Volume 25, Number 2 August 2018
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(sixteen times annually)

Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

The main purpose of "J.Computational Analysis and Applications" is to publish high quality research articles from all subareas of Computational Mathematical Analysis and its many potential applications and connections to other areas of Mathematical Sciences. Any paper whose approach and proofs are computational,using methods from Mathematical Analysis in the broadest sense is suitable and welcome for consideration in our journal, except from Applied Numerical Analysis articles. Also plain word articles without formulas and proofs are excluded. The list of possibly connected mathematical areas with this publication includes, but is not restricted to: Applied Analysis, Applied Functional Analysis, Approximation Theory, Asymptotic Analysis, Difference Equations, Differential Equations, Partial Differential Equations, Fourier Analysis, Fractals, Fuzzy Sets, Harmonic Analysis, Inequalities, Integral Equations, Measure Theory, Moment Theory, Neural Networks, Numerical Functional Analysis, Potential Theory, Probability Theory, Real and Complex Analysis, Signal Analysis, Special Functions, Splines, Stochastic Analysis, Stochastic Processes, Summability, Tomography, Wavelets, any combination of the above, e.t.c.

"J.Computational Analysis and Applications" is a peer-reviewed Journal. See the instructions for preparation and submission of articles to JoCAAA. Assistant to the Editor: Dr.Razvan Mezei,mezei_razvan@yahoo.com, Madison,WI,USA.

Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $150 more(handling and postages) to the above prices for Print. No credit card payments.

JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

Editorial Board
Associate Editors of Journal of Computational Analysis and Applications

Dartmouth College
8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
sever.dragomir@vu.edu.au
Inequalities, Functional Analysis, Numerical Analysis, Approximations, Information Theory, Stochastics.

Oktay Duman
TOBB University of Economics and Technology,
Department of Mathematics, TR-06530,
Ankara, Turkey,
oduman@etu.edu.tr
Classical Approximation Theory, Summability Theory, Statistical Convergence and its Applications

Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
Approximation Theory, Computer Aided Geometric Design

John R. Graef
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
Ordinary and functional differential equations, difference equations, impulsive systems, differential inclusions, dynamic equations on time scales, control theory and their applications

Weimin Han
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

Computational Mathematics

Choonkil Park
Department of Mathematics
Hanyang University
Seoul 133-791
S. Korea, baak@hanyang.ac.kr
Functional Equations

Svetlozar (Zari) Rachev,
Professor of Finance, College of Business, and Director of Quantitative Finance Program,
Department of Applied Mathematics & Statistics
Stonybrook University
312 Harriman Hall, Stony Brook, NY 11794-3775
tel: +1-631-632-1998,
svetlozar.rachev@stonybrook.edu

Alexander G. Ramm
Mathematics Department
Kansas State University
Manhattan, KS 66506-2602
e-mail: ramm@math.ksu.edu
Inverse and Ill-posed Problems, Scattering Theory, Operator Theory, Theoretical Numerical Analysis, Wave Propagation, Signal Processing and Tomography

Tomasz Rychlik
Polish Academy of Sciences
Instytut Matematyczny PAN
00-956 Warszawa, skr. poczt. 21
ul. Śniadeckich 8
Poland
trychlik@impan.pl
Mathematical Statistics, Probabilistic Inequalities

Boris Shekhtman
Department of Mathematics
University of South Florida
Tampa, FL 33620, USA
Tel 813-974-9710
shekhtma@usf.edu
Approximation Theory, Banach spaces, Classical Analysis

T. E. Simos
Department of Computer Science and Technology
Faculty of Sciences and Technology
University of Peloponnese
GR-221 00 Tripolis, Greece
Postal Address:
26 Menelaou St.
Anfithea - Paleon Faliron
GR-175 64 Athens, Greece
tsimos@mail.ariadne-t.gr
Numerical Analysis

H. M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3R4
Canada
tel.250-472-5313; office,250-477-6960 home, fax 250-721-8962
harimsri@math.uvic.ca
Real and Complex Analysis, Fractional Calculus and Appl., Integral Equations and Transforms, Higher Transcendental Functions and Appl.,q-Series and q-Polynomials, Analytic Number Th.

I. P. Stavroulakis
Department of Mathematics
University of Ioannina
451-10 Ioannina, Greece
ipstav@cc.uoi.gr
Differential Equations
Phone +3-065-109-8283

Manfred Tasche
Department of Mathematics
University of Rostock
D-18051 Rostock, Germany
manfred.tasche@mathematik.uni-rostock.de
Numerical Fourier Analysis, Fourier Analysis, Harmonic Analysis, Signal Analysis, Spectral Methods, Wavelets, Splines, Approximation Theory

Roberto Triggiani
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
PDE, Control Theory, Functional Analysis, rtrggani@memphis.edu

Juan J. Trujillo
University of La Laguna
Departamento de Analisis Matematico
C/Astr.Fco.Sanchez s/n
38271. LaLaguna. Tenerife.
SPAIN
Tel/Fax 34-922-318209
Juan.Trujillo@ull.es

Fractional: Differential Equations-Operators-Fourier Transforms, Special functions, Approximations, and Applications

Ram Verma
International Publications
1200 Dallas Drive #824 Denton,
TX 76205, USA
Verma99@msn.com
Applied Nonlinear Analysis, Numerical Analysis, Variational Inequalities, Optimization Theory, Computational Mathematics, Operator Theory

Xiang Ming Yu
Department of Mathematical Sciences
Southwest Missouri State University
Springfield, MO 65804-0094
417-836-5931
xmy944f@missouristate.edu
Classical Approximation Theory, Wavelets

Lotfi A. Zadeh
Professor in the Graduate School and Director, Computer Initiative, Soft Computing (BISC)
Computer Science Division
University of California at Berkeley
Berkeley, CA 94720
Office: 510-642-4959
Sec: 510-642-8271
Home: 510-526-2569
FAX: 510-642-1712
zadeh@cs.berkeley.edu
Fuzzyness, Artificial Intelligence, Natural language processing, Fuzzy logic

Richard A. Zalik
Department of Mathematics
Auburn University
Auburn University, AL 36849-5310
USA.
Tel 334-844-6557 office
678-642-8703 home
Fax 334-844-6555
zalik@auburn.edu
Approximation Theory, Chebychev Systems, Wavelet Theory

Ahmed I. Zayed
Department of Mathematical Sciences
DePaul University
2320 N. Kenmore Ave.
Chicago, IL 60614-3250
773-325-7808
e-mail: azayed@condor.depaul.edu
Shannon sampling theory, Harmonic analysis and wavelets, Special functions and orthogonal polynomials, Integral transforms

Ding-Xuan Zhou
Department Of Mathematics
City University of Hong Kong
83 Tat Chee Avenue
Kowloon, Hong Kong
852-2788 9708,Fax:852-2788 8561
e-mail: mazhou@cityu.edu.hk
Approximation Theory, Spline functions, Wavelets

Xin-long Zhou
Fachbereich Mathematik, Fachgebiet Informatik
Gerhard-Mercator-Universitat Duisburg
Lotharstr.65, D-47048 Duisburg, Germany
e-mail:Xzhou@informatik.uni-duisburg.de
Fourier Analysis, Computer-Aided Geometric Design, Computational Complexity, Multivariate Approximation Theory, Approximation and Interpolation Theory

Jessada Tariboon
Department of Mathematics,
King Mongkut's University of Technology N. Bangkok
1518 Pracharat 1 Rd., Wongsawang,
Bangsue, Bangkok, Thailand 10800
jessada.t@sci.kmutnb.ac.th, Time scales, Differential/Difference Equations, Fractional Differential Equations

Instructions to Contributors
Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

1. Manuscripts files in Latex and PDF and in English, should be submitted via email to the Editor-in-Chief:

Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
Tel. 901.678.3144
e-mail: ganastss@memphis.edu

Authors may want to recommend an associate editor the most related to the submission to possibly handle it.

Also authors may want to submit a list of six possible referees, to be used in case we cannot find related referees by ourselves.

2. Manuscripts should be typed using any of TEX,LaTEX,AMS-TEX,or AMS-LaTEX and according to EUDOXUS PRESS, LLC. LATEX STYLE FILE. (Click HERE to save a copy of the style file.)They should be carefully prepared in all respects. Submitted articles should be brightly typed (not dot-matrix), double spaced, in ten point type size and in 8(1/2)x11 inch area per page. Manuscripts should have generous margins on all sides and should not exceed 24 pages.

3. Submission is a representation that the manuscript has not been published previously in this or any other similar form and is not currently under consideration for publication elsewhere. A statement transferring from the authors(or their employers,if they hold the copyright) to Eudoxus Press, LLC, will be required before the manuscript can be accepted for publication.The Editor-in-Chief will supply the necessary forms for this transfer.Such a written transfer of copyright,which previously was assumed to be implicit in the act of submitting a manuscript,is necessary under the U.S.Copyright Law in order for the publisher to carry through the dissemination of research results and reviews as widely and effective as possible.

4. The paper starts with the title of the article, author's name(s) (no titles or degrees), author's affiliation(s) and e-mail addresses. The affiliation should comprise the department, institution (usually university or company), city, state (and/or nation) and mail code.

The following items, 5 and 6, should be on page no. 1 of the paper.

5. An abstract is to be provided, preferably no longer than 150 words.

6. A list of 5 key words is to be provided directly below the abstract. Key words should express the precise content of the manuscript, as they are used for indexing purposes.

The main body of the paper should begin on page no. 1, if possible.

7. All sections should be numbered with Arabic numerals (such as: 1. INTRODUCTION) .
Subsections should be identified with section and subsection numbers (such as 6.1. Second-Value Subheading).
If applicable, an independent single-number system (one for each category) should be used to label all theorems, lemmas, propositions, corollaries, definitions, remarks, examples, etc. The label (such as Lemma 7) should be typed with paragraph indentation, followed by a period and the lemma itself.

8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbed pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

10. List references alphabetically at the end of the paper and number them consecutively. Each must be cited in the text by the appropriate Arabic numeral in square brackets on the baseline.
References should include (in the following order):
initials of first and middle name, last name of author(s)
title of article,

name of publication, volume number, inclusive pages, and year of publication.

Authors should follow these examples:

Journal Article

1. H.H.Gonska,Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

Book

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea,New York,1986.

Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

12. Footnotes should be avoided. When their use is absolutely necessary, footnotes should be numbered consecutively using Arabic numerals and should be typed at the bottom of the page to which they refer. Place a line above the footnote, so that it is set off from the text. Use the appropriate superscript numeral for citation in the text.

13. After each revision is made please again submit via email Latex and PDF files of the revised manuscript, including the final one.

14. Effective 1 Nov. 2009 for current journal page charges, contact the Editor in Chief. Upon acceptance of the paper an invoice will be sent to the contact author. The fee payment will be due one month from the invoice date. The article will proceed to publication only after the fee is paid. The charges are to be sent, by money order or certified check, in US dollars, payable to Eudoxus Press, LLC, to the address shown on the Eudoxus homepage.

No galleys will be sent and the contact author will receive one (1) electronic copy of the journal issue in which the article appears.

15. This journal will consider for publication only papers that contain proofs for their listed results.

COUPLED FIXED POINT THEOREM IN PARTIALLY ORDERED MODULAR METRIC SPACES AND ITS AN APPLICATION

ALİ MUTLU, KÜBRA ÖZKAN, AND UTKU GÜRDAL

Abstract. In this article, we extend certain coupled fixed theorem which was introduced for mappings having the mixed monotone property in various metric spaces to partially ordered modular metric spaces. In addition to this, we express some results about this theorem. Finally, we show using our main theorem that there exists a unique solution for a given nonlinear integral equation.

1. Introduction

The first time in literature, Guo and Lakshmikantham [12] introduced the concept of coupled fixed point in 1987. After that, Bhaskar and Lakshmikantham [5] introduced the concept of the mixed monotone property and expressed certain coupled fixed point theorems which are considered as the most interesting fixed point theorems for mappings having this property in ordered metric spaces. They showed the existence of a unique solution for a periodic boundary value problem. Since the coupled fixed point theorems in the study of nonlinear integral equations and differential equations are important tools, many researcher have studied them in various partially ordered metric spaces, e.g. [3, 4, 6, 13, 16, 18, 20, 21, 22, 23, 24].

Lately, a lot of significant results related to fixed point theorems have been extended to modular metric spaces which was introduced by Chistyakov via F-modular [7] in 2008 and developed the theory of this spaces in 2010 [8]. And then, many authors made various studies on these structures, e.g. [1, 2, 9, 10, 11, 14, 15, 17].

In this article, we extend certain coupled fixed theorem which was introduced for mappings having the mixed monotone property in various metric spaces to partially ordered modular metric spaces. In addition to this, we investigate some results about this theorem. Finally, we show using our main theorem that there exists a unique solution for a given nonlinear integral equation.

2. Modular Metric Spaces

Here, we express a series of definitions of some fundamental notions related to modular metric spaces.

Definition 2.1. [19] Let X be a vector space on $\mathbb{R}$ and $\rho : X \to [0,\infty]$ be a function. If ρ satisfies the following conditions, we call that ρ is a modular on X:

(1) $\rho(0) = 0$;
(2) If $a \in X$ and $\rho(\gamma a) = 0$ for all numbers $\gamma > 0$, then $a = 0$;

2010 *Mathematics Subject Classification.* 46A80, 47H10, 54H25.
Key words and phrases. Modular metric space, coupled fixed point, complete modular metric.

(3) $\rho(-a) = \rho(a)$, for all $a \in X$;
(4) $\rho(\gamma a + \theta b) \leq \rho(a) + \rho(b)$ for all $\gamma, \theta \geq 0$ with $\gamma + \theta = 1$ and $a, b \in X$.

Let $X \neq \emptyset$ and $\omega : (0, \infty) \times X \times X \to [0, \infty]$ be a function where $\lambda \in (0, \infty)$. Throughout this article, the value $\omega(\lambda, a, b)$ is denoted by $\omega_\lambda(a, b)$ for all $a, b \in X$ and $\lambda > 0$.

Definition 2.2. [8] Let $X \neq \emptyset$ and $\omega : (0, \infty) \times X \times X \to [0, \infty]$ be a function. If ω satisfies the following conditions for all $a, b, c \in X$, we call that ω is a metric modular on X:

(m1) $\omega_\lambda(a, b) = 0$ for all $\lambda > 0 \Leftrightarrow a = b$;
(m2) $\omega_\lambda(a, b) = \omega_\lambda(b, a)$ for all $\lambda > 0$;
(m3) $\omega_{\lambda+\mu}(a, b) \leq \omega_\lambda(a, c) + \omega_\mu(c, b)$ for all $\lambda, \mu > 0$.

From [8, 9], we know that as fix $a_0 \in X$, the two sets

$$X_\omega = X_\omega(a_0) = \{a \in X : \omega_\lambda(a, a_0) \to 0 \text{ as } \lambda \to \infty\}$$

and

$$X^*_\omega = X^*_\omega(a_0) = \{a \in X : \exists \lambda = \lambda(a) > 0 \text{ such that } \omega_\lambda(a, a_0) < \infty\}$$

are said to be modular spaces.

From [8, 9], the modular space X_ω can be equipped by a metric

$$d_\omega(a, b) = \inf\{\lambda > 0 : \omega_\lambda(a, b) \leq \lambda\}$$

which is generated by ω for any $a, b \in X_\omega$ where ω is a modular on X.

Definition 2.3. [15] Let X_ω be a modular metric space, $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in X_ω and $C \subseteq X_\omega$. Then,

(1) $\{a_n\}_{n\in\mathbb{N}}$ is called a modular convergent sequence such that $a_n \to a,\ a \in X_\omega$ if
$$\omega_\lambda(a_n, a) \to 0 \text{ as } n \to \infty$$
for all $\lambda > 0$.
(2) $\{a_n\}_{n\in\mathbb{N}}$ is called a modular Cauchy sequence if and only if for all $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that $\omega_\lambda(a_n, a_m) < \epsilon$ for each $m, n \geq n(\epsilon)$ and $\lambda > 0$.
(3) C is called complete modular if every modular Cauchy sequence $\{a_n\}$ in C is a modular convergent in C.

3. Main Results

Let $\leq$ be a ordered relation and X_ω be a modular metric space. Throughout this article, $(X_\omega, \leq)$ denotes partially ordered modular metric space.

Definition 3.1. Let $(X_\omega, \leq)$ be a partially ordered modular metric space. The mapping $F : X_\omega \times X_\omega \to X_\omega$ has the mixed monotone property if F holds the following conditions for any $a, b \in X_\omega$

$$a_1 \leq a_2 \Rightarrow F(a_1, b) \leq F(a_2, b),\ a_1, a_2 \in X_\omega$$

and

$$b_1 \geq b_2 \Rightarrow F(a, b_1) \leq F(a, b_2),\ b_1, b_2 \in X_\omega.$$

These imply that F is monotone non-decreasing in a and monotone non-increasing in b.

Definition 3.2. Let X_ω be a modular metric space, $(a,b) \in X_\omega \times X_\omega$ and $F : X_\omega \times X_\omega \to X_\omega$ be a mapping. (a,b) is called a coupled fixed point of the mapping if

$$F(a,b) = a \text{ and } F(b,a) = b.$$

Theorem 3.3. *Let $(X_\omega, \leq)$ be a partially ordered complete modular metric space, the mapping $F : X_\omega \times X_\omega \to X_\omega$ has the mixed monotone property in X_ω and k, l be non-negative constants such that $k + l < 1$. Suppose that we have the following condition for all $a, b, p, q \in X_\omega$ and $\lambda > 0$*

$$\omega_\lambda(F(a,b), F(p,q)) \leq k\omega_\lambda(a,p) + l\omega_\lambda(b,q) \tag{3.1}$$

where $a \geq p$, $q \geq b$.

If there exist $a_0, b_0 \in X_\omega$ with $a_0 \leq F(a_0,b_0)$ and $b_0 \geq F(b_0,a_0)$, then F has a unique coupled fixed point.

Proof. Let $a_0, b_0 \in X_\omega$ with $a_0 \leq F(a_0,b_0)$ and $b_0 \geq F(b_0,a_0)$. We take $a_1, b_1 \in X_\omega$ with $a_1 = F(a_0,b_0)$ and $b_1 = F(b_0,a_0)$. Again we take $a_2, b_2 \in X_\omega$ with $a_2 = F(a_1,b_1)$ and $b_2 = F(b_1,a_1)$. Repeating this way, we obtain sequences $\{a_n\}$ and $\{b_n\}$ in X_ω with

$$a_{n+1} = F(a_n,b_n) \text{ and } b_{n+1} = F(b_n,a_n)$$

for all $n \in \mathbb{N}^+$. In view of mixed monotone property of F, we get

$$a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots$$

and

$$b_0 \geq b_1 \geq b_2 \geq \cdots \geq b_n \geq b_{n+1} \geq \cdots .$$

Then, by (3.1), we get

$$\begin{aligned} \omega_\lambda(a_n, a_{n+1}) &= \omega_\lambda(F(a_{n-1}, b_{n-1}), F(a_n, b_n)), \\ &\leq k\omega_\lambda(a_{n-1}, a_n) + l\omega_\lambda(b_{n-1}, b_n) \end{aligned} \tag{3.2}$$

for all $n \in \mathbb{N}^+$, $\lambda > 0$ and $k + l < 1$. Similarly,

$$\begin{aligned} \omega_\lambda(b_n, b_{n+1}) &= \omega_\lambda(F(b_{n-1}, a_{n-1}), F(b_n, a_n)), \\ &\leq k\omega_\lambda(b_{n-1}, b_n) + l\omega_\lambda(a_{n-1}, a_n) \end{aligned} \tag{3.3}$$

for all $n \in \mathbb{N}^+$, $\lambda > 0$ and $k + l < 1$. Therefore, letting

$$e_n = \omega_\lambda(a_n, a_{n+1}) + \omega_\lambda(b_n, b_{n+1})$$

for all $n \in \mathbb{N}^+$, $\lambda > 0$. Using equations (3.2) and (3.3), we get

$$\begin{aligned} e_n &= \omega_\lambda(a_n, a_{n+1}) + \omega_\lambda(b_n, b_{n+1}) \\ &\leq k\omega_\lambda(a_{n-1}, a_n) + l\omega_\lambda(b_{n-1}, b_n) + k\omega_\lambda(b_{n-1}, b_n) + l\omega_\lambda(a_{n-1}, a_n) \\ &= (k+l)(\omega_\lambda(a_{n-1}, a_n) + \omega_\lambda(b_{n-1}, b_n)) \\ &= (k+l)e_{n-1}. \end{aligned}$$

Then, we obtain that

$$0 \leq e_n \leq (k+l)e_{n-1} \leq (k+l)^2 e_{n-2} \leq \cdots \leq (k+l)^n e_0. \tag{3.4}$$

If $e_0 = 0$, then $e_0 = \omega_\lambda(a_0,a_1) + \omega_\lambda(b_0,b_1) = 0$. Therefore, we get $\omega_\lambda(a_0,a_1) = 0$ and $\omega_\lambda(b_0,b_1) = 0$. So, from condition (m2) of modular metric spaces, we get

$$a_0 \leq a_1 = F(a_0,b_0) \quad \text{and} \quad b_0 \geq b_1 = F(b_0,a_0).$$

This implies that (a_0,b_0) is a coupled fixed point of F.

Now, let $e_0 > 0$. Preserving the generality, suppose that for $m, n \in \mathbb{N}$ and $n < m$, there exist $n_{\frac{\lambda}{m-n}} \in \mathbb{N}$ satisfying

$$\omega_{\frac{\lambda}{m-n}}(a_n, a_{n+1}) + \omega_{\frac{\lambda}{m-n}}(b_n, b_{n+1}) = e_n$$

for all $\frac{\lambda}{m-n} > 0$ and $n \geq n_{\frac{\lambda}{m-n}}$. We get

(3.5)
$$\begin{aligned}
\omega_\lambda(a_n, a_m) &\leq \omega_{\frac{\lambda}{m-n}}(a_n, a_{n+1}) + \omega_{\frac{\lambda}{m-n}}(a_{n+1}, a_{n+2}) + \cdots + \omega_{\frac{\lambda}{m-n}}(a_{m-1}, a_m), \\
\omega_\lambda(b_n, b_m) &\leq \omega_{\frac{\lambda}{m-n}}(b_n, b_{n+1}) + \omega_{\frac{\lambda}{m-n}}(b_{n+1}, b_{n+2}) + \cdots + \omega_{\frac{\lambda}{m-n}}(b_{m-1}, b_m)
\end{aligned}$$

for each $n < m$. Thus, from (3.4) and (3.5), we get

$$\begin{aligned}
\omega_\lambda(a_n, a_m) + \omega_\lambda(b_n, b_m) &\leq (\omega_{\frac{\lambda}{m-n}}(a_n, a_{n+1}) + \omega_{\frac{\lambda}{m-n}}(b_n, b_{n+1})) + \cdots \\
&\quad + (\omega_{\frac{\lambda}{m-n}}(a_{m-1}, a_m) + \omega_{\frac{\lambda}{m-n}}(b_{m-1}, b_m)), \\
&= e_n + e_{n+1} + \cdots + e_{m-1}, \\
&\leq (k+l)^n e_0 + (k+l)^{n+1} e_0 + \cdots + (k+l)^{m-1} e_0, \\
&= ((k+l)^n + (k+l)^{n+1} + \cdots + (k+l)^{m-1}) e_0, \\
&\leq \frac{(k+l)^n}{1-(k+l)} e_0 \qquad (3.6)
\end{aligned}$$

for $n < m$ and $\lambda > 0$. Let $k + l = \delta$. Since there exists n_0 such that $\frac{\delta^{n_0}}{1-\delta} e_0 < \epsilon$, from (3.6), we have for each $n, m \geq n_0$ that

$$\omega_\lambda(a_n, a_m) + \omega_\lambda(b_n, b_m) < \epsilon$$

for an arbitrary $\epsilon > 0$. Then, $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences in X_ω. Using completeness of X_ω, we can talk about existence of $a, b \in X_\omega$ with

$$\lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} b_n = b.$$

There exists $n_0 \in \mathbb{N}$ with $\omega_{\frac{\lambda}{2}}(a_n, a) < \frac{\epsilon}{2}$ and $\omega_{\frac{\lambda}{2}}(b_n, b) < \frac{\epsilon}{2}$ for all $n \geq n_0$, $\lambda > 0$ and every $\epsilon > 0$. So, from (3.1), we get

$$\begin{aligned}
\omega_\lambda(F(a, b), a) &\leq \omega_{\frac{\lambda}{2}}(F(a, b), a_{n+1}) + \omega_{\frac{\lambda}{2}}(a_{n+1}, a) \\
&= \omega_{\frac{\lambda}{2}}(F(a, b), F(a_n, b_n)) + \omega_{\frac{\lambda}{2}}(a_{n+1}, a) \\
&\leq k\omega_{\frac{\lambda}{2}}(a_n, a) + l\omega_{\frac{\lambda}{2}}(b_n, b) + \omega_{\frac{\lambda}{2}}(a_{n+1}, a) \\
&< k\frac{\epsilon}{2} + l\frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= (k+l)\frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

for all $\lambda > 0$ and each $n \in \mathbb{N}$. Then, $\omega_\lambda(F(a, b), a) = 0$. So, $F(a, b) = a$. In similar way, we get $F(b, a) = b$. These imply that (a, b) is a coupled fixed point of F. We assume that F has an another coupled fixed point (a^*, b^*). Then, for $\lambda > 0$ we get

$$\omega_\lambda(a^*, a) = \omega_\lambda(F(a^*, b^*), F(a, b)) \leq k\omega_\lambda(a^*, a) + l\omega_\lambda(b^*, b)$$

and

$$\omega_\lambda(b^*, b) = \omega_\lambda(F(b^*, a^*), F(b, a)) \leq k\omega_\lambda(b^*, b) + l\omega_\lambda(a^*, a).$$

Therefore, we get

$$\omega_\lambda(a^*, a) + \omega_\lambda(b^*, b) \leq (k+l)(\omega_\lambda(a^*, a) + \omega_\lambda(b^*, b)). \qquad (3.7)$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 25, NO.2, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

Since $k+l<1$, from (3.7) we get $\omega_\lambda(a^*,a)+\omega_\lambda(b^*,b)=0$ for all $\lambda>0$. Hence, we obtain that

$$\omega_\lambda(a^*,a)=0 \Leftrightarrow a^*=a \text{ and } \omega_\lambda(b^*,b)=0 \Leftrightarrow b^*=b.$$

Therefore, (a,b) is a unique coupled fixed point of F. □

If we take equal the constants k, l in Theorem (3.3), the following corollary is obtained.

Corollary 3.4. Let $(X_\omega,\leq)$ be a partially ordered complete modular metric space, the mapping $F: X_\omega\times X_\omega\to X_\omega$ has the mixed monotone property in X_ω and $k\in[0,1)$. Suppose that we have the following condition for all $a,b,p,q\in X_\omega$ and $\lambda>0$

$$\omega_\lambda(F(a,b),F(p,q))\leq \frac{k}{2}(\omega_\lambda(a,p)+\omega_\lambda(b,q)) \tag{3.8}$$

where $a\geq p$, $q\geq b$.

If there exist $a_0,b_0\in X_\omega$ with

$$a_0\leq F(a_0,b_0) \text{ and } b_0\geq F(b_0,a_0),$$

then F has a unique coupled fixed point.

Theorem 3.5. *Let $(X_\omega,\leq)$ be a partially ordered complete modular metric space. Suppose that X_ω satisfies the following conditions*

(i) *if a non-decreasing sequence $a_n\to a$, then $a_n\leq a$ for all n,*

(ii) *if a non-increasing sequence $b_n\to b$, then $b_n\geq b$ for all n.*

Let the mapping $F: X_\omega\times X_\omega\to X_\omega$ has the mixed monotone property in X_ω and k, l be non-negative constants such that $k+l<1$. Suppose that we have the following condition for all $a,b,p,q\in X_\omega$ and $\lambda>0$

$$\omega_\lambda(F(a,b),(p,q))\leq k\omega_\lambda(a,p)+l\omega_\lambda(b,q)$$

where $a\geq p$, $q\geq b$.

If there exist $a_0,b_0\in X_\omega$ with

$$a_0\leq F(a_0,b_0) \text{ and } b_0\geq F(b_0,a_0),$$

then F has a unique coupled fixed point.

Proof. This proof can be made in analogy to the proof of Theorem (3.3). Here, it will be enough only to show that $F(a,b)=a$ and $F(b,a)=b$ for proof. Let $\epsilon>0$. Since $F(a_{n-1},b_{n-1})=a_n\to a$ and $F(b_{n-1},a_{n-1})=b_n\to b$, there exists $n_0\in\mathbb{N}$ with

$$\begin{array}{rcl}\omega_\lambda(a_n,a) & = & \omega_\lambda(F(a_{n-1},b_{n-1}),a)<\frac{\epsilon}{3}\\ \omega_\lambda(b_n,b) & = & \omega_\lambda(F(b_{n-1},a_{n-1}),b)<\frac{\epsilon}{3}\end{array} \tag{3.9}$$

for all $n\geq n_0$. Letting $n\geq n_0$ and using the equations (3.1) and (3.9), we get

$$\begin{array}{rcl}\omega_\lambda(F(a,b),a) & \leq & \omega_{\frac{\lambda}{2}}(F(a,b),a_{n+1})+\omega_{\frac{\lambda}{2}}(a_{n+1},a)\\ & = & \omega_{\frac{\lambda}{2}}(F(a,b),F(a_n,b_n))+\omega_{\frac{\lambda}{2}}(a_{n+1},a)\\ & \leq & k\omega_{\frac{\lambda}{2}}(a_n,a)+l\omega_{\frac{\lambda}{2}}(b_n,b)+\omega_{\frac{\lambda}{2}}(a_{n+1},a)\\ & < & k\dfrac{\epsilon}{3}+l\dfrac{\epsilon}{3}+\dfrac{\epsilon}{3}\\ & = & (k+l)\dfrac{\epsilon}{3}+\dfrac{\epsilon}{3}<\epsilon.\end{array}$$

Then, we get $F(a,b) = a$. In a similar way, we obtain that $\omega_\lambda(F(a,b),a) < \epsilon$ implies $F(b,a) = b$. On the other hand, uniqueness of the coupled fixed point of F can be shown in a similar way with the proof of Theorem (3.3). □

Corollary 3.6. Let $(X_\omega, \leq)$ be a partially ordered complete modular metric space. Suppose that X_ω satisfies the following conditions;

(i) if a non-decreasing sequence $a_n \to a$, then $a_n \leq a$ for all n,

(ii) if a non-increasing sequence $b_n \to b$, then $b_n \geq b$ for all n.

Let the mapping $F : X_\omega \times X_\omega \to X_\omega$ has the mixed monotone property in X_ω and $k \in [0,1)$. Suppose that we have the following condition for all $a, b, p, q \in X_\omega$ and $\lambda > 0$

$$\omega_\lambda(F(a,b),(p,q)) \leq \frac{k}{2}(\omega_\lambda(a,p) + \omega_\lambda(b,q))$$

where $a \geq p$, $q \geq b$. If there exist $a_0, b_0 \in X_\omega$ with

$$a_0 \leq F(a_0,b_0) \text{ and } b_0 \geq F(b_0,a_0),$$

then F has a unique coupled fixed point.

Example 3.7. Let $X_\omega = \mathbb{R}$. If we take the usual partial order $\leq$ in $\mathbb{R}$, then $(\mathbb{R}, \leq)$ is a partially ordered set. We define a mapping $\omega : (0,\infty) \times \mathbb{R} \times \mathbb{R} \to [0,\infty)$ by $\omega_\lambda(a,b) = \frac{|a-b|}{\lambda}$ for all $a, b \in \mathbb{R}$ and $\lambda > 0$. It can be said that X_ω is a complete modular metric space. We take a mapping $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $F(a,b) = \frac{a+b}{6}$. We easily see that F has the mixed monotone property. Then, we have

$$\begin{aligned}
\omega_\lambda(F(a,b),F(p,q)) &= \omega_\lambda(\frac{a+b}{6}, \frac{p+q}{6}) \\
&= \frac{|\frac{a+b}{6} - \frac{p+q}{6})|}{\lambda} \\
&= \frac{1}{6}\frac{|a-p+b-q|}{\lambda} \\
&\leq \frac{1}{6}(\frac{|a-p|}{\lambda} + \frac{|b-q|}{\lambda}) \\
&= \frac{1}{6}(\omega_\lambda(a,p) + \omega_\lambda(b,q))
\end{aligned}$$

for any $a, b, p, q \in X_\omega$ So, the equation (3.8) is satisfied for $k = \frac{1}{3}$. Therefore, from Corollary (3.4), F has a unique coupled fixed point. Also, there are $a_0 = 0 \leq F(0,0) = F(a_0,b_0)$ and $b_0 = 0 \geq F(0,0) = F(b_0,a_0)$. It is obvious that $(0,0)$ is the coupled fixed point of F.

On the other hand, if $F : X_\omega \times X_\omega \to X_\omega$ is defined by $F(a,b) = \frac{a+b}{2}$, then F satisfies the condition (3.8) for $k = 1$. Then, we get

$$\begin{aligned}
\omega_\lambda(F(a,b),F(p,q)) &= \omega_\lambda(\frac{a+b}{2}, \frac{p+q}{2}) \\
&= \frac{|\frac{a+b}{2} - \frac{p+q}{2})|}{\lambda} \\
&= \frac{1}{2}\frac{|a-p+b-q|}{\lambda} \\
&\leq \frac{1}{2}(\frac{|a-p|}{\lambda} + \frac{|b-q|}{\lambda}) \\
&= \frac{1}{2}(\omega_\lambda(a,p) + \omega_\lambda(b,q)).
\end{aligned}$$

Therefore, coupled fixed points of F are both $(0,0)$ and $(1,1)$. Namely, F has not a unique the coupled fixed point. Then, the conditions $k<1$ in Corollary (3.4) and $k+l<1$ in Theorem (3.3) are the most appropriate conditions for satisfying the uniqueness of coupled fixed point.

4. Application

Here, we show that there exists a unique solution of a nonlinear integral equation using the Theorem (3.3).

We consider the following nonlinear integral equations:

$$\begin{aligned} a(s) &= \int_0^S f(s,a(t),b(t))dt,\ s\in[0,S]=I \\ b(s) &= \int_0^S f(s,b(t),a(t))dt,\ s\in[0,S]=I \end{aligned} \tag{4.1}$$

where $S\in\mathbb{R}^+$ $(S>0)$ and $f:I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$.

Let $X_\omega=C(I,\mathbb{R})$ and X_ω be a partially ordered set. We define the order relation as follows:

$$a\le b\Leftrightarrow a(s)\le b(s)$$

for $a,b\in C(I,\mathbb{R})$ and all $s\in I$. We can easily see that X_ω is a complete modular metric space such that

$$\omega_\lambda(a,b)=\sup_{s\in I}\frac{|a(s)-b(s)|}{\lambda},$$

for all $a,b\in X$ and $\lambda>0$.

Assumption 4.1. There exist two non-negative constants k and l with $k+l<1$ such that

$$0\le f(s,a,b)-f(s,p,q)\le\frac{1}{S}\left(\frac{k(a-p)+l(q-b)}{\lambda}\right) \tag{4.2}$$

for all $s\in I$, $a,b,p,q\in X_\omega$ and $\lambda>0$ where $a\ge p$, $q\ge b$.

Definition 4.2. $(\alpha,\beta)\in C(I,\mathbb{R})\times C(I,\mathbb{R})$ is called a coupled lower and upper solution of the integral equations (4.1) if $\alpha(s)\le\beta(s)$ and

$$\alpha(s)\le\int_0^S f(s,\alpha(t),\beta(t))dt$$

$$\beta(s)\le\int_0^S f(s,\beta(t),\alpha(t))dt$$

for all $s\in I$.

Theorem 4.3. *We suppose that the Assumption* (4.1) *is satisfied. The integral equations* (4.1) *have a unique solution in* $C(I,\mathbb{R})$ *if there exists a coupled lower and upper solution for equations* (4.1).

Proof. Let $X_\omega=C(I,\mathbb{R})$. X_ω is a partially ordered set if we define the order relation such that for $a,b\in C(I,\mathbb{R})$ and all $s\in I$

$$a\le b\Leftrightarrow a(s)\le b(s).$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 25, NO.2, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

It is obvious that X_ω is a complete modular metric space such that

$$\omega_\lambda(a,b) = \sup_{s\in I} \frac{|a(s)-b(s)|}{\lambda} \tag{4.3}$$

for $a,b \in C(I,\mathbb{R})$ and all $\lambda > 0$. Also, we define a partial order on $X_\omega \times X_\omega = C(I,\mathbb{R}) \times C(I,\mathbb{R})$ such that

$$(a,b) \leq (p,q) \Rightarrow a(s) \leq p(s) \text{ and } q(s) \leq b(s)$$

for $(a,b),(p,q) \in X_\omega \times X_\omega$ and all $s \in I$.

Now, we define $F : X_\omega \times X_\omega \to X_\omega$ with

$$F(a,b)(s) = \int_0^S f(s,a(t),b(t))dt \tag{4.4}$$

for $a,b \in C(I,\mathbb{R})$ and $s \in I$. We need to show that F has the mixed monotone property. If $a_1 \leq a_2$, that is, $a_1(s) \leq a_2(s)$ for all $s \in I$, by Assumption (4.1) we get

$$\begin{aligned} F(a_1,b)(s) - F(a_2,b)(s) &= \int_0^S f(s,a_1(t),b(t))dt - \int_0^S f(s,a_2(t),b(t))dt \\ &= \int_0^S (f(s,a_1(t),b(t)) - f(s,a_2(t),b(t)))dt \\ &\leq 0. \end{aligned}$$

Then, $F(a_1,b)(s) \leq F(a_2,b)(s)$ for all $s \in I$. That is, $F(a_1,b) \leq F(a_2,b)$.

Similarly, if $b_1 \geq b_2$, that is, $b_1(s) \geq b_2(s)$ for all $s \in I$, by Assumption (4.1), we get

$$\begin{aligned} F(a,b_1)(s) - F(a,b_2)(s) &= \int_0^S f(s,a(t),b_1(t))dt - \int_0^S f(s,a(t),b_2(t))dt \\ &= \int_0^S (f(s,a(t),b_1(t)) - f(s,a(t),b_2(t)))dt \\ &\leq 0. \end{aligned}$$

Then, $F(a,b_1)(s) \leq F(a,b_2)(s)$ for all $s \in I$. That is, $F(a,b_1) \leq F(a,b_2)$. Therefore, $F(a,b)$ is monotone nondecreasing in a and monotone nonincreasing in b.

Now, we show that F has a coupled fixed point. Let $a \geq p$ and $q \geq b$. Then, $a(s) \geq p(s)$ and $q(s) \geq b(s)$ for all $s \in I$. From equation (4.2), (4.3) and (4.4), we obtain that

$$\begin{aligned}
\frac{|F(a,b)(s)-F(p,q)(s)|}{\lambda} &= \frac{|\int_0^S f(s,a(t),b(t))dt - \int_0^S f(s,p(t),q(t))dt|}{\lambda} \\
&= \frac{|\int_0^S (f(s,a(t),b(t)) - f(s,p(t),q(t)))dt|}{\lambda} \\
&\leq \frac{\int_0^S |f(s,a(t),b(t)) - f(s,p(t),q(t))|dt}{\lambda} \\
&\leq \frac{1}{S}\int_0^S (\frac{|k(a(t)-p(t)) + l(q(t)-b(t))|}{\lambda})dt \\
&\leq \frac{1}{S}\int_0^S (k\frac{|a(t)-p(t)|}{\lambda} + l\frac{|b(t)-q(t)|}{\lambda})dt \\
&\leq \frac{1}{S}\int_0^S (k\sup_{z\in I}\frac{|a(z)-p(z)|}{\lambda} + l\sup_{z\in I}\frac{|b(z)-q(z)|}{\lambda})dt \\
&= \frac{1}{S}\cdot S\cdot (k\sup_{z\in I}\frac{|a(z)-p(z)|}{\lambda} + l\sup_{z\in I}\frac{|b(z)-q(z)|}{\lambda}) \\
&= k\sup_{z\in I}\frac{|a(z)-p(z)|}{\lambda} + l\sup_{z\in I}\frac{|b(z)-q(z)|}{\lambda}
\end{aligned}$$

for all $\lambda > 0$ which implies that

$$\frac{|F(a,b)(s)-F(p,q)(s)|}{\lambda} \leq k\sup_{z\in I}\frac{|a(z)-p(z)|}{\lambda} + l\sup_{z\in I}\frac{|b(z)-q(z)|}{\lambda}.$$

Therefore, we obtain that

$$\omega_\lambda(F(a,b),F(p,q)) \leq k\omega_\lambda(a,p) + l\omega_\lambda(b,q)$$

for $a,b,p,q \in X_\omega$ and all $\lambda > 0$. On the other hand, let (α,β) be a coupled lower and upper solution of the integral equations (4.1), then we get

$$\alpha(s) \leq F(\alpha,\beta)(s) \text{ and } \beta(s) \geq F(\beta,\alpha)(s)$$

for all $s \in I$ where $\alpha,\beta \in I$. Then,

$$\alpha \leq F(\alpha,\beta) \text{ and } \beta \geq F(\beta,\alpha).$$

We obtain that Theorem (3.3) is satisfied. Therefore, from Theorem (3.3), we get a unique solution $(a,b) \in X_\omega \times X_\omega$ of integral equations (4.1). □

References

1. A.A.N. Abdou and M.A. Khamsi, *Fixed point results of pointwise contractions in modular metric spaces*, Fixed Point Theory Appl. **2013** (2013), Article ID 163.
2. A.A.N. Abdou and M.A. Khamsi, *Fixed points of multivalued contraction mappings in modular metric spaces*, Fixed Point Theory Appl. **2014** (2014), Article ID 249 .
3. I. Altun and G. Durmaz, *Some fixed point theorems on ordered cone metric spaces*, Rend. Circ. Mat. Palermo **58** (2009), 319-325.
4. I. Altun and H. Simsek, *Some fixed point theorems on ordered metric spaces and application*, Fixed Point Theory Appl. **2010** (2010), 17 pages, Article ID 621492.
5. T.G. Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. **65** (2006), 1379-1393.
6. L.j. Círíc and V. Lakshmikantham, *Coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Stoch. Anal. Appl. 27 (2009), 1246-1259.

7. V.V. Chistyakov, *Modular metric spaces generated by F-modulars*, Folia Math. **14** (2008), 3-25.
8. V.V. Chistyakov, *Modular metric spaces I basic concepts*, Nonlinear Anal. **72** (2010), 1-14.
9. V.V. Chistyakov, *Fixed points of modular contractive maps*, Doklady Mathematics **86** (2012), no.1, 515-518.
10. Y.J. Cho, R. Saadati and G. Sadeghi, *Quasi-contractive mappings in modular metric spaces*, J. Appl. Math. **2012** (2012), 5 pages Article ID 907951.
11. M.E. Ege and C. Alaca, *Some properties of modular S-metric spaces and its fixed point results*, Journal of Computational Analysis and Applications. **20** (2016), no.1, 24–33.
12. D. Guo and V. Lakshmikantham, *Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal. **11** (1987), 623-632.
13. E. Karapinar, *Coupled fixed point theorems for nonlinear contractions in cone metric spaces*, Comput. Math. Appl. **59** (2010), 3656-3668.
14. E. Kilinc and C. Alaca, *A Fixed point theorem in modular metric spaces*, Adv. Fixed Point Theory **4** (2014), 199–206.
15. C. Mongkolkeha, W. Sintunavarat and P. Kumam, *Fixed point theorems for contraction mappings in modular metric spaces*, Fixed Point Theory Appl. **2011** (2011), 2011:93.
16. V. Lakshmikantham and L.j. Círíc, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal. **70** (2009), 4341-4349.
17. P. Kumam, *Fixed point theorems for nonexpansive mapping in modular spaces*, Arch Math. (Brno) **40** (2004), 345–353.
18. N.V. Luong and N.X. Thuan, *Coupled fixed points in partially ordered metric spaces and application*, Nonlinear Anal. **74** (2011), 983992.
19. J. Musielak and W. Orlicz, *On modular spaces*, Studia Math. **18** (1959), 49–65.
20. A. Mutlu, N. Yolcu, B. Mutlu and N. Bildik, *On Common Coupled Fixed Point Theorems for Comparable Mappings in Ordered Partially Metric Spaces*, Abstract and Applied Analysis **2014** (2014), 6 pages Article ID 486384.
21. A. Mutlu, N. Yolcu and B. Mutlu, *Coupled Fixed Point Theorem for Mixed Monotone Mappings on Partially Ordered Dislocated Quasi Metric Spaces*, Global Journal of Mathematics **1** (2015), no.1, 12-17.
22. A. Mutlu, N. Yolcu and B. Mutlu, *Fixed Point Theorems in Partially Ordered Rectangular Metric Spaces*, British Journal of Mathematics & Computer Science **15** (2016), no.2, 1-9.
23. F. Sabetghadam, H.P. Masiha and A.H. Sanatpour, *Some coupled fixed point theorems in cone metric spaces*, Fixed Point Theory Appl. (2009) 8 pages Article ID 125426 doi:10.1155/2009/125426.
24. B. Samet, *Coupled fixed point theorems for a generalized MeirKeeler contraction in partially ordered metric spaces*, Nonlinear Anal. **72** (2010), 4508-4517.

Manisa Celal Bayar University , Faculty of Science and Arts, Department of Mathematics Muradiye Campus, 45047, Manisa/TURKEY
E-mail address: ali.mutlu@cbu.edu.tr, abgamutlu@gmail.com

Manisa Celal Bayar University , Faculty of Science and Arts, Department of Mathematics Muradiye Campus, 45047, Manisa/TURKEY
E-mail address: kubra.ozkan@hotmail.com

Manisa Celal Bayar University , Faculty of Science and Arts, Department of Mathematics Muradiye Campus, 45047, Manisa/TURKEY
E-mail address: utkugurdal@gmail.com

Connected $m-K_n-$residual graph and its application in cryptology

Huiming Duan[1,2], Xinping Xiao[1],*Congjun Rao[1]

1.College of Science,Wuhan University of Technology, Wuhan, 430070, China

2.College of Science, Chongqing University of Posts and Telecommunications, Nanan, Chongqing, 400065, China

Abstract

The minimum order and extremal graph of a connected $m-K_n-$residual graph were first proposed by Erdös, Harary and Klawe, in addition to two important conjectures related to the graph. They addressed said conjectures solely from the perspective of K_n-residual graphs, and did not study the $m-K_n-$residual graphs $m>1$. In this study, we found that with $n>2m+2$, the minimum order of connected $m-K_n-$residual graph was $(m+n)(m+1)$ and the unique extremal graph was $K_{m+n}\times K_{m+1}$. This conclusion agreed with the conjectures proposed by Erdös, Harary and Klawe. Moreover, with $n=2m+2$, we identified at least two non isomorphic $m-K_n-$residual graphs; this did not correspond with the conjecture. The $m-K_n-$residual graph was determined only by m,n and the maximum connected branch r. The relationship among m,n and r was similar to that among the parameters of Hill cryptosystem implementation steps. According to these observations and principle knowledge regarding Hill cryptosystem implementation, the novel binary cryptosystem related $m-K_n-$residual graphs was established. We also built a Hill password encryption algorithm that ensures the binary cryptosystem is effective. The complexity of the minimum order and extremal graphs of connected $m-K_n-$residual graphs make the ciphertext, plaintext, and relationship between the keys highly complex and give the binary cryptosystem favorable performance.

Keywords: Residual graph; Minimum order; Extremal graph; Isomorphic.

1 Introduction

The residual graph, which represents an important branch of graph theory, was first proposed by Erdös, Harary and Klawe[1], and has since been widely applied throughout information science, networking, computer science, and other fields[2-7]. By definition, a residual graph is built by removing points in the closed neighborhood $N(u)$ that are isomorphic with the original graph; each removed point the closed neighborhood $N(u)$ in the graph has the same nature as its counterpart the original graph. For example, K_3 is a highly stable triangle that retains its original shape but with even higher stability after the adjacent edges and vertices of $m-K_3-$residual graph are removed for m times. In cryptology, an important component of information security[8-9], known information is denoted by graphs and kept in cipher form. Corresponding residual graphs are constructed according to their relevant definitions. Many residual graphs can be constructed, but it is difficult to select only one as the ciphertext: Only a sufficiently complex(i.e.,unique)representative residual graph can ensure information security. The complete graph K_n is often utilized in the computer networking and computer aided design fields[10-12], making it a useful research object in regards to the minimum order and extremal graph of connected $m-K_n-$residual graph.

Erdös, Harary and Klawe originally defined this concept[1] and they concluded that as $n>1, n\neq 2$, the minimum order of connected $m-Kn-$residual graph is $2(n+1)$; as $n\neq 2,3,4$, $K_{n+1}\times K_2$ is the unique connected K_n-residual graph with the minimum order. They also proposed the following two conjectures for connected $m-K_n-$residual graphs.

Conjecture 1: *If $n\neq 2$, then every connected $m-K_n-$residual graph has at least min$\{2n(m+1),(n+m)(m+1)\}$ vertices.*

Conjecture 2: *If n is large, there is a unique smallest connected $m-K_n-$residual graph.*

*Corresponding author. Tel:+86-13507164166; E-mail:xpingxiao@163.com (X.P. Xiao).

It is difficult to study the minimum order and residua graph of connected $m-K_n-$residual graph. As shown in Fig.1, for example, we successfully constructed a residual graph with order of 6 from $m=2$ and $n=1$, but it was very difficult to prove that the minimum order was 6 or even that the graph as-drawn was the unique extremal graph.

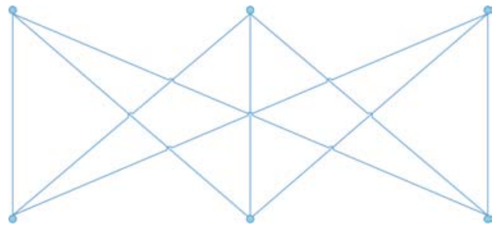

Fig.**1** $2-K_1-$residual graph.

There has been notable progress towards resolving problems related to the minimum order and extremal graph of connected $m-K_n-$residual graphs. Author [13], for example, investigated complete residual graphs of odd order to find that there is no K_n-residual graph of odd order for any odd n. Author [14] studied the K_n-residual graph to determine several other important properties and ultimately obtain the minimum order and extremal graph with $n=2,3,4$. Author [15] explored the $2-K_n-$residual graph to find that when $n>5, n\neq 6$, the results are in accordance with the two conjectures listed above. Other researchers [16] found that when $n\geq 9$, the $3-K_n-$residual graph is also in accordance with the conjectures. Author [17] examined the nature of a connected residual graph when $n=3,4$. In this study, we explored a connected $m-K_n-$residual graph to find that when $n>2m+2$, the minimum order was $(m+n)(m+1)$ and the unique extremal graph was $K_{m+n}\times K_{m+1}$. This also agreed well with the conjectures described above. In addition, when $n=2m+2$, there were two isomorphic connected $m-K_n-$residual graphs as least$-$this does not align with the conjectures.

Recent years have seen rapid advancements in modern cryptography [18-23]. Information security has become especially important with the advent of the internet. Protecting information via secure, efficient cryptosystems is a popular research subject, to this effect. Hill cryptosystems [23-24] are widely used in encryption and digital signature applications, but do contain loopholes. Researchers are still looking for a more effective cryptosystem. At present, the relationship between graph theory and cryptography is understood primarily as a relationship between the block cipher and DNA algorithm components of graph theory [25-27]. In other words, graph theory is an important theoretical basis for cryptography.

Again, there are some loopholes in the security of the traditional Hill cryptosystem [28-30]. Despite notable achievements in improving cryptosystem security, there is much room for further improvement. In this paper, plaintext is denoted by m, n, and the maximum connected branch r. The parameters of the Hill cryptosystem and minimum order and extremal graph of the connected residual graph were carefully assessed, and a novel Hill password encryption algorithm was established. The binary cryptosystem was found to be secure due to the complexity of the minimum order and extremal graphs of connected $m-K_n-$residual graphs, which makes the ciphertext, plaintext, and relationship between the keys highly complex. The security of this binary cryptosystem was also found to be adjustable.

The remainder of this paper is organized as follows. Section 2 provides some background information on the residual graph concept, and Section 3 introduces the minimum order and extremal graph of the connected $m-K_n-$residual graph. In Section 4, the binary cipher is proposed according to several parameters of the connected $m-K_n-$residual graph and the security of the resulting system is discussed. Section 5 concludes the paper.

2 Preliminaries

The following concepts and results for solving the $m-K_n-$residual graphs.

Definition 2.1 *$V(G)$ is the number of vertices in a graph G, $N(u)$ is the closed neighborhood of vertex $u\in V(G)$, and $N^*(u)$ are the original neighborhood and closed neighborhood of u in G.*

Definition 2.2 *For $u\in V(G)$, define $G_u=G-N[u]$. For convenience, we use notation $\langle S\rangle$ to mean $G[S]$(the subgraph induced by S in G).*

Definition 2.3 *Let $F \subset G$, then the degree of F in G is the cardinality of its boundary $d_G(F) = \sum d_G(x) - d_F(x), x \in F$.*

Definition 2.4 *A graph G is said to be a $F-$residual graph if for every vertex v in G, the graph obtained from G by removing the closed neighborhood of v is isomorphic to F. We inductively define a multiply $-F-$residual graph by saying that G is an $m-F-$residual graph if the removal of the closed neighborhood of any vertex of G results in a $(m-1)-F-$residual graph, where of course a $1-F-$residual graph is simply said to be a $F-$residual graph.*

Definition 2.5 *Let $X, Y \subset V(G)$, X is said to be adjacent to Y, and viceversa, if there exist $x \in X$ and $y \in Y$, then $xy \in E(G)$. If $xy \in E(G)$ for all $x \in X$ and $y \in Y$, then X is said to be complete adjacent to Y, and viceversa, for example, X and Y are said to be nonadjacent if there are no edges between them.*

Definition 2.6 *Let G_1 and G_2 are two disjoint graphs the join, $G = G_1 + G_2$, of G_1 and G_2 is defined as follows $V(G) = V(G_1) \cup V(G_2)$ two vertex u and v are adjacent to each other, if and only if, $u \in V(G_1)$ and $v \in V(G_2)$, or $uv \in E(G_1)$ or $uv \in E(G_2)$.*

The known supporting results are summarized in the following Lemma.

Lemma 2.1 *[1] If G is a connected F-residual graph, then for any vertex u in G, the degree $d(u) = \nu(G) - \nu(F) - 1$.*

Lemma 2.2 *[17] If $G = G_1 + G_2$, then G is $m - F-$residual graph, if and only if, both G_1 and G_2 are $m - F-$residual graphs.*

Lemma 2.3 *[15] If G is a $2 - K_n-$residual graph, when $\nu(G) = 3n + t$, $1 \leq 2n$, then it hasn't $u \in G$, which makes $d(u) = n + t - 1$, when $n \geq 5, n \neq 6$, then $\nu(G) = 3n + 6$, $G \cong K_{n+2} \times K_3$.*

Lemma 2.4 *[16] If G is a $3 - K_n-$residual graph, when $n \geq 11$, then $\nu(G) = 4n + 12$, $G \cong K_{n+3} \times K_4$.*

Lemma 2.5 *[17] Assume G is an mK_n-residual graph, $G \neq (m+1)K_n$, then $\nu(G) \geq 2(m+1)n$, and $K_{m+1,m+1}[K_n]$ is the unique extremal graphs.*

3 On connected $m - K_n-$residual graphs

In order to obtain the minimum order and extremal graph of the connected $m - K_n-$residual graph, we need the following Lemma.

Lemma 3.1 *Assume G is an $m - K_n-$residual graph, $m \geq 2$, $u \in G$, and $F \subset G$, F is the maximum connected subgraph, and F is a $r - K_n-$residual graph, $(0 \leq r \leq m-2)$, if $\nu(G) < \nu(F) + 2(m-r)n$, then there certainly a $v \in G_u$, which makes $G_v = G - N[v]$ connected.*

Proof. In the following, by reductio absurdum, we suppose there don't exist vertexes $v \in G_u$, which make $G_v = G - N[v]$ connected, then we need to prove $\nu(G) \geq \nu(F_1) + 2(m-r)n$. Set $H = \{F | F \subset G_v, v \in G_u, u \in F\}$. Let $F_1 \in H$, $\nu(F_1) = max\{\nu(F) | F \in H\}$, and $G_v = F_1 \cup G_1$, then F_1 is a $r - K_n-$residual graph. When $r = 0$, $F_1 \cong K_n$, then G_1 is an $(m-r-2) - K_n-$residual graph.

For every $w \in (G - N(F_1)) \subset G_u$, let $G_w = F_2 \cup G_2$, where $F_1 \subset G_w$, hence $F_1 \subset F_2$, because F_1 is the biggest connected components, then $F_2 = F_1$, $G_w = F_1 \cup G_2$, $N(F) \cap G_2 = \emptyset$, $G_2 \subset [G - N(F_1)]$, G_2 is an $(m-r-2) - K_n-$residual graph too, $G - N(F_1)$ is a $(m-r-1) - K_n-$residual graph, and $\nu(G - N(F_1)) \geq (m-r)n$.

Let $N(F_1) - F_1 = X$, then $G - N(F_1)$ is complete adjacent to X, in the following, we have discussed F_1 according to three conditions.

Case 1. F_1 is complete adjacent to X, then

$$G = \langle N(F_1) \cup (G - N(F_1)) \rangle = \langle X \cup F_1 \cup (G - N(F_1)) \rangle$$
$$= \langle X \rangle + \langle F_1 \cup (G - N(F_1)) \rangle = \langle X \rangle + \langle G - X \rangle,$$

because $|X| \geq (m+1)n$, then

$$\nu(G) \geq (m+1)n + \nu(F_1) + (m-r)n \geq \nu(F_1) + 2(m-r)n.$$

Case 2. F_1 is a $(r+1)$ independent, let $\{u_0, u_1, \cdots, u_r\} \subset F_1$, let $X - N(u_0, u_1, \cdots, u_r\ ) = X_1$, then $X_1 \neq \emptyset$, hence we have

$$G' = G - N(u_0, u_1, \cdots, u_r) = \langle X \rangle + \langle G - N(F_1) \rangle,$$

by Lemma 2.2, then G' is a $(m-r-1) - K_n-$residual graph, and $|X_1| \geq (m-r)n$, according to the proving in Case 1, we know that $\nu(G) \geq \nu(F_1) + 2(m-r)n$.

Case 3. There is a l independent set and $l < r+1$. Set $u_1, u_2, \cdots, u_l\} \subset F_1$, let $X - N(u_1, u_2, \cdots, u_l) = X_2$, then $X_2 \neq \emptyset$. Let

$$G'' = G - N(u_1, u_2, \cdots, u_l) = \langle X_1 \rangle + \langle G'' \cup (G - N(F_1)) \rangle,$$

because$|X| \geq |X_2| \geq (m-l+1)n > (m-r)n$, according to the proving in Case 1, we know that $\nu(G) \geq \nu(F_1) + 2(m-r)n$.

From the proves above, we know that if $\nu(G) < \nu(F) + 2(m-r)n$, then there certainly exist a vertices $v \in G_u$, which makes $G_v = G - N[v]$ connected.

Theorem 3.1 *Assume G is an $m-K_n-$residual graph, $m \geq 2$, if $\delta(G) = n$, then $\nu(G) \geq (m+3)n+m-1$, and the Fig.3 is the only extremal graph.*

Proof. We take induction to m, let G_1 be a $2 - K_n-$residual graph, if exist $w \in G_{1u}$, which makes $G_w = G_1 - N(w)$ connected, then $d_{G_1}(w) = n+t-2$, $d(w) = n+t-1$, $G_w = H_1 \cup H_2 \cong 2K_n$ and $u \in H_1 \cong K_n, i = 1, 2$. By Lemma 3.1,

$$\nu(G_1) \geq \delta(G) + \nu(G_w) + 1 \geq n + 4n + 1 = 5n + 1,$$

G_w is a K_n-residual graph, when $\nu(G_1) = 5n+1$, we can specify the Fig.1 is the minimum $2-K_n-$residual graph.

Suppose that Theorem is true for $m \geq 2$, and assume G is an $(m+1) - K_n-$residual graph, and $\delta(G) = n$, let $u \in G$, $d(u) = n$, and assume $\nu(G) \leq (m+4)n+m$. In the following, we prove that it must exist a vertex $v \in G_u$ with the case of the above assumptions, which makes G_v connected. By contrary, by Lemma 3.1, we have $\nu(G) \geq \nu(F) + 2(m-r+1)n$, and F is a connected $r - K_n-$residual graph, if $F \cong K_n$. We have

$$\begin{aligned}\nu(G) &\geq n + 2(m+1)n = (m+4)n + (m-1)n \\ &= (m+4)n + m + (m-1)n - 1) - 1 > (m+4)n + m,\end{aligned}$$

this is a contradiction. Since $u \in F$, $d(w) = n$, we have $r \neq 1$. If $r \geq 2$, by induction hypothesis of F is a connected $r - K_n-$residual graph with $\delta(F) = n$, hence $\nu(F) \geq (r+3)n + r - 1$ and

$$\begin{aligned}\nu(G) &\geq (r+3)n + r - 1 + 2(m+1-r)n \\ &= (m+4)n + (m+1-r)n + r - 1 \\ &> (m+4)n + m,\end{aligned}$$

which contradicts that $\nu(G) \leq (m+4)n+m$. So we have there must be a vertex $v \in G_u$, which makes G_v connected. Let $u \in G_v$, then $n \leq \delta(G_v) \leq d_{G_v}(u) \leq d(u) = n$, so we have $\delta(G_v) = n$, and by induction hypothesis, we have $\nu(G_u) \leq (m+3)n + m - 1$, and by Lemma 2.1,

$$\begin{aligned}\nu(G) &= \nu(G_u) + d(v) + 1 \\ &\geq (m+3)n + m - 1 + n + 1 \\ &= (m+4)n + m.\end{aligned}$$

So we have $\nu(G) = (m+4)n + m$, and $\nu(G) < (m+4)n + m$ is not true, then $\nu(G) \geq (m+4)n + m$. When $\nu(G) = (m+4)n+m$, we can construct $(m-1) - K_n-$residual graph and $m - K_n-$residual graph, the Fig.2 is the minimum $(m-1) - K_n-$residual graph, and the Fig.3 is the minimum $m - K_n-$residual graph.

In the following, we have proved the Fig.3 is the only extremal graph. Let $v \in G_u$, $G_v = X_1 \cup X_2 \cup Y_1 \cup Y_2 \cup V_j$, $j = 3, 4, \cdots, m+1$, $X_i \cong Y_i, i = 1, 2$, $V_j \cong K_{n+1}$. Let $V_j = \{x_j, y_j\} \cup C_j$, $j = 3, 4, \cdots, m+1$, and let $X = X_1 \cup X_2 \cup \{x_3, x_4, \cdots, x_{m+1}\}$, $Y = Y_1 \cup Y_2 \cup \{y_3, y_4, \cdots, y_{m+1}\}$, then X is complete adjacent to Y. Suppose $u \in C_3$, G_u is the Fig.2, we have $G = \langle N(v) \cup G_v \rangle$, $G_u = \langle (G_v - N(u)) \cup N(v) \rangle = \langle ((G_u) - V_3) \cup V_{m+2} \rangle$, where $V_{m+2} = N(u) = \{x_{m+2}, y_{m+2}\} \subset C_{m+2}$, x_{m+2} is complete adjacent to $Y - \{y_3\}$, y_{m+2} is complete adjacent to $X - \{x_3\}$, hence we have let $X^* = X \cup \{x_{m+2}\}$, $Y^* = Y \cup \{y_{m+2}\}$, then X^* is complete adjacent to Y^*, so that the Fig.3 of G is the only extremal graph. □

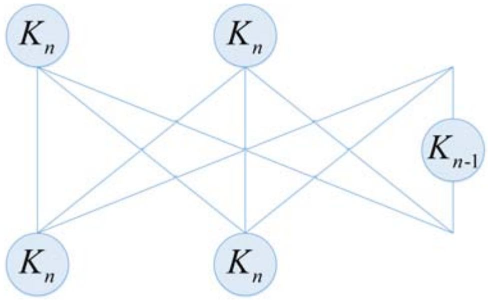

Fig.**1** $2 - K_n-$residual graph with order $5n + 1$

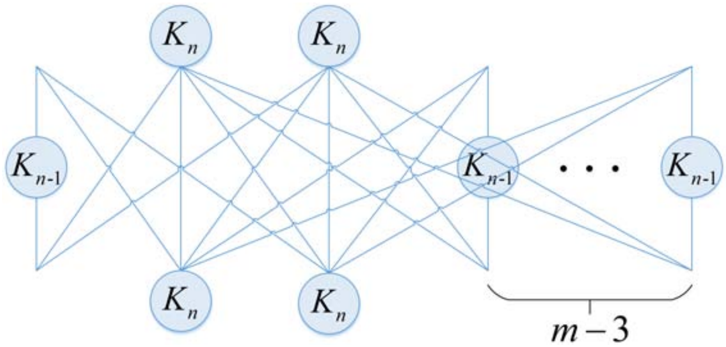

Fig.**2** $(m - 1) - K_n$-residual graph with order $(m + 2)n + m - 2$

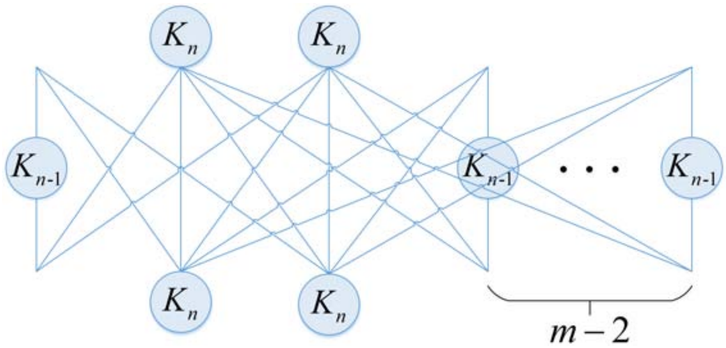

Fig.**3** $m - K_n$-residual graph with order $(m + 3)n + m - 1$

On the basis of the conclusion from literature [15] and the definition of $m - K_n-$residual graph, we have

Theorem 3.2 *Let G be a connected $m - K_n-$residual graph, $\nu(G) = (m + 1)n + t$, $1 \leq t \leq 2n$, $m > 2$, $n > 4$, then G has no vertex of degree $n + t - 1$.*

Proof. We take induction to m, when $m = 2$, by Lemma 2.3, G has no vertex of degree $n + t - 1$. When $m = 3$, G is a connected $3 - K_n-$residual graph $\nu(G) = 4n + t, 1 \leq t \leq 2n$. Supposing there exist such vertices like $u \in G$, which make $d(u) = n + t - 1$ and $G_u = H_1 \cup H_2 \cup H_3 = 3K_n$, because $\nu(G) = 4n + t, 1 \leq t \leq 2n$, then $\nu(G) \leq 6n \leq 6n + 2$. By Lemma 3.1, there exists at least $v \in G_u$, and we might as well suppose $v \in H_3$, which makes G_v connected, then $G_v - N(u) = H_1 \cup H_2 \cong 2K_n$, and by Lemma 2.1, $\nu(G_v) = \nu(G) - d(v) - 1$, let $d(v) = n - r$, $\nu(G_v) = 4n + t - n - r - 1 = 3n + t_1$, let $t_1 = n - r - 1$, $\nu(G_v) = 3n + t_1, 1 \leq t_1 \leq 2n$, because G_v is $2 - K_n-$residual graph, it is contradictory to the conclusion of literature [15] that there are no vertexes makes $d(u) = n + t - 1$.

We suppose let G be a connected $m - K_n-$residual graph, and there doesn't exist such vertices like $u \in G$, which make $d(u) = n + t - 1$. For G be a connected $(m + 1) - K_n-$residual graph, $\nu(G) = (m + 2)n + t$, $1 \leq t \leq 2n$, $m > 2$, $n > 4$. Supposing there exists $u \in G$, which makes $d(u) = n+t-1$, then $G_u = H_1 \cup H_2 \cup \cdots \cup H_{m+1} = (m+1)K_n$, because $\nu(G) = (m+2)n+t$,$1 \leq t \leq 2n$, then

$$\begin{aligned} \nu(G) &\leq (m + 2)n + 2n = mn + 4n \\ &< mn + 4n + m = (m + 4)n + m, \end{aligned}$$

by Lemma 2.3 and Lemma 3.1, there exists $v \in G_u$, let $v \in H_{m+1}$, which makes G_v is connected then

5

$$G_v - N(u) = H_1 \cup H_2 \cup \cdots \cup H_m \cong mK_n,$$

and by Lemma2.1, $\nu(G_v) = \nu(G) - d(v) - 1$, let $d(v) = n - r$, $\nu(G_v) = (m+2)n + t - n - r - 1 = (m+1)n + t_1$, $t_1 = n - r - 1$, it is contradictory to the induction and assumption that G_v is an $m - K_n-$residual graph, then there don't exist vertices, which make $d(u) = n + t - 1$, $u \in G$.□

Theorem 3.3 *There are at least two non isomorphic minimum $m-K_{n_1}-$residual graph, One is $K_{m+n_1} \times K_{m+1}$, and the other is $G_m[K_{\frac{n_1}{2}}]$, where $G_m \cong m - K_2-$residual graph, $n_1 = 2m + 2$, and this example doesn't meet the conclusion Erdös and Harary and Klawe made in [1].*

Proof. Let $G \cong K_{m+n_1} \times K_{m+1}$, $\nu(G) = (m + n_1)(m + 1)$, and $G_m \cong m - K_2-$residual graph, by literature [1], $\nu(G_m) \geq 3m + 2$, $\nu(G_m[K_{\frac{n_1}{2}}]) = \frac{n_1}{2}(3m + 2)$, and $n_1 = 2m + 2$, so when $n = n_1$, it doesn't meet the conclusion Erdös and Harary and Klawe made that only one $K_{m+n_1} \times K_{m+1}$ is an $m - K_{n_1}-$residual graph. □

Theorem 3.4 *Let G be a connected $m - K_n-$residual graph, if $n > 2m + 2$, $m \geq 3$, then $\nu(G) \geq (m + n)(m + 1)$, and when $\nu(G) = (m + n)(m + 1)$, $G \cong K_{m+n} \times K_{m+1}$ is a connected $m - K_n-$residual graph of minimum order, it is only such graph, so we show that the conjectures are true, when $n > 2m+2$.*

Proof. At first, we prove $\nu(G) \geq (m + n)(m + 1)$ and construct $m - K_n-$residual graph, and show the conjecture [1] is true.

Let G_1 be $2 - K_n-$residual graph, when $n > 6$, by Lemma 2.3, $G_1 \cong K_{n+2} \times K_3$, and G_1 is the Fig.4, all points in the same square are mutually adjacent, and two vertices, which are joined by a line, are adjacent. Let $u \in G$, we have G_u is $(m - 1) - K_n-$residual graph, suppose $\nu(G_u) = (m + n - 1)m$, and $G_u \cong K_{m+n-1} \times K_m$. Let

$$G_u = \langle \overline{H_1} \cup \overline{H_2} \cdots \cup \overline{H_m} \rangle = \{x_r^j|_{r=1,2,\cdots,m}^{j=1,2,\cdots,n+m-1}\}, \tag{3.1}$$

where $\overline{H_r} = \langle x_r^1, x_r^2, \cdots, x_r^{n+m-1} \rangle$, and if $i = j$, then x_l^i is adjacent to x_k^j, if $i \neq j$, x_l^i is nonadjacent to x_m^j, $l \neq k, k, l = 1, 2, \cdots, m$, adjacent to the Fig.5, all points in the same square are mutually adjacent, and two vertices, which are joined by a line, are adjacent. According to the induction and assumption that $\nu(G_u) \geq (m + n - 1)m$. When G_u is disconnected, by Lemma 2.5, let $G_u = (m - 1)nK_n$, and $\nu(G_u) \geq 2(m - 1)n$. Because $n \geq 2m + 2, m \geq 3$, then $\nu(G_u) \geq 2(m - 1)n > (m + n - 1)m$, by definition of residual graph, if and only if, when G_u is connected, $\nu(G)$ is minimum. According to the Fig.5, we can construct $m - K_n-$residual graph in the Fig.6, when $\nu(G) = (m + n)(m + 1)$. The Fig.6, all points in the same square are mutually adjacent, and two vertices, which are joined by a line, are adjacent. By G_u is connected and Lemma 2.1, $\nu(G) = \nu(G_u) + d(v) + 1$, according to the induction and assumption that $\nu(G_u) \geq (m + n - 1)m$, and by the Fig.6, $d(u) \geq n + 2m - 1$, and by Lemma 2.1, hence

$$\begin{aligned}
\nu(G) &= \nu(G_u) + d(v) + 1 \\
&\geq (m + n - 1)m + n + 2m - 1 + 1 \\
&= (m + n)(m + 1).
\end{aligned}$$

In the following, exist a vertex $u \in G$, which makes G_u connected. Because $\nu(G) = (m + n)(m + 1)$, by Lemma 3.1, when F is $r - K_n-$residual graph, $0 \leq r \leq (m - 2)$, $\nu(G) = \nu(F) + 2(m - r)n$. In the following, we prove $\nu(G) = \nu(F) + 2(m - r)n \geq (m + n)(m + 1)$, when $F \cong K_n$, and $n > 2m + 2$, we have $\nu(G) \geq n + 2mn > (m + n)(m + 1)$, if $r \geq 1$ by induction hypothesis F is a connected $r - K_n-$residual graph. So $\nu(F) \geq (r + n)(r + 1)$, and

$$\nu(G) = \nu(F) + 2(m - r)n = (r + n)(r + 1) + 2(m - r)n,$$

because of $(r + n)(r + 1) + 2(m - r)n - (m + n)(m + 1) > (m - r)^2 + m - r > 0$, then

$$\begin{aligned}
\nu(G) &= \nu(F) + 2(m - r)n \\
&= (r + n)(r + 1) + 2(m - r)n \\
&> (m + n)(m + 1).
\end{aligned}$$

So it must exist a vertex $u \in G$, which makes G_u connected.

From the proves above, we know $\nu(G) \geq (m + n)(m + 1)$, when $n > 2m + 2$, because $2n(m + 1) > (m + n)(m + 1)$, so we show the conjecture [1] is true.

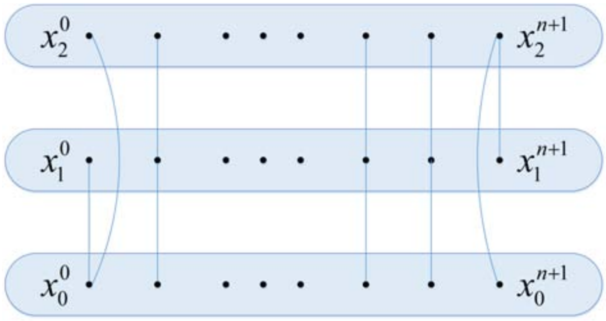

Fig.**4** $2-K_n-$residual graph with minimum order $3n+6$

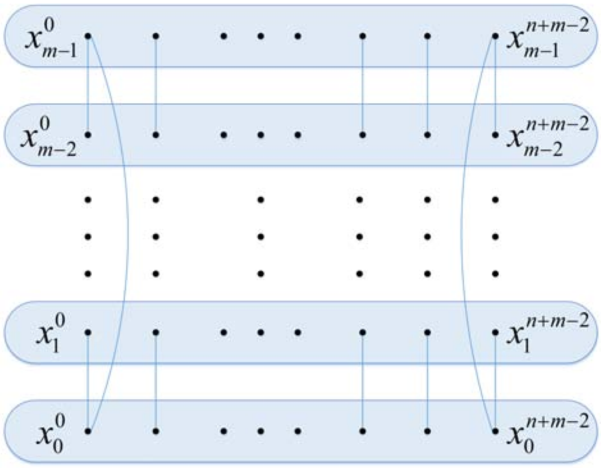

Fig.**5** $(m-1)-K_n-$residual graph with minimum order $(m+n-1)m$

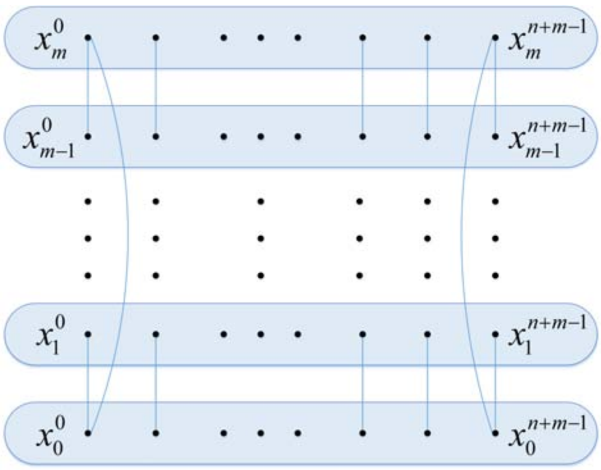

Fig.**6** $m-K_n-$residual graph with minimum order $(m+n)(m+1)$

In the following, we prove uniqueness, we have a discussion according to five conditions, so we show that the conjecture [2] is true.

Fact 1. Let

$$F = H_1 \cup H_2 \cup \cdots \cup H_m \cong K_{m+n-1} \times K_m,$$

where $H_i \cong K_{n+m-1}, i = 1, 2, \cdots, m$, then H_i and H_j have bijection

$$\theta : V(H_i) \to V(H_j),\ i, j = 1, 2, \cdots, m,$$

and $u_i \in H_i$ is adjacent to $\theta(u_i) \in H_j$, where $i \neq j$, $i, j = 1, 2, \cdots, m$. If $H \subset F$, and $H \subset K_s$, $3 \leq s \leq n+m-1$, then $H \subset H_w$, $1 \leq w \leq n$.

Fact 2. Let $u \in G$, $\nu(G_u) = (m+n-1)(m-1)$, hence $G_u \cong K_{m+n-1} \times K_m$. Adjacent to (3.1), let

7

$G_2 = G - N[x_2^{n+m-1}] = \langle H_0^* \cup H_1^* \cup H_3^* \cup \cdots \cup H_m^* \rangle \cong K_{m+n-1} \times K_m$,
we have

$$K_{m+n-2} \cong \overline{H_1} - x_1^{n+m-1} = \langle x_1^1, x_1^2, \cdots, x_1^{n+m-2} \rangle \subset G_2$$

by Fact 1, without loss of generality, we may assume that

$$\langle x_1^1, x_1^1, \cdots, x_1^{n+m-2} \rangle \subset H_1^* = \langle x_0^0, x_1^1, \cdots, x_1^{n+m-2} \rangle$$
$$\langle x_2^1, x_2^2, \cdots, x_2^{n+m-2} \rangle \subset H_2^* = \langle x_2^0, x_2^1, \cdots, x_2^{n+m-2} \rangle$$
$$\cdots\cdots$$
$$\langle x_m^1, x_m^2, \cdots, x_m^{n+m-2} \rangle \subset H_m^* = \langle x_m^0, x_m^1, \cdots, x_m^{n+m-2} \rangle \tag{3.2}$$

If x_0^j is adjacent to x_1^j, where $j = 0, 1, \cdots, n+m-1$. Obvious $x_0^0 = u$, then $H_0^* = \langle x_0^0, x_0^1, \cdots, x_0^{n+m-2} \rangle$.

We now prove x_1^0 is adjacent to x_1^{n+m-1}. Suppose the contrary, let $G_3 = G - N[x_1^{n+m-1}]$, $x_1^0 \in G_3$, by (3.1), (3.2), we have x_1^0 is adjacent to $\{x_1^1, x_1^2, \cdots, x_1^{n+m-1}\} \subset N[x_1^{n+m-1}]$. Thus

$$\begin{aligned} d(x_1^0) &\geq d_{G_3}(x_1^0) + n + m - 2 \\ &= n + m - 1 + n + m - 2 \\ &> n + m - 1 - m = n + 2m - 1. \end{aligned}$$

So x_1^0 is adjacent to x_1^{n+m-1}, x_1^0 is adjacent to $\overline{H_1}$. Set

$$H_1 = \langle x_1^0, x_1^1, \cdots, x_1^{n+m-1} \rangle \cong K_{m+n}.$$

Similarly, x_w^0 is adjacent to $\overline{H_w}$, set

$$H_w = \langle x_w^0, x_w^1, \cdots, x_w^{n+m-1} \rangle \cong K_{m+n}, w = 1, 2, \cdots, m.$$

Similarly, in

$$G - N^*[x_2^{n+m-1}] = \langle H_0^* \cup H_1^* \cup H_3^* \cup \cdots \cup H_m^* \rangle,$$

we have $x_0^{n+m-1} \in N^*[x_2^{n+m-1}] = \langle H_0^* \cup H_1^* \cup H_3^* \cup \cdots \cup H_m^* \rangle$ complete adjacent to $\overline{H_0^*}$. Obvious,

$$x_0^{n+m-1} \in (H_2 \cup x_1^{n+m-1} \cup x_3^{n+m-1} \cup \cdots \cup x_m^{n+m-1}) \subset N^*[x_2^{n+m-1}],$$

hence $H_0 = \langle x_0^0, x_0^1, \cdots, x_0^{n+m-1} \rangle \cong K_{m+n}$.

Fact 3. Any vertex in H_r is adjacent to single vertex in $H_s, r \neq s$. Suppose the contrary, let $x_0^j \in H_0$ be nonadjacent to H_m, then

$$G^* = G - N[x_0^j] \cong K_{n+m-1} \times K_m,$$

but $H_m \cong K_{m+n}$, $H_m^* \subset G^*$, contrary to $G^* \cong K_{n+m-1} \times K_m$, hence x_0^j is adjacent to H_m. If H_m has two vertices adjacent to x_0^j, by

$$d_{H_0}(x_0^j) = n + m + m - 1 = n + 2m - 1,$$
$$d(x_0^j) = n + m + m + 1 = n + 2m + 1,$$

we have this x_0^j is nonadjacent to one of H_w, which is a contradiction. Similarly, x_{i-1}^j is adjacent to $H_i, i = 1, 2, \cdots, m$, and is adjacent to only one vertex in H_i.

Fact 4. By Fact 3, we have x_1^0 adjacent to H_2. If x_1^0 is adjacent to $x_2^j, j \neq 0$, by $x_2^j, j \neq 0$ is adjacent to x_1^j, thus H_1 has two vertices adjacent to $x_2^j, j \neq 0$, contrary to Fact 3, so x_1^0 adjacent to x_2^0. Similar, so x_i^0 adjacent to x_j^0, and x_i^{n+m-1} adjacent to $x_0^{n+m-1}, i = 1, 2, \cdots, m$.

Fact 5. Since x_0^j is adjacent to x_m^j for $j = 0, n+m-1$, let x_0^j be nonadjacent to x_m^j for $j \neq 0, n+m-1$, by Fact 3, we have x_0^j is adjacent to $x_m^i, i \neq j$. Since x_l^j is adjacent to $x_k^j, l \neq k$, $l, k = 1, 2, \cdots, m$, set

$$G - N[x_0^j] = \langle (H_1 - x_1^j) \cup (H_2 - x_2^j) \cup \cdots \cup (H_m - x_m^j) \rangle \cong K_{n+m-1} \times K_m \tag{3.3}$$

By Fact 4, we have x_1^t adjacent to x_m^t, $t \neq i, j$, by (3.3), we have x_1^i adjacent to x_m^j, contrary (3.1), hence x_0^j is adjacent to x_m^j. Similarly, x_0^j is adjacent to x_m^j.

Hence

$$G = \langle X \rangle = \langle x_r^j |_{r=0,1,2,\cdots,m}^{j=0,1,2,\cdots,n+m-1} \rangle,$$

where x_r^i is adjacent to x_s^j, if and only if $r = s, i \neq j$, or $i = j, r \neq s$, thus $G \cong K_{m+n} \times K_{m+1}$.□

4 Binary cryptosystem of connected $m - K_n-$residual graphs

The implementation principle and steps of the traditional Hill cryptosystem were followed to establish the novel, binary cryptosystem proposed here. The Hill cryptosystem is a symmetrical cipher that is effectively resistant to frequency analysis that was first proposed by Prof. Lester S. Hill in 1929. It is implemented in the following steps.

(1) The digitization of plaintext $M = [m_1, m_2, \cdots, m_k]$ (41 characters, including 26 letters, the figures from 0 to 9 and punctuation corresponded to the figures from 0 to 40, respectively), with t components taken as a row vector (if the amount could not reach t in the last row, the space is required for supplementation.) that constitute a nt matrix, written as M .

(2) Matrix A of $t \cdot t$ is constructed in Z_{41} where $gcd(det(A), 41) = 1$ is required as the encryption key.

(3) Encryption operation $C = E(M) \equiv MA(mod41)$ is carried out to obtain the ciphertext.

(4) Decryption operation $M = D(C) \equiv CA^{-1}(mod41)$ is carried out to recover the plaintext.

It is difficult for the Hill cryptosystem to withstand a plaintext attack. After attackers intercept the ciphertext C, they are able to guess certain words used in the plaintext to attempt to ascertain the key K, then can calculate MK to determine whether ciphertext C can be generated. A large amount of information is stored in a computer system in a form of figure and transmitted through a public signal channel. Unfortunately, these computer systems and signal channels are very susceptible to attack in an open environment. The complexities of the minimum order and extremal graph of the connected $m - K_n-$residual graph can be used to denote the plaintext with different K_n, thus forming the proposed binary cryptosystem of the connected $m - K_n-$residual graph. The corresponding encryption algorithm is as follows.

(1) Each character in plaintext M is translated into the corresponding figure n_i; 41 characters, (26 letters, figures from 0 to 9, and punctuation corresponded to the figures from 0 to 40, respectively, where $n_i \in Z_{41}$.)

(2)For encryption operation, each figure n_i corresponds to a K_{n_i}. The sequence $[m_1, m_2, \cdots, m_k]$ represents the multiples, where $m_k \in Z$. Finally, the largest connected branch r can be identified and assigned to an extremal graph, i.e., the ciphertext$C$.

(3) For decryption operation, each ciphertext C equals an extremal graph and the complete graph K_{n_j} is determined according to the multiple sequences $[m_1, m_2, \cdots, m_k]$ and the largest connected branch r. The corresponding n_j can also be identified, $n_j \in Z_{41}$(i.e., $j(j = 1, 2, \cdots, 41)$); the resulting graph is the plaintext M.

These implementation steps are depicted in Fig. 8.

The binary cryptosystem of the connected $m - K_n-$residual graph makes full use of all available complexity in constructing the minimum order and extremal graph of the connected $m - K_n-$residual graph, which ensures optimal binary cryptosystem security. Even if attackers intercept the ciphertext C and knew that the minimum order and extremal graph of the connected $m - K_n-$residual graph have been used for encryption, they are unable to solve the decryption operation and thus cannot obtain the plaintext.

5 Conclusions

(1) The minimum order and extremal graphs of connected $m - K_n-$residual graphs form an open question put forward by Erdös, Harary, and Klawe which was addressed in this study.

(2) It is difficult to determine the minimum order and extremal graphs of connected $m - K_n-$residual graphs.

(3) We confirmed that the m and n values identified are in accordance with the conjectures of Erdös, Harary, and Klawe; namely, that the minimum order and extremal graphs of connected $m - K_n-$residual graphs do exist. This had important practical significance for establishing the proposed binary cryptosystem.

(4) A new binary password system comprised of an image-encryption-based password system and encryption algorithm was proposed here. This system represents an enhanced relationship between graph theory and cryptography.

(5) According to the complexity of the minimum order and extremal graph of the connected $m - K_n-$residual graph, the ciphertext, plaintext, and relationship between the keys is highly complex and the binary cryptosystem performs well. The security of the binary cryptosystem can be effectively adjusted according to these factors.

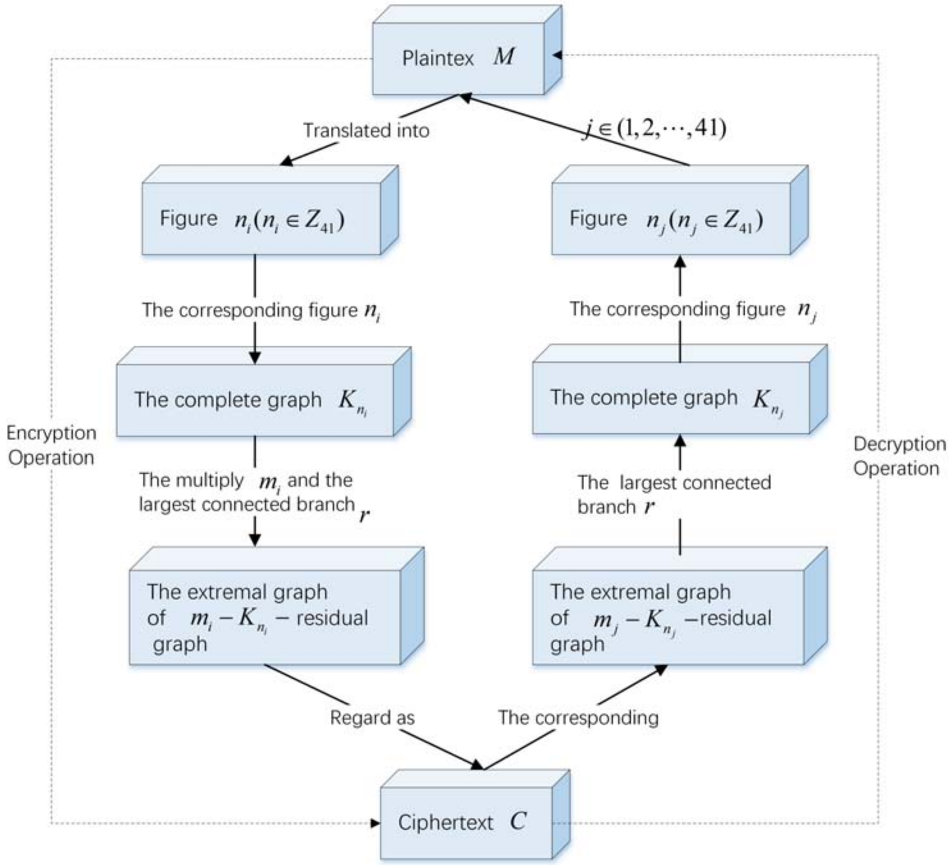

Fig.**8** Encryption and decryption operation

6 Acknowledgements

The authors are grateful to the editor for their valuable comments. This work was supported by National Natural Science Foundation of China (Grant no. 11671001,51479151, 71671135, 71540027,61472056) ,Chongqing Frontier and Applied Basic Research Project (cstc2015jcyjA00034, cstc2015jcyjA00015), and The Science and Technology Research Program of Chongqing Municipal Educational Committee(KJ1600425,KJ150 12024).

References

[1] P. Erdös, F. Harary and M. Klawe, Residually-Complete Graphs, Annals of Discrete Mathematics 6 (1980) 117-123.

[2] A.Matthias, S. Friedl, Residual properties of graph manifold groups, Topology and its Applications 158(10) (2011) 1179-1191.

[3] M.S. Lin, C.C. Ting, A polynomial-time algorithm for computing K-terminal residual reliability of d-trapezoid graphs, Information Processing Letters 115(2)(2015)371-376.

[4] A.A.Chernyak, Residual reliability of P-threshold graphs, Discrete Applied Mathematics 135(1) (2004) 83–95.

[5] B. Trotta, Residual Properties of Simple Graphs, Bulletin of the Australian Mathematical Society 82(3) (2008) 488-504.

[6] M.S.Lin, C.C.Ting, A polynomial-time algorithm for computing K-terminal residual reliability of d-trapezoid graphs, Information processing letters 115 (2015) 371-376.

[7] P. Luksic, T. Pisanski, Distance-residual graphs, Mathematics 9(3) (2006) 104-111.

[8] K. Patra, B. Nemade, D. P. Mishra and P. P. Satapathy, Cued-Click Point Graphical Password Using Circular Tolerance to Increase Password Space and Persuasive Features, Procedia Computer Science 79 (2016) 490-498.

[9] T.Y. Chang, C.J. Tsai and J.H. Lin, A graphical-based password keystroke dynamic authentication system for touch screen handheld mobile devices, Journal of Systems and Software 85(5) (2012) 1157-1165.

[10] F. Septyanto, K. A. Sugeng, On the Rainbow and Strong Rainbow Connection Numbers of the m-Splitting of the Complete Graph K_n, Procedia Computer Science 74(2015)155-161.

[11] R. El-Shanawany, H. O.F. Gronau, M. Grttmller, Orthogonal double covers of K_n,n by small graphs, Discrete Applied Mathematics 138(1-2) (2004) 47-63.

[12] A. Aguado, Structures, relative stabilities, and electronic properties of potassium clusters K_n $(13 \leq n \leq 80)$ 1021(1) (2013) 135-143.

[13] S. Yang, H. Duan, Residually Complete Graph with odd order, Acta Mathematics Applicatae Sinica 34(5) 2011 778-785.

[14] H. Duan, Y. Li, On connected K_n- Residual Graphs, Operations Research Transactions 20(2) (2016) 38-48.

[15] J. Liao, S. Yang, Y. deng, On connected $2 - K_n-$residual graphs, Mediterranean Journal of Mathematics 6 (2012) 12–27.

[16] H. Duan, B. Zeng, Z. Dou, On connected three multiply K_n- Residual Graphs, Operations Research Transactions 18(2) (2014) 59-68.

[17] J. Liao, S. Yang, Two improvement on the Erdös, Harary and Klawe conjecture, Mediterranean Journal of Mathematics 3 (2014) 1-16.

[18] A. Thomas, E. M. Manuel, Embedment of Montgomery Algorithm on Elliptic Curve Cryptography over RSA Public Key Cryptography, Procedia Technology 24 (2016) 911-917.

[19] E. Bellini, N. Murru, An efficient and secure RSA-like cryptosystem exploiting Rdei rational functions over conics, Finite Fields and Their Applications 39 (2016) 179-194.

[20] R. Rizk, Y. Alkady, Two-phase hybrid cryptography algorithm for wireless sensor networks, Journal of Electrical Systems and Information Technology 2(3) (2015) 296-313.

[21] P. Singh, V. Neema, S. Daulatabad and A. P. Shah, Subthreshold Circuit Designing and Implementation of Finite Field Multiplier for Cryptography Application, Procedia Computer Science 79 (2016) 597-602.

[22] S. Chandra, B. Mandal, Sk. S. Alam and S. Bhattacharyya, Content Based Double Encryption Algorithm Using Symmetric Key Cryptography, Procedia Computer Science 57 (2015) 1228-1234.

[23] K. Adinarayana Reddy, B. Vishnuvardhan, Madhuviswanatham, A.V.N. Krishna, A Modified Hill Cipher Based on Circulant Matrices,Procedia Technology 4 (2012) 114-118.

[24] B. Acharya, M. Dhar Sharma, S. Tiwari, V. K. Minz, Privacy protection of biometric traits using modified hill cipher with involutory key and robust cryptosystem, Procedia Computer Science 2 (2010) 242-247.

[25] Bakhtiari, M. Aizaini, Serious Security Weakness in RSA Cryptosystem, International Journal of Computer Science Issues 9(1) (2012) 175-178.

[26] Bassam J. Mohd, Thaier Hayajneh, Athanasios V. Vasilakos, A survey on lightweight block ciphers for low-resource devices: Comparative study and open issues, Journal of Network and Computer Applications 58 (2015) 73-93.

[27] X.J. Tong, Z. Wang, Y. Liu, M. Zhang, L.J. Xu, A novel compound chaotic block cipher for wireless sensor networks,Communications in Nonlinear Science and Numerical Simulation. 22 (2015) 120-133.

[28] X. Xu, L. He, A. Shimada, R. Taniguchi, H.M. Lu, Learning unified binary codes for cross-modal retrieval via latent semantic hashing, Neurocomputing 213(12) (2016) 191-203.

[29] Y.C. Wan, M.W. Wang, Z.W. Ye, X.D. Lai, A feature selection method based on modified binary coded ant colony optimization algorithm, Applied Soft Computing 49 (2016) 248-258.

[30] V. Thayananthan, A. Albeshri, Big Data Security Issues Based on Quantum Cryptography and Privacy with Authentication for Mobile Data Center, Procedia Computer Science 50 (2015) 149-156.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 25, NO.2, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

A comparative analysis of the Harry Dym model with and without singular kernel.

E.F. Doungmo Goufo*, P. Tchepmo, Z. Ali and A. Kubeka
Department of Mathematical Sciences, University of South Africa, Florida, 0003 South Africa
* Corresponding author email: dgoufef@unisa.ac.za

Abstract
The question that raised after the recent introduction of the derivative with no singular kernel was: **Does this non-singularity have a comprehensive impact on real life phenomena like wave motion or other related motions?** We comprehensively analyze the Harry Dym model generalized with two types of derivatives, namely a derivative with singular kernel, the Caputo derivative and the other one without singular kernel called the Caputo-Fabrizio derivative. Using Picard L-stability combined with the fixed-point theorem, the well-posedness of both models are proved together with their existence and uniqueness results existence. Techniques to approximate numerical solutions are provided for each of the two models with graphical representations performed and compared for several values given to the derivative order α. Similar behaviors are noticed for soliton waves related to close values of α and are compared to the soliton wave of the standard first order ($\alpha = 1$) Harry Dym model.

Keywords: Harry Dym equation; existence, uniqueness, derivative with and without singular kernel, approximated solutions

AMS Mathematics Subject Classification: 35F10, 26A33, 35D05.

1. Introduction

It is well known that one of the most popularly used derivative with fractional order is the one introduced in 1967 by Michele Caputo and called the Caputo derivative [1] given by

$$^{c}D_t^{\alpha} r(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{d}{d\tau} r(x,\tau)\, d\tau, \tag{1.1}$$

$0 < \alpha \leq 1$, with its associated anti-derivative well known to be defined by

$$I_0^{-\alpha} r(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{r(x,\tau)}{(t-\tau)^{1-\alpha}} d\tau. \tag{1.2}$$

However in April 2015, Caputo and Fabrizio [5] observed that the Caputo fractional derivative is unable to properly described some features related to some behavior happening in the fields of classical thermal media, classical viscoelastic materials or electromagnetic. Hence they proposed a new definition, the fractional derivative with no singular kernel in order to address the noticed unsolved issues. The new version called the Caputo-Fabrizio derivative differentiate from the Caputo derivative by having no singular kernel in the integral part. It reads as

$$^{cf}D_t^{\alpha}r(x,t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}\int_0^t \frac{d}{d\tau}r(x,\tau)\exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right)d\tau. \tag{1.3}$$

$0 < \alpha \leq 1$, where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$. Its associated fractional integral (anti-derivative) is given by [8]:

$$^{cf}I_t^{\alpha}r(x,t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}r(x,t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t r(x,\tau)\,d\tau, \tag{1.4}$$

$\alpha \in [0,1]$ $t \geq 0$. This expression shows the integral as a sort of average between the function r and its integral. Since then, many authors have improved the concepts. Starting with Losada and Nieto [8] who developed the associated fractional integral (1.3) and Doungmo Goufo [11] or Doungmo Goufo and Atangana [12] who proposed the related Riemann-Liouville version.

Nevertheless, the literature is full of several other definitions of fractional derivative. General formulation is done in [9] and many useful properties are intensively analyzed in [1–3, 10], especially in the analysis of the spread of diseases [14–17]. The main goal of this work is to apply both Caputo and Caputo-Fabrizio derivatives to the same Harry Dym equation and see their impact on the output behavior of the solutions. In other words, does the non-singularity have a significant influence on a real life process like wave motion or other motion? Recall that unlike Caputo derivative, there is no singularity at $t = \tau$ for Caputo-Fabrizio derivative and the following equality for Caputo-Fabrizio fractional derivative is satisfied:

$$\lim_{\alpha\to 1} {}^{cf}D_t^{\alpha}r(x,t) = \frac{\partial}{\partial t}r(x,t) \tag{1.5}$$

and

$$\lim_{\alpha\to 0} {}^{cf}D_t^{\alpha}r(x,t) = r(x,t) - r(x,0). \tag{1.6}$$

This paper investigates the non-linear third-order Harry Dym differential equation within the scope of the two types of derivatives mentioned above. Note that the traditional Harry Dym model can be solved using the Lax operator [18–20] and is associated with the Sturm-Liouville operator.

2. Solvability with Caputo fractional derivative

Let $\Omega = (a,b)$, $\mathbb{R} \ni T > 0$ $\mathbb{R} \ni b > a \in \mathbb{R}$ and $r \in C^0[\Omega \times [0,T]]$. Let $\alpha \in [0;1], \beta \in (0,+\infty)$ then, the non-linear Dym equation expressed with the Caputo time fractional derivative is investigated in this section. Existence and uniqueness of the exact solution are shown for the model under investigation that reads as

$$^{C}D_t^{\alpha}r(x,t) = r^3 r_{xxx}(x,t), \tag{2.1}$$

subject to the initial condition

$$r(x,0) = g(x) \tag{2.2}$$

with $^{C}D_t^{\alpha}r(x,t)$ the Caputo derivative as defined in (1.1) and $g:\ \Omega \longmapsto \mathbb{R}_+$. We start by transforming (2.1) into an integral form by applying the anti-derivative integral (1.2) on both sides to get

$$r(x,t)-r(x,0)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1}(r^3 r_{xxx}(x,t))dy \tag{2.3}$$

Let us proceed by simplicity and consider the operator with three variables

$$\Xi(x,t,r)=r^3 r_{xxx}(x,t). \tag{2.4}$$

The next goal is to show that operator Ξ with respect to variable r verifies the Lipschitz condition. For that,

$$\|\Xi(x,t,r)-\Xi(x,t,v)\|=\left\|r^3 r_{xxx}(x,t)-v^3 v_{xxx}(x,t)\right\|_{C^0[\Omega\times[0,T]]} \tag{2.5}$$

Assuming that r and v are bounded functions, there is a positive constants $k_1>0$ and $k_2>0$ such that

$$\|r\|^3_{C^0[\Omega\times[0,T]]}\leq k_1 \qquad \text{and} \qquad \|v\|^3_{C^0[\Omega\times[0,T]]}\leq k_2. \tag{2.6}$$

Furthermore, using the properties of the norm and the Lipschitz condition for the first order derivative function ∂_x there is a positive constant ϑ such that (2.5) becomes

$$\|\Xi(x,t,r)-\Xi(x,t,v)\|_{C^0[\Omega\times[0,T]]}\leq k_1k_2\vartheta^3\,\|r-v\|_{C^0[\Omega\times[0,T]]}\,. \tag{2.7}$$

Putting

$$K=k_1k_2\vartheta^3,$$

we finally get

$$\|\Xi(x,t,r)-\Xi(x,t,v)\|_{C^0[\Omega\times[0,T]]}\leq K\,\|r-v\|_{C^0[\Omega\times[0,T]]}\,,$$

which therefore proves the desired Lipschitz condition. This enables us to evaluate the following norm

$$\|r(x,t)\|_{C^0[\Omega\times[0,T]]}\,,\ t\in[0,T].$$

Assuming the existence and the boundedness of the initial condition g, there is a positive constant C such that $\|g(x)\|_{C^0[\Omega\times[0,T]]}\leq C$ for any $x\in\Omega$. Whence,

$$\begin{array}{l}\|r(x,t)\|_{C^0[\Omega\times[0,T]]}\leq\|r(x,0)\|_{C^0[\Omega\times[0,T]]}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-y)^{\alpha-1}\Xi(x,y,r(x,y))dy\\ \leq\|g(x)\|_{C^0[\Omega\times[0,T]]}+\frac{K}{\Gamma(\alpha)}\int_0^t(t-y)^{\alpha-1}dy\\ \leq C+\frac{KT^\alpha}{\alpha\Gamma(\alpha)}\end{array}, \tag{2.8}$$

which yields the following propositions:

Proposition 2.1. *Assuming that g given in* (2.2) *is bounded, let $0<\alpha<1$ and $\Xi[x,t,r(x,t)]$: $[\Omega\times[0,T]]\times B\longrightarrow A$ (with $A\supset B$) be a continuous function with respect to t for any fixed $x\in\Omega,\ r\in\overline{B}$. If $r(x,t)\in C^0\left[\Omega\times[0,T]\right]$, then the function $r(x,t)$ verifies the model* (2.1)-(2.2) *if and only if $r(x,t)$ verifies the corresponding Volterra integral equation* (2.3).

Proof. To prove the necessity condition we assume that $r(x,t) \in C^0[\Omega \times [0,T]]$ satisfies the equations (2.1)-(2.2). Because $\Xi(x,y,r(x,y) \in C[\Omega \times [0,T] \times B]$ for any $r \in \bar{B}$ then (2.1) means there exists the Caputo fractional derivative of r in $C[\Omega \times [0,T]]$. However

$$ {}_0^C D_t^\alpha r = \frac{\partial}{\partial t}\left(I_0^{-\alpha}\right)[r(x,t) - r(x,0)]. \tag{2.9}$$

Exploiting

$$I_0^{-\alpha}[r(x,t) - r(x,0)] \in C^0[\Omega \times [0,T]]$$

and applying the results in [13] for $\gamma = 0$ to

$$V(x,t) = r(x,t) - r(x,o),$$

yields

$$\begin{aligned} I_0^\alpha \, {}_0^C D_t^\alpha r(x,t) &= I_0^\alpha \, {}_0^C D_0^\alpha [r(x,t) - r(x,0)] \\ &= r(x,t) - r(x,0) - \sum_{j=1}^{1} \frac{r_{1-\alpha}^{i-j}(x,0)_{\alpha-j}}{\Gamma(\alpha - j + 1)} \end{aligned} \tag{2.10}$$

with $r_{1-\alpha}(x,t) = I_0^{1-\alpha}[r(x,t) - r(x,0)]$. Using the integration by parts in (2.10) and differentiating the resulting expression give

$$r_{1-\alpha}^{(1-j)}(x,t) = \frac{\partial}{\partial t}\left(I_0^{2-\alpha}[\partial_t r(x,t) - r(x,0)]\right) \tag{2.11}$$

Changing of variable $t = \beta + \rho(y - \beta)$ leads to

$$r_{1-\alpha}^{(1-j)}(x,t) = \frac{(y-\beta)^{1-\alpha}}{\Gamma(1-\alpha)} \int_0^1 (1-\rho)^{-\alpha}\left(r^{1-j}[\beta + \rho(y-\beta)]\right). \tag{2.12}$$

Recalling $0 < \alpha < 1$ and $r^{1-j}(x,t) \in C[\Omega \times [0,T]]$, equation (2.12) and (2.10) take the form

$$I_0^\alpha \, {}_0^C D_t^\alpha r(x,t) = r(x,t) - r(x,0) \tag{2.13}$$

Since $I_0^\alpha \Xi(x,\tau,r(x,\tau)) \in C^0[\Omega \times [0,T]]$ and using the Lipschitz condition of Ξ, we obtain

$$\|I_0^\alpha \Xi(x,\tau,r(x,\tau)\|_{C^0[\Omega \times [0,T]]} \leq \frac{KT^\alpha}{\alpha\Gamma(\alpha)}.$$

Nonetheless, applying I_0^α on both sides of (2.1) and making use of the initial condition we prove the necessity condition by recovering the Volterra version (2.3).

Conversely for the sufficient condition, assume $r(x,t) \in C[\Omega \times [0,T]]$ verifies the Volterra version (2.3) of equations (2.1)-(2.2). It suffices to show that $r(x,t)$ satisfies the initial condition (2.2).

The differentiation of the two sides of the Volterra version yields

$$\partial_t r(x,t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t \Xi(x,\tau,r(x,\tau))(t-\tau)^{\alpha} d\tau.$$

Changing again the variable $t = \beta + \rho(y-\beta)$ in the Volterra expression for $k = 1$, leads to

$$r^k(x,t) = \frac{(y-\beta)}{\Gamma(\alpha-k)} \int_0^t \frac{\Xi(x,\beta+\rho(y-\beta), r(x,\beta+\rho(y-\beta)))}{(1-\beta)^{1-\alpha+k}} dy$$

Passing to the limit as $y \longrightarrow \beta^+$ and making use of the continuity of K show that the sought initial condition is verified, and the sufficient condition is proved. □

Proposition 2.2. *Considering* $0 < \alpha < 1$ *and the Lipschitz condition for* Ξ *then, there is a unique solution for equation* (2.3) *in the space*

$$C^{0,\alpha}[0,T] \times \Omega$$

Proof. From the above analysis in Theorem 2.1, it is sufficient to show the existence of the unique solution $r(x,t) \in C^0[\Omega \times [0,T]]$ of the Volterra equation (2.3). Indeed, the model (2.3) holds in any interval $[0,\tau] \subseteq [0,T]$. Hence, we select the adequate $t_1 \in [0,T]$ so that

$$\left| \frac{Kt_1^{\alpha}}{\alpha\Gamma(\alpha)} \right| < 1$$

and then, prove the desired existence result of a unique $r(x,t) \in C^0[\Omega \times [0,t_1]]$. We can exploit the technique of successive approximation and set

$$\begin{aligned} r_0(x,t) &= r(x,0) \\ r_n(x,t) &= r_{n-1}(x,0) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\Xi(x,\tau,r_{n-1}(x,\tau))}{(t-\tau)^{1-\alpha}} d\tau, \quad n \in \mathbb{N}. \end{aligned} \tag{2.14}$$

Obviously, $r(x,0) \in C^0[0,T]$ and the differentiation of (2.14) with respect to t yields

$$\partial_t r_n(x,t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t \frac{\Xi(x,\tau,r_{n-1}(x,\tau))}{(t-\tau)^{-\alpha}} d\tau.$$

Using the differentiability of $r(x,0)$ with respect to t leads to $r_n(x,t) \in C^0[0,T]$.
The quantity $\|r_n(x,t) - r_{n-1}(x,t)\|_{C^0[\Omega\times[0,t_1]]}$ can be evaluated for $n \in \mathbb{N}$. Whence,

$$\|r_n(x,t) - r_{n-1}(x,t)\|_{C^0[\Omega\times[0,t_1]]} \leq \frac{Kt_1^{\alpha}}{\alpha\Gamma(\alpha)}.$$

Furthermore,

$$\begin{aligned}\|r_2(x,t)-r_1(x,t)\|_{C^0[\Omega\times[0,t_1]]} &\leq \|I_0^{\alpha}\Xi(x,\tau,r_1(x,\tau))-\Xi(x,\tau,r_0(x,\tau))\|_{C^0[\Omega\times[0,t_1]]}\\ &\leq \frac{t_1^{\alpha}}{\alpha\Gamma(\alpha)}\|r_1(x,t)-r_0(x,t)\|_{C^0[\Omega\times[0,t_1]]}\\ &\leq \frac{Kt_1^{\alpha}}{\alpha\Gamma(\alpha)}\cdot\frac{t_1^{\alpha}}{\alpha\Gamma(\alpha)}\end{aligned}$$

The above iteration is repeated n-times and yields

$$\|r_2(x,t)-r_1(x,t)\|_{C^0[\Omega\times[0,t_1]]} \leq \left(\frac{Kt_1^{\alpha}}{\alpha\Gamma(\alpha)}\right)^{n-1}\cdot\frac{t_1^{\alpha}}{\alpha\Gamma(\alpha)}$$

Hence, the sequence $\{r_n(x,t)\}_{n\in\mathbb{N}}$ has $r(x,t)\in C^0[\Omega\times[0,t_1]]$ as its limit function. Moreover, the assumption

$$\left|\frac{t_1^{\alpha}}{\alpha\Gamma(\alpha)}\right|<1$$

gives $\lim\limits_{n\longrightarrow\infty}\|r_n(x,t)-r_{n-1}(x,t)\|_{C^0[\Omega\times[0,t_1]]}=0$. However by considering $t_1=T$, we can estimate

$$\Xi(x,t,r_n(x,\tau))-\Xi(x,t,r(x,t)).$$

Taking into account the Lipschitz condition of Ξ leads to

$$\left\|\frac{1}{\Gamma(\alpha)}\int_0^t\frac{\Xi(x,\tau,r_n(x,\tau))}{(t-\tau)^{1-\alpha}}d\tau-\frac{1}{\Gamma(\alpha)}\int_0^t\frac{\Xi(x,\tau,r(x,\tau))}{(t-\tau)^{1-\alpha}}d\tau\right\|_{C^0[\Omega\times[0,T]]}$$

$$\leq\left(\frac{Lt_1^{\alpha}}{\alpha\Gamma(\alpha)}\right)\|r_n(x,t)-r(x,t)\|_{C^0[\Omega\times[0,t_1]]}$$

and

$$\lim_{n\longrightarrow\infty}\left\|\frac{1}{\Gamma(\alpha)}\int_0^t\frac{\Xi(x,\tau,r_n(x,\tau))}{(t-\tau)^{1-\alpha}}d\tau-\frac{1}{\Gamma(\alpha)}\int_0^t\frac{\Xi(x,\tau,r(x,\tau))}{(t-\tau)^{1-\alpha}}d\tau\right\|_{C^0[\Omega\times[0,T]]}=0,$$

which prove that $r(x,t)$ satisfies (2.3) in the space $C^0[\Omega\times[0,T]]$.

Uniqueness result

Considering now that there are two separate solutions $r_1(x,t)$ and $r_2(x,t)$ verifying (2.3) on $[0,t_1]$ then,

$$\|r_1(x,t)-r_2(x,t)\|_{C^0[\Omega\times[0,t_1]]}\leq\frac{Kt_1^{\alpha}}{\alpha\Gamma(\alpha)}\|r_1(x,t)-r_2(x,t)\|_{C^0[\Omega\times[0,t_1]]}.$$

This gives $1\leq\frac{Kt_1^{\alpha}}{\alpha\Gamma(\alpha)}$ which is a contradiction. The solution is unique in $C^0[\Omega\times[0,t_1]]$
We consider now the closed interval $[t_1,t_2]$ with $t_2=t_1+h_1$ where $h_1>0$ and $t_2<T$. For $t\in[t_1,t_2]$,

we have

$$r(x,t)=\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\frac{\Xi(x,\tau,r_n(x,\tau))}{(t-\tau)^{1-\alpha}}d\tau+r(x,0)+\frac{1}{\Gamma(\alpha)}\int_{0}^{t_1}\frac{\Xi(x,\tau,r_n(x,\tau))}{(t-\tau)^{1-\alpha}}d\tau.$$

Taking into account the uniqueness result on $[0,t_1]$, then

$$r(x,t)=r_1^*(x,t)+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\frac{\Xi(x,\tau,r_n(x,\tau))}{(t-\tau)^{1-\alpha}}d\tau$$

where $r_1^*(x,t)=r(x,0)+\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\frac{\Xi(x,\tau,r_n(x,\tau))}{(t-\tau)^{1-\alpha}}d\tau$ is a given function.
we repeat the same analysis presented above and there is an unique solution $r(x,t)\in C^0\left[\Omega\times[t_1,t_2]\right]$ for (2.3). Taking another interval $[t_2,t_3]$ so that $t_3=t_2+h_2$ with $h_2>0$ and $t_3<T$, the same analysis is performed to finally obtain the existence of an unique solution $r(x,t)\in C^0\left[\Omega\times[0,T]\right]$ of equation (2.3) and therefore, proves the existence of an unique solution of equation (2.1) in the space $C^{0,\alpha}\left[\Omega\times[0,T]\right]$. □

Corollary 2.3. *Assume Ξ satisfies Theorem 2.1. If the inequality*

$$\left|\frac{KT^{\alpha}}{\alpha\Gamma(\alpha)}\right|\leq 1$$

holds, then the sequence $r_n(x,t)$, $(n\in\mathbb{N})$ tends to the exact solution $r(x,t)$. Moreover, we have for any $n\in\mathbb{N}$,

$$\|r(x,t)-r_n(x,t)\|_{C^0[\Omega\times[0,t_1]]}\leq\frac{T^{\alpha}}{\alpha\Gamma(\alpha)}\frac{K^n}{1-\frac{T^{\alpha}}{\alpha\Gamma(\alpha)}}$$

Proof. The proof is done by the mathematical induction on n.

For

$$\|r_1(x,t)-r_0(x,t)\|_{C^0[\Omega\times[0,t_1]]}\leq\frac{KT^{\alpha}}{\alpha\Gamma(\alpha)}$$

$$\|r_2(x,t)-r_1(x,t)\|_{C^0[\Omega\times[0,t_1]]}\leq K\left(\frac{T}{\alpha\Gamma(\alpha)}\right)^2$$

Thus by induction,

$$\|r_n(x,t)-r_{n-1}(x,t)\|_{C^0[\Omega\times[0,t_1]]}\leq\frac{T^{\alpha}}{\alpha\Gamma(\alpha)}\left(\frac{KT^{\alpha}}{\alpha\Gamma(\alpha)}\right)^{n-1}$$

but,

$$
\begin{aligned}
&\|r(x,t)-r_n(x,t)\|_{C^0[\Omega\times[0,t_1]]} = \lim_{j\longrightarrow\infty}\|r_{n+j}(x,t)-r_n(x,t)\|_{C^0[\Omega\times[0,t_1]]}\\
&= \|r_{n+1}(x,t)-r_n(x,t)\|_{C^0[\Omega\times[0,t_1]]} + \|r_{n+2}(x,t)-r_{n+1}(x,t)\|_{C^0[\Omega\times[0,t_1]]} + \cdots\\
&\leq K^n\left(\frac{T^\alpha}{\alpha\Gamma(\alpha)}\right)^{n+1} + K^{n+1}\left(\frac{T^\alpha}{\alpha\Gamma(\alpha)}\right)^{n+2} + \cdots \qquad (2.15)\\
&= K^n\left(\frac{T^\alpha}{\alpha\Gamma(\alpha)}\right)^{n+1}\left[\sum_{k=0}^{\infty}\left(\frac{KT^\alpha}{\alpha\Gamma(\alpha)}\right)^k\right]
\end{aligned}
$$

and the corollary is proved. □

This implies the following existence and uniqueness results for our model (2.1)-(2.2):

Corollary 2.4. *The function $r(x,t)$ is the strong solution of the sequence $r_n(x,t)$ given by* (2.14).

Proof. This result is an immediate consequence of Propositions 2.2 and 2.1 □

3. Analyzis with Caputo-Fabrizio fractional derivative

3.1. Introduction and formulation

Definition 3.1 (Piccard's *L*-stability).
Consider the Banach space $(B,\|\|)$, the self-map L of B and the recursive technique $\sigma_{n+1} = g(L,\sigma_n)$. Let us assume that $B(L)$, containing all the fixed points of L has at least one element and σ_n converges to an element b of $B(L)$. Let $\{u_n\}\subseteq B$ and set $d_n = \|u_{n+1} - g(L,u_n)\|$. Therefore $\lim_{n\longrightarrow\infty} d_n = 0$ leads to $\lim_{n\longrightarrow\infty} u_n = b$. In this case, we say that the reccurence formula $\sigma_{n+1} = g(L,\sigma_n)$ is *L*-stable.

Remark 3.1. Assuming that $\{u_n\}$ has a upper boundary, $\sigma_{n+1} = L\sigma_n$ is called Piccard's iteration if the conditions of Definition 3.1 are satisfied and therefore, will be *L*-stable.

Lemma 3.1. *Let $(B,\|\|)$ be a Banach space and L a self-map of B verifying*

$$\|Lx-Ly\| \leq C\|x-Lx\| + C\|x-y\|$$

for all $x,y\in B$, where $0\leq C$, $0\leq\alpha\leq 1$. If L admits a fixed point f then, L is Picard L-stable.

Proof. [4, Theorem 3.1] □

Proposition 3.2. *The self-map L expressed as*

$$
\begin{aligned}
L(r_n(x,t)) &= r_{n+1}(x,t)\\
&= r_n(x,t) + {}^{cf}I_t^\alpha[r^3 r_{xxx}(x,t))]
\end{aligned}
$$

is L-stable in $L^2(a,b)$

Proof. The first step is to prove that L has a fixed-point. For that set $i, j \in \mathbb{N}$ then,

$$\|Lr_i(x,t)-Lr_j(x,t)\| = \|r_{i+1}(x,t)-r_{j+1}(x,t)\| = \left\|r_i(x,t) +^{cf} I_t^{\alpha}\left[r_i^3\partial_{x^3}^3 r_i\right] - r_j(x,t) -^{cf} I_t^{\alpha}\left[r_j^3\partial_{x^3}^3 r_j\right]\right\|$$

with

$$^{cf}I_t^{\alpha} r(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} r(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t r(\tau)\,d\tau,$$

the anti-derivative associated to the Caputo-Fabrizio derivative as given in (1.4). Then, making use of the boundedness the function u, the Lipschitz condition for the first order operator ∂_x with the same constants $k_1,\ k_2,\ \vartheta$ as in the previous section, we have

$$\|Lr_i(x,t)-Lr_j(x,t)\| \le \|r_i(x,t)-r_j(x,t)\| + \|^{cf}I_t^{\alpha}[r_i^3\partial_{x^3}^3 r_i - r_j^3\partial_{x^3}^3 r_j]\|$$

$$\le \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} k_1k_2\vartheta^3\|r_i-r_j\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} k_1k_2\vartheta^3\|r_i-r_j\|.$$

Thus,

$$\|Lr_i(x,t)-Lr_j(x,t)\| \le \mathscr{K}\,\|r_i(x,t)-r_j(x,t)\|$$

With

$$\mathscr{K} = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} k_1k_2\vartheta^3\|r_i-r_j\| + \frac{2\alpha}{(2-\alpha)M(\alpha)} k_1k_2\vartheta^3.$$

Consequently L is Lipschitz continuous with respect to r and this means the non-linear operator L has a fixed point. To conclude the proof, it is necessary to remark that taking $C=0$ and $C=P$, the conditions of Lemma 3.1 are verified for L and then, L is Picard L-stable. □

4. Numerical Solvability

This section deals with some numerical schemes associated with both models. So a technique to determine the solution for each model using integral iterative methods is presented. We The model with Caputo derivative is iteratively solved making use of Laplace transform while the model with the Caputo-Fabrizio derivative exploit the Sumudu transform. Similar results are obtained as shown below.

4.1. Numerical Approximations with Caputo derivative

Applying the Laplace transform $\mathscr{L}$ on both sides of the model (2.1)-(2.2) iteratively yields

$$pr(x,p) - g(x) = \mathscr{L}\left[r^3 r_{xxx}(x,t)\right](p),$$

equivalently

$$r(x,p)$$

$$= \frac{g(x)}{p} + \frac{1}{p}\mathscr{L}\left[r^3 r_{xxx}(x,t)\right](p).$$

Taking the inverse Laplace transform $\mathscr{L}^{-1}$ yields

$$r(x,t)$$

$$= g(x) + \mathscr{L}^{-1}\left(\frac{1}{p}\mathscr{L}\left[r^3 r_{xxx}(x,t)\right]\right)(t).$$

Now, we can introduce the following iterative formula

$$\begin{aligned} r_0(x,t) &= g(x) \\ r_{n+1}(x,t) &= r_n(x,t) + \mathscr{L}^{-1}\left(\frac{1}{p}\mathscr{L}\left[r^3 r_{xxx}(x,t)\right]\right)(t). \end{aligned}$$

the above formula leads to a numerical approximation with Caputo derivative and the approximate solution reads as

$$r(x,t) = \lim_{n\longrightarrow\infty} r_n(x,t)$$

4.2. Numerical Approximations Caputo-Fabrizio derivative

Here we first recall the following important relation

$$\mathscr{S}\left({}_0^{cf}D_t^{\alpha} f(t)\right) = M(\alpha)\frac{pF(p) - f(o)}{1-\alpha-\alpha p}$$

where $F(p) = \mathscr{S}(f(t))$ is the Sumudu transform of $f(t)$. Applying the Sumudu transform $\mathscr{S}$ on both sides of equation (2.1) yields

$$\frac{pr(x,p) - g(x)}{1-\alpha-\alpha p} = \mathscr{S}\left[r^3 r_{xxx}(x,t)\right](p),$$

equivalently

$$r(x,p)$$

$$= \frac{g(x)}{p} + (1-\alpha-\alpha p)\mathscr{S}\left[r^3 r_{xxx}(x,t)\right](p).$$

The inverse Sumudu transform $\mathscr{S}^{-1}$ yields

$$r(x,t)$$

$$= g(x) + \mathscr{S}^{-1}\left((1-\alpha-\alpha p)\mathscr{S}\left[r^3 r_{xxx}(x,t)\right]\right)(t).$$

repeating as above, the following iterative formula introduced

$$\begin{aligned} r_0(x,t) &= g(x) \\ r_{n+1}(x,t) &= r_n(x,t) + \mathscr{S}^{-1}\left((1-\alpha-\alpha p)\mathscr{S}\left[r^3 r_{xxx}(x,t)\right]\right)(t). \end{aligned}$$

which leads to a numerical approximation with Caputo-Fabrizio derivative and the approximate solution reads as

$$r(x,t) = \lim_{n\longrightarrow\infty} r_n(x,t).$$

Those recurrence schemes are used and numerical representations of both models can be depicted in Fig.1 to Fig.5 for different values of the order α. The graphics in Fig.1, Fig.2, Fig.3, Fig.4 and Fig.5, performed respectively for $\alpha = 0.2,\ \alpha = 0.3,\ \alpha = 0.4,\ \alpha = 0.8$ and $\alpha = 1.0$ using the Caputo derivative and compared to one using the Caputo-Fabrizio derivative show the standard well-known wave solution of the Harry Dym equation. It is clear that the figures show similar behavior for solutions of both models.

5. Concluding remarks

We have proved the existence and uniqueness of the solution to for the nonlinear Harry Dym equation modelled with both the classical Caputo derivative and the newly introcuced derivative of fractional order with no singular kernel. It is the first time that the same model of Harry Dym is analyzed using both derivatives in the same work. This proves that there is a possible way to extent the nonlinear Harry Dym model to the scope of fractional calculus.Two numerical methods suitable to approximate the solutions of model with both derivatives have been presented with numerical simulations performed for $\alpha = 0.2,\ \alpha = 0.3,\ \alpha = 0.4,\ \alpha = 0.8$ and $\alpha = 1.0$. Each figure exhibits solution with similar behavior for the involved wave associated to the Dym equation. This paper innovates by pointing out another concrete application of the Caputo-Fabrizio derivative, new in the literature and till under investigation. More complex investigation will certainly follow.

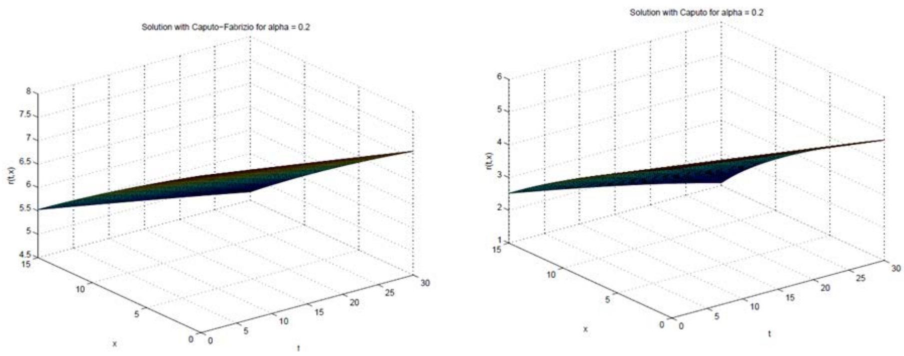

Fig. 1: Representation of the solution $r(x,t)$ when $\alpha = 0.2$ with both derivatives.

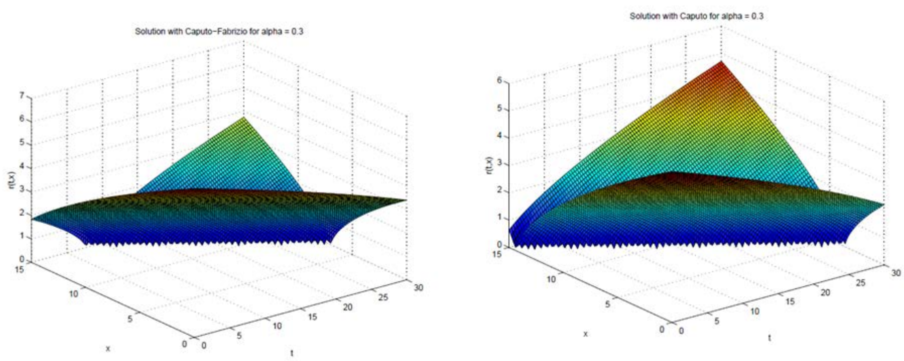

Fig. 2: Representation of the solution $r(x,t)$ when $\alpha = 0.3$ with both derivatives.

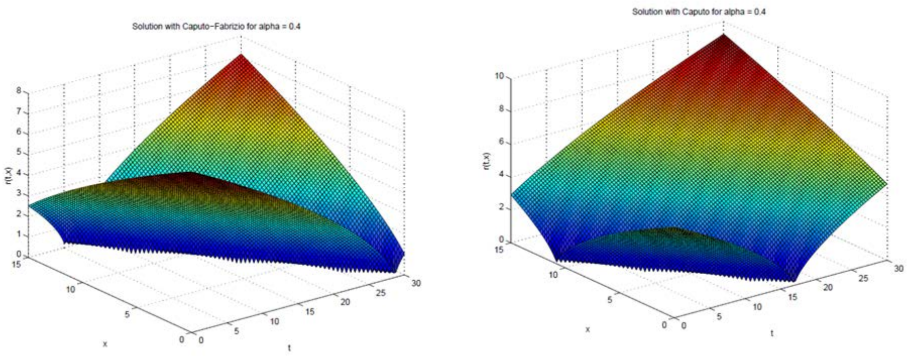

Fig. 3: Representation of the solution $r(x,t)$ when $\alpha = 0.4$ with both derivatives.

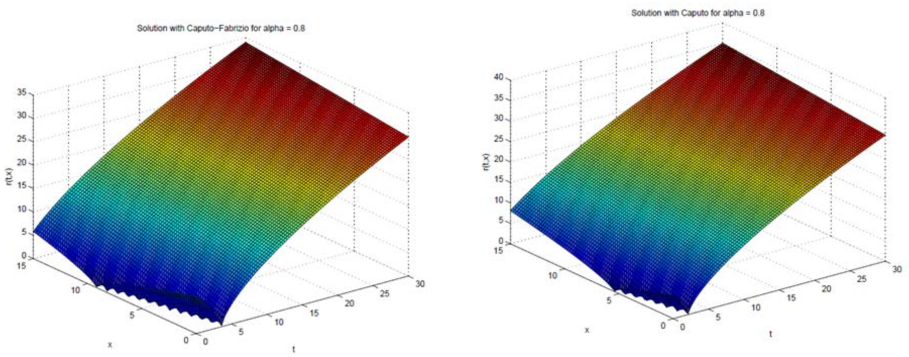

Fig. 4: Representation of the solution $r(x,t)$ when $\alpha = 0.8$ with both derivatives.

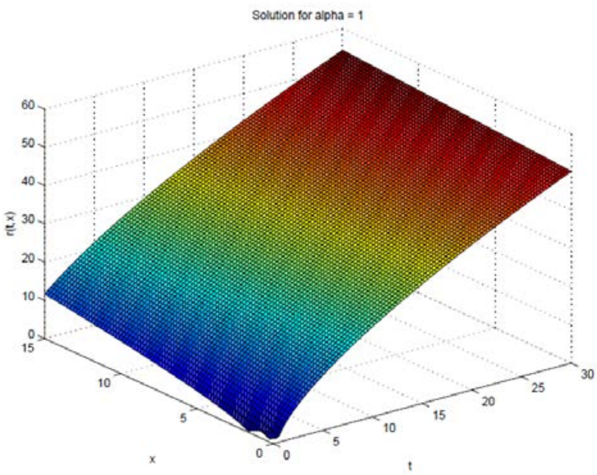

Fig. 5: Representation of the solution $r(x,t)$ when $\alpha = 1.0$ with both derivatives.

References

[1] Caputo M., Linear models of dissipation whose Q is almost frequency independent II, Geophys. J. R. Ast. Soc. 13 (5), 529-539, 1967; Reprinted in: Fract. Calc. Appl. Anal. 11 (1), 3-14, (2008)

[2] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, vol. 204, Elsevier Science B.V., Amsterdam, The Netherlands, 2006.

[3] Atangana A. and Doungmo Goufo E.F., "Extension of Matched Asymptotic Method to Fractional Boundary Layers Problems," Mathematical Problems in Engineering, vol. 2014, Article ID 107535, 7 pages, 2014. doi:10.1155/2014/107535

[4] D. Eberly, Stability Analysis for Systems of Differential Equations, Geometric Tools, LLC, (2008). http://www.geometrictools.com.

[5] Caputo M, Fabrizio M, A new definition of fractional derivative without singular kernel, Progr. Fract. Differ. Appl. 1 (2), (2015) 1–13

[6] Caputo Michele , Carcione José M., Botelho Marco A. B., Modeling Extreme-Event Precursors with the Fractional Difusion Equation, Fractional Calculus and Applied Analysis, 18 (1) 208–222, (2015).

[7] Doungmo Goufo EF, A biomathematical view on the fractional dynamics of cellulose degradation, Fractional Calculus and Applied Analysis, Vol. 18, No 3, 554–564, (2015). DOI: 10.1515/fca-2015-0034.

[8] Losada J, Nieto JJ, Properties of the new fractional derivative without singular Kernel. Progr. Fract. Differ. Appl, 1(2), (2015) 87–92

[9] Tenreiro Machado J., Mainardi F., and Kiryakova V. Fractional calculus: quo vadimus? (Where are we going?). Fract. Calc. Appl. Anal., 18(2):495–526, 2015.

[10] Doungmo Goufo EF, A biomathematical view on the fractional dynamics of cellulose degradation, Fractional Calculus and Applied Analysis, Vol. 18, No 3, 554–564, (2015). DOI: 10.1515/fca-2015-0034

[11] Doungmo Goufo EF, Application of the Caputo-Fabrizio fractional derivative without singular kernel to Korteweg–de Vries–Bergers equation, Mathematical Modelling and Analysis, Vol. 21, Issue 2, 188-198, 2016, http://dx.doi.org/10.3846/13926292.2016.1145607.

[12] Doungmo Goufo EF, Atangana A, Analytical and numerical schemes for a derivative with filtering property and no singular kernel with applications to diffusion, The European Physical Journal Plus, Vol. 131, Issue 8, 2016. DOI: 10.1140/epjp/i2016-16269-1

[13] Luchko Y. and Gorenflo R., The initial value problem for some fractional differential equations with the Caputo derivative. Preprint Series A0898, Freic Universitat Berlin, 1998, Fachbreich Mathematik and Informatik, 1998.

[14] Area I, Batarfi H, Losada J, Nieto JJ, Shammakh W, Torres A. On a Fractional Order Ebola Epidemic Model. Adv Difference Equ. 2015; Art. ID 278, 12 pp.

[15] Rachah A, Torres DFM. Mathematical modelling, simulation, and optimal control of the 2014 Ebola outbreak in West Africa. Discrete Dyn Nat Soc. 2015; Art. ID 842792, 9 pp.

[16] Area I, Losada J, Ndaïrou F, Nieto JJ, Tcheutia DD. Mathematical modeling of 2014 Ebola outbreak. Math Method Appl Sci.; in press.

[17] Du, M, Wang, Z, Hu, H: Measuringmemorywith the order of fractional derivative. Sci. Rep. 3, 3431 (2013)

[18] Beals R. and Coifman R., Scattering and Inverse Scattering for First Order Systems, Commun. Pure and Appl. Math. 37 (1984) 39-90

[19] Gerdjikov V. and Yanovski A., Completeness of the Eigenfunctions for the Caudrey-Beals-Coifman System, J. Math. Phys. 35 (1994) 3687-3725

[20] Z. Yan, The Riccati equation with variable coefficients expansion algorithm to find more exact solutions of nonlinear differential equation, MMRC, AMSS, Academis Sinica, Beijing 22(2003), 275-284.

[21] Khan Y., Sayevand K., Fardi M., Ghasemi M., A novel computing multi-parametric homotopy approach for system of linear and nonlinear Fredholm integral equations, Applied Mathematics and Computation 2014, 249, 229–236.

Product-type Operators from Weighted Bergman Spaces to Bloch-Orlicz Spaces

Zhi-jie Jiang

(*School of Mathematics and Statistics, Sichuan University of Science and Engineering,*

Zigong, Sichuan, 643000, P. R. China)

Abstract: Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. By constructing some suitable test functions in weighted Bergman space, in this paper the boundedness and compactness of the product-type operators $D^nM_uC_\varphi$, $D^nC_\varphi M_u$, $C_\varphi D^nM_u$, $M_uD^nC_\varphi$, $M_uC_\varphi D^n$ and $C_\varphi M_uD^n$ from weighted Bergman space to Bloch-Orlicz space are characterized in terms of the symbol functions u and φ.

Keywords: Weighted Bergman-type space; Bloch-Orlicz space; product-type operator; boundedness; compactness

MR (2010) Subject Classification: 47B38; 47B33, 47B37

Chinese Library Classification: O177.2

1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. The weighted composition operator $W_{\varphi,u}$ on $H(\mathbb{D})$ is defined by

$$W_{\varphi,u}f(z) = u(z)f(\varphi(z)),\ z \in \mathbb{D}.$$

If $u \equiv 1$, it becomes the composition operator, usually denoted by C_φ. If $\varphi(z) = z$, it becomes the multiplication operator, usually denoted by M_u. Since $W_{\varphi,u} = M_uC_\varphi$, it is a product-type operator. A standard problem is to provide function theoretic characterizations when φ and u induce a bounded or compact weighted composition operator (see, for example, [2, 4, 8, 13, 15, 23, 26, 27] and the references therein).

Let $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The nth differentiation operator D^n on $H(\mathbb{D})$ is defined by

$$D^nf(z) = f^{(n)}(z),\ \ z \in \mathbb{D},$$

Foundation item: Supported by the Sichuan Province University Key Laboratory of Bridge Non-destruction Detecting and Engineering Computing (No.2016QZJ01), the Key Fund Project of Sichuan Provincial Department of Education (No.15ZA0221) and the Cultivation Project of Sichuan University of Science and Engineering (No.2015PY04).

E-mail: matjzj@126.com

where $f^{(0)} = f$. If $n = 1$, it is the differentiation operator D. A systematic study of some other product-type operators started by Stević et al. since the publication of papers [11] and [12]. Before that there were a few papers in the topic, e.g., [5]. The publication of paper [11] first attracted some attention in product-type operators DC_φ and $C_\varphi D$ (see, e.g., [14, 18, 20] and the references therein). The publication of paper [12] attracted some attention in product-type operators involving integral-type ones (see, e.g., [9, 19, 21] and the references therein). Now there is a great interest in various product-type operators (see, e.g., [6, 7, 10, 16, 30, 31] and the references therein).

By using multiplication, composition and the nth differentiation operators, we define the product-type operators in the following six ways

$$D^n M_u C_\varphi,\ D^n C_\varphi M_u,\ C_\varphi D^n M_u,\ M_u D^n C_\varphi,\ M_u C_\varphi D^n,\ C_\varphi M_u D^n. \tag{1.1}$$

When $n = 1$, they were studied by Sharma in [17]. They were also studied on the weighted Bergman space in a unified manner by Stević et al. in [24] and [25].

By constructing some test functions in weighted Bergman space, here we characterize the boundedness and compactness of the product-type operators in (1.1) from weighted Bergman space to Bloch-Orlicz space. Because some more suitable test functions were not found in weighted Bergman space, before this work we didn't find any result on these operators from weighted Bergman space to Bloch-Orlicz space.

Let $dA(z) = \frac{1}{\pi}dxdy$ be the normalized Lebesgue measure on $\mathbb{D}$. For $\alpha > -1$, let $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$ be the weighted Lebesgue measure on $\mathbb{D}$. For $p \geq 1$, the famous weighted Bergman space A^p_α consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{A^p_\alpha} = \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) < \infty.$$

It is well known that the weighted Bergman space A^p_α with the norm $\|\cdot\|_{A^p_\alpha}$ is a Banach space. For some results of the weighted Bergman space, see, for example, [28, 29].

Let Ψ be a strictly increasing convex function on $[0, +\infty)$ such that $\Psi(0) = 0$. The Bloch-Orlicz space $\mathcal{B}^\Psi$ was introduced in [15] by Ramos Fernández, is the class of all $f \in H(\mathbb{D})$ such that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\Psi(\lambda|f'(z)|) < \infty$$

for some $\lambda > 0$ depending on f. Ramos Fernández in [15] proved that $\mathcal{B}^\Psi$ is isometrically equal to μ_Ψ-Bloch space, where

$$\mu_\Psi(z) = \frac{1}{\Psi^{-1}(\frac{1}{1-|z|^2})},\ \ z \in \mathbb{D}.$$

Hence, $\mathcal{B}^\Psi$ is a Banach space with the norm given by

$$\|f\|_{\mathcal{B}^\Psi} = |f(0)| + \sup_{z\in\mathbb{D}} \mu_\Psi(z)|f'(z)|.$$

This space generalizes some other spaces. For example, if $\Psi(t) = t^p$ with $p > 0$, then the space $\mathcal{B}^\Psi$ coincides with the weighted Bloch space $\mathcal{B}^\alpha$, where $\alpha = 1/p$. Also, if $\Psi(t) = t\log(1+t)$, then $\mathcal{B}^\Psi$ coincides with the Log-Bloch space (see [1]).

Let X and Y be Banach spaces. A linear operator $L : X \to Y$ is bounded if there exists a positive constant K such that

$$\|Lf\|_Y \le K\|f\|_X$$

for all $f \in X$. The operator $L : X \to Y$ is compact if it maps bounded sets into relatively compact sets.

In this paper, the letter C denotes a positive constant which may differ from one occurrence to the other. The notation $a \lesssim b$ means that there exists a positive constant C such that $a \le Cb$. When $a \lesssim b$ and $b \lesssim a$, we write $a \asymp b$.

2 Prerequisites

The first result is a alternative to Proposition 3.11 in [3], which characterizes the compactness in terms of sequential convergence. So the proof is omitted.

Lemma 2.1. *Let T be one of the operators in (1.1). Then the bounded operator $T : A^p_\alpha \to \mathcal{B}^\Psi$ is compact if and only if for every bounded sequence $\{f_j\}_{j\in\mathbb{N}}$ in A^p_α such that $f_j \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $j \to \infty$, it follows that*

$$\lim_{j\to\infty} \|Tf_j\|_{\mathcal{B}^\Psi} = 0.$$

For $k = 0$, the following lemma was proved in [29], while for $k \ge 1$ it essentially follows from the Jensen's inequality (see [6]).

Lemma 2.2. *Let $\alpha > -1$ and $p \ge 1$. Then for each $k \in \mathbb{N}_0$, there exists a positive constant $C_k = C(\alpha, p, k)$ independent of $f \in A^p_\alpha$ and $z \in \mathbb{D}$ such that*

$$|f^{(k)}(z)| \le \frac{C_k\|f\|_{A^p_\alpha}}{(1-|z|^2)^{k+\frac{\alpha+2}{p}}}.$$

In order to construct some test functions in weighted Bergman space, for a fixed $w \in \mathbb{D}$ and $i \in \mathbb{N}_0$ we define the following function

$$k_{w,i}(z) = \frac{(1-|w|^2)^{i+\frac{\alpha+2}{p}}}{(1-\overline{w}z)^{i+\frac{2\alpha+4}{p}}},\ z \in \mathbb{D}.$$

Then from [6], we know that $f_{w,i} \in A^p_\alpha$ and

$$\sup_{w\in\mathbb{D}} \|k_{w,i}\|_{A^p_\alpha} \lesssim 1. \tag{2.1}$$

By using some suitable linear combinations of the functions $k_{w,i}$, we obtain the test function in A^p_α in the following result, which will be used in the proofs of our main results.

Lemma 2.3. *Let $w \in \mathbb{D}$ and $n \in \mathbb{N}$. Then for each fixed $k \in \{0, 1, \ldots, n+1\}$, there exist constants $a_{0,k}$, $a_{1,k}$, $\ldots$, $a_{n+1,k}$ such that the function*

$$f_{w,k}(z) = \sum_{i=0}^{n+1} a_{i,k} k_{w,i}(z)$$

satisfies

$$f^{(k)}_{w,k}(w) = \frac{\overline{w}^k}{(1-|w|^2)^{k+\frac{\alpha+2}{p}}} \quad and \quad f^{(j)}_{w,k}(w) = 0 \tag{2.2}$$

for each $j \in \{0, 1, \ldots, n+1\} \setminus \{k\}$. Moreover,

$$\sup_{w\in\mathbb{D}} \|f_{w,k}\|_{A^p_\alpha} \lesssim 1. \tag{2.3}$$

Proof. We write $a = (2\alpha + 4)/p$. From a direct calculation, it follows that the system (2.2) is equivalent to the following system

$$\begin{cases} \sum\limits_{i=0}^{n+1} (a+i) a_{i,k} = 0 \\ \sum\limits_{i=0}^{n+1} (a+i)(\alpha+i+1) a_{i,k} = 0 \\ \qquad \cdots\cdots\cdots \\ \sum\limits_{i=0}^{n+1} \prod\limits_{j=0}^{k-1} (a+i+j) a_{i,k} = 1 \\ \qquad \cdots\cdots\cdots \\ \sum\limits_{i=0}^{n+1} \prod\limits_{j=0}^{n} (a+i+j) a_{i,k} = 0. \end{cases} \tag{2.4}$$

Hence we only need to prove that there exist constants $a_{0,k}$, $a_{1,k}$, $\ldots$, $a_{n+1,k}$ such that the system (2.4) holds. By Lemma 3 in [22], the determinant of the system (2.4) equals $\prod_{j=1}^{n+1} j!$, which is different from zero. So there exist constants $a_{0,k}$, $a_{1,k}$, $\ldots$, $a_{n+1,k}$ such that the system (2.4) holds. From (2.1) the asymptotic expression of $\sup_{w\in\mathbb{D}} \|f_{w,k}\|_{A^p_\alpha} \lesssim 1$ is obvious. $\square$

Remark 2.1. *It is not hard to see that $f_{w,k} \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $|w| \to 1^-$.*

Stević in [22] used the Faà di Bruno's formula of the following version

$$(f \circ \varphi)^{(n)}(z) = \sum_{k=0}^{n} f^{(k)}(\varphi(z)) B_{n,k}(\varphi'(z), \ldots, \varphi^{(n-k+1)}(z)), \tag{2.5}$$

where $B_{n,k}(x_1, ..., x_{n-k+1})$ is the Bell polynomial. For $n \in \mathbb{N}$ the sum can go from $k = 1$ since $B_{n,0}(\varphi'(z), ..., \varphi^{(n-k+1)}(z)) = 0$, however we will keep the summation since for $n = 0$ the only existing term $B_{0,0}$ is equal to 1. From (2.5) and the Leibnitz formula the next Lemma 2.4 follows.

Lemma 2.4. *Let f, $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then*

$$\big(u(z)f(\varphi(z))\big)^{(n+1)} = \sum_{k=0}^{n+1} f^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big).$$

3 Boundedness of the product-type operators

First we characterize the boundedness of the operator $D^n M_u C_\varphi : A_\alpha^p \to \mathcal{B}^\Psi$.

Theorem 3.1. *Let $\alpha > -1$, $p \geq 1$, φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then the following statements hold.*

(i) The operator $D^n M_u C_\varphi : A_\alpha^p \to \mathcal{B}^\Psi$ is bounded.

(ii) The functions u and φ satisfy the following conditions:

$$I_k := \sup_{z\in\mathbb{D}} \frac{\mu_\Psi(z)\Big|\sum\limits_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big)\Big|}{(1-|\varphi(z)|^2)^{k+\frac{\alpha+2}{p}}} < \infty$$

for each $k \in \{0, 1, \dots, n+1\}$.

Proof. $(i) \Rightarrow (ii)$. Let $h_0(z) \equiv 1 \in A_\alpha^p$. Then we get

$$L_0 = \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=0}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,0}(\varphi'(z), \dots, \varphi^{(j+1)}(z))\Big| \leq C\|D^n M_u C_\varphi\|. \tag{3.1}$$

Let $h_k(z) = z^k \in A_\alpha^p$, $k = 1, 2, \dots, n+1$. Assume now that we have proved the following inequalities

$$\sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{j=l}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,l}(\varphi'(z), \dots, \varphi^{(j-l+1)}(z))\Big| \leq C\|D^n M_u C_\varphi\| \tag{3.2}$$

for each $l \in \{0, 1, ..., k-1\}$ and a $k \leq n+1$. Applying Lemma 2.4 to the function h_k, and noticing that $h_k^{(s)}(z) \equiv 0$ for $s > k$, we get

$$\big(D^n M_u C_\varphi h_k\big)'(z) = \sum_{j=0}^{k} h_k^{(j)}(\varphi(z)) \sum_{i=j}^{n+1} C_{n+1}^{i} u^{(n+1-i)}(z) B_{i,j}(\varphi'(z), \dots, \varphi^{(i-j+1)}(z))$$

$$= \sum_{j=0}^{k} k\cdots(k-j+1)(\varphi(z))^{k-j} \sum_{i=j}^{n+1} C_{n+1}^{i} u^{(n+1-i)}(z) B_{i,j}(\varphi'(z), \dots, \varphi^{(i-j+1)}(z)). \tag{3.3}$$

From (3.3), the boundedness of function φ and the triangle inequality, by noticing that the coefficient at

$$\sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z))$$

is independent of z and finally using hypothesis (3.2) we easily obtain

$$L_k := \sup_{z\in\mathbb{D}} \mu_\Psi(z) \Big| \sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)) \Big| \le C\|D^n M_u C_\varphi\|. \quad (3.4)$$

By induction we see that (3.4) holds for each $k \in \{0, 1, \dots, n+1\}$.

For a fixed $w \in \mathbb{D}$ and $k \in \{0, 1, \dots, n+1\}$, by Lemma 2.3 there exist constants $a_{0,k}$, $a_{1,k}$, $\dots$, $a_{n+1,k}$ such that the function

$$f_{\varphi(w),k}(z) = \sum_{i=0}^{n+1} a_{i,k} k_{\varphi(w),i}(z),$$

satisfies

$$f_{\varphi(w),k}^{(k)}(\varphi(w)) = \frac{\overline{\varphi(w)}^{k}}{(1-|\varphi(w)|^2)^{k+\frac{\alpha+2}{p}}} \quad \text{and} \quad f_{\varphi(w),k}^{(j)}(\varphi(w)) = 0 \quad (3.5)$$

for each $j \in \{0, 1, \dots, n+1\} \setminus \{k\}$. Moreover,

$$\sup_{w\in\mathbb{D}} \|f_{\varphi(w),k}\|_{A^p_\alpha} \le C. \quad (3.6)$$

Then from (3.5), (3.6) and the boundedness of $D^n M_u C_\varphi : A^p_\alpha \to \mathcal{B}^\Psi$, we have

$$\begin{aligned} I_k(w) &:= \frac{\mu_\Psi(w)|\varphi(w)|^k \Big| \sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(w) B_{j,k}\big(\varphi'(w), \dots, \varphi^{(j-k+1)}(w)\big)\Big|}{(1-|\varphi(w)|^2)^{k+\frac{\alpha+2}{p}}} \\ &\le \|D^n M_u C_\varphi f_{\varphi(w),k}\|_{\mathcal{B}^\Psi} \le C\|D^n M_u C_\varphi\|. \end{aligned} \quad (3.7)$$

From (3.7) we see that

$$\sup_{z\in\mathbb{D}} I_k(z) \le C\big\|D^n M_u C_\varphi\big\|,$$

from which we obtain

$$\sup_{|\varphi(z)|>1/2} \frac{\mu_\Psi(z)\Big| \sum_{j=k}^{n+1} C_{n+1}^{j} u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z))\Big|}{(1-|\varphi(z)|^2)^{k+\frac{\alpha+2}{p}}} \le C\|D^n M_u C_\varphi\|. \quad (3.8)$$

On the other hand, from (3.4) we get

$$\sup_{|\varphi(z)|\leq 1/2}\frac{\mu_\Psi(z)\Big|\sum\limits_{j=k}^{n+1}C_{n+1}^{j}u^{(n+1-j)}(z)B_{j,k}(\varphi'(z),\dots,\varphi^{(j-k+1)}(z))\Big|}{(1-|\varphi(z)|^2)^{k+\frac{\alpha+2}{p}}}\leq CL_k\leq C\|D^nM_uC_\varphi\|. \tag{3.9}$$

Hence from (3.8) and (3.9) we see that $I_k<\infty$ for each $k\in\{0,1,...,n+1\}$.

$(ii)\Rightarrow(i)$. From Lemma 2.2 and Lemma 2.4, for all $f\in A_\alpha^p$ we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}\mu_\Psi(z)\big|\big(D^nM_uC_\varphi f\big)'(z)\big|\\
&=\sup_{z\in\mathbb{D}}\mu_\Psi(z)\Big|\sum_{k=0}^{n+1}f^{(k)}(\varphi(z))\sum_{j=k}^{n+1}C_{n+1}^{j}u^{(n+1-j)}(z)B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big|\\
&\leq\sup_{z\in\mathbb{D}}\mu_\Psi(z)\sum_{k=0}^{n+1}\big|f^{(k)}(\varphi(z))\big|\Big|\sum_{j=k}^{n+1}C_{n+1}^{j}u^{(n+1-j)}(z)B_{j,k}\big(\varphi'(z),\dots,\varphi^{(j-k+1)}(z)\big)\Big|\\
&\leq\sum_{k=0}^{n+1}C_kI_k\|f\|_{A_\alpha^p}.
\end{aligned}\tag{3.10}$$

It is clear that

$$|(D^nM_uC_\varphi f)(0)|\leq C\|f\|_{A_\alpha^p}. \tag{3.11}$$

Hence from (3.10) and (3.11) it follows that $D^nM_uC_\varphi:A_\alpha^p\to\mathcal{B}^\Psi$ is bounded. □

Remark 3.1. *If $D^nC_\varphi M_u:A_\alpha^p\to\mathcal{B}^\Psi$ is a zero operator, then it is obvious that $\|D^nC_\varphi M_u\|=0$. Hence, the case is usually excluded from such considerations.*

Remark 3.2. *Since $D^nC_\varphi M_u=D^nM_{u\circ\varphi}C_\varphi$, the characterization of the boundedness of $D^nC_\varphi M_u:A_\alpha^p\to\mathcal{B}^\Psi$ can be directly obtained from Theorem 3.1. So we omit here.*

Noticing that

$$(C_\varphi D^nM_uf)'(z)=\sum_{k=0}^{n+1}C_{n+1}^{k}u^{(n+1-k)}(\varphi(z))\varphi'(z)f^{(k)}(\varphi(z)),$$

we can obtain the following result whose proof is similar to that of Theorem 3.1. So we also omit.

Theorem 3.2. *Let $\alpha>-1$, $p\geq1$, φ be an analytic self-map of $\mathbb{D}$ and $u\in H(\mathbb{D})$. Then the following statements hold.*

(i) The operator $C_\varphi D^nM_u:A_\alpha^p\to\mathcal{B}^\Psi$ is bounded.

(ii) The functions u and φ satisfy the following conditions:

$$J_k := \sup_{z\in\mathbb{D}} \frac{\mu_\Psi(z)|u^{(n+1-k)}(\varphi(z))||\varphi'(z)|}{(1-|\varphi(z)|^2)^{k+\frac{\alpha+2}{p}}} < \infty$$

for each $k \in \{0, 1, \ldots, n+1\}$.

From a calculation, we have

$$\begin{aligned}(M_u D^n C_\varphi f)'(z) = \sum_{k=0}^{n} f^{(k)}(\varphi(z))\big[u'(z)B_{n,k}(\varphi'(z), \ldots, \varphi^{(n-k+1)}(z))\\ + u(z)B_{n+1,k}(\varphi'(z), \ldots, \varphi^{(n-k+2)}(z))\big] + u(z)(\varphi'(z))^{n+1} f^{(n+1)}(\varphi(z)),\end{aligned}$$

and then we have the next result.

Theorem 3.3. *Let $\alpha > -1$, $p \geq 1$, φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then the following statements hold.*

(i) The operator $M_u D^n C_\varphi : A^p_\alpha \to \mathcal{B}^\Psi$ is bounded.

(ii) The functions u and φ satisfy the following conditions:

$$M_k := \sup_{z\in\mathbb{D}} \frac{\mu_\Psi(z)\big|u'(z)B_{n,k}(\varphi'(z), \ldots, \varphi^{(n-k+1)}(z)) + u(z)B_{n+1,k}(\varphi'(z), \ldots, \varphi^{(n-k+2)}(z))\big|}{(1-|\varphi(z)|^2)^{k+\frac{\alpha+2}{p}}} < \infty$$

for each $k \in \{0, 1, \ldots, n\}$, and

$$M_{n+1} := \sup_{z\in\mathbb{D}} \frac{\mu_\Psi(z)|u(z)||\varphi'(z)|^{n+1}}{(1-|\varphi(z)|^2)^{n+1+\frac{\alpha+2}{p}}} < \infty.$$

Since $(M_u C_\varphi D^n f)'(z) = u'(z)f^{(n)}(\varphi(z)) + u(z)\varphi'(z)f^{(n+1)}(\varphi(z))$, we have the following result.

Theorem 3.4. *Let $\alpha > -1$, $p \geq 1$, φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then the following statements hold.*

(i) The operator $M_u C_\varphi D^n : A^p_\alpha \to \mathcal{B}^\Psi$ is bounded.

(ii) The functions u and φ satisfy the following conditions:

$$R_{:} = \sup_{z\in\mathbb{D}} \frac{\mu_\Psi(z)|u'(z)|}{(1-|\varphi(z)|^2)^{n+\frac{\alpha+2}{p}}} < \infty,$$

and

$$S := \sup_{z\in\mathbb{D}} \frac{\mu_\Psi(z)|u(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1+\frac{\alpha+2}{p}}} < \infty.$$

Remark 3.3. *Noticing that* $C_\varphi M_u D^n = M_{u\circ\varphi} C_\varphi D^n$*, we can obtain the characterization of the boundedness of* $C_\varphi M_u D^n : A^p_\alpha \to \mathcal{B}^\Psi$ *from Theorem 3.4. Here we omit.*

4 Compactness of the product-type operators

We first characterize the compactness of the operator $D^n M_u C_\varphi : A^p_\alpha \to \mathcal{B}^\Psi$.

Theorem 4.1. *Let* $\alpha > -1$*,* $p \geq 1$*,* φ *be an analytic self-map of* $\mathbb{D}$ *and* $u \in H(\mathbb{D})$*. Then the following statements hold.*

(i) The operator $D^n M_u C_\varphi : A^p_\alpha \to \mathcal{B}^\Psi$ *is compact.*

(ii) The functions u *and* φ *satisfy* $L_k < \infty$ *and*

$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu_\Psi(z)\Big|\sum\limits_{j=k}^{n+1} C^j_{n+1} u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \ldots, \varphi^{(j-k+1)}(z)\big)\Big|}{(1-|\varphi(z)|^2)^{k+\frac{\alpha+2}{p}}} = 0$$

for each $k \in \{0, 1, \ldots, n+1\}$*.*

Proof. $(i) \Rightarrow (ii)$. Suppose that the operator $D^n M_u C_\varphi : A^p_\alpha \to \mathcal{B}^\Psi$ is compact. Then it is clear that the operator $D^n M_u C_\varphi : A^p_\alpha \to \mathcal{B}^\Psi$ is bounded. Hence from the proof of Theorem 3.1 it follows that $L_k < \infty$ for each $k \in \{0, 1, \ldots, n+1\}$. Consider a sequence $\{\varphi(z_i)\}_{i\in\mathbb{N}}$ in $\mathbb{D}$ such that $|\varphi(z_i)| \to 1^-$ as $i \to \infty$. If such a sequence does not exist, then the last condition in (ii) obviously holds. Without loss of generality, we may suppose that $|\varphi(z_i)| > 1/2$ for all $i \in \mathbb{N}$. For each fixed $k \in \{0, 1, \ldots, n+1\}$, by using this sequence we define the function sequence $f_{i,k}(z) = f_{\varphi(z_i),k}(z),\ i \in \mathbb{N}$. Then from Lemma 2.3 and Remark 2.1, we see that $\sup_{i\in\mathbb{N}} \|f_{i,k}\|_{A^p_\alpha} \leq C$ and $f_{i,k} \to 0$ uniformly on every compact subset of $\mathbb{D}$ as $i \to \infty$, moreover

$$f^{(k)}_{i,k}(\varphi(z_i)) = \frac{\overline{\varphi(z_i)}^k}{(1-|\varphi(z_i)|^2)^{k+\frac{\alpha+2}{p}}} \quad \text{and} \quad f^{(j)}_{i,k}(\varphi(z_i)) = 0 \tag{4.1}$$

for each $j \in \{0, 1, \ldots, n+1\} \setminus \{k\}$. Then from Lemma 2.1 we have

$$\lim_{i\to\infty} \|D^n M_u C_\varphi f_{i,k}\|_{\mathcal{B}^\Psi} = 0. \tag{4.2}$$

Combing (4.1) and (4.2), for each fixed $k \in \{0, 1, \ldots, n+1\}$ we get

$$\lim_{i\to\infty} \frac{\mu_\Psi(z_i)\Big|\sum\limits_{j=k}^{n+1} C^j_{n+1} u^{(n+1-j)}(z_i) B_{j,k}(\varphi'(z_i), \ldots, \varphi^{(j-k+1)}(z_i))\Big|}{(1-|\varphi(z_i)|^2)^{k+\frac{\alpha+2}{p}}} = 0. \tag{4.3}$$

$(ii) \Rightarrow (i)$. We first prove that $D^n M_u C_\varphi : A^p_\alpha \to \mathcal{B}^\Psi$ is bounded. We observe that the last condition in (ii) implies that for every $\varepsilon > 0$, there is an $\eta \in (0,1)$, such that for any $z \in K = \{z \in \mathbb{D} : |\varphi(z)| > \eta\}$

$$I_k(z) = \frac{\mu_\Psi(z)\Big|\sum\limits_{j=k}^{n+1} C^j_{n+1} u^{(n+1-j)}(z) B_{j,k}(\varphi'(z), \dots, \varphi^{(j-k+1)}(z))\Big|}{(1-|\varphi(z)|^2)^{k+\frac{\alpha+2}{p}}} < \varepsilon \tag{4.4}$$

for each $k \in \{0,1,\dots,n+1\}$. From the fact $L_k < \infty$ for each $k \in \{0,1,\dots,n+1\}$ and (4.4), we have

$$I_k \le \varepsilon + \frac{L_k}{(1-\eta^2)^{k+\frac{\alpha+2}{p}}}. \tag{4.5}$$

From (4.5) and Theorem 3.1, we see that the operator $D^n M_u C_\varphi : A^p_\alpha \to \mathcal{B}^\Psi$ is bounded.

In order to prove that $D^n M_u C_\varphi : A^p_\alpha \to \mathcal{B}^\Psi$ is compact, by Lemma 2.1 we only need to prove that, if $\{f_i\}_{i\in\mathbb{N}}$ is a sequence in A^p_α such that $\sup_{i\in\mathbb{N}} \|f_i\|_{A^p_\alpha} \le M$ and $f_i \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $i \to \infty$, then

$$\lim_{i\to\infty} \|D^n M_u C_\varphi f_i\|_{\mathcal{B}^\Psi} = 0.$$

For such chosen ε and η, by using (4.4), Lemma 2.2 and Lemma 2.4, we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|\big(D^n M_u C_\varphi f_i\big)'(z)\big| \\
&= \sup_{z\in\mathbb{D}} \mu_\Psi(z)\Big|\sum_{k=0}^{n+1} f_i^{(k)}(\varphi(z)) \sum_{j=k}^{n+1} C^j_{n+1} u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sup_{z\in\mathbb{D}} \mu_\Psi(z) \sum_{k=0}^{n+1} \big|f_i^{(k)}(\varphi(z))\big|\Big|\sum_{j=k}^{n+1} C^j_{n+1} u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \big(\sup_{z\in K} + \sup_{z\in\mathbb{D}\setminus K}\big)\mu_\Psi(z) \sum_{k=0}^{n+1} \big|f_i^{(k)}(\varphi(z))\big|\Big|\sum_{j=k}^{n+1} C^j_{n+1} u^{(n+1-j)}(z) B_{j,k}\big(\varphi'(z), \dots, \varphi^{(j-k+1)}(z)\big)\Big| \\
&\le \sum_{k=0}^{n+1} L_k \sup_{|z|\le\eta} |f_i^{(k)}(z)| + M \sum_{k=0}^{n+1} C_k \varepsilon.
\end{aligned} \tag{4.6}$$

From (4.6), Lemma 2.1 and the fact $f_i \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$ implies that for each $k \in \mathbb{N}$, $f_i^{(k)} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $i \to \infty$, we finally get

$$\lim_{i\to\infty} \sup_{z\in\mathbb{D}} \mu_\Psi(z)\big|(D^n M_u C_\varphi f_i)'(z)\big| = 0. \tag{4.7}$$

It is clear that

$$\lim_{i\to\infty} \big|(D^n M_u C_\varphi f_i)(0)\big| = 0. \tag{4.8}$$

From (4.7) and (4.8) we obtain

$$\lim_{i\to\infty}\|D^nM_uC_\varphi f_i\|_{\mathcal{B}^\Psi}=0.$$

This shows that the operator $D^nM_uC_\varphi : A^p_\alpha \to \mathcal{B}^\Psi$ is compact. □

Remark 4.1. *Since $D^nC_\varphi M_u = D^nM_{u\circ\varphi}C_\varphi$, the characterization of the compactness of $D^nC_\varphi M_u : A^p_\alpha \to \mathcal{B}^\Psi$ can be directly obtained from Theorem 4.1. So we omit here.*

Similar to Theorems 3.2, 3.3 and 3.4, we have the following results.

Theorem 4.2. *Let $\alpha > -1$, $p \geq 1$, φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then the following statements hold.*

(i) The operator $C_\varphi D^nM_u : A^p_\alpha \to \mathcal{B}^\Psi$ is compact.

(ii) The functions u and φ satisfy the following conditions:

$$\sup_{z\in\mathbb{D}}\mu_\Psi(z)|u^{(n+1-k)}(\varphi(z))||\varphi'(z)|<\infty$$

and

$$\lim_{|\varphi(z)|\to1^-}\frac{\mu_\Psi(z)|u^{(n+1-k)}(\varphi(z))||\varphi'(z)|}{(1-|\varphi(z)|^2)^{k+\frac{\alpha+2}{p}}}=0$$

for each $k \in \{0, 1, \ldots, n+1\}$.

Theorem 4.3. *Let $\alpha > -1$, $p \geq 1$, φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then the following statements hold.*

(i) The operator $M_uD^nC_\varphi : A^p_\alpha \to \mathcal{B}^\Psi$ is compact.

(ii) The functions u and φ satisfy the following conditions:

$$\sup_{z\in\mathbb{D}}\mu_\Psi(z)\big|u'(z)B_{n,k}(\varphi'(z),\ldots,\varphi^{(n-k+1)}(z))+u(z)B_{n+1,k}(\varphi'(z),\ldots,\varphi^{(n-k+2)}(z))\big|<\infty,$$

$$\lim_{|\varphi(z)|\to1^-}\frac{\mu_\Psi(z)\big|u'(z)B_{n,k}(\varphi'(z),\ldots,\varphi^{(n-k+1)}(z))+u(z)B_{n+1,k}(\varphi'(z),\ldots,\varphi^{(n-k+2)}(z))\big|}{(1-|\varphi(z)|^2)^{k+\frac{\alpha+2}{p}}}=0$$

for each $k \in \{0, 1, \ldots, n\}$,

$$\sup_{z\in\mathbb{D}}\mu_\Psi(z)|u(z)||\varphi'(z)|^{n+1}<\infty,$$

and

$$\lim_{|\varphi(z)|\to1^-}\frac{\mu_\Psi(z)|u(z)||\varphi'(z)|^{n+1}}{(1-|\varphi(z)|^2)^{n+1+\frac{\alpha+2}{p}}}=0.$$

Theorem 4.4. *Let $\alpha > -1$, $p \geq 1$, φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. Then the following statements hold.*

(i) The operator $M_u C_\varphi D^n : A^p_\alpha \to \mathcal{B}^\Psi$ is compact.

(ii) The functions u and φ are such that $u \in \mathcal{B}^\Psi$,

$$\sup_{z\in\mathbb{D}} \mu_\Psi(z)|u(z)||\varphi'(z)| < \infty,$$

$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu_\Psi(z)|u'(z)|}{(1-|\varphi(z)|^2)^{n+\frac{\alpha+2}{p}}} = 0,$$

and

$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu_\Psi(z)|u(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1+\frac{\alpha+2}{p}}} = 0.$$

Remark 4.2. *Noticing that $C_\varphi M_u D^n = M_{u\circ\varphi} C_\varphi D^n$, we can obtain the characterization of the compactness of $C_\varphi M_u D^n : A^p_\alpha \to \mathcal{B}^\Psi$ from Theorem 4.4. Here we omit.*

References

[1] Attele, K., Toeplitz and Hankel operators on Bergman spaces, *Hokkaido Math. J.,* 1992, 21:279-293.

[2] Colonna, F. and Li, S., Weighted composition operators from the minimal Möbius invariant space into the Bloch space, *Mediter. J. Math.,* 2013, 10(1):395-409.

[3] Cowen, C. C. and MacCluer, B. D., Composition operators on spaces of analytic functions, Boca Roton: CRC Press, 1995.

[4] Esmaeili, K. and Lindström, M., Weighted composition operators between Zygmund type spaces and their essential norms, *Integral Equ. Oper. Theory.,* 2013, 75:473-490.

[5] Hibschweiler, R. A. and Portnoy, N., Composition followed by differentiation between Bergman and Hardy spaces, *Rocky Mountain J. Math.,* 2005, 35(3):843-855.

[6] Jiang, Z. J., On a class of operators from weighted Bergman spaces to some spaces of analytic functions, *Taiwan. J. Math.,* 2011, 15(5):2095-2121.

[7] Jiang, Z. J., On a product-type operator from weighted Bergman-Orlicz space to some weighted type spaces, *Appl. Math. Comput.,* 2015, 256:37-51.

[8] Jiang, Z. J., Completely continuous composition operators on Orlicz spaces, *Adv. Math. (China),* 2015, 44(1):111-116 (in Chinese).

[9] Krantz, S. and Stević, S., On the iterated logarithmic Bloch space on the unit ball, *Nonlinear Anal.,* 2009, 71:1772-1795.

[10] Liu, Y. and Yu, Y., Products of composition, multiplication and radial derivative operators from logarithmic Bloch spaces to weighted-type spaces on the unit ball, *J. Math. Anal. Appl.,* 2015, 423(1):76-93.

[11] Li, S. and Stević, S., Composition followed by differentiation between Bloch type spaces, *J. Comput. Anal. Appl.,* 2007, 9(2):195-205.

[12] Li, S. and Stević, S., Products of composition and integral type operators from H^∞ to the Bloch space, *Complex Var. Elliptic Equ.,* 2008, 53(5):463-474.

[13] Madigan, K. and Matheson, A., Compact composition operators on the Bloch space, *Trans. Amer. Math. Soc.,* 1995, 347:2679-2687.

[14] Ohno, S., Products of composition and differentiation on Bloch spaces, *Bull. Korean Math. Soc.,* 2009, 46(6):1135-1140.

[15] Ramos Fernández, J. C., Composition operators on Bloch-Orlicz type spaces, *Appl. Math. Comput.,* 2010, 217:3392-3402.

[16] Sehba, B. and Stević, S., On some product-type operators from Hardy-Orlicz and Bergman-Orlicz spaces to weighted-type spaces, *Appl. Math. Comput.,* 2014, 233:565-581.

[17] Sharma, A. K., Products of composition multiplication and differentiation between Bergman and Bloch type spaces, *Turkish. J. Math.,* 2011, 35:275-291.

[18] Stević, S., Norm and essential norm of composition followed by differentiation from α-Bloch spaces to H_μ^∞, *Appl. Math. Comput.,* 2009, 207:225-229.

[19] Stević, S., On an integral-type operator from logarithmic Bloch-type and mixed-norm spaces to Bloch-type spaces, *Nonlinear Anal.,* 2009, 71:6323-6342.

[20] Stević, S., Products of composition and differentiation operators on the weighted Bergman space, *Bull. Belg. Math. Soc.,* 2009, 16:623-635.

[21] Stević, S., Products of integral-type operators and composition operators from the mixed norm space to Bloch-type spaces, *Siberian Math. J.,* 2009, 50(4):726-736.

[22] Stević, S., Weighted differentiation composition operators from H^∞ and Bloch spaces to nth weighted-type spaces on the unit disk, *Appl. Math. Comput.,* 2010, 216:3634-3641.

[23] Stević, S., Chen, R. and Zhou, Z., Weighted composition operators between Bloch type spaces in the polydisc, *Mat. Sb.,* 2010, 201(2):289-319.

[24] Stević, S., Sharma, A. K. and Bhat, A., Products of multiplication composition and differentiation operators on weighted Bergman spaces, *Appl. Math. Comput.,* 2011, 217:8115-8125.

[25] Stević, S., Sharma, A. K. and Bhat, A., Essential norm of multiplication composition and differentiation operators on weighted Bergman spaces, *Appl. Math. Comput.,* 2011, 218:2386-2397.

[26] Zhang, X.J., Guan, Y. and Li, M., Composition operators between logarithmic weighted Bloch type spaces on the unit ball, *Adv. Math. (China),* 2015, 44(3):553-561 (in Chinese).

[27] Zhao, L. K., Fredholm weighted composition operators on the weighted Dirichlet space, *Adv. Math. (China),* 2014, 43(3):419-424 (in Chinese).

[28] Zhu, K., Operator theory in function space, New York: Marcel Dekker, 1990.

[29] Zhu, K., Spaces of holomorphic functions in the unit ball, New York: Springer-Verlag, 2005.

[30] Zhu, X., Products of differentiation, composition and multiplication operator from Bergman type spaces to Bers spaces, *Integral Transforms Spec. Funct.,* 2007, 18:223-231.

[31] Zhu, X., Generalized weighted composition operators from Bloch spaces into Bers-type spaces, *Filomat.,* 2012, 26:1163-1169.

On some recent fixed point results for α-admissible mappings in b-metric spaces

Huaping Huang[1], Guantie Deng[1,*], Zhanmei Chen[1], Stojan Radenović[2]

1. School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, PR China

2. Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120, Beograd, Serbia

Abstract: The purpose of this paper is to present some fixed point theorems for weak α-admissible mappings type in the setting of b-metric spaces. The results greatly optimize and improve some fixed point results in the existing literature. Moreover, we highlight our assertions by utilizing an example. In addition, we use our results to obtain the existence of solution for a class of nonlinear integral equations.

Keywords: α-admissible mapping, α-contraction mapping, rational α-Geraghty contraction of type, fixed point, integral equation

1 Introduction

Since Polish mathematician Banach proved the well-known Banach contraction mapping principle in metric spaces in 1922 (see [1]), fixed point theory occupies a prominent place in strong research activity. Due to its applications in finding the existence of solutions for the nonlinear Volterra integral equations, nonlinear integro-differential equations and existence of equilibria in game theory as well, it has become the most celebrated tool in nonlinear analysis. During the past decades, scholars extend this principle towards different spaces, such as G-metric spaces, 2-metric spaces, fuzzy metric spaces, probabilistic metric spaces, cone metric spaces, partial metric spaces, modular metric spaces, b-metric spaces, *etc* (see [2-10]). Whereas, the most influential spaces among them, *i.e.*, b-metric spaces, or metric type spaces called by some authors, introduced by Bakhtin [9] or Czerwik [10], have a rapid development. Compared with other spaces, people are willing to deal with fixed point problems or the variational principle for single-valued or multi-valued operators in b-metric spaces, based on the fact that b-metrics have no continuity in general.

On the other hand, people fascinate fixed point results by substituting the Banach contractive mapping, such as Kannan contraction mapping, Chatterjea contraction mapping, α-ψ-contractive type mapping, cyclic contractive mapping, multivalued contraction

*Correspondence: denggt@bnu.edu.cn (G. Deng)

mapping, and so on (see [6-14]). Recently, Samet *et al.* [12] introduced the notion of α-admissible mapping in the framework of metric spaces, and very recently, Sintunavarat [15] introduced the concepts of α-admissible mapping type S, weak α-admissible mapping, weak α-admissible mapping type S, as some generalizations of α-admissible mapping. Moreover, [15] proved fixed point theorems based on his new types of α-admissibility in the setup of b-metric spaces. In this paper, inspired by [15], we introduce the notion of α-admissibility mapping, and obtain some fixed point theorems, as compared to the main results of [15], with much simpler conditions and more straightforward proofs. Furthermore, we cope with some fixed point results for the mappings on rational α-Geraghty contraction of type in terms of α-admissibility in b-metric spaces. In addition, we give an application in the existence of a solution for a class of nonlinear integral equations. Our conditions are weak and applicable compared to the applications from [15].

For the sake of reader, the following definitions and results will be needed in the sequel.

Definition 1.1([16]). A mapping $\varphi : [0, \infty) \to [0, \infty)$ is said to be an altering distance function if it holds:

(1) φ is nondecreasing and continuous;

(2) $\varphi(t) = 0$ if and only if $t = 0$.

Definition 1.2([15]). Let X be a nonempty set and $s \geq 1$ a given real number. Let $\alpha : X \times X \to [0, \infty)$ and $f : X \to X$ be mappings. We say f is an α-admissible mapping type S if for all $x, y \in X$, $\alpha(x, y) \geq s$ leads to $\alpha(fx, fy) \geq s$. In particular, f is called α-admissible mapping if $s = 1$.

Remark 1.3 Usually, use $\mathcal{A}(X, \alpha)$ and $\mathcal{A}_s(X, \alpha)$ to denote the collection of all α-admissible mappings on X and the collection of all α-admissible mappings type S on X. It is worth reminding that the class of α-admissible mappings and the class of α-admissible mappings type S are independent, in other words, $\mathcal{A}(X, \alpha) \neq \mathcal{A}_s(X, \alpha)$ in general case.

Definition 1.4([15]). Let X be a nonempty set and $s \geq 1$ a given real number. Let $\alpha : X \times X \to [0, \infty)$ and $f : X \to X$ be mappings. We say f is a weak α-admissible mapping type S if for all $x \in X$, $\alpha(x, fx) \geq s$ leads to $\alpha(fx, ffx) \geq s$. In particular, f is called weak α-admissible mapping if $s = 1$.

Remark 1.5. Customarily, utilize $\mathcal{WA}(X, \alpha)$ and $\mathcal{WA}_s(X, \alpha)$ to denote the collection of all weak α-admissible mappings on X and the collection of all weak α-admissible mappings type S on X. Clearly, $\mathcal{A}(X, \alpha) \subseteq \mathcal{WA}(X, \alpha)$ and $\mathcal{A}_s(X, \alpha) \subseteq \mathcal{WA}_s(X, \alpha)$.

Definition 1.6([10]). Let X be a nonempty set and $s \geq 1$ a real number. A mapping $d : X \times X \to [0, \infty)$ is called a b-metric if for all $x, y, z \in X$, the following conditions are satisfied:

(b1) $d(x, y) = 0$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$;

(d3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, (X, d) is called a b-metric space.

Definition 1.7([17]). Let (X, d) be a b-metric space, $x \in X$ and $\{x_n\}$ a sequence in X.

Then we say

(i) $\{x_n\}$ b-converges to x if $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(ii) $\{x_n\}$ is a b-Cauchy sequence if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

(iii) (X, d) is b-complete if every b-Cauchy sequence is b-convergent in X.

(iv) a function $f : X \to Y$ is b-continuous at a point $x \in X$ if $\{x_n\} \subset X$ b-converges to x, then $\{fx_n\}$ b-converges to fx, where (Y, ρ) is a b-metric space.

Throughout this paper, unless otherwise specified, X is a nonempty set, $f : X \to X$ is a mapping, $\mathrm{Fix}(f)$ denotes the set of all fixed points of f on X, that is,

$$\mathrm{Fix}(f) := \{x \in X | fx = x\}.$$

Also, for each elements x and y in a b-metric space (X, d) with coefficient $s \geq 1$, let

$$M_s(x, y) := \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2s}\right\}.$$

Lemma 1.8([18]). Let (X, d) be a b-metric space with coefficient $s \geq 1$ and let $\{x_n\}$ and $\{y_n\}$ be b-convergent to points $x, y \in X$, respectively. Then we have

$$\frac{1}{s^2}d(x, y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x, y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s}d(x, z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq sd(x, z).$$

Definition 1.9([15]). Let (X, d) be a b-metric space with coefficient $s \geq 1$, let $\alpha : X \times X \to [0, \infty)$ be a mapping and let $\psi, \varphi : [0, \infty) \to [0, \infty)$ be two altering distance functions. A mapping $f : X \to X$ is said to be an $(\alpha, \psi, \varphi)_s$-contraction mapping if

$$x, y \in X \text{ with } \alpha(x, y) \geq s \Longrightarrow \psi(s^3 d(fx, fy)) \leq \psi(M_s(x, y)) - \varphi(M_s(x, y)). \tag{1.1}$$

In this case, write $\Omega_s(X, \alpha, \psi, \varphi)$ as the collection of all $(\alpha, \psi, \varphi)_s$-contraction mappings.

Theorem 1.10([15]). Let (X, d) be a b-complete b-metric space with coefficient $s \geq 1$, let $\psi, \varphi : [0, \infty) \to [0, \infty)$ be two altering distance functions and let $\alpha : X \times X \to [0, \infty)$ and $f : X \to X$ be given mappings. Suppose that the following conditions hold:

(S_1) $f \in \Omega_s(X, \alpha, \psi, \varphi) \cap \mathcal{WA}_s(X, \alpha)$;

(S_2) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq s$;

(S_3) α has a transitive property type S, that is, for $x, y, z \in X$,

$$\alpha(x, y) \geq s \text{ and } \alpha(y, z) \geq s \Rightarrow \alpha(x, z) \geq s;$$

(S_4) f is b-continuous.

Then $\mathrm{Fix}(f) \neq \emptyset$.

Theorem 1.11([15]). Let (X, d) be a b-complete b-metric space with coefficient $s \geq 1$, let $\psi, \varphi : [0, \infty) \to [0, \infty)$ be two altering distance functions and let $\alpha : X \times X \to [0, \infty)$ and $f : X \to X$ be given mappings. Suppose that the following conditions hold:

(S_1) $f \in \Omega_s(X, \alpha, \psi, \varphi) \cap \mathcal{WA}_s(X, \alpha)$;
(S_2) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq s$;
(S_3) α has a transitive property type S;
$(\widetilde{S_4})$ X is α_s-regular, that is, if $\{x_n\}$ is a sequence in X such that

$$\alpha(x_n, x_{n+1}) \geq s$$

for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq s$ for all $n \in \mathbb{N}$.
Then $\mathrm{Fix}(f) \neq \emptyset$.

Corollary 1.12([15]). Let (X, d) be a b-complete b-metric space with coefficient $s \geq 1$, let $\psi, \varphi : [0, \infty) \to [0, \infty)$ be two altering distance functions and let $\alpha : X \times X \to [0, \infty)$ and $f : X \to X$ be given mappings. Suppose that the following conditions hold:
$(\widetilde{S_1})$ $f \in \Omega_s(X, \alpha, \psi, \varphi) \cap \mathcal{A}_s(X, \alpha)$;
(S_2) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq s$;
(S_3) α has a transitive property type S;
(S_4) f is b-continuous.
Then $\mathrm{Fix}(f) \neq \emptyset$.

Corollary 1.13([15]). Let (X, d) be a b-complete b-metric space with coefficient $s \geq 1$, let $\psi, \varphi : [0, \infty) \to [0, \infty)$ be two altering distance functions and let $\alpha : X \times X \to [0, \infty)$ and $f : X \to X$ be given mappings. Suppose that the following conditions hold:
$(\widetilde{S_1})$ $f \in \Omega_s(X, \alpha, \psi, \varphi) \cap \mathcal{A}_s(X, \alpha)$;
(S_2) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq s$;
(S_3) α has a transitive property type S;
$(\widetilde{S_4})$ X is α_s-regular.
Then $\mathrm{Fix}(f) \neq \emptyset$.

2 Main results

Definition 2.1. Let (X, d) be a b-metric space with coefficient $s \geq 1$, let $\alpha : X \times X \to [0, \infty)$ be a mapping and $\varepsilon > 1$ be a constant. A mapping $f : X \to X$ is said to be an α-contraction mapping if

$$x, y \in X \text{ with } \alpha(x, y) \geq s \Longrightarrow s^{\varepsilon} d(fx, fy) \leq M_s(x, y). \tag{2.1}$$

In this case, write $\Omega_s(X, \alpha)$ as the collection of all α-contraction mappings.

Theorem 2.2. Let (X, d) be a b-complete b-metric space with coefficient $s > 1$. Let $\alpha : X \times X \to [0, \infty)$ and $f : X \to X$ be given mappings. Suppose that the following conditions hold:
(S_1) $f \in \Omega_s(X, \alpha) \cap \mathcal{WA}_s(X, \alpha)$;
(S_2) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq s$;
(S_3) f is b-continuous.
Then $\mathrm{Fix}(f) \neq \emptyset$.

Proof. By (S_2), for $x_0 \in X$, construct a Picard iteration sequence $\{x_n\}$ satisfying $x_{n+1} = fx_n$, $n \in \mathbb{N}$. Assume that $x_{n_0} = x_{n_0+1}$ for some n_0, then $\mathrm{Fix}(f) = \{x_{n_0}\} \neq \emptyset$, in this case, the conclusion is satisfied. So set $x_n \neq x_{n+1}$ for all n, that is, $d(x_n, x_{n+1}) > 0$ for all n. Let us prove the following inequality:

$$d(x_{n+1}, x_{n+2}) \leq \lambda d(x_n, x_{n+1}), \tag{2.2}$$

where $\lambda \in [0, \frac{1}{s})$ is a constant.

Indeed, in view of $f \in \mathcal{WA}_s(X, \alpha)$ and $\alpha(x_0, fx_0) \geq s$, it implies that

$$\alpha(x_1, x_2) = \alpha(fx_0, ffx_0) \geq s.$$

Repeating this process, we make a conclusion that

$$\alpha(x_n, x_{n+1}) \geq s$$

for all n. Making the most of (2.1), we have

$$\begin{aligned} s^{\varepsilon} d(x_{n+1}, x_{n+2}) &= s^{\varepsilon} d(fx_n, fx_{n+1}) \\ &\leq M_s(x_n, x_{n+1}) \\ &= \max\Bigg\{ d(x_n, x_{n+1}), d(x_n, fx_n), d(x_{n+1}, fx_{n+1}), \\ &\quad \frac{d(x_n, fx_{n+1}) + d(x_{n+1}, fx_n)}{2s} \Bigg\} \\ &= \max\Bigg\{ d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \\ &\quad \frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2s} \Bigg\} \\ &\leq \max\left\{ d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2} \right\} \\ &= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}. \end{aligned} \tag{2.3}$$

If $d(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$, then by (2.3), it follows that

$$s^{\varepsilon} d(x_{n+1}, x_{n+2}) \leq d(x_{n+1}, x_{n+2}).$$

Hence, $d(x_{n+1}, x_{n+2}) = 0$, it is a contradiction. If $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1})$, then by (2.3), it establishes that

$$s^{\varepsilon} d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}).$$

As a result, (2.2) holds, where $\lambda = \frac{1}{s^{\varepsilon}} \in [0, \frac{1}{s})$.

Now by [11, Lemma 3.1], taking advantage of (2.2), we claim that $\{x_n\}$ is a b-Cauchy sequence. Since (X, d) is b-complete, we know that $\{x_n\}$ b-converges to some point $x \in X$. Finally, we show $x \in \mathrm{Fix}(f)$. Actually, by using (S_3), it is not hard to verify that

$$d(fx, x) \leq s[d(fx, fx_n) + d(fx_n, x)] = s[d(fx, fx_n) + d(x_{n+1}, x)] \to 0 \text{ as } n \to \infty.$$

Therefore, $d(fx, x) = 0$, that is to say, $x \in \mathrm{Fix}(f)$. □

Theorem 2.3. Let (X, d) be a b-complete b-metric space with coefficient $s > 1$. Let $\alpha : X \times X \to [0, \infty)$ and $f : X \to X$ be given mappings. Suppose that the following conditions hold:

(S_1) $f \in \Omega_s(X, \alpha) \cap \mathcal{WA}_s(X, \alpha)$;

(S_2) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq s$;

$(\widetilde{S_3})$ X is α_s-regular.

Then $\mathrm{Fix}(f) \neq \emptyset$.

Proof. Making full use of the proof of Theorem 2.2, we obtain a sequence $\{x_n\}$ satisfying $x_{n+1} = fx_n \to x \in X$ as $n \to \infty$. Then by $(\widetilde{S_3})$, we get $\alpha(x_n, x) \geq s$ for all $n \in \mathbb{N}$. By virtue of (S_1), we have

$$\begin{aligned} s^{\varepsilon} d(fx_n, fx) &\leq M_s(x_n, x) \\ &= \max\Big\{ d(x_n, x), d(x_n, fx_n), d(x, fx), \\ &\qquad \frac{d(x_n, fx) + d(x, fx_n)}{2s} \Big\} \\ &\leq \max\Big\{ d(x_n, x), s[d(x_n, x) + d(x_{n+1}, x)], d(x, fx), \\ &\qquad \frac{d(x_n, x) + d(x, fx)}{2} + \frac{d(x, x_{n+1})}{2s} \Big\} \\ &\to \max\left\{0, 0, d(x, fx), \frac{d(x, fx)}{2}\right\} = d(x, fx) \ (n \to \infty), \end{aligned}$$

which implies that

$$\lim_{n\to\infty} d(fx_n, fx) \leq \frac{1}{s^{\varepsilon}} d(x, fx). \tag{2.4}$$

Note that

$$\frac{1}{s} d(x, fx) \leq d(x, fx_n) + d(fx_n, fx) = d(x, x_{n+1}) + d(fx_n, fx). \tag{2.5}$$

Taking the limit as the above inequality (2.5) and utilizing (2.4), we speculate that

$$\frac{1}{s} d(x, fx) \leq \frac{1}{s^{\varepsilon}} d(x, fx),$$

which follows that $d(x, fx) = 0$, that is, $x \in \mathrm{Fix}(f)$. □

Corollary 2.4. Let (X, d) be a b-complete b-metric space with coefficient $s > 1$. Let $\alpha : X \times X \to [0, \infty)$ and $f : X \to X$ be given mappings. Suppose that the following conditions hold:

$(\widetilde{S_1})$ $f \in \Omega_s(X, \alpha) \cap \mathcal{A}_s(X, \alpha)$;

(S_2) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq s$;

(S_3) f is b-continuous.

Then $\mathrm{Fix}(f) \neq \emptyset$.

Corollary 2.5. Let (X,d) be a b-complete b-metric space with coefficient $s>1$. Let $\alpha : X\times X \to [0,\infty)$ and $f : X \to X$ be given mappings. Suppose that the following conditions hold:

$(\widetilde{S}_1)$ $f \in \Omega_s(X,\alpha)\cap \mathcal{A}_s(X,\alpha)$;

(S_2) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq s$;

$(\widetilde{S}_3)$ X is α_s-regular.

Then $\mathrm{Fix}(f) \neq \emptyset$.

Remark 2.6. Theorem 2.2, Theorem 2.3, Corollary 2.4 and Corollary 2.5 greatly optimize and improve Sintunavarat's theorems, *i.e.*, Theorem 1.10, Theorem 1.11, Corollary 1.12 and Corollary 1.13, respectively. Actually, on the one hand, compared with (1.1), (2.1) not only deletes the limitation of the altering distance functions ψ and φ, but also it dispenses with an item $\varphi(M_s(x,y))$ which makes the condition become much wider. These are some great improvements. Moreover, our index $\varepsilon > 1$ is arbitrary, and it clearly contains $\varepsilon = 3$. Hence, our range $\varepsilon > 1$ is much larger and more applicable. On the other hand, our theorems dismiss the condition of transitive property type S for the mapping α. That is to say, the conditions of our theorems are weaker than Sintunavarat's theorems. Therefore, our conclusions may be more convenient than Sintunavarat's in applications.

Remark 2.7. From the proofs of our theorems, it is easy to see that we do not use Lemma 1.8. Our proofs are much simpler since we do not refer to b-discontinuity of b-metric. Whereas, in order to overcome the difficulty of the b-discontinuity of b-metric, the proofs of Sintunavarat's theorems are very comprehensive based on the fact of depending on Lemma 1.8 strongly.

Example 2.8. Let $X = \mathbb{R}$ and define

$$d(x,y) = |x-y|^2$$

for all $x, y \in X$. Then (X,d) is a b-complete b-metric space with coefficient $s = 2$. Define mappings $f : X \to X$ and $\alpha : X \times X \to [0,\infty)$ by

$$fx = \begin{cases} \frac{x}{4}, & x \in [0, \frac{16}{3}], \\ \frac{1}{8}x + \frac{2}{3}, & x \in (\frac{16}{3}, \infty), \end{cases}$$

and

$$\alpha(x,y) = \begin{cases} \frac{5}{4} + \frac{1}{|x-y|}, & x,y \in [0, \frac{16}{3}], \\ 0, & \text{otherwise}. \end{cases}$$

Let us prove $f \in \Omega_s(X,\alpha)$. Actually, assume that $x, y \in X$ with $\alpha(x,y) \geq s = 2$ and hence $x, y \in [0, \frac{16}{3}]$ with $|x-y| \leq \frac{4}{3}$. Let $1 < \varepsilon \leq 4$ be a constant. Then

$$\begin{aligned} 2^{\varepsilon} d(fx,fy) &= 2^{\varepsilon}|fx-fy|^2 \\ &= 2^{\varepsilon}\left|\frac{x}{4}-\frac{y}{4}\right|^2 \\ &= 2^{\varepsilon-4}|x-y|^2 \\ &\leq M_s(x,y). \end{aligned}$$

As a consequence, $f \in \Omega_s(X, \alpha)$.

We also can verify $f \in \mathcal{WA}_s(X, \alpha)$. Indeed, if $x \in X$ such that

$$\alpha(x, fx) \geq s = 2,$$

then $x, fx \in [0, \frac{16}{3}]$ and $|x - fx| \leq \frac{4}{3}$. Thus $x \in [0, \frac{16}{9}]$. This indicates that $ffx \in [0, \frac{1}{9}]$ and hence

$$\alpha(fx, ffx) = \frac{5}{4} + \frac{16}{3|x|} \geq \frac{17}{4} > s = 2.$$

Otherwise, it is obvious that f is b-continuous and there exists $x_0 = 1$ such that

$$\alpha(x_0, fx_0) = \alpha(1, f1) = \frac{5}{4} + \frac{1}{|1 - f1|} = \frac{31}{12} \geq 2 = s.$$

Consequently, all the conditions of Theorem 2.2 hold. Thus $\mathrm{Fix}(f) = \{0\} \neq \emptyset$.

However, we cannot use Theorem 1.10 to get $\mathrm{Fix}(f) \neq \emptyset$, since α is unsuitable for the condition (S_3) of this theorem. Indeed, put $x = 4$, $y = 3$, $z = 2$. Though $\alpha(x, y) = \frac{9}{4} > 2$ and $\alpha(y, z) = \frac{9}{4} > 2$, whereas, $\alpha(x, z) = \frac{7}{4} < 2$. So (S_3) does not hold in this example. In other words, Theorem 2.2 is more superior than Theorem 1.10. □

In the sequel, let $s \geq 1$ be a constant and let $\mathcal{F}_s$ denote the class of all functions $\beta : [0, \infty) \to [0, \frac{1}{s})$ satisfying the following condition:

$$\limsup_{n\to\infty} \beta(t_n) = \frac{1}{s} \text{ implies that } t_n \to 0 \text{ as } n \to \infty.$$

Definition 2.9. Let (X, d) be a b-metric space with coefficient $s \geq 1$, and let $\alpha : X \times X \to [0, \infty)$ be a function. A mapping $f : X \to X$ is called a rational α-Geraghty contraction of type $\mathbf{I}_{\varepsilon,\beta}$ if there exist $\varepsilon > 0$ and $\beta \in \mathcal{F}_s$ such that

$$x, y \in X \text{ with } \alpha(x, y) \geq s \Longrightarrow \alpha(x, y)s^{\varepsilon} d(fx, fy) \leq \beta(M_I(x, y))M_I(x, y), \tag{2.6}$$

where

$$M_I(x, y) = \max\left\{d(x, y), \frac{d(x, fx)\, d(y, fy)}{1 + d(x, y)}, \frac{d(x, fx)\, d(y, fy)}{1 + d(fx, fy)}\right\}.$$

Definition 2.10. Let (X, d) be a b-metric space with coefficient $s \geq 1$, and let $\alpha : X \times X \to [0, \infty)$ be a function. A mapping $f : X \to X$ is called a rational α-Geraghty contraction of type $\mathbf{II}_{\varepsilon,\beta}$ if there exist $\varepsilon > 0$ and $\beta \in \mathcal{F}_s$ such that

$$x, y \in X \text{ with } \alpha(x, y) \geq s \Longrightarrow \alpha(x, y)s^{\varepsilon} d(fx, fy) \leq \beta(M_{II}(x, y))M_{II}(x, y), \tag{2.7}$$

where

$$M_{II}(x, y) = \max\left\{d(x, y), \frac{d(x, fx)\, d(x, fy) + d(y, fy)\, d(y, fx)}{1 + s[d(x, fx) + d(y, fy)]}, \right.$$
$$\left. \frac{d(x, fx)\, d(x, fy) + d(y, fy)\, d(y, fx)}{1 + s[d(x, fy) + d(y, fx)]}\right\}.$$

Definition 2.11. Let (X,d) be a b-metric space with coefficient $s \geq 1$, and let $\alpha : X \times X \to [0,\infty)$ be a function. A mapping $f : X \to X$ is called a rational α-Geraghty contraction of type $\mathbf{III}_{\varepsilon,\beta}$ if there exist $\varepsilon > 0$ and $\beta \in \mathcal{F}_s$ such that

$$x, y \in X \text{ with } \alpha(x,y) \geq s \Longrightarrow \alpha(x,y)s^{\varepsilon}d(fx,fy) \leq \beta(M_{III}(x,y))M_{III}(x,y), \tag{2.8}$$

where

$$M_{III}(x,y) = \max\left\{d(x,y), \frac{d(x,fx)\,d(y,fy)}{1+s\left[d(x,y)+d(x,fy)+d(y,fx)\right]}, \frac{d(x,fy)\,d(x,y)}{1+sd(x,fx)+s^3\left[d(y,fx)+d(y,fy)\right]}\right\}.$$

Theorem 2.12. Let (X,d) be a b-complete b-metric space with coefficient $s > 1$, and let $\alpha : X \times X \to [0,\infty)$ be a function and $f : X \to X$ be a mapping. Suppose that the following conditions hold:

(i) f is a rational α-Geraghty contraction of type $\mathbf{I}_{\varepsilon,\beta}$ (resp. type $\mathbf{II}_{\varepsilon,\beta}$ or type $\mathbf{III}_{\varepsilon,\beta}$);

(ii) $f \in \mathcal{A}_s(X,\alpha)$ and there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq s$;

(iii) f is b-continuous or X is α_s-regular.

Then $\mathrm{Fix}(f) \neq \emptyset$.

Proof. By (ii) and the proof of Theorem 2.2, we can construct a Picard iteration sequence $\{x_n\}$ satisfying $x_{n+1} = fx_n$ and

$$\alpha(x_n, x_{n+1}) \geq s$$

for all $n \in \mathbb{N}$. Let us prove that

$$d(x_{n+1}, x_{n+2}) \leq \lambda d(x_n, x_{n+1}) \tag{2.9}$$

for all $n \in \mathbb{N}$, where $\lambda \in [0, \frac{1}{s})$.

First of all, let f be a rational α-Geraghty contraction of type $\mathbf{I}_{\varepsilon,\beta}$. Then by (2.6), we have

$$\begin{aligned} s^{\varepsilon}d(x_{n+1},x_{n+2}) &= s^{\varepsilon}d(fx_n, fx_{n+1}) \\ &\leq \alpha(x_n,x_{n+1})s^{\varepsilon}d(fx_n,fx_{n+1}) \\ &\leq \beta(M_I(x_n,x_{n+1}))M_I(x_n,x_{n+1}) \\ &\leq \frac{1}{s}M_I(x_n,x_{n+1}), \end{aligned} \tag{2.10}$$

which follows that

$$\begin{aligned} &s^{\varepsilon+1}d(x_{n+1},x_{n+2}) \leq M_I(x_n,x_{n+1}) \\ &= \max\left\{d(x_n,x_{n+1}), \frac{d(x_n,x_{n+1})\,d(x_{n+1},x_{n+2})}{1+d(x_n,x_{n+1})}, \frac{d(x_n,x_{n+1})\,d(x_{n+1},x_{n+2})}{1+d(x_{n+1},x_{n+2})}\right\} \\ &\leq \max\left\{d(x_n,x_{n+1}), \frac{d(x_n,x_{n+1})\,d(x_{n+1},x_{n+2})}{d(x_n,x_{n+1})}, \frac{d(x_n,x_{n+1})\,d(x_{n+1},x_{n+2})}{d(x_{n+1},x_{n+2})}\right\} \\ &= \max\left\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\right\}. \end{aligned} \tag{2.11}$$

If $d\left(x_n, x_{n+1}\right) \leq d(x_{n+1}, x_{n+2})$, then from (2.11), it leads to

$$d\left(x_{n+1}, x_{n+2}\right) \leq \frac{1}{s^{\varepsilon+1}} d\left(x_{n+1}, x_{n+2}\right) < d\left(x_{n+1}, x_{n+2}\right).$$

This is a contradiction. So

$$d(x_{n+1}, x_{n+2}) \leq \frac{1}{s^{\varepsilon+1}} d\left(x_n, x_{n+1}\right).$$

In this case, (2.9) is satisfied, where $\lambda = \frac{1}{s^{\varepsilon+1}} \in [0, \frac{1}{s})$.

Secondly, let f be a rational α-Geraghty contraction of type $\mathbf{II}_{\varepsilon,\beta}$. Then similarly by (2.10), we have

$$s^{\varepsilon} d(x_{n+1}, x_{n+2}) \leq \frac{1}{s} M_{II}(x_n, x_{n+1}),$$

which establishes that

$$\begin{aligned}
&s^{\varepsilon+1} d(x_{n+1}, x_{n+2}) \leq M_{II}(x_n, x_{n+1})\\
&= \max\left\{d\left(x_n, x_{n+1}\right), \frac{d\left(x_n, x_{n+1}\right) d\left(x_n, x_{n+2}\right) + d\left(x_{n+1}, x_{n+2}\right) d\left(x_{n+1}, x_{n+1}\right)}{1 + s\left[d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, x_{n+2}\right)\right]},\right.\\
&\qquad \left.\frac{d\left(x_n, x_{n+1}\right) d\left(x_n, x_{n+2}\right) + d\left(x_{n+1}, x_{n+2}\right) d\left(x_{n+1}, x_{n+1}\right)}{1 + s\left[d\left(x_n, x_{n+2}\right) + d\left(x_{n+1}, x_{n+1}\right)\right]}\right\}\\
&= \max\left\{d\left(x_n, x_{n+1}\right), \frac{d\left(x_n, x_{n+1}\right) d\left(x_n, x_{n+2}\right)}{1 + s\left[d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, x_{n+2}\right)\right]}, \frac{d\left(x_n, x_{n+1}\right) d\left(x_n, x_{n+2}\right)}{1 + sd\left(x_n, x_{n+2}\right)}\right\}\\
&\leq \max\left\{d\left(x_n, x_{n+1}\right), \frac{sd\left(x_n, x_{n+1}\right)\left[d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, x_{n+2}\right)\right]}{1 + s\left[d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, x_{n+2}\right)\right]}, \frac{d(x_n, x_{n+1}) d(x_n, x_{n+2})}{sd(x_n, x_{n+2})}\right\}\\
&\leq \max\left\{d\left(x_n, x_{n+1}\right), \frac{sd\left(x_n, x_{n+1}\right)\left[d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, x_{n+2}\right)\right]}{s\left[d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, x_{n+2}\right)\right]}, \frac{d(x_n, x_{n+1})}{s}\right\}\\
&\leq d\left(x_n, x_{n+1}\right).
\end{aligned}$$

Accordingly, (2.9) is also satisfied, where $\lambda = \frac{1}{s^{\varepsilon+1}} \in [0, \frac{1}{s})$.

Thirdly, let f be a rational α-Geraghty contraction of type $\mathbf{III}_{\varepsilon,\beta}$. Then similarly by (2.13), we have

$$s^{\varepsilon} d(x_{n+1}, x_{n+2}) \leq \frac{1}{s} M_{III}(x_n, x_{n+1}),$$

which implies that

$$\begin{aligned}
&s^{\varepsilon+1} d(x_{n+1}, x_{n+2}) \leq M_{III}(x_n, x_{n+1})\\
&= \max\left\{d\left(x_n, x_{n+1}\right), \frac{d\left(x_n, x_{n+1}\right) d\left(x_{n+1}, x_{n+2}\right)}{1 + s\left[d\left(x_n, x_{n+1}\right) + d\left(x_n, x_{n+2}\right) + d\left(x_{n+1}, x_{n+1}\right)\right]},\right.\\
&\qquad \left.\frac{d\left(x_n, x_{n+2}\right) d\left(x_n, x_{n+1}\right)}{1 + sd\left(x_n, x_{n+1}\right) + s^3\left[d\left(x_{n+1}, x_{n+1}\right) + d\left(x_{n+1}, x_{n+2}\right)\right]}\right\}\\
&\leq \max\left\{d\left(x_n, x_{n+1}\right), \frac{d\left(x_n, x_{n+1}\right) d\left(x_{n+1}, x_{n+2}\right)}{1 + s\left[d\left(x_n, x_{n+1}\right) + d\left(x_n, x_{n+2}\right)\right]},\right.\\
&\qquad \left.\frac{s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] d(x_n, x_{n+1})}{1 + sd(x_n, x_{n+1}) + s^3 d(x_{n+1}, x_{n+2})}\right\}\\
&\leq \max\left\{d\left(x_n, x_{n+1}\right), \frac{d\left(x_n, x_{n+1}\right) d\left(x_{n+1}, x_{n+2}\right)}{d\left(x_n, x_{n+1}\right)}, \frac{s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] d(x_n, x_{n+1})}{s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]}\right\}\\
&= \max\left\{d\left(x_n, x_{n+1}\right), d\left(x_{n+1}, x_{n+2}\right)\right\}.
\end{aligned}$$

A similar discussion as to (2.11), we can get (2.9), too.

In a word, under the conditions of (i) and (ii), we always acquire (2.9). As a consequence, by using [11, Lemma 3.1] and the b-completeness of (X, d), there exists a point $x \in X$ such that $x_n \to x$ as $n \to \infty$. Now by (iii), if f is b-continuous, then

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = fx,$$

that is, $x \in \mathrm{Fix}(f)$. If X is α_s-regular, then $\alpha(x_n, x) \geq s$. Put

$$M(x_n, x) \in \{M_I(x_n, x), M_{II}(x_n, x), M_{III}(x_n, x)\},$$

it follows immediately from (2.6), (2.7) and (2.8) that

$$\alpha(x_n, x)s^\varepsilon d(fx_n, fx) \leq \beta(M(x_n, x))M(x_n, x). \tag{2.12}$$

We show that

$$\lim_{n\to\infty} M(x_n, x) = 0. \tag{2.13}$$

Indeed, for one thing,

$$\begin{aligned} M_I(x_n, x) &= \max\left\{d(x_n, x), \frac{d(x_n, x_{n+1})\, d(x, fx)}{1 + d(x_n, x)}, \frac{d(x_n, x_{n+1})\, d(x, fx)}{1 + d(x_{n+1}, fx)}\right\} \\ &\to \max\{0, 0, 0\} = 0, \quad \text{as } n \to \infty. \end{aligned}$$

For another thing,

$$\begin{aligned} M_{II}(x_n, x) &= \max\left\{d(x_n, x), \frac{d(x_n, x_{n+1})\, d(x_n, fx) + d(x, fx)\, d(x, x_{n+1})}{1 + s\left[d(x_n, x_{n+1}) + d(x, fx)\right]},\right. \\ &\qquad \left.\frac{d(x_n, x_{n+1})\, d(x_n, fx) + d(x, fx)\, d(x, x_{n+1})}{1 + s\left[d(x_n, fx) + d(x, x_{n+1})\right]}\right\} \\ &\leq \max\left\{d(x_n, x), d(x_n, x_{n+1})\, s\left[d(x_n, x) + d(x, fx)\right] + d(x, fx)\, d(x, x_{n+1}),\right. \\ &\qquad \left. d(x_n, x_{n+1})\, s\left[d(x_n, x) + d(x, fx)\right] + d(x, fx)\, d(x, x_{n+1})\right\} \\ &\to \max\{0, 0, 0\} = 0, \quad \text{as } n \to \infty. \end{aligned}$$

For the third thing,

$$\begin{aligned} M_{III}(x_n, x) &= \max\left\{d(x_n, x), \frac{d(x_n, x_{n+1})d(x, fx)}{1 + s[d(x_n, x) + d(x_n, fx) + d(x, x_{n+1})]},\right. \\ &\qquad \left.\frac{d(x_n, fx)d(x_n, x)}{1 + sd(x_n, x_{n+1}) + s^3[d(x, x_{n+1}) + d(x, fx)]}\right\} \\ &\to \max\{0, 0, 0\} = 0, \quad \text{as } n \to \infty. \end{aligned}$$

Thus (2.13) holds.

Using (2.12) and (2.13), we speculate that

$$\begin{aligned} s^\varepsilon d(x, fx) &\leq s^{\varepsilon+1}[d(x, fx_n) + d(fx_n, fx)] \\ &\leq s^{\varepsilon+1} d(x, x_{n+1}) + s^{\varepsilon+1}\alpha(x_n, x)d(fx_n, fx) \\ &\leq s^{\varepsilon+1} d(x, x_{n+1}) + s\beta(M(x_n, x))M(x_n, x) \\ &\leq s^{\varepsilon+1} d(x, x_{n+1}) + M(x_n, x) \to 0, \quad \text{as } n \to \infty, \end{aligned}$$

which establishes that $d(x, fx) = 0$, that is to say, $x \in \mathrm{Fix}(f)$. □

3 Application

In this section, we prove an existence theorem for a solution of the following nonlinear integral equation by using our results in the previous section:

$$x(c) = \phi(c) + \int_a^b K(c, r, x(r))\mathrm{d}r, \tag{3.1}$$

where $a, b \in \mathbb{R}$, $x \in C[a, b]$ (the set of all continuous functions from $[a, b]$ into $\mathbb{R}$, $\phi : [a, b] \to \mathbb{R}$ and $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are given mappings.

The following theorem greatly improves Theorem 3.1 of [15] with simpler conditions, which illustrates the superiority of our results.

Theorem 3.1. Consider the nonlinear integral equation (3.1). Suppose that the following conditions hold:

(i) $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing in the third order;

(ii) there exists $p > 1$ satisfying the following condition: for each $r, c \in [a, b]$ and $x, y \in C[a, b]$ with $x(w) \le y(w)$ for all $w \in [a, b]$, we have

$$|K(c, r, x(r)) - K(c, r, y(r))| \le \zeta(c, r)|x(r) - y(r)|, \tag{3.2}$$

where $\zeta : [a, b] \times [a, b] \to [0, \infty)$ is a continuous function satisfying

$$\sup_{c \in [a,b]} \left(\int_a^b \zeta(c, r)^p \mathrm{d}r \right) \le \frac{1}{2^{\varepsilon p - \varepsilon}(b - a)^{p-1}}$$

and $\varepsilon > 1$ is a constant.

(iii) there exists $x_0 \in C[a, b]$ such that $x_0(c) \le \phi(c) + \int_a^b K(c, r, x_0(r))\mathrm{d}r$ for all $c \in [a, b]$.

Then the nonlinear integral equation (3.1) has a solution.

Proof. Put $X = C[a, b]$ and define a mapping $f : X \to X$ by

$$(fx)(c) = \phi(c) + \int_a^b K(c, r, x(r))\mathrm{d}r$$

for all $x \in X$ and $c \in [a, b]$. Define a mapping $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \sup_{c \in [a,b]} |x(c) - y(c)|^p \quad (p > 1)$$

for all $x, y \in X$. Then (X, d) is a b-complete b-metric space with coefficient $s = 2^{p-1}$. Define a mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 2^{p-1}, & x(c) \le y(c) \text{ for all } c \in [a, b], \\ \tau, & \text{otherwise}, \end{cases}$$

where $0 < \tau < 2^{p-1}$. Since K is nondecreasing in the third order, we get $f \in \mathcal{A}_s(X, \alpha) \subset \mathcal{WA}_s(X, \alpha)$. By (iii), it infers (S_2) in Theorem 2.3 is satisfied. Also, we get that condition $(\widetilde{S}_3)$ in Theorem 2.3 also holds (see [21]).

Finally, we show $f \in \Omega_s(X, \alpha)$. To prove this fact, we first choose $q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $x, y \in X$ such that $\alpha(x, y) \geq s = 2^{p-1}$, that is, $x(c) \leq y(c)$ for all $c \in [a, b]$. From (ii) and the Hölder inequality, for each $c \in [a, b]$ we get

$$\begin{aligned}
& 2^{\varepsilon p-\varepsilon}|(fx)(c) - (fy)(c)|^p \\
\leq\ & 2^{\varepsilon p-\varepsilon}\left(\int_a^b |K(c, r, x(r)) - K(c, r, y(r))|\mathrm{d}r\right)^p \\
\leq\ & 2^{\varepsilon p-\varepsilon}\left[\left(\int_a^b 1^q \mathrm{d}r\right)^{\frac{1}{q}}\left(\int_a^b |K(c, r, x(r)) - K(c, r, y(r))|^p \mathrm{d}r\right)^{\frac{1}{p}}\right]^p \\
\leq\ & 2^{\varepsilon p-\varepsilon}(b-a)^{\frac{p}{q}}\left(\int_a^b \zeta(c, r)^p |x(r) - y(r)|^p \mathrm{d}r\right) \\
\leq\ & 2^{\varepsilon p-\varepsilon}(b-a)^{\frac{p}{q}}\left(\int_a^b \zeta(c, r)^p d(x, y)\mathrm{d}r\right) \\
\leq\ & 2^{\varepsilon p-\varepsilon}(b-a)^{p-1} M_s(x, y)\left(\int_a^b \zeta(c, r)^p \mathrm{d}r\right) \\
\leq\ & M_s(x, y).
\end{aligned}$$

This implies that $s^\varepsilon d(fx, fy) \leq M_s(x, y)$. Hence $f \in \Omega_s(X, \alpha)$. Thus all the conditions of Theorem 2.3 are satisfied and hence f has a fixed point in X. It follows that the nonlinear integral equation (3.1) has a solution. □

Remark 3.2. Compared with [15, Theorem 3.1], our Theorem 3.1 has many superiorities. First, our condition (ii) is much simpler than (ii) from [15, Theorem 3.1]. Indeed, our condition (3.2) is weaker than the corresponding condition of [15, Theorem 3.1]. Moreover, we delete the function $\Upsilon(t)$. Whereas, $\Upsilon(t)$ is a complex function with very strong conditions. Otherwise, our function $\zeta(c, r)$ satisfies the wider condition since ε is arbitrary. Further, even if $\varepsilon = 3$, our condition for $\zeta(c, r)$ is also much weaker.

Remark 3.3. In [15, Theorem 3.1], there exist some mistakes. For instance, the incorrect equality from the proof of [15, Theorem 3.1] appears as follows:

$$\left(s^3 d(fx, fy)\right)^p = \left(2^{3p-3} \sup_{t \in [a,b]} |(fx)(t) - (fy)(t)|\right)^p.$$

In fact, it should be the following:

$$\left(s^3 d(fx, fy)\right)^p = \left(2^{3p-3} \sup_{t \in [a,b]} |(fx)(t) - (fy)(t)|^p\right)^p.$$

Due to such mistake, the conditions from [15, Theorem 3.1] need some revisions. Similar revisions should be done in Corollary 3.2 and Corollary 3.3 from [15, Theorem 3.1].

Acknowledgements

The research was partially supported by the National Natural Science Foundation of China (11271045).

References

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leurs applications aux équations intégrales., Fund. Math., **3** (1922), 133-181.

[2] Z. Mustafa, F. Awawdeh, W. Shatanawi, Fixed point theorem for expansive mappings in G-metric Spaces, Int. J. Contemp. Math. Sciences, **5**(50) (2010), 2463-2472.

[3] Mona. S. Bakry, H. M. Abu-Donia, Fixed-point theorems for a probabilistic 2-metric spaces, J. King Saud University (Science), **22** (2010), 217-221.

[4] C. Ionescu, S. Rezapour, M.-E. Samei, Fixed points of some new contractions on intuitionistic fuzzy metric spaces, Fixed Point Theory Appl., **2013**, 2013: 168.

[5] G. Cain, R. Kasriel, Fixed and periodic points of local contraction mappings on probabilistic metric spaces, Theory Comput. Systems, **9**(3) (1975), 289-297.

[6] L. -G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl., **332**(2) (2007), 1468-1476.

[7] T. Abdeljawad, J. Alzabut, A. Mukheimer, Y Zaidan, Banach contraction principle for cyclical mappings on partial metric spaces, Fixed Point Theory Appl., **2012**, 2012: 154.

[8] A.-A. Abdou, M.-A Khamsi, Fixed points of multivalued contraction mappings in modular metric spaces, Fixed Point Theory Appl., **2014**, 2014: 249.

[9] I. A. Bakhtin, The contraction principle in quasi-metric spaces, Funct. Anal., **30** (1989), 26-37.

[10] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inform., Univ. Ostrav., **1** (1993), 5-11.

[11] M. Jovanović, Z. Kadelburg, S. Radenović, Common fixed point results in metric type spaces, Fixed Point Theory Appl., **2010**, Article ID 978121, 15 pages.

[12] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings., Nonlinear Anal., **75** (2012), 2154-2165.

[13] R. Kannan, Some results on fixed points. II. Amer. Math. Monthly, **76** (1969), 405-408.

[14] M. Imdad, A. Erduran, Suzuki-type generalization of Chatterjea contraction mappings on complete partial metric spaces, J. Oper., **2013** (2013), Article ID 923843, 5 pages.

[15] W. Sintunavarat, Nonlinear integral equations with new admissibility types in b-metric spaces, J. Fixed Point Theory Appl., DOI 10.1007/s11784-015-0276-6, 20 pages.

[16] H. Huang, S. Radenović, J. Vujaković, On some recent coincidence and immediate consequences in partially ordered b-metric spaces, Fixed Point Theory and Applications, **2015**, 2015: 63.

[17] M. Boriceanu, M. Bota and A. Petrusel, Mutivalued fractals in b-metric spaces, Cent. Eur. J. Math., **8** (2010), 367-377.

[18] A. Aghajani, M. Abbas, J. R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces, Math. Slovaca, **64**(4) (2014), 941-960.

[19] H. Huang, L. Paunović, S. Radenović, On some new fixed point results for rational Geraghty contractive mappings in ordered b-metric spaces, J. Nonlinear Science Appl., **8**(5) (2015), 800-807.

[20] N. Hussain, E. Karapınar, P. Salimi, F. Akbar, α-admissible mappings and related fixed point theorems, Fixed Point Theory Appl., **2013**, 2013: 114.

[21] J. J. Nieto, R. Rodíguez-López, Existence and uniqueness of fixed points in partially ordered sets and applications to ordinary differential equations. Acta Math. Sin. (Engl. Ser.) **23** (2007), 2205-2212.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 25, NO.2, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

Anti Implicative IF-Ideals in BCK/BCI-algebras

Servet Kutukcu* **and** **Adnan Tuna**
(Dedicated to the 79th birthday of Ivo G. Rosenberg.)

Department of Mathematics, Faculty of Science and Arts Ondokuz Mayıs University, 55139, Samsun, Turkey
Department of Mathematics, Faculty of Science and Arts University of Niğde, 51240, Niğde, Turkey

Abstract: Using triangular norms, we present a new classification of fuzzy subalgebras, ideals and implicative ideals in BCK/BCI-algebras.

Keywords: t-norm; anti if-ideal; anti implicative if-ideal; BCK/BCI-algebra.

1 Introduction

BCK/BCI-algebras are an important class of logical algebras introduced by Imai and Iseki [7], and was extensively investigated by several researches. BCK/BCI-algebras generalize, on the one hand, the notion of the algebra of sets with the set subtraction as the only fundemental non-nullary operation and, on the other hand, the notion of the implication algebra (see [7]). In 1965, Zadeh [16] introduced the notion of fuzzy sets and in 1991, Xi [15] applied this notion to BCK/BCI-algebras. In 1990, Biswas [4] introduced the notion of anti fuzzy subgroups of groups and in 2008, modifying Biswas' idea, Kutukcu and Sharma [10] introduced the notion of anti fuzzy ideals in BCC-algebras.

In the present paper, we introduce the notions of anti if-subalgebras, anti if-ideals and anti implicative if-ideals of BCK/BCI-algebras with respect to arbitrary t-conorms and t-norms. Illustrating with examples, we prove that our definitions are more general than the classical ones. We also prove that an if-subset of a BCK/BCI-algebra is an anti if-ideal if and only if the complement of this if-subset is an anti if-ideal. We also discuss some relationships between such notions.

Let us recall [7,8] that a BCI-algebra is an algebra $(X, *, 0)$ of type $(2, 0)$ which satisfies the following conditions, for all $x, y, z \in X$: (i) $((x * y) * (x * z)) * (z * y) = 0$; (ii) $(x * (x * y)) * y = 0$; (iii) $x * x = 0$; (iv) $x * y = 0$ and $y * x = 0$ imply $x = y$. A BCI-algebra X satisfying the additional condition (v) for all $x \in X$, $0 * x = 0$ is called a BCK-algebra.

We can define a partial ordering $\leq$ on X by $x \leq y$ if and only if $x * y = 0$. Furthermore, in any BCK/BCI-algebra X, the following properties hold, for all $x, y, z \in X$: (i) $(x * y) * z = (x * z) * y$; (ii)

*Corresponding Author
E-mail: skutukcu@omu.edu.tr (S. Kutukcu), atuna@nigde.edu.tr (A. Tuna)
2010 MR Subject Classification: 06F35, 03G25, 03E72, 94D05

$x*(x*(x*y)) = x*y$; (iii) $x*y \le x$; (iv) $x*0 = x$; (v) $(x*z)*(y*z) \le x*y$; (vi) $x \le y$ implies $x*z \le y*z$ and $z*y \le z*x$.

A non-empty subset A of a BCK/BCI-algebra X is called an ideal of X if $0 \in A$, and $x*y \in A$ and $y \in A$ imply $x \in A$ for all $x, y \in X$.

A non-empty subset A of a BCK/BCI-algebra X is called an implicative ideal of X if $0 \in A$, and $(x*(y*x))*z \in A$ and $z \in A$ imply $x \in A$ for all $x, y, z \in X$. Any implicative ideal is an ideal, but not conversely.

A mapping f of a BCK/BCI-algebra X into a BCK/BCI-algebra Y is called a homomorphism if $f(x*y) = f(x)*f(y)$ for all $x, y \in X$.

By a triangular conorm (shortly t-conorm) S [14], we mean a binary operation on the unit interval $[0,1]$ which satisfies the following conditions, for all $x, y, z \in [0,1]$: (i) $S(x,0) = x$; (ii) $S(x,y) \le S(x,z)$ if $y \le z$; (iii) $S(x,y) = S(y,x)$; (iv) $S(x, S(y,z)) = S(S(x,y),z)$. Some important examples of t-conorms are $S_L(x,y) = \min\{x+y, 1\}$, $S_P(x,y) = x+y-xy$ and $S_M(x,y) = \max\{x,y\}$.

By a triangular norm (shortly t-norm) T [14], we mean a binary operation on the unit interval $[0,1]$ which satisfies the following conditions, for all $x, y, z \in [0,1]$: (i) $T(x,1) = x$; (ii) $T(x,y) \le T(x,z)$ if $y \le z$; (iii) $T(x,y) = T(y,x)$; (iv) $T(x, T(y,z)) = T(T(x,y),z)$. Some important examples of t-norms are $T_L(x,y) = \max\{x+y-1, 0\}$, $T_P(x,y) = xy$ and $T_M(x,y) = \min\{x,y\}$.

A t-conorm S and a t-norm T are called associated [11], i.e. $S(x,y) = 1 - T(1-x, 1-y)$ for all $x, y \in [0,1]$. For example, t-conorm S_M and t-norm T_M are associated [6,9-11]. Also, it is well known [6,9] that if S is a t-conorm and T is a t-norm, then $\max\{x,y\} \le S(x,y)$ and $\min\{x,y\} \ge T(x,y)$ for all $x, y \in [0,1]$, respectively.

Note that, the concepts of t-conorms and t-norms are known as the axiomatic skeletons that we use for characterizing fuzzy unions and intersections, respectively. These concepts were originally introduced by Menger [13] and several properties and examples for these concepts were proposed by many authors (see [6,9-11,13,14]).

A fuzzy subset A in an arbitrary non-empty set X is a function $\mu_A : X \to [0,1]$. The complement of μ_A, denoted by μ_A^c, is the fuzzy subset in X given by $\mu_A^c(x) = 1 - \mu_A(x)$ for all $x \in X$.

Definition 1.1 ([15]) *A fuzzy subset A in a BCK/BCI-algebra X is called a fuzzy BCK/BCI-subalgebra of X if*

$$\mu_A(x*y) \ge \min\{\mu_A(x), \mu_A(y)\}$$

for all $x, y \in X$.

Definition 1.2 ([15]) *A fuzzy subset A in a BCK/BCI-algebra X is called a fuzzy ideal of X if*

$$\mu_A(0) \ge \mu_A(x) \ge \min\{\mu_A(x*y), \mu_A(y)\}$$

for all $x, y \in X$.

Definition 1.3 ([10,12]) *A fuzzy subset A in a BCK/BCI-algebra X is called a implicative fuzzy ideal of X if*

$$\mu_A(0) \ge \mu_A(x) \ge \min\{\mu_A((x*(y*x))*z), \mu_A(z)\}$$

for all $x, y, z \in X$.

Definition 1.4 ([12]) *A fuzzy subset A in a BCK/BCI-algebra X is called an anti fuzzy BCK/BCI-subalgebra of X if*

$$\mu_A(x*y) \le \max\{\mu_A(x), \mu_A(y)\}$$

for all $x, y \in X$.

2

Definition 1.5 ([12]) *A fuzzy subset A in a BCK/BCI-algebra X is called an anti fuzzy ideal of X if*

$$\mu_A(0) \leq \mu_A(x) \leq \max\{\mu_A(x*y), \mu_A(y)\}$$

for all $x, y \in X$.

As a generalization of the notion of fuzzy subsets in X, Atanassov [2] introduced the concept of intuitionistic fuzzy subsets (or simply if-sets) defined on X as objects having the form

$$A = \{(x, \mu_A(x), \lambda_A(x)) : x \in X\}$$

where the functions $\mu_A : X \to [0,1]$ and $\lambda_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\lambda_A(x)$) of each element x in X to the set A, respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all x in X.

In [3], for every two if-subsets A and B in X, we have

(i) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\lambda_A(x) \geq \lambda_B(x)$ for all $x \in X$,

(ii) $\square A = \{(x, \mu_A(x), \mu_A^c(x)) : x \in X\}$,

(iii) $\Diamond A = \{(x, \lambda_A^c(x), \lambda_A(x)) : x \in X\}$.

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \lambda_A)$ for the if-subset $A = \{(x, \mu_A(x), \lambda_A(x)) : x \in X\}$.

2 Anti IF-Ideals

Definition 2.1 *An if-subset $A = (\mu_A, \lambda_A)$ in a BCK/BCI-algebra X is said to be an anti intuitionistic fuzzy BCK/BCI-subalgebra of X (or simply, an anti if-BCK/BCI-subalgebra of X) if*

(i) $\mu_A(x*y) \leq \max\{\mu_A(x), \mu_A(y)\}$,

(ii) $\lambda_A(x*y) \geq \min\{\lambda_A(x), \lambda_A(y)\}$

for all $x, y \in X$.

Definition 2.2 *An if-subset $A = (\mu_A, \lambda_A)$ in a BCK/BCI-algebra X is said to be an anti intuitionistic fuzzy BCK/BCI-subalgebra of X with respect to a t-conorm S and a t-norm T (or simply, an (S,T)-anti if-BCK/BCI-subalgebra of X) if*

(i) $\mu_A(x*y) \leq S(\mu_A(x), \mu_A(y))$

(ii) $\lambda_A(x*y) \geq T(\lambda_A(x), \lambda_A(y))$

for all $x, y \in X$.

Remark 2.3 *Every anti if-BCK/BCI-subalgebra of a BCK/BCI-algebra is an (S,T)-anti if-BCK/BCI-subalgebra of X such that $S = S_M$ and $T = T_M$, but it is clear that the converse is not true. If $\lambda_A(x) = 1 - \mu_A(x)$ for all $x \in X$, then every anti if-BCK/BCI-subalgebra of a BCK/BCI-algebra X is an anti fuzzy BCK/BCI-subalgebra of X. Also, if $\lambda_A(x) = 1 - \mu_A(x)$ for all $x \in X$, $S = S_M$ and $T = T_M$, then every (S,T)-anti if-BCK/BCI-subalgebra of a BCK/BCI-algebra X is an anti fuzzy BCK/BCI-subalgebra of X.*

Definition 2.4 *An if-subset $A = (\mu_A, \lambda_A)$ in a BCK/BCI-algebra X is said to be an anti if-ideal of X if*

(i) $\mu_A(0) \leq \mu_A(x)$ and $\lambda_A(0) \geq \lambda_A(x)$,

(ii) $\mu_A(x) \leq \max\{\mu_A(x * y), \mu_A(y)\}$,

(iii) $\lambda_A(x) \geq \min\{\lambda_A(x * y), \lambda_A(y)\}$

for all $x, y \in X$.

Definition 2.5 *An if-subset $A = (\mu_A, \lambda_A)$ in a BCK/BCI-algebra X is said to be an anti if-ideal of X with respect to a t-conorm S and a t-norm T (or simply, an (S,T)-anti if-ideal of X) if*

(i) $\mu_A(0) \leq \mu_A(x)$ and $\lambda_A(0) \geq \lambda_A(x)$,

(ii) $\mu_A(x) \leq S(\mu_A(x * y), \mu_A(y))$,

(iii) $\lambda_A(x) \geq T(\lambda_A(x * y), \lambda_A(y))$

for all $x, y \in X$.

Remark 2.6 *Every anti if-ideal of a BCK/BCI-algebra is an (S,T)-anti if-ideal of X such that $S = S_M$ and $T = T_M$, but it is clear that the converse is not true. If $\lambda_A(x) = 1 - \mu_A(x)$ for all $x \in X$, then every anti if- ideal of a BCK/BCI-algebra X is an anti fuzzy ideal of X. Also, if $\lambda_A(x) = 1 - \mu_A(x)$ for all $x \in X$, $S = S_M$ and $T = T_M$, then every (S,T)-anti if-ideal of a BCK/BCI-algebra X is an anti fuzzy ideal of X.*

Example 2.7 *Let $X = \{0, 1, 2, 3\}$ be a BCK-algebra with the Cayley table as follows*

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Define an if-set $A = (\mu_A, \lambda_A)$ in X by

$$\mu_A(x) = \begin{cases} 0, & x = 0 \\ 1/2, & x = 1 \text{ or } 2 \\ 1, & x = 3 \end{cases} \quad \text{and } \lambda_A(x) = \begin{cases} 1, & x = 0 \\ 1/3, & x = 1 \text{ or } 2 \\ 0, & x = 3 \end{cases}$$

It is easy to check that $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$, $\mu_A(0) \leq \mu_A(x)$ and *$\lambda_A(0) \geq \lambda_A(x)$.* Also, *$\mu_A(x) \leq S_M(\mu_A(x*y), \mu_A(y))$ and $\lambda_A(x) \geq T_L(\lambda_A(x*y), \lambda_A(y))$ for all $x, y \in X$. Hence $A = (\mu_A, \lambda_A)$ is an (S_M, T_L)-anti if-ideal of X. Also note that t-conorm S_M and t-norm T_L are not associated.*

Remark 2.8 *Note that, the above example holds even with the t-conorm S_M and t-norm T_M, and hence $A = (\mu_A, \lambda_A)$ is also an (S_M, T_M)-anti if-ideal of X. Therefore, every anti if-ideal of X is an (S,T)-anti if-ideal but the converse is not true.*

Example 2.9 *Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the Cayley table as follows*

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	b	a	0	0
d	d	d	d	d	0

Define an if-set $A = (\mu_A, \lambda_A)$ in X by

$$\mu_A(x) = \begin{cases} 1/2, & x \in \{0, a, b\} \\ 3/4, & \text{otherwise} \end{cases} \quad \text{and } \lambda_A(x) = \begin{cases} 1/3, & x \in \{0, a, b\} \\ 1/4, & \text{otherwise.} \end{cases}$$

It is easy to check that $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$, $\mu_A(0) \leq \mu_A(x)$ and $\lambda_A(0) \geq \lambda_A(x)$. Also, $\mu_A(x) \leq S_L(\mu_A(x*y), \mu_A(y))$ and $\lambda_A(x) \geq T_P(\lambda_A(x*y), \lambda_A(y))$ for all $x, y \in X$. Hence $A = (\mu_A, \lambda_A)$ is an (S_L, T_P)-anti if-ideal of X. But $A = (\mu_A, \lambda_A)$ is not an anti if-ideal of X.

Lemma 2.10 *If $A = (\mu_A, \lambda_A)$ is an (S, T)-anti if-ideal of a BCK/BCI-algebra X, then so is $\Box A = (\mu_A, \mu_A^c)$ such that t-conorm S and t-norm T are associated.*

Proof. Since $A = (\mu_A, \lambda_A)$ is an (S, T)-anti if-ideal of X, then $\mu_A(0) \leq \mu_A(x)$ for all $x \in X$ and so $1 - \mu_A^c(0) \leq 1 - \mu_A^c(x)$, hence $\mu_A^c(0) \geq \mu_A^c(x)$. Also, for all $x, y \in X$, we have

$$\mu_A(x) \leq S(\mu_A(x * y), \mu_A(y))$$

and so

$$1 - \mu_A^c(x) \leq S(1 - \mu_A^c(x * y), 1 - \mu_A^c(y))$$

which implies

$$\mu_A^c(x) \geq 1 - S(1 - \mu_A^c(x * y), 1 - \mu_A^c(y)).$$

Since S and T are associated, we have

$$\mu_A^c(x) \geq T(\mu_A^c(x * y), \mu_A^c(y)).$$

Thus, $\Box A = (\mu_A, \mu_A^c)$ is an (S, T)-anti if-ideal of X. ■

Lemma 2.11 *If $A = (\mu_A, \lambda_A)$ is an (S, T)-anti if-ideal of a BCK/BCI-algebra X, then so is $\Diamond A = (\lambda_A^c, \lambda_A)$ such that t-conorm S and t-norm T are associated.*

Proof. The proof is similar to the proof of Lemma 2.10. ■

Combining the above two lemmas, it is easy to see that the following theorem is valid.

Theorem 2.12 *$A = (\mu_A, \lambda_A)$ is an (S, T)-anti if-ideal of a BCK/BCI-algebra X if and only if $\Box A$ and $\Diamond A$ are (S, T)-anti if-ideals of X such that t-conorm S and t-norm T are associated.*

Corollary 2.13 *$A = (\mu_A, \lambda_A)$ is an (S, T)-anti if-ideal of a BCK/BCI-algebra X if and only if μ_A and λ_A^c are anti fuzzy ideals of X such that t-conorm S and t-norm T are associated.*

Lemma 2.14 *Let $A = (\mu_A, \lambda_A)$ be an (S, T)-anti if-ideal of a BCK/BCI-algebra X. If $\leq$ is a partial ordering on X then $\mu_A(x) \leq \mu_A(y)$ and $\lambda_A(y) \leq \lambda_A(x)$ for all $x, y \in X$ such that $x \leq y$.*

Proof. Let X be a BCK/BCI-algebra. It is known [8] that $\leq$ is a partial ordering on X defined by $x \leq y$ if and only if $x * y = 0$ for all $x, y \in X$. Let $A = (\mu_A, \lambda_A)$ be an (S,T)-anti if-ideal of X. Then

$$\mu_A(x) \leq S(\mu_A(x*y), \mu_A(y)) = S(\mu_A(0), \mu_A(y)) \leq \mu_A(y)$$

and

$$\lambda_A(x) \geq T(\lambda_A(x*y), \lambda_A(y)) = T(\lambda_A(0), \lambda_A(y)) \geq \lambda_A(y).$$

These complete the proof. ■

Theorem 2.15 *Let $A = (\mu_A, \lambda_A)$ be an (S,T)-anti if-ideal of a BCK/BCI-algebra X. A is an (S,T)-anti if-BCK/BCI-subalgebra of X.*

Proof. Let $A = (\mu_A, \lambda_A)$ be an (S,T)-anti if-ideal *of* X. Since $x * y \leq x$ for all $x, y \in X$, it follows from Lemma 2.14 that $\mu_A(x*y) \leq \mu_A(x)$ and $\lambda_A(x) \leq \lambda_A(x*y)$. Then

$$\mu_A(x*y) \leq \mu_A(x) \leq S(\mu_A(x*y), \mu_A(y)) \leq S(\mu_A(x), \mu_A(y))$$

and

$$\lambda_A(x*y) \geq \lambda_A(x) \geq T(\lambda_A(x*y), \lambda_A(y)) \geq T(\lambda_A(x), \lambda_A(y))$$

and so A is an (S,T)-anti if-BCK/BCI-subalgebra of X. ■

Remark 2.16 *The converse of the above theorem does not hold in general. In fact, suppose that X be the BCK-algebra in Example 2.7. Define an if-set $A = (\mu_A, \lambda_A)$ in X by*

$$\mu_A(x) = \begin{cases} 0, & x = 0 \\ 1/2, & x = 1 \\ 1, & x = 2 \text{ } or \text{ } 3 \end{cases} \quad and \quad \lambda_A(x) = \begin{cases} 1, & x = 0 \\ 1/3, & x = 1 \\ 0, & x = 2 \text{ } or \text{ } 3 \end{cases}$$

By routine calculations, we know that $A = (\mu_A, \lambda_A)$ is an (S_M, T_M)-anti if-BCK-subalgebra of X but not an (S_M, T_M)-anti if-ideal of X because $\mu_A(2) = 1 > \max\{\mu_A(2*1), \mu_A(1)\}$ and $\lambda_A(2) = 0 < \min\{\lambda_A(2*1), \lambda_A(1)\}$.

If $A = (\mu_A, \lambda_A)$ is an if-subset in a BCK/BCI-algebra X and f is a self mapping of X, we define mappings $\mu_A[f] : X \to [0,1]$ by $\mu_A[f](x) = \mu_A(f(x))$ and $\lambda_A[f] : X \to [0,1]$ by $\lambda_A[f](x) = \lambda_A(f(x))$ for all $x \in X$, respectively.

Proposition 2.17 *If $A = (\mu_A, \lambda_A)$ is an (S,T)-anti if-ideal of a BCK/BCI-algebra X and f is an increasing endomorphism of X, then $(\mu_A[f], \lambda_A[f])$ is an (S,T)-anti if-ideal of X.*

Proof. For any given $x, y \in X$, we have

$$\begin{aligned} \mu_A[f](x) &= \mu_A(f(x)) \leq S(\mu_A(f(x)*f(y)), \mu_A(f(y))) \\ &= S(\mu_A(f(x*y)), \mu_A(f(y))) \\ &= S(\mu_A[f](x*y), \mu_A[f](y)), \end{aligned}$$

$$\begin{aligned} \lambda_A[f](x) &= \lambda_A(f(x)) \geq T(\lambda_A(f(x)*f(y)), \lambda_A(f(y))) \\ &= T(\lambda_A(f(x*y)), \lambda_A(f(y))) \\ &= T(\lambda_A[f](x*y), \lambda_A[f](y)). \end{aligned}$$

Also, since X is a BCK/BCI-algebra, we have $0 * x = 0$ and so $0 \leq x$ for all $x \in X$. Since f is increasing, we have $f(0) \leq f(x)$ for all $x \in X$ and, from Lemma 2.14, $\mu_A(f(0)) \leq \mu_A(f(x))$ and $\lambda_A(f(x)) \leq \lambda_A(f(0))$ i.e., $\mu_A[f](0) \leq \mu_A[f](x)$ and $\lambda_A[f](x) \leq \lambda_A[f](0)$ for all $x \in X$. This completes the proof. ■

If f is a self mapping of a BCK/BCI-algebra X and $B = (\mu_B, \lambda_B)$ is an if-subset in $f(X)$, then the if-subset $A = (\mu_A, \lambda_A)$ in X defined by $\mu_A = \mu_B \circ f$ and $\lambda_A = \lambda_B \circ f$ (i.e., $\mu_A(x) = \mu_B(f(x))$ and $\lambda_A(x) = \lambda_B(f(x))$ for all $x \in X$) is called the *preimage* of B under f.

Theorem 2.18 *An onto increasing homomorphic preimage of an (S,T)-anti if-ideal is an (S,T)-anti if-ideal.*

Proof. Let $f : X \to Y$ be an onto homomorphism of BCK/BCI-algebras, $B = (\mu_B, \lambda_B)$ be an (S,T)-anti if-ideal of Y, and $A = (\mu_A, \lambda_A)$ be preimage of B under f. Then, we have

$$\begin{aligned}
\mu_A(x) &= \mu_B(f(x)) \leq S(\mu_B(f(x) * f(y)), \mu_B(f(y))) \\
&= S(\mu_B(f(x * y)), \mu_B(f(y))) \\
&= S(\mu_A(x * y), \mu_A(y)),
\end{aligned}$$

$$\begin{aligned}
\lambda_A(x) &= \lambda_B(f(x)) \geq T(\lambda_B(f(x) * f(y)), \lambda_B(f(y))) \\
&= T(\lambda_B(f(x * y)), \lambda_B(f(y))) \\
&= T(\lambda_A(x * y), \lambda_A(y))
\end{aligned}$$

for all $x, y \in X$. Also, $\mu_A(0) = \mu_B(f(0)) \leq \mu_B(f(x)) = \mu_A(x)$ and $\lambda_A(0) = \lambda_B(f(0)) \geq \lambda_B(f(x)) = \lambda_A(x)$ for all $x \in X$. Hence, $A = (\mu_A, \lambda_A)$ is an (S,T)-anti if-ideal of X. ■

Lemma 2.19 ([9]) *Let S and T be a t-conorm and a t-norm, respectively. Then*

$$S(S(x,y), S(z,t)) = S(S(x,z), S(y,t)),$$

$$T(T(x,y), T(z,t)) = T(T(x,z), T(y,t))$$

for all $x, y, z, t \in [0,1]$.

Theorem 2.20 *Let S be a t-conorm, T be a t-norm and $X = X_1 \times X_2$ be the direct product BCK/BCI-algebra of BCK/BCI-algebras X_1 and X_2. If $A_1 = (\mu_{A_1}, \lambda_{A_1})$ (resp. $A_2 = (\mu_{A_2}, \lambda_{A_2})$) is an (S,T)-anti if-ideal of X_1 (resp. X_2), then $A = (\mu_A, \lambda_A)$ is an (S,T)-anti if-ideal of X defined by $\mu_A = \mu_{A_1} \times \mu_{A_2}$ and $\lambda_A = \lambda_{A_1} \times \lambda_{A_2}$ such that*

$$\mu_A(x_1, x_2) = (\mu_{A_1} \times \mu_{A_2})(x_1, x_2) = S(\mu_{A_1}(x_1), \mu_{A_2}(x_2)),$$

$$\lambda_A(x_1, x_2) = (\lambda_{A_1} \times \lambda_{A_2})(x_1, x_2) = T(\lambda_{A_1}(x_1), \lambda_{A_2}(x_2))$$

for all $(x_1, x_2) \in X$.

Proof. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be any elements of X. Since X is a BCK/BCI-algebra, we have

$$\begin{aligned}
\mu_A(x) &= (\mu_{A_1} \times \mu_{A_2})(x_1, x_2) \\
&= S(\mu_{A_1}(x_1), \mu_{A_2}(x_2)) \\
&\leq S(S(\mu_{A_1}(x_1 * y_1), \mu_{A_1}(y_1)), S(\mu_{A_2}(x_2 * y_2), \mu_{A_2}(y_2))) \\
&= S(S(\mu_{A_1}(x_1 * y_1), \mu_{A_2}(x_2 * y_2)), S(\mu_{A_1}(y_1), \mu_{A_2}(y_2))) \\
&= S((\mu_{A_1} \times \mu_{A_2})(x_1 * y_1, x_2 * y_2), (\mu_{A_1} \times \mu_{A_2})(y_1, y_2)) \\
&= S((\mu_{A_1} \times \mu_{A_2})((x_1, x_2) * (y_1, y_2)), (\mu_{A_1} \times \mu_{A_2})(y_1, y_2)) \\
&= S(\mu_A(x * y), \mu_A(y)),
\end{aligned}$$

$$\begin{aligned}
\lambda_A(x) &= (\lambda_{A_1} \times \lambda_{A_2})(x_1, x_2) \\
&= T(\lambda_{A_1}(x_1), \lambda_{A_2}(x_2)) \\
&\geq T(T(\lambda_{A_1}(x_1 * y_1), \lambda_{A_1}(y_1)), T(\lambda_{A_2}(x_2 * y_2), \lambda_{A_2}(y_2))) \\
&= T(T(\lambda_{A_1}(x_1 * y_1), \lambda_{A_2}(x_2 * y_2)), T(\lambda_{A_1}(y_1), \lambda_{A_2}(y_2))) \\
&= T((\lambda_{A_1} \times \lambda_{A_2})(x_1 * y_1, x_2 * y_2), (\lambda_{A_1} \times \lambda_{A_2})(y_1, y_2)) \\
&= T((\lambda_{A_1} \times \lambda_{A_2})((x_1, x_2) * (y_1, y_2)), (\lambda_{A_1} \times \lambda_{A_2})(y_1, y_2)) \\
&= T(\lambda_A(x * y), \lambda_A(y))
\end{aligned}$$

Also,

$$\begin{aligned}
\mu_A(0) &= (\mu_{A_1} \times \mu_{A_2})(0, 0) = S(\mu_{A_1}(0), \mu_{A_2}(0)) \\
&\leq S(\mu_{A_1}(x_1), \mu_{A_2}(x_2)) = (\mu_{A_1} \times \mu_{A_2})(x_1, x_2) \\
&= \mu_A(x),
\end{aligned}$$

$$\begin{aligned}
\lambda_A(0) &= (\lambda_{A_1} \times \lambda_{A_2})(0, 0) = T(\lambda_{A_1}(0), \lambda_{A_2}(0)) \\
&\geq T(\lambda_{A_1}(x_1), \lambda_{A_2}(x_2)) = (\lambda_{A_1} \times \lambda_{A_2})(x_1, x_2) \\
&= \lambda_A(x).
\end{aligned}$$

This completes the proof. ■

Definition 2.21 *Let S be a t-conorm and T be a t-norm, and let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be if-sets in a BCK/BCI-algebra X. Then S-product of μ_A and μ_B, and T-product of λ_A and λ_B, written $[\mu_A.\mu_B]_S$ and $[\lambda_A.\lambda_B]_T$, are defined by*

$$[\mu_A.\mu_B]_S(x) = S(\mu_A(x), \mu_B(x)),$$

$$[\lambda_A.\lambda_B]_T(x) = T(\lambda_A(x), \lambda_B(x))$$

for all $x \in X$, respectively.

Theorem 2.22 *Let S be a t-conorm and T be a t-norm, and let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be (S, T)-anti if-ideals of a BCK/BCI-algebra X. If S_1 is a t-conorm which dominates S, that is,*

$$S_1(S(x, y), S(z, t)) \leq S(S_1(x, z), S_1(y, t))$$

and T_1 is a t-norm which dominates T, that is,

$$T_1(T(x,y),T(z,t)) \geq T(T_1(x,z),T_1(y,t))$$

for all $x,y,z,t \in [0,1]$, then $([\mu_A.\mu_B]_{S_1},[\lambda_A.\lambda_B]_{T_1})$ is an (S,T)-anti if-ideal of X.

Proof. For any $x,y \in X$, we have

$$\begin{aligned}
[\mu_A.\mu_B]_{S_1}(x) &= S_1(\mu_A(x),\mu_B(x)) \\
&\leq S_1(S(\mu_A(x*y),\mu_A(y)),S(\mu_B(x*y),\mu_B(y))) \\
&\leq S(S_1(\mu_A(x*y),\mu_B(x*y)),S_1(\mu_A(y),\mu_B(y))) \\
&= S([\mu_A.\mu_B]_{S_1}(x*y),[\mu_A.\mu_B]_{S_1}(y)),
\end{aligned}$$

$$\begin{aligned}
[\lambda_A.\lambda_B]_{T_1}(x) &= T_1(\lambda_A(x),\lambda_B(x)) \\
&\geq T_1(T(\lambda_A(x*y),\lambda_A(y)),T(\lambda_B(x*y),\lambda_B(y))) \\
&\geq T(T_1(\lambda_A(x*y),\lambda_B(x*y)),T_1(\lambda_A(y),\lambda_B(y))) \\
&= T([\lambda_A.\lambda_B]_{T_1}(x*y),[\lambda_A.\lambda_B]_{T_1}(y)).
\end{aligned}$$

Also,

$$[\mu_A.\mu_B]_{S_1}(0) = S_1(\mu_A(0),\mu_B(0)) \leq S_1(\mu_A(x),\mu_B(x)) = [\mu_A.\mu_B]_{S_1}(x),$$

$$[\lambda_A.\lambda_B]_{T_1}(0) = T_1(\lambda_A(0),\lambda_B(0)) \geq T_1(\lambda_A(x),\lambda_B(x)) = [\lambda_A.\lambda_B]_{T_1}(x)$$

This completes the proof. ■

3 Anti Implicative IF-Ideals

Definition 3.1 *A fuzzy subset A in a BCK/BCI-algebra X is said to be an anti implicative fuzzy ideal of X if*

(i) $\mu_A(0) \leq \mu_A(x)$,

(ii) $\mu_A(x) \leq \max\{\mu_A((x*(y*x))*z),\mu_A(z)\}$

for all $x,y,z \in X$.

Definition 3.2 *An if-subset $A = (\mu_A,\lambda_A)$ in a BCK/BCI-algebra X is said to be an anti implicative if-ideal of X if*

(i) $\mu_A(0) \leq \mu_A(x)$ and $\lambda_A(0) \geq \lambda_A(x)$,

(ii) $\mu_A(x) \leq \max\{\mu_A((x*(y*x))*z),\mu_A(z)\}$,

(iii) $\lambda_A(x) \geq \min\{\lambda_A(x*(y*x))*z),\lambda_A(z)\}$

for all $x,y,z \in X$.

Definition 3.3 *An if-subset $A = (\mu_A, \lambda_A)$ in a BCK/BCI-algebra X is said to be an anti implicative if-ideal of X with respect to a t-conorm S and a t-norm T (or simply, an (S,T)-anti implicative if-ideal of X) if*

(i) $\mu_A(0) \leq \mu_A(x)$ and $\lambda_A(0) \geq \lambda_A(x)$,

(ii) $\mu_A(x) \leq S(\mu_A((x*(y*x))*z), \mu_A(z))$,

(iii) $\lambda_A(x) \geq T(\lambda_A(x*(y*x))*z), \lambda_A(z))$

for all $x, y, z \in X$.

Remark 3.4 *Every anti implicative if-ideal of a BCK/BCI-algebra is an (S,T)-anti implicative if-ideal of X such that $S = S_M$ and $T = T_M$, but it is clear that the converse is not true. If $\lambda_A(x) = 1 - \mu_A(x)$ for all $x \in X$, then every anti implicative if-ideal of a BCK/BCI-algebra X is an anti implicative fuzzy ideal of X. Also, if $\lambda_A(x) = 1 - \mu_A(x)$ for all $x \in X$, $S = S_M$ and $T = T_M$, then every (S,T)-anti implicative if-ideal of a BCK/BCI-algebra X is an anti implicative fuzzy ideal of X.*

Example 3.5 *In Example 2.9, it is easy to see that $A = (\mu_A, \lambda_A)$ is also an (S_L, T_P)-anti implicative if-ideal of X.*

Remark 3.6 *An (S,T)-anti if-ideal of a BCK/BCI-algebra X need not to be (S,T)-anti implicative if-ideal. For instance, in Example 2.7, we know that $A = (\mu_A, \lambda_A)$ is an (S_M, T_L)-anti if-ideal of X but it is not an (S_M, T_L)-anti implicative if-ideal of X, because $\mu_A(1) > S_M(\mu_A((1*(2*1))*0), \mu_A(0))$ and $\lambda_A(1) < T_L(\lambda_A((1*(2*1))*0), \lambda_A(0))$.*

Theorem 3.7 *Any (S,T)-anti implicative if-ideal of a BCK/BCI-algebra X is an (S,T)-anti if-ideal of X.*

Proof. In Definition 3.3, let $z = y$ and $y = x$. Hence $\mu_A(x) \leq S(\mu_A((x*(x*x))*y), \mu_A(y))$ and $\lambda_A(x) \geq T(\lambda_A((x*(x*x))*y), \lambda_A(y))$. Since $x*x = 0$ and $x*0 = x$, we obtain (ii) and (iii) in Definition 2.5. This completes the proof. ∎

Theorem 3.8 *Let $A = (\mu_A, \lambda_A)$ be an (S,T)-anti if-ideal of a BCK/BCI-algebra X. Then $A = (\mu_A, \lambda_A)$ is an (S,T)-anti implicative if-ideal of X iff $\mu_A(x) \leq \mu_A(x*(y*x))$ and $\lambda_A(x) \geq \lambda_A(x*(y*x))$ for all $x, y \in X$.*

Proof. Assume that $A = (\mu_A, \lambda_A)$ is an (S,T)-anti implicative if-ideal. Taking $z = 0$ in (ii) and (iii), and using (i) in Definition 3.3, we get the inequalities. Conversely, since $A = (\mu_A, \lambda_A)$ is an (S,T)-anti if-ideal, hence

$$\mu_A(x) \leq \mu_A(x*(y*x)) \leq S(\mu_A((x*(y*x))*z), \mu_A(z)),$$

$$\lambda_A(x) \geq \lambda_A(x*(y*x)) \geq T(\lambda_A((x*(y*x))*z), \lambda_A(z)).$$

This completes the proof. ∎

Lemma 3.9 *Let $A = (\mu_A, \lambda_A)$ be an (S,T)-anti if-ideal of a BCK/BCI-algebra X. If $x*y \leq z$ holds in X, then $\mu_A(x) \leq S(\mu_A(y), \mu_A(z))$ and $\lambda_A(x) \geq T(\lambda_A(y), \lambda_A(z))$ for all $x, y \in X$.*

Proof. Since $x * y \leq z$ holds *for all* $x, y \in X$, we have

$$\begin{aligned} \mu_A(x*y) &\leq S(\mu_A((x*y)*z), \mu_A(z)) \\ &\leq S(\mu_A(z*z), \mu_A(z)) \\ &= S(\mu_A(0), \mu_A(z)) \\ &\leq \mu_A(z) \end{aligned}$$

it follows that

$$\mu_A(x) \leq S(\mu_A(x*y), \mu_A(y)) \leq S(\mu_A(y), \mu_A(z)),$$

$$\begin{aligned} \lambda_A(x*y) &\geq T(\lambda_A((x*y)*z), \lambda_A(z)) \\ &\geq T(\lambda_A(z*z), \lambda_A(z)) \\ &= T(\lambda_A(0), \lambda_A(z)) \\ &\geq \lambda_A(z) \end{aligned}$$

hence

$$\lambda_A(x) \geq T(\lambda_A(x*y), \lambda_A(y)) \geq T(\lambda_A(y), \lambda_A(z)).$$

This completes the proof. ■

Theorem 3.10 ([12]) *A BCK-algebra is implicative iff it is both commutative and positive implicative.*

Theorem 3.11 ([12]) *If X is an implicative BCK-algebra, then $x * ((x * (y * x)) * z) \leq z$ for all $x, y, z \in X$.*

Theorem 3.12 *In an implicative BCK/BCI-algebra, every (S,T)-anti if-ideal is an (S,T)-anti implicative if-ideal.*

Proof. The proof is easily follows from Lemma 3.9 and Theorem 3.11. ■

Theorem 3.13 *The intersection of any set of (S,T)-anti implicative if-ideals of a BCK/BCI-algebra X is also an (S,T)-anti implicative if-ideal whenever S and T are continuous norms.*

Proof. Let $\left\{A_i = (\mu_{A_i}, \lambda_{A_i})\right\}_{i \in I}$ be a family of (S,T)-anti implicative if-ideals of X. Then, for any $x, y, z \in X$,

$$\begin{aligned} \left(\cap\ \mu_{A_i}\right)(0) = \inf\left\{\mu_{A_i}(0)\right\} \leq \inf\left\{\mu_{A_i}(x)\right\} = \left(\cap\ \mu_{A_i}\right)(x), \\ \left(\cap\ \lambda_{A_i}\right)(0) = \sup\left\{\lambda_{A_i}(0)\right\} \geq \sup\left\{\lambda_{A_i}(x)\right\} = \left(\cap\ \lambda_{A_i}\right)(x). \end{aligned}$$

Also,

$$\begin{aligned} \left(\cap\ \mu_{A_i}\right)(x) &= \inf\left\{\mu_{A_i}(x)\right\} \leq \inf\left\{S(\mu_{A_i}((x*y)*z), \mu_{A_i}(z))\right\} \\ &= S(\inf\left\{\mu_{A_i}((x*y)*z)\right\}, \inf\left\{\mu_{A_i}(z)\right\}) \\ &= S(\left(\cap\ \mu_{A_i}\right)((x*y)*z), \left(\cap\ \mu_{A_i}\right)(z)), \end{aligned}$$

$$\begin{aligned} \left(\cap\ \lambda_{A_i}\right)(x) &= \sup\left\{\mu_{A_i}(x)\right\} \geq \sup\left\{T(\lambda_{A_i}((x*y)*z), \lambda_{A_i}(z))\right\} \\ &= T(\sup\left\{\lambda_{A_i}((x*y)*z)\right\}, \sup\left\{\lambda_{A_i}(z)\right\}) \\ &= T(\left(\cap\ \lambda_{A_i}\right)((x*y)*z), \left(\cap\ \lambda_{A_i}\right)(z)). \end{aligned}$$

This completes the proof. ■

4 Conclusions

In this work, we introduce the notions of anti intuitionistic fuzzy BCK/BCI- subalgebras, anti intuitionistic fuzzy ideals and anti implicative intuitionistic fuzzy ideals with the help of arbitrary t-conorms and t-norms, and discuss some properties such as product, direct product and relations between them. But there are still some open problems. How can we define the notions of anti intuitionistic fuzzy filters and anti intuitionistic fuzzy congruences with recpect to arbitrary t-conorms and t-norms on a BCK/BCI-algebra? What are the relations between such notions, between the cosets of an anti intuitionistic fuzzy filter and anti intuitionistic fuzzy congruences? These could be a topic of further research. Furthermore, using that generalizations, one could define the notion anti intuitionistic fuzzy subgroups in BCK/BCI-algebras with respect to arbitrary t-conorms and t-norms in the sense of [1] and [7]. Using the idea of Dudek et al. [5], one could also generalize the notion of fuzzy topological anti BCK/BCI-algebras to intuitionistic fuzzy structures.

The notions given in this paper can be fundamental to other sciences. For instance, in the last decade, most of researchers are focused on Content Based Image Retrieval, shortly CBIR, and managing uncertainty becomes a fundamental topic in image database. Intuitionistic fuzzy set theory can be ideally suited to deal with this kind of uncertainty. This fuzziness is mainly due to similarity of media features, imperfection in the feature extraction algorithms, esc. Using the concept of this paper, one could develop an anti intuitionistic fuzzy model for image data and provide an anti intuitionistic fuzzy subalgebra for dealing with such data. Moreover, new anti intuitionistic fuzzy algebraic operators could be defined in order to capture the fuzziness related to the semantic descriptors of an image, and built thematic categorizations of multimedia documents using ontological information and anti intuitionistic fuzzy subalgebra in triangular norm systems.

Problem. Can we replace in the statement of Theorem 3.13, the condition "S and T are continuous" with "$\inf_{a>0} S(a,a)=0$ and $\sup_{b<1} T(b,b)=1$"?

References

[1] M.T. Abu Osman, On some product of fuzzy subgroups, Fuzzy Sets and Systems 24(1987), 79-86.

[2] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20(1986), 87-96.

[3] K. Atanassov, New operations defined over the intuitionistic fuzzy sets, Fuzzy Sets and Systems 61(1994), 137-142.

[4] R. Biswas, Fuzzy subgroups and anti fuzzy subgroups, Fuzzy Sets and Systems 35(1990), 121-124.

[5] W.A. Dudek, J.B. Jun, S.M. Hong, On fuzzy topological BCC-algebras, Discuss Math. Gen. Algebra Appl. 20(1)(2000), 77-86.

[6] O. Hadzic, E. Pap, Fixed point theory in probabilistic metric spaces, Dordrecht, Kluwer Acad. Publishers, 2001.

[7] Y. Imai, K. Iseki, On axiom systems of propositional calculi, Proc. Japan Academy 42(1966), 19-22.

[8] K. Iseki, S. Tanaka, An introduction to the theory of BCK-algebras, Math. Japon. 22(1978), 1-26.

[9] E.P. Klement, R. Mesiar, E. Pap, Triangular norms, Dordrecht, Kluwer Acad. Publishers, 2000.

[10] S. Kutukcu, S. Sharma, On Anti Fuzzy Structures in BCC-Algebras, J. Concr. Appl. Math. 6(1)(2008), 9-19.

[11] R. Lowen, Fuzzy set theory, Dordrecht, Kluwer Acad. Publishers, 1996.

[12] M.X. Luo, Fuzzy implicative ideals, J. Shaanxi Normal Univ. nat. Sci. Ed. 29(2001), 24-28.

[13] K. Menger, Statistical metrics, Proc. Nat. Acad. Sci. 28(1942), 535-537.

[14] B. Schweizer, A. Sklar, Statistical metric spaces, Pacific J. Math. 10(1960), 314-334.

[15] O.G. Xi, Fuzzy BCK-algebras, Math. Japon. 36(5)(1991), 935-942.

[16] L.A. Zadeh, Fuzzy sets, Inform. Control 8(1965), 338-353.r

On (IT)-commutativity condition in fixed point consideration of set-valued and single-valued mappings

Servet Kutukcu*, **Sushil Sharma and Bhavana Deshpande**

Department of Mathematics, Faculty of Science Arts, Ondokuz Mayıs University, 55139, Samsun, Turkey
Department of Mathematics, Madhav Science College, Ujjain (M.P.), India
Department of Mathematics, Govt Arts Science, P. G. College Ratlam (M. P.), India

Abstract: The concept of (IT)-commutativity and property (E.A) between set-valued mappings and single-valued mappings are used to prove some common fixed point theorems on metric spaces without taking condition of continuity.

Keywords: (IT)-commutativity, (E.A) property, fixed point.

1 Introduction

Banach contraction principle or Banach's fixed point theorem provides a technique for solving a variety of applied problems in mathematical science and engineering. Many authors have extended, generalized and improved Banach's fixed point theorem in different ways. In [9] Jungck introduced more generalized commuting mappings, called compatible mappings, which are more general than commuting and weakly commuting mappings. This concept has been useful for obtaining more comprehensive fixed point theorems (Jeong and Rhoades [8], Jungck [9,10], Jungck and Rhoades [11], Kang and Rhoades [13]).

Recently, Jungck and Rhoades [11,12] defined the concepts of $\delta-$compatible mappings and weakly compatible mappings, respectively, which extend the concept of compatible mappings in single-valued settings to set-valued mappings. They showed that compatible mappings and $\delta-$compatible mappings are weakly compatible but the converse does not need to be true. Several authors used these concepts to prove some common fixed point theorems (Ahmed [2], Chang[3], Rashwan and Ahmed [20,21], Rhoades [22]). Pant [16-19] initiated the study of noncompatible maps and introduced pointwise R-weak commutativity of mappings. He also showed that pointwise R- weak commutativity is equivalent to weak compatibility at coincidence points for single-valued mappings. Following Itoh and Takahshi [7], Singh and Mishra [23] introduced the notion of (IT)-commutativity for single-valued and multi-valued mappings. They showed that (IT)-commutativity of hybrid pair is more general than their

*Corresponding Author
E-mail: skutukcu@omu.edu.tr (S. Kutukcu), sksharma2005@yahoo.com (S. Sharma), bhavnadeshpande@yahoo.com (B. Deshpande)
2010 MR Subject Classification: 47H10, 54H25

weak compatibility at the same point. More recently, Aamri and Moutawakil [1] defined a property (E.A) for self single valued maps and obtained some fixed point theorems for such mappings under strict contractive conditions. The class of (E.A) maps contains the class of noncompatible maps.

In this paper, the concept of (IT)-commutativity and property (E.A) between single-valued mappings and set-valued mappings are used to prove some common fixed point theorems on metric spaces without taking any mapping continuous. We improve and generalize the results of Ahmed [2], Fisher [4] and Khan et al. [15].

2 Preliminaries

In the sequel, (X, d) denotes a metric space and $B(X)$ is the set of all non empty bounded subsets of X. As in [3] and [6], we define

$$\begin{aligned} \delta(A,B) &= \sup\{d(a,b) : a \in A, b \in B\}, \\ D(A,B) &= \inf\{d(a,b) : a \in A, b \in B\}, \\ H(A,B) &= \inf\{r > 0 : A_r \supset B, B_r \supset A\} \end{aligned}$$

for all A, B in $B(X)$ where $A_r = \{x \in X : d(x,a) < r$ for some $a \in A\}$ and $B_r = \{y \in X : d(y,b) < r$ for some $b \in B\}$.

If $A = \{a\}$ for some $a \in A$, we denote $\delta(a,B)$, $D(a,B)$ and $H(a,B)$ for $\delta(A,B)$, $D(A,B)$ and $H(A,B)$ respectively. Also, if $B = \{b\}$ and $A = \{a\}$, one can deduce that $\delta(A,B) = D(A,B) = H(A,B) = d(a,b)$.

It follows immediately from the definition of $\delta(A,B)$ that $\delta(A,B) = \delta(B,A) \geq 0$, $\delta(A,B) \leq \delta(A,C) + \delta(C,B)$, $\delta(A,B) = 0$ iff $A = B = \{a\}$, $\delta(A,A) = diamA$ for all $A, B, C \in B(X)$.

Definition 2.1 ([6]) *A sequence $\{A_n\}$ of nonempty subsets of X is said to be convergent to a subset A of X if*

(i) Each point a in A is the limit of a convergent sequence $\{a_n\}$, where a_n is in A_n for all $n \in N$.

(ii) For arbitrary $\epsilon > 0$, there exists an integer m such that $A_n \subseteq A_\epsilon$ for $n > m$, where A_ϵ denotes the set of all points x in X for which there exists a point a in A, depending on x, such that $d(x,a) < \epsilon$.

A is said to be the limit of the sequence $\{A_n\}$.

Lemma 2.2 ([6]) *If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ converging to A and B in $B(X)$, respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A,B)$.*

Lemma 2.3 ([6]) *If $\{A_n\}$ is a sequence in $B(X)$ and y is a point in X such that $\delta(A_n, y) \to 0$, then the sequence $\{A_n\}$ converges to the set $\{y\}$ in $B(X)$.*

Lemma 2.4 ([3]) *For $A, B, C, D \in B(X)$, we have*

$$\delta(A,B) \leq H(A,C) + \delta(C,D) + H(D,B).$$

Definition 2.5 ([6]) *The mappings $I : X \to X$ and $F : X \to B(X)$ are said to be weakly commuting if $IFx \in B(X)$ and $\delta(FIx, IFy) \leq \max\{\delta(Ix, Fx), diamIFx\}$ for all x in X.*

Definition 2.6 ([11]) *The mappings $I : X \to X$ and $F : X \to B(X)$ are δ-compatible if the limit $lim_{n\to\infty}\delta(FIx_n, IFx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $IFx_n \in B(X)$, $Fx_n \to \{t\}$ and $Ix_n \to t$ for some t in X.*

Definition 2.7 ([12]) *The mappings $I : X \to X$ and $F : X \to B(X)$ are weakly compatible if they commute at coincidence points, i.e. for each point $u \in X$ such that $Fu = \{Iu\}$, we have $FIu = IFu$ (Note that the equation $Fu = \{Iu\}$ implies that Fu is singleton).*

Itoh and Takahsi [7], and Singh and Mishra [23] defined the (IT)-commutativity for single-valued and set-valued mappings is as follows:

Definition 2.8 *The mappings $I : X \to X$ and $F : X \to B(X)$ are said to be (IT)-commuting (Itoh-Takahasi commutativity is simply called (IT)-commuting) at $x \in X$ if $IFx \subset FIx$. I and F are (IT)-commuting on X if they are (IT)-commuting at each point of X.*

In [14], the property (E.A) for single-valued and set-valued mappings is defined as follows:

Definition 2.9 *The mappings $I : X \to X$ and $F : X \to B(X)$ are said to satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X such that $lim_{n\to\infty}Ix_n = t$ and $lim_{n\to\infty}Fx_n = \{t\}$ for some $t \in X$.*

Remark 2.10 *Let X is a metric space, $I : X \to X$ and $F : X \to B(X)$. Then it is clear from Jungck and Rhoade's [11] definition that I and F will not be δ-compatible if there exists at least one sequence $\{x_n\}$ in X such that $lim_{n\to\infty}Ix_n = t$ and $lim_{n\to\infty}Fx_n = \{t\}$ for some $t \in X$, but $\delta(FIx_n, IFx_n)$ is either non- zero or non existent. Thus two non δ-compatible maps satisfy the property (E.A).*

Example 2.11 *Let $X = [1,\infty]$ with the usual metric. Define $I : X \to X$, $F : X \to B(X)$ by $Ix = x + 1$ and $Fx = [1, x + 1]$ for all $x \in X$. Then $IF_1 \subset FI_1$. Therefore, I and F are IT-commuting at $x = 1$. But I and F are not weakly compatible since $IF_1 \neq FI_1$. Consider the sequence $\{x_n\} = \{1/n\}$. Clearly $lim_{n\to\infty}Ix_n = 1$ and $lim_{n\to\infty}Fx_n = \{1\}$. Thus I and F satisfy the property (E.A). But I and F are not δ-compatible.*

Remark 2.12 *It is clear from Remark 1 and Example 1 that if $I : X \to X$ and $F : X \to B(X)$ are (IT)-commuting maps then I and F satisfy the property (E.A).*

Let $I : X \to X$ and $F : X \to B(X)$. In all that follows, $C(F, I)$ stands for the set of coincidence points of the maps F and I, that is $C(F, I) = \{z : \{Iz\} = Fz\}$.

3 Main Results

Theorem 3.1 *Let (X, d) be a complete metric space. Let I, J be mappings of X into itself and F, G of X into $B(X)$ such that*

(1.1) $\delta(Fx, Gy) \leq \max\{cd(Ix, Jy), c\delta(Ix, Fx), c\delta(Jy, Gy), aD(Ix, Gy)+bD(Jy, Fx)\}$ for all $x, y \in X$, where $0 \leq c < 1$, $a, b \geq 0$, $a + b < 1$, $c\max\{\frac{a}{1-a}, \frac{b}{1-b}\} < 1$,

(1.2) $\cup F(X) \subset J(X)$ and $\cup G(X) \subset I(X)$, then

3

(i) F and I have a coincidence point.
(ii) G and J have a coincidence point.
Further if

(1.3) F and I are (IT)- commuting at $p \in C(F, I)$, G and J are (IT)- commuting at $q \in C(G, J)$, then

(iii) I, J, F and G have a unique common fixed point $u \in X$.

Proof. Let x_0 be an arbitrary point in X. By (1.2), we choose a point x_1 in X such that $Jx_1 \in Fx_0 = Z_0$ and for this point x_1, there exists a point x_2 in X such taht $Ix_2 \in Gx_1 = Z_1$ and so on. Continuing in this manner, we can define a sequence $\{x_n\}$ as follows:

(1.4) $Jx_{2n+1} \in Fx_{2n} = Z_{2n}$ and $Ix_{2n+2} \in Gx_{2n+1} = Z_{2n+1}$, $n \in N \cup \{0\}$.

For simplicity, we put $V_n = \delta(Z_n, Z_{n+1})$ for $n \in N \cup \{0\}$. By (1.1) and (1.4), we have

$$\begin{aligned} V_{2n} &= \delta(Z_{2n}, Z_{2n+1}) = \delta(Fx_{2n}, Gx_{2n+1}) \\ &\leq \max\{cd(Ix_{2n}, Jx_{2n+1}), c\delta(Ix_{2n}, Fx_{2n}), c\delta(Jx_{2n+1}, Gx_{2n+1}), \\ &\quad aD(Ix_{2n}, Gx_{2n+1}) + bD(Jx_{2n+1}, Fx_{2n})\} \\ &\leq \max\{cV_{2n-1}, cV_{2n}, a(V_{2n-1} + V_{2n})\} \\ &\leq \max\{c, \frac{a}{1-a}\}V_{2n-1} \end{aligned}$$

for $n \in N$. Similarly, one can show that

$$V_{2n+1} = \delta(Z_{2n+1}, Z_{2n+2}) = \delta(Gx_{2n+1}, Fx_{2n+2}) \leq \max\{c, \frac{b}{1-b}\}V_{2n}$$

for $n \in N$. If we put $\beta = \max\{c, \frac{a}{1-a}\}. \max\{c, \frac{b}{1-b}\}$, then by hypothesis it can be easily seen that $0 \leq \beta < 1$. So we deduce that

(1.5) $V_{2n} \leq \beta V_{2n-2} \leq .... \leq \beta^n V_0$ and $V_{2n+1} \leq \beta V_{2n-1} \leq .... \leq \beta^n V_1$

for $n \in N$. Put $M = \max\{V_0, V_1\}$. It follows from the inequality (1.5) that if z_n is an arbitrary point in the set Z_n for $n \in N$, then we obtain that $d(z_{2n}, z_{2n+1}) \leq \delta(Z_{2n}, Z_{2n+1}) \leq \beta^n M$ and $d(z_{2n+1}, z_{2n+2}) \leq \delta(Z_{2n+1}, Z_{2n+2}) \leq \beta^n M$. This implies that $\{z_n\}$ is a Cauchy sequence in the complete metric space X. Hence it converges to a point $u \in X$, which does not depend upon the particular choice of each z_n. In particular, the sequences $\{Ix_{2n}\}$ and $\{Jx_{2n+1}\}$ converge to u and the sequences of sets $\{Fx_{2n}\}$ and $\{Gx_{2n+1}\}$ converge to the set $\{u\}$.

Since $\cup G(X) \subset I(X)$, there exists a point $p \in X$ such that $\{u\} = \{Ip\}$. Then using (1.1), we have

$$\begin{aligned} \delta(Fp, Gx_{2n+1}) &\leq \max\{cd(Ip, Jx_{2n+1}), c\delta(Ip, Fp), c\delta(Jx_{2n+1}, Gx_{2n+1}), \\ &\quad aD(Ip, Gx_{2n+1}) + bD(Jx_{2n+1}, Fp)\} \\ &\leq \max\{cd(u, Jx_{2n+1}), c\delta(u, Fp), c\delta(Jx_{2n+1}, Gx_{2n+1}), \\ &\quad a\delta(u, Gx_{2n+1}) + b\delta(Jx_{2n+1}, Fp)\}. \end{aligned}$$

Taking the limit as $n \to \infty$, we have

$$\delta(Fp, u) \leq \max\{c\delta(u, Fp), b\delta(u, Fp)\} = \max\{c, b\}\delta(Fp, u)$$

and hence $Fp = \{u\}$, since $\max\{c, b\} < 1$. Therefore $\{Ip\} = \{u\} = Fp$. Thus Fp is singleton and $p \in C(F, I)$.

Since $\cup F(X) \subset J(X)$, there exists a point $q \in X$ such that $\{u\} = \{Jq\}$. Then using (1.1), we have

$$\begin{aligned}
\delta(u, Gq) &= \delta(Fp, Gq) \\
&\leq \max\{cd(u,u), c\delta(u,u), c\delta(u,Gq), a\delta(u,Gq) + b\delta(u,u)\} \\
&= \max\{c\delta(u,Gq), a\delta(u,Gq)\} \\
&= \max\{c,a\}\delta(u,Gq).
\end{aligned}$$

Since $\max\{c,a\} < 1$, then $\{u\} = Gq = \{Jq\}$. Thus Gq is singleton and $q \in C(G,J)$.

Since F and I are (IT)-commuting at $p \in C(F,I)$, so $IFp \subset FIp$, that is, $\{Iu\} \subset Fu$. Using (1.1), we have

$$\begin{aligned}
\delta(Fu, Gx_{2n+1}) &\leq \max\{cd(Iu, Jx_{2n+1}), c\delta(Iu, Fu), c\delta(Jx_{2n+1}, Gx_{2n+1}), \\
&\quad aD(Iu, Gx_{2n+1}) + bD(Jx_{2n+1}, Fu)\} \\
&\leq \max\{c\delta(Fu, Jx_{2n+1}), c\delta(Jx_{2n+1}, Gx_{2n+1}), \\
&\quad a\delta(Fu, Gx_{2n+1}) + b\delta(Jx_{2n+1}, Fu)\}.
\end{aligned}$$

Taking the limit as $n \to \infty$, we have

$$\begin{aligned}
\delta(Fu, u) &\leq \max\{c\delta(Fu,u), a\delta(Fu,u) + b\delta(u,Fu)\} \\
&= \max\{c, a+b\}\delta(Fu,u).
\end{aligned}$$

Since $\max\{c, a+b\} < 1$, then $\{u\} = Fu$. Since $\{u\} = \{Iu\}$, $\{Iu\} = Fu = \{u\}$. Similarly we can prove that $Gu = \{Ju\} = \{u\}$. Thus u is a common fixed point of I, J, F and G.

Now suppose there exists $w \in X, u \neq w$ such that $Fw = \{w\} = \{Iw\}$. Using (1.1), we obtain that

$$\begin{aligned}
d(w,u) &\leq \delta(Fw, Gu) \\
&\leq \max\{cd(w,u), ad(w,u) + bd(u,w)\} \\
&= \max\{c, a+b\}d(w,u).
\end{aligned}$$

Since $\max\{c, a+b\} < 1$, it follows that $u = w$. So u is unique common fixed point of F and I such that $Fu = \{u\} = \{Iu\}$. Similarly, it can be shown that u is the unique common fixed point of G and J such that $Gu = \{u\} = \{Ju\}$. This completes the proof. ■

By using the property (E.A) and by removing the completeness of the space with a set of alternative conditions in Theorem 3.1, we prove the following:

Theorem 3.2 *Let (X,d) be a metric space. Let I, J be mappings of X into itself and F, G of X into $B(X)$ satisfying condition (1.1), (1.2) and*

(2.1) $\{F, I\}$ or $\{G, J\}$ satisfy the property (E.A),

(2.2) if the range of one of $I(X)$, $J(X)$, $F(X)$ or $G(X)$ is a complete subspace of X, then

(i) F and I have a coincidence point

(ii) G and J have a coincidence point.

Further if (1.3) is satisfied then

5

(iii) I, J, F and G have a unique common fixed point.

Proof. Suppose that the pair $\{F, I\}$ satisfies the property (E.A). Then there exists a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} Ix_n = u$ and $lim_{n\to\infty} Fx_n = \{u\}$ for some $u \in X$. Since $\cup F(X) \subset J(X)$, there exists a sequence $\{y_n\}$ in X such that $lim_{n\to\infty} Fx_n = lim_{n\to\infty} Jy_n$. Hence $lim_{n\to\infty} Jy_n = u$. Let us now show that $lim_{n\to\infty} Gy_n = \{u\}$. Indeed in view of (1.1), we have

$$\begin{aligned}\delta(Fx_n, Gy_n) &\leq \max\{cd(Ix_n, Jy_n), c\delta(Ix_n, Fx_n), c\delta(Jy_n, Gy_n),\\ &\quad aD(Ix_n, Gy_n) + bD(Jy_n, Fx_n)\}.\end{aligned}$$

Taking the limit as $n \to \infty$ and since $\max\{a, c\} < 1$, we have $lim_{n\to\infty} Gy_n = \{u\}$. Suppose that $J(X)$ is complete subspace of X. Then $u = Jq$ for some $q \in X$. Subsequently, we have $lim_{n\to\infty} Gy_n = \{Jq\} = lim_{n\to\infty} Fx_n$ and $lim_{n\to\infty} Ix_n = lim_{n\to\infty} Jy_n = Jq$. Using (1.1), we have

$$\begin{aligned}\delta(Fx_n, Gq) &\leq \max\{cd(Ix_n, Jq), c\delta(Ix_n, Fx_n), c\delta(Jq, Gq),\\ &\quad aD(Ix_n, Gq) + bD(Jq, Fx_n)\}.\end{aligned}$$

Taking the limit as $n \to \infty$ and since $\max\{a, c\} < 1$, we have $Gq = \{Jq\} = \{u\}$. Thus Gq is singleton and $q \in C(G, q)$. On the other hand $\cup G(X) \subset I(X)$, there exists $p \in X$ such that $\{Ip\} = Gq$. Using (1.1), we have

$$\begin{aligned}\delta(Fp, Gq) &\leq \max\{cd(Ip, Jq), c\delta(Ip, Fp), c\delta(Jq, Gq),\\ &\quad aD(Ip, Gq) + bD(Jq, Fp)\}\\ &\leq \max\{c, b\}\delta(Ip, Fp).\end{aligned}$$

Since $\max\{c, b\} < 1$, so $\{Ip\} = Fp$ that is $p \in C(F, I)$ and $\{Ip\} = Fp = \{u\} = \{Jq\} = Gq$. Since F and I are (IT)-commuting at $p \in X$. Therefore $IFp \subset FIp$ which implies $\{Iu\} \subset Fu$. Similarly $\{Ju\} \subset Gu$. By using (1.1), we have

$$\begin{aligned}\delta(Fu, Gy_n) &\leq \max\{cd(Iu, Jy_n), c\delta(Iu, Fu), c\delta(Jy_n, Gy_n),\\ &\quad aD(Iu, Gy_n) + bD(Jy_n, Fu)\}\\ &\leq \max\{c\delta(Fu, Jy_n), c\delta(Fu, Fu), c\delta(Jy_n, Gy_n),\\ &\quad a\delta(Fu, Gy_n) + b\delta(Jy_n, Fu)\}.\end{aligned}$$

Taking the limit as $n \to \infty$, we have $\delta(Fu, u) \leq \max\{c, a + b\}\delta(Fu, u)$, then $Fu = \{u\}$. Since $\{Iu\} \subset Fu$ and Fu is singleton, $Iu = u$. Therefore u is common fixed point of F and I. Similarly we can prove that u is common fixed point of G and J.

The proof is similar when $I(X)$ is assumed to be complete subspace of X. The cases in which $F(X)$ or $G(X)$ is a complete subspace of X are similar to the cases in which $J(X)$ or $I(X)$, respectively, is complete since $\cup F(X) \subset J(X)$ and $\cup G(X) \subset I(X)$. Hence u is common fixed point of I, J, F and G. The uniqueness of common fixed point $u \in X$ follows from (1.1). This completes the proof. ■

With the help of Remark 2.12 and Theorem 3.2, we get the following:

Corollary 3.3 *Let (X, d) be a metric space. Let I, J be mappings of X into itself and F, G of X into $B(X)$ satisfying condition (1.1), (1.2), (2.2) and the pairs $\{F, I\}$, $\{G, J\}$ are (IT)-commuting. Then I, J, F and G have a unique common fixed point.*

The conditions (1.2) and (2.2) can be removed by taking $I(X)$ and $J(X)$ as closed subspaces of X. We prove the following:

6

Theorem 3.4 *Let (X, d) be a metric space. Let I, J be mappings of X into itself and F, G of X into $B(X)$ satisfying condition (1.1) and*

(3.1) the pairs $\{F, I\}$ and $\{G, J\}$ satisfy the property (E.A),

(3.2) $I(X)$ and $J(X)$ are closed subspaces of X, then

(i) F and I have a coincidence point,
(ii) G and J have a coincidence point.
Further if (1.3) is satisfied then
(iii) I, J, F and G have a unique common fixed point.

Proof. Since the pair $\{F, I\}$ satisfies the property (E.A), then there exists a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} Ix_n = u$ and $lim_{n\to\infty} Fx_n = \{u\}$ for some $u \in X$. Again since the pair $\{G, J\}$ satisfies the property (E.A), then there exists a sequence $\{y_n\}$ in X such that $lim_{n\to\infty} Jy_n = w$ and $lim_{n\to\infty} Gy_n = \{w\}$ for some $w \in X$. If $w \neq u$, then by using (1.1), we have

$$\begin{aligned} \delta(Fx_n, Gy_n) &\leq \max\{cd(Ix_n, Jy_n), c\delta(Ix_n, Fx_n), c\delta(Jy_n, Gy_n), \\ &\qquad aD(Ix_n, Gy_n) + bD(Jy_n, Fx_n)\}. \end{aligned}$$

Taking the limit as $n \to \infty$, we have

$$\begin{aligned} \delta(u, w) &\leq \max\{cd(u,\ w), ad(u, w) + bd(u, w)\} \\ &= \max\{c, a + b\}d(u, w). \end{aligned}$$

Since $\max\{c, a + b\} < 1$, it follows that $u = w$. Since $I(X)$ is closed, we have $lim_{n\to\infty} Ix_n = Ip$ for some $p \in X$. Thus $Ip = u$. Subsequently, we have $lim_{n\to\infty} Fx_n = \{Ip\}$. From (1.1), we have

$$\begin{aligned} \delta(Fp, Gy_n) &\leq \max\{cd(Ip, Jy_n), c\delta(Ip, Fp), c\delta(Jy_n, Gy_n), \\ &\qquad aD(Ip, Gy_n) + bD(Jy_n, Fp)\} \end{aligned}$$

Taking the limit as $n \to \infty$, we have $\delta(Fp, u) \leq \max\{c, b\}\delta(u, Fp)$ and hence $Fp = \{u\} = \{Ip\}$, since $\max\{c, b\} < 1$. Since $J(X)$ is closed, we have $lim_{n\to\infty} Jy_n = Jq$ for some $q \in X$. Thus $Jq = u$. Subsequently, we have $lim_{n\to\infty} Gy_n = \{Jq\} = \{u\}$. Using (1.1), we have

$$\begin{aligned} \delta(u, Gq) &= \delta(Fp, Gq) \leq \max\{cd(Ip, Jq), c\delta(Ip, Fp), c\delta(Jq, Gq), \\ &\qquad aD(Ip, Gq) + bD(Jq, Fp)\} \\ &= \max\{c\delta(u, Gq), a\delta(u, Gq)\} \\ &= \max\{c, a\}\delta(u, Gq). \end{aligned}$$

Since $\max\{c, a\} < 1$ then $\{u\} = Gq$. Thus Gq is singleton and $q \in C(G, J)$. The remaining part of the proof is same as that of Theorem 3.1. This completes the proof. ■

With the help of Remark 2.12 and Theorem 3.4, we get the following:

Corollary 3.5 *Let (X, d) be a metric space. Let I, J be mappings of X into itself and F, G of X into $B(X)$ satisfying condition (1.1), (3.2) and the pairs $\{F, I\}$ and $\{G, J\}$ are (IT)-commuting then I, J, F and G have a fixed point.*

References

[1] M. Aamri, D. El. Moutawakil, Some new common fixed point theorems under strict contractive conditions. J. Math. Anal. Appl. 270(2002), 181-188.

[2] M. A. Ahmed, Common fixed point theorems for set- valued and single-valued. Demonstratio Math. Vol. 36(2)(2003), 471-481.

[3] T. H. Chang, Fixed point theorems for contractive type set- valued mappings, Math. Japonica 38(4)(1993), 675-690.

[4] B. Fisher, Common fixed points of mappings and set-valued mappings on a metric spaces.Kyungpook Math. J. 25(1985), 35-42.

[5] B. Fisher and S. Sessa, Common fixed points of two pairs of weakly commutings, Univ. U. Novom Sadu. Zb. Rad. Period Mat. Fak. Ser. Mat 18(1986), 45-59.

[6] B. Fisher and S. Sessa, Two common fixed point theorems for weakly commuting mappings, Periodica Math. Hungraica, 20(3)(1989), 207-218.

[7] S. Itoh, W. Takahashi, Single-valued mappings, Multivalued mappings and fixed point theorems, J. Math. Anal. Appl. 59(1977), 514-521.

[8] G. S. Jeong and B. E. Rhoades, Some remarks for involving fixed point theorems for more than two maps, Ind. J. Pure Appl. Math. 28(9)(1997), 1171-1196.

[9] G. Jungck, Compatible mappings and common fixed points, Internat J. Math. Sci., 9(1986), 771-779.

[10] G. Jungck, Common fixed points of commuting and compatible maps on compacta, Proc. Amar. Math. Soc. 103(1988), 977-983.

[11] G. Jungck and B. E. Rhoades, Some fixed point theorems for compatible maps, Internat J. Math. and Math. Sci. 16(1993), 417-428.

[12] G. Jungck and B. E. Rhoades, Fixed points for set-valued functions without continuity, Ind. J. Pure and Appl. Math. 16(3)(1998), 227-238.

[13] S. M. Kang and B. E. Rhoades, Fixed points for four mappings, Math. Japonica 37(1992), 1053-1059.

[14] T. Kamran, Coincidence and fixed points for hybrid strict contraction, J. Math. Anal. Appl. 299(2004), 235-241.

[15] M. S. Khan, I. Kubiaczyk and S. Sessa, On set-valued and single-valued mappings with common fixed points, Bull. Malaysian Math. Soc. 2(10)(1987), 71-81.

[16] R. P. Pant, Common fixed points of non-commuting mappings, J. Math. Anal. Appl.. 188(1994), 436-440.

[17] R. P. Pant, Common fixed point theorems for contractive maps, J. Math. Anal. Appl. 226-(1998), 251-258.

[18] R. P. Pant, Common fixed points of Lipschitz type mapping pairs, J. Math. Anal. Appl. 240(1999), 280-283.

[19] R. P. Pant, Discontinuity and fixed points, J. Math. Anal. Appl. 240(1999), 284-289.

[20] R. A. Rashwan and M. A. Ahmed, Common fixed points for δ-compatible mappings, Southwest J. Pure Appl. Math. 1(1996), 51-61.

[21] R. A. Rashwan and M. A. Ahmed, Fixed points of single and set-valued mappings, Kyungpook Math. J. 38(1998), 29-37.

[22] B. E. Rhoades, Common fixed points of compatible set-valued mappings, Publ. Math. Debrecen 48, No. 3-4(1996), 237-240.

[23] S. L. Singh and S. N. Mishra, Coincidences and fixed points of nonself hybrid contraction, J. Math. Anal. Appl. 256(2001), 486-497.

Some Generalizations in Intuitionistic Fuzzy Metric Spaces

Servet Kutukcu*   and Sunny Chauhan

Department of Mathematics, Faculty of Science and Arts, Ondokuz Mayıs University, 55139, Samsun, Turkey
Near Nehru Training Centre, H.No:274, Nai Basti B-14, Bijnor-246701
Uttar Pradesh, India

Abstract: The aim of this paper is to give some new fixed point theorems for contractive type mappings in intuitionistic fuzzy metric spaces. The results presented improve and generalize some well known results in literature.

Keywords: Intuitionistic fuzzy metric space; fuzzy metric space; complete; compact; fixed point.

1 Introduction

In [6], the well-known fixed point theorems of Banach [1] and Edelstein [3] were extended to fuzzy metric spaces obtaining the following two theorems,

Theorem 1.1 *Let $(X, M, *)$ be a complete fuzzy metric space. Let $T : X \to X$ be a mapping satisfying*

$$M(Tx, Ty, kt) \geq M(x, y, t)$$

for all $x, y \in X$ where $0 < k < 1$. Then T has a unique fixed point.

Theorem 1.2 *Let $(X, M, *)$ be a compact fuzzy metric space. Let $T : X \to X$ be a mapping satisfying*

$$M(Tx, Ty, .) > M(x, y, .)$$

for all $x \neq y$ (i.e. $M(Tx, Ty, .) \geq M(x, y, .)$ and $M(Tx, Ty, .) \neq M(x, y, .)$ for all $x \neq y$). Then T has a unique fixed point.

In this paper, we give some new fixed point theorems in intuitionistic fuzzy metric spaces. The results not only improve and generalize Theorems 1.1 and 1.2, but also unify and extend some main results of [5-8]. Let us call some basic definitions,

*Corresponding Author
E-mail: skutukcu@omu.edu.tr (S. Kutukcu), sun.gkv@gmail.com (S. Chauhan)
2010 MR Subject Classification: 47H10, 54H25

Definition 1.3 ([2]) *The 5-tuble $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$,*

(i) $M(x, y, t) + N(x, y, t) \leq 1$;

(ii) $M(x, y, 0) = 0$;

(iii) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$;

(iv) $M(x, y, t) = M(y, x, t)$;

(v) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $t, s > 0$;

(vi) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous;

(vii) $\lim_{t\to\infty} M(x, y, t) = 1$;

(viii) $N(x, y, 0) = 1$;

(ix) $N(x, y, t) = 0$ for all $t > 0$ iff $x = y$;

(x) $N(x, y, t) = N(y, x, t)$;

(xi) $N(x, z, t + s) \leq N(x, y, t) \Diamond N(y, z, s)$ for all $t, s > 0$;

(xii) $N(x, y, .) : [0, \infty) \to [0, 1]$ is right continuous;

(xiii) $\lim_{t\to\infty} N(x, y, t) = 0$.

Remark 1.4 *By (iii) and (v), it is easy to show that $M(x, y, .)$ is non-decreasing and by (ix) and (xi), it is easy to show that $N(x, y, .)$ is non-increasing for all $x, y \in X$.*

Remark 1.5 *Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1-M, *, \Diamond)$ such that t-norm $*$ and t-conorm $\Diamond$ are assosiated, i.e. $x\Diamond y = 1-((1-x)*(1-y))$ for any $x, y \in [0, 1]$. Moreover, Theorem 1.1 is a fuzzy generalization of Theorem 2 in [4].*

Definition 1.6 ([2]) *A sequence $\{x_n\}$ in an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is a Cauchy sequence iff $\lim_n M(x_{n+p}, x_n, t) = 1$ and $\lim_n N(x_{n+p}, x_n, t) = 0$ for each $t > 0$ and $p \in \mathbb{N}$. A sequence $\{x_n\}$ in X is convergent to $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ and $\lim_{n\to\infty} N(x_n, x, t) = 0$ for each $t > 0$. An intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is called complete, if every Cauchy sequence in X is convergent. It is called compact, if every sequence in X contains a convergent subsequence.*

Lemma 1.7 ([2]) *Let $\lim_n x_n = x$ and $\lim_n y_n = y$. Then, for all $t > 0$,*

$$\lim_{n\to\infty} \inf M(x_n, y_n, t) \geq M(x, y, t) \text{ and } \lim_{n\to\infty} \sup N(x_n, y_n, t) \leq N(x, y, t),$$

$$\lim_{n\to\infty} \sup M(x_n, y_n, t) \leq M(x, y, t) \text{ and } \lim_{n\to\infty} \inf N(x_n, y_n, t) \geq N(x, y, t).$$

Particularly, if $M(x, y, .)$ and $N(x, y, .)$ are continuous at point t, then

$$\lim_{n\to\infty} M(x_n, y_n, t) = M(x, y, t) \text{ and } \lim_{n\to\infty} N(x_n, y_n, t) = N(x, y, t).$$

2 Main Results

Definition 2.1 *Let the function $\phi : [0, \infty) \to [0, \infty)$ satisfies the following conditions:*

(ϕ_1) *$\phi(t)$ is strictly increasing, $\phi(0) = 0$ and $\lim_n \phi^n(t) = \infty$ for all $t > 0$, where $\phi^n(t)$ denotes the n-th iterative function of $\phi(t)$. Then $\phi(t) > t$* and *$\phi^n(t) > \phi^{n-1}(t)$ for $t > 0$* and *$n = 1, 2, ...$*

(ϕ_2) $\lim_{t\to\infty}[\phi(t) - t] = \infty$.

Lemma 2.2 *Let $*$ and $\Diamond$ be continuous t-norm and t-conorm, respectively. Then for each $\lambda \in (0, 1)$, there is a sequence $\{\lambda_n\}$ in $(0, 1)$ such that*

$$(1 - \lambda_n) * (1 - \lambda_n) > 1 - \lambda_{n-1} \text{ and } \lambda_n \Diamond \lambda_n < \lambda_{n-1},\ n = 1, 2, ...,$$

where $\lambda_0 = \lambda$ (obviously, the sequence $\{\lambda_n\}$ satisfying the condition is decreasing).

Proof. Since $*$ is continuous at point $(1, 1)$ and $a * b \le 1 * 1 = 1$, and since $\Diamond$ is continuous at point $(0, 0)$ and $a \Diamond b \ge 0 \Diamond 0 = 0$ for all $a, b \in [0, 1]$, we have

$$\sup_{0<\mu<1}[(1 - \mu) * (1 - \mu)] = 1 \text{ and } \inf_{0<\mu<1} \mu \Diamond \mu = 0,$$

respectively. Thus, for each $\lambda \in (0, 1)$, there exists $\lambda_1 \in (0, 1)$ such that

$$(1 - \lambda_1) * (1 - \lambda_1) > 1 - \lambda \text{ and } \lambda_1 \Diamond \lambda_1 < \lambda.$$

Similarly, there exists $\lambda_2 \in (0, 1)$ such that

$$(1 - \lambda_2) * (1 - \lambda_2) > 1 - \lambda_1 \text{ and } \lambda_2 \Diamond \lambda_2 < \lambda_1.$$

Continuing this procedure, we can obtain a sequence $\{\lambda_n\} \subset (0, 1)$ satisfying the condition. ■

Lemma 2.3 *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. Let $T : X \to X$ be a mapping satisfying*

$$M(Tx, Ty, t_1) > M(x, y, t_1) \text{ and } N(Tx, Ty, t_1) < N(x, y, t_1),$$

where t_1 is a fixed positive number. Then there exist a continuity point t_0 of $M(x, y, .)$ and $N(x, y, .)$ such that

$$M(Tx, Ty, t_0) > M(x, y, t_0) \text{ and } N(Tx, Ty, t_0) < N(x, y, t_0).$$

Proof. Since $M(Tx, Ty, .) - M(x, y, .)$ is left continuous and $N(Tx, Ty, .) - N(x, y, .)$ is right continuous at point t_1, there exists $t_2 > 0$ such that

$$M(Tx, Ty, t) > M(x, y, t) \text{ and } N(Tx, Ty, t) < N(x, y, t)$$

for all $t \in [t_2, t_1]$. Note that the set of discontinuous points of $M(x, y, .)$ and $N(x, y, .)$ is countable at most. Hence, there exists $t_0 \in [t_2, t_1]$ such that $M(x, y, .)$ and $N(x, y, .)$ are continuous at t_0. ■

Theorem 2.4 *Let $(X, M, N, *, \Diamond)$ be a complete intuitionistic fuzzy metric space. Let $T : X \to X$ be a mapping satisfying the following conditions:*

(i) there exists $x_0 \in X$ such that

$$\lim_{n\to\infty} M(x_0, T^i x_0, t) = 1,\ \lim_{n\to\infty} N(x_0, T^i x_0, t) = 0, i = 1, 2, ...; \tag{2.1}$$

(ii) there exists a mapping $m : X \to \mathbb{N}$ such that for any $x, y \in X$

$$\begin{aligned} M(T^{m(x)}x, T^{m(x)}y, t) &\geq M(x, y, \phi(t)) \text{ and} \\ N(T^{m(x)}x, T^{m(x)}y, t) &\leq N(x, y, \phi(t)) \end{aligned} \tag{2.2}$$

where the function $\phi(t)$ satisfies conditions (ϕ_1) and (ϕ_2).

Then T has a unique fixed point x_* and the quasi-iterative sequence $\{x_n = T^{m(x_{n-1})}x_{n-1}\}$ converges to x_*.

Proof. First, we prove that

$$\sup_{s>0} \inf_{x\in O_T(x_0)} M(x_0, x, s) = 1 \text{ and } \inf_{s>0} \sup_{x\in O_T(x_0)} N(x_0, x, s) = 0 \tag{2.3}$$

where $O_T(x_0) = \{x_0, Tx_0, T^2x_0, ...\}$ is called the orbit of x_0 for T. For any $n \in \mathbb{N}$ with $n > m(x_0)$, we can denote

$$n = km(x_0) + s \text{ where } 0 \leq s < m(x_0).$$

Note that $\phi(t) > t$ for all $t > 0$ and $\lim_{t\to\infty}[\phi(t) - t] = \infty$. By (2.1), we have

$$\lim_{t\to\infty} M(x_0, T^i x_0, \phi(t)) = 1 \text{ and } \lim_{t\to\infty} N(x_0, T^i x_0, \phi(t)) = 0 \tag{2.4}$$

for $i = 1, 2, ..., m(x_0)$ and

$$\lim_{t\to\infty} M(x_0, T^i x_0, \phi(t) - t) = 1 \text{ and } \lim_{t\to\infty} N(x_0, T^i x_0, \phi(t) - t) = 0. \tag{2.5}$$

Moreover, by Lemma 2.2, for any $\lambda \in (0, 1)$, there is a sequence $\{\lambda_n\}$ in $(0, 1)$ such that

$$(1 - \lambda_n) * (1 - \lambda_n) > 1 - \lambda_{n-1} \text{ and } \lambda_n \Diamond \lambda_n < \lambda_{n-1},\ n = 1, 2, ...$$

Thus, it follows from (2.4) and (2.5) that for given λ_k, there exists $t_0 > 0$ such that

$$\min_{1\leq i\leq m(x_0)} M(x_0, T^i x_0, \phi(t)) > 1 - \lambda_k \text{ and } \max_{1\leq i\leq m(x_0)} N(x_0, T^i x_0, \phi(t)) < \lambda_k,$$

$$M(x_0, T^{m(x_0)}x_0, \phi(t) - t) > 1 - \lambda_k \text{ and } N(x_0, T^{m(x_0)}x_0, \phi(t) - t) < \lambda_k, \forall t > t_0.$$

So, by (2.2), for all $t > t_0$, we have

$$\begin{aligned} M(x_0, T^n x_0, \phi(t)) &= M(x_0, T^{km(x_0)+s}x_0, \phi(t)) \\ &\geq M(x_0, T^{m(x_0)}x_0, \phi(t) - t) \\ &\quad *M(T^{m(x_0)}x_0, T^{km(x_0)+s}x_0, t) \\ &\geq M(x_0, T^{m(x_0)}x_0, \phi(t) - t) \\ &\quad *M(x_0, T^{(k-1)m(x_0)+s}x_0, \phi(t)) \\ &\geq ... \geq M(x_0, T^{m(x_0)}x_0, \phi(t) - t) \\ &\quad *\overset{(k)}{\cdots} * M(x_0, T^{m(x_0)}x_0, \phi(t) - t) * M(x_0, T^s x_0, \phi(t)) \\ &> (1 - \lambda_k) * \overset{(k+1)}{\cdots} * (1 - \lambda_k) \\ &> (1 - \lambda_{k-1}) * \overset{(k)}{\cdots} * (1 - \lambda_{k-1}) \\ &> ... > (1 - \lambda_1) * (1 - \lambda_1) > 1 - \lambda, \end{aligned}$$

4

$$\begin{aligned}
N(x_0, T^n x_0, \phi(t)) &= N(x_0, T^{km(x_0)+s}x_0, \phi(t)) \\
&\leq N(x_0, T^{m(x_0)}x_0, \phi(t)-t) \\
&\quad \Diamond N(T^{m(x_0)}x_0, T^{km(x_0)+s}x_0, t) \\
&\leq N(x_0, T^{m(x_0)}x_0, \phi(t)-t) \\
&\quad \Diamond N(x_0, T^{(k-1)m(x_0)+s}x_0, \phi(t)) \\
&\leq ... \leq N(x_0, T^{m(x_0)}x_0, \phi(t)-t) \\
&\quad \Diamond \overset{(k)}{\cdots} \Diamond N(x_0, T^{m(x_0)}x_0, \phi(t)-t) \Diamond N(x_0, T^s x_0, \phi(t)) \\
&< \lambda_k \Diamond \overset{(k+1)}{\cdots} \Diamond \lambda_k \\
&< \lambda_{k-1} \Diamond \overset{(k+1)}{\cdots} \Diamond \lambda_{k-1} \\
&< ... < \lambda_1 \Diamond \lambda_1 < \lambda.
\end{aligned}$$

Therefore, for all $t > t_0$,

$$\inf_{x \in O_T(x_0)} M(x_0, x, \phi(t)) \geq 1 - \lambda \text{ and } \sup_{x \in O_T(x_0)} N(x_0, x, \phi(t)) \leq \lambda$$

and hence

$$\sup_{s>0} \inf_{x \in O_T(x_0)} M(x_0, x, s) \geq 1 - \lambda \text{ and } \inf_{s>0} \sup_{x \in O_T(x_0)} N(x_0, x, s) \leq \lambda.$$

By the arbitrariness of λ, we have

$$\sup_{s>0} \inf_{x \in O_T(x_0)} M(x_0, x, s) = 1 \text{ and } \inf_{s>0} \sup_{x \in O_T(x_0)} N(x_0, x, s) = 0.$$

Next, we prove that the quasi-iterative sequence $\left\{x_n = T^{m(x_{n-1})}x_{n-1}\right\}_{n=1}^{\infty}$ is a Cauchy sequence. For convenience, put $m_i = m(x_i), i = 0, 1, 2, ...$ Then, by (2.1), for all $t > 0$, we have

$$\begin{aligned}
M(x_n, x_{n+p}, t) &= M(T^{m_{n-1}}x_{n-1}, T^{m_{n+p-1}+m_{n+p-2}+...+m_{n-1}}x_{n-1}, t) \\
&\geq M(x_{n-1}, T^{m_{n+p-1}+...+m_n}x_{n-1}, \phi(t)) \\
&\geq M(x_{n-2}, T^{m_{n+p-1}+...+m_n}x_{n-2}, \phi^2(t)) \\
&\geq ... \geq M(x_0, T^{m_{n+p-1}+...+m_n}x_0, \phi^n(t)) \\
&\geq \inf_{x \in O_T(x_0)} M(x_0, x, \phi^n(t)) \\
&\geq \sup_{0<s<\phi^n(t)} \inf_{x \in O_T(x_0)} M(x_0, x, s),
\end{aligned}$$

$$\begin{aligned}
N(x_n, x_{n+p}, t) &= N(T^{m_{n-1}}x_{n-1}, T^{m_{n+p-1}+m_{n+p-2}+...+m_{n-1}}x_{n-1}, t) \\
&\leq N(x_{n-1}, T^{m_{n+p-1}+...+m_n}x_{n-1}, \phi(t)) \\
&\leq N(x_{n-2}, T^{m_{n+p-1}+...+m_n}x_{n-2}, \phi^2(t)) \\
&\leq ... \leq N(x_0, T^{m_{n+p-1}+...+m_n}x_0, \phi^n(t)) \\
&\leq \sup_{x \in O_T(x_0)} N(x_0, x, \phi^n(t)) \\
&\leq \inf_{0<s<\phi^n(t)} \sup_{x \in O_T(x_0)} N(x_0, x, s).
\end{aligned}$$

Then, by condition (ϕ_1) and (2.3)

$$\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1 \text{ and } \lim_{n\to\infty} N(x_n, x_{n+p}, t) = 0$$

for all $t > 0$. This means that $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists $\lim_n x_n = x_* \in X$. Now, we prove that x_* is the unique fixed point of T^{m_*}, where $m_* = m(x_*)$. By (v) and (xi) in Definition 1.3, and (2.2), we have

$$\begin{aligned} M(x_*, T^{m_*}x_*, t) &\geq M(x_*, T^{m_*}x_n, \frac{t}{2}) * M(T^{m_*}x_n, T^{m_*}x_*, \frac{t}{2}) \\ &\geq M(x_*, T^{m_*}x_n, \frac{t}{2}) * M(x_n, T^{m_*}x_*, \phi(\frac{t}{2})), \end{aligned} \tag{2.6}$$

$$\begin{aligned} N(x_*, T^{m_*}x_*, t) &\leq N(x_*, T^{m_*}x_n, \frac{t}{2}) \Diamond N(T^{m_*}x_n, T^{m_*}x_*, \frac{t}{2}) \\ &\leq N(x_*, T^{m_*}x_n, \frac{t}{2}) \Diamond N(x_n, T^{m_*}x_*, \phi(\frac{t}{2})). \end{aligned} \tag{2.7}$$

It is easy to prove that

$$\lim_{n\to\infty} M(x_*, T^{m_*}x_n, u) = 1 \text{ and } \lim_{n\to\infty} N(x_*, T^{m_*}x_n, u) = 0$$

for all $u > 0$. In fact,

$$\begin{aligned} M(x_*, T^{m_*}x_n, u) &\geq M(x_*, x_n, \frac{u}{2}) * M(x_n, T^{m_*}x_n, \frac{u}{2}) \\ &= M(x_*, x_n, \frac{u}{2}) * M(T^{m_{n-1}}x_{n-1}, T^{m_{n-1}+m_*}x_{n-1}, \frac{u}{2}) \\ &\geq M(x_*, x_n, \frac{u}{2}) * M(x_{n-1}, T^{m_*}x_{n-1}, \phi(\frac{u}{2})) \\ &\geq ... \geq M(x_*, x_n, \frac{u}{2}) * M(x_0, T^{m_*}x_0, \phi^n(\frac{u}{2})) \to 1, \end{aligned}$$

$$\begin{aligned} N(x_*, T^{m_*}x_n, u) &\leq N(x_*, x_n, \frac{u}{2}) \Diamond N(x_n, T^{m_*}x_n, \frac{u}{2}) \\ &= N(x_*, x_n, \frac{u}{2}) \Diamond N(T^{m_{n-1}}x_{n-1}, T^{m_{n-1}+m_*}x_{n-1}, \frac{u}{2}) \\ &\leq N(x_*, x_n, \frac{u}{2}) \Diamond N(x_{n-1}, T^{m_*}x_{n-1}, \phi(\frac{u}{2})) \\ &\leq ... \leq N(x_*, x_n, \frac{u}{2}) \Diamond N(x_0, T^{m_*}x_0, \phi^n(\frac{u}{2})) \to 0. \end{aligned}$$

Thus, letting $n \to \infty$ on the right sides of (2.6) and (2.7), and noting the continuity of $*$ and $\Diamond$, we have

$$M(x_*, T^{m_*}x_*, t) = 1 \text{ and } N(x_*, T^{m_*}x_*, t) = 0$$

for all $t > 0$. This implies that $T^{m_*}x_* = x_*$, i.e. x_* is a fixed point of $T^{m(x_*)}$. To show uniqueness, assume that $T^{m(x_*)}y = y$ for some $y \in X$. Then

$$\begin{aligned} M(x_*, y, t) &= M(T^{m(x_*)}x_*, T^{m(x_*)}y, t) \geq M(x_*, y, \phi(t)), \\ N(x_*, y, t) &= N(T^{m(x_*)}x_*, T^{m(x_*)}y, t) \leq N(x_*, y, \phi(t)). \end{aligned}$$

6

On the other hand, as $M(x_*, y, t)$ is non-decreasing and $N(x_*, y, t)$ is non-increasing, we have $M(x_*, y, t) \leq M(x_*, y, \phi(t))$ and $N(x_*, y, t) \geq N(x_*, y, \phi(t))$, respectively. Thus

$$\begin{aligned} M(x_*, y, t) &= M(x_*, y, \phi(t)) = M(x_*, y, \phi^n(t)), \\ N(x_*, y, t) &= N(x_*, y, \phi(t)) = N(x_*, y, \phi^n(t)) \end{aligned}$$

for all $t > 0$. Hence, by condition (ϕ_1), and (vii) and (xiii) in Definition 1.3, we have

$$M(x_*, y, t) = 1 \text{ and } N(x_*, y, t) = 0,$$

i.e. $x_* = y$. Finally, we prove x_* is the unique fixed point of T, too. In fact, since $T^{m(x_*)}x_* = x_*$, it follows that $Tx_* = T(T^{m_*}x_*) = T^{m_*}(Tx_*)$. Hence, $Tx_* = x_*$. Uniqueness is obvious. This completes the proof. ■

From Theorem 2.4, we can obtain the following consequence immediately.

Corollary 2.5 *Let $(X, M, N, *, \Diamond)$ be a complete intuitionistic fuzzy metric space. Let $T : X \to X$ be a mapping satisfying the following conditions:*

(i) there exists $x_0 \in X$ such that

$$\lim_{n\to\infty} M(x_0, T^i x_0, t) = 1, \lim_{n\to\infty} N(x_0, T^i x_0, t) = 0, i = 1, 2, ...;$$

(ii) there exists a mapping $m : X \to \mathbb{N}$ such that for any $x, y \in X$

$$M(T^{m(x)}x, T^{m(x)}y, t) \geq M(x, y, t/k), N(T^{m(x)}x, T^{m(x)}y, t) \leq N(x, y, t/k)$$

where $0 < k < 1$.

Then the conclusion of Theorem 2.4 remains true.

Proof. Taking $\phi(t) = t/k$. Obviously, $\phi(t)$ satisfies the conditions (ϕ_1) and (ϕ_2). Theorefore the conclusion follows from Theorem 2.4 directly. ■

Corollary 2.6 *Let $(X, M, N, *, \Diamond)$ be a complete intuitionistic fuzzy metric space. Let $T : X \to X$ be a mapping. If there exists a mapping $m : X \to \mathbb{N}$ such that for any $x, y \in X$,*

$$M(T^{m(x)}x, T^{m(x)}y, t) \geq M(x, y, \phi(t)), N(T^{m(x)}x, T^{m(x)}y, t) \leq N(x, y, \phi(t)),$$

where the function $\phi(t)$ satisfies the conditions (ϕ_1) and (ϕ_2). Then T has a unique fixed point x_* and the iterative sequence $\{T^n x\}$ converges to x_* for every $x \in X$.

Proof. By Theorem 2.4, we need only to show that the iterative sequence $\{T^n x\}$ converges to x_*. For $n \in \mathbb{N}$ with $n > m(x_*)$,

$$n = km(x_*) + s, 0 \leq s < m(x_*).$$

Since

$$\begin{aligned} M(x_*, T^n x, t) &= M(T^{m(x_*)}x_*, T^{km(x_*)+s}x, t) \\ &\geq M(x_*, T^{(k-1)m(x_*)+s}x, \phi(t)) \\ &\geq ... \geq M(x_*, T^s x, \phi^k(t)) \to 1, \end{aligned}$$

$$\begin{aligned} N(x_*, T^n x, t) &= N(T^{m(x_*)}x_*, T^{km(x_*)+s}x, t) \\ &\leq N(x_*, T^{(k-1)m(x_*)+s}x, \phi(t)) \\ &\leq \ldots \leq N(x_*, T^s x, \phi^k(t)) \to 0 \end{aligned}$$

so $\lim_n M(x_*, T^n x, t) = 1$ and $\lim_n N(x_*, T^n x, t) = 0$ for all $t > 0$, i.e. $T^n x \to x_*$. ∎

Remark 2.7 *Taking $\phi(t) = t/k$ $(0 < k < 1)$ and $m(x) = 1$ in Corollary 2.6, we at once obtain Theorem 1.1. Hence Theorem 1.1 is a special case of Corollary 2.6. In the view of Remark 1.5, taking $\phi(t) = t/k$, $N = 1 - M$, and $* = \min$ and $\Diamond = \max$ (i.e. $a * b = \min(a, b)$ and $a \Diamond b = \max(a, b)$) in Corollary 2.6, we obtain that the main results of [5-8] are special cases of Corollary 2.6.*

Theorem 2.8 *Let $(X, M, N, *, \Diamond)$ be a compact intuitionistic fuzzy metric space. Let $T : X \to X$ be a continuous mapping satisfying*

$$M(Tx, Ty, .) > \min\{M(x, Tx, .), M(y, Ty, .), M(x, y, .)\}, \tag{2.8}$$
$$N(Tx, Ty, .) < \max\{N(x, Tx, .), N(y, Ty, .), N(x, y, .)\} \tag{2.9}$$

for all $x \neq y$. If there exists $x_0 \in X$ such that $\{T^n x_0\}_{n=0}^{\infty}$ has an accumulation point $x_* \in X$ and for all $t > 0, n = 1, 2, \ldots$

$$\begin{aligned} M(T^{n-1}x_0, T^n x_0, t) &\leq M(T^n x_0, T^{n+1}x_0, t), \\ N(T^{n-1}x_0, T^n x_0, t) &\geq N(T^n x_0, T^{n+1}x_0, t) \end{aligned}$$

then x_* is the unique fixed point of T and $\lim_n T^n x_0 = x_*$.

Proof. Assume $T^n x_0 \neq T^{n+1}x_0$ for each $n \in \mathbb{N}$. (If not, there is $n_0 \in \mathbb{N}$ such that $T^{n_0}x_0 = T^{n_0+1}x_0$. This means that $x_* = T^{n_0}x_0$ is a fixed point of T and $\lim_n T^n x_0 = x_*$). Since $\{T^n x_0\}_{n=0}^{\infty}$ has an accumulation point $x_* \in X$, there exists a subsequence $\{T^{n_i}x_0\}$, $\lim_i T^{n_i}x_0 = x_*$. $\{M(T^n x_0, T^{n+1}x_0, t)\}$ and $\{N(T^n x_0, T^{n+1}x_0, t)\}$ are non-decreasing and non-increasing, respectively, and also bounded, thus $\{M(T^{n_i}x_0, T^{n_i+1}x_0, t)\}$, $\{M(T^{n_i+1}x_0, T^{n_i+2}x_0, t)\}$ and $\{N(T^{n_i}x_0, T^{n_i+1}x_0, t)\}$, $\{N(T^{n_i+1}x_0, T^{n_i+2}x_0, t)\}$ are convergent to a common limit, i.e.

$$\begin{aligned} \lim_{i\to\infty} M(T^{n_i}x_0, T^{n_i+1}x_0, t) &= \lim_{i\to\infty} M(T^{n_i+1}x_0, T^{n_i+2}x_0, t), \\ \lim_{i\to\infty} N(T^{n_i}x_0, T^{n_i+1}x_0, t) &= \lim_{i\to\infty} N(T^{n_i+1}x_0, T^{n_i+2}x_0, t) \end{aligned}$$

for all $t > 0$. By the continuity of T, we have

$$\lim_{i\to\infty} T^{n_i+1}x_0 = \lim_{i\to\infty} T(T^{n_i}x_0) = Tx_*.$$

Suppose $x_* \neq Tx_*$. Putting $y = Tx$ in (2.8) and (2.9), we have

$$M(x, Tx, .) < M(Tx, T^2x, .) \text{ and } N(x, Tx, .) > N(Tx, T^2x, .)$$

for every $x \neq Tx$. So, by Lemma 2.3, there exists a continuous point t_0 of $M(x_*, Tx_*, .)$ and $N(x_*, Tx_*, .)$ such that $M(Tx_*, T^2x_*, t_0) > M(x_*, Tx_*, t_0)$ and $N(Tx_*, T^2x_*, t_0) < N(x_*, Tx_*, t_0)$. On the other hand, by Lemma 1.7

$$\begin{aligned} M(x_*, Tx_*, t_0) &= \lim_{i\to\infty} M(T^{n_i}x_0, T(T^{n_i}x_0), t_0) \\ &= \lim_{i\to\infty} M(T^{n_i+1}x_0, T^{n_i+2}x_0, t_0) \\ &\geq M(Tx_*, T^2x_*, t_0), \end{aligned}$$

8

$$\begin{aligned} N(x_*, Tx_*, t_0) &= \lim_{i\to\infty} N(T^{n_i}x_0, T(T^{n_i}x_0), t_0) \\ &= \lim_{i\to\infty} N(T^{n_i+1}x_0, T^{n_i+2}x_0, t_0) \\ &\leq N(Tx_*, T^2x_*, t_0) \end{aligned}$$

are contradictions. Therefore $x_* = Tx_*$, i.e. x_* is a fixed point of T. Uniqueness follows from (2.8) and (2.9). Finally, we prove that $\lim_n T^n x_0 = x_*$. Since $\lim_i T^{n_i}x_0 = x_*$ and $\lim_i T^{n_i+1}x_0 = Tx_* = x_*$, by Lemma 1.7,

$$\begin{aligned} \lim_{i\to\infty}\inf M(T^{n_i}x_0, T^{n_i+1}x_0, t) &\geq M(x_*, x_*, t) = 1, \\ \lim_{i\to\infty}\sup N(T^{n_i}x_0, T^{n_i+1}x_0, t) &\leq N(x_*, x_*, t) = 0 \end{aligned}$$

for all $t > 0$. So, $\lim_i M(T^{n_i}x_0, T^{n_i+1}x_0, t) = 1$ and $\lim_i N(T^{n_i}x_0, T^{n_i+1}x_0, t) = 0$ for all $t > 0$. For any $n \in \mathbb{N}$ with $n > n_1$, there exists n_i with $n_{i+1} \geq n > n_i$. By (2.8) and (2.9), we have

$$\begin{aligned} M(T^n x_0, x_*, t) &\geq \min\left\{M(T^{n-1}x_0, T^n x_0, t), 1, M(T^{n-1}x_0, x_*, t)\right\} \\ &\geq \min\left\{\begin{array}{c} M(T^{n-1}x_0, T^n x_0, t), M(T^{n-2}x_0, T^{n-1}x_0, t) \\ , M(T^{n-2}x_0, x_*, t) \end{array}\right\} \\ &= \min\left\{M(T^{n-2}x_0, T^{n-1}x_0, t), M(T^{n-2}x_0, x_*, t)\right\} \\ &\geq \dots \geq \min\left\{M(T^{n_i}x_0, T^{n_i+1}x_0, t), M(T^{n_i}x_0, x_*, t)\right\}, \end{aligned}$$

$$\begin{aligned} N(T^n x_0, x_*, t) &\leq \max\left\{N(T^{n-1}x_0, T^n x_0, t), 0, N(T^{n-1}x_0, x_*, t)\right\} \\ &\leq \max\left\{\begin{array}{c} N(T^{n-1}x_0, T^n x_0, t), N(T^{n-2}x_0, T^{n-1}x_0, t) \\ , N(T^{n-2}x_0, x_*, t) \end{array}\right\} \\ &= \max\left\{N(T^{n-2}x_0, T^{n-1}x_0, t), N(T^{n-2}x_0, x_*, t)\right\} \\ &\leq \dots \leq \max\left\{N(T^{n_i}x_0, T^{n_i+1}x_0, t), N(T^{n_i}x_0, x_*, t)\right\}. \end{aligned}$$

Letting $n \to \infty$ ($n_i \to \infty$), we have

$$\lim_{n\to\infty} M(T^n x_0, x_*, t) \geq 1 \text{ and } \lim_{n\to\infty} N(T^n x_0, x_*, t) \leq 0$$

for all $t > 0$. Hence, $\lim_n T^n x_0 = x_*$. ∎

Remark 2.9 *Theorem 1.2 is the immediate consequence of Theorem 2.8. In fact, by Theorem 1.2, it is easy to see that T is continuous and (2.8), (2.9) hold for any $x_0 \in X$. In addition, by the compactness of X, $\{T^n x_0\}$ has an accumulation point. Hence Theorem 1.2 follows immediately from Theorem 2.8.*

References

[1] Banach S. Theorie des operations lineaires. Warsaw: Monografje Mathematyczne; 1932.

[2] Chauhan S, Kutukcu S, Dhiman N, Kumar S. Variants of R-weakly commuting mappings and common fixed point theorems in intuitionistic fuzzy metric spaces. Kuwait J Sci 2014;41:49-64.

[3] Edelstein M. On fixed and periodic points under contractive mapping. J London Math Soc 1962;37:74-9.

[4] Fang J. A fixed point theorem of ϕ-contraction mapping on probabilistic metric spaces. J Xinjiang Univ 1988;5(3):21-8.

[5] Fang J. On fixed point theorems in fuzzy metric spaces. Fuzzy Sets Syst 1992;46:107-13.

[6] Grabiec M. Fixed point in fuzzy metric spaces. Fuzzy Sets Syst 1988;27:385-9.

[7] Istratescu I. A fixed point theorem for mappings with a probabilistic contractive iterate. Rev Roumaine Math Pures Appl 1981;26:431-5.

[8] Sehgal VM, Bharucha-Reid AT. Fixed points of contraction mappings on probabilistic metric spaces. Math Systems Theory 1972;6:97-102.

Existence and uniqueness of positive solutions for singular higher order fractional differential equations with infinite-point boundary value conditions*

Qianqian Leng, Jiandong Yin†, Pinghua Yan

Department of Mathematics, Nanchang University, Nanchang 330031, PR China.

E-mail: 13517914026@163.com(Q. Leng); yjdaxf@163.com(J. Yin), 18270803044@163.com (P. Yan)

Abstract

In this paper, we consider the singular problem for higher order fractional differential equations with infinite-point boundary value conditions. First, we get some properties of the Green function, then by a monotone iterative technique, we establish the existence and uniqueness of the positive solutions for singular higher order fractional differential equations with infinite-point boundary value conditions. Moreover the iterative sequences of positive solution and error estimation are also given. In the end, an example is given to illustrate one of the main results.

Keywords: fractional differential equation, positive solution, monotone iterative technique, Green function, existence and uniqueness,completely continuous operator

1 Introduction

In this paper, we consider the following singular problem for a higher order fractional differential equation with infinite-point boundary value conditions.

$$\begin{cases} -\mathcal{D}^{\alpha}x(t) = q(t)f(x(t), \mathcal{D}^{\mu_1}x(t), \mathcal{D}^{\mu_2}x(t), \cdots, \mathcal{D}^{\mu_{n-1}}x(t)), \quad t \in (0,1) \\ x(0) = \mathcal{D}^{\mu_i}x(0) = 0, \mathcal{D}^{\mu}x(1) = \displaystyle\sum_{j=1}^{\infty} \alpha_j \mathcal{D}^{\mu}x(\xi_j), \quad 1 \le i \le n-1, \end{cases} \tag{1.1}$$

where $n \geq 3$, $n-1 < \alpha < n$, $n-k-1 < \alpha - \mu_k < n-k$, for $k = 1, 2, \cdots, n-2$, $\mu - \mu_{n-1} > 0$, $\alpha - \mu_{n-1} \leq 2$, $\alpha - \mu > 1$, $\alpha_j \geq 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1 (j = 1, 2, \ldots)$, $0 < \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-\mu-1} < 1$, $\mathcal{D}^{\alpha}$ is the standard Riemann-Liouville derivative. Recently, fractional differential equations have been applied in variety of different areas such as chemical physics, engineering, electrical networks, mechanics, see [1, 3–10] and references cited therein for details. Zhang et.al [11], by establishing eigenvalue interval for the existence of positive solutions from Schauder's fixed point theorem and the upper and lower solutions, obtained a multiple positive solution of the following fractional differential equation.

$$\begin{cases} -\mathcal{D}^{\alpha}x(t) = \lambda f(x(t), \mathcal{D}^{\mu_1}x(t), \mathcal{D}^{\mu_2}x(t), \cdots, \mathcal{D}^{\mu_{n-1}}x(t)), t \in (0,1) \\ x(0) = \mathcal{D}^{\mu_i}x(0) = 0, \mathcal{D}^{\mu}x(1) = \displaystyle\sum_{j=1}^{p-2} \alpha_j \mathcal{D}^{\mu}x(\xi_j), 1 \le i \le n-1, \end{cases} \tag{1.2}$$

*The work was supported by the Foundation of Department of Education of Jiangxi Province(No. GJJ152008).

†Corresponding author

where $n \geq 3$, $n-1 < \alpha < n$, $n-k-1 < \alpha - \mu_k < n-k$, for $k = 1, 2, \cdots, n-2$, $\mu - \mu_{n-1} > 0$, $\alpha - \mu_{n-1} \leq 2$, $\alpha - \mu > 1$, $\alpha_j \geq 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{p-3} < \xi_{p-2} < 1 (j = 1, 2, \ldots, p-2)$, $0 < \sum_{j=1}^{p-2} \alpha_j \xi_j^{\alpha-\mu-1} < 1$, $\mathcal{D}^\alpha$ is the standard Riemann-Liouville derivative. In [12], Zhang and Han concerned the following singular nonlinear $(n-1, 1)$ conjugate-type fractional differential equation with one nonlocal term

$$\begin{cases} -\mathcal{D}^\alpha x(t) = f(t, x(t)), t \in (0,1), \alpha \in (n-1, n], \\ x^k(0) = 0, 0 \leq k \leq n-2, x(1) = \displaystyle\int_0^1 x(s) dA(s), \end{cases} \tag{1.3}$$

where $\alpha \geq 2$, $\mathcal{D}^\alpha$ is the standard Riemann-Liouville derivative,A is a function of bounded variation and $\int_0^1 x(s)dA(s)$ denotes the Riemann-Stieltjes integral of x with respect to A, dA is a signed measure. By using a monotone iterative technique, Zhang and Han [12] established the existence and uniqueness of this positive solution.

Motivated by the above articles, we study the existence and uniqueness of positive solution of fractional differential equation (1.1). By means of a monotone iterative technique, we obtain the existence and uniqueness of the positive solutions of (1.1).

In the rest of this article, we suppose that following assumptions hold:

(L1) $q(t) : (0,1) \to [0, +\infty)$ is continuous and does not vanish identically of $(0,1)$ and

$$0 < \int_0^1 q(s) ds < +\infty;$$

(L2) $f : (0, +\infty)^n \to [0, +\infty)$ is continuous and is nondecreasing at $x_i > 0$ for $i = 1, 2, \cdots, n$;

(L3) for all $r \in (0,1)$, there exists a constant $0 < \lambda < 1$ such that for each $(x_1, x_2, \cdots, x_n) \in (0, +\infty)^n$,

$$f(rx_1, rx_2, \cdots, rx_n) \geq r^\lambda f(x_1, x_2, \cdots, x_n).$$

Remark 1.1. *If (L3) holds, then for $r \geq 1$, there exists a constant $0 < \lambda < 1$ such that for each $(x_1, x_2, \cdots, x_n) \in (0, +\infty)^n$,*

$$f(rx_1, rx_2, \cdots, rx_n) \leq r^\lambda f(x_1, x_2, \cdots, x_n).$$

2 Preliminaries

For the convenience of the reader, we present the necessary definitions and lemmas from the fractional calculus theory. These definitions and lemmas can be found in monograph [3, 5, 6, 8, 10, 11].

Definition 2.1. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} f(s) ds,$$

provided that the right-hand side is point wise defined on $(0, +\infty)$.

Definition 2.2. *The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, +\infty) \to \mathbb{R}$ is given by*

$$\mathcal{D}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t (t-s)^{(n-\alpha-1)} f(s) ds,$$

where $n - 1 \leq \alpha < n$, provided that the right-hand side is point wise defined on $(0, +\infty)$.

Lemma 2.1. *(1) If $x \in C(0,1)\bigcap L(0,1)$, $\nu > \sigma > 0$, then*

$$I^{\nu} I^{\sigma} x(t) = I^{\nu+\sigma} x(t), \mathcal{D}^{\sigma} I^{\nu} x(t) = I^{\nu-\sigma} x(t), \mathcal{D}^{\sigma} I^{\sigma} x(t) = x(t).$$

(2)If $\nu > 0$,$\sigma > 0$, then

$$\mathcal{D}^{\nu} t^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma-\nu)} t^{\sigma-\nu-1}.$$

Lemma 2.2. *Assume that $x \in C(0,1)\bigcap L(0,1)$, with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1)\bigcap L^1(0,1)$. Then*

$$I^{\alpha}\mathcal{D}^{\alpha} x(t) = x(t) + c_1 t^{\alpha-1} + c_1 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in \mathbb{R}(i = 1, 2, \cdots, n)$, n is the smallest integer greater than or equal to α.

Remark 2.1. *There are many kinds of functions satisfying conditions $(L2)$ and $(L3)$. In fact, let*
$K = \{f(x_1, x_2, \cdots, x_n) : f(x_1, x_2, \cdots, x_n)$ satisfies conditions (L2) and (L3)$\}$
and $a_i(t)$ be nonnegative continuous on $(0, +\infty)$, $(i = 1, 2, \cdots, n)$.
Then it is easily verified directly that the following facts hold:
(1) $\sum_{1}^{n} a_i(t) x_i^{b_i} \in K$, where $b_i \in (0,1)$ are constants, $i = 1, 2, \cdots, n$.
(2)$\sum_{1}^{n} [a_i(t) x_i^{\mu_i}]^{\frac{1}{\mu}} \in K$ $(i = 1, 2, \cdots, n)$ and $\mu \geq \max\limits_{1\leq i\leq n}\{\mu_i\}$,
(3)If $f(x_1, x_2, \cdots, x_n) \in K$ then $a_i(t) f(x_1, x_2, \cdots, x_n) \in K(i = 1, 2, \cdots, n)$.
(4)If $f_i(x_1, x_2, \cdots, x_n) \in K$ $(i = 1, 2, \cdots, n)$, then

$$\max_{1\leq i\leq n}\{f_i(x_1, x_2, \cdots, x_n)\} \in K, \min_{1\leq i\leq n}\{f_i(x_1, x_2, \cdots, x_n)\} \in K$$

and

$$\max_{1\leq i\leq n}\{f_i(x_1, x_2, \cdots, x_n)\} + \min_{1\leq i\leq n}\{f_i(x_1, x_2, \cdots, x_n)\} \in K.$$

Let

$$G_1(t,s) = \begin{cases} \dfrac{t^{\alpha-\mu_{n-1}-1}(1-s)^{\alpha-\mu-1} - (t-s)^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}, 0 \leq s \leq t \leq 1 \\ \dfrac{t^{\alpha-\mu_{n-1}-1}(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-1})}, 0 \leq t \leq s \leq 1 \end{cases} \tag{2.1}$$

and

$$G_2(t,s) = \begin{cases} \dfrac{t^{\alpha-\mu-1}(1-s)^{\alpha-\mu-1} - (t-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-1})}, 0 \leq s \leq t \leq 1 \\ \dfrac{t^{\alpha-\mu-1}(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-1})}, 0 \leq t \leq s \leq 1. \end{cases} \tag{2.2}$$

Obviously for $t, s \in [0,1]$, we have

$$0 \leq G_1(t,s) \leq \frac{t^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})}, \tag{2.3}$$

$$0 \leq \frac{t^{\alpha-\mu-1}(1-t)s(1-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-1})} \leq G_2(t,s) \leq \frac{t^{\alpha-\mu-1}}{\Gamma(\alpha-\mu_{n-1})}. \tag{2.4}$$

Lemma 2.3. *If $h(t) \in L^1(0,1)$ then the boundary value problem*

$$\begin{cases} -\mathcal{D}^{\alpha-\mu_{n-1}} w(t) = h(t), t \in (0,1) \\ w(0) = 0, \mathcal{D}^{\alpha-\mu_{n-1}} w(1) = \sum_{j=1}^{\infty} \alpha_j \mathcal{D}^{\alpha-\mu_{n-1}} w(\xi_j) \end{cases} \tag{2.5}$$

has the unique solution

$$w(t) = \int_0^1 G(t,s)h(s)ds,$$

where

$$G(t,s) = G_1(t,s) + \frac{t^{\alpha-\mu_{n-1}-1}}{1 - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-\mu-1}} \sum_{j=1}^{\infty} \alpha_j G_2(\xi_j, s) \tag{2.6}$$

is the Green function of the boundary value problem (2.5)

Proof. The proof is similar to Lemma 2.3 of [11] if replacing $\sum_{j=1}^{p-2} \alpha_j \mathcal{D}^{\alpha-\mu_{n-1}} w(\xi_j)$ with $\sum_{j=1}^{\infty} \alpha_j \mathcal{D}^{\alpha-\mu_{n-1}} w(\xi_j)$, so we omit the details. □

Lemma 2.4. *The function $G(t,s)$ has the following properties;*
(1) $G(t,s) > 0$, for $t,s \in (0,1)$;
(2) $ct^{\alpha-\mu_{n-1}-1}(1-s)^{\alpha-\mu-1}s \le G(t,s) \le dt^{\alpha-\mu_{n-1}-1}$, where

$$c = \frac{\sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-\mu-1}(1-\xi_j)}{(1 - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-\mu-1})\Gamma(\alpha-\mu_{n-1})}, \quad d = \frac{1}{(1 - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-\mu-1})\Gamma(\alpha-\mu_{n-1})}.$$

Proof. The first result is obvious so we only prove the second one.

From (2.3), (2.4), (2.6) we have

$$\begin{aligned} G(t,s) &\ge \frac{t^{\alpha-\mu_{n-1}-1}}{1 - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-\mu-1}} \sum_{j=1}^{\infty} \alpha_j G_2(\xi_j, s) \\ &\ge t^{\alpha-\mu_{n-1}-1} \frac{\sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-\mu-1}(1-\xi_j)(1-s)^{\alpha-\mu-1}s}{(1 - \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-\mu-1})\Gamma(\alpha-\mu_{n-1})} \\ &= ct^{\alpha-\mu_{n-1}-1}(1-s)^{\alpha-\mu-1}s \end{aligned}$$

and

$$\begin{aligned} G(t,s) &= G_1(t,s) + \frac{t^{\alpha-\mu_{n-1}-1}}{1-\sum_{j=1}^{\infty}\alpha_j\xi_j^{\alpha-\mu-1}}\sum_{j=1}^{\infty}\alpha_j G_2(\xi_j,s) \\ &\leq \frac{t^{\alpha-\mu_{n-1}-1}}{\Gamma(\alpha-\mu_{n-1})} + \frac{t^{\alpha-\mu_{n-1}-1}}{1-\sum_{j=1}^{\infty}\alpha_j\xi_j^{\alpha-\mu-1}}\sum_{j=1}^{\infty}\alpha_j G_2(\xi_j,s) \\ &\leq \frac{t^{\alpha-\mu_{n-1}-1}}{(1-\sum_{j=1}^{\infty}\alpha_j\xi_j^{\alpha-\mu-1})\Gamma(\alpha-\mu_{n-1})} = dt^{\alpha-\mu_{n-1}-1}. \end{aligned}$$

□

For the convenience of expression in the rest of the paper, we let $\mu_0 = 0$, $E = C[0,1]$. Now let us consider the following modified problem of (1.1):

$$\begin{cases} -\mathcal{D}^{\alpha-\mu_{n-1}}v(t) = q(t)f(I^{\mu_{n-1}-\mu_0}v(t), I^{\mu_{n-1}-\mu_1}v(t), \cdots, I^{\mu_{n-1}-\mu_{n-2}}v(t), v(t)), t\in(0,1) \\ v(0)=0, \mathcal{D}^{\mu-\mu_{n-1}}v(1) = \sum_{j=1}^{\infty}\alpha_j\mathcal{D}^{\mu-\mu_{n-1}}v(\xi_j), \end{cases} \tag{2.7}$$

Lemma 2.5. *Let $x(t) = I^{\mu_{n-1}-\mu_0}v(t)$, $v(t)\in E$. Then we can transform (1.1) into (2.7). Moreover, if $v\in C[(0,1),(0,+\infty)]$ is a solution of (2.7), then the function $x(t) = I^{\mu_{n-1}-\mu_0}v(t)$ is a positive solution of (1.1).*

Proof. The proof is similar to Lemma 2.5 of [11], so we omit the details. □

Let $e_{n-1}(t) = t^{\alpha-\mu_{n-1}-1}$, and for $i = 0,1,2,\cdots,n-2$, define

$$e_i(t) = I^{\mu_{n-1}-\mu_i}e_{n-1}(t) = \frac{\Gamma(\alpha-\mu_{n-1})}{\Gamma(\alpha-\mu_i)}t^{\alpha-\mu_i-1}.$$

Let $P = \{v\in E : v(t)\geq 0, t\in[0,1]\}$. Clearly P is a normal cone in E. Now let us define a sub-cone of P as follows:

$D = \{v(t)\in P$: there exist two positive numbers $L_v \geq l_v$ such that $l_v e_{n-1}(t) \leq v(t) \leq L_v e_{n-1}(t)\}$. Obviously, D is a nonempty set since $e_{n-1}(t)\in P$.

Lemma 2.6. *Let the operator $T : D \to E$ be defined by*

$$(Tv)(t) = \int_0^1 G(t,s)q(s)f(I^{\mu_{n-1}-\mu_0}v(s), I^{\mu_{n-1}-\mu_1}v(s), \cdots, I^{\mu_{n-1}-\mu_{n-2}}v(s), v(s))ds,$$

and $0 < \int_0^1 q(s)f(e_0(s), e_1(s), \cdots, e_{n-2}(s), e_{n-1}(s))ds < +\infty$, then any fixed point of T is a solution of (2.7), and T is a well defined completely continuous operator and $T : D\to D$.

Proof. For each $v\in D$, there exist two positive numbers $L_v > 1 > l_v$ such that

$$l_v e_{n-1}(t) \leq v(t) \leq L_v e_{n-1}(t), t\in[0,1]. \tag{2.8}$$

According to (2.8), (L2), (L3), Lemma 2.1(2) and Remark 1.1, we have

$$\begin{aligned}(Tv)(t) &\leq \int_0^1 G(t,s)q(s)f(I^{\mu_{n-1}-\mu_0}L_v e_{n-1}(s), I^{\mu_{n-1}-\mu_1}L_v e_{n-1}(s), \cdots, I^{\mu_{n-1}-\mu_{n-2}}L_v e_{n-1}(s), L_v e_{n-1}(s))ds \\ &\leq L_v^\lambda \int_0^1 G(t,s)q(s)f(I^{\mu_{n-1}-\mu_0}e_{n-1}(s), I^{\mu_{n-1}-\mu_1}e_{n-1}(s), \cdots, I^{\mu_{n-1}-\mu_{n-2}}e_{n-1}(s), e_{n-1}(s))ds \\ &\leq dL_v^\lambda \int_0^1 q(s)f(e_0(s), e_1(s), \cdots, e_{n-2}(s), e_{n-1}(s))ds e_{n-1}(t) < +\infty,\end{aligned}$$

and

$$\begin{aligned}(Tv)(t) &\geq \int_0^1 G(t,s)q(s)f(I^{\mu_{n-1}-\mu_0}l_v e_{n-1}(s), I^{\mu_{n-1}-\mu_1}l_v e_{n-1}(s), \cdots, I^{\mu_{n-1}-\mu_{n-2}}l_v e_{n-1}(s), l_v e_{n-1}(s))ds \\ &\geq l_v^\lambda \int_0^1 G(t,s)q(s)f(I^{\mu_{n-1}-\mu_0}e_{n-1}(s), I^{\mu_{n-1}-\mu_1}e_{n-1}(s), \cdots, I^{\mu_{n-1}-\mu_{n-2}}e_{n-1}(s), e_{n-1}(s))ds \\ &\geq cl_v^\lambda \int_0^1 s(1-s)^{\alpha-\mu-1}q(s)f(e_0(s), e_1(s), \cdots, e_{n-2}(s), e_{n-1}(s))ds e_{n-1}(t).\end{aligned}$$

From above it is not difficulty to prove that $T : D \to D$ is a completely continuous operator. □

3 Main results

Theorem 3.1. *Suppose that* (L1)-(L3) *hold and*

$$0 < \int_0^1 q(s)f(e_0(s), e_1(s), \cdots, e_{n-2}(s), e_{n-1}(s))ds < +\infty.$$

Then the equation (2.7) has a unique positive solution ν^* in D, and for each initial value $\nu_0 \in D$, the iterative sequence $\nu_n = T\nu_{n-1}(n = 1, 2, \cdots)$ converges to ν^* as $n \to +\infty$. Meanwhile $I^{\mu_{n-1}-\mu_0}v^*$ is the unique solution of (1.1). Furthermore we have the following error estimation

$$\|I^{\mu_{n-1}-\mu_0}\nu_n - \nu^*\| \leq 2(1-(t_0^2)^{\lambda^n})\|I^{\mu_{n-1}-\mu_0}\omega_0\|,$$

and the rate of convergence is $o(1-(t_0^2)^{\lambda^n})$, where t_0 is determined by ν_0, and $t_0 \in (0,1)$.

Proof. From Lemma 2.6 for any $\nu_0 \in D$, then there exist four positive constants l_{ν_0}, L_{ν_0}, $l_{\nu_0}^-$, $L_{\nu_0}^-$ such that

$$l_{\nu_0}e_{n-1}(t) \leq \nu_0 \leq L_{\nu_0}e_{n-1}(t), l_{\nu_0}^- e_{n-1}(t) \leq T\nu_0 \leq L_{\nu_0}^- e_{n-1}(t).$$

So

$$\frac{l_{\nu_0}^-}{L_{\nu_0}}\nu_0 \leq T\nu_0 \leq \frac{L_{\nu_0}^-}{l_{\nu_0}}\nu_0.$$

Let

$$t_0 = \min\left\{\left(\frac{l_{\nu_0}^-}{L_{\nu_0}}\right)^{\frac{1}{1-\lambda}}, \left(\frac{l_{\nu_0}}{L_{\nu_0}^-}\right)^{\frac{1}{1-\lambda}}\right\},$$

then $t_0 \in (0,1)$ and

$$t_0^{1-\lambda}\nu_0 \leq T\nu_0 \leq \left(\frac{1}{t_0}\right)^{1-\lambda}\nu_0. \tag{3.1}$$

Let $\mu_0 = t_0\nu_0$, $\omega_0 = \frac{1}{t_0}\nu_0$, then $\mu_0 \leq \nu_0 \leq \omega_0$. Now we define

$$\mu_n = T\mu_{n-1}, \omega_n = T\omega_{n-1}(n = 1, 2, \cdots). \tag{3.2}$$

According to (L3)and Remark 1.1 we have

$$\begin{aligned}(Trv)(t) &= \int_0^1 G(t,s)q(s)f(I^{\mu_{n-1}-\mu_0}rv(s), I^{\mu_{n-1}-\mu_1}rv(s), \cdots, I^{\mu_{n-1}-\mu_{n-2}}rv(s), rv(s))ds \\ &\geq r^\lambda T(v), \text{if } \ 0<\lambda<1 \\ (Trv)(t) &= \int_0^1 G(t,s)q(s)f(I^{\mu_{n-1}-\mu_0}rv(s), I^{\mu_{n-1}-\mu_1}rv(s), \cdots, I^{\mu_{n-1}-\mu_{n-2}}rv(s), rv(s))ds \\ &\leq r^\lambda T(v), \text{if } \ \lambda>1.\end{aligned}$$

By a careful calculation and (L2) we can obtain

$$T\mu_0 \geq t_0^\lambda T\nu_0 \geq t_0\nu_0 = \mu_0, T\omega_0 \leq \left(\frac{1}{t_0}\right)^\lambda T\nu_0 \leq \frac{1}{t_0}\nu_0 = \omega_0. \tag{3.3}$$

By induction we can get

$$\mu_0 \leq \mu_1 \leq \cdots \leq \mu_n \leq \cdots \leq \omega_n \leq \cdots \leq \omega_1 \leq \omega_0. \tag{3.4}$$

Notice that $\mu_0 = t_0^2\omega_0$, by induction it is easy to get $\mu_n \geq (t_0^2)^{\lambda^n}\omega_n, (n=1,2,\cdots)$. Since P is a normal cone with the normality constant 1, and $\mu_{n+m} - \mu_n \leq \omega_n - \mu_n$ for each $m \in \mathbb{N}$, we obtain

$$\|\mu_{n+m} - \mu_n\| \leq \|\omega_n - \mu_n\| \leq (1-(t_0^2)^{\lambda^n})\|\omega_0\| \to 0, (n \to +\infty). \tag{3.5}$$

It means that $\{\mu_n\}$ is a Cauchy-sequence, so μ_n converges to some $\nu^* \in D$, and $T\nu^* \geq \nu^*$ in the other hand we have

$$\|\omega_n - \nu^*\| \leq \|\omega_n - \mu_n\| + \|\mu_n - \nu^*\| \to 0, (n \to +\infty).$$

This implies that ω_n converges to ν^*, and and $T\nu^* \leq \nu^*$. So ν^* is a fixed point of T, and $\nu^* \in [\mu_0, \omega_0]$. For each initial value $\nu_0 \in D$, we have $\mu_0 \leq \nu_0 \leq \omega_0$ and $\mu_n \leq \nu_n \leq \omega_n$, $(n=1,2,\cdots)$. So

$$\|\nu_n - \nu^*\| \leq \|\nu_n - \mu_n\| + \|\mu_n - \nu^*\| \leq 2\|\omega_n - \mu_n\| \leq 2(1-(t_0^2)^{\lambda^n})\|\omega_0\|,$$

which implies that ν_n converges to ν^*. From Lemma 2.5 we get $I^{\mu_{n-1}-\mu_0}\nu^*$ is a positive solution of equation (1.1) satisfying

$$I^{\mu_{n-1}-\mu_0}\nu_n - I^{\mu_{n-1}-\mu_0}\nu^* = I^{\mu_{n-1}-\mu_0}(\nu_n - \nu^*) \leq 2(1-(t_0^2)^{\lambda^n})I^{\mu_{n-1}-\mu_0}\omega_0.$$

Furthermore we have the error estimation

$$\|I^{\mu_{n-1}-\mu_0}\nu_n - I^{\mu_{n-1}-\mu_0}\nu^*\| \leq 2(1-(t_0^2)^{\lambda^n})\|I^{\mu_{n-1}-\mu_0}\omega_0\|.$$

So the rate of convergence is $o(1-(t_0^2)^{\lambda^n})$, where t_0 is determined by ν_0.

Now we shall prove the uniqueness of positive solution of(2.7).

For any fixed point $\nu^- \in D$ of T, from the definition of D, let

$$t_1 = inf\{t>0 : \nu^- \leq t\nu^*\}.$$

Then $0 < t_1 < +\infty$. Now we prove $t_1 \leq 1$. Otherwise, then

$$\nu^- = T\nu^- \leq T(t_1\nu^*) \leq t_1^\lambda T(\nu^*) = t_1^\lambda \nu^*.$$

It implies $t_1^\lambda < t_1$, which contradicts the definition of t_1. So $t_1 \leq 1$ and $\nu^- \leq \nu^*$. In the other hand, let $t_2 = inf\{t>0 : \nu^* \leq t\nu^-\}$. In the same way we obtain $\nu^* \leq \nu^-$, thus $\nu^- = \nu^*$, that is ν^* is the unique fixed point of $T \in D$. Of course, it is also the unique positive solution of (2.7).

Thus according to Lemma 2.5, $I^{\mu_{n-1}-\mu_0}v^*$ is the unique solution of (1.1). □

Example 3.1 Consider the following equation

$$\begin{cases} \mathcal{D}^{\frac{7}{2}}x(t)+t^2(x^{\frac{4}{9}}(t)+(\mathcal{D}^{\frac{3}{4}}x(t))^{\frac{1}{3}}+(\mathcal{D}^{\frac{7}{4}}x(t))^{\frac{3}{7}}+(\mathcal{D}^{\frac{7}{3}}x(t))^{\frac{3}{8}})=0, t\in[0,1] \\ x(0)=\mathcal{D}^{\frac{3}{4}}x(0)=\mathcal{D}^{\frac{7}{4}}x(0)=\mathcal{D}^{\frac{7}{3}}x(0)=0 \\ \mathcal{D}^{\frac{12}{5}}x(1)=\sum_{j=1}^{\infty}\frac{3}{4}(\frac{1}{3})^j\mathcal{D}^{\frac{12}{5}}x([1-(\frac{1}{3})^j]^{10}), \end{cases} \tag{3.6}$$

where $n=4$, $\mu_1=\frac{3}{4}$, $\mu_2=\frac{7}{4}$, $\mu_3=\frac{7}{3}$, $q(t)=t^2$, $\alpha=\frac{7}{2}$,

$$f(x_1,x_2,x_3,x_4)=x_1^{\frac{4}{9}}(t)+x_2^{\frac{1}{3}}(t)+x_3^{\frac{3}{7}}(t)+x_4^{\frac{3}{8}}(t).$$

By a careful calculation we get $\alpha-\mu-1=\frac{1}{10}$, $\alpha-\mu_{n-1}-1=\frac{1}{6}$,

$$f(x_1,x_2,x_3,x_4)=x_1^{\frac{4}{9}}(t)+x_2^{\frac{1}{3}}(t)+x_3^{\frac{3}{7}}(t)+x_4^{\frac{3}{8}}(t),$$

is increasing on $[0,1]$.

And $f(rx_1,rx_2,rx_3,rx_4)=r^{\frac{4}{9}}f(x_1,x_2,x_3,x_4)$, $0<\frac{4}{9}<1$, $1>\frac{3}{4}=\sum_{j=1}^{\infty}\alpha_j\xi_j^{\alpha-\mu-1}>0$.

$$e_0=I^{\frac{7}{3}}t^{\frac{1}{6}}=\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}}, e_1=I^{\frac{19}{12}}t^{\frac{1}{6}}=\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{11}{4})}t^{\frac{7}{4}}, e_2=I^{\frac{7}{12}}t^{\frac{1}{6}}=\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{7}{4})}t^{\frac{3}{4}}, e_3=t^{\frac{1}{6}}.$$

$$\begin{aligned} 0 &< \int_0^1 q(s)f(e_0(s),e_1(s),\cdots,e_{n-2}(s),e_{n-1}(s))ds \\ &= \int_0^1 s^2 f(\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{7}{2})}s^{\frac{5}{2}},\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{11}{4})}s^{\frac{7}{4}},\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{7}{4})}t^{\frac{3}{4}},s^{\frac{1}{6}}) \\ &\le \int_0^1 f(\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{7}{2})},\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{11}{4})},\frac{\Gamma(\frac{7}{6})}{\Gamma(\frac{7}{4})},1)<+\infty. \end{aligned}$$

From Theorem 3.1, the equation (3.6) has a unique positive solution.

References

[1] B. Ahmad, A. Alsaedi. Existence and uniqueness of solutions for coupled systems of higher-order nonlinear fractional differential equations. Fixed Point Theory Appl.2010(2010)(Article ID364560,17pages).

[2] Y. Cui. Uniqueness of solution for boundary value problems for fractional differential equations. Appl. Math. Lett. 51(2016)48-54.

[3] C. Goodrich. Existence of a positive solution to systems of differential equations of fractional order. Comput. Math. Appl. 62(2011)1251-128.

[4] J.Li,B.Liu,L.Liu.Solutions for a boundary value problem at resonance on $[0,+\infty)$. Math. Comput. Model. 58(2013) 1769-1776.

[5] R. MillerK. An introduction to the fractional calculus and fractional differential equations. NewYork: Wiley; 1993.

[6] I. Podlubny. Fractional Differential Equations. Mathematics in Science and Engineering, Academic Press, NewYork: London, Toronto. 1999.

[7] H. Qu, X. Liu. Existence of nonnegative solutions for a fractional m-point boundary value problem at resonance. Bound.ValueProbl. 127(2013)415-425.

[8] Y. Wang, L. Liu, Y. Wu. Positive solutions for a nonlocal fractional differential equation. Nonlinear. Anal. 74(2011)3599-3605.

[9] L. Yang, W. Zhang. Positive solution for second order multi-point boundary value problem at resonance.Fixed Point Theory.2(2011)155-166

[10] S. Zhang. Positive solutions to singular boundary value problem for nonlinear fractional differential equation. Comput. Math. Appl. 59(2010)1300-1309.

[11] X. Zhang, L. Liu, Y. Wu. The eigenvalue problem for a singular higher order fractional diferential equation involving fractional derivatives. Appl. Math. Comput. 218(2012)8526-8536.

[12] X. Zhang, Y. Han. Existence and uniqueness of positive solutions for higher order nonlocal fractional differential equations.Appl.Math.Lett. 25(2012)555-560.

[13] M.Zima, P. Drygaś. Existence of positive solutions for a kind of periodic boundary value problem at resonance.Bound.ValueProbl.1(2013)1-10

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 25, NO.2, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

ON THE (h, q)-EULER POLYNOMIALS AND NUMBERS

LEE-CHAE JANG, JEONG GON LEE

Graduate School of Education,
Konkuk University, Seoul 143-701, Republic of Korea
E-mail : lcjang@konkuk.ac.kr

Division of Mathematics and Informational Statistics,
and Nanoscale Science and Technology Institute,
Wonkwang University Republic of Korea, Iksan, 570-749, Republic of Korea
E-mail : jukolee@wku.ac.kr

Abstract. In this paper, we consider the fermionic p-adic q-integral on $\mathbb{Z}_p$ which was defined by Kim [T. Kim, On p-adic interpolating function for q-Euler numbers and its derivatives, J. Math. Anal. Appl. 339(2008) 598-608]. By using this integral, we define the (h, q)-Euler polynomials and numbers and give some interesting identities of these polynomials and numbers.

1. Introduction

Let p be a fixed prime number with $p \equiv 1 (\text{mod } 2)$. Throughout this work, $\mathbb{Z}_p$, $\mathbb{C}$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic integers, the complex field, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized by $|p|_p = \frac{1}{p}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p < 1$. We use the notation

$$[x]_q = \frac{1-q^x}{1-q} \text{ and } [x]_{-q} = \frac{1-(-q)^x}{1+q}, \qquad (\text{see } [1,4,5,11,12,15,21,22,24]). \tag{1}$$

Let $C(\mathbb{Z}_p)$ be the set of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \qquad (\text{see } [2,3,6-25]). \tag{2}$$

It is well-known that the classical Euler polynomials $E_n(x)$ are defined by means of the following generating function:

$$\int_{\mathbb{Z}_p} e^{t(x+y)} d\mu_{-q}(y) = \frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \qquad (\text{see } [8-13,15,16,19,22-25]). \tag{3}$$

1991 *Mathematics Subject Classification.* 05A10, 11B68, 11S80, 05A19.

1

For a fixed $h \in \mathbb{C}_p$ with $|1-h|_p < p^{-\frac{1}{1-p}}$, we define the (h,q)-Euler polynomials as

$$\mathcal{E}_{n,q}^{(h)}(x) = \int_{\mathbb{Z}_p} [y+x]_q^n h^y d\mu_{-q}(y). \tag{4}$$

When $x = 0$, $\mathcal{E}_{n,q}^{(h)} = \mathcal{E}_{n,q}^{(h)}(0)$ are called the (h,q)-Euler numbers. When $q = 1$, $E_n^{(h)}(x) = \mathcal{E}_{n,1}^{(h)}(x)$ is called the (h)-Euler polynomials as follows

$$E_n^{(h)}(x) = \int_{\mathbb{Z}_p} (y+x)^n h^y d\mu_{-q}(y), \tag{5}$$

and $E_n^{(h)} = E_n^{(h)}(0)$ is called the (h)-Euler numbers.

In this paper, by using the fermionic p-adic q-integral on $\mathbb{Z}_p$, we define the (h,q)-Euler polynomials and discuss their properties. Furthermore, we give some interesting identities of these polynomials and numbers.

2. The (h,q)-Euler polynomials

From (4), we observe that

$$\mathcal{E}_{n,q}^{(h)}(x) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} q^{lx} (-1)^l \int_{\mathbb{Z}_p} (q^l h)^y d\mu_{-q}(y). \tag{6}$$

We note that

$$\begin{aligned}
\int_{\mathbb{Z}_p} (q^l h)^y d\mu_{-q}(y) &= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} (q^{l+1}h)^y (-1)^y \\
&= \frac{[2]_q}{2} \lim_{N\to\infty} \frac{1+(q^{l+1}h)^{p^N}}{1+q^{l+1}h} \\
&= \frac{[2]_q}{1+q^{l+1}h}.
\end{aligned} \tag{7}$$

By (6) and (7), we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{N} \cup \{0\}$, we have*

$$\mathcal{E}_{n,q}^{(h)}(x) = \frac{[2]_q}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} q^{lx} (-1)^l \frac{1}{1+q^{l+1}h}. \tag{8}$$

Now, we observe that

$$\begin{aligned}
&\mathcal{E}_{n,q}^{(h)}(x) \\
&= \frac{[2]_q}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} q^{lx} (-1)^l \sum_{m=0}^{\infty} (-1)^m q^{(l+1)m} h^m \\
&= \frac{[2]_q}{(1-q)^n} \sum_{m=0}^{\infty} (-1)^m h^m q^m \sum_{l=0}^{n} \binom{n}{l} q^{l(x+m)} (-1)^l \\
&= [2]_q \sum_{m=0}^{\infty} (-1)^m h^m q^m \frac{1}{(1-q)^n} (q^{x+m}-1)^n
\end{aligned}$$

$$= \quad [2]_q \sum_{m=0}^{\infty} (-1)^m h^m q^m [x+m]_q^n. \tag{9}$$

By (9), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{N} \cup \{0\}$, we have*

$$\mathcal{E}_{n,q}^{(h)}(x) = [2]_q \sum_{m=0}^{\infty} (-1)^m h^m q^m [x+m]_q^n. \tag{10}$$

Let us consider the generating function of the (h,q)-Euler polynomials as $F_{h,q}(x,t) = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{(h)}(x)\frac{t^n}{n!}$, where $t \in \mathbb{C}_p$ with $|1-t|_p < 1$ or $t \in \mathbb{C}$ with $|t| < 1$. Then we see that

$$\begin{aligned}
F_{h,q}(x,t) &= \sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{(h)}(x)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left([2]_q \sum_{m=0}^{\infty} (-1)^m h^m q^m [x+m]_q^n \right) \frac{t^n}{n!} \\
&= [2]_q \sum_{m=0}^{\infty} (-1)^m h^m q^m \sum_{n=0}^{\infty} [x+m]_q^n \frac{t^n}{n!} \\
&= [2]_q \sum_{m=0}^{\infty} (-1)^m h^m q^m e^{[x+m]_q t}.
\end{aligned} \tag{11}$$

By (11), we obtain the following theorem.

Theorem 2.3. *Let $F_{h,q}(x,t)$ be the generating function of (h,q)-Euler polynomials as $F_q(x,t) = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}^{(h)}(x)\frac{t^n}{n!}$. Then we have*

$$F_{h,q}(x,t) = [2]_q \sum_{m=0}^{\infty} (-1)^m h^m q^m e^{[x+m]_q t}. \tag{12}$$

From Theorem 2.3, we note that

$$\begin{aligned}
lim_{q \to 1} F_{h.q}(x,t) &= 2\sum_{m=0}^{\infty} (-1)^m h^m e^{(x+m)t} \\
&= 2e^{xt} \sum_{m=0}^{\infty} (-1)^m h^m e^{mt} \\
&= \frac{2}{he^t+1} e^{xt}
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
&\int_{\mathbb{Z}_p} h^y e^{t(x+y)} d\mu_{-q}(y) \\
&= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} h^y (x+y)^n d\mu_{-q}(y) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} E_n^{(h)}(x) \frac{t^n}{n!},
\end{aligned} \tag{14}$$

where $E_n^{(h)}(x)$ are the (h)-Euler polynomials. We also observe that if $f_1(x) = f(x+1)$, then we have

$$\begin{aligned} qI_{-q}(f_1) &= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} (-1)^x f(x+1) q^{x+1} \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=1}^{p^N} (-1)^{x-1} f(x) q^x \\ &= -\lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} (-1)^x f(x) q^x \\ &\quad + \lim_{N\to\infty} \frac{[2]_q}{2} \left((-1)^{p^N-1} q^{p^N} f(p^N) + f(0) \right) \\ &= -\lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} (-1)^{x-1} f(x) q^x + [2]_q f(0) \\ ] &= -I_{-q}(f) + [2]_q f(0). \end{aligned} \tag{15}$$

By (15), we obtain the following integral equation.

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0). \tag{16}$$

We note that if $f(y) = h^y e^{(x+y)t}$, by (5) and (16), then we get

$$\int_{\mathbb{Z}_p} h^y e^{t(x+y)} d\mu_{-q}(y) = \frac{2}{he^t+1} e^{xt}. \tag{17}$$

From (13), (14) and (17)

Theorem 2.4. *Let $E_n^{(h)}(x)$ be (h)-Euler polynomials with the generating function $F_{h,1}(x,t) = \sum_{n=0}^{\infty} E_n^{(h)}(x)\frac{t^n}{n!}$. Then we have*

$$F_{h,1}(x,t) = \frac{2}{he^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n^{(h)}(x) \frac{t^n}{n!}, \tag{18}$$

In particular, when $h = 1$ and $q = 1$, we get

$$F_{1,1}(x,t) = \frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \tag{19}$$

where $E_n(x)$ are the classical Euler polynomials.

From (16), we note that if $f(x) = h^x [x]_q^n$, then we have

$$q \int_{\mathbb{Z}_p} h^{x+1} [x+1]_q^n d\mu_{-q}(x) + \int_{\mathbb{Z}_p} h^x [x]_q^n d\mu_{-q}(x) = [2]_q 0_q^n. \tag{20}$$

By (20), we obtain the following theorem.

Theorem 2.5. *For $n \in \mathbb{N} \cup \{0\}$, we have*

$$qh\mathcal{E}_{n,q}^{(h)}(1) + \mathcal{E}_{n,q}^{(h)} = [2]_q \delta_{n,0}, \tag{21}$$

where $\delta_{n,0}$ is the Kronecker's symbol.

We note that

$$\begin{aligned} [x+y]_q^n &= ([x]_q + q^x[y]_q)^n \\ &= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} [y]_q^l. \end{aligned} \tag{22}$$

By (22), we get

$$\begin{aligned} \mathcal{E}_{n,q}^{(h)}(x) &= \int_{\mathbb{Z}_p} h^y [x+y]_q^n d\mu_{-q}(y) \\ &= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} \int_{\mathbb{Z}_p} [y]_q^l d\mu_{-q}(y) \\ &= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} \mathcal{E}l, q^{(h)}(x) \\ &= \left(\mathcal{E}_q^{(h)}(x) + [x]_q\right)^n, \end{aligned} \tag{23}$$

with the usual convention about replacing $(\mathcal{E}_q^{(h)}(x))^l = \mathcal{E}_{n,q}^{(h)}(x)$. Thus, by (21) and (23), we obtain the following theorem.

Theorem 2.6. *For $n \in \mathbb{N} \cup \{0\}$, we have*

$$qh\left(q\mathcal{E}_q^{(h)}(1) + 1\right)^n + \mathcal{E}_{n,q}^{(h)} = [2]_q \delta_{n,0} \tag{24}$$

with the usual convention about replacing $(\mathcal{E}_q^{(h)}(x))^l = \mathcal{E}_{n,q}^{(h)}(x)$, where $\delta_{n,0}$ is the Kronecker's symbol.

3. Remarks

In this section, we assume that $q, h \in \mathbb{C}$ with $|q| < 1$ and $|h| < 1$. For $s \in \mathbb{C}$, we define the (h, q)-Zeta functions as follows:

$$\zeta_q^{(h)}(s, x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n h^h}{[n+x]_q^s}, \quad x \neq 0, -1, -2, \cdots. \tag{25}$$

By (10) and (25), we obtain the following theorem.

Theorem 3.1. *For $m \in \mathbb{N}$, we have*

$$\zeta_q^{(h)}(-m, x) = \mathcal{E}_{m,q}^{(h)}(x). \tag{26}$$

We note that if $d \in \mathbb{N}$ with $d \equiv 1 (\text{mod } 2)$, then we have

$$\begin{aligned} &\int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) \\ = &\lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x \end{aligned}$$

$$\begin{aligned}
&= \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{a=0}^{d-1} \sum_{x=0}^{p^N-1} f(a+dx)(-q)^{a+dx} \\
&= \lim_{N\to\infty} \frac{1+q^d}{1+q^{dp^N}} \cdot \frac{1+q}{1+q^d} \sum_{a=0}^{d-1} \sum_{x=0}^{p^N-1} f(a+dx)(-q)^{a+dx} \\
&= \frac{1}{[d]_{-q}} \lim_{N\to\infty} \frac{1}{[p^N]_{-q^d}} \sum_{a=0}^{d-1} \sum_{x=0}^{p^N-1} f(x)f(a+dx)(-q)^{a+dx} \\
&= \frac{1}{[d]_{-q}} \sum_{a=0}^{d-1} (-q)^a \lim_{N\to\infty} \frac{1}{[p^N]_{-q^d}} \sum_{x=0}^{p^N-1} f(x)f(a+dx)(-q^d)^x \\
&= \frac{1}{[d]_{-q}} \sum_{a=0}^{d-1} (-q)^a \int_{\mathbb{Z}_p} f(a+dx) d\mu_{-q}(x). \qquad (27)
\end{aligned}$$

By (27), we obtain the following theorem.

Theorem 3.2. *Let $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\ 2)$. Then, we have*

$$\int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \frac{1}{[d]_{-q}} \sum_{a=0}^{d-1} (-q)^a \int_{\mathbb{Z}_p} f(a+dx) d\mu_{-q}(x). \qquad (28)$$

We note that if $f(y) = h^y [x+y]_q^n$, then we derive

$$\begin{aligned}
&\int_{\mathbb{Z}_p} h^y [x+y]_q^n d\mu_{-q}(y) \\
&= \frac{1}{[d]_{-q}} \sum_{a=0}^{d-1} (-q)^a \int_{\mathbb{Z}_p} h^{a+dy} [a+dy+x]_q^n d\mu_{-q}(y) \\
&= \frac{1}{[d]_{-q}} \sum_{a=0}^{d-1} (-q)^a h^a \int_{\mathbb{Z}_p} h^{dy} [a+dy+x]_q^n d\mu_{-q}(y) \\
&= \frac{[d]_q^n}{[d]_{-q}} \sum_{a=0}^{d-1} (-q)^a h^a \int_{\mathbb{Z}_p} h^{dy} [\frac{a+x}{d} + y]_{q^d}^n d\mu_{-q}(y) \\
&= \frac{[d]_q^n}{[d]_{-q}} \sum_{a=0}^{d-1} (-q)^a h^a \mathcal{E}_{n,q^d}^{(h)} \left(\frac{a+x}{d} \right), \qquad (29)
\end{aligned}$$

where $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\ 2)$.

Theorem 3.3. *For $n \in \mathbb{N} \cup \{0\}$, we have*

$$\mathcal{E}_{n,q}^{(h)}(x) = \frac{[d]_q^n}{[d]_{-q}} \sum_{a=0}^{d-1} (-q)^a h^a \mathcal{E}_{n,q^d}^{(h)} \left(\frac{a+x}{d} \right). \qquad (30)$$

4. Results and discussion

The results of our works are some interesting identities of the (h,q)-Euler polynomials as follows. The first result is to fine the generating function of the (h,q)-Euler polynomials in

Theorem 2.3. We note that the (h, q)-Euler polynomial is the generalization of the q-Euler polynomial and the (h)-Euler polynomials (see Theorem 2.4). The second result is to find the recurrence formula of the (h, q)-Euler polynomials in Theorem 2.6. The third result is to compare with the (h, q)-Euler polynomials and the (h, q)-Zeta functions in Theorem 3.1. The fourth result is to find the distribution equation of the (h, q)-Euler polynomials in Theorem 3.3.

In the future, we will work to find some interesting identities between the (h, q)-Euler polynomials and the q-Euler polynomials and to compare with the zeros of the (h, q)-Euler polynomials and the zeros of the q-Euler polynomials.

5. Conclusions

We defined (h, q)-Euler polynomials which are q-Euler polynomials with some weight function in Eq.(6) and remarked that if $h = 1$, $(1, q)$-Euler polynomials are q-Euler polynomials. In Theorems 2.1 and 2.2, we found some properties of (h, q)-Euler polynomials. In Theorem 2.3, we obtained some specific type of the generating function for (h, q)-Euler polynomials and noted that when $h = q$, we saw the more specific type of the generating function of (h, q)-Euler polynomials.

Furthermore, we investigated very interesting recursion formula of (h, q)-Euler polynomials in Theorem 3.1. In section 3, we defined the (h, q)-Zeta functions and gave some relation with the (h, q)-Zeta funtion and (h, q)-Euler polynomials. Finally, we obtained a distribution identity equation of (h, q)-Euler polynomials in Theorem 3.3.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

Acknowledgement
The authors are grateful to the anonymous referees for their careful reading of the paper and for their interesting suggestions and comments which helped us to improve the presentation considerably.

References

[1] W. A. Al-Salem, *q-Bernoulli numbers and polynomials*, *Math. Nachr.* **17** (1959) 239–260 (1959).
[2] A. Bayad, J. Chikhi, Non linear recurrences for Apostol-Bernoulli-Euler numbers of higher order,, *Adv. Stud. Contemp. Math (Kyungshang)* **22** (2012), no. 1, 1–6.
[3] A. Bayad, T. Kim, *Identities for Apostol-type Frobenius-Euler polynomials resulting from the study of a nonlinear operator*, *Russ. J. MAth. Phys.* **23** (2016), no. 2, 164–171.
[4] L. Carlitz, *Expansion of q-Bernoulli numbers*, *Duke Math. J.* **25** (1958) 355–364.
[5] L. Carlitz, *q-Bernoulli and Eulerian numbers*, *Trans. Amer. Math. Soc* **76** (1954). 332–350.
[6] L. Carlitz, *q-Bernoulli and polynomials*, *Duke Math. J.* **15** (1948). 987–1000.
[7] D. V. Dolgy, D. J. Kang, T. Kim, B. Lee, *Some new identities on the twisted (h, q)-Euler numbers and q-Bernstein polynomials*, *J. Comput. Anal. Appl.* **14** (2012), no. 5, 974–984.
[8] L. C. Jang, J. G. Lee, *A note on the Barnes-type q-Euler polynomials*, *Adv. Difference Equ.* **2015**, 2015:250, 7 pp.

[9] D. S. Kim, T. Kim, *Barnes-type Boole polynomials*, *Contrib. Discrete Math.* **11** (2016), no. 1, 7–15.
[10] D. S. Kim, T. Kim, *Some identities of degenerate Euler polynomials arising from p-adic fermionic integral on* $\mathbb{Z}_p$, *Integral Transforms Spec. Funct.* **26** (2015), no. 4, 295–302.
[11] D. S. Kim, T. Kim, *Symmetric identities of higher-order degenerate q-Euler polynomials*, *J. Nonlinear Sci. Appl.* **9** (2016), no. 2, 443–451.
[12] T. Kim, D. S. Kim, H.-I. Kwon, J.-J. Seo, D. V. Dolgy, *Some identities of q-Euler polynomials under the symmetric group of degree n*, *J. Nonlinear Sci. Appl.* **9** (2016), no. 3, 1077–1082.
[13] T. Kim, *q-Euler numbers and polynomials associated with p-adic q-integrals*, *J. Nonlinear Math. Phys.* **14** (2007), no. 1, 15–27.
[14] T. Kim, *Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on* $\mathbb{Z}_p$, *Russ. J. Math. Phys.* **16** (2009), no .1, 93–96.
[15] T. Kim, *Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on* $\mathbb{Z}_p$, *Russ. J. Math. Phys.* **16** (2009), no. 4, 484–491.
[16] T. Kim, D. S. Kim, H.-I. Kwon, J.-J. Seo, *Some identities for degenerate Frobenius-Euler numbers arising from nonlinear differential equations*, *Italian J. Pure and Appl. Math.* **36** (2016), 843–850.
[17] T. Kim, D. S. Kim, *Differential equations associated with Catalan-Daehee numbers and their applications*, *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Mathemáticas* **2016** (2016), 1–11.
[18] T. Kim, D. S. Kim, *A note on nonlinear Changhee differential equations*, *Russ. J. Math. Phys.* **23** (2016), no. 1, 88–92.
[19] T. Kim, D. V. Dolgy, L. C. Jang, H. I. Kwon, *Some identities of degenerate q-Euler polynomials under the symmetry group n*, *J. Nonlinear Sci. Appl.* **9** (2016), no. 6, 4707–4712.
[20] T. Kim, *q-Volkenborn integration*, *Russ. J. Math. Phys.* **9** (2002), no. 3, 288-299.
[21] T. Kim, D. S. Kim, D. V. Dolgy, *Degenerate q-Euler polynomials*, *Adv. Difference Equ.* **2015** 2015:246, 11 pp.
[22] T. Kim, *Some properties on the integral of the product of several Euler polynomials*, *Quaest. Math.* **38** (2015), no. 4, 553–562.
[23] A. Sharma, *q-Bernoulli and Euler numbers of higher order*, *Duke Math. J.* **25** (1958) 343–353.
[24] K. Shiratani, S. Yokoyama, *An application of p-adic convolutions*, *Mem. Fac. Sci. Kyushu Univ. Ser. A* **36** (1982), no. 1, 73–83.
[25] S. L. Sobolev, *Roots of Euler polynomials*, *Theory of cubature formulas and numerical mathematics (Proc. Conf., Novosibirsk, 1978) (Russian)* pp. 125–141, 253,

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 25, NO.2, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

HERMITE-HADAMARD TYPE INEQUALITIES FOR n-TIMES DIFFERENTIABLE AND α-LOGARITHMICALLY PREINVEX FUNCTIONS

SHUHONG WANG*

ABSTRACT: In the paper, the concept of α-logarithmically preinvex function is introduced, and by creating an integral identity involving an n-times differentiable function, some new Hermite-Hadamard type inequalities for α-logarithmically preinvex functions are established.

KEY WORDS: Integral identity, Hermite-Hadamard type inequality, α-logarithmically preinvex function.

1. Introduction

Throughout this paper, let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = (0, \infty)$, $\mathbb{N}$ denote the set of all positive integers, I denote the interval in $\mathbb{R}$, and A denote the set in $\mathbb{R}^n, n \in \mathbb{N}$.

First, let us recall some definitions of various convex functions.

Definition 1.1. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$. If the inequality (1.1) reverses, then f is said to be concave on I.

Definition 1.2 ([1]). A set $A \subseteq \mathbb{R}^n$ is said to be invex with respect to the map $\eta : A \times A \to \mathbb{R}^n$, if for every $x, y \in A$ and $t \in [0, 1]$

$$y + t\eta(x, y) \in A. \tag{1.2}$$

The invex set A is also called a η-connected set.

It is obvious that every convex set is invex with respect to the map $\eta(x, y) = x - y$, but there exist invex sets which are not convex (see [1], for example).

Definition 1.3 ([1]). Let $A \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : A \times A \to \mathbb{R}^n$. For every $x, y \in A$, the η-path P_{xv} joining the points x and $v = x + \eta(y, x)$ is defined by

$$P_{xv} = \{v | v = x + t\eta(y, x), t \in [0, 1]\}. \tag{1.3}$$

Definition 1.4 ([13]). Let $A \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : A \times A \to \mathbb{R}^n$. A function $f : A \to \mathbb{R}$ is said to be preinvex with respect to η, if for every $x, y \in A$ and $t \in [0, 1]$

$$f(y + t\eta(x, y)) \le tf(x) + (1-t)f(y). \tag{1.4}$$

The function f is said to be preincave if and only if $-f$ is preinvex.

Every convex function is preinvex with respect to the map $\eta(x, y) = x - y$, but not conversely (see [13], for instance).

*College of Mathematics, Inner Mongolia University for Nationalities, Tongliao 028043, China.
*To whom correspondence should be addressed. e-mail:shuhong7682@163.com
This paper was typeset using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX.

Definition 1.5 ([10]). Let $A \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : A \times A \to \mathbb{R}^n$. The function $f : A \to \mathbb{R}_+$ on the set A is said to be logarithmically preinvex with respect to η, if for every $x, y \in A$ and $t \in [0, 1]$

$$f(y + t\eta(x, y)) \le \big[f(x)\big]^t \big[f(y)\big]^{1-t}. \tag{1.5}$$

For properties and applications of preinvex and logarithmically preinvex functions, please refer to [8, 9, 14, 16] and closely related references therein.

The most important inequality in the theory of convex functions, the well known Hermite-Hadamard's integral inequality, may be stated as follows. Let $I \subseteq \mathbb{R}$ and $a, b \in I$ with $a < b$. If $f : [a, b] \subseteq I \to \mathbb{R}$ is a convex function on $[a, b]$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2}. \tag{1.6}$$

If f is concave on $[a, b]$, then the inequality (1.6) is reversed.

The inequality (1.6) has been generalized by many mathematicians. Some of them may be recited as follows.

Theorem 1.1 ([5]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I and $a, b \in I$ with $a < b$. If $|f'(x)|^q$ for $q \ge 1$ is convex function on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{1/q} \tag{1.7}$$

and

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{1/q}. \tag{1.8}$$

Theorem 1.2 ([6]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on I, $a, b \in I$ with $a < b$, and $p > 1$. If $|f'(x)|^{p/(p-1)}$ is convex on $[a, b]$, then*

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| &\le \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p} \\ &\times \left\{\Big[|f'(a)|^{p/(p-1)} + 3|f'(b)|^{p/(p-1)}\Big]^{1-1/p} + \Big[3|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}\Big]^{1-1/p}\right\}. \end{aligned} \tag{1.9}$$

Theorem 1.3 ([9]). *Let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $\eta(a, b) > 0$ for all $a \ne b$. If $f : A \to \mathbb{R}_+$ is a preinvex function on A, then the following inequality holds*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \le \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2}. \tag{1.10}$$

Theorem 1.4 ([11]). *Let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $\eta(a, b) > 0$ for all $a \ne b$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function. If $|f^{(n)}|^q, q > 1, q \in \mathbb{R}$, is a log-preinvex function on A, then we have the inequality*

$$\begin{aligned}&\left|f\left(a + \frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,\mathrm{d}x\right| \\ &\quad\le \frac{\eta(b,a)\sqrt{|f'(a)|}}{2^{1/p}(p+1)^{1/p}q^{1/q}}\left[\frac{\big(|f'(b)|\big)^{q/2} - \big(|f'(a)|\big)^{q/2}}{\big(\ln\big(|f'(b)|\big) - \ln\big(|f'(a)|\big)\big)}\right]^{1/q}, \end{aligned}\tag{1.11}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 25, NO.2, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

Theorem 1.5 ([7]). *Let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $\eta(a,b) > 0$ for all $a \neq b$. Suppose $f : A \to \mathbb{R}$ is a function such that $f^{(n)}$ exits on A and $f^{(n)}$ is integrable on $[a, a+\eta(b,a)]$ for $n \in \mathbb{N}, n \geq 1$. If $|f^{(n)}|^q$ is log-preinvex on A for $n \in \mathbb{N}, n \geq 1, q \geq 1$, then we have the following inequality*

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}\left(a+\frac{1}{2}\eta(b,a)\right) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,\mathrm{d}x\right|$$
$$\leq \frac{(\eta(b,a))^n\left|f^{(n)}(a)\right|(n!)^{1/q-1}}{2^{(n+1)(q-1)/q}(n+1)^{1-1/q}}\left\{[E_2(n,q)]^{1/q}+[E_3(n,q)]^{1/q}\right\}, \quad (1.12)$$

where

$$E_2(n,q) = \frac{(-1)^{n+1}}{q^{n+1}\left[\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right]^{n+1}} + \left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{q/2}\sum_{k=0}^{n}\frac{(-1)^k}{(n-k)!q^{k+1}2^{n-k}\left[\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right]^{k+1}} \quad (1.13)$$

and

$$E_3(n,q) = \frac{\left|f^{(n)}(b)\right|^q}{q^{n+1}\left[\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right]^{n+1}\left|f^{(n)}(a)\right|^q} - \left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{q/2}\sum_{k=0}^{n}\frac{1}{(n-k)!q^{k+1}2^{n-k}\left[\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right]^{k+1}}. \quad (1.14)$$

Recently, some related inequalities for preinvex functions were also obtained in [4, 15, 16, 17].

2. New Definitions and lemmas

Now we introduce the concept of α-logarithmic preinvex function.

Definition 2.1. Let $A \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : A \times A \to \mathbb{R}^n$. The function $f : A \to \mathbb{R}_+$ on the set A is said to be α-logarithmically preinvex function with respect to η, if for every $x, y \in A, t \in [0,1]$ and some $\alpha \in (0,1]$

$$f(y+t\eta(x,y)) \leq \left[f(x)\right]^{t^\alpha}\left[f(y)\right]^{1-t^\alpha}. \quad (2.1)$$

Clearly, when taking $\alpha = 1$ in (2.1), then f becomes the standard logarithmically preinvex convex function on A.

In order to obtain our main results, we need the following lemmas.

Lemma 2.1. *For $n \in \mathbb{N}$, let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $\eta(a,b) > 0$ for all $a \neq b$. If $f : A \to \mathbb{R}$ is an n-times differentiable function on A and $f^{(n)}$ is integrable on the η-path P_{bc} for $c = b + \eta(a,b)$, then*

$$\sum_{k=0}^{n-1}\frac{1}{(k+1)!\eta(a,b)}\left[(b+\eta(a,b)-x)^{k+1}-(b-x)^{k+1}\right]f^{(k)}(x) - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x$$
$$= \frac{(-1)^{n+1}[\eta(a,b)]^n}{n!}\left[\int_0^{\frac{x-b}{\eta(a,b)}} t^n f^{(n)}(b+t\eta(a,b))\,\mathrm{d}t + \int_{\frac{x-b}{\eta(a,b)}}^1 (t-1)^n f^{(n)}(b+t\eta(a,b))\,\mathrm{d}t\right], \quad (2.2)$$

where $x \in [b, b+\eta(b,a)]$.

Proof. Since $a, b \in A$ and A is an invex set with respect to η, for every $t \in [0,1]$, it follows that $b + t\eta(a,b) \in A$. When $n = 1$, by integrating by part in the right-hand side of (2.2), it turns out that

$$\begin{aligned}
&\eta(a,b)\left[\int_0^{\frac{x-b}{\eta(a,b)}} tf'(b+t\eta(a,b))\,\mathrm{d}t + \int_{\frac{x-b}{\eta(a,b)}}^1 (t-1)f'(b+t\eta(a,b))\,\mathrm{d}t\right]\\
&= \frac{x-b}{\eta(a,b)}f(x) - \int_0^{\frac{x-b}{\eta(a,b)}} f(b+t\eta(a,b))\,\mathrm{d}t - \frac{x-b-\eta(a,b)}{\eta(a,b)}f(x) - \int_{\frac{x-b}{\eta(a,b)}}^1 f(b+t\eta(a,b))\,\mathrm{d}t\\
&= f(x) - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x.
\end{aligned}$$

Hence, the identity (2.2) holds for $n = 1$.

When $n = m$ and $m \ge 1$, suppose that the identity (2.2) is valid.

When $n = m + 1$, by the hypothesis, we have

$$\begin{aligned}
&\frac{(-1)^{m+2}[\eta(a,b)]^{m+1}}{(m+1)!}\left[\int_0^{\frac{x-b}{\eta(a,b)}} t^{m+1}f^{(m+1)}(b+t\eta(a,b))\,\mathrm{d}t\right.\\
&\left.+\int_{\frac{x-b}{\eta(a,b)}}^1 (t-1)^{m+1}f^{(m+1)}(b+t\eta(a,b))\,\mathrm{d}t\right]\\
&= \frac{(-1)^{m+2}[\eta(a,b)]^m}{(m+1)!}\left[\left(\frac{x-b}{\eta(a,b)}\right)^{m+1} f^{(m)}(x) - (m+1)\int_0^{\frac{x-b}{\eta(a,b)}} t^m f^{(m)}(b+t\eta(a,b))\,\mathrm{d}t\right.\\
&\left.-\left(\frac{x-b-\eta(a,b)}{\eta(a,b)}\right)^{m+1} f^{(m)}(x) - (m+1)\int_{\frac{x-b}{\eta(a,b)}}^1 (t-1)^m f^{(m)}(b+t\eta(a,b))\,\mathrm{d}t\right]\\
&= \frac{(-1)^{m+2}f^{(m)}(x)}{(m+1)!\eta(a,b)}\left[(x-b)^{m+1} - (x-b-\eta(a,b))^{m+1}\right]\\
&+\frac{(-1)^{m+1}[\eta(a,b)]^m}{m!}\left[\int_0^{\frac{x-b}{\eta(a,b)}} t^m f^{(m)}(b+t\eta(a,b))\,\mathrm{d}t + \int_{\frac{x-b}{\eta(a,b)}}^1 (t-1)^m f^{(m)}(b+t\eta(a,b))\,\mathrm{d}t\right]\\
&= \sum_{k=0}^m \frac{1}{(k+1)!\eta(a,b)}\left[(b+\eta(a,b)-x)^{k+1} - (b-x)^{k+1}\right]f^{(k)}(x) - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x.
\end{aligned}$$

Therefore, when $n = m + 1$, the identity (2.2) holds. By induction, the proof of Lemma 2.1 is complete. □

Remark 2.1. Under the conditions of Lemma 2.1, taking $x = b + \frac{\eta(a,b)}{2}$, we get

$$\begin{aligned}
&\sum_{k=0}^{n-1} \frac{[\eta(a,b)]^k[1+(-1)^k]}{(k+1)!2^{k+1}} f^{(k)}\left(b+\frac{\eta(a,b)}{2}\right) - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x\\
&\quad= \frac{(-1)^{n+1}[\eta(a,b)]^n}{n!}\left[\int_0^{\frac12} t^n f^{(n)}(b+t\eta(a,b))\,\mathrm{d}t + \int_{\frac12}^1 (t-1)^n f^{(n)}(b+t\eta(a,b))\,\mathrm{d}t\right],
\end{aligned}\tag{2.3}$$

which may be found in [7].

Lemma 2.2 ([14]). *Let $\mu > 0$ and $x \ge 0$. Then*

$$E(n;\mu,x) \triangleq \int_0^x t^n \mu^t\,\mathrm{d}t = \begin{cases} \frac{(-1)^{n+1}n!}{(\ln\mu)^{n+1}} + n!\mu^x \sum_{k=0}^n \frac{(-1)^k x^{n-k}}{(n-k)!(\ln\mu)^{k+1}}, & \mu \ne 1,\\ \frac{x^{n+1}}{n+1}, & \mu = 1 \end{cases}\tag{2.4}$$

for $n \geq 0, n \in \mathbb{N}$.

Lemma 2.3 ([12]). *Let* $r > 0$, $\mu > 0$ *and* $x \geq 0$. *Then*

$$F(r;\mu,x) \triangleq \int_0^x t^{r-1}\mu^t \,\mathrm{d}t = \begin{cases} x^r\mu^x \sum_{k=1}^{\infty} \frac{(-x\ln\mu)^{k-1}}{(r)_k} < \infty, & \mu \neq 1, \\ \frac{x^r}{r}, & \mu = 1, \end{cases} \tag{2.5}$$

where $(r)_k = r(r+1)(r+2)\cdots(r+k-1)$.

Lemma 2.4 ([3]). *Let* $0 < \phi \leq 1 \leq \psi$ *and* $0 < t, s \leq 1$. *Then*

$$\phi^{t^s} \leq \phi^{st} \tag{2.6}$$

and

$$\psi^{t^s} \leq \psi^{st+1-s}. \tag{2.7}$$

.

3. Hermite-Hadamard type inequalities

Now we start out to establish some new Hermite-Hadamard type inequalities for n-times differentiable and α-logarithmically preinvex functions.

Theorem 3.1. *For* $n \in \mathbb{N}$ *and* $n \geq 1$, *let* $A \subseteq \mathbb{R}$ *be an open invex set with respect to* $\eta : A \times A \to \mathbb{R}$ *and* $a, b \in A$ *with* $\eta(a,b) > 0$ *for all* $a \neq b$. *Suppose that* $f : A \to \mathbb{R}$ *is an* n*-times differentiable function on* A *and* $f^{(n)}$ *is integrable on the* η*-path* P_{bc} *for* $c = b + \eta(a,b)$. *If* $\left|f^{(n)}\right|^q$ *is* α*-logarithmically preinvex function on* A *for* $q \geq 1$, *then for* $\alpha \in (0,1]$ *and* $x \in [b, b+\eta(a,b)]$

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1} \frac{1}{(k+1)!\eta(a,b)}\left[(b+\eta(a,b)-x)^{k+1}-(b-x)^{k+1}\right] f^{(k)}(x) - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x\right| \\
&\leq \frac{[\eta(a,b)]^{(n+1-q)/q}}{n!(n+1)^{1-1/q}}\left\{(x-b)^{(n+1)(q-1)/q}\left|f^{(n)}(a)\right|^{\xi}\left|f^{(n)}(b)\right|^{\zeta}\left[E\left(n;\mu^{q\alpha},\frac{x-b}{\eta(a,b)}\right)\right]^{1/q}\right. \\
&\quad \left. + (b+\eta(a,b)-x)^{(n+1)(q-1)/q}\left|f^{(n)}(a)\right|^{\delta}\left|f^{(n)}(b)\right|^{\sigma}\left[E\left(n;\mu^{-q\alpha},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right]^{1/q}\right\}, \qquad (3.1)
\end{aligned}$$

where $\mu = \frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}$,

$$\{\xi,\zeta\} = \begin{cases} \{0,1\}, & \textit{if } 0 < \mu \leq 1, \\ \{1-\alpha,\alpha\}, & \textit{if } \mu > 1, \end{cases} \tag{3.2}$$

$$\{\delta,\sigma\} = \begin{cases} \{\alpha,1-\alpha\}, & \textit{if } 0 < \mu \leq 1, \\ \{1,0\}, & \textit{if } \mu > 1, \end{cases} \tag{3.3}$$

and $E(n;\mu,x)$ *is defined in* (2.4).

Proof. Since $a, b \in A$ and A is an invex set with respect to η, for every $t \in [0,1]$, we have $b + t\eta(a,b) \in A$. Using Lemma 2.1, Hölder's inequality and α-logarithmic preinvexity of $\left|f^{(n)}\right|^q$, we deduce that

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1} \frac{1}{(k+1)!\eta(a,b)}\left[(b+\eta(a,b)-x)^{k+1}-(b-x)^{k+1}\right] f^{(k)}(x) - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x\right| \\
&\leq \frac{[\eta(a,b)]^n}{n!}\left[\int_0^{\frac{x-b}{\eta(a,b)}} t^n\left|f^{(n)}(b+t\eta(a,b))\right|\mathrm{d}t + \int_{\frac{x-b}{\eta(a,b)}}^1 (1-t)^n\left|f^{(n)}(b+t\eta(a,b))\right|\mathrm{d}t\right]
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{[\eta(a,b)]^n}{n!}\Biggl\{\biggl[\int_0^{\frac{x-b}{\eta(a,b)}} t^n \,\mathrm{d}t\biggr]^{1-1/q}\biggl[\int_0^{\frac{x-b}{\eta(a,b)}} t^n\bigl|f^{(n)}(b+t\eta(a,b))\bigr|^q \,\mathrm{d}t\biggr]^{1/q}\\
&\quad+\biggl[\int_{\frac{x-b}{\eta(a,b)}}^1 (1-t)^n \,\mathrm{d}t\biggr]^{1-1/q}\biggl[\int_{\frac{x-b}{\eta(a,b)}}^1 (1-t)^n\bigl|f^{(n)}(b+t\eta(a,b))\bigr|^q \,\mathrm{d}t\biggr]^{1/q}\Biggr\}\\
&\le \frac{[\eta(a,b)]^{(n+1-q)/q}}{n!(n+1)^{1-1/q}}\Biggl\{(x-b)^{(n+1)(q-1)/q}\biggl[\int_0^{\frac{x-b}{\eta(a,b)}} t^n\bigl|f^{(n)}(a)\bigr|^{qt^\alpha}\bigl|f^{(n)}(b)\bigr|^{q(1-t^\alpha)} \,\mathrm{d}t\biggr]^{1/q}\\
&\quad+(b+\eta(a,b)-x)^{(n+1)(q-1)/q}\biggl[\int_{\frac{x-b}{\eta(a,b)}}^1 (1-t)^n\bigl|f^{(n)}(a)\bigr|^{qt^\alpha}\bigl|f^{(n)}(b)\bigr|^{q(1-t^\alpha)} \,\mathrm{d}t\biggr]^{1/q}\Biggr\}\\
&= \frac{[\eta(a,b)]^{(n+1-q)/q}\bigl|f^{(n)}(b)\bigr|}{n!(n+1)^{1-1/q}}\Biggl\{(x-b)^{(n+1)(q-1)/q}\biggl[\int_0^{\frac{x-b}{\eta(a,b)}} t^n\mu^{qt^\alpha} \,\mathrm{d}t\biggr]^{1/q}\\
&\quad+(b+\eta(a,b)-x)^{(n+1)(q-1)/q}\biggl[\int_{\frac{x-b}{\eta(a,b)}}^1 (1-t)^n\mu^{qt^\alpha} \,\mathrm{d}t\biggr]^{1/q}\Biggr\},
\end{aligned}
\tag{3.4}
$$

where $\mu = \frac{|f^{(n)}(a)|}{|f^{(n)}(b)|}$.

By Lemma 2.2 and Lemma 2.4, for $0<\mu\le 1$, we give

$$
\int_0^{\frac{x-b}{\eta(a,b)}} t^n\mu^{qt^\alpha} \,\mathrm{d}t \le \int_0^{\frac{x-b}{\eta(a,b)}} t^n\mu^{q\alpha t} \,\mathrm{d}t = E\biggl(n;\mu^{q\alpha},\frac{x-b}{\eta(a,b)}\biggr)
\tag{3.5}
$$

and

$$
\begin{aligned}
&\int_{\frac{x-b}{\eta(a,b)}}^1 (1-t)^n\mu^{qt^\alpha} \,\mathrm{d}t \le \int_{\frac{x-b}{\eta(a,b)}}^1 (1-t)^n\mu^{q\alpha t} \,\mathrm{d}t\\
&= \int_0^{\frac{b+\eta(a,b)-x}{\eta(a,b)}} t^n\mu^{q\alpha(1-t)} \,\mathrm{d}t = \mu^{q\alpha}E\biggl(n;\mu^{-q\alpha},\frac{b+\eta(a,b)-x}{\eta(a,b)}\biggr);
\end{aligned}
\tag{3.6}
$$

for $\mu>1$, we obtain

$$
\int_0^{\frac{x-b}{\eta(a,b)}} t^n\mu^{qt^\alpha} \,\mathrm{d}t \le \int_0^{\frac{x-b}{\eta(a,b)}} t^n\mu^{q(\alpha t+1-\alpha)} \,\mathrm{d}t = \mu^{q(1-\alpha)}E\biggl(n;\mu^{q\alpha},\frac{x-b}{\eta(a,b)}\biggr)
\tag{3.7}
$$

and

$$
\begin{aligned}
&\int_{\frac{x-b}{\eta(a,b)}}^1 (1-t)^n\mu^{qt^\alpha} \,\mathrm{d}t \le \int_{\frac{x-b}{\eta(a,b)}}^1 (1-t)^n\mu^{q(\alpha t+1-\alpha)} \,\mathrm{d}t\\
&= \int_0^{\frac{b+\eta(a,b)-x}{\eta(a,b)}} t^n\mu^{q(\alpha(1-t)+1-\alpha)} \,\mathrm{d}t = \mu^{q}E\biggl(n;\mu^{-q\alpha},\frac{b+\eta(a,b)-x}{\eta(a,b)}\biggr).
\end{aligned}
\tag{3.8}
$$

Substituting (3.5) to (3.8) into (3.4), we get the inequality (3.1).
Theorem 3.1 is thus proved. □

Corollary 3.1.1. *Under the assumptions of Theorem 3.1,*

(1) *if $q=1$, we know*

$$
\begin{aligned}
&\left|\sum_{k=0}^{n-1}\frac{1}{(k+1)!\eta(a,b)}\bigl[(b+\eta(a,b)-x)^{k+1}-(b-x)^{k+1}\bigr]f^{(k)}(x)-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x\right|\\
&\le \frac{[\eta(a,b)]^n}{n!}\Biggl\{\bigl|f^{(n)}(a)\bigr|^\xi\bigl|f^{(n)}(b)\bigr|^\zeta E\biggl(n;\mu^\alpha,\frac{x-b}{\eta(a,b)}\biggr)
\end{aligned}
$$

$$+\big|f^{(n)}(a)\big|^{\delta}\big|f^{(n)}(b)\big|^{\sigma}E\bigg(n;\mu^{-\alpha},\frac{b+\eta(a,b)-x}{\eta(a,b)}\bigg)\bigg\}; \tag{3.9}$$

(2) *if* $\alpha=1$, *we get*

$$\bigg|\sum_{k=0}^{n-1}\frac{1}{(k+1)!\eta(a,b)}\big[(b+\eta(a,b)-x)^{k+1}-(b-x)^{k+1}\big]f^{(k)}(x)-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,\mathrm{d}x\bigg|$$
$$\le\frac{[\eta(a,b)]^{(n+1-q)/q}}{n!(n+1)^{1-1/q}}\bigg\{(x-b)^{(n+1)(q-1)/q}\big|f^{(n)}(b)\big|\bigg[E\bigg(n;\mu^{q},\frac{x-b}{\eta(a,b)}\bigg)\bigg]^{1/q}$$
$$+(b+\eta(a,b)-x)^{(n+1)(q-1)/q}\big|f^{(n)}(a)\big|\bigg[E\bigg(n;\mu^{-q},\frac{b+\eta(a,b)-x}{\eta(a,b)}\bigg)\bigg]^{1/q}\bigg\}; \tag{3.10}$$

(3) *if* $\alpha=1$ *and* $q=1$, *we have*

$$\bigg|\sum_{k=0}^{n-1}\frac{1}{(k+1)!\eta(a,b)}\big[(b+\eta(a,b)-x)^{k+1}-(b-x)^{k+1}\big]f^{(k)}(x)-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,\mathrm{d}x\bigg|$$
$$\le\frac{[\eta(a,b)]^{n}}{n!}\bigg\{\big|f^{(n)}(b)\big|E\bigg(n;\mu,\frac{x-b}{\eta(a,b)}\bigg)+\big|f^{(n)}(a)\big|E\bigg(n;\mu,\frac{b+\eta(a,b)-x}{\eta(a,b)}\bigg)\bigg\}. \tag{3.11}$$

Theorem 3.2. *For* $n\in\mathbb{N}$ *and* $n\ge1$, *let* $A\subseteq\mathbb{R}$ *be an open invex set with respect to* $\eta:A\times A\to\mathbb{R}$ *and* $a,b\in A$ *with* $\eta(a,b)>0$ *for all* $a\ne b$. *Suppose that* $f:A\to\mathbb{R}$ *is an n-times differentiable function on A and* $f^{(n)}$ *is integrable on the* η*-path* P_{bc} *for* $c=b+\eta(a,b)$. *If* $\big|f^{(n)}\big|^q$ *is* α*-logarithmically preinvex function on A for* $q>1$, *then for* $\alpha\in(0,1]$, $x\in[b,b+\eta(a,b)]$ *and* $0\le r\le nq$, *we get*

$$\bigg|\sum_{k=0}^{n-1}\frac{1}{(k+1)!\eta(a,b)}\big[(b+\eta(a,b)-x)^{k+1}-(b-x)^{k+1}\big]f^{(k)}(x)-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,\mathrm{d}x\bigg|$$
$$\le\frac{[\eta(a,b)]^{(r+1-q)/q}}{n!}\bigg(\frac{q-1}{nq+q-r-1}\bigg)^{1-1/q}$$
$$\times\bigg\{(x-b)^{n+1-(r+1)/q}\big|f^{(n)}(a)\big|^{\xi}\big|f^{(n)}(b)\big|^{\zeta}\bigg[F\bigg(r+1;\mu^{q\alpha},\frac{x-b}{\eta(a,b)}\bigg)\bigg]^{1/q}$$
$$+(b+\eta(a,b)-x)^{n+1-(r+1)/q}\big|f^{(n)}(a)\big|^{\delta}\big|f^{(n)}(b)\big|^{\sigma}\bigg[F\bigg(r+1;\mu^{-q\alpha},\frac{b+\eta(a,b)-x}{\eta(a,b)}\bigg)\bigg]^{1/q}\bigg\}, \tag{3.12}$$

where μ, $\{\xi,\zeta\}$ *and* $\{\delta,\sigma\}$ *are given in Theorem 3.1, and* $F(r;\mu,x)$ *is defined by* (2.5).

Proof. Since $a,b\in A$ and A is an invex set with respect to η, for every $t\in[0,1]$, it is easy to see that $b+t\eta(a,b)\in A$. From Lemma 2.1, Hölder's inequality and α-logarithmic preinvexity of $\big|f^{(n)}\big|^q$, one has that

$$\bigg|\sum_{k=0}^{n-1}\frac{1}{(k+1)!\eta(a,b)}\big[(b+\eta(a,b)-x)^{k+1}-(b-x)^{k+1}\big]f^{(k)}(x)-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,\mathrm{d}x\bigg|$$
$$\le\frac{[\eta(a,b)]^{n}}{n!}\bigg[\int_0^{\frac{x-b}{\eta(a,b)}}t^{n}\big|f^{(n)}(b+t\eta(a,b))\big|\,\mathrm{d}t+\int_{\frac{x-b}{\eta(a,b)}}^{1}(1-t)^{n}\big|f^{(n)}(b+t\eta(a,b))\big|\,\mathrm{d}t\bigg]$$
$$\le\frac{[\eta(a,b)]^{n}}{n!}\bigg\{\bigg[\int_0^{\frac{x-b}{\eta(a,b)}}t^{(nq-r)/(q-1)}\,\mathrm{d}t\bigg]^{1-1/q}\bigg[\int_0^{\frac{x-b}{\eta(a,b)}}t^{r}\big|f^{(n)}(b+t\eta(a,b))\big|^{q}\,\mathrm{d}t\bigg]^{1/q}$$

$$
\begin{aligned}
&+\left[\int_{\frac{x-b}{\eta(a,b)}}^{1}(1-t)^{(nq-r)/(q-1)}\,\mathrm{d}t\right]^{1-1/q}\left[\int_{\frac{x-b}{\eta(a,b)}}^{1}(1-t)^{r}\left|f^{(n)}(b+t\eta(a,b))\right|^{q}\mathrm{d}t\right]^{1/q}\Bigg\}\\
&\le\frac{[\eta(a,b)]^{(r+1-q)/q}}{n!}\left(\frac{q-1}{nq+q-r-1}\right)^{1-1/q}\\
&\quad\times\Bigg\{(x-b)^{n+1-(r+1)/q}\left[\int_{0}^{\frac{x-b}{\eta(a,b)}}t^{r}\left|f^{(n)}(a)\right|^{qt^{\alpha}}\left|f^{(n)}(b)\right|^{q(1-t^{\alpha})}\mathrm{d}t\right]^{1/q}\\
&\quad+(b+\eta(a,b)-x)^{n+1-(r+1)/q}\left[\int_{\frac{x-b}{\eta(a,b)}}^{1}(1-t)^{r}\left|f^{(n)}(a)\right|^{qt^{\alpha}}\left|f^{(n)}(b)\right|^{q(1-t^{\alpha})}\mathrm{d}t\right]^{1/q}\Bigg\}\\
&=\frac{[\eta(a,b)]^{(r+1-q)/q}\left|f^{(n)}(b)\right|}{n!}\left(\frac{q-1}{nq+q-r-1}\right)^{1-1/q}\Bigg\{(x-b)^{n+1-(r+1)/q}\left[\int_{0}^{\frac{x-b}{\eta(a,b)}}t^{r}\mu^{qt^{\alpha}}\mathrm{d}t\right]^{1/q}\\
&\quad+(b+\eta(a,b)-x)^{n+1-(r+1)/q}\left[\int_{\frac{x-b}{\eta(a,b)}}^{1}(1-t)^{r}\mu^{qt^{\alpha}}\mathrm{d}t\right]^{1/q}\Bigg\},
\end{aligned}\tag{3.13}
$$

where $\mu=\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}(b)\right|}$.

By Lemma 2.3 and Lemma 2.4, for $0<\mu\le1$, we show

$$
\int_{0}^{\frac{x-b}{\eta(a,b)}}t^{r}\mu^{qt^{\alpha}}\mathrm{d}t\le\int_{0}^{\frac{x-b}{\eta(a,b)}}t^{r}\mu^{q\alpha t}\mathrm{d}t=F\left(r+1;\mu^{q\alpha},\frac{x-b}{\eta(a,b)}\right)\tag{3.14}
$$

and

$$
\begin{aligned}
&\int_{\frac{x-b}{\eta(a,b)}}^{1}(1-t)^{r}\mu^{qt^{\alpha}}\mathrm{d}t\le\int_{\frac{x-b}{\eta(a,b)}}^{1}(1-t)^{r}\mu^{q\alpha t}\mathrm{d}t\\
&=\int_{0}^{\frac{b+\eta(a,b)-x}{\eta(a,b)}}t^{r}\mu^{q\alpha(1-t)}\mathrm{d}t=\mu^{q\alpha}F\left(r+1;\mu^{-q\alpha},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right);
\end{aligned}\tag{3.15}
$$

for $\mu>1$, we state

$$
\int_{0}^{\frac{x-b}{\eta(a,b)}}t^{r}\mu^{qt^{\alpha}}\mathrm{d}t\le\int_{0}^{\frac{x-b}{\eta(a,b)}}t^{r}\mu^{q(\alpha t+1-\alpha)}\mathrm{d}t=\mu^{q(1-\alpha)}F\left(r+1;\mu^{q\alpha},\frac{x-b}{\eta(a,b)}\right)\tag{3.16}
$$

and

$$
\begin{aligned}
&\int_{\frac{x-b}{\eta(a,b)}}^{1}(1-t)^{r}\mu^{qt^{\alpha}}\mathrm{d}t\le\int_{\frac{x-b}{\eta(a,b)}}^{1}(1-t)^{r}\mu^{q(\alpha t+1-\alpha)}\mathrm{d}t\\
&=\int_{0}^{\frac{b+\eta(a,b)-x}{\eta(a,b)}}t^{r}\mu^{q(\alpha(1-t)+1-\alpha)}\mathrm{d}t=\mu^{q}F\left(r+1;\mu^{-q\alpha},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right).
\end{aligned}\tag{3.17}
$$

Substituting (3.14) to (3.17) into (3.13), we get to the inequality (3.12).
The proof of Theorem 3.2 is established. □

Corollary 3.2.1. *Under the assumptions of Theorem 3.2,*

(1) *if* $\alpha=1$, *we write*

$$
\begin{aligned}
&\left|\sum_{k=0}^{n-1}\frac{1}{(k+1)!\eta(a,b)}\left[(b+\eta(a,b)-x)^{k+1}-(b-x)^{k+1}\right]f^{(k)}(x)-\frac{1}{\eta(a,b)}\int_{b}^{b+\eta(a,b)}f(x)\,\mathrm{d}x\right|\\
&\le\frac{[\eta(a,b)]^{(r+1-q)/q}}{n!}\left(\frac{q-1}{nq+q-r-1}\right)^{1-1/q}
\end{aligned}
$$

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 25, NO.2, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

$$\times\left\{(x-b)^{n+1-(r+1)/q}\big|f^{(n)}(b)\big|\left[F\left(r+1;\mu^{q},\frac{x-b}{\eta(a,b)}\right)\right]^{1/q}\right.$$
$$\left.+(b+\eta(a,b)-x)^{n+1-(r+1)/q}\big|f^{(n)}(a)\big|\left[F\left(r+1;\mu^{-q},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right]^{1/q}\right\}; \tag{3.18}$$

(2) *if $r=nq$, we find*

$$\left|\sum_{k=0}^{n-1}\frac{1}{(k+1)!\eta(a,b)}\big[(b+\eta(a,b)-x)^{k+1}-(b-x)^{k+1}\big]f^{(k)}(x)-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,\mathrm{d}x\right|$$
$$\le\frac{[\eta(a,b)]^{(nq+1-q)/q}}{n!}\left\{(x-b)^{1-1/q}\big|f^{(n)}(a)\big|^{\xi}\big|f^{(n)}(b)\big|^{\zeta}\left[F\left(nq+1;\mu^{q\alpha},\frac{x-b}{\eta(a,b)}\right)\right]^{1/q}\right.$$
$$\left.+(b+\eta(a,b)-x)^{1-1/q}\big|f^{(n)}(a)\big|^{\delta}\big|f^{(n)}(b)\big|^{\sigma}\left[F\left(nq+1;\mu^{-q\alpha},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right]^{1/q}\right\}. \tag{3.19}$$

In particular, if $r=nq$ and $\alpha=1$, one has

$$\left|\sum_{k=0}^{n-1}\frac{1}{(k+1)!\eta(a,b)}\big[(b+\eta(a,b)-x)^{k+1}-(b-x)^{k+1}\big]f^{(k)}(x)-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,\mathrm{d}x\right|$$
$$\le\frac{[\eta(a,b)]^{(nq+1-q)/q}}{n!}\left\{(x-b)^{1-1/q}\big|f^{(n)}(b)\big|\left[F\left(nq+1;\mu^{q},\frac{x-b}{\eta(a,b)}\right)\right]^{1/q}\right.$$
$$\left.+(b+\eta(a,b)-x)^{1-1/q}\big|f^{(n)}(a)\big|\left[F\left(nq+1;\mu^{-q},\frac{b+\eta(a,b)-x}{\eta(a,b)}\right)\right]^{1/q}\right\}; \tag{3.20}$$

(3) *if $r=0$, we observe*

$$\left|\sum_{k=0}^{n-1}\frac{1}{(k+1)!\eta(a,b)}\big[(b+\eta(a,b)-x)^{k+1}-(b-x)^{k+1}\big]f^{(k)}(x)-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,\mathrm{d}x\right|$$
$$\le\begin{cases}\frac{[\eta(a,b)]^{(1-q)/q}}{n!(q\alpha)^{1/q}}\left(\frac{q-1}{nq+q-1}\right)^{1-1/q}\Big\{(x-b)^{n+1-1/q}\big|f^{(n)}(a)\big|^{\xi}\\ \times\big|f^{(n)}(b)\big|^{\zeta-\alpha(x-b)/\eta(a,b)}\left[\frac{|f^{(n)}(a)|^{(x-b)q\alpha/\eta(a,b)}-|f^{(n)}(b)|^{(x-b)q\alpha/\eta(a,b)}}{\ln|f^{(n)}(a)|-\ln|f^{(n)}(b)|}\right]^{1/q}\\ +(b+\eta(a,b)-x)^{n+1-1/q}\big|f^{(n)}(a)\big|^{\delta-\alpha(b+\eta(a,b)-x)/\eta(a,b)}\big|f^{(n)}(b)\big|^{\sigma}\\ \times\left[\frac{|f^{(n)}(b)|^{(b+\eta(a,b)-x)q\alpha/\eta(a,b)}-|f^{(n)}(a)|^{(b+\eta(a,b)-x)q\alpha/\eta(a,b)}}{\ln|f^{(n)}(b)|-\ln|f^{(n)}(a)|}\right]^{1/q}\Big\}, & \text{if } \mu\ne1,\\ \frac{|f^{(n)}(b)|}{n!\eta(a,b)}\left(\frac{q-1}{nq+q-1}\right)^{1-1/q}\big[(x-b)^{n+1}+(b+\eta(a,b)-x)^{n+1}\big], & \text{if } \mu=1.\end{cases} \tag{3.21}$$

Specially, if $r=0$ and $\alpha=1$, one has

$$\left|\sum_{k=0}^{n-1}\frac{1}{(k+1)!\eta(a,b)}\big[(b+\eta(a,b)-x)^{k+1}-(b-x)^{k+1}\big]f^{(k)}(x)-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,\mathrm{d}x\right|$$

$$
\leq \begin{cases} \frac{[\eta(a,b)]^{(1-q)/q}}{n!q^{1/q}}\left(\frac{q-1}{nq+q-1}\right)^{1-1/q}\Big\{(x-b)^{n+1-1/q}\left|f^{(n)}(b)\right|^{(b+\eta(a,b)-x)/\eta(a,b)} \\ \times\left[\frac{\left|f^{(n)}(a)\right|^{(x-b)q/\eta(a,b)}-\left|f^{(n)}(b)\right|^{(x-b)q/\eta(a,b)}}{\ln\left|f^{(n)}(a)\right|-\ln\left|f^{(n)}(b)\right|}\right]^{1/q} \\ +(b+\eta(a,b)-x)^{n+1-1/q}\left|f^{(n)}(a)\right|^{(x-b)/\eta(a,b)} \\ \times\left[\frac{\left|f^{(n)}(b)\right|^{(b+\eta(a,b)-x)q/\eta(a,b)}-\left|f^{(n)}(a)\right|^{(b+\eta(a,b)-x)q/\eta(a,b)}}{\ln\left|f^{(n)}(b)\right|-\ln\left|f^{(n)}(a)\right|}\right]^{1/q}\Big\}, & \text{if } \mu\neq 1, \\ \frac{\left|f^{(n)}(b)\right|}{n!\eta(a,b)}\left(\frac{q-1}{nq+q-1}\right)^{1-1/q}\left[(x-b)^{n+1}+(b+\eta(a,b)-x)^{n+1}\right], & \text{if } \mu=1. \end{cases} \tag{3.22}
$$

Acknowledgements. This work was supported by the Foundation of the Research Program of Science and Technology at Universities of Inner Mongolia Autonomous Region under Grant No. NJZZ16175.

Competing interests. The authors declare that they have no competing interests.

References

[1] T. Antczak, *Mean value in invexity analysis*, Nonlinear Anal. **60** (2005), no. 8, 1471–1484; Available online at http://dx.doi.org/10.1016/j.na.2004.11.005.

[2] A. Barani, A. G. Ghazanfariand, and S. S. Dragomir, *Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex*, J. Inequal. Appl. **2012** (2012):247; Available online at http://dx.doi.org/10.1186/1029-242X-2012-247.

[3] R. F. Bai, F. Qi, B. Y. Xi, *Hermite-Hadamard type inequalities for the m- and (α, m)-logarithmically convex functions*, Filomat **27:1** (2013), 1–7; Available online at http://dx.doi.org/10.2298/FIL1301001B.

[4] A. Çelik, B. Dal, M. Kisa, *On integral inequalities of hermite-hadamard type for logarithmically H-Preinvex functions via fractional integrals*, International Journal of Pure and Applied Mathematics, **16:1** (2016), 279–289.

[5] S. S. Dragomir and R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett. **11** (1998), no. 5, 91–95; Available online at http://dx.doi.org/10.1016/S0893-9659(98)00086-X.

[6] U. S. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula*, Appl. Math. Comput. **147** (2004), no. 1, 137–146.

[7] M. A. Latif and S. S. Dragomir, *On Hermite-Hadamard type integral inequalities for n-times differentiable log-preinvex functions*, Miskolc Mathematical Notes, **16**, no. 1 (2016), 339–342.

[8] S. R. Mohan and S. K Neogy, *On invex sets and preinvex function*, J. Math. Anal. Appl. **189** (1995), no. 3, 901–908; Available online at http://dx.doi.org/10.1006/jmaa.1995.1057.

[9] M. A. Noor, *Hermite-Hadamard integral inequalities for log-preinvex functions*, J. Math. Anal. Approx. Theory **2** (2007),126–131.

[10] M. A. Noor, *Variational-like inequalities*, Optimization **30** (1994),323–330.

[11] M. Z. Sarikaya, H. Bozkurt, and N. Alp, *On Haermite-Hadamard type integral inequalities for preinvex and log-preinvex functions*, Contemporary Analysis and Applied Mathematics **1** (2013), no. 2, 237–252.

[12] J. Wang, J. Deng and M. Fečkan, *Haermite-Hadamard type inequalities for r-convex functions via Riemann-Liouville fractional integrals*, Ukrainian Math. J. **65** (2013), 193–211.

[13] T. Weir and B. Mond, *Preinvex functions in multiple objective optimization*, Journal of Mathematical Analysis and Applications **136** (1998), 29–38.

[14] S. H. Wang and X. M. Liu, *New Hermite-Hadamard type integral inequalities for n-times differentiable and s-logarithmically preinvex functions*, Abstact and Appllied Analysis, **2014** (2014), Article ID 725987, 11 pages; Available online at http://dx.doi.org/10.1155/2014/725987.

[15] S. H. Wang and F. Qi, *Hermite-Hadamard type integral inequalities for n-times differentiable and preinvex functions*, Journal of Inequalities and Applications **49**(2014); Available online at http://dx.doi.org/10.1186/1029-242X-2014-49.

[16] Y. Wang, S.-H. Wang, and F. Qi, *Simpson type integral inequalities in which the power of the absolute value of the first derivative of the integrand is s-preinvex*, Facta Univ. Ser. Math. Inform. **28** (2013), no. 2, 151–159.

[17] Y. Wang, M. M. Zheng, and F. Qi, *Integral inequalities of Hermite-Hadamard type for functions whose derivatives are α-preinvex*, Journal of Inequalities and Applications **2014:97** (2014); Available online at http://dx.doi.org/10.1186/1029-242X-2014-97.

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 25, NO.2, 2018, COPYRIGHT 2018 EUDOXUS PRESS, LLC

Existence of solutions for boundary value problems of fractional differential equation in Banach spaces*

Yabing Gao, Pengyu Chen†

Department of Mathematics, Northwest Normal University,
Lanzhou 730070, People's Republic of China

Abstract

In this paper, with the help of a new estimation technique for the measure of noncompactness, under more general conditions of growth and noncompactness measure, combining with the Sadovskii's fixed point theorem and Leray-Schauder type fixed point theorem of condensing mapping, we obtain the existence of solutions for the following boundary value problems of fractional differential equation in Banach spaces

$$\begin{cases} -{}^{C}D_{0^+}^{\beta}u(t) = f(t, u(t)), & t \in J, \\ u(0) = u(1) = \theta, \end{cases}$$

where $1 < \beta \leq 2$ is a real number, $J = [0, 1]$, ${}^{C}D_{0^+}^{\beta}$ is the Caputo fractional derivative of order β, $f : J \times E \to E$ is continuous, E is a Banach spaces, θ is the zero element of E.

Keywords: Banach spaces; fractional differential equation; measure of noncompactness; condensing mapping

Mathematics Subject Classification(2010): 34A08; 34B15; 47H08; 49K27

1 Introduction

Differential equations of fractional order occur more frequently in difference research areas and engineering, such as physics, chemistry. For the details, one can

*Research supported by NNSFs of China (11501455, 11661071), Key Project of Gansu Provincial National Science Foundation (1606RJZA015) and Project of NWNU-LKQN-14-6.

†Corresponding author. Tel.: +86 0931 7971111. E-mail addresses: gyabing1610@163.com, chpengyu123@163.com (P. Chen).

see the monographs [1-5]. Recently, many people pay attention to the boundary value problem for fractional order ordinary differential equation involving Caputo's derivative, see [8-29] and the references therein. However, as far as we know, there are few papers studied the existence of solution to boundary value problems for fractional differential equation involving Caputo's derivative in abstract space.

In 1988, Guo and Lakshmikantham [6] firstly studied the existence of solutions for the following boundary value problem of nonlinear second order differential equation in Banach spaces

$$\begin{cases} -u''(t) = f(t, u(t)), \quad t \in [0, 1], \\ u(0) = u(1) = \theta, \end{cases} \tag{1}$$

where $f : J \times E \to E$ is continuous, E is a Banach spaces, θ is the zero element of E.

In 2008, Li and Guo [7] obtained an existence result of positive solution of BVP (1) by employing a new estimate of noncompactness measure and the fixed point index theory of condensing mapping.

Motivated by the above mentioned works, in this paper, with the help of a new estimation technique of noncompactness measure, under more general conditions of growth and noncompactness measure, combining with the Sadovskii's fixed point theorem and Leray-Schauder type fixed point theorem of condensing mapping, we consider the existence of solutions for the following boundary value problem (BVP) of fractional differential equation in Banach space E

$$\begin{cases} -{}^{C}D_{0+}^{\beta} u(t) = f(t, u(t)), \quad t \in J, \\ u(0) = u(1) = \theta, \end{cases} \tag{2}$$

where $1 < \beta \leq 2$ is real number, $J = [0, 1]$, ${}^{C}D_{0+}^{\beta}$ is the Caputo fractional derivative of order β, $f : J \times E \to E$ is continuous, θ is the zero element of Banach space E. It is easy to see that if $\beta = 2$, then BVP (2) will degrade into the problem (1), which was studied in [6] and [7].

2 Preliminaries

Throughout this paper, we denote by $C(J, E)$ the Banach space of all continuous E-value functions on interval J with the supnorm $\|u\|_C = \sup\limits_{t \in J} \|u(t)\|$.

Definition 2.1 *The fractional integral of order $q > 0$ with the lower limit zero for a function g is defined as*

$$I_{0+}^{q} g(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s)ds, \quad t > 0,$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 *The Caputo fractional derivative of order $q > 0$ with the lower limit zero for a function g is defined as*

$${}^{C}D_{0+}^{q} g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)ds, \quad t > 0, \quad 0 \leq n-1 < q < n,$$

where the function $g(t)$ has absolutely continuous derivatives up to order $n-1$.

If g is an abstract function with values in E, then the integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner's sense.

Next, we recall some properties of the measure of noncompactness that will be used in the proof of our main results. In order to no confusion may occur, we denote by $\alpha(\cdot)$ the Kuratowski measure of noncompactness on both the bounded sets of E and $C(J, E)$. For the details of the definition and properties of the measure of noncompactness, we refer to the monographs [30] and [31]. For any $D \subset C(J, E)$ and $t \in J$, set $D(t) = \{u(t) \mid u \in D\} \subset E$. If $D \subset C(J, E)$ is bounded, then $D(t)$ is bounded in E and $\alpha(D(t)) \leq \alpha(D)$.

Lemma 2.3 ([36]). *Let E be a Banach space, and let $D \subset C(J, E)$ be bounded and equicontinuous. Then $\alpha(D(t))$ is continuous on J, and*

$$\max_{t \in J} \alpha(D(t)) = \alpha(D) = \alpha(D(J)).$$

Lemma 2.4 ([35]). *Let E be a Banach space, and let $D = \{u_n\} \subset C(J, E)$ be a bounded and countable set. Then $\alpha(D(t))$ is Lebesgue integral on J, and*

$$\alpha\Big(\Big\{ \int_J u_n(t)dt \mid n \in \mathbb{N} \Big\}\Big) \leq 2 \int_J \alpha(D(t))dt.$$

Lemma 2.5 ([32,33]). *Let E be a Banach space, and let $D \subset E$ be bounded. Then there exists a countable set $D_0 \subset D$, such that $\alpha(D) \leq 2\alpha(D_0)$.*

3

Lemma 2.6 ([34]). *Let $u \in C(J, E)$, $\psi \in L(J, \mathbb{R}^+)$. Then*

$$\int_J u(s)\psi(s)ds \in \int_J \psi(s)ds\overline{co}\, u(J),$$

where $u(J) = \{u(t) \mid t \in J\}$.

Lemma 2.7 ([31]). *Let E be a Banach space. Assume that $D \subset E$ is a bounded closed and convex set on E, $Q : D \to D$ is condensing. Then Q has at least one fixed point in D.*

Lemma 2.8 ([36]). *Let E be a Banach space. Assume that $Q : D \to D$ is condensing, if $\{x \mid x = \lambda Qx, 0 < \lambda < 1\}$ is a bounded set. Then Q has at least one fixed point in D.*

With the help of the existence of solutions for fractional differential equation in real number space, we firstly get the corresponding existence and uniqueness of the fractional linear differential equation in Banach spaces.

Lemma 2.9. *Assume that $1 < \beta \leq 2$ and $J = [0, 1]$. Then for any $h \in C(J, E)$, the boundary value problems of fractional linear differential equation in Banach spaces (LBVP)*

$$\begin{cases} -{}^C D_{0+}^{\beta} u(t) = h(t), \quad t \in J, \\ u(0) = u(1) = \theta, \end{cases} \tag{3}$$

has a unique solution

$$u(t) = \int_0^1 G(t, s)h(s)ds := Th(t), \tag{4}$$

where

$$G(t, s) = \frac{1}{\Gamma(\beta)} \begin{cases} t(1-s)^{\beta-1} - (t-s)^{\beta-1}, & 0 \leq s \leq t \leq 1, \\ t(1-s)^{\beta-1}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{5}$$

Proof. From the expression (4), we first easily know that $u(t)$ is a solution of LBVP (3). Next, we prove $u(t)$ is uniqueness solution.

Suppose $u_1(t), u_2(t) \in C(J, E)$ is solutions of LBVP (3). For any $\varphi \in E^*$, where E^* is conjugate space of E. Let $r(t) = \varphi(u_1(t) - u_2(t))$. Then $r(t)$ is a solution

4

about pure variable t of fractional linear differential equation

$$\begin{cases} -{}^C D_{0+}^{\beta} r(t) = 0, \quad 1 < \beta \le 2, \quad t \in J, \\ r(0) = r(1) = 0. \end{cases}$$

Evidently, $r(t) \equiv 0$, according to the arbitrary of $\varphi \in E^*$, we know $u_1(t) - u_2(t) \equiv \theta$, also, $u_1(t) \equiv u_2(t)$ on J. Thus, $u(t)$ expressed by (4) is a unique solution of LBVP (3). This completes the proof of Lemma 2.9. $\square$

Lemma 2.10. *The integral operator $T : C(J,E) \to C(J,E)$ defined by (4) satisfies the following inequality of norm*

$$\|T\| \le \frac{1}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big).$$

Proof. For any $h \in C(J,E)$, by the definition of T, we have

$$\|Th(t)\| \le \|\int_0^1 G(t,s)h(s)ds\|,$$

replace above $G(t,s)$ with the expression of $G(t,s)$ defined by (5), we obtain

$$\|Th(t)\| \le \Big(\frac{t}{\Gamma(\beta+1)} - \frac{t^{\beta}}{\Gamma(\beta+1)}\Big)\|h\|_C,$$

and since, we easily know

$$\frac{1}{\Gamma(\beta+1)}\Big(t - t^{\beta}\Big)\|h\|_C \le \frac{1}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)\|h\|_C$$

Hence

$$\|Th\| \le \frac{1}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)\|h\|_C,$$

namely,

$$\|T\| \le \frac{1}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big).$$

This proof is completed. $\square$

Let us introduce the following assumptions which are used hereafter.

(F1) There exist constants $c_0, c_1 > 0$ satisfying $c_1 < \Big(\frac{1}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)\Big)^{-1}$, such that for any $t \in J$ and $x \in E$, $\|f(t,x)\| \le c_1\|x\| + c_0$;

(F2) There exist constant L satisfying $0 < L < \frac{\left(\frac{1}{\Gamma(\beta+1)}\left(\beta^{\frac{1}{1-\beta}}-\beta^{\frac{\beta}{1-\beta}}\right)\right)^{-1}}{4}$, such that for any $t \in J$ and bounded set $D \subset E$, $\alpha(f(t,D)) \leq L\alpha(D)$;

(F3) There exist constant L satisfying $0 < L < \frac{\left(\frac{1}{\Gamma(\beta+1)}\left(\beta^{\frac{1}{1-\beta}}-\beta^{\frac{\beta}{1-\beta}}\right)\right)^{-1}}{2}$, such that for every bounded set $D \subset E$, $\alpha(f(J,D)) < L\alpha(D)$;

(F4) There exist constant L satisfying $0 < L < \left(\frac{1}{\Gamma(\beta+1)}\left(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\right)\right)^{-1}$, such that for every bounded set $D \subset E$, $\alpha(f(J,D)) < L\alpha(D)$.

Defined by the following integral operator $Q : C(J,E) \to C(J,E)$

$$Qu(t) = \int_0^1 G(t,s)f(s,u(s))ds. \tag{6}$$

To be convenient for next prove procedure, we firstly introduce the following two Lemmas.

Lemma 2.11. *Assume that (F1) and (F2) hold. Then $Q : C(J,E) \to C(J,E)$ is a condensing mapping.*

Proof. By (6) we easily know that Q is a continuous operator. Next, we show that Q is a condensing mapping.

When (F1) hold, we see easily that Q maps bounded set in $C(J,E)$ into bounded and equicontinuous set in $C(J,E)$. For every non-relatively compact bounded set $D \subset C(J,E)$, we will prove in the following that $\alpha(Q(D)) < \alpha(D)$.

For every bounded set $D \subset C(J,E)$, by the Lemma 2.5, there exists a countable subset $D_1 = \{u_n\} \subset D$, such that $\alpha(Q(D)) \leq 2\alpha(Q(D_1))$. Since $\alpha(Q(D_1))$ is a bounded and equicontinuous set, by the Lemma 2.3, we have $\alpha(Q(D_1)) = \max_{t\in J} \alpha(Q(D_1(t)))$. By the Lemma 2.4, for any $t \in J$, we can obtain

$$\begin{aligned} \alpha(Q(D_1(t))) &= \alpha\Big(\Big\{ \int_0^1 G(t,s)f(s,u_n(s))ds \mid n \in \mathbb{N}\Big\}\Big) \\ &\leq 2\int_0^1 \alpha\Big(\Big\{G(t,s)f(s,u_n(s))ds \mid n \in \mathbb{N}\Big\}\Big)ds \\ &\leq 2\int_0^1 G(t,s)\alpha\Big(f(s,D_1(s))\Big)ds. \end{aligned}$$

6

According to properties of the measure of noncompactness and condition (F2), we have

$$\alpha(f(s, D_1(s)) \leq L\alpha(D_1(s)) \leq L\alpha(D_1) \leq L\alpha(D).$$

Further

$$\alpha(Q(D_1(t))) \leq 2L\alpha(D) \int_0^1 G(t,s)ds.$$

From this inequality, we have

$$\begin{aligned}
\alpha(Q(D_1)) &= \max_{t\in J} \alpha(Q(D_1(t))) \\
&\leq 2L\alpha(D) \max_{t\in J} \int_0^1 G(t,s)ds \\
&\leq \frac{2L}{\Gamma(\beta+1)} \Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)\alpha(D).
\end{aligned}$$

From the condition (F2), we get that

$$\alpha(Q(D)) \leq 2\alpha(Q(D_1)) \leq \frac{4L}{\Gamma(\beta+1)} \Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)\alpha(D) < \alpha(D).$$

Then, $Q : C(J,E) \to C(J,E)$ is a condensing mapping. This completes the proof of Lemma 2.11. $\square$

Lemma 2.12. *Assume that (F1) and (F3) hold. Then $Q : C(J,E) \to C(J,E)$ is a condensing mapping.*

Proof. By Q defined by (6) and condition (F1), we easily know that Q maps bounded set in $Q : C(J,E)$ into bounded and equicontinuous set in $Q : C(J,E)$. For every non-relatively compact bounded set $D \subset C(J,E)$, we hope prove that $\alpha(Q(D)) < \alpha(D)$.

For every bounded set $D \subset C(J,E)$, since $Q(D)$ is bounded and equicontinuous set, by the Lemma 2.3, we know $\alpha(Q(D)) = \max_{t\in J} \alpha(Q(D(t)))$.

Denote $B = D(J)$, then $B \subset E$ is bounded and $\alpha(B) \leq 2\alpha(D)$. For any $t \in J, u \in D$, by the Lemma 2.6, we obtain

$$Qu(t) = \int_0^1 G(t,s)f(s,u(s)) \in \int_0^1 G(t,s)ds\overline{\text{co}}\, f(J \times B),$$

$$Q(D)(t) \subset \int_0^1 G(t,s)ds\overline{\text{co}}\, f(J \times B).$$

7

Combining with properties of the measure of noncompactness and condition (F3), we have

$$\begin{aligned}\alpha(Q(D)(t)) &\leq \int_0^1 G(t,s)ds\alpha\Big(\overline{\mathrm{co}}\, f(J\times B)\Big)\\ &= \int_0^1 G(t,s)ds\alpha\Big(f(J\times B)\Big)\\ &\leq L\int_0^1 G(t,s)ds\alpha(B)\\ &\leq 2L\int_0^1 G(t,s)ds\alpha(D)\\ &\leq \frac{2L}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}}-\beta^{\frac{\beta}{1-\beta}}\Big)\alpha(D).\end{aligned}$$

From condition (F3), make the maximum value on both sides of this inequality, we know

$$\alpha(Q(D)) = \max_{t\in J}\alpha(Q(D(t))) \leq \frac{2L}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}}-\beta^{\frac{\beta}{1-\beta}}\Big)\alpha(D) < \alpha(D).$$

Therefore,$Q : C(J,E)\to C(J,E)$ is a condensing mapping. This proof is completed. □

By the definition of Q, we know that the solution of BVP (2) is equivalent to the fixed point of Q. We will find the fixed point of Q by using the Leray-Schauder fixed point theorem and the Sadovskii's fixed point theorem about condensing mapping.

3 Main results

Theorem 3.1. *Assume that $f : J\times E\to E$ is continuous and satisfies the conditions (F1) and (F2). Then BVP (2) has at least one solution.*

Proof. We firstly prove that $\Omega = \{u\in C(J,E)\mid u=\lambda Qu, 0<\lambda<1\}$ is a bounded set in $C(J,E)$. In fact, for any $u\in\Omega$ satisfies

$$u(t) = \lambda Qu(t) = \lambda\int_0^1 G(t,s)f(s,u(s))ds,\ 0<\lambda<1.$$

Combining with condition (F1), we have

$$\begin{aligned}\|u(t)\| &= \lambda\| \int_0^1 G(t,s)f(s,u(s))ds\| \\ &\le \int_0^1 G(t,s)(c_0 + c_1\|u(s)\|)ds \\ &\le c_0 \int_0^1 G(t,s)ds + c_1 T_0 \|u(t)\|,\end{aligned}$$

where $T\|u(t)\| = \int_0^1 G(t,s)\|u(s)\|ds$ is defined by (3). Since $\|T\| \le \frac{1}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)$, we can obtain $\|c_1 T\| < \frac{c_1}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big) < 1$, and by the famous perturbation theorem of unit operator, we easily know that $I - c_1T$ exists a bounded inverse operator $(I - c_1T)^{-1}$, and

$$(I - c_1T)^{-1} = \sum_{n=0}^{\infty} (c_1T)^n$$

is positive. Hence

$$\begin{aligned}\|u\|_C &\le \frac{c_0}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)(I - c_1T)^{-1}(1) \\ &= \frac{c_0}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)\sum_{n=0}^{\infty}(c_1T)^n(1).\end{aligned}$$

Since $c_1 < \Big(\frac{1}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)\Big)^{-1}$, then there exist $\varepsilon > 0$ such that $c_1 + \varepsilon < \Big(\frac{1}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)\Big)^{-1}$. Namely

$$\|c_1^n T^n\| \le c_1^n \|T\|^n \le \Big(\frac{c_1}{c_1+\varepsilon}\Big)^n.$$

By the fact that $\sum_{n=0}^{\infty}\Big(\frac{c_1}{c_1+\varepsilon}\Big)^n$ is convergence, we know that $\sum_{n=0}^{\infty} c_1^n\|T^n\|$ is convergence. Denote $M_0 = \sum_{n=0}^{\infty} c_1^n\|T^n\| < +\infty$. Then $\|u\|_C \le \frac{c_0}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)M_0$. Therefore, combined with Lemma 2.8 and Lemma 2.11, we get that the operator Q has a fixed point, which is the solution of BVP (2). This proof is completed. □

Similar with the Proof of Theorem 3.1, we can obtain the following result.

Theorem 3.2. *Assume that $f : J \times E \to E$ is continuous and satisfies the conditions (F1) and (F3). Then BVP (2) has at least one solution.*

Theorem 3.3. *Assume that $f : J \times E \to E$ is continuous and satisfies the conditions (F1) and (F4). Then BVP (2) has at least one solution.*

Proof. Firstly, when (F1) hold, we easily know Q maps bounded set in $C(J, E)$ into bounded and equicontinuous set in $C(J, E)$.

Secondly, let $\Omega_R = \overline{B_C}(\theta, R)$ be a closed ball in $C(J, E)$, where

$$R \geq \frac{c_0\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)}{\Gamma(\beta+1) - c_1\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)}.$$

Clearly, Ω_R is a bounded set in $C(J, E)$. For any $t \in J, u \in \Omega_R$, according to the condition (F1), we have

$$\begin{aligned} \|(Qu)(t)\| &\leq \int_0^1 G(t,s)\|f(s,u(s))\|ds \\ &\leq \int_0^1 G(t,s)(c_0 + c_1\|u(s)\|)ds \\ &\leq \frac{c_0}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big) + \frac{c_1}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}} - \beta^{\frac{\beta}{1-\beta}}\Big)R \\ &\leq R, \end{aligned}$$

then $Q(\Omega_R) \subset \Omega_R$ and it is continuous operator. Let $\Omega_0 = \overline{\text{co}}\, Q(\Omega_R)$, obvious Ω_0 is a bounded closed and convex set in $C(J, E)$ and it is equicontinuous. Since $\Omega_0 \subset \Omega_R$, then we have $Q(\Omega_0) \subset Q(\Omega_R) \subset \Omega_0$ and $Q : \Omega_0 \to \Omega_0$ is a continuous operator.

Finally, we will prove $Q : \Omega_0 \to \Omega_0$ is a condensing operator.

For any non-relatively compact bounded B in Ω_0, we can obtain that both B and $Q(B)$ is a bounded set and equicontinuous set. By the Lemma 2.3, we have

$$\alpha(B) = \max_{t\in J} \alpha(B(t)) = \alpha(B(J)),\ \alpha(Q(B)) = \max_{t\in J} \alpha(Q(B(t))).$$

By the proof of the Lemma 2.12 and the Lemma 2.6, for any $t \in J$, $u \in B$, we have

$$Q(B)(t) \subset \int_0^1 G(t,s)ds\overline{\text{co}}\, f(J \times B(J)).$$

From the properties of the measure of noncompactness and condition (F4), we easily obtain

$$\alpha(Q(B)(t)) \leq \int_0^1 G(t,s)ds\alpha(\overline{\text{co}}\, f(J \times B(J)))$$

10

$$
\begin{aligned}
&\leq L\int_0^1 G(t,s)ds\alpha(B(J))\\
&= L\int_0^1 G(t,s)ds\alpha(B).
\end{aligned}
$$

Combining with condition (F4), we have

$$
\begin{aligned}
\alpha(Q(B)(t)) &= \max_{t\in J}\alpha(Q(B(t)))\\
&\leq L\alpha(B)\max_{t\in J}\int_0^1 G(t,s)ds\\
&\leq \frac{L}{\Gamma(\beta+1)}\Big(\beta^{\frac{1}{1-\beta}}-\beta^{\frac{\beta}{1-\beta}}\Big)\alpha(B)\\
&< \alpha(B).
\end{aligned}
$$

Therefore, $Q:\Omega_0\to\Omega_0$ is a condensing operator. By the Lemma 2.7, we obtain that the Q has at least one fixed point, which is the solution of BVP (2). This completes the proof of Theorem 3.3. □

References

[1] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Differential Equations, Wiley, New York, 1993.

[2] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[3] R. Hilfer, Applications of Fractional Calcus in Physics, World Scientific, Singapore, 2000.

[4] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, in: North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V., Amsterdam, 2006.

[5] S. Heikkila, V. Lakshmikantham, Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations, MaRcel Dekker, Inc.,New York, 1994.

[6] D. Guo, V. Lakshmikantham, Multiple solutions of two-point boundary value problems of ordinary differential equations in Banach spaces, J. Math. Anal. Appl. 129 (1988) 211-222.

[7] Y. Li, C. Guo, Positive solutions to nonlinear second-order boundary value problems in order Banach spaces, J. Lanzhou Univ. Nat. Sci. 44(6) (2008) 120-123. (in Chinese).

[8] Z. Bai, H. L*ü*, Positive solutions for boundary value problem of nonlinear fractional differential equations, J. Math. Anal. Appl. 311 (2005) 495-505.

[9] S. Zhang, Existence of solution for a boundary value problem of fractional order, Acta Math. Sci. Ser. B Engl. Ed. 26 (2006) 220-228.

[10] V. Lakshmikantham, A.S. Vatsala, Basic theory of fractional differential equations, Nonlinear Anal. 69 (2008) 2677-2682.

[11] V. Lakshmikanthama, S. Leela, A Krasnoselskii-Krein-type uniqueness result for fractional differential equations, Nonlinear Anal. 71 (2009) 3421-3424.

[12] X. Su, L. Liu, Existence of solution for boundary value problem of nonlinear fractional differential equation, Appl. Math. J. Chinese. Univ. Ser. B 22 (2007) 291-298.

[13] X. Su, S. Zhang, Solutions to boundary value problems for nonlinear differential equations of fractional order, Electron, J. Differerential Equations, 2009, No. 26. 15pp.

[14] M. Benchohra, S. Hamani, S.K. Ntouyas, Boundary value problems for differential equations with fractional order and nonlocal conditions, Nonlinear Anal. 71 (2009) 2391-2396.

[15] B. Ahmad, S.K. Ntouyas, A. Assolami, Caputo type fractional differential equations with nonlocal Riemann-Liouville integral boundary conditions, J. Appl. Math. Comput. 41 (2013) 339-350.

[16] J. Tariboon, S.K. Ntouyas, W. Sudsutad, Fractional integral problems for fractional differential equations via Caputo derivative, Adv. Difference Equ. 2014, 181, 17pp.

[17] J. Tariboon, S.K. Ntouyas, A. Singubol, Boundary value problems for fractional differential equations with fractional multiterm integral conditions, J. Appl. Math. 2014, Art.ID 806156, 10pp.

[18] B. Ahmad, S.K. Ntouyas, Existence results for a coupled system of Caputo type sequential fractional differential equations with nonlocal integral boundary conditions, Appl. Math. and Comput. 266 (2015) 615-622.

[19] A. Guezane-Lakoud, R. Khaldi, Solvability of a fractional boundary value problem with fractional integral condition, Nonlinear Anal. 75 (2012) 2692-2700.

[20] X. Yang, Z. Wei, W. Dong, Existence of positive solutions for the boundary value problem of nonlinear fractional differential equations, Commun. Nonlinear Sci. Numer. Simul. 17 (2012) 85-92.

[21] Z. Wei, C. Pang, Y. Ding, Positive solutions of singular Caputo fractional differential equations with integral boundary conditions, Commun. Nonlinear Sci. Numer. Simul. 17 (2012) 3148-3160.

[22] Z. Bai, Eigenvalue intervals for a class of fractional boundary value problem, Comput. Math. Appl.64 (2012) 3253-3257.

[23] G. Wang, Monotone iterative technique for boundary value problems of a nonlinear fractional differential equation with deviating arguments, J. Comput. Appl. Math. 236 (2012) 2425-2430.

[24] R. P. Agarwal, B. Ahmad, Existence theorey for anti-periodic boundary value problems of fractional differential equations and inclusions, Comput. Math. Appl. 62 (2011) 1200-1214.

[25] J. Yue, J. Sun, S. Zhang, Existence of positive solution for BVP of nonlinear fractional differential equation, Discrete Dyn. Nat. Soc. 2015, Art.ID 736108, 6pp.

[26] B. Ahmad, S. Sivasundaram, On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order, Appl. Math. Comput. 217 (2010) 480-487.

[27] Z.W. Lv, J. Liang, T.J. Xiao, Solutions to the Cauchy problem for differential equations in Banach spaces with fractional order, Comput. Math. Appl. 62 (2011) 1303-1311.

[28] V. Lakshmikantham1, J.V. Devi, Theory of fractional differential equations in a Banach space, Eur. J. Pure Appl. Math. 1 (2008) 38-45.

[29] R.P. Agarwal, V. Lakshmikantham, J.J. Nieto, On the concept of solutions for fractional differential equations with uncertainly, Nonlinear Anal. 72 (2010) 2859-2862.

[30] J. Banas, K. Goebel, Measure of Noncompactness in Banach Spaces, Lect. Notes Pure Appl. Math., vol. 60, Marcel Pekker, New York, 1980.

[31] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, New York, 1985.

[32] Y. Li, Existence of solutions of initial value problems for abstract semilinear evolution equations, Acta Math. Sin. 48 (2005) 1089-1094 (in Chinese).

[33] P. Chen, Y. Li, Monotone iterative technique for a class of semilinear evolution equations with nonlocal conditions, Results Math. 63 (2013) 731-744.

[34] Y. Li, X. Wang, Positive periodic solutions for ordinary differential equations in ordered Banach spaces, Xibei Shifan Daxue Xuebao Ziran Kexue Ban 38(1) (2002) 1-5. (in Chinese).

[35] H.P. Heinz, On the behaviour of measure of noncompactness with respect to differentiation and integration of vector-valued functions, Nonlinear Anal. 7 (1983) 1351-1371.

[36] D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, New York, 1988.

On the Higher Order Difference equation $x_{n+1} = Ax_n + Bx_{n-l} + Cx_{n-k} + \frac{\gamma x_{n-k}}{Dx_{n-s} + Ex_{n-t}}$

M. M. El-Dessoky[1,2] and M. A. El-Moneam[3]
[1]King Abdulaziz University, Faculty of Science, Mathematics Department,
P. O. Box 80203, Jeddah 21589, Saudi Arabia.
[2]Department of Mathematics, Faculty of Science, Mansoura University,
Mansoura 35516, Egypt.
[3]Department of Mathematics, Faculty of Science, Jazan University, Saudi Arabia.
E-mail: dessokym@mans.edu.eg; mabdelmeneam2014@yahoo.com

ABSTRACT

The main objective of this paper is to study the global stability of the positive solutions, the boundedness and the periodic character of the difference equation

$$x_{n+1} = Ax_n + Bx_{n-l} + Cx_{n-k} + \frac{\gamma x_{n-k}}{Dx_{n-s} + Ex_{n-t}}, \qquad n = 0,\ 1,\ ...,$$

where the parameters $A,\ B,\ C,\ D,\ E,\ \gamma \in (0,\ \infty)$ and the initial conditions $x_{-\sigma}, x_{-\sigma+1}, ..., x_{-1},\ x_0$ are positive real numbers where $\sigma = max\{l,\ k,\ s,\ t\}$. Numerical examples will be given to explicate our results.

Keywords: Difference equations, Stability, Global stability, Periodic solutions, Boundedness.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

Difference equations appeared much earlier than differential equations. But it is only recently that difference equations gained the place they deserve. There is no doubt the interest in difference calculus is related to computers which let effectively apply approximate methods to solve nonlinear difference equations and systems of difference equations [1 - 6]. Particularly, the boundedness, persistence, local asymptotic stability, global character, and the existence of positive periodic solutions can be discussed in many papers [7 - 36].

In [7] El-Dessoky investigated the global stability character and the periodicity of solutions of the recursive sequence

$$x_{n+1} = \frac{ax_{n-l} + bx_{n-k}}{c + dx_{n-l}x_{n-k}}.$$

Xiu-Mei Jia et al. [8] studied the dynamical behavior of rational difference equation,

$$y_{n+1} = \frac{r + py_n + y_{n-k}}{qy_n + y_{n-k}}.$$

Kosmala et al. [9] obtained the periodic and the global stability of solutions of rational difference equation

$$y_{n+1} = \frac{p + y_{n-1}}{qy_n + y_{n-1}}.$$

In [10] Abo-Zeid studied the global asymptotic stability of all solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-2}}{B + Cx_n x_{n-1} x_{n-2}}.$$

In [11] Devault et al. investigated the dynamics of the difference equation

$$x_{n+1} = p + \frac{x_{n-k}}{x_n}.$$

Elsayed et al. [12] investigated the global stability and the periodicity of solutions of the difference equation

$$x_{n+1} = ax_n + \frac{b+cx_{n-1}}{d+ex_{n-1}}.$$

Zayed et al. [13] obtained some qualitative behavior of the solutions of the difference equation,

$$x_{n+1} = \gamma x_{n-k} + \frac{ax_n+bx_{n-k}}{cx_n-dx_{n-k}}.$$

In [14] El-Dessoky obtained the global stability character, the boundedness and the periodicity of the positive solutions of the difference equation

$$x_{n+1} = ax_n + bx_{n-k} + cx_{n-l} - \frac{dx_{n-s}}{ex_{n-s}-\alpha x_{n-t}}.$$

Our goal is to investigate some qualitative behavior of the positive solutions of the difference equation

$$x_{n+1} = Ax_n + Bx_{n-l} + Cx_{n-k} + \frac{\gamma x_{n-k}}{Dx_{n-s}+Ex_{n-t}}, \qquad n = 0,\ 1,\ ..., \tag{1}$$

where the initial conditions $x_{-\sigma}, x_{-\sigma+1}, ..., x_{-1}, x_0$ are positive real numbers where $\sigma = max\{l,\ k,\ s,\ t\}$ and the coefficients $A,\ B,\ C,\ D,\ E,\ \gamma \in (0,\ \infty)$.

Let J be some interval of real numbers and let

$$G : J^{\sigma+1} \to J,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-\sigma}, x_{-\sigma+1}, ..., x_0 \in J$, the difference equation

$$x_{n+1} = G(x_n, x_{n-1}, ..., x_{n-\sigma}), \quad n = 0, 1, ..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-\sigma}^{\infty}$.

DEFINITION 1.1. *The linearized equation of Eq. (2) about the equilibrium $\overline{x}$ is the linear difference equation*

$$y_{n+1} = q_1 y_n + q_2 y_{n-l} + q_3 y_{n-k} + q_4 y_{n-s} + q_5 y_{n-t}. \tag{3}$$

$$q_1 = \frac{\partial G(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial x_n},\ q_2 = \frac{\partial G(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial x_{n-l}},\ q_3 = \frac{\partial G(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial x_{n-k}},\ q_4 = \frac{\partial G(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial x_{n-s}},\ q_5 = \frac{\partial G(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial x_{n-t}}.$$

The characteristic equation associated with Eq. (3) is

$$q(\lambda) = q_1 + q_2\lambda^{\sigma-l} + q_3\lambda^{\sigma-k} + q_4\lambda^{\sigma-s} + q_5\lambda^{\sigma-t} = 0, \tag{4}$$

THEOREM 1.2. ***[1]**: Assume that $q_1,\ q_2,\ q_3,\ q_4$ and $q_5 \in R$. Then*

$$|q_1| + |q_2| + |q_3| + |q_4| + |q_5| < 1, \tag{4}$$

is a sufficient condition for the locally asymptotically stability of the Eq. (2).

THEOREM 1.3. *[4, 5]: Let $h : [\zeta, \eta]^{\sigma+1} \to [\zeta, \eta]$, be a continuous function, where σ is a positive integer, and where $[\zeta, \eta]$ is an interval of real numbers. Consider the difference equation*

$$x_{n+1} = h(x_n,\ x_{n-1}, ...,\ x_{n-\sigma}), \qquad n = 0, 1, ... \tag{5}$$

Suppose that g satisfies the following conditions.

(1) For each integer i with $1 \leq i \leq \sigma + 1$; the function $h(z_1,\ z_2,\ ...,\ z_{\sigma+1})$ is weakly monotonic in z_i for fixed $z_1,\ z_2,\ ...,\ z_{i-1},\ z_{i+1},\ ...,\ z_{\sigma+1}$.

(2) If $m,\ M$ is a solution of the system

$$m = h(m_1,\ m_2,\ ...,\ m_{\sigma+1}), \qquad M = h(M_1,\ M_2,\ ...,\ M_{\sigma+1}),$$

then $m = M$, where for each $i = 1,\ 2,\ ...,\ \sigma + 1$, we set

$$m_i = \begin{cases} m, & \text{if } h \text{ is non-decreasing in } z_i \\ M, & \text{if } h \text{ is non-increasing in } z_i \end{cases}$$

and

$$M_i = \begin{cases} M, & \text{if } h \text{ is non-decreasing in } z_i \\ m, & \text{if } h \text{ is non-increasing in } z_i. \end{cases}$$

Then there exists exactly one equilibrium point $\overline{x}$ of Eq. (5), and every solution of Eq. (5) converges to $\overline{x}$.

2. LOCAL STABILITY OF EQ. (1)

Eq. (1) has equilibrium point and is given by

$$\overline{x} = A\overline{x} + B\overline{x} + C\overline{x} + \tfrac{\gamma\overline{x}}{D\overline{x}+E\overline{x}},$$

or

$$\overline{x}(D + E)\,(1 - A - B - C) = \gamma.$$

If $A + B + c < 1$; then the only positive equilibrium point $\overline{x}$ of Eq. (1) is given by

$$\overline{x} = \tfrac{\gamma}{(D+E)(1-A-B-C)}.$$

THEOREM 2.1. *The equilibrium point $\overline{x}$ of Eq. (1) is locally asymptotically stable if*

$$A + B + C < 1.$$

Proof: Suppose that $G : (0,\ \infty)^5 \longrightarrow (0,\ \infty)$ be a continuous function defined by

$$G(u_1,\ u_2,\ u_3,\ u_4,\ u_5) = Au_1 + Bu_2 + Cu_3 + \tfrac{\gamma u_3}{Du_4+Eu_5}. \tag{6}$$

Therefore, it follows that

$$\begin{aligned} \tfrac{\partial G(u_1,\ u_2,\ u_3,\ u_4,\ u_5)}{\partial u_1} &= A, \quad \tfrac{\partial G(u_1,\ u_2,\ u_3,\ u_4,\ u_5)}{\partial u_2} = B, \quad \tfrac{\partial G(u_1,\ u_2,\ u_3,\ u_4,\ u_5)}{\partial u_3} = C + \tfrac{\gamma}{Du_4+Eu_5}, \\ \tfrac{\partial G(u_1,\ u_2,\ u_3,\ u_4,\ u_5)}{\partial u_4} &= -\tfrac{D\gamma u_3}{(Du_4+Eu_5)^2}, \quad \tfrac{\partial G(u_1,\ u_2,\ u_3,\ u_4,\ u_5)}{\partial u_5} = -\tfrac{E\gamma u_3}{(Du_4+Eu_5)^2}. \end{aligned}$$

So, we can write

$$\begin{aligned} \tfrac{\partial G(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial u_1} &= A = p_1, \quad \tfrac{\partial G(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial u_2} = B = p_2, \quad \tfrac{\partial G(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial u_3} = 1 - A - B = p_3, \\ \tfrac{\partial G(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial u_4} &= \tfrac{-D\gamma\left(\tfrac{\gamma}{(D+E)(1-A-B-C)}\right)}{(D+E)^2\left(\tfrac{\gamma}{(D+E)(1-A-B-C)}\right)^2} = \tfrac{D(A+B+C-1)}{(D+E)} = p_4, \\ \tfrac{\partial G(\overline{x},\ \overline{x},\ \overline{x},\ \overline{x},\ \overline{x})}{\partial u_5} &= \tfrac{-E\gamma\left(\tfrac{\gamma}{(D+E)(1-A-B-C)}\right)}{(D+E)^2\left(\tfrac{\gamma}{(D+E)(1-A-B-C)}\right)^2} = \tfrac{E(A+B+C-1)}{(D+E)} = p_5. \end{aligned}$$

Then the linearized equation of Eq. (1) about $\overline{x}$ is

$$y_{n+1} - p_1 y_n - p_2 y_{n-l} - p_3 y_{n-k} - p_4 y_{n-s} - p_5 y_{n-t} = 0.$$

It follows by Theorem 1 that, Eq. (1) is local asymptotically stable if and only if

$$|p_1| + |p_2| + |p_3| + |p_4| + |p_5| < 1.$$

Thus,

$$\begin{aligned}|p_1| + |p_2| + |p_3| + |p_4| + |p_5| &= |A| + |B| \\ &\quad + |1 - A - B| + \left|\frac{D(A+B+C-1)}{(D+E)}\right| + \left|\frac{E(A+B+C-1)}{(D+E)}\right| \\ &= 1 + A + B + C - 1 = A + B + C,\end{aligned}$$

for

$$A + B + C < 1.$$

The proof is complete.

Theorem 2.2. *The equilibrium $\overline{x}$ of Eq. (1) is unstable if $A + B + C > 1$.*

Example 1. The solution of Eq. (1) is local stability if $l = 2,\ k = s = 3,\ t = 4,\ A = B = \gamma = 0.1,\ C = 0.2,\ D = 1$ and $E = 0.6$ and the initial conditions $x_{-4} = 0.2,\ x_{-3} = 0.7,\ x_{-2} = 0.3,\ x_{-1} = 2.1$ and $x_0 = 1.1$ (See Fig. 1).

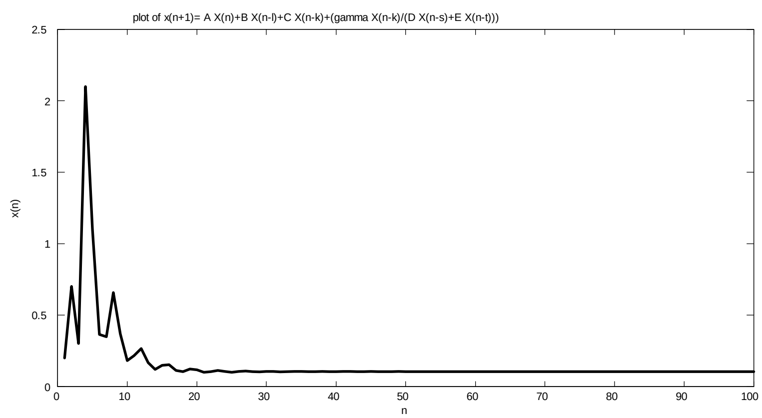

Figure 1. Plot the behavior of the solution of Eq. (1).

Example 2. The solution of Eq. (1) is local stability if $l = 4,\ k = t = 2,\ s = 3,\ A = 0.09,\ B = C = 0.01,\ D = 0.3,\ E = 0.5$ and $\gamma = 0.6$ and the initial conditions $x_{-4} = 0.2,\ x_{-3} = 0.7,\ x_{-2} = 0.2,\ x_{-1} = 2.1$ and $x_0 = 1.1$ (See Fig. 2).

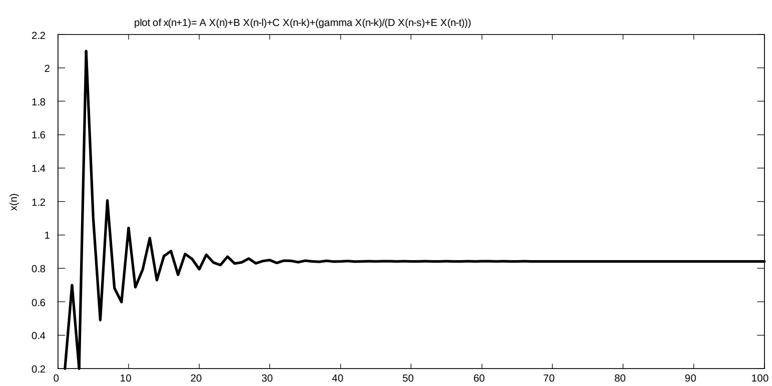

Figure 2. Plot the behavior of the solution of Eq. (1).

Example 3. The solution of Eq. (1) is unstable if $l=4,\ k=t=2,\ s=3,\ A=C=0.2,\ B=\gamma=0.7,\ D=0.1$ and $E=0.5$ and the initial conditions $x_{-4}=0.2,\ x_{-3}=0.7,\ x_{-2}=0.3,\ x_{-1}=2.1$ and $x_0=1.1$ (See Fig. 3).

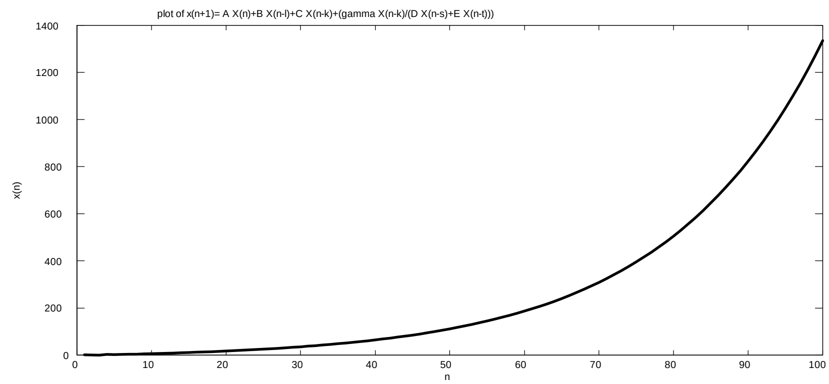

Figure 3. Plot the behavior of the solution of Eq. (1) is unstable.

3. GLOBAL ATTRACTIVITY OF EQ. (1)

Theorem 3.1. *The equilibrium point $\overline{x}$ is a global attractor of Eq. (1) if $A+B+C<1$ and $\gamma>1$.*

Proof: Suppose that ζ and η are real numbers and assume that $h:[\zeta,\eta]^5\longrightarrow[\zeta,\eta]$ is a function defined by

$$h(v_1,\ v_2,\ v_3,\ v_4,\ v_5)=Av_1+Bv_2+Cv_3+\tfrac{\gamma v_3}{Dv_4+Ev_5}. \tag{7}$$

Then

$$\begin{aligned}\tfrac{\partial h(v_1,\ v_2,\ v_3,\ v_4,\ v_5)}{\partial v_1} &= A,\ \tfrac{\partial h(v_1,\ v_2,\ v_3,\ v_4,\ v_5)}{\partial v_2}=B,\ \tfrac{\partial h(v_1,\ v_2,\ v_3,\ v_4,\ v_5)}{\partial v_3}=C+\tfrac{\gamma}{Dv_4+Ev_5},\\ \tfrac{\partial h(v_1,\ v_2,\ v_3,\ v_4,\ v_5)}{\partial v_4} &= -\tfrac{D\gamma v_3}{(Dv_4+Ev_5)^2},\ \tfrac{\partial h(v_1,\ v_2,\ v_3,\ v_4,\ v_5)}{\partial v_5}=-\tfrac{E\gamma v_3}{(Dv_4+Ev_5)^2}.\end{aligned}$$

First, we can see that the function $h(v_1,\ v_2,\ v_3,\ v_4,\ v_5)$ increasing in $v_1,\ v_2,\ v_3$ and decreasing in $v_4,\ v_5$ for $C+\dfrac{\gamma}{Dv_4+Ev_5}>0$.

Let (m,M) be a solution of the system $M=h(M,\ M,\ M,\ m,\ m)$ and $m=h(m,\ m,\ m,\ M,\ M)$. Then from Eq. (1), we see that

$$M=AM+BM+CM+\tfrac{\gamma M}{Dm+Em}\text{ and }m=Am+Bm+Cm+\tfrac{\gamma m}{DM+EM},$$

and then

$$M(1-A-B-C)=\tfrac{\gamma M}{Dm+Em}\text{ and }m(1-A-B-C)=\tfrac{\gamma m}{DM+EM},$$

thus

$$(1-A-B-C)\,(D+E)\,Mm=\gamma M$$

and

$$(1-A-B-C)\,(D+E)\,Mm=\gamma m.$$

Subtracting we obtain

$$\gamma(M-m)=0,$$

then

$$M=m.$$

It follows by Theorem 2 that $\overline{x}$ is a global attractor of Eq. (1). This completes the proof.

Second, we can see that the function $h(v_1,\ v_2,\ v_3,\ v_4,\ v_5)$ increasing in $v_1,\ v_2$ and decreasing in $v_3,\ v_4,\ v_5$ for $C+\dfrac{\gamma}{Dv_4+Ev_5}<0.$

Let (m, M) be a solution of the system $M = h(M,\ M,\ m,\ m,\ m)$ and $m = h(m,\ m,\ M,\ M,\ M)$. Then from Eq. (1), we see that

$$M = AM + BM + Cm + \tfrac{\gamma m}{Dm+Em} \text{ and } m = Am + Bm + CM + \tfrac{\gamma M}{DM+EM},$$

and then

$$M(1-A-B) - Cm = \tfrac{\gamma}{D+E},$$

and

$$m(1-A-B) - CM = \tfrac{\gamma}{D+E}.$$

Subtracting we obtain

$$(M-m)(1-A-B-C) = 0,$$

under the condition $A+B+C<1$, we see that

$$M = m.$$

It follows by Theorem 2 that $\overline{x}$ is a global attractor of Eq. (1). This completes the proof.

Example 4. The solution of Eq. (1) is global stability if $l = 2,\ k = s = 3,\ t = 4,\ A = 0.5,\ B = 0.07,\ C = 0.01,\ D = 2,\ E = 0.6$ and $\gamma = 0.1$ and the initial conditions $x_{-4} = 0.2,\ x_{-3} = 0.7,\ x_{-2} = 0.3,\ x_{-1} = 2.1$ and $x_0 = 1.1$ (See Fig. 4).

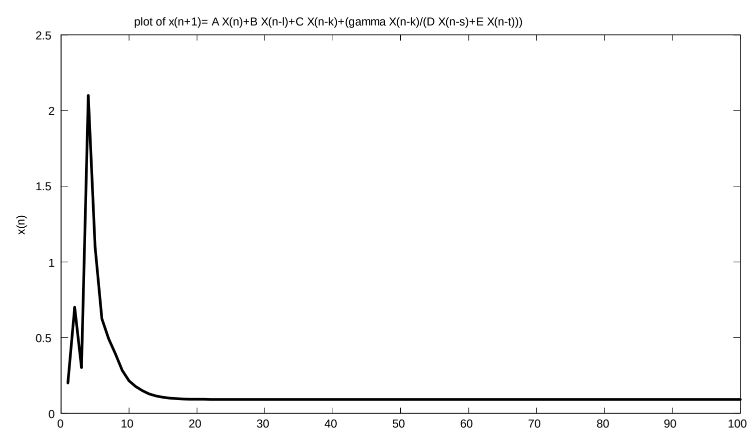

Figure 4. Plot the behavior of the solution of Eq. (1) is global stability.

4. EXISTENCE OF PERIODIC SOLUTIONS

THEOREM 4.1. *If l, k, t are an even and s is an odd then Eq. (1) has a prime period two solutions if and only if*

$$(i) \qquad (E-D)(A+B+C+1)+4D>0.$$

Proof: First Let there exists a prime period two solution

$$...P,\ Q,\ P,\ Q,\ ...,$$

of Eq. (1). We see from Eq. (1) when $l,\ k$ and t are even and s odd that

$$P = AQ + BQ + CQ + \tfrac{\gamma Q}{DP+EQ} \text{ and } Q = AP + BP + CP + \tfrac{\gamma P}{DQ+EP}.$$

Therefore,

$$DP^2 + EPQ = D(A+B+C)PQ + E(A+B+C)Q^2 + \gamma Q, \tag{8}$$

and

$$DQ^2 + EPQ = D(A+B+C)PQ + E(A+B+C)P^2 + \gamma P. \tag{9}$$

Subtracting (9) from (8) gives

$$D(P^2 - Q^2) + E(A+B+C)(P^2 - Q^2) + \gamma(P - Q) = 0,$$

then

$$(P - Q)\left[(D + E(A+B+C))(P+Q) + \gamma\right] = 0$$

Since $P \neq Q$, then

$$P + Q = \tfrac{-\gamma}{D+E(A+B+C)}. \tag{10}$$

Again, adding (8) and (9) yields

$$D(Q^2 + P^2) + 2EPQ = 2D(A+B+C)PQ + E(A+B+C)(Q^2 + P^2) + \gamma(Q + P),$$

then

$$2(E - D(A+B+C))PQ = (E(A+B+C) - D)(Q^2 + P^2) + \gamma(Q + P). \tag{11}$$

By using (10), (11) and the relation

$$P^2 + Q^2 = (P+Q)^2 - 2PQ \quad \text{for all } \ P,\ Q \in R,$$

we obtain

$$(E(A+B+C) - D)((P+Q)^2 - 2PQ) + \gamma(Q+P) = 2(E - D(A+B+C))PQ,$$
$$2\left[E - D(A+B+C) + E(A+B+C) - D\right]PQ = (E(A+B+C) - D)(P+Q)^2 + \gamma(Q+P),$$
$$2(E-D)(A+B+C+1)PQ = \left(\tfrac{-\gamma}{D+E(A+B+C)}\right)^2 (E(A+B+C) - D) + \gamma\left(\tfrac{-\gamma}{D+E(A+B+C)}\right),$$

$$\begin{aligned} 2(E-D)(A+B+C+1)PQ &= \tfrac{\gamma^2(E(A+B+C)-D) - \gamma^2(D+E(A+B+C))}{(D+E(A+B+C))^2}, \\ 2(E-D)(A+B+C+1)PQ &= -\tfrac{2D\gamma^2}{(D+E(A+B+C))^2}. \end{aligned}$$

Then,

$$PQ = -\left(\tfrac{D\gamma^2}{(D+E(A+B+C))^2}\right)\left(\tfrac{1}{(E-D)(A+B+C+1)}\right). \tag{12}$$

Now it is obvious from Eq. (10) and Eq. (12) that P and Q are the two distinct roots of the quadratic equation

$$t^2 + \tfrac{\gamma}{D+E(A+B+C)}t - \left(\tfrac{D\gamma^2}{(D+E(A+B+C))^2}\right)\left(\tfrac{1}{(E-D)(A+B+C+1)}\right) = 0,$$
$$(D + E(A+B+C))\,t^2 + \gamma t - \tfrac{D\gamma^2}{(E-D)(A+B+C+1)(D+E(A+B+C))} = 0, \tag{13}$$

and so

$$\begin{aligned}\left(\frac{\gamma}{D+E(A+B+C)}\right)^2+4\left(\frac{D\gamma^2}{(D+E(A+B+C))^2}\right)\left(\frac{1}{(E-D)(A+B+C+1)}\right) &> 0,\\ 1+\left(\frac{4D}{(E-D)(A+B+C+1)}\right) &> 0,\end{aligned}$$

or

$$(E-D)(A+B+C+1)+4D>0.$$

For $E>D$ then the Inequality (i) holds.

Second suppose that Inequality (i) is true. We will show that Eq. (1) has a prime period two solution.

Suppose that

$$P=\frac{\gamma(\zeta-1)}{2(D+E\alpha)} \text{ and } Q=-\frac{\gamma(1+\zeta)}{2(D+E\alpha)},$$

where $\zeta=\sqrt{1+\frac{4D}{(E-D)(\alpha+1)}}$and $\alpha=A+B+C$.

We see from the inequality (i) that

$$(E-D)(A+B+C+1)+4D>0,$$

which equivalents to

$$(E-D)(\alpha+1)+4D>0.$$

Therefore P and Q are distinct real numbers.

Set

$$x_{-l}=Q,\ x_{-k}=Q,\ x_{-s}=P,\ x_{-t}=Q,...,\ x_{-3}=P,\ x_{-2}=Q,\ x_{-1}=P,\ x_0=Q.$$

We would like to show that

$$x_1=x_{-1}=P \quad\text{and}\quad x_2=x_0=Q.$$

It follows from Eq. (1) that

$$x_1=AQ+BQ+CQ+\frac{\gamma Q}{DP+EQ}=\alpha\left(\frac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)+\frac{\gamma\left(\frac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)}{D\left(\frac{\gamma(-1+\zeta)}{2(D+E\alpha)}\right)+E\left(\frac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)}.$$

Dividing the denominator and numerator by $2\,(D+E\alpha)$ we get

$$x_1=\alpha\left(\frac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)+\frac{\gamma(-1-\zeta)}{D(-1+\zeta)+E(-1-\zeta)}=\alpha\left(\frac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)+\frac{\gamma(1+\zeta)}{(D+E)+(E-D)\zeta}.$$

Multiplying the denominator and numerator of the right side by $(D+E)-(D-E)\,\zeta$

$$\begin{aligned}x_1 &= \alpha\left(\frac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)+\frac{\gamma(1+\zeta)((D+E)-(E-D)\zeta)}{((D+E)+(E-D)\zeta)((D+E)-(E-D)\zeta)}=\alpha\left(\frac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)+\frac{\gamma\left((D+E)-(E-D)\zeta+(D+E)\zeta-(E-D)\zeta^2\right)}{(D+E)^2-(E-D)^2\zeta^2},\\
&= \alpha\left(\frac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)+\frac{\gamma\left((D+E)+2D\zeta-(E-D)\zeta^2\right)}{(D+E)^2-(E-D)^2\zeta^2}=\alpha\left(\frac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)+\frac{\gamma\left((D+E)+2D\zeta-(E-D)\left(\frac{(E-D)(\alpha+1)+4D}{(E-D)(\alpha+1)}\right)\right)}{(D+E)^2-(E-D)^2\left(\frac{(E-D)(\alpha+1)+4D}{(E-D)(\alpha+1)}\right)},\\
&= \alpha\left(\frac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)+\frac{\gamma\left((D+E)+2D\zeta-\left(\frac{(E-D)(\alpha+1)+4D}{(\alpha+1)}\right)\right)}{(D+E)^2-(E-D)\left(\frac{(E-D)(\alpha+1)+4D}{(\alpha+1)}\right)}=\alpha\left(\frac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)+\frac{\gamma\left((D+E)+2D\zeta-\left((E-D)+\frac{4D}{(\alpha+1)}\right)\right)}{(D+E)^2-(E-D)\left((E-D)+4\frac{D}{(\alpha+1)}\right)},\\
&= \alpha\left(\frac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)+\frac{\gamma\left(D+E+2D\zeta-E+D-\frac{4D}{(\alpha+1)}\right)}{(D+E)^2-(E-D)^2-\frac{4(E-D)D}{(\alpha+1)}}=\alpha\left(\frac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)+\frac{\gamma D\left(2+2\zeta-\frac{4}{(\alpha+1)}\right)}{4DE-\frac{4(E-D)D}{(\alpha+1)}},\\
&= \alpha\left(\frac{\gamma(-1-\zeta)}{2(D+E\alpha)}\right)+\frac{\gamma((1+\zeta)(\alpha+1)-2)}{2E(\alpha+1)-2(E-D)}=\gamma\left(\frac{\alpha(-1-\zeta)+(1+\zeta)(\alpha+1)-2}{2(D+E\alpha)}\right)=\frac{\gamma(\zeta-1)}{2(D+E\alpha)}=P.\end{aligned}$$

Similarly as before, it is easy to show that
$$x_2 = Q.$$
Then by induction we get
$$x_{2n} = Q \quad \text{and} \quad x_{2n+1} = P \quad \text{for all} \quad n \geq -2.$$
Thus Eq. (1) has the prime period two solution
$$...,P,\ Q,\ P,\ Q,...,$$
where P and Q are the distinct roots of the quadratic equation (13) and the proof is complete.

Example 5. Figure (5) shows the Eq. (1) has a prime period two solution when $l = 2,\ k = t = 4,\ s = 3,$ $A = 0.01,\ B = 0.02,\ C = 0.03,\ D = 0.2,\ E = 0.3$ and $\gamma = 0.1$ and the initial conditions $x_{-4} = x_{-2} = x_0 = Q$ and $x_{-3} = x_{-1} = P$.

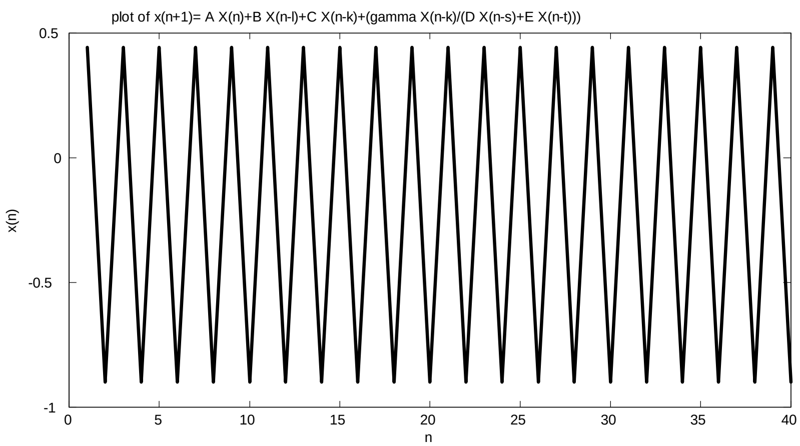

Figure 5. Plot the solution of Eq. (1) has a periodic solution.

Theorem 4.2. *Eq. (1) has a prime period two solutions if and only if*
$$(D - E)(A + B + C + 1) + 4E > 0\ , l, k, s - even\ and\ t - odd.$$

Theorem 4.3. *Eq. (1) has a prime period two solutions if and only if*
$$1 - C - 3A - 3B > 0 \quad l, s, t - even\ and\ k - odd.$$

Theorem 4.4. *Eq. (1) has a prime period two solutions if and only if*
$$A + B + C > 3\ l, k - even\ and\ s, t - odd.$$

Theorem 4.5. *Eq. (1) has a prime period two solutions if and only if*
$$(D - E)(A + B + 1 - C) > 4D(A + B) \quad l, s - even\ and\ k, t - odd.$$

Theorem 4.6. *Eq. (1) has a prime period two solutions if and only if*

$$(E-D)(1+A+B-C) > 4E(A+B) \quad l,t-\textit{even and } k,s-\textit{odd}.$$

Theorem 4.7. *Eq. (1) has a prime period two solutions if and only if*

$$(D-E)(1+A+C-B)+4E(1-B) > 0, \ k,s-\textit{even and } l,t-\textit{odd}.$$

Theorem 4.8. *Eq. (1) has a prime period two solutions if and only if*

$$(E-D)(1+A+C-B)+4D(1-B) > 0, \ k,t-\textit{even and } l,s-\textit{odd}.$$

Theorem 4.9. *Eq. (1) has a prime period two solutions if and only if*

$$A+C+3B > 3, \ k-\textit{even and } l,s,t-\textit{odd}.$$

Theorem 4.10. *Eq. (1) has a prime period two solutions if and only if*

$$(D-E)(1+A-C-B) > 4DA, \ s-\textit{even and } l,k,t-\textit{odd}.$$

Theorem 4.11. *Eq. (1) has a prime period two solutions if and only if*

$$(E-D)(1+A-C-B) > 4EA, \ t-\textit{even and } l,k,s-\textit{odd}.$$

Theorem 4.12. *Eq. (1) has a prime period two solutions if and only if*

$$3A+B+C < 1, \ s,t-\textit{even and } l,k-\textit{odd}.$$

Theorem 4.13. *Eq. (1) has no prime period two solutions if* l, k , t *and* s *are an even when* $1+A+B+C \neq 0$.

Proof. Let there exists a prime period two solution $...P,\ Q,\ P,\ Q,\ ...$, of Eq. (1). We see from Eq. (1) when l, k , t and s are even

$$P = AQ + BQ + CQ + \tfrac{\gamma Q}{DQ+EQ}, \tag{14}$$

and

$$Q = AP + BP + CP + \tfrac{\gamma P}{DP+EP}. \tag{15}$$

Subtracting (14) from (15) gives

$$(1+A+B+C)(P-Q) = 0,$$

then

$$(P-Q)\left[(D+E)(A+B+C)(P+Q)+\gamma\right] = 0$$

Since $1+A+B+C \neq 0$, then $P=Q$. This is a contradiction. Thus, the proof is completed.

Example 6. Figure (6) shows the Eq. (1) has no period two solution when $l = 2, k = s = t = 4$, $A = 0.1$, $B = D = 0.2$, $C = 0.6$, $E = 0.3$ and $\gamma = 0.1$ and the initial conditions $x_{-4} = 0.2$, $x_{-3} = 0.7$, $x_{-2} = 0.3$, $x_{-1} = 2.1$ and $x_0 = 1.1$.

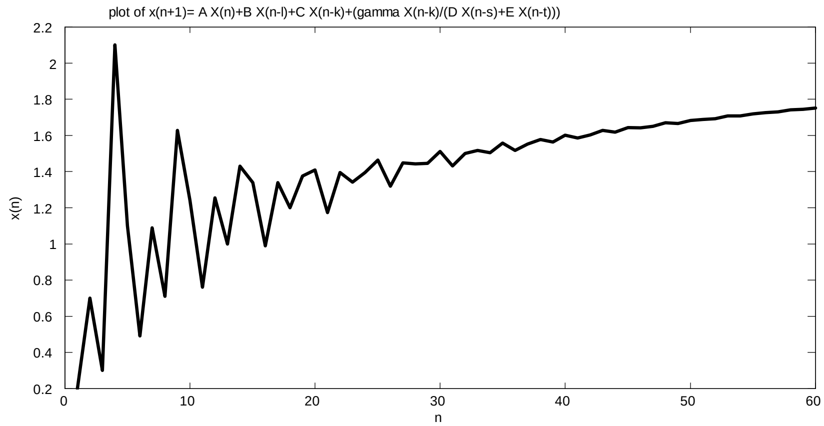

Figure 6. Plot the solution of Eq. (1) has no periodic.

THEOREM 4.14. *Eq. (1) has no prime period two solutions if l is an even and k, s, t are an odd when $A + B - C + 1 \neq 0$.*

THEOREM 4.15. *Eq. (1) has no a prime period two solutions if l, k, s, t are an odd when $1 - A + B + C \neq 0$.*

THEOREM 4.16. *Eq. (1) has no a prime period two solutions if k, t, s are an even and l is an odd when $1 + A + C - B \neq 0$.*

5. BOUNDEDNESS OF THE SOLUTIONS OF EQ. (1)

THEOREM 5.1. *Let $\{x_n\}$ be a solution of Eq. (1). Then the following statements are true:*
(i) Suppose $\gamma < D$ and for some $N \geq 0$, the intial conditions

$$x_{N-\sigma+1},\ ...,\ x_{N-1},\ x_N \in \left[\tfrac{\gamma}{D},\ 1\right],$$

are valid, then we have the inequality

$$\tfrac{\gamma}{D}(A + B + C) + \tfrac{\gamma^2}{(D^2+\gamma E)} \leq x_n \leq (A + B + C) + \tfrac{\gamma}{(\gamma+E)}, \tag{16}$$

for all $n \geq N$.
(ii) Suppose $\gamma > D$ and for some $N \geq 0$, the intial conditions

$$x_{N-\sigma+1},\ ...,\ x_{N-1},\ x_N \in \left[1,\ \tfrac{\gamma}{D}\right],$$

are valid, then we have the inequality

$$(A + B + C) + \tfrac{\gamma}{(\gamma+E)} \leq x_n \leq \frac{\gamma}{D}(A + B + C) + \tfrac{\gamma^2}{(D^2+\gamma E)}, \tag{17}$$

for all $n \geq N$.

Proof: First of all, if for some $N > 0$, $\frac{\gamma}{D} \leq x_N \leq 1$, we have

$$x_{N+1} = Ax_N + Bx_{N-l} + Cx_{N-k} + \tfrac{\gamma x_{N-k}}{Dx_{N-s}+Ex_{N-t}} \leq A + B + C + \tfrac{\gamma x_{N-k}}{Dx_{N-s}+Ex_{N-t}}.$$

But, it is easy to see that $Dx_{N-s}+Ex_{N-t}\geq\gamma+E$, then we get

$$x_{N+1}\leq A+B+C+\tfrac{\gamma}{\gamma+E}. \tag{18}$$

Similarly, we can show that

$$x_{N+1}\geq\tfrac{\gamma}{D}\left(A+B+C\right)+\tfrac{\gamma x_{N-k}}{Dx_{N-s}+Ex_{N-t}} \tag{19}$$

But, one can see that $Dx_{N-s}+Ex_{N-t}\leq\frac{D^2+\gamma E}{D}$, then

$$x_{N+1}\geq\tfrac{\gamma}{D}\left(A+B+C\right)+\tfrac{\gamma^2}{D^2+\gamma E} \tag{20}$$

From (18) and (20) we deduce for all $n\geq N$ that the inequality (16) is valid. Hence, the proof of part (i) is completed.

Similarly, if $1\leq x_N\leq\frac{\gamma}{D}$, then we can prove part (ii) which is omitted here for convenience. Thus, the proof is now completed.

Acknowledgements

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR technical and financial support.

REFERENCES

1. V. L. Kocic and G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic Publishers, Dordrecht, 1993.
2. W. G. Kelley, A. C. Peterson, Difference Equations: An Introduction With Applications, Academic Press, New York, 1991
3. R. P. Agarwal, Difference Equations and Inequalities, Marcel Dekker, New York, 1992.
4. E. A. Grove, G. Ladas, Periodicities in Nonlinear Difference Equations, Chapman & Hall/CRC Press, Boca Raton, FL, 2005.
5. M. R. S. Kulenović, G. Ladas, Dynamics of Second Order Rational Difference Equations; With Open Problems and Conjectures, Chapman & Hall/CRC Press, Boca Raton, FL, 2001.
6. S. N. Elaydi, An Introduction to Difference Equations, Springer-Verlag New York Inc., 1996.
7. M. M. El-Dessoky, *Qualitative behavior of rational difference equation of big Order*, Discrete Dyn. Nat. Soc., 2013, (2013), Article ID 495838, 6 pages.
8. Xiu-Mei Jia and Wan-Tong Li, *Boundedness and Global Attractivity of a Higher-Order Nonlinear Difference Equation*, Discrete Dyn. Nat. Soc., 2010, (2010), Article ID 610467, 17 pages.
9. Chang-you Wang, Shu Wang, Zhi-wei Wang, Fei Gong, and Rui-fang Wang, *Asymptotic Stability for a Class of Nonlinear Difference Equations*, Discrete Dyn. Nat. Soc., 2010, (2010), Article ID 791610, 10 pages.
10. R. Abo-Zeid, *On the oscillation of a third order rational difference equation*, J. Egypt. Math. Soc., 23, (2015), 62–66.
11. R. Devault, C., Kent, W. Kosmala, *On the recursive sequence* $x_{n+1}=p+\dfrac{x_{n-k}}{x_n}$, J. Difference Equ. Appl., 9, (2003), 721–730.
12. E. M. Elsayed, M. M. El-Dessoky, Asim Asiri, *Dynamics and Behavior of a Second Order Rational Difference equation*, J. Comput. Anal. Appl., 16, (2014), 794-807.
13. E. M. E. Zayed, M. A. El-Moneam, *On the Rational Recursive Sequence* $X_{n+1}=\gamma X_{n-k}+\frac{aX_n+bX_{n-k}}{cX_n-dX_{n-k}}$, Bulletin of the Iranian Mathematical Society, 36, (2010), 103-115.
14. M. M. El-Dessoky, *Dynamics and Behavior of the Higher Order Rational Difference equation*, J. Comput. Anal. Appl., 21, (2016), (4), 743-760.

15. R. Abo-Zeid, *Attractivity of two nonlinear third order difference equations*, J. Egypt. Math. Soc., 21, (2013), 241–247.
16. G. Papaschinopoulos, C. J., Schinas, G. Stefanidou, *On the nonautonomous difference equation* $x_{n+1} = A_n + \frac{x_{n-1}^p}{x_n^q}$, Appl. Math. Comput., 217, (2011), 5573-5580.
17. Mehmet Gűműş, *The Periodicity of Positive Solutions of the Nonlinear Difference Equation* $x_{n+1} = \alpha + \frac{x_{n-k}^p}{x_n^p}$, Discrete Dyn. Nat. Soc., 2013, (2013), Article ID 742912, 3 pages.
18. M. A. El-Moneam, *On the Dynamics of the Higher Order Nonlinear Rational Difference Equation*, Math. Sci. Lett., 3, (2014), 121-129.
19. S. Basu, O. Merino, *Global behavior of solutions to two classes of second-order rational difference equations*, Adv. Differ. Equ., 2009, (2009), Article ID 128602, 27 pages.
20. Mehmet Gűműş and Őzkan Őcalan, *Some Notes on the Difference Equation* $x_{n+1} = \alpha + \frac{x_{n-1}}{x_n^k}$, Discrete Dyn. Nat. Soc., 2012, (2012), Article ID 258502, 12 pages.
21. S. Ebru Das., *Dynamics of a nonlinear rational difference equation*, Hacet. J. Math. Stat., 42, (2013), 9-14.
22. Miron B. Bekker, Martin J. Bohner, Hristo D. Voulov, *Asymptotic behavior of solutions of a rational system of difference equations*, J. Nonlinear Sci. Appl., 7, (2014), 379-382.
23. You-Hui Su and Wan-Tong Li, *Global attractivity of a higher order nonlinear difference equation*, J. Difference Equ. Appl., 11, (2005), 947-958.
24. M. M. El-Dessoky, On the dynamics of higher Order difference equations $x_{n+1} = ax_n + \frac{\alpha x_n x_{n-l}}{\beta x_n + \gamma x_{n-k}}$, J. Comput. Anal. Appl., 22, (2017), 1309-1322.
25. Huili Ma, Hui Feng, Jiaofeng Wang, Wandi Ding, *Boundedness and asymptotic behavior of positive solutions for di erence equations of exponential form*, J. Nonlinear Sci. Appl., 8, (2015), 893-899.
26. E. M. Elsayed, M. M. El-Dessoky, Asim Asiri, Dynamics and Behavior of a Second Order Rational Difference equation, J. Comput. Anal. Appl., 16, (2014), 794-807.
27. M. M. El-Dessoky, On the dynamics of a higher Order rational difference equations, J. Egypt. Math. Soc. (to apper).
28. E. M. Elsayed, *Dynamics and behavior of a higher order rational difference equation*, J. Nonlinear Sci. Appl., 9, (2016), 1463-1474.
29. N. Touafek, *On a second order rational difference equation*, Hacet. J. Math. Stat., 41, (2012), 867-874.
30. M. M. El-Dessoky, Qualitative behavior of rational difference equation of big Order, Discrete Dyn Nat Soc, Vol. 2013, (2013), Article ID 495838, 6 pages.
31. E. M. Elsayed and M. M. El-Dessoky, Dynamics and Global Behavior for a Fourth − Order Rational Difference Equation, Hacet. J. Math. Stat., 42(5), (2013), 479-494.
32. E. M. E. Zayed, *Dynamics of the nonlinear rational difference equation* $x_{n+1} = Ax_n + Bx_{n-k} + \frac{px_n + x_{n-k}}{q + x_{n-k}}$, Eur. J. Pure Appl. Math., 3, (2010), 254-268.
33. E. M. Elsayed, M. M. El-Dessoky, E. O. Alzahrani, The Form of The Solution and Dynamics of a Rational Recursive Sequence, J. Comput. Anal. Appl., 17, (2014), 172-186.
34. M. A. El-Moneam, E. M. E. Zayed, On the dynamics of the nonlinear rational difference equation $x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-l} + \frac{bx_{n-k}}{dx_{n-k} - ex_{n-l}}$, J. Egypt. Math. Soc., 23, (2015), 494–499.
35. A. M. Ahmed, N. A. Eshtewy, Attractivity of the recursive sequence $x_{n+1} = \frac{A - Bx_{n-2}}{C + Dx_{n-1}}$, J. Egypt. Math. Soc., 24, (2016), 392–395.
36. M. M. El-Dessoky and E. O. Alzahrani, On the Difference equation $x_{n+1} = Ax_n + \frac{B\sum_{i=0}^{k} x_{n-i}}{C + D\prod_{i=0}^{k} x_{n-i}}$, J. Comput. Anal. Appl., 23, (2017), 78-91.

AN ADDITIVE (α, β)-FUNCTIONAL EQUATION AND LINEAR MAPPINGS IN BANACH SPACES

CHOONKIL PARK, SUN YOUNG JANG*, AND YOUNG CHO

ABSTRACT. In this paper, we investigate the additive (α, β)-functional equation

$$f(x) + \overline{\alpha} f(\alpha y) = \beta^{-1} f(\beta(x+y)) \tag{0.1}$$

for all complex numbers α with $|\alpha| = 1$ and for a fixed nonzero complex number β.

Using the fixed point method and the direct method, we prove the Hyers-Ulam stability of the additive (α, β)-functional equation (0.1) in complex Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [21] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [9] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [17] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. See [5, 7, 13, 14, 15, 18, 19, 20, 22] for more information on functional equations.

We recall a fundamental result in fixed point theory.

Theorem 1.1. [2, 6] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

2010 *Mathematics Subject Classification.* Primary 39B52, 39B62, 47H10.

Key words and phrases. Hyers-Ulam stability; additive (α, β)-functional equation; $\mathbb{C}$-linear mapping; fixed point method; direct method; complex Banach space.

*Corresponding author .

(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all* $y \in Y$.

In 1996, Isac and Rassias [10] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [3, 4, 16]).

In Section 2, we solve the additive (α, β)-functional equation (0.1) in vector spaces and prove the Hyers-Ulam stability of the additive (α, β)-functional equation (0.1) in Banach spaces by using the fixed point method.

In Section 3, we prove the Hyers-Ulam stability of the additive (α, β)-functional equation (0.1) in Banach spaces by using the direct method.

Throughout this paper, assume that X is a complex normed space and that Y is a complex Banach space. Let β be a fixed nonzero complex number.

2. Additive (α, β)-functional equation (0.1) in complex Banach spaces I

We solve the additive (α, β)-functional equation (0.1) in complex vector spaces.

Lemma 2.1. *Let X and Y be complex vector spaces. If a mapping $f : X \to Y$ satisfies*

$$f(x) + \overline{\alpha} f(\alpha y) = \beta^{-1} f(\beta(x + y)) \tag{2.1}$$

for all $x, y \in X$ and all $\alpha \in \mathbb{T} := \{\mu \in \mathbb{C} \mid |\mu| = 1\}$, then $f : X \to Y$ is $\mathbb{C}$-linear.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $(1 + \overline{\alpha}) f(0) = \beta^{-1} f(0)$ for all $\alpha \in \mathbb{T}$. So $f(0) = 0$.

Letting $\alpha = 1$, $y = -x$ in (2.1), we get $f(x) + f(-x) = 0$ and so $f(-x) = -f(x)$ for all $x \in X$.

Letting $\alpha = 1$, $x = 0$ and replacing y by $x + y$ in (2.1), we get

$$f(x + y) = \beta^{-1} f(\beta(x + y))$$

for all $x, y \in X$. Letting $\alpha = 1$ in (2.1), we get $f(x) + f(y) = \beta^{-1} f(\beta(x + y))$ and so

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in X$.

Letting $y = -x$ in (2.1), we get $f(x) + \overline{\alpha} f(-\alpha x) = 0$ and so $f(\alpha x) = \alpha f(x)$ for all $x \in X$ and all $\alpha \in \mathbb{T}$. By the same reasoning as in the proof of [12, Theorem 2.1], the mapping $f : X \to Y$ is $\mathbb{C}$-linear. □

Using the fixed point method, we prove the Hyers-Ulam stability of the additive (α, β)-functional equation (2.1) in complex Banach spaces.

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \le \frac{L}{2}\varphi(x, y) \tag{2.2}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\left\| f(x) + \overline{\alpha} f(\alpha y) - \beta^{-1} f(\beta(x+y)) \right\| \le \varphi(x, y) \tag{2.3}$$

for all $x, y \in X$ and all $\alpha \in \mathbb{T}$. Then there exists a unique $\mathbb{C}$-linear mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{L}{2(1-L)}(\varphi(0, 2x) + \varphi(x, x)) \tag{2.4}$$

for all $x \in X$.

Proof. Let $\alpha = 1$.

Letting $y = x$ in (2.3), we get

$$\left\| 2f(x) - \beta^{-1} f(2\beta x) \right\| \le \varphi(x, x) \tag{2.5}$$

for all $x \in X$.

Replacing y by $2x$ and letting $x = 0$ in (2.3), we get

$$\left\| f(2x) - \beta^{-1} f(2\beta x) \right\| \le \varphi(0, 2x) \tag{2.6}$$

for all $x \in X$.

It follows from (2.5) and (2.6) that

$$\|f(2x) - 2f(x)\| \le \varphi(0, 2x) + \varphi(x, x) \tag{2.7}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf \{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \le \mu(\varphi(0, 2x) + \varphi(x, x)), \ \forall x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [11]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \le \varepsilon(\varphi(0, 2x) + \varphi(x, x))$$

for all $x \in X$. Hence

$$\begin{aligned}\|Jg(x) - Jh(x)\| &= \left\|2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right)\right\| \le 2\varepsilon\left(\varphi(0,x) + \varphi\left(\frac{x}{2},\frac{x}{2}\right)\right)\\ &\le 2\varepsilon\frac{L}{2}(\varphi(0,2x) + \varphi(x,x)) = L\varepsilon(\varphi(0,2x) + \varphi(x,x))\end{aligned}$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg,Jh) \le L\varepsilon$. This means that

$$d(Jg,Jh) \le Ld(g,h)$$

for all $g,h \in S$.

It follows from (2.7) that

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \le \varphi(0,x) + \varphi\left(\tfrac{x}{2},\tfrac{x}{2}\right) \le \tfrac{L}{2}(\varphi(0,2x) + \varphi(x,x))$$

for all $x \in X$. So $d(f,Jf) \le \frac{L}{2}$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A(x) = 2A\left(\frac{x}{2}\right) \tag{2.8}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.8) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|f(x) - A(x)\| \le \mu(\varphi(0,2x) + \varphi(x,x))$$

for all $x \in X$;

(2) $d(J^l f, A) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f,A) \le \frac{1}{1-L}d(f,Jf)$, which implies

$$\|f(x) - A(x)\| \le \frac{L}{2(1-L)}(\varphi(0,2x) + \varphi(x,x))$$

for all $x \in X$.

It follows from (2.2) and (2.3) that

$$\begin{aligned}&\left\|A(x) + \overline{\alpha}A(\alpha y) - \beta^{-1}A(\beta(x+y))\right\|\\ &= \lim_{n\to\infty} 2^n \left\|f\left(\frac{x}{2^n}\right) + \overline{\alpha}f\left(\frac{\alpha y}{2^n}\right) - \beta^{-1}f\left(\beta\left(\frac{x+y}{2^n}\right)\right)\right\|\\ &\le \lim_{n\to\infty} 2^n\varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right) = 0\end{aligned}$$

for all $x, y \in X$ and all $\alpha \in \mathbb{T}$. So

$$A(x) + \overline{\alpha}A(\alpha y) - \beta^{-1}A\left(\beta(x+y)\right) = 0$$

for all $x, y \in X$ and all $\alpha \in \mathbb{T}$. By Lemma 2.1, the mapping $A : X \to Y$ is $\mathbb{C}$-linear. □

Corollary 2.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying*

$$\left\|f(x) + \overline{\alpha}f(\alpha y) - \beta^{-1}f(\beta(x+y))\right\| \le \theta(\|x\|^r + \|y\|^r) \tag{2.9}$$

for all $x, y \in X$ and all $\alpha \in \mathbb{T}$. Then there exists a unique $\mathbb{C}$-linear mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2^r + 2}{2^r - 2}\theta\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{1-r}$ and we get the desired result. □

Theorem 2.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique $\mathbb{C}$-linear mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2(1-L)}(\varphi(0, 2x) + \varphi(x, x))$$

for all $x \in X$.

Proof. It follows from (2.7) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \le \frac{1}{2}(\varphi(0, 2x) + \varphi(x, x))$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (2.9). Then there exists a unique $\mathbb{C}$-linear mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2 + 2^r}{2 - 2^r}\theta\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = 2^{r-1}$ and we get desired result. $\square$

3. Additive (α, β)-functional equation (0.1) in complex Banach spaces II

In this section, using the direct method, we prove the Hyers-Ulam stability of the additive (α, β)-functional equation (2.1) in complex Banach spaces.

Theorem 3.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\Psi(x, y) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) \ < \ \infty,$$

$$\left\| f(x) + \overline{\alpha} f(\alpha y) - \beta^{-1} f(\beta(x + y)) \right\| \ \leq \ \varphi(x, y) \tag{3.1}$$

for all $x, y \in X$ and all $\alpha \in \mathbb{T}$. Then there exists a unique $\mathbb{C}$-linear mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{2}(\Psi(0, 2x) + \Psi(x, x)) \tag{3.2}$$

for all $x \in X$.

Proof. Let $\alpha = 1$.

It follows from (2.7) that

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \leq \varphi(0, x) + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| \ &\leq \ \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \\ &\leq \ \sum_{j=l}^{m-1} \left(2^j \varphi\left(0, \frac{x}{2^j}\right) + 2^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) \right) \end{aligned} \tag{3.3}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.3) that the sequence $\{2^k f(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a Banach space, the sequence $\{2^k f(\frac{x}{2^k})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{k \to \infty} 2^k f\left(\frac{x}{2^k}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.3), we get (3.2).

Now, let $T : X \to Y$ be another additive mapping satisfying (3.2). Then we have

$$\begin{aligned}
\|A(x) - T(x)\| &= \left\| 2^q A\left(\frac{x}{2^q}\right) - 2^q T\left(\frac{x}{2^q}\right) \right\| \\
&\le \left\| 2^q A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right) \right\| + \left\| 2^q T\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{x}{2^q}\right) \right\| \\
&\le 2^q \Psi\left(0, \frac{2x}{2^q}\right) + 2^q \Psi\left(\frac{x}{2^q}, \frac{x}{2^q}\right),
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A.

The rest of the proof is similar to the proof of Theorem 2.2. $\square$

Corollary 3.2. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.9). *Then there exists a unique $\mathbb{C}$-linear mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{2^r + 2}{2^r - 2} \theta \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.1 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. $\square$

Theorem 3.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$,* (3.1) *and*

$$\Psi(x, y) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique $\mathbb{C}$-linear mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{1}{2}(\Psi(0, 2x) + \Psi(x, x)) \tag{3.4}$$

for all $x \in X$.

Proof. It follows from (2.7) that

$$\left\| f(x) - \frac{1}{2} f(2x) \right\| \le (\varphi(0, 2x) + \varphi(x, x))$$

for all $x \in X$. Hence

$$\begin{aligned}
\left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\| &\le \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f\left(2^j x\right) - \frac{1}{2^{j+1}} f\left(2^{j+1} x\right) \right\| \qquad (3.5) \\
&\le \sum_{j=l}^{m-1} \left(\frac{1}{2^{j+1}} \varphi(0, 2^{j+1} x) + \frac{1}{2^{j+1}} \varphi(2^j x, 2^j x) \right)
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.5) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.5), we get (3.4).

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.1. □

Corollary 3.4. *Let $r < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.9). *Then there exists a unique $\mathbb{C}$-linear mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2 + 2^r}{2 - 2^r} \theta \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.3 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. □

Acknowledgement

S. Y. Jang was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2013007226).

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
[2] L. Cădariu, V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math. **4**, no. 1, Art. ID 4 (2003).
[3] L. Cădariu, V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber. **346** (2004), 43–52.
[4] L. Cădariu, V. Radu, *Fixed point methods for the generalized stability of functional equations in a single variable*, Fixed Point Theory Appl. **2008**, Art. ID 749392 (2008).
[5] A. Chahbi, N. Bounader, *On the generalized stability of d'Alembert functional equation*, J. Nonlinear Sci. Appl. **6** (2013), 198–204.
[6] J. Diaz, B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Am. Math. Soc. **74** (1968), 305–309.
[7] G. Z. Eskandani, P. Găvruta, *Hyers-Ulam-Rassias stability of pexiderized Cauchy functional equation in 2-Banach spaces,* J. Nonlinear Sci. Appl. **5** (2012), 459–465.
[8] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–43.
[9] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. **27** (1941), 222–224.
[10] G. Isac, Th. M. Rassias, *Stability of ψ-additive mappings: Applications to nonlinear analysis*, Internat. J. Math. Math. Sci. **19** (1996), 219–228.

[11] D. Miheţ, V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008), 567–572.
[12] C. Park, *Homomorphisms between Poisson JC^*-algebras*, Bull. Braz. Math. Soc. **36** (2005), 79–97.
[13] C. Park, *Orthogonal stability of a cubic-quartic functional equation*, J. Nonlinear Sci. Appl. **5** (2012), 28–36.
[14] C. Park, K. Ghasemi, S. G. Ghaleh, S. Jang, *Approximate n-Jordan $*$-homomorphisms in C^*-algebras,* J. Comput. Anal. Appl. **15** (2013), 365-368.
[15] C. Park, A. Najati, S. Jang, *Fixed points and fuzzy stability of an additive-quadratic functional equation,* J. Comput. Anal. Appl. **15** (2013), 452–462.
[16] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.
[17] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Am. Math. Soc. **72** (1978), 297–300.
[18] K. Ravi, E. Thandapani, B. V. Senthil Kumar, *Solution and stability of a reciprocal type functional equation in several variables,* J. Nonlinear Sci. Appl. **7** (2014), 18–27.
[19] D. Shin, C. Park, Sh. Farhadabadi, *On the superstability of ternary Jordan C^*-homomorphisms,* J. Comput. Anal. Appl. **16** (2014), 964–973.
[20] D. Shin, C. Park, Sh. Farhadabadi, *Stability and superstability of J^*-homomorphisms and J^*-derivations for a generalized Cauchy-Jensen equation,* J. Comput. Anal. Appl. **17** (2014), 125–134.
[21] S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.
[22] C. Zaharia, *On the probabilistic stability of the monomial functional equation,* J. Nonlinear Sci. Appl. **6** (2013), 51–59.

Research Institute for Natural Sciences, Hanyang University, Seoul 04763, Republic of Korea
E-mail address: baak@hanyang.ac.kr

Department of Mathematics, University of Ulsan, Ulsan 44610, Republic of Korea
E-mail address: jsym@ulsan.ac.kr

Faculty of Electrical and Electronics Engineering, Ulsan College West Campus, Ulsan , Republic of Korea
E-mail address: ycho@uc.ac.kr

Riemann-Liouville fractional Hermite-Hadamard inequalities for h-preinvex functions

Tingsong Du[1*] Shanhe Wu[2] Shoujiang Zhao[1] Muhammad Uzair Awan[3]

[1] Department of Mathematics, College of Science, China Three Gorges University, Yichang 443002, Hubei, P. R. China.
[2] Department of Mathematics, Longyan University, Longyan 364012, Fujian, P. R. China.
[3] Department of Mathematics, Government College University, Faisalabad, Pakistan.
[*] Corresponding author E-mail: tingsongdu@ctgu.edu.cn

Abstract

In this paper, we first prove two Hermite-Hadamard type inequalities for h-preinvex functions via Riemann-Liouville fractional integrals, and then, by introducing an integral identity including the second order derivatives of a given function, we establish some Hermite-Hadamard type inequalities for functions whose second order derivatives are h-preinvex via Riemann-Liouville fractional integrals.

2010 Mathematics Subject Classification: Primary 26A33; 26A51; Secondary 26D07, 26D20, 41A55.

Key words and phrases: Hermite-Hadamard's inequality; h-preinvex functions; Riemann-Liouville fractional integrals.

1 Introduction

In [27], Sarikaya et al. considered Hermite-Hadamard type inequalities involving Riemann-Liouville fractional integrals and established the following interesting inequalities.

Theorem 1.1 *Let $f : [u, v] \to \mathbb{R}$ be a positive function with $0 \leq u < v$ and let $f \in L^1[u, v]$. Suppose f is a convex function on $[u, v]$, then the following inequalities for fractional integrals hold:*

$$f\Big(\frac{u+v}{2}\Big) \leq \frac{\Gamma(\alpha+1)}{2(v-u)^\alpha}[J^\alpha_{u^+}f(v) + J^\alpha_{v^-}f(u)] \leq \frac{f(u)+f(v)}{2}, \qquad (1.1)$$

where the symbol $J^{\alpha}_{u^+}f$ and $J^{\alpha}_{v^-}f$ denote respectively the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \in \mathbb{R}^+$ defined by

$$J^{\alpha}_{u^+}f(x) = \frac{1}{\Gamma(\alpha)} \int_u^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \quad u < x$$

and

$$J^{\alpha}_{v^-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^v (t-x)^{\alpha-1} f(t)\mathrm{d}t, \quad x < v.$$

Here, $\Gamma(\alpha)$ is the Gamma function and its definition is

$$\Gamma(\alpha) = \int_0^{\infty} e^{-\mu}\mu^{\alpha-1}\mathrm{d}\mu.$$

We observe that, for $\alpha = 1$, the inequality (1.1) can be reduced to the following termed Hermite-Hadamard inequality

$$f\left(\frac{u+v}{2}\right) \le \frac{1}{v-u}\int_u^v f(x)\mathrm{d}x \le \frac{f(u)+f(v)}{2}, \tag{1.2}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping on the interval I of real numbers and $u, v \in I$ with $u < v$.

In recent years, many researchers have studied error estimations with respect to the inequality (1.2); for refinements, counterparts, generalization please refer to [7, 16, 17, 22, 35–37].

We evoke, now, some basic definitions as follows.

Definition 1.1 *([34]) A set $S \subseteq \mathbb{R}^n$ is said to be invex set with respect to the mapping $\eta : S \times S \to \mathbb{R}^n$, if $x + t\eta(y, x) \in S$ for every $x, y \in S$ and $t \in [0, 1]$. The invex set S is also termed an η-connected set.*

Definition 1.2 *([34]) A function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex respecting η, if*

$$f\big(x + t\eta(y, x)\big) \le (1-t)f(x) + tf(y), \forall x, y \in K, t \in [0, 1]. \tag{1.3}$$

The function f is said to be preincave if and only if $-f$ is preinvex.

Very recently, some new generalizations of integral inequalities in connection with the preinvexity were explored by Du, Liao and Li [6], Hussain and Qaisar [8], Latif and Dragomir [14], Li and Du [15], respectively.

Definition 1.3 *([31]) Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function, $h \not\equiv 0$. We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is h-convex, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$ and $t \in (0, 1)$, one has*

$$f\big(tx + (1-t)y\big) \le h(t)f(x) + h(1-t)f(y). \tag{1.4}$$

If inequality (1.4) is reversed, then f is called h-concave, i.e. $f \in SV(h, I)$.

If $h(t) = t$, then any non-negative convex mapping belongs to $SX(h, I)$ and each non-negative concave mapping belongs to $SV(h, I)$; if $h(t) = \frac{1}{t}$, then $SX(h, I) = Q(I)$; if $h(t) = 1$, then $SX(h, I) \supseteq P(I)$; and if $h(t) = t^s$ for $s \in (0, 1]$, then $SX(h, I) \supseteq K_s^2$.

Definition 1.4 *([19]) Let $h : [0, 1] \to \mathbb{R}$ be a non-negative function and $h \neq 0$. The function f on the invex set X is said to be h-preinvex with respect to η, if*

$$f\big(x + t\eta(y, x)\big) \leq h(1-t)f(x) + h(t)f(y) \tag{1.5}$$

for each $x, y \in X$ and $t \in [0, 1]$ where $f(\cdot) > 0$.

Definition 1.5 *([31]) We say that $h : J \subseteq \mathbb{R} \to \mathbb{R}$ is a super-multiplicative function, if for any $x, y \in J$ with $xy \in J$, one has*

$$h(xy) \geq h(x)h(y).$$

Definition 1.6 *([1]) We say that $h : J \subseteq \mathbb{R} \to \mathbb{R}$ is a super-additive function, if for every $x, y \in J$ with $x + y \in J$, one has*

$$h(x + y) \geq h(x) + h(y).$$

In order to prove some of our results in the present paper, we need the following Condition C given by Mohan and Neogy in [24].

Condition C: Let $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, we say that the mapping η satisfies the condition C if for any $x, y \in \mathbb{R}^n$,

(C_1) $\quad \eta\big(x, x + t\eta(y, x)\big) = -t\eta(y, x)$,

(C_2) $\quad \eta\big(y, x + t\eta(y, x)\big) = (1-t)\eta(y, x)$,

for all $t \in [0, 1]$.

Note that for any $x, y \in \mathbb{R}^n$, $t_1, t_2 \in [0, 1]$ and from Condition C, we can deduce

$$\eta\Big(x + t_2\eta(y, x), x + t_1\eta(y, x)\Big) = (t_2 - t_1)\eta(y, x).$$

Also, if $t_1 = 0$, we have

$$\eta\Big(x + t_2\eta(y, x), x\Big) = t_2\eta(y, x).$$

In [21], Noor et al. proved the following variant of the Fejér-Hermite-Hadamard inequality type under h-preinvexity.

Theorem 1.2 *Let $f : I \subseteq \mathbb{R} \to (0, \infty)$ be an h-preinvex function with $\eta(v, u) > 0$, $h(\frac{1}{2}) \neq 0$ and let $w : [u; u+\eta(v, u)] \to \mathbb{R}$ is a non-negative, integrable function and symmetric regarding $u + \frac{1}{2}\eta(v, u)$, then recurring to Condition C, we have*

$$\begin{aligned}\frac{1}{2h(\frac{1}{2})} f\Big(\frac{2u + \eta(v, u)}{2}\Big) \int_u^{u+\eta(v,u)} w(x)\mathrm{d}x &\leq \int_u^{u+\eta(v,u)} f(x)w(x)\mathrm{d}x \\ &\leq \frac{f(u) + f(v)}{2}[h(t) + h(1-t)] \int_u^{u+\eta(v,u)} w(x)\mathrm{d}x\end{aligned} \tag{1.6}$$

Corollary 1.1 *In Theorem 1.2, letting $w(x) = 1$ or $\frac{1}{\eta(v,u)}$, we can obtain*

$$\frac{1}{2h(\frac{1}{2})} f\Big(\frac{2u+\eta(v,u)}{2}\Big) \leq \frac{1}{\eta(v,u)} \int\limits_{u}^{u+\eta(v,u)} f(x)\mathrm{d}x \leq \frac{f(u)+f(v)}{2}[h(t)+h(1-t)], \tag{1.7}$$

specially for $\eta(v,u) = v-u$, we can get

$$\frac{1}{2h(\frac{1}{2})} f\Big(\frac{u+v}{2}\Big) \leq \frac{1}{v-u} \int\limits_{u}^{v} f(x)\mathrm{d}x \leq \frac{f(u)+f(v)}{2}[h(t)+h(1-t)]. \tag{1.8}$$

Corollary 1.2 *In Theorem 1.2, if f is an h-preincave function and $w(x) = 1$ or $\frac{1}{\eta(v,u)}$, we can get*

$$\frac{f(u)+f(v)}{2}[h(t)+h(1-t)] \leq \frac{1}{\eta(v,u)} \int\limits_{u}^{u+\eta(v,u)} f(x)\mathrm{d}x \leq \frac{1}{2h(\frac{1}{2})} f\Big(\frac{2u+\eta(v,u)}{2}\Big). \tag{1.9}$$

In some of the recent literatures, Riemann-Liouville fractional Hermite-Hadamard type inequalities are applied widely in the field of analysis, and many new results of fractional Hermite-Hadamard type inequalities are gained based on the original Hermite-Hadamard inequalities for functions of different classes. For example, refer to for convex functions [4, 9–11, 25, 27, 28], for m-convex functions [32, 38] and (s,m)-convex functions [2], for r-convex functions [33], for harmonically convex functions [3, 13], for quasi-geometrically convex functions [12], for GA-s-convex functions [18], for preinvex functions [23, 26], for (α,m)-preinvex functions [5], for h-convex functions [20] and references cited therein.

Motivated and inspired by these results and the recent developments in this area, in the present paper, two Hermite-Hadamard's inequalities for h-preinvex functions via fractional integrals are firstly established and the obtained results of [30] are generalized. Secondly, a second-order new identity for fractional integrals is found. By virtue of this integral identity, we present the left-sided new Hermite-Hadamard type inequalities for h-preinvex functions and h-preincave functions via Riemann-Liouville fractional integrals. Some results proved in this paper can be viewed as generalization of several known results of [29, 30].

2 Main Results

In this section, we are going to prove our main results.

Theorem 2.1 *Let $f : I \subseteq \mathbb{R} \to (0,\infty)$ be an h-preinvex function with $\eta(b,a) > 0$ and $f \in L_1[a,b]$. Then one has the following inequality recurring to Condition*

C via fractional integrals

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{\eta^{\alpha}(b,a)}\Big[J_{a^+}^{\alpha}f\big(a+\eta(b,a)\big)+J_{(a+\eta(b,a))^-}^{\alpha}f(a)\Big]\\
&\le\big[f(a)+f\big(a+\eta(b,a)\big)\big]\int_0^1 t^{\alpha-1}[h(t)+h(1-t)]\mathrm{d}t \qquad (2.1)\\
&\le\frac{2\big[f(a)+f\big(a+\eta(b,a)\big)\big]}{(\alpha p-p+1)^{\frac{1}{p}}}\bigg(\int_0^1\big(h(t)\big)^q\mathrm{d}t\bigg)^{\frac{1}{q}},
\end{aligned}$$

where $p^{-1}+q^{-1}=1$ *with* $p,q>1$.

Proof. Since f is h-preinvex, we have

$$f\big(x+(1-t)\eta(y,x)\big)\le h(t)f(x)+h(1-t)f(y)$$

and

$$f\big(x+t\eta(y,x)\big)\le h(1-t)f(x)+h(t)f(y).$$

By adding these inequalities we can deduce

$$f\big(x+(1-t)\eta(y,x)\big)+f\big(x+t\eta(y,x)\big)\le[h(t)+h(1-t)][f(x)+f(y)]. \qquad (2.2)$$

By employing (2.2) with $x=a$ and $y=a+\eta(b,a)$ we have

$$\begin{aligned}
&f\Big(a+(1-t)\eta\big(a+\eta(b,a),a\big)\Big)+f\Big(a+t\eta\big(a+\eta(b,a),a\big)\Big)\\
&\le\big[h(t)+h(1-t)\big]\big[f(a)+f\big(a+\eta(b,a)\big)\big].
\end{aligned} \qquad (2.3)$$

By making use of Condition C for the left hand side of (2.3), we can get

$$\begin{aligned}
&f\Big(a+(1-t)\eta\big(a+\eta(b,a),a\big)\Big)+f\Big(a+t\eta\big(a+\eta(b,a),a\big)\Big)\\
&=f\big(a+(1-t)\eta(b,a)\big)+f\big(a+t\eta(b,a)\big).
\end{aligned} \qquad (2.4)$$

Utilizing (2.4) in (2.3), we have

$$\begin{aligned}
&f\Big(a+(1-t)\eta(b,a)\Big)+f\Big(a+t\eta(b,a)\Big)\\
&\le\big[h(t)+h(1-t)\big]\Big[f(a)+f\Big(a+\eta(b,a)\Big)\Big].
\end{aligned} \qquad (2.5)$$

Then multiplying both sides of (2.5) by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 t^{\alpha-1}\Big[f\Big(a+(1-t)\eta(b,a)\Big)+f\Big(a+t\eta(b,a),a\Big)\Big]\mathrm{d}t\\
&\le\int_0^1 t^{\alpha-1}\big[h(t)+h(1-t)\big]\big[f(a)+f\big(a+\eta(b,a)\big)\big]\mathrm{d}t
\end{aligned}$$

and

$$\begin{aligned}&\frac{\Gamma(\alpha)}{\eta^{\alpha}(b,a)}\Big[J_{a^+}^{\alpha}f\big(a+\eta(b,a)\big)+J_{(a+\eta(b,a))^-}^{\alpha}f(a)\Big]\\&\le\big[f(a)+f\big(a+\eta(b,a)\big)\big]\int_0^1 t^{\alpha-1}[h(t)+h(1-t)]\mathrm{d}t\end{aligned}\tag{2.6}$$

and therefore the first inequality of (2.1) is proved.

To prove the second inequality in (2.1), by virtue of Hölder's inequality for the right hand side of (2.6), we get

$$\begin{aligned}\int_0^1 t^{\alpha-1}[h(t)+h(1-t)]\mathrm{d}t&\le\Big(\int_0^1\big(t^{\alpha-1}\big)^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1\big(h(t)+h(1-t)\big)^q\mathrm{d}t\Big)^{\frac{1}{q}}\\&=\Big(\frac{1}{\alpha p-p+1}\Big)^{\frac{1}{p}}\Big(\int_0^1\big(h(t)+h(1-t)\big)^q\mathrm{d}t\Big)^{\frac{1}{q}}.\end{aligned}$$

Due to the Minkowski inequality, we may deduce

$$\begin{aligned}&\Big(\frac{1}{\alpha p-p+1}\Big)^{\frac{1}{p}}\Big(\int_0^1\big(h(t)+h(1-t)\big)^q\mathrm{d}t\Big)^{\frac{1}{q}}\\&\le\Big(\frac{1}{\alpha p-p+1}\Big)^{\frac{1}{p}}\Big[\Big(\int_0^1\big(h(t)\big)^q\mathrm{d}t\Big)^{\frac{1}{q}}+\Big(\int_0^1\big(h(1-t)\big)^q\mathrm{d}t\Big)^{\frac{1}{q}}\Big]\\&=\frac{2}{(\alpha p-p+1)^{\frac{1}{p}}}\Big(\int_0^1\big(h(t)\big)^q\mathrm{d}t\Big)^{\frac{1}{q}},\end{aligned}$$

where the proof is completed.

We point out, now, some special cases of Theorem 2.1.

Corollary 2.1 *Letting $\eta(b,a)=b-a$, the inequalities (2.1) reduce to Theorem 2.1 given by Tunç in [30, Page 561].*

Corollary 2.2 *In Theorem 2.1, letting $\alpha=1$, we can deduce*

$$\begin{aligned}\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x&\le\Big[f(a)+f\big(a+\eta(b,a)\big)\Big]\int_0^1 h(t)\mathrm{d}t\\&\le\Big[f(a)+f\big(a+\eta(b,a)\big)\Big]\Big(\int_0^1\big(h(t)\big)^q\mathrm{d}t\Big)^{\frac{1}{q}}\end{aligned}$$

for h-preinvex functions.

Corollary 2.3 *(1) If we take $h(t)=t$ in Corollary 2.2, we get*

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\le\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}\le\frac{f(a)+f\big(a+\eta(b,a)\big)}{(q+1)^{\frac{1}{q}}}$$

6

for preinvex functions.
(2) If we take $h(t) = 1$ in Corollary 2.2, we have

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \le f(a) + f\big(a+\eta(b,a)\big)$$

for P-preinvex functions.
(3) If we take $h(t) = t^s$ in Corollary 2.2, we also obtain

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \le \frac{f(a) + f\big(a+\eta(b,a)\big)}{s+1} \le \frac{f(a) + f\big(a+\eta(b,a)\big)}{(sq+1)^{\frac{1}{q}}}$$

for s-preinvex functions in the second sense with $s \in (0,1]$.

Theorem 2.2 *Let $f : I \to (0,\infty)$ be an h-preinvex function with $\eta(b,a) > 0$, h be super-additive on I and $f \in L_1[a,b]$, $h \in L_1[0,1]$. Then one has inequality for h-preinnvex functions via fractional integrals*

$$\frac{\Gamma(\alpha)}{\eta^\alpha(b,a)}\Big[J^\alpha_{a^+} f\big(a+\eta(b,a)\big) + J^\alpha_{(a+\eta(b,a))^-} f(a)\Big] \le \frac{h(1)}{\alpha}\big[f(a) + f\big(a+\eta(b,a)\big)\big]. \tag{2.7}$$

Proof. Since f is h-preinvex and h is super-additive, by virtue of (2.5), we have

$$\begin{aligned} f\big(a+(1-t)\eta(b,a)\big) + f\big(a+t\eta(b,a)\big) &\le \big[h(t)+h(1-t)\big]\big[f(a) + f\big(a+\eta(b,a)\big)\big] \\ &\le h(1)\big[f(a) + f\big(a+\eta(b,a)\big)\big]. \end{aligned} \tag{2.8}$$

Then multiplying both sides of (2.8) by $t^{\alpha-1}$ and integrating the resulting inequality respecting t over $[0,1]$, we get

$$\begin{aligned} &\int_0^1 t^{\alpha-1}\Big[f\big(a+(1-t)\eta(b,a)\big) + f\big(a+t\eta(b,a)\big)\Big]\mathrm{d}t \\ &\le \int_0^1 t^{\alpha-1}h(1)\Big[f(a) + f\big(a+\eta(b,a)\big)\Big]\mathrm{d}t \end{aligned}$$

and

$$\begin{aligned} &\frac{\Gamma(\alpha)}{\eta^\alpha(b,a)}\Big[J^\alpha_{a^+} f\big(a+\eta(b,a)\big) + J^\alpha_{(a+\eta(b,a))^-} f(a)\Big] \\ &\le h(1)\Big[f(a) + f\big(a+\eta(b,a)\big)\Big]\int_0^1 t^{\alpha-1}\mathrm{d}t, \end{aligned} \tag{2.9}$$

which completes the proof.

As special cases, we provide the following results for the Theorem 2.2.

Corollary 2.4 *Letting $\eta(b,a) = b-a$, the inequality* (2.7) *reduces to Theorem* 2.4 *proven by Tunç in [30, Page 562].*

Corollary 2.5 *If we take $\alpha = 1$ with $\eta(b,a) = b - a$ in Theorem 2.2, then the inequality (2.7) becomes a special version of right hand side of (1.8).*

We prove, next, some Hermite-Hadamard type inequalities for mappings whose derivatives are differentiable h-preinvex via fractional integrals. To do this, we present the following lemma.

Lemma 2.1 *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $\eta(b,a) > 0$. Suppose that $f : A \to \mathbb{R}$ be a twice differentiable function on A. If $f'' \in L[a, a+\eta(b,a)]$, then the following identity for Riemann-Liouville fractional integrals with $\alpha > 0$ holds:*

$$\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a)) + J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big] - f\Big(\frac{2a+\eta(b,a)}{2}\Big)$$
$$= \frac{\eta^2(b,a)}{2}\Big[\int_0^{\frac{1}{2}}\big(t-\lambda(t)\big)f''\big(a+t\eta(b,a)\big)\mathrm{d}t + \int_{\frac{1}{2}}^{1}\big((1-t)-\lambda(t)\big)f''\big(a+t\eta(b,a)\big)\mathrm{d}t\Big], \tag{2.10}$$

where

$$\lambda(t) = \frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{\alpha+1}. \tag{2.11}$$

Proof. Set

$$I_1 = \frac{\eta^2(b,a)}{2}\int_0^{\frac{1}{2}} tf''\big(a+t\eta(b,a)\big)\mathrm{d}t,$$
$$I_2 = \frac{\eta^2(b,a)}{2}\int_{\frac{1}{2}}^{1}(1-t)f''\big(a+t\eta(b,a)\big)\mathrm{d}t,$$

and

$$I_3 = \frac{\eta^2(b,a)}{2}\int_0^1 \frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{\alpha+1}f''\big(a+t\eta(b,a)\big)\mathrm{d}t.$$

Since $a, b \in A$ and A is an invex set regarding η, for every $t \in [0,1]$, we have $a + t\eta(b,a) \in A$. Integrating by part yields that

$$\begin{aligned} I_1 &= \frac{\eta^2(b,a)}{2}\Big[\frac{1}{\eta(b,a)}tf'\big(a+t\eta(b,a)\big)\Big|_0^{\frac{1}{2}} - \frac{1}{\eta(b,a)}\int_0^{\frac{1}{2}} f'\big(a+t\eta(b,a)\big)\mathrm{d}t\Big] \\ &= \frac{\eta(b,a)}{4}f'\big(\frac{2a+\eta(b,a)}{2}\big) - \frac{1}{2}f\big(a+t\eta(b,a)\big)\Big|_0^{\frac{1}{2}} \\ &= \frac{\eta(b,a)}{4}f'\big(\frac{2a+\eta(b,a)}{2}\big) - \frac{1}{2}\Big[f\Big(\frac{2a+\eta(b,a)}{2}\Big) - f(a)\Big]. \end{aligned}$$

Analogously we also have

$$I_2 = -\frac{\eta(b,a)}{4}f'\big(\frac{2a+\eta(b,a)}{2}\big) + \frac{1}{2}\Big[f(a+\eta(b,a)) - f\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big].$$

8

On the other hand, integrating by part, it yields that

$$\begin{aligned}
I_3 =&\frac{\eta^2(b,a)}{2}\Bigg[\frac{1-t^{\alpha+1}-(1-t)^{\alpha+1}}{(\alpha+1)\eta(b,a)}f'\big(a+t\eta(b,a)\big)\Big|_0^1\\
&-\int_0^1\frac{-(\alpha+1)t^{\alpha}+(\alpha+1)(1-t)^{\alpha}}{(\alpha+1)\eta(b,a)}f'\big(a+t\eta(b,a)\big)\mathrm{d}t\Bigg]\\
=&\frac{\eta^2(b,a)}{2}\Bigg[\frac{t^{\alpha}-(1-t)^{\alpha}}{\eta^2(b,a)}f\big(a+t\eta(b,a)\big)\Big|_0^1\\
&-\int_0^1\frac{\alpha t^{\alpha-1}+\alpha(1-t)^{\alpha-1}}{\eta^2(b,a)}f\big(a+t\eta(b,a)\big)\mathrm{d}t\Bigg]\\
=&\frac{f(a)+f\big(a+t\eta(b,a)\big)}{2}-\frac{\alpha}{2}\Bigg[\int_0^1\Big(t^{\alpha-1}+(1-t)^{\alpha-1}\Big)f\big(a+t\eta(b,a)\big)\mathrm{d}t\Bigg].
\end{aligned}\tag{2.12}$$

Let $u=a+t\eta(b,a)$ and using the reduction formula $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)(\alpha>0)$ for Euler gamma function, we get

$$\frac{\alpha}{2}\int_0^1 t^{\alpha-1}f\big(a+t\eta(b,a)\big)\mathrm{d}t=\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}J^{\alpha}_{(a+\eta(b,a))^-}f(a)\tag{2.13}$$

and similarly we also have

$$\frac{\alpha}{2}\int_0^1(1-t)^{\alpha-1}f\big(a+t\eta(b,a)\big)\mathrm{d}t=\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}J^{\alpha}_{a^+}f\big(a+t\eta(b,a)\big).\tag{2.14}$$

Applying (2.13) and (2.14) to (2.12), we obtain

$$I_3=\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f\big(a+\eta(b,a)\big)+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big].$$

From I_1, I_2 and I_3, it follows that

$$I_1+I_2-I_3=\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big),$$

which is the desired result. The proof of Lemma 2.1 is completed.

Corollary 2.6 *Letting $\eta(b,a)=b-a$, the formula (2.10) reduces to lemma 2.1 given by Zhang and Wang in [38, page 4]. Clearly, the obtained Lemma 2.1 in the present paper is an extension of a result proved by Zhang et al. in [38].*

With the aid of Lemma 2.1, let us begin with our next results involving fractional integral inequalities for h-preinvex functions.

Theorem 2.3 *Let $h: J\subseteq\mathbb{R}\to\mathbb{R}$ ($[0,1]\subseteq J$) be a non-negative function and $h(t)\geq t$ for $0\leq t\leq 1$. Suppose that $f:[a,a+\eta(b,a)]\subseteq[0,\infty)\to\mathbb{R}$ be a twice differentiable mapping on $(a,a+\eta(b,a))$ with $\eta(b,a)>0$ such that f''*

9

$\in L_1[a, a+\eta(b,a)]$. If $|f''|$ is h-preinvex on $[a, a+\eta(b,a)]$, then the following inequality for fractional integrals with $\alpha > 0$ hold:

$$\left|\frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\Big[J^\alpha_{a^+}f\big(a+\eta(b,a)\big)+J^\alpha_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right|$$
$$\le \frac{\eta^2(b,a)}{2}\Big[\int_0^{\frac{1}{2}}\Big(t-\lambda(t)\Big)h(t)\mathrm{d}t+\int_{\frac{1}{2}}^1\Big((1-t)-\lambda(t)\Big)h(t)\mathrm{d}t\Big]\Big(|f''(a)|+|f''(b)|\Big), \tag{2.15}$$

where $\lambda(t)$ is defined by (2.11).

Proof. From Lemma 2.1 and properties of absolute value, we have

$$\left|\frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\Big[J^\alpha_{a^+}f\big(a+\eta(b,a)\big)+J^\alpha_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right|$$
$$\le\frac{\eta^2(b,a)}{2}\left[\int_0^{\frac{1}{2}}\big|t-\lambda(t)\big|\big|f''\big(a+t\eta(b,a)\big)\big|\mathrm{d}t+\int_{\frac{1}{2}}^1\big|(1-t)-\lambda(t)\big|\big|f''\big(a+t\eta(b,a)\big)\big|\mathrm{d}t\right]$$
$$=\frac{\eta^2(b,a)}{2}\left[\int_0^{\frac{1}{2}}t\big|f''\big(a+t\eta(b,a)\big)\big|\mathrm{d}t+\int_{\frac{1}{2}}^1(1-t)\big|f''\big(a+t\eta(b,a)\big)\big|\mathrm{d}t\right.$$
$$\left.-\int_0^1\lambda(t)|f''\big(a+t\eta(b,a)\big)|\mathrm{d}t\right]. \tag{2.16}$$

According to the h-preinvexity of $|f''|$ on $[a, a+\eta(b,a)]$, we have

$$\left|\frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\Big[J^\alpha_{a^+}f\big(a+\eta(b,a)\big)+J^\alpha_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right|$$
$$\le\frac{\eta^2(b,a)}{2}\left\{\int_0^{\frac{1}{2}}t\Big[h(1-t)|f''(a)|+h(t)|f''(b)|\Big]\mathrm{d}t+\int_{\frac{1}{2}}^1(1-t)\Big[h(1-t)|f''(a)|+h(t)|f''(b)|\Big]\mathrm{d}t\right.$$
$$\left.-\int_0^1\lambda(t)\Big[h(1-t)|f''(a)|+h(t)|f''(b)|\Big]\mathrm{d}t\right\}$$
$$=\frac{\eta^2(b,a)}{2}\left\{|f''(a)|\Big[\int_0^{\frac{1}{2}}th(1-t)\mathrm{d}t+\int_{\frac{1}{2}}^1(1-t)h(1-t)\mathrm{d}t-\int_0^1\lambda(t)h(1-t)\mathrm{d}t\Big]\right.$$
$$\left.+|f''(b)|\Big[\int_0^{\frac{1}{2}}th(t)\mathrm{d}t+\int_{\frac{1}{2}}^1(1-t)h(t)\mathrm{d}t-\int_0^1\lambda(t)h(t)\mathrm{d}t\Big]\right\}$$
$$=\frac{\eta^2(b,a)}{2}\Big[\int_0^{\frac{1}{2}}\Big(t-\lambda(t)\Big)h(t)\mathrm{d}t+\int_{\frac{1}{2}}^1\Big((1-t)-\lambda(t)\Big)h(t)\mathrm{d}t\Big]\Big(|f''(a)|+|f''(b)|\Big), \tag{2.17}$$

where we use the following fact that

$$\int_0^{\frac{1}{2}}th(1-t)\mathrm{d}t=\int_{\frac{1}{2}}^1(1-t)h(t)\mathrm{d}t,$$

$$\int_0^{\frac{1}{2}} th(t)\mathrm{d}t = \int_{\frac{1}{2}}^1 (1-t)h(1-t)\mathrm{d}t$$

and

$$\int_0^1 \lambda(t)h(1-t)\mathrm{d}t = \int_0^1 \lambda(t)h(t)\mathrm{d}t.$$

Hence, we obtain the required inequality (2.15). This completes the proof of Theorem 2.3.

Let us discuss some special cases of Theorem 2.3.

Corollary 2.7 *Under conditions of Theorem 2.3, if we choose $h(t) = t$, then (2.15) becomes the following inequality for preinvex functions*

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\Big[J^\alpha_{a^+}f\big(a+\eta(b,a)\big) + J^\alpha_{(a+\eta(b,a))^-}f(a)\Big] - f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right|\\ &\le \frac{\eta^2(b,a)}{16}\Big[\frac{\alpha^2-\alpha+2}{(\alpha+1)(\alpha+2)}\Big]\Big[|f''(a)|+|f''(b)|\Big],\end{aligned}$$

specially for $\alpha = 1$ and $\eta(b,a) = b-a$, we get the following inequality for convex functions

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f\Big(\frac{a+b}{2}\Big)\right| \le \frac{(b-a)^2}{24}\Big[\frac{|f''(a)|+|f''(b)|}{2}\Big],$$

which is is the same as the inequality established by Sarikaya, Saglam, and Yildirim in [29, Theorem 3].
Also, in Theorem 2.3, letting $h(t) = 1$, then we get

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\Big[J^\alpha_{a^+}f\big(a+\eta(b,a)\big) + J^\alpha_{(a+\eta(b,a))^-}f(a)\Big] - f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right|\\ &\le \frac{\eta^2(b,a)}{2}\Big\{\Big[|f''(a)|+|f''(b)|\Big]\Big(\frac{1}{4}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\Big)\Big\}\end{aligned}$$

for P-preinvex functions.

Corollary 2.8 *In given conditions of Theorem 2.3, if we take $h(t) = t^s$, then (2.15) reduces to the following inequality for s-preinvex functions*

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\Big[J^\alpha_{a^+}f\big(a+\eta(b,a)\big) + J^\alpha_{(a+\eta(b,a))^-}f(a)\Big] - f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right|\\ &\le \frac{\eta^2(b,a)}{2}\Big\{\Big[|f''(a)|+|f''(b)|\Big]\Big[\frac{2^{s+1}-1}{2^{s+1}(s+1)(s+2)} - \frac{1}{(s+1)(s+\alpha+2)} + \frac{\beta(s+1,\alpha+2)}{\alpha+1}\Big]\Big\}.\end{aligned}$$

Corollary 2.9 *In Theorem 2.3, if $|f''(x)| \le M$, then (2.15) becomes the following inequality*

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\Big[J^\alpha_{a^+}f\big(a+\eta(b,a)\big) + J^\alpha_{(a+\eta(b,a))^-}f(a)\Big] - f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right|\\ &\le M\eta^2(b,a)\Big[\int_0^{\frac{1}{2}} th(t)\mathrm{d}t + \int_{\frac{1}{2}}^1(1-t)h(t)\mathrm{d}t - \int_0^1\lambda(t)h(t)\mathrm{d}t\Big],\end{aligned}$$

11

where $\lambda(t)$ is defined by (2.11).

Theorem 2.4 *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ ($[0,1] \subseteq J$) be a non-negative and $h(t) \geq t$ for $0 \leq t \leq 1$. Suppose $f : [a, a+\eta(b,a)] \subseteq [0,\infty) \to \mathbb{R}$ be a twice differentiable mapping on $(a, a+\eta(b,a))$ with $\eta(b,a) > 0$ such that h^q, $f'' \in L_1[a, a+\eta(b,a)]$ and $p^{-1} + q^{-1} = 1$ with $p, q > 1$. If $|f''|$ is h-preinvex on $[a, a+\eta(b,a)]$, then the following inequality for fractional integrals with $\alpha > 0$ hold:*

$$
\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a)) + J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big] - f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right| \\
\leq&\frac{\eta^2(b,a)}{2}\Big[|f''(a)|+|f''(b)|\Big]\Bigg\{\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\frac{1}{2}\Big)^{1+\frac{1}{p}}\Bigg[\Big(\int_0^{\frac{1}{2}} h^q(t)\mathrm{d}t\Big)^{\frac{1}{q}} + \Big(\int_{\frac{1}{2}}^1 h^q(t)\mathrm{d}t\Big)^{\frac{1}{q}}\Bigg] \\
&+ \frac{1}{\alpha+1}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\Big(\int_0^1 h^q(t)\mathrm{d}t\Big)^{\frac{1}{q}}\Bigg\}.
\end{aligned}
\tag{2.18}
$$

Proof. Continuing from inequality (2.17) in the proof of Theorem 2.3, using properties of absolute value again, recurring to definition of $\lambda(t)$ and Hölder's inequality, we have

$$
\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a)) + J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big] - f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right| \\
\leq&\frac{\eta^2(b,a)}{2}\Big[|f''(a)|+|f''(b)|\Big] \\
&\times\Bigg\{\Big(\int_0^{\frac{1}{2}} t^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^{\frac{1}{2}} h^q(t)\mathrm{d}t\Big)^{\frac{1}{q}} + \Big(\int_{\frac{1}{2}}^1 (1-t)^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_{\frac{1}{2}}^1 h^q(t)\mathrm{d}t\Big)^{\frac{1}{q}} \\
&+ \frac{1}{\alpha+1}\Big(\int_0^1 \big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\big)^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1 h^q(t)\mathrm{d}t\Big)^{\frac{1}{q}}\Bigg\} \\
\leq&\frac{\eta^2(b,a)}{2}\Big[|f''(a)|+|f''(b)|\Big] \\
&\times\Bigg\{\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\frac{1}{2}\Big)^{1+\frac{1}{p}}\Bigg[\Big(\int_0^{\frac{1}{2}} h^q(t)\mathrm{d}t\Big)^{\frac{1}{q}} + \Big(\int_{\frac{1}{2}}^1 h^q(t)\mathrm{d}t\Big)^{\frac{1}{q}}\Bigg] \\
&+ \frac{1}{\alpha+1}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\Big(\int_0^1 h^q(t)\mathrm{d}t\Big)^{\frac{1}{q}}\Bigg\}.
\end{aligned}
$$

To prove the second inequality above, we use the following fact that

$$
\int_0^{\frac{1}{2}} t^p\mathrm{d}t = \int_{\frac{1}{2}}^1 (1-t)^p\mathrm{d}t = \frac{1}{2^{p+1}(p+1)} \tag{2.19}
$$

and

$$
\int_0^1 \Big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\Big)^p\mathrm{d}t \leq \frac{p\alpha+p-1}{p\alpha+p+1}, \tag{2.20}
$$

where we use the following inequality

$$\big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\big)^p \le 1-t^{p(\alpha+1)}-(1-t)^{p(\alpha+1)}$$

for any $t\in[0,1]$, which follows from

$$(A-B)^p \le A^p - B^p$$

for any $A>B\ge 0$ and $p>1$.
Hence, we can get the desired result (2.18).

We give, now, some special cases of Theorem 2.4.

Corollary 2.10 *In Theorem 2.4, if we take $p=q=2$, then we get*

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right|\\ \le&\frac{\eta^2(b,a)}{2}\Big[|f''(a)|+|f''(b)|\Big]\left\{\frac{1}{2\sqrt{6}}\left[\Big(\int_0^{\frac12}h(t^2)\mathrm{d}t\Big)^{\frac12}+\Big(\int_{\frac12}^{1}h(t^2)\mathrm{d}t\Big)^{\frac12}\right]\right.\\ &\left.+\frac{1}{\alpha+1}\Big(\frac{2\alpha+1}{2\alpha+3}\Big)^{\frac12}\Big(\int_0^1 h(t^2)\mathrm{d}t\Big)^{\frac12}\right\},\end{aligned}$$

where h is super-multiplicative.

Corollary 2.11 *Under assumptions of Theorem 2.4, letting $h(t)=t$, then the inequality (2.18) becomes the following inequality for the preinvex function*

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right|\\ \le&\frac{\eta^2(b,a)}{2}\Big(\frac{1}{q+1}\Big)^{\frac1q}\Big[|f''(a)|+|f''(b)|\Big]\\ &\times\left\{\Big(\frac{1}{p+1}\Big)^{\frac1p}\Big(\frac12\Big)^{1+\frac1p}\left[\Big(\frac12\Big)^{1+\frac1q}+\Big(1-\frac{1}{2^{q+1}}\Big)^{\frac1q}\right]+\frac{1}{\alpha+1}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac1p}\right\},\end{aligned}$$

specially for $\alpha=1$, we get the following inequality for preinvex functions

$$\begin{aligned}&\left|\frac{1}{2\eta(b,a)}\int_a^{a+\eta(b,a)}f(t)\mathrm{d}t-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right|\\ \le&\frac{\eta^2(b,a)}{4}\Big(\frac{1}{q+1}\Big)^{\frac1q}\Big[|f''(a)|+|f''(b)|\Big]\\ &\times\left\{\Big(\frac{1}{2p+2}\Big)^{\frac1p}\left[\Big(\frac12\Big)^{1+\frac1q}+\Big(1-\frac{1}{2^{q+1}}\Big)^{\frac1q}\right]+\Big(\frac{2p-1}{2p+1}\Big)^{\frac1p}\right\}.\end{aligned}$$

13

In Theorem 2.4, if we choose $h(t)=1$, then we get

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right|\\
&\leq\frac{\eta^2(b,a)}{2}\Big[|f''(a)|+|f''(b)|\Big]\left\{\frac{1}{2}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}+\frac{1}{\alpha+1}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\right\},
\end{aligned}$$

for P-preinvex functions.

Corollary 2.12 *In Theorem 2.4, if we take $h(t)=t^s$, then the inequality (2.18) becomes the following inequality for s-preinvex functions*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right|\\
\leq&\frac{\eta^2(b,a)}{2}\Big(\frac{1}{sq+1}\Big)^{\frac{1}{q}}\Big[|f''(a)|+|f''(b)|\Big]\\
&\times\left\{\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\frac{1}{2}\Big)^{1+\frac{1}{p}}\left[\Big(\frac{1}{2}\Big)^{s+\frac{1}{q}}+\Big(1-\frac{1}{2^{sq+1}}\Big)^{\frac{1}{q}}\right]+\frac{1}{\alpha+1}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\right\}.
\end{aligned}$$

Theorem 2.5 *Let $h:J\subseteq\mathbb{R}\to\mathbb{R}$ ($[0,1]\subseteq J$) be a non-negative and super-multiplicative function, $h(t)\geq t$ for $0\leq t\leq 1$. Assume that $f:[a,a+\eta(b,a)]\subseteq[0,\infty)\to\mathbb{R}$ be a twice differentiable mapping on $(a,a+\eta(b,a))$ with $\eta(b,a)>0$ such that $f''\in L_1[a,a+\eta(b,a)]$. If $|f''|^q$ is h-preinvex on $[a,a+\eta(b,a)]$, $q\geq 1$ and $|f''(x)|\leq M$, $x\in[a,a+\eta(b,a)]$, then the following inequalities for fractional integrals with $\alpha>0$ hold:*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\big(\frac{2a+\eta(b,a)}{2}\big)\right|\\
\leq&\frac{M\eta^2(b,a)}{2}\left\{\Big(\frac{1}{8}\Big)^{1-\frac{1}{q}}\left[\Big(\int_0^{\frac{1}{2}}t[h(t)+h(1-t)]\mathrm{d}t\Big)^{\frac{1}{q}}+\Big(\int_{\frac{1}{2}}^1(1-t)[h(t)+h(1-t)]\mathrm{d}t\Big)^{\frac{1}{q}}\right]\right.\\
&\left.+\frac{2^{q\alpha}-1}{2^{q\alpha}(\alpha+1)}\Big(\int_0^1\big[h(t)+h(1-t)\big]\mathrm{d}t\Big)^{\frac{1}{q}}\right\} \qquad (2.21)\\
\leq&\frac{M\eta^2(b,a)}{2}\left\{\Big(\frac{1}{8}\Big)^{1-\frac{1}{q}}\left[\Big(\int_0^{\frac{1}{2}}[h(t^2)+h(t-t^2)]\mathrm{d}t\Big)^{\frac{1}{q}}+\Big(\int_{\frac{1}{2}}^1\Big[h(t-t^2)+h\Big((1-t)^2\Big)\Big]\mathrm{d}t\Big)^{\frac{1}{q}}\right]\right.\\
&\left.+\frac{2^{q\alpha}-1}{2^{q\alpha}(\alpha+1)}\Big(\int_0^1\big[h(t)+h(1-t)\big]\mathrm{d}t\Big)^{\frac{1}{q}}\right\}. \qquad (2.22)
\end{aligned}$$

Proof. Continuing from inequality (2.16) in the proof of Theorem 2.3, using properties of absolute value again, recurring to definition of $\lambda(t)$ and power

14

mean inequality for $q \geq 1$, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\Big[J^\alpha_{a^+}f(a+\eta(b,a))+J^\alpha_{(a+\eta(b,a))^-}f(a)\Big]-f\big(\frac{2a+\eta(b,a)}{2}\big)\right|\\
\leq&\frac{\eta^2(b,a)}{2}\Big\{\Big(\int_0^{\frac12}t\mathrm{d}t\Big)^{1-\frac1q}\Big(\int_0^{\frac12}t\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\Big)^{\frac1q}\\
&+\Big(\int_{\frac12}^1(1-t)\mathrm{d}t\Big)^{1-\frac1q}\Big(\int_{\frac12}^1(1-t)\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\Big)^{\frac1q}\\
&+\frac{1}{\alpha+1}\bigg(\int_0^1 1\mathrm{d}t\bigg)^{1-\frac1q}\bigg[\int_0^1\big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\big)^q\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\bigg]^{\frac1q}\bigg\}.
\end{aligned}$$

According to the h-preinvexity of $|f''|^q$ and $|f''|\leq M$, we get

$$\begin{aligned}
\int_0^{\frac12}t\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\leq&\int_0^{\frac12}t\Big[h(1-t)|f''(a)|^q+h(t)|f''(b)|^q\Big]\mathrm{d}t\\
\leq&M^q\int_0^{\frac12}t[h(t)+h(1-t)]\mathrm{d}t.
\end{aligned}$$

Similarly, we also have

$$\int_{\frac12}^1(1-t)\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\leq M^q\int_{\frac12}^1(1-t)[h(t)+h(1-t)]\mathrm{d}t$$

and

$$\begin{aligned}
&\int_0^1\Big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\Big)^q\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\\
\leq&\Big(1-\frac{1}{2^{q\alpha}}\Big)M^q\int_0^1[h(t)+h(1-t)]\mathrm{d}t,
\end{aligned}$$

where we use the fact that

$$\big(1-(1-t)^{\alpha+1}-t^{\alpha+1}\big)^q\leq1-\big[(1-t)^{\alpha+1}+t^{\alpha+1}\big]^q\leq1-(2^{-\alpha})^q=1-\frac{1}{2^{q\alpha}}$$

for any $t\in[0,1]$ with $q\geq1$.
Also

$$\int_0^{\frac12}t\mathrm{d}t=\int_{\frac12}^1(1-t)\mathrm{d}t=\frac18.$$

Using these results, we see that the inequality (2.21) is proved. To prove (2.22), and using the additional properties of h in hypothetical conditions, we further have

$$\int_0^{\frac12}t[h(t)+h(1-t)]\mathrm{d}t\leq\int_0^{\frac12}[h(t^2)+h(t-t^2)]\mathrm{d}t$$

15

and

$$\int_{\frac{1}{2}}^{1}(1-t)[h(t)+h(1-t)]\mathrm{d}t \le \int_{\frac{1}{2}}^{1}\Big[h(t-t^2)+h\Big((1-t)^2\Big)\Big]\mathrm{d}t.$$

Hence, the proof of (2.22) is completed.

Elementary calculation yields the following result.

Corollary 2.13 *In Theorem 2.5, if we choose $h(t)=t$, we can obtain*

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right|\\&\le\frac{M\eta^2(b,a)}{2}\Big(\frac{1}{4}+\frac{2^{q\alpha}-1}{2^{q\alpha}(\alpha+1)}\Big),\end{aligned}$$

specially for $h(t)=1$, we get

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right|\\&\le\frac{M\eta^2(b,a)}{2^{1-\frac{1}{q}}}\Big(\frac{1}{4}+\frac{2^{q\alpha}-1}{2^{q\alpha}(\alpha+1)}\Big).\end{aligned}$$

Theorem 2.6 *Let $h: J\subseteq\mathbb{R}\to\mathbb{R}$ ($[0,1]\subseteq J$) be a non-negative and super-additive function, $h(t)\ge t$ for $0\le t\le 1$. Suppose that $f:[a,a+\eta(b,a)]\subseteq[0,\infty)\to\mathbb{R}$ be a twice differentiable mapping on $(a,a+\eta(b,a))$ with $\eta(b,a)>0$ such that $f''\in L_1[a,a+\eta(b,a)]$. If $|f''|^q$ is h-preinvex on $[a,a+\eta(b,a)]$, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $|f''(x)|\le M$, $x\in[a,a+\eta(b,a)]$, then the following inequalities for fractional integrals with $\alpha>0$ hold:*

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right|\\&\le\frac{M\eta^2(b,a)}{2}\Bigg\{\Big(\frac{1}{2}\Big)^{1+\frac{1}{p}}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big[\Big(\int_0^{\frac{1}{2}}[h(t)+h(1-t)]\mathrm{d}t\Big)^{\frac{1}{q}}+\Big(\int_{\frac{1}{2}}^{1}[h(t)+h(1-t)]\mathrm{d}t\Big)^{\frac{1}{q}}\Big]\\&\quad+\frac{1}{\alpha+1}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\Big(\int_0^1[h(t)+h(1-t)]\mathrm{d}t\Big)^{\frac{1}{q}}\Bigg\} \qquad (2.23)\\&\le\frac{M\eta^2(b,a)}{2}\Big[\frac{1}{2}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}h^{\frac{1}{q}}(1)+\frac{1}{\alpha+1}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}h^{\frac{1}{q}}(1)\Big]. \qquad (2.24)\end{aligned}$$

Proof. Continuing from inequality (2.16) in the proof of Theorem 2.3, using properties of absolute value again, recurring to definition of $\lambda(t)$ and Hölder's

16

inequality for $q>1$, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right|\\
\leq&\frac{\eta^2(b,a)}{2}\bigg\{\Big(\int_0^{\frac12}t^p\mathrm{d}t\Big)^{\frac1p}\Big[\int_0^{\frac12}\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\Big]^{\frac1q}\\
&+\Big(\int_{\frac12}^1(1-t)^p\mathrm{d}t\Big)^{\frac1p}\Big[\int_{\frac12}^1\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\Big]^{\frac1q}\\
&+\frac{1}{\alpha+1}\Big[\int_0^1\Big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\Big)^p\mathrm{d}t\Big]^{\frac1p}\Big[\int_0^1\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\Big]^{\frac1q}\bigg\}.
\end{aligned}$$

According to the h-preinvexity of $|f''|^q$ and $|f''|\leq M$, we can get

$$\begin{aligned}
\int_0^{\frac12}\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t&\leq|f''(a)|^q\int_0^{\frac12}h(1-t)\mathrm{d}t+|f''(b)|^q\int_0^{\frac12}h(t)\mathrm{d}t\\
&\leq M^q\int_0^{\frac12}[h(t)+h(1-t)]\mathrm{d}t.
\end{aligned}$$

Similarly we also have

$$\int_{\frac12}^1\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\leq M^q\int_{\frac12}^1[h(t)+h(1-t)]\mathrm{d}t$$

and

$$\int_0^1\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\leq M^q\int_0^1[h(t)+h(1-t)]\mathrm{d}t.$$

By virtue of the above results and the fact (2.19) and the inequality (2.20), we complete the proof of (2.23).
Using the supper-additive property of h in the assumptions, we further have

$$h(t)+h(1-t)\leq h(1).$$

Hence, the proof of (2.24) is completed.

Finally we shall obtain estimate of Riemann-Liouville fractional Hermite-Hadamard inequality for for h-preincave functions.

Theorem 2.7 *Let $h:J\subseteq\mathbb{R}\to\mathbb{R}$ ($[0,1]\subseteq J$) be a non-negative function, $h(t)\geq t$ for $0\leq t\leq 1$ and $f:[a,a+\eta(b,a)]\subseteq[0,\infty)\to\mathbb{R}$ be a twice differentiable mapping on $(a,a+\eta(b,a))$ with $\eta(b,a)>0$ such that $f''\in L_1[a,a+\eta(b,a)]$. If $|f''|^q$ is h-preincave on $[a,a+\eta(b,a)]$, $p,q>1$, $p^{-1}+q^{-1}=1$, then the following inequality for fractional integrals with $\alpha>0$ hold:*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right|\\
&\leq\frac{\eta^2(b,a)}{2}\Big[\Big(\frac{1}{2p+2}\Big)^{\frac1p}+\frac{1}{\alpha+1}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac1p}\Big]\Big[\frac{1}{2h(\frac12)}\Big]^{\frac1q}\Big|f''\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big|
\end{aligned}$$

17

Proof. Continuing from inequality (2.16) in the proof of Theorem 2.3, using properties of absolute value again, recurring to definition of $\lambda(t)$ and Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right|\\
\leq&\frac{\eta^2(b,a)}{2}\bigg\{\Big(\int_0^{\frac{1}{2}}t^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big[\int_0^{\frac{1}{2}}\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\Big]^{\frac{1}{q}}\\
&+\Big(\int_{\frac{1}{2}}^1(1-t)^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big[\int_{\frac{1}{2}}^1\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\Big]^{\frac{1}{q}}\\
&+\frac{1}{\alpha+1}\Big[\int_0^1\Big(1-t^{\alpha+1}-(1-t)^{\alpha+1}\Big)^p\mathrm{d}t\Big]^{\frac{1}{p}}\Big[\int_0^1\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\Big]^{\frac{1}{q}}\bigg\}\\
\leq&\frac{\eta^2(b,a)}{2}\bigg\{\Big(\frac{1}{2^{p+1}(p+1)}\Big)^{\frac{1}{p}}\Big[\Big(\int_0^{\frac{1}{2}}\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\Big)^{\frac{1}{q}}+\Big(\int_{\frac{1}{2}}^1\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\Big]\\
&+\frac{1}{\alpha+1}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\Big[\int_0^1\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\Big]^{\frac{1}{q}}\bigg\}.
\end{aligned}$$

To prove the second inequality above, we here use the fact (2.19) and the inequality(2.20) again.

Also, $|f''|^q$ is h-preincave on $[a,a+\eta(b,a)]$, by inequalities (1.9) we have

$$\int_0^{\frac{1}{2}}\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\leq\int_0^1\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\leq\frac{1}{2h(\frac{1}{2})}\Big|f''\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big|^q.$$

Similarly, we also have

$$\int_{\frac{1}{2}}^1\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\leq\frac{1}{2h(\frac{1}{2})}\Big|f''\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big|^q$$

and

$$\int_0^1\big|f''\big(a+t\eta(b,a)\big)\big|^q\mathrm{d}t\leq\frac{1}{2h(\frac{1}{2})}\Big|f''\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big|^q.$$

Therefore, we can get

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right|\\
\leq&\frac{\eta^2(b,a)}{2}\Big[\Big(\frac{1}{2p+2}\Big)^{\frac{1}{p}}+\frac{1}{\alpha+1}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\Big]\Big[\frac{1}{2h(\frac{1}{2})}\Big]^{\frac{1}{q}}\Big|f''\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big|.
\end{aligned}$$

Direct computation provides the following corollary.

Corollary 2.14 *In given conditions of Theorem 2.7, if we take $h(t)=t$, we obtain the following inequality for the preincave functions*

$$\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\Big]-f\Big(\frac{2a+\eta(b,a)}{2}\Big)\right|\\
\leq&\frac{\eta^2(b,a)}{2}\Big[\Big(\frac{1}{2p+2}\Big)^{\frac{1}{p}}+\frac{1}{\alpha+1}\Big(\frac{p\alpha+p-1}{p\alpha+p+1}\Big)^{\frac{1}{p}}\Big]\Big|f''\Big(\frac{2a+\eta(b,a)}{2}\Big)\Big|.
\end{aligned}$$

Acknowledgements

This work was supported by the National Natural Science Foundation of China (No. 11601267) and the Natural Science Foundation of Fujian Province of China(No. 2016J01023).

References

[1] H. Alzer, A superadditive property of Hadamard's gamma function, *Abh. Math. Semin. Univ. Hambg.*, **79**, no. 1, 11-23, (2009).

[2] G. A. Anastassiou, Generalised fractional Hermite-Hadamard inequalities involving m-convexity and (s, m)–convexity, *Facta Universitatis (Niš) Ser. Math. Inform.*, **28**, 107-126, (2013).

[3] F. Chen, Extensions of the Hermite-Hadamard inequality for harmonically convex functions via fractional integrals, *Appl. Math. Comput.*, **268**, 121-128, (2015).

[4] S. S. Dragomir, M. I. Bhatti, M. Iqbal, and M. Muddassar, Some new Hermite-Hadamard's type fractional integral inequalities, *J. Comput. Anal. Appl.*, **18**, 655-661, (2015).

[5] T. S, Du, J. G. Liao, L. Z. Chen, and M. U. Awan, Properties and Riemann-Liouville fractional Hermite-Hadamard inequalities for the generalized (α, m)-preinvex functions, *J. Inequal. Appl.*, **2016**, Article Number 306, 24 pages, (2016).

[6] T. -S. Du, J. -G. Liao, and Y. -J. Li, Properties and integral inequalities of Hadamard-Simpson type for the generalized (s, m)-preinvex functions, *J. Nonlinear Sci. Appl.*, **9**, 3112-3126, (2016).

[7] T. Du, Y. Li, and Z. Yang, A generalization of Simpson's inequality via differentiable mapping using extended (s, m)-convex functions, *Appl. Math. Comput.*, **293**, 358-369, (2017).

[8] S. Hussain and S. Qaisar, More results on Hermite-Hadamard type inequality through (α, m)-preinvexity, *J. Appl. Anal. Comput.*, **6**, 293-305, (2016).

[9] S. -R. Hwang, K. -L. Tseng, and K. -C. Hsu, New inequalities for fractional integrals and their applications, *Turkish J. Math.*, **40**, 471-486, (2016).

[10] S. -R. Hwang, S. -Y. Yeh, and K. -L. Tseng, Refinements and similar extensions of Hermite-Hadamard inequality for fractional integrals and their applications, *Appl. Math. Comput.*, **249**, 103-113, (2014).

[11] M. Iqbal, M. I. Bhatti, and K. Nazeer, Generalization of inequalities analogous to Hermite-Hadamard inequality via fractional integrals, *Bull. Korean Math. Soc.*, **52**, 707-716, (2015).

[12] İ. İşcan, New general integral inequalities for quasi-geometrically convex functions via fractional integrals, *J. Inequal. Appl.*, **2013**, Article Number 491, 15 pages, (2013).

[13] İ. İşcan and S. Wu, Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals, *Appl. Math. Comput.*, **238**, 237-244, (2014).

[14] M. A. Latif and S. S. Dragomir, Generalization of Hermite-Hadamard type inequalities for n-times differentiable functions which are s-preinvex in the second sense with applications, *Hacet. J. Math. Stat.*, **44**, 839-853, (2015).

[15] Y. J. Li and T. S. Du, A generalization of Simpson type inequality via differentiable functions using extended $(s,m)_\phi$-preinvex functions, *J. Comput. Anal. Appl.*, **22**, 613-632, (2017).

[16] Y. J. Li and T. S. Du, Some Simpson type integral inequalities for functions whose third derivatives are (α,m)-GA-convex functions, *J. Egyptian Math. Soc.*, **24**, no. 2, 175-180, (2016).

[17] Y. J. Li, T. S. Du, and B. Yu, Some new integral inequalities of Hadamard-Simpson type for extended (s,m)-preinvex functions, *Ital. J. Pure Appl. Math.*, **36**, 583-600, (2016).

[18] Y. M. Liao, J. H. Deng, and J. R. Wang, Riemann-Liouville fractional Hermite-Hadamard inequalities. Part II: for twice differentiable geometric-arithmetically s-convex functions, *J. Inequal. Appl.*, **2013**, Article Number 517, 13 pages, (2013).

[19] M. Matłoka, Inequalities for h-preinvex functions, *Appl. Math. Comput.*, **234**, 52-57, (2014).

[20] M. Matłoka, Some inequalities of Hadamard type for mappings whose second derivatives are h-convex via fractional integrals, *J. Fract. Calc. Appl.*, **6**, no. 1, 110-119, (2015).

[21] M. A. Noor, K. I. Noor, M. U. Awan, and J. Li, On Hermite-Hadamard inequalities for h-preinvex functions, *Filomat*, **28**, no. 7, 1463-1474, (2014).

[22] M. A. Noor, K. I. Noor, M. U. Awan, and S. Costache, Some integral inequalities for harmonically h-convex functions, *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.*, **77**, no. 1, 5-16, (2015).

[23] M. A. Noor, K. I. Noor, M. V. Mihai, and M. U. Awan, Fractional Hermite-Hadamard inequalities for some classes of differentiable preinvex functions, *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.*, **78**, no. 3, 163-174, (2016).

[24] S. R. Mohan and S. K. Neogy, On invex sets and preinvex functions, *J. Math. Anal. Appl.*, **189**, 901-908, (1995).

[25] M. E. Özdemir, S. S. Dragomir, and Ç. Yildiz, The Hadamard inequality for convex function via fractional integrals, *Acta Math. Sci. Ser. B Engl. Ed.*, **33B**, 1293-1299, (2013).

[26] S. Qaisar, M. Iqbal, and M. Muddassar, New Hermite-Hadamard's inequalities for preinvex functions via fractional integrals, *J. Comput. Anal. Appl.*, **20**, 1318-1328, (2016).

[27] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Başak, Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. Comput. Modelling*, **57**, 2403-2407, (2013).

[28] M. Z. Sarikaya, H. Filiz, and M. E. Kiris, On some generalized integral inequalities for Riemann-Liouville fractional integrals, *Filomat*, **29**, 1307-1314, (2015).

[29] M. Z. Sarikaya, A. Saglam, and H. Yildirim, New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are convex and quasi-convex, https://arxiv.org/abs/1005.0451v1, (2010).

[30] M. Tunç, On new inequalities for h-convex functions via Riemann-Liouville fractional integration, *Filomat*, **27**, no. 4, 559-565, (2013).

[31] S. Varošanec, On h-convexity, *J. Math. Anal. Appl.*, **326**, no. 1, 303-311, (2007).

[32] J. R. Wang, X. Z. Li, M. Fečkan, and Y. Zhou, Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity, *Appl. Anal.*, **92**, no. 11, 2241-2253, (2013).

[33] J. Wang, J. Deng, and M. Fečkan, Hermite-Hadamard-type inequalities for r-convex functions based on the use of Riemann-Liouville frantional integrals, *Ukrainian Math. J.*, **65**, 193-211, (2013).

[34] T. Weir and B. Mond, Pre-invex functions in multiple objective optimization, *J. Math. Anal. Appl.*, **136**, 29-38, (1988).

[35] S. Wu, On the weighted generalization of the Hermite-Hadamard inequality and its applications, *Rocky Mountain J. Math.*, **39**, 1741-1749, (2009).

[36] S. Wu, L. Debnath, Inequalities for convex sequences and their applications, *Comput. Math. Appl.*, **54** , 525-534, (2007).

[37] S. -H. Wu, B. Sroysang, J. -S. Xie, and Y. -M. Chu, Parametrized inequality of Hermite-Hadamard type for functions whose third derivative absolute values are quasi-convex, *SpringerPlus*, **4** , Article Number 831, 9 pages, (2015).

[38] Y. R. Zhang and J. R. Wang, On some new Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals, *J. Inequal. Appl.*, **2013**, Article Number 220, 27 pages, (2013).

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 25, NO. 2, 2018

TABLE OF CONTENTS, JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 25, NO. 2, 2018

(continued)