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A novel approach for solving fully fuzzy linear programming problem with LR flat fuzzy numbers[†]

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Abstract: The fuzzy linear programming problem with triangular fuzzy numbers in its objective functions or constraints has been discussed by many scholars based on using Zadeh's decomposition theorem of fuzzy numbers and transforming it into some crisp linear programming problems. However, existed methods and results are limited if the fuzzy linear programming problem with generalized fuzzy numbers in its objective functions and constraints. In this paper, we discuss fully fuzzy linear programming (FFLP) problems of which all parameters are LR flat fuzzy numbers and a simple but practical method is developed to solve it. In this method, the approximate representation of fully fuzzy constraints is investigated by means of the arithmetic operations on LR flat fuzzy numbers space firstly. Meanwhile, we constructed a auxiliary multi-objective programming to solve the FFLP problems. After that, three approaches are proposed to solve the constructed auxiliary multi-objective programming, including optimistic approach, pessimistic approach and linear sum approach based on membership function. Finally, the obtained results are compared with the existing works and numerical example is given to illustrate the effectiveness of the proposed method.

Keywords: LR flat fuzzy number; multi-objective linear programming; fully fuzzy linear programming

1. Introduction

Linear programming (LP) is an essential mathematical tool in science and technology. Although, it has been investigated and expanded for more than six decades by many researchers from various point of views, it is still useful to develop new approaches in order to better fit the real world problems within the framework of linear programming. In conventional approach, parameters of linear programming models must be well defined and precise. However, in real world environment, this is not a realistic assumption. Usually, most of information is not deterministic and in this situation human has a capability to make a rational decision based on this uncertainty. This is hard challenge for decision makers to design an intelligent system which make a decision the same as the human. In fact, some of parameters of the system may be represented by fuzzy quantities rather than crisp ones in practice. Hence, it is necessary to develop mathematical theory and numerical schemes to handle fuzzy linear programming (FLP) problems. Bellman and Zadeh [1] proposed the concept of decision making in fuzzy environments. Since then, a number of researchers have exhibited their interest to various types of the FLP problems and proposed several approaches for solving these problems [2-11]. FLP model with triangular fuzzy numbers (TFNs) [6]. Lai and Hwang [6] developed a new approach to some possibilistic linear programming problems with TFNs, they transformed the fuzzy linear programming into a multi-objective linear programming model, involving three objective functions: minimizing the low loss, maximizing the most possible value and maximizing the upper the profit. FLP model with trapezoidal fuzzy numbers (TrFNs) [7 - 11]. For example, Ganesan [7] and Ebrahimnejad [8] studied the fuzzy linear programs with TrFNs, but the constructed fuzzy linear programming models are only suitable for the symmetrical TrFNs. Mahadavi-Amiri [9] and Campos [10] utilized the ranking function to solve the fuzzy linear programming models with TrFNs. Wan [11] developed a method which taken advantage of multi-objective linear programming model to solve FLP problems with TrFNs. However, in all of the above mentioned works, those cases of

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FLP problems have been studied in which not all parts of the problem were assumed to be fuzzy, e.g., only the right hand side or the objective function coefficients were fuzzy but the variables were not fuzzy. The FLP problems in which all the parameters as well as the variables are represented by fuzzy numbers are known as fully fuzzy linear programming (FFLP) problems. Many authors [12-14] have proposed different methods for solving FFLP problems.

Lotfi [12] proposed a novel method to obtain the approximate solution of FFLP problems by using the concept of the symmetric triangular fuzzy numbers and introduce an approach to defuzzify a general fuzzy quantity. In Kumar's article [13], an exact optimal solution is achieved using a linear ranking function. In this method, the linear ranking function has been used to convert the fuzzy objective function to a crisp objective function. The shortcoming exists of it is that the fuzziness of objective function has been neglected by the linear ranking function. Ezzati [14] proposed a new algorithm to solve FFLP problems with TFNs. To the best of our knowledge, till now there is no method in the literature to obtain the exact solution of FFLP problems in which all the parameters as well as the variables are represented by LR flat fuzzy numbers. The LR flat fuzzy number and its operations were first introduced by Dubois [15]. We know that triangular fuzzy numbers are just specious cases of LR flat fuzzy numbers. In 2006, Dehgham [16] discussed the computational methods for fully fuzzy linear systems whose coefficient matrix and the right-hand side vector are denoted by LR fuzzy numbers. In this paper, the approximate representation of fully fuzzy constraints ($\max(\min) \tilde{c}^T \otimes \tilde{x}$, s.t. $\tilde{A} \otimes \tilde{x} = \tilde{b}$, $\tilde{x} \geq 0$, where \tilde{c}^T , \tilde{A} , \tilde{b} and \tilde{c} are fuzzy matrixs which consist of LR flat fuzzy numbers) is investigated by means of the arithmetic operations on LR flat fuzzy numbers space firstly. Meanwhile, we constructed a auxiliary multi-objective programming to solve the FFLP problems. After that, three approaches are proposed to solve the constructed auxiliary multi-objective programming, including optimistic approach, pessimistic approach and linear sum approach based on membership function. Finally, the results obtained are compared with the existing works and numerical example is given to illustrate the effectiveness of the proposed method.

The structure of this paper is organized as follows. In Section 2, we review some basic concepts and introduce the interval objective programming. In Section 3, the approximate representation of fully fuzzy constraints is given. The method for solving the FFLP problem is discussed in Section 4. In Section 5, the numerical example is solved and the obtained results are compared with the existing works. Conclusion is drawn in Section 6.

2. Preliminaries

2.1 Basic definitions and arithmetic operations

In this section, some basic definitions and arithmetic operations of LR flat fuzzy numbers are presented.

Definition 2.1.^[17] A fuzzy number \tilde{A} , defined on universal set of real numbers R , denoted as $\tilde{A} = (m, n, \alpha, \beta)_{LR}$, is said to be an LR flat fuzzy number if its membership function $\mu_{\tilde{A}}(x)$ is given by

$$u_{\tilde{A}}(x) = \begin{cases} L(\frac{m-x}{\alpha}) & x < m, \alpha > 0, \\ R(\frac{x-n}{\beta}) & x > n, \beta > 0, \\ 1 & m \leq x \leq n, \end{cases}$$

where the closed interval $[m, n]$ is the mode of \tilde{A} , α and β are the left and right spreads, respectively. The function $L(\cdot)$, which is called the left shape function satisfying:

- (1) $L(x) = L(-x)$;
- (2) $L(0) = 1$ and $L(1) = 0$;
- (3) $L(x)$ is non-increasing on $[0, +\infty)$.

The definition of the right shape function $R(\cdot)$ is usually similar to that of $L(\cdot)$.

Usually, we could define the predetermined left spreads shape functions $L(\cdot)$ and right spreads shape functions $R(\cdot)$ as follows according to the need of mathematical modeling:

- (1) $L(x) = \max\{0, 1 - |x|^p\} (p > 0)$;
- (2) $L(x) = \max\{0, \frac{1}{e-1}(e^{1-|x|^p} - 1)\} (p > 0)$;
- (3) $L(x) = \max\{0, \frac{2}{1+|x|^p} - 1\} (p > 0)$,

and similarly to $R(\cdot)$.

Definition 2.2.^[18] An LR flat fuzzy number $\tilde{A} = (m, n, \alpha, \beta)_{LR}$ is said to be non-negative LR flat fuzzy number if $m - \alpha \geq 0$ and is said to be non-positive LR flat fuzzy number if $n + \beta \leq 0$.

Definition 2.3.^[17] LR flat fuzzy numbers $\tilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}$ and $\tilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ are said to be equal, i.e., $\tilde{A}_1 = \tilde{A}_2$ if and only if $m_1 = m_2, n_1 = n_2, \alpha_1 = \alpha_2, \beta_1 = \beta_2$.

Remark 2.1. If $m = n$, then an LR flat fuzzy number $(m, n, \alpha, \beta)_{LR}$ is said to be an LR fuzzy number and is denoted as $(m, \alpha, \beta)_{LR}$ or $(n, \alpha, \beta)_{LR}$.

Remark 2.2. If $m \neq n$ and $L(x) = R(x) = \max\{0, 1 - |x|\}$, then an LR flat fuzzy number $(m, n, \alpha, \beta)_{LR}$ is said to be a trapezoidal fuzzy number and is denoted as (m, n, α, β) .

Remark 2.3. If $m = n$ and $L(x) = R(x) = \max\{0, 1 - |x|\}$, then an LR flat fuzzy number $(m, n, \alpha, \beta)_{LR}$ is said to be a triangular fuzzy number and is denoted as (m, α, β) .

The arithmetic operations between two LR flat fuzzy numbers are defined by the extension principle as follows^[17]:

Let $\tilde{A}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR}, \tilde{A}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR}$ be any LR flat fuzzy numbers and $\tilde{A}_3 = (m_3, n_3, \alpha_3, \beta_3)_{RL}$ be any RL flat fuzzy number. Then,

- (i) $\tilde{A}_1 \oplus \tilde{A}_2 = (m_1 + m_2, n_1 + n_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)_{LR}$.
- (ii) $\tilde{A}_1 \ominus \tilde{A}_3 = (m_1 - n_3, n_1 - m_3, \alpha_1 + \beta_3, \alpha_3 + \beta_1)_{LR}$.
- (iii) If \tilde{A}_1 and \tilde{A}_2 are non-negative, then $\tilde{A}_1 \otimes \tilde{A}_2 \cong (m_1 m_2, n_1 n_2, m_1 \alpha_2 + m_2 \alpha_1, n_1 \beta_2 + n_2 \beta_1)_{LR}$.
- (iv) If \tilde{A}_1 is non-positive and \tilde{A}_2 is non-negative, then $\tilde{A}_1 \otimes \tilde{A}_2 \cong (m_1 n_2, n_1 m_2, \alpha_1 n_2 - m_1 \beta_2, m_2 \beta_1 - n_1 \alpha_2)_{LR}$.
- (v) $\lambda \otimes \tilde{A}_1 \cong \begin{cases} (\lambda m_1, \lambda n_1, \lambda \alpha_1, \lambda \beta_1)_{LR}, & \lambda \geq 0, \\ (\lambda n_1, \lambda m_1, -\lambda \beta_1, -\lambda \alpha_1)_{RL}, & \lambda < 0. \end{cases}$

Definition 2.4. A matrix $\tilde{\mathbf{A}} = (\tilde{a}_{ij})_{m \times n}$ ($i = 1, 2, \dots, m, j = 1, 2, \dots, n$) is said to be a fuzzy matrix if each element of $\tilde{\mathbf{A}}$ is a fuzzy number. If for every element $\tilde{a}_{ij} \geq 0$ (or $\tilde{a}_{ij} \leq 0$), then $\tilde{\mathbf{A}}$ is said to be a non-negative (or non-positive) fuzzy matrix, denoted by $\tilde{\mathbf{A}} \geq \mathbf{0}$ (or $\tilde{\mathbf{A}} \leq \mathbf{0}$).

Let all element of $\tilde{\mathbf{A}} = (\tilde{a}_{ij})_{m \times n}$ ($i = 1, 2, \dots, m, j = 1, 2, \dots, n$) are LR flat fuzzy numbers, i.e., $\tilde{a}_{ij} = (m_{ij}, n_{ij}, \alpha_{ij}, \beta_{ij})_{LR}$. Then we can represent $m \times n$ fuzzy matrix $\tilde{\mathbf{A}}$ with new notation $\tilde{\mathbf{A}} = (\mathbf{A}, \mathbf{B}, \mathbf{M}, \mathbf{N})$, where $\mathbf{A} = (m_{ij}), \mathbf{B} = (n_{ij}), \mathbf{M} = (\alpha_{ij})$ and $\mathbf{N} = (\beta_{ij})$ are four $m \times n$ matrices.

For brevity, a crisp matrix consisting of real numbers is written as a matrix directly throughout this paper, similar to linear systems, linear equations, matrix equations, and so on.

Definition 2.5.^[16,19] Let $\tilde{\mathbf{A}} = (\tilde{a}_{ij})_{m \times n}$ and $\tilde{\mathbf{B}} = (\tilde{b}_{ij})_{n \times p}$. Then

$$\tilde{\mathbf{A}} \otimes \tilde{\mathbf{B}} = \tilde{\mathbf{D}} = (\tilde{d}_{ij})_{m \times p},$$

where

$$\tilde{d}_{ij} = \bigoplus_{k=1, \dots, n} \tilde{a}_{ik} \otimes \tilde{b}_{kj}.$$

2.2 Interval objective programming

Ishibuchi and Tanaka [20] gave the definitions of the maximization and minimization problems with the interval objective functions, which are introduced in Definitions 2.6 and 2.7 as follows.

Definition 2.6.^[20] Let $\tilde{a} = [a_l, a_u]$ be an interval. The maximization problem with the interval objective function is described as follow:

$$\begin{aligned} \max \quad & \{\tilde{a}\}, \\ \text{s.t.} \quad & \tilde{a} \in \Omega, \end{aligned}$$

which is equivalent to the following bi-objective mathematical programming problem:

$$\begin{aligned} \max \quad & \{a_l, \frac{1}{2}(a_l + a_u)\}, \\ \text{s.t.} \quad & \tilde{a} \in \Omega, \end{aligned}$$

where Ω is a set of constraints in which the variable \tilde{a} should satisfy according to requirements in real situations.

Definition 2.7.^[20] Let $\tilde{a} = [a_l, a_u]$ be an interval. The minimization problem with the interval objective function is described as follow:

$$\begin{aligned} \min \quad & \{\tilde{a}\}, \\ \text{s.t.} \quad & \tilde{a} \in \Omega, \end{aligned}$$

which is equivalent to the following bi-objective mathematical programming problem:

$$\begin{aligned} \min \quad & \{a_u, \frac{1}{2}(a_l + a_u)\}, \\ \text{s.t.} \quad & \tilde{a} \in \Omega. \end{aligned}$$

3. Fully fuzzy linear programming problem

Linear programming is one of the most frequently applied operations research technique. In the conventional approach, the value of the parameters and variables of linear programming models must be well defined and precise. However, this is not a realistic assumption in the real world environment. There may exists uncertainty about the parameters and variables in the real life problems. So, the fuzzy numbers and fuzzy variables should be used in the LP problem. Therefore, we encounter with the FFLP problems.

Consider the standard form of FFLP problem which with m constraints and n variables as follows:

$$\begin{aligned} \max(\min) \tilde{Z} &= \tilde{\mathbf{c}}^T \otimes \tilde{\mathbf{x}}, \\ \text{s.t.} \quad \tilde{\mathbf{A}} \otimes \tilde{\mathbf{x}} &= \tilde{\mathbf{b}}, \\ \tilde{\mathbf{x}} &\text{ is non-negative fuzzy number vector,} \end{aligned} \tag{0.1}$$

where $\tilde{\mathbf{A}} = (\tilde{a}_{ij})_{m \times n}$, $\tilde{\mathbf{c}}^T = (\tilde{c}_j)_{1 \times n}$, $\tilde{\mathbf{x}} = (\tilde{x}_j)_{n \times 1}$, $\tilde{\mathbf{b}} = (\tilde{b}_i)_{m \times 1}$, and $\tilde{a}_{ij} \geq 0$ (or $\tilde{a}_{ij} < 0$), $\tilde{c}_j \geq 0$ (or $\tilde{c}_j < 0$), \tilde{b}_i, \tilde{x}_j are LR flat fuzzy numbers.

It should be noted that $\tilde{\mathbf{A}} \otimes \tilde{\mathbf{x}} \leq \tilde{\mathbf{b}}$ and $\tilde{\mathbf{A}} \otimes \tilde{\mathbf{x}} \geq \tilde{\mathbf{b}}$ can be transformed to the standard form by introducing a vector variable $\tilde{\mathbf{T}} = (\tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_m)$, where \tilde{t}_j ($j = 1, 2, \dots, m$) are LR flat fuzzy numbers, as $\tilde{\mathbf{A}} \otimes \tilde{\mathbf{x}} \oplus \tilde{\mathbf{T}} = \tilde{\mathbf{b}}$ and $\tilde{\mathbf{A}} \otimes \tilde{\mathbf{x}} \ominus \tilde{\mathbf{T}} = \tilde{\mathbf{b}}$, respectively.

Definition 3.1. The fuzzy exact optimal solution of FFLP problem (1) will be a fuzzy number vector $\tilde{\mathbf{x}}^*$ if it satisfies the following characteristics:

- (1) $\tilde{\mathbf{x}}^* = (\tilde{x}_j^*)_{n \times 1} \geq \mathbf{0}$, where \tilde{x}_j^* ($j = 1, 2, \dots, n$) are LR flat fuzzy numbers;
- (2) $\tilde{\mathbf{A}} \otimes \tilde{\mathbf{x}}^* = \tilde{\mathbf{b}}$;

(3) For any $\tilde{\mathbf{x}} \in \tilde{\mathbf{S}} = \{\tilde{\mathbf{x}} | \tilde{\mathbf{A}} \otimes \tilde{\mathbf{x}} = \tilde{\mathbf{b}}, \tilde{\mathbf{x}} = (\tilde{x}_j)_{n \times 1} \geq \mathbf{0}, \text{ where } \tilde{x}_j \text{ are LR flat fuzzy numbers}\}$, we have $\tilde{\mathbf{c}}^T \otimes \tilde{\mathbf{x}}^* \geq \tilde{\mathbf{c}}^T \otimes \tilde{\mathbf{x}}$ (in case of maximization problem), $\tilde{\mathbf{c}}^T \otimes \tilde{\mathbf{x}}^* \leq \tilde{\mathbf{c}}^T \otimes \tilde{\mathbf{x}}$ (in case of minimization problem).

Remark 3.1. Let $\tilde{\mathbf{x}}^*$ be an exact optimal solution of FFLP problem (1). If there exists an $\tilde{\mathbf{x}}' \in \tilde{\mathbf{S}}$ such that $\tilde{\mathbf{c}}^T \otimes \tilde{\mathbf{x}}^* = \tilde{\mathbf{c}}^T \otimes \tilde{\mathbf{x}}'$, then $\tilde{\mathbf{x}}'$ is also an exact optimal solution of FFLP problem (1) and is called an alternative exact optimal solution.

Note that the elements (\tilde{a}_{ij}) in coefficient matrix $\tilde{\mathbf{A}}$ of the FFLP problem (1) have two forms, i.e., (1) $\tilde{a}_{ij} \geq 0$, (2) $\tilde{a}_{ij} < 0$. So we define the coefficient matrix $\tilde{\mathbf{A}}$ as follows:

$$\begin{aligned} (\tilde{\mathbf{A}}_1)_{ij} &= \begin{cases} \tilde{a}_{ij}, & \tilde{a}_{ij} \geq 0, \\ 0, & \tilde{a}_{ij} < 0; \end{cases} \\ (\tilde{\mathbf{A}}_2)_{ij} &= \begin{cases} \tilde{a}_{ij}, & \tilde{a}_{ij} \leq 0, \\ 0, & \tilde{a}_{ij} > 0, \end{cases} \end{aligned}$$

where $1 \leq i \leq m, 1 \leq j \leq n$.

Obviously,

$$\tilde{\mathbf{A}} = \tilde{\mathbf{A}}_1 \oplus \tilde{\mathbf{A}}_2,$$

$$\tilde{\mathbf{A}} \otimes \tilde{\mathbf{x}} = \tilde{\mathbf{A}}_1 \otimes \tilde{\mathbf{x}} \oplus \tilde{\mathbf{A}}_2 \otimes \tilde{\mathbf{x}}.$$

Similarly, we define the coefficient matrix $(\tilde{\mathbf{c}}^T)$ of objective function of FFLP problem (1) as follows:

$$(\tilde{\mathbf{c}}_1^T)_j = \begin{cases} \tilde{c}_j, & \tilde{c}_j \geq 0, \\ 0, & \tilde{c}_j < 0; \end{cases}$$

$$(\tilde{\mathbf{c}}_2^T)_j = \begin{cases} \tilde{c}_j, & \tilde{c}_j \leq 0, \\ 0, & \tilde{c}_j > 0, \end{cases}$$

where $1 \leq j \leq n$.

Obviously,

$$\begin{aligned} \tilde{\mathbf{c}}^T &= \tilde{\mathbf{c}}_1^T \oplus \tilde{\mathbf{c}}_2^T, \\ \tilde{\mathbf{c}}^T \otimes \tilde{\mathbf{x}} &= \tilde{\mathbf{c}}_1^T \otimes \tilde{\mathbf{x}} \oplus \tilde{\mathbf{c}}_2^T \otimes \tilde{\mathbf{x}}. \end{aligned}$$

In the FFLP problems (1), let $\tilde{\mathbf{c}} = (\mathbf{c}, \mathbf{d}, \mathbf{p}, \mathbf{q})_{LR} = \tilde{\mathbf{c}}_1 \oplus \tilde{\mathbf{c}}_2$ (where $\tilde{\mathbf{c}}_1 = (\mathbf{c}_1, \mathbf{d}_1, \mathbf{p}_1, \mathbf{q}_1)_{LR} \geq \mathbf{0}$, $\tilde{\mathbf{c}}_2 = (\mathbf{c}_2, \mathbf{d}_2, \mathbf{p}_2, \mathbf{q}_2)_{LR} \leq \mathbf{0}$), $\tilde{\mathbf{x}} = (\mathbf{x}, \mathbf{y}, \mathbf{s}, \mathbf{t})_{LR}$. Then, according to the operations of the LR flat fuzzy numbers, we have

$$\begin{aligned} \tilde{\mathbf{c}}^T \otimes \tilde{\mathbf{x}} &= \tilde{\mathbf{c}}_1^T \otimes \tilde{\mathbf{x}} \oplus \tilde{\mathbf{c}}_2^T \otimes \tilde{\mathbf{x}} = (\mathbf{c}_1, \mathbf{d}_1, \mathbf{p}_1, \mathbf{q}_1)_{LR} \otimes (\mathbf{x}, \mathbf{y}, \mathbf{s}, \mathbf{t})_{LR} \oplus (\mathbf{c}_2, \mathbf{d}_2, \mathbf{p}_2, \mathbf{q}_2)_{LR} \otimes (\mathbf{x}, \mathbf{y}, \mathbf{s}, \mathbf{t})_{LR} \\ &\cong (\mathbf{c}_1\mathbf{x}, \mathbf{d}_1\mathbf{y}, \mathbf{c}_1\mathbf{s} + \mathbf{p}_1\mathbf{x}, \mathbf{d}_1\mathbf{t} + \mathbf{q}_1\mathbf{y})_{LR} \oplus (\mathbf{c}_2\mathbf{y}, \mathbf{d}_2\mathbf{x}, \mathbf{p}_2\mathbf{y} - \mathbf{c}_2\mathbf{t}, \mathbf{q}_2\mathbf{x} - \mathbf{d}_2\mathbf{s})_{LR} \\ &= (\mathbf{c}_1\mathbf{x} + \mathbf{c}_2\mathbf{y}, \mathbf{d}_1\mathbf{y} + \mathbf{d}_2\mathbf{x}, \mathbf{c}_1\mathbf{s} - \mathbf{c}_2\mathbf{t} + \mathbf{p}_1\mathbf{x} + \mathbf{p}_2\mathbf{y}, \mathbf{q}_2\mathbf{x} + \mathbf{q}_1\mathbf{y} + \mathbf{d}_1\mathbf{t} - \mathbf{d}_2\mathbf{s})_{LR}. \end{aligned}$$

For brevity, we substitute $(Z_m, Z_n, Z_\alpha, Z_\beta)_{LR}$ for $(\mathbf{c}_1\mathbf{x} + \mathbf{c}_2\mathbf{y}, \mathbf{d}_1\mathbf{y} + \mathbf{d}_2\mathbf{x}, \mathbf{c}_1\mathbf{s} - \mathbf{c}_2\mathbf{t} + \mathbf{p}_1\mathbf{x} + \mathbf{p}_2\mathbf{y}, \mathbf{q}_2\mathbf{x} + \mathbf{q}_1\mathbf{y} + \mathbf{d}_1\mathbf{t} - \mathbf{d}_2\mathbf{s})_{LR}$ throughout this paper.

Theorem 3.1. (Approximate representation of fully fuzzy constraints) Let $\tilde{\mathbf{A}} = (\mathbf{A}, \mathbf{B}, \mathbf{M}, \mathbf{N})_{LR} = \tilde{\mathbf{A}}_1 \oplus \tilde{\mathbf{A}}_2$, $\tilde{\mathbf{b}} = (\mathbf{b}, \mathbf{g}, \mathbf{h}, \mathbf{f})_{LR}$, $\tilde{\mathbf{x}} = (\mathbf{x}, \mathbf{y}, \mathbf{s}, \mathbf{t})_{LR} \geq \mathbf{0}$, where $\tilde{\mathbf{A}}_1 = (\mathbf{A}_1, \mathbf{B}_1, \mathbf{M}_1, \mathbf{N}_1)_{LR} \geq \mathbf{0}$, $\tilde{\mathbf{A}}_2 = (\mathbf{A}_2, \mathbf{B}_2, \mathbf{M}_2, \mathbf{N}_2)_{LR} \leq \mathbf{0}$. Then $\tilde{\mathbf{A}} \otimes \tilde{\mathbf{x}} = \tilde{\mathbf{b}}$ can be represented approximately as follows:

$$\begin{cases} \mathbf{A}_1\mathbf{x} + \mathbf{A}_2\mathbf{y} = \mathbf{b}, \\ \mathbf{B}_1\mathbf{y} + \mathbf{B}_2\mathbf{x} = \mathbf{g}, \\ \mathbf{M}_1\mathbf{x} + \mathbf{M}_2\mathbf{y} + \mathbf{A}_1\mathbf{s} - \mathbf{A}_2\mathbf{t} = \mathbf{h}, \\ \mathbf{N}_2\mathbf{x} + \mathbf{N}_1\mathbf{y} + \mathbf{B}_1\mathbf{t} - \mathbf{B}_2\mathbf{s} = \mathbf{f}. \end{cases} \quad (0.2)$$

Proof. Since $\tilde{\mathbf{A}} = (\mathbf{A}, \mathbf{B}, \mathbf{M}, \mathbf{N})_{LR} = \tilde{\mathbf{A}}_1 \oplus \tilde{\mathbf{A}}_2$, $\tilde{\mathbf{A}}_1 = (\mathbf{A}_1, \mathbf{B}_1, \mathbf{M}_1, \mathbf{N}_1)_{LR} \geq \mathbf{0}$, $\tilde{\mathbf{A}}_2 = (\mathbf{A}_2, \mathbf{B}_2, \mathbf{M}_2, \mathbf{N}_2)_{LR} \leq \mathbf{0}$, $\tilde{\mathbf{x}} = (\mathbf{x}, \mathbf{y}, \mathbf{s}, \mathbf{t})_{LR} \geq \mathbf{0}$, we have

$$\begin{aligned} \tilde{\mathbf{A}} \otimes \tilde{\mathbf{x}} &= (\tilde{\mathbf{A}}_1 \oplus \tilde{\mathbf{A}}_2) \otimes \tilde{\mathbf{x}} = \tilde{\mathbf{A}}_1 \otimes \tilde{\mathbf{x}} \oplus \tilde{\mathbf{A}}_2 \otimes \tilde{\mathbf{x}} \\ &= (\mathbf{A}_1, \mathbf{B}_1, \mathbf{M}_1, \mathbf{N}_1)_{LR} \otimes (\mathbf{x}, \mathbf{y}, \mathbf{s}, \mathbf{t})_{LR} \oplus (\mathbf{A}_2, \mathbf{B}_2, \mathbf{M}_2, \mathbf{N}_2)_{LR} \otimes (\mathbf{x}, \mathbf{y}, \mathbf{s}, \mathbf{t})_{LR} \\ &\cong (\mathbf{A}_1\mathbf{x}, \mathbf{B}_1\mathbf{y}, \mathbf{A}_1\mathbf{s} + \mathbf{M}_1\mathbf{x}, \mathbf{B}_1\mathbf{t} + \mathbf{N}_1\mathbf{y})_{LR} \oplus (\mathbf{A}_2\mathbf{y}, \mathbf{B}_2\mathbf{x}, \mathbf{M}_2\mathbf{y} - \mathbf{A}_2\mathbf{t}, \mathbf{N}_2\mathbf{x} - \mathbf{B}_2\mathbf{s})_{LR} \\ &= (\mathbf{A}_1\mathbf{x} + \mathbf{A}_2\mathbf{y}, \mathbf{B}_1\mathbf{y} + \mathbf{B}_2\mathbf{x}, \mathbf{A}_1\mathbf{s} - \mathbf{A}_2\mathbf{t} + \mathbf{M}_1\mathbf{x} + \mathbf{M}_2\mathbf{y}, \mathbf{N}_2\mathbf{x} + \mathbf{N}_1\mathbf{y} + \mathbf{B}_1\mathbf{t} - \mathbf{B}_2\mathbf{s})_{LR} \\ &= \tilde{\mathbf{b}} = (\mathbf{b}, \mathbf{g}, \mathbf{h}, \mathbf{f})_{LR}. \end{aligned}$$

According to Definition 2.3, we have

$$\begin{aligned} \mathbf{A}_1\mathbf{x} + \mathbf{A}_2\mathbf{y} &= \mathbf{b}, & \mathbf{B}_1\mathbf{y} + \mathbf{B}_2\mathbf{x} &= \mathbf{g}, \\ \mathbf{M}_1\mathbf{x} + \mathbf{M}_2\mathbf{y} + \mathbf{A}_1\mathbf{s} - \mathbf{A}_2\mathbf{t} &= \mathbf{h}, & \mathbf{N}_2\mathbf{x} + \mathbf{N}_1\mathbf{y} + \mathbf{B}_1\mathbf{t} - \mathbf{B}_2\mathbf{s} &= \mathbf{f}. \end{aligned}$$

Hence, $\tilde{\mathbf{A}} \otimes \tilde{\mathbf{x}} = \tilde{\mathbf{b}}$ can be represented approximately as (2).

4. Proposed method to find the fuzzy exact optimal solution of FFLP problem

In this section, in order to find an effective fuzzy solution of the type of Eq. (1) problem, we are going to introduce a method based on the definition of the interval objective programming and the arithmetic operations of LR flat fuzzy numbers.

We consider the case of maximizing fuzzy objective function at first.

In Eq. (1), the fuzzy objective $\tilde{Z} = \tilde{c}^T \otimes \tilde{x} = (c_1x + c_2y, d_1y + d_2x, c_1s - c_2t + p_1x + p_2y, q_2x + q_1y + d_1t - d_2s)_{LR} = (Z_m, Z_n, Z_\alpha, Z_\beta)_{LR}$ is an LR flat fuzzy number. This fuzzy objective is fully defined by four corner points $(Z_m, 1)$, $(Z_n, 1)$, $(Z_m - Z_\alpha, 0)$ and $(Z_n + Z_\beta, 0)$, geometrically. Thus, maximizing the fuzzy objective can be obtained by pushing these four critical points in the direction of the right-hand side. Fortunately, the vertical coordinates of the critical points are fixed at either 1 or 0. The only considerations then are the four horizontal coordinates. Therefore, our problem is to solve

$$\begin{aligned} \max \quad & \{Z_m - Z_\alpha, Z_m, Z_n, Z_n + Z_\beta\}, \\ \text{s.t.} \quad & \tilde{A} \otimes \tilde{x} = \tilde{b}, \\ & \tilde{x} \geq 0. \end{aligned} \tag{0.3}$$

However, the above four objectives $(Z_m - Z_\alpha)$, Z_m , Z_n and $(Z_n + Z_\beta)$ should always preserve the form of the LR flat fuzzy number $(Z_m, Z_n, Z_\alpha, Z_\beta)_{LR}$ during the optimization process. Thus, in order to keep the LR flat fuzzy number shape (normal and convex) of the possibility distribution, it is necessary to make a little change.

For the mode of LR flat fuzzy number $(Z_m, Z_n, Z_\alpha, Z_\beta)_{LR}$, since the objective function of Eq. (1) is to maximize \tilde{Z} , it is natural to maximize the interval $[Z_m, Z_n]$ for this objective function. According to Definition 2.6, in order to maximize the interval $[Z_m, Z_n]$, we need maximize the left endpoint Z_m and maximize the middle point $\frac{1}{2}(Z_m + Z_n)$ of this interval simultaneously. For the lower and upper limits of LR flat fuzzy number $(Z_m, Z_n, Z_\alpha, Z_\beta)_{LR}$, we minimize Z_α and maximize Z_β instead of maximizing the lower $Z_m - Z_\alpha$ and the upper $Z_n + Z_\beta$, respectively.

Therefore, combining Theorem 3.1, Eq. (3) can be transformed into the following multi-objective programming model:

$$\begin{aligned} \min \quad & Z_1 = Z_\alpha, \\ \max \quad & Z_2 = Z_m, \\ \max \quad & Z_3 = \frac{1}{2}(Z_m + Z_n), \\ \max \quad & Z_4 = Z_\beta, \\ \text{s.t.} \quad & \mathbf{A}_1\mathbf{x} + \mathbf{A}_2\mathbf{y} = \mathbf{b}, \\ & \mathbf{B}_1\mathbf{y} + \mathbf{B}_2\mathbf{x} = \mathbf{g}, \\ & \mathbf{M}_1\mathbf{x} + \mathbf{M}_2\mathbf{y} + \mathbf{A}_1\mathbf{s} - \mathbf{A}_2\mathbf{t} = \mathbf{h}, \\ & \mathbf{N}_2\mathbf{x} + \mathbf{N}_1\mathbf{y} + \mathbf{B}_1\mathbf{t} - \mathbf{B}_2\mathbf{s} = \mathbf{f}, \\ & \mathbf{y} - \mathbf{x} \geq \mathbf{0}, \mathbf{x} - \mathbf{s} \geq \mathbf{0}, \mathbf{s} \geq \mathbf{0}, \mathbf{t} \geq \mathbf{0}. \end{aligned} \tag{0.4}$$

Although Eq. (4) is also a multi-objective linear programming model, it can effectively keep the LR flat fuzzy number shape of objective function \tilde{Z} . To solve Eq. (4), we may use any MOLP technique [21] such as utility theory, goal programming fuzzy programming or interactive approaches. In this paper, we propose three kinds of approaches to solve this multi-objective linear programming model. Since the objective function Z_i is the function of the decision variable vector $\tilde{x} = (x, y, s, t)_{LR}$, simply denote by $Z_i = Z_i(\tilde{x})$ ($i = 1, 2, 3, 4$). Let Z_1^{\min} , Z_i^{\max} ($i = 2, 3, 4$) and \tilde{x}_i' ($i = 1, 2, 3, 4$) respectively be the minimum objective, maximum objective value and the optimal solution for the following single objective linear programming model:

$$\begin{aligned} \min \quad & Z_1 = Z_1(\tilde{x}), \\ \text{s.t.} \quad & \mathbf{A}_1\mathbf{x} + \mathbf{A}_2\mathbf{y} = \mathbf{b}, \\ & \mathbf{B}_1\mathbf{y} + \mathbf{B}_2\mathbf{x} = \mathbf{g}, \\ & \mathbf{M}_1\mathbf{x} + \mathbf{M}_2\mathbf{y} + \mathbf{A}_1\mathbf{s} - \mathbf{A}_2\mathbf{t} = \mathbf{h}, \\ & \mathbf{N}_2\mathbf{x} + \mathbf{N}_1\mathbf{y} + \mathbf{B}_1\mathbf{t} - \mathbf{B}_2\mathbf{s} = \mathbf{f}, \\ & \mathbf{y} - \mathbf{x} \geq \mathbf{0}, \mathbf{x} - \mathbf{s} \geq \mathbf{0}, \mathbf{s} \geq \mathbf{0}, \mathbf{t} \geq \mathbf{0}, \end{aligned} \tag{0.5}$$

and

$$\begin{aligned}
 \max \quad & Z_i = Z_i(\tilde{\mathbf{x}}), \quad (i = 2, 3, 4) \\
 \text{s.t.} \quad & \mathbf{A}_1 \mathbf{x} + \mathbf{A}_2 \mathbf{y} = \mathbf{b}, \\
 & \mathbf{B}_1 \mathbf{y} + \mathbf{B}_2 \mathbf{x} = \mathbf{g}, \\
 & \mathbf{M}_1 \mathbf{x} + \mathbf{M}_2 \mathbf{y} + \mathbf{A}_1 \mathbf{s} - \mathbf{A}_2 \mathbf{t} = \mathbf{h}, \\
 & \mathbf{N}_2 \mathbf{x} + \mathbf{N}_1 \mathbf{y} + \mathbf{B}_1 \mathbf{t} - \mathbf{B}_2 \mathbf{s} = \mathbf{f}, \\
 & \mathbf{y} - \mathbf{x} \geq \mathbf{0}, \quad \mathbf{x} - \mathbf{s} \geq \mathbf{0}, \quad \mathbf{s} \geq \mathbf{0}, \quad \mathbf{t} \geq \mathbf{0}.
 \end{aligned} \tag{0.6}$$

Then, set $Z_1^{\max} = \max\{Z_1(\tilde{\mathbf{x}}_1'), Z_1(\tilde{\mathbf{x}}_2'), Z_1(\tilde{\mathbf{x}}_3'), Z_1(\tilde{\mathbf{x}}_4')\}$, $Z_i^{\min} = \min\{Z_i(\tilde{\mathbf{x}}_1'), Z_i(\tilde{\mathbf{x}}_2'), Z_i(\tilde{\mathbf{x}}_3'), Z_i(\tilde{\mathbf{x}}_4')\}$ ($i = 2, 3, 4$). The linear membership function of the objective function Z_1 can be calculated as follows:

$$\mu_{z_1}(\tilde{\mathbf{x}}) = \begin{cases} 1 & \text{if } Z_1 < Z_1^{\min}, \\ \frac{Z_1^{\max} - Z_1}{Z_1^{\max} - Z_1^{\min}} & \text{if } Z_1^{\min} \leq Z_1 \leq Z_1^{\max}, \\ 0 & \text{if } Z_1 > Z_1^{\max}. \end{cases}$$

The linear membership function of the objective function $Z_i (i = 2, 3, 4)$ can be calculated as follows:

$$\mu_{z_i}(\tilde{\mathbf{x}}) = \begin{cases} 0 & \text{if } Z_i < Z_i^{\min}, \\ \frac{Z_i - Z_i^{\min}}{Z_i^{\max} - Z_i^{\min}} & \text{if } Z_i^{\min} \leq Z_i \leq Z_i^{\max}, \\ 1 & \text{if } Z_i > Z_i^{\max}. \end{cases}$$

Thus, Eq. (4) can be solved by the following linear programming model:

$$\begin{aligned}
 \max \quad & \mu, \\
 \text{s.t.} \quad & 4\mu_{z_i}(\tilde{\mathbf{x}}) + \sum_{i=1}^4 \mu_{z_i}(\tilde{\mathbf{x}}) \geq 8\mu \quad (i = 1, 2, 3, 4), \\
 & \mathbf{A}_1 \mathbf{x} + \mathbf{A}_2 \mathbf{y} = \mathbf{b}, \\
 & \mathbf{B}_1 \mathbf{y} + \mathbf{B}_2 \mathbf{x} = \mathbf{g}, \\
 & \mathbf{M}_1 \mathbf{x} + \mathbf{M}_2 \mathbf{y} + \mathbf{A}_1 \mathbf{s} - \mathbf{A}_2 \mathbf{t} = \mathbf{h}, \\
 & \mathbf{N}_2 \mathbf{x} + \mathbf{N}_1 \mathbf{y} + \mathbf{B}_1 \mathbf{t} - \mathbf{B}_2 \mathbf{s} = \mathbf{f}, \\
 & \mathbf{y} - \mathbf{x} \geq \mathbf{0}, \quad \mathbf{x} - \mathbf{s} \geq \mathbf{0}, \quad \mathbf{s} \geq \mathbf{0}, \quad \mathbf{t} \geq \mathbf{0},
 \end{aligned} \tag{0.7}$$

or

$$\begin{aligned}
 \max \quad & \mu, \\
 \text{s.t.} \quad & 4\mu_{z_i}(\tilde{\mathbf{x}}) + \sum_{i=1}^4 \mu_{z_i}(\tilde{\mathbf{x}}) \leq 8\mu \quad (i = 1, 2, 3, 4), \\
 & \mathbf{A}_1 \mathbf{x} + \mathbf{A}_2 \mathbf{y} = \mathbf{b}, \\
 & \mathbf{B}_1 \mathbf{y} + \mathbf{B}_2 \mathbf{x} = \mathbf{g}, \\
 & \mathbf{M}_1 \mathbf{x} + \mathbf{M}_2 \mathbf{y} + \mathbf{A}_1 \mathbf{s} - \mathbf{A}_2 \mathbf{t} = \mathbf{h}, \\
 & \mathbf{N}_2 \mathbf{x} + \mathbf{N}_1 \mathbf{y} + \mathbf{B}_1 \mathbf{t} - \mathbf{B}_2 \mathbf{s} = \mathbf{f}, \\
 & \mathbf{y} - \mathbf{x} \geq \mathbf{0}, \quad \mathbf{x} - \mathbf{s} \geq \mathbf{0}, \quad \mathbf{s} \geq \mathbf{0}, \quad \mathbf{t} \geq \mathbf{0},
 \end{aligned} \tag{0.8}$$

or

$$\begin{aligned}
 \max \quad & w_1 \mu_{z_1}(\tilde{\mathbf{x}}) + w_2 \mu_{z_2}(\tilde{\mathbf{x}}) + w_3 \mu_{z_3}(\tilde{\mathbf{x}}) + w_4 \mu_{z_4}(\tilde{\mathbf{x}}), \\
 \text{s.t.} \quad & \mathbf{A}_1 \mathbf{x} + \mathbf{A}_2 \mathbf{y} = \mathbf{b}, \\
 & \mathbf{B}_1 \mathbf{y} + \mathbf{B}_2 \mathbf{x} = \mathbf{g}, \\
 & \mathbf{M}_1 \mathbf{x} + \mathbf{M}_2 \mathbf{y} + \mathbf{A}_1 \mathbf{s} - \mathbf{A}_2 \mathbf{t} = \mathbf{h}, \\
 & \mathbf{N}_2 \mathbf{x} + \mathbf{N}_1 \mathbf{y} + \mathbf{B}_1 \mathbf{t} - \mathbf{B}_2 \mathbf{s} = \mathbf{f}, \\
 & \mathbf{y} - \mathbf{x} \geq \mathbf{0}, \quad \mathbf{x} - \mathbf{s} \geq \mathbf{0}, \quad \mathbf{s} \geq \mathbf{0}, \quad \mathbf{t} \geq \mathbf{0},
 \end{aligned} \tag{0.9}$$

where $\mathbf{w} = (w_1, w_2, w_3, w_4)^T$ is the weight vector of objective Z_i ($i = 1, 2, 3, 4$), satisfies that $w_i \geq 0$ ($i = 1, 2, 3, 4$) and $\sum_{i=1}^4 w_i = 1$. Eq. (7) is a kind of pessimistic approach which shows that the decision maker (DM) is very conservative, whereas Eq. (8) is a kind of optimistic approach which shows that the (DM) is very aggressive. Eq. (9) is the linear sum approach based on membership function.

Analogously, for the case of minimization fuzzy objective function programming, using Definition 2.7, it can be transformed into the following multi-objective programming model:

$$\begin{aligned}
 & \max \quad Z_1 = Z_\alpha, \\
 & \min \quad Z_2 = Z_n, \\
 & \min \quad Z_3 = \frac{1}{2}(Z_m + Z_n), \\
 & \min \quad Z_4 = Z_\beta, \\
 & \text{s.t.} \quad \mathbf{A}_1 \mathbf{x} + \mathbf{A}_2 \mathbf{y} = \mathbf{b}, \\
 & \quad \quad \mathbf{B}_1 \mathbf{y} + \mathbf{B}_2 \mathbf{x} = \mathbf{g}, \\
 & \quad \quad \mathbf{M}_1 \mathbf{x} + \mathbf{M}_2 \mathbf{y} + \mathbf{A}_1 \mathbf{s} - \mathbf{A}_2 \mathbf{t} = \mathbf{h}, \\
 & \quad \quad \mathbf{N}_2 \mathbf{x} + \mathbf{N}_1 \mathbf{y} + \mathbf{B}_1 \mathbf{t} - \mathbf{B}_2 \mathbf{s} = \mathbf{f}, \\
 & \quad \quad \mathbf{y} - \mathbf{x} \geq \mathbf{0}, \quad \mathbf{x} - \mathbf{s} \geq \mathbf{0}, \quad \mathbf{s} \geq \mathbf{0}, \quad \mathbf{t} \geq \mathbf{0},
 \end{aligned} \tag{0.10}$$

which can be solved by using a similar approach (as previously described).

5. Examples

In this section, we will demonstrate efficiency and superiority of the proposed method using numerical examples. At the same time, the shortcomings of the existing methods [11-14] for solving FFLP problems with equality constraints are pointed out.

Example 5.1. Let $L(x) = \max\{0, 1 - |x|\}$, $R(x) = \max\{0, \frac{1}{e-1}(e^{1-x^2} - 1)\}$. Consider the following FFLP:

$$\begin{aligned}
 & \max \quad Z = (2, 3, 1, 3) \otimes \tilde{x}_1 \oplus (-3, -2, 2, 1) \otimes \tilde{x}_2 \oplus (0, 1, 0, 0) \otimes \tilde{x}_3, \\
 & \text{s.t.} \quad (2, 2, 1, 0) \otimes \tilde{x}_1 \oplus (-2, -1, 2, 1) \otimes \tilde{x}_2 \oplus (-2, -1, 1, 0) \otimes \tilde{x}_3 = (2, 9, 16, 4), \\
 & \quad \quad (-3, -1, 1, 1) \otimes \tilde{x}_1 \oplus (2, 2, 1, 1) \otimes \tilde{x}_2 \oplus (-2, -1, 0, 1) \otimes \tilde{x}_3 = (-13, 2, 11, 12), \\
 & \quad \quad \tilde{x}_1 \geq 0, \quad \tilde{x}_2 \geq 0, \quad \tilde{x}_3 \geq 0.
 \end{aligned} \tag{0.11}$$

Let $\tilde{x}_1 = (x_1, y_1, s_1, t_1)_{LR}$, $\tilde{x}_2 = (x_2, y_2, s_2, t_2)_{LR}$, $\tilde{x}_3 = (x_3, y_3, s_3, t_3)_{LR}$. Then

$$\begin{aligned}
 \tilde{\mathbf{c}}^T &= ((2, 3, 1, 3), (-3, -2, 2, 1), (0, 1, 0, 0)), & \tilde{\mathbf{b}}^T &= ((2, 9, 16, 4), (-13, 2, 11, 12)), \\
 \tilde{\mathbf{A}} &= \begin{pmatrix} (2, 2, 1, 0) & (-2, -1, 2, 1) & (-2, -1, 1, 0) \\ (-3, -1, 1, 1) & (2, 2, 1, 1) & (-2, -1, 0, 1) \end{pmatrix}, \\
 \mathbf{A} &= \begin{pmatrix} 2, -2, -2 \\ -3, 2, -2 \end{pmatrix}, & \mathbf{B} &= \begin{pmatrix} 2, -1, -1 \\ -1, 2, -1 \end{pmatrix}, & \mathbf{M} &= \begin{pmatrix} 1, 2, 1 \\ 1, 1, 0 \end{pmatrix}, & \mathbf{N} &= \begin{pmatrix} 0, 1, 0 \\ 1, 1, 1 \end{pmatrix}, \\
 \mathbf{A}_1 &= \begin{pmatrix} 2, 0, 0 \\ 0, 2, 0 \end{pmatrix}, & \mathbf{A}_2 &= \begin{pmatrix} 0, -2, -2 \\ -3, 0, -2 \end{pmatrix}, & \mathbf{B}_1 &= \begin{pmatrix} 2, 0, 0 \\ 0, 2, 0 \end{pmatrix}, & \mathbf{B}_2 &= \begin{pmatrix} 0, -1, -1 \\ -1, 0, -1 \end{pmatrix}, \\
 \mathbf{M}_1 &= \begin{pmatrix} 1, 0, 0 \\ 0, 1, 0 \end{pmatrix}, & \mathbf{M}_2 &= \begin{pmatrix} 0, 2, 1 \\ 1, 0, 0 \end{pmatrix}, & \mathbf{N}_1 &= \begin{pmatrix} 0, 0, 0 \\ 0, 1, 0 \end{pmatrix}, & \mathbf{N}_2 &= \begin{pmatrix} 0, 1, 0 \\ 1, 0, 1 \end{pmatrix}. \\
 \mathbf{c}^T &= (2, -3, 0), & \mathbf{d}^T &= (3, -2, 1), & \mathbf{p}^T &= (1, 2, 0), & \mathbf{q}^T &= (3, 1, 0), \\
 \mathbf{c}_1^T &= (2, 0, 0), & \mathbf{c}_2^T &= (0, -3, 0), & \mathbf{d}_1^T &= (3, 0, 1), & \mathbf{d}_2^T &= (0, -2, 0), \\
 \mathbf{p}_1^T &= (1, 0, 0), & \mathbf{p}_2^T &= (0, 2, 0), & \mathbf{q}_1^T &= (3, 0, 0), & \mathbf{q}_2^T &= (0, 1, 0), \\
 \mathbf{b}^T &= (2, -13), & \mathbf{g}^T &= (9, 2), & \mathbf{h}^T &= (16, 11), & \mathbf{f}^T &= (4, 12).
 \end{aligned}$$

By Eq.(4), Eq. (11) can be solved by the auxiliary MOLP model as follows:

$$\begin{aligned}
 & \min Z_1 = x_1 + 2y_2 + 2s_1 + 3t_2, \\
 & \max Z_2 = 2x_1 - 3y_2, \\
 & \max Z_3 = x_1 - x_2 + 1.5y_1 - 1.5y_2 + 0.5y_3, \\
 & \max Z_4 = x_2 + 3y_1 + 3t_1 + t_3 + 2s_2, \\
 & \text{s.t. } 2x_1 - 2y_2 - 2y_3 = 2, \\
 & \quad 2x_2 - 3y_1 - 2y_3 = -13, \\
 & \quad 2y_1 - x_2 - x_3 = 9, \\
 & \quad 2y_2 - x_1 - x_3 = 2, \\
 & \quad x_1 + 2y_2 + y_3 + 2s_1 + 2t_2 + 2t_3 = 16, \\
 & \quad x_2 + y_1 + 2s_2 + 3t_1 + 2t_3 = 11, \\
 & \quad x_2 + 2t_1 + s_2 + s_3 = 4, \\
 & \quad x_1 + x_3 + y_2 + s_1 + s_3 + 2t_2 = 12, \\
 & \quad y_i - x_i \geq 0, x_i - s_i \geq 0, s_i \geq 0, t_i \geq 0 \ (i = 1, 2, 3, 4).
 \end{aligned} \tag{0.12}$$

Using Eq. (5,6), it follows that

$$\begin{aligned}
 Z_1^{\max} &= 17.20, & Z_2^{\max} &= -0.40, & Z_3^{\max} &= 6.10, & Z_4^{\max} &= 23.86, \\
 Z_1^{\min} &= 16.35, & Z_2^{\min} &= -0.85, & Z_3^{\min} &= 5.93, & Z_4^{\min} &= 22.60.
 \end{aligned}$$

Then, we can get

$$\begin{aligned}
 \mu_{z_1}(\tilde{\mathbf{x}}) &= \begin{cases} 1 & \text{if } Z_1 < 16.35, \\ \frac{17.20-Z_1}{0.85} & \text{if } 16.35 \leq Z_1 \leq 17.20, \\ 0 & \text{if } Z_1 > 17.20; \end{cases} \\
 \mu_{z_2}(\tilde{\mathbf{x}}) &= \begin{cases} 1 & \text{if } Z_2 > -0.40, \\ \frac{Z_2+0.85}{0.45} & \text{if } -0.85 \leq Z_2 \leq -0.40, \\ 0 & \text{if } Z_2 < -0.85; \end{cases} \\
 \mu_{z_3}(\tilde{\mathbf{x}}) &= \begin{cases} 1 & \text{if } Z_3 > 6.10, \\ \frac{Z_3-5.93}{0.17} & \text{if } 5.93 \leq Z_3 \leq 6.10, \\ 0 & \text{if } Z_3 < 5.93; \end{cases} \\
 \mu_{z_4}(\tilde{\mathbf{x}}) &= \begin{cases} 1 & \text{if } Z_4 > 23.86, \\ \frac{Z_4-22.60}{1.26} & \text{if } 22.60 \leq Z_4 \leq 23.86, \\ 0 & \text{if } Z_4 < 22.60. \end{cases}
 \end{aligned}$$

Next, we use three approach (i.e., Eqs. (7)-(9)) to solving MOLP (12), respectively.

First, according to Eq. (7) (i.e., pessimistic approach), we can obtain the optimal solution of the MOLP (12) as follows:

$$\begin{aligned}
 x_1^* &= 5.28, x_2^* = 2.96, x_3^* = 0.15, y_1^* = 6.06, y_2^* = 3.72, y_3^* = 0.38, \\
 s_1^* &= 0.00, s_2^* = 0.88, s_3^* = 0.15, t_1^* = 0.00, t_2^* = 1.35, t_3^* = 0.11.
 \end{aligned}$$

Therefore, the optimal solution of the FFLP (11) is:

$$\tilde{\mathbf{x}}^* = \begin{pmatrix} (5.28, 6.06, 0.00, 0.00)_{LR} \\ (2.96, 3.72, 0.88, 1.35)_{LR} \\ (0.15, 0.38, 0.15, 0.11)_{LR} \end{pmatrix}.$$

Substituted the above optimal solution into the objective function of FFLP (11), the optimal objective value is obtained as $(-0.60, 12.64, 16.77, 23.01)_{LR}$. Namely, the most likely value of the objective is between -0.60 and 12.64, the upper and lower limits of the objective value are -17.37 and 35.65, respectively. Its membership function is given by

$$u_{\tilde{Z}}(t) = \begin{cases} L\left(\frac{-0.60-t}{16.77}\right) & t < -0.60, \\ 1 & -0.60 \leq t \leq 12.64, \\ R\left(\frac{t-12.64}{23.01}\right) & t > 12.64. \end{cases}$$

That is

$$u_{\tilde{z}}(t) = \begin{cases} 1 + \frac{0.60+t}{16.77} & -17.37 < t < -0.60, \\ 1 & -0.60 \leq t \leq 12.64, \\ \frac{1}{e-1} \left(e^{1 - \frac{(t-12.64)^2}{23.01^2}} - 1 \right) & 12.64 < t < 35.65, \\ 0, & \text{others.} \end{cases}$$

Second, solving the MOLP (12) by using Eq. (8) (i.e., optimistic approach), we obtain the optimal solution as follows:

$$\begin{aligned} x_1^* &= 4.09, \quad x_2^* = 1.14, \quad x_3^* = 0.01, \quad y_1^* = 5.08, \quad y_2^* = 3.05, \quad y_3^* = 0.02, \\ s_1^* &= 0.72, \quad s_2^* = 0.54, \quad s_3^* = 0.00, \quad t_1^* = 1.16, \quad t_2^* = 2.07, \quad t_3^* = 0.11. \end{aligned}$$

Therefore, the optimal solution of the FFLP (11) is:

$$\tilde{\mathbf{x}}^* = \begin{pmatrix} (4.09, 5.08, 0.72, 1.16)_{LR} \\ (1.14, 3.05, 0.54, 2.07)_{LR} \\ (0.01, 0.02, 0.00, 0.11)_{LR} \end{pmatrix}.$$

Substituted the above optimal solution into the objective function of FFLP (11), the optimal objective value is obtained as $(-0.97, 12.98, 17.84, 21.05)_{LR}$. Namely, the most likely value of the objective is between -0.97 and 12.98, the upper and lower limits of the objective value are -18.81 and 34.03, respectively. Its membership function is given by

$$u_{\tilde{z}}(t) = \begin{cases} L\left(\frac{-0.97-t}{17.84}\right) & t < -0.97, \\ 1 & -0.97 \leq t \leq 12.98, \\ R\left(\frac{t-12.98}{21.05}\right) & t > 12.98, \end{cases}$$

That is

$$u_{\tilde{z}}(t) = \begin{cases} 1 + \frac{0.97+t}{17.84} & -18.81 < t < -0.97, \\ 1 & -0.97 \leq t \leq 12.98, \\ \frac{1}{e-1} \left(e^{1 - \frac{(t-12.98)^2}{21.05^2}} - 1 \right) & 12.98 < t < 34.03, \\ 0, & \text{others.} \end{cases}$$

Finally, solving the MOLP (12) by using Eq. (9) (i.e., linear sum approach based on membership function), we obtain the optimal solution for $\mathbf{w} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^T$ as follows:

$$\begin{aligned} x_1^* &= 5.32, \quad x_2^* = 2.84, \quad x_3^* = 0.05, \quad y_1^* = 5.95, \quad y_2^* = 3.68, \quad y_3^* = 0.42, \\ s_1^* &= 0.00, \quad s_2^* = 1.11, \quad s_3^* = 0.05, \quad t_1^* = 0.00, \quad t_2^* = 1.45, \quad t_3^* = 0.00. \end{aligned}$$

Therefore, the optimal solution of the FFLP (11) is:

$$\tilde{\mathbf{x}}^* = \begin{pmatrix} (5.32, 5.95, 0.00, 0.00)_{LR} \\ (2.84, 3.68, 1.11, 1.45)_{LR} \\ (0.05, 0.42, 0.05, 0.00)_{LR} \end{pmatrix}.$$

Substituted the above optimal solution into the objective function of FFLP (11), the optimal objective value is obtained as $(-0.40, 12.59, 17.03, 22.91)_{LR}$. Namely, the most likely value of the objective is between -0.40 and 12.59, the upper and lower limits of the objective value are -17.43 and 35.50, respectively. Its membership function is given by

$$u_{\tilde{z}}(t) = \begin{cases} L\left(\frac{-0.40-t}{17.03}\right) & t < -0.40, \\ 1 & -0.40 \leq t \leq 12.59, \\ R\left(\frac{t-12.59}{22.91}\right) & t > 12.59, \end{cases}$$

That is

$$u_{\tilde{z}}(t) = \begin{cases} 1 + \frac{0.40+t}{17.03} & -17.43 < t < -0.40, \\ 1 & -0.40 \leq t \leq 12.59, \\ \frac{1}{e-1} \left(e^{1 - \frac{(t-12.59)^2}{22.91^2}} - 1 \right) & 12.59 < t < 35.50, \\ 0, & \text{others.} \end{cases}$$

To illustrate the influence of the weight vector w on the optimal value in this example, we use a different weight vector w to solve the MOLP (12) according to Eq. (9). Generally speaking, we need to consider some special cases. One is an average weight, i.e., $w = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^T$. Secondly, the objective functions Z_2 and Z_3 are relative to the mode of the flat fuzzy number Z , which are the most possible values of Z . Thus, more weights should be assigned to Z_2 and Z_3 , i.e., $w = (\frac{1}{6}, \frac{2}{6}, \frac{2}{6}, \frac{1}{6})^T$. Thirdly, similar to the Olympic games, which discarded a maximum point and a minimum point, we can set $w = (0, \frac{1}{2}, \frac{1}{2}, 0)^T$. Finally, $w = (\frac{2}{6}, \frac{1}{6}, \frac{1}{6}, \frac{2}{6})^T$ emphasizes both ends and reduces the middle. All the computation results are shown in Table 1.

Table 1: The optimal solution and optimal objective value for FFLP (11) with different approaches.

variable	x_1	x_2	x_3	y_1	y_2	y_3	s_1	s_2	s_3	t_1	t_2	t_3	optimal value Z
by Eq.(6)	5.28	2.96	0.15	6.06	3.72	0.38	0.00	0.88	0.15	0.00	1.35	0.11	$(-0.60, 12.64, 16.77, 23.01)_{LR}$
by Eq.(7)	4.09	1.14	0.01	5.08	3.05	0.02	0.72	0.54	0.00	1.16	2.07	0.11	$(-0.97, 12.98, 17.84, 21.05)_{LR}$
by Eq.(8)													
$w = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^T$	5.32	2.84	0.05	5.95	3.68	0.42	0.00	1.11	0.05	0.00	1.45	0.00	$(-0.40, 12.59, 17.03, 22.91)_{LR}$
$w = (\frac{1}{6}, \frac{2}{6}, \frac{2}{6}, \frac{1}{6})^T$	5.32	2.84	0.05	5.95	3.68	0.42	0.00	1.11	0.05	0.00	1.45	0.00	$(-0.40, 12.59, 17.03, 22.91)_{LR}$
$w = (0, \frac{1}{2}, \frac{1}{2}, 0)^T$	5.20	2.60	0.00	5.80	3.60	0.40	0.00	1.00	0.00	0.20	1.60	0.00	$(-0.40, 12.60, 17.20, 22.60)_{LR}$
$w = (\frac{2}{6}, \frac{1}{6}, \frac{1}{6}, \frac{2}{6})^T$	5.45	3.55	0.36	6.45	3.91	0.36	0.00	0.45	0.00	0.00	1.14	0.05	$(-0.83, 12.61, 16.69, 23.85)_{LR}$

It can be seen from Table 1 that, the optimal solutions and optimal objective value of Z for $w = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^T$, $w = (\frac{1}{6}, \frac{2}{6}, \frac{2}{6}, \frac{1}{6})^T$ and $w = (0, \frac{1}{2}, \frac{1}{2}, 0)^T$ are completely identical. The optimal solution and optimal objective value of Z for $w = (\frac{2}{6}, \frac{1}{6}, \frac{1}{6}, \frac{2}{6})^T$ are not the same as that for the other weight vectors. This signifies that the weight vector w does affect the optimal solutions and optimal objective value.

In addition, Table 1 shows that applying different approaches to solving the multi-objective programming may result in different optimal solutions and optimal objective values. The DM can choose the proper approach to solving the multi-objective programming according to his/her risk preference and actual requirements.

Note that for a fully fuzzy linear programming problem, the fuzzy optimal solution \tilde{x}^* is more general than the result which are calculated in Lotfi [12], Kumar [13] and Ezzati [14] when the left spreads shape functions $L(\cdot)$ and right spreads shape functions $R(\cdot)$ are linear functions. Furthermore, even the uncertain elements in a fuzzy linear programming problem were extended fuzzy numbers, we can construct a corresponding auxiliary multi-objective programming problem and solve it by use of three approaches (i.e., optimistic approach, pessimistic approach and linear sum approach based on membership function).

In Wan [11], the FLP problems with trapezoidal fuzzy numbers have been studied in which not all parts of the problem were assumed to be fuzzy (the variables were not fuzzy). Lotfi [12] proposed a novel method to find the optimal solution of FFLP problems, this method was obtained by using the concept of the symmetric triangular fuzzy numbers and introduce an approach to defuzzify a general fuzzy quantity. However, it can be applied only if the elements of the coefficient matrix are symmetric fuzzy numbers. In Kumar's article [13], the fuzzy optimal solution of FFLP problem can be obtained by using the arithmetic operations of triangular fuzzy numbers and linear ranking function which is used to convert the fuzzy objective function to the real objective function. Although the ranking function is convenient for the specific numerical computation, the fuzziness of the objective function is neglected by it. Ezzati [14] proposed a new algorithm to solve FFLP problems with TFNs. It can be applied only if all the parameters as well as the variables are represented by TFNs which is the specious case of LR flat fuzzy number, for the generalized fuzzy numbers, it is unsuitable.

6. Conclusion

In this paper, we propose a simple and practical method to solve a full fuzzy linear programming problem. The corresponding auxiliary MOLP problem is constructed and we solve it by use of three approaches. By numerical example, the obtained results of proposed algorithm with Wan [11], Lotfi [12], Kumar [13] and R. Ezzati [14] have been compared and shown the reliability and applicability of our

algorithm.

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Generalized Bateman’s G –function and its bounds

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Abstract

In this paper, we presented some functional equations of the generalized Bateman’s G –function $G_h(x)$ and its relation with the hypergeometric series ${}_3F_2$. We deduced an asymptotic expansion of the function $G_h(x)$ and studied the completely monotonic property of some functions involving it. Also, we presented some new bounds of the function $G_h(x)$ and the double inequality

$$\ln\left(1 + \frac{h}{x + \beta}\right) < G_h(x) - \frac{2h}{x(x+h)} < \ln\left(1 + \frac{h}{x + \alpha}\right), \quad x > 0; 0 < h < 2$$

where the constants $\alpha = 1$ and $\beta = \frac{h}{e^{\gamma + \frac{2}{h} + \psi(\frac{h}{2})} - 1}$ are the best possible.

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Key Words: Psi function, Bateman’s G-function, functional equation, asymptotic formula, Laplace transform, inequality, monotonicity, best possible bound.

1 Introduction.

The ordinary gamma function $\Gamma(x)$ is defined by [3]

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0$$

and the Psi or digamma function $\psi(x)$ is given by

$$\psi(x) = \frac{d}{dx} \log \Gamma(x).$$

The gamma function and its logarithmic derivatives $\psi^{(n)}(x)$ are of the most widely used special functions encountered in advanced mathematics . For more details about the properties of

these functions and their bounds, please refer to [2], [3], [8], [9], [15]-[18], [23]-[27] and plenty of references therein.

The Bateman's G -function is defined by [7]

$$G(x) = \psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right), \quad x \neq 0, -1, -2, \dots \quad (1)$$

The function $G(x)$ is very useful in estimating and summing certain numerical and algebraic series. For more details about the properties, bounds and applications of the $G(x)$, please refer to [7], [12]-[14], [15], [19], [28] and the references therein.

The function $G(x)$ satisfies the following relations [7]

$$G(x) = 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{k+x}, \quad (2)$$

$$G(x+1) + G(x) = 2x^{-1}, \quad (3)$$

$$G(nx) = 2n^{-1} \sum_{k=0}^{n-1} (-1)^{k+1} \psi\left(x + \frac{k}{n}\right), \quad n = 2, 4, 6, \dots \quad (4)$$

$$G(nx) = n^{-1} \sum_{k=0}^{n-1} (-1)^k G\left(x + \frac{k}{n}\right), \quad n = 1, 3, 5, \dots \quad (5)$$

$$G(x) = 2 \int_0^{\infty} \frac{e^{-xt}}{1+e^{-t}} dt, \quad x > 0 \quad (6)$$

$$G(x) = 2x^{-1} {}_2F_1(1, x; x+1; -1), \quad (7)$$

where

$${}_rF_s(\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_s; x) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_r)_k x^k}{(\beta_1)_k \dots (\beta_s)_k k!}$$

is the generalized hypergeometric series [3] defined for $r, s \in \mathbb{N}$, $\alpha_j, \beta_j \in \mathbb{C}$, $\beta_j \neq 0, -1, -2, \dots$ and the Pochhammer or shifted symbol $(\alpha)_n$ is given by

$$(\alpha)_0 = 1 \quad \text{and} \quad (\alpha)_m = \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)}, \quad m \geq 1.$$

Qiu and Vuorinen [28] presented the double inequality

$$\frac{4(3/2 - \ln 4)}{x^2} < G(x) - x^{-1} < \frac{1}{2x^2}, \quad x > 0.5. \quad (8)$$

Mahmoud and Agarwal [12] deduced the following asymptotic formula for Bateman's G -function

$$G(x) \sim x^{-1} + \sum_{k=1}^{\infty} \frac{(2^{2k} - 1)B_{2k}}{k} x^{-2k}, \quad x \rightarrow \infty \quad (9)$$

and they presented the inequality

$$\frac{1}{2x^2 + 1.5} < G(x) - x^{-1} < \frac{1}{2x^2}, \quad x > 0 \tag{10}$$

which improved the lower bound of the inequality (8) for $x > \sqrt{\frac{9-6 \ln 4}{8 \ln 4 - 11}}$. Also, Mahmoud and Almuashi [13] proved the following double inequality of the Bateman's G -function

$$\sum_{n=1}^{2m} \frac{(2^n - 1)B_{2n}}{n} x^{-2n} < G(x) - x^{-1} < \sum_{n=1}^{2m-1} \frac{(2^n - 1)B_{2n}}{n} x^{-2n}, \quad m \in \mathbb{N} \tag{11}$$

with the best possible bounds, where B_m 's are the Bernoulli numbers [11].

Mortici [15] presented the double inequality

$$0 < \psi(x + \lambda) - \psi(x) \leq \psi(\lambda) + \gamma - \lambda + \lambda^{-1}, \quad x \geq 1; 0 < \lambda < 1 \tag{12}$$

where γ is the Euler constant, which also improves the double inequality (8). Also, Alzer deduced the inequality [2]

$$x^{-1} - T_r(\lambda; x) - \omega_r(\lambda; x) < \psi(x + \lambda) - \psi(x) < x^{-1} - T_r(\lambda; x),$$

where $x > 0, r = 0, 1, 2, \dots, 0 < \lambda < 1,$

$$T_r(\lambda; x) = (1 - \lambda) \left[\frac{1}{\lambda + r + 1} + \sum_{i=0}^{r-1} \frac{1}{(x + i + 1)(x + i + \lambda)} \right]$$

and

$$\omega_r(\lambda; x) = \frac{1}{x + r + \lambda} \log \frac{(x + r)^{(x+r)(1-\lambda)}(x + r + 1)^{(x+r+1)\lambda}}{(x + r + \lambda)^{x+r+\lambda}}.$$

Mahmoud, Talat and Moustafa [14] presented the following family of approximations of the function $G(x)$

$$M(\mu, x) = \ln \left(1 + \frac{1}{x + \mu} \right) + \frac{2}{x(x + 1)}, \quad x > 0; 1 \leq \mu \leq 2$$

which is of an order of convergence of $O \left(\ln \frac{(x+2)[(e^2-4)x+4]}{(x+1)[(e^2-4)x+e^2]} \right)$ for $x > 2$ and $\mu \in (1, \frac{4}{e^2-4})$ and is asymptotically equivalent to $G(x)$ as $x \rightarrow \infty$.

In this paper, we presented some functional equations of the generalized Bateman's G -function

$$G_h(x) = \psi \left(\frac{x + h}{2} \right) - \psi \left(\frac{x}{2} \right), \quad 0 < h < 2; x \neq -2m, -2m - h \text{ for } m = 0, 1, 2, \dots \tag{13}$$

and its relation with the hypergeometric function ${}_3F_2$. We deduced an asymptotic expansion of the function $G_h(x)$ and studied the completely monotonic property of the function $G_h(x) - \frac{s}{x^r}$ for different values of the parameter s, r and h for $x > 0$. Also, some new bounds and best possible bounds of the generalized Bateman's G -function are given.

2 Some relations of the function $G_h(x)$.

Lemma 2.1. *The function $G_h(x)$ satisfies the functional equation*

$$G_h(x + 1) + G_h(x) = 2(\psi(x + h) - \psi(x)), \quad x > 0. \tag{14}$$

Proof. Using the integral representation [3]

$$\psi(z) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-tz}}{1 - e^{-t}} dt, \quad R(z) > 0 \tag{15}$$

we get

$$G_h(x) = 2 \int_0^\infty \frac{1 - e^{-ht}}{1 - e^{-2t}} e^{-xt} dt, \quad x > 0. \tag{16}$$

Also,

$$\begin{aligned} \psi(x + h) - \psi(x) &= \int_0^\infty \frac{1 - e^{-ht}}{1 - e^{-t}} e^{-xt} dt \\ &= \int_0^\infty \frac{1 - e^{-ht}}{1 - e^{-2t}} e^{-(x+1)t} dt + \int_0^\infty \frac{1 - e^{-ht}}{1 - e^{-2t}} e^{-xt} dt \\ &= \frac{1}{2} [G_h(x + 1) + G_h(x)]. \end{aligned}$$

□

In case of $h = 1$ and using the functional equation $\psi(x + 1) = \frac{1}{x} + \psi(x)$, we get the relation (3).

Lemma 2.2. *The function $G_h(x)$ satisfies the functional equation*

$$G_h(mx) = \frac{1}{m} \sum_{r=0}^{m-1} G_{\frac{h}{m}} \left(x + \frac{2r}{m} \right), \quad x > 0; m \in \mathbb{N}. \tag{17}$$

Proof.

$$\begin{aligned} \sum_{r=0}^{m-1} G_{\frac{h}{m}} \left(x + \frac{2r}{m} \right) &= \int_0^\infty \left(\sum_{r=0}^{m-1} e^{\frac{-2rt}{m}} \right) \frac{1 - e^{\frac{-ht}{m}}}{1 - e^{-2t}} e^{-xt} dt \\ &= \int_0^\infty \left(\frac{1 - e^{-2t}}{1 - e^{\frac{-2t}{m}}} \right) \frac{1 - e^{\frac{-ht}{m}}}{1 - e^{-2t}} e^{-xt} dt \\ &= \int_0^\infty \frac{1 - e^{\frac{-ht}{m}}}{1 - e^{\frac{-2t}{m}}} e^{-xt} dt \\ &= m G_h(mx). \end{aligned}$$

□

Remark 1. The following new functional equation of the ordinary function $G(x)$ in terms of the generalized function $G_h(x)$ can be obtained in case of $h = 1$.

Corollary 2.3. *The function $G(x)$ satisfies the functional equation*

$$G(mx) = \frac{1}{m} \sum_{r=0}^{m-1} G_{\frac{1}{m}} \left(x + \frac{2r}{m} \right), \quad x > 0; m \in \mathbb{N}. \tag{18}$$

The following result relates the function $G_h(x)$ and the hypergeometric function ${}_3F_2$.

Lemma 2.4. *The function $G_h(x)$ satisfies*

$$G_h(x) = \frac{h}{x+h} {}_3F_2 \left(1, 1, \frac{h+2}{2}; 2, \frac{x+h+2}{2}; 1 \right), \quad x > 0. \tag{19}$$

Proof. Using the integral representation [3]

$$\psi(z) = -\gamma + \int_0^1 \frac{1-t^{z-1}}{1-t} dt, \quad \operatorname{Re}(z) > 0$$

we get

$$G_h(x) = \int_0^1 \frac{t^{\frac{x-2}{2}} - t^{\frac{x+h-2}{2}}}{1-t} dt = \int_0^1 \left(t^{\frac{x-2}{2}} - t^{\frac{x+h-2}{2}} \right) \left(\sum_{n=0}^{\infty} t^n \right) dt, \quad x > 0$$

and then

$$G_h(x) = \sum_{n=0}^{\infty} \frac{2h}{(x+2n)(x+h+2n)}, \quad x > 0. \tag{20}$$

Using the relation

$$x+n = \frac{x(x+1)_n}{(x)_n},$$

we obtain

$$\begin{aligned} G_h(x) &= \frac{2h}{x(x+h)} \sum_{n=0}^{\infty} \frac{\left(\frac{x+h}{2}\right)_n \left(\frac{x}{2}\right)_n}{\left(\frac{x+h+2}{2}\right)_n \left(\frac{x+2}{2}\right)_n} \\ &= \frac{2h}{x(x+h)} {}_3F_2 \left(1, \frac{x}{2}, \frac{x+h}{2}; \frac{x+2}{2}, \frac{x+h+2}{2}; 1 \right), \quad x > 0. \end{aligned}$$

Now using the two-term Thomae transformation formula [30], [21]

$${}_3F_2(\alpha, \beta, \sigma; \delta, \eta; 1) = \frac{\Gamma(\delta)\Gamma(\theta - \sigma)}{\Gamma(\theta)\Gamma(\delta - \sigma)} {}_3F_2(\eta - \alpha, \eta - \beta, \sigma; \theta, \eta; 1), \quad \theta = \delta + \eta - \alpha - \beta$$

with

$$\alpha = \frac{x}{2}, \quad \beta = \frac{x+h}{2}, \quad \sigma = 1, \quad \eta = \frac{x+h+2}{2}, \quad \delta = \frac{x+2}{2}$$

we have

$${}_3F_2 \left(1, \frac{x}{2}, \frac{x+h}{2}; \frac{x+2}{2}, \frac{x+h+2}{2}; 1 \right) = \frac{x}{2} {}_3F_2 \left(1, 1, \frac{h+2}{2}; 2, \frac{x+h+2}{2}; 1 \right),$$

which complete the proof. □

Remark 2. From the formulas (7) and (19) for $h = 1$, we can conclude that

$${}_3F_2 \left(1, 1, 3/2; 2, \frac{x+3}{2}; 1 \right) = \frac{2(x+1)}{x} {}_2F_1(1, x; x+1; -1), \quad x > 0. \tag{21}$$

3 An asymptotic expansion of the function $G_h(x)$.

It is well known that the Psi function has the asymptotic expansion [6]

$$\psi(z) \sim \ln z - \sum_{k=1}^{\infty} \frac{(-1)^k B_k}{k} \frac{1}{z^k}$$

and its generalization is given by

$$\psi(z+l) \sim \ln z - \sum_{k=1}^{\infty} \frac{(-1)^k B_k(l)}{k} \frac{1}{z^k},$$

where $B_k(l)$ are the Bernoulli polynomials defined by the generating function [11]

$$\frac{ze^{zt}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(l)}{k!} z^k$$

and the Bernoulli constants $B_k = B_k(0)$. Using the operations of the asymptotic expansions [5]; [20], we obtain

$$\psi(z+l) - \psi(z) \sim \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} [B_k(l) - B_k] \frac{1}{z^k}.$$

For more details about the general theory of the asymptotic expansion of the function $f(z+t)$ by the asymptotic expansion of the function $f(z)$ using Appell polynomials, we refer to [4]. Now, using the identity [11]

$$B_k(l) = \sum_{r=0}^k \binom{k}{r} B_r l^{k-r},$$

we get

$$\psi(z+l) - \psi(z) \sim \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left[\sum_{r=0}^{k-1} \binom{k}{r} B_r l^{k-r} \right] \frac{1}{z^k}.$$

Then we obtain the following result.

Lemma 3.1. *The following asymptotic series holds:*

$$G_h(x) \sim \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^n}{n} \left[B_r \left(\frac{h}{2} \right) - B_r \right] \frac{1}{x^n}, \quad x \rightarrow \infty. \tag{22}$$

or

$$G_h(x) \sim \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[\sum_{r=0}^{n-1} \binom{n}{r} 2^r B_r h^{n-r} \right] \frac{1}{x^n}, \quad x \rightarrow \infty. \tag{23}$$

Remark 3. As a special case at $h = 1$, we obtain

$$G(x) \sim \frac{1}{x} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 2^n}{n} \left[B_r \left(\frac{1}{2} \right) - B_r \right] \frac{1}{x^n}, \quad x \rightarrow \infty$$

and using the identities [1]

$$B_n \left(\frac{1}{2} \right) = (2^{1-n} - 1) B_n, \quad n = 0, 1, 2, \dots$$

and

$$B_{2n+1} = 0, \quad n = 1, 2, \dots$$

then we get the asymptotic series (9).

Now we will study the completely monotonic property of the function $G_h(x) - \frac{s}{x^r}$ for different values of the parameters s, h and r for positive values of x .

Lemma 3.2. *The functions*

$$\chi_{s,1}(x, h) = G_h(x) - \frac{s}{x} \quad s \leq h; \quad x > 0; \quad 0 < h < 2, \quad (24)$$

and

$$\chi_{s,r}(x, h) = G_h(x) - \frac{s}{x^r} \quad s < 0; \quad x > 0; \quad 0 < h < 2; \quad r = 2, 3, 4, \dots \quad (25)$$

are strictly completely monotonic.

Proof. Using the relation (16) and the known formula

$$(r-1)! x^{-m} = \int_0^\infty v^{m-1} e^{-xv} dv, \quad m \in \mathbb{N} \quad (26)$$

we get

$$(-1)^n \chi_{s,r}^{(n)}(x, h) = \int_0^\infty \phi_{h,s}(r, t) \frac{t^n e^{-xt}}{e^{2t} - 1} dt, \quad n = 0, 1, 2, 3, \dots \quad (27)$$

where

$$\phi_{h,s}(r, t) = 2(e^{2t} - e^{(2-h)t}) - \frac{s t^{r-1}}{(r-1)!} (e^{2t} - 1).$$

Then

$$\phi_{h,s}(r, t) = \sum_{k=1}^\infty \frac{2^{k+1} t^k}{k!} P_{h,s}(r, t),$$

where

$$P_{h,s,k}(r, t) = 1 - \left(1 - \frac{h}{2}\right)^k - \frac{s}{2(r-1)!} t^{r-1}.$$

Firstly, if $r = 1$, we obtain

$$P_{h,s,k}(1, t) = 1 - \left(1 - \frac{h}{2}\right)^k - \frac{s}{2} > 0 \quad \text{iff} \quad \frac{s}{2} \leq 1 - \left(1 - \frac{h}{2}\right)^k \quad k = 1, 2, 3, \dots$$

But

$$\frac{h}{2} \leq 1 - \left(1 - \frac{h}{2}\right)^k \quad 0 < h < 2; \quad k = 1, 2, 3, \dots$$

and thus, $\phi_{h,s}(1, t) > 0$ for all $t \geq 0$ iff $s \leq h$. Secondly, when $r = 2, 3, 4, \dots$, then $P_{h,s,k}(r, t)$ is increasing as a function of t if $s < 0$ with $P_{h,s,k}(r, 0) = 1 - \left(1 - \frac{h}{2}\right)^k > 0$ for $0 < h < 2$ and $k = 1, 2, 3, \dots$. Thus $\phi_{h,s}(r, t) > 0$ for all $t \geq 0, r = 2, 3, \dots$ iff $s < 0$. \square

As a result of the strictly completely monotonicity of the function $\chi_{s,1}(x, h)$ and the relation (23), we obtain

$$\chi_{s,1}(x, h) > \lim_{x \rightarrow \infty} (\chi_{s,1}(x, h)) = 0, \quad s \leq h.$$

Hence, we have the following result:

Corollary 3.3. *The following inequality holds*

$$G_h(x) > \frac{h}{x}, \quad x > 0; 0 < h < 2. \tag{28}$$

4 Some Bounds of the function $G_h(x)$.

Lemma 4.1.

$$G_h(x) < \frac{2}{x} + \frac{h(2-h)}{2x^2}, \quad x > 0; 0 < h < 2. \tag{29}$$

Proof. By using the formulas (16) and (26), we get for $x > 0$ that

$$\begin{aligned} G_h(x) - \frac{2}{x} - \frac{h(2-h)}{2x^2} &= \int_0^\infty \left(2(e^{2t} - e^{(2-h)t}) - 2(e^{2t} - 1) - \frac{h(2-h)}{2}(e^{2t} - 1)t \right) \frac{e^{-xt}}{e^{2t} - 1} dt \\ &= \int_0^\infty \left(2(1 - e^{(2-h)t}) - \frac{h(2-h)}{2}(e^{2t} - 1)t \right) \frac{e^{-xt}}{e^{2t} - 1} dt \\ &< 0 \quad \text{for } 0 < h < 2. \end{aligned}$$

□

Theorem 1.

$$G_h(x) < \frac{h}{x} + \frac{h(2-h)}{2x^2}, \quad x > 0; 1 \leq h < 2. \tag{30}$$

Proof. Using the two formulas (16) and (26), we have

$$G_h(x) - \frac{h}{x} - \frac{h(2-h)}{2x^2} = \int_0^\infty \rho_h(t) \frac{e^{-xt}}{e^{2t} - 1} dt,$$

where

$$\rho_h(t) = 2(e^{2t} - e^{(2-h)t}) - h(e^{2t} - 1) - \frac{h(2-h)}{2}(e^{2t} - 1)t \quad t > 0.$$

Then

$$\rho_h''(t) = 2(h-2)e^{(2-h)t}Q_h(t)$$

with

$$Q_h(t) = 2 - h + e^{ht}(h - 2 + ht).$$

The function $Q_h(t)$ is convex function with minimum value at $t_0 = \frac{1-h}{h}$, which is non positive for $1 \leq h < 2$ and $Q_h(0) = 0$. Hence $Q_h(t) > 0$ for $1 \leq h < 2$. Hence $\rho_h(t)$ is concave for $1 \leq h < 2$ and its has maximum value at $t = 0$. Then

$$\rho_h(t) < 0, \quad 1 \leq h < 2; t > 0.$$

Then the function $G_h(x) - \frac{h}{x} - \frac{h(2-h)}{2x^2}$ is strictly increasing function for $1 \leq h < 2$ and $x > 0$ and using the asymptotic expansion (23), we get

$$\lim_{x \rightarrow \infty} \left(G_h(x) - \frac{h}{x} - \frac{h(2-h)}{2x^2} \right) = 0,$$

which complete the proof. □

Remark 4. In case of $h = 1$, the inequality (30) will prove the right-hand side of the inequality (8).

To obtain our next result, we will apply the following monotone form of L'Hôpital's rule [10] (see also [22] and [29]).

Theorem 2. Let $-\infty < \alpha < \beta < \infty$ and $L, U : [\alpha, \beta] \rightarrow \mathbb{R}$ be continuous on $[\alpha, \beta]$ and differentiable on (α, β) . Let $U'(x) \neq 0$ on (α, β) . If $L'(x)/U'(x)$ is increasing (decreasing) on (α, β) , then so are

$$\frac{L(x) - L(\alpha)}{U(x) - U(\alpha)} \quad \text{and} \quad \frac{L(x) - L(\beta)}{U(x) - U(\beta)}. \tag{31}$$

If $L'(x)/U'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Theorem 3.

$$G_h(x) > \frac{h}{x} + \frac{h(2-h)}{2(x^2 + 3h^2)}, \quad x > 0; 0 < h < 2. \tag{32}$$

Proof. Using the two formulas (16) and (26) and the Laplace transformation of the sine function, we get

$$G_h(x) - \frac{h}{x} - \frac{h(2-h)}{2(x^2 + 3h^2)} = \int_0^\infty \xi_h(t) \frac{e^{-xt}}{6(e^{2t} - 1)} dt,$$

where

$$\xi_h(t) = 6(-2e^{2t} - e^{(2+h)t}(-2+h) + he^{ht}) + \sqrt{3}e^{ht}(-1 + e^{2t})(-2+h) \sin(\sqrt{3}ht).$$

Now consider the function

$$\tau_h(t) = \frac{2\sqrt{3}e^{-ht}(-2e^{2t} - e^{(2+h)t}(-2+h) + he^{ht})}{(-1 + e^{2t})(2-h)} \quad t > 0; 0 < h < 2.$$

The function

$$\frac{2\sqrt{3} \frac{d}{dt} (e^{-ht}(-2e^{2t} - e^{(2+h)t}(-2+h) + he^{ht}))}{\frac{d}{dt}((-1 + e^{2t})(2-h))} = 2\sqrt{3}e^{-ht}(-1 + e^{ht})$$

is increasing function for $t > 0$. Using the monotone form of L'Hôpital's rule, we get that the function $\tau_h(t)$ is increasing. Similarly, the function

$$H_h(t) = \frac{\tau_h(t)}{\sqrt{3}ht}, \quad t > 0; 0 < h < 2.$$

is increasing function and

$$\lim_{t \rightarrow \infty} H_h(t) = 1.$$

Then

$$2\sqrt{3}e^{-ht} (-2e^{2t} - e^{(2+h)t}(-2+h) + he^{ht}) > ht (-1 + e^{2t}) (2-h), \quad t > 0; 0 < h < 2$$

and using Jordan's inequality

$$\frac{2z}{\pi} \leq \sin z \leq z, \quad x \in [0, \pi/2]$$

we have

$$2\sqrt{3}e^{-ht} (-2e^{2t} - e^{(2+h)t}(-2+h) + he^{ht}) > ht (-1 + e^{2t}) (2-h) \sin(\sqrt{3}ht), \quad t > 0; 0 < h < 2.$$

Hence

$$\xi_h(t) > 0, \quad t > 0; 0 < h < 2.$$

Then the function $G_h(x) - \frac{h}{x} - \frac{h(2-h)}{2(x^2+3h^2)}$ is strictly decreasing function for $0 \leq h < 2$ and $x > 0$. Also, using the asymptotic expansion (23), we get

$$\lim_{x \rightarrow \infty} \left(G_h(x) - \frac{h}{x} - \frac{h(2-h)}{2(x^2+3h^2)} \right) = 0,$$

which complete the proof. □

Remark 5. In case of $h = 1$, the inequality (32) will prove the left-hand side of the inequality (10).

Remark 6. Using the inequalities (29), (30) and (32) with the relation (20), we get the following estimations

$$\frac{1}{2x} + \frac{2-h}{4(x^2+3h^2)} < \sum_{n=0}^{\infty} \frac{1}{(x+2n)(x+h+2n)} < \frac{1}{2x} + \frac{2-h}{4x^2}, \quad x > 0; 1 \leq h < 2$$

and

$$\frac{1}{2x} + \frac{2-h}{4(x^2+3h^2)} < \sum_{n=0}^{\infty} \frac{1}{(x+2n)(x+h+2n)} < \frac{1}{hx} + \frac{2-h}{4x^2}, \quad x > 0; 0 < h < 2.$$

5 Sharp double inequality of the function $G_h(x)$.

Theorem 4. For $x > 0$, we have

$$\ln \left(1 + \frac{h}{x+2} \right) + \frac{2h}{x(x+h)} \leq G_h(x) \leq \ln \left(1 + \frac{h}{x} \right) + \frac{2h}{x(x+h)}. \tag{33}$$

Proof. Consider the following function for $x > 0$ and $0 < h < 2$

$$A_{x,h}(t) = \frac{2h}{(x+h+2t)(x+2t)}, \quad t \geq 0$$

which is strictly decreasing since

$$\frac{d}{dt}A_{x,h}(t) = -\frac{4h(2x+h+4t)}{(x+2t)^2(x+h+2t)^2}.$$

The function $A_{x,h}(t)$ is continuous for $t \geq 0$, then its lower Riemann sum $L(A_{x,h}(t), P)$ with respect to the partition $P = \{0, 1, \dots, n\}$ of the interval $[0, n]$, $n \in N$, satisfies

$$L(A_{x,h}(t), P) < \int_0^n A_{x,h}(t)dt.$$

Hence

$$\sum_{i=1}^n A_{x,h}(i) < \ln \left(\frac{(x+h)(x+2n)}{x(x+h+2n)} \right)$$

or

$$\sum_{i=0}^n \frac{2h}{(x+h+2i)(x+2i)} < \ln \left(\frac{(x+h)(x+2n)}{x(x+h+2n)} \right) + \frac{2h}{x(x+h)}.$$

Then as $n \rightarrow \infty$, we get

$$G_h(x) \leq \ln \left(1 + \frac{h}{x} \right) + \frac{2h}{x(x+h)}.$$

Also, the function $A_{x,h}(t+1)$ is continuous for $t \geq -1$, then its upper Riemann sum $U(A_{x,h}(t+1), P)$ with respect to the partition P of the interval $[0, n]$, $n \in N$, satisfies

$$\int_0^n A_{x,h}(t+1)dt < U(A_{x,h}(t+1), P).$$

Hence

$$\ln \left(\frac{(x+h+2)(x+2+2n)}{(x+2)(x+h+2+2n)} \right) < \sum_{i=0}^{n-1} A_{x,h}(i+1) = \sum_{i=1}^n A_{x,h}(i)$$

or

$$\ln \left(\frac{(x+h+2)(x+2+2n)}{(x+2)(x+h+2+2n)} \right) + \frac{2h}{x(x+h)} < \sum_{i=0}^n \frac{2h}{(x+h+2i)(x+2i)}.$$

Then as $n \rightarrow \infty$, we get

$$\ln \left(1 + \frac{h}{x+2} \right) + \frac{2h}{x(x+h)} \leq G_h(x).$$

□

Our next step, will be the investigation of the best bounds of the double inequality (33). To obtain this, we will deduce some auxiliary Lemmas.

Lemma 5.1. *The function*

$$f_h(x) = e^{G_h(x) - \frac{2h}{x(x+h)}} - 1, \quad x > 0; 0 < h < 2$$

satisfies

$$\lim_{x \rightarrow \infty} \left(\frac{h}{f_h(x)} - x \right) = 1, \tag{34}$$

$$\lim_{x \rightarrow 0} \left(\frac{h}{f_h(x)} - x \right) = \frac{h}{e^{\gamma + \frac{2}{h} + \psi(\frac{h}{2})}}, \tag{35}$$

$$\lim_{x \rightarrow \infty} \frac{d}{dx} \left(\frac{h}{f_h(x)} \right) = 1, \tag{36}$$

and

$$\lim_{x \rightarrow 0} \frac{d}{dx} \left(\frac{h}{f_h(x)} \right) = \delta(h), \tag{37}$$

where

$$\delta(h) = \frac{e^{\gamma + \frac{2}{h} + \psi(\frac{h}{2})} (24 + h^2\pi^2 - 6h^2\psi'(\frac{h}{2}))}{12h (e^{\gamma + \frac{2}{h} + \psi(\frac{h}{2})} - 1)^2}.$$

Proof. Using the expansion

$$f_h(x) = \frac{h}{x} - \frac{h}{x^2} + O(x^{-3}), \tag{38}$$

we get

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(\frac{h}{f_h(x)} - x \right) &= \lim_{x \rightarrow \infty} \left(\frac{h}{\frac{h}{x} - \frac{h}{x^2} + O(x^{-3})} - x \right) \\ &= \lim_{x \rightarrow \infty} \left(\frac{\frac{h}{x} - O(x^{-2})}{\frac{h}{x} - \frac{h}{x^2} + O(x^{-3})} \right) = 1. \end{aligned}$$

The expansion

$$G_h(x) - \frac{2h}{x(x+h)} = \gamma + \frac{2}{h} + \psi\left(\frac{h}{2}\right) + \left(-\frac{2}{h^2} - \frac{\pi^2}{12} + \frac{1}{2}\psi'\left(\frac{h}{2}\right)\right)x + O(x^2) \tag{39}$$

gives us that

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{h}{f_h(x)} - x \right) &= \lim_{x \rightarrow 0} \left(\frac{h}{e^{\gamma + \frac{2}{h} + \psi(\frac{h}{2}) + (-\frac{2}{h^2} - \frac{\pi^2}{12} + \frac{1}{2}\psi'(\frac{h}{2}))x + O(x^2)} - 1} - x \right) \\ &= \frac{h}{e^{\gamma + \frac{2}{h} + \psi(\frac{h}{2})} - 1}. \end{aligned}$$

Now using the expansion (38) and the expansion

$$G'_h(x) + \frac{2h(h+2x)}{x^2(x+h)^2} = -\frac{h}{x^2} + O(x^{-3}),$$

we have

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{d}{dx} \left(\frac{h}{f_h(x)} \right) &= \lim_{x \rightarrow \infty} \left(\frac{-h}{\left[\frac{h}{x} - \frac{h}{x^2} + O(x^{-3}) \right]^2} \left[1 + \frac{h}{x} - \frac{h}{x^2} + O(x^{-3}) \right] \left[-\frac{h}{x^2} + O(x^{-3}) \right] \right) \\ &= \lim_{x \rightarrow \infty} \left(\frac{h^2 + O\left(\frac{1}{x}\right)}{h^2 - O\left(\frac{1}{x}\right)} \right) \\ &= 1. \end{aligned}$$

Also, using the expansions

$$e^{G_h(x) - \frac{2h}{x(x+h)}} = e^{\gamma + \frac{2}{h} + \psi\left(\frac{h}{2}\right)} + O(x)$$

and

$$G'_h(x) + \frac{2h(h+2x)}{x^2(x+h)^2} = \frac{-2}{h^2} - \frac{\pi^2}{12} + \frac{1}{2} + \psi'\left(\frac{h}{2}\right) + O(x),$$

we obtain

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{d}{dx} \left(\frac{h}{f_h(x)} \right) &= \lim_{x \rightarrow 0} \left(\frac{-h}{\left[e^{\gamma + \frac{2}{h} + \psi\left(\frac{h}{2}\right)} - 1 + O(x) \right]^2} \left[e^{\gamma + \frac{2}{h} + \psi\left(\frac{h}{2}\right)} + O(x) \right] \right. \\ &\quad \left. \left[\frac{-2}{h^2} - \frac{\pi^2}{12} + \frac{1}{2} + \psi'\left(\frac{h}{2}\right) + O(x) \right] \right) \\ &= \delta(h). \end{aligned}$$

□

Lemma 5.2.

$$\delta(h) = \frac{e^{\gamma + \frac{2}{h} + \psi\left(\frac{h}{2}\right)} (24 + h^2\pi^2 - 6h^2\psi'\left(\frac{h}{2}\right))}{12h \left(e^{\gamma + \frac{2}{h} + \psi\left(\frac{h}{2}\right)} - 1 \right)^2} < 1, \quad 0 < h < 2. \quad (40)$$

Proof. Consider the two functions

$$P_1(h) = \gamma + \frac{2}{h} + \psi\left(\frac{h}{2}\right)$$

and

$$P_2(h) = 24 + h^2\pi^2 - 6h^2\psi'\left(\frac{h}{2}\right).$$

Using the integral representation (15), we have

$$P'_1(h) = \frac{-2}{h^2} + \frac{1}{2}\psi'\left(\frac{h}{2}\right) = \int_0^\infty \frac{2t}{e^{2t} - 1} e^{-xt} dt > 0$$

and then the function $P_1(h)$ is increasing and positive since

$$\lim_{h \rightarrow 0} P_1(h) = 0,$$

where

$$\psi\left(\frac{h}{2}\right) = -\gamma - \frac{2}{h} + \frac{\pi^2}{12}h + O(h^2).$$

Also,

$$P_2'(h) = hP_3(h),$$

where

$$P_3(h) = 2\pi^2 - 12\psi'\left(\frac{h}{2}\right) - 3h\psi''\left(\frac{h}{2}\right).$$

Using the integral representation (15), we obtain

$$P_3(h) = 2\pi^2 - \int_0^\infty \frac{24e^{2t}(e^{2t} - 1 + t)}{(e^{2t} - 1)^2} e^{-xt} dt$$

and then

$$P_3'(h) = \int_0^\infty \frac{24e^{2t}(e^{2t} - 1 + t)}{(e^{2t} - 1)^2} te^{-xt} dt > 0.$$

Hence the function $P_3(h)$ is strictly increasing and using the expansions

$$\psi'\left(\frac{h}{2}\right) = \frac{4}{h^2} + \frac{\pi^2}{6} + O(h)$$

and

$$\psi''\left(\frac{h}{2}\right) = \frac{-16}{h^3} + \psi'(1) + O(h),$$

we get

$$\lim_{h \rightarrow 0} P_3(h) = 0.$$

Then $P_3(h) > 0$ and hence the function $P_2(h)$ is strictly increasing and positive since

$$\lim_{h \rightarrow 0} P_2(h) = \lim_{h \rightarrow 0} \left(24 + h^2\pi^2 - 6h^2 \left(\frac{4}{h^2} + \frac{\pi^2}{6} + O(h) \right) \right) = 0.$$

Now consider the two functions

$$P_4(h) = 24(e - 1)^2 e^{P_1(h)} P_2(h)$$

and

$$P_5(h) = 288eh(e^{P_1(h)} - 1)^2.$$

Using the properties of the two functions $P_1(h)$ and $P_2(h)$, we conclude that $P_4(h)$ and $P_5(h)$ are strictly increasing positive functions and

$$P_4(0) = P_5(0), \quad P_4(2) = P_5(2) = 576e(e - 1)^2,$$

$$P_4(1) = 12(e - 1)^2(\pi^2 - 12)e^2 < P_5(1) = 18e(e^2 - 4)^2,$$

where $\psi\left(\frac{1}{2}\right) = -\gamma - \ln 4$ (see [3]). Then we get

$$P_4(h) < P_5(h), \quad 0 < h < 2$$

which complete the proof. □

Now, we are in position to prove the following result.

Theorem 5. For all $x \in (0, \infty)$ and a fixed $h \in (0, 2)$, we have

$$\ln \left(1 + \frac{h}{x + \beta} \right) < G_h(x) - \frac{2h}{x(x+h)} < \ln \left(1 + \frac{h}{x + \alpha} \right), \tag{41}$$

where the constants $\alpha = 1$ and $\beta = \frac{h}{e^{\gamma + \frac{2}{h} + \psi(\frac{h}{2})} - 1}$ are the best possible.

Proof. Using the inequality (33), we have

$$0 < \frac{h}{e^{G_h(x) - \frac{2h}{x(x+h)}} - 1} - x < 2.$$

Let

$$M_h(x) = \frac{h}{f_h(x)} - x, \quad x > 0; \quad 0 < h < 2.$$

Using the relation

$$f'_h(x) = e^{G_h(x) - \frac{2h}{x(x+h)}} \left(G'_h(x) + \frac{2h(h+2x)}{x^2(x+h)^2} \right)$$

and the integral representation

$$G_h(x) = 2 \int_0^\infty \frac{1 - e^{-ht}}{1 - e^{-2t}} e^{-xt} dt, \quad x > 0$$

we have

$$G'_h(x) + \frac{2h(h+2x)}{x^2(x+h)^2} = - \int_0^\infty \frac{2e^{-ht}(e^{ht} - 1)t}{e^{2t} - 1} e^{-xt} dt < 0$$

then $f_h(x)$ is strictly decreasing positive function. Hence $\frac{h}{f_h(x)}$ is strictly increasing positive function. Also, $f_h(x)$ is strictly convex function since

$$f''_h(x) = e^{G_h(x) - \frac{2h}{x(x+h)}} \left[\left(G'_h(x) + \frac{2h(h+2x)}{x^2(x+h)^2} \right)^2 + \int_0^\infty \frac{2e^{-ht}(e^{ht} - 1)t^2}{e^{2t} - 1} e^{-xt} dt \right] > 0.$$

From the relations (36) and (37) with the inequality (40), we conclude that the function $\frac{h}{f_h(x)}$ is convex and $\frac{d}{dx} \left(\frac{h}{f_h(x)} \right)$ is increasing function. Thus we get

$$\frac{d}{dx} \left(\frac{h}{f_h(x)} \right) < \lim_{x \rightarrow \infty} \frac{d}{dx} \left(\frac{h}{f_h(x)} \right) = 1, \quad x > 0.$$

Then $M_h(x)$ is strictly decreasing function for all $x > 0$, where

$$\frac{d}{dx} M_h(x) = \frac{d}{dx} \frac{h}{f_h(x)} - 1 < 0.$$

Hence $\lim_{x \rightarrow \infty} M_h(x) < M_h(x) < \lim_{x \rightarrow 0^+} M_h(x)$ and using the limits (34) and (35), we have

$$1 < M_h(x) < \frac{h}{e^{\gamma + \frac{2}{h} + \psi(\frac{h}{2})}} \tag{42}$$

with best bounds. □

Remark 7. Using the double inequality (41) with the relation (19) or the relation (20), we obtain the following estimation

$$\begin{aligned} \left(\frac{x+h}{h}\right) \ln\left(1+\frac{h}{x+\beta}\right) + \frac{2}{x} &< {}_3F_2\left(1, 1, \frac{h+2}{2}; 2, \frac{x+h+2}{2}; 1\right) \\ &< \left(\frac{x+h}{h}\right) \ln\left(1+\frac{h}{x+\alpha}\right) + \frac{2}{x} \quad x > 0; 0 < h < 2 \end{aligned}$$

or

$$\begin{aligned} \frac{1}{2h} \ln\left(1+\frac{h}{x+\beta}\right) + \frac{1}{x(x+h)} &< \sum_{k=0}^{\infty} \frac{1}{(x+2k)(x+h+2k)} \\ &< \frac{1}{2h} \ln\left(1+\frac{h}{x+\alpha}\right) + \frac{1}{x(x+h)}, \quad x > 0; 0 < h < 2 \end{aligned}$$

with sharp bounds $\alpha = 1$ and $\beta = \frac{h}{e^{\gamma+\frac{2}{h}+\psi(\frac{h}{2})}-1}$.

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Interval-valued fuzzy quasi-metric spaces ^{*}

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Abstract — Under the context of quasi-metric, promoted the concept of interval-valued fuzzy metric space, the main results are as follows : (1) topology induced by quasi-metric is consistent with which induced via a standard interval-valued fuzzy quasi-metric; (2) proved that every quasi-metrizable topological space admits a compatible interval-valued fuzzy quasi-metric. On the contrary, topology generated by interval-valued fuzzy quasi-metric is quasi-metrizable; (3) discussed some properties of interval-valued fuzzy quasi-metric space which is bicompletion. proved that if an interval-valued fuzzy quasi-metric space has bicompletion, then it is unique up to isometry. In addition, we define a fuzzy contraction mapping of interval-valued fuzzy metric space, promote the Banach and Edelstein fixed point theorem to interval fuzzy metric space.

Keywords — Interval-valued fuzzy quasi-metric spaces; Quasi-metric; Quasi-uniformity; Bicompletion; Isometry

AMS classification 54A40

1 Introduction

In 1975, Kramosil and Michalek introduced the concept of fuzzy metric spaces which is closely related with probabilistic metric spaces in [1], it is also known as generalized Menger space. Currently, many mathematical workers use the concept of fuzzy sets giving different fuzzy metric space ideas [1-5]. Among them, A.George and P.Veeramani improved the concept of fuzzy metric space defined by Kramosil and Michalek in [2, 6], proposed the concept of continuous t-norm, and they used fuzzy sets to represent the uncertainty of the distance between two points in a fuzzy metric space, thus obtained a stronger form of fuzzy metric which has a Hausdorff topology. It is clearly that

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the concept of fuzzy metric space is different with the paper [1, 3, 4, 5] defines. In particular, every metric induces a fuzzy metric space which is under A.George and P.Veeramani significance. Instead, each fuzzy metric space in A.George and P.Veeramani significance generate a metrizable topology [7, 8]. As early as in 1975, Zadeh has given the concept of interval-valued fuzzy sets [9]. It characterizes fuzzy set by interval-valued membership functions, which is another generalization of fuzzy set. In 2012, based on the concept of A.George and P.Veeramani's fuzzy metric space, Shen [10] given a conception of continuous interval value t- norm and interval-valued fuzzy metric space, and also discussed some topological properties of such metric spaces. In addition, it is known that quasi-metric space constitutes a very effective tool in discuss and resolve topology algebraic, approximation theory, theory computational science and other aspects [11,18]. In the context of quasi-metric, this article generalized interval-valued fuzzy metric that the paper [10] defined, proposed the concept of interval-valued fuzzy quasi-metric, as a basis for the study. Structure of the article is divided into five parts. The first part is to introduce the background knowledge; the second part is prior knowledge; the third section discuss the quasi-metrizable of interval-valued fuzzy metric space; part IV discuss the bicompletion of interval-valued fuzzy metric space; section V discuss the fixed point theorem of interval-valued fuzzy metric space.

2 Preliminaries

Interval analysis (see <http://www.cs.utep.edu/interval-comp/main.html>) led by interval numbers is an area of active research in mathematics, numerical analysis and computer science began in the late 1950s. An interval number is a point $\langle a^-, a^+ \rangle$ in the 2-dimensional Euclidean space R^2 which satisfies $a^- \leq a^+$. The set of all interval numbers is denoted by $\mathbb{I}(R)$ ¹. For any $\langle a^-, a^+ \rangle, \langle b^-, b^+ \rangle \in \mathbb{I}(R)$ and each nonnegative real number r , define $\langle a^-, a^+ \rangle \oplus \langle b^-, b^+ \rangle = \langle a^- + b^-, a^+ + b^+ \rangle$, $\langle a^-, a^+ \rangle \ominus \langle b^-, b^+ \rangle = \langle \min\{a^- - b^-, a^+ - b^+\}, \max\{a^- - b^-, a^+ - b^+\} \rangle$, and $r\langle a^-, a^+ \rangle = \langle ra^-, ra^+ \rangle$. We write $\langle a^-, a^+ \rangle = \mathbf{a}$ when $a^- = a^+ = a$, and $\mathbb{I}(I) = \{\langle a^-, a^+ \rangle \in \mathbb{I}(R) \mid a^-, a^+ \in I\}$ (where $I = [0, 1]$ is the ordinary closed unit interval). In this paper we only involve the notion of interval-valued fuzzy set used generally to dispose uncertainty (for an overview of

¹ Unless confusion arise, we identify a real number $a \in R$ with an interval number \mathbf{a} , and identify a closed interval $[a, b]$ of R and a point $\langle a, b \rangle$ in R^2 since there exists a nature one-to-one correspondence between the set of all closed intervals of R and $\mathbb{I}(R)$; we will also use $\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$ to denote interval numbers.

interval analysis, its relationship to fuzzy set theory, and possible areas of further fruitful research, please see [10,12,13,14,15,17,19]).

Definition 2.1 An interval-valued fuzzy set (resp., fuzzy set) on a set X is exactly a mapping $A : X \longrightarrow \mathbb{I}(I)$ (resp., $A : X \longrightarrow I$). The set of all interval-valued fuzzy sets on X is denoted by $\text{IVF}(X)$.

Each $A \in \text{IVF}(X)$ induces two fuzzy sets $A^- : X \longrightarrow I$ and $A^+ : X \longrightarrow I$ whose values are determined by $A(x) = \langle A^-(x), A^+(x) \rangle$ ($\forall x \in X$). For $A, B \in \text{IVF}(X)$, the infimum $A \wedge B \in \text{IVF}(X)$ of A and B in $\text{IVF}(X)$ is given by

$$(A \wedge B)(x) = \langle \min\{A^-(x), B^-(x)\}, \min\{A^+(x), B^+(x)\} \rangle \quad (\forall x \in X),$$

the supremum $A \vee B \in \text{IVF}(X)$ of A and B in $\text{IVF}(X)$ is given by

$$(A \vee B)(x) = \langle \max\{A^-(x), B^-(x)\}, \max\{A^+(x), B^+(x)\} \rangle \quad (\forall x \in X),$$

and the complement $A' \in \text{IVF}(X)$ of A in $\text{IVF}(X)$ is given by

$$A'(x) = \langle 1 - A^+(x), 1 - A^-(x) \rangle \quad (\forall x \in X).$$

Definition 2.2^[10] An interval-valued t-norm is a binary operation $* : \mathbb{I}(I) \times \mathbb{I}(I) \longrightarrow \mathbb{I}(I)$ which satisfies the following conditions:

- (i) $\mathbf{a} * \mathbf{b} = \mathbf{b} * \mathbf{a}$ ($\forall \mathbf{a}, \mathbf{b} \in \mathbb{I}(I)$).
- (ii) $\mathbf{a} * (\mathbf{b} * \mathbf{c}) = (\mathbf{a} * \mathbf{b}) * \mathbf{c}$ ($\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{I}(I)$).
- (iii) $\mathbf{a} * \mathbf{b} \leq \mathbf{a} * \mathbf{c}$ whenever $\mathbf{b} \leq \mathbf{c}$.
- (iv) $\mathbf{a} * \mathbf{1} = \mathbf{a}$ ($\forall \mathbf{a} \in \mathbb{I}(I)$); $\langle a^-, a^+ \rangle * \langle 0, 1 \rangle = \langle 0, a^+ \rangle$ ($\forall \langle a^-, a^+ \rangle \in \mathbb{I}(I)$).

Proposition 2.3^[10] An interval-valued t-norm $*$ has the following properties:

- (i) $\mathbf{0} * \mathbf{a} = \mathbf{a} * \mathbf{0} = \mathbf{0}$; $\langle 0, 1 \rangle * \langle a^-, a^+ \rangle = \langle 0, a^+ \rangle$; $\mathbf{1} * \mathbf{a} = \mathbf{a}$.
- (ii) $\mathbf{a} * \mathbf{b} \leq \mathbf{c} * \mathbf{d}$ whenever $\mathbf{a} \leq \mathbf{c}$ and $\mathbf{b} \leq \mathbf{d}$.
- (iii) $\mathbf{0} = \mathbf{a} * \mathbf{0} \leq \mathbf{a} * \mathbf{b} \leq \mathbf{a} * \mathbf{1} = \mathbf{a}$, $\mathbf{0} = \mathbf{a} * \mathbf{0} \leq \mathbf{0} * \mathbf{b} \leq \mathbf{1} * \mathbf{b} = \mathbf{b}$, $\mathbf{0} = \mathbf{a} * \mathbf{0} \leq \mathbf{a} * \mathbf{b} \leq \mathbf{a} \wedge \mathbf{b}$.

Definition 2.4^[10] (1) Let $\mathbf{a} = \langle a^-, a^+ \rangle$ and $\mathbf{a}_n = \langle a_n^-, a_n^+ \rangle$ are in $\mathbb{I}(I)$ for each $n \in N$ (the natural number set). $\{\mathbf{a}_n\}_{n \in N}$ (briefly, $\{\mathbf{a}_n\}$) is said to be convergent to \mathbf{a} (write as $\lim_{n \rightarrow \infty} \mathbf{a}_n = \mathbf{a}$) if $\lim_{n \rightarrow \infty} a_n^- = a^-$ and $\lim_{n \rightarrow \infty} a_n^+ = a^+$.

(2) An interval-valued t-norm $*$ is said to be continuous in its first component if $\lim_{n \rightarrow \infty} (\mathbf{a}_n * \mathbf{b}) = (\lim_{n \rightarrow \infty} \mathbf{a}_n) * \mathbf{b} = \mathbf{a} * \mathbf{b}$ for all $\mathbf{b} \in \mathbb{I}(I)$ whenever $\lim_{n \rightarrow \infty} \mathbf{a}_n = \mathbf{a}$ ($\{\mathbf{a}_n\}_{n \in N} \subseteq \mathbb{I}(I)$, $\mathbf{a} \in \mathbb{I}(I)$).

Proposition 2.5^[10] The followings hold for a continuous interval-valued t-norm $*$:

- (i) For any $a_1, a_2 \in \mathbb{I}(I)$ with $a_1 > a_2$, there exists a $a_3 \in \mathbb{I}(I)$ such that $a_1 * a_3 \geq a_2$.
- (ii) For any $a_4 \in \mathbb{I}(I)$, there exists a $a_5 \in \mathbb{I}(I)$ such that $a_5 * a_5 \geq a_4$.

Definition 2.6^[10] An interval-valued fuzzy metric space is a triple $(X, M, *)$, where $*$ is a continuous t-norm and M (called an interval-valued fuzzy metric on X) is an interval-valued fuzzy set on $X^2 \times (0, \infty)$ having the following properties ($x, y, z \in X$):

- (IV1) $M(x, y, t) > \mathbf{0}$.
- (IV2) $M(x, y, t) = \mathbf{1}$ if and only if $x = y$.
- (IV3) $M(x, y, t) = M(y, x, t)$.
- (IV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $t, s > 0$.

(IV5) $M(x, y, \cdot) : (0, \infty) \longrightarrow \mathbb{I}(I) - \{\mathbf{0}\}$ is continuous, i.e. both $M^-(x, y, \cdot) = p_1 \circ M(x, y, \cdot) : (0, \infty) \longrightarrow (0, 1]$ and $M^+(x, y, \cdot) = p_2 \circ M(x, y, \cdot) : (0, \infty) \longrightarrow (0, 1]$ are continuous, where $p_i : I^2 \longrightarrow I$ is the projective mapping ($i = 1, 2$), and $M^-(x, y, t)$ and $M^+(x, y, t)$ denote the lower nearness and upper nearness degree between x and y with respect to t , respectively.

- (IV6) $\lim_{t \rightarrow \infty} M(x, y, t) = \mathbf{1}$.

Remark 2.7^[10] (1) An interval-valued fuzzy metric space $(X, M, *)$ will degenerate into an ordinary fuzzy metric space if $M^-(x, y, t) = M^+(x, y, t)$ for all $t > 0$.

(2) Let $(X, M, *)$ be an interval-valued fuzzy metric space.

(i) The set $B(x, r, t) = \{y \in X \mid M(x, y, t) > \mathbf{1} \ominus r\}$ is called a open ball with center $x \in X$, where $t > 0$ and $r \in \mathbb{I}(I) - \{\mathbf{0}, \mathbf{1}\}$.

(ii) $\mathcal{J}_M = \{A \subseteq X \mid \forall x \in A, \exists t_x > 0, \exists r_x \in \mathbb{I}(I) - \{\mathbf{0}, \mathbf{1}\}, B(x, r_x, t_x) \subseteq A\}$ is a topology on X , and $\{B(x, r, t) \mid x \in X, r \in \mathbb{I}(I) - \{\mathbf{0}, \mathbf{1}\}, t > 0\}$ is a base of \mathcal{J}_M .

(iii) For each $(x, y, t, r) \in X^2 \times (0, +\infty) \times (\mathbb{I}(I) - \{\mathbf{0}, \mathbf{1}\})$, there exists a $t_0 \in (0, t)$ such that $M(x, y, t_0) > \mathbf{1} \ominus r$ whenever $M(x, y, t) > \mathbf{1} \ominus r$.

(iv) For a sequence $\{x_n\} \subseteq X$, $\lim_{n \rightarrow \infty} x_n = x$ in (X, \mathcal{J}_M) if and only if $\lim_{n \rightarrow \infty} M(x, x_n, t) = \mathbf{1}$.

(v) A sequence $\{x_n\}$ in $(X, M, *)$ is called a Cauchy sequence if for all $\varepsilon > \mathbf{0}$ and $t > 0$, there exists an $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > \mathbf{1} \ominus \varepsilon$ for all $n, m \geq n_0$.

(vi) $(X, M, *)$ is said to be complete if every Cauchy sequence in it is convergent.

Remark 2.8^[16] (1) A quasi-pseudo-metric space is a pair (X, d) , where X is a set and d (called a quasi-pseudo-metric on X) is a mapping $d : X \longrightarrow [0, +\infty)$ satisfying the following conditions:

- (i) $d(x, x) = 0 \quad (\forall x \in X)$;

(ii) $d(x, z) \leq d(x, y) + d(y, z) \quad (\forall x, y, z \in X)$.

(2) A quasi-metric space is a pair (X, d) , where X is a set and d (called a quasi-metric on X) is a quasi-pseudo-metric on X satisfying the following condition:

(iii) $x = y \iff d(x, y) = d(y, x) = 0$.

Each quasi-metric d on X induces a T_0 topology \mathcal{J}_d on X which has as a base the family of open balls $\{B_d(x, r) \mid x \in X, r > 0\}$, where $B_d(x, r) = \{y \in X \mid d(x, y) < r\}$ ($x \in X, r > 0$).

(3) A topological space (X, \mathcal{J}) is said to be quasi-metrizable if there is a quasi-metric d on X such that $\mathcal{J}_d = \mathcal{J}$ (in this case, we say that d is compatible with \mathcal{J} , and that \mathcal{J} is a quasi-metrizable topology on X), where $\mathcal{J}_d = \langle\langle \{B_d(x, r) \mid x \in X, r > 0\} \rangle\rangle$ (i.e. \mathcal{J}_d is the smallest topology on X containing $\{B_d(x, r) \mid x \in X, r > 0\}$).

(4) If d is a quasi-pseudo-metric on X , then d^{-1} (called the conjugate of d which is defined by $d^{-1}(x, y) = d(y, x) \quad (\forall x, y \in X)$) and d^s (defined by $d^s(x, y) = \max\{d(x, y), d^{-1}(x, y)\} \quad (\forall x, y \in X)$) are also quasi-pseudo-metrics on X . A quasi-metric space (X, d) is said to be bicomplete if (X, d^s) is a complete metric space. In this case we say that d is a bicomplete quasi-metric on X .

3 Quasi-metrizable of interval-valued fuzzy quasi-metric space

Definition 3.1 Let X be a set, $*$ a continuous t-norm, and M an interval-valued fuzzy set on $X^2 \times (0, \infty)$ having the following properties:

(IV1) $M(x, y, t) > \mathbf{0} \quad (x, y \in X, t \in (0, \infty))$.

(IV2) $M(x, x, t) = \mathbf{1} \quad (x \in X, t \in (0, \infty))$.

(IV3) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s) \quad (x, y, z \in X, s, t \in (0, \infty))$.

(IV4) $M(x, y, \cdot) : (0, \infty) \longrightarrow \mathbb{I}(I) - \{\mathbf{0}\}$ is continuous ($x, y \in X$).

(IV5) $\lim_{t \rightarrow \infty} M(x, y, t) = \mathbf{1} \quad (x, y \in X)$.

(1) Such a triple $(X, M, *)$ is called an interval-valued fuzzy quasi-pseudo-metric space, and $(M, *)$ is called an interval-valued fuzzy quasi-pseudo-metric on X .

(2) This triple $(X, M, *)$ is called an interval-valued fuzzy quasi-metric space (and $(M, *)$ is called an interval-valued fuzzy quasi-metric on X) if M satisfies the following condition:

(IV1') $x = y$ if and only if $M(x, y, t) = M(y, x, t) = \mathbf{1}$ for all $t > 0$.

(3) This triple $(X, M, *)$ is called a T_1 interval-valued fuzzy quasi-metric space (and $(M, *)$ is called a T_1 interval-valued fuzzy quasi-metric on X) if M satisfies the following condition:

$$(IV2') \quad x = y \text{ if and only if } M(x, y, t) = \mathbf{1} \text{ for all } t > 0,$$

(4) This triple $(X, M, *)$ is called an interval-valued fuzzy pseudo-metric space (and $(M, *)$ is called an interval-valued fuzzy pseudo-metric on X) if M satisfies the following condition:

$$(IV6) \quad M(x, y, t) = M(y, x, t) \text{ for all } t > 0.$$

Definition 3.1' An interval-valued fuzzy (pseudo-) metric on X is a interval-valued fuzzy quasi-(pseudo-)metric $(M, *)$ on X such that for each $x, y \in X$:

$$(IV6) \quad M(x, y, t) = M(y, x, t) \text{ for all } t > 0.$$

Remark 3.2 (1) There are many interval-valued fuzzy quasi-metric spaces which are not interval-valued fuzzy metric space.

(2) There are many interval-valued fuzzy pseudo-metric spaces which are not interval-valued fuzzy metric space.

Interval-valued fuzzy quasi-pseudo metric space is a weakened form of the interval value fuzzy metric Spaces, which is a generalization of the interval-valued fuzzy metric Spaces, will have a greater scope in the application.

Remark 3.4 It is clear that every interval-valued fuzzy metric is a T_1 interval-valued fuzzy quasi-metric; every T_1 interval-valued fuzzy quasi-metric is an interval-valued fuzzy quasi-metric, and every interval-valued fuzzy quasi-metric is an interval-valued fuzzy quasi-pseudo-metric.

Definition 3.5 An interval-valued fuzzy quasi-(pseudo-)metric space is a triple $(X, M, *)$ such that X is a (nonempty) set and $(M, *)$ is an interval-valued fuzzy quasi-(pseudo-)metric on X .

The notions of a T_1 interval-valued fuzzy quasi-metric space and of a interval-valued fuzzy metric space are defined in the obvious manner.

Remark 3.6 If $(M, *)$ is an interval-valued fuzzy quasi-(pseudo-)metric on a set X , it is immediate to show that $(M^{-1}, *)$ is also an interval-valued fuzzy quasi-(pseudo-)metric on X , where M^{-1} is the interval-valued fuzzy set in $X \times X \times (0, +\infty)$ defined by $M^{-1}(x, y, t) = M(y, x, t)$. Moreover, if we denote M^i the interval-valued fuzzy set in $X \times X \times (0, +\infty)$ given by $M^i(x, y, t) = \min\{M(x, y, t), M^{-1}(x, y, t)\}$, then $(M^i, *)$ is, clearly, an interval-valued fuzzy (pseudo-)metric on X .

Thus, conditions (IV1') of the definition 3.2 above is equivalent to the following:

$M(x, x, t) = \mathbf{1}$ for all $x \in X$ and $t > 0$, and $M^i(x, y, t) < \mathbf{1}$ for all $x \neq y$ and $t > 0$.

Definition 3.7 Let $(X, M, *)$ be an interval-valued fuzzy quasi-pseudo-metric space. We define open ball $B_M(x, \mathfrak{r}, t) = \{y \in X | M(x, y, t) > \mathbf{1} - \mathfrak{r}\}$ for $t > 0$ with center $x \in X$ and the interval number \mathfrak{r} , $\mathbf{0} < \mathfrak{r} < \mathbf{1}$, $t > 0$.

Similar to the [10], we get the following conclusion:

Proposition 3.8 Let $(X, M, *)$ be an interval-valued fuzzy quasi-pseudo-metric space. Then every open ball $B_M(x, \mathfrak{r}, t)$ is an open set.

Proof. Let $B_M(x, \mathfrak{r}, t)$ is an open ball. For any $y \in B_M(x, \mathfrak{r}, t)$, we know that $M(x, y, t) > \mathbf{1} - \mathfrak{r}$. Therefore, there exists a $t_0 \in (0, t)$ such that $M(x, y, t_0) > \mathbf{1} - \mathfrak{r}$. Set $\mathfrak{r}_0 = M(x, y, t_0)$. Since $\mathfrak{r}_0 > \mathbf{1} - \mathfrak{r}$, there exists a $\mathfrak{s} \in \mathbb{I}(I)$ such that $\mathfrak{r}_0 > \mathbf{1} - \mathfrak{s} > \mathbf{1} - \mathfrak{r}$. Now for given \mathfrak{r}_0 and \mathfrak{s} with $\mathfrak{r}_0 > \mathbf{1} - \mathfrak{s}$, there exists a $\mathfrak{r}_1 \in \mathbb{I}(I)$ such that $\mathfrak{r}_0 *_I \mathfrak{r}_1 \geq \mathbf{1} - \mathfrak{s}$. Consider the open ball $B_M(y, \mathbf{1} - \mathfrak{r}_1, t - t_0)$, we will obtain that $B_M(y, \mathbf{1} - \mathfrak{r}_1, t - t_0) \subset B_M(x, \mathfrak{r}, t)$. In fact, for every $z \in B_M(y, \mathbf{1} - \mathfrak{r}_1, t - t_0)$, we have $M(y, z, t - t_0) > \mathfrak{r}_1$. Therefore,

$$M(x, z, t) \geq M(x, y, t_0) *_I M(y, z, t - t_0) \geq \mathfrak{r}_0 *_I \mathfrak{r}_1 \geq \mathbf{1} - \mathfrak{s} > \mathbf{1} - \mathfrak{r}$$

Thus $x \in B_M(x, \mathfrak{r}, t)$, and hence $B_M(y, \mathbf{1} - \mathfrak{r}_1, t - t_0) \subset B_M(x, \mathfrak{r}, t)$.

Similar to the [10], we get the following conclusion:

Theorem 3.9 Let $(X, M, *)$ be an interval-valued fuzzy quasi-pseudo-metric space. Define

$$\tau_M = \{A \subset X | \forall x \in A, \exists \mathfrak{r} \in \mathbb{I}(I) - \{\mathbf{0}, \mathbf{1}\}, \text{ and } t > 0 \text{ such that } B_M(x, \mathfrak{r}, t) \subset A\}$$

Then τ_M is a topology on X .

Proposition 3.10 A sequence $\{x_n\}_n$ in an interval-valued fuzzy quasi-pseudo-metric space $(X, M, *)$ is said to be a Cauchy sequence if and only if $\lim_n M(x_{n+p}, x_n, t) = \mathbf{1}$, for all $p > 0, t > 0$.

A sequence $\{x_n\}_n$ in an interval-valued fuzzy quasi-pseudo-metric space $(X, M, *)$ is converging to x in X , denoted by $x_n \longrightarrow x$, if and only if $\lim_n M(x, x_n, t) = \mathbf{1}$ for all $t > 0$.

An interval-valued fuzzy quasi-pseudo-metric space $(X, M, *)$ is said to be complete if and only if every Cauchy sequence is convergent.

Proposition 3.11 Let $(X, M, *)$ be an interval-valued fuzzy quasi-pseudo-metric space. Then, for each $x, y \in X$ the function $M(x, y, -)$ is nondecreasing.

Proof. Let $x, y \in X$ and $0 \leq t < s$. Then

$$M(x, y, s) \geq M(x, x, s - t) * M(x, y, t) = \mathbf{1} * M(x, y, t) = M(x, y, t).$$

Proposition 3.12 Let $(X, M, *)$ be an interval-valued fuzzy quasi-metric space. Then, for each $t > 0$ the function

$$M(-, -, t) : (X \times X, \tau_{M^i} \times \tau_{M^i}) \longrightarrow \mathbb{I}(I) - \{\mathbf{0}\}$$

is continuous.

Proof. Fix $t > 0$. Let $x, y \in X$ and let $(x'_n, y'_n)_n$ be a sequence in $X \times X$ that converges to (x, y) with respect to $\tau_{M^i} \times \tau_{M^i}$. Then, it will be sufficient to show that $M(x, y, t) = \lim_n M(x_n, y_n, t)$ for some subsequence $(x_n, y_n)_n$ of $(x'_n, y'_n)_n$.

Indeed, since $(M(x'_n, y'_n, t))_n$ is sequence in $\mathbb{I}(I) - \{\mathbf{0}\}$, there is a subsequence $(x_n, y_n)_n$ of $(x'_n, y'_n)_n$ such that the sequence $(M(x_n, y_n, t))_n$ converges to some ε of $\mathbb{I}(I)$. Fix $\delta > 0$ such that $2\delta < t$. Then

$$M(x_n, y_n, t) \geq M(x_n, x, \delta) * M(x, y, t - 2\delta) * M(y, y_n, \delta),$$

and

$$M(x, y, t + 2\delta) \geq M(x, x_n, \delta) * M(x_n, y_n, t) * M(y_n, y, \delta).$$

Since $\lim_n M^i(x, x_n, \delta) = \lim_n M^i(y, y_n, \delta) = \mathbf{1}$, we deduce that

$$M(x, y, t + 2\delta) \geq \lim_n M(x_n, y_n, t) \geq M(x, y, t - 2\delta).$$

Finally, it follows from the continuity of $M(x, y, -)$ that $M(x, y, t) = \lim_n M(x_n, y_n, t)$.

This completes the proof.

Definition 3.13 Let (X, d) be a quasi-metric space. Define the interval-valued t-norm

$$a * b = \langle a^- \wedge b^-, a^+ \wedge b^+ \rangle$$

and interval-valued fuzzy quasi-metric

$$M_d(x, y, t) = \langle M^-(x, y, t), M^+(x, y, t) \rangle = \left\langle \frac{t}{t + ld(x, y)}, \frac{t}{t + md(x, y)} \right\rangle$$

$$(\forall x, y \in X, \forall t, l, m \in \mathbb{R}^+).$$

Then $(X, M_d, *)$ is an interval-valued fuzzy quasi-metric space called the standard interval-valued fuzzy quasi-metric space and $(M_d, *)$ is the interval-valued fuzzy quasi-metric induced by d .

Furthermore, it is easy to check that $(M_d)^{-1} = M_{d^{-1}}$ and $(M_d)^i = M_{d^s}$, where $d^{-1}(x, y) = d(y, x)$, $d^s(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$.

In the paper [10], it is proved that every metric can induce an interval-valued fuzzy metric. Moreover, if (X, d) is a metric space, then the topology generated by d coincides with the topology τ_{M_d} generated by the interval-valued fuzzy metric. Finally, from proposition 3.8, 3.10, 3.11 and definition 3.13, it follows that the topology τ_d , generated by d , coincides with the topology τ_{M_d} generated by the induced standard interval-valued fuzzy quasi-metric $(M_d, *)$.

Definition 3.14 We say that a topological space (X, τ) admits a compatible interval-valued fuzzy quasi-metric if there is an interval-valued fuzzy quasi-metric $(M, *)$ on X such that $\tau = \tau_M$.

It follows from definition 3.11 that every quasi-metrizable topological space admits a compatible interval-valued fuzzy quasi-metric. Then, conversely, the topology generated by an interval-valued fuzzy quasi-metric space is quasi-metrizable.

Lemma 3.15 Let $(X, M_d, *)$ be an interval-valued fuzzy quasi-metric space. Then $\{U_n : n = 2, 3, \dots\}$ is a base for a quasi-uniformity \mathcal{U}_M on X compatible with τ_M , where $U_n = \{(x, y) \in X \times X : M(x, y, 1/n) > \langle 1 - 1/n, 1 \rangle\}$, for $n = 2, 3, \dots$.

Moreover the conjugate quasi-uniformity $(u_M)^{-1}$ coincides with $u_{M^{-1}}$ and it is compatible with $\tau_{M^{-1}}$. From definition 3.11, Lemma 3.13 and the well-known result that the topology generated by a quasi-uniformity with a countable base is quasi-pseudo-metrizable, we immediately deduce the following.

Theorem 3.16 For a topological space (X, τ) the following are equivalent .

- (i) (X, τ) is quasi-metrizable.
- (ii) (X, τ) admits a compatible interval-valued fuzzy quasi-metric.

4 Bicomplete interval-valued fuzzy quasi-metric space

Definition 4.1 An interval-valued fuzzy quasi-metric space $(X, M, *)$ is called bicomplete if $(X, M^i, *)$ is a complete interval-valued fuzzy metric space. In this case, we say that $(M, *)$ is a bicomplete interval-valued fuzzy quasi-metric on X .

Proposition 4.2

(1) Let $(X, M, *)$ be a bicomplete interval-valued fuzzy quasi-metric space. Then (X, τ_M) admits a compatible bicomplete quasi-metric space.

(2)) Let (X, d) be a bicomplete quasi-metric space. Then $(X, M_d, *)$ is a bicomplete interval-valued fuzzy quasi-metric space.

Proof. (1) Let d be a quasi-metric on X inducing the quasi-uniformity \mathcal{U}_M . Then d is compatible with τ_M . Now let $(x_n)_n$ be a Cauchy sequence in (X, d^s) . Clearly $(x_n)_n$ is a Cauchy sequence in the interval-valued fuzzy metric space $(X, M^i, *)$. So it converges to a point $y \in X$ with respect to τ_{M^i} . Hence $(x_n)_n$ converges to y with respect to τ_{d^s} . Consequently d is bicomplete.

(2) This part is almost obvious because $(M_d)^i = M_{d^s}$ (see definition 3.11) and thus each Cauchy sequence in $(X, (M_d)^i, *)$ is clearly a Cauchy sequence in (X, d^s) .

Definition 4.3 Let $(X, M, *)$ and (Y, N, \star_I) be two interval-valued fuzzy quasi-metric space. Then

(i) A mapping f from X to Y is called an isometry if for each $x, y \in X$ and each $t > 0$, $M(x, y, t) = N(f(x), f(y), t)$.

(ii) $(X, M, *)$ and (Y, N, \star_I) are called isometric if there is an isometry from X onto Y .

Definition 4.4 Let $(X, M, *)$ be an interval-valued fuzzy quasi-metric space. An interval-valued fuzzy quasi-metric bicompletion of $(X, M, *)$ is a bicomplete interval-valued fuzzy quasi-metric space (Y, N, \star_I) such that $(X, M, *)$ is isometric to a τ_{N^i} -dense subspace of Y .

Lemma 4.5 Let $(X, M, *)$ be an interval-valued fuzzy quasi-metric space and (Y, N, \star_I) a bicomplete interval-valued fuzzy quasi-metric space. If there is a τ_{M^i} -dense subset A of X and an isometry $f : (A, M, *) \longrightarrow (Y, N, \star_I)$, then there exists a unique isometry $F : (X, M, *) \longrightarrow (Y, N, \star_I)$ such that $F|_A = f$.

Proof. It is clear that f is a quasi-uniformly continuous mapping from the quasi-uniform space $(A, \mathcal{U}_M|_{\mathcal{A} \times \mathcal{A}})$ to the quasi-uniform space (Y, \mathcal{U}_N) . By Theorem 3.29 of [16], f has a unique quasi-uniformly continuous extension $F : (X, \mathcal{U}_M) \longrightarrow (Y, \mathcal{U}_N)$. We shall show that actually F is an isometry from $(X, M, *)$ to (Y, N, \star_I) . Indeed, let $x, y \in X$ and $t > 0$. Then, there exist two sequences $(x_n)_n$ and $(y_n)_n$ in A such that $(x_n)_n \longrightarrow x$ and $(y_n)_n \longrightarrow y$ with respect to τ_{M^i} . Thus $F(x_n) \longrightarrow F(x)$ and $F(y_n) \longrightarrow F(y)$ with respect to τ_{N^i} . Choose $\varepsilon \in \mathbb{I}(I) - \{\mathbf{0}, \mathbf{1}\}$ with $0 < \varepsilon < t$. Therefore, there is n_ε such that for $n > n_\varepsilon$,

$$M(x, x_n, \varepsilon/2) > \mathbf{1} - \varepsilon, M(y_n, y, \varepsilon/2) > \mathbf{1} - \varepsilon,$$

$$N(F(x_n), F(x), \varepsilon/2) > \mathbf{1} - \varepsilon, N(F(y_n), F(y), \varepsilon/2) > \mathbf{1} - \varepsilon.$$

Thus

$$\begin{aligned} M(x, y, t) &\geq M(x, x_n, \varepsilon/2) * M(x_n, y_n, t - \varepsilon) * M(y_n, y, \varepsilon/2) \\ &\geq (\mathbf{1} - \varepsilon) * N(F(x_n), F(y_n), t - \varepsilon) * (\mathbf{1} - \varepsilon) \\ &\geq (\mathbf{1} - \varepsilon) * \langle (\mathbf{1} - \varepsilon) \star_I N(F(x), F(y), t - 2\varepsilon) \star_I (\mathbf{1} - \varepsilon) \rangle * (\mathbf{1} - \varepsilon) \end{aligned}$$

By continuity of $*$ and \star_I and by continuity of $N(F(x), F(y), -)$, it follows that $M(x, y, t) \geq N(F(x), F(y), t)$. Similarly we can show that $N(F(x), F(y), t) \geq M(x, y, t)$. Consequently F is an isometry from $(X, M, *)$ to (Y, N, \star_I) .

Theorem 4.6 Suppose that (Y_1, N_1, \star_{I_1}) and (Y_2, N_2, \star_{I_2}) are two interval-valued fuzzy quasi-metric bicompletions of $(X, M, *)$. Then (Y_1, N_1, \star_{I_1}) and (Y_2, N_2, \star_{I_2}) are isometric. Thus, if an interval-valued fuzzy quasi-metric space has an interval-valued fuzzy quasi-metric bicompletion, it is unique in the mean of isometry.

Proof. Since (Y_2, N_2, \star_{I_2}) is an interval-valued fuzzy quasi-metric bicompletion of $(X, M, *)$, there is an isometry f from $(X, M, *)$ onto a dense subspace of (Y_2, N_2, \star_{I_2}) . By Lemma 4.5, f admits a (unique) extension F to (Y_1, N_1, \star_{I_1}) which is also an isometry. So, it remains to see that F is onto. But this fact follows from standard arguments. Indeed, given $y_2 \in Y_2$, there is a sequence $(x_n)_n$ in X such that $F(x_n) \rightarrow y_2$. Since F is an isometry, $(x_n)_n$ is a Cauchy sequence, so it converges to some point $y_1 \in Y_1$. Consequently $F(y_1) = y_2$. The proof is complete.

5 Fixed point theorems in interval-valued fuzzy metric spaces

We will discuss the interval-valued fuzzy metric space fixed point problem in this section. Firstly, the definition of interval-valued fuzzy contraction mapping in interval-valued fuzzy metric space, then a generalization of Banach and Edelstein fixed point theorems is given.

In an interval-valued fuzzy metric space $(X, M, *)$, whenever $M(x, y, t) > \mathbf{1} - r$ for all $x, y \in X, t > 0, r \in \mathbb{I}(I) - \{\mathbf{0}, \mathbf{1}\}$, there exists a t_0 with $0 < t_0 < t$ such that $M(x, y, t_0) > \mathbf{1} - r$.

Definition 5.1 Let $(X, M, *)$ be an interval-valued fuzzy metric space. We will say the mapping $f : X \rightarrow X$ is t -uniformly continuous if for each $\varepsilon \in \mathbb{I}(I)$, there exists a $r \in (I)$ such that $M(x, y, t) > \mathbf{1} - r$ implies $M(f(x), f(y), t) \geq \mathbf{1} - \varepsilon$, for each $x, y \in X$ and $t > 0$.

Clearly if f is t -uniformly continuous it is uniformly continuous for the uniformity generated by M , and then continuous for the topology deduced from M .

Definition 5.2 Let $(X, M, *)$ be an interval-valued fuzzy metric space. We will say the mapping $f : X \rightarrow X$ is fuzzy contractive if there exists a $k \in (0, 1)$ such that for each $x, y \in X$, and $t > 0$,

$$M(x, y, t) * (\mathbf{1} - M(f(x), f(y), t)) \leq k(\mathbf{1} - M(x, y, t)) * M(f(x), f(y), t),$$

where k is called the contractive constant of f .

The above definition is justified by the next Proposition 5.4.

Proposition 5.3 Let $(X, M, *)$ be an interval-valued fuzzy metric space. If $f : X \rightarrow X$ is fuzzy contractive then f is t -uniformly continuous.

Proposition 5.4 Let (X, d) be a metric space. The mapping $f : X \rightarrow X$ is contractive (a contraction) on the metric space (X, d) with contractive constant k if and only if f is fuzzy contractive, with contractive constant k , on the standard interval-valued fuzzy metric space $(X, M_d, *)$, induced by d .

Recall that a sequence (x_n) in a metric space (X, d) is said to be contractive if there exists a $k \in (0, 1)$ such that $d(x_{n+1}, x_{n+2}) \leq kd(x_n, x_{n+1})$, for all $n \in \mathbb{N}$. Now, we give the following definition (compare with Definition 5.2).

Definition 5.5 Let $(X, M, *)$ be an interval-valued fuzzy metric space. We will say that the sequence (x_n) in X is fuzzy contractive if there exists $k \in (0, 1)$ such that for all $t > 0, n \in \mathbb{N}$,

$$M(x_n, x_{n+1}, t) * (\mathbf{1} - M(x_{n+1}, x_{n+2}, t)) \leq k(\mathbf{1} - M(x_n, x_{n+1}, t)) * M(x_{n+1}, x_{n+2}, t).$$

Proposition 5.6 Let $(X, M_d, *)$ be the standard interval-valued fuzzy metric space induced by the metric d on X . The sequence (x_n) in X is contractive in (X, d) if and only if (x_n) is fuzzy contractive in $(X, M_d, *)$.

Research of the fixed point theorem in fuzzy metric space has attracted the attention of many scholars [20,23-29], below we will discuss the fuzzy metric space fixed point theorems in interval-valued fuzzy metric space.

Definition 5.7 A sequence $\{x_n\}$ in an interval-valued fuzzy metric space $(X, M, *)$ is a Cauchy sequence if and only if for each $\varepsilon > \mathbf{0}$ and each $t > 0$ there exists a $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > \mathbf{1} - \varepsilon$ for all $n, m \geq n_0$.

Definition 5.8 An interval-valued fuzzy metric space $(X, M, *)$ in which every Cauchy sequence is convergent is called a complete fuzzy metric space.

Theorem 5.9 A sequence $\{x_n\}$ in an interval-valued fuzzy metric space $(X, M, *)$ converges to x if and only if $M(x, x_n, t) \rightarrow \mathbf{1}$ as $n \rightarrow \infty$.

Result 5.10 The metric space (X, d) is complete if and only if the standard interval-valued fuzzy metric space $(X, M_d, *)$ is complete.

Proof may refer to A. George and P. Veeramani literature [5].

Next, we extend the Banach fixed point theorem to fuzzy contractive mappings of interval-valued fuzzy metric spaces.

Theorem 5.11 (fuzzy Banach contraction theorem) Let $(X, M, *)$ be an interval-valued fuzzy metric space in which fuzzy contractive sequences are Cauchy. Let $T : X \rightarrow X$ be a fuzzy contractive mapping being k the contractive constant. Then T has a unique fixed point.

Proof. Fix $x \in X$. Let $x_n = T^n(x), n \in \mathbb{N}$. We have for $t > 0$,

$$M(x, x_1, t) * (\mathbf{1} - M(T(x), T^2(x), t)) \leq k(\mathbf{1} - M(x, x_1, t)) * M(T(x), T^2(x), t),$$

and by induction, for any $n \in \mathbb{N}$,

$$M(x_n, x_{n+1}, t) * (\mathbf{1} - M(x_{n+1}, x_{n+2}, t)) \leq k(\mathbf{1} - M(x_n, x_{n+1}, t)) * M(x_{n+1}, x_{n+2}, t),$$

Then (x_n) is a fuzzy contractive sequence, so it is a Cauchy sequence and, hence, (x_n) converges to y , for some $y \in X$. We will see y is a fixed point for T . By Theorem 5.9, we have

$$M(y, x_n, t) * (\mathbf{1} - M(T(y), T(x_n), t)) \leq k(\mathbf{1} - M(y, x_n, t)) * M(T(y), T(x_n), t) \longrightarrow \mathbf{0}$$

as $n \longrightarrow \infty$, then $\lim_n M(T(y), T(x_n), t) = \mathbf{1}$ for each $t > 0$, and, therefore, $\lim_n T(x_n) = T(y)$, i.e., $\lim_n x_{n+1} = T(y)$ and then $T(y) = y$.

To show uniqueness, assume $T(z) = z$ for some $z \in X$. Then for $t > 0$ we have

$$\begin{aligned} & M(y, z, t) * (\mathbf{1} - M(y, z, t)) \\ &= M(y, z, t) * (\mathbf{1} - M(T(y), T(z), t)) \\ &\leq k[M(T(y), T(z), t) * (\mathbf{1} - M(y, z, t))] \\ &= k[M(y, z, t) * (\mathbf{1} - M(T(y), T(z), t))] \\ &\leq k^2[M(T(y), T(z), t) * (\mathbf{1} - M(y, z, t))] \\ &\leq \dots \\ &\leq k^n[M(T(y), T(z), t) * (\mathbf{1} - M(y, z, t))] \end{aligned}$$

then $M(y, z, t) * (\mathbf{1} - M(y, z, t)) \rightarrow \mathbf{0}$ as $n \rightarrow \infty$.

Hence, $M(y, z, t) = \mathbf{1}$ and then $y = z$.

Now suppose $(X, M_d, *)$ is a complete standard interval-valued fuzzy metric space induced by the metric d on X . From Result 5.10 (X, d) is complete, then if (x_n) is a fuzzy contractive sequence, by Proposition 5.6 it is contractive in (X, d) , hence convergent.

So, from Theorem 5.11 we have the following corollary, which can be considered as the fuzzy version of the classic Banach contraction theorem on complete interval-valued metric spaces.

Corollary 5.12 Let $(X, M_d, *)$ be a complete standard interval-valued fuzzy metric space and let $T : X \rightarrow X$ a fuzzy contractive mapping. Then T has a unique fixed point.

Definition 5.13^[10] An interval-valued fuzzy metric space $(X, M, *)$ is compact space if X is a compact set.

Lemma 5.14 Let $(X, M_d, *)$ be an interval-valued fuzzy metric space. If $\lim_{n \in \mathbb{N}} x_n = x$ and $\lim_{n \in \mathbb{N}} y_n = y$, then for all $t > 0$ and $0 < \varepsilon < \frac{t}{2}$,

$$M(x, y, t - \varepsilon) \leq \lim_{n \in \mathbb{N}} M(x_n, y_n, t) \leq M(x, y, t + \varepsilon).$$

Proof. By Definition 3.1(IV3),

$$M(x_n, y_n, t) \geq M(x_n, x, \frac{\varepsilon}{2}) * M(x, y, t - \varepsilon) * M(y, y_n, \frac{\varepsilon}{2}).$$

Thus,

$$\lim_{n \in \mathbb{N}} M(x_n, y_n, t) \geq \mathbf{1} * M(x, y, t - \varepsilon) * \mathbf{1} = M(x, y, t - \varepsilon).$$

On the other hand,

$$M(x, y, t + \varepsilon) \geq M(x, y, t - \varepsilon) * M(x_n, y_n, t) * M(y_n, y, \frac{\varepsilon}{2}),$$

hence

$$M(x, y, t + \varepsilon) \geq \lim_{n \in \mathbb{N}} M(x_n, y_n, t).$$

Corollary 5.15 Let $(X, M, *)$ be an interval-valued fuzzy metric space. If $\lim_{n \in \mathbb{N}} x_n = x$ and $\lim_{n \in \mathbb{N}} y_n = y$, then $\lim_{n \in \mathbb{N}} M(x_n, y_n, t) = M(x, y, t)$ for all $t > 0$.

Theorem 5.16 (fuzzy Edelstein contraction theorem). Let $(X, M_d, *)$ be a compact interval-valued fuzzy metric space. Let $T : X \rightarrow X$ be a mapping satisfying for all $x \neq y$ and $t > 0$,

$$M(Tx, Ty, t) > M(x, y, t),$$

then T has a unique fixed point.

Proof. Let $x \in X$ and $x_n = T^n x (n \in \mathbb{N})$. Assume $x_n \neq x_{n+1}$ for each n (if not, $T(x_n) = x_n$). Now, assume $x_n \neq x_m (n \neq m)$. Otherwise we get

$$M(x_n, x_{n+1}, t) = M(x_m, x_{m+1}, t) > M(x_{m-1}, x_{m+1}, t) > \cdots > M(x_n, x_{n+1}, t),$$

where $m > n$, a contradiction. Since X is compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$. Let $y = \lim_{i \in \mathbb{N}} x_{n_i}$. We also assume that $y, Ty \notin \{x_n : n \in \mathbb{N}\}$ (if not, choose a subsequence with such a property). According to the above assumptions we may now write

$$M(Tx_{n_i}, Ty, t) > M(x_{n_i}, y, t)$$

for all $i \in \mathbb{N}$ and $t > 0$. Since $M(x, y, \cdot)$ is continuous for all $x, y \in X$, by Corollary 5.15 we obtain

$$\lim_{i \in \mathbb{N}} M(Tx_{n_i}, Ty, t) \geq \lim_{i \in \mathbb{N}} M(x_{n_i}, y, t) = \mathbf{1}$$

for each $t > 0$, hence

$$\lim_{i \in \mathbb{N}} Tx_{n_i} = Ty, \dots \dots \dots (1).$$

Similarly, we obtain

$$\lim_{i \in \mathbb{N}} T^2x_{n_i} = T^2y, \dots \dots \dots (2),$$

since $Ty \neq Tx_{n_i}$ for all i . Now, observe that

$$M(x_{n_1}, Tx_{n_1}, t) < M(Tx_{n_1}, Tx_{n_1}^2, t) < \dots < M(x_{n_i}, Tx_{n_i}, t) < M(Tx_{n_i}, T^2x_{n_i}, t) < \dots < M(x_{n_{i+1}}, Tx_{n_{i+1}}, t) < M(Tx_{n_{i+1}}, T^2x_{n_{i+1}}, t) < \dots < \mathbf{1} \text{ for all } t > 0.$$

Thus $\{M(x_{n_i}, Tx_{n_i}, t)\}$ and $\{M(Tx_{n_i}, T^2x_{n_i}, t)\}$ ($t > 0$) are convergent to a common limit [cf.[29)]. So, by (1),(2) and Corollary 5.15 we get

$$\begin{aligned} M(y, Ty, t) &= M(\lim_i x_{n_i}, T(\lim_i x_{n_i}), t) \\ &= \lim_i M(x_{n_i}, Tx_{n_i}, t) \\ &= \lim_i M(Tx_{n_i}, T^2x_{n_i}, t) \\ &= M(\lim_i Tx_{n_i}, \lim_i T^2x_{n_i}, t) \\ &= M(Ty, T^2y, t) \end{aligned}$$

for all $t > 0$. Suppose $y \neq Ty$. Then, by $M(Tx, Ty, t) > M(x, y, t)$, $M(y, Ty, t) > M(Ty, T^2y, t), (t > 0)$, a contradiction. Hence $y = Ty$, a fixed point. Uniqueness follows at once from $M(Tx, Ty, t) > M(x, y, t)$.

6 Concluding remarks

In this paper, the concept of interval-valued fuzzy metric space is defined, and it is proved that the topology induced by quasi-metric is consistent with which induced via a standard interval-valued fuzzy quasi-metric, every quasi-metrizable topological space admits a compatible interval-valued fuzzy quasi-metric. On the contrary, topology generated by interval-valued fuzzy quasi-metric is quasi-metrizable. Furthermore,

some properties of interval-valued fuzzy quasi-metric space which is bicompletion are discussed, and it is proved that if an interval-valued fuzzy quasi-metric space has bicompletion, then it is unique in the mean of isometry. Finally, a fuzzy contraction mapping of interval-valued fuzzy metric space is defined, and the Banach and Edelstein fixed point theorem to interval fuzzy metric space are promoted.

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**BI-UNIVALENT FUNCTIONS OF COMPLEX ORDER
 BASED ON QUASI-SUBORDINATE CONDITIONS
 INVOLVING WRIGHT HYPERGEOMETRIC FUNCTIONS**

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ABSTRACT. In the present paper, we introduce and investigate a new subclass of bi-univalent functions of complex order defined in the open unit disk, which are associated with Wright hypergeometric functions and satisfying quasi-subordinate conditions. Furthermore, we find estimates on the second and the third coefficients of the Taylor-Maclaurin series for functions in the new subclass. Several special consequences of the results are also pointed out.

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1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Let \mathcal{A} denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Further, by \mathcal{S} we shall denote the class of all functions in \mathcal{A} which are univalent in \mathbb{U} . Some of the important and well-investigated subclasses of the univalent function class \mathcal{S} include (for example) the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in \mathbb{U} and the class $\mathcal{K}(\alpha)$ of convex functions of order α in \mathbb{U} .

The study of operators plays an important role in the geometric function theory and its related fields. Many differential and integral operators can be written in terms of convolution of certain analytic functions. It is observed that this formalism brings an ease in further mathematical exploration and also helps to understand the geometric properties of such operators better.

The convolution or Hadamard product of two functions $f, h \in \mathcal{A}$ is denoted by $f * h$ and is defined as

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \tag{1.2}$$

where $f(z)$ is given by (1.1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$.

Now we briefly recall the definitions of the special functions and operators used in this paper.

For complex parameters $\alpha_1, \dots, \alpha_l$ ($\frac{\alpha_j}{A_j} \neq 0, -1, \dots; j = 1, 2, \dots, l$) and β_1, \dots, β_m ($\frac{\beta_j}{B_j} \neq 0, -1, \dots; j = 1, 2, \dots, m$), Fox's H-function (for details, see [23]) we mean the Wright's generalized hypergeometric functions ${}_l\Psi_m$ with $A_j, B_j > 0$, give (rather general and typical examples of H -functions, not reducible to G -functions):

$$\begin{aligned} {}_l\Psi_m \left(\begin{matrix} (\alpha_1, A_1), \dots, (\alpha_l, A_l) \\ (\beta_1, B_1), \dots, (\beta_m, B_m) \end{matrix} : z \right) &= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + nA_1) \dots \Gamma(\alpha_l + nA_l)}{\Gamma(\beta_1 + nB_1) \dots \Gamma(\beta_m + nB_m)} \frac{z^n}{n!} \\ &= H_{l,m+1}^{1,l} \left[-z \mid \begin{matrix} (1 - \alpha_1, A_1), \dots, (1 - \alpha_l, A_l) \\ (0, 1), (1 - \beta_1, B_1), \dots, (1 - \beta_m, B_m) \end{matrix} \right], \end{aligned} \tag{1.3}$$

where $1 + \sum_{n=1}^m B_n - \sum_{n=1}^l A_n \geq 0$, ($l, m \in \mathbb{N} = \{1, 2, \dots\}$) and for suitably bounded values of $|z|$.

We note that when $A_1 = \dots = A_l = B_1 = \dots = B_m = 1$, they turn into the generalized hypergeometric functions

$$\begin{aligned} {}_l\Psi_m \left(\begin{matrix} (\alpha_1, 1), \dots, (\alpha_l, 1) \\ (\beta_1, 1), \dots, (\beta_m, 1) \end{matrix} : z \right) &= \left[\frac{\prod_{j=1}^l \Gamma(\alpha_j)}{\prod_{i=1}^m \Gamma(\beta_j)} \right] {}_lF_m(\alpha_1, \dots, \alpha_l : \beta_1, \dots, \beta_m : z) \tag{1.4} \\ &(l \leq m + 1; l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U}). \end{aligned}$$

Now we state the linear operator due to Srivastava [23] (see [6]) and Wright [24] in terms of the Hadamard product (or convolution) involving the generalized hypergeometric function. Let $l, m \in \mathbb{N}$ and suppose that the parameters $\alpha_1, A_1, \dots, \alpha_l, A_l$ and $\beta_1, B_1, \dots, \beta_m, B_m$ are also positive real numbers. Then, corresponding to a function

$${}_l\Phi_m[(\alpha_j, A_j)_{1,l}; (\beta_j, B_j)_{1,m}; z]$$

defined by

$${}_l\Phi_m[(\alpha_j, A_j)_{1,l}; (\beta_j, B_j)_{1,m}; z] = \Omega z {}_l\Psi_m[(\alpha_j, A_j)_{1,l}(\beta_j, B_j)_{1,m}; z] \tag{1.5}$$

where $\Omega = \left(\prod_{j=1}^l \Gamma(\alpha_j) \right)^{-1} \left(\prod_{j=1}^m \Gamma(\beta_j) \right)$, we consider a linear operator

$$\mathcal{W}[(\alpha_j, A_j)_{1,l}; (\beta_j, B_j)_{1,m}] : \mathcal{A} \rightarrow \mathcal{A}$$

defined by the following Hadamard product (or convolution)

$$\mathcal{W}[(\alpha_j, A_j)_{1,l}; (\beta_j, B_j)_{1,m}]f(z) := z {}_l\Psi_m[(\alpha_j, A_j)_{1,l}; (\beta_j, B_j)_{1,m}; z] * f(z).$$

We observe that , for $f(z)$ of the form(1.1),we have

$$\mathcal{W}[(\alpha_j, A_j)_{1,l}; (\beta_j, B_j)_{1,m}]f(z) = z + \sum_{n=2}^{\infty} \varphi_n a_n z^n \tag{1.6}$$

where

$$\varphi_n = \frac{\Omega \Gamma(\alpha_1 + A_1(n - 1)) \dots \Gamma(\alpha_l + A_l(n - 1))}{(n - 1)! \Gamma(\beta_1 + B_1(n - 1)) \dots \Gamma(\beta_m + B_m(n - 1))} \tag{1.7}$$

If, for convenience, we write

$$\mathcal{W}_m^l f(z) = \mathcal{W}[(\alpha_1, A_1), \dots, (\alpha_l, A_l); (\beta_1, B_1), \dots, (\beta_m, B_m)]f(z). \tag{1.8}$$

We state the following remark due to Srivastava [23] (see [6]) and Wright [24].

Remark 1.1. Other interesting and useful special cases of the Fox-Wright generalized hypergeometric function ${}_l\Psi_m$ defined by (1.3) include (for example) the generalized Bessel function

$${}_0\Psi_1(-; (\nu + 1, \mu); -z) \equiv \mathcal{J}_\mu^\nu = \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(n\mu + \nu + 1)}.$$

For $\mu = 1$, corresponds essentially to the classical Bessel function $\mathcal{J}^\nu(z)$, and the generalized Mittag-Leffler function

$${}_1\Psi_1((1, 1); (\mu, \lambda); z) \equiv \mathbf{E}_\mu^\lambda = \sum_{n=0}^{\infty} \frac{(z)^n}{\Gamma(n\lambda + \mu)}.$$

Remark 1.2. By setting $A_j = 1(j = 1, \dots, l)$ and $B_j = 1(j = 1, \dots, m)$ in (1.5), we are led immediately to the *generalized hypergeometric function* ${}_lF_m(z)$ is defined by

$$\Omega_l F_m(z) \equiv \Omega_l F_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_l)_n}{(\beta_1)_n \dots (\beta_m)_n} \frac{z^n}{n!} \tag{1.9}$$

$$(l \leq m + 1; l, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; z \in \mathbb{U}),$$

where $(\alpha)_n$ is the Pochhammer symbol.

In view of the relationship (1.9), the linear operator(1.6) includes the Dziok-Srivastava operator (see [5]), so that it includes (as its special cases) various other linear operators introduced and studied by Bernardi [3], Carlson and Shaffer [4], Libera [11], Livingston [12], Ruscheweyh [19] and Srivastava and Owa [22].

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1} , defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4} \right),$$

where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots \quad (1.10)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in \mathbb{U} if both $f(z)$ and $f^{-1}(z)$ are univalent in \mathbb{U} . Let Σ denote the class of bi-univalent functions in \mathbb{U} given by (1.1).

For some intriguing examples of functions and characterization of the class Σ , one could refer Srivastava et al., [21] and the references stated therein (see also, [9]). Recently there has been triggering interest to study the bi-univalent function class Σ (see [8, 9, 13, 15, 17, 21]) and obtain non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ for $n \in \mathbb{N} \setminus \{1, 2\}$ is presumably still an open problem.

In 1970 Robertson[18] introduced the concept of quasi-subordination. An analytic function $f(z)$ is quasi-subordinate to an analytic function $\phi(z)$, in the open unit disk if there exist analytic functions $h(z)$ and w , with $w(0) = 0$ such that $|h(z)| \leq 1$, $|w(z)| < 1$ and $f(z) = h(z)\phi[w(z)]$. Then we write $f(z) \prec_{\tilde{q}} \phi(z)$. If $h(z) = 1$, then the quasi-subordination reduces to the subordination. Also, if $w(z) = z$ then $f(z) = h(z)\phi(z)$ and in this case we say that $f(z)$ is majorized by $\phi(z)$ and it is written as $f(z) \ll \phi(z)$ in \mathbb{U} . Hence it is obvious that quasi-subordination is the generalization of subordination as well as majorization. It is unfortunate that the concept quasi-subordination is so far an underlying concept in the area of complex function theory although it deserves much attention as it unifies the concept of both subordination and majorization.

Through out this paper it is assumed that ϕ is analytic in \mathbb{U} with $\phi(0) = 1$ and let

$$\phi(z) = 1 + C_1z + C_2z^2 + C_3z^3 + \dots \quad (C_1 > 0). \quad (1.11)$$

also let

$$\psi(z) = D_0 + D_1z + D_2z^2 + D_3z^3 + \dots \quad (|\psi(z)| \leq 1; z \in \mathbb{U}). \quad (1.12)$$

Motivated by the earlier work of Deniz [7] (see [1, 17, 20]) in the present paper, we introduce new subclass of the function class Σ of complex order $\gamma \in \mathbb{C} \setminus \{0\}$, involving Wright hypergeometric functions \mathcal{W}_m^l , and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the new subclasses of function class Σ . Several related classes are also considered, and connection to earlier known results are made.

Definition 1.3. A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{G}_\Sigma^{l,m}(\gamma, \lambda, \phi)$ if the following conditions are satisfied:

$$\frac{1}{\gamma} \left(\frac{z(\mathcal{W}_m^l f(z))'}{(1-\lambda)\mathcal{W}_m^l f(z) + \lambda z(\mathcal{W}_m^l f(z))'} - 1 \right) \prec_{\tilde{q}} (\phi(z) - 1) \quad (1.13)$$

and

$$\frac{1}{\gamma} \left(\frac{w(\mathcal{W}_m^l g(w))'}{(1-\lambda)\mathcal{W}_m^l g(w) + \lambda z(\mathcal{W}_m^l g(w))'} - 1 \right) \prec_{\tilde{q}} (\phi(w) - 1), \quad (1.14)$$

where $\gamma \in \mathbb{C} \setminus \{0\}$, $0 \leq \lambda < 1$, $z \in \mathbb{U}$ and the function g is given by (1.10).

Example 1. For $\lambda = 0$ and $\gamma \in \mathbb{C} \setminus \{0\}$, a function $f \in \Sigma$, given by (1.1) is said to be in the class $\mathfrak{S}_{\Sigma}^{l,m}(\gamma, \phi)$ if the following conditions are satisfied:

$$\frac{1}{\gamma} \left(\frac{z(\mathcal{W}_m^l f(z))'}{\mathcal{W}_m^l f(z)} - 1 \right) \prec_{\tilde{q}} (\phi(z) - 1) \tag{1.15}$$

and

$$\frac{1}{\gamma} \left(\frac{w(\mathcal{W}_m^l g(w))'}{\mathcal{W}_m^l g(w)} - 1 \right) \prec_{\tilde{q}} (\phi(w) - 1), \tag{1.16}$$

where $z, w \in \mathbb{U}$ and the function g is given by (1.10).

On specializing the parameters l, m one can state the various new subclasses of Σ (or $\mathfrak{G}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$), as illustrations, we present some examples for the case with $A_j = 1$ ($j = 1, 2, \dots, l$); $B_j = 1$ ($j = 1, 2, \dots, m$).

Example 2. If $l \leq m + 1$, $l, m \in N_0 := N \cup \{0\}$, and $\gamma \in \mathbb{C} \setminus \{0\}$, then a function $f \in \Sigma$, given by (1.1) is said to be in the class $\mathfrak{S}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ if the following conditions are satisfied:

$$\frac{1}{\gamma} \left(\frac{z(\mathcal{H}_m^l f(z))'}{(1 - \lambda)(\mathcal{H}_m^l f(z)) + \lambda(\mathcal{H}_m^l f(z))'} - 1 \right) \prec_{\tilde{q}} (\phi(z) - 1)$$

and

$$\frac{1}{\gamma} \left(\frac{w(\mathcal{H}_m^l g(w))'}{(1 - \lambda)(\mathcal{H}_m^l g(w)) + \lambda(\mathcal{H}_m^l g(w))'} - 1 \right) \prec_{\tilde{q}} (\phi(w) - 1), \quad (0 \leq \lambda < 1),$$

where $\mathcal{H}_m^l f(z) := \left(z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_l)_n}{(\beta_1)_n \dots (\beta_m)_n} \frac{z^n}{n!} \right) * f(z) \equiv \mathcal{W}[(\alpha_t, 1)_{1,l}; (\beta_t, 1)_{1,m}]f(z)$ is a well-known Dziok-Srivastava operator [5], the function g is given by (1.10) and $z, w \in \mathbb{U}$.

Example 3. If $l = 2$ and $m = 1$ with $\alpha_1 = a$ ($a > 0$), $\alpha_2 = b$, ($b > 0$) $\beta_1 = c$ ($c > 0$), and $\gamma \in \mathbb{C} \setminus \{0\}$, a function $f \in \Sigma$, given by (1.1) is said to be in the class $\mathfrak{S}_{\Sigma}^{a,c,b}(\gamma, \lambda, \phi)$, if the following conditions are satisfied:

$$\frac{1}{\gamma} \left(\frac{z(\mathcal{J}_c^{a,b} f(z))'}{(1 - \lambda)(\mathcal{J}_c^{a,b} f(z)) + \lambda(\mathcal{J}_c^{a,b} f(z))'} - 1 \right) \prec_{\tilde{q}} (\phi(z) - 1)$$

and

$$\frac{1}{\gamma} \left(\frac{w(\mathcal{J}_c^{a,b} g(w))'}{(1 - \lambda)(\mathcal{J}_c^{a,b} g(w)) + \lambda(\mathcal{J}_c^{a,b} g(w))'} - 1 \right) \prec_{\tilde{q}} (\phi(w) - 1), \quad (0 \leq \lambda < 1; z, w \in \mathbb{U}),$$

where $\mathcal{J}_c^{a,b} f(z) := \left(z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} z^n \right) * f(z) \equiv \mathcal{H}_1^2(a, b; c)f(z)$, is a well-known Hohlov operator [10] and the function g is given by (1.10).

Example 4. If $l = 2$ and $m = 1$ with $\alpha_1 = a$ ($a > 0$), $\alpha_2 = 1$, $\beta_1 = c$ ($c > 0$), and $\gamma \in \mathbb{C} \setminus \{0\}$, a function $f \in \Sigma$, given by (1.1) is said to be in the class $\mathfrak{S}_{\Sigma}^{a,c}(\gamma, \lambda, \phi)$, if the following conditions are satisfied:

$$\frac{1}{\gamma} \left(\frac{z(\mathcal{L}(a, c)f(z))'}{(1 - \lambda)(\mathcal{L}(a, c)f(z)) + \lambda(\mathcal{L}(a, c)f(z))'} - 1 \right) \prec_{\tilde{q}} (\phi(z) - 1)$$

and

$$\frac{1}{\gamma} \left(\frac{w(\mathcal{L}(a, c)g(w))'}{(1 - \lambda)(\mathcal{L}(a, c)g(w)) + \lambda(\mathcal{L}(a, c)g(w))'} - 1 \right) \prec_{\tilde{q}} (\phi(w) - 1), \quad (0 \leq \lambda < 1; z, w \in \mathbb{U}),$$

where $\mathcal{L}(a, c)f(z) := \left(z + \sum_{n=2}^{\infty} \frac{(a)_n}{(c)_n} z^n \right) * f(z) \equiv \mathcal{H}_1^2(a, 1; c)f(z)$, is a well-known Carlson-Shaffer operator [4] and the function g is given by (1.10).

Example 5. If $l = 2$ and $m = 1$ with $\alpha_1 = \delta + 1$ ($\delta \geq -1$), $\alpha_2 = 1$, $\beta_1 = 1$, and $\gamma \in \mathbb{C} \setminus \{0\}$, a function $f \in \Sigma$, given by (1.1) is said to be in the class $\mathfrak{S}_{\Sigma}^{\delta}(\gamma, \lambda, \phi)$ if the following conditions are satisfied:

$$\frac{1}{\gamma} \left(\frac{z(\mathcal{D}^{\delta}f(z))'}{(1 - \lambda)(\mathcal{D}^{\delta}f(z)) + \lambda(\mathcal{D}^{\delta}f(z))'} - 1 \right) \prec_{\tilde{q}} (\phi(z) - 1)$$

and

$$\frac{1}{\gamma} \left(\frac{w(\mathcal{D}^{\delta}g(w))'}{(1 - \lambda)(\mathcal{D}^{\delta}g(w)) + \lambda(\mathcal{D}^{\delta}g(w))'} - 1 \right) \prec_{\tilde{q}} (\phi(w) - 1), \quad (0 \leq \lambda < 1; z, w \in \mathbb{U}),$$

where \mathcal{D}^{δ} is called Ruscheweyh derivative[19] of order δ ($\delta \geq -1$) and $\mathcal{D}^{\delta}f(z) := \frac{z}{(1-z)^{\delta+1}} * f(z) \equiv \mathcal{H}_1^2(\delta + 1, 1; 1)f(z)$ and the function g is given by (1.10).

Example 6. If $l = 2$ and $m = 1$ with $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$, and $\gamma \in \mathbb{C} \setminus \{0\}$, a function $f \in \Sigma$, given by (1.1) is said to be in the class $\mathfrak{S}_{\Sigma}^*(\gamma, \lambda, \phi)$ if the following conditions are satisfied:

$$\frac{1}{\gamma} \left(\frac{zf'(z)}{(1 - \lambda)f(z) + \lambda f'(z)} - 1 \right) \prec_{\tilde{q}} (\phi(z) - 1)$$

and

$$\frac{1}{\gamma} \left(\frac{wg'(w)}{(1 - \lambda)g(w) + \lambda g'(w)} - 1 \right) \prec_{\tilde{q}} (\phi(w) - 1),$$

where $0 \leq \lambda < 1$, $z, w \in \mathbb{U}$ and the function g is given by (1.10).

Example 7. If $l = 2$ and $m = 1$ with $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$, and $\lambda = 0$; $\gamma \in \mathbb{C} \setminus \{0\}$, a function $f \in \Sigma$, given by (1.1) is said to be in the class $\mathfrak{S}_{\Sigma}^*(\gamma, \phi)$ if the following conditions are satisfied:

$$\frac{1}{\gamma} \left(\frac{zf'(z)}{f(z)} - 1 \right) \prec_{\tilde{q}} (\phi(z) - 1)$$

and

$$\frac{1}{\gamma} \left(\frac{wg'(w)}{g(w)} - 1 \right) \prec_{\tilde{q}} (\phi(w) - 1),$$

where $0 \leq \lambda < 1$, $z, w \in \mathbb{U}$ and the function g is given by (1.10).

Remark 1.4. For $\lambda = 0$ and $\gamma \in \mathbb{C} \setminus \{0\}$, a function $f \in \Sigma$, given by (1.1), as in Example 1, one can state various analogous subclasses defined in Examples 2 to 4.

Definition 1.5. A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{B}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ if the following conditions are satisfied:

$$\frac{1}{\gamma} \left(\frac{z^{1-\lambda}(\mathcal{W}_m^l f(z))'}{[\mathcal{W}_m^l f(z)]^{1-\lambda}} - 1 \right) \prec_{\tilde{q}} (\phi(z) - 1) \tag{1.17}$$

and

$$\frac{1}{\gamma} \left(\frac{w^{1-\lambda}(\mathcal{W}_m^l g(w))'}{[\mathcal{W}_m^l g(w)]^{1-\lambda}} - 1 \right) \prec_{\tilde{q}} (\phi(w) - 1), \tag{1.18}$$

where $\gamma \in \mathbb{C} \setminus \{0\}$, $\lambda \geq 0$, $z, w \in \mathbb{U}$ and the function g is given by(1.10).

On specializing the parameters λ one can define the various new subclasses of Σ associated with Wright hypergeometric functions \mathcal{W}_m^l , as illustrated in the following examples.

Example 8. For $\lambda = 0$ and a function $f \in \Sigma$, given by (1.1), $\mathcal{B}_{\Sigma}^{l,m}(\gamma, 0, \phi) \equiv \mathcal{S}_{\Sigma}^{l,m}(\gamma, \phi)$.

Example 9. For $\lambda = 1$, a function $f \in \Sigma$, given by (1.1) is said to be in the class $\mathcal{H}_{\Sigma}^{l,m}(\gamma, \phi)$ if the following conditions are satisfied:

$$\frac{1}{\gamma} (\mathcal{W}_m^l f(z))' - 1 \prec_{\tilde{q}} (\phi(z) - 1) \tag{1.19}$$

and

$$\frac{1}{\gamma} ((\mathcal{W}_m^l g(w))' - 1) \prec_{\tilde{q}} (\phi(w) - 1), \tag{1.20}$$

where $\gamma \in \mathbb{C} \setminus \{0\}$; $z, w \in \mathbb{U}$ and the function g is given by(1.10).

It is of interest to note that for $\gamma = 1$ the class $\mathcal{B}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ reduces to the following new subclass $\mathcal{B}_{\Sigma}^{l,m}(\lambda, \phi)$.

Example 10. A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{B}_{\Sigma}^{l,m}(\lambda, \phi)$ if the following conditions are satisfied:

$$\left(\frac{z^{1-\lambda}(\mathcal{W}_m^l f(z))'}{[\mathcal{W}_m^l f(z)]^{1-\lambda}} - 1 \right) \prec_{\tilde{q}} (\phi(z) - 1) \tag{1.21}$$

and

$$\left(\frac{w^{1-\lambda}(\mathcal{W}_m^l g(w))'}{[\mathcal{W}_m^l g(w)]^{1-\lambda}} - 1 \right) \prec_{\tilde{q}} (\phi(w) - 1), \tag{1.22}$$

where $\lambda \geq 0$, $z, w \in \mathbb{U}$ and the function g is given by (1.10).

Remark 1.6. On specializing the parameters l, m one can state the various new subclasses of $\mathcal{B}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$, as illustrated in Examples 1 to Examples 7, with $A_j = 1$ ($j = 1, 2, \dots, l$); $B_j = 1$ ($j = 1, 2, \dots, m$).

In the following section, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the above-defined subclasses $\mathcal{G}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ and $\mathcal{B}_{\Sigma}^{l,m}(\gamma, \lambda, \phi)$ of the function class Σ .

2. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS $\mathcal{G}_\Sigma^{l,m}(\gamma, \lambda, \phi)$

In order to derive our main results, we shall need the following lemma.

Lemma 2.1. (see [16]) *If $h \in \mathcal{P}$, then $|c_k| \leq 2$ for each k , where \mathcal{P} is the family of all functions h , analytic in \mathbb{U} , for which*

$$\Re\{h(z)\} > 0 \quad (z \in \mathbb{U}),$$

where

$$h(z) = 1 + c_1z + c_2z^2 + \dots \quad (z \in \mathbb{U}).$$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{G}_\Sigma^{l,m}(\gamma, \lambda, \phi)$. Define the functions $p(z)$ and $q(z)$ by

$$p(z) := \frac{1 + u(z)}{1 - u(z)} = 1 + p_1z + p_2z^2 + \dots$$

and

$$q(z) := \frac{1 + v(z)}{1 - v(z)} = 1 + q_1z + q_2z^2 + \dots$$

or, equivalently,

$$u(z) := \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left[p_1z + \left(p_2 - \frac{p_1^2}{2} \right) z^2 + \dots \right]$$

and

$$v(z) := \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2} \left[q_1z + \left(q_2 - \frac{q_1^2}{2} \right) z^2 + \dots \right].$$

Then $p(z)$ and $q(z)$ are analytic in \mathbb{U} with $p(0) = 1 = q(0)$. Since $u, v : \mathbb{U} \rightarrow \mathbb{U}$, the functions $p(z)$ and $q(z)$ have a positive real part in \mathbb{U} , and $|p_i| \leq 2$ and $|q_i| \leq 2$.

Theorem 2.2. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{G}_\Sigma^{l,m}(\gamma, \lambda, \phi)$. Then*

$$|a_2| \leq \frac{|\gamma| |D_0| C_1 \sqrt{C_1}}{\sqrt{|\gamma| D_0 (\lambda^2 - 1) C_1^2 + (1 - \lambda)^2 (C_1 - C_2) \varphi_2^2 + 2\gamma(1 - \lambda) D_0 C_1^2 \varphi_3}} \quad (2.1)$$

and

$$|a_3| \leq \frac{|\gamma D_0|^2 C_1^2}{(1 - \lambda)^2 \varphi_2^2} + \frac{|\gamma D_0| C_1}{2(1 - \lambda) \varphi_3} + \frac{|\gamma D_1| C_1}{2(1 - \lambda) \varphi_3}. \quad (2.2)$$

Proof. It follows from (1.17) and (1.18) that

$$\frac{1}{\gamma} \left(\frac{z(\mathcal{W}_m^l f(z))'}{(1 - \lambda)\mathcal{W}_m^l f(z) + \lambda z(\mathcal{W}_m^l f(z))'} - 1 \right) = \psi(z)[\phi(u(z)) - 1] \quad (2.3)$$

and

$$\frac{1}{\gamma} \left(\frac{w(\mathcal{W}_m^l g(w))'}{(1 - \lambda)\mathcal{W}_m^l g(w) + \lambda z(\mathcal{W}_m^l g(w))'} - 1 \right) = \psi(w)[\phi(v(w)) - 1], \quad (2.4)$$

where $p(z)$ and $q(w)$ in \mathcal{P} and have the following forms:

$$\psi(z)[\phi(u(z)) - 1] = \frac{1}{2}D_0C_1p_1z + \left[\frac{1}{2}D_1 C_1 p_1 + \frac{1}{2}D_0C_1 \left(p_2 - \frac{p_1^2}{2} \right) + \frac{D_0C_2}{4}p_1^2 \right] z^2 + \dots \tag{2.5}$$

and

$$\psi(w)[\phi(v(w)) - 1] = \frac{1}{2}D_0C_1q_1w + \left[\frac{1}{2}D_1 C_1 q_1 + \frac{1}{2}D_0C_1 \left(q_2 - \frac{q_1^2}{2} \right) + \frac{D_0C_2}{4}q_1^2 \right] w^2 + \dots \tag{2.6}$$

respectively. Now, equating the coefficients in (2.3) and (2.4), we get

$$\frac{(1 - \lambda)}{\gamma} \varphi_2 a_2 = \frac{1}{2}D_0C_1p_1, \tag{2.7}$$

$$\frac{(\lambda^2 - 1)}{\gamma} \varphi_2^2 a_2^2 + \frac{2(1 - \lambda)}{\gamma} \varphi_3 a_3 = \frac{1}{2}D_1 C_1 p_1 + \frac{1}{2}D_0C_1 \left(p_2 - \frac{p_1^2}{2} \right) + \frac{D_0C_2}{4}p_1^2, \tag{2.8}$$

$$- \frac{(1 - \lambda)}{\gamma} \varphi_2 a_2 = \frac{1}{2}D_0C_1q_1 \tag{2.9}$$

and

$$\frac{(\lambda^2 - 1)}{\gamma} \varphi_2^2 a_2^2 + \frac{2(1 - \lambda)}{\gamma} \varphi_3 (2a_2^2 - a_3) = \frac{1}{2}D_1 C_1 q_1 + \frac{1}{2}D_0C_1 \left(q_2 - \frac{q_1^2}{2} \right) + \frac{D_0C_2}{4}q_1^2. \tag{2.10}$$

From (2.7) and (2.9), we find that

$$a_2 = \frac{\gamma D_0 C_1 p_1}{2(1 - \lambda) \varphi_2} = \frac{-\gamma D_0 C_1 q_1}{2(1 - \lambda) \varphi_2}, \tag{2.11}$$

which implies

$$p_1 = -q_1. \tag{2.12}$$

and

$$8(1 - \lambda)^2 \varphi_2^2 a_2^2 = \gamma^2 D_0^2 C_1^2 (p_1^2 + q_1^2). \tag{2.13}$$

Adding (2.8) and (2.10), by using(2.12)and(2.13) ,we obtain

$$4([\gamma D_0(\lambda^2 - 1)C_1^2 + (1 - \lambda)^2(C_1 - C_2)]\varphi_2^2 + 2\gamma D_0(1 - \lambda)C_1^2\varphi_3) a_2^2 = \gamma^2 D_0^2 C_1^3 (p_2 + q_2). \tag{2.14}$$

Thus,

$$a_2^2 = \frac{\gamma^2 D_0^2 C_1^3 (p_2 + q_2)}{4([\gamma D_0(\lambda^2 - 1)C_1^2 + (1 - \lambda)^2(C_1 - C_2)]\varphi_2^2 + 2\gamma D_0(1 - \lambda)C_1^2\varphi_3)}. \tag{2.15}$$

Applying Lemma 2.1 for the coefficients p_2 and q_2 , we immediately have

$$|a_2|^2 \leq \frac{|\gamma|^2 |D_0|^2 C_1^3}{|[\gamma D_0(\lambda^2 - 1)C_1^2 + (1 - \lambda)^2(C_1 - C_2)]\varphi_2^2 + 2\gamma D_0(1 - \lambda)C_1^2\varphi_3|}. \tag{2.16}$$

Since $C_1 > 0$,the last inequality gives the desired estimate on $|a_2|$ given in (2.1).

Next, in order to find the bound on $|a_3|$, by subtracting (2.10) from (2.8), we get

$$\begin{aligned} \frac{4(1-\lambda)}{\gamma} \varphi_3 a_3 - \frac{4(1-\lambda)}{\gamma} \varphi_3 a_2^2 \\ = \frac{D_1 C_1(p_1 - q_1)}{2} + \frac{D_0 C_1(p_2 - q_2)}{2} + \frac{D_0(C_2 - C_1)(p_1^2 - q_1^2)}{4}. \end{aligned} \quad (2.17)$$

It follows from (2.11), (2.12) and (2.17) that

$$a_3 = \frac{\gamma^2 D_0^2 C_1^2 (p_1^2 + q_1^2)}{8(1-\lambda)^2 \varphi_2^2} + \frac{\gamma D_1 C_1 (p_1 - q_1)}{8(1-\lambda) \varphi_3} + \frac{\gamma C_1 D_0 (p_2 - q_2)}{8(1-\lambda) \varphi_3}.$$

Applying Lemma 2.1 once again for the coefficients p_2 and q_2 , we readily get

$$|a_3| \leq \frac{|\gamma D_0|^2 C_1^2}{(1-\lambda)^2 \varphi_2^2} + \frac{|\gamma D_0| C_1}{2(1-\lambda) \varphi_3} + \frac{|\gamma D_1| C_1}{2(1-\lambda) \varphi_3}.$$

This completes the proof of Theorem 2.2. □

Putting $\lambda = 0$ in Theorem 2.2, we have the following corollary.

Corollary 2.3. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{S}_\Sigma^{l,m}(\gamma, \phi)$. Then*

$$|a_2| \leq \frac{|\gamma D_0| C_1 \sqrt{C_1}}{\sqrt{|(C_1 - C_2) - \gamma D_0 C_1^2| \varphi_2^2 + 2\gamma D_0 C_1^2 \varphi_3}} \quad (2.18)$$

and

$$|a_3| \leq \frac{|\gamma D_0|^2 C_1^2}{\varphi_2^2} + \frac{|\gamma D_0| C_1}{2\varphi_3} + \frac{|\gamma D_1| C_1}{2\varphi_3}. \quad (2.19)$$

Now we state the following corollaries for the function classes $\mathcal{S}_\Sigma^*(\gamma, \lambda, \phi)$ and $\mathcal{S}_\Sigma^*(\gamma, \lambda)$ defined in Example 5 and Example 6, respectively.

Corollary 2.4. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{S}_\Sigma^*(\gamma, \lambda, \phi)$. Then*

$$|a_2| \leq \frac{|\gamma D_0| C_1 \sqrt{C_1}}{(1-\lambda) \sqrt{|(C_1 - C_2) + \gamma D_0 C_1^2|}} \quad (2.20)$$

and

$$|a_3| \leq \frac{|\gamma D_0|^2 C_1^2}{(1-\lambda)^2} + \frac{|\gamma D_0| C_1}{2(1-\lambda)} + \frac{|\gamma D_1| C_1}{2(1-\lambda)}. \quad (2.21)$$

Corollary 2.5. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{S}_\Sigma^*(\gamma, \phi)$. Then*

$$|a_2| \leq \frac{|\gamma D_0| C_1 \sqrt{C_1}}{\sqrt{|(C_1 - C_2) + \gamma D_0 C_1^2|}} \quad (2.22)$$

and

$$|a_3| \leq |\gamma D_0|^2 C_1^2 + \frac{|\gamma D_0| C_1}{2} + \frac{|\gamma D_1| C_1}{2}. \quad (2.23)$$

3. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS $\mathcal{B}_\Sigma^{l,m}(\gamma, \lambda, \phi)$

Theorem 3.1. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma^{\mu,b}(\gamma, \lambda, \phi)$. Then*

$$|a_2| \leq \frac{|\gamma| |D_0| C_1 \sqrt{2C_1}}{\sqrt{|\gamma D_0 C_1^2 [(\lambda - 1)(\lambda + 2)\varphi_2^2 + 2(\lambda + 2)\varphi_3] - 2(C_2 - C_1)(1 + \lambda)^2 \varphi_2^2|}} \quad (3.1)$$

and

$$|a_3| \leq \left(\frac{|\gamma D_0| C_1}{(1 + \lambda)\varphi_2} \right)^2 + \frac{|\gamma D_0| C_1}{(\lambda + 2)\varphi_3} + \frac{|\gamma D_1| C_1}{2(1 + \lambda)\varphi_3}. \quad (3.2)$$

Proof. Let $f \in \mathcal{B}_\Sigma^{l,m}(\gamma, \lambda, \phi)$ and $g = f^{-1}$. Then there are analytic functions $u, v : \Delta \rightarrow \Delta$ with $u(0) = 0 = v(0)$, satisfying

$$\frac{1}{\gamma} \left(\frac{z^{1-\lambda} (\mathcal{W}_m^l f(z))'}{[\mathcal{W}_m^l f(z)]^{1-\lambda}} - 1 \right) = \psi(z) [\phi(u(z)) - 1] \quad (3.3)$$

and

$$\frac{1}{\gamma} \left(\frac{w^{1-\lambda} (\mathcal{W}_m^l g(w))'}{[\mathcal{W}_m^l g(w)]^{1-\lambda}} - 1 \right) = \psi(w) [\phi(u(w)) - 1]. \quad (3.4)$$

In light of (1.1) - (1.11), from (2.5) and (2.6), it is evident that

$$\begin{aligned} & \frac{(\lambda + 1)}{\gamma} \varphi_2 a_2 z + \frac{1}{\gamma} \left[(\lambda + 2)\varphi_3 a_3 + \frac{(\lambda - 1)(\lambda + 2)}{2} \varphi_2^2 a_2^2 \right] z^2 + \dots \\ & = \frac{1}{2} D_0 C_1 p_1 z + \left[\frac{1}{2} D_1 C_1 p_1 + \frac{1}{2} D_0 C_1 \left(p_2 - \frac{p_1^2}{2} \right) + \frac{D_0 C_2}{4} p_1^2 \right] z^2 + \dots \end{aligned}$$

and

$$\begin{aligned} & - \frac{(\lambda + 1)}{\gamma} \varphi_2 a_2 w + \frac{1}{\gamma} \left[-(\lambda + 2)\varphi_3 a_3 + \left(\frac{(\lambda - 1)(\lambda + 2)}{2} \varphi_2^2 + 2(\lambda + 2)\varphi_3 \right) a_2^2 \right] w^2 + \dots \\ & = \frac{1}{2} D_0 C_1 q_1 w + \left[\frac{1}{2} D_1 C_1 q_1 + \frac{1}{2} D_0 C_1 \left(q_2 - \frac{q_1^2}{2} \right) + \frac{D_0 C_2}{4} q_1^2 \right] w^2 + \dots \end{aligned}$$

Now proceeding on lines similar to Theorem 2.2 we get the desired results. □

Choosing $\lambda = 0$ and $\lambda = 1$ we state the initial Taylor coefficients for the function classes $\mathcal{S}_\Sigma^{l,m}(\gamma, \phi)$ and $\mathcal{H}_\Sigma^{l,m}(\gamma, \phi)$.

4. CONCLUDING REMARKS

For the class of strongly starlike functions, the function ϕ is given by

$$\phi(z) = \left(\frac{1 + z}{1 - z} \right)^\alpha = 1 + 2\alpha z + 2\alpha^2 z^2 + \dots \quad (0 < \alpha \leq 1), \quad (4.1)$$

which gives

$$C_1 = 2\alpha \quad \text{and} \quad C_2 = 2\alpha^2.$$

On the other hand, for $-1 \leq B \leq A < 1$ if we take

$$\phi(z) = \frac{1 + Az}{1 + Bz} = 1 + (A - B)z - B(A - B)z^2 + B^2(A - B)z^3 + \dots, \quad (4.2)$$

then we have

$$C_1 = (A - B), \quad C_2 = -B(A - B).$$

By taking, $A = (1 - 2\beta)$ where $0 \leq \beta < 1$ and $B = -1$ in (4.2), we get

$$\begin{aligned} \phi(z) &= \frac{1 + (1 - 2\beta)z}{1 - z} \\ &= 1 + 2(1 - \beta)z + 2(1 - \beta)z^2 + 2(1 - \beta)z^3 + \dots \end{aligned} \quad (4.3)$$

Hence, we have

$$C_1 = C_2 = 2(1 - \beta).$$

Further, by taking $\beta = 0$, in (4.3), we get

$$\phi(z) = \frac{1 + z}{1 - z} = 1 + 2z + 2z^2 + 2z^3 + \dots, \quad (4.4)$$

Hence,

$$C_1 = C_2 = 2.$$

Various Choices of ϕ as mentioned above and suitably choosing the values of C_1 and C_2 , we state some interesting results analogous to Theorem 2.2, Theorem 3.1 and the Corollaries 2.3 to 2.5 for various new subclasses of Σ .

Remark 4.1. Setting $A_j = 1$ ($j = 1, \dots, l$) and $B_j = 1$ ($j = 1, \dots, m$) and if $l = 2$ and $m = 1$ with $\alpha_1 = \mu + 1$ ($\mu > -1$), $\alpha_2 = 1$, $\beta_1 = \mu + 2$, where \mathcal{J}_μ is a Bernardi operator [3] defined by

$$\mathcal{J}_\mu f(z) := \frac{\mu + 1}{z^\mu} \int_0^z t^{\mu-1} f(t) dt \equiv H_1^2(\mu + 1, 1; \mu + 2) f(z).$$

Note that the operator J_1 was studied earlier by Libera [11] and Livingston [12] and various other interesting corollaries and consequences of our main results (which are asserted by Theorem 2.2 and Theorem 3.1 above) can be derived similarly. Further, by setting $A_j = 1$ ($j = 1, \dots, l$) and $B_j = 1$ ($j = 1, \dots, m$) and suitably choosing l, m, λ , we can state the various results for the new classes defined in Examples 1 to 10, but the details involved may be left as an exercise for the interested reader.

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On uni-soft mighty filters of BE -algebras

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Abstract. The notion of uni-soft mighty filters of a BE -algebra is introduced, and related properties are investigated. The problem of classifying uni-soft mighty by their γ -exclusive filter is solved. We construct a new quotient structure of a transitive BE -algebra using a unit soft filter and consider some properties of it.

1. Introduction

Kim and Kim [7] introduced the notion of a BE -algebra, and investigated several properties. In [2], Ahn and So introduced the notion of ideals in BE -algebras. They gave several descriptions of ideals in BE -algebras.

Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [12]. In response to this situation Zadeh [13] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [14]. To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [10]. Maji et al. [9] and Molodtsov [10] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [10] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [9] described the application of soft set

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theory to a decision making problem. Maji et al. [8] also studied several operations on the theory of soft sets.

H. R. Lee and S. S. Ahn [6] introduced the notion of a mighty filter in BE -algebras, and investigated some properties of it. Jun et al. [5] introduced the notion of a uni-soft filter of a BE -algebra, and studied their properties.

In this paper, we introduce the notion of a uni-soft mighty filter of a BE -algebra, and investigate their properties. We solve the problem of classifying uni-soft mighty by their γ -exclusive filter. Also we construct a new quotient structure of a transitive BE -algebra using a unit soft filter and study some properties of it.

2. PRELIMINARIES

We recall some definitions and results discussed in [7].

An algebra $(X; *, 1)$ of type $(2, 0)$ is called a BE -algebra if

- (BE1) $x * x = 1$ for all $x \in X$;
- (BE2) $x * 1 = 1$ for all $x \in X$;
- (BE3) $1 * x = x$ for all $x \in X$;
- (BE4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$ (*exchange*)

We introduce a relation “ \leq ” on a BE -algebra X by $x \leq y$ if and only if $x * y = 1$. A non-empty subset S of a BE -algebra X is said to be a *subalgebra* of X if it is closed under the operation “ $*$ ”. Noticing that $x * x = 1$ for all $x \in X$, it is clear that $1 \in S$. A BE -algebra $(X; *, 1)$ is said to be *self distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$.

Definition 2.1. Let $(X; *, 1)$ be a BE -algebra and let F be a non-empty subset of X . Then F is called a *filter* of X if

- (F1) $1 \in F$;
- (F2) $x * y \in F$ and $x \in F$ imply $y \in F$ for all $x, y \in X$.

F is a *mighty filter* [6] of X if it satisfies (F1) and

- (F3) $z * (y * x) \in F$ and $z \in F$ imply $((x * y) * y) * x \in F$ for all $x, y, z \in X$.

Proposition 2.2. Let $(X; *, 1)$ be a BE -algebra and let F be a filter of X . If $x \leq y$ and $x \in F$ for any $y \in X$, then $y \in F$.

Proposition 2.3. Let $(X; *, 1)$ be a self distributive BE -algebra. Then following hold: for any $x, y, z \in X$,

- (i) if $x \leq y$, then $z * x \leq z * y$ and $y * z \leq x * z$.
- (ii) $y * z \leq (z * x) * (y * z)$.
- (iii) $y * z \leq (x * y) * (x * z)$.

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A BE -algebra $(X; *, 1)$ is said to be *transitive* if it satisfies Proposition 2.3(iii).

Theorem 2.4.([6]) *A filter F of a BE -algebra X is mighty if and only if*

$$(2.1) \quad (\forall x, y \in X)(y * x \in F \Rightarrow ((x * y) * y) * x \in F).$$

A soft set theory is introduced by Molodtsov [10]. In what follows, let U be an initial universe set and X be a set of parameters. Let $\mathcal{P}(U)$ denote the power set of U and $A, B, C, \dots \subseteq X$.

Definition 2.5. A *soft set* (f, A) of X over U is defined to be the set of ordered pairs

$$(f, A) := \{(x, f(x)) : x \in X, f(x) \in \mathcal{P}(U)\},$$

where $f : X \rightarrow \mathcal{P}(U)$ such that $f(x) = \emptyset$ if $x \notin A$.

For a soft set (f, A) of X and a subset γ of U , the γ -*exclusive set* of (f, A) , denoted by $e_A(f; \gamma)$, is defined to be the set

$$e_A(f; \gamma) := \{x \in A \mid f(x) \subseteq \gamma\}.$$

For any soft sets (f, X) and (g, X) of X , we call (f, X) a *soft subset* of (g, X) , denoted by $(f, X) \tilde{\subseteq} (g, X)$, if $f(x) \subseteq g(x)$ for all $x \in X$. The *soft union* of (f, X) and (g, X) , denoted by $(f, X) \tilde{\cup} (g, X)$, is defined to be the soft set $(f \tilde{\cup} g, X)$ of X over U in which $f \tilde{\cup} g$ is defined by

$$(f \tilde{\cup} g)(x) = f(x) \cup g(x) \text{ for all } x \in X.$$

The *soft intersection* of (f, X) and (g, X) , denoted by $(f, X) \tilde{\cap} (g, X)$, is defined to be the soft set $(f \tilde{\cap} g, M)$ of X over U in which $f \tilde{\cap} g$ is defined by

$$(f \tilde{\cap} g)(x) = f(x) \cap g(x) \text{ for all } x \in M.$$

3. UNI-SOFT MIGHTY FILTERS

In what follows, we take a BE -algebra X , as a set of parameters unless specified.

Definition 3.1.([5]) A soft set (f, X) of X over U is called a *union-soft filter* (briefly, *uni-soft filter*) of X if it satisfies:

- (US1) $(\forall x \in X)(f(1) \subseteq f(x))$,
- (US2) $(\forall x, y \in X)(f(y) \subseteq f(x * y) \cup f(x))$.

Proposition 3.2.([5]) Every uni-soft filter (f, X) of X over U satisfies the following properties:

- (i) $(\forall x, y \in X)(x \leq y \Rightarrow f(y) \subseteq f(x))$.
- (ii) $(\forall x, y, z \in X)(z \leq x * y \Rightarrow f(y) \subseteq f(x) \cup f(z))$.

Definition 3.3. A soft set (f, X) of X over U is called a *union-soft mighty filter* (briefly, *uni-soft mighty filter*) of X if it satisfies (US1) and

$$(US3) \quad (\forall x, y, z \in X)(f(((x * y) * y) * x) \subseteq f(z * (y * x)) \cup f(z)).$$

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Example 3.4. Let $E = X$ be the set of parameters where $X := \{1, a, b, c, d, 0\}$ is a BE -algebra [7] with the following Cayley table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Let (f, X) be a soft set of X over $U := \mathbb{Z}$ defined as follows:

$$f : X \rightarrow \mathcal{P}(U), x \mapsto \begin{cases} \mathbb{N} & \text{if } x \in \{1, a, b\} \\ \mathbb{Z} & \text{if } x \in \{c, d, 0\}. \end{cases}$$

It is easy to check that (f, X) is a uni-soft mighty filter of X .

Proposition 3.5. *Every uni-soft mighty filter of a BE -algebra X is a uni-soft filter of X .*

Proof. Let (f, X) be a uni-soft mighty filter of X . Setting $y := 1$ in (US3), we have $f(x) = (((x * 1) * 1) * x) \subseteq f(z * (1 * x)) \cup f(z) = f(z * x) \cup (z)$. Hence (US2) holds. Therefore (f, X) is a uni-soft filter of X . \square

The converse of Proposition 3.5 is not true in general as seen in the following example.

Example 3.6. Let $E = X$ be the set of parameters where $X := \{1, a, b, c, d\}$ is a BE -algebra [7] with the following Cayley table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Let (f, X) be a soft set of X over $U := \mathbb{Z}$ defined as follows:

$$f : X \rightarrow \mathcal{P}(U), x \mapsto \begin{cases} 3\mathbb{N} & \text{if } x \in \{1\} \\ 4\mathbb{N} & \text{if } x \in \{a\} \\ 3\mathbb{Z} & \text{if } x \in \{b, c, d\}. \end{cases}$$

It is easy to check that (f, X) is a uni-soft filter of X . But it is not a uni-soft mighty filter of X , since $f(((a * c) * c) * a) = f(a) = 4\mathbb{N} \not\subseteq f(1 * (c * a)) \cup f(1) = f(1) = 3\mathbb{N}$.

We provide conditions for a uni-soft filter to be a uni-soft mighty filter.

Theorem 3.7. *Any uni-soft filter of a BE -algebra X is mighty if and only if it satisfies*

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$$(3.1) \quad (\forall x, y \in X)(f(((x * y) * y) * x)) \subseteq f(y * x).$$

Proof. Assume that a uni-soft filter (f, X) of X is mighty. Putting $z := 1$ in (US3), we have $f(((x * y) * y) * x) \subseteq f(1 * (y * x)) \cup f(1) = f(y * x)$. Hence (3.1) holds.

Conversely, suppose that the uni-soft filter (f, X) of X satisfies the condition (3.1). Using (3.1) and (US2), we have $f(((x * y) * y) * x) \subseteq f(y * x) \subseteq f(z * (y * x)) \cup f(z)$ for any $x, y \in X$. Hence (f, X) is an uni-soft mighty filter of X . \square

Proposition 3.8. *Let (f, X) be a uni-soft mighty filter of a BE -algebra X . Then $X_f := \{x \in X \mid f(x) = f(1)\}$ is a mighty filter of X .*

Proof. Clearly, $1 \in X_f$. Let $z * (y * x), z \in X_f$. Then $f(z * (y * x)) = f(1)$ and $f(z) = f(1)$. It follows from (US3) that $f(((x * y) * y) * x) \subseteq f(z * (y * x)) \cup f(z) = f(1)$. By (US1), we get $f(((x * y) * y) * x) = f(1)$. Hence $((x * y) * y) * x \in X_f$. Therefore X_f is a mighty filter of X . \square

Theorem 3.9. *Let (f, X) and (g, X) be uni-soft filters of a transitive BE -algebra such that $(f, X) \tilde{\subseteq} (g, X)$ and $f(1) = g(1)$. If (g, X) is mighty, then so is (f, X) .*

Proof. Let $x, y \in X$. Note that $y * ((y * x) * x) = (y * x) * (y * x) = 1$. Since (g, X) is a uni-soft mighty filter of a BE -algebra X , by (3.1) and $(f, X) \tilde{\subseteq} (g, X)$ we have $g(1) = g(y * ((y * x) * x)) \supseteq g((((y * x) * x) * y) * y * ((y * x) * x)) \supseteq f((((y * x) * x) * y) * y * ((y * x) * x))$. Since $f(1) = g(1)$, we get $f(y * x * (((y * x) * x) * y) * y * x) = f((((y * x) * x) * y) * y * ((y * x) * x)) = f(1)$. It follows from (US1) and (US2) that

$$\begin{aligned} f(y * x) &= f(1) \cup f(y * x) \\ &= f((y * x) * (((y * x) * x) * y) * y * x) \cup f(y * x) \\ &\supseteq f((((y * x) * x) * y) * y * x). \end{aligned} \tag{3.2}$$

Since X is transitive, we get

$$\begin{aligned} [((((y * x) * x) * y) * y) * x] & [((x * y) * y) * x] \\ & \geq ((x * y) * y) * (((y * x) * x) * y) * y \\ & \geq (((y * x) * x) * y) * (x * y) \\ & \geq x * ((y * x) * x) \\ & = (y * x) * (x * x) \\ & = (y * x) * 1 = 1. \end{aligned}$$

It follows from Proposition 3.2 that $f((((y * x) * x) * y) * y) * x \cup f(1) = f((((y * x) * x) * y) * y) * x \supseteq f(((x * y) * y) * x)$. Using (3.2), we have $f(y * x) \supseteq f((((y * x) * x) * y) * y) * x \supseteq f(((x * y) * y) * x)$. Therefore $f(y * x) \supseteq f(((x * y) * y) * x)$. By Theorem 3.7, (f, X) is a uni-soft mighty filter of X . \square

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Let (f, X) be a uni-soft filter of a transitive BE -algebra X . Define a binary relation “ \sim^f ” on X by putting $x \sim^f y$ if and only if $f(x * y) = f(y * x) = f(1)$ for any $x, y \in X$.

Lemma 3.10. *The relation “ \sim^f ” is an equivalence relation on a transitive BE -algebra X .*

Proof. For any $x \in X$, $x * x = 1$ by (BE1). So $f(x * x) = f(1)$, hence $x \sim^f x$, which \sim^f is reflexive. Suppose that $x \sim^f y$ for any $x, y \in X$. Then $f(x * y) = f(y * x) = f(1)$. Hence \sim^f is symmetric. Assume that $x \sim^f y$ and $y \sim^f z$ for any $x, y, z \in X$. Then $f(x * y) = f(y * x) = f(1)$ and $f(y * z) = f(z * y) = f(1)$. By transitivity, $(x * y) * [(y * z) * (x * z)] = 1$ and $(z * y) * [(y * x) * (z * x)] = 1$. By Proposition 3.2, we have $f(x * z) \subseteq f(y * z) \cup f(x * y) = f(1)$ and $f(z * x) \subseteq f(y * x) \cup f(z * y) = f(1)$. Hence $f(z * x) = f(x * z) = f(1)$, i.e., $x \sim^f z$. Thus \sim^f is an equivalence relation on X . \square

Lemma 3.11. *The relation “ \sim^f ” is a congruence relation on a transitive BE -algebra X .*

Proof. If $x \sim^f y$ and $u \sim^f v$ for any $x, y, u, v \in X$, then $f(x * y) = f(y * x) = f(1)$ and $f(u * v) = f(v * u) = f(1)$. By transitivity, $(u * v) * [(x * u) * (x * v)] = 1$ and $(v * u) * [(x * v) * (x * u)] = 1$, it follows from Proposition 3.2 that $f(1) = f(u * v) \supseteq f((x * u) * (x * v))$ and $f(1) = f(v * u) \supseteq f((x * v) * (x * u))$. Hence $f((x * u) * (x * v)) = f(1)$ and $f((x * v) * (x * u)) = f(1)$. Therefore $x * u \sim^f x * v$. By a similar way, we can prove that $x * v \sim^f y * v$. Hence $x * u \sim^f y * v$. Therefore \sim^f is a congruence relation on X . \square

X is decomposed by the congruence relation \sim^f . The class containing x is denoted by f_x . Denote $X/f := \{f_x | x \in X\}$. We define a binary relation \bullet on X/f by $f_x \bullet f_y := f_{x * y}$. This definition is well defined since \sim^f is a congruence relation on X .

Lemma 3.12. $f_1 = X_f$.

Proof. $f_1 = \{x \in X | 1 \sim^f x\} = \{x \in X | f(1 * x) = f(x * 1) = f(1)\} = \{x \in X | f(x) = f(1)\} = X_f$. \square

Theorem 3.13. *Let (X, f) be a uni-soft filter of a transitive BE -algebra X . Then $(X/f; \bullet, f_1)$ is a transitive BE -algebra.*

Proof. It is straightforward. \square

Theorem 3.14. *Let X be a transitive BE -algebra. If every filter of the quotient algebra X/f is mighty, then a uni-soft filter of X is mighty.*

Proof. Suppose that every filter of the quotient algebra X/f is mighty and let $x, y \in X$ be such that $y * x \in f_1$. Then $f(y * x) = f(1)$ and so $f_y \bullet f_x \in f_1$. Since $\{f_1\}$ is a mighty filter of X/f , it follows from Theorem 2.4 that $f_{((x * y) * y) * x} = ((f_x \bullet f_y) \bullet f_y) \bullet f_x \in f_1$. Hence $f((((x * y) * y) * x) * x) = f(1)$. Therefore $f(y * x) = f((((x * y) * y) * x)$. Thus (f, X) is a uni-soft mighty filter by Theorem 3.7. \square

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Theorem 3.15. *A uni-soft set (X, f) of a BE -algebra X is a uni-soft mighty filter of X if and only if the set $e_X(f; \gamma) := \{x \in X | f(x) \subseteq \gamma\}$ is a mighty filter of X for all $\gamma \in \mathcal{P}([0, 1])$ whenever it is nonempty.*

Proof. Suppose that (f, X) is a uni-soft mighty filter of X . Let $x, y, z \in X$ and $\gamma \in \mathcal{P}([0, 1])$ be such that $z * (y * x) \in e_X(f; \gamma)$ and $z \in e_X(f; \gamma)$. Then $f(z * (y * x)) \subseteq \gamma$ and $f(z) \subseteq \gamma$. It follows from (US1) and (US3) that $f(1) \subseteq f_X(((x * y) * y) * x) \subseteq f(z * (y * x)) \cup f(z) \subseteq \gamma$. Hence $1 \in e_X(f; \gamma)$ and $((x * y) * y) * x \in e_X(f; \gamma)$, and therefore $e_X(f; \gamma)$ is a mighty filter of X .

Conversely, assume that $e_X(f; \gamma)$ is a mighty filter of X for all $\gamma \in \mathcal{P}([0, 1])$ with $e_X(f; \gamma) \neq \emptyset$. For any $x \in X$, let $f(x) = \gamma$. Then $x \in e_X(f; \gamma)$. Since $e_X(f; \gamma)$ is a mighty filter of X , we have $1 \in e_X(f; \gamma)$ and so $f(x) = \gamma \supseteq f(1)$. For any $x, y, z \in X$, let $f(z * (y * x)) = \gamma_{z*(y*x)}$ and $f(z) = \gamma_z$. Let $\gamma := \gamma_{z*(y*x)} \cup \gamma_z$. Then $z * (y * x) \in e_X(f; \gamma)$ and $z \in e_X(f; \gamma)$ which imply that $((x * y) * y) * x \in e_X(f; \gamma)$. Hence $f(((x * y) * y) * x) \subseteq \gamma = \gamma_{z*(y*x)} \cup \gamma_z = f(z * (y * x)) \cup f(z)$. Thus (f, X) is a uni-soft mighty filter of X . \square

We call $e_X(f; \gamma)$ a γ -exclusive filter of a BE -algebra X .

Theorem 3.16. *Every filter of a BE -algebra can be represented as a γ -exclusive set of a uni-soft mighty filter, i.e., given a filter F a BE -algebra X , there exists a uni-soft mighty filter (f, X) of X over U such that F is the γ -exclusive set of (f, X) for a non-empty subset γ of U .*

Proof. Let F be a filter of a BE -algebra X . For a subset γ of U , define a soft set (f, X) over U by

$$f : X \rightarrow \mathcal{P}(U), x \mapsto \begin{cases} \gamma & \text{if } x \in F, \\ U & \text{if } x \notin F. \end{cases}$$

Obviously, $F = e_X(f; \gamma)$. We now prove that (f, X) is a uni-soft mighty filter of X . Since $1 \in F = e_X(f; \gamma)$, we have $f(1) = \gamma \subseteq f(x)$ for all $x \in X$. Let $x, y, z \in X$. If $z * (y * x), z \in F$, then $((x * y) * y) * x \in F$ because F is a mighty filter of X . Hence $f(z * (y * x)) = f(z) = f(((x * y) * y) * x) = \gamma$, and so $f(z * (y * x)) \cup f(z) \supseteq f(((x * y) * y) * x)$. If $z * (y * x) \in F$ and $z \notin F$, then $f(z * (y * x)) = \gamma$ and $f(z) = U$ which imply that $f(z * (y * x)) \cup f(z) = \gamma \cup U = U \supseteq f(((x * y) * y) * x)$. Similarly, if $z * (y * x) \notin F$ and $z \in F$, then $f(z * (y * x)) \cup f(z) \supseteq f(((x * y) * y) * x)$. Obviously, if $z * (y * x) \notin F$ and $z \notin F$, then $f(z * (y * x)) \cup f(z) \supseteq f(((x * y) * y) * x)$. Therefore (f, X) is a uni-soft mighty filter of X . \square

Any filter of a BE -algebra X cannot be represented as a γ -exclusive set of a uni-soft mighty filter (f, X) of X over U (see Example 3.17).

Example 3.17. Let $X = \{1, a, b, c, d, 0\}$ be the BE -algebra as in Example 3.4. Given $U = X$, consider a soft set (f, X) over U which is defined by

$$f : X \rightarrow \mathcal{P}(U), x \mapsto \begin{cases} \{c\} & \text{if } x \in \{1, a, b\}, \\ \{1, c\} & \text{if } x \in \{c, d, 0\}. \end{cases}$$

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Then (f, X) is a uni-soft mighty filter of X . The γ -exclusive sets of (f, X) are described as follows:

$$e_X(f; \gamma) = \begin{cases} X & \text{if } \gamma \supseteq \{1, c\}, \\ \{1, a, b\} & \text{if } \{c\} \subseteq \gamma \subsetneq \{1, c\} \\ \emptyset & \text{otherwise.} \end{cases}$$

The filter $\{1\}$ cannot be a γ -exclusive set $e_X(f; \gamma)$, since there is no $\gamma \subseteq U$ such that $e_X(f; \gamma) = \{1\}$.

Proposition 3.18. *Let (f, X) be a uni-soft filter of a transitive BE-algebra X . The mapping $\gamma : X \rightarrow X/f$, given by $\gamma(x) := f_x$, is a surjective homomorphism, and $\text{Ker } \gamma = \{x \in X | \gamma(x) = f_1\} = X_f$.*

Proof. Let $f_x \in X/f$. Then there exists an element $x \in X$ such that $\gamma(x) = f_x$. Hence γ is surjective. For any $x, y \in X$, we have

$$\gamma(x * y) = f_{x*y} = f_x \bullet f_y = \gamma(x) \bullet \gamma(y).$$

Thus γ is a homomorphism. Moreover, $\text{Ker } \gamma = \{x \in X | \gamma(x) = f_1\} = \{x \in X | x \sim^f 1\} = \{x \in X | f(x) = f(1)\} = X_f$. □

Example 3.19. Let $E = X$ be the set of parameters where $X := \{1, a, b, c, d, 0\}$ is a transitive BE-algebra [6] with the following Cayley table:

*	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	b	c	b	c
b	1	a	1	b	a	d
c	1	a	1	1	a	a
d	1	1	1	b	1	b
0	1	1	1	1	1	1

Let (f, X) be a soft set of X over U defined as follows:

$$f : X \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \gamma_1 & \text{if } x \in \{1, b, c\} \\ \gamma_2 & \text{if } x \in \{a, d, 0\}, \end{cases}$$

where γ_1 and γ_2 are subsets of U with $\gamma_1 \subsetneq \gamma_2$. Then (f, X) is a uni-soft mighty filter over U . By Proposition 3.5, it is a uni-soft filter over U . Then $X_f = \{x \in X | f(x) = f(1)\} = \{1, b, c\}$. Define $x \sim^f y$ if and only if $f(x * y) = f(y * x) = f(1)$. Then $f_1 = \{x \in X | x \sim^f 1\} = \{x \in X | f(x * 1) = f(1 * x) = f(1)\} = \{1, b, c\}$, $f_a = \{x \in X | x \sim^f a\} = \{x \in X | f(x * a) = f(a * x) = f(1)\} = \{a\}$, $f_d = \{x \in X | x \sim^f d\} = \{x \in X | f(x * d) = f(d * x) = f(1)\} = \{d\}$, and $f_0 = \{x \in X | x \sim^f 0\} = \{x \in X | f(x * 0) = f(0 * x) = f(1)\} = \{0\}$. Hence $X/f = \{f_1, f_a, f_d, f_0\}$. Let $\varphi : X \rightarrow X/f$ be a map defined by $\varphi(1) = \varphi(b) = \varphi(c) = f_1$, $\varphi(a) = f_a$, $\varphi(d) = f_d$, $\varphi(0) = f_0$. It is easy to check

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that φ is a homomorphism and $Ker\varphi = \{x \in X | \varphi(x) = f_1\} = \{x \in X | x \sim^f 1\} = \{x \in X | f(x) = f(1)\} = X_f$.

Proposition 3.20. *Let $\mu : X \rightarrow Y$ be an epimorphism of BE -algebras. If (f, Y) is a uni-soft filter of Y , then $(f \circ \mu, X)$ is a uni-soft filter of X .*

Proof. For any $x \in X$, we have $(f \circ \mu)(x) = f(\mu(x)) \supseteq f(1_Y) = f(\mu(1_X)) = (f \circ \mu)(1_X)$ and $(f \circ \mu)(y) = f(\mu(y)) \subseteq f(a *_Y \mu(y)) \cup f(a)$ for any $a \in Y$. Let x be any preimage of a under μ . Then

$$\begin{aligned} (f \circ \mu)(y) &\subseteq f(a *_Y \mu(y)) \cup f(a) \\ &= f(\mu(x) *_Y \mu(y)) \cup f(\mu(x)) \\ &= f(\mu(x *_X y)) \cup f(\mu(x)) \\ &= (f \circ \mu)(x *_X y) \cup (f \circ \mu)(x). \end{aligned}$$

Therefore $(f \circ \mu, X)$ is a uni-soft filter of X . □

Proposition 3.21. *Let (f, X) be a uni-soft filter of a transitive BE -algebra X . If J is a filter of X , then J/f is a filter of X/f .*

Proof. Let (f, X) be a uni-soft filter of X and let J be a filter of X . For any $x * y, x \in J$, we obtain $y \in J$. Hence for any $f_x \bullet f_y = f_{x*y}, f_x \in J/f$, we have $f_y \in J/f$. Thus J/f is a filter of X/f . □

Theorem 3.22. *Let (f, X) be a uni-soft filter of a transitive BE -algebra X . If J^* is a filter of a transitive BE -algebra X/f , then there exists a filter $J = \{x \in X | f_x \in J^*\}$ in X such that $J/f = J^*$.*

Proof. Since J^* is a filter of X/f , so $f_x \bullet f_y = f_{x*y}, f_x \in J^*$ imply $f_y \in J^*$ for any $f_x, f_y \in J^*$. Therefore $x * y, x \in J$ imply $y \in J$ for any $x, y \in J$. Therefore J is a filter of X . By Proposition 3.18, we have

$$\begin{aligned} J/f &= \{f_j | j \in J\} \\ &= \{f_j | \exists f_x \in J^* \text{ such that } j \sim^f x\} \\ &= \{f_j | \exists f_x \in J^* \text{ such that } f_x = f_j\} \\ &= \{f_j | f_j \in J^*\} = J^*. \end{aligned}$$

Theorem 3.23. *Let (f, X) be a uni-soft filter of a transitive BE -algebra X . If J is a filter of X , then $\frac{X/f}{J/f} \cong X/J$.*

Proof. Note that $\frac{X/f}{J/f} = \{[f_x]_{J/f} | f_x \in X/f\}$. If we define $\varphi : \frac{X/f}{J/f} \rightarrow X/J$ by $\varphi([f_x]_{J/f}) = [x]_J = \{y \in X | x \sim^J y\}$, then it is well defined. In fact, suppose that $[f_x]_{J/f} = [f_y]_{J/f}$. Then

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$f_x \sim^{J/f} f_y$ and so $f_{x*y} = f_x \bullet f_y \in J/f$. Hence $x * y \in J$. Therefore $x \sim^J y$, i.e., $[x]_J = [y]_J$. Given $[f_x]_{J/f}, [f_y]_{J/f} \in \frac{X/f}{J/f}$, we have

$$\begin{aligned} \varphi([f_x]_{J/f} \bullet [f_y]_{J/f}) &= \varphi([f_x \bullet f_y]_{J/f}) \\ &= [x * y]_J = [x]_J * [y]_J \\ &= \varphi([f_x]_{J/f}) * \varphi([f_y]_{J/f}). \end{aligned}$$

Hence φ is a homomorphism.

Obviously, φ is onto. Finally, we show that φ is one-to-one. If $\varphi([f_x]_{J/f}) = \varphi([f_y]_{J/f})$, then $[x]_J = [y]_J$, i.e., $x \sim^J y$. If $f_a \in [f_x]_{J/f}$, then $f_a \sim^{J/f} f_x$ and hence $f_{a*x} \in J/f$. It follows that $a * x \in J$, i.e., $a \sim^J x$. Since \sim^J is an equivalence relation, $a \sim^J y$ and so $J_a = J_y$. Hence $a * y \in J$ and so $f_{a*y} \in J/f$. Therefore $f_a \sim^{J/f} f_y$. Hence $f_a \in [f_y]_{J/f}$. Thus $[f_x]_{J/f} \subseteq [f_y]_{J/f}$. Similarly, we obtain $[f_y]_{J/f} \subseteq [f_x]_{J/f}$. Therefore $[f_x]_{J/f} = [f_y]_{J/f}$. It is completes the proof. \square

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INCLUSION RELATIONSHIPS FOR SOME SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH GENERALIZED BESSEL FUNCTIONS

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ABSTRACT. This paper introduces new subclasses of analytic functions and investigate the inclusion properties of these subclasses using the generalized Bessel functions of the first kind. We also derive a variety of special cases and corollaries of the main results.

1. Introduction

Let $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ be an open disk and let \mathcal{A} be the class of functions f of the form

$$f(z) = z + \sum_{n=1}^{\infty} a_{n+1} z^{n+1}, \tag{1.1}$$

which are analytic in \mathcal{U} and satisfy the following normalization condition:

$$f(0) = f'(0) - 1 = 0.$$

Let \mathcal{S} subclasses of \mathcal{A} containing all functions which are univalent, \mathcal{C} is the close-to-convex, $\mathcal{S}^*(\alpha)$ starlike of order α and $\mathcal{K}(\alpha)$ is the convex of order α in \mathcal{U} . For functions $f_j \in \mathcal{A}$ given by

$$f_j(z) = z + \sum_{n=1}^{\infty} a_{n+1,j} z^{n+1}, \quad (j = 1, 2)$$

we state the Hadamard product of f_1 and f_2 by

$$(f_1 * f_2)(z) := z + \sum_{n=1}^{\infty} a_{n+1,1} a_{n+1,2} z^{n+1}, \quad (z \in \mathcal{U}).$$

$f(z)$ is named subordinate to $g(z)$ if $\exists w(z)$ analytic in \mathcal{U} in such a way that

$$w(0) = 0, \quad |w(z)| < 1 \quad (z \in \mathcal{U}) \quad \text{and} \quad f(z) = g(w(z)) \quad (z \in \mathcal{U}).$$

and we write $f(z) \prec g(z)$. Let \mathcal{M} be the class of analytic functions $\varphi(z)$ in \mathcal{U} with $\varphi(0) = 1$. We consider that \mathcal{S} denote the subclasses of \mathcal{A} containing all functions which are univalent, \mathcal{C} is the close-to-convex, $\mathcal{S}^*(\alpha)$ starlike of order α and $\mathcal{K}(\alpha)$ denote the convex of order α in \mathcal{U} . Using the subordination between analytic functions, now we

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define the subclasses of \mathcal{A} for $0 \leq \alpha, \beta < 1$ and $\varphi, \psi \in \mathcal{N}$ (cf., [4, 5, 9, 11]):

$$\mathcal{S}^*(\alpha; \varphi) := \left\{ f \in \mathcal{A} : \frac{1}{1-\alpha} \left(\frac{zf'(z)}{f(z)} - \alpha \right) \prec \varphi(z), z \in \mathcal{U} \right\},$$

$$\mathcal{K}(\alpha; \varphi) := \left\{ f \in \mathcal{A} : \frac{1}{1-\alpha} \left(1 + \frac{zf''(z)}{f'(z)} - \alpha \right) \prec \varphi(z), z \in \mathcal{U} \right\}$$

and

$$\mathcal{C}(\alpha, \beta; \varphi, \psi) := \left\{ f \in \mathcal{A} : \text{for some } g \in \mathcal{S}^*(\alpha; \varphi) \text{ s.t. } \frac{1}{1-\beta} \left(\frac{zf'(z)}{g(z)} - \beta \right) \prec \psi(z), z \in \mathcal{U} \right\}.$$

clearly

$$f(z) \in \mathcal{K}(\alpha; \varphi) \iff zf'(z) \in \mathcal{S}^*(\alpha; \varphi).$$

The particular choices of φ and ψ yields the familier subclasses of \mathcal{A} as:

$$\mathcal{S}^* \left(\alpha; \frac{1+z}{1-z} \right) = \mathcal{S}^*(\alpha), \quad \mathcal{K} \left(\alpha; \frac{1+z}{1-z} \right) = \mathcal{K}(\alpha) \quad \text{and} \quad \mathcal{C} \left(0, 0; \frac{1+z}{1-z}, \frac{1+z}{1-z} \right) = \mathcal{C}.$$

Recently, Deniz et al. [7] gave the transformation $\phi_{l,b,c}(z)$ of generalized Bessel function of first kind of order l (cf. [1]):

$$\mathcal{W}_{l,b,c}(z) = \sum_{k=0}^{\infty} \frac{(-c)^k}{k! \Gamma(l+k+\frac{b+1}{2})} \left(\frac{z}{2} \right)^{2k+l} \quad (z \in \mathbb{C}, l, b, c \in \mathbb{R}). \tag{1.2}$$

by

$$\begin{aligned} \phi_{l,b,c}(z) &= 2^l \Gamma \left(l + \frac{b+1}{2} \right) z^{1-l/2} \mathcal{W}_{l,b,c}(z^{1/2}) \\ &= z + \sum_{k=1}^{\infty} \frac{(-c)^k}{4^k \cdot (\nu)_k} \frac{z^{k+1}}{k!} \quad \left(\nu = l + \frac{b+1}{2} \notin \mathbb{Z}_0^- := \{0, -1, -2, \dots\} \right), \end{aligned}$$

where $(\lambda)_k$ represents the Pochhammer symbol given by

$$(\lambda)_k = \lambda(\lambda+1)(\lambda+2) \cdots (\lambda+k-1) \quad (k \in \mathbb{N} := \{1, 2, 3, \dots\}) \quad \text{and} \quad (\lambda)_0 = 1$$

Subsequently, by using $\phi_{l,b,c}(z)$, Deniz [6] developed the operator B_ν^c as follows:

$$B_\nu^c f(z) = \phi_{l,b,c}(z) * f(z) = z + \sum_{k=1}^{\infty} \frac{(-c)^k a_{k+1}}{4^k \cdot (\nu)_k} \frac{z^{k+1}}{k!} \quad (z \in \mathbb{C}). \tag{1.3}$$

clearly from (1.3),

$$z (B_{\nu+1}^c f(z))' = \nu B_\nu^c f(z) - (\nu-1) B_{\nu+1}^c f(z), \tag{1.4}$$

where

$$\nu = l + \frac{b+1}{2} \notin \mathbb{Z}_0^-.$$

Indeed, the operator B_ν^c given by (1.3) provides an elementary transform of the generalized hypergeometric function,

$$B_\nu^c f(z) = z {}_0F_1 \left(- ; \nu; -\frac{c}{4} z \right) * f(z)$$

and

$$\phi_{\nu,c} \left(-\frac{c}{4} z \right) = z {}_0F_1(- ; \nu; z).$$

In this article, we investigate several inclusion relationships for each of the following subclasses of \mathcal{A} , associated with $\mathcal{W}_{l,b,c}(z)$ (see also [15], [16] and [17] for inclusion relationships of various other function classes). Indeed, for $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-, 0 \leq \alpha, \beta < 1$ and $\varphi, \psi \in \mathcal{N}$ we define

$$\begin{aligned} \mathcal{S}_\nu^c(\alpha; \varphi) &:= \{f \in \mathcal{A} : B_\nu^c f(z) \in \mathcal{S}^*(\alpha; \varphi), z \in \mathcal{U}\}, \\ \mathcal{K}_\nu^c(\alpha; \varphi) &:= \{f \in \mathcal{A} : B_\nu^c f(z) \in \mathcal{K}(\alpha; \varphi), z \in \mathcal{U}\} \\ \mathcal{C}_\nu^c(\alpha, \beta; \varphi, \psi) &:= \{f \in \mathcal{A} : B_\nu^c f(z) \in \mathcal{C}(\alpha, \beta; \varphi, \psi), z \in \mathcal{U}\}. \end{aligned}$$

Also,

$$f(z) \in \mathcal{K}_\nu^c(\alpha; \varphi) \iff z f'(z) \in \mathcal{S}_\nu^c(\alpha; \varphi). \tag{1.5}$$

Particularly, we set

$$\mathcal{S}_\nu^c\left(\alpha; \left(\frac{1+Az}{1+Bz}\right)^\delta\right) = \mathcal{S}_\nu^c(\alpha; A, B; \delta), \quad (0 < \delta \leq 1; -1 \leq B < A \leq 1)$$

and

$$\mathcal{K}_\nu^c\left(\alpha; \left(\frac{1+Az}{1+Bz}\right)^\delta\right) = \mathcal{K}_\nu^c(\alpha; A, B; \delta), \quad (0 < \delta \leq 1; -1 \leq B < A \leq 1).$$

We need the following results for the investigation of our inclusion properties.

Lemma 1. [8] *Let $\phi(z)$ be analytic and convex univalent in \mathcal{U} with $\phi(0) = 1$ and $Re\{\eta\phi(z) + \sigma\} > 0$ ($\eta, \sigma \in \mathbb{C}$). If $p(z)$ is analytic in \mathcal{U} with $p(0) = 1$, then the subordination*

$$p(z) + \frac{z p'(z)}{\eta p(z) + \sigma} \prec \phi(z) \quad (z \in \mathcal{U})$$

implies that

$$p(z) \prec \phi(z) \quad (z \in \mathcal{U}).$$

Lemma 2. [12] *Let $h(z)$ be convex in \mathcal{U} with $h(0) = 1$. Let $Q(z)$ be analytic in \mathcal{U} with $Re\{Q(z)\} \geq 0$ ($z \in \mathcal{U}$). If $q(z)$ is analytic in \mathcal{U} such that $q(0) = h(0)$, then we have*

$$q(z) + Q(z)zq'(z) \prec h(z) \quad (z \in \mathcal{U})$$

which implies that

$$q(z) \prec h(z) \quad (z \in \mathcal{U}).$$

2. The Main Inclusion Relationships

Theorem 3. *Let $f \in \mathcal{A}$, $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ and $\alpha + \nu > 1$ ($0 \leq \alpha < 1$). Then*

$$f \in \mathcal{S}_\nu^c(\alpha; \varphi) \implies f \in \mathcal{S}_{\nu+1}^c(\alpha; \varphi)$$

or equivalently,

$$\mathcal{S}_\nu^c(\alpha; \varphi) \subset \mathcal{S}_{\nu+1}^c(\alpha; \varphi) \quad (\varphi \in \mathcal{N}).$$

Proof. Let $f \in \mathcal{S}_\nu^c(\alpha; \varphi)$ and set

$$q(z) = \frac{1}{1-\alpha} \left(\frac{z(B_{\nu+1}^c f(z))'}{B_{\nu+1}^c f(z)} - \alpha \right), \tag{2.1}$$

where $q(z)$ is analytic in \mathcal{U} with $q(0) = 1$. From (1.4) we get,

$$\nu \frac{B_\nu^c f(z)}{B_{\nu+1}^c f(z)} = \frac{z(B_{\nu+1}^c f(z))'}{B_{\nu+1}^c f(z)} + (\nu - 1). \tag{2.2}$$

By using (2.1) and (2.2), we have

$$\frac{B_\nu^c f(z)}{B_{\nu+1}^c f(z)} = \frac{1}{\nu} [(1 - \alpha)q(z) + \alpha + \nu - 1]. \tag{2.3}$$

Now, by applying logarithmic differentiation on (2.3), we get

$$\frac{z(B_\nu^c f(z))'}{B_\nu^c f(z)} = \frac{z(B_{\nu+1}^c f(z))'}{B_{\nu+1}^c f(z)} + \frac{(1 - \alpha)zq'(z)}{(1 - \alpha)q(z) + \alpha + \nu - 1}$$

in view of (2.1), yields

$$\frac{1}{1 - \alpha} \left(\frac{z(B_\nu^c f(z))'}{B_\nu^c f(z)} - \alpha \right) = q(z) + \frac{zq'(z)}{(1 - \alpha)q(z) + \alpha + \nu - 1} \quad (z \in \mathcal{U}). \tag{2.4}$$

Finally, by applying Lemma 1 and (2.4), we have $q(z) \prec \varphi(z)$, hence $f \in \mathcal{S}_{\nu+1}^c(\alpha; \varphi)$. \square

Theorem 4. *Let $f \in \mathcal{A}$, $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ and $\alpha + \nu > 1$ ($0 \leq \alpha < 1$). Then*

$$f \in \mathcal{K}_\nu^c(\alpha; \varphi) \implies f \in \mathcal{K}_{\nu+1}^c(\alpha; \varphi)$$

or equivalently,

$$\mathcal{K}_\nu^c(\alpha; \varphi) \subset \mathcal{K}_{\nu+1}^c(\alpha; \varphi) \quad (\varphi \in \mathcal{N}).$$

Proof. Using (1.5) and Theorem 3, we have

$$\begin{aligned} f(z) \in \mathcal{K}_\nu^c(\alpha; \varphi) &\iff B_\nu^c f(z) \in \mathcal{K}(\alpha; \varphi) \\ &\iff z(B_\nu^c f(z))' \in \mathcal{S}^*(\alpha; \varphi) \\ &\iff B_\nu^c(zf'(z)) \in \mathcal{S}^*(\alpha; \varphi) \\ &\iff zf'(z) \in \mathcal{S}_\nu^c(\alpha; \varphi) \\ &\implies zf'(z) \in \mathcal{S}_{\nu+1}^c(\alpha; \varphi) \\ &\iff B_{\nu+1}^c(zf'(z)) \in \mathcal{S}^*(\alpha; \varphi) \\ &\iff z(B_{\nu+1}^c f(z))' \in \mathcal{S}^*(\alpha; \varphi) \\ &\iff B_{\nu+1}^c f(z) \in \mathcal{K}(\alpha; \varphi) \\ &\iff f(z) \in \mathcal{K}_{\nu+1}^c(\alpha; \varphi). \end{aligned}$$

\square

If we take

$$\varphi(z) = \left(\frac{1 + Az}{1 + Bz} \right)^\delta \quad (-1 \leq B < A \leq 1; 0 < \delta \leq 1; z \in \mathcal{U})$$

in Theorems 3 and 4, then we obtain the corollaries as:

Corollary 5. Let $f \in \mathcal{A}$, $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ and $\alpha + \nu > 1$ ($0 \leq \alpha < 1$). Then

$$\mathcal{S}_\nu^c(\alpha; A, B; \delta) \subset \mathcal{S}_{\nu+1}^c(\alpha; A, B; \delta) \quad (-1 \leq B < A \leq 1; 0 < \delta \leq 1)$$

and

$$\mathcal{K}_\nu^c(\alpha; A, B; \delta) \subset \mathcal{K}_{\nu+1}^c(\alpha; A, B; \delta) \quad (-1 \leq B < A \leq 1; 0 < \delta \leq 1).$$

Theorem 6. Let $f \in \mathcal{A}$, $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ and $\alpha + \nu > 1$ ($0 \leq \alpha < 1$). Then

$$f \in \mathcal{C}_\nu^c(\alpha, \beta; \varphi, \psi) \implies f \in \mathcal{C}_{\nu+1}^c(\alpha, \beta; \varphi, \psi)$$

or equivalently,

$$\mathcal{C}_\nu^c(\alpha, \beta; \varphi, \psi) \subset \mathcal{C}_{\nu+1}^c(\alpha, \beta; \varphi, \psi) \quad (0 \leq \alpha, \beta < 1; \varphi, \psi \in \mathcal{N}).$$

Proof. Let $f \in \mathcal{C}_\nu^c(\alpha, \beta; \varphi, \psi)$. Then \exists a function $g \in \mathcal{S}_\nu^c(\alpha; \varphi) \ni$

$$\frac{1}{1-\beta} \left(\frac{z(B_\nu^c f(z))'}{B_\nu^c g(z)} - \beta \right) \prec \psi(z), \quad (0 \leq \beta < 1, z \in \mathcal{U}).$$

Now let

$$\omega(z) = \frac{1}{1-\beta} \left(\frac{z(B_{\nu+1}^c f(z))'}{B_{\nu+1}^c g(z)} - \beta \right), \tag{2.5}$$

where $\omega(z)$ is analytic in \mathcal{U} with $\omega(0) = 1$. Making use of (1.4) we also have

$$\begin{aligned} \frac{z(B_\nu^c f(z))'}{B_\nu^c g(z)} &= \frac{B_\nu^c(zf'(z))}{B_\nu^c g(z)} \\ &= \frac{z(B_{\nu+1}^c(zf'(z)))' + (\nu-1)B_{\nu+1}^c(zf'(z))}{z(B_{\nu+1}^c g(z))' + (\nu-1)B_{\nu+1}^c g(z)} \\ &= \left[\frac{z(B_{\nu+1}^c(zf'(z)))'}{B_{\nu+1}^c g(z)} + (\nu-1)\frac{B_{\nu+1}^c(zf'(z))}{B_{\nu+1}^c g(z)} \right] \cdot \left[\frac{z(B_{\nu+1}^c g(z))'}{B_{\nu+1}^c g(z)} + \nu - 1 \right]^{-1} \end{aligned} \tag{2.6}$$

By Theorem 3,

$$g \in \mathcal{S}_\nu^c(\alpha; \varphi) \implies g \in \mathcal{S}_{\nu+1}^c(\alpha; \varphi),$$

therefore, we set

$$\vartheta(z) = \frac{1}{1-\alpha} \left(\frac{z(B_{\nu+1}^c g(z))'}{B_{\nu+1}^c g(z)} - \alpha \right), \tag{2.7}$$

where $\Re(\vartheta(z)) > 0$ ($z \in \mathcal{U}$). Applying (2.5) and (2.7) into (2.6) we have

$$\frac{z(B_\nu^c f(z))'}{B_\nu^c g(z)} = \frac{\left[z(B_{\nu+1}^c(zf'(z)))' \right] \cdot [B_{\nu+1}^c g(z)]^{-1} + (\nu-1)[(1-\beta)\omega(z) + \beta]}{(1-\alpha)\vartheta(z) + \alpha + \nu - 1} \tag{2.8}$$

The logarithmic differentiation of (2.5) gives

$$\frac{z(B_{\nu+1}^c(zf'(z)))'}{B_{\nu+1}^c g(z)} = (1-\beta)z\omega'(z) + [(1-\alpha)\vartheta(z) + \alpha] \cdot [(1-\beta)\omega(z) + \beta],$$

which, in conjunction with (2.8), yields

$$\frac{1}{1 - \beta} \left(\frac{z(B_\nu^c f(z))'}{B_\nu^c g(z)} - \beta \right) = \omega(z) + \frac{z\omega'(z)}{(1 - \alpha)\vartheta(z) + \alpha + \nu - 1}.$$

Since $\alpha + \nu > 1$ and $\vartheta(z) \prec \varphi(z)$ in \mathcal{U} ,

$$\Re\{(1 - \alpha)\vartheta(z) + \alpha + \nu - 1\} > 0 \quad (z \in \mathcal{U}).$$

Finally, by applying Lemma 2, we have $\omega(z) \prec \psi(z)$, so that $f \in \mathcal{C}_{\nu+1}^c(\alpha, \beta; \varphi, \psi)$. \square

3. Inclusion Relationships Involving the Integral Operator F_σ

The generalized Bernardi-Libera-Livingston integral operator F_σ ($\sigma > -1$) (cf. [3, 10, 13]) considered here and is defined by

$$F_\sigma(f) := F_\sigma(f)(z) = \frac{\sigma + 1}{z^\sigma} \int_0^z t^{\sigma-1} f(t) dt \quad (f \in \mathcal{A}; \sigma > -1). \quad (3.1)$$

Theorem 7. *Let $f(z) \in \mathcal{A}$, $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ and $\sigma \geq 0$. If $f \in \mathcal{S}_\nu^c(\alpha; \varphi)$ ($0 \leq \alpha < 1$; $\varphi \in \mathcal{N}$), then $F_\sigma(f) \in \mathcal{S}_\nu^c(\alpha; \varphi)$ ($0 \leq \alpha < 1$; $\varphi \in \mathcal{N}$).*

Proof. Let $f \in \mathcal{S}_\nu^c(\alpha; \varphi)$ and set

$$q(z) = \frac{1}{1 - \alpha} \left(\frac{z(B_{\nu+1}^c f(z))'}{B_{\nu+1}^c f(z)} - \alpha \right), \quad (3.2)$$

such that $q(z)$ is analytic in \mathcal{U} with $q(0) = 1$. Utilizing (3.1) we obtain,

$$\nu \frac{B_\nu^c f(z)}{B_{\nu+1}^c f(z)} = \frac{z(B_{\nu+1}^c f(z))'}{B_{\nu+1}^c f(z)} + (\nu - 1). \quad (3.3)$$

By using (3.2) and (3.3), we conclude that

$$\frac{B_\nu^c f(z)}{B_{\nu+1}^c f(z)} = \frac{1}{\nu} [(1 - \alpha)q(z) + \alpha + \nu - 1]. \quad (3.4)$$

By applying logarithmic differentiation on (3.4), we get

$$\frac{z(B_\nu^c f(z))'}{B_\nu^c f(z)} = \frac{z(B_{\nu+1}^c f(z))'}{B_{\nu+1}^c f(z)} + \frac{(1 - \alpha)zq'(z)}{(1 - \alpha)q(z) + \alpha + \nu - 1}$$

in view of (3.2), yields

$$\frac{1}{1 - \alpha} \left(\frac{z(B_\nu^c f(z))'}{B_\nu^c f(z)} - \alpha \right) = q(z) + \frac{zq'(z)}{(1 - \alpha)q(z) + \alpha + \nu - 1} \quad (z \in \mathcal{U}). \quad (3.5)$$

Finally, by applying Lemma 1 and (3.5), we have $q(z) \prec \varphi(z)$, hence $f \in \mathcal{S}_{\nu+1}^c(\alpha; \varphi)$. \square

Theorem 8. *Let $f \in \mathcal{A}$, $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ and $\sigma \geq 0$. If $f \in \mathcal{K}_\nu^c(\alpha; \varphi)$ ($0 \leq \alpha < 1$; $\varphi \in \mathcal{N}$), then $F_\sigma(f) \in \mathcal{K}_\nu^c(\alpha; \varphi)$ ($0 \leq \alpha < 1$; $\varphi \in \mathcal{N}$).*

Proof. Using (1.5) along with Theorem 7, we see that

$$\begin{aligned} f \in \mathcal{K}_\nu^c(\alpha; \varphi) &\iff zf'(z) \in \mathcal{S}_\nu^c(\alpha; \varphi) \\ &\implies F_\sigma(zf'(z)) \in \mathcal{S}_\nu^c(\alpha; \varphi) \\ &\iff z(F_\sigma(f)(z))' \in \mathcal{S}_\nu^c(\alpha; \varphi) \\ &\iff F_\sigma(f)(z) \in \mathcal{K}_\nu^c(\alpha; \varphi). \end{aligned}$$

□

Following corollaries due to Theorem 7 and Theorem 8 given as:

Corollary 9. *Let $f(z) \in \mathcal{A}$, $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ and $\sigma \geq 0$. If $f \in \mathcal{S}_\nu^c(\alpha; A, B; \delta)$ ($0 \leq \alpha < 1; -1 \leq B < A \leq 1; 0 < \delta \leq 1$), then $F_\sigma(f) \in \mathcal{S}_\nu^c(\alpha; A, B; \delta)$ ($0 \leq \alpha < 1; -1 \leq B < A \leq 1; 0 < \delta \leq 1$).*

Corollary 10. *Let $f(z) \in \mathcal{A}$, $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ and $\sigma \geq 0$. If $f \in \mathcal{K}_\nu^c(\alpha; A, B; \delta)$ ($0 \leq \alpha < 1; -1 \leq B < A \leq 1; 0 < \delta \leq 1$), then $F_\sigma(f) \in \mathcal{K}_\nu^c(\alpha; A, B; \delta)$ ($0 \leq \alpha < 1; -1 \leq B < A \leq 1; 0 < \delta \leq 1$).*

Theorem 11. *Let $f(z) \in \mathcal{A}$, $c \in \mathbb{C}$, $\nu \in \mathbb{R} \setminus \mathbb{Z}_0^-$ and $\sigma \geq 0$. If $f \in \mathcal{C}_\nu^c(\alpha, \beta; \varphi, \psi)$, then $F_\sigma(f) \in \mathcal{C}_\nu^c(\alpha, \beta; \varphi, \psi)$ ($0 \leq \alpha, \beta < 1; \varphi, \psi \in \mathcal{N}$).*

Proof. Let $f \in \mathcal{C}_\nu^c(\alpha, \beta; \varphi, \psi)$. Then, $\exists g \in \mathcal{S}_\nu^c(\alpha; \varphi) \ni$

$$\frac{1}{1-\beta} \left(\frac{z(B_\nu^c f(z))'}{B_\nu^c g(z)} - \beta \right) \prec \psi(z), \quad (0 \leq \beta < 1, z \in \mathcal{U}).$$

Thus, we set

$$Q(z) = \frac{1}{1-\beta} \left(\frac{z(B_\nu^c F_\sigma(f)(z))'}{B_\nu^c F_\sigma(g)(z)} - \beta \right), \tag{3.6}$$

where $Q(z)$ is analytic in \mathcal{U} and $Q(0) = 1$. The use of (3.3) results

$$\begin{aligned} \frac{z(B_\nu^c f(z))'}{B_\nu^c g(z)} &= \frac{B_\nu^c(zf'(z))}{B_\nu^c g(z)} \\ &= \frac{z(B_\nu^c F_\sigma(zf'(z)))' + \sigma B_\nu^c F_\sigma(zf'(z))}{z(B_\nu^c F_\sigma(g)(z))' + \sigma B_\nu^c F_\sigma(g)(z)} \\ &= \frac{\frac{z(B_\nu^c F_\sigma(zf'(z)))'}{B_\nu^c F_\sigma(g)(z)} + \sigma \frac{B_\nu^c F_\sigma(zf'(z))}{B_\nu^c F_\sigma(g)(z)}}{\frac{z(B_\nu^c F_\sigma(g)(z))'}{B_\nu^c F_\sigma(g)(z)} + \sigma}, \end{aligned} \tag{3.7}$$

By Theorem 7,

$$g \in \mathcal{S}_\nu^c(\alpha; \varphi) \implies F_\sigma(g) \in \mathcal{S}_\nu^c(\alpha; \varphi),$$

so that we can set

$$H(z) = \frac{1}{1-\alpha} \left(\frac{z(B_\nu^c F_\sigma(g)(z))'}{B_\nu^c F_\sigma(g)(z)} - \alpha \right),$$

where $Re(H(z)) > 0$ ($z \in \mathcal{U}$). As parallel as the method used to prove the Theorem 6, from (3.6) and (3.7), we have

$$\frac{1}{1-\beta} \left(\frac{z(B_\nu^c f(z))'}{B_\nu^c g(z)} - \beta \right) = Q(z) + \frac{zQ'(z)}{(1-\alpha)H(z) + \alpha + \sigma}.$$

Hence, by using Lemma 2, we notice that $Q(z) \prec \psi(z)$, so $F_\sigma(f) \in \mathcal{C}_\nu^c(\alpha, \beta; \varphi, \psi)$. □

4. Remarks and Observations

The study of inclusion relationships for some subclasses of analytic functions with $\mathcal{W}_{l,b,c}(z)$ permits the study of Bessel ($J_\nu(z)$), modified Bessel ($I_\nu(z)$) and spherical Bessel functions ($j_n(z)$) together. By specializing the parameters in the operator B_ν^c we obtain the following new operators associated with $J_\nu(z)$, $I_\nu(z)$ and $j_n(z)$ (see, [2] and [6]):

- Choosing $b = c = 1$ in (1.3) we get the operator $\mathcal{J}_l : \mathcal{A} \rightarrow \mathcal{A}$ associated with $J_l(z)$ as:

$$\begin{aligned} \mathcal{J}_l f(z) &= \phi_{l,1,1}(z) * f(z) = \left[2^l \Gamma(l+1) z^{1-l/2} J_l(z^{1/2}) \right] * f(z) \\ &= z + \sum_{k=1}^{\infty} \frac{(-1)^k a_{k+1} z^{k+1}}{4^k (l+1)_k k!}. \end{aligned}$$

- Taking $b = 1$ and $c = -1$ in (1.3) we have $\mathcal{I}_l : \mathcal{A} \rightarrow \mathcal{A}$ associated with $I_l(z)$ as:

$$\begin{aligned} \mathcal{I}_l f(z) &= \phi_{l,1,-1}(z) * f(z) = \left[2^l \Gamma(l+1) z^{1-l/2} I_l(z^{1/2}) \right] * f(z) \\ &= z + \sum_{k=1}^{\infty} \frac{a_{k+1} z^{k+1}}{4^k (l+1)_k k!}. \end{aligned}$$

- Letting $b = 2$ and $c = 1$ in (1.3) we get $\mathcal{Q}_l : \mathcal{A} \rightarrow \mathcal{A}$ associated with $j_l(z)$ as:

$$\begin{aligned} \mathcal{Q}_l f(z) &= \phi_{l,2,1}(z) * f(z) = \left[\pi^{-1/2} 2^{l+1/2} \Gamma(l+3/2) z^{1-l/2} j_l(z^{1/2}) \right] * f(z) \\ &= z + \sum_{k=1}^{\infty} \frac{(-1)^k a_{k+1} z^{k+1}}{4^k (l+3/2)_k k!}. \end{aligned}$$

Thus, our inclusion results (Theorems 3-6) can be applied with a view of deducing the following corollaries.

Corollary 12. Let $f \in \mathcal{A}$, $p \in \mathbb{R} \setminus \mathbb{Z}^-$ and $\alpha + p > 0$ ($0 \leq \alpha < 1$). Then

$$f \in \mathcal{S}_{p+1}^1(\alpha; \varphi) \implies f \in \mathcal{S}_{p+2}^1(\alpha; \varphi) \quad (\varphi \in \mathcal{N})$$

or equivalently,

$$\mathcal{J}_p f(z) \in \mathcal{S}^*(\alpha; \varphi) \text{ then } f(z) \in \mathcal{S}_{p+n}^1(\alpha; \varphi) \quad (n \in \mathbb{N} \setminus \{1\}; \varphi \in \mathcal{N}).$$

Corollary 13. Let $f \in \mathcal{A}$, $p \in \mathbb{R} \setminus \mathbb{Z}^-$ and $\alpha + p > 0$ ($0 \leq \alpha < 1$). Then

$$f \in \mathcal{K}_{p+1}^1(\alpha; \varphi) \implies f \in \mathcal{K}_{p+2}^1(\alpha; \varphi) \quad (\varphi \in \mathcal{N})$$

or equivalently,

$$\mathcal{J}_p f(z) \in \mathcal{K}(\alpha; \varphi) \text{ then } f(z) \in \mathcal{K}_{p+n}^1(\alpha; \varphi) \quad (n \in \mathbb{N} \setminus \{1\}; \varphi \in \mathcal{N}).$$

Corollary 14. Let $f \in \mathcal{A}$, $p \in \mathbb{R} \setminus \mathbb{Z}^-$ and $\alpha + p > 0$ ($0 \leq \alpha < 1$). Then

$$f \in \mathcal{C}_{p+1}^1(\alpha, \beta; \varphi, \psi) \implies f \in \mathcal{C}_{p+2}^1(\alpha, \beta; \varphi, \psi) \quad (0 \leq \beta < 1; \varphi, \psi \in \mathcal{N})$$

or equivalently,

$$\mathcal{I}_p f(z) \in \mathcal{C}(\alpha, \beta; \varphi, \psi) \text{ then } f(z) \in \mathcal{C}_{p+n}^1(\alpha, \beta; \varphi, \psi) \quad (n \in \mathbb{N} \setminus \{1\}; 0 \leq \beta < 1; \varphi, \psi \in \mathcal{N}).$$

Finally, we remark that similar results can be obtained involving the operators \mathcal{I}_p and \mathcal{Q}_p by specializing the parameter in Theorems 3-6. We also remark that several other applications and corollaries of our main results (Theorems 3-6) can indeed be derived similarly.

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Fixed point properties of Suzuki generalized nonexpansive set-valued mappings in complete CAT(0) spaces [☆]

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Abstract

In this paper, we introduce the notion of Suzuki generalized nonexpansive set-valued mappings from traditional Banach spaces to CAT(0) spaces. Moreover we discuss fixed point properties including the existence, Δ -convergence and strong convergence for this kind of mappings in complete CAT(0) spaces. Our results extend the results of B. Nanjaras [12] for the Suzuki generalized nonexpansive single-valued mappings.

Key words: Suzuki generalized nonexpansive set-valued mapping, CAT(0) space, Existence theorem, Convergence theorem, Fixed point

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1. Introduction

In recent years, more and more classical fixed point theorems for single-valued nonexpansive mappings are extended by set-valued nonexpansive mappings. As a result, fixed point theory for set-valued nonexpansive mappings get rapid development[see [6–10]].

In 2008, Suzuki [11] brought in a new kind of mappings which was called satisfying the condition (C) in Banach spaces. In 2010, B. Nanjaras, B. Panyanak and W. Phuengrattana [12] established fixed point theorems for the mappings satisfying the condition (C) in CAT(0) spaces. In 2011, Abkar and Eslamian [14] gave the definition of set-valued mappings for the condition (C) and then they proved the existence of fixed point in uniformly convex Banach spaces [13].

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $x' \in X$ is a map c from a closed interval $[0, h] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(h) = x'$ and $d(c(l), c(l')) = |l - l'|$, for all $l, l' \in [0, h]$. In particular, c is an isometry and $d(x, x') = h$. The image γ of c is called a geodesic (or metric) segment joining x and x' denoted by $[x, x']$ whenever it is unique. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and x' for each $x, x' \in X$.

A geodesic space is said to be a CAT(0) space if the following CAT(0) inequality

$$d\left(z, \frac{x_1 \oplus x_2}{2}\right)^2 \leq \frac{1}{2}d(z, x_1)^2 + \frac{1}{2}d(z, x_2)^2 - \frac{1}{4}d(x_1, x_2)^2$$

satisfies for all $x_1, x_2, z \in X$. This is the (CN) inequality of Bruhat and Tits [1](More details spaces see [2]).

Lemma 1.1. *Let (X, d) be a CAT(0) space.*

(i)[3, Lemma 2.1(iv)] *For each $x_1, x_2 \in X$ and $\alpha \in [0, 1]$, there exists a unique point $y \in [x_1, x_2]$ such that*

$$d(x_1, y) = \alpha d(x_1, x_2), \quad d(x_2, y) = (1 - \alpha)d(x_1, x_2).$$

Denote $y = (1 - \alpha)x_1 \oplus \alpha x_2$ in the above equations conveniently.

(ii)[3, Lemma 2.4] *For each $x_1, x_2, y \in X$ and $\alpha \in [0, 1]$. We have*

$$d((1 - \alpha)x_1 \oplus \alpha x_2, y) \leq (1 - \alpha)d(x_1, y) + \alpha d(x_2, y). \quad (1.1)$$

Obviously, Lemma 1.1. shows the convexness of CAT(0) spaces.

Lemma 1.2. ([4, Lemma 2.5]) *Let (X, d) be a CAT(0) space. Then*

$$d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x, z)^2 + td(y, z)^2 - t(1-t)d(x, y)^2$$

for all $t \in [0, 1]$ and $x, y, z \in X$.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X . For $z \in X$, we set

$$r(z, \{x_n\}) = \limsup_{n \rightarrow \infty} d(z, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(z, \{x_n\}) : z \in X\}.$$

The asymptotic radius $r_D(\{x_n\})$ of $\{x_n\}$ with respect to $D \subset X$ is given by

$$r_D(\{x_n\}) = \inf\{r(z, \{x_n\}) : z \in D\}.$$

The asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{z \in X : r(z, \{x_n\}) = r(\{x_n\})\}.$$

And the asymptotic center $A_D(\{x_n\})$ of $\{x_n\}$ with respect to $D \subset X$ is the set

$$A_D(\{x_n\}) = \{z \in D : r(z, \{x_n\}) = r(\{x_n\})\}.$$

It follows from [5, Proposition 7]) that $A(\{x_n\})$ consists of exactly one point in a CAT(0) space. In 1976, Lim [1] introduced the concept of Δ -convergence in a general metric space. In 2008, Kirk and Panyanak [4] brought in Δ -convergence to CAT(0) spaces and proved that there is an analogy between Δ -convergence and weak convergence.

Definition 1.3. ([2]) A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ -converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n \rightarrow \infty} x_n = x$ and call x the Δ -limit of $\{x_n\}$.

Lemma 1.4. ([2]) *If D is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in D , then the asymptotic center asymptotic center of $\{x_n\}$ is in D .*

Lemma 1.5. ([2]) *Every bounded sequence in a complete CAT(0) space always has a Δ -convergent subsequence.*

Lemma 1.6. ([4]) *If $\{x_n\}$ is a bounded sequence in a complete CAT(0) space with $A(\{x_n\}) = \{p\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$, and the sequence $\{d(x_n, u)\}$ converges, then $p = u$.*

The purpose of this paper is to bring in the concept of Suzuki generalized nonexpansive set-valued mappings in CAT(0) spaces. Then we shall prove a common fixed point theorem for commuting pairs consisting of a single-valued and a set-valued mapping both satisfying the condition (C) which is analogous to the results in Banach spaces [13]. Furthermore, we also establish Δ -convergence and strong convergence of Mann iteration in CAT(0) spaces.

2. Preliminaries

Let D be a nonempty subset of a CAT(0) space X . We denote by $B(D)$ the collection of all nonempty bounded closed subsets of D and $C(D)$ the collection of all nonempty compact subsets of D . Suppose H is the Hausdorff metric with respect to d , that is,

$$H(U, V) := \max \left\{ \sup_{u \in U} \text{dist}(u, V), \sup_{v \in V} \text{dist}(v, U) \right\}, \quad U, V \in B(X)$$

where $\text{dist}(u, V) = \inf_{v \in V} d(u, v)$ is the distance from the point u to the set V .

Let $T : X \rightarrow 2^X$ be a set-valued mapping. If an element $x \in X$ satisfies $x \in Tx$, then x is called a fixed point of T . The set of fixed points of T is denoted by $\text{Fix}(T)$.

Definition 2.1. A set-valued mapping $T : X \rightarrow B(X)$ is

(i) nonexpansive provided

$$H(Tx, Ty) \leq d(x, y), \quad x, y \in X;$$

(ii) quasi nonexpansive if $\text{Fix}(T) \neq \emptyset$ and

$$H(Tx, p) \leq d(x, p), \quad x \in X,$$

for all $p \in \text{Fix}(T)$.

Definition 2.2. A set-valued mapping $T : X \rightarrow B(X)$ is called to satisfy the condition (C) if

$$\frac{1}{2}dist(x, Tx) \leq d(x, y) \quad \text{implies} \quad H(Tx, Ty) \leq d(x, y)$$

for all $x, y \in X$.

This kind of set-valued mappings satisfying the condition (C) can be called Suzuki generalized nonexpansive as well.

Proposition 2.3. *Suppose a set-valued mapping $T : X \rightarrow B(X)$ satisfies the condition (C) with the nonempty fixed point set. Then T is a quasi-nonexpansive set-valued mapping.*

Theorem 2.4. *([12, Theorem 4.1]) Suppose D is a nonempty bounded closed convex subset of a complete $CAT(0)$ space X and $t : D \rightarrow D$ is a mapping which satisfies the condition (C). Then $F(t)$ is nonempty in D .*

Corollary 2.5. *([12, Corollary 4.2]) Suppose D is a nonempty bounded closed convex subset of a complete $CAT(0)$ space X and $t : D \rightarrow D$ is a mapping which satisfies the condition (C). Then $F(t)$ is nonempty closed, convex and hence contractible.*

Definition 2.6. Suppose D is a nonempty bounded closed convex subset of a $CAT(0)$ space X . Let $t : D \rightarrow D$ and $T : D \rightarrow B(D)$ be a single-valued mapping and a set-valued mapping respectively. Then t and T are said to be commuting mappings if for every $x, y \in D$ such that $x \in Ty$ and $ty \in D$, we have $tx \in Tty$.

Definition 2.7. A set-valued mapping $T : X \rightarrow B(X)$ is said to satisfy the condition (E_μ) provided that

$$dist(x, Ty) \leq \mu dist(x, Tx) + d(x, y), \quad x, y \in X.$$

We say that T satisfies the condition (E) whenever T satisfies (E_μ) for some $\mu \geq 1$.

Proposition 2.8. *Let $T : X \rightarrow B(X)$ be a set-valued mapping which satisfies the condition (C). Then T satisfies the condition (E_3) .*

PROOF. Given $x \in X$. Since for any $z \in Tx$,

$$\text{dist}(x, Tx) \leq d(x, z) \tag{2.1}$$

then

$$\frac{1}{2}\text{dist}(x, Tx) \leq \frac{1}{2}d(x, z) \leq d(x, z).$$

From our assumption, we have

$$H(Tx, Tz) \leq d(x, z) \tag{2.2}$$

for any $x \in X$ and $z \in Tx$. Then for $x, y \in X$ and $z \in Tx$ either $\frac{1}{2}\text{dist}(x, Tx) \leq d(x, y)$ or $\frac{1}{2}H(Tx, Tz) \leq d(y, z)$ holds. On the contrary, together with (2.1) and (2.2), we get

$$\begin{aligned} d(x, z) &\leq d(x, y) + d(y, z) \\ &< \frac{1}{2}\text{dist}(x, Tx) + \frac{1}{2}H(Tx, Tz) \\ &\leq \frac{1}{2}d(x, z) + \frac{1}{2}d(x, z) \\ &= d(x, z) \end{aligned}$$

which is a contradiction.

Case I. $\frac{1}{2}\text{dist}(x, Tx) \leq d(x, y)$ holds.

From the above hypothesis, we can obtain $H(Tx, Ty) \leq d(x, y)$ by the condition (C). Hence,

$$\begin{aligned} \text{dist}(x, Ty) &\leq \text{dist}(x, Tx) + H(Tx, Ty) \\ &\leq \text{dist}(x, Tx) + d(x, y) \\ &\leq 3\text{dist}(x, Tx) + d(x, y). \end{aligned}$$

Case II. $\frac{1}{2}H(Tx, Tz) \leq d(y, z)$ holds. Then,

$$\begin{aligned} \frac{1}{2}\text{dist}(z, Tz) &\leq \frac{1}{2} \sup_{u \in Tx} \text{dist}(u, Tz) \\ &\leq \frac{1}{2} \max\{ \sup_{u \in Tx} \text{dist}(u, Tz), \sup_{v \in Tz} \text{dist}(v, Tx) \} \\ &= \frac{1}{2}H(Tx, Tz) \\ &\leq d(y, z). \end{aligned}$$

From the assumption, we have

$$H(Ty, Tz) \leq d(y, z). \tag{2.3}$$

Together with (2.1), (2.2) and (2.3), we can obtain

$$\begin{aligned} \text{dist}(x, Ty) &\leq \text{dist}(x, Tx) + H(Tx, Ty) \\ &\leq \text{dist}(x, Tx) + H(Tx, Tz) + H(Tz, Ty) \\ &\leq d(x, z) + d(x, z) + d(y, z) \\ &\leq 2d(x, z) + d(x, y) + d(x, z) \\ &= 3d(x, z) + d(x, y). \end{aligned}$$

Because this is applied for any $z \in Tx$, we get

$$\text{dist}(x, Ty) \leq 3\text{dist}(x, Tx) + d(x, y).$$

By overall consideration, T satisfies the condition (E_3) . \square

Lemma 2.9. ([12, Lemma 2.5]) *Let $\{x_n\}, \{y_n\}$ be bounded sequences in a CAT(0) space X and let $\{\alpha_n\} \in [0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_n \alpha_n < 1$. Suppose that $x_{n+1} = \alpha_n y_n \oplus (1 - \alpha_n) x_n$ and $d(y_{n+1}, y_n) \leq d(x_{n+1}, x_n)$ for all $n \in N$. Then*

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = 0.$$

3. Fixed point properties of Suzuki generalized nonexpansive set-valued mappings

In this section, we denote by D a nonempty bounded closed convex subset of a complete CAT(0) space X . Firstly, we shall discuss the existence of a common fixed point.

Theorem 3.1. *Let $t : D \rightarrow D$ and $T : D \rightarrow C(D)$ be a single-valued mapping and a set-valued mapping respectively. If both t and T satisfy the condition (C) and in the meantime, they are commuting, then they have a common fixed point, that is, there exists a point $z \in D$ such that $z = tz \in Tz$.*

PROOF. By Theorem 2.4 and Corollary 2.5, we know that the mapping t has a fixed point set $Fix(t)$ which is a nonempty closed convex subset of X . Let $p \in Fix(t)$. As t and T are commuting, we have $tq \in Ttp = Tp$ for each

$q \in Tp$. Therefore, Tp is invariant under t for each $p \in Fix(t)$. Since Tp is a bounded closed convex subset of X , we can obtain that t has a fixed point in Tp . Hence, $Tp \cap Fix(t) \neq \emptyset$ for $p \in Fix(t)$.

Take $x_1 \in Fix(t)$. Because $Tx_1 \cap Fix(t) \neq \emptyset$, we are able to select $y_1 \in Tx_1 \cap Fix(t)$. Define $x_2 = \frac{1}{2}(x_1 + y_1)$. Due to the fact that $Fix(t)$ is a convex set, we have $x_2 \in Fix(t)$. Let $y_2 \in Tx_2$ be selected as follows

$$d(y_1, y_2) = dist(y_1, Tx_2).$$

Since $\frac{1}{2}d(y_1, ty_1) = 0 \leq d(y_1, y_2)$, from the assumption, we know

$$d(y_1, ty_2) = d(ty_1, ty_2) \leq d(y_1, y_2).$$

Note that $ty_2 \in Tx_2$, hence, there comes a contradiction. Therefore, $y_2 \in Fix(t)$. In such a way, we can find a sequence $\{x_n\}$ in $Fix(t)$ such that

$$x_{n+1} = \frac{1}{2}(x_n + y_n), \quad n \geq 1$$

where $y_n \in Tx_n \cap Fix(t)$ and $d(y_{n-1}, y_n) = dist(y_{n-1}, Tx_n)$. Therefore, by Lemma 1.1.(i) we get

$$\frac{1}{2}d(x_n, y_n) = d(x_n, x_{n+1})$$

from which we can conclude

$$\frac{1}{2}dist(x_n, Tx_n) \leq \frac{1}{2}d(x_n, y_n) = d(x_n, x_{n+1}), \quad n \geq 1.$$

Hence,

$$H(Tx_n, Tx_{n+1}) \leq d(x_n, x_{n+1}), \quad n \geq 1$$

which implies

$$d(y_n, y_{n+1}) = dist(y_n, Tx_{n+1}) \leq H(Tx_n, Tx_{n+1}) \leq d(x_n, x_{n+1}).$$

We now apply Lemma 2.9. to obtain that

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$$

where $y_n \in Tx_n$, i.e., we find an approximate fixed point sequence for T in $Fix(t)$.

Assume that $z = A(\{x_n\})$. By Lemma 1.4. we know that $z \in Fix(t)$. For each $n \geq 1$, we choose $z_n \in Tz$ such that

$$d(y_n, z_n) = dist(y_n, Tz).$$

Because $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$, there exists n_0 such that

$$d(x_n, y_n) \leq d(x_n, z), \quad n \geq n_0.$$

This implies that

$$\frac{1}{2}dist(x_n, Tx_n) \leq d(x_n, z),$$

and hence,

$$H(Tx_n, Tz) \leq d(x_n, z), \quad n \geq n_0.$$

Therefore,

$$d(y_n, z_n) \leq H(Tx_n, Tz) \leq d(x_n, z), \quad n \geq n_0.$$

As $\frac{1}{2}d(y_n, Ty_n) = 0 \leq d(y_n, z_n)$ for each $n \geq 1$, we get

$$d(y_n, tz_n) = d(ty_n, tz_n) \leq d(y_n, z_n).$$

Since $z \in Fix(t)$ and $z_n \in Tz$, by the fact that the mappings t and T are commuting, we can obtain that $tz_n \in Ttz_n = Tz$. Now by the uniqueness of z_n as the nearest point to y_n , we get $tz_n = z_n \in Fix(t)$.

As Tz is compact, the sequence $\{z_n\}$ has a convergent subsequence $\{z_{n_k}\}$ with $\lim_{k \rightarrow \infty} z_{n_k} = z^* \in Tz$. Because $z_{n_k} \in Fix(t)$ for all n , and $Fix(t)$ is closed, we can obtain that $z^* \in Fix(t)$. Take the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\{y_{n_k}\}$ of $\{y_n\}$, we have

$$d(x_{n_k}, z^*) \leq d(x_{n_k}, y_{n_k}) + d(y_{n_k}, z_{n_k}) + d(z_{n_k}, z^*) \tag{3.1}$$

and for all $n_k \geq n_0$,

$$d(y_{n_k}, z_{n_k}) \leq d(x_{n_k}, z).$$

Letting $k \rightarrow \infty$ and taking superior limit on the both sides of (3.1), we have

$$\limsup_{k \rightarrow \infty} d(x_{n_k}, z^*) \leq \limsup_{k \rightarrow \infty} d(x_{n_k}, z).$$

By the uniqueness of asymptotic centers, this shows $z = z^* \in Tz$ and hence $z = tz \in Tz$. \square

Corollary 3.2. *Let $t : D \rightarrow D$ and $T : D \rightarrow C(D)$ be a single-valued mapping and a set-valued mapping respectively. Assume that t and T are commuting mappings. Then there exists a point $z \in D$ such that $z = tz \in Tz$.*

Corollary 3.3. *Suppose $T : D \rightarrow C(D)$ is set-valued mapping which satisfies the condition (C). Then T has a fixed point.*

We will prove Δ -convergence and strong convergence theorems in the following. Before that, we discuss some Lemmas which will be used in the main proofs.

Lemma 3.4. *Suppose $T : D \rightarrow C(D)$ is a set-valued mapping which satisfies the condition (C). Define a sequence $\{x_n\}$ by $x_1 \in D$ and*

$$x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n y_n, \quad \alpha_n \in \left[\frac{1}{2}, 1\right), n \geq 1$$

where $y_n \in Tx_n$ such that $d(y_{n-1}, y_n) = \text{dist}(y_{n-1}, Tx_n)$. Then

$$\lim_{n \rightarrow \infty} \text{dist}(Tx_n, x_n) = 0.$$

PROOF. It follows from Lemma 1.1.(i) that for every natural number $n \geq 1$

$$\frac{1}{2} \text{dist}(x_n, Tx_n) \leq \alpha_n \text{dist}(x_n, Tx_n) \leq \alpha_n d(x_n, y_n) = d(x_n, x_{n+1})$$

From our assumption, we get

$$H(Tx_n, Tx_{n+1}) \leq d(x_n, x_{n+1}), \quad n \geq 1.$$

Hence

$$d(y_n, y_{n+1}) = \text{dist}(y_n, Tx_{n+1}) \leq H(Tx_n, Tx_{n+1}) \leq d(x_n, x_{n+1}), \quad n \geq 1.$$

We can conclude that $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$ where $y_n \in Tx_n$ by Lemma 2.9., i.e., $\lim_{n \rightarrow \infty} \text{dist}(Tx_n, x_n) \leq \lim_{n \rightarrow \infty} d(x_n, y_n) = 0 \quad \square$

Lemma 3.5. *Suppose $T : D \rightarrow C(D)$ is a set-valued mapping which satisfies the condition (C). Define a sequence $\{x_n\}$ as in Lemma 3.4. Then $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for all $p \in \text{Fix}(T)$.*

PROOF. By Theorem 3.1. $F(T)$ is nonempty. Take $p \in \text{Fix}(t)$, by Lemma 1.2. and Proposition 2.3. we have

$$\begin{aligned} d^2(x_{n+1}, p) &\leq (1 - \alpha_n)d^2(x_n, p) + \alpha_n d^2(y_n, p) - \alpha_n(1 - \alpha_n)d^2(x_n, y_n) \\ &\leq (1 - \alpha_n)d^2(x_n, p) + \alpha_n H^2(Tx_n, Tp) - \alpha_n(1 - \alpha_n)d^2(x_n, y_n) \\ &\leq (1 - \alpha_n)d^2(x_n, p) + \alpha_n d^2(x_n, p) - \alpha_n(1 - \alpha_n)d^2(x_n, y_n) \\ &= d^2(x_n, p) - \alpha_n(1 - \alpha_n)d^2(x_n, y_n). \end{aligned}$$

This entails

$$d^2(x_{n+1}, p) \leq d^2(x_n, p).$$

Therefore, $d(x_{n+1}, p) \leq d(x_n, p)$ for all $n \geq 1$ which implies $\{d(x_n, p)\}_{n=1}^\infty$ is bounded and decreasing. Hence, $\lim_{n \rightarrow \infty} d(x_n, p)$ exists for each $p \in \text{Fix}(T)$. \square

Lemma 3.6. *Let $T : D \rightarrow C(D)$ be a set-valued mapping which satisfies the condition (C). If $\{x_n\}$ is a sequence in D such that $\text{dist}(Tx_n, x_n) \rightarrow 0$ and Δ -converges to some $\omega \in X$. Then $\omega \in D$ and $\omega \in T\omega$.*

PROOF. We first note that $\omega \in D$ by Lemma 1.4. For each $n \geq 1$, we select $\omega_n \in T\omega$ such that $d(x_n, \omega_n) = \text{dist}(x_n, T\omega)$. By the compactness of $T\omega$, there exists a subsequence $\{\omega_{n_k}\}$ of $\{\omega_n\}$ such that $\{\omega_{n_k}\} \rightarrow \omega^* \in T\omega$. Taking the subsequence $\{x_{n_k}\}$ of $\{x_n\}$, it follows from Proposition 2.8. that

$$\text{dist}(x_{n_k}, T\omega) \leq 3\text{dist}(x_{n_k}, Tx_{n_k}) + d(x_{n_k}, \omega).$$

Note that

$$\begin{aligned} d(x_{n_k}, \omega^*) &\leq d(x_{n_k}, \omega_{n_k}) + d(\omega_{n_k}, \omega^*) \\ &\leq 3\text{dist}(x_{n_k}, Tx_{n_k}) + d(x_{n_k}, \omega) + d(\omega_{n_k}, \omega^*). \end{aligned}$$

Letting $k \rightarrow \infty$ and taking superior limit on the both sides of the above inequation, we have

$$\limsup_{k \rightarrow \infty} d(x_{n_k}, \omega^*) \leq \limsup_{k \rightarrow \infty} d(x_{n_k}, \omega).$$

By the uniqueness of asymptotic centers, we have $\omega = \omega^* \in T\omega$. \square

Theorem 3.7. *Let $T : D \rightarrow C(D)$ be a set-valued mapping which satisfies the condition (C). Define a sequence $\{x_n\}$ as in Lemma 3.4. Then $\{x_n\}$ Δ -converges to a fixed point of T .*

PROOF. By Lemma 3.4., $\lim_{n \rightarrow \infty} dist(Tx_n, x_n) = 0$. Now we prove that

$$W_\omega(x_n) := \bigcup_{\{\mu_n\} \subset \{x_n\}} A(\{\mu_n\}) \subset Fix(T) \tag{3.2}$$

and $W_\omega(x_n)$ consists of exactly one point.

In fact, let $\mu \in W_\omega(x_n)$, then there exists a sequence $\{\mu_n\}$ of $\{x_n\}$ such that $A(\{\mu_n\}) = \{\mu\}$. By Lemma 1.4. and 1.5., there exists a subsequence $\{\nu_n\}$ of $\{\mu_n\}$ such that $\Delta\text{-}\lim_{n \rightarrow \infty} \nu_n = \nu \in D$. Since $\lim_{n \rightarrow \infty} dist(T\nu_n, \nu_n) = 0$, then $\nu \in Fix(T)$ by Lemma 3.6. and $\lim_{n \rightarrow \infty} d(x_n, \nu)$ exists by Lemma 3.5. By Lemma 1.6. $\mu = \nu$. This implies that $W_\omega(x_n) \subset Fix(T)$.

Next we prove that $W_\omega(x_n)$ consists of exactly one point. Let $\{\mu_n\}$ be a subsequence of $\{x_n\}$ with $A(\{\mu_n\}) = \{\mu\}$ and let $A(\{x_n\}) = \{x\}$. Since $\mu \in W_\omega(x_n) \subset Fix(T)$, from Lemma 3.5. we know that $\{d(x_n, \mu)\}$ is convergent. In view of Lemma 1.6, $x = \mu$. \square

Finally, we shall give the strong convergence for Suzuki generalized non-expansive set-valued mappings in complete CAT(0) spaces.

Theorem 3.8. *Suppose D is a nonempty compact convex subset of a complete CAT(0) space and $T : D \rightarrow C(D)$ is a set-valued mapping which satisfies the condition (C). Define a sequence $\{x_n\}$ as in Lemma 3.4. Then $\{x_n\}$ converges strongly to a fixed point of T .*

PROOF. By Lemma 3.4., $\lim_{n \rightarrow \infty} dist(Tx_n, x_n) = 0$. Since D is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow z$ for some $z \in D$. By Proposition 2.8., we have

$$\begin{aligned} dist(z, Tz) &\leq d(z, x_{n_k}) + dist(x_{n_k}, Tz) \\ &\leq d(z, x_{n_k}) + 3dist(x_{n_k}, Tx_{n_k}) + d(x_{n_k}, z) \\ &= 2d(z, x_{n_k}) + 3dist(x_{n_k}, Tx_{n_k}). \end{aligned}$$

Letting $k \rightarrow \infty$, we have $z \in Fix(T)$. Since $\{x_{n_k}\}$ converges strongly to z and $\lim_{n \rightarrow \infty} d(x_n, z)$ exists, we can conclude that $\{x_{n_k}\}$ converges strongly to z . \square

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WEIGHTED COMPOSITION FOLLOWED AND PROCEEDED BY DIFFERENTIATION OPERATORS FROM ZYGMUND SPACES TO BERS-TYPE SPACES

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ABSTRACT. In this paper, we investigate boundedness and compactness of the weighted composition followed and proceeded by differentiation operators from Zygmund spaces to Bers-type spaces and little Bers-type spaces. Some sufficient and necessary conditions for the boundedness and compactness of these operators are obtained.

1. INTRODUCTION

Let $\Delta = \{z : |z| < 1\}$ be the open unit disc in the complex plane \mathbb{C} , and let $H(\Delta)$ be the class of all analytic functions on Δ .

Assume that μ is a positive continuous function on $[0, 1)$, having the property that there exist positive numbers s and t , $0 < s < t$, and $\delta \in [0, 1)$, such that

$$\begin{aligned} \frac{\mu(r)}{(1-r)^s} \text{ is decreasing on } [\delta, 1), \lim_{r \rightarrow 1} \frac{\mu(r)}{(1-r)^s} = 0, \\ \frac{\mu(r)}{(1-r)^t} \text{ is increasing on } [\delta, 1), \lim_{r \rightarrow 1} \frac{\mu(r)}{(1-r)^t} = \infty. \end{aligned}$$

Then μ is called a normal function (see [6], [18]).

An analytic function f on Δ is said to belong to the Bers-type space, denoted by H_μ^∞ , if

$$\|f\|_{H_\mu^\infty} = \sup_{z \in \Delta} \mu(|z|)|f(z)| < \infty,$$

and it is said to belong to the little Bers-type space $H_{\mu,0}^\infty$ if

$$\lim_{|z| \rightarrow 1} \mu(|z|)|f(z)| = 0.$$

It is clear that both H_μ^∞ and $H_{\mu,0}^\infty$ are Banach spaces with the norm $\|\cdot\|_{H_\mu^\infty}$, and $H_{\mu,0}^\infty$ is a closed subspace of H_μ^∞ . When $\mu \equiv 1$, the space H_μ^∞ is just H^∞ , which is defined by

$$H^\infty = \{f \in H(\Delta) : \|f\|_\infty = \sup_{z \in \Delta} |f(z)| < \infty\}.$$

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See [25] for more about Bers-type space.

An f in $H(\Delta)$ is said to belong to the Zygmund space, denoted by \mathcal{Z} , if

$$\sup \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty,$$

where the supremum is taken over all $e^{i\theta} \in \partial\Delta$ and $h > 0$. By Theorem 5.3 in [2], we see that $f \in \mathcal{Z}$ if and only if

$$\|f\|_{\mathcal{Z}} = |f(0)| + |f'(0)| + \sup_{z \in \Delta} (1 - |z|^2)|f''(z)| < \infty.$$

It is easy to check that \mathcal{Z} is a Banach space under the above norm. For every $f \in \mathcal{Z}$, by using a result in [9], we have that

$$|f'(z)| \leq C\|f\|_{\mathcal{Z}} \ln \frac{e}{1 - |z|^2}.$$

Let \mathcal{Z}_0 denote the subspace of \mathcal{Z} consisting of those $f \in \mathcal{Z}$ for which

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)|f''(z)| = 0.$$

The space \mathcal{Z}_0 is called the little Zygmund space.

Let φ be a nonconstant analytic self-map of Δ , and let ϕ be an analytic function in Δ . For $f \in H(\Delta)$, we define the linear operators

$$\phi C_{\varphi} Df = \phi(f' \circ \varphi) = \phi f'(\varphi)$$

and

$$\phi DC_{\varphi} f = \phi(f \circ \varphi)' = \phi f'(\varphi) \varphi'.$$

They are called weighted composition followed and proceeded by differentiation operators respectively, where C_{φ} and D are composition and differentiation operators respectively. Associated with φ is the composition operator $C_{\varphi} f = f \circ \varphi$ and weighted composition operator $\phi C_{\varphi} f = \phi f \circ \varphi$ for $\phi \in H(\Delta)$ and $f \in H(\Delta)$. It is interesting to provide a function theoretic characterization for φ inducing a bounded or compact composition operator, weighted composition operator and related ones on various spaces (see, *e.g.*, [1, 3, 10, 15, 17, 19-21, 23-24, 26]). For example, it is well known that C_{φ} is bounded on the classical Hardy, Bloch and Bergman spaces. Operators DC_{φ} and $C_{\varphi} D$ as well as some other products of linear operators were studied, for example, in [5, 7-8, 11, 13, 16, 22] (see also the references therein). There has been some considerable recent interest in investigation various type of operators from or to Zygmund type spaces (see, [4, 9-12, 27]).

In this paper, we investigate the operators ϕDC_{φ} and $\phi C_{\varphi} D$ from Zygmund spaces to Bers-type spaces and little Bers-type spaces by using the

similar ways in [14]. Some sufficient and necessary conditions for the boundedness and compactness of these operators are given.

Throughout this paper, constants are denoted by C , they are positive and may differ from one occurrence to the other. The notation $A \approx B$ means that there is a positive constant C such that $\frac{B}{C} \leq A \leq CB$.

2. MAIN RESULTS AND PROOFS

In this section, we state and prove our main results. In order to formulate our main results, we quote several lemmas which will be used in the proofs of the main results in this paper. The following lemma can be proved in a standard way (see, e.g., Proposition 3.11 in [1]). Hence we omit the details.

Lemma 2.1. *Let φ be an analytic self-map of Δ , ϕ be an analytic function in Δ . Suppose that μ is normal. Then ϕDC_φ (or $\phi C_\varphi D$): \mathcal{Z} (or \mathcal{Z}_0) $\rightarrow H_\mu^\infty$ is compact if and only if ϕDC_φ (or $\phi C_\varphi D$): \mathcal{Z} (or \mathcal{Z}_0) $\rightarrow H_\mu^\infty$ is bounded and for any bounded sequence $\{f_n\}_{n \in \mathbb{N}}$ in \mathcal{Z} (or \mathcal{Z}_0) which converges to zero uniformly on compact subsets of Δ as $n \rightarrow \infty$, and $\|\phi DC_\varphi f_n\|_{H_\mu^\infty} \rightarrow 0$ (or $\|\phi C_\varphi D f_n\|_{H_\mu^\infty} \rightarrow 0$) as $n \rightarrow \infty$.*

Lemma 2.2. *A closed set \mathbb{K} of $H_{\mu,0}^\infty$ is compact if and only if it is bounded and satisfies*

$$(2.1) \quad \limsup_{|z| \rightarrow 1} \sup_{f \in \mathbb{K}} \mu(|z|)|f(z)| = 0.$$

Proof. First of all, we suppose that \mathbb{K} is compact and let $\varepsilon > 0$. By the definition of $H_{\mu,0}^\infty$, we can choose an $\frac{\varepsilon}{2}$ -net which center at f_1, f_2, \dots, f_n in \mathbb{K} respectively, and a positive number r ($0 < r < 1$), such that $\mu(|z|)|f_i(z)| < \frac{\varepsilon}{2}$, for $1 \leq i \leq n$ and $|z| > r$. If $f \in \mathbb{K}$, $\|f - f_i\|_{H_\mu^\infty} < \frac{\varepsilon}{2}$ for some f_i , so we have

$$\mu(|z|)|f(z)| \leq \|f - f_i\|_{H_\mu^\infty} + \mu(|z|)|f_i(z)| < \varepsilon,$$

for $|z| > r$. This establishes (2.1).

On the other hand, if \mathbb{K} is a closed bounded set which satisfies (2.1) and $\{f_n\}$ is a sequence in \mathbb{K} , then by the Montel's theorem, there is a subsequence $\{f_{n_k}\}$ which converges uniformly on compact subsets of Δ to some analytic function f . According to (2.1), for every $\varepsilon > 0$, there is an r , $0 < r < 1$, such that for all $g \in \mathbb{K}$, $\mu(|z|)|g(z)| < \frac{\varepsilon}{2}$, if $|z| > r$. It follows that $\mu(|z|)|f(z)| < \frac{\varepsilon}{2}$, if $|z| > r$. Since $\{f_{n_k}\}$ converges uniformly to f on $|z| \leq r$, it follows that $\lim_{k \rightarrow \infty} \sup \|f_{n_k} - f\|_{H_\mu^\infty} \leq \varepsilon$, i.e $\lim_{k \rightarrow \infty} \|f_{n_k} - f\|_{H_\mu^\infty} = 0$, so that \mathbb{K} is compact. □

Theorem 2.3. *Let φ be an analytic self-map of Δ , and ϕ be an analytic function in Δ . Suppose that μ is normal. Then the following statements are equivalent.*

- (i) $\phi DC_\varphi : \mathcal{Z} \rightarrow H_\mu^\infty$ is bounded;
- (ii) $\phi DC_\varphi : \mathcal{Z}_0 \rightarrow H_\mu^\infty$ is bounded;
- (iii)

$$(2.2) \quad \sup_{z \in \Delta} \mu(|z|) |\phi(z)\varphi'(z)| \ln \frac{e}{1 - |\varphi(z)|^2} < \infty.$$

Proof. (i) \Rightarrow (ii). This implication is obvious.

(ii) \Rightarrow (iii). Assume that $\phi DC_\varphi : \mathcal{Z}_0 \rightarrow H_\mu^\infty$ is bounded, i.e., there exists a constant C such that

$$\|\phi DC_\varphi f\|_{H_\mu^\infty} \leq C \|f\|_{\mathcal{Z}}$$

for all $f \in \mathcal{Z}_0$. Taking the function $f(z) = z \in \mathcal{Z}_0$, we get

$$(2.3) \quad \sup_{z \in \Delta} \mu(|z|) |\phi(z)\varphi'(z)| < \infty.$$

Set

$$h(z) = (z - 1) \left[\left(1 + \ln \frac{1}{1 - z} \right)^2 + 1 \right]$$

and

$$(2.4) \quad h_a(z) = \frac{h(\bar{a}z)}{\bar{a}} \left(\ln \frac{1}{1 - |a|^2} \right)^{-1}$$

for $a \in \Delta \setminus \{0\}$. It is known that $h_a \in \mathcal{Z}_0$ (see [9]). Since

$$(2.5) \quad h'_a(z) = \left(\ln \frac{1}{1 - \bar{a}z} \right)^2 \left(\ln \frac{1}{1 - |a|^2} \right)^{-1},$$

for $|\varphi(\lambda)| > \frac{1}{2}$, we have

$$C \|\phi DC_\varphi\|_{\mathcal{Z}_0 \rightarrow H_\mu^\infty} \geq \|\phi DC_\varphi h_{\varphi(\lambda)}\|_{H_\mu^\infty} \geq \mu(|\lambda|) |\phi(\lambda)\varphi'(\lambda)| \ln \frac{1}{1 - |\varphi(\lambda)|^2}.$$

Hence, we have that

$$(2.6) \quad \sup_{|\varphi(\lambda)| > \frac{1}{2}} \mu(|\lambda|) |\phi(\lambda)\varphi'(\lambda)| \ln \frac{1}{1 - |\varphi(\lambda)|^2} < \infty.$$

On the other hand, from the inequality (2.3) we have that

$$(2.7) \quad \sup_{|\varphi(\lambda)| \leq \frac{1}{2}} \mu(|\lambda|) |\phi(\lambda)\varphi'(\lambda)| \ln \frac{1}{1 - |\varphi(\lambda)|^2} \leq \sup_{\lambda \in \Delta} \mu(|\lambda|) |\phi(\lambda)\varphi'(\lambda)| \ln \frac{4}{3} < \infty.$$

Hence, from (2.3), (2.6) and (2.7), we obtain (2.2).

(iii) \Rightarrow (i). Assume that (2.2) holds. Then, for every $f \in \mathcal{Z}$, we have

$$(2.8) \quad \begin{aligned} \mu(|z|)|(\phi DC_\varphi f)(z)| &= \mu(|z|)|\phi(z)\varphi'(z)f'(\varphi(z))| \\ &\leq C\mu(|z|)|\phi(z)\varphi'(z)| \ln \frac{e}{1-|\varphi(z)|^2} \|f\|_{\mathcal{Z}}. \end{aligned}$$

Taking the supremum in (2.8) for $z \in \Delta$, and employing (2.2), we deduce that $\phi DC_\varphi : \mathcal{Z} \rightarrow H_\mu^\infty$ is bounded. The proof of Theorem 2.3 is completed. \square

Theorem 2.4. *Let φ be an analytic self-map of Δ , and ϕ be an analytic function in Δ . Suppose that μ is normal. Then the following statements are equivalent.*

- (i) $\phi DC_\varphi : \mathcal{Z} \rightarrow H_\mu^\infty$ is compact;
- (ii) $\phi DC_\varphi : \mathcal{Z}_0 \rightarrow H_\mu^\infty$ is compact;
- (iii) $\phi DC_\varphi : \mathcal{Z} \rightarrow H_\mu^\infty$ is bounded, and

$$(2.9) \quad \lim_{|\varphi(z)| \rightarrow 1} \mu(|z|)|\phi(z)\varphi'(z)| \ln \frac{e}{1-|\varphi(z)|^2} = 0.$$

Proof. (i) \Rightarrow (ii). This implication is clear.

(ii) \Rightarrow (iii). Assume that $\phi DC_\varphi : \mathcal{Z}_0 \rightarrow H_\mu^\infty$ is compact. Then it is clear that $\phi DC_\varphi : \mathcal{Z}_0 \rightarrow H_\mu^\infty$ is bounded. By Theorem 2.3 we know that $\phi DC_\varphi : \mathcal{Z} \rightarrow H_\mu^\infty$ is bounded. Let $(z_n)_{n \in \mathbb{N}}$ be a sequence in Δ such that $|\varphi(z_n)| \rightarrow 1$ as $n \rightarrow \infty$ and $\varphi(z_n) \neq 0$, $n \in \mathbb{N}$ (if such a sequence does not exist then (2.9) is vacuously satisfied). Set

$$(2.10) \quad h_n(z) = \frac{h(\overline{\varphi(z_n)}z)}{\varphi(z_n)} \left(\ln \frac{1}{1-|\varphi(z_n)|^2} \right)^{-1}, \quad n \in \mathbb{N}.$$

Then from the proof of Theorem 2.3, we see that $h_n \in \mathcal{Z}_0$ for each $n \in \mathbb{N}$. Moreover $h_n \rightarrow 0$ uniformly on compact subsets of Δ as $n \rightarrow \infty$ and

$$h'_n(\varphi(z_n)) = \ln \frac{1}{1-|\varphi(z_n)|^2}.$$

Since $\phi DC_\varphi : \mathcal{Z}_0 \rightarrow H_\mu^\infty$ is compact, by Lemma 2.1, we have

$$\lim_{n \rightarrow \infty} \|\phi DC_\varphi h_n\|_{H_\mu^\infty} = 0.$$

Hence,

$$(2.11) \quad \lim_{n \rightarrow \infty} \mu(|z_n|)|\phi(z_n)\varphi'(z_n)| \ln \frac{1}{1-|\varphi(z_n)|^2} = 0.$$

From (2.11) easily follows that $\lim_{n \rightarrow \infty} \mu(|z_n|)|\phi(z_n)\varphi'(z_n)| = 0$, which altogether imply (2.9).

(iii) \Rightarrow (i). Suppose that $\phi DC_\varphi : \mathcal{Z} \rightarrow H_\mu^\infty$ is bounded and that conditions (2.9) holds. From Theorem 2.3, we know that

$$(2.12) \quad C = \sup_{z \in \Delta} \mu(|z|)|\phi(z)\varphi'(z)| < \infty.$$

By the assumption, for every $\varepsilon > 0$, there is a $\delta \in (0, 1)$, such that

$$(2.13) \quad \mu(|z|)|\phi(z)\varphi'(z)| \ln \frac{e}{1 - |\varphi(z)|^2} < \varepsilon,$$

whenever $\delta < |\varphi(z)| < 1$.

Assume that $(f_k)_{k \in \mathbb{N}}$ is a sequence in \mathcal{Z} such that $\sup_{k \in \mathbb{N}} \|f_k\|_{\mathcal{Z}} \leq L$ and f_k converges to 0 uniformly on compact subsets of Δ as $k \rightarrow \infty$. Let $K = \{z \in \Delta : |\varphi(z)| \leq \delta\}$. Then by (2.12) and (2.13), we have that

$$\begin{aligned} \sup_{z \in \Delta} \mu(|z|)|(\phi DC_{\varphi} f_k)(z)| &= \sup_{z \in \Delta} \mu(|z|)|\phi(z)\varphi'(z)f'_k(\varphi(z))| \\ &\leq \sup_{z \in K} \mu(|z|)|\phi(z)\varphi'(z)f'_k(\varphi(z))| + \sup_{z \in \Delta \setminus K} \mu(|z|)|\phi(z)\varphi'(z)f'_k(\varphi(z))| \\ &\leq \sup_{z \in K} \mu(|z|)|\phi(z)\varphi'(z)f'_k(\varphi(z))| \\ &\quad + C \sup_{z \in \Delta \setminus K} \mu(|z|)|\phi(z)\varphi'(z)| \ln \frac{e}{1 - |\varphi(z)|^2} \|f_k\|_{\mathcal{Z}} \\ &\leq C \sup_{|\omega| \leq \delta} |f'_k(\omega)| + C\varepsilon \|f_k\|_{\mathcal{Z}}, \end{aligned}$$

i.e., we obtain

$$(2.14) \quad \|\phi DC_{\varphi} f_k\|_{H_{\mu}^{\infty}} \leq C \sup_{|\omega| \leq \delta} |f'_k(\omega)| + C\varepsilon \|f_k\|_{\mathcal{Z}} + |\phi(0)||f'_k(\varphi(0))||\varphi'(0)|.$$

Since f_k converges to 0 uniformly on compact subsets of Δ as $k \rightarrow \infty$, Cauchy's estimate gives that $f'_k \rightarrow 0$ as $k \rightarrow \infty$ on compact subsets of Δ . Hence, letting $k \rightarrow \infty$ in (2.14), and using the fact that ε is an arbitrary positive number, we obtain

$$\lim_{k \rightarrow \infty} \|\phi DC_{\varphi} f_k\|_{H_{\mu}^{\infty}} = 0.$$

Combining this with Lemma 2.1 the result easily follows. The proof of Theorem 2.4 is completed. \square

Theorem 2.5. *Let φ be an analytic self-map of Δ , and ϕ be an analytic function in Δ . Suppose that μ is normal. Then $\phi DC_{\varphi} : \mathcal{Z}_0 \rightarrow H_{\mu,0}^{\infty}$ is bounded if and only if $\phi DC_{\varphi} : \mathcal{Z}_0 \rightarrow H_{\mu}^{\infty}$ is bounded and*

$$(2.15) \quad \lim_{|z| \rightarrow 1} \mu(|z|)|\phi(z)\varphi'(z)| = 0.$$

Proof. Assume that $\phi DC_{\varphi} : \mathcal{Z}_0 \rightarrow H_{\mu,0}^{\infty}$ is bounded. Then, it is clear that $\phi DC_{\varphi} : \mathcal{Z}_0 \rightarrow H_{\mu}^{\infty}$ is bounded. Taking the test function $f(z) = z$, we obtain (2.15).

Conversely, assume that $\phi DC_{\varphi} : \mathcal{Z}_0 \rightarrow H_{\mu}^{\infty}$ is bounded and (2.15) holds. Then for each polynomial p , we have that

$$(2.16) \quad \mu(|z|)|(\phi DC_{\varphi} p)(z)| \leq \mu(|z|)|\phi(z)\varphi'(z)p'(\varphi(z))|.$$

In view of the facts that

$$\sup_{\omega \in \Delta} |p'(\omega)| < \infty,$$

from (2.15) and (2.16), it follows that $\phi DC_\varphi p \in H_{\mu,0}^\infty$. Since the set of all polynomials is dense in \mathcal{Z}_0 (see [11]), we have that for every $f \in \mathcal{Z}_0$, there is a sequence of polynomials $(p_n)_{n \in \mathbb{N}}$ such that $\|f - p_n\|_{\mathcal{Z}} \rightarrow 0$ as $n \rightarrow \infty$. Hence

$$\|\phi DC_\varphi f - \phi DC_\varphi p_n\|_{H_\mu^\infty} \leq \|\phi DC_\varphi\|_{\mathcal{Z}_0 \rightarrow H_\mu^\infty} \|f - p_n\|_{\mathcal{Z}} \rightarrow 0$$

as $n \rightarrow \infty$. Since the operator $\phi DC_\varphi : \mathcal{Z}_0 \rightarrow H_\mu^\infty$ is bounded, so $\phi DC_\varphi(\mathcal{Z}_0) \subseteq H_{\mu,0}^\infty$, which implies the boundedness of $\phi DC_\varphi : \mathcal{Z}_0 \rightarrow H_{\mu,0}^\infty$.

Theorem 2.6. *Let φ be an analytic self-map of Δ , and ϕ be an analytic function in Δ . Suppose that μ is normal. Then the following statements are equivalent.*

- (i) $\phi DC_\varphi : \mathcal{Z} \rightarrow H_{\mu,0}^\infty$ is compact;
- (ii) $\phi DC_\varphi : \mathcal{Z}_0 \rightarrow H_{\mu,0}^\infty$ is compact;
- (iii)

$$(2.17) \quad \lim_{|z| \rightarrow 1} \mu(|z|) |\phi(z)\varphi'(z)| \ln \frac{e}{1 - |\varphi(z)|^2} = 0.$$

Proof. (i) \Rightarrow (ii). This implication is trivial.

(ii) \Rightarrow (iii). Assume that $\phi DC_\varphi : \mathcal{Z}_0 \rightarrow H_{\mu,0}^\infty$ is compact. Then $\phi DC_\varphi : \mathcal{Z}_0 \rightarrow H_{\mu,0}^\infty$ is bounded. From the proof of Theorem 2.5, we know that

$$(2.18) \quad \lim_{|z| \rightarrow 1} \mu(|z|) |\phi(z)\varphi'(z)| = 0.$$

Hence, if $\|\varphi\|_\infty < 1$, from (2.18), we obtain that

$$\lim_{|z| \rightarrow 1} \mu(|z|) |\phi(z)\varphi'(z)| \ln \frac{e}{1 - |\varphi(z)|^2} \leq \ln \frac{e}{1 - \|\varphi\|_\infty^2} \lim_{|z| \rightarrow 1} \mu(|z|) |\phi(z)\varphi'(z)| = 0,$$

from which the result follows in this case.

Now assume that $\|\varphi\|_\infty = 1$. Let $(z_k)_{k \in \mathbb{N}}$ be a sequence such that $|\varphi(z_k)| \rightarrow 1$ as $k \rightarrow \infty$. Since $\phi DC_\varphi : \mathcal{Z}_0 \rightarrow H_\mu^\infty$ is compact, by Theorem 2.4,

$$(2.19) \quad \lim_{|\varphi(z)| \rightarrow 1} \mu(|z|) |\phi(z)\varphi'(z)| \ln \frac{e}{1 - |\varphi(z)|^2} = 0.$$

From (2.18) and (2.19), we have that for every $\varepsilon > 0$, there exists an $r \in (0, 1)$ such that

$$\mu(|z|) |\phi(z)\varphi'(z)| \ln \frac{e}{1 - |\varphi(z)|^2} < \varepsilon,$$

when $r < |\varphi(z)| < 1$, and there exists a $\sigma \in (0, 1)$ such that

$$\mu(|z|)|\phi(z)\varphi'(z)| \leq \frac{\varepsilon}{\ln \frac{e}{1-r^2}},$$

when $\sigma < |z| < 1$. Therefore, when $\sigma < |z| < 1$ and $r < |\varphi(z)| < 1$, we have

$$(2.20) \quad \mu(|z|)|\phi(z)\varphi'(z)| \ln \frac{e}{1-|\varphi(z)|^2} < \varepsilon.$$

On the other hand, if $\sigma < |z| < 1$ and $|\varphi(z)| \leq r$, we obtain

$$(2.21) \quad \mu(|z|)|\phi(z)\varphi'(z)| \ln \frac{e}{1-|\varphi(z)|^2} < \mu(|z|)|\phi(z)\varphi'(z)| \ln \frac{e}{1-r^2} < \varepsilon.$$

Inequality (2.20) together with (2.21) gives the (2.17).

(iii) \Rightarrow (i). Let $f \in \mathcal{Z}$. we have

$$\mu(|z|)|(\phi DC_\varphi f)(z)| \leq C\mu(|z|)|\phi(z)\varphi'(z)| \ln \frac{e}{1-|\varphi(z)|^2} \|f\|_{\mathcal{Z}}.$$

Taking the supremum in this inequality over all $f \in \mathcal{Z}$ such that $\|f\|_{\mathcal{Z}} \leq 1$, then letting $|z| \rightarrow 1$, and using (2.17), we obtain that

$$\lim_{|z| \rightarrow 1} \sup_{\|f\|_{\mathcal{Z}} \leq 1} \mu(|z|)|(\phi DC_\varphi f)(z)| = 0.$$

Using Lemma 2.2 we obtain that the operator $\phi DC_\varphi: \mathcal{Z} \rightarrow H_{\mu,0}^\infty$ is compact. \square

Similarly to the proofs of Theorems 2.3-2.6, we can get the following results, we omit the proof.

Theorem 2.7. *Let φ be an analytic self-map of Δ , and ϕ be an analytic function in Δ . Suppose that μ is normal. Then the following statements are equivalent.*

- (i) $\phi C_\varphi D : \mathcal{Z} \rightarrow H_\mu^\infty$ is bounded;
- (ii) $\phi C_\varphi D : \mathcal{Z}_0 \rightarrow H_\mu^\infty$ is bounded;
- (iii)

$$\sup_{z \in \Delta} \mu(|z|)|\phi(z)| \ln \frac{e}{1-|\varphi(z)|^2} < \infty.$$

Theorem 2.8. *Let φ be an analytic self-map of Δ , and ϕ be an analytic function in Δ . Suppose that μ is normal. Then the following statements are equivalent.*

- (i) $\phi C_\varphi D : \mathcal{Z} \rightarrow H_\mu^\infty$ is compact;
- (ii) $\phi C_\varphi D : \mathcal{Z}_0 \rightarrow H_\mu^\infty$ is compact;
- (iii) $\phi C_\varphi D : \mathcal{Z} \rightarrow H_\mu^\infty$ is bounded,

$$\lim_{|\varphi(z)| \rightarrow 1} \mu(|z|)|\phi(z)| \ln \frac{e}{1-|\varphi(z)|^2} = 0.$$

Theorem 2.9. *Let φ be an analytic self-map of Δ , and ϕ be an analytic function in Δ . Suppose that μ is normal. Then $\phi C_\varphi D : \mathcal{Z}_0 \rightarrow H_{\mu,0}^\infty$ is bounded if and only if $\phi C_\varphi D : \mathcal{Z}_0 \rightarrow H_\mu^\infty$ is bounded and $\phi(z) \in H_{\mu,0}^\infty$.*

Theorem 2.10. *Let φ be an analytic self-map of Δ , and ϕ be an analytic function in Δ . Suppose that μ is normal. Then the following statements are equivalent.*

- (i) $\phi C_\varphi D : \mathcal{Z} \rightarrow H_{\mu,0}^\infty$ is compact;
- (ii) $\phi C_\varphi D : \mathcal{Z}_0 \rightarrow H_{\mu,0}^\infty$ is compact;
- (iii)

$$\lim_{|z| \rightarrow 1} \mu(|z|) |\phi(z)| \ln \frac{e}{1 - |\varphi(z)|^2} = 0.$$

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Global stability in n -dimensional stochastic difference equations for predator-prey models

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Abstract

There are relatively few theoretical papers to consider the positivity of solutions of discrete time stochastic difference equations (DSDEs), compared to many publications on theoretical analysis of solutions of deterministic difference equations and stochastic differential equations. Additionally, no papers theoretically investigate the global stability of nontrivial solutions of n -dimensional DSDEs. In this paper, we consider the Euler-Maruyama scheme for n -dimensional stochastic difference equations that are a generalization of a two-dimensional model of stochastic predator-prey interactions, and show the positivity and the global stability of nontrivial solutions of the scheme by applying a new discretized version of the Itô formula. Numerical simulations are introduced to support the results.

Key words: Euler-Maruyama scheme, Positivity, Global stability, Stochastic difference equations.

1. Introduction

Stochastic differential equation (SDE) models have been increasingly used in a range of application areas, including biology, chemistry, mechanics, economics, and finance. In general, the exact solutions of SDEs are not known, so one has to numerically solve these SDEs. This leads us to consider and analyze discrete time stochastic difference equations (DSDEs), which can be also viewed as stochastically perturbed versions of deterministic difference equations (DDEs). There are many publications on estimations of the difference between solutions of SDEs and DSDEs. The global asymptotic stability of the trivial solution of DSDEs has been also widely addressed (see [1], [2], [3] and references therein). However, relatively few studies theoretically consider the positivity of solutions of DSDEs that are scalar equations on a finite time interval (see [4] references therein). In particular, to the best of our knowledge, there is no paper that theoretically deals with the global stability of nontrivial solutions of DSDEs, except [5], in which two-dimensional DSDEs are treated with a new discretized version of the Itô formula. Therefore, the aim of this paper is to extend the method used in [5] for investigating the positivity and the global stability of nontrivial solutions of n -dimensional DSDEs on an infinite time interval with stochastic predator-prey models.

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We generalize the two dimensional predator-prey model in [6]

$$\begin{aligned} dx(t) &= x(t)\{r_1 - a_{11}x(t) - a_{12}y(t)\}dt + \sigma_1x(t)dW_1(t), \\ dy(t) &= y(t)\{-r_2 + a_{21}x(t) - a_{22}y(t)\}dt + \sigma_2y(t)dW_2(t), \end{aligned} \tag{1}$$

to the n -dimensional stochastic differential equations

$$dx^i(t) = x^i(t) \left(r_i + \sum_{j=1}^{i-1} a_{ij}x^j(t) - \sum_{j=i}^n a_{ij}x^j(t) \right) dt + \sigma_i x^i(t) dW_i(t), \tag{2}$$

where W_i are independent and real valued Wiener processes on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Although all the parameters r_i, a_{ij} and σ_i in (1) are positive, we weaken the conditions on the signs in (2) such that

$$r_i \in \mathbb{R}, a_{ii} > 0, a_{ij} \geq 0, \sigma_i > 0 \ (1 \leq i, j \leq n, i \neq j).$$

Consider the Euler-Maruyama scheme for (2)

$$x_{k+1}^i = x_k^i \left\{ 1 + \Delta t \left(r_i + \sum_{j=1}^{i-1} a_{ij}x_k^j - \sum_{j=i}^n a_{ij}x_k^j \right) + \sqrt{\Delta t} \sigma_i \xi_{k+1}^i \right\}, \tag{3}$$

where $1 \leq i \leq n, k \geq 0, x_0^i > 0, \Delta t = \frac{1}{N}$ for $N \in \mathbb{N}, t_k = k\Delta t$, and discrete Wiener processes $W_i(t_{k+1}) - W_i(t_k)$ are $\sqrt{\Delta t} \xi_{k+1}^i$ with a mutually independent and identically distributed sequence $(\xi_k^1, \dots, \xi_k^n)_{k=1}^\infty$ of the standard normal random variables. The solutions of (3) are defined with respect to a complete, filtered probability space $(\Omega_N, \mathcal{F}_N, \{\mathcal{F}_k\}_{k=1}^\infty, \mathbb{P}_N)$, where $\{\mathcal{F}_k\}_{k=1}^\infty$ is the natural filtration generated by the stochastic sequence $(\xi_k^1, \dots, \xi_k^n)_{k=1}^\infty$.

The positivity of the solutions of continuous time SDEs (1) is obtained in the infinite time interval $[0, \infty)$ without the assumption of boundedness of the noises $W_i(t)$ by using the concept of explosion time (see [7] and [6]). However, for obtaining the positivity of the solutions of discrete time DSDEs (3) in the infinite time interval, we restrict noises to bounded noises, which means that ξ_k^i are assumed to be truncated standard normal random variables with support $[-\varsigma, \varsigma]$ for a positive constant ς that satisfies

$$E(\xi_k^i) = 0, \ E [(\xi_k^i)^2] = 1 - \eta_\varsigma. \tag{4}$$

The positive value $\eta_\varsigma = \frac{2\varsigma\phi(\varsigma)}{\Phi(\varsigma) - \Phi(-\varsigma)}$ can be assumed to be sufficiently close to 0, where ϕ and Φ are the probability density and the cumulative distribution functions of the standard normal random variable, respectively. For example, when $\varsigma = 20$, we have $0 < \eta_\varsigma < 10^{-85}$.

The paper is organized as follows. Section 2 gives the positivity and the boundedness of the solutions of (3). In Section 3, we introduce the new discrete Itô formula developed in [5] by using a known discrete Itô formula for stochastic difference equations (see [8] [9] and [10]), which is the main tool for finding conditions for the global stability of the solutions of (3). Section 4 introduces auxiliary equations, the solutions of which are used for upper bounds of the solutions of (3). We show the global stability of the solutions of the two-dimensional model (3) in Section 5. The properties of the solutions of the auxiliary equations are used in Section 6 to find conditions for at least one of the solutions of (3) to converge to zero. In addition, the approaches in Section 5 are extended into the n -dimensional model (3) for finding conditions for the global stability the solutions of (3). Section 7 gives simulation results to confirm the results obtained in this paper.

2. Positivity and boundedness of solutions of DSDEs

In this section, we show the positivity and boundedness of solutions of the n -dimensional model (3) by applying the approach used in the DDE model (3) with $\sigma_1 = \sigma_2 = 0$ (see [11]). For simplicity, we use the notations for every constant a ,

$$\hat{a} = a \cdot \Delta t, \quad \tilde{a} = a \cdot \sqrt{\Delta t}$$

and

$$\mathbf{x}_k^i = (x_k^1, \dots, x_k^{i-1}, x_k^{i+1}, \dots, x_k^n).$$

Then the n -dimensional discrete model (3) can be written as

$$x_{k+1}^i = F_{\mathbf{x}_k^i}(x_k^i),$$

where

$$F_{\mathbf{x}_k^i}(x_k^i) = x_k^i \left(1 + \hat{r}_i + \sum_{1 \leq j \leq i-1} \hat{a}_{ij} x_k^j - \sum_{i \leq j \leq n} \hat{a}_{ij} x_k^j + \tilde{\sigma}_i \zeta_{k+1}^i \right). \quad (5)$$

For a vector $\zeta_k^i = (\zeta_k^1, \dots, \zeta_k^{i-1}, \zeta_k^{i+1}, \dots, \zeta_k^n)$ of real numbers, define

$$V(\zeta_k^i) = (2\hat{a}_{ii})^{-1} \left(1 + \hat{r}_i + \sum_{j=1}^{i-1} \hat{a}_{ij} \zeta_k^j - \sum_{j=i+1}^n \hat{a}_{ij} \zeta_k^j + \tilde{\sigma}_i \zeta_{k+1}^i \right). \quad (6)$$

If $V(\zeta_k^i) > 0$, then (5) gives

$$F_{\zeta_k^i}(x) \text{ is increasing on } 0 \leq x < V(\zeta_k^i). \quad (7)$$

For $1 \leq i \leq n$, denote

$$\chi_i = \tilde{a}_{ii}^{-1} \left(\tilde{r}_i + \sum_{1 \leq j \leq i-1} \tilde{a}_{ij} \chi_j + \sigma_i \varsigma \right) \quad (8)$$

and assume that

$$\chi_i \leq (2\hat{a}_{ii})^{-1} \left(1 + \hat{r}_i - \sum_{i+1 \leq j \leq n} \hat{a}_{ij} \chi_j - \tilde{\sigma}_i \varsigma \right), \quad (9)$$

$$\hat{r}_i + \sum_{1 \leq j \leq i-1} \hat{a}_{ij} \chi_j + \tilde{\sigma}_i \varsigma \leq 1. \quad (10)$$

The model (3) is also assumed to satisfy the initial condition

$$(x_0^1, \dots, x_0^n) \in \prod_{1 \leq i \leq n} (0, \chi_i). \quad (11)$$

Theorem 1. *Let x_k^i be the solutions of (3) and χ_i be defined in (8). Then*

$$(x_k^1, \dots, x_k^n) \in \prod_{1 \leq i \leq n} (0, \chi_i), \quad k \geq 0.$$

Proof. It follows from (11), (9) and (6) that for $1 \leq i \leq n$

$$0 < x_0^i < \chi_i < (2\hat{a}_{ii})^{-1} \left(1 + \hat{r}_i - \sum_{i+1 \leq j \leq n} \hat{a}_{ij} \chi_j - \tilde{\sigma}_i \varsigma \right) < V(\mathbf{x}_0^i).$$

Then letting $\zeta_0^i = \mathbf{x}_0^i$ in (7) gives the positivity

$$x_1^i = F_{\mathbf{x}_0^i}(x_0^i) > F_{\mathbf{x}_0^i}(0) = 0, \quad 1 \leq i \leq n.$$

Let $\omega \in \Omega_N$. If $\hat{r}_i + \sum_{1 \leq j \leq i-1} \hat{a}_{ij}x_0^j(\omega) - \sum_{i \leq j \leq n} \hat{a}_{ij}x_0^j(\omega) + \tilde{\sigma}_i \xi_1^i(\omega) \leq 0$, then

$$x_1^i(\omega) = F_{x_0^i}(x_0^i)(\omega) \leq x_0^i(\omega) < \chi_i,$$

and otherwise, we have $0 < x_0^i(\omega) < f(x_0^i)(\omega)$ with

$$f(x_0^i) = \tilde{a}_{ii}^{-1} \left(\tilde{r}_i + \sum_{1 \leq j \leq i-1} \tilde{a}_{ij}x_0^j - \sum_{i+1 \leq j \leq n} \tilde{a}_{ij}x_0^j + \sigma_i \xi_1^i \right).$$

Since $0 < f(x_0^i) < V(x_0^i)$ by (10), using both (7) with $\zeta_k^i = x_0^i$ and (8) with (11), we have

$$x_1^i(\omega) = F_{x_0^i}(x_0^i)(\omega) < F_{x_0^i}(f(x_0^i))(\omega) = f(x_0^i)(\omega) < \chi_i, 1 \leq i \leq n.$$

Therefore if $(x_0^1, \dots, x_0^n) \in \prod_{1 \leq i \leq n} (0, \chi_i)$, then

$$(x_1^1, \dots, x_1^n) \in \prod_{1 \leq i \leq n} (0, \chi_i),$$

and hence applying mathematical induction, we can complete the proof. □

Remark 1. Since (8) and (9) can be written as

$$\begin{aligned} \chi_i &= (\Delta t)^{-0.5} a_{ii}^{-1} \left(\tilde{r}_i + \sum_{1 \leq j \leq i-1} \tilde{a}_{ij} \chi_j + \sigma_i \varsigma \right), \\ \chi_i &\leq (\Delta t)^{-1} (2a_{ii})^{-1} \left(1 + \hat{r}_i - \sum_{i+1 \leq j \leq n} \hat{a}_{ij} \chi_j - \tilde{\sigma}_i \varsigma \right), \end{aligned}$$

the conditions (9) and (10) can be satisfied when taking small values of Δt . For example, take $n = 3$ in (3). The definition (8) gives $\chi_1 = \tilde{a}_{11}^{-1} (\tilde{r}_1 + \sigma_1 \varsigma)$, $\chi_2 = \tilde{a}_{22}^{-1} (\tilde{r}_2 + \tilde{a}_{21} \chi_1 + \sigma_2 \varsigma)$ and $\chi_3 = \tilde{a}_{33}^{-1} (\tilde{r}_3 + \tilde{a}_{31} \chi_1 + \tilde{a}_{32} \chi_2 + \sigma_3 \varsigma)$. Let $\Delta t = 0.0001$, $\varsigma = 20$, $r_i = a_{ij} = \sigma_i = 1$ for $1 \leq i, j \leq 3$. Then the conditions (9) and (10) are satisfied.

3. A new discretized version of the Itô formula

In order to find conditions for the stability of the solutions of (3), we need a discretized form of the Itô formula. Although there are discretized versions of the Itô formula (see [8], [9] and [10]), we developed a variant which is suitable for our purpose in [5]. For the completeness of this paper, we include the proof of the new discretized version of the Itô formula in Appendix below.

We write $q_1(h) = O(q_2(h))$ (or $q_1(h) = O(q_2(h))$ for $h \rightarrow 0$ to be more precise) if there exist positive constants C and h_0 such that $|q_1(h)| \leq C|q_2(h)|$ for all h with $0 < h \leq h_0$. We make the following assumptions about the noise ξ :

- (a) The noise ξ satisfies that for some C and μ with $0 < \mu < 1$

$$E(\xi) = 0, E(\xi^2) = 1 - \mu, E(|\xi|^\ell) \leq C \quad (\ell = 1, 3).$$

- (b) The probability density function p exists and satisfies that for some C and all sufficiently large $|x|$

$$|x|^3 p(x) \leq \frac{C}{|x|}.$$

The truncated Normal random variables satisfy both (a) and (b) with $\mu = \eta_\zeta$ in (4). By $C^3(\mathbb{R})$, we denote the set of all functions defined on \mathbb{R} that are continuously differentiable up to the order 3.

Lemma 1. Consider functions $\phi, \varphi : \mathbb{R} \rightarrow \mathbb{R}$ satisfying for some $\delta > 0$,

- (i) $\varphi = \phi$ on $[1 - \delta, 1 + \delta]$.
- (ii) $\varphi \in C^3(\mathbb{R})$ and $|\varphi'''(x)| \leq C$ for some C and all $x \in \mathbb{R}$.
- (iii) $\int_{\mathbb{R}} |\varphi(x) - \phi(x)| dx < \infty$.

Let ξ be an \mathcal{F} -independent random variable satisfying (a) and (b). Let f and g be \mathcal{F} -measurable random variables satisfying that for some positive constants ε and C ,

$$\max\{h|f|, \sqrt{h}|g|\} \leq Ch^\varepsilon \tag{12}$$

Then

$$\begin{aligned} E \left[\phi \left(1 + hf + \sqrt{h}g\xi \right) \middle| \mathcal{F} \right] \\ = \phi(1) + \phi'(1)hf + \frac{1}{2}\phi''(1)hg^2 \cdot (1 - \mu) + hfO(h^\varepsilon) + hg^2O(h^\varepsilon). \end{aligned}$$

Remark 2. For the solutions x_k^i of (3), let $f = r_i + \sum_{j=1}^{i-1} a_{ij}x_k^j - \sum_{j=i}^n a_{ij}x_k^j$ and $g = \sigma_i$. Then f and g satisfy (12) with $\varepsilon = 0.5$ due to $0 < x_k^i \leq \chi_i = O(h^{-0.5})$ for $1 \leq i \leq n$ and $k \geq 0$.

Remark 3. In order to construct φ corresponding to the function

$$\phi(x) = \begin{cases} \ln|x| & (|x| > 0) \\ 0 & (x = 0) \end{cases},$$

we modify the function φ in [2]. Define the function φ as follows:

$$\varphi(x) = \begin{cases} \ln|x| & (|x| \geq e^{-1}) \\ -\frac{1}{4}e^4x^4 + e^2x^2 - \frac{7}{4} + \frac{e^6}{6}(x - e^{-1})^3(x + e^{-1})^3 & (|x| \leq e^{-1}) \end{cases}.$$

Then φ satisfies all the conditions in Lemma 1.

Notation 1. For simplicity, we use the notations for $1 \leq i \leq n$ and $k > 0$,

$$\bar{E}(x_k^i) = \frac{1}{k} \sum_{s=0}^{k-1} E(x_s^i),$$

and for every constant a and η_ζ in (4)

$$\dot{a} = a \cdot \{1 + O(h^{0.5})\}, \quad a_\eta = a \cdot (1 - \eta_\zeta), \quad r_{i\sigma} = r_i - 0.5\sigma_{i\eta}^2.$$

Remark 4. Since the solutions x_{k+1}^i of (3) are positive, we can take logarithm of (3). Then applying Lemma 1 with (ϕ, φ) in Remark 3, the \mathcal{F}_k -independent and normally truncated random variable ξ_{k+1} and (f, g) in Remark 2, we have

$$E[\ln x_{k+1}^i | \mathcal{F}_k] = E[\ln x_k^i | \mathcal{F}_k] + E \left[\phi \left(1 + hf + \sqrt{h}g\xi_{k+1} \right) \middle| \mathcal{F}_k \right],$$

which gives

$$\begin{aligned} E(\ln x_{k+1}^i) &= \ln x_k^i + hf + \frac{1}{2}hg^2 \cdot (1 - \eta_\zeta u) + hfO(h^{0.5}) + hg^2O(h^{0.5}) \\ &= \ln x_k^i + \mathring{h} \left(r_i - \frac{1}{2}\sigma_{i\eta}^2 + \sum_{j=1}^{i-1} a_{ij}x_k^j - \sum_{j=i}^n a_{ij}x_k^j \right). \end{aligned} \tag{13}$$

Taking expectation of (13) and adding the result, we can obtain

$$E(\ln x_k^i) = E(\ln x_0^i) + k\mathring{h} \left(r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\bar{E}(x_k^j) - \sum_{j=i}^n a_{ij}\bar{E}(x_k^j) \right). \tag{14}$$

4. Auxiliary equations

In order to find upper bounds of x_k^i , we consider the auxiliary equations for $1 \leq i \leq n$ and $k \geq 0$

$$z_{k+1}^i = z_k^i \left(1 + \hat{r}_i + \sum_{j=1}^{i-1} \hat{a}_{ij}z_k^j - \hat{a}_{ii}z_k^i + \tilde{\sigma}_i \xi_{k+1}^i \right), \quad z_0^i = x_0^i. \tag{15}$$

Since (15) is the system (3) with $a_{ij} = 0$ for $1 \leq i < j \leq n$, Theorem 1 gives

$$(z_k^1, \dots, z_k^n) \in \prod_{1 \leq i \leq n} (0, \chi_i), \quad k \geq 0. \tag{16}$$

Let β_i be the solutions of the equations

$$r_{i\sigma} + \sum_{1 \leq j \leq i-1} a_{ij}\beta_j - a_{ii}\beta_i = 0, \quad 1 \leq i \leq n. \tag{17}$$

Note that (13) and (14) with $a_{1j} = 0$ ($2 \leq j \leq n$) become

$$E(\ln z_{k+1}^1) = \ln z_k^1 + \mathring{\Delta}t (r_{1\sigma} - a_{11}z_k^1), \tag{18}$$

$$\begin{aligned} E(\ln z_k^1) &= E(\ln z_0^1) + k\mathring{\Delta}t \{r_{1\sigma} - a_{11}\bar{E}(z_s^1)\} \\ &= E(\ln z_0^1) + k\mathring{\Delta}ta_{11} \left\{ \beta_1 - k^{-1} \sum_{s=0}^{k-1} E(z_s^1) \right\}, \end{aligned} \tag{19}$$

due to $\beta_1 = r_{1\sigma}a_{11}^{-1}$ in (17).

The proofs of Lemma 2 and 3 were given in [5]. For the completeness of this paper, we write the proofs again.

Lemma 2. *Let z_k^1 and β_1 be the solutions of (15) and (17), respectively. If $\beta_1 > 0$, then for every $\epsilon > 0$ and some integer $N_\epsilon > 0$*

$$k^{-1} \sum_{s=0}^{k-1} E(z_s^1) \leq \beta_1 + \epsilon, \quad k \geq N_\epsilon.$$

Proof. Suppose that the theorem is false, which means that there exist a constant $\varepsilon_0 > 0$ and an infinite increasing sequence $\{k_m\}$ satisfying both for all k_m

$$k_m^{-1} \sum_{s=0}^{k_m-1} E(z_s^1) \geq \beta_1 + \varepsilon_0, \tag{20}$$

and for all k with $k \neq k_m$

$$k^{-1} \sum_{s=0}^{k-1} E(z_s^1) < \beta_1 + \varepsilon_0. \tag{21}$$

Combining (20) and (19), we have

$$\lim_{m \rightarrow \infty} E(\ln z_{k_m}^1) = -\infty. \tag{22}$$

Substituting (22) and the boundedness of z_k^1 into (18) gives

$$\lim_{m \rightarrow \infty} \ln z_{k_m-1}^1 = -\infty, \text{ a.s.}$$

and then

$$\lim_{m \rightarrow \infty} z_{k_m-1}^1 = 0, \text{ a.s.} \tag{23}$$

Thus the dominated convergence theorem with (16) leads to

$$\lim_{m \rightarrow \infty} E(z_{k_m-1}^1) = 0. \tag{24}$$

In order to obtain a contraction we show the following Claim 1 and 2.

Claim 1: For all sufficiently large k

$$k^{-1} \sum_{s=0}^{k-1} E(z_s^1) \geq \beta_1 + \varepsilon_0. \tag{25}$$

Assume that there exists $k = k_m - 1$ satisfying (21). The system of (20) and (21) becomes

$$\begin{aligned} \sum_{s=0}^{k_m-1} E(z_s^1) &\geq k_m (\beta_1 + \varepsilon_0), \\ \sum_{s=0}^{k_m-2} E(z_s^1) &< (k_m - 1) (\beta_1 + \varepsilon_0), \end{aligned}$$

which gives

$$E(z_{k_m-1}^1) > \beta_1 + \varepsilon_0. \tag{26}$$

Hence there exist finitely many k satisfying (21) due to (24) and (26).

Claim 2: As (20) implies (24), the equation (25) implies

$$\lim_{k \rightarrow \infty} E(z_k^1) = 0,$$

which is contradictive to (25) due to $\beta_1 + \varepsilon_0 > 0$. This contradiction completes the proof. \square

Lemma 3. Let (z_k^1, z_k^2) and (β_1, β_2) be the solutions of (15) and (17), respectively.

- (a) If $r_{i\sigma} < 0$ ($i = 1, 2$), then $\lim_{k \rightarrow \infty} z_k^i = 0$ ($i = 1, 2$), a.s.
- (b) Assume $r_{1\sigma} > 0$. Then $\lim_{k \rightarrow \infty} k^{-1} \sum_{s=0}^{k-1} E(z_s^1) = \beta_1$.
 - (i) If $r_{2\sigma} + a_{21}\beta_1 < 0$, then $\lim_{k \rightarrow \infty} z_k^2 = 0$, a.s.

(ii) If $r_{2\sigma} + a_{21}\beta_1 > 0$, then $\lim_{k \rightarrow \infty} k^{-1} \sum_{s=0}^{k-1} E(z_s^2) = \beta_2$.

Proof. (a) Since $r_{1\sigma} < 0$ is equivalent to $\beta_1 < 0$, the equation (19) with the positivity of z_k^1 gives that if $r_{1\sigma} < 0$, then $\lim_{k \rightarrow \infty} E(\ln z_k^1) = -\infty$ and so we have $\lim_{k \rightarrow \infty} z_k^1 = 0$, *a.s.* Hence the dominated convergence theorem yields $\lim_{k \rightarrow \infty} E(z_k^1) = 0$, which implies

$$\lim_{k \rightarrow \infty} \overline{E}(z_k^1) = 0. \tag{27}$$

It remains to show that $\lim_{k \rightarrow \infty} z_k^2 = 0$, *a.s.* Using (13) and (14) with $a_{2j} = 0$ ($3 \leq j \leq n$), we have

$$E(\ln z_{k+1}^2) = \ln z_k^2 + \Delta t (r_{2\sigma} + a_{21}z_k^1 - a_{22}z_k^2), \tag{28}$$

$$\begin{aligned} E(\ln z_k^2) &= E(\ln z_0^2) + k\Delta t \{r_{2\sigma} + a_{21}\overline{E}(z_k^1) - a_{22}\overline{E}(z_k^2)\} \\ &= E(\ln z_0^2) + k\Delta t a_{22} [a_{22}^{-1} \{r_{2\sigma} + a_{21}\overline{E}(z_k^1)\} - \overline{E}(z_k^2)] \end{aligned} \tag{29}$$

Combining (27) and (29) with $r_{2\sigma} < 0$ and the positivity of z_k^2 , we have

$$\lim_{k \rightarrow \infty} E(\ln z_k^2) = -\infty$$

Therefore, as (22) implies (23), we can obtain that if $r_{2\sigma} < 0$, then $\lim_{k \rightarrow \infty} z_k^2 = 0$, *a.s.*

(b) Assume $r_{1\sigma} > 0$, which gives $\beta_1 = r_{1\sigma}a_{11}^{-1} > 0$.

Due to Lemma 2, it is enough to prove that for all $\epsilon > 0$ there exists an integer N_ϵ satisfying

$$\beta_1 - \epsilon \leq k^{-1} \sum_{s=0}^{k-1} E(z_s^1), \quad k \geq N_\epsilon. \tag{30}$$

Suppose that (30) is false, which means that there exist a constant $\epsilon_0 > 0$ and an infinite increasing sequence $\{k_m\}$ such that

$$\beta_1 - \epsilon_0 > k_m^{-1} \sum_{s=0}^{k_m-1} E(z_s^1).$$

Then the boundedness of z_k^1 and (19) imply that for all k_m

$$\infty > E(\ln z_{k_m}^1) > E(\ln z_0^1) + k_m \Delta t a_{11} \epsilon_0,$$

which is a contradiction. Therefore (30) is true and then Lemma 2 gives

$$\lim_{k \rightarrow \infty} k^{-1} \sum_{s=0}^{k-1} E(z_s^1) = \beta_1. \tag{31}$$

(b)-(i) Assume that $r_{1\sigma} > 0$ and $r_{2\sigma} + a_{21}\beta_1 < 0$.

Applying (31) to (29) with $r_{2\sigma} + a_{21}\beta_1 < 0$ and the positivity of z_k^2 , we have

$$\lim_{k \rightarrow \infty} E(\ln z_k^2) = -\infty.$$

Therefore, as (22) implies (23), we can obtain $\lim_{k \rightarrow \infty} z_k^2 = 0$, *a.s.*

(b)-(ii) Assume that $r_{1\sigma} > 0$ and $r_{2\sigma} + a_{21}\beta_1 > 0$ and then $\beta_2 = a_{22}^{-1}(r_{2\sigma} + a_{21}\beta_1) > 0$.

Following the proof of Lemma 2, we can obtain that for every $\epsilon > 0$ and some integer $N_\epsilon > 0$

$$k^{-1} \sum_{s=0}^{k-1} E(z_s^2) \leq \beta_2 + \epsilon, \quad k \geq N_\epsilon, \tag{32}$$

by replacing z_k^1, β_1 , (18) and (19) in Lemma 2 with z_k^2, β_2 , (28) and (29), respectively, and using (31).

On the other hand, following the proof of (30) with (29) instead of (19), we can obtain that for all $\epsilon > 0$ there exists an integer N_ϵ satisfying

$$\beta_2 - \epsilon \leq k^{-1} \sum_{s=0}^{k-1} E(z_s^2), \quad k \geq N_\epsilon. \tag{33}$$

Combining (32) and (33), we obtain $\lim_{k \rightarrow \infty} k^{-1} \sum_{s=0}^{k-1} E(z_s^2) = \beta_2$. □

The proofs of Remark 5 and 6 are given in Appendix below.

Remark 5. Using the idea in Lemma 3, we can find conditions under which the solution z_k^3 of (15) converges. Let $\tilde{\beta}_3 = a_{33}^{-1}(r_{3\sigma} + a_{31}\beta_1)$, which is equal to β_3 when $\beta_2 = 0$.

- (a) Assume that $r_{1\sigma} > 0$ and $r_{2\sigma} + a_{21}\beta_1 < 0$.
 - (i) If $r_{3\sigma} + a_{31}\beta_1 < 0$, then $\lim_{k \rightarrow \infty} z_k^3 = 0, a.s.$
 - (ii) If $r_{3\sigma} + a_{31}\beta_1 > 0$, then $\lim_{k \rightarrow \infty} k^{-1} \sum_{s=0}^{k-1} E(z_s^3) = \tilde{\beta}_3$.
- (b) Assume that $r_{1\sigma} > 0$ and $r_{2\sigma} + a_{21}\beta_1 > 0$.
 - (i) If $r_{3\sigma} + \sum_{j=1}^2 a_{3j}\beta_j < 0$, then $\lim_{k \rightarrow \infty} z_k^3 = 0, a.s.$
 - (ii) If $r_{3\sigma} + \sum_{j=1}^2 a_{3j}\beta_j > 0$, then $\lim_{k \rightarrow \infty} k^{-1} \sum_{s=0}^{k-1} E(z_s^3) = \beta_3$.

Remark 6. Replacing (3) with (15), the equation (14) becomes

$$E(\ln z_k^i) = E(\ln z_0^i) + k\Delta t \left\{ r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\bar{E}(z_k^j) - a_{ii}\bar{E}(z_k^i) \right\}. \tag{34}$$

Substituting (17) to (34) yields

$$E(\ln z_k^i) = E(\ln z_0^i) + k\Delta t \left[\sum_{j=1}^{i-1} a_{ij} \{ \bar{E}(z_k^j) - \beta_j \} - a_{ii} \{ \bar{E}(z_k^i) - \beta_i \} \right], \tag{35}$$

so that we can extend (b)-(ii) in both Lemma 3 and Remark 5 to the n -dimensional case: If $r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\beta_j > 0$ ($1 \leq i \leq n$), then $\lim_{k \rightarrow \infty} \bar{E}(z_k^i) = \beta_i$ ($1 \leq i \leq n$) and, as a result, (35) yields

$$\lim_{k \rightarrow \infty} \frac{1}{k} E(\ln z_k^i) = 0, \quad 1 \leq i \leq n. \tag{36}$$

Lemma 4. Let x_k^i and z_k^i be the solutions of (3) and (15), respectively. Then

$$0 < x_k^i \leq z_k^i, \quad 1 \leq i \leq n, \quad k \geq 0.$$

Proof. Let $\mathbf{0}_0^k$ be the zero vector of $n - 1$ entries for $k \geq 0$. Since $x_0^1 > 0$ and $F_{\zeta_0^1}(x)$ in (5) is decreasing in ζ_0^1 , we have

$$x_1^1 = F_{x_0^1}(x_0^1) \leq F_{\mathbf{0}_0^1}(x_0^1). \tag{37}$$

Using (6) and (9), we have

$$0 < x_0^1 \leq z_0^1 \leq \chi_1 \leq V(\mathbf{0}_0^1)$$

and then (7) yields

$$F_{\mathbf{0}_0^1}(x_0^1) \leq F_{\mathbf{0}_0^1}(z_0^1) = z_1^1. \tag{38}$$

Hence combining (37) and (38) gives

$$x_1^1 \leq z_1^1. \tag{39}$$

Note that for $k \geq 1$

$$z_k^1 > 0, \quad z_k^1 \leq \chi_1 \leq V(\mathbf{0}_0^k) \tag{40}$$

by Theorem 1, (16), (6), (9). Following the proof of (39), we can obtain

$$x_k^1 \leq z_k^1 \quad (k \geq 0)$$

by using mathematical induction with (40) and $\mathbf{0}_0^k$.

Similarly, letting $\mathbf{z}_0^2 = (z_0^1, 0, \dots, 0)$ and using $0 < x_0^2 \leq z_0^2 \leq \chi_2 \leq V(\mathbf{z}_0^2)$, we have

$$x_1^2 = F_{\mathbf{z}_0^2}(x_0^2) \leq F_{\mathbf{0}_0^2}(x_0^2) \leq F_{\mathbf{0}_0^2}(z_0^2) = z_1^2. \tag{41}$$

Hence mathematical induction with (41) and $\mathbf{z}_{k-1}^2 = (z_{k-1}^1, 0, \dots, 0)$ gives

$$x_k^2 \leq z_k^2 \quad (k \geq 0).$$

Therefore, using mathematical induction with $\mathbf{z}_k^i = (z_k^1, \dots, z_k^{i-1}, 0, \dots, 0)$, we can complete the proof. □

Remark 7. If $r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\beta_j > 0$ ($1 \leq i \leq n$), then both Lemma 4 and (36) imply that for every $\epsilon > 0$ there exists an integer N_ϵ such that

$$\frac{1}{k}E(\ln x_k^i) \leq \epsilon, \quad k \geq N_\epsilon, \tag{42}$$

which will be used in Section 5 and 6 to show that all the solutions of (3) converge to positive values.

5. Global stability of the n-dimensional DSDEs

In this section, we first find conditions under which at least one of the solutions of (3) converges to zero, a.s. and then another conditions under which all $\bar{E}(x_k^i)$ ($1 \leq i \leq n$) converge to positive values. The extension of the approach used for the global stability of the two-dimensional DSDEs can lead us to the global stability of the n -dimensional DSDEs, so that we again treat the two-dimensional DSDEs in [5].

5.1. Global stability of the two-dimensional DSDEs

In this section, we consider the two-dimensional model (3) with $n = 2$

$$\begin{aligned} x_{k+1}^1 &= x_k^1(1 + \hat{r}_1 - \hat{a}_{11}x_k^1 - \hat{a}_{12}x_k^2 + \tilde{\sigma}_1\xi_{k+1}^1), \\ x_{k+1}^2 &= x_k^2(1 + \hat{r}_2 + \hat{a}_{21}x_k^1 - \hat{a}_{22}x_k^2 + \tilde{\sigma}_2\xi_{k+1}^2). \end{aligned} \tag{43}$$

Define A_2 and D_2^i ($i = 1, 2$) by

$$A_2 = \begin{pmatrix} a_{11} & a_{12} \\ -a_{21} & a_{22} \end{pmatrix} \text{ and } \begin{pmatrix} r_{1\sigma} \\ r_{2\sigma} \end{pmatrix} = A_2 \begin{pmatrix} D_2^1 \\ D_2^2 \end{pmatrix}. \tag{44}$$

Theorem 2. Let x_k^i and β_i be the solutions of (43) and (17), respectively.

- (a) If $r_{i\sigma} < 0$ ($i = 1, 2$), then $\lim_{k \rightarrow \infty} x_k^i = 0$ ($i = 1, 2$), a.s.
- (b) If $r_{1\sigma} > 0$, $r_{2\sigma} + a_{21}\beta_1 < 0$, then $\lim_{k \rightarrow \infty} \overline{E}(x_k^1) = \beta_1$ and $\lim_{k \rightarrow \infty} x_k^2 = 0$, a.s.
- (c) If $r_{1\sigma} > a_{22}^{-1}a_{12}r_{2\sigma}$ and $r_{2\sigma} + a_{21}\beta_1 > 0$, then $\lim_{k \rightarrow \infty} \overline{E}(x_k^i) = D_2^i$ ($i = 1, 2$), where D_2^i is defined in (44).

Proof. (a) The proof is followed by applying Lemma 3-(a) and Lemma 4.

(b) From Lemma 3-(b)-(i) and Lemma 4, we have $\lim_{k \rightarrow \infty} x_k^2 = 0$, a.s. which gives that for every $\varepsilon > 0$, there exists an integer N_ε satisfying

$$0 < x_k^2 \leq \varepsilon, \quad k \geq N_\varepsilon. \tag{45}$$

Consider the system

$$\begin{aligned} u_{k+1}^L &= u_k^L(1 + \hat{r}_1 - \hat{a}_{11}u_k^L - \hat{a}_{12}\varepsilon + \tilde{\sigma}_1\xi_{k+1}^1), \quad u_0^L = x_0^1, \\ u_{k+1}^U &= u_k^U(1 + \hat{r}_1 - \hat{a}_{11}u_k^U + \hat{a}_{12}\varepsilon + \tilde{\sigma}_1\xi_{k+1}^1), \quad u_0^U = x_0^1. \end{aligned}$$

Following the proofs of both Lemma 4 and (45), we can have

$$0 < u_k^L \leq x_k^1 \leq u_k^U \quad \text{for } k \geq N_\varepsilon,$$

and then due to (31), we can obtain

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{s=0}^{k-1} E(u_s^L) = \frac{r_{1\sigma} - a_{12}\varepsilon}{a_{11}}, \quad \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{s=0}^{k-1} E(u_s^U) = \frac{r_{1\sigma} + a_{12}\varepsilon}{a_{11}},$$

which gives the desired result.

(c) Let $|A_2|$ be the determinant of A_2 in (44), which is positive. Then

$$\begin{pmatrix} D_2^1 \\ D_2^2 \end{pmatrix} = A_2^{-1} \begin{pmatrix} r_{1\sigma} \\ r_{2\sigma} \end{pmatrix} = |A_2|^{-1} \begin{pmatrix} a_{22}r_{1\sigma} - a_{12}r_{2\sigma} \\ a_{11}(a_{21}\beta_1 + r_{2\sigma}) \end{pmatrix} \tag{46}$$

and hence the conditions in (c) imply $D_2^i > 0$ for $i = 1, 2$.

Applying (44) to (14) with $n = 2$ gives

$$\begin{pmatrix} E(\ln x_k^1) \\ E(\ln x_k^2) \end{pmatrix} = \begin{pmatrix} E(\ln x_0^1) \\ E(\ln x_0^2) \end{pmatrix} + k\Delta t A_2 \begin{pmatrix} D_2^1 - \overline{E}(x_k^1) \\ D_2^2 - \overline{E}(x_k^2) \end{pmatrix}, \tag{47}$$

and multiplying the matrix $|A_2|A_2^{-1}$ to (47), we have

$$a_{22}E(\ln x_k^1) - a_{12}E(\ln x_k^2) = C_1 + k\Delta t|A_2| (D_2^1 - \bar{E}(x_k^1)), \tag{48}$$

$$a_{21}E(\ln x_k^1) + a_{11}E(\ln x_k^2) = C_2 + k\Delta t|A_2| (D_2^2 - \bar{E}(x_k^2)), \tag{49}$$

where $C_1 = a_{22}E(\ln x_0^1) - a_{12}E(\ln x_0^2)$ and $C_2 = a_{21}E(\ln x_0^1) + a_{11}E(\ln x_0^2)$.

Since $|A_2| > 0$, $D_2^1 > 0$ in (48), and $-a_{12}E(\ln x_k^2) > -\infty$ from Theorem 1, we can follow the proof of Lemma 2 with (48) instead (19) and, as a result, obtain that for every $\epsilon_2^1 > 0$ there exists an integer $N_{\epsilon_2^1}$ such that

$$\bar{E}(x_k^1) \leq D_2^1 + \epsilon_2^1, \quad k \geq N_{\epsilon_2^1}. \tag{50}$$

Substituting (42) into (49) gives that for every $\epsilon_2^2 > 0$ there exists an integer $N_{\epsilon_2^2}$ satisfying

$$\bar{E}(x_k^2) \geq D_2^2 - \epsilon_2^2, \quad k \geq N_{\epsilon_2^2}. \tag{51}$$

Applying (50) to the second equation in (47), we have for $k \geq N_{\epsilon_2^1}$

$$E(\ln x_k^2) \leq E(\ln x_0^2) + k\Delta t a_{22} \{a_{22}^{-1}a_{21}\epsilon_2^1 + D_2^2 - \bar{E}(x_k^2)\}. \tag{52}$$

Instead of (19) and (18), using (52) and (14) with $n = i = 2$ and $D_2^2 > 0$, and following the proof of Lemma 2, we can obtain that for every $\epsilon_2^2 > 0$ there exists an integer $N_{\epsilon_2^2}$ such that

$$\bar{E}(x_k^2) \leq D_2^2 + \epsilon_2^2, \quad k \geq N_{\epsilon_2^2}. \tag{53}$$

Hence (51) and (53) imply

$$\lim_{k \rightarrow \infty} \bar{E}(x_k^2) = D_2^2. \tag{54}$$

It remains to show $\lim_{k \rightarrow \infty} \bar{E}(x_k^1) = D_2^1$. Applying (54) to (49) with (42) yields

$$\lim_{k \rightarrow \infty} \frac{1}{k} E(\ln x_k^1) = \lim_{k \rightarrow \infty} \frac{1}{k} E(\ln x_k^2) = 0,$$

with which (48) gives the desired result. □

Remark 8. Assume $r_{1\sigma} < 0$. Then it follows from Lemma 3-(a) and Lemma 4 that $\lim_{k \rightarrow \infty} x_k^1 = 0, a.s.$ In addition, if $r_{2\sigma} > 0$, then we can have

$$\lim_{k \rightarrow \infty} \bar{E}(x_k^2) = a_{22}^{-1}r_{2\sigma}$$

by following the proof of Theorem 2-(b) with $\lim_{k \rightarrow \infty} x_k^1 = 0, a.s.$, instead of $\lim_{k \rightarrow \infty} x_k^2 = 0, a.s.$

Remark 9. Since $|A_2| > 0$, the identity (46) gives that the two conditions in Theorem 2-(c) are equivalent to $D_2^1 > 0$ and $D_2^2 > 0$. This equivalence with $|A_2| > 0$ is used to find conditions for the global stability of solutions of the n -dimensional model (3) in the next section.

5.2. Model reduction

Using Lemma 3, Remark 5 and Lemma 4, we can find conditions under which at least one of the solutions of (3) converges to zero, a.s. For example, the three-dimensional model (3) with $n = 3$ can be reduced to the two-dimensional model (3) and hence we can apply Theorem 2 as follows.

Theorem 3. Let x_k^i and β_i be the solutions of (3) and (17) with $n = 3$.

(a) If $r_{1\sigma} < 0$, then $\lim_{k \rightarrow \infty} x_k^1 = 0$, a.s.

(b) Assume $r_{1\sigma} > 0$.

(i) Let $r_{i\sigma} + a_{i1}\beta_i < 0$ ($i = 2, 3$). Then $\lim_{k \rightarrow \infty} x_k^i = 0$, a.s. ($i = 2, 3$) and $\lim_{k \rightarrow \infty} \bar{E}(x_k^1) = \beta_1$.

(ii) Let $r_{2\sigma} + a_{21}\beta_1 < 0$ and $r_{3\sigma} + a_{31}\beta_1 > 0$. Then $\lim_{k \rightarrow \infty} x_k^2 = 0$, a.s.

Furthermore, if $r_{1\sigma} > a_{33}^{-1}a_{13}r_{3\sigma}$, then $\lim_{k \rightarrow \infty} (\bar{E}(x_k^1), \bar{E}(x_k^3)) = (\check{D}_2^1, \check{D}_2^3)$.

(iii) Let $r_{2\sigma} + a_{21}\beta_1 > 0$ and $r_{3\sigma} + \sum_{i=1}^2 a_{3i}\beta_i < 0$. Then $\lim_{k \rightarrow \infty} x_k^3 = 0$, a.s.

Furthermore, if $r_{1\sigma} > a_{22}^{-1}a_{12}r_{2\sigma}$, then $\lim_{k \rightarrow \infty} (\bar{E}(x_k^1), \bar{E}(x_k^2)) = (D_2^1, D_2^2)$.

Here $(\check{D}_2^1, \check{D}_2^3)$ is equal to (D_2^1, D_2^2) in (44) when a_{12}, a_{21} and a_{22} are replaced with a_{13}, a_{31} and a_{33} , respectively.

Remark 10. Assume $r_{1\sigma} < 0$, which gives $\lim_{k \rightarrow \infty} x_k^1 = 0$, a.s. by Lemma 3-(a) and Lemma 4. Then the three-dimensional model (3) can be considered as the two-dimensional model (3) with two state variables x_k^2, x_k^3 and $a_{21} = a_{31} = 0$. Therefore, we can apply Theorem 2 and Remark 8.

Remark 11. Assume that $r_3 < 0$ and $a_{31} > 0$ in Theorem 3-(ii). Then $r_{3\sigma} < 0$ and hence $r_{3\sigma} + a_{31}\beta_1 = r_{3\sigma} + a_{31}a_{11}^{-1}r_{1\sigma} > 0$ gives $r_{1\sigma} > 0 \geq a_{33}^{-1}a_{13}r_{3\sigma}$. Therefore Theorem 3-(ii) is satisfied without using the condition $r_{1\sigma} > a_{33}^{-1}a_{13}r_{3\sigma}$ if $r_3 < 0$. Similarly, if $r_2 < 0$ and $a_{21} > 0$, then Theorem 3-(iii) is satisfied without the condition $r_{1\sigma} > a_{22}^{-1}a_{12}r_{2\sigma}$.

5.3. Convergence of all state variables to nonzero values

Now, we find conditions under which $\bar{E}(x_k^i)$ ($1 \leq i \leq n$) converge to positive values D_n^i . The conditions are extensions of the following three conditions used in the proof of Theorem 2-(c):

(A1) $|A_2| = \begin{vmatrix} a_{11} & a_{12} \\ -a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} + a_{12}a_{21} > 0$.

(A2) $D_2^i > 0$, $i = 1, 2$ (see Remark 9).

(A3) The system of equations (48) and (49) has the sign-pattern matrix

$$\text{sgn} \begin{pmatrix} a_{22} & -a_{12} \\ a_{21} & a_{11} \end{pmatrix} = \begin{pmatrix} + & -0 \\ +0 & + \end{pmatrix},$$

which means that the signs of $a_{22}, -a_{12}, a_{21}$, and a_{11} are positive, non-positive, non-negative, and positive, respectively.

Define A_n and D_n^i , extensions of (44), by

$$A_n = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(n-1)} & a_{1n} \\ -a_{21} & a_{22} & \cdots & a_{2(n-1)} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & -a_{n(n-1)} & a_{nn} \end{pmatrix}, \quad \begin{pmatrix} r_{1\sigma} \\ r_{2\sigma} \\ \vdots \\ r_{n\sigma} \end{pmatrix} = A_n \begin{pmatrix} D_n^1 \\ D_n^2 \\ \vdots \\ D_n^n \end{pmatrix}, \quad (55)$$

with the assumptions corresponding to (A1) and (A2):

(A1') $|A_n| > 0$.

(A2') $D_n^i > 0$ for $1 \leq i \leq n$.

Using (55), we can write (14) as

$$E(\ln x_k^i) = E(\ln x_0^i) + k\Delta t \left\{ -\sum_{j=1}^{i-1} a_{ij} (D_n^j - \bar{E}(x_k^j)) + \sum_{j=i}^n a_{ij} (D_n^j - \bar{E}(x_k^j)) \right\},$$

which is corresponding to (47). Hence (48) and (49) are extended as

$$\sum_{1 \leq j \leq n} C_{ji} E(\ln x_k^j) = \sum_{1 \leq j \leq n} C_{ji} E(\ln x_0^j) + k\Delta t |A_n| \{D_n^i - \bar{E}(x_k^i)\}, \quad (56)$$

where cofactors C_{ij} of A_n satisfy the assumption corresponding to (A3):

$$(A3') \operatorname{sgn} \begin{pmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & \ddots & \vdots \\ C_{1n} & \cdots & C_{nn} \end{pmatrix} = \begin{pmatrix} + & -0 & \cdots & -0 & -0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -0 & -0 & \cdots & + & -0 \\ +0 & +0 & \cdots & +0 & + \end{pmatrix}.$$

Remark 12. It follows from (17) and (55) that (A2') gives

$$\begin{aligned} a_{11}\beta_1 &= r_{1\sigma} = \sum_{1 \leq j \leq n} a_{1j} D_n^j \geq a_{11} D_n^1 > 0, \\ a_{22}\beta_2 &= r_{2\sigma} + a_{21}\beta_1 \geq r_{2\sigma} + a_{21} D_n^1 = \sum_{2 \leq j \leq n} a_{2j} D_n^j \geq a_{22} D_n^2. \end{aligned}$$

Repeating this process, we can conclude that (A2') implies $r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\beta_j > 0$ ($1 \leq i \leq n$), and then we can use (42) in the proof of the following theorem.

Now we can extend the proof of Theorem 2-(c) to the n -dimensional model (3).

Theorem 4. Let x_k^i be the solutions of (3) and D_n^i be defined in (55). Assume that (A1')–(A3') are satisfied. Then

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{s=0}^{k-1} E(x_s^i) = D_n^i, \quad 1 \leq i \leq n.$$

Proof. Instead of (48), using (56) for $1 \leq i \leq n - 1$ and following the proof of (50) with (A1') and (A2'), we can obtain that for every $\epsilon_n^i > 0$ there exists an integer $N_{\epsilon_n^i}$ such that

$$\bar{E}(x_k^i) \leq D_n^i + \epsilon_n^i, \quad 1 \leq i \leq n - 1, \quad k \geq N_{\epsilon_n^i}. \quad (57)$$

Substituting (42) into (56) with $i = n$ yields that for every $\epsilon_n^n > 0$ there exists an integer $N_{\epsilon_n^n}$ such that

$$\bar{E}(x_k^n) \geq D_n^n - \epsilon_n^n, \quad k \geq N_{\epsilon_n^n}. \tag{58}$$

As in the proof of (53), we can have that for every $\epsilon_n^m > 0$ there exists an integer $N_{\epsilon_n^m}$ such that

$$\bar{E}(x_k^n) \leq D_n^n + \epsilon_n^m, \quad k \geq N_{\epsilon_n^m} \tag{59}$$

by replacing both (50) and the second equation in (47) with (57) and

$$E(\ln x_k^n) = E(\ln x_0^n) + k\Delta t \left\{ \sum_{j=1}^{n-1} a_{nj} (\bar{E}(x_k^j) - D_n^j) - a_{nn} (\bar{E}(x_k^n) - D_n^n) \right\},$$

respectively. Hence (58) and (59) give

$$\lim_{k \rightarrow \infty} \bar{E}(x_k^n) = D_n^n. \tag{60}$$

Now it remains to show $\lim_{k \rightarrow \infty} \bar{E}(x_k^i) = D_n^i$ ($1 \leq i \leq n - 1$). Substituting (60) to (56) with $i = n$ and using both $C_{in} \geq 0$ ($1 \leq i \leq n$) in (A3') and (42), we can obtain

$$\lim_{k \rightarrow \infty} \frac{1}{k} E(\ln x_k^i) = 0, \quad 1 \leq i \leq n - 1,$$

with which (56) gives the desired result. □

Remark 13. Following the proof of Lemma 1 under the assumptions about the noise ξ , we can obtain the new discretized Itô formula for the Milstein method

$$\begin{aligned} & E \left[\phi \left(1 + hf + \sqrt{h}g\xi + \frac{1}{2}hg^2 \cdot (\xi^2 - 1) \right) \middle| \mathcal{F} \right] \\ &= \phi(1) + \phi'(1) \left(hf - \frac{1}{2}hg^2\mu \right) + \frac{1}{2}\phi''(1)hg^2 \cdot (1 - \mu) + h(f + g^2)O(h^\epsilon), \end{aligned}$$

which gives

$$\begin{aligned} & E \left[\ln \left(1 + hf + \sqrt{h}g\xi + 0.5hg^2 \cdot (\xi^2 - 1) \right) \middle| \mathcal{F} \right] \\ &= h \{ (f - 0.5g^2) + (f + g^2)O(h^\epsilon) \}. \end{aligned}$$

Therefore, replacing $r_{i\sigma} = r_i - 0.5\sigma_{in}^2$ and $\sqrt{\Delta t}\sigma_i\xi_k^i$ with $r_{i\sigma} = r_i - 0.5\sigma_i^2$ and $\sqrt{\Delta t}\sigma_i\xi_k^i + 0.5\Delta t\sigma_i^2 \{(\xi_k^i)^2 - 1\}$ for $1 \leq i \leq n$, respectively, we can conclude that the solutions of the Milstein scheme for (2) satisfy all the results in this paper.

6. Numerical examples

In this section, we provide some simulations that illustrate our results in Theorem 3 and 4 for the three-dimensional model (3) with $\Delta t = 0.001$ and $\zeta = 20$ in (4). Let $r_1 = 1.1, a_{11} = 2.1, a_{12} = 0.2, a_{13} = 0.1, a_{21} = 0.5, a_{22} = 1.1, a_{23} = 0.1, a_{31} = 3.1, a_{32} = 5.1, a_{33} = 0.5$ and $\sigma_2 = \sigma_3 = 0.1$ in the model (3) with $n = 3$. The model is simulated 1000 and 5000 times in Figure 1 and 2, respectively, for calculating the values of expectations.

Example 1. Figure 1-(a) denotes three plots in the first column of Figure 1. Each figure in the i th row of Figure 1 is the sequence of the 1000 realizations of x_k^i ($i = 1, 2, 3$). In Figure 1-(a), we let $(-r_2, -r_3) = (0.1, 2.1)$ and $\sigma_1 = 2$. The values $(-r_2, -r_3)$ in Figure 1-(b), (c) and (d) are $(5.1, 5.1), (5.1, 1.1), (0.1, 5.1)$, respectively, with $\sigma_1 = 0.1$. Then the values of parameters in Figure 1-(a), (b), (c) and (d) satisfy the conditions in Theorems 3-(a), (b)-(i), (ii) and (iii), respectively, and Figure 1 demonstrates Theorem 3: For example, all x_k^i ($i = 1, 2, 3$) in Figure 1-(a) converge to zero. At $k = 2 \cdot 10^6$, which means time 2000, the errors $|\bar{E}(x_k^1) - a_{11}^{-1}r_{1\sigma}|$ in Figure 1-(b), $|\bar{E}(x_k^i) - \check{D}_2^i|$ ($i = 1, 3$) in Figure 1-(c), and $|\bar{E}(x_k^i) - D_2^i|$ ($i = 1, 2$) in Figure 1-(d) are 0.00012, 0.000075, 0.0019, 0.000038 and 0.00068, respectively.

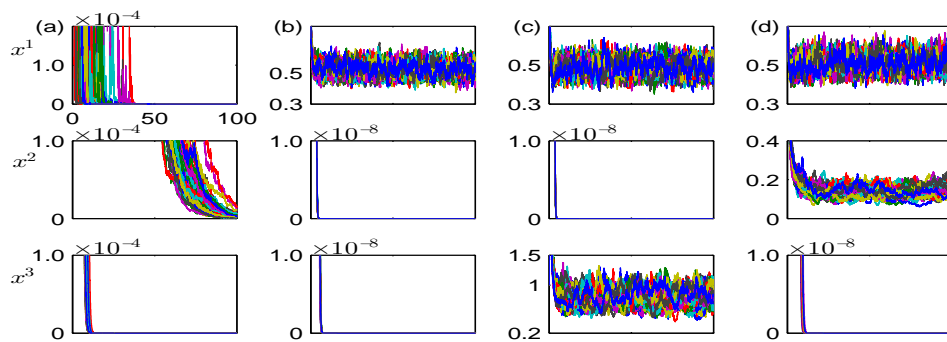


Figure 1: All the x -axes denote time $k\Delta t$ from 0 to 100. The curves in each figure are 1000 realizations of one of the solutions x_k^i ($1 \leq i \leq 3$) of DSDEs. The solutions x_k^i ($1 \leq i \leq 3$) in (a), x_k^i ($i = 2, 3$) in (b), x_k^2 in (c), x_k^3 in (d) are all convergent to zero.

Example 2. Let $r_2 = -0.1, r_3 = -2.1$ and $\sigma_1 = 0.1$. Then all the conditions in Theorem 4 are satisfied. Consequently, the solutions x_k^i ($1 \leq i \leq 3$) are positive, which is demonstrated in Figure 2-(a), and $\bar{E}(x_k^i)$ are convergent to positive values D_3^i , which is demonstrated in Figure 2-(b). At $k = 5 \cdot 10^6$, which means time 5000, the error vector $(|\bar{E}(x_k^1) - D_3^1|, |\bar{E}(x_k^2) - D_3^2|, |\bar{E}(x_k^3) - D_3^3|)$ is marked with the larger star in Figure 2-(b) and equal to $(0.000028, 0.00011, 0.0018)$.

7. Conclusion

In this paper, dealing with the n -dimensional stochastic difference model, we have extended the new approach to obtain the global stability of the fixed point of the two-dimensional stochastic and discrete predator-prey system. As in the two-dimensional model, we have found the conditions under which at least one discrete solution converges to

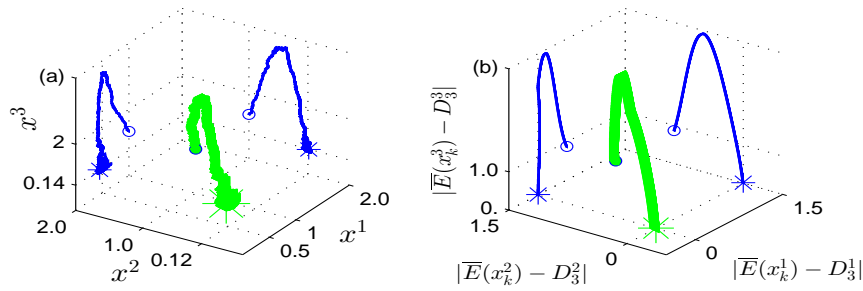


Figure 2: The start and the final points of each curve are denoted by the circle and the star symbols, respectively. Two thicker curves in (a) and (b) show one realization of (x_k^1, x_k^2, x_k^3) ($0 \leq k \leq 10^6$) and error vectors $(|\bar{E}(x_k^1) - D_3^1|, |\bar{E}(x_k^2) - D_3^2|, |\bar{E}(x_k^3) - D_3^3|)$ ($0 \leq k \leq 5 \cdot 10^6$), respectively. Each of the other curves is the projection of one of the thicker curves onto one plane: The two planes in Figure 2-(a) are $x^1 = 2$ and $x^2 = 2$.

zero by using a model reduction method. Additionally, we have also found the conditions under which all expectations of solutions globally converge to non-zero fixed values. While proving the global stability of solutions of the two- and n -dimensional stochastic difference models, we have used our new discretized form of the Itô formula and therefore found the possibility that the new discrete Itô formula will be applied to other stochastic difference models.

Appendix

A.1. The proof of Lemma 1

By Taylor expansion,

$$\varphi(1+x) = \varphi(1) + \varphi'(1)x + \frac{\varphi''(1)}{2}x^2 + \frac{\varphi'''(\theta)}{6}x^3$$

with θ lying between 1 and x . We substitute $x = hf + \sqrt{hg}\xi$ and take expectations. Since ξ is \mathcal{F} -independent with $E(\xi) = 0$ and $E(\xi^2) = 1 - \mu$, and f, g are \mathcal{F} -measurable, we have

$$E(x|\mathcal{F}) = hf, \quad E(x^2|\mathcal{F}) = (hf)^2 + hg^2 \cdot (1 - \mu).$$

Note that

$$\left| E\left(\frac{\varphi'''(\theta)}{6}x^3 \mid \mathcal{F}\right) \right| \leq \frac{C}{6} E(|x^3| \mid \mathcal{F}) = hfO(h^\varepsilon) + hg^2O(h^\varepsilon)$$

by expanding x^3 and using $E(|\xi^i|) < \infty$ ($i = 1, 2, 3$) and (12). Therefore

$$E(\varphi(1+x) \mid \mathcal{F}) \tag{61}$$

$$= \varphi(1) + \varphi'(1)hf + \frac{\varphi''(1)}{2}hg^2 \cdot (1 - \mu) + hfO(h^\varepsilon) + hg^2O(h^\varepsilon). \tag{62}$$

Now it is enough to show

$$E\left(\varphi\left(1 + hf + \sqrt{hg}\xi\right) - \varphi\left(1 + hf + \sqrt{hg}\xi\right) \mid \mathcal{F}\right) = hg^2O(h^\varepsilon).$$

Since f, g are \mathcal{F} -measurable and ξ is \mathcal{F} -independent, letting $c_1 = 1 + hf$, $c_2 = \sqrt{hg}$ and $U_\delta = [1 - \delta, 1 + \delta]$ gives

$$\begin{aligned} & E \left(\phi \left(1 + hf + \sqrt{hg}\xi \right) - \varphi \left(1 + hf + \sqrt{hg}\xi \right) \mid \mathcal{F} \right) \\ &= \int_{\mathbb{R}} \{ \phi(c_1 + c_2x) - \varphi(c_1 + c_2x) \} p(x) dx \\ &= \int_{\mathbb{R} - U_\delta} \{ \phi(s) - \varphi(s) \} p \left(\frac{s - c_1}{c_2} \right) \frac{ds}{|c_2|} \end{aligned} \tag{63}$$

where p is the probability density function of ξ .

Note that

$$\begin{aligned} & \left| \int_{\mathbb{R} - U_\delta} \{ \phi(s) - \varphi(s) \} p \left(\frac{s - c_1}{c_2} \right) \frac{ds}{|c_2|} \right| \\ & \leq \left\{ \int_{\mathbb{R} - U_\delta} |\phi(s) - \varphi(s)| \frac{ds}{|c_2|} \right\} \sup_{s \notin U_\delta} \left\{ p \left(\frac{s - c_1}{c_2} \right) \frac{1}{|c_2|} \right\} \\ & \leq C \cdot |c_2|^2 \sup_{s \notin U_\delta} \left\{ p \left(\frac{s - c_1}{c_2} \right) \frac{1}{|c_2|^3} \right\} \\ & = C \cdot hg^2 \sup_{s \notin U_\delta} \left\{ p \left(\frac{s - 1 - hf}{\sqrt{hg}} \right) \frac{1}{|\sqrt{hg}|^3} \right\}. \end{aligned}$$

Here letting $y = \frac{s - 1 - hf}{\sqrt{hg}}$, we have

$$\sup_{s \notin U_\delta} \left\{ p \left(\frac{s - 1 - hf}{\sqrt{hg}} \right) \frac{1}{|\sqrt{hg}|^3} \right\} = \sup_{s \notin U_\delta} \left\{ \frac{p(y) |y|^3}{|s - 1 - hf|^3} \right\}$$

For all $s \notin U_\delta$, there exists some δ_0 such that for all sufficiently small $h > 0$

$$|s - 1 - hf| > |s - 1| - h|f| > \delta - Ch^\epsilon > \delta_0 > 0 \tag{64}$$

and then we have

$$|y| \equiv \frac{|s - 1 - hf|}{\sqrt{h}|g|} \geq \frac{\delta_0}{O(h^\epsilon)}. \tag{65}$$

Hence it follows from (64), (65) and the assumption (b) that

$$\sup_{s \notin U_\delta} \left\{ \frac{p(y) |y|^3}{|s - 1 - hf|^3} \right\} = O(h^\epsilon),$$

which gives

$$\left| \int_{\mathbb{R} - U_\delta} \{ \phi(s) - \varphi(s) \} p \left(\frac{s - c_1}{c_2} \right) \frac{ds}{|c_2|} \right| = hg^2 O(h^\epsilon). \tag{66}$$

Therefore using (61), (63) and (66), we obtain the desired result.

A.2. The proof of Remark 5

Using (13) and (14) with $a_{3j} = 0$ ($4 \leq j \leq n$), we have

$$E(\ln z_{k+1}^3) = \ln z_k^3 + \Delta t (r_{3\sigma} + a_{31}z_k^1 + a_{32}z_k^2 - a_{33}z_k^3), \tag{67}$$

$$E(\ln z_k^3) = E(\ln z_0^3) + k\Delta t a_{33} [a_{33}^{-1} \{r_{3\sigma} + a_{31}\bar{E}(z_k^1) + a_{32}\bar{E}(z_k^2)\} - \bar{E}(z_k^3)]. \tag{68}$$

A.2.1. The proof of Remark 5-(a)-(i)

Assume that

$$r_{1\sigma} > 0, r_{2\sigma} + a_{21}\beta_1 < 0, r_{3\sigma} + a_{31}\beta_1 < 0.$$

Lemma 3-(b) and (b)-(i) give that

$$\lim_{k \rightarrow \infty} \bar{E}(z_k^1) = \beta_1 \tag{69}$$

and

$$\lim_{k \rightarrow \infty} z_k^2 = 0, \tag{70}$$

respectively. Applying the dominated convergence theorem with (70), we have

$$\lim_{k \rightarrow \infty} E(z_k^2) = 0$$

and then

$$\lim_{k \rightarrow \infty} \bar{E}(z_k^2) = 0. \tag{71}$$

Substituting (69) and (71) into (68) and using the positivity of z_k^3 , we obtain

$$\lim_{k \rightarrow \infty} E(\ln z_k^3) = 0. \tag{72}$$

Therefore combining (72) and (67) with the boundedness of z_k^i ($i = 1, 2, 3$) in (16), we can obtain the desired result: $\lim_{k \rightarrow \infty} z_k^3 = 0, a.s.$

A.2.2. The proof of Remark 5-(a)-(ii)

Assume that

$$r_{1\sigma} > 0, r_{2\sigma} + a_{21}\beta_1 < 0, r_{3\sigma} + a_{31}\beta_1 > 0$$

and then

$$\lim_{k \rightarrow \infty} \bar{E}(z_k^1) = \beta_1, \lim_{k \rightarrow \infty} \bar{E}(z_k^2) = 0, \tilde{\beta}_3 = a_{33}^{-1} (r_{3\sigma} + a_{31}\beta_1) > 0. \tag{73}$$

Following the proof of Lemma 2, we can obtain that for every $\epsilon > 0$ and some $N_\epsilon > 0$

$$k^{-1} \sum_{s=0}^{k-1} E(z_s^3) \leq \tilde{\beta}_3 + \epsilon, k \geq N_\epsilon, \tag{74}$$

by replacing z_k^1, β_1 , (18) and (19) in Lemma 2 with $z_k^3, \tilde{\beta}_3$, (67) and (68), respectively and using (73). Hence it remains to show that for every $\epsilon > 0$ and some $N_\epsilon > 0$

$$\tilde{\beta}_3 - \epsilon \leq k^{-1} \sum_{s=0}^{k-1} E(z_s^3), k \geq N_\epsilon. \tag{75}$$

Following the proof of (30) with (68) instead of (31), we can obtain (75), which gives the desired result : $\lim_{k \rightarrow \infty} k^{-1} \sum_{s=0}^{k-1} E(z_s^3) = \tilde{\beta}_3.$

A.2.3. The proof of Remark 5-(b)-(i)

Assume that

$$r_{1\sigma} > 0, r_{2\sigma} + a_{21}\beta_1 > 0, r_{3\sigma} + \sum_{j=1}^2 a_{3j}\beta_j < 0$$

and then

$$\lim_{k \rightarrow \infty} \bar{E}(z_k^i) = \beta_i \quad (i = 1, 2), \quad \beta_3 = a_{33}^{-1} \left(r_{3\sigma} + \sum_{j=1}^2 a_{3j}\beta_j \right) < 0, \tag{76}$$

due to Lemma 3-(b) and (b)-(ii).

Combining (76) and (68) with the positivity of z_k^3 , we can have

$$\lim_{k \rightarrow \infty} E(\ln z_k^3) = -\infty.$$

Therefore, as (22) implies (23), we can obtain the desired result: $\lim_{k \rightarrow \infty} z_k^3 = 0, a.s.$

A.3. The proofs of Remark 5-(b)-(ii) and Remark 6

Using mathematical induction, we prove that

$$\text{if } r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\beta_j > 0 \quad (1 \leq i \leq n), \text{ then } \lim_{k \rightarrow \infty} \bar{E}(z_k^i) = \beta_i \quad (1 \leq i \leq n).$$

Lemma 3-(a) gives that

$$\text{if } r_{1\sigma} > 0, \text{ then } \lim_{k \rightarrow \infty} \bar{E}(z_k^1) = \beta_1.$$

Assume that for some $\ell \in \{2, \dots, n\}$

$$r_{i\sigma} + \sum_{j=1}^{i-1} a_{ij}\beta_j > 0 \quad (1 \leq i \leq \ell) \text{ and } \lim_{k \rightarrow \infty} \bar{E}(z_k^i) = \beta_i \quad (1 \leq i \leq \ell - 1). \tag{77}$$

Using (15) with $i = \ell$ instead of (3), the equations (13) and (14) become

$$E(\ln z_{k+1}^\ell) = \ln z_k^\ell + \Delta t \left(r_{\ell\sigma} + \sum_{j=1}^{\ell-1} a_{\ell j} z_k^j - a_{\ell\ell} z_k^\ell \right), \tag{78}$$

$$E(\ln z_k^\ell) = E(\ln z_0^\ell) + k \Delta t a_{\ell\ell} \left\{ a_{\ell\ell}^{-1} \sum_{j=1}^{\ell-1} a_{\ell j} \bar{E}(z_k^j) - \bar{E}(z_k^\ell) \right\}. \tag{79}$$

Following the proof of Lemma 2, we can obtain that for every $\epsilon > 0$ and some $N_\epsilon > 0$

$$\lim_{k \rightarrow \infty} \bar{E}(z_k^\ell) \leq \beta_\ell + \epsilon, \quad k \geq N_\epsilon, \tag{80}$$

by replacing $z_k^1, \beta_1, (18)$ and (19) in Lemma 2 with $z_k^\ell, \beta_\ell, (78)$ and (79) , respectively and using (77). Hence it remains to show that for every $\epsilon > 0$ and some $N_\epsilon > 0$

$$\beta_\ell - \epsilon \leq \lim_{k \rightarrow \infty} \bar{E}(z_k^\ell), \quad k \geq N_\epsilon. \tag{81}$$

Following the proof of (30) with (79) instead of (31), we can obtain (81), which gives

$$\lim_{k \rightarrow \infty} \bar{E}(z_k^\ell) = \beta_\ell.$$

Therefore the desired result is obtained.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Entire solutions of certain type of nonlinear differential equations and differential-difference equations *

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Abstract. In this paper, we investigate the transcendental entire solutions of finite order of the differential equations or differential-difference equations

$$f^2(z) + P(f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z},$$

where $p_1, p_2, \alpha_1, \alpha_2$ are nonzero constants with $\alpha_1 \neq \alpha_2$, and $P(f)$ denotes a differential polynomial or differential-difference polynomial in $f(z)$ with degree 1. And we partially answer a question proposed by Li [10] (P. Li, Entire solutions of certain type of differential equations II, J. Math. Anal. Appl. 375(2011), 310 – 319).

Mathematics Subject Classification (2010). 39B32, 34M05, 30D35.

Keywords. Entire solutions, Differential equations, Differential-difference equations, finite order.

1 Introduction and Results

In this paper, we assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna theory [1, 2]. In addition, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$, as $r \rightarrow \infty$, outside of a possible exceptional set of finite logarithmic measure.

Recently, a number of papers (including [3 – 12]) have focused on solvability and existence of meromorphic solutions of differential equations, difference equations or differential-difference equations in complex plane. Specifically, it shows in [4] that the equation $4f^3 + 3f'' = -\sin 3z$ has exactly three nonconstant entire solutions, namely $f_1(z) = \sin z$, $f_2(z) = \frac{\sqrt{3}}{2} \cos z - \frac{1}{2} \sin z$ and $f_3(z) = -\frac{\sqrt{3}}{2} \cos z - \frac{1}{2} \sin z$. The following two Theorems obtained more general results.

Theorem A. (See [5]) *Let $n \geq 2$ be an integer, $P(f)$ be a differential polynomial in $f(z)$ of degree at most $n - 2$, and λ, p_1, p_2 be three nonzero constants. If $f(z)$ is an entire solution of the following equation*

$$f^n(z) + P(f) = p_1 e^{\lambda z} + p_2 e^{-\lambda z}, \tag{1.1}$$

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then

$$f(z) = c_1 e^{\lambda z/n} + c_2 e^{-\lambda z/n},$$

where c_1 and c_2 are constants and $c_i^2 = p_i, i = 1, 2$.

Theorem B. (See [10]) *Let $n \geq 2$ be an integer, $P(f)$ be a differential polynomial in $f(z)$ of degree at most $n - 2$, and $p_1, p_2, \alpha_1, \alpha_2$ be nonzero constants and $\alpha_1 \neq \alpha_2$. If $f(z)$ is a transcendental meromorphic solution of the following equation*

$$f^n(z) + P(f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \tag{1.2}$$

and satisfying $N(r, f) = S(r, f)$, then one of the following holds:

- (i) $f(z) = c_0(z) + c_1 e^{\alpha_1 z/n}$;
- (ii) $f(z) = c_0(z) + c_2 e^{\alpha_2 z/n}$;
- (iii) $f(z) = c_1 e^{\alpha_1 z/n} + c_2 e^{\alpha_2 z/n}$ and $\alpha_1 + \alpha_2 = 0$,

where $c_0(z)$ is a small function of $f(z)$ and c_1, c_2 are constants satisfying $c_i^2 = p_i, i = 1, 2$.

For further study, Li [10] proposed the following question.

Question. How to find the solutions of (1.2) under the condition $\deg P(f) = n - 1$?

In this paper, we study this question and partially answer this question, and obtain the following result.

Theorem 1.1. *Let $p_1, p_2, \alpha_1, \alpha_2$ be nonzero constants such that $\alpha_1 \neq \alpha_2$, $a(z)$ be a nonzero polynomial. If $f(z)$ is a transcendental entire solution of finite order of the differential equation*

$$f^2(z) + a(z)f'(z) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \tag{1.3}$$

and satisfying $N\left(r, \frac{1}{f}\right) = S(r, f)$, then $a(z)$ must be a constant, and one of the following holds:

- (i) $f(z) = c_1 e^{\alpha_1 z/2}$, $ac_1 \alpha_1 = 2p_2$, $\alpha_1 = 2\alpha_2$;
- (ii) $f(z) = c_2 e^{\alpha_2 z/2}$, $ac_2 \alpha_2 = 2p_1$, $\alpha_2 = 2\alpha_1$;

where c_1, c_2 are constants satisfying $c_i^2 = p_i, i = 1, 2$.

Remark 1.1. *We conjecture that the condition $N\left(r, \frac{1}{f}\right) = S(r, f)$ in Theorem 1.1 can be omitted using another method although we have not found a suitable one yet.*

In [11], Wang and Li investigated the existence of entire solutions of differential-difference equation

$$f^n(z) + q(z)f^{(k)}(z+c) = ae^{ibz} + de^{-ibz}, \tag{1.4}$$

and obtained the following Theorem.

Theorem C. (See [11]) *For two integers $n \geq 3, k \geq 0$ and a nonlinear differential-difference equation (1.4), where $q(z)$ is a polynomial and a, b, c, d are constants such that $|a| + |d| \neq 0, bc \neq 0$.*

(i) *Let $n = 3$. If $q(z)$ is nonconstant, then the equation (1.4) does not admit entire solutions of finite order. If $q := q(z)$ is a constant, then equation (1.4) admits three distinct transcendental entire solutions of finite order, provided that*

$$bc = 3m\pi \ (m \neq 0, \text{ if } q \neq 0), \ q^3 = (-1)^{m+1} \left(\frac{3i}{b}\right)^{3k} 27ad,$$

when k is an even, or

$$bc = \frac{3\pi}{2} + 3m\pi \text{ (if } q \neq 0), \quad q^3 = i(-1)^m \left(\frac{3i}{b}\right)^{3k} 27ad,$$

when k is an odd, for an integer m .

(ii) Let $n \geq 3$. If $ad \neq 0$, then the equation (1.4) does not admit entire solutions of finite order. If $ad = 0$, then equation (1.4) admits n distinct transcendental entire solutions of finite order, provided that $q := q(z) \equiv 0$.

In this paper, we shall tackle differential-difference equations in the form (1.4) with $n = 2$, and obtain the following result.

Theorem 1.2. Let p_1, p_2, λ, c be nonzero constants, $k \geq 0$ be an integer and $a(z)$ be a nonzero polynomial. If $f(z)$ is a transcendental entire solution of finite order of the differential-difference equation

$$f^2(z) + a(z)f^{(k)}(z+c) = p_1e^{\lambda z} + p_2e^{-\lambda z}, \tag{1.5}$$

then $a(z)$ must be a constant, and satisfying one of the following relations:

- (i) $f(z) = \pm \frac{i}{2}a\left(\frac{\lambda}{2}\right)^k + c_1e^{\lambda z/2} + c_2e^{-\lambda z/2}$, $e^{\lambda c} = -1$, when k is an odd;
- (ii) $f(z) = \pm \frac{1}{2}a\left(\frac{\lambda}{2}\right)^k + c_1e^{\lambda z/2} + c_2e^{-\lambda z/2}$, $e^{\lambda c} = 1$, when k is an even and $k > 0$, where a, c_1, c_2 are constants with $\frac{1}{64}a^4\left(\frac{\lambda}{2}\right)^{4k} = p_1p_2$ and $c_i^2 = p_i$, $i = 1, 2$;
- (iii) $f(z) = \pm \frac{1}{2}a + c_1e^{\lambda z/2} + c_2e^{-\lambda z/2}$, $e^{\lambda c} = 1$, when $k = 0$, where a, c_1, c_2 are constants with $\frac{1}{64}a^4 = p_1p_2$ or $\frac{9}{64}a^4 = p_1p_2$ and $c_i^2 = p_i$, $i = 1, 2$.

Example 1.1. $f(z) = e^z$ is a transcendental entire solution of finite order of the differential equation

$$f^2(z) + f'(z) = e^{2z} + e^z.$$

Example 1.2. The differential-difference equation

$$f^2(z) + f(z + \pi i/2) = \frac{1}{4}e^{4z} + \frac{9}{16}e^{-4z}.$$

has exactly two entire solutions, namely $f_1(z) = \frac{1}{2} + \frac{1}{2}e^{2z} - \frac{3}{4}e^{-2z}$ and $f_2(z) = \frac{1}{2} - \frac{1}{2}e^{2z} + \frac{3}{4}e^{-2z}$.

Example 1.3. The differential-difference equation

$$f^2(z) - f'(z + \pi i/2) = \frac{1}{4}e^{2z} + \frac{1}{16}e^{-2z}.$$

has exactly two entire solutions, namely $f_1(z) = \frac{i}{2} + \frac{1}{2}e^z + \frac{1}{4}e^{-z}$ and $f_2(z) = \frac{i}{2} - \frac{1}{2}e^z - \frac{1}{4}e^{-z}$.

Example 1.4. The differential-difference equation

$$f^2(z) - 2f''(z + \pi i) = \frac{1}{4}e^{2z} + e^{-2z}.$$

has exactly two entire solutions, namely $f_1(z) = -1 + \frac{1}{2}e^z - e^{-z}$ and $f_2(z) = -1 - \frac{1}{2}e^z + e^{-z}$.

2 Lemmas for the Proof of Theorems

Lemma 2.1. (*Clunie's Theorem*)(See [2, 13]) Let $f(z)$ be a transcendental meromorphic solution of

$$f^n(z)P(z, f) = Q(z, f),$$

where $P(z, f)$ and $Q(z, f)$ are polynomials in $f(z)$ and its derivatives with meromorphic coefficients, say $\{a_\lambda | \lambda \in I\}$, such that $m(r, a_\lambda) = S(r, f)$ for all $\lambda \in I$. If the total degree of $Q(z, f)$ as a polynomial in $f(z)$ and its derivatives is $\leq n$, then

$$m(r, P(z, f)) = S(r, f). \tag{2.1}$$

Lemma 2.2. (See [14]) Let $f(z)$ be a nonconstant finite order meromorphic solution of

$$f^n(z)P(z, f) = Q(z, f),$$

where $P(z, f)$ and $Q(z, f)$ are polynomials in $f(z)$ and its shifts with small meromorphic coefficients, and let $c \in \mathbb{C}$, $\delta < 1$. If the total degree of $Q(z, f)$ as a polynomial in $f(z)$ and its shifts is $\leq n$, then

$$m(r, P(z, f)) = o\left(\frac{T(r + |c|, f)}{r^\delta}\right) + o(T(r, f)) \tag{2.2}$$

for all r outside of a possible exceptional set with finite logarithmic measure.

Remark 2.1. In Lemma 2.2, if $f(z)$ is transcendental with finite order, and $P(z, f), Q(z, f)$ are differential-difference polynomials in $f(z)$, then by using a similar method as in the proof of Lemma 2.1, we see that a similar conclusion of Lemma 2.2 holds. Moreover, we know that if the coefficients of $P(z, f)$ and $Q(z, f)$ are polynomials $a_j(z)$, $j = 1, \dots, k$, then (2.2) can be replaced by

$$m(r, P(z, f)) = S(r, f) + O\left(\sum_{j=1}^k m(r, a_j(z))\right), \tag{2.3}$$

where r is sufficiently large.

Lemma 2.3. (See [1, 2]) Let $f(z)$ be a transcendental meromorphic function and $k \geq 1$ be an integer. Then

$$m\left(r, \frac{f^{(k)}(z)}{f(z)}\right) = S(r, f). \tag{2.4}$$

Lemma 2.4. Let α be a nonzero constant, and $H(z)$ be a nonvanishing polynomial. Then the differential equation

$$\alpha f(z) - 2f'(z) = H(z) \tag{2.5}$$

has a special solution $c_0(z)$ which is a nonzero polynomial.

Proof. Similarly to the proof of [Lemma 2.3, 8]. If $H(z)$ is a nonzero constant, then $c_0(z) = \frac{H(z)}{\alpha}$ is a special solution of (2.5). Now suppose that

$$H(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0,$$

where $n \geq 1$ is an integer, $a_n \neq 0, a_{n-1}, \dots, a_1, a_0$ are constants.

We use the method of undetermined coefficients to derive the polynomial solution $c_0(z)$ satisfying (2.5) by $\alpha, a_n, a_{n-1}, \dots, a_1, a_0$. By (2.5), we see that $\deg c_0(z) = \deg H(z)$. For $n = 1$, equation (2.5) has a polynomial solution $c_0(z) = \frac{1}{\alpha}a_1z + \frac{1}{\alpha}(a_0 + \frac{2}{\alpha}a_1)$.

In a general case, for $n \geq 2$, (2.5) has a polynomial solution

$$c_0(z) = b_n z^n + \dots + b_j z^j + \dots + b_1 z + b_0,$$

where

$$b_n = \frac{1}{\alpha}a_n, \quad b_j = \frac{1}{\alpha}a_j + \frac{2(j+1)}{\alpha}b_{j+1}, \quad j = n-1, \dots, 0.$$

Therefore, (2.5) has a nonzero polynomial solution $c_0(z)$.

Lemma 2.5. (See [8]) *Let λ be a nonzero constant, and $H(z)$ be a nonvanishing polynomial. Then the differential equation*

$$4f''(z) - \lambda^2 f(z) = H(z) \tag{2.6}$$

has a special solution $c_0(z)$ which is a nonzero polynomial.

Lemma 2.6. (See [15]) *Suppose that $f_1(z), f_2(z), \dots, f_n(z) (n \geq 2)$ are meromorphic functions and $g_1(z), g_2(z), \dots, g_n(z)$ are entire functions satisfying the following conditions,*

- (1) $\sum_{j=1}^n f_j(z)e^{g_j(z)} \equiv 0$;
- (2) $g_j(z) - g_k(z)$ are not constants for $1 \leq j < k \leq n$;
- (3) For $1 \leq j \leq n, 1 \leq h < k \leq n, T(r, f_j(z)) = o(T(r, e^{g_n(z)-g_k(z)}))(r \rightarrow \infty, r \notin E)$, where $E \subset [1, \infty)$ is finite linear measure or finite logarithmic measure.

Then $f_j(z) \equiv 0 (j = 1, \dots, n)$.

Lemma 2.7. *Let $P(z)$ be a nonzero polynomial, A, B, c be nonzero constants and $A \neq 1$. If*

$$P^2(z) \equiv AP(z)P(z+c) + B, \tag{2.7}$$

then $P(z)$ must be a constant, $P(z) := p(\text{constant})$.

Proof. If $P(z)$ is a nonconstant polynomial, then $\deg P(z) \geq 1$. Now suppose that

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0,$$

where $n \geq 1$ is an integer, $a_n \neq 0, a_{n-1}, \dots, a_0$ are constants, then

$$\begin{aligned} P(z+c) &= a_n(z+c)^n + a_{n-1}(z+c)^{n-1} + \dots + a_0, \\ P^2(z) &= a_n^2 z^{2n} + 2a_n a_{n-1} z^{2n-1} + \dots, \\ P(z)P(z+c) &= a_n^2 z^{2n} + (na_n^2 c + 2a_n a_{n-1})z^{2n-1} + \dots, \\ P^2(z) - AP(z)P(z+c) &= (1-A)a_n^2 z^{2n} + \dots. \end{aligned}$$

From $A \neq 1$, we see that $\deg[P^2(z) - AP(z)P(z+c)] = 2n \geq 2$, it contradicts with (2.7). Thus $P(z)$ must be a nonzero constant, set $P(z) := p(\neq 0)$.

3 Proof of Theorems

Proof of Theorem 1.1

Denote $P = a(z)f'(z)$. Suppose that $f(z)$ is a transcendental entire solution of finite order of equation (1.3). By differentiating (1.3), we have

$$2ff' + P' = \alpha_1 p_1 e^{\alpha_1 z} + \alpha_2 p_2 e^{\alpha_2 z}. \tag{3.1}$$

Eliminating $e^{\alpha_1 z}, e^{\alpha_2 z}$ from (1.3) and (3.1), respectively, we have

$$\alpha_1 f^2 - 2ff' + \alpha_1 P - P' = (\alpha_1 - \alpha_2) p_2 e^{\alpha_2 z}, \tag{3.2}$$

$$\alpha_2 f^2 - 2ff' + \alpha_2 P - P' = (\alpha_2 - \alpha_1) p_1 e^{\alpha_1 z}. \tag{3.3}$$

Differentiating (3.3) yields

$$2\alpha_2 ff' - 2(f')^2 - 2ff'' + \alpha_2 P' - P'' = \alpha_1(\alpha_2 - \alpha_1) p_1 e^{\alpha_1 z}. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\varphi = -Q, \tag{3.5}$$

where

$$\varphi = \alpha_1 \alpha_2 f^2 - 2(\alpha_1 + \alpha_2)ff' + 2(f')^2 + 2ff'', \tag{3.6}$$

and

$$Q = \alpha_1 \alpha_2 P - (\alpha_1 + \alpha_2)P' + P''. \tag{3.7}$$

If $\varphi \neq 0$, by (3.5) – (3.7) and Lemma 2.3, we see that

$$m\left(r, \frac{\varphi}{f}\right) = S(r, f) \quad \text{and} \quad m\left(r, \frac{\varphi}{f^2}\right) = S(r, f). \tag{3.8}$$

From $N\left(r, \frac{1}{f}\right) = S(r, f)$ and (3.8), we see that

$$\begin{aligned} T(r, f) + S(r, f) &= m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{\varphi}{f}\right) + m\left(r, \frac{1}{\varphi}\right) \\ &\leq T(r, \varphi) + S(r, f) = m(r, \varphi) + S(r, f) \\ &\leq m\left(r, \frac{\varphi}{f}\right) + m(r, f) + S(r, f) \\ &= T(r, f) + S(r, f), \end{aligned}$$

that is,

$$T(r, \varphi) = T(r, f) + S(r, f). \tag{3.9}$$

From (3.8) and (3.9), we see that

$$\begin{aligned} 2T(r, f) + S(r, f) &= 2m\left(r, \frac{1}{f}\right) = m\left(r, \frac{1}{f^2}\right) \\ &\leq m\left(r, \frac{\varphi}{f^2}\right) + m\left(r, \frac{1}{\varphi}\right) \\ &\leq T(r, \varphi) + S(r, f) \\ &= T(r, f) + S(r, f), \end{aligned}$$

that is $T(r, f) \leq S(r, f)$, a contradiction. Then we have $\varphi \equiv 0$, that is $Q \equiv 0$. By (3.7), we have

$$\alpha_1\alpha_2P - (\alpha_1 + \alpha_2)P' + P'' \equiv 0. \tag{3.10}$$

Since $P = a(z)f'(z)$, it is obvious that $P \not\equiv 0$. Since $\alpha_1 \neq \alpha_2$, we see that $\alpha_1P - P' \equiv 0$ and $\alpha_2P - P' \equiv 0$ cannot hold simultaneously. Suppose $\alpha_1P - P' \not\equiv 0$. By (3.10), we get

$$\alpha_1P - P' = Ae^{\alpha_2z}, \tag{3.11}$$

where A is a nonzero constant. Substituting (3.11) into (3.2), we have

$$f(\alpha_1f - 2f') = \frac{[(\alpha_1 - \alpha_2)p_2 - A]\alpha_1}{A}P - \frac{(\alpha_1 - \alpha_2)p_2 - A}{A}P'. \tag{3.12}$$

Since the right-hand side of (3.12) is a differential polynomial in $f(z)$ of degree ≤ 1 . By Lemma 2.1, we have

$$m(r, \alpha_1f - 2f') = S(r, f). \tag{3.13}$$

If $\alpha_1f - 2f' \not\equiv 0$, since $f(z)$ is a transcendental entire function of finite order, by (3.13), we see that

$$m(r, \alpha_1f - 2f') = S(r, f) = O(\log r),$$

which implies that $\alpha_1f - 2f'$ is a nonzero polynomial. Then we have

$$\alpha_1f - 2f' = H(z), \tag{3.14}$$

where $H(z)$ is a nonvanishing polynomial, but $H(z)$ may be a nonzero constant. By Lemma 2.4, we know that (3.14) has a nonzero polynomial solution, say, $c_0(z)$.

Since the differential equation

$$\alpha_1f - 2f' = 0 \tag{3.15}$$

has a fundamental solution $f(z) = e^{\alpha_1z/2}$. Then the general entire solution $f(z)$ of (3.14) can be expressed as

$$f(z) = c_0(z) + c_1e^{\alpha_1z/2}, \tag{3.16}$$

where c_1 is a constant, $c_0(z)$ is a nonzero polynomial.

Substituting (3.16) into (1.3), we have

$$(c_1^2 - p_1)e^{\alpha_1z} - p_2e^{\alpha_2z} + \left(2c_0(z) + \frac{\alpha_1}{2}a(z)\right)c_1e^{\alpha_1z/2} + c_0^2(z) + a(z)c_0'(z) = 0. \tag{3.17}$$

By $\alpha_1 \neq \alpha_2$, if $\alpha_2 \neq \frac{\alpha_1}{2}$, by Lemma 2.6, we see that $p_2 = 0$, a contradiction.

If $\alpha_2 = \frac{\alpha_1}{2}$, then (3.17) can be rewritten as

$$(c_1^2 - p_1)e^{\alpha_1z} + \left[\left(2c_0(z) + \frac{\alpha_1}{2}a(z)\right)c_1 - p_2\right]e^{\alpha_2z} + c_0^2(z) + a(z)c_0'(z) = 0. \tag{3.18}$$

By $\alpha_1 \neq \alpha_2$ and Lemma 2.6, we have

$$\left(2c_0(z) + \frac{\alpha_1}{2}a(z)\right)c_1 - p_2 \equiv 0, \quad c_0^2(z) + a(z)c_0'(z) \equiv 0, \tag{3.19}$$

then $c_0(z) \equiv 0$, a contradiction.

Therefore, we have $\alpha_1f - 2f' \equiv 0$, which yields

$$f^2 = Be^{\alpha_1z}, \tag{3.20}$$

where B is a nonzero constant. By (1.3), (3.11) and (3.20), we have

$$\left(1 - \frac{p_1}{B}\right) f^2 = \frac{\alpha_1 p_2 - A}{A} P - \frac{p_2}{A} P'. \tag{3.21}$$

If $p_1 \neq B$, then by (3.21) and Lemma 2.1, we get $T(r, f) = S(r, f)$, which is impossible. Therefore, $p_1 = B$ and $f(z) = c_1 e^{\alpha_1 z/2}$, where c_1 is a nonzero constant satisfying $c_1^2 = p_1$. Substituting $f(z) = c_1 e^{\alpha_1 z/2}$ and $c_1^2 = p_1$ into (1.3), we have

$$\frac{c_1 \alpha_1}{2} a(z) e^{\alpha_1 z/2} - p_2 e^{\alpha_2 z} = 0. \tag{3.22}$$

By $\alpha_1 \neq \alpha_2$ and Lemma 2.6, we see that $\alpha_1 = 2\alpha_2$ and $c_1 \alpha_1 a(z) \equiv 2p_2$, then $a(z)$ must be a constant, set $a(z) := a$.

If $\alpha_2 P - P' \neq 0$, then by a similar method, we can deduce that $f(z) = c_2 e^{\alpha_2 z/2}$, $\alpha_2 = 2\alpha_1$, $c_2 \alpha_2 a = 2p_1$, $c_2^2 = p_2$. This completes the proof of Theorem 1.1.

Proof of Theorem 1.2

Denote $P_1(f) = a(z)f^{(k)}(z + c)$. Suppose that $f(z)$ be a transcendental entire solution of finite order of equation (1.5). By differentiating both sides of equation (1.5), we have

$$2ff' + P_1'(f) = \lambda(p_1 e^{\lambda z} - p_2 e^{-\lambda z}). \tag{3.23}$$

Differentiating (3.23), we obtain

$$2(f')^2 + 2ff'' + P_1''(f) = \lambda^2(p_1 e^{\lambda z} + p_2 e^{-\lambda z}). \tag{3.24}$$

Combining (1.5) with (3.24), we have

$$(f')^2 = \frac{1}{2}\lambda^2 f^2 - ff'' + Q_1(f), \tag{3.25}$$

where $Q_1(f) = \frac{1}{2}(\lambda^2 P_1(f) - P_1''(f))$.

Eliminating $e^{\lambda z}$, $e^{-\lambda z}$ from (1.5) and (3.23), we obtain

$$\lambda^2(f^2 + P_1(f))^2 - (2ff' + P_1'(f))^2 = 4p_1 p_2 \lambda^2,$$

which implies that

$$\lambda^2 f^4 - 4f^2(f')^2 = R_3(f), \tag{3.26}$$

where

$$R_3(f) = -[2\lambda^2 f^2 P_1(f) + \lambda^2 (P_1(f))^2 - 4ff'P_1'(f) - (P_1'(f))^2 - 4p_1 p_2 \lambda^2].$$

Substituting (3.25) into (3.26), we see that

$$f^3(4f'' - \lambda^2 f) = T_3(f), \tag{3.27}$$

where $T_3(f) = 4f^2 Q_1(f) + R_3(f)$.

Now, we consider two cases, case 1: $T_3(f) \equiv 0$ and case 2: $T_3(f) \neq 0$.

Case 1: $T_3(f) \equiv 0$. By (3.27), we have

$$4f'' - \lambda^2 f \equiv 0. \tag{3.28}$$

Every entire solution $f(z) (\neq 0)$ of (3.28) can be expressed as

$$f(z) = c_1 e^{\lambda z/2} + c_2 e^{-\lambda z/2}, \tag{3.29}$$

where c_1, c_2 are constants, and at least one of them being not equal to zero.

Then

$$f^{(k)}(z+c) = c_1 \left(\frac{\lambda}{2}\right)^k e^{\lambda c/2} e^{\lambda z/2} + (-1)^{k+1} c_2 \left(\frac{\lambda}{2}\right)^k e^{-\lambda c/2} e^{-\lambda z/2}. \tag{3.30}$$

Substituting (3.29) and (3.30) into (1.5), we obtain

$$(c_1^2 - p_1) e^{\lambda z} + (c_2^2 - p_2) e^{-\lambda z} + a(z) c_1 \left(\frac{\lambda}{2}\right)^k e^{\lambda c/2} e^{\lambda z/2} + (-1)^{k+1} a(z) c_2 \left(\frac{\lambda}{2}\right)^k e^{-\lambda c/2} e^{-\lambda z/2} + 2c_1 c_2 = 0. \tag{3.31}$$

By Lemma 2.6, we see that

$$a(z) c_1 \left(\frac{\lambda}{2}\right)^k e^{\lambda c/2} \equiv 0, \quad (-1)^{k+1} a(z) c_2 \left(\frac{\lambda}{2}\right)^k e^{-\lambda c/2} \equiv 0. \tag{3.32}$$

From $\lambda \neq 0$ and $a(z) \not\equiv 0$, then $c_1 = c_2 = 0$, a contradiction.

Case 2: $T_3(f) \neq 0$. Since $f(z)$ is a transcendental entire function of finite order, we see that (3.27) satisfies the conditions of Lemma 2.2 and Remark 2.1. Thus we have

$$m(r, 4f'' - \lambda^2 f) = S(r, f) + O(m(r, a(z))) = O(\log r),$$

which implies that $4f'' - \lambda^2 f$ is a polynomial. Thus, from (3.27) and $T_3(f) \neq 0$, we have

$$4f'' - \lambda^2 f = H(z), \tag{3.33}$$

where $H(z)$ is a nonvanishing polynomial. By Lemma 2.5, we see that (3.33) must have a nonzero polynomial solution, say, $c_0(z)$.

Since the differential equation

$$4f'' - \lambda^2 f = 0, \tag{3.34}$$

has two fundamental solutions

$$f_1(z) = e^{\frac{\lambda}{2}z}, \quad f_2(z) = e^{-\frac{\lambda}{2}z}.$$

Then the general entire solution $f(z) (\neq 0)$ of (3.33) can be expressed as

$$f(z) = c_0(z) + c_1 e^{\frac{\lambda}{2}z} + c_2 e^{-\frac{\lambda}{2}z}, \tag{3.35}$$

where c_1, c_2 are constants, $c_0(z)$ is a nonzero polynomial.

If k is an odd, then

$$f^{(k)}(z+c) = c_0^{(k)}(z+c) + c_1 \left(\frac{\lambda}{2}\right)^k e^{\lambda c/2} e^{\lambda z/2} - c_2 \left(\frac{\lambda}{2}\right)^k e^{-\lambda c/2} e^{-\lambda z/2}. \tag{3.36}$$

Substituting (3.35) and (3.36) into (1.5), we obtain

$$\begin{aligned} & (c_1^2 - p_1) e^{\lambda z} + (c_2^2 - p_2) e^{-\lambda z} + \left(2c_1 c_0(z) + a(z) c_1 \left(\frac{\lambda}{2}\right)^k e^{\lambda c/2} \right) e^{\lambda z/2} \\ & + \left(2c_2 c_0(z) - a(z) c_2 \left(\frac{\lambda}{2}\right)^k e^{-\lambda c/2} \right) e^{-\lambda z/2} + c_0^2(z) + a(z) c_0^{(k)}(z+c) + 2c_1 c_2 = 0. \end{aligned} \tag{3.37}$$

By Lemma 2.6, we see that

$$\begin{cases} c_1^2 = p_1, c_2^2 = p_2; \\ 2c_1c_0(z) + a(z)c_1\left(\frac{\lambda}{2}\right)^k e^{\lambda c/2} \equiv 0; \\ 2c_2c_0(z) - a(z)c_2\left(\frac{\lambda}{2}\right)^k e^{-\lambda c/2} \equiv 0; \\ c_0^2(z) + a(z)c_0^{(k)}(z+c) + 2c_1c_2 \equiv 0. \end{cases} \quad (3.38)$$

From (3.38), we know that

$$a(z) \equiv a(\text{constant}), \quad c_0(z) \equiv c_0(\text{constant}),$$

and then

$$c_0 = \pm \frac{i}{2} a \left(\frac{\lambda}{2}\right)^k, \quad \frac{1}{64} a^4 \left(\frac{\lambda}{2}\right)^{4k} = p_1 p_2, \quad e^{\lambda c} = -1, \quad c_i^2 = p_i, \quad i = 1, 2.$$

If k is an even, then

$$f^{(k)}(z+c) = c_0^{(k)}(z+c) + c_1 \left(\frac{\lambda}{2}\right)^k e^{\lambda c/2} e^{\lambda z/2} + c_2 \left(\frac{\lambda}{2}\right)^k e^{-\lambda c/2} e^{-\lambda z/2}. \quad (3.39)$$

Substituting (3.35) and (3.39) into (1.5), we obtain

$$\begin{aligned} & (c_1^2 - p_1)e^{\lambda z} + (c_2^2 - p_2)e^{-\lambda z} + \left(2c_1c_0(z) + a(z)c_1 \left(\frac{\lambda}{2}\right)^k e^{\lambda c/2}\right) e^{\lambda z/2} \\ & + \left(2c_2c_0(z) + a(z)c_2 \left(\frac{\lambda}{2}\right)^k e^{-\lambda c/2}\right) e^{-\lambda z/2} + c_0^2(z) + a(z)c_0^{(k)}(z+c) + 2c_1c_2 = 0. \end{aligned} \quad (3.40)$$

By Lemma 2.6, we see that

$$\begin{cases} c_1^2 = p_1, c_2^2 = p_2; \\ 2c_1c_0(z) + a(z)c_1\left(\frac{\lambda}{2}\right)^k e^{\lambda c/2} \equiv 0; \\ 2c_2c_0(z) + a(z)c_2\left(\frac{\lambda}{2}\right)^k e^{-\lambda c/2} \equiv 0; \\ c_0^2(z) + a(z)c_0^{(k)}(z+c) + 2c_1c_2 \equiv 0. \end{cases} \quad (3.41)$$

If k is even and $k > 0$, from (3.41), we know that

$$a(z) \equiv a(\text{constant}), \quad c_0(z) \equiv c_0(\text{constant}),$$

and then

$$c_0 = \pm \frac{1}{2} a \left(\frac{\lambda}{2}\right)^k, \quad \frac{1}{64} a^4 \left(\frac{\lambda}{2}\right)^{4k} = p_1 p_2, \quad e^{\lambda c} = 1, \quad c_i^2 = p_i, \quad i = 1, 2.$$

If $k = 0$, from (3.41), we see that

$$c_0^2(z) \equiv \pm 2c_0(z)c_0(z+c) - 2c_1c_2. \quad (3.42)$$

By (3.42) and Lemma 2.7, we know that $c_0(z) \equiv c_0(\text{constant})$. By (3.41), we have $a(z) \equiv a(\text{constant})$, and

$$c_0 = \pm \frac{1}{2} a, \quad \frac{1}{64} a^4 = p_1 p_2 \text{ or } \frac{9}{64} a^4 = p_1 p_2, \quad e^{\lambda c} = 1, \quad c_i^2 = p_i, \quad i = 1, 2.$$

This completes the proof of Theorem 1.2.

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APPROXIMATE GENERALIZED QUADRATIC MAPPINGS IN (β, p) -BANACH SPACES

HARK-MAHN KIM AND HONG-MEI LIANG

ABSTRACT. In this article, we present general solution of a generalized quadratic functional equation with several variables, and then obtain its generalized Hyers–Ulam stability results for approximate generalized quadratic mappings in (β, p) -Banach spaces.

1. INTRODUCTION

In 1940, S. M. Ulam [30] gave the following question associated with the stability of group homomorphisms: Let G be a group and let G' be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : G \rightarrow G'$ satisfies the inequality

$$d(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G$, then there exists a homomorphism $F : G \rightarrow G'$ with $d(f(x), F(x)) < \varepsilon$ for all $x \in G$?

The question of Ulam was first solved by D. H. Hyers [12] for approximate Cauchy additive mappings on Banach spaces. Th. M. Rassias [22] provided a generalized Hyers–Ulam stability for the unbounded Cauchy difference controlled by a sum of unbounded function $\varepsilon(\|x\|^p + \|y\|^p)$ for case $0 \leq p < 1$. And then Z. Gajda [8] provided a generalized Hyers–Ulam stability for the Cauchy difference controlled by the same unbounded function $\varepsilon(\|x\|^p + \|y\|^p)$ for case $p > 1$. In 1984, J. M. Rassias [23] gave a similar stability for the unbounded Cauchy difference controlled by a product of unbounded function $\varepsilon\|x\|^p\|y\|^q$, $p+q \neq 1$. More generally, Găvruta [10] established a generalized Hyers–Ulam stability under replacing the bound of Cauchy difference controlled by an integrated general control function with regular condition.

The following functional equation

$$f(x + y) + f(x - y) = 2[f(x) + f(y)]$$

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is called a quadratic functional equation since it may be originated from the important parallelogram equality

$$\|x + y\|^2 + \|x - y\|^2 = 2[\|x\|^2 + \|y\|^2]$$

in inner product spaces. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. The Hyers–Ulam stability problem for the quadratic functional equation was first proved by F. Skof [29] for a mapping $f : X \rightarrow Y$, where X is a normed space and Y is a Banach space. In 1992, S. Czerwik [6] demonstrated the Hyers–Ulam stability of the quadratic functional equation with the sum of powers of norms in the sense of Th. M. Rassias approach using the direct method. In the same year, J. M. Rassias [24] certified the Hyers–Ulam stability of the quadratic functional equation with the product of powers of norms using the direct method. In 1995, C. Borelli and G. L. Forti [2] have verified the generalized Hyers–Ulam stability theorem of the quadratic functional equation. V. Radu [21], L. Cădariu and V. Radu [3, 4] have proposed to investigate the stability of functional equations using the fixed point method which is based on the alternative fixed point theorem. Since then, the stability of several functional equations using the fixed point method has been extensively investigated by several mathematicians [9, 11, 14, 16, 17].

Recently, A. Zivari-Kazempour and M. Eshaghi Gordji [32] have determined the general solution of the quadratic functional equation

$$f(x + 2y) + f(y + 2z) + f(z + 2x) = 2f(x + y + z) + 3[f(x) + f(y) + f(z)],$$

and then have investigated its generalized Hyers–Ulam stability. Motivated from this quadratic functional equation, we now consider a generalized functional equation

$$\begin{aligned} (1.1) \quad & \sum_{i=1}^n \sum_{1 \leq i_1 \leq \dots \leq i_n \leq n} f\left(kx_i + \sum_{j=1, j \neq i}^{n-k} x_{i_j}\right) \\ & = (n + k - 1) \binom{n-2}{n-k-1} f\left(\sum_{i=1}^n x_i\right) + \frac{nk(k-1)}{n-k} \binom{n-2}{n-k-1} \sum_{i=1}^n f(x_i), \end{aligned}$$

where n, k are fixed integers with $n \geq 3$ and $2 \leq k \leq n - 1$. Kim and Liang [15] have presented the classical stability results of quadratic functional equation (1.1) by using the fixed point approach in normed spaces. In this article, we give the general solution of the functional equation (1.1), and then investigate generalized Hyers–Ulam stability results for approximate generalized quadratic mappings in (β, p) -Banach spaces.

2. THE GENERAL SOLUTION OF EQUATION (1.1)

First of all, we introduce basic definitions, notations and preliminary theorems in the sequel. Let \mathbb{N} be the set of all natural numbers, $n \in \mathbb{N}$ and let X and Y be vector spaces. A

mapping $A_n : X^n \rightarrow Y$ is called n -additive if it is additive in each of its variables. A mapping A_n is called symmetric if $A_n(x_1, \dots, x_n) = A_n(x_{\pi(1)}, \dots, x_{\pi(n)})$ for every permutation $\{\pi(1), \dots, \pi(n)\}$ of $\{1, \dots, n\}$ and all $x_1, \dots, x_n \in X$. If $A_n(x_1, \dots, x_n)$ is an n -additive symmetric map, then $A^n(x)$ will denote the diagonal $A_n(x, \dots, x)$ for all $x \in X$, which will be called a monomial mapping of degree n . Note that $A^n(rx) = r^n A^n(x)$ for all $x \in X$ and all rational number $r \in \mathbb{Q}$. A mapping $p : X \rightarrow Y$ is called a generalized polynomial of degree $n \in \mathbb{N}$ provided that there exist an i -additive symmetric mappings $A_i : X^i \rightarrow Y$ ($1 \leq i \leq n$) such that $p(x) = \sum_{i=0}^n A^i(x)$ for all $x \in X$, where $A^n \neq 0$ and $A^0(x) = A^0 \in Y$ is a constant.

For $f : X \rightarrow Y$, let Δ_h be the difference operator defined as follows: $\Delta_h f(x) = f(x + h) - f(x)$ for all $x, h \in X$. We notice that these difference operators satisfy commutativity: $\Delta_{h_1} \circ \Delta_{h_2} f(x) = \Delta_{h_2} \circ \Delta_{h_1} f(x)$ for all $x, h_1, h_2 \in X$, where $\Delta_{h_1} \circ \Delta_{h_2}$ denotes the composition of the operators Δ_{h_1} and Δ_{h_2} . Furthermore, let $\Delta_h^0 f(x) = f(x)$, $\Delta_h^1 f(x) = \Delta_h f(x)$ and $\Delta_h^{n+1} f(x) = \Delta_h \circ \Delta_h^n f(x)$ for all $n \in \mathbb{N}$ and all $x, h \in X$. In explicit form, the functional equation $\Delta_h^{n+1} f(x) = 0$ can be written as

$$\Delta_h^{n+1} f(x) = \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} f(x + jh) = 0$$

for all $x, h \in X$. For any given $n \in \mathbb{N}$, the following is well-known Fréchet functional equation

$$\Delta_{h_{n+1}} \circ \dots \circ \Delta_{h_1} f(x) = 0$$

for all $x, h_1, \dots, h_{n+1} \in X$.

The following theorem was proved by Mazur and Orlicz [19, 20] and by Djoković [7] in greater generality.

Theorem 2.1. *Let X and Y be vector spaces, $n \in \mathbb{N}$ and $f : X \rightarrow Y$, then the followings are equivalent.*

- (1) $\Delta_h^{n+1} f(x) = 0$ for all $x, h \in X$.
- (2) $\Delta_{h_{n+1}} \circ \dots \circ \Delta_{h_1} f(x) = 0$ for all $x, h_1, \dots, h_{n+1} \in X$.
- (3) $f(x) = A^n(x) + \dots + A^0(x)$ for all $x \in X$, where $A^0(x) = A^0$ is an arbitrary element of Y and $A^i(x)$ is the diagonal of an i -additive symmetric mapping $A_i : X^i \rightarrow Y$ ($i = 1, \dots, n$).

The following two theorems ([27]) are need for us to establish general solution of the functional equation (1.1) (see also [28, 31] for details).

Theorem 2.2. *Let G be a commutative semigroup with identity, S a commutative group and n a nonnegative integer. Let the multiplication by $n!$ be bijective in S . The function $f : G \rightarrow S$ is a solution of Fréchet functional equation $\Delta_{h_{n+1}} \circ \dots \circ \Delta_{h_1} f(x) = 0$ for all $x, h_1, \dots, h_{n+1} \in G$ if and only if f is a generalized polynomial of degree at most n .*

Theorem 2.3. *Let G and S be commutative groups, n a nonnegative integer, φ_i, ψ_i additive functions from G into G and $\varphi_i(G) \subseteq \psi_i(G)$ ($i = 1, \dots, n+1$). If the functions $f, f_i : G \rightarrow S$ ($i = 1, \dots, n+1$) satisfy*

$$f(x) + \sum_{i=1}^{n+1} f_i(\varphi_i(x) + \psi_i(y)) = 0$$

for all $x, y \in G$, then f satisfies Fréchet functional equation $\Delta_{h_{n+1}} \circ \dots \circ \Delta_{h_1} f(x) = 0$ for all $x, h_1, \dots, h_{n+1} \in G$.

Before taking up our subject, we note that a mapping f is quadratic if and only if f satisfies the functional equation

$$f(2x + y) + f(x + 2y) = 4f(x + y) + f(x) + f(y),$$

which is equivalent to the functional equation

$$f(3x + y) + f(x + 3y) = 6f(x + y) + 4f(x) + 4f(y)$$

for all $x, y \in X$ [5]. Moreover, it is very meaningful and elementary to see that the functional equation

$$(2.1) \quad f(kx + y) + f(x + ky) = 2kf(x + y) + (k - 1)^2[f(x) + f(y)]$$

between vector spaces, where k is a fixed positive integer with $k \geq 2$, is equivalent to the quadratic functional equation

$$(2.2) \quad f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

for all $x, y \in X$ using Theorem 2.2 and Theorem 2.3 together with $f(kx) = k^2f(x)$ for all $x \in X$, and so the mapping f satisfying the equation (2.1) is quadratic.

Now, we are ready to present the general solution of the functional equation (1.1).

Theorem 2.4. *Let X and Y be vector spaces. If a mapping $f : X \rightarrow Y$ is solution of the functional equation (1.1), then f satisfies the functional equation (2.2) and so it is quadratic.*

Proof. Assume that a mapping $f : X \rightarrow Y$ satisfies the functional equation (1.1). Letting $x_1 = \dots = x_n := 0$ in (1.1), we obtain

$$n \binom{n-1}{n-k} f(0) = (n+k-1) \binom{n-2}{n-k-1} f(0) + \frac{n^2k(k-1)}{n-k} \binom{n-2}{n-k-1} f(0),$$

which yields $f(0) = 0$.

On the other hand, one has the following identity by setting $x_1 := x, x_2 = \dots = x_n := 0$ in (1.1)

$$\begin{aligned} & \binom{n-1}{n-k} f(kx) + (n-1) \binom{n-2}{n-k-1} f(x) \\ &= (n+k-1) \binom{n-2}{n-k-1} f(x) + \frac{nk(k-1)}{n-k} \binom{n-2}{n-k-1} f(x) \end{aligned}$$

for all $x \in X$. Thus $f(kx) = k^2 f(x)$ for all $x \in X$.

Putting $x_1 := x, x_2 := y, x_3 = \dots = x_n := 0$ in (1.1), we get

$$\begin{aligned} & \binom{n-2}{n-k-1} [f(kx+y) + f(x+ky)] + \binom{n-2}{n-k} [f(kx) + f(ky)] \\ &+ (n-2) \binom{n-3}{n-k-2} f(x+y) + (n-2) \binom{n-3}{n-k-1} [f(x) + f(y)] \\ &= (n+k-1) \binom{n-2}{n-k-1} f(x+y) + \frac{nk(k-1)}{n-k} \binom{n-2}{n-k-1} [f(x) + f(y)], \end{aligned}$$

from which it follows that

$$f(kx+y) + f(x+ky) = 2kf(x+y) + (k-1)^2[f(x) + f(y)]$$

for all $x, y \in X$. Hence f is a quadratic mapping. □

3. THE GENERALIZED HYERS–ULAM STABILITY OF EQUATION (1.1)

We recall some basic facts concerning the (β, p) -normed spaces [25].

Let β be a fixed real number with $0 < \beta \leq 1$ and X a linear space over \mathbb{K} , where \mathbb{K} denote either \mathbb{R} or \mathbb{C} . A quasi- β -norm is a real-valued function on X satisfying the following:

- (1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;
- (2) $\|\lambda x\| = |\lambda|^\beta \|x\|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$;
- (3) There is a constant $M \geq 1$ such that $\|x+y\| \leq M(\|x\| + \|y\|)$ for all $x, y \in X$.

In this case, the pair $(X, \|\cdot\|)$ is called a quasi- β -normed space. A quasi- β -Banach space is a complete quasi- β -normed space. Let p be a real number with $0 < p \leq 1$. The quasi- β -norm $\|\cdot\|$ on X is called a (β, p) -norm if, moreover, $\|\cdot\|^p$ satisfies the following triangle inequality

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi- β -Banach space is called a (β, p) -Banach space. We notice that quasi-1-normed spaces are equivalent to quasi-normed spaces and that $(1, p)$ -Banach spaces with $(1, p)$ -norm are equivalent to p -Banach spaces with p -norm. We may

refer to [1, 26] for the concept of quasi-normed spaces and p -Banach spaces. Given a p -norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on X . By the Aoki–Rolewicz theorem [26], each quasi-norm is equivalent to some p -norm [1].

Now, we study the generalized Hyers–Ulam stability of the functional equation (1.1) by using the direct method. For convenience, we use the following abbreviations:

$$\begin{aligned}
 & Df(x_1, \dots, x_n) \\
 & := \sum_{i=1}^n \sum_{1 \leq i_1 \leq \dots \leq i_n \leq n} f\left(kx_i + \sum_{j=1, i_j \neq i}^{n-k} x_{i_j}\right) - (n+k-1) \binom{n-2}{n-k-1} f\left(\sum_{i=1}^n x_i\right) \\
 & \quad - \frac{nk(k-1)}{n-k} \binom{n-2}{n-k-1} \sum_{i=1}^n f(x_i), \quad (x_1, \dots, x_n) \in X^n := X \times \dots \times X; \\
 M & := \frac{n-k}{k(k-1) \binom{n-2}{n-k-1}}.
 \end{aligned}$$

Theorem 3.1. *Let X be a vector space and Y a (β, p) -Banach space. Assume that $\varphi : X^n \rightarrow [0, \infty)$ is a function satisfying*

$$(3.1) \quad \Phi_1(x_1, \dots, x_n) := \sum_{j=0}^{\infty} \frac{\varphi(n^j x_1, \dots, n^j x_n)^p}{n^{2j\beta p}} < \infty$$

for all $x_1, \dots, x_n \in X$. If a mapping $f : X \rightarrow Y$ satisfies the functional inequality

$$(3.2) \quad \|Df(x_1, \dots, x_n)\| \leq \varphi(x_1, \dots, x_n)$$

for all $x_1, \dots, x_n \in X$, then there exists a unique quadratic mapping $Q_1 : X \rightarrow Y$ such that

$$(3.3) \quad \|f(x) - Q_1(x)\| \leq \frac{M^\beta}{n^{2\beta}} \Phi_1(x, \dots, x)^{\frac{1}{p}}$$

for all $x \in X$.

Proof. Letting $x_1 = \dots = x_n := x$ in (3.2), we obtain

$$\|f(nx) - n^2 f(x)\| \leq M^\beta \varphi(x, \dots, x)$$

for all $x \in X$. Dividing the above inequality by $n^{2\beta}$, one has

$$\left\| \frac{f(nx)}{n^2} - f(x) \right\| \leq \frac{M^\beta}{n^{2\beta}} \varphi(x, \dots, x)$$

for all $x \in X$. Replacing x by $n^l x$ and dividing by $n^{2l\beta}$, we get

$$\left\| \frac{f(n^{l+1}x)}{n^{2(l+1)}} - \frac{f(n^l x)}{n^{2l}} \right\|^p \leq \frac{M^{\beta p}}{n^{2(l+1)\beta p}} \varphi(n^l x, \dots, n^l x)^p$$

for all $x \in X$ and all $l \in \mathbb{N} \cup \{0\}$. Thus, for any $m \in \mathbb{N}$, one deduces that

$$(3.4) \quad \left\| \frac{f(n^{l+m}x)}{n^{2(l+m)}} - \frac{f(n^l x)}{n^{2l}} \right\|^p \leq \frac{M^{\beta p}}{n^{2\beta p}} \sum_{i=l}^{l+m-1} \frac{\varphi(n^i x, \dots, n^i x)^p}{n^{2i\beta p}}$$

for all $x \in X$ and all $l \in \mathbb{N} \cup \{0\}$. In view of (3.4), it is easily checked that the sequence $\{\frac{f(n^l x)}{n^{2l}}\}$ is Cauchy in Y . Since Y is complete, the sequence is convergent in Y . Hence, we may define a mapping $Q_1 : X \rightarrow Y$ by

$$(3.5) \quad Q_1(x) := \lim_{l \rightarrow \infty} \frac{f(n^l x)}{n^{2l}}$$

for all $x \in X$. Moreover, if we take $l = 0$ and $m \rightarrow \infty$ in (3.4), then we arrive at the approximation (3.3). By (3.2) and (3.5), we see that

$$\|DQ_1(x_1, \dots, x_n)\|^p = \lim_{l \rightarrow \infty} \left\| \frac{Df(n^l x_1, \dots, n^l x_n)}{n^{2l}} \right\|^p \leq \lim_{l \rightarrow \infty} \frac{\varphi(n^l x_1, \dots, n^l x_n)^p}{n^{2l\beta p}} = 0$$

for all $x_1, \dots, x_n \in X$. Hence Q_1 satisfies the functional equation (1.1), and so it is quadratic.

To show the uniqueness of Q_1 , we suppose there exists another quadratic mapping $Q'_1 : X \rightarrow Y$ which satisfies the functional inequality

$$\|f(x) - Q'_1(x)\| \leq \frac{M^\beta}{n^{2\beta}} \Phi_1(x, \dots, x)^{\frac{1}{p}}$$

for all $x \in X$. Since Q_1 and Q'_1 are quadratic mappings, we see that

$$\begin{aligned} \|Q_1(x) - Q'_1(x)\|^p &= \frac{1}{n^{2l\beta p}} \|Q_1(n^l x) - Q'_1(n^l x)\|^p \\ &\leq \frac{1}{n^{2l\beta p}} [\|Q_1(n^l x) - f(n^l x)\|^p + \|f(n^l x) - Q'_1(n^l x)\|^p] \\ &\leq \frac{2M^{\beta p}}{n^{2(l+1)\beta p}} \sum_{j=0}^{\infty} \frac{\varphi(n^{j+l} x, \dots, n^{j+l} x)^p}{n^{2j\beta p}} = \frac{2M^{\beta p}}{n^{2\beta p}} \sum_{j=l}^{\infty} \frac{\varphi(n^j x, \dots, n^j x)^p}{n^{2j\beta p}} = 0 \end{aligned}$$

for all $x \in X$. Hence Q_1 is a unique quadratic mapping satisfying (3.3). □

Theorem 3.2. *Let X be a vector space and Y a (β, p) -Banach space. Suppose there exists a function $\varphi : X^n \rightarrow [0, \infty)$ satisfying*

$$\Phi_2(x_1, \dots, x_n) := \sum_{i=1}^{\infty} n^{2i\beta p} \varphi\left(\frac{x_1}{n^i}, \dots, \frac{x_n}{n^i}\right)^p < \infty$$

for all $x_1, \dots, x_n \in X$. If a mapping $f : X \rightarrow Y$ satisfies the functional inequality (3.2), then there exists a unique quadratic mapping $Q_2 : X \rightarrow Y$ such that

$$\|f(x) - Q_2(x)\| \leq \frac{M^\beta}{n^{2\beta}} \Phi_2(x, \dots, x)^{\frac{1}{p}}$$

for all $x \in X$.

Proof. We see from (3.2) that

$$\|f(nx) - n^2 f(x)\|^p \leq M^{\beta p} \varphi(x, \dots, x)^p$$

for all $x \in X$. Thus, it follows that

$$\|n^{2l} f(\frac{x}{n^l}) - f(x)\|^p \leq \frac{M^{\beta p}}{n^{2\beta p}} \sum_{i=1}^l n^{2i\beta p} \varphi(\frac{x}{n^i}, \dots, \frac{x}{n^i})^p$$

for all $x \in X$ and all $l \in \mathbb{N} \cup \{0\}$. The remaining assertion goes in a similar way as the corresponding part of Theorem 3.1. □

For the moment, we recall the fixed point alternative theorem from [18].

Theorem 3.3. *Let (X, d) be a generalized complete metric space and let $J : X \rightarrow X$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n , or there exists a positive integer n_0 such that

- (1) $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;
- (2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;
- (3) y^* is the unique fixed point of J in the set

$$Y = \{y \in X | d(J^{n_0} x, y) < \infty\};$$

- (4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In the following, we are going to apply the fixed point method to investigate the generalized Hyers–Ulam stability of the functional equation (1.1).

Theorem 3.4. *Let X be a vector space and Y a (β, p) -Banach space. Suppose there exists a positive number L_1 with $0 < L_1 < 1$ for which a function $\varphi : X^n \rightarrow [0, \infty)$ satisfies*

$$(3.6) \quad \varphi(nx_1, \dots, nx_n) \leq n^{2\beta} L_1 \varphi(x_1, \dots, x_n)$$

for all $x_1, \dots, x_n \in X$. If a mapping $f : X \rightarrow Y$ satisfies the functional inequality (3.2), then there exists a unique quadratic mapping $Q_1 : X \rightarrow Y$ such that

$$(3.7) \quad \|f(x) - Q_1(x)\| \leq \frac{M^\beta}{n^{2\beta}(1-L_1)} \varphi(x, \dots, x)$$

for all $x \in X$.

Proof. First, we denote a set of mappings from X to Y by

$$Y^X := \{g : X \rightarrow Y \text{ a mapping}\},$$

and define a generalized metric d on Y^X as follows:

$$d(g, h) := \inf\{\alpha \in [0, \infty] : \|g(x) - h(x)\| \leq \alpha\varphi(x, \dots, x) \forall x \in X\}$$

for all $g, h \in Y^X$. Then we may show that (Y^X, d) is a complete generalized metric space (see [13]).

Now, we define a mapping $J_1 : Y^X \rightarrow Y^X$ by

$$J_1g(x) := \frac{g(nx)}{n^2}$$

for all $g \in Y^X$ and all $x \in X$. Given $g, h \in Y^X$, letting $c_{gh} \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq c_{gh}$, we can write

$$\|J_1g(x) - J_1h(x)\| = \left\| \frac{g(nx)}{n^2} - \frac{h(nx)}{n^2} \right\| \leq \frac{1}{n^{2\beta}} c_{gh} \varphi(nx, \dots, nx) \leq L_1 c_{gh} \varphi(x, \dots, x),$$

which yields $d(J_1g, J_1h) \leq L_1 c_{gh}$ and so $d(J_1g, J_1h) \leq L_1 d(g, h)$ by letting $c_{gh} \rightarrow d(g, h)^+$. Hence J_1 is a strictly contractive mapping with Lipschitz constant L_1 on Y^X .

Second, if we set $x_1 = \dots = x_n := x$ in the hypothesis (3.2) and divide both sides by $n^{2\beta}$, then we have

$$\left\| f(x) - \frac{f(nx)}{n^2} \right\| \leq \frac{M^\beta}{n^{2\beta}} \varphi(x, \dots, x)$$

for all $x \in X$, which implies

$$(f, J_1f) \leq \frac{M^\beta}{n^{2\beta}} < \infty,$$

and so

$$d(J_1^k f, J_1^{k+1} f) \leq d(f, J_1f) \leq \frac{M^\beta}{n^{2\beta}} < \infty$$

for all $k \in \mathbb{N}$. Thus according to Theorem 3.3, J_1 has a unique fixed point $Q_1 : X \rightarrow Y$ in the set

$$\Delta = \{g \in Y^X : d(f, g) < \infty\},$$

where Q_1 is defined by

$$(3.8) \quad Q_1(x) := \lim_{k \rightarrow \infty} J_1^k f(x) = \lim_{k \rightarrow \infty} \frac{f(n^k x)}{n^{2k}}$$

for all $x \in X$.

Moreover,

$$d(f, Q_1) \leq \frac{1}{1 - L_1} d(f, J_1f) \leq \frac{M^\beta}{n^{2\beta}(1 - L_1)},$$

this means that (3.7) holds.

Finally, we prove that the mapping Q_1 is quadratic. It follows from (3.2) and (3.8) that

$$\begin{aligned} \|DQ_1(x_1, \dots, x_n)\| &= \lim_{k \rightarrow \infty} \frac{1}{n^{2k\beta}} \|Df(n^k x_1, \dots, n^k x_n)\| \\ &\leq \lim_{k \rightarrow \infty} \frac{1}{n^{2k\beta}} \varphi(n^k x_1, \dots, n^k x_n) \leq \lim_{k \rightarrow \infty} L_1^k \varphi(x_1, \dots, x_n) = 0 \end{aligned}$$

for all $x_1, \dots, x_n \in X$. By Theorem 2.4, the mapping $Q_1 : X \rightarrow Y$ is quadratic, as desired. \square

Theorem 3.5. *Let X be a vector space and Y a (β, p) -Banach space. Assume that $\varphi : X^n \rightarrow [0, \infty)$ is a function satisfying*

$$\varphi\left(\frac{x_1}{n}, \dots, \frac{x_n}{n}\right) \leq \frac{L_2}{n^{2\beta}} \varphi(x_1, \dots, x_n)$$

for some real number L_2 with $0 < L_2 < 1$ and all $x_1, \dots, x_n \in X$. If a mapping $f : X \rightarrow Y$ satisfies the functional inequality (3.2), then there exists a unique quadratic mapping $Q_2 : X \rightarrow Y$ such that

$$\|f(x) - Q_2(x)\| \leq \frac{M^\beta L_2}{n^{2\beta}(1 - L_2)} \varphi(x, \dots, x)$$

for all $x \in X$.

Proof. Now, we define a mapping $J_2 : Y^X \rightarrow Y^X$ on the function space (Y^X, d) by

$$J_2 g(x) := n^2 g\left(\frac{x}{n}\right)$$

for all $g \in Y^X$ and all $x \in X$. Then we see from (3.2) that

$$\|f(x) - n^2 f\left(\frac{x}{n}\right)\| \leq M^\beta \varphi\left(\frac{x}{n}, \dots, \frac{x}{n}\right) \leq \frac{M^\beta L_2}{n^{2\beta}} \varphi(x, \dots, x)$$

for all $x \in X$, which induces

$$d(f, J_2 f) \leq \frac{M^\beta L_2}{n^{2\beta}} < \infty$$

by definition.

The rest of proof follows from the similar argument to the corresponding part of Theorem 3.4. \square

Corollary 3.6. *Let X be a quasi- α -normed space and Y a (β, p) -Banach space. If a mapping $f : X \rightarrow Y$ satisfies the functional inequality*

$$\|Df(x_1, \dots, x_n)\| \leq \theta \left(\sum_{i=1}^n \|x_i\|^r \right)$$

for all $x_1, \dots, x_n \in X$, where $r, \theta > 0, r\alpha \neq 2\beta$, then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{M^\beta n \theta}{|n^{2\beta p} - n^{\alpha r p}|^{\frac{1}{p}}} \|x\|^r$$

for all $x \in X$.

Corollary 3.7. Let X be a quasi- α -normed space and Y a (β, p) -Banach space, and assume that θ is a given positive real number and r_1, \dots, r_n are real numbers with $\sum_{i=1}^n r_i := r > 0$ and $r\alpha \neq 2\beta$. If a mapping $f : X \rightarrow Y$ satisfies the functional inequality

$$\|Df(x_1, \dots, x_n)\| \leq \theta \left(\prod_{i=1}^n \|x_i\|^{r_i} \right)$$

for all $x_1, \dots, x_n \in X$, then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{M^\beta \theta}{|n^{2\beta p} - n^{\alpha r p}|^{\frac{1}{p}}} \|x\|^r$$

for all $x \in X$.

Corollary 3.8. Let X be a vector space and Y a (β, p) -Banach space, and assume $\varepsilon \geq 0$ is any given real number. If a mapping $f : X \rightarrow Y$ satisfies the functional inequality

$$\|Df(x_1, \dots, x_n)\| \leq \varepsilon$$

for all $x_1, \dots, x_n \in X$, then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{M^\beta}{(n^{2\beta p} - 1)^{\frac{1}{p}}} \varepsilon$$

for all $x \in X$.

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FIXED POINT THEOREMS FOR GENERALIZED HYBRID MAPPINGS IN FUZZY HILBERT SPACES

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ABSTRACT. The aim of the study is to investigate nonexpansive, nonspreading, hybrid and contractive mappings in fuzzy Hilbert spaces and to prove some fixed point theorems.

1. INTRODUCTION AND PRELIMINARIES

It was Katsaras [17], who while studying fuzzy topological vector spaces, was the first to introduce the idea of fuzzy norm on a linear space in 1984. Later on many other mathematicians like Felbin [13], Cheng & Mordeson [11], Bag & Samanta [1] etc. introduced definitions of fuzzy normed linear spaces in different approach. A large number of paper have been published in fuzzy normed linear spaces, for reference please see [2, 3, 4, 5, 6, 7, 14, 15]. On the other hand studies on fuzzy inner product spaces are relatively recent and few works have been done in fuzzy inner product spaces. Biswas [8], El-Abyed & Hamouly [12] were among the first who gave a meaningful definition of fuzzy inner product space and associated fuzzy norm function. Later on, Kohli & Kumar [20] modified the definition of inner product space introduced by Biswas. We introduce a broad class of nonlinear mappings in fuzzy Hilbert spaces and then we prove some fixed point theorems for the class of such mappings.

Definition 1.1. Let U be a real linear space. A fuzzy subset N of $U \times \mathbb{R}$ is called a fuzzy norm on U if, for all $x, u \in U$ and $c \in F$, the following conditions are satisfied:

- (N1) $\forall t \in \mathbb{R}, t \leq 0; N(x, t) = 0;$
- (N2) $\forall t \in \mathbb{R}, t > 0; N(x, t) = 1$ if and only if $x = 0;$
- (N3) $\forall t \in \mathbb{R}, t > 0; N(x, t) = N(x, \frac{t}{|c|})$ if $c \neq 0;$
- (N4) $\forall s, t \in \mathbb{R}, x, u \in U;$

$$N(x + u, s + t) \geq \min \{N(x, s), N(u, t)\}$$

- (N5) $N(x, \cdot)$ is a non-decreasing function of \mathbb{R} and $\lim_{t \rightarrow \infty} N(x, t) = 1.$

The pair (U, N) will be referred to as a fuzzy normed linear space.

Theorem 1.2. Let (U, N) be a fuzzy normed linear space. Assume further that,

- (N6) $\forall t > 0, N(x, t) > 0$ implies $x = 0.$

Define $\|x\|_\alpha = \wedge \{t > 0 : N(x, t) \geq \alpha\}, \alpha \in (0, 1).$

Then $\{\|\cdot\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of norms on U and they are called α -norms on U corresponding to the fuzzy norm N on $U.$

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Theorem 1.3. *Let (U, N) be a fuzzy normed linear space satisfying (N6). Assume further that*

(N7) for $x \neq 0$, $N(x, \cdot)$ is a continuous function of \mathbb{R} .

Let $\|x\|_\alpha = \wedge \{t > 0 : N(x, t) \geq \alpha\}$, $\alpha \in (0, 1)$ and $N' : U \times \mathbb{R} \rightarrow [0, 1]$ be a function defined by

$$N'(x, t) = \begin{cases} \wedge \{\alpha \in (0, 1) : \|x\|_\alpha \leq t\}, & \text{if } (x, t) \neq (0, 0) \\ 0 & \text{if } (x, t) = (0, 0). \end{cases}$$

Then (i) $\{\|\cdot\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of norms on U .

(ii) N' is a fuzzy norm on U .

(iii) $N' = N$

In a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively, it is known that

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha \|x\|^2 + (1 - \alpha) \|y\|^2 - \alpha(1 - \alpha) \|x - y\|^2 \tag{1.1}$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$ (see [31]). Furthermore, in a Hilbert space, we have that

$$2 \langle x - y, z - w \rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2 \tag{1.2}$$

for all $x, y, z, w \in H$. Using means and the Riesz theorem, we can obtain the following result (see [22, 25, 26, 27]).

Lemma 1.4. *Let H be a Hilbert space, $\{x_n\}$ a bounded sequence in H and let μ be a mean on l^∞ . Then there exists a unique point $z_0 \in \overline{\text{co}}\{x_n | n \in \mathbb{N}\}$ such that*

$$\mu_n \langle x_n, y \rangle = \langle z_0, y \rangle, \forall y \in H.$$

We can define the following nonlinear mappings (see [9, 16, 18, 19, 21, 28, 29]) in fuzzy Hilbert spaces.

Let H be a fuzzy Hilbert space with inner product $\{\langle \cdot, \cdot \rangle_\alpha : \alpha \in (0, 1)\}$ and norm $\{\|\cdot\|_\alpha : \alpha \in (0, 1)\}$. Let C be a nonempty subset of H . A mapping $T : C \rightarrow H$ is said to be *nonexpansive*, *non-spreading*, and *hybrid* if

$$\|Tx - Ty\|_\alpha \leq \|x - y\|_\alpha,$$

$$2 \|Tx - Ty\|_\alpha^2 \leq \|Tx - y\|_\alpha^2 + \|Ty - x\|_\alpha^2$$

and

$$3 \|Tx - Ty\|_\alpha^2 \leq \|x - y\|_\alpha^2 + \|Tx - y\|_\alpha^2 + \|Ty - x\|_\alpha^2$$

for all $x, y \in C$, respectively. A mapping $F : C \rightarrow H$ is said to be *firmly nonexpansive* if

$$\|Fx - Fy\|_\alpha^2 \leq \langle x - y, Fx - Fy \rangle_\alpha$$

for all $x, y \in C$. A mapping T from C into H is said to be *widely generalized hybrid* if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha \|Tx - Ty\|_\alpha^2 + \beta \|x - Ty\|_\alpha^2 + \gamma \|Tx - y\|_\alpha^2 + \delta \|x - y\|_\alpha^2 \\ &+ \max \left\{ \varepsilon \|x - Tx\|_\alpha^2, \zeta \|y - Ty\|_\alpha^2 \right\} \leq 0 \end{aligned}$$

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for all $x, y \in C$ and T is called symmetric generalized hybrid if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha \|Tx - Ty\|_\alpha^2 + \beta(\|x - Ty\|_\alpha^2 + \|Tx - y\|_\alpha^2) \\ &+ \gamma \|x - y\|_\alpha^2 + \delta(\|x - Tx\|_\alpha^2 + \|y - Ty\|_\alpha^2) \leq 0 \end{aligned} \tag{1.3}$$

for all $x, y \in C$. Such a mapping T is also called $(\alpha, \beta, \gamma, \delta)$ -symmetric generalized hybrid. If $\alpha = 1, \beta = \delta = 0$ and $\gamma = -1$ in (1.3), then the mapping T is *nonexpansive*. If $\alpha = 2, \beta = -1$ and $\gamma = \delta = 0$ in (1.3), then the mapping T is *nonspreading*. Furthermore, if $\alpha = 3, \beta = \gamma = -1$ and $\delta = 0$ in (1.3), then the mapping T is *hybrid*.

Let H be a fuzzy Hilbert space and let C be a nonempty subset of H . Then $T : C \rightarrow H$ is called a *widely strict pseudo-contraction* if there exists $r \in \mathbb{R}$ with $r < 1$ such that

$$\|Tx - Ty\|_\alpha^2 \leq \|x - y\|_\alpha^2 + r \|(I - T)x - (I - T)y\|_\alpha^2, \quad \forall x, y \in C.$$

We call such T a *widely r -strict pseudo-contraction*. If $0 \leq r < 1$, then T is a *strict pseudo-contraction* (see [10, 23, 24]). Furthermore, if $r = 0$, then T is nonexpansive. Conversely, let $S : C \rightarrow H$ be a nonexpansive mapping and define $T : C \rightarrow H$ by $T = \frac{1}{1+n}S + \frac{n}{1+n}I$ for all $x \in C$ and $n \in \mathbb{N}$. Then T is a widely $(-n)$ -strict pseudocontraction. In fact, from the definition of T , it follows that $S = (1 + n)T - nI$. Since S is nonexpansive, we have that for any $x, y \in C$,

$$\|(1 + n)Tx - nx - ((1 + n)Ty - ny)\|_\alpha^2 \leq \|x - y\|_\alpha^2$$

and hence

$$\|Tx - Ty\|_\alpha^2 \leq \|x - y\|_\alpha^2 + n \|(I - T)x - (I - T)y\|_\alpha^2.$$

We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \rightarrow x$ and $x_n \rightharpoonup x$, respectively. Let A be a nonempty subset of H . We denote by $\overline{\text{co}}A$ the closure of the convex hull of A . Let T be a mapping from C into H . We denote by $F(T)$ the set of fixed points of T .

2. FIXED POINT THEOREMS

Theorem 2.1. *Let H be a fuzzy Hilbert space, C a nonempty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta)$ -symmetric generalized hybrid mapping from C into itself such that the conditions (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta > 0$ and (3) $\delta \geq 0$ hold. Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z : n = 0, 1, \dots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + 2\beta + \gamma > 0$ on the condition (1).*

Proof. Suppose that T has a fixed point z . Then $\{T^n z : n = 0, 1, \dots\} = \{z\}$ and hence $\{T^n z : n = 0, 1, \dots\}$ is bounded. Conversely, suppose that there exists $z \in C$ such that $\{T^n z : n = 0, 1, \dots\}$ is bounded. Since T is an $(\alpha, \beta, \gamma, \delta)$ -symmetric generalized hybrid mapping of C into itself, we have that

$$\begin{aligned} &\alpha \|Tx - T^{n+1}z\|_\alpha^2 + \beta(\|x - T^{n+1}z\|_\alpha^2 + \|Tx - T^n z\|_\alpha^2) \\ &+ \gamma \|x - T^n z\|_\alpha^2 + \delta(\|x - Tx\|_\alpha^2 + \|T^n z - T^{n+1}z\|_\alpha^2) \leq 0 \end{aligned}$$

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for all $n \in \mathbb{N} \cup \{0\}$ and $x \in C$. Since $\{T^n z\}$ is bounded, we can apply a Banach limit μ to both sides of the inequality. Since $\mu_n \|Tx - T^n z\|_\alpha^2 = \mu_n \|Tx - T^{n+1} z\|_\alpha^2$ and $\mu_n \|x - T^{n+1} z\|_\alpha^2 = \mu_n \|x - T^n z\|_\alpha^2$, we have that

$$(\alpha + \beta)\mu_n \|Tx - T^n z\|_\alpha^2 + (\beta + \gamma)\mu_n \|x - T^n z\|_\alpha^2 + \delta(\|x - Tx\|_\alpha^2 + \mu_n \|T^n z - T^{n+1} z\|_\alpha^2) \leq 0.$$

Furthermore, since

$$\mu_n \|Tx - T^n z\|_\alpha^2 = \|Tx - x\|_\alpha^2 + 2\mu_n \langle Tx - x, x - T^n z \rangle_\alpha + \mu_n \|x - T^n z\|_\alpha^2,$$

we have that

$$(\alpha + \beta + \delta) \|Tx - x\|_\alpha^2 + 2(\alpha + \beta)\mu_n \langle Tx - x, x - T^n z \rangle_\alpha + (\alpha + 2\beta + \gamma)\mu_n \|x - T^n z\|_\alpha^2 + \delta\mu_n \|T^n z - T^{n+1} z\|_\alpha^2 \leq 0.$$

From (1) $\alpha + 2\beta + \gamma \geq 0$ and (3) $\delta \geq 0$, we have that

$$(\alpha + \beta + \delta) \|Tx - x\|_\alpha^2 + 2(\alpha + \beta)\mu_n \langle Tx - x, x - T^n z \rangle_\alpha \leq 0. \tag{2.1}$$

Since there exists $p \in H$ from Lemma 1.4 such that

$$\mu_n \langle y, T^n z \rangle_\alpha = \langle y, p \rangle_\alpha$$

for all $y \in H$, we have from (2.1) that

$$(\alpha + \beta + \delta) \|Tx - x\|_\alpha^2 + 2(\alpha + \beta) \langle Tx - x, x - p \rangle_\alpha \leq 0. \tag{2.2}$$

Since C is closed and convex, we have that

$$p \in \overline{co}\{T^n x : n \in \mathbb{N}\} \subset C.$$

Putting $x = p$, we obtain from (2.2) that

$$(\alpha + \beta + \delta) \|Tp - p\|_\alpha^2 \leq 0. \tag{2.3}$$

We have from (2) $\alpha + \beta + \delta > 0$ that $\|Tp - p\|_\alpha^2 \leq 0$. This implies that p is a fixed point in T .

Next suppose that $\alpha + 2\beta + \gamma > 0$. Let p_1 and p_2 be fixed points of T . Then we have that

$$\alpha \|Tp_1 - Tp_2\|_\alpha^2 + \beta(\|p_1 - Tp_2\|_\alpha^2 + \|Tp_1 - p_2\|_\alpha^2) + \gamma \|p_1 - p_2\|_\alpha^2 + \delta(\|p_1 - Tp_1\|_\alpha^2 + \|p_2 - Tp_2\|_\alpha^2) \leq 0$$

and hence $(\alpha + 2\beta + \gamma) \|p_1 - p_2\|_\alpha^2 \leq 0$. We have from $\alpha + 2\beta + \gamma > 0$ that $p_1 = p_2$. Therefore a fixed point of T is unique. This completes the proof. \square

We can derive the following theorem from Theorem 2.1.

Theorem 2.2. *Let H be a fuzzy Hilbert space, C a nonempty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta)$ -symmetric generalized hybrid mapping from C into itself such that the conditions (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta > 0$ and (3) $\delta \geq 0$ hold. Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha + 2\beta + \gamma > 0$ on the condition (1).*

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A mapping T from C into H is called $(\alpha, \beta, \gamma, \delta, \zeta)$ -symmetric more generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \zeta \in \mathbb{R}$ such that

$$\begin{aligned} & \alpha \|Tx - Ty\|_{\alpha}^2 + \beta(\|x - Ty\|_{\alpha}^2 + \|Tx - y\|_{\alpha}^2) + \gamma \|x - y\|_{\alpha}^2 \\ & + \delta(\|x - Tx\|_{\alpha}^2 + \|y - Ty\|_{\alpha}^2) + \zeta \|x - y - (Tx - Ty)\|_{\alpha}^2 \leq 0 \end{aligned} \tag{2.4}$$

for all $x, y \in C$.

Theorem 2.3. *Let H be a fuzzy Hilbert space, C a nonempty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \zeta)$ -symmetric more generalized hybrid mapping from C into itself such that the conditions (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta + \zeta > 0$ and (3) $\delta + \zeta \geq 0$ hold. Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z : n = 0, 1, \dots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + 2\beta + \gamma > 0$ on the condition (1).*

Proof. Since $T : C \rightarrow C$ is an $(\alpha, \beta, \gamma, \delta, \zeta)$ -symmetric more generalized hybrid mapping, there exist $\alpha, \beta, \gamma, \delta, \zeta \in \mathbb{R}$ satisfying (2.4). We also have that

$$\begin{aligned} & \|x - y - (Tx - Ty)\|_{\alpha}^2 = \|x - Tx\|_{\alpha}^2 + \|y - Ty\|_{\alpha}^2 \\ & - \|x - Ty\|_{\alpha}^2 - \|y - Tx\|_{\alpha}^2 + \|x - y\|_{\alpha}^2 + \|Tx - Ty\|_{\alpha}^2 \end{aligned} \tag{2.5}$$

for all $x, y \in C$. Thus we obtain from (2.4) that

$$\begin{aligned} & (\alpha + \zeta) \|Tx - Ty\|_{\alpha}^2 + (\beta - \zeta)(\|x - Ty\|_{\alpha}^2 + \|Tx - y\|_{\alpha}^2) \\ & + (\gamma + \zeta) \|x - y\|_{\alpha}^2 + (\delta + \zeta)(\|x - Tx\|_{\alpha}^2 + \|y - Ty\|_{\alpha}^2) \leq 0. \end{aligned} \tag{2.6}$$

The conditions (1) $\alpha + 2\beta + \gamma \geq 0$ and (2) $\alpha + \beta + \delta + \zeta > 0$ are equivalent to $(\alpha + \zeta) + 2(\beta - \zeta) + (\gamma + \zeta) \geq 0$ and $(\alpha + \zeta) + (\beta - \zeta) + (\delta + \zeta) > 0$, respectively. Furthermore, since (3) $\delta + \zeta \geq 0$ holds, we have the desired result from Theorem 2.1. \square

As a direct consequence of Theorem 2.3, we obtain the following.

Theorem 2.4. *Let H be a fuzzy Hilbert space, C a nonempty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \zeta)$ -symmetric more generalized hybrid mapping from C into itself such that the conditions (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta + \zeta > 0$ and (3) $\delta + \zeta \geq 0$ hold. Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z : n = 0, 1, \dots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + 2\beta + \gamma > 0$ on the condition (1).*

We can extend the above theorem as follows.

Theorem 2.5. *Let H be a fuzzy Hilbert space, C a nonempty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \zeta)$ -symmetric more generalized hybrid mapping from C into itself which satisfies the conditions (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta + \zeta > 0$ and (3) there exists $\lambda \in [0, 1)$ such that $(\alpha + \beta)\lambda + \delta + \zeta \geq 0$. Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha + 2\beta + \gamma > 0$ on the condition (1).*

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Proof. Since $T : C \rightarrow C$ is an $(\alpha, \beta, \gamma, \delta, \zeta)$ -symmetric more generalized hybrid mapping, we obtain that

$$\begin{aligned} & \alpha \|Tx - T^{n+1}z\|_{\alpha}^2 + \beta(\|x - T^{n+1}z\|_{\alpha}^2 + \|Tx - T^n z\|_{\alpha}^2) + \gamma \|x - T^n z\|_{\alpha}^2 \\ & + \delta(\|x - Tx\|_{\alpha}^2 + \|T^n z - T^{n+1}z\|_{\alpha}^2) + \zeta \|(x - Tx) - (T^n z - T^{n+1}z)\|_{\alpha}^2 \leq 0 \end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$ and all $x \in C$.

Let $\lambda \in [0, 1) \cap \{\lambda : (\alpha + \beta)\lambda + \zeta + \eta \geq 0\}$ and define $S = (1 - \lambda)T + \lambda I$. Since C is convex, S is a mapping from C into itself. Since C is bounded, $\{S^n z : n = 0, 1, \dots\}$ is bounded for all $z \in C$. Since $\lambda \neq 1$, we obtain that $F(S) = F(T)$. Moreover, from $T = \frac{1}{1-\lambda}S - \frac{\lambda}{1-\lambda}I$ and (2.1), we have that

$$\begin{aligned} & \alpha \left\| \left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x \right) - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y \right) \right\|_{\alpha}^2 \\ & + \beta \left\| x - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y \right) \right\|_{\alpha}^2 + \beta \left\| \left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x \right) - y \right\|_{\alpha}^2 + \gamma \|x - y\|_{\alpha}^2 \\ & + \delta \left\| x - \left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x \right) \right\|_{\alpha}^2 + \delta \left\| y - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y \right) \right\|_{\alpha}^2 \\ & + \zeta \left\| \left(x - \left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x \right) \right) - \left(y - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y \right) \right) \right\|_{\alpha}^2 \\ = & \alpha \left\| \left(\frac{1}{1-\lambda}(Sx - Sy) - \frac{\lambda}{1-\lambda}(x - y) \right) \right\|_{\alpha}^2 \\ & + \beta \left\| x - \left(\frac{1}{1-\lambda}(x - Sy) - \frac{\lambda}{1-\lambda}(x - y) \right) \right\|_{\alpha}^2 \\ & + \beta \left\| x - \left(\frac{1}{1-\lambda}(Sx - y) - \frac{\lambda}{1-\lambda}(x - y) \right) \right\|_{\alpha}^2 + \gamma \|x - y\|_{\alpha}^2 \\ & + \delta \left\| \frac{1}{1-\lambda}(x - Sx) \right\|_{\alpha}^2 + \delta \left\| \frac{1}{1-\lambda}(y - Sy) \right\|_{\alpha}^2 \\ & + \zeta \left\| \frac{1}{1-\lambda}(x - Sx) - \frac{1}{1-\lambda}(y - Sy) \right\|_{\alpha}^2 \\ = & \frac{\alpha}{1-\lambda} \|Sx - Sy\|_{\alpha}^2 + \frac{\beta}{1-\lambda} \|x - Sy\|_{\alpha}^2 \\ & + \frac{\beta}{1-\lambda} \|Sx - y\|_{\alpha}^2 + \left(-\frac{\lambda}{1-\lambda}(\alpha + 2\beta) + \gamma \right) \|x - y\|_{\alpha}^2 \\ & + \frac{\delta + \beta\lambda}{(1-\lambda)^2} \|x - Sx\|_{\alpha}^2 + \frac{\delta + \beta\lambda}{(1-\lambda)^2} \|y - Sy\|_{\alpha}^2 \\ & + \frac{\zeta + \alpha\lambda}{(1-\lambda)^2} \|(x - Sx) - (y - Sy)\|_{\alpha}^2 \leq 0 \end{aligned}$$

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Therefore, S is an $\left(\frac{\alpha}{1-\lambda}, \frac{\beta}{1-\lambda}, -\frac{\lambda}{1-\lambda}(\alpha + 2\beta) + \gamma, \frac{\delta + \beta\lambda}{(1-\lambda)^2}, \frac{\zeta + \alpha\lambda}{(1-\lambda)^2}\right)$ -symmetric more generalized hybrid mapping. Furthermore, we obtain that

$$\begin{aligned} \frac{\alpha}{1-\lambda} + \frac{2\beta}{1-\lambda} + \frac{\gamma}{1-\lambda} - \frac{\lambda}{1-\lambda}(\alpha + 2\beta) + \gamma &= \alpha + 2\beta + \gamma \geq 0, \\ \frac{\alpha}{1-\lambda} + \frac{\beta}{1-\lambda} + \frac{\delta + \beta\lambda}{(1-\lambda)^2} + \frac{\zeta + \alpha\lambda}{(1-\lambda)^2} &= \frac{\alpha + \beta + \delta + \zeta}{(1-\lambda)^2} > 0, \\ \frac{\delta + \beta\lambda}{(1-\lambda)^2} + \frac{\zeta + \alpha\lambda}{(1-\lambda)^2} &= \frac{(\alpha + \beta)\lambda + \delta + \zeta}{(1-\lambda)^2} \geq 0. \end{aligned}$$

Therefore, by Theorem 2.4, we obtain $F(S) \neq \phi$.

Next, suppose that $\alpha + 2\beta + \gamma > 0$. Let p_1 and p_2 be fixed points of T . Then

$$\begin{aligned} \alpha \|Tp_1 - Tp_2\|_\alpha^2 + \beta(\|p_1 - Tp_2\|_\alpha^2 + \|Tp_1 - p_2\|_\alpha^2) + \gamma \|p_1 - p_2\|_\alpha^2 \\ \delta(\|p_1 - Tp_1\|_\alpha^2 + \|p_2 - Tp_2\|_\alpha^2) + \zeta \|(p_1 - Tp_1) + (p_2 - Tp_2)\|_\alpha^2 \\ = (\alpha + 2\beta + \gamma) \|p_1 - p_2\|_\alpha^2 \leq 0 \end{aligned}$$

and hence $p_1 = p_2$. Therefore a fixed point of T is unique. □

For the case $\beta + \delta = 0$ in Theorem 2.5, we have the following theorem.

Theorem 2.6. *Let H be a fuzzy Hilbert space, C a nonempty bounded closed convex subset of H and let T be an $(\alpha, -\beta, \gamma, \beta, \zeta)$ -symmetric more generalized hybrid mapping from C into itself, i.e., there exist $\alpha, \beta, \gamma, \zeta \in \mathbb{R}$ such that*

$$\begin{aligned} \alpha \|Tx - Ty\|_\alpha^2 + \beta(\|x - Ty\|_\alpha^2 + \|Tx - y\|_\alpha^2) + \gamma \|x - y\|_\alpha^2 \\ - \beta(\|x - Tx\|_\alpha^2 + \|y - Ty\|_\alpha^2) + \zeta \|x - y - (Tx - Ty)\|_\alpha^2 \leq 0 \end{aligned} \tag{2.7}$$

for all $x, y \in C$. Furthermore, suppose that T satisfies the following conditions (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \zeta > 0$ and (3) there exists $\lambda \in [0, 1)$ such that $(\alpha + \beta)\lambda + \delta + \zeta \geq 0$. Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha + 2\beta + \gamma > 0$ on the condition (1).

Using Theorem 2.3, we prove the following fixed point theorem.

Theorem 2.7. *Let H be a fuzzy Hilbert space, C a nonempty bounded closed convex subset of H and let T be a widely strict pseudo-contraction from C into itself, i.e., there exists $r \in \mathbb{R}$ with $r < 1$ such that*

$$\|Tx - Ty\|_\alpha^2 \leq \|x - y\|_\alpha^2 + r \|(I - T)x - (I - T)y\|_\alpha^2, \forall x, y \in C. \tag{2.8}$$

Then T has a fixed point in C .

Proof. We first assume that $r \leq 0$. We have from (2.8) that for all $x, y \in C$,

$$\|Tx - Ty\|_\alpha^2 - \|x - y\|_\alpha^2 - r \|(I - T)x - (I - T)y\|_\alpha^2 \leq 0 \tag{2.9}$$

Then T is a $(1, 0, -1, 0, -r)$ -symmetric more generalized hybrid mapping. Furthermore, (1) $\alpha + 2\beta + \gamma = 1 - 1 \geq 0$, (2) $\alpha + \beta + \delta + \zeta = 1 - r > 0$ and (3) $\delta + \zeta = -r \geq 0$ in Theorem 1.4

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are satisfied. Thus T has a fixed point from Theorem 2.3. Assume that $0 \leq r < 1$ and define a mapping T as follows:

$$Sx = \lambda x + (1 - \lambda)Tx, \forall x \in C,$$

where $r \leq \lambda < 1$. Then S is a mapping from C into itself and $F(S) = F(T)$. From $Sx = \lambda x + (1 - \lambda)Tx$, we also have that

$$Tx = \frac{1}{1 - \lambda}Sx - \frac{\lambda}{1 - \lambda}x.$$

Thus we have

$$\begin{aligned} 0 &\geq \left\| \frac{1}{1 - \lambda}Sx - \frac{\lambda}{1 - \lambda}x - \left(\frac{1}{1 - \lambda}Sy - \frac{\lambda}{1 - \lambda}y \right) \right\|_{\alpha}^2 \\ &\quad - \|x - y\|_{\alpha}^2 - r \left\| x - y - \left\{ \frac{1}{1 - \lambda}Sx - \frac{\lambda}{1 - \lambda}x - \left(\frac{1}{1 - \lambda}Sy - \frac{\lambda}{1 - \lambda}y \right) \right\} \right\|_{\alpha}^2 \\ &= \left\| \frac{1}{1 - \lambda}(Sx - Sy) - \frac{\lambda}{1 - \lambda}(x - y) \right\|_{\alpha}^2 \\ &\quad - \|x - y\|_{\alpha}^2 - r \left\| \frac{1}{1 - \lambda}(x - y) - \frac{1}{1 - \lambda}(Sx - Sy) \right\|_{\alpha}^2 \\ &= \frac{1}{1 - \lambda} \|Sx - Sy\|_{\alpha}^2 - \frac{\lambda}{1 - \lambda} \|x - y\|_{\alpha}^2 \\ &\quad + \frac{1}{1 - \lambda} \cdot \frac{\lambda}{1 - \lambda} \|x - y - (Sx - Sy)\|_{\alpha}^2 - \|x - y\|_{\alpha}^2 \\ &\quad - \frac{r}{(1 - \lambda)^2} \|x - y - (Sx - Sy)\|_{\alpha}^2 \\ &= \frac{1}{1 - \lambda} \|Sx - Sy\|_{\alpha}^2 - \frac{1}{1 - \lambda} \|x - y\|_{\alpha}^2 + \frac{\lambda - r}{(1 - \lambda)^2} \|x - y - (Sx - Sy)\|_{\alpha}^2. \end{aligned}$$

Then S is a $\left(\frac{1}{1 - \lambda}, 0, -\frac{1}{1 - \lambda}, 0, \frac{\lambda - r}{(1 - \lambda)^2}\right)$ -symmetric more generalized hybrid. From

$$\begin{aligned} \frac{1}{1 - \lambda} + 2.0 - \frac{1}{1 - \lambda} &= 0, \\ \frac{1}{1 - \lambda} + \frac{\lambda - r}{(1 - \lambda)^2} &> 0 \quad \text{and} \\ \frac{\lambda - r}{(1 - \lambda)^2} &\geq 0, \end{aligned}$$

(1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta + \delta + \zeta > 0$ and (3) $\delta + \zeta \geq 0$ in Theorem 1.4 are satisfied. Thus S has a fixed point in C from Theorem 2.3 and hence T has a fixed point. This completes the proof. \square

Let H be a fuzzy Hilbert space and let C be a nonempty subset of H . Let T be a mapping of C into H . For $u \in H$ and $s, t \in (0, 1)$, we define the following mapping:

$$Sx = tx + (1 - t)(su + (1 - s)Tx)$$

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for all $x \in C$. We call such S a TWY mapping generated by u, T, s, t . Since $Sx = tx + s(1 - t)u + (1 - t)(1 - s)Tx$, we have that for all $x, y \in C$,

$$\begin{aligned} \|Sx - Sy\|_{\alpha}^2 &= \|t(x - y) + (1 - t)(1 - s)(Tx - Ty)\|_{\alpha}^2 & (2.10) \\ &= t^2 \|x - y\|_{\alpha}^2 + (1 - t)^2(1 - s)^2 \|Tx - Ty\|_{\alpha}^2 \\ &\quad + 2t(1 - t)(1 - s) \langle x - y, Tx - Ty \rangle_{\alpha} \\ &= t^2 \|x - y\|_{\alpha}^2 + (1 - t)^2(1 - s)^2 \|Tx - Ty\|_{\alpha}^2 \\ &\quad + t(1 - t)(1 - s)(\|x - Ty\|_{\alpha}^2 + \|y - Tx\|_{\alpha}^2 \\ &\quad - \|x - Tx\|_{\alpha}^2 - \|y - Ty\|_{\alpha}^2) \\ &= t^2 \|x - y\|_{\alpha}^2 + (1 - t)^2(1 - s)^2 \|Tx - Ty\|_{\alpha}^2 \\ &\quad + t(1 - t)(1 - s)(\|x - Ty\|_{\alpha}^2 + \|y - Tx\|_{\alpha}^2) \\ &\quad - t(1 - t)(1 - s)(\|x - Tx\|_{\alpha}^2 - \|y - Ty\|_{\alpha}^2). \end{aligned}$$

Similarly, we have that

$$\begin{aligned} &\|x - Sy\|_{\alpha}^2 + \|y - Sx\|_{\alpha}^2 & (2.11) \\ = &s(1 - t)^2 (\|u - x\|_{\alpha}^2 + \|u - y\|_{\alpha}^2) \\ &- s(1 - s)(1 - t)^2 (\|u - Tx\|_{\alpha}^2 + \|u - Ty\|_{\alpha}^2) \\ &- t(1 - t)(1 - s)(\|x - Tx\|_{\alpha}^2 + \|y - Ty\|_{\alpha}^2) \\ &+ (1 - t)(1 - s)(\|x - Tx\|_{\alpha}^2 + \|y - Ty\|_{\alpha}^2) + 2t \|x - y\|_{\alpha}^2, \end{aligned}$$

and

$$\begin{aligned} &\|x - Sx\|_{\alpha}^2 + \|y - Sy\|_{\alpha}^2 \\ = &s(1 - t)^2 (\|u - x\|_{\alpha}^2 + \|u - y\|_{\alpha}^2) \\ &- s(1 - s)(1 - t)^2 (\|u - Tx\|_{\alpha}^2 + \|u - Ty\|_{\alpha}^2) \\ &+ s(1 - s)(1 - t)^2 (\|x - Tx\|_{\alpha}^2 + \|y - Ty\|_{\alpha}^2). \end{aligned}$$

We also have that

$$\begin{aligned} &\|x - y - Sx - Sy\|_{\alpha}^2 & (2.12) \\ = &(1 - s)(1 - t)^2 (\|x - Tx\|_{\alpha}^2 + \|y - Ty\|_{\alpha}^2) \\ &- (1 - s)(1 - t)^2 (\|x - Ty\|_{\alpha}^2 + \|y - Tx\|_{\alpha}^2) \\ &+ (1 - t)^2 \|x - y\|_{\alpha}^2 + (1 - t)^2(1 - s)^2 \|Tx - Ty\|_{\alpha}^2. \end{aligned}$$

Using (2.11) and (2.12), we have that

$$\begin{aligned} &\|x - Sx\|_{\alpha}^2 + \|y - Sy\|_{\alpha}^2 - \|x - Sy\|_{\alpha}^2 - \|y - Sx\|_{\alpha}^2 & (2.13) \\ = &(1 - s)(1 - t)(\|x - Tx\|_{\alpha}^2 + \|y - Ty\|_{\alpha}^2 \\ &- \|x - Ty\|_{\alpha}^2 + \|y - Tx\|_{\alpha}^2 - 2t \|x - y\|_{\alpha}^2). \end{aligned}$$

Using (2.10) and (2.13), we have the following theorem.

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Theorem 2.8. *Let H be a fuzzy Hilbert space, C a nonempty bounded closed convex subset of H and let T be a widely strict pseudo-contraction from C into itself, i.e., there exists $r \in \mathbb{R}$ with $r < 1$ such that*

$$\|Tx - Ty\|_{\alpha}^2 \leq \|x - y\|_{\alpha}^2 + r \|(I - T)x - (I - T)y\|_{\alpha}^2 \quad \forall x, y \in C. \quad (2.14)$$

Let $u \in C$ and $s \in (0, 1)$. Define a mapping $U : C \rightarrow C$ as follows:

$$Ux = su + (1 - s)Tx, \forall x \in C.$$

Then U has a unique fixed point in C .

Proof. Since T is a widely r -strict pseudo-contraction from C into itself, we have that for all $x, y \in C$,

$$\|Tx - Ty\|_{\alpha}^2 - \|x - y\|_{\alpha}^2 - r \|(I - T)x - (I - T)y\|_{\alpha}^2 \leq 0.$$

If $r \leq 0$, then T is a nonexpansive mapping. Therefore U is a contractive mapping. Using the fixed point theorem for contractive mappings, we have that U has a unique fixed point in C . Let $0 < r < 1$. Since

$$\begin{aligned} \|x - y - (Tx - Ty)\|_{\alpha}^2 &= \|x - Tx\|_{\alpha}^2 + \|y - Ty\|_{\alpha}^2 \\ &\quad - \|x - Ty\|_{\alpha}^2 - \|y - Tx\|_{\alpha}^2 + \|x - y\|_{\alpha}^2 + \|Tx - Ty\|_{\alpha}^2, \end{aligned}$$

we have that

$$\begin{aligned} &(1 - r) \|Tx - Ty\|_{\alpha}^2 - (1 + r) \|x - y\|_{\alpha}^2 \\ &- r \|x - Tx\|_{\alpha}^2 + \|y - Ty\|_{\alpha}^2 - \|x - Ty\|_{\alpha}^2 - \|y - Tx\|_{\alpha}^2 \leq 0. \end{aligned}$$

For u, T and $s, r \in (0, 1)$, define a TWY mapping S as follows:

$$Sx = rx + (1 - r)(su + (1 - s)Tx), \forall x \in C.$$

Then we have from (2.10) that

$$\begin{aligned} &\frac{1}{(1 - r)(1 - s)^2} \|Sx - Sy\|_{\alpha}^2 - \frac{r^2}{(1 - r)(1 - s)^2} \|x - y\|_{\alpha}^2 \\ &+ \frac{r}{(1 - s)} \left(\|x - Tx\|_{\alpha}^2 + \|y - Ty\|_{\alpha}^2 - \|x - Ty\|_{\alpha}^2 - \|y - Tx\|_{\alpha}^2 \right) \\ &- (1 + r) \|x - y\|_{\alpha}^2 \\ &- r(\|x - Tx\|_{\alpha}^2 + \|y - Ty\|_{\alpha}^2 - \|x - Ty\|_{\alpha}^2 - \|y - Tx\|_{\alpha}^2) \leq 0. \end{aligned}$$

We have from (2.13) that

$$\begin{aligned} &\frac{1}{(1 - r)(1 - s)^2} \|Sx - Sy\|_{\alpha}^2 - \frac{r^2}{(1 - r)(1 - s)^2} \|x - y\|_{\alpha}^2 \\ &+ \frac{r}{(1 - r)(1 - s)^2} \left(\|x - Sx\|_{\alpha}^2 + \|y - Sy\|_{\alpha}^2 - \|x - Sy\|_{\alpha}^2 - \|y - Sx\|_{\alpha}^2 \right) \\ &+ \frac{2r^2}{(1 - r)(1 - s)^2} \|x - y\|_{\alpha}^2 - (1 + r) \|x - y\|_{\alpha}^2 \\ &- \frac{r}{(1 - r)(1 - s)} \left(\|x - Sx\|_{\alpha}^2 + \|y - Sy\|_{\alpha}^2 - \|x - Sy\|_{\alpha}^2 - \|y - Sx\|_{\alpha}^2 \right) \end{aligned}$$

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$$-\frac{2r^2}{(1-r)(1-s)} \|x - y\|_\alpha^2 \leq 0$$

and hence

$$\begin{aligned} & \frac{1}{(1-r)(1-s)^2} \|Sx - Sy\|_\alpha^2 \\ & - \frac{rs}{(1-r)(1-s)^2} \left(\|x - Sy\|_\alpha^2 + \|y - Sx\|_\alpha^2 \right) \\ & + \left(\frac{r^2}{(1-r)(1-s)^2} - \frac{1-s+r^2(1+s)}{(1-r)(1-s)} \right) \|x - y\|_\alpha^2 \\ & + \frac{rs}{(1-r)(1-s)^2} \left(\|x - Sx\|_\alpha^2 + \|y - Sy\|_\alpha^2 \right) \leq 0. \end{aligned}$$

For this inequality, we apply Theorem 2.2. We first obtain that

$$\begin{aligned} & \frac{1}{(1-r)(1-s)^2} - \frac{2rs}{(1-r)(1-s)^2} + \frac{r^2}{(1-r)(1-s)^2} - \frac{1-s+r^2(1+s)}{(1-r)(1-s)} \\ = & \frac{s(1+r)(2-s(1-r))}{(1-r)(1-s)^2} > 0. \end{aligned}$$

Furthermore, we have that

$$\begin{aligned} & \frac{1}{(1-r)(1-s)^2} - \frac{rs}{(1-r)(1-s)^2} + \frac{rs}{(1-r)(1-s)^2} \\ = & \frac{1}{(1-r)(1-s)^2} > 0, \\ & \frac{rs}{(1-r)(1-s)^2} \geq 0. \end{aligned}$$

Thus S has a unique fixed point z in C from Theorem 1.3. Since z is a fixed point of S , we have $z = rz + (1-r)(su + (1-s)Tz)$. From $1-r \neq 0$, we have that

$$z = su + (1-s)Tz.$$

This completes the proof. □

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Some properties on Dirichlet-Hadamard product of Dirichlet series *

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Abstract

By constructing a new form Dirichlet-Hadamard product of Dirichlet series, we investigate the relation about the growth of Dirichlet series and obtain some estimates on the upper and the lower bounds of the (lower) q -order and the (lower) q -type of Dirichlet-Hadamard product of Dirichlet series. We also study the growth on scalar multiplication and shift of Dirichlet series. Our results of this paper are improvements of the previous theorems given by Kong and Deng.

Key words: Dirichlet-Hadamard product, growth, scalar multiplication, Dirichlet series.

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1 Introduction and basic notes

Consider Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \quad s = \sigma + it, \tag{1}$$

where

$$0 \leq \lambda_1 < \lambda_2 < \dots < \lambda_n < \dots, \lambda_n \rightarrow \infty, \text{ as } n \rightarrow \infty;$$

$s = \sigma + it$ (σ, t are real variables); a_n are nonzero complex numbers. Let $f(s)$ satisfy

$$\limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n} = 0, \tag{2}$$

$$\limsup_{n \rightarrow \infty} \frac{\log |a_n|}{\lambda_n} = -\infty, \tag{3}$$

then we have the abscissas of convergence and absolute convergence are $+\infty$ by applying the Valion's formula (see [4]), that is, $f(s)$ is an analytic function in the whole plane \mathbb{C} . We denote D to be the class of all functions $f(s)$ satisfying (2),(3).

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Definition 1.1 (see [16]). Let $f(s) \in D$, the order of $f(s)$ is defined by

$$\rho = \limsup_{\sigma \rightarrow +\infty} \frac{\log \log M(\sigma, f)}{\sigma},$$

where

$$M(\sigma, f) = \sup_{-\infty < t \in \mathbb{R} < +\infty} \{|f(\sigma + it)|\}, \quad \text{for } \sigma \in \mathbb{R}.$$

For $\rho = 0, 0 < \rho < \infty, \rho = \infty$, $f(s)$ can be called zero order, finite order, infinite order Dirichlet series, respectively. For infinite order Dirichlet series, we will introduce the definition of q -order as follows.

Definition 1.2 (see [16]). Let $f(s) \in D$, we define the q -order ρ and lower q -order χ of $f(s)$ as follows

$$\rho = \rho_{[q]} = \limsup_{\sigma \rightarrow +\infty} \frac{\log^{[q]} M(\sigma, f)}{\sigma}, \quad \chi = \chi_{[q]} = \liminf_{\sigma \rightarrow +\infty} \frac{\log^{[q]} M(\sigma, f)}{\sigma}.$$

In addition, if $\rho \in (0, +\infty)$, the q -type T and lower q -type τ can be defined as follows:

$$T = \limsup_{\sigma \rightarrow +\infty} \frac{\log^{[q-1]} M(\sigma, f)}{e^{\sigma \rho}}, \quad \tau = \liminf_{\sigma \rightarrow +\infty} \frac{\log^{[q-1]} M(\sigma, f)}{e^{\sigma \rho}},$$

where $q = 2, 3, \dots$, $\log^{[0]} x = x, \log^{[k]} = \log(\log^{[k-1]} x)$.

Definition 1.3 (see [16]). If $\rho = \chi$, $f(s)$ is called $\rho_{[q]}$ -regular growth, and if $\tau = T$, $f(s)$ is called $\rho_{[q]}$ -perfectly regular growth.

In the past several decades, considerable attention has been paid to the growth and the value distribution of Dirichlet series; see [4, 16] for some results. For examples, J. R. Yu, D. C. Sun and Z. S. Gao investigated the growth and value distribution of entire functions defined by Dirichlet series (see [1, 2, 3, 5, 9, 12, 15]); M. N. Sheremeta, A. Nautiyal, and H. Y. Xu studied the problem on the approximation of Dirichlet series (see [8, 10, 13, 14]); Y. Y. Kong, K. A. M. Sayyed, M. S. Metwally and M. T. Mohamed studied the growth of Hadamard-product of Dirichlet series (see [6, 7, 11]), and so on. We list several classical results as follows.

Theorem 1.1 Let $f(s) \in D$ be of order ρ , then

$$\rho = \limsup_{n \rightarrow +\infty} \frac{\lambda_n \log^{[q-1]} \lambda_n}{\log |a_n|^{-1}}, \tag{4}$$

$$T = \limsup_{n \rightarrow +\infty} |a_n|^{\frac{\rho}{\lambda_n}} \log^{[q-2]} \left(\frac{\lambda_n}{e^\rho} \right). \tag{5}$$

Theorem 1.2 Let $f(s) \in D$ be of lower order χ , then

$$\chi \leq \liminf_{n \rightarrow +\infty} \frac{\lambda_n \log^{[q-1]} \lambda_{n-1}}{\log |a_n|^{-1}}, \tag{6}$$

where $q = 2, 3, \dots$, the equality holds if and only if

$$\varphi(n) = \frac{\log |a_n| - \log |a_{n+1}|}{\lambda_{n+1} - \lambda_n}$$

is a non-decreasing function with n .

Theorem 1.3 Let $f(s) \in D$ be of order $\rho(0 < \rho < +\infty)$ and of type τ , then

$$\tau \leq \liminf_{n \rightarrow +\infty} |a_n|^{\frac{\rho}{\lambda_n}} \log^{[q-2]} \left(\frac{\lambda_{n-1}}{e\rho} \right), \tag{7}$$

where $q = 2, 3, \dots$, the equality holds if and only if $\varphi(n)$ is a non-decreasing function with n , and $\log^{[q-2]} \lambda_{n-1} \sim \log^{[q-2]} \lambda_n, n \rightarrow +\infty$.

In 2009 and 2014, Y. Y. Kong and G. T. Deng investigated their growth of Dirichlet series by defining the Dirichlet-Hadamard product (see [5, 6]) and obtained some interesting results.

Definition 1.4 (see [5, 6]). Let $f_1(s) = \sum_{n=1}^{\infty} a_n e^{\gamma_n s}$, $f_2(s) = \sum_{n=1}^{\infty} b_n e^{\xi_n s}$ and $f_1(s), f_2(s) \in D$, the Dirichlet-Hadamard product function $F_1(s)$ of $f_1(s), f_2(s)$ is

$$F_1(s) = (f_1 \Delta f_2)(\mu, \nu; s) = \sum_{n=1}^{\infty} c_n e^{\lambda_n s},$$

where $c_n = a_n^\mu b_n^\nu$, a_n, b_n are nonzero complex numbers; $\lambda_n = \frac{\gamma_n + \xi_n}{2}$, μ, ν are positive numbers, $0 < \gamma_n, \xi_n \uparrow +\infty$.

Theorem 1.4 (see [6, Theorem 2.1]). Let $f_1(s), f_2(s) \in D$ be of q -order ρ_1, ρ_2 , respectively, and satisfy

$$\gamma_n = \eta \xi_n, \tag{8}$$

then the q -order ρ of the Dirichlet-Hadamard product $F_1(s)$ satisfies

$$\rho \leq \frac{\rho_1 \rho_1 (2 + \frac{1}{\eta} + \eta)}{2\nu \rho_1 (\frac{1}{\eta} + 1) + 2\mu \rho_2 (1 + \eta)}, \quad \rho_1, \rho_2 \in [0, +\infty).$$

Theorem 1.5 (see [6, Theorem 2.2]). Let $f_1(s), f_2(s) \in D$ be of lower q -order χ_1, χ_2 , respectively, and satisfy the condition of Lemma 2.2, then the lower q -order χ of the Dirichlet-Hadamard product $F_1(s)$ satisfies

$$\chi \geq \frac{(2 + \frac{1}{\eta} + \eta) \chi_1 \chi_2}{2\nu \chi_1 (\frac{1}{\eta} + 1) + 2\mu \chi_2 (1 + \eta)}.$$

In this paper, we will introduce a more general form Dirichlet-Hadamard product of Dirichlet series, which is improvement of Kong's definition [6].

Definition 1.5 Let $f_1(s) = \sum_{n=1}^{\infty} a_n e^{\gamma_n s}$, $f_2(s) = \sum_{n=1}^{\infty} b_n e^{\xi_n s}$ and $f_1(s), f_2(s) \in D$, the generalized Dirichlet-Hadamard product can be defined by

$$F(s) = (f_1 \Delta f_2)(\mu, \nu; \alpha, \beta; s) = \sum_{n=1}^{\infty} c_n e^{\lambda_n s}, \quad c_n = a_n^\mu b_n^\nu, \quad \lambda_n = \alpha \gamma_n + \beta \xi_n,$$

where α, β, μ, ν are positive numbers, a_n, b_n are nonzero complex numbers, $0 < \gamma_n, \xi_n \uparrow +\infty$.

Remark 1.1 When $\alpha = \beta = \frac{1}{2}$, the generalized Dirichlet-Hadamard product is the Dirichlet-Hadamard product by Kong.

2 Some Lemmas and the (lower) q -order of the generalized Dirichlet-Hadamard product

In this paper, we prove some theorems about the upper and the lower bounds of the (lower) q -order and the (lower) q -type of generalized Dirichlet-Hadamard product.

Lemma 2.1 *Let $f_1(s), f_2(s) \in D$ and satisfy (8), then $F(s)$ is analytic in the whole complex plane, that is, $F(s)$ is an entire function.*

Proof:

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\log n}{\lambda_n} &= \limsup_{n \rightarrow \infty} \frac{\log n}{\alpha\gamma_n + \beta\xi_n} \\ &\leq \frac{1}{\alpha} \limsup_{n \rightarrow \infty} \frac{\log n}{\gamma_n} = 0, \end{aligned}$$

and

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\log |c_n|}{\lambda_n} &= \limsup_{n \rightarrow \infty} \frac{\mu \log |a_n| + \nu \log |b_n|}{\alpha\gamma_n + \beta\xi_n} \\ &\leq \limsup_{n \rightarrow \infty} \frac{\mu \log |a_n|}{\alpha\gamma_n + \beta\xi_n} = -\infty. \end{aligned}$$

This completes the proof of Lemma 2.1. □

Lemma 2.2 *Let γ_n, ξ_n satisfy (8), and*

$$\varphi_1(n) = (\gamma_{n+1} - \gamma_n)^{-1} \log \left| \frac{a_n}{a_{n+1}} \right|, \quad \varphi_2(n) = (\xi_{n+1} - \xi_n)^{-1} \log \left| \frac{b_n}{b_{n+1}} \right|$$

be non-decreasing functions with n , then

$$\varphi(n) = (\lambda_{n+1} - \lambda_n)^{-1} \log \left| \frac{c_n}{c_{n+1}} \right|$$

is a non-decreasing function with n .

Proof:

$$\begin{aligned} \varphi(n) &= \frac{1}{(\alpha\gamma_{n+1} + \beta\xi_{n+1}) - (\alpha\gamma_n + \beta\xi_n)} \log \left| \frac{a_n^\mu b_n^\nu}{a_{n+1}^\mu b_{n+1}^\nu} \right| \\ &= \frac{\mu \log \left| \frac{a_n}{a_{n+1}} \right|}{(\gamma_{n+1} - \gamma_n)(\alpha + \beta \frac{\xi_{n+1} - \xi_n}{\gamma_{n+1} - \gamma_n})} + \frac{\nu \log \left| \frac{b_n}{b_{n+1}} \right|}{(\xi_{n+1} - \xi_n)(\alpha \frac{\gamma_{n+1} - \gamma_n}{\xi_{n+1} - \xi_n} + \beta)} \\ &= \frac{\mu\varphi_1(n)}{\alpha + \frac{\beta}{\eta}} + \frac{\nu\varphi_2(n)}{\beta + \alpha\eta}. \end{aligned} \tag{9}$$

Since $\varphi_1(n), \varphi_2(n)$ are non-decreasing functions, thus it follows from (9) that $\varphi(n)$ is a non-decreasing function. □

Theorem 2.1 *Let $f_1(s), f_2(s) \in D$ be of q -order ρ_1, ρ_2 , respectively. If $f_1(s), f_2(s)$ satisfy (8), then $F(s)$ is of q -order ρ satisfying*

$$\rho \leq \frac{(\alpha\eta + \beta)\rho_1\rho_2}{\mu\eta\rho_2 + \nu\rho_1}.$$

Proof: We only prove the case $\rho_1, \rho_2 \in (0, +\infty)$. From (4), for any $\varepsilon > 0$, there exist two positive integers N_1, N_2 such that $n > N = \max\{N_1, N_2\}$, we have

$$\frac{\gamma_n \log^{[q-1]} \gamma_n}{\log |a_n|^{-1}} < \rho_1 + \varepsilon, \quad \frac{\xi_n \log^{[q-1]} \xi_n}{\log |b_n|^{-1}} < \rho_2 + \varepsilon. \tag{10}$$

Since $c_n = a_n^\mu b_n^\nu$, it follows from (10) that

$$\log |c_n|^{-1} = \mu \log |a_n|^{-1} + \nu \log |b_n|^{-1} > \frac{\mu \gamma_n \log^{[q-1]} \gamma_n}{\rho_1 + \varepsilon} + \frac{\nu \xi_n \log^{[q-1]} \xi_n}{\rho_2 + \varepsilon}, \tag{11}$$

then from (11) we have

$$\frac{\lambda_n \log^{[q-1]} \lambda_n}{\log |c_n|^{-1}} < \frac{\lambda_n \log^{[q-1]} \lambda_n}{\frac{\mu}{\rho_1 + \varepsilon} \gamma_n \log^{[q-1]} \gamma_n + \frac{\nu}{\rho_2 + \varepsilon} \xi_n \log^{[q-1]} \xi_n}. \tag{12}$$

Since $q = 2, 3, \dots$ and $\xi_n = \frac{1}{\alpha\eta + \beta} \lambda_n, \gamma_n = \frac{1}{\alpha + \frac{\beta}{\eta}} \lambda_n$, it follows

$$\log^{[q-1]} \gamma_n \sim \log^{[q-1]} \xi_n \sim \log^{[q-1]} \lambda_n. \tag{13}$$

Since ε is arbitrary, it follows from (12) and (13) that

$$\rho = \limsup_{n \rightarrow \infty} \frac{\lambda_n \log^{[q-1]} \lambda_n}{\log |c_n|^{-1}} \leq \frac{1}{\frac{\mu}{\rho_1} \frac{1}{\alpha + \frac{\beta}{\eta}} + \frac{\nu}{\rho_2} \frac{1}{\alpha\eta + \beta}} = \frac{(\alpha\eta + \beta)\rho_1\rho_2}{\mu\eta\rho_2 + \nu\rho_1}.$$

□

Theorem 2.2 *Let $f_1(s), f_2(s) (\in D)$ be of lower q -order χ_1, χ_2 , respectively. If $f_1(s), f_2(s)$ satisfy the conditions of Lemma 2.2, then the lower q -order χ of $F(s)$ satisfies*

$$\chi \geq \frac{(\alpha\eta + \beta)\chi_1\chi_2}{\eta\mu\chi_2 + \nu\chi_1}.$$

Proof: Suppose that $\chi_1, \chi_2 > 0$. From Theorem 1.2, for any $\varepsilon > 0$, there exists a positive number $N \in \mathbb{N}_+$ such that $n > N$, we have

$$\frac{\gamma_n \log^{[q-1]} \gamma_{n-1}}{\log |a_n|^{-1}} > \chi_1 - \varepsilon, \quad \frac{\xi_n \log^{[q-1]} \xi_{n-1}}{\log |b_n|^{-1}} > \chi_2 - \varepsilon. \tag{14}$$

Since $c_n = a_n^\mu b_n^\nu$, it follows from (14) that

$$\begin{aligned} \log |c_n|^{-1} &= \mu \log |a_n|^{-1} + \nu \log |b_n|^{-1} \\ &< \frac{\mu}{\chi_1 - \varepsilon} (\gamma_n \log^{[q-1]} \gamma_{n-1}) + \frac{\nu}{\chi_2 - \varepsilon} (\xi_n \log^{[q-1]} \xi_{n-1}). \end{aligned} \tag{15}$$

Thus, from (13) and (15) we have

$$\frac{\lambda_n \log^{[q-1]} \lambda_{n-1}}{\log |c_n|^{-1}} > \frac{\lambda_n \log^{[q-1]} \lambda_{n-1}}{\frac{\mu}{\chi_1 - \varepsilon} (\gamma_n \log^{[q-1]} \gamma_{n-1}) + \frac{\nu}{\chi_2 - \varepsilon} (\xi_n \log^{[q-1]} \xi_{n-1})}. \tag{16}$$

By Lemma 2.2, we have $\varphi(n)$ is a non-decreasing function. And since ε is arbitrary, it follows from (16) that

$$\chi = \liminf_{n \rightarrow \infty} \frac{\lambda_n \log^{[q-1]} \lambda_{n-1}}{\log |c_n|^{-1}} \geq \left(\frac{1}{\alpha + \frac{\beta}{\eta}} \frac{\mu}{\chi_1} + \frac{1}{\alpha\eta + \beta} \frac{\nu}{\chi_2} \right)^{-1} = \frac{(\alpha\eta + \beta)\chi_1\chi_2}{\mu\eta\chi_2 + \nu\chi_1}.$$

This completes the proof of Theorem 2.2. □

Theorem 2.3 Suppose that $f_1(s), f_2(s)$ are two $\rho_{[q]}$ -regular growth functions and of q -order ρ_1, ρ_2 , respectively, and if $f_1(s), f_2(s)$ satisfy the conditions of Lemma 2.2.

(i) Then $F(s)$ is also $\rho_{[q]}$ -regular growth function, and of q -order ρ satisfying

$$\rho = \frac{(\alpha + \beta)\rho_1\rho_2}{\mu\eta\rho_2 + \nu\rho_1}, \quad \rho_1, \rho_2 \in [0, +\infty).$$

(ii) If $\rho_1, \rho_2 \in (0, +\infty)$ and $f_1(s), f_2(s)$ are of q -type T_1, T_2 , respectively, then q -type T of $F(s)$ satisfy

$$T \leq \begin{cases} T_1^{\frac{\mu\rho\eta}{(\alpha\eta+\beta)\rho_1}} T_2^{\frac{\nu\rho}{(\alpha\eta+\beta)\rho_2}}, & q = 3, 4, 5, \dots, \\ \frac{\alpha\eta + \beta}{\rho\eta^{\frac{\mu\rho\eta}{(\alpha\eta+\beta)\rho_1}}} (\rho_1 T_1)^{\frac{\mu\rho\eta}{\rho_1(\alpha\eta+\beta)}} (\rho_2 T_2)^{\frac{\nu\rho}{\rho_2(\alpha\eta+\beta)}}, & q = 2. \end{cases}$$

Proof: (i) Since $f_1(s), f_2(s)$ are two $\rho_{[q]}$ -regular growth functions, thus $\chi_1 = \rho_1, \chi_2 = \rho_2$, where χ_1, χ_2 are the lower q -order of $f_1(s), f_2(s)$, respectively. Thus, it follows by Theorem 2.1 and Theorem 2.2 that

$$\rho = \frac{(\alpha\eta + \beta)\rho_1\rho_2}{\mu\eta\rho_2 + \nu\rho_1}, \quad \rho_1, \rho_2 \in [0, +\infty).$$

This proves (i).

(ii) From Theorem 1.1, we have

$$T_1 = \limsup_{n \rightarrow \infty} |a_n|^{\frac{\rho_1}{\gamma_n}} \log^{[q-2]} \left(\frac{\gamma_n}{e\rho_1} \right), \quad T_2 = \limsup_{n \rightarrow \infty} |b_n|^{\frac{\rho_2}{\xi_n}} \log^{[q-2]} \left(\frac{\xi_n}{e\rho_2} \right).$$

So for any $\varepsilon > 0$, there exists a positive number $N \in \mathbb{N}_+$ such that $n > N$, we have

$$|a_n|^{\frac{\rho_1}{\gamma_n}} \leq \frac{T_1 + \varepsilon}{\log^{[q-2]} \left(\frac{\gamma_n}{e\rho_1} \right)}, \quad |b_n|^{\frac{\rho_2}{\xi_n}} \leq \frac{T_2 + \varepsilon}{\log^{[q-2]} \left(\frac{\xi_n}{e\rho_2} \right)}.$$

If $q = 3, 4, 5, \dots$, we have

$$\log^{[q-2]} \left(\frac{\gamma_n}{e\rho_1} \right) \sim \log^{[q-2]} \gamma_n, \quad \log^{[q-2]} \left(\frac{\xi_n}{e\rho_2} \right) \sim \log^{[q-2]} \xi_n. \tag{17}$$

And $c_n = a_n^\mu b_n^\nu$, then it follows from (17) that

$$\begin{aligned} |c_n|^{\frac{\rho}{\lambda_n}} \log^{[q-2]} \left(\frac{\lambda_n}{e\rho} \right) &= (|a_n|^\mu |b_n|^\nu)^{\frac{\rho}{\lambda_n}} \log^{[q-2]} \left(\frac{\lambda_n}{e\rho} \right) \\ &= (|a_n|^{\frac{\rho_1}{\gamma_n}})^{\frac{\mu\rho\gamma_n}{\rho_1\lambda_n}} (|b_n|^{\frac{\rho_2}{\xi_n}})^{\frac{\nu\rho\xi_n}{\rho_2\lambda_n}} \log^{[q-2]} \left(\frac{\lambda_n}{e\rho} \right) \\ &\leq \left(\frac{T_1 + \varepsilon}{\log^{[q-2]} \left(\frac{\gamma_n}{e\rho_1} \right)} \right)^{\frac{\mu\rho\gamma_n}{\rho_1\lambda_n}} \left(\frac{T_2 + \varepsilon}{\log^{[q-2]} \left(\frac{\xi_n}{e\rho_2} \right)} \right)^{\frac{\nu\rho\xi_n}{\rho_2\lambda_n}} \log^{[q-2]} \left(\frac{\lambda_n}{e\rho} \right) \\ &= \left(\frac{T_1 + \varepsilon}{\log^{[q-2]} \left(\frac{\gamma_n}{e\rho_1} \right)} \right)^{\frac{\mu\eta\rho}{(\alpha\eta+\beta)\rho_1}} \left(\frac{T_2 + \varepsilon}{\log^{[q-2]} \left(\frac{\xi_n}{e\rho_2} \right)} \right)^{\frac{\nu\rho}{(\alpha\eta+\beta)\rho_2}} \log^{[q-2]} \left(\frac{\lambda_n}{e\rho} \right). \end{aligned} \tag{18}$$

Thus, from (18) we have

$$T = \limsup_{n \rightarrow \infty} |c_n|^{\frac{\rho}{\lambda_n}} \log^{[q-2]} \left(\frac{\lambda_n}{e\rho} \right) \leq (T_1 + \varepsilon)^{\frac{\mu\eta\rho}{(\alpha\eta+\beta)\rho_1}} (T_2 + \varepsilon)^{\frac{\nu\rho}{(\alpha\eta+\beta)\rho_2}}. \tag{19}$$

If $q = 2$, from (5), we have for any $\varepsilon > 0$, there exists a positive number $N \in \mathbb{N}_+$ such that $n > N$

$$|a_n|^{\frac{\rho_1}{\gamma_n}} \leq \frac{T_1 + \varepsilon}{\frac{\gamma_n}{e\rho_1}}, \quad |b_n|^{\frac{\rho_2}{\xi_n}} \leq \frac{T_2 + \varepsilon}{\frac{\xi_n}{e\rho_2}}, \tag{20}$$

Since $c_n = a_n^\mu b_n^\nu$, it follows from (20) that

$$\begin{aligned} |c_n|^{\frac{\rho}{\lambda_n}} \frac{\lambda_n}{e\rho} &= (|a_n|^{\frac{\rho_1}{\gamma_n}})^{\frac{\mu\rho\gamma_n}{\rho_1\lambda_n}} (|b_n|^{\frac{\rho_2}{\xi_n}})^{\frac{\nu\rho\xi_n}{\rho_2\lambda_n}} \frac{\lambda_n}{e\rho} \\ &\leq \left(\frac{T_1 + \varepsilon}{\frac{\gamma_n}{e\rho_1}}\right)^{\frac{\mu\rho\eta}{(\alpha\eta + \beta)\rho_1}} \left(\frac{T_2 + \varepsilon}{\frac{\xi_n}{e\rho_2}}\right)^{\frac{\nu\rho}{(\alpha\eta + \beta)\rho_2}} \left(\frac{\lambda_n}{e\rho}\right). \end{aligned} \tag{21}$$

Since ε is arbitrary, it follows from (19) and (21) that

$$T \leq \begin{cases} T_1^{\frac{\mu\rho\eta}{(\alpha\eta + \beta)\rho_1}} T_2^{\frac{\nu\rho}{(\alpha\eta + \beta)\rho_2}}, & q = 3, 4, 5, \dots, \\ \frac{\alpha\eta + \beta}{\rho\eta^{\frac{\mu\rho\eta}{(\alpha\eta + \beta)\rho_1}}} (\rho_1 T_1)^{\frac{\mu\rho\eta}{\rho_1(\alpha\eta + \beta)}} (\rho_2 T_2)^{\frac{\nu\rho}{\rho_2(\alpha\eta + \beta)}}, & q = 2. \end{cases}$$

Thus, this completes the proof of Theorem 2.3. □

Theorem 2.4 Let $f_1(s), f_2(s)$ be two $\rho_{[q]}$ -perfectly regular growth functions, and satisfy (8), the condition of lemma 2.2 and

$$\log^{[q-2]} \gamma_{n-1} \sim \log^{[q-2]} \gamma_n, \quad \log^{[q-2]} \xi_{n-1} \sim \log^{[q-2]} \xi_n, \quad n \rightarrow \infty. \tag{22}$$

Set $\rho_1, \rho_2, T_1, T_2, \tau_1$ and τ_2 be the q -order, q -type and lower q -type of $f_1(s), f_2(s)$, then $F(s)$ is of $\rho_{[q]}$ -perfectly regular growth ρ and its q -type T satisfies

$$T = \begin{cases} T_1^{\frac{\mu\rho\eta}{(\alpha\eta + \beta)\rho_1}} T_2^{\frac{\nu\rho}{(\alpha\eta + \beta)\rho_2}}, & q = 3, 4, 5, \dots, \\ \frac{\alpha\eta + \beta}{\rho\eta^{\frac{\mu\rho\eta}{(\alpha\eta + \beta)\rho_1}}} (\rho_1 T_1)^{\frac{\mu\rho\eta}{\rho_1(\alpha\eta + \beta)}} (\rho_2 T_2)^{\frac{\nu\rho}{\rho_2(\alpha\eta + \beta)}}, & q = 2. \end{cases}$$

Proof: Suppose that $\rho_1, \rho_2 \in (0, +\infty), \tau_1, \tau_2 < +\infty$. From Theorem 1.3, for any $\varepsilon > 0$, there exists a positive number $N \in \mathbb{N}_+$ such that $n > N$, we have

$$|a_n|^{\frac{\rho_1}{\gamma_n}} \log^{[q-2]} \left(\frac{\gamma_{n-1}}{e\rho_1}\right) \geq \tau_1 - \varepsilon, \quad |b_n|^{\frac{\rho_2}{\xi_n}} \log^{[q-2]} \left(\frac{\xi_{n-1}}{e\rho_2}\right) \geq \tau_2 - \varepsilon.$$

If $q \geq 3$, it follows

$$\begin{aligned} \tau &= \liminf_{n \rightarrow \infty} |c_n|^{\frac{\rho}{\lambda_n}} \log^{[q-2]} \left(\frac{\lambda_{n-1}}{e\rho}\right) \\ &= \liminf_{n \rightarrow \infty} (|a_n|^\mu |b_n|^\nu)^{\frac{\rho}{\lambda_n}} \log^{[q-2]} \left(\frac{\lambda_{n-1}}{e\rho}\right) \\ &\geq \liminf_{n \rightarrow \infty} \left\{ \left[\frac{\tau_1 - \varepsilon}{\log^{[q-2]} \left(\frac{\gamma_{n-1}}{e\rho_1}\right)}\right]^{\frac{\mu\gamma_n}{\rho_1}} \left[\frac{\tau_2 - \varepsilon}{\log^{[q-2]} \left(\frac{\xi_{n-1}}{e\rho_2}\right)}\right]^{\frac{\nu\xi_n}{\rho_2}} \right\}^{\frac{\rho}{\lambda_n}} \log^{[q-2]} \left(\frac{\lambda_{n-1}}{e\rho}\right) \\ &\geq (\tau_1 - \varepsilon)^{\frac{\mu\rho\eta}{(\alpha\eta + \beta)\rho_1}} (\tau_2 - \varepsilon)^{\frac{\nu\rho}{(\alpha\eta + \beta)\rho_2}}. \end{aligned} \tag{23}$$

From Theorem 2.3 and since $\varepsilon >$ is arbitrary, it follows from (23) that

$$T_1^{\frac{\mu\rho\eta}{(\alpha\eta + \beta)\rho_1}} T_2^{\frac{\nu\rho}{(\alpha\eta + \beta)\rho_2}} \geq T \geq \tau \geq (\tau_1)^{\frac{\mu\rho\eta}{(\alpha\eta + \beta)\rho_1}} (\tau_2)^{\frac{\nu\rho}{(\alpha\eta + \beta)\rho_2}}. \tag{24}$$

If $q = 2$, from (22) we have

$$\begin{aligned}
 \tau &= \liminf_{n \rightarrow \infty} |c_n|^{\frac{\rho}{\lambda_n}} \frac{\lambda_{n-1}}{e\rho} \\
 &\geq \liminf_{n \rightarrow \infty} \left(\frac{\tau_1 - \varepsilon}{\frac{\gamma_{n-1}}{e\rho_1}} \right)^{\frac{\mu\rho\gamma_n}{\rho_1\lambda_n}} \left(\frac{\tau_2 - \varepsilon}{\frac{\xi_{n-1}}{e\rho_2}} \right)^{\frac{\nu\rho\xi_n}{\rho_2\lambda_n}} \\
 &\geq \frac{1}{\rho} \left[\frac{(\alpha\eta + \beta)\rho_1\tau_1}{\eta} \right]^{\frac{\mu\rho\eta}{(\alpha\eta + \beta)\rho_1}} \left[(\alpha\eta + \beta)\rho_2\tau_2 \right]^{\frac{\nu\rho}{(\alpha\eta + \beta)\rho_2}} \\
 &= \frac{\alpha\eta + \beta}{\rho\eta^{\frac{\mu\rho\eta}{(\alpha\eta + \beta)\rho_1}}} (\rho_1\tau_1)^{\frac{\mu\rho\eta}{\rho_1(\alpha\eta + \beta)}} (\rho_2\tau_2)^{\frac{\nu\rho}{\rho_2(\alpha\eta + \beta)}} \tag{25}
 \end{aligned}$$

From Theorem 2.3 and since $\varepsilon > 0$ is arbitrary, it follows from (25) that

$$\begin{aligned}
 \frac{\alpha\eta + \beta}{\rho\eta^{\frac{\mu\rho\eta}{(\alpha\eta + \beta)\rho_1}}} (\rho_1T_1)^{\frac{\mu\rho\eta}{\rho_1(\alpha\eta + \beta)}} (\rho_2T_2)^{\frac{\nu\rho}{\rho_2(\alpha\eta + \beta)}} &\geq T \geq \tau \\
 &\geq \frac{\alpha\eta + \beta}{\rho\eta^{\frac{\mu\rho\eta}{(\alpha\eta + \beta)\rho_1}}} (\rho_1T_1)^{\frac{\mu\rho\eta}{\rho_1(\alpha\eta + \beta)}} (\rho_2T_2)^{\frac{\nu\rho}{\rho_2(\alpha\eta + \beta)}} \tag{26}
 \end{aligned}$$

Since $f_1(s), f_2(s)$ are $\rho_{[q]}$ -perfectly regular growth and $\tau_j = T_j, j = 1, 2$, from (24) and (26), it is easy to get the conclusions of Theorem 2.4.

Thus, we complete the proof of Theorem 2.4. □

3 The linear substitution of Dirichlet series

Next, we define the scalar multiplication of Dirichlet series as follows

Definition 3.1 Let k be a positive number, we define the scalar multiplication of Dirichlet series as follows

$$H(s) = f(ks) = \sum_{n=1}^{\infty} a_n e^{\lambda_n(ks)} = \sum_{n=1}^{\infty} a_n e^{\zeta_n s}, \zeta_n = k\lambda_n.$$

Theorem 3.1 Let $f(s) \in D$, then $H(s) \in D$. Furthermore, if $\varphi(n)$ is a non-decreasing function with n , then the q -order ρ^* and the lower q -order χ^* of $H(s)$ satisfy $\rho^* = k\rho$ and $\chi^* = k\chi$.

Proof: Since

$$\limsup_{n \rightarrow \infty} \frac{\log |a_n|}{\zeta_n} = \limsup_{n \rightarrow \infty} \frac{\log |a_n|}{k\lambda_n} = -\infty,$$

thus, we have $H(s) \in D$.

Furthermore, we have

$$\begin{aligned}
 \rho^* &= \limsup_{n \rightarrow \infty} \frac{\zeta_n \log^{[q-1]} \zeta_n}{\log |a_n|^{-1}} = \limsup_{n \rightarrow \infty} \frac{k\lambda_n \log^{[q-1]} k\lambda_n}{\log |a_n|^{-1}} \\
 &= \limsup_{n \rightarrow \infty} k \frac{\lambda_n \log^{[q-1]} \lambda_n}{\log |a_n|^{-1}} = k\rho,
 \end{aligned}$$

and

$$\begin{aligned}
 \chi^* &= \liminf_{n \rightarrow \infty} \frac{\zeta_n \log^{[q-1]} \zeta_{n-1}}{\log |a_n|^{-1}} = \liminf_{n \rightarrow \infty} \frac{k\lambda_n \log^{[q-1]} k\lambda_{n-1}}{\log |a_n|^{-1}} \\
 &= \liminf_{n \rightarrow \infty} k \frac{\lambda_n \log^{[q-1]} \lambda_{n-1}}{\log |a_n|^{-1}} = k\chi.
 \end{aligned}$$

This completes the proof of Theorem 3.1. □

From Theorem 3.1, we can obtain the following result easily.

Theorem 3.2 *Let $f_1(s), f_2(s) \in D$ and satisfy the conditions of Lemma 2.2. Set ρ, χ be the q -order and the lower q -order of $F(s)$, then the q -order ρ^* and the lower q -order χ^* of $H^*(s) = F(ks)$ satisfy $\rho^* = k\rho, \chi^* = k\chi$.*

Let $f_1(s), f_2(s) \in D$ and k, m be positive numbers. Set $H_1(s) = f_1(ks), H_2(s) = f_2(ms)$ and

$$H^{**}(s) = (H_1 \triangle H_2)(\mu, \nu, s) = \sum_{n=1}^{\infty} c_n e^{\lambda_n s}, \quad c_n = a_n^\mu b_n^\nu, \quad \lambda_n = \alpha k \gamma_n + \beta m \xi_n,$$

where α, β, μ, ν are positive numbers, a_n, b_n are nonzero complex numbers, $0 < \gamma_n, \xi_n \uparrow +\infty$. The following result is about the growth of $H^{**}(s)$.

Theorem 3.3 *Let $f_1(s), f_2(s) \in D$ satisfy (8) and the conditions of Lemma 2.2. Let ρ, χ be the q -order and lower q -order of $F(s)$, then the q -order ρ^{**} and the lower q -order χ^{**} of $H^{**}(s)$ satisfy*

$$\rho^{**} = \frac{k\alpha\eta + m\beta}{\alpha\eta + \beta} \rho, \quad \chi^{**} = \frac{k\alpha\eta + m\beta}{\alpha\eta + \beta} \chi.$$

Proof: Since

$$\begin{aligned} \rho^{**} &= \limsup_{n \rightarrow \infty} \frac{\lambda_n \log^{[q-1]} \lambda_n}{\log |c_n|^{-1}} \\ &= \limsup_{n \rightarrow \infty} \frac{(k\alpha\gamma_n + m\beta\xi_n) \log^{[q-1]}(k\alpha\gamma_n + m\beta\xi_n)}{\log |c_n|^{-1}} \\ &= \limsup_{n \rightarrow \infty} \frac{k\alpha\eta + m\beta}{\alpha\eta + \beta} \frac{\lambda_n \log^{[q-1]}(k\alpha\gamma_n + m\beta\xi_n)}{\log |c_n|^{-1}} \\ &= \frac{k\alpha\eta + m\beta}{\alpha\eta + \beta} \rho, \end{aligned} \tag{27}$$

and

$$\begin{aligned} \chi^{**} &= \liminf_{n \rightarrow \infty} \frac{\lambda_n \log^{[q-1]} \lambda_{n-1}}{\log |c_n|^{-1}} \\ &= \liminf_{n \rightarrow \infty} \frac{(k\alpha\gamma_n + m\beta\xi_n) \log^{[q-1]}(k\alpha\gamma_{n-1} + m\beta\xi_{n-1})}{\log |c_n|^{-1}} \\ &= \liminf_{n \rightarrow \infty} \frac{k\alpha\eta + m\beta}{\alpha\eta + \beta} \frac{\lambda_n \log^{[q-1]}(k\alpha\gamma_{n-1} + m\beta\xi_{n-1})}{\log |c_n|^{-1}} \\ &= \frac{k\alpha\eta + m\beta}{\alpha\eta + \beta} \chi, \end{aligned} \tag{28}$$

thus from (27) and (28) we can prove the conclusions of Theorem 3.3. □

Remark 3.1 *From Theorem 3.3, we can get that $\rho^* = \rho^{**}, \chi^* = \chi^{**}$ if $k = m$.*

In 2008, Kong defined the shift of Dirichlet series (see [6]).

Definition 3.2 (see [6]). *Let α be a real number, the shift of Dirichlet series can be defined as follows*

$$G(s) = f(s + \alpha) = \sum_{n=1}^{\infty} a_n e^{\lambda_n(s+\alpha)} = \sum_{n=1}^{\infty} a'_n e^{\lambda_n s},$$

where $a'_n = a_n e^{\lambda_n \alpha}$.

Let $f_1(s), f_2(s) \in D$, α_1, α_2 be two real numbers and k be a positive number. Set $G_1(s) = f_1(s + \alpha_1), G_2(s) = f_2(s + \alpha_2), H_1(s) = f_1(ks)$, and

$$G^*(s) = (G_1 \Delta G_2)(\mu, \nu, s) = \sum_{n=1}^{\infty} c_n e^{\lambda_n s}, \quad c_n = (a'_n)^\mu (b'_n)^\nu, \quad \lambda_n = \alpha \gamma_n + \beta \xi_n,$$

and

$$G^{**}(s) = (H_1 \Delta G_2)(\mu, \nu, s) = \sum_{n=1}^{\infty} c_n e^{\lambda_n s}, \quad c_n = a_n^\mu (b'_n)^\nu, \quad \lambda_n = \alpha k \gamma_n + \beta \xi_n,$$

where α, β, μ, ν are positive numbers, a_n, b_n are nonzero complex numbers, $0 < \gamma_n, \xi_n \uparrow +\infty$. We investigate the growth of $G^*(s), G^{**}(s)$ and obtain the following results

Theorem 3.4 *Let $f_1(s), f_2(s) \in D$ satisfy (8) and the conditions of Lemma 2.2. Let ρ, χ be the q -order and lower q -order of $F(s)$, then the q -order ρ_1^{**} and the lower q -order χ_1^{**} of $G^*(s)$ satisfy $\rho_1^{**} = \rho, \chi_1^{**} = \chi$.*

Proof: Since

$$\rho_1^{**} = \limsup_{n \rightarrow \infty} \frac{\lambda_n \log^{[q-1]} \lambda_n}{\log |c_n|^{-1}} \tag{29}$$

$$= \limsup_{n \rightarrow \infty} \frac{\lambda_n \log^{[q-1]} \lambda_n}{\log |c_n|^{-1} - (\mu \gamma_n \alpha_1 + \nu \xi_n \alpha_2)} \tag{30}$$

$$= \limsup_{n \rightarrow \infty} \frac{\lambda_n \log^{[q-1]} \lambda_n}{\log |c_n|^{-1} (1 + \frac{\mu \gamma_n \alpha_1 + \nu \xi_n \alpha_2}{\log |c_n|})} \tag{31}$$

$$= \rho, \tag{32}$$

that is, $\rho_1^{**} = \rho$.

Similarly, we have $\chi_1^{**} = \chi$. □

Theorem 3.5 *Let $f_1(s), f_2(s) \in D$ satisfy (8) and the conditions of Lemma 2.2. Let ρ, χ be the q -order and lower q -order of $F(s)$, then the q -order ρ_2^{**} and the lower q -order χ_2^{**} of $G^{**}(s)$ satisfy*

$$\rho_2^{**} = \frac{k\alpha\eta + \beta}{\alpha\eta + \beta} \rho, \chi_2^{**} = \frac{k\alpha\eta + \beta}{\alpha\eta + \beta} \chi.$$

Proof: Since

$$\rho_2^{**} = \limsup_{n \rightarrow \infty} \frac{\lambda_n \log^{[q-1]} \lambda_n}{\log |c_n|^{-1}} \tag{33}$$

$$= \limsup_{n \rightarrow \infty} \frac{(k\alpha\gamma_n + \beta\xi_n) \log^{[n-1]} (k\alpha\gamma_n + \beta\xi_n)}{\log |c_n|^{-1} - (\nu \xi_n \alpha_2)} \tag{34}$$

$$= \limsup_{n \rightarrow \infty} \frac{\frac{k\alpha\eta + \beta}{\alpha\eta + \beta} \lambda_n \log^{[n-1]} (\alpha k \gamma_n + \beta \xi_n)}{\log |c_n|^{-1} (1 + \frac{\nu \xi_n \alpha_2}{\log |c_n|})} \tag{35}$$

$$= \frac{k\alpha\eta + \beta}{\alpha\eta + \beta} \rho. \tag{36}$$

that is, $\rho_2^{**} = \frac{k\alpha\eta + \beta}{\alpha\eta + \beta} \rho$.

Similarly, we have $\chi_2^{**} = \frac{k\alpha\eta + \beta}{\alpha\eta + \beta} \chi$. □

Remark 3.2 *From Theorem 3.5, we can get that $\rho_2^{**} = \rho_1^{**}, \chi_2^{**} = \chi_1^{**}$ if $k = 1$.*

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The differential and subdifferential for fuzzy mappings based on the generalized difference of n -cell fuzzy-numbers

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Abstract We use the concept of generalized difference for fuzzy n -cell numbers which is presented in this paper to introduce and study the differential and gradient of fuzzy n -cell mappings. At the same time, some connections between gradient, *boundary-function-wise* gradient and *level-wise* gradient of fuzzy n -cell mappings are established. Furthermore, the subdifferential for fuzzy n -cell mappings based on the ordering \preceq_c is discussed.

Keywords: Fuzzy numbers; fuzzy n -cell mappings; gradient; subdifferential.

1. Introduction

Since the concept and operations of fuzzy set were introduced by Zadeh [1], enormous researchers have been dedicated on development of various aspects of the theory and applications of fuzzy sets. Soon after, Zadeh proposed the notion of fuzzy numbers in [2,3,4]. Since then, fuzzy numbers have been extensively investigated by many authors.

The importance of the derivative of a function in the study of mathematical programming and fuzzy differential equations is well-known. It is necessary to introduce a concept of differentiability for fuzzy mappings. Toward this end, in fuzzy analysis, there are a variety of notions of derivative for fuzzy mappings. The concept of fuzzy derivative first introduced by Chang and Zadeh [5] in 1972. Since then, numerous definitions of the differentiability of fuzzy mappings have been presented. In 1983, Puri and Ralescu [6] defined the derivative and G -derivative of fuzzy mappings from an open subset of a normed space into n -dimension fuzzy number space E^n by using embedding theorem (which shows how to isometrically embed E^n into a Banach space as a closed convex cone of vertex zero) and Hukuhara difference. In 1987, Kaleva [7] discussed the G -derivative, obtained a sufficient condition of the H -differentiability of the fuzzy mappings from $[a, b]$ into E^n and a necessary condition for the H -differentiability of fuzzy mapping from $[a, b]$ into E^1 . In 2003, Wang and Wu [8] put forward the concepts of directional derivative, differential and sub-differential of fuzzy mappings from R^n into E^1 by using Hukuhara difference. However, the usual Hukuhara difference between two fuzzy numbers exists only under very restrictive conditions [7] and the H -difference of two fuzzy numbers does not always exist [9]. The g -difference proposed in [9] overcomes these shortcomings of the above discussed concepts and the g -difference of two fuzzy numbers always exists. Based on the novel generalizations of the Hukuhara difference for fuzzy sets, Bede [9] introduced and studied new generalized differentiability concepts for fuzzy valued functions in 2013, in particular, a new very general fuzzy differentiability concept was defined and studied, the so-called g -derivative, and it was shown that the g -derivative is the most general among all similar definitions.

Motivated both by [9] and the importance of the concept of differential for fuzzy analysis, the concept of differential and gradient for fuzzy n -cell mappings is introduced, which is based on the novel generalizations difference of fuzzy n -cell numbers presented in this paper.

The remainder of the paper is organised as follows: First of all, we give the preliminary terminology used in the present paper. And then, in Section 3, we present the concept of generalized difference for fuzzy n -cell numbers and discuss several properties for it. We use the generalized difference for fuzzy n -cell numbers to introduce and study differential and gradient for fuzzy n -cell mappings in Section 4. At

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last, using the concept of the ordering \preceq_c for fuzzy n -cell numbers, section 5 deals with the subdifferential for fuzzy n -cell mappings.

2. Preliminaries

Throughout this paper, R^n denote the n -dimensional Euclidean space and $F(R^n)$ denote the set of all fuzzy subsets on R^n . A fuzzy subset of (in short, a fuzzy set) R^n is a function $u : R^n \rightarrow [0, 1]$. For each fuzzy sets u , we denote by $[u]^r = \{x \in R^n : u(x) \geq r\}$, for any $r \in (0, 1]$, its r -level set. By $\text{supp}u = \{x \in R^n : u(x) > 0\}$ we represent the support of u . Suppose $u \in F(R^n)$, satisfies the following conditions:

- (1) u is a normal fuzzy set, i.e., there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,
- (2) u is a convex fuzzy set, i.e., $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n$ and $\lambda \in [0, 1]$,
- (3) u is upper semicontinuous ,
- (4) $[u]^0 = \{x \in R^n : u(x) > 0\} = \overline{\bigcup_{r \in (0,1]} [u]^r}$ is compact, here \overline{A} denotes the closure of A .

Then u is called a fuzzy number. We use E^n to denote the fuzzy number space [10,11,12,13].

It is clear that each $u \in R^n$ can be considered as a fuzzy number u defined by

$$\tilde{u}(x) = \begin{cases} 1, & x = u, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, the fuzzy number $\tilde{0}$ is defined as $\tilde{0}(x) = 1$ if $x = 0$, and $\tilde{0}(x) = 0$ otherwise.

Definition 2.1. [14] If $u \in E^n$, and $[u]^r$ is a cell, i.e., for any $r \in [0, 1]$,

$$[u]^r = \prod_{i=1}^n [u_i^-(r), u_i^+(r)] = [u_1^-(r), u_1^+(r)] \times [u_2^-(r), u_2^+(r)] \times \cdots \times [u_n^-(r), u_n^+(r)],$$

where $u_i^-(r), u_i^+(r) \in R$ with $u_i^-(r) \leq u_i^+(r)$ ($i = 1, 2, \dots, n$), then we call u a fuzzy n -cell number. Denote the collection of all fuzzy n -cell numbers by $L(E^n)$.

For any $r \in [0, 1]$, $l_i[u]^r = u_i^+(r) - u_i^-(r)$ ($i = 1, 2, \dots, n$) is called the r -level length of a fuzzy n -cell number u with respect to the i th component.

Theorem 2.1. [14] (Representation theorem). If $u \in L(E^n)$, then for $i = 1, 2, \dots, n$, $u_i^-(r), u_i^+(r)$ are real-valued functions on $[0, 1]$, and satisfy

- (1) $u_i^-(r)$ are non-decreasing, left continuous at $r \in (0, 1]$ and right continuous at $r = 0$,
- (2) $u_i^+(r)$ are non-increasing, left continuous at $r \in (0, 1]$ and right continuous at $r = 0$,
- (3) $u_i^-(r) \leq u_i^+(r)$ (it is equivalent to $u_i^-(1) \leq u_i^+(1)$).

Conversely if $a_i(r), b_i(r)$ ($i = 1, 2, \dots, n$) are real-valued functions on $[0, 1]$ which satisfy conditions (1)-(3), then there exists a unique $u \in L(E^n)$ such that $[u]^r = \prod_{i=1}^n [a_i(r), b_i(r)]$ for any $r \in [0, 1]$.

Theorem 2.2. [14] Let $u, v \in L(E^n)$ and $k \in R$. Then for any $r \in [0, 1]$,

- (1) $[u + v]^r = [u]^r + [v]^r = \prod_{i=1}^n [u_i^-(r) + v_i^-(r), u_i^+(r) + v_i^+(r)]$,
- (2) $[ku]^r = k[u]^r = \begin{cases} \prod_{i=1}^n [ku_i^-(r), ku_i^+(r)], & k \geq 0, \\ \prod_{i=1}^n [ku_i^+(r), ku_i^-(r)], & k < 0, \end{cases}$
- (3) $[uv]^r = \prod_{i=1}^n [\min\{u_i^-(r)v_i^-(r), u_i^-(r)v_i^+(r), u_i^+(r)v_i^-(r), u_i^+(r)v_i^+(r)\}, \max\{u_i^-(r)v_i^-(r), u_i^-(r)v_i^+(r), u_i^+(r)v_i^-(r), u_i^+(r)v_i^+(r)\}]$.

Given $u, v \in L(E^n)$, the distance $D : L(E^n) \times L(E^n) \rightarrow [0, +\infty)$ between u and v is defined by the equation

$$\begin{aligned} D(u, v) &= \sup_{r \in [0,1]} d([u]^r, [v]^r) \\ &= \sup_{r \in [0,1]} \max_{1 \leq i \leq n} \{|u_i^-(r) - v_i^-(r)|, |u_i^+(r) - v_i^+(r)|\}. \end{aligned}$$

Then $(L(E^n), D)$ is a complete metric space, and satisfies $D(u + w, v + w) = D(u, v)$, $D(ku, kv) = |k|D(u, v)$ for any $u, v, w \in L(E^n)$, $k \in R$.

In recent years, several authors have discussed different ordering relation of fuzzy numbers [15]. To the best of our knowledge, very few investigations have been appeared to study ordering relation of fuzzy n -cell numbers. For this reason, an ordering \preceq_c of fuzzy n -cell numbers will be introduced and be applied to solve fuzzy constrained minimization problem.

Definition 2.2. Let $\tau : L(E^n) \rightarrow R^n$ be a vector-valued function defined by

$$\begin{aligned} \tau(u) &= (2 \int_0^1 r \frac{\int_{[u]^r} x_1 dx_1 dx_2 \dots dx_n}{\int_{[u]^r} 1 dx_1 dx_2 \dots dx_n} dr, 2 \int_0^1 r \frac{\int_{[u]^r} x_2 dx_1 dx_2 \dots dx_n}{\int_{[u]^r} 1 dx_1 dx_2 \dots dx_n} dr, \dots, 2 \int_0^1 r \frac{\int_{[u]^r} x_n dx_1 dx_2 \dots dx_n}{\int_{[u]^r} 1 dx_1 dx_2 \dots dx_n} dr) \\ &= (\int_0^1 r(u_1^+(r) + u_1^-(r))dr, \int_0^1 r(u_2^+(r) + u_2^-(r))dr, \dots, \int_0^1 r(u_n^+(r) + u_n^-(r))dr), \end{aligned}$$

where $\int_0^1 r \frac{\int_{[u]^r} x_i dx_1 dx_2 \dots dx_n}{\int_{[u]^r} 1 dx_1 dx_2 \dots dx_n} dr$ ($i = 1, 2, \dots, n$) is the Lebesgue integral of $r \frac{\int_{[u]^r} x_i dx_1 dx_2 \dots dx_n}{\int_{[u]^r} 1 dx_1 dx_2 \dots dx_n}$ ($i = 1, 2, \dots, n$) on $[0, 1]$. The vector-valued function τ is called a ranking value function defined on $L(E^n)$.

In this case $\tau(u)$ represents a centroid of the fuzzy n -cell number u . From the ranking value function $\tau(u)$, we consider the following ordering relation \preceq_c on $L(E^n)$.

Definition 2.3. Let $u, v \in L(E^n)$, $C \subseteq R^n$ be a closed convex cone with $0 \in C$ and $C \neq R^n$. We say that $u \preceq_c v$ (u precedes v) if

$$\tau(v) \in \tau(u) + C \quad (\tau(v) - \tau(u) \in C).$$

Obviously the order relation \preceq_c is reflexive and transitive, and \preceq_c is a partial order relation on $L(E^n)$. If $u, v \in E^1$, $C = [0, +\infty) \subseteq R$, then Definition 2.3 coincides with Definition 2.5 of reference [15].

We say that $u \prec_c v$ if $u \preceq_c v$ and $\tau(u) \neq \tau(v)$. Sometimes we may write $v \succeq_c u$ (resp. $v \succ_c u$) instead of $u \preceq_c v$ (resp. $u \prec_c v$).

Remark 2.1. Let $u, v \in L(E^n)$, $k_1, k_2 \in R$. According to Theorem 2.2 and Definition 2.2, it is easy to verify that $\tau(k_1 u + k_2 v) = k_1 \tau(u) + k_2 \tau(v)$.

Theorem 2.3. Let $u_1, u_2, v_1, v_2 \in L(E^n)$, $k_1, k_2 \in [0, +\infty]$, $C \subseteq R^n$ be a closed convex cone with $0 \in C$ and $C \neq R^n$. If $u_1 \preceq_c v_1$ and $u_2 \preceq_c v_2$, then $k_1 u_1 + k_2 u_2 \preceq_c k_1 v_1 + k_2 v_2$.

Proof. It follows from Definition 2.3 that $\tau(v_1) - \tau(u_1) \in C$ and $\tau(v_2) - \tau(u_2) \in C$. On the other hand, closed convex cone C is closed under addition and positive scalar multiplication, thus

$$k_1(\tau(v_1) - \tau(u_1)) + k_2(\tau(v_2) - \tau(u_2)) \in C,$$

which implies that $k_1 \tau(v_1) + k_2 \tau(v_2) \in k_1 \tau(u_1) + k_2 \tau(u_2) + C$. It is obvious from Remark 2.1 that $\tau(k_1 v_1 + k_2 v_2) \in \tau(k_1 u_1 + k_2 u_2) + C$, then $k_1 u_1 + k_2 u_2 \preceq_c k_1 v_1 + k_2 v_2$.

3. Generalized difference for fuzzy n -cell numbers

We denote by \mathcal{K}_C^n the family of all nonempty compact convex subsets of R^n , that is $\{\mathcal{K}_C^n \subset R^n : A \neq \emptyset \text{ is compact and convex}\}$. Stefanini [16] defined the generalized Hukuhara difference of two sets $A \in \mathcal{K}_C^n$ and $B \in \mathcal{K}_C^n$ as follows:

$$A \ominus_{gH} B = C \iff \begin{cases} (1) A = B + C, \\ \text{or } (2) B = A + (-1)C. \end{cases}$$

For any $A \in \mathcal{K}_C^n$ and $B \in \mathcal{K}_C^n$, if the generalized Hukuhara difference $C = A \ominus_{gH} B$ exists, it is unique.

Lemma 3.1. [16] Let $A = \prod_{i=1}^n A_i$, $B = \prod_{i=1}^n B_i$, where $A_i = [a_i^-, a_i^+]$ and $B_i = [b_i^-, b_i^+]$ are real compact intervals ($\prod_{i=1}^n$ denotes the cartesian product). If $A \ominus_{gH} B$ exists, then

$$A \ominus_{gH} B = \prod_{i=1}^n (A_i \ominus_{gH} B_i) = \prod_{i=1}^n [\min\{a_i^- - b_i^-, a_i^+ - b_i^+\}, \max\{a_i^- - b_i^-, a_i^+ - b_i^+\}].$$

Lemma 3.2. [16] The gH -difference $A \ominus_{gH} B$ exists if and only if one of the two conditions is satisfied:

(i) $a_i^- - b_i^- \leq a_i^+ - b_i^+, i = 1, 2, \dots, n$

or

(ii) $a_i^- - b_i^- \geq a_i^+ - b_i^+, i = 1, 2, \dots, n.$

According to Lemma 3.2, the definition of generalized Hukuhara difference for real compact intervals is extended to the fuzzy case.

Definition 3.1. Let $u, v \in L(E^n)$. If $l_i[u]^r \leq l_i[v]^r$ or $l_i[u]^r \geq l_i[v]^r$ for any $r \in [0, 1]$ and $i = 1, 2, \dots, n$, then the generalized difference (g -difference for short) is given by its level sets as

$$[u \ominus_g v]^r = \prod_{i=1}^n [\inf_{\beta \geq r} \min\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\}, \sup_{\beta \geq r} \max\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\}],$$

where $\beta \in [r, 1]$.

Remark 3.1. If $u, v \in E^1$, we have

$$[u \ominus_g v]^r = [\inf_{\beta \geq r} \min\{u^-(\beta) - v^-(\beta), u^+(\beta) - v^+(\beta)\}, \sup_{\beta \geq r} \max\{u^-(\beta) - v^-(\beta), u^+(\beta) - v^+(\beta)\}],$$

which coincides with Definition 7 of reference [9].

Theorem 3.1. Let $u, v \in L(E^n)$. If $l_i[u]^r \leq l_i[v]^r$ or $l_i[u]^r \geq l_i[v]^r$ for any $r \in [0, 1]$ and $i = 1, 2, \dots, n$, then the g -difference $u \ominus_g v$ exists and $u \ominus_g v \in L(E^n)$.

Proof. Assume that

$$\begin{aligned} [w]^r &= [u \ominus_g v]^r \\ &= \prod_{i=1}^n [\inf_{\beta \geq r} \min\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\}, \\ &\quad \sup_{\beta \geq r} \max\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\}], \end{aligned}$$

for any $r \in [0, 1]$. We can prove that the class of sets $[w]^r$ determines a fuzzy n -cell number.

For any $r \in [0, 1]$, we have

$$\begin{aligned} w_i^-(r) &= \inf_{\beta \geq r} \min\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\} \\ &\leq \sup_{\beta \geq r} \max\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\} \\ &= w_i^+(r). \end{aligned}$$

It can be easily seen that $w_i^-(r)$ are non-decreasing while $w_i^+(r)$ are non-increasing, $w_i^-(r)$ and $w_i^+(r)$ are left continuous on $(0, 1]$ and right continuous at 0. It follows from Theorem 2.1 that the g -difference $u \ominus_g v$ exists and $u \ominus_g v = w \in L(E^n)$.

From now on, throughout this paper, we will assume that the g -difference $u \ominus_g v$ for any fuzzy n -cell numbers u and v exists.

Theorem 3.2. For any $u, v, w \in L(E^n)$, we have

- (1) $u \ominus_g u = \tilde{0}$, $u \ominus_g \tilde{0} = u$, $\tilde{0} \ominus_g u = -u$,
- (2) $u \ominus_g v = -(v \ominus_g u)$,
- (3) $(u + v) \ominus_g (u + w) = v \ominus_g w$,
- (4) $k(u \ominus_g v) = ku \ominus_g kv$, $k \in R$,
- (5) $(u + v) \ominus_g v = u$,
- (6) $\tilde{0} \ominus_g (u \ominus_g v) = v \ominus_g u = (-u) \ominus_g (-v)$,
- (7) $u \ominus_g v = v \ominus_g u = w$ if and only if $w = -w$, furthermore, $w = \tilde{0}$ if and only if $u = v$.

Proof. The proof of (1), (3) and (4) are immediate.

(2) According to Definition 3.1, we have

$$\begin{aligned} &-[v \ominus_g u]^r \\ &= -\prod_{i=1}^n [\inf_{\beta \geq r} \min\{v_i^-(\beta) - u_i^-(\beta), v_i^+(\beta) - u_i^+(\beta)\}, \\ &\quad \sup_{\beta \geq r} \max\{v_i^-(\beta) - u_i^-(\beta), v_i^+(\beta) - u_i^+(\beta)\}] \\ &= \prod_{i=1}^n [-\sup_{\beta \geq r} \max\{v_i^-(\beta) - u_i^-(\beta), v_i^+(\beta) - u_i^+(\beta)\}, \\ &\quad -\inf_{\beta \geq r} \min\{v_i^-(\beta) - u_i^-(\beta), v_i^+(\beta) - u_i^+(\beta)\}] \\ &= \prod_{i=1}^n [-\sup_{\beta \geq r} (-\min\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\}), \\ &\quad -\inf_{\beta \geq r} (-\max\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\})] \\ &= \prod_{i=1}^n [\inf_{\beta \geq r} \min\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\}, \\ &\quad \sup_{\beta \geq r} \max\{u_i^-(\beta) - v_i^-(\beta), u_i^+(\beta) - v_i^+(\beta)\}] \\ &= [u \ominus_g v]^r, \end{aligned}$$

for any $r \in [0, 1]$. It follows from Theorem 2.2 that $u \ominus_g v = -(v \ominus_g u)$.

(5) We have from Theorem 2.2 that

$$\begin{aligned} & [(u + v) \ominus_g v]^r \\ &= \prod_{i=1}^n [\inf_{\beta \geq r} \min\{(u_i^-(\beta) + v_i^-(\beta)) - v_i^-(\beta), (u_i^+(\beta) + v_i^+(\beta)) - v_i^+(\beta)\}, \\ & \quad \sup_{\beta \geq r} \max\{(u_i^-(\beta) + v_i^-(\beta)) - v_i^-(\beta), (u_i^+(\beta) + v_i^+(\beta)) - v_i^+(\beta)\}] \\ &= \prod_{i=1}^n [\inf_{\beta \geq r} \min\{u_i^-(\beta), u_i^+(\beta)\}, \sup_{\beta \geq r} \max\{u_i^-(\beta), u_i^+(\beta)\}] \\ &= \prod_{i=1}^n [\inf_{\beta \geq r} u_i^-(\beta), \sup_{\beta \geq r} u_i^+(\beta)] \\ &= \prod_{i=1}^n [u_i^-(r), u_i^+(r)] \\ &= [u]^r, \end{aligned}$$

for any $r \in [0, 1]$. Then $(u + v) \ominus_g v = u$.

(6) It follows from (1), (2) and (3) that the proof of (6) is immediate.

(7) We have from (2) that the proof of (7) is immediate.

4. The differential and gradient for fuzzy n -cell mappings

In this work, let M be a convex set of m -dimensional Euclidean space R^m . We consider mappings F from M into $L(E^n)$. Such a mapping is called a fuzzy n -cell mapping. For the sake of brevity, F is called a fuzzy mapping. Let $\tilde{F} : M \rightarrow L(E^n)$, for any $r \in [0, 1]$, we denote $F_r(t) = \prod_{i=1}^n [F_i^-(r, t), F_i^+(r, t)]$.

Definition 4.1. Let $\tilde{F} : M \rightarrow L(E^n)$, $t_0 = (t_1^0, t_2^0, \dots, t_m^0) \in \text{int}M$, $t = (t_1, t_2, \dots, t_m) \in \text{int}M$. If g -difference $\tilde{F}(t) \ominus_g \tilde{F}(t_0)$ exists and there exist $u_j \in L(E^n)$ ($j = 1, 2, \dots, m$), such that

$$\lim_{t \rightarrow t_0} \frac{D(\tilde{F}(t) \ominus_g \tilde{F}(t_0), \sum_{j=1}^m u_j(t_j - t_j^0))}{d(t, t_0)} = 0,$$

then we say that \tilde{F} is differentiable at t_0 and (u_1, u_2, \dots, u_m) is the gradient of \tilde{F} at t_0 , denoted by $\nabla \tilde{F}(t_0)$, i.e., $\nabla \tilde{F}(t_0) = (u_1, u_2, \dots, u_m)$.

Remark 4.1. Let $\tilde{F} : M \rightarrow L(E^n)$, $t_0 \in M$. Then the gradient $\nabla \tilde{F}(t_0)$ exists at t_0 if and only if $\tilde{F}(t) \ominus_g \tilde{F}(t_0)$ exists and there are $u_j \in L(E^n)$ ($j = 1, 2, \dots, m$), such that

$$u_j = \lim_{h \rightarrow 0} \frac{\tilde{F}(t_1^0, \dots, t_j^0 + h, \dots, t_m^0) \ominus_g \tilde{F}(t_1^0, \dots, t_j^0, \dots, t_m^0)}{h},$$

where $h \in R$ and $t = (t_1^0, \dots, t_j^0 + h, \dots, t_m^0) \in \text{int}M$.

Here the limit is taken in the metric space $(L(E^n), D)$.

Definition 4.2. Let $\tilde{F} : M \rightarrow L(E^n)$, $t_0 = (t_1^0, t_2^0, \dots, t_m^0) \in \text{int}M$, $t = (t_1, t_2, \dots, t_m) \in \text{int}M$. If there exists $u_{ij}^-(r), u_{ij}^+(r) \in R$ ($i = 1, 2, \dots, n, j = 1, 2, \dots, m$), such that

$$\lim_{t \rightarrow t_0} \frac{|F_i^-(r, t) - F_i^-(r, t_0) - \sum_{j=1}^m u_{ij}^-(r)(t_j - t_j^0)|}{d(t, t_0)} = 0 \quad (i = 1, 2, \dots, n),$$

and

$$\lim_{t \rightarrow t_0} \frac{|F_i^+(r, t) - F_i^+(r, t_0) - \sum_{j=1}^m u_{ij}^+(r)(t_j - t_j^0)|}{d(t, t_0)} = 0 \quad (i = 1, 2, \dots, n),$$

uniformly for $r \in [0, 1]$, then we say that \tilde{F} is *boundary-function-wise* differentiable (*b-differentiable* for short) at t_0 .

Theorem 4.1. Let $\tilde{F} : M \rightarrow L(E^n)$ be a fuzzy mapping. If \tilde{F} is *b-differentiable* at $t_0 = (t_1^0, t_2^0, \dots, t_m^0) \in \text{int}M$, then there exist $u_j \in L(E^n)$, such that for any $r \in [0, 1]$,

$$[u_j]^r = \prod_{i=1}^n [\inf_{\beta \geq r} \min\{u_{ij}^-(\beta), u_{ij}^+(\beta)\}, \sup_{\beta \geq r} \max\{u_{ij}^-(\beta), u_{ij}^+(\beta)\}] \quad (j = 1, 2, \dots, m).$$

Proof. For any $r \in [0, 1]$, we can show that the class of sets

$$\prod_{i=1}^n [\inf_{\beta \geq r} \min\{u_{ij}^-(\beta), u_{ij}^+(\beta)\}, \sup_{\beta \geq r} \max\{u_{ij}^-(\beta), u_{ij}^+(\beta)\}] \quad (j = 1, 2, \dots, m)$$

satisfies the conditions of Theorem 2.1. According to Definition 4.2, if there exists $\delta > 0$, such that for any $|h| < \delta$ with $t = (t_1^0, \dots, t_j^0 + h, \dots, t_m^0) \in \text{int}M$, we have

$$u_{ij}^-(r) = \lim_{h \rightarrow 0} \frac{F_i^-(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h},$$

$$u_{ij}^+(r) = \lim_{h \rightarrow 0} \frac{F_i^+(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h},$$

for all $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$. Since $F_i^-(r, t)$ and $F_i^+(r, t)$ are left continuous with respect $r \in (0, 1]$ and right continuous at $r = 0$,

$$\frac{F_i^-(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}$$

and

$$\frac{F_i^+(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}$$

are left continuous at $r \in (0, 1]$ and right continuous at $r = 0$. Thus for any $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$,

$$\inf_{\beta \geq r} \min\left\{ \frac{F_i^-(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}, \right. \\ \left. \frac{F_i^+(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h} \right\},$$

$$\sup_{\beta \geq r} \max\left\{ \frac{F_i^-(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}, \right. \\ \left. \frac{F_i^+(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h} \right\}$$

are left continuous at $r \in (0, 1]$ and right continuous at $r = 0$. Therefore, for any $r \in [0, 1]$, $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$, we have

(1) $\inf_{\beta \geq r} \min\{u_{ij}^-(\beta), u_{ij}^+(\beta)\}$ are non-decreasing and left continuous at $r \in (0, 1]$ and right continuous at $r = 0$,

(2) $\sup_{\beta \geq r} \max\{u_{ij}^-(\beta), u_{ij}^+(\beta)\}$ are non-increasing and left continuous at $r \in (0, 1]$ and right continuous at $r = 0$,

(3) $\inf_{\beta \geq r} \min\{u_{ij}^-(\beta), u_{ij}^+(\beta)\} \leq \sup_{\beta \geq r} \max\{u_{ij}^-(\beta), u_{ij}^+(\beta)\}$.

Consequently, there exist $u_j \in L(E^n)$ ($j = 1, 2, \dots, m$), such that for any $r \in [0, 1]$,

$$[u_j]^r = \prod_{i=1}^n [\inf_{\beta \geq r} \min\{u_{ij}^-(\beta), u_{ij}^+(\beta)\}, \sup_{\beta \geq r} \max\{u_{ij}^-(\beta), u_{ij}^+(\beta)\}] \quad (j = 1, 2, \dots, m).$$

Definition 4.3. Let $\tilde{F} : M \rightarrow L(E^n)$ is b -differentiable at t_0 . For any $r \in [0, 1]$, we denote

$$[u_j]^r = \prod_{i=1}^n [\inf_{\beta \geq r} \min\{u_{ij}^-(\beta), u_{ij}^+(\beta)\}, \sup_{\beta \geq r} \max\{u_{ij}^-(\beta), u_{ij}^+(\beta)\}] \quad (j = 1, 2, \dots, m),$$

then we say that (u_1, u_2, \dots, u_m) is the *boundary-function-wise gradient* (b -gradient for short) of \tilde{F} at t_0 , denoted by $\nabla_b \tilde{F}(t_0)$, i.e.,

$$\nabla_b \tilde{F}(t_0) = (u_1, u_2, \dots, u_m).$$

Remark 4.2. Let $\tilde{F} : M \rightarrow L(E^n)$, $t_0 \in M$. Then the b -gradient $\nabla_b \tilde{F}(t_0)$ exists at t_0 if and only if there are $u_{ij}^-(r)$, $u_{ij}^+(r) \in R$ ($i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$), such that

$$u_{ij}^-(r) = \lim_{h \rightarrow 0} \frac{F_i^-(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}$$

and

$$u_{ij}^+(r) = \lim_{h \rightarrow 0} \frac{F_i^+(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}$$

uniformly for $r \in [0, 1]$, where $h \in R$ with $t = (t_1^0, \dots, t_j^0 + h, \dots, t_m^0) \in \text{int}M$ and

$$[u_j]^r = \prod_{i=1}^n [\inf_{\beta \geq r} \min\{u_{ij}^-(\beta), u_{ij}^+(\beta)\}, \sup_{\beta \geq r} \max\{u_{ij}^-(\beta), u_{ij}^+(\beta)\}] \quad (j = 1, 2, \dots, m).$$

Theorem 4.2. Let the b -gradient $\nabla_b \tilde{F}(t_0)$ of fuzzy mapping $\tilde{F} : M \rightarrow L(E^n)$ be exist at $t_0 \in \text{int}M$. If $\tilde{F}(t) \ominus_g \tilde{F}(t_0)$ exists, then the gradient $\nabla \tilde{F}(t_0)$ of \tilde{F} at t_0 exists and we have $u_j = v_j$ ($j = 1, 2, \dots, m$), where $\nabla \tilde{F}(t_0) = (u_1, u_2, \dots, u_m)$, $\nabla_b \tilde{F}(t_0) = (v_1, v_2, \dots, v_m)$.

Proof. Let $t_0 = (t_1^0, \dots, t_j^0, \dots, t_m^0) \in \text{int}M$, $h \in R$ and $t = (t_1^0, \dots, t_j^0 + h, \dots, t_m^0) \in \text{int}M$. According to Theorem 2.2 and Definition 3.1, for any $r \in [0, 1]$, we have

$$\begin{aligned} & \left[\frac{\tilde{F}(t) \ominus_g \tilde{F}(t_0)}{h} \right]^r \\ &= \frac{1}{h} [\tilde{F}(t) \ominus_g \tilde{F}(t_0)]^r \\ &= \frac{1}{h} \prod_{i=1}^n [\inf_{\beta \geq r} \min\{F_i^-(\beta, t) - F_i^-(\beta, t_0), F_i^+(\beta, t) - F_i^+(\beta, t_0)\}, \\ & \quad \sup_{\beta \geq r} \max\{F_i^-(\beta, t) - F_i^-(\beta, t_0), F_i^+(\beta, t) - F_i^+(\beta, t_0)\}] \\ &= \prod_{i=1}^n [\inf_{\beta \geq r} \min\left\{ \frac{F_i^-(\beta, t) - F_i^-(\beta, t_0)}{h}, \frac{F_i^+(\beta, t) - F_i^+(\beta, t_0)}{h} \right\}, \\ & \quad \sup_{\beta \geq r} \max\left\{ \frac{F_i^-(\beta, t) - F_i^-(\beta, t_0)}{h}, \frac{F_i^+(\beta, t) - F_i^+(\beta, t_0)}{h} \right\}]. \end{aligned}$$

Because the b -gradient $\nabla_b \tilde{F}(t_0)$ of \tilde{F} be exist at $t_0 \in \text{int}M$, for any $r \in [0, 1]$, we have

$$\begin{aligned} & \lim_{h \rightarrow 0} \left[\frac{\tilde{F}(t_1^0, \dots, t_j^0 + h, \dots, t_m^0) \ominus_g \tilde{F}(t_1^0, \dots, t_j^0, \dots, t_m^0)}{h} \right]^r \\ &= \lim_{h \rightarrow 0} \prod_{i=1}^n [\inf_{\beta \geq r} \min\left\{ \frac{F_i^-(\beta, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(\beta, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}, \right. \\ & \quad \left. \frac{F_i^+(\beta, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(\beta, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h} \right\}, \\ & \quad \sup_{\beta \geq r} \max\left\{ \frac{F_i^-(\beta, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(\beta, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}, \right. \\ & \quad \left. \frac{F_i^+(\beta, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(\beta, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h} \right\}] \\ &= \prod_{i=1}^n [\inf_{\beta \geq r} \min\left\{ \lim_{h \rightarrow 0} \frac{F_i^-(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}, \right. \\ & \quad \left. \lim_{h \rightarrow 0} \frac{F_i^+(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h} \right\}, \\ & \quad \sup_{\beta \geq r} \max\left\{ \lim_{h \rightarrow 0} \frac{F_i^-(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}, \right. \\ & \quad \left. \lim_{h \rightarrow 0} \frac{F_i^+(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h} \right\}] \\ &= \prod_{i=1}^n [\inf_{\beta \geq r} \min\{u_{ij}^-(\beta), u_{ij}^+(\beta)\}, \sup_{\beta \geq r} \max\{u_{ij}^-(\beta), u_{ij}^+(\beta)\}] \\ &= [v_j]^r. \end{aligned}$$

Therefore, there exist $u_j = v_j \in L(E^n)$, such that

$$\lim_{h \rightarrow 0} \frac{\tilde{F}(t_1^0, \dots, t_j^0 + h, \dots, t_m^0) \ominus_g \tilde{F}(t_1^0, \dots, t_j^0, \dots, t_m^0)}{h} = u_j \quad (j = 1, 2, \dots, m),$$

which implies that the gradient $\nabla \tilde{F}(t_0)$ of \tilde{F} at t_0 exists and $u_j = v_j$ ($j = 1, 2, \dots, m$).

Definition 4.4. Let $\tilde{F} : M \rightarrow L(E^n)$, $t_0 = (t_1^0, t_2^0, \dots, t_m^0) \in \text{int}M$ and $t = (t_1, t_2, \dots, t_m) \in \text{int}M$. If the gH -difference $F_r(t) \ominus_{gH} F_r(t_0)$ exist for all $r \in [0, 1]$, and there exist $[u_j]^r = \prod_{i=1}^n [u_{ij}^-(r), u_{ij}^+(r)] \subseteq R^n$ ($j = 1, 2, \dots, m$), such that

$$\lim_{t \rightarrow t_0} \frac{d(F_r(t) \ominus_{gH} F_r(t_0), \sum_{j=1}^m [u_j]^r (t_j - t_j^0))}{d(t, t_0)} = 0$$

uniformly for $r \in [0, 1]$, then we say that \tilde{F} is *level-wise* differentiable (l -differentiable for short) at t_0 .

Theorem 4.3. Let $\tilde{F} : M \rightarrow L(E^n)$ be a fuzzy mapping. If \tilde{F} is l -differentiable at $t_0 = (t_1^0, t_2^0, \dots, t_m^0) \in \text{int}M$, then there exist $u_j \in L(E^n)$, such that for any $r \in [0, 1]$,

$$[v_j]^r = \prod_{i=1}^n [\inf_{\beta \geq r} u_{ij}^-(\beta), \sup_{\beta \geq r} u_{ij}^+(\beta)] \quad (j = 1, 2, \dots, m).$$

Proof. For any $r \in [0, 1]$, We can show that the class of sets

$$\prod_{i=1}^n [\inf_{\beta \geq r} u_{ij}^-(\beta), \sup_{\beta \geq r} u_{ij}^+(\beta)] \quad (j = 1, 2, \dots, m)$$

satisfies the conditions of Theorem 2.1. According to Lemma 3.1 and Definition 4.4, if there exists $\delta > 0$, such that for any $|h| < \delta$ with $t = (t_1^0, \dots, t_j^0 + h, \dots, t_m^0) \in \text{int}M$, we have

$$\begin{aligned} u_{ij}^-(r) &= \min \left\{ \lim_{h \rightarrow 0} \frac{F_i^-(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}, \right. \\ &\quad \left. \lim_{h \rightarrow 0} \frac{F_i^+(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h} \right\}, \\ u_{ij}^+(r) &= \max \left\{ \lim_{h \rightarrow 0} \frac{F_i^-(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}, \right. \\ &\quad \left. \lim_{h \rightarrow 0} \frac{F_i^+(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h} \right\}, \end{aligned}$$

for all $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$. Since $F_i^-(r, t)$ and $F_i^+(r, t)$ are left continuous with respect $r \in (0, 1]$ and right continuous at $r = 0$,

$$\frac{F_i^-(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}$$

and

$$\frac{F_i^+(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}$$

are left continuous at $r \in (0, 1]$ and right continuous at $r = 0$. Thus, for any $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$,

$$\inf_{\beta \geq r} \min \left\{ \frac{F_i^-(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}, \frac{F_i^+(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h} \right\},$$

and

$$\sup_{\beta \geq r} \max \left\{ \frac{F_i^-(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}, \frac{F_i^+(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h} \right\}$$

are left continuous at $r \in (0, 1]$ and right continuous at $r = 0$. Therefore, for any $r \in [0, 1]$, $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$, we have

$$(1) \quad \inf_{\beta \geq r} \min \left\{ \lim_{h \rightarrow 0} \frac{F_i^-(\beta, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(\beta, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}, \right. \\ \left. \lim_{h \rightarrow 0} \frac{F_i^+(\beta, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(\beta, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h} \right\}$$

are non-decreasing and left continuous at $r \in (0, 1]$ and right continuous at $r = 0$,

$$(2) \quad \sup_{\beta \geq r} \max \left\{ \lim_{h \rightarrow 0} \frac{F_i^-(\beta, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(\beta, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}, \right. \\ \left. \lim_{h \rightarrow 0} \frac{F_i^+(\beta, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(\beta, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h} \right\}$$

are non-increasing and left continuous at $r \in (0, 1]$ and right continuous at $r = 0$,

$$(3) \quad \inf_{\beta \geq r} \min \left\{ \lim_{h \rightarrow 0} \frac{F_i^-(\beta, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(\beta, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}, \right. \\ \left. \lim_{h \rightarrow 0} \frac{F_i^+(\beta, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(\beta, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h} \right\} \\ \leq \sup_{\beta \geq r} \max \left\{ \lim_{h \rightarrow 0} \frac{F_i^-(\beta, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(\beta, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}, \right. \\ \left. \lim_{h \rightarrow 0} \frac{F_i^+(\beta, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(\beta, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h} \right\}.$$

Consequently, there exist $v_j \in L(E^n)$ ($j = 1, 2, \dots, m$), such that for any $r \in [0, 1]$,

$$[v_j]^r = \prod_{i=1}^n \left[\inf_{\beta \geq r} u_{ij}^-(\beta), \sup_{\beta \geq r} u_{ij}^+(\beta) \right] \quad (j = 1, 2, \dots, m).$$

Definition 4.5. Let $\tilde{F} : M \rightarrow L(E^n)$ is l -differentiable at t_0 , for any $r \in [0, 1]$, we denote

$$[v_j]^r = \prod_{i=1}^n \left[\inf_{\beta \geq r} u_{ij}^-(\beta), \sup_{\beta \geq r} u_{ij}^+(\beta) \right] \quad (j = 1, 2, \dots, m),$$

then we say that (v_1, v_2, \dots, v_m) is the *level-wise gradient* (l -gradient for short) of \tilde{F} at t_0 , denoted by $\nabla_l \tilde{F}(t_0)$, i.e.,

$$\nabla_l \tilde{F}(t_0) = (v_1, v_2, \dots, v_m).$$

Remark 4.3. Let $\tilde{F} : M \rightarrow L(E^n)$, $t_0 \in M$. Then the l -gradient $\nabla_l \tilde{F}(t_0)$ exists at t_0 if and only if $F_r(t) \ominus_{gH} F_r(t_0)$ exist and there are $[u_j]^r = \prod_{i=1}^n [u_{ij}^-(r), u_{ij}^+(r)] \subseteq R^n$ ($j = 1, 2, \dots, m$), such that

$$u_{ij}^-(r) = \min \left\{ \lim_{h \rightarrow 0} \frac{F_i^-(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}, \right. \\ \left. \lim_{h \rightarrow 0} \frac{F_i^+(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h} \right\}$$

and

$$u_{ij}^+(r) = \max \left\{ \lim_{h \rightarrow 0} \frac{F_i^-(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}, \right. \\ \left. \lim_{h \rightarrow 0} \frac{F_i^+(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h} \right\}$$

uniformly for $r \in [0, 1]$, where $h \in R$ with $t = (t_1^0, \dots, t_j^0 + h, \dots, t_m^0) \in \text{int}M$ and

$$[u_j]^r = \prod_{i=1}^n \left[\inf_{\beta \geq r} \min \{u_{ij}^-(\beta), u_{ij}^+(\beta)\}, \sup_{\beta \geq r} \max \{u_{ij}^-(\beta), u_{ij}^+(\beta)\} \right] \quad (j = 1, 2, \dots, m).$$

Theorem 4.4. Let the l -gradient $\nabla_l \tilde{F}(t_0)$ of fuzzy mapping $\tilde{F} : M \rightarrow L(E^n)$ be exist at $t_0 \in \text{int}M$. If $\tilde{F}(t) \ominus_g \tilde{F}(t_0)$ exists, then the gradient $\nabla \tilde{F}(t_0)$ of \tilde{F} at t_0 exists and we have $u_j = v_j$ ($j = 1, 2, \dots, m$), where $\nabla \tilde{F}(t_0) = (u_1, u_2, \dots, u_m)$, $\nabla_l \tilde{F}(t_0) = (v_1, v_2, \dots, v_m)$.

Proof. Let $t_0 = (t_1^0, \dots, t_j^0, \dots, t_m^0) \in \text{int}M$, $h \in R$ and $t = (t_1^0, \dots, t_j^0 + h, \dots, t_m^0) \in \text{int}M$. We denote $[u_j]^r = \prod_{i=1}^n [\inf_{\beta \geq r} u_{ij}^-(\beta), \sup_{\beta \geq r} u_{ij}^+(\beta)]$, then $u_j \in L(E^n)$ and

$$u_{ij}^-(r) = \min \left\{ \lim_{h \rightarrow 0} \frac{F_i^-(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}, \right. \\ \left. \lim_{h \rightarrow 0} \frac{F_i^+(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h} \right\},$$

$$u_{ij}^+(r) = \max \left\{ \lim_{h \rightarrow 0} \frac{F_i^-(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}, \right. \\ \left. \lim_{h \rightarrow 0} \frac{F_i^+(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(r, t_1^0, \dots, t_j^0, \dots, t_m^0)}{h} \right\},$$

for all $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$. It follows from Theorem 2.2 and Lemma 3.1 that

$$D\left(\frac{\tilde{F}(t_1^0, \dots, t_j^0 + h, \dots, t_m^0) \ominus_g \tilde{F}(t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}, u_j\right) \\ = \sup_{r \in [0, 1]} d\left(\frac{1}{h} \prod_{i=1}^n [\inf_{\beta \geq r} \min\{F_i^-(\beta, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(\beta, t_1^0, \dots, t_j^0, \dots, t_m^0), \right. \\ \left. F_i^+(\beta, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(\beta, t_1^0, \dots, t_j^0, \dots, t_m^0)\}, \right. \\ \left. \sup_{\beta \geq r} \max\{F_i^-(\beta, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(\beta, t_1^0, \dots, t_j^0, \dots, t_m^0), \right. \\ \left. F_i^+(\beta, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(\beta, t_1^0, \dots, t_j^0, \dots, t_m^0)\}\right], \\ \prod_{i=1}^n [\inf_{\beta \geq r} u_{ij}^-(\beta), \sup_{\beta \geq r} u_{ij}^+(\beta)]) \\ \leq \sup_{r \in [0, 1]} d\left(\frac{1}{h} \prod_{i=1}^n [\min\{F_i^-(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(r, t_1^0, \dots, t_j^0, \dots, t_m^0), \right. \\ \left. F_i^+(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(r, t_1^0, \dots, t_j^0, \dots, t_m^0)\}, \right. \\ \left. \max\{F_i^-(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^-(r, t_1^0, \dots, t_j^0, \dots, t_m^0), \right. \\ \left. F_i^+(r, t_1^0, \dots, t_j^0 + h, \dots, t_m^0) - F_i^+(r, t_1^0, \dots, t_j^0, \dots, t_m^0)\}\right], \\ \prod_{i=1}^n [\inf_{\beta \geq r} u_{ij}^-(\beta), \sup_{\beta \geq r} u_{ij}^+(\beta)]) \\ = \sup_{r \in [0, 1]} d\left(\frac{F_r(t_1^0, \dots, t_j^0 + h, \dots, t_m^0) \ominus_{gH} F_r(t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}, [u_j]^r\right).$$

Because

$$\lim_{h \rightarrow 0} \frac{F_r(t_1^0, \dots, t_j^0 + h, \dots, t_m^0) \ominus_{gH} F_r(t_1^0, \dots, t_j^0, \dots, t_m^0)}{h} = [u_j]^r$$

uniformly for $r \in [0, 1]$, for any $\varepsilon > 0$, there exists $\delta > 0$, when $|h| < \delta$, we have

$$D\left(\frac{\tilde{F}(t_1^0, \dots, t_j^0 + h, \dots, t_m^0) \ominus_g \tilde{F}(t_1^0, \dots, t_j^0, \dots, t_m^0)}{h}, u_j\right) \leq \sup_{r \in [0, 1]} d\left(\frac{F_r(t_0 + h) \ominus_{gH} F_r(t_0)}{h}, [u_j]^r\right) < \varepsilon.$$

Therefore, the gradient $\nabla \tilde{F}(t_0)$ of \tilde{F} at t_0 exists and $\nabla \tilde{F}(t_0) = (u_1, u_2, \dots, u_m) = \nabla_l \tilde{F}(t_0)$.

5. The subdifferential for fuzzy n -cell mappings

In recent years, nonsmooth analysis has increasingly come to play a role in functional analysis, optimization, optimal design, differential equations and control theory. The subdifferential is an important tool, used widely in nonsmooth analysis and optimization, thus we will discuss subdifferential concept for fuzzy n -cell mappings based on the ordering \preceq_c .

Definition 5.1. [17] Let $\tilde{F} : M \rightarrow L(E^n)$ be a fuzzy n -cell mapping. \tilde{F} is said to be convex (c.) on M if

$$\tilde{F}(\lambda t + (1 - \lambda)t') \preceq_c \lambda \tilde{F}(t) + (1 - \lambda)\tilde{F}(t')$$

for any $t, t \in M$ and $\lambda \in [0, 1]$.

The convex fuzzy n -cell mappings in the following arguments are assumed to be comparable.

Definition 5.2. Let $\tilde{F} : M \rightarrow L(E^n)$ be a convex fuzzy n -cell mapping on M , $t_0 = (t_1^0, t_2^0, \dots, t_m^0) \in \text{int}M$, $t = (t_1^0, \dots, t_j^0 + h, \dots, t_m^0) \in \text{int}M$. if there exist $u_j \in L(E^n)$ ($j = 1, 2, \dots, m$), such that

$$\tilde{F}(t_1^0, \dots, t_j^0 + h, \dots, t_m^0) \ominus_g \tilde{F}(t_1^0, \dots, t_j^0, \dots, t_m^0) \succeq_C hu_j,$$

then we call (u_1, u_2, \dots, u_m) a subgradient of \tilde{F} at t_0 , and say the set of all subgradients of \tilde{F} at t_0 to be subdifferential of \tilde{F} at t_0 , denoted by $\partial\tilde{F}(t_0)$, i.e.,

$$\partial\tilde{F}(t_0) = \{(u_1, u_2, \dots, u_m) : \tilde{F}(t_1^0, \dots, t_j^0 + h, \dots, t_m^0) \ominus_g \tilde{F}(t_1^0, \dots, t_j^0, \dots, t_m^0) \succeq_C hu_j, u_j \in L(E^n)\}.$$

According to Theorem 2.3, it is easy to verify the following conclusion.

Theorem 5.1. Let $\tilde{F} : M \rightarrow L(E^n)$ be a convex fuzzy n -cell mapping on M . Then, we have

$$\partial(\alpha\tilde{F}(t) + \beta\tilde{G}(t)) = \alpha\partial\tilde{F}(t) + \beta\partial\tilde{G}(t),$$

for any $t \in \text{int}M$ and $\alpha, \beta \geq 0$.

Theorem 5.2. Let $\tilde{F} : M \rightarrow L(E^n)$ be a convex fuzzy n -cell mapping on M . Then the subdifferential $\partial\tilde{F}(t)$ is a convex set in $L(E^n)$.

Proof. For an empty subdifferential, the assertion is trivial. Take two arbitrary subgradients

$$(u_1, u_2, \dots, u_m), (v_1, v_2, \dots, v_m) \in \partial\tilde{F}(t_1, \dots, t_j, \dots, t_m).$$

When $(t_1, \dots, t_j + h, \dots, t_m) \in \text{int}M$, we have

$$\lambda(\tilde{F}(t_1, \dots, t_j + h, \dots, t_m) \ominus_g \tilde{F}(t_1, \dots, t_j, \dots, t_m)) \succeq_C \lambda hu_j,$$

$$(1 - \lambda)(\tilde{F}(t_1, \dots, t_j + h, \dots, t_m) \ominus_g \tilde{F}(t_1, \dots, t_j, \dots, t_m)) \succeq_C (1 - \lambda)hv_j,$$

for any $\lambda \in [0, 1]$. It follows from Theorem 2.3 that

$$\tilde{F}(t_1, \dots, t_j + h, \dots, t_m) \ominus_g \tilde{F}(t_1, \dots, t_j, \dots, t_m) \succeq_C h(\lambda u_j + (1 - \lambda)v_j)$$

for any $j = 1, 2, \dots, m$. Therefore,

$$\lambda(u_1, u_2, \dots, u_m) + (1 - \lambda)(v_1, v_2, \dots, v_m) \in \partial\tilde{F}(t),$$

which implies that the subdifferential $\partial\tilde{F}(t)$ is a convex set in E^n .

Next, We study the problems of minimizing and maximizing a convex fuzzy n -cell mapping and discuss the necessary and sufficient conditions for optimality.

Let $\tilde{F} : M \rightarrow L(E^n)$ be a fuzzy n -cell mapping. We consider an unconstrained fuzzy minimization problem (FMP):

$$\text{Minimize } \tilde{F}(t),$$

$$\text{Subject to } t \in \text{int}M.$$

A point $t_0 \in \text{int}M$ is called a feasible solution to the problem, if for no $t \in \text{int}M$ such that $\tilde{F}(t) \preceq_c \tilde{F}(t_0)$, then t_0 is called an optimal solution, or a global minimum point.

With the aid of definitions for subdifferential and optimal solution we can immediately present a necessary and sufficient optimality condition. This theorem is formulated without proof because it is an obvious consequence of the definition of the subdifferential.

Theorem 5.3. Let $\tilde{F} : M \rightarrow L(E^n)$ be a convex fuzzy n -cell mapping on M . Then t_0 is a global minimum point if and only if $(\tilde{0}, \tilde{0}, \dots, \tilde{0}) \in \partial\tilde{F}(t_0)$.

6. Conclusion

In this paper, we introduce the concept of generalized difference of n -cell fuzzy-numbers and an ordering relation on the fuzzy n -cell number space is considered. Using the generalized difference of n -cell fuzzy-numbers the generalized differential and gradient concepts for fuzzy n -cell mappings are discussed. Furthermore, we have used the ordering relation \preceq_c to obtain the subdifferential for fuzzy n -cell mappings based on the generalized difference of n -cell fuzzy-numbers.

Future research includes studying optimality conditions for fuzzy constrained minimization problem. One alternative is to define the concept of invex function using g -differentiability and the ordering relation \preceq_c for fuzzy n -cell mappings.

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