Cld(X)

Given a fuzzy metric space $(X, M, *)$, we will denote by $\mathcal{P}(X)$, Cld(X) and Fin(X), the set of nonempty subsets, the set of nonempty closed subsets and the set of nonempty finite subsets of (X, τ_M), respectively. For every $C \in \mathcal{P}(X)$, $a \in X$ and $t > 0$, let $M(a, C, t) := \sup_{c \in C} M(a, c, t)$, $M(C, a, t) := \sup_{c \in C} M(c, a, t)$ (see Definition 2.4 of [20]). It is clear that $M(a, C, t) = M(C, a, t)$.

Lemma 3.1 *Let $(X, M, *)$ be a fuzzy metric space, $a, c \in X$, $D \in \mathcal{P}(X)$ and $t, s \in (0, \infty)$. Then $M(a, D, t + s) \geq M(a, c, t) * M(c, D, s)$.*

Proof Note that, for each $d \in D$,

$$M(a, D, t + s) \geq M(a, d, t + s) \geq M(a, c, t) * M(c, d, s).$$

It follows from continuity of $*$ that

$$M(a, D, t + s) \geq M(a, c, t) * M(c, D, s).$$

Let $(X, M, *)$ be a fuzzy metric space, $A, C \in$ Cld(X) and $t > 0$, define H_M: Cld$(X) \times$ Cld$(X) \times (0, \infty) \to [0, 1]$ by

$$H_M(A, C, t) = \min\{\inf_{a \in A} M(a, C, t), \inf_{c \in C} M(A, c, t)\}.$$

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If $(X, M, *)$ is a stationary fuzzy metric space, then we write $H_M(A, C)$, $M(a, C)$ and $M(A, c)$ instead of $H_M(A, C, t)$, $M(a, C, t)$ and $M(A, c, t)$, respectively.

Theorem 3.2 *Let $(X, M, *)$ be a stationary and F-bounded fuzzy metric space. Then (Cld(X),$H_M, *$) is a fuzzy metric space.*

Proof Let $A, C, D \in \text{Cld}(X)$.

Obviously, (i), (ii), (iii) and (v) in Definition 2.2 hold.

Now, we are going to prove that (iv) in Definition 2.2 is satisfied, i.e., $H_M(A, D) \geq H_M(A, C) * H_M(C, D)$. Let $a \in A$. Then we can choose a sequence $\{c_n^a\}_{n \in \mathbf{N}}$ in C such that

$$\lim_{n \to +\infty} M(a, c_n^a) = M(a, C).$$

Since $\{M(c_n^a, D)\}_{n \in \mathbf{N}}$ is a sequence in [0,1], there is a subsequence $\{c_{n_l}^a\}_{l \in \mathbf{N}}$ of $\{c_n^a\}_{n \in \mathbf{N}}$ such that the sequence $\{M(c_{n_l}^a, D)\}_{l \in \mathbf{N}}$ converges to some point of [0,1]. It follows from Lemma 3.1 that

$$M(a, D) \geq M(a, c_{n_k}^a) * M(c_{n_k}^a, D)$$

for every $c_{n_k}^a \in \{c_{n_l}^a | l \in \mathbf{N}\}$. Therefore, by continuity of $*$, we have

$$\begin{aligned} M(a, D) &\geq \lim_{k \to +\infty} M(a, c_{n_k}^a) * \lim_{k \to +\infty} M(c_{n_k}^a, D) \\ &= M(a, C) * \lim_{k \to +\infty} M(c_{n_k}^a, D). \end{aligned}$$

According to continuity of $*$, we deduce that

$$\begin{aligned} \inf_{a \in A} M(a, D) &\geq \inf_{a \in A} M(a, C) * \inf_{a \in A} \lim_{k \to +\infty} M(c_{n_k}^a, D) \\ &\geq \inf_{a \in A} M(a, C) * \inf_{a \in A} \{M(c_{n_k}^a, D) \mid k \in \mathbf{N}\} \\ &\geq \inf_{a \in A} M(a, C) * \inf_{c \in C} M(c, D) \\ &\geq H_M(A, C) * H_M(C, D). \end{aligned}$$

Analogously, we get

$$\inf_{d \in D} M(A, d) \geq H_M(A, C) * H_M(C, D).$$

Hence

$$H_M(A, D) \geq H_M(A, C) * H_M(C, D).$$

$(H_M, *)$ will be called *the Hausdorff fuzzy metric on Cld(X).*

Next we will give two examples.

Example 3.3 Let $X = (1, 10]$. Denote $a * b = a \cdot b$ for all $a, b \in [0, 1]$. Define M by

$$M(x, y, t) = \frac{\min\{x, y\}}{\max\{x, y\}}$$

for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a stationary and F-bounded fuzzy metric space (see [7]). So, by Theorem 3.2, (Cld(X),$H_M, *$) is a fuzzy metric space.

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Example 3.4 Let $X = [3, 5]$. Denote $a * b = \max\{a + b - 1, 0\}$ for all $a, b \in [0, 1]$ and let M be a fuzzy set on $X \times X \times (0, \infty)$ defined as follows:

$$M(x, y, t) = \begin{cases} 1 & x = y, \\ \frac{1}{x} + \frac{1}{y} & x \neq y, \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a stationary and F-bounded fuzzy metric space (see [18]). Thus, according to Theorem 3.2, (Cld(X),H_M, $*$) is a fuzzy metric space.

More examples of stationary and F-bounded fuzzy metric spaces may be found in [7, 11, 18].

4 Precompactness of the Hausdorff fuzzy metric

We start this section by recalling the concept of precompact.

Definition 4.1 [8] A fuzzy metric space $(X, M, *)$ is called *precompact* if for each $r \in (0, 1)$ and $t > 0$, there is a finite subset A of X such that $X = \bigcup_{a \in A} B_M(a, r, t)$.

Theorem 4.2 *Let Y be a dense subspace of a stationary and F-bounded fuzzy metric space $(X, M, *)$. Then Fin(Y) is dense in (Cld(X), H_M, $*$) if and only if $(X, M, *)$ is precompact.*

Proof Assume that Fin(Y) is dense in (Cld(X), H_M, $*$). Let $r \in (0, 1)$. Since Fin(Y) $\subseteq$ Fin(X), we get

$$\text{Fin}(X) \cap B_{H_M}(X, r) \neq \emptyset.$$

Take $A \in \text{Fin}(X) \cap B_{H_M}(X, r)$. Then $A \in \text{Fin}(X)$ and $H_M(X, A) > 1 - r$. Hence

$$\inf_{x \in X} M(x, A) > 1 - r.$$

Let $x \in X$. Then $M(x, A) > 1 - r$. Since $A \in \text{Fin}(X)$, there exists an $a \in A$ such that

$$M(x, a) = M(x, A) > 1 - r.$$

We have

$$x \in B_M(a, r).$$

So

$$X \subset \bigcup_{a \in A} B_M(a, r).$$

It follows that $(X, M, *)$ is precompact.

Conversely, assume that $(X, M, *)$ is precompact. Let $D \in \text{Cld}(X)$ and $\varepsilon \in (0, 1)$. Then, by the continuity of $*$, there exists a $\delta \in (0, \varepsilon)$ such that

$$(1 - \delta) * (1 - \delta) > 1 - \varepsilon.$$

We only need to verify that $\text{Fin}(Y) \cap B_{H_M}(D, \varepsilon) \neq \emptyset$. Since $(X, M, *)$ is precompact, there exists a $C = \{c_1, c_2, \cdots, c_n\} \in \text{Fin}(X)$ such that

$$X = \bigcup_{i=1}^{n} B_M(c_i, \delta).$$

Now, we can find $C' = \{c_{n_1}, c_{n_2}, \cdots, c_{n_k}\} \subset C$ such that

$$D \subset \bigcup_{i=1}^{k} B_M(c_{n_i}, \delta)$$

and

$$D \cap B_M(c_{n_i}, \delta) \neq \emptyset (i = 1, 2, \cdots, k).$$

Take $a_{n_i} \in D \cap B_M(c_{n_i}, \delta)$ $(i = 1, 2, \cdots, k)$. Denote $A = \{a_{n_1}, a_{n_2}, \cdots, a_{n_k}\}$. Since Y is a dense subspace of X, we can find an $e_{n_i} \in B_M(c_{n_i}, \delta) \cap Y$ for every $i \in \{1, 2, \cdots, k\}$. So $E = \{e_{n_1}, e_{n_2}, \cdots, e_{n_k}\} \in \text{Fin}(Y)$. For $e_{n_i} \in E$ $(i = 1, 2, \cdots, k)$, we have

$$M(D, e_{n_i}) \geq M(a_{n_i}, e_{n_i}) \geq M(a_{n_i}, c_{n_i}) * M(c_{n_i}, e_{n_i}) \geq (1 - \delta) * (1 - \delta) > 1 - \varepsilon.$$

Hence

$$\inf_{e \in E} M(D, e) = \min\{M(D, e_{n_i}) | i = 1, 2, \cdots, k\} > 1 - \varepsilon.$$

On the other hand, let $d \in D$. Then there exists a $c_{n_i} \in C'$ such that

$$d \in B_M(c_{n_i}, \delta).$$

Hence

$$M(d, E) \geq M(d, e_{n_i}) \geq M(d, c_{n_i}) * M(c_{n_i}, e_{n_i}) \geq (1 - \delta) * (1 - \delta).$$

So

$$\inf_{d \in D} M(d, E) \geq (1 - \delta) * (1 - \delta) > 1 - \varepsilon.$$

Hence

$$H_M(D, E) = \min\{\inf_{d \in D} M(d, E), \inf_{e \in E} M(D, e)\} > 1 - \varepsilon,$$

that is, $E \in B_{H_M}(D, \varepsilon)$. Consequently,

$$E \in \text{Fin}(Y) \cap B_{H_M}(D, \varepsilon).$$

We complete the proof.

Let $(X, M, *)$ be a fuzzy metric space and $A \subset X$. We will denote by $M|_{A \times A \times (0, \infty)}$ the restriction of M on $A \times A \times (0, \infty)$. It is easy to see that $M|_{A \times A \times (0, \infty)}$ is a fuzzy metric on A. We will simply write $M|_A$ instead of $M|_{A \times A \times (0, \infty)}$ when confusion is not possible.

Lemma 4.3 *Let $(X, M, *)$ be a precompact fuzzy metric space and $A \subset X$. Then $(A, M|_A, *)$ is precompact.*

Proof Let $\varepsilon \in (0, 1)$ and $t > 0$. Then there exists a $\delta \in (0, \varepsilon)$ such that $(1-\delta) * (1-\delta) > 1-\varepsilon$. Because of precompactness of $(X, M, *)$, we can find finite points $x_1, x_2, \cdots, x_n$ in X such that

$$X = \bigcup_{i=1}^{n} B_M(x_i, \delta, \frac{t}{2}).$$

Since $A \subset X$, there exists $\{x_{n_1}, x_{n_2}, \cdots, x_{n_k}\} \subset \{x_1, x_2, \cdots, x_n\}$ such that

$$A \subset \bigcup_{j=1}^{k} B_M(x_{n_j}, \delta, \frac{t}{2})$$

and

$$A \cap B_M(x_{n_j}, \delta, \frac{t}{2}) \neq \emptyset (j = 1, 2, \cdots, k).$$

Choose $y_{n_j} \in A \cap B_M(x_{n_j}, \delta, \frac{t}{2})$ $(j = 1, 2, \cdots, k)$. Then, for each $z \in B_M(x_{n_j}, \delta, \frac{t}{2})$, we have

$$M(z, y_{n_j}, t) \geq M(z, x_{n_j}, \frac{t}{2}) * M(x_{n_j}, y_{n_j}, \frac{t}{2}) \geq (1-\delta) * (1-\delta) > 1-\varepsilon.$$

So

$$B_M(x_{n_j}, \delta, \frac{t}{2}) \subset B_M(y_{n_j}, \varepsilon, t).$$

Hence

$$A \subset \bigcup_{j=1}^{k} B_M(x_{n_j}, \delta, \frac{t}{2}) \subset \bigcup_{j=1}^{k} B_M(y_{n_j}, \varepsilon, t).$$

Whence

$$A = (\bigcup_{j=1}^{k} B_M(y_{n_j}, \varepsilon, t)) \cap A = \bigcup_{j=1}^{k} (B_M(y_{n_j}, \varepsilon, t) \cap A) = \bigcup_{j=1}^{k} B_{M|_A}(y_{n_j}, \varepsilon, t).$$

We are done.

Theorem 4.4 *Let $(X, M, *)$ be a stationary and F-bounded fuzzy metric space. Then $(Cld(X), H_M, *)$ is precompact if and only if $(X, M, *)$ is precompact.*

Proof Suppose that $(\mathrm{Cld}(X), H_M, *)$ is precompact. For each $x, y \in X$, we have $H_M(\{x\}, \{y\}) = M(x, y)$. So we can regard X as a subset of $\mathrm{Cld}(X)$ and M as $H_M|_{\{\{x\}|x \in X\}}$. It follows from Lemma 4.3 that $(X, M, *)$ is precompact.

Conversely, suppose that $(X, M, *)$ is precompact. Let $\varepsilon \in (0, 1)$ and $D \in \mathrm{Cld}(X)$. Then, by precompactness of $(X, M, *)$, we can find an $F \in \mathrm{Fin}(X)$ such that

$$X = \bigcup_{x \in F} B_M(x, \frac{\varepsilon}{2}).$$

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Therefore, there exists an $F_D \subset F$ such that

$$D \subset \bigcup_{x \in F_D} B_M(x, \frac{\varepsilon}{2}).$$

Also, we can find $x_D \in B_M(x, \frac{\varepsilon}{2}) \cap D$ for every $x \in F_D$. Note that there exists an $x_y \in F_D$ such that

$$y \in B_M(x_y, \frac{\varepsilon}{2})$$

for every $y \in D$. It follows that

$$M(y, F_D) \geq M(y, x_y) > 1 - \frac{\varepsilon}{2}.$$

Hence

$$\inf_{y \in D} M(y, F_D) \geq 1 - \tfrac{\varepsilon}{2} > 1 - \varepsilon.$$

On the other hand, for each $x \in F_D$, we get

$$M(D, x) \geq M(x_D, x) > 1 - \frac{\varepsilon}{2}.$$

Hence

$$\inf_{x \in F_D} M(D, x) \geq 1 - \frac{\varepsilon}{2} > 1 - \varepsilon.$$

So

$$H_M(D, F_D) > 1 - \varepsilon,$$

i.e., $D \in B_{H_M}(F_D, \varepsilon)$. Since F is a finite set, we see that $\mathcal{F} = \{F_D | D \in \text{Cld}(X)\}$ is a finite family. Observe that

$$\text{Cld}(X) = \bigcup_{F_D \in \mathcal{F}} B_{H_M}(F_D, \varepsilon).$$

It follows that $(\text{Cld}(X), H_M, *)$ is precompact.

Definition 4.5 Let $(X, M, *)$ be a fuzzy metric space, $Y \subset X$, $r \in (0, 1)$ and $t > 0$. Y is said to be *fuzzy r discrete with respect to t* if $M(x, y, t) < 1 - r$ whenever $x, y \in Y$ and $x \neq y$.

Definition 4.6 Let $(X, M, *)$ be a fuzzy metric space and $Y \subset X$. Y is called a *fuzzy uniformly discrete set* provided that there exist $r \in (0, 1)$ and $t > 0$ such that Y is fuzzy r discrete with respect to t.

According to Zorn's lemma, it is straightforward to show that, by the inclusion relationship of the sets, X has a maximal subset which is fuzzy r discrete with respect to t for all $r \in (0, 1)$ and $t > 0$.

Lemma 4.7 *Let $(X, M, *)$ be a fuzzy metric space and Y be a fuzzy uniformly discrete subset of X. Then Y is a closed set in (X, τ_M).*

Proof By assumption, we can find some $r_0 \in (0,1)$ and $t_0 > 0$ such that Y is fuzzy r_0 discrete with respect to t_0. According to the continuity of $*$, there exists a $\varepsilon \in (0, r_0)$ such that

$$(1-\varepsilon) * (1-\varepsilon) > 1 - r_0.$$

Let $x \notin Y$. To complete our proof, it suffices to prove that there exists a open set U of x in X such that $U \cap Y = \emptyset$. For every $y, z \in B_M(x, \varepsilon, \frac{t_0}{2})$, we have

$$M(y, z, t_0) \geq M(x, y, \frac{t_0}{2}) * M(x, z, \frac{t_0}{2}) \geq (1-\varepsilon) * (1-\varepsilon) > 1 - r_0.$$

So $B_M(x, \varepsilon, \frac{t_0}{2})$ contains at most one point of Y. If $B_M(x, \varepsilon, \frac{t_0}{2}) \cap Y = \emptyset$, then $B_M(x, \varepsilon, \frac{t_0}{2})$ is the required open set. If $B_M(x, \varepsilon, \frac{t_0}{2}) \cap Y = \{a\}$, then, by the Hausdorffness of $(X, M, *)$, we can choose an $n \in \mathbf{N}$ such that

$$a \notin B_M(x, \frac{1}{n}, \frac{1}{n}).$$

So we get

$$x \in B_M(x, \varepsilon, \frac{t_0}{2}) \cap B_M(x, \frac{1}{n}, \frac{1}{n}),$$

with

$$B_M(x, \varepsilon, \frac{t_0}{2}) \cap B_M(x, \frac{1}{n}, \frac{1}{n}) \cap Y = \emptyset,$$

which implies that $B_M(x, \varepsilon, \frac{t_0}{2}) \cap B_M(x, \frac{1}{n}, \frac{1}{n})$ is the required open set. We are done.

Lemma 4.8 *Let $(X, M, *)$ be a fuzzy metric space and Y be an uncountable fuzzy uniformly discrete subset of X. Then X is not separable.*

Proof Suppose that X is separable. Then we take a countable and dense subset A in X. According to assumption, we can find some $r_0 \in (0,1)$ and $t_0 > 0$ such that $M(x, y, t_0) < 1 - r_0$ for all $x, y \in Y$ and $x \neq y$. According to continuity of $*$, there exists a $\varepsilon \in (0, r_0)$ such that

$$(1-\varepsilon) * (1-\varepsilon) > 1 - r_0.$$

Pick $a, b \in Y$, with $a \neq b$. We conclude that

$$B_M(a, \varepsilon, \frac{t_0}{2}) \cap B_M(b, \varepsilon, \frac{t_0}{2}) = \emptyset.$$

Indeed, otherwise, we can take a $c \in B_M(a, \varepsilon, \frac{t_0}{2}) \cap B_M(b, \varepsilon, \frac{t_0}{2})$. Then

$$M(a, b, t_0) \geq M(a, c, \frac{t_0}{2}) * M(c, b, \frac{t_0}{2}) \geq (1-\varepsilon) * (1-\varepsilon) > 1 - r_0,$$

which contradicts $M(a, b, t_0) < 1 - r_0$. So $\{B_M(y, \varepsilon, \frac{t_0}{2}) | y \in Y\}$ is an uncountable and pair-wise disjoint open family. Since A is dense in X, we see that

$B_M(y, \varepsilon, \frac{t_0}{2}) \cap A \neq \emptyset$ for all $y \in Y$. This shows that A is uncountable, which is a contradiction. We complete the proof.

Lemma 4.9 [8] *Let $(X, M, *)$ be a precompact fuzzy metric space. Then it is separable.*

Theorem 4.10 *Let $(X, M, *)$ be a stationary and F-bounded fuzzy metric space. Then $(X, M, *)$ is precompact if and only if (Cld(X),$H_M, *$) is separable.*

Proof Suppose that $(X, M, *)$ is precompact. then, according to Theorem 4.4, we deduce that (Cld(X),$H_M, *$) is precompact. So, by Lemma 4.9, we see that (Cld(X),$H_M, *$) is separable.

Conversely, Let (Cld(X),$H_M, *$) be separable. Suppose that $(X, M, *)$ fails to be precompact. Then there exists an infinite fuzzy uniformly discrete subset Y of X. Observe that $\mathcal{P}(Y)$ is the set of nonempty subsets of Y. Then $\mathcal{P}(Y)$ is uncountable. Take $A, C \in \mathcal{P}(Y)$. Then, by Lemma 4.7, we see that $A, C \in$ Cld(X). Now, for each $a \in A$ and $c \in C$, we can find $r_0 \in (0, 1)$ such that

$$M(a, c) < 1 - r_0.$$

So

$$M(a, C) \leq 1 - r_0.$$

It follows that

$$\inf_{a \in A} M(a, C) \leq 1 - r_0.$$

Similarly, we obtain

$$\inf_{c \in C} M(A, c) \leq 1 - r_0.$$

Hence

$$H_M(A, C) \leq 1 - r_0 < 1 - \frac{r_0}{2},$$

which implies that $\mathcal{P}(Y)$ is a fuzzy uniformly discrete subset of Cld(X). Thus, by Lemma 4.8, Cld(X) fails to be separable, which is a contradiction. Consequently, $(X, M, *)$ is precompact.

From Theorem 4.2, Theorem 4.4 and Theorem 4.10 we immediately deduce the next corollary.

Corollary 4.11 *Let Y be a dense subspace of a stationary and F-bounded fuzzy metric space $(X, M, *)$. Then the following are equivalent.*

(i) *$(X, M, *)$ is precompact.*

(ii) *Fin(Y) is dense in (Cld$(X), H_M, *$).*

(iii) *(Cld$(X), H_M, *$) is precompact.*

(iv) *(Cld(X),$H_M, *$) is separable.*

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Semilocal convergence of a modified Chebyshev-like's method for solving nonlinear equations under generalized weak condition *

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Abstract: In this paper, the semilocal convergence of the modified Chebyshev-like's method is established by using recurrence relations under generalized weak condition. We prove an existence-uniqueness theorem and give a priori error bound which demonstrates the R-order convergence of the method. Moreover, the dynamical behavior of this method is also studied. Finally, numerical examples are presented to demonstrate our approach.

Keywords: Nonlinear equations; Chebyshev-like's method; Recurrence relations; Semilocal convergence; A priori error bounds; Dynamics.

1 Introduction

A number of problems arisen from scientific and engineering areas often needs to find the solution of nonlinear equations in Banach spaces

$$F(x) = 0, \tag{1.1}$$

where F is a third-order Fréchet-differentiable operator defined on a convex subset Ω of a Banach space X with values in a Banach space Y.

Generally, iterative methods are often used to solve this problem [1]. Newton's method being a second-order method is one of best known of these methods. The convergence of Newton's method in Banach spaces was established by Kantorovich in [2]. The convergence of the sequence obtained by the iterative expression is derived from the convergence of majorizing sequences. This technique has been used by many authors in order to establish the order of convergence of the variants of Newton's methods [3-9]. An alternative approach is developed to establish this convergence by using recurrence relations. The approach is also a very popular technique to

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establish the convergence of iterative methods. For example, it has been successfully applied to the convergence analysis of Newton's method and some high-order methods [10-23].

In [9], we introduce a modified Chebyshev-like's method given by

$$x_{n+1} = x_n - \Big[I + \frac{1}{2}K_F(x_n) + \frac{1}{2}K_F^2(x_n)\Big]\Gamma_n F(x_n), \tag{1.2}$$

where $\Gamma_n = [F'(x_n)]^{-1}$, $K_F(x_n) = \Gamma_n F''(u_n)\Gamma_n F(x_n)$ and $u_n = x_n - \frac{1}{3}\Gamma_n F(x_n)$.

By assuming that

(A1) Γ_0 exist and $\|\Gamma_0 F(x_0)\| \le \eta$,

(A2) $\|\Gamma_0\| \le \beta$,

(A3) $\|F''(x)\| \le M, x \in \Omega$,

(A4) $\|F'''(x)\| \le N, x \in \Omega$,

(A5) $\|F'''(x) - F'''(y)\| \le L\|x - y\|, \ \ \forall\ x, y \in \Omega$,

we have analyzed the semilocal convergence of the method given by (1.2) by majorizing sequences and proved the R-order is improved to four, the computation efficiency and error estimate were also given. Numerical applications shows this method can solve some equations successfully.

But under assumptions (A1)-(A5), we can not study the solution of some equations. Such as the nonlinear integral equation of mixed Hammerstein type, which is given by

$$x(s) + \sum_{i=1}^{m}\int_a^b G_i(s,t)H_i(x(t))dt = u(s), \qquad s \in [a,b], \tag{1.3}$$

where $-\infty < a < b < \infty$, u, H_i and G_i, for $i = 1, 2, \cdots, m$, are known functions and x is a solution to be determined. The problem is from the dynamic model of a chemical reactor [24]. On the condition that $H_i'''(x(t))$ is (L_i, q_i)-Hölder continuous in Ω, $i = 1, 2, \ldots, m$, then corresponding operator $F : \Omega \subseteq C[0,1] \to C[0,1]$,

$$[F(x)](s) = x(s) + \sum_{i=1}^{m}\int_a^b G_i(s,t)H_i(x(t))dt - u(s), \qquad s \in [a,b], \tag{1.4}$$

is such that its third Fréchet derivative is neither Lipschitz continuous nor Hölder continuous in Ω while, for an example, we consider the max-norm. For this case,

$$\|F'''(x) - F'''(y)\| \le \sum_{i=1}^{m} L_i\|x - y\|^{q_i}, \ L_i \ge 0, q_i \in [0,1], \ \ x, y \in \Omega. \tag{1.5}$$

Because of the importance of nonlinear integral equation of mixed Hammerstein type, several authors [18-20] have considered a mild condition

$$\|F'''(x) - F'''\| \le \omega(\|x - y\|), \ x, \ y \in \Omega, \tag{1.6}$$

where $\omega(z)$ is nondecreasing continuous real valued function for $z > 0$, such that $\omega(0) \ge 0$, on F''' to study the semilocal convergence of some iterative methods.

In the paper, the semilocal convergence of the modified Chebyshev-like's method is established by using recurrence relations under the assumption that F''' satisfies the ω-continuity condition (1.6), An existence-uniqueness theorem is also established for the solution along with its a priori error bounds. Moreover, the dynamical behavior of modified Chebyshev-like's method is also studied. Finally numerical examples are presented to demonstrate our approach.

2 Recurrence relations

Let $x_0 \in \Omega$ and the nonlinear operator $F : \Omega \subset X \to Y$ be continuously third-order Fréchet differentiable where Ω is an open set and X and Y are Banach spaces. We assume that

(C1) Γ_0 exist and $\|\Gamma_0 F(x_0)\| \leq \eta$,

(C2) $\|\Gamma_0\| \leq \beta$,

(C3) $\|F''(x)\| \leq M, x \in \Omega$,

(C4) $\|F'''(x)\| \leq N, x \in \Omega$,

(C5) $\|F'''(x) - F'''(y)\| \leq \omega(\|x-y\|), \ \ \forall\, x, y \in \Omega$, where $\omega(z)$ is non-decreasing continuous real function for $z > 0$ and satisfy $\omega(0) \geq 0$,

(C6) there exists a non-negative real function $\varphi(t) \leq 1$, such that $\omega(tz) \leq \varphi(t)\omega(z)$, for $t \in [0,1]$, $z \in (0, +\infty)$.

Notice that condition (C6) is not restrictive, since we can always consider $\varphi(t) = 1$, as a consequence of ω is non-decreasing function, but its interest is to sharp the priori error bounds.

Firstly we give the following lemma to show an approximation of operator F, which will be used in the latter developments.

Lemma 1 Assume that the nonlinear operator $F : \Omega \subset X \to Y$ is continuously third-order Fréchet differentiable where Ω is an open set and X and Y are Banach spaces. Then we have

$$\begin{aligned} F(x_{n+1}) =& -\frac{1}{2}\big[F''(u_n) - F''(x_n)\big](y_n - x_n)K_F(x_n)(y_n - x_n) \\ &+ \frac{1}{2}F''(x_n)(y_n - x_n)K_F^2(x_n)(y_n - x_n) \\ &+ \int_0^1 F''\big(y_n + t(x_{n+1} - y_n)\big)(1-t)dt(x_{n+1} - y_n)^2 \\ &- \frac{1}{6}\int_0^1 \Big[F'''\Big(x_n + \frac{1}{3}t(y_n - x_n)\Big) - F'''(x_n)\Big]dt(y_n - x_n)^3 \\ &+ \frac{1}{2}\int_0^1 \Big[F'''\big(x_n + t(y_n - x_n)\big) - F'''(x_n)\Big](1-t)^2 dt(y_n - x_n)^3 \\ &+ \int_0^1 F'''\big(x_n + t(y_n - x_n)\big)(1-t)dt(y_n - x_n)^2(x_{n+1} - y_n). \end{aligned} \tag{2.1}$$

Now we consider the following scalar functions

$$g(t) = 1 + \frac{1}{2}t + \frac{1}{2}t^2, \tag{2.2}$$

$$h(t) = \frac{1}{1 - tg(t)}, \tag{2.3}$$

$$\ell(t, u, v) = \frac{1}{8}t^3(t^2 + 2t + 5) + \frac{1}{12}(3t + 5)tu + \frac{I_1 + 3I_2}{6}v, \tag{2.4}$$

where $I_1 = \int_0^1 \varphi(\frac{1}{3}t)dt$ and $I_2 = \int_0^1 \varphi(t)(1-t)^2 dt$.

Let $\Phi(t) = g(t)t - 1$. Since $\Phi(0) = -1$ and $\Phi(1) = 1 > 0$, then $\Phi(t)$ has at least a zero in $(0, 1)$. Let s is the smallest positive zero of the scalar function $g(t)t - 1$.

Denote $a_0 = M\beta\eta$, $b_0 = N\beta\eta^2$, $c_0 = w(\eta)\beta\eta^2$ and $d_0 = h(a_0)\ell(a_0, b_0, c_0)$. Next, some properties of the functions g, h, ℓ defined in (2.2)-(2.4) are given in the following lemma.

Lemma 2 Let the real functions g, h and ℓ be given in (2.2)-(2.4). Then
(i) $g(t)$ and $h(t)$ are increasing and $g(t) > 1, h(t) > 1$ for $t \in (0, s)$,
(ii) $\ell(t, u, v)$ is increasing for $t \in (0, s), u > 0, v > 0$.

For $n = 0$, the existence of Γ_0 implies the existence of y_0, u_0. This gives us

$$\|y_0 - x_0\| = \|\Gamma_0 F(x_0)\| \leq \eta,$$

and

$$\|u_0 - x_0\| = \frac{1}{3}\Big\|\Gamma_0 F(x_0)\Big\| \leq \frac{1}{3}\eta.$$

This means that $y_0,\ u_0 \in B(x_0, R\eta)$, where $R = g(a_0)/\big(1 - d_0\big)$. Furthermore, we have

$$\begin{aligned} \|K_F(x_0)\| &= \|\Gamma_0 F''(u_0)\Gamma_0 F(x_0)\| \\ &\leq \|\Gamma_0\|\|F''(u_0)\|\|\Gamma_0 F(x_0)\| \\ &\leq M\beta\eta = a_0. \end{aligned}$$

We can obtain

$$\begin{aligned} \|x_1 - x_0\| &= \Big\|I + \frac{1}{2}K_F(x_0) + \frac{1}{2}K_F^2(x_0)\Big\|\|\Gamma_0 F(x_0)\| \\ &\leq \Big[1 + \frac{1}{2}a_0 + \frac{1}{2}a_0^2\Big]\|y_0 - x_0\| = g(a_0)\|y_0 - x_0\|. \end{aligned} \tag{2.5}$$

From the assumption $d_0 < 1/h(a_0) < 1$, it follows that $x_1 \in B(x_0, R\eta)$. We also have

$$\begin{aligned} \|x_1 - y_0\| &\leq \Big\|\frac{1}{2}K_F(x_0) + \frac{1}{2}K_F^2(x_0)\Big\|\|\Gamma_0 F(x_0)\| \\ &\leq \frac{a_0(1 + a_0)}{2}\|y_0 - x_0\|. \end{aligned} \tag{2.6}$$

By $a_0 < s$ and $g(a_0) < g(s)$, we have

$$\begin{aligned} \|I - \Gamma_0 F'(x_1)\| &\leq \|\Gamma_0\|\|F'(x_0) - F'(x_1)\| \\ &\leq M\beta\|x_1 - x_0\| \leq a_0 g(a_0) < 1. \end{aligned}$$

It follows by the Banach lemma that $\Gamma_1 = [F'(x_1)]^{-1}$ exists and

$$\|\Gamma_1\| \leq \frac{1}{1 - a_0 g(a_0)}\|\Gamma_0\| = h(a_0)\|\Gamma_0\|. \tag{2.7}$$

By Lemma 1, we can get

$$\begin{aligned} \|F(x_1)\| &\leq \frac{a_0^2}{2}M\eta^2 + \frac{a_0}{6}N\eta^3 + \frac{1}{2}M\|x_1 - y_0\|^2 \\ &\quad + \frac{1}{6}I_1 w(\eta)\eta^3 + \frac{1}{2}I_2 w(\eta)\eta^3 + \frac{1}{2}N\eta^2\|x_1 - y_0\|. \end{aligned} \tag{2.8}$$

Then from (2.7) and (2.8), we have

$$\begin{aligned} \|y_1 - x_1\| &= \|\Gamma_1 F(x_1)\| \leq \|\Gamma_1\|\|F(x_1)\| \\ &\leq d_0\|y_0 - x_0\|. \end{aligned} \tag{2.9}$$

Because of $g(a_0) > 1$, we obtain

$$\begin{aligned}\|y_1 - x_0\| &\leq \|y_1 - x_1\| + \|x_1 - x_0\| \\ &\leq \Big(g(a_0) + d_0\Big)\eta \\ &< g(a_0)\Big(1 + d_0\Big)\eta < R\eta,\end{aligned} \tag{2.10}$$

which shows $y_1 \in B(x_0, R\eta)$.

In addition, we have

$$M\|\Gamma_1\|\|\Gamma_1 F(x_1)\| \leq Mh(a_0)\|\Gamma_0\|d_0\|y_0 - x_0\| = a_0 h(a_0) d_0, \tag{2.11}$$

$$N\|\Gamma_1\|\|\Gamma_1 F(x_1)\|^2 \leq Nh(a_0)\|\Gamma_0\|d_0^2\|y_0 - x_0\|^2 = b_0 h(a_0) d_0^2, \tag{2.12}$$

$$\begin{aligned}\omega(\|y_1 - x_1\|)\|\Gamma_1\|\|y_1 - x_1\|^2 &\leq h(a_0)\|\Gamma_0\|\omega(d_0\|y_0 - x_0\|)d_0^2\|y_0 - x_0\|^2 \\ &\leq c_0 h(a_0) d_0^2 \varphi(d_0),\end{aligned} \tag{2.13}$$

$$\begin{aligned}\|x_2 - x_0\| &\leq \|x_2 - x_1\| + \|x_1 - x_0\| \\ &\leq g(a_1)\|y_1 - x_1\| + g(a_0)\|y_0 - x_0\| \\ &\leq (1 + d_0) g(a_0)\|y_0 - x_0\| < R\eta.\end{aligned} \tag{2.14}$$

Since

$$\begin{aligned}&\|I - \Gamma_1 F'(x_2)\| \leq \|\Gamma_1\|\|F'(x_1) - F'(x_2)\| \\ &\leq M\|\Gamma_1\|\|x_2 - x_1\| \leq a_0 h(a_0) d_0 g(a_0 h(a_0) d_0) < 1,\end{aligned}$$

and by the Banach lemma, $\Gamma_2 = [F'(x_2)]^{-1}$ exists and

$$\|\Gamma_2\| \leq h(a_0 h(a_0) d_0)\|\Gamma_1\|. \tag{2.15}$$

Hence x_2 is well defined.

We now write $a_0 h(a_0) d_0 = a_1$, $b_0 h(a_0) d_0^2 = b_1$, $c_0 h(a_0) d_0^2 \varphi(d_0) = c_1$ and define for $n \geq 0$

$$\begin{aligned}a_{n+1} &= a_n h(a_n) d_n, \\ b_{n+1} &= b_n h(a_n) d_n^2, \\ c_{n+1} &= c_n h(a_n) d_n^2 \varphi(d_n), \\ d_{n+1} &= h(a_{n+1}) \ell(a_{n+1}, b_{n+1}, c_{n+1}).\end{aligned} \tag{2.16}$$

Later developments will require the following lemma, where some properties of the previous scalar sequences are given.

Lemma 3 Let the real functions g, h and ℓ be given in (2.2)-(2.4). If

$$0 < a_0 < s \;\text{ and }\; h(a_0) d_0 < 1, \tag{2.17}$$

then we have

(i) $h(a_n) > 1$ and $h(a_n) d_n < 1$ for $n \geq 0$,

(ii) the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ are decreasing,

(iii) $g(a_n) a_n < 1$ and $h(a_n) d_n < 1$ for $n \geq 0$.

Our next goal is to guarantee that (1.2) is well defined. To do this, the system of recurrence relations given in the next lemma must be satisfied. The proof follows by using a similar way that the above-mentioned and invoking the induction hypothesis.

Lemma 4 Let the assumptions of Lemma 3 and the conditions (C1)-(C6) hold. Then the following items are true for all $n \geq 1$:

(i) There exists $\Gamma_n = [F'(x_n)]^{-1}$ and $\|\Gamma_n\| \leq h(a_{n-1})\|\Gamma_{n-1}\|$,

(ii) $\|y_n - x_n\| = \|\Gamma_n F(x_n)\| \leq d_{n-1}\|y_{n-1} - x_{n-1}\| \leq d_0^n\|y_0 - x_0\| < \eta$,

(iii) $M\|\Gamma_n\|\|\Gamma_n F(x_n)\| \leq a_n$,

(iv) $N\|\Gamma_n\|\|\Gamma_n F(x_n)\|^2 \leq b_n$,

(v) $\omega(\|y_n - x_n\|)\|\Gamma_n\|\|y_n - x_n\|^2 \leq c_n$,

(vi) $\|x_{n+1} - x_n\| \leq g(a_n)\|y_n - x_n\|$,

(vii) $\|y_n - x_0\| \leq R\eta$ and $\|x_{n+1} - x_0\| \leq g(a_0)\frac{1-d_0^{n+1}}{1-d_0}\|y_0 - x_0\| < R\eta$, where $R = \frac{g(a_0)}{1-d_0}$.

3 Semilocal convergence

We are now interested in proving that sequence (1.2) is convergent. To do this, we will see that (1.2) is a Cauchy sequence. We will give some properties of the scalar sequence $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ in the following lemma.

Lemma 5 Let the real functions g, h and ℓ be given in (2.2)-(2.4). Let $\tau \in (0, 1)$, then $g(\tau t) < g(t), h(\tau t) < h(t)$ and $\ell(\tau t, \tau^2 u, \tau^2 v) < \tau^2 \ell(t, u, v)$ for $t \in (0, s)$.

Lemma 6 Under the assumptions of Lemma 3. Let $\gamma = h(a_0)d_0$, $\delta = 1/h(a_0)$. For $n \geq 0$, we have

$$a_{n+1} \leq \gamma^{3^n} a_n \leq \gamma^{\frac{3^{n+1}-1}{2}} a_0, \tag{3.1}$$

$$b_{n+1} < \left(\gamma^{3^n}\right)^2 b_n < \gamma^{3^{n+1}-1} b_0, \tag{3.2}$$

$$c_{n+1} < \left(\gamma^{3^n}\right)^2 c_n < \gamma^{3^{n+1}-1} c_0, \tag{3.3}$$

$$d_{n+1} = h(a_{n+1})\ell(a_{n+1}, b_{n+1}, c_{n+1}) \leq \delta\gamma^{3^{n+1}}. \tag{3.4}$$

Proof The Lemma will be proved by induction. Since $a_1 = \gamma a_0$, by the above-mentioned assumption, we get $\gamma < 1$. We also get

$$b_1 = b_0 h(a_0) d_0^2 < \gamma^2 b_0,$$

and

$$c_1 = c_0 h(a_0) d_0^2 \varphi(d_0) < \gamma^2 c_0,$$

as $\varphi(d_0) \leq 1$.

Suppose (3.1)-(3.3) hold for $n = k$, then

$$\begin{aligned} a_{k+1} &= a_k h(a_k) d_k = a_k h^2(a_k)\ell(a_k, b_k, c_k) \\ &\leq \gamma^{3^{k-1}} a_{k-1} h^2\left(\gamma^{3^{k-1}} a_{k-1}\right)\ell\Big(\gamma^{3^{k-1}} a_{k-1}, \left(\gamma^{3^{k-1}} b_{k-1}\right)^2, \left(\gamma^{3^{k-1}} c_{k-1}\right)^2\Big) \\ &\leq \gamma^{3^{k-1}} a_{k-1} h^2(a_{k-1})\left(\gamma^{3^{k-1}}\right)^2 \ell(a_{k-1}, b_{k-1}, c_{k-1}) = \gamma^{3^k} a_k. \end{aligned}$$

We also have

$$b_{k+1} = b_k h(a_k) d_k^2 < \Big(\frac{a_{k+1}}{a_k}\Big)^2 b_k < \Big(\gamma^{3^k}\Big)^2 b_k,$$

and

$$c_{k+1} = c_k h(a_k) d_k^2 \varphi(d_k) < \Big(\frac{a_{k+1}}{a_k}\Big)^2 c_k \varphi(d_k) \le \Big(\gamma^{3^k}\Big)^2 c_k,$$

as $\varphi(d_k) \le 1$. Hence

$$\begin{aligned} a_{k+1} &\le \gamma^{3^k} a_k \le \gamma^{3^k}\gamma^{3^{k-1}}\cdots\gamma^{3^0} a_0 = \gamma^{\frac{3^{k+1}-1}{2}} a_0, \\ b_{k+1} &< \Big(\gamma^{3^k}\Big)^2 b_k < \Big(\gamma^{3^k}\Big)^2\Big(\gamma^{3^{k-1}}\Big)^2\cdots\Big(\gamma^{3^0}\Big)^2 b_0 = \gamma^{3^{k+1}-1} b_0, \\ c_{k+1} &< \Big(\gamma^{3^k}\Big)^2 c_k < \Big(\gamma^{3^k}\Big)^2\Big(\gamma^{3^{k-1}}\Big)^2\cdots\Big(\gamma^{3^0}\Big)^2 c_0 = \gamma^{3^{k+1}-1} c_0. \end{aligned}$$

Thus (3.1)-(3.3) hold by induction. Furthermore

$$\begin{aligned} d_{n+1} &= h(a_{n+1})\ell(a_{n+1}, b_{n+1}, c_{n+1}) \le h\Big(\gamma^{\frac{3^{n+1}-1}{2}} a_0\Big)\ell\Big(\gamma^{\frac{3^{n+1}-1}{2}} a_0, \gamma^{3^{n+1}-1} b_0, \gamma^{3^{n+1}-1} c_0\Big) \\ &= \gamma^{3^{n+1}} \frac{h(a_0)\ell(a_0, b_0, c_0)}{\gamma} = \delta\gamma^{3^{n+1}}. \end{aligned}$$

The proof is completed.

3.1 Convergence theorem

Now we give a theorem to show the existence and uniqueness of the solution and the domain in which it is located, along with a priori error bounds, which lead to the R-order of convergence of iteration (1.2).

Theorem 1 Let X and Y be two Banach spaces and $F : \Omega \subseteq X \to Y$ be a third-order Fréchet differentiable on a non-empty open convex subset Ω. g, h, ℓ are defined by (2.2)-(2.4). Let $a_0 = M\beta\eta, b_0 = N\beta\eta^2$ and $c_0 = w(\eta)\beta\eta^2$ satisfy $0 < a_0 < s$ and $h(a_0)d_0 < 1$, $\overline{B(x_0, R\eta)} \in \Omega$ where $R = g(a_0)/\big(1-d_0\big)$. Assume that $x_0 \in \Omega$ and all conditions (C1)-(C6) hold. Then starting from x_0, the sequence $\{x_n\}$ generated by the modified Chebyshev-like's method (1.2) converges to a solution x^* of $F(x) = 0$ with x_n, x^* belong to $\overline{B(x_0, R\eta)}$ and x^* is the unique solution of $F(x)$ in $B(x_0, \frac{2}{M\beta} - R\eta) \cap \Omega$. Moreover, a priori error estimate is given by

$$\|x_n - x^*\| \le g(a_0)\eta\delta^n\gamma^{\frac{3^n-1}{2}} \frac{1}{1-\delta\gamma^{3^n}}, \tag{3.5}$$

where $\gamma = h(a_0)d_0$ and $\delta = 1/h(a_0)$.

Proof It is sufficient to show that $\{x_n\}$ is a Cauchy sequence in order to establish the convergence of $\{x_n\}$.

From Lemma 4 and Lemma 6, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\le g(a_n)\|y_n - x_n\| \le g(a_n)d_{n-1}\|y_{n-1} - x_{n-1}\| \\ &\le \cdots \le g(a_n)\|y_0 - x_0\| \prod_{j=0}^{n-1} d_j \\ &\le g(a_n)\eta \prod_{j=0}^{n-1}(\delta\gamma^{3^j}) = g(a_n)\eta\delta^n\gamma^{\frac{3^n-1}{2}}, \end{aligned}$$

where $\gamma = h(a_0)d_0 < 1$ and $\delta = 1/h(a_0) < 1$. Hence,

$$\begin{aligned}
\|x_{m+n} - x_n\| &\leq \|x_{m+n} - x_{m+n-1}\| + \cdots + \|x_{n+1} - x_n\| \\
&\leq g(a_{m+n-1})\|y_{m+n-1} - x_{m+n-1}\| + \cdots + g(a_n)\|y_n - x_n\| \\
&\leq g(a_{m+n-1})\gamma^{\frac{3^{m+n-1}-1}{2}}\delta^{m+n-1}\eta + \cdots + g(a_n)\gamma^{\frac{3^n-1}{2}}\delta^n\eta \\
&\leq g(a_n)\delta^n\big[\gamma^{\frac{3^{m+n-1}-1}{2}}\delta^{m-1} + \cdots + \gamma^{\frac{3^n-1}{2}}\big]\eta \\
&= g(a_n)\delta^n\gamma^{\frac{3^n-1}{2}}\Big[\gamma^{\frac{3^n[3^{m-1}-1]}{2}}\delta^{m-1} + \cdots + \gamma^{\frac{3^n[3-1]}{2}}\delta + 1\Big]\eta.
\end{aligned}$$

By Bernoulli's inequality, for every real number $x > -1$ and every integer $k \geq 0$, we have $(1+x)^k - 1 \geq kx$. Thus,

$$\|x_{m+n} - x_n\| \leq g(a_0)\delta^n\gamma^{\frac{3^n-1}{2}}\ \frac{1-\gamma^{m\cdot 3^n}\delta^m}{1-\gamma^{3^n}\delta}\eta. \tag{3.6}$$

It follows that $\{x_n\}$ is a Cauchy sequence. So there exists a x^* such that $\lim\limits_{n\to\infty} x_n = x^*$.

For $m \geq 1$ and $n = 0$, we get

$$\|x_m - x_0\| \leq g(a_0)\frac{1-\gamma^m\delta^m}{1-\gamma\delta}\eta < R\eta.$$

Hence, $x_m \in B(x_0, R\eta)$, for all $m \geq 0$. By letting $n = 0$, $m \to \infty$ in (3.6), we obtain

$$\|x^* - x_0\| \leq R\eta. \tag{3.7}$$

This shows $x^* \in \overline{B(x_0, R\eta)}$.

Now we prove that x^* is a solution of $F(x) = 0$. Since

$$\begin{aligned}
\|F'(x_n)\| &\leq \|F'(x_0)\| + \|F'(x_n) - F'(x_0)\| \\
&\leq \|F'(x_0)\| + M\|x_n - x_0\| \\
&\leq \|F'(x_0)\| + MR\eta,
\end{aligned}$$

we can obtain

$$\|F(x_n)\| \leq \|F'(x_n)\|\|\Gamma_n F(x_n)\| \leq (\|F'(x_0)\| + MR\eta)\|\Gamma_n F(x_n)\|. \tag{3.8}$$

Since

$$\|\Gamma_n F(x_n)\| \leq d_{n-1}\|y_{n-1} - x_{n-1}\| = \cdots = \eta\Big(\prod_{i=0}^{n-1} d_i\Big) \leq \eta\delta^n\gamma^{\frac{3^n-1}{2}},$$

by letting $n \to \infty$, we obtain $\|\Gamma_n F(x_n)\| \to 0$, and $\|F(x_n)\| \to 0$ in (3.8). Hence, by the continuity of F in Ω, we obtain $F(x^*) = 0$.

Now we prove the uniqueness of x^* in $B(x_0, \frac{2}{M\beta} - R\eta) \cap \Omega$. Firstly we can obtain $x^* \in B(x_0, \frac{2}{M\beta} - R\eta) \cap \Omega$. Since $R = g(a_0)/\big(1 - d_0\big) < 1/a_0$, then we have

$$\frac{2}{M\beta} - R\eta = \Big(\frac{2}{a_0} - R\Big)\eta > \frac{1}{a_0}\eta > R\eta,$$

and then $\overline{B(x_0, R\eta)} \subseteq B(x_0, \frac{2}{M\beta} - R\eta) \cap \Omega$.

Let x^{**} be another zero of $F(x)$ in $B(x_0, \frac{2}{M\beta} - R\eta) \cap \Omega$. By Taylor theorem, we have

$$0 = F(x^{**}) - F(x^*) = \int_0^1 F^{'}\big((1-t)x^* + tx^{**}\big)dt(x^{**} - x^*). \tag{3.9}$$

Since

$$\begin{aligned}
&\|\Gamma_0\|\Big\|\int_0^1 \Big[F'\big((1-t)x^* + tx^{**}\big) - F'(x_0)\Big]dt\Big\| \\
&\leq M\beta\int_0^1 \big[(1-t)\|x^* - x_0\| + t\|x^{**} - x_0\|\big]dt \\
&< \frac{M\beta}{2}\Big[R\eta + \frac{2}{M\beta} - R\eta\Big] = 1,
\end{aligned}$$

it follows by the Banach lemma that $\int_0^1 F'\big((1-t)x^* + tx^{**}\big)dt$ is invertible and hence $x^{**} = x^*$.

By letting $m \to \infty$ in (3.6), we obtain (3.5) and furthermore

$$\|x_n - x^*\| \leq \frac{g(a_0)\eta}{\gamma^{1/3}(1-d_0)}\big(\gamma^{1/3}\big)^{3^n}. \tag{3.10}$$

This means that the modified Chebyshev-like's method given by (1.2) is of R-order of convergence at least three.

3.2 R-order of convergence

Now we consider the following mixed condition

$$\|F'''(x) - F'''(y)\| \leq \sum_{i=1}^m L_i\|x-y\|^{q_i}, L_i \geq 0, q_i \in [0,1], x, y \in \Omega.$$

By choosing $w(\mu) = \sum_{i=1}^m (L_i\mu^{q_i})$, we have $w(t\mu) = \sum_{i=1}^m (L_i t^{q_i}\mu^{q_i})$, since $t \in [0,1]$, $q_i \in [0,1]$, then $\varphi(t) = t^p$, where $p = min\{q_1, q_2, \cdots, q_m\}$.

Now we consider that $\varphi(t) = t^p$, where $p \in (0,1]$. In this situation

$$I_1 = \int_0^1 \varphi\big(\frac{1}{3}t\big)dt = \frac{1}{3^p}\cdot\frac{1}{1+p}, \tag{3.11}$$

and

$$I_2 = \int_0^1 \varphi(t)(1-t)^2 dt = \frac{1}{(1+p)(2+p)(3+p)}. \tag{3.12}$$

The sequence $\{c_n\}$ is reduced to

$$c_{n+1} = c_n h(a_n)d_n^{2+p}, \quad n \geq 1.$$

Moveover

$$\ell(\tau t, \tau^2 u, \tau^{2+p}v) < \tau^{3+p}\ell(t,u,v), \quad \text{for } \tau \in (0,1),\ p \in [0,1].$$

Then

$$\begin{aligned}
a_{n+1} &\leq \gamma^{(3+p)^n}a_n \leq \gamma^{\frac{(3+p)^{n+1}-1}{2+p}}a_0, \quad n \geq 0, \\
b_{n+1} &< \gamma^{2(3+p)^n}b_n < \gamma^{\frac{2[(3+p)^{n+1}-1]}{2+p}}b_0, \quad n \geq 0, \\
c_{n+1} &< \gamma^{(2+p)(3+p)^n}c_n < \gamma^{(3+p)^{n+1}-1}c_0, \quad n \geq 0.
\end{aligned}$$

Consequently, the new a prior error estimates for iteration (1.2) are:

$$\|x^* - x_n\| \le g(a_0)\eta\gamma^{\frac{(3+p)^n-1}{2+p}} \frac{\delta^n}{1-\gamma^{(3+p)^n}\delta}, \quad n \ge 0,$$

so that

$$\|x^* - x_n\| \le \frac{g(a_0)\eta}{\gamma^{\frac{1}{2+p}}(1-\gamma\delta)} \left(\gamma^{\frac{1}{2+p}}\right)^{(3+p)^n}, \quad n \ge 0,$$

and the R-order of convergence is then at least $3+p$.

Remark Notice that $w(z) = Lz,\ L \ge 0$, if F''' is Lipschitz continuous in Ω and the R-order of convergence of iteration (1.2) is now at least four. And if F''' is (L,p)-Hölder continuous in Ω, then $w(z) = Lz^p,\ L \ge 0$, so that (1.2) is of R-order of convergence at least $3+p$.

4 Application

We illustrate the previous study with an application to the following nonlinear integral equations.

Example 1. Consider the mixed Hammerstein type integral equation [24]:

$$x(s) = 1 + \frac{1}{3}\int_0^1 G(s,t)\Big(x(t)^{10/3} + x(t)^4\Big)dt, \qquad s,t \in [0,1], \tag{4.1}$$

where $x \in X$. Here $X = C[0,1]$ is the space of continuous functions on $[0,1]$ with the max-norm

$$\|x\| = \max_{s\in[0,1]} |x(s)|.$$

And the kernel G is the Green function

$$G(s,t) = \begin{cases} (1-s)t, & t \le s, \\ s(1-t), & s \le t. \end{cases}$$

The analysis and computation of these types of equations are justified by the dynamic model of a chemical reactor.

Solving (4.1) is equivalent to solve $F(x) = 0$, where $F : \Omega \subseteq C[0,1] \to C[0,1]$,

$$[F(x)](s) = x(s) - 1 - \frac{1}{3}\int_0^1 G(s,t)\Big(x(t)^{10/3} + x(t)^4\Big)dt, \quad s \in [0,1], \tag{4.2}$$

and Ω is a suitable non-empty open convex domain. Note that the first, second and third Fréchet derivatives of the operator F are given by

$$[F'(x)y](s) = y(s) - \frac{1}{3}\int_0^1 G(s,t)\Big(\frac{10}{3}x(t)^{7/3} + 4x(t)^3\Big)y(t)dt, \tag{4.3}$$

$$[F''(x)yz](s) = -\frac{1}{3}\int_0^1 G(s,t)\Big(\frac{70}{9}x(t)^{4/3} + 12x(t)^2\Big)z(t)y(t)dt, \tag{4.4}$$

$$[F'''(x)yzu](s) = -\frac{1}{3}\int_0^1 G(s,t)\Big(\frac{280}{27}x(t)^{1/3} + 24x(t)\Big)u(t)z(t)y(t)dt. \tag{4.5}$$

Observe that F''' is neither Lipschitz continuous nor Hölder continuous in Ω, but the operator F satisfies the assumptions of Theorem 1, so that a solution of (4.1) can be approximated by (1.2).

Now we consider $\Omega = B(0, 3/2) \subseteq X$ as an open convex nonempty domain and choose $x_0(s) = 1$ as an initial approximation solution. One can easily obtain

$$\|\Gamma_0\| \le 36/25 = \beta, \quad \|\Gamma_0 F(x_0)\| \le 3/25 = \eta,$$

$$\|F''(x)\| \le 1.6815 = M, \quad \|F'''(x)\| \le 1.9946 = N,$$

$$\omega(z) = \frac{35}{81}\sqrt[3]{z} + z, \ \varphi(t) = \sqrt[3]{t}, \ I_1 = 0.5200, \quad I_2 = 0.0964.$$

Hence, $a_0 = M\beta\eta = 0.2906$, $\quad b_0 = N\beta\eta^2 = 0.0414$ and $c_0 = \omega(\eta)\beta\eta^2 = 0.0069$, so that

$$\Phi(a_0) = a_0 g(a_0) - 1 \simeq -0.6550 < 0, \quad \text{and} \quad h(a_0)d_0 \simeq 0.0564 < 1.$$

Besides, the solution x^* belongs to $\overline{B(x_0, R\eta)} = \overline{B(1, 0.1480\cdots)} \subseteq \Omega$ and it is unique in $B(1, 0.6780\cdots) \cap \Omega$.

Finally, we discretize (4.1) to transform it into a finite dimensional problem and we apply (1.2) to approximate a solution. This procedure consists of approximating the integral appearing in (4.1) by a numerical quadrature formula. We consider the following Gauss-Legendre formula

$$\int_0^1 v(t)dt \simeq \sum_{i=1}^m w_i v(t_i),$$

where the nodes t_i and the weights w_i can be easily computed.

If we denote $x_j = x(t_j)$, $(j = 1, 2, \cdots, m)$, (4.1) becomes the following nonlinear system of equations

$$x_j = 1 + \frac{1}{3}\sum_{k=1}^m \alpha_{jk}(x_k^{10/3} + x_k^4), \qquad j = 1, 2, \cdots, m, \tag{4.6}$$

where

$$\alpha_{jk} = \begin{cases} w_k t_k(1 - t_j) & if\ k \le j, \\ w_k t_j(1 - t_k) & if\ k < j. \end{cases}$$

Now we apply the method given by (1.2) to compute (4.6) and compare it with the Chebyshev-like's method in [30]. Taking into account that we have previously considered the starting function $x_0(s) = 1$, we now choose the vector $x_0 = (1, 1, \cdots, 1)^T$ as the initial iterate. All computations are carried out with double arithmetic precision. In the tests, we take $m = 10, 20$ in (4.6) respectively. Displayed in Tables 1 and 2 is the max-norm of vector functions at each iterative step. The stopping criterion that we consider is $\|F(x_n)\|_\infty \le 10^{-15}$.

From the numerical results, we can see that the performance of the method (1.2) is better. This means that our method can be of practical interest.

Table 1: Results of system (4.6) with $m = 10$

n	Chebyshev-like method	the method (1.2)
1	3.098738e-004	6.359533e-005
2	1.645750e-011	6.661338e-016
3	6.661338e-016	

Table 2: Results of system (4.6) with $m = 20$

n	Chebyshev-like method	the method (1.2)
1	2.956476e-004	5.838933e-005
2	1.313716e-011	9.992007e-016
3	1.443290e-015	
4	4.440892e-016	

5 Dynamics

The dynamical analysis of a method is becoming a trend in recent publications [25-28] on iterative methods because it allows us to classify the various iterative formulas, not only from the point of view of its order of convergence, but also analyzing how these formulas behave as function of the initial estimate that is taken. Let us first recall some dynamical concepts. Consider a Fréchet differential function $G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. The orbit of $x \in \mathbb{R}^n$ is defined as:

$$x, G(x), G^2(x), \cdots, G^p(x), \cdots.$$

A point x_f is a fixed point of G if $G(x_f) = x_f$. The basin of attraction of x_f is the set of points whose orbits tend to this fixed point

$$\mathcal{A}(x_f) = \{x \in \mathbb{R}^n : G^p(x) \longrightarrow x_f \quad \text{for} \quad p \to \infty\}.$$

In this section we study the dynamics of the method (1.2) when applied to the solution of a 2×2 nonlinear system and compare it with the dynamics of Chebyshev-like's method in [30]. We show that the method is generally convergent and depict the attraction basins.

Example 2. Consider the following system [29]

$$\begin{cases} f_1(x) = \dfrac{1}{2} sin(x_1 x_2) - \dfrac{1}{4\pi} x_2 - \dfrac{1}{2} x_1 = 0 \\ f_2(x) = \left(1 - \dfrac{1}{4\pi}\right)\left(e^{2x_1} - e\right) + \dfrac{e}{\pi} x_2 - 2e x_1 = 0 \end{cases}$$

where $0.25 \le x_1 \le 1$ and $1.5 \le x_2 \le 2\pi$. The exact solutions are $x^* = (\frac{1}{2}, \pi)^T$ and $x^* = (0.29945, 2.83693)^T$.

For the comparisons, we have run the methods with tolerance 10^{-5}, performing a maximum of 20 iterations. The starting points form a uniform grid of size 600×600 in a rectangle of the real plane. The attraction basins have been colored according to the corresponding fixed point.

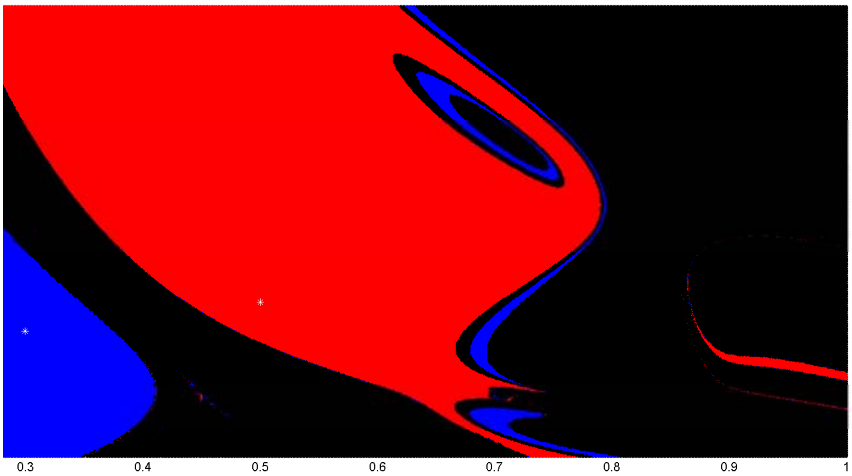

Figure 1: Attraction basins for Chebyshev-like's method

Figures 1 and 2 show the attraction basins of the method (1.2) and Chebyshev-like's method in [30], respectively. According to the figure, for any starting point that arise from the red or blue regions, the methods are converge to the solution in that region, while starting points from other region failed to convergence. The presented basin of attraction show the good performance of the method (1.2) as compared to Chebyshev-like's method in [30].

6 Conclusion

In this paper, the semilocal convergence of the modified Chebyshev-like's method for solving nonlinear equations in Banach spaces is established by using recurrence relations under the assumption that F''' satisfies ω-continuity condition. An existence-uniqueness theorem is given to show the R-order convergence of the method. Also a priori error bounds is given. From the numerical results, we can observe that the performance of our method in this paper is better.

The dynamical behavior of the method (1.2) has been compared with that of Chebyshev-like's method in [30]. The presented basin of attraction have also confirmed better performance of the method (1.2).

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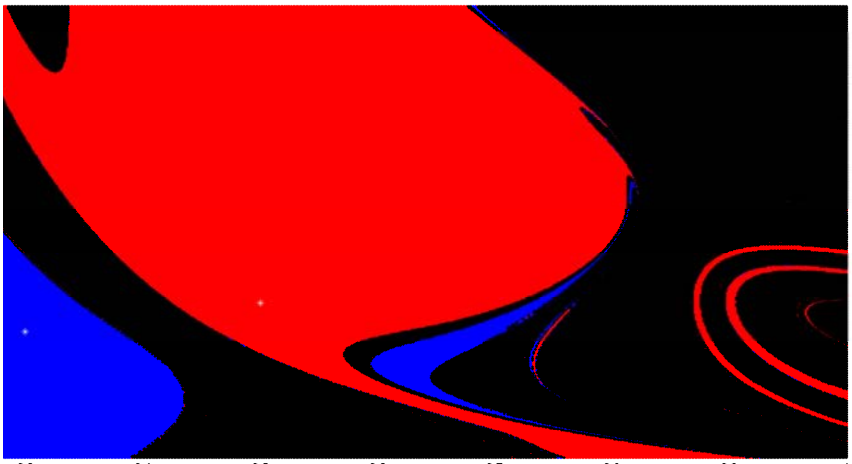

Figure 2: Attraction basins for Modified Chebyshev-like's method

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Generalized Hyers–Ulam stability of sextic functional equation in random normed spaces

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Abstract

By using the direct and fixed point methods, we establish the general solution and generalized Hyers–Ulam stability of the following sextic functional equation

$$f(nx+y)+f(nx-y)+f(x+ny)+f(x-ny)=(n^4+n^2)[f(x+y)+f(x-y)] + 2(n^6-n^4-n^2+1)[f(x)+f(y)]$$

in random normed spaces. Also, we present an illustrative example with the Lukasiewicz t-norm that can be a suitable approximation using this sextic function.

1. Introduction

If the values of norms are probability distribution functions, then we have a generalized notion of normed space named random normed space, that was introduced by Sherstnev in [31] and extended by Alsina, Schweizer and Sklar in [1]. The theory of random normed spaces have significant applications in quantum particle physics (see [20]). Also, it has very useful applications in many fields like population dynamics, chaos control, computer programming, nonlinear dynamical system, nonlinear operators, statistical convergence, etc.

On the other hand, one of the most important issues in the theory of functional equations, concerning the famous Ulam stability problem is: when a mapping satisfying a functional equation approximately, must be close to an exact solution of a given functional equation?

Ulam [35] in 1940 raised the first stability problem concerning group homomorphisms. Hyers [12] was the first mathematician to present an affirmative partial answer to the question of Ulam for Banach spaces. Subsequently, Hyers' theorem was generalized by Aoki [2] for additive mappings and by Rassias [28] for linear mappings by considering an unbounded Cauchy difference. Gavruta [10] obtained a generalization of Rassias' theorem, which allows the Cauchy difference to be controlled by a general unbounded function.

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The stability problems of a wide class of functional equations have been investigated by a number of authors, and there are many interesting results concerning those problem (see, e.g., [3, 11, 13, 15, 25, 28, 32–34, 36]). Also by using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see, e.g., [5–7, 19, 26]).

The generalized Hyers–Ulam stability of different functional equations in random normed spaces, paranormed spaces, quasi-normed spaces and quasi-β-normed spaces has been studied by many authors (see , e.g., [8, 14, 18, 22–24]). Park and Lee [23] proved the Hyers–Ulam stability of the following additive-quadratic-cubic-quatric functional equation

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x)+f(2y)+f(-2y)-4f(y)-4f(-y) \tag{1.1}$$

in paranormed spaces. The general solution of quintic and sextic functional equations

$$f(x+3y)-5f(x+2y)+10f(x+y)-10f(x)+5f(x-y)-f(x-2y)=120y \tag{1.2}$$

and

$$f(x+3y)-6f(x+2y)+15f(x+y)-20f(x)+15f(x-y)-6f(x-2y)+f(x-3y)=720f(y) \tag{1.3}$$

was introduced and investigated on the generalized Hyers–Ulam stability in quasi-β-normed spaces via fixed point method by Xu et al. [37].

The general solution and the generalized Hyers–Ulam stability of the sextic functional equation

$$\begin{aligned} f(nx+y)+f(nx-y)+f(x+ny)+f(x-ny) &= (n^4+n^2)[f(x+y)+f(x-y)] \\ &\quad + 2(n^6-n^4-n^2+1)[f(x)+f(y)] \end{aligned} \tag{1.4}$$

in paranormed spaces was discussed by Ravi and Sabarinathan [29].

In this paper, we present the general solution and generalized Hyers–Ulam stability of the sextic functional equation (1.4) under arbitrary t-norms by direct method and under Min t-norm by fixed point method in random normed spaces and provide an example for random normed spaces with the Lukasiewicz t-norm, by direct method.

2. Preliminaries

Before giving the main result, we present some basic facts related to random normed spaces and some preliminary results. We say that $f:\mathbb{R}\longrightarrow[0,1]$ is a distribution function if and only if it is a monotone, nondecreasing, left continuous, $\inf f(x)=0$ and $\sup f(x)=1$. By $\triangle^+$ we denote a collection of all distribution functions and by D^+ the set of all distribution functions such that $f(x)=0$. If $a\in\mathbb{R}_0$, then $H_a\in D^+$ where

$$H_a(t):=\begin{cases}0 & \text{if } t\le a,\\ 1 & \text{if } t>a.\end{cases}$$

It is obvious that $H_0 \geq f$ for all $f \in D^+$.

Definition 2.1 ([8, 30]). A mapping $T : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm (briefly a t-norm) if T satisfies the following conditions:

1. T is commutative and associative;
2. T is continuous;
3. $T(a, 1) = a$ for all $a \in [0, 1]$;
4. $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$.

Typical examples of continuous t-norms are $T_p(a, b) = ab$, $T_M(a, b) = \min(a, b)$ and $T_L(a, b) = \max(a+b-1, 0)$ (the Lukasiewicz t-norm).

Recall (see [8, 11]) that if T is a t-norm and x_n is a given sequence of numbers in $[0, 1]$, $T^n_{i=1}x_i$ is defined recurrently by $T^1_{i=1}x_i = x_1$ and $T^n_{i=1}x_i = T(T^{n-1}_{i=1}x_i, x_n)$ for $n \geq 2$.

It is known [11] that for the Lukasiewicz t-norm the following implication holds:

$$\lim_{n\to\infty}(T_L)^{\infty}_{i=1}x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty}(1 - x_n) < \infty.$$

Definition 2.2 ([31]). A random normed space (briefly RN-space) is a triple (X, μ, T) where X is a vector space, T is a continuous t-norm, and μ is a mapping from X into D^+ such that the following conditions hold:

1. $\mu_x(t) = H_0(t)$ for all $t > 0$ iff $x = 0$;
2. $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$, $t > 0$ and $\alpha \neq 0$;
3. $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y, z \in X$ and $t, s \geq 0$.

Definition 2.3 ([18]). Let (X, μ, T) be an RN-space. Then

1. A sequence $\{x_n\}$ in X is said to be convergent to x in X if, for every $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x}(\varepsilon) > 1 - \lambda$, whenever $n \geq N$.
2. A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for every $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(\varepsilon) > 1 - \lambda$, whenever $n \geq m \geq N$.
3. An RN-space (X, μ, T) is said to be complete if and only if every Cauchy sequence in X is convergent to a point in X.

Theorem 2.4 ([30]). *If (X, μ, T) is an RN-space and x_n is a sequence such that $x_n \longrightarrow x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

Definition 2.5 ([16]). Let X be a set. A function $d : X \times X \longrightarrow [0, \infty]$ is called a generalized metric on X if it satisfies

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 2.6 ([4, 9]). *Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n, or there exists a positive integer n_0 such that

1. *$d(J^n x, J^{n+1}x) < \infty$, $\forall n \geq n_0$;*
2. *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
3. *y^* is the unique fixed point of J in the set $Y = \{y \in X | d(J^{n_0}x, y) < \infty\}$;*
4. *$d(y, y^*) \leqslant \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.*

3. Hyers–Ulam stability of the sextic functional equation (1.4) by direct method

In this section, using the direct method, we prove the generalized Hyers–Ulam stability of the sextic functional equation (1.4) in complete RN-spaces. Also, we present an illustrative example with the Lukasiewicz t-norm that can be suitable approximation using this sextic function.

Theorem 3.1. *Let X be a real liner space, (Y, μ, T) a complete RN-space and $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$ for which there is $\phi : X^2 \longrightarrow D^+$ ($\phi(x, y)$ is denoted by $\phi_{x,y}$) such that*

$$\mu_{D_s f(x,y)}(t) \geqslant \phi_{x,y}(t), \tag{3.1}$$

where

$$\begin{aligned} D_s f(x, y) :=& f(nx + y) + f(nx - y) + f(x + ny) + f(x - ny) \\ &- (n^4 + n^2)[f(x + y) + f(x - y)] - 2(n^6 - n^4 - n^2 + 1)[f(x) + f(y)] \end{aligned}$$

for all $x, y \in X$ and $t > 0$. If

$$\lim_{m\to\infty} T_{i=1}^{\infty}(\phi_{n^{i+m-1}x,0}(n^{6m+5i}t)) = 1, \tag{3.2}$$

and

$$\lim_{m\to\infty} \phi_{n^m x, n^m y}(n^{6m}t) = 1 \tag{3.3}$$

for all $x, y \in X$ and $t > 0$, then there exists a unique sextic mapping $S : X \longrightarrow Y$ satisfying (1.4) and the inequality

$$\mu_{f(x)-s(x)}(t) \geq T_{i=1}^{\infty}(\phi_{n^{i-1}x,0}(n^{5i}t) \tag{3.4}$$

for all $x \in X$ and $t > 0$.

Proof. Letting $y = 0$ in (3.1), we get

$$\mu_{f(nx)-n^6 f(x)}(t) \geq \phi_{x,0}(2t) \geq \phi_{x,0}(t) \tag{3.5}$$

for all $x \in X$. Then we get

$$\mu_{\frac{f(nx)}{n^6} - f(x)}(t) \geq \phi_{x,0}(n^6 t), \tag{3.6}$$

therefore,

$$\mu_{\frac{f(n^{k+1}x)}{n^{6k+6}} - \frac{f(n^k x)}{n^{6k}}}(t) \geq \phi_{n^k x,0}(n^{6k+6}t), \tag{3.7}$$

that is,

$$\mu_{\frac{f(n^{k+1}x)}{n^{6k+6}} - \frac{f(n^k x)}{n^{6k}}}(\frac{t}{n^{k+1}}) \geq \phi_{n^k x,0}(n^{5(k+1)}t) \tag{3.8}$$

for every $k \in N$, $t > 0$, n positive integer, $n > 1$. As

$$1 > \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} + ... + \frac{1}{n^k},$$

by the triangle inequality it follows:

$$\begin{aligned} \mu_{\frac{f(n^m x)}{n^{6m}} - f(x)}(t) &\geq \mu_{\frac{f(n^m x)}{n^{6m}} - f(x)}(\sum_{k=0}^{m-1} \frac{1}{n^{k+1}}t) \\ &\geq T_{k=0}^{m-1}\left(\mu_{\frac{f(n^{k+1}x)}{n^{6k+6}} - \frac{f(n^k x)}{n^{6k}}}(\frac{1}{n^{k+1}}t)\right) \\ &\geqslant T_{k=0}^{m-1}(\phi_{n^k x,0}(n^{5k+5}t) \\ &= T_{i=1}^{m}\left(\phi_{n^{i-1}x,0}(n^{5i}t)\right), \end{aligned} \tag{3.9}$$

$x \in X$, $t > 0$, and $n > 1$. In order to prove the convergence of the sequence $\{\frac{f(n^j x)}{n^{6j}}\}$, we replace x by $n^j x$, and multiplying the left-hand side of (3.9) by $\frac{n^{6j}}{n^{6j}}$, we get

$$\mu_{\frac{f(n^{m+j}x)}{n^{6m+6j}} - \frac{f(n^j x)}{n^{6j}}}(t) \geq T_{i=1}^{m}\left(\phi_{n^{j+i-1}x,0}(n^{6j+5i}t)\right). \tag{3.10}$$

Since the right-hand side of the inequality (3.10) tends to 1 as m and j tend to infinity, the sequence $\{\frac{f(n^j x)}{n^{6j}}\}$ is a Cauchy sequence. Therefore, we may define

$$S(x) = \lim_{j \longrightarrow \infty} \frac{f(n^j x)}{n^{6j}}$$

for all $x \in X$.

Replacing x, y by $n^m x$ and $n^m y$, respectively, in (3.1), then multiplying the right hand-side by $\frac{n^{6m}}{n^{6m}}$, it follows that

$$\mu_{\frac{1}{n^{6m}} D_s f(n^m x, n^m y)}(t) \geq \phi_{n^m x, n^m y}(n^{6m}t)$$

for all $x, y \in X$, and positive integer n, $n > 1$. Taking the limit as $m \to \infty$ we find that S satisfies (1.4), that is, S is a sextic map. To prove (3.4) take the limit as $m \to \infty$ in (3.9).

Finally, to prove the uniqueness of the sextic function S, let us assume that there exists a sextic function r which satisfies (3.4) and equation (1.4). Therefore

$$\begin{aligned}\mu_{r(x)-s(x)}(t) &= \mu_{r(x) - \frac{f(n^j x)}{n^{6j}} + \frac{f(n^j)}{n^{6j}} - s(x)}(t) \\ &\geq T(\mu_{r(x) - \frac{f(n^j x)}{n^{6j}}}(\frac{t}{2}), \mu_{\frac{f(n^j x)}{n^{6j}} - s(x)}(\frac{t}{2})).\end{aligned}$$

Taking the limit as $j \to \infty$, we find $\mu_{r(x)-s(x)}(t) = 1$. Therefore $r = s$. □

Corollary 3.2. *Let X be a real liner space and (Y, μ, T) a complete RN-space such that ($T = T_M$, T_p or T_L) and $f : X \longrightarrow Y$ be a mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant 1 - \frac{\|x\|}{t + \|x\|} \tag{3.11}$$

for all $x \in X$, $t > 0$. Then there exists a unique sextic mapping $S : X \longrightarrow Y$ satisfying (1.4) and

$$\mu_{f(x)-s(x)}(t) \geqslant T_{i=1}^{\infty}(1 - \frac{\|x\|}{n^{4i+1}t + \|x\|})$$

for every $x \in X$, and $t > 0$.

Proof. It is enough to put,

$$\phi_{x,y}(t) = 1 - \frac{\|x\|}{t + \|x\|}$$

for all $x, y \in X$ and $t > 0$, in Theorem 3.1. □

Corollary 3.3. *Let X be a real liner space and (Y, μ, T) a complete RN-space such that ($T = T_M$, T_p or T_L) and $f : X \longrightarrow Y$ be a mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + \varepsilon\|x_0\|},$$

$x_0 \in X$, $t > 0$, and $\varepsilon > 0$. Then there exists a unique sextic mapping $S : X \longrightarrow Y$ satisfying (1.4) and

$$\mu_{f(x)-s(x)}(t) \geqslant T_{i=1}^{\infty}(\frac{n^{5i}t}{n^{5i}t + \varepsilon\|x_0\|}).$$

Proof. It is enough to put,

$$\phi_{x,y}(t) = \frac{t}{t + \varepsilon\|x_0\|}$$

for all $x, y \in X$ and $t > 0$, in Theorem 3.1. □

Corollary 3.4. *Let X be a real linear space and (Y, μ, T) a complete RN-space such that ($T = T_M$, T_p or T_L) and let $L \geq 0$ and p be a real number with $0 < p < 5$ and $f : X \longrightarrow Y$ be a mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + L(\| x \|^p + \| y \|^p)}$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique sextic mapping $S : X \longrightarrow Y$ satisfying (1.4) and

$$\mu_{f(x)-s(x)}(t) \geq T_{i=1}^{\infty}\left(\frac{t}{t + Ln^{i(p-5)-p} \| x \|^p}\right)$$

for every $x \in X$ and $t > 0$.

Proof. It is enough to put

$$\phi_{x,y}(t) = \frac{t}{t + L(\| x \|^p + \| y \|^p)}$$

for all $x, y \in X$ and $t > 0$, in Theorem 3.1. □

Example 3.5. Let $(X, \|.\|)$ be a Banach algebra and

$$\mu_x(t) = \begin{cases} \max\{1 - \frac{\|x\|}{t}, 0\} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0, \end{cases}$$

for all $x, y \in X$ and $t > 0$. Let

$$\varphi_{x,y}(t) = \begin{cases} \max\{1 - \frac{(8n^6)(\|x\|+\|y\|)}{t}, 0\} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

We note that $\varphi_{x,y}(t)$ is a distribution function and $\lim_{j\to\infty} \varphi_{n^j x, n^j y}(n^{6j}t) = 1$ for all $x, y \in X$ and $t > 0$.

It is easy to show that (X, μ, T_L) is an RN-space (this was essentially proved by Mushtari in [21], see also [27]). Indeed, $\mu_x(t) = 1$, $\forall t > 0$ implies $\frac{\|x\|}{t} = 0$ and hence $x = 0$ for all $x \in X$ and $t > 0$. Obviously, $\mu_{\lambda x}(t) = \mu_x(\frac{t}{\lambda})$ for all $x \in X$ and $t > 0$. Next, for all $x, y \in X$ and $t, s > 0$, we have

$$\begin{aligned} \mu_{x+y}(t+s) &= \max\{1 - \frac{\|x+y\|}{t+s}, 0\} \\ &= \max\{1 - \|\frac{x+y}{t+s}\|, 0\} \\ &= \max\{1 - \|\frac{x}{t+s} + \frac{y}{t+s}\|, 0\} \\ &\geq \max\{1 - \|\frac{x}{t}\| - \|\frac{y}{s}\|, 0\} \\ &= T_L(\mu_x(t), \mu_y(s)). \end{aligned}$$

It is easy to see that (X, μ, T_L) is complete, for

$$\mu_{x-y}(t) \geq 1 - \frac{\|x-y\|}{t}, \quad \forall x, y \in X,$$

and $t > 0$ and $(X, \|.\|)$ is complete. Define a mapping $f : X \longrightarrow X$ by $f(x) = x^6 + \|x\|x_0$ for all $x \in X$, where x_0 is a unit vector in X. A simple computation shows that

$$\begin{aligned}
&\|f(nx+y) + f(nx-y) + f(x+ny) + f(x-ny) \\
&\quad - (n^4+n^2)[f(x+y)+f(x-y)] - 2(n^6-n^4-n^2+1)[f(x)+f(y)]\| \\
&= |\parallel nx+y \parallel + \parallel nx-y \parallel + \parallel x+ny \parallel + \parallel x-ny \parallel \\
&\quad - (n^2+n^4)[\parallel x+y \parallel + \parallel x-y \parallel] \\
&\quad - 2(n^6-n^4-n^2+1)[\parallel x \parallel + \parallel y \parallel]| \\
&\leq 2(n^6+n+2)(\parallel x \parallel + \parallel y \parallel) \leq 8n^6(\parallel x \parallel + \parallel y \parallel)
\end{aligned}$$

for all $x, y \in X$. Hence $\mu_{D_s f(x,y)}(t) \geq \phi_{x,y}(t)$ for all $x, y \in X$ and $t > 0$. Fix $x \in X$ and $t > 0$. Then it follows that,

$$\begin{aligned}
(T_L)_{i=1}^{\infty} \left(\phi_{n^{i+j-1}x,0}(n^{6j+5i}t)\right) &= \max\left\{\sum_{i=1}^{\infty} \left(\phi_{n^{i+j-1}x,0}(n^{6j+5i}t) - 1\right) + 1, 0\right\} \\
&= \max\left\{1 - \frac{8n^5\|x\|}{n^{5j}(n^4-1)t}, 0\right\}
\end{aligned}$$

for all $x \in X$, $n \in \mathbb{N}$ and $t > 0$. Hence

$$\lim_{j\to\infty} (T_L)_{i=1}^{\infty} \left(\varphi_{n^{i+j-1}x,0}(n^{6j+5i}t)\right) = 1$$

for all $x \in X$ and $t > 0$. Thus, all the conditions of Theorem 3.1 hold. Since

$$(T_L)_{i=1}^{\infty} \left(\phi_{n^{i-1}x,0}(n^{5i}t)\right) = \max\{1 - \frac{8n^5\|x\|}{(n^4-1)t}, 0\}$$

for all $x \in X$ and $t > 0$, we can deduce that $S(x) = x^6$ is the unique sextic mapping $S : X \longrightarrow X$ such that

$$\mu_{f(x)-s(x)}(t) \geq \max\{1 - \frac{8n^5\|x\|}{(n^4-1)t}, 0\}$$

for all $x \in X$ and $t > 0$.

4. Hyers–Ulam stability of the sextic functional equation (1.4) by fixed point method

In this section, using the fixed point method, we prove the generalized Hyers–Ulam stability of the sextic functional equation (1.4) in complete RN-spaces.

Theorem 4.1. *Let X be a real liner space and (Y, μ, T_M) be a complete RN-space and $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$ for which there is $\phi : X^2 \longrightarrow D^+$ ($\phi(x, y)$ is denoted by $\phi_{x,y}$) such that*

$$\phi_{nx,ny}(\alpha t) \geq \phi_{x,y}(t), \quad 0 < \alpha < n^6,$$

and

$$\mu_{D_s f(x,y)}(t) \geqslant \phi_{x,y}(t) \tag{4.1}$$

for all $x, y \in X$, and $t > 0$, where

$$D_s f(x, y) := f(nx + y) + f(nx - y) + f(x + ny) + f(x - ny)$$

$$- (n^4 + n^2)[f(x + y) + f(x - y)] - 2(n^6 - n^4 - n^2 + 1)[f(x) + f(y)]$$

for all $x, y \in X$ and $t > 0$. Then there exists a unique sextic mapping $g : X \longrightarrow Y$ such that

$$\mu_{f(x)-g(x)}(t) \geqslant \phi_{x,0}(2(n^6 - \alpha)t) \tag{4.2}$$

for all $x \in X$ and $t > 0$. Moreover, we have

$$g(x) = lim_{m\longrightarrow\infty} \frac{f(n^m x)}{n^{6m}}.$$

Proof. Let $y = 0$ in (4.1); we get

$$\mu_{2f(nx)-2n^6 f(x)}(t) \geq \phi_{x,0}(t) \tag{4.3}$$

for all $x \in X$ and $t > 0$ and hence

$$\mu_{\frac{f(nx)}{n^6}-f(x)}(t) \geq \phi_{x,0}(2n^6 t). \tag{4.4}$$

Consider the set

$$E := \{g : X \to Y : \; g(0) = 0\},$$

and the mapping d_G defined on $E \times E$ by

$$d_G(g, h) = \texttt{inf}\{\epsilon > 0 : \mu_{g(x)-h(x)}(\epsilon t) \geq \phi_{x,0}(2n^6 t)\},$$

for all $x \in X$, $t > 0$. Then (E, d_G) is a complete generalized metric space (see the proof of [17, Lemma 2.1]). Now, let us consider the linear mapping $J : E \to E$ defined by

$$Jg(x) = \frac{g(nx)}{n^6}.$$

Now, we show that J is a strictly contractive self-mapping of E with the Lipschitz constant $k = \frac{\alpha}{n^6}$. Indeed, let $g, h \in E$ be the mappings such that $d_G(g, h) < \epsilon$. Then we have

$$\mu_{g(x)-h(x)}(\epsilon t) \geq \phi_{x,0}(2n^6 t)$$

for all $x \in X$ and $t > 0$ and hence

$$\begin{aligned}\mu_{Jg(x)-Jh(x)}(\frac{\epsilon\alpha t}{n^6}) &= \mu_{\frac{g(nx)}{n^6}-\frac{h(nx)}{n^6}}(\frac{\epsilon\alpha t}{n^6}) \\ &= \mu_{g(nx)-h(nx)}(\alpha\varepsilon t) \\ &\geq \phi_{nx,0}(2\alpha n^6 t)\end{aligned}$$

for all $x \in X$ and $t > 0$. Since

$$\phi_{nx,ny}(\alpha t) \geq \phi_{x,y}(t), \quad 0 < \alpha < n^6,$$

we have

$$\mu_{Jg(x)-Jh(x)}(\frac{\epsilon\alpha t}{n^6}) \geq \phi_{x,0}(2n^6 t),$$

that is,

$$d_G(g, h) < \epsilon \Longrightarrow d_G(Jg, Jh) < \frac{\alpha}{n^6}\epsilon.$$

This means that

$$d_G(Jg, Jh) < \frac{\alpha}{n^6} d_G(g, h),$$

for all $g, h \in E$. Next, from

$$\mu_{\frac{f(nx)}{n^6}-f(x)}(t) \geq \phi_{x,0}(2n^6 t),$$

it follows that $d_G(f, Jf) \leq 1$. Using Theorem 2.6, we show the existence of a fixed point of J, that is, the existence of a mapping $g : X \longrightarrow Y$ such that $g(nx) = n^6 g(x)$ for all $x \in X$. Since, for all $x \in X$ and $t > 0$,

$$d_G(u, v) < \epsilon \Longrightarrow \mu_{u(x)-v(x)}(t) \geq \phi_{x,0}(\frac{2n^6 t}{\epsilon}),$$

it follows from $d_G(J^n f, g) \longrightarrow 0$ that $lim_{m\longrightarrow\infty} \frac{f(n^m x)}{n^{6m}} = g(x)$ for all $x \in X$. Also from

$$d_G(f, g) \leq \frac{1}{1 - L} d(f, Jf)$$

for all $g, h \in E$, we have $d_G(f, g) \leq \frac{1}{1-\frac{\alpha}{n^6}}$, and it immediately follows that

$$\mu_{g(x)-f(x)}(\frac{n^6}{n^6 - \alpha} t) \geqslant \phi_{x,0}(2n^6 t)$$

for all $x \in X$ and $t > 0$. This means that

$$\mu_{g(x)-f(x)}(t) \geqslant \phi_{x,0}(2(n^6 - \alpha)t)$$

for all $x \in X$ and $t > 0$. Finally, the uniqueness of g follows from the fact that g is the unique fixed point of J such that there exists $C \in (0, \infty)$ satisfying

$$\mu_{g(x)-f(x)}(Ct) \geqslant \phi_{x,0}(2n^6 t)$$

for all $x \in X$ and $t > 0$. This completes the proof. □

Corollary 4.2. *Let X be a real liner space, (Y, μ, T_M) a complete RN-space, and $f : X \longrightarrow Y$ a mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant 1 - \frac{\|x\|}{t + \|x\|} \tag{4.5}$$

for all $x \in X$, $t > 0$. Then there exists a unique sextic mapping $s : X \longrightarrow Y$ satisfying (1.4) *and*

$$\mu_{f(x)-s(x)}(t) \geqslant 1 - \frac{\|x\|}{2(n^6 - \alpha)t + \|x\|}$$

for every $x \in X$, $t > 0$, and n positive integer. Moreover, we have

$$s(x) = \lim_{m\longrightarrow\infty} \frac{f(n^m x)}{n^{6m}}.$$

Proof. It is enough to put,

$$\phi_{x,y}(t) = 1 - \frac{\|x\|}{t + \|x\|}$$

for all $x \in X$ and $t > 0$ in Theorem 4.1. Then we can choose $n < \alpha < n^6$ and so we get the desired result. □

Corollary 4.3. *Let X be a real liner space, (Y, μ, T_M) a complete RN-space and $f : X \longrightarrow Y$ a mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + \varepsilon\|x_0\|},$$

$x_0 \in X$, $t > 0$, and $\varepsilon > 0$. Then there exists a unique sextic mapping $s : X \longrightarrow Y$ satisfying (1.4) *and*

$$\mu_{f(x)-s(x)}(t) \geqslant \frac{2(n^6 - \alpha)t}{2(n^6 - \alpha)t + \varepsilon\|x_0\|}$$

for every $x \in X$, $t > 0$, and n positive integer. Moreover, we have

$$s(x) = \lim_{m\longrightarrow\infty} \frac{f(n^m x)}{n^{6m}}.$$

Proof. It is enough to put

$$\phi_{x,y}(t) = \frac{t}{t + \varepsilon\|x_0\|}$$

for all $x \in X$, and $t > 0$ in Theorem 4.1. Then we can choose $n < \alpha < n^6$ and so we get the desired result. □

Corollary 4.4. *Let X be a real liner space, (Y, μ, T_M) a complete RN-space and $f : X \longrightarrow Y$ a mapping satisfying*

$$\mu_{D_s f(x,y)}(t) \geqslant \frac{t}{t + \theta(\| x \|^p + \| y \|^p)}$$

for all $x, y \in X$, $t > 0$, $\theta > 0$, and $0 < p < 6$. Then there exists a unique sextic mapping $s : X \longrightarrow Y$ satisfying (1.4) *and*

$$\mu_{f(x)-s(x)}(t) \geq \frac{2(n^6 - \alpha)t}{2(n^6 - \alpha)t + \theta \| x \|^p}$$

for every $x \in X$ and $t > 0$. Moreover, we have

$$s(x) = lim_{m \longrightarrow \infty} \frac{f(n^m x)}{n^{6m}}.$$

Proof. It is enough to put

$$\phi_{x,y}(t) = \frac{t}{t + \theta(\| x \|^p + \| y \|^p)}$$

for all $x, y \in X$ and $t > 0$ in Theorem 4.1. Then we can choose $n^p < \alpha < n^6$ and so we get the desired result. □

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On convergence theorem of a finite family of nonlinear mappings in uniformly convex metric spaces

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Abstract

In this paper, we introduce the $S-$mapping generated by a finite family of nonexpansive mapping and real numbers in convex metric space by using concept of the $S-$mapping defined by Kangtunyakarn and Suantai [1]. Then, we prove convergence of Ishikawa iteration generated by the $S-$mapping to a common fixed point of a finite family of nonexpansive mappings in uniformly convex metric space.
Keywords: Convex metric space; Nonexpansive mapping; $S-$mapping.

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1 Introduction

Throughout this paper, we assume that (X, d) is a complete metric space and C is a nonempty closed convex subset of (X, d). A point x is called a fixed point of T if $Tx = x$. We use $F(T)$ to denote the set of fixed point of T. Recalled the following definitions;

Definition 1.1. *The mapping $T : C \to C$ is said to be nonexpansive if*

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in C.$$

In 1970, Takahashi [9] introduce the following definition as follows:

Definition 1.2. *Let (X, d) be a metric space. A mapping $W : X \times X \times [0, 1] \to X$ is said to be a convex structure on X if for each $(x, y, \lambda) \in X \times X \times [0, 1]$ and for all $u \in X$,*

$$d\big(u, W(x, y, \lambda)\big) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

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We observe that $W(x, y, \lambda) = \lambda x + (1-\lambda)y$ is a convex structure on a normed linear space. A metric space (X, d) together with a convex structure W is called a *convex metric space* denoted by (X, d, W). A nonempty subset C of X is said to be *convex* if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$.

Two classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one is introduced by Mann [10] and is defined as follows:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad \forall n \geq 1,$$

where $x_1 \in C$, $\{\alpha_n\} \subseteq [0, 1]$.

The second iteration process is referred to as Ishikawa's iteration process [5] which is defined recursively by

$$\begin{cases} x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n \\ y_n = \beta_n x_n + (1-\beta_n)Tx_n, \quad \forall n \geq 1, \end{cases} \tag{1.1}$$

where $x_1 \in C$, $\{\alpha_n\}, \{\beta_n\} \subseteq [0, 1]$.

In 2009, Kangtunyakarn and Suantai [1] introduced the mapping generated by a finite family of nonexpansive mapping and family of real numbers as follows:

Definition 1.3. *Let C be a nonempty convex subset of real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpanxive mappings of C into itself. For each $j = 1, 2, ..., N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I \in [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. They define the mapping $S : C \to C$ as follows:*

$$\begin{aligned} U_0 &= I, \\ U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I, \\ U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I, \\ U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I, \\ &\;\;\vdots \\ U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I, \\ S &= U_N = \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I. \end{aligned} \tag{1.2}$$

This mapping is called *S-mapping* generated by $T_1, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$.

In this paper, by using the concept of the $S-$mapping in Definition 1.3, we define the $S-$mapping generated by a finite family of nonexpansive mappings and real numbers in convex metric space. Then, we prove convergence of Ishikawa iteration generated by the $S-$mapping to a common fixed point of a finite family of nonexpansive mappings in uniformly convex metric space.

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2 Preliminaries

In this section, we recall some lemmas and definitions to prove our main result as follows:

Definition 2.1. *(See [7]) A convex metric space (X, d, W) is said to be uniformly convex if for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that for all $r > 0$ and $x, y, z \in X$ with $d(z, x) < r, d(z, y) < r$ and $d(x, y) \geq r\epsilon$,*

$$d\big(z, W(x, y, \frac{1}{2})\big) \leq (1 - \delta)r.$$

Lemma 2.1. *(See [11], [3]) Let (X, d, W) be a convex metric space. For each $x, y \in X$ and $\lambda, \lambda_1, \lambda_2 \in [0, 1]$, we have the following.*
(i) $W(x, x, \lambda) = x$, $W(x, y, 0) = y$ and $W(x, y, 1) = x$.
(ii) $d\big(x, W(x, y, \lambda)\big) = (1 - \lambda)d(x, y)$ and $d\big(y, W(x, y, \lambda)\big) = \lambda d(x, y)$.
(iii) $d(x, y) = d\big(x, W(x, y, \lambda)\big) + d\big(W(x, y, \lambda), y\big)$.
(iv) $|\lambda_1 - \lambda_2|d(x, y) \leq d\big(W(x, y, \lambda_1), W(x, y, \lambda_2)\big)$.

We say that a convex metric space (X, d, W) has the property:
(C) if $W(x, y, \lambda) = W(y, x, 1 - \lambda)$ for all $x, y \in X$ and $\lambda \in [0, 1]$,
(I) if $d\big(W(x, y, \lambda_1), W(x, y, \lambda_2)\big) \leq |\lambda_1 - \lambda_2|d(x, y)$ for all $x, y \in X$ and $\lambda_1, \lambda_2 \in [0, 1]$,
(H) if $d\big(W(x, y, \lambda), W(x, z, \lambda)\big) \leq (1-\lambda)d(y, z)$ for all $x, y, z \in X$ and $\lambda \in [0, 1]$,
(S) if $d\big(W(x, y, \lambda), W(z, w, \lambda)\big) \leq \lambda d(x, z) + (1 - \lambda)d(y, w)$ for all $x, y, z, w \in X$ and $\lambda \in [0, 1]$.

Remark 2.2. It is easy to see that the property (C) and (H) imply continuity of a convex structure $W : X \times X \times [0, 1] \to X$ and the property (S) implies the property (H). In 2005, Aoyama et al. [3] proved that a convex metric space with property (C) and (H) has the property (S).

In 2011, Phuengrattana and Suantai [8] proved the following lemma as follows;

Lemma 2.3. *(See [8]) Property (C) holds in uniformly convex metric space.*

Remark 2.4. (See [8]) From Lemma 2.3, a uniformly convex metric space (X, d, W) with the property (H) has the property S and the convex structure W is also continuous.

Lemma 2.5. *(See [6]) Let (X, d, W) be a uniformly convex metric space with continuous convex structure. Then for arbitrary positive number ϵ, there exists $\eta = \eta(\epsilon) > 0$ such that*

$$d\big(z, W(x, y, \lambda)\big) \leq (1 - 2\min\{\lambda, 1 - \lambda\}\eta)r$$

for all $r > 0$ and $x, y, z \in X, d(z, x) \leq r, d(z, y) \leq r, d(x, y) \geq r\varepsilon$ and $\lambda \in [0, 1]$.

Lemma 2.6. *(See [2],[4]) Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n, \ n \geq 1.$$

If

$$\sum_{n=1}^{\infty} \delta_n < \infty, \quad and \quad \sum_{n=1}^{\infty} b_n < \infty,$$

then $\lim_{n\to\infty} a_n$ exists.

We introduce the following definition to use in the next section.

Definition 2.2. *Let (X, d, W) be a complete convex metric space and C be a nonempty closed convex subset of (X, d, W). Let $\{T_i\}_{i=1}^N$ be a finite family of mappings of C into C. For each $j = 1, 2, \cdots, N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$ where $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. For every $x \in C$, we define the mapping $S : C \times C \times [0,1] \to C$ as follows:*

$$\begin{aligned}
U_0 x &= x, \\
U_1 x &= W\big(T_1 U_0 x, W(U_0 x, x, \frac{\alpha_2^1}{1-\alpha_1^1}), \alpha_1^1\big), \\
U_2 x &= W\big(T_2 U_1 x, W(U_1 x, x, \frac{\alpha_2^2}{1-\alpha_1^2}), \alpha_1^2\big), \\
&\vdots \\
U_{N-1} x &= W\big(T_{N-1} U_{N-2} x, W(U_{N-2} x, x, \frac{\alpha_2^{N-1}}{1-\alpha_1^{N-1}}), \alpha_1^{N-1}\big), \\
Sx &= U_N x = W\big(T_N U_{N-1} x, W(U_{N-1} x, x, \frac{\alpha_2^N}{1-\alpha_1^N}), \alpha_1^N\big).
\end{aligned}$$

This mapping is called $S-$mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$.

Lemma 2.7. *Let C be a nonempty closed convex subset of a complete uniformly convex metric space (X, d, W) with property (H). Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpanxive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, 3, ..., N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1)$ for all $j = 1, 2, ..., N-1$, $\alpha_1^N \in (0,1]$ α_2^j, $\alpha_3^j \in [0,1)$ for all $j = 1, 2, ..., N$. Let S be the mapping generated by $T_1, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$.*

Proof. From Lemma 2.1 and definition of $S-$mapping, it is easy to see that $\bigcap_{i=1}^N F(T_i) \subseteq F(S)$. Next, we show that $F(S) \subseteq \bigcap_{i=1}^N F(T_i)$. To show this let $x_0 \in F(S)$ and $q \in \bigcap_{i=1}^N F(T_i)$, we have

$$\begin{aligned}
d(q, Sx_0) &= d\Big(q, W\big(T_N U_{N-1} x_0, W(U_{N-1} x_0, x_0, \frac{\alpha_2^N}{1-\alpha_1^N}), \alpha_1^N\big)\Big) \\
&\leq \alpha_1^N d(q, T_N U_{N-1} x_0) + (1-\alpha_1^N) d\Big(q, W(U_{N-1} x_0, x_0, \frac{\alpha_2^N}{1-\alpha_1^N})\Big)
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_1^N d(q, T_N U_{N-1}x_0) + (1-\alpha_1^N)\Big(\frac{\alpha_2^N}{1-\alpha_1^N} d(q, U_{N-1}x_0) \\
&+ (1-\frac{\alpha_2^N}{1-\alpha_1^N})d(q,x_0)\Big) \\
&= \alpha_1^N d(q, T_N U_{N-1}x_0) + \alpha_2^N d(q, U_{N-1}x_0) + \alpha_3^N d(q,x_0) \\
&\leq (1-\alpha_3^N) d(q, U_{N-1}x_0) + \alpha_3^N d(q,x_0) \\
&\leq (1-\alpha_3^N)\Big((1-\alpha_3^{N-1})d(q, U_{N-2}x_0) + \alpha_3^{N-1} d(q,x_0)\Big) \\
&\quad +\alpha_3^N d(q,x_0) \\
&= (1-\alpha_3^N)(1-\alpha_3^{N-1})d(q, U_{N-2}x_0) + \alpha_3^{N-1}(1-\alpha_3^N) d(q,x_0) \\
&\quad +\alpha_3^N d(q,x_0) \\
&= \Pi_{j=N-1}^N (1-\alpha_3^j) d(q, U_{N-2}x_0) + \big(1-\Pi_{j=N-1}^N(1-\alpha_3^j)\big) d(q,x_0) \\
&\leq \\
&\vdots \\
&\leq \Pi_{j=3}^N (1-\alpha_3^j) d(q, U_2x_0) + \big(1-\Pi_{j=3}^N(1-\alpha_3^j)\big) d(q,x_0) \\
&= \Pi_{j=3}^N (1-\alpha_3^j) d\big(q, W\big(T_2U_1x_0, W(U_1x_0, x_0, \frac{\alpha_2^2}{1-\alpha_1^2}), \alpha_1^2\big)\big) \\
&\quad +\big(1-\Pi_{j=3}^N(1-\alpha_3^j)\big) d(q,x_0) \\
&\leq \Pi_{j=3}^N (1-\alpha_3^j)\Big(\alpha_1^2 d(q, T_2U_1x_0) + (1-\alpha_1^2) d\big(q, W(U_1x_0, x_0, \frac{\alpha_2^2}{1-\alpha_1^2})\big)\Big) \\
&\quad +\big(1-\Pi_{j=3}^N(1-\alpha_3^j)\big) d(q,x_0) \\
&\leq \Pi_{j=3}^N (1-\alpha_3^j)\Big(\alpha_1^2 d(q, T_2U_1x_0) + (1-\alpha_1^2)\big(\frac{\alpha_2^2}{1-\alpha_1^2} d(q, U_1x_0) \\
&\quad +(1-\frac{\alpha_2^2}{1-\alpha_1^2}) d(q,x_0)\big)\Big) + \big(1-\Pi_{j=3}^N(1-\alpha_3^j)\big) d(q,x_0) \\
&= \Pi_{j=3}^N (1-\alpha_3^j)\Big(\alpha_1^2 d(q, T_2U_1x_0) + \alpha_2^2 d(q, U_1x_0) + \alpha_3^2 d(q,x_0)\Big) \\
&\quad +\big(1-\Pi_{j=3}^N(1-\alpha_3^j)\big) d(q,x_0) \\
&\leq \Pi_{j=3}^N (1-\alpha_3^j)\Big((1-\alpha_3^2) d(q, U_1x_0) + \alpha_3^2 d(q,x_0))\Big) \\
&\quad +\big(1-\Pi_{j=3}^N(1-\alpha_3^j)\big) d(q,x_0) \\
&= \Pi_{j=2}^N (1-\alpha_3^j) d(q, U_1x_0) + \big(1-\Pi_{j=2}^N(1-\alpha_3^j)\big) d(q,x_0) \\
&= \Pi_{j=2}^N (1-\alpha_3^j) d\big(q, W\big(T_1U_0x_0, W(U_0x_0, x_0, \frac{\alpha_2^1}{1-\alpha_1^1}), \alpha_1^1\big)\big) \\
&\quad +\big(1-\Pi_{j=2}^N(1-\alpha_3^j)\big) d(q,x_0)
\end{aligned}
$$

$$\begin{aligned}
&= \Pi_{j=2}^{N}(1-\alpha_3^j)d\big(q, W\big(T_1x_0, x_0, \alpha_1^1\big)\big) + \big(1-\Pi_{j=2}^{N}(1-\alpha_3^j)\big)d(q, x_0) \\
&\leq \Pi_{j=2}^{N}(1-\alpha_3^j)\big(\alpha_1^1 d(q, T_1x_0) + (1-\alpha_1^1)d(q, x_0)\big) + \big(1-\Pi_{j=2}^{N}(1-\alpha_3^j)\big)d(q, x_0) \\
&\leq \Pi_{j=2}^{N}(1-\alpha_3^j)d(q, x_0) + \big(1-\Pi_{j=2}^{N}(1-\alpha_3^j)\big)d(q, x_0) \\
&= d(q, x_0). \qquad (2.1)
\end{aligned}$$

From (2.1), we have

$$d(q, U_1x_0) = d\big(q, W(T_1x_0, x_0, \alpha_1^1)\big) = d(q, x_0) \text{ and } d(q, T_1x_0) = d(q, x_0).$$

Suppose $x_0 \neq T_1x_0$, then we have $d(x_0, T_1x_0) > 0$. Choose $r = d(q, x_0) > 0$ and $\epsilon = \frac{d(x_0, T_1x_0)}{r}$, we have $d(q, T_1x_0) \leq d(q, x_0) = r$, $d(q, x_0) \leq r$ and $d(x_0, T_1x_0) \geq r\epsilon$. From Lemma 2.5, we have

$$d\big(q, W(T_1x_0, x_0, \alpha_1^1)\big) < d(q, x_0) \text{ for } \alpha_1^1 \in (0, 1).$$

This is a contradiction, we have $x_0 \in T_1x_0$, that is, $x_0 \in F(T_1)$. Since $x_0 = T_1x_0$ definition of U_1 and Lemma 2.1, we have $U_1x_0 = x_0$, that is, $x_0 \in F(U_1)$. From (2.1) and $x_0 = U_1x_0$, we have

$$d(q, U_2x_0) = d\big(q, W\big(T_2x_0, x_0, \alpha_1^2\big)\big) = d(q, x_0) \text{ and } d(q, T_2x_0) = d(q, x_0).$$

Suppose $x_0 \neq T_2x_0$, then we have $d(x_0, T_2x_0) > 0$. Choose $r_1 = d(q, x_0) > 0$ and $\epsilon = \frac{d(x_0, T_2x_0)}{r_1}$, we have $d(q, T_2x_0) \leq d(q, x_0) = r_1$, $d(q, x_0) \leq r_1$ and $d(x_0, T_2x_0) \geq r_1\epsilon$. From Lemma 2.5, we have

$$d\big(q, W(T_2x_0, x_0, \alpha_1^2)\big) < d(q, x_0) \text{ for } \alpha_1^2 \in (0, 1).$$

This is a contradiction, we have $x_0 = T_2x_0$, that is, $x_0 \in F(T_2)$. Since $x_0 = T_2x_0$ definition of U_2 and Lemma 2.1, we have $U_2x_0 = x_0$, that is, $x_0 \in F(U_2)$.
By continuing on this way we can conclude that $x_0 \in F(T_i)$ and $x_0 \in F(U_i)$ for all $i = 1, 2, \ldots, N-1$.
Finally, we show that $x_0 \in F(T_N)$. From definition of S and Lemma 2.1, we have

$$Sx_0 = W\big(T_NU_{N-1}x_0, W(U_{N-1}x_0, x_0, \frac{\alpha_2^N}{1-\alpha_1^N}), \alpha_1^N\big) = W(T_Nx_0, x_0, \alpha_1^N).$$

Since

$$0 = d(x_0, Sx_0) = d(x_0, W(T_Nx_0, x_0, \alpha_1^N)) = \alpha_1^N d(T_Nx_0, x_0),$$

we have $x_0 = T_Nx_0$, that is, $x_0 \in F(T_N)$. Hence $F(S) \subseteq \bigcap_{i=1}^{N} F(T_i)$. □

Remark 2.8. From Theorem 2.7, we have the mapping S is nonexpansive. To show this, let $x, y \in C$. By remark 2.4, we have

$$\begin{aligned}
d(Sx,Sy) &= d\Big(W\big(T_NU_{N-1}x, W(U_{N-1}x, x, \frac{\alpha_2^N}{1-\alpha_1^N}), \alpha_1^N\big),\\
&\quad W\big(T_NU_{N-1}y, W(U_{N-1}y, y, \frac{\alpha_2^N}{1-\alpha_1^N}), \alpha_1^N\big)\Big)\\
&\leq \alpha_1^N d(T_NU_{N-1}x, T_NU_{N-1}y)\\
&\quad +(1-\alpha_1^N)d\big(W(U_{N-1}x, x, \frac{\alpha_2^N}{1-\alpha_1^N}), W(U_{N-1}y, y, \frac{\alpha_2^N}{1-\alpha_1^N})\big)\\
&\leq \alpha_1^N d(T_NU_{N-1}x, T_NU_{N-1}y)\\
&\quad +(1-\alpha_1^N)\Big(\frac{\alpha_2^N}{1-\alpha_1^N}d(U_{N-1}x, U_{N-1}y) + \big(1-\frac{\alpha_2^N}{1-\alpha_1^N}\big)d(x,y)\Big)\\
&\leq \alpha_1^N d(U_{N-1}x, U_{N-1}y) + \alpha_2^N d(U_{N-1}x, U_{N-1}y) + \alpha_3^N d(x,y)\\
&= (1-\alpha_3^N)d(U_{N-1}x, U_{N-1}y) + \alpha_3^N d(x,y)\\
&\leq (1-\alpha_3^N)\big((1-\alpha_3^{N-1})d(U_{N-2}x, U_{N-2}y) + \alpha_3^{N-1}d(x,y)\big) + \alpha_3^N d(x,y)\\
&= \Pi_{j=n-1}^N(1-\alpha_3^j)d(U_{N-2}x, U_{N-2}y) + \big(1-\Pi_{j=N-1}^N(1-\alpha_3^j)\big)d(x,y)\\
&\leq\\
&\vdots\\
&= \Pi_{j=1}^N(1-\alpha_3^j)d(U_0x, U_0y) + \big(1-\Pi_{j=1}^N(1-\alpha_3^j)\big)d(x,y)\\
&= d(x,y).
\end{aligned}$$

3 Main results

Theorem 3.1. *Let C be a nonempty compact closed convex subset of a complete uniformly convex metric space (X, d, W) with property (H). Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, 3, ..., N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0,1)$ for all $j = 1, 2, ..., N-1$, $\alpha_1^N \in (0,1]$ α_2^j, $\alpha_3^j \in [0,1)$ for all $j = 1, 2, ..., N$. Let S be the mapping generated by $T_1, ..., T_N$ and $\alpha_1, \alpha_2, ..., \alpha_N$. Let $x_1 \in C$ and let $\{x_n\}, \{y_n\}$ be sequences generated by*

$$\begin{cases} x_{n+1} = W(x_n, Sy_n, \gamma_n),\\ y_n = W(x_n, Sx_n, \beta_n) \end{cases} \tag{3.1}$$

for all $n \geq 1$ where $\{\gamma_n\}, \{\beta_n\}$ are sequences in $[0,1]$ satisfying $0 < a \leq \gamma_n \leq b < 1$ and $\sum_{n=1}^\infty \beta_n(1-\beta_n) = \infty$. Then the sequence $\{x_n\}$ converges to $z \in \bigcap_{i=1}^N F(T_i)$.

Proof. First, we show that $\inf_{n\in\mathbb{N}} d(x_n, Sx_n) = 0$. Assume that $\inf_{n\in\mathbb{N}} d(x_n, Sx_n) =$

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$r' > 0$. Let $p \in \bigcap_{i=1}^{N} F(T_i)$, by nonexpansiveness of $S-$mapping, we have

$$\begin{aligned} d(p, x_{n+1}) &= d\big(p, W(x_n, Sy_n, \gamma_n)\big) \\ &\leq \gamma_n d(p, x_n) + (1-\gamma_n) d(p, Sy_n) \\ &\leq \gamma_n d(p, x_n) + (1-\gamma_n) d(p, y_n) \\ &= \gamma_n d(p, x_n) + (1-\gamma_n)\Big(d\big(p, W(x_n, Sx_n, \beta_n)\big)\Big) \\ &\leq \gamma_n d(p, x_n) + (1-\gamma_n)\Big(\beta_n d(p, x_n) + (1-\beta_n) d(p, Sx_n)\Big) \\ &\leq d(p, x_n). \end{aligned}$$

It implies by Lemma 2.6 that $\lim_{n\to\infty} d(p, x_n)$ exists. Then, we have $\lim_{n\to\infty} d(p, x_n) = r'' > 0$. By nonexpansiveness of S, we have $d(p, Sx_n) \leq d(p, x_n)$. Since $\{d(p, x_n)\}$ is a nonincreasing and $\inf_{n\in\mathbb{N}} d(x_n, Sx_n) = r' > 0$, we have

$$\begin{aligned} d(x_n, Sx_n) &\geq r' \\ &\geq \frac{r'}{d(p, x_n)} d(p, x_n) \\ &\geq \frac{r'}{d(p, x_1)} d(p, x_n) \\ &> 0, \quad \forall n \in \mathbb{N}. \end{aligned}$$

By Lemma 2.5, there exists $\eta = \eta\big(\frac{r'}{d(p,x_1)}\big) > 0$ such that

$$\begin{aligned} d(p, x_{n+1}) &\leq \gamma_n d(p, x_n) + (1-\gamma_n) d\big(p, W(x_n, Sx_n, \beta_n)\big) \\ &\leq \gamma_n d(p, x_n) + (1-\gamma_n)\Big(\big(1 - 2\min\{\beta_n, 1-\beta_n\}\eta\big) d(p, x_n)\Big) \\ &= \gamma_n d(p, x_n) + (1-\gamma_n) d(p, x_n) - 2(1-\gamma_n)\min\{\beta_n, 1-\beta_n\}\eta d(p, x_n) \\ &\leq \gamma_n d(p, x_n) + (1-\gamma_n) d(p, x_n) - 2(1-\gamma_n)\beta_n(1-\beta_n)\eta d(p, x_n) \\ &= d(p, x_n) - 2(1-\gamma_n)\beta_n(1-\beta_n)\eta d(p, x_n), \end{aligned}$$

which follows that

$$2(1-b)\beta_n(1-\beta_n)\eta r'' \leq 2(1-\gamma_n)\beta_n(1-\beta_n)\eta d(p, x_n) \leq d(p, x_n) - d(p, x_{n+1}). \quad (3.2)$$

From (3.2), it implies that

$$2(1-b)\eta \sum_{n=1}^{k} \beta_n(1-\beta_n) r'' \leq d(p, x_1) - d(p, x_{k+1}) \quad (3.3)$$

for all $k \geq 1$. Letting $k \to \infty$ in (3.3) and , we have $\infty \leq d(p, x_1) - r'' < \infty$. This is a contradiction, then we have $\inf_{n\in\mathbb{N}} d(x_n, Sx_n) = 0$. Then, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} d(x_{n_j}, Sx_{n_j}) = 0$. Since C is compact, then there exists a subsequence $\{x_{n_{j_l}}\}$ of $\{x_{n_j}\}$ and p such that $\lim_{l\to\infty} x_{n_{j_l}} = p$. From nonexpansiveness of S, we have

$$\begin{aligned} d(p, Sp) &\leq d(p, x_{n_{j_l}}) + d(x_{n_{j_l}}, Sx_{n_{j_l}}) + d(Sx_{n_{j_l}}, Sp) \\ &\leq 2d(p, x_{n_{j_l}}) + d(x_{n_{j_l}}, Sx_{n_{j_l}}). \end{aligned}$$

Taking $l \to \infty$, it implies that $p \in F(S)$. From Lemma 2.7, we have $p \in \bigcap_{i=1}^{N} F(T_i)$. Since $\lim_{n\to\infty} d(p, x_n)$ exists, we can conclude that $\{x_n\}$ converges to $p \in \bigcap_{i=1}^{N} F(T_i)$.

□

We can prove the following results by using Theorem 3.1.

Corollary 3.2. *Let C be a nonempty compact closed convex subset of a complete uniformly convex metric space (X, d, W) with property (H). Let $T : C \to C$ be a nonexpansive mappings with $F(T) \neq \emptyset$. Let $x_1 \in C$ and let $\{x_n\}, \{y_n\}$ be sequences generated by*

$$\begin{cases} x_{n+1} = W(x_n, Ty_n, \gamma_n), \\ y_n = W(x_n, Tx_n, \beta_n) \end{cases} \tag{3.4}$$

for all $n \geq 1$ where $\{\gamma_n\}, \{\beta_n\}$ are sequences in $[0, 1]$ satisfying $0 < a \leq \gamma_n \leq b < 1$ and $\sum_{n=1}^{\infty} \beta_n(1 - \beta_n) = \infty$. Then the sequence $\{x_n\}$ converges to $z \in F(T)$.

Proof. Put $N = 1$ in Theorem 3.1, we obtain the desired result. □

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