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# Kantorovich Type Integral Inequalities for Tensor Products of Continuous Fields of Positive Operators

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**Abstract**

This paper establishes a number of Kantorovich type integral inequalities involving tensor products of continuous fields of positive operators parametrized by a locally compact Hausdorff space. Such integrals appear as Bochner integrals with respect to a finite Radon measure on that space. Kantorovich type inequalities in which the operator product are replaced by an operator mean are also investigated.

**Keywords:** continuous field of operators, Bochner integral, tensor product, operator mean, operator monotone function
**Mathematics Subject Classifications 2010:** 26D15, 46G10, 47A63, 47A64, 47A80.

## 1 Introduction

One of well-known analytic inequalities is the classical Kantorovich inequality [10], which states that for real numbers $a_i$ and $w_i$ such that $0 < a \leqslant a_i \leqslant b$ and $w_i \geqslant 0$ for all $1 \leqslant i \leqslant n$, we have

$$\left(\sum_{i=1}^{n} w_i a_i\right)\left(\sum_{i=1}^{n} \frac{w_i}{a_i}\right) \leqslant \frac{(a+b)^2}{4ab}\left(\sum_{i=1}^{n} w_i\right)^2. \tag{1.1}$$

This inequality can be regarded as a reverse version of weighted arithmetic-harmonic mean inequality. Applications of this inequality arise in convergence analysis for numerical methods and statistics. Various generalizations, variations, refinements and equivalences of this inequality in several settings have been investigated. Let us focus on an integral version of (1.1):

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**Theorem 1.1** (see *e.g.* [4])**.** *Let $J$ be a real interval equipped with a probability measure $\mu$. For any continuous function $f : J \to \mathbb{R}$ such that* $\mathrm{Range}(f) \subseteq [a, b]$ *for some $a, b > 0$, it holds that*

$$\int_J f^2 \, d\mu \leqslant \frac{(a+b)^2}{4ab} \left( \int_J f \, d\mu \right)^2. \tag{1.2}$$

Over the years, Kantorovich type inequalities were obtained in the contexts of matrices and operators, see *e.g.* [5, 7, 12, 14] and references therein. A matrix analogue of the inequality (1.1) involving Hadamard product (entrywise product, denoted by $\odot$) is given as follows.

**Theorem 1.2** ([13], Theorem 2.2)**.** *For each $i = 1, 2, \ldots, n$, let $A_i$ and $W_i$ be positive definite matrices of the same size such that $0 < aI \leqslant A_i \leqslant bI$. Then*

$$\sum_{i=1}^{n} W_i^{\frac{1}{2}} A_i W_i^{\frac{1}{2}} \odot \sum_{i=1}^{n} W_i^{\frac{1}{2}} A_i^{-1} W_i^{\frac{1}{2}} \leqslant \frac{a^2+b^2}{2ab} \left( \sum_{i=1}^{n} W_i \odot \sum_{i=1}^{n} W_i \right). \tag{1.3}$$

Note that the constant bound $(a^2+b^2)/(2ab)$ of the matrix case (1.3) is slightly different to that of scalar case $(a^2+b^2)/(4ab)$ in (1.1) and (1.2). The inequality (1.3) can be viewed as a reverse of the Fiedler's inequality

$$A \circ A^{-1} \geqslant I$$

which holds for any positive definite matrix $A$ (see [6]). Kantorovich type inequality in which the operator product is replaced by an operator mean was considered in [15, 17].

In this paper, we establish certain integral inequalities of Kantorovich type for continuous fields of positive operators on a Hilbert space. The inequalities (1.1) and (1.2) are generalized in many ways in terms of Bochner integrals of operator-valued functions defined on a locally compact Hausdorff space equipped with a finite Radon measure. Instead of the Hadamard product in Theorem 1.2, we consider the (Hilbert) tensor product and Kubo-Ando operator mean. Our results include discrete inequalities as special cases.

This paper consists of four sections. Section 2 provides fundamental facts about continuous fields of operators and its integrability. Section 3 deals with Kantorovich type integral inequalities involving tensor products of continuous fields of operators. In Section 4, we recall Kubo-Ando theory of operator means and then derive Kantorovich type inequalities involving operator means.

# 2 Continuous field of operators and Bochner integrability

Throughout, let $\mathcal{H}$ be a complex Hilbert space. Denote by $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{B}(\mathcal{H})^+$ the C*-algebra of all bounded linear operators on $\mathcal{H}$ and its positive cone, respectively. Let $\mathbb{A}$ and $\mathbb{A}^+$ be a unital C*-subalgebra of $\mathfrak{B}(\mathcal{H})$ and its positive cone, respectively. Capital letters always denote operators on a Hilbert space. In particular, $I$ denotes the identity operator, where the underlying space is clear from the context. The spectrum of an operator $A$ is expressed as $\mathrm{Sp}(A)$. As usual, the operator norm of an operator $A$ is denoted by $\|A\|$. For selfadjoint elements $A, B \in \mathbb{A}$, the expression $A \leqslant B$ indicates that $B - A$ is a positive element, while $A > 0$ means that $A$ is positive and invertible.

Let us denote the supremum norm of a real-valued function $f$ defined on a set $E$ by $\|f\|_{\infty,E}$. The symbol $\|\cdot\|_1$ denotes the $L^1$-norm on a given set, which is clear from the context.

The next lemma asserts the continuity of the map $A \mapsto f(A)$. Here, $f(A)$ is the continuous functional calculus of $f$ on $\mathrm{Sp}(A)$.

**Lemma 2.1.** *Let $\Delta$ be a nonempty compact subset of $\mathbb{C}$ and let $f : \Delta \to \mathbb{C}$ be a continuous function. Let $\mathcal{A}$ be the subset of $\mathbb{A}$ consisting of all operators whose spectra are contained in $\Delta$. Then the map sending $A \in \mathcal{A}$ to $f(A) \in \mathbb{A}$ is continuous.*

*Proof.* Let $\epsilon > 0$. Weierstrass' approximation theorem guarantees the existence of a polynomial $p$ such that

$$\|f - p\|_{\infty,\Delta} < \frac{\epsilon}{3}.$$

Since the map $X \mapsto p(X)$ is continuous on $\mathcal{A}$, there is a positive constant $\delta$ such that $\|p(A) - p(B)\| < \frac{\epsilon}{3}$ whenever $\|A - B\| < \delta$. For any operators $A, B \in \mathcal{A}$ such that $\|A - B\| < \delta$, we have

$$\begin{aligned} \|f(A) - f(B)\| &\leqslant \|f(A) - p(A)\| + \|p(A) - p(B)\| + \|p(B) - f(B)\| \\ &= \|f - p\|_{\infty,\sigma(A)} + \|p(A) - p(B)\| + \|f - p\|_{\infty,\sigma(B)} \\ &\leqslant \|f - p\|_{\infty,\Delta} + \|p(A) - p(B)\| + \|f - p\|_{\infty,\Delta} \\ &< \epsilon. \end{aligned}$$

Note that the above equality holds since the Gelfand transform $f \mapsto f(A)$ is an isometry. Therefore the map $A \mapsto f(A)$ is continuous. $\square$

From now on, let $\Omega$ be a locally compact Hausdorff space. Equip $\Omega$ with a Radon measure $\mu$, *i.e.*, $\mu$ is a Borel measure on $\Omega$ that is finite on all compact subsets, outer regular on all Borel subsets, and inner regular on all open subsets. A family $(A_t)_{t\in\Omega}$ of operators in $\mathbb{A}$ is said to be a *continuous field of operators* if the parametrization $t \mapsto A_t$ is continuous on $\Omega$. If we further assume the Lebesgue integrability of the function $t \mapsto \|A_t\|$, then the Bochner integral $\int_\Omega A_t \, d\mu(t)$ is well-defined as the element in $\mathbb{A}$ satisfying

$$\phi\left(\int_\Omega A_t \, d\mu(t)\right) = \int_\Omega \phi(A_t) \, d\mu(t)$$

for every $\phi$ in the norm dual of $\mathbb{A}$ (see *e.g.* [16]). Let $\mathcal{C}(\Omega; \mathbb{A}^+)$ be the set of all continuous fields$(A_t)_{t\in\Omega}$ such that $A_t \in \mathbb{A}^+$ for all $t \in \Omega$. If we want to

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specify that $\mathrm{Sp}(A_t) \subseteq J$, for some subset $J \subseteq [0,\infty)$, we shall use the notation $\mathcal{C}(\Omega; \mathbb{A}^+, J)$.

The next lemma is useful for integrating any vector-valued function on a finite measure space.

**Lemma 2.2** (see *e.g.* [1], Theorem 11.44)**.** *Let $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ be a Banach space, and let $(\Gamma, \nu)$ be a finite measure space. Suppose that $f : \Gamma \to \mathbb{X}$ is a measurable function (here, $\mathbb{X}$ is equipped with the Borel $\sigma$-algebra). Then $f$ is Bochner integrable if and only if its norm function $\|f\|$ is Lebesgue integrable, i.e.,*

$$\int_\Gamma \|f\| \, d\nu \; < \; \infty.$$

*Here, $\|f\|$ is defined by $\|f\|(x) = \|f(x)\|_{\mathbb{X}}$ for any $x \in \mathbb{X}$.*

In what follows, suppose that $\mu$ is a finite Radon measure on $\Omega$. The integrability of a real-valued function is always in the sense of Lebesgue.

**Proposition 2.3.** *Let $\Delta$ be a nonempty compact subset of $\mathbb{C}$ and let $f : \Delta \to \mathbb{C}$ be a continuous function. Let $(A_t)_{t\in\Omega}$ be a continuous field of normal operators in $\mathbb{A}$ whose spectra are contained in $\Delta$. Let $(W_t)_{t\in\Omega}$ be a field in $\mathcal{C}(\Omega; \mathbb{A}^+)$. Suppose that the function $t \mapsto \|W_t\|$ is integrable on $\Omega$. Then we can form the Bochner integral*

$$\int_\Omega W_t^{\frac{1}{2}} f(A_t) W_t^{\frac{1}{2}} \, d\mu(t). \tag{2.1}$$

*In addition, if $f$ is nonnegative on $\Delta$, then the operator* (2.1) *is positive.*

*Proof.* By Lemma 2.2, it suffices to prove the integrability of the norm function $t \mapsto \|W_t^{\frac{1}{2}} f(A_t) W_t^{\frac{1}{2}}\|$. Since $t \mapsto A_t$ is continuous, the map $t \mapsto f(A_t)$ is continuous by Lemma 2.1, and hence so is the map $t \mapsto W_t^{\frac{1}{2}} f(A_t) W_t^{\frac{1}{2}}$. Thus

$$\begin{aligned}
\int_\Omega \|W_t^{\frac{1}{2}} f(A_t) W_t^{\frac{1}{2}}\| \, d\mu(t) \;&\leqslant\; \int_\Omega \|W_t^{\frac{1}{2}}\| \cdot \|f(A_t)\| \cdot \|W_t^{\frac{1}{2}}\| \, d\mu(t) \\
&=\; \int_\Omega \|W_t\|^{\frac{1}{2}} \cdot \|f\|_{\infty, \mathrm{Sp}(A_t)} \cdot \|W_t\|^{\frac{1}{2}} \, d\mu(t) \\
&\leqslant\; \int_\Omega \|f\|_{\infty,\Delta} \cdot \|W_t\| \, d\mu(t) \\
&=\; \|f\|_{\infty,\Delta} \int_\Omega \|W_t\| \, d\mu(t) \\
&<\; \infty.
\end{aligned}$$

Now, suppose $f(\Delta) \subseteq [0,\infty)$. The spectral mapping theorem implies that $f(A_t)$ is a positive element for all $t \in \Omega$. Therefore the resulting integral (2.1) is positive. □

**Remark 2.4.** For convenience, we may assume that $\Omega$ is a compact Hausdorff space. In this case, any Radon measure on $\Omega$ is always finite and hence every continuous field $(X_t)_{t\in\Omega}$ of operators is automatically Bochner integrable. Indeed, its norm function $t \mapsto \|X_t\|$ is bounded and, thus, integrable. It follows that the map $t \mapsto X_t$ is Bochner integrable by Lemma 2.2.

**Lemma 2.5** (see *e.g.* [1], Lemma 11.45)**.** *Let $\mathbb{X}$ and $\mathbb{Y}$ be Banach spaces and let $(\Gamma, \nu)$ be a measure space. Suppose that a function $f : \Gamma \to \mathbb{X}$ is Bochner integrable. If $T : \mathbb{X} \to \mathbb{Y}$ be a bounded linear operator, then the composition $T \circ f$ is also Bochner integrable and*

$$\int_\Gamma (T \circ f)\, d\nu \;=\; T\left(\int_\Gamma f\, d\nu\right).$$

The next proposition will be useful in later discussions.

**Proposition 2.6.** *Let $(A_t)_{t\in\Omega}$ be a bounded continuous field of operators in $\mathbb{A}$. For any $X \in \mathbb{A}$, we have*

$$\int_\Omega A_t\, d\mu(t) \otimes X \;=\; \int_\Omega (A_t \otimes X)\, d\mu(t), \tag{2.2}$$

$$X \otimes \int_\Omega A_t\, d\mu(t) \;=\; \int_\Omega (X \otimes A_t)\, d\mu(t). \tag{2.3}$$

*Proof.* By Lemma 2.2, the map $t \mapsto A_t$ is Bochner integrable on $\Omega$ since it is continuous and bounded. Note that the maps $T \mapsto T \otimes X$ and $T \mapsto X \otimes T$ are bounded linear operators from $\mathfrak{B}(\mathcal{H})$ to $\mathfrak{B}(\mathcal{H} \otimes \mathcal{H})$. It follows from Lemma 2.5 that the maps $t \mapsto A_t \otimes X$ and $t \mapsto X \otimes A_t$ are Bochner integrable on $\Omega$, and the properties (2.2) and (2.3) hold. $\square$

# 3 Integral inequalities of Kantorovich type for tensor products of operators

In this section, we derive operators integral inequalities of Kantorovich type in which the operator product is given by the tensor product. From now on, let $a, b$ be constants such that $0 < a \leqslant b$. For each $A, B \in \mathfrak{B}(\mathcal{H})$, we denote

$$A \otimes_s B \;=\; \frac{1}{2}(A \otimes B + B \otimes A).$$

Recall that the tensor power $A^{\otimes 2}$ is defined to be $A \otimes A$.

**Lemma 3.1.** *The minimum constant $k$ for which the inequality*

$$A \otimes B^{-1} + A^{-1} \otimes B \;\leqslant\; kI. \tag{3.1}$$

*holds for all positive elements $A, B \in \mathbb{A}$ whose spectra are contained in $[a, b]$ is determined by $k = (a^2 + b^2)/(ab)$. Here, $I$ denotes the identity on $\mathcal{H} \otimes \mathcal{H}$.*

*Proof.* Since $\mathrm{Sp}(A), \mathrm{Sp}(B) \subseteq [a,b]$, we have

$$\begin{aligned}\mathrm{Sp}(A \otimes B^{-1}) &= \{xy : x \in \mathrm{Sp}(A), y \in \mathrm{Sp}(B^{-1})\} \\ &= \{xz^{-1} : x \in \mathrm{Sp}(A), z \in \mathrm{Sp}(B)\} \\ &\subseteq [a/b, b/a].\end{aligned}$$

Note also that $(A \otimes B^{-1})^{-1} = A^{-1} \otimes B$. Let us denote by $r(\cdot)$ the spectral radius of an operator. Spectral mapping theorem now implies that

$$\begin{aligned}\|A \otimes B^{-1} + A^{-1} \otimes B\| &= r(A \otimes B^{-1} + A^{-1} \otimes B) \\ &= \sup\{\lambda + \lambda^{-1} : \lambda \in \mathrm{Sp}(A \otimes B^{-1})\} \\ &\leqslant \sup\{\lambda + \lambda^{-1} : \lambda \in [a/b, b/a]\} \\ &= \frac{a^2+b^2}{ab}.\end{aligned}$$

Thus, we arrive at inequality (3.1). The constant $(a^2+b^2)/(ab)$ cannot be improved since the case $A = aI_{\mathcal{H}}$ and $B = bI_{\mathcal{H}}$ is reduced to the scalar case. $\square$

The following theorem is an integral inequality of Kantorovich type.

**Theorem 3.2.** *Let $(A_t)_{t\in\Omega}$ be a field in $\mathcal{C}(\Omega; \mathbb{A}^+, [a,b])$. Let $(W_t)_{t\in\Omega}$ be a field in $\mathcal{C}(\Omega; \mathbb{A}^+)$ such that the function $t \mapsto \|W_t\|$ is integrable on $\Omega$. Then*

$$\int_\Omega W_t^{\frac{1}{2}} A_t W_t^{\frac{1}{2}}\, d\mu(t) \otimes_s \int_\Omega W_t^{\frac{1}{2}} A_t^{-1} W_t^{\frac{1}{2}}\, d\mu(t) \leqslant \frac{a^2+b^2}{2ab} \left( \int_\Omega W_t\, d\mu(t) \right)^{\otimes 2}. \quad (3.2)$$

*Moreover, the constant $(a^2+b^2)/(2ab)$ is best possible.*

*Proof.* For convenience, let us denote

$$X = \int_\Omega {W_t}^{\frac{1}{2}} A_t {W_t}^{\frac{1}{2}}\, d\mu(t) \text{ and } Y = \int_\Omega {W_t}^{\frac{1}{2}} {A_t}^{-1} {W_t}^{\frac{1}{2}}\, d\mu(t).$$

By Proposition 2.3 with $\Delta = [a,b]$, the operators $X, Y$ and $\int_\Omega W_t\, d\mu(t)$ are well-defined and positive via putting $f(x) = x$, $f(x) = 1/x$, and $f(x) = 1$, respectively. Using properties (2.2) and (2.3) in Proposition 2.6, we obtain

$$\begin{aligned}X \otimes Y &= \int_\Omega \left( W_t^{\frac{1}{2}} A_t^{-1} W_t^{\frac{1}{2}} \otimes \int_\Omega W_r^{\frac{1}{2}} A_r^{-1} W_r^{\frac{1}{2}}\, d\mu(r) \right) d\mu(t) \\ &= \iint_{\Omega^2} \left( W_t^{\frac{1}{2}} A_t W_t^{\frac{1}{2}} \otimes W_r^{\frac{1}{2}} A_r^{-1} W_r^{\frac{1}{2}} \right) d\mu(r)\, d\mu(t).\end{aligned}$$

Similarly, we have

$$Y \otimes X = \iint_{\Omega^2} \left( W_t^{\frac{1}{2}} {A_t}^{-1} W_t^{\frac{1}{2}} \otimes W_r^{\frac{1}{2}} A_r W_r^{\frac{1}{2}} \right) d\mu(r)\, d\mu(t).$$

It follows that

$$\begin{aligned} 2(X \otimes_s Y) &= \iint_{\Omega^2} \left( W_t^{\frac{1}{2}} A_t W_t^{\frac{1}{2}} \otimes W_r^{\frac{1}{2}} A_r^{-1} W_r^{\frac{1}{2}} + W_t^{\frac{1}{2}} A_t^{-1} W_t^{\frac{1}{2}} \otimes W_r^{\frac{1}{2}} A_r W_r^{\frac{1}{2}} \right) d\mu(r)\, d\mu(t) \\ &= \iint_{\Omega^2} (W_t \otimes W_r)^{\frac{1}{2}} \left( A_t \otimes A_r^{-1} + A_t^{-1} \otimes A_r \right) (W_t \otimes W_r)^{\frac{1}{2}}\, d\mu(r)\, d\mu(t). \end{aligned}$$

Lemma 3.1 together with Proposition 2.6 imply that

$$\begin{aligned} X \otimes_s Y &\leqslant \frac{1}{2} \int_\Omega \int_\Omega \frac{a^2+b^2}{ab} (W_t \otimes W_r)\, d\mu(r)\, d\mu(t) \\ &= \frac{a^2+b^2}{2ab} \int_\Omega \left( \int_\Omega W_r\, d\mu(r) \right) \otimes W_t\, d\mu(t) \\ &= \frac{a^2+b^2}{2ab} \int_\Omega W_t\, d\mu(t) \otimes \int_\Omega W_t\, d\mu(t). \end{aligned}$$

Therefore, we arrive at (3.2). The best possibility of the constant $(a^2+b^2)/(2ab)$ also comes from Lemma 3.1. $\square$

The next result is an integral inequality of Kantorovich type in which the weights are scalars.

**Corollary 3.3.** *Let $(A_t)_{t\in\Omega}$ be a field in $\mathcal{C}(\Omega; \mathbb{A}^+, [a,b])$. Let $w : \Omega \to [0,\infty)$ be a continuous integrable function. Then*

$$\int_\Omega w(t) A_t\, d\mu(t) \otimes_s \int_\Omega w(t) A_t^{-1}\, d\mu(t) \leqslant \frac{a^2+b^2}{2ab} \|w\|_1^2 I. \tag{3.3}$$

*Proof.* From Theorem 3.2, put $W_t = w(t)I$ for each $t \in \Omega$. $\square$

**Corollary 3.4.** *Let $(A_t)_{t\in\Omega}$ and $(B_t)_{t\in\Omega}$ be fields in $\mathcal{C}(\Omega; \mathbb{A}^+)$ such that*

*i)* $\mathrm{Sp}(A_t) \subseteq [a,b] \subseteq (0,\infty)$ *for each $t \in \Omega$,*

*ii) the function $t \mapsto \|B_t\|$ is integrable on $\Omega$, and*

*iii) $A_t B_t = B_t A_t$ for each $t \in \Omega$.*

*Then*

$$\int_\Omega A_t B_t\, d\mu(t) \otimes_s \int_\Omega A_t^{-1} B_t\, d\mu(t) \leqslant \frac{a^2+b^2}{2ab} \left( \int_\Omega B_t\, d\mu(t) \right)^2 I. \tag{3.4}$$

*Proof.* From Theorem 3.2, put $W_t = B_t$ for each $t \in \Omega$. $\square$

**Corollary 3.5.** *Let $f, \phi : \Omega \to [0,\infty)$ be continuous functions. Assume that* $\mathrm{Range}(f) \subseteq [a,b] \subseteq (0,\infty)$ *and $\phi$ is integrable with $\int_\Omega \phi\, d\mu = 1$. Then*

$$\|\phi f\|_1 \leqslant \frac{a^2+b^2}{2ab} \frac{1}{\|\phi/f\|_1}.$$

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*Proof.* From Corollary 3.4, put $\mathbb{A} = \mathbb{C}$. Note that $\int_\Omega (\phi/f)\, d\mu > 0$. □

Theorem 3.2 can be extended in the following way:

**Theorem 3.6.** *Let $(A_t)_{t\in\Omega}$ be a filed in $\mathcal{C}(\Omega; \mathbb{A}^+, [a, b])$. Let $(W_t)_{t\in\Omega}$ be a field in $\mathcal{C}(\Omega; \mathbb{A}^+)$ such that the function $t \mapsto \|W_t\|$ is integrable on $\Omega$. Let $f$ be a continuous real-valued function defined on $[a, b] \cup [1/b, 1/a]$ such that*

*(i) $f(x)f(1/x) \leqslant 1$ for all $x \in [a, b]$,*

*(ii) $f([a, b]) \subseteq [a, b]$ or $f([a, b]) \subseteq [1/b, 1/a]$.*

*Then*

$$\int_\Omega W_t^{\frac{1}{2}} f(A_t) W_t^{\frac{1}{2}}\, d\mu(t) \otimes_s \int_\Omega W_t^{\frac{1}{2}} f(A_t^{-1}) W_t^{\frac{1}{2}}\, d\mu(t) \leqslant \frac{a^2+b^2}{2ab} \left( \int_\Omega W_t\, d\mu(t) \right)^{\otimes 2}. \tag{3.5}$$

*Proof.* Since $\mathrm{Sp}(A_t^{-1}) \subseteq [1/b, 1/a]$ for each $t$, the function $t \mapsto W_t^{\frac{1}{2}} f(A_t^{-1}) W_t^{\frac{1}{2}}$ is Bochner integrable by Proposition 2.3. The assumption also implies that

$$f(A_t^{-1}) \leqslant f(A_t)^{-1}$$

for each $t \in \Omega$. The inequality (3.5) now follows from Theorem 3.2. Note that the constant $(a^2 + b^2)/(2ab)$ is not affected. □

Theorem 3.6 is reduced to Theorem 3.2 by setting $f(x) = x$ or $f(x) = 1/x$.

**Corollary 3.7.** *Let $0 < a < b$. Consider three continuous functions $\phi : \Omega \to [a, b]$, $g : [a, b] \to (0, \infty)$ and $f : [a, b] \cup [1/b, 1/a] \to \mathbb{R}$. Suppose that*

*(i) $f(x)f(1/x) \leqslant 1$ for all $x \in [a, b]$,*

*(ii) $f([a, b]) \subseteq [a, b]$ or $f([a, b]) \subseteq [1/b, 1/a]$.*

*Then we have the bound*

$$\|(fg) \circ \phi\|_1 \leqslant \frac{a^2+b^2}{2ab} \frac{\|g \circ \phi\|_1^2}{\|(f \circ \frac{1}{\phi})(g \circ \phi)\|_1}.$$

*Proof.* It is a special case of Theorem 3.6 when $\mathbb{A} = \mathbb{C}$. □

# 4 Kantorovich type integral inequalities involving operator means

In this section, we establish integral analogues of Kantorovich inequality involving operator means. First of all, we recall some fundamental facts about operator means [11]; see also [9, Ch. 5].

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**Definition 4.1.** A binary operation $\sigma : \mathfrak{B}(\mathcal{H})^+ \times \mathfrak{B}(\mathcal{H})^+ \to \mathfrak{B}(\mathcal{H})^+$ is called a *connection* if the following conditions hold for all $A, B, C, D \in \mathfrak{B}(\mathcal{H})^+$:

(i) (joint) monotonicity: $A \leqslant C, B \leqslant D \implies A \,\sigma\, B \leqslant C \,\sigma\, D$

(ii) transformer inequality: $C(A \,\sigma\, B)C \leqslant (CAC) \,\sigma\, (CBC)$

(iii) (joint) continuity from above: for any sequences $(A_n), (B_n)$ in $\mathfrak{B}(\mathcal{H})^+$, if $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n \,\sigma\, B_n \downarrow A \,\sigma\, B$. Here, $X_n \downarrow X$ indicates that $(X_n)$ is a decreasing sequence converging strongly to $X$.

It follows that every connection $\sigma$ satisfies the following properties:

$$X(A \,\sigma\, B)X \;=\; (XAX) \,\sigma\, (XBX), \tag{4.1}$$
$$(A + B) \,\sigma\, (C + D) \;\geqslant\; (A \,\sigma\, C) + (B \,\sigma\, D) \tag{4.2}$$

for all $A, B, C, D \geqslant 0$ and $X > 0$. A *mean* is a connection $\sigma$ with idempotent property $A \,\sigma\, A = A$ for all $A \geqslant 0$.

Recall also that a continuous function $f : [0, \infty) \to \mathbb{R}$ is said to be *operator monotone* if the condition $0 \leqslant A \leqslant B$ implies $f(A) \leqslant f(B)$. Such $f$ is said to be *super-multiplicative* if $f(xy) \geqslant f(x)f(y)$ for all $x, y \geqslant 0$.

**Proposition 4.2** ([11])**.** *There is a one-to-one correspondence between operator connections and operator monotone functions from* $[0, \infty)$ *to itself such that*

$$f(A) \;=\; I \,\sigma\, A, \quad A \in \mathfrak{B}(\mathcal{H})^+. \tag{4.3}$$

*Moreover,* $\sigma$ *is an operator mean if and only if* $f(1) = 1$.

Such $f$ in this proposition is called the *representing function* of $\sigma$. Every operator connection $\sigma$ admits an integral representation (see *e.g.* [3])

$$A \,\sigma\, B \;=\; \int_0^1 A \,!_t\, B \, d\nu(t), \qquad A, B \in \mathfrak{B}(\mathcal{H})^+$$

for some finite Radon measure $\nu$ on the interval $[0, 1]$. Here, $!_t$ denotes the $t$-weighted harmonic mean. Hence if $A, B \in \mathbb{A}^+$, then $A \,\sigma\, B \in \mathbb{A}^+$ since the integral is a limit of finite sums.

**Lemma 4.3** ([2])**.** *For any operator connection* $\sigma$ *and* $A, B \in \mathfrak{B}(\mathcal{H})^+$*, we have*

$$\|A \,\sigma\, B\| \leqslant \|A\| \,\sigma\, \|B\|.$$

*Here,* $\sigma$ *on the right hand side is the induced connection on* $[0, \infty)$ *defined by* $(a \,\sigma\, b)I = aI \,\sigma\, bI$ *for any* $a, b \in [0, \infty)$.

**Lemma 4.4.** *Let* $\sigma$ *be an operator connection with associated super-multiplicative operator-monotone function. Then for all positive operators* $A, B, C, D$*, we have*

$$(A \,\sigma\, C) \otimes_s (B \,\sigma\, D) \leqslant (A \otimes_s B) \,\sigma\, (C \otimes_s D). \tag{4.4}$$

*Proof.* By a continuity argument using the monotonicity and the continuity from above of a connection, we may assume that $A, B > 0$. Putting $X = A^{-\frac{1}{2}}CA^{-\frac{1}{2}}$ and $Y = B^{-\frac{1}{2}}DB^{-\frac{1}{2}}$, we have from properties (4.1) and (4.3) that

$$\begin{aligned}(A\,\sigma\,C) \otimes (B\,\sigma\,D) &= (A \otimes B)^{\frac{1}{2}}[(I\,\sigma\,X) \otimes (I\,\sigma\,Y)](A \otimes B)^{\frac{1}{2}} \\ &= (A \otimes B)^{\frac{1}{2}}[f(X) \otimes f(Y)](A \otimes B)^{\frac{1}{2}} \\ &\leqslant (A \otimes B)^{\frac{1}{2}}[f(X \otimes Y)](A \otimes B)^{\frac{1}{2}} \\ &= (A \otimes B)^{\frac{1}{2}}[I\,\sigma\,(X \otimes Y)](A \otimes B)^{\frac{1}{2}} \\ &= (A \otimes B)\,\sigma\,(C \otimes D).\end{aligned}$$

Now, using property (4.2) yields

$$\begin{aligned}&(A\,\sigma\,C) \otimes (B\,\sigma\,D) + (B\,\sigma\,D) \otimes (A\,\sigma\,C) \\ &\qquad \leqslant (A \otimes B)\,\sigma\,(C \otimes D) + (B \otimes A)\,\sigma\,(D \otimes C) \\ &\qquad \leqslant [(A \otimes B) + (B \otimes A)]\ \sigma\ [(C \otimes D) + (D \otimes C)]\,.\end{aligned}$$

□

The following result can be regarded as a Kantorovich type integral inequality concerning an operator mean.

**Theorem 4.5.** *Let $(A_t)_{t\in\Omega}$ be a filed in $\mathcal{C}(\Omega;\mathbb{A}^+,[a,b])$. Let $(W_t)_{t\in\Omega}$ be a field in $\mathcal{C}(\Omega;\mathbb{A}^+)$ such that the function $t \mapsto \|W_t\|$ is integrable on $\Omega$. Let $\sigma$ be a mean associated with a super-multiplicative representing function. Then*

$$\begin{aligned}\int_\Omega W_t^{\frac{1}{2}}(A_t\,\sigma\,B_t)W_t^{\frac{1}{2}}\,d\mu(t)\ \otimes_s\ \int_\Omega W_t^{\frac{1}{2}}(A_t^{-1}\,\sigma\,B_t^{-1})W_t^{\frac{1}{2}}\,d\mu(t) \\ \leqslant \frac{a^2+b^2}{2ab}\left(\int_\Omega W_t\,d\mu(t)\right)^{\otimes 2}.\end{aligned} \tag{4.5}$$

*Proof.* The upper semicontinuity of $\sigma$, and the continuity of the maps $t \mapsto A_t$ and $t \mapsto B_t$ together imply the measurability of the map $t \mapsto A_t\sigma B_t$. Note that $\|A_t\sigma B_t\| \leqslant b$ by the monotonicity and the idempotency of $\sigma$, and the norm estimation in Lemma 4.3. It follows that the map $t \mapsto W_t^{\frac{1}{2}}(A_t\,\sigma\,B_t)W_t^{\frac{1}{2}}$ is Bochner integrable by Lemma 2.2. Similarly, the map $t \mapsto W_t^{\frac{1}{2}}(A_t^{-1}\,\sigma\,B_t^{-1})W_t^{\frac{1}{2}}$

is Bochner integrable. Now, we have

$$
\begin{aligned}
&\int_\Omega W_t^{\frac{1}{2}}(A_t \sigma B_t) W_t^{\frac{1}{2}}\, d\mu(t) \otimes_s \int_\Omega W_t^{\frac{1}{2}}(A_t^{-1}\,\sigma\, B_t^{-1}) W_t^{\frac{1}{2}}\, d\mu(t) \\
&\leqslant \int_\Omega \left( W_t^{\frac{1}{2}} A_t W_t^{\frac{1}{2}} \,\sigma\, W_t^{\frac{1}{2}} B_t W_t^{\frac{1}{2}} \right) d\mu(t) \otimes_s \int_\Omega \left( W_t^{\frac{1}{2}} A_t^{-1} W_t^{\frac{1}{2}} \,\sigma\, W_t^{\frac{1}{2}} B_t^{-1} W_t^{\frac{1}{2}} \right) d\mu(t) \\
&\qquad \text{(since } \sigma \text{ satisfies the transformer inequality)} \\
&\leqslant \left[ \int_\Omega {W_t}^{\frac{1}{2}} A_t {W_t}^{\frac{1}{2}}\, d\mu(t)\, \sigma \int_\Omega {W_t}^{\frac{1}{2}} B_t {W_t}^{\frac{1}{2}}\, d\mu(t) \right] \\
&\qquad \otimes_s \left[ \int_\Omega {W_t}^{\frac{1}{2}} A_t^{-1} {W_t}^{\frac{1}{2}}\, d\mu(t)\, \sigma \int_\Omega {W_t}^{\frac{1}{2}} B_t^{-1} {W_t}^{\frac{1}{2}}\, d\mu(t) \right] && \text{(by (4.2))} \\
&\leqslant \left[ \int_\Omega {W_t}^{\frac{1}{2}} A_t {W_t}^{\frac{1}{2}}\, d\mu(t) \otimes_s \int_\Omega {W_t}^{\frac{1}{2}} A_t^{-1} {W_t}^{\frac{1}{2}}\, d\mu(t) \right] \\
&\qquad \sigma \left[ \int_\Omega {W_t}^{\frac{1}{2}} B_t {W_t}^{\frac{1}{2}}\, d\mu(t) \otimes_s \int_\Omega {W_t}^{\frac{1}{2}} B_t^{-1} {W_t}^{\frac{1}{2}}\, d\mu(t) \right] && \text{(by Lemma 4.4)} \\
&\leqslant \frac{a^2+b^2}{2ab} \left( \int_\Omega W_t\, d\mu(t) \right)^{\otimes 2} \sigma\ \frac{a^2+b^2}{2ab} \left( \int_\Omega W_t\, d\mu(t) \right)^{\otimes 2} && \text{(by Theorem 3.2)} \\
&= \frac{a^2+b^2}{2ab} \left( \int_\Omega W_t\, d\mu(t) \right)^{\otimes 2}. && \square
\end{aligned}
$$

Theorem 4.5 is reduced to Theorem 3.2 by putting $A_t = B_t$ for all $t \in \Omega$.

**Corollary 4.6.** *Let $(A_t)_{t\in\Omega}$ and $(B_t)_{t\in\Omega}$ be two fields in $\mathcal{C}(\Omega; \mathbb{A}^+, [a,b])$. Let $w : \Omega \to [0,\infty)$ be an integrable continuous function. Let $\sigma$ be an operator mean associated with a super-multiplicative representing function. Then*

$$\int_\Omega w(t)(A_t \,\sigma\, B_t)\, d\mu(t) \ \otimes_s \int_\Omega w(t)(A_t^{-1} \,\sigma\, B_t^{-1})\, d\mu(t) \ \leqslant\ \frac{a^2+b^2}{2ab} \|w\|_1^2 I. \tag{4.6}$$

*Proof.* From Theorem 4.5, put $W_t = w(t)I$ for all $t \in \Omega$. $\square$

**Theorem 4.7.** *Let $0 < a \leqslant 1 \leqslant b$. Let $(A_t)_{t\in\Omega}$ be a field in $\mathcal{C}(\Omega; \mathbb{A}^+, [a,b])$. Let $(W_t)_{t\in\Omega}$ be a field in $\mathcal{C}(\Omega; \mathbb{A}^+)$ such that the function $t \mapsto \|W_t\|$ is integrable. For any super-multiplicative operator-monotone function $f : [0,\infty) \to [0,\infty)$ such that $f(1) = 1$, we have*

$$\int_\Omega W_t^{\frac{1}{2}} f(A_t) W_t^{\frac{1}{2}}\, d\mu(t) \otimes_s \int_\Omega W_t^{\frac{1}{2}} f(A_t^{-1}) W_t^{\frac{1}{2}}\, d\mu(t) \leqslant \frac{a^2+b^2}{2ab} \left( \int_\Omega W_t\, d\mu(t) \right)^{\otimes 2}. \tag{4.7}$$

*Proof.* Proposition 4.2 guarantees the existence of an operator mean $\sigma$ such that $f(A) = I \,\sigma\, A$ for all $A \geqslant 0$. The inequality (4.7) now follows from Theorem 4.5 by considering $I \,\sigma A_t$ instead of $A_t \,\sigma\, B_t$. $\square$

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**Corollary 4.8.** *Let $0 < a \leqslant 1 \leqslant b$ and $\alpha \in [-1,1]$. Let $(A_t)_{t\in\Omega}$ be a field in $\mathcal{C}(\Omega,;\mathbb{A}^+,[a,b])$, and let $(B_t)_{t\in\Omega}$ be a field in $\mathcal{C}(\Omega,;\mathbb{A}^+)$ such that $A_tB_t = B_tA_t$ for each $t \in \Omega$. Then*

$$\int_\Omega A_t^\alpha B_t \, d\mu(t) \otimes_s \int_\Omega A_t^{-\alpha} B_t \, d\mu(t) \;\leqslant\; \frac{a^2+b^2}{2ab}\left(\int_\Omega B_t \, d\mu(t)\right)^{\otimes 2}. \tag{4.8}$$

*Proof.* It suffices to assume that $\alpha \in [0,1]$. The famous Löwner-Heinz states that the function $f(x) = x^\alpha$ is operator monotone (see *e.g.* [9, Ch.4]). Note that $f$ is also super-multiplicative and $f(1) = 1$. The inequality 4.8 now follows by replacing $W_t$ by $B_t$ in Theorem 4.7. □

The case $\mathbb{A} = \mathbb{C}$ in Corollary 4.8 reads as follows.

**Corollary 4.9.** *Let $0 < a \leqslant 1 \leqslant b$ and $\alpha \in [-1,1]$. Let $\phi : \Omega \to [a,b]$ and $g : \Omega \to (0,\infty)$ be continuous functions. We have*

$$\|g\phi^\alpha\|_1 \;\leqslant\; \frac{a^2+b^2}{2ab}\frac{\|g\|_1^2}{\|g\phi^{-\alpha}\|_1}.$$

The next result is a generalization of Theorem 1.2 in the context of operators in which the constant bound is given by $(a^2+b^2)/(2ab)$.

**Corollary 4.10.** *Let $(A_t)_{t\in\Omega}$ be a field in $\mathcal{C}(\Omega;\mathbb{A}^+,[a,b])$. If $\mu(\Omega) = 1$, then*

$$\int_\Omega A_t^2 \, d\mu(t) \otimes_s I \;\leqslant\; \frac{a^2+b^2}{2ab}\left(\int_\Omega A_t \, d\mu(t)\right)^{\otimes 2}. \tag{4.9}$$

*Proof.* From Corollary 4.8, put $\alpha = 1$ and $A_t = B_t$ for all $t \in \Omega$. □

**Remark 4.11.** Discrete versions for all results in this paper can be obtained by putting $\Omega$ to be a finite space endowed with the counting measure. For example, a discrete version of Theorem 4.5 is as follows: For each $i = 1,2,\ldots,n$, let $A_i$ and $B_i$ be operators in $\mathbb{A}^+$ whose spectra are contained in $[a,b]$, and let $W_i \in \mathbb{A}^+$. Let $\sigma$ be an operator mean associated with a super-multiplicative operator-monotone function. Then

$$\sum_{i=1}^n W_i^{\frac{1}{2}}(A_i \,\sigma\, B_i)W_i^{\frac{1}{2}} \;\otimes_s\; \sum_{i=1}^n W_i^{\frac{1}{2}}(A_i^{-1} \,\sigma\, B_i^{-1})W_i^{\frac{1}{2}} \;\leqslant\; \frac{a^2+b^2}{2ab}\left(\sum_{i=1}^n W_i\right)^{\otimes 2}.$$

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# Characteristic fuzzy sets and conditional fuzzy subalgebras

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**Abstract.** The notion of characteristic fuzzy sets is introduced. Using this notion, conditions for a subset of $BCK/BCI$-algebra to be a subalgebra are discussed. The notion of conditional fuzzy subalgebras is introduced, and several properties are investigated. Given a subalgebra of $BCK/BCI$-algebras, conditions for the characteristic fuzzy set to be a conditional fuzzy subalgebra of several types are provided.

## 1. Introduction

The notions of "membership" and "quasicoincidence" of fuzzy points and fuzzy sets were introduced by Pu and Liu in [13]. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, played a vital role to generate some different types of fuzzy subgroups, called $(\alpha, \beta)$-fuzzy subgroups, introduced by Bhakat and Das [1]. In particular, $(\in, \in \vee\, q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. Recently, these notions are applied to several algebraic structures, for example, near rings (see [2]), hypernear-rings (see [3]), hemirings (see [4]), lattices (see [9]), pseudo-$BL$ algebras (see [15]), and $BL$-algebras (see [16]) etc. In $BCK/BCI$-algebras, many research articles have been published on $(\alpha, \beta)$-fuzzy subalgebras (see [6], [7], [8], [11], [12] and [14]) which is an important and useful generalization of the well-known concepts, called fuzzy subalgebras.

In this paper, we define characteristic fuzzy sets, as a generalization of crisp characteristic function, and conditional fuzzy subalgebra. Using this notion, we discuss conditions

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for a subset of $BCK/BCI$-algebra to be a subalgebra. Given a subalgebra of $BCK/BCI$-algebras, we provide conditions for the characteristic fuzzy set to be a conditional $(\in, q)$-fuzzy subalgebra, a conditional $(q, \in)$-fuzzy subalgebra, a conditional $(q, q)$-fuzzy subalgebra, a conditional $(\in, \in \wedge\, q\,)$-fuzzy subalgebra, and a conditional $(q, \in \wedge\, q\,)$-fuzzy subalgebra.

## 2. Preliminaries

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms:

(a1) $((x * y) * (x * z)) * (z * y) = 0$,
(a2) $(x * (x * y)) * y = 0$,
(a3) $x * x = 0$,
(a4) $x * y = y * x = 0 \;\Rightarrow\; x = y$,

for all $x, y, z \in X$. We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. If a $BCI$-algebra $X$ satisfies the axiom

(a5) $0 * x = 0$ for all $x \in X$,

then we say that $X$ is a *BCK-algebra*. A nonempty subset $S$ of a $BCK/BCI$-algebra $X$ is called a *subalgebra* of $X$ if $x * y \in S$ for all $x, y \in S$. We refer the reader to the books [5] and [10] for further information regarding $BCK/BCI$-algebras.

A fuzzy set $\mu$ in a set $X$ of the form

$$\mu(y) := \begin{cases} t \in (0, 1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support $x$ and value $t$ and is denoted by $x_t$.

For a fuzzy point $x_t$ and a fuzzy set $\mu$ in a set $X$, Pu and Liu [13] introduced the symbol $x_t \alpha \mu$, where $\alpha \in \{\in, q\,, \in \vee\, q\,, \in \wedge\, q\,\}$. To say that $x_t \in \mu$ (resp. $x_t\; q\; \mu$), we mean $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$), and in this case, $x_t$ is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set $\mu$. To say that $x_t \in \vee\, q\; \mu$ (resp. $x_t \in \wedge\, q\; \mu$), we mean $x_t \in \mu$ or $x_t\, q\, \mu$ (resp. $x_t \in \mu$ and $x_t\, q\, \mu$). To say that $x_t\, \overline{\alpha}\, \mu$, we mean $x_t \alpha \mu$ does not hold, where $\alpha \in \{\in, q, \in \vee\, q\,, \in \wedge\, q\,\}$.

A fuzzy set $\mu$ in a $BCK/BCI$-algebra $X$ is called a *fuzzy subalgebra* of $X$ if it satisfies:

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\} \tag{2.1}$$

for all $x, y \in X$.

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A fuzzy set $\mu$ in $X$ is said to be an $(\alpha, \beta)$-*fuzzy subalgebra* of $X$, where $\alpha, \beta \in \{\in, q, \in\vee q, \in\wedge q\}$ and $\alpha \neq \in\wedge q$, if it satisfies the following condition:

$$x_{t_1}\alpha\,\mu,\; y_{t_2}\alpha\,\mu \;\Rightarrow\; (x*y)_{\min\{t_1,t_2\}}\,\beta\,\mu. \tag{2.2}$$

for all $x, y \in X$ and $t_1, t_2 \in (0, 1]$.

**Lemma 2.1** ([7]). *A fuzzy set $\mu$ in $X$ is an $(\in, \in\vee q)$-fuzzy subalgebra of $X$ if and only if it satisfies:*

$$(\forall x, y \in X)\,(\mu(x*y) \geq \min\{\mu(x), \mu(y), 0.5\})\,. \tag{2.3}$$

## 3. Characteristic fuzzy sets

In what follows, let $X$ denote a $BCK/BCI$-algebra and $\varepsilon, \delta \in [0, 1]$ with $\varepsilon > \delta$ unless otherwise specified.

For a non-empty subset $S$ of $X$, define a *characteristic fuzzy set* $\mu_S^{(\varepsilon,\delta)}$ in $X$ as follows:

$$\mu_S^{(\varepsilon,\delta)}(x) := \begin{cases} \varepsilon & \text{if } x \in S, \\ \delta & \text{otherwise.} \end{cases}$$

In particular, the characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ in $X$ with $\varepsilon = 1$ and $\delta = 0$ is the characteristic function $\chi_S$ of $S$ in $X$.

**Theorem 3.1.** *For any non-empty subset $S$ of $X$, the following are equivalent:*

(1) *$S$ is a subalgebra of $X$.*
(2) *The characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is a fuzzy subalgebra of $X$.*

*Proof.* Assume that $S$ is a subalgebra of $X$ and let $x, y \in X$. If $x, y \in S$, then $x * y \in S$ and so

$$\mu_S^{(\varepsilon,\delta)}(x*y) = \varepsilon = \min\left\{\mu_S^{(\varepsilon,\delta)}(x), \mu_S^{(\varepsilon,\delta)}(y)\right\}.$$

If $x \notin S$ or $y \notin S$, then $\mu_S^{(\varepsilon,\delta)}(x) = \delta$ or $\mu_S^{(\varepsilon,\delta)}(y) = \delta$. Hence

$$\mu_S^{(\varepsilon,\delta)}(x*y) \geq \delta = \min\left\{\mu_S^{(\varepsilon,\delta)}(x), \mu_S^{(\varepsilon,\delta)}(y)\right\}.$$

Therefore $\mu_S^{(\varepsilon,\delta)}$ is a fuzzy subalgebra of $X$.

Conversely, suppose that (2) is valid. Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon$ and $\mu_S^{(\varepsilon,\delta)}(y) = \varepsilon$. It follows that $\mu_S^{(\varepsilon,\delta)}(x * y) \geq \min\left\{\mu_S^{(\varepsilon,\delta)}(x), \mu_S^{(\varepsilon,\delta)}(y)\right\} = \varepsilon$. Thus $x * y \in S$, and therefore $S$ is a subalgebra of $X$. □

**Theorem 3.2.** *If $S$ is a subalgebra of $X$, then the characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in\vee q)$-fuzzy subalgebra of $X$.*

*Proof.* Assume that $S$ is a subalgebra of $X$. For any $x, y \in X$, if $x, y \in S$, then $x * y \in S$ and so

$$\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon \geq \min\left\{\mu_S^{(\varepsilon,\delta)}(x), \mu_S^{(\varepsilon,\delta)}(y), 0.5\right\}.$$

If $x \notin S$ or $y \notin S$, then $\mu_S^{(\varepsilon,\delta)}(x) = \delta$ or $\mu_S^{(\varepsilon,\delta)}(y) = \delta$. Hence

$$\mu_S^{(\varepsilon,\delta)}(x * y) \geq \delta \geq \min\left\{\mu_S^{(\varepsilon,\delta)}(x), \mu_S^{(\varepsilon,\delta)}(y), 0.5\right\}.$$

It follows from Lemma 2.1 that $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in\vee q)$-fuzzy subalgebra of $X$. □

The converse of Theorem 3.2 is not true in general as seen in the following example.

**Example 3.3.** Let $X = \{0, a, b, c, d\}$ be a $BCK$-algebra with the following Cayley table:

| $*$ | $0$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $a$ | $0$ | $0$ | $0$ | $0$ |
| $b$ | $b$ | $b$ | $0$ | $0$ | $0$ |
| $c$ | $c$ | $c$ | $b$ | $0$ | $0$ |
| $d$ | $d$ | $d$ | $c$ | $b$ | $0$ |

For a subset $S = \{0, c, d\}$ of $X$, consider a characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ in $X$ with $\varepsilon = 0.7$ and $\delta = 0.5$. Then $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in\vee q)$-fuzzy subalgebra of $X$, but $S$ is not a subalgebra of $X$ since $d * c = b \notin S$.

**Theorem 3.4.** *Assume that $\varepsilon \leq 0.5$. If the characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in\vee q)$-fuzzy subalgebra of $X$ then $S$ is a subalgebra of $X$.*

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*Proof.* Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$. Using Lemma 2.1, we have

$$\mu_S^{(\varepsilon,\delta)}(x * y) \geq \left\{\mu_S^{(\varepsilon,\delta)}(x), \mu_S^{(\varepsilon,\delta)}(y), 0.5\right\} = \{\varepsilon, 0.5\} = \varepsilon,$$

and so $x * y \in S$. Therefore $S$ is a subalgebra of $X$. □

**Corollary 3.5.** *A non-empty subset $S$ of $X$ is a subalgebra of $X$ if and only if the characteristic function $\chi_S$ of $S$ is an $(\in, \in\vee q)$-fuzzy subalgebra of $X$.*

*Proof.* Clearly, we can find the necessity by taking $\varepsilon = 1$ and $\delta = 0$ in Theorem3.2.

Conversely, suppose that the characteristic function $\chi_S$ of $S$ is an $(\in, \in \vee q)$-fuzzy subalgebra of $X$. Let $x, y \in S$. Then $\chi_S(x) = 1 = \chi_S(y)$, which implies from (2.3) that

$$\chi_S(x * y) \geq \min\{\chi_S(x), \chi_S(y), 0.5\} = \min\{1, 0.5\} = 0.5.$$

Hence $x * y \in S$, and therefore $S$ is a subalgebra of $X$. □

**Theorem 3.6.** *For a subset $S$ of $X$, let $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, q)$-fuzzy subalgebra of $X$. If $\delta \leq 0.5$ or $\varepsilon + \delta \leq 1$, then $S$ is a subalgebra of $X$.*

*Proof.* Let $x, y \in S$ and assume that $\delta \leq 0.5$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon > \delta$ and $\mu_S^{(\varepsilon,\delta)}(y) = \varepsilon > \delta$, that is, $x_\delta \in \mu_S^{(\varepsilon,\delta)}$ and $y_\delta \in \mu_S^{(\varepsilon,\delta)}$. Hence $(x * y)_\delta = (x * y)_{\min\{\delta,\delta\}}\, q\, \mu_S^{(\varepsilon,\delta)}$, which implies that $\mu_S^{(\varepsilon,\delta)}(x * y) + \delta > 1$. Since $\delta \leq 0, 5$, it follows that $\mu_S^{(\varepsilon,\delta)}(x * y) > 1 - \delta \geq \delta$. Thus $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$ and $x * y \in S$. Therefore $S$ is a subalgebra of $X$.

Now, suppose that $\varepsilon + \delta \leq 1$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, and so $x_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$. Hence $(x * y)_\varepsilon = (x * y)_{\min\{\varepsilon,\varepsilon\}}\, q\, \mu_S^{(\varepsilon,\delta)}$, which implies that $\mu_S^{(\varepsilon,\delta)}(x * y) + \varepsilon > 1$. Therefore $\mu_S^{(\varepsilon,\delta)}(x*y) > 1-\varepsilon \geq \delta$, and thus $\mu_S^{(\varepsilon,\delta)}(x*y) = \varepsilon$, that is, $x*y \in S$. Consequently, $S$ is a subalgebra of $X$. □

**Theorem 3.7.** *Let $\varepsilon > 0.5$. If the characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in)$-fuzzy subalgebra of $X$, then $S$ is a subalgebra of $X$.*

*Proof.* Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, which implies that

$$\mu_S^{(\varepsilon,\delta)}(x) + \varepsilon = \varepsilon + \varepsilon > 1 \text{ and } \mu_S^{(\varepsilon,\delta)}(y) + \varepsilon = \varepsilon + \varepsilon > 1,$$

that is, $x_\varepsilon\, q\, \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon\, q\, \mu_S^{(\varepsilon,\delta)}$. Since $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in)$-fuzzy subalgebra of $X$, it follows that $(x * y)_\varepsilon = (x * y)_{\min\{\varepsilon,\varepsilon\}} \in \mu_S^{(\varepsilon,\delta)}$ and so that $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$, that is, $x * y \in S$. Therefore $S$ is a subalgebra of $X$. □

**Theorem 3.8.** *Assume that $\varepsilon > 0.5$ and $\varepsilon + \delta \leq 1$. If the characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is a $(q, q)$-fuzzy subalgebra of $X$, then $S$ is a subalgebra of $X$.*

*Proof.* Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, which implies that

$$\mu_S^{(\varepsilon,\delta)}(x) + \varepsilon = \varepsilon + \varepsilon > 1 \text{ and } \mu_S^{(\varepsilon,\delta)}(y) + \varepsilon = \varepsilon + \varepsilon > 1,$$

that is, $x_\varepsilon \, q \, \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon \, q \, \mu_S^{(\varepsilon,\delta)}$. Since $\mu_S^{(\varepsilon,\delta)}$ is a $(q, q)$-fuzzy subalgebra of $X$, it follows that $(x*y)_\varepsilon = (x*y)_{\min\{\varepsilon,\varepsilon\}} \, q \, \mu_S^{(\varepsilon,\delta)}$. Hence $\mu_S^{(\varepsilon,\delta)}(x*y) > 1-\varepsilon \geq \delta$, and therefore $\mu_S^{(\varepsilon,\delta)}(x*y) = \varepsilon$. This proves that $x * y \in S$, and $S$ is a subalgebra of $X$. □

**Theorem 3.9.** *Assume that* $\varepsilon + \delta \leq 1$. *If the characteristic fuzzy set* $\mu_S^{(\varepsilon,\delta)}$ *is an* $(\in, \in\wedge\, q\,)$*-fuzzy subalgebra of* $X$*, then* $S$ *is a subalgebra of* $X$.

*Proof.* Assume that $\varepsilon + \delta \leq 1$ and the characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in\wedge\, q\,)$-fuzzy subalgebra of $X$. Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, and so $x_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$. Hence $(x*y)_\varepsilon = (x*y)_{\min\{\varepsilon,\varepsilon\}} \in\wedge\, q \, \mu_S^{(\varepsilon,\delta)}$, that is, $(x*y)_\varepsilon = (x*y)_{\min\{\varepsilon,\varepsilon\}} \in \mu_S^{(\varepsilon,\delta)}$ and $(x * y)_\varepsilon = (x * y)_{\min\{\varepsilon,\varepsilon\}} \, q \, \mu_S^{(\varepsilon,\delta)}$. Hence $\mu_S^{(\varepsilon,\delta)}(x * y) \geq \varepsilon$ and $\mu_S^{(\varepsilon,\delta)}(x * y) + \varepsilon > 1$. If $\mu_S^{(\varepsilon,\delta)}(x * y) \geq \varepsilon$, then $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$ and thus $x * y \in S$. If $\mu_S^{(\varepsilon,\delta)}(x * y) + \varepsilon > 1$, then $\mu_S^{(\varepsilon,\delta)}(x * y) > 1 - \varepsilon \geq \delta$ and so $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$, which shows that $x * y \in S$. Therefore $S$ is a subalgebra of $X$. □

**Theorem 3.10.** *Assume that* $\varepsilon > 0.5$ *and* $\varepsilon + \delta \leq 1$. *If the characteristic fuzzy set* $\mu_S^{(\varepsilon,\delta)}$ *is a* $(q, \in\wedge\, q\,)$*-fuzzy subalgebra or a* $(q, \in\vee\, q\,)$*-fuzzy subalgebra of* $X$*, then* $S$ *is a subalgebra of* $X$.

*Proof.* Let $x, y \in S$. Then $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, which implies that

$$\mu_S^{(\varepsilon,\delta)}(x) + \varepsilon = \varepsilon + \varepsilon > 1 \text{ and } \mu_S^{(\varepsilon,\delta)}(y) + \varepsilon = \varepsilon + \varepsilon > 1,$$

that is, $x_\varepsilon \, q \, \mu_S^{(\varepsilon,\delta)}$ and $y_\varepsilon \, q \, \mu_S^{(\varepsilon,\delta)}$. If $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in\wedge\, q\,)$-fuzzy subalgebra of $X$, then

$$(x * y)_\varepsilon = (x * y)_{\min\{\varepsilon,\varepsilon\}} \in\wedge\, q \, \mu_S^{(\varepsilon,\delta)},$$

that is, $\mu_S^{(\varepsilon,\delta)}(x * y) \geq \varepsilon$ and $\mu_S^{(\varepsilon,\delta)}(x * y) + \varepsilon > 1$. If $\mu_S^{(\varepsilon,\delta)}(x * y) \geq \varepsilon$, then $x * y \in S$. If $\mu_S^{(\varepsilon,\delta)}(x * y) + \varepsilon > 1$, then $\mu_S^{(\varepsilon,\delta)}(x * y) > 1 - \varepsilon \geq \delta$ and so $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$. Thus $x * y \in S$, and therefore $S$ is a subalgebra of $X$.

If $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in\vee\, q\,)$-fuzzy subalgebra of $X$, then $(x * y)_\varepsilon = (x * y)_{\min\{\varepsilon,\varepsilon\}} \in\vee\, q \, \mu_S^{(\varepsilon,\delta)}$, and so that $(x * y)_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$ or $(x * y)_\varepsilon \, q \, \mu_S^{(\varepsilon,\delta)}$. If $(x * y)_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$, then $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$ and so $x * y \in S$. If $(x * y)_\varepsilon \, q \, \mu_S^{(\varepsilon,\delta)}$, then $\mu_S^{(\varepsilon,\delta)}(x * y) + \varepsilon > 1$. Since $\varepsilon + \delta \leq 1$, it follows that $\mu_S^{(\varepsilon,\delta)}(x * y) > 1 - \varepsilon \geq \delta$ and so that $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$. Thus $x * y \in S$. Therefore $S$ is a subalgebra of $X$. □

**Lemma 3.11.** *We have the following relations among the types of* $(\in, \in\vee q)$, $(\in\vee q, \in)$, $(\in\vee q, q)$, $(\in\vee q, \in\wedge q)$, *and* $(\in\vee q, \in\vee q)$*:*

$$\begin{tikzcd} & (\in, \in\vee q) & \\ & (\in\vee q, \in\vee q) \arrow[u, Rightarrow] & \\ (\in\vee q, \in) \arrow[ur, Rightarrow] & (\in\vee q, \in\wedge q) \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[u, Rightarrow] & (\in\vee q, q) \arrow[ul, Rightarrow] \end{tikzcd} \tag{3.1}$$

Combining Lemma 3.11 and Theorem 3.4, we have the following corollary.

**Corollary 3.12.** *Assume that* $\varepsilon \leq 0.5$. *If the characteristic fuzzy set* $\mu_S^{(\varepsilon,\delta)}$ *is any one of an* $(\alpha, \beta)$*-fuzzy subalgebra of* $X$ *with* $(\alpha, \beta) \in \{(\in\vee q, \in), (\in\vee q, \in\wedge q), (\in\vee q, \in\vee q)\}$, *then* $S$ *is a subalgebra of* $X$.

**Theorem 3.13.** *Assume that* $\varepsilon + \delta \leq 1$. *If the characteristic fuzzy set* $\mu_S^{(\varepsilon,\delta)}$ *is a* $(\in\vee q, q)$*-fuzzy subalgebra of* $X$, *then* $S$ *is a subalgebra of* $X$.

*Proof.* If $S$ is not a subalgebra of $X$, then there exists $a, b \in S$ such that $a * b \notin S$. Thus $\mu_S^{(\varepsilon,\delta)}(a) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(b)$ and $\mu_S^{(\varepsilon,\delta)}(a * b) = \delta$. Hence $a_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$ and $b_\varepsilon \in \mu_S^{(\varepsilon,\delta)}$, which imply that $a_\varepsilon \in\vee q\, \mu_S^{(\varepsilon,\delta)}$ and $b_\varepsilon \in\vee q\, \mu_S^{(\varepsilon,\delta)}$. Since $\mu_S^{(\varepsilon,\delta)}(a * b) + \varepsilon = \delta + \varepsilon \leq 1$, we have $(a * b)_\varepsilon\, \overline{q}\, \mu_S^{(\varepsilon,\delta)}$. This is a contradiction, and so $S$ is a subalgebra of $X$. □

## 4. Conditional $(\alpha, \beta)$-fuzzy subalgebras

We begin with a definition.

**Definition 4.1.** Let $\mathcal{R} := \{\rho \in (0, 1] \mid \rho \text{ has relations to } \varepsilon \text{ and/or } \delta\}$. A characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ in $X$ is called an $\mathcal{R}$*-conditional* $(\alpha, \beta)$*-fuzzy subalgebra* of $X$, where $\alpha, \beta \in \{\in, q, \in\vee q, \in\wedge q\}$ and $\alpha \neq \in\wedge q$, if it satisfies the following condition:

$$(\forall x, y \in X)\,(\forall \rho_1, \rho_2 \in \mathcal{R})\left(x_{\rho_1} \alpha\, \mu_S^{(\varepsilon,\delta)},\ y_{\rho_2} \alpha\, \mu_S^{(\varepsilon,\delta)} \ \Rightarrow\ (x * y)_{\min\{\rho_1, \rho_2\}}\, \beta\, \mu_S^{(\varepsilon,\delta)}\right). \tag{4.1}$$

**Example 4.2.** (1) Let $X = \{0, 1, 2, 3, 4\}$ be a set with the following Cayley table:

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 | 1 |
| 2 | 2 | 2 | 0 | 2 | 0 |
| 3 | 3 | 1 | 3 | 0 | 3 |
| 4 | 4 | 4 | 2 | 4 | 0 |

Then $X$ is a $BCK$-algebra (see [10]). If we take $\mathcal{R} = \{\rho \in (0,1] \mid 0.3 < \rho \leq 0.7\}$, then $\mu_S^{(\varepsilon,\delta)}$ with $S := \{0,2,4\}$ is an $\mathcal{R}$-conditional $(\in, \in\wedge q)$-fuzzy subalgebra of $X$ where $\delta = 0.2$ and $\varepsilon = 0.7$.

(2) For a fixed element $a$ of a $BCI$-algebra $X$, let

$$S := \{x \in X \mid a * (a * x) = x\}.$$

For $\delta = 0.3$ and $\varepsilon = 0.6$, if we consider $\mathcal{R}_1 = \{\rho \in (0,1] \mid \rho > 0.4\}$ then $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}_1$-conditional $(\in, q)$-fuzzy subalgebra of $X$. If we take $\mathcal{R}_2 = \{\rho \in (0,1] \mid \rho \leq 0.6\}$ then $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}_2$-conditional $(q, \in)$-fuzzy subalgebra of $X$.

(3) Let $X = \{0,1,2,a,b\}$ be a set with the following Cayley table:

| $*$ | 0 | 1 | 2 | $a$ | $b$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | $a$ | $a$ |
| 1 | 1 | 0 | 1 | $a$ | $a$ |
| 2 | 2 | 2 | 0 | $a$ | $a$ |
| $a$ | $a$ | $a$ | $a$ | 0 | 0 |
| $b$ | $b$ | $a$ | $b$ | 1 | 0 |

Then $X$ is a $BCI$-algebra (see [5, 10]). Consider $\mathcal{R} = \{\rho \in (0,1] \mid 0.3 < \rho \leq 0.9\}$, Then $\mu_S^{(\varepsilon,\delta)}$ with $S := \{0,1,2\}$ is an $\mathcal{R}$-conditional $(q,q)$-fuzzy subalgebra of $X$ where $\delta = 0.1$ and $\varepsilon = 0.7$.

(4) Let $X$ be a $BCI$-algebra and let $S := \{x * a \mid x \in X\}$ for a fixed element $a \in X$. Consider $\mathcal{R} = \{\rho \in (0,1] \mid 0.3 < \rho \leq 0.7\}$. Then $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional $(q, \in\wedge q)$-fuzzy subalgebra of $X$ with $\delta = 0.1$ and $\varepsilon = 0.7$.

**Theorem 4.3.** *Let $\mathcal{R} := \{\rho \in (0,1] \mid \rho > \delta \text{ and } \varepsilon + \rho > 1\}$. If $S$ is a subalgebra of $X$, then $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional $(\in, q)$-fuzzy subalgebra of $X$.*

*Proof.* Let $x, y \in X$ and $\rho_1, \rho_2 \in \mathcal{R}$ be such that $x_{\rho_1} \alpha\, \mu_S^{(\varepsilon,\delta)}$ and $y_{\rho_2} \alpha\, \mu_S^{(\varepsilon,\delta)}$. Then $\mu_S^{(\varepsilon,\delta)}(x) \geq \rho_1 > \delta$ and $\mu_S^{(\varepsilon,\delta)}(y) \geq \rho_2 > \delta$, which imply that $x, y \in S$. Thus $x * y \in S$, and so $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$. Hence

$$\mu_S^{(\varepsilon,\delta)}(x * y) + \min\{\rho_1, \rho_2\} = \varepsilon + \min\{\rho_1, \rho_2\} > 1,$$

that is, $(x * y)_{\min\{\rho_1,\rho_2\}}\, q\, \mu_S^{(\varepsilon,\delta)}$. Therefore $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional $(\in, q)$-fuzzy subalgebra of $X$. $\square$

If we take $\varepsilon = 1$ and $\delta = 0$ in Theorems 3.6 and 4.3, then we have the following corollary.

**Corollary 4.4.** *A non-empty subset $S$ of $X$ is a subalgebra of $X$ if and only if the characteristic function $\chi_S$ of $S$ is an $(\in, q)$-fuzzy subalgebra of $X$.*

**Theorem 4.5.** *Let $\mathcal{R} := \{\rho \in (0,1] \mid \varepsilon \geq \rho \text{ and } \delta \leq 1-\rho\}$. If $S$ is a subalgebra of $X$, then $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional $(q, \in)$-fuzzy subalgebra of $X$.*

*Proof.* Let $x, y \in X$ and $\rho_1, \rho_2 \in \mathcal{R}$ be such that $x_{\rho_1} \, q \, \mu_S^{(\varepsilon,\delta)}$ and $y_{\rho_2} \, q \, \mu_S^{(\varepsilon,\delta)}$. Then $\mu_S^{(\varepsilon,\delta)}(x) + \rho_1 > 1$ and $\mu_S^{(\varepsilon,\delta)}(y) + \rho_2 > 1$, which imply that $\mu_S^{(\varepsilon,\delta)}(x) > 1 - \rho_1 \geq \delta$ and $\mu_S^{(\varepsilon,\delta)}(u) > 1 - \rho_2 \geq \delta$. Hence $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, and so $x, y \in S$. Since $S$ is a subalgebra of $X$, we have $x * y \in S$. Thus $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon \geq \min\{\rho_1, \rho_2\}$, and hence $(x * y)_{\min\{\rho_1,\rho_2\}} \in \mu_S^{(\varepsilon,\delta)}$. Therefore $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional $(q, \in)$-fuzzy subalgebra of $X$. □

If we take $\varepsilon = 1$ and $\delta = 0$ in Theorems 3.7 and 4.5, then we have the following corollary.

**Corollary 4.6.** *A non-empty subset $S$ of $X$ is a subalgebra of $X$ if and only if the characteristic function $\chi_S$ of $S$ is a $(q, \in)$-fuzzy subalgebra of $X$.*

**Theorem 4.7.** *Let $\mathcal{R} := \{\rho \in (0,1] \mid \delta \leq 1 - \rho < \varepsilon\}$. If $S$ is a subalgebra of $X$, then the characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional $(q, q)$-fuzzy subalgebra of $X$.*

*Proof.* Let $x, y \in X$ and $\rho_1, \rho_2 \in \mathcal{R}$ be such that $x_{\rho_1} \, q \, \mu_S^{(\varepsilon,\delta)}$ and $y_{\rho_2} \, q \, \mu_S^{(\varepsilon,\delta)}$. Then $\mu_S^{(\varepsilon,\delta)}(x) + \rho_1 > 1$ and $\mu_S^{(\varepsilon,\delta)}(y) + \rho_2 > 1$, which imply that $\mu_S^{(\varepsilon,\delta)}(x) > 1 - \rho_1 \geq \delta$ and $\mu_S^{(\varepsilon,\delta)}(y) > 1 - \rho_2 \geq \delta$. It follows that $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$ and so that $x, y \in S$. Since $S$ is a subalgebra of $X$, we have $x * y \in S$ and so $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon$. Thus

$$\mu_S^{(\varepsilon,\delta)}(x * y) + \min\{\rho_1, \rho_2\} = \varepsilon + \min\{\rho_1, \rho_2\} > 1,$$

that is, $(x * y)_{\min\{\rho_1,\rho_2\}} \, q \, \mu_S^{(\varepsilon,\delta)}$. This shows that $\mu_S^{(\varepsilon,\delta)}$ is a $(q, q)$-fuzzy subalgebra of $X$. □

If we take $\varepsilon = 1$ and $\delta = 0$ in Theorems 3.8 and 4.7, then we have the following corollary.

**Corollary 4.8.** *A non-empty subset $S$ of $X$ is a subalgebra of $X$ if and only if the characteristic function $\chi_S$ of $S$ is a $(q, q)$-fuzzy subalgebra of $X$.*

Since the $(q, \in\vee\, q)$-fuzzy subalgebra is induced by a $(q, \in)$-fuzzy subalgebra or a $(q, q)$-fuzzy subalgebra, we have the following corollary by using Theorems 4.5 and 4.7.

**Corollary 4.9.** *Let $\mathcal{R}$ be any one of*

$$\{\rho \in (0,1] \mid \delta \leq 1 - \rho < \varepsilon\} \text{ and } \{\rho \in (0,1] \mid \varepsilon \geq \rho \text{ and } \delta \leq 1 - \rho\}.$$

*If $S$ is a subalgebra of $X$, then the characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional $(q, \in\vee\, q)$-fuzzy subalgebra of $X$.*

**Theorem 4.10.** *Let $\mathcal{R} := \{\rho \in (0,1] \mid \delta < \rho \leq \varepsilon$ and $1-\rho < \varepsilon\}$. If $S$ is a subalgebra of $X$, then the characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional $(\in, \in\wedge q)$-fuzzy subalgebra of $X$.*

*Proof.* Let $x, y \in X$ and $\rho_1, \rho_2 \in \mathcal{R}$ be such that $x_{\rho_1} \in \mu_S^{(\varepsilon,\delta)}$ and $x_{\rho_2} \in \mu_S^{(\varepsilon,\delta)}$. Then $\mu_S^{(\varepsilon,\delta)}(x) \geq \rho_1 > \delta$ and $\mu_S^{(\varepsilon,\delta)}(y) \geq \rho_2 > \delta$, which imply that $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$. Hence $x, y \in S$. Since $S$ is a subalgebra of $X$, we have $x * y \in S$. Hence $\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon \geq \min\{\rho_1, \rho_2\}$, i.e., $(x * y)_{\min\{\rho_1,\rho_2\}} \in \mu_S^{(\varepsilon,\delta)}$. Now,

$$\mu_S^{(\varepsilon,\delta)}(x * y) + \min\{\rho_1, \rho_2\} = \varepsilon + \min\{\rho_1, \rho_2\} > 1$$

and so $(x * y)_{\min\{\rho_1,\rho_2\}}\, q\, \mu_S^{(\varepsilon,\delta)}$. Therefore $(x * y)_{\min\{\rho_1,\rho_2\}} \in\wedge q\, \mu_S^{(\varepsilon,\delta)}$, and consequently $\mu_S^{(\varepsilon,\delta)}$ is an $(\in, \in\wedge q)$-fuzzy subalgebra of $X$. □

If we take $\varepsilon = 1$ and $\delta = 0$ in Theorems 3.9 and 4.10, then we have the following corollary.

**Corollary 4.11.** *A non-empty subset $S$ of $X$ is a subalgebra of $X$ if and only if the characteristic function $\chi_S$ of $S$ is an $(\in, \in\wedge q)$-fuzzy subalgebra of $X$.*

**Theorem 4.12.** *Let $\mathcal{R} := \{\rho \in (0,1] \mid \varepsilon \geq \rho$ and $\varepsilon + \rho > 1 \geq \delta + \rho\}$. If $S$ is a subalgebra of $X$, then the characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ is an $\mathcal{R}$-conditional $(q, \in\wedge q)$-fuzzy subalgebra of $X$.*

*Proof.* Let $x, y \in X$ and $\rho_1, \rho_2 \in \mathcal{R}$ be such that $x_{\rho_1}\, q\, \mu_S^{(\varepsilon,\delta)}$ and $y_{\rho_2}\, q\, \mu_S^{(\varepsilon,\delta)}$. Then $\mu_S^{(\varepsilon,\delta)}(x) + \rho_1 > 1$ and $\mu_S^{(\varepsilon,\delta)}(y) + \rho_2 > 1$, which imply that $\mu_S^{(\varepsilon,\delta)}(x) > 1 - \rho_1 \geq \delta$ and $\mu_S^{(\varepsilon,\delta)}(y) > 1 - \rho_2 \geq \delta$. Hence $\mu_S^{(\varepsilon,\delta)}(x) = \varepsilon = \mu_S^{(\varepsilon,\delta)}(y)$, and so $x, y \in S$. Since $S$ is a subalgebra of $X$, we have $x * y \in S$ and thus

$$\mu_S^{(\varepsilon,\delta)}(x * y) = \varepsilon \geq \min\{\rho_1, \rho_2\},$$

that is, $(x * y)_{\min\{\rho_1,\rho_2\}} \in \mu_S^{(\varepsilon,\delta)}$. Now, $\mu_S^{(\varepsilon,\delta)}(x * y) + \min\{\rho_1, \rho_2\} = \varepsilon + \min\{\rho_1, \rho_2\} > 1$, and so $(x * y)_{\min\{\rho_1,\rho_2\}}\, q\, \mu_S^{(\varepsilon,\delta)}$. Hence $(x * y)_{\min\{\rho_1,\rho_2\}} \in\wedge q\, \mu_S^{(\varepsilon,\delta)}$, and $\mu_S^{(\varepsilon,\delta)}$ is a $(q, \in\wedge q)$-fuzzy subalgebra of $X$. □

If we take $\varepsilon = 1$ and $\delta = 0$ in Theorems 3.10 and 4.12, then we have the following corollary.

**Corollary 4.13.** *A non-empty subset $S$ of $X$ is a subalgebra of $X$ if and only if the characteristic function $\chi_S$ of $S$ is an $(q, \in\wedge q)$-fuzzy subalgebra of $X$.*

Before ending our research, we pose an open question.

**Question.** *Given a subalgebra $S$ of $X$, when will the characteristic fuzzy set $\mu_S^{(\varepsilon,\delta)}$ in $X$ be a*

(1) *conditional* $(\in\vee q, \in)$*-fuzzy subalgebra of* $X$*?*
(2) *conditional* $(\in\vee q, q)$*-fuzzy subalgebra of* $X$*?*
(3) *conditional* $(\in\vee q, \in\vee q)$*-fuzzy subalgebra of* $X$*?*
(4) *conditional* $(\in\vee q, \in\wedge q)$*-fuzzy subalgebra of* $X$*?*

## 5. Conclusion

we have introduced the notions of characteristic fuzzy sets, as a generalization of crisp characteristic function, and conditional fuzzy subalgebra. Using this notion, we have discussed conditions for a subset of $BCK/BCI$-algebra to be a subalgebra. Given a subalgebra of $BCK/BCI$-algebras, we have provided conditions for the characteristic fuzzy set to be a conditional $(\in, q)$-fuzzy subalgebra, a conditional $(q, \in)$-fuzzy subalgebra, a conditional $(q, q)$-fuzzy subalgebra, a conditional $(\in, \in\wedge q)$-fuzzy subalgebra, and a conditional $(q, \in\wedge q)$-fuzzy subalgebra.

On the basis of these results, we will apply the notions of characteristic fuzzy sets and conditional fuzzy substructures to ideal and filter theory in several algebraic structures, for example, $BCK/BCI$-algebras, $MV$-algebras, $BL$-algebras, $MTL$-algebras, residuated lattices, $R_0$-algebras, lattice implication algebras, $EQ$-algebras etc.


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# *Some common fixed point theorems for two pairs of self maps in dislocated metric spaces*

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**Abstract.** The purpose of this paper is to establish some common fixed point theorems for two pairs of self mappings in dislocated metric spaces which generalize, extend, and improve related results in the literature. Some applications of new results are provided.

**Keywords:** Dislocated metric space; fixed point; weakly compatible maps; common fixed point.

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# 1 Introduction and preliminaries

In 2000, Hitzler and Seda [4] presented the concept of dislocated metric space and generalized the well-known Banach contraction mapping principle in complete dislocated metric spaces. In recent years, the study of dislocated metric spaces has always attracted interest of researchers; see, for instance, [1–3, 5–8, 10–16] and the references cited therein. One of the main reasons for this lies in the fact that dislocated metric spaces play very important roles not only in topology but also in other branches of science involving mathematics especially in logic programming and electronics engineering.

In the present paper, we prove some common fixed point theorems in the setting of dislocated metric spaces for two pairs of weakly compatible self mappings which generalize, extend, and improve related results reported in the literature. We need the following auxiliary definitions and results.

**Definition 1.1** [4] *Let $\mathcal{X}$ be a nonempty set and let $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a function satisfying the following conditions:*
$(c_1)$ $d(\xi, \eta) = d(\eta, \xi)$ *for all* $\xi, \eta \in \mathcal{X}$;
$(c_2)$ $d(\xi, \eta) = d(\eta, \xi) = 0$ *implies that* $\xi = \eta$;
$(c_3)$ $d(\xi, \eta) \leq d(\xi, \zeta) + d(\zeta, \eta)$ *for all* $\xi, \eta, \zeta \in \mathcal{X}$.
*Then $d$ is called dislocated metric (or d-metric) on $\mathcal{X}$. The nonempty set $\mathcal{X}$ together with d-metric, i.e., $(\mathcal{X}, d)$, is called a dislocated metric space.*

**Definition 1.2** [4] *A sequence $\{\xi_n\}$ in a d-metric space $(\mathcal{X}, d)$ is called Cauchy sequence if for given $\epsilon > 0$, there exists an $n_0 \in N$ such that $d(\xi_m, \xi_n) < \epsilon$ for all $m, n \geq n_0$.*

**Definition 1.3** [4] *A sequence $\{\xi_n\}$ in a d-metric space $(\mathcal{X}, d)$ converges with respect to $d$ (or in $d$) if there exists a $\xi \in \mathcal{X}$ such that*

$$\lim_{n\to\infty} d(\xi_n, \xi) = 0.$$

*In this case, $\xi$ is called a limit of sequence $\{\xi_n\}$ and we write $\xi_n \to \xi$ as $n \to \infty$.*

**Definition 1.4** [4] *A d-metric space $(\mathcal{X}, d)$ is called complete if every Cauchy sequence in it is convergent with respect to $d$.*

**Definition 1.5** [4] *Let $(\mathcal{X}, d)$ be a d-metric space. A mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is called contraction if there exists a $\lambda \in [0, 1)$ such that $d(\mathcal{T}\xi, \mathcal{T}\eta) \leq \lambda d(\xi, \eta)$ for all $\xi, \eta \in \mathcal{X}$.*

**Lemma 1.6** [11] *Let $(\mathcal{X}, d)$ be a d-metric space. If $g : \mathcal{X} \to \mathcal{X}$ is a contraction function, then $\{g^n(\xi_0)\}$ is a Cauchy sequence for each $\xi_0 \in \mathcal{X}$.*

**Lemma 1.7** [4] *Limits in a d-metric space are unique.*

**Definition 1.8** [9] *Let $A$ and $S$ be mappings from a metric space $(\mathcal{X}, d)$ into itself. Then, $A$ and $S$ are said to be weakly compatible if they commute at their coincident points; that is, $A\xi = S\xi$ for some $\xi \in \mathcal{X}$ yields $AS\xi = SA\xi$.*

**Theorem 1.9** [4] *Let $(\mathcal{X}, d)$ be a complete dislocated metric space and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a contraction mapping. Then, $\mathcal{T}$ has a unique fixed point.*

**Remark 1.10** [3] *It is easy to verify that in a d-metric space, the following statements hold.*
*$(i)$ A subsequence of a Cauchy sequence in d-metric space is a Cauchy sequence.*
*$(ii)$ A Cauchy sequence in d-metric space with a convergent subsequence is also convergent.*
*$(iii)$ Limits of a convergent sequence are unique.*
*$(iv)$ A d-metric $d$ is continuous, i.e., $\xi_n \to \xi$ and $\eta_n \to \eta$ imply that $d(\xi_n, \eta_n) \to d(\xi, \eta)$ as $n \to \infty$.*

# 2 Main results

In this section, we prove some fixed point theorems in d-metric spaces.

**Theorem 2.1** *Let $(\mathcal{X}, d)$ be a complete dislocated metric space. Assume that $A, B, S, T : \mathcal{X} \to \mathcal{X}$ are continuous self mappings satisfying the conditions:*
*$(i)$ $T(\mathcal{X}) \subset A(\mathcal{X})$ and $S(\mathcal{X}) \subset B(\mathcal{X})$;*
*$(ii)$ the pairs $(S, A)$ and $(T, B)$ are weakly compatible;*
*$(iii)$ $d(S\xi, T\eta) \leq a_1[d(A\xi, T\eta) + d(B\eta, S\xi)] + a_2[d(B\eta, T\eta) + d(A\xi, S\xi)] + a_3 d(A\xi, B\eta) + a_4[d(A\xi, S\xi) + d(A\xi, T\eta)] + a_5[d(A\xi, B\eta) + d(B\eta, T\eta)]$ for all $\xi, \eta \in \mathcal{X}$, where $a_1, a_2, a_3, a_4, a_5 \geq 0$ and $0 \leq 4a_1 + 2a_2 + a_3 + 3a_4 + 2a_5 < 1$. Then $A$, $B$, $S$, and $T$ have a unique common fixed point.*

**Proof.** Define two sequences $\{\xi_n\}$ and $\{\eta_n\}$ by

$$\eta_{2n} := B\xi_{2n+1} = S\xi_{2n} \text{ and } \eta_{2n+1} := A\xi_{2n+2} = T\xi_{2n+1} \text{ for } n = 0, 1, 2, \ldots.$$

If $\eta_{2n} = \eta_{2n+1}$ for some $n$, then $B\xi_{2n+1} = T\xi_{2n+1}$. Therefore, $\xi_{2n+1}$ is a coincident point of $B$ and $T$. Also, if $\eta_{2n+1} = \eta_{2n+2}$ for some $n$, then $A\xi_{2n+2} = S\xi_{2n+2}$. Hence, $\xi_{2n+2}$ is a coincidence point of $A$ and $S$. Suppose now that $\eta_{2n} \neq \eta_{2n+1}$ for all $n$. Then, we conclude that

$$\begin{aligned}
d(\eta_{2n}, \eta_{2n+1}) &= d(S\xi_{2n}, T\xi_{2n+1}) \\
&\leq a_1[d(A\xi_{2n}, T\xi_{2n+1}) + d(B\xi_{2n+1}, S\xi_{2n})] \\
&\quad +a_2[d(B\xi_{2n+1}, T\xi_{2n+1}) + d(A\xi_{2n}, S\xi_{2n})] \\
&\quad +a_3 d(A\xi_{2n}, B\xi_{2n+1}) + a_4[d(A\xi_{2n}, S\xi_{2n}) + d(A\xi_{2n}, T\xi_{2n+1})] \\
&\quad +a_5[d(A\xi_{2n}, B\xi_{2n+1}) + d(B\xi_{2n+1}, T\xi_{2n+1})] \\
&\leq a_1[d(\eta_{2n-1}, \eta_{2n+1}) + d(\eta_{2n}, \eta_{2n})] \\
&\quad +a_2[d(\eta_{2n}, \eta_{2n+1}) + d(\eta_{2n-1}, \eta_{2n})] \\
&\quad +a_3 d(\eta_{2n-1}, \eta_{2n}) + a_4[d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n-1}, \eta_{2n+1})] \\
&\quad +a_5[d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n}, \eta_{2n+1})] \\
&\leq a_1[d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n}, \eta_{2n+1}) \\
&\quad +d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n}, \eta_{2n+1})] \\
&\quad +a_2[d(\eta_{2n}, \eta_{2n+1}) + d(\eta_{2n-1}, \eta_{2n})] + a_3 d(\eta_{2n-1}, \eta_{2n}) \\
&\quad +a_4[d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n}, \eta_{2n+1})] \\
&\quad +a_5[d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n}, \eta_{2n+1})] \\
&= (2a_1 + a_2 + a_3 + 2a_4 + a_5) d(\eta_{2n-1}, \eta_{2n}) \\
&\quad +(2a_1 + a_2 + a_4 + a_5) d(\eta_{2n}, \eta_{2n+1}).
\end{aligned}$$

Therefore, we get

$$d(\eta_{2n}, \eta_{2n+1}) \leq \frac{2a_1 + a_2 + a_3 + 2a_4 + a_5}{1 - (2a_1 + a_2 + a_4 + a_5)} d(\eta_{2n-1}, \eta_{2n}).$$

Let

$$h = \frac{2a_1 + a_2 + a_3 + 2a_4 + a_5}{1 - (2a_1 + a_2 + a_4 + a_5)} < 1.$$

Then

$$d(\eta_n, \eta_{n+1}) \leq h d(\eta_{n-1}, \eta_n).$$

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Similarly, we have

$$d(\eta_{n-1},\eta_n) \le hd(\eta_{n-2},\eta_{n-1}).$$

Continuing this process, we obtain

$$d(\eta_n,\eta_{n+1}) \le h^n d(\eta_0,\eta_1).$$

Now, for any $m,n$ satisfying $m > n$, using triangle inequality, we get

$$\begin{aligned} d(\eta_n,\eta_m) &\le d(\eta_n,\eta_{n+1}) + d(\eta_{n+1},\eta_{n+2}) + \cdots + d(\eta_{m-1},\eta_m) \\ &\le h^n d(\eta_0,\eta_1) + h^{n+1} d(\eta_0,\eta_1) + \cdots + h^{m-1} d(\eta_0,\eta_1) \\ &\le (h^n + h^{n+1} + h^{n+2} + \cdots) d(\eta_0,\eta_1) \\ &= \frac{h^n}{1-h} d(\eta_0,\eta_1). \end{aligned}$$

Since $h \in [0,1)$, $h^n \to 0$ as $n \to \infty$, which shows that $\{\eta_n\}$ is a Cauchy sequence in the complete dislocated metric space $(\mathcal{X},d)$. Hence, there exists a point $p \in \mathcal{X}$ such that $\lim_{n\to\infty} \eta_n = p$ and

$$\lim_{n\to\infty} S\xi_{2n} = \lim_{n\to\infty} B\xi_{2n+1} = \lim_{n\to\infty} T\xi_{2n+1} = \lim_{n\to\infty} A\xi_{2n+2} = p.$$

Since $T(\mathcal{X}) \subset A(\mathcal{X})$, there exists a point $v \in \mathcal{X}$ such that $p = Av$. Therefore,

$$\begin{aligned} d(Sv,p) &= d(Sv,T\xi_{2n+1}) \\ &\le a_1[d(Av,T\xi_{2n+1}) + d(B\xi_{2n+1},Sv)] \\ &\quad +a_2[d(B\xi_{2n+1},T\xi_{2n+1}) + d(Av,Sv)] \\ &\quad +a_3 d(Av,Sv) + a_4[d(Av,Sv) + d(Au,T\xi_{2n+1})] \\ &\quad +a_5[d(Av,B\xi_{2n+1}) + d(B\xi_{2n+1},T\xi_{2n+1})]. \end{aligned}$$

Taking $n \to \infty$, we get

$$\begin{aligned} d(Sv,p) &\le a_1[d(p,p) + d(p,Sv)] + a_2[d(p,p) + d(p,Sv)] \\ &\quad +a_3 d(p,Sv) + a_4[d(p,Sv) + d(p,p)] + a_5[d(p,p) + d(p,p)] \\ &\le (2a_1 + 2a_2 + 2a_4 + 4a_5) d(p,Sv) + (a_1 + a_2 + a_3 + a_4) d(p,Sv) \\ &= (3a_1 + 3a_2 + a_3 + 3a_4 + 4a_5) d(p,Sv), \end{aligned}$$

which is a contradiction, and so $Sv = Av = p$. Again, since $S(\mathcal{X}) \subset B(\mathcal{X})$, there exists a point $u \in \mathcal{X}$ such that $p = Bu$. We claim now that $p = Tu$. If

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$p \neq Tu$, then

$$\begin{aligned}
d(p,Tu) &= d(Sv,Tu) \\
&\leq a_1[d(Av,Tu)+d(Bu,Sv)]+a_2[d(Bu,Tu)+d(Av,Sv)] \\
&\quad +a_3 d(Av,Bu)+a_4[d(Av,Sv)+d(Av,Tu)] \\
&\quad +a_5[d(Av,Bu)+d(Bu,Tu)] \\
&= a_1[d(p,Tu)+d(p,p)]+a_2[d(p,Tu)+d(p,p)]+a_3 d(p,p) \\
&\quad +a_4[d(p,p)+d(p,Tu)]+a_5[d(p,p)+d(p,Tu)] \\
&\leq (3a_1+3a_2+2a_3+3a_4+3a_5)d(p,Tu),
\end{aligned}$$

which is a contradiction, and thus $p = Tu$. Hence, we have $Sv = Av = Tu = Bu = p$. Since $(S, A)$ are weakly compatible, $SAv = ASv$ implies that $Sp = Ap$. Next, we show that $p$ is the fixed point of $S$. If $Sp \neq p$, then

$$\begin{aligned}
d(Sp,p) &= d(Sp,Tu) \\
&\leq a_1[d(Ap,Tu)+d(Bu,Sp)]+a_2[d(Bu,Tu)+d(Ap,Sp)] \\
&\quad +a_3 d(Ap,Bu)+a_4[d(Ap,Sp)+d(Ap,Tu)] \\
&\quad +a_5[d(Ap,Bu)+d(Bu,Tu)] \\
&= a_1[d(Sp,p)+d(p,Sp)]+a_2[d(p,p)+d(Sp,Sp)]+a_3 d(Sp,p) \\
&\quad +a_4[d(Sp,Sp)+d(Sp,p)]+a_5[d(Sp,p)+d(p,p)] \\
&\leq (2a_1+4a_2+a_3+3a_4+3a_5)d(Sp,p),
\end{aligned}$$

which is a contradiction, and so $Sp = p$. This yields $Ap = Sp = p$. Again, $(T, B)$ are weakly compatible, and hence $TBu = BTu$ implies that $Tp = Bp$. Now, we show that $p$ is the fixed point of $T$. If $Tp \neq p$, then

$$\begin{aligned}
d(p,Tp) &= d(Sp,Tp) \\
&\leq a_1[d(Ap,Tp)+d(Bp,Sp)]+a_2[d(Bp,Tp)+d(Ap,Sp)] \\
&\quad +a_3 d(Ap,Sp)+a_4[d(Ap,Sp)+d(Ap,Tp)] \\
&\quad +a_5[d(Ap,Bp)+d(Bp,Tp)] \\
&= a_1[d(p,Tp)+d(Tp,p)]+a_2[d(Tp,Tp)+d(p,p)] \\
&\quad +a_3 d(p,Tp)+a_4[d(p,Tp)+d(p,Tp)] \\
&\quad +a_5[d(p,Tp)+d(Tp,Tp)] \\
&\leq (2a_1+4a_2+a_3+2a_4+3a_5)d(p,Tp),
\end{aligned}$$

which is a contradiction, and hence $p = Tp$. Therefore, we have $Ap = Bp = Sp = Tp = p$, which shows that $p$ is the common fixed point of the self mappings $A$, $B$, $S$, and $T$.

**Uniqueness**. Suppose that $v \neq u$ are two common fixed points of the mappings $A$, $B$, $S$, and $T$. Then, we have

$$\begin{aligned}
d(v,u) &= d(Sv,Tu) \\
&\leq a_1[d(Av,Tu)+d(Bu,Sv)]+a_2[d(Bu,Tu)+d(Av,Sv)] \\
&\quad +a_3d(Av,Bu)+a_4[d(Av,Sv)+d(Av,Tu)] \\
&\quad +a_5[d(Av,Bu)+d(Bu,Tu)] \\
&= a_1[d(v,u)+d(u,v)]+a_2[d(u,u)+d(v,v)]+a_3d(v,u) \\
&\quad +a_4[d(v,v)+d(v,u)]+a_5[d(v,u)+d(u,u)] \\
&\leq (2a_1+4a_2+a_3+3a_4+3a_5)d(v,u),
\end{aligned}$$

which is a contradiction, and therefore $v = u$. The proof is complete. □

Letting $A = B = I$ (an identity mapping), we can derive the following result from Theorem 2.1.

**Corollary 2.2** *Let $(\mathcal{X}, d)$ be a complete dislocated metric space. If $S, T : \mathcal{X} \to \mathcal{X}$ are continuous self mappings satisfying*

$$\begin{aligned}
d(S\xi,T\eta) &\leq a_1[d(\xi,T\eta)+d(\eta,S\xi)]+a_2[d(\eta,T\eta)+d(\xi,S\xi)]+a_3d(\xi,\eta) \\
&\quad +a_4[d(\xi,S\xi)+d(\xi,T\eta)]+a_5[d(\xi,\eta)+d(\eta,T\eta)]
\end{aligned}$$

*for all $\xi, \eta \in \mathcal{X}$, where $a_1, a_2, a_3, a_4, a_5 \geq 0$ and $0 \leq 4a_1+2a_2+a_3+3a_4+2a_5 < 1$, then $S$ and $T$ have a unique common fixed point.*

If $a_4 = a_5 = 0$ and $S = T$, then Corollary 2.2 reduces to the following result obtained by Isufati [6].

**Corollary 2.3** *Let $(\mathcal{X}, d)$ be a complete dislocated metric space. If $T : \mathcal{X} \to \mathcal{X}$ is a continuous self mapping satisfying*

$$d(T\xi,T\eta) \leq a_1[d(\xi,T\eta)+d(\eta,T\xi)]+a_2[d(\eta,T\eta)+d(\xi,T\xi)]+a_3d(\xi,\eta)$$

*for all $\xi, \eta \in \mathcal{X}$, where $a_1, a_2, a_3 \geq 0$ and $0 \leq 4a_1+2a_2+a_3 < 1$, then $T$ has a unique fixed point.*

Letting $a_4 = a_5 = 0$ in Theorem 2.1, we get the following result reported by Panthi and Jha [11].

**Corollary 2.4** *Let $(\mathcal{X}, d)$ be a complete dislocated metric space. If $A, B, S, T : \mathcal{X} \to \mathcal{X}$ are continuous self mappings satisfying the conditions:*
*$(i)$ $T(\mathcal{X}) \subset A(\mathcal{X})$ and $S(\mathcal{X}) \subset B(\mathcal{X})$;*
*$(ii)$ the pairs $(S, A)$ and $(T, B)$ are weakly compatible;*
*$(iii)$ $d(S\xi, T\eta) \leq a_1[d(A\xi, T\eta) + d(B\eta, S\xi)] + a_2[d(B\eta, T\eta) + d(A\xi, S\xi)] + a_3 d(A\xi, B\eta)$ for all $\xi, \eta \in \mathcal{X}$, where $a_1, a_2, a_3 \geq 0$ and $0 \leq 4a_1 + 2a_2 + a_3 < 1$, then $A$, $B$, $S$, and $T$ have a unique common fixed point.*

**Remark 2.5** *Our results improve those obtained by Aage and Salunke [1, 2], Jha and Panthi [7], Jha et al. [8], Rao and Rangaswamy [12], and Shrivastava et al. [14].*

# 3 Further results without any continuity requirement

In this section, we prove some fixed point theorems without any continuity requirement in d-metric spaces.

**Theorem 3.1** *Let $(\mathcal{X}, d)$ be a complete dislocated metric space. Suppose that $A, B, S, T : \mathcal{X} \to \mathcal{X}$ are self mappings satisfying the conditions:*
*$(i)$ $T(\mathcal{X}) \subset A(\mathcal{X})$ and $S(\mathcal{X}) \subset B(\mathcal{X})$;*
*$(ii)$ the pairs $(S, A)$ and $(T, B)$ are weakly compatible;*
*$(iii)$ $d(S\xi, T\eta) \leq a_1[d(A\xi, T\eta) + d(B\eta, S\xi)] + a_2[d(B\eta, T\eta) + d(A\xi, S\xi)] + a_3 d(A\xi, B\eta) + a_4[d(A\xi, S\xi) + d(A\xi, T\eta)] + a_5[d(A\xi, B\eta) + d(B\eta, T\eta)]$ for all $\xi, \eta \in \mathcal{X}$, where $a_1, a_2, a_3, a_4, a_5 \geq 0$ and $0 \leq 4a_1 + 2a_2 + a_3 + 3a_4 + 2a_5 < 1$, then $A$, $B$, $S$, and $T$ have a unique common fixed point.*

**Proof.** Let $\xi_0 \in \mathcal{X}$ be arbitrary. Choose $\xi_1 \in \mathcal{X}$ such that $B\xi_1 = S\xi_0$. Again, choose $\xi_2 \in \mathcal{X}$ such that $A\xi_2 = T\xi_1$. Continuing this process, choose $\xi_n \in \mathcal{X}$ such that $S\xi_{2n} = B\xi_{2n+1}$ and $T\xi_{2n+1} = A\xi_{2n+2}$ for $n = 0, 1, 2, \ldots$. To simplify, we consider the sequence $\{\eta_n\}$ which is defined by $\eta_{2n} := S\xi_{2n}$ and $\eta_{2n+1} := T\xi_{2n+1}$ for $n = 0, 1, 2, \ldots$. Next, we claim that $\{\eta_n\}$ is a Cauchy sequence. Indeed, for $n \geq 1$, we have

$$\begin{aligned} d(\eta_{2n}, \eta_{2n+1}) &= d(S\xi_{2n}, T\xi_{2n+1}) \\ &\leq a_1[d(A\xi_{2n}, T\xi_{2n+1}) + d(B\xi_{2n+1}, S\xi_{2n})] \\ &\quad +a_2[d(A\xi_{2n}, S\xi_{2n}) + d(B\xi_{2n+1}, T\xi_{2n+1})] \\ &\quad +a_3 d(A\xi_{2n}, B\xi_{2n+1}) + a_4[d(A\xi_{2n}, S\xi_{2n}) + d(A\xi_{2n}, T\xi_{2n+1})] \\ &\quad +a_5[d(A\xi_{2n}, B\xi_{2n+1}) + d(B\xi_{2n+1}, T\xi_{2n+1})] \\ &\leq a_1[d(\eta_{2n-1}, \eta_{2n+1}) + d(\eta_{2n}, \eta_{2n})] \\ &\quad +a_2[d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n}, \eta_{2n+1})] \\ &\quad +a_3 d(\eta_{2n-1}, \eta_{2n}) + a_4[d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n-1}, \eta_{2n+1})] \\ &\quad +a_5[d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n}, \eta_{2n+1})] \\ &\leq a_1[d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n}, \eta_{2n+1}) \\ &\quad +d(\eta_{2n}, \eta_{2n+1}) + d(\eta_{2n+1}, \eta_{2n})] \\ &\quad +a_2[d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n}, \eta_{2n+1})] + a_3 d(\eta_{2n-1}, \eta_{2n}) \\ &\quad +a_4[d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n}, \eta_{2n+1})] \\ &\quad +a_5[d(\eta_{2n-1}, \eta_{2n}) + d(\eta_{2n}, \eta_{2n+1})] \\ &= (a_1 + a_2 + a_3 + 2a_4 + a_5) d(\eta_{2n-1}, \eta_{2n}) \\ &\quad +(3a_1 + a_2 + a_4 + a_5) d(\eta_{2n}, \eta_{2n+1}). \end{aligned}$$

Hence, we conclude that

$$d(\eta_{2n}, \eta_{2n+1}) \leq h d(\eta_{2n-1}, \eta_{2n}),$$

where

$$h = \frac{a_1 + a_2 + a_3 + 2a_4 + a_5}{1 - (3a_1 + a_2 + a_4 + a_5)} \in [0, 1).$$

This implies that $\{\eta_n\}$ is a Cauchy sequence in $\mathcal{X}$. Then, by Remark 1.10, $\{S\xi_{2n}\}$, $\{B\xi_{2n+1}\}$, $\{T\xi_{2n+1}\}$, and $\{A\xi_{2n+2}\}$ are also Cauchy sequences. Assume that $S\xi$ is a complete subspace of $\mathcal{X}$, the sequence $\{S\xi_{2n}\}$ converges to some $Sa$ such that $a \in \mathcal{X}$. So, $\{\eta_n\}$, $\{B\xi_{2n+1}\}$, $\{T\xi_{2n+1}\}$, and $\{A\xi_{2n+2}\}$ also converge to $Sa$. Since $S\mathcal{X} \subset B\mathcal{X}$, there exists a $v \in \mathcal{X}$ such that $Sa = Bv$. We show that $Bv = Tv$. In fact, we have

$$\begin{aligned} d(S\xi_{2n}, Tv) &\leq a_1[d(A\xi_{2n}, Tv) + d(Bv, S\xi_{2n})] \\ &\quad +a_2[d(A\xi_{2n}, S\xi_{2n}) + d(Bv, Tv)] \\ &\quad +a_3 d(A\xi_{2n}, Bv) + a_4[d(A\xi_{2n}, S\xi_{2n}) + d(A\xi_{2n}, Tv)] \\ &\quad +a_5[d(A\xi_{2n}, Bv) + d(Bv, Tv)]. \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} d(Bv, Tv) &\leq a_1[d(Bv, Tv) + d(Bv, Bv)] + a_2[d(Bv, Bv) + d(Bv, Tv)] \\ &\quad + a_3 d(Bv, Bv) + a_4[d(Bv, Bv) + d(Bv, Tv)] \\ &\quad + a_5[d(Bv, Bv) + d(Bv, Tv)] \\ &\leq (a_1 + a_2 + a_3 + a_4 + a_5) d(Bv, Bv) \\ &\quad + (a_1 + a_2 + a_4 + a_5) d(Bv, Tv) \\ &\leq (3a_1 + 3a_2 + 2a_3 + 3a_4 + 3a_5) d(Bv, Tv). \end{aligned}$$

Therefore, $d(Bv, Tv) = 0$, which implies that $Tv = Bv$. Since $T\mathcal{X} \subset A\mathcal{X}$, there exists a $u \in \mathcal{X}$ such that $Tv = Au$. We show that $Su = Au$. Indeed, we have

$$\begin{aligned} d(Su, Au) &= d(Su, Tv) \\ &\leq a_1[d(Au, Tv) + d(Bv, Su)] + a_2[d(Au, Su) + d(Bv, Tv)] \\ &\quad + a_3 d(Au, Bv) + a_4[d(Au, Su) + d(Au, Tv)] \\ &\quad + a_5[d(Au, Bv) + d(Bv, Tv)] \\ &\leq a_1[d(Au, Au) + d(Au, Su)] + a_2[d(Au, Su) + d(Au, Au)] \\ &\quad + a_3 d(Au, Au) + a_4[d(Au, Su) + d(Au, Au)] \\ &\quad + a_5[d(Au, Au) + d(Au, Au)] \\ &\leq a_1[d(Au, Su) + d(Su, Au) + d(Au, Su)] \\ &\quad + a_2[d(Au, Su) + d(Au, Su) + d(Su, Au)] \\ &\quad + a_3[d(Au, Su) + d(Su, Au)] \\ &\quad + a_4[d(Au, Su) + d(Au, Su) + d(Su, Au)] \\ &\quad + a_5[d(Au, Su) + d(Su, Au) + d(Au, Su) + d(Su, Au)] \\ &= (3a_1 + 3a_2 + 2a_3 + 3a_4 + 4a_5) d(Au, Su). \end{aligned}$$

Hence, $d(Su, Au) = 0$, which yields $Au = Su$, and so $Bv = Tv = Au = Su$. By virtue of the fact that $(S, A)$ are weakly compatible, we deduce that $ASu = SAu$, which yields $AAu = ASu = SAu = SSu$. The weak compatibility of $B$ and $T$ implies that $BTv = TBv$, from which it follows that $BBv = BTv = TBv = TTv$. Let us show that $Bv$ is a fixed point of $T$.

In fact, we have

$$\begin{aligned}
d(Bv, TBv) &= d(Su, TBv) \\
&\leq a_1[d(Au, TBv) + d(BBv, Su)] \\
&\quad +a_2[d(Au, Su) + d(BBv, TBv)] \\
&\quad +a_3d(Au, BBv) + a_4[d(Au, Su) + d(Au, TBv)] \\
&\quad +a_5[d(Au, BBv) + d(BBv, TBv)] \\
&\leq a_1[d(Bv, TBv) + d(TBv, Bv)] \\
&\quad +a_2[d(Bv, Bv) + d(TBv, TBv)] \\
&\quad +a_3d(Bv, TBv) + a_4[d(Bv, Bv) + d(Bv, TBv)] \\
&\quad +a_5[d(Bv, TBv) + d(TBv, TBv)] \\
&\leq 2a_1d(Bv, TBv) + a_2[d(Bv, TBv) + d(TBv, Bv) \\
&\quad +d(TBv, Bv) + d(Bv, TBv)] + a_3d(Bv, TBv) \\
&\quad +a_4[d(Bv, TBv) + d(TBv, Bv) + d(Bv, TBv)] \\
&\quad +a_5[d(Bv, TBv) + d(TBv, Bv) + d(Bv, TBv)] \\
&= (2a_1 + 4a_2 + a_3 + 3a_4 + 3a_5)d(Bv, TBv),
\end{aligned}$$

which yields $d(Bv, TBv) = 0$, and so $TBv = Bv$. Therefore, $Bv$ is a fixed point of $T$. It follows that $BBv = TBv = Bv$, which implies that $Bv$ is also a fixed point of $B$. On the other hand, we get

$$\begin{aligned}
d(SBv, Bv) &= d(SBv, TBv) \\
&\leq a_1[d(ABv, TBv) + d(BBv, SBv)] + a_2[d(ABv, SBv) \\
&\quad +d(BBv, TBv)] + a_3d(ABv, BBv) + a_4[d(ABv, SBv) \\
&\quad +d(ABv, TBv)] + a_5[d(ABv, BBv) + d(BBv, TBv)] \\
&\leq a_1[d(Bv, Bv) + d(Bv, SBv)] + a_2[d(SBv, Bv) \\
&\quad +d(Bv, SBv)] + a_3d(Bv, Bv) + a_4[d(Bv, Bv) \\
&\quad +d(Bv, Bv)] + a_5[d(Bv, Bv) + d(Bv, Bv)] \\
&\leq a_1[d(Bv, SBv) + d(SBv, Bv) + d(Bv, SBv)] \\
&\quad +a_2[d(SBv, Bv) + d(Bv, SBv)] + a_3[d(Bv, SBv) \\
&\quad +d(SBv, Bv)] + a_4[d(Bv, SBv) + d(SBv, Bv) \\
&\quad +d(Bv, SBv) + d(SBv, Bv)] + a_5[d(Bv, SBv) \\
&\quad +d(SBv, Bv) + d(Bv, SBv) + d(SBv, Bv)] \\
&= (3a_1 + 2a_2 + 2a_3 + 4a_4 + 4a_5)d(SBv, Bv),
\end{aligned}$$

which implies that $d(Bv, SBv) = 0$, and hence $SBv = Bv$. Therefore, $Bv$ is a fixed point of $S$. It follows that $ABv = SBv = Bv$, which shows that $Bv$ is also a fixed point of $A$. Then $Bv$ is a common fixed point of $A$, $B$, $S$, and $T$.

**Uniqueness**. Let $w, u \in \mathcal{X}$ be two fixed points such that $Aw = Bw = Sw = Tw$ and $Au = Bu = Su = Tu$. If $d(w, u) \neq 0$, then

$$\begin{aligned} d(w,u) &= d(Sw, Tu) \\ &\leq a_1[d(Aw, Tu) + d(Bu, Sw)] + a_2[d(Bu, Tu) + d(Aw, Sw)] \\ &\quad +a_3 d(Aw, Bu) + a_4[d(Aw, Sw) + d(Aw, Tu)] \\ &\quad +a_5[d(Aw, Bu) + d(Bu, Tu)] \\ &= a_1[d(w,u) + d(u,w)] + a_2[d(u,u) + d(w,w)] + a_3 d(w,u) \\ &\quad +a_4[d(w,w) + d(w,u)] + a_5[d(w,u) + d(u,u)] \\ &\leq (2a_1 + 4a_2 + a_3 + 3a_4 + 3a_5) d(w,u), \end{aligned}$$

which is a contradiction. Hence, $d(v, u) = 0$, which implies that $v = u$. The proof is complete. □

**Remark 3.2** *One can derive from Theorem 3.1 a number of fixed point theorems for self mappings $A$, $B$, $S$, and $T$. For example, we have the following result by letting $a_4 = a_5 = 0$.*

**Corollary 3.3** *Let $(\mathcal{X}, d)$ be a complete dislocated metric space. If $A, B, S, T : \mathcal{X} \to \mathcal{X}$ are self mappings satisfying the conditions:*
*(i) $T(\mathcal{X}) \subset A(\mathcal{X})$ and $S(\mathcal{X}) \subset B(\mathcal{X})$;*
*(ii) the pairs $(S, A)$ and $(T, B)$ are weakly compatible;*
*(iii) $d(S\xi, T\eta) \leq a_1[d(A\xi, T\eta) + d(B\eta, S\xi)] + a_2[d(B\eta, T\eta) + d(A\xi, S\xi)] + a_3 d(A\xi, B\eta)$ for all $\xi, \eta \in \mathcal{X}$, where $a_1, a_2, a_3 \geq 0$ and $0 \leq 4a_1 + 2a_2 + a_3 < 1$, then $A$, $B$, $S$, and $T$ have a unique common fixed point.*

**Remark 3.4** *Our results generalize, extend, and improve those obtained by Bennani et al. [3].*

# 4 Examples

The following examples illustrate theoretical results obtained in the previous sections.

**Example 4.1** *Assume that* $\mathcal{X} = [0, 1]$, $d$ *is a usual metric, and define the mappings* $A$, $B$, $S$, *and* $T$ *by*

$$A\xi = \xi, \quad B\xi = \xi, \quad S\xi = 0, \quad \textit{and} \quad T\xi = \frac{1}{12}\xi.$$

*Let*

$$a_1 = \frac{1}{20}, \quad a_2 = \frac{1}{24}, \quad a_3 = \frac{1}{28}, \quad a_4 = \frac{1}{30}, \quad \textit{and} \quad a_5 = \frac{1}{34}.$$

*Then* $A$, $B$, $S$, *and* $T$ *satisfy all assumptions of Theorem 2.1. As a matter of fact,* $0 \in \mathcal{X}$ *is the unique common fixed point of the mappings* $A$, $B$, $S$, *and* $T$.

**Example 4.2** *Let* $\mathcal{X} = [0, 1]$, $d(\xi, \eta) = |\xi| + |\eta|$, *and define the mappings* $A$, $B$, $S$, *and* $T$ *by*

$$A\xi = \xi, \quad B\xi = \xi, \quad S\xi = 0, \quad \textit{and} \quad T\xi = \frac{\xi}{6}.$$

*Set*

$$a_1 = \frac{1}{25}, \quad a_2 = \frac{1}{28}, \quad a_3 = \frac{1}{32}, \quad a_4 = \frac{1}{36}, \quad \textit{and} \quad a_5 = \frac{1}{40}.$$

*Then* $A$, $B$, $S$, *and* $T$ *satisfy all assumptions of Theorem 3.1. In fact,* $0 \in \mathcal{X}$ *is the unique common fixed point of the mappings* $A$, $B$, $S$, *and* $T$.

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# QUADRATIC $\rho$-FUNCTIONAL INEQUALITIES IN NON-ARCHIMEDEAN BANACH SPACES

SUNGSIK YUN

ABSTRACT. In this paper, we solve the quadratic $\rho$-functional inequalities

$$\begin{aligned} &\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\ &\leq \left\|\rho\left(4f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)-2f(y)\right)\right\|, \end{aligned} \tag{0.1}$$

where $\rho$ is a fixed non-Archimedean number with $|\rho|<|2|$, and

$$\begin{aligned} &\left\|4f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)-2f(y)\right\| \\ &\leq \|\rho(f(x+y)+f(x-y)-2f(x)-2f(y))\|, \end{aligned} \tag{0.2}$$

where $\rho$ is a fixed non-Archimedean number with $|\rho|<1$.

Furthermore, we prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequalities (0.1) and (0.2) in non-Archimedean Banach spaces.

## 1. INTRODUCTION AND PRELIMINARIES

A *valuation* is a function $|\cdot|$ from a field $K$ into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs|=|r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s|\leq|r|+|s|, \qquad \forall r,s\in K.$$

A field $K$ is called a *valued field* if $K$ carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s|\leq\max\{|r|,|s|\}, \qquad \forall r,s\in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1|=|-1|=1$ and $|n|\leq 1$ for all $n\in\mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0|=0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

**Definition 1.1.** ([8]) Let $X$ be a vector space over a field $K$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\|: X\to[0,\infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\|=0$ if and only if $x=0$;

(ii) $\|rx\|=|r|\|x\| \qquad (r\in K, x\in X)$;

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(iii) the strong triangle inequality

$$\|x+y\| \le \max\{\|x\|, \|y\|\}, \qquad \forall x, y \in X$$

holds. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

**Definition 1.2.** (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space $X$. Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$ there is a positive integer $N$ such that

$$\|x_n - x_m\| \le \varepsilon$$

for all $n, m \ge N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space $X$. Then the sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$ there are a positive integer $N$ and an $x \in X$ such that

$$\|x_n - x\| \le \varepsilon$$

for all $n \ge N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} x_n = x$.

(iii) If every Cauchy sequence in $X$ converges, then the non-Archimedean normed space $X$ is called a *non-Archimedean Banach space*.

The stability problem of functional equations originated from a question of Ulam [18] concerning the stability of group homomorphisms. The functional equation $f(x+y) = f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [7] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [11] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [6] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The functional equation $f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called the *Jensen equation*.

The functional equation $f(x+y)+f(x-y) = 2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [17] for mappings $f : E_1 \to E_2$, where $E_1$ is a normed space and $E_2$ is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain $E_1$ is replaced by an Abelian group. The functional equation $2f\left(\frac{x+y}{2}\right) + 2\left(\frac{x-y}{2}\right) = f(x) + f(y)$ is called a *Jensen type quadratic equation*. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 3, 4, 9, 10, 12, 13, 14, 15, 16, 19, 20]).

In Section 2, we solve the quadratic $\rho$-functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequality (0.1) in non-Archimedean Banach spaces.

In Section 3, we solve the quadratic $\rho$-functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequality (0.2) in non-Archimedean Banach spaces.

Throughout this paper, assume that $X$ is a non-Archimedean normed space and that $Y$ is a non-Archimedean Banach space. Let $|2| \ne 1$.

## 2. Quadratic $\rho$-functional inequality (0.1) in non-Archimedean normed spaces

Throughout this section, assume that $\rho$ is a fixed non-Archimedean number with $|\rho| < |2|$.

In this section, we solve the quadratic $\rho$-functional inequality (0.1) in non-Archimedean normed spaces.

**Lemma 2.1.** *If a mapping $f : G \to Y$ satisfies*

$$\begin{aligned}\|f(x+y)+f(x-y)-2f(x)-2f(y)\| & \\ \leq \left\|\rho\left(4f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)-2f(y)\right)\right\| &\end{aligned} \tag{2.1}$$

*for all $x, y \in G$, then $f : G \to Y$ is quadratic.*

*Proof.* Assume that $f : G \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $\|2f(0)\| \leq |\rho|\|f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.1), we get $\|f(2x) - 4f(x)\| \leq 0$ and so $f(2x) = 4f(x)$ for all $x \in G$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{2.2}$$

for all $x \in G$.

It follows from (2.1) and (2.2) that

$$\begin{aligned}\|f(x+y)+f(x-y)-2f(x)-2f(y)\| & \\ \leq \left\|\rho\left(4f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)-2f(y)\right)\right\| & \\ = |\rho|\|f(x+y)+f(x-y)-2f(x)-2f(y)\| &\end{aligned}$$

and so

$$f(x+y)+f(x-y) = 2f(x)+2f(y)$$

for all $x, y \in G$. □

Now, we prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequality (2.1) in non-Archimedean Banach spaces.

**Theorem 2.2.** *Let $r < 2$ and $\theta$ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying*

$$\begin{aligned}\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \ &\leq \ \left\|\rho\left(4f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)-2f(y)\right)\right\| \\ &+ \ \theta(\|x\|^r+\|y\|^r)\end{aligned} \tag{2.3}$$

*for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x)-Q(x)\| \leq \frac{2}{|2|^r}\theta\|x\|^r \tag{2.4}$$

*for all $x \in X$.*

*Proof.* Letting $x = y = 0$ in (2.3), we get $\|f(0)\| \leq |\rho|\|2f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.3), we get

$$\|f(2x)-4f(x)\| \leq 2\theta\|x\|^r \tag{2.5}$$

for all $x \in X$. So $\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \le \frac{2}{|2|^r}\theta\|x\|^r$ for all $x \in X$. Hence

$$\begin{aligned}
&\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \qquad (2.6)\\
&\le \max\left\{\left\|4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&= \max\left\{|4|^l \left\|f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |4|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\le \max\left\{\frac{|4|^l}{|2|^{rl}}, \cdots, \frac{|4|^{m-1}}{|2|^{r(m-1)}}\right\}\frac{2}{|2|^r}\theta\|x\|^r = \frac{\theta}{|2|^{(r-2)l}}\frac{2}{|2|^r}\|x\|^r
\end{aligned}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.6) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.4).

It follows from (2.3) that

$$\begin{aligned}
&\|Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y)\|\\
&= \lim_{n\to\infty} |4|^n \left\|f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right\|\\
&\le \lim_{n\to\infty} |4|^n|\rho| \left\|4f\left(\frac{x+y}{2^{n+1}}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\right\| + \lim_{n\to\infty}\frac{|4|^n\theta}{|2|^{nr}}(\|x\|^r + \|y\|^r)\\
&= |\rho| \left\|4Q\left(\frac{x+y}{2}\right) + Q(x-y) - 2Q(x) - 2Q(y)\right\|
\end{aligned}$$

for all $x, y \in X$. So

$$\|Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y)\| \le \left\|\rho\left(4Q\left(\frac{x+y}{2}\right) + Q(x-y) - 2Q(x) - 2Q(y)\right)\right\|$$

for all $x, y \in X$. By Lemma 2.1, the mapping $h : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.4). Then we have

$$\begin{aligned}
\|Q(x) - T(x)\| &= \left\|4^q Q\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right)\right\|\\
&\le \max\left\{\left\|4^q Q\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\|, \left\|4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\|\right\} \le \frac{2}{|2|^{(r-2)q+r}}\theta\|x\|^r,
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of $Q$. Thus the mapping $Q : X \to Y$ is a unique quadratic mapping satisfying (2.4). □

**Theorem 2.3.** *Let $r > 2$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.3). *Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{2\theta}{|4|}\|x\|^r$$

*for all $x \in X$.*

*Proof.* It follows from (2.5) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{2\theta}{|4|}\|x\|^r$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

## 3. Quadratic $\rho$-functional inequality (0.2)

Throughout this section, assume that $\rho$ is a fixed non-Archimedean number with $|\rho| < 1$.

In this section, we solve the quadratic $\rho$-functional inequality (0.2) in non-Archimedean normed spaces.

**Lemma 3.1.** *If a mapping $f : G \to Y$ satisfies*

$$\begin{aligned} &\left\|4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right\| \\ &\leq \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| \end{aligned} \tag{3.1}$$

*for all $x, y \in G$, then $f : G \to Y$ is quadratic.*

*Proof.* Assume that $f : G \to Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $\|f(0)\| \leq |\rho|\|2f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.1), we get $\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq 0$ and so

$$4f\left(\frac{x}{2}\right) = f(x) \tag{3.2}$$

for all $x \in G$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &= \left\|4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right\| \\ &\leq |\rho|\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in G$. □

Now, we prove the Hyers-Ulam stability of the quadratic $\rho$-functional inequality (3.1) in non-Archimedean Banach spaces.

**Theorem 3.2.** *Let $r < 2$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying*

$$\begin{aligned} \left\|4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right\| &\leq \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| \\ &+ \theta(\|x\|^r + \|y\|^r) \end{aligned} \tag{3.3}$$

*for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leq \theta\|x\|^r \tag{3.4}$$

*for all $x \in X$.*

*Proof.* Letting $x = y = 0$ in (3.3), we get $\|2f(0)\| \le |\rho|\|f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.3), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \le \theta\|x\|^r \tag{3.5}$$

for all $x \in X$. So

$$\begin{aligned}
&\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \qquad (3.6)\\
&\quad \le \max\left\{\left\|4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\quad = \max\left\{|4|^l \left\|f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |4|^{m-1} \left\|f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\quad \le \max\left\{\frac{|4|^l}{|2|^{rl}}, \cdots, \frac{|4|^{m-1}}{|2|^{r(m-1)}}\right\}\theta\|x\|^r = \frac{\theta}{|2|^{(r-2)l}}\|x\|^r
\end{aligned}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (3.6) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.4).

The rest of the proof is similar to the proof of Theorem 2.2. □

**Theorem 3.3.** *Let $r > 2$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (3.3). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{|2|^r\theta}{|4|}\|x\|^r \tag{3.7}$$

*for all $x \in X$.*

*Proof.* It follows from (3.5) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{|2|^r\theta}{|4|}\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}
&\left\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\right\| \qquad (3.8)\\
&\quad \le \max\left\{\left\|\frac{1}{4^l}f\left(2^l x\right) - \frac{1}{4^{l+1}}f\left(2^{l+1}x\right)\right\|, \cdots, \left\|\frac{1}{4^{m-1}}f\left(2^{m-1}x\right) - \frac{1}{4^m}f\left(2^m x\right)\right\|\right\}\\
&\quad = \max\left\{\frac{1}{|4|^l}\left\|f\left(2^l x\right) - \frac{1}{4}f\left(2^{l+1}x\right)\right\|, \cdots, \frac{1}{|4|^{m-1}}\left\|f\left(2^{m-1}x\right) - \frac{1}{4}f\left(2^m x\right)\right\|\right\}\\
&\quad \le \max\left\{\frac{|2|^{rl}}{|4|^{l+1}}, \cdots, \frac{|2|^{r(m-1)}}{|4|^{(m-1)+1}}\right\}|2|^r\theta\|x\|^r = \frac{|2|^r\theta}{|2|^{(2-r)l+2}}\|x\|^r
\end{aligned}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (3.8) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence

$\{\frac{1}{4^n}f(2^nx)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get (3.7).

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.2. □

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# A Fast Inversion Free Iterative Algorithm for Solving $X + A^*X^{-1}A = I$

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**Abstract**

We introduce a new inversion free variant of the basic fixed point iteration method for obtaining a maximal positive definite solution of the nonlinear matrix equation $X + A^*X^{-1}A = I$ with $A$ normal. It has fewer operations and matrix-matrix multiplications than the existing algorithms. We derive convergence conditions for the iteration and some numerical results to illustrate the behavior of the new algorithm.

**AMS classification**: 15A24; 65F10; 65F35

**Key words:** Nonlinear matrix equation; Hermitian positive definite solution; Fixed point iteration method; Convergence rate.

## 1 Introduction

Consider the nonlinear matrix equation

$$X + A^*X^{-1}A = I, \tag{1.1}$$

where $A$ is an $n \times n$ complex normal matrix and $I$ the identity matrix. Here $A^*$ stands for the conjugate transpose of $A$.

Nonlinear matrix equation (1.1) has many applications. It often arises in control theory, dynamic programming, ladder networks, stochastic filtering, statistics, and etc.; see [1, 3–11, 14–23] and the references therein. It is well known that $X$ is a solution of (1.1) if and only if it solves

$$X = I - A^*(I - A^*X^{-1}A)^{-1}A.$$

Assuming that $A$ is invertible, we can write the above equation as

$$X = F^*(R + X^{-1})^{-1}F + I,$$

where $F = A^{-*}A$ and $R = -A^{-*}A^{-1}$. This is a special case of the discrete algebraic Riccati equation

$$X - F^*(R + X^{-1})^{-1}F - I = 0,$$

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where $I = I^*$ and $R = R^*$ is invertible. For more details about the discrete algebraic Riccati equation, we refer to [2, 13].

In [17], Zhan proposed the following inversion free iteration

$$M1 : \begin{cases} X_{n+1} = I - A^* Y_n A, \\ Y_{n+1} = Y_n(2I - X_n Y_n), \end{cases}$$

starting from $X_0 = Y_0 = I$.

In [11], Guo and Lancaster proposed the following inversion free iteration

$$M2 : \begin{cases} Y_{n+1} = Y_n(2I - X_n Y_n), \\ X_{n+1} = I - A^* Y_{n+1} A, \end{cases}$$

starting from $X_0 = Y_0 = I$.

When $A$ is a nonsingular matrix, Monsalve and Raydan proposed in [16] the following inversion free iteration

$$M3 : \begin{cases} X_0 = AA^*, \\ X_{n+1} = 2X_n - X_n A^{-*}(I - X_n)A^{-1}X_n, \quad n = 0, 1, \ldots \end{cases}$$

to solve the minimal solution. The maximal solution of (1.1) can be obtained through $X_+ = I - Y_-$, where $Y_-$ is the minimal solution of the dual equation $Y + AY^{-1}A^* = I$. M3 generates a Hermitian sequence of $X_n$. The implementation of iteration M3 involves three matrix-matrix multiplications per iteration and the inverse operation of $A$ at the beginning only.

In [8], El-Sayed and Al-Dbiban proposed an algorithm that avoids the matrix inversion for every iteration, called an inversion free variant of the basic fixed point iteration.

$$M4 : \begin{cases} Y_{n+1} = (I - X_n)Y_n + I, \\ X_{n+1} = I - A^* Y_{n+1} A, \end{cases}$$

starting from $X_0 = Y_0 = I$.

It is important to notice that M1 and M2 generate a Hermitian sequence and require four matrix-matrix multiplications per iteration, while M4 requires three matrix-matrix multiplications per iteration but does not generate a Hermitian sequence. If $A$ is a normal matrix, we will prove that M4 generates a Hermitian sequence. And we propose the following algorithm.

$$M5 : \begin{cases} Y_0 = I, \\ Y_{n+1} = (AY_n)^*(AY_n) + I, \quad n = 0, 1, \ldots. \end{cases}$$

The algorithm indicated by M5 is an inverse-free iterative method. Notice that it only requires to compute $X = I - A^* Y A$ at the end of the process, and only needs two matrix-matrix multiplications per iteration. Therefore it is clearly inexpensive. By an inductive argument, it is also worth noticing that in M5, $Y_n$ is a Hermitian matrix for all $n$.

The following notations will be used throughout the paper. Let $\mathbb{C}^{n\times n}$ be the set of $n \times n$ complex matrices. The notation $B \geq 0 (B > 0)$ means that $B$ is a Hermitian positive semi-definite (definite) matrix. Moreover, $B \geq C (B > C)$ is used as a different notation for $B - C \geq 0 (B - C > 0)$. This induces a partial ordering on the Hermitian matrices. The symbols $\rho(A)$ and $\|A\|$ denote the spectral radius and the spectral norm of a square matrix $A$, respectively.

## 2 Conditions for the Existence of Solutions

The following lemmas are needed for our purpose.

**Lemma 2.1.** *For Algorithm M4, if $A$ is normal, then*

$$AX_n = X_n A, \quad AY_n = Y_n A,$$

$$A^* X_n = X_n A^*, \quad A^* Y_n = Y_n A^*$$

*for $n = 0, 1, \ldots$.*

*Proof.* Since $Y_1 = Y_0 = X_0 = I$,

$$AY_0 = Y_0 A, \quad AY_1 = Y_1 A, \quad AX_0 = X_0 A.$$

Because

$$X_1 = I - A^* Y_1 A = I - A^* A,$$
$$Y_2 = (I - X_1) Y_1 + I = A^* A + I,$$

and $AA^* = A^* A$,

$$AY_2 = A(A^* A + I) = (A^* A + I) A = Y_2 A.$$

That is, $AY_n = Y_n A$ is true for n=0, 1, 2. So, assume that $AY_n = Y_n A$ is true for $n = k$. Now we prove that $AY_n = Y_n A$ when $n = k + 1$. In fact

$$AY_{k+1} = A((I - X_k) Y_k + I) = AA^* Y_k A Y_k + A = Y_{k+1} A.$$

This completes the induction for $n = k + 1$. Therefore,

$$AY_n = Y_n A$$

for $n = 0, 1, 2, \ldots$. We also have

$$AX_{n+1} = A(I - A^* Y_{n+1} A) = A - A^* Y_{n+1} AA + A = X_{n+1} A$$

for $n = 1, 2, \ldots$. The proof of $A^* X_n = X_n A^*$ and $A^* Y_n = Y_n A^*$ are similar to that of $AX_n = X_n A$ and $AY_n = Y_n A$, respectively. □

**Lemma 2.2.** *If $A$ is normal, then Algorithm M4 generates a Hermitian sequence.*

*Proof.* Since $Y_1 = Y_0 = X_0 = I$, $Y_1$, $Y_0$, and $X_0$ are Hermitian. Assume that $Y_n$ is Hermitian for $n = k$. Then

$$Y_{k+1}^* = ((I - X_k) Y_k + I)^* = (A^* Y_k A Y_k + I)^* = Y_k A^* Y_k A + I.$$

If $A$ is a normal matrix, then according to Lemma 2.1,

$$Y_{k+1}^* = A^* Y_k A Y_k + I = Y_{k+1}.$$

This completes the induction for $n = k + 1$. Since $X_n = I - A^* Y_n A$,

$$X_n^* = I - A^* Y_n^* A = I - A^* Y_n A = X_n$$

for $n = 1, 2, \ldots$. This completes the proof. □

**Lemma 2.3.** *For Algorithm M4, if $A$ is normal, then $\{X_n, Y_n, X_{n+1}, Y_{n+1}\}$ is a commuting family, $n = 0, 1, 2, \ldots$.*

*Proof.* Since $Y_1 = Y_0 = X_0 = I$, $X_1 = I - A^* A$, it is easy to check that $\{X_0, Y_0, X_1, Y_1\}$ is a commuting family. Now we prove that $\{X_n, Y_n, X_{n+1}, Y_{n+1}\}$ is a commuting family for $n = 1, 2, \ldots$. Since $X_n = I - A^* Y_n A$,

$$Y_{n+1} = (I - X_n) Y_n + I = A^* Y_n A Y_n + I.$$

According to Lemma 2.1,

$$\begin{aligned} Y_{n+1} Y_n &= (A^* Y_n A Y_n + I) Y_n \\ &= A^* Y_n A Y_n Y_n + Y_n \\ &= Y_n A^* Y_n A Y_n + Y_n \\ &= Y_n Y_{n+1}, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned}Y_{n+1}X_{n+1} &= Y_{n+1}(I - A^*Y_{n+1}A)\\ &= Y_{n+1} - Y_{n+1}A^*Y_{n+1}A\\ &= Y_{n+1} - A^*Y_{n+1}AY_{n+1}\\ &= X_{n+1}Y_{n+1}.\end{aligned}$$

This implies that $Y_nX_n = X_nY_n$ for $n = 1, 2, \ldots$. From (2.1) and Lemma 2.1,

$$\begin{aligned}Y_{n+1}X_n &= Y_{n+1}(I - A^*Y_nA)\\ &= Y_{n+1} - Y_{n+1}A^*Y_nA\\ &= Y_{n+1} - A^*Y_nAY_{n+1}\\ &= X_nY_{n+1},\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}X_{n+1}Y_n &= (I - A^*Y_{n+1}A)Y_n\\ &= Y_n - A^*Y_{n+1}AY_n\\ &= Y_n - Y_nA^*Y_{n+1}A\\ &= Y_nX_{n+1}.\end{aligned}$$

It follows from (2.2) and Lemma 2.1 that

$$\begin{aligned}X_{n+1}X_n &= (I - A^*Y_{n+1}A)X_n\\ &= X_n - A^*Y_{n+1}AX_n\\ &= X_n - X_nA^*Y_{n+1}A\\ &= X_nX_{n+1}.\end{aligned}$$

This completes the proof. □

**Lemma 2.4.** *If $0 < M \le N$, $0 < P \le Q$, and $\{M, N, P, Q\}$ is a commuting family, then*

$$MP \le NQ.$$

*Proof.* Since $M$, $N$, $P$, $Q$ are positive definite matrices, and $\{M, N, P, Q\}$ is a commuting family. By Theorem 2.5.5 in [12], there is a unitary $U$ such that

$$U^*MU = \Lambda, \quad U^*NU = \Omega, \quad U^*PU = \Sigma, \quad U^*MU = \Gamma,$$

where $\Lambda$, $\Omega$, $\Sigma$, and $\Gamma$ are diagonal. Since $M$, $N$, $P$, $Q$ are positive definite matrices, $\Lambda$, $\Omega$, $\Sigma$, and $\Gamma$ are positive diagonal matrices. Because $\{M, N, P, Q\}$ is a commuting family,

$$MP = U\Lambda\Sigma U = U\Lambda^{\frac{1}{2}}\Sigma\Lambda^{\frac{1}{2}}U = P^{\frac{1}{2}}MP^{\frac{1}{2}} \le P^{\frac{1}{2}}NP^{\frac{1}{2}} = N^{\frac{1}{2}}PN^{\frac{1}{2}}. \tag{2.3}$$

Since $0 < P \le Q$,

$$N^{\frac{1}{2}}PN^{\frac{1}{2}} \le N^{\frac{1}{2}}QN^{\frac{1}{2}} = NQ. \tag{2.4}$$

Combining (2.3) and (2.4), we have

$$MP \le NQ.$$

This completes the proof. □

Now, we prove that the sequence $\{X_n\}$ in Algorithm M4 is monotone decreasing and converges to the maximal solution $X_+$, and the sequence $\{Y_n\}$ in Algorithm M4 is monotone increasing and converges to $X_+^{-1}$.

**Theorem 2.5.** *Let $A$ be normal. If the nonlinear matrix equation (1.1) has a positive definite solution, and the two sequences $\{X_n\}$ and $\{Y_n\}$ are determined by Algorithm M4, then $\{X_n\}$ is monotone decreasing and converges to the maximal solution $X_+$, and $\{Y_n\}$ is monotone increasing and converges to $X_+^{-1}$.*

*Proof.* We will prove that

$$I = X_0 \geq X_1 \geq \cdots \geq X_n \geq X_+$$

and

$$I = Y_0 \leq Y_1 \leq \cdots \leq Y_n \leq X_+^{-1}.$$

Since $X_+$ is a solution of (1.1), i.e.,

$$X_+ = I - A^* X_+^{-1} A,$$

$X_0 = I \geq X_+$ and $I \leq X_+^{-1}$. Also

$$X_1 = I - A^* A \leq I = X_0$$

and

$$X_1 = I - A^* A \geq I - A^* X_+^{-1} A = X_+,$$

i.e., $I = X_0 \geq X_1 \geq X_+$.

For the sequence $\{Y_n\}$ we have $Y_0 = Y_1 = I$, and since $I \leq X_+^{-1}$, then $Y_0 = Y_1 \leq X_+^{-1}$. From Lemmas 2.3 and 2.4, on the one hand,

$$Y_2 = (I - X_1)Y_1 + I = A^* A + I \geq I = Y_1 = Y_0;$$

on the other hand

$$Y_2 = (I - X_1)Y_1 + I \leq (I - X_+)X_+^{-1} + I = X_+^{-1},$$

i.e., $Y_0 = Y_1 \leq Y_2 \leq X_+^{-1}$.

Assume that the above inequalities are true for $n = k$, i.e.,

$$I = X_0 \geq X_1 \geq \cdots \geq X_k \geq X_+ \tag{2.5}$$

and

$$I = Y_0 \leq Y_1 \leq \cdots \leq Y_k \leq X_+^{-1}. \tag{2.6}$$

Now we prove inequalities for $n = k+1$. From (2.5) and (2.6), the sequences $\{X_n\}$ and $\{Y_n\}$ are Hermitian positive definite for $i = 0, 1, \ldots, k$. According to Algorithm M4, Lemmas 2.3 and 2.4, (2.5), and (2.6), we get

$$Y_{k+1} = (I - X_k)Y_k + I \geq (I - X_{k-1})Y_{k-1} + I = Y_k,$$

$$Y_{k+1} = (I - X_k)Y_k + I \leq (I - X_+)X_+^{-1} + I = X_+^{-1},$$

i.e., $Y_k \leq Y_{k+1} \leq X_+^{-1}$. Concerning the sequence $\{X_n\}$, we have

$$X_k - X_{k+1} = A^*(Y_{k+1} - Y_k)A$$

since $Y_{k+1} \geq Y_k$. Hence $X_k \geq X_{k+1}$. Therefore,

$$X_{k+1} = I - A^* Y_{k+1} A \geq I - A^* X_+^{-1} A = X_+,$$

i.e., $X_k \geq X_{k+1} \leq X_+$.

This completes the induction for $n = k+1$. Thus,

$$I = X_0 \geq X_1 \geq \cdots \geq X_n \geq X_+$$

and

$$I = Y_0 \leq Y_1 \leq \cdots \leq Y_n \leq X_+^{-1}$$

are true for all $n$. Therefore, These are convergent sequences, i.e., $\lim_{n\to\infty} X_n$ and $\lim_{n\to\infty} Y_n$ exist. Taking limit in Algorithm M4 leads to $Y = X^{-1}$ and $X = I - A^* X^{-1} A$. Moreover, as each $X_n \geq X_+$ and $Y_n \leq X_+^{-1}$, then $X = X_+$ and $Y = X_+^{-1}$, respectively. This completes the proof. $\square$

According to Theorem 2.5, we know that the sequence $\{Y_n\}$ determined by Algorithm M4 is monotone increasing and converges to $X_+^{-1}$. From Lemmas 2.1 and 2.3, if $A$ is normal, then $\{A, Y_n, Y_{n+1}\}$ is a commuting family, $n = 0, 1, 2, \ldots$. So we can amend Algorithm M4 and obtain the following algorithm.

$$M5 : \begin{cases} Y_0 = I, \\ Y_{n+1} = (AY_n)^*(AY_n) + I, \quad n = 0, 1, \ldots. \end{cases}$$

In this algorithm, M5 generates a Hermitian sequence, and requires two matrix-matrix multiplications per iteration. Notice that it only requires to compute $X = I - A^*YA$ at the end of the process. It is an inverse-free iterative method.

If $A$ is normal, from [9, Theorem 11], the nonlinear matrix equation (1.1) has a solution if and only if $\rho(A) \leq \frac{1}{2}$. Therefore, the nonlinear matrix equation (1.1) has a solution if and only if $\|A\| \leq \frac{1}{2}$.

**Lemma 2.6.** *Let $A$ be normal. Assume that nonlinear matrix equation (1.1) has a positive definite solution and the sequence $\{Y_n\}$ is determined by Algorithm M5. Then $\{Y_n\}$ satisfies $\|AY_n\| < 1$ for every $n = 0, 1, \ldots$.*

*Proof.* Since $A$ is normal and the nonlinear matrix equation (1.1) has a positive definite solution, $\|A\| \leq \frac{1}{2}$. Because $Y_0 = I$, $\|AY_0\| = \|A\| \leq \frac{1}{2} < 1$. For $Y_1$ we have $Y_1 = A^*A + I$, thus $\|AY_1\| = \|A(A^*A + I)\| \leq \|A\|^3 + \|A\| < 1$. That is, the inequality holds for $n = 0, 1$. So, assume that the inequality satisfies $n = k$, i.e., $\|AY_k\| < 1$. Now we prove the inequality when $n = k + 1$.

$$\begin{aligned} \|AY_{k+1}\| &= \|A\left((AY_k)^*(AY_k) + I\right)\| \\ &= \|A(AY_k)^*(AY_k) + A\| \\ &\leq \|A\|\|AY_k\|^2 + \|A\| \\ &< \|A\| + \|A\| \\ &\leq 1. \end{aligned}$$

This completes the induction for $n = k + 1$ and the lemma. □

**Lemma 2.7.** *Let $A$ be normal. The maximal solution $X_+$ of the nonlinear matrix equation (1.1) commutes with $A$.*

*Proof.* If $A$ is normal, by [18], we have

$$X_+ = \frac{1}{2}\left[I + (I - 4A^*A)^{1/2}\right].$$

So, $AX_+ = X_+A$. This completes the proof. □

**Theorem 2.8.** *Let $A$ be normal. If the nonlinear matrix equation (1.1) has a positive definite solution, then the sequence $\{Y_n\}$ determined by Algorithm M5 satisfies*

$$\|Y_{n+1} - X_+^{-1}\| \leq \|AX_+^{-1}\|\|Y_n - X_+^{-1}\|$$

*for all $n$ large enough.*

*Proof.* Since the nonlinear matrix equation (1.1) has a positive definite solution, $X_+ = \frac{1}{2}\left[I + (I - 4A^*A)^{1/2}\right]$ is the maximal solution. Then

$$X_+ + A^*X_+^{-1}A = I.$$

Multiplying by $X_+^{-1}$ on the right, we obtain

$$X_+^{-1} = (X_+^{-1}A)^*AX_+^{-1} + I.$$

By Lemma 2.7,

$$X_+^{-1} = (AX_+^{-1})^*(AX_+^{-1}) + I.$$

By Algorithm M5,

$$Y_{n+1} = (AY_n)^*(AY_n) + I. \tag{2.7}$$

Subtracting $X_+^{-1}$ from both sides of (2.7) we have that

$$\begin{aligned} Y_{n+1} - X_+^{-1} &= (AY_n)^*(AY_n) + I - X_+^{-1} \\ &= (AY_n)^*(AY_n) + I - \left((AX_+^{-1})^*(AX_+^{-1}) + I\right) \\ &= (AY_n)^*(AY_n) - (AX_+^{-1})^*(AX_+^{-1}) \\ &= (AY_n)^*(AY_n) - (AX_+^{-1})^*(AY_n) + (AX_+^{-1})^*(AY_n) - (AX_+^{-1})^*(AX_+^{-1}) \\ &= (Y_n - X_+^{-1})^*A^*(AY_n) + (AX_+^{-1})^*A(Y_n - X_+^{-1}). \end{aligned} \tag{2.8}$$

Taking norms in (2.8) and recalling that $\lim\limits_{n\to\infty} Y_n = X_+^{-1}$ and $\|A\| \le \frac{1}{2}$, we have that

$$\begin{aligned} \|Y_{n+1} - X_+^{-1}\| &= \|(Y_n - X_+^{-1})^*A^*(AY_n) + (AX_+^{-1})^*A(Y_n - X_+^{-1})\| \\ &\le \|Y_n - X_+^{-1}\|\|A\| \left(\|AY_n\| + \|AX_+^{-1}\|\right) \\ &\le \|AX_+^{-1}\|\|Y_n - X_+^{-1}\|. \end{aligned}$$

This completes the proof. □

# 3 Numerical Experiments

To illustrate the performance of our method described in the previous section, in this section several interesting examples are given, which were carried out using MATLAB on a PC computer. We report the number of required iterations (denoted as IT), the norm of the residual (denoted as Res), the computing time in seconds (denoted as CPU), and the number of matrix-matrix (denoted as MM) multiplications required when the process is stopped. Assume $A$ is normal and (1.1) has a solution. Then

$$X_+ = \frac{1}{2}\left[I + (I - 4A^*A)^{1/2}\right] \tag{3.1}$$

is the maximal solution, and if $A$ is nonsingular,

$$X_- = \frac{1}{2}\left[I - (I - 4A^*A)^{1/2}\right]. \tag{3.2}$$

is the minimal solution [18, Theorem 4.1]. This allows us to test the local convergence behavior of the five methods by taking the computed $X_+$ by using (3.1) as the accurate maximal solution and $X_-$ by using (3.2) as the accurate minimal solution.

In our implementation,

$$\|Y_n - X_+^{-1}\| \le \|X_+^{-1}\| \times 10^{-9}$$

is used as the termination criterion for M1, M2, M4, and M5, and

$$\|X_n - X_-\| \le \|X_-\| \times 10^{-9}$$

is used as the termination criterion for M3. We compare our iteration M5, with the inverse-free methods M1, M2, M3 and M4 for solving (1.1). In all cases we describe also the initial guess for which convergence is guaranteed.

Since M3 converges to the minimal solution, we use M3 to find the minimal solution $Y_-$ of the dual equation $Y + AY^{-1}A^* = I$. We can obtain the maximal solution of (1.1) through $X_+ = I - Y_-$. The value of "Res" in our tables reports $\|F(X_n)\|_F = \|X_n + A^*X_n^{-1}A - I\|_F$ for M1, M2, M3, M4 and M5 when the process is stopped.

**Experiment 3.1.** *In this test, the matrix $A$ is from [8] using Example 3.1*

$$A = \frac{1}{32}\begin{pmatrix} 0.2 & -0.1 & -0.5 & 0.1 \\ -0.1 & 0.6 & -0.5 & 0.7 \\ -0.5 & -0.5 & 0.1 & 0.8 \\ 0.1 & 0.7 & 0.8 & 0.5 \end{pmatrix}$$

Table 1: Performance of M1, M2, M3, M4 and M5 to solve (1) for Experiment 3.1.

| Scheme | IT | $\|F(X_n)\|_F$ | MM |
|---|---|---|---|
| M1 | 6 | 8.7390e-12 | 24 |
| M2 | 4 | 2.9322e-14 | 16 |
| M3 | 7 | 1.7493e-11 | 21 |
| M4 | 4 | 1.1655e-13 | 12 |
| M5 | 3 | 1.1655e-13 | 6 |

*Since $A$ is normal and $\|A\| = 0.0412 \leq 0.5$, from [9, Theorem 11], the nonlinear matrix equation (1.1) has a solution. In Table 1, we can see that M1 requires more matrix-matrix multiplications than the other methods to achieve convergence. M3 requires more iterations than the other four methods to satisfy the stopping criterion. M5 carries out fewer iterations and matrixCmatrix multiplications than all the other methods. For this experiment, we could say that M5 is the best option.*

**Experiment 3.2.** *In this test, let*

$$A = \frac{1}{14}\begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{pmatrix} \in \mathbb{C}^{m\times m}.$$

Table 2: Iterations and the number of matrix-matrix products for Experiment 3.2.

| m | $\|A\|$ | M1 | | M2 | | M3 | | M4 | | M5 | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | IT | MM | IT | MM | IT | MM | IT | MM | IT | MM |
| 4 | 0.4013 | 29 | 116 | 16 | 64 | 15 | 45 | 22 | 66 | 21 | 42 |
| 8 | 0.4200 | 33 | 132 | 18 | 72 | 17 | 51 | 26 | 78 | 25 | 50 |
| 16 | 0.4261 | 34 | 136 | 19 | 76 | 18 | 54 | 27 | 81 | 26 | 52 |
| 32 | 0.4279 | 35 | 140 | 19 | 76 | 18 | 54 | 28 | 84 | 27 | 54 |
| 64 | 0.4284 | 35 | 140 | 19 | 76 | 18 | 54 | 28 | 84 | 27 | 54 |
| 128 | 0.4285 | 35 | 140 | 19 | 76 | 18 | 54 | 28 | 84 | 27 | 54 |
| 256 | 0.4286 | 35 | 140 | 19 | 76 | 18 | 54 | 28 | 84 | 27 | 54 |
| 512 | 0.4286 | 35 | 140 | 19 | 76 | 18 | 54 | 28 | 84 | 27 | 54 |
| 1024 | 0.4286 | 35 | 140 | 19 | 76 | - | - | 28 | 84 | 27 | 54 |

Table 3: CPU time(s) for Experiment 3.2.

| m | M1 | M2 | M3 | M4 | M5 |
|---|---|---|---|---|---|
| 4 | 0.002191 | 0.000315 | 0.000634 | 0.000986 | 0.000246 |
| 8 | 0.002479 | 0.000821 | 0.000937 | 0.001263 | 0.000568 |
| 16 | 0.005854 | 0.001868 | 0.003523 | 0.002451 | 0.000956 |
| 32 | 0.012124 | 0.008553 | 0.006721 | 0.006701 | 0.004385 |
| 64 | 0.019125 | 0.014864 | 0.017950 | 0.022326 | 0.013876 |
| 128 | 0.075103 | 0.046237 | 0.075727 | 0.065126 | 0.049925 |
| 256 | 0.522263 | 0.300470 | 0.479174 | 0.383668 | 0.290019 |
| 512 | 3.557083 | 2.176928 | 20.980815 | 2.392791 | 1.886120 |
| 1024 | 46.442456 | 24.082342 | - | 27.282594 | 21.436297 |

*According to Table 2, we know that $\|A\| \leq 0.5$. Since $A$ is normal and $A$ is nonsingular, the nonlinear matrix equation $X + A^*X^{-1}A = I$ has a solution. In this test, the matrix sequence of $X_n$ in M3 is badly scaled*

Table 4: Errors for Experiment 3.2.

| m | M1 | M2 | M3 | M4 | M5 |
|---|---|---|---|---|---|
| 4 | 1.9737e-10 | 5.8016e-11 | 2.2961e-10 | 1.3549e-10 | 1.3549e-10 |
| 8 | 1.8449e-10 | 6.3914e-11 | 2.1623e-10 | 9.5667e-11 | 9.5667e-11 |
| 16 | 3.2512e-10 | 5.5185e-11 | 1.8543e-10 | 1.4121e-10 | 1.4121e-10 |
| 32 | 2.5631e-10 | 9.4206e-11 | 3.0952e-10 | 1.1751e-10 | 1.1751e-10 |
| 64 | 3.8878e-10 | 1.4330e-10 | 4.6222e-10 | 1.8003e-10 | 1.8003e-10 |
| 128 | 5.6750e-10 | 2.0941e-10 | 6.7030e-10 | 2.6386e-10 | 2.6386e-10 |
| 256 | 8.1479e-10 | 3.0082e-10 | 9.5949e-10 | 3.7953e-10 | 3.7953e-10 |
| 512 | 1.1608e-9 | 4.2868e-10 | 1.3654e-09 | 5.4118e-10 | 5.4118e-10 |
| 1024 | 1.647e-19 | 6.0853e-10 | - | 7.6847e-10 | 7.6847e-10 |

*when $m = 1024$. In Table 2, we can see that M1 requires more iterations and matrix-matrix multiplications than the other methods to satisfy the stopping criterion. We can also observe that M5 needs more iterations than M2 and M3 to reach convergence, but it carries out fewer matrix-matrix multiplications than M1, M2, M4. Table 2 shows that M5 and M3 require the same number of matrix-matrix multiplications, but M5 considerably outperforms M3 in CPU time from Table 3. From Table 3 we observe that M5 outperforms the other methods in CPU time. From our numerical results, we can see that M1 is the most expensive iteration out of the five methods. For this experiment, we could say that M5 is the best option.*

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# On the Ulam-Hyers Stability of Some Differential Equations involving Hadamard Fractional Derivatives☆

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**Abstract**

In this paper, we show the Ulam-Hyers stability and Ulam-Hyers-Rassias stability criterion for nonlinear Hadamard fractional relaxation differential equations on compact and unbounded time intervals. More explicit Ulam-Hyers stability and Ulam-Hyers-Rassias results are presented by virtue of estimation of Mittag-Leffler functions.

*Keywords:* Hadamard fractional derivative, Relaxation differential equations, Ulam-Hyers stability, Ulam-Hyers-Rassias stability, Mittag-Leffler functions.

## 1. Introduction

The widely application of fractional differential equations arise in various areas of physics and engineering (see [1, 2, 3, 4]). During the past decades, fractional differential equations has been more and more recognized as an alternative model to the classical differential equations. There are many interesting advance on the theory analysis for Caputo type and Riemann-Liouville type fractional differential equations as well as Hadamard type fractional differential equations (see, for example, [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]).

The well-known Ulam stability problem of functional equations originated are posed in 1940. Numerous monographs and special issues have appeared devoted to the theory of Ulam stability for functional equations and differential equations (see for example [16, 17, 18, 19, 20, 21, 22, 23]). Recently, Li and Wang [24] explore some fundamental properties of continuity, integrable estimation, asymptotic property on Mittag-Leffler functions for a Hadamard fractional differential equation with constant coefficient and present existence results for such equation by using fixed point theorems. However, to our knowledge, Ulam's stability results for nonlinear Hadamard fractional differential equation with constant coefficient have not been investigated extensively. Especially, there are few research on the Ulam's stability for this kind of equation on noncompact interval.

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In this paper, we investigate Ulam's type stability of Hadamard fractional differential equations with constant coefficient $\lambda \in \mathbb{R} \setminus \{0\}$ of the type:

$$ {}_H D_{1^+}^{\alpha} y(x) = \lambda y(x) + f(x, y(x)),\ 0 < \alpha < 1,\ x \in J = (1, e] \text{ or } (e, \infty), \tag{1} $$

where ${}_H D_{1^+}^{\alpha}$ denotes the left-sided Hadamard fractional derivative of order $\alpha$ with the low limit 1 (see Definition 2.2), and nonlinear term $f : J \times \mathbb{R} \to \mathbb{R}$ is a given function satisfying some certain conditions. Let $\varepsilon > 0$ and $\varphi : J \to \mathbb{R}^+$ be a continuous function.

Set $\bar{J} := [1, e]$ or $[e, \infty)$. Consider equation (1) and the following inequalities:

$$ |{}_H D_{1^+}^{\alpha} z(x) - \lambda z(x) - f(x, z(x))| \leq \varepsilon,\ 0 < \alpha < 1,\ x \in J, \tag{2} $$

and

$$ |{}_H D_{1^+}^{\alpha} z(x) - \lambda z(x) - f(x, z(x))| \leq \varepsilon \varphi(x),\ 0 < \alpha < 1,\ x \in J. \tag{3} $$

**Definition 1.1.** *Equation (1) is Ulam-Hyers stable if there exists a constant $c > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in C_{\gamma,\ln}(\bar{J}, \mathbb{R})$ of inequality (2) there exists a solution $y \in C_{\gamma,\ln}(\bar{J}, \mathbb{R})$ of equation (1) with*

$$ |z(x) - y(x)| \leq c\varepsilon,\ x \in J. $$

**Remark 1.2.** *A function $z \in C_{\gamma,\ln}(\bar{J}, \mathbb{R})$ is a solution of inequality (2) if and only if there exists a function $h \in C_{\gamma,\ln}(\bar{J}, \mathbb{R})$ such that (i) $|h(x)| \leq \varepsilon, x \in J$, (ii) ${}_H D_{1^+}^{\alpha} z(x) = \lambda z(x) + f(x, z(x)) + h(x), x \in J$.*

**Definition 1.3.** *Equation (1) is Ulam-Hyers-Rassias stable stable if there exists a constant $c > 0$ such that for each $\varepsilon > 0$ and for each solution $z \in C_{\gamma,\ln}(\bar{J}, \mathbb{R})$ of inequality (3) there exists a solution $y \in C_{\gamma,\ln}(\bar{J}, \mathbb{R})$ of equation (1) with*

$$ |z(x) - y(x)| \leq c\varepsilon\varphi(x),\ x \in J. $$

**Remark 1.4.** *A function $z \in C_{\gamma,\ln}(\bar{J}, \mathbb{R})$ is a solution of inequality (3) if and only if there exists a function $\tilde{h} \in C_{\gamma,\ln}(\bar{J}, \mathbb{R})$ such that (i) $|\tilde{h}(x)| \leq \varepsilon\varphi(x), x \in J$, (ii) ${}_H D_{1^+}^{\alpha} z(x) = \lambda z(x) + f(x, z(x)) + \tilde{h}(x), x \in J$.*

The rest of this paper is organized as follows. In Section 2, some notations and preparation results are given. In Section 3, some useful remarks on bounded and unbounded time intervals are presented. Section 4 is devoted to to give Ulam-Hyers stability and Ulam-Hyers-Rassias stability criteria of the equation (1) on bounded and unbounded time intervals respectively. Finally, the reason on the equation (1) is not necessary Ulam-Hyers-Rassias stable is analysed.

## 2. Preliminaries

Let $Y$ be a Banach space endowed with the norm $\|\cdot\|_Y$. Set $J := (1, e]$ or $(e, \infty)$. Denote $C(J, Y)$ be the Banach space of all continuous functions from $J$ into $Y$ with the norm $\|y\|_C = \sup_{x\in J} \|y(x)\|_Y$. For $0 < \mu < 1$, we denote the set $C_{\mu,\ln}(\bar{J}, Y) := \{y(x) : y : J \to Y$ is continuous such that $\left(\ln \frac{x}{a}\right)^{\mu} y(x) \in C(\bar{J}, Y)\}$. Following [1, Theorem 3.29], $C_{\mu,\ln}(\bar{J}, Y)$ is a Banach space with the norm $\|y\|_{C_{\mu,\ln}} = \|(\ln x)^{\mu} y(x)\|_C = \sup\limits_{x\in\bar{J}} \|(\ln x)^{\mu} y(x)\|_Y$.

The following definitions and lemmas will be used in this paper.

**Definition 2.1.** *(see [1, p.110, (2.7.1)]) The left-sided Hadamard fractional integral of order $\alpha \in \mathbb{R}^+$ of function $y(t)$ are defined by*

$$({}_HJ^{\alpha}_{a^+}y)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} y(s)\frac{ds}{s},\ (0 < a < t \le b),$$

*where $\Gamma(\cdot)$ is the Gamma function.*

**Definition 2.2.** *(see [1, p.111, (2.7.7)]) The left-sided Hadamard fractional derivative of order $\alpha \in [n-1, n)$, $n \in \mathbb{Z}^+$ of function $y(t)$ are defined by*

$$({}_HD^{\alpha}_{a^+}y)(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_a^t \left(\ln \frac{t}{s}\right)^{n-\alpha+1} y(s)\frac{ds}{s},\ (0 < a < t \le b).$$

**Lemma 2.3.** *(see [4, Theorem 2.3]) Let $\alpha, \beta \in (0, 1]$ and $\beta < 1+\alpha$ be arbitrary. Then the following statements hold:*

*(i) For all $z > 0$, we have*

$$\mathbb{E}_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} = \frac{1}{\alpha} z^{\frac{1-\beta}{\alpha}} \exp(z^{\frac{1}{\alpha}}) + \int_0^{\infty} S(v, z)dv,$$

*where*

$$S(v, z) = \frac{1}{\pi\alpha} v^{\frac{1-\beta}{\alpha}} \exp(-v^{\frac{1}{\alpha}}) \frac{v \sin(\pi(1-\beta)) - z \sin(\pi(1-\beta+\alpha))}{v^2 - 2vz \cos(\pi\alpha) + z^2}.$$

*(ii) For all $z < 0$, we have*

$$\mathbb{E}_{\alpha,\beta}(z) = \int_0^{\infty} S(v, z)dv,$$

*where*

$$S(v, z) = \frac{1}{\pi\alpha} v^{\frac{1-\beta}{\alpha}} \exp(-v^{\frac{1}{\alpha}}) \frac{v \sin(\pi(1-\beta)) - z \sin(\pi(1-\beta+\alpha))}{v^2 - 2vz \cos(\pi\alpha) + z^2}.$$

*We note that $\mathbb{E}_{\alpha}(z) = \mathbb{E}_{\alpha,1}(z)$.*

By virtue of Lemma 2.3, Li and Wang [24] derived the following useful results for two-parameter Mittag-Leffler function.

**Lemma 2.4.** *(see [24, Theorem 2.11]) Let $\lambda > 0$ be arbitrary, $\alpha, \beta \in (0, 1]$ and $\beta < 1+\alpha$. Denote*

$$\omega(\alpha, \beta, \lambda) = \max\left\{\frac{\alpha \sin(\beta\pi)\Gamma(2\alpha-\beta+1)}{\lambda^2\alpha\pi \sin^2(\pi\alpha)}, \frac{\alpha|\sin(\pi(\beta-\alpha))|\Gamma(\alpha-\beta+1)}{\lambda\alpha\pi \sin^2(\pi\alpha)}\right\}.$$

*For all $x \in (1, \infty)$, we have*

$$\left|(\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha}) - \frac{1}{\alpha}\lambda^{\frac{1-\alpha}{\alpha}}\exp(\lambda^{\frac{1}{\alpha}}\ln x)\right| \leq \frac{\omega(\alpha,\alpha,\lambda)}{(\ln x)^{\alpha+1}}.$$

*In particular, for all $x \in (1, \infty)$,*

$$\left|\mathbb{E}_{\alpha}(\lambda(\ln x)^{\alpha}) - \frac{1}{\alpha}\exp(\lambda^{\frac{1}{\alpha}}\ln x)\right| \leq \frac{\omega(\alpha,1,\lambda)}{(\ln x)^{\alpha}}.$$

Further, we give the following integral estimation.

**Lemma 2.5.** *Let $\lambda > 0$ be arbitrary, $\alpha \in (0, 1]$, we have*

*$(i)$ For all $x \in (1, e]$, we have*

$$\begin{aligned} J_1 &:= \left|\int_1^x \left((\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) - \lambda^{\frac{1-\alpha}{\alpha}}\mathbb{E}_{\alpha}(\lambda(\ln x)^{\alpha})\exp(-\lambda^{\frac{1}{\alpha}}\ln t)\right)\frac{dt}{t}\right| \\ &\leq \frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{\mathbb{E}_{\alpha}(\lambda)}{\lambda}. \end{aligned}$$

*$(ii)$ For all $x \in (e, \infty)$, we have*

$$\begin{aligned} J_2 &:= \left|\int_1^x \left((\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})\exp(-\lambda^{\frac{1}{\alpha}}\ln t)\right)\frac{dt}{t}\right| \\ &\leq \frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{\exp(\lambda^{\frac{1}{\alpha}})}{\alpha\lambda} + \omega(\alpha,\alpha,\lambda)\left(\frac{1}{\alpha} + \frac{2}{\lambda^{\frac{1}{\alpha}}}\right) := M(\alpha,\lambda). \end{aligned}$$

*Proof.* $(i)$ For all $x \in (1, e]$, we obtain

$$\begin{aligned} J_1 &\leq \mathbb{E}_{\alpha,\alpha}(\lambda)\left|\int_1^x (\ln x - \ln t)^{\alpha-1}\frac{dt}{t}\right| + \lambda^{\frac{1-\alpha}{\alpha}}\mathbb{E}_{\alpha}(\lambda)\left|\int_1^x \exp(-\lambda^{\frac{1}{\alpha}}\ln t)\frac{dt}{t}\right| \\ &\leq \frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{\mathbb{E}_{\alpha}(\lambda)}{\lambda}, \end{aligned}$$

where we use the decreasing property of $\mathbb{E}_{\alpha,\alpha}(z)$ for $z > 0$.

$(ii)$ For all $x > e$, by using Lemma 2.4, we have

$$\begin{aligned} I_1 &:= \left|\int_1^{\frac{x}{e}} \left((\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})\exp(-\lambda^{\frac{1}{\alpha}}\ln t)\right)\frac{dt}{t}\right| \\ &\leq \left|\int_1^{\frac{x}{e}} \left((\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) - \frac{1}{\alpha}\lambda^{\frac{1-\alpha}{\alpha}}\exp(\lambda^{\frac{1}{\alpha}}(\ln x - \ln t))\right)\frac{dt}{t}\right| \\ &\quad + \left|\int_1^{\frac{x}{e}} \frac{1}{\alpha}\lambda^{\frac{1-\alpha}{\alpha}}\exp(\lambda^{\frac{1}{\alpha}}(\ln x - \ln t)) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})\exp(-\lambda^{\frac{1}{\alpha}}\ln t)\right)\frac{dt}{t}\right| \\ &\leq \int_1^{\frac{x}{e}} \frac{\omega(\alpha,\alpha,\lambda)}{(\ln x - \ln t)^{\alpha+1}}\frac{dt}{t} + \int_1^{\frac{x}{e}} \frac{\omega(\alpha,\alpha,\lambda)}{(\ln x)^{\alpha+1}}\exp(-\lambda^{\frac{1}{\alpha}}\ln t)\frac{dt}{t} \\ &\leq \int_1^{\frac{x}{e}} \frac{\omega(\alpha,\alpha,\lambda)}{(\ln x - \ln t)^{\alpha+1}}\frac{dt}{t} + \omega(\alpha,\alpha,\lambda)\int_1^{\frac{x}{e}} \exp(-\lambda^{\frac{1}{\alpha}}\ln t)\frac{dt}{t} \\ &\leq \frac{\omega(\alpha,\alpha,\lambda)}{\alpha} + \frac{\omega(\alpha,\alpha,\lambda)}{\lambda^{\frac{1}{\alpha}}} \\ &= \omega(\alpha,\alpha,\lambda)\left(\frac{1}{\alpha} + \frac{1}{\lambda^{\frac{1}{\alpha}}}\right). \end{aligned}$$

According to Lemma 2.4 again, we have

$$(\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha}) \le \frac{\omega(\alpha,\alpha,\lambda)}{(\ln x)^{\alpha+1}} + \frac{1}{\alpha}\lambda^{\frac{1-\alpha}{\alpha}}\exp(\lambda^{\frac{1}{\alpha}}\ln x).$$

Thus, using the decreasing property of $\mathbb{E}_{\alpha,\alpha}(z)$ for $z>0$ again, one has

$$\begin{aligned}
I_2 &:= \left|\int_{\frac{x}{e}}^{x}\Big((\ln x-\ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x-\ln t)^{\alpha}) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})\exp(-\lambda^{\frac{1}{\alpha}}\ln t)\Big)\frac{dt}{t}\right| \\
&\le \left|\int_{\frac{x}{e}}^{x}(\ln x-\ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x-\ln t)^{\alpha})\frac{dt}{t}\right| + \left|\int_{\frac{x}{e}}^{x}(\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})\exp(-\lambda^{\frac{1}{\alpha}}\ln t)\frac{dt}{t}\right| \\
&\le \mathbb{E}_{\alpha,\alpha}(\lambda)\int_{\frac{x}{e}}^{x}(\ln x-\ln t)^{\alpha-1}\frac{dt}{t} \\
&\quad + \left|\int_{\frac{x}{e}}^{x}\left(\frac{\omega(\alpha,\alpha,\lambda)}{(\ln x)^{\alpha+1}}\exp(-\lambda^{\frac{1}{\alpha}}\ln t) + \frac{\lambda^{\frac{1-\alpha}{\alpha}}}{\alpha}\exp(\lambda^{\frac{1}{\alpha}}(\ln x-\ln t))\right)\frac{dt}{t}\right| \\
&\le \frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{\omega(\alpha,\alpha,\lambda)}{(\ln x)^{\alpha+1}}\int_{\frac{x}{e}}^{x}\exp(-\lambda^{\frac{1}{\alpha}}\ln t)\frac{dt}{t} + \frac{1}{\alpha}\lambda^{\frac{1-\alpha}{\alpha}}\int_{\frac{x}{e}}^{x}\exp(\lambda^{\frac{1}{\alpha}}(\ln x-\ln t))\frac{dt}{t} \\
&\le \frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{\omega(\alpha,\alpha,\lambda)}{\lambda^{\frac{1}{\alpha}}} - \frac{\exp(-\lambda^{\frac{1}{\alpha}})}{\lambda^{\frac{1}{\alpha}}} + \frac{1}{\alpha}\lambda^{\frac{1-\alpha}{\alpha}}\lambda^{-\frac{1}{\alpha}}\exp(\lambda^{\frac{1}{\alpha}}) \\
&\le \frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{\omega(\alpha,\alpha,\lambda)}{\lambda^{\frac{1}{\alpha}}} + \frac{\exp(\lambda^{\frac{1}{\alpha}})}{\alpha\lambda}.
\end{aligned}$$

From above, we obtain

$$J_2 \le I_1 + I_2 \le \frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{\exp(\lambda^{\frac{1}{\alpha}})}{\alpha\lambda} + \omega(\alpha,\alpha,\lambda)\left(\frac{1}{\alpha}+\frac{2}{\lambda^{\frac{1}{\alpha}}}\right).$$

The proof is completed □

To end this section, we recall the following inequality which will be used in the sequel.

**Lemma 2.6.** *(see [25, Lemma 23]) If $\lambda, v, w > 0$, then for any $t > a, a > 0$, we have*

$$\left(\ln\frac{t}{a}\right)^{1-v}\int_a^t\left(\ln\frac{t}{s}\right)^{v-1}\left(\ln\frac{s}{a}\right)^{\lambda-1}\left(\frac{s}{a}\right)^{-w}\frac{ds}{s} \le Cw^{-\lambda},$$

*where $C$ is a positive constant independent of the time variable $t$.*

## 3. Some useful lemmas and remarks

Now we plan to give the following integral estimation.

**Lemma 3.1.** *Let $\lambda > 0$, $z \in C_{\gamma,\ln}([1,\infty),\mathbb{R})$ be a solution of inequality (2). Then $z$ is a solution of the following inequality:*

$$\begin{aligned}
&\left|z(x) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})c_0 - \int_1^x(\ln x-\ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x-\ln t)^{\alpha})f(t,z(t))\frac{dt}{t}\right| \\
&\le \int_1^x\frac{\omega(\alpha,\alpha,\lambda)\varepsilon}{(\ln x-\ln t)^{\alpha+1}}\frac{dt}{t} + \frac{\varepsilon x^{\lambda^{\frac{1}{\alpha}}}}{\alpha\lambda},
\end{aligned}$$

*where $c_0 = {}_HJ_{1^+}^{1-\alpha}z(1^+)$.*

*Proof.* According to the Remark 1.2, we have

$$ {}_H D^{\alpha}_{1^+} z(x) = \lambda z(x) + f(x, z(x)) + h(x),\ x \in (1, \infty). $$

By [1, p.234,(4.1.89)-(4.1.95)] or [26, p.182, (7.2.60)-(7.2.64)], we obtain

$$ \begin{aligned} z(x) &= (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})c_0 + \int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) f(t, z(t))\frac{dt}{t} \\ &\quad + \int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) h(t)\frac{dt}{t},\ x \in (1, \infty). \end{aligned} $$

For all $1 < x < \infty$, using Lemma 2.4, one has

$$ \begin{aligned} & \left| z(x) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})c_0 - \int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) f(t, z(t))\frac{dt}{t} \right| \\ = & \left| \int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) h(t)\frac{dt}{t} \right| \\ \le & \left| \int_1^x \left( (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) - \frac{1}{\alpha}\lambda^{\frac{1-\alpha}{\alpha}} \exp(\lambda^{\frac{1}{\alpha}}(\ln x - \ln t)) \right) h(t)\frac{dt}{t} \right| \\ & + \left| \int_1^x \frac{1}{\alpha}\lambda^{\frac{1-\alpha}{\alpha}} \exp(\lambda^{\frac{1}{\alpha}}(\ln x - \ln t)) h(t)\frac{dt}{t} \right| \\ \le & \int_1^x \frac{\omega(\alpha,\alpha,\lambda)\varepsilon}{(\ln x - \ln t)^{\alpha+1}}\frac{dt}{t} + \varepsilon \int_1^x \frac{1}{\alpha}\lambda^{\frac{1-\alpha}{\alpha}} \exp(\lambda^{\frac{1}{\alpha}}(\ln x - \ln t))\frac{dt}{t} \\ \le & \int_1^x \frac{\omega(\alpha,\alpha,\lambda)\varepsilon}{(\ln x - \ln t)^{\alpha+1}}\frac{dt}{t} + \frac{\varepsilon}{\alpha\lambda}\exp(\lambda^{\frac{1}{\alpha}}\ln x) - \frac{\varepsilon}{\lambda^{\frac{1}{\alpha}}} \\ \le & \int_1^x \frac{\omega(\alpha,\alpha,\lambda)\varepsilon}{(\ln x - \ln t)^{\alpha+1}}\frac{dt}{t} + \frac{\varepsilon x^{\lambda^{\frac{1}{\alpha}}}}{\alpha\lambda}. \end{aligned} $$

The proof is completed. □

**Remark 3.2.** *Note that for some fixed point $x_0$ and $x_0 > \delta > 1$, we have*

$$ \int_1^{x_0} \frac{1}{(\ln x_0 - \ln t)^{\alpha+1}}\frac{dt}{t} = \lim_{\delta \to x_0} \int_1^{\delta} \frac{1}{(\ln x_0 - \ln t)^{\alpha+1}}\frac{dt}{t} = \lim_{\delta \to x_0} \frac{1}{\alpha}[(\ln x_0 - \ln \delta)^{-\alpha} - (\ln x_0)^{-\alpha}] = \infty, $$

*which yields that it is not possible to obtain some explicit estimation in this case.*

Next, we divide our time interval $(1, \infty)$ into two subintervals $(1, e]$ and $(e, \infty)$.

**Remark 3.3.** *Let $\lambda > 0$, $z \in C_{\gamma,\ln}([1, e], \mathbb{R})$ be a solution of inequality (2). Then $z$ is a solution of the following inequality:*

$$ \begin{aligned} & \left| z(x) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})c_0 - \int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) f(t, z(t))\frac{dt}{t} \right| \\ \le & \left| \int_1^x \left( (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) - \lambda^{\frac{1-\alpha}{\alpha}}\mathbb{E}_{\alpha}(\lambda(\ln x)^{\alpha}) \exp(-\lambda^{\frac{1}{\alpha}}\ln t) \right) h(t)\frac{dt}{t} \right| \\ & + \left| \int_1^x \lambda^{\frac{1-\alpha}{\alpha}}\mathbb{E}_{\alpha}(\lambda(\ln x)^{\alpha}) \exp(-\lambda^{\frac{1}{\alpha}}\ln t) h(t)\frac{dt}{t} \right| \\ \le & \ \varepsilon\left( \frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{\mathbb{E}_{\alpha}(\lambda)}{\lambda} \right) + \frac{\varepsilon\mathbb{E}_{\alpha}(\lambda)}{\lambda} \\ \le & \ \varepsilon\left( \frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{2\mathbb{E}_{\alpha}(\lambda)}{\lambda} \right),\ x \in (1, e], \end{aligned} $$

*where we use Lemma 2.5(i), Remark 1.2 and $\mathbb{E}_{\alpha,\alpha}(z)$ is an increasing function for $z > 0$.*

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**Remark 3.4.** *Let $\lambda > 0$, $z \in C_{\gamma,\ln}([e,\infty),\mathbb{R})$ be a solution of inequality (2). Then $z$ is a solution of the following inequality:*

$$\begin{aligned}
&\left| z(x) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^\alpha)c_0 - \int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^\alpha) f(t,z(t))\frac{dt}{t}\right| \\
\leq\ & \left|\int_1^x \left((\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^\alpha) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^\alpha)\exp(-\lambda^{\frac{1}{\alpha}}\ln t)\right) h(t)\frac{dt}{t}\right| \\
& + \left|\int_1^x (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^\alpha)\exp(-\lambda^{\frac{1}{\alpha}}\ln t) h(t)\frac{dt}{t}\right| \\
\leq\ & \varepsilon M(\alpha,\lambda) + \varepsilon\left|\int_1^x \left(\frac{\omega(\alpha,\alpha,\lambda)}{(\ln x)^{\alpha+1}}\exp(-\lambda^{\frac{1}{\alpha}}\ln t) + \frac{1}{\alpha}\lambda^{\frac{1-\alpha}{\alpha}}\exp(\lambda^{\frac{1}{\alpha}}(\ln x - \ln t))\right)\frac{dt}{t}\right| \\
\leq\ & \varepsilon M(\alpha,\lambda) + \frac{\varepsilon\omega(\alpha,\alpha,\lambda)}{\lambda^{\frac{1}{\alpha}}} + \frac{\varepsilon x^{\frac{1}{\alpha}}}{\lambda^{\frac{1}{\alpha}}} \\
=\ & \varepsilon\left(M(\alpha,\lambda) + \frac{\omega(\alpha,\alpha,\lambda)}{\lambda^{\frac{1}{\alpha}}} + \frac{x^{\frac{1}{\alpha}}}{\lambda^{\frac{1}{\alpha}}}\right),
\end{aligned}$$

*where we use Lemma 2.4, Lemma 2.5(ii) and Remark 1.2.*

**Remark 3.5.** *Let $\lambda < 0$, $z \in C_{\gamma,\ln}([1,\infty),\mathbb{R})$ be a solution of inequality (2). Then $z$ is a solution of the following inequality:*

$$\begin{aligned}
&\left| z(x) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^\alpha)c_0 - \int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^\alpha) f(t,z(t))\frac{dt}{t}\right| \\
\leq\ & \frac{\varepsilon(\ln x)^\alpha}{\Gamma(\alpha+1)},\ x \in (1,\infty),
\end{aligned}$$

*where we use the fact $\mathbb{E}_{\alpha,\alpha}(z) \leq \frac{1}{\Gamma(\alpha)}$ for $z < 0$.*

**Remark 3.6.** *Let $\lambda < 0$, $z \in C_{\gamma,\ln}([1,\infty),\mathbb{R})$ be a solution of inequality (3). Then $z$ is a solution of the following inequality:*

$$\begin{aligned}
&\left| z(x) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^\alpha)c_0 - \int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^\alpha) f(t,z(t))\frac{dt}{t}\right| \\
=\ & \left|\int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^\alpha)\tilde{h}(t)\frac{dt}{t}\right| \\
\leq\ & \frac{\varepsilon}{\Gamma(\alpha)}\int_1^x (\ln x - \ln t)^{\alpha-1}\varphi(t)\frac{dt}{t},\ x \in (1,\infty),
\end{aligned}$$

*where we use the fact $\mathbb{E}_{\alpha,\alpha}(z) \leq \frac{1}{\Gamma(\alpha)}$ for $z < 0$ again.*

## 4. Main results

*4.1. Ulam-Hyers stability results*

We introduce the following assumptions:

$(A_1)$ $f : (1,e] \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(\cdot, y(\cdot)) \in C_{\gamma\ln}[1,e]$, $1-\alpha \leq \gamma < 1$.

$(A_2)$ There exists $L > 0$ such that

$$|f(x,y) - f(x,z)| \leq L|y-z| \text{ for each } x \in J \text{ and all } y, z \in \mathbb{R}.$$

$(A_3)$ $\omega = 1 - L\mathbb{E}_{\alpha,\alpha}(\lambda)B[1-\gamma,\alpha] > 0$.

**Theorem 4.1.** *Assume that $(A_1)$, $(A_2)$, and $(A_3)$ are satisfied. Then equation (1) with $\lambda > 0$ is Ulam-Hyers stable on $J = (1, e]$.*

*Proof.* Let $z \in C_{\gamma,\ln}([1,e],\mathbb{R})$ be a solution of inequality (2). By $(A_1)$, $(A_2)$, and $(A_3)$, one can apply Banach fixed point theorem to derive

$$\begin{cases} {}_HD^{\alpha}_{1^+}y(x) = \lambda y(x) + f(x, y(x)), \ 0 < \alpha < 1, \ x \in J, \\ {}_HJ^{1-\alpha}_{1^+}z(1^+) = c_0, \end{cases}$$

has the unique solution

$$y(x) = (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})c_0 + \int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha})f(t, y(t))\frac{dt}{t}.$$

By using Lemma 2.5$(i)$ and Remark 3.3, we have

$$\begin{aligned}
& |(z(x) - y(x))(\ln x)^{\gamma}| \\
= & \left|(\ln x)^{\gamma}\left(z(x) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})c_0 - \int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha})f(t, y(t))\frac{dt}{t}\right)\right| \\
\leq & \left|(\ln x)^{\gamma}\left(z(x) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})c_0 - \int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha})f(t, z(t))\frac{dt}{t}\right)\right| \\
+ & \left|\int_1^x (\ln x)^{\gamma}(\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha})(f(t, z(t)) - f(t, y(t)))\frac{dt}{t}\right| \\
\leq & \ \varepsilon(\ln x)^{\gamma}\left(\frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{2\mathbb{E}_{\alpha}(\lambda)}{\lambda}\right) + L\mathbb{E}_{\alpha,\alpha}(\lambda)\int_1^x (\ln x)^{\gamma}(\ln x - \ln t)^{\alpha-1}(\ln t)^{-\gamma}\frac{dt}{t}\|z - y\|_{C_{\gamma,\ln}} \\
\leq & \ \varepsilon(\ln x)^{\gamma}\left(\frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{2\mathbb{E}_{\alpha}(\lambda)}{\lambda}\right) + L\mathbb{E}_{\alpha,\alpha}(\lambda)B[1-\gamma, \alpha]\|z - y\|_{C_{\gamma,\ln}},
\end{aligned}$$

which yields that

$$\|z - y\|_{C_{\gamma,\ln}} \leq \frac{\varepsilon}{\omega}\left(\frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{2\mathbb{E}_{\alpha}(\lambda)}{\lambda}\right)(\ln x)^{\gamma}.$$

Thus,

$$|z(x) - y(x)| \leq c\varepsilon, \quad c = \frac{1}{\omega}\left(\frac{\mathbb{E}_{\alpha,\alpha}(\lambda)}{\alpha} + \frac{2\mathbb{E}_{\alpha}(\lambda)}{\lambda}\right) > 0.$$

The proof is completed. □

**Remark 4.2.** *Let $\lambda < 0$. Assume that $(A_1)$ and $(A_2)$ are satisfied. One can use the above similar methods via Remark 3.5 to check that the equation (1) is Ulam-Hyers-Rassias stable on $J = (1, \infty)$ provided by $\rho = 1 - \frac{LB[1-\gamma,\alpha]}{\Gamma(\alpha)} > 0$. That is,*

$$|z(x) - y(x)| \leq c\varepsilon\varphi(x), \ x \in (1, \infty), \quad c = \frac{\varepsilon}{\rho\Gamma(\alpha+1)} > 0, \ \varphi(x) = (\ln x)^{\alpha}.$$

**Example 4.3.** *Let $\alpha = \frac{2}{3}$, $\lambda = \frac{1}{2}$ and $\gamma = \frac{1}{2}$. Consider the fractional order differential equation*

$$ {}_HD^{\frac{2}{3}}_{1^+}y(x) = \frac{1}{2}y(x) + \frac{1}{l}\sin^2 y(x), \ x \in (1, e], l > 0, \tag{4}$$

*and the inequality*

$$|{}_HD^{\frac{2}{3}}_{1^+}z(x) - \frac{1}{2}z(x) - \frac{1}{l}\sin^2 z(x)| \leq \varepsilon, \ x \in (1, e]. \tag{5}$$

*Let* $z \in C_{\gamma,\ln}([1,e],\mathbb{R})$ *be a solution of inequality (5). Then there exists a function* $h(x) = \varepsilon \ln x \in C_{\gamma,\ln}([1,e],\mathbb{R})$ *such that* $|h(x)| \le \varepsilon, x \in (1,e]$, *and* ${}_HD^{\frac{2}{3}}_{1^+} z(x) = \frac{1}{2} z(x) + \frac{1}{l}\sin^2 z(x) + h(x),\ x \in (1,e]$.

*Define* $f(x,y(x)) = \frac{1}{l}\sin^2(x),\ x \in (1,e]$ *and* $L = \frac{2}{l}$. *Obviously,* $(A_1)$ *and* $(A_2)$ *hold. Moreover, we choose* $l = 6\mathbb{E}_{\frac{2}{3},\frac{2}{3}}\left(\frac{1}{2}\right) B\left[\frac{1}{2},\frac{2}{3}\right]$, *then*

$$\omega = 1 - \frac{2}{l}\mathbb{E}_{\frac{2}{3},\frac{2}{3}}\left(\frac{1}{2}\right) B\left[\frac{1}{2},\frac{2}{3}\right] = \frac{2}{3} > 0,$$

*which implies that* $(A_3)$ *holds. According to Theorem 4.2, we have*

$$|z(x) - y(x)| \le \frac{3\varepsilon}{2}\left(1.5\mathbb{E}_{\frac{2}{3},\frac{2}{3}}\left(\frac{1}{2}\right) + 4\mathbb{E}_{\frac{2}{3}}\left(\frac{1}{2}\right)\right).$$

*Thus, equation (4) is Ulam-Hyers stable on* $(1,e]$ *with* $c = \frac{3}{2}\left(1.5\mathbb{E}_{\frac{2}{3},\frac{2}{3}}\left(\frac{1}{2}\right) + 4\mathbb{E}_{\frac{2}{3}}\left(\frac{1}{2}\right)\right)$.

*4.2. Ulam-Hyers-Rassias stability result*

Next, we introduce the following assumptions:

$(B_1)$ Let $\lambda < 0$ and $\gamma = 1 - \alpha$.

$(B_2)$ $f : J \times \mathbb{R} \to \mathbb{R}$ is jointly continuous and there exists $L(\cdot) \in C([1,\infty),\mathbb{R}^+)$ such that

$$|f(x,y) - f(x,z)| \le L(x)|y - z| \ \text{ for each } x \in J \text{ and all } y, z \in \mathbb{R},$$

where $L(\cdot)$ satisfying

$$\int_1^x (\ln x)^{\gamma}(\ln x - \ln t)^{\alpha-1}(\ln t)^{-\gamma} L(t)\frac{dt}{t} \le \widetilde{C}(\ln x)^{\alpha-1},\ \widetilde{C} > 0. \tag{6}$$

$(B_3)$ There exists a $\varphi(\cdot) \in C([1,\infty),\mathbb{R}^+)$ such that

$$\int_1^x (\ln x - \ln t)^{\alpha-1}\varphi(t)\frac{dt}{t} \le \widehat{C}\varphi(t),\ \widehat{C} > 0. \tag{7}$$

$(B_4)$ $\omega' = 1 - \frac{\widetilde{C}}{\Gamma(\alpha)} > 0$.

**Theorem 4.4.** *Let* $\lambda < 0$. *Assume that* $(B_1)$, $(B_2)$, $(B_3)$ *and* $(B_4)$ *are satisfied. Then equation (1) is Ulam-Hyers-Rassias stable on* $J = (1,\infty)$.

*Proof.* Note that the fact $\mathbb{E}_{\alpha,\alpha}(z) \le \frac{1}{\Gamma(\alpha)}$ for $z < 0$. By Remark 3.6, one can obtain

$$\begin{aligned}
& |(z(x) - y(x))(\ln x)^{\gamma}| \\
\le & \left|(\ln x)^{\gamma}\left(z(x) - (\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})c_0 - \int_1^x (\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha}) f(t,z(t))\frac{dt}{t}\right)\right| \\
& + \left|\int_1^x (\ln x)^{\gamma}(\ln x - \ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x - \ln t)^{\alpha})(f(t,z(t)) - f(t,y(t)))\frac{dt}{t}\right| \\
\le & \frac{1}{\Gamma(\alpha)}\left|\int_1^x (\ln x)^{\gamma}(\ln x - \ln t)^{\alpha-1}\varepsilon\varphi(t)\frac{dt}{t}\right| \\
& + \frac{1}{\Gamma(\alpha)}\int_1^x (\ln x)^{\gamma}(\ln x - \ln t)^{\alpha-1}L(t)|y - z|\frac{dt}{t} \\
\le & \frac{\varepsilon(\ln x)^{\gamma}}{\Gamma(\alpha)}\widehat{C}\varphi(x) + \frac{1}{\Gamma(\alpha)}\int_1^x (\ln x)^{\gamma}(\ln x - \ln t)^{\alpha-1}(\ln t)^{-\gamma}L(t)\frac{dt}{t}\|y - z\|_{C_{\gamma,\ln}} \\
\le & \frac{\varepsilon(\ln x)^{\gamma}}{\Gamma(\alpha)}\widehat{C}\varphi(x) + \frac{\widetilde{C}}{\Gamma(\alpha)}(\ln x)^{\gamma+\alpha-1}\|y - z\|_{C_{\gamma,\ln}}.
\end{aligned}$$

This yields that

$$\left(1-\frac{\widetilde{C}}{\Gamma(\alpha)}\right)\|y-z\|_{C_{\gamma,\ln}}\leq\frac{\varepsilon(\ln x)^{\gamma}}{\Gamma(\alpha)}\widehat{C}\varphi(x).$$

This implies that

$$|y(x)-z(x)|\leq\frac{\widehat{C}\varepsilon}{\omega'\Gamma(\alpha)}\varphi(x),\ x\in J.$$

The proof is completed. $\square$

**Example 4.5.** *Let $\alpha=\frac{1}{2}$, $\lambda=-\frac{1}{2}$ and $\gamma=\frac{1}{2}$. Consider the fractional order differential equation*

$$_HD^{\frac{1}{2}}_{1^+}y(x)=-\frac{1}{2}y(x)+\frac{1}{lx^3}\sin^2 y(x),\ x\in(1,\infty),l>0,\tag{8}$$

*and the inequality*

$$|_HD^{\frac{1}{2}}_{1^+}z(x)-\frac{1}{2}z(x)-\frac{1}{lx^3}\sin^2 z(x)|\leq\varepsilon\varphi(x),\ x\in(1,\infty).\tag{9}$$

*Define $f(x,y(x))=\frac{1}{lx^3}\sin^2(x)$ and $L(x)=\frac{1}{lx^3}$, $x\in(1,\infty)$. Let $z\in C_{\gamma,\ln}([1,\infty),\mathbb{R})$ be a solution of inequality (9). There exists a function $h(x)=\frac{\varepsilon}{x}(\ln x)^{\alpha-1-\gamma}\in C_{\gamma,\ln}([1,\infty),\mathbb{R})$ such that $|h(x)|\leq\varepsilon(\ln x)^{\alpha-1-\gamma}:=\varphi(x)$, $x\in(1,\infty)$. Moreover, (5) via Lemma 2.6 reduces to*

$$\int_1^x(\ln x)^{\frac{1}{2}}(\ln x-\ln t)^{-\frac{1}{2}}(\ln t)^{-\frac{1}{2}}\frac{2}{lt^3}\frac{dt}{t}\leq\frac{2C}{l}\sqrt{3}(\ln x)^{-\frac{1}{2}},\ x\in(1,\infty),$$

*and (9) reduces to*

$$\int_1^x(\ln x)^{\gamma}(\ln x-\ln t)^{\alpha-1}\varepsilon(\ln x)^{\alpha-1-\gamma}\frac{dt}{t}\leq\hat{C}\varepsilon(\ln x)^{\alpha-1-\gamma}=\hat{C}\varepsilon(\ln x)^{-1},\ \hat{C}>0,\ x\in(1,\infty).$$

*From above, $(B_1)$, $(B_2)$ and $(B_3)$ hold. Now we choose $l=\frac{\Gamma(0.5)}{4\sqrt{3}C}$ and $\omega'=1-\frac{2C\sqrt{3}}{l}=\frac{1}{2}>0$, which implies that $(B_4)$ holds. According to Theorem 4.4, we have*

$$|z(x)-y(x)|\leq\frac{2\hat{C}\varepsilon}{\Gamma(0.5)}(\ln x)^{-1}.$$

*Thus, equation (8) is Ulam-Hyers-Rassias stable on $(1,\infty)$ with $c=\frac{2\hat{C}}{\Gamma(0.5)}$ and $\varphi(x)=\varepsilon(\ln x)^{-1}$, $x\in(1,\infty)$.*

*4.3. Final remarks*

Let $\lambda>0$. Assume that $(B_1)$ and $(B_2)$ are satisfied. It seems that we can not use the above approach to discuss Ulam-Hyers-Rassias stability of the equation (1) with $\lambda>0$ on $J=(e,\infty)$.

In fact, by Remark 3.4, one has

$$\begin{aligned}|(z(x)-y(x))(\ln x)^{\gamma}| &\leq \left(\varepsilon M(\alpha,\lambda)+\frac{\varepsilon\omega(\alpha,\alpha,\lambda)}{\lambda^{\frac{1}{\alpha}}}+\frac{\varepsilon x^{\frac{1}{\alpha}}}{\lambda^{\frac{1}{\alpha}}}\right)(\ln x)^{\gamma}\\ &\quad+\left|\int_1^x(\ln x)^{\gamma}(\ln x-\ln t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x-\ln t)^{\alpha})L(t)|y(t)-z(t)|\frac{dt}{t}\right|\\ &\leq \left(\varepsilon M(\alpha,\lambda)+\frac{\varepsilon\omega(\alpha,\alpha,\lambda)}{\lambda^{\frac{1}{\alpha}}}+\frac{\varepsilon x^{\frac{1}{\alpha}}}{\lambda^{\frac{1}{\alpha}}}\right)(\ln x)^{\gamma}+\Upsilon(x)\|y-z\|_{C_{\gamma,\ln}},\end{aligned}\tag{10}$$

where

$$\Upsilon(x) = (\ln x)^{\gamma}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})\bigg|\int_1^x (\ln x - \ln t)^{\alpha-1}L(t)(\ln t)^{-\gamma}\frac{dt}{t}\bigg|.$$

Set $L(t) = t^{-w},\ w > 0$. According to Lemma 2.6, we have

$$\begin{aligned}
\Upsilon(x) &= (\ln x)^{\gamma}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})\bigg|\int_1^x (\ln x - \ln t)^{\alpha-1}(\ln t)^{-\gamma}t^{-w}\frac{dt}{t}\bigg| \\
&= (\ln x)^{\gamma}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})\bigg|\int_1^x (\ln x - \ln t)^{\alpha-1}(\ln t)^{(1-\gamma)-1}t^{-w}\frac{dt}{t}\bigg| \\
&\leq Cw^{1-\gamma}\frac{\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})}{(\ln x)^{1-\alpha-\gamma}} \quad (C > 0) \\
&\leq \frac{\lambda^{\frac{1-\alpha}{\alpha}}Cw^{1-\gamma}}{\alpha}x^{\lambda^{\frac{1}{\alpha}}+\gamma},
\end{aligned}$$

where we use the fact

$$\lim_{x\to\infty}\frac{\frac{1}{\alpha}\lambda^{\frac{1-\alpha}{\alpha}}\exp(\lambda^{\frac{1}{\alpha}}\ln x)}{(\ln x)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\lambda(\ln x)^{\alpha})} = 1.$$

Dividing $(\ln x)^{\gamma}$ by (10) we obtain

$$|(z(x) - y(x))| \leq \left(\varepsilon M(\alpha,\lambda) + \frac{\varepsilon\omega(\alpha,\alpha,\lambda)}{\lambda^{\frac{1}{\alpha}}} + \frac{\varepsilon x^{\frac{1}{\alpha}}}{\lambda^{\frac{1}{\alpha}}}\right) + \frac{\Upsilon(x)}{(\ln x)^{\gamma}}\|y - z\|_{C_{\gamma,\ln}}. \qquad (11)$$

Obviously,

$$\lim_{x\to\infty}\frac{\Upsilon(x)}{(\ln x)^{\gamma}} = \frac{\lambda^{\frac{1-\alpha}{\alpha}}cw^{1-\gamma}}{\alpha}\lim_{x\to\infty}\frac{x^{\lambda^{\frac{1}{\alpha}}+\gamma}}{(\ln x)^{\gamma}} = \infty.$$

Thus, the term $\frac{\Upsilon(x)}{(\ln x)^{\gamma}}\|y - z\|_{C_{\gamma,\ln}}$ in (11) does not vanish. Therefore, the equation (1) with $\lambda > 0$ is not necessary Ulam-Hyers-Rassias stable on $J = (e, \infty)$.

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# LINEARLY STABLE PERIODIC SOLUTIONS FOR LAGRANGIAN EQUATION

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Abstract. In this paper we study the existence and uniqueness of linearly stable periodic solutions for the Lagrangian equation. The proof is based on the eigenvalue theory combined with degree theory. Compared with those results in the literature, our conditions are weaker.

## 1. Introduction

This paper is devoted to the study of the existence and uniqueness of linear stablity of periodic solutions for the following nonlinear scalar Lagrangian equation

$$\ddot{x} + g(t, x) = 0, \tag{1.1}$$

where $g(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a $T$-periodic function in $t$ and is semilinear in the following sense: There exist $T$-periodic functions $\phi, \Phi \in L^1(0, T)$ such that

$$\phi(t) \leq g_x(t, x) \leq \Phi(t),$$

uniformly in $t \in [0, T]$. We say that a $T$-periodic solution $\psi(t)$ of (1.1) is *linearly stable* if the linearized equation

$$\ddot{y} + (g_x(t, \psi(t))y = 0 \tag{1.2}$$

is stable. But it is not sufficient to guarantee that $\psi(t)$ is Lyapunov stable as (1.1) is a conservative system, Lyapunov stability of $\psi(t)$ cannot be determined by linearized equation (1.2) and involves higher order approximations of (1.1). Based on this idea, a practical method, now known as the third order approximation, has been developed by Ortega based on the Birkhoff normal forms and the Moser's twist theorem [17]. After there has been considerable progress on this topic. We refer the reader to [2, 3, 4, 5]. However, an "almost" necessary condition for $\psi(t)$ to be stable is that it is linearly stable. In this direction, it is worth mentioning the example found by Chu [3]. That is, the equilibrium $x(t) = 0$ of the motion of

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a pendulum with variable length and relativistic effects

$$\Big(\frac{x'}{\sqrt{1-x'^2}}\Big)' + l(t)\sin x = 0, \quad l(t) > 0, \quad l \in C(\mathbb{R}/T\mathbb{Z}),$$

is stable if its linearized equation

$$\ddot{x} + l(t)x = 0$$

is stable. In this paper it is shown how a topological invariant, the index of an oscillation, can be used to obtain linear stability results. Actually, it will be proved that the index characterizes linearly stable in certain case (see section 2). Our study is mainly motivated by [7], where a result is the following.

**Theorem 1.1.** *If there exists $a > 0$ and $\Phi \in L^1(0,T)$ such that*

$$a < g_x(t,x) \le \Phi(t), \tag{1.3}$$

*for all $x$ and a.e. $t \in [0,T]$, and $\|\Phi\|_p < K(2p^*)$ for some $p \in [1,+\infty]$, then (1.1) has a unique $T$-periodic solution which is linearly stable.*

It must be noticed that the strict positiveness assumption of $g_x(t,x)$ is crucial for this result since this implied the monotonicity of the nonlinearity and then a method of lower and upper coupled with the monotone iterative technique was used to get existence and linear stability. Unfortunately, in some interesting problems we find that the constant $g_x(t,x)$ changes sign and Theorem 1.1 cannot be applied. It should be pointed out that Ortega has presented the index characterized asymptotic stability in dissipative case. See the references [15, 16] and the surveys [18, 19]. In this paper, we try to obtain similar results for the conservative case following the ideas in [15, 16, 18, 19], see Theorem 2.3 below.

The purpose of the present paper is to extend Theorem 1.1 which hold when $g_x(t,x)$ changes sign, we prove the following theorems.

**Theorem 1.2.** *Suppose that $g(t,x) \in C^1(\mathbb{R}\times\mathbb{R})$ satisfies the following semilinearity condition: there exist $T$-periodic functions $\phi, \Phi \in L^1(0,T)$ such that*

$$\phi(t) \le g_x(t,x) \le \Phi(t), \tag{1.4}$$

*uniformly in $t$. Furthermore, assume*

$$\overline{\phi(t)} > 0 \text{ and } \underline{\lambda}_1(\Phi) > 0, \tag{1.5}$$

*here $\overline{\phi(t)}$ denotes the average of $\phi(t)$ over a period and $\underline{\lambda}_1$ is antiperiodic eigenvalue. Then (1.1) has a unique $T$-periodic solution which is linearly stable.*

Let us recall a lower bound for $\underline{\lambda}_1(\Phi)$ from [20]. Let us define the positive part of a function $\Phi$ as $\Phi^+ = \max\{\Phi, 0\}$. If the $L^p$ norm $\|\Phi^+\|_p$ satisfies

$$\|\Phi^+\|_p \le K(2p^*),\ p^* = p/p-1,$$

then (see (13) in [20])

$$\underline{\lambda}_1(\Phi) \ge \Big(\frac{\pi}{T}\Big)^2\Big(1 - \frac{\|\Phi^+\|_p}{K(2p^*)}\Big).$$

Here $K(q)$ is the best Sobolev constant in the following inequality:

$$C\|x\|_q^2 \le \|\dot{x}\|_2^2 \ \text{ for all } x \in H_0^1(0,T),$$

where $H_0^1(0,T)$ is a Sobolev space of all the $T$-periodic absolutely continuous functions $x$ such that $\int_0^T \dot{x}^2(t)dt < \infty$ with the norm

$$\|x\|_{1,T} = \Big(\int_0^T \dot{x}^2(t)dt + \int_0^T x^2(t)dt\Big)^{\frac{1}{2}}.$$

Explicitly,

$$K(q) = \begin{cases} \frac{2\pi}{qT^{1+2/q}}\Big(\frac{2}{2+q}\Big)^{1-2/q}\Big(\frac{\Gamma(\frac{1}{q})}{\Gamma(\frac{1}{2}+\frac{1}{q})}\Big)^2, & \text{if } 1 \le q < \infty, \\ \frac{4}{T}, & \text{if } q = \infty. \end{cases}$$

See Talenti [13]. Thus we have the following

**Corollary 1.3.** *Assume all conditions of Theorem 1.2 hold except (1.5) and assume*

$$\overline{\phi(t)} > 0 \text{ and } \|\Phi^+\|_p < K(2p^*), \quad 1 \le p \le +\infty.$$

*Then the same conclusion holds. In particular, when $p = +\infty$, we arrive at the usual criterion*

$$\|\Phi^+\|_\infty < K(2) = \frac{\pi^2}{T^2}.$$

## 2. Linear stability and index

2.1. **Hill's equation and eigenvalue theory.** To each function $a \in L^1(\mathbb{R}/T\mathbb{Z})$, we associate a linear equation

$$\ddot{x} + a(t)x = 0, \tag{2.1}$$

which is called Hill's equation and there are many studies about it. The book by Magnus and Winkler [11] is a classical reference.

Now we recall some standard notions in the theory of Hill's equations. Denote by $\Psi(t) = \phi_1(t) + i\phi_2(t)$ the complex-valued solution of (2.1) with the initial data: $\Psi(0) = 1$ and $\Psi'(0) = i$, where $\phi_1$ and $\phi_2$ are respectively the real and imaginary parts of $\Psi$. Let

$$M(t) = \begin{pmatrix} \phi_1(t) & \phi_2(t) \\ \dot{\phi}_1(t) & \dot{\phi}_2(t) \end{pmatrix}$$

be employed for the fundamental matrix solution of

$$\dot{X} = A(t)X, \ X(0) = I_2,$$

where the column vector function $X(t) = (x(t), x'(t))^T$, $I_2$ is the $2 \times 2$ identity matrix and $A(t)$ is the matrix function

$$A(t) = \begin{pmatrix} 0 & 1 \\ -a(t) & 0 \end{pmatrix}.$$

Liouville's theorem implies that the matrix solution $M(t)$ always satisfies

$$\det M(t) = 1.$$

This property motivates our interest in the symplectic group. The monodromy matrix associated with (2.1) is

$$M(T) = \begin{pmatrix} \phi_1(T) & \phi_2(T) \\ \dot{\phi}_1(T) & \dot{\phi}_2(T) \end{pmatrix}.$$

Then $M$ is symplectic, i.e., $\det M = 1$. The eigenvalues $\rho_i$, $i = 1, 2$, of $M$ are called the *Floquet multipliers* of (2.1). They satisfy $\rho_1 \cdot \rho_2 = 1$. We can classify (2.1) into three types, according to the Floquet multipliers, as either *hyperbolic* when $|\rho_{1,2}| \neq 1$, or *elliptic* when $|\rho_{1,2}| = 1$ but $\rho_{1,2} \neq \pm 1$, or *parabolic* when $\rho_{1,2} = \pm 1$, respectively.

Next we introduce some notations on eigenvalues. Consider the eigenvalue problems

$$x'' + (\lambda + a(t))x = 0 \tag{2.2}$$

subject to the periodic boundary condition

$$x(0) - x(T) = x'(0) - x'(T) = 0, \tag{2.3}$$

or to the anti-periodic boundary condition

$$x(0) + x(T) = x'(0) + x'(T) = 0. \tag{2.4}$$

We use

$$\lambda_1^D(a) < \lambda_2^D(a) < \cdots < \lambda_n^D(a) \cdots$$

to denote all eigenvalues of (2.2) with the Dirichlet boundary condition $(D)$ :

$$x(0) = x(T) = 0. \tag{2.5}$$

The following are standard results for eigenvalue theory. See, e.g. Reference [11]. A partial generalization of these results to the one-dimensional $p$-Laplacian with periodic potentials is given in Reference [14].

**Theorem 2.1.** *There exist two sequences $\{\underline{\lambda}_n(a) : n \in \mathbb{N}\}$ and $\{\overline{\lambda}_n(a) : n \in \mathbb{Z}^+\}$ of the reals such that*

*$(P_1)$ they have the following order:*

$$-\infty < \overline{\lambda}_0(a) < \underline{\lambda}_1(a) \leq \overline{\lambda}_1(a) < \cdots < \underline{\lambda}_n(a) \leq \overline{\lambda}_n(a) < \cdots$$

*and $\underline{\lambda}_n(a) \to +\infty$, $\overline{\lambda}_n(a) \to +\infty$ as $n \to \infty$.*

*$(P_2)$ $\lambda$ is an eigenvalue of (2.2)-(2.3) if and only if $\lambda = \underline{\lambda}_n(a)$ or $\overline{\lambda}_n(a)$ for some even integer $n$; $\lambda$ is an eigenvalue of (2.2)-(2.4) if and only if $\lambda = \underline{\lambda}_n(a)$ or $\overline{\lambda}_n(a)$ for some odd integer $n$.*

*$(P_3)$ (Continuity) $\lambda_n^D(a), \underline{\lambda}_n(a)$, and $\overline{\lambda}_n(a)$ are continuous functions of $q$ with respect to the $L^1$-metric on $q$'s: $d(a_1, a_2) = \int_0^T |a_1(t) - a_2(t)|dt$.*

*$(P_4)$ the eigenvalues $\underline{\lambda}_n(a)$ and $\overline{\lambda}_n(a)$ can be recovered from the Dirichlet eigenvalues in the following way: for any $n \in \mathbb{N}$,*

$$\underline{\lambda}_n(a) = \min\{\lambda_n^D(a_{t_0}) : t_0 \in \mathbb{R}\},\ \overline{\lambda}_n(a) = \max\{\lambda_n^D(a_{t_0}) : t_0 \in \mathbb{R}\},$$

*here $a_{t_0}(t)$ denotes the translation of $a(t) : a_{t_0}(t) \equiv a(t + t_0)$.*

*$(P_5)$ (Comparison) the comparison results hold for all of these eigenvalues. If $a_1 \geq a_2$ then*

$$\underline{\lambda}_n(a_1) \leq \underline{\lambda}_n(a_2),\ \overline{\lambda}_n(a_1) \leq \overline{\lambda}_n(a_1),\ \lambda_n^D(q_1) \leq \lambda_n^D(q_2), \tag{2.6}$$

*for any $n \in \mathbb{N}$. If $a_1(t) \geq a_2(t)$ for all $t$, and $a_1(t) > a_2(t)$ for $t$ in a subset of positive measure, then all of the inequalities in (2.6) are strict.*

*$(P_6)$ (Nodal structure) The eigenfunction of $\overline{\lambda}_0(a)$ do not vanish everywhere. For $n \in \mathbb{N}$, the eigenfunctions of $\underline{\lambda}_n(a)$ or $\overline{\lambda}_n(a)$ have exactly $n-1$ zeros in the intervals of the form $(t_0, t_0 + T)$.*

2.2. **Definition of the index via the Poincaré map.** In this subsection we assume uniqueness for the initial value problem associated to (1.1). Given $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, let $x(t;\xi)$ be the solution of (1.1) satisfying

$$x(0) = \xi_1, \quad x'(0) = \xi_2.$$

The Poincaré map is defined as the mapping

$$P_T : D_T \subset \mathbb{R}^2 \to \mathbb{R}^2, \qquad P_T(\xi) = (x(T;\xi), \dot{x}(T;\xi)),$$

where $D_T = \{\xi \in \mathbb{R}^2 : x(t;\xi)$ is defined in $[0,T]\}$.

The standard theory of the Cauchy problem says that $D_T$ is open in $\mathbb{R}^2$ and $P_T$ is a homeomorphism between $D_T$ and $P_T(D_T)$. In addition, the fixed points of $P_T$ correspond to the initial conditions of the $T$-periodic solutions and the search of $T$-periodic solutions is reduced to the study of the equation in $\mathbb{R}^2$,

$$\xi = P_T(\xi).$$

Let $x$ be a $T$-periodic solution of 1.1 and $\xi_0 = (x(0), x'(0))$. The solution $x$ is said to be isolated (periodic $T$) if $\xi_0$ is an isolated fixed point of $P_T$. In such case the index of $x$ is defined in terms of the following formula

$$\mathrm{ind}_T(x) = i[P_T, \xi_0],$$

where $i$ refers to the definition of the local fixed point index in the plane employed in [1]. For more information on the index of periodic solutions see [8] and [12].

2.3. **Connection between linear stability and index.** In this subsection we proof a few lemmas that are crucial for the proofs of the main results. First we need the following.

**Definition 2.2.** Given a $T$-periodic solution $x(t)$ of (1.1). It will be said that $x(t)$ is *non-degenerate* of periodic $T$ if the variational equation is

$$\ddot{y} + g_x(t,x)y = 0 \tag{2.7}$$

has no periodic solutions different from zero of periodic $T$.

**Theorem 2.3.** *Assume that $x$ is a nondegenerate $T$-periodic solution of (1.1) such that the inequality below holds*

$$\underline{\lambda}_1(g_x(t,x)) \geq 0, \tag{2.8}$$

*for $t \in \mathbb{R}$. Then $x$ is linearly stable if and only if* $\mathrm{ind}_T$*($x$)=1.*

**Remark 2.4.** Notice that a nondegenerate solution is always isolated and it is assumed that $x$ is a nondegenerate in order to employ linearization techniques. We do not know if Theorem 2.3 is still valid when nondegenerate is replaced by degenerate. Because the computation of the index in the degenerate case

$$\rho_1 = \rho_2 = 1$$

is more delicate the previous technique does not work and the index of $x$ depends not only on (2.7) but also on the nonlinear terms of the Taylor expansion of $g$. Some methods about computation in the degenerate case can be found in [**?**] or [9] for more details.

The crucial step in proof of Theorem 1.2 is the following observation on the Hill equation (2.1).

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**Lemma 2.5.** *Assume that*

$$\underline{\lambda}_1(a) > 0. \tag{2.9}$$

*Then problem (2.1) does not admit any negative Floquet multipliers. In particular, (2.1) does not admit any nontrivial subharmonic periodic solution of order 2.*

**Proof.** Suppose that there is a nontrivial solution of (2.1) with a negative Floquet multiplier, i.e. $x(t+T) = \rho x(t)$, $t \in \mathbb{R}$ for some $\rho < 0$. Hence, there exists $t_0 \in [0, T]$ with $x(t_0) = x(t_0 + T) = 0$. Thus the corresponding $x$ is a nontrivial solution of the following Dirichlet boundary value problem

$$\begin{cases} \ddot{x}(t) + a(t)x = 0, \\ x(t_0) = x(T + t_0) = 0. \end{cases}$$

That is, $x$ is an eigenfunction associated with eigenvalue $\lambda_k^D(a) = 0$ for some $k \geq 1$ of

$$\begin{cases} \ddot{x}(t) + (\lambda + a(t))x = 0, \\ x(t_0) = x(T + t_0) = 0, \end{cases}$$

and hence $\underline{\lambda}_1(a) \leq \lambda_1^D(a) \leq \lambda_k^D(a) = 0$, contradicting (2.9).

**Proof of Theorem 2.3.** When the inequality in (2.8) is not strict it is elementary to show that $x$ is linearly stable and $\mathrm{ind}_T(x) = 1$. Therefore it will be assumed that the inequality in (2.8) is strict, at least on a set of positive measure. Denote by $\rho_1, \rho_2$ ($|\rho_1| \geq |\rho_2|$) the Floquet multipliers of (2.1). By Lemma 2.5 the multipliers are either conjugate complex or real and positive. In the elliptic case,

$$\rho_1 = \overline{\rho_2}$$

if and only if

$$\mathrm{ind}_T(x) = \mathrm{sign}\{\det(I_2 - M(T))\} = \mathrm{sign}\{|1 - \rho_1|^2\} = 1.$$

In the hyperbolic case,

$$0 < \rho_1 < 1 < \rho_2$$

if and only if

$$\mathrm{ind}_T(x) = \mathrm{sign}\{\det(I_2 - M(T))\} = \mathrm{sign}\{(1 - \rho_1)(1 - \rho_2)\} = -1.$$

The parabolic case is excluded because $x$ is nondegenerate 1 cannot be a Floquet multipler.

The conclusion now follows from the well-known principle of stability for Hill equation (see Theorem 7.2 in [6]) that periodic system (2.1) is stable in the sense of Lyapunov if and only if (2.1) is elliptic, or is parabolic ($\rho_1 = \rho_2 = \pm 1$) with further property that all solutions of (2.1) satisfy $x(t+T) = x(t)$, the $T$-periodic solutions in case $\rho_1 = \rho_2 = 1$, or $x(t+T) = -x(t)$, the $T$-anti-periodic solutions in case $\rho_1 = \rho_2 = -1$.

## 3. Proof of Theorem 1.2

The proof of existence is based on the following two lemmas.

**Lemma 3.1.** *Assume that*

$$\overline{a(t)} > 0 \text{ and } \underline{\lambda}_1(a) > 0. \tag{3.1}$$

*Then Hill equation (2.1) has only the trivial $T$-periodic solution.*

**Proof.** Suppose on the contrary that (2.1) admits a nontrivial $T$-periodic solution $x(t)$. We claim that $x(t)$ vanishes at some $t_0 \in [0, T]$. If not, then $x(t) \neq 0$ for all $t$ in $\mathbb{R}$. By the periodic boundary conditions, we have $\dot{x}(T) = \dot{x}(0)$ and $\frac{\dot{x}(T)}{x(T)} = \frac{\dot{x}(0)}{x(0)}$. Dividing (2.1) by $x(t)$ and integrating by part gives that

$$\int_0^T \frac{\dot{x}(t)^2}{x(t)^2} + \int_0^T a(t)dt = 0,$$

which contradicts the hypothesis $\overline{a(t)} > 0$. So $x(t)$ has a zero in [0,T]. We may assume that $x(0) = 0$ so that $x(0) = x(T) = 0$. Thus the corresponding $x$ is a nontrivial solution of the Dirichlet boundary value problem (2.1)-(2.5) That is, $x$ is an eigenfunction associated with eigenvalue $\lambda_k^D(a) = 0$ for some $k \geq 1$ of (2.2)-(2.5), and hence $\underline{\lambda}_1(a) \leq \lambda_1^D(a) \leq \lambda_k^D(a) = 0$, contradicting (3.1).

**Lemma 3.2.** *Under the conditions of Lemma 3.1, the Hill equation (2.1) is stable.*

**Proof.** The proof will be completed using Theorem 2.3. To this end, we compute the local index and consider following parametric equation

$$L_\lambda = \ddot{x} + [\lambda a(t) + (1-\lambda)a_0]x = 0, \quad \lambda \in [0,1],$$

where $0 < a_0 < (\pi/T)^2$.

Let $a_\lambda = \lambda a(t) + (1-\lambda)a_0$. Since

$$\overline{a_\lambda} = \lambda\overline{a(t)} + (1-\lambda)a_0 > 0,$$

and

$$\underline{\lambda}_1(a_\lambda) \geq \lambda\ \underline{\lambda}_1(a) + (1-\lambda)\underline{\lambda}_1(a_0) > 0,$$

it follows from Lemma 3.1 that $L_\lambda x = 0$ does not admit a nontrivial $T$-periodic solution. Let $B_\epsilon$ be the $\epsilon$-ball of 0, then $L_\lambda x = 0$ has no $T$-periodic solution on $\partial B_\epsilon$ for $\lambda \in [0,1]$. By the homotopy invariance properties of the topological degree, we have that

$$\begin{aligned} \operatorname{ind}(L_1, 0) &= \deg(L_1, B_\epsilon, 0) = \deg(L_\lambda, B_\epsilon, 0) \\ &= \deg(L_0, B_\epsilon, 0) = \operatorname{sgn}\begin{vmatrix} 0 & 1 \\ -a_0 & 0 \end{vmatrix} = 1. \end{aligned}$$

The conclusion follows from Theorem 2.3.

**Proof of Theorem 1.2** In order to show that the conditions are sufficient, we divide the proof into two steps.

*Step 1: Uniqueness.* Suppose that $x_1(t)$ and $x_2(t)$ are two $T$-periodic solutions of (1.1). Then

$$[x_1(t) - x_2(t)]'' + [g(t, x_1(t)) - g(t, x_2(t))] = 0. \tag{3.2}$$

Setting $\widetilde{x}(t) = x_1(t) - x_2(t)$, we obtain, from (3.2), that

$$\widetilde{x}''(t) + \beta(t)\widetilde{x}(t) = 0, \tag{3.3}$$

where $\beta(t) = \frac{g(t,x_1) - g(t,x_2)}{x_1 - x_2}$. It follows from Lemma 3.1 that $\widetilde{x}(t) \equiv 0$, which implies that $x_1(t) \equiv x_2(t)$ for all $t \in \mathbb{R}$.

*Step 2: Existence and linearly stable.* Without loss of generality, we may assume that $g(t, 0) = 0$, for otherwise we can reduce both sides of Eq. (1.1) by $g(t, 0)$. A natural choice for the parametrized equation in applying homotopy invariance property is to take $H$ defined by

$$H_\lambda(x) = \ddot{x}(t) + \lambda g(t, x) + (1-\lambda)\Phi(t)x = 0, \tag{3.4}$$

in which $\Phi(t)$ is as in Theorem 1.2.

We claim that there is $R > 0$ such that equation (3.4) has no solution on $\partial B_R$ in $L^\infty[0,T]$ for all $\lambda \in [0,1]$. Suppose the assertion is not true. Let $x_n = x_n(t)$ be a sequence of $T$-periodic solutions such that $\|x_n\| \to \infty$ and $\lambda_n \in [0,1]$ be the corresponding sequence. Let $z_n(t) = \frac{x_n(t)}{\|x_n\|}$. First, dividing (3.4) by $\|x_n\|$, then multiplying by $\varphi(t) \in C_T^2$ and finally integrating by parts we have that

$$\int_0^T \Big\{z_n\ddot{\varphi} + \Big[\lambda_n g(t,x_n) + (1-\lambda_n)\Phi(t)x_n\Big]\cdot\varphi/\|x_n\|\Big\}dt = 0.$$

The conditions of Theorem 1.2 imply that $\{[\lambda_n g(t,x_n) + (1-\lambda_n)\Phi(t)x_n]/\|x_n\|\}$ is bounded and hence is pre-compact in weak star topology in $L^1[0,T]$. Thus there is a subsequence such that $g(t,x_n)/x_n \rightharpoonup \alpha(t)$ and $\lambda_n \to \lambda$. Taking the limit as $n \to \infty$, one obtains that $z_n \to z$,

$$\int_0^T (z\ddot{\varphi} + z\omega(t)\varphi)dt = 0,$$

where $\omega(t) = \lambda\alpha(t) + (1-\lambda)\Phi(t)$ satisfying the conditions of Lemma 3.1. It follows from Lemma 3.1 that $z(t) \equiv 0$, which contradicts $\|z(t)\| = 1$. This shows the boundedness of the periodic solutions of (3.4). By the standard argument one can verify that the $C^1$-norm is bounded independently of $\lambda$. Next, by applying the homotopy invariance property, we have that

$$\deg(H_1, B_R, 0) = \deg(H_0, B_R, 0).$$

From the same reasonings of Lemma 3.2 one proves that $\deg(H_0, B_R, 0) = 1$. This completes the existence of $T$-periodic solution $x$, and linear stability can be obtained by Lemma 3.2.

## 4. Final remark

Theorem 2.3 can be applied to other kinds of problems. In particular, to the piecewise linear equation

$$\ddot{x} + \mu(t)x^+ - \nu(t)x^- = 0, \tag{4.1}$$

where $x^+ = \max\{x,0\}, x^- = \max\{-x,0\}, \mu,\nu \in L^1(0,T)$. The equation (4.1) is very popular since a series of works of Lazer and McKenna [10] as a simple mathematical model for vertical oscillations of a long-span suspension bridge. The following result can be proved in a way similar to Theorem 1.2

*Equation (4.1) has a unique $T$-periodic solution which is linearly stable if $\overline{\mu(t)} > 0, \overline{\nu(t)} > 0$ and*

$$\|\max_{t\in\mathbb{R}}\{\mu(t),\nu(t)\}\|_p < K(2p^*), \quad 1 \le p \le +\infty.$$

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# On a system of three max-type nonlinear difference equations

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**Abstract:** The primary focus of this paper is to investigate the boundedness and asymptotic behavior of the following symmetric system of max-type difference equations

$$x_{n+1}=\max\{c,\frac{y_n^p}{z_{n-1}^q}\},\ y_{n+1}=\max\{c,\frac{z_n^p}{x_{n-1}^q}\},\ z_{n+1}=\max\{c,\frac{x_n^p}{y_{n-1}^q}\},\quad n\in\mathbb{N}_0,$$

where the parameters $c,p,q\in(0,\infty)$ and the initial conditions $x_{-1},x_0,y_{-1},y_0,z_{-1},z_0$ are arbitrary positive real numbers. Our main results considerably improve results appearing in the literature (see, Stević, (2014) [29]).

**Keywords:** max-type system, difference equations, boundedness, global attractivity.

## 1. Introduction

In last few decades there has been a great interest in studying nonlinear difference equations and systems for developing some new techniques which can be used in investigating the models describing real life situations in biology, control theory, economics, etc. (see, e.g., [1-15] and the references therein). Recently, the so-called max-type difference equation has attracted more and more attention. However, the maxima operator is not a smooth function in n-dimensional real vector space so that the techniques which use

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derivatives could be of almost no use, so the study of max-type systems of difference equations become more difficult. Some studies of these difference equations have been presented in [16-25].

Paper [26] is one of the first such papers on max-type difference equations. It studies positive solutions of the difference equation

$$x_{n+1} = \max\{a, \frac{x_n^p}{x_{n-1}^p}\}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where initial values $x_{-1}, x_0$, and parameters $a$ and $p$ are positive numbers.

In [27], Stevo Stević studied the boundedness character of positive solutions to the following max-type difference equation,

$$x_n = \max\{A, \frac{x_{n-1}^p}{x_{n-k}^r}\}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $k \in N \setminus \{1\}$, the parameters $A$ and $r$ are positive and $p$ is a nonnegative real number.

As an extension of (1.2), Stevo Stević studied the boundedness character and global attractivity of positive solutions of the following symmetric system of max-type difference equation

$$x_{n+1} = \max\{c, \frac{y_n^p}{x_{n-1}^p}\}, \quad y_{n+1} = \max\{c, \frac{x_n^p}{y_{n-1}^p}\}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

where $c, p \in (0, +\infty)$ (see [28]).

Above results motivated Stevo Stević to continuously investigate the behavior of positive solutions of the following max-type system of differences

$$x_{n+1} = \max\{c, \frac{y_n^p}{z_{n-1}^p}\}, y_{n+1} = \max\{c, \frac{z_n^p}{x_{n-1}^p}\}, z_{n+1} = \max\{c, \frac{x_n^p}{y_{n-1}^p}\}, \quad n \in \mathbb{N}_0, \tag{1.4}$$

where the parameters $c$ and $p$ are positive real numbers. It is proved that system (1.4) is permanent when $p \in (0,4)$ and so forth (see [29]).

Motivated by works [26-29], the primary focus of this paper is to investigate the boundedness character and global attractivity of the following max-type difference equations

$$x_{n+1} = \max\{c, \frac{y_n^p}{z_{n-1}^q}\}, y_{n+1} = \max\{c, \frac{z_n^p}{x_{n-1}^q}\}, z_{n+1} = \max\{c, \frac{x_n^p}{y_{n-1}^q}\}, \quad n \in \mathbb{N}_0, \tag{1.5}$$

where $c, p, q \in (0,+\infty)$ . It is obvious that the paper can be considered as a continuation of studying special cases of the next systems of difference equations

$$x_{n+1} = \max\{A_n, \frac{y_{n-m}^p}{z_{n-k}^q}\}, y_{n+1} = \max\{A_n, \frac{z_{n-m}^p}{x_{n-k}^q}\}, z_{n+1} = \max\{A_n, \frac{x_{n-m}^p}{y_{n-k}^q}\}, \quad n \in \mathbb{N}_0,$$

where $m, k \in N$ , $p, q \in (0,+\infty)$ , and $\left(A_n\right)_{n\in N_0}$ is a sequence of positive numbers. For more related papers in this research area, see, for example, [30-33] and the references therein.

The rest of the paper is organized as follows. In Section 2, we will focus our attention on the buondedness character of solutions of system (1.5) by developing new iterative method and inequality technique. In Section 3, we will investigate the asymptotic behavior of solutions of system (1.5). Then we show an example and carry out numerical simulations in Section 4, from which it can be seen that all simulations agree with the theoretical results. We finally conclude our paper in Section 5.

## 2. Boundedness character of solutions

This section is devoted to analyzing the boundedness of the positive solutions to the max-type difference systems (1.5).

**Theorem 2.1.** Assume that $f(\lambda) = \lambda^2 - p\lambda + q$ and (a) there is $\lambda_1 > 1$ such that $f(\lambda_1) = 0$, or (b) there is $\lambda_1 = \lambda_2 = 1$ such that $f(\lambda_1) = f(\lambda_2) = 0$ . Then the system (1.5) has positive unbounded solutions.

**Proof.** Obviously, from (1.5), we can easily see that

$$x_{n+1} \geq \frac{y_n^p}{z_{n-1}^q}, \quad y_{n+1} \geq \frac{z_n^p}{x_{n-1}^q}, \quad z_{n+1} \geq \frac{x_n^p}{y_{n-1}^q}. \tag{2.1}$$

By taking logarithm in (2.1), for any $n \in \mathbb{N}_0$ , we obtain

$$\ln x_{n+1} \geq p \ln y_n - q \ln z_{n-1}, \quad \ln y_{n+1} \geq p \ln z_n - q \ln x_{n-1}, \quad \ln z_{n+1} \geq p \ln x_n - q \ln y_{n-1}. \tag{2.2}$$

Moreover, it follows that

$$\ln x_{n+1} y_{n+1} z_{n+1} \geq p \ln x_n y_n z_n - q \ln x_{n-1} y_{n-1} z_{n-1}. \tag{2.3}$$

Let $v_n = \ln x_n y_n z_n$ , where $n \geq -1$, then inequality (2.3) becomes

$$v_{n+1} \geq p v_n - q v_{n-1}, \quad n \in \mathbb{N}_0. \tag{2.4}$$

By hypothesis (a), we have that $f(\lambda_1) = 0$ and $\lambda_1 > 1$.

Let

$$f_1(\lambda) = \frac{f(\lambda)}{\lambda - \lambda_1} = \lambda + a, \tag{2.5}$$

then it follows that

$$f(\lambda) = (\lambda + a)(\lambda - \lambda_1). \tag{2.6}$$

Thus, we can obtain $p = \lambda_1 - a$ and $q = -a\lambda_1$.

Set

$$u_n = v_n + av_{n-1}, \quad n \in \mathbb{N}_0. \tag{2.7}$$

Then inequation (2.4) can be written in the following form

$$\begin{aligned} v_{n+1} - pv_n + qv_{n-1} &= v_{n+1} - (\lambda_1 - a)v_n - a\lambda_1 v_{n-1} \\ &= v_{n+1} + av_n - \lambda_1(v_n + av_{n-1}) \\ &= u_{n+1} - \lambda_1 u_n \\ &\geq 0. \end{aligned} \tag{2.8}$$

That is

$$u_{n+1} \geq \lambda_1 u_n. \tag{2.9}$$

Let $v_{-1}, v_0$ be chosen such that

$$v_0 \geq |a||v_{-1}|. \tag{2.10}$$

This, along with (2.9), yields to

$$u_{n+1} \geq \lambda_1^n u_0, \text{ and } u_0 > 0. \tag{2.11}$$

Letting $n \to \infty$ in (2.11), from assumption (a) $\lambda_1 > 1$ and $u_0 > 0$, it follows that

$$u_n = v_n + av_{n-1} \to +\infty \text{ as } n \to +\infty. \tag{2.12}$$

Hence $\{v_n\}_{n \geq -1}$ is unbounded. As $v_n = \ln x_n y_n z_n$, it follows that

$$x_n y_n z_n \to \infty \text{ as } n \to \infty, \tag{2.13}$$

which along with $\sqrt{x_n^2 + y_n^2 + z_n^2} \geq \sqrt{3}\sqrt[3]{x_n y_n z_n}$ implies

$$\sqrt{x_n^2 + y_n^2 + z_n^2} \to +\infty, \tag{2.14}$$

from which it follows that the sequence $\{(x_n, y_n, z_n)\}_{n \geq -1}$ is unbounded.

By hypothesis (b), we have $p = 2,\ q = 1$. Then from (2.1) we get

$$\frac{x_{n+1}}{y_n} \geq \frac{y_n}{z_{n-1}}, \quad \frac{y_{n+1}}{z_n} \geq \frac{z_n}{x_{n-1}}, \quad \frac{z_{n+1}}{x_n} \geq \frac{x_n}{y_{n-1}}. \tag{2.15}$$

Moreover, one has

$$\frac{x_{n+1}y_{n+1}y_{n+1}}{x_n y_n z_n} \geq \frac{x_n y_n z_n}{x_{n-1}y_{n-1}z_{n-1}} \geq \cdots \geq \frac{x_0 y_0 z_0}{x_{-1}y_{-1}z_{-1}},\ n \in N_0. \tag{2.16}$$

and consequently

$$x_n y_n z_n \geq \left(\frac{y_0 x_0 z_0}{x_{-1}y_{-1}z_{-1}}\right)^n x_0 y_0 z_0,\ n \in N_0. \tag{2.17}$$

If we choose the initial conditions $x_{-1}, y_{-1}, z_{-1}, x_0, y_0, z_0$ such that $x_0 y_0 z_0 > x_{-1}y_{-1}z_{-1} > 0$ then we obtain (2.13) and consequently (2.14), which implies that the sequence $\{(x_n, y_n, z_n)\}_{n\geq -1}$ is unbounded, and then the proof of Theorem 2.1 is completed.

Next, we study the different cases concerning with the boundedness of positive solutions to the systems (1.5).

**Theorem 2.2.** If $c > 0$, $p > 0$ and $p^2 < 4q$, then the solutions to system (1.5) are bounded.

**Proof.** Assume that $(x_n, y_n, z_n)_{n\geq -1}$ is a positive solution to systems (1.5). Then the following estimate obviously holds

$$\min\{x_n, y_n, z_n\} \geq c, \quad n \in \mathbb{N}_0. \tag{2.18}$$

Due to the symmetry among $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$, as long as we prove the boundedness of $\{x_n\}$, other sequences $\{y_n\}$ and $\{z_n\}$ can be proved as well.

From systems (1.5), it follows that

$$x_{n+1} = \max\{c, \frac{y_n^p}{z_{n-1}^q}\} = \max\{c, \frac{c^p}{z_{n-1}^q}, \frac{z_{n-1}^{p^2-q}}{x_{n-2}^{pq}}\}, n \in \mathbb{N}_0. \tag{2.19}$$

**Case1.** When $p^2 \leq q$, we get

$$x_{n+1} \leq \max\{c, \frac{1}{c^{q-p}}, \frac{1}{c^{pq-p^2+q}}\}. \tag{2.20}$$

Thus, the sequence $\{x_n\}_{n\geq -1}$ is bounded.

**Case2.** When $p^2 > q$, let sequence $\{a_l\}_{l\geq 0}$ be defined as follows

$$a_{l+1} = q/(p-a_l), \quad a_0 = 0, \quad l \in N_0 \cdot \tag{2.21}$$

From (1.5) and (2.21), we have

$$\begin{aligned}
x_{n+1} &= \max\{c, \frac{y_n^p}{z_{n-1}^q}\} = \max\{c, \frac{c^p}{z_{n-1}^q}, \frac{z_{n-1}^{p^2-q}}{x_{n-2}^{pq}}\} \\
&= \max\{c, (\frac{c}{z_{n-1}^{q/p}})^p, (\frac{z_{n-1}}{x_{n-2}^{q/(p-q/p)}})^{(p-q/p)p}\} \\
&= \max\{c, (\frac{c}{z_{n-1}^{q/p}})^p, (\frac{c}{x_{n-2}^{q/(p-q/p)}})^{(p-q/p)p}, (\frac{x_{n-2}^{p-q/(p-q/p)}}{y_{n-3}^q})^{(p-q/p)p}\} \\
&= \cdots\cdots \\
&= \max\{c, (\frac{c}{z_{n-1}^{q/p}}, (\frac{c}{x_{n-2}^{q/(p-q/p)}}, \cdots, (\frac{x_{n-(3k-1)}^{p-a_{3k-1}}}{y_{n-3k}^q})^{p-a_{3k-2}}, \cdots)^{(p-q/p)})^p\} \\
&= \max\{c, (\frac{c}{z_{n-1}^{q/p}}, (\frac{c}{x_{n-2}^{q/(p-q/p)}}, \cdots, (\frac{c^{p-a_{3k-1}}}{y_{n-3k}^q}, \frac{y_{n-3k}^{p(p-a_{3k-1})-q}}{z_{n-(3k+1)}^q})^{p-a_{3k-2}}, \cdots)^{(p-q/p)})^p\} \\
&= \max\{c, (\frac{c}{z_{n-1}^{q/p}}, (\frac{c}{x_{n-2}^{q/(p-q/p)}}, \cdots, (\frac{y_{n-3k}^{p-a_{3k}}}{z_{n-(3k+1)}^q})^{p-a_{3k-1}}, \cdots)^{(p-q/p)})^p\} \\
&= \max\{c, (\frac{c}{z_{n-1}^{q/p}}, (\frac{c}{x_{n-2}^{q/(p-q/p)}}, \cdots, (\frac{z_{n-(3k+1)}^{p-a_{3k+1}}}{x_{n-(3k+2)}^q})^{p-a_{3k}}, \cdots)^{(p-q/p)})^p\}.
\end{aligned} \tag{2.22}$$

From the monotonicity of $g(x) = q/(p-x)$ on the interval $(0, p)$ along with the fact $0 = a_0 < a_1 = q/p$, it follows that the sequence $\{a_l\}$ is increasing as far as $a_l \le p$ for every $l \in \mathrm{N}_0$. Hence, we have $\lim_{l\to+\infty} a_l = x^*$, $x^* \in (0, p]$ and $x^*$ is the solution of the following equation

$$f(x) = x(p-x) - q = 0. \tag{2.23}$$

However the equation (2.23) has no real roots existing in $(0, p]$ when $p^2 < 4q$, which is contradiction. Hence there is $l_0 \in \mathbb{N}$ such that $a_{l_0-1} < p$ and $a_{l_0} \ge p$.

If $l_0 = 3k$, then by using (2.18) in (2.22) it follows that

$$\begin{aligned}
x_{n+1} &= \max\{c, (\frac{c}{z_{n-1}^{q/p}}, (\frac{c}{x_{n-2}^{q/(p-q/p)}}, \cdots, (\frac{y_{n-3k}^{p-a_{3k}}}{z_{n-(3k+1)}^q})^{p-a_{3k-1}}, \cdots)^{(p-q/p)})^p\} \\
&\le \max\{c, (\frac{c}{c^{a_1}}, (\frac{c}{c^{a_2}}, \cdots, (\frac{1}{c^{q-p+a_{3k}}})^{p-a_{3k-1}}, \cdots)^{(p-a_1)})^p\}
\end{aligned} \tag{2.24}$$

for $n > 3k$, from which the boundedness of $\{x_n\}_{n\ge-1}$ follows in this case.

If $l_0 = 3k+1$, then it follows that

$$\begin{aligned} x_{n+1} &= \max\{c, (\frac{c}{z_{n-1}^{q/p}}, (\frac{c}{x_{n-2}^{q/(p-q/p)}}, \cdots, (\frac{z_{n-(3k+1)}^{p-a_{3k+1}}}{x_{n-(3k+2)}^{q}})^{p-a_{3k}}, \cdots)^{(p-q/p)})^{p}\} \\ &\le \max\{c, (\frac{c}{c^{a_1}}, (\frac{c}{c^{a_2}}, \cdots, (\frac{1}{c^{q-p+a_{3k+1}}})^{p-a_{3k}}, \cdots)^{(p-a_1)})^{p}\} \end{aligned} \tag{2.25}$$

for $n \ge 3k+1$, from which the boundedness of $\{x_n\}_{n\ge -1}$ follows in this case.

If $l_0 = 3k+2$, then it follows that

$$\begin{aligned} x_{n+1} &= \max\{c, (\frac{c}{z_{n-1}^{q/p}}, (\frac{c}{x_{n-2}^{q/(p-q/p)}}, \cdots, (\frac{x_{n-(3k+2)}^{p-a_{3k+2}}}{y_{n-3(k+1)}^{q}})^{p-a_{3k+1}}, \cdots)^{(p-q/p)})^{p}\} \\ &\le \max\{c, (\frac{c}{c^{a_1}}, (\frac{c}{c^{a_2}}, \cdots, (\frac{1}{c^{q-p+a_{3k+2}}})^{p-a_{3k+1}}, \cdots)^{(p-a_1)})^{p}\} \end{aligned} \tag{2.26}$$

for $n \ge 3k+1$, from which the boundedness of $\{x_n\}_{n\ge -1}$ follows in this case.

Combined the case 1 $p^2 \le q$ and the case 2 $q < p^2 < 4q$, we can obtain that the sequence $\{x_n\}_{n\ge -1}$ is bounded when $p^2 < 4q$. In the same way, we can prove that the sequence $\{y_n\}$ and $\{z_n\}$ are bounded as well. Hence, every solution to systems (1.5) is bounded when $p^2 < 4q$.

**Theorem 2.3.** Assume that $c > 0$, $q > 0$ and $p = 1$. Then the solutions to systems (1.5) are bounded.

**Proof**. Assume that $\{(x_n, y_n, z_n)\}$ is any positive solution to systems (1.5) in particular $p = 1$. We can easily know that $x_n \ge c, y_n \ge c, z_n \ge c$. Therefore, we have

$$x_{n+1} \le \max\{c, \frac{y_n}{c^q}\}, \quad y_{n+1} \le \max\{c, \frac{z_n}{c^q}\}, \quad z_{n+1} \le \max\{c, \frac{x_n}{c^q}\}, \quad n \in N_0. \tag{2.27}$$

From the above (2.27), it follows that

$$x_{n+1} \le \max\{c, \frac{y_n}{c^q}\} \le \max\{c, \frac{c}{c^q}, \frac{z_{n-1}}{c^{2q}}\} \le \max\{c, \frac{c}{c^q}, \frac{c}{c^{2q}}, \frac{x_{n-2}}{c^{3q}}\}. \tag{2.28}$$

Set

$$v_{n+1} = \max\{c, \frac{c}{c^q}, \frac{c}{c^{2q}}, \frac{v_{n-2}}{c^{3q}}\}, n = 1, 2, \cdots \ and \ v_1 = x_1, v_0 = x_0, v_{-1} = x_{-1}. \tag{2.29}$$

Assume that $\{v_n\}$ is the solution to (2.29). Then $v_n$ is greater than $x_n$ for any $n > 2$.

**Case 1**. $c > 1$.

**(a).** If $v_{-1} \le c^{3q+1}$, $v_0 \le c^{3q+1}$ and $v_1 \le c^{3q+1}$, from (2.29) we can obtain that $c^q > 1$ and $\frac{v_{n-2}}{c^{3q}} < c$, so $v_2 = c, v_5 = c, v_8 = c, \cdots$, which implies that $v_{3n-1} = c$. Moreover, $v_4 = c$, $v_7 = c$, $v_{10} = c, \cdots$, which implies that $v_{3n+1} = c$ and similarly $v_{3n} = c$. Hence, the boundedness of $\{v_n\}_{n\ge -1}$ follows in this case.

**(b).** If $v_{-1} > c^{3q+1}, v_0 > c^{3q+1}$ and $v_1 > c^{3q+1}$, from (2.29) we can obtain that $c < z_5 < z_2 < z_{-1}$. Through iteration, we can get that $\{v_{3n-1}\}$ is monotonically decreasing. Additionally, $v_n \ge c$ for any $n = 3k-1, k \in N$, we can obtain that $\{v_{3n-1}\}$ is bounded. Similarly, $\{v_{3n}\}$ and $\{v_{3n+1}\}$ are bounded as well. Hence, the boundedness of $\{v_n\}_{n\ge -1}$ follows in this case.

**(c).** If $v_{-1} \le c^{3q+1}$, $v_0 \le c^{3q+1}$ and $v_1 > c^{3q+1}$, from above proof we can obtain that $v_{3n-1} = c$, $v_{3n} = c$ and $\{v_{3n+1}\}$ is monotonically decreasing. Additionally $v_n \ge c$, we can obtain the boundedness of $\{v_n\}_{n\ge -1}$ follows in this case.

**(d).** If $v_{-1} \le c^{3q+1}$, $v_0 > c^{3q+1}$ and $v_1 \le c^{3q+1}$, from above proof we can obtain that $v_{3n-1} = c$, $v_{3n+1} = c$ and $\{v_{3n}\}$ is monotonically decreasing. Additionally $v_n \ge c$, we can obtain the boundedness of $\{v_n\}_{n\ge -1}$ follows in this case.

**(e).** If $v_{-1} > c^{3q+1}, v_0 \le c^{3q+1}$ and $v_1 \le c^{3q+1}$, from above proof we can obtain that $v_{3n} = c$, $v_{3n+1} = c$ and $\{v_{3n-1}\}$ is monotonically decreasing. Additionally $v_n \ge c$, we can obtain the boundedness of $\{v_n\}_{n\ge -1}$ follows in this case.

**(f).** If $v_{-1} \le c^{3q+1}$, $v_0 > c^{3q+1}$ and $v_1 > c^{3q+1}$, from above proof we can obtain that $v_{3n-1} = c$, $\{v_{3n}\}$ and $\{v_{3n+1}\}$ are monotonically decreasing. Additionally $v_n \ge c$, we can obtain the boundedness of $\{v_n\}_{n\ge -1}$ follows in this case.

**(g).** If $v_{-1} > c^{3q+1}$, $v_0 \le c^{3q+1}$ and $v_1 > c^{3q+1}$, from above proof we can obtain that $v_{3n} = c$, $\{v_{3n-1}\}$ and $\{v_{3n+1}\}$ are monotonically decreasing. Additionally $v_n \ge c$, we can obtain the boundedness of $\{v_n\}_{n\ge -1}$ follows in this case.

**(h).** If $v_{-1} > c^{3q+1}$, $v_0 > c^{3q+1}$ and $v_1 \le c^{3q+1}$, from above proof we can obtain that $v_{3n+1} = c$, $\{v_{3n-1}\}$ and $\{v_{3n}\}$ are monotonically decreasing. Additionally $v_n \ge c$, we can obtain the boundedness of $\{v_n\}_{n\ge -1}$ follows in this case.

Due to the bounededness of $\{v_n\}_{n\geq -1}$ and $x_n \leq v_n$, we can obtain the boundedness of $\{x_n\}$. Similarly $\{y_n\}$ and $\{z_n\}$ are bounded as well. Hence, every positive solution to systems (1.5) is bounded.

**Case 2**. $0 < c \leq 1$.

**(a)** If $q \geq 1$, in fact $x_n \geq c$, from (1.5) it follows that

$$\begin{aligned} x_{n+1} &= \max\{c, \frac{y_n}{z_{n-1}^q}\} = \max\{c, \frac{c}{z_{n-1}^q}, \frac{z_{n-1}}{z_{n-1}^q x_{n-2}^q}\} \\ &= \max\{c, \frac{c}{z_{n-1}^q}, \frac{1}{z_{n-1}^{q-1} x_{n-2}^q}\} \leq \max\{c, \frac{1}{c^{q-1}}, \frac{1}{c^{2q-1}}\} \end{aligned} \tag{2.30}$$

for $n \in N$, which means that $\{x_n\}$ is bounded.

**(b)**. If $0 < q < 1$, let sequence $\{a_l\}_{l\geq 0}$ be defined as follows

$$a_{l+1} = a_l - b_l, \quad b_{l+1} = qa_l, \quad a_1 = 1-q, b_1 = q, \quad l \in \mathbb{N}. \tag{2.31}$$

Thus, from (1.5) we have

$$\begin{aligned} x_{n+1} &= \max\{c, \frac{c}{z_{n-1}^q}, \frac{z_{n-1}^{1-q}}{x_{n-2}^q}\} = \max\{c, \frac{c}{z_{n-1}^q}, \frac{z_{n-1}^{a_1}}{x_{n-2}^{b_1}}\} \\ &= \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{1-q}}{x_{n-2}^q}, \frac{x_{n-2}^{(1-q)-q}}{y_{n-3}^{q(1-q)}}\} = \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{a_1}}{x_{n-2}^{b_1}}, \frac{x_{n-2}^{a_1-b_1}}{y_{n-3}^{qa_1}}\} \\ &= \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{1-q}}{x_{n-2}^q}, \frac{c^{(1-q)-q}}{y_{n-3}^{q(1-q)}}, \frac{y_{n-3}^{[(1-q)-q]-q(1-q)}}{z_{n-4}^{q[(1-q)-q]}}\} = \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{a_1}}{x_{n-2}^{b_1}}, \frac{c^{a_1-b_1}}{y_{n-3}^{qa_1}}, \frac{y_{n-3}^{a_2-b_2}}{z_{n-4}^{qa_2}}\} \\ &= \cdots\cdots \\ &= \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{a_1}}{x_{n-2}^{b_1}}, \frac{c^{a_1-b_1}}{y_{n-3}^{b_2}}, \frac{c^{a_2-b_2}}{z_{n-4}^{b_3}} \cdots, \frac{x_{n-(3k-1)}^{a_{3k-2}-b_{3k-2}}}{y_{n-3k}^{b_{3k-1}}}\} = \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{a_1}}{x_{n-2}^{b_1}}, \frac{c^{a_2}}{y_{n-3}^{b_2}}, \frac{c^{a_3}}{z_{n-4}^{b_3}} \cdots, \frac{x_{n-(3k-1)}^{a_{3k-1}}}{y_{n-3k}^{b_{3k-1}}}\} \\ &= \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{a_1}}{x_{n-2}^{b_1}}, \frac{c^{a_1-b_1}}{y_{n-3}^{b_2}}, \frac{c^{a_2-b_2}}{z_{n-4}^{b_3}} \cdots, \frac{y_{n-3k}^{a_{3k-1}-b_{3k-1}}}{z_{n-(3k+1)}^{b_{3k}}}\} = \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{a_1}}{x_{n-2}^{b_1}}, \frac{c^{a_2}}{y_{n-3}^{b_2}}, \frac{c^{a_3}}{z_{n-4}^{b_3}} \cdots, \frac{y_{n-3k}^{a_{3k}}}{z_{n-(3k+1)}^{b_{3k}}}\} \\ &= \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{a_1}}{x_{n-2}^{b_1}}, \frac{c^{a_1-b_1}}{y_{n-3}^{b_2}}, \frac{c^{a_2-b_2}}{z_{n-4}^{b_3}} \cdots, \frac{z_{n-(3k+1)}^{a_{3k}-b_{3k}}}{x_{n-(3k+2)}^{b_{3k+1}}}\} = \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{a_1}}{x_{n-2}^{b_1}}, \frac{c^{a_2}}{y_{n-3}^{b_2}}, \frac{c^{a_3}}{z_{n-4}^{b_3}} \cdots, \frac{z_{n-3k+1}^{a_{3k+1}}}{x_{n-(3k+2)}^{b_{3k+1}}}\} \end{aligned} \tag{2.32}$$

for every $k \in N$.

From (2.31), we can deduce

$$a_{l+1} - a_l + qa_{l-1} = 0, l \in N. \tag{2.33}$$

It is easy to see that the general solution of difference equation (2.33) is

$$a_l = c_1\lambda_1^l + c_2\lambda_2^l, \quad c_1, c_2 \in R, \tag{2.34}$$

where $\lambda_{1,2} = (1 \pm \sqrt{1-4q})/2$ . The fact $|\lambda_{1,2}| < 1$ along with (2.34) implies that the sequence $\lim_{l\to+\infty} a_l = 0$. From this and (2.31) we get $\lim_{l\to+\infty} b_l = 0$ .

Now note that from (2.32) it follows that

$$x_{n+1} \le \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{a_1}}{x_{n-2}^{b_1}}, \frac{c^{a_2}}{y_{n-3}^{b_2}}, \frac{c^{a_3}}{z_{n-4}^{b_3}} \cdots, \frac{x_{n-(3k-1)}^{a_{3k-1}}}{y_{n-3k}^{b_{3k-1}}}\}, \tag{2.35}$$

or

$$x_{n+1} \le \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{a_1}}{x_{n-2}^{b_1}}, \frac{c^{a_2}}{y_{n-3}^{b_2}}, \frac{c^{a_3}}{z_{n-4}^{b_3}} \cdots, \frac{y_{n-3k}^{a_{3k}}}{z_{n-(3k+1)}^{b_{3k}}}\}, \tag{2.36}$$

or

$$x_{n+1} \le \max\{c, \frac{c}{z_{n-1}^q}, \frac{c^{a_1}}{x_{n-2}^{b_1}}, \frac{c^{a_2}}{y_{n-3}^{b_2}}, \frac{c^{a_3}}{z_{n-4}^{b_3}} \cdots, \frac{z_{n-(3k+1)}^{a_{3k+1}}}{x_{n-(3k+2)}^{b_{3k+1}}}\}. \tag{2.37}$$

The convergence of $\{a_l\}_{l\ge-1}$ and $\{b_l\}_{l\ge-1}$ along with (2.35)-(2.37) implies the boundedness of $\{x_n\}_{n\ge-1}$ . Since systems (1.5) is symmetric, the boundedness of $\{x_n\}_{n\ge-1}$ implies the boundedness of $\{y_n\}_{n\ge-1}$ and $\{z_n\}_{n\ge-1}$, as claimed.

**Theorem 2.4.** Assume that $c > 0$, $p \in (0,1)$, then the solutions to systems (1.5) are bounded.

**Proof.** Assume that $(x_n, y_n, z_n)_{n\ge-1}$ is a positive solution to systems (1.5). Then the following estimate obviously holds

$$\min\{x_n, y_n, z_n\} \ge c, \quad n \in \mathbb{N}_0. \tag{2.38}$$

Hence

$$\begin{aligned} x_{n+1} &\le \max\{c, \frac{y_n^p}{c^q}\} \le \max\{c, \frac{c^p}{c^q}, \frac{z_{n-1}^{p^2}}{c^{pq+q}}\} \\ &\le \max\{c, \frac{c^p}{c^q}, \frac{c^{p^2}}{c^{pq+q}}, \frac{x_{n-2}^{p^3}}{c^{p^2q+pq+q}}\}. \end{aligned} \tag{2.39}$$

Let $\{v_n\}$ be the solutions of the following difference equation (2.40) and $v_{-1} = x_{-1}$, $v_0 = x_0, v_1 = x_1$.

$$v_{n+1} = \max\{c, \frac{1}{c^{q-p}}, \frac{1}{c^{pq-p^2+q}}, \frac{v_{n-2}^{p^3}}{c^{p^2q+pq+q}}\}, \quad n = 1, 2, 3, \cdots. \tag{2.40}$$

Since $p \in (0,1)$, the following function

$$f(x) = \max\{c, \frac{1}{c^{q-p}}, \frac{1}{c^{pq-p^2+q}}, \frac{x^{p^3}}{c^{p^2q+pq+q}}\} \tag{2.41}$$

is a concave function for sufficiently large $x$. Thus, it follows that there is a fixed point $x^*$, such that $f(x) < x$ for $x > x^*$. It is easy to see that if $v_2 \in (0, x^*]$, the sequence $\{v_n\}_{n\geq 2}$ is bounded above by $x^*$ and if $v_2 > x^*$, it is non-increasing and bounded below by $x^*$. Hence the sequence $\{v_n\}$ is bounded and consequently the sequence $\{x_n\}$ is bounded too. In the same way, we can prove that the sequence $\{y_n\}$ and $\{z_n\}$ are bounded as well. Hence, every solution to systems (1.5) is bounded as claimed.

## 3. Asymptotic behavior of solutions

This section is devoted to analyzing the asymptotic behavior of solutions to system (1.5) for $c > 1$ and $c \in (0,1]$.

**Theorem 3.1.** Assuming $0 < p \leq 1$, when $c \in (0,1]$ then $\{(x_n, y_n, z_n)\}$ converges to $(x^*, y^*, z^*) = (1,1,1)$, while $c > 1$, $\{(x_n, y_n, z_n)\}$ converges to $(x^*, y^*, z^*) = (c,c,c)$.

**Proof.** **Case 1.** $c \in (0,1]$.

Due to the positivity of the solution to $\{(x_n, y_n, z_n)\}$ with initial data $x_0, x_{-1} > 0$, $y_0, y_{-1} > 0$ and $z_0, z_{-1} > 0$, the systems (1.5) can be transformed to the following systems (3.1) by the change of $x_n = e^{s_n}$, $y_n = e^{t_n}$, $z_n = e^{r_n}$, where $s_n \geq 0,\ t_n \geq 0,\ r_n \geq 0,\ n > k$ and $\ln c < 0$.

$$\begin{aligned} s_{n+1} &= \max\{\ln c, pt_n - qr_{n-1}\} \leq \max\{\ln c, pt_n\} \leq pt_n, \\ t_{n+1} &= \max\{\ln c, pr_n - qs_{n-1}\} \leq \max\{\ln c, pr_n\} \leq pr_n, \\ r_{n+1} &= \max\{\ln c, ps_n - qt_{n-1}\} \leq \max\{\ln c, ps_n\} \leq ps_n. \end{aligned} \tag{3.1}$$

Obviously, from systems (3.1), inequality (3.2) follows

$$s_{n+3} \leq p^3 s_n, \quad t_{n+3} \leq p^3 t_n, \quad r_{n+3} \leq p^3 r_n. \tag{3.2}$$

For $p \leq 1$, $s_{n+3} < s_n$ can be obtained from inequality (3.2). The sequences $\{s_{3n}\}$, $\{s_{3n+1}\}$ and $\{s_{3n+2}\}$ are monotone decreasing. In addition, as $s_n > 0$, we can get that $\lim_{n\to\infty} s_n = 0$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} e^{s_n} = 1$. Then $\{x_n\}$ converges to 1. $\{y_n\}$ and $\{z_n\}$ can be proved in the same way. Therefore, when $c \in (0,1]$, if $p \le 1$, then $\{(x_n, y_n, z_n)\}$ converges to $(x^*, y^*, z^*) = (1,1,1)$.

**Case 2. c>1.**

Correspondingly, systems (1.5) can be transformed to systems (3.3) by the change of $x_n = c^{s_n}, y_n = c^{t_n}, z_n = c^{r_n}$, where $s_n \ge 1, t_n \ge 1, r_n \ge 1$ due to $x_n \ge c, y_n \ge c, z_n \ge c$.

$$\begin{aligned} s_{n+1} &= \max\{1, pt_n - qr_{n-1}\}, \\ t_{n+1} &= \max\{1, pr_n - qs_{n-1}\}, \\ r_{n+1} &= \max\{1, ps_n - qt_{n-1}\}. \end{aligned} \tag{3.3}$$

Furthermore, systems (3.3) can be written as systems (3.4).

$$\begin{aligned} s_{n+1} - 1 &= \max\{0, pt_n - qr_{n-1} - 1\}, \\ t_{n+1} - 1 &= \max\{0, pr_n - qs_{n-1} - 1\}, \\ r_{n+1} - 1 &= \max\{0, ps_n - qt_{n-1} - 1\}. \end{aligned} \tag{3.4}$$

Similar proof has been fully given in case $c \in (0,1]$ and here the rest proof is omitted. Then the result is much alike with the one in above case:

$$\lim_{n\to\infty} s_n = 1, \quad \lim_{n\to\infty} t_n = 1, \quad \lim_{n\to\infty} r_n = 1, \tag{3.5}$$

and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} c^{s_n} = c, \quad \lim_{n\to\infty} y_n = \lim_{n\to\infty} c^{t_n} = c, \quad \lim_{n\to\infty} z_n = \lim_{n\to\infty} c^{r_n} = c. \tag{3.6}$$

Therefore, assume $0 < p \le 1$ and $c > 1$, then $(x_n, y_n, z_n)$ converges to $(x^*, y^*, z^*) = (c,c,c)$. Hence Theorem 3.1 is proved completely.

## 4. Simulation experiment

In this section, some numerical simulations are given to support our theoretical analysis. As examples, we consider the following difference equations

$$x_{n+1} = \max\{0.5, \frac{y_n^{0.5}}{z_{n-1}^2}\}, y_{n+1} = \max\{0.5, \frac{z_n^{0.5}}{x_{n-1}^2}\}, y_{n+1} = \max\{0.5, \frac{z_n^{0.5}}{x_{n-1}^2}\}, n \in \mathbb{N}_0, \tag{4.1}$$

$$x_{n+1} = \max\{1, \frac{y_n}{z_{n-1}^2}\}, \quad y_{n+1} = \max\{1, \frac{z_n}{x_{n-1}^2}\}, \quad z_{n+1} = \max\{1, \frac{x_n}{y_{n-1}^2}\}, \quad n \in \mathbb{N}_0, \tag{4.2}$$

and

$$x_{n+1} = \max\{1.5, \frac{y_n^{0.5}}{z_{n-1}^2}\}, y_{n+1} = \max\{1.5, \frac{z_n^{0.5}}{x_{n-1}^2}\}, z_{n+1} = \max\{1.5, \frac{x_n^{0.5}}{y_{n-1}^2}\}, \quad n \in \mathbb{N}_0. \tag{4.3}$$

By employing Matlab R2013b, we solve the numerical solutions of the above equations, which are shown respectively in the following Figures.

More precisely, the initial conditions of (4.1) are that $x_{-1}=1.5$, $x_0=1.2$, $y_{-1}=0.5$, $y_0=0.8$, $z_{-1}=1$, and $z_0=1.5$. It is easy to show that the equations (4.1) satisfy the conditions of Theorem 2.2. Fig.4.1 shows that the solutions of the equations (4.1) are bounded. The initial conditions of (4.2) are that $x_{-1}=1.5$, $x_0=1.2$, $y_{-1}=0.5$, $y_0=0.8$, $z_{-1}=2.5$, and $z_0=3$. It is easy to show that the equations (4.2) satisfy the conditions of Theorem 2.3 and Theorem 3.1. Figure 4.2 shows the solutions to equations (4.2) are bounded and globally attractive. The initial conditions of equations (4.3) are that $x_{-1}=1.5$, $x_0=1.2$, $y_{-1}=0.5$, $y_0=0.8$, $z_{-1}=1$, and $z_0=1.5$. It is easy to show that the equations (4.3) satisfy the conditions of Theorem 2.4 and Theorem 3.1. Figure 4.3 shows the solutions to the equations (4.3) are bounded and globally attractive.

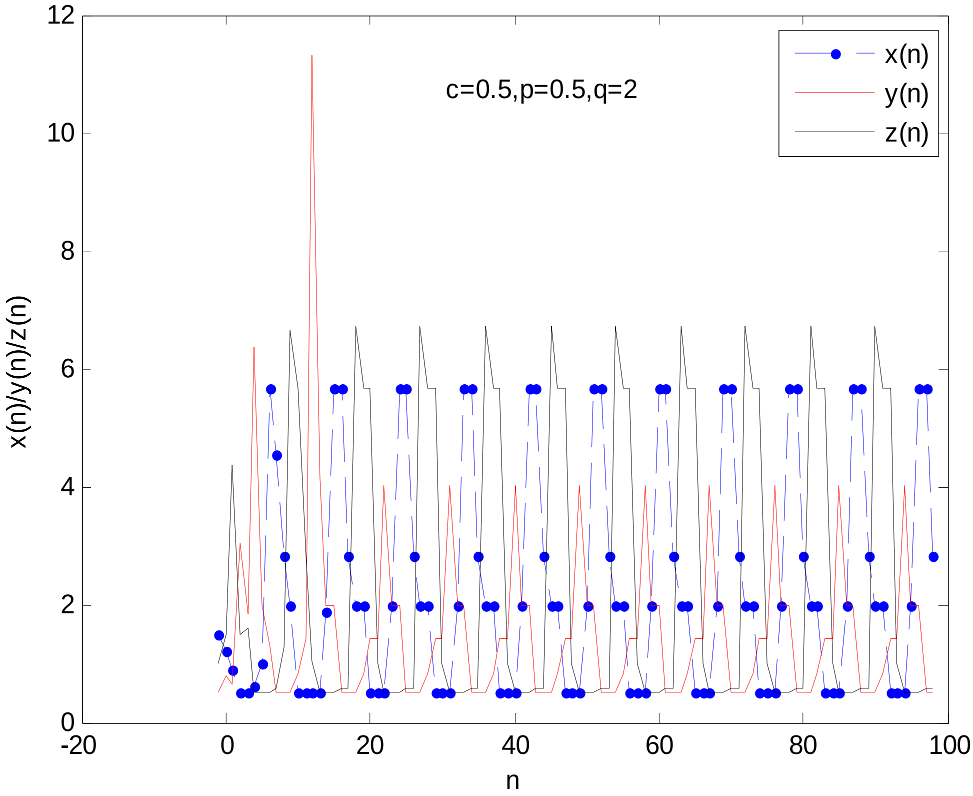


Figure 4.1. the solutions to equation (4.1)

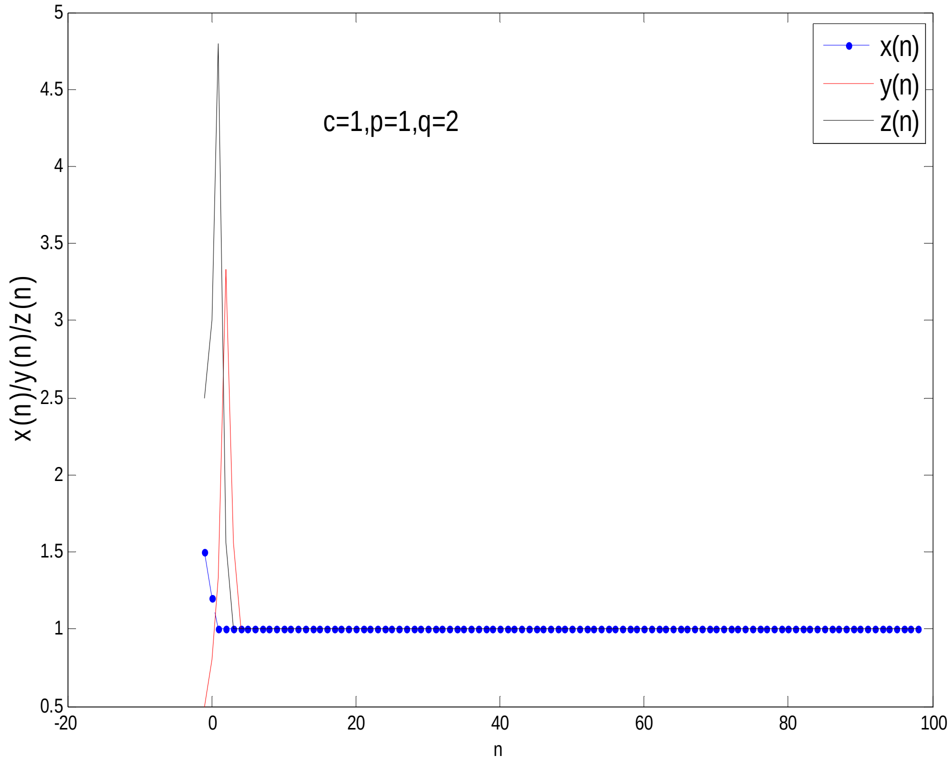


Figure4.2. the solutions to equation (4.2)

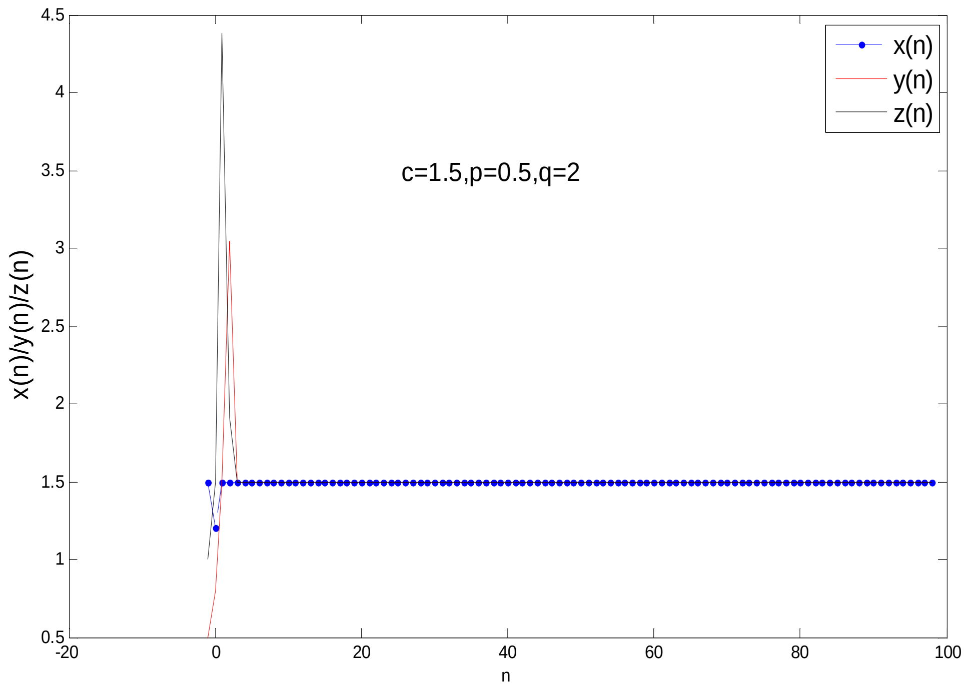


Figure 4.3. the solutions to equation (4.3)

## 5. Conclusion

It is obvious that the system of three max-type difference equations (1.5) is the extension of the models in [26-29]. In this paper, we have dealt with the problem of boundedness character and global attractivity for a class of max-type difference system. And we have obtained some sufficient conditions which ensure the boundedness character and global attractivity of the max-type system. Especially, the sufficient conditions that we obtained are very simple, which provide flexibility for the application and analysis of max-type difference system. These results generalize and improve some previous works. In addition, we present the use of a new iteration method for symmetric systems of max-type difference equations. This technique is a powerful tool for solving various difference equations and it can be applied to other nonlinear differential equations in mathematical physics. Computations are performed using the software package Matlab R2013b. Finally, some numerical examples are given to show the validity of the obtained theoretic results

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# The convexity of n-dimensional fuzzy mappings and the saddle point conditions of the fuzzy optimization problems

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**Abstract** The purpose of this work is to consider the optimization problem of n-dimensional fuzzy number valued functions. Firstly, the differentiability and convexity of n-dimensional fuzzy number valued function are discussed by means of the support function and a new order relation, which is built in the aid of the support function and the order of vector. Secondly, the fuzzy Lagrange function of fuzzy nonlinear programming is presented and weak duality theorems are obtained. At last, the saddle point of fuzzy lagrangian function is defined, the sufficient and necessity conditions of saddle point are given.

**Keywords** Fuzzy numbers; fuzzy programming; saddle point; duality theorem

## 1. Introduction

Since the concept and operations of fuzzy set were introduced by Zadeh, many studies have focused on the theoretical aspects and applications of fuzzy sets, one of the main stream is the fuzzy optimization in operation research. In 1970, Bellman and Zadeh[1] inspired the development of fuzzy optimization by providing the aggregation operators, which combined the fuzzy goals and fuzzy decision space. After this motivation and inspiration, there came out a lot of articles dealing with the fuzzy optimization problems and the insightful survey can be seen in [3, 7, 11].

The duality of fuzzy linear programming was first studied by Rodder and Zimmermann[9] who considered the economic interpretation of the dual variables. Zhong and Shi[20] presented a parametric approach for duality in fuzzy multi criteria and multi constrainted linear programming which extended fuzzy linear programming approaches. Wu[14]formulate the fuzzy primal and dual linear programming problems with fuzzy coefficients by using fuzzy scalar product, prove the weak and strong duality theorems. Wu[15] discuss the saddle-point optimality conditions in fuzzy optimization problems by introducing the fuzzy scalar product. In Wu[16], under a general setting partial ordering, the duality theorems and saddle point optimality of fuzzy nonlinear programming problems are derived. Zhang[21]discuss the saddle-points and minimax theorems under fuzzy environment, obtain the KT conditions for fuzzy programming and consider the "perturbed" convex fuzzy programming. Gong[5] propose the fuzzy Lagrangian function of a fuzzy optimization problem by considering a total ordering on the set of fuzzy numbers, and the saddle point of fuzzy Lagrangian function with its optimality condition were dicussed.Howere,the fuzzy number in these research above is on the real line,which is one dimensional.There are few studies on n-dimensional fuzzy numbers, maybe the ranking of n-dimensional fuzzy numbers has been a bottleneck for researchers.

The differentiability of fuzzy mappings from an open subset of a normed space into the n-dimension fuzzy number space $E^n$ was developed by Puri and Ralescu[8], which generalized and extended the concept of Hukuhara differentiability for set mappings. Wang and Wu[12] proposed the directional derivative,differential and sub-differential of fuzzy mappings from $R^n$ into $E$ using Hukuhara difference. Hai[6] characterize the generalized difference of n-dimensional fuzzy number valued functions by means of support functions and give the order relation $\preceq_s$by the aid of support function. Yan[18] give the order relation on $E$ considering the left and right endpoints and weights, which is a total order. Based on this, we give the order relation of n- dimensional fuzzy numbers by means of the support function and the order of vector, this order is partial and practical.

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The purpose of this work is to consider the optimization problem of n-dimensional fuzzy number valued function. First, we present the terminology used in the present paper, and give the order relation of n-dimensional fuzzy-number-valued function. In section 3, the differentiability and convexity are introduced and its relations is studied. For nonlinear fuzzy programming problem, the weak duality theorem the saddle point of fuzzy lagrangian function is is presented, further, the sufficient and necessity condition of saddle point are obtained.

## 2. Definitions and preliminaries

In this section,basic definitions and operations for fuzzy numbers are presented.

**Definition 2.1**[17] $X = R^n, n \geq 1$ is the real n-dimensional Euclidean space, A fuzzy number is a mapping $\widetilde{u} : R^n \to [0,1]$ with the following properties:

(1) $\widetilde{u}$ is a normal fuzzy set, i.e.there exists $x_0 \in R^n$ such that $\widetilde{u}(x_0) = 1$,

(2) $\widetilde{u}$ is a convex fuzzy set, i.e. $\widetilde{u}(\lambda x + (1-\lambda)y) \geq \min\{\widetilde{u}(x), \widetilde{u}(y)\}$ for any $x, y \in R^n$ and $\lambda \in [0,1]$.

(3) $\widetilde{u}$ is upper semi-continuous.

(4) $[\widetilde{u}]^0 = cl(\mathrm{supp}\widetilde{u}) = \overline{\{x \in R^n : \widetilde{u}(x) > 0\}}$ is compact.

we will denote $E^n$ the set of fuzzy numbers.It is clear that any $\widetilde{u} \in E^n, r \in [0,1], [\widetilde{u}]^r = \{x \in R^n : \widetilde{u}(x) \geq r\}$ denoted as $r-$level cut is a compact convex set. Further, we give the representation theorem of these compact convex sets.

**Theorem 2.2**[17] Let $\widetilde{u} \in E^n$, then

(1)$[\widetilde{u}]^r$ is a nonempty compact convex subset of $R^n$ for any $r \in [0,1]$,

(2)$[\widetilde{u}]^{r_1} \subseteq [\widetilde{u}]^{r_2}$, for $0 \leq r_2 \leq r_1 \leq 1$,

(3)If $r_k > 0$ and $r_k$ is a nondecreasing sequence converging to $r \in (0,1]$, then $\bigcap_{k=1}^{\infty} [\widetilde{u}]^{r_k} = [\widetilde{u}]^r$,

Conversely, if $\{[A]^r \subseteq R^n : r \in [0,1]\}$ satisfies the conditions (1)-(3),then there exists a unique $\widetilde{u} \in E^n$ such that $[\widetilde{u}]^r = [A]^r$ for each $r \in (0,1]$ and $[\widetilde{u}]^0 = cl(\bigcup_{r\in(0,1]} [\widetilde{u}]^r) \subseteq [A]^0$.

Let $\widetilde{u}, \widetilde{v} \in E^n$ and $k \in R$, the addition $\widetilde{u}+\widetilde{v}$ and scalar multiplication $k\widetilde{u}$ is defined as: for any $x \in R^n$,

$$(\widetilde{u}+\widetilde{v})(x) = \sup_{s+t=x} \min\{\widetilde{u}(s), \widetilde{v}(t)\},$$

$$(k\widetilde{u})(x) = \widetilde{u}(\frac{x}{k}), k \neq 0, 0\widetilde{u} = \widetilde{0},$$

where $\widetilde{0}(x) = 1$ when $x = 0, \widetilde{0}(x) = 0$ when $x \neq 0$.

It is easy to get that the addition $\widetilde{u}+\widetilde{v}$ and scalar multiplication $k\widetilde{u}$ have the level cut:

$$[\widetilde{u}+\widetilde{v}]^r = [\widetilde{u}]^r + [\widetilde{v}]^r = \{x+y : x \in [\widetilde{u}]^r, y \in [\widetilde{u}]^r\},$$

$$[k\widetilde{u}]^r = k[\widetilde{u}]^r = \{kx : x \in [\widetilde{u}]^r\}.$$

The Hausdorff distance $D : E^n \times E^n \to [0, +\infty)$ is defined by

$$D(\widetilde{u}, \widetilde{v}) = \sup_{r\in[0,1]} d([\widetilde{u}]^r, \widetilde{v}]^r),$$

where $d$ is Harsdorff metric given by $d(A,B) = \inf\{\varepsilon : N(A,\varepsilon) \supset B, N(B,\varepsilon) \supset A\}$, and $N(A,\varepsilon) = \{x \in R^n : d(x,A) = \inf_{y\in A} d(x,y) \leq \varepsilon\}$is the $\varepsilon-$ neighborhood of $A$. Then $(E^n, D)$ is a complete metric space, and satisfies $D(\widetilde{u}+\widetilde{w}, \widetilde{v}+\widetilde{w}) = D(\widetilde{u}, \widetilde{v}), D(k\widetilde{u}, k\widetilde{v}) = |k|D(\widetilde{u}, \widetilde{v})$ for any $\widetilde{u}, \widetilde{v}, \widetilde{w} \in E^n$ and $k \in R$.

**Definition 2.3**[17] Let $\widetilde{u} \in E^n$, the support function of $\widetilde{u}$ is defined by

$$\widetilde{u}^*(r,p) = \sup_{a\in[\widetilde{u}]^r} \langle a, p\rangle, (r,p) \in I \times S^{n-1},$$

where $I = [0,1], S^{n-1} = \{x \in R^n : \|x\| = 1\}$ be the unit sphere of $R^n$ and $\langle\cdot,\cdot\rangle$be the inner product in $R^n$, that is $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$, where $x = (x_1, x_2, \cdots, x_n) \in R^n, y = (y_1, y_2, \cdots, y_n) \in R^n$. Also, assume that $\widetilde{u} = (\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_n)$, then $\langle x, \widetilde{u}\rangle = \sum_{i=1}^n x_i \widetilde{u}_i, \widetilde{u}_i \in E^n$.

**Theorem 2.4**[6,17]Let $\widetilde{u} \in E^n$, then support function$\widetilde{u}^*$ satisfy:

(1)$\widetilde{u}^*(r, p+q) \leq \widetilde{u}^*(r,p) + \widetilde{u}^*(r,q)$ for $p, q \in S^{n-1}$.

(2)$\widetilde{u}^*(r, kp) = k\widetilde{u}^*(r,p), k \geq 0$.

(3)$\widetilde{u}^*$ is uniformly bounded on $I \times S^{n-1}$, and $|\widetilde{u}^*(r,p)| \leq \sup_{a\in[\widetilde{u}]^0} \|a\|$,

(4)$\widetilde{u}^*(r,p)$ is nonincreasing and left continuous on $r \in [0,1]$, right continuous at $r = 0$ for each fixed $p \in S^{n-1}$.

(5)$\widetilde{u}^*(r,\cdot)$ is uniformly Lipschitz continuous for $r \in [0,1]$, that is

$$|\widetilde{u}^*(r,p) - \widetilde{v}^*(r,q)| \leq (\sup_{a\in[\widetilde{u}]^0} \|a\|)\|x-y\|,$$

(6)

$$d([\widetilde{u}]^r, [\widetilde{v}]^r) = \sup_{p\in S^{n-1}} |\widetilde{u}^*(r,p) - \widetilde{v}^*(r,p)|,$$

for any $r \in [0,1], \widetilde{u}, \widetilde{u} \in E^n$.

(7)$(-\widetilde{u})^*(r,p) = \widetilde{u}^*(r,-p)$.

**Theorem 2.5** Let $\widetilde{u}, \widetilde{v} \in E^n$, then

(1)$(s\widetilde{u} + t\widetilde{v})^*(r,p) = s\widetilde{u}^*(r,p) + t\widetilde{v}^*(r,p), s,t \geq 0$,

(2)$D(\widetilde{u},\widetilde{v}) = \sup_{r\in[0,1]} \|\widetilde{u}^*(r,p) - \widetilde{v}^*(r,p)\| = \sup_{r\in[0,1]} \sup_{p\in S^{n-1}} |\widetilde{u}^*(r,p) - \widetilde{v}^*(r,p)|$.

**Proof** (1)We prove that $(\widetilde{u}+\widetilde{v})^*(r,p) = \widetilde{u}^*(r,p)+\widetilde{v}^*(r,p)$ firstly. From the definition of support function,

$$(\widetilde{u}+\widetilde{v})^*(r,p) = \sup_{a\in[\widetilde{u}+\widetilde{v}]^r} \langle a,p\rangle = \sup_{a\in[\widetilde{u}]^r+[\widetilde{v}]^r} \langle a,p\rangle = \sup_{b\in[\widetilde{u}]^r, c\in[\widetilde{v}]^r} \langle b+c,p\rangle$$
$$= \sup_{b\in[\widetilde{u}]^r, c\in[\widetilde{v}]^r} (\langle b,p\rangle + \langle c,p\rangle) = \widetilde{u}^*(r,p) + \widetilde{v}^*(r,p).$$

in addition,for any $k \geq 0$,

$$(k\widetilde{u})^*(r,p) = \sup_{a\in[k\widetilde{u}]^r} \langle a,p\rangle = \sup_{a\in k[\widetilde{u}]^r} \langle a,p\rangle = \sup_{\frac{a}{k}\in[\widetilde{u}]^r} k\langle \frac{a}{k},p\rangle = k\widetilde{u}^*(r,p),$$

therefore,we get (1).

(2)

$$D(\widetilde{u},\widetilde{v}) = \sup_{r\in[0,1]} d([\widetilde{u}]^r,[\widetilde{u}]^r) = \sup_{r\in[0,1]} \sup_{p\in S^{n-1}} |\widetilde{u}^*(r,p) - \widetilde{v}^*(r,p)| = \sup_{r\in[0,1]} \|\widetilde{u}^*(r,p) - \widetilde{v}^*(r,p)\|.$$

we denote by $\mathcal{K}^n$ and $\mathcal{K}^n_{\mathfrak{C}}$ the spaces of (nonempty) compact convex sets of $R^n$ respectively.The generalized Hukuhara difference of two set $A, B \in \mathcal{K}^n_{\mathfrak{C}}$(gH-difference for short)is defined in[4,6,10] as follows:

$$A \ominus_{gH} B = C \Leftrightarrow \begin{cases} (a)A = B + C, \\ \text{or(b)B=A+(-1)C} \end{cases}$$

where $A + B = \{x + y : x \in A, y \in B\}, kA = \{kx : x \in A\}, k \in R$. Stefanini[10] extent the generalized Hukuhara difference to the fuzzy case. For any $\widetilde{u},\widetilde{v} \in E^n$, the generalized Hukuhara difference(gH-difference for short)is the fuzzy number $\widetilde{w}$, if it exist,then

$$\widetilde{u} \ominus_{gH} \widetilde{v} = \widetilde{w} \Leftrightarrow \begin{cases} (a)\widetilde{u} = \widetilde{v} + \widetilde{w}, \\ \text{or}(b)\widetilde{v} = \widetilde{u} + (-1)\widetilde{w}. \end{cases}$$

From the theorem 2.4,it is easy to have the follows:

**Theorem 2.6** Let $\widetilde{u},\widetilde{v} \in E^n, \widetilde{u} \ominus_{gH} \widetilde{v} = \widetilde{w}$, Then $\widetilde{w}^*(r,p) = \widetilde{u}^*(r,p) - \widetilde{v}^*(r,p), r \in [0,1], p \in S^{n-1}$

**Proof** Since $\widetilde{u} \ominus_{gH} \widetilde{v} = \widetilde{w}$, then either $(a)\widetilde{u} = \widetilde{v} + \widetilde{w}$, or $(b)\widetilde{v} = \widetilde{u} + (-1)\widetilde{w}$. For $(a)$, from 2.5(1),we have

$$\widetilde{u}^*(r,p) = \widetilde{v}^*(r,p) + \widetilde{w}^*(r,p).$$

For (b),

$$\widetilde{v}^*(r,p) = \widetilde{u}^*(r,p) + (-1)\widetilde{w}^*(r,p) = \widetilde{u}^*(r,p) + \widetilde{w}^*(r,-p),$$

then

$$\widetilde{w}^*(r,p) = -\widetilde{w}^*(r,-p) = \widetilde{u}^*(r,p) - \widetilde{v}^*(r,p)$$

for any $r \in [0,1], p \in S^{n-1}$.

**Definition 2.7**[18] Let $\widetilde{u}, \widetilde{v} \in E, \widetilde{u} \preceq \widetilde{v}$ denoted as

$$\int_0^1 r(\widetilde{u}^-(r) + \widetilde{u}^+(r))dr \le \int_0^1 r(\widetilde{v}^-(r) + \widetilde{v}^+(r))dr,$$

where $[\widetilde{u}]^r = [\widetilde{u}^-(r), \widetilde{u}^+(r)], [\widetilde{v}]^r = [\widetilde{v}^-(r), \widetilde{v}^+(r)], r \in [0,1]$.

For any $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n) \in R^n$, we define: $x \le y$ if and only if $x_i \le y_i$for any $i (i = 1, 2, \cdots, n)$, and $x < y$ means $x \le y$ and there exist $m$, such that $x_m < y_m (m = 1, 2, \cdots, n)$.

**Definition 2.8** For any $\widetilde{u}, \widetilde{v} \in E^n$, we say that $\widetilde{u} \preceq \widetilde{v}$ if

$$(\tau(\widetilde{u}_1), \tau(\widetilde{u}_2), \cdots, \tau(\widetilde{u}_n) \le (\tau(\widetilde{v}_1), \tau(\widetilde{v}_2), \cdots, \tau(\widetilde{v}_n),$$

where $\tau(\widetilde{u}_i) = \int_0^1 r(\widetilde{u}^*(r, e_i^+) - \widetilde{u}^*(r, e_i^-))dr, e_i^+ = (x_1, x_2, \cdots, x_n) \in S^{n-1}, e_i^- = (y_1, y_2, \cdots, y_n) \in S^{n-1}$ and$x_j = 1, y_j = -1$ when $j = i$, $x_j = y_j = 0$ when $j \ne i (i, j = 1, 2, \cdots, n)$. we say that $\widetilde{u} \prec \widetilde{v}$ if $\widetilde{u} \preceq \widetilde{v}$ and there exist $i (i = 1, 2, \cdots, n)$, such that $\tau(\widetilde{u}_i) < \tau(\widetilde{v}_i)$. Particularly, Def 2.8 is just as Def 2.7 when $n = 1$, it means the Def 2.8 is the extension of the Def 2.7. $\widetilde{u} \preceq \widetilde{v}$ also denoted as $\widetilde{v} \succeq \widetilde{u}$.

$\min(\widetilde{u}, \widetilde{v}) = \widetilde{w}$ if and only if $\tau(\widetilde{w}_i) = \min(\tau(\widetilde{u}_i), \tau(\widetilde{v}_i)) (i = 1, 2, \cdots, n)$. In the follows, we denote $(\tau(\widetilde{u}_1), \tau(\widetilde{u}_2), \cdots, \tau(\widetilde{u}_n)) = H[\widetilde{u}]$,

## 3. Differentiability and convexity

Wang and Wu[12] present the directional derivative of the fuzzy mapping $F : R^n \to E$, that is characterized by the directional derivative of the real function. Below we give the differentiability of $F : R^n \to E^n$, and transformed it into the differentiability of functional in Banach space, then defined gradient and convexity and studied its relations.

**Definition 3.1** Let $F : M(\subset R^n) \to E^n$ be a fuzzy-number-valued function, $x^0 = (x_1^0, x_2^0, \cdots, x_n^0) \in$ intM. If there exist $\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_n \in E^n$, such that

$$\lim_{x \to x^0} \frac{D(F(x), F(x^0) + \sum_{j=1}^n (x_j - x_j^0)\widetilde{u}_j)}{d(x, x^0)} = 0,$$

then we call $F$ is differentiable at $x^0$, and denote$\bigtriangledown F(x^0) = (\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_n)$ the gradient of $F$ at $x^0$, where $x = (x_1, x_2, \cdots, x_n)$.

**Theorem 3.2** Let $F : M(\subset R^n) \to E^n$ be a fuzzy-number-valued function, $x^0 = (x_1^0, x_2^0, \cdots, x_n^0) \in$ intM. $F$ is differentiable at $x^0$ if and only if

$$F(x)^*(r, p) = F(x^0)^*(r, p) + \sum_{j=1}^n (x_j - x_j^0)\widetilde{u}_j^*(r, p) + o(d(x, x^0)) \tag{3.1}$$

for any $r \in [0,1]$ and $p \in S^{n-1}$.

**Proof** $F$ is differentiable at $x^0$, if and only if

$$\lim_{x \to x^0} \frac{D(F(x), F(x^0) + \sum_{j=1}^n (x_j - x_j^0)\widetilde{u}_j)}{d(x, x^0)} = 0,$$

if and only if

$$\lim_{x \to x^0} \frac{\sup_{r \in [0,1]} \sup_{p \in S^{n-1}} |F(x)^*(r, p) - F(x^0)^*(r, p) - \sum_{j=1}^n (x_j - x_j^0)\widetilde{u}_j^*(r, p)|}{d(x, x^0)} = 0,$$

if and only if

$$\lim_{x \to x^0} \frac{|F(x)^*(r, p) - F(x^0)^*(r, p) - \sum_{j=1}^n (x_j - x_j^0)\widetilde{u}_j^*(r, p)|}{d(x, x^0)} = 0$$

for any $r \in [0,1]$ and $p \in S^{n-1}$, if and only if

$$F(x)^*(r, p) = F(x^0)^*(r, p) + \sum_{j=1}^n (x_j - x_j^0)\widetilde{u}_j^*(r, p) + o(d(x, x^0))$$

for any $r \in [0,1]$ and $p \in S^{n-1}$.

**Theorem 3.3** Let $F : M(\subset R^n) \to E^n$ be a fuzzy-number-valued function, $F$ is differentiable at $\overline{x}$, if $\overline{x}$ is a local minimum solution, then $\nabla F(\overline{x}) = (\widetilde{0}, \widetilde{0}, \cdots, \widetilde{0})$.

**Proof** Since $\overline{x}$ is a local minimum solution, then there exist a $\delta > 0$, such that $F(x) \succeq F(\overline{x})$ for any $x \in \bigcup(\overline{x}, \delta) \cap M$. that is

$$\tau(F(x)_i) = \int_0^1 r[F(x)^*(r, e_i^+) - F(x)^*(r, e_i^-)]dr \geq \int_0^1 r[F(\overline{x})^*(r, e_i^+) - F(\overline{x})^*(r, e_i^-)]dr = \tau(F(\overline{x})_i)$$

for $1 \leq i \leq n$. From the theorem 3.2 and arbitrariness of $x$,

$$\tau((\widetilde{u}_j)_i) = \int_0^1 r[\widetilde{u}_j^*(r, e_i^+) - \widetilde{u}_j^*(r, e_i^-)]dr = 0 (i, j = 1, 2, \cdots, n),$$

thus $\widetilde{u}_j = \widetilde{0}$, and $\nabla F(\overline{x}) = (\widetilde{0}, \widetilde{0}, \cdots, \widetilde{0})$.

**Definition 3.4** Let $F : M(\subset R^n) \to E^n$ be a fuzzy-number-valued function,$M$ be a convex set. Then $F(x)$ is said to be a convex fuzzy-number-valued function on $M$ if for any $x, y \in M, \lambda \in [0,1]$, such that $\lambda x + (1-\lambda)y \in M$, we have

$$F(\lambda x + (1-\lambda)y) \preceq \lambda F(x) + (1-\lambda)F(y). \tag{3.2}$$

we call $F(x)$ is a strictly convex fuzzy-number-valued function on $M$, if for any $x, y \in M, x \neq y, \lambda \in [0,1]$,such that $\lambda x + (1-\lambda)y \in M$, we have

$$F(\lambda x + (1-\lambda)y) \prec \lambda F(x) + (1-\lambda)F(y).$$

**Theorem 3.5** Let $M$ be an open convex set, $F : M \to E^n$, and $F$ is differentiable, then $F$ is convex if and only if

$$F(x) \succeq F(y) + \langle \nabla F(y), x - y \rangle \tag{3.3}$$

for any $x, y \in M$.

**Proof** Assume that $F$ is convex, then for any $x, y \in M, \lambda \in (0,1)$, we have

$$F(\lambda x + (1-\lambda)y) \preceq \lambda F(x) + (1-\lambda)F(y),$$

that is

$$H[F(\lambda x + (1-\lambda)y)] \leq \lambda H[F(x)] + (1-\lambda)H[F(y)].$$

thus

$$\tau(F(\lambda x + (1-\lambda)y)_i) \leq \lambda\tau(F(x)_i) + (1-\lambda)\tau(F(y)_i)$$

for any $1 \leq i \leq n$, that is

$$\int_0^1 r[F(\lambda x + (1-\lambda)y)^*(r, e_i^+) - F(\lambda x + (1-\lambda)y)^*(r, e_i^-)]dr$$

$$\leq \lambda \int_0^1 r[F(x)^*(r, e_i^+) - F(x)^*(r, e_i^-)]dr + (1-\lambda) \int_0^1 r[F(y)^*(r, e_i^+) - F(y)^*(r, e_i^-)]dr.$$

Since $F$ is differentiable, then

$$F(\lambda x + (1-\lambda)y)^*(r, p) - F(y)^*(r, p) = \sum_{j=1}^n \lambda(x_j - y_j)\widetilde{v}_j^*(r, p) + \lambda o\|x - y\|,$$

for any $r \in [0,1]$ and $p \in S^{n-1}$, where $\nabla F(y) = (\widetilde{v}_1, \widetilde{v}_2, \cdots, \widetilde{v}_n)$, therefore,

$$\int_0^1 r[F(\lambda x + (1-\lambda)y)^*(r, e_i^+) - F(\lambda x + (1-\lambda)y)^*(r, e_i^-)]dr - \int_0^1 r[F(y)^*(r, e_i^+) - F(y)^*(r, e_i^-)]dr$$

$$\leq \lambda(\int_0^1 r[F(x)^*(r,e_i^+)-F(x)^*(r,e_i^-)]dr-\int_0^1 r[F(y)^*(r,e_i^+)-F(y)^*(r,e_i^-)]dr),$$

thus

$$\int_0^1 r[\sum_{j=1}^n \lambda(x_j-y_j)\widetilde{v}_j^*(r,e_i^+)-\sum_{j=1}^n \lambda(x_j-y_j)\widetilde{v}_j^*(r,e_i^-)]dr$$

$$\leq \lambda(\int_0^1 r[F(x)^*(r,e_i^+)-F(x)^*(r,e_i^-)]dr-\int_0^1 r[F(y)^*(r,e_i^+)-F(y)^*(r,e_i^-)]dr),$$

that is

$$\tau(\sum_{j=1}^n (x_j-y_j)(\widetilde{v}_j)_i)\leq \tau(F(x)_i)-\tau(F(y)_i),$$

We have

$$\sum_{j=1}^n (x_j-y_j)\widetilde{v}_j \preceq F(x)-F(y),$$

therefore

$$F(x)\succeq F(y)+\langle \nabla F(y), x-y\rangle.$$

Conversely, assume that for any $x^{(1)}, x^{(2)}\in M$, we have

$$F(x^{(2)})\succeq F(x^{(1)})+\langle \nabla F(x^{(1)}), x^{(2)}-x^{(1)}\rangle.$$

Let $y$ be a point between the $x^{(1)}$ and $x^{(2)}$, then $y=\lambda x^{(1)}+(1-\lambda)x^{(2)}$ for some $\lambda\in(0,1)$, and $y\in M$ since $M$ is a convex set. Based on the assumption, we have

$$F(x^{(1)})\succeq F(y)+\langle \nabla F(y), x^{(1)}-y\rangle,$$

$$F(x^{(2)})\succeq F(y)+\langle \nabla F(y), x^{(2)}-y\rangle,$$

that is

$$H[F(x^{(1)})]\geq H[F(y)]+H[\langle \nabla F(y), x^{(1)}-y\rangle], \tag{3.4}$$

$$H[F(x^{(2)})]\geq H[F(y)]+H[\langle \nabla F(y), x^{(2)}-y\rangle]. \tag{3.5}$$

From (3.4)and (3.5),

$$\tau(F(x^{(1)})_i)\geq \tau(F(y)_i)+\tau(\langle \nabla F(y), x^{(1)}-y\rangle)_i), \tag{3.6}$$

$$\tau(F(x^{(2)})_i)\geq \tau(F(y)_i)+\tau(\langle \nabla F(y), x^{(2)}-y\rangle)_i) \tag{3.7}$$

for $1\leq i<n$. Multiple(3.6),(3.7)by $\lambda,(1-\lambda)$ respectively, and then add the result, we have

$$\lambda\tau(F(x^{(1)})_i)+(1-\lambda)\tau(F(x^{(2)})_i)\geq \tau(F(y)_i),$$

that is

$$F(\lambda x^{(1)}+(1-\lambda)x^{(2)})\preceq \lambda F(x^{(1)})+(1-\lambda)F(x^{(2)}),$$

thus $F$ is a convex fuzzy-number-valued function.

## 4. The duality and the saddle point

Duality plays an important role in the development of optimization theory and algorithm. In this section, the duality theory of fuzzy nonlinear programming is introduced, and the weak duality theorems are obtained. At the same time, the Lagrange function of fuzzy nonlinear programming and saddle point are defined, and then discusses the relation between the saddle point of Lagrange function and the optimal solution of prime problem and dual problem and given saddle point optimality conditions.

Let $X \subset R^n$ be an open set, $F(x), G_i(x)(i = 1, 2, \cdots, m)$ be fuzzy-valued functions on $X$, now we consider the following primal fuzzy optimization:

$$(FP) \quad \begin{cases} \min F(x), \\ G_j(x) \preceq \widetilde{0}(j = 1, 2, \cdots, m), \end{cases} \tag{4.1}$$

where $S = \{x \in X | G_j(x) \preceq \widetilde{0}(j = 1, 2, \cdots, m)\}$ is the set of feasible solutions for problem $(FP)$, and denote $x \in S$ the feasible solution for problem $(FP)$.

We define the fuzzy-valued Lagrangian function for the primal problem as follow:

$$L(x, u) = F(x) + \sum_{j=1}^{m} u_j G_j(x)$$

for all $x \in S$ and all $(u_1, u_2, \cdots, u_m) \in R^m, u_j \geq 0(j = 1, 2, \cdots, m)$.

Now we define the dual fuzzy optimization problem as follow:

$$(FD) \quad \begin{cases} \max L(u) \\ u_j \geq 0(j = 1, 2, \cdots, m). \end{cases} \tag{4.2}$$

where $L(u) = \min_x L(x, u), u = (u_1, u_2, \cdots, u_m) \in R^m$.

**Theorem 4.1** (Weak Duality Theorem) Let $x \in X(\subset R^n), u \in Y(\subset R^m)$ be the feasible solution of problems (FP) and (FD) respectively. then

$$F(x) \succeq L(u).$$

**Proof** From the definition of $L(u)$, we have

$$L(u) = \min_x L(x, u) = \min_x (F(x) + \sum_{j=1}^{m} u_j G_j(x)) \preceq F(x) + \sum_{j=1}^{m} u_j G_j(x). \tag{4.3}$$

that is

$$\tau(L(u)_i) \leq \tau(F(x)_i) + \sum_{j=1}^{m} u_j \tau(G_j(x)_i),$$

for $1 \leq i < n$. Since $x$ and $u$ is the feasible solution of problems (FP) and (FD) respectively, that is $u_j \geq 0$ and $G_j(x) \preceq \widetilde{0}(j = 1, 2, \cdots, m)$, thus $\tau(G_j(x)_i) \leq 0(i = 1, 2, \cdots, n)$, then we have $\tau(L(u)_i) \leq \tau(F(x)_i)$for $1 \leq i < n$, then $L(u) \preceq F(x)$.

From above it follows easily:

**Proposition 4.2** For the problems (FP) and (FD), we have

$$\min\{F(x) | G_j(x) \preceq \widetilde{0}, x \in X, j = 1, 2, \cdots, m\} \succeq \max\{L(u) | u \geq 0\}.$$

**Proposition 4.3** Assume that

$$F(\overline{x}) \preceq L(\overline{u}),$$

where $\overline{x} \in \{x | G_j(x) \preceq \widetilde{0}, x \in X, j = 1, 2, \cdots, m\}$, $\overline{u} \geq 0$, then $\overline{x}$ and $\overline{u}$ are the optimal solutions of problems (FP) and (FD) respectively.

**Definition 4.4** Let $\overline{x} \in X(\subset R^n), \overline{u} \in Y(\subset R^m)$, then $(\overline{x}, \overline{u})$ is called a saddle point of the fuzzy-valued Lagrangian function $L : X \times Y \to E^n$ if and only if

$$L(\overline{x}, u) \preceq L(\overline{x}, \overline{u}) \preceq L(x, \overline{u}) \tag{4.4}$$

holds for every $(x, u) \in X \times Y$.

**Theorem 4.5** Let $(\overline{x}, \overline{u})$ be a saddle point of the fuzzy-valued Lagrangian function $L(x, u)$, then $\overline{x}$ and $\overline{u}$ are the optimal solutions of problems (FP) and (FD) respectively.

**Proof** Assume that $(\overline{x}, \overline{u})$ be a saddle point, we are going to prove $\overline{x} \in S$ firstly. It follows easily from the definition of saddle point, $L(\overline{x}, u) \preceq L(\overline{x}, \overline{u})$ holds for all $u \in R^m$, that is

$$F(\overline{x}) + \sum_{j=1}^{m} u_j G_j(\overline{x}) \preceq F(\overline{x}) + \sum_{j=1}^{m} \overline{u}_j G_j(\overline{x}),$$

so

$$\tau(F(\overline{x})_i) + \sum_{j=1}^{m} u_j \tau(G_j(\overline{x})_i) \leq \tau(F(\overline{x})_i) + \sum_{j=1}^{m} \overline{u}_j \tau(G_j(\overline{x})_i),$$

$$\sum_{j=1}^{m} (u_j - \overline{u}_j) \tau(G_j(\overline{x})_i) \leq 0. \tag{4.5}$$

holds for any $1 \leq i < n$, Let $u_k = \overline{u}_k + 1$ and $u_j = \overline{u}_j, j \neq k$, from (5.5)we have $\tau(G_k(\overline{x})_i) \leq 0 (k = 1, 2, \cdots, m)$.It says that $\overline{x}$ is a feasible solution of (FP). Now we'll prove $\overline{x}$ and $\overline{u}$ are the optimal solutions of problems (FP) and (FD) respectively. Let $u_j (j = 1, 2, \cdots, m)$ in (4.5) be taken as 0, then $\sum_{j=1}^{m} (-\overline{u}_j) \tau(G_j(\overline{x})_i) \leq 0$. since $\overline{u}_j \geq 0, \tau(G_j(\overline{x})_i) \leq 0$, then we have

$$\sum_{j=1}^{m} (-\overline{u}_j) \tau(G_j(\overline{x})_i) = 0. \tag{4.6}$$

From the right inequality of (4.4), $\tau(F(\overline{x})_i) \leq \tau(F(x)_i) + \sum_{j=1}^{m} (\overline{u}_j \tau(G_j(x)_i)$ holds for all $x \in S$. So $F(\overline{x}) \succeq L(\overline{u})$. That is $\overline{x}$ and $\overline{u}$ are the optimal solutions of problems (FP) and (FD) from the proposition 4.3.

**Lemma 4.6**[16] Let $X$ be a nonempty convex set in a real vector space $R^n$, $F : X \to R, G = (G_1, G_2, \cdots, G_n), G_i : R^n \to R (i = 1, 2, \cdots, n)$ be convex functions. We consider the following conditions.

Condition a: $F(x) < 0$ and $G(x) \leq 0$ for some $x \in X$;

Condition b: $u_0 F(x) + \langle u, G(x) \rangle \geq 0$ for all $x \in X, (u_0, u) \geq 0$ and $(u_0, u) \neq 0$.

If $\overline{x}$ does not satisfy Condition a, then Condition b has a solution $(u_0, u)$ when $\overline{x}$ substitute $x$.

**Theorem 4.7** Let $X \subset R^n$ be a nonempty convex sets, $F : X \to E^n, G_j : X \to E^n (j = 1, 2, \cdots, m)$ be convex fuzzy-valued functions, $\overline{x}$ be an optimal solution of problem $(FP)$, assume that there exist $x$ , such that $G_j(x) \preceq \widetilde{0}$, then there exists $\overline{u} \geq 0$,such that

$$L(x, \overline{u}) \succeq F(\overline{x})$$

holds for every $x \in X$.

**Proof** $\overline{x}$ be an optimal solution of $(FP)$, then $F(x) \succeq F(\overline{x})$ for any $x \in X$, that is

$$\tau(F(x)_i) = \int_0^1 r[F(x)^*(r, e_i^+) - F(x)^*(r, e_i^-)]dr \geq \int_0^1 r[F(\overline{x})^*(r, e_i^+) - F(\overline{x})^*(r, e_i^-)]dr = \tau(F(\overline{x})_i)$$

for $1 \leq i < n$, Since $F : X \to E^n, G_j : X \to E^n (j = 1, 2, \cdots, m)$ be convex fuzzy-valued functions, then $\tau(F(x)_i), \tau(G_j(x)_i) (j = 1, 2, \cdots, m, i = 1, 2, \cdots, n)$ are convex real-valued functions. Therefore we consider the following systems:

$$\int_0^1 r[F(x)^*(r, e_i^+) - F(x)^*(r, e_i^-)]dr - \int_0^1 r[F(\overline{x})^*(r, e_i^+) - F(\overline{x})^*(r, e_i^-)]dr < 0,$$

$$\int_0^1 r[G_j^*(r, e_i^+) - G_j^*(r, e_i^-)]dr \leq 0 (j = 1, 2, \cdots, m),$$

the system has no solution on $X$, then from lemma 4.6, there exists $(u_0, u) \geq 0$ and $(u_0, u) \neq 0$ such that

$$u_0(\tau(F(x)_i) - \tau(F(\overline{x})_i)) + \sum_{j=1}^{m} u_j \tau(G_j(x)_i) \geq 0$$

for every $x \in X$. assume that $u_0 = 0$, then $\sum_{j=1}^{m} u_j \tau(G_j(x)_i) \geq 0$ holds for every $x \in X$, since there exists $x$, such that $\tau(G_j(x)_i) \leq 0$ for $1 \leq i \leq n$, then $u_j = 0 (j = 1, 2, \cdots, m)$, it contracts $(u_0, u) \neq 0$, thus $u_0 > 0$. dividing the inequality by $u_0$, we have

$$\tau(F(x)_i) - \tau(F(\overline{x})_i) + \sum_{j=1}^{m} u'_j \tau(G_j(x)_i) \geq 0,$$

where $u'_j = \frac{u_j}{u_0} (j = 1, 2, \cdots, m)$. then

$$F(x) + \sum_{j=1}^{m} u'_j G_j(x) \succeq F(\overline{x}).$$

denote $\overline{u}$ by $u'$, and then there exists $\overline{u} \geq 0$, such that $L(x, \overline{u}) \succeq F(\overline{x})$.

**Theorem 4.8** Let $X \subset R^n$ be a nonempty convex set, $F : X \to E^n, G_j : X \to E^n (j = 1, 2, \cdots, m)$ be convex fuzzy-valued functions, $\overline{x}$ be a optimal solution of problem $(FP)$, assume there exists $x$, such that $G_j(x) \preceq \widetilde{0}$, then there exists $\overline{u} \geq 0$, such that $(\overline{x}, \overline{u})$ be a saddle point of the fuzzy-valued Lagrangian function $L(x, u)$.

**Proof** Let $\overline{x}$ be an optimal solution of problem $(FP)$, from the theorem 4.7, there exists $\overline{u} \geq 0$ such that

$$L(x, \overline{u}) \succeq F(\overline{x}) \tag{4.7}$$

holds for every $x \in X$. then

$$L(\overline{x}, \overline{u}) = F(\overline{x}) + \sum_{j=1}^{m} \overline{u}_j G_j(\overline{x}) \succeq F(\overline{x}),$$

and $\overline{u}_j \geq 0, G_j(\overline{x}) \preceq \widetilde{0} (j = 1, 2, \cdots, m)$, so

$$\sum_{j=1}^{m} \overline{u}_j G_j(\overline{x}) = \widetilde{0}. \tag{4.8}$$

That is

$$L(\overline{x}, \overline{u}) = F(\overline{x}) + \sum_{j=1}^{m} \overline{u}_j G_j(\overline{x}) = F(\overline{x}),$$

From (4.7), we have

$$L(\overline{x}, \overline{u}) \preceq L(x, \overline{u}). \tag{4.9}$$

From the definition of $L(x, u)$, we have

$$L(\overline{x}, u) = F(\overline{x}) + \sum_{j=1}^{m} u_j G_j(\overline{x}),$$

and $G_j(\overline{x}) \preceq \widetilde{0}, u_j \geq 0 (j = 1, 2, \cdots, m)$, then

$$L(\overline{x}, u) \preceq F(\overline{x}) = L(\overline{x}, \overline{u}). \tag{4.10}$$

the (4.9),(4.10) indicate that $(\overline{x}, \overline{u})$ be a saddle point of the fuzzy-valued Lagrangian function $L(x, u)$.

## 5. Conclusion

In this article, we introduced the convexity and differentiability of n-dimensional fuzzy-number-valued function by means of a new order relation, which Pave a way for n-dimensional fuzzy optimization problem, so the saddle point optimal condition can be implemented.The n-dimensional fuzzy number valued function has been embedded into a complete Banach space, which is expressed by its support function, but the order of its support function is improper for fuzzy number, and establish this new order relationship more grasp the location information of fuzzy numbers. In the future work, we will discuss the application of n-dimensional fuzzy optimization in practice.

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# Complex harmonic poles in the evolution of macromolecules depolymerization

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**Abstract**

The full comprehension and handling of the phenomenon of shattering, sometime happening during the process of polymer chain degradation [29, 32], remains unsolved when using the traditional evolution equations describing the degradation. This traditional model has been proved to be very hard to handle as it involves evolution of two intertwined quantities. moreover, the explicit form of its solution is, in general, impossible to obtain. In this article, we explore the possibility of generalizing evolution equation modeling the polymer chain degradation and analyze the model with $\beta$ derivative. We consider the general case where the breakup rate depends on the size of the chain breaking up. In the process, the alternative version of Sumudu integral transform is used to provide an explicit form of the general solution representing the evolution of polymer sizes distribution. In particular, we show that this evolution exhibits existence of complex periodic properties due to the presence of cosine and sine functions governing the solutions. Numerical simulations are performed for some particular cases and proves that such a system describing the polymer chain degradation contains complex and simple harmonic poles whose effects are given by these functions or a combination of them. This result may be crucial in the ongoing research to better handle and explain the phenomenon of shattering.

**Keywords:** $\beta$- derivative; depolymerization; replicated fractional poles; simple and complex harmonic motion; shattering

## 1 Introduction, motivation and Justification

Depolymerization is the process where polymers or biopolymers are converted into monomers or mixtures of monomers. Polymers range from familiar synthetic plastics such as polystyrene (also called styrofoam) to natural biopolymers such as DNA and proteins that are fundamental to biological structure and function. Historically, products arising from the linkage of repeating units by covalent chemical bonds have been the primary focus of polymer science; emerging important areas of the science now focus on non-covalent links. Polyisoprene of latex rubber and the polystyrene of styrofoam are examples of polymeric natural/biological and synthetic polymers, respectively. In biological contexts, essentially all biological macromolecules, i.e. proteins (polyamides), nucleic

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acids (polynucleotides), and polysaccharides are purely polymeric, composed in large part of polymeric components, for instance, isoprenylated/lipid-modified glycoproteins, where small lipidic molecule and oligosaccharide modifications occur on the polyamide backbone of the protein.

Today, it is widely known that the Newtonian concept of derivative can no longer satisfy all the complexity of the natural occurrences. A couple of complex phenomena and features happening in some areas of sciences or engineering are still (partially) unexplained by the traditional existing methods and remain open problems. Usually in mathematical modeling of a natural phenomenon that changes, the evolution is described by a family of time-parameter operators, that map an initial given state of the system to all subsequent states that takes the system during the evolution. A widely devotion has been predominantly offered to way of looking at that evolution in which time's change is described as transitions from one state to another. Hence, this is how the theory of semigroups was developed [16, 25], providing the mathematicians with very interesting tools to investigate and analyze resulting mathematical models. However, most of the phenomena scientists try to analyze and describe mathematically are complex and very hard to handle. Some of them like depolymerization, the rock fractures and fragmentation processes are difficult to analyze [11, 33] and often involve evolution of two intertwined quantities: the number of particles and the distribution of mass among the particles in the ensemble [15, 20, 28]. Then, though linear, they display non-linear features such as phase transition (called "shattering") causing the appearance of a "dust" of "zero-size" particles with nonzero mass. The phenomena of "shattering" remain (partially) unexplained by traditional models.

Another example is the groundwater flowing within a leaky aquifer. Recall that an aquifer is an underground layer of water-bearing permeable rock or unconsolidated materials (gravel, sand, or silt) from which groundwater can be extracted using a water well. Then, how do we explain accurately the observed movement of water within the leaky aquifer? As an attempt to answer this question, Hantush [17, 18] proposed an equation with the same name and his model has since been used by many hydro-geologists around the world. However, it is necessary to note that the model does not take into account all the non-usual details surrounding the movement of water through a leaky geological formation. Indeed, due to the deformation of some aquifers, the Hantush equation is not able to account for the effect of the changes in the mathematical formulation. Hence, all those non-usual features are beyond the usual models' resolutions and need other techniques and methods of modeling with more parameters involved.

Furthermore, time's evolution and changes occurring in some systems do not happen on the same manner after a fixed or constant interval of time and do not follow the same routine as one would expect. For instance, a huge variation can occur in a fraction of second causing a major change that may affect the whole system's state forever. Indeed, it has turned out recently that many phenomena in different fields, including sciences, engineering and technology can be described very successfully by the models using fractional order differential equations [4, 6, 9, 10, 13, 14, 19, 22, 27]. Hence, differential equations with fractional derivative have become a useful tool for describing nonlinear phenomena that are involved in many branches of chemistry, engineering, biology, ecology and numerous domains of applied sciences. Many mathematical models, including those in acoustic dissipation, mathematical epidemiology, continuous time random walk, biomedical engineering, fractional signal and image processing, control theory, Levy statistics, fractional phase-locked loops, fractional Brownian, porous media, fractional filters motion and non-local phenomena have proved to provide a better description of the phenomenon under investigation than models with the conventional integer-order derivative [6, 22, 26].

One of the attempts to enhance mathematical models was to introduce the concept of derivative with fractional order. There exist in the literature number of definitions of fractional derivatives, including Riemann–Liouville and Caputo derivatives respectively defined as

$$D_x^{\alpha}\left(f(x)\right)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n}\int_0^x (x-t)^{n-\alpha-1} f(t)\,dt, \tag{1}$$

$n-1<\alpha\leq n$ and

$$D_x^{\alpha}\left(f(x)\right)=\frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}\left(\frac{d}{dt}\right)^{n} f(t)\,dt, \tag{2}$$

$n-1<\alpha\leq n$. A new fractional derivative with no singular kernel was recently proposed by Caputo et al. in [7]. However, Caputo fractional derivative [8], for instance, is the one mostly used for modelling real world problems in the field [4, 6, 13–15, 20, 28]. However, this derivative exhibits some limitations like not obeying the traditional chain rule; which chain rule represents one of the key elements of the match asymptotic method [20, 28]. Recall that the match asymptotic method has never been used to solve any kind of fractional differential equations because of the nature and properties of fractional derivatives. Hence, the conformable fractional derivative was proposed [2, 21]. This fractional derivative is theoretically very easier to handle and obeys the chain rule. But it also exhibits a huge failure that is expressed by the fact that the fractional derivative of any differentiable function at the point zero is zero. This does not make any sense in a physical point of view and then, a modified new version, the $\beta$–derivative was proposed in order to skirt the noticed weakness. The main aim of this new derivative was, first of all, to extend the well-known match asymptotic method to the scope of the fractional differential equation and later to describe the boundary layers problems within the folder of fractional calculus. The $\beta$–derivative was defined as [1, 15, 20]:

$${}_0^A D_t^{\beta} g(t)=\begin{cases} \lim\limits_{\varepsilon\to 0}\dfrac{g\left(t+\varepsilon\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)-g(t)}{\varepsilon} & \text{for all } t\geq 0,\quad 0<\beta\leq 1 \\ g(t) & \text{for all } t\geq 0, \beta=0, \end{cases} \tag{3}$$

where $g$ is a function such that $g:[0, \ \infty)\to\mathbb{R}$ and $\Gamma$ the gamma-function

$$\Gamma(\zeta)=\int_0^\infty t^{\zeta-1}e^{-t}dt.$$

If the above limit of exists then $g$ is said to be $\beta-$differentiable.

Note that for $\beta=1$, we have ${}_0^AD_t^\beta g(t)=\frac{d}{dt}g(t)$. Moreover, unlike other derivatives with fractional parameters, the $\beta$–derivative of a function can be locally defined at a certain point, the same way like the first order derivative. For a general order, let us say $m\beta$, the $m\beta$–derivative of $g$ is defined as

$$ {}_0^AD_t^{m\beta}g(t)={}_0^AD_t^\beta\left({}_0^AD_t^{(m-1)\beta}g(t)\right) \quad \text{for all } t\geq 0, m\in\mathbb{N},\ 0<\beta\leq 1 \tag{4}$$

Notice that the $m\beta$–derivative of a given function provides information about the previous $n-1$–derivatives of the same function. For instance we have

$$\begin{aligned}
{}_0^AD_t^{2\beta}g(t) &={}_0^AD_t^\beta\left({}_0^AD_t^\beta g(t)\right) \\
&=\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\left[(1-\beta)\left(t+\frac{1}{\Gamma(\beta)}\right)^{-\beta}g'+\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}g''\right].
\end{aligned} \tag{5}$$

This gives the $\beta$–derivative a unique property of memory, that is not provided by any other derivative. It is also easy to verify that for $\beta=1$, we recover the second derivative of $g$. For more properties and details on this new derivative, the readers can consult the reference [1, 15, 20, 28].

### 1.1 The kinetic equation

The evolution of the sizes distribution occurring during polymer chain degradation is well known [12, 15, 32] to be described by the following integrodifferential equation

$$\frac{\partial}{\partial t}g(x,t)=-g(x,t)\int_0^x H(y,x-y)dy+2\int_x^\infty g(y,t)H(x,y-x)dy, \qquad x,\, t>0. \tag{6}$$

Expressing the solution of equation (6) in its explicit form is very hard since fragmentation (or polymer chain degradation) processes, as explained in the previous section, are difficult to analyse as they involve evolution of two intertwined quantities: the distribution of mass among the particles in the ensemble and the number of particles in it. That is why, though linear, they display non-linear features such as “shattering” phenomena which they cannot fully explain [11, 15, 33]. Then, in order to have a broader idea about the evolution of polymer chain degradation and maybe trying to understand the phenomenon of shattering as described here above, we explore the possibility of extending the analysis by considering the $\beta$–derivative defined in the previous section. This yields the following integrodifferential equation:

$$ {}_{0}^{A}D_t^{\beta} g(x,t) = -g(x,t)\int_0^x H_\beta(y,x-y)dy + 2\int_x^{\infty} g(y,t)H_\beta(x,y-x)dy, \qquad x,\, t>0. \tag{7}$$

subject to the initial condition

$$ g(x,0) = g_0(x), \qquad x>0 \tag{8}$$

where $g(x,t)$ represents the density of $x$-groups (i.e. groups of size $x$) at time $t$ and $H_\beta(x,y)$ gives the average fragmentation rate, that is, the average number at which clusters of size $x+y$ undergo splitting to form an $x$-group and a $y$-group.

## 2 Some useful properties in the $\beta-$differentiation

Recall that there is a growing problem about the choice of the type of fractional derivative to use among the large number of its existing versions. We already mentioned the incapacity of most of them to explicitly provide the variation of the functions. Moreover, many models using fractional derivatives are not easy to handle analytically. The $\beta$–derivative allows us to palliate some insufficiencies of other fractional derivatives and then, we were able to successfully extend the well-known match asymptotic method [20, 28] to the scope of the fractional differential equation and also describe the boundary layers problems within the scope of fractional calculus. Next we recall some properties of the $\beta$–derivative all proved in [15, 20, 28].

**Theorem** 2.1. *Assuming that, a given function, say $g\ :[a,\,\infty)\to\mathbb{R}$ is $\beta-$differentiable at a given point, say $t_0\geq a,\ \beta\in(0,\ 1]$, then $g$ is also continuous at $t_0$.*

**Theorem** 2.2. *Assuming that $f$ is $\beta-$differentiable on an open interval $(a,\ b)$ then*

1. *If ${}_{0}^{A}D_t^{\beta} f(t)<0$ for all $t\in(a,\ b)$ then $f$ is decreasing on $(a,b)$;*
2. *If ${}_{0}^{A}D_t^{\beta} f(t)>0$ for all $t\in(a,\ b)$ then $f$ is increasing on $(a,b)$;*
3. *If ${}_{0}^{A}D_t^{\beta} f(t)=0$ for all $t\in(a,\ b)$ then $f$ is constant on $(a,b)$.*

**Theorem** 2.3. *Assuming that, $g\neq 0$ and $f$ are two $\beta-$differentiable functions with $\beta\in(0,1]$ then the following relations are satisfied*

1. ${}_{0}^{A}D_t^{\beta}\left(af(t)+bg(t)\right) = a\,{}_{0}^{A}D_t^{\beta}\left(f(t)\right) + b\,{}_{0}^{A}D_t^{\beta}\left(g(t)\right)$ *for all real numbers $a$ and $b$;*
2. ${}_{0}^{A}D_t^{\beta}(c)=0$ *for any given constant $c$;*
3. ${}_{0}^{A}D_t^{\beta}\left(f(t)\,g(t)\right) = g(t)\,{}_{0}^{A}D_t^{\beta}\left(f(t)\right) + f(t)\,{}_{0}^{A}D_t^{\beta}\left(g(t)\right)$;
4. ${}_{0}^{A}D_t^{\beta}\left(\frac{f(t)}{g(t)}\right) = \frac{g(t)\,{}_{0}^{A}D_t^{\beta}(f(t)) - f(t)\,{}_{0}^{A}D_t^{\beta}(g(t))}{g^2(t)}$.

**Theorem** 2.4. *Let $f:[a,\infty)\to\mathbb{R}$ be a function such that $f$ is differentiable and also $\beta-$differentiable. Let $g$ be a function defined in the range of $f$ and also differentiable, then we have the following rule*

$$ {}_0^AD_t^\beta\left(gof(t)\right)=\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}f'(t)g'\left(f(t)\right) \tag{9}$$

**Definition 2.1.** Let $f:[a,\infty)\to\mathbb{R}$ be a given function, then we propose that the $\beta-$integral of $f$ is

$$ {}_a^AI_t^\beta\left(f(t)\right)=\int_a^t\left(\xi+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}f(\xi)d\xi \tag{10}$$

The above operator is the inverse operator of the proposed fractional derivative. We shall present to underpin this statement by the following theorem.

**Theorem** 2.5. ${}_0^AD_t^\beta\left[{}_0^AI_t^\beta f(t)\right]=f(t)$ for all $t\geq 0$ with $f$ a given continuous and differentiable function.

*Proof.* [1, Theorem 7] □

**Theorem** 2.6.

$$ {}_a^AI_t^\beta\left[D_t^\beta f(t)\right]=f(t)-f(a) \tag{11}$$

for all $t\geq a$ with $f$ a given continuous and differentiable function.

*Proof.* [1, Theorem 8] □

# 3 Solutions to the model

Note that these above models (6) and (7) are well applicable in many branches of natural sciences, including physics, chemistry, engineering, biology, ecology, just to name a few, and in numerous domains of applied sciences, such as the rock fractures and break of droplets. Various types of fragmentation equations have been comprehensively analyzed in numerous works (see, e.g., [12, 30, 33]). In the domain of polymer science, the fragmentation dynamics has also been of considerable interest, since degradation of bonds or depolymerisation results in fragmentation, see [5, 23, 32]. In [23], the authors used statistical arguments to find and analyze the size distribution of the model. The authors in [5] analysed the model in combination with the inverse process, that is, the coagulation process, and provided a similar result for the size distribution. However, the classical fragmentation model (6) has been proved to be unable to fully describe some bizarre phenomena observed in such a degradation process, like for instance shattering as described above and also in [11, 23, 32, 33]. Recall that shattering is a phenomenon seen as an explosive or dishonest Markov process, see e.g. [3, 24] and has been associated with an infinite cascade of breakup events creating a ‘dust’ of particles of zero size which, however, carry non-zero mass. Hence, to have explicit solutions to the model, we consider the case where the breakup rate depends on the size of the chain breaking and takes the form

$$H_\beta(x,y) = (x+y)^\nu, \quad \nu \in \mathbb{R} \tag{12}$$

Substituting in equation (7) yields

$$D_t^\beta(g(x,t)) = -x^{\nu+1}g(x,t) + 2\int_x^\infty y^\nu g(y,t)dy, \quad 0 \le \beta \le 1 \tag{13}$$

Taking the the modified Sumudu transform $S_\beta$ (see the Appendix below) of both sides of equation (13) yields

$$S_\beta\left(D_t^\beta g(x,t), r\right) = -x^{\nu+1}G_s^\beta(x,r) + 2\int_x^\infty y^\nu G_s^\beta(y,r)dy,$$

where $G_s^\beta(x,r)$ represents the the modified Sumudu transform $S_\beta(g(x,t),r)$ of $g(x,t)$. Using the relation (23) of Appendix, we obtain

$$r^{-2}(G_s^\beta(x,r) - g_0(x)) = -x^{\nu+1}G_s^\beta(x,r) + 2\int_x^\infty y^\nu G_s^\beta(y,r)dy,$$

rearranged to have

$$\left(1 + x^{\nu+1}r^2\right)G_s^\beta(x,r) - 2r^2\int_x^\infty y^\nu G_s^\beta(y,r)dy = g_0(x). \tag{14}$$

Next, it is important to mention that considering the differential equation (13), it is implicitly required that the function $\xi \longrightarrow g(\xi,t)$ is integrable, in the sense of Lebesgue, on any interval $[\epsilon,\infty)$ for $\epsilon > 0$ and almost every $\xi > 0$. Obviously, the same assertion applies to the functions $\xi \longrightarrow g_0(\xi)$ and $\xi \longrightarrow G_s^\beta(\xi,r), \quad 0 \le \beta \le 1$.

This allows us to put

$$Z(x,r) = -2r^2\int_x^\infty y^\nu G_s^\beta(y,r)dy \tag{15}$$

knowing that the integrand will be integrable over any interval $[\epsilon,\infty)$ and the integral will be absolutely continuous at each $x > 0$. The substitution of $Z(x,r)$ into (14) yields the partial differential equation

$$\left(\frac{1 + x^{\nu+1}r^2}{1 + r^2x^\nu}\right)\partial_x Z(x,r) + Z(x,r) = g_0(x). \tag{16}$$

Choosing the constant in the general solution so as to have solutions converging to zero at $\infty$, we obtain its solution given as

$$Z(x,r) = 2r^2e^{-\sigma_{r,\nu}(x)}\int_x^\infty \frac{\xi^\nu g_0(\xi)}{1 + r^2\xi^{\nu+1}}e^{\sigma_{r,\nu}(\xi)}d\xi$$

where

$$\sigma_{r,\nu}(x)=\int_0^x \frac{2r^2\xi^\nu}{1+r^2\xi^{\nu+1}}d\xi=\ln\left(1+r^2x^{\nu+1}\right)^{\frac{2}{\nu+1}}. \tag{17}$$

Thus, substituting $Z(x,r)$ into (15) yields the solution of (14) given as

$$\begin{aligned} G_s^\beta(x,r) &= \frac{-1}{x^\nu}\left(\frac{2r^2x^\nu}{1+r^2x^{\nu+1}}e^{-\sigma_{r,\nu}(x)}\right)\int_\infty^x \frac{\xi^\nu g_0(\xi)}{1+r^2\xi^{\nu+1}}e^{\sigma_{r,\nu}(\xi)}d\xi+\frac{g_0(x)}{1+r^2x^{\nu+1}} \\ &= \frac{g_0(x)}{1+r^2x^{\nu+1}}-\frac{2r^2}{\left(1+r^2x^{\nu+1}\right)^{\frac{2}{\nu+1}+1}}\int_\infty^x \xi^\nu\left(1+r^2\xi^{\nu+1}\right)^{\frac{2}{\nu+1}-1}g_0(\xi)d\xi \end{aligned} \tag{18}$$

Applying the inverse of the modified Sumudu transform, which coincides with the inverse Sumudu transform, we are finally lead to the solution of the model (13), given by

$$\begin{aligned} g(x,t) &= S_\beta^{-1}(G_s^\beta(x,r),t) \\ &= g_0(x)S_\beta^{-1}\left(\frac{1}{1+r^2x^{\nu+1}},t\right)-2\int_\infty^x \xi^\nu g_0(\xi)S_\beta^{-1}\left(\frac{r^2\left(1+r^2\xi^{\nu+1}\right)^{\frac{2}{\nu+1}-1}}{\left(1+r^2x^{\nu+1}\right)^{\frac{2}{\nu+1}+1}},t\right)d\xi \\ &= g_0(x)\cos(t\sqrt{x^{\nu+1}})-2\int_\infty^x \xi^\nu g_0(\xi)S_\beta^{-1}\left(\frac{r^2\left(1+r^2\xi^{\nu+1}\right)^{\frac{2}{\nu+1}-1}}{\left(1+r^2x^{\nu+1}\right)^{\frac{2}{\nu+1}+1}},t\right)d\xi \end{aligned} \tag{19}$$

***Remark*** *3.1.* The expression $g(x,t)$ in (19) is well-defined only if the integral

$$\int_\infty^x \xi^\nu g_0(\xi)S_\beta^{-1}\left(\frac{r^2\left(1+r^2\xi^{\nu+1}\right)^{\frac{2}{\nu+1}-1}}{\left(1+r^2x^{\nu+1}\right)^{\frac{2}{\nu+1}+1}},t\right)d\xi$$

converges.

We are now capable of taking some specific values of $\nu$ to see the exact expression of the solution.

- For $\nu=1$, expression (19) becomes

$$\begin{aligned} g(x,t) &= g_0(x)S_\beta^{-1}\left(\frac{1}{1+r^2x^2},t\right)-2\int_\infty^x \xi g_0(\xi)S_\beta^{-1}\left(\frac{r^2}{\left(1+r^2x^2\right)^2},t\right)d\xi \\ &= g_0(x)\cos xt-\frac{t\sin xt}{x}\int_\infty^x \xi g_0(\xi)d\xi \end{aligned} \tag{20}$$

- For $\nu = -3$, expression (19) becomes

$$g(x,t) = g_0(x) S_\beta^{-1}\left(\frac{1}{1+r^2x^{-2}}, t\right) - 2\int_\infty^x \xi g_0(\xi) S_\beta^{-1}\left(r^2\left(1+r^2\xi^{-2}\right)^{-2}, t\right) d\xi$$
$$= g_0(x)\cos\frac{t}{x} - 2\int_\infty^x \xi g_0(\xi) \frac{\xi t \sin\frac{t}{\xi}}{2} d\xi \qquad (21)$$
$$= g_0(x)\cos\frac{t}{x} - \int_\infty^x t\xi^2 g_0(\xi)\sin\frac{t}{\xi} d\xi$$

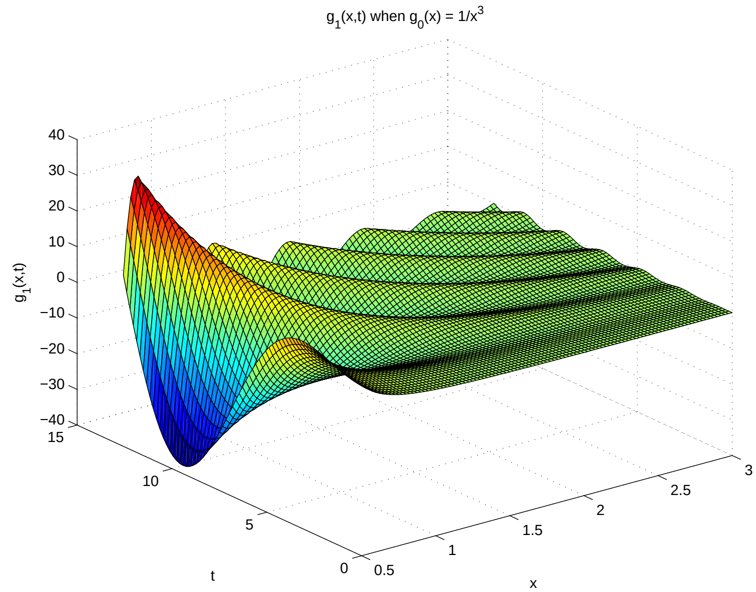


**Fig.** 1. $g(x,t)$ when $\nu = 1$ and $g_0(x) = 1/x^3$

# 4 Concluding remarks

We have explored the possibility of using new and alternative methods to generalize evolution equation modeling the polymer chain degradation. In the process, a modified version of the Sumudu transform is exploited to perform analysis of the system endowed the $\beta-$derivative and where the breakup rate depends on the size of the chain breaking up. Explicit forms of the solutions in some particular cases showed that the dynamics of this evolution exhibits complex periodic properties due to the presence of cosine and sine functions, as shown in Figs. 1 to 6, plotted for a positive value ($\nu = 1$) and a negative value ($\nu = -3$) of $\nu$. Figs. 1 to 3 represent the solution for $\nu = 1$ with initial condition $g_0(x) = 1/x^3$ : Fig.1 is the 2$-$D surface plot while Fig. 2 and 3 are respectively its cross

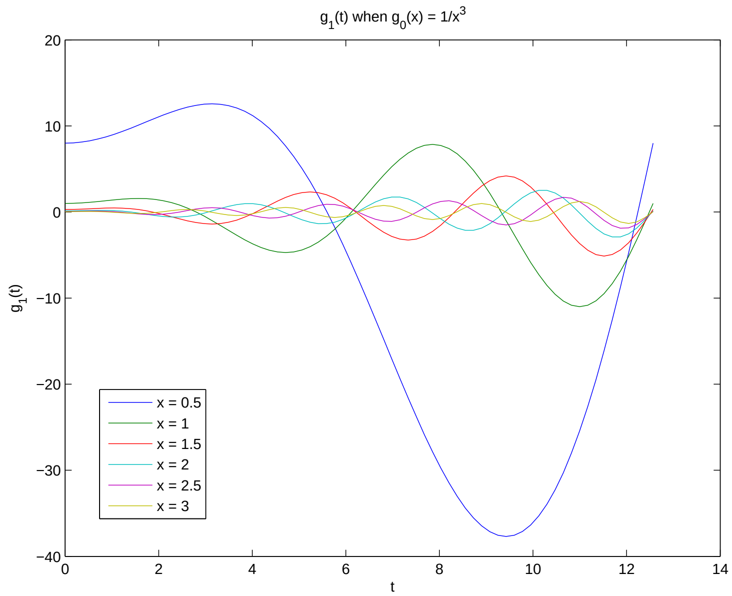


**Fig.** 2. $g(x,t)$ as a function of $t$ when $\nu = 1$ and $g_0(x) = 1/x^3$, for a few values of $x$

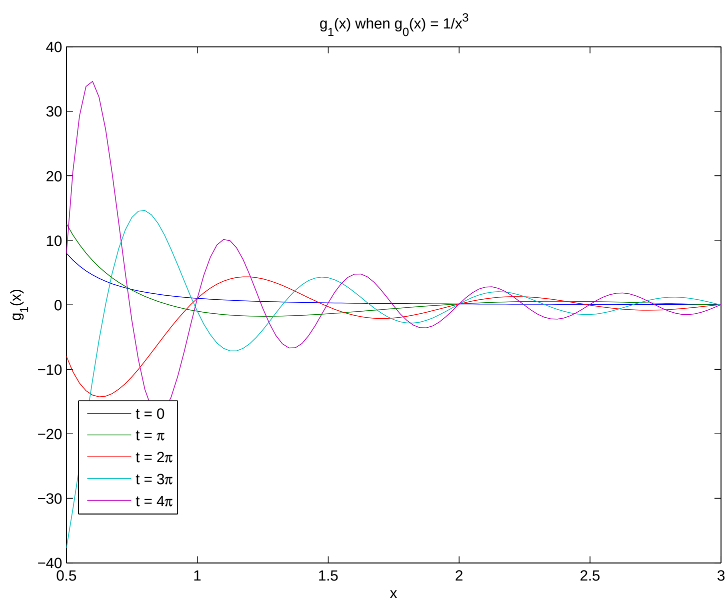


**Fig.** 3. $g(x,t)$ as a function of $x$ when $\nu = 1$ and $g_0(x) = 1/x^3$, for a few values of $t$: $0, \pi, 2\pi, 3\pi, 4\pi$

section and longitudinal section drawn for some specific values of the size $x$ and time $t$. A similar reasoning applies to Figs. 4 to 6, but this time with $\nu = -3$. This infers existence of complex and simple harmonic poles in the dynamics of polymer chain degradation whose effects are characterized by these functions or a combination of them. This work improved the preceding one with the inclusion of a more general expression of the breakup rate derivative and $\beta-$derivative. This work might be a breakthrough that may lead to

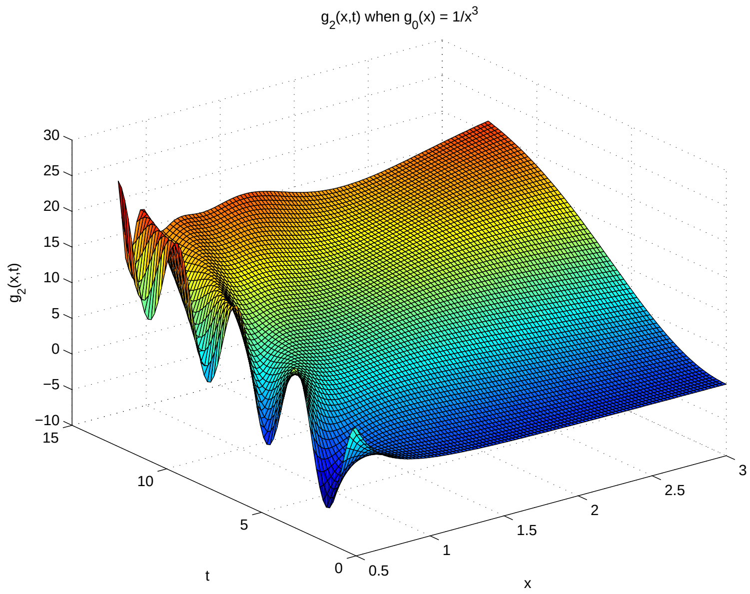


**Fig.** 4. $g(x,t)$ when $\nu = -3$ and $g_0(x) = 1/x^3$

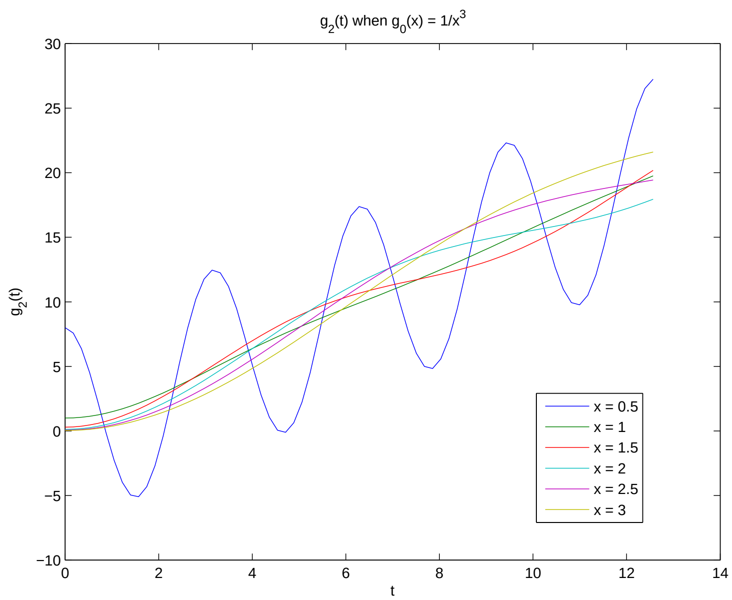


**Fig.** 5. $g(x,t)$ as a function of $t$ when $\nu = -3$ and $g_0(x) = 1/x^3$, for a few values of $x$

a better understanding of bizarre phenomena happening in some dynamics such as the phenomenon of shattering.

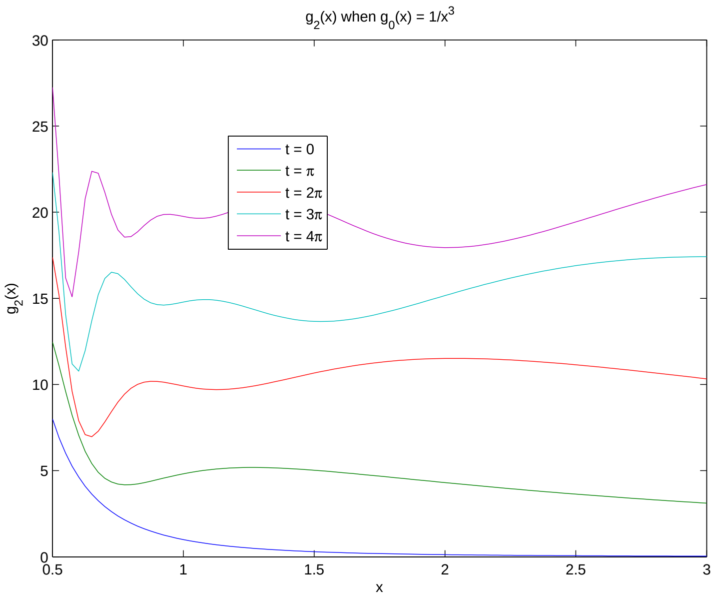


**Fig.** 6. $g(x,t)$ as a function of $x$ when $\nu=-3$ and $g_0(x)=1/x^3$, for a few values of $t$: $0,\pi,2\pi,3\pi,4\pi$

## Appendix: The new Sumudu integral transform

**Definition:** Let $g$ be a function defined in $(0,\infty)$, then, we define the modified Sumudu transform of $g$ as

$$S_\beta(g(t),u)=\int_0^\infty\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta-\lceil\beta\rceil}\frac{1}{u}e^{-\frac{t}{u}}g(t)dt, \tag{22}$$

where $\lceil\beta\rceil$ is the smallest integer greater or equal to $\beta$. Since $\beta\in(0,1]$ in this article then, $\beta-\lceil\beta\rceil=\beta-1$.

## An important property of the modified Sumudu transform:

If $S(g(t),u)$ is the well known Sumudu transform of $g$ defined in [31] as

$$S(g(t),u)=\int_0^\infty\frac{1}{u}exp\left[-\frac{t}{u}\right]g(t)dt,$$

then, we have the following relation:

$$S_\beta({}_0^AD_t^\beta g^{n-1}(t),u)=\frac{1}{u^n}S(g(t),u)-\sum_{k=0}^{n-1}\frac{1}{u^{n-k}}g^{(k)}(0) \tag{23}$$

*Proof.* By definition we have

$$S_\beta({}_0^AD_t^\beta g^{n-1}(t),u)=\int_0^\infty\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}$$
$$\frac{1}{u}exp\left[-\frac{t}{u}\right]\left(\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\lim_{\varepsilon\to0}\frac{g^{n-1}\left(t+\varepsilon\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)-g^{n-1}(t)}{\varepsilon}\right)dt \tag{24}$$
$$=\int_0^\infty\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\frac{1}{u}exp\left[-\frac{t}{u}\right]\left(\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\lim_{\eta\to0}\frac{g^{n-1}(t+\eta)-g^{n-1}(t)}{\eta}\right)dt$$

where we have put $\eta=\varepsilon\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\longrightarrow0$ as $\varepsilon\longrightarrow0$. Hence, making use of the well known property of Sumudu transform $S(g(t),u)$ [31], we obtain

$$S_\beta({}_0^AD_t^\beta g^{n-1}(t),u)=S(g^n(t),u)=\frac{1}{u^n}S(g(t),u)-\sum_{k=0}^{n-1}\frac{1}{u^{n-k}}g^{(k)}(0),$$

which concludes the proof. □


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# A fixed point convergence theorem with applications in left multivariate fractional calculus

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**Abstract**

A fixed point theorem is given under general conditions on the operators involved in a Banach space setting. The results find applications in left multivariate fractional calculus.

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**Key Words and Phrases:** Fixed point, Banach space, semi-local convergence, left multivariate fractional calculus.

## 1 Introductions

Numerous problems can be formulated as an equation like

$$R(x) = 0, \tag{1.1}$$

where $R$ is a continuous operator defined on a subset $\Omega$ of a Banach space $B_1$ with values in a Banach space $B_2$ using Mathematical Modelling [1], [7], [11], [12], [16], [18]. The solutions denoted by $x^*$ can be found in explicit form only in special cases. That is why most solution methods for these equations are usually iterative. Let $\mathcal{L}(B_1, B_2)$ denote the space of bounded linear operators from $B_1$ into $B_2$. Let also $A(\cdot) : \Omega \to \mathcal{L}(B_1, B_1)$ be a continuous operator. Set

$$F = LR, \tag{1.2}$$

where $L \in \mathcal{L}(B_2, B_1)$. We shall approximate $x^*$ using a sequence $\{x_n\}$ generated by the fixed point scheme:

$$\begin{aligned} & x_{n+1} := x_n + z_n, \quad A(x_n) z_n + F(x_n) = 0 \\ & \Leftrightarrow z_n = Q(z_n) := (I - A(x_n)) z_n - F(x_n), \end{aligned} \tag{1.3}$$

where $x_0 \in \Omega$. The sequence $\{x_n\}$ defined by

$$x_{n+1} = Q(x_n) = Q^{(n+1)}(x_0) \tag{1.4}$$

exists. In case of convergence we write:

$$Q^{\infty}(x_0) := \lim_{n \to \infty} (Q^n(x_0)) = \lim_{n \to \infty} x_n. \tag{1.5}$$

Many methods in the literature can be considered special cases of method (1.3). We can choose $A$ to be: $A(x) = F'(x)$ (Newton's method), $A(x) = F'(x_0)$ (Modified Newton's method), $A(x) = [x, g(x); F]$, $g : \Omega \to B_1$ (Steffensen's method). Many other choices for $A$ can be found in [1-20] and the references there in. Therefore, it is important to study the convergence of method (1.3) under generalized conditions. In particular, we present the semi-local convergence of method (1.3) using only continuity assumptions on operator $F$ and for a so general operator $A$ as to allow applications to left multivariate fractional calculus and other areas.

The rest of the paper is organized as follows: Section 2 contains the semi-local convergence of method (1.3). In the concluding Section 3, we suggest some applications to left multivariate fractional calculus.

# 2 Convergence

Let $B(w, \xi)$, $\overline{B}(w, \xi)$ stand, respectively for the open and closed balls in $B_1$ with center $w \in B_1$ and of radius $\xi > 0$.

We present the semi-local convergence of method (1.3) in this section.

**Theorem 2.1** *Let $F : \Omega \subset B_1 \to B_2$, $A(\cdot) : \Omega \to \mathcal{L}(B_1, B_1)$ and $x_0 \in \Omega$ be as defined in the Introduction. Suppose: there exist $\delta_0 \in (0,1)$, $\delta_1 \in (0,1)$, $\eta \geq 0$ such that for each $x, y \in \Omega$*

$$\delta := \delta_0 + \delta_1 < 1, \tag{2.1}$$

$$\|F(x_0)\| \leq \eta, \tag{2.2}$$

$$\|I - A(x)\| \leq \delta_0, \tag{2.3}$$

$$\|F(y) - F(x) - A(x)(y - x)\| \leq \delta_1 \|y - x\| \tag{2.4}$$

*and*

$$\overline{B}(x_0, \delta) \subseteq \Omega, \tag{2.5}$$

*where*

$$\rho = \frac{\eta}{1-\delta}. \tag{2.6}$$

*Then, sequence* $\{x_n\}$ *generated for* $x_0 \in \Omega$ *by*

$$x_{n+1} = x_n + Q^{\infty}(0), \quad Q_n(z) := (I - A(x_n))z - F(x_n) \tag{2.7}$$

*is well defined in* $\overline{B}(x_0, \rho)$, *remains in* $\overline{B}(x_0, \rho)$ *for each* $n = 0, 1, 2, ...$ *and converges to* $x^*$ *which is the only solution of equation* $F(x) = 0$ *in* $\overline{B}(x_0, \rho)$. *Moreover, an apriori error estimate is given by the sequence* $\{\rho_n\}$ *defined by*

$$\rho_0 := \rho, \quad \rho_n = T_n^{\infty}(0), \quad T_n(t) = \delta_0 + \delta_1 \rho_{n-1} \tag{2.8}$$

*for each* $n = 1, 2, ...$ *and satisfying*

$$\lim_{n\to\infty} \rho_n = 0. \tag{2.9}$$

*Furthermore, an aposteriori error estimate is given by the sequence* $\{\sigma_n\}$ *derfined by*

$$\sigma_n := H_n^{\infty}(0), \; H_n(t) = \delta t + \delta_1 p_{n-1}, \tag{2.10}$$

$$q_n := \|x_n - x_0\| \leq \rho - \rho_n \leq \rho, \tag{2.11}$$

*where*

$$p_{n-1} := \|x_n - x_{n-1}\| \quad \textit{for each} \;\; n = 1, 2, ... \tag{2.12}$$

**Proof.** We shall show using mathematical induction the following assertion is true:

$(A_n)$ $x_n \in X$ and $\rho_n \geq 0$ are well defined and such that

$$\rho_n + p_{n-1} \leq \rho_{n-1}. \tag{2.13}$$

By the definition of $\rho$, (2.3)-(2.6) we have that there exists $r \leq \rho$ (Lemma 1.4 [7, pp. 3]) such that

$$\delta_0 \tau + \|F(x_0)\| = r$$

and

$$\delta_0^k r \leq \delta_0^k \rho \to 0 \text{ as } k \to \infty.$$

That is (Lemma 1.5 [7, pp. 4]) $x_1$ is well defined and $p_0 \leq r$.

We need the estimate:

$$T_1(\rho - r) = \delta_0(\rho - r) + \delta_1 \rho_0 =$$

$$\delta_0 \rho - \delta_0 r + \delta_1 \rho = G_0(\rho) - r = \rho - r.$$

That is (Lemma 1.4 [7, pp. 3]) $\rho_1$ exists and satisfies

$$\rho_1 + p_0 \leq \rho - r + r = \rho = \rho_0.$$

Hence $(I_0)$ is true. Suppose that for each $k = 1, 2, ..., n$, assertion $(I_k)$ is true. We must show: $x_{k+1}$ exists and find a bound $r$ for $p_k$. Indeed, we have in turn that

$$\begin{aligned}\delta_0\rho_k + \delta_1\left(\rho_{k-1} - \rho_k\right) &= \delta_0\rho_k + \delta_1\rho_{k-1} - \delta_1\rho_k \\ &= T_k\left(\rho_k\right) - \delta_1\rho_k \leq \rho_k.\end{aligned}$$

That is there exists $r \leq \rho_k$ such that

$$r = \delta_0 r + \delta_1\left(\rho_{k-1} - \rho_k\right) \quad \text{and} \quad \left(\delta_0 + \delta_1\right)^i r \to 0 \tag{2.14}$$

as $i \to \infty$.

The induction hypothesis gives that

$$q_k \leq \sum_{m=0}^{k-1} p_m \leq \sum_{m=0}^{k-1}\left(\rho_m - \rho_{m+1}\right) = \rho - \rho_k \leq \rho,$$

so $x_k \in \overline{B}\left(x_0, \rho\right) \subseteq \Omega$ and $x_1$ satisfies $\|I - A\left(x_1\right)\| \leq \delta_0$ (by (2.3)).

Using the induction hypothesis, (1.3) and (2.4), we get

$$\begin{aligned}\|F\left(x_k\right)\| &= \|F\left(x_k\right) - F\left(x_{k-1}\right) - A\left(x_{k-1}\right)\left(x_k - x_{k-1}\right)\| \\ &\leq \delta_1 p_{k-1} \leq \delta_1\left(\rho_{k-1} - \rho_k\right)\end{aligned} \tag{2.15}$$

leading together with (2.14) to:

$$\delta_0 r + \|F\left(x_k\right)\| \leq r,$$

which implies $x_{k+1}$ exists and $p_k \leq r \leq \rho_k$. It follows from the definition of $\rho_{k+1}$ that

$$T_{k+1}\left(\rho_k - r\right) = T_k\left(\rho_k\right) - r = \rho_k - r,$$

so $\rho_{k+1}$ exists and satisfies

$$\rho_{k+1} + p_k \leq \rho_k - r + r = \rho_k$$

so the induction for $(I_n)$ is completed.

Let $j \geq k$. Then, we obtain in turn that

$$\|x_{j+k} - x_k\| \leq \sum_{i=k}^{j} p_i \leq \sum_{i=k}^{j}\left(\rho_j - \rho_{j+1}\right) = \rho_k - \rho_{j+k} \leq \rho_k. \tag{2.16}$$

We also have using induction that

$$\rho_{k+1} = T_{k+1}\left(\rho_{k+1}\right) \leq T_{k+1}\left(\rho_k\right) \leq \delta\rho_k \leq ... \leq \delta^{k+1}\rho. \tag{2.17}$$

4

Hence, by (2.1) and (2.17) $\lim_{k\to\infty} \rho_k = 0$, so $\{x_k\}$ is a complete sequence in a Banach space $X$ and as such it converges to some $x^*$. By letting $j \to \infty$ in (2.16), we conclude that $x^* \in \overline{B}(x_k, \rho_k)$. Moreover, by letting $k \to \infty$ in (2.15) and using the continuity of $F$ we get that $F(x^*) = 0$. Notice that

$$H_k(\rho_k) \leq T_k(\rho_k) \leq \rho_k,$$

so the apriori bound exists. That is $\sigma_k$ is smaller in general than $\rho_k$. Clearly, the conditions of the theorem are satisfied for $x_k$ replacing $x_0$ (by (2.16)). Hence, by (2.8) $x^* \in \overline{B}(x_n, \sigma_n)$, which completes the proof for the aposteriori bound. ■

**Remark 2.2** *(a) It follows from the proof of Theorem 2.1 that the conclusions hold, if $A(\cdot)$ is replaced by a more general continuous operator $A : \Omega \to B_1$.*

*(b) In the next section some applications are suggested for special choices of the "A" operators with $\gamma_0 := \delta_0$ and $\gamma_1 := \delta_1$.*

# 3 Applications to left multivariate fractional calculus

Our presented earlier semi-local convergence results, see Theorem 2.1, apply in the next two multivariate fractional settings given that the following inequalities are fulfilled:

$$\|1 - A(x)\|_\infty \leq \gamma_0 \in (0,1), \tag{3.1}$$

and

$$\left\|(F(y) - F(x))\overrightarrow{i} - A(x)(y-x)\right\| \leq \gamma_1 \|y - x\|, \tag{3.2}$$

where $\gamma_0, \gamma_1 \in (0,1)$, furthermore

$$\gamma = \gamma_0 + \gamma_1 \in (0,1), \tag{3.3}$$

for all $x, y \in \prod_{i=1}^{k} [a_i^*, b_i^*]$, where $a_i < a_i^* < b_i^* < b_i$, $i = 1, ..., k$.

Above $\overrightarrow{i}$ is the unit vector in $\mathbb{R}^k$, $k \in \mathbb{N}$, $\left\|\overrightarrow{i}\right\| = 1$, and $\|\cdot\|$ is a norm in $\mathbb{R}^k$.

The specific functions $A(x)$, $F(x)$ will be described next.

I) Consider the left multidimensional Riemann-Liouville fractional integral of order $\alpha = (\alpha_1, ..., \alpha_k)$ ($\alpha_i > 0$, $i = 1, ..., k$):

$$\left(I_{a+}^{\alpha} f\right)(x) = \frac{1}{\prod_{i=1}^{k} \Gamma(\alpha_i)} \int_{a_1}^{x_1} ... \int_{a_k}^{x_k} \prod_{i=1}^{k} (x_i - t_i)^{\alpha_i - 1} f(t_1, ..., t_k)\, dt_1 ... dt_k, \tag{3.4}$$

5

where $\Gamma$ is the gamma function, $f \in L_{\infty}\left(\prod_{i=1}^{k}[a_i,b_i]\right)$, $a=(a_1,...,a_k)$, and $x=(x_1,...,x_k)\in\prod_{i=1}^{k}[a_i,b_i]$.

By [6], we get that $\left(I_{a+}^{\alpha}f\right)$ is a continuous function on $\prod_{i=1}^{k}[a_i,b_i]$. Furthermore by [6] we get that $I_{a+}^{\alpha}$ is a bounded linear operator, which is a positive operator, plus that $\left(I_{a+}^{\alpha}f\right)(a)=0$.

In particular, $\left(I_{a+}^{\alpha}f\right)$ is continuous on $\prod_{i=1}^{k}[a_i^*,b_i^*]$.

Thus there exist $x_1,x_2\in\prod_{i=1}^{k}[a_i^*,b_i^*]$ such that

$$\begin{aligned}\left(I_{a+}^{\alpha}f\right)(x_1)&=\min\left(I_{a+}^{\alpha}f\right)(x),\\ \left(I_{a+}^{\alpha}f\right)(x_2)&=\max\left(I_{a+}^{\alpha}f\right)(x),\end{aligned}\tag{3.5}$$

over all $x\in\prod_{i=1}^{k}[a_i^*,b_i^*]$.

We assume that

$$\left(I_{a+}^{\alpha}f\right)(x_1)>0.\tag{3.6}$$

Hence

$$\left\|I_{a+}^{\alpha}f\right\|_{\infty,\prod_{i=1}^{k}[a_i^*,b_i^*]}=\left(I_{a+}^{\alpha}f\right)(x_2)>0.\tag{3.7}$$

Here, we define

$$Jf(x)=mf(x),\ \ 0<m<\frac{1}{2},\tag{3.8}$$

for any $x\in\prod_{i=1}^{k}[a_i^*,b_i^*]$.

Therefore the equation

$$Jf(x)=0,\ \ x\in\prod_{i=1}^{k}[a_i^*,b_i^*],\tag{3.9}$$

has the same solutions as the equation

$$F(x):=\frac{Jf(x)}{2\left(I_{a+}^{\alpha}f\right)(x_2)}=0,\quad x\in\prod_{i=1}^{k}[a_i^*,b_i^*].\tag{3.10}$$

Notice that

$$I_{a+}^{\alpha}\left(\frac{f}{2\left(I_{a+}^{\alpha}f\right)(x_2)}\right)(x)=\frac{\left(I_{a+}^{\alpha}f\right)(x)}{2\left(I_{a+}^{\alpha}f\right)(x_2)}\le\frac{1}{2}<1,\quad x\in\prod_{i=1}^{k}[a_i^*,b_i^*].\tag{3.11}$$

Call

$$A(x) := \frac{\left(I_{a+}^{\alpha}f\right)(x)}{2\left(I_{a+}^{\alpha}f\right)(x_2)}, \ \ \forall\, x \in \prod_{i=1}^{k}\left[a_i^*, b_i^*\right]. \tag{3.12}$$

We notice that

$$0 < \frac{\left(I_{a+}^{\alpha}f\right)(x_1)}{2\left(I_{a+}^{\alpha}f\right)(x_2)} \leq A(x) \leq \frac{1}{2}, \ \ \forall\, x \in \prod_{i=1}^{k}\left[a_i^*, b_i^*\right]. \tag{3.13}$$

Hence, the first condition (3.1) is fulfilled by

$$|1 - A(x)| = 1 - A(x) \leq 1 - \frac{\left(I_{a+}^{\alpha}f\right)(x_1)}{2\left(I_{a+}^{\alpha}f\right)(x_2)} =: \gamma_0, \ \ \forall\, x \in \prod_{i=1}^{k}\left[a_i^*, b_i^*\right]. \tag{3.14}$$

Hence, $\|1 - A(x)\|_{\infty} \leq \gamma_0$, where $\|\cdot\|_{\infty}$ is over $\prod_{i=1}^{k}\left[a_i^*, b_i^*\right]$. Clearly $\gamma_0 \in (0,1)$.

Next, we assume that $\frac{f(x)}{2\left(I_{a+}^{\alpha}f\right)(x_2)}$ is a contraction, that is

$$\left|\frac{f(x)}{2\left(I_{a+}^{\alpha}f\right)(x_2)} - \frac{f(y)}{2\left(I_{a+}^{\alpha}f\right)(x_2)}\right| \leq \theta\,\|x-y\|, \ \ \text{all } x, y \in \prod_{i=1}^{k}\left[a_i^*, b_i^*\right], \ \ 0 < \theta < 1. \tag{3.15}$$

Hence

$$\left|\frac{mf(x)}{2\left(I_{a+}^{\alpha}f\right)(x_2)} - \frac{mf(y)}{2\left(I_{a+}^{\alpha}f\right)(x_2)}\right| \leq m\theta\,\|x-y\| \leq \frac{\theta}{2}\,\|x-y\|, \ \ \text{all } x, y \in \prod_{i=1}^{k}\left[a_i^*, b_i^*\right]. \tag{3.16}$$

Set $\lambda = \frac{\theta}{2}$, it is $0 < \lambda < \frac{1}{2}$. We have that

$$|F(x) - F(y)| \leq \lambda\,\|x-y\|, \tag{3.17}$$

all $x, y \in \prod_{i=1}^{k}\left[a_i^*, b_i^*\right]$.

Equivalently we have

$$|Jf(x) - Jf(y)| \leq 2\lambda\left(I_{a+}^{\alpha}f\right)(x_2)\,\|x-y\|, \ \ \text{all } x, y \in \prod_{i=1}^{k}\left[a_i^*, b_i^*\right]. \tag{3.18}$$

We observe that

$$\left\|\left(F(y) - F(x)\right)\overrightarrow{i} - A(x)(y-x)\right\| \leq$$

$$|F(y) - F(x)| + |A(x)|\,\|y-x\| \leq \tag{3.19}$$

$$\lambda\,\|y-x\| + |A(x)|\,\|y-x\| = (\lambda + |A(x)|)\,\|y-x\| =: (\psi_1), \ \ \forall\, x, y \in \prod_{i=1}^{k}\left[a_i^*, b_i^*\right].$$

By [6], we have that

$$\left|\left(I_{a+}^{\alpha}f\right)(x)\right| \leq \left(\prod_{i=1}^{k}\frac{(b_i-a_i)^{\alpha_i}}{\Gamma(\alpha_i+1)}\right)\|f\|_{\infty}, \tag{3.20}$$

$\forall\ x \in \prod_{i=1}^{k}[a_i^*, b_i^*]$, where $\|\cdot\|_{\infty}$ now is over $\prod_{i=1}^{k}[a_i, b_i]$.

Hence

$$|A(x)| = \frac{\left|\left(I_{a+}^{\alpha}f\right)(x)\right|}{2\left(I_{a+}^{\alpha}f\right)(x_2)} \leq \frac{1}{2\left(I_{a+}^{\alpha}f\right)(x_2)}\left(\prod_{i=1}^{k}\frac{(b_i-a_i)^{\alpha_i}}{\Gamma(\alpha_i+1)}\right)\|f\|_{\infty} < \infty, \tag{3.21}$$

$\forall\ x \in \prod_{i=1}^{k}[a_i^*, b_i^*]$.

Therefore we get

$$(\psi_1) \leq \left(\lambda + \frac{1}{2\left(I_{a+}^{\alpha}f\right)(x_2)}\left(\prod_{i=1}^{k}\frac{(b_i-a_i)^{\alpha_i}}{\Gamma(\alpha_i+1)}\right)\|f\|_{\infty}\right)\|y-x\|, \tag{3.22}$$

$\forall\ x, y \in \prod_{i=1}^{k}[a_i^*, b_i^*]$.

Call

$$0 < \gamma_1 := \lambda + \frac{1}{2\left(I_{a+}^{\alpha}f\right)(x_2)}\left(\prod_{i=1}^{k}\frac{(b_i-a_i)^{\alpha_i}}{\Gamma(\alpha_i+1)}\right)\|f\|_{\infty}, \tag{3.23}$$

and by choosing $(b_i - a_i)$ small enough, $i = 1, ..., k$, we can make $\gamma_1 \in (0,1)$, fulfilling (3.2).

Next, we call and we need that

$$0 < \gamma := \gamma_0 + \gamma_1 = \left(1 - \frac{\left(I_{a+}^{\alpha}f\right)(x_1)}{2\left(I_{a+}^{\alpha}f\right)(x_2)}\right) +$$

$$\left(\lambda + \frac{1}{2\left(I_{a+}^{\alpha}f\right)(x_2)}\left(\prod_{i=1}^{k}\frac{(b_i-a_i)^{\alpha_i}}{\Gamma(\alpha_i+1)}\right)\|f\|_{\infty}\right) < 1, \tag{3.24}$$

equivalently,

$$\lambda + \frac{1}{2\left(I_{a+}^{\alpha}f\right)(x_2)}\left(\prod_{i=1}^{k}\frac{(b_i-a_i)^{\alpha_i}}{\Gamma(\alpha_i+1)}\right)\|f\|_{\infty} < \frac{\left(I_{a+}^{\alpha}f\right)(x_1)}{2\left(I_{a+}^{\alpha}f\right)(x_2)}, \tag{3.25}$$

equivalently,

$$2\lambda\left(I_{a+}^{\alpha}f\right)(x_2) + \left(\prod_{i=1}^{k}\frac{(b_i-a_i)^{\alpha_i}}{\Gamma(\alpha_i+1)}\right)\|f\|_{\infty} < \left(I_{a+}^{\alpha}f\right)(x_1), \tag{3.26}$$

which is possible for small $\lambda$ and small $(b_i - a_i)$, all $i = 1, ..., k$. That is $\gamma \in (0,1)$, fulfilling (3.3). So our numerical method converges and solves (3.9).

II) Let $\alpha = (\alpha_1, ..., \alpha_k)$, $\alpha_i > 0$, $m_i = \lceil \alpha_i \rceil$ ($\lceil \cdot \rceil$ ceiling function), $\alpha_i \notin \mathbb{N}$, $i = 1, ..., k \in \mathbb{N}$, and $G \in C^{\sum_{i=1}^{k} m_i - 1}\left(\prod_{i=1}^{k} [a_i, b_i]\right)$, such that

$$0 \neq \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} ... \partial x_k^{m_k}} \in L_\infty\left(\prod_{i=1}^{k} [a_i, b_i]\right).$$

Here we consider the multivariate left Caputo type fractional mixed partial derivative of order $\alpha$:

$$D_{*a}^{\alpha} G(x) = \frac{1}{\prod_{i=1}^{k} \Gamma(m_i - \alpha_i)} \int_{a_1}^{x_1} ... \int_{a_k}^{x_k} \prod_{i=1}^{k} (x_i - t_i)^{m_i - \alpha_i - 1} \cdot \tag{3.27}$$

$$\frac{\partial^{\sum_{i=1}^{k} m_i} G(t_1, ..., t_k)}{\partial t_1^{m_1} ... \partial t_k^{m_k}} dt_1 ... dt_k,$$

where again $\Gamma$ is the gamma function, $a = (a_1, ..., a_k)$, $\forall\ x = (x_1, ..., x_k) \in \prod_{i=1}^{k} [a_i, b_i]$. Notice here that $m_i - \alpha_i > 0$, $i = 1, ..., k$.

By [6], we get that $D_{*a}^{\alpha} G$ is a continuous function on $\prod_{i=1}^{k} [a_i, b_i]$, and it holds that $D_{*a}^{\alpha} G(a) = 0$.

In particular $D_{*a}^{\alpha} G$ is continuous on $\prod_{i=1}^{k} [a_i^*, b_i^*]$, where $a_i < a_i^* < b_i^* < b_i$, $i = 1, ..., k$.

Therefore there exist $x_1, x_2 \in \prod_{i=1}^{k} [a_i^*, b_i^*]$ such that

$$\begin{aligned} (D_{*a}^{\alpha} G)(x_1) &= \min (D_{*a}^{\alpha} G)(x), \\ (D_{*a}^{\alpha} G)(x_2) &= \max (D_{*a}^{\alpha} G)(x), \end{aligned} \tag{3.28}$$

over all $x \in \prod_{i=1}^{k} [a_i^*, b_i^*]$.

We assume that

$$(D_{*a}^{\alpha} G)(x_1) > 0. \tag{3.29}$$

Hence

$$\|D_{*a}^{\alpha} G\|_{\infty, \prod_{i=1}^{k} [a_i^*, b_i^*]} = (D_{*a}^{\alpha} G)(x_2) > 0. \tag{3.30}$$

Here we define

$$JG(x) = mG(x), \quad 0 < m < \frac{1}{2}, \tag{3.31}$$

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for any $x \in \prod_{i=1}^{k} [a_i^*, b_i^*]$.

Therefore the equation

$$JG(x) = 0, \quad x \in \prod_{i=1}^{k} [a_i^*, b_i^*], \tag{3.32}$$

has the same solutions as the equation

$$F(x) := \frac{JG(x)}{2D_{*a}^{\alpha} G(x_2)} = 0, \quad x \in \prod_{i=1}^{k} [a_i^*, b_i^*]. \tag{3.33}$$

Notice that

$$D_{*a}^{\alpha} \left( \frac{G(x)}{2D_{*a}^{\alpha} G(x_2)} \right) = \frac{D_{*a}^{\alpha} G(x)}{2D_{*a}^{\alpha} G(x_2)} \leq \frac{1}{2} < 1, \quad x \in \prod_{i=1}^{k} [a_i^*, b_i^*]. \tag{3.34}$$

We call

$$A(x) := \frac{D_{*a}^{\alpha} G(x)}{2D_{*a}^{\alpha} G(x_2)}, \ \forall\, x \in \prod_{i=1}^{k} [a_i^*, b_i^*]. \tag{3.35}$$

We notice that

$$0 < \frac{D_{*a}^{\alpha} G(x_1)}{2D_{*a}^{\alpha} G(x_2)} \leq A(x) \leq \frac{1}{2}. \tag{3.36}$$

Hence, the first condition (3.1) is fulfilled by

$$|1 - A(x)| = 1 - A(x) \leq 1 - \frac{D_{*a}^{\alpha} G(x_1)}{2D_{*a}^{\alpha} G(x_2)} =: \gamma_0, \ \forall\, x \in \prod_{i=1}^{k} [a_i^*, b_i^*]. \tag{3.37}$$

Hence

$$\|1 - A(x)\|_{\infty} \leq \gamma_0, \tag{3.38}$$

where $\|\cdot\|_{\infty}$ is over $\prod_{i=1}^{k} [a_i^*, b_i^*]$.

Clearly $\gamma_0 \in (0, 1)$.

Next we assume that $\frac{G(x)}{2(D_{*a}^{\alpha} G)(x_2)}$ is a contraction, that is

$$\left| \frac{G(x)}{2(D_{*a}^{\alpha} G)(x_2)} - \frac{G(y)}{2D_{*a}^{\alpha} G(x_2)} \right| \leq \theta \|x - y\|, \quad \text{all } x, y \in \prod_{i=1}^{k} [a_i^*, b_i^*], \tag{3.39}$$

with $0 < \theta < 1$.

Hence

$$\left| \frac{mG(x)}{2(D_{*a}^{\alpha} G)(x_2)} - \frac{mG(y)}{2(D_{*a}^{\alpha} G)(x_2)} \right| \leq m\theta \|x - y\| \leq \frac{\theta}{2} \|x - y\|, \tag{3.40}$$

all $x, y \in \prod_{i=1}^{k} [a_i^*, b_i^*]$.

Set $\lambda = \frac{\theta}{2}$, it is $0 < \lambda < \frac{1}{2}$. We have that

$$|F(x) - F(y)| \leq \lambda \|x - y\|, \tag{3.41}$$

all $x, y \in \prod_{i=1}^{k} [a_i^*, b_i^*]$.

Equivalently we have

$$|JG(x) - JG(y)| \leq 2\lambda (D_{*a}^{\alpha} G)(x_2) \|x - y\|, \quad \text{all } x, y \in \prod_{i=1}^{k} [a_i^*, b_i^*]. \tag{3.42}$$

We observe that

$$\left\| (F(y) - F(x)) \overrightarrow{i} - A(x)(y - x) \right\| \leq$$

$$|F(y) - F(x)| + |A(x)| \|y - x\| \leq$$

$$\lambda \|y - x\| + |A(x)| \|y - x\| = (\lambda + |A(x)|) \|y - x\| =: (\psi_2), \tag{3.43}$$

$\forall\ x, y \in \prod_{i=1}^{k} [a_i^*, b_i^*]$.

By (3.27), we notice that

$$|D_{*a}^{\alpha} G(x)| \leq \frac{1}{\prod_{i=1}^{k} \Gamma(m_i - \alpha_i)} \cdot$$

$$\left( \int_{a_1}^{x_1} ... \int_{a_k}^{x_k} \prod_{i=1}^{k} (x_i - t_i)^{m_i - \alpha_i - 1} dt_1 ... dt_k \right) \left\| \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} ... \partial x_k^{m_k}} \right\|_{\infty}$$

$$= \frac{1}{\prod_{i=1}^{k} \Gamma(m_i - \alpha_i)} \left( \prod_{i=1}^{k} \frac{(x_i - a_i)^{m_i - \alpha_i}}{m_i - \alpha_i} \right) \left\| \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} ... \partial x_k^{m_k}} \right\|_{\infty}$$

$$= \left( \prod_{i=1}^{k} \frac{(x_i - a_i)^{m_i - \alpha_i}}{\Gamma(m_i - \alpha_i + 1)} \right) \left\| \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} ... \partial x_k^{m_k}} \right\|_{\infty}. \tag{3.44}$$

We have proved that

$$|D_{*a}^{\alpha} G(x)| \leq \left( \prod_{i=1}^{k} \frac{(b_i - a_i)^{m_i - \alpha_i}}{\Gamma(m_i - \alpha_i + 1)} \right) \left\| \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} ... \partial x_k^{m_k}} \right\|_{\infty}, \tag{3.45}$$

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$\forall\ x \in \prod_{i=1}^{k} [a_i^*, b_i^*]$, where $\|\cdot\|_\infty$ now is over $\prod_{i=1}^{k} [a_i, b_i]$.

Hence we get

$$|A(x)| \leq \frac{1}{2D_{*a}^{\alpha} G(x_2)} \left( \prod_{i=1}^{k} \frac{(b_i - a_i)^{m_i - \alpha_i}}{\Gamma(m_i - \alpha_i + 1)} \right) \left\| \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} ... \partial x_k^{m_k}} \right\|_\infty < \infty, \quad (3.46)$$

$\forall\ x \in \prod_{i=1}^{k} [a_i^*, b_i^*]$.

Therefore we obtain

$$(\psi_2) \leq \left( \lambda + \frac{1}{2D_{*a}^{\alpha} G(x_2)} \left( \prod_{i=1}^{k} \frac{(b_i - a_i)^{m_i - \alpha_i}}{\Gamma(m_i - \alpha_i + 1)} \right) \left\| \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} ... \partial x_k^{m_k}} \right\|_\infty \right) \|y - x\|, \quad (3.47)$$

$\forall\ x, y \in \prod_{i=1}^{k} [a_i^*, b_i^*]$.

Call

$$0 < \gamma_1 := \lambda + \frac{1}{2D_{*a}^{\alpha} G(x_2)} \left( \prod_{i=1}^{k} \frac{(b_i - a_i)^{m_i - \alpha_i}}{\Gamma(m_i - \alpha_i + 1)} \right) \left\| \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} ... \partial x_k^{m_k}} \right\|_\infty, \quad (3.48)$$

and by choosing $(b_i - a_i)$ small enough, $i = 1, ..., k$, we can make $\gamma_1 \in (0, 1)$, fulfilling (3.2).

Next we call and we need that

$$0 < \gamma := \gamma_0 + \gamma_1 = \left( 1 - \frac{D_{*a}^{\alpha} G(x_1)}{2D_{*a}^{\alpha} G(x_2)} \right) +$$

$$\left\{ \lambda + \frac{1}{2D_{*a}^{\alpha} G(x_2)} \left( \prod_{i=1}^{k} \frac{(b_i - a_i)^{m_i - \alpha_i}}{\Gamma(m_i - \alpha_i + 1)} \right) \left\| \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} ... \partial x_k^{m_k}} \right\|_\infty \right\} < 1, \quad (3.49)$$

equivalently,

$$\lambda + \frac{1}{2D_{*a}^{\alpha} G(x_2)} \left( \prod_{i=1}^{k} \frac{(b_i - a_i)^{m_i - \alpha_i}}{\Gamma(m_i - \alpha_i + 1)} \right) \left\| \frac{\partial^{\sum_{i=1}^{k} m_i} G}{\partial x_1^{m_1} ... \partial x_k^{m_k}} \right\|_\infty < \frac{D_{*a}^{\alpha} G(x_1)}{2D_{*a}^{\alpha} G(x_2)}, \quad (3.50)$$

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equivalently,

$$2\lambda D_{*a}^{\alpha}G\left(x_2\right)+\left(\prod_{i=1}^{k}\frac{\left(b_i-a_i\right)^{m_i-\alpha_i}}{\Gamma\left(m_i-\alpha_i+1\right)}\right)\left\|\frac{\partial^{\sum_{i=1}^{k}m_i}G}{\partial x_1^{m_1}...\partial x_k^{m_k}}\right\|_{\infty}<D_{*a}^{\alpha}G\left(x_1\right), \quad (3.51)$$

which is possible for small $\lambda$ and small $(b_i-a_i)$, all $i=1,...,k$. That is $\gamma\in(0,1)$, fulfilling (3.3). So our numerical method converges and solves (3.32).

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# Stability of delay-distributed virus dynamics model with cell-to-cell transmission and CTL immune response

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**Abstract**

In this paper, we study the stability analysis of a virus dynamics model with CTL immune response and with both cell-to-cell and virus-to-cell transmissions. The model contains three types of distributed time delays. The existence and global stability of all steady states of the model are determined by two parameters, the basic reproduction number ($R_0$) and the CTL immune response activation number ($R_1$). By using suitable Lyapunov functionals, we show that if $R_0 \leq 1$, then the infection-free steady state $E_0$ is globally asymptotically stable; if $R_1 \leq 1 < R_0$, then the CTL-inactivated infection steady state $E_1$ is globally asymptotically stable; if $R_1 > 1$, then the CTL-activated infection steady state $E_2$ is globally asymptotically stable. Numerical simulations are conducted to support the theoretical results.

**Keywords:** Virus dynamics; CTL immune response; Global stability; time delay; cell-to-cell transmission.

## 1 Introduction

During the past decades, several mathematical models have been proposed to describe the dynamical behavior of many human viruses such as HIV, HBV, HCV and HTLV-I (see e.g. [1]-[27]). Studying the global stability of the model's equilibria has become one of the most important features which help us to better understanding of the virus dynamics. Thus, several researchers have devoted extensive efforts to study the global stability of virus dynamics models (see e.g. [2]-[13]). All the above mentioned works focus on cell-free viral spread in a compartment such as the bloodstream. Recently, some viral infection models have been proposed to model both virus-to-cell and cell-to-cell transmissions (see [28]-[29]). The viral infection model with cell-to-cell transmission and distributed time delay has been proposed in [29] as:

$$\dot{T}(t) = \lambda - dT(t) - \beta_1 T(t)V(t) - \beta_2 T(t)T^*(t), \tag{1}$$

$$\dot{T}^*(t) = \int_0^\infty \left[\beta_1 T(t-s)V(t-s)ds + \beta_2 T(t-s)T^*(t-s)\right] f(s)e^{-\mu_1 s}ds - \mu_1 T^*(t), \tag{2}$$

$$\dot{V}(t) = bT^*(t-s)ds - cV(t), \tag{3}$$

where, $T(t)$, $T^*(t)$ and$V(t)$ are the concentrations of the uninfected cells which are susceptible to infection, infected cells that produces viruses, and free virus particles at time $t$, respectively; $\beta_1$ is the virus-to-cell infection rate constant; $\beta_2$ is the cell-to-cell infection rate constant; $\mu_1$ and $c$ are death rate constants of the infected cells and viruses, respectively; $b$ is the average number of viruses that bud out from an infected cell. $e^{-\mu_1 s}$ is the survival rate of infected cells during the time delay $s$, where $s$ is assumed to be distributed according to a probability distribution $f(s)$.

It is observed that, all the viral infection models with cell-to-cell transmission did not consider the effect of immune response. The immune response is universal and necessary to eliminate or control the disease after viral infection. The Cytotoxic T Lymphocyte (CTL) cells are responsible to attack and kill the infected cells. Several viral infection models have been introduced in the literature to model the CTL immune response to several diseases [23]-[27]. However, in [23]-[27], only virus-to-cell transmission has been considered. Therefore, our aim in this paper is to propose and analyze a delay-distributed virus dynamics model with virus-to-cell and cell-to-cell transmissions and takes into account the CTL immune response.

# 2 The model

In this section, we propose a virus dynamics model with cell-to-cell transmission and CTL immune response.

$$\dot{T}(t) = \lambda - dT(t) - \beta_1 T(t)V(t) - \beta_2 T(t)T^*(t), \tag{4}$$

$$\dot{T}^*(t) = \int_0^\infty f_1(s)e^{-\mu_1 s}\left[\beta_1 T(t-s)V(t-s) + \beta_2 T(t-s)T^*(t-s)\right] ds - \mu_1 T^*(t) - pT^*(t)Z(t), \tag{5}$$

$$\dot{V}(t) = b\int_0^\infty e^{-\mu_2 s} f_2(s)T^*(t-s)ds - cV(t), \tag{6}$$

$$\dot{Z}(t) = kT^*(t)Z(t) - qZ(t), \tag{7}$$

where, $Z(t)$ is the concentration of CTL immune cells at time $t$. The infected cells are killed by the CTL immune response with rate $pT^*(t)Z(t)$, where $p$ is constant. The CTLs are proliferated at a rate $kT^*(t)Z(t)$ and die at a rate $qZ(t)$. All the other variables and parameters of the model have the same meanings as given in (1)-(3).

Let us assume that the probability distribution function $f_i(s)$ satisfy $f_i(s) > 0$, $i = 1, 2$ and

$$\int_0^\infty f_i(s)ds = 1, \quad \int_0^\infty f_i(u)e^{\ell u}du < \infty, \quad i = 1, 2,$$

where $\ell > 0$. Denote

$$\eta_i = \int_0^\infty f_i(s)e^{-\mu_i s}ds, \quad i = 1, 2.$$

Thus $0 < \eta_i \le 1$. Define the Banach space of fading memory type

$$C_\alpha = \{\phi \in C((-\infty, 0], \mathbb{R}) : \phi(\theta)e^{\alpha\theta} \text{ is uniformly continuous for } \theta \in (-\infty, 0] \text{ and } \|\phi\| < \infty\}$$

where $\alpha$ is a positive constant and $\|\phi\| = \sup_{\theta\le 0} |\phi(\theta)|\, e^{\alpha\theta}$. Let

$$C_\alpha^+ = \{\phi \in C_\alpha : \phi(\theta) \ge 0 \ for\ \theta \in (-\infty, 0]\}.$$

The initial conditions for system (4)-(7) are given as:

$$\begin{aligned} &T(\theta) = \varphi_1(\theta),\ T^*(\theta) = \varphi_2(\theta), V(\theta) = \varphi_3(\theta),\ Z(\theta) = \varphi_4(\theta), \quad \text{for } \theta \in (-\infty, 0], \\ &\varphi_i \in C_\alpha^+,\ \ i = 1, ..., 4. \end{aligned} \tag{8}$$

By the fundamental theory of functional differential equations [33], system (4)-(7) with initial conditions (8) has a unique solution.

## 2.1 Non-negativity and boundedness of solutions

We show the non-negativity and boundedness of the solutions of model (4)-(7).

**Lemma 1**. The solutions $(T(t), T^*(t), V(t), Z(t))$ of model (4)-(7) with initial conditions (8) are non-negative and ultimately bounded.

**Proof**: First we prove $T(t) > 0$ for all $t \geq 0$. Assume the contrary and let $t_1 > 0$ such that $T(t_1) = 0$. Then from Eq. (4), we have $\dot{T}(t_1) = \lambda > 0$. Therefore $T(t) < 0$ for $t \in (t_1 - \varepsilon, t_1)$ and $\varepsilon > 0$ is sufficiently small. This contradicts with the fact of $T(t) > 0$ for $t \in [0, t_1)$. It follows that $T(t) > 0$ for $t \geq 0$. From Eqs. (5)-(7), we have

$$\begin{aligned}
T^*(t) &= \varphi_2(0)e^{-\int_0^t(\mu_1+pZ(\zeta))d\zeta} \\
&\quad + \int_0^t e^{-\int_\eta^t(\mu_1+pZ(\zeta))d\zeta} \int_0^\infty f_1(s)e^{-\mu_1 s}\left[\beta_1 T(\eta-s)V(\eta-s) + \beta_2 T(\eta-s)T^*(\eta-s)\right] ds d\eta, \\
V(t) &= \varphi_3(0)e^{-ct} + b\int_0^t e^{-c(t-\zeta)} \int_0^\infty f_2(s)e^{-\mu_2 s}T^*(\zeta-s)dsd\zeta, \\
Z(t) &= \varphi_4(0)e^{-\int_0^t(q-kT^*(\zeta))d\zeta},
\end{aligned}$$

which yield that $T^*(t) \geq 0$, $V(t) \geq 0$ and $Z(t) \geq 0$for all $t \geq 0$.

Next we show the boundedness of the solutions. From Eq. (7) we have $\lim\limits_{t\to\infty} \sup T(t) \leq \dfrac{\lambda}{d}$. Let

$$F(t) = \int_0^\infty f_1(s)e^{-\mu_1 s}T(t-s)ds + T^*(t) + \frac{p}{k}Z(t).$$

Then

$$\begin{aligned}
\dot{F}(t) &= \int_0^\infty f_1(s)e^{-\mu_1 s}\left[\lambda - dT(t-s) - \beta_1 T(t-s)V(t-s) - \beta_2 T(t-s)T^*(t-s)\right] ds - \\
&\quad + \int_0^\infty \beta_1 T(t-s)V(t-s)f_1(s)e^{-\mu_1 s}ds + \int_0^\infty \beta_2 T(t-s)T^*(t-s)f_1(s)e^{-\mu_1 s}ds - \mu_1 T^*(t) - \frac{pq}{k}Z(t) \\
&= \lambda\eta_1 - d\int_0^\infty f_1(s)e^{-\mu_1 s}T(t-s)ds - \mu_1 T^*(t) - \frac{pq}{k}Z(t) \\
&\leq \lambda - \sigma\left(\int_0^\infty f_1(s)e^{-\mu_1 s}T(t-s)ds + T^*(t) + \frac{p}{k}Z(t)\right) = \lambda - \sigma F(t),
\end{aligned}$$

where, $\sigma = \min\{d, \mu_1, q\}$. Hence, $\limsup_{t\to\infty} F(t) \leq \dfrac{\lambda}{\sigma}$. Since $\int_0^\infty f_1(s)e^{-\mu_1 s}T(t-s)ds > 0$ , $T^* \geq 0$ and $Z \geq 0$, then $\limsup_{t\to\infty} T^*(t) \leq L_1$ and $\limsup_{t\to\infty} Z(t) \leq L_2$, where $L_1 = \dfrac{\lambda}{\sigma}$ and $L_2 = \frac{k}{p}L_1$. From Eq. (6) we have

$$\dot{V} = b\int_0^\infty e^{-\mu_2 s}f_2(s)T^*(t-s)ds - cV(t) \leq b\eta_2 L_1 - cV(t) \leq bL_1 - cV(t).$$

Thus $\limsup\limits_{t\to\infty} V(t) \leq L_3$, where $L_3 = \dfrac{bL_1}{c}$. Therefore, $T(t), T^*(t), V(t)$ and $Z(t)$ are ultimately bounded. □

## 2.2 Steady States

**Lemma 1.**

(i) If $R_0 \leq 1$, then there exists only positive steady state $E_0$,

(i) if $R_1 \leq 1 < R_0$, then there exist only two positive steady states $E_0$ and $E_1$,

(ii) if $R_1 > 1$, then there exist three positive steady states $E_0$, $E_1$ and $E_2$.

**The proof.** Let the R.H.S of system (4)-(7) be equal zero

$$0 = \lambda - dT - \beta_1 TV - \beta_2 TT^*, \tag{9}$$

$$0 = \eta_1\left(\beta_1 TV + \beta_2 TT^*\right) - \mu_1 T^* - pT^*Z, \tag{10}$$

$$0 = \eta_2 bT^* - cV, \tag{11}$$

$$0 = kT^*Z - qZ. \tag{12}$$

Solving Eqs. (9)-(12) we find that the system has three steady states, infection-free steady state $E_0 = (T_0, 0, 0, 0, 0)$, where $T_0 = \frac{\lambda}{d}$, CTL-inactivated infection steady state $E_1(T_1, T_1^*, V_1, 0)$ and CTL-activated infection steady state $E_2(T_2, T_2^*, V_2, Z_2)$, where

$$T_1 = \frac{T_0}{R_0}, \qquad T_1^* = \frac{dc}{(b\beta_1\eta_2 + c\beta_2)}(R_0 - 1), \qquad V_1 = \frac{b\eta_2 T_1^*}{c},$$
$$T_2 = \frac{k\lambda c}{kdc + bq\beta_1\eta_2 + q\beta_2 c}, \quad T_2^* = \frac{q}{k}, \quad V_2 = \frac{b\eta_2}{c}T_2^*, \quad Z_2 = \frac{\mu_1}{p}(R_1 - 1),$$

and

$$R_0 = \frac{T_0\eta_1}{\mu_1 c}(b\beta_1\eta_2 + \beta_2 c), \qquad R_1 = \frac{kdc}{q(b\beta_1\eta_2 + \beta_2 c) + kdc}R_0,$$

where $R_0$ represents the basic infection reproduction number which describes the average number of newly infected cells generated from one infected cell at the beginning of the infectious process and $R_1$ represents the immune response activation number which expresses the CTL load during the lifespan of a CTL cell. Clearly $R_0 > R_1$. □

## 2.3 Global stability analysis

In this section, we study the global stability of all the steady states of system (4)-(7) employing the method of Lyapunov function. We will use the follwing function $g(x) = x - 1 - \ln x$ and the notation $(T, T^*, V, Z) = (T(t), T^*(t), V(t), Z(t))$.

**Theorem 1.** If $R_0 \le 1$, then $E_0$ is GAS.

**Proof.** Define a Lyapunov functional $L$ as follows:

$$\begin{aligned} L(T, T^*, V, Z) &= T_0 g\left(\frac{T}{T_0}\right) + \frac{1}{\eta_1}T^* + \frac{\beta_1 T_0}{c}V + \frac{p}{\eta_1 k}Z \\ &+ \frac{1}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\int_0^s [\beta_1 T(t-\theta)V(t-\theta) + \beta_2 T(t-\theta)T^*(t-\theta)]d\theta ds \\ &+ \frac{b\beta_1 T_0}{c}\int_0^\infty f_2(s)e^{-\mu_2 s}\int_0^s T^*(t-\theta)d\theta ds. \end{aligned}$$

Calculating the derivative of $L$ along the solutions of the system (4)-(7), we obtain

$$\begin{aligned} \frac{dL}{dt} &= \left(1 - \frac{T_0}{T}\right)(\lambda - dT - \beta_1 TV - \beta_2 TT^*) \\ &+ \frac{1}{\eta_1}\left[\int_0^\infty f_1(s)e^{-\mu_1 s}[\beta_1 T(t-s)V(t-s) + \beta_2 T(t-s)T^*(t-s)]\,ds - \mu_1 T^* - pT^*Z\right] \\ &+ \frac{\beta_1 T_0}{c}\left[b\int_0^\infty e^{-\mu_2 s}f_2(s)T^*(t-s)ds - cV\right] + \frac{p}{\eta_1 k}[kT^*Z - qZ] \\ &+ \frac{1}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}[\beta_1 TV + \beta_2 TT^* - \beta_1 T(t-s)V(t-s) - \beta_2 T(t-s)T^*(t-s)]\,ds \\ &+ \frac{b\beta_1 T_0}{c}\int_0^\infty f_2(s)e^{-\mu_2 s}[T^* - T^*(t-s)]\,ds \\ &= \left(1 - \frac{T_0}{T}\right)(\lambda - dT) + \left(\beta_2 T_0 + \frac{b\beta_1 T_0\eta_2}{c} - \frac{\mu_1}{\eta_1}\right)T^* - \frac{pq}{\eta_1 k}Z \\ &= -d\frac{(T - T_0)^2}{T} + \frac{\mu_1}{\eta_1}(R_0 - 1)T^* - \frac{pq}{\eta_1 k}Z. \end{aligned} \tag{13}$$

If $R_0 \le 1$, then $\frac{dL}{dt} \le 0$ for all $T, T^*, Z > 0$. Thus the solutions of system (4)-(7) limit to $M$, the largest invariant subset of $\left\{(T, T^*, V, Z) : \frac{dL}{dt} = 0\right\}$. Clearly, it follows from Eq. (13) that $\frac{dL}{dt} = 0$ if and only if $T = T_0$,

$T^*=0$ and $Z=0$. Noting that $M$ is invariant, for each element of $M$ we have $T^*=0$ and $Z=0$, then $\dot{T}^*=0$. From Eq. (5) we drive that

$$0=\dot{T}^*=\int_0^\infty f_1(s)e^{-\mu_1 s}\beta_1 T_0 V(t-s)ds.$$

It follows that $V=0$. Hence $\frac{dL}{dt}=0$ if and only if $T=T_0$, $T^*=0$, $V=0$ and $Z=0$. LaSalle's invariance principle implies that $E_0$ is GAS when $R_0\le 1$. □

**Theorem 2.** If $R_1 \le 1 < R_0$ , then $E_1$ is GAS.

**Proof.** Define the following Lyapunov functional

$$\begin{aligned}
U(T,T^*,V,Z) &= T_1 g\left(\frac{T}{T_1}\right)+\frac{1}{\eta_1}T_1^* g\left(\frac{T^*}{T_1^*}\right)+\frac{\beta_1 T_1 V_1}{b\eta_2 T_1^*}V_1 g\left(\frac{V}{V_1}\right)+\frac{p}{\eta_1 k}Z\\
&+\frac{\beta_1 T_1 V_1}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\int_0^s g\left(\frac{T(t-\theta)V(t-\theta)}{T_1 V_1}\right)d\theta ds\\
&+\frac{\beta_2 T_1 T_1^*}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\int_0^s g\left(\frac{T(t-\theta)T^*(t-\theta)}{T_1 T_1^*}\right)d\theta ds\\
&+\frac{\beta_1 T_1 V_1}{\eta_2}\int_0^\infty f_2(s)e^{-\mu_2 s}\int_0^s g\left(\frac{T^*(t-\theta)}{T_1^*}\right)d\theta ds.
\end{aligned}$$

The time derivative of $U$ along the trajectories of (4)-(7) is given by

$$\begin{aligned}
\frac{dU}{dt} &= \left(1-\frac{T_1}{T}\right)(\lambda-dT-\beta_1 TV-\beta_2 TT^*)\\
&+\frac{1}{\eta_1}\left(1-\frac{T_1^*}{T^*}\right)\left(\int_0^\infty f_1(s)e^{-\mu_1 s}\left[\beta_1 T(t-s)V(t-s)+\beta_2 T(t-s)T^*(t-s)\right]ds-\mu_1 T^*-pT^*Z\right)\\
&+\frac{\beta_1 T_1 V_1}{b\eta_2 T_1^*}\left(1-\frac{V_1}{V}\right)\left(b\int_0^\infty f_2(s)e^{-\mu_2 s}T^*(t-s)ds-cV\right)+\frac{p}{\eta_1 k}(kT^*Z-qZ)\\
&+\frac{\beta_1 T_1 V_1}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\left(\frac{TV}{T_1 V_1}-\frac{T(t-s)V(t-s)}{T_1 V_1}+\ln\left(\frac{T(t-s)V(t-s)}{TV}\right)\right)ds\\
&+\frac{\beta_2 T_1 T_1^*}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\left(\frac{TT^*}{T_1 T_1^*}-\frac{T(t-s)T^*(t-s)}{T_1 T_1^*}+\ln\left(\frac{T(t-s)T^*(t-s)}{TT^*}\right)\right)ds\\
&+\frac{\beta_1 T_1 V_1}{\eta_2}\int_0^\infty f_2(s)e^{-\mu_2 s}\left(\frac{T^*}{T_1^*}-\frac{T^*(t-s)}{T_1^*}+\ln\left(\frac{T^*(t-s)}{T^*}\right)\right)ds. \qquad (14)
\end{aligned}$$

Collecting terms of Eq. (14) and applying the steady state condtions for $E_1$:

$$\lambda-dT_1=\beta_1 T_1 V_1+\beta_2 T_1 T_1^*=\frac{\mu_1}{\eta_1}T_1^*=\frac{c\mu_1}{b\eta_1\eta_2}V_1,$$

we get

$$\begin{aligned}
\frac{dU}{dt} &= -\frac{d}{T}(T-T_1)^2+(\beta_1 T_1 V_1+\beta_2 T_1 T_1^*)\left(1-\frac{T_1}{T}\right)\\
&-\frac{\beta_1 T_1 V_1}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\frac{T(t-s)V(t-s)T_1^*}{T_1 V_1 T^*}ds-\frac{\beta_2 T_1 T_1^*}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\frac{T(t-s)T^*(t-s)}{T_1 T^*}ds\\
&-\frac{\beta_1 T_1 V_1}{\eta_2}\int_0^\infty f_2(s)e^{-\mu_2 s}\frac{V_1 T^*(t-s)}{VT_1^*}ds+\frac{\beta_1 T_1 V_1}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\ln\left(\frac{T(t-s)V(t-s)}{TV}\right)ds\\
&+\frac{\beta_2 T_1 T_1^*}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\ln\left(\frac{T(t-s)T^*(t-s)}{TT^*}\right)ds\\
&+\frac{\beta_1 T_1 V_1}{\eta_2}\int_0^\infty f_2(s)e^{-\mu_2 s}\ln\left(\frac{T^*(t-s)}{T^*}\right)ds+\frac{p}{\eta_1}\left(T_1^*-\frac{q}{k}\right)Z+2\beta_1 T_1 V_1+\beta_2 T_1 T_1^*.
\end{aligned}$$

Consider the following equalities

$$\begin{aligned}
\ln\left(\frac{T(t-s)V(t-s)}{TV}\right) &= \ln\left(\frac{T(t-s)V(t-s)T_i^*}{T_iV_iT^*}\right) + \ln\left(\frac{T_i}{T}\right) + \ln\left(\frac{V_iT^*}{VT_i^*}\right), \\
\ln\left(\frac{T(t-s)T^*(t-s)}{TT^*}\right) &= \ln\left(\frac{T(t-s)T^*(t-s)}{T_iT^*}\right) + \ln\left(\frac{T_i}{T}\right), \\
\ln\left(\frac{T^*(t-s)}{T^*}\right) &= \ln\left(\frac{V_iT^*(t-s)}{VT_i^*}\right) + \ln\left(\frac{VT_i^*}{V_iT^*}\right), \qquad i=1,2. \qquad (15)
\end{aligned}$$

Using Eq. (15) with $i = 1$ we get

$$\begin{aligned}
\frac{dU}{dt} &= -\frac{d}{T}(T-T_1)^2 - (\beta_1T_1V_1 + \beta_2T_1T_1^*)\left(\frac{T_1}{T} - 1 - \ln\left(\frac{T_1}{T}\right)\right) \\
&- \frac{\beta_1T_1V_1}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\left[\frac{T(t-s)V(t-s)T_1^*}{T_1V_1T^*} - 1 - \ln\left(\frac{T(t-s)V(t-s)T_1^*}{T_1V_1T^*}\right)\right]ds \\
&- \frac{\beta_2T_1T_1^*}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\left[\frac{T(t-s)T^*(t-s)}{T_1T^*} - 1 - \ln\left(\frac{T(t-s)T^*(t-s)}{T_1T^*}\right)\right]ds \\
&- \frac{\beta_1T_1V_1}{\eta_2}\int_0^\infty f_2(s)e^{-\mu_2 s}\left[\frac{V_1T^*(t-s)}{VT_1^*} - 1 - \ln\left(\frac{V_1T^*(t-s)}{VT_1^*}\right)\right]ds + \frac{p}{\eta_1}\left(T_1^* - \frac{q}{k}\right)Z \\
&= -\frac{d}{T}(T-T_1)^2 - (\beta_1T_1V_1 + \beta_2T_1T_1^*)\,g\left(\frac{T_1}{T}\right) - \frac{\beta_1T_1V_1}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}g\left(\frac{T(t-s)V(t-s)T_1^*}{T_1V_1T^*}\right)ds \\
&- \frac{\beta_2T_1T_1^*}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}g\left(\frac{T(t-s)T^*(t-s)}{T_1T^*}\right)ds - \frac{\beta_1T_1V_1}{\eta_2}\int_0^\infty f_2(s)e^{-\mu_2 s}g\left(\frac{V_1T^*(t-s)}{VT_1^*}\right)ds \\
&+ \frac{p}{\eta_1}\frac{\beta_1bq\eta_2 + \beta_2qc + dck}{(\beta_1bq\eta_2 + \beta_2qc)k}(R_1-1)\,Z.
\end{aligned}$$

Hence, if $R_1 \le 1$, then we obtain that $\frac{dU}{dt} \le 0$ and then solutions of system (4)-(7) limit to $M$, the largest invariant subset of $\left\{(T, T^*, V, Z) : \frac{dU}{dt} = 0\right\}$. It can be seen that, $\frac{dU}{dt} = 0$ if and only if

$$\frac{T_1}{T} = \frac{T(t-s)V(t-s)T_1^*}{T_1V_1T^*} = \frac{T(t-s)T^*(t-s)}{T_1T^*} = \frac{V_1T^*(t-s)}{VT_1^*} = 1.$$

LaSalle's invariance principle implies the global stability of $E_1$. □

**Theorem 3.** If $R_1 > 1$, then $E_2$ is GAS.

Define the following Lyapunov functional

$$\begin{aligned}
W(T,T^*,V,Z) &= T_2g\left(\frac{T}{T_2}\right) + \frac{1}{\eta_1}T_2^*g\left(\frac{T^*}{T_2^*}\right) + \frac{\beta_1T_2V_2}{b\eta_2T_2^*}V_2g\left(\frac{V}{V_2}\right) + \frac{p}{\eta_1k}Z_2g\left(\frac{Z}{Z_2}\right) \\
&+ \frac{\beta_1T_2V_2}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\int_0^s g\left(\frac{T(t-\theta)V(t-\theta)}{T_2V_2}\right)d\theta ds \\
&+ \frac{\beta_2T_2T_2^*}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\int_0^s g\left(\frac{T(t-\theta)T^*(t-\theta)}{T_2T_2^*}\right)d\theta ds \\
&+ \frac{\beta_1T_2V_2}{\eta_2}\int_0^\infty f_2(s)e^{-\mu_2 s}\int_0^s g\left(\frac{T^*(t-\theta)}{T_2^*}\right)d\theta ds.
\end{aligned}$$

The time derivative of $W$ along the trajectories of (4)-(7) is given by

$$\begin{aligned}
\frac{dW}{dt} &= \left(1-\frac{T_2}{T}\right)(\lambda - dT - \beta_1 TV - \beta_2 TT^*) \\
&+ \frac{1}{\eta_1}\left(1-\frac{T_2^*}{T^*}\right)\left(\int_0^\infty f_1(s)e^{-\mu_1 s}\left(\beta_1 T(t-s)V(t-s) + \beta_2 T(t-s)T^*(t-s)\right)ds - \mu_1 T^* - pT^*Z\right) \\
&+ \frac{\beta_1 T_2 V_2}{b\eta_2 T_2^*}\left(1-\frac{V_2}{V}\right)\left(b\int_0^\infty f_2(s)e^{-\mu_2 s}T^*(t-s)ds - cV\right) + \frac{p}{\eta_1 k}\left(1-\frac{Z_2}{Z}\right)(kT^*Z - qZ) \\
&+ \frac{\beta_1 T_2 V_2}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\left(\frac{TV}{T_2V_2} - \frac{T(t-s)V(t-s)}{T_2V_2} + \ln\left(\frac{T(t-s)V(t-s)}{TV}\right)\right)ds \\
&+ \frac{\beta_2 T_2 T_2^*}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\left(\frac{TT^*}{T_2T_2^*} - \frac{T(t-s)T^*(t-s)}{T_2T_2^*} + \ln\left(\frac{T(t-s)T^*(t-s)}{TT^*}\right)\right)ds \\
&+ \frac{\beta_1 T_2 V_2}{\eta_2}\int_0^\infty f_2(s)e^{-\mu_2 s}\left(\frac{T^*}{T_2^*} - \frac{T^*(t-s)}{T_2^*} + \ln\left(\frac{T^*(t-s)}{T^*}\right)\right)ds.
\end{aligned}$$

Using the following steady state conditions for $E_2$

$$\lambda - dT_2 = \beta_1 T_2 V_2 + \beta_2 T_2 T_2^* = \frac{p}{\eta_1}T_2^* Z_2 + \frac{\mu_1}{\eta_1}T_2^*, \qquad T_2^* = \frac{q}{k}, \qquad V_2 = \frac{bq\eta_2}{ck},$$

we get

$$\begin{aligned}
\frac{dW}{dt} &= -\frac{d}{T}(T-T_2)^2 + (\beta_1 T_2V_2 + \beta_2 T_2 T_2^*)\left(1-\frac{T_2}{T}\right) \\
&- \frac{\beta_1 T_2 V_2}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\frac{T(t-s)V(t-s)T_2^*}{T_2V_2T^*}ds - \frac{\beta_2 T_2 T_2^*}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\frac{T(t-s)T^*(t-s)}{T_2T^*}ds \\
&- \frac{\beta_1 T_2 V_2}{\eta_2}\int_0^\infty f_2(s)e^{-\mu_2 s}\frac{V_2T^*(t-s)}{VT_2}ds + \frac{\beta_1 T_2 V_2}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\ln\left(\frac{T(t-s)V(t-s)}{TV}\right)ds \\
&+ \frac{\beta_2 T_2 T_2^*}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s}\ln\left(\frac{T(t-s)T^*(t-s)}{TT^*}\right)ds \\
&+ \frac{\beta_1 T_2 V_2}{\eta_2}\int_0^\infty f_2(s)e^{-\mu_2 s}\ln\left(\frac{T^*(t-s)}{T^*}\right)ds + 2\beta_1 T_2 V_2 + \beta_2 T_2 T_2^*.
\end{aligned}$$

Using Eq. (15) with $i=2$ we get

$$\begin{aligned}
\frac{dW}{dt} &= -\frac{d}{T}(T-T_2)^2 - (\beta_1 T_2V_2 + \beta_2 T_2T_2^*)\, g\left(\frac{T_2}{T}\right) - \frac{\beta_1 T_2 V_2}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s} g\left(\frac{T(t-s)V(t-s)T_2^*}{T_2V_2T^*}\right)ds \\
&- \frac{\beta_2 T_2 T_2^*}{\eta_1}\int_0^\infty f_1(s)e^{-\mu_1 s} g\left(\frac{T(t-s)T^*(t-s)}{T_2T^*}\right)ds - \frac{\beta_1 T_2 V_2}{\eta_2}\int_0^\infty f_2(s)e^{-\mu_2 s} g\left(\frac{V_2T^*(t-s)}{VT_2^*}\right)ds.
\end{aligned}$$

Noting that $T, T^*, V, Z > 0$, we have that $\frac{dW}{dt} \le 0$. The solutions of model (4)-(7) converge to $M$, the largest invariant subset of $\left\{(T, T^*, V, Z) : \frac{dW}{dt} = 0\right\}$. We have $\frac{dW}{dt} = 0$ if and only if $T = T_2$ and $g = 0$ i.e.,

$$\frac{T_2}{T} = \frac{T(t-s)V(t-s)T_2^*}{T_2V_2T^*} = \frac{T(t-s)T^*(t-s)}{T_2T^*} = \frac{V_2T^*(t-s)}{VT_2^*} = 1. \tag{16}$$

If $T = T_2$, then from Eq. (16) we get $T^* = T_2^*$ and $V = V_2$. The set $M$ is invariant and for any element belongs to $M$ satisfies $T^* = T_2^*$ and

$$\dot{T}^* = 0 = \eta_1(\beta_1 T_2 V_2 + \beta_2 T_2 T_2^*) - \mu_1 T_2^* - pT_2^* Z,$$

which gives $Z = Z_2$. Therefore, $\frac{dW}{dt} = 0$ if and only if $T = T_2, T^* = T_2^*, V = V_2$ and $Z = Z_2$. The global asymptotic stability of $E_2$ follows from LaSalle's invariance principle. □

# 3 Numerical simulations

In this section, we perform numerical simulations for the model (4)-(7) with particular distribution functions $f_1(s)$ and $f_2(s)$ as:

$$f_1(s) = \delta(s - s_1), \;\; f_2(s) = \delta(s - s_2),$$

where $\delta(.)$ is the dirac delta function, $s_1$ and $s_2$ are positive constants. Then, we can see that,

$$\int_0^\infty f_i(s)ds = 1, \quad \eta_i = \int_0^\infty \delta(s - s_i)e^{-\mu_i s}ds = e^{-\mu_i s_i}, \; i = 1, 2,$$

$$\int_0^\infty \delta(s - s_1)e^{-\mu s}\phi(t - s)ds = e^{-\mu s_1}\phi(t - s_1),$$

for any function $\phi$. With such choice, model (4)-(7) leads to:

$$\begin{aligned}
\dot{T}(t) &= \lambda - dT(t) - \beta_1 T(t)V(t) - \beta_2 T(t)T^*(t), && (17)\\
\dot{T}^*(t) &= \left[\beta_1 T(t - s_1)V(t - s_1) + \beta_2 T(t - s_1)T^*(t - s_1)\right] e^{-\mu_1 s_1} - \mu_1 T^*(t) - pT^*(t)Z(t), &&\\
\dot{V}(t) &= be^{-\mu_2 s_2}T^*(t - s_2) - cV(t), && (18)\\
\dot{Z}(t) &= kT^*(t)Z(t) - qZ(t). && (19)
\end{aligned}$$

The parameters $R_0$ and $R_1$ become $R_0 = \dfrac{e^{-\mu_1 s_1}\lambda(b\beta_1 e^{-\mu_2 s_2} + \beta_2 c)}{c\mu_1 d}$, $R_1 = \dfrac{kdc}{q(b\beta e^{-\mu_2 s_2} + \beta_2 c) + kdc}R_0$.

Now we perform some numerical simulations for model (17)-(19) with parameters values given in Table 1.

Table 1: The values of the parameters of model (17 )-(19).

| Parameter | Value | Parameter | Value |
|---|---|---|---|
| $\lambda$ | 10 | $c$ | 3 |
| $d$ | 0.01 | $q$ | 0.1 |
| $p$ | 0.1 | $\beta_2$ | 0.0001 |
| $b$ | 10 | $\mu_1$ | 0.9 |
| $s_1$ | Varied | $\mu_2$ | 0.1 |
| $s_2$ | Varied | $\beta_1, k$ | Varied |

## 3.1 Effect of the parameters $\beta_1$ and $k$ on the stability of steady states

To show the global stablity of the steady states we consider three different initial conditions:

IC1: $\varphi_1(\theta) = 600$, $\varphi_2(\theta) = 1, \varphi_3(\theta) = 1, \varphi_4(\theta) = 10$,

IC2: $\varphi_1(\theta) = 200$, $\varphi_2(\theta) = 0.5, \varphi_3(\theta) = 3, \varphi_4(\theta) = 5$,

IC3: $\varphi_1(\theta) = 700$, $\varphi_2(\theta) = 5, \varphi_3(\theta) = 9, \varphi_4(\theta) = 12$,

where, $\theta \in [-\max\{s_1, s_2\}, 0]$.

In this case we choose $s_1 = 0.5$, $s_2 = 0.9$ and study the following subcases:

**(i)**: $R_0 < 1$. We choose, $\beta_1 = 0.0001$ and $k = 0.008$, then we compute $R_0 = 0.295489$ and $R_1 = 0.194228$. From Lemma 2 we have that the system has one steady state $E_0$. From Figures 1-4 we can see that, the concentration of uninfected cells is increasing and tends its normal value $\lambda/d = 1000$, while the concentrations of infected cells, free viruses and CTLs are decaying and approaching zero. It means that, $E_0$ is GAS and the virus will be removed. This result support the result of Theorem 1.

**(ii)**: $R_1 \leq 1 < R_0$. We choose $\beta_1 = 0.001$ and $k = 0.008$, and then, $R_0 = 2.317257$ and $R_1 = 0.455395$. Lemma 2 state that the system has two steady states $E_0$ and $E_1$. Figures 5-8 show that the numerical results are consistent with Theorem 2. We can see that, the solution of the system converges to the steady $E_1(431.54, 4.03, 12.77, 0)$ for the initial conditions IC1-IC3.

**(iii)**: $R_1 > 1$: In this case, we choose $\beta_1 = 0.001$ and $k = 0.03$ and then $R_1 = 1.108600 > 1$. According to Lemma 2, the system has three steady states $E_0, E_1$ and $E_2$. From Figures 9-12 we can see that, the solutions of the system approach the steady state $E_2(478.41, 3.33, 10.57, 0.98)$ for large $t$ and for the initial conditions IC1-IC3. This support the result of Theorems 3.

## 3.2 Effect of the time delays on the stability of steady states

In this case, we consider the initial condition IC2. We take the values $\beta_1 = 0.001$ and $k = 0.03$. Without loss of generality we let $S = s_1 = s_2$. In Table 2, we present the values of $R_0$, $R_1$ and the steady states of system (17 )-(19) with different values of $S$.

From Table 2 we can see that, the values of $R_0$ and $R_1$ are decreased as $S$ is increased. Using the values of the parameters given in Table 1, we obtain that the following:

(i) if $0 \leq S < 0.8447$, then $E_2$ exists and it is GAS,

(ii) if $0.8447 \leq S < 0.8868$, then $E_1$ exists and it is GAS,

(iii) if $S \geq 0.8868$, then $E_0$ is GAS.

Figures 13-16 show that the numerical results are also compatible with the results of Theorems 1-3. From a biological point of view, the intracellular delay plays a similar role as an antiviral treatment in eliminating the virus. We observe that, sufficiently large delay suppresses viral replication and clears the virus. This gives us some suggestions on new drugs to prolong the increase the intracellular delay period.

Table 2: The values of steady states, $R_0$ and $R_1$ for model (17)-(19) with different values of the delay parameter $S$.

| Delay parameter | Steady states | $R_0$ | $R_1$ |
|---|---|---|---|
| $S = 0.0$ | $E_2(578.98, 1.11, 1.23, 27.89)$ | 6.54 | 3.79 |
| $S = 0.2$ | $E_2(626.03, 1.11, 1.01, 17.56)$ | 4.40 | 2.76 |
| $S = 0.4$ | $E_2(670.65, 1.11, 0.83, 9.87)$ | 2.96 | 1.99 |
| $S = 0.7$ | $E_2(731.69, 1.11, 0.61, 1.99)$ | 1.64 | 1.2 |
| $S = 0.80$ | $E_2(751.30, 1.11, 0.55, 0)$ | 1.33 | 1 |
| $S = 0.9$ | $E_1(904.23, 0.39, 0.18, 0)$ | 1.11 | 0.85 |
| $S = 0.95$ | $E_0(1000, 0, 0, 0)$ | 1 | 0.78 |
| $S = 1$ | $E_0(1000, 0, 0, 0)$ | 0.91 | 0.71 |
| $S = 1.5$ | $E_0(1000, 0, 0, 0)$ | 0.34 | 0.29 |
| $S = 2$ | $E_0(1000, 0, 0, 0)$ | 0.13 | 0.12 |

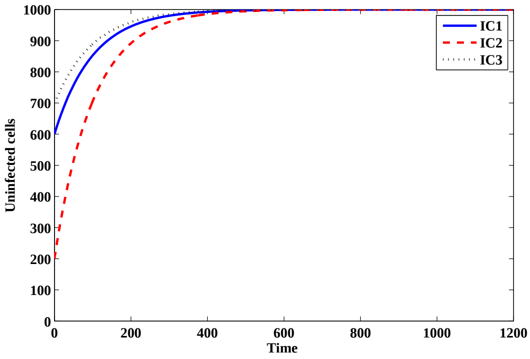


Figure 1: The evolution of uninfected cells with initial IC1-IC3 in case of $R_0 \leq 1$.

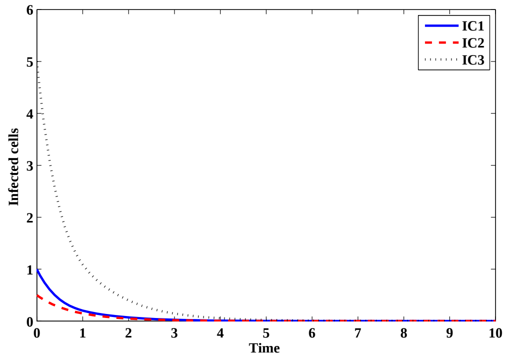


Figure 2: The evolution of infected cells with initial IC1-IC3 in case of $R_0 \leq 1$.

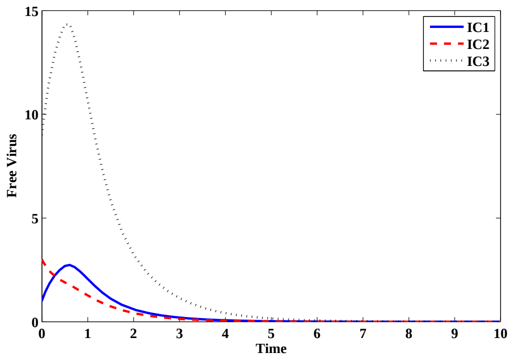


Figure 3: The evolution of free viruses with initial IC1-IC3 in case of $R_0 \leq 1$.

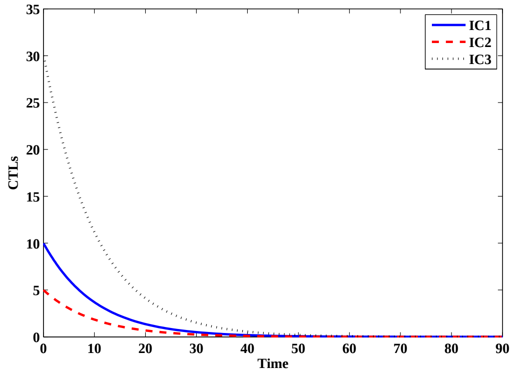


Figure 4: The evolution of CTLs with initial IC1-IC3 in case of $R_0 \leq 1$.

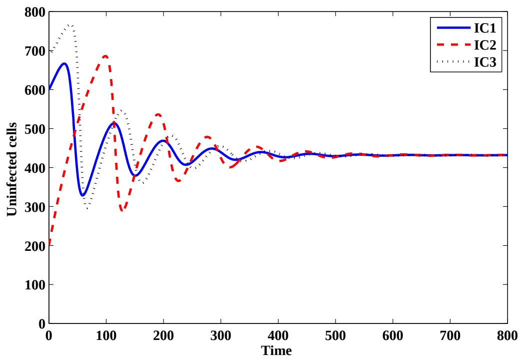


Figure 5: The evolution of uninfected cells with initial IC1-IC3 in case of $R_1 \leq 1 < R_0$.

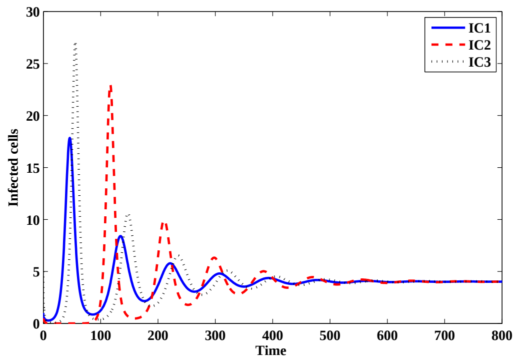


Figure 6: The evolution of infected cells with initial IC1-IC3 in case of $R_1 \leq 1 < R_0$.

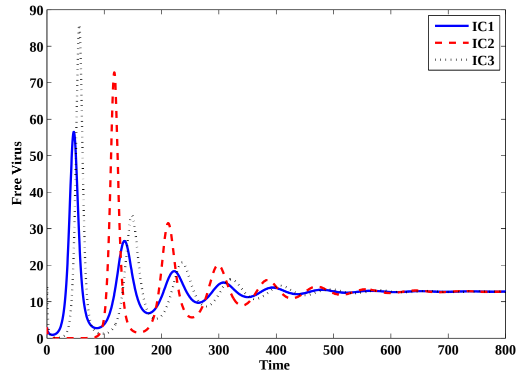


Figure 7: The evolution of free viruses with initial IC1-IC3 in case of $R_1 \le 1 < R_0$.

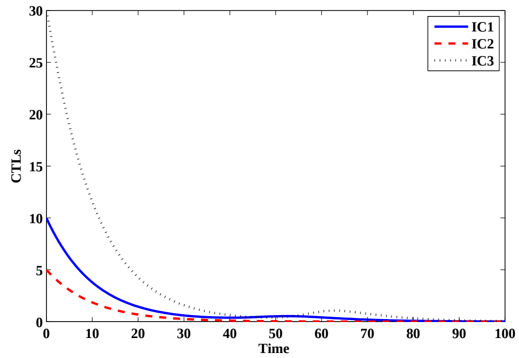


Figure 8: The evolution of CTLs with initial IC1-IC3 in case of $R_1 \le 1 < R_0$.

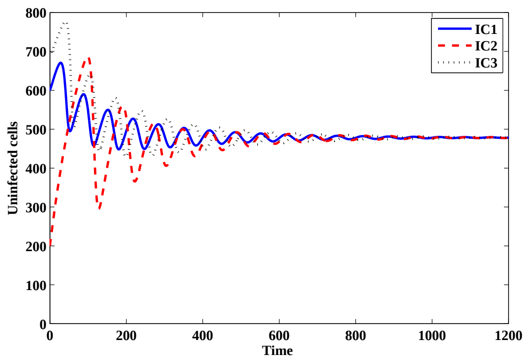


Figure 9: The evolution of uninfected cells with initial IC1-IC3 in case of $R_1 > 1$.

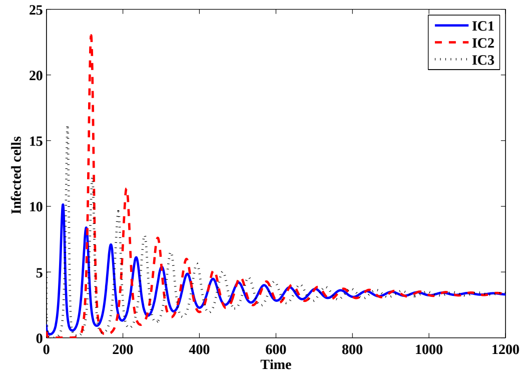


Figure 10: The evolution of infected cells with initial IC1-IC3 in case of $R_1 > 1$.

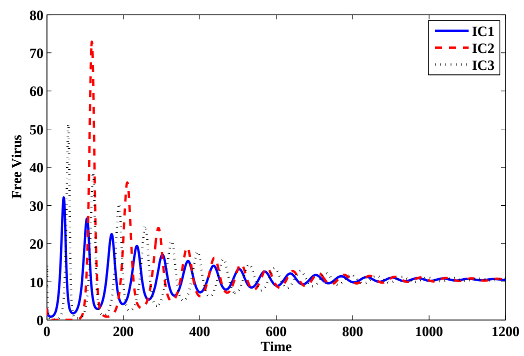


Figure 11: The evolution of free viruses with initial IC1-IC3 in case of $R_1 > 1$.

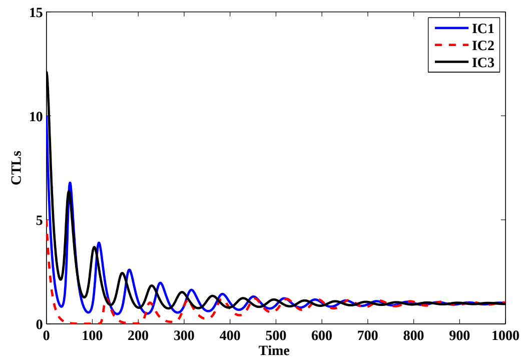


Figure 12: The evolution of CTLs with initial IC1-IC3 in case of $R_1 > 1$.

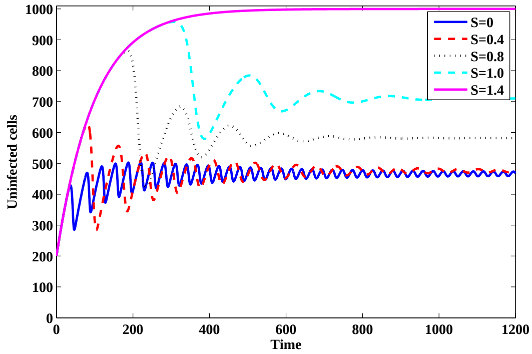


Figure 13: The evolution of uninfected cells with different delay parameter $S$.

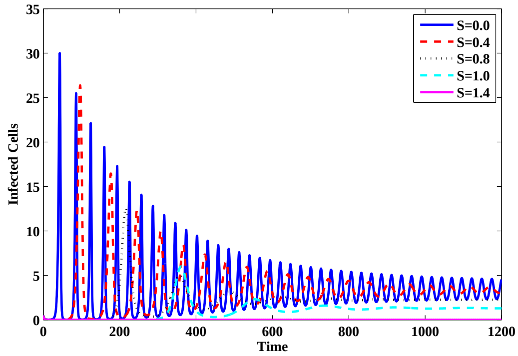


Figure 14: The evolution of infected cells with different delay parameter $S$.

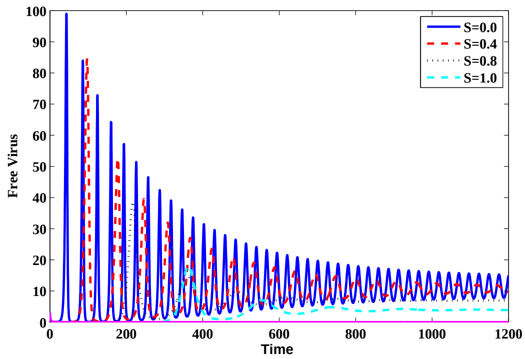


Figure 15: The evolution of free viruses with different delay parameter $S$.

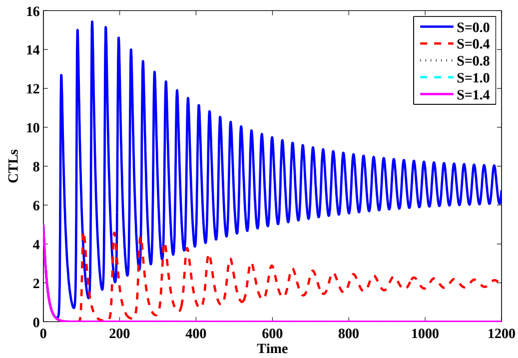


Figure 16: The evolution of CTLs with different delay parameter $S$.

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# Dynamical behavior of HIV-1 infection with saturated virus-target and infected-target incidences and delays

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**Abstract**

This paper study the dynamical behavior of HIV-1 infection model with saturated virus-target and infected-target incidences. The model is incorporated by two types of intracellular distributed time delays. The model generalizes all the existing HIV-1 infection models with cell-to-cell transmission presented in the literature by considering saturated incidence rate. The nonnegativity and boundedness of the solutions of the model as well as global stability of the steady states are studied. The global stability are established using Lyapunov method. Using MATLAB we conduct some numerical simulations to confirm our results. The effect of the saturated incidence of the HIV-1 dynamics is shown.

**Keywords:** HIV-1 dynamics; Global stability; time delay; cell-to-cell transfer.

## 1 Introduction

It is known that human immunodeficiency virus type 1 (HIV-1) infects the CD4$^+$ T cells which play the central role in the immune system of the human body. Mathematical models that describe the dynamics of HIV-1 are helpful in understanding the virus dynamics and improving diagnosis and treatment strategies. The basic HIV-1 infection model has been given in [1] as:

$$\dot{T} = \rho - dT - \beta TV \tag{1}$$

$$\dot{T}^* = \beta TV - \mu T^*, \tag{2}$$

$$\dot{V} = bT^* - cV, \tag{3}$$

where, $T$, $T^*$ and $V$ are the concentrations of the uninfected CD4$^+$ T cells, infected cells, and free HIV-1 particles, respectively. The CD4$^+$ T cells are replenished at rate $\rho$, die at rate $dT$ and become infected at rate $\beta TV$, where $\beta$ is the virus-target incidence rate constant. The infected cells are die at rate $\mu$. The HIV-1 particles are produced from infected cells at rate $bT^*$ and cleared at rate $cV$. Parameters $\rho, d, \beta, \mu, b$ and $c$ are all positive.

In model (1)-(3), the infection rate is given by bilinear incidence $\beta TV$. In case when the concentration of the viruses is high, this bilinear incidence may not describe the HIV-1 dynamics accurately. Therefore, the model has been modified to incorporate the saturated incidence rate [2]:

$$\dot{T} = \rho - dT - \beta T\left(\frac{V}{1+\alpha V}\right) \tag{4}$$

$$\dot{T}^* = \beta T\left(\frac{V}{1+\alpha V}\right) - \mu T^*, \tag{5}$$

$$\dot{V} = bT^* - cV, \tag{6}$$

where, $\alpha$ is the saturation constant. Moreover, several works have been done to modify the basic model (1)-(3) by considering different effects such as: CTL immune response [3]-[5], humoral immune response [6]-[8], nonlinear incidence rate [9]-[11], intracellular time delay [10], [12], [13], [15], antiviral treatments [15]-[17], latently infected cells [18]-[19] and two types of target cells [20]-[22]. All the these works assume that the uninfected CD4$^+$ T cells becomes infected due to HIV-1 contacts. Recently, it has been reported that the uninfected CD4$^+$ T cells can also become infected due to direct contact with infected cells (see [23]-[26]). However, in [23]-[26], the rates of virus-target and infected-target infection are based on the mass action principle.

The aim of this paper is to study the dynamical behavior of HIV-1 infection model with saturated virus-target and infected-target incidences. Both discrete and distributed time delays are incorporated. We study the global stability analysis of the model using Lyapunov method.

# 2 HIV-1 model with discrete delays

We formulate an HIV-1 infection model with saturated virus-target and infected-target incidences and two types of discrete time delays as:

$$\dot{T}(t) = \rho - dT(t) - \frac{\beta_1 T(t)V(t)}{1+\alpha_1 V(t)} - \frac{\beta_2 T(t)T^*(t)}{1+\alpha_2 T^*(t)}, \tag{7}$$

$$\dot{T}^*(t) = e^{-\delta_1\tau_1}\left[\frac{\beta_1 T(t-\tau_1)V(t-\tau_1)}{1+\alpha_1 V(t-\tau_1)} + \frac{\beta_2 T(t-\tau_1)T^*(t-\tau_1)}{1+\alpha_2 T^*(t-\tau_1)}\right] - \mu T^*(t), \tag{8}$$

$$\dot{V}(t) = be^{-\delta_2\tau_2}T^*(t-\tau_2) - cV(t). \tag{9}$$

Parameter $\tau_1$ represents for the time between the virus or the infected cell contacts with an uninfected CD4$^+$ T cell, until it becomes infected but not yet producer cell. The parameter $\tau_2$ represents the time needed for new HIV-1 to be mature. The factor $e^{-\delta_1\tau_1}$ is the loss of CD4$^+$ T cells during the interval $[t-\tau_1, t]$ while, $e^{-\delta_2\tau_2}$ represents the loss of infected cells during the interval $[t-\tau_2, t]$, where $\delta_1$ and $\delta_2$ are positive constants.

The initial conditions for system (7)-(9) are given as:

$$\begin{aligned} &T(\eta) = \varphi_1(\eta),\ T^*(\eta) = \varphi_2(\eta), V(\eta) = \varphi_3(\eta), \\ &\varphi_j(\eta) \geq 0, \quad \eta \in [-\tau, 0], \quad\ \ j = 1,2,3 \end{aligned} \tag{10}$$

where $\tau = max\{\tau_1, \tau_2\}$ and $(\varphi_1(\eta), \varphi_2(\eta), \varphi_3(\eta)) \in C([-\tau:0), \mathbb{R}^3_+)$, where $C$ is the Banach space of continuous functions mapping the interval $[-\tau, 0)$ into $\mathbb{R}^3_+$. System (7)-(9) with initial conditions (10) has a unique solution [27].

## 2.1 Basic properties

The non-negativity and boundedness of the solutions of system (7)-(9) is established in the following lemma:

**Lemma 1**. All solutions $(T(t), T^*(t), V(t))$ of model (7)-(9) with initial conditions (10) are non-negative and ultimately bounded.

**Proof**: From Eq. (7), we have $\dot{T}\mid_{T=0} = \rho > 0$, therefore $T(t) > 0$ for $t \in (0, \varpi_1)$ where $(0, \varpi_1)$ is the maximal interval of existence of solution of system (7)-(9) with (10). Moreover, from Eqs. (8)-(9), we have

$$\begin{aligned} T^*(t) &= e^{-\mu t}\varphi_2(0) + e^{-\delta_1\tau_1}\int_0^t e^{-\mu(t-\eta)}\left[\frac{\beta_1 T(\eta-\tau_1)V(\eta-\tau_1)}{1+\alpha_1 V(\eta-\tau_1)} + \frac{\beta_2 T(\eta-\tau_1)T^*(\eta-\tau_1)}{1+\alpha_2 T^*(\eta-\tau_1)}\right] d\eta \geq 0, \\ V(t) &= e^{-ct}\varphi_3(0) + be^{-\delta_2\tau_2}\int_0^t e^{-c(t-\eta)}T^*(\eta-\tau_2)d\eta \geq 0, \end{aligned}$$

for $t \in [0, \tau]$. By recursive argument we obtain $T^*(t), V(t) \geq 0$ for all $t \geq 0$.

From Eq. (7) we know $\lim_{t\to\infty}\sup T(t) \leq \frac{\rho}{d}$. Let $F_1(t) = e^{-\delta_1\tau_1}T(t-\tau_1) + T^*(t)$. Then

$$\begin{aligned}
\dot{F}_1(t) &= e^{-\delta_1\tau_1}\left[\rho - dT(t-\tau_1) - \frac{\beta_1 T(t-\tau_1)V(t-\tau_1)}{1+\alpha_1 V(t-\tau_1)} - \frac{\beta_2 T(t-\tau_1)T^*(t-\tau_1)}{1+\alpha_2 T^*(t-\tau_1)}\right] \\
&+ e^{-\delta_1\tau_1}\left[\frac{\beta_1 T(t-\tau_1)V(t-\tau_1)}{1+\alpha_1 V(t-\tau_1)} + \frac{\beta_2 T(t-\tau_1)T^*(t-\tau_1)}{1+\alpha_2 T^*(t-\tau_1)}\right] - \mu T^*(t) \\
&= \rho e^{-\delta_1\tau_1} - de^{-\delta_1\tau_1}T(t-\tau_1) - \mu T^*(t) \\
&\leq \rho - \sigma\left(e^{-\delta_1\tau_1}T(t-\tau_1) + T^*(t)\right) = \rho - \sigma F_1(t),
\end{aligned}$$

where, $\sigma = \min\{d,\mu\}$. Hence, $\limsup_{t\to\infty} F_1(t) \leq \frac{\rho}{\sigma}$ and then $\limsup_{t\to\infty} T^*(t) \leq \frac{\rho}{\sigma}$ . From Eq. (9) we have

$$\dot{V}(t) = be^{-\delta_2\tau_2}T^*(t-\tau_2) - cV(t) \leq be^{-\delta_2\tau_2}\frac{\rho}{\sigma} - cV(t) < b\frac{\rho}{\sigma} - cV(t).$$

Thus $\limsup_{t\to\infty} V(t) \leq \frac{b\rho}{c\sigma}$. Therefore, $T(t), T^*(t)$ and $V(t)$ are all ultimately bounded.□

Now we prove the existence of the steady state of the model (7)-(9).

**Lemma 2.**

(i) If $\mathcal{R}_0 \leq 1$, then there exists only positive steady state $S_0$,

(ii) If $1 < \mathcal{R}_0$, then there exist two positive steady states $S_0$ and $S_1$.

**The proof.** Let the R.H.S of system (7)-(9) equal to zero

$$0 = \rho - dT - \frac{\beta_1 TV}{1+\alpha_1 V} - \frac{\beta_2 TT^*}{1+\alpha_2 T^*}, \tag{11}$$

$$0 = e^{-\delta_1\tau_1}\left(\frac{\beta_1 TV}{1+\alpha_1 V} + \frac{\beta_2 TT^*}{1+\alpha_2 T^*}\right) - \mu T^*, \tag{12}$$

$$0 = e^{-\delta_2\tau_2}bT^* - cV, \tag{13}$$

Solving Eqs. (11)-(13) we find that the system has two steady states, disease-free steady state $S_0 = (T_0, 0, 0)$, where $T_0 = \frac{\rho}{d}$ and endemic steady state $S_1(T_1, T_1^*, V_1)$, where

$$\begin{aligned}
T_1 &= \frac{\mu c\left(1+\alpha_1 V_1\right)\left(be^{-\delta_2\tau_2} + \alpha_2 cV_1\right)}{be^{-(\delta_1\tau_1+\delta_2\tau_2)}\left[\beta_1\left(be^{-\delta_2\tau_2} + \alpha_2 cV_1\right) + \beta_2 c\left(1+\alpha_1 V_1\right)\right]}, \qquad T_1^* = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \\
V_1 &= \frac{be^{-\delta_2\tau_2}T_1^*}{c},
\end{aligned}$$

where

$$\begin{aligned}
A &= \mu be^{-\delta_2\tau_2}\left(d\alpha_1\alpha_2 + \beta_1\alpha_2 + \beta_2\alpha_1\right), \\
B &= \beta_2(\mu c - \rho\alpha_1 be^{-(\delta_1\tau_1+\delta_2\tau_2)}) + \beta_1 be^{-\delta_2\tau_2}(\mu - \rho\alpha_2 e^{-\delta_1\tau_1}) + d\mu(c\alpha_2 + \alpha_1 be^{-\delta_2\tau_2}), \\
C &= d\mu c\left(1 - \mathcal{R}_0\right).
\end{aligned} \tag{14}$$

and

$$\mathcal{R}_0 = \frac{T_0 e^{-\delta_1\tau_1}\left(b\beta_1 e^{-\delta_2\tau_2} + \beta_2 c\right)}{\mu c},$$

where $\mathcal{R}_0$ represents the basic infection reproduction number. □

## 2.2 Global properties

In the following we established the global stability of the two steady states by of system (7)-(9) by constructing suitable Lyapunov functionals. Through the paper we will use the following function $g(x) = x - 1 - \ln x$ and the notation $(T, T^*, V) = (T(t), T^*(t), V(t))$.

**Theorem 1.** If $\mathcal{R}_0 \leq 1$, then $S_0$ is globally asymptotically stable.

**Proof.** Define a Lyapunov functional

$$\begin{aligned} L_1(T,T^*,V) &= T_0 g\left(\frac{T}{T_0}\right) + \frac{1}{e^{-\delta_1\tau_1}}T^* + \frac{\beta_1 T_0}{c}V + \int_0^{\tau_1}\left[\frac{\beta_1 T(t-\eta)V(t-\eta)}{1+\alpha_1 V(t-\eta)} + \frac{\beta_2 T(t-\eta)T^*(t-\eta)}{1+\alpha_2 T^*(t-\eta)}\right]d\eta \\ &+ \frac{b\beta_1 T_0}{c}e^{-\delta_2\tau_2}\int_0^{\tau_2} T^*(t-\eta)d\eta. \end{aligned}$$

We evaluate $\frac{dL_1}{dt}$ along the solutions of the system (7)-(9),

$$\begin{aligned} \frac{dL_1}{dt} &= \left(1-\frac{T_0}{T}\right)\left(\rho - dT - \frac{\beta_1 TV}{1+\alpha_1 V} - \frac{\beta_2 TT^*}{1+\alpha_2 T^*}\right) \\ &+ \frac{1}{e^{-\delta_1\tau_1}}\left[e^{-\delta_1\tau_1}\left(\frac{\beta_1 T(t-\tau_1)V(t-\tau_1)}{1+\alpha_1 V(t-\tau_1)} + \frac{\beta_2 T(t-\tau_1)T^*(t-\tau_1)}{1+\alpha_2 T^*(t-\tau_1)}\right) - \mu T^*\right] \\ &+ \frac{\beta_1 T_0}{c}\left[be^{-\delta_2\tau_2}T^*(t-\tau_2) - cV\right] + \frac{\beta_1 TV}{1+\alpha_1 V} + \frac{\beta_2 TT^*}{1+\alpha_2 T^*} \\ &- \frac{\beta_1 T(t-\tau_1)V(t-\tau_1)}{1+\alpha_1 V(t-\tau_1)} - \frac{\beta_2 T(t-\tau)T^*(t-\tau_1)}{1+\alpha_2 T^*(t-\tau_1)} + \frac{b\beta_1 T_0}{c}e^{-\delta_2\tau_2}\left[T^* - T^*(t-\tau_2)\right] \\ &= \left(1-\frac{T_0}{T}\right)(\rho - dT) - \alpha_1\beta_1 T_0\frac{V^2}{1+\alpha_1 V} - \alpha_2\beta_2 T_0\frac{T^{*2}}{1+\alpha_2 T^*} \\ &+ \frac{\mu}{e^{-\delta_1\tau_1}}\left(\frac{T_0 b\beta_1 e^{-(\delta_1\tau_1+\delta_2\tau_2)}}{\mu c} + \frac{T_0\beta_2 e^{-\delta_1\tau_1}}{\mu} - 1\right)T^* \\ &= -d\frac{(T-T_0)^2}{T} - \alpha_1\beta_1 T_0\frac{V^2}{1+\alpha_1 V} - \alpha_2\beta_2 T_0\frac{T^{*2}}{1+\alpha_2 T^*} + \frac{\mu}{e^{-\delta_1\tau_1}}(\mathcal{R}_0 - 1)T^*. \end{aligned} \tag{15}$$

If $\mathcal{R}_0 \leq 1$, then $\frac{dL_1}{dt} \leq 0$ for all $T, T^*, V > 0$ and $\frac{dL_1}{dt} = 0$ if and only if $T = T_0$ , $T^* = 0$ and $V = 0$. Let $D_0 = \left\{(T,T^*,V) : \frac{dL_1}{dt} = 0\right\}$. It is easy to show that $S_0$ is the largest invariant subset of $D_0$. LaSalle's invariance principle implies that $S_0$ is globally asymptotically stable when $\mathcal{R}_0 \leq 1$.□

**Theorem 2.** If $1 < \mathcal{R}_0$, then $S_1$ is globally asymptotically stable.

**Proof.** Define

$$\begin{aligned} U(T,T^*,V) &= T_1 g\left(\frac{T}{T_1}\right) + \frac{1}{e^{-\delta_1\tau_1}}T_1^* g\left(\frac{T^*}{T_1^*}\right) + \frac{\beta_1 T_1 V_1}{be^{-\delta_2\tau_2}T_1^*\left(1+\alpha_1 V_1\right)}V_1 g\left(\frac{V}{V_1}\right) \\ &+ \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\int_0^{\tau_1} g\left(\frac{T(t-\eta)V(t-\eta)\left(1+\alpha_1 V_1\right)}{T_1 V_1\left(1+\alpha_1 V(t-\eta)\right)}\right)d\eta \\ &+ \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\int_0^{\tau_1} g\left(\frac{T(t-\eta)T^*(t-\eta)\left(1+\alpha_2 T_1^*\right)}{T_1 T_1^*\left(1+\alpha_2 T^*(t-\eta)\right)}\right)d\eta + \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\int_0^{\tau_2} g\left(\frac{T^*(t-\eta)}{T_1^*}\right)d\eta. \end{aligned}$$

Evaluating $\frac{dU_1}{dt}$ along the trajectories of (7)-(9) as:

$$\begin{aligned}
\frac{dU_1}{dt} &= \left(1-\frac{T_1}{T}\right)\left(\rho - dT - \frac{\beta_1 TV}{1+\alpha_1 V} - \frac{\beta_2 TT^*}{1+\alpha_2 T^*}\right) \\
&+ \frac{1}{e^{-\delta_1\tau_1}}\left(1-\frac{T_1^*}{T^*}\right)\left(e^{-\delta_1\tau_1}\left[\frac{\beta_1 T(t-\tau_1)V(t-\tau_1)}{1+\alpha_1 V(t-\tau_1)} + \frac{\beta_2 T(t-\tau_1)T^*(t-\tau_1)}{1+\alpha_2 T^*(t-\tau_1)}\right] - \mu T^*\right) \\
&+ \frac{\beta_1 T_1 V_1}{be^{-\delta_2\tau_2}T_1^*\left(1+\alpha_1 V_1\right)}\left(1-\frac{V_1}{V}\right)\left(be^{-\delta_2\tau_2}T^*(t-\tau_2) - cV\right) \\
&+ \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\left(\frac{TV\left(1+\alpha_1 V_1\right)}{T_1 V_1\left(1+\alpha_1 V\right)} - \frac{T(t-\tau_1)V(t-\tau_1)\left(1+\alpha_1 V_1\right)}{T_1 V_1\left(1+\alpha_1 V(t-\tau_1)\right)}\right) \\
&+ \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\ln\left(\frac{T(t-\tau_1)V(t-\tau_1)\left(1+\alpha_1 V\right)}{TV\left(1+\alpha_1 V(t-\tau_1)\right)}\right) \\
&+ \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\left(\frac{TT^*\left(1+\alpha_2 T_1^*\right)}{T_1 T_1^*\left(1+\alpha_2 T^*\right)} - \frac{T(t-\tau_1)T^*(t-\tau_1)\left(1+\alpha_2 T_1^*\right)}{T_1 T_1^*\left(1+\alpha_2 T^*(t-\tau_1)\right)}\right) \\
&+ \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\ln\left(\frac{T(t-\tau_1)T^*(t-\tau_1)\left(1+\alpha_2 T^*\right)}{TT^*\left(1+\alpha_2 T^*(t-\tau_1)\right)}\right) \\
&+ \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\left(\frac{T^*}{T_1^*} - \frac{T^*(t-\tau_2)}{T_1^*} + \ln\left(\frac{T^*(t-\tau_2)}{T^*}\right)\right). \qquad (16)
\end{aligned}$$

Collecting terms of Eq. (16) and applying the steady state conditions for $S_1$:

$$\rho - dT_1 = \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1} + \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*} = \frac{\mu}{e^{-\delta_1\tau_1}}T_1^* = \frac{c\mu}{be^{-(\delta_1\tau_1+\delta_2\tau_2)}}V_1,$$

we get

$$\begin{aligned}
\frac{dU_1}{dt} &= -\frac{d}{T}\left(T-T_1\right)^2 + \left(1-\frac{T_1}{T}\right)\left(\frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1} + \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\right) + \frac{\beta_1 T_1 V}{1+\alpha_1 V} + \frac{\beta_2 T_1 T^*}{1+\alpha_2 T^*} \\
&- \frac{\beta_1 T_1^* T(t-\tau_1)V(t-\tau_1)}{T^*\left(1+\alpha_1 V(t-\tau_1)\right)} - \frac{\beta_2 T_1^* T(t-\tau_1)T^*(t-\tau_1)}{T^*\left(1+\alpha_2 T^*(t-\tau_1)\right)} - \frac{\beta_2 T_1 T^*}{1+\alpha_2 T_1^*} + \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1} \\
&+ \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*} - \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\frac{V_1 T^*(t-\tau_2)}{VT_1^*} - \frac{\beta_1 T_1 V}{1+\alpha_1 V_1} + \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1} \\
&+ \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\ln\left(\frac{T(t-\tau_1)V(t-\tau_1)\left(1+\alpha_1 V\right)}{TV\left(1+\alpha_1 V(t-\tau_1)\right)}\right) + \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\ln\left(\frac{T(t-\tau_1)T^*(t-\tau_1)\left(1+\alpha_2 T^*\right)}{TT^*\left(1+\alpha_2 T^*(t-\tau_1)\right)}\right) \\
&+ \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\ln\left(\frac{T^*(t-\tau_2)}{T^*}\right).
\end{aligned}$$

Consider the following equalities:

$$\begin{aligned}
\ln\left(\frac{T(t-\tau_1)V(t-\tau_1)\left(1+\alpha_1 V\right)}{TV\left(1+\alpha_1 V(t-\tau_1)\right)}\right) &= \ln\left(\frac{T(t-\tau_1)V(t-\tau_1)\left(1+\alpha_1 V_1\right)T_1^*}{T_1 V_1\left(1+\alpha_1 V(t-\tau_1)\right)T^*}\right) + \ln\left(\frac{T_1}{T}\right) \\
&+ \ln\left(\frac{1+\alpha_1 V}{1+\alpha_1 V_1}\right) + \ln\left(\frac{V_1 T^*}{VT_1^*}\right), \\
\ln\left(\frac{T(t-\tau_1)T^*(t-\tau_1)\left(1+\alpha_2 T^*\right)}{TT^*\left(1+\alpha_2 T^*(t-\tau_1)\right)}\right) &= \ln\left(\frac{T(t-\tau_1)T^*(t-\tau_1)\left(1+\alpha_2 T_1^*\right)}{T_1 T^*\left(1+\alpha_2 T^*(t-\tau_1)\right)}\right) + \ln\left(\frac{T_1}{T}\right) \qquad (17) \\
&+ \ln\left(\frac{1+\alpha_2 T^*}{1+\alpha_2 T_1^*}\right), \\
\ln\left(\frac{T^*(t-\tau_2)}{T^*}\right) &= \ln\left(\frac{V_1 T^*(t-\tau_2)}{VT_1^*}\right) + \ln\left(\frac{VT_1^*}{V_1 T^*}\right).
\end{aligned}$$

Using Eqs. (17) we get

$$\begin{aligned}
\frac{dU_1}{dt} &= -\frac{d}{T}(T-T_1)^2 + \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\left[\frac{(1+\alpha_1 V_1)V}{(1+\alpha_1 V)V_1} - \frac{V}{V_1} - 1 + \frac{1+\alpha_1 V}{1+\alpha_1 V_1}\right] \\
&+ \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\left[\frac{(1+\alpha_2 T_1^*)T^*}{(1+\alpha_2 T^*)T_1^*} - \frac{T^*}{T_1^*} - 1 + \frac{1+\alpha_2 T^*}{1+\alpha_2 T_1^*}\right] - \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\left[\frac{T_1}{T} - 1 - \ln\left(\frac{T_1}{T}\right)\right] \\
&- \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\left[\frac{V_1 T^*(t-\tau_2)}{V T_1^*} - 1 - \ln\left(\frac{V_1 T^*(t-\tau_2)}{V T_1^*}\right) + \frac{1+\alpha_1 V}{1+\alpha_1 V_1} - 1 - \ln\left(\frac{1+\alpha_1 V}{1+\alpha_1 V_1}\right)\right] \\
&- \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\left[\frac{T_1}{T} - 1 - \ln\left(\frac{T_1}{T}\right)\right] - \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\left[\frac{1+\alpha_2 T^*}{1+\alpha_2 T_1^*} - 1 - \ln\left(\frac{1+\alpha_2 T^*}{1+\alpha_2 T_1^*}\right)\right] \\
&- \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\left[\frac{T(t-\tau_1)V(t-\tau_1)(1+\alpha_1 V_1)T_1^*}{T_1 V_1(1+\alpha_1 V(t-\tau_1))T^*} - 1 - \ln\left(\frac{T(t-\tau_1)V(t-\tau_1)(1+\alpha_1 V_1)T_1^*}{T_1 V_1(1+\alpha_1 V(t-\tau_1))T^*}\right)\right] \\
&- \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\left[\frac{T(t-\tau_1)T^*(t-\tau_1)(1+\alpha_2 T_1^*)}{T_1 T^*(1+\alpha_2 T^*(t-\tau_1))} - 1 - \ln\left(\frac{T(t-\tau_1)T^*(t-\tau_1)(1+\alpha_2 T_1^*)}{T_1 T^*(1+\alpha_2 T^*(t-\tau_1))}\right)\right]
\end{aligned}$$

Then

$$\begin{aligned}
\frac{dU_1}{dt} &= -\frac{d}{T}(T-T_1)^2 - \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\left[\frac{\alpha_1 (V-V_1)^2}{(1+\alpha_1 V)(1+\alpha_1 V_1)V_1}\right] - \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\left[\frac{\alpha_2 (T^*-T_1^*)^2}{(1+\alpha_2 T^*)(1+\alpha_2 T_1^*)T_1^*}\right] \\
&- \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\left[g\left(\frac{T_1}{T}\right) + g\left(\frac{T(t-\tau_1)V(t-\tau_1)(1+\alpha_1 V_1)T_1^*}{T_1 V_1(1+\alpha_1 V(t-\tau_1))T^*}\right) + g\left(\frac{1+\alpha_1 V}{1+\alpha_1 V_1}\right) + g\left(\frac{V_1 T^*(t-\tau_2)}{V T_1^*}\right)\right] \\
&- \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\left[g\left(\frac{T_1}{T}\right) + g\left(\frac{T(t-\tau_1)T^*(t-\tau_1)(1+\alpha_2 T_1^*)}{T_1 T^*(1+\alpha_2 T^*(t-\tau_1))}\right) + g\left(\frac{1+\alpha_2 T^*}{1+\alpha_2 T_1^*}\right)\right]. \qquad (18)
\end{aligned}$$

Since $\mathcal{R}_0 > 1$, then $T, T^*, V > 0$. From Eq. (18) we have $\frac{dU_1}{dt} \leq 0$ and $\frac{dU_1}{dt} = 0$ ocurs at $S_1$. Let $D_1 = \left\{(T, T^*, V) : \frac{dU1}{dt} = 0\right\}$. It is clear that $S_1$ is the largest invariant subset of $D_1$. Using LaSalle's invariance principle we obtain that $S_1$ is globally asymptotically stable when $\mathcal{R}_0 > 1$. □

# 3 HIV-1 model with distributed delays

In this section, we formulate an HIV-1 infection model with saturated virus-target and infected-target incidences and two types of distributed time delays:

$$\dot{T}(t) = \rho - dT(t) - \frac{\beta_1 T(t)V(t)}{1+\alpha_1 V(t)} - \frac{\beta_2 T(t)T^*(t)}{1+\alpha_2 T^*(t)}, \qquad (19)$$

$$\dot{T}^*(t) = \int_0^\infty f_1(s)e^{-\delta_1 s}\left[\frac{\beta_1 T(t-s)V(t-s)}{1+\alpha_1 V(t-s)} + \frac{\beta_2 T(t-s)T^*(t-s)}{1+\alpha_2 T^*(t-s)}\right]ds - \mu T^*(t), \qquad (20)$$

$$\dot{V}(t) = b\int_0^\infty f_2(s)e^{-\delta_2 s}T^*(t-s)ds - cV(t). \qquad (21)$$

Let us assume that the probability distribution functions $f_i(s)$ satisfy $f_i(s) > 0$, $i = 1, 2$ and

$$\int_0^\infty f_i(s)ds = 1, \quad \int_0^\infty f_i(u)e^{\ell u}du < \infty, \quad i = 1, 2,$$

where $\ell > 0$. Denote $\eta_i = \int_0^\infty f_i(s)e^{-\delta_i s}ds, \quad i = 1, 2$, thus, $0 < \eta_i \leq 1$. Define the Banach space of fading memory type

$$C_\gamma = \{\phi \in C((-\infty, 0], \mathbb{R}) : e^{\alpha\eta}\phi(\eta) \text{ is uniformly continuous for } \eta \in (-\infty, 0] \text{ and } \|\phi\| < \infty\}$$

where $\gamma$ is a positive constant and $\|\phi\| = \sup_{\eta\leq 0} |\phi(\eta)|\, e^{\gamma\eta}$. Let

$$C_\gamma^+ = \{\phi \in C_\gamma : \phi(\eta) \geq 0 \; for \;\; \eta \in (-\infty, 0]\}.$$

The initial conditions for system (19)-(21) are given as:

$$\begin{aligned}&T(\eta)=\varphi_1(\eta),\ T^*(\eta)=\varphi_2(\eta), V(\eta)=\varphi_3(\eta), \quad \text{for } \eta\in(-\infty,0],\\ &\varphi_i\in C_\gamma^+,\ \ i=1,2,3.\end{aligned} \tag{22}$$

System (7)-(9) with initial conditions (22) has a unique solution [27].

## 3.1 Basic properties

The non-negativity and boundedness of the solutions of model (19)-(21) will be established in the next lemma.

**Lemma 3**. The solutions $(T(t),T^*(t),V(t))$ of model (19)-(21) with initial conditions (22) are non-negative and ultimately bounded.

**Proof**: Similar to the proof of Lemma 1, one can show $T(t)>0$ for all $T(t)>0$ for $t\in(0,\varpi_2)$, where $(0,\varpi_2)$ is the maximal interval of existence of solution of system (19)-(21) with (22). From Eqs. (20)-(21), we have

$$\begin{aligned}T^*(t)&=e^{-\mu t}\varphi_2(0)+\int_0^t e^{-\mu(t-\eta)}\int_0^\infty f_1(s)e^{-\delta_1 s}\left[\frac{\beta_1T(\eta-s)V(\eta-s)}{1+\alpha_1V(\eta-s)}+\frac{\beta_2T(\eta-s)T^*(\eta-s)}{1+\alpha_2T^*(\eta-s)}\right]dsd\eta\geq 0,\\ V(t)&=e^{-ct}\varphi_3(0)+b\int_0^t e^{-c(t-\eta)}\int_0^\infty f_2(s)e^{-\delta_2 s}T^*(\eta-s)dsd\eta\geq 0.\end{aligned}$$

From Eq. (19) we have $\lim\limits_{t\to\infty}\sup T(t)\leq\dfrac{\rho}{d}$. Let $F(t)=\int_0^\infty f_1(s)e^{-\delta_1 s}T(t-s)ds+T^*(t)$. Then

$$\begin{aligned}\dot F_2(t)&=\int_0^\infty f_1(s)e^{-\delta_1 s}\left[\rho-dT(t-s)-\frac{\beta_1T(t-s)V(t-s)}{1+\alpha_1V(t-s)}-\frac{\beta_2T(t-s)T^*(t-s)}{1+\alpha_2T^*(t-s)}\right]ds\\ &+\int_0^\infty f_1(s)e^{-\delta_1 s}\left[\frac{\beta_1T(t-s)V(t-s)}{1+\alpha_1V(t-s)}+\frac{\beta_2T(t-s)T^*(t-s)}{1+\alpha_2T^*(t-s)}\right]ds-\mu T^*(t)\\ &=\rho\eta_1-d\int_0^\infty f_1(s)e^{-\delta_1 s}T(t-s)ds-\mu T^*(t)\\ &\leq\rho-\sigma\left(\int_0^\infty f_1(s)e^{-\delta_1 s}T(t-s)ds+T^*(t)\right)=\rho-\sigma F_2(t),\end{aligned}$$

where, $\sigma=\min\{d,\mu\}$. Hence, $\limsup_{t\to\infty}F_2(t)\leq\dfrac{\rho}{\sigma}$. Since $\int_0^\infty f_1(s)e^{-\delta_1 s}T(t-s)ds>0$ and $T^*\geq 0$ , then $\limsup_{t\to\infty}T^*(t)\leq\dfrac{\rho}{\sigma}$ . From Eq. (21) we have

$$\dot V(t)=b\int_0^\infty f_2(s)e^{-\delta_2 s}T^*(t-s)ds-cV(t)\leq b\eta_2\frac{\rho}{\sigma}-cV(t)\leq b\frac{\rho}{\sigma}-cV(t).$$

Thus $\limsup\limits_{t\to\infty}V(t)\leq\dfrac{b\rho}{c\sigma}$. Therefore, $T(t),T^*(t)$ and $V(t)$ are ultimately bounded.□

The existence of the steady state of the model (19)-(21) will be shown in the next lemma.

**Lemma 4.**

(i) If $\mathcal{R}_0\leq 1$, then there exists only positive steady state $S_0$,

(ii) if $1<\mathcal{R}_0$, then there exist only two positive steady states $S_0$ and $S_1$.

**The proof.** Let the R.H.S of system (19)-(21) be equal zero

$$0=\rho-dT-\frac{\beta_1TV}{1+\alpha_1V}-\frac{\beta_2TT^*}{1+\alpha_2T^*}, \tag{23}$$

$$0=\eta_1\left(\frac{\beta_1TV}{1+\alpha_1V}+\frac{\beta_2TT^*}{1+\alpha_2T^*}\right)-\mu T^*, \tag{24}$$

$$0=\eta_2bT^*-cV, \tag{25}$$

Solving Eqs. (23)-(25) we find that the system has two steady states, disease-free steady state $S_0 = (T_0, 0, 0, 0)$, where $T_0 = \frac{\rho}{d}$, and endemic steady state $S_1(T_1, T_1^*, V_1)$, where

$$T_1 = \frac{\mu c\,(1+\alpha_1 V_1)\,(b\eta_2 + \alpha_2 c V_1)}{b\eta_1\eta_2\,[\beta_1\,(b\eta_2+\alpha_2 c V_1) + \beta_2 c\,(1+\alpha_1 V_1)]}, \qquad T_1^* = \frac{-B+\sqrt{B^2-4AC}}{2A}, \qquad V_1 = \frac{b\eta_2 T_1^*}{c},$$

where

$$\begin{aligned} A &= \mu b\eta_2\,(d\alpha_1\alpha_2 + \beta_1\alpha_2 + \beta_2\alpha_1)\,, \\ B &= \beta_2(\mu c - \rho\alpha_1 b\eta_1\eta_2) + \beta_1 b\eta_2(\mu - \rho\alpha_2\eta_1) + d\mu(c\alpha_2 + \alpha_1 b\eta_2), \\ C &= d\mu c\,(1-\mathcal{R}_0)\,, \end{aligned} \tag{26}$$

and

$$\mathcal{R}_0 = \frac{T_0\eta_1}{\mu c}\,(b\beta_1\eta_2 + \beta_2 c)\,,$$

where $\mathcal{R}_0$ represents the basic infection reproduction number. □

## 3.2 Global properties

In this section, we study the global stability of all the steady states of system (19)-(21) employing the method of Lyapunov function.

**Theorem 3.** If $\mathcal{R}_0 \le 1$, then $S_0$ is globally asymptotically stable.

**Proof.** Define

$$\begin{aligned} L_2(T,T^*,V) &= T_0 g\left(\frac{T}{T_0}\right) + \frac{1}{\eta_1}T^* + \frac{\beta_1 T_0}{c}V \\ &+ \frac{1}{\eta_1}\int_0^\infty f_1(s)e^{-\delta_1 s}\int_0^s \left[\frac{\beta_1 T(t-\eta)V(t-\eta)}{1+\alpha_1 V(t-\eta)} + \frac{\beta_2 T(t-\eta)T^*(t-\eta)}{1+\alpha_2 T^*(t-\eta)}\right] d\eta ds \\ &+ \frac{b\beta_1 T_0}{c}\int_0^\infty f_2(s)e^{-\delta_2 s}\int_0^s T^*(t-\eta)d\eta ds. \end{aligned}$$

Calculating $\frac{dL_2}{dt}$ along the solutions of the system (19)-(21), we obtain

$$\begin{aligned} \frac{dL_2}{dt} &= \left(1-\frac{T_0}{T}\right)\left(\rho - dT - \frac{\beta_1 TV}{1+\alpha_1 V} - \frac{\beta_2 TT^*}{1+\alpha_2 T^*}\right) \\ &+ \frac{1}{\eta_1}\left[\int_0^\infty f_1(s)e^{-\delta_1 s}\left[\frac{\beta_1 T(t-s)V(t-s)}{1+\alpha_1 V(t-s)} + \frac{\beta_2 T(t-s)T^*(t-s)}{1+\alpha_2 T^*(t-s)}\right] ds - \mu T^*\right] \\ &+ \frac{\beta_1 T_0}{c}\left[b\int_0^\infty f_2(s)e^{-\delta_2 s}T^*(t-s)ds - cV\right] \\ &+ \frac{1}{\eta_1}\int_0^\infty f_1(s)e^{-\delta_1 s}\left[\frac{\beta_1 TV}{1+\alpha_1 V} + \frac{\beta_2 TT^*}{1+\alpha_2 T^*} - \frac{\beta_1 T(t-s)V(t-s)}{1+\alpha_1 V(t-s)} - \frac{\beta_2 T(t-s)T^*(t-s)}{1+\alpha_2 T^*(t-s)}\right] ds \\ &+ \frac{b\beta_1 T_0}{c}\int_0^\infty f_2(s)e^{-\delta_2 s}\left[T^* - T^*(t-s)\right] ds \\ &= \left(1-\frac{T_0}{T}\right)(\rho - dT) - \alpha_1\beta_1 T_0\frac{V^2}{1+\alpha_1 V} - \alpha_2\beta_2 T_0\frac{T^{*2}}{1+\alpha_2 T^*} + \frac{\mu}{\eta_1}\left(\frac{T_0 b\beta_1\eta_1\eta_2}{\mu c} + \frac{T_0\beta_2\eta_1}{\mu} - 1\right)T^* \\ &= -d\frac{(T-T_0)^2}{T} - \alpha_1\beta_1 T_0\frac{V^2}{1+\alpha_1 V} - \alpha_2\beta_2 T_0\frac{T^{*2}}{1+\alpha_2 T^*} + \frac{\mu}{\eta_1}(\mathcal{R}_0 - 1)T^*. \end{aligned} \tag{27}$$

If $\mathcal{R}_0 \le 1$, then $\frac{dL_2}{dt} \le 0$ for all $T, T^*, V > 0$. Similar to the proof of Theorem 1 one can easily show that $S_0$ is globally asymptotically stable when $\mathcal{R}_0 \le 1$.□

**Theorem 4.** If $1 < \mathcal{R}_0$ , then $S_1$ is globally asymptotically stable.

**Proof.** We construct the following Lyapunov functional

$$
\begin{aligned}
U_2(T,T^*,V,Z) &= T_1 g\left(\frac{T}{T_1}\right) + \frac{1}{\eta_1}T_1^* g\left(\frac{T^*}{T_1^*}\right) + \frac{\beta_1 T_1 V_1}{b\eta_2 T_1^*\left(1+\alpha_1 V_1\right)} V_1 g\left(\frac{V}{V_1}\right) \\
&+ \frac{\beta_1 T_1 V_1}{\eta_1\left(1+\alpha_1 V_1\right)} \int_0^\infty f_1(s) e^{-\delta_1 s} \int_0^s g\left(\frac{T(t-\eta)V(t-\eta)\left(1+\alpha_1 V_1\right)}{T_1 V_1\left(1+\alpha_1 V(t-\eta)\right)}\right) d\eta ds \\
&+ \frac{\beta_2 T_1 T_1^*}{\eta_1\left(1+\alpha_2 T_1^*\right)} \int_0^\infty f_1(s) e^{-\delta_1 s} \int_0^s g\left(\frac{T(t-\eta)T^*(t-\eta)\left(1+\alpha_2 T_1^*\right)}{T_1 T_1^*\left(1+\alpha_2 T^*(t-\eta)\right)}\right) d\eta ds \\
&+ \frac{\beta_1 T_1 V_1}{\eta_2\left(1+\alpha_1 V_1\right)} \int_0^\infty f_2(s) e^{-\delta_2 s} \int_0^s g\left(\frac{T^*(t-\eta)}{T_1^*}\right) d\eta ds.
\end{aligned}
$$

We evaluate $\frac{dU_2}{dt}$ along the trajectories of (19)-(21) is given by

$$
\begin{aligned}
\frac{dU_2}{dt} &= \left(1-\frac{T_1}{T}\right)\left(\rho - dT - \frac{\beta_1 TV}{1+\alpha_1 V} - \frac{\beta_2 TT^*}{1+\alpha_2 T^*}\right) \\
&+ \frac{1}{\eta_1}\left(1-\frac{T_1^*}{T^*}\right)\left(\int_0^\infty f_1(s)e^{-\delta_1 s}\left[\frac{\beta_1 T(t-s)V(t-s)}{1+\alpha_1 V(t-s)} + \frac{\beta_2 T(t-s)T^*(t-s)}{1+\alpha_2 T^*(t-s)}\right] ds - \mu T^*\right) \\
&+ \frac{\beta_1 T_1 V_1}{b\eta_2 T_1^*\left(1+\alpha_1 V_1\right)}\left(1-\frac{V_1}{V}\right)\left(b\int_0^\infty f_2(s)e^{-\delta_2 s}T^*(t-s)ds - cV\right) \\
&+ \frac{\beta_1 T_1 V_1}{\eta_1\left(1+\alpha_1 V_1\right)} \int_0^\infty f_1(s)e^{-\delta_1 s}\left(\frac{TV\left(1+\alpha_1 V_1\right)}{T_1 V_1\left(1+\alpha_1 V\right)} - \frac{T(t-s)V(t-s)\left(1+\alpha_1 V_1\right)}{T_1 V_1\left(1+\alpha_1 V(t-s)\right)}\right) ds \\
&+ \frac{\beta_1 T_1 V_1}{\eta_1\left(1+\alpha_1 V_1\right)} \int_0^\infty f_1(s)e^{-\delta_1 s}\ln\left(\frac{T(t-s)V(t-s)\left(1+\alpha_1 V\right)}{TV\left(1+\alpha_1 V(t-s)\right)}\right) ds \\
&+ \frac{\beta_2 T_1 T_1^*}{\eta_1\left(1+\alpha_2 T_1^*\right)} \int_0^\infty f_1(s)e^{-\delta_1 s}\left(\frac{TT^*\left(1+\alpha_2 T_1^*\right)}{T_1 T_1^*\left(1+\alpha_2 T^*\right)} - \frac{T(t-s)T^*(t-s)\left(1+\alpha_2 T_1^*\right)}{T_1 T_1^*\left(1+\alpha_2 T^*(t-s)\right)}\right) ds \\
&+ \frac{\beta_2 T_1 T_1^*}{\eta_1\left(1+\alpha_2 T_1^*\right)} \int_0^\infty f_1(s)e^{-\delta_1 s}\ln\left(\frac{T(t-s)T^*(t-s)\left(1+\alpha_2 T^*\right)}{TT^*\left(1+\alpha_2 T^*(t-s)\right)}\right) ds \\
&+ \frac{\beta_1 T_1 V_1}{\eta_2\left(1+\alpha_1 V_1\right)} \int_0^\infty f_2(s)e^{-\delta_2 s}\left(\frac{T^*}{T_1^*} - \frac{T^*(t-s)}{T_1^*} + \ln\left(\frac{T^*(t-s)}{T^*}\right)\right) ds. \qquad (28)
\end{aligned}
$$

Collecting terms of Eq. (28) and applying the steady state conditions for $S_1$:

$$
\rho - dT_1 = \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1} + \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*} = \frac{\mu}{\eta_1}T_1^* = \frac{c\mu}{b\eta_1\eta_2}V_1,
$$

we get

$$
\begin{aligned}
\frac{dU_2}{dt} &= -\frac{d}{T}\left(T-T_1\right)^2 + \left(1-\frac{T_1}{T}\right)\left(\frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1} + \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\right) + \frac{\beta_1 T_1 V}{1+\alpha_1 V} \\
&+ \frac{\beta_2 T_1 T^*}{1+\alpha_2 T^*} - \frac{T_1^*}{\eta_1 T^*}\int_0^\infty f_1(s)e^{-\delta_1 s}\frac{\beta_1 T(t-s)V(t-s)}{1+\alpha_1 V(t-s)} ds \\
&- \frac{T_1^*}{\eta_1 T^*}\int_0^\infty f_1(s)e^{-\delta_1 s}\frac{\beta_2 T(t-s)T^*(t-s)}{1+\alpha_2 T^*(t-s)} ds - \frac{\beta_2 T_1 T^*}{1+\alpha_2 T_1^*} + \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1} \\
&+ \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*} - \frac{\beta_1 T_1 V_1}{\eta_2\left(1+\alpha_1 V_1\right)}\int_0^\infty f_2(s)e^{-\delta_2 s}\frac{V_1 T^*(t-s)}{VT_1^*} ds - \frac{\beta_1 T_1 V}{1+\alpha_1 V_1} \\
&+ \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1} + \frac{\beta_1 T_1 V_1}{\eta_1\left(1+\alpha_1 V_1\right)}\int_0^\infty f_1(s)e^{-\delta_1 s}\ln\left(\frac{T(t-s)V(t-s)\left(1+\alpha_1 V\right)}{TV\left(1+\alpha_1 V(t-s)\right)}\right) ds \\
&+ \frac{\beta_2 T_1 T_1^*}{\eta_1\left(1+\alpha_2 T_1^*\right)}\int_0^\infty f_1(s)e^{-\delta_1 s}\ln\left(\frac{T(t-s)T^*(t-s)\left(1+\alpha_2 T^*\right)}{TT^*\left(1+\alpha_2 T^*(t-s)\right)}\right) ds \\
&+ \frac{\beta_1 T_1 V_1}{\eta_2\left(1+\alpha_1 V_1\right)}\int_0^\infty f_2(s)e^{-\delta_2 s}\ln\left(\frac{T^*(t-s)}{T^*}\right) ds.
\end{aligned}
$$

Using Eq. (17) we get

$$\begin{aligned}
\frac{dU_2}{dt} &= -\frac{d}{T}(T-T_1)^2 + \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\left[\frac{(1+\alpha_1 V_1)V}{(1+\alpha_1 V)V_1} - \frac{V}{V_1} - 1 + \frac{1+\alpha_1 V}{1+\alpha_1 V_1}\right] \\
&+ \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\left[\frac{(1+\alpha_2 T_1^*)T^*}{(1+\alpha_2 T^*)T_1^*} - \frac{T^*}{T_1^*} - 1 + \frac{1+\alpha_2 T^*}{1+\alpha_2 T_1^*}\right] \\
&- \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\left[\frac{T_1}{T} - 1 - \ln\left(\frac{T_1}{T}\right)\right] - \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\left[\frac{T_1}{T} - 1 - \ln\left(\frac{T_1}{T}\right)\right] \\
&- \frac{\beta_1 T_1 V_1}{\eta_1(1+\alpha_1 V_1)}\int_0^\infty f_1(s)e^{-\delta_1 s}\left[\frac{T(t-s)V(t-s)(1+\alpha_1 V_1)T_1^*}{T_1 V_1(1+\alpha_1 V(t-s))T^*} - 1\right]ds \\
&+ \frac{\beta_1 T_1 V_1}{\eta_1(1+\alpha_1 V_1)}\int_0^\infty f_1(s)e^{-\delta_1 s}\ln\left(\frac{T(t-s)V(t-s)(1+\alpha_1 V_1)T_1^*}{T_1 V_1(1+\alpha_1 V(t-s))T^*}\right)ds \\
&- \frac{\beta_2 T_1 T_1^*}{\eta_1(1+\alpha_2 T_1^*)}\int_0^\infty f_1(s)e^{-\delta_1 s}\left[\frac{T(t-s)T^*(t-s)(1+\alpha_2 T_1^*)}{T_1 T^*(1+\alpha_2 T^*(t-s))} - 1\right]ds \\
&+ \frac{\beta_2 T_1 T_1^*}{\eta_1(1+\alpha_2 T_1^*)}\int_0^\infty f_1(s)e^{-\delta_1 s}\ln\left(\frac{T(t-s)T^*(t-s)(1+\alpha_2 T_1^*)}{T_1 T^*(1+\alpha_2 T^*(t-s))}\right)ds \\
&- \frac{\beta_1 T_1 V_1}{\eta_2(1+\alpha_1 V_1)}\int_0^\infty f_2(s)e^{-\delta_2 s}\left[\frac{V_1 T^*(t-s)}{V T_1^*} - 1 - \ln\left(\frac{V_1 T^*(t-s)}{V T_1^*}\right)\right]ds \\
&- \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\left[\frac{1+\alpha_1 V}{1+\alpha_1 V_1} - 1 - \ln\left(\frac{1+\alpha_1 V}{1+\alpha_1 V_1}\right)\right] - \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\left[\frac{1+\alpha_2 T^*}{1+\alpha_2 T_1^*} - 1 - \ln\left(\frac{1+\alpha_2 T^*}{1+\alpha_2 T_1^*}\right)\right] \\
&= -\frac{d}{T}(T-T_1)^2 - \frac{\beta_1 T_1 V_1}{1+\alpha_1 V_1}\left[\frac{\alpha_1(V-V_1)^2}{(1+\alpha_1 V)(1+\alpha_1 V_1)V_1}\right] - \frac{\beta_2 T_1 T_1^*}{1+\alpha_2 T_1^*}\left[\frac{\alpha_2(T^*-T_1^*)^2}{(1+\alpha_2 T^*)(1+\alpha_2 T_1^*)T_1^*}\right] \\
&- \frac{\beta_1 T_1 V_1}{\eta_1(1+\alpha_1 V_1)}\int_0^\infty f_1(s)e^{-\delta_1 s}\left[g\left(\frac{T_1}{T}\right) + g\left(\frac{T(t-s)V(t-s)(1+\alpha_1 V_1)T_1^*}{T_1 V_1(1+\alpha_1 V(t-s))T^*}\right) + g\left(\frac{1+\alpha_1 V}{1+\alpha_1 V_1}\right)\right]ds \\
&- \frac{\beta_2 T_1 T_1^*}{\eta_1(1+\alpha_2 T_1^*)}\int_0^\infty f_1(s)e^{-\delta_1 s}\left[g\left(\frac{T_1}{T}\right) + g\left(\frac{T(t-s)T^*(t-s)(1+\alpha_2 T_1^*)}{T_1 T^*(1+\alpha_2 T^*(t-s))}\right) + g\left(\frac{1+\alpha_2 T^*}{1+\alpha_2 T_1^*}\right)\right]ds \\
&- \frac{\beta_1 T_1 V_1}{\eta_2(1+\alpha_1 V_1)}\int_0^\infty f_2(s)e^{-\delta_2 s} g\left(\frac{V_1 T^*(t-s)}{V T_1^*}\right)ds.
\end{aligned}$$

Similar to the proof of Theorem 2, one can easily show that $S_1$ is globally asymptotically stable. □

# 4 Numerical simulations

In order to illustrate our theoretical results, we will perform numerical simulations for system (7)-(9). We use the data given in Table 1.

Table 1: The data of system (7)-(9).

| Parameter | Value | Parameter | Parameter |
|---|---|---|---|
| $\lambda$ | 10 cells mm$^{-3}$day$^{-1}$ | $\tau_1$ | Varied |
| $d$ | 0.01 day$^{-1}$ | $\tau_2$ | Varied |
| $\beta_1$ | Varied | $\delta_1$ | 0.9 day$^{-1}$ |
| $\beta_2$ | 0.0001 cells$^{-1}$ mm$^3$day$^{-1}$ | $\delta_2$ | 0.1 day$^{-1}$ |
| $\alpha_1$ | Varied | $b$ | 10 virus cells$^{-1}$day$^{-1}$ |
| $\alpha_2$ | Varied | $c$ | 3 day$^{-1}$ |
| $\mu$ | 0.9 day$^{-1}$ | | |

### 4.1 Effect of the parameter $\beta_1$ on the stability of steady states

To show the global stability of the steady states we consider three different initial conditions:

IC1: $\varphi_1(\eta) = 600$, $\varphi_2(\eta) = 1, \varphi_3(\eta) = 1$,

IC2: $\varphi_1(\eta) = 200$, $\varphi_2(\eta) = 0.5, \varphi_3(\eta) = 3$,

IC3: $\varphi_1(\eta) = 700$, $\varphi_2(\eta) = 5, \varphi_3(\eta) = 9$,

where, $\eta \in [-\max\{\tau_1, \tau_2\}, 0]$.

In this case we choose $\tau_1 = 0.5$ day, $\tau_2 = 0.9$ day, $\alpha_1 = 0.009$ virus $^{-1}$ mm$^3$, $\alpha_2 = 0.005$ cells $^{-1}$ mm$^3$ and study the following subcases for the initial conditions IC1-IC3.:

**(i)** $\mathcal{R}_0 \leq 1$. We choose, $\beta_1 = 0.0001$ virus $^{-1}$ mm$^3$day$^{-1}$, then we compute $\mathcal{R}_0 = 0.2867 < 1$. From Lemma 2 we have that the system has one steady state $S_0$. From Figures 1-3 we can see that, the concentration of uninfected CD4$^+$ T cells is increasing and tends its normal value $\rho/d = 1000$, while the concentrations of infected cells and free HIV-1 are decaying and approaching zero for all the three initial conditions IC1-IC3. It means that, $S_0$ is globally asymptotically stable and the virus will be removed. This result support the result of Theorem 1.

**(ii)** $\mathcal{R}_0 > 1$. We take $\beta_1 = 0.001$ virus $^{-1}$ mm$^3$day$^{-1}$, and then, $\mathcal{R}_0 = 2.2292 > 1$. Lemma 2 state that the system has two positive steady states $S_0$ and $S_1$. It is clear from Figures 4-6 that, both the numerical results and the theoretical results given in Theorem 2 are consistent. It is seen that, the solutions of the system converges to the steady $S_1(491.6543, 3.6015, 10.9717)$, for all the three initial conditions IC1-IC3.

### 4.2 Effect of the saturation infection on the HIV-1 dynamics

In this case, we consider the initial condition IC2. We take the values $\tau_1 = 0.5$ day, $\tau_2 = 0.9$ day and$\beta_1 = 0.001$ virus$^{-1}$ mm$^3$day$^{-1}$. Figures 7-9 show the effect of saturation infection. We observe that, as $\alpha_1$ and $\alpha_2$ are increased, both the virus-target and infected-target infection rates are decreased, and then the concentration of the CD4$^+$ T cells are increased, while the concentrations of the infected cells and free HIV-1 particles are decreased.

### 4.3 Effect of the time delays on the stability of steady states

In this case, we consider the initial condition IC2. We take the values $\beta_1 = 0.001$ virus $^{-1}$ mm$^3$day$^{-1}, \alpha_1 = 0.009$ virus$^{-1}$ mm$^3$ and $\alpha_2 = 0.005$ cells $^{-1}$ mm$^3$. Let us consider the case $\tau = \tau_1 = \tau_2$. The values of $\mathcal{R}_0$ and the steady states of system (7)-(9) with different values of $\tau$ are presented in Table 2.

Table 2: The values of steady states, $R_0$ for model (7)-(9) with different values of the delay parameter $\tau$.

| Delay parameter | Steady states | $R_0$ |
|---|---|---|
| $\tau = 0.0$ | $E_1 = (319.8688, 7.5570, 25.1900)$ | 3.8148 |
| $\tau = 0.2$ | $E_1 = (373.1935, 5.8172, 19.0069)$ | 3.1251 |
| $\tau = 0.6$ | $E_1 = (517.7076, 3.1228, 9.8032)$ | 2.0973 |
| $\tau = 0.9$ | $E_1 = (670.8935, 1.6267, 4.9557)$ | 1.5552 |
| $\tau = 1.$ | $E_1 = (733.0161, 1.2061, 3.6377)$ | 1.4077 |
| $\tau$ =1.3431 | $E_0 = (1000, 0, 0, 0)$ | 1 |
| $\tau = 1.5$ | $E_0 = (1000, 0, 0, 0)$ | 0.8552 |
| $\tau = 2$ | $E_0 = (1000, 0, 0, 0)$ | 0.5196 |
| $\tau = 2.5$ | $E_0 = (1000, 0, 0, 0)$ | 0.3157 |

From Table 2 we can see that, $\mathcal{R}_0$ is decreased as $\tau$ is increased. Using the data given in Table 1, we get:

(i) if $0 \leq \tau < 1.343070098$, then $S_1$ exists and it is globally asymptotically stable,

(ii) if $\tau \geq 1.343070098$, then $S_0$ is globally asymptotically stable.

Figures 10-12 show that the numerical results are also compatible with the results of Theorems 1 and 2. It can be seen that when the time delay is increased, the system can be stabilized around the disease-free steady state $S_0$. This means that the delay plays a similar job as the antiviral treatment in clearing the HIV-1 from the plasma.

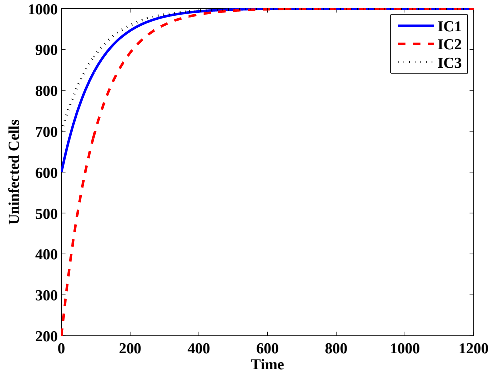


Figure 1: The evolution of uninfected CD4$^+$ T cells with initial IC1-IC3 in case of $R_0 \leq 1$.

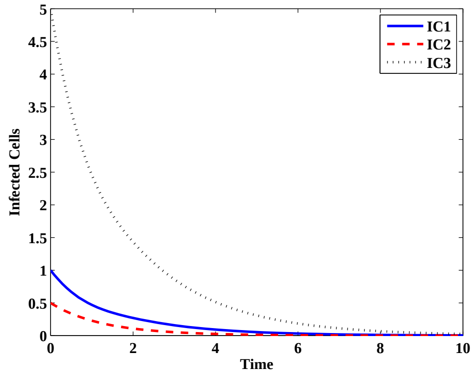


Figure 2: The evolution of infected cells with initial IC1-IC3 in case of $R_0 \leq 1$.

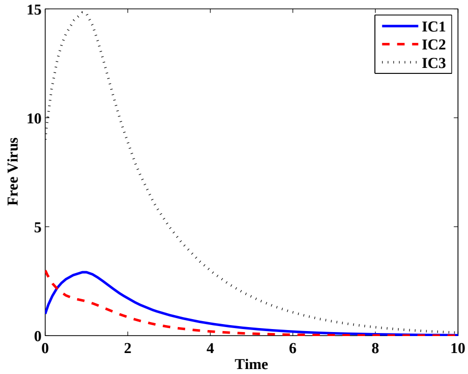


Figure 3: The evolution of free HIV-1 with initial IC1-IC3 in case of $R_0 \leq 1$.

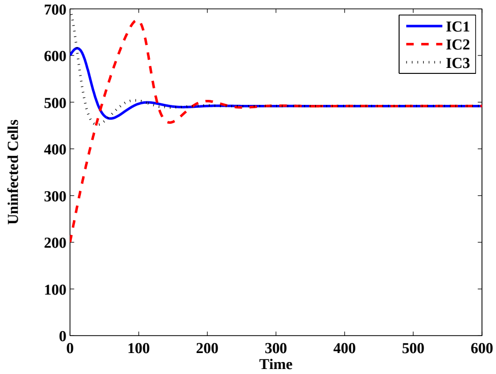


Figure 4: The evolution of uninfected CD4$^+$ T cells with initial IC1-IC3 in case of $R_0 > 1$.

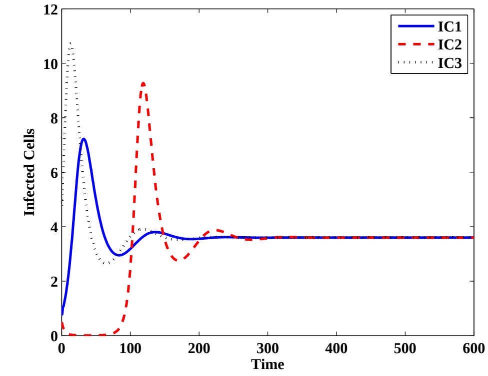


Figure 5: The evolution of infected cells with initial IC1-IC3 in case of $R_0 > 1$.

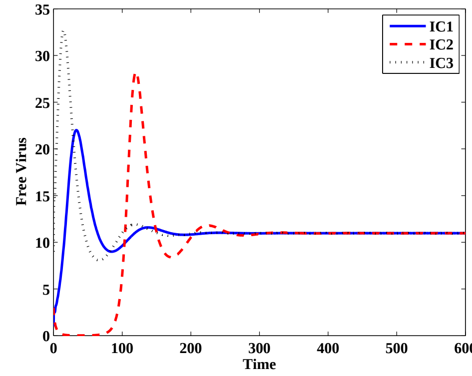


Figure 6: The evolution of free HIV-1 with initial IC1-IC3 in case of $R_0 > 1$.

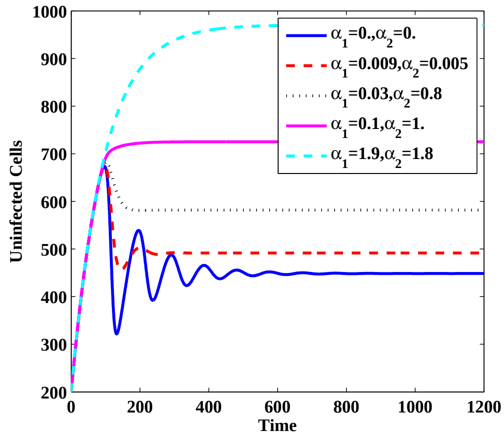


Figure 7: The evolution of uninfected CD4$^+$ T cells with different saturation parameters $\alpha_1$,$\alpha_2$.

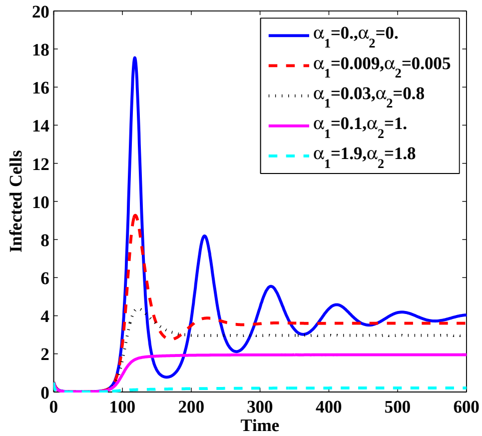


Figure 8: The evolution of infected cells with different saturation parameters $\alpha_1$,$\alpha_2$.

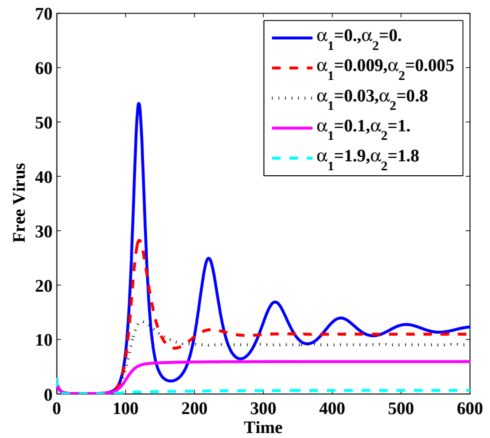


Figure 9: The evolution of free HIV-1 with different saturation parameters $\alpha_1$,$\alpha_2$.

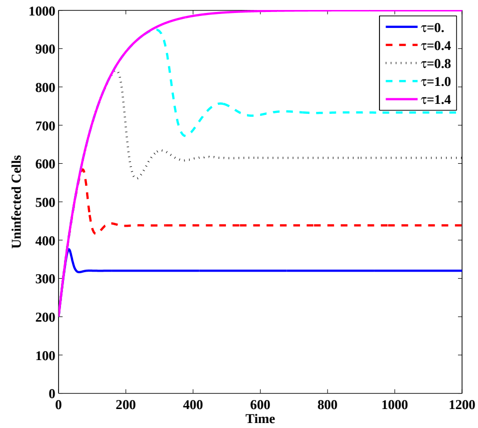


Figure 10: The evolution of uninfected CD4$^+$ T cells with different delay parameter $\tau$.

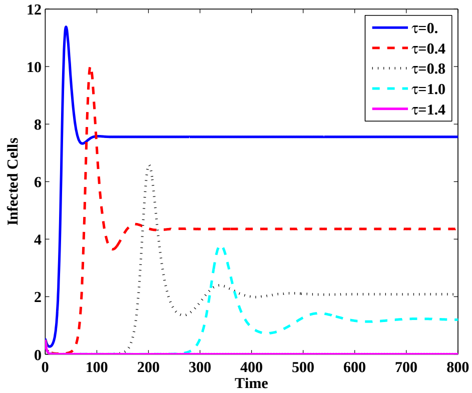


Figure 11: The evolution of infected cells with different delay parameter $\tau$.

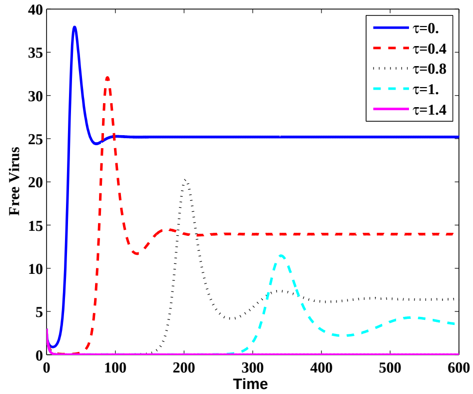


Figure 12: The evolution of free HIV-1 with different delay parameter $\tau$.

## 5 Acknowledgment

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# Equations on Banach space valued functions of abstract $g$-fractional calculus

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**Abstract**

The aim of this paper is utilize proper iterative methods for solving equations on Banach spaces. The differentiability of the operator involved is not assumed neither the convexity of its domain. Applications of the semi-local convergence are suggested including Banach space valued functions of fractional calculus, where all integrals are of Bochner-type.

**2010 AMS Subject Classification Codes:** 65G99, 65H10, 26A33, 46B25, 47J25, 47J05.

**Key Words and Phrases:** iterative method, Banach space, semi-local convergence, Fractional Calculus, Bochner-type integral.

## 1 Introduction

Let $B_1, B_2$ stand for Banach space and let $\Omega$ stand for an open subset of $B_1$. Let also $U(z,\rho) := \{u \in B_1 : \|u - z\| < \rho\}$ and let $\overline{U}(z,\rho)$ stand for the closure of $U(z,\rho)$.

Many problems in Computational Sciences, Engineering, Mathematical Chemistry, Mathematical Physics, Mathematical Economics and other disciplines can written as

$$F(x) = 0 \tag{1.1}$$

using Mathematical Modeling [1]-[17], where $F : \Omega \to B_2$ is a continuous operator. The solution $x^*$ of equation (1.1) is sought in closed form, but this is attainable only in special cases. That explains why most solution methods for such equations are usually iterative. There is a plethora of iterative methods for solving equation (1.1), more the [2, 6, 7, 9 - 13, 15, 16].

**Newton's method** [6, 7, 11, 15, 16]:

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n). \tag{1.2}$$

Secant method:

$$x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1} F(x_n), \tag{1.3}$$

where $[\cdot, \cdot; F]$ denotes a divided difference of order one on $\Omega \times \Omega$ [7, 15, 16].

Newton-like method:

$$x_{n+1} = x_n - E_n^{-1} F(x_n), \tag{1.4}$$

where $E_n = E(F)(x_n)$ and $E : \Omega \to \mathcal{L}(B_1, B_2)$ the space of bounded linear operators from $B_1$ into $B_2$. Other methods can be found in [7], [11], [15], [16] and the references therein.

In the present study we consider the new method defined for each $n = 0, 1, 2, \ldots$ by

$$\begin{gathered} x_{n+1} = G(x_n) \\ G(x_{n+1}) = G(x_n) - A_n^{-1} F(x_n), \end{gathered} \tag{1.5}$$

where $x_0 \in \Omega$ is an initial point, $G : B_3 \to \Omega$ ($B_3$ a Banach space), $A_n = A(F)(x_{n+1}, x_n) = A(x_{n+1}, x_n)$ and $A : \Omega \times \Omega \to \mathcal{L}(B_1, B_2)$. Method (1.5) generates a sequence which we shall show converges to $x^*$ under some Lipschitz-type conditions (to be precised in Section 2). Although method (1.5) (and Section 2) is of independent interest, it is nevertheless designed especially to be used in $g$-Abstract Fractional Calculus (to be precised in Section 3). As far as we know such iterative methods have not yet appeared in connection to solve equations in Abstract Fractional Calculus.

In this paper we present the semi-local convergence of method (1.5) in Section 2. Some applications to Abstract $g$-Fractional Calculus are suggested in Section 3 on a certain Banach space valued functions, where all the integrals are of Bochner-type [8], [14].

## 2 Semi-local Convergence analysis

We present the semi-local convergence analysis of method (1.5) using conditions $(M)$:

$(m_1)$ $F : \Omega \subset B_1 \to B_2$ is continuous, $G : B_3 \to \Omega$ is continuous and $A(x, y) \in \mathcal{L}(B_1, B_2)$ for each $(x, y) \in \Omega \times \Omega$.

2

$(m_2)$ There exist $\beta > 0$ and $\Omega_0 \subset B_1$ such that $A(x,y)^{-1} \in \mathcal{L}(B_2, B_1)$ for each $(x,y) \in \Omega_0 \times \Omega_0$ and

$$\left\| A(x,y)^{-1} \right\| \leq \beta^{-1}.$$

Set $\Omega_1 = \Omega \cap \Omega_0$.

$(m_3)$ There exists a continuous and nondecreasing function $\psi : [0,+\infty)^3 \to [0,+\infty)$ such that for each $x, y \in \Omega_1$

$$\|F(x) - F(y) - A(x,y)(G(x) - G(y))\| \leq$$

$$\beta\psi(\|x-y\|, \|x-x_0\|, \|y-x_0\|) \|G(x) - G(y)\|.$$

$(m_4)$ There exists a continuous and nondecreasing function $\psi_0 : [0,+\infty) \to [0,+\infty)$ such that for each $x \in \Omega_1$

$$\|G(x) - G(x_0)\| \leq \psi_0(\|x-x_0\|) \|x-x_0\|.$$

$(m_5)$ For $x_0 \in \Omega_0$ and $x_1 = G(x_0) \in \Omega_0$ there exists $\eta \geq 0$ such that

$$\left\| A(x_1, x_0)^{-1} F(x_0) \right\| \leq \eta.$$

$(m_6)$ There exists $s > 0$ such that

$$\psi(\eta, s, s) < 1,$$

$$\psi_0(s) < 1$$

and

$$\|G(x_0) - x_0\| \leq s \leq \frac{\eta}{1-q_0},$$

where $q_0 = \psi(\eta, s, s)$.

$(m_7)$ $\overline{U}(x_0, s) \subset \Omega$.

Next, we present the semi-local convergence analysis for method (1.5) using the conditions $(M)$ and the preceding notation.

**Theorem 2.1** *Assume that the conditions $(M)$ hold. Then, sequence $\{x_n\}$ generated by method (1.5) starting at $x_0 \in \Omega$ is well defined in $U(x_0, s)$, remains in $U(x_0, s)$ for each $n = 0, 1, 2, \ldots$ and converges to a solution $x^* \in \overline{U}(x_0, s)$ of equation $F(x) = 0$. The limit point $x^*$ is the unique solution of equation $F(x) = 0$ in $\overline{U}(x_0, s)$.*

**Proof.** By the definition of $s$ and $(m_5)$, we have $x_1 \in U(x_0, s)$. The proof is based on mathematical induction on $k$. Suppose that $\|x_k - x_{k-1}\| \leq q_0^{k-1}\eta$ and $\|x_k - x_0\| \leq s$.

We get by (1.5), $(m_2)-(m_5)$ in turn that

$$\|G(x_{k+1})-G(x_k)\| = \left\|A_k^{-1}F(x_k)\right\| =$$
$$\left\|A_k^{-1}\left(F(x_k)-F(x_{k-1})-A_{k-1}\left(G(x_k)-G(x_{k-1})\right)\right)\right\|$$
$$\leq \left\|A_k^{-1}\right\|\left\|F(x_k)-F(x_{k-1})-A_{k-1}\left(G(x_k)-G(x_{k-1})\right)\right\| \leq$$
$$\beta^{-1}\beta\psi\left(\|x_k-x_{k-1}\|,\|x_{k-1}-x_0\|,\|y_k-x_0\|\right)\|G(x_k)-G(x_{k-1})\| \leq$$
$$\psi(\eta,s,s)\|G(x_k)-G(x_{k-1})\| = q_0\|G(x_k)-G(x_{k-1})\| \leq q_0^k\|x_1-x_0\| \leq q_0^k\eta \tag{2.1}$$

and by $(m_6)$

$$\|x_{k+1}-x_0\| = \|G(x_k)-x_0\| \leq \|G(x_k)-G(x_0)\| + \|G(x_0)-x_0\|$$
$$\leq \psi_0(\|x_k-x_0\|)\|x_k-x_0\| + \|G(x_0)-x_0\|$$
$$\leq \psi_0(s)s + \|G(x_0)-x_0\| \leq s.$$

The induction is completed. Moreover, we have by (2.1) that for $m=0,1,2,...$

$$\|x_{k+m}-x_k\| \leq \frac{1-q_0^m}{1-q_0}q_0^k\eta.$$

It follows from the preceding inequation that sequence $\{G(x_k)\}$ is complete in a Banach space $B_1$ and as such it converges to some $x^* \in \overline{U}(x_0,s)$ (since $\overline{U}(x_0,s)$ is a closed ball). By letting $k\to+\infty$ in (2.1) we get $F(x^*)=0$. We also get by (1.5) that $G(x^*)=x^*$. To show the uniqueness part, let $x^{**}\in U(x_0,s)$ be a solution of equation $F(x)=0$ and $G(x^{**})=x^{**}$. By using (1.5), we obtain in turn that

$$\|x^{**}-G(x_{k+1})\| = \left\|x^{**}-G(x_k)+A_k^{-1}F(x_k)-A_k^{-1}F(x^{**})\right\| \leq$$
$$\left\|A_k^{-1}\right\|\left\|F(x^{**})-F(x_k)-A_k\left(G(x^{**})-G(x_k)\right)\right\| \leq$$
$$\beta^{-1}\beta\psi_0\left(\|x^{**}-x_k\|,\|x_{k+1}-x_0\|,\|x_k-x_0\|\right)\|G(x^{**})-G(x_k)\| \leq$$
$$q_0\|G(x^{**})-G(x_k)\| \leq q_0^{k+1}\|x^{**}-x_0\|,$$

so $\lim\limits_{k\to+\infty}x_k=x^{**}$. We have shown that $\lim\limits_{k\to+\infty}x_k=x^*$, so $x^*=x^{**}$. ■

**Remark 2.2** *(1) Condition $(m_2)$ can become part of condition $(m_3)$ by considering*

$(m_3)'$ *There exists a continuous and nondecreasing function $\varphi:[0,+\infty)^3\to[0,+\infty)$ such that for each $x,y\in\Omega_1$*

$$\left\|A(x,y)^{-1}\left[F(x)-F(y)-A(x,y)\left(G(x)-G(y)\right)\right]\right\| \leq$$

4

$$\varphi\left(\|x-y\|,\|x-x_0\|,\|y-x_0\|\right)\|G(x)-G(y)\|.$$

*Notice that*

$$\varphi(u_1,u_2,u_3)\leq\psi(u_1,u_2,u_3)$$

*for each* $u_1\geq 0$, $u_2\geq 0$ *and* $u_3\geq 0$. *Similarly, a function* $\varphi_1$ *can replace* $\psi_1$ *for the uniqueness of the solution part. These replacements are of Mysovskii-type [6], [11], [15] and influence the weaking of the convergence criterion in* $(m_6)$, *error bounds and the precision of* $s$.

*(2) Suppose that there exist* $\beta>0$, $\beta_1>0$ *and* $L\in\mathcal{L}(B_1,B_2)$ *with* $L^{-1}\in\mathcal{L}(B_2,B_1)$ *such that*

$$\left\|L^{-1}\right\|\leq\beta^{-1}$$

$$\|A(x,y)-L\|\leq\beta_1$$

*and*

$$\beta_2:=\beta^{-1}\beta_1<1.$$

*Then, it follows from the Banach lemma on invertible operators [11], and*

$$\left\|L^{-1}\right\|\|A(x,y)-L\|\leq\beta^{-1}\beta_1=\beta_2<1$$

*that* $A(x,y)^{-1}\in\mathcal{L}(B_2,B_1)$. *Let* $\beta=\frac{\beta^{-1}}{1-\beta_2}$. *Then, under these replacements, condition* $(m_2)$ *is implied, therefore it can be dropped from the conditions* $(M)$.

**Remark 2.3** *Section 2 has an interest independent of Section 3. It is worth noticing that the results especially of Theorem 2.1 can apply in Abstract g-Fractional Calculus as illustrated in Section 3. By specializing function* $\psi$, *we can apply the results of say Theorem 2.1 in the examples suggested in Section 3. In particular for (3.21), we choose for* $u_1\geq 0$, $u_2\geq 0$, $u_3\geq 0$

$$\psi(u_1,u_2,u_3)=\frac{\lambda\mu_1^{\nu}}{\beta\Gamma(\nu)(\nu+1)},$$

*if* $|g(x)-g(y)|\leq\mu_1$ *for each* $x,y\in[a,b]$;

$$\psi(u_1,u_2,u_3)=\frac{\lambda\mu_2^{\nu}}{\beta\Gamma(\nu)(\nu+1)},$$

*if* $|g(x)-g(y)|\leq\xi_2\|x-y\|$ *for each* $x,y\in[a,b]$ *and* $\mu_2=\xi_2|b-a|$;

$$\psi(u_1,u_2,u_3)=\frac{\lambda\mu_3^{\nu}}{\beta\Gamma(\nu)(\nu+1)},$$

*if* $|g(x)|\leq\xi_3$ *for each* $x,y\in[a,b]$ *and* $\mu_3=2\xi_3$, *where* $\lambda,\nu$ *and* $F$ *are defined in Section 3. Other choices of function* $\psi$ *are also possible.*

*Notice that with these choices of function* $\psi$ *and* $f=F$ *and* $g=G$, *crucial condition* $(m_3)$ *is satisfied, which justifies our definition of method (1.5). We can provide similar choices for the other examples of Section 3.*

# 3 Applications to $X$-valued $g$-Fractional Calculus of Canavati type

Here we deal with Banach space $(X, \|\cdot\|)$ valued functions $f$ of real domain $[a, b]$. All integrals here are of Bochner-type, see [14]. The derivatives of $f$ are defined similarly to numerical ones, see [17], pp. 83-86 and p. 93.

Here both needed backgrounds come from [5].

Let $\nu > 1$, $\nu \notin \mathbb{N}$, with integral part $[\nu] = n \in \mathbb{N}$. Let $g : [a, b] \to \mathbb{R}$ be a strictly increasing function, such that $g \in C^1([a, b])$, $g^{-1} \in C^n([g(a), g(b)])$, and let $f \in C^n([a, b], X)$. It clear then we obtain that $\left(f \circ g^{-1}\right) \in C^n([g(a), g(b)], X)$. Let $\alpha := \nu - [\nu] = \nu - n$ $(0 < \alpha < 1)$.

(I) See [5]. Let $h \in C([g(a), g(b)], X)$, we define the $X$-valued left Riemann-Liouville fractional integral as

$$\left(J_\nu^{z_0} h\right)(z) := \frac{1}{\Gamma(\nu)} \int_{z_0}^{z} (z - t)^{\nu - 1} h(t)\, dt, \tag{3.1}$$

for $g(a) \le z_0 \le z \le g(b)$, where $\Gamma$ is the gamma function.

We define the subspace $C_{g(x)}^{\nu}([g(a), g(b)], X)$ of $C^n([g(a), g(b)], X)$, where $x \in [a, b]$:

$$C_{g(x)}^{\nu}([g(a), g(b)], X) := \left\{ h \in C^n([g(a), g(b)], X) : J_{1-\alpha}^{g(x)} h^{(n)} \in C^1([g(x), g(b)], X) \right\}. \tag{3.2}$$

So let $h \in C_{g(x)}^{\nu}([g(a), g(b)], X)$; we define the $X$-valued left $g$-generalized fractional derivative of $h$ of order $\nu$, of Canavati type, over $[g(x), g(b)]$ as

$$D_{g(x)}^{\nu} h := \left(J_{1-\alpha}^{g(x)} h^{(n)}\right)'. \tag{3.3}$$

Clearly, for $h \in C_{g(x)}^{\nu}([g(a), g(b)], X)$, there exists

$$\left(D_{g(x)}^{\nu} h\right)(z) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dz} \int_{g(x)}^{z} (z - t)^{-\alpha} h^{(n)}(t)\, dt, \tag{3.4}$$

for all $g(x) \le z \le g(b)$.

In particular, when $f \circ g^{-1} \in C_{g(x)}^{\nu}([g(a), g(b)], X)$ we have that

$$\left(D_{g(x)}^{\nu}\left(f \circ g^{-1}\right)\right)(z) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dz} \int_{g(x)}^{z} (z - t)^{-\alpha} \left(f \circ g^{-1}\right)^{(n)}(t)\, dt, \tag{3.5}$$

for all $z : g(x) \le z \le g(b)$.

We have that $D_{g(x)}^{n}\left(f \circ g^{-1}\right) = \left(f \circ g^{-1}\right)^{(n)}$ and $D_{g(x)}^{0}\left(f \circ g^{-1}\right) = f \circ g^{-1}$.

From [5] we have for $\left(f \circ g^{-1}\right) \in C_{g(x)}^{\nu}([g(a), g(b)], X)$, where $x \in [a, b]$, ($X$-valued left fractional Taylor's formula) that

$$f(y) - f(x) = \sum_{k=1}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(x))}{k!} (g(y) - g(x))^k + \tag{3.6}$$

$$\frac{1}{\Gamma(\nu)}\int_{g(x)}^{g(y)}(g(y)-t)^{\nu-1}\left(D_{g(x)}^{\nu}\left(f\circ g^{-1}\right)\right)(t)\,dt, \quad \text{for all } y\in[a,b]: y\geq x.$$

Alternatively, for $\left(f\circ g^{-1}\right)\in C_{g(y)}^{\nu}([g(a),g(b)],X)$, where $y\in[a,b]$, we can write (again $X$-valued left fractional Taylor's formula) that:

$$f(x)-f(y)=\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!}(g(x)-g(y))^{k}+ \tag{3.7}$$

$$\frac{1}{\Gamma(\nu)}\int_{g(y)}^{g(x)}(g(x)-t)^{\nu-1}\left(D_{g(y)}^{\nu}\left(f\circ g^{-1}\right)\right)(t)\,dt, \quad \text{for all } x\in[a,b]: x\geq y.$$

Here we consider $f\in C^{n}([a,b],X)$, such that $\left(f\circ g^{-1}\right)\in C_{g(x)}^{\nu}([g(a),g(b)],X)$, for every $x\in[a,b]$; which is the same as $\left(f\circ g^{-1}\right)\in C_{g(y)}^{\nu}([g(a),g(b)],X)$, for every $y\in[a,b]$ (i.e. exchange roles of $x$ and $y$); we write that as $\left(f\circ g^{-1}\right)\in C_{g+}^{\nu}([g(a),g(b)],X)$.

We have that

$$\left(D_{g(y)}^{\nu}\left(f\circ g^{-1}\right)\right)(z)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\int_{g(y)}^{z}(z-t)^{-\alpha}\left(f\circ g^{-1}\right)^{(n)}(t)\,dt, \tag{3.8}$$

for all $z: g(y)\leq z\leq g(b)$.

So here we work with $f\in C^{n}([a,b],X)$, such that $\left(f\circ g^{-1}\right)\in C_{g+}^{\nu}([g(a),g(b)],X)$.

We define the $X$-valued left linear fractional operator

$$(A_1(f))(x,y):=\begin{cases}\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(x))}{k!}(g(y)-g(x))^{k-1}+\\ \left(D_{g(x)}^{\nu}\left(f\circ g^{-1}\right)\right)(g(y))\frac{(g(y)-g(x))^{\nu-1}}{\Gamma(\nu+1)}, \quad y>x,\\ \\ \sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!}(g(x)-g(y))^{k-1}+\\ \left(D_{g(y)}^{\nu}\left(f\circ g^{-1}\right)\right)(g(x))\frac{(g(x)-g(y))^{\nu-1}}{\Gamma(\nu+1)}, \quad x>y,\\ \\ f^{(n)}(x), \quad x=y.\end{cases} \tag{3.9}$$

We may assume that (see [12], p. 3)

$$\|(A_1(f))(x,x)-(A_1(f))(y,y)\|=\left\|f^{(n)}(x)-f^{(n)}(y)\right\|=$$

$$\left\|\left(f^{(n)}\circ g^{-1}\right)(g(x))-\left(f^{(n)}\circ g^{-1}\right)(g(y))\right\|\leq\Phi\,|g(x)-g(y)|, \tag{3.10}$$

where $\Phi>0$; for any $x,y\in[a,b]$.

We make the following estimations:

(i) case of $y>x$: We have that

$$\|f(y)-f(x)-(A_1(f))(x,y)(g(y)-g(x))\|=$$

$$\left\| \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(y)} (g(y)-t)^{\nu-1} \left( D_{g(x)}^{\nu} \left( f \circ g^{-1} \right) \right)(t)\, dt - \right.$$
$$\left. \left( D_{g(x)}^{\nu} \left( f \circ g^{-1} \right) \right)(g(y)) \frac{(g(y)-g(x))^{\nu}}{\Gamma(\nu+1)} \right\|$$

(by [1], p. 426, Theorem 11.43)

$$= \frac{1}{\Gamma(\nu)} \left\| \int_{g(x)}^{g(y)} (g(y)-t)^{\nu-1} \left( \left( D_{g(x)}^{\nu} \left( f \circ g^{-1} \right) \right)(t) - \left( D_{g(x)}^{\nu} \left( f \circ g^{-1} \right) \right)(g(y)) \right) dt \right\| \tag{3.11}$$

(by [8])

$$\leq \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(y)} (g(y)-t)^{\nu-1} \left\| \left( D_{g(x)}^{\nu} \left( f \circ g^{-1} \right) \right)(t) - \left( D_{g(x)}^{\nu} \left( f \circ g^{-1} \right) \right)(g(y)) \right\| dt$$

(we assume here that

$$\left\| \left( D_{g(x)}^{\nu} \left( f \circ g^{-1} \right) \right)(t) - \left( D_{g(x)}^{\nu} \left( f \circ g^{-1} \right) \right)(g(y)) \right\| \leq \lambda_1 |t-g(y)|, \tag{3.12}$$

for every $t, g(y), g(x) \in [g(a), g(b)]$ such that $g(y) \geq t \geq g(x)$; $\lambda_1 > 0$)

$$\leq \frac{\lambda_1}{\Gamma(\nu)} \int_{g(x)}^{g(y)} (g(y)-t)^{\nu-1} (g(y)-t)\, dt = \tag{3.13}$$

$$\frac{\lambda_1}{\Gamma(\nu)} \int_{g(x)}^{g(y)} (g(y)-t)^{\nu}\, dt = \frac{\lambda_1}{\Gamma(\nu)} \frac{(g(y)-g(x))^{\nu+1}}{(\nu+1)}. \tag{3.14}$$

We have proved that

$$\|f(y)-f(x)-(A_1(f))(x,y)(g(y)-g(x))\| \leq \frac{\lambda_1}{\Gamma(\nu)} \frac{(g(y)-g(x))^{\nu+1}}{(\nu+1)}, \tag{3.15}$$

for all $x, y \in [a,b] : y > x$.

(ii) Case of $x > y$ : We observe that

$$\|f(y)-f(x)-(A_1(f))(x,y)(g(y)-g(x))\| =$$

$$\|f(x)-f(y)-(A_1(f))(x,y)(g(x)-g(y))\| =$$

$$\left\| \frac{1}{\Gamma(\nu)} \int_{g(y)}^{g(x)} (g(x)-t)^{\nu-1} \left( D_{g(y)}^{\nu} \left( f \circ g^{-1} \right) \right)(t)\, dt - \right.$$
$$\left. \left( D_{g(y)}^{\nu} \left( f \circ g^{-1} \right) \right)(g(x)) \frac{(g(x)-g(y))^{\nu}}{\Gamma(\nu+1)} \right\| = \tag{3.16}$$

$$\frac{1}{\Gamma(\nu)} \left\| \int_{g(y)}^{g(x)} (g(x)-t)^{\nu-1} \left( \left( D_{g(y)}^{\nu} \left( f \circ g^{-1} \right) \right)(t) - \left( D_{g(y)}^{\nu} \left( f \circ g^{-1} \right) \right)(g(x)) \right) dt \right\| \leq$$

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$$\frac{1}{\Gamma(\nu)}\int_{g(y)}^{g(x)}(g(x)-t)^{\nu-1}\left\|\left(D_{g(y)}^{\nu}\left(f\circ g^{-1}\right)\right)(t)-\left(D_{g(y)}^{\nu}\left(f\circ g^{-1}\right)\right)(g(x))\right\|dt \tag{3.17}$$

(we assume that

$$\left\|\left(D_{g(y)}^{\nu}\left(f\circ g^{-1}\right)\right)(t)-\left(D_{g(y)}^{\nu}\left(f\circ g^{-1}\right)\right)(g(x))\right\|\leq\lambda_2\left|t-g(x)\right|, \tag{3.18}$$

for all $t, g(x), g(y) \in [g(a), g(b)]$ such that $g(x) \geq t \geq g(y)$; $\lambda_2 > 0$)

$$\leq\frac{\lambda_2}{\Gamma(\nu)}\int_{g(y)}^{g(x)}(g(x)-t)^{\nu-1}(g(x)-t)\,dt= \tag{3.19}$$

$$\frac{\lambda_2}{\Gamma(\nu)}\int_{g(y)}^{g(x)}(g(x)-t)^{\nu}\,dt=\frac{\lambda_2}{\Gamma(\nu)}\frac{(g(x)-g(y))^{\nu+1}}{(\nu+1)}.$$

We have proved that

$$\|f(y)-f(x)-(A_1(f))(x,y)(g(y)-g(x))\|\leq\frac{\lambda_2}{\Gamma(\nu)}\frac{(g(x)-g(y))^{\nu+1}}{(\nu+1)}, \tag{3.20}$$

for any $x, y \in [a, b] : x > y$.

**Conclusion 3.1** *Set* $\lambda := \max(\lambda_1, \lambda_2)$. *Then*

$$\|f(y)-f(x)-(A_1(f))(x,y)(g(y)-g(x))\|\leq\frac{\lambda}{\Gamma(\nu)}\frac{|g(y)-g(x)|^{\nu+1}}{(\nu+1)}, \tag{3.21}$$

$\forall\ x, y \in [a, b]$ *(the case of* $x = y$ *is trivially true).*

We may choose that $\frac{\lambda}{\Gamma(\nu)} < 1$.

Also we notice here that $\nu + 1 > 2$.

(II) See [5] again. Let $h \in C([g(a), g(b)], X)$, we define the $X$-valued right Riemann-Liouville fractional integral as

$$\left(J_{z_0-}^{\nu}h\right)(z):=\frac{1}{\Gamma(\nu)}\int_{z}^{z_0}(t-z)^{\nu-1}h(t)\,dt, \tag{3.22}$$

for $g(a) \leq z \leq z_0 \leq g(b)$.

We define the subspace $C_{g(x)-}^{\nu}([g(a), g(b)], X)$ of $C^n([g(a), g(b)], X)$, where $x \in [a, b]$ :

$$C_{g(x)-}^{\nu}([g(a),g(b)],X):=\left\{h\in C^n([g(a),g(b)],X):J_{g(x)-}^{1-\alpha}h^{(n)}\in C^1([g(a),g(x)],X)\right\}. \tag{3.23}$$

So let $h \in C_{g(x)-}^{\nu}([g(a), g(b)], X)$; we define the $X$-valued right $g$-generalized fractional derivative of $h$ of order $\nu$, of Canavati type, over $[g(a), g(x)]$ as

$$D_{g(x)-}^{\nu}h:=(-1)^{n-1}\left(J_{g(x)-}^{1-\alpha}h^{(n)}\right)'. \tag{3.24}$$

Clearly, for $h \in C^{\nu}_{g(x)-}([g(a), g(b)], X)$, there exists

$$\left(D^{\nu}_{g(x)-}h\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)}\frac{d}{dz}\int_z^{g(x)} (t-z)^{-\alpha} h^{(n)}(t)\,dt, \tag{3.25}$$

for all $g(a) \leq z \leq g(x) \leq g(b)$.

In particular, when $f \circ g^{-1} \in C^{\nu}_{g(x)-}([g(a), g(b)], X)$ we have that

$$\left(D^{\nu}_{g(x)-}\left(f \circ g^{-1}\right)\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)}\frac{d}{dz}\int_z^{g(x)} (t-z)^{-\alpha} \left(f \circ g^{-1}\right)^{(n)}(t)\,dt, \tag{3.26}$$

for all $g(a) \leq z \leq g(x) \leq g(b)$.

We get that

$$\left(D^{n}_{g(x)-}\left(f \circ g^{-1}\right)\right)(z) = (-1)^n \left(f \circ g^{-1}\right)^{(n)}(z), \tag{3.27}$$

and

$$\left(D^{0}_{g(x)-}\left(f \circ g^{-1}\right)\right)(z) = \left(f \circ g^{-1}\right)(z), \tag{3.28}$$

for all $z \in [g(a), g(x)]$, see [5].

From [5] we have, for $\left(f \circ g^{-1}\right) \in C^{\nu}_{g(x)-}([g(a), g(b)], X)$, where $x \in [a,b]$, $\nu \geq 1$ ($X$-valued right fractional Taylor's formula) that:

$$f(y) - f(x) = \sum_{k=1}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(x))}{k!}(g(y) - g(x))^k +$$

$$\frac{1}{\Gamma(\nu)}\int_{g(y)}^{g(x)} (t - g(y))^{\nu-1}\left(D^{\nu}_{g(x)-}\left(f \circ g^{-1}\right)\right)(t)\,dt, \quad \text{all } a \leq y \leq x. \tag{3.29}$$

Alternatively, for $\left(f \circ g^{-1}\right) \in C^{\nu}_{g(y)-}([g(a), g(b)], X)$, where $y \in [a,b]$, $\nu \geq 1$ (again $X$-valued right fractional Taylor's formula) that:

$$f(x) - f(y) = \sum_{k=1}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(y))}{k!}(g(x) - g(y))^k +$$

$$\frac{1}{\Gamma(\nu)}\int_{g(x)}^{g(y)} (t - g(x))^{\nu-1}\left(D^{\nu}_{g(y)-}\left(f \circ g^{-1}\right)\right)(t)\,dt, \quad \text{all } a \leq x \leq y. \tag{3.30}$$

Here we consider $f \in C^n([a,b], X)$, such that $\left(f \circ g^{-1}\right) \in C^{\nu}_{g(x)-}([g(a), g(b)], X)$, for every $x \in [a,b]$; which is the same as $\left(f \circ g^{-1}\right) \in C^{\nu}_{g(y)-}([g(a), g(b)], X)$, for every $y \in [a,b]$; (i.e. exchange roles of $x$ and $y$) we write that as $\left(f \circ g^{-1}\right) \in C^{\nu}_{g-}([g(a), g(b)], X)$.

We have that

$$\left(D^{\nu}_{g(y)-}\left(f \circ g^{-1}\right)\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)}\frac{d}{dz}\int_z^{g(y)} (t-z)^{-\alpha} \left(f \circ g^{-1}\right)^{(n)}(t)\,dt, \tag{3.31}$$

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for all $g(a) \leq z \leq g(y) \leq g(b)$.

So here we work with $f \in C^n([a,b], X)$, such that $\left(f \circ g^{-1}\right) \in C^{\nu}_{g-}([g(a), g(b)], X)$.

We define the $X$-valued right linear fractional operator

$$(A_2(f))(x,y) := \begin{cases} \sum_{k=1}^{n-1} \frac{\left(f\circ g^{-1}\right)^{(k)}(g(x))}{k!}(g(y)-g(x))^{k-1} - \\ \left(D^{\nu}_{g(x)-}\left(f\circ g^{-1}\right)\right)(g(y))\frac{(g(x)-g(y))^{\nu-1}}{\Gamma(\nu+1)}, \quad x>y, \\ \\ \sum_{k=1}^{n-1} \frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!}(g(x)-g(y))^{k-1} - \\ \left(D^{\nu}_{g(y)-}\left(f\circ g^{-1}\right)\right)(g(x))\frac{(g(y)-g(x))^{\nu-1}}{\Gamma(\nu+1)}, \quad y>x, \\ \\ f^{(n)}(x), \quad x=y. \end{cases} \tag{3.32}$$

We may assume that ([12], p. 3)

$$\|(A_2(f))(x,x) - (A_2(f))(y,y)\| = \left\|f^{(n)}(x) - f^{(n)}(y)\right\| \leq \Phi^* |g(x) - g(y)|, \tag{3.33}$$

where $\Phi^* > 0$; for any $x, y \in [a,b]$.

We make the following estimations:

(i) case of $x > y$ : We have that

$$\begin{gathered} \|f(x) - f(y) - (A_2(f))(x,y)(g(x) - g(y))\| = \\ \|f(y) - f(x) - (A_2(f))(x,y)(g(y) - g(x))\| = \\ \|f(y) - f(x) + (A_2(f))(x,y)(g(x) - g(y))\| = \end{gathered} \tag{3.34}$$

$$\left\|\frac{1}{\Gamma(\nu)}\int_{g(y)}^{g(x)}(t-g(y))^{\nu-1}\left(D^{\nu}_{g(x)-}\left(f\circ g^{-1}\right)\right)(t)\,dt - \left(D^{\nu}_{g(x)-}\left(f\circ g^{-1}\right)\right)(g(y))\frac{(g(x)-g(y))^{\nu}}{\Gamma(\nu+1)}\right\| \tag{3.35}$$

(by [1], p. 426, Theorem 11.43)

$$= \frac{1}{\Gamma(\nu)}\left\|\int_{g(y)}^{g(x)}(t-g(y))^{\nu-1}\left(\left(D^{\nu}_{g(x)-}\left(f\circ g^{-1}\right)\right)(t) - \left(D^{\nu}_{g(x)-}\left(f\circ g^{-1}\right)\right)(g(y))\right)dt\right\|$$

(by [8])

$$\leq \frac{1}{\Gamma(\nu)}\int_{g(y)}^{g(x)}(t-g(y))^{\nu-1}\left\|\left(D^{\nu}_{g(x)-}\left(f\circ g^{-1}\right)\right)(t) - \left(D^{\nu}_{g(x)-}\left(f\circ g^{-1}\right)\right)(g(y))\right\|dt \tag{3.36}$$

(we assume here that

$$\left\|\left(D^{\nu}_{g(x)-}\left(f\circ g^{-1}\right)\right)(t) - \left(D^{\nu}_{g(x)-}\left(f\circ g^{-1}\right)\right)(g(y))\right\| \leq \rho_1 |t - g(y)|, \tag{3.37}$$

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for every $t, g(y), g(x) \in [g(a), g(b)]$ such that $g(x) \geq t \geq g(y)$; $\rho_1 > 0$)

$$\leq \frac{\rho_1}{\Gamma(\nu)} \int_{g(y)}^{g(x)} (t - g(y))^{\nu-1} (t - g(y)) \, dt =$$

$$\frac{\rho_1}{\Gamma(\nu)} \int_{g(y)}^{g(x)} (t - g(y))^{\nu} \, dt = \frac{\rho_1}{\Gamma(\nu)} \frac{(g(x) - g(y))^{\nu+1}}{(\nu+1)}. \tag{3.38}$$

We have proved that

$$\|f(x) - f(y) - (A_2(f))(x,y)(g(x) - g(y))\| \leq \frac{\rho_1}{\Gamma(\nu)} \frac{(g(x) - g(y))^{\nu+1}}{(\nu+1)}, \tag{3.39}$$

$\forall\, x, y \in [a,b] : x > y$.

(ii) Case of $x < y$ : We have that

$$\|f(x) - f(y) - (A_2(f))(x,y)(g(x) - g(y))\| =$$

$$\|f(x) - f(y) + (A_2(f))(x,y)(g(y) - g(x))\| = \tag{3.40}$$

$$\left\| \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(y)} (t - g(x))^{\nu-1} \left( D_{g(y)-}^{\nu} \left( f \circ g^{-1} \right) \right)(t) \, dt - \right.$$

$$\left. \left( D_{g(y)-}^{\nu} \left( f \circ g^{-1} \right) \right)(g(x)) \frac{(g(y) - g(x))^{\nu}}{\Gamma(\nu+1)} \right\| =$$

$$\frac{1}{\Gamma(\nu)} \left\| \int_{g(x)}^{g(y)} (t - g(x))^{\nu-1} \left( \left( D_{g(y)-}^{\nu} \left( f \circ g^{-1} \right) \right)(t) - \left( D_{g(y)-}^{\nu} \left( f \circ g^{-1} \right) \right)(g(x)) \right) dt \right\| \leq$$

$$\frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(y)} (t - g(x))^{\nu-1} \left\| \left( D_{g(y)-}^{\nu} \left( f \circ g^{-1} \right) \right)(t) - \left( D_{g(y)-}^{\nu} \left( f \circ g^{-1} \right) \right)(g(x)) \right\| dt \tag{3.41}$$

(we assume that

$$\left\| \left( D_{g(y)-}^{\nu} \left( f \circ g^{-1} \right) \right)(t) - \left( D_{g(y)-}^{\nu} \left( f \circ g^{-1} \right) \right)(g(x)) \right\| \leq \rho_2 |t - g(x)|, \tag{3.42}$$

for any $t, g(x), g(y) \in [g(a), g(b)] : g(y) \geq t \geq g(x)$; $\rho_2 > 0$)

$$\leq \frac{\rho_2}{\Gamma(\nu)} \int_{g(x)}^{g(y)} (t - g(x))^{\nu-1} (t - g(x)) \, dt =$$

$$\frac{\rho_2}{\Gamma(\nu)} \int_{g(x)}^{g(y)} (t - g(x))^{\nu} \, dt = \tag{3.43}$$

$$\frac{\rho_2}{\Gamma(\nu)} \frac{(g(y) - g(x))^{\nu+1}}{(\nu+1)}. \tag{3.44}$$

We have proved that

$$\|f(x)-f(y)-(A_2(f))(x,y)(g(x)-g(y))\| \leq \frac{\rho_2}{\Gamma(\nu)} \frac{(g(y)-g(x))^{\nu+1}}{(\nu+1)}, \tag{3.45}$$

$\forall\ x,y \in [a,b] : x < y$.

**Conclusion 3.2** *Set* $\rho := \max(\rho_1, \rho_2)$. *Then*

$$\|f(x)-f(y)-(A_2(f))(x,y)(g(x)-g(y))\| \leq \frac{\rho}{\Gamma(\nu)} \frac{|g(x)-g(y)|^{\nu+1}}{(\nu+1)}, \tag{3.46}$$

$\forall\ x,y \in [a,b]$ *((3.46) is trivially true when* $x = y$*).*

One may choose $\frac{\rho}{\Gamma(\nu)} < 1$.
Here again $\nu + 1 > 2$.

**Conclusion 3.3** *Based on (3.10) and (3.21) of (I), and based on (3.33) and (3.46) of (II), using our numerical results presented earlier, we can solve numerically* $f(x) = 0$.

Some examples for $g$ follow:

$g(x) = e^x,\ \ x \in [a,b] \subset \mathbb{R}$,
$g(x) = \sin x$,
$g(x) = \tan x$,
where $x \in \left[-\frac{\pi}{2}+\varepsilon, \frac{\pi}{2}-\varepsilon\right]$, with $\varepsilon > 0$ small.

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