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On some applications of differential subordinations

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Abstract

The purpose of the presented paper is to derive several properties and characteristics of analytic functions, by applying the differential subordination techniques. Several consequences of the results stated here are also pointed out.

Key words: analytic functions; univalent functions; starlike functions; convex functions; differential subordination; best dominant.

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1 Introduction

Let $\mathcal{A}$ be the class of analytic functions which are analytic in the open unit disk $U = \{z : |z| < 1\}$, of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad z \in U. \tag{1.1}$$

By S we denote the subclass of $\mathcal{A}$ consisting of univalent functions.
Principle of Subordination (see [6]): If f and g are two analytic functions in U, we say that f is subordinate to g, written as $f \prec g$, if there exists a Schwarz function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, for all $z \in U$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.
Important results in the theory of differential subordinations were elaborated by Miller and Mocanu in ([6]) and ([5]).

Definition 1.1 *[6] Denote by Q the set of all functions q that are analytic and injective on $\overline{U} \backslash E(q)$, where $E(q) = \{\zeta \in \partial U : \lim_{z\to\zeta} q(z) = \infty\}$, and are such that $q'(\zeta) \neq 0$, for $\zeta \in \partial U \backslash E(q)$.*

If $q \in Q$, then $\Delta = q(U)$ is a simply connected domain.

In 1915, J. Alexander (see [1]) introduced the class of starlike functions defined as:

$$S^* = \left\{ f \in \mathcal{A} : \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > 0,\ z \in U \right\}. \tag{1.2}$$

Let S^*_α denote the class of starlike univalent functions of order α, $0 \le \alpha < 1$, defined as:

$$S^*(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > \alpha,\ z \in U,\ 0 \le \alpha < 1 \right\}. \tag{1.3}$$

A function $f(z) \in \mathcal{A}$ is said to be close-to-convex if there exists a function $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$ in S^* such that

$$\operatorname{Re}\left[\frac{zf'(z)}{g(z)}\right] > 0, \text{ for all } z \in U. \tag{1.4}$$

This class of functions $f(z) \in \mathcal{A}$ with the condition (1.4) is denoted by K and called the class of close-to-convex functions. The class K was introduced by Kaplan [3], who showed that all close-to-convex functions are univalent. A function $f(z) \in \mathcal{A}$ is said to belongs to the class $K(\alpha)$ of close-to-convex functions of order α in U, if $g(z) \in S^*(\alpha)$ and satisfy

$$\mathrm{Re}\left[\frac{zf^{'}(z)}{g(z)}\right] > \alpha, \text{ for all } z \in U,\ 0 \leq \alpha < 1. \tag{1.5}$$

In the present investigation, we will obtain different properties of $\frac{zf^{'}(z)}{g(z)}$, for $f \in \mathcal{A}$, given by (1.1) and $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$ in S^*, by using differential subordination. Many studies of the classes involving $\frac{zf^{'}(z)}{g(z)}$ were appeared in the literature (see [2]). Several authors obtained several applications in the geometric functions theory by using differential subordination (see [4], [8], [7], [11], [9], [10], [12]). Also, sufficient conditions for close-to-convex and close-to-convex of order α of a function $f \in \mathcal{A}$ are obtained. Some consequences of the main results are mentioned.

The following lemmas will be required in order to prove our main results.

Lemma 1.1 *[6] Let the function q be univalent in the unit disk U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$, when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that*

1. Q is starlike univalent in the unit disk U
and
2. $\mathrm{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \mathrm{Re}\left[\frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right] > 0$, for $z \in U$.
If p is analytic in U, with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z), \tag{1.6}$$

then $p(z) \prec q(z)$ and q is the best dominant of (1.6).

Lemma 1.2 *[6] Let the function $q \in Q$, with $q(0) = a$ and let $p(z) = a + a_n z^n + \ldots$ be analytic in U with $p(z) \neq a$ and $n \geq 1$. If p is not subordinate to q, then there exist points $z_0 = r_0 e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U \backslash E(q)$ and an $m \geq n \geq 1$, for which $p(U_{r_0}) \subset q(U)$,*

$$(1)\ p(z_0) = q(\zeta_0), \tag{1.7}$$
$$(2)\ z_0 p'(z_0) = m\zeta_0 q'(\zeta_0), \tag{1.8}$$
$$(3)\ \mathrm{Re}\left[\frac{z_0 p''(z_0)}{p'(z_0)} + 1\right] \geq m\mathrm{Re}\left[\frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1\right]. \tag{1.9}$$

2 Main results

Theorem 2.1 *Let $f(z) = z + a_2 z^2 + \ldots$ and $g(z) = z + b_2 z^2 + \ldots$ be analytic functions in U, with $f(0) = g(0) = 0$, $g(z) \in S^*$, with $g(z) \neq zf^{'}(z)$ and*

$$\mathrm{Re}\left[\frac{2(f'(z))^2}{f'(z)g'(z) - f''(z)g(z)} - \frac{zf'(z)}{g(z)}\right] > 0, \quad z \in U. \tag{2.1}$$

Then $f \in K$.

Proof. At the begining let us note that $f \in K$ is equivalent with the condition

$$\mathrm{Re}\left[\frac{zf^{'}(z)}{g(z)}\right] > 0, \quad z \in U. \tag{2.2}$$

Let

$$p(z) = \frac{zf^{'}(z)}{g(z)} = 1 + p_1 z + p_2 z^2 + \ldots,\ z \in U, \tag{2.3}$$

with $p(z) \neq 1$, and let

$$q(z) = \frac{1+z}{1-z} = 1 + q_1 z + q_2 z^2 + \ldots,\ z \in U, \tag{2.4}$$

with $q(0) = 1$.
So, $\Delta = q(U) = \{\omega : \mathrm{Re}\omega > 0\}$ and $E(q) = \{1\}$, and $q \in Q$.
We have to prove that $\mathrm{Re}\{p(z)\} > 0$, for all $z \in U$.

This condition is equivalent with

$$p(z) \prec q(z), \ z \in U. \tag{2.5}$$

From (2.3), perform calculations, we obtain:

$$\frac{2(f'(z))^2}{f'(z)g'(z) - f''(z)g(z)} - \frac{zf'(z)}{g(z)} = \frac{p(z)\left[p(z) + zp'(z)\right]}{p(z) - zp'(z)}. \tag{2.6}$$

Let suppose that (2.5) does not hold, i.e.

$$p(z) \not\prec q(z) = \frac{1+z}{1-z}, \ z \in U. \tag{2.7}$$

Hence from Lemma (1.2), there exists a point $z_0 \in U$ and a point $\zeta_0 \in \partial U \setminus \{1\}$, $|\zeta_0| = 1$, such that

$$\begin{aligned} p(z_0) &= q(\zeta_0); \ z_0 p'(z_0) = m\zeta_0 q'(\zeta_0), \text{ for some } m \geq 1, \\ &\text{and} \\ \mathrm{Re}[\,p(z)] &\geq 0, \text{ for all } z \in U_{|z_0|}\,. \end{aligned}$$

This implies that

$$\mathrm{Re}[\,p(z_0)] = \mathrm{Re}\left[\frac{1+\zeta_0}{1-\zeta_0}\right] = 0, \tag{2.8}$$

so, we can take $p(z_0)$ on the form

$$p(z_0) := iy, \ y \in \mathbb{R}. \tag{2.9}$$

By (2.4) and (2.9), we have

$$\zeta_0 = q^{-1}(p(z_0)) = \frac{p(z_0) - 1}{p(z_0) + 1} = \frac{iy - 1}{iy + 1} = \frac{y^2 + 2iy - 1}{1 + y^2}. \tag{2.10}$$

From (2.4) and (2.10), we get

$$q'(\zeta_0) = \frac{2}{(1-\zeta_0)^2} = \frac{\left(1+y^2\right)^2}{4(1-iy)^2}. \tag{2.11}$$

We obtain

$$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0) = -\frac{m}{2}\left(1+y^2\right), \text{ for some } m \geq 1.$$

Therefore, we have

$$\mathrm{Re}\left[\frac{2(f'(z_0))^2}{f'(z_0)g'(z_0) - f''(z_0)g(z_0)} - \frac{zf'(z_0)}{g(z_0)}\right] = \mathrm{Re}\left[\frac{p(z_0)\left[p(z_0) + zp'(z_0)\right]}{p(z_0) - zp'(z_0)}\right] =$$

$$\mathrm{Re}\left[\frac{iy\left[iy - \frac{m}{2}\left(1+y^2\right)\right]}{iy + \frac{m}{2}\left(1+y^2\right)}\right] = -\frac{my^2\left(1+y^2\right)}{\frac{m^2}{4}\left(1+y^2\right)^2 + y^2} < 0, \text{ for } y \in \mathbb{R} \text{ and } m \geq 1,$$

which contradicts the condition (2.1) from the hypothesis of the theorem, therefore $p \prec q$, i.e.

$$\mathrm{Re}\left[\frac{zf'(z)}{g(z)}\right] > 0, \quad z \in U, \tag{2.12}$$

and the proof is complete.

Theorem 2.2 *Let $f(z) = z + a_2 z^2 + \ldots$ and $g(z) = z + b_2 z^2 + \ldots$ be analytic functions in U, with $f(0) = g(0) = 0$, $g(z) \in S^*$, with $g(z) \neq zf'(z)$ and let $0 < \mu \leq 1$. If*

$$\left| \frac{2(f'(z))^2}{f'(z)g'(z) - f''(z)g(z)} - \frac{zf'(z)}{g(z)} - 1 \right| < \mu, \ z \in U, \tag{2.13}$$

then

$$\left| \frac{zf'(z)}{g(z)} - 1 \right| < \mu, \ z \in U. \tag{2.14}$$

Proof. Let

$$p(z) = \frac{zf'(z)}{g(z)} = 1 + p_1 z + p_2 z^2 + ..., \ z \in U. \tag{2.15}$$

Taking account (2.15), the conditions (2.13) and (2.14) can be rewritten as

$$\mid \frac{p(z)\left[p(z) + zp'(z)\right]}{p(z) - zp'(z)} - 1 \mid < \mu \Rightarrow \mid p(z) - 1 \mid < \mu, \tag{2.16}$$

with $p(z) \neq 1$, and let

$$q(z) = 1 + \mu z, \text{ with } q(0) = 1. \tag{2.17}$$

We have $\Delta = q(U) = \{\omega : \mid \omega - 1 \mid < \mu\}$, $E(q) = \varnothing$ and $q \in Q$.
The relation $\mid p(z) - 1 \mid < \mu$ is equivalent to

$$p(z) \prec q(z), \ z \in U. \tag{2.18}$$

Let suppose that (2.18) does not hold, i.e.

$$p(z) \not\prec q(z) = 1 + \mu z, \ z \in U.$$

From Lemma (1.2), there exists a point $z_0 \in U$ and a point $\zeta_0 \in \partial U$, such that

$$\begin{aligned} p(z_0) &= q(\zeta_0); \ z_0 p'(z_0) = m\zeta_0 q'(\zeta_0), \text{ for some } m \geq 1, \\ & \text{and} \\ \mid & p(z) - 1 \mid < \mu, \text{ for all } z \in U_{|z_0|}. \end{aligned}$$

This implies that

$$\mid p(z_0) - 1 \mid = \mid q(\zeta_0) - 1 \mid = \mu,$$

so, we can take $p(z_0)$ on the form

$$p(z_0) := 1 + \mu e^{i\theta}, \ \theta \in \mathbb{R}. \tag{2.19}$$

We have

$$\zeta_0 = q^{-1}(p(z_0)) = \frac{p(z_0) - 1}{\mu} = e^{i\theta}. \tag{2.20}$$

So,

$$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0) = m\mu e^{i\theta}, \text{ for some } m \geq 1.$$

Therefore, we have

$$\begin{aligned} \mid & \frac{p(z_0)\left[p(z_0) + zp'(z_0)\right]}{p(z_0) - zp'(z_0)} - 1 \mid = \mid \mu e^{i\theta} \mid \mid \frac{1 + \mu e^{i\theta} + m\mu e^{i\theta} + 2m}{1 + \mu e^{i\theta} - m\mu e^{i\theta}} \mid = \\ = & \ \mu \mid \frac{1 + 2m + (1+m)\mu e^{i\theta}}{1 + (1-m)\mu e^{i\theta}} \mid, \text{ for some } m \geq 1. \end{aligned}$$

We will prove that the last expression is greater or equal to μ, that is equivalent with

$$\mid \frac{1 + 2m + (1+m)\mu e^{i\theta}}{1 + (1-m)\mu e^{i\theta}} \mid \geq 1, \quad z \in U,$$

or

$$\mid 1 + 2m + (1+m)\mu e^{i\theta} \mid^2 \geq \mid 1 + (1-m)\mu e^{i\theta} \mid^2, \quad z \in U.$$

The last inequality is equivalent to

$$\mu^2 + (m+2)\mu\cos\theta + m + 1 \geq \mu^2 - (m+2)\mu + m + 1 \geq 0, \quad z \in U,$$

or

$$(\mu - 1)\left[\mu - (m+1)\right] \geq 0, \quad z \in U.$$

The above inequality holds for every $m \geq 1$ and $0 < \mu \leq 1$.
This contradicts with the condition (2.13) from the hypothesis of the theorem, therefore $p \prec q$.
So, $\mid p(z) - 1 \mid < \mu$ and the proof of the theorem is complete.

Theorem 2.3 *Let the function $q(z)$ be analytic and univalent in U such that $q(0) = 1$ and $q(z) \neq 0$, for all $z \in U$ and satisfies*

$$\operatorname{Re}\left[1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right] > 0, \quad z \in U. \tag{2.21}$$

If $f, g \in \mathcal{A}$ with $f(0) = f'(0) - 1 = g(0) = g'(0) - 1 = 0$, $g(z) \in S^*$ and

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} \prec \frac{zq'(z)}{q(z)} = h(z), \tag{2.22}$$

then

$$\frac{zf'(z)}{g(z)} \prec q(z), \quad z \in U, \tag{2.23}$$

and $q(z)$ is the best dominant of (2.22).

Proof. Let the function p be defined by

$$p(z) := \frac{zf'(z)}{g(z)}, \quad z \in U, \tag{2.24}$$

where $f, g \in \mathcal{A}$ with $f(0) = f'(0) - 1 = g(0) = g'(0) - 1 = 0$, $g(z) \in S^*$. The function p is analytic in U and $p(0) = 1$.
Differentiating this function with respect to g, we get
$zp'(z) = \frac{zf'(z)}{g(z)} + \frac{z^2 f''(z)}{g(z)} - \frac{z^2 f'(z) g'(z)}{g^2(z)}, \quad z \in U.$
By setting $\theta(w) := 0$ and $\phi(w) := \frac{1}{w}$, it can be easily verified that ϕ is analytic in a domain $D = \mathbb{C}\backslash\{0\}$, which contains $q(U)$ and $\phi(w) \neq 0$, where $w \in q(U)$.
Also, by letting $Q(z) = zq'(z)\phi(q(z)) = \frac{zq'(z)}{q(z)}$, we find that $Q(z)$ is starlike in U, since
$\operatorname{Re}\left[\frac{zQ'(z)}{Q(z)}\right] = \operatorname{Re}\left[1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right] > 0.$
For the function $h(z) = \theta(q(z)) + Q(z) = \frac{zq'(z)}{q(z)}$, we have $\operatorname{Re}\left[\frac{zh'(z)}{Q(z)}\right] = \operatorname{Re}\left[\frac{zQ'(z)}{Q(z)}\right] > 0$.
Therefore, the conditions of Lemma 1.1 are met and we find that, if

$$\frac{zp'(z)}{p(z)} = \theta(p(z)) + zp'(z)\phi(p(z)) \prec \frac{zq'(z)}{q(z)}, \quad z \in U, \tag{2.25}$$

then

$$\frac{zf'(z)}{g(z)} = p(z) \prec q(z), \quad z \in U, \tag{2.26}$$

and q is the best dominant of (2.22). And the proof is complete.

Corollary 2.4 *Let $f, g \in \mathcal{A}$ with $f(0) = f'(0) - 1 = g(0) = g'(0) - 1 = 0$, $g(z) \in S^*$ and $0 < \mu \leq 1$. If*

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} \prec \frac{\mu z}{1 + \mu z} = h_1(z), \tag{2.27}$$

then

$$\frac{zf'(z)}{g(z)} \prec 1 + \mu z, \quad z \in U, \tag{2.28}$$

and $1 + \mu z$ is the best dominant of (2.27). Furthermore,

$$\left| \frac{zf'(z)}{g(z)} - 1 \right| < \mu, \quad z \in U, \tag{2.29}$$

and this result is sharp.

Proof. Let the function $q(z) := 1 + \mu z$, $z \in U$, $0 < \mu \leq 1$, satisfies all the conditions of Theorem 2.3.
So,

$$h_1(z) = Q(z) = zq'(z)\phi(q(z)) = \frac{zq'(z)}{q(z)} = \frac{\mu z}{1 + \mu z}. \tag{2.30}$$

From (2.30), we find that the subordinations (2.22) and (2.30) are equivalent.
Hence, it follows directly from Theorem 2.3 that

$$\frac{zf'(z)}{g(z)} \prec 1+\mu z, \quad z \in U, \tag{2.31}$$

and $1+\mu z$ is the best dominant of (2.27).
Now, let us assume that the subordination (2.27) and inequality

$$\left| \frac{zf'(z)}{g(z)} - 1 \right| < \mu_1, \quad z \in U,\, 0 < \mu_1 \le 1, \tag{2.32}$$

hold, i.e.

$$\frac{zf'(z)}{g(z)} \prec 1+\mu_1 z, \quad z \in U,\, 0 < \mu_1 \le 1. \tag{2.33}$$

But the function $1+\mu z$ is the best dominant of (2.27), meaning that $1+\mu z \prec 1+\mu_1 z$, i.e. $\mu \prec \mu_1$.
Therefore the conclusion is sharp, i.e. in the inequality (2.29) μ cannot be replaced by a smaller number such that the implication holds. This completes the proof of the corollary.

Remark 2.1 *It is easy to verify that for* $0 < \mu \le 1$, $h_1(z) = \frac{\mu z}{1+\mu z}$ *is an open disk with center* $c = \frac{h_1(1)+h_1(-1)}{2} = \frac{\frac{\mu}{1+\mu} - \frac{\mu}{1-\mu}}{2} = -\frac{\mu^2}{1-\mu^2}$, *and radius* $r = h_1(1) - c = \frac{\mu}{1+\mu} + \frac{\mu^2}{1-\mu^2} = \frac{\mu}{1-\mu^2}$. *For* $\mu = 1$, $h_1(U) = \{x+iy : x < 1, y \in \mathbb{R}\}$.

Therefore, the above corollary can be rewritten as:

Corollary 2.5 *Let* $f, g \in \mathcal{A}$ *with* $f(0) = f'(0) - 1 = g(0) = g'(0) - 1 = 0$, $g(z) \in S^*$ *and* $0 < \mu \le 1$.
(i) If $0 < \mu < 1$ *and*

$$\left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} + \frac{\mu^2}{1-\mu^2} \right| < \frac{\mu}{1-\mu^2}, z \in U \tag{2.34}$$

then

$$\left| \frac{zf'(z)}{g(z)} - 1 \right| < \mu, \quad z \in U. \tag{2.35}$$

This result is sharp, i.e. the radius of the open disk from the conclusion is the smallest possible so that the corresponding implication holds.
(ii) If

$$\mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\right] < 1, \quad z \in U, \tag{2.36}$$

then

$$\left| \frac{zf'(z)}{g(z)} - 1 \right| < 1, \tag{2.37}$$

and these conclusions are sharp.

From Corrolary 2.4 and Corrolary 2.5 we obtain the following result:

Corollary 2.6 *Let* $f, g \in \mathcal{A}$ *with* $f(0) = f'(0) - 1 = g(0) = g'(0) - 1 = 0$, $g(z) \in S^*$ *and* $0 < \zeta \le \frac{1}{2}$. *If*

$$\left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} \right| < \zeta, \quad \text{for all } z \in U, \tag{2.38}$$

then

$$\left| \frac{zf'(z)}{g(z)} - 1 \right| < \frac{\zeta}{1-\zeta}, \quad z \in U. \tag{2.39}$$

Proof. Let $\frac{\zeta}{1-\zeta} = \mu$.
Then

$$\left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} \right| < \zeta = \frac{\mu}{1+\mu}, \quad \text{for all } z \in U. \tag{2.40}$$

If $0 < \zeta < \frac{1}{2}$, then $0 < \mu < 1$. We have

$$\mid 1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} + \frac{\mu^2}{1-\mu^2} - \frac{\mu^2}{1-\mu^2} \mid < \frac{\mu}{1+\mu}, \quad z \in U, \tag{2.41}$$

i.e.

$$\mid 1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} + \frac{\mu^2}{1-\mu^2} \mid < \frac{\mu}{1+\mu} + \frac{\mu^2}{1-\mu^2} = \frac{\mu}{1-\mu^2}, \quad z \in U. \tag{2.42}$$

From Corrolary 2.5, we get

$$\mid \frac{zf'(z)}{g(z)} - 1 \mid < \mu = \frac{\zeta}{1-\zeta}, \quad z \in U. \tag{2.43}$$

If $\zeta = \frac{1}{2}$, then $\mu = 1$. We have

$$\mid 1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} \mid < 1, \text{ for all } z \in U, \tag{2.44}$$

i.e.

$$\operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\right] < 1, \quad z \in U. \tag{2.45}$$

From Corrolary 2.5, we get

$$\mid \frac{zf'(z)}{g(z)} - 1 \mid < 1, \quad z \in U, \tag{2.46}$$

and the proof is complete.

Corollary 2.7 *If $f, g \in \mathcal{A}$ with $f(0) = f'(0) - 1 = g(0) = g'(0) - 1 = 0$, $g(z) \in S^*$. If*

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} \prec \frac{2z}{1-z^2} = h(z), \tag{2.47}$$

then $f \in K$.

Proof. Let $q(z) = \frac{1+z}{1-z} = 1 + q_1 z + ...$, with $q(0) = 1$ satisfies the conditions of Theorem 2.3.
We have

$$h(z) = \theta(q(z)) + Q(z) = \frac{zq'(z)}{q(z)}, \quad z \in U. \tag{2.48}$$

So,

$$h(z) = \frac{2z}{1-z^2}, \quad z \in U. \tag{2.49}$$

From (2.49), we find that the subordinations (2.22) and (2.47) are equivalent. Therefore, it follows directly from Theorem 2.3 that

$$\frac{zf'(z)}{g(z)} \prec \frac{1+z}{1-z}, \quad z \in U, \tag{2.50}$$

and $\frac{1+z}{1-z}$ is the best dominant of (2.47).
Since the subordination (2.22) is equivalent with the condition

$$\operatorname{Re}\left[\frac{zf'(z)}{g(z)}\right] > 0, \quad z \in U,$$

we get that $f \in K$, which completes the proof.

Corollary 2.8 *Let $f, g \in \mathcal{A}$ with $f(0) = f'(0) - 1 = g(0) = g'(0) - 1 = 0$, $g(z) \in S^*$ and $0 \le \alpha < 1$. If*

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} \prec \frac{2z(1-\alpha)}{(1-z)[1+(1-2\alpha)z]}, \quad z \in U, \tag{2.51}$$

then $f \in K(\alpha)$.

Proof. For $0 \leq \alpha < 1$ and $z \in U$, setting $q(z) = \frac{1+(1-2\alpha)z}{1-z} = 1 + q_1 z + ...$, with $q(0) = 1$ in Theorem 2.3, we obtain

$$\operatorname{Re}\left[1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right] = \operatorname{Re}\left[\frac{1}{1-z} + \frac{1}{1+(1-2\alpha)z} - 1\right] > 0, \quad z \in U.$$

We have

$$h(z) = \frac{zq'(z)}{q(z)} = \frac{2z(1-\alpha)}{(1-z)[1+(1-2\alpha)z]}, \quad z \in U.$$

Hence, the hypotheses of Theorem 2.3 are satisfied. Therefore, it follows directly from Theorem 2.3 that

$$\frac{zf'(z)}{g(z)} \prec \frac{1+(1-2\alpha)z}{1-z}, \quad z \in U, 0 \leq \alpha < 1, \tag{2.52}$$

i.e.

$$\operatorname{Re}\left[\frac{zf'(z)}{g(z)}\right] > \alpha, \quad z \in U,$$

i.e. $f \in K(\alpha)$, and the proof is now complete.

References

[1] Alexander, J.W., Functions which maps the interior of the unit circle upon simple regions, *Ann. of Math.*, 17, 12-22 (1915).

[2] R. Bucur, L. Andrei, D. Breaz, Geometric Properties of a New Integral Operator, *Abstract and Applied Analysis*, Article ID 430197 (2015).

[3] W. Kaplan, Close-to-convex schlicht functions, *Mich. Math. J.*, 169-185 (1952).

[4] S.Kanas, A.E. Tudor, Differential Subordinations and Harmonic Means, *Bull. Malays. Math. Sci. Soc.*, 38, 1243–1253 (2015).

[5] S.S. Miller, P.T. Mocanu, Subordinants of Differential Superordinations, *Complex Variables,* vol. 48, no. 10, 815-826, October (2003).

[6] S.S. Miller, P.T. Mocanu, *Differential Subordinations: Theory and Applications,* Marcel Dekker Inc., New York, 2000.

[7] M. H. Mohd, M. Darus, On a class of spiral-like functions with respect to a boundary point related to subordination, *Journal of Inequalities and Applications*, 274 (2013).

[8] M. Nunokawa, N. E. Cho, O.S. Kwon, J. Sokół, On differential subordinations in the complex plane, *Journal of Inequalities and Applications*, 7 (2015).

[9] H. Shiraishi, S. Owa, T. Hayami, K. Kuroki and H. M. Srivastava, Starlike problems for certain analytic functions concerned with subordinations, *Pan Amer. Math. J.,* 21, 63–77 (2011).

[10] H. M. Srivastava and S. S. Eker, Some applications of a subordination theorem for a class of analytic functions, *Appl. Math. Lett.,* 21, 394–399 (2008).

[11] N. Tuneski, M. Obradović, Some properties of certain expressions of analytic functions, *Computers and Mathematics with Applications,* 62, 3438–3445 (2011).

[12] Q.-H. Xu, C.-B. Lv, N.-C. Luo and H. M. Srivastava, Sharp coefficient estimates for a certain general class of spirallike functions by means of differential subordination, *Filomat,* 27, 1351–1356 (2013).

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1

DUALISTIC CONTRACTIONS OF RATIONAL TYPE AND FIXED POINT THEOREMS

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ABSTRACT. In this paper, we introduce a dualistic contraction of rational type and use it to obtain some new fixed point results in dualistic partial metric spaces for dominating and dominated mappings. These results extend various comparable results existing in literature from set of positive real numbers to set of real numbers. Moreover, we give an example that shows the usefulness and effectiveness of these results among corresponding fixed point theorems established in partial metric spaces (nonnegative real numbers).

Keywords: fixed point; dualistic partial metric; dualistic contraction of rational type.
AMS 2010 Subject Classification: 47H10; 54H25.

1. Introduction and preliminaries

In [7], Matthews introduced the concept of partial metric space as a suitable mathematical tool for program verification and proved an analogue of Banach fixed point theorem in complete partial metric spaces. Fixed point theorems in complete partial metric spaces have been investigated in [3, 4, 11, 12, 14]. O'Neill [10] introduced the concept of dualistic partial metric, which is more general than partial metric and established a robust relationship between dualistic partial metric and quasi metric. In [13], Oltra and Valero presented a Banach fixed point theorem on complete dualistic partial metric spaces. Following Oltra and Valero, Nazam et al. [2, 8] established some fixed point results in dualistic partial metric spaces.

In this paper, we shall extend the following two fixed point theorems from nonnegative real numbers to real numbers to the theorems from partial metric spaces to dualistic partial metric spaces.

Harjani et al. [5] extended Banach's fixed point principle as follows:

Theorem 1. Let M be a complete ordered metric space and $T : M \to M$ be a continuous and nondecreasing rational type contraction mapping. Then T has a unique fixed point $m^* \in M$. Moreover, the Picard iterative sequence $\{T^n(j)\}_{n\in\mathbb{N}}$ converges to m^* for every $j \in M$.

Isik and Tukroglu [6] established an ordered partial metric space version of Theorem 1, stated below.

Theorem 2. Let M be a complete ordered partial metric space and $T : M \to M$ be a continuous and nondecreasing rational type contraction mapping. Then T has a unique fixed point $m^* \in M$. Moreover, the Picard iterative sequence $\{T^n(j)\}_{n\in\mathbb{N}}$ converges to m^* for every $j \in M$.

We need some mathematical concepts of dualistic partial metric space and results to make this paper self sufficient.

Throughout, in this paper, the letters $\mathbb{R}_0^+$, $\mathbb{R}$ and $\mathbb{N}$ will represent the set of nonnegative real numbers, set of real numbers and set of natural numbers, respectively.

Definition 1. [1]

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(1) Let M be a nonempty set and $T : M \to M$ be a self mapping. A point $m^* \in M$ is called a fixed point of T if $m^* = T(m^*)$.
(2) Let $(M, \preceq)$ be an ordered set and $T : M \to M$ be a self mapping defined on M satisfying the property $j \preceq T(j)$ for all $j \in M$. Then T is called a dominating mapping.
(3) Let $(M, \preceq)$ be an ordered set and $T : M \to M$ be a self mapping defined on M satisfying the property $T(j) \preceq j$ for all $j \in M$. Then T is called a dominated mapping.

According to O'Neill, a dualistic partial metric can be defined as follow:

Definition 2. [10] Let M be a nonempty set. A function $D : M \times M \to \mathbb{R}$ is called a dualistic partial metric if for any $j, k, l \in M$, the following conditions hold:
(D_1) $j = k \Leftrightarrow D(j, j) = D(k, k) = D(j, k)$.
(D_2) $D(j, j) \leq D(j, k)$.
(D_3) $D(j, k) = D(k, j)$.
(D_4) $D(j, l) \leq D(j, k) + D(k, l) - D(k, k)$.

We observe that, as in the metric case, if D is a dualistic partial metric then $D(j, k) = 0$ implies $j = k$. In case $D(j, k) \in \mathbb{R}_0^+$ for all $j, k \in M$, then D is a partial metric on M. If (M, D) is a dualistic partial metric space, then the function $d_D : M \times M \to \mathbb{R}_0^+$ defined by

$$d_D(j, k) = D(j, k) - D(j, j),$$

is a quasi metric on M such that $\tau(D) = \tau(d_D)$. In this case, $d_D^s(j, k) = \max\{d_D(j, k), d_D(k, j)\}$ defines a metric on M.

Remark 1. It is obvious that every partial metric is a dualistic partial metric but the converse is not true. To support this comment, define $D_\vee : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$D_\vee(j, k) = j \vee k = \sup\{j, k\} \;\; \forall\, j, k \in \mathbb{R}.$$

It is easy to check that $D_\vee$ is a dualistic partial metric. Note that $D_\vee$ is not a partial metric, since $D_\vee(-1, -2) = -1 \notin \mathbb{R}_0^+$. Nevertheless, the restriction of $D_\vee$ to $\mathbb{R}_0^+$, $D_\vee|_{\mathbb{R}_0^+}$, is a partial metric.

Definition 3. [2] Let M be a nonempty set. Then $(M, \preceq, D)$ is said to be an ordered dualistic partial metric space if
(i) $(M, \preceq)$ is a partially ordered set.
(ii) (M, D) is a dualistic partial metric space.

Example 1. If $(M, \preceq, d)$ is an ordered metric space and $c \in \mathbb{R}$ is an arbitrary constant, then $D_c : M \times M \to \mathbb{R}$ given by

$$D_c(j, k) = d(j, k) + c,$$

defines an ordered dualistic partial metric on M and $(M, \preceq, D_c)$ is an ordered dualistic partial metric space.

Following [10], each dualistic partial metric D on M generates a T_0 topology $\tau(D)$ on M. The elements of the topology $\tau(D)$ are open balls of the form $\{B_D(j, \epsilon) : j \in M, \epsilon > 0\}$, where $B_D(j, \epsilon) = \{k \in M : D(j, k) < \epsilon + D(j, j)\}$. A sequence $\{j_n\}_{n \in \mathbb{N}}$ in (M, D) converges to a point $j \in M$ if and only if $D(j, j) = \lim_{n \to \infty} D(j, j_n)$.

Definition 4. [8] Let (M, D) be a dualistic partial metric space.
(1) A sequence $\{j_n\}_{n \in \mathbb{N}}$ in (M, D) is called a Cauchy sequence if $\lim_{n,m \to \infty} D(j_n, j_m)$ exists and is finite.

(2) A dualistic partial metric space (M, D) is said to be complete if every Cauchy sequence $\{j_n\}_{n\in\mathbb{N}}$ in M converges, with respect to $\tau(D)$, to a point $j \in M$ such that $D(j,j) = \lim_{n,m\to\infty} D(j_n, j_m)$.

The following lemma will be helpful in the sequel.

Lemma 1. [8, 9]

(1) *A dualistic partial metric (M, D) is complete if and only if the metric space (M, d_D^s) is complete.*
(2) *A sequence $\{j_n\}_{n\in\mathbb{N}}$ in M converges to a point $j \in M$, with respect to $\tau(d_D^s)$ if and only if $\lim_{n\to\infty} D(j, j_n) = D(j,j) = \lim_{n\to\infty} D(j_n, j_m)$.*
(3) *If $\lim_{n\to\infty} j_n = \upsilon$ such that $D(\upsilon,\upsilon) = 0$, then $\lim_{n\to\infty} D(j_n, k) = D(\upsilon, k)$ for every $k \in M$.*

Motivated by the results presented by Isik and Tukroglu [6] and Valero [9], we present a new fixed point theorem in an ordered dualistic partial metric space for both dominating and dominated mappings.

2. Main results

We begin with the following definition.

Definition 5. Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space. A mapping $T : M \to M$ is said to be a dualistic contraction of rational type if

$$|D(T(j), T(k))| \leq \alpha \left| \frac{D(j, T(j)) \cdot D(k, T(k))}{D(j,k)} \right| + \beta |D(j,k)| \tag{2.1}$$

holds for all comparable elements $j, k \in M$ and $0 < \alpha + \beta < 1$.

Theorem 3. *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *T is a dualistic contraction of rational type;*
(2) *T is a dominating mapping;*
(3) *T is a continuous mapping.*

Then T has a fixed point m^*. Moreover, $D(m^*, m^*) = 0$.

Proof. Let j_0 be an initial point in M and $j_n = T(j_{n-1})$ be an iterative sequence. If there exists a positive integer r such that $j_{r+1} = j_r$, then j_r is the fixed point of T and it completes the proof. If $j_n \neq j_{n+1}$ for any $n \in \mathbb{N}$, then since T is dominating, $j_n \preceq T(j_n)$ for all $n \in \mathbb{N}$. Therefore,

$$j_0 \preceq j_1 \preceq j_2 \preceq j_3 \preceq \cdots \preceq j_n \preceq j_{n+1} \cdots .$$

Now since $j_n \preceq j_{n+1}$, by (2.1), we have

$$\begin{aligned}
|D(j_n, j_{n+1})| &= |D(T(j_{n-1}), T(j_n))| \\
&\leq \alpha \left| \frac{D(j_{n-1}, j_n) \cdot D(j_n, j_{n+1})}{D(j_{n-1}, j_n)} \right| + \beta |D(j_{n-1}, j_n)| \\
&\leq \alpha |D(j_n, j_{n+1})| + \beta |D(j_{n-1}, j_n)|, \\
|D(j_n, j_{n+1})| - \alpha |D(j_n, j_{n+1})| &\leq \beta |D(j_{n-1}, j_n)|, \\
(1-\alpha)|D(j_n, j_{n+1})| &\leq \beta |D(j_{n-1}, j_n)|, \\
|D(j_n, j_{n+1})| &\leq (\frac{\beta}{1-\alpha}) |D(j_{n-1}, j_n)|.
\end{aligned}$$

If $\gamma = \dfrac{\beta}{1-\alpha}$, then $0 < \gamma < 1$ and we have

$$|D(j_n, j_{n+1})| \leq \gamma |D(j_{n-1}, j_n)|.$$

Thus

$$|D(j_n, j_{n+1})| \leq \gamma |D(j_{n-1}, j_n)| \leq \gamma^2 |D(j_{n-2}, j_{n-1})| \leq \cdots \leq \gamma^n |D(j_0, j_1)|.$$

Since $j_n \preceq j_n$, for each $n \in \mathbb{N}$, by (2.1), we have

$$\begin{aligned}
|D(j_n, j_n)| = |D(T(j_{n-1}), T(j_{n-1}))| &\leq \frac{\alpha |D(j_{n-1}, j_n)|^2}{|D(j_{n-1}, j_{n-1})|} + \beta |D(j_{n-1}, j_{n-1})| \\
&\leq |D(j_{n-1}, j_{n-1})| \left\{ \frac{\alpha |D(j_{n-1}, j_n)|^2}{|D(j_{n-1}, j_{n-1})|^2} + \beta \right\} \\
&\leq (\alpha + \beta) |D(j_{n-1}, j_{n-1})|.
\end{aligned}$$

Indeed, $\left| \dfrac{D(j_{n-1}, j_n)^2}{D(j_{n-1}, j_{n-1})^2} \right| = 1$. Thus we obtain that

$$|D(j_n, j_n)| \leq (\alpha + \beta)^n |D(j_0, j_0)|. \tag{2.2}$$

Now we show that $\{j_n\}$ is a Cauchy sequence in (M, d_D^s). Note that $d_D(j_n, j_{n+1}) = D(j_n, j_{n+1}) - D(j_n, j_n)$, that is, $d_D(j_n, j_{n+1}) + D(j_n, j_n) = D(j_n, j_{n+1}) \leq |D(j_n, j_{n+1})|$. Thus we have

$$\begin{aligned}
d_D(j_n, j_{n+1}) + D(j_n, j_n) &\leq \gamma^n |D(j_0, j_1)|, \\
d_D(j_n, j_{n+1}) &\leq \gamma^n |D(j_0, j_1)| + |D(j_n, j_n)| \\
&\leq \gamma^n |D(j_0, j_1)| + (\alpha + \beta)^n |D(j_0, j_0)|.
\end{aligned}$$

Continuing this way, we obtain that

$$d_D(j_{n+k-1}, j_{n+k}) \leq \gamma^{n+k-1} |D(j_0, j_1)| + (\alpha + \beta)^{n+k-1} |D(j_0, j_0)|.$$

Now

$$\begin{aligned}
d_D(j_n, j_{n+k}) &\leq d_D(j_n, j_{n+1}) + d_D(j_{n+1}, j_{n+2}) + \cdots + d_D(j_{n+k-1}, j_{n+k}) \\
&\leq \{\gamma^n + \gamma^{n+1} + \cdots + \gamma^{n+k-1}\} |D(j_0, j_1)| \\
&+ \{(\alpha + \beta)^n + (\alpha + \beta)^{n+1} + \cdots + (\alpha + \beta)^{n+k-1}\} |D(j_0, j_0)|.
\end{aligned}$$

Thus, for $n + k = m > n$,

$$d_D(j_n, j_m) \leq \frac{\gamma^n}{1-\gamma} |D(j_0, j_1)| + \frac{(\alpha + \beta)^n}{1 - (\alpha + \beta)} |D(j_0, j_0)|.$$

Similarly, we have

$$d_D(j_m, j_n) \leq \frac{\gamma^n}{1-\gamma} |D(j_1, j_0)| + \frac{(\alpha + \beta)^n}{1 - (\alpha + \beta)} |D(j_0, j_0)|.$$

Taking limit as $n, m \to \infty$, we have

$$\lim_{n,m\to\infty} d_D(j_m, j_n) = 0 = \lim_{n,m\to\infty} d_D(j_n, j_m) \text{ and hence } \lim_{n,m\to\infty} d_D^s(j_m, j_n) = 0.$$

Thus $\{j_n\}$ is a Cauchy sequence in (M, d_D^s). Since (M, D) is a complete dualistic partial metric space, by Lemma 1, the metric space (M, d_D^s) is also complete. Therefore, there exists $m^* \in M$ such that $\lim_{n\to\infty} d_D^s(j_n, m^*) = 0$. Again from Lemma 1, we have

$$\lim_{n\to\infty} d_D^s(j_n, m^*) = 0 \iff D(m^*, m^*) = \lim_{n\to\infty} D(j_n, m^*) = \lim_{n,m\to\infty} D(j_m, j_n).$$

Now $\lim_{n,m\to\infty} d_D(j_m, j_n) = 0$ implies that $\lim_{n,m\to\infty}[D(j_m, j_n) - D(j_n, j_n)] = 0$ and hence $\lim_{n,m\to\infty} D(j_n, j_m) = \lim_{n\to\infty} D(j_n, j_n)$. By (2.2), we have $\lim_{n\to\infty} D(j_n, j_n) = 0$. Consequently, $\lim_{n,m\to\infty} D(j_n, j_m) = 0$ and $\{j_n\}$ is a Cauchy sequence in (M, D). Thus

$$D(m^*, m^*) = \lim_{n\to\infty} D(j_n, m^*) = 0. \tag{2.3}$$

Now $d_D(m^*, T(m^*)) = D(m^*, T(m^*)) - D(m^*, m^*) = D(m^*, T(m^*))$ implies that $D(m^*, T(m^*)) \geq 0$. Since T is continuous, for a given $\epsilon > 0$, there exists a $\delta > 0$ such that $T(B_D(m^*, \delta)) \subseteq B_D(T(m^*), \epsilon)$. Since $\lim_{n\to\infty} D(j_{n+1}, m^*) = D(m^*, m^*) = 0$, there exists an $r \in \mathbb{N}$ such that $D(j_n, m^*) < D(m^*, m^*) + \delta$ for all $n \geq r$, and so $\{j_n\} \subset B_D(m^*, \delta)$ for all $n \geq r$. This implies that $T(j_n) \in T(B_D(m^*, \delta) \subseteq B_D(T(m^*), \epsilon)$ and so $D(T(j_n), T(m^*)) < D(T(m^*), T(m^*)) + \epsilon$ for all $n \geq r$. Now for any $\epsilon > 0$, we know that

$$-\epsilon + D(T(m^*), T(m^*)) < D(T(m^*), T(m^*)) \leq D(j_{n+1}, T(m^*)),$$

which implies that

$$|D(j_{n+1}, T(m^*)) - D(T(m^*), T(m^*))| < \epsilon.$$

That is, $D(T(m^*), T(m^*)) = \lim_{n\to\infty} D(j_{n+1}, T(m^*))$. The uniqueness of limit in $\mathbb{R}$ implies

$$\lim_{n\to\infty} D(j_{n+1}, T(m^*)) = D(T(m^*), T(m^*)) = D(m^*, T(m^*)). \tag{2.4}$$

Finally, we have $D(T(m^*), m^*) = \lim_{n\to\infty} D(T(j_n), j_n) = \lim_{n\to\infty} D(j_{n+1}, j_n) = 0$. This shows that $D(m^*, T(m^*)) = 0$. So from (2.3) and (2.4) we deduce that

$$D(m^*, T(m^*)) = D(T(m^*), T(m^*)) = D(m^*, m^*).$$

This leads us to conclude that $m^* = T(m^*)$ and hence m^* is a fixed point of T. □

In order to prove the uniqueness of fixed point of a mapping T in the above theorem, we give the following theorem.

Theorem 4. *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space and $T : M \to M$ be a mapping which satisfies all the conditions of Theorem 3. Then T has a unique fixed point provided that for each fixed point m^*, n^* of T, there exists $\omega \in M$ which is comparable to both m^* and n^*.*

Proof. From Theorem 3, it follows that the set of fixed points of T is nonempty.
Assume that n^* is another fixed point of T, that is, $n^* = T(n^*)$.

Case I: m^* and n^* are comparable.
In this case, we have

$$\begin{aligned}
|D(m^*, n^*)| &= |D(T(m^*), T(n^*))| \\
&\leq \alpha \left| \frac{D(m^*, T(m^*)) \cdot D(n^*, T(n^*))}{D(m^*, n^*)} \right| + \beta |D(m^*, n^*)| \\
&\leq \alpha \left| \frac{D(m^*, m^*) \cdot D(n^*, n^*)}{D(m^*, n^*)} \right| + \beta |D(m^*, n^*)|.
\end{aligned}$$

That is, $(1 - \beta)|D(m^*, n^*)| \leq 0$ which implies that $|D(m^*, n^*)| \leq 0$ and hence $D(m^*, n^*) = 0 = D(m^*, m^*) = D(n^*, n^*)$. So m^* is a unique fixed point of T.

Case II: m^* and n^* are incomparable.
In this case, there exists ω which is comparable to both m^*, n^*. Without any loss of generality,

we assume that $m^* \preceq \omega$ and $n^* \preceq \omega$. Since T is dominating, $m^* \preceq T(\omega)$ and $n^* \preceq T(\omega)$ imply that $m^* \preceq T^{n-1}(\omega)$ and $n^* \preceq T^{n-1}(\omega)$. Thus

$$\begin{aligned} |D(m^*, T^n(\omega))| &\leq \alpha \left| \frac{D(T^{n-1}(m^*), T^n(m^*)) \cdot D(T^{n-1}(\omega), T^n(\omega))}{D(T^{n-1}(m^*), T^{n-1}(\omega))} \right| \\ &+ \beta |D(T^{n-1}(m^*), T^{n-1}(\omega))|, \end{aligned}$$

which implies that $|D(m^*, T^n(\omega))| \leq \beta |D(m^*, T^{n-1}(\omega))|$. Thus $\lim_{n\to\infty} D(m^*, T^n(\omega)) = 0$. Similarly, $\lim_{n\to\infty} D(n^*, T^n(\omega)) = 0$. Moreover, by D_4,

$$\begin{aligned} D(n^*, m^*) &\leq D(n^*, T^n(\omega)) + D(T^n(\omega), m^*) - D(T^n(\omega), T^n(\omega)) \\ &\leq D(n^*, T^n(\omega)) + D(T^n(\omega), m^*) - D(T^n(\omega), m^*) - D(m^*, T^n(\omega)) + D(m^*, m^*). \end{aligned}$$

Letting $n \to \infty$, we obtain that $D(n^*, m^*) \leq 0$ but $d_D(m^*, m^*) = D(n^*, m^*) - D(n^*, n^*)$ implies that $D(n^*, m^*) \geq 0$. Hence $D(n^*, m^*) = 0$ which gives that $n^* = m^*$. $\square$

We provide an example to explain the above result.

Example 2. Let $M = \mathbb{R}^2$. Define $D_\vee : M \times M \to \mathbb{R}$ by $D_\vee(j, k) = j_1 \vee k_1$, where $j = (j_1, j_2)$ and $k = (k_1, k_2)$. Note that $(M, D_\vee)$ is a complete dualistic partial metric space. Let $T : M \to M$ be given by

$$T(j) = e^{|D_\vee(j,j)|} \quad \text{for all } j \in M.$$

In M, we define the relation $\succeq$ in the following way:

$$j \succeq k \text{ if and only if } j_1 \geq k_1, \text{ where } j = (j_1, j_2) \text{ and } k = (k_1, k_2).$$

Clearly, $\succeq$ is a partial order on M and T is a continuous, dominating mapping with respect to $\succeq$ and satisfies the contractive condition (2.1).

Remark 2. Since every partial metric is a dualistic partial metric D with $D(j, k) \in \mathbb{R}_0^+$ for all $j, k \in M$, Theorem 3 is an extension of Theorem 2.

There arises the following natural question:

Whether the contractive condition (2.1) in the statement of Theorem 3 can be replaced by the contractive condition in Theorem 2?

The following example provides a negative answer to the above question.

Example 3. *Define the mapping $T_0 : \mathbb{R} \to \mathbb{R}$ by*

$$T_0(j) = \begin{cases} 0 & \text{if } j > 1 \\ -\frac{1}{2} & \text{if } j = -5 \end{cases}.$$

Clearly, for any $j, k \in \mathbb{R}$ and $\alpha = 0.1$ and $\beta = 0.09$, the following contractive condition is satisfied

$$D_\vee(T_0(j), T_0(k)) \leq \frac{\alpha D_\vee(j, T_0(j)) \cdot D_\vee(k, T_0(k))}{D_\vee(j, k)} + \beta D_\vee(j, k),$$

where $D_\vee$ is a complete dualistic partial metric on $\mathbb{R}$. Here T has no fixed point. Thus a fixed point free mapping satisfies the contractive condition of Theorem 2. On the other hand, for $\alpha = 0.1$ and $\beta = 0.09$, we have

$$\begin{aligned} 0.5 = |D_\vee(-\frac{1}{2}, -\frac{1}{2})| &= |D_\vee(T_0(-5), T_0(-5))| \\ > 0.455 &= \left| \frac{\alpha D_\vee(-5, T_0(-5)) \cdot D(-5, T_0(-5))}{D_\vee(-5, -5)} \right| + \beta |D_\vee(-5, -5)|. \end{aligned}$$

Thus the contractive condition (2.1) of Theorem 3 does not hold.

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In next theorem, we show that the conclusion of Theorem 3 remains the same if the continuity of the mapping T is replaced with the following condition:

(H): If $\{j_n\}$ is a nondecreasing sequence in M such that $j_n \to \upsilon$, then $j_n \preceq \upsilon$ for all $n \in \mathbb{N}$. For dominated mappings, the following condition will be needed:

(Q): If $\{j_n\}$ is a nonincreasing sequence in M such that $j_n \to \upsilon$, then $j_n \succeq \upsilon$ for all $n \in \mathbb{N}$.

Theorem 5. *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *T is a dualistic contraction of rational type;*
(2) *T is a dominating mapping;*
(3) *(H) holds.*

Then T has a fixed point m^*. Moreover, $D(m^*, m^*) = 0$.

Proof. By the arguments similar to those in the proof of Theorem 3, we obtain that $\{j_n\}$ is a nondecreasing sequence in M such that $j_n \to m^*$. By (H), we have $j_n \preceq m^*$. Since T is a dominating mapping, we have $j_n \preceq T(m^*)$ and

$$m^* \preceq T(m^*). \tag{2.5}$$

From the proof of Theorem 3, we deduce that $\{T^n(m^*)\}$ is a nondecreasing sequence. Suppose that $\lim_{n\to+\infty} T^n(m^*) = \mu$ for some $\mu \in M$. Now $j_n \preceq m^*$ implies $j_n \preceq T^n(m^*)$ for all $n \geq 1$. Thus we have

$$j_n \preceq m^* \preceq T(m^*) \preceq T^n(m^*) \quad n \geq 1.$$

From (2.1), we have

$$\begin{aligned} |D(j_{n+1}, T^{n+1}(m^*))| &= |D(T(j_n), T(T^n(m^*)))| \\ &\leq \left| \frac{\alpha D(j_n, j_{n+1}) \cdot D(T^n(m^*), T^{n+1}(m^*))}{D(j_n, T^n(m^*))} \right| + \beta |D(j_n, T^n(m^*))|. \end{aligned}$$

Taking limit as $n \to +\infty$, we obtain that

$$|D(m^*, \mu)| \leq \beta |D(m^*, \mu)|,$$

which implies that $m^* = \mu$. Thus $\lim_{n\to+\infty} T^n(m^*) = \mu$ implies that $\lim_{n\to+\infty} T^n(m^*) = m^*$. Hence

$$T(m^*) \preceq m^*. \tag{2.6}$$

From (2.5) and (2.6), it follows that $m^* = T(m^*)$. ☐

Now we present some important consequences of our results.

Corollary 1. *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *T is a dualistic contraction of rational type with $\beta = 0$;*
(2) *T is a dominating mapping;*
(3) *T is a continuous mapping.*

Then T has a fixed point m^*. Moreover, $D(m^*, m^*) = 0$.

Corollary 2. *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *T is a dualistic contraction of rational type with $\beta = 0$;*
(2) *T is a dominating mapping;*
(3) *(H) holds.*

Then T has a fixed point m^*. Moreover, $D(m^*, m^*) = 0$.

Corollary 3. *Let $(M,\preceq,D)$ be a complete ordered dualistic partial metric space. Suppose that $T:M\to M$ is a mapping such that*

(1) *T is a dualistic contraction of rational type with $\alpha=0$;*
(2) *T is a dominating mapping;*
(3) *T is a continuous mapping.*

Then T has a fixed point m^*. Moreover, $D(m^*,m^*)=0$.

Corollary 4. *Let $(M,\preceq,D)$ be a complete ordered dualistic partial metric space. Suppose that $T:M\to M$ is a mapping such that*

(1) *T is a dualistic contraction of rational type with $\alpha=0$;*
(2) *T is a dominating mapping;*
(3) *(H) holds.*

Then T has a fixed point m^*. Moreover, $D(m^*,m^*)=0$.

For dominated mappings, we present the following results.

Theorem 6. *Let $(M,\preceq,D)$ be a complete ordered dualistic partial metric space. Suppose that $T:M\to M$ is a mapping such that*

(1) *T is a dualistic contraction of rational type;*
(2) *T is a dominated mapping;*
(3) *T is a continuous mapping.*

Then T has a fixed point m^*. Moreover, $D(m^*,m^*)=0$.

Proof. Let $j_0\in M$ be an initial element and $j_n=T(j_{n-1})$ for all $n\geq 1$. If there exists a positive integer r such that $j_{r+1}=j_r$ then $j_r=T(j_r)$, and so we are done. Suppose that $j_n\neq j_{n+1}$ for all $n\in\mathbb{N}$. Since T is a dominated mapping, $j_0\succeq T(j_0)=j_1$, and $j_1\succeq T(j_1)$ implies $j_1\succeq j_2$, and $j_2\succeq T(j_2)$ implies $j_2\succeq j_3$. Continuing in the similar way, we get

$$j_0\succeq j_1\succeq j_2\succeq j_3\succeq\cdots\succeq j_n\succeq j_{n+1}\succeq j_{n+2}\succeq\cdots.$$

The remaining part of the proof follows from the proof of Theorem 3. □

The following example shall illustrate Theorem 6.

Example 4. Let $M=\mathbb{R}^2$. Define $D_\vee:M\times M\to\mathbb{R}$ by $D_\vee(j,k)=j_1\vee k_1$, where $j=(j_1,j_2)$ and $k=(k_1,k_2)$. Note that $(M,D_\vee)$ is a complete dualistic partial metric space. Let $T:M\to M$ be given by

$$T(j)=\frac{j}{2}\quad\text{for all } j\in M.$$

In M, we define the relation $\succeq$ in the following way:

$$j\succeq k \text{ if and only if } j_1\geq k_1, \text{ where } j=(j_1,j_2) \text{ and } k=(k_1,k_2).$$

Clearly, $\succeq$ is a partial order on M and T is a continuous, dominated mapping with respect to $\succeq$. Moreover, $T(-1,0)\succeq(-1,0)$. We shall show that for all $j,k\in M$, the contractive condition (2.1) is satisfied. For this, note that

$$\begin{aligned}
|D_\vee(T(j),T(k))|=\left|D_\vee\left(\frac{j}{2},\frac{k}{2}\right)\right| &= \left|\frac{j_1}{2}\right| \text{ for all } j_1\geq k_1,\\
|D_\vee(j,T(j))|=\left|D_\vee\left(j,\frac{j}{2}\right)\right| &= \begin{cases}\left|\frac{j_1}{2}\right| & \text{if } j_1\leq 0\\ |j_1| & \text{if } j_1\geq 0,\end{cases}\\
|D_\vee(k,T(k))|=\left|D_\vee\left(k,\frac{k}{2}\right)\right| &= \begin{cases}\left|\frac{k_1}{2}\right| & \text{if } k_1\leq 0\\ |k_1| & \text{if } k_1\geq 0,\end{cases}\\
\text{and } |D_\vee(j,k)| &= |j_1| \text{ for all } j_1\geq k_1.
\end{aligned}$$

Now for $\alpha = \frac{1}{3}$, $\beta = \frac{1}{2}$. If $j_1 \leq 0$, $k_1 \leq 0$, then

$$|D_\vee(T(j), T(k))| \leq \alpha \left| \frac{D_\vee(j, T(j)) \cdot D_\vee(k, T(k))}{D_\vee(j, k)} \right| + \beta |D_\vee(j, k)|$$

holds for all $j \succeq k$ if and only if $6|j_1| \leq |k_1| + 6|j_1|$.

If $j_1 \geq 0$, $k_1 \geq 0$, then the contractive condition

$$|D_\vee(T(j), T(k))| \leq \alpha \left| \frac{D_\vee(j, T(j)) \cdot D_\vee(k, T(k))}{D_\vee(j, k)} \right| + \beta |D_\vee(j, k)|$$

holds for all $j \succeq k$ if and only if $j_1 \leq \frac{2}{3} k_1 + j_1$.

Finally, if $j_1 \geq 0$, $k_1 \leq 0$, then

$$|D_\vee(T(j), T(k))| \leq \alpha \left| \frac{D_\vee(j, T(j)) \cdot D_\vee(k, T(k))}{D_\vee(j, k)} \right| + \beta |D_\vee(j, k)|$$

holds for all $j \succeq k$ if and only if $3j_1 \leq |k_1| + 3j_1$. Thus all the conditions of Theorem 5 are satisfied. Moreover, $(0, 0)$ is a fixed point of T.

Theorem 7. *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space. Suppose that $T : M \to M$ is a mapping such that*

1. *T is a dualistic contraction of rational type;*
2. *T is a dominated mapping;*
3. *(Q) holds.*

Then T has a fixed point m^*. Moreover, $D(m^*, m^*) = 0$.

The proof can be obtained by the proofs of Theorems 5 and 6. Some consequences of Theorems 6 and 7 are given below.

Corollary 5. *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space. Suppose that $T : M \to M$ is a mapping such that*

1. *T is a dualistic contraction of rational type with $\beta = 0$;*
2. *T is a dominated mapping;*
3. *T is a continuous mapping.*

Then T has a fixed point m^*. Moreover, $D(m^*, m^*) = 0$.

Corollary 6. *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space. Suppose that $T : M \to M$ is a mapping such that*

1. *T is a dualistic contraction of rational type with $\beta = 0$;*
2. *T is a dominated mapping;*
3. *(Q) holds.*

Then T has a fixed point m^*. Moreover, $D(m^*, m^*) = 0$.

Corollary 7. *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space. Suppose that $T : M \to M$ is a mapping such that*

1. *T is a dualistic contraction of rational type with $\alpha = 0$;*
2. *T is a dominated mapping;*
3. *T is a continuous mapping.*

Then T has a fixed point m^*. Moreover, $D(m^*, m^*) = 0$.

Corollary 8. *Let $(M, \preceq, D)$ be a complete ordered dualistic partial metric space. Suppose that $T : M \to M$ is a mapping such that*

(1) *T is a dualistic contraction of rational type with $\alpha = 0$;*
(2) *T is a dominated mapping;*
(3) *(Q) holds.*

Then T has a fixed point m^*. Moreover, $D(m^*, m^*) = 0$.

References

[1] M. Arshad, A. Azam, M. Abbas, A. Shoaib, *Fixed point results of dominated mappings on a closed ball in ordered partial metric spaces without continuity,* U.P.B. Sci. Bull. Series A **76** (2014), 123–134.

[2] M. Arshad, M. Nazam, I. Beg, *Fixed point theorems in ordered dualistic partial metric spaces*, Korean J. Math. **24** (2016), 169–179.

[3] X. Fan, Z. Wang, *Some fixed point theorems in generalized quasi-partial metric spaces*, J. Nonlinear Sci. Appl. **9** (2016), 1658–1674.

[4] A. Farajzadeh, P. Chuadchawna, A. Kaewcharoen, *Fixed point theorems for $(\alpha, \eta, \psi, \xi)$-contractive multi-valued mappings on α-η-complete partial metric spaces*, J. Nonlinear Sci. Appl. **9** (2016), 1977–1990.

[5] J. Harjani, B. Lopez, K. Sadarangani, *A fixed point theorem for mappings satisfying a contractive condition of rational type on a partially ordered metric space*, Abstr. Appl. Anal. **2010**, Art. ID 190701 (2010).

[6] H. Isik, D. Tukroglu *Some fixed point theorems in ordered partial metric spaces,* J. Inequal. Spec. Funct. **4** (2013), No. 2, 13–18.

[7] S.G. Matthews, *Partial metric topology*, in Proceedings of the $11^t h$ Summer Conference on General Topology and Applications **728**, pp. 183–197, New York Acad. Sci., New York, 1994.

[8] M. Nazam, M. Arshad, *On a fixed point theorem with application to integral equations*, Int. J. Anal. **2016**, Art. ID 9843207 (2016).

[9] S. Oltra, O. Valero, *Banach's fixed point theorem for partial metric spaces*, Rend. Ist. Mat. Univ. Trieste **36** (2004),17–26.

[10] S.J. O'Neill, *Partial metric, valuations and domain theory*, Ann. New York Acad. Sci. **806** (1996), 304–315.

[11] A. Shoaib, *α-η dominated mappings and related common fixed point results in closed ball*, J. Concrete Appl. Math. **13** (2015), 152–170.

[12] A. Shoaib, M. Arshad, M. A. Kutbi, *Common fixed points of a pair of Hardy Rogers type mappings on a closed ball in ordered partial metric spaces*, J. Comput. Anal. Appl. **17** (2014), 255–264.

[13] O. Valero, *On Banach fixed point theorems for partial metric spaces*, Appl. Gen. Topol. **6** (2005), 229–240.

[14] H-H. Zheng, F. Gu, *Some results of common fixed point for four self-maps satisfying a new Ψ-contractive condition in partial metric spaces*, J. Nonlinear Sci. Appl. **9** (2016), 2258–2272.

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HOMOMORPHISMS AND DERIVATIONS IN PROPER LIE CQ^*-ALGEBRAS

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Abstract. In this paper, we investigate homomorphisms in proper Lie CQ^*-algebras and derivations on proper Lie CQ^*-algebras associated with the following Pexiderized functional equation

$$f(x+y) = f_0(x) + f_1(y).$$

Moreover, we prove the Hyers-Ulam stability of homomorphisms in proper Lie CQ^*-algebras and of derivations on proper Lie CQ^*-algebras.

1. Introduction and preliminaries

Ulam [24] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$?

Hyers [10] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \le \epsilon.$$

Rassias [16] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.* See [18, 19, 20, 21, 22] for more information on functional equations and their stability.

Let A be a linear space and A_0 is a $*$-algebra contained in A as a subspace.We say that A is a quasi $*$-algebra over A_0 if

(i) the right and left multiplications of an element of A and an element of A_0 are defined and linear;

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(ii) $x_1(x_2a) = (x_1x_2)a, (ax_1)x_2 = a(x_1x_2)$ and $x_1(ax_2) = (x_1a)x_2$ for all $x_1, x_2 \in A_0$ and all $a \in A$;

(iii) an involution $*$, which extends the involution of A_0, is defined in A with the property $(ab)^* = b^*a^*$ whenever the multiplication is defined.

A quasi $*$-algebra (A, A_0) is said to be a locally convex quasi $*$-algebra if in A a locally convex topology τ is defined such that

(i) the involution is continuous and the multiplications are separately continuous;

(ii) A_0 is dense in $A[\tau]$.

Throughout this paper, we suppose that a locally convex quasi $*$-algebra (A, A_0) is complete. For an overview on partial $*$-algebra and related topics we refer to [1].

In a series of papers [2, 4, 5, 6] many authors have considered a special class of quasi $*$-algebras, called proper CQ^*-algebras, which arise as completions of C^*-algebras. They can be introduced in the following way:

Let A be a Banach module over the C^*-algebra A_0 with involution $*$ and C^*-norm $\| \cdot \|_0$ such that $A_0 \subset A$. We say that (A, A_0) is a proper CQ^*-algebra if

(i) A_0 is dense in A with respect to its norm $\| \cdot \|$;

(ii) $(ab)^* = b^*a^*$ whenever the multiplication is defined;

(iii) $\| y \|_0 = sup_{a\in A,\|a\|\le 1} \| ay \|$ for all $y \in A_0$.

Definition 1.1. A proper CQ^*-algebra (A, A_0), endowed with the Lie product

$$[z, x] = \frac{zx - xz}{2}$$

for all $x \in A$ and all $z \in A_0$, is called a proper Lie CQ^*-algebra.

Definition 1.2. Let (A, A_0) and (B, B_0) be proper Lie CQ^*-algebras.

(i) A $\mathbb{C}$-linear mapping $H : A \longrightarrow B$ is called a proper Lie CQ^*-algebra homomorphism if $H(z) \in B_0$ and $H([z, x]) = [H(z), H(x)]$ for all $z \in A_0$ and all $x \in A$.

(ii) A $\mathbb{C}$-linear mapping $\delta : A_0 \to A$ is called a Lie derivation if

$$\delta([z, x]) = [\delta(z), x] + [z, \delta(x)]$$

for all $x, z \in A_0$ (see [12]).

Park and Rassias [14] investigated homomorphisms in proper JCQ^*-triples and derivations on proper JCQ^*-triples. Park [15] investigated homomorphisms in proper CQ^*-ternary algebras and derivations on proper CQ^*-ternary algebras. Najati and Park [11] investigated homomorphisms in quasi-Banach algebras associated to the Pexiderized Cauchy function equation.

In this paper, we investigate homomorphisms in proper Lie CQ^*-algebras and derivations on proper Lie CQ^*-algebras associated with the following Pexiderized functional equation

$$f(x + y) = f_0(x) + f_1(y).$$

This is applied to investigate homomorphisms in proper Lie CQ^*-algebras.

Throughout this paper, assume that k is a fixed positive integer.

This paper is organized as follows: In Sections 2 and 3, we investigate homomorphisms in proper Lie CQ^*-algebras and derivations in proper Lie CQ^*-algebras.

In Sections 4 and 5, we prove the Hyers-Ulam stability of homomorphisms in proper Lie CQ^*-algebras and stability of derivations on proper Lie CQ^*-algebras.

2. Homomorphism in proper Lie CQ^*-algebra

Throughout this section, assume that (A, A_0) is a proper Lie CQ^*-algebra with C^*-norm $\| . \|_{A_0}$ and norm $\| . \|_A$, and that (B, B_0) is a proper Lie CQ^*-algebras with C^*-norm $\| . \|_{B_0}$ and norm $\| . \|_B$.

Theorem 2.1. *Let $\varphi : A_0 \times A \longrightarrow [0, +\infty)$ be a mapping such that*

$$\lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n w, x) = 0 \tag{2.1}$$

for all $w \in A_0$ and all $x \in A$, and let $f, f_0, f_1 : A \longrightarrow B$ be mappings with $f(0) = 0$ and $f(w), f_0(0) \in B_0$ for all $w \in A_0$ and

$$\|\mu f(x) - f_0(y) - f_1(z)\|_B \leq \| kf(\frac{\mu x + y + z}{k}) \|_B, \tag{2.2}$$

$$\|f([w, x]) - [f_0(w), f_1(x)]\|_B \leq \varphi(w, x) \tag{2.3}$$

for all $\mu \in \mathbb{T}^1 := \{\mu \in \mathbb{C} : |\mu| = 1\}$, all $w \in A_0$ and all $x, y, z \in A$. Then the mapping $f : A \longrightarrow B$ is a proper Lie CQ^*-algebra homomorphism. Moreover,

$$f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x)$$

for all $x \in A$.

Proof. Letting $\mu = 1$, $y = -x$ and $z = 0$ in (2.2), we get

$$f(x) = f_0(-x) - f_0(0)$$

for all $x \in A$. So $f_0(w) \in B_0$ for all $w \in A_0$. Similarly, we have

$$f(x) = f_1(-x) - f_1(0)$$

for all $x \in A$. By (2.2), we have

$$\|f(x + y) - f(x) - f(y)\|_B \leq \|f(x + y) - f_0(-x) - f_1(-y)\|_B = 0$$

for all $x, y \in A$. So the mapping $f : A \longrightarrow B$ is additive. Letting $y = -\mu x$, and $z = 0$ in (2.2), we get

$$\mu f(x) = f_0(-\mu x) + f_1(0) = f(\mu x)$$

for all $x \in A$. By the same reasoning as in the proof of [13, Theorem 2.1], the mapping $f : A \longrightarrow B$ is $\mathbb{C}$-linear. By (2.1) and (2.3), we have

$$\begin{aligned}\|f([w, x]) - [f(w), f(x)]\|_B &= \lim_{n\to\infty} \frac{1}{2^n}\|f(2^n[w, x]) - [f_0(2^n w), f_1(x)]\|_B \\ &\leq \lim_{n\to\infty} \frac{\varphi(2^n w, x)}{2^n} = 0\end{aligned}$$

for all $w \in A_0$ and all $x \in A$. So

$$f([w, x]) = [f(w), f(x)]$$

for all $w \in A_0$ and all $x \in A$.

Therefore, the mapping $f : A \longrightarrow B$ is a proper Lie CQ^*-algebra homomorphism, as desired. □

Remark 2.2. We can formulate a similar theorem if we replace the condition (2.1) by

$$\lim_{n\to\infty} \frac{1}{2^n}\varphi(w, 2^n x) = 0$$

for all $w \in A_0$ and all $x \in A$.

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Corollary 2.3. *Let θ, r_0, r_1 be non-negative real numbers such that $r_j \in [0,1)$ for some $0 \le j \le 1$ and let $f, f_0, f_1 : A \longrightarrow B$ be mappings with $f(0) = 0$ and $f(w), f_0(0) \in B_0$ for all $w \in A_0$ and satisfying (2.2) and*

$$\|f([w,x]) - [f_0(w), f_1(x)]\|_B \le \theta \|w\|_A^{r_0} \|x\|_A^{r_1}$$

for all $w \in A_0$ and all $x \in A$ (by putting $\|.\|_A^0 = 1$). Then the mapping $f : A \longrightarrow B$ is a proper Lie CQ^*-algebra homomorphism. Moreover,

$$f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x)$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.1. □

Corollary 2.4. *Let θ, r_0, r_1 be non-negative real numbers such that $r_j \in [0,1)$ for some $0 \le j \le 1$ and let $f, f_0, f_1 : A \longrightarrow B$ be mappings with $f(0) = 0$ and $f(w), f_0(0) \in B_0$ for all $w \in A_0$ and satisfying (2.2) and*

$$\|f([w,x]) - [f_0(w), f_1(x)]\|_B \le \theta(\|w\|_{A_0}^{r_0} + \|x\|_A^{r_1})$$

for all $w \in A_0$ and all $x \in A$ (by putting $\|.\|_A^0 = 1$). Then the mapping $f : A \longrightarrow B$ is a proper Lie CQ^*-algebra homomorphism. Moreover,

$$f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x)$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.1. □

3. Derivations on proper Lie CQ^*-algebras

Throughout this section, assume that (A, A_0) is a proper Lie CQ^*-algebra with C^*-norm $\| . \|_{A_0}$ and norm $\| . \|_A$.

Theorem 3.1. *Let $\varphi : A_0 \times A_0 \to [0, +\infty)$ be a mapping such that*

$$\lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n w_0, w_1) = 0 \tag{3.1}$$

for all $w_0, w_1 \in A_0$ and let $f, f_0, f_1 : A_0 \longrightarrow A$ be mappings with $f(0) = 0$ and $f(w), f_0(0) \in A_0$ for all $w \in A_0$ and

$$\|\mu f(w) - f_0(w_0) - f_1(w_1)\|_A \le \| k f(\frac{\mu w + w_0 + w_1}{k}) \|_A,$$

$$\|f([w_0, w_1]) + [f_0(w_0), w_1] + [w_0, f_1(w_1)]\,\|_A \le \varphi(w_0, w_1) \tag{3.2}$$

for all $\mu \in \mathbb{T}^1$ and all $w, w_0, w_1 \in A_0$. Then the mapping $f : A_0 \longrightarrow A$ is a Lie derivation. Moreover,

$$f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x)$$

for all $x \in A_0$.

Proof. By the same reasoning as in the proof of Theorem 2.1, the mapping $f : A_0 \longrightarrow A$ is $\mathbb{C}$-linear and $f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x)$ for all $x \in A_0$. By (3.1) and (3.2), we have

$$\begin{aligned}
&\|f([w_0, w_1]) - [f(w_0), w_1] - [w_0, f(w_1)]\|_A \\
&= \lim_{n\to\infty} \frac{1}{2^n}\|f(2^n[w_0, w_1]) + [f_0(2^n w_0), w_1] + [2^n w_0, f_1(w_1)]\|_A \\
&\le \lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n w_0, w_1) = 0
\end{aligned}$$

for all $w_0, w_1 \in A_0$. So

$$f([w_0, w_1]) = [f(w_0), w_1] + [w_0, f(w_1)]$$

for all $w_0, w_1 \in A_0$. Therefore, the mapping $f : A_0 \longrightarrow A$ is a Lie derivation, as desired. □

Remark 3.2. We can formulate a similar theorem if we replace the condition (3.1) by

$$\lim_{n\to\infty} \frac{1}{2^n}\varphi(w_0, 2^n w_1) = 0$$

for all $w_0, w_1 \in A_0$.

Corollary 3.3. *Let θ, r_0, r_1 be non-negative real numbers such that $r_j \in [0, 1)$ for some $0 \le j \le 1$ and let $f, f_0, f_1 : A \longrightarrow B$ be mappings with $f(0) = 0$,$f(w), f_0(0) \in A_0$ for all $w \in A_0$ and satisfying (3.2) such that*

$$\big\|f([w_0, w_1]) + [f_0(w_0), w_1] + [w_0, f_1(w_1)]\big\|_A \le \theta(\|w_0\|_{A_0}^{r_0} + \|w_1\|_{A_0}^{r_1})$$

for all $w_0, w_1 \in A_0$ (by putting $\|.\|_A^0 = 1$). Then the mapping $f : A_0 \longrightarrow A$ is a Lie derivation. Moreover,

$$f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x)$$

for all $x \in A_0$.

Proof. The proof follows from Theorem 3.1. □

Corollary 3.4. *Let θ, r_0, r_1 be non-negative real numbers such that $r_j \in [0, 1)$ for some $0 \le j \le 1$ and let $f, f_0, f_1 : A \longrightarrow B$ be mappings with $f(0) = 0$ and satisfying (3.2) such that*

$$\big\|f([w_0, w_1]) + [f_0(w_0), w_1] + [w_0, f_1(w_1)]\big\|_A \le \theta\|w_0\|_{A_0}^{r_0}\|w_1\|_{A_0}^{r_1}$$

for all $w_0, w_1 \in A_0$ (by putting $\|.\|_A^0 = 1$). Then the mapping $f : A_0 \longrightarrow A$ is a Lie derivation. Moreover,

$$f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x)$$

for all $x \in A_0$.

Proof. The proof follows from Theorem 3.1. □

4. Stability of homomorphism in proper Lie CQ^*-algebras

Using an idea of Găvruta [7], we prove the Hyers-Ulam stability of homomorphisms in proper Lie CQ^*-algebras.

Theorem 4.1. *Let $\varphi : A \times A \to [0, +\infty)$ be a mapping such that $\varphi(0, 0) = 0$ and*

$$\lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x, 2^n y) = 0, \tag{4.1}$$

$$\widetilde{\varphi}(x) := \sum_{i=0}^{\infty} \frac{1}{2^i}\Big[\varphi(2^i x, 2^i x) + \varphi(2^i x, 0) + \varphi(0, 2^i x)\Big] < \infty \tag{4.2}$$

for all $x, y \in A$ and let $f, f_0, f_1 : A \to B$ be mappings with $f(0) = 0$ and $f(w), f_0(w), f_1(w) \in B_0$ for all $w \in A_0$ such that

$$\|f(\mu x + \mu y) - \mu f_0(x) - \mu f_1(y)\|_B \le \varphi(x, y), \tag{4.3}$$

$$\|f(w_0 + w_1) - f_0(w_0) - f_1(w_1)\|_{B_0} \le \varphi(w_0, w_1), \tag{4.4}$$

$$\|f([w_0, x]) - [f_0(w_0), f_1(x)]\|_B \le \varphi(w_0, x) \tag{4.5}$$

for all $\mu \in \mathbb{T}^1$, all $w_0, w_1 \in A_0$ and all $x, y \in A$. Then there exists a unique proper Lie CQ^*-algebra homomorphism $H : A \longrightarrow B$ such that

$$\begin{aligned} \|f(x) - H(x)\|_B &\le \frac{1}{2}\widetilde{\varphi}(x), \\ \|f_0(x) - f_0(0) - H(x)\|_B &\le \frac{1}{2}\widetilde{\varphi}(x) + \varphi(x, 0), \\ \|f_1(x) - f_1(0) - H(x)\|_B &\le \frac{1}{2}\widetilde{\varphi}(x) + \varphi(0, x) \end{aligned} \tag{4.6}$$

for all $x \in A$.

Proof. Letting $y = 0$ and $\mu = 1$ in (4.3), we get

$$\|f(x) - f_0(x) - f_1(0)\|_B \le \varphi(x, 0) \tag{4.7}$$

Similarly, we get

$$\|f(y) - f_1(y) - f_0(0)\|_B \le \varphi(0, y) \tag{4.8}$$

for all $x, y \in A$. Using (4.7) and (4.8), we get

$$\|f(\mu x + \mu y) - \mu f(x) - \mu f(y)\|_B \le \psi(x, y) \tag{4.9}$$

where

$$\psi(x, y) := \varphi(x, y) + \varphi(x, 0) + \varphi(0, y)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. Letting $x = y$ and $\mu = 1$ in (4.9), we get

$$\|f(2x) - 2f(x)\|_B \le \psi(x, x) \tag{4.10}$$

Replacing x by $2^n x$ in (4.10) and dividing both sides of (4.10) by 2^{n+1}, we get

$$\Big\|\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^n x)}{2^n}\Big\|_B \le \frac{\psi(2^n x, 2^n x)}{2^{n+1}} \tag{4.11}$$

for all $x \in A$ and all non-negative integers n. By (4.11), we have

$$\Big\|\frac{f(2^{n+1}x)}{2^{n+1}} - \frac{f(2^m x)}{2^m}\Big\|_B \le \frac{1}{2}\sum_{i=m}^{n} \frac{\psi(2^i x, 2^i x)}{2^i} \tag{4.12}$$

for all $x \in A$ and all non-negative integers n and m with $n \ge m$. Thus we conclude from (4.2) and (4.12) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is Cauchy in B for all $x \in A$. Since B is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges in B for all $x \in A$. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x) = \lim_{n\to\infty} \frac{1}{2^n} f_i(2^n x) \quad (i = 0, 1) \tag{4.13}$$

for all $x \in A$. Letting $m = 0$ and passing the limit when $n \to \infty$ in (4.12), we get (4.6). It follows from (4.1), (4.3) and (4.13) that

$$\begin{aligned}
&\left\|H(\mu x + \mu y) - \mu H(x) - \mu H(y)\right\|_B \\
&= \lim_{n\to\infty} \frac{1}{2^n}\left\|f(2^n \mu x + 2^n \mu y) - \mu f_0(2^n x) - \mu f_1(2^n y)\right\|_B \\
&\le \lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{2^n} = 0
\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. Hence

$$H(\mu x + \mu y) = \mu H(x) + \mu H(y)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. By the same reasoning as in the proof of [13, Theorem 2.1], the mapping $H : A \to B$ is $\mathbb{C}$-linear. It follows from (4.4) and (4.13) that $H(w) \in B_0$ for all $w \in A_0$. It follows from (4.1), (4.5) and (4.13) that

$$\begin{aligned}
\|H([w,x]) - [H(w), H(x)]\|_B &\le \lim_{n\to\infty} \frac{1}{4^n}\left\|f(4^n[w,x]) - [f_0(2^n w), f_1(2^n x)]\right\|_B \\
&\le \lim_{n\to\infty} \frac{\varphi(2^n w, 2^n x)}{4^n} = 0
\end{aligned}$$

for all $w \in A_0$ and all $x \in A$. Hence

$$H([w,x]) = [H(w), H(x)]$$

for all $w \in A_0$ and all $x \in A$. So $H : A \longrightarrow B$ is a proper Lie CQ^*-algebra homomorphism. Now, we show that $H : A \longrightarrow B$ is unique. Let $T : A \to B$ be another proper Lie CQ^*-algebra homomorphism satisfying (4.6). It follows from (4.2), (4.6) and (4.13) that

$$\begin{aligned}
\|H(x) - T(x)\|_B &= \lim_{n\to\infty} \frac{1}{2^n}\|f(2^n x) - T(2^n x)\|_B \\
&\le \frac{1}{2}\lim_{n\to\infty} \frac{1}{2^n}\widetilde{\varphi}(2^n x) = 0
\end{aligned}$$

for all $x \in A$. So $H = T$. □

Corollary 4.2. *Let θ, r_0, r_1 be non-negative real numbers such that $0 \le r_0, r_1 < 1$ and let $f, f_0, f_1 : A \longrightarrow B$ be mappings with $f(0) = 0$ and $f(w), f_0(w), f_1(w) \in B_0$ for all $w \in A_0$ and*

$$\|f(\mu x + \mu y) - \mu f_0(x) - \mu f_1(y)\|_B \le \theta(\|x\|_A^{r_0} + \|y\|_A^{r_1}),$$

$$\|f(w_0 + w_1) - f_0(w_0) - f_1(w_1)\|_{B_0} \le \theta(\|w_0\|_A^{r_0} + \|w_1\|_A^{r_1}),$$

$$\|f([w_0, x]) - [f_0(w_0), f_1(x)]\|_B \le \theta(\|w_0\|_A^{r_0} + \|x\|_A^{r_1})$$

for all $\mu \in \mathbb{T}^1$, all $w_0, w_1 \in A_0$ and all $x, y \in A$ (by putting $\|.\|_A^0 = 1$). Then there exists a unique proper Lie CQ^*-algebra homomorphism $H : A \longrightarrow B$ such that

$$\|f(x) - H(x)\|_B \le 2\left(\frac{\|x\|_A^{r_0}}{2 - 2^{r_0}} + \frac{\|x\|_A^{r_1}}{2 - 2^{r_1}}\right)\theta$$

$$\|f_i(x) - f_i(0) - H(x)\|_B \le 2\left(\frac{\|x\|_A^{r_0}}{2 - 2^{r_0}} + \frac{\|x\|_A^{r_1}}{2 - 2^{r_1}}\right)\theta + \theta\|x\|_A^{r_i} \quad (i = 0, 1)$$

for all $x \in A$.

Proof. The proof follows from Theorem 4.1. □

Corollary 4.3. *Let θ, r_0, r_1 be non-negative real numbers such that $\lambda := r_0 + r_1 < 1$ and let $f, f_0, f_1 : A \longrightarrow B$ be mappings with $f(0) = 0$ and $f(w), f_0(w), f_1(w) \in B_0$ for all $w \in A_0$ and*

$$\|f(\mu x + \mu y) - \mu f_0(x) - \mu f_1(y)\|_B \le \theta \|x\|_A^{r_0} \|y\|_A^{r_1},$$

$$\|f(w_0 + w_1) - f_0(w_0) - f_1(w_1)\|_{B_0} \le \theta \|w_0\|_A^{r_0} \|w_1\|_A^{r_1},$$

$$\|f([w_0, x]) - [f_0(w_0), f_1(x)]\|_B \le \theta \|w_0\|_A^{r_0} \|x\|_A^{r_1}$$

for all $\mu \in \mathbb{T}^1$, all $w_0, w_1 \in A_0$ and all $x, y \in A$ (by putting $\|.\|_A^0 = 1$). Then there exists a unique proper Lie CQ^*-algebra homomorphism $H : A \longrightarrow B$ such that

$$\|f(x) - H(x)\|_B \le \frac{\theta}{2 - 2^\lambda} \|x\|_A^\lambda,$$
$$\|f_i(x) - f_i(0) - H(x)\|_B \le \frac{\theta}{2 - 2^\lambda} \|x\|_A^\lambda \quad (i = 0, 1)$$

for all $x \in A$.

Proof. The proof follows from Theorem 4.1. □

For $r_0 = r_1 = 0$, we have the following theorem:

Theorem 4.4. *Let θ be non-negative real number and let $f, f_0, f_1 : A \longrightarrow B$ be mappings such that $f(w), f_0(w), f_1(w) \in B_0$ for all $w \in A_0$ and*

$$\|f(\mu x + \mu y) - \mu f_0(x) - \mu f_1(y)\|_B \le \theta,$$

$$\|f(w_0 + w_1) - f_0(w_0) - f_1(w_1)\|_{B_0} \le \theta,$$

$$\|f([w_0, x]) - [f_0(w_0), f_1(x)]\|_B \le \theta$$

for all $\mu \in \mathbb{T}^1$, all $w_0, w_1 \in A_0$ and all $x, y \in A$. Then there exists a unique proper Lie CQ^*-algebra homomorphism $H : A \longrightarrow B$ such that

$$\|f(x) - H(x)\|_B \le 3\theta + M,$$

$$\|f_0(x) + f_1(0) - H(x)\|_B \le 4\theta + M,$$

$$\|f_1(x) + f_0(0) - H(x)\|_B \le 4\theta + M$$

for all $x \in A$, where $M = \|f_0(0) + f_1(0)\|_B$.

Proof. The proof is similar to the proof of Theorem 4.1. □

5. Stability of derivation on proper Lie CQ^*-algebras

We prove the Hyers-Ulam stability of derivations on proper Lie CQ^*-algebras.

Theorem 5.1. *Let $\varphi : A_0 \times A_0 \longrightarrow [0, +\infty)$ be a mapping satisfying (4.1) and (4.2) for all $x, y \in A_0$ such that $\varphi(0,0) = 0$ and let $f, f_0, f_1 : A_0 \longrightarrow A$ be mappings with $f(0) = 0$ and $f(w), f_0(w), f_1(w) \in A_0$ for all $w \in A_0$ such that*

$$\|f(\mu w_0 + \mu w_1) - \mu f_0(w_0) - \mu f_1(w_1)\|_{A_0} \le \varphi(w_0, w_1),$$

$$\big\|f([w_0, w_1]) - [f_0(w_0), w_1] - [w_0, f_1(w_1)]\big\|_A \le \varphi(w_0, w_1) \tag{5.1}$$

for all $\mu \in \mathbb{T}^1$ and all $w_0, w_1 \in A_0$. Then there exists a unique Lie derivation $\delta : A_0 \longrightarrow A$ such that

$$\begin{aligned} \|f(x) - \delta(x)\|_A &\le \frac{1}{2}\widetilde{\varphi}(x), \\ \|f_0(x) - f_0(0) - \delta(x)\|_A &\le \frac{1}{2}\widetilde{\varphi}(x) + \varphi(x, 0), \\ \|f_1(x) - f_1(0) - \delta(x)\|_A &\le \frac{1}{2}\widetilde{\varphi}(x) + \varphi(0, x) \end{aligned} \tag{5.2}$$

for all $x \in A_0$.

Proof. By the same reasoning as in the proof of Theorem 4.1, there exists a unique $\mathbb{C}$-linear mapping $\delta : A_0 \longrightarrow A$ satisfying (5.2). The mapping $\delta : A_0 \longrightarrow A$ is defined by

$$\delta(w) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n w) = \lim_{n\to\infty} \frac{1}{2^n} f_i(2^n w) \quad (i = 0, 1) \tag{5.3}$$

for all $w \in A_0$ and $\delta(w) \in A_0$ for all $w \in A_0$. It follows from (4.1), (5.1) and (5.3) that

$$\begin{aligned} &\big\|\delta([w_0, w_1]) - [\delta(w_0), w_1] - [w_0, \delta(w_1)]\big\|_A \\ &= \lim_{n\to\infty} \frac{1}{4^n} \big\|f(4^n[w_0, w_1]) - [f_0(2^n w_0), 2^n w_1] - [2^n w_0, f_1(2^n w_1)]\big\|_A \\ &\le \lim_{n\to\infty} \frac{\varphi(2^n w_0, 2^n w_1)}{4^n} = 0 \end{aligned}$$

for all $w_0, w_1 \in A_0$. So

$$\delta([w_0, w_1]) = [\delta(w_0), w_1] + [w_0, \delta(w_1)]$$

for all $w_0, w_1 \in A_0$. Hence the mapping $\delta : A_0 \to A$ is a unique Lie derivation satisfying (5.2). □

Corollary 5.2. *Let θ, r_0, r_1 be non-negative real numbers such that $0 \le r_0, r_1 < 1$ and let $f, f_0, f_1 : A_0 \to A$ be mappings with $f(0) = 0$ and $f(w), f_0(w), f_1(w) \in A_0$ for all $w \in A_0$ such that*

$$\|f(\mu w_0 + \mu w_1) - \mu f_0(w_0) - \mu f_1(w_1)\|_{A_0} \le \theta(\|w_0\|_{A_0}^{r_0} + \|w_1\|_{A_0}^{r_1}),$$

$$\big\|f([w_0, w_1]) - [f_0(w_0), w_1] - [w_0, f_1(w_1)]\big\|_A \le \theta(\|w_0\|_{A_0}^{r_0} + \|w_1\|_{A_0}^{r_1})$$

for all $\mu \in \mathbb{T}^1$ and all $w_0, w_1 \in A_0$ (by putting $\|.\|_A^0 = 1$). Then there exists a unique Lie derivation $\delta : A_0 \to A$ such that

$$\|f(x) - \delta(x)\|_A \le 2\Big(\frac{\|x\|_{A_0}^{r_0}}{2 - 2^{r_0}} + \frac{\|x\|_{A_0}^{r_1}}{2 - 2^{r_1}}\Big)\theta,$$

$$\|f_i(x) - f_i(0) - \delta(x)\|_A \le 2\Big(\frac{\|x\|_{A_0}^{r_0}}{2 - 2^{r_0}} + \frac{\|x\|_{A_0}^{r_1}}{2 - 2^{r_1}}\Big)\theta + \theta\|x\|_{A_0}^{r_i} \qquad (i = 0, 1)$$

for all $x \in A_0$.

Proof. The proof follows from Theorem 5.1. □

Corollary 5.3. *Let* θ, r_0, r_1 *be non-negative real numbers such that* $0 \le \lambda < 1$ *and let* $f, f_0, f_1 : A_0 \longrightarrow A$ *be mappings with* $f(0) = 0$ *and* $f(w), f_0(w), f_1(w) \in A_0$ *for all* $w \in A_0$ *such that*

$$\|f(\mu w_0 + \mu w_1) - \mu f_0(w_0) - \mu f_1(w_1)\|_{A_0} \le \theta \|w_0\|_{A_0}^{r_0} \|w_1\|_{A_0}^{r_1},$$

$$\big\|f([w_0, w_1]) - [f_0(w_0), w_1] - [w_0, f_1(w_1)]\big\|_A \le \theta \|w_0\|_{A_0}^{r_0} \|w_1\|_{A_0}^{r_1}$$

for all $\mu \in \mathbb{T}^1$ *and all* $w_0, w_1 \in A_0$*(by putting* $\|.\|_A^0 = 1$*). Then there exists a unique Lie derivation* $\delta : A_0 \longrightarrow A$ *such that*

$$\|f(x) - \delta(x)\|_A \le \frac{\theta}{2 - 2^\lambda} \|x\|_{A_0}^\lambda,$$

$$\|f_i(x) - f_i(0) - \delta(x)\|_A \le \frac{\theta}{2 - 2^\lambda} \|x\|_{A_0}^\lambda \qquad (i = 0, 1)$$

for all $x \in A_0$.

Proof. The proof follows from Theorem 5.1. □

For $r_0 = r_1 = 0$, we have the following theorem:

Theorem 5.4. *Let* θ *be non-negative real number and let* $f, f_0, f_1 : A_0 \longrightarrow A$ *be mappings and* $f(w), f_0(w), f_1(w) \in A_0$ *for all* $w \in A_0$ *and*

$$\|f(\mu w_0 + \mu w_1) - \mu f_0(w_0) - \mu f_1(w_1)\|_{A_0} \le \theta,$$

$$\|f([w_0, w_1]) - [f_0(w_0), f_1(w_1)]\|_A \le \theta$$

for all $\mu \in \mathbb{T}^1$ *and all* $w_0, w_1 \in A_0$. *Then there exists a unique Lie derivation* $\delta : A \longrightarrow B$ *such that*

$$\|f(x) - \delta(x)\|_A \le 3\theta + M,$$

$$\|f_0(x) + f_1(0) - \delta(x)\|_A \le 4\theta + M,$$

$$\|f_1(x) + f_0(0) - \delta(x)\|_A \le 4\theta + M$$

for all $x \in A_0$*, where* $M = \|f_0(0) + f_1(0)\|_A$.

Proof. The proof is similar to Theorem 5.1. □

References

[1] J.P. Antoine, A. Inoue, C. Trapani, Partial $*$-Algebras and Their Operator Realizations, Kluwer, Dordrecht, 2002.

[2] F. Bagarello, A. Inoue, C. Trapani, Some classes of topological quasi $*$-algebras, Proc. Amer. Math. Soc. **129** (2001), 2973–2980.

[3] F. Bagarello, G. Morchio, Dynamics of mean-field spin models from basic results in abstract differential equations, J. Stat. Phys. **66** (1992), 849–866.

[4] F. Bagarello C.Trapani, States and representations of CQ^*-algebras, Ann. Inst. H. Poincaré **61** (1994), 103–133.

[5] F. Bagarello, C. Trapani, CQ^*-algebras: Structure properties, Publ. Res. Inst. Math. Sci. **32** (1996), 85–116.

[6] F. Bagarello, C. Trapani, Morphisms of certain Banach C^*-modules, Publ. Res. Inst. Math. Sci. **36** (2000), 681–705.

[7] P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. **184** (1994), 431–436.
[8] G.O.S. Ekhaguere, Partial W^*-dynamical systems, in: Current Topics in Operator Algebras, Proceedings of the Satellite Conference of ICM- 90, World Scientific, Singapore, 1991, pp. 202–217.
[9] R. Haag, D. Kastler, An algebraic approach to quantum field theory, J. Math. Phys. **5** (1964), 848–861.
[10] D.H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. USA **27** (1941), 222–224.
[11] A. Najati, C. Park, Hyers-Ulam-Rassias stability of homomrphisims in quasi-Banach algebras associated to the Pexiderized Cauchy functional equation, J. Math. Anal. Appl. **335** (2007), 763–778.
[12] C. Park, Homomorphisms between Lie JC^*-algebras and Cauchy-Rassias stability of Lie JC^*-algebra derivations, J. Lie Theory **15** (2005), 393–414.
[13] C. Park, Homomorphisms between Poisson JC^*-algebras, Bull. Braz. Math. Soc. **36** (2005), 79–97.
[14] C. Park, Th.M. Rassias, Homomorphisms and derivations in proper JCQ^*-triples, J. Math. Anal. Appl. **337** (2008), 1404–1414.
[15] C. Park, Proper CQ^*-ternary algebras, J. Nonlinear Sci. Appl. **6** (2013), 152–161.
[16] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[17] G.L. Sewell, Quantum Mechanics and its Emergent Macrophysics, Princeton Univ. Press, Princeton, NJ, 2002.
[18] S. Schin, D. Ki, J. Chang, M. Kim, *Random stability of quadratic functional equations: a fixed point approach*, J. Nonlinear Sci. Appl. **4** (2011), 37–49.
[19] S. Shagholi, M. Bavand Savadkouhi, M. Eshaghi Gordji, *Nearly ternary cubic homomorphism in ternary Fréchet algebras,* J. Comput. Anal. Appl. **13** (2011), 1106–1114.
[20] S. Shagholi, M. Eshaghi Gordji, M. Bavand Savadkouhi, *Stability of ternary quadratic derivation on ternary Banach algebras,* J. Comput. Anal. Appl. **13** (2011), 1097–1105.
[21] D. Shin, C. Park, Sh. Farhadabadi, *On the superstability of ternary Jordan C^*-homomorphisms,* J. Comput. Anal. Appl. **16** (2014), 964–973.
[22] D. Shin, C. Park, Sh. Farhadabadi, *Stability and superstability of J^*-homomorphisms and J^*-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.
[23] C. Trapani, Quasi-$*$-algebras of operators and their applications, Rev. Math. Phys. **7** (1995), 1303–1332.
[24] S.M. Ulam, A Collection of the Mathematical Problems, Interscience Publ., New York, 1960.

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WOVEN FRAMES IN HILBERT C^*-MODULES

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Abstract. In this paper we introduce woven frames in Hilbert C^*-modules. We then investigate some properties of woven frames and prove some results. For a given frame $\{\varphi_i\}_{i\in I}$ in a Hilbert C^*-module U, we obtain some conditions on a perturbed family $\{\psi_i\}_{i\in I}$ of U which imply that $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ are woven.

1. Introduction

1.1. **Hilbert C^*-modules.** Let $\mathcal{A}$ be a C^*-algebra. A pre-Hilbert $\mathcal{A}$-module is a complex vector space U which is also a left $\mathcal{A}$-module equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : U \times U \to \mathcal{A}$ such that

(1) $\langle x, x\rangle \geqslant 0, \quad x \in U$,
(2) $\langle x, x\rangle = 0 \Leftrightarrow x = 0, \quad x \in U$,
(3) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle, \quad \alpha, \beta \in \mathbb{C}$ and $x, y, z \in U$,
(4) $\langle ax, y\rangle = a\langle x, y\rangle, \quad x, y \in U, a \in \mathcal{A}$,
(5) $\langle x, y\rangle = \langle y, x\rangle^*, \quad x, y \in U$.

It is easy to see that scalar multiplication and the left $\mathcal{A}$-module structure of a pre-Hilbert $\mathcal{A}$-module U are compatible in the sense that

$$(\lambda a)x = a(\lambda x) = \lambda(ax), \quad \lambda \in \mathbb{C},\ a \in A,\ x \in U.$$

For $x \in U$, we set $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. It is well known that U is a normed space with $\|.\|$ (see [10]). If U is complete with norm $\|.\|$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$. We are especially interested in finitely and countably generated C^*-modules over unital C^*-algebras $\mathcal{A}$. A Hilbert $\mathcal{A}$-module U is called *countably generated* if there exists a countable set $\{x_i : i \in I\} \subseteq U$ such that U equals the closed linear span of the set $\{a_i x_i : i \in I, a_i \in \mathcal{A}\}$. A Hilbert $\mathcal{A}$-module U is *finitely generated* if there exists a finite set $\{x_1, x_2, \cdots, x_m\} \subseteq U$ such that each $x \in U$ can be expressed as an $\mathcal{A}$-linear combination of $\{x_1, x_2, \cdots, x_m\}$.

For a unital C^*-algebra $\mathcal{A}$, let $\ell^2(\mathcal{A}) = \{\{a_i\}_{i=1}^{\infty} \subseteq \mathcal{A} : \sum_{i=1}^{\infty} a_i a_i^*$ converges in norm in $\mathcal{A}\}$. The pointwise operations

$$\{a_i\}_{i=1}^{\infty} + \{b_i\}_{i=1}^{\infty} := \{a_i + b_i\}_{i=1}^{\infty}, \quad a\{a_i\}_{i=1}^{\infty} := \{aa_i\}_{i=1}^{\infty}$$

and the inner product

$$\langle \{a_i\}_{i=1}^{\infty}, \{b_i\}_{i=1}^{\infty}\rangle := \sum_{i=1}^{\infty} a_i b_i^*$$

turn $\ell^2(\mathcal{A})$ into a Hilbert $\mathcal{A}$-module. The sequence $\{e_i\}_{i=1}^{\infty}$, where e_i takes value $1_{\mathcal{A}}$ at i and $0_{\mathcal{A}}$ everywhere else, is an orthonormal basis for $\ell^2(\mathcal{A})$ and it is called the *standard orthonormal basis.*

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For Hilbert $\mathcal{A}$-modules U and V, a map $T : U \to V$ is called *bounded $\mathcal{A}$-linear* if

$$T(\lambda x + y) = \lambda T(x) + T(y), \quad T(ax) = aT(x), \quad \lambda \in \mathbb{C},\ x, y \in U,\ a \in \mathcal{A},$$

and if

$$\|T\| := \sup\Big\{\|T(x)\| :\ x \in U,\ \|x\| \leqslant 1\Big\} < \infty.$$

For Hilbert C^*-modules U and V, a map $T : U \to V$ is called *adjointable* if there is a map $T^* : V \to U$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle, \quad x \in U, y \in V.$$

We denote by $\mathbf{L}(U, V)$ the set of all adjointable maps from U to V. It is easy to show that every $T \in \mathbf{L}(U, V)$ is $\mathcal{A}$-linear and bounded. In fact $\mathbf{L}(U, V)$ is a Banach space with respect to the operator norm. Moreover, $\mathbf{L}(U, U)$ is a C^*-algebra and we will denote it by $\mathbf{L}(U)$ [10]. In spite of the fact that every (linear) bounded operator in Hilbert space is adjointable, not all bounded linear operators on Hilbert C^*-modules are adjointable, since Riesz representation theorem for continuous functionals in Hilbert spaces does not hold for general Hilbert C^*-modules (see [7]).

Proposition 1.1. [10] *Let U be a Hilbert $\mathcal{A}$-module and let T be a bounded $\mathcal{A}$-linear operator on U. Then the following conditions are equivalent:*

(1) *T is a positive element of $\mathbf{L}(U)$.*
(2) *$\langle Tx, x\rangle \geqslant 0$ for all $x \in U$.*

Proposition 1.2. *Let $\{e_i\}_{i=1}^{\infty}$ be the standard orthonormal basis of $\ell^2(\mathcal{A})$. For $\sigma \subset \mathbb{N}$, we define $P_\sigma : \ell^2(\mathcal{A}) \to \ell^2(\mathcal{A})$ by $P_\sigma(\sum_{i=1}^{\infty} a_ie_i) = \sum_{i\in\sigma} a_ie_i$. Then P_σ is an adjointable projection onto $\overline{span}\{e_i\}_{i\in\sigma}$.*

Proof. We have $P_\sigma^2 = P_\sigma$, and so P_σ is a projection. We show that P_σ is adjointable and $P_\sigma^* = P_\sigma$. For this, let $a = \sum_{i=1}^{\infty} a_ie_i, b = \sum_{i=1}^{\infty} b_ie_i \in \ell^2(\mathcal{A})$. Then

$$\begin{aligned}
\langle P_\sigma a, b\rangle &= \langle \sum_{i\in\sigma} a_ie_i, \sum_{j=1}^{\infty} b_je_j\rangle = \sum_{i\in\sigma}\langle a_ie_i, \sum_{j=1}^{\infty} b_je_j\rangle \\
&= \sum_{i\in\sigma}\langle a_ie_i, b_ie_i\rangle = \sum_{i\in\sigma}\langle\sum_{j=1}^{\infty} a_je_j, b_ie_i\rangle \\
&= \langle\sum_{j=1}^{\infty} a_je_j, \sum_{i\in\sigma} b_ie_i\rangle = \langle a, P_\sigma b\rangle.
\end{aligned}$$

This completes the proof. □

1.2. **Frames in Hilbert C^*-modules.** The concept of frame for a separable Hilbert space H was introduced by Duffin and Schaeffer [4], and it was defined as a finite or countable sequence $\{\varphi_i\}_{i\in I}$ in H such that there exist constants $A, B > 0$ satisfying

$$A\|x\|^2 \leqslant \sum_{i\in I} \|\langle x, \varphi_i\rangle\|^2 \leqslant B\|x\|^2, \quad x \in H.$$

Frames for Hilbert spaces have natural analogues for Hilbert C^*-modules. Frames for Hilbert C^*-modules were introduced by Frank and Larson [7]. Hilbert C^*-module frames are generalization of Hilbert space frames. A *frame* for a countably (or finitely) generated Hilbert C^*-module U is a sequence $\{\varphi_i\}_{i\in I}$ (I is a finite or countable subset of $\mathbb{N}$) in U for which there are constants $A, B > 0$ such that

$$A\langle x, x\rangle \leqslant \sum_{i\in I}\langle x, \varphi_i\rangle\langle\varphi_i, x\rangle \leqslant B\langle x, x\rangle, \quad x \in U. \tag{1.1}$$

We consider *standard frames* for which $\sum_{i\in I}\langle x,\varphi_i\rangle\langle\varphi_i,x\rangle$ converges in norm for every $x\in U$. The constants A, B are called the lower and upper frame bounds, respectively. If only the second inequality in (1.1) is satisfied, we say that $\{\varphi_i\}_{i\in I}$ is a *Bessel sequence* with a *Bessel bound* B. A frame $\{\varphi_i\}_{i\in I}$ is called a *tight frame* if we can choose $A=B$ and a *Parseval frame* (or *normalized tight frame*) if $A=B=1$.

Frank and Larson [7] showed that standard frames always exist in a countably (or finitely) generated Hilbert C^*-module over a unital C^*-algebra. Li [11] showed that every infinite-dimensional commutative unital C^*-algebra has a Hilbert C^*-module admitting no frames. Many related concepts of frames in Hilbert spaces such as g-frame, perturbation of frames and Riesz bases and stability of g-frames were introduced and investigated in Hilbert C^*-module spaces [5, 6, 8, 9, 14, 15].

In this paper we introduce the concept of woven frames and investigate whose relation with perturbation of frames in Hilbert C^*-modules.

Arambašić proved the following result which states that the condition (1.1) from the definition of standard frames can be replaced with a weaker one.

Theorem 1.3. [1] *Let $\mathcal{A}$ be a C^*-algebra, U a countably generated Hilbert $\mathcal{A}$-module, and $\{\varphi_i\}_{i\in I}$ a sequence in U such that $\sum_{i\in I}\langle x,\varphi_i\rangle\langle\varphi_i,x\rangle$ converges in norm for every $x\in U$. Then $\{\varphi_i\}_{i\in I}$ is a standard frame for U if and only if there are constants $C, D>0$ such that*

$$C\|x\|^2\leqslant\left\|\sum_{i\in I}\langle x,\varphi_i\rangle\langle\varphi_i,x\rangle\right\|\leqslant D\|x\|^2,\quad x\in U.$$

Let $\{\varphi_i\}_{i\in I}$ be a standard frame for a Hilbert C^*-module U. We define the *synthesis operator* $T:\ell^2(\mathcal{A})\to U$ by $T(\{a_i\}_{i\in I})=\sum_{i\in I}a_i\varphi_i$. It is well known that T is adjointable and its adjoint $T^*:U\to\ell^2(\mathcal{A})$ which is called the *analysis operator* fulfills $T^*x=\{\langle x,\varphi_i\rangle\}_{i\in I}$. By composing T and T^*, we obtain the frame operator $S:U\to U$ which is given by

$$Sx=TT^*x=\sum_{i\in I}\langle x,\varphi_i\rangle\varphi_i,\quad x\in U.$$

The frame operator is positive, invertible, and is the unique operator in $\mathbf{L}(U)$ such that the reconstruction formula

$$x=\sum_{i\in I}\langle x,S^{-1}\varphi_i\rangle\varphi_i=\sum_{i\in I}\langle x,\varphi_i\rangle S^{-1}\varphi_i,$$

holds for all $x\in U$. It is easy to see that the sequences $\{S^{-1}\varphi_i\}_{i\in I}$ and $\{S^{-\frac{1}{2}}\varphi_i\}_{i\in I}$ are frames for U. The frame $\{S^{-1}\varphi_i\}_{i\in I}$ is said to be the *canonical dual frame* of $\{\varphi_i\}_{i\in I}$ and the frame $\{S^{-\frac{1}{2}}\varphi_i\}_{i\in I}$ is said to be the *canonical Parseval frame* of $\{\varphi_i\}_{i\in I}$ [7].

Theorem 1.4. [1] *Let $\mathcal{A}$ be a C^*-algebra, U a countably generated Hilbert $\mathcal{A}$-module, $\{\varphi_i\}_{i\in I}$ a sequence in U, and $\theta x=\{\langle x,\varphi_i\rangle\}_{i\in I}$ for $x\in U$. The following statements are mutually equivalent:*

1. *$\{\varphi_i\}_{i\in I}$ is a standard frame for U.*
2. *$\theta\in\mathbf{L}(U,\ell^2(A))$ and θ is bounded below with respect to norm, i.e., there is $m>0$ such that $\|\theta x\|\geqslant m\|x\|$ for all $x\in U$.*
3. *$\theta\in\mathbf{L}(U,\ell^2(A))$ and θ is bounded below with respect to the inner product, i.e., there is $m'>0$ such that $\langle\theta x,\theta x\rangle\geqslant m'\langle x,x\rangle$ for all $x\in U$.*
4. *$\theta\in\mathbf{L}(U,\ell^2(A))$ and θ^* is surjective.*

Theorem 1.5. [1] *Let $\mathcal{A}$ be a C^*-algebra, V and W countably generated Hilbert $\mathcal{A}$-modules, and $T\in\mathbf{L}(V,W)$ surjective. If $\{\varphi_i : i\in I\}$ is a standard frame for V with frame bounds C and D, then $\{T\varphi_i : i\in I\}$ is a standard frame for W with frame bounds $\frac{C}{\|(TT^*)^{-1}\|}$ and $D\|T\|^2$.*

For the converse, we have the following.

Proposition 1.6. *Let $\{\varphi_i\}_{i\in I}$ be a frame for U and P be an adjointable operator on U such that $\{P\varphi_i\}_{i\in I}$ is a frame for U. Then P is a surjective operator.*

Proof. Let S be the frame operator of $\{\varphi_i\}_{i\in I}$. Then

$$\sum_{i\in I}\langle x, P\varphi_i\rangle P\varphi_i = P\left(\sum_{i\in I}\langle P^*x, \varphi_i\rangle\varphi_i\right) = PSP^*(x), \quad x\in U.$$

So PSP^* is the frame operator of $\{P\varphi_i\}_{i\in I}$. Since PSP^* is invertible, we infer that P is surjective. $\square$

Definition 1.7. A frame $\{\varphi_i\}_{i\in I}$ for a Hilbert $\mathcal{A}$-module U is said to be a *(standard) Riesz basis* for U if it satisfies

(*i*) $\varphi_i \neq 0$ for all $i\in I$;

(*ii*) if $\sum_{j\in J} a_j\varphi_j = 0$ for $\{a_j : j\in J\}\subset\mathcal{A}$ and $J\subset I$, then every summand a_jx_j is equal to zero.

We need the following lemmas to obtain optimal frame bounds.

Lemma 1.8. [12] *Let a, b be positive elements of a unital C^*-algebra $\mathcal{A}$. If $a\leqslant b$, then $0\leqslant b^{-1}\leqslant a^{-1}$.*

Lemma 1.9. [13] *Let T be a bounded and $\mathcal{A}$-linear map on Hilbert $\mathcal{A}$-module U. Then*

$$\|T\| = \inf\{K^{\frac{1}{2}} : \langle Tx, Tx\rangle \leqslant K\langle x, x\rangle, \quad x\in U\}.$$

In the rest of paper we suppose that $\mathcal{A}$ is a unital C^*-algebra and U is a countably or finitely generated Hilbert $\mathcal{A}$-module.

Theorem 1.10. *Let $\{\varphi_i\}_{i\in I}$ be a frame for U with lower and upper frame bounds A and B, respectively. Let S be the frame operator of $\{\varphi_i\}_{i\in I}$. Then*

(1) *$\{S^{-1}\varphi_i\}_{i\in I}$ is a frame with lower and upper frame bounds B^{-1} and A^{-1}, respectively.*

(2) *The optimal frame bounds A, B for $\{\varphi_i\}_{i\in I}$ are given by*

$$A = \|S^{-1}\|^{-1}, \quad B = \|S\|.$$

Proof. (1) Since S^{-1} is an adjointable and invertible operator, by Theorem 1.5, $\{S^{-1}\varphi_i\}_{i\in I}$ is a frame for U. We show that B^{-1} and A^{-1} are frame bounds of $\{S^{-1}\varphi_i\}_{i\in I}$. Since $AId\leqslant S\leqslant BId$, we have $B^{-1}Id\leqslant S^{-1}\leqslant A^{-1}Id$ by Lemma 1.8. This means $B^{-1}\langle x, x\rangle \leqslant \langle S^{-1}x, x\rangle \leqslant A^{-1}\langle x, x\rangle$ for all $x\in U$.

On the other hand, for every $x\in U$ we have

$$\sum_{i\in I}\langle x, S^{-1}\varphi_i\rangle\langle S^{-1}\varphi_i, x\rangle = \sum_{i\in I}\langle S^{-1}x, \varphi_i\rangle\langle\varphi_i, S^{-1}x\rangle = \langle SS^{-1}x, S^{-1}x\rangle = \langle S^{-1}x, x\rangle.$$

Therefore,

$$B^{-1}\langle x, x\rangle \leqslant \sum_{i\in I}\langle x, S^{-1}\varphi_i\rangle\langle S^{-1}\varphi_i, x\rangle \leqslant A^{-1}\langle x, x\rangle, \quad x\in U.$$

(2) It is clear that $\langle S^{\frac{1}{2}}x, S^{\frac{1}{2}}x\rangle \leqslant B\langle x, x\rangle$ for all $x\in U$. By Lemma 1.9, we have

$$\|S^{\frac{1}{2}}\| = \inf\{K^{\frac{1}{2}} : \langle S^{\frac{1}{2}}x, S^{\frac{1}{2}}\rangle \leqslant K\langle x, x\rangle \quad , x\in U\}.$$

So $\|S\| = \|S^{\frac{1}{2}}S^{\frac{1}{2}}\| = \|S^{\frac{1}{2}}\|^2$ is the optimal upper frame bound for $\{\varphi_i\}_{i\in I}$

$$\|S\| = \|S^{\frac{1}{2}}S^{\frac{1}{2}}\| = \|S^{\frac{1}{2}}\|^2 = B.$$

For the optimal lower frame bound, since the frame operator of $\{S^{-1}\varphi_i\}_{i\in I}$ is S^{-1}, we get $\|S^{-1}\|$ is the optimal upper frame bound for $\{S^{-1}\varphi_i\}_{i\in I}$. Thus $\|S^{-1}\|^{-1}$ is the optimal lower frame bound for $\{\varphi_i\}_{i\in I}$.

This completes the proof. $\square$

Proposition 1.11. *Let $\{\varphi_i\}_{i\in I}$ be a Bessel sequence in U and $S : U \to U$ be given by*

$$Sx = \sum_{i\in I} \langle x, \varphi_i\rangle \varphi_i, \quad x \in U.$$

If there exists a constant $A > 0$ such that

$$\|Sx\| \geqslant A\|x\|, \quad x \in U, \tag{1.2}$$

then $\{\varphi_i\}_{i\in I}$ is a frame for U with lower frame bound A.

Proof. We define the operator $T : \ell^2(\mathcal{A}) \to U$ by $T(\{a_i\}_{i\in I}) = \sum_{i\in I} a_i\varphi_i$. Since $\{\varphi_i\}_{i\in I}$ is a Bessel sequence in U, T is well defined, adjointable, and $S = TT^*$. By (1.2), we have

$$\|T^*x\| \geqslant \frac{A}{\|T\|}\|x\|, \quad x \in U.$$

Hence by Theorem 1.4, $\{\varphi_i\}_{i\in I}$ is a frame for U. Therefore S is invertible and $\|S^{-1}\| \leqslant A^{-1}$. Since the optimal lower frame bound for $\{\varphi_i\}_{i\in I}$ is $\|S^{-1}\|^{-1}$, we get A is a lower frame bound for $\{\varphi_i\}_{i\in I}$. □

If $\{\varphi_i\}_{i\in I}$ is a frame for U with lower bound A and frame operator S, then

$$\|Sx\| = \sup_{\|y\|=1} \|\langle Sx, y\rangle\| \geqslant \frac{1}{\|x\|}\|\langle Sx, x\rangle\| \geqslant A\|x\|, \quad x \in U \setminus \{0\}.$$

Therefore, $\|Sx\| \geqslant A\|x\|$ for all $x \in U$.

2. Woven frames

The concept of woven frames for Hilbert spaces was introduced in [2].

In this section we introduce woven frames for finitely or countably generated Hilbert C^*-modules.

We use the notation $[m] = \{1, ..., m\}$ in the rest of the paper for $m \in \mathbb{N}$.

Definition 2.1. A family $\left\{\{\varphi_{ij}\}_{i\in I}\right\}_{j\in[M]}$ of frames for U is called *woven* if there exist universal constants $0 < A < B < \infty$ such that for every partition $\{\sigma_j\}_{j\in[M]}$ of I, the family $\{\varphi_{ij}\}_{i\in\sigma_j, j\in[M]}$ is a frame for U with lower and upper frame bounds A and B, respectively. Each family $\{\varphi_{ij}\}_{i\in\sigma_j, j\in[M]}$ is called a *weaving.*

To verify that a family of frames is woven, it is enough to check that there exists a universal lower frame bound, since every weaving automatically has a universal upper frame bound.

Proposition 2.2. *If $\left\{\{\varphi_{ij}\}_{i\in I}\right\}_{j\in[M]}$ is a family of Bessel sequences with bounds B_j for $j \in [M]$, then every weaving is a Bessel sequence with Bessel bound $\sum_{j=1}^M B_j$.*

Proof. For every partition $\{\sigma_j\}_{j\in[M]}$ of I and every $x \in U$, we have

$$\sum_{j=1}^{M} \sum_{i\in\sigma_j} \langle x, \varphi_{ij}\rangle\langle\varphi_{ij}, x\rangle \leqslant \sum_{j=1}^{M} \sum_{i\in I} \langle x, \varphi_{ij}\rangle\langle\varphi_{ij}, x\rangle \leqslant \sum_{j=1}^{M} B_j\langle x, x\rangle.$$

So $\sum_{j=1}^M B_j$ is a Bessel bound for every weaving. □

By Theorem 1.5 and Proposition 1.6, we get the following.

Proposition 2.3. *Let $\left\{\{\varphi_{ij}\}_{i\in I}\right\}_{j\in[M]}$ be a woven family of frames for U and $P : U \to U$ be an adjointable operator. Then $\left\{\{P\varphi_{ij}\}_{i\in I}\right\}_{j\in[M]}$ are woven frames if and only if P is surjective.*

Corollary 2.4. *Let $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ be two woven frames. We can always assume that one of this two woven frames is Parseval.*

Proof. Let S be the frame operator of $\{\varphi_i\}_{i\in I}$. Then by Proposition 2.3, $\{S^{-\frac{1}{2}}\varphi_i\}_{i\in I}$ and $\{S^{-\frac{1}{2}}\psi_i\}_{i\in I}$ are woven frames, where $\{S^{-\frac{1}{2}}\varphi_i\}_{i\in I}$ is a Parseval frame. $\square$

In the following proposition, we assume that $\mathcal{Z}(\mathcal{A}) = \{a \in \mathcal{A} : ab = ba, \forall b \in \mathcal{A}\}$.

Proposition 2.5. *Let $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ be woven frames for Hilbert $\mathcal{A}$-module U with universal lower and upper bounds A, B, respectively. Assume that $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$ are sequences in $\mathcal{Z}(\mathcal{A})$ such that $0 < C1_{\mathcal{A}} \leqslant a_i^*a_i \leqslant D1_{\mathcal{A}}$ and $0 < C1_{\mathcal{A}} \leqslant b_i^*b_i \leqslant D1_{\mathcal{A}}$ for all $i \in I$. Then $\{a_i\varphi_i\}_{i\in I}$ and $\{b_i\psi_i\}_{i\in I}$ are woven frames for U with universal lower and upper bounds AC and BD, respectively.*

Proof. Let $\sigma \subset I$ and $x \in U$. Then

$$\begin{aligned}
&\sum_{i\in\sigma}\langle x, a_i\varphi_i\rangle\langle a_i\varphi_i, x\rangle + \sum_{i\in\sigma^c}\langle x, b_i\psi_i\rangle\langle b_i\psi_i, x\rangle \\
&= \sum_{i\in\sigma} a_i^*\langle x, \varphi_i\rangle a_i\langle \varphi_i, x\rangle + \sum_{i\in\sigma^c} b_i^*\langle x, \psi_i\rangle b_i\langle \psi_i, x\rangle \\
&= \sum_{i\in\sigma} a_i^*a_i\langle x, \varphi_i\rangle\langle \varphi_i, x\rangle + \sum_{i\in\sigma^c} b_i^*b_i\langle x, \psi_i\rangle\langle \psi_i, x\rangle.
\end{aligned}$$

The last equality implies that

$$AC\langle x, x\rangle \leqslant \sum_{i\in\sigma}\langle x, a_i\varphi_i\rangle\langle a_i\varphi_i, x\rangle + \sum_{i\in\sigma^c}\langle x, b_i\psi_i\rangle\langle b_i\psi_i, x\rangle \leqslant BD\langle x, x\rangle.$$

Thus two frames $\{a_i\varphi_i\}_{i\in I}$ and $\{b_i\psi_i\}_{i\in I}$ are woven. $\square$

In the following result the commutativity of coefficients is not necessary.

Corollary 2.6. *Let $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ be woven frames for U with universal lower and upper bounds A, B, respectively. Assume that $\{\lambda_i\}_{i\in I}$ and $\{\mu_i\}_{i\in I}$ are sequences in $\mathbb{C}$ such that $0 < C \leqslant |\lambda_i|^2 \leqslant D < \infty$ and $0 < C \leqslant |\mu_i|^2 \leqslant D < \infty$ for all $i \in I$. Then $\{\lambda_i\varphi_i\}_{i\in I}$ and $\{\mu_i\psi_i\}_{i\in I}$ are woven frames for U with universal lower and upper bounds AC and BD, respectively.*

Proposition 2.7. *Assume that $\{\varphi_i\}_{i\in J}$ and $\{\psi_i\}_{i\in J}$ are woven frames. If $J \subset I$ such that $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ are Bessel sequences, then $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ are woven.*

Proof. By Proposition 2.2, it is enough to show that there is a universal lower bound. Let $\sigma \subset I$ and $x \in U$. Then

$$A\langle x, x\rangle \leqslant \sum_{i\in\sigma\cap J}\langle x, \varphi_i\rangle\langle \varphi_i, x\rangle + \sum_{i\in\sigma^c\cap J}\langle x, \psi_i\rangle\langle \psi_i, x\rangle \leqslant \sum_{i\in\sigma}\langle x, \varphi_i\rangle\langle \varphi_i, x\rangle + \sum_{i\in\sigma^c}\langle x, \psi_i\rangle\langle \psi_i, x\rangle.$$

This completes the proof. $\square$

The next result shows that we can remove some vectors from woven frames.

Proposition 2.8. *Let $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ be woven frames for U with universal lower and upper frame bounds A and B, respectively. If $J \subset I$ and there exists $0 < D < A$ such that*

$$\sum_{i\in J}\langle x, \varphi_i\rangle\langle \varphi_i, x\rangle \leqslant D\langle x, x\rangle, \quad x \in U,$$

then $\{\varphi_i\}_{i\in(I\setminus J)}$ and $\{\psi_i\}_{i\in(I\setminus J)}$ are woven frames for U with universal lower and upper frame bounds $A - D$ and B, respectively.

Proof. Suppose that $\sigma \subset (I \setminus J)$. Then

$$\begin{aligned}\sum_{i\in\sigma}\langle x,\varphi_i\rangle\langle\varphi_i,x\rangle+\sum_{i\in(I\setminus J)\setminus\sigma}\langle x,\psi_i\rangle\langle\psi_i,x\rangle &\leqslant \sum_{i\in\sigma\cup J}\langle x,\varphi_i\rangle\langle\varphi_i,x\rangle+\sum_{i\in(I\setminus J)\setminus\sigma}\langle x,\psi_i\rangle\langle\psi_i,x\rangle\\ &\leqslant B\langle x,x\rangle,\end{aligned}$$

and

$$\begin{aligned}&\sum_{i\in\sigma}\langle x,\varphi_i\rangle\langle\varphi_i,x\rangle+\sum_{i\in(I\setminus J)\setminus\sigma}\langle x,\psi_i\rangle\langle\psi_i,x\rangle\\ &=\left(\sum_{i\in\sigma\cup J}\langle x,\varphi_i\rangle\langle\varphi_i,x\rangle-\sum_{i\in J}\langle x,\varphi_i\rangle\langle\varphi_i,x\rangle\right)+\sum_{i\in(I\setminus J)\setminus\sigma}\langle x,\psi_i\rangle\langle\psi_i,x\rangle\\ &\geqslant (A-D)\langle x,x\rangle\end{aligned}$$

for all $x \in U$. So $A - D$ and B are lower and upper weaving bounds as desired. If we set $\sigma = J^c$ (respectively $\sigma = \emptyset$), then $\{\varphi_i\}_{i\in(I\setminus J)}$ (respectively $\{\psi_i\}_{i\in(I\setminus J)}$) is a frame for U. □

Since every frame is woven with itself, we have the following.

Corollary 2.9. *Let $\{\varphi_i\}_{i\in I}$ be a frame with lower frame bound A for U. If $J \subset I$ and there exists $0 < D < A$ such that*

$$\sum_{i\in J}\langle x,\varphi_i\rangle\langle\varphi_i,x\rangle \leqslant D\langle x,x\rangle,\quad x\in U,$$

then $\{\varphi_i\}_{i\in I\setminus J}$ is a frame with lower frame bound $A - D$.

In the next example, we show that if a frame is woven with a Riesz basis, it may be not a Riesz basis. However in Hilbert spaces a Riesz basis only is woven with a Riesz basis and it cannot be woven with a frame which is not a Riesz basis [2].

Example 2.10. Let l^∞ be the set of all bounded complex-valued sequences. If the multiplication is defined pointwise and the involution is defined by conjugate, then $\mathcal{A} = l^\infty$ will be a C^*-algebra with supremum norm. Let $U = c_0$ be the set of all sequences converging to zero. For any $u = \{u_i\}_{i=1}^\infty, v = \{v_i\}_{i=1}^\infty \in U$ and $\{c_i\}_{i=1}^\infty \in \mathcal{A}$, we define

$$\langle u,v\rangle = uv^* = \{u_i\overline{v_i}\}_{i=1}^\infty,\quad \{c_i\}_{i=1}^\infty\{u_i\}_{i=1}^\infty=\{c_iu_i\}_{i=1}^\infty.$$

Then U is a Hilbert $\mathcal{A}$-module. Obviously, $\{e_i\}_{i\in\mathbb{N}}$, where e_i takes value 1 at i and 0 everywhere else, is an orthonormal basis of U. Now let $x_i = e_i$ for each i, and

$$y_i=\begin{cases}e_1+e_2 & \text{if } i=1,2\\ e_i & \text{if } i\neq 1,2.\end{cases}$$

Then $\{x_i\}_{i\in\mathbb{N}}$ is a Riesz basis that is woven with $\{y_i\}_{i\in\mathbb{N}}$, but $\{y_i\}_{i\in\mathbb{N}}$ is not a Riesz basis.

3. Perturbation and woven frames

In this section we show that under some conditions frames that are perturbation of each other are woven.

Theorem 3.1. [8] *Let U be a finitely or countably generated Hilbert $\mathcal{A}$-module over a unital C^*-algebra $\mathcal{A}$, and $\{\varphi_i\}_{i\in I}$ be a frame for U with frame bounds A and B. Suppose that $\{\psi_i\}_{i\in I}$ is a* *sequence of U and that there exist $\lambda_1, \lambda_2, \mu > 0$ such that $\max\{\lambda_1 + \frac{\mu}{\sqrt{A}}, \lambda_2\} < 1$. Then $\{\psi_i\}_{i\in I}$ is also a frame for U with frame bounds*

$$\left(\frac{(1-\lambda_1)\sqrt{A}-\mu}{1+\lambda_2}\right)^2 \quad \textit{and} \quad \left(\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2}\right)^2,$$

if one of the following conditions is fulfilled for any finite sequence $\{a_i\}_{i=1}^n \subseteq \mathcal{A}$ and all $x \in U$:

$$\begin{aligned}\left\|\sum_{i=1}^n \langle x, \varphi_i - \psi_i\rangle\langle \varphi_i - \psi_i, x\rangle\right\|^{\frac{1}{2}} &\leqslant \lambda_1 \left\|\sum_{i=1}^n \langle x, \varphi_i\rangle\langle \varphi_i, x\rangle\right\|^{\frac{1}{2}} \\ &\quad + \lambda_2 \left\|\sum_{i=1}^n \langle x, \psi_i\rangle\langle \psi_i, x\rangle\right\|^{\frac{1}{2}} + \mu\|x\|;\end{aligned}$$

or

$$\left\|\sum_{i=1}^n a_i(\varphi_i - \psi_i)\right\| \leqslant \lambda_1 \left\|\sum_{i=1}^n a_i\varphi_i\right\| + \lambda_2 \left\|\sum_{i=1}^n a_i\psi_i\right\| + \mu \left\|\sum_{i=1}^n a_i a_i^*\right\|^{\frac{1}{2}}.$$

We show that if $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ have the above perturbations, then they may be woven frames.

Theorem 3.2. *Let $\{\varphi_i\}_{i\in I}$ be a frame for U with frame bounds A, B and $\{\psi_i\}_{i\in I}$ be a sequence in U. If for all sequences $\{a_i\}_{i\in I} \in \ell^2(\mathcal{A})$*

$$\left\|\sum_{i\in I} a_i(\varphi_i - \psi_i)\right\| \leqslant \lambda_1 \left\|\sum_{i\in I} a_i\varphi_i\right\| + \lambda_2 \left\|\sum_{i\in I} a_i\psi_i\right\| + \mu \left\|\sum_{i\in I} a_i a_i^*\right\|^{\frac{1}{2}}, \tag{3.1}$$

or

$$\begin{aligned}\left\|\sum_{i\in I} \langle x, \varphi_i - \psi_i\rangle\langle \varphi_i - \psi_i, x\rangle\right\|^{\frac{1}{2}} &\leqslant \lambda_1 \left\|\sum_{i\in I} \langle x, \varphi_i\rangle\langle \varphi_i, x\rangle\right\|^{\frac{1}{2}} \\ &\quad + \lambda_2 \left\|\sum_{i\in I} \langle x, \psi_i\rangle\langle \psi_i, x\rangle\right\|^{\frac{1}{2}} + \mu\|x\|\end{aligned} \tag{3.2}$$

for some $\lambda_1, \lambda_2, \mu > 0$ such that $\max\{\lambda_1 + \frac{\mu}{\sqrt{A}}, \lambda_2\} < 1$ and

$$\left(\lambda_1\sqrt{B} + \lambda_2\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2} + \mu\right)\left(\sqrt{B} + \frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2}\right) \leqslant \alpha < A,$$

then $\{\psi_i\}_{i\in I}$ is a frame for U, and two frames $\{\psi_i\}_{i\in I}$ and $\{\varphi_i\}_{i\in I}$ are woven.

Proof. By Theorem 3.1, $\{\psi_i\}_{i\in I}$ is a frame for U with bounds

$$\left(\frac{(1-\lambda_1)\sqrt{A}-\mu}{1+\lambda_2}\right)^2 \quad \text{and} \quad \left(\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2}\right)^2.$$

Suppose that $\sigma \subset I$ and $P_\sigma : \ell^2(\mathcal{A}) \to \ell^2(\mathcal{A})$ denotes the projection described in Proposition 1.2. Let T, R be the synthesis operators for the frames $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$, respectively. We define $T_\sigma, R_\sigma : \ell^2(\mathcal{A}) \to U$ by $T_\sigma = TP_\sigma$ and $R_\sigma = RP_\sigma$. It is clear that

$$T_\sigma(\{a_i\}_{i\in I}) = TP_\sigma(\{a_i\}_{i\in I}) = \sum_{i\in\sigma} a_i\varphi_i,$$

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and

$$R_\sigma(\{a_i\}_{i\in I}) = RP_\sigma(\{a_i\}_{i\in I}) = \sum_{i\in\sigma} a_i\psi_i.$$

Since $\|P_\sigma\| \leqslant 1$, we have $\|T_\sigma - R_\sigma\| \leqslant \|T - R\|, \|T_\sigma\| \leqslant \|T\|$ and $\|R_\sigma\| \leqslant \|R\|$. Also the inequalities in (3.1) and (3.2) become

$$\|T - R\| \leqslant \lambda_1\sqrt{B} + \lambda_2\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2} + \mu,$$

and

$$\|T^* - R^*\| \leqslant \lambda_1\sqrt{B} + \lambda_2\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2} + \mu.$$

For every $x \in U$, we have

$$\begin{aligned}
&\left\|\sum_{i\in\sigma}\langle x,\varphi_i\rangle\varphi_i - \sum_{i\in\sigma}\langle x,\psi_i\rangle\psi_i\right\| \\
&= \|T_\sigma T_\sigma^* x - R_\sigma R_\sigma^* x\| \\
&\leqslant \|(T_\sigma T_\sigma^* - T_\sigma R_\sigma^*)x\| + \|(T_\sigma R_\sigma^* - R_\sigma R_\sigma^*)x\| \\
&\leqslant \|T_\sigma\|\|T_\sigma^* - R_\sigma^*\|\|x\| + \|T_\sigma - R_\sigma\|\|R_\sigma^*\|\|x\| \\
&\leqslant \|T\|\|T-R\|\|x\| + \|T-R\|\|R\|\|x\| \\
&= \|T-R\|(\|T\|+\|R\|)\|x\| \\
&\leqslant \left(\lambda_1\sqrt{B} + \lambda_2\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2} + \mu\right)(\|T\|+\|R\|)\|x\| \\
&\leqslant \left(\lambda_1\sqrt{B} + \lambda_2\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2} + \mu\right)\left(\sqrt{B} + \frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2}\right)\|x\| \\
&\leqslant \alpha\|x\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left\|\sum_{i\in\sigma}\langle x,\psi_i\rangle\psi_i + \sum_{i\in\sigma^c}\langle x,\varphi_i\rangle\varphi_i\right\| &= \left\|\sum_{i\in I}\langle x,\varphi_i\rangle\varphi_i + \left(\sum_{i\in\sigma}\langle x,\psi_i\rangle\psi_i - \sum_{i\in\sigma}\langle x,\varphi_i\rangle\varphi_i\right)\right\| \\
&\geqslant \left\|\sum_{i\in I}\langle x,\varphi_i\rangle\varphi_i\right\| - \left\|\sum_{i\in\sigma}\langle x,\psi_i\rangle\psi_i - \sum_{i\in\sigma}\langle x,\varphi_i\rangle\varphi_i\right\| \\
&\geqslant A\|x\| - \left\|\sum_{i\in\sigma}\langle x,\psi_i\rangle\psi_i - \sum_{i\in\sigma}\langle x,\varphi_i\rangle\varphi_i\right\| \\
&\geqslant A\|x\| - \alpha\|x\| = (A-\alpha)\|x\|
\end{aligned}$$

for all $x \in U$. So by Proposition 1.11, $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ are woven under the perturbations (3.1) and (3.2). ☐

Theorem 3.3. *Let $\{\varphi_i\}_{i\in I}$ and $\{\psi_{ij}\}_{i\in I}$ be frames with bounds A, B and A_j, B_j for $j = 1, ..., n-1$, respectively. Suppose that $\lambda \in (0,1)$ such that $\lambda(\sqrt{B} + \sqrt{B_j}) \leqslant \alpha$ for all $j = 1, \ldots, n-1$, where $(n-1)\alpha < A$. If*

$$\left\|\sum_{i\in I} a_i(\varphi_i - \psi_{ij})\right\| \leqslant \lambda\|\{a_i\}_{i\in I}\|, \quad j = 1, \ldots, n-1 \tag{3.3}$$

for all $\{a_i\}_{i\in I} \in \ell^2(\mathcal{A})$, *then* $\{\varphi_i\}_{i\in I}$ *and* $\{\psi_{ij}\}_{i\in I}$ *for* $j = 1, ..., n-1$ *are woven frames with bounds* $A-(n-1)\alpha$ *and* $B + \sum_{j=1}^{n-1} B_j$.

Proof. Suppose that $P_\sigma : \ell^2(\mathcal{A}) \to \ell^2(\mathcal{A})$ denotes the projection (with respect to $\sigma \subset I$) described in Proposition 1.2. Let T and T_j be the synthesis operators for the frames $\{\varphi_i\}_{i\in I}$ and $\{\psi_{ij}\}_{i\in I}$, respectively. Let $\sigma \cup \{\sigma_j\}_{j=1}^{n-1}$ be an arbitrary partition of I. We define

$$T_\sigma = TP_\sigma, \quad T_{j_{\sigma_j}} = T_j P_{\sigma_j}, \quad j = 1, \dots, n-1.$$

It follows from (3.3) that

$$\|T - T_j\| \leqslant \lambda.$$

For every $x \in U$, we have

$$\begin{aligned}\left\|\sum_{j=1}^{n-1}\left(\sum_{i\in\sigma_j}\langle x,\psi_{ij}\rangle\psi_{ij} - \sum_{i\in\sigma_j}\langle x,\varphi_i\rangle\varphi_i\right)\right\| &= \left\|\sum_{j=1}^{n-1}\left(T_{j_{\sigma_j}}T^*_{j_{\sigma_j}}(x) - T_{\sigma_j}T^*_{\sigma_j}(x)\right)\right\| \\ &\leqslant \sum_{j=1}^{n-1}\|T_{j_{\sigma_j}}T^*_{j_{\sigma_j}}(x) - T_{\sigma_j}T^*_{\sigma_j}(x)\|,\end{aligned}$$

and

$$\begin{aligned}&\sum_{j=1}^{n-1}\|T_{j_{\sigma_j}}T^*_{j_{\sigma_j}}(x) - T_{\sigma_j}T^*_{\sigma_j}(x)\| \\ &\leqslant \sum_{j=1}^{n-1}\|T_{j_{\sigma_j}}T^*_{j_{\sigma_j}}(x) - T_{j_{\sigma_j}}T^*_{\sigma_j}(x)\| + \sum_{j=1}^{n-1}\|T_{j_{\sigma_j}}T^*_{\sigma_j}(x) - T_{\sigma_j}T^*_{\sigma_j}(x)\| \\ &\leqslant \sum_{j=1}^{n-1}\|T_j\|\|T-T_j\|\|x\| + \sum_{j=1}^{n-1}\|T\|\|T-T_j\|\|x\| \\ &\leqslant \sum_{j=1}^{n-1}\lambda(\sqrt{B}+\sqrt{B_j})\|x\| \\ &\leqslant (n-1)\alpha\|x\|.\end{aligned}$$

Hence

$$\left\|\sum_{j=1}^{n-1}\left(\sum_{i\in\sigma_j}\langle x,\psi_{ij}\rangle\psi_{ij} - \sum_{i\in\sigma_j}\langle x,\varphi_i\rangle\varphi_i\right)\right\| \leqslant (n-1)\alpha\|x\|.$$

Therefore,

$$\begin{aligned}&\left\|\sum_{j=1}^{n-1}\sum_{i\in\sigma_j}\langle x,\psi_{ij}\rangle\psi_{ij} + \sum_{i\in\sigma}\langle x,\varphi_i\rangle\varphi_i\right\| \\ &= \left\|\sum_{i\in I}\langle x,\varphi_i\rangle\varphi_i + \sum_{j=1}^{n-1}\left(\sum_{i\in\sigma_j}\langle x,\psi_{ij}\rangle\psi_{ij} - \sum_{i\in\sigma_j}\langle x,\varphi_i\rangle\varphi_i\right)\right\| \\ &\geqslant \left\|\sum_{i\in I}\langle x,\varphi_i\rangle\varphi_i\right\| - \left\|\sum_{j=1}^{n-1}\left(\sum_{i\in\sigma_j}\langle x,\psi_{ij}\rangle\psi_{ij} - \sum_{i\in\sigma_j}\langle x,\varphi_i\rangle\varphi_i\right)\right\| \\ &\geqslant A\|x\| - (n-1)\alpha\|x\| = [A-(n-1)\alpha]\|x\|\end{aligned}$$

for all $x \in U$. So by Proposition 1.11, we get the result. □

Proposition 3.4. *Let $\{\varphi_i\}_{i\in I}$ be a frame for U with frame bounds A, B and $\{\psi_i\}_{i\in I}$ be a sequence in U. Let $\lambda_1, \lambda_2, \mu > 0$ such that $\max\{\lambda_1 + \frac{\mu}{\sqrt{A}}, \lambda_2\} < 1$ and*

$$\lambda_1\sqrt{B} + \lambda_2\left(\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2}\right) + \mu < \sqrt{A}.$$

If

$$\begin{aligned}\left\|\sum_{i\in I}\langle x, \varphi_i - \psi_i\rangle\langle \varphi_i - \psi_i, x\rangle\right\|^{\frac{1}{2}} &\leqslant \lambda_1\left\|\sum_{i\in I}\langle x, \varphi_i\rangle\langle \varphi_i, x\rangle\right\|^{\frac{1}{2}} \\ &\quad + \lambda_2\left\|\sum_{i\in I}\langle x, \psi_i\rangle\langle \psi_i, x\rangle\right\|^{\frac{1}{2}} + \mu\|x\|,\end{aligned} \tag{3.4}$$

or

$$\left\|\sum_{i\in I} a_i(\varphi_i - \psi_i)\right\| \leqslant \lambda_1\left\|\sum_{i\in I} a_i\varphi_i\right\| + \lambda_2\left\|\sum_{i\in I} a_i\psi_i\right\| + \mu\left\|\sum_{i\in I} a_i a_i^*\right\|^{\frac{1}{2}} \tag{3.5}$$

for all $x \in U$ and all $\{a_i\}_i \in \ell^2(\mathcal{A})$, then $\{\psi_i\}_{i\in I}$ is a frame and it is woven with $\{\varphi_i\}_{i\in I}$.

Proof. It follows from Theorem 3.2 that $\{\psi_i\}_{i\in I}$ is a frame for U. We only show that $\{\psi_i\}_{i\in I}$ is woven with $\{\varphi_i\}_{i\in I}$. Suppose that $\sigma \subset I$. If the inequality (3.4) holds, then we have

$$\begin{aligned}&\left\|\sum_{i\in\sigma}\langle x, \varphi_i\rangle\langle \varphi_i, x\rangle + \sum_{i\in\sigma^c}\langle x, \psi_i\rangle\langle \psi_i, x\rangle\right\|^{\frac{1}{2}} \\ &= \|\{\langle x, \varphi_i\rangle\}_{i\in\sigma} \cup \{\langle x, \psi_i\rangle\}_{i\in\sigma^c}\| \\ &= \|\{\langle x, \varphi_i\rangle\}_{i\in\sigma} \cup \{\langle x, \varphi_i\rangle - \langle x, \varphi_i - \psi_i\rangle\}_{i\in\sigma^c}\| \\ &= \|\{\langle x, \varphi_i\rangle\}_{i\in I} - \{\langle x, \varphi_i - \psi_i\rangle\}_{i\in\sigma^c}\| \\ &\geqslant \|\{\langle x, \varphi_i\rangle\}_{i\in I}\| - \|\{\langle x, \varphi_i - \psi_i\rangle\}_{i\in\sigma^c}\| \\ &\geqslant \sqrt{A}\|x\| - \left(\lambda_1\sqrt{B} + \lambda_2\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2} + \mu\right)\|x\| \\ &= \left[\sqrt{A} - \left(\lambda_1\sqrt{B} + \lambda_2\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2} + \mu\right)\right]\|x\|\end{aligned}$$

for all $x \in U$.

If the inequality (3.5) holds, then $T : \ell^2(\mathcal{A}) \to U$ given by $T(\{a_i\}_i) = \sum_{i\in I} a_i(\varphi_i - \psi_i)$ is well defined and bounded with $\|T\| \leqslant \lambda_1\sqrt{B} + \lambda_2\left(\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2}\right) + \mu$. It is easy to see $T^*(x) = \{\langle x, \varphi_i - \psi_i\rangle\}_{i\in I}$

for all $x \in U$. Therefore,

$$\begin{aligned}
&\left\|\sum_{i\in\sigma}\langle x,\varphi_i\rangle\langle\varphi_i,x\rangle+\sum_{i\in\sigma^c}\langle x,\psi_i\rangle\langle\psi_i,x\rangle\right\|^{\frac{1}{2}}\\
&=\|\{\langle x,\varphi_i\rangle\}_{i\in\sigma}\cup\{\langle x,\psi_i\rangle\}_{i\in\sigma^c}\|\\
&=\|\{\langle x,\varphi_i\rangle\}_{i\in\sigma}\cup\{\langle x,\varphi_i\rangle-\langle x,\varphi_i-\psi_i\rangle\}_{i\in\sigma^c}\|\\
&=\|\{\langle x,\varphi_i\rangle\}_{i\in I}-\{\langle x,\varphi_i-\psi_i\rangle\}_{i\in\sigma^c}\|\\
&\geqslant\|\{\langle x,\varphi_i\rangle\}_{i\in I}\|-\|\{\langle x,\varphi_i-\psi_i\rangle\}_{i\in\sigma^c}\|\\
&\geqslant\sqrt{A}\|x\|-\|T^*\|\|x\|\\
&=\left[\sqrt{A}-\left(\lambda_1\sqrt{B}+\lambda_2\frac{(1+\lambda_1)\sqrt{B}+\mu}{1-\lambda_2}+\mu\right)\right]\|x\|
\end{aligned}$$

for all $x \in U$. Hence $\{\varphi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ are woven by Proposition 1.11. □

Corollary 3.5. *Let $\{\varphi_i\}_{i\in I}$ be a frame for U with bounds A, B, and let $\{\psi_i\}_{i\in I}$ be a sequence in U. If there exists a constant $R < A$ such that*

$$\left\|\sum_{i\in I}\langle x,\varphi_i-\psi_i\rangle\langle\varphi_i-\psi_i,x\rangle\right\|\leqslant R\|x\|^2,\quad x\in U,$$

then $\{\psi_i\}_{i\in I}$ is a frame for U and $\{\psi_i\}_{i\in I}$ is woven with $\{\varphi_i\}_{i\in I}$.

Lemma 3.6. [3] *Let X be a Banach space and let $T : X \to X$ be a linear operator. Assume that there exist constants $\lambda_1, \lambda_2 \in [0, 1)$ such that*

$$\|Tx-x\|\leqslant\lambda_1\|x\|+\lambda_2\|Tx\|,\quad x\in X.$$

Then T is a bounded and invertible operator on X, and

$$\frac{1-\lambda_1}{1+\lambda_2}\|x\|\leqslant\|Tx\|\leqslant\frac{1+\lambda_1}{1-\lambda_2}\|x\|,\quad \frac{1-\lambda_2}{1+\lambda_1}\|x\|\leqslant\|T^{-1}x\|\leqslant\frac{1+\lambda_2}{1-\lambda_1}\|x\|,\quad x\in X.$$

Theorem 3.7. *Let $\{\varphi_i\}_{i\in I}$ be a frame for U with frame bounds A, B and let $T : U \to U$ be an adjointable operator. Assume that $\lambda_1, \lambda_2 \in [0, 1)$ such that $\sqrt{B}(\lambda_1 + \lambda_2\|T\|) < \sqrt{A}$. If $\|T - Id\| < \lambda_1 + \lambda_2\|T\|$, then $\{\varphi_i\}_{i\in I}$ and $\{T\varphi_i\}_{i\in I}$ are woven.*

Proof. By Lemma 3.6, T is invertible and we get that $\{T\varphi_i\}_{i\in I}$ is a frame for U. For every $\sigma \subset I$ and for every $x \in U$, we have

$$\begin{aligned}
\left\|\sum_{i\in\sigma}\langle x,\varphi_i\rangle\langle\varphi_i,x\rangle+\sum_{i\in\sigma^c}\langle x,T\varphi_i\rangle\langle T\varphi_i,x\rangle\right\|^{\frac{1}{2}}&=\|\{\langle x,\varphi_i\rangle\}_{i\in\sigma}\cup\{\langle x,T\varphi_i\rangle\}_{i\in\sigma^c}\|\\
&=\|\{\langle x,\varphi_i\rangle\}_{i\in\sigma}\cup\{\langle x,\varphi_i\rangle-\langle(Id-T^*)x,\varphi_i\rangle\}_{i\in\sigma^c}\|\\
&=\|\{\langle x,\varphi_i\rangle\}_{i\in I}-\{\langle(Id-T^*)x,\varphi_i\rangle\}_{i\in\sigma^c}\|\\
&\geqslant\|\{\langle x,\varphi_i\rangle\}_{i\in I}\|-\|\{\langle(Id-T^*)x,\varphi_i\rangle\}_{i\in\sigma^c}\|\\
&\geqslant\sqrt{A}\|x\|-\sqrt{B}\|(Id-T^*)x\|\\
&\geqslant(\sqrt{A}-\sqrt{B}\|Id-T^*\|)\|x\|\\
&\geqslant(\sqrt{A}-\sqrt{B}(\lambda_1+\lambda_2\|T\|))\|x\|.
\end{aligned}$$

So by Theorem 1.3, we get the result. □

Corollary 3.8. *Let $\{\varphi_i\}_{i\in I}$ be a frame for U with frame bounds A, B and let $T : U \to U$ be an adjointable operator. If $\|Id - T\|^2 \le \alpha$ with $\alpha < \frac{A}{B}$, then $\{\varphi_i\}_{i\in I}$ and $\{T\varphi_i\}_{i\in I}$ are woven.*

Remark 3.9. Corollary 3.8 can be generalized to a finite number of invertible operators, and in this case we assume that the sum over all j of $\|Id - T_j\|$ is less than $\sqrt{\frac{A}{B}}$.

Corollary 3.10. *Let $\{\varphi_i\}_{i\in I}$ be a frame with frame bounds A, B and frame operator S. If $\frac{B}{A} < 2$, then $\{\varphi_i\}_{i\in I}$ is woven with $\{\frac{2AB}{A+B}S^{-1}\varphi_i\}_{i\in I}$ and if $\frac{B}{A} \leqslant (\sqrt{2}+1)^2$, then $\{\varphi_i\}_{i\in I}$ is woven with $\{\frac{2\sqrt{AB}}{\sqrt{A}+\sqrt{B}}S^{-\frac{1}{2}}\varphi_i\}_{i\in I}$.*

References

[1] L. Arambašić, *On frames for countably generated Hilbert C^*-modules,* Proc. Am. Math. Soc. **135** (2007), 469–478.

[2] T. Bemrose, P. G. Casazza, K. Gröchenig, M. C. Lammers, R. G. Lynch, *Weaving frames,* Oper. Matrices **10** (2016), 1093–1116.

[3] P. G. Casazza, O. Christensen, *Perturbation of operators and applications to frame theory,* J. Fourier Anal. Appl. **3** (1997), 543–557.

[4] R. J. Duffin, A. C. Schaeffer, *A class of nonharmonic Fourier series,* Trans. Am. Math. Soc. **72** (1952), 341–366.

[5] Y. Feng, Y. Li, Y. Shen, S. Yang, *Construction of nonseparable symmetric tight wavelet frames of $L_2(\mathbb{R}^n)$,* J. Comput. Anal. Appl. **15** (2013), 300–307.

[6] Y. Feng, D. Yuan, S. Yang, *Construction of orthogonal shearlet t tight frames with symmetry,* J. Comput. Anal. Appl. **15** (2013), 300–307.

[7] M. Frank, D. R. Larson, *Frames in Hilbert C^*-modules and C^*-algebras,* J. Operator Theory **48** (2002), 273–314.

[8] D. Han, W. Jing, R. Mohapatra, *Perturbation of frames and Riesz bases in Hilbert C^*-modules,* Linear Algebra Appl. **431** (2009), 746–759.

[9] A. Khosravi, B. Khosravi, *g-Frames and modular Riesz bases in Hilbert C^*-modules,* Int. J. Wavelets Multiresolut. Inf. Process. **10** (2012), Art. ID 1250013, 12 pp.

[10] E. C. Lance, *Hilbert C^*-modules - a toolkit for operator algebraists,* London Mathematical Society Lecture Note Series, vol. 210, Cambridge University Press, England, 1995.

[11] H. Li, *A Hilbert C^*-module admitting no frames,* Bull. London Math. Soc. **42** (2010), 388–394.

[12] G. J. Murphy, *C^*-algebras and operator theory,* Academic Press, San Diego, 1990.

[13] W. Paschke, *Inner product modules over B^*-algebras,* Trans. Am. Math. Soc. **182** (1973), 443–468.

[14] M. Rashidi-Kouchi, A. Nazari, M. Amini, *On stability of g-frames and g-Riesz bases in Hilbert C^*-modules,* Int. J. Wavelets Multiresolut. Inf. Process. **12** (2014), Art. ID 1450036, 16 pp.

[15] Z.-Q. Xiang, *A note on the stability of g-frames in Hilbert C^*-modules,* Int. J. Wavelets Multiresolut. Inf. Process. **14** (2016), Art. ID 1650031, 9 pp.

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THE GLOBAL ATTRACTIVITY OF SOME RATIONAL DIFFERENCE EQUATIONS

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Abstract

In this paper we investigate the asymptotic behavior of the solutions and the global attractivity of the equilibrium point of the following rational difference equation which was conjectured in (**[6],** *Conjecture 5.170.1***),**

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{Cx_{n-1} + Dx_{n-2}},\ n = 0, 1, \cdots, \tag{1}$$

with positive parametres α, β, γ, C, D and with arbitrary positive initial conditions $x_{-2},\ x_{-1},\ x_0$.

Key Words : Difference equation, equilibrium point, locally asymptotically stable, global attractor.

Mathematics Subject Classification : 39A10

1 Introduction and Preliminaries

Recently it is very interesting to investigate the asymptotic behavior of solutions of a rational difference equations and there has been a lot of interest in studying the global attractivity of their equilibrium points. One of the reasons is that difference equations have been applied in several mathematical models in biology, economics, genetics, physiology, ecology, physics etc. See, for example, **[1], [2], [3], [4] , [5], [8].**

We begin by introducing some basic definitions and some theorems needed in the sequel. For details, see **[7], [9].** Firstly, we investigate the local asymptotic stability of the equilibrium of the normalized form of Eq.(1) and then we study the global attractor of the equilibrium point of this rational difference equation. Finally, some numerical examples are presented to verify our theoretical results and graphed by Mathematica.

Let I be some interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function. A difference equation of order $(k + 1)$ is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, \cdots, x_{n-k}), \qquad n = 0, 1, \cdots. \tag{2}$$

A solution of Eq.(2) is a sequence $\{x_n\}_{n=-k}^{\infty}$ that satisfies Eq.(2) for all $n \geq -k$. As a special case of Eq.(2), for every set of initial conditions $x_0, x_{-1}, x_{-2} \in I$, the third order difference equation

$$x_{n+1} = f(x_n, x_{n-1}, x_{n-2}), \qquad n = 0, 1, \cdots, \tag{3}$$

has a unique solution $\{x_n\}_{n=-2}^{\infty}$.

Definition 1 *A solution of Eq.(2) that is constant for all $n \geq -k$ is called an equilibrium solution of Eq.(2). If*

$$x_n = \overline{x}, \text{ for all } n \geq -k$$

is an equilibrium solution of Eq.(2), then $\overline{x}$ is called an equilibrium point, or simply an equilibrium of Eq.(2).

So a point $\overline{x} \in I$ is called an equilibrium point of Eq.(2) if

$$\overline{x} = f\left(\overline{x}, \overline{x}, \cdots, \overline{x}\right),$$

that is,

$$x_n = \overline{x} \quad \text{for} \quad n \geq -k$$

is a solution of Eq.(2).

Definition 2 (Stability) *Let $\overline{x}$ an equilibrium point of Eq.(2).*

(a) *An equilibrium point $\overline{x}$ of Eq.(2) is called locally stable if, for every $\varepsilon > 0$; there exists $\delta > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(2) with*

$$|x_{-k} - \overline{x}| + |x_{1-k} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \delta,$$

then

$$|x_n - \overline{x}| < \varepsilon, \quad \text{for all } n \geq -k.$$

(b) *An equilibrium point $\overline{x}$ of Eq.(2) is called locally asymptotically stable if, it is locally stable, and if in addition there exists $\gamma > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(2) with*

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \cdots + |x_0 - \overline{x}| < \gamma,$$

then we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(c) *An equilibrium point $\overline{x}$ of Eq.(2) is called a global attractor if, for every solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(2), we have*

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(d) *An equilibrium point $\overline{x}$ of Eq.(2) is called globally asymptotically stable if it is locally stable, and a global attractor.*

(e) *An equilibrium point $\overline{x}$ of Eq.(2) is called unstable if it is not locally stable.*

Now we present some important results and definitions which will be useful for our investigation.

Suppose that the function f is continuously differentiable in some open neighborhood of an equilibrium point $\overline{x}$. Let

$$q_i = \frac{\partial f}{\partial u_i}(\overline{x}, \overline{x}, \cdots, \overline{x}), \quad \text{for } i = 0, 1, \cdots, k$$

denote the partial derivative of $f(u_0, u_1, \cdots, u_k)$ with respect to u_i evaluated at the equilibrium point $\overline{x}$ of Eq.(2)

Definition 3 *The equation*

$$z_{n+1} = q_0 z_n + q_1 z_{n-1} + \cdots + q_k z_{n-k}, \; n = 0, 1, \cdots, \tag{4}$$

is called the linearized equation of Eq.(2) about the equilibrium point $\overline{x}$*, and the equation*

$$\lambda^{k+1} - q_0 \lambda^k - \cdots - q_{k-1}\lambda - q_k = 0 \tag{5}$$

is called the characteristic equation of Eq.(4) about $\overline{x}$*.*

Theorem 4 (Clark Theorem) *([6], p.6) Assume that* $q_0, q_1, ..., q_k$ *are real numbers such that*

$$|q_0| + |q_1| + \cdots + |q_k| < 1.$$

Then all roots of Eq.(5) lie inside the unit disk.

We give the following theorems without proofs. The *Linearized Stability Theorem*, is very useful in determining the local stability character of the equilibrium point $\overline{x}$ of Eq.(2)

Theorem 5 (The Linearized Stability Theorem) *([6], p.5) Assume that the function* f *is a continuously differentiable function defined on some open neighborhood of an equilibrium point* $\overline{x}$*. Then the following statements are true:*

(a) *When all the roots of Eq.(5) have absolute value less than one, then the equilibrium point* $\overline{x}$ *of Eq.(2) is locally asymptotically stable.*

(b) *If at least one root of Eq.(5) has absolute value greater than one, then the equilibrium point* $\overline{x}$ *of Eq.(2) is unstable.*

2 Local Stability Analysis

In this section we investigate the local asymptotic stability of the positive equilibrium $\overline{x}$ of the normalized form of Eq.(1).

Lemma 6

(a) *Eq.(1) can be written in the normalized form*

$$x_{n+1} = \frac{\alpha + \beta x_n + x_{n-1}}{x_{n-1} + Dx_{n-2}}, \ n = 0, 1, \cdots, \tag{6}$$

with positive parameters α, β, D *and with arbitrary positive initial conditions* x_{-2}, x_{-1}, x_0.

(b) *Positive equilibrium point of Eq.(6) is*

$$\overline{x} = \frac{(\beta+1) + \sqrt{(\beta+1)^2 + 4\alpha(1+D)}}{2(1+D)}.$$

(c) *The linearized equation of Eq.(6) about its positive equilibrium* $\overline{x}$ *is*

$$z_{n+1} - \frac{\beta}{(1+D)\overline{x}} z_n + \frac{\overline{x}-1}{(1+D)\overline{x}} z_{n-1} + \frac{D}{1+D} z_{n-2} = 0.$$

Proof.

(a) The Eq.(1) which by the change of variables

$$x_n = \frac{\gamma}{C} y_n$$

reduces to the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + x_{n-1}}{x_{n-1} + Dx_{n-2}}, \qquad n = 0, 1, \cdots,$$

where

$$\alpha := \frac{\alpha C}{\gamma^2}, \quad \beta := \frac{\beta}{\gamma}, \quad D := \frac{D}{C}.$$

(b) The positive equilibrium points of Eq.(6) are the non-negative solutions of the equation

$$\overline{x} = \frac{\alpha + \beta\overline{x} + \overline{x}}{\overline{x} + D\overline{x}}$$

or equivalently

$$(1+D)\overline{x}^2 - (1+\beta)\overline{x} - \alpha = 0. \tag{7}$$

Hence, one can easily obtain the solutions of Eq.(7) are

$$\overline{x} = \frac{1 + \beta + \sqrt{(1+\beta)^2 + 4\alpha(1+D)}}{2(1+D)} \tag{8}$$

and

$$\overline{x} = \frac{1 + \beta - \sqrt{(1+\beta)^2 + 4\alpha(1+D)}}{2(1+D)}$$

So, the positive equilibrium point of Eq.(6) is unique and is given by (8).

(c) Now, let I be some interval of real numbers and let

$$f : I \times I \times I \to I$$

be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}, x_{n-2}) = \frac{\alpha + \beta x_n + x_{n-1}}{x_{n-1} + Dx_{n-2}}.$$

Thus, we obtain that

$$\begin{aligned} q_0 &= \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}, \overline{x}) = \left[\frac{\beta.(x_{n-1} + Dx_{n-2})}{(x_{n-1} + Dx_{n-2})^2}\right](\overline{x}, \overline{x}, \overline{x}) \\ &= \frac{\beta.(\overline{x} + D\overline{x})}{\overline{x}^2(1+D)^2} = \frac{\beta\overline{x}(1+D)}{\overline{x}^2(1+D)^2} = \frac{\beta}{\overline{x}(1+D)} \end{aligned}$$

$$\begin{aligned} q_1 &= \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}, \overline{x}) = \left[\frac{(x_{n-1} + Dx_{n-2}) - (\alpha + \beta x_n + x_{n-1})}{(x_{n-1} + Dx_{n-2})^2}\right](\overline{x}, \overline{x}, \overline{x}) \\ &= \frac{\overline{x} + D\overline{x} - \overline{x}.(\overline{x} + D\overline{x})}{\overline{x}^2(1+D)^2} = \frac{(\overline{x} + D\overline{x})(1 - \overline{x})}{\overline{x}^2(1+D)^2} = \frac{1 - \overline{x}}{\overline{x}(1+D)}. \end{aligned}$$

and

$$\begin{aligned} q_2 &= \frac{\partial f}{\partial x_{n-2}}(\overline{x}, \overline{x}, \overline{x}) = \left[\frac{-(\alpha + \beta x_n + x_{n-1}).D}{(x_{n-1} + Dx_{n-2})^2}\right](\overline{x}, \overline{x}, \overline{x}) \\ &= \frac{-(\alpha + \beta\overline{x} + \overline{x}).D}{(\overline{x} + D\overline{x})^2} = \frac{-\overline{x}.(\overline{x} + D\overline{x}).D}{(\overline{x} + D\overline{x})^2} = \frac{-D}{1+D} \end{aligned}$$

If $\overline{x}$ denotes an equilibrium point of Eq.(6), then the linearized equation associated with Eq.(6) about the equilibrium point $\overline{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1} + q_2 z_{n-2}$$

or

$$z_{n+1} - \frac{\beta}{(1+D)\overline{x}} z_n + \frac{\overline{x} - 1}{(1+D)\overline{x}} z_{n-1} + \frac{D}{1+D} z_{n-2} = 0. \tag{9}$$

Lemma 7 *The positive equilibrium $\overline{x}$ of Eq.(6) is locally asymptotically stable when*

$$\frac{(1+\beta)^2 (D-1)}{4} < \alpha \quad \text{and} \quad \beta < 1.$$

Proof. From Theorem 4 it follows that all roots of the characteristic equation of Eq.(9) lie in an open disc $|\lambda| < 1$, if

$$|q_0| + |q_1| + |q_2| < 1.$$

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This implies that

$$\left|\frac{\beta}{(1+D)\overline{x}}\right|+\left|\frac{1-\overline{x}}{(1+D)\overline{x}}\right|+\left|\frac{-D}{1+D}\right|<1.$$

Hence

$$\begin{aligned}
\frac{\beta}{(1+D)\overline{x}}+\frac{|1-\overline{x}|}{(1+D)\overline{x}}+\frac{D\overline{x}}{(1+D)\,\overline{x}} &< 1 && (10)\\
\beta+D\overline{x}+|1-\overline{x}| &< (1+D)\overline{x} && (11)\\
|1-\overline{x}| &< \overline{x}+D\overline{x}-\beta-D\overline{x} && (12)\\
|1-\overline{x}| &< \overline{x}-\beta && (13)
\end{aligned}$$

and so we have two following two cases to consider.

Case I : Since

$$1-\overline{x}<\overline{x}-\beta$$

we have

$$\frac{1+\beta}{2}<\overline{x}.$$

$$\frac{1+\beta}{2}<\frac{(\beta+1)+\sqrt{(\beta+1)^2+4\alpha(1+D)}}{2(1+D)}$$

$$\frac{(1+\beta)^2\,(D-1)}{4}<\alpha.$$

Case II : Since

$$-\overline{x}+\beta<1-\overline{x}$$

it follows that

$$\beta<1.$$

3 Main Result

In this section we are concerned with the global attractor of Eq.(6). The following two theorems will be needed in the main result of this paper.

Theorem 8 *(**[10]**, p.205) Let $[a,b]$ be an interval of real numbers and assume that*

$$f:[a,b]\times[a,b]\times[a,b]\longrightarrow[a,b]$$

is a continuous function satisfying the following properties:

(a) *$f(x,y,z)$ is non-decreasing in x for each y and $z\in[a,b]$ and is non-increasing in y and z for each $x\in[a,b]$ of its arguments;*

(b) *If* $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$M = f(M, m, m) \quad \text{and} \quad m = f(m, M, M),$$

then $m = M$.

Then Eq.(3) has a unique equilibrium $\overline{x} \in [a, b]$ *and every solution of Eq.(3) converges to* $\overline{x}$.

Theorem 9 *(**[10]**, p.202) Let* $[a, b]$ *be an interval of real numbers and assume that*

$$f : [a, b] \times [a, b] \times [a, b] \longrightarrow [a, b]$$

is a continuous function satisfying the following properties:

(a) $f(x, y, z)$ *is non-decreasing in* x *and* $y \in [a, b]$ *for each* $z \in [a, b]$, *and is non-increasing in* $z \in [a, b]$ *for each* x *and* $y \in [a, b]$

(b) *If* $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$m = f(m, m, M) \quad \text{and} \quad M = f(M, M, m),$$

then $m = M$.

Then Eq.(3) has a unique equilibrium $\overline{x} \in [a, b]$ *and every solution of Eq.(3) converges to* $\overline{x}$.

We are now in a position to give the main result of this work.

Lemma 10 *The equilibrium point* $\overline{x}$ *is a global attractor of Eq.(6) if one of the following statements holds:*

$$(i)\ Dw < \alpha + \beta u \text{ and } \beta \neq 1, \tag{14}$$

$$(ii)\ Dw > \alpha + \beta u \text{ and } D > 1,\ \frac{(\beta+1)^2 (D-1)}{4} < \alpha. \tag{15}$$

Proof. *Let* α *and* β *be real numbers and assume that* $g : [\alpha, \beta]^3 \to [\alpha, \beta]$ *is a function defined by*

$$g(u, v, w) = \frac{\alpha + \beta u + v}{v + Dw}.$$

Then it follows that

$$\begin{aligned}
\frac{\partial g(u, v, w)}{\partial u} &= \frac{\beta}{v + Dw}, \\
\frac{\partial g(u, v, w)}{\partial v} &= \frac{Dw - \alpha - \beta u}{(v + Dw)^2}, \\
\frac{\partial g(u, v, w)}{\partial w} &= \frac{-(\alpha + \beta u + v)D}{(v + Dw)^2}.
\end{aligned}$$

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We consider two cases :

Case I : *If $Dw < \alpha+\beta u$ then we can easily see that the function $g(u,v,w)$ is increasing in u and decreasing in v,w.*

Suppose that (m,M) is a solution of the system $M = g(M,m,m)$ and $m = g(m,M,M)$ then from (6),we see that

$$M = \frac{\alpha+\beta M+m}{m+Dm}, \qquad m = \frac{\alpha+\beta m+M}{M+DM}.$$

Since

$$Mm + DMm - \beta M - m - \alpha = 0$$
$$Mm + DMm - \beta m - M - \alpha = 0$$

we have

$$(m-M)(\beta-1) = 0$$

When $\beta \neq 1$, we have

$$M = m$$

which the result follows.

It follows from Theorem 8 that $\overline{x}$ is global attractor of Eq.(6) and then the proof is complete.

Case II : *If $Dw > \alpha+\beta u$, then we can easily see that the function $g(u,v,w)$ is increasing in u,v and decreasing in w.*

Suppose that (m,M) is a solution of the system $M = g(M,M,m)$ and $m = g(m,m,M)$. Then from (6), we see that

$$M = \frac{\alpha+\beta M+M}{M+Dm}, \qquad m = \frac{\alpha+\beta m+m}{m+DM}.$$

Since

$$M^2 + DMm - M(\beta+1) - \alpha = 0$$
$$m^2 + DMm - m(\beta+1) - \alpha = 0$$

we have

$$(m-M)((m+M)-(\beta+1)) = 0$$

with simple calculations. Now if $m+M \neq \beta+1$, then $M = m$. On the other hand if $m+M = \beta+1$ then m and M satisfy the equation

$$m^2 + Dm(\beta+1-m) = \alpha+\beta m+m$$

and so

$$m^2(1-D) + (\beta+1)(D-1)m - \alpha = 0. \tag{16}$$

The discriminant of the Eq.(16)

$$\begin{aligned}\Delta &= [(\beta+1)(D-1)]^2 + 4(1-D)\alpha \\ &= (D-1)\left[(\beta+1)^2(D-1) - 4\alpha\right]\end{aligned}$$

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is negative when

$$D > 1 \text{ and } (\beta + 1)^2 (D - 1) < 4\alpha$$

then we have

$$M = m$$

which the result follows.

It follows from Theorem 9 that $\overline{x}$ is global attractor of Eq.(6) and then the proof is complete. ■

4 Numerical Examples

In this section we give some numerical examples to support our theoretical discussion which was mentioned in the previous section.

Example 11 *Consider the equation $x_{n+1} = \dfrac{1 + 3x_n + 2x_{n-1}}{2x_{n-1} + x_{n-2}}$ with initial conditions $x_{-2} = 0.1$, $x_{-1} = 0.2$, $x_0 = 0.4$ to verify our theoretical results. (See Fig. 1)*

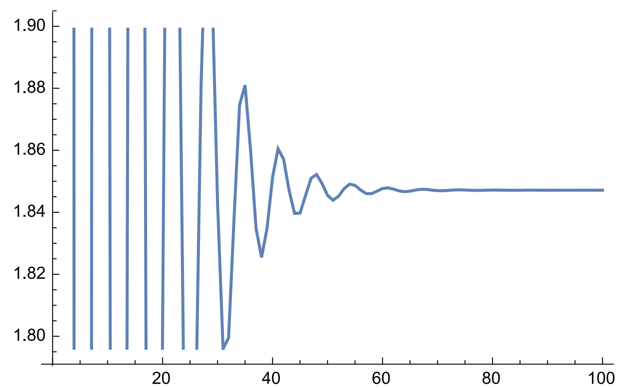

Figure 1: Plot of the difference equation $x_{n+1} = \frac{1+3x_n+2x_{n-1}}{2x_{n-1}+x_{n-2}}$.

Example 12 *Consider the equation $x_{n+1} = \dfrac{0.5 + 2x_n + x_{n-1}}{x_{n-1} + (0.1)\, x_{n-2}}$ with initial conditions $x_{-2} = 20$, $x_{-1} = 1/3$, $x_0 = 0.1$ to verify our theoretical results. (See Fig. 2)*

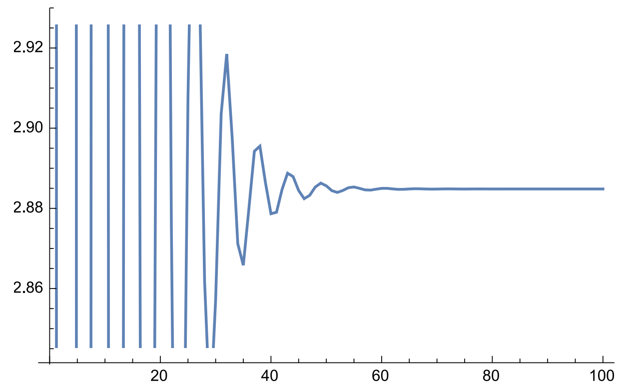

Figure 2: Plot of the difference equation $x_{n+1} = \frac{0.5+2x_n+x_{n-1}}{x_{n-1}+(0.1)x_{n-2}}$

References

[1] R. P. Agarwal. *Difference Equations and Inequalities*, 1st edition, Marcel Dekker, New York, 1992, 2nd edition, 2000.

[2] A. M. Amleh, V. Kirk, and G. Ladas. On the dynamics of $x_{n+1} = \dfrac{a+bx_{n-1}}{A+Bx_{n-2}}$, *Math. Sci. Res. Hot-Line*, **5** (2001), 1-15.

[3] A. Breet, E. J. Janowski and M. R. S. Kulenovic. Global asymptotic stability for linear fractional difference equation. *Journal of Difference Equations,* vol.2014, Article ID 275312, 11 pages, 2014.

[4] C. H. Gibbons, M. R. S. Kulenovic, and G. Ladas, On the dynamics of $x_{n+1} = \dfrac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n}$, *Proceedings of the Fifth International Conference on Difference Equations and Applications*, Temuco, Chile, Taylor & Francis, London 2002, 141-158.

[5] E. Camouzis and G. Ladas, "When does local asymptoticstability imply global attractivity in rational equations?", *Journal of Difference Equations and Applications*, vol. 12, no. 8, 2006, pp. 863–885.

[6] E. Camouzis and G. Ladas**.** *Dynamics of third order rational difference equations with open problems and conjectures, volume 5 of Advances in Discrete Mathematics and Applications.* Chapman & Hall/CRC, Boca Raton, FL, 2008.

[7] S. Elaydi. *An Introduction to Difference Equations*, Springer-Verlag, New York, Inc, 1996.

[8] G. Ladas, On the recursive sequence $x_{n+1} = \dfrac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}$, *J. Difference Equ. Appl.*, **1** (1995), 317-321.

[9] V. L. Kocić and G. Ladas. *Global behavior of nonlinear difference equations of higher order with applications*, Kluwer Academic Publishers, Dordrecht, (1993).

[10] M. R. S. Kulenovic and G. Ladas. *Dynamics of second order rational difference equations.* Chapman & Hall/CRC, Boca Raton, FL, *With Open Problems and Conjectures*, 2002.

Hybrid pair via α-admissible Geraghty F-contraction with applications

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Abstract: In this paper, we introduce a notion of hybrid pair (f, T), which is α_*-admissible with respect to η for generalized multivalued Geraghty F-contraction mappings and obtain coincidence and common fixed point results for such mappings. We provide some examples to support our results and give applications to dynamic programming and integral equations. Our results provide extension as well as substantial generalizations and improvements of several well known results in the existing comparable literature.

1 Introduction and preliminaries

Let (X, d) be a metric space. Let $CB(X)$ $(CL(X))$ be the family of all nonempty closed and bounded (nonempty closed) subsets of X. For $A, B \in CL(X)$, define

$$E_{A,B} = \{\varepsilon > 0 : A \subseteq N_\varepsilon(B), B \subseteq N_\varepsilon(A)\}.$$

The Hausdorff metric H on $CL(X)$ induced by metric d is given as:

$$H(A, B) = \begin{cases} \inf E_{A,B} & \text{if } E_{A,B} \neq \emptyset \\ \infty & \text{if } E_{A,B} = \emptyset \end{cases}.$$

Let $f : X \to X$ and $T : X \to CL(X)$. A hybrid pair $\{f, T\}$ is said to satisfy a *range inclusion condition* if $f(X) \subseteq T(X)$.

A point x in X is called a fixed point of T if $x \in Tx$. The set of all fixed points of T is denoted by $F(T)$. Furthermore, a point x in X is called a coincidence point of f and T if $fx \in Tx$. The set of all such points is denoted by $C(f, T)$. If for some point x in X, we have $x = fx \in Tx$, then a point x is called a common fixed point of f and T. We denote the set of all common fixed points of f and T by $F(f, T)$. A mapping $T : X \to CL(X)$ is said to be continuous at $p \in X$ if for any sequence $\{x_n\}$ in X with $\lim\limits_{n\to\infty} d(x_n, p) = 0$, we have $\lim\limits_{n\to\infty} H(Tx_n, Tp) = 0$.

Definition 1.1 [28] *Let $\alpha : X \times X \to [0, \infty)$. A self mapping T on X is said to be α-admissible if for any $x, y \in X$, $\alpha(x, y) \geq 1$ implies that $\alpha(Tx, Ty) \geq 1$.*

Hussain et al. [17] introduced the notion of α_*-admissible mappings as follows:

Definition 1.2 [17] *Let $\alpha, \eta : X \times X \to [0, \infty)$ be two functions where η is bounded. A mapping $T : X \to 2^X$ is called α_*-admissible with respect to η if $\alpha(x, y) \geq \eta(x, y)$ implies $\alpha_*(Tx, Ty) \geq \eta_*(Tx, Ty)$, $x, y \in X$, where $\alpha_*(A, B) = \inf\{\alpha(x, y) : x \in A,\ y \in B\}$ and $\eta_*(A, B) = \sup\{\eta(x, y) : x \in A,\ y \in B\}$.*

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If $\eta(x,y)=1$ for all $x,y\in X$, then the above definition reduces to [17, Definition 4.1].

In Definition 1.2, if $\alpha(x,y)=1$ for all $x,y\in X$, then T is called an η_*-subadmissible mapping.

We extend Definition 1.2 to hybrid pair of mappings as follows:

Definition 1.3 *Let $f:X\to X$, $T:X\to CL(X)$, and $\alpha,\eta:X\times X\to[0,\infty)$ be two functions where η is bounded. We say that the hybrid pair (f,T) is α_*-admissible with respect to η if $\alpha(fx,fy)\geq\eta(fx,fy)$ implies $\alpha_*(Tx,Ty)\geq\eta_*(Tx,Ty)$, $x,y\in X$, where*

$$\alpha_*(A,B)=\inf\{\alpha(fx,fy):fx\in A,\ fy\in B\}\ \text{and}\ \eta_*(A,B)=\sup\{\eta(fx,fy):fx\in A,\ fy\in B\}.$$

Definition 1.4 *Let $f,T:X\to X$ and $\alpha,\eta:X\times X\to[0,+\infty)$. We say that the pair (f,T) is an α-admissible mapping with respect to η if $\alpha(fx,fy)\geq(fx,fy)$ implies $\alpha(Tx,Ty)\geq\eta(Tx,Ty)$, $x,y\in X$.*

In 1973, Geraghty [14] studied most interesting generalization of Banach contraction principle.

Theorem 1.5 [14] *Let (X,d) be a metric space. Let $T:X\to X$ be a self mapping. Suppose that there exists $\beta\in\Omega$ such that for all $x,y\in X$,*

$$d(Tx,Ty)\leq\beta\left(d(x,y)\right)d(x,y),$$

then S has a fixed unique point $p\in X$ and $\{T^n x\}$ converges to p for each $x\in X$.

We denote by Ω the family of all functions $\beta:[0,+\infty)\to[0,1)$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n)\to1$ implies $t_n\to0$.

For more discussion on Geraghty contraction mappings, we refer to [22, 23] and references therein.

Berinde and Berinde [12] extended the notion of weak contraction mappings as follows:

Definition 1.6 [12, 13] *A mapping $T:X\to CL(X)$ is called a multivalued weak contraction if there exist two constants $\theta\in(0,1)$ and $L\geq0$ such that*

$$H(Tx,Ty)\leq\theta d(x,y)+Ld(y,Tx)$$

holds for all x,y in X.

The following definition of a generalized multivalued (θ,L)-strict almost contraction mapping is due to Berinde and Păcurar [13].

Definition 1.7 [13] *A mapping $T:X\to CL(X)$ is called a generalized multivalued (θ,L)-strict almost contraction mapping if there exist two constants $\theta\in(0,1)$ and $L\geq0$ such that*

$$H(Tx,Ty)\leq\theta d(x,y)+L\min\{d(y,Tx),d(x,Ty),d(x,Tx),d(y,Ty)\}$$

holds for all x,y in X.

We have the following fixed point theorem given in [13].

Theorem 1.8 *Let (X,d) be a complete metric space and $T:X\to CL(X)$ a generalized multivalued (θ,L)-strict almost contraction mapping. Then $F(T)\neq\emptyset$. Moreover, for any $p\in F(T)$, T is continuous at p.*

Kamran [21] extended the notion of a multivalued weak contraction mapping to a hybrid pair $\{f,T\}$ of single valued mapping f and multivalued mapping T. For more discussion on multivalued mappings, we refer to [4, 15] and references therein.

Definition 1.9 *Let (X,d) be a metric space and f a self map on X. A multivalued mapping $T:X\to CL(X)$ is called a generalized multivalued (f,θ,L)-weak contraction mapping if there exist two constants $\theta\in(0,1)$ and $L\geq0$ such that*

$$H(Tx,Ty)\leq\theta d(fx,fy)+Ld(fy,Tx)$$

holds for all x,y in X.

Abbas [1] extended the above definition as follows.

Definition 1.10 [1] *Let (X,d) be a metric space and f a self mapping on X. A multivalued mapping $T : X \to CL(X)$ is called a generalized multivalued (f,θ,L)-almost contraction mapping if there exist two constants $\theta \in (0,1)$ and $L \geq 0$ such that*

$$H(Tx,Ty) \leq \theta M(x,y) + LN(x,y)$$

holds for all x,y in X, where

$$\begin{aligned} M(x,y) &= \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2}\}, \\ N(x,y) &= \min\{d(fx,Tx), d(fy,Ty), d(fx,Ty), d(fy,Tx)\}. \end{aligned}$$

Let $\digamma$ be the collection of all mappings $F : \mathbb{R}_+ \to \mathbb{R}$ which satisfy the following conditions:
$(C1)$ F is strictly increasing, that is, for all $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha < \beta \Rightarrow F(\alpha) < F(\beta)$;
$(C2)$ For every sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
$(C3)$ There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.
Wardowski [32] introduced the following concept of F-contraction mappings.

Definition 1.11 [32] *Let (X,d) be a metric space. A self mapping f on X is said to be an F-contraction on X if there exists $\tau > 0$ such that*

$$d(fx,fy) > 0 \Rightarrow \tau + F(d(fx,fy)) \leq F(d(x,y))$$

for all $x,y \in X$, where $F \in \digamma$.

Remark 1.12 [32] *Every F-contraction mapping is continuous.*

Abbas et al. [2] extended the concept of F-contraction mapping and obtained common fixed point results. They employed their results to obtain fixed points of a generalized nonexpansive mappings on star shaped subsets of normed linear spaces. Recently, Minak [24] proved some fixed point results for Ćirić type generalized F-contractions on complete metric spaces.

Sgroi and Vetro [29] proved the following result to obtain fixed point of multivalued mappings as a generalization of Nadler's Theorem [25].

Theorem 1.13 [29] *Let (X,d) be a complete metric space and $T : X \to CL(X)$ a multivalued mapping. Assume that there exist $F \in \digamma$ and $\tau \in \mathbb{R}_+$ such that*

$$2\tau + F(H(Tx,Ty)) \leq F(\alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) + \delta d(x,Ty) + Ld(y,Tx))$$

for all $x,y \in X$, with $Tx \neq Ty$, where $\alpha,\beta,\gamma,\delta,L \geq 0$, $\alpha+\beta+\gamma+2\delta = 1$ and $\gamma \neq 1$. Then T has a fixed point.

Acar et al. [3] proved the following result.

Theorem 1.14 [3] *Let (X,d) be a complete metric space and $T : X \to K(X)$ (i.e., compact subsets of X). Assume that there exist an $F \in \digamma$ and $\tau \in \mathbb{R}_+$ such that for any $x,y \in X$, we have*

$$H(Tx,Ty) > 0 \Longrightarrow \tau + F(H(Tx,Ty)) \leq F(M(x,y)),$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}.$$

Then T has a fixed point if T or F is continuous,

Recently, Altun et al. [6] proved the following result.

Theorem 1.15 [6] *Let (X,d) be a complete metric space and $T : X \to CB(X)$. Assume that there exist an $F \in \digamma$ and $\tau, \lambda \in \mathbb{R}_+$ such that for any $x, y \in X$, we have*

$$H(Tx,Ty) > 0 \text{ implies that } \tau + F(H(Tx,Ty)) \le F(d(x,y) + \lambda d(y,Tx)).$$

Then the mapping T is a multivalued weakly Picard operator.

For the definition of multivalued weakly Picard operator and the related results, we refer to [12].

Definition 1.16 *Let f be a self mapping on a metric space X and $T : X \to CL(X)$ a multivalued mapping. Then T is called a generalized multivalued (f,L)-almost F-contraction mapping if there exist $F \in \digamma$ and $\tau \in \mathbb{R}_+$ and $L \ge 0$ such that*

$$2\tau + F(H(Tx,Ty)) \le F(M(x,y) + LN(x,y)) \tag{1}$$

for all x, y in X with $Tx \ne Ty$ and

$$\begin{aligned} M(x,y) &= \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2}\}, \\ N(x,y) &= \min\{d(fx,Tx), d(fy,Ty), d(fx,Ty), d(fy,Tx)\}). \end{aligned}$$

Remark 1.17 *Take $F(x) = \ln x$ in Definition 1.16. Then (1) becomes*

$$2\tau + \ln(H(Tx,Ty)) \le \ln(M(x,y) + LN(x,y),$$

that is,

$$\begin{aligned} H(Tx,Ty)) &\le e^{-2\tau}M(x,y) + e^{-2\tau}LN(x,y) \\ &= \theta_1 M(x,y) + L_1 N(x,y), \end{aligned}$$

where $\theta_1 = e^{-2\tau} \in (0,1)$ and $L_1 = e^{-2\tau}L \ge 0$. Thus we obtain the generalized multivalued (f,θ_1,L_1)-almost contraction mapping [1].

Remark 1.18 *Take $\alpha = \beta = \gamma = \frac{1}{4}$, $\delta = \frac{1}{8} = L$. Note that $\alpha+\beta+\gamma+2\delta = 1$. Then the contraction condition in Theorem 1.13 becomes*

$$\begin{aligned} 2\tau + F(H(Tx,Ty)) &\le F\left(\frac{1}{4}\left(d(x,y) + (d(x,Tx)+d(y,Ty)) + \frac{d(x,Ty)+d(y,Tx)}{2}\right)\right) \\ &\le F\left(\frac{1}{4}(4M(x,y))\right) = F((M(x,y) + 0N(x,y))) \end{aligned}$$

for all $x, y \in X$ with $Tx \ne Ty$. Thus, for $L = 0$ and $f = I_X$,

$$\begin{aligned} M(x,y) &= \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2}\}, \\ N(x,y) &= \min\{d(fx,fy), d(fx,Tx), d(fy,Ty)\}, \end{aligned}$$

and the contraction condition in Theorem 1.14 is an $(f,0)$-almost F-contraction, a special case of generalized multivalued (f,L)-almost F-contraction (for $L = 0$ and $\tau = 2\tau_1$).

Now, we give the following definition.

Definition 1.19 *Let f and T be a pair of self mappings on a metric space X. Suppose that $\alpha, \eta : X \times X \to [0,\infty)$ and $\beta : [0,+\infty) \to [0,1)$ be three functions. Then T is called a Geraghty F-contraction with respect to η if for any $x, y \in X$ with $\alpha(Tx,Ty) \ge \eta(Tx,Ty)$, $Tx \ne Ty$, we have*

$$2\tau + F(d(Tx,Ty)) \le F(\beta(M(x,y))M(x,y)),$$

where

$$M(x,y) = \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2}\},$$

for $F \in \digamma, \beta \in \Omega$ and $\tau \in \mathbb{R}_+$.

Definition 1.20 *Let f be a self mapping on a metric space X and $T : X \to CL(X)$ a multivalued mapping. Suppose that $\alpha, \eta : X \times X \to [0, \infty)$ and $\beta : [0, +\infty) \to [0, 1)$ be three functions. Then T is called a generalized multivalued Geraghty F-contraction with respect to η if for any $x, y \in X$ with $\eta_*(Tx, Ty) \leq \alpha_*(Tx, Ty)$ and $Tx \neq Ty$, we have*

$$2\tau + F(H(Tx, Ty)) \leq F(\beta (M(x, y)) M(x, y)),$$

where

$$M(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx, Ty) + d(fy, Tx)}{2}\}$$

for $F \in \mathcal{F}, \beta \in \Omega$ and $\tau \in \mathbb{R}_+$.

Definition 1.21 [20] *Let $f : X \to X$ and $T : X \to CL(X)$ a multivalued mapping. The pair (f, T) is called* (a) *commuting if $Tfx = fTx$ for all $x \in X$* (b) *weakly compatible if they commute at their coincidence points, that is, $fTx = Tfx$ whenever $x \in C(f, T)$.*

The mapping f is called T-*weakly commuting* at $x \in X$ if $f^2x \in Tfx$ If a hybrid pair (f, T) is weakly compatible at $x \in C(f, T)$, then f is T-weakly commuting at x and hence $f^n(x) \in C(f, T)$. However, the converse is not true in general. For detailed discussion on the above mentioned notions and their implications, we refer to [5, 16, 18, 19, 20, 30, 31] and references therein.

2 Main results

Throughout this section, we assume that the mapping F is right continuous.

Now we state our main result.

Theorem 2.1 *Let f be a self mapping on a metric space (X, d) and $T : X \to CL(X)$ a multivalued mapping with $\overline{T(X)} \subseteq f(X)$ satisfying the following assertions:*

(i) *the pair (f, T) is an α_*-admissible mapping with respect to η;*

(ii) *T is a generalized multivalued Geraghty F-contraction with respect to η;*

(iii) *there exists $x_0, x_1 \in X$ such that $fx_1 \in Tx_0$ and $\alpha(fx_0, fx_1) \geq \eta(fx_0, fx_1)$;*

(iv) *if $\{fx_n\}$ is a sequence in $f(X)$ such that $\alpha(fx_n, fx_{n+1}) \geq \eta(fx_n, fx_{n+1})$ for all $n \in \mathbb{N}$ and $fx_n \to fu^*$, then $\alpha(fx_n, fu^*) \geq \eta(fx_n, fu^*)$ for all $n \in \mathbb{N}$.*

If $\overline{T(X)}$ is complete, then $C(f, T) \neq \phi$ provided that F is continuous. Moreover, $F(f, T) \neq \emptyset$ if one of the following conditions holds:

(a) *for some $x \in C(f, T)$ with f is T-weakly commuting at x, $f^2x = fx$;*

(b) *$f(C(f, T))$ is a singleton subset of $C(f, T)$.*

Proof. We first note that, by Definition 1.16, $H(Tx, Ty) < \infty$ for all $x, y \in X$. Now we shall show that $C(f, T) \neq \emptyset$. Let x_0 and x_1 be given points in X such that $fx_1 \in Tx_0$ and $\alpha(fx_0, fx_1) \geq \eta(fx_0, fx_1)$. If $H(Tx_0, Tx_1) = 0$, then $Tx_0 = Tx_1$ and so $x_1 \in C(f, T)$. Assume that $H(Tx_0, Tx_1) > 0$. Since F is right continuous, there exists $h > 1$ such that

$$F(hH(Tx_0, Tx_1)) < F(H(Tx_0, Tx_1)) + \tau.$$

Then $d(fx_1, Tx_1) \leq H(Tx_0, Tx_1)$, and hence there exists $y_1 \in Tx_1$ such that

$$d(fx_1, y_1) < hH(Tx_0, Tx_1).$$

Pick an element x_2 in X such that $fx_2 = y_1$. Then the above inequality becomes

$$d(fx_1, fx_2) < hH(Tx_0, Tx_1).$$

If $fx_1 = fx_2$, then $fx_1 \in Tx_1$. In this case x_1 becomes a coincidence point of f and T and the proof is finished.

Assume that $fx_1 \neq fx_2$, that is, $d(fx_1, fx_2) > 0$. Since F is strictly increasing, we obtain

$$F(d(fx_1, fx_2)) < F(hH(Tx_0, Tx_1)) < F(H(Tx_0, Tx_1)) + \tau.$$

Since (f, T) is an α_*-admissible mapping with respect to η, $\alpha(fx_0, fx_1) \geq \eta(fx_0, fx_1)$ implies that $\alpha_*(Tx_0, Tx_1) \geq \eta_*(Tx_0, Tx_1)$. Since T is a generalized multivalued Geraghty $F-$contraction with respect to η, we have

$$\begin{aligned}
F(d(fx_1, fx_2)) &< F(H(Tx_0, Tx_1)) + \tau \\
&\leq F(\beta\,(M(x_0, x_1))\,M(x_0, x_1)) - 2\tau + \tau \\
&= F(\beta \left(\max\{d(fx_0, fx_1), d(fx_0, Tx_0), d(fx_1, Tx_1), \frac{d(fx_0, Tx_1) + d(fx_1, Tx_0)}{2}\} \right) \\
&\quad \times \max\{d(fx_0, fx_1), d(fx_0, Tx_0), d(fx_1, Tx_1), \frac{d(fx_0, Tx_1) + d(fx_1, Tx_0)}{2}\}) - \tau \\
&\leq F(\beta \left(\max\{d(fx_0, fx_1), d(fx_0, fx_1), d(fx_1, fx_2), \frac{d(fx_0, fx_2) + d(fx_1, fx_1)}{2}\} \right) \\
&\quad \times \max\{d(fx_0, fx_1), d(fx_0, fx_1), d(fx_1, fx_2), \frac{d(fx_0, fx_2) + d(fx_1, fx_1)}{2}\}) - \tau \\
&\leq F(\beta \left(\max\{d(fx_0, fx_1), d(fx_1, fx_2), \frac{d(fx_0, fx_1) + d(fx_1, fx_2)}{2}\} \right) \\
&\quad \times \max\{d(fx_0, fx_1), d(fx_1, fx_2), \frac{d(fx_0, fx_1) + d(fx_1, fx_2)}{2}\}) - \tau \\
&= F(\beta\,(\max\{d(fx_0, fx_1), d(fx_1, fx_2)\})\max\{d(fx_0, fx_1), d(fx_1, fx_2)\}) - \tau.
\end{aligned}$$

Suppose that $d(fx_1, fx_2) \nless d(fx_0, fx_1)$. Then we obtain

$$F(d(fx_2, fx_1)) < F(\beta\,(d(fx_1, fx_2))\,d(fx_1, fx_2) - \tau.$$

Since $\beta \in \Omega$, we have

$$F(d(fx_2, fx_1)) < F(d(fx_1, fx_2)) - \tau,$$

which implies $\tau \leq 0$, a contradiction. Hence $d(fx_1, fx_2) < d(fx_0, fx_1)$ and so

$$\tau + F(d(fx_2, fx_1)) \leq F(\beta\,(d(fx_0, fx_1))\,d(fx_0, fx_1)\}.$$

Note that $\alpha(fx_1, fx_2) \geq \alpha_*(Tx_0, Tx_1) \geq \eta_*(Tx_0, Tx_1) \geq \eta(fx_1, fx_2)$. That is, $\alpha(fx_1, fx_2) \geq \eta(fx_1, fx_2)$ which further implies that $\alpha_*(Tx_1, Tx_2) \geq \eta_*(Tx_1, Tx_2)$. If $H(Tx_1, Tx_2) = 0$, then $Tx_1 = Tx_2$ and so $x_2 \in C(f, T)$. Assume that $H(Tx_1, Tx_2) > 0$. Since F is right continuous, there exists $h > 1$ such that

$$F(hH(Tx_1, Tx_2)) < F(H(Tx_1, Tx_2)) + \tau.$$

Then $d(fx_2, Tx_2) \leq H(Tx_1, Tx_2)$, and hence there exists $y_2 \in Tx_2$ such that

$$d(fx_2, y_2) < hH(Tx_1, Tx_2).$$

Pick an element x_3 in X such that $fx_3 = y_2$. Then the above inequality becomes

$$d(fx_2, fx_3) < hH(Tx_1, Tx_2).$$

If $fx_2 = fx_3$, then $fx_2 \in Tx_2$. In this case x_2 becomes a coincidence point of f and T and the proof is finished.

Assume that $fx_2 \neq fx_3$, that is, $d(fx_2, fx_3) > 0$. Since F is strictly increasing, we obtain

$$F(d(fx_2, fx_3)) < F(hH(Tx_1, Tx_2)) < F(H(Tx_1, Tx_2)) + \tau.$$

Now $\alpha_*(Tx_1, Tx_2) \geq \eta_*(Tx_1, Tx_2)$ implies that

$$\begin{aligned}
F(d(fx_2, fx_3)) &< F(H(Tx_1, Tx_2)) + \tau. \\
&\leq F(\beta(M(x_1, x_2))\, M(x_1, x_2)) - 2\tau + \tau \\
&= F(\beta\left(\max\{d(fx_1, fx_2), d(fx_1, Tx_1), d(fx_2, Tx_2), \frac{d(fx_1, Tx_2) + d(fx_2, Tx_1)}{2}\}\right) \\
&\quad \times \max\{d(fx_1, fx_2), d(fx_1, Tx_1), d(fx_2, Tx_2), \frac{d(fx_1, Tx_2) + d(fx_2, Tx_1)}{2}\}) - \tau \\
&\leq F(\beta\left(\max\{d(fx_1, fx_2), d(fx_1, fx_2), d(fx_2, fx_3), \frac{d(fx_1, fx_3) + d(fx_2, fx_2)}{2}\}\right) \\
&\quad \times \max\{d(fx_1, fx_2), d(fx_1, fx_2), d(fx_2, fx_3), \frac{d(fx_1, fx_3) + d(fx_2, fx_2)}{2}\}) - \tau \\
&\leq F(\beta\left(\max\{d(fx_1, fx_2), d(fx_2, fx_3), \frac{d(fx_1, fx_2) + d(fx_2, fx_3)}{2}\}\right) \\
&\quad \times \max\{d(fx_1, fx_2), d(fx_2, fx_3), \frac{d(fx_1, fx_2) + d(fx_2, fx_3)}{2}\}) - \tau \\
&= F(\beta(\max\{d(fx_1, fx_2), d(fx_2, fx_3)\})\max\{d(fx_1, fx_2), d(fx_2, fx_3)) - \tau.
\end{aligned}$$

Thus we have

$$\tau + F(d(fx_3, fx_2)) < F(d(fx_1, fx_2)).$$

Note that $\alpha(fx_2, fx_3) \geq \alpha_*(Tx_1, Tx_2) \geq \eta_*(Tx_1, Tx_2) \geq \eta(fx_2, fx_3)$. That is, $\alpha(fx_2, fx_3) \geq \eta(fx_2, fx_3)$ which further implies that $\alpha_*(Tx_2, Tx_3) \geq \eta_*(Tx_2, Tx_3)$. By continuing this process, we obtain a sequence $\{fx_n\} \subset f(X)$ such that $fx_n \in Tx_{n-1}$,

$$\alpha(fx_{n-1}, fx_n) \geq \eta(fx_{n-1}, fx_n) \text{ implies that } \alpha_*(Tx_{n-1}, Tx_n) \geq \eta_*(Tx_{n-1}, Tx_n)$$

and we have

$$\begin{aligned}
&F(d(fx_n, fx_{n+1})) < F(H(Tx_{n-1}, Tx_n)) + \tau \\
&\leq F(\beta\left(\max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n), \frac{d(fx_{n-1}, Tx_n) + d(fx_n, Tx_{n-1})}{2}\}\right) \\
&\times \max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, Tx_{n-1}), d(fx_n, Tx_n), \frac{d(fx_{n-1}, Tx_n) + d(fx_n, Tx_{n-1})}{2}\}) - \tau \\
&\leq F(\beta\left(\max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \frac{d(fx_{n-1}, fx_{n+1}) + d(fx_n, fx_n)}{2}\}\right) \\
&\times \max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \frac{d(fx_{n-1}, fx_{n+1}) + d(fx_n, fx_n)}{2}\}) - \tau \\
&\leq F(\beta\left(\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \frac{d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1})}{2}\}\right) \\
&\times \max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \frac{d(fx_{n-1}, fx_n) + d(fx_n, fx_{n+1})}{2}\}) - \tau \\
&= F(\beta(\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\})\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\}) - \tau
\end{aligned}$$

and

$$\tau + F(d(fx_n, fx_{n+1})) \leq F(\beta(\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\})\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\})$$

for all $n \in \mathbb{N}$. Since F is strictly increasing, we have

$$d(fx_n, fx_{n+1}) < \beta(\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\})\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1})\}.$$

If

$$\max\{d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}) = d(fx_n, fx_{n+1})$$

for some n, then

$$d(fx_n, fx_{n+1}) < d(fx_n, fx_{n+1}),$$

which is a contradiction since $\beta \in \Omega$ and hence we have

$$d(fx_n, fx_{n+1}) < d(fx_{n-1}, fx_n).$$

Consequently,

$$\tau + F(d(fx_n, fx_{n+1})) \leq F(d(fx_{n-1}, fx_n))$$

for all $n \in \mathbb{N}$. Thus we obtain that

$$\begin{aligned} F(d(fx_n, fx_{n+1})) &\leq F(d(fx_{n-1}, fx_n)) - \tau \\ &\leq F(d(fx_{n-2}, fx_{n-1})) - 2\tau \\ &\;\;\vdots \\ &\leq F(d(fx_0, fx_1)) - n\tau. \end{aligned}$$

Taking limit as $n \to \infty$, we have $\lim_{n\to\infty} F(d(fx_n, fx_{n+1})) = -\infty$. By (C2), $\lim_{n\to\infty} d(fx_n, fx_{n+1}) = 0$. By (C3), there exists an $r \in (0, 1)$ such that

$$\lim_{n\to\infty} \{d(fx_n, fx_{n+1})\}^r F(d(fx_n, fx_{n+1})) = -\infty.$$

Hence it follows that

$$\begin{aligned} & \{d(fx_n, fx_{n+1})\}^r F(d(fx_n, fx_{n+1})) - \{d(fx_n, fx_{n+1})\}^r F(d(fx_0, fx_1)) \\ \leq\; & d(fx_n, fx_{n+1})^r [F(d(fx_0, fx_1) - n\tau)] - d(x_n, x_{n+1})^r F(d(fx_0, fx_1)) \\ =\; & -n\tau [d(fx_n, fx_{n+1})]^r \leq 0. \end{aligned}$$

Taking limit as n tends to ∞, we obtain that $\lim_{n\to\infty} n \{d(fx_n, fx_{n+1})\}^r = 0$, i.e., $\lim_{n\to\infty} n^{1/r} d(fx_n, fx_{n+1}) = 0$. This implies that $\sum_{n=1}^{\infty} d(fx_n, fx_{n+1})$ is convergent and hence the sequence $\{fx_n\}_{n\geq 1}$ is a Cauchy sequence in $T(X) \subseteq \overline{T(X)}$. Since $\overline{T(X)}$ is complete, there exists $p \in \overline{T(X)}$ such that $\lim_{n\to\infty} fx_n = p$. Now $\overline{T(X)} \subseteq f(X)$ implies that there exists u^* in X such that $fu^* = p$. Since $\{fx_n\}$ is a sequence in $f(X)$ such that $\alpha(fx_n, fx_{n+1}) \geq \eta(fx_n, fx_{n+1})$ for all $n \in \mathbb{N}$ and $fx_n \to fu^*$, $\alpha(fx_n, fu^*) \geq \eta(fx_n, fu^*)$ for all $n \in \mathbb{N}$. Since the hybrid pair (f, T) is α_*-admissible with respect to η, we have $\alpha_*(Tx_n, Tu^*) \geq \eta_*(Tx_n, Tu^*)$, which implies that

$$\begin{aligned} 2\tau + F(d(fx_{n+1}, Tu^*)) &\leq 2\tau + F(H(Tx_n, Tu^*)) \\ &\leq F(\beta(M(x_n, u^*)) M(x_n, u^*) \end{aligned}$$

for all $n \in \mathbb{N}$ by the contractive condition.

Next suppose that F is continuous. Then

$$\lim_{n\to\infty} d(fx_n, Tu^*) = d(fu^*, Tu^*). \tag{2}$$

Note that

$$\begin{aligned} d(fu^*, Tu^*) &\leq M(x_n, u^*) = \max\left\{d(fx_n, fu^*), d(fx_n, Tx_n), d(fu^*, Tu^*), \frac{d(fx_n, Tu^*) + d(fu^*, Tx_n)}{2}\right\} \\ &\leq \max\left\{d(fx_n, fu^*), d(fx_n, fx_{n+1}), d(fu^*, Tu^*), \frac{d(fx_n, Tu^*) + d(fu^*, fx_{n+1})}{2}\right\}. \end{aligned}$$

Taking limit as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} M(x_n, u^*) = d(fu^*, Tu^*). \tag{3}$$

Since $\beta \in \Omega$, it follows from (2), (3) and the continuity of F that

$$2\tau + F(d(fu^*, Tu^*)) \leq F(d(fu^*, Tu^*)).$$

that is, $d(fu^*, Tu^*) = 0$ and thus $fu^* \in Tu^*$.

Now let (a) hold, that is, for $x \in C(f,T)$, f is T-weakly commuting at x. So we get $f^2x \in Tfx$. By the given hypothesis, $fx = f^2x$ and hence $fx = f^2x \in Tfx$. Consequently, $fx \in F(f,T)$.

Let (b) hold. Since $f(C(f,T)) = \{x\}$ and $x \in C(f,T)$, $x = fx \in Tx$. Thus $F(f,T) \neq \emptyset$. ∎

Example 2.2 *Let $X = [1,\infty)$ be the usual metric space. Define $f : X \to X$, $\alpha, \eta : X \times X \to [0,\infty)$, $\beta : [0,+\infty) \to [0,1)$ and $T : X \to CL(X)$ by $fx = x^2$, $Tx = [x+2,\infty)$ for all $x \in X$, $\alpha(x,y) = \begin{cases} e^{xy} & \text{if } x,y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$, $\eta(x,y) = \begin{cases} e^{x-y} & \text{if } x \geq y \\ 0 & \text{otherwise.} \end{cases}$, $\beta(t) = \frac{1}{t+1}$, $\forall t > 0$, where $\tau = \ln\sqrt{2}$, and $F(t) = \ln(t)$ for all $t > 0$. Note that $\overline{T(X)} = T(X) = [3,\infty)$ and $\alpha(fx,fy) \geq \eta(fx,fy)$ imply that $\alpha_*(Tx,Ty) \geq \eta_*(Tx,Ty)$ for all $x,y \in X$ with $Tx \neq Ty$ (equivalently, $x \neq y$) and we have*

$$2\tau + F(H(Tx,Ty)) \leq F\left(\beta\left(M(x,y)\right) M(x,y)\right).$$

Thus all the conditions of Theorem 2.1 are satisfied.

If we take $\eta(x,y) = 1$ in Theorem 2.1, then we have the following result.

Theorem 2.3 *Let f be a self mapping on a metric space (X,d) and $T : X \to CL(X)$ a multivalued mapping with $\overline{T(X)} \subseteq f(X)$ satisfying the following assertions:*

(i) *the pair (f,T) is an α_*-admissible mapping;*

(ii) *T is a generalized multivalued Geraghty F-contraction;*

(iii) *there exists $x_0, x_1 \in X$ such that $fx_1 \in Tx_0$ and $\alpha(fx_0, fx_1) \geq 1$;*

(iv) *if $\{fx_n\}$ is a sequence in $f(X)$ such that $\alpha(fx_n, fx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $fx_n \to fu^*$, then $\alpha(fx_n, fu^*) \geq 1$ for all $n \in \mathbb{N}$.*

If $\overline{T(X)}$ is complete, then $C(f,T) \neq \phi$ provided that F is continuous. Moreover, $F(f,T) \neq \emptyset$ if one of the following conditions holds:

(a) *for some $x \in C(f,T)$ with f is T-weakly commuting at x, $f^2x = fx$;*

(b) *$f(C(f,T))$ is a singleton subset of $C(f,T)$.*

Corollary 2.4 *Let f and T be a self mapping on a metric space (X,d) such that $T(X) = f(X)$. Suppose that*

(i) *the pair (f,T) is an α-admissible mapping with respect to η;*

(ii) *T is a Geraghty F-contraction with respect to η;*

(iii) *there exists $x_0, x_1 \in X$ such that $fx_1 = Tx_0$ and $\alpha(fx_0, fx_1) \geq \eta(fx_0, fx_1)$;*

(iv) *if $\{fx_n\}$ is a sequence in $f(X)$ such that $\alpha(fx_n, fx_{n+1}) \geq \eta(fx_n, fx_{n+1})$ for all $n \in \mathbb{N}$ and $fx_n \to fu^*$, then $\alpha(fx_n, fu^*) \geq \eta(fx_n, fu^*)$ for all $n \in \mathbb{N}$.*

If $T(X)$ is complete, then $C(f,T) \neq \phi$ provided that F is continuous. Moreover, $F(f,T) \neq \emptyset$ if f and T commute at their coincidence point.

Proof. If we take $X = CL(X)$ in Theorem 2.1, then we obtain the required result. ∎

3 Applications

In this section, we obtain the existence and uniqueness of common solution of: (I) system of functional equations arising in dynamical programing problems and (II) system of integral equations.

(I) Application in dynamic programming

Decision space and a state space are two basic components of dynamic programming problems. State space is a set of states including initial states, action states and transitional states. So a state space is set of parameters representing different states. A decision space is the set of possible actions that can be taken to solve the problem. These general settings allow us to formulate many problems in mathematical optimization and computer programming. In particular, the problem of dynamic programming related to multistage process reduces to the problem of solving functional equations

$$p(x) = \sup_{y\in D}\{g(x,y)+G_1(x,y,p(\xi(x,y)))\}, \text{ for } x\in W, \tag{4}$$

$$q(x) = \sup_{y\in D}\{g'(x,y)+G_2(x,y,q(\xi(x,y)))\}, \text{ for } x\in W, \tag{5}$$

where U and V are Banach spaces, $W\subseteq U$ and $D\subseteq V$ and

$$\begin{aligned} \xi &: W\times D\longrightarrow W \\ g,g' &: W\times D\longrightarrow \mathbb{R} \\ G_1,G_2 &: W\times D\times\mathbb{R}\longrightarrow\mathbb{R} \\ \beta &: \mathbb{R}^+\to[0,1)\,. \end{aligned}$$

For more discussions and results on dynamic programming problems, we refer to [8, 9, 10, 11, 27] and references mentioned therein. Suppose that W and D are the state and decision spaces, respectively. We aim to give the existence and uniqueness of common and bounded solution of functional equations given in (4) and (5). Let $B(W)$ denote the set of all bounded real valued functions on W. For an arbitrary $h\in B(W)$, define $\|h\|=\sup_{x\in W}|h(x)|$. Then $(B(W),\|\cdot\|)$ is a Banach space endowed with the metric d given by

$$d(h,k)=\sup_{x\in W}|h(x)-k(x)|\,.$$

Suppose that the following conditions hold:

$(C1)$ G_1,G_2,g and g' are bounded.

$(C2)$ For $x\in W$, $h\in B(W)$ and $b>0$, define

$$Kh(x)=\sup\nolimits_{y\in D}\{g(x,y)+G_1(x,y,h(\xi(x,y)))\}, \tag{6}$$

$$Jh(x)=\sup\nolimits_{y\in D}\{g'(x,y)+G_2(x,y,h(\xi(x,y)))\}. \tag{7}$$

Moreover, assume that there exist $\tau>0$ and $L\geq 0$ such that for every $(x,y)\in W\times D$, $h,k\in B(W)$ and $t\in W$ imply

$$|G_1(x,y,h(t))-G_1(x,y,k(t))|\leq e^{-2\tau}[\beta\left(M(h,k)\right)M(h,k)], \tag{8}$$

where

$$M((h,k))=\max\{d(Jh,Jk),d(Jk,Kk),d(Jh,Kh),\frac{d(Jh,Kk)+d(Jk,Kh)}{2}\}.$$

$(C3)$ For any $h\in B(W)$, there exists $k\in B(W)$ such that $Kh(x)=Jk(x)$ for $x\in W$.

$(C4)$ There exists $h\in B(W)$ such that $Kh(x)=Jh(x)$ implies that $JKh(x)=KJh(x)$.

$(C5)$ There exist $\alpha,\eta:B(W)\times B(W)\to\mathbb{R}^+$ such that $\alpha(Jh_1,Jh_2)\geq\eta(Jh_1,Jh_2)$.

Theorem 3.1 *Assume that the conditions* $(C1)-(C5)$ *are satisfied. If* $J(B(W))$ *is a closed convex subspace of* $B(W)$*, then the functional equations* (4) *and* (5) *have a unique, common and bounded solution.*

Proof. Note that $(B(W),d)$ is a complete metric space. By $(C1)$, J,K are self-mappings of $B(W)$ and each element is mapped into a singleton set. The condition $(C3)$ implies that $K(B(W))\subseteq J(B(W))$ is satisfied. It follows from $(C4)$ that J and K commute at their coincidence points. Let λ be an arbitrary positive number and $h_1,h_2\in B(W)$. Choose $x\in W$ and $y_1,y_2\in D$ such that

$$Kh_j<g(x,y_j)+G_1(x,y_j,h_j(x_j)+\lambda, \tag{9}$$

where $x_j = \xi(x, y_j)$, $j = 1, 2$. Further, from (6) and (7), we have

$$\begin{aligned} Kh_1 &\geq g(x, y_2) + G_1(x, y_2, h_1(x_2)), \\ Kh_2 &\geq g(x, y_1) + G_1(x, y_1, h_2(x_1)). \end{aligned} \tag{10}$$

Then (9) and (10) together with (8) imply that

$$\begin{aligned} Kh_1(x) - Kh_2(x) &< G_1(x, y_1, h_1(x_1)) - G_1(x, y_1, h_2(x_2)) + \lambda \\ &\leq |G_1(x, y_1, h_1(x_1)) - G_1(x, y_1, h_2(x_2))| + \lambda \\ &\leq e^{-2\tau}(\beta\,(M(h,k))\,M(h,k)) + \lambda. \end{aligned} \tag{11}$$

Then (8) and (9) together with (7) imply that

$$\begin{aligned} Kh_2(x) - Kh_1(x) &\leq G_1(x, y_1, h_2(x_2)) - G_1(x, y_1, h_1(x_1)) \\ &\leq |G_1(x, y_1, h_1(x_1)) - G_1(x, y_1, h_2(x_2))| \\ &\leq e^{-2\tau}(\beta\,(M(h,k))\,M(h,k)). \end{aligned} \tag{12}$$

From (11) and (12), we have

$$|Kh_1(x) - Kh_2(x)| \leq e^{-2\tau}(\beta\,(M(h,k))\,M(h,k)). \tag{13}$$

The inequality (13) implies

$$d(Kh_1, Kh_2) \leq e^{-2\tau}[(\beta\,(M(h,k))\,M(h,k))].$$

Since $\alpha(Jh_1(x), Jh_2(x)) \geq \eta(Jh_1(x), Jh_2(x))$ implies that $\alpha(Kh_1(x), Kh_2(x)) \geq \eta(Kh_1(x), Kh_2(x))$ and so we have

$$2\tau + \ln[d(Kh_1(x) - Kh_2(x))] \leq \ln[(\beta\,(M(h(t), k(t)))\,M(h(t), k(t)))].$$

Therefore, by Corollary 2.4, the pair (K, J) has a common fixed point h^*, that is, $h^*(x)$ is a unique, bounded and common solution of (4) and (5). ■

(II) Application of integral equations

Now we discuss an application of fixed point theorem, proved in the previous section, to solve the system of Volterra type integral equations. Such system is given by the following equations:

$$u(t) = \int_0^t K_1(t, s, u(s))ds + g(t), \tag{14}$$

$$w(t) = \int_0^t K_2(t, s, w(s))ds + f(t). \tag{15}$$

for $t \in [0, a]$, where $a > 0$. We find the solution of the system (14) and (15). Let $C([0, a], \mathbb{R})$ be the space of all continuous functions defined on $[0, a]$. For $u \in C([0, a], \mathbb{R})$, define supremum norm as: $\|u\|_\tau = \sup_{t\in[0,a]} \{u(t)e^{-\tau(t)t}\}$, where $\tau > 0$. Let $C([0, a], \mathbb{R})$ be endowed with the metric

$$d_\tau(u, v) = \sup_{t\in[0,a]} \|\,|u(t) - v(t)|\,e^{-\tau(t)t}\|_\tau$$

for all $u, v \in C([0, a], \mathbb{R})$. With these setting $C([0, a], \mathbb{R}, \|\cdot\|_\tau)$ becomes a Banach space.

Now we prove the following theorem to ensure the existence of solution of system of integral equations. For more details on such applications, we refer the reader to [7, 26].

Theorem 3.2 *Assume the following conditions are satisfied:*

(i) $K_1, K_2 : [0,a] \times [0,a] \times \mathbb{R} \to \mathbb{R}$ *and* $f, g : [0,a] \to \mathbb{R}$ *are continuous;*

(ii) *Define*

$$\begin{aligned} Tu(t) &= \int_0^t K_1(t,s,u(s))ds + g(t), \\ Su(t) &= \int_0^t K_2(t,s,u(s))ds + f(t). \end{aligned}$$

Suppose $\alpha, \eta : [0,a] \times [0,a] \to \mathbb{R}^+, \beta : [0,a] \to [0,1)$. *Then there exist* $\tau \geq 1$ *and* $L \geq 0$ *such that*

$$|K_1(t,s,u) - K_2(t,s,v)| \leq \tau e^{-2\tau}[\beta\left(M(u,v)\right) M(u,v)]$$

for all $t, s \in [0,a]$ *and* $u, v \in C([0,a], \mathbb{R})$, *where*

$$M(u,v) = \max\{|Su(t) - Sv(t)|, |Sv(t) - Tv(t)|, |Su(t) - Tu(t)|, \frac{|Su(t) - Tv(t)| + |Sv(t) - Tu(t)|}{2}\};$$

(iii) *There exists* $u \in C([0,a], \mathbb{R})$ *such that* $Tu(t) = Su(t)$ *implies* $TSu(t) = STu(t)$.

Then the system of the integral equations (14) *and* (15) *has a solution.*

Proof. Define the mapping $\alpha, \eta : [0,a] \times [0,a] \to \mathbb{R}^+$ by

$$\alpha(u,v) = \begin{cases} e^{uv} \text{ if } u,v \in [0,a] \\ 0, \quad \text{otherwise} \end{cases} \text{ and } \eta(u,v) = \begin{cases} e^{u-v} \text{ if } u,v \in [0,a] \\ 0, \quad \text{otherwise.} \end{cases}$$

Then $\alpha(u,v) \geq \eta(u,v)$ and so $\alpha(Su,Sv) \geq \eta(Su,Sv)$. By the assumption (iii),

$$\begin{aligned} |Tu(t) - Tv(t)| &= \int_0^t |K_1(t,s,u(s) - K_2(t,s,v(s)))|\,ds \\ &\leq \int_0^t \tau e^{-2\tau}([\beta\left(M(u,v)\right) M(u,v)]e^{-\tau s})e^{\tau s}ds \\ &\leq \int_0^t \tau e^{-2\tau}\|\beta\left(M(u,v)\right) M(u,v)\|_\tau e^{\tau s}ds \\ &\leq \tau e^{-2\tau}\|\beta\left(M(u,v)\right) M(u,v)\|_\tau \int_0^t e^{\tau s}ds \\ &\leq \tau e^{-2\tau}\|\beta\left(M(u,v)\right) M(u,v)\|_\tau \frac{1}{\tau}e^{\tau t} \\ &\leq e^{-2\tau}\|\beta\left(M(u,v)\right) M(u,v)\|_\tau e^{\tau t}, \end{aligned}$$

which implies

$$|Tu(t) - Tv(t)|\, e^{-\tau t} \leq e^{-2\tau}\|\beta\left(M(u,v)\right) M(u,v)\|_\tau.$$

That is,

$$\|Tu(t) - Tv(t)\|_\tau \leq e^{-2\tau}\|\beta\left(M(u,v)\right) M(u,v)\|_\tau$$

and

$$\alpha(Su,Sv) \geq \eta(Su,Sv) \text{ implies } \alpha(Tu,Tv) \geq \eta(Tu,Tv)$$

and so we have

$$2\tau + \ln\|Tu(t) - Tv(t)\|_\tau \leq \ln\|\beta\left(M(u,v)\right) M(u,v)\|_\tau.$$

So all the conditions of Corollary 2.4 are satisfied. Hence the system of integral equations (14) and (15) has a unique common solution. ■

References

[1] M. Abbas, Coincidence points of multivalued f-almost nonexpansive mappings, Fixed Point Theory **13** (2012), 3–10.

[2] M. Abbas, B. Ali, S. Romaguera, Fixed and periodic points of generalized contractions in metric spaces, Fixed Point Theory Appl. **2013**, 2013:243.

[3] O. Acar, G. Durmaz, G. Minak, Generalized multivalued F-contraction on complete metric spaces, Bull. Iranian Math. Soc. **40** (2014), 1469–1478.

[4] J. Ahmad, N. Hussain, A. R. Khan, A. Azam, Fixed point results for generalized multi-valued contractions, J. Nonlinear Sci. Appl. **8** (2015), 909–918.

[5] M. A. Al-Thagafi, N. Shahzad, Coincidence points, generalized I-nonexpansive multimaps and applications, Nonlinear Anal. **67** (2007), 2180–2188.

[6] I. Altun, G. Mınak, H. Dag, Multivalued F-contraction on complete metric spaces, J. Nonlinear Convex Anal. (in press).

[7] A. Augustynowicz, Existence and uniqueness of solutions for partial differential-functional equations of the first order with deviating argument of the derivative of unknown function, Serdica Math. J. **23** (1997), 203–210.

[8] R. Baskaran, P.V. Subrahmanyam, A note on the solution of a class of functional equations. Appl. Anal. **22** (1986), 235–241.

[9] R. Bellman, Methods of Nonlinear Analysis. Vol. II. Mathematics in Science and Engineering, vol. 61. Academic Press, New York 1973.

[10] R. Bellman, E.S. Lee, Functional equations in dynamic programming. Aequationes. Math. **17** (1978), 1–18.

[11] P.C. Bhakta, S. Mitra, Some existence theorems for functional equations arising in dynamic programming. J. Math. Anal. Appl. **98** (1984), 348–362 .

[12] M. Berinde, V. Berinde, On general class of multivalued weakly Picard mappings, J. Math. Anal. Appl. **326** (2007), 772–782.

[13] V. Berinde, M. Păcurar, Fixed points and continuity of almost contractions, Fixed Point Theory **9** (2008), 23–34.

[14] M. Geraghty, On contractive mappings, Proc. Am. Math. Soc. **40** (1973), 604–608.

[15] N. Hussain, J. Ahmad, L. Ćirić, A. Azam, Coincidence point theorems for generalized contractions with application to integral equations, Fixed Point Theory Appl. **2015**, 2015:78.

[16] N. Hussain, G. Jungck, Common fixed point and invariant approximation results for noncommuting generalized (f, g)-nonexpansive maps, J. Math. Anal. Appl. **321** (2006), 851–861.

[17] N. Hussain, P Salimi, A. Latif, Fixed point results for single and set-valued α-η-ψ-contractive mappings, Fixed Point Theory Appl. **2013**, 2013:212.

[18] G. Jungck, Commuting mappings and fixed points, Am. Math. Monthly **83** (1976), 261–263.

[19] G. Jungck, Common fixed points for commuting and compatible maps on compacta, Proc. Am. Math. Soc. **103** (1988), 977–983.

[20] G. Jungck, B. E. Rhoades, Fixed points for set valued functions without continuity, Indian J. Pure Appl. Math. **29** (1998), 227–238.

[21] T. Kamran, Multivalued f-weakly Picard mappings, Nonlinear Anal. **67** (2007), 2289–2296.

[22] E. Karapinar, α-ψ-Geraghty contraction type mappings and some related fixed point results, Filomat **28** (2014), 37–48.

[23] E. Karapinar, A discussion on "α-ψ-Geraghty contraction type mappings, Filomat **28** (2014), 761–766.

[24] G. Minak, A. Halvaci, I. Altun, Ćirić type generalized F-contractions on complete metric spaces and fixed point results, Filomat **28** (2014), 1143–1151.

[25] S. B. Nadler, Multivalued contraction mappings, Pacific J. Math. **30** (1969), 475–488.

[26] D. O'Regan, A. Petruşel, Fixed point theorems for generalized contractions in ordered metric spaces, J. Math. Anal. Appl. **341** (2008), 1241–1252.

[27] H. K. Pathak, Y. Cho, S. Kang, B. Lee, Fixed point theorems for compatible mappings of type P and applications to dynamic programming. Matematiche **50** (1995), 15–33.

[28] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, Nonlinear Anal. **75** (2012), 2154–2165.

[29] M. Sgroi, C. Vetro, Multi-valued F-contractions and the solution of certain functional and integral equations, Filomat **27** (2013), 1259–1268.

[30] A. Shoaib, α-η dominated mappings and related common fixed point results in closed ball, J. Concrete Appl. Math. **13** (2015), 152–170.

[31] A. Shoaib, M. Arshad, M. A. Kutbi, Common fixed points of a pair of Hardy Rogers type mappings on a closed ball in ordered partial metric spaces, J. Comput. Anal. Appl. **17** (2014), 255–264.

[32] D. Wardowski, Fixed points of new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. **2012**, 2012:94.

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α-ψ-GERAGHTY CONTRACTIONS IN GENERALIZED METRIC SPACES VIA NEW FUNCTIONS

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Abstract. In this paper, we introduce the class of α-h-F-ψ-Greaghty contractions where the pair (F, h) is up class of type I and establish several fixed point theorems for this newly introduced class. Our results extend some recent results of Asadi et al. [1] as well as other corresponding results.

1. Introduction

Banach contraction principle is one of the most pivotal results of fixed point theory. Since this principle plays a very crucial role in solving various kinds of nonlinear equations, it has been extended in several possible ways. Geraghty [3] generalized the Banach contraction mapping principle by introducing the class of auxiliary function in the following way:

Let $F^{\#}$ denote all functions $\beta : [0, \infty) \to [0, 1)$ which satisfy the condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \text{ implies } \lim_{n\to\infty} t_n = 0.$$

By using the function $\beta \in F^{\#}$, Geraghty [3] proved the following remarkable theorem.

Theorem 1.1. [3] *Let (X, d) be a complete metric space and $T : X \to X$ be an operator. Suppose that there exists $\beta : [0, \infty) \to [0, 1)$ satisfying the condition*

$$\beta(t_n) \to 1 \text{ implies } t_n \to 0$$

If T satisfies the following inequality

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y) \tag{1.1}$$

for all $x, y \in X$, then T has a unique fixed point

Recently, Samet et al. [8] introduced the class of α-ψ contractive mappings and established several fixed point theorems for such mappings in the set up of complete metric space which generalized and extend the Banach contraction principle as well as many other well known fixed point theorems existing in the literature. Branciari [2] replaces the triangle inequality in the metric space with the quadrilateral inequality and introduced a new space called generalized metric space or rectangular metric space. He extended the Banach contraction principle to this newly defined space. Very recently, Asadi et al. [1] utilized the concept of Geraghty [3] and Samet et al. [8] and introduced the notion of α-ψ-Greaghty contractions in the context of generalized metric space and extended several well known contractions existing in the literature. For more information, see [9, 10].

In the present work, we extend the notion α-ψ-Geraghty contractions announced by Asadi et al. [1] by introducing the class of α-h-F-ψ-Geraghty contractions and investigate the existence and uniqueness of fixed points for this newly introduced class in the setting of generalized metric space.

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Before going to the main work, we recall some useful definitions and auxiliary results that will be needed in the sequel. Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and the set of real numbers, respectively.

Definition 1.2. [2] Let X be a nonempty set and let $d : X \times X \to [0,\infty]$ satisfy the following conditions for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y:

$$\begin{array}{ll} (1) & d(x,y) = 0 \text{ if and only if } x = y \\ (2) & d(x,y) = d(y,x) \\ (3) & d(x,y) \leq d(x,u) + d(u,v) + d(v,y) \end{array} \tag{1.2}$$

Then the pair (X,d) is called a generalized metric space and abbreviated as GMS. Every metric space is GMS but the converse need not be true (see [1, Example 39]).

Given a generalized metric d on X and $\epsilon > 0$, we call $B_d(x,\epsilon) = \{y \in X | d(x,y) < \epsilon\}$ as ϵ-ball centered at x. It is to be noted that a GMS becomes a topological space, when a subset U of X is said to be open if to each $a \in U$, there exists a positive number ϵ_a such that $B_d(a,\epsilon_a) \subseteq U$. The concepts of convergence, Cauchy sequence, completeness and continuity on a GMS are defined below.

Definition 1.3. Let (X,d) be a generalized metric space.

(1) A sequence $\{x_n\}$ in (X,d) is GMS convergent to a limit x if and only if $d(x_n,x) \to 0$ as $n \to \infty$.
(2) A sequence $\{x_n\}$ in (X,d) is GMS Cauchy if and only if for every $\epsilon > 0$ there exists a positive integer $N(\epsilon)$ such that $d(x_n,x_m) < \epsilon$ for all $n > m > N(\epsilon)$.
(3) A space (X,d) is called complete if every GMS Cauchy sequence in X is GMS convergent.
(4) A mapping $T : (X,d) \to (X,d)$ is continuous if for any sequence $\{x_n\}$ in X such that $d(x_n,x) \to 0$ as $n \to \infty$, we have $d(Tx_n,Tx) \to 0$ as $n \to \infty$.

It is to be noted that any generalized metric space need not be complete, neither the respective topology needs to be Hausdorff and a convergent sequence in GMS need not be Cauchy.

Lemma 1.4. [4] *Let (X,d) be a generalized metric space and let $\{x_n\}$ be a Cauchy sequence in X such that $x_m \neq x_n$ whenever $m \neq n$. Then $\{x_n\}$ can converge to at most one point.*

Lemma 1.5. [4] *Let (X,d) be a generalized metric space and let $\{x_n\}$ be a sequence in X with distinct elements ($x_n \neq x_m$ for $n \neq m$). Suppose that $d(x_n,x_{n+1})$ and $d(x_n,x_{n+2})$ tend to 0 as $n \to \infty$ and that $\{x_n\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and the following four sequences*

$$d(x_{m_k},x_{n_k}), \quad d(x_{m_k},x_{n_{k+1}}), \quad d(x_{m_{k-1}},x_{n_k}), \quad d(x_{m_{k-1}},x_{n_{k+1}}) \tag{1.3}$$

tend to ϵ as $k \to \infty$.

Proposition 1.6. [7] *Suppose that $\{x_n\}$ is a Cauchy sequence in a GMS (X,d) with $\lim_{n\to\infty} d(x_n,u) = 0$, where $u \in X$. Then $\lim_{n\to\infty} d(x_n,z) = d(u,z)$ for all $z \in X$.*

Samet et al. [8] introduced the notion of α-admissible mappings as follows.

Definition 1.7. Let X be a nonempty set, and let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be mappings. Then T is called α-admissible if for all $x, y \in X$, we have

$$\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1. \tag{1.4}$$

Recently, Karapinar et al. [5] defined the notion of triangular α-admissible mappings as follows.

Definition 1.8. Let X be a nonempty set, and let $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}$ be mappings. Then T is called triangular α-admissible if

(1) $x, y \in X,\ \alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1$;
(2) $x, y, z \in X,\ \alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1 \Rightarrow \alpha(x,y) \geq 1$.

Lemma 1.9. [5] *Let $T : X \to X$ be a triangular α-admissible map. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Now we recall the following class of auxiliary functions (or altering distance functions) (see [6]) which will be used densely in the sequel: Let Ψ denote the class of the functions $\psi : [0,\infty) \to [0,\infty)$ which satisfy the following conditions:
(a) ψ is nondecreasing;
(b) ψ is continuous;
(c) $\psi(t) = 0 \Leftrightarrow t = 0$.

2. Main results

Definition 2.1. A function $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is of subclass of type I if it is continuous and satisfies the following:
for $y, z \in \mathbb{R}^+, y \geq 1 \Longrightarrow h(1,z) \leq h(y,z)$.

Example 2.2. The following functions $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ are of subclass of type I. For $y, z \in \mathbb{R}^+$,
(1) $h(y,z) = (z+l)^y, l > 1$;
(2) $h(y,z) = (y+l)^z, l > 1$;
(3) $h(y,z) = yz$;
(4) $h(y,z) = (\frac{1+y}{2})z$;
(5) $h(y,z) = y^k z$;
(6) $h(y,z) = z$;
(7) $h(y,z) = \frac{1+2y}{3}z$;
(8) $h(y,z) = (\frac{\sum_{i=0}^{n} y^i}{n+1})z$;
(9) $h(y,z) = (\frac{\sum_{i=0}^{n} y^i}{n+1} + l)^z, l > 1$.

Definition 2.3. Let $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a function. We say that the pair (F,h) is a upclass of type I if F is continuous, h is a function of subclass of type I and the following hold:
(1) for $0 \leq s \leq 1,\ t \in \mathbb{R}^+ \Longrightarrow F(s,t) \leq F(1,t)$;
(2) for $z, s, t \in \mathbb{R}^+$ if $h(1,z) \leq F(s,t) \Longrightarrow z \leq st$.

Example 2.4. The following functions $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ are of upclass of type I. For $x, y, z, s, t \in \mathbb{R}^+$,
(1) $h(y,z) = (z+l)^y, l > 1, F(s,t) = st + l$;
(2) $h(y,z) = (xy+l)^z, l > 1, F(s,t) = (1+l)^{st}$;
(3) $h(y,z) = yz, F(s,t) = st$;
(4) $h(y,z) = (\frac{1+y}{2})z, F(s,t) = st$;
(5) $h(y,z) = y^k z, F(s,t) = t$;
(6) $h(y,z) = \frac{1+2y}{3}z, F(s,t)) = st$;
(7) $h(y,z) = (\frac{1+y}{2})z, F(s,t) = st$;
(8) $h(y,z) = (\frac{\sum_{i=0}^{n} y^i}{n+1})z, F(s,t) = st$;
(9) $h(y,z) = (\frac{\sum_{i=0}^{n} y^i}{n+1} + l)^z, l > 1, F(s,t) = (1+l)^{st}$.

We start to this section with the following definition.

Definition 2.5. Let (X, d) be a generalized metric space and let $\alpha : X \times X \to \mathbb{R}$ be a function. A map $T : X \to X$ is called an α-h-F-ψ-Geraghty contraction type map if there exists $\beta \in F^{\#}$ such that for all $x, y \in X$,

$$h(\alpha(x,y), \psi(d(Tx,Ty))) \leq F(\beta(\psi d(x,y)), \psi(d(x,y))), \tag{2.1}$$

where $\psi \in \Psi$ and the pair (F, h) is a upclass of type I.

Theorem 2.6. *Let (X, d) be a complete generalized metric space, $\alpha : X \times X \to \mathbb{R}$ a function, and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

1. *T is an α-h-F-ψ-Geraghty contraction type map;*
2. *T is triangular α-admissible;*
3. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*
4. *T is continuous.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*.

Proof. By (3), from $x_0 \in X$, construct the sequence $\{x_n\}$ as $x_{n+1} = Tx_n$, $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then $x^* = x_n$ is a fixed point of T. Assume further that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Since T is triangular α-admissible, it follows from (3) that

$$\alpha(x_1, x_2) = \alpha(x_1, Tx_1) \geq 1 \quad \text{and} \quad \alpha(x_1, x_3) = \alpha(x_1, T^2x_1) \geq 1.$$

By Lemma 1.9, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{and} \quad \alpha(x_n, x_{n+2}) \geq 1. \tag{2.2}$$

for all $n \in \mathbb{N}$. Notice that we also find $\alpha(x_n, x_{n+m}) \geq 1$ for each $m, n \in \mathbb{N}$.

Now we shall prove that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. By taking $x = x_{n-1}$ and $y = x_n$ in (2.1) and regarding (2.2), we get that

$$\begin{aligned} h(1, \psi(d(x_n, x_{n+1}))) &\leq h(\alpha(x_{n-1}, x_n), \psi(d(Tx_{n-1}, Tx_n))) \\ &\leq F(\beta(\psi(d(x_{n-1}, x_n))), \psi(d(x_{n-1}, x_n))) \Longrightarrow \\ \psi(d(x_n, x_{n+1})) &\leq \beta(\psi(d(x_{n-1}, x_n))), \psi(d(x_{n-1}, x_n)) < \psi(d(x_{n-1}, x_n)) \end{aligned} \tag{2.3}$$

for each $n \in \mathbb{N}$.

Since ψ is nondecreasing, we conclude from (2.3) that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$$

for each $n \in \mathbb{N}$. Thus we conclude that the sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and nonincreasing. As a result, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. We claim that $r = 0$. Suppose, on the contrary, that $r > 0$. Then, on account of (2.3), we get that

$$\frac{\psi(d(x_n, x_{n+1}))}{\psi(d(x_{n-1}, x_n))} \leq \beta(\psi(d(x_{n-1}, x_n))) < 1,$$

which yields that $\lim_{n\to\infty} \beta(\psi(d(x_n, x_{n+1}))) = 1$. We obtain

$$\lim_{n\to\infty} \psi(d(x_n, x_{n+1})) = 0, \tag{2.4}$$

due to the fact that $\beta \in F^{\#}$. On the other hand, the continuity of ψ together with (2.4) yield that

$$r = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.5}$$

Analogously, we shall prove that $\lim_{n\to\infty} d(x_n, x_{n+2}) = 0$. By substituting $x = x_{n-1}$ and $y = x_{n+1}$ in (2.1) and taking (2.2) into account, we find that

$$\begin{aligned} h(1, \psi(d(x_n, x_{n+2}))) &\leq h(\alpha(x_{n-1}, x_{n+1}), \psi(d(Tx_{n-1}, Tx_{n+1})) \\ &\leq F(\beta(\psi(d(x_{n-1}, x_{n+1}))), \psi(d(x_{n-1}, x_{n+1}))) \Longrightarrow \\ \psi(d(x_n, x_{n+2})) &\leq \beta(\psi(d(x_{n-1}, x_{n+1})))\psi(d(x_{n-1}, x_{n+1})) < \psi(d(x_{n-1}, x_{n+1})) \end{aligned} \tag{2.6}$$

for each $n \in \mathbb{N}$. Since ψ is nondecreasing, we conclude from (2.6) that

$$d(x_n, x_{n+2}) < d(x_{n-1}, x_{n+1})$$

for each $n \in \mathbb{N}$. Thus we observe that the sequence $\{d(x_{n-1}, x_{n+1})\}$ is nonnegative and nonincreasing. Consequently, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = r$. We assert that $r = 0$. Suppose, on the contrary, that $r > 0$. Then, by regarding (2.6), we get that

$$\frac{\psi(d(x_n, x_{n+2}))}{\psi(d(x_{n-1}, x_{n+1}))} \leq \beta(\psi(d(x_{n-1}, x_{n+1}))) < 1,$$

which implies that $\lim_{n\to\infty} \beta(\psi(d(x_{n-1}, x_{n+1}))) = 1$. We obtain

$$\lim_{n\to\infty} \psi(d(x_{n-1}, x_{n+1})) = 0, \tag{2.7}$$

due to the fact that $\beta \in F^{\#}$. On the other hand, the continuity of ψ together with (2.7) yield that

$$r = \lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = 0 = \lim_{n\to\infty} d(x_n, x_{n+2}). \tag{2.8}$$

Suppose that $x_n = x_m$ for some $m, n \in \mathbb{N}$, $m < n$. Then

$$\begin{aligned} \psi(d(x_m, x_{m+1})) &= \psi(d(x_n, x_{n+1})) \\ &\leq \beta(\psi(d(x_{n-1}, x_n)))\psi(d(x_{n-1}, x_n)) \\ &< \psi(d(x_{n-1}, x_n)) \\ &\leq \psi^{n-m}(d(x_m, x_{m+1})) \\ &< \psi(d(x_m, x_{m+1})), \end{aligned}$$

a contradiction. Hence all elements of the sequence $\{x_n\}$ are distinct.

We are ready to prove that $\{x_n\}$ is a Cauchy sequence in (X, d). Suppose, on the contrary, that we have

$$\varepsilon = \limsup_{m,n\to\infty} d(x_n, x_m) > 0. \tag{2.9}$$

Regarding the quadrilateral inequality, we need to examine two possible cases:

Case 1. Suppose $k = n - m$ is odd, where $k \geq 1$. Then we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m) \\ &= d(x_n, x_{n+1}) + d(Tx_n, Tx_m) + d(x_{m+1}, x_m), \end{aligned} \tag{2.10}$$

which is equivalent to

$$d(x_n, x_m) - d(x_n, x_{n+1}) - d(x_{m+1}, x_m) \leq d(Tx_n, Tx_m) \tag{2.11}$$

Since T is triangular α-admissible, by applying ψ, we get that

$$\psi(d(x_n, x_m) - d(x_n, x_{n+1}) - d(x_{m+1}, x_m)) \leq \psi(d(Tx_n, Tx_m))$$

and

$$\begin{aligned} h(1, \psi(d(Tx_n, Tx_m))) &\leq h(\alpha(x_n, x_m), \psi(d(Tx_n, Tx_m))) \\ &\leq F(\beta(\psi(d(x_n, x_m))), \psi(d(x_n, x_m))) \Longrightarrow \end{aligned}$$

$$\psi(d(Tx_n, Tx_m))) \leq \beta(\psi(d(x_n, x_m))), \psi(d(x_n, x_m)). \tag{2.12}$$

Letting $m, n \to \infty$, we deduce that

$$\lim_{m,n\to\infty} \psi(d(x_n, x_m) - d(x_n, x_{n+1}) - d(x_{m+1}, x_m)) \leq \lim_{m,n\to\infty} \beta(\psi(d(x_n, x_m))) \lim_{m,n\to\infty} \psi(d(x_m, x_n)).$$

So, by using (2.5), (2.9) and the continuity of ψ, we get

$$1 \leq \lim_{m,n\to\infty} \beta(\psi(d(x_n, x_m))),$$

which implies that $\lim_{m,n\to\infty} \beta(\psi(d(x_n, x_m))) = 1$. Consequently, we get $\lim_{m,n\to\infty} d(x_n, x_m) = 0$, which is a contradiction.

Case 2. Suppose $k = n - m$ is even, where $k \geq 1$. So we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+2}) + d(x_{n+2}, x_{m+2}) + d(x_{m+2}, x_m) \\ &= d(x_n, x_{n+2}) + d(Tx_{n+1}, Tx_{m+1}) + d(x_{m+2}, x_m), \end{aligned} \tag{2.13}$$

that can be written as

$$d(x_n, x_m) - d(x_n, x_{n+2}) - d(x_{m+2}, x_m) \leq d(Tx_{n+1}, Tx_{m+1}). \tag{2.14}$$

Due to the fact that T is triangular α-admissible, by applying ψ, we obtain that

$$\psi(d(x_n, x_m) - d(x_n, x_{n+2}) - d(x_{m+2}, x_m)) \leq \psi(d(Tx_{n+1}, Tx_{m+1}))$$

and

$$\begin{aligned} h(1, \psi(d(Tx_{n+1}, Tx_{m+1}))) &\leq h(\alpha(x_{n+1}, x_{m+1}), \psi(d(Tx_{n+1}, Tx_{m+1}))) \\ &\leq F(\beta(\psi(d(x_{n+1}, x_{m+1}))), \psi(d(x_{n+1}, x_{m+1}))) \Longrightarrow \end{aligned}$$

$$\psi(d(Tx_{n+1}, Tx_{m+1})) \leq \beta(\psi(d(x_{n+1}, x_{m+1}))) \psi(d(x_{n+1}, x_{m+1})). \tag{2.15}$$

Letting $m, n \to \infty$, we find that

$$\lim_{m,n\to\infty} \psi(d(x_n, x_m) - d(x_n, x_{n+2}) - d(x_{m+2}, x_m)) \leq \lim_{m,n\to\infty} \beta(\psi(d(x_{n+1}, x_{m+1}))) \lim_{m,n\to\infty} \psi(d(x_{n+1}, x_{m+1}))$$

So, by using (2.8), (2.9) and the continuity of ψ, we observe

$$1 \leq \lim_{m,n\to\infty} \beta(\psi(d(x_{n+1}, x_{m+1}))),$$

which implies that $\lim_{m,n\to\infty} \beta(\psi(d(x_{n+1}, x_{m+1}))) = 1$. Thus we conclude that $\lim_{m,n\to\infty} d(x_{n+1}, x_{m+1}) = 0$, which is a contradiction.

From Case 1 and Case 2, we conclude that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete generalized metric space, there exists $x^* \in X$ such that $\lim_{n\to\infty} d(x_n, x^*) = 0$. Since T is continuous, we have

$$\lim_{n\to\infty} d(Tx_n, x^*) = \lim_{n\to\infty} d(x_{n+1}, Tx^*) = 0.$$

By Lemma 1.4, we have that $Tx^* = x^*$. □

It is also possible to remove the continuity of the mapping T by replacing it with a weaker condition:

Definition 2.7. Let (X, d) be a complete generalized metric space, $\alpha : X \times X \to \mathbb{R}$ a function and let $T : X \to X$ be a map. We say that the sequence $\{x_n\}$ is α-regular if the following condition is satisfied:

If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to +\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Theorem 2.8. *Let (X, d) be a complete generalized metric space, $\alpha : X \times X \to \mathbb{R}$ a function and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

(1) *T is an α-h-F-ψ-Geraghty contraction type map;*
(2) *T is triangular α-admissible;*
(3) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*
(4) *either T is continuous or $\{x_n\}$ is α-regular.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*.

Proof. Following the proof of Theorem 2.6, we know that the sequence $\{x_n\}$, defined by $x_{n+1} = Tx_n$ for all $n \geq 0$, converges to some $x^* \in X$. From (2.2) and the assumption (4) of the theorem, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq 1$. Applying (2.1), for all k, we get that

$$\begin{aligned} h(1, \psi(d(x_{n(k)+1}, Tx^*))) &\leq h(\alpha(x_{n(k)}, x^*), \psi(d(Tx_{n(k)}, Tx^*)) \\ &\leq F(\beta(\psi(d(x_{n(k)}, x^*))), \psi(d(x_{n(k)}, x^*))) \\ &\Longrightarrow \end{aligned}$$

$$\psi(d(x_{n(k)+1}, Tx^*)) \leq \beta(\psi(d(x_{n(k)}, x^*)))\psi(d(x_{n(k)}, x^*)) < \psi(d(x_{n(k)}, x^*)). \tag{2.16}$$

Letting $k \to \infty$ in (2.16), we have

$$\lim_{k\to\infty} \psi(d(x_{n(k)+1}, Tx^*)) \leq 0.$$

Therefore, in view of Proposition 1.6, we obtain $x^* = Tx^*$. □

Now we introduce the notion of generalized α-h-F-ψ-Geraghty contraction.

Definition 2.9. Let (X, d) be a generalized metric space and let $\alpha : X \times X \to \mathbb{R}$ be a function. A map $T : X \to X$ is called an α-h-F-ψ-Geraghty contraction type map if there exists $\beta \in F^{\#}$ such that for all $x, y \in X$,

$$h(\alpha(x, y), \psi(d(Tx, Ty))) \leq F(\beta(\psi(M(x, y)), \psi(M(x, y))), \tag{2.17}$$

where $M(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty)\}$, $\psi \in \Psi$ and the pair (F, h) is a upclass of type I.

Theorem 2.10. *Let (X, d) be a complete generalized metric space, $\alpha : X \times X \to \mathbb{R}$ a function and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

(1) *T is a generalized α-h-F-ψ-Geraghty contraction type map;*
(2) *T is triangular α-admissible;*
(3) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*
(4) *T is continuous.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*.

Proof. By (3), from $x_0 \in X$, construct the sequence $\{x_n\}$ as $x_{n+1} = Tx_n$, $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then $x^* = x_n$ is a fixed point of T. Assume further that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Since T is triangular α-admissible, it follows from (3) that

$$\alpha(x_1, x_2) = \alpha(x_1, Tx_1) \geq 1 \quad \text{and} \quad \alpha(x_1, x_3) = \alpha(x_1, T^2x_1) \geq 1$$

and so by induction, we get

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{and} \quad \alpha(x_n, x_{n+2}) \geq 1 \tag{2.18}$$

for $n \in \mathbb{N}$. And we also find $\alpha(x_n, x_{n+m}) \geq 1$ for each $m, n \in \mathbb{N}$.
Therefore, by (2.17)

$$\begin{aligned} h(1, \psi(d(x_n, x_{n+1}))) &\leq h(\alpha(x_{n-1}, x_n), \psi(d(Tx_{n-1}, Tx_n))) \\ &\leq F(\beta(\psi(M(x_{n-1}, x_n))), \psi(M(x_{n-1}, x_n))) \Longrightarrow \\ \psi(d(x_n, x_{n+1})) &\leq \beta(\psi(M(x_{n-1}, x_n)))\psi(M(x_{n-1}, x_n)) < \psi(M(x_{n-1}, x_n)) \end{aligned} \tag{2.19}$$

for each $n \geq 1$, where

$$M(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \tag{2.20}$$

If $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$, then by (2.19), we get

$$\psi(d(x_n, x_{n+1})) < \psi(d(x_n, x_{n+1})),$$

which is a contradiction. Hence $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n)$ and (2.19) gives

$$\psi(d(x_n, x_{n+1})) < \psi(d(x_{n-1}, x_n)), \text{ for all } n \in \mathbb{N}. \tag{2.21}$$

This yields that, for each $n \in \mathbb{N}$,

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n). \tag{2.22}$$

Thus we conclude that the sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and nonincreasing. As a result, there exists $t \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = t$. We claim that $t = 0$. Suppose, on the contrary, that $t > 0$. Then, on account of (2.19), we get that

$$\frac{\psi(d(x_n, x_{n+1}))}{\psi(M(x_{n-1}, x_n))} \leq \beta(\psi(M(x_{n-1}, x_n))) < 1,$$

which yields that $\lim_{n\to\infty} \beta(\psi(d(x_n, x_{n+1}))) = 1$. We obtain

$$\lim_{n\to\infty} \psi(d(x_n, x_{n+1})) = 0, \tag{2.23}$$

due to the fact that $\beta \in F^{\#}$. On the other hand, the continuity of ψ together with (2.23) yield that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.24}$$

Now we shall show

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{2.25}$$

Regarding (2.17) and (2.18), we find that

$$\begin{aligned} h(1, \psi(d(x_n, x_{n+2}))) &\leq h(\alpha(x_{n-1}, x_{n+1})\psi(d(Tx_{n-1}, Tx_{n+1})) \\ &\leq F(\beta(\psi(M(x_{n-1}, x_{n+1}))), \psi(M(x_{n-1}, x_{n+1}))) \Longrightarrow \end{aligned}$$

$$\psi(d(x_n, x_{n+2}))) \leq \beta(\psi(M(x_{n-1}, x_{n+1})))\psi(M(x_{n-1}, x_{n+1}))) < \psi(M(x_{n-1}, x_{n+1})) \tag{2.26}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} M(x_{n-1}, x_{n+1}) &= \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, Tx_{n-1}), d(x_{n+1}, Tx_{n+1})\} \\ &= \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2})\}. \end{aligned} \tag{2.27}$$

In view of (2.22), we obtain

$$M(x_{n-1}, x_{n+1}) = \max\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n)\}.$$

Define $a_n = d(x_n, x_{n+2})$ and $b_n = d(x_n, x_{n+1})$. Then, taking (2.26) into account, we get

$$\psi(a_n) < \psi(\max\{a_{n-1}, b_{n-1}\}).$$

This yields that, for each $n \in \mathbb{N}$,

$$a_n < \max\{a_{n-1}, b_{n-1}\}. \tag{2.28}$$

By (2.22), we have

$$b_n < \max\{a_{n-1}, b_{n-1}\}. \tag{2.29}$$

Therefore,

$$\max\{a_n, b_n\} < \max\{a_{n-1}, b_{n-1}\}$$

for all $n \in \mathbb{N}$. Thus the sequence $\max\{a_n, b_n\}$ is nonegative and nonincreasing and so it converges to some $r \geq 0$. Clearly, by (2.24),

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \max\{a_n, b_n\} = r.$$

Now we will show that $r = 0$. If $r > 0$, then in view of (2.26), we have

$$\frac{\psi(d(x_n, x_{n+2}))}{\psi(M(x_{n-1}, x_{n+1}))} \leq \beta(\psi(M(x_{n-1}, x_{n+1}))) < 1,$$

which yields that $\lim_{n\to\infty} \beta(\psi(M(x_{n-1}, x_{n+1}))) = 1$. We obtain

$$\lim_{n\to\infty} \psi(M(x_{n-1}, x_{n+1})) = 0, \tag{2.30}$$

due to the fact that $\beta \in F^{\#}$. On the other hand, the continuity of ψ together with (2.30) yield that

$$\psi(r) = \psi(\lim_{n\to\infty} \max\{a_{n-1}, b_{n-1}\}) = \lim_{n\to\infty} \psi(\max\{a_{n-1}, b_{n-1}\}) = 0,$$

which is a contradiction and hence $r = 0$.

Suppose that $x_n = x_m$ for some $m, n \in \mathbb{N}$, $m < n$. Then

$$\begin{aligned}
\psi(d(x_m, x_{m+1})) &= \psi(d(x_n, x_{n+1})) \\
&\leq \beta(\psi(M(x_{n-1}, x_n)))\psi(M(x_{n-1}, x_n)) \\
&< \psi(d(x_{n-1}, x_n)) \\
&\leq \psi^{n-m}(d(x_m, x_{m+1})) \\
&< \psi(d(x_m, x_{m+1})),
\end{aligned}$$

a contradiction. Hence all elements of the sequence $\{x_n\}$ are distinct.

In order to prove that $\{x_n\}$ is a Cauchy sequence in (X, d), suppose that it is not. Then by Lemma 1.5, using (2.24) and (2.25), we assert that there exist $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and the sequences (1.3) tend to ϵ as $k \to \infty$. By substituting $x = x_{m_k}$ and $y = x_{n_{k+1}}$ in (2.17) and taking (2.18) into account, we obtain

$$\begin{aligned}
h(1, \psi(d(x_{m_k}, x_{n_{k+1}}))) &\leq h(\alpha(x_{m_{k-1}}, x_{n_k}), \psi(d(Tx_{m_{k-1}}, Tx_{n_k}))) \\
&\leq F(\beta(\psi(M(x_{m_{k-1}}, x_{n_k}))), \psi(M(x_{m_{k-1}}, x_{n_k}))).
\end{aligned} \tag{2.31}$$

On the other hand, we have

$$\begin{aligned}
M(x_{m_{k-1}}, x_{n_k}) &= \max\{d(x_{m_{k-1}}, x_{n_k}), d(x_{m_{k-1}}, Tx_{m_{k-1}}), d(x_{n_k}, Tx_{n_k})\} \\
&= \max\{d(x_{m_{k-1}}, x_{n_k}), d(x_{m_{k-1}}, x_{m_k}), d(x_{n_k}, x_{n_{k+1}})\}
\end{aligned} \tag{2.32}$$

and hence

$$\lim_{k\to\infty} \psi(M(x_{m_{k-1}}, x_{n_k})) = \psi(\epsilon). \tag{2.33}$$

From (2.31), we have

$$\frac{\psi(d(x_{m_k}, x_{n_{k+1}}))}{\psi(M(x_{m_{k-1}}, x_{n_k}))} \leq \beta(\psi(M(x_{m_{k-1}}, x_{n_k}))) < 1.$$

Letting $k \to \infty$, we get

$$\lim_{k\to\infty} \beta(\psi(M(x_{m_{k-1}}, x_{n_k}))) = 1.$$

Thus $\lim_{k\to\infty} \psi(M(x_{m_{k-1}}, x_{n_k})) = 0$ and so (2.33) gives $\psi(\epsilon) = 0$, which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete generalized metric space, there exists $x^* \in X$ such that $\lim\limits_{n\to\infty} d(x_n, x^*) = 0$. Since T is continuous, we have

$$\lim_{n\to\infty} d(Tx_n, x^*) = \lim_{n\to\infty} d(x_{n+1}, Tx^*) = 0.$$

By Lemma 1.4, we get that $Tx^* = x^*$. □

Theorem 2.11. *Let (X, d) be a complete generalized metric space, $\alpha : X \times X \to \mathbb{R}$ a function and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:*

(1) *T is a generalized α-h-F-ψ-Geraghty contraction type map;*
(2) *T is triangular α-admissible;*
(3) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$;*
(4) *$\{x_n\}$ is α-regular.*

Then T has a fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to x^*.

Proof. Following the proof of Theorem 2.10, we know that the sequence $\{x_n\}$, defined by $x_{n+1} = Tx_n$ for all $n \geq 0$, converges to some $x^* \in X$. Now we shall show that $Tx^* = x^*$. Suppose, on the contrary, that $Tx^* \neq x^*$, i.e,, $d(x^*, Tx^*) > 0$. Since x_n is α-regular, from (2.18), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq 1$. Applying (2.17), for all k, we get that

$$\begin{aligned} h(1, \psi(d(x_{n(k)+1}, Tx^*))) &\leq h(\alpha(x_{n(k)}, x^*), \psi(d(Tx_{n(k)}, Tx^*))) \\ &\leq F(\beta(\psi(M(x_{n(k)}, x^*))), \psi(M(x_{n(k)}, x^*))), \end{aligned} \tag{2.34}$$

where $M(x_{n(k)}, x^*) = \max\{d(x_{n(k)}, x^*), d(x_{n(k)}, Tx_{n(k)}), d(x^*, Tx^*)\}$.

Letting $k \to \infty$ in (2.34), we have

$$\lim_{n\to\infty} \psi(d(x_{n(k)+1}, Tx^*)) < \psi(d(x^*, Tx^*)).$$

In view of Proposition 1.6, we get a contradiction and hence $x^* = Tx^*$. □

For the uniqueness of a fixed point of α-ψ-Geraghty contractive mapping, we will consider the following condition.

Condition (U): For all $x, y \in F(T)$, we have $\alpha(x, y) \geq 1$, where $F(T)$ denotes the set of fixed points of T.

Theorem 2.12. *If the condition (U) is added to the hypothesis of Theorem 2.6 (respectively, Theorem 2.8), then we obtain that u is the unique fixed point of T.*

Proof. We will show that u is a unique fixed point of T. Let v be another fixed point of T with $v \neq u$. By hypothesis (U),

$$1 \leq \alpha(u, v) = \alpha(Tu, Tv).$$

Now, using (2.5), we have

$$\begin{aligned} h(1, \psi(d(u, v))) &\leq h(\alpha(u, v), \psi(d(Tu, Tv))) \\ &\leq F(\beta(\psi(d(u, v))), \psi(d(u, v))) \Longrightarrow \end{aligned}$$

$$\psi(d(u, v)) \leq \beta(\psi(d(u, v)))\psi(d(u, v)) < \psi(d(u, v)),$$

which is a contradiction. Hence $u = v$. □

Theorem 2.13. *If the condition (U) is added to the hypothesis of Theorem 2.10 (respectively, Theorem 2.11), then we obtain that u is the unique fixed point of T.*

Proof. Let v be another fixed point of T with $v \neq u$. Then by the assumption (U),

$$1 \leq \alpha(u,v) = \alpha(Tu,Tv).$$

Now, using (2.17), we have

$$\begin{aligned} h(1,\psi(d(u,v))) &\leq h(\alpha(u,v),\psi(d(Tu,Tv))) \\ &\leq F(\beta(\psi(M(u,v)))\psi(M(u,v))), \end{aligned}$$

$$\psi(d(u,v))) \leq \beta(\psi(M(u,v)))\psi(M(u,v)) < \psi(M(u,v)),$$

where

$$M(u,v) = \max\{d(u,v), d(u,Tu), d(v,Tv)\}.$$

Therefore,

$$\psi(d(u,v)) < \psi(d(u,v)),$$

which is a contradiction. Hence $u = v$. $\square$

Now we give a useful example.

Example 2.14. Let $X = A \cup B$, where $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ and $B = [\frac{3}{4}, 1]$. Define the function $d : X \times X \to \mathbb{R}$ as follows:

$$\begin{aligned} d\left(\frac{1}{2},\frac{1}{3}\right) &= \frac{1}{2}, \\ d\left(\frac{1}{2},\frac{1}{4}\right) &= d\left(\frac{1}{3},\frac{1}{4}\right) = \frac{1}{5}, \\ d\left(\frac{1}{4},\frac{1}{5}\right) &= d\left(\frac{1}{2},\frac{1}{5}\right) = d\left(\frac{1}{3},\frac{1}{5}\right) = \frac{1}{6}, \\ d(x,x) &= 0 \text{ for all } x \in A, \\ d(x,y) &= d(y,x) \text{ for all } x,y \in A \end{aligned}$$

and $d(x,y) = |x-y|$, whenever $(x \in A, y \in B)$ or $(x \in B, y \in A)$ or $(x, y \in B)$. It is easy to check that (X,d) is a generalized metric space. Let $T : X \to X$ be a mapping defined by

$$T(x) = \begin{cases} \frac{2x+1}{x+2} & \text{if } x \in B \\ \frac{1}{5} & \text{otherwise} \end{cases}$$

and the function $\alpha : X \times X \to [0,\infty)$ defined by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in B \\ 0 & \text{otherwise.} \end{cases}$$

Define $\beta(t) = \frac{1}{1+t}$. Using routine calculation it is easy to check that T is an α-h-F-ψ-Geraghty contraction for $h(y,z) = yz$, $F(s,t) = st$ and $\psi(t) = t$. For $x_0 = \frac{3}{4}$, we have $\alpha(x_0, Tx_0) \geq 1$ and $\alpha\left(x_0, T^2x_0\right) \geq 1$. Using the definition of the maps α and T, we observe that $\alpha(x,y) \geq 1 \Longrightarrow \alpha(Tx,Ty) \geq 1$ for all $x,y \in X$ and also $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1 \Longrightarrow \alpha(x,y) \geq 1$ and so T is triangular α-admissible. Moreover, if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n with $x_n \to x$ as $n \to \infty$, then the definition of α gives $x_n \in B$ for each n. Since B is closed, it follows that $x \in B$. Thus $\alpha(x_n, x) = 1$ for each n and hence the sequence $\{x_n\}$ is α-regular. So all the hypotheses of Theorem 2.8 are satisfied and therefore T has a fixed point. Here $x = 1$ is such a fixed point.

Remark 2.15. If the self map T is an α-ψ-Geraghty contraction type map (see [1]), then T is an α-h-F-ψ-Geraghty contraction type map for $h(y, z) = yz$ and $F(s, t) = st$. Therefore the results of Asadi et al. [1] can be obtained as a particular case of our results.

References

[1] M. Asadi, E. Karapinar, A. Kumar, *α-ψ-Geraghty contractions on generalized metric spaces*, J. Inequal. Appl. **2014**, 2014:423.

[2] A. Branciari, Â fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, Publ. Math. Debrecen **57** (2000), 31–37.

[3] M, Geraghty, *On contractive mappings*, Proc. Amer. Math. Soc. **40** (1973), 604–608.

[4] Z. Kadelburg, S. Radenović, *Fixed point results in generalized metric spaces without Hausdorff property*, Math. Sci. **8** (2014), Art. ID 125.

[5] E. Karapinar, P. Kumam, P. Salimi, *On α-ψ-Meir-Keeler contractive mappings*, Fixed Point Theory Appl. **2013**, 2013:94.

[6] M.S. Khan, M. Swaleh, S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Aust. Math. Soc. **30** (1984), 1–9.

[7] W.A. Kirk, N. Shahzad, Generalized metrics and Caristi's theorem, Fixed Point Theory Appl. **2013**, 2013:129.

[8] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, Nonlinear Anal. **75** (2012), 2154–2165.

[9] A. Shoaib, *α-η dominated mappings and related common fixed point results in closed ball*, J. Concrete Appl. Math. **13** (2015), 152–170.

[10] A. Shoaib, M. Arshad, M. A. Kutbi, *Common fixed points of a pair of Hardy Rogers type mappings on a closed ball in ordered partial metric spaces*, J. Comput. Anal. Appl. **17** (2014), 255–264.

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Oscillation of nth-order nonlinear dynamic equations on time scales

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Absract In this paper, we investigate the oscillation of the following nth-order nonlinear dynamic equation

$$(r(t)\Phi_\gamma(a_{n-1}(t)(a_{n-2}(t)(\cdots(a_1(t)x^\Delta(t))^\Delta\cdots)^\Delta)^\Delta)^\Delta+\sum_{i=0}^{k}q_i(t)\Phi_{\alpha_i}(x(\delta_i(t)))=0$$

on a time scale $\mathbb{T}$ with $n\geq 2$. We obtain some new oscillation criteria of the above equation.

Keywords: Oscillation; Dynamic equation; Time scale

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1. Introduction

In this paper, we study the following nth-order nonlinear dynamic equation

$$(r(t)\Phi_\gamma(a_{n-1}(t)(a_{n-2}(t)(\cdots(a_1(t)x^\Delta(t))^\Delta\cdots)^\Delta)^\Delta)^\Delta+\sum_{i=0}^{k}q_i(t)\Phi_{\alpha_i}(x(\delta_i(t)))=0, \qquad (1.1)$$

on a time scale $\mathbb{T}$ satisfying $\inf\mathbb{T}=t_0$ and $\sup\mathbb{T}=\infty$, where $n,k\in\mathbb{N}\equiv\{1,2,\cdots\}$ with $n\geq 2$ and $\gamma,\alpha_i>0$, $i=0,1,2,\cdots,k$, are ratios of odd numbers, $q_i(t)\in C_{rd}(\mathbb{T},[0,\infty))$ and $q_i(t)\not\equiv 0$, $i=0,1,2,\cdots,k$. And we also assume the following conditions are satisfied:

(H1) $\Phi_p(u)=|u|^{p-1}u$ for any $p>0$.

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(H2) There is an integer $m \in [1, k)$ such that

$$\alpha_1 > \alpha_2 > \cdots > \alpha_m > \alpha_0 = \gamma > \alpha_{m+1} > \cdots > \alpha_k. \tag{1.2}$$

(H3) $a_j(t) \in C_{rd}(\mathbb{T}, (0, \infty)),\ 1 \leq j \leq n-1,\ r(t) \in C_{rd}(\mathbb{T}, (0, \infty))$ satisfying

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\gamma}(t) a_{n-1}(t)} \Delta t = \infty.$$

(H4) $\delta_i(t) \in C_{rd}(\mathbb{T}, \mathbb{T})$ such that $\delta_i^{\Delta}(t) > 0$ and $\lim_{t \to \infty} \delta_i(t) = \infty$.

Write

$$S_l(t) = \begin{cases} x(t), & l = 0, \\ a_l(t) S_{l-1}^{\Delta}(t), & 1 \leq l \leq n-1. \end{cases}$$

Then (1.1) reduces to the equation

$$(r(t)\Phi_\gamma(S_{n-1}(t)))^{\Delta} + \sum_{i=0}^{k} q_i(t) \Phi_{\alpha_i}(x(\delta_i(t))) = 0. \tag{1.3}$$

Few decades, much attention has been paid to various dynamic equations, see [1-5]. In recent years, there has been more interest in obtaining conditions for oscillation of solutions of dynamic equations on time scales, see [6-10]. Erbe et al.[1] researched the oscillation of the equation

$$x^{\Delta^n}(t) + p(t) x^{\alpha}(\sigma(t)) = 0, t \in \mathbb{T},$$

where $\alpha > 1$ is a ratio of odd numbers; and Jia et al.[2] investigated oscillation for the linear equation

$$x^{\Delta^n}(t) + p(t) x(t) = 0, t \in \mathbb{T}.$$

On a general time scale $\mathbb{T}$. Recently, Zhang et al.[3] established the oscillation criteria for the equation

$$(r(t)\Phi_\gamma(x^{\Delta^{n-1}}(t))^{\Delta} + \sum_{i=0}^{k} q_i(t) \Phi_{\alpha_i}(x(\delta_i(t))) = 0, \tag{1.4}$$

where $\Phi_p(u), \gamma, \alpha_i, \delta_i(t)$ is the same as (1.1).

The purpose of this paper is to generalize results in [3] to more general equation and obtain some new oscillation criteria of (1.1).

2. Preliminary results

Lemma 2.1[3] Assume (1.2) holds. Then there exist $\eta_i \in (0, 1),\ i = 1, \cdots, k$, such that

$$\sum_{i=1}^{k} \alpha_i \eta_i = \gamma \quad \text{and} \quad \sum_{i=1}^{k} \eta_i = 1. \qquad \square$$

Lemma 2.2[4] Assume that

$$\int_{t_0}^{\infty} \frac{\triangle s}{a_i(s)} = \infty, \quad 1 \le i \le n-2, \tag{2.1}$$

and $1 \le m \le n-2$. Then,

(1) $\liminf_{t\to\infty} S_m(t) > 0$ implies $\lim_{t\to\infty} S_i(t) = +\infty$, $0 \le i \le m-1$;

(2) $\limsup_{t\to\infty} S_m(t) < 0$ implies $\lim_{t\to\infty} S_i(t) = -\infty$, $0 \le i \le m-1$. □

Lemma 2.3 Assume (2.1) holds, and suppose that $x(t) > 0$ and $(r(t)\Phi_\gamma(S_{n-1}(t)))^\Delta < 0$ for $t \ge t_0$. Then there exist an integer $m \in [0, n-1]$ with $m+n-1$ even and a sufficiently large $T \in \mathbb{T}$ such that, for any $t \ge T$,

(1) $(-1)^{m+i}S_i(t) > 0$, $m \le i \le n-1$;

(2) $S_i(t) > 0$, $1 \le i \le m-1$, when $m > 1$.

Proof First we claim that $S_{n-1}(t) > 0$ for $t > t_0$. Otherwise, there exists some $t_1 > t_0$ such that $S_{n-1}(t_1) < 0$. By (1.3) and (H1), $r(t)S_{n-1}^\gamma(t)$ is strictly decreasing on $[t_0,\infty)_\mathbb{T}$. It follows that $r^{\frac{1}{\gamma}}(t)(-S_{n-1}(t))$ is positive and strictly increasing on $[t_1,\infty)_\mathbb{T}$. Thus,

$$\begin{aligned} S_{n-2}(t) &= S_{n-2}(t_1) - \int_{t_1}^{t} \frac{r^{\frac{1}{\gamma}}(s)(-S_{n-1}(s))}{r^{\frac{1}{\gamma}}(s)a_{n-1}(s)}\Delta s \\ &\le S_{n-2}(t_1) - r^{\frac{1}{\gamma}}(t_1)(-S_{n-1}(t_1)) \int_{t_1}^{t} \frac{1}{r^{\frac{1}{\gamma}}(s)a_{n-1}(s)}\Delta s. \end{aligned}$$

By (H3), we have $\lim_{t\to\infty} S_{n-2}(t) = -\infty$. From Lemma 2.2, we get $\lim_{t\to\infty} S_0(t) = -\infty$, i.e., $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the fact that $x(t) > 0$ for $t \ge t_0$. Consequently, $S_{n-1}(t) > 0$ for $t > t_0$. Then we get two cases as follows:

(i) $S_i(t) > 0,\ 0 \le i \le n-1$;

(ii) there exists an integer $j \in [1,\ n-2]$ such that $S_j(t) < 0$.

For case (ii), let m be the smallest integer $m \in [0, n-1]$ with $m+n-1$ even such that $(-1)^{m+i}S_i(t) > 0$ for $t \ge t_0$ and $m \le i \le n-1$. Note that $S_{m-1}^\Delta(t) = \frac{S_m(t)}{a_m(t)} > 0$ for $t \ge t_0$. So, when $m > 1$, we have that either $S_{m-1}(t) < 0$ for $t \ge t_0$ or there is $t_2 \in \mathbb{T}$ such that $S_{m-1}(t) \ge S_{m-1}(t_2) > 0$ for $t \in \ge t_2$.

If $S_{m-1}(t) < 0$ for $t \ge t_0$, then using the above arguments similar to the case of $S_{n-1}(t) < 0$, we have $S_{m-2}(t) > 0$ for $t \ge t_0$, which is a contradiction to the definition of m.

If $S_{m-1}(t) \ge S_{m-1}(t_2) > 0$ for $t \ge t_2$, then from (1) of Lemma 2.2 we have $\lim_{t\to\infty} S_i(t) = \infty$ for $0 \le i \le m-1$. This completes the proof. □

Lemma 2.4 Assume that (2.1) holds and one of the following two conditions is satisfied:

$$\int_{t_0}^{\infty} \sum_{i=0}^{k} q_i(s)\Delta s = \infty \tag{2.2}$$

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or

$$\int_{t_0}^{\infty}\Big[\Big(\int_{u_2}^{\infty}\frac{(\int_s^{\infty}\sum_{i=0}^{k}q_i(u_1)\Delta u_1)^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)a_{n-1}(s)}\Delta s\Big)\Big/a_{n-2}(u_2)\Big]\Delta u_2=\infty. \tag{2.3}$$

Let $x(t)$ be an eventually positive solution of $Eq.(1.3)$. Then there exists $T\in\mathbb{T}$ such that, for any $t\geq T$, $(r(t)\Phi_\gamma(S_{n-1}(t)))^{\Delta}<0$, and the following statements hold:

(1) when n is even,

$$S_j(t)>0,\quad j=1,2,\cdots,n-1. \tag{2.4}$$

(2) when n is odd, either (2.4) holds or $\lim_{t\to\infty}x(t)=0$.

Proof Let $x(t)$ be an eventually positive solution of $Eq.(1.3)$, then by (H4), we may assume that $x(t)>0$ and $x(\delta_i(t))>0$ $(i=0,1,\cdots,k)$ for $t\geq t_0$. By $Eq.(1.3)$ and (H1), we have

$$(r(t)\Phi_\gamma(S_{n-1}(t)))^{\Delta}=(r(t)(S_{n-1}(t))^{\gamma})^{\Delta}<0,\quad t\geq t_0.$$

From Lemma 2.3, there exists $t_1\in\mathbb{T}$ such that

$$S_j(t)>0\ \text{ for }\ t\geq t_1 \text{ and } j\in[1,m-1], \tag{2.5}$$

$$(-1)^{m+j}S_j(t)>0\ \text{ for }\ t\geq t_1 \text{ and } j\in[m,n-1]. \tag{2.6}$$

When n is even, by Lemma 2.3, m must be an odd number. By (2.5), $x^{\Delta}(t)=\frac{S_1(t)}{a_1(t)}>0$. Hence, $\lim_{t\to\infty}x(t)$ exists, and it's positive or $\lim_{t\to\infty}x(t)=\infty$. In this case, we will show that $m=n-1$. Otherwise, the odd integer $m\leq n-3$. Now (2.6) implies that, for any $t\geq t_1$,

$$S_{n-2}(t)<0\ \text{ and }\ S_{n-3}(t)>0.$$

Note that there exist $T\geq t_1$ and $a>0$ such that $x(t)\geq a$ and $x(\delta_i(t))\geq a$ $(i=0,1,\cdots,k)$ for $t\geq T$. Taking $b:=\min_{0\leq i\leq k}\{a^{\alpha_i}\}$, we have

$$(r(t)(S_{n-1}(t))^{\gamma})^{\Delta}\leq-\sum_{i=0}^{k}q_i(t)a^{\alpha_i}\leq-b\sum_{i=0}^{k}q_i(t). \tag{2.7}$$

If (2.2) holds, integrating (2.7) from T to t with $t\geq T$, we obtain that

$$r(t)(S_{n-1}(t))^{\gamma}\leq r(T)(S_{n-1}(T))^{\gamma}-b\int_T^t\sum_{i=0}^{k}q_i(s)\Delta s\to-\infty,\quad \text{as } t\to\infty,$$

which contradicts the fact that $S_{n-1}(t)>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. Hence, (2.4) holds. If (2.3) holds, integrating (2.7) from t to u_1 with $T\leq t\leq u_1$, we obtain that

$$r(t)(S_{n-1}(t))^{\gamma}\geq r(u_1)(S_{n-1}(u_1))^{\gamma}+b\int_t^{u_1}\sum_{i=0}^{k}q_i(s)\Delta s\geq b\int_t^{u_1}\sum_{i=0}^{k}q_i(s)\Delta s.$$

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Taking $u_1 \to \infty$, we have

$$S_{n-1}(t) \geq b^{\frac{1}{\gamma}} \frac{(\int_t^\infty \sum_{i=0}^k q_i(s)\Delta s)^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(t)} = b^{\frac{1}{\gamma}} \left(r^{-1}(t) \int_t^\infty \sum_{i=0}^k q_i(s)\Delta s \right)^{\frac{1}{\gamma}}. \tag{2.8}$$

Since $S_{n-2}(t) < 0$, integrating (2.8) from t to u_2 with $T \leq t \leq u_2$, we have

$$-S_{n-2}(t) \geq S_{n-2}(u_2) - S_{n-2}(t) \geq b^{\frac{1}{\gamma}} \int_t^{u_2} \frac{(\int_t^\infty \sum_{i=0}^k q_i(u_1)\Delta u_1)^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s) a_{n-1}(s)} \Delta s.$$

Taking $u_2 \to \infty$, we have

$$-S_{n-2}(t) \geq b^{\frac{1}{\gamma}} \int_t^\infty \frac{(\int_t^\infty \sum_{i=0}^k q_i(u_1)\Delta u_1)^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s) a_{n-1}(s)} \Delta s. \tag{2.9}$$

Since $S_{n-3}(t) > 0$, integrating (2.9) from T to t with $t \geq T$, we get

$$S_{n-3}(T) \geq -S_{n-3}(t) + S_{n-3}(T) \geq b^{\frac{1}{\gamma}} \int_T^t \left[\left(\int_t^\infty \frac{(\int_t^\infty \sum_{i=0}^k q_i(u_1)\Delta u_1)^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s) a_{n-1}(s)} \Delta s \right) \Big/ a_{n-2}(u_2) \right] \Delta u_2.$$

Let $t \to \infty$, we obtain

$$\int_T^\infty \left[\left(\int_t^\infty \frac{(\int_t^\infty \sum_{i=0}^k q_i(u_1)\Delta u)^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s) a_{n-1}(s)} \Delta s \right) \Big/ a_{n-2}(u_2) \right] \Delta u_2 \leq b^{-\frac{1}{\gamma}} S_{n-3}(T) < \infty,$$

which contradicts (2.3). Hence, $m = n-1$ and (2.4) holds.

When n is odd, by Lemma 2.3, m must be even. From (2.5) and (2.6), either $x^\Delta(t) > 0$ or $x^\Delta(t) < 0$ hold, which implies $\lim_{t\to\infty} x(t) = c \geq 0$. If $c > 0$, we claim that $m = n-1$. Otherwise, $m \leq n-3$. Similar as above, we can arrive a contradiction. This completes the proof. □

In the sequel, for any $n \in \mathbb{N}$ and $t, T \in \mathbb{T}$ with $t \geq T$, we define $\beta_k(t, T)$ as follows:

$$\beta_i(t, T) = \int_T^t \frac{\beta_{i-1}(s, T)}{a_{n-i}(s)} \Delta s, \quad i = 1, 2, \cdots, k,$$

where $\beta_0(t, T) = \frac{1}{r^\gamma(t)}$.

Lemma 2.5 Suppose that (2.1) and either (2.2) or (2.3) hold. Let $x(t)$ be an eventually positive solution of $Eq.$(1.3) satisfied (2.4). Then there exists $T \in \mathbb{T}$ such that, for $t \in [T, \infty)_\mathbb{T}$,

$$S_1(t) \geq r^{\frac{1}{\gamma}}(t) S_{n-1}(t) \beta_{n-2}(t, T) \quad \text{and} \quad x(t) \geq r^{\frac{1}{\gamma}}(t) S_{n-1}(t) \beta_{n-1}(t, T).$$

Proof By the hypothesis and (H4), there exists $T \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0,\ x(\delta_i(t)) > 0$ $(i = 0, 1, \cdots, k)$ and (2.4) hold for $t \geq T$. It's easy to see, from Lemma 2.4, that $r^{\frac{1}{\gamma}}(t)S_{n-1}(t)$ is decreasing on $[T, \infty)_{\mathbb{T}}$. From (2.4), we have

$$\begin{aligned} S_{n-2}(t) &= S_{n-2}(T) + \int_T^t \frac{S_{n-1}(s)}{a_{n-1}(s)}\Delta s \\ &= S_{n-2}(T) + \int_T^t \frac{r^{\frac{1}{\gamma}}(s)S_{n-1}(s)}{r^{\frac{1}{\gamma}}(s)a_{n-1}(s)}\Delta s \\ &\geq r^{\frac{1}{\gamma}}(t)S_{n-1}(t)\int_T^t \frac{1}{r^{\frac{1}{\gamma}}(s)a_{n-1}(s)}\Delta s \\ &= r^{\frac{1}{\gamma}}(t)S_{n-1}(t)\beta_1(t, T). \end{aligned}$$

Integrating above inequality from T to t for $t \geq T$, we have

$$\begin{aligned} S_{n-3}(t) &= S_{n-3}(T) + \int_T^t \frac{S_{n-2}(s)}{a_{n-2}(s)}\Delta s \\ &\geq r^{\frac{1}{\gamma}}(t)S_{n-1}(t)\int_T^t \frac{\beta_1(s, T)}{a_{n-2}(s)}\Delta s \\ &= r^{\frac{1}{\gamma}}(t)S_{n-1}(t)\beta_2(t, T). \end{aligned}$$

By induction, we can show that

$$S_1(t) \geq r^{\frac{1}{\gamma}}(t)S_{n-1}(t)\beta_{n-2}(t, T), x(t) \geq r^{\frac{1}{\gamma}}(t)S_{n-1}(t)\beta_{n-1}(t, T).$$

This completes the proof. □

Lemma 2.6[3] Let $g(y) = By - Ay^{\frac{\gamma+1}{\gamma}}$, where A, B and y are positive numbers. Then $g(y)$ attains its maximum value on $[0, \infty)$ at $y^* = (\frac{B\gamma}{A(\gamma+1)})^\gamma$, and

$$\max_{y\in[0,\infty)} g(y) = g(y^*) = \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}}{A^\gamma}. \qquad \square$$

3. Main Results

For the convenience of presentation, we give some notations. Write $\mathbb{D} = \{(t, s) \in \mathbb{T}^2 :\ t \geq s \geq 0\}$, for any $z \in C^1_{rd}(\mathbb{T}, (0, \infty))$, we denote

$$\mathcal{H} = \{H(t, s) \in C^1_{rd}(\mathbb{D}, [0, \infty)) : H^\Delta_s(t, s) \leq 0 \text{ and } H(t, s) = 0 \text{ iff } t = s\},$$

$$C(t, s) = H^\Delta_s(t, s)z^\sigma(s) + H(t, s)z^\Delta(s),$$

and

$$C_+(t,s) = \max\{H_s^\Delta(t,s)z^\sigma(s) + H(t,s)z_+^\Delta(s), 0\},$$

where $H(t,s) \in \mathcal{H}$ and $z_+^\Delta(s) = \max\{z^\Delta(s), 0\}$.

Theorem 3.1 Suppose that (2.1) and either (2.2) or (2.3) hold, and η_i $(i = 1, \cdots, k)$ are defined as in Lemma 2.1. Assume that for sufficiently large $T \in \mathbb{T}$, one of the following conditions is satisfied:

(C1) either

$$\int_T^\infty Q(s)\Delta s = \infty,$$

or

$$\int_T^\infty Q(s)\Delta s < \infty \quad \text{and} \quad \limsup_{t\to\infty} \beta_{n-1}^\gamma(\delta(t), T) \int_t^\infty Q(s)\Delta s > 1,$$

(C2) there is a function $z \in C_{rd}^1(\mathbb{T}, (0,\infty))$ such that

$$\limsup_{t\to\infty} \int_T^t \left[Q(s)z(s) - \frac{z_+^\Delta(s)}{\beta_{n-1}^\gamma(\delta^\sigma(s), T)}\right]\Delta s = \infty,$$

(C3) there is a function $z \in C_{rd}^1(\mathbb{T}, (0,\infty))$ such that

$$\limsup_{t\to\infty} \int_T^t \left[Q(s)z(s) - \frac{1}{(\gamma+1)^{\gamma+1}} \frac{a_1^\gamma(s)(z_+^\Delta(s))^{\gamma+1}}{z^\gamma(s)\beta_{n-2}^\gamma(\delta(s), T)(\delta^\Delta(s))^\gamma}\right]\Delta s = \infty,$$

(C4) there are a function $z \in C_{rd}^1(\mathbb{T}, (0,\infty))$ and $H \in \mathcal{H}$ such that

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \Bigg[H(t,s)Q(s)z(s) - \frac{a_1^\gamma(s)C_+^{\gamma+1}(t,s)}{H^\gamma(t,s)(\gamma+1)^{\gamma+1}z^\gamma(s)\beta_{n-2}^\gamma(\delta(s), T)(\delta^\Delta(s))^\gamma}\Bigg]\Delta s = \infty$$

where $Q(t) = q_0(t) + \prod_{i=1}^k \left(\eta_i^{-1}q_i(t)\right)^{\eta_i}$ and $\delta(t) = \min\{t, \delta_i(t),\ i = 0, 1, 2, \cdots, k\}$. Then

(i) every solution of $Eq.(1.3)$ either oscillates or tends to zero as $t \to \infty$ when n is odd,

(ii) every solution of $Eq.(1.3)$ is oscillatory when n is even.

Proof Assume that $Eq.(1.3)$ has a nonoscillatory solution $x(t)$. Without loss of generality, we may suppose that $x(t)$ is eventually positive. Then, Lemma 2.4 and Lemma 2.5 hold, and by (H4), there exists a sufficiently large $T \in \mathbb{T}$ such that $x(t) > 0$ and $x(\delta_i(t)) > 0$ $(i = 0, 1, \cdots, k)$ for $t \geq T$.

When n is odd, from Lemma 2.4, we have that (2.4) holds or $\lim_{t\to\infty} x(t) = 0$. If (2.4) holds, $Eq.(1.3)$ reduces to

$$(r(t)(S_{n-1}(t))^\gamma)^\Delta + \sum_{i=0}^k q_i(t)x^{\alpha_i}(\delta_i(t)) = 0. \tag{3.1}$$

We consider four parts corresponding conditions (C1)-(C4).

Part 1: Suppose that (C1) holds.

Write $\psi(t) = r(t)(S_{n-1}(t))^\gamma$, then $\psi(t) > 0$ and $\psi^\Delta(t) + \sum_{i=0}^k q_i(t)x^{\alpha_i}(\delta(t)) \leq 0$ for $t \geq T$. So we have $\psi^\Delta(t) < 0$ and $\lim_{t\to\infty}\psi(t) = b \geq 0$. Put $a_i = \eta_i^{-1} q_i(t)x^{\alpha_i}(\delta(t))$, by the arithmetic-geometric mean inequality [11], we get

$$\sum_{i=1}^k \eta_i a_i \geq \prod_{i=1}^k a_i^{\eta_i} \text{ and } a_i \geq 0.$$

So we have

$$\sum_{i=0}^k q_i(t)x^{\alpha_i}(\delta(t)) \geq q_0(t)x^\gamma(\delta(t)) + (\eta_i^{-1}q_i(t))^{\eta_i}x^\gamma(\delta(t)) = Q(t)x^\gamma(\delta(t)). \tag{3.2}$$

Consequently,

$$\psi^\Delta(t) + Q(t)x^\gamma(\delta(t)) \leq 0. \tag{3.3}$$

Integrating (3.3) from t to ∞, we have

$$b - \psi(t) + \int_t^\infty Q(s)x^\gamma(\delta(s))\Delta s \leq 0.$$

If $\int_t^\infty Q(s)\Delta s = \infty$, using (2.4), we can arrive a contradiction. If $\int_t^\infty Q(s)\Delta s < \infty$, then

$$\psi(\delta(t)) \geq \psi(t) \geq \int_t^\infty Q(s)x^\gamma(\delta(s))\Delta s \geq x^\gamma(\delta(t))\int_t^\infty Q(s)\Delta s.$$

From Lemma 2.5, we have

$$\beta_{n-1}^\gamma(\delta(t), T)\int_t^\infty Q(s)\Delta s \leq 1,$$

which contradicts to (C1). Therefore, every solution of $Eq.$(1.3) either oscillates or tends to zero as $t \to \infty$.

Part 2: Suppose that (C2) holds.

Define

$$w(t) = \frac{z(t)r(t)(S_{n-1}(t))^\gamma}{x^\gamma(\delta(t))}, \qquad t \geq T. \tag{3.4}$$

We get $w(t) > 0$ immediately. From Lemma 2.5, we have

$$\begin{aligned} w^\Delta(t) &= \Big(r(t)(S_{n-1}(t))^\gamma\Big)^\Delta\left(\frac{z(t)}{x^\gamma(\delta(t))}\right) + \Big(r(t)(S_{n-1}(t))^\gamma\Big)^\sigma\left(\frac{z(t)}{x^\gamma(\delta(t))}\right)^\Delta \\ &= \Big(r(t)(S_{n-1}(t))^\gamma\Big)^\sigma\left[\frac{z^\Delta(t)x^\gamma(\delta(t)) - z(t)(x^\gamma(\delta(t)))^\Delta}{x^\gamma(\delta(t))x^\gamma(\delta^\sigma(t))}\right] \end{aligned}$$

$$\begin{aligned}
&+ z(t)\frac{-\sum_{i=0}^{k} q_i(t)x^{\alpha_i}(\delta_i(t))}{x^\gamma(\delta(t))}\\
\le\; &\frac{z_+^\Delta(t)(r(t)(S_{n-1}(t))^\gamma)^\sigma}{x^\gamma(\delta^\sigma(t))} - z(t)\frac{\sum_{i=0}^{k} q_i(t)x^{\alpha_i}(\delta(t))}{x^\gamma(\delta(t))}\\
&- \frac{(r(t)(S_{n-1}(t))^\gamma)^\sigma z(t)(x^\gamma(\delta(t)))^\Delta}{x^\gamma(\delta(t)x^\gamma(\delta^\sigma(t)))}. \qquad (3.5)
\end{aligned}$$

Noting that $x^\Delta(t) = \frac{S_1(t)}{a_1(t)} > 0$. When $\gamma \ge 1$, by Keller's chain rule, we have

$$\begin{aligned}
(x^\gamma(t))^\Delta &= \gamma\Big[\int_0^1 (x(t) + h\mu(t)x^\Delta(t))^{\gamma-1}dh\Big]x^\Delta(t)\\
&\ge \gamma x^\Delta(t)\int_0^1 ((1-h)x(t) + hx(t))^{\gamma-1}dh = \gamma x^{\gamma-1}(t)x^\Delta(t).
\end{aligned}$$

So $(x^\gamma(\delta(t)))^\Delta \ge \gamma x^{\gamma-1}(\delta(t))(x(\delta(t)))^\Delta$. It's easy to see, from (H4), that $\delta^\Delta(t) > 0$ for $t \in \mathbb{T}$. Hence, from [12, Theorem 1.93], we have

$$(x^\gamma(\delta(t)))^\Delta \ge \gamma x^{\gamma-1}(\delta(t))x^\Delta(\delta(t))\delta^\Delta(t) \ge 0. \qquad (3.6)$$

When $0 < \gamma < 1$,

$$(x^\gamma(t))^\Delta \ge \gamma x^\Delta(t)\int_0^1 ((1-h)x^\sigma(t) + hx^\sigma(t))^{\gamma-1}dh = \gamma(x^\sigma(t))^{\gamma-1}x^\Delta(t).$$

So $(x^\gamma(\delta(t)))^\Delta \ge \gamma x^{\gamma-1}(\delta^\sigma(t))(x(\delta(t)))^\Delta$. From [12, Theorem 1.93], we have

$$(x^\gamma(\delta(t)))^\Delta \ge \gamma x^{\gamma-1}(\delta^\sigma(t))x^\Delta(\delta(t))\delta^\Delta(t) \ge 0. \qquad (3.7)$$

Noting that $r(t) > 0$. From (3.6) and (3.7), we get

$$\frac{(r(t)S_{n-1}^\gamma(t))^\sigma z(t)(x^\gamma(\delta(t)))^\Delta}{x^\gamma(\delta(t))x^\gamma(\delta^\sigma(t))} \ge 0.$$

Since $(r(t)S_{n-1}^\gamma(t))^\Delta < 0,\ \delta(t) \le t \le \sigma(t)$ and $\delta^\sigma(t) \le \sigma(t)$, we have

$$r(\sigma(t))S_{n-1}^\gamma(\sigma(t)) \le r(t)S_{n-1}^\gamma(t) \le r(\delta(t))S_{n-1}^\gamma(\delta(t)). \qquad (3.8)$$

and

$$r(\sigma(t))S_{n-1}^\gamma(\sigma(t)) \le r(\delta^\sigma(t))S_{n-1}^\gamma(\delta^\sigma(t)). \qquad (3.9)$$

Hence, from (3.2), (3.9), Lemma 2.5 and $x^\Delta(t) > 0$, we have

$$w^\Delta(t) \le -z(t)Q(t) + \frac{z_+^\Delta(t)}{\beta_{n-1}^\gamma(\delta^\sigma(t), T)}.$$

Integrating the above inequality from T to t for $t \geq T$, we get

$$\int_T^t \left[z(s)Q(s) - \frac{z_+^\Delta(s)}{\beta_{n-1}^\gamma(\delta^\sigma(s), T)} \right] \Delta s \leq w(T) - w(t) < w(T).$$

Taking the lim sup on both sides as $t \to \infty$, we obtain a contradiction to (C2). Therefore, every solution of $Eq.$(1.3) either oscillates or tends to zero as $t \to \infty$.

Part 3: Suppose that (C3) holds.

When $\gamma \geq 1$, from (2.4), (3.5) and (3.6), we have

$$w^\Delta(t) \leq -z(t)Q(t) + \frac{z_+^\Delta(t)}{z^\sigma(t)} w^\sigma(t) - (r(t)S_{n-1}^\gamma(t))^\sigma \frac{z(t)\gamma x^\Delta(\delta(t))\delta^\Delta(t)}{x^{\gamma+1}(\delta^\sigma(t))}. \tag{3.10}$$

Since $x^\Delta(t) = \frac{S_1(t)}{a_1(t)} \geq \frac{r^{\frac{1}{\gamma}(t)} S_{n-1}(t)\beta_{n-2}(t,T))}{a_1(t)}$, from (3.8) we get

$$\begin{aligned}
-(r(t)S_{n-1}^\gamma(t))^\sigma \frac{z(t)\gamma x^\Delta(\delta(t))\delta^\Delta(t)}{x^{\gamma+1}(\delta^\sigma(t))} &= \frac{-(r^\sigma(t))^{\frac{\gamma+1}{\gamma}} S_{n-1}^{\gamma+1}(t)}{x^{\gamma+1}(\delta^\sigma(t))} \frac{z(t)\gamma x^\Delta(\delta(t))\delta^\Delta(t)}{r^{\frac{1}{\gamma}}(\sigma(t))S_{n-1}(\sigma(t))} \\
&\leq \frac{-(r^\sigma(t))^{\frac{\gamma+1}{\gamma}} S_{n-1}^{\gamma+1}(t)}{x^{\gamma+1}(\delta^\sigma(t))} \frac{z(t)\gamma x^\Delta(\delta(t))\delta^\Delta(t)}{r^{\frac{1}{\gamma}}(\delta(t))S_{n-1}(\delta(t))} \\
&\leq -\frac{z(t)\gamma\beta_{n-2}(t,T)\delta^\Delta(t)}{a_1(t)z^{\frac{\gamma+1}{\gamma}}(\sigma(t))} w^{\frac{\gamma+1}{\gamma}}(\sigma(t)).
\end{aligned}$$

Then

$$w^\Delta(t) \leq -z(t)Q(t) + \frac{z_+^\Delta(t)}{z^\sigma(t)} w^\sigma(t) - \frac{z(t)\gamma\beta_{n-2}(t,T)\delta^\Delta(t)}{a_1(t)z^{\frac{\gamma+1}{\gamma}}(\sigma(t))} w^{\frac{\gamma+1}{\gamma}}(\sigma(t)). \tag{3.11}$$

When $0 < \gamma < 1$, by (3.2), (3.5) and (3.7), we have

$$w^\Delta(t) \leq -z(t)Q(t) + \frac{z_+^\Delta(t)}{z^\sigma(t)} w^\sigma(t) - (r(t)S_{n-1}^\gamma(t))^\sigma \frac{z(t)\gamma(x(\delta^\sigma(t)))^{\gamma-1} x^\Delta(\delta(t))\delta^\Delta(t)}{x^\gamma(\delta(t))(x(\delta^\sigma(t)))^\gamma}.$$

Then, by (3.8) and Lemma 2.5, we have

$$\begin{aligned}
&-(r(t)S_{n-1}^\gamma(t))^\sigma \frac{z(t)\gamma(x(\delta^\sigma(t)))^{\gamma-1} x^\Delta(\delta(t))\delta^\Delta(t)}{x^\gamma(\delta(t))(x(\delta^\sigma(t)))^\gamma} \\
= \; & \frac{-(r^\sigma(t))^{\frac{\gamma+1}{\gamma}} S_{n-1}^{\gamma+1}(t)}{x^\gamma(\delta(t))x(\delta^\sigma(t))} \frac{z(t)\gamma x^\Delta(\delta(t))\delta^\Delta(t)}{r^{\frac{1}{\gamma}}(\sigma(t))S_{n-1}(\sigma(t))} \\
\leq \; & \frac{-(r^\sigma(t))^{\frac{\gamma+1}{\gamma}} S_{n-1}^{\gamma+1}(t)}{x^\gamma(\delta(t))x(\delta^\sigma(t))} \frac{z(t)\gamma x^\Delta(\delta(t))\delta^\Delta(t)}{r^{\frac{1}{\gamma}}(\delta(t))S_{n-1}(\delta(t))} \\
\leq \; & -\frac{z(t)\gamma\beta_{n-2}(t,T)\delta^\Delta(t)}{a_1(t)z^{\frac{\gamma+1}{\gamma}}(\sigma(t))} w^{\frac{\gamma+1}{\gamma}}(\sigma(t)).
\end{aligned}$$

It follows that

$$w^{\Delta}(t) \leq -z(t)Q(t) + \frac{z_{+}^{\Delta}(t)}{z^{\sigma}(t)}w^{\sigma}(t) - \frac{z(t)\gamma\beta_{n-2}(t,T)\delta^{\Delta}(t)}{a_1(t)z^{\frac{\gamma+1}{\gamma}}(\sigma(t))}w^{\frac{\gamma+1}{\gamma}}(\sigma(t)). \tag{3.12}$$

Let

$$B = \frac{z_{+}^{\Delta}(t)}{z^{\sigma}(t)}, \quad A = \frac{z(t)\gamma\beta_{n-2}(t,T)\delta^{\Delta}(t)}{a_1(t)z^{\frac{\gamma+1}{\gamma}}(\sigma(t))}, \quad y = w^{\sigma}(t).$$

Then, by Lemma 2.6 and (3.12), for all $t \geq T$,

$$w^{\Delta}(t) \leq -z(t)Q(t) + \frac{1}{(\gamma+1)^{\gamma+1}}\frac{a_1^{\gamma}(t)(z_{+}^{\Delta}(t))^{\gamma+1}}{z^{\gamma}(t)\beta_{n-2}^{\gamma}(t,T)(\delta^{\Delta}(t))^{\gamma}}.$$

Integrating the above inequality from T to t for $t \geq T$, we get

$$\int_T^t \left[z(s)Q(s) - \frac{1}{(\gamma+1)^{\gamma+1}}\frac{a_1^{\gamma}(s)(z_{+}^{\Delta}(s))^{\gamma+1}}{z^{\gamma}(s)\beta_{n-2}^{\gamma}(s,T)(\delta^{\Delta}(s))^{\gamma}}\right]\Delta s \leq w(T) - w(t) < w(T).$$

By taking the lim sup on both sides as $t \to \infty$, we obtain a contradiction to (C3). Therefore, every solution of $Eq.$(1.3) either oscillates or tends to zero as $t \to \infty$.

Part 4: Suppose that (C4) holds.

From (3.11) and (3.12), we have that for $H \in \mathcal{H}$ and $t \geq T$

$$\begin{aligned}
\int_T^t H(t,s)z(s)Q(s)\Delta s \;&\leq\; -\int_T^t H(t,s)w^{\Delta}(s)\Delta s + \int_T^t H(t,s)w^{\sigma}(s)\frac{z_{+}^{\Delta}(s)}{z^{\sigma}(s)}\Delta s \\
&\quad - \int_T^t H(t,s)\frac{z(s)\gamma\beta_{n-2}(\delta(s),T)\delta^{\Delta}(s)}{a_1(s)z^{\frac{\gamma+1}{\gamma}}(\sigma(s))}w^{\frac{\gamma+1}{\gamma}}(\sigma(s))\Delta s \\
&\leq\; H(t,T)w(T) + \int_T^t \left[H_s^{\Delta}(t,s) + H(t,s)\frac{z_{+}^{\Delta}(s)}{z^{\sigma}(s)}\right]w^{\sigma}(s)\Delta s \\
&\quad - \int_T^t H(t,s)\frac{z(s)\gamma\beta_{n-2}(\delta(s),T)\delta^{\Delta}(s)}{a_1(s)z^{\frac{\gamma+1}{\gamma}}(\sigma(s))}w^{\frac{\gamma+1}{\gamma}}(\sigma(s))\Delta s \\
&\leq\; H(t,T)w(T) + \int_T^t \left[C_{+}(t,s)\right]w^{\sigma}(s)\Delta s \\
&\quad - \int_T^t H(t,s)\frac{z(s)\gamma\beta_{n-2}(\delta(s),T)\delta^{\Delta}(s)}{a_1(s)z^{\frac{\gamma+1}{\gamma}}(\sigma(s))}w^{\frac{\gamma+1}{\gamma}}(\sigma(s))\Delta s.
\end{aligned}$$

Let

$$B = C_{+}(t,s), \quad A = H(t,s)\frac{z(s)\gamma\beta_{n-2}(\delta(s),T)\delta^{\Delta}(s)}{a_1(s)z^{\frac{\gamma+1}{\gamma}}(\sigma(s))}, \quad y = w^{\sigma}(s).$$

By Lemma 2.6, for all $t \geq T$,

$$\begin{aligned}\int_T^t H(t,s)z(s)Q(s)\Delta s &\leq H(t,T)w(T)\\ &\quad + \int_T^t \frac{[C_+(t,s)]^{\gamma+1}(z^\sigma(s))^{\gamma+1}a_1^\gamma(s)}{\beta_{n-2}^\gamma(\delta(s),T)(\delta^\Delta(s))^\gamma H^\gamma(t,s)(\gamma+1)^{\gamma+1}z^\gamma(s)}\Delta s,\end{aligned}$$

i.e.,

$$\begin{aligned}w(T) &\geq \frac{1}{H(t,T)}\int_T^t \Bigg[H(t,s)z(s)Q(s)\\ &\quad - \frac{[C_+(t,s)]^{\gamma+1}a_1^\gamma(s)}{\beta_{n-2}^\gamma(\delta(s),T)(\delta^\Delta(s))^\gamma H^\gamma(t,s)(\gamma+1)^{\gamma+1}z^\gamma(s)}\Bigg]\Delta s.\end{aligned}$$

Taking the lim sup on both sides as $t \to \infty$, we obtain a contradiction to (C4). Therefore, every solution of $Eq.$(1.3) either oscillates or tends to zero as $t \to \infty$.

When n is even. From Lemma 2.4, (2.4) holds. Similar to above four parts, we can show that every solution of $Eq.$(1.3) is oscillatory. This completes the proof. □

Theorem 3.2 Let $\gamma \geq 1$. Suppose that (2.1) and either (2.2) or (2.3) hold. For sufficiently large $T \in \mathbb{T}$ and $z \in C_{rd}^1(\mathbb{T},(0,\infty))$, one of the following two conditions is satisfied:

(I) $\limsup_{t\to\infty} \int_T^t \left[z(s)Q(s) - \frac{a_1(s)(z^\Delta(s))^2}{4\gamma\beta^*(\delta(s),T)z(s)\delta^\Delta(s)}\right]\Delta s = \infty,$

(II) there exists $H \in \mathcal{H}$ such that

$$\limsup_{t\to\infty} \frac{1}{H(t,T)}\int_T^t \left[H(t,s)z(s)Q(s) - \frac{a_1(s)C^2(t,s)}{4\gamma z(s)\delta^\Delta(s)\beta^*(\delta(s),T)H(t,s)}\right]\Delta s = \infty,$$

where $\beta^*(t,T) = \beta_{n-1}^{\gamma-1}(t,T)\beta_{n-2}(t,T)$. Then

(i) every solution of $Eq.$(1.3) either oscillates or tends to zero as $t \to \infty$ when n is odd,

(ii) every solution of $Eq.$(1.3) is oscillatory when n is even.

Proof Assume that $Eq.$(1.3) has a non-oscillatory solution $x(t)$. Without loss of generality, we may suppose that $x(t)$ is eventually positive. Then, Lemma 2.4 and Lemma 2.5 hold, and by (H4) there exists a sufficiently large $T \in [t_0,\infty)$ such that $x(t) > 0$ and $x(\delta_i(t)) > 0$ ($i = 0,1,\cdots,k$) for $t \geq T$.

When n is odd, from Lemma 2.4 we see that (2.4) holds or $\lim_{t\to\infty} x(t) = 0$. If (2.4) holds, we separate the rest of the proof into two parts.

Part one: Suppose that (I) holds.

Define $w(t)$ as in (5.4). Because $x(t) > 0$, $\sigma(t) \geq t$, by (3.2), (3.5) and (3.6), we get that

$$w^\Delta(t) \leq -z(t)Q(t) + \frac{z^\Delta(t)}{z^\sigma(t)}w^\sigma(t) - (r(t)S_{n-1}^\gamma(t))^\sigma \frac{z(t)\gamma x^{\gamma-1}(\delta(t))x^\Delta(\delta(t))\delta^\Delta(t)}{x^\gamma(\delta^\sigma(t))x^\gamma(\delta(t))}$$

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$$\leq \quad -z(t)Q(t)+\frac{z^{\Delta}(t)}{z^{\sigma}(t)}w^{\sigma}(t)-\frac{z(t)\gamma x^{\gamma-1}(\delta(t))x^{\Delta}(\delta(t))\delta^{\Delta}(t)}{(r(t)S_{n-1}^{\gamma}(t))^{\sigma}(z^{\sigma}(t))^2}(w^{\sigma}(t))^2.$$

From (3.8) and Lemma 2.4, we get

$$\begin{aligned} w^{\Delta}(t) &\leq -z(t)Q(t)+\frac{z^{\Delta}(t)}{z^{\sigma}(t)}w^{\sigma}(t)-\frac{z(t)\delta^{\Delta}(t)}{(z^{\sigma}(t))^2 r(\sigma(t))}\frac{x^{\gamma-1}(\delta(t))}{S_{n-1}^{\gamma-1}(\delta(t))}\frac{x^{\Delta}(\delta(t))}{S_{n-1}(\delta(t))}(w^{\sigma}(t))^2 \\ &\leq -z(t)Q(t)+\frac{z^{\Delta}(t)}{z^{\sigma}(t)}w^{\sigma}(t)-\frac{z(t)\delta^{\Delta}(t)}{a_1(t)(z^{\sigma}(t))^2}\beta^*(\delta(t),T)(w^{\sigma}(t))^2. \end{aligned}$$

By completing the square for $w^{\sigma}(t)$ on the right-hand side, we have

$$w^{\Delta}(t)\leq -z(t)Q(t)+\frac{a_1(t)(z^{\Delta}(t))^2}{4\gamma\beta^*(\delta(t),T)z(t)\delta^{\Delta}(t)}.$$

Integrating the above inequality from T to t for $t\geq T$, we get

$$\int_T^t\left[z(s)Q(s)-\frac{a_1(s)(z^{\Delta}(s))^2}{4\gamma\beta^*(\delta(s),T)z(s)\delta^{\Delta}(s)}\right]\Delta s\leq w(T)-w(t)<w(T).$$

Taking the lim sup on both sides as as $t\to\infty$, we obtain a contradiction to (I). Therefore, every solution of $Eq.$(1.3) either oscillates or tends to zero as $t\to\infty$.

Part two: Suppose that (II) holds.

The proof is similar to Part 4 of Theorem 3.1 and Part one of Theorem 3.2.

When n is even. From Lemma 2.4, (2.4) holds. Similarly, we can show that every solution of $Eq.$(1.3) is oscillatory. The proof is completed. □

4. Examples

Example 4.1 Consider the equation

$$\left(t^{-1}\Phi_{\frac{1}{2}}(S_n(t))\right)^{\Delta}+t^{-\frac{1}{2}}\Phi_{\frac{1}{2}}(x(t+1))+t^{-2}\Phi_{\frac{19}{3}}(x(t-1))+t^{-3}\Phi_{\frac{1}{3}}(x(t+2))=0,\quad t\in\mathbb{T}$$

where $S_n(t)$ satisfies $Eq.$(1.3), $a_i(t)=t^{-1}$ $(1\leq i\leq n)$, $\mathbb{T}=[2,\infty)_{\mathbb{R}}$. Here, we have

(1) $n\geq 2,\ \gamma=\alpha_0=\frac{1}{2},\ \alpha_1=\frac{19}{3},\ \alpha_2=\frac{1}{3}$;

(2) $r(t)=t^{-1},\ q_0(t)=t^{-\frac{1}{2}},\ q_1(t)=t^{-2},\ q_2(t)=t^{-3}$;

(3) $\delta_0(t)=t+1,\ \delta_1(t)=t-1$ and $\delta(t)=t-1$.

Clearly,

$$\int_{t_0}^{\infty}\frac{1}{r^{1/\gamma}(t)a_n(t)}\Delta t=\int_2^{\infty}t^3\Delta t=\infty,$$

$$\int_{t_0}^{\infty}\frac{1}{a_i(t)}\Delta t=\int_2^{\infty}t\Delta t=\infty,\quad 1\leq i\leq n,$$

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hence (H1)-(H4) hold. Note that

$$\int_2^\infty \sum_{i=0}^{2} q_0(s)\Delta s = \int_2^\infty \left(s^{-\frac{1}{2}} + s^{-2} + s^{-3}\right)\Delta s \geq \int_2^\infty s^{-\frac{1}{2}}\Delta s = \infty,$$

i.e., (2.2) holds. Let $\eta_1 = \frac{1}{36},\ \eta_2 = \frac{35}{36}$. Then η_1, η_2 satisfies Lemma 2.1. With $z(t) = 1$ we see that for sufficiently large $T \in \mathbb{T}$,

$$\begin{aligned}
&\limsup_{t\to\infty} \int_T^t \left[Q(s)z(s) - \frac{1}{(\gamma+1)^{\gamma+1}} \frac{a_1^\gamma(s)(z_+^\Delta(s))^{\gamma+1}}{z^\gamma(s)\beta_{n-2}^\gamma(\delta(s),T)(\delta^\Delta(s))^\gamma}\right]\Delta s \\
= &\limsup_{t\to\infty} \int_T^t \left[s^{-\frac{1}{2}} + \left(36s^{-2}\right)^{\frac{1}{36}} \left(\frac{36}{35}s^{-3}\right)^{\frac{35}{36}}\right]\Delta s \\
\geq &\limsup_{t\to\infty} \int_T^t s^{-\frac{1}{2}}\Delta s = \infty.
\end{aligned}$$

Hence, the condition (C3) of Theorem 3.1 is satisfied.

By Theorem 3.1, every solution of $Eq.$(1.3) either oscillates or tends to zero as $t \to \infty$ when n is odd and oscillatory when n is even. □

Example 4.2 Consider the equation

$$\left(t\Phi_2(S_n(t))\right)^\Delta + \Phi_2(x(t)) + 2\Phi_3(x(t-1)) + 3t^{-3}\Phi_{\frac{3}{2}}(x(t+2)) = 0, \quad t \in \mathbb{T}$$

where $S_n(t)$ satisfies $Eq.$(1.3), $\mathbb{T} = [1,\infty)_\mathbb{R},\ a_i(t) = t^{-4}(1 \leq i \leq n)$. Here, we have

(1) $n \geq 2,\ k = 2,\ \gamma = \alpha_0 = 3,\ \alpha_1 = 3,\ \alpha_2 = \frac{3}{2}$;

(2) $r(t) = t,\ q_0(t) = 1,\ q_1(t) = 2,\ q_2(t) = 3t^{-3}$;

(3) $\delta_0(t) = t,\ \delta_1(t) = t-1,\ \delta_2(t) = t+2$ and $\delta(t) = t-1$.

Clearly,

$$\int_{t_0}^\infty \frac{1}{r^{1/\gamma}(t)a_n(t)}\Delta t = \int_1^\infty t^{\frac{7}{2}}\Delta t = \infty,$$

$$\int_{t_0}^\infty \frac{1}{a_i(t)}\Delta t = \int_1^\infty t^4\Delta t = \infty, (1 \leq i \leq n)$$

hence (H1)-(H4) hold. Note that

$$\begin{aligned}
&\int_1^\infty \left[\left(\int_v^\infty \frac{(\int_s^\infty \sum_{i=0}^k q_i(u)\Delta u)^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)a_{n-1}(s)}\Delta s\right)\Big/ a_{n-2}(v)\right]\Delta v \\
= &\int_1^\infty \left[\left(\int_v^\infty \frac{(\int_s^\infty (1+2+3u^{-3})\Delta u)^{\frac{1}{2}}}{s^{\frac{1}{2}}s^{-4}}\Delta s\right)\Big/ v^{-4}\right]\Delta v \\
\geq &\int_1^\infty \left(\int_v^\infty \frac{(\int_s^\infty (3u^{-3})\Delta u)^{\frac{1}{2}}}{s^{\frac{1}{2}}s^{-4}}\Delta s\right)v^4\Delta v
\end{aligned}$$

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$$\geq \int_1^\infty \Big(\int_v^\infty s^{\frac{5}{2}}\Delta s\Big)v^4\Delta v = \infty,$$

i.e., (2.3) holds. Let $\eta_1 = \frac{1}{3}$, $\eta_2 = \frac{2}{3}$. Then η_1, η_2 satisfies Lemma 2.1. With $z(t) = 1$ we see that for sufficiently large $T \in \mathbb{T}$,

$$\limsup_{t\to\infty} \int_T^t Q(s)\Delta s = \limsup_{t\to\infty} \int_T^t \Big[1 + (3\times 2)^{\frac{1}{3}}\Big(\frac{3}{2}\times 3s^{-3}\Big)^{\frac{2}{3}}\Big]\Delta s = \infty.$$

Hence, the condition (I) of Theorem 3.2 is satisfied. By Theorem 3.2, every solution of $Eq.$(1.3) either oscillates or tends to zero as $t \to \infty$ when n is odd and is oscillatory when n is even. □

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REFERENCES

[1] L.Erbe, B.Jia, A.Peterson, Oscillation of nth-order superlinear dynamic equations on time scales, Rocky Mt. J. Math., 41(2011), 471-491.

[2] B.Jia, L.Erbe, A.Peterson, Asymptotic behavior of nth-order sublinear dynamic equations on time scales, Common. Appl. Anal., 15(2011), 183-194.

[3] S.Zhang, Q.Wang, Q.Kong, Asymptotic and oscillation of nth-order nonlinear dynamic equations on time scales, Applied Mathematics and computation, 275(2016), 324-334.

[4] T.Sun, Q.He, H.Xi, W.Y, Oscillation for higher order dynamic equations on time scales, Abstract and Applied Analysis, Volume 2013, Article ID 267281, 8 pages.

[5] R.P.Agarwal, M.Bohner, Basic calculus on time scales and some of its applications, Results Math., 35(1999), 3-22.

[6] T.Sun, W.Y, H.Xi, Oscillation behavior and comparison for higher order nonlinear dynamic equations on time scales, J. Appl. Math. Informatics, 30(2012), 289-304.

[7] S.R.Grace, R.P.Agarwal, A.Zafer, Oscillation of higher order nonlinear dynamic equations on time scales, Adv. Differ.Eq., 1(2012), 1-18.

[8] Q.R.Wang, Oscillation criteria for even order nonlinear damped differential equations, Acta Math. Hung., 95(3)(2002), 169-178.

[9] S.Zhang, Q.Wang, Oscillation of second-order nonlinear neutral dynamic equations on time scales, Appl. Math. Comput., 216(2010), 2837-2848.

[10] S.R.Grace, On the oscillation of nth order dynamic equations on time scales, Mediterr. J. Math., 10(1)(2013), 147-156.

[11] E.F.Beckenbach, R.Bellman, Inequalities, Springer, Berlin, 1961.

[12] M.Bohner, A.Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhuser, Boston, 2001.

Normality of meromorphic functions concerning sharing values

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Abstract: Let n, k be positive integers satisfying $n \geq 2k+4$, $a \neq 0$ be a complex number, and let $\mathcal{F}$ be a family of functions meromorphic in a domain D, if for each $f \in \mathcal{F}$, $f^n + af^{(k)}$ and $g^n + ag^{(k)}$ share b, and all zeros have multiplicity at least $k+1$, then $\mathcal{F}$ is normal in D.

Keywords: meromorphic function; normal family; shared value.

1. Introduction and results

In this paper, we denote by $\mathbb{C}$ the whole complex plane. Let f be a meromorphic function in a domain $D \subset \mathbb{C}$. For $a \in \mathbb{C}$, set $\overline{E}_f(a) = \{z \in D : f(z) = a\}$. We say that two meromorphic functions f and g share the value a provided that $\overline{E}_f(a) = \overline{E}_g(a)$ in D. When $a = \infty$ the zeros of $f - a$ means the poles of f.

Let $\mathcal{F}$ a family of meromorphic functions defined on $D \subset \mathbb{C}$. $\mathcal{F}$ is said to be normal on D, in the sense of Montel, if for any sequence $f_j \in \mathcal{F}$ there exists a subsequence f_{n_j} converges spherically locally uniformly on D, to a meromorphic function or ∞ (see [1], [2], [3]).

According to Bloch's principle, every condition which reduces a meromorphic function in the plane $\mathbb{C}$ to a constant, makes a family of meromorphic functions in a domain D normal. It is also more interesting to find normality criteria from the point of view of shared values. In this area, Schwick[4] first proved an interesting result that a family

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of meromorphic functions in a domain is normal if in which every function shares three distinct finite complex numbers with its first derivatives. And later, Sun[5] proved that a family of meromorphic functions in a domain is normal if in which each pair of functions share three fixed distinct values, which is an improvement of the famous Montel's Normal Criterion [6] by the idea of shared values. More results about normality criteria concerning shared values can be found, for instance, in [7-9] and so on.

In 2008, Zhang[10] proved

Theorem A. (see [10]). let $\mathcal{F}$ be a family of functions meromorphic in a domain D, n be a positive integer and a, b be two constants such that $a \neq 0, \infty$ and $b \neq \infty$. If $n \geq 4$ and for each f and g in $\mathcal{F}$, $f' - af^n$ and $g' - ag^n$ share the value b, then $\mathcal{F}$ is normal in D.

In this paper,we replace f' by $f^{(k)}$ in Theorem A and obtain the following theorem.

Theorem 1. Let n, k be a positive integers satisfying $n \geq 2k+4$, $a \neq 0, \infty$ and $b \neq \infty$ be complex numbers, and let $\mathcal{F}$ be a family of functions meromorphic in a domain D. If for each $f, g \in \mathcal{F}$, $f^n + af^{(k)}$ and $g^n + ag^{(k)}$ share b, and all zeros have multiplicity at least $k+1$, then $\mathcal{F}$ is normal in D.

Example: Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_n\}$ where

$$f_n(z) = \frac{1}{n\sqrt[4]{z}}, z \in D, n = 1, 2, 3, \ldots.$$

Clearly $f_n'' + f_n^9 = \frac{5n^8+5}{8n^9\sqrt[4]{z^9}}$. So for each pair m, n, $f_n'' + f_n^7$ and $f_m'' + f_m^7$ share 0 in D, but $\mathcal{F}$ is not normal at the point $z = 0$ since $f_n^{\sharp}(\frac{1}{n^4}) = \frac{n^4}{4(1+n)} \to \infty (n \to \infty)$. This example show that Theorem 1 is not valid if f doesn't satisfy that all zeros have multiplicity at least $k+1$.

2.Lemmas

In this section, we present some lemmas which will be needed in the sequel.

Lemma 2.1([8]). Let $\mathcal{F}$ be a family of functions meromorphic on the unit disc, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. Then if $\mathcal{F}$ is not normal, there exist, for each $0 \leq \alpha \leq k$,

a) a number $0 < r < 1$;

b) points z_n, $|z_n| < r$;

c) functions $f_n \in F$; and

d) positive numbers $\rho_n \to 0$

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n\xi) = g_n(\xi) \to g(\xi)$ locally uniformly with respect to the spherical

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metric, where g is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k, such that $g^\sharp(\xi) \leq g^\sharp(0) = kA + 1$. In particular, g has order at most two; and, in case g is an entire function, it is of exponential type.

Lemma 2.2 [11] Let f be a meromorphic function, then we have

$$T(r,f) < \overline{N}(r,f) + N(r,\frac{1}{f}) + N(r,\frac{1}{f-1}) - N(r,\frac{1}{f'}) + S(r,f), \tag{2.1}$$

where

$$S(r,f) = 2m(r,\frac{f'}{f}) + m(r,\frac{f'}{f-1}) + log|\frac{192f(0)(f(0)-1)}{f'(0)}|.$$

Lemma 2.3 Let f be a transcendental meromorphic function, n and k be two integers and $n \geq 2k+4$, and all zeros of f are of multiplicity greater than $k+1$, then $f^n + af^{(k)}$ assumes every finite complex value b infinitely often.

Proof: Let $\psi(z) = \frac{af^{(k)}(z)-b}{f^n(z)}$, and suppose that $\psi(z) = -1$ has only finite number of roots. Then by Lemma 2.2 we have

$$T(r,\psi) < \overline{N}(r,\psi) + \overline{N}(r,\frac{1}{\psi}) + \overline{N}(r,\frac{1}{\psi+1}) + S(r,\psi). \tag{2.2}$$

Now the poles of $\psi(z)$ occur only at zeros of $f(z)$, and those poles which are not simultaneously zeros of $af^{(k)} - b$ have multiplicity at least n. Zeros of $\psi(z)$ can occur only at zeros of $af^{(k)} - b$ which are not poles of $f(z)$. Thus

$$\begin{aligned}\overline{N}(r,\psi) + \overline{N}(r,\frac{1}{\psi}) &\leq \frac{1}{n}N(r,\psi) + \overline{N}(r,\frac{1}{af^{(k)}-b}) + \overline{N}(r,f)\\ &\leq \frac{1}{n}T(r,\psi) + T(r,f^{(k)}) + \overline{N}(r,f).\end{aligned} \tag{2.3}$$

By the first fundamental theorem, we have

$$\begin{aligned}\overline{N}(r,\psi) + \overline{N}(r,\frac{1}{\psi}) &\leq \frac{1}{n}T(r,\psi) + T(r,f) + (k+1)\overline{N}(r,f) + S(r,f)\\ &\leq \frac{1}{n}T(r,\psi) + (k+2)T(r,f) + S(r,f).\end{aligned} \tag{2.4}$$

Take (2.4) into (2.1), we have

$$(1-\frac{1}{n})T(r,\psi) < (k+2)T(r,f) + S(r,f). \tag{2.5}$$

On the other hand,

$$\begin{aligned}nT(r,f) &= T(r,f^n) = T(r,\frac{af^{(k)}-b}{\psi})\\ &\leq T(r,f^{(k)}) + T(r,\psi) + O(1)\\ &\leq (k+1)T(r,f) + T(r,\psi) + S(r,f),\end{aligned}$$

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that is

$$(n-k-1)T(r,f) \leq T(r,\psi) + S(r,f). \tag{2.6}$$

Combining (3.5) and (3.6) we obtain

$$(1-\frac{1}{n})T(r,\psi) < [\frac{k+2}{n-k-1} + O(1)]T(r,\psi) \tag{2.7}$$

which contradicts with the condition $n \geq 2k+4$.

Thus we proved Lemma 2.3.

Lemma 2.4 Let f be a nonconstant rational function, n and k be two integers and $n \geq 2k+4$, and all zeros of f are of multiplicity greater than $k+1$, then $f^n + af^{(k)}$ has at least two distinct zeros.

Proof: Case 1. If $f^n + af^{(k)}$ has no zeros, it is easy to see that f is not a polynomial, then f is rational function but not a polynomial.

Let $f = \frac{P}{(z-z_1)^{m_1}(z-z_2)^{m_2}\cdots(z-z_t)^{m_t}} = \frac{P}{Q}$, $f^{(k)} = \frac{P_1}{Q_1}$. We denote $p = degP$, and $q = degQ$, then $degQ_1 = q + kt$, $degP_1 = p + k(t-1)$.

$$f^n + af^{(k)} = \frac{P^nQ_1 + aP_1Q^n}{Q^nQ_1}. \tag{2.8}$$

$deg(P^nQ_1) = np + q + kt$, $deg(P_1Q^n) = p + k(t-1) + nq$.

If $p \geq q$, then $np + q + kt - (p + k(t-1) + nq) = (n-1)(p-q) + k > 0$, that is $deg(P^nQ_1) > deg(P_1Q^n)$; If $p < q$, since $n \geq 2k+4$, then $np+q+kt-(p+k(t-1)+nq) = (n-1)(p-q)+k < 0$, that is $deg(P^nQ_1) < deg(P_1Q^n)$, thus $f^n + af^{(k)}$ have zeros, which is a contradiction.

Case 2. $f^n + af^{(k)}$ has only one distinct zero z_0.

If f is a polynomial, then $f^n + af^{(k)} = A(z-z_0)^l$. From the condition that all zeros of f are of multiplicity greater than $k+1$, we can deduce that z_0 is the only zero of f, so $f = b(z-z_0)^m, m \geq k+1$, where b is a constant and m is a positive integer.

$$f^{(k)} = bm(m-1)\cdots(m-k+1)(z-z_0)^{m-k} \tag{2.9}$$

and

$$f^n + af^{(k)} = A^n(z-z_0)^{nl} + c(z-z_0)^{m-k} = (z-z_0)^{m-k}[A(z-z_0)^{nl-m+k} + C], \tag{2.10}$$

thus $f^n + af^{(k)}$ has two distinct zeros, contradiction.

So f is a non-polynomial rational function, then we can assume that

$$f(z) = \frac{A(z-z_0)^m}{(z-z_1)^{n_1}(z-z_2)^{n_2}\cdots(z-z_s)^{n_s}}. \tag{2.11}$$

where A is a constant, and s is a positive integer. By integration (2.4) we get

$$f^{(k)} = \frac{(z-z_0)^{m-k}g(z)}{(z-z_1)^{n_1+1}(z-z_2)^{n_2+1}\cdots(z-z_s)^{n_s+1}}, \tag{2.12}$$

From (2.11) and (2.12) we obtain

$$\begin{aligned} f^n + af^{(k)} &= \frac{A^n(z-z_0)^{nm} + a(z-z_0)^{m-k}g(z)(z-z_1)^{(n-1)n_1-k}\cdots(z-z_s)^{(n-1)n_s-k}}{(z-z_1)^{n_1}(z-z_2)^{n_2}\cdots(z-z_s)^{n_s}} \\ &= \frac{(z-z_0)^{m-k}[A^n(z-z_0)^{nm-m+k} + ag(z)(z-z_1)^{(n-1)n_1-k}\cdots(z-z_s)^{(n-1)n_s-k}]}{(z-z_1)^{n_1}(z-z_2)^{n_2}\cdots(z-z_s)^{n_s}}. \end{aligned} \tag{2.13}$$

On the other hand, since $f^n + af^{(k)}$ has only one zero, we have

$$f^n + af^{(k)} = \frac{C(z-z_0)^l}{(z-z_1)^{n_1}(z-z_2)^{n_2}\cdots(z-z_s)^{n_s}}. \tag{2.14}$$

Combining (2.13) and (2.14) we have

$$C(z-z_0)^l = (z-z_0)^{m-k}g_1(z),$$

where $g_1(z) = A^n(z-z_0)^{nm-m+k} + ag(z)(z-z_1)^{(n-1)n_1-k}\cdots(z-z_s)^{(n-1)n_s-k}$. If $l > m-k$, then $g_1(z)$ has a zero z_0, which is impossible. If $l = m-k$, $g_1(z) = C$, that is $A^n(z-z_0)^{nm-m+k} + ag(z)(z-z_1)^{(n-1)n_1-k}\cdots(z-z_s)^{(n-1)n_s-k} = C$, thus $nm-m+k = (n-1)N$, where $N = n_1+n_2+\cdots+n_s$, thus $(n-1)(N-m) = k$, which is impossible since $n \geq 2k+4$.

The proof of Lemma 2.4 is completed.

3. The Proof of Theorem 1

We may assume that $D = \Delta$, the unit disc. Suppose that F is not normal on Δ. Then by Lemma1, we can find $f_j \in F, z_j \in \Delta$, and $\rho_j \to 0^+$ such that $g_j(\xi) = \rho_n^{\frac{k}{n-1}} f_j(z_j + \rho_j\xi)$ converges locally uniformly with respect to the sphericity metric to a nonconstant meromorphic function g on $\mathbb{C}$, all of whose zeros have multiplicity at least k, which satisfies $g^\sharp(\xi) \leq g^\sharp(0) = kA+1$, in particular, g has order at most two.

On every compact subset of $\mathbb{C}$ we have

$$\rho_j^{\frac{nk}{n-1}}[f_j^n + af_j^{(k)} - b] = g_j^n(\xi) + ag_j^{(k)}(\xi) - \rho_j^{\frac{nk}{n-1}}b \to g^n(\xi) + ag^{(k)}(\xi). \tag{3.1}$$

If $g^n(\xi) + ag^{(k)}(\xi) \equiv 0$, then g has no poles and g is not a polynomial, thus g is a transcendental entire function. From $g^n(\xi) + ag^{(k)}(\xi) \equiv 0$ we obtain $g^{n-1} = -a\frac{g^{(k)}}{g}$, by the first fundamental theorem we have

$$(n-1)T(r,g) = (n-1)m(r,g) = m(r,g^{n-1}) = m(r,-a\frac{g^{(k)}}{g}) = S(r,g),$$

since $n \geq 2k+4$, we obtain $T(r,g) = S(r,g)$, which is a contradiction, thus $g^n(\xi) + ag^{(k)}(\xi) \not\equiv 0$.

By Lemma 2.3 and Lemma 2.4 we obtain that $g^n + ag^{(k)}$ has at least two distinct zeros.

Next we prove that $g^n + ag^{(k)}$ has only one distinct zero.

Let ξ_0 and ξ_0^* be two distinct zeros of $g^n + ag^{(k)}$. We choose a small $\delta > 0$ such that $D_1 \cap D_2 = \emptyset$, where $D_1 = \xi \in \mathbb{C} : |\xi - \xi_0| < \delta$ and $D_2 = \xi \in \mathbb{C} : |\xi - \xi_0^*| < \delta$.

From (3.1), Hurwitz's theorem implied that there exist points $\xi_j \in D_1$ and $\xi_j^* \in D_2$ such that for sufficiently large j

$$f_j^n(z_j + \rho_j\xi_j) + af_j^{(k)}(z_j + \rho_j\xi_j) = b,$$

$$f_j^n(z_j + \rho_j\xi_j^*) + af_j^{(k)}(z_j + \rho_j\xi_j^*) = b.$$

By the assumption of Theorem 1, we see that for each $f_m \in F$

$$f_m^n(z_j + \rho_j\xi_j) + af_m^{(k)}(z_j + \rho_j\xi_j) = b,$$

$$f_m^n(z_j + \rho_j\xi_j^*) + af_m^{(k)}(z_j + \rho_j\xi_j^*) = b.$$

Fix m and let $j \to \infty$, we have $z_j + \rho_j\xi_j \to z_0$, and $z_j + \rho_j\xi_j^* \to z_0$, and

$$f_m^n(z_0) + af_m^{(k)}(z_0) = b.$$

Since the zeros of $f_m^n + af_m^{(k)} - b$ have no accumulation points, we deduce that $z_j + \rho_j\xi_j = z_0$ and $z_j + \rho_j\xi_j^* = z_0$, for sufficiently large j. Hence $\xi_j = \xi_j^* = (z_0 - z_j)/\rho_j$, which contradicts the fact that $\xi_j \in D_1$, $\xi_j^* \in D_2$ and $D_1 \cap D_2 = \emptyset$.

Thus we complete the proof of Theorem 1.

References

[1] Hayman, W.K.; The spherical derivative of integral and meromorphic functions. Comment Math. Helv. 40. 117-148(1966).

[2] Schiff. J., Normal Families. Springer. Berlin (1993).

[3] Y. L.: Value distribution theory, Springer. Berlin (1993).

[4] Schwick. W.: Sharing values and normality. Arch. Math. 59, 50-54 (1992).

[5] D. C. Sun, On the normal criteriion of shared values, J. Wuhan Univ. Natur. Sci. Ed. 3(1994)9-12(in Chinese).

[6] P. Montel, Sur les familles de fonctions analytiques qui admettent des valeurs exceptionnelles dans un domaine, Ann. Sci. Ecole Norm. Sup.29(1912)487-535.

[7] M. Fang, L. Zalcman, A note on normality and shared values, J. Aust. Math. Soc. 32(2004)141-150.

[8] X. Pang, L. Zalcman, Normal families and shared values, Bull. London. Math. Soc. 32(2000)325-331.

[9] Q. Zhang, Normal criteria concerning sharing values, Kodai Math. J. 25(2002)8-14.

[10] Q. C. Zhang, Normal families of meromoephic functions concerning shared values, J. Math. Anal. Appl. 338(2008)545-551.

[11] W.K. Hayman, Picard values of meromorphic functions and their derivatives, Ann. of Math. 70(1959)9-42.

A general iterative algorithm for solving a class of variational inequalities over the common fixed points set of a finite family of nonexpansive mappings in Banach spaces

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Abstract In this paper, we introduce a general iterative algorithm for finding a common element of the set of common fixed points of an finite family of nonexpansive mappings and the set of solutions of class of variational inequalities in uniformly convex and q-uniformly smooth Banach space. We prove that the sequence generated by the iterative algorithm converges strongly to the unique solution of the variational inequality under suitable conditions. Our result improves and extends the recent results announced by many others.

Keywords Strong convergence, Fixed point, nonexpansive mapping, Variational inequality, Banach space.
Mathematics Subject Classification(2010) 47H09; 47H10.

1. Introduction

Throughout this paper, we denote by X and X^* a real Banach space and the dual space of X, respectively. Let C be a subset of X and T be a mapping. We denote the fixed points of T by $F(T) = \{x \in C : Tx = x\}$ and denote $\to$ and $\rightharpoonup$ by strong and weak convergence, respectively. Let $q > 1$ be a real number.

A Banach space X is said to be strictly convex, if whenever x and y are not colinear, then

$$\|x + y\| < \|x\| + \|y\|.$$

The modulus of convexity of X is defined by

$$\delta_X(\epsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : \|x\|, \|y\| \le 1, \|x - y\| \ge \epsilon\right\}$$

for all $\epsilon \in [0, 2]$. X is said to be uniformly convex if $\delta_X(0) = 0$, and $\delta_X(\epsilon) > 0$ for all $0 < \epsilon \le 2$. It is known that every uniformly convex Banach space is strictly convex and reflexive(see [1]).

Let $S(X) = \{x \in X : \|x\| = 1\}$. Then the norm of X is said to be Gâteaux differentiable if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S(X)$. In this case, X is said to be smooth. Let $\rho_X : [0, \infty) \longrightarrow [0, \infty)$ be the modulus of smoothness of X defined by

$$\rho_X(t) = \sup\left\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x \in S(X), \|y\| \le t\right\}.$$

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A Banach space X is said to be uniformly smooth if $\dfrac{\rho_X(t)}{t} \to 0$ as $t \to 0$. A Banach space X is said to be q-uniformly smooth($q > 1$), if there exists a fixed constant $c > 0$ such that $\rho_X(t) \leq ct^q$. It is easy to see that X is q-uniformly smooth, then $q \leq 2$ and X is uniformly smooth. The(generalized)duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \left\{ x^* \in X^* : \langle x, x^* \rangle = \|x\|^q, \ \|x^*\| = \|x\|^{q-1} \right\}, \ \forall x \in X,$$

where $\langle \cdot, \cdot \rangle$ denote the duality pairing between X and X^*. In particular, $J = J_2$ is called the normalized duality mapping and $J_q(x) = \|x\|^{q-2} J_2(x)$ for $x \neq 0$. If X is a Hilbert space, then $J = I$ where I is the identity mapping. Further, we have the following properties of the generalized duality mapping J_q:

(1) $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \in X$ with $x \neq 0$.

(2) $J_q(tx) = t^{q-1} J_q(x)$ for all $x \in X$ and $t \in [0, \infty)$.

(3) $J_q(-x) = -J_q(x)$ for all $x \in X$.

It is well-known that if X is smooth, then J_q is single-valued, which is denoted by j_q. The duality mapping J_q from a smooth Banach space X into X^* is said to be weakly sequentially continuous generalized duality mapping if for all $\{x_n\} \subset X$ with $x_n \rightharpoonup x$ implies $J_q(x_n) \rightharpoonup J_q(x)$.

Definition 1.1 A mapping $T : C \to X$ is said to be:
(1) L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L \|x - y\|, \ \forall x, y \in C. \tag{1.1}$$

If $0 < L < 1$, then T is a contraction and if $L = 1$, then T is a nonexpansive mapping.
(2) α-strongly accretive if there exists $j_q(x - y) \in J_q(x - y)$ and a constant $\alpha > 0$ such that

$$\langle Ax - Ay, j_q(x - y) \rangle \geq \alpha \|x - y\|^q, \ \forall x, y \in C. \tag{1.2}$$

Remark 1.2 If $X := H$ is a real Hilbert space, accretive and strongly accretive mappings coincide with monotone and strongly monotone mappings, respectively.

Recall that the normal Manns iterative algorithm was introduced by Mann [1] in 1953. Since then, construction of fixed points for nonexpansive mappings and strict pseudo-contractions via the normal Manns iterative algorithm has extensively investigated by many authors (see, e.g. [2-8]).

variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral, free, moving, equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework. Several numerical methods have been developed for solving variational inequalities and related optimization problems, see [9-14,25-28] and the references therein.

In 2010, Tian [11] introduced the general steepest-descent method

$$x_{n+1} = \alpha_n \rho f(x_n) + (I - \mu\alpha_n F)Tx_n, \forall n \geq 0, \tag{1.3}$$

where F is an L-Lipschitzian and η-strongly monotone operator. Under certain approximate conditions, the sequence $\{x_n\}$ generated by (1.5) converges strongly to a fixed point of T, which solves the variational inequality:

$$\langle(\rho f-\mu F)x^*, x-x^*\rangle \leq 0, \forall x \in Fix(T).$$

Recently, Zhang et al. [12] proposed the following iterative algorithm:

$$x_{k+1}=\alpha_k\gamma V(x_k)+(I-\mu\alpha_k F)T_N^k\cdots T_1^k(x_k), \forall k \geq 0, \tag{1.4}$$

where V is a L-Lipschitzian mapping, and F is a Lipschitzian and strongly monotone mapping,

$$T_i^k := (1-\beta_k^i)I+\beta_k^i T_i.$$

$\{T_i\}_{i=1}^N$ is a finite family of nonexpansive mappings of H. They obtained that under some approximate assumptions on the operators and parameters, the sequence $\{x_k\}$ generated by (1.6) converges strongly to the unique solution of the variational inequality:

$$\langle(\mu F-\gamma V)x^*, (x-x^*)\rangle \geq 0, \forall x \in C=\bigcap_{i=1}^N Fix(T_i).$$

On the other hand, Ceng et al. [16] investigated implicit and explicit iterative schemes for finding the fixed points of a nonexpansive mapping T on a nonempty, closed and convex subset C in a real Hilbert space H as follows:

$$x_t = P_C[t\gamma V x_t + (I-t\mu F)Tx_t] \tag{1.5}$$

and

$$x_{n+1}=P_C[\alpha_n\gamma V x_n+(I-\alpha_n\mu F)Tx_n], \forall n \geq 0, \tag{1.6}$$

where V is a L-Lipschitzian mapping and F is a κ-Lipschitzian and η-strongly monotone operator. Then they proved that under some approximate assumptions on the operators and parameters, the sequences generated by (1.7) and (1.8) converge strongly to the unique solution of the variational inequality

$$\langle(\mu F-\gamma V)x^*, x-x^*\rangle \geq 0, \forall x \in Fix(T). \tag{1.7}$$

Pongsakorn et al. [17] introduced the following iterative process:

$$x_{n+1}=\beta_n x_n+(1-\beta_n)P_C[\alpha_n\gamma S x_n+(I-\alpha_n F)Tx_n], \forall n \geq 1, \tag{1.8}$$

where P_C is a metric projection from H onto C, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$, $S: C \to H$ is a Lipschitzian mapping, $F: C \to H$ is an invertible positive linear operator, and $T: C \to C$ is a nonexpansive mapping. Then they proved strong convergence theorems under different control conditions on $\{\alpha_n\}$ and $\{\beta_n\}$, the sequence $\{x_n\}$ generated by (1.10) converges to a fixed point of T, which is a unique solution of some variational inequalities.

The following questions naturally arise in connection with above results:

Question 1. Can theorem of Zhang et al. [12] be extended from a real Hilbert space to a general Banach space? such as q-uniformly smooth Banach space.

Question 2. Can we extend the iterative scheme of algorithm (1.6) to a more general iterative scheme?

The purpose of this paper is to give the affirmative answers to these questions mentioned above, motivated by Tian [11], Zhang et al. [12], Ceng et al. [16] and Pongsakorn et al. [17], we introduce a general iterative method. Under some suitable assumptions, we prove the strong convergence theorems of such iterative scheme in q-uniformly smooth Banach space which admits a weakly sequentially continuous generalized duality mapping. The results presented in this article extend and generalize the corresponding results announced by many others in the literature.

2. Preliminaries

Let C and D be nonempty subsets of a Banach space X such that C is nonempty closed convex and $D \subset C$, then a mapping $Q : C \to D$ is sunny provided $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in C$ and $t \geq 0$, whenever $x + t(x - Q(x)) \in C$. A mapping $Q : C \to D$ is retraction if $Qx = x$ for all $x \in D$. Furthermore, Q is a sunny nonexpansive retraction from C onto D if Q is a retraction from C onto D which is also sunny and nonexpansive. It is well known that if $X := H$ is a real Hilbert space, then a sunny nonexpansive retraction Q_C is coincident with the metric projection from X onto C.

A subset D of C is called of a sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C onto D.

In order to prove our main results, we need the following lemmas.

Lemma 2.1 ([15]). Let $1 < p < \infty$, $q \in (1, 2]$, $r > 0$ be given.
(i) If X is uniformly convex, then there exists a continuous, strictly increasing and convex function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda \|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)\varphi(\|x - y\|), \quad x, y \in \mathfrak{B}_r, \quad 0 \leq \lambda \leq 1,$$

where $W_p(\lambda) = \lambda^p(1-\lambda) + (1-\lambda)^p\lambda$, $\mathfrak{B}_r = \{z \in X : \|z\| \leq r\}$.
(ii) Let X be a real q-uniformly smooth Banach space, then there exists a constant $C_q > 0$ such that

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q x\rangle + C_q \|y\|^q$$

for all $x, y \in X$.

Lemma 2.2 ([21]). Assume $\{\alpha_n\}$ is a sequence of nonnegative real numbers satisfying the property

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and δ_n is a sequence such that
(i) $\sum\limits_{n=0}^{\infty} \gamma_n = \infty$,
(ii) $\limsup\limits_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ or $\sum\limits_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim\limits_{n\to\infty} \alpha_n = 0$.

Lemma 2.3 ([7]). Let C be a nonempty convex subset of a real uniformly convex Banach space X and $T : C \to X$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Then $I - T$ is demiclosed at zero.

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Lemma 2.4 ([20]). Let $C \subseteq X$ be a nonempty set. $T : C \to X$ is said to be α-averaged mapping if there exists some number $\alpha \in (0,1)$ such that

$$T = (1-\alpha)I + \alpha S$$

for some nonexpansive mapping S.

(i) The composite of finitely many averaged mappings is averaged. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0,1)$, then both T_1T_2 and T_2T_1 are α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.

(ii) If the mapping $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then

$$\bigcap_{i=1}^{N} F(T_i) = F(T_1 \cdots T_N).$$

In particular, if $N = 2$, we have $F(T_1) \bigcap F(T_2) = F(T_1T_2) = F(T_2T_1)$.

Lemma 2.5 ([21], p.63). Let $q > 1$. Then the following inequality holds:

$$ab \le \frac{1}{q}a^q + \frac{q-1}{q}b^{\frac{q}{q-1}}$$

for arbitrary positive real numbers a, b.

Lemma 2.6 ([22]). Let C be a nonempty closed convex subset of a real q-uniformly smooth Banach space X. Let $V : C \to X$ be a k-Lipschitz and η-strongly accretive operator with constants $k, \eta > 0$. Let $0 < \mu < (\frac{q\eta}{c_q k^q})^{\frac{1}{q-1}}$ and $\tau = \mu(\eta - \frac{C_q \mu^{q-1} k^q}{q})$. Then for $t \in \big(0, \min\{1, \frac{1}{\tau}\}\big)$, the mapping $S : C \to X$ define by $S := (I - t\mu V)$ is a contraction with a constant $1 - t\tau$.

Lemma 2.7 ([23]). Let C be a nonempty closed convex subset of a real smooth Banach space X. Let $\tilde{C}$ be a nonempty subset of C. Let $Q_C : C \to \tilde{C}$ be a retraction, and let j, j_q be the normalized duality mapping and generalized duality mapping on X, respectively. Then the following are equivalent:

(i) Q_C is sunny and nonexpensive;

(ii) $\|Q_C x - Q_C y\|^2 \le \langle x - y, j(Q_C x - Q_C y)\rangle$, $\forall x, y \in C$;

(iii) $\langle x - Q_C x, j(y - Q_C x)\rangle \le 0$, $\forall x \in C, y \in \tilde{C}$;

(iv) $\langle x - Q_C x, j_q(y - Q_C x)\rangle \le 0$, $\forall x \in C, y \in \tilde{C}$.

3. Main results

Let C be a nonempty, closed and convex subset of a real q-uniformly smooth Banach space X. Let Q_C be a sunny nonexpansive retraction from X onto C. Let $F : C \to X$ be a L-Lipschitz and η-strongly accretive operator, $V : C \to X$ be α-Lipschitzian mapping and T be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $0 < \mu < (\frac{q\eta}{c_q L^q})^{\frac{1}{q-1}}$, $0 < \gamma\alpha < \tau$ with $\tau = \mu(\eta - \frac{C_q \mu^{q-1} L^q}{q})$. For each $t \in \big(0, \min\{1, \frac{1}{\tau}\}\big)$, we define the mapping $S_t : C \to C$ by

$$S_t x := Q_C[t\gamma V x + (I - t\mu F)Tx], \forall x \in C.$$

It is easy to see that S_t is a contraction. Indeed, in terms of Lemma 2.6, we have

$$\begin{aligned}
\|S_t x - S_t y\| &= \|Q_C[t\gamma V x + (I - t\mu F)Tx] - Q_C[t\gamma V y + (I - t\mu F)Ty]\| \\
&\leq \|[t\gamma V x + (I - t\mu F)Tx] - [t\gamma V y + (I - t\mu F)Ty]\| \\
&\leq t\gamma \|Vx - Vy\| + \|(I - t\mu F)Tx - (I - t\mu F)Ty\| \\
&\leq t\gamma\alpha \|x - y\| + (1 - t\tau)\|x - y\| \\
&= (1 - t(\tau - \gamma\alpha))\|x - y\|.
\end{aligned}$$

Hence, S_t has a unique fixed point, denoted by x_t, which uniquely solve the fixed point equation

$$x_t = Q_C[t\gamma V x_t + (I - t\mu F)Tx_t]. \tag{3.1}$$

Lemma 3.1 Let C be a nonempty, closed and convex subset of a real uniformly convex and q-uniformly smooth Banach space X which admits a weakly sequentially continuous generalized duality mapping j_q from x into X^*. Let Q_C be a sunny nonexpansive retraction. Let $F : C \to X$ be a L-Lipschitz and η-strongly accretive operator, $V : C \to X$ be α-Lipschitzian mapping and $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $0 < \mu < (\frac{q\eta}{c_q L^q})^{\frac{1}{q-1}}$ and $0 < \gamma\alpha < \tau$ with $\tau = \mu(\eta - \frac{C_q \mu^{q-1} L^q}{q})$. For each $t \in \left(0, \min\left\{1, \frac{1}{\tau}\right\}\right)$, let $\{x_t\}$ defined by (3.1), then $\{x_t\}$ converges strongly to $x^* \in F(T)$ as $t \to 0$, which x^* is the unique solution of the variational inequality

$$\langle (\gamma V - \mu F)x^*, j_q(x - x^*)\rangle \leq 0, \quad \forall x \in F(T). \tag{3.2}$$

Proof. We observe that

$$\begin{aligned}
\frac{C_q \mu^{q-1} L^q}{q} > 0 &\Leftrightarrow \eta - \frac{C_q \mu^{q-1} L^q}{q} < \eta \\
&\Leftrightarrow \mu\left(\eta - \frac{C_q \mu^{q-1} L^q}{q}\right) < \mu\eta \\
&\Leftrightarrow \tau < \mu\eta.
\end{aligned} \tag{3.3}$$

It follows that

$$0 < \gamma\alpha < \tau < \mu\eta. \tag{3.4}$$

First, we show the uniqueness of solution of the variational inequality (3.2). Suppose that $\tilde{x}$, x^* are solutions of (3.2), then

$$\langle (\gamma V - \mu F)x^*, j_q(x - x^*)\rangle \leq 0, \quad \forall x \in F(T). \tag{3.5}$$

and

$$\langle (\gamma V - \mu F)\tilde{x}, j_q(x - \tilde{x})\rangle \leq 0, \quad \forall x \in F(T). \tag{3.6}$$

Adding up (3.5) and (3.6), we have

$$\begin{aligned}0 &\geq \langle(\mu F-\gamma V)\tilde{x}-(\mu F-\gamma V)x^*, j_q(\tilde{x}-x^*)\rangle\\ &= \mu\langle F\tilde{x}-Fx^*, j_q(\tilde{x}-x^*)\rangle-\gamma\langle V\tilde{x}-Vx^*, j_q(\tilde{x}-x^*)\rangle\\ &\geq \mu\eta\|\tilde{x}-x^*\|^q-\gamma\alpha\|\tilde{x}-x^*\|^q\\ &\geq (\mu\eta-\gamma\alpha)\|\tilde{x}-x^*\|^q.\end{aligned}$$

Therefore, $\tilde{x}=x^*$ and the uniqueness is proved. Below we use x^* to denote the unique solution of (3.2).

We observe that $\{x_t\}$ is bounded. Indeed, taking $\bar{x}\in F(T)$, then, we have

$$\begin{aligned}\|x_t-\bar{x}\| &\leq \|t\gamma Vx_t+(I-t\mu F)Tx_t-\bar{x}\|\\ &= \|t(\gamma Vx_t-\mu F\bar{x})+(I-t\mu F)Tx_t-(I-t\mu F)\bar{x}\|\\ &\leq t\gamma\|Vx_t-V\bar{x}\|+t\|\gamma V\bar{x}-\mu F\bar{x}\|+(1-t\tau)\|Tx_t-\bar{x}\|\\ &\leq (1-t(\tau-\gamma\alpha))\|x_t-\bar{x}\|+t\|\gamma V\bar{x}-\mu F\bar{x}\|.\end{aligned}$$

therefore, $\|x_t-\bar{x}\|\leq\frac{\|\gamma V\bar{x}-\mu F\bar{x}\|}{\tau-\gamma\alpha}$, so are $\{Vx_t\}$ and $\{FTx_t\}$.

By the definition of $\{x_t\}$, we have

$$\begin{aligned}\|x_t-Tx_t\| &= \|Q_C[t\gamma Vx_t+(I-t\mu F)Tx_t-Q_CTx_t]\|\\ &\leq \|[t\gamma Vx_t+(I-t\mu F)Tx_t-Tx_t]\|\\ &= t\|\gamma Vx_t-\mu FTx_t\|\to 0 \ as\ t\to 0.\end{aligned}\tag{3.7}$$

Next, we prove that $x_t\to\tilde{x}$ as $t\to 0$.

Setting $y_t=t\gamma Vx_t+(I-t\mu F)Tx_t$, we obtain $x_t=Q_Cy_t$. Assume that $\{t_n\}\subset(0,1)$ is a sequence such that $t_n\to 0$ as $n\to\infty$. Since $\{x_t\}$ is bounded and X is reflexive, there exists a subsequence $\{x_{t_n}\}$ of $\{x_t\}$ such that $x_{t_n}\rightharpoonup\tilde{x}$.

we claim $\|x_{t_n}-\tilde{x}\|\to 0$.

$$\begin{aligned}\|x_{t_m}-\tilde{x}\|^q &= \langle x_{t_m}-\tilde{x}, j_q(x_{t_m}-\tilde{x})\rangle\\ &= \langle Q_Cy_{t_m}-y_{t_m}+y_{t_m}-\tilde{x}, j_q(x_{t_m}-\tilde{x})\rangle\\ &\leq \langle t_m\gamma Vx_{t_m}+(I-t_m\mu F)Tx_{t_m}-\tilde{x}, j_q(x_{t_m}-\tilde{x})\rangle\\ &= \langle t_m(\gamma Vx_{t_m}-\mu F\tilde{x})+(I-t_m\mu F)Tx_{t_m}-(I-t_m\mu F)\tilde{x}, j_q(x_{t_m}-\tilde{x})\rangle\\ &= t_m\langle\gamma Vx_{t_m}-\mu F\tilde{x}, j_q(x_{t_m}-\tilde{x})\rangle+\langle(I-t_m\mu F)Tx_{t_m}-(I-t_m\mu F)\tilde{x}, j_q(x_{t_m}-\tilde{x})\rangle\\ &\leq t_m\langle\gamma Vx_{t_m}-\mu F\tilde{x}, j_q(x_{t_m}-\tilde{x})\rangle+(1-t_m\tau)\|x_{t_m}-\tilde{x}\|^q.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|x_{t_m}-\tilde{x}\|^q &\leq \frac{1}{\tau}\langle\gamma Vx_{t_m}-\mu F\tilde{x}, j_q(x_{t_m}-\tilde{x})\rangle\\ &= \frac{1}{\tau}\langle\gamma(Vx_{t_m}-V\tilde{x})+(\gamma V\tilde{x}-\mu F\tilde{x}), j_q(x_{t_m}-\tilde{x})\rangle\\ &\leq \frac{1}{\tau}\gamma\alpha\|x_{t_m}-\tilde{x}\|^q+\frac{1}{\tau}\langle\gamma V\tilde{x}-\mu F\tilde{x}, j_q(x_{t_m}-\tilde{x})\rangle\end{aligned}$$

which implies that

$$\|x_{t_n} - \tilde{x}\|^q \leq \frac{1}{\tau - \gamma\alpha} \langle \gamma V\tilde{x} - \mu F\tilde{x}, j_q(x_{t_n} - \tilde{x})\rangle . \tag{3.8}$$

By (3.8), we get that $\|x_{t_n} - \tilde{x}\| \to 0$. From (3.7), (3.8) and Lemma 2.3, we have $\bar{x} \in F(T)$. Next, we show that x^* solves the variational inequality (3.2). Since

$$x_t = Q_C y_t = Q_C y_t - y_t + y_t,$$

we derive that

$$(\mu F - \gamma V)x_t = \frac{1}{t}(Q_C y_t - y_t) - \frac{1}{t}(I - T)x_t + \mu(Fx_t - FTx_t). \tag{3.9}$$

Note that $I - T$ is accretive(i.e.$\langle (I-T)x - (I-T)y, j_q(x-y)\rangle \geq 0$, for all $x, y \in C$). For all $z \in F(T)$, it follows from (3.9) and Lemma 2.7 that

$$\begin{aligned}\langle (\mu F - \gamma V)x_t, j_q(x_t - z)\rangle &= \frac{1}{t}\langle Q_C y_t - y_t, j_q(Q_C y_t - z)\rangle - \frac{1}{t}\langle (I-T)x_t - (I-T)z, j_q(x_t - z)\rangle \\ &\quad + \mu \langle Fx_t - FTx_t, j_q(x_t - z)\rangle \\ &\leq \mu \langle Fx_t - FTx_t, j_q(x_t - z)\rangle \\ &\leq \mu \|Fx_t - FTx_t\| \|x_t - z\|^{q-1} \\ &\leq \|x_t - Tx_t\| M \end{aligned} \tag{3.10}$$

where $M = \sup\left\{\mu L\|x_t - z\|^{q-1}\right\}$, where $t \in \left(0, \min\left\{1, \frac{1}{\tau}\right\}\right)$.

Now replacing t in (3.10) with t_n and taking the limit as $n \to \infty$, we noticing that $x_{t_n} - Tx_{t_n} \to \bar{x} - T\bar{x} = 0$ for $\bar{x} \in F(T)$, we obtain $\langle (\mu F - \gamma V)\bar{x} - (\mu F - \gamma V)\bar{x}, j_q(\bar{x} - z)\rangle \leq 0$. That is, $\bar{x} \in F(T)$ is the solution of variational inequality (3.2). Consequently, $\bar{x} = x^*$ by uniqueness. Therefore $x_t \to x^*$ as $t \to 0$. This completes the proof. □

Theorem 3.2 Let C be a nonempty, closed and convex subset of a real uniform convex and q-uniformly smooth Banach space X which admits a weakly sequentially continuous generalized duality mapping j_q from X into X^*. Let Q_C be a sunny nonexpansive retraction. Let $F : C \to X$ be a L-Lipschitz and η-strongly accretive operator, $V : C \to X$ be α-Lipschitzian mapping and $\{S_i\}_{i=1}^N : C \to C$ be a finite family of nonexpansive mappings such that $\Omega = \bigcap_{i=1}^N F(S_i) \neq \emptyset$. Define $T_i^k := (1 - \beta_k^i)I + \beta_k^i S_i$, where $\beta_k^i \in (a,b) \subset (0,1)$. For arbitrarily given $x_0 \in X$ and $0 < \mu < (\frac{q\eta}{c_q L^q})^{\frac{1}{q-1}}$, let $\{x_k\}$ be the sequence generated iteratively by :

$$x_{k+1} = \beta_k x_k + (1 - \beta_k)Q_C[\alpha_k \gamma V x_k + (I - \mu\alpha_k F)T_N^k \cdots T_1^k x_k], \quad \forall k \geq 0 \tag{3.11}$$

where $0 < \gamma\alpha < \tau$ with $\tau = \mu(\eta - \frac{C_q \mu^{q-1} L^q}{q})$. Assume that $\{\alpha_k\} \subset (0,1)$ and $\{\beta_k\} \subset [0,1)$ satisfying the following conditions:

(i) $\lim_{k\to\infty} \alpha_k = 0$ and $\sum_{k=1}^{\infty} \alpha_k = \infty$;

(ii) $\sum_{k=1}^{\infty} |\beta_{k+1} - \beta_k| < \infty$, $\limsup_{n\to\infty} \beta_k < 1$ and $\sum_{k=1}^{\infty} |\alpha_{k+1} - \alpha_k| < \infty$;

(iii) $\sum_{k=1}^{\infty} \left|\beta_{k+1}^i - \beta_k^i\right| < \infty$.

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Then the sequence $\{x_k\}$ converges strongly to the unique solution x^* of the variational inequality:

$$\langle(\gamma V-\mu F)x^*, j_q(x-x^*)\rangle \leq 0, \forall x\in\Omega. \tag{3.12}$$

Proof. Set $y_k = Q_C[\alpha_n\gamma V x_k + (I-\mu\alpha_k F)T_N^k\cdots T_1^k x_k]$, then

$$x_{k+1} = \beta_k x_k + (1-\beta_k)y_k. \tag{3.13}$$

Since our methods easily deduce the general case, we prove theorem 3.2 for $N=2$. The proof is divided into five steps.

Step 1. First we show that $\{x_k\}$ is bounded. Indeed, taking some $p\in\Omega$, from (3.13) and Lemma 2.6, we have

$$\begin{aligned}
&\|y_k-p\| = \left\|Q_C[\alpha_k\gamma V x_k + (I-\mu\alpha_k F)T_2^kT_1^k x_k] - Q_C(p)\right\|\\
&\leq \left\|\alpha_k\gamma V x_k + (I-\mu\alpha_k F)T_2^kT_1^k x_k - p\right\|\\
&= \left\|(I-\mu\alpha_k F)T_2^kT_1^k x_k - (I-\mu\alpha_k F)p + \alpha_k(\gamma V x_k - \gamma V p) + \alpha_k(\gamma V p - \mu F p)\right\|\\
&\leq \left\|(I-\mu\alpha_k F)T_2^kT_1^k x_k - (I-\mu\alpha_k F)p\right\| + \alpha_k\|\gamma V x_k - \gamma V p\| + \alpha_k\|\gamma V p - \mu F p\|\\
&\leq (1-\alpha_k\tau)\|x_k-p\| + \alpha_k\gamma\alpha\|x_k-p\| + \alpha_k\|\gamma V p - \mu F p\|\\
&\leq [1-\alpha_k(\tau-\gamma\alpha)]\|x_k-p\| + \alpha_k\|\gamma V p - \mu F p\|.
\end{aligned}$$

Also, it follows that

$$\begin{aligned}
&\|x_{k+1}-p\| = \|\beta_k x_k + (1-\beta_k)y_k - p\|\\
&= \|\beta_k(x_k-p) + (1-\beta_k)(y_k-p)\|\\
&\leq \beta_k\|x_k-p\| + (1-\beta_k)\|y_k-p\|\\
&\leq \beta_k\|x_k-p\| + (1-\beta_k)[1-\alpha_k(\tau-\gamma\alpha)]\|x_k-p\| + (1-\beta_k)\alpha_k\|\gamma V p - \mu F p\|\\
&= [1-\alpha_k(1-\beta_k)(\tau-\gamma\alpha)]\|x_k-p\| + \alpha_k(1-\beta_k)\|\gamma V p - \mu F p\|\\
&= [1-\alpha_k(1-\beta_k)(\tau-\gamma\alpha)]\|x_k-p\| + \alpha_k(1-\beta_k)(\tau-\gamma\alpha)\frac{\|\gamma V p - \mu F p\|}{\tau-\gamma\alpha}.
\end{aligned}$$

By induction, we have

$$\|x_{k+1}-p\| \leq \max\left\{\|x_0-p\|, \frac{\|\gamma V p - \mu F p\|}{\tau-\gamma\alpha}\right\}, \forall k\geq 0.$$

Hence $\{x_k\}$ is bounded. So are $\left\{T_2^kT_1^k(x_k)\right\}, \left\{FT_2^kT_1^k(x_k)\right\}$ and $\{V(x_k)\}$.

Step 2. We claim that

$$\lim_{k\to\infty}\|x_{k+1}-x_k\| = 0.$$

We note that

$$
\begin{aligned}
&\|y_{k+1}-y_k\| \\
&= \left\|Q_C[\alpha_{k+1}\gamma V x_{k+1}+(I-\mu\alpha_{k+1}F)T_2^{k+1}T_1^{k+1}x_{k+1}]-Q_C[\alpha_k\gamma V x_k+(I-\mu\alpha_k F)T_2^kT_1^kx_k]\right\| \\
&\leq \left\|[\alpha_{k+1}\gamma V x_{k+1}+(I-\mu\alpha_{k+1}F)T_2^{k+1}T_1^{k+1}x_{k+1}]-[\alpha_k\gamma V x_k+(I-\mu\alpha_k F)T_2^kT_1^kx_k]\right\| \\
&\leq \|\alpha_{k+1}\gamma(Vx_{k+1}-Vx_k)\|+\left\|(I-\mu\alpha_{k+1}F)T_2^{k+1}T_1^{k+1}x_{k+1}-(I-\mu\alpha_{k+1}F)T_2^kT_1^kx_k\right\| \\
&\quad+\left\|(\alpha_{k+1}-\alpha_k)(\gamma Vx_k-\mu FT_2^kT_1^kx_k)\right\| \\
&\leq \alpha_{k+1}\gamma\alpha\,\|x_{k+1}-x_k\|+(1-\alpha_{k+1}\tau)[\|x_{k+1}-x_k\|+\left\|T_2^{k+1}T_1^kx_k-T_2^kT_1^kx_k\right\| \\
&\quad+\left\|T_1^{k+1}x_k-T_1^kx_k\right\|]+|\alpha_{k+1}-\alpha_k|\cdot M_1 \\
&= [1-\alpha_{k+1}(\tau-\gamma\alpha)]\,\|x_{k+1}-x_k\|+[(1-\alpha_{k+1}\tau)[\left\|T_2^{k+1}T_1^kx_k-T_2^kT_1^kx_k\right\| \\
&\quad+\left\|T_1^{k+1}x_k-T_1^kx_k\right\|]+|\alpha_{k+1}-\alpha_k|\cdot M_1
\end{aligned}
\tag{3.14}
$$

where M_1 is a fixed constant satisfying

$$M_1 \geq \sup_{k\geq 1}\{\left\|\gamma Vx_k-\mu FT_2^kT_1^kx_k\right\|\}.$$

It follows from (3.13) and (3.14) that

$$
\begin{aligned}
&\|x_{k+2}-x_{k+1}\| = \|\beta_{k+1}x_{k+1}+(1-\beta_{k+1})y_{k+1}-\beta_kx_k-(1-\beta_k)y_k\| \\
&= \|\beta_{k+1}(x_{k+1}-x_k)+(1-\beta_{k+1})(y_{k+1}-y_k)+(\beta_{k+1}-\beta_k)(x_k-y_k)\| \\
&\leq \beta_{k+1}\,\|x_{k+1}-x_k\|+(1-\beta_{k+1})\,\|y_{k+1}-y_k\|+|\beta_{k+1}-\beta_k|\,\|x_k-y_k\| \\
&\leq \beta_{k+1}\,\|x_{k+1}-x_k\|+(1-\beta_{k+1})[1-\alpha_{k+1}(\tau-\gamma\alpha)]\,\|x_{k+1}-x_k\| \\
&\quad+(1-\alpha_{k+1}\tau)(1-\beta_{k+1})[\left\|T_2^{k+1}T_1^kx_k-T_2^kT_1^kx_k\right\|+\left\|T_1^{k+1}x_k-T_1^kx_k\right\|] \\
&\quad+(1-\beta_{k+1})\,|\alpha_k-\alpha_{k-1}|\cdot M_1+|\beta_{k+1}-\beta_k|\cdot M_2 \\
&\leq [1-\alpha_{k+1}(1-\beta_{k+1})(\tau-\gamma\alpha)]\,\|x_{k+1}-x_k\|+(1-\beta_{k+1})(1-\alpha_{k+1}\tau) \\
&\quad[\left\|T_2^{k+1}T_1^kx_k-T_2^kT_1^kx_k\right\|+\left\|T_1^{k+1}x_k-T_1^kx_k\right\|] \\
&\quad+(1-\beta_{k+1})\,|\alpha_k-\alpha_{k-1}|\cdot M_1+|\beta_{k+1}-\beta_k|\cdot M_2
\end{aligned}
\tag{3.15}
$$

where $M_2 \geq \sup_{k\geq 1}\{\|x_k-y_k\|\}$, then (3.5) reduces to the following:

$$\|x_{k+2}-x_{k+1}\| \leq (1-\gamma_k)\,\|x_{k+1}-x_k\|+\delta_k.$$

where $\gamma_k = \alpha_{k+1}(1-\beta_{k+1})(\tau-\gamma\alpha)$, $\delta_k = (1-\beta_{k+1})(1-\alpha_{k+1}\tau)[\left\|T_2^{k+1}T_1^kx_k-T_2^kT_1^kx_k\right\|+\left\|T_1^{k+1}x_k-T_1^kx_k\right\|]+(1-\beta_{k+1})\,|\alpha_k-\alpha_{k-1}|\cdot M_1+|\beta_{k+1}-\beta_k|\cdot M_2$. It is easily seen that

$$\sum_{k=1}^{\infty}\gamma_k=\infty.$$

Note that

$$
\begin{aligned}
\left\|T_1^{k+1}x_k-T_1^kx_k\right\| &= \left\|(1-\beta_{k+1}^1)x_k+\beta_{k+1}^1S_1x_k-(1-\beta_k^1)x_k+\beta_k^1S_1x_k\right\| \\
&= \left\|(\beta_{k+1}^1-\beta_k^1)(S_1x_k-x_k)\right\| \\
&\leq \left|\beta_{k+1}^1-\beta_k^1\right|(\|S_1x_k\|+\|x_k\|).
\end{aligned}
\tag{3.16}
$$

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Similarly,

$$\left\|T_2^{k+1}T_1^k x_k - T_2^k T_1^k x_k\right\| \leq \left|\beta_{k+1}^2 - \beta_k^2\right| (\left\|S_2 T_1^k x_k\right\| + \left\|T_1^k x_k\right\|). \tag{3.17}$$

From condition (iii), (3.16) and (3.17), we get that

$$\sum_{k=1}^{\infty} \left\|T_1^{k+1}x_k - T_1^k x_k\right\| < \infty, \sum_{k=1}^{\infty} \left\|T_2^{k+1}T_1^k x_k - T_2^k T_1^k x_k\right\| < \infty. \tag{3.18}$$

Then we have that $\sum_{n=1}^{\infty} |\delta_k| < \infty$. Therefore, from Lemma 2.2, it follows that

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{3.19}$$

Step 3. We show that

$$\lim_{k\to\infty} \left\|x_k - T_2^k T_1^k x_k\right\| = 0.$$

Since

$$\begin{aligned}\|x_k - y_k\| &\leq \|x_k - x_{k+1}\| + \|x_{k+1} - y_k\| \\ &\leq \|x_k - x_{k+1}\| + \beta_k \|x_k - y_k\|,\end{aligned}$$

that is,

$$\|x_k - y_k\| \leq \frac{1}{1-\beta_k} \|x_k - x_{k+1}\|.$$

It follows from (ii) and (3.19) that

$$\lim_{k\to\infty} \|x_k - y_k\| = 0. \tag{3.20}$$

By (i), we have

$$\begin{aligned}\left\|y_k - T_2^k T_1^k x_k\right\| &= \left\|P_C[\alpha_k \gamma V x_k + (I - \mu\alpha_k F)T_2^k T_1^k x_k] - P_C T_2^k T_1^k x_k\right\| \\ &\leq \left\|\alpha_k \gamma V x_k + (I - \mu\alpha_k F)T_2^k T_1^k x_k - T_2^k T_1^k x_k\right\| \\ &\leq \alpha_k \left\|\gamma V x_k - \mu F T_2^k T_1^k x_k\right\| \to 0 \quad as \quad k \to \infty. \end{aligned} \tag{3.21}$$

Moreover, we observe that

$$\left\|x_k - T_2^k T_1^k x_k\right\| \leq \|x_k - y_k\| + \left\|y_k - T_2^k T_1^k x_k\right\|, \tag{3.22}$$

it follows from (3.20), (3.21) and (3.22) that

$$\lim_{k\to\infty} \left\|x_k - T_2^k T_1^k x_k\right\| = 0. \tag{3.23}$$

Step 4. We will show that

$$\limsup_{k\to\infty} \langle(\gamma V - \mu F)x^*, j_q(x_k - x^*)\rangle \leq 0, \tag{3.24}$$

where x^* is the unique solution of (3.12). To show this, we take a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that

$$\limsup_{k\to\infty} \langle(\gamma V - \mu F)x^*, j_q(x_k - x^*)\rangle = \lim_{j\to\infty} \left\langle(\gamma V - \mu F)x^*, j_q(x_{k_j} - x^*)\right\rangle.$$

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Without loss of generality, we may further assume that $x_{k_j} \rightharpoonup \hat{x}$ as $j \to \infty$ due to reflexivity of the Banach space X and boundness of $\{x_k\}$. Since $\{\beta_k^i\}$ is bounded for $i = 1, 2$, we can assume that $\beta_{k_j}^i \to \beta_\infty^i$ as $j \to \infty$ where $0 < a \le \beta_\infty^i \le b < 1$ for $i = 1, 2$. Define $T_i^\infty = (1 - \beta_\infty^i)I + \beta_\infty^i S_i (i = 1, 2)$. From Lemma 2.4, we have

$$F(T_1^\infty T_2^\infty) = F(T_1^\infty) \cap F(T_2^\infty) = F(S_1) \cap F(S_2).$$

Note that

$$\left\| T_i^{k_j} x - T_i^\infty x \right\| \le \left| \beta_{k_j}^i - \beta_\infty^i \right| (\|S_i x\| + \|x\|).$$

Hence, we deduce that

$$\lim_{j\to\infty} \sup_{x\in D} \left\| T_i^{k_j} x - T_i^\infty x \right\| = 0, \tag{3.25}$$

where D is an arbitrary bounded subset of X. Combine (3.23) and (3.25), we obtain

$$\begin{aligned}
&\|x_{k_j} - T_2^\infty T_1^\infty x_{k_j}\| \\
&\le \left\| x_{k_j} - T_2^{k_j} T_1^{k_j} x_{k_j} \right\| + \left\| T_2^{k_j} T_1^{k_j} x_{k_j} - T_2^\infty T_1^{k_j} x_{k_j} \right\| + \left\| T_2^\infty T_1^{k_j} x_{k_j} - T_2^\infty T_1^\infty x_{k_j} \right\| \\
&\le \left\| x_{k_j} - T_2^{k_j} T_1^{k_j} x_{k_j} \right\| + \sup_{x\in D'} \left\| T_2^{k_j} x - T_2^\infty x \right\| + \sup_{x\in D'} \left\| T_1^{k_j} x - T_1^\infty x \right\|,
\end{aligned}$$

where D' is a bounded subset including $\{T_1^{k_j} x_{k_j}\}$ and D'' is a bounded subset including $\{x_{k_j}\}$. Hence $\lim\limits_{j\to\infty} \|x_{k_j} - T_2^\infty T_1^\infty x_{k_j}\| = 0$. From Lemma 2.3, we have $\hat{x} \in F(T_1^\infty T_2^\infty)$. Since Banach space X has a weakly sequentially continuous generalized duality mapping $j_q : X \to X^*$, we obtain that

$$\begin{aligned}
\limsup_{k\to\infty} \langle (\gamma V - \mu F)x^*, j_q(y_k - x^*) \rangle &= \limsup_{k\to\infty} \langle (\gamma V - \mu F)x^*, j_q(x_k - x^*) \rangle \\
&= \lim_{j\to\infty} \left\langle (\gamma V - \mu F)x^*, j_q(x_{k_j} - x^*) \right\rangle \\
&= \langle (\gamma V - \mu F)x^*, j_q(\hat{x} - x^*) \rangle \le 0.
\end{aligned}$$

Step 5. Finally we show that $x_k \to x^*$ as $k \to \infty$.
Set $y_k = Q_C u_k$ for all $k \ge 0$, where $u_k = \alpha_k \gamma V x_k + (I - \mu\alpha_k F) T_2^k T_1^k x_k$. From Lemma 2.7, we have

$$\begin{aligned}
&\|y_k - x^*\|^q = \langle Q_C u_k - x^*, j_q(Q_C u_k - x^*) \rangle \\
&= \langle Q_C u_k - u_k, j_q(Q_C u_k - x^*) \rangle + \langle u_k - x^*, j_q(Q_C u_k - x^*) \rangle \\
&\le \langle u_k - x^*, j_q(y_k - x^*) \rangle \\
&= \left\langle \alpha_k \gamma V x_k + (I - \mu\alpha_k F) T_2^k T_1^k x_k - x^*, j_q(y_k - x^*) \right\rangle \\
&= \alpha_k \gamma \langle V x_k - V x^*, j_q(y_k - x^*) \rangle + \alpha_k \langle \gamma V x^* - \mu F x^*, j_q(y_k - x^*) \rangle \\
&\quad + \left\langle (I - \mu\alpha_k F) T_2^k T_1^k x_k - (I - \mu\alpha_k F)x^*, j_q(y_k - x^*) \right\rangle \\
&\le \alpha_k \alpha\gamma \|x_k - x^*\| \|y_k - x^*\|^{q-1} + \alpha_k \langle \gamma V x^* - \mu F x^*, j_q(y_k - x^*) \rangle \\
&\quad + (1 - \alpha_k \tau) \|x_k - x^*\| \|y_k - x^*\|^{q-1} \\
&= [1 - \alpha_k(\tau - \alpha\gamma)] \|x_k - x^*\| \|y_k - x^*\|^{q-1} + \alpha_k \langle \gamma V(x^*) - \mu F(x^*), j_q(y_k - x^*) \rangle \\
&= [1 - \alpha_k(\tau - \alpha\gamma)](\frac{1}{q}\|x_k - x^*\|^q + \frac{q-1}{q}\|y_k - x^*\|^q) + \alpha_k \langle \gamma V(x^*) - \mu F(x^*), j_q(y_k - x^*) \rangle,
\end{aligned}$$

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which implies that

$$\|y_k - x^*\|^q \le [1 - \alpha_k(\tau - \alpha\gamma)]\|x_k - x^*\|^q + q\alpha_k \langle \gamma V(x^*) - \mu F(x^*), j_q(y_k - x^*)\rangle . \tag{3.26}$$

Again from (3.26) and Lemma 2.1, we obtain

$$\begin{aligned}
&\|x_{k+1} - x^*\|^q = \|\beta_k x_k + (1-\beta_k)y_k - x^*\|^q \\
&= \|\beta_k(x_k - x^*) + (1-\beta_k)(y_k - x^*)\|^q \\
&\le \beta_k\|x_k - x^*\|^q + (1-\beta_k)\|y_k - x^*\|^q \\
&\le \beta_k\|x_k - x^*\|^q + (1-\beta_k)[1-\alpha_k(\tau-\alpha\gamma)]\|x_k - x^*\|^q + q\alpha_k(1-\beta_k)\langle \gamma V(x^*) - \mu F(x^*), j_q(y_k - x^*)\rangle \\
&\le [1 - \alpha_k(1-\beta_k)(\tau - \alpha\gamma)]\|x_k - x^*\|^q + q\alpha_k(1-\beta_k)\langle \gamma V(x^*) - \mu F(x^*), j_q(y_k - x^*)\rangle , \quad (3.27)
\end{aligned}$$

Put $a_k = \alpha_k(1-\beta_k)(\tau - \alpha\gamma)$ and $\delta_k = q\alpha_k(1-\beta_k)\langle \gamma V(x^*) - \mu F(x^*), j_q(y_k - x^*)\rangle$, thus (3.27) reduce to the following:

$$\|x_{k+1} - x^*\|^q \le (1 - a_k)\|x_k - x^*\|^q + \delta_k. \tag{3.28}$$

Apply Lemma 2.2 to (3.28), we obtain $x_k \to x^*$ as $k \to \infty$. This completes the proof. □

Remark 3.3 Theorem 3.2 extend and generalize Theorem 3.2 of Pongsakorn et al. [17] in the following aspects:

(i) From a real Hilbert space to a real q-uniformly smooth Banach space which admits a weakly sequentially continuous generalized duality mapping.

(ii) From an invertible positive linear operator $F : C \to H$ to the Lipschitzian and strongly accretive operator $F : C \to X$.

(iii) From nonexpansive mapping $T : C \to C$ to a finite family of nonexpansive mappings.

Remark 3.4 Compared with the known results in the literature, our results are very different from Theorem 3.1 of Zhang et al. [12] in the following aspects:

(i) Theorem 3.2 extends and improves Theorem 3.1 of Zhang et al. [12] from real Hilbert space to a real q-uniformly smooth Banach space which admits a weakly sequentially continuous generalized duality mapping.

(ii) We generalize the iteration process in theorem 3.1 of Zhang et al. [12].

References

[1] W. R. Mann, Mean Value Methods in Iteration, Proc. Amer. Math. Soc., (1953), **4(3)**:506-506.

[2] G. Marino, H. K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl., (2007), **329**:336-349.

[3] H. Y. Zhou, Nonexpansive mappings and iterative methods in uniformly convex Banach spaces, Acta Mathematica Sinica(English Series), (2004), **20**:829-836.

[4] H. Y. Zhou, Convergence theorems for λ-strict pseudo-contractions in 2-uniformly smooth Banach spaces, Nolinear Anal., (2008), **69**:3160-3173.

[5] H. Y. Zhou, Convergence theorems for λ-strict pseudocontractions in q-uniformly smooth Banach spaces, Acta Mathematica Sinica(English Series), (2010), **26**:743-758.

[6] H. Zhang, Y. F. Su, Strong convergence theorems for strict pseudo-contractions in q-uniformly smooth Banach spaces, Nolinear Anal., (2009), **70**:3236-3242.

[7] Q. L. Dong, S. N. He, F. Su, Strong convergence of an iterative algorithm for an infinite family of strict pseudo-contractions in Banach spaces, Appl. Math. Comput., (2010), **216**:959-969.

[8] Y. S. Song, Strong convergence of viscosity approximation methods with strong pseudocontraction for Lipschitz pseudocontractive mappings, Positivity, (2009), **13**:643-655.

[9] Y. H. Yao, M. A. Noor, K. I. Noor, Y.-C. Liou, H. Yaqoob, Modified extragradient methods for a system of variational inequalities in Banach spaces, Acta Appl. Math., (2010), **110**:1211-1224.

[10] L. C. Ceng, C. Wang, J.C. Yao, Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities, Math. Methods Oper. Res., (2008), **67**:375-390.

[11] M. Tian, A general iterative algorithm for nonexpansive mappings in Hilbert spaces, Nolinear Anal., (2010), **73**:689-694.

[12] H. K. Xu, Viscosity approximation methods for nonexpansive mappings, J. Math. Anal. Appl., (2004), **298**:279-291.

[13] Y. H. Wang, W. Xu, strong convergence of a modified iterative algorithm for hierarchial fixed point problems and variational inequalities, Fixed Point Theory Appl., (2013), **2013**:121.

[14] C. J. Zhang, C.P. Yang, A new explicit iterative algorithm for solving a class of variational inequalities over the common fixed points set of a finite family of nonexpansive mappings, Fixed Point Theory Appl., (2014), **2014**:60.

[15] H. K. Xu, Inequalities in Banach spaces with applications, Nonlinear Anal-Theor., (1991), **16**:1127-1138.

[16] L. C. Ceng, Q. H. Anasri, J. C. Yao, some iterative methods for finding fixed points and for solving constrained convex minimization problems, Nolinear Anal., (2011), **74**:5289-5302.

[17] P. Sunthrayuth, Y. J. Cho, P. Kumam, general iterative algorithms approach to variational inequalities and minimum-norm fixed point for minimization and split feasibility problems, Opsearch, (2014), **51**:400-415.

[18] R. E. B. Jr, Properties of fixed point sets of nonexpansive mappings in Banach spaces, Trans. Amer. Math. Soc. (1973), **179**:251-262.

[19] A. Y. Matsushita, W. Takahashi, Strong convergence theorems for nonexpansive nonself-mappings without boundary conditions, Nolinear Anal., (2008), **68**:412-419.

[20] G. L*ó*pez, V. Martin, H. K. Xu, Iterative algorithm for the multiple-sets split feasibility problem, Biomedical Math., (2009), 243-279.

[21] D. S. Mitrinović, Analytic Inequalities, Springer-Verlag, New York, (1970).

[22] K. Aoyama, Y. Kimura, W. Takahashi, M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, Nonlinear Anal., (2007), **67**:2350-2360.

[23] J. L. Guan, C. S. Hu, X. wang, a new general iterative algorithm with Meir-Keeler contractions for variational inequality problems in q-uniformly smooth Banach spaces, Fixed Point Theory Appl., (2013), **2013**:292.

[24] T. Suzuki, Strong convergence of Krasnoselskii and Mann'type sequences for one parameter nonex-pansive semigroups without Bochner integral, J. Math. Anal. Appl., (2005), **35**:227-239.

[25] M. A. Noor, K. I. Noor, E. Al-Said, Iterative methods for solving general quasi-vatiational inequalities, Optim Lett., (2010), **4**:513-530.

[26] M. A. Noor, On a system of general mixed variational inequalities, Optim Lett., (2009), **3**:437-451.

[27] M. A. Noor, Projection methods for nonconvex variational inequalities, Optim Lett., (2009), **3**:411-418.

[28] Y. H. Yao, N. Shahzad, Strong convergence of a proximal point algorithm with general errors, Optim Lett., (2012), **6**:621-628.

FOURIER SERIES OF FUNCTIONS ASSOCIATED WITH POLY-BERNOULLI POLYNOMIALS

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Abstract. In this paper, we consider three types of functions associated with poly-Bernoulli functions and derive their Fourier series expansions. In addition, we will express each of them in terms of Bernoulli functions.

1. Introduction

As is well known, the *Bernoulli polynomials* $B_m(x)$ are given by the generating function

$$\frac{t}{e^t-1}e^{xt}=\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!},\quad \text{(see [4,14,16,19])}. \tag{1.1}$$

For any integer r, the poly-Bernoulli polynomials $\mathbf{B}_m^{(r)}(x)$ of index r are given by the generating function

$$\frac{Li_r(1-e^{-t})}{e^t-1}e^{xt}=\sum_{m=0}^{\infty}\mathbf{B}_m^{(r)}(x)\frac{t^m}{m!},\quad \text{(see [2,3,8,10,13,20])}, \tag{1.2}$$

where $Li_r(x)=\sum_{m=1}^{\infty}\frac{x^m}{m^r}$ is the rth polylogarithmic function for $r\geq 1$, and a rational function for $r\leq 0$.

We observe here that

$$\frac{d}{dx}(Li_{r+1}(x))=\frac{1}{x}Li_r(x). \tag{1.3}$$

We need to note the following as to the poly-Bernoulli polynomials:

$$\frac{d}{dx}\mathbf{B}_m^{(r)}(x)=m\mathbf{B}_{m-1}^{(r)}(x),(m\geq 1), \tag{1.4}$$

$$\begin{aligned}&\mathbf{B}_m^{(1)}(x)=B_m(x),\mathbf{B}_0^{(r)}(x)=1,\mathbf{B}_m^{(0)}(x)=x^m,\\&\mathbf{B}_m^{(0)}=\delta_{m,0},\mathbf{B}_m^{(r+1)}(1)-\mathbf{B}_m^{(r+1)}(0)=\mathbf{B}_{m-1}^{(r)},\ (m\geq 1).\end{aligned} \tag{1.5}$$

For any real number x, let

$$<x>=x-[x]\in[0,1) \tag{1.6}$$

denote the fractional part of x.

Fourier series expansion of higher-order Bernoulli functions were treated in the recent paper [15]. Here we will study three types of functions associated with poly-Bernoulli functions and derive their Fourier series expansions. In addition, we will express each of them in terms of Bernoulli functions.

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(1) $\alpha_m(< x >) = \sum_{k=0}^{m} \mathbb{B}_k^{(r+1)}(< x >) < x >^{m-k},\ (m \geq 1)$;
(2) $\beta_m(< x >) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!}\mathbb{B}_k^{(r+1)}(< x >) < x >^{m-k},\ (m \geq 1)$;
(3) $\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\mathbb{B}_k^{(r+1)}(< x >) < x >^{m-k},\ (m \geq 2)$.

For elementary facts about Fourier analysis, the reader may refer to any book (for example, see [1,17,21]).

As to $\gamma_m(< x >)$, we note that the next polynomial identity follows immediately from Theorems 4.1 and 4.2, which is in turn derived from the Fourier series expansion of $\gamma_m(< x >)$:

$$\begin{aligned}&\sum_{k=1}^{m-1} \frac{1}{k(m-k)}\mathbf{B}_k^{(r+1)}(x)x^{m-k},\\ &= \frac{1}{m}\sum_{s=0}^{m}\binom{m}{s}\Big(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(\mathbf{B}_{m-s}^{(r)}+1)\Big)B_s(x)\end{aligned}$$

where $H_l = \sum_{j=1}^{l}\frac{1}{j}$ are the harmonic numbers and

$$\Lambda_l = \sum_{k=1}^{l-1}\frac{\mathbf{B}_k^{(r+1)}}{k(l-k)} + \sum_{k=1}^{l-1}\frac{\mathbf{B}_{k-1}^{(r)}}{k(l-k)}.$$

The obvious polynomial identities can be derived also for $\alpha_m(< x >)$ and $\beta_m(< x >)$ from Theorems 2.1 and 2.2, and Theorems 3.1 and 3.2, respectively. It is remarkable that from the Fourier series expansion of the function $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}B_k(< x >)B_{m-k}(< x >)$ we can derive the Faber-Pandharipande-Zagier identity (see [6,9,11,12]) and the Miki's identity (see [5,7,9,11,12,18]).

2. The function $\alpha_m(< x >)$

For integers r, m with $m \geq 1$, we let

$$\alpha_m(x) = \sum_{k=0}^{m}\mathbf{B}_k^{(r+1)}(x)x^{m-k}.$$

Now, we consider the function

$$\alpha_m(< x >) = \sum_{k=0}^{m}\mathbf{B}_k^{(r+1)}(< x >) < x >^{m-k},\ (m \geq 1), \tag{2.1}$$

defined on $(-\infty, \infty)$, which is periodic with period 1.

The Fourier series of $\alpha_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi i n x},$$

where

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(< x >)e^{-2\pi inx}dx \\ &= \int_0^1 \alpha_m(x)e^{-2\pi inx}dx. \end{aligned} \tag{2.2}$$

Before proceeding any further, we need to observe the following.

$$\begin{aligned} \alpha_m'(x) &= \sum_{k=0}^{m} \left(k\mathbf{B}_{k-1}^{(r+1)}(x)x^{m-k} + (m-k)\mathbf{B}_k^{(r+1)}(x)x^{m-k-1} \right) \\ &= \sum_{k=1}^{m} k\mathbf{B}_{k-1}^{(r+1)}(x)x^{m-k} + \sum_{k=0}^{m-1}(m-k)\mathbf{B}_k^{(r+1)}(x)x^{m-k-1} \\ &= \sum_{k=0}^{m-1}(k+1)\mathbf{B}_k^{(r+1)}(x)x^{m-1-k} + \sum_{k=0}^{m-1}(m-k)\mathbf{B}_k^{(r+1)}(x)x^{m-1-k} \\ &= (m+1)\sum_{k=0}^{m-1}\mathbf{B}_k^{(r+1)}(x)x^{m-1-k} \\ &= (m+1)\alpha_{m-1}(x). \end{aligned} \tag{2.3}$$

From this, we see that

$$\left(\frac{\alpha_{m+1}(x)}{m+2} \right)' = \alpha_m(x), \tag{2.4}$$

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\left(\alpha_{m+1}(1) - \alpha_{m+1}(0)\right). \tag{2.5}$$

For $m \geq 1$, we put

$$\begin{aligned} \Delta_m &= \alpha_m(1) - \alpha_m(0) \\ &= \sum_{k=0}^{m} \left(\mathbf{B}_k^{(r+1)}(1) - \mathbf{B}_k^{(r+1)}\delta_{m,k} \right) \\ &= \mathbf{B}_0^{(r+1)}(1) + \sum_{k=1}^{m}\mathbf{B}_k^{(r+1)}(1) - \sum_{k=0}^{m}\mathbf{B}_k^{(r+1)}\delta_{m,k} \\ &= 1 + \sum_{k=1}^{m}\left(\mathbf{B}_k^{(r+1)} + \mathbf{B}_{k-1}^{(r)} \right) - \mathbf{B}_m^{(r+1)} \\ &= \sum_{k=0}^{m-1}\mathbf{B}_k^{(r+1)} + \sum_{k=0}^{m-1}\mathbf{B}_k^{(r)}. \end{aligned} \tag{2.6}$$

Then $\alpha_m(1) = \alpha_m(0) \iff \Delta_m = 0$, and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}.$$

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We are now ready to determine the Fourier coefficients $A_n^{(m)}$.
$Case 1 : n \neq 0$.

$$\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\left[\alpha_m(x) e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 \alpha_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\left(\alpha_m(1) - \alpha_m(0)\right) + \frac{m+1}{2\pi i n}\int_0^1 \alpha_{m-1}(x) e^{-2\pi i n x} dx \\
&= \frac{m+1}{2\pi i n} A_n^{(m-1)} - \frac{1}{2\pi i n}\Delta_m \\
&= \frac{m+1}{2\pi i n}\left(\frac{m}{2\pi i n} A_n^{(m-2)} - \frac{1}{2\pi i n}\Delta_{m-1}\right) - \frac{1}{2\pi i n}\Delta_m \\
&= \frac{(m+1)_2}{(2\pi i n)^2} A_n^{(m-2)} - \sum_{j=1}^{2}\frac{(m+1)_{j-1}}{(2\pi i n)^j}\Delta_{m-j+1} \\
&= \cdots \\
&= \frac{(m+1)!}{(2\pi i n)^m} A_n^{(0)} - \sum_{j=1}^{m}\frac{(m+1)_{j-1}}{(2\pi i n)^j}\Delta_{m-j+1} \\
&= -\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi i n)^j}\Delta_{m-j+1},
\end{aligned} \tag{2.7}$$

where $A_n^{(0)} = \int_0^1 e^{-2\pi i n x} dx = 0$.

$Case 2 : n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x) dx = \frac{1}{m+2}\Delta_{m+1}. \tag{2.8}$$

Here we recall the following facts about Bernoulli functions $B_m(< x >)$:
(a) for $m \geq 2$,

$$B_m(< x >) = -m! \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^m}. \tag{2.9}$$

(b) for $m = 1$,

$$-\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \tag{2.10}$$

$\alpha_m(< x >), (m \geq 1)$ is piecewise C^∞. Moreover, $\alpha_m(< x >)$ is continuous for those positive integers m with $\Delta_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers m with $\Delta_m \neq 0$.

Assume first that m is a positive integer with $\Delta_m = 0$. Then $\alpha_m(1) = \alpha_m(0)$. Hence $\alpha_m(< x >)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\alpha_m(< x >)$ converges uniformly to $\alpha_m(< x >)$, and

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$$\begin{aligned}
\alpha_m(<x>) &= \frac{1}{m+2}\Delta_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx} \\
&= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=1}^{m}\binom{m+2}{j}\Delta_{m-j+1} \\
&\times\left(-j!\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right) \\
&= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>) \\
&+ \Delta_m \times \begin{cases} B_1(<x>), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{2.11}$$

Now, we can state our first result.

Theorem 2.1. *For each positive integer l, let*

$$\Delta_l = \sum_{k=0}^{l-1}\boldsymbol{B}_k^{(r+1)} + \sum_{k=0}^{l-1}\boldsymbol{B}_k^{(r)}.$$

Assume that $\Delta_m = 0$, for a positive integer m. Then we have the following.

(a) $\sum_{k=0}^{m}\boldsymbol{B}_k^{(r+1)}(<x>)<x>^{m-k}$ *has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=0}^{m}\boldsymbol{B}_k^{(r+1)}(<x>)<x>^{m-k} \\
&= \frac{1}{m+2}\Delta_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx},
\end{aligned}$$

for all $x \in (-\infty, \infty)$, where the convergence is uniform.

$$\begin{aligned}
\text{(b)} &\sum_{k=0}^{m}\boldsymbol{B}_k^{(r+1)}(<x>)<x>^{m-k} \\
&= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>),
\end{aligned}$$

for all $x \in (-\infty, \infty)$, where $B_j(<x>)$ is the Bernoulli function.

Assume next that m is a positive integer with $\Delta_m \neq 0$. Then $\alpha_m(1) \neq \alpha_m(0)$. Hence $\alpha_m(<x>)$ is piecewise C^∞, and discontinuous with jump discontinuities at

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integers. The Fourier series of $\alpha_m(<x>)$ converges pointwise to $\alpha_m(<x>)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\alpha_m(0)+\alpha_m(1)\right)=\alpha_m(0)+\frac{1}{2}\Delta_m,$$

for $x \in \mathbb{Z}$.

Next, we can state the second result.

Theorem 2.2. *For each positive integer l, let*

$$\Delta_l=\sum_{k=0}^{l-1}\boldsymbol{B}_k^{(r+1)}+\sum_{k=0}^{l-1}\boldsymbol{B}_k^{(r)}.$$

Assume that $\Delta_m \neq 0$, for a positive integer m. Then we have the following.

$$\begin{aligned}&(a)\frac{1}{m+2}\Delta_{m+1}\\&+\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx}\\&=\begin{cases}\sum_{k=0}^{m}\boldsymbol{B}_k^{(r+1)}(<x>)<x>^{m-k}, & \text{for } x\notin\mathbb{Z},\\ \boldsymbol{B}_m^{(r+1)}+\frac{1}{2}\Delta_m, & \text{for } x\in\mathbb{Z}.\end{cases}\end{aligned}$$

$$\begin{aligned}&(b)\frac{1}{m+2}\Delta_{m+1}+\frac{1}{m+2}\sum_{j=1}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>)\\&=\sum_{k=0}^{m}\boldsymbol{B}_k^{(r+1)}(<x>)<x>^{m-k},\ for\ x\notin\mathbb{Z};\end{aligned}$$

$$\begin{aligned}&\frac{1}{m+2}\Delta_{m+1}+\frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>)\\&=\boldsymbol{B}_m^{(r+1)}+\frac{1}{2}\Delta_m,\ for\ x\in\mathbb{Z}.\end{aligned}$$

3. The fuction $\beta_m(<x>)$

Let $\beta_m(x)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\mathbf{B}_k^{(r+1)}(x)x^{m-k},\ (m\geq 1)$.
Before proceeding further, we need to observe the following.

$$\begin{aligned}\beta'_m(x) &= \sum_{k=0}^{m}\left\{\frac{k}{k!(m-k)!}\mathbf{B}_{k-1}^{(r+1)}(x)x^{m-k}(x)\right.\\ &\left.+\frac{m-k}{k!(m-k)!}\mathbf{B}_{k}^{(r+1)}(x)x^{m-k-1}\right\}\\ &=\sum_{k=1}^{m}\frac{1}{(k-1)!(m-k)!}\mathbf{B}_{k-1}^{(r+1)}x^{m-k}\\ &+\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}\mathbf{B}_{k}^{(r+1)}(x)x^{m-k-1}\\ &=2\sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}\mathbf{B}_{k}^{(r+1)}(x)x^{m-1-k}\\ &=2\beta_{m-1}(x).\end{aligned} \tag{3.1}$$

From this, we obtain $\left(\frac{\beta_{m+1}(x)}{2}\right)' = \beta_m(x)$,
and

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\Big(\beta_{m+1}(1)-\beta_{m+1}(0)\Big). \tag{3.2}$$

For $m \geq 1$, we have

$$\begin{aligned}\Omega_m &= \beta_m(1)-\beta_m(0)\\ &=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\mathbf{B}_{k}^{(r+1)}(1)-\frac{1}{m!}\mathbf{B}_{m}^{(r+1)}\\ &=\frac{1}{m!}+\sum_{k=1}^{m}\frac{1}{k!(m-k)!}\mathbf{B}_{k}^{(r+1)}(1)-\frac{1}{m!}\mathbf{B}_{m}^{(r+1)}\\ &=\frac{1}{m!}+\sum_{k=1}^{m}\frac{1}{k!(m-k)!}(\mathbf{B}_{k}^{(r+1)}+\mathbf{B}_{k-1}^{(r)})-\frac{1}{m!}\mathbf{B}_{m}^{(r+1)}\\ &=\sum_{k=0}^{m-1}\frac{1}{k!(m-k)!}\mathbf{B}_{k}^{(r+1)}+\sum_{k=1}^{m}\frac{1}{k!(m-k)!}\mathbf{B}_{k-1}^{(r)}\end{aligned} \tag{3.3}$$

Then $\beta_m(1)=\beta_m(0) \Longleftrightarrow \Omega_m = 0$.
Also,

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}.$$

Now, we are going to consider the function

$$\beta_m(<x>) = \sum_{k=0}^{m}\frac{1}{k!(m-k)!}\mathbf{B}_{k}^{(r+1)}(<x>)<x>^{m-k},\ (m\geq 1),$$

defined on $(-\infty,\infty)$, which is periodic with period 1.

The Fourier series of $\beta_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi i n x},$$

where

$$B_n^{(m)} = \int_0^1 \beta_m(<x>) e^{-2\pi i n x} dx = \int_0^1 \beta_m(x) e^{-2\pi i n x} dx.$$

Next, we want to determine the Fourier coefficients $B_n^{(m)}$.

Case 1:$n \neq 0$.

$$\begin{aligned}
B_n^{(m)} &= \int_0^1 \beta_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\Big[\beta_m(x) e^{-2\pi i n x}\Big]_0^1 + \frac{1}{2\pi i n}\int_0^1 \beta_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\Big(\beta_m(1) - \beta_m(0)\Big) + \frac{1}{\pi i n}\int_0^1 \beta_{m-1}(x) e^{-2\pi i n x} dx \\
&= \frac{2}{2\pi i n} B_n^{(m-1)} - \frac{1}{2\pi i n}\Omega_m \\
&= \frac{2}{2\pi i n}\Big(\frac{2}{2\pi i n} B_n^{(m-2)} - \frac{1}{2\pi i n}\Omega_{m-1}\Big) - \frac{1}{2\pi i n}\Omega_m \\
&= \Big(\frac{2}{2\pi i n}\Big)^2 B_n^{(m-2)} - \sum_{j=1}^{2} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1} \\
&= \cdots \\
&= \Big(\frac{2}{2\pi i n}\Big)^m B_n^{(0)} - \sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1} \\
&= -\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1},
\end{aligned} \tag{3.4}$$

where $B_n^{(0)} = \int_0^1 e^{-2\pi i n x} dx = 0$.

Case 2: $n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x) dx = \frac{1}{2}\Omega_{m+1}. \tag{3.5}$$

$\beta_m(<x>)$, $(m \geq 1)$ is piecewise C^∞. Moreover, $\beta_m(<x>)$ is continuous for those positive integers m with $\Omega_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers m with $\Omega_m \neq 0$.

Assume first that m is a positive integer with $\Omega_m = 0$. Then $\beta_m(1) = \beta_m(0)$. Hence $\beta_m(<x>)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\beta_m(<x>)$ converges uniformly to $\beta_m(<x>)$, and

$$\begin{aligned}
&\beta_m(<x>)\\
&=\frac{1}{2}\Omega_{m+1}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\Big(-\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\Big)e^{2\pi inx}\\
&=\frac{1}{2}\Omega_{m+1}+\sum_{j=1}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}\times\Big(-j!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\Big)\\
&=\frac{1}{2}\Omega_{m+1}+\sum_{j=2}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(<x>)\\
&+\Omega_m\times\begin{cases}B_1(<x>), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}.\end{cases}
\end{aligned}\tag{3.6}$$

Now, we can state our first result.

Theorem 3.1. *For each positive integer l, let*

$$\Omega_l=\sum_{k=0}^{l-1}\frac{1}{k!(l-k)!}\boldsymbol{B}_k^{(r+1)}+\sum_{k=1}^{l}\frac{1}{k!(l-k)!}\boldsymbol{B}_{k-1}^{(r)}.$$

Assume that $\Omega_m=0$, for a positive integer m. Then we have the following.
(a) $\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\boldsymbol{B}_k^{(r+1)}(<x>)<x>^{m-k}$ *has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\boldsymbol{B}_k^{(r+1)}(<x>)<x>^{m-k}\\
&=\frac{1}{2}\Omega_{m+1}-\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\Big(-\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\Big)e^{2\pi inx},
\end{aligned}\tag{3.7}$$

for all $x\in(-\infty,\infty)$, where the convergence is uniform.

(b) $$\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\boldsymbol{B}_k^{(r+1)}(<x>)<x>^{m-k}=\frac{1}{2}\Omega_{m+1}+\sum_{j=2}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(<x>),$$

for all $x\in(-\infty,\infty)$, where $B_j(<x>)$ is the Bernoulli function.

Assume next that m is a positive integer with $\Omega_m\neq 0$. Then, $\beta_m(1)\neq\beta_m(0)$. Hence $\beta_m(<x>)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\beta_m(<x>)$ converges pointwise to $\beta_m(<x>)$, for $x\notin\mathbb{Z}$, and converges to

$$\frac{1}{2}(\beta_m(0)+\beta_m(1))=\beta_m(0)+\frac{1}{2}\Omega_m=\frac{1}{m!}\mathbf{B}_m^{(r+1)}+\frac{1}{2}\Omega_m,\tag{3.8}$$

for $x\in\mathbb{Z}$.

Now, we can state our second result.

Theorem 3.2. *For each positive integer l, let*

$$\Omega_l=\sum_{k=0}^{l-1}\frac{1}{k!(l-k)!}\boldsymbol{B}_k^{(r+1)}+\sum_{k=1}^{l}\frac{1}{k!(l-k)!}\boldsymbol{B}_{k-1}^{(r)}.$$

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Assume that $\Omega_m \neq 0$, for a positive integer m. Then we have the following.

(a)
$$\frac{1}{2}\Omega_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \Big(-\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}\Big) e^{2\pi i n x}$$
$$= \begin{cases} \sum_{k=0}^{m} \frac{1}{k!(m-k)!} \boldsymbol{B}_k^{(r+1)}(< x >) < x >^{m-k}, & \text{for } x \notin \mathbb{Z}, \\ \frac{1}{m!}\boldsymbol{B}_m^{(r+1)} + \frac{1}{2}\Omega_m, & \text{for } x \in \mathbb{Z}. \end{cases}$$

(b)
$$\frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m} \frac{2^{j-1}}{j!}\Omega_{m-j+1} B_j(< x >)$$
$$= \sum_{k=0}^{m} \frac{1}{k!(m-k)!} \boldsymbol{B}_k^{(r+1)}(< x >) < x >^{m-k}, \quad \text{for } x \notin \mathbb{Z};$$
$$\frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m} \frac{2^{j-1}}{j!}\Omega_{m-j+1} B_j(< x >)$$
$$= \frac{1}{m!}\boldsymbol{B}_m^{(r+1)} + \frac{1}{2}\Omega_m, \quad \text{for } x \in \mathbb{Z}.$$

4. The function $\gamma_m(< x >)$

Let $\gamma_m(x) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \mathbf{B}_k^{(r+1)}(x) x^{m-k}$, $(m \geq 2)$.

$$\gamma_m'(x) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \Big(k\mathbf{B}_{k-1}^{(r+1)}(x) x^{m-k} + (m-k)\mathbf{B}_k^{(r+1)}(x) x^{m-k-1} \Big)$$
$$= \sum_{k=0}^{m-2} \frac{1}{m-1-k}\mathbf{B}_k^{(r+1)}(x) x^{m-1-k} + \sum_{k=1}^{m-1} \frac{1}{k}\mathbf{B}_k^{(r+1)}(x) x^{m-1-k}$$
$$= \frac{1}{m-1}x^{m-1} + \sum_{k=1}^{m-2} \frac{1}{m-1-k}\mathbf{B}_k^{(r+1)}(x) x^{m-1-k} + \frac{1}{m-1}\mathbf{B}_{m-1}^{(r+1)}(x)$$
$$+ \sum_{k=1}^{m-2} \frac{1}{k}\mathbf{B}_k^{(r+1)}(x) x^{m-1-k}$$
$$= \frac{1}{m-1}\Big(x^{m-1} + \mathbf{B}_{m-1}^{(r+1)}(x)\Big) + (m-1)\sum_{k=1}^{m-2} \frac{1}{k(m-1-k)}\mathbf{B}_k^{(r+1)}(x) x^{m-1-k}$$
$$= \frac{1}{m-1}\Big(x^{m-1} + \mathbf{B}_{m-1}^{(r+1)}(x)\Big) + (m-1)\gamma_{m-1}(x). \tag{4.1}$$

Thus,
$$\gamma_m'(x) = \frac{1}{m-1}\Big(x^{m-1} + \mathbf{B}_{m-1}^{(r+1)}(x)\Big) + (m-1)\gamma_{m-1}(x).$$

From this, we obtain

$$\left(\frac{1}{m}\Big(\gamma_{m+1}(x)-\frac{1}{m(m+1)}\mathbf{B}_{m+1}^{(r+1)}(x)-\frac{1}{m(m+1)}x^{m+1}\Big)\right)'=\gamma_m(x).$$

$$\begin{aligned}
&\int_0^1\gamma_m(x)dx\\
&=\frac{1}{m}\Big[\gamma_{m+1}(x)-\frac{1}{m(m+1)}\mathbf{B}_{m+1}^{(r+1)}(x)-\frac{1}{m(m+1)}x^{m+1}\Big]_0^1\\
&=\frac{1}{m}\Big(\gamma_{m+1}(1)-\gamma_{m+1}(0)-\frac{1}{m(m+1)}\Big(\mathbf{B}_{m+1}^{(r+1)}(1)-\mathbf{B}_{m+1}^{(r+1)}(0)\Big)\\
&-\frac{1}{m(m+1)}\Big)\\
&=\frac{1}{m}\Big(\gamma_{m+1}(1)-\gamma_{m+1}(0)-\frac{1}{m(m+1)}\mathbf{B}_{m}^{(r)}-\frac{1}{m(m+1)}\Big).
\end{aligned}\tag{4.2}$$

For $m\geq 2$, we let

$$\begin{aligned}
\Lambda_m=\Lambda_m(r)&=\gamma_m(1)-\gamma_m(0)\\
&=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbf{B}_k^{(r+1)}(1)\\
&=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\Big(\mathbf{B}_k^{(r+1)}+\mathbf{B}_{k-1}^{(r)}\Big)\\
&=\sum_{k=1}^{m-1}\frac{\mathbf{B}_k^{(r+1)}}{k(m-k)}+\sum_{k=1}^{m-1}\frac{\mathbf{B}_{k-1}^{(r)}}{k(m-k)}.
\end{aligned}\tag{4.3}$$

Then,

$$\gamma_m(1)=\gamma_m(0)\ \Leftrightarrow\ \Lambda_m=0,\tag{4.4}$$

and

$$\int_0^1\gamma_m(x)dx=\frac{1}{m}\left(\Lambda_{m+1}-\frac{1}{m(m+1)}\mathbf{B}_m^{(r)}-\frac{1}{m(m+1)}\right).\tag{4.5}$$

We are going to consider

$$\gamma_m(<x>)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbf{B}_k^{(r+1)}(<x>)<x>^{m-k},\tag{4.6}$$

defined on $(-\infty,\infty)$, which is periodic with period 1.

The Fourier series of $\gamma_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty}C_n^{(m)}e^{2\pi inx},\tag{4.7}$$

where

$$C_n^{(m)}=\int_0^1\gamma_m(<x>)e^{-2\pi inx}dx=\int_0^1\gamma_m(x)e^{-2\pi inx}dx.\tag{4.8}$$

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Now, we want to determine the Fourier coefficients $C_n^{(m)}$.
Case 1: $n \neq 0$. We can easily show that, for $l \geq 1$,

$$\begin{aligned}
&\int_0^1 \mathbf{B}_l^{(r+1)}(x)e^{-2\pi inx}dx \\
&= \begin{cases} -\sum_{k=1}^{l} \frac{(l)_{k-1}}{(2\pi in)^k}\mathbf{B}_{l-k}^{(r)}(< x >), & \text{for } n \neq 0, \\ \frac{1}{l+1}\mathbf{B}_l^{(r)}, & \text{for } n = 0, \end{cases}
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
&\int_0^1 x^l e^{-2\pi inx}dx \\
&= \begin{cases} -\sum_{k=1}^{l} \frac{(l)_{k-1}}{(2\pi in)^k}, & \text{for } n \neq 0, \\ \frac{1}{l+1}, & \text{for } n = 0. \end{cases}
\end{aligned} \tag{4.10}$$

By using (4.9) and (4.10), we can obtain the following recursive relation.

$$\begin{aligned}
&C_n^{(m)} = \int_0^1 \gamma_m(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}\Big[\gamma_m(x)e^{-2\pi inx}\Big]_0^1 + \frac{1}{2\pi in}\int_0^1 \gamma_m'(x)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}\Big(\gamma_m(1) - \gamma_m(0)\Big) \\
&+ \frac{1}{2\pi in}\int_0^1 \Big((m-1)\gamma_{m-1}(x) + \frac{1}{m-1}(x^{m-1} + \mathbf{B}_{m-1}^{(r+1)}(x))\Big)e^{-2\pi inx}dx \\
&= -\frac{1}{2\pi in}\Lambda_m + \frac{m-1}{2\pi in}C_n^{(m-1)} + \frac{1}{2\pi in(m-1)}\int_0^1 \mathbf{B}_{m-1}^{(r+1)}(x)e^{-2\pi inx}dx \\
&+ \frac{1}{2\pi in(m-1)}\int_0^1 x^{m-1}e^{-2\pi inx}dx \\
&= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Theta_m - \frac{1}{2\pi in(m-1)}\Phi_m,
\end{aligned} \tag{4.11}$$

where, for $m \geq 2$,

$$\begin{aligned}
&\Lambda_m = \gamma_m(1) - \gamma_m(0) = \sum_{k=1}^{m-1}\frac{\mathbf{B}_k^{(r+1)}}{k(m-k)} + \sum_{k=1}^{m-1}\frac{\mathbf{B}_{k-1}^{(r)}}{k(m-k)}, \\
&\Theta_m = \sum_{k=1}^{m-1}\frac{(m-1)_{k-1}}{(2\pi in)^k}\mathbf{B}_{m-k-1}^{(r)}, \\
&\Phi_m = \sum_{k=1}^{m-1}\frac{(m-1)_{k-1}}{(2\pi in)^k}.
\end{aligned} \tag{4.12}$$

From the relation (4.11), we can get an expression for $C_n^{(m)}$.

$$
\begin{aligned}
C_n^{(m)} &= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Phi_m - \frac{1}{2\pi in(m-1)}\Theta_m \\
&= \frac{m-1}{2\pi in}\Big(\frac{m-2}{2\pi in}C_n^{(m-2)} - \frac{1}{2\pi in}\Lambda_{m-1} - \frac{1}{2\pi in(m-2)}\Phi_{m-1} - \frac{1}{2\pi in(m-2)}\Theta_{m-1}\Big) \\
&- \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Phi_m - \frac{1}{2\pi in(m-1)}\Theta_m \\
&= \frac{(m-1)_2}{(2\pi in)^2}C_n^{(m-2)} - \sum_{j=1}^{2}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} - \sum_{j=1}^{2}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\
&- \sum_{j=1}^{2}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1} \\
&= \cdots \\
&= \frac{(m-1)!}{(2\pi in)^{m-1}}C_n^{(1)} - \sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} - \sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\
&- \sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1},
\end{aligned} \tag{4.13}
$$

where

$$C_n^{(1)} = \int_0^1 \gamma_1(x)e^{-2\pi inx}dx = 0. \tag{4.14}$$

Before proceeding further, we note the following:

$$
\begin{aligned}
&\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1} \\
&= \sum_{j=1}^{m-1}\frac{1}{m-j}\sum_{k=1}^{m-j}\frac{(m-1)_{j+k-2}}{(2\pi in)^{j+k}}\mathbf{B}_{m-j-k}^{(r)} \\
&= \sum_{j=1}^{m-1}\frac{1}{m-j}\sum_{s=j+1}^{m}\frac{(m-1)_{s-2}}{(2\pi in)^s}\mathbf{B}_{m-s}^{(r)} \\
&= \frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\frac{H_{m-1}-H_{m-s}}{m-s+1}\mathbf{B}_{m-s}^{(r)}.
\end{aligned} \tag{4.15}
$$

Putting everything altogether, we obtain

$$C_n^{(m)} = -\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\Big(\Lambda_{m-s+1} \quad +\frac{H_{m-1}-H_{m-s}}{m-s+1}(\mathbf{B}_{m-s}^{(r)}+1)\Big). \tag{4.16}$$

Case 2: $n = 0$.

$$C_0^{(m)} = \int_0^1 \gamma_m(x)dx = \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbf{B}_m^{(r)} - \frac{1}{m(m+1)}\right). \tag{4.17}$$

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$\gamma_m(< x >)$, $(m \geq 2)$ is piecewise C^∞. Moreover, $\gamma_m(< x >)$ is continuous for those positive integers $m \geq 2$ with $\Lambda_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers ≥ 2 with $\Lambda_m \neq 0$.

Assume first that $\Lambda_m = 0$. Then $\gamma_m(1) = \gamma_m(0)$. Hence $\gamma_m(< x >)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\gamma_m(< x >)$ converges uniformly to $\gamma_m(< x >)$, and

$$\begin{aligned}
&\gamma_m(< x >) \\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbf{B}_m^{(r)} - \frac{1}{m(m+1)}\right) \\
&- \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi i n)^s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(\mathbf{B}_{m-s}^{(r)}+1)\right)\right)e^{2\pi i n x} \\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbf{B}_m^{(r)} - \frac{1}{m(m+1)}\right) \\
&+ \frac{1}{m}\sum_{s=1}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(\mathbf{B}_{m-s}^{(r)}+1)\right) \\
&\times\left(-s!\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^s}\right) \\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbf{B}_m^{(r)} - \frac{1}{m(m+1)}\right) \\
&+ \frac{1}{m}\sum_{s=2}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(\mathbf{B}_{m-s}^{(r)}+1)\right)B_s(< x >) \\
&+ \Lambda_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z} \end{cases} \\
&= \frac{1}{m}\sum_{\substack{s=0 \\ s\neq 1}}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(\mathbf{B}_{m-s}^{(r)}+1)\right)B_s(< x >) \\
&+ \Lambda_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{4.18}$$

Now, we are ready to state our first result.

Theorem 4.1. *For each integer $l \geq 2$, let*

$$\Lambda_l = \sum_{k=1}^{l-1}\frac{\boldsymbol{B}_k^{(r+1)}}{k(l-k)} + \sum_{k=1}^{l-1}\frac{\boldsymbol{B}_{k-1}^{(r)}}{k(l-k)},$$

with $\Lambda_1 = 0$.

Assume that $\Lambda_m = 0$, *for an integer* $m \geq 2$. *Then we have the following.*
(a) $\sum_{k=1}^{m-1} \frac{1}{k(m-k)} \boldsymbol{B}_k^{(r+1)}(<x>) <x>^{m-k}$ *has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} \boldsymbol{B}_k^{(r+1)}(<x>) <x>^{m-k} \\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)} \boldsymbol{B}_m^{(r)} - \frac{1}{m(m+1)}\right) \\
&- \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \Big(\frac{1}{m} \sum_{s=1}^{m} \frac{(m)_s}{(2\pi i n)^s} \Big(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1} (\boldsymbol{B}_{m-s}^{(r)} + 1)\Big)\Big) e^{2\pi i n x},
\end{aligned}$$

for all $x \in (-\infty, \infty)$, *where the convergence is uniform.*

(b)

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} \boldsymbol{B}_k^{(r+1)}(<x>) <x>^{m-k} \\
&= \frac{1}{m} \sum_{\substack{s=0 \\ s \neq 1}}^{m} \binom{m}{s} \Big(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1} (\boldsymbol{B}_{m-s}^{(r)} + 1)\Big) B_s(<x>),
\end{aligned}$$

for all $x \in (-\infty, \infty)$, *where* $B_k(<x>)$ *is the Bernoulli function.*

Assume next that m is a positive integer ≥ 2, with $\Lambda_m \neq 0$. Then, $\gamma_m(1) \neq \gamma_m(0)$. Hence $\gamma_m(<x>)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\gamma_m(<x>)$ convergence pointwise to $\gamma_m(<x>)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}(\gamma_m(0) + \gamma_m(1)) = \gamma_m(0) + \frac{1}{2}\Lambda_m = \frac{1}{2}\Lambda_m,$$

for $x \in \mathbb{Z}$.

Next, we can state the second result.

Theorem 4.2. *For each integer* $l \geq 2$, *let*

$$\Lambda_l = \sum_{k=1}^{l-1} \frac{\boldsymbol{B}_k^{(r+1)}}{k(l-k)} + \sum_{k=1}^{l-1} \frac{\boldsymbol{B}_{k-1}^{(r)}}{k(l-k)},$$

with $\Lambda_1 = 0$.
Assume that $\Lambda_m \neq 0$, *for the an integer* $m \geq 2$. *Then we have the following.*
(a)

$$\begin{aligned}
&\frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)} \boldsymbol{B}_m^{(r)} - \frac{1}{m(m+1)}\right) \\
&- \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \Big(\frac{1}{m} \sum_{s=1}^{m} \frac{(m)_s}{(2\pi i n)^s} \Big(\Lambda_{m-s+1} + \frac{H_{m-1} - H_{m-s}}{m-s+1} (\boldsymbol{B}_{m-s}^{(r)} + 1)\Big)\Big) e^{2\pi i n x} \\
&= \begin{cases} \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \boldsymbol{B}_k^{(r+1)}(<x>) <x>^{m-k}, & \text{for } x \notin \mathbb{Z}, \\ \frac{1}{2}\Lambda_m, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned}$$

(b)

$$\frac{1}{m}\sum_{s=0}^{m}\binom{m}{s}\Big(\Lambda_{m-s+1}+\frac{H_{m-1}-H_{m-s}}{m-s+1}(\boldsymbol{B}_{m-s}^{(r)}+1)\Big)B_s(<x>)$$
$$=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\boldsymbol{B}_k^{(r+1)}(<x>)<x>^{m-k},\ for\ x\notin\mathbb{Z};$$

$$\frac{1}{m}\sum_{\substack{s=0\\ s\neq 1}}^{m}\binom{m}{s}\Big(\Lambda_{m-s+1}+\frac{H_{m-1}-H_{m-s}}{m-s+1}(\boldsymbol{B}_{m-s}^{(r)}+1)\Big)B_s(<x>)$$
$$=\frac{1}{2}\Lambda_m,\ for\ x\in\mathbb{Z}.$$

References

1. M. Abramowitz, IA. Stegun, *Handbook of Mathematical Functions*, Dover, New York, **1970**.
2. T. Arakawa, M. Kaneko, *On poly-Bernoulli numbers,* Comment. Math. Univ. St. Paul. **48**(1999), no. 2, 159–167.
3. A. Bayad, Y. Hamahata, *Multiple polylogarithms and multi-poly-Bernoulli polynomials,* Funct. Approx. Comment. Math.**46**(2012), part 1, 45–61.
4. L. Carlitz , *A note on Bernoulli numbers and polynomials*, Elem. Math., **29**(1974), 90-92.
5. G. V. Dunne, C. Schubert, *Bernoulli number identities from quantum field theory and topological string theory,* Commun. Number Theory Phys. **7(2)**(2013), 225-249.
6. C. Faber, R. Pandharipande, *Hodge integrals and Gromov-Witten theory,* Invent. Math. **139(1)**(2000), 173-199.
7. I. M. Gessel, *On Miki's identities for Bernoulli numbers,* J. Number Theory **110(1)**(2005), 75-82.
8. M. Kaneko, *Poly-Bernoulli numbers,* J. Theor. Nombres Bordeaux **9**(1997), no. 1, 221–228.
9. D.S. Kim, D. V. Dolgy, T. Kim, S.-H. Rim, *Some Formulae for the Product of Two Bernoulli and Euler Polynomials,* Abstr. Appl. Anal. 2012, Art. ID 784307.
10. D. S. Kim, T. Kim, *A note on poly-Bernoulli and higher-order poly-Bernoulli polynomials,* Russ. J. Math. Phys., **22(1)** (2015), 26–33.
11. D.S. Kim, T. Kim, *Bernoulli basis and the product of several Bernoulli polynomials,* Int. J. Math. Math. Sci. 2012, Art. ID 463659.
12. D.S. Kim, T. Kim, *Some identities of higher order Euler polynomials arising from Euler basis,* Integral Transforms Spec. Funct., **24(9)** (2013), 734-738.
13. D. S. Kim, T. Kim, *Higher-order Bernoulli and poly-Bernoulli mixed type polynomials,* Georgian Math. J. **22**(2015), no. 2, 265–272.
14. T. Kim, *Some identities for the Bernoulli, the Euler and Genocchi numbers and polynomials*, Adv. Stud. Contemp. Math., **20**(2010), no.1, 23-28.
15. T. Kim, D.S. Kim, S.-H. Rim, D.-V. Dolgy, *Fourier series of higher-order Bernoulli functions and their applications,* J. Inequal. Appl. 2017, 2017:8, 7pp.
16. H. Liu, and W. Wang, *Some identities on the the Bernoulli, Euler and Genocchi poloynomials via power sums and alternate power sums*, Disc. Math., **309**(2009), 3346-3363.
17. J. E. Marsden, *Elementary classical analysis,* W. H. Freeman and Company, 1974.
18. K. Shiratani, S. Yokoyama, *An application of p-adic convolutions,* Mem. Fac. Sci. Kyushu Univ. Ser. A **36(1)**(1982), 7383.
19. H. M. Srivastava, *Some generalizations and basic extensions of the Bernoulli, Euler and Genocchi polynomials*, Appl. Math. and Inf. Sci., **5**(2011), no. 3, 390-414.
20. P. T. Young, *Bernoulli and poly-Bernoulli polynomial convolutions and identities of p-adic Arakawa-Kaneko zeta functions*, J. Number Theory, **12**, no. 5, (2016), 1295-1309.
21. D. G. Zill, M. R. Cullen, *Advanced Engineering Mathematics,* Jones and Bartlett Publishers 2006.

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Fixed point results under constraint inequalities in Menger PM-spaces

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Abstract

In this paper, we discuss the existence of a fixed point of a self-mapping T on a Menger PM-space under two constraint inequalities with respect to two partial orders under two kinds of contractive conditions and obtain some new fixed point results. We then present several useful consequences of our main results. We also give examples to show the validity of our main results.
Keywords: Menger PM-space; fixed point; constraint inequalities; partial order; implicit contraction

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1 Introduction and preliminaries

In 1942, Menger introduced the concept of a probabilistic metric space which give birth to a new branch called probabilistic analysis [1]-[2]. The theory of PM-spaces and its applications has attracted much attention since then and fixed point theory for nonlinear operators in the setting of PM-spaces was studied by many authors [3]-[11].

The research on the existence of fixed points for mappings in a metric space equipped with a partial order was initiated by Turinici [12]. Ran and Reurings [13] obtained fixed point results for continuous monotone operators in a partially ordered metric space and applied them to study the existence of positive solutions to certain classes of nonlinear matrix equations. The results in [13] were generalized by many authors from different aspects (see *e.g.* [14]-[17]). On the other hand, the fixed point problems for mappings in partially ordered Menger PM-spaces were also extensively studied (see *e.g.* [18]-[21]).

Let $(X, \mathscr{F}, \Delta)$ be a Menger PM-space and X be endowed with two partial orders $\preceq_1$ and $\preceq_2$. Consider five self-mappings $T, A, B, C, D : X \to X$. In this paper, we are interested in the following problem: Find $x \in X$, such that

$$\begin{cases} x = Tx, \\ Ax \preceq_1 Bx, \\ Cx \preceq_2 Dx. \end{cases} \tag{1.1}$$

In [22], Jleli and Samet considered the existence of solutions to (1.1) in the framework of metric spaces. They introduced the concepts of d-regularity and $(A, B, C, D, \preceq_1, \preceq_2)$-stability, and obtained a fixed point result which guaranteed the existence of a fixed point of T under two constraint inequalities. In [23], Ansari *et al.* argued that the result of Jleli and Samet holds by assuming that only A and B are continuous (or only C and D are continuous), and proved that (1.1) has a unique solution. Moreover, they considered the existence of solutions to (1.1) under a certain implicit contraction by introducing a more general class of functions.

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In this paper, we will discuss the existence of a fixed point of a self-mapping T in the setting of Menger PM-spaces under two constraint inequalities with respect to two partial orders and obtain a new fixed point result, which extends the main results of [22] and [23] from metric spaces to Menger PM-spaces. As a consequence, we derive some corollaries of our main result. Also, some examples are given to show the validity of the new results.

We now recall some basic definitions in the theory of Menger PM-spaces.

A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a *distribution function* if it is nondecreasing left-continuous with $\sup_{t\in\mathbb{R}} F(t) = 1$ and $\inf_{t\in\mathbb{R}} F(t) = 0$.

We will denote by $\mathscr{D}$ the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \le 0, \\ 1, & t > 0. \end{cases}$$

Definition 1.1[5] A mapping $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a *triangular norm* (for short, a t-norm) if the following conditions are satisfied: $\Delta(a,1) = a$; $\Delta(a,b) = \Delta(b,a)$; $\Delta(a,c) \ge \Delta(b,d)$ for $a \ge b, c \ge d$;$\Delta(a,\Delta(b,c)) = \Delta(\Delta(a,b),c)$.

A typical example of a t-norm is Δ_{min} which is defined by $\Delta_{min}(a,b) = \min\{a,b\}$ for all $a,b \in [0,1]$.

Definition 1.2[5] A triplet $(X, \mathscr{F}, \Delta)$ is called a *Menger probabilistic metric space* (for short, a *Menger PM-space*) if X is a nonempty set, Δ is a t-norm and $\mathscr{F}$ is a mapping from $X \times X$ into $\mathscr{D}$ satisfying the following conditions (we denote $\mathscr{F}(x,y)$ by $F_{x,y}$):

(PM-1) $F_{x,y}(0) = 0$;

(PM-2) $F_{x,y}(t) = H(t)$ for all $t \in \mathbb{R}$ if and only if $x = y$;

(PM-3) $F_{x,y}(t) = F_{y,x}(t)$ for all $t \in \mathbb{R}$;

(PM-4) $F_{x,y}(t+s) \ge \Delta(F_{x,z}(t), F_{z,y}(s))$ for all $x,y,z \in X$ and $t,s \ge 0$.

Remark 1.1[5] If a Menger PM-space $(X, \mathscr{F}, \Delta)$ satisfies the condition $\sup_{0<t<1} \Delta(t,t) = 1$, then $(X, \mathscr{F}, \Delta)$ is a Hausdorff topological space in the (ϵ, λ)-topology $\mathscr{T}$, *i.e.*, the family of sets $\{U_x(\epsilon,\lambda) : \epsilon > 0, \lambda \in (0,1]\}(x \in X)$ is a basis of neighborhoods of a point x for $\mathscr{T}$, where $U_x(\epsilon,\lambda) = \{y \in X : F_{x,y}(\epsilon) > 1 - \lambda)\}$.

By virtue of the topology $\mathscr{T}$, a sequence $\{x_n\}$ is said to be $\mathscr{T}$-convergent to $x \in X$(we write $x_n \xrightarrow{\mathscr{T}} x(n \to \infty)$) if for any given $\epsilon > 0$ and $\lambda \in (0,1]$, there exists a positive integer $N = N(\epsilon,\lambda)$ such that $F_{x_n,x}(\epsilon) > 1 - \lambda$ whenever $n \ge N$, which is equivalent to $\lim_{n\to\infty} F_{x_n,x}(t) = 1$ for all $t > 0$; $\{x_n\}$ is called a $\mathscr{T}$-Cauchy sequence in $(X, \mathscr{F}, \Delta)$ if for any given $\epsilon > 0$ and $\lambda \in (0,1]$, there exists a positive integer $N = N(\epsilon,\lambda)$ such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ whenever $n,m \ge N$; $(X, \mathscr{F}, \Delta)$ is said to be $\mathscr{T}$-complete if each $\mathscr{T}$-Cauchy sequence in X is $\mathscr{T}$-convergent in X. It is worth noting that in a Menger PM-space, $\lim_{n\to\infty} x_n = x$ implies that $x_n \xrightarrow{\mathscr{T}} x(n \to \infty)$.

Remark 1.2[5] Let (X,d) be a metric space and $\mathscr{F} : X \times X \to \mathscr{D}$ be defined by

$$\mathscr{F}(x,y)(t) = F_{x,y}(t) = H(t - d(x,y)), \forall x,y \in X \text{ and } t > 0. \tag{1.2}$$

Then $(X, \mathscr{F}, \Delta_{min})$ is a $\mathscr{T}$-complete Menger PM-space induced by (X,d).

The following definitions and notations will be needed in the sequel.

Definition 1.3 Let $(X, \mathscr{F}, \Delta)$ be a Menger PM-space and $\preceq$ be partial order on X. $\preceq$ is called *F-regular*, if for any sequences $\{a_n\}, \{b_n\} \subset X$, we have

$$\lim_{n\to\infty} F_{a_n,a}(t) = \lim_{n\to\infty} F_{b_n,b}(t) = 1, \forall t > 0, \text{ and } a_n \preceq b_n \text{ for all } n \in \mathbb{N} \Longrightarrow a \preceq b,$$

where $(a,b) \in X \times X$.

Definition 1.4 Let X be a nonempty set endowed with two partial orders $\preceq_1$ and $\preceq_2$. Let $T, A, B, C, D : X \to X$ be five self-mappings. The mapping T is called $(A, B, C, D, \preceq_1, \preceq_2)$-*stable*, if the following condition is satisfied:

$$x \in X, Ax \preceq_1 Bx \Longrightarrow CTx \preceq_2 DTx.$$

Denote by Ψ the set of functions $\psi : [0, 1] \to [0, 1]$ satisfying the following conditions:

(ψ_1) ψ is lower semi-continuous;

(ψ_2) $\psi(t) = 0$ if and only if $t = 1$.

Denote by Φ the set of functions $\phi : [0, 1] \to [1, \infty)$ satisfying the following conditions:

(ϕ_1) ϕ is lower semi-continuous;

(ϕ_2) $\phi(t) = 1$ if and only if $t = 1$.

2 Main results

We are now ready to state and prove our main results.

Theorem 2.1 Let $(X, \mathscr{F}, \Delta)$ be a $\mathscr{T}$-complete Menger PM-space endowed with two partial orders $\preceq_1$ and $\preceq_2$, and $T, A, B, C, D : X \to X$ be self-mappings. Suppose that the following conditions are satisfied:

(i) $\preceq_i$ is F-regular, $i = 1, 2$;

(ii) A and B are $\mathscr{T}$-continuous or C and D are $\mathscr{T}$-continuous;

(iii) there exists $x_0 \in X$, such that $Ax_0 \preceq_1 Bx_0$;

(iv) T is $(A, B, C, D, \preceq_1, \preceq_2)$-stable;

(v) T is $(C, D, A, B, \preceq_2, \preceq_1)$-stable;

(vi) there exists $\psi \in \Psi$ such that

$$Ax \preceq_1 Bx, Cy \preceq_2 Dy \Longrightarrow F_{Tx,Ty}(t) \geq F_{x,y}(t) + \psi(F_{x,y}(t)), \forall t > 0.$$

Then the sequence $\{T^n x_0\}$ converges to some $x^* \in X$, which is a solution to (1.1). Moreover, x^* is the unique solution to (1.1).

Proof. By (iii), there exists $x_0 \in X$ such that $Ax_0 \preceq_1 Bx_0$. Construct a sequence $\{x_n\} \subset X$ by

$$x_n = T^n x_0, \;\; n = 0, 1, 2, \cdots .$$

By (iv), we have $CTx_0 \preceq_2 DTx_0$, that is, $Cx_1 \preceq_2 Dx_1$. Thus, we get $Ax_0 \preceq_1 Bx_0$ and $Cx_1 \preceq_2 Dx_1$. By (v), we have $ATx_1 \preceq_1 BTx_1$, that is, $Ax_2 \preceq_1 Bx_2$. Again, by (iv), we obtain $CTx_2 \preceq_2 DTx_2$, that is, $Cx_3 \preceq_2 Dx_3$. Thus, we get $Ax_2 \preceq_1 Bx_2$ and $Cx_3 \preceq_2 Dx_3$. Continuing this process, we obtain

$$Ax_{2n} \preceq_1 Bx_{2n} \text{ and } Cx_{2n+1} \preceq_2 Dx_{2n+1}, n = 0, 1, 2, \cdots . \tag{2.1}$$

By (2.1) and (vi), we have

$$F_{x_{n+1},x_n}(t) = F_{Tx_n,Tx_{n-1}}(t) \geq F_{x_n,x_{n-1}}(t) + \psi(F_{x_n,x_{n-1}}(t)), \forall t > 0, n = 1, 2, \cdots , \tag{2.2}$$

which yields that

$$F_{x_{n+1},x_n}(t) \geq F_{x_n,x_{n-1}}(t), \forall t > 0, n = 1, 2, \cdots .$$

Thus, $\{F_{x_{n+1},x_n}(t)\}$ is an increasing sequence of positive numbers for each $t > 0$. Therefore, there exists some $r(t) \in [0, 1]$, such that

$$\lim_{n\to\infty} F_{x_{n+1},x_n}(t) = r(t), \forall t > 0. \tag{2.3}$$

By (2.2), we have

$$\liminf_{n\to\infty} F_{x_{n+1},x_n}(t) \geq \liminf_{n\to\infty}(F_{x_n,x_{n-1}}(t) + \psi(F_{x_n,x_{n-1}}(t))), \forall t > 0.$$

Using (2.3) and (ψ_1), we obtain

$$r(t) \geq r(t) + \psi(r(t)), \forall t > 0,$$

which implies that $\psi(r(t)) = 0, \forall t > 0$. By (ψ_2), we get $r(t) = 1, \forall t > 0$, i.e.,

$$\lim_{n\to\infty} F_{x_{n+1},x_n}(t) = 1, \forall t > 0. \tag{2.4}$$

We now show that $\{x_n\}$ is a $\mathscr{T}$-Cauchy sequence in $(X, \mathscr{F}, \Delta)$. Suppose that this is not true. Then there exists $\epsilon_0 > 0$ and $\lambda_0 \in (0, 1]$, for which we can find two sequences of positive integers $\{m_k\}$ and $\{n_k\}$, such that for all positive integers k, we have

$$n(k) > m(k) > k, \;\; F_{x_{m(k)},x_{n(k)}}(\epsilon_0) \leq 1 - \lambda_0, \;\; F_{x_{m(k)},x_{n(k)-1}}(\epsilon_0) > 1 - \lambda_0. \tag{2.5}$$

For any $\delta \in (0, \epsilon_0)$, we have

$$F_{x_{m(k)},x_{n(k)}}(\epsilon_0) \geq \Delta(F_{x_{m(k)},x_{n(k)-1}}(\epsilon_0 - \delta), F_{x_{n(k)-1},x_{n(k)}}(\delta)).$$

Letting $k \to \infty$, by (2.4), we have

$$\liminf_{k\to\infty} F_{x_{m(k)},x_{n(k)}}(\epsilon_0) \geq \Delta(\liminf_{k\to\infty} F_{x_{m(k)},x_{n(k)-1}}(\epsilon_0 - \delta), 1) = \liminf_{k\to\infty} F_{x_{m(k)},x_{n(k)-1}}(\epsilon_0 - \delta).$$

Letting $\delta \to 0$, by the left-continuity of the distribution function and (2.5), we obtain

$$1 - \lambda_0 \geq \liminf_{k\to\infty} F_{x_{m(k)},x_{n(k)}}(\epsilon_0) \geq \liminf_{k\to\infty} F_{x_{m(k)},x_{n(k)-1}}(\epsilon_0) \geq 1 - \lambda_0,$$

which implies that

$$\liminf_{k\to\infty} F_{x_{m(k)},x_{n(k)}}(\epsilon_0) = 1 - \lambda_0. \tag{2.6}$$

On the other hand, for any $\delta \in (0, \epsilon_0)$, we have

$$F_{x_{m(k)},x_{n(k)+1}}(\epsilon_0) \geq \Delta(F_{x_{m(k)},x_{n(k)}}(\epsilon_0 - \delta), F_{x_{n(k)},x_{n(k)+1}}(\delta))$$

and

$$F_{x_{m(k)},x_{n(k)}}(\epsilon_0) \geq \Delta(F_{x_{m(k)},x_{n(k)+1}}(\epsilon_0 - \delta), F_{x_{n(k)+1},x_{n(k)}}(\delta)).$$

Letting $k \to \infty$, by (2.4), (2.6) and the left-continuity of the distribution function, we have

$$\liminf_{k\to\infty} F_{x_{m(k)},x_{n(k)+1}}(\epsilon_0) \geq 1 - \lambda_0 \;\; \text{and} \;\; 1 - \lambda_0 \geq \liminf_{k\to\infty} F_{x_{m(k)},x_{n(k)+1}}(\epsilon_0),$$

which implies that

$$\liminf_{k\to\infty} F_{x_{m(k)},x_{n(k)+1}}(\epsilon_0) = 1 - \lambda_0. \tag{2.7}$$

Similarly, for any $\delta \in (0, \epsilon_0)$, we have

$$F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0) \geq \Delta(F_{x_{m(k)},x_{n(k)}}(\epsilon_0 - \delta), F_{x_{m(k)-1},x_{m(k)}}(\delta))$$

and

$$F_{x_{m(k)},x_{n(k)}}(\epsilon_0) \geq \Delta(F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0 - \delta), F_{x_{m(k)-1},x_{m(k)}}(\delta)).$$

Letting $k \to \infty$, by (2.4), (2.6) and the left-continuity of the distribution function, we have

$$\liminf_{k\to\infty} F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0) \geq 1-\lambda_0 \ \text{ and } \ 1-\lambda_0 \geq \liminf_{k\to\infty} F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0),$$

which implies that

$$\liminf_{k\to\infty} F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0) = 1-\lambda_0. \tag{2.8}$$

Also, for any $\delta \in (0, \epsilon_0)$, we have

$$F_{x_{n(k)+1},x_{m(k)}}(\epsilon_0) \geq \Delta(F_{x_{n(k)+1},x_{m(k)+1}}(\epsilon_0-\delta), F_{x_{m(k)},x_{m(k)+1}}(\delta))$$

and

$$F_{x_{n(k)+1},x_{m(k)+1}}(\epsilon_0) \geq \Delta(F_{x_{n(k)+1},x_{m(k)}}(\epsilon_0-\delta), F_{x_{m(k)},x_{m(k)+1}}(\delta)).$$

Letting $k \to \infty$, we can similarly obtain

$$\liminf_{k\to\infty} F_{x_{n(k)+1},x_{m(k)+1}}(\epsilon_0) = 1-\lambda_0. \tag{2.9}$$

Note that for all k, there exists a positive integer $0 \leq i(k) \leq 1$ such that

$$n(k) - m(k) + i(k) \equiv 1(2).$$

By (2.1), for all $k > 1$, we have

$$Ax_{n(k)} \preceq_1 Bx_{n(k)} \ \text{ and } \ Cx_{m(k)-i(k)} \preceq_2 Dx_{m(k)-i(k)}$$

or

$$Ax_{m(k)-i(k)} \preceq_1 Bx_{m(k)-i(k)} \ \text{ and } \ Cx_{n(k)} \preceq_2 Dx_{n(k)}.$$

Then from (vi), we have

$$F_{Tx_{n(k)},Tx_{m(k)-i(k)}}(t) \geq F_{x_{n(k)},x_{m(k)-i(k)}}(t) + \psi(F_{x_{n(k)},x_{m(k)-i(k)}}(t)), \forall k \in \mathbb{Z}^+ \text{ and } t > 0,$$

that is,

$$F_{x_{n(k)+1},x_{m(k)-i(k)+1}}(t) \geq F_{x_{n(k)},x_{m(k)-i(k)}}(t) + \psi(F_{x_{n(k)},x_{m(k)-i(k)}}(t)), \forall k \in \mathbb{Z}^+ \text{ and } t > 0. \tag{2.10}$$

Set

$$\Gamma_1 := \{k > 1 | i(k) = 0\} \ \text{ and } \ \Gamma_2 := \{k > 1 | i(k) = 1\}.$$

Now consider the following two cases.

• Case 1. Γ_1 is a countably infinite set. By (2.10), we get

$$F_{x_{n(k)+1},x_{m(k)+1}}(\epsilon_0) \geq F_{x_{n(k)},x_{m(k)}}(\epsilon_0) + \psi(F_{x_{n(k)},x_{m(k)}}(\epsilon_0)), \forall k \in \Gamma_1,$$

which yields that

$$\liminf_{k\to\infty} F_{x_{n(k)+1},x_{m(k)+1}}(\epsilon_0) \geq \liminf_{k\to\infty}(F_{x_{n(k)},x_{m(k)}}(\epsilon_0) + \psi(F_{x_{n(k)},x_{m(k)}}(\epsilon_0))).$$

Combining (2.6), (2.9) and (ψ_1), we obtain

$$1-\lambda_0 \geq 1-\lambda_0 + \psi(1-\lambda_0),$$

which implies that $\psi(1-\lambda_0) = 0$. By (ψ_2), we get $\lambda_0 = 0$, which is in contradiction with $\lambda_0 > 0$.

- Case 2. Γ_1 is a finite set. In this case, Γ_2 must be a countably infinite set. By (2.10), we get

$$F_{x_{n(k)+1},x_{m(k)}}(\epsilon_0) \geq F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0) + \psi(F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0)), \forall k \in \Gamma_2,$$

which yields that

$$\liminf_{k\to\infty} F_{x_{n(k)+1},x_{m(k)}}(\epsilon_0) \geq \liminf_{k\to\infty}(F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0) + \psi(F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0))).$$

Combining (2.7), (2.8) and (ψ_1), we obtain

$$1-\lambda_0 \geq 1-\lambda_0+\psi(1-\lambda_0),$$

which implies that $\psi(1-\lambda_0)=0$. By (ψ_2), we get $\lambda_0=0$, which is in contradiction with $\lambda_0>0$.

Therefore, we deduce that $\{x_n\}$ is a $\mathscr{T}$-Cauchy sequence in $(X,\mathscr{F},\Delta)$. Since $(X,\mathscr{F},\Delta)$ is $\mathscr{T}$-complete, there exists $x^* \in X$, such that $x_n \xrightarrow{\mathscr{T}} x^*(n\to\infty)$, i.e.,

$$\lim_{n\to\infty} F_{x_n,x^*}(t) = 1, \forall t>0. \tag{2.11}$$

On the other hand, by (2.1), we have

$$Ax_{2n} \preceq_1 Bx_{2n}, n=0,1,2,\cdots.$$

Suppose first that A and B are $\mathscr{T}$-continuous, by (2.11), we have $Ax_{2n} \xrightarrow{\mathscr{T}} Ax^*(n\to\infty)$ and $Bx_{2n} \xrightarrow{\mathscr{T}} Bx^*(n\to\infty)$, i.e.,

$$\lim_{n\to\infty} F_{Ax_{2n},Ax^*}(t) = \lim_{n\to\infty} F_{Bx_{2n},Bx^*}(t) = 1, \forall t>0.$$

Since $\preceq_1$ is F-regular, we get

$$Ax^* \preceq_1 Bx^*. \tag{2.12}$$

On the other hand, by (2.1), (2.12) and condition (vi), for all $t>0$ and any $\delta\in(0,t)$, we obtain

$$\begin{aligned} F_{Tx^*,x^*}(t) &\geq \Delta(F_{Tx^*,Tx_{2n+1}}(t-\delta), F_{x_{2n+2},x^*}(\delta)) \\ &\geq \Delta(F_{x^*,x_{2n+1}}(t-\delta)+\psi(F_{x^*,x_{2n+1}}(t-\delta)), F_{x_{2n+2},x^*}(\delta)), n=0,1,2,\cdots. \end{aligned}$$

Therefore, we get

$$F_{Tx^*,x^*}(t) \geq \Delta(\liminf_{n\to\infty} F_{x^*,x_{2n+1}}(t-\delta)+\psi(F_{x^*,x_{2n+1}}(t-\delta)), \liminf_{n\to\infty} F_{x_{2n+2},x^*}(\delta)), n=0,1,2,\cdots.$$

Using (ψ_1), (ψ_2) and (2.11), we obtain

$$F_{Tx^*,x^*}(t) \geq \Delta(1+\psi(1),1) = \Delta(1,1) = 1, \forall t>0,$$

which implies that

$$Tx^* = x^*. \tag{2.13}$$

Now, since T is $(A,B,C,D,\preceq_1,\preceq_2)$-stable, from (2.12), we get

$$CTx^* \preceq DTx^*,$$

which implies from (2.13) that

$$Cx^* \preceq Dx^*. \tag{2.14}$$

As a consequence, it follows from (2.12), (2.13) and (2.14) that $x^* \in X$ is a solution to (1.1).

Suppose now that $y^* \in X$ is another solution to (1.1), that is,

$$Ty^* = y^*, Ay^* \preceq_1 By^*, Cy^* \preceq_2 Dy^*, \text{ and } F_{x^*,y^*}(t_0) < 1 \text{ for some } t_0 > 0.$$

By condition (vi) and (ψ_2), we obtain

$$F_{x^*,y^*}(t_0) = F_{Tx^*,Ty^*}(t_0) \geq F_{x^*,y^*}(t_0) + \psi(F_{x^*,y^*}(t_0)) > F_{x^*,y^*}(t_0),$$

which is a contradiction. Therefore, x^* is the unique solution to (1.1).

If we require only the $\mathscr{T}$-continuity of C and D by condition (ii), we can also deduce the conclusions using similar arguments. This completes the proof.

Example 2.1 Let $X = \mathbb{R}$ be the set of all real numbers equipped with the standard order $\leq$. Take $\preceq_1 = \preceq_2 = \leq$. Let X be endowed with the standard metric $d(x,y) = |x-y|, (x,y) \in X \times X$ and $\mathscr{F} : X \times X \to \mathscr{D}$ is defined by (1.2). Then $(X, \mathscr{F}, \Delta_{min})$ is a $\mathscr{T}$-complete Menger PM-space induced by (X,d). Let $T : X \to X$ be defined by

$$Tx = \begin{cases} -1, & \text{if } x < 0, \\ 2, & \text{if } x \geq 0, \end{cases}$$

and $A, B, C, D : X \to X$ be defined by

$$Ax = 2x^2, \; Bx = 6x, \; Cx = 2,$$

$$Dx = \begin{cases} 0, & \text{if } x < 2, \\ x, & \text{if } x \geq 2. \end{cases}$$

It is obvious that $\preceq_i$ is F-regular, $i = 1, 2$. Moreover, A and B are $\mathscr{T}$-continuous. Also, for $x_0 = 1$, we have $Ax_0 = 2 \leq 6 = Bx_0$. If for some $x \in X$, we have $Ax \leq Bx$, then $0 \leq x \leq 3$, which yields $CTx = 2 \leq 3 = D3 = DTx$. So T is $(A, B, C, D, \preceq_1, \preceq_2)$-stable. If for some $x \in X$, we have $Cx \leq Dx$, then $x \geq 2$, which yields $ATx = A3 = 18 = B3 = BTx$. So T is $(C, D, A, B, \preceq_1, \preceq_2)$-stable. Note that for any $(x,y) \in X \times X$, we have

$$Ax \leq Bx, Cy \leq Dy \Longrightarrow (x,y) \in [0,3] \times [2,\infty) \Longrightarrow (Tx, Ty) = (3,3).$$

Therefore,

$$Ax \preceq_1 Bx, Cy \preceq_2 Dy \Longrightarrow F_{Tx,Ty}(t) = 1 \geq F_{x,y}(t) + \psi(F_{x,y}(t)), \forall t > 0,$$

where $\psi(x) = 1 - x$ for $x \in [0,1]$. By Theorem 2.1, (1.1) has a unique solution, which is $x^* = 3$. On the other hand, observe that D is not $\mathscr{T}$-continuous. So we don't need A, B, C, D are all $\mathscr{T}$-continuous to guarantee the existence of the solution.

Remark 2.1 Note that although the conditions of Theorem 2.1 are sufficient to guarantee the existence of a unique solution to (1.1), T need not have a unique fixed point, as the above example shows (in fact, the mapping T has two fixed points -2 and 3).

Denote by L the class of all functions $f : [0,1] \times [1, \infty) \to \mathbb{R}$ satisfying

(L_1) f is $\mathscr{T}$-continuous;

(L_2) $f(a,b) \geq a, \forall a \in [0,1]$ and $b \in [1,\infty)$;

(L_3) $a \in [0,1], b \in [1,\infty), f(a,b) = a \Longrightarrow b = 1$;

(L_4) $\forall a \in [0,1], b_1 \geq b_2 \Longrightarrow f(a,b_1) \geq f(a,b_2)$.

We give the second main result of this paper by utilizing this new class of functions.

Theorem 2.2 Let $(X, \mathscr{F}, \Delta)$ be a $\mathscr{T}$-complete Menger PM-space endowed with two partial orders $\preceq_1$ and $\preceq_2$, and $T, A, B, C, D : X \to X$ be self-mappings. Suppose that the following conditions are satisfied:

(i) $\preceq_i$ is F-regular, $i = 1, 2$;

(ii) A and B are $\mathscr{T}$-continuous or C and D are $\mathscr{T}$-continuous;

(iii) there exists $x_0 \in X$, such that $Ax_0 \preceq_1 Bx_0$;

(iv) T is $(A, B, C, D, \preceq_1, \preceq_2)$-stable;

(v) T is $(C, D, A, B, \preceq_2, \preceq_1)$-stable;

(vi) there exists $\phi \in \Phi$ and $f \in L$ such that

$$Ax \preceq_1 Bx, Cy \preceq_2 Dy \Longrightarrow F_{Tx,Ty}(t) \geq f(F_{x,y}(t), \phi(F_{x,y}(t))), \forall t > 0.$$

Then the sequence $\{T^n x_0\}$ converges to some $x^* \in X$, which is a solution to (1.1). Moreover, x^* is the unique solution to (1.1).

Proof. By (iii), there exists $x_0 \in X$ such that $Ax_0 \preceq_1 Bx_0$. Construct a sequence $\{x_n\} \subset X$ by

$$x_n = T^n x_0, \quad n = 0, 1, 2, \cdots.$$

By similar arguments in the proof of Theorem 2.1, we obtain (2.1).

By (2.1) and (vi), we have

$$F_{x_{n+1},x_n}(t) = F_{Tx_n,Tx_{n-1}}(t) \geq f(F_{x_n,x_{n-1}}(t), \phi(F_{x_n,x_{n-1}}(t))), \forall t > 0, n = 1, 2, \cdots, \tag{2.15}$$

which by (L_2) yields that

$$F_{x_{n+1},x_n}(t) \geq F_{x_n,x_{n-1}}(t), \forall t > 0, n = 1, 2, \cdots.$$

Thus, $\{F_{x_{n+1},x_n}(t)\}$ is an increasing sequence of positive numbers for each $t > 0$. Therefore, there exists some $p(t) \in [0, 1]$, such that

$$\lim_{n\to\infty} F_{x_{n+1},x_n}(t) = p(t), \forall t > 0. \tag{2.16}$$

By the properties of f and ϕ, we obtain

$$p(t) \geq f(p(t), \phi(p(t))) \geq p(t),$$

which yields that

$$f(p(t), \phi(p(t))) = p(t).$$

Hence, by (L_1), we get $\phi(p(t)) = 1, \forall t > 0$, and thus $p(t) = 1, \forall t > 0$, that is,

$$\lim_{n\to\infty} F_{x_{n+1},x_n}(t) = 1, \forall t > 0. \tag{2.17}$$

We now show that $\{x_n\}$ is a $\mathscr{T}$-Cauchy sequence in $(X, \mathscr{F}, \Delta)$. Suppose that this is not true. Then there exists $\epsilon_0 > 0$ and $\lambda_0 \in (0, 1]$, for which we can find two sequences of positive integers $\{m_k\}$ and $\{n_k\}$, such that for all positive integers k, (2.5) holds. By similar arguments in the proof of Theorem 2.1, (2.6)−(2.9) hold.

Note that for all k, there exists a positive integer $0 \leq i(k) \leq 1$ such that

$$n(k) - m(k) + i(k) \equiv 1(2).$$

Similarly, by (2.1) and (vi), we have

$$F_{x_{n(k)+1},x_{m(k)-i(k)+1}}(t) \geq f(F_{x_{n(k)},x_{m(k)-i(k)}}(t) + \phi(F_{x_{n(k)},x_{m(k)-i(k)}}(t))), \forall k \in \mathbb{Z}^+ \text{ and } t > 0. \tag{2.18}$$

Set

$$\Gamma_1 := \{k > 1 | i(k) = 0\} \text{ and } \Gamma_2 := \{k > 1 | i(k) = 1\}.$$

Now consider the following two cases.

• Case 1. Γ_1 is a countably infinite set. By (2.18), we get

$$F_{x_{n(k)+1},x_{m(k)+1}}(\epsilon_0) \geq f(F_{x_{n(k)},x_{m(k)}}(\epsilon_0), \phi(F_{x_{n(k)},x_{m(k)}}(\epsilon_0))), \forall k \in \Gamma_1,$$

which yields that

$$\liminf_{k\to\infty} F_{x_{n(k)+1},x_{m(k)+1}}(\epsilon_0) \geq \liminf_{k\to\infty} f(F_{x_{n(k)},x_{m(k)}}(\epsilon_0), \phi(F_{x_{n(k)},x_{m(k)}}(\epsilon_0))).$$

Combining (2.6), (2.9) and the properties of f and ϕ, we obtain

$$1-\lambda_0 \geq f(1-\lambda_0, \phi(1-\lambda_0)) \geq 1-\lambda_0,$$

which implies that

$$f(1-\lambda_0, \phi(1-\lambda_0)) = 1-\lambda_0.$$

Again, by (L_3) and (ϕ_2), we get $\lambda_0 = 0$, which is in contradiction with $\lambda_0 > 0$.

• Case 2. Γ_1 is a finite set. In this case, Γ_2 must be a countably infinite set. By (2.18), we get

$$F_{x_{n(k)+1},x_{m(k)}}(\epsilon_0) \geq f(F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0), \phi(F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0))), \forall k \in \Gamma_2,$$

which yields that

$$\liminf_{k\to\infty} F_{x_{n(k)+1},x_{m(k)}}(\epsilon_0) \geq \liminf_{k\to\infty} f(F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0) + \phi(F_{x_{n(k)},x_{m(k)-1}}(\epsilon_0))).$$

Combining (2.7), (2.8) and (ϕ_1), we also obtain

$$1-\lambda_0 \geq f(1-\lambda_0, \phi(1-\lambda_0)) \geq 1-\lambda_0.$$

It follows from the properties of f and ϕ that $\lambda_0 = 0$, which is a contradiction.

Therefore, we deduce that $\{x_n\}$ is a $\mathscr{T}$-Cauchy sequence in $(X, \mathscr{F}, \Delta)$. Since $(X, \mathscr{F}, \Delta)$ is $\mathscr{T}$-complete, there exists $x^* \in X$, such that $x_n \stackrel{\mathscr{T}}{\to} x^*(n \to \infty)$, i.e.,

$$\lim_{n\to\infty} F_{x_n,x^*}(t) = 1, \forall t > 0. \tag{2.19}$$

We only consider the case when A and B are $\mathscr{T}$-continuous (the proof is similar when C and D are $\mathscr{T}$-continuous). By (2.19), we have

$$\lim_{n\to\infty} F_{Ax_{2n},Ax^*}(t) = \lim_{n\to\infty} F_{Bx_{2n},Bx^*}(t) = 1, \forall t > 0.$$

Since $\preceq_1$ is F-regular, noting that (2.1) holds, we get

$$Ax^* \preceq_1 Bx^*. \tag{2.20}$$

On the other hand, by (2.1), (2.12) and condition (vi), for all $t > 0$ and any $\delta \in (0, t)$, we obtain

$$\begin{aligned} F_{Tx^*,x^*}(t) &\geq \Delta(F_{Tx^*,Tx_{2n+1}}(t-\delta), F_{x_{2n+2},x^*}(\delta)) \\ &\geq \Delta(f(F_{x^*,x_{2n+1}}(t-\delta), \phi(F_{x^*,x_{2n+1}}(t-\delta))), F_{x_{2n+2},x^*}(\delta)), n = 0, 1, 2, \cdots. \end{aligned}$$

By (2.19) and the properties of f and ϕ, we obtain

$$F_{Tx^*,x^*}(t) \geq \Delta(f(1, \phi(1)), 1) = f(1, \phi(1)) = f(1, 1) \geq 1, \forall t > 0,$$

which implies that

$$Tx^* = x^*. \tag{2.21}$$

Now, since T is $(A, B, C, D, \preceq_1, \preceq_2)$-stable, from (2.20), we get

$$CTx^* \preceq DTx^*,$$

which implies from (2.21) that

$$Cx^* \preceq Dx^*. \tag{2.22}$$

As a consequence, it follows from (2.20), (2.21) and (2.22) that $x^* \in X$ is a solution to (1.1).

Suppose now that $y^* \in X$ is another solution to (1.1), that is,

$$Ty^* = y^*, Ay^* \preceq_1 By^*, Cy^* \preceq_2 Dy^*, \text{ and } F_{x^*,y^*}(t_0) < 1 \text{ for some } t_0 > 0.$$

By condition (vi) and (ϕ_2), we obtain

$$F_{x^*,y^*}(t_0) = F_{Tx^*,Ty^*}(t_0) \geq f(F_{x^*,y^*}(t_0), \phi(F_{x^*,y^*}(t_0))) \geq F_{x^*,y^*}(t_0),$$

which yields that

$$f(F_{x^*,y^*}(t_0), \phi(F_{x^*,y^*}(t_0))) = F_{x^*,y^*}(t_0).$$

By (L_3) and (ϕ_2), we get $F_{x^*,y^*}(t_0) = 1$, which is a contradiction. Therefore, x^* is the unique solution to (1.1). This completes the proof.

3 Some consequences

Setting $\preceq_1 = \preceq_2 = \preceq$, $C = B$ and $D = A$, problem (1.1) becomes the following one: Find $x \in X$, such that

$$\begin{cases} x = Tx, \\ Ax = Bx, \end{cases} \tag{3.1}$$

where $T, A, B : X \to X$ are given self-mappings and $(X, \mathscr{F}, \Delta)$ is a Menger PM-space endowed with a partial order $\preceq$.

Letting $\preceq_1 = \preceq_2 = \preceq$, $C = B$ and $D = A$, we can obtain the following corollary from Theorem 2.2.

Corollary 3.1 Let $(X, \mathscr{F}, \Delta)$ be a $\mathscr{T}$-complete Menger PM-space endowed with a partial order $\preceq$, and $T, A, B : X \to X$ be self-mappings. Suppose that the following conditions are satisfied:

(i) $\preceq$ is F-regular;

(ii) A and B are $\mathscr{T}$-continuous;

(iii) there exists $x_0 \in X$, such that $Ax_0 \preceq_1 Bx_0$;

(iv) $x \in X, Ax \preceq Bx \Longrightarrow BTx \preceq ATx$;

(v) $x \in X, Bx \preceq Ax \Longrightarrow ATx \preceq BTx$;

(vi) there exists $\phi \in \Phi$ and $f \in L$ such that

$$Ax \preceq Bx, By \preceq_2 Ay \Longrightarrow F_{Tx,Ty}(t) \geq f(F_{x,y}(t), \phi(F_{x,y}(t))), \forall t > 0.$$

Then the sequence $\{T^n x_0\}$ converges to some $x^* \in X$, which is a solution to (3.1). Moreover, x^* is the unique solution to (3.1).

Next, we provide an example to show the validity of Corollary 3.1.

Example 3.1 Let $X = \{(6, 2), (2, 2), (8, 2), (5, 2), (5, 3)\} \subset \mathbb{R}^2$ and $\preceq$ be the partial order on X defined by

$$(x_1, y_1), (x_2, y_2) \in X, \ \ (x_1, y_1) \preceq (x_2, y_2) \Longleftrightarrow x_1 \leq x_2, \ y_1 \leq y_2.$$

Let X be endowed with the metric d defined by

$$d((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2},\ \text{for } x=(x_1,y_1), y=(x_2,y_2)\in X.$$

Let $\mathscr{F}: X\times X\to\mathscr{D}$ be defined by (1.2). Then it follows from Remark 1.2 that $(X,\mathscr{F},\Delta_{min})$ is a $\mathscr{T}$-complete Menger PM-space.

Let $T,A,B : X\to X$ be the mappings defined by

$$\begin{cases} T(6,2)=(6,2),\\ T(2,2)=(8,2),\\ T(8,2)=(5,2),\\ T(5,2)=(6,2),\\ T(5,3)=(2,2),\end{cases} \quad \begin{cases} A(6,2)=(6,2),\\ A(2,2)=(4,2),\\ A(8,2)=(3,2),\\ A(5,2)=(7,2),\\ A(5,3)=(4,3),\end{cases} \quad \text{and} \quad \begin{cases} B(6,2)=(6,2),\\ B(2,2)=(5,2),\\ B(8,2)=(2,2),\\ B(5,2)=(8,2),\\ B(5,3)=(6,2).\end{cases}$$

It is obvious that

$$u\in X,\ u\preceq Bu \Longleftrightarrow u\in\{(6,2),(2,2),(2,2)\}$$

and

$$v\in X,\ Bv\preceq v \Longleftrightarrow v\in\{(6,2),(8,2)\}.$$

By the above definitions, it is easy to check that

$$u\in X, Au\preceq Bu \Longrightarrow BTu\preceq ATu$$

and

$$v\in X, Bv\preceq Av \Longrightarrow ATv\preceq BTv.$$

Now, let $(u,v)\in X\times X$ satisfies $Au\preceq Bu$ and $Bv\preceq Av$. Then we have

$$(u,v)\in\{((6,2),(6,2)),((6,2),(8,2)),((2,2),(6,2)),((2,2),(8,2)),((5,2),(6,2)),((5,2),(8,2))\}.$$

For $(u,v)=((6,2),(6,2))$, we have

$$F_{Tu,Tv}(t)=H(t-d(Tu,Tv))=H(t)=1\geq 1\cdot\phi(1)=F_{u,v}(t)\cdot\phi(F_{u,v}(t)), \forall t>0.$$

For $(u,v)=((6,2),(8,2))$, we have

$$F_{Tu,Tv}(t)=H(t-d(Tu,Tv))=H(t-1)\geq H(t-2)\cdot\phi(H(t-2))=F_{u,v}(t)\cdot\phi(F_{u,v}(t)), \forall t>0.$$

For $(u,v)=((2,2),(6,2))$, we have

$$F_{Tu,Tv}(t)=H(t-d(Tu,Tv))=H(t-2)\geq H(t-4)\cdot\phi(H(t-4))=F_{u,v}(t)\cdot\phi(F_{u,v}(t)), \forall t>0.$$

For $(u,v)=((2,2),(8,2))$, we have

$$F_{Tu,Tv}(t)=H(t-d(Tu,Tv))=H(t-3)\geq H(t-6))\cdot\phi(H(t-6))=F_{u,v}(t)\cdot\phi(F_{u,v}(t)), \forall t>0.$$

For $(u,v)=((5,2),(6,2))$, we have

$$F_{Tu,Tv}(t)=H(t-d(Tu,Tv))=H(t)=1\geq H(t-1)\cdot\phi(H(t-1))=F_{u,v}(t)\cdot\phi(F_{u,v}(t)), \forall t>0.$$

For $(u,v)=((5,2),(8,2))$, we have

$$F_{Tu,Tv}(t)=H(t-d(Tu,Tv))=H(t-1)\geq H(t-3)\cdot\phi(H(t-3))=F_{u,v}(t)\cdot\phi(F_{u,v}(t)), \forall t>0.$$

Then we have

$$Au \preceq Bu, Bv \preceq Av \Longrightarrow F_{Tu,Tv}(t) \geq f(F_{u,v}(t), \psi(F_{u,v}(t))), \forall t > 0,$$

where $f(a,b) = a \cdot b, \forall a \in [0,1]$ and $b \in [1,\infty)$. Therefore, all the conditions of Corollary 3.1 are satisfied and thus (3.1) has a unique solution. In fact, we observe that $x^* = (6,2)$ is the unique solution to (3.1). We would like to point out that there exists $t_0 = 3 > 0$, such that

$$F_{T(6,2),T(5,3)}(t_0) = H(t_0 - 4) = 0 < 1 = H(t_0 - \sqrt{2}) = F_{(6,2),(5,3)}(t_0),$$

which shows that T is not a SB-contraction on $(X, \mathscr{F}, \Delta)$.

Furthermore, setting $A = I_X$ in (3.1), where I_X denotes the identity mapping on X, we get another problem: Find $x \in X$, such that

$$\begin{cases} x = Tx, \\ x = Bx, \end{cases} \tag{3.2}$$

where $T, B : X \to X$ are given self-mappings and $(X, \mathscr{F}, \Delta)$ is a Menger PM-space endowed with a partial order $\preceq$. This problem is to find a common fixed point of two self-mappings on a partially ordered Menger PM-space.

Taking $A = I_X$ in Corollary 3.1, we obtain the following result immediately.

Corollary 3.2 Let $(X, \mathscr{F}, \Delta)$ be a $\mathscr{T}$-complete Menger PM-space endowed with a partial order $\preceq$, and $T, B : X \to X$ be self-mappings. Suppose that the following conditions are satisfied:

(i) $\preceq$ is F-regular;

(ii) B is $\mathscr{T}$-continuous;

(iii) there exists $x_0 \in X$, such that $x_0 \preceq_1 Bx_0$;

(iv) $x \in X, x \preceq Bx \Longrightarrow BTx \preceq Tx$;

(v) $x \in X, Bx \preceq x \Longrightarrow Tx \preceq BTx$;

(vi) there exists $\phi \in \Phi$ and $f \in L$ such that

$$x \preceq Bx, By \preceq y \Longrightarrow F_{Tx,Ty}(t) \geq f(F_{x,y}(t), \phi(F_{x,y}(t))), \forall t > 0.$$

Then the sequence $\{T^n x_0\}$ converges to some $x^* \in X$, which is a solution to (3.2). Moreover, x^* is the unique solution to (3.2).

Furthermore, by taking $B = T$ in Corollary 3.2, we obtain a fixed point theorem of a self-mapping T on a partially ordered Menger PM-space.

References

[1] K. Menger, *Statistical metrics*, Proc. Natl. Acad. Sci. USA **28** (1942) 535-537.

[2] B. Schweizer, A. Sklar, *Statistical metric spaces*, Pacific J. Math. **10** (1960) 313-334.

[3] B. Schweizer, A. Sklar, *Probabilistic Metric Spaces*, North-Holland, Amsterdam, 1983.

[4] O. Hadžić, E. Pap, *Fixed Point Theory in Probabilistic Metric Spaces*, Kluwer Academic Publishers, Dordrecht, 2001.

[5] S. Chang, Y. J. Cho, S. M. Kang, *Nonlinear Operator Theory in Probabilistic Metric Spaces*, Nova Science Publishers Inc., New York, 2001.

[6] B. Schweizer, *Commentary on Probabilistic Geometry*, in Karl Menger, Selecta Mathematica, edited by B. Schweizer, A. Sklar, K. Sigmund, et al. Springer/ Wien/ New York, **2** (2003) 409-432.

[7] C. Zhu, *Several nonlinear operator problems in the Menger PN space*, Nonlinear Anal. **65** (2006) 1281-1284.

[8] C. Zhu, *Research on some problems for nonlinear operators*, Nonlinear Anal. **71** (2009) 4568-4571.

[9] Z. Wu, C. Zhu, J. Li, *Common fixed point theorems for two hybrid pairs of mappings satisfying the common property (E.A) in Menger PM-spaces*, Fixed Point Theory Appl. **2013** (2013) 25.

[10] Z. Wu, C. Zhu, J. Wang, *Common fixed point theorems for occasionally weakly biased mappings under contractive conditions in Menger PM-spaces*, J. Comput. Anal. Appl. **19(4)** (2015) 661-669.

[11] Z. Wu, C. Zhu, X. Zhang, *Some new fixed point theorems for single and set-valued admissible mappings in Menger PM-spaces*, Revista de la Real Academia de Ciencias Exactas, F́sicas y Naturales. Serie A. Matem t́icas, **110** (2016) 755-769.

[12] M. Turinici, *Abstract comparison principles and multivariable Gronwall-Bellman inequalities*, J. Math. Anal. Appl. **117** (1986) 100-127.

[13] A. C. M. Ran, M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc. **132** (2004) 1435-1443.

[14] J. Harjani, K. Sadarangani, *Fixed point theorems for weakly contractive mappings in partially ordered sets*, Nonlinear Anal. **71** (2009) 3403-3410.

[15] E. Karapinar, *Couple fixed point theorems for nonlinear contractions in cone metric spaces*, Comput. Math. Appl. **59** (2010) 3656-3668.

[16] V. Lakshmikantham, L. Ćirić, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal. **70** (2009) 4341-4349.

[17] B. Samet, C. Vetro, *Coupled fixed point theorems for multi-valued nonlinear contraction mappings in partially ordered metric spaces*, Nonlinear Anal. **74** (2011) 4260-4268.

[18] L. Ćirić, D. Mihet, R. Saadati, *Monotone generalized contractions in partially ordered probabilistic metric spaces*, Topol. Appl. **156** (2009) 2838-2844.

[19] J. Wu, *Some fixed point theorems for mixed monotone operators in partially ordered probabilistic metric spaces*, Fixed Point Theory Appl. **2014** (2014) 49.

[20] W. Xu, C. Zhu, Z. Wu, L. Zhu, *Fixed point theorems for two new types of cyclic weakly contractive mappings in partially ordered Menger PM-spaces*, J. Nonlinear Sci. Appl. **8** (2015) 412-422.

[21] Q. Tu, C. Zhu, Z. Wu, X. Mu, *Some new coupled fixed point theorems in partially ordered complete probabilistic metric spaces*, J. Nonlinear Sci. Appl. **9** (2016) 1116-1128.

[22] M. Jleli and B. Samet, *A fixed point problem under two constraint inequalities*, Fixed Point Theory Appl. **2016** (2016) 18.

[23] A. H. Ansari, P. Kumam and B. Samet, *A fixed point problem with constraint inequalities via an implict contraction*, J. Fixed Point Theory Appl. (2016) DOI 10.1007/s11784-016-0320-1.

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Fixed Point Theorems for several types of Meir-Keeler contraction mappings in M_s-metric spaces

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Abstract: In this paper, we first introduce several types of Meir-Keeler contractive mappings in the structure of M_S-metric spaces. Then we study some existence and uniqueness fixed point theorems for these types of MKC mappings in M_s-metric spaces via Gupta-Saxena type contraction and other fraction version type contractions. Also, we extend and improve very recent results in fixed point theory.

MSC: 47H10;54H25

Keywords: Fixed point; Meir-Keeler contraction mappings; M_s-metric space

1. Introduction and Preliminaries

In 2014, Nabil [1] established an extension of S-metric spaces to partial S-metric spaces and pointed out that every S-metric space is a partial S-metric space. Also, they obtained some fixed point results under certain contractive principle in partial $S-$metric spaces. Recently, Nabil et al.[2] have extended the concept of a partial $S-$metric space to the concept of an M_s-metric space. They gave a more general extension of almost any metric space with three dimensions and that is not by defining the self "distance" in a metric as in partial metric spaces, but they assumed that is not necessary that the self "distance" is less than the value of the metric among three distinct elements.

In 1969, Meir and Keeler [3] established a fixed point theorem in a metric space (X, d) for mappings satisfying the condition that for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq d(x, y) < \epsilon + \delta \quad implies \quad d(Tx, Ty) < \epsilon, \tag{1}$$

$\forall x, y \in X$. This condition is called the Meri-Keeler contractive (MKC, for short) type condition. Since then, many authors extended and improved this condition and established fixed point results for new generalized MKC mappings, see [4]-[7].

In this paper, we establish some of the fixed point theorem for some types of MKC mappings in M_s-metric spaces. Also, we extend and improve very recent results in fixed point theory.

Next, we remind the reader of some definitions, notions, lemmas which are useful in the sequel.

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Now, we present the definition of an M_s-metric space, but first we introduce the following notations which are useful in the sequel:

(1) $m_{s_{x,y,z}} = \min\{m_s(x,x,x), m_s(y,y,y), m_s(z,z,z)\}$;
(2) $M_{s_{x,y,z}} = \max\{m_s(x,x,x), m_s(y,y,y), m_s(z,z,z)\}$.

Definition 1.1. [2] An M_s-metric on a nonempty set X is a function $m_s : X^3 \mapsto \mathbb{R}^+$ that satisfies the following conditions: for $\forall x, y, z, t \in X$,

(m_s1) $m_s(x,x,x) = m_s(y,y,y) = m_s(z,z,z) = m_s(x,y,z)$ if and only if $x = y = z$;
(m_s2) $m_{s_{x,y,z}} \leq m_s(x,y,z)$;
(m_s3) $m_s(x,x,y) = m_s(y,y,x)$;
(m_s4) $(m_s(x,y,z) - m_{s_{x,y,z}}) \leq (m_s(x,x,t) - m_{s_{x,x,t}}) + (m_s(y,y,t) - m_{s_{y,y,t}}) + (m_s(z,z,t) - m_{s_{z,z,t}})$.

Then the pair (X, m_s) is called an M_s-*metric space*.

Immediate examples of such M_s-metric space are:

(1) Let $X = [0,\infty)$ and $m_s : X^3 \mapsto \mathbb{R}^+$ be a mapping defined by

$$m_s(x,y,z) = \max\{x,y,z\} - \min\{x,y,z\},$$

for $\forall x, y, z \in X$. Then m_s is an M_s-metric on X.

(2) Let X be a nonempty set and d be the ordinary metric on X. Define mapping $m_s : X^3 \mapsto [0,\infty)$ by

$$m_s(x,y,z) = d(x,y) + d(x,z) + d(y,z),$$

for $\forall x, y, z \in X$. Then m_s is an M_s-metric on X.

(3) Let $X = \{1,2,3\}$ and define a mapping m_s on X by

$$\begin{aligned} &m_s(1,2,3) = 6, \ \ m_s(1,1,2) = m_s(2,2,1) = 10, \\ &m_s(1,1,3) = m_s(3,3,1) = m_s(2,2,3) = m_s(3,3,2) = 7, \\ &m_s(2,2,2) = 9, \ \ m_s(3,3,3) = 5, \ \ m_s(1,1,1) = 8. \end{aligned}$$

Then m_s is an M_s-metric on X.

Remark 1.1. If m_s is an M_s-metric on a nonempty set X, then two mappings $m_s^w, m_s^* : X^3 \mapsto \mathbb{R}^+$ defined by

$$m_s^w(x,y,z) = m_s(x,y,z) - 2m_{s_{x,y,z}} + M_{s_{x,y,z}}$$

and

$$m_s^*(x,y,z) = \begin{cases} m_s(x,y,z) - m_{s_{x,y,z}}, & x \neq y \neq z, \\ 0, & x = y = z = 0, \end{cases}$$

for all $x, y, z \in X$ are two ordinary S-metrics on X. In fact, if $m_s^w(x,y,z) = 0$, then we have

$$m_s(x,y,z) = 2m_{s_{x,y,z}} - M_{s_{x,y,z}}.$$

But, from the equation defined above and (m_s2), it follows that

$$m_{s_{x,y,z}} = M_{s_{x,y,z}} = m_s(x,x,x) = m_s(y,y,y) = m_s(z,z,z).$$

So, by the equation above, we have that $m_s(x,y,z) = m_s(x,x,x) = m_s(y,y,y) = m_s(z,z,z)$ and so $x = y = z$. We can get the inequality property in the definition of an S-metric from Lemma 1.1 (7) and (m_s4).

Lemma 1.1. Let (X, m_s) be an M_s-metric space. Then, for all $x, y, z, t \in X$,

(1) $m_s(x,y,y) \le m_s(x,x,y)$;
(2) $m_s(x,y,x) \le 2m_s(x,x,y)$;
(3) $m_s(x,y,z) - m_{s_{x,y,z}} \le (m_s(x,x,z) - m_{s_{x,x,z}}) + (m_s(y,y,z) - m_{s_{y,y,z}})$;
(4) $m_s(x,y,z) - m_{s_{x,y,z}} \le (m_s(x,x,y) - m_{s_{x,x,y}}) + (m_s(z,z,y) - m_{s_{z,z,y}})$;
(5) $m_s(x,y,z) - m_{s_{x,y,z}} \le (m_s(y,y,x) - m_{s_{y,y,x}}) + (m_s(z,z,x) - m_{s_{z,z,x}})$;
(6) $m_s(x,y,z) - m_{s_{x,y,z}} \le \frac{2}{3}[(m_s(x,x,z) - m_{s_{x,x,z}}) + (m_s(z,z,y) - m_{s_{z,z,y}}) + (m_s(y,y,x) - m_{s_{y,y,x}})]$;
(7) $(M_{s_{x,y,z}} - m_{s_{x,y,z}}) \le (M_{s_{x,x,t}} - m_{s_{x,x,t}}) + (M_{s_{y,y,t}} - m_{s_{y,y,t}}) + (M_{s_{z,z,t}} - m_{s_{z,z,t}})$.

Proof. (1)-(7) can be directly obtained from Definition 1.1. □

2. Topology for M_s-metric

It is clear that each M_s-metric m_s on X generates a topology τ_{m_s} on X. The set

$$\{B_{m_s}(x,\epsilon) : x \in X, \epsilon > 0\},$$

where

$$B_{m_s}(x,\epsilon) := \{y \in X : m_s(x,x,y) - m_{s_{x,x,y}} < \epsilon\},$$

for $\forall x \in X$ and $\epsilon > 0$, forms a base of τ_{m_s}.

Definition 2.1. Let (X, m_s) be an M_s-metric space. Then:

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point x if

$$\lim_{n\to\infty}(m_s(x_n,x_n,x) - m_{s_{x_n,x_n,x}}) = 0.$$

(2) A sequence $\{x_n\}$ in X is called an M_s-*Cauchy sequence* if

$$\lim_{n,m\to\infty}(m_s(x_n,x_n,x_m) - m_{s_{x_n,x_n,x_m}}), \lim_{n,m\to\infty}(M_{s_{x_n,x_n,x_m}} - m_{s_{x_n,x_n,x_m}})$$

exist and finite.

(3) An M_s-metric space is said to be *complete* if every m_s-Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_{m_s}, to a point $x \in X$ such that

$$\lim_{n\to\infty}(m_s(x_n,x_n,x) - m_{s_{x_n,x_n,x}}) = 0, \quad \lim_{n\to\infty}(M_{s_{x_n,x_n,x}} - m_{s_{x_n,x_n,x}}) = 0.$$

Lemma 2.1. Let (X, m_s) be an M_s-metric space. Then:

(1) $\{x_n\}$ is an M_s-Cauchy sequence in (X, m_s) if and only if it is an S-Cauchy sequence in the S-metric space (X, m_s^w).

(2) An M_s-metric space (X, m_s) is complete if and only if the S-metric space (X, m_s^w) is complete. Furthermore,

$$\begin{aligned} &\lim_{n\to\infty} m_s^w(x_n, x_n, x) = 0 \\ &\Longrightarrow \lim_{n\to\infty}(m_s(x_n, x_n, x) - m_{s_{x_n,x_n,x}}) = 0, \ \lim_{n\to\infty}(M_{s_{x_n,x_n,x}} - m_{s_{x_n,x_n,x}}) = 0. \end{aligned}$$

Proof. It is obviously follows from the definitions of M_s-Cauchy sequence, M_s-completeness, S-Cauchy sequence and S-completeness. □

Meanwhile, the above assertions are true for m_s^*.

Lemma 2.2. Assume that $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in an M_s-metric space (X, m_s). Then:

$$\lim_{n\to\infty} m_s(x_n, x_n, y_n) - m_{s_{x_n,x_n,y_n}} = m_s(x, x, y) - m_{s_{x,x,y}}.$$

Proof. We have

$$\begin{aligned} &|(m_s(x_n, x_n, y_n) - m_{s_{x_n,x_n,y_n}}) - (m_s(x, x, y) - m_{s_{x,x,y}})| \\ \leq\ &2|m_s(x_n, x_n, x) - m_{s_{x_n,x_n,x}}| + 2|m_s(y_n, y_n, y) - m_{s_{y_n,y_n,y}}|. \end{aligned}$$

□

From Lemma 2.2, we can deduce the following lemma:

Lemma 2.3. Assume that $x_n \to x$ as $n \to \infty$ in an M_s-metric space (X, m_s). Then:

$$\lim_{n\to\infty} m_s(x_n, x_n, y) - m_{s_{x_n,x_n,y}} = m_s(x, x, y) - m_{s_{x,x,y}},$$

$\forall y \in X$.

Lemma 2.4. Assume that $x_n \to x$ and $x_n \to y$ as $n \to \infty$ in an M_s-metric space (X, m_s). Then: $m_s(x, x, y) = m_{s_{x,x,y}}$. Furthermore, if $m_s(x, x, x) = m_s(y, y, y)$, then $x = y$.

Proof. By Lemma 2.2, we have

$$0 = \lim_{n\to\infty} m_s(x_n, x_n, x_n) - m_{s_{x_n,x_n,x_n}} = m_s(x, x, y) - m_{s_{x,x,y}}.$$

□

Lemma 2.5. Let $\{x_n\}$ be a sequence in M_s-metric space (X, m_s) such that there exists $r \in [0, 1)$ such that

$$m_s(x_{n+1}, x_{n+1}, x_n) \leq r m_s(x_n, x_n, x_{n-1}), \tag{2}$$

for $\forall n \in \mathbb{N}$. Then we have

(1) $\lim_{n\to\infty} m_s(x_n, x_n, x_{n-1}) = 0$.
(2) $\lim_{n\to\infty} m_s(x_n, x_n, x_n) = 0$.
(3) $\lim_{n,m\to\infty} m_{s_{x_n,x_n,x_m}} = 0$.
(4) $\{x_n\}$ is an M_s-Cauchy sequence.

Proof. From the equation (2), we have

$$m_s(x_n, x_n, x_{n-1}) \leq r m_s(x_{n-1}, x_{n-1}, x_{n-2}) \leq r^2 m_s(x_{n-2}, x_{n-2}, x_{n-3}) \leq \cdots \leq r^n m_s(x_1, x_1, x_0)$$

and so $\lim_{n\to\infty} m_s(x_n, x_n, x_{n-1}) = 0$, which implies that (A) holds. From (m_s2) and (1), we have

$$\lim_{n\to\infty} m_{s_{x_n,x_n,x_{n-1}}} \leq \lim_{n\to\infty} m_s(x_n, x_n, x_{n-1}) = 0,$$

that is, (2) holds. Clearly, (3) and (4) hold. □

Theorem 2.1. The topology τ_{m_s} is not Hausdorff.

Proof. Let $x, y, z \in X$ be such that $a := m_s(x, x, x) < m_s(z, z, z) = \frac{a+b}{2} < b := m_s(y, y, y)$ with

$$\frac{b}{2} < \frac{k}{2} < m_s(y, y, y) < M_{s_{x,x,y}} = b, \ \ r = 2m_s(x, x, y) - a - b > 0,$$

$$\max\{m_s(x, x, z), m_s(z, z, y)\} \leq (2m_s(x, x, y) - k)\frac{\epsilon}{r}.$$

Without loss of generality, we assume that, for each $\epsilon > 0$, $\epsilon < r$. Now, we need to prove that the intersection of the following neighborhoods is not empty:

$$U_x = \{z \in X : m_s(x, x, z) - m_{s_{x,x,z}} < \epsilon\}, \ \ V_y = \{z \in X : m_s(y, y, z) - m_{s_{y,y,z}} < \epsilon\}.$$

To prove $z \in U_x$, we have

$$m_s(x, x, z) < (2m_s(x, x, y) - k)\frac{\epsilon}{r},$$

$$\begin{aligned} m_s(x, x, z) - m_{s_{x,x,z}} &< (2m_s(x, x, y) - k)\frac{\epsilon}{r} - a \\ &< (2m_s(x, x, y) - k - a)\frac{\epsilon}{r} \\ &< (2m_s(x, x, y) - a - b)\frac{\epsilon}{r} \\ &= \epsilon, \end{aligned}$$

and, for any $z \in V_y$, we also have

$$m_s(y, y, z) < \ \ (2m_s(x, x, y) - k)\tfrac{\epsilon}{r},$$

$$\begin{aligned} m_s(y, y, z) - m_{s_{y,y,z}} &< (2m_s(x, x, y) - k)\frac{\epsilon}{r} - \frac{a+b}{2} \\ &< (2m_s(x, x, y) - k)\frac{\epsilon}{r} - \frac{a+b}{2}\frac{\epsilon}{r} \\ &= (2m_s(x, x, y) - k - \frac{a+b}{2})\frac{\epsilon}{r} \\ &< (2m_s(x, x, y) - a - b)\frac{\epsilon}{r} \\ &= \epsilon. \end{aligned}$$

So, we can find $x, y \in X$ such that, for all nonempty neighborhoods U_x of x and V_y of y, $U_x \cap V_y \neq \emptyset$. This completes the proof. □

3. Main Results

The following definition is new version of definition in [3] for an M_s-metric space.

Definition 3.1. A Meir-Keeler mapping is a mapping $T : X \mapsto X$ on an M_s-metric space (X, m_s) such that

$$\forall \epsilon > 0 \quad \exists \delta(\epsilon) > 0 \quad such \quad that \quad \forall x, y \in X \epsilon \leq m_s(x, x, y) < \epsilon + \delta(\epsilon) \quad implies \quad m_s(Tx, Tx, Ty) < \epsilon. \quad (3)$$

Theorem 3.1. Let (X, m_s) be a complete M_s-metric space and let T be a mapping from X into itself satisfying the following condition:

$$\forall \epsilon > 0 \quad \exists \delta(\epsilon) > 0 \quad such \quad that \quad \forall x, y \in X \epsilon \leq m_s(x, x, y) < \epsilon + \delta(\epsilon) \Rightarrow m_s(Tx, Tx, Ty) < \epsilon.$$

Then, T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$ the sequence $\{T^n(x)\}$ converges to x^*.

Proof. It is easy to check that T is a strict contractive mapping, i.e.,

$$x \neq y \Rightarrow m_s(Tx, Tx, Ty) < m_s(x, x, y). \quad (4)$$

Let $x_0 \in X$ and define the sequence $\{x_n\}$ by $x_n = T^n(x_0), n \in \mathbb{N}$. So we have

$$m_s(x_n, x_n, x_{n-1}) = m_s(Tx_{n-1}, Tx_{n-1}, Tx_{n-2}) < m_s(x_{n-1}, x_{n-1}, x_{n-2}), \quad \forall n \in \mathbb{N}. \quad (5)$$

So the sequence $\{m_s(x_n, x_n, x_{n-1})\}$ is bounded below and decreasing.
Then, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} m_s(x_n, x_n, x_{n-1}) = \lim_{n\to\infty} m_s(x_{n-1}, x_{n-1}, x_n) = r.$$

Next, we will show that $r = 0$. If $r > 0$, therefore, $m_s(x_n, x_n, x_{n-1}) \geq r$, for $\forall n \in \mathbb{N}$.
On the other hand, for $r > 0$, by the hypothesis that T is MKC mapping, there exists $\delta(r) > 0$ such that

$$r \leq m_s(x_{n-1}, x_{n-1}, x_{n-2}) < r + \delta(r) \Rightarrow m_s(Tx_{n-1}, Tx_{n-1}, Tx_{n-2}) = m_s(x_n, x_n, x_{n-1}) < r,$$

which implies that it is a contradiction. Hence, $r = 0$. Then, we have that

$$\lim_{n\to\infty} m_s(x_n, x_n, x_{n-1}) = \lim_{n\to\infty} m_s(x_{n-1}, x_{n-1}, x_n) = 0. \quad (6)$$

and

$$\lim_{n\to\infty} \min\{m_s(x_{n-1}, x_{n-1}, x_{n-1}), m_s(x_n, x_n, x_n)\} = \lim_{n\to\infty} m_{s_{x_n, x_n, x_{n-1}}} < \lim_{n\to\infty} m_s(x_n, x_n, x_{n-1}) = 0. \quad (7)$$

And then, we also have that

$$\lim_{m,n\to\infty} m_{s_{x_m, x_m, x_n}} = 0 \quad and \quad \lim_{m,n\to\infty} M_{s_{x_m, x_m, x_n}} = 0. \quad (8)$$

Next, we claim that $\lim_{m,n\to\infty} m_s(x_m, x_m, x_n) = 0$, that is
for every $\epsilon > 0$, there exists $N \in \mathbb{N}$, such that

$$m_s(x_l, x_l, x_{l+k}) < \epsilon, \quad (9)$$

for all $l \geq N$ and $k \in \mathbb{N}$.
For $\forall \epsilon > 0$.
By (7), we can choose $N_1 \in \mathbb{N}$ such that for all $m, n > N_1$,

$$m_{s_{x_m,x_m,x_n}} < \frac{\epsilon}{4}. \tag{10}$$

Since $\{m_s(x_{n-1}, x_{n-1}, x_n\}$ converges to 0, as $n \to \infty$, for every $\delta > 0$, there exists $N_2 \in \mathbb{N}$ such that

$$m_s(x_{n-1}, x_{n-1}, x_n) < \frac{\delta}{4}, \quad \forall n > N_2 + 1. \tag{11}$$

Choose δ such as $\delta < \epsilon$. We will prove (8) by using mathematic induction on k.
For $k = 1$, (8) becomes that

$$m_s(x_l, x_l, x_{l+1}) < \frac{\delta}{4} < \frac{\epsilon}{4} < \epsilon,$$

and clearly holds for all $l > N_2 + 1$, due to (11) and the choice of δ.
Assume that the inequality (9) holds for some $k = m$, that is

$$m_s(x_l, x_l, x_{l+m}) < \frac{\delta}{4} < \frac{\epsilon}{4} < \epsilon,$$

for $\forall l \geq N_2 + 1$.
For $k = m + 1$, we have to show that $m_s(x_l, x_l, x_{l+m}) < \epsilon$, for some $l \geq N$.
Take $N = \max\{N_1, N_2 + 1\}$. For all $l \geq N$, we have that

$$\begin{aligned}
& m_s(x_l, x_l, x_{l+m+1}) \\
\leq\ & (m_s(x_l, x_l, x_{l+m}) - m_{s_{x_l,x_l,x_{l+m}}}) + (m_s(x_l, x_l, x_{l+m}) - m_{s_{x_l,x_l,x_{l+m}}}) \\
+\ & (m_s(x_{l+m}, x_{l+m}, x_{l+m+1}) - m_{s_{x_{l+m},x_{l+m},x_{l+m+1}}}) + m_{s_{x_l,x_l,x_{l+m}}} \\
\leq\ & 2m_s(x_l, x_l, x_{l+m}) + m_s(x_{l+m}, x_{l+m}, x_{l+m+1}) + m_{s_{x_l,x_l,x_{l+m}}} \\
<\ & 2 \cdot \frac{\delta}{4} + \frac{\delta}{4} + \frac{\epsilon}{4} \\
<\ & 2 \cdot \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} \\
=\ & \epsilon.
\end{aligned}$$

Hence, (9) holds for $k = m + 1$.
Therefore, the claim is true.
So by (8) and $\lim_{m,n\to\infty} m_s(x_m, x_m, x_n) = 0$, we see that the sequence $\{x_n\}$ is a Cauchy sequence and by completeness of X, $x_n \to x^*$ in m_s for some $x^* \in X$, that is

$$\lim_{n\to\infty} (m_s(x_n, x_n, x^*) - m_{s_{x_n,x_n,x^*}}) = 0. \tag{12}$$

But $m_{s_{x_n,x_n,x^*}} \to 0$, as $n \to \infty$, due to $m_s(x_n, x_n, x_n) \to 0$. So $m_s(x_n, x_n, x^*) \to 0$, as $n \to \infty$.
Thus, by the hypothesis, we have that $m_s(Tx_n, Tx_n, Tx^*) < m_s(x_n, x_n, x^*) \to 0$, as $n \to \infty$. Hence, by (m_s2), we have that

$$m_{s_{Tx_n,Tx_n,Tx^*}} \leq m_s(Tx_n, Tx_n, Tx^*) \to 0.$$

Therefore, $Tx_n \to Tx^*$, as $n \to \infty$.
Equation (6) implies that $m_s(x_n, x_n, Tx_n) \to 0$, as $n \to \infty$. Since $m_{s_{x_n,x_n,Tx_n}} \to 0$, as $n \to \infty$, by

Lemma 2.2, we get $m_s(x^*, x^*, Tx^*) = m_{s_{x^*,x^*,Tx^*}}$.
On the other hand, by Lemma 2.2 and

$$Tx_{n-1} = x_n \to x^* \quad and \quad x_{n+1} = Tx_n \to Tx^*,$$

we have

$$\begin{aligned} 0 &= \lim_{n\to\infty}(m_s(x_n, x_n, Tx_n) - m_{s_{x_n,x_n,Tx_n}}) \\ &= \lim_{n\to\infty}(m_s(x_n, x_n, x_{n+1}) - m_{s_{x_n,x_n,Tx_n}}) \\ &= m_s(x^*, x^*, x^*) - m_{s_{x^*,x^*,Tx^*}} \\ &= m_s(Tx^*, Tx^*, Tx^*) - m_{s_{x^*,x^*,Tx^*}}. \end{aligned}$$

Thus, $m_s(x^*, x^*, x^*) = m_{s_{x^*,x^*,Tx^*}} = m_s(Tx^*, Tx^*, Tx^*)$.
And since

$$m_s(x^*, x^*, Tx^*) = m_s(Tx^*, Tx^*, x^*) = m_{s_{x^*,x^*,Tx^*}} = m_s(x^*, x^*, x^*) = m_s(Tx^*, Tx^*, Tx^*),$$

then, by Lemma 2.4, we have that $x^* = Tx^*$.
Uniqueness by the contraction (4) is clear. □

Next, we establish a fixed point theorem for a MKC mapping in M_s-metric space via Gupta-Saxena type contraction.

Put

$$C(x, x, y) = m_s(x, x, y) + \frac{(1 + m_s(x, x, Tx))m_s(y, y, Ty)}{1 + m_s(x, x, y)} + \frac{m_s(x, x, Tx)m_s(y, y, Ty)}{m_s(x, x, y)}.$$

Theorem 3.2. Let (X, m_s) be a complete M_s-metric space and let T be a continuous mapping from X into itself satisfying the following condition:

$$\forall\epsilon > 0 \quad \exists\delta(\epsilon) > 0 \quad such \quad that \quad \forall x, y \in X \epsilon \leq kC(x, x, y) < \epsilon + \delta(\epsilon) \Rightarrow m_s(Tx, Tx, Ty) < \epsilon, \quad (13)$$

for some $0 < k < \frac{1}{3}$.
Then, T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$ the sequence $\{T^n(x)\}$ converges to x^*.

Proof. From definition of $C(x, x, y)$, it follows that T is a strict contraction, i.e.,

$$x \neq y \Rightarrow m_s(Tx, Tx, Ty) < kC(x, x, y). \quad (14)$$

Let $x_0 \in X$ and define the sequence $\{x_n\}$ by $x_n = T^n(x_0), n \in \mathbb{N}$. So we have

$$\begin{aligned} C(x_{n-1}, x_{n-1}, x_n) &= m_s(x_{n-1}, x_{n-1}, x_n) + \frac{(1 + m_s(x_{n-1}, x_{n-1}, x_n))m_s(x_n, x_n, x_{n+1})}{1 + m_s(x_{n-1}, x_{n-1}, x_n)} \\ &+ \frac{m_s(x_{n-1}, x_{n-1}, x_n)m_s(x_n, x_n, x_{n+1})}{m_s(x_{n-1}, x_{n-1}, x_n)} \\ &= m_s(x_{n-1}, x_{n-1}, x_n) + 2m_s(x_n, x_n, x_{n+1}), \end{aligned}$$

and

$$\begin{aligned} m_s(x_n, x_n, x_{n+1}) &= m_s(Tx_{n-1}, Tx_{n-1}, Tx_n) \\ &< kC(x_{n-1}, x_{n-1}, x_n) \\ &= k(m_s(x_{n-1}, x_{n-1}, x_n) + 2m_s(x_n, x_n, x_{n+1})). \end{aligned}$$

Therefore,

$$m_s(x_n, x_n, x_{n+1}) < rm_s(x_{n-1}, x_{n-1}, x_n),$$

where $r = \frac{k}{1-2k} \in [0, 1)$ by the choice of k. Now, by Lemma 2.5, $\{x_n\}$ is a Cauchy sequence, and by completeness of X, $Tx_{n-1} = x_n \to x^*$ as $n \to \infty$ in m_s for some $x^* \in X$. Since T is a continuous mapping, so $x_n = Tx_{n-1} \to Tx^*$ as $n \to \infty$ in m_s. By Lemma 2.4, we find

$$m_s(x^*, x^*, Tx^*) = m_{s_{x^*, x^*, Tx^*}},$$

and

$$\begin{aligned} 0 &= \lim_{n\to\infty} (m_s(x_n, x_n, Tx_n) - m_{s_{x_n, x_n, Tx_n}}) \\ &= m_s(x^*, x^*, Tx^*) - m_{s_{x^*, x^*, Tx^*}} \\ &= m_s(Tx^*, Tx^*, Tx^*) - m_{s_{x^*, x^*, Tx^*}}. \end{aligned}$$

By Lemma 2.2, we have that

$$m_s(x^*, x^*, Tx^*) = m_{s_{x^*, x^*, Tx^*}} = m_s(x^*, x^*, x^*) = m_s(Tx^*, Tx^*, Tx^*).$$

Then, by Lemma 2.4, we have that $x^* = Tx^*$.
Uniqueness by the contraction (14) is clear. □

Theorem 3.3. Let (X, m_s) be an M_s-metric space and let T be a self-mapping defined on X. Assume that there exists a function $\varphi(t) : [0, \infty) \mapsto [0, \infty)$ satisfying the following conditions:

(1) $\varphi(0) = 0$ and $t > 0 \Rightarrow \varphi(t) > 0$;
(2) φ is nondecreasing and right continuous;
(3) for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \le \varphi(kC(x, x, y)) < \epsilon + \delta \Rightarrow \varphi(m_s(Tx, Tx, Ty)) < \epsilon, \tag{15}$$

for some $0 < k < \frac{1}{3}$ and for all $x, y \in X$ with $x \neq y$.
Then, (13) is satisfied.

Proof. Fix $\epsilon > 0$, so $\varphi(\epsilon) > 0$. By (15) there exists $\delta > 0$ such that

$$\forall x, y \in X, x \neq y, \varphi(\epsilon) < \varphi(kC(x, x, y)) < \varphi(\epsilon) + \delta \Rightarrow \varphi(m_s(Tx, Tx, Ty)) < \varphi(\epsilon).$$

In view of the fact that φ is right continuous, then there exists $\delta' > 0$ such that

$$\varphi(\epsilon + \delta) < \varphi(\epsilon) + \delta'.$$

Now, for $x, y \in X$ with $x \neq y$ and fixed

$$\epsilon \leq kC(x,x,y) < \epsilon + \delta,$$

Since φ is a nondecreasing mapping, we have

$$\varphi(\epsilon) \leq \varphi(kC(x,x,y)) < \varphi(\epsilon+\delta) < \varphi(\epsilon) + \delta'.$$

So we get

$$\varphi(m_s(Tx,Tx,Ty)) < \varphi(\epsilon),$$

which implies that $m_s(Tx,Tx,Ty) < \epsilon$, i.e., (13) is satisfied. □

Corollary 3.1. Let (X, m_s) be an M_s-metric space and let T be a self-mapping defined on X. Assume that there exists a function $h(t) : [0,\infty) \mapsto [0,\infty)$ is a locally integrable satisfying the following conditions:

(1) $\int_0^t h(s)ds > 0$ for all $t > 0$;
(2) for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\frac{1}{k}\epsilon \leq \int_0^{C(x,x,y)} h(s)ds < \frac{1}{k}\epsilon + \delta \Rightarrow \int_0^{\frac{1}{k}m_s(Tx,Tx,Ty)} h(s)ds < \frac{1}{k}\epsilon,$$

for some $0 < k < \frac{1}{3}$ and for all $x, u \in X$ with $x \neq y$.
Then (13) is satisfied.

Proof. Defining $\varphi(t)$ in Theorem 3.3 by $\varphi(t) = \int_0^t h(s)ds$, then we can draw the conclusion. □

Next, we establish a fixed point theorems for a MKC mapping in M_s-metric spaces via other fractional type contractions.

Put

$$M_A(x,y,z) = m_s(x,y,z) + \frac{m_s(x,Tx,Tx)m_s(y,Ty,Ty)m_s(z,Tz,Tz)}{m_s(x,y,z)m_s(Tx,Ty,Tz)} + \frac{m_s(x,Tx,Tx)m_s(y,Ty,Ty)m_s(z,Tz,Tz)}{m_s(x,y,z)^2}.$$

Theorem 3.4. Let (X, m_s) be a complete M_s-metric space and let T be a continuous mapping from X into itself satisfying the following condition:

$$\forall \epsilon > 0 \quad \exists \delta(\epsilon) > 0 \quad such \quad that \quad \forall x,y,z \in X \epsilon \leq kM_A(x,y,z) < \epsilon + \delta(\epsilon) \Rightarrow m_s(Tx,Ty,Tz) < \epsilon. \quad (16)$$

for some $0 < k < \frac{1}{3}$.
Then, T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$ the sequence $\{T^n(x)\}$ converges to x^*.

Proof. From definition of $M_A(x,y,z)$, it follows that T is a strict contraction, i.e.,

$$x \neq y \neq z \Rightarrow m_s(Tx,Ty,Tz) < kM_A(x,y,z). \quad (17)$$

Let $x_0 \in X$ and define the sequence $\{x_n\}$ by $x_n = T^n(x_0), n \in \mathbb{N}$. So we have

$$\begin{aligned}
M_A(x_{n-1}, x_{n-1}, x_n) &= m_s(x_{n-1}, x_{n-1}, x_n) + \frac{m_s(x_{n-1}, x_n, x_n)^2 m_s(x_n, x_{n+1}, x_{n+1})}{m_s(x_{n-1}, x_{n-1}, x_n) m_s(x_n, x_n, x_{n+1})} \\
&+ \frac{m_s(x_{n-1}, x_n, x_n)^2 m_s(x_n, x_{n+1}, x_{n+1})}{m_s(x_{n-1}, x_{n-1}, x_n)^2} \\
&\leq m_s(x_{n-1}, x_{n-1}, x_n) + \frac{m_s(x_{n-1}, x_{n-1}, x_n)^2 m_s(x_n, x_n, x_{n+1})}{m_s(x_{n-1}, x_{n-1}, x_n) m_s(x_n, x_n, x_{n+1})} \\
&+ \frac{m_s(x_{n-1}, x_n, x_n)^2 m_s(x_n, x_n, x_{n+1})}{m_s(x_{n-1}, x_{n-1}, x_n)^2} \quad ('' \leq '' by Lemma 1.1(1)) \\
&= m_s(x_{n-1}, x_{n-1}, x_n) + 2m_s(x_n, x_n, x_{n+1}),
\end{aligned}$$

and

$$\begin{aligned}
m_s(x_n, x_n, x_{n+1}) &= m_s(Tx_{n-1}, Tx_{n-1}, Tx_n) \\
&< kM_A(x_{n-1}, x_{n-1}, x_n) \\
&< k(m_s(x_{n-1}, x_{n-1}, x_n) + 2m_s(x_n, x_n, x_{n+1})).
\end{aligned}$$

Therefore,

$$m_s(x_n, x_n, x_{n+1}) < r m_s(x_{n-1}, x_{n-1}, x_n),$$

where $r = \frac{k}{1-2k} \in [0,1)$ by the choice of k. Now, by Lemma 2.5, $\{x_n\}$ is a Cauchy sequence, and by completeness of X, $Tx_{n-1} = x_n \to x^*$ as $n \to \infty$ in m_s for some $x^* \in X$. Since T is a continuous mapping, so $x_n = Tx_{n-1} \to Tx^*$ as $n \to \infty$ in m_s. By Lemma 2.4, we find

$$m_s(x^*, x^*, Tx^*) = m_{s_{x^*, x^*, Tx^*}},$$

and

$$\begin{aligned}
0 &= \lim_{n\to\infty} (m_s(x_n, x_n, Tx_n) - m_{s_{x_n, x_n, Tx_n}}) \\
&= m_s(x^*, x^*, Tx^*) - m_{s_{x^*, x^*, Tx^*}} \\
&= m_s(Tx^*, Tx^*, Tx^*) - m_{s_{x^*, x^*, Tx^*}}.
\end{aligned}$$

By Lemma 2.2, we have that

$$m_s(x^*, x^*, Tx^*) = m_{s_{x^*, x^*, Tx^*}} = m_s(x^*, x^*, x^*) = m_s(Tx^*, Tx^*, Tx^*).$$

Then, by Lemma 2.4, we have that $x^* = Tx^*$.
Uniqueness by the contraction (17) is clear. □

Put

$$\begin{aligned}
M_B(x, y, z) &= m_s(x, y, z) + \frac{(1 + m_s(x, x, Tx)) m_s(y, y, Ty)}{1 + m_s(x, y, z)} \\
&+ \frac{(1 + m_s(y, y, Ty)) m_s(z, z, Tz)}{1 + m_s(x, y, z)} + \frac{m_s(x, x, Tx) m_s(y, y, Ty) m_s(z, z, Tz)}{m_s(x, y, z)^2}.
\end{aligned}$$

Theorem 3.5. Let (X, m_s) be a complete M_s-metric space and let T be a continuous mapping from X into itself satisfying the following condition:

$$\forall \epsilon > 0 \quad \exists \delta(\epsilon) > 0 \quad such \quad that \quad \forall x, y, z \in X \epsilon \leq kM_B(x, y, z) < \epsilon + \delta(\epsilon) \Rightarrow m_s(Tx, Ty, Tz) < \epsilon. \tag{18}$$

for some $0 < k < \frac{1}{4}$.
Then, T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$ the sequence $\{T^n(x)\}$ converges to x^*.

Proof. From definition of $M_B(x,y,z)$, it follows that T is a strict contraction, i.e.,

$$x \neq y \neq z \Rightarrow m_s(Tx,Ty,Tz) < kM_B(x,y,z). \tag{19}$$

Let $x_0 \in X$ and define the sequence $\{x_n\}$ by $x_n = T^n(x_0), n \in \mathbb{N}$. So we have

$$\begin{aligned} M_B(x_{n-1},x_{n-1},x_n) &= m_s(x_{n-1},x_{n-1},x_n) + \frac{(1+m_s(x_{n-1},x_{n-1},x_n))m_s(x_{n-1},x_{n-1},x_n)}{1+m_s(x_{n-1},x_{n-1},x_n)} \\ &+ \frac{(1+m_s(x_{n-1},x_{n-1},x_n))m_s(x_n,x_n,x_{n+1})}{1+m_s(x_{n-1},x_{n-1},x_n)} + \frac{m_s(x_{n-1},x_{n-1},x_n)^2 m_s(x_n,x_n,x_{n+1})}{m_s(x_{n-1},x_{n-1},x_n)^2} \\ &= 2(m_s(x_{n-1},x_{n-1},x_n) + m_s(x_n,x_n,x_{n+1})), \end{aligned}$$

and

$$\begin{aligned} m_s(x_n,x_n,x_{n+1}) &= m_s(Tx_{n-1},Tx_{n-1},Tx_n) \\ &< kM_B(x_{n-1},x_{n-1},x_n) \\ &< 2k(m_s(x_{n-1},x_{n-1},x_n) + m_s(x_n,x_n,x_{n+1})). \end{aligned}$$

Therefore,

$$m_s(x_n,x_n,x_{n+1}) < rm_s(x_{n-1},x_{n-1},x_n),$$

where $r = \frac{2k}{1-2k} \in [0,1)$ by the choice of k.
Then, the conclusion we can directly obtain by using the similar arguments to the proof of Theorem 3.4. □

Replacing $C(x,x,y)$ by $M_A(x,y,z)$ or $M_B(x,y,z)$ in Theorem 3.3 and Corollary 3.1, we can obtain the following theorem and corollary.

Theorem 3.6. Let (X,m_s) be an M_s-metric space and let T be a self-mapping defined on X. Assume that there exists a function $\varphi(t) : [0,\infty) \mapsto [0,\infty)$ satisfying the following conditions:

(1) $\varphi(0) = 0$ and $t > 0 \Rightarrow \varphi(t) > 0$;
(2) φ is nondecreasing and right continuous;
(3) for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \leq \varphi(kM_A(x,x,y)) < \epsilon + \delta \Rightarrow \varphi(m_s(Tx,Tx,Ty)) < \epsilon,$$

for some $0 < k < \frac{1}{3}$ and for all $x,y,z \in X$ with $x \neq y \neq z$.
Then, (16) is satisfied.

Theorem 3.7. Let (X,m_s) be an M_s-metric space and let T be a self-mapping defined on X. Assume that there exists a function $\varphi(t) : [0,\infty) \mapsto [0,\infty)$ satisfying the following conditions:

(1) $\varphi(0) = 0$ and $t > 0 \Rightarrow \varphi(t) > 0$;
(2) φ is nondecreasing and right continuous;

(3) for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \leq \varphi(kM_B(x,x,y)) < \epsilon + \delta \Rightarrow \varphi(m_s(Tx,Tx,Ty)) < \epsilon,$$

for some $0 < k < \frac{1}{4}$ and for all $x, y, z \in X$ with $x \neq y \neq z$.
Then, (18) is satisfied.

Corollary 3.2. Let (X, m_s) be an M_s-metric space and let T be a self-mapping defined on X. Assume that there exists a function $h(t) : [0,\infty) \mapsto [0,\infty)$ is a locally integrable satisfying the following conditions:

(1) $\int_0^t h(s)ds > 0$ for all $t > 0$;
(2) for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\frac{1}{k}\epsilon \leq \int_0^{M_A(x,y,z)} h(s)ds < \frac{1}{k}\epsilon + \delta \Rightarrow \int_0^{\frac{1}{k}m_s(Tx,Ty,Tz)} h(s)ds < \frac{1}{k}\epsilon,$$

for some $0 < k < \frac{1}{3}$ and for all $x, u \in X$ with $x \neq y \neq z$.
Then (16) is satisfied.

Corollary 3.3. Let (X, m_s) be an M_s-metric space and let T be a self-mapping defined on X. Assume that there exists a function $h(t) : [0,\infty) \mapsto [0,\infty)$ is a locally integrable satisfying the following conditions:

(1) $\int_0^t h(s)ds > 0$ for all $t > 0$;
(2) for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\frac{1}{k}\epsilon \leq \int_0^{M_B(x,y,z)} h(s)ds < \frac{1}{k}\epsilon + \delta \Rightarrow \int_0^{\frac{1}{k}m_s(Tx,Ty,Tz)} h(s)ds < \frac{1}{k}\epsilon,$$

for some $0 < k < \frac{1}{4}$ and for all $x, u \in X$ with $x \neq y \neq z$.
Then (18) is satisfied.

We are now in a position to define two new types of Meir-Keeler contractions on M_s-metric spaces, say type A and type B.

Definition 3.2. Let (X, m_s) ba an M_s-metric space. A self-mapping $T : X \mapsto X$ is said to be a Meir-Keeler contraction mapping of type A if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq M(x,y) < \epsilon + \delta(\epsilon) \quad implies \quad m_s(Tx,Tx,Ty) < \epsilon, \tag{20}$$

where $M(x,y) = \min\{m_s(x,x,y), m_s(x,x,Tx), m_s(y,y,Ty)\}$, for all $x, y \in X$.

Definition 3.3. Let (X, m_s) ba an M_s-metric space. A self-mapping $T : X \mapsto X$ is said to be a Meir-Keeler contraction mapping of type B if for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that

$$\epsilon \leq N(x,y) < \epsilon + \delta(\epsilon) \quad implies \quad m_s(Tx,Tx,Ty) < \epsilon, \tag{21}$$

where $N(x,y) = \min\{m_s(x,x,y), \frac{1}{2}[m_s(x,x,Tx) + m_s(y,y,Ty)]\}$, for all $x, y \in X$.

Remark 3.1. (1) Suppose that $T : X \mapsto X$ is a Meir-Keeler contraction mapping of type A (respectively, type B). Then

$$m_s(Tx, Tx, Ty) < M(x, y) \quad (respectively, N(x, y)),$$

for all $x, y \in X$ with $x \neq y$.

(2) It is readily verified that $M(x, y) \leq N(x, y)$ for all $x, y \in X$, where $M(x, y), N(x, y)$ are defined in Definition 3.2 and Definition 3.3.

Theorem 3.8. Let (X, m_s) be a complete M_s-metric space and let T be a mapping from X into itself satisfying the following condition:

$$\forall \epsilon > 0 \quad \exists \delta(\epsilon) > 0 \quad such \quad that \quad \forall x, y \in X \epsilon \leq M(x, y) < \epsilon + \delta(\epsilon) \Rightarrow m_s(Tx, Tx, Ty) < \epsilon, \tag{22}$$

where $M(x, y) = \min\{m_s(x, x, y), m_s(x, x, Tx), m_s(y, y, Ty)\}$.
Then, T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$ the sequence $\{T^n(x)\}$ converges to x^*.

Proof. It is readily verified that T is a strict contractive mapping, i.e.,

$$x \neq y \Rightarrow m_s(Tx, Tx, Ty) < M(x, y), \tag{23}$$

where $M(x, y) = \min\{m_s(x, x, y), m_s(x, x, Tx), m_s(y, y, Ty)\}$.
Let $x_0 \in X$ and define the sequence $\{x_n\}$ by $x_n = T^n(x_0), n \in \mathbb{N}$. So we have

$$m_s(x_n, x_n, x_{n-1}) = m_s(Tx_{n-1}, Tx_{n-1}, Tx_{n-2}) < M(x_{n-1}, x_{n-2}).$$

where $M(x_{n-1}, x_{n-2}) = \min\{m_s(x_{n-1}, x_{n-1}, x_{n-2}), m_s(x_{n-1}, x_{n-1}, x_n), m_s(x_{n-2}, x_{n-2}, x_{n-1})\}$.
In what follows, we examine three cases.
Case1. Assume that $M(x_{n-1}, x_{n-2}) = m_s(x_{n-1}, x_{n-1}, x_{n-2})$. Then (22) becomes that

$$\epsilon \leq m_s(x_{n-1}, x_{n-1}, x_{n-2}) < \epsilon + \delta \Rightarrow m_s(x_n, x_n, x_{n-1}) < \epsilon.$$

Therefore, we deduce that

$$m_s(x_n, x_n, x_{n-1}) < \epsilon \leq m_s(x_{n-1}, x_{n-1}, x_{n-2}),$$

for all $n \in \mathbb{N}$. That is $\{m_s(x_n, x_n, x_{n-1})\}$ is a bounded below and decreasing, and it converges to some $r \geq 0$. To show that $r = 0$, we assume the contrary that $r > 0$. Then we must have

$$0 < r \leq m_s(x_n, x_n, x_{n-1}), \quad \forall n \in \mathbb{N}.$$

On the other hand, for $r > 0$, by the hypothesis that T is MKC mapping of type A, there exists $\delta(r) > 0$ such that

$$r \leq m_s(x_{n-1}, x_{n-1}, x_{n-2}) < r + \delta(r) \Rightarrow m_s(Tx_{n-1}, Tx_{n-1}, Tx_{n-2}) = m_s(x_n, x_n, x_{n-1}) < r,$$

which is a contradiction. Hence, $r = 0$. Therefore, we get

$$\lim_{n\to\infty} m_s(x_n, x_n, x_{n-1}) = \lim_{n\to\infty} m_s(x_{n-1}, x_{n-1}, x_n) = 0. \tag{24}$$

Case2. Assume that $M(x_{n-1}, x_{n-2}) = m_s(x_{n-1}, x_{n-1}, x_n)$. Then (22) becomes that

$$\epsilon \leq m_s(x_{n-1}, x_{n-1}, x_n) < \epsilon + \delta \Rightarrow m_s(x_n, x_n, x_{n+1}) < \epsilon.$$

Therefore, we deduce that

$$m_s(x_n, x_n, x_{n-1}) < \epsilon \leq m_s(x_{n-1}, x_{n-1}, x_{n-2}),$$

for all $n \in \mathbb{N}$. That is $\{m_s(x_n, x_n, x_{n+1})\}$ is a bounded below and decreasing, and it converges to some $L \geq 0$. In fact, the limit L of this sequence is 0, which can be shown by mimicking the proof of (24) done above.
Case3. Assume that $M(x_{n-1}, x_{n-2}) = m_s(x_{n-2}, x_{n-2}, x_{n-1})$. Then (22) becomes that

$$\epsilon \leq m_s(x_{n-2}, x_{n-2}, x_{n-1}) < \epsilon + \delta \Rightarrow m_s(x_{n-1}, x_{n-1}, x_n) < \epsilon.$$

Therefore, we deduce that

$$m_s(x_{n-1}, x_{n-1}, x_n) < \epsilon \leq m_s(x_{n-2}, x_{n-2}, x_{n-1}),$$

for all $n \in \mathbb{N}$. As in two cases above, the sequence $\{m_s(x_{n-1}, x_{n-1}, x_n)\}$ is a bounded below and decreasing, hence it converges to 0.
As a result, we see all three cases, the sequence $\{m_s(x_n, x_n, x_{n-1})\}$ converges to 0.
Form (21), we get that

$$\lim_{n\to\infty} \min\{m_s(x_{n-1}, x_{n-1}, x_{n-1}), m_s(x_n, x_n, x_n)\} = \lim_{n\to\infty} m_{s_{x_n, x_n, x_{n-1}}} < \lim_{n\to\infty} m_s(x_n, x_n, x_{n-1}) = 0. \quad (25)$$

and

$$\lim_{m,n\to\infty} m_{s_{x_m, x_m, x_n}} = 0 \quad and \quad \lim_{m,n\to\infty} M_{s_{x_m, x_m, x_n}} = 0. \tag{26}$$

Using similar arguments as in proof of Theorem 3.1, it can be shown that

$$\lim_{m,n\to\infty} m_s(x_m, x_m, x_n) = 0.$$

So, by (26) and $\lim_{m,n\to\infty} m_s(x_m, x_m, x_n) = 0$, we get that the sequence $\{x_n\}$ is a Cauchy sequence and by completeness of X, $x_n \to x^*$ in m_s for some $x^* \in X$, that is

$$\lim_{n\to\infty} (m_s(x_n, x_n, x^*) - m_{s_{x_n, x_n, x^*}}) = 0.$$

But $m_{s_{x_n, x_n, x^*}} \to 0$, as $n \to \infty$, due to $m_s(x_n, x_n, x_n) \to 0$. So $m_s(x_n, x_n, x^*) \to 0$, as $n \to \infty$.
Thus, by the hypothesis, we have that $m_s(Tx_n, Tx_n, Tx^*) < M(x_n, x^*) \to 0$, as $n \to \infty$. Hence, by $(m_s 2)$, we have that

$$m_{s_{Tx_n, Tx_n, Tx^*}} \leq m_s(Tx_n, Tx_n, Tx^*) \to 0.$$

Therefore, $Tx_n \to Tx^*$, as $n \to \infty$.
Equation (24) implies that $m_s(x_n, x_n, Tx_n) \to 0$, as $n \to \infty$. Since $m_{s_{x_n, x_n, Tx_n}} \to 0$, as $n \to \infty$, by Lemma 2.2, we get $m_s(x^*, x^*, Tx^*) = m_{s_{x^*, x^*, Tx^*}}$.
On the other hand, by Lemma 2.2 and

$$Tx_{n-1} = x_n \to x^* \quad and \quad x_{n+1} = Tx_n \to Tx^*,$$

we have

$$\begin{aligned} 0 &= \lim_{n\to\infty}(m_s(x_n, x_n, Tx_n) - m_{s_{x_n,x_n,Tx_n}}) \\ &= \lim_{n\to\infty}(m_s(x_n, x_n, x_{n+1}) - m_{s_{x_n,x_n,Tx_n}}) \\ &= m_s(x^*, x^*, x^*) - m_{s_{x^*,x^*,Tx^*}} \\ &= m_s(Tx^*, Tx^*, Tx^*) - m_{s_{x^*,x^*,Tx^*}}. \end{aligned}$$

Thus, $m_s(x^*, x^*, x^*) = m_{s_{x^*,x^*,Tx^*}} = m_s(Tx^*, Tx^*, Tx^*)$.
And since

$$m_s(x^*, x^*, Tx^*) = m_s(Tx^*, Tx^*, x^*) = m_{s_{x^*,x^*,Tx^*}} = m_s(x^*, x^*, x^*) = m_s(Tx^*, Tx^*, Tx^*),$$

then, by Lemma 2.4, we have that $x^* = Tx^*$.
Uniqueness by the contraction (23) is clear. □

In what follows, we present an existence and uniqueness theorem for fixed point of Meir-Keeler contraction of type B. Taking Remark 3.1 into account, we observe that the proof of this is similar to the proof of Theorem 3.8.

Theorem 3.9. Let (X, m_s) be a complete M_s-metric space and let T be a mapping from X into itself satisfying the following condition:

$$\forall\epsilon > 0 \quad \exists\delta(\epsilon) > 0 \quad such \quad that \quad \forall x, y \in X \epsilon \leq N(x, y) < \epsilon + \delta(\epsilon) \Rightarrow m_s(Tx, Tx, Ty) < \epsilon, \tag{27}$$

where $N(x, y) = \min\{m_s(x, x, y), \frac{1}{2}[m_s(x, x, Tx) + m_s(y, y, Ty)]\}$.
Then, T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$ the sequence $\{T^n(x)\}$ converges to x^*.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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References

[1] M. Nabil, *A contraction principle in partial S-metric spaces*, Universal J. Math. and Mathmatical, **5**(2014), 109–119.

[2] M. Nabil, S. Nizar, A. Thabet, *A new extension of S-metric spaces*, J. Nonlinear Sci. Appl., in press.

[3] A. Meir, E. Keeler, *A theorem on contraction mappings*, J.Math.Anal.Appl., **28**(1969), 326–329.

[4] M. Maiti, T. K. Pal, *Generalizations of two fixed points theorems*, Bull. Calcutta Math. Soc., **70**(1978), 57–61.

[5] S. Park, B. E. Rhoades, *Meir-Keeler type contractive conditions*, Math. Japn., **26**(1)(1981), 13–20.

[6] C. Mongkolkeha, P. Kumam, *Best proximity points for asympototic proximal pointwise weaker Meir-Keeler-type $\psi-$contraction mappings*, J. Egypt. Math. Soc., **26**(1)(1981), 13–20.

[7] A. Latif, M. E. Gordji, E. Karapinar, W. Sintunavarat, *Fixed point results for generalized $(\alpha-\psi)-$Meir-Keeler contractive mappings and applications*, Journal of Inequalities and Applications, **68**(2014), 11 pages.

Tender and naive weak closure operations on lower BCK-semilattices

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Abstract. The notion of tender and naive weak closure operation is introduced, and their relations and properties are investigated. Using a weak closure operation "cl" and an ideal A of a lower BCK-semilattice X with the greatest element 1, a new ideal K of X containing the ideal A^{cl} of X is established. Using this ideal K, a new function

$$cl_t : \mathcal{I}(X) \to \mathcal{I}(X),\ A \mapsto K$$

is given, and related properties are considered. We show that if "cl" is a tender (resp., naive) weak closure operation on $\mathcal{I}(X)$, then so are "cl_t" and "cl_f".

1. Introduction

In [2], Bordbar et al. introduced a weak closure operation, which is more general form than closure operation, on ideals of BCK-algebras, and investigated related properties. Regarding weak closure operation, they defined finite type and (strong) quasi-primeness, and investigated related properties. They also discussed positive implicative (resp., commutative and implicative) weak closure operations, and provided several examples to illustrate notions and properties.

In this paper, we introduce the notion of tender and naive weak closure operation, and investigate their relations and properties. Using a weak closure operation "cl" and an ideal A of a lower BCK-semilattice X with the greatest element 1, we construct a new ideal K of X containing the ideal A^{cl} of X. Using this ideal K, we define a new function

$$cl_t : \mathcal{I}(X) \to \mathcal{I}(X),\ A \mapsto K$$

and investigate related properties. We show that if "cl" is a tender (resp., naive) weak closure operation on $\mathcal{I}(X)$, then so are "cl_t" and "cl_f".

2. Preliminaries

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A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X;*,0)$ of type $(2,0)$ is called a *BCI-algebra* if it satisfies the following conditions

(I) $(\forall x,y,z \in X)\ (((x*y)*(x*z))*(z*y)=0)$,

(II) $(\forall x,y \in X)\ ((x*(x*y))*y=0)$,

(III) $(\forall x \in X)\ (x*x=0)$,

(IV) $(\forall x,y \in X)\ (x*y=0,\ y*x=0\ \Rightarrow\ x=y)$.

If a BCI-algebra X satisfies the following identity

(V) $(\forall x \in X)\ (0*x=0)$,

then X is called a *BCK-algebra*.

Any BCK/BCI-algebra X satisfies the following conditions

(a1) $(\forall x \in X)\ (x*0=x)$,

(a2) $(\forall x,y,z \in X)\ (x \le y\ \Rightarrow\ x*z \le y*z,\ z*y \le z*x)$,

(a3) $(\forall x,y,z \in X)\ ((x*y)*z=(x*z)*y)$,

(a4) $(\forall x,y,z \in X)\ ((x*z)*(y*z) \le x*y)$

where $x \le y$ if and only if $x*y=0$. A BCK-algebra X is called a *lower BCK-semilattice* (see [6]) if X is a lower semilattice with respect to the BCK-order.

A subset A of a BCK/BCI-algebra X is called an *ideal* of X (see [6]) if it satisfies

$$0 \in A, \tag{2.1}$$

$$(\forall x \in X)\,(\forall y \in A)\,(x*y \in A\ \Rightarrow\ x \in A)\,. \tag{2.2}$$

Note that every ideal A of a BCK/BCI-algebra X satisfies the following implication (see [6]).

$$(\forall x,y \in X)\,(x \le y,\ y \in A\ \Rightarrow\ x \in A)\,. \tag{2.3}$$

For any subset A of X, the ideal generated by A is defined to be the intersection of all ideals of X containing A, and it is denoted by $\langle A\rangle$. If A is finite, then we say that $\langle A\rangle$ is *finitely generated ideal* of X (see [6]).

Let $\mathcal{I}(X)$ and $\mathcal{I}_f(X)$ be a set of all ideals of X and a set of all finitely generated ideals of X, respectively.

We refer the reader to the books [5, 6] for further information regarding BCK/BCI-algebras.

3. **Tender and naive weak closure operations**

In what follows, let X be a lower BCK-semilattice unless otherwise specified. For any $a,b \in X$, denote by $a \wedge b$ the greatest lower bound of a and b.

Definition 3.1 ([2]). An element x of X is called a *zeromeet element* of X if the condition

$$(\exists\, y \in X \setminus \{0\})\,(x \wedge y = 0)$$

is valid. Otherwise, x is called a *non-zeromeet element* of X.

Denote by $Z(X)$ the set of all zeromeet elements of X, that is,

$$Z(X) = \{x \in X \mid x \wedge y = 0 \text{ for some nonzero element } y \in X\}.$$

Obviously, $0 \in Z(X)$ and if X has the greatest element 1, then $1 \in X \setminus Z(X)$.

Lemma 3.2 ([2])**.** *For any $x, y \in X$, if $x, y \notin Z(X)$, then $x \wedge y \notin Z(X)$, that is, the set $X \setminus Z(X)$ is closed under the operation $\wedge$.*

Definition 3.3 ([4])**.** For any nonempty subsets A and B of X, we denote

$$A \wedge B := \langle \{a \wedge b \mid a \in A, b \in B\} \rangle$$

which is called the *meet ideal* of X generated by A and B. In this case, we say that the operation "$\wedge$" is a *meet operation*. If $A = \{a\}$, then $\{a\} \wedge B$ is denoted by $a \wedge B$. Also, if $B = \{b\}$, then $A \wedge \{b\}$ is denoted by $A \wedge b$.

Definition 3.4 ([3])**.** For any nonempty subsets A and B of X, we define a set

$$(A :_\wedge B) := \{x \in X \mid x \wedge B \subseteq A\}$$

whenever $x \wedge B$ exists for all $x \in X$, and it is called the *relative annihilator* of B with respect to A.

Lemma 3.5 ([3])**.** *If A and B are ideals of a lower BCK-semilattice X, then the relative annihilator $(A :_\wedge B)$ of B with respect to A is an ideal of X.*

Definition 3.6 ([2])**.** A mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ is called a *weak closure operation* on $\mathcal{I}(X)$ if the following conditions are valid.

$$(\forall A \in \mathcal{I}(X)) \, (A \subseteq cl(A)) \,, \tag{3.1}$$

$$(\forall A, B \in \mathcal{I}(X)) \, (A \subseteq B \;\Rightarrow\; cl(A) \subseteq cl(B)) \,. \tag{3.2}$$

If a weak closure operation $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ satisfies the condition

$$(\forall A \in \mathcal{I}(X)) \, (cl(cl(A)) = cl(A)) \,, \tag{3.3}$$

then we say that "cl" is a closure operation on $\mathcal{I}(X)$ (see [1]). In what follows, we use A^{cl} instead of $cl(A)$.

For non-zeromeet elements a and b of X and $A \in \mathcal{I}(X)$, we consider two ideals

$$((a \wedge A)^{cl} :_\wedge \langle b \rangle) \text{ and } A^{cl},$$

and investigate their relations where "cl" is a weak closure operation on $\mathcal{I}(X)$. In the following example, we will check that there are following relations:

(1) $((a \wedge A)^{cl} :_\wedge \langle b \rangle) \subseteq A^{cl}$ for some $A \in \mathcal{I}(X)$ and some non-zeromeet elements a and b of X,

(2) $((a \wedge A)^{cl} :_\wedge \langle b \rangle) \supseteq A^{cl}$ for some $A \in \mathcal{I}(X)$ and some non-zeromeet elements a and b of X,

(3) They have no inclusion relation, that is,

$$((a \wedge A)^{cl} :_\wedge \langle b\rangle) \nsubseteq A^{cl} \text{ and } ((a \wedge A)^{cl} :_\wedge \langle b\rangle) \nsupseteq A^{cl}$$

for some $A \in \mathcal{I}(X)$ and some non-zeromeet elements a and b of X.

Example 3.7. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	0
3	3	3	3	0	3
4	4	4	4	4	0

We have 6 ideals of X, and they are $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 1, 2\}$, $A_3 = \{0, 1, 2, 3\}$, $A_4 = \{0, 1, 2, 4\}$ and $A_5 = X$. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_1$, $A_1^{cl} = A_2$, $A_2^{cl} = A_2$, $A_3^{cl} = A_5$, $A_4^{cl} = A_5$ and $A_5^{cl} = A_5$. Then "cl" is a weak closure operation on $\mathcal{I}(X)$. For non-zeromeet elements 1 and 3 of X, we have

$$((1 \wedge A_3)^{cl} :_\wedge \langle 3\rangle) = (A_2 :_\wedge \{0, 1, 2, 3\}) = \{0, 1, 2, 4\} = A_4 \subsetneq A_5 = A_3^{cl}.$$

Example 3.8. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	0
3	3	3	3	0	3
4	4	4	4	4	0

We have 5 ideals of X, and they are $A_0 = \{0\}$, $A_1 = \{0, 1, 2\}$, $A_2 = \{0, 1, 2, 3\}$, $A_3 = \{0, 1, 2, 4\}$ and $A_4 = X$. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_0$, $A_1^{cl} = A_2$, $A_2^{cl} = A_4$, $A_3^{cl} = A_4$ and $A_4^{cl} = A_4$. Then "cl" is a weak closure operation on $\mathcal{I}(X)$. For non-zeromeet elements 1 and 2 of X, we have

$$((2 \wedge A_1)^{cl} :_\wedge \langle 1\rangle) = (A_2 :_\wedge \{0, 1, 2\}) = \{0, 1, 2, 3, 4\} = X \supsetneq A_2 = A_1^{cl}.$$

Example 3.9. Let $X = \{0, 1, 2, 3, 4\}$ be a lower BCK-semilattice which is given in Example 3.7. If we define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_0$, $A_1^{cl} = A_2$, $A_2^{cl} = A_2$, $A_3^{cl} = A_5$, $A_4^{cl} = A_4$, $A_5^{cl} = A_5$. Then "cl" is a weak closure operation on $\mathcal{I}(X)$. For non-zeromeet elements 3 and 4, we have

$$((3 \wedge A_4)^{cl} :_\wedge \langle 4\rangle) = (A_2 :_\wedge \{0, 1, 2, 4\}) = \{0, 1, 2, 3\} = A_3$$

and $A_4^{cl} = A_4$. Therefore

$$((3 \wedge A_4)^{cl} :_\wedge \langle 4\rangle) \nsubseteq A_4^{cl} \text{ and } ((3 \wedge A_4)^{cl} :_\wedge \langle 4\rangle) \nsupseteq A_4^{cl}.$$

We consider the equality of $((a \wedge A)^{cl} :_\wedge \langle b\rangle)$ and A^{cl}, that is,

$$((a \wedge A)^{cl} :_\wedge \langle b\rangle) = A^{cl}. \tag{3.4}$$

Proposition 3.10. *If X has the greatest element 1, then every weak closure operation "cl" on $\mathcal{I}(X)$ satisfies the equality (3.4) for some $A \in \mathcal{I}(X)$ and non-zeromeet elements a and b of X.*

Proof. The ideals $A = \{0\}$ and a non-zeromeet element $b = 1$ satisfy the equality (3.4) for all non-zeromeet element a of X. □

The following example shows that the converse of Proposition 3.10 is not true in general.

Example 3.11. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	4	4	0

Note that 4 is the greatest element and there are 6 ideals of X, that is, $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 1, 2\}$, $A_3 = \{0, 1, 3\}$, $A_4 = \{0, 1, 2, 3\}$ and $A_5 = X$. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_0$, $A_1^{cl} = A_3$, $A_2^{cl} = A_2$, $A_3^{cl} = A_4$, $A_4^{cl} = A_5$ and $A_5^{cl} = A_5$. Note that 0 is the only zeromeet element of X. For non-zeromeet elements 2 and 3 of X, we have $(3 \wedge A_1)^{cl} = A_1^{cl} = A_3$. Hence

$$((3 \wedge A_1)^{cl} :_\wedge \langle 2 \rangle) = (A_3 :_\wedge A_2) = \{0, 1, 3\} = A_3 = A_1^{cl}.$$

Therefore "cl" satisfies the equality (3.4) for ideal A_1 and non-zeromeet elements 2 and 3 of X. But "cl" is not a weak closure operation because $A_1 \subseteq A_2$, but

$$A_1^{cl} = A_3 \nsubseteq A_2 = A_2^{cl}.$$

Definition 3.12. A weak closure operation "cl" on $\mathcal{I}(X)$ is said to be

- *tender* if for any $A \in \mathcal{I}(X)$ and any non-zeromeet elements a and b of X, the equality (3.4) is valid,
- *naive* if for any $A \in \mathcal{I}(X)$ there exist non-zeromeet elements a and b of X such that the equality (3.4) is valid.

Example 3.13. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3\}$ with the following Cayley table.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	3	0

We have 5 ideals of X, and they are $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2\}$, and $A_4 = X$. Note that 3 is only the non-zeromeet element of X. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$

by $A_0^{cl} = A_0$, $A_1^{cl} = A_3$, $A_2^{cl} = A_3$, $A_3^{cl} = A_3$ and $A_4^{cl} = A_4$. Then "cl" is a weak closure operation on $\mathcal{I}(X)$ and

$(3 \wedge A_0)^{cl} :_\wedge \langle 3\rangle) = (A_0^{cl} :_\wedge \langle 3\rangle) = (A_0 :_\wedge \langle 3\rangle) = A_0 = A_0^{cl}$,
$(3 \wedge A_1)^{cl} :_\wedge \langle 3\rangle) = (A_1^{cl} :_\wedge \langle 3\rangle) = (A_3 :_\wedge \langle 3\rangle) = A_3 = A_1^{cl}$,
$(3 \wedge A_2)^{cl} :_\wedge \langle 3\rangle) = (A_2^{cl} :_\wedge \langle 3\rangle) = (A_3 :_\wedge \langle 3\rangle) = A_3 = A_2^{cl}$,
$(3 \wedge A_3)^{cl} :_\wedge \langle 3\rangle) = (A_3^{cl} :_\wedge \langle 3\rangle) = (A_3 :_\wedge \langle 3\rangle) = A_3 = A_3^{cl}$,
$(3 \wedge A_4)^{cl} :_\wedge \langle 3\rangle) = (A_4^{cl} :_\wedge \langle 3\rangle) = (A_4 :_\wedge \langle 3\rangle) = A_4 = A_4^{cl}$.

Therefore "cl" is a tender weak closure operation on $\mathcal{I}(X)$. Also, it is a naive weak closure operation on $\mathcal{I}(X)$.

Obviously, every tender weak closure operation is a native weak closure operation. But the converse is not true in general as seen in the following example.

Example 3.14. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3\}$ with the following Cayley table.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	1
3	3	3	3	0

We have 3 ideals of X, and they are $A_0 = \{0\}$, $A_1 = \{0, 1, 2\}$ and $A_2 = X$. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_1^{cl} = A_1$ and $A_2^{cl} = A_2$. We can easily check that "cl" is a naive weak closure operation on $\mathcal{I}(X)$. But, it is not a tender weak closure operation on $\mathcal{I}(X)$. In fact, we know that there are two non-zeromeet elements 2 and 3. Thus

$$((3 \wedge A_1)^{cl} :_\wedge \langle 2\rangle) = (A_1^{cl} :_\wedge \langle 2\rangle) = (A_1 :_\wedge A_1) = X \neq A_1 = A_1^{cl}.$$

Definition 3.15 ([2])**.** Given a (weak) closure operation "cl" on $\mathcal{I}(X)$, we define a new operation $cl_f : \mathcal{I}(X) \to \mathcal{I}(X)$ by

$$(\forall A \in \mathcal{I}(X)) \left(A^{cl_f} = \cup\{B^{cl} \mid B \subseteq A,\ B \in \mathcal{I}_f(X)\}\right), \tag{3.5}$$

where $\mathcal{I}_f(X)$ is the set of all finitely generated ideals of X.

Definition 3.16 ([2])**.** A (weak) closure operation "cl" on $\mathcal{I}(X)$ is said to be of *finite type* if the following assertion is valid.

$$(\forall A \in \mathcal{I}(X)) \left(A^{cl} = A^{cl_f}\right). \tag{3.6}$$

Theorem 3.17. *If "cl" is a tender weak closure operation on $\mathcal{I}(X)$, then*

$$((a \wedge A)^{cl_f} :_\wedge \langle b\rangle) \subseteq A^{cl_f}. \tag{3.7}$$

for all $A \in \mathcal{I}(X)$ and every non-zeromeet elements a and b of X.

Proof. Suppose that "cl" is a tender weak closure operation on $\mathcal{I}(X)$. Let $A \in \mathcal{I}(X)$ and consider non-zeromeet elements a and b of X. If $x \in ((a \wedge A)^{cl_f} :_\wedge \langle b \rangle)$, then $x \wedge \langle b \rangle \subseteq (a \wedge A)^{cl_f}$ and so $x \wedge z \in (a \wedge A)^{cl_f}$ for all $z \in \langle b \rangle$. Thus there exists a finitely generated ideal B of X such that $B \subseteq a \wedge A$ and $x \wedge z \in B^{cl}$. Since

$$B = \langle x_1, x_2, \cdots, x_n \rangle$$

for some $x_1, x_2, \cdots, x_n \in X$, it follows that $x_i \in a \wedge A$ and so that $x_i = a \wedge a_i$ for $a_i \in A$ and $i \in \{1, 2, \cdots, n\}$. Thus

$$B = \langle a \wedge a_1, a \wedge a_2, \cdots, a \wedge a_n \rangle.$$

Now put $C = \langle a_1, a_2, \cdots, a_n \rangle$. Then $C \subseteq A$. If $x \in B$, then

$$(\cdots((x * (a \wedge a_1)) * (a \wedge a_2)) * \cdots) * (a \wedge a_n) = 0. \tag{3.8}$$

Since $a_1, a_2, \cdots, a_n \in C$, we have

$$a \wedge a_i \in \{a \wedge c \mid c \in C\} \text{ for } i = 1, 2, \cdots, n.$$

Since $a \wedge C = \langle \{a \wedge c \mid c \in C\} \rangle$, it follows from (3.8) that $x \in a \wedge C$. Thus $B \subseteq a \wedge C$, and hence $x \wedge z \in B^{cl} \subseteq (a \wedge C)^{cl}$ which means that $x \in ((a \wedge C)^{cl} :_\wedge z)$. Since z is an arbitrary element of $\langle b \rangle$, we have

$$x \in ((a \wedge C)^{cl} :_\wedge \langle b \rangle) = C^{cl}.$$

Since C is a finitely generated ideal of X which is contained in A, we have $x \in A^{cl_f}$. Therefore $((a \wedge A)^{cl_f} :_\wedge \langle b \rangle) \subseteq A^{cl_f}$. □

If the condition "tender" in Theorem 3.17 is omitted, then Theorem 3.17 is not true as seen in the following example.

Example 3.18. Consider the lower BCK-semilattice X and the weak closure operation "cl" on $\mathcal{I}(X)$ as in Example 3.14. Then "cl" is a naive weak closure operation but it is not tender. In fact, for ideal A_1 and $2, 3 \in X \setminus Z(X)$, we have

$$((3 \wedge A_1)^{cl_f} = \cup\{B^{cl} \mid B \subseteq 3 \wedge A_1, B \in \mathcal{I}_f(X)\} = A_1.$$

Thus $((3 \wedge A_1)^{cl_f} :_\wedge \langle 2 \rangle) = (A_1 :_\wedge A_1) = X$. But,

$$A_1^{cl_f} = \cup\{B^{cl} \mid B \subseteq A_1, B \in \mathcal{I}_f(X)\} = A_1.$$

Therefore $((3 \wedge A_1)^{cl_f} :_\wedge \langle 2 \rangle) = X \nsubseteq A_1 = A_1^{cl_f}$, that is, the condition (3.7) is not true.

Remark 3.19. Example 3.18 also shows that the condition (3.7) does not hold whenever we use a naive weak closure operation instead of a tender weak closure operation on $\mathcal{I}(X)$.

Lemma 3.20 ([3])**.** *If A is an ideal of X, then $(A :_\wedge X) = A$ and $(A :_\wedge A) = X$.*

Theorem 3.21. *Suppose that X has the greatest element 1. If "cl" is a naive weak closure operation on $\mathcal{I}(X)$, then so is "cl_f".*

Proof. Note that if "*cl*" is a weak closure operation on $\mathcal{I}(X)$, then so is "cl_f" (see [1, Lemma 4.1]). Suppose that A is an ideal of X. Since $1 \wedge A = A$ and $\langle 1 \rangle = X$, it follows from Lemma 3.20 that

$$A^{cl_f} = ((1 \wedge A)^{cl_f} :_\wedge \langle 1 \rangle).$$

Therefore "cl_f" is naive weak closure operation on $\mathcal{I}(X)$. □

Corollary 3.22. *Suppose that X has the greatest element 1. If "cl" is a tender weak closure operation on $\mathcal{I}(X)$, then "cl_f" is a naive weak closure operation on $\mathcal{I}(X)$.*

Corollary 3.23. *Suppose that X has the greatest element 1. If "cl" is a naive weak closure operation on $\mathcal{I}(X)$, then*

$$(\forall A \in \mathcal{I}(X))(\exists a, b \in X \setminus Z(X)) \left(((a \wedge A)^{cl_f} :_\wedge \langle b \rangle) = A^{cl_f}\right).$$

Lemma 3.24 ([5])**.** *Every commutative BCK-algebra X satisfies the identity:*

$$(\forall x, y, z \in X)\,((x \wedge y) * (x \wedge z) = (x \wedge y) * z).$$

Theorem 3.25. *In a commutative BCK-algebra X with the greatest element 1, if "cl" is a tender weak closure operation on $\mathcal{I}(X)$, then so is "cl_f".*

Proof. Suppose that "*cl*" is a tender weak closure operation on $\mathcal{I}(X)$. Note that if "*cl*" is a weak closure operation on $\mathcal{I}(X)$, then so is "cl_f" (see [1, Lemma 4.1]). For any $A \in \mathcal{I}(X)$, if $x \in A^{cl_f}$, then there exists $B \in \mathcal{I}_f(X)$ such that $B \subseteq A$ and $x \in B^{cl}$. Since "*cl*" is a tender weak closure operation, we have

$$B^{cl} = ((a \wedge B)^{cl} :_\wedge \langle b \rangle)$$

for every elements $a, b \in X \setminus Z(X)$. Thus $x \in ((a \wedge B)^{cl} :_\wedge \langle b \rangle)$ which means that

$$x \wedge \langle b \rangle \subseteq (a \wedge B)^{cl}.$$

Now we will show that $a \wedge B \subseteq C$ where $C = \langle a \wedge x_1, a \wedge x_2, \cdots, a \wedge x_n \rangle$. Let $p \in a \wedge B$. Then $p = a \wedge q$ for some $q \in B$. Since B is finitely generated, we have

$$(\cdots((q * x_1) * x_2) * \cdots) * x_n = 0$$

for some $x_1, x_2, \cdots, x_n \in X$. It follows from Lemma 3.24 and (a3) that

$$\begin{aligned} ((a \wedge q) * (a \wedge x_1)) * (a \wedge x_2) &= ((a \wedge q) * x_1) * (a \wedge x_2) \\ &= ((a \wedge q) * (a \wedge x_2)) * x_1 \\ &= ((a \wedge q) * x_2) * x_1 \\ &= ((a \wedge q) * x_1) * x_2 \end{aligned}$$

and so from Lemma 3.24 and (a3) again that

$$\begin{aligned}&(((a \wedge q) * (a \wedge x_1)) * (a \wedge x_2)) * (a \wedge x_3)\\&= (((a \wedge q) * x_1) * x_2) * (a \wedge x_3)\\&= (((a \wedge q) * (a \wedge x_3)) * x_1) * x_2\\&= (((a \wedge q) * x_3) * x_1) * x_2\\&= (((a \wedge q) * x_1) * x_2) * x_3.\end{aligned}$$

By the mathematical induction, we conclude that

$$\begin{aligned}&(\cdots(((a \wedge q) * (a \wedge x_1)) * (a \wedge x_2)) * \cdots) * (a \wedge x_n)\\&= (\cdots(((a \wedge q) * x_1) * x_2) * \cdots) * x_n.\end{aligned} \tag{3.9}$$

The inequality $a \wedge q \leq q$ implies from (a2) that

$$(\cdots(((a \wedge q) * x_1) * x_2) * \cdots) * x_n \leq (\cdots((q * x_1) * x_2) * \cdots) * x_n = 0$$

which implies that

$$\begin{aligned}0 &= (\cdots(((a \wedge q) * x_1) * x_2) * \cdots) * x_n\\&= (\cdots(((a \wedge q) * (a \wedge x_1)) * (a \wedge x_2)) * \cdots) * (a \wedge x_n).\end{aligned}$$

Hence $p = a \wedge q \in C$, and so $a \wedge B \subseteq C$. Thus

$$x \wedge \langle b \rangle \subseteq (a \wedge B)^{cl} \subseteq C^{cl}.$$

Since C is a finitely generated ideal of X which is contained in $a \wedge A$, it follows that $x \wedge \langle b \rangle \subseteq (a \wedge A)^{cl_f}$, that is, $x \in ((a \wedge A)^{cl_f} :_\wedge \langle b \rangle)$. Thus $A^{cl_f} \subseteq ((a \wedge A)^{cl_f} :_\wedge \langle b \rangle)$. Now let $x \in ((a \wedge A)^{cl_f} :_\wedge \langle b \rangle)$. Then $x \wedge \langle b \rangle \subseteq (a \wedge A)^{cl_f}$, and so there exists $B \in \mathcal{I}_f(X)$ such that $B \subseteq a \wedge A$ and $x \wedge z \in B^{cl}$ for every $z \in \langle b \rangle$. Hence

$$x \in ((1 \wedge B)^{cl} :_\wedge \langle b \rangle).$$

Since "cl" is a tender weak closure operation on $\mathcal{I}(X)$, we have $((1 \wedge B)^{cl} :_\wedge \langle b \rangle) = B^{cl}$ and $x \in B^{cl}$. Since $B \in \mathcal{I}_f(X)$ and $B \subseteq a \wedge A \subseteq A$, we get $x \in A^{cl_f}$. Therefore

$$A^{cl_f} = ((a \wedge A)^{cl_f} :_\wedge \langle b \rangle). \tag{3.10}$$

Consequently we know that "cl_f" is a tender weak closure operation on $\mathcal{I}(X)$. □

Lemma 3.26 ([3])**.** *For any nonempty subsets A, B_1 and B_2 of X, we have*

$$B_1 \subseteq B_2 \;\Rightarrow\; (A :_\wedge B_2) \subseteq (A :_\wedge B_1). \tag{3.11}$$

Lemma 3.27 ([3])**.** *For any ideal A of X, we have*

$$(\forall x, y \in X)\,(x \leq y \;\Rightarrow\; x \wedge A \subseteq y \wedge A)\,. \tag{3.12}$$

Theorem 3.28. *Assume that X has the greatest element 1. For every weak closure operation "cl" on $\mathcal{I}(X)$ and every ideal A of X, let*

$$K := \cup\{((a \wedge A)^{cl} :_{\wedge} \langle b\rangle) \mid a, b \in X \setminus Z(X)\}. \tag{3.13}$$

Then K is an ideal of X containing A^{cl}.

Proof. Suppose that A is an ideal of X. Then $((a \wedge A)^{cl} :_{\wedge} \langle b\rangle)$ is an ideal of X for every $a, b \in X \setminus Z(X)$ by Definition 3.3 and Lemma 3.5. Hence it is clear that $0 \in K$. Let $x, y \in X$ be such that $x \in K$ and $y * x \in K$. Then there exist $a_1, a_2, b_1, b_2 \in X \setminus Z(X)$ such that

$$x \in ((a_1 \wedge A)^{cl} :_{\wedge} \langle b_1\rangle) \text{ and } y * x \in ((a_2 \wedge A)^{cl} :_{\wedge} \langle b_2\rangle).$$

Thus $x \wedge \langle b_1\rangle \subseteq (a_1 \wedge A)^{cl}$ and $(y * x) \wedge \langle b_2\rangle \subseteq (a_2 \wedge A)^{cl}$. Since $a_i \leq 1$ for $i = 1, 2$, it follows from Lemma 3.27 that $a_i \wedge A \subseteq 1 \wedge A = A$ for $i = 1, 2$. Hence

$$(a_i \wedge A)^{cl} \subseteq (1 \wedge A)^{cl}$$

for $i = 1, 2$, which implies that $x \wedge \langle b_1\rangle \subseteq (1 \wedge A)^{cl}$ and $(y * x) \wedge \langle b_2\rangle \subseteq (1 \wedge A)^{cl}$. Therefore $x \in ((1 \wedge A)^{cl} :_{\wedge} \langle b_1\rangle)$ and $y * x \in ((1 \wedge A)^{cl} :_{\wedge} \langle b_2\rangle)$. Note that $\langle b_1 \wedge b_2\rangle \subseteq \langle b_1\rangle$ and $\langle b_1 \wedge b_2\rangle \subseteq \langle b_2\rangle$, and so

$$((1 \wedge A)^{cl} :_{\wedge} \langle b_1\rangle) \subseteq ((1 \wedge A)^{cl} :_{\wedge} \langle b_1 \wedge b_2\rangle)$$

and

$$((1 \wedge A)^{cl} :_{\wedge} \langle b_2\rangle) \subseteq ((1 \wedge A)^{cl} :_{\wedge} \langle b_1 \wedge b_2\rangle).$$

Therefore

$$x \in ((1 \wedge A)^{cl} :_{\wedge} \langle b_1 \wedge b_2\rangle)$$

and

$$y * x \in ((1 \wedge A)^{cl} :_{\wedge} \langle b_1 \wedge b_2\rangle).$$

It follows that $y \in ((1 \wedge A)^{cl} :_{\wedge} \langle b_1 \wedge b_2\rangle)$. Since $X \setminus Z(X)$ is closed under the operation $\wedge$ by Lemma 3.2, then $b_1 \wedge b_2 \in X \setminus Z(X)$. Therefore $y \in K$ and K is an ideal of X. Obviously $A^{cl} \subseteq K$. □

Assume that X has the greatest element 1 and define a new function

$$cl_t : \mathcal{I}(X) \to \mathcal{I}(X),\ A \mapsto K \tag{3.14}$$

where K is the ideal in Theorem 3.28.

Theorem 3.29. *Assume that X has the greatest element 1. If "cl" is a weak closure operation on $\mathcal{I}(X)$, then so is the function "cl_t" in (3.14).*

Proof. Let "cl" be a weak closure operation on $\mathcal{I}(X)$. For any ideal A of X, we have

$$A \subseteq A^{cl} = ((1 \wedge A)^{cl} :_\wedge \langle 1 \rangle) \subseteq A^{cl_t}.$$

Let A and B be ideals of X. Then

$$\begin{aligned} A \subseteq B &\Rightarrow a \wedge A \subseteq a \wedge B \Rightarrow (a \wedge A)^{cl} \subseteq (a \wedge B)^{cl} \\ &\Rightarrow ((a \wedge A)^{cl} :_\wedge \langle b \rangle) \subseteq ((a \wedge B)^{cl} :_\wedge \langle b \rangle), \end{aligned}$$

and so $A^{cl_t} \subseteq B^{cl_t}$. Therefore "$cl_t$" is a weak closure operation on $\mathcal{I}(X)$. □

Theorem 3.30. *Assume that X has the greatest element 1. If "cl" is a finite type weak closure operation on $\mathcal{I}(X)$, then so is the function "cl_t" in (3.14).*

Proof. Assume that "cl" is a finite type weak closure operation on $\mathcal{I}(X)$. Then "cl_t" is a weak closure operation on $\mathcal{I}(X)$ by Theorem 3.29. For any ideal A of X, if $x \in A^{cl_t}$ then there exist non-zeromeet elements a and b of X such that

$$x \in ((a \wedge A)^{cl} :_\wedge \langle b \rangle).$$

Thus $x \wedge \langle b \rangle \subseteq (a \wedge A)^{cl}$, and so $x \wedge z \in (a \wedge A)^{cl}$ for every element $z \in \langle b \rangle$. Since "cl" is of finite type, there exists a finitely generated ideal B such that $B \subseteq a \wedge A$ and $x \wedge z \in B^{cl}$. On the other hand, we know that $B \subseteq a \wedge C$ for some finite generated ideal C of X since B is finitely generated. Thus $x \wedge z \in B^{cl}$ implies that $x \wedge z \in (a \wedge C)^{cl}$, that is, $x \in ((a \wedge C)^{cl} :_\wedge z)$. Since $z \in \langle b \rangle$, it follows that

$$x \in ((a \wedge C)^{cl} :_\wedge \langle b \rangle)$$

and so that $x \in C^{cl_t}$. Therefore $A^{cl_t} \subseteq A^{(cl_t)_f}$. Obviously, $A^{(cl_t)_f} \subseteq A^{cl_t}$ and therefore A^{cl_t} is a finite type weak closure operation on $\mathcal{I}(X)$. □

Theorem 3.31. *Assume that X has the greatest element 1. If "cl" is a naive weak closure operation on $\mathcal{I}(X)$, then so is the function "cl_t" in (3.14).*

Proof. Suppose that "cl" is a naive weak closure operation on $\mathcal{I}(X)$. Then "cl_t" is a weak closure operation on $\mathcal{I}(X)$ by Theorem 3.29. For any $A \in \mathcal{I}(X)$, if $x \in A^{cl_t}$, then there exists $a, b \in X \setminus Z(X)$ such that $x \in ((a \wedge A)^{cl} :_\wedge \langle b \rangle)$. Thus

$$x \wedge \langle b \rangle \subseteq (a \wedge A)^{cl}.$$

Since $a \wedge A \in \mathcal{I}(X)$ and "cl" is naive, there exists $p, q \in X \setminus Z(X)$ such that

$$(a \wedge A)^{cl} = ((p \wedge (a \wedge A))^{cl} :_\wedge \langle q \rangle).$$

Hence $x \wedge \langle b \rangle \subseteq ((p \wedge (a \wedge A))^{cl} :_\wedge \langle q \rangle)$. Also

$$((p \wedge (a \wedge A))^{cl} :_\wedge \langle q \rangle) \subseteq (a \wedge A))^{cl_t}.$$

Therefore $x \in ((a \wedge A)^{cl_t} :_\wedge \langle b \rangle)$.

Conversely, suppose that $x \in ((a \wedge A)^{cl_t} :_\wedge \langle b \rangle)$. Then $x \wedge z \in (a \wedge A)^{cl_t}$ for every $z \in \langle b \rangle$, and so there exist $p, q \in X \setminus Z(X)$ such that

$$x \wedge z \in ((p \wedge (a \wedge A))^{cl} :_\wedge \langle q \rangle).$$

Hence $x \wedge \langle b \rangle \wedge \langle q \rangle \subseteq ((p \wedge a) \wedge A)^{cl}$. Also $\langle b \wedge q \rangle \subseteq \langle b \rangle \wedge \langle q \rangle$. Therefore

$$x \wedge \langle b \wedge q \rangle \subseteq ((p \wedge a) \wedge A)^{cl}$$

which means that

$$x \in (((p \wedge a) \wedge A)^{cl} :_\wedge \langle b \wedge q \rangle).$$

Since $X \setminus Z(X)$ is closed under the operation $\wedge$ by Lemma 3.2, we have $p \wedge a$, $b \wedge q \in X \setminus Z(X)$ and therefore $x \in A^{cl_t}$. Consequently, "cl_t" is a naive weak closure operation on $\mathcal{I}(X)$. □

Theorem 3.32. *Assume that X has the greatest element 1. If "cl" is a tender weak closure operation on $\mathcal{I}(X)$, then so is the function "cl_t" in (3.14).*

Proof. It is similar to the proof of Theorem 3.31. □

References

[1] H. Bordbar and M. M. Zahedi, A finite type of closure operations on BCK-algebra, Appl. Math. Inf. Sci. Lett. 4 (2016), no. 2, 1–9.

[2] H. Bordbar, M. M. Zahedi, S. S. Ahn and Y. B. Jun, Weak closure operations on ideals of BCK-algebras, J. Comput. Anal. Appl. (in press).

[3] H. Bordbar, M. M. Zahedi and Y. B. Jun, Relative annihilators in lower BCK-semilattices, Mathematical Sciences Letters (submitted).

[4] H. Bordbar, Sun Shin Ahn, M. M. Zahedi and Y. B. Jun, Semiring structures based on meet and plus ideals in lower BCK-semilattices, J. Comput. Anal. Appl. (in press).

[5] Y. Huang, BCI-algebra, Science Press, Beijing 2006.

[6] J. Meng and Y. B. Jun, BCK-algebras, Kyung Moon Sa Co., Seoul 1994.

Some Functional Inequalities for Generalized Error Function

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Abstract

In the paper, the authors present some functional inequalities for the generalized error function. Concretely speaking, the authors present several inequalities of the generalized error function in terms of the arithmetic, logarithmic, and exponential means, find monotonicity, convexity, and concavity of the generalized error function, and, consequently, derive two Grünbaum type and Turán type inequalities.

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1 Introduction

It is well known that the error function $\operatorname{erf}(x)$ is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \,\mathrm{d}\, t, \quad x \in \mathbb{R}.$$

The function $\operatorname{erf}(x)$ has numerous applications in statistics, probability, partial differential equations, and so on. It is not difficult to directly verify that $\operatorname{erf}(x)$ is odd on $(-\infty, \infty)$, convex on $(-\infty, 0)$, concave on $[0, \infty)$, strictly increasing on $\mathbb{R}$, and $\operatorname{erf}(0) = 0$. Moreover, the limit $\lim_{x\to\infty} \operatorname{erf}(x) = 1$ can be found in [1, 12, 17, 18, 23, 28] and the closely related references therein.

In 1955, Chu [13] established a double inequality

$$\sqrt{1 - e^{-ax^2}} \le \operatorname{erf}(x) \le \sqrt{1 - e^{-bx^2}}\,,$$

where $0 \le a \le 1$ and $b \ge \frac{4}{\pi}$ for $x \ge 0$. Later, Cao et al. [12, 17] obtained the double inequality

$$\frac{1}{\sqrt{1 + (9\pi/16 - 1)/(n^2)}} \le \operatorname{erf}(n) < \frac{1}{\sqrt{1 - 3/(4n^2)}}$$

for $n \in \mathbb{N}$ and derived the probability integral $\int_0^\infty e^{-x^2} \,\mathrm{d}\, x = \frac{\sqrt{\pi}}{2}$. See also related texts in [18, p. 31]. In 1968, Mitrinović and Weinacht [21] obtained

$$\operatorname{erf}(x) + \operatorname{erf}(y) \le \operatorname{erf}(x+y) + \operatorname{erf}(x)\operatorname{erf}(y), \quad x, y \ge 0.$$

In 2003 and 2009, Alzer [2, 3] proved for all $x \ge y > 0$ the sharp double inequalities

$$\operatorname{erf}(1) < \frac{\operatorname{erf}(x+\operatorname{erf}(y))}{\operatorname{erf}(y+\operatorname{erf}(x))} < \frac{2}{\sqrt{\pi}} \quad \text{and} \quad 0 < \frac{\operatorname{erf}(x\operatorname{erf}(y))}{\operatorname{erf}(y\operatorname{erf}(x))} \le 1.$$

For $p > 0$ and $x \in (0, \infty)$, the generalized error function $\operatorname{erf}_p(x)$ is defined by

$$\operatorname{erf}_p(x) = \frac{p}{\Gamma(1/p)} \int_0^x e^{-t^p} \,\mathrm{d}\, t,$$

where $\Gamma(z)$ for $\Re(z) > 0$ stands for the classical Euler gamma function. It is easy to see that $\operatorname{erf}_2(x) = \operatorname{erf}(x)$. By an easy computation, we can see that $\operatorname{erf}_p(x)$ is odd on $(-\infty, \infty)$, convex on $(-\infty, 0)$, concave on $[0, \infty)$, strictly increasing on $\mathbb{R}$, $\operatorname{erf}_p(0) = 0$, and $\lim_{x\to\infty} \operatorname{erf}_p(x) = 1$.

In 1997, Alzer [5] established the double inequality

$$\Gamma\left(1+\frac{1}{p}\right)\left(1-e^{-bx^p}\right)^{1/p} < \int_0^x e^{-t^p} \,\mathrm{d}\, t < \Gamma\left(1+\frac{1}{p}\right)\left(1-e^{-ax^p}\right)^{1/p}, \quad x > 0,$$

where

$$\begin{cases} a = 1 \quad \text{and} \quad b = \left[\Gamma\left(1+\dfrac{1}{p}\right)\right]^{-p}, & 0 < p < 1; \\ a = \left[\Gamma\left(1+\dfrac{1}{p}\right)\right]^{-p} \quad \text{and} \quad b = 1, & p > 1 \end{cases}$$

are best possible constants. In 1999, Qi and Guo [26, Theorem 1] proved that

$$xe^{-(x/2)^p} < \int_0^x e^{-t^p} \,\mathrm{d}\, t < \frac{x}{2}\left(1+e^{-x^p}\right) \tag{1.1}$$

for $p \in (0, 1]$ and $x \in (0, \infty)$; if $p > 1$ and $0 < x < \left(1 - \frac{1}{p}\right)^{1/p}$, the double inequality (1.1) reverses. In 1999, Qi et al [25, 27] constructed

$$\int_0^{\pi/2} e^{-t^2 \sin^2 x} \sin x \,\mathrm{d}\, x \le \frac{1-e^{-t^2}}{t^2}, \quad \int_0^x e^{-t^2} \,\mathrm{d}\, t \ge \frac{1-e^{-x^2}}{x}, \quad \int_0^x e^{-t^\alpha} \,\mathrm{d}\, t \ge \frac{1-e^{-x^\alpha}}{x^{\alpha-1}},$$

$$\int_0^t e^{x^2} \,\mathrm{d}\, x < \frac{e^{t^2}-1}{t}, \quad \int_x^\infty e^{-t^\alpha} \,\mathrm{d}\, t \le \frac{e^{-x^\alpha}}{\alpha x^{\alpha-1}}, \quad \int_x^\infty e^{-t^3} \,\mathrm{d}\, t \ge \frac{e^{-x^3}}{3(x+1)^2}, \quad \int_0^x e^{t^\alpha} \,\mathrm{d}\, t \le \frac{e^{x^\alpha}-1}{x^{\alpha-1}},$$

where $\alpha \ge 1$. There are more results on erf_p in the papers [8, 16, 20, 22, 24, 25, 27] and the closely related references therein.

For two distinct positive numbers x and y, the arithmetic, geometric, harmonic, logarithmic, exponential means and the power mean of order $t \in \mathbb{R}$ are respectively defined by $A(x, y) = \frac{x+y}{2}$, $G(x, y) = \sqrt{xy}$, $H(x, y) = \frac{1}{A(1/x, 1/y)}$, and

$$L(x, y) = \frac{x-y}{\ln x - \ln y}, \qquad I(x, y) = \frac{1}{e}\left(\frac{x^x}{y^y}\right)^{1/(x-y)}, \qquad M_t(x, y) = \left(\frac{x^t+y^t}{2}\right)^{1/t}.$$

Recently, several mathematicians began to study inequalities for the error function $\operatorname{erf}(x)$ with respect to all kinds of means, including the arithmetic mean, harmonic mean, and the power mean. See the papers [4, 14, 15] and the closely related references therein.

Let $f : J \subseteq (0,\infty) \to (0,\infty)$ be continuous. Let M and N be means defined above. If

$$f(M(x,y)) \lesseqgtr N(f(x), f(y))$$

for all $x, y \in J$, we call f MN-convex (or MN-concave, respectively).

In 2007, Anderson et al [6] studied the generalized convexity (or concavity, respectively) with respect to general means. In 2010, Baricz [7] presented that if f is differentiable, then it is (a,b)-convex (or (a,b)-concave, respectively) on J if and only if $\frac{x^{1-a}f'(x)}{f^{1-b}(x)}$ is increasing (or decreasing, respectively). We observe that the $(1,1)$-convexity means the AA-convexity, the $(1,0)$-convexity is the AG-convexity, and the $(0,0)$-convexity implies GG-convexity. It is easy to see that a function $f : J \subseteq (0,\infty) \to (0,\infty)$ is said to be geometrically convex if it is convex with respect to the geometric mean, that is, the inequality $f\big(x^\lambda y^{1-\lambda}\big) \le [f(x)]^\lambda [f(y)]^{1-\lambda}$ holds for all $x, y > 0$ and $\lambda \in (0,1)$; if the above inequality is reversed, the function f is called geometrically concave. We note that a differentiable function f is geometrically convex (or concave, respectively) if and only if $\frac{xf'(x)}{f(x)}$ is increasing (or decreasing, respectively).

The first aim of this paper is to present several inequalities of the generalized error function $\operatorname{erf}_p(x)$ in terms of the arithmetic, logarithmic, and exponential means $A(a,b)$, $L(a,b)$ and $I(a,b)$. The second aim is to find monotonicity, convexity, and concavity of the generalized error function $\operatorname{erf}_p(x)$ and, consequently, derive two Grünbaum type and Turán type inequalities.

Our main results can be stated as following theorems.

Theorem 1.1. *For all $p \in (0,\infty)$ and $x, y \in (0,\infty)$, we have*

$$L\big(\operatorname{erf}_p(x), \operatorname{erf}_p(y)\big) < \operatorname{erf}_p(L(x,y)) \quad \text{and} \quad I\big(\operatorname{erf}_p(x), \operatorname{erf}_p(y)\big) < \operatorname{erf}_p(A(x,y)).$$

Theorem 1.2. *For all $p \in [1,\infty)$ and $x, y \in (0,\infty)$, we have*

$$L\big(x^{p-1}\operatorname{erf}_p(x), y^{p-1}\operatorname{erf}_p(y)\big) < L^{p-1}(x,y)\operatorname{erf}_p(L(x,y))$$

and

$$I\big(x^{p-1}\operatorname{erf}_p(x), y^{p-1}\operatorname{erf}_p(y)\big) < A^{p-1}(x,y)\operatorname{erf}_p(A(x,y)).$$

Theorem 1.3. *Let $x, y, z \in (0,\infty)$ such that $z^2 = x^2 + y^2$. Then we have the Grünbaum type inequality*

$$1 + \operatorname{erf}_p\big(z^2\big) \ge \operatorname{erf}_p\big(x^2\big) + \operatorname{erf}_p\big(y^2\big).$$

Theorem 1.4. *For all $p \in (0,\infty)$, the function $x \mapsto \frac{\operatorname{erf}_p(x)}{\arctan_p x}$ is strictly decreasing in $x \in (0,\infty)$, where $\arctan_p(x)$ is defined by $\arctan_p x = \int_0^x \frac{1}{1+t^p}\,\mathrm{d}\,t$. Consequently, we have*

$$\frac{2}{\pi_p}\arctan_p x < \operatorname{erf}_p(x) < \frac{p}{\Gamma(1/p)}\arctan_p x.$$

where $\frac{2}{\pi_p}$ and $\frac{p}{\Gamma(1/p)}$ are best possible constants.

Theorem 1.5. *For all $x \in (0,\infty)$ fixed, the function $p \mapsto \operatorname{erf}_p(x)$ is strictly logarithmically concave in $p \in \left(0, \frac{1}{x_0}\right)$, where $x_0 = 1.461632\ldots$ is the only positive root of the digamma function $\psi(x)$. Consequently, the Turán type inequality*

$$\operatorname{erf}_p^2(x) > \operatorname{erf}_{p-1}(x)\operatorname{erf}_{p+1}(x)$$

is valid for all $p \in \left(0, \frac{1}{x_0}\right)$ and $x \in (0,\infty)$.

2 Lemmas

In order to prove our main results, we need the following lemmas.

Lemma 2.1 ([10, Theorem 1, p. 138])**.** *Let $f : J \subseteq (0,\infty) \to (0,\infty)$.*

1. *If f is increasing and logarithmically convex (or logarithmically concave, respectively), then it is LL-convex (or concave, respectively);*
2. *If f is increasing and logarithmically convex (or logarithmically concave, respectively), then it is AL-convex (or concave, respectively).*

Lemma 2.2 ([11, Theorem 1, p. 6])**.** *Let $f : J \subseteq (0,\infty) \to (0,\infty)$. If $f(x)$ is continuously differentiable, increasing, and logarithmically convex (or logarithmically concave, respectively), then*

$$I(f(x), f(y)) \geq f(I(x,y)) \quad \text{and} \quad I(f(x), f(y)) \leq f(A(x,y)).$$

Lemma 2.3 ([29, Example 152, p. 124])**.** *If $f(x)$ and $g(x)$ are convex functions on $(0,\infty)$ such that $f_1(x) \geq 0$, $f_2(x) \geq 0$, and $f_1(0) = f_2(0) = 0$, then $\frac{f_1(x)f_2(x)}{x}$ is also convex on $(0,\infty)$.*

Lemma 2.4 ([9, Lemma 3, p. 246])**.** *Let $f : (a,\infty) \to \mathbb{R}$ for $a \geq 0$ and $g(x) = \frac{f(x)-1}{x}$ be increasing on (a,∞). Then $h(x) = f\left(x^2\right)$ satisfies the Grünbaum type inequality*

$$1 + h(z) \geq h(x) + h(y) \tag{2.1}$$

for $x, y \geq a$ and $z^2 = x^2 + y^2$. If $g(x)$ is decreasing, then the inequality (2.1) is reversed.

Lemma 2.5 ([19, Lemma 3.2, p. 523])**.** *Let $f(x), g(x)$ be continuous on $[a,b]$, differentiable on (a,b), $g'(x) \neq 0$ on (a,b), and $f(a) = g(a) = 0$ or $f(b) = g(b) = 0$. If $\frac{f'(x)}{g'(x)}$ is increasing (or decreasing, respectively) on (a,b), then the ratio $\frac{f(x)}{g(x)}$ is also increasing (or decreasing, respectively) on (a,b).*

3 Proofs of main results

Now we are in a position to prove our main results.

Proof of Theorem 1.1. A simple computation yields

$$\operatorname{erf}_p'(x) = \frac{p}{\Gamma(1/p)} e^{-x^p} > 0 \quad \text{and} \quad \operatorname{erf}_p''(x) = -\frac{p^2}{\Gamma(1/p)} x^{p-1} e^{-x^p} < 0.$$

Hence, the function $\operatorname{erf}_p(x)$ is strictly increasing and concave on $(0,\infty)$. Since the concavity implies the logarithmic concavity, considering Lemmas 2.1 and 2.2, Theorem 1.1 is thus proved. □

Proof of Theorem 1.2. Considering Lemmas 2.1 and 2.2, it suffices to prove that $x^{p-1}\operatorname{erf}_p(x)$ is strictly increasing and concave on $(0,\infty)$. This follows readily from a simple computation

$$\left[x^{p-1}\operatorname{erf}_p(x)\right]' = x^{p-2}\left[(p-1)\operatorname{erf}_p(x) + x\operatorname{erf}_p'(x)\right] > 0.$$

On the other hand, the function $f_1(x) = x^p$ and $f_2(x) = -\operatorname{erf}_p(x)$ are convex on $(0,\infty)$ and $f_1(0) = f_2(0) = 0$. Making use of Lemma 2.3 reveals that $-x^{p-1}\operatorname{erf}_p(x)$ is also convex on $(0,\infty)$. So, the function $x^{p-1}\operatorname{erf}_p(x)$ is strictly increasing and concave on $(0,\infty)$. The proof of Theorem 1.2 is complete. □

Proof of Theorem 1.3. By virtue of Lemma 2.4, we only need to prove that the function $\frac{\operatorname{erf}_p(x)-1}{x}$ is strictly increasing on $(0,\infty)$. A direct computation results in

$$\left[\frac{\operatorname{erf}_p(x)-1}{x}\right]' = \frac{1}{x^2}\left[\frac{p}{\Gamma(1/p)}e^{-x^p} - \operatorname{erf}_p(x) + 1\right] \triangleq \frac{g(x)}{x^2} \quad \text{and} \quad g'(x) = -\frac{p^2}{\Gamma(1/p)}x^p e^{-x^p} < 0.$$

Accordingly, the function $g(x)$ is strictly decreasing on $(0,\infty)$. Therefore, from the inequality $g(x) > \lim_{x\to\infty} g(x) = 0$, Theorem 1.3 follows immediately. □

Proof of Theorem 1.4. For proving the monotonicity of $\frac{\operatorname{erf}_p(x)}{\arctan_p x}$, we denote $g_1(x) = \operatorname{erf}_p(x)$ and $g_2(x) = \arctan_p x$. Then $g_1(0) = g_2(0) = 0$ and

$$\frac{g_1'(x)}{g_2'(x)} = \frac{p}{\Gamma(1/p)}(1+x^p)e^{-x^p} \triangleq \frac{p}{\Gamma(1/p)}q(x).$$

By a direct differentiation, it follows that $q'(x) = -px^{2p-1}e^{-x^p} < 0$ which implies that the function $q(x)$ is strictly decreasing on $(0,\infty)$. By Lemma 2.5, we complete the required proof. □

Proof of Theorem 1.5. Let $t \in (0,x)$ and $x \in (0,\infty)$ be fixed. Define the function

$$\alpha(p) = \ln\left[\frac{p}{\Gamma(1/p)}e^{-t^p}\right] = \ln p - \ln\left[\Gamma\left(\frac{1}{p}\right)\right] - t^p.$$

By a direct computation, we have

$$\alpha'(p) = \frac{1}{p} + \frac{1}{p^2}\psi\left(\frac{1}{p}\right) - t^p\ln t \quad \text{and} \quad \alpha''(p) = -\frac{1}{p^2} - \frac{2}{p^3}\psi\left(\frac{1}{p}\right) - \frac{1}{p^4}\psi'\left(\frac{1}{p}\right) - t^p\ln^2 t < 0.$$

Therefore, the function $\alpha(p)$ is strictly logarithmically concave. By the fact that integrating preserves the monotonicity and logarithmic concavity, it follows that the function $p \mapsto \operatorname{erf}_p(x)$ is strictly logarithmically concave in $p \in \left(0, \frac{1}{x_0}\right)$. □

References

[1] M. Abramowitz and I. A. Stegun(Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 10th printing, Dover Publications, New York and Washington, 1972.

[2] H. Alzer, *Functional inequalities for the error function*, Aequationes Math. **66** (2003), no. 1-2, 119–127; Available online at http://dx.doi.org/10.1007/s00010-003-2683-9.

[3] H. Alzer, *Functional inequalities for the error function, II*, Aequationes Math. **78** (2009), no. 1-2, 113–121; Available online at http://dx.doi.org/10.1007/s00010-009-2963-0.

[4] H. Alzer, *Error function inequalities*, Adv. Comput. Math. **33** (2010), no. 3, 349–379; Available online at http://dx.doi.org/10.1007/s10444-009-9139-2.

[5] H. Alzer, *On some inequalities for the incomplete gamma function*, Math. Comp. **66** (1997), no. 218, 771–778; Available online at http://dx.doi.org/10.1090/S0025-5718-97-00814-4.

[6] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, *Generalized convexity and inequalities*, J. Math. Anal. Appl. **335** (2007), no. 2, 1294–1308; Available online at http://dx.doi.org/10.1016/j.jmaa.2007.02.016.

[7] Á. Baricz, *Geometrically concave univariate distributions*, J. Math. Anal. Appl. **363** (2010), no. 1, 182–196; Available online at http://dx.doi.org/10.1016/j.jmaa.2009.08.029.

[8] Á. Baricz, B. A. Bhayo, and T. K. Pogány, *Functional inequalities for generalized inverse trigonometric and hyperbolic functions*, J. Math. Anal. Appl. **417** (2014), no. 1, 244–259; Available online at http://dx.doi.org/10.1016/j.jmaa.2014.03.039.

[9] Á. Baricz, B. A. Bhayo, and K. Riku, *Convexity properties of generalized trigonometric and hyperbolic functions*, Aequationes Math. **89** (2015), no. 3, 473–484; Available online at http://dx.doi.org/10.1007/s00010-013-0222-x.

[10] B. Bhayo and L. Yin, *Logarithmic mean inequality for generalized trigonometric and hyperbolic functions*, Acta Univ. Sapientiae Math. **6** (2014), no. 2, 135–145; Available online at http://dx.doi.org/10.1515/ausm-2015-0002.

[11] B. A. Bhayo and L. Yin, *On the generalized convexity and concavity*, Probl. Anal. Issues Anal. **4(22)** (2015), no. 1, 3–10; Available online at http://dx.doi.org/10.15393/j3.art.2015.2709.

[12] J. Cao, D.-W. Niu, and F. Qi, *A Wallis type inequality and a double inequality for probability integral*, Aust. J. Math. Anal. Appl. **5** (2007), no. 1, Art. 3; Available online at http://ajmaa.org/cgi-bin/paper.pl?string=v4n1/V4I1P3.tex.

[13] J. T. Chu, *On bounds for the normal integral*, Biometrika **42** (1955), 263–265.

[14] Y.-M. Chu, Y.-M. Li, W.-F. Xia, and X.-H. Zhang, *Best possible inequalities for the harmonic mean of error function*, J. Inequal. Appl. **2014**, 2014:525, 9 pages; Available online at http://dx.doi.org/10.1186/1029-242X-2014-525.

[15] Y.-M. Chu and T.-H. Zhao, *Concavity of the error function with respect to Hölder mean*, Math. Inequal. Appl. **19** (2016), no. 2, 589–595; Available online at http://dx.doi.org/10.7153/mia-19-43.

[16] Á. Elbert and A. Laforgia, *An inequality for the product of two integrals relating to the incomplete Gamma function*, J. Inequal. Appl. **5** (2000), no. 1, 39–51; Available online at `http://dx.doi.org/10.1155/S1025583400000035`.

[17] B.-N. Guo and F. Qi, *A class of completely monotonic functions involving divided differences of the psi and tri-gamma functions and some applications*, J. Korean Math. Soc. **48** (2011), no. 3, 655–667; Available online at `http://dx.doi.org/10.4134/JKMS.2011.48.3.655`.

[18] B.-N. Guo and F. Qi, *On the Wallis formula*, Internat. J. Anal. Appl. **8** (2015), no. 1, 30–38.

[19] R. Klén, M. Vuorinen and X.-H. Zhang, *Inequalities for the generalized trigonometric and hyperbolic functions*, J. Math. Anal. Appl. **409** (2014), no. 1, 521–529; Available online at `http://dx.doi.org/10.1016/j.jmaa.2013.07.021`.

[20] A. Laforgia and P. Natalini, *Supplements to known monotonicity results and inequalities for the gamma and incomplete gamma functions*, J. Inequal. Appl. **2006**, Article ID 48727, 8 pages; Available online at `http://dx.doi.org/10.1155/JIA/2006/48727`.

[21] D. S. Mitrinović and R. J. Weinacht, *Problems and Solutions: Solutions of Advanced Problems: 5555*, Amer. Math. Monthly **75** (1968), no. 10, 1129–1130; Available online at `http://dx.doi.org/10.2307/2315774`.

[22] P. Natalini and B. Palumbo, *Inequalities for the incomplete gamma function*, Math. Inequal. Appl. **3** (2000), no. 1, 69–77; Available online at `http://dx.doi.org/10.7153/mia-03-08`.

[23] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark(eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, 2010; Available online at `http://dlmf.nist.gov/`.

[24] F. Qi, *Monotonicity results and inequalities for the gamma and incomplete gamma functions*, Math. Inequal. Appl. **5** (2002), no. 1, 61–67; Available online at `http://dx.doi.org/10.7153/mia-05-08`.

[25] F. Qi, L.-H. Cui, and S.-L. Xu, *Some inequalities constructed by Tchebysheff's integral inequality*, Math. Inequal. Appl. **2** (1999), no. 4, 517–528; Available online at `http://dx.doi.org/10.7153/mia-02-42`.

[26] F. Qi and S.-L. Guo, *Inequalities for the incomplete gamma and related functions*, Math. Inequal. Appl. **2** (1999), no. 1, 47–53; Available online at `http://dx.doi.org/10.7153/mia-02-05`.

[27] F. Qi and J.-Q. Mei, *Some inequalities of the incomplete gamma and related functions*, Z. Anal. Anwendungen **18** (1999), no. 3, 793–799; Available online at `http://dx.doi.org/10.4171/ZAA/914`.

[28] K. Oldham, J. Myland, and J. Spanier, *An Atlas of Functions*, 2nd Ed., Springer, New York, 2009; Available online at `http://dx.doi.org/10.1007/978-0-387-48807-3`.

[29] C.-H. Xue and S.-L. Xu, *Mathematical Analysis: Selected Exercises and Their Solutions*, Vol. 1, Tsinghua University Press, Beijing, 2009. (Chinese)

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