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# Locally and globally small Riemann sums and Henstock integral of fuzzy-number-valued functions

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**Abstract** In this paper, we first define and discuss the locally small Riemann sums (*LSRS*) for fuzzy-number-valued functions. In addition the necessary and sufficient conditions have been obtained for a fuzzy-number-valued function which has (*LSRS*), i.e., if a fuzzy-number-valued function is Henstock (*H*) integrable on  $[a, b]$  then it has (*LSRS*) and the converse is always true. Secondly, the globally small Riemann sums (*GSRS*) for fuzzy-number-valued functions is defined and discussed, and the necessary and sufficient conditions have been given for a fuzzy-number-valued function which has (*GSRS*), i.e., if a fuzzy-number-valued function is (*H*) integrable on  $[a, b]$  then it has (*GSRS*) and the converse is always true. Finally, by Egorov's Theorem, we obtain the dominated convergence theorem for globally small Riemann sums (*GSRS*) of fuzzy-number-valued functions.

**Keywords:** Fuzzy numbers; fuzzy integrals; (*H*) integral; (*LSRS*); (*GSRS*).

## 1 Introduction

Since the concept of fuzzy sets was firstly introduced by Zadeh in 1965 [22], it has been studied extensively from many different aspects of the theory and applications, such as fuzzy topology, fuzzy analysis, fuzzy decision making and fuzzy logic, information science and so on. fuzzy integrals of fuzzy-number-valued functions have been studied by many authors from different points of views, including Goetschel [9], Nanda [15], Kaleva [12], Wu [18, 19] and other authors [1, 3, 4, 5, 6, 8]. The locally and globally small Riemann sums have been introduced by many authors from different points of views. In 1986, Schurle characterized the Lebesgue integral in (*LSRS*) (locally small Riemann sums) property [16]. The (*LSRS*) property has been used to characterized the Perron (*P*) integral on  $[a, b]$  [17]. By considering the equivalency between the (*P*) integral and the Henstock-Kurzweil (*HK*) integral, the (*LSRS*) property has been used to characterized the (*HK*) integral on  $[a, b]$  [13].

The (*LSRS*) property brought a research to have global characterization on the Riemann sums of an (*HK*) integrable function on  $[a, b]$ . This research has been done by considering the following fact: Every (*HK*) integrable function on  $[a, b]$  is measurable, however, there is no guarantee the boundedness of the function. A measurable function  $f$  is (*HK*) integrable on  $[a, b]$  depends on it behaves on the set of  $x$  in which  $|f(x)|$  is large, i.e.  $|f(x)| \geq N$  for some  $N$ . This fact has been characterized in (*GSRS*) (globally small Riemann sums) property [13].

The (*GSRS*) property involves one characteristic of the primitive of an (*HK*) integrable function. That is the primitive of the (*HK*) integral on  $[a, b]$  is *ACG\** (generalized strongly absolutely continuous) on  $[a, b]$ . This is not a simple concept.

In 2015, Indrati [11] introduced a countably Lipschitz condition of a function which is simpler than the *ACG\**, and proved that the (*HK*) integrable function or it's primitive could be characterized in countably Lipschitz condition. Also, by considering the characterization of the (*HK*) integral in the (*GSRS*) property, it showed that the relationship between (*GSRS*) property and countably Lipschitz condition of an (*HK*) integrable function on  $[a, b]$ .

In this paper, we first define and discuss the locally small Riemann sums (*LSRS*) for fuzzy-number-valued functions. In addition the necessary and sufficient conditions have been obtained for a fuzzy-number-valued

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function which has (*LSRS*), i.e., if a fuzzy-number-valued function is (*H*) integrable on  $[a, b]$  then it has (*LSRS*) and the converse is always true. Secondly, the globally small Riemann sums (*GSRS*) for fuzzy-number-valued functions is defined and discussed, and the necessary and sufficient conditions have been given for a fuzzy-number-valued function which has (*GSRS*), i.e., if a fuzzy-number-valued function is (*H*) integrable on  $[a, b]$  then it has (*GSRS*) and the converse is always true. Finally, by Egorov's Theorem, we obtain the dominated convergence theorem for globally small Riemann sums (*GSRS*) of fuzzy-number-valued functions.

The paper is organized as follows, in Section 2 we shall review the relevant concepts and properties of fuzzy sets and the definition of (*H*) integrals for fuzzy-number-valued functions. Section 3 is devoted to discussing the locally small Riemann sums (*LSRS*) of fuzzy-number-valued functions. In section 4 we shall investigate the globally small Riemann sums (*GSRS*) of fuzzy-number-valued functions by Egorov's Theorem, we obtain the dominated convergence theorem for globally small Riemann sums (*GSRS*) of fuzzy-number-valued functions. The last section provides Conclusions.

## 2 Preliminaries

**Definition 2.1** [10, 13] Let  $\delta : [a, b] \rightarrow \mathbb{R}^+$  be a positive real-valued function.  $P = \{[x_{i-1}, x_i]; \xi_i\}$  is said to be a  $\delta$ -fine division, if the following conditions are satisfied:

- (1)  $a = x_0 < x_1 < x_2 < \dots < x_n = b$ ;
- (2)  $\xi_i \in [x_{i-1}, x_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)) (i = 1, 2, \dots, n)$ .

For brevity, we write  $P = \{[u, v]; \xi\}$ , where  $[u, v]$  denotes a typical interval in  $P$  and  $\xi$  is the associated point of  $[u, v]$ .

**Definition 2.2** [10, 13] A real-valued function  $f(x)$  is said to be (*H*) integrable to  $G$  on  $[a, b]$  if for every  $\varepsilon > 0$  there is a function  $\delta(\xi) > 0$  such that for any  $\delta$ -fine division  $P = \{[u, v]; \xi\}$  we have

$$\left| \sum_{(P)} f(\xi)(v - u) - G \right| < \varepsilon \tag{2.1}$$

As usual, we write (*RH*)  $\int_a^b f(x)dx = G$  and  $f(x) \in RH[a, b]$ .

For the results about fuzzy number space  $E^1$ . we recall that  $E^1 = \{u : R \rightarrow [0, 1] : u \text{ satisfies (1)-(4) below}\}$ :

- (1)  $u$  is normal, i.e., there exists a  $x_0 \in R$  such that  $u(x_0) = 1$ ;
- (2)  $u$  is a convex fuzzy set, i.e.,  $u(rx + (1 - r)y) \geq \min(u(x), u(y))$ ,  $x, y \in R$ ,  $r \in [0, 1]$ ;
- (3)  $u$  is upper semi-continuous;
- (4)  $cl\{x \in R : u(x) > 0\}$  is compact, where  $clA$  denotes the closure of  $A$ .

For  $0 < r \leq 1$ , denote  $[u]^r = \{x : u(x) \geq r\}$ . Then from (1)-(4), it follows that the  $r$ -level set  $[u]^r$  is a close interval for all  $r \in [0, 1]$  (refer to [2, 7, 9, 12, 14, 20, 21]). We write  $u^r = [u]^r = [u^-_r, u^+_r]$  or  $[u_-(r), u_+(r)]$ .

For  $u, v \in E^1$ ,  $k \in \mathbb{R}$ , the addition and scalar multiplication are defined by the equations:

$$[u + v]^r = [u]^r + [v]^r, \text{ i.e., } u^-_r + v^-_r = [u + v]^-_r \text{ and } u^+_r + v^+_r = [u + v]^+_r;$$

$$[ku]^r = k[u]^r, \text{ i.e., } [ku]^-_r = \min\{ku^-_r, ku^+_r\} \text{ and } [ku]^+_r = \max\{ku^-_r, ku^+_r\},$$

respectively.

Define  $D(u, v) = \sup_{r \in [0, 1]} d([u]^r, [v]^r) = \sup_{r \in [0, 1]} \max\{|u^-_r - v^-_r|, |u^+_r - v^+_r|\}$ , where  $d$  is Hausdorff metric. Furthermore, we write

$$\|\tilde{u}\|_{E^1} = D(\tilde{u}, \tilde{0}) = \sup_{\lambda \in [0, 1]} \max\{|u^-_\lambda|, |u^+_\lambda|\}.$$

Notice that  $\|\cdot\|_{E^1} = D(\cdot, \tilde{0})$  doesn't stands for the norm of  $E^1$ .

For  $u, v \in E^1$ ,  $u \leq v$  means  $u^-_r \leq v^-_r, u^+_r \leq v^+_r$  (see [2, 7, 9, 12, 14, 20, 21]).

Using the results of [2, 7, 9, 12, 14, 20, 21], we recall that:

- (1)  $(E^1, D)$  is a complete metric space,
- (2)  $D(u + w, v + w) = D(u, v)$ ,
- (3)  $D(u + v, w + e) \leq D(u, w) + D(v, e)$ ,
- (4)  $D(ku, kv) = |k|D(u, v)$ ,  $k \in R$ ,

- (5)  $D(u + v, \tilde{0}) \leq D(u, \tilde{0}) + D(v, \tilde{0})$ ,  
(6)  $D(u + v, w) \leq D(u, w) + D(v, \tilde{0})$ , where  $\tilde{0} = \chi_{\{0\}}$  and  $u, v, w, e \in E^1$ .

**Definition 2.3** [18] A fuzzy-number-valued function  $\tilde{f}(x)$  is said to be  $(H)$  integrable to  $\tilde{A} \in E^1$  if for every  $\varepsilon > 0$  there is a function  $\delta(\xi) > 0$  such that for any  $\delta$ -fine division  $P = \{[u, v]; \xi\}$  of  $[a, b]$ , we have

$$D\left(\sum \tilde{f}(\xi)(v - u), \tilde{A}\right) < \varepsilon \quad (2.2)$$

$$(FH) \int_a^b \tilde{f}(x) dx = \tilde{A} \quad \text{and} \quad \tilde{f}(x) \in FH[a, b]. \quad (2.3)$$

**Lemma 2.1** [18] Let  $\tilde{f} : [a, b] \rightarrow E^1$  be a fuzzy-number-valued function. Then  $\tilde{f} \in FH[a, b]$  iff  $f_-^r(x), f_+^r(x) \in H[a, b]$  uniformly for any  $r \in [0, 1]$ , i.e.,  $\delta(\xi)$  in Definition 2.2 is independent of  $r \in [0, 1]$ .

### 3 Locally small Riemann sums and Henstock $(H)$ integral of fuzzy-number-valued functions

In this section, we shall define locally small Riemann sums or in short  $(LSRS)$  and show that it's the necessary and sufficient condition for  $\tilde{f}(x)$  to be Henstock  $(H)$  integrable on  $[a, b]$ .

**Definition 3.1** A fuzzy-number-valued function  $\tilde{f} : [a, b] \rightarrow E^1$  is said to be have locally small Riemann sums or  $(LSRS)$  if for every  $\varepsilon > 0$  there is a  $\delta(\xi) > 0$  such that for every  $t \in [a, b]$ , we have

$$\left\| \sum \tilde{f}(\xi)(v - u) \right\|_{E^1} < \varepsilon, \quad (3.1)$$

whenever  $P = \{[u, v]; \xi\}$  is a  $\delta$ -fine division of an interval  $[r, s] \subset (t - \delta(t), t + \delta(t))$ ,  $t \in [r, s]$  and  $\Sigma$  sums over  $P$ .

If there exists a  $z \in E^1$  such that  $x = y + z$ , then we call  $z$  the  $H$ - difference of  $x$  and  $y$ , denoted by  $x - y$ .

**Lemma 3.1** [18] Let  $\tilde{f} \in FH[a, b]$  and  $\tilde{F}$  be the primitive of  $\tilde{f}(x)$  then  $\tilde{F}$  satisfies the  $H$ - difference.

**Lemma 3.2** (Henstock Lemma). If a fuzzy-number-valued function  $\tilde{f} : [a, b] \rightarrow E^1$  is  $(H)$  integrable on  $[a, b]$  with primitive  $\tilde{F}$ , i.e., for every  $\varepsilon > 0$  there is a positive function  $\delta(\xi) > 0$  such that for any  $\delta$ -fine division  $P = \{[u, v]; \xi\}$  of  $[a, b]$ , we have

$$D\left(\sum \tilde{f}(\xi)(v - u), \sum \tilde{F}(u, v)\right) < \varepsilon. \quad (3.2)$$

Then for any sum of parts  $\sum_1$  from  $\sum$ , we have

$$D\left(\sum_1 \tilde{f}(\xi)(v - u), \sum_1 \tilde{F}(u, v)\right) < \varepsilon. \quad (3.3)$$

The proof is similar to the Theorem 3.7 [13].

**Theorem 3.1** If  $\tilde{f}(x)$  is  $(H)$  integrable on  $[a, b]$  then it has  $LSRS$ .

**Proof** Let  $\tilde{F}$  be the primitive of  $\tilde{f}(x)$ . Given  $\varepsilon > 0$  there is a  $\delta(\xi) > 0$  such that for any  $\delta$ -fine division  $P = \{[u, v]; \xi\}$  of  $[a, b]$ , we have

$$D\left(\sum \tilde{f}(\xi)(v - u), \sum \tilde{F}(u, v)\right) < \varepsilon. \quad (3.4)$$

Where  $\tilde{F}(u, v) = \tilde{F}(v) - \tilde{F}(u)$ . By the continuity of  $\tilde{F}$  at  $\xi$ ,

$$D(\tilde{F}(u), \tilde{F}(v)) < \varepsilon \quad \text{whenever} \quad [u, v] \subset (\xi - \delta(\xi), \xi + \delta(\xi)).$$

Therefore for  $t \in [a, b]$  and any  $\delta$ -fine division  $P = \{[u, v]; \xi\}$  of  $[r, s] \subset (t - \delta(t), t + \delta(t))$ , we have

$$\begin{aligned} \left\| \sum \tilde{f}(\xi)(v - u) \right\|_{E^1} &\leq D\left(\sum \tilde{f}(\xi)(v - u), \sum \tilde{F}(u, v)\right) + D(\tilde{F}(r), \tilde{F}(s)) \\ &< 2\varepsilon. \end{aligned}$$

That is  $\tilde{f}(x)$  has  $LSRS$ .

This completes the proof. □

**Lemma 3.3** [18] (Cauchy criterion). A fuzzy-number-valued function  $\tilde{f} : [a, b] \rightarrow E^1$  is  $(H)$  integrable on  $[a, b]$  iff for every  $\varepsilon > 0$  there is a positive function  $\delta(\xi) > 0$  such that whenever  $P_1 = \{[u_1, v_1]; \xi_1\}$ ,  $P_2 = \{[u_2, v_2]; \xi_2\}$  are two  $\delta$ -fine divisions, we have

$$D\left(\sum_{(P_1)} \tilde{f}(\xi_1)(v_1 - u_1), \sum_{(P_2)} \tilde{f}(\xi_2)(v_2 - u_2)\right) < \varepsilon. \quad (3.5)$$

**Theorem 3.2** If a fuzzy-number-valued function  $\tilde{f} : [a, b] \rightarrow E^1$  has  $LSRS$  on  $[a, b]$  then  $\tilde{f}(x)$  is  $(H)$  integrable on any closed sub-interval  $C \subset (a, b)$ . (Where  $C = [r, s]$ ).

**Proof** A fuzzy-number-valued function  $\tilde{f} : [a, b] \rightarrow E^1$  has  $LSRS$  means that for every  $\varepsilon > 0$  there is a  $\delta(\xi) > 0$  such that for every  $t \in [a, b]$ , we have

$$\left\| \sum \tilde{f}(\xi)(v - u) \right\|_{E^1} < \varepsilon, \quad (3.6)$$

whenever  $P = \{[u, v]; \xi\}$  is a  $\delta$ -fine division of an interval  $C \subset (t - \delta(t), t + \delta(t))$ ,  $t \in C$  and  $\Sigma$  sums over  $P$ .

(i) If there  $t \in [a, b]$  with  $C \subset (t - \delta(t), t + \delta(t))$  we have the following discussion:

(1) If  $t \in C$  then for every  $\varepsilon > 0$  there is a two  $\delta$ -fine divisions  $P_1 = \{[u_1, v_1]; \xi_1\}$ ,  $P_2 = \{[u_2, v_2]; \xi_2\}$  on  $C$ , such that

$$D\left(\sum_{(P_1)} \tilde{f}(\xi_1)(v_1 - u_1), \sum_{(P_2)} \tilde{f}(\xi_2)(v_2 - u_2)\right) < \varepsilon. \quad (3.7)$$

According to the Cauchy criterion, then  $\tilde{f}(x)$  is  $(H)$  integrable on  $C$ .

(2) If  $t \notin C$  then there is a closed interval  $E \subset (t - \delta(t), t + \delta(t))$ , with the result that  $t \in E$  and  $C \subset E$  (where  $E = [g, h]$ ). As a result, for every  $\varepsilon > 0$  there is a two  $\delta$ -fine divisions  $P_1 = \{[u_1, v_1]; \xi_1\}$ ,  $P_2 = \{[u_2, v_2]; \xi_2\}$  on  $E$ , such that

$$D\left(\sum_{(P_1)} \tilde{f}(\xi_1)(v_1 - u_1), \sum_{(P_2)} \tilde{f}(\xi_2)(v_2 - u_2)\right) < \varepsilon. \quad (3.8)$$

According to the Cauchy criterion, then  $\tilde{f}(x)$  is  $(H)$  integrable on  $E$ . Because  $C \subset E$  and  $\tilde{f}(x)$  is  $(H)$  integrable on  $E$  then  $\tilde{f}(x)$  is  $(H)$  integrable on  $C$ .

(ii) If  $C \not\subseteq (t - \delta(t), t + \delta(t))$  then there is a positive function  $\delta$  on  $[a, b]$  which resulted in the presence that  $P = \{(C_i, t_i) : i = 1, 2, \dots, k\}$  is a  $\delta$ -fine division of the interval  $C$ . It follows that  $\tilde{f}(x)$  is  $(H)$  integrable on  $C_i$  for  $i = 1, 2, \dots, k$ .

Then  $\tilde{f}(x)$  is  $(H)$  integrable on  $C$ .

This completes the proof. □

**Corollary 3.1** If a fuzzy-number-valued function  $\tilde{f} : [a, b] \rightarrow E^1$  has  $LSRS$  on  $[a, b]$  then  $\tilde{f}(x)$  is  $(H)$  integrable on  $C$  for any simple set  $C \subset (a, b)$ .

Notice that a simple set  $C$  means that there exists finite closed sub-interval  $C_i$  which belongs to  $(a, b)$  such that  $C = \bigcup_{i=1}^k C_i$ .

**Theorem 3.3** If a fuzzy-number-valued function  $\tilde{f} : [a, b] \rightarrow E^1$  has  $LSRS$  on  $[a, b]$  then  $\tilde{f}(x)$  is  $(H)$  integrable on  $[a, b]$ .

**Proof** A fuzzy-number-valued function  $\tilde{f} : [a, b] \rightarrow E^1$  has  $LSRS$  then for every  $\varepsilon > 0$  there is  $\delta^*(\xi) > 0$  such that for every  $t \in [a, b]$ , we have

$$\left\| \sum \tilde{f}(\xi)(v - u) \right\|_{E^1} < \varepsilon, \quad (3.9)$$

whenever  $P = \{[u, v]; \xi\}$  is a  $\delta^*$ -fine division of an interval  $C \subset (t - \delta(t), t + \delta(t))$ ,  $t \in C$  and  $\Sigma$  sums over  $P$ . According to the Corollary 3.1,  $\tilde{f}(x)$  is  $(H)$  integrable on  $C$  for any simple set  $C \subset (a, b)$ .

Rows set  $\{E_i\}$ ,  $E_i \cap E_j = \phi$ ,  $\forall i \neq j$  with property  $(a, b) = \bigcup E_i$ ,  $E_i$  is a closed interval. Thus for above  $\varepsilon > 0$ , there is a positive numbers  $n_0$  with property

$$\mu\{[a, b] - \bigcup_{i \leq n_0} E_i\} < \varepsilon, \quad (3.10)$$

where  $\mu$  is Lebesgue measure.

For any  $i$ , there is a positive function  $\delta_i$  such that for any  $\delta_i$ -fine division on  $E_i$ , we have

$$D(\sum \tilde{f}(\xi)(v-u), (H) \int_{E_i} \tilde{f}(x)dx) < \varepsilon. \tag{3.11}$$

Define a positive function  $\delta$  by the formula:

$$\delta(\xi) = \begin{cases} \min\{\delta^*(\xi), \frac{1}{2}d(\xi, \partial[a, b])\} & \text{if } \xi \in \bigcup_{i > n_0} E_i, \\ \min\{\delta^*(\xi), \delta_i(\xi)\}, & \text{if } \xi \in \bigcup_{i \leq n_0} E_i. \end{cases}$$

For each  $\mathcal{C} = \{C\} = \{C_1, C_2, \dots, C_k\}$  with  $C_j = E_i \cap Q$  (where  $Q = [u, v]$ ), for one  $i \leq n_0$  and one  $Q$  with  $\{[u, v]; \xi\}$  is a  $\delta$ -fine division and  $\xi \in (a, b)$ , we have

(i) If  $C_j = E_i$  for  $i \leq n_0$ . Because  $\tilde{f}(x)$  is  $(H)$  integrable on  $E_i$  and  $\tilde{f}(x)$  is  $(H)$  integrable on  $C_j$  consequently  $\tilde{f}(x)$  is  $(H)$  integrable on  $\bigcup_{j=1}^k C_j$ . Selected a positive function  $\delta_*$  with  $\delta_*(\xi) = \min\{\delta_j(\xi) : j = 1, 2, \dots, k\}$ , then for each  $\delta_*$ -fine division  $P = \{[u, v]; \xi\}$  on  $\bigcup_{j=1}^k C_j$ , we have

$$D((H) \int_{\bigcup_{j=1}^k C_j} \tilde{f}(x)dx, \sum \tilde{f}(\xi)(v-u)) < \varepsilon. \tag{3.12}$$

Thus obtained:

$$\begin{aligned} \|\mathcal{C} \sum (H) \int_C \tilde{f}(x)dx\|_{E^1} &\leq D((H) \int_{\bigcup_{j=1}^k C_j} \tilde{f}(x)dx, \sum \tilde{f}(\xi)(v-u)) + \sum_{j=1}^k \|\sum \tilde{f}(\xi)(v-u)\|_{E^1} \\ &< \varepsilon + k\varepsilon. \end{aligned}$$

According to the properties of Cauchy,  $\tilde{f}(x)$  is  $(H)$  integrable on  $[a, b]$ .

(ii) If  $C_j = E_i \cap Q$ , for  $i \leq n_0$  and one  $\delta$ -fine  $Q$  with  $\{[u, v]; \xi\}$  and  $\xi \in (a, b)$  then  $C_j \subset (\xi - \delta(\xi), \xi + \delta(\xi))$ . According to the Theorem 3.2, then  $\tilde{f}(x)$  is  $(H)$  integrable on  $C_j$ . As the result  $\tilde{f}(x)$  is  $(H)$  integrable on  $\bigcup_{j=1}^k C_j$ . Selected a positive function  $\delta_1$  with property  $\delta_1(\xi) \leq \delta(\xi)$  then for each  $\delta_*$ -fine division  $P = \{[u, v]; \xi\}$  on  $\bigcup_{j=1}^k C_j$ , we have

$$D((H) \int_{\bigcup_{j=1}^k C_j} \tilde{f}(x)dx, \sum \tilde{f}(\xi)(v-u)) < \varepsilon. \tag{3.13}$$

Thus obtained:

$$\begin{aligned} \|\mathcal{C} \sum (H) \int_C \tilde{f}(x)dx\|_{E^1} &\leq D((H) \int_{\bigcup_{j=1}^k C_j} \tilde{f}(x)dx, \sum \tilde{f}(\xi)(v-u)) + \sum_{j=1}^k \|\sum \tilde{f}(\xi)(v-u)\|_{E^1} \\ &< \varepsilon + k\varepsilon. \end{aligned}$$

According to the properties of Cauchy,  $\tilde{f}(x)$  is  $(H)$  integrable on  $[a, b]$ .

This completes the proof. □

**Corollary 3.2** A fuzzy-number-valued function  $\tilde{f} : [a, b] \rightarrow E^1$  has *LSRS* on  $[a, b]$  iff  $\tilde{f}(x)$  is  $(H)$  integrable on  $[a, b]$ .

## 4 Globally small Riemann sums and Henstock $(H)$ integral of fuzzy-number-valued functions

In this section, we shall define globally small Riemann sums or in short *GSRS* and show that it's the necessary and sufficient condition for  $\tilde{f}(x)$  to be Henstock  $(H)$  integrable on  $[a, b]$ .

**Definition 4.1** A fuzzy-number-valued function  $\tilde{f} : [a, b] \rightarrow E^1$  is said to be have globally small Riemann sums or (*GSRS*) if for every  $\varepsilon > 0$  there exists a positive integer  $N$  such that for every  $n \geq N$  there is a  $\delta_n(\xi) > 0$  and for every  $\delta_n$ -fine division  $P = \{[u, v]; \xi\}$  of  $[a, b]$ , we have

$$\left\| \sum_{\|\tilde{f}(\xi)\|_{E^1} > n} \tilde{f}(\xi)(v - u) \right\|_{E^1} < \varepsilon, \quad (4.1)$$

where the  $\sum$  is taken over  $P$  and for which  $\|\tilde{f}(\xi)\|_{E^1} > n$ .

**Theorem 4.1** Let  $\tilde{f}(x)$  be (*H*) integrable to  $\tilde{F}(a, b)$  on  $FH[a, b]$  and  $\tilde{F}_n(a, b)$  the integral of  $\tilde{f}_n(x)$  on  $FH[a, b]$ , where  $\tilde{f}_n(x) = \tilde{f}(x)$  when  $\|\tilde{f}(x)\|_{E^1} \leq n$  and  $\tilde{0}$  otherwise. If  $\tilde{F}_n(a, b) \rightarrow \tilde{F}(a, b)$  as  $n \rightarrow \infty$  then  $\tilde{f}(x)$  has *GSRS*.

**Proof** Given  $\varepsilon > 0$  there is a  $\delta_n(\xi) > 0$  such that for every  $\delta_n$ -fine division  $P = \{[u, v]; \xi\}$  of  $[a, b]$ , we have

$$D(\sum \tilde{f}_n(\xi)(v - u), \tilde{F}_n(a, b)) < \varepsilon, \quad (4.2)$$

where  $(FH) \int_a^b \tilde{f}_n(x) dx = \tilde{F}_n(a, b)$ .

$$D(\sum \tilde{f}(\xi)(v - u), \tilde{F}(a, b)) < \varepsilon, \quad (4.3)$$

where  $(FH) \int_a^b \tilde{f}(x) dx = \tilde{F}(a, b)$ .

Choose  $N$  so that whenever  $n \geq N$

$$D(\tilde{F}_n(a, b), \tilde{F}(a, b)) < \varepsilon. \quad (4.4)$$

Therefore for  $n \geq N$  and  $\delta_n$ -fine division  $P = \{[u, v]; \xi\}$  of  $[a, b]$ , we have

$$\begin{aligned} & \left\| \sum_{\|\tilde{f}(\xi)\|_{E^1} > n} \tilde{f}(\xi)(v - u) \right\|_{E^1} = D(\sum \tilde{f}_n(\xi)(v - u), \sum \tilde{f}(\xi)(v - u)) \\ & \leq D(\sum \tilde{f}_n(\xi)(v - u), \tilde{F}_n(a, b)) + D(\tilde{F}_n(a, b), \tilde{F}(a, b)) + D(\tilde{F}(a, b), \sum \tilde{f}(\xi)(v - u)) \\ & < 3\varepsilon. \end{aligned}$$

Hence  $\tilde{f}(x)$  has *GSRS*.

This completes the proof.  $\square$

**Theorem 4.2** A fuzzy-number-valued function  $\tilde{f}(x)$  has *GSRS* iff  $\tilde{f}(x)$  is (*H*) integrable on  $[a, b]$  and  $\tilde{F}_n(a, b) \rightarrow \tilde{F}(a, b)$  as  $n \rightarrow \infty$  where  $\tilde{F}_n(a, b)$  and  $\tilde{F}(a, b)$  are defined as in Theorem 4.1.

**Proof** Theorem 4.1 proves the sufficiency. We shall prove only the necessity. Suppose  $\tilde{f}(x)$  has *GSRS*. Note that  $\tilde{f}_n(x)$ , as defined in Theorem 4.1, is (*FH*) integrable on  $[a, b]$  for all  $n$ . Then for  $n, m \geq N$  and a suitably chosen  $\delta$ -fine division  $P = \{[u, v]; \xi\}$ , we have

$$\begin{aligned} & D(\tilde{F}_n(a, b), \tilde{F}_m(a, b)) \\ & \leq D(\tilde{F}_n(a, b), \sum_{\|\tilde{f}(\xi)\|_{E^1} \leq n} \tilde{f}(\xi)(v - u)) + D(\sum_{\|\tilde{f}(\xi)\|_{E^1} \leq m} \tilde{f}(\xi)(v - u), \tilde{F}_m(a, b)) \\ & + \left\| \sum_{\|\tilde{f}(\xi)\|_{E^1} > n} \tilde{f}(\xi)(v - u) \right\|_{E^1} + \left\| \sum_{\|\tilde{f}(\xi)\|_{E^1} > m} \tilde{f}(\xi)(v - u) \right\|_{E^1} \\ & < 4\varepsilon. \end{aligned}$$

That is,  $\tilde{F}_n(a, b)$  converge to a fuzzy number, say  $\tilde{F}(a, b)$ , as  $n \rightarrow \infty$ . Again, for suitably chosen  $N$  and  $\delta(\xi)$  and for every  $\delta$ -fine division  $P = \{[u, v]; \xi\}$ , we have

$$\begin{aligned} & D(\sum \tilde{f}(\xi)(v - u), \tilde{F}(a, b)) \leq D(\tilde{F}(a, b), \tilde{F}_N(a, b)) \\ & + D(\tilde{F}_N(a, b), \sum_{\|\tilde{f}(\xi)\|_{E^1} \leq N} \tilde{f}(\xi)(v - u)) + \left\| \sum_{\|\tilde{f}(\xi)\|_{E^1} > N} \tilde{f}(\xi)(v - u) \right\|_{E^1} \\ & < 3\varepsilon. \end{aligned}$$

That is,  $\tilde{f}(x)$  is (*FH*) integrable on  $[a, b]$ .

This completes the proof.  $\square$



**Theorem 4.3** Let  $\tilde{f}_n(x) \in FH[a, b]$ ,  $n = 1, 2, 3 \dots$  and satisfy:

- (1)  $\lim_{n \rightarrow \infty} \tilde{f}_n(x) = \tilde{f}(x)$  almost everywhere in  $[a, b]$ ;
- (2) there exists a Lebesgue ( $L$ ) integrable ( $H$  integrable) function  $h(x)$  on  $[a, b]$  such that

$$D(\tilde{f}_n(x), \tilde{f}_m(x)) < h(x). \quad (4.5)$$

Then,  $\tilde{f}_n(x)$  has *GSRS* on  $[a, b]$  uniformly for any  $n$ . Naturally,  $\tilde{f}$  is (*SFH*) integrable on  $[a, b]$ . Furthermore,

$$\lim_{n \rightarrow \infty} (SFH) \int_a^b \tilde{f}_n(x) dx = (SFH) \int_a^b \tilde{f}(x) dx. \quad (4.6)$$

**Proof** Let  $\varepsilon > 0$ . Since  $H(x) = (L) \int_a^x h(t) dt$  is absolutely continuous on  $[a, b]$ , there exists a positive number  $\eta > 0$  such that  $\sum |H(b_i) - H(a_i)| < \varepsilon$  whenever  $\{[a_i, b_i]\}$  is a finite collection of non-overlapping intervals in  $[a, b]$  that satisfy  $\sum (b_i - a_i) < \eta$ . Since  $\lim_{n \rightarrow \infty} \tilde{f}_n(x) = \tilde{f}(x)$  almost everywhere in  $[a, b]$ , and

$$\begin{aligned} D(\tilde{f}_n, \tilde{f}) &= \sup_{r \in [0, 1]} \max\{|(f_n(x))_r^- - (f(x))_r^-|, |(f_n(x))_r^+ - (f(x))_r^+|\} \\ &= \sup_{r_k \in [0, 1]} \max\{|(f_n(x))_{r_k}^- - (f(x))_{r_k}^-|, |(f_n(x))_{r_k}^+ - (f(x))_{r_k}^+|\} \end{aligned}$$

is a sequence of Lebesgue ( $L$ ) measurable functions, where  $r_k \in [0, 1]$  is the set of rational numbers, by Egorov's Theorem, there exists an open set  $G$  with  $L(G) < \eta$  such that  $\lim_{n \rightarrow \infty} \tilde{f}_n(x) = \tilde{f}(x)$  uniformly for  $x \in [a, b] \setminus G$ . Then, there is a natural number  $N$ , such that for any  $n, m > N$ , and for any  $x \in [a, b] \setminus G$ , we have  $D(\tilde{f}_n(x), \tilde{f}_m(x)) < \varepsilon$ . Since  $h(x)$  is ( $H$ ) integrable on  $[a, b]$ , there is a  $\delta_h(\xi) > 0$  such that for any  $\delta_h$ -fine division  $P = \{[u, v]; \xi\}$  of  $[a, b]$ , we have

$$\left| \sum h(\xi)(v - u) - (L) \int_a^b h(t) dt \right| < \varepsilon. \quad (4.7)$$

Define

$$\delta(\xi) = \begin{cases} \delta_h(\xi), & \text{if } \xi \in [a, b] \setminus G, \\ \delta(\xi), \text{ satisfying } (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)) \subset G, & \text{if } \xi \in [a, b]. \end{cases}$$

Then, it follows that for a  $\delta$ -fine division  $P_0 = \{[x_{i-1}, x_i]; \xi_i\}$  of  $[a, b]$ ,

$$\begin{aligned} & D\left(\sum \tilde{f}_n(\xi_i)(x_i - x_{i-1}), \sum \tilde{f}_m(\xi_i)(x_i - x_{i-1})\right) \\ & \leq D\left(\sum_{\xi_i \in [a, b] \setminus G} \tilde{f}_n(\xi_i)(x_i - x_{i-1}), \sum_{\xi_i \in [a, b] \setminus G} \tilde{f}_m(\xi_i)(x_i - x_{i-1})\right) \\ & + D\left(\sum_{\xi_i \in G} \tilde{f}_n(\xi_i)(x_i - x_{i-1}), \sum_{\xi_i \in G} \tilde{f}_m(\xi_i)(x_i - x_{i-1})\right) \\ & \leq \sum_{\xi_i \in [a, b] \setminus G} D(\tilde{f}_n(\xi_i), \tilde{f}_m(\xi_i))(x_i - x_{i-1}) + \sum_{\xi_i \in G} D(\tilde{f}_n(\xi_i), \tilde{f}_m(\xi_i))(x_i - x_{i-1}) \\ & < \varepsilon(b - a) + \left| \sum_{\xi_i \in G} h(\xi_i)(x_i - x_{i-1}) - \int_G h(t) dt \right| + \left| \int_G h(t) dt \right| \\ & < \varepsilon(b - a) + 3\varepsilon. \end{aligned}$$

Hence, there is a natural number  $N$  such that for any  $n, m > N$ , we have

$$\begin{aligned} & D(\tilde{F}_n[a, b], \tilde{F}_m[a, b]) \\ & \leq D(\tilde{F}_n[a, b], \sum \tilde{f}_n(\xi_i)(x_i - x_{i-1})) + D(\tilde{F}_m[a, b], \sum \tilde{f}_m(\xi_i)(x_i - x_{i-1})) \\ & + D(\sum \tilde{f}_n(\xi_i)(x_i - x_{i-1}), \sum \tilde{f}_m(\xi_i)(x_i - x_{i-1})) \\ & < 3\varepsilon. \end{aligned}$$

Thus,  $\tilde{F}_n[a, b]$  is a Cauchy sequence, and there is a natural number  $N_1$  such that for any  $n > N_1$ , we have  $D(\tilde{F}_n[a, b], \tilde{A}) < \varepsilon$ . According to the ( $FH$ ) integrability of  $\tilde{f}_{N_1}(x)$ , there is a  $\delta_{N_1}(\xi) > 0$  such that for any  $\delta_{N_1}$ -fine division  $P = \{[u, v]; \xi\}$  of  $[a, b]$ , for any  $n > N_{N_1}$ , we have

$$\begin{aligned} & D\left(\sum \tilde{f}_n(\xi)(v - u), \tilde{F}_n[a, b]\right) \\ & \leq D(\tilde{F}_n[a, b], \tilde{F}_{N_1}[a, b]) + D\left(\sum \tilde{f}_{N_1}(\xi)(v - u), \tilde{F}_{N_1}[a, b]\right) \\ & + D\left(\sum \tilde{f}_n(\xi)(v - u), \sum \tilde{f}_{N_1}(\xi)(v - u)\right) \\ & < 3\varepsilon. \end{aligned}$$

This completes the proof. □

## 5 conclusions

In this paper, we introduced locally and globally small Riemann sums for fuzzy-number-valued functions. We proved that a fuzzy-number-valued functions is  $(H)$  integrable on  $[a, b]$  iff it has  $(LSRS)$ . Also it is proved that a fuzzy-number-valued functions is  $(H)$  integrable on  $[a, b]$  iff it has  $(GSRS)$ . Finally, by Egorov's Theorem, we obtained the dominated convergence theorem for  $(GSRS)$  of fuzzy-number-valued functions.

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**THE GENERAL ITERATIVE METHODS FOR SPLIT VARIATIONAL  
INCLUSION PROBLEM AND FIXED POINT PROBLEM  
IN HILBERT SPACES**

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**Abstract.** In this paper, we consider and analyze a general iterative method to approximate a common solution of split variational inclusion problem and fixed point problem for a nonexpansive mapping, which is the unique solution for the variational inequality in real Hilbert spaces. Furthermore, under reasonable conditions, the sequence generated by the proposed iterative scheme converges strongly to a common solution of split variational inclusion problem and fixed point problem for a nonexpansive mapping, which is a solution of a certain optimization problem related to a strongly positive linear operator. The results presented in this paper improve and extend the corresponding results reported by some authors recently.

**Keywords:** Split variational inclusion problem, Nonexpansive mapping, Fixed point problem, Hilbert space.

**AMS Subject Classification:** 47H10, 47H09.

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1. INTRODUCTION

Throughout the paper unless otherwise stated, let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be two real Hilbert spaces with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\| \cdot \|$ . Let  $C$  and  $Q$  be nonempty closed convex subsets of  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively. A mapping  $S : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  is called contraction, if there exists a constant  $\alpha \in (0, 1)$  such that

$$\|Sx - Sy\| \leq \alpha \|x - y\|, \forall x, y \in \mathcal{H}_1.$$

If  $\alpha = 1$ ,  $S$  is called nonexpansive, that is,

$$\|Sx - Sy\| \leq \|x - y\|, \forall x, y \in \mathcal{H}_1.$$

Further, we consider the the following fixed point problem (in short, FPP) for a nonexpansive mapping  $S : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  : Find  $x \in \mathcal{H}_1$  such that

$$Sx = x. \tag{1.1}$$

The solution set of FPP (1.1) is denoted by  $\text{Fix}(S)$ . It is well known that if  $\text{Fix}(S) \neq \emptyset$ ,  $\text{Fix}(S)$  is closed and convex. Next, let  $T : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a single-valued mapping. We recall the following definitions:

(i)  $T$  is said to be monotone, if

$$\langle Tx - Ty, x - y \rangle \geq 0, \forall x, y \in \mathcal{H}_1.$$

(ii)  $T$  is said to be  $\alpha$ -strongly monotone, if there exists a constant  $\alpha > 0$  such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha \|x - y\|^2, \forall x, y \in \mathcal{H}_1.$$

(iii)  $T$  is said to be  $\beta$ -inverse strongly monotone(or,  $\beta$ -ism), if there exists a constant  $\beta > 0$  such that

$$\langle Tx - Ty, x - y \rangle \geq \beta \|Tx - Ty\|^2, \forall x, y \in \mathcal{H}_1.$$

(iv)  $T$  is said to be firmly nonexpansive, if

$$\langle Tx - Ty, x - y \rangle \geq \|Tx - Ty\|^2, \forall x, y \in \mathcal{H}_1.$$

Next, let  $M : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$  be a multi-valued mappings. We recall the following definitions:

- $M$  is called monotone if for all  $x, y \in \mathcal{H}_1, u \in Mx$  and  $v \in My$  such that  $\langle x - y, u - v \rangle \geq 0$ .

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- A monotone mappings  $M : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$  is maximal if the  $\text{Graph}(M)$  is not properly contained in the graph of any other monotone mapping.

For more precisely, a monotone mappings  $M$  is maximal if and only if for  $(x, u) \in \mathcal{H}_1 \times \mathcal{H}_1, \langle x-y, u-v \rangle \geq 0$ , for every  $(y, v) \in \text{Graph}(M)$  implies that  $u \in Mx$ , where  $\text{Graph}(M) := \{(x, y) \in \mathcal{H}_1 \times \mathcal{H}_1 : y \in Mx\}$ . Let  $M : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$  be a multi-valued mappings. Then, the resolvent mapping associated with  $M$ , is defined by

$$J_\lambda^M(x) := (I + \lambda M)^{-1}(x), \forall x \in \mathcal{H}_1$$

for some  $\lambda > 0$ , where  $I$  stands identity operator on  $\mathcal{H}_1$ . We note that for all  $\lambda > 0$  the resolvent operator  $J_\lambda^M$  is single-valued, nonexpansive and firmly nonexpansive.

For a given single-valued operator  $F : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ , Hartman and Stampacchia [12] introduced the variational inequality problem (in short, VIP) :

$$\text{(VIP)} \quad \begin{cases} \text{Find } x^* \in C \text{ such that} \\ \langle F(x^*), x - x^* \rangle \geq 0, \forall x \in C. \end{cases}$$

The VIP is a powerful tool to investigate and study a wide class of unrelated problems arising in industrial, regional, physical, pure and applied sciences in a unified and general framework. Variational inequalities have been extended and generalised in several direction using novel and new techniques. The following existence result of solutions for VIP can be found in [12]. Let  $\mathcal{H}_1$  be a real Hilbert space,  $C$  a nonempty, compact and convex subset  $\mathcal{H}_1$ . Then, if  $F : C \rightarrow \mathcal{H}_1$  is continuous, there exists  $x^* \in C$  such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \forall x \in C.$$

Recently, in 2011, Moudafi [24] introduced the following split monotone variational inclusion problem (in short, SMVIP):

$$\text{(SMVIP)} \quad \begin{cases} \text{Find } x^* \in \mathcal{H}_1 \text{ such that } 0 \in f_1(x^*) + B_1(x^*), \\ y^* = Ax^* \in \mathcal{H}_2 \text{ solves } 0 \in f_2(y^*) + B_2(y^*). \end{cases}$$

where  $B_1 : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$  is a multi-valued mappings on a Hilbert space  $\mathcal{H}_1$ ,  $B_2 : \mathcal{H}_2 \rightarrow 2^{\mathcal{H}_2}$  is a multi-valued mappings on a Hilbert space  $\mathcal{H}_2$ ,  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  is a bounded linear operator,  $f_1 : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  and  $f_2 : \mathcal{H}_2 \rightarrow \mathcal{H}_2$  are two given single-valued operators. If  $f_1 \equiv 0$  and  $f_2 \equiv 0$ , then SMVIP reduces to the following split variational inclusion problem (in short, SVIP) : Find  $x^* \in \mathcal{H}_1$  such that

$$0 \in B_1(x^*), \tag{1.2}$$

and

$$y^* = Ax^* \in \mathcal{H}_2 \text{ solves } 0 \in B_2(y^*). \tag{1.3}$$

When looked separately, (1.2) is the variational inclusion problem and we denoted its solution set by  $\text{SOLVIP}(B_1)$ . The SVIP (1.2)-(1.3) constitutes a pair of variational inclusion problems which have to be solved so that the image  $y^* = Ax^*$  under a given bounded linear operator  $A$ , of the solution  $x^*$  of SVIP (1.2) in  $\mathcal{H}_1$  is the solution of another SVIP (1.3) in another space  $\mathcal{H}_2$ , we denote the solution set of SVIP (1.3) by  $\text{SOLVIP}(B_2)$ . The solution set of The SVIP (1.2)-(1.3) is denoted by

$$\Gamma : \{x^* \in \mathcal{H}_1 : x^* \in \text{SOLVIP}(B_1) \text{ and } Ax^* \in \text{SOLVIP}(B_2)\}.$$

l,Recently, Byrne et al. [3] studied the weak and strong convergence of the following iterative method for SVIP. For given  $x_0 \in \mathcal{H}_1$ , compute iterative sequence  $\{x_n\}$  generated by the following scheme.

$$x_{n+1} = J_\lambda^{B_1}(x) \left( x_n + \gamma A^* (J_\lambda^{B_2} - I) Ax_n \right), \tag{1.4}$$

for  $\lambda > 0$  and  $A^*$  is the adjoint of  $A, L = \|A^*A\|$  and  $\gamma \in (0, \frac{2}{L})$ . It is proved, in [3], that the sequence  $\{x_n\}$  generated by (1.4) converges strongly to  $x^*$  which is the solution of SVIP.

Very recently, Kazami and Rizvi [13] studied and analyzed the strong convergence of the iterative method for approximating a common solution of SVIP and FPP for a nonexpansive mapping in a real Hilbert space. Let  $g : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a contraction mapping with constant  $\alpha \in (0, 1)$  and  $S : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a nonexpansive mapping. For a given  $x_0 \in \mathcal{H}_1$  arbitrarily, let  $\{u_n\}$  and  $\{x_n\}$  be generated by

$$\begin{aligned} u_n &= J_\lambda^{B_1} \left( x_n + \gamma A^* (J_\lambda^{B_2} - I) Ax_n \right); \\ x_{n+1} &= \alpha_n g(x_n) + (1 - \alpha_n) Su_n, \end{aligned} \tag{1.5}$$

where  $\lambda > 0$  and  $\gamma \in (0, \frac{1}{L})$ ,  $L$  is the spectral radius of the operator  $A^*A$  and  $A^*$  is the adjoint of  $A$  and  $\{\alpha_n\}$  is a sequence in  $(0, 1)$ . They proved that, under some certain conditions imposed on the parameters  $\{\alpha_n\}$ , the sequences  $\{u_n\}$  and  $\{x_n\}$  both converge strongly to  $z \in \text{Fix}(S) \cap \Gamma$ , where  $z = P_{\text{Fix}(S) \cap \Gamma}g(z)$ .

On the other hand, iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, e.g., [11, 27, 28, 29] and the references therein. Convex minimization problems have a great impact and influence in the development of almost all branches of pure and applied sciences. A typical problem is to minimize a quadratic function over the set of the fixed points a nonexpansive mapping on a real Hilbert space:

$$\theta(x) = \min_{x \in C} \frac{1}{2} \langle Gx, x \rangle - \langle x, b \rangle, \tag{1.6}$$

where  $G$  is a linear bounded operator,  $C$  is the fixed point set of a nonexpansive mapping  $T$  and  $b$  is a given point in  $H$ . Let  $H$  be a real Hilbert space. Recall that a linear bounded operator  $B$  is strongly positive if there is a constant  $\bar{\gamma} > 0$  with property

$$\langle Gx, x \rangle \geq \bar{\gamma} \|x\|^2 \text{ for all } x \in H. \tag{1.7}$$

Marino and Xu [25] introduced the following general iterative scheme basing on the viscosity approximation method introduced by Moudafi [14]:

$$x_{n+1} = (I - \alpha_n G)Tx_n + \alpha_n \beta f(x_n), n \geq 0. \tag{1.8}$$

where  $G$  is a strongly positive bounded linear operator on  $H$ . They proved that if the sequence  $\{\alpha_n\}$  of parameters satisfies appropriate conditions, then the sequence  $\{x_n\}$  generated by (1.8) converges strongly to the unique solution of the variational inequality

$$\langle (G - \beta f)x^*, x - x^* \rangle \geq 0, x \in C \tag{1.9}$$

which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2} \langle Gx, x \rangle - h(x),$$

where  $h$  is a potential function for  $\beta f$  (i.e.,  $h'(x) = \beta f(x)$  for  $x \in H$ ).

Motivated by the work of Kazmi and Rizvi [13] and Moudafi [24] and Marino and Xu [25] and by the ongoing research in this direction, we suggest and analyze a general iterative method for approximating a common solution of SVIP and FPP which solves the variational inequality (1.9). More precisely, let  $g : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a contraction mapping with constant  $\alpha \in (0, 1)$ ,  $S : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a nonexpansive mapping and  $G : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a strongly positive, bounded linear operator with constant  $\mu$  and  $0 < \beta < \frac{\mu}{\alpha}$ . For a given  $x_0 \in \mathcal{H}_1$  arbitrarily, let  $\{u_n\}$  and  $\{x_n\}$  generated by

$$\begin{aligned} u_n &= J_\lambda^{B_1} \left( x_n + \gamma A^* (J_\lambda^{B_2} - I)Ax_n \right); \\ x_{n+1} &= \alpha_n \beta f(x_n) + (I - \alpha_n G)Su_n, \end{aligned} \tag{1.10}$$

where  $\lambda > 0$  and  $\gamma \in (0, \frac{1}{L})$ ,  $L$  is the spectral radius of the operator  $A^*A$  and  $A^*$  is the adjoint of  $A$  and  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  and  $B_1 : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$ ,  $B_2 : \mathcal{H}_2 \rightarrow 2^{\mathcal{H}_2}$  two multi-valued mappings on  $\mathcal{H}_1$ , and  $\mathcal{H}_2$ , respectively. We prove that the iterative method (1.10) converges strongly to a common element of SVIP and FPP for a nonexpansive mapping, which is a solution of a certain optimization problem related to a strongly positive linear operator. The result presented in this paper generalize the corresponding results of Kazmi and Rizvi [13] and Moudafi [24], Marino and Xu [25] and many others.

## 2. PRELIMINARIES

For a real Hilbert space  $\mathcal{H}_1$  with the norm  $\|\cdot\|$  and the inner product  $\langle \cdot, \cdot \rangle$ , it is well known that for any  $\lambda \in (0, 1)$ ,

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 - \lambda(1 - \lambda) \|x - y\|^2, \quad \forall x, y \in \mathcal{H}_1. \tag{2.1}$$

Further, every nonexpansive operator  $T : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  satisfies, for all  $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_1$ , the inequality

$$\langle (x - T(x)) - (y - T(y)), T(y) - T(x) \rangle \leq \frac{1}{2} \|(T(x) - x) - (T(y) - y)\|^2 \tag{2.2}$$

and therefore, we get, for all  $(x, y) \in \mathcal{H}_1 \times \text{Fix}(T)$ ,

$$\langle (x - T(x), (y - T(x)) \rangle \leq \frac{1}{2} \|T(x) - x\|^2. \tag{2.3}$$

A mapping  $T : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  is said to be averaged if and only if it can be written as the average of the identity mapping and a nonexpansive mapping, i.e.,

$$T := (1 - \alpha)I + \alpha S$$

where  $\alpha \in (0, 1)$  and  $S : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  is nonexpansive and  $I$  is the identity operator on  $\mathcal{H}_1$

**Proposition 2.1.** (i) If  $T = (1 - \alpha)S + \alpha V$ , where  $S : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  is averaged,  $V : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  is nonexpansive and  $\alpha \in (0, 1)$ , then  $T$  is averaged.

(ii) The composite of finitely many averaged mapping is averaged.

(iii) If the mapping are averaged and have a nonempty common fixed point, then

$$\bigcap_{i=1}^N \text{Fix}(T_i) = \text{Fix}(T_1, T_2, \dots, T_N).$$

(iv) If  $T$  is  $\tau$ -ism, then for  $\gamma > 0$ ,  $\gamma T$  is  $\frac{\tau}{\gamma}$ -ism

(v)  $T$  is averaged if and only if, its complement  $I - T$  is  $\tau$ -ism for some  $\tau > \frac{1}{2}$

For every point  $x \in \mathcal{H}_1$ , there exists a unique nearest point in  $C$ , denoted by  $P_C x$ , such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

$P_C$  is called the (nearest point or metric) projection of  $\mathcal{H}_1$  onto  $C$ . In addition,  $P_C x$  is characterized by the following properties:  $P_C x \in C$  and

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \tag{2.4}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in \mathcal{H}_1, y \in C. \tag{2.5}$$

Recall that a mapping  $T : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  is said to be firmly nonexpansive mapping if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle, \quad \forall x, y \in \mathcal{H}_1.$$

It is well known that  $P_C$  is a firmly nonexpansive mapping of  $\mathcal{H}_1$  onto  $C$  and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle, \quad \forall x, y \in \mathcal{H}_1. \tag{2.6}$$

If  $G$  an  $\alpha$ -inverse-strongly monotone mapping of  $C$  into  $\mathcal{H}_1$ , then it is obvious that  $G$  is  $\frac{1}{\alpha}$ -Lipschitz continuous. We also have that for all  $x, y \in C$  and  $\lambda > 0$ ,

$$\begin{aligned} \|(I - \lambda G)x - (I - \lambda G)y\|^2 &= \|x - y - \lambda(Gx - Gy)\|^2 \\ &= \|x - y\|^2 - 2\lambda \langle Gx - Gy, x - y \rangle + \lambda^2 \|Gx - Gy\|^2 \\ &\leq \|x - y\|^2 + \lambda(\lambda - 2\alpha) \|Gx - Gy\|^2 \end{aligned} \tag{2.7}$$

So, if  $\lambda \leq 2\alpha$ , then  $I - \lambda G$  is a nonexpansive mapping of  $C$  into  $\mathcal{H}_1$ .

Next, we denote weak convergence and strong convergence by notations  $\rightharpoonup$  and  $\rightarrow$ , respectively. A space  $X$  is said to satisfy Opial's condition [31] if for each sequence  $\{x_n\}$  in  $X$  which converges weakly to a point  $x \in X$ , we have

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|, \quad \forall y \in X, y \neq x.$$

**Lemma 2.2.** [25] Let  $\mathcal{H}_1$  be a Hilbert space,  $C$  be a nonempty closed convex subset of  $H$ , and  $f : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a contraction with coefficient  $0 < \alpha < 1$ , and  $G$  be a strongly positive linear bounded operator with coefficient  $\bar{\gamma} > 0$ . Then, for  $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ ,

$$\langle x - y, (G - \gamma f)x - (G - \gamma f)y \rangle \geq (\bar{\gamma} - \gamma\alpha) \|x - y\|^2, \quad x, y \in \mathcal{H}_1.$$

That is,  $G - \gamma f$  is strongly monotone with coefficient  $\bar{\gamma} - \gamma\alpha$ .

**Lemma 2.3.** [25] Assume  $G$  is a strongly positive linear bounded operator on a Hilbert space  $\mathcal{H}_1$  with coefficient  $\bar{\gamma} > 0$  and  $0 < \rho \leq \|G\|^{-1}$ . Then  $\|I - \rho G\| \leq 1 - \rho\bar{\gamma}$ .

**Lemma 2.4.** [23] *Let  $\{x_n\}$  and  $\{y_n\}$  be bounded sequences in a Banach space  $X$  and let  $\{\beta_n\}$  be a sequence in  $[0, 1]$  with  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$ . Suppose that  $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$  for all integers  $n \geq 0$  and  $\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$ . Then  $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$ .*

**Lemma 2.5.** [31] *Let  $\mathcal{H}_1$  be a Hilbert space,  $C$  a closed convex subset of  $\mathcal{H}_1$ , and  $S : C \rightarrow C$  a nonexpansive mapping with  $F(S) \neq \emptyset$ . If  $\{x_n\}$  is a sequence in  $C$  weakly converging to  $x \in C$  and if  $\{(I - S)x_n\}$  converges strongly to  $y$ , then  $(I - S)x = y$ ; in particular, if  $y = 0$ , then  $x \in \text{Fix}(S)$ .*

**Lemma 2.6.** [26] *Assume  $\{a_n\}$  is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \sigma_n)a_n + \delta_n,$$

where  $\{\sigma_n\}$  is a sequence in  $(0, 1)$  and  $\{\delta_n\}$  is a sequence in  $\mathbb{R}$  such that

- (1)  $\sum_{n=1}^{\infty} \sigma_n = \infty$ ;
- (2)  $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\sigma_n} \leq 0$  or  $\sum_{n=1}^{\infty} |\delta_n \sigma_n| < \infty$ .

Then  $\lim_{n \rightarrow \infty} a_n = 0$ .

### 3. MAIN RESULTS

In this section, we prove a strong convergence theorem for the general iterative methods for approximating the common element of SVIP and FPP which is the unique solution for the variational inequality (1.9). First, we have the following technical lemma, which is immediately consequence of the definition of resolvent mapping:

**Lemma 3.1.** *SVIP is equivalent to find  $x^* \in \mathcal{H}_1$  such that  $y^* = Ax^* \in \mathcal{H}_2$ ,  $x^* = J_{\lambda}^{B_1}(x^*)$  and  $y^* = J_{\lambda}^{B_2}(y^*)$  for some  $\lambda > 0$ .*

**Theorem 3.2.** *Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be two real Hilbert spaces. Let  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded linear operator. Assume that  $B_1 : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$  and  $B_2 : \mathcal{H}_2 \rightarrow 2^{\mathcal{H}_2}$  are maximal monotone mappings. Let  $S : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a nonexpansive mapping such that  $\text{Fix}(S) \cap \Gamma \neq \emptyset$ . Let  $f : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a contraction mapping with constant  $\alpha \in (0, 1)$  and  $G : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  a strongly positive, bounded linear operator with constant  $\mu$  such that  $\|G\| = 1$ , and  $0 < \beta < \frac{\mu}{\alpha}$ . For given  $\forall x_0 \in \mathcal{H}_1$ , let the sequences  $\{u_n\}$  and  $\{x_n\}$  be generated by (1.10), where  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  satisfying the following conditions :*

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ;
- (ii)  $\sum_{n=1}^{\infty} \alpha_n = \infty$  and
- (iii)  $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ .

Then the sequences  $\{u_n\}$  and  $\{x_n\}$  both converge strongly to  $z \in \text{Fix}(S) \cap \Gamma$ , where  $z = P_{\text{Fix}(S) \cap \Gamma}(I - G + \beta f)(z)$ . Moreover,  $z$  is a unique solution of the variational inequality (1.9).

**Proof** We observe that  $P_{\text{Fix}(S) \cap \Gamma}(I - G + \beta f)$  is a contraction. Indeed, applying Lemma 2.3 with  $\|G\| = 1$ , we have,

$$\begin{aligned} \|P_{\text{Fix}(S) \cap \Gamma}(I - G + \beta f)(x) - P_{\text{Fix}(S) \cap \Gamma}(I - G + \beta f)(y)\| &\leq \|(\beta f + (I - G))(x) - (\beta f + (I - G))(y)\| \\ &\leq \beta \|f(x) - f(y)\| + \|I - G\| \|x - y\| \\ &\leq \gamma \alpha \|x - y\| + (1 - \bar{\gamma}) \|x - y\| \\ &\leq (1 - (\bar{\gamma} - \alpha \beta)) \|x - y\|, \end{aligned}$$

for all  $x, y \in \mathcal{H}_1$ . Therefore, Banach's Contraction Mapping Principle guarantees that  $P_{\text{Fix}(S) \cap \Gamma}(I - G + \beta f)$  has a unique fixed point, say  $z \in \mathcal{H}_1$ . That is,  $z = Q_F(\gamma f + (I - G))(z)$ . Next, we divide the proof into five steps as follows.

**Step 1.** We first show that the sequences  $\{x_n\}$  is bounded. Let  $p \in \text{Fix}(S) \cap \Gamma$ , then we have that  $p = J_{\lambda}^{B_1} p$ ,  $Ap = J_{\lambda}^{B_2}(Ap)$  and  $Sp = p$ . So, we have

$$\begin{aligned} \|u_n - p\|^2 &= \left\| J_{\lambda}^{B_1}(x_n + \gamma A^*(J_{\lambda}^{B_2} - I)Ax_n) - p \right\|^2 \\ &= \left\| J_{\lambda}^{B_1}(x_n + \gamma A^*(J_{\lambda}^{B_2} - I)Ax_n) - J_{\lambda}^{B_1} p \right\|^2 \\ &\leq \left\| x_n + \gamma A^*(J_{\lambda}^{B_2} - I)Ax_n - p \right\|^2 \end{aligned}$$

$$\leq \|x_n - p\|^2 + \gamma^2 \left\| A^* \left( J_\lambda^{B_2} - I \right) Ax_n \right\|^2 + 2\gamma \left\langle x_n - p, A^* \left( J_\lambda^{B_2} - I \right) Ax_n \right\rangle. \quad (3.1)$$

It follow that

$$\begin{aligned} \|u_n - p\|^2 &\leq \|x_n - p\|^2 + \gamma^2 \left\langle \left( J_\lambda^{B_2} - I \right) Ax_n, AA^* \left( J_\lambda^{B_2} - I \right) Ax_n \right\rangle \\ &\quad + 2\gamma \left\langle x_n - p, A^* \left( J_\lambda^{B_2} - I \right) Ax_n \right\rangle. \end{aligned} \quad (3.2)$$

Since

$$\begin{aligned} \gamma^2 \left\langle \left( J_\lambda^{B_2} - I \right) Ax_n, AA^* \left( J_\lambda^{B_2} - I \right) Ax_n \right\rangle &\leq L\gamma^2 \left\langle \left( J_\lambda^{B_2} - I \right) Ax_n, \left( J_\lambda^{B_2} - I \right) Ax_n \right\rangle \\ &= L\gamma^2 \left\| \left( J_\lambda^{B_2} - I \right) Ax_n \right\|^2 \end{aligned} \quad (3.3)$$

and using (2.3), we have

$$\begin{aligned} 2\gamma \left\langle x_n - p, A^* \left( J_\lambda^{B_2} - I \right) Ax_n \right\rangle &= 2\gamma \left\langle A(x_n - p), \left( J_\lambda^{B_2} - I \right) Ax_n \right\rangle \\ &= 2\gamma \left\langle A(x_n - p) + \left( J_\lambda^{B_2} - I \right) Ax_n - \left( J_\lambda^{B_2} - I \right) Ax_n, \left( J_\lambda^{B_2} - I \right) Ax_n \right\rangle \\ &= 2\gamma \left\{ \left\langle J_\lambda^{B_2} Ax_n - Ap, \left( J_\lambda^{B_2} - I \right) Ax_n \right\rangle - \left\| \left( J_\lambda^{B_2} - I \right) Ax_n \right\|^2 \right\} \\ &\leq 2\gamma \left\{ \frac{1}{2} \left\| \left( J_\lambda^{B_2} - I \right) Ax_n \right\|^2 - \left\| \left( J_\lambda^{B_2} - I \right) Ax_n \right\|^2 \right\} \\ &\leq -\gamma \left\| \left( J_\lambda^{B_2} - I \right) Ax_n \right\|^2. \end{aligned} \quad (3.4)$$

From the inequalities (3.2), (3.3) and (3.4), we can conclude that

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 + \gamma(L\gamma - 1) \left\| \left( J_\lambda^{B_2} - I \right) Ax_n \right\|^2. \quad (3.5)$$

Since  $\gamma \in (0, \frac{1}{L})$ , we obtain

$$\|u_n - p\|^2 \leq \|x_n - p\|^2,$$

which implies that

$$\|u_n - p\| \leq \|x_n - p\|. \quad (3.6)$$

Therefore

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n \beta f(x_n) + (I - \alpha_n G)Su_n - p\| \\ &\leq \beta \alpha_n \|f(x_n) - f(p)\| + \|(I - \alpha_n G)\| \|Su_n - p\| + \alpha_n \|\beta f(p) - Gp\| \\ &\leq \beta \alpha_n \alpha \|x_n - p\| + (1 - \alpha_n \mu) \|u_n - p\| + \alpha_n \|\beta f(p) - Gp\| \\ &\leq \beta \alpha_n \alpha \|x_n - p\| + (1 - \alpha_n \mu) \|x_n - p\| + \alpha_n \|\beta f(p) - Gp\| \\ &= (\beta \alpha_n \alpha + (1 - \alpha_n \mu)) \|x_n - p\| + \alpha_n \|\beta f(p) - Gp\| \\ &= (1 - \alpha_n (\mu - \beta \alpha)) \|x_n - p\| + \alpha_n (\mu - \beta \alpha) \frac{\|\beta f(p) - Gp\|}{(\mu - \beta \alpha)} \\ &\leq \max \left\{ \|x_n - p\|, \frac{\|\beta f(p) - Gp\|}{\mu - \beta \alpha} \right\}. \end{aligned}$$

By induction, we have

$$\|x_n - p\| \leq \max \left\{ \|x_1 - p\|, \frac{\|\beta f(p) - Gp\|}{\mu - \beta \alpha} \right\}, \quad \forall n \geq 1. \quad (3.7)$$

Hence  $\{x_n\}$  is bounded and consequently, we deduce that  $\{u_n\}, \{f(x_n)\}$  and  $\{Su_n\}$  are bounded.

**Step 2.** We show that the sequences  $\{x_n\}$  is asymptotically regular, i.e.,

$$\|x_{n+1} - x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$



For each  $n \in \mathbb{N}$ , we notice that

$$\begin{aligned}
 \|x_{n+1} - x_n\| &= \|\alpha_n \beta f(x_n) + (I - \alpha_n G)Su_n - (\alpha_{n-1} \beta f(x_{n-1}) + (I - \alpha_{n-1} G)Su_{n-1})\| \\
 &\leq \|(I - \alpha_n G)(Su_n - Su_{n-1}) - (\alpha_n - \alpha_{n-1})GSu_{n-1} \\
 &\quad + \beta \alpha_n (f(x_n) - f(x_{n-1})) + \beta (\alpha_n - \alpha_{n-1})f(x_{n-1})\| \\
 &\leq (1 - \alpha_n \mu) \|Su_n - Su_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|GSu_{n-1}\| \\
 &\quad + \beta \alpha_n \|f(x_n) - f(x_{n-1})\| + \beta |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| \\
 &\leq (1 - \alpha_n \mu) \|u_n - u_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|GSu_{n-1}\| \\
 &\quad + \beta \alpha_n \|x_n - x_{n-1}\| + \beta |\alpha_n - \alpha_{n-1}| \|f(x_{n-1})\| \\
 &= (1 - \alpha_n \mu) \|u_n - u_{n-1}\| + \beta \alpha_n \|x_n - x_{n-1}\| \\
 &\quad + |\alpha_n - \alpha_{n-1}| (\|GSu_{n-1}\| + \beta \|f(x_{n-1})\|) \\
 &\leq (1 - \alpha_n \mu) \|u_n - u_{n-1}\| + \beta \alpha_n \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| K
 \end{aligned} \tag{3.8}$$

where  $K = \sup \{\|GSu_{n-1}\| + \beta \|f(x_{n-1})\| : n \in \mathbb{N}\}$ . Since, for  $\gamma \in (0, \frac{1}{L})$ , the mapping  $J_\lambda^{B_1}(I + \gamma A^*(J_\lambda^{B_2} - I)A)$  is averaged and hence nonexpansive, therefore

$$\begin{aligned}
 \|u_n - u_{n-1}\| &= \left\| J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n) - J_\lambda^{B_1}(x_{n-1} + \gamma A^*(J_\lambda^{B_2} - I)Ax_{n-1}) \right\| \\
 &\leq \left\| J_\lambda^{B_1}(I + \gamma A^*(J_\lambda^{B_2} - I)A)x_n - J_\lambda^{B_1}(I + \gamma A^*(J_\lambda^{B_2} - I)A)x_{n-1} \right\| \\
 &\leq \|x_n - x_{n-1}\|.
 \end{aligned} \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\|x_{n+1} - x_n\| \leq (1 - \alpha_n(\mu - \beta\alpha)) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| K.$$

By applying Lemma 2.6 with  $\beta_n = \alpha_n(\mu - \beta\alpha)$  and  $\delta_n = |\alpha_n - \alpha_{n-1}| K$ , we obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{3.10}$$

**Step 3.** We show that

$$\|x_{n+1} - p\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

For each  $n \in \mathbb{N}$ ,

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &= \|\alpha_n \beta f(x_n) + (I - \alpha_n G)Su_n - p\|^2 \\
 &= \|\alpha_n (\beta f(x_n) - Gp) + (I - \alpha_n G)(Su_n - p)\|^2 \\
 &\leq \|(I - \alpha_n G)(Su_n - p)\|^2 + 2\alpha_n \langle \beta f(x_n) - Gp, x_{n+1} - p \rangle \\
 &\leq (1 - \alpha_n \mu)^2 \|Su_n - p\|^2 + 2\alpha_n \beta \langle f(x_n) - f(p), x_{n+1} - p \rangle \\
 &\quad + 2\alpha_n \langle \beta f(x_n) - Gp, x_{n+1} - p \rangle \\
 &\leq (1 - \alpha_n \mu)^2 \|u_n - p\|^2 + 2\alpha_n \beta \langle f(x_n) - f(p), x_{n+1} - p \rangle \\
 &\quad + 2\alpha_n \langle \beta f(p) - Gp, x_{n+1} - p \rangle \\
 &\leq (1 - \alpha_n \mu)^2 \|u_n - p\|^2 + 2\alpha_n \beta \alpha \|x_n - p\| \|x_{n+1} - p\| \\
 &\quad + 2\alpha_n \|\beta f(p) - Gp\| \|x_{n+1} - p\|.
 \end{aligned} \tag{3.11}$$

Thus, from (3.5), we obtain

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &\leq (1 - \alpha_n \mu)^2 \left\{ \|x_n - p\|^2 + \gamma(L\gamma - 1) \left\| (J_\lambda^{B_2} - I)Ax_n \right\|^2 \right\} \\
 &\quad + 2\alpha_n \beta \alpha \|x_n - p\| \|x_{n+1} - p\| \\
 &\quad + 2\alpha_n \|\beta f(p) - Gp\| \|x_{n+1} - p\| \\
 &= (1 - 2\alpha_n \mu + (\alpha_n \mu)^2) \|x_n - p\|^2 + (1 - \alpha_n \mu)^2 (\gamma(L\gamma - 1) \left\| (J_\lambda^{B_2} - I)Ax_n \right\|^2) \\
 &\quad + 2\alpha_n \beta \alpha \|x_n - p\| \|x_{n+1} - p\| \\
 &\quad + 2\alpha_n \|\beta f(p) - Gp\| \|x_{n+1} - p\| \\
 &\leq \|x_n - p\|^2 + \alpha_n \mu^2 \|x_n - p\|^2 - (1 - \alpha_n \mu)^2 (\gamma(1 - L\gamma) \left\| (J_\lambda^{B_2} - I)Ax_n \right\|^2)
 \end{aligned}$$

$$\begin{aligned} &+2\alpha_n\beta\alpha\|x_n - p\|\|x_{n+1} - p\| \\ &+2\alpha_n\|\beta f(p) - Gp\|\|x_{n+1} - p\|. \end{aligned} \tag{3.12}$$

Therefore,

$$\begin{aligned} (1 - \alpha_n\mu)^2(\gamma(1 - L\gamma))\left\|\left(J_\lambda^{B_2} - I\right)Ax_n\right\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\mu^2\|x_n - p\|^2 \\ &\quad + 2\alpha_n\beta\alpha\|x_n - p\|\|x_{n+1} - p\| \\ &\quad + 2\alpha_n\|\beta f(p) - Gp\|\|x_{n+1} - p\| \\ &\leq \|x_{n+1} - x_n\|(\|x_n - p\| + \|x_{n+1} - p\|) + \alpha_n\mu^2\|x_n - p\|^2 \\ &\quad + 2\alpha_n\beta\alpha\|x_n - p\|\|x_{n+1} - p\| \\ &\quad + 2\alpha_n\|\beta f(p) - Gp\|\|x_{n+1} - p\|. \end{aligned}$$

Since  $\gamma(1 - L\gamma) > 0$ , and  $\alpha_n \rightarrow 0$  and  $\|x_{n+1} - x_n\| \rightarrow 0$  as  $n \rightarrow \infty$ , we have

$$\lim_{n \rightarrow \infty} \left\|\left(J_\lambda^{B_2} - I\right)Ax_n\right\| = 0. \tag{3.13}$$

Furthermore, using (3.7), (3.11) and  $\gamma \in (0, \frac{1}{L})$ , we notice that

$$\begin{aligned} \|u_n - p\|^2 &= \left\|J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n) - p\right\|^2 \\ &= \left\|J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n) - J_\lambda^{B_1}p\right\|^2 \\ &\leq \left\langle u_n - p, x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n - p \right\rangle \\ &= \frac{1}{2} \left\{ \|u_n - p\|^2 + \|x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n - p\|^2 \right. \\ &\quad \left. - \|(u_n - p) - (x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n - p)\| \right\} \\ &= \frac{1}{2} \left\{ \|u_n - p\|^2 + \|x_n - p\|^2 + \gamma(L\gamma - 1)\left\|\left(J_\lambda^{B_2} - I\right)Ax_n\right\|^2 \right. \\ &\quad \left. - \|(u_n - x_n) - \gamma A^*(J_\lambda^{B_2} - I)Ax_n\|^2 \right\} \\ &\leq \frac{1}{2} \left\{ \|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2 \right. \\ &\quad \left. + \gamma^2\|A^*(J_\lambda^{B_2} - I)Ax_n\|^2 - 2\gamma\langle u_n - x_n, A^*(J_\lambda^{B_2} - I)Ax_n \rangle \right\} \\ &\leq \frac{1}{2} \left\{ \|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2 \right. \\ &\quad \left. + 2\gamma\|A(u_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\| \right\}. \end{aligned}$$

Thus, we obtain

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\gamma\|A(u_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\|. \tag{3.14}$$

It follows from (3.11) and (3.14) that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq (1 - \alpha_n\mu)^2 \left[ \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\gamma\|A(u_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\| \right] \\ &\quad + 2\alpha_n\beta\alpha\|x_n - p\|\|x_{n+1} - p\| \\ &\quad + 2\alpha_n\|\beta f(p) - Gp\|\|x_{n+1} - p\| \\ &= (1 - 2\alpha_n\mu + (\alpha_n\mu)^2)\|x_n - p\|^2 - (1 - \alpha_n\mu)^2\|u_n - x_n\|^2 \\ &\quad + (1 - \alpha_n\mu)^2 2\gamma\|A(u_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\| \\ &\quad + 2\alpha_n\beta\alpha\|x_n - p\|\|x_{n+1} - p\| \\ &\quad + 2\alpha_n\|\beta f(p) - Gp\|\|x_{n+1} - p\| \\ &\leq \|x_n - p\|^2 + \alpha_n\mu^2\|x_n - p\|^2 - (1 - \alpha_n\mu)^2\|u_n - x_n\|^2 \\ &\quad + (1 - \alpha_n\mu)^2 2\gamma\|A(u_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\| \end{aligned}$$

$$\begin{aligned}
 &+ 2\alpha_n\beta\alpha\|x_n - p\|\|x_{n+1} - p\| \\
 &+ 2\alpha_n\|\beta f(p) - Gp\|\|x_{n+1} - p\|.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 (1 - \alpha_n\mu)^2\|u_n - x_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\mu^2\|x_n - p\|^2 \\
 &\quad + (1 - \alpha_n\mu)^2 2\gamma\|A(u_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\| \\
 &\quad + 2\alpha_n\beta\alpha\|x_n - p\|\|x_{n+1} - p\| \\
 &\quad + 2\alpha_n\|\beta f(p) - Gp\|\|x_{n+1} - p\| \\
 &\leq \|x_{n+1} - x_n\|(\|x_n - p\| + \|x_{n+1} - p\|) + \alpha_n\mu^2\|x_n - p\|^2 \\
 &\quad + (1 - \alpha_n\mu)^2 2\gamma\|A(u_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\| \\
 &\quad + 2\alpha_n\beta\alpha\|x_n - p\|\|x_{n+1} - p\| \\
 &\quad + 2\alpha_n\|\beta f(p) - Gp\|\|x_{n+1} - p\|.
 \end{aligned}$$

Since  $\alpha_n \rightarrow 0$  as  $n \rightarrow \infty$ , and from (3.10) and (3.13), we obtain

$$\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0. \tag{3.15}$$

Since  $\|Su_n - u_n\| \leq \|Su_n - x_n\| + \|x_n - u_n\|$ , it follows that

$$\|Su_n - u_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{3.16}$$

**Step 4.** We will show that

$$\limsup_{n \rightarrow \infty} \langle (G - \beta f)z, x_n - z \rangle \leq 0, \text{ where } z = P_{\text{Fix}(S) \cap \Gamma}(I - G + \beta f)(z).$$

Since  $\{u_n\}$  is bounded, we consider a weak cluster point  $w$  of  $\{u_n\}$ . Hence, there exists a subsequence  $\{u_{n_i}\}$  of  $\{u_n\}$ , which converges weakly to  $w$ . Now,  $S$  being nonexpansive, by (3.16) and Lemma 2.5, we obtain that  $w \in \text{Fix}(S)$ . On the other hand,  $u_{n_i} = J_\lambda^{B_1} \left( x_{n_i} + \gamma A^*(J_\lambda^{B_2} - I)Ax_{n_i} \right)$  can be rewritten as

$$\frac{(x_{n_i} - u_{n_i}) + A^*(J_\lambda^{B_2} - I)Ax_{n_i}}{\lambda} \in B_1u_{n_i}. \tag{3.17}$$

By passing to limit  $i \rightarrow \infty$  in (3.17) and by taking into account (3.13), (3.15) and the fact that the graph of a maximal monotone operator is weakly-strongly closed, we obtain  $0 \in B_1(w)$ . Furthermore, since  $\{u_n\}$  and  $\{x_n\}$  have the same asymptotical behavior,  $\{Ax_{n_i}\}$  weakly converges to  $Aw$ . Again, by (3.13) and the fact that the resolvent  $J_\lambda^{B_2}$  is nonexpansive and Lemma 3.1, we obtain that  $Aw \in B_2(Aw)$ . Thus  $w \in \text{Fix}(S) \cap \Gamma$ . Since  $z = P_{\text{Fix}(S) \cap \Gamma}(I - G + \beta f)(z)$ . Indeed, we have

$$\begin{aligned}
 \limsup_{n \rightarrow \infty} \langle (G - \beta f)z, z - x_n \rangle &= \lim_{i \rightarrow \infty} \langle (G - \beta f)z, z - x_{n_i} \rangle \\
 &= \langle (G - \beta f)z, z - w \rangle \leq 0.
 \end{aligned} \tag{3.18}$$

**Step 5.** Finally, we will show that  $x_n \rightarrow z$  as  $n \rightarrow \infty$ . We have

$$\begin{aligned}
 \|x_{n+1} - z\|^2 &= \|\alpha_n\beta f(x_n) + (I - \alpha_n G)Su_n - z\|^2 \\
 &= \|\alpha_n(\beta f(x_n) - Gz) + (I - \alpha_n G)(Su_n - z)\|^2 \\
 &\leq \|(I - \alpha_n G)(Su_n - z)\|^2 + 2\alpha_n\langle \beta f(x_n) - Gz, x_{n+1} - z \rangle \\
 &\leq (1 - \alpha_n\mu)^2\|u_n - z\|^2 + 2\alpha_n\beta\langle f(x_n) - f(z), x_{n+1} - z \rangle \\
 &\quad + 2\alpha_n\langle \beta f(x_n) - Gz, x_{n+1} - z \rangle \\
 &\leq (1 - \alpha_n\mu)^2\|x_n - z\|^2 + 2\alpha_n\beta\langle f(x_n) - f(z), x_{n+1} - z \rangle \\
 &\quad + 2\alpha_n\langle \beta f(z) - Gz, x_{n+1} - z \rangle \\
 &\leq (1 - \alpha_n\mu)^2\|x_n - z\|^2 + 2\alpha_n\beta\alpha\|x_n - z\|\|x_{n+1} - z\| \\
 &\quad + 2\alpha_n\|\beta f(z) - Gz\|\|x_{n+1} - z\| \\
 &\leq (1 - \alpha_n\mu)^2\|x_n - z\|^2 + \alpha_n\beta\alpha[\|x_n - z\|^2 + \|x_{n+1} - z\|^2] \\
 &\quad + 2\alpha_n\|\beta f(z) - Gz\|\|x_{n+1} - z\|
 \end{aligned}$$

$$\begin{aligned} &\leq ((1 - \alpha_n \mu)^2 + \alpha_n \beta \alpha) \|x_n - z\|^2 + \alpha_n \beta \alpha \|x_n - z\|^2 + \|x_{n+1} - z\|^2 \\ &\quad + 2\alpha_n \|\beta f(z) - Gz\| \|x_{n+1} - z\|, \end{aligned}$$

which implies that

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq \frac{1 - 2\alpha_n \mu + (\alpha_n \mu)^2 + \alpha_n \beta \alpha}{1 - \alpha_n \beta \alpha} \|x_n - z\|^2 \\ &\quad + \frac{2\alpha_n}{1 - \alpha_n \beta \alpha} \langle \beta f(z) - Gz, x_{n+1} - z \rangle \\ &= \left[ 1 - \frac{2(\mu - \beta \alpha) \alpha_n}{1 - \alpha_n \beta \alpha} \right] \|x_n - z\|^2 + \frac{(\alpha_n \mu)^2}{1 - \alpha_n \beta \alpha} \|x_n - z\|^2 \\ &\quad + \frac{2\alpha_n}{1 - \alpha_n \beta \alpha} \langle \beta f(z) - Gz, x_{n+1} - z \rangle \\ &\leq \left[ 1 - \frac{2(\mu - \beta \alpha) \alpha_n}{1 - \alpha_n \beta \alpha} \right] \|x_n - z\|^2 \\ &\quad + \frac{2(\mu - \beta \alpha) \alpha_n}{1 - \alpha_n \beta \alpha} \left[ \frac{(\alpha_n \mu^2) M}{2(\mu - \beta \alpha)} + \frac{1}{\mu - \beta \alpha} \langle \beta f(z) - Gz, x_{n+1} - z \rangle \right] \\ &= (1 - \sigma_n) \|x_n - z\|^2 + \sigma_n \delta_n, \end{aligned}$$

where  $M = \sup\{\|x_n - z\|^2 : n \in \mathbb{N}\}$ ,  $\sigma_n = \frac{2(\mu - \beta \alpha) \alpha_n}{1 - \alpha_n \beta \alpha}$  and  $\delta_n = \frac{(\alpha_n \mu^2) M}{2(\mu - \beta \alpha)} + \frac{1}{\mu - \beta \alpha} \langle \beta f(z) - Gz, x_{n+1} - z \rangle$ . It is easily to see that  $\sigma_n \rightarrow 0$ ,  $\sum_{n=1}^{\infty} \sigma_n = \infty$  and  $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\sigma_n} \leq 0$  by (3.18). Thus, by Lemma 2.6, we deduce that  $x_n \rightarrow z$  as  $n \rightarrow \infty$ . Further it follows from  $\|u_n - x_n\| \rightarrow 0$ ,  $u_n \rightarrow w \in \text{Fix}(S) \cap \Gamma$  and  $x_n \rightarrow z$  as  $n \rightarrow \infty$ , that is  $z = w$ . This completes the proof.  $\square$

**Remark 3.3.** In general case, if  $G$  is any strongly positive bounded linear operator with coefficient  $\bar{\gamma}$  and  $0 < \gamma < \frac{\mu}{\alpha}$ . We define a bounded linear operator  $\bar{G}$  on  $E$  by

$$\bar{G} = \|G\|^{-1} G.$$

It is easy to see that  $\bar{G}$  is a strongly positive with coefficient  $\|G\|^{-1} \mu > 0$  such that  $\|\bar{G}\| = 1$  and

$$0 < \|G\|^{-1} \gamma < \frac{\|G\|^{-1} \mu}{\alpha}.$$

Let the sequence  $\{x_n\}$  be defined by, for any  $x_0 \in E$ ,

$$\begin{aligned} u_n &= J_{\lambda}^{B_1} \left( x_n + \gamma A^* (J_{\lambda}^{B_2} - I) A x_n \right); \\ x_{n+1} &= \alpha_n \beta \|G\|^{-1} f(x_n) + (I - \alpha_n \bar{G}) S u_n, \end{aligned} \tag{3.19}$$

Replacing  $G$  with  $\bar{G}$  in Theorem 3.2, we obtain the following result.

**Theorem 3.4.** Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be two real Hilbert spaces. Let  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded linear operator. Assume that  $B_1 : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$  and  $B_2 : \mathcal{H}_2 \rightarrow 2^{\mathcal{H}_2}$  are maximal monotone mappings. Let  $S : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a nonexpansive mapping such that  $\text{Fix}(S) \cap \Gamma \neq \emptyset$ . Let  $f : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a contraction mapping with constant  $\alpha \in (0, 1)$  and  $G : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  a strongly positive, bounded linear operator with constant  $\mu$  and  $0 < \beta < \frac{\mu}{\alpha}$ . For given  $\forall x_0 \in \mathcal{H}_1$ , let the sequences  $\{u_n\}$  and  $\{x_n\}$  be generated by (3.19), where  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  satisfying the following conditions :

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ;
- (ii)  $\sum_{n=1}^{\infty} \alpha_n = \infty$  and
- (iii)  $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ .

Then the sequences  $\{u_n\}$  and  $\{x_n\}$  both converge strongly to  $z \in \text{Fix}(S) \cap \Gamma$ , where

$$z = P_{\text{Fix}(S) \cap \Gamma} \left( (I - \|G\|^{-1} (G + \gamma f) \bar{x}) z \right).$$

Moreover,  $z$  is also a unique solution of the variational inequality (1.9).

*Proof.* From Theorem 3.2, we have that  $\{x_n\}$  converges strongly, as  $n \rightarrow \infty$ , to a point  $z$  satisfying

$$z = P_{\text{Fix}(S) \cap \Gamma} (I - \|G\|^{-1}(G + \gamma f)) z,$$

which is a unique solution of the variational inequality:

$$\|G\|^{-1} \langle (G - \beta f)z, x - z \rangle \geq 0, x \in \mathcal{H}_1 \tag{3.20}$$

It is easy to see that (3.20) is equivalent to (1.9). Hence  $z$  is a unique solution of the variational inequality (1.9). □

Putting  $G = I$  and  $\beta = 1$  in Theorem 3.2, we have the following results immediately.

**Theorem 3.5.** [13] *Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be two real Hilbert spaces. Let  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded linear operator. Assume that  $B_1 : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$  and  $B_2 : \mathcal{H}_2 \rightarrow 2^{\mathcal{H}_2}$  are maximal monotone mappings. Let  $S : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a nonexpansive mapping such that  $\text{Fix}(S) \cap \Gamma \neq \emptyset$ . Let  $f : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a contraction mapping with constant  $\alpha \in (0, 1)$ . For any given  $x_0 \in \mathcal{H}_1$ , let the sequences  $\{u_n\}$  and  $\{x_n\}$  generated by*

$$\begin{aligned} u_n &= J_\lambda^{B_1} (x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n); \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)Su_n, \end{aligned} \tag{3.21}$$

where  $\lambda > 0$  and  $\gamma \in (0, \frac{1}{L})$ ,  $L$  is the spectral radius of the operator  $A^*A$  and  $A^*$  is the adjoint of  $A$  and  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  such that

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ;
- (ii)  $\sum_{n=1}^\infty \alpha_n = \infty$  and
- (iii)  $\sum_{n=1}^\infty |\alpha_n - \alpha_{n-1}| < \infty$ .

Then the sequences  $\{u_n\}$  and  $\{x_n\}$  both converge strongly to  $z \in \text{Fix}(S) \cap \Gamma$ , where  $z = P_{\text{Fix}(S) \cap \Gamma} f(z)$ .

Applying Theorem 3.2, we can establish the strong convergence for new iterative method as the following theorem.

**Theorem 3.6.** *Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be two real Hilbert spaces. Let  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a bounded linear operator. Assume that  $B_1 : \mathcal{H}_1 \rightarrow 2^{\mathcal{H}_1}$  and  $B_2 : \mathcal{H}_2 \rightarrow 2^{\mathcal{H}_2}$  are maximal monotone mappings. Let  $S : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a nonexpansive mapping such that  $\text{Fix}(S) \cap \Gamma \neq \emptyset$ . Let  $f : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  be a contraction mapping with constant  $\alpha \in (0, 1)$  and  $G : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  a strongly positive, bounded linear operator with constant  $\mu$  and  $0 < \beta < \frac{\mu}{\alpha}$ . For any given  $y_1 \in \mathcal{H}_1$ , let the sequences  $\{u'_n\}$  and  $\{y_n\}$  generated by*

$$\begin{aligned} u'_n &= J_\lambda^{B_1} (y_n + \gamma A^*(J_\lambda^{B_2} - I)Ay_n); \\ y_{n+1} &= \alpha_n \beta f(Su'_n) + (I - \alpha_n G)Su'_n, n \geq 1, \end{aligned} \tag{3.22}$$

where  $\lambda > 0$  and  $\gamma \in (0, \frac{1}{L})$ ,  $L$  is the spectral radius of the operator  $A^*A$  and  $A^*$  is the adjoint of  $A$  and  $\{\alpha_n\}$  is a sequence in  $(0, 1)$  such that

- (i)  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ;
- (ii)  $\sum_{n=1}^\infty \alpha_n = \infty$  and
- (iii)  $\sum_{n=1}^\infty |\alpha_n - \alpha_{n-1}| < \infty$ .

Then the sequences  $\{u'_n\}$  and  $\{y_n\}$  both converge strongly to  $z$  obtained in Theorem 3.2.

*Proof.* Let  $\{x_n\}$  be the sequence given by  $x_1 = y_1$  and

$$\begin{aligned} u_n &= J_\lambda^{B_1} (x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n); \\ x_{n+1} &= \alpha_n \beta f(x_n) + (I - \alpha_n G)Su_n, n \geq 1. \end{aligned} \tag{3.23}$$

From Theorem 3.2,  $x_n \rightarrow z$ . Next, we claim that  $y_n \rightarrow z$ . Since  $J_\lambda^{B_1}$  and  $J_\lambda^{B_2}$  both are firmly nonexpansive, they are averaged. For  $\gamma \in (0, \frac{1}{L})$ ,  $L$ , the mapping  $(I + \gamma A^*(J_\lambda^{B_2} I)A)$  is averaged, see [15]. It follows from Proposition 2.1 (ii) that the mapping  $J_\lambda^{B_1}(I + \gamma A^*(J_\lambda^{B_2} I)A)$  is averaged and hence nonexpansive. For each  $n \geq 1$ , we can estimate the following

$$\begin{aligned} \|x_{n+1} - y_{n+1}\| &= \|\alpha_n \beta f(x_n) + (I - \alpha_n G)Su_n - \alpha_n \beta f(Su'_n) - (I - \alpha_n G)Su'_n\| \\ &\leq \|\alpha_n \beta f(x_n) - \alpha_n \beta f(Su'_n)\| + \|(I - \alpha_n G)Su_n - (I - \alpha_n G)Su'_n\| \\ &\leq \alpha_n \beta \alpha \|Su'_n - x_n\| + (1 - \alpha_n \mu) \|u_n - u'_n\| \end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n \beta \alpha \|Su'_n - Sz\| + \alpha_n \beta \alpha \|Sz - x_n\| + (1 - \alpha_n \mu) \|x_n - y_n\| \\
&\leq \alpha_n \beta \alpha \|u'_n - z\| + \alpha_n \beta \alpha \|z - x_n\| + (1 - \alpha_n \mu) \|x_n - y_n\| \\
&\leq \alpha_n \beta \alpha \|y_n - z\| + \alpha_n \beta \alpha \|z - x_n\| + (1 - \alpha_n \mu) \|x_n - y_n\| \\
&\leq \alpha_n \beta \alpha \|y_n - x_n\| + \alpha_n \beta \alpha \|x_n - z\| + \alpha_n \beta \alpha \|z - x_n\| + (1 - \alpha_n \mu) \|x_n - y_n\| \\
&= (1 - \alpha_n(\mu - \beta\alpha)) \|x_n - y_n\| + \alpha_n(\mu - \beta\alpha) \frac{2\beta\alpha}{\mu - \beta\alpha} \|x_n - z\|.
\end{aligned}$$

It follows from  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,  $\lim_{n \rightarrow \infty} \|x_n - z\| = 0$  and Lemma 2.6 that  $\|x_n - y_n\| \rightarrow 0$ . Consequently,  $y_n \rightarrow z$  as required.  $\square$

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## A FIXED POINT ALTERNATIVE TO THE STABILITY OF AN ADDITIVE $\rho$ -FUNCTIONAL INEQUALITIES IN FUZZY BANACH SPACES

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ABSTRACT. In this paper, we solve the following additive  $\rho$ -functional inequalities

$$N\left(f(x+y) - f(x) - f(y) - \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right), t\right) \geq \frac{t}{t + \varphi(x, y)} \quad (0.1)$$

and

$$N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho(f(x+y) - f(x) - f(y)), t\right) \geq \frac{t}{t + \varphi(x, y)} \quad (0.2)$$

in fuzzy normed spaces, where  $\rho$  is a fixed real number with  $\rho \neq 1$ .

Using the fixed point method, we prove the Hyers-Ulam stability of the additive  $\rho$ -functional inequalities (0.1) and (0.2) in fuzzy Banach spaces.

### 1. INTRODUCTION AND PRELIMINARIES

Katsaras [23] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [14, 27, 52]. In particular, Bag and Samanta [3], following Cheng and Mordeson [10], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [26]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [4].

We use the definition of fuzzy normed spaces given in [3, 31, 32] to investigate the Hyers-Ulam stability of additive  $\rho$ -functional inequalities in fuzzy Banach spaces.

**Definition 1.1.** [3, 31, 32, 33] Let  $X$  be a real vector space. A function  $N : X \times \mathbb{R} \rightarrow [0, 1]$  is called a *fuzzy norm* on  $X$  if for all  $x, y \in X$  and all  $s, t \in \mathbb{R}$ ,

- ( $N_1$ )  $N(x, t) = 0$  for  $t \leq 0$ ;
- ( $N_2$ )  $x = 0$  if and only if  $N(x, t) = 1$  for all  $t > 0$ ;
- ( $N_3$ )  $N(cx, t) = N(x, \frac{t}{|c|})$  if  $c \neq 0$ ;
- ( $N_4$ )  $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$ ;
- ( $N_5$ )  $N(x, \cdot)$  is a non-decreasing function of  $\mathbb{R}$  and  $\lim_{t \rightarrow \infty} N(x, t) = 1$ .
- ( $N_6$ ) for  $x \neq 0$ ,  $N(x, \cdot)$  is continuous on  $\mathbb{R}$ .

The pair  $(X, N)$  is called a *fuzzy normed vector space*.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [30, 31].

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**Definition 1.2.** [3, 31, 32, 33] Let  $(X, N)$  be a fuzzy normed vector space. A sequence  $\{x_n\}$  in  $X$  is said to be *convergent* or *converge* if there exists an  $x \in X$  such that  $\lim_{n \rightarrow \infty} N(x_n - x, t) = 1$  for all  $t > 0$ . In this case,  $x$  is called the *limit* of the sequence  $\{x_n\}$  and we denote it by  $N\text{-}\lim_{n \rightarrow \infty} x_n = x$ .

**Definition 1.3.** [3, 31, 32, 33] Let  $(X, N)$  be a fuzzy normed vector space. A sequence  $\{x_n\}$  in  $X$  is called *Cauchy* if for each  $\varepsilon > 0$  and each  $t > 0$  there exists an  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$  and all  $p > 0$ , we have  $N(x_{n+p} - x_n, t) > 1 - \varepsilon$ .

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping  $f : X \rightarrow Y$  between fuzzy normed vector spaces  $X$  and  $Y$  is continuous at a point  $x_0 \in X$  if for each sequence  $\{x_n\}$  converging to  $x_0$  in  $X$ , then the sequence  $\{f(x_n)\}$  converges to  $f(x_0)$ . If  $f : X \rightarrow Y$  is continuous at each  $x \in X$ , then  $f : X \rightarrow Y$  is said to be *continuous* on  $X$  (see [4]).

The stability problem of functional equations originated from a question of Ulam [51] concerning the stability of group homomorphisms.

The functional equation  $f(x + y) = f(x) + f(y)$  is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [19] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Th.M. Rassias [43] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [15] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

The functional equation  $f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$  is called the *Jensen equation*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [9, 20, 22, 24, 25, 28, 39, 40, 41, 45, 46, 47, 48, 49, 50]).

Gilányi [17] showed that if  $f$  satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x - y)\| \leq \|f(x + y)\| \tag{1.1}$$

then  $f$  satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x + y) + f(x - y).$$

See also [44]. Fechner [13] and Gilányi [18] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [38] investigated the Cauchy additive functional inequality

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\| \tag{1.2}$$

and the Cauchy-Jensen additive functional inequality

$$\|f(x) + f(y) + 2f(z)\| \leq \left\| 2f\left(\frac{x + y}{2} + z\right) \right\| \tag{1.3}$$

and proved the Hyers-Ulam stability of the functional inequalities (1.2) and (1.3) in Banach spaces.

Park [36, 37] defined additive  $\rho$ -functional inequalities and proved the Hyers-Ulam stability of the additive  $\rho$ -functional inequalities in Banach spaces and non-Archimedean Banach spaces.

We recall a fundamental result in fixed point theory.

Let  $X$  be a set. A function  $d : X \times X \rightarrow [0, \infty]$  is called a *generalized metric* on  $X$  if  $d$  satisfies

- (1)  $d(x, y) = 0$  if and only if  $x = y$ ;
- (2)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ;
- (3)  $d(x, z) \leq d(x, y) + d(y, z)$  for all  $x, y, z \in X$ .

**Theorem 1.4.** [6, 11] Let  $(X, d)$  be a complete generalized metric space and let  $J : X \rightarrow X$  be a strictly contractive mapping with Lipschitz constant  $L < 1$ . Then for each given element  $x \in X$ , either

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers  $n$  or there exists a positive integer  $n_0$  such that

- (1)  $d(J^n x, J^{n+1} x) < \infty, \quad \forall n \geq n_0$ ;
- (2) the sequence  $\{J^n x\}$  converges to a fixed point  $y^*$  of  $J$ ;
- (3)  $y^*$  is the unique fixed point of  $J$  in the set  $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$ ;
- (4)  $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$  for all  $y \in Y$ .

In 1996, G. Isac and Th.M. Rassias [21] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [1, 5, 7, 8, 12, 16, 30, 34, 35, 41, 42]).

In Section 2, we solve the additive  $\rho$ -functional inequality (0.1) and prove the Hyers-Ulam stability of the additive  $\rho$ -functional inequality (0.1) in fuzzy Banach spaces by using the fixed point method.

In Section 3, we solve the additive  $\rho$ -functional inequality (0.2) and prove the Hyers-Ulam stability of the additive  $\rho$ -functional inequality (0.2) in fuzzy Banach spaces by using the fixed point method.

Throughout this paper, assume that  $X$  is a real vector space and  $(Y, N)$  is a fuzzy Banach space.

## 2. ADDITIVE $\rho$ -FUNCTIONAL INEQUALITY (0.1)

In this section, we prove the Hyers-Ulam stability of the additive  $\rho$ -functional inequality (0.1) in fuzzy Banach spaces. Let  $\rho$  be a real number with  $\rho \neq 1$ . We need the following lemma to prove the main results.

**Lemma 2.1.** *Let  $f : X \rightarrow Y$  be a mapping satisfying*

$$f(x + y) - f(x) - f(y) = \rho \left( 2f \left( \frac{x + y}{2} \right) - f(x) - f(y) \right) \tag{2.1}$$

for all  $x, y \in X$ . Then  $f : X \rightarrow Y$  is additive.

*Proof.* Letting  $x = y = 0$  in (2.1), we get  $-f(0) = 0$  and so  $f(0) = 0$ .

Replacing  $y$  by  $x$  in (2.1), we get  $f(2x) - 2f(x) = 0$  and so  $f(2x) = 2f(x)$  for all  $x \in X$ . Thus

$$f(x + y) - f(x) - f(y) = \rho \left( 2f \left( \frac{x + y}{2} \right) - f(x) - f(y) \right) = \rho(f(x + y) - f(x) - f(y))$$

and so  $f(x + y) = f(x) + f(y)$  for all  $x, y \in X$ . □

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**Theorem 2.2.** Let  $\varphi : X^2 \rightarrow [0, \infty)$  be a function such that there exists an  $L < 1$  with

$$\varphi(x, y) \leq \frac{L}{2}\varphi(2x, 2y)$$

for all  $x, y \in X$ . Let  $f : X \rightarrow Y$  be a mapping satisfying

$$N\left(f(x+y) - f(x) - f(y) - \rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right), t\right) \geq \frac{t}{t + \varphi(x, y)} \quad (2.2)$$

for all  $x, y \in X$  and all  $t > 0$ . Then  $A(x) := N\text{-}\lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right)$  exists for each  $x \in X$  and defines an additive mapping  $A : X \rightarrow Y$  such that

$$N(f(x) - A(x), t) \geq \frac{(2 - 2L)t}{(2 - 2L)t + L\varphi(x, x)} \quad (2.3)$$

for all  $x \in X$  and all  $t > 0$ .

*Proof.* Letting  $y = x$  in (2.2), we get

$$N(f(2x) - 2f(x), t) \geq \frac{t}{t + \varphi(x, x)} \quad (2.4)$$

for all  $x \in X$ .

Consider the set

$$S := \{g : X \rightarrow Y\}$$

and introduce the generalized metric on  $S$ :

$$d(g, h) = \inf \left\{ \mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(x, x)}, \forall x \in X, \forall t > 0 \right\},$$

where, as usual,  $\inf \phi = +\infty$ . It is easy to show that  $(S, d)$  is complete (see [29, Lemma 2.1]).

Now we consider the linear mapping  $J : S \rightarrow S$  such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all  $x \in X$ .

Let  $g, h \in S$  be given such that  $d(g, h) = \varepsilon$ . Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, x)}$$

for all  $x \in X$  and all  $t > 0$ . Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right), L\varepsilon t\right) \\ &= N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{2}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\varphi(x, x)} \\ &= \frac{t}{t + \varphi(x, x)} \end{aligned}$$

for all  $x \in X$  and all  $t > 0$ . So  $d(g, h) = \varepsilon$  implies that  $d(Jg, Jh) \leq L\varepsilon$ . This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all  $g, h \in S$ .

It follows from (2.4) that

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), \frac{L}{2}t\right) \geq \frac{t}{t + \varphi(x, x)}$$

for all  $x \in X$  and all  $t > 0$ . So  $d(f, Jf) \leq \frac{L}{2}$ .

By Theorem 1.4, there exists a mapping  $A : X \rightarrow Y$  satisfying the following:

(1)  $A$  is a fixed point of  $J$ , i.e.,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{2.5}$$

for all  $x \in X$ . Since  $f : X \rightarrow Y$  is odd,  $A : X \rightarrow Y$  is an odd mapping. The mapping  $A$  is a unique fixed point of  $J$  in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that  $A$  is a unique mapping satisfying (2.5) such that there exists a  $\mu \in (0, \infty)$  satisfying

$$N(f(x) - A(x), \mu t) \geq \frac{t}{t + \varphi(x, x)}$$

for all  $x \in X$ ;

(2)  $d(J^n f, A) \rightarrow 0$  as  $n \rightarrow \infty$ . This implies the equality

$$N\text{-}\lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all  $x \in X$ ;

(3)  $d(f, A) \leq \frac{1}{1-L}d(f, Jf)$ , which implies the inequality

$$d(f, A) \leq \frac{L}{2 - 2L}.$$

This implies that the inequality (2.3) holds.

By (2.2),

$$\begin{aligned} & N\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right) \right. \\ & \quad \left. - \rho\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right)\right), 2^n t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)} \end{aligned}$$

for all  $x, y \in X$ , all  $t > 0$  and all  $n \in \mathbb{N}$ . So

$$\begin{aligned} & N\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right) \right. \\ & \quad \left. - \rho\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right)\right), t\right) \geq \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n}\varphi(x, y)} \end{aligned}$$

for all  $x, y \in X$ , all  $t > 0$  and all  $n \in \mathbb{N}$ . Since  $\lim_{n \rightarrow \infty} \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n}\varphi(x, y)} = 1$  for all  $x, y \in X$  and all  $t > 0$ ,

$$A(x+y) - A(x) - A(y) = \rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right)$$

for all  $x, y \in X$ . By Lemma 2.1, the mapping  $A : X \rightarrow Y$  is Cauchy additive, as desired.  $\square$

**Corollary 2.3.** *Let  $\theta \geq 0$  and let  $p$  be a real number with  $p > 1$ . Let  $X$  be a normed vector space with norm  $\| \cdot \|$ . Let  $f : X \rightarrow Y$  be a mapping satisfying*

$$N \left( f(x+y) - f(x) - f(y) - \rho \left( 2f \left( \frac{x+y}{2} \right) - f(x) - f(y) \right), t \right) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{2.6}$$

for all  $x, y \in X$  and all  $t > 0$ . Then  $A(x) := N\text{-}\lim_{n \rightarrow \infty} 2^n f(\frac{x}{2^n})$  exists for each  $x \in X$  and defines an additive mapping  $A : X \rightarrow Y$  such that

$$N(f(x) - A(x), t) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 2\theta\|x\|^p}$$

for all  $x \in X$  and all  $t > 0$ .

*Proof.* The proof follows from Theorem 2.2 by taking  $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$  for all  $x, y \in X$ . Then we can choose  $L = 2^{1-p}$ , and we get the desired result.  $\square$

**Theorem 2.4.** *Let  $\varphi : X^2 \rightarrow [0, \infty)$  be a function such that there exists an  $L < 1$  with*

$$\varphi(x, y) \leq 2L\varphi \left( \frac{x}{2}, \frac{y}{2} \right)$$

for all  $x, y \in X$ . Let  $f : X \rightarrow Y$  be a mapping satisfying (2.2). Then  $A(x) := N\text{-}\lim_{n \rightarrow \infty} \frac{1}{2^n} f(2^n x)$  exists for each  $x \in X$  and defines an additive mapping  $A : X \rightarrow Y$  such that

$$N(f(x) - A(x), t) \geq \frac{(2 - 2L)t}{(2 - 2L)t + \varphi(x, x)} \tag{2.7}$$

for all  $x \in X$  and all  $t > 0$ .

*Proof.* Let  $(S, d)$  be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping  $J : S \rightarrow S$  such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all  $x \in X$ .

It follows from (2.4) that

$$N \left( f(x) - \frac{1}{2}f(2x), \frac{1}{2}t \right) \geq \frac{t}{t + \varphi(x, x)}$$

for all  $x \in X$  and all  $t > 0$ . So  $d(f, Jf) \leq \frac{1}{2}$ . Hence

$$d(f, A) \leq \frac{1}{2 - 2L},$$

which implies that the inequality (2.7) holds.

The rest of the proof is similar to the proof of Theorem 2.2.  $\square$

**Corollary 2.5.** *Let  $\theta \geq 0$  and let  $p$  be a real number with  $0 < p < 1$ . Let  $X$  be a normed vector space with norm  $\| \cdot \|$ . Then  $A(x) := N\text{-}\lim_{n \rightarrow \infty} \frac{1}{2^n} f(2^n x)$  exists for each  $x \in X$  and defines an additive mapping  $A : X \rightarrow Y$  such that*

$$N(f(x) - A(x), t) \geq \frac{(2 - 2^p)t}{(2 - 2^p)t + 2\theta\|x\|^p}$$

for all  $x \in X$  and all  $t > 0$ .

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*Proof.* The proof follows from Theorem 2.4 by taking  $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$  for all  $x, y \in X$ . Then we can choose  $L = 2^{p-1}$ , and we get the desired result.  $\square$

3. ADDITIVE  $\rho$ -FUNCTIONAL INEQUALITY (0.2)

In this section, we prove the Hyers-Ulam stability of the additive  $\rho$ -functional inequality (0.2) in fuzzy Banach spaces. Let  $\rho$  be a fuzzy number with  $\rho \neq 1$ .

**Lemma 3.1.** *Let  $f : X \rightarrow Y$  be a mapping satisfying  $f(0) = 0$  and*

$$2f\left(\frac{x+y}{2}\right) - f(x) - f(y) = \rho(f(x+y) - f(x) - f(y)) \tag{3.1}$$

for all  $x, y \in X$ . Then  $f : X \rightarrow Y$  is additive.

*Proof.* Letting  $y = 0$  in (3.1), we get  $2f\left(\frac{x}{2}\right) - f(x) = 0$  and so  $f(2x) = 2f(x)$  for all  $x \in X$ . Thus

$$f(x+y) - f(x) - f(y) = 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) = \rho(f(x+y) - f(x) - f(y))$$

and so  $f(x+y) = f(x) + f(y)$  for all  $x, y \in X$ .  $\square$

**Theorem 3.2.** *Let  $\varphi : X^2 \rightarrow [0, \infty)$  be a function such that there exists an  $L < 1$  with*

$$\varphi(x, y) \leq \frac{L}{2}\varphi(2x, 2y)$$

for all  $x, y \in X$ . Let  $f : X \rightarrow Y$  be a mapping satisfying  $f(0) = 0$  and

$$N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho(f(x+y) - f(x) - f(y)), t\right) \geq \frac{t}{t + \varphi(x, y)} \tag{3.2}$$

for all  $x, y \in X$  and all  $t > 0$ . Then  $A(x) := N\text{-}\lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right)$  exists for each  $x \in X$  and defines an additive mapping  $A : X \rightarrow Y$  such that

$$N(f(x) - A(x), t) \geq \frac{(1-L)t}{(1-L)t + \varphi(x, 0)} \tag{3.3}$$

for all  $x \in X$  and all  $t > 0$ .

*Proof.* Letting  $y = 0$  in (3.2), we get

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) = N\left(2f\left(\frac{x}{2}\right) - f(x), t\right) \geq \frac{t}{t + \varphi(x, 0)} \tag{3.4}$$

for all  $x \in X$ .

Consider the set

$$S := \{g : X \rightarrow Y\}$$

and introduce the generalized metric on  $S$ :

$$d(g, h) = \inf \left\{ \mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(x, 0)}, \forall x \in X, \forall t > 0 \right\},$$

where, as usual,  $\inf \phi = +\infty$ . It is easy to show that  $(S, d)$  is complete (see [29, Lemma 2.1]).

Now we consider the linear mapping  $J : S \rightarrow S$  such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all  $x \in X$ .

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Let  $g, h \in S$  be given such that  $d(g, h) = \varepsilon$ . Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, 0)}$$

for all  $x \in X$  and all  $t > 0$ . Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right), L\varepsilon t\right) \\ &= N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{2}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi\left(\frac{x}{2}, 0\right)} \geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\varphi(x, 0)} \\ &= \frac{t}{t + \varphi(x, 0)} \end{aligned}$$

for all  $x \in X$  and all  $t > 0$ . So  $d(g, h) = \varepsilon$  implies that  $d(Jg, Jh) \leq L\varepsilon$ . This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all  $g, h \in S$ .

It follows from (3.4) that

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) \geq \frac{t}{t + \varphi(x, 0)}$$

for all  $x \in X$  and all  $t > 0$ . So  $d(f, Jf) \leq 1$ .

By Theorem 1.4, there exists a mapping  $A : X \rightarrow Y$  satisfying the following:

(1)  $A$  is a fixed point of  $J$ , i.e.,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{3.5}$$

for all  $x \in X$ . The mapping  $A$  is a unique fixed point of  $J$  in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that  $A$  is a unique mapping satisfying (3.5) such that there exists a  $\mu \in (0, \infty)$  satisfying

$$N(f(x) - A(x), \mu t) \geq \frac{t}{t + \varphi(x, 0)}$$

for all  $x \in X$ ;

(2)  $d(J^n f, A) \rightarrow 0$  as  $n \rightarrow \infty$ . This implies the equality

$$N\text{-}\lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all  $x \in X$ ;

(3)  $d(f, A) \leq \frac{1}{1-L}d(f, Jf)$ , which implies the inequality

$$d(f, A) \leq \frac{1}{1-L}.$$

This implies that the inequality (3.3) holds.

By (3.2),

$$N\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right) - \rho\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right), 2^nt\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}$$

for all  $x, y \in X$ , all  $t > 0$  and all  $n \in \mathbb{N}$ . So

$$N\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right) - \rho\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right), t\right) \geq \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n}\varphi(x, y)}$$

for all  $x, y \in X$ , all  $t > 0$  and all  $n \in \mathbb{N}$ . Since  $\lim_{n \rightarrow \infty} \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n}\varphi(x, y)} = 1$  for all  $x, y \in X$  and all  $t > 0$ ,

$$2A\left(\frac{x+y}{2}\right) - A(x) - A(y) = \rho(A(x+y) - A(x) - A(y))$$

for all  $x, y \in X$ . By Lemma 3.1, the mapping  $A : X \rightarrow Y$  is Cauchy additive, as desired.  $\square$

**Corollary 3.3.** *Let  $\theta \geq 0$  and let  $p$  be a real number with  $p > 1$ . Let  $X$  be a normed vector space with norm  $\|\cdot\|$ . Let  $f : X \rightarrow Y$  be a mapping satisfying  $f(0) = 0$  and*

$$N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y) - \rho(f(x+y) - f(x) - f(y)), t\right) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \quad (3.6)$$

for all  $x, y \in X$  and all  $t > 0$ . Then  $A(x) := N\text{-}\lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right)$  exists for each  $x \in X$  and defines an additive mapping  $A : X \rightarrow Y$  such that

$$N(f(x) - A(x), t) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 2^p\theta\|x\|^p}$$

for all  $x \in X$  and all  $t > 0$ .

*Proof.* The proof follows from Theorem 3.2 by taking  $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$  for all  $x, y \in X$ . Then we can choose  $L = 2^{1-p}$ , and we get the desired result.  $\square$

**Theorem 3.4.** *Let  $\varphi : X^2 \rightarrow [0, \infty)$  be a function such that there exists an  $L < 1$  with*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all  $x, y \in X$ . Let  $f : X \rightarrow Y$  be a mapping satisfying  $f(0) = 0$  and (3.3). Then  $A(x) := N\text{-}\lim_{n \rightarrow \infty} \frac{1}{2^n} f(2^n x)$  exists for each  $x \in X$  and defines an additive mapping  $A : X \rightarrow Y$  such that

$$N(f(x) - A(x), t) \geq \frac{(1 - L)t}{(1 - L)t + L\varphi(x, 0)} \quad (3.7)$$

for all  $x \in X$  and all  $t > 0$ .

*Proof.* Let  $(S, d)$  be the generalized metric space defined in the proof of Theorem 3.2.

Now we consider the linear mapping  $J : S \rightarrow S$  such that

$$Jg(x) := \frac{1}{2}g(2x)$$



for all  $x \in X$ .

It follows from (3.4) that

$$N\left(f(x) - \frac{1}{2}f(2x), Lt\right) \geq \frac{t}{t + \varphi(x, 0)}$$

for all  $x \in X$  and all  $t > 0$ . So  $d(f, Jf) \leq L$ . Hence

$$d(f, A) \leq \frac{L}{1 - L},$$

which implies that the inequality (3.7) holds.

The rest of the proof is similar to the proof of Theorem 3.2.  $\square$

**Corollary 3.5.** *Let  $\theta \geq 0$  and let  $p$  be a real number with  $0 < p < 1$ . Let  $X$  be a normed vector space with the norm  $\|\cdot\|$ . Let  $f : X \rightarrow Y$  be a mapping satisfying  $f(0) = 0$  and (3.6). Then  $A(x) := N\text{-}\lim_{n \rightarrow \infty} \frac{1}{2^n} f(2^n x)$  exists for each  $x \in X$  and defines an additive mapping  $A : X \rightarrow Y$  such that*

$$N(f(x) - A(x), t) \geq \frac{(2 - 2^p)t}{(2 - 2^p)t + 2^p\theta\|x\|^p}$$

for all  $x \in X$ .

*Proof.* The proof follows from Theorem 3.4 by taking  $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$  for all  $x, y \in X$ . Then we can choose  $L = 2^{p-1}$ , and we get the desired result.  $\square$

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## FOURIER SERIES OF HIGHER-ORDER GENOCCHI FUNCTIONS AND THEIR APPLICATIONS

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ABSTRACT. In this paper, we derive some identities for higher-order Genocchi functions arising from Fourier series for them . In addition, we give some application of these identities related to Bernoulli function.

### 1. Introduction

The numbers  $G_k$ , ( $k \geq 0$ ), in the Taylor expansion

$$\frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \quad (\text{see [2 - 11]}), \quad (1.1)$$

are known as the Genocchi numbers. These numbers arise in the series expansion of trigonometric functions, and are extremely important in the number theory and analysis. The Genocchi polynomials  $G_n(x)$ , ( $n \geq 0$ ), are defined by the generating function

$$\left( \frac{2t}{e^t + 1} \right) e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad (\text{see [5, 12, 13]}). \quad (1.2)$$

Note that  $G_n(x) \in \mathbb{Z}[x]$  with  $\deg G_n(x) = n - 1$ , for  $n \geq 1$ . Let  $f(x)$  be a square integrable function defined on  $[-p, p]$ . Then the Fourier series of  $f(x)$  is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right), \quad (1.3)$$

where

$$a_0 = \frac{1}{p} \int_{-p}^p f(x) dx, \quad a_n = \frac{1}{p} \int_{-p}^p f(x) \cos \frac{n\pi}{p} x dx, \quad (1.4)$$

and

$$b_n = \frac{1}{p} \int_{-p}^p f(x) \sin \frac{n\pi}{p} x dx, \quad (\text{see [6, 7]}). \quad (1.5)$$

The Fourier series in (1.3) can be alternatively given as follows:

$$\sum_{n=-\infty}^{\infty} C_n e^{\frac{n\pi i}{p} x}, \quad (i = \sqrt{-1}), \quad (1.6)$$

where

$$C_n = \frac{1}{2p} \int_{-p}^p f(x) e^{-\frac{n\pi i}{p} x} dx, \quad (\text{see [6, 7]}). \quad (1.7)$$

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For  $r \in \mathbb{N}$ , the higher-order Genocchi polynomials are defined by the generating function to be

$$\left(\frac{2t}{e^t + 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} G_n^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see [5]}). \tag{1.8}$$

When  $x = 0$ ,  $G_n^{(r)} = G_n^{(r)}(0)$  are called the higher-order Genocchi numbers.

For any real number  $x$ , we define

$$\langle x \rangle = x - [x] \in [0, 1), \tag{1.9}$$

where  $[ \cdot ]$  is the Gauss symbol. Note that  $\langle x \rangle$  is the fractional part of  $x$ . Thus,  $G_m^{(r)}(\langle x \rangle)$  are functions defined on  $(-\infty, \infty)$  and periodic with period 1, which are called Genocchi functions of order  $r$ .

In this paper, we derive some identities of Genocchi functions of order  $r$  arising from Fourier series for them. In addition, we give some application of these identities related to Bernoulli functions.

## 2. Fourier series of higher-order Genocchi functions and their applications

From (1.8), we note that

$$G_m^{(r)}(x) = 0, \text{ for } 0 \leq m \leq r - 1, \text{ and } G_r^{(r)}(x) = r!. \tag{2.1}$$

Now, we assume that  $m \geq r + 1 \geq 2$ . We first observe that

$$G_m^{(r)}(x + 1) = 2mG_{m-1}^{(r-1)}(x) - G_m^{(r)}(x), \quad (m \geq 0). \tag{2.2}$$

Indeed,

$$\begin{aligned} \sum_{m=0}^{\infty} G_m^{(r)}(x + 1) \frac{t^m}{m!} &= \left(\frac{2t}{e^t + 1}\right)^r e^{(x+1)t} = \left(\frac{2t}{e^t + 1}\right)^r e^{xt}(e^t + 1 - 1) \\ &= \left(\frac{2t}{e^t + 1}\right)^{r-1} e^{xt} \left(\frac{2t}{e^t + 1}\right) (e^t + 1) - \left(\frac{2t}{e^t + 1}\right)^r e^{xt} \\ &= 2t \sum_{m=0}^{\infty} G_m^{(r-1)}(x) \frac{t^m}{m!} - \sum_{m=0}^{\infty} G_m^{(r)}(x) \frac{t^m}{m!} \\ &= 2 \sum_{m=0}^{\infty} mG_{m-1}^{(r-1)}(x) \frac{t^m}{m!} - \sum_{m=0}^{\infty} G_m^{(r)}(x) \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left(2mG_{m-1}^{(r-1)}(x) - G_m^{(r)}(x)\right) \frac{t^m}{m!}. \end{aligned} \tag{2.3}$$

For  $x = 0$  in , we have

$$G_m^{(r)}(1) = 2mG_{m-1}^{(r-1)}(0) - G_m^{(r)}(0). \tag{2.4}$$

By (2.4), we get

$$G_m^{(r)}(1) = G_m^{(r)}(0) \Leftrightarrow G_m^{(r)}(0) = mG_{m-1}^{(r-1)}(0). \tag{2.5}$$

$G_m^{(r)}(\langle x \rangle)$  is piecewise  $C^\infty$ . Moreover,  $G_m^{(r)}(\langle x \rangle)$  is continuous for those  $(r, m)$  with  $G_m^{(r)}(0) = mG_{m-1}^{(r-1)}(0)$ , and discontinuous with jump discontinuities at integers for those  $(r, m)$  with  $G_m^{(r)}(0) \neq$

$mG_{m-1}^{(r-1)}(0)$ . The Fourier series of  $G_m^{(r)}(< x >)$  is

$$\sum_{n=-\infty}^{\infty} C_n^{(r,m)} e^{2\pi inx}, \quad (i = \sqrt{-1}), \tag{2.6}$$

where

$$\begin{aligned} C_n^{(r,m)} &= \int_0^1 G_m^{(r)}(< x >) e^{-2\pi inx} dx \\ &= \int_0^1 G_m^{(r)}(x) e^{-2\pi inx} dx. \end{aligned} \tag{2.7}$$

Now, we observe that

$$\begin{aligned} C_n^{(r,m)} &= \int_0^1 G_m^{(r)}(x) e^{-2\pi inx} dx \\ &= \frac{1}{m+1} \left[ G_{m+1}^{(r)} e^{-2\pi inx} \right]_0^1 + \frac{2\pi in}{m+1} \int_0^1 G_{m+1}^{(r)}(x) e^{-2\pi inx} dx \\ &= \frac{1}{m+1} \left( G_{m+1}^{(r)}(0) - G_{m+1}^{(r)}(0) \right) + \frac{2\pi in}{m+1} C_n^{(r,m+1)} \\ &= \frac{2}{m+1} \left( (m+1)G_m^{(r-1)}(0) - G_{m+1}^{(r)}(0) \right) + \frac{2\pi in}{m+1} C_n^{(r,m+1)}. \end{aligned} \tag{2.8}$$

Replacing  $m$  by  $m - 1$  in (2.8), we have

$$\frac{2\pi in}{m} C_n^{(r,m)} = C_n^{(r,m-1)} + \frac{2}{m} \left( G_m^{(r)}(0) - mG_{m-1}^{(r-1)}(0) \right). \tag{2.9}$$

Assume first that  $n \neq 0$ . Then we have

$$\begin{aligned} C_n^{(r,m)} &= \frac{m}{2\pi in} C_n^{(r,m-1)} + \frac{1}{\pi in} \left( G_m^{(r)}(0) - mG_{m-1}^{(r-1)}(0) \right) \\ &= \frac{m}{2\pi in} \left( \frac{m-1}{2\pi in} C_n^{(r,m-2)} + \frac{1}{\pi in} \left( G_{m-1}^{(r)}(0) - (m-1)G_{m-2}^{(r-1)}(0) \right) \right) \\ &\quad + \frac{1}{\pi in} \left( G_m^{(r)}(0) - mG_{m-1}^{(r-1)}(0) \right) \\ &= \frac{m(m-1)}{(2\pi in)^2} C_n^{(r,m-2)} + \frac{m}{2} \frac{1}{(\pi in)^2} \left( G_{m-1}^{(r)}(0) - (m-1)G_{m-2}^{(r-1)}(0) \right) \\ &\quad + \frac{1}{\pi in} \left( G_m^{(r)}(0) - mG_{m-1}^{(r-1)}(0) \right) \\ &= \frac{m(m-1)}{(2\pi in)^2} \left\{ \frac{m-2}{2\pi in} C_n^{(r,m-3)} + \frac{1}{\pi in} \left( G_{m-2}^{(r)}(0) - (m-2)G_{m-3}^{(r-1)}(0) \right) \right\} \\ &\quad + \frac{m}{2} \frac{1}{(\pi in)^2} \left( G_{m-1}^{(r)}(0) - (m-1)G_{m-2}^{(r-1)}(0) \right) + \frac{1}{\pi in} \left( G_m^{(r)}(0) - mG_{m-1}^{(r-1)}(0) \right) \\ &= \frac{m(m-1)(m-2)}{(2\pi in)^3} C_n^{(r,m-3)} + \frac{m(m-1)}{2^2} \frac{1}{(\pi in)^3} \left( G_{m-2}^{(r)}(0) - (m-2)G_{m-3}^{(r-1)}(0) \right) \\ &\quad + \frac{m}{2} \frac{1}{(\pi in)^2} \left( G_{m-1}^{(r)}(0) - (m-1)G_{m-2}^{(r-1)}(0) \right) + \frac{1}{\pi in} \left( G_m^{(r)}(0) - mG_{m-1}^{(r-1)}(0) \right) \\ &= \dots \\ &= \frac{m!}{(2\pi in)^{m-1}} C_n^{(r,1)} + \sum_{k=1}^{m-1} \frac{\binom{m}{k-1}}{2^{k-1}} \frac{1}{(\pi in)^k} \left( G_{m+1-k}^{(r)}(0) - (m+1-k)G_{m-k}^{(r-1)}(0) \right). \end{aligned} \tag{2.10}$$

Note that

$$C_n^{(r,1)} = \int_0^1 G_1^{(r)}(x)e^{-2\pi inx} dx = 0, \tag{2.11}$$

since  $G_1^{(r)}(x) = 0$ , for  $r \geq 2$  and  $G_1^{(1)}(x) = 1$ . From (2.10) and (2.11), we have

$$\begin{aligned} C_n^{(r,m)} &= \sum_{k=1}^{m-1} \frac{(m)_{k-1}}{2^{k-1}} \frac{1}{(\pi in)^k} \left( G_{m-k+1}^{(r)}(0) - (m-k+1)G_{m-k}^{(r-1)}(0) \right) \\ &= \sum_{k=1}^{\min\{m+1-r, m-1\}} \frac{(m)_{k-1}}{2^{k-1}} \frac{1}{(\pi in)^k} \left( G_{m-k+1}^{(r)}(0) - (m-k+1)G_{m-k}^{(r-1)}(0) \right) \end{aligned} \tag{2.12}$$

Here we used the fact that  $G_m^{(r)} = 0$ , for  $0 \leq m \leq r-1$ . Assume next that  $n = 0$ . Then we have

$$\begin{aligned} C_0^{(r,m)} &= \int_0^1 G_m^{(r)}(x) dx = \frac{1}{m+1} \left[ G_{m+1}^{(r)}(x) \right]_0^1 = \frac{1}{m+1} \left( G_{m+1}^{(r)}(1) - G_{m+1}^{(r)}(0) \right) \\ &= \frac{2}{m+1} \left( (m+1)G_m^{(r-1)}(0) - G_{m+1}^{(r)}(0) \right) \end{aligned} \tag{2.13}$$

Before proceeding further, we recall the following facts about Bernoulli functions  $B_n(\langle x \rangle)$  :

$$B_m(\langle x \rangle) = -m! \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^m}, \text{ for } m \geq 2, \quad (\text{see [1]}), \tag{2.14}$$

and

$$- \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{2\pi inx}}{2\pi in} = \begin{cases} B_1(\langle x \rangle), & \text{for } x \notin \mathbb{Z} \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \tag{2.15}$$

The series in (2.14) converges uniformly, but that in (2.15) converges only pointwise. Assume first that  $G_m^{(r)}(0) = mG_{m-1}^{(r-1)}(0)$ . Then  $G_m^{(r)}(1) = G_m^{(r)}(0)$ .  $G_m^{(r)}(\langle x \rangle)$  is piecewise  $C^\infty$ , and continuous. Hence the Fourier series of  $G_m^{(r)}(\langle x \rangle)$  converges uniformly to  $G_m^{(r)}(\langle x \rangle)$ , and we have

$$\begin{aligned} G_m^{(r)}(\langle x \rangle) &= \frac{2}{m+1} \left( (m+1)G_m^{(r-1)}(0) - G_{m+1}^{(r)}(0) \right) \\ &+ \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left( \sum_{k=1}^{\min\{m+1-r, m-1\}} \frac{(m)_{k-1}}{2^{k-1}} \frac{1}{(\pi in)^k} \left( G_{m-k+1}^{(r)}(0) - (m-k+1)G_{m-k}^{(r-1)}(0) \right) \right) e^{2\pi inx}, \end{aligned} \tag{2.16}$$

for all  $x \in (-\infty, \infty)$ . In addition, we can express this in terms of Bernoulli functions  $B_m(\langle x \rangle)$ .

$$\begin{aligned}
 G_m^{(r)}(\langle x \rangle) &= \frac{2}{m+1} \left( (m+1)G_m^{(r-1)}(0) - G_{m+1}^{(r)}(0) \right) \\
 &\quad - \sum_{k=1}^{\min\{m+1-r, m-1\}} \frac{2(m)_{k-1}}{k!} \left( G_{m-k+1}^{(r)}(0) - (m-k+1)G_{m-k}^{(r-1)}(0) \right) \left( -k! \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^k} \right) \\
 &= \frac{2}{m+1} \left( (m+1)G_m^{(r-1)}(0) - G_{m+1}^{(r)}(0) \right) \\
 &\quad + \sum_{k=2}^{\min\{m+1-r, m-1\}} \frac{2(m)_{k-1}}{k!} \left( (m+1-k)G_{m-k}^{(r-1)}(0) - G_{m+1-k}^{(r)}(0) \right) B_k(\langle x \rangle) \\
 &\quad + 2 \left( mG_{m-1}^{(r-1)}(0) - G_m^{(r)}(0) \right) \times \begin{cases} B_1(\langle x \rangle), & \text{for } x \notin \mathbb{Z} \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
 \end{aligned} \tag{2.17}$$

Therefore, we obtain the following theorem.

**Theorem 2.1.** *Let  $m \geq r + 1 \geq 2$ . Assume that*

$$G_m^{(r)}(0) = mG_{m-1}^{(r-1)}(0).$$

*Then we have*

$$\begin{aligned}
 (a) \quad G_m^{(r)}(\langle x \rangle) &= \frac{2}{m+1} \left( (m+1)G_m^{(r-1)}(0) - G_{m+1}^{(r)}(0) \right) \\
 &\quad + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left( \sum_{k=1}^{\min\{m+1-r, m-1\}} \frac{(m)_{k-1}}{2^{k-1}} \frac{1}{(\pi i n)^k} \left( G_{m-k+1}^{(r)}(0) - (m-k+1)G_{m-k}^{(r-1)}(0) \right) \right) e^{2\pi i n x},
 \end{aligned}$$

*for all  $x \in (-\infty, \infty)$ , where the convergence is uniform.*

$$\begin{aligned}
 (b) \quad G_m^{(r)}(\langle x \rangle) &= \frac{2}{m+1} \left( (m+1)G_m^{(r-1)}(0) - G_{m+1}^{(r)}(0) \right) \\
 &\quad + \sum_{k=2}^{\min\{m+1-r, m-1\}} \frac{2(m)_{k-1}}{k!} \left( (m+1-k)G_{m-k}^{(r-1)}(0) - G_{m+1-k}^{(r)}(0) \right) B_k(\langle x \rangle),
 \end{aligned}$$

*for all  $x \in (-\infty, \infty)$ , where  $B_k(\langle x \rangle)$  is the Bernoulli function.*

Assume next that  $G_m^{(r)}(0) \neq mG_{m-1}^{(r-1)}(0)$ . Then  $G_m^{(r)}(1) \neq G_m^{(r)}(0)$ , and hence  $G_m^{(r)}(\langle x \rangle)$  is piecewise  $C^\infty$  and discontinuous with jump discontinuities at integers. Thus, the Fourier series of  $G_m^{(r)}(\langle x \rangle)$  converges pointwise to  $G_m^{(r)}(\langle x \rangle)$ , for  $x \notin \mathbb{Z}$ , and converges to  $\frac{1}{2} \left( G_m^{(r)}(1) + G_m^{(r)}(0) \right) = mG_{m-1}^{(r-1)}(0)$ , for  $x \in \mathbb{Z}$ . Therefore, we obtain the following theorem.



**Theorem 2.2.** *Let  $m \geq r + 1 \geq 2$ . Assume that  $G_m^{(r)}(0) \neq mG_{m-1}^{(r)}(0)$ . Then we have*

$$(a) \frac{2}{m+1} \left( (m+1)G_m^{(r-1)}(0) - G_{m+1}^{(r)}(0) \right) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left( \sum_{k=1}^{\min\{m+1-r, m-1\}} \frac{(m)_{k-1}}{2^{k-1}} \left( \frac{1}{\pi i n} \right)^k \right. \\ \left. \times \left( G_{m+1-k}^{(r)}(0) - (m+1-k)G_{m-k}^{(r-1)}(0) \right) \right) e^{2\pi i n x} = \begin{cases} G_m^{(r)}(\langle x \rangle), & \text{for } x \notin \mathbb{Z}, \\ mG_{m-1}^{(r-1)}(0), & \text{for } x \in \mathbb{Z}. \end{cases}$$

Here the convergence is pointwise.

$$(b) \frac{2}{m+1} \left( (m+1)G_m^{(r-1)}(0) - G_{m+1}^{(r)}(0) \right) \\ + \sum_{k=1}^{\min\{m+1-r, m-1\}} \frac{2(m)_{k-1}}{k!} \left( (m+1-k)G_{m-k}^{(r-1)}(0) - G_{m+1-k}^{(r)}(0) \right) B_k(\langle x \rangle) \\ = G_m^{(r)}(\langle x \rangle), \quad \text{for } x \notin \mathbb{Z},$$

and

$$\frac{2}{m+1} \left( (m+1)G_m^{(r-1)}(0) - G_{m+1}^{(r)}(0) \right) \\ + \sum_{k=2}^{\min\{m+1-r, m-1\}} \frac{2(m)_{k-1}}{k!} \left( (m+1-k)G_{m-k}^{(r-1)}(0) - G_{m+1-k}^{(r)}(0) \right) B_k(\langle x \rangle) \\ = mG_{m-1}^{(r-1)}(0), \quad \text{for } x \in \mathbb{Z}.$$

Here  $B_k(\langle x \rangle)$  is the Bernoulli function.

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$(g, \varphi_{h,m})$  – CONVEX AND  $(g, \log \varphi)$  – CONVEX DOMINATED  
 FUNCTIONS AND HADAMARD TYPE INEQUALITIES  
 RELATED TO THEM

MUSTAFA GÜRBÜZ

ABSTRACT. In this paper, we present the notion of  $(g, \varphi_{h,m})$ –convex and  $(g, \log \varphi)$ –convex dominated function and present some properties of them. Besides, we attain some Hermite-Hadamard-type inequalities for  $(g, \varphi_{h,m})$  –convex and  $(g, \log \varphi)$  – convex dominated functions. Our results generalize some findings about Hermite-Hadamard-type inequalities in the literature.

1. INTRODUCTION

The inequality

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}$$

which holds for every convex function  $f$ , from a closed set  $[a, b]$  to  $\mathbb{R}$ , is known in the literature as Hermite-Hadamard’s inequality (see [13]).

In [1], Dragomir and Ionescu introduced the following class of functions.

**Definition 1.** Let  $g : I \rightarrow \mathbb{R}$  be a convex function on the interval  $I$ . The function  $f : I \rightarrow \mathbb{R}$  is called  $g$ –convex dominated on  $I$  if the following condition is satisfied:

$$(1.2) \quad \begin{aligned} & |\lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1-\lambda)y)| \\ & \leq \lambda g(x) + (1-\lambda)g(y) - g(\lambda x + (1-\lambda)y) \end{aligned}$$

for all  $x, y \in I$  and  $\lambda \in [0, 1]$ .

In [1] and [2], the authors connect together some disparate threads through a Hermite-Hadamard motif. The first of these threads is the unifying concept of a  $g$ –convex-dominated function. In [3], Hwang *et al.* established some inequalities of Fejér type for  $g$ –convex-dominated functions. Finally, in [4], [5] and [6] authors introduced several new different kinds of convex -dominated functions and then gave Hermite-Hadamard-type inequalities for this classes of functions.

In [7], S. Varošanec introduced the following class of functions.

$I$  and  $J$  are intervals in  $\mathbb{R}$ ,  $(0, 1) \subseteq J$  and functions  $h$  and  $f$  are real non-negative functions defined on  $J$  and  $I$ , respectively.

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**Definition 2.** Let  $h$  be a non-negative function from  $J$  which is a subset of  $\mathbb{R}$  to  $\mathbb{R}$ ,  $h \neq 0$ .  $f : I \rightarrow \mathbb{R}$  is called an  $h$ -convex function, or that  $f$  belongs to the class  $SX(h, I)$ , if  $f$  is non-negative and for all  $x, y \in I$  and  $\alpha \in (0, 1]$ , we get

$$(1.3) \quad f(\alpha x + (1 - \alpha)y) \leq h(\alpha) f(x) + h(1 - \alpha) f(y).$$

If the inequality (1.3) is reversed, then  $f$  is said to be  $h$ -concave, i.e.  $f \in SV(h, I)$ .

Youness have defined the  $\varphi$ -convex functions in [9]. A function  $\varphi : [a, b] \rightarrow [c, d]$  where  $[a, b] \subset \mathbb{R}$ :

**Definition 3.** A function  $f : [a, b] \rightarrow \mathbb{R}$  is said to be  $\varphi$ -convex on  $[a, b]$  if for every two points  $x \in [a, b]$ ,  $y \in [a, b]$  and  $t \in [0, 1]$  the following inequality holds:

$$(1.4) \quad f(t\varphi(x) + (1 - t)\varphi(y)) \leq tf(\varphi(x)) + (1 - t)f(\varphi(y)).$$

In [8], Sarıkaya defined a new kind of  $\varphi$ -convexity using  $h$ -convexity as following:

**Definition 4.** Let  $I$  be an interval in  $\mathbb{R}$  and  $h : (0, 1) \rightarrow (0, \infty)$  be a given function. We say that a function  $f : I \rightarrow [0, \infty)$  is  $\varphi_h$ -convex if

$$(1.5) \quad f(t\varphi(x) + (1 - t)\varphi(y)) \leq h(t)f(\varphi(x)) + h(1 - t)f(\varphi(y))$$

for all  $x, y \in I$  and  $t \in (0, 1)$ .

If inequality (1.5) is reversed, then  $f$  is said to be  $\varphi_h$ -concave. In particular, if  $f$  satisfies (1.5) with  $h(t) = t$ ,  $h(t) = t^s$  ( $s \in (0, 1)$ ),  $h(t) = \frac{1}{t}$ , and  $h(t) = 1$ , then  $f$  is said to be  $\varphi$ -convex,  $\varphi_s$ -convex,  $\varphi$ -Godunova-Levin function and  $\varphi$ - $P$ -function, respectively.

In [10], Özdemir *et al.* defined  $(h - m)$ -convexity and obtained Hermite-Hadamard-type inequalities as following.

**Definition 5.** Let  $h : J \subset \mathbb{R} \rightarrow \mathbb{R}$  be a non-negative function. We say that  $f : [0, b] \rightarrow \mathbb{R}$  is a  $(h - m)$ -convex function, if  $f$  is non-negative and for all  $x, y \in [0, b]$ ,  $m \in [0, 1]$  and  $\alpha \in (0, 1)$ , we have

$$(1.6) \quad f(\alpha x + m(1 - \alpha)y) \leq h(\alpha) f(x) + mh(1 - \alpha) f(y).$$

If the inequality is reversed, then  $f$  is said to be  $(h - m)$ -concave function on  $[0, b]$ .

In [2], Dragomir *et al.* proved the following theorem for  $g$ -convex dominated functions related to (1.1):

**Definition 6.** A function  $f : I \rightarrow [0, \infty)$  is said to be  $\log$ -convex or multiplicatively convex if  $\log t$  is convex, or, equivalently, if for all  $x, y \in I$  and  $t \in [0, 1]$  one has the inequality

$$(1.7) \quad f(tx + (1 - t)y) \leq [f(x)]^t [f(y)]^{1-t}.$$

We note that if  $f$  and  $g$  are convex and  $g$  is increasing, then  $g \circ f$  is convex; moreover, since  $f = \exp(\log f)$ , it follows that a  $\log$ -convex function is convex, but the converse may not necessarily be true [12]. This follows directly from (1.7) because, by the arithmetic-geometric mean inequality, we have

$$[f(x)]^t [f(y)]^{1-t} \leq tf(x) + (1 - t)f(y)$$

for all  $x, y \in I$  and  $t \in [0, 1]$ .

For some results related to this classical results, (see [13], [14], [15], [16], [17]) and the references therein.

In [17], Sarıkaya has defined the log  $-\varphi$ -convex function as following.

**Definition 7.** Let us consider a  $\varphi : [a, b] \rightarrow [a, b]$  where  $[a, b] \subset \mathbb{R}$  and  $I$  stands for a convex subset of  $\mathbb{R}$ . We say that a function  $f : I \rightarrow \mathbb{R}^+$  is a log  $-\varphi$ -convex if

$$(1.8) \quad f(t\varphi(x) + (1-t)\varphi(y)) \leq [f(\varphi(x))]^t [f(\varphi(y))]^{1-t}$$

for all  $x, y \in I$  and  $t \in [0, 1]$ .

**Theorem 1.** Let  $g : I \rightarrow \mathbb{R}$  be a convex function and  $f : I \rightarrow \mathbb{R}$  a  $g$ -convex dominated mapping. Then for all  $a, b \in I$  with  $a < b$ ,

$$(1.9) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{1}{b-a} \int_a^b g(x) dx - g\left(\frac{a+b}{2}\right)$$

and

$$(1.10) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{g(a) + g(b)}{2} - \frac{1}{b-a} \int_a^b g(x) dx.$$

In [4], Kavurmacı *et al.* proved the next theorem:

**Theorem 2.** Let  $h : J \rightarrow \mathbb{R}$  be a non-negative function,  $h \neq 0$ ,  $g : I \rightarrow \mathbb{R}$  be an  $h$ -convex function and the real function  $f : I \rightarrow \mathbb{R}$  be a  $(g, h)$ -convex dominated on  $I$ . Then one has the inequalities:

$$(1.11) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{b-a} \int_a^b g(x) dx - \frac{1}{2h\left(\frac{1}{2}\right)} g\left(\frac{a+b}{2}\right)$$

and

$$(1.12) \quad \left| [f(a) + f(b)] \int_0^1 h(\lambda) d\lambda - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq [g(a) + g(b)] \int_0^1 h(\lambda) d\lambda - \frac{1}{b-a} \int_a^b g(x) dx$$

for all  $x, y \in I$  and  $\lambda \in [0, 1]$ .

In [6], Özdemir *et al.* proved the following theorem:

**Theorem 3.** Let a nonnegative function  $g : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  belong to the class of  $P(I)$ . The real function  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is  $(g, P(I))$ -convex dominated on  $I$ . If  $a, b \in I$  with  $a < b$  and  $f, g \in L_1[a, b]$ , then one has the inequalities:

$$(1.13) \quad \left| \frac{2}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{2}{b-a} \int_a^b g(x) dx - g\left(\frac{a+b}{2}\right)$$

and

$$(1.14) \quad \left| [f(a) + f(b)] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq [g(a) + g(b)] - \frac{1}{b-a} \int_a^b g(x) dx$$

for all  $x, y \in I$ .

In [11], Özdemir *et al.* proved the following theorem:

**Theorem 4.** Let  $h : (0, 1) \rightarrow (0, \infty)$  be a given function,  $g : I \rightarrow [0, \infty)$  be a given  $\varphi_h$ -convex function. If  $f : I \rightarrow [0, \infty)$  is Lebesgue integrable and  $(g, \varphi_h)$ -convex dominated on  $I$  for linear continuous, non-constant function  $\varphi : [a, b] \rightarrow [a, b]$ , then the following inequalities hold:

$$(1.15) \quad \left| \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) dx - \frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right| \leq \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} g(x) dx - \frac{1}{2h\left(\frac{1}{2}\right)} g\left(\frac{\varphi(a) + \varphi(b)}{2}\right)$$

and

$$(1.16) \quad \left| [f(\varphi(a)) + f(\varphi(b))] \int_0^1 h(t) dt - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) dx \right| \leq [g(\varphi(a)) + g(\varphi(b))] \int_0^1 h(t) dt - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} g(x) dx$$

for all  $x, y, a \in [0, b]$ ,  $t \in (0, 1)$  and  $m \in (0, 1]$ .

In the following sections our main results are given: We introduce the notion of  $(g, \varphi_{h,m})$ -convex and  $(g, \log \varphi)$ -convex dominated function and present some properties of them. Besides, we present some Hermite-Hadamard-type inequalities for  $(g, \varphi_{h,m})$ -convex and  $(g, \log \varphi)$ -convex dominated functions. Our results generalize the Hermite-Hadamard-type inequalities in [2], [4], [6] and [11].

## 2. $(g, \varphi_{h,m})$ -CONVEX DOMINATED FUNCTIONS

**Definition 8.** Let  $h : (0, 1) \rightarrow \mathbb{R}$  be a non-negative function,  $h \neq 0$ ,  $g : [0, b] \subseteq [0, \infty) \rightarrow \mathbb{R}^+$  be a given  $\varphi_{h,m}$ -convex function. The real function  $f : [0, b] \rightarrow \mathbb{R}^+$  is called  $(g, \varphi_{h,m})$ -convex dominated on  $[0, b]$  if the following condition is satisfied

$$(2.1) \quad |h(t)f(\varphi(x)) + mh(1-t)f(\varphi(y)) - f(t\varphi(x) + m(1-t)\varphi(y))| \leq h(t)g(\varphi(x)) + mh(1-t)g(\varphi(y)) - g(t\varphi(x) + m(1-t)\varphi(y))$$

for all  $x, y \in [0, b]$ ,  $t \in (0, 1)$  and  $m \in [0, 1]$ .

In particular, if  $f$  satisfies (2.1) with  $m = 1$ , then  $f$  is said to be  $(g, \varphi_h)$ -convex dominated function. If the inequality (2.1) is reversed, then  $f$  is said to be  $\varphi_{h,m}$ -concave dominated function on  $[0, b]$ .

The next simple characterisation of  $(g, \varphi_{h,m})$ -convex dominated functions holds.

**Lemma 1.** Let  $h : (0, 1) \rightarrow (0, \infty)$  be a given function,  $g : [0, b] \subseteq [0, \infty) \rightarrow \mathbb{R}^+$  be a given  $\varphi_{h,m}$ -convex function and  $f : [0, b] \rightarrow \mathbb{R}^+$  be a real function. The following statements are equivalent:

- (1)  $f$  is  $(g, \varphi_{h,m})$ -convex dominated on  $[0, b]$ .
- (2) The mappings  $g - f$  and  $g + f$  are  $\varphi_{h,m}$ -convex on  $[0, b]$ .
- (3) There exist two  $\varphi_{h,m}$ -convex mappings  $l, k$  defined on  $[0, b]$  such that

$$f = \frac{1}{2}(l - k) \quad \text{and} \quad g = \frac{1}{2}(l + k) .$$

*Proof.*  $1 \iff 2$  The condition (2.1) is equivalent to

$$\begin{aligned} & g(t\varphi(x) + m(1-t)\varphi(y)) - h(t)g(\varphi(x)) - mh(1-t)g(\varphi(y)) \\ & \leq h(t)f(\varphi(x)) + mh(1-t)f(\varphi(y)) - f(t\varphi(x) + m(1-t)\varphi(y)) \\ & \leq h(t)g(\varphi(x)) + mh(1-t)g(\varphi(y)) - g(t\varphi(x) + m(1-t)\varphi(y)) \end{aligned}$$

for all  $x, y \in [0, b]$  and  $t \in (0, 1)$ . The two inequalities may be rearranged as

$$\begin{aligned} & (g + f)(t\varphi(x) + m(1-t)\varphi(y)) \\ & \leq h(t)(g + f)(\varphi(x)) + mh(1-t)(g + f)(\varphi(y)) \end{aligned}$$

and

$$\begin{aligned} & (g - f)(t\varphi(x) + m(1-t)\varphi(y)) \\ & \leq h(t)(g - f)(\varphi(x)) + mh(1-t)(g - f)(\varphi(y)) \end{aligned}$$

which are equivalent to the  $\varphi_{h,m}$ -convexity of  $g + f$  and  $g - f$ , respectively.

$2 \iff 3$  Let we define the mappings  $f, g$  as  $f = \frac{1}{2}(l - k)$  and  $g = \frac{1}{2}(l + k)$ . Then if we sum and subtract  $f$  and  $g$ , respectively, we have  $g + f = l$  and  $g - f = k$ . By the condition 2 in Lemma 1, the mappings  $g - f$  and  $g + f$  are  $\varphi_{h,m}$ -convex on  $[0, b]$ , so  $l, k$  are  $\varphi_{h,m}$ -convex mappings on  $[0, b]$  too.  $\square$

**Theorem 5.** Let  $h : (0, 1) \rightarrow (0, \infty)$  be a given function,  $g : [0, b] \subseteq [0, \infty) \rightarrow \mathbb{R}^+$  be a given  $\varphi_{h,m}$ -convex function. If  $f$  is defined from  $[0, b]$  to  $[0, \infty)$  and it is Lebesgue integrable with  $(g, \varphi_{h,m})$ -convex dominated on  $[0, b]$  for linear continuous, non-constant function  $\varphi : [0, b] \rightarrow [0, b]$ , then the following inequalities hold:

$$\begin{aligned} & \left| \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) dx + \frac{m^2}{\varphi(b) - \varphi(a)} \int_{\frac{\varphi(a)}{m}}^{\frac{\varphi(b)}{m}} f(x) dx - \frac{1}{h\left(\frac{1}{2}\right)} f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right| \\ & \quad (2.2) \\ & \leq \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} g(x) dx + \frac{m^2}{\varphi(b) - \varphi(a)} \int_{\frac{\varphi(a)}{m}}^{\frac{\varphi(b)}{m}} g(x) dx - \frac{1}{h\left(\frac{1}{2}\right)} g\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \end{aligned}$$

and

$$\begin{aligned} & \left| \left[ f(\varphi(a)) + m \left\{ f\left(\varphi\left(\frac{a}{m}\right)\right) + f\left(\varphi\left(\frac{b}{m}\right)\right) + mf\left(\varphi\left(\frac{b}{m^2}\right)\right) \right\} \right] \int_0^1 h(t) dt \right. \\ & \quad (2.3) \left. - \left[ \frac{1}{m\varphi\left(\frac{b}{m}\right) - \varphi(a)} \int_{\varphi(a)}^{m\varphi\left(\frac{b}{m}\right)} f(x) dx + \frac{m}{m\varphi\left(\frac{b}{m^2}\right) - \varphi\left(\frac{a}{m}\right)} \int_{\varphi\left(\frac{a}{m}\right)}^{m\varphi\left(\frac{b}{m^2}\right)} f(x) dx \right] \right| \\ & \leq \left[ g(\varphi(a)) + m \left\{ g\left(\varphi\left(\frac{a}{m}\right)\right) + g\left(\varphi\left(\frac{b}{m}\right)\right) + mg\left(\varphi\left(\frac{b}{m^2}\right)\right) \right\} \right] \int_0^1 h(t) dt \\ & \quad - \left[ \frac{1}{m\varphi\left(\frac{b}{m}\right) - \varphi(a)} \int_{\varphi(a)}^{m\varphi\left(\frac{b}{m}\right)} g(x) dx + \frac{m}{m\varphi\left(\frac{b}{m^2}\right) - \varphi\left(\frac{a}{m}\right)} \int_{\varphi\left(\frac{a}{m}\right)}^{m\varphi\left(\frac{b}{m^2}\right)} g(x) dx \right] \end{aligned}$$

for all  $x, y, a \in [0, b]$ ,  $t \in (0, 1)$  and  $m \in (0, 1]$ .

*Proof.* By the Definition 8 with  $t = \frac{1}{2}$ ,  $x = \lambda a + (1 - \lambda)b$ ,  $y = (1 - \lambda) \frac{a}{m} + \lambda \frac{b}{m}$ ,  $\lambda \in [0, 1]$  and  $m \in (0, 1]$ , as the mapping  $f$  is  $(g, \varphi_{h,m})$ -convex dominated function, we have that

$$\begin{aligned} & \left| h\left(\frac{1}{2}\right) \left[ f\left(\varphi\left(\lambda a + (1 - \lambda)b\right)\right) + mf\left(\varphi\left((1 - \lambda)\frac{a}{m} + \lambda\frac{b}{m}\right)\right) \right] \right. \\ & \quad \left. - f\left(\frac{\varphi(\lambda a + (1 - \lambda)b) + m\varphi\left((1 - \lambda)\frac{a}{m} + \lambda\frac{b}{m}\right)}{2}\right) \right| \\ & \leq \\ & h\left(\frac{1}{2}\right) \left[ g\left(\varphi\left(\lambda a + (1 - \lambda)b\right)\right) + mg\left(\varphi\left((1 - \lambda)\frac{a}{m} + \lambda\frac{b}{m}\right)\right) \right] \\ & \quad - g\left(\frac{\varphi(\lambda a + (1 - \lambda)b) + m\varphi\left((1 - \lambda)\frac{a}{m} + \lambda\frac{b}{m}\right)}{2}\right). \end{aligned}$$

Then using the linearity of  $\varphi$ -function, we have

$$\begin{aligned} & \left| h\left(\frac{1}{2}\right) \left[ f\left(\lambda\varphi(a) + (1 - \lambda)\varphi(b)\right) + mf\left(\frac{1 - \lambda}{m}\varphi(a) + \frac{\lambda}{m}\varphi(b)\right) \right] - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right| \\ & \leq \\ & h\left(\frac{1}{2}\right) \left[ g\left(\lambda\varphi(a) + (1 - \lambda)\varphi(b)\right) + mg\left(\frac{1 - \lambda}{m}\varphi(a) + \frac{\lambda}{m}\varphi(b)\right) \right] - g\left(\frac{\varphi(a) + \varphi(b)}{2}\right). \end{aligned}$$

If we integrate the above inequality with respect to  $\lambda$  over  $[0, 1]$ , the inequality (2.2) is proved.

Since  $f$  is  $(g, \varphi_{h,m})$ -convex dominated on  $[0, b]$ , we have

$$\begin{aligned} & |h(t)f(\varphi(x)) + mh(1 - t)f(\varphi(y)) - f(t\varphi(x) + m(1 - t)\varphi(y))| \\ & \leq h(t)g(\varphi(x)) + mh(1 - t)g(\varphi(y)) - g(t\varphi(x) + m(1 - t)\varphi(y)) \end{aligned}$$

for all  $x, y > 0$  which gives for  $x = a$  and  $y = b$

$$\begin{aligned} & \left| h(t)f(\varphi(a)) + mh(1 - t)f\left(\varphi\left(\frac{b}{m}\right)\right) - f\left(t\varphi(a) + m(1 - t)\varphi\left(\frac{b}{m}\right)\right) \right| \\ & \leq \\ & h(t)g(\varphi(a)) + mh(1 - t)g\left(\varphi\left(\frac{b}{m}\right)\right) - g\left(t\varphi(a) + m(1 - t)\varphi\left(\frac{b}{m}\right)\right) \end{aligned}$$

and for  $x = \frac{a}{m}$ ,  $y = \frac{b}{m^2}$ , multiplying with  $m$ ,

$$\begin{aligned} & \left| mh(t)f\left(\varphi\left(\frac{a}{m}\right)\right) + m^2h(1 - t)f\left(\varphi\left(\frac{b}{m^2}\right)\right) - mf\left(t\varphi\left(\frac{a}{m}\right) + m(1 - t)\varphi\left(\frac{b}{m^2}\right)\right) \right| \\ & \leq \\ & mh(t)g\left(\varphi\left(\frac{a}{m}\right)\right) + m^2h(1 - t)g\left(\varphi\left(\frac{b}{m^2}\right)\right) - mg\left(t\varphi\left(\frac{a}{m}\right) + m(1 - t)\varphi\left(\frac{b}{m^2}\right)\right) \end{aligned}$$



for all  $t \in (0, 1)$ . By properties of modulus and adding the above inequalities side by side we get,

$$\begin{aligned} & \left| h(t) \left[ f\left(\varphi\left(\frac{a}{m}\right)\right) + mf\left(\varphi\left(\frac{a}{m}\right)\right) \right] + mh(1-t) \left[ f\left(\varphi\left(\frac{b}{m}\right)\right) + mf\left(\varphi\left(\frac{b}{m^2}\right)\right) \right] \right. \\ & \left. - \left[ f\left(t\varphi\left(\frac{a}{m}\right) + m(1-t)\varphi\left(\frac{b}{m}\right)\right) + mf\left(t\varphi\left(\frac{a}{m}\right) + m(1-t)\varphi\left(\frac{b}{m^2}\right)\right) \right] \right| \\ & \leq \\ & h(t) \left[ g\left(\varphi\left(\frac{a}{m}\right)\right) + mg\left(\varphi\left(\frac{a}{m}\right)\right) \right] + mh(1-t) \left[ g\left(\varphi\left(\frac{b}{m}\right)\right) + mg\left(\varphi\left(\frac{b}{m^2}\right)\right) \right] \\ & - \left[ g\left(t\varphi\left(\frac{a}{m}\right) + m(1-t)\varphi\left(\frac{b}{m}\right)\right) + mg\left(t\varphi\left(\frac{a}{m}\right) + m(1-t)\varphi\left(\frac{b}{m^2}\right)\right) \right] \end{aligned}$$

Thus, integrating over  $t$  on  $[0, 1]$  we obtain the inequality (2.3). □

**Remark 1.** Under the assumptions of Theorem 5, if we choose  $m = 1$ , the inequalities (2.2) and (2.3) reduce to Hermite-Hadamard type inequalities for  $(g, \varphi_h)$ -convex dominated functions given as (1.15) and (1.16) in [11].

**Remark 2.** Under the assumptions of Theorem 5, if we choose  $m = 1$ ,  $h(t) = t$ ,  $t \in (0, 1)$  and the function  $\varphi$  is the identity, then the inequalities (2.2) and (2.3) reduce to Hermite-Hadamard type inequalities for convex-dominated functions given as (1.9) and (1.10) in [2].

**Remark 3.** Under the assumptions of Theorem 5, if we choose  $m = 1$ ,  $h(t) = t^s$ ,  $t, s \in (0, 1)$  and the function  $\varphi$  is the identity, then the inequalities (2.2) and (2.3) reduce to Hermite-Hadamard type inequalities for  $(g, s)$ -convex-dominated functions given as (1.9) and (1.10) in [4].

**Remark 4.** Under the assumptions of Theorem 5, if we choose  $m = 1$ ,  $h(t) = \frac{1}{t}$ ,  $t \in (0, 1)$  and the function  $\varphi$  is the identity, then the inequalities (2.2) and (2.3) reduce to Hermite-Hadamard type inequalities for  $(g, Q(I))$ -convex-dominated functions given as (1.9) and (1.10) in [6].

### 3. $(g, \log \varphi)$ -CONVEX DOMINATED FUNCTIONS

**Definition 9.** Let  $g : [a, b] \subseteq \mathbb{R} \rightarrow (0, \infty)$  be a given  $\log - \varphi$ -convex mapping where  $\varphi : [a, b] \rightarrow [a, b]$ . The real function  $f : [a, b] \rightarrow (0, \infty)$  is called  $(g, \log - \varphi)$ -convex dominated on  $[a, b]$  if it holds

$$\begin{aligned} (3.1) \quad & \left| [f(\varphi(x))]^t [f(\varphi(y))]^{1-t} - f(t\varphi(x) + (1-t)\varphi(y)) \right| \\ & \leq [g(\varphi(x))]^t [g(\varphi(y))]^{1-t} - g(t\varphi(x) + (1-t)\varphi(y)) \end{aligned}$$

for all  $x, y \in [a, b]$  and  $t \in [0, 1]$ .

**Proposition 1.** Let  $g : [a, b] \subseteq \mathbb{R} \rightarrow (0, \infty)$  be a given  $\log - \varphi$ -convex mapping where  $\varphi : [a, b] \rightarrow [a, b]$  and  $f : [a, b] \rightarrow (0, \infty)$  be a  $(g, \log - \varphi)$ -convex dominated function on  $[a, b]$ . Then the mapping  $g + f$  is  $\varphi$ -convex on  $[a, b]$ .

*Proof.* The condition (3.1) is equivalent to

$$\begin{aligned} & g(t\varphi(x) + (1-t)\varphi(y)) - [g(\varphi(x))]^t [g(\varphi(y))]^{1-t} \\ & \leq [f(\varphi(x))]^t [f(\varphi(y))]^{1-t} - f(t\varphi(x) + (1-t)\varphi(y)) \\ & \leq [g(\varphi(x))]^t [g(\varphi(y))]^{1-t} - g(t\varphi(x) + (1-t)\varphi(y)) \end{aligned}$$

for all  $x, y \in [a, b]$  and  $t \in [0, 1]$ . The left side of the inequality may be rearranged as

$$\begin{aligned} & (g + f)(t\varphi(x) + (1-t)\varphi(y)) \\ & \leq [f(\varphi(x))]^t [f(\varphi(y))]^{1-t} + [g(\varphi(x))]^t [g(\varphi(y))]^{1-t} \\ & \leq tf(\varphi(x)) + (1-t)f(\varphi(y)) + tg(\varphi(x)) + (1-t)g(\varphi(y)) \\ & = t(f + g)(\varphi(x)) + (1-t)(f + g)(\varphi(y)) \end{aligned}$$

which is equivalent to the  $\varphi$ -convexity of  $f + g$ . □

**Theorem 6.** Let  $g : [a, b] \subseteq \mathbb{R} \rightarrow (0, \infty)$  be a given  $\log - \varphi$ -convex mapping and  $f : [a, b] \rightarrow (0, \infty)$  is Lebesgue integrable and  $(g, \log - \varphi)$ -convex dominated function on  $[a, b]$  for linear continuous function  $\varphi : [a, b] \rightarrow [a, b]$ , then the following inequalities hold:

$$\begin{aligned} (3.2) \quad & \left| \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} G(f(x), f(\varphi(a) + \varphi(b) - x)) dx - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right| \\ & \leq \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} G(g(x), g(\varphi(a) + \varphi(b) - x)) dx - g\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \end{aligned}$$

$$\begin{aligned} (3.3) \quad & \left| \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} G(f(x), f(\varphi(a) + \varphi(b) - x)) dx - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right| \\ & \leq \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} g(x) dx - g\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \end{aligned}$$

and

$$\begin{aligned} (3.4) \quad & \left| L(f(\varphi(b)), f(\varphi(a))) - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(x) dx \right| \\ & \leq L(g(\varphi(b)), g(\varphi(a))) - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} g(x) dx \end{aligned}$$

for all  $x, y \in [a, b]$ .

*Proof.* By the Definition 3.1 with  $t = \frac{1}{2}$ ,  $x = \lambda a + (1 - \lambda)b$ ,  $y = (1 - \lambda)a + \lambda b$  and  $\lambda \in [0, 1]$ , as the mapping  $f$  is  $(g, \log -\varphi)$ -convex dominated function, we have that

$$\begin{aligned} & \left| [f(\varphi(\lambda a + (1 - \lambda)b))]^{\frac{1}{2}} [f(\varphi((1 - \lambda)a + \lambda b))]^{\frac{1}{2}} - f\left(\frac{\varphi(\lambda a + (1 - \lambda)b) + \varphi((1 - \lambda)a + \lambda b)}{2}\right) \right| \\ & \leq [g(\varphi(\lambda a + (1 - \lambda)b))]^{\frac{1}{2}} [g(\varphi((1 - \lambda)a + \lambda b))]^{\frac{1}{2}} - g\left(\frac{\varphi(\lambda a + (1 - \lambda)b) + \varphi((1 - \lambda)a + \lambda b)}{2}\right). \end{aligned}$$

Then using the linearity of  $\varphi$ -function we have

$$\begin{aligned} (3.5) \quad & \left| [f(\lambda\varphi(a) + (1 - \lambda)\varphi(b))]^{\frac{1}{2}} [f((1 - \lambda)\varphi(a) + \lambda\varphi(b))]^{\frac{1}{2}} - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right| \\ & \leq [g(\lambda\varphi(a) + (1 - \lambda)\varphi(b))]^{\frac{1}{2}} [g((1 - \lambda)\varphi(a) + \lambda\varphi(b))]^{\frac{1}{2}} - g\left(\frac{\varphi(a) + \varphi(b)}{2}\right). \end{aligned}$$

If we integrate the above inequality with respect to  $\lambda$  over  $[0, 1]$ , the inequality in (3.2) is proved.

On the other hand, if we use the inequality  $\sqrt{ab} \leq \frac{1}{2}(a + b)$  for  $a, b > 0$  on (3.5) we have

$$\begin{aligned} & \left| [f(\lambda\varphi(a) + (1 - \lambda)\varphi(b))]^{\frac{1}{2}} [f((1 - \lambda)\varphi(a) + \lambda\varphi(b))]^{\frac{1}{2}} - f\left(\frac{\varphi(a) + \varphi(b)}{2}\right) \right| \\ & \leq \frac{g(\lambda\varphi(a) + (1 - \lambda)\varphi(b)) + g((1 - \lambda)\varphi(a) + \lambda\varphi(b))}{2} - g\left(\frac{\varphi(a) + \varphi(b)}{2}\right). \end{aligned}$$

If we integrate the above inequality with respect to  $\lambda$  over  $[0, 1]$ , the inequality in (3.3) is proved.

To prove the inequality in (3.4), firstly we use the Definition 3.1 for  $x = a$  and  $y = b$ , we have

$$\begin{aligned} & \left| [f(\varphi(a))]^t [f(\varphi(b))]^{1-t} - f(t\varphi(a) + (1 - t)\varphi(b)) \right| \\ & \leq [g(\varphi(a))]^t [g(\varphi(b))]^{1-t} - g(t\varphi(a) + (1 - t)\varphi(b)) \end{aligned}$$

Then, we integrate the above inequality with respect to  $t$  over  $[0, 1]$ , we get

$$\begin{aligned} & \left| f(\varphi(b)) \int_0^1 \left[ \frac{f(\varphi(a))}{f(\varphi(b))} \right]^t dt - \int_0^1 f(t\varphi(a) + (1 - t)\varphi(b)) dt \right| \\ & \leq g(\varphi(b)) \int_0^1 \left[ \frac{g(\varphi(a))}{g(\varphi(b))} \right]^t dt - \int_0^1 g(t\varphi(a) + (1 - t)\varphi(b)) dt \\ & = g(\varphi(b)) \left( \frac{g(\varphi(a))}{g(\varphi(b))} - 1 \right) \frac{1}{\log g(\varphi(a)) - \log g(\varphi(b))} - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} g(x) dx \\ & = \frac{g(\varphi(b)) - g(\varphi(a))}{\log f(\varphi(b)) - \log f(\varphi(a))} - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} g(x) dx \\ & = L(g(\varphi(b)), g(\varphi(a))) - \frac{1}{\varphi(b) - \varphi(a)} \int_{\varphi(a)}^{\varphi(b)} g(x) dx. \end{aligned}$$

If similar calculation is made for the function  $f$ , the proof is completed.  $\square$

**Corollary 1.** *If function  $\varphi$  is the identity in (3.2), (3.3) and (3.4), then we have*

$$(3.6) \quad \left| \frac{1}{b-a} \int_a^b G(f(x), f(a+b-x)) dx - f\left(\frac{a+b}{2}\right) \right|$$

$$\leq \frac{1}{b-a} \int_a^b G(g(x), g(a+b-x)) dx - g\left(\frac{a+b}{2}\right)$$

$$(3.7) \quad \left| \frac{1}{b-a} \int_a^b G(f(x), f(a+b-x)) dx - f\left(\frac{a+b}{2}\right) \right|$$

$$\leq \frac{1}{b-a} \int_a^b g(x) dx - g\left(\frac{a+b}{2}\right)$$

and

$$(3.8) \quad \left| L(f(b), f(a)) - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\leq L(g(b), g(a)) - \frac{1}{b-a} \int_a^b g(x) dx$$

for all  $x, y \in [a, b]$ .

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**FIXED POINT THEOREM AND A UNIQUENESS THEOREM  
CONCERNING THE STABILITY OF FUNCTIONAL EQUATIONS  
IN MODULAR SPACES**

CHANGIL KIM

ABSTRACT. In this paper, we investigate a fixed point theorem in modular spaces, whose induced modular is lower semi-continuous, for a mapping with some conditions in place of the condition of bounded orbit. Using this fixed point theorem, we prove the generalized Hyers-Ulam stability for the following additive-quadratic functional equation

$$f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 2f(2x) + 4f(x) = 0$$

in modular spaces.

1. INTRODUCTION AND PRELIMINARIES

A problem that mathematicians has dealt with, for almost fifty years, is how to generalize the classical function space  $L^p$ . A first attempt was made by Birnbaum and Orlicz in 1931. This generalization found many applications in differential and integral equations with kernels of nonpower types. The more abstract generalization was given by Nakano [13] based on replacing the particular integral form of the functional by an abstract one that satisfies some good properties. This functional was called modular. This idea was refined and generalized by Musielak and Orlicz [11] in 1959. Modular spaces have been studied for almost forty years and there is a large set of known applications of them in various parts of analysis([6], [7], [9], [10], [12], [14], [17], [20]).

It is well known that fixed point theories are one of powerful tools in solving mathematical problems. Banach's contraction principle is one of the pivotal results in fixed point theories and they have a board set of applications. Khamsi, Kozowski and Reich [4] investigated the fixed point theorem in modular spaces. In [5], Khamsi proved a series of fixed point theorems in modular spaces, where the modulars do not satisfy  $\Delta_2$ -conditions.

**Lemma 1.1.** [5] *Let  $X_\rho$  be a modular space whose induced modular is lower semi-continuous and let  $C \subseteq X_\rho$  be a  $\rho$ -complete subset. If  $T : C \rightarrow C$  is a  $\rho$ -contraction, that is, there is a constant  $L \in [0, 1)$  such that*

$$\rho(Tx - Ty) \leq L\rho(x - y), \quad \forall x, y \in C$$

*and  $T$  has a bounded orbit at a point  $x_0 \in C$ , then the sequence  $\{T^n x_0\}$  is  $\rho$ -convergent to a point  $w \in C$ .*

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The stability problem for functional equations first was planed in 1940 by Ulam [18].

“Let  $G_1$  be a group and  $G_2$  a metric group with the metric  $d$ . Given a constant  $\delta > 0$ , does there exist a constant  $c > 0$  such that if a mapping  $f : G_1 \rightarrow G_2$  satisfies  $d(f(xy), f(x)f(y)) < c$  for all  $x, y \in G_1$ , then there exists a unique homomorphism  $h : G_1 \rightarrow G_2$  with  $d(f(x), h(x)) < \delta$  for all  $x \in G_1$ ?”

In the next year, Hyers [3] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers’ theorem was generalized by Aoki [1] for additive mappings and by Rassias [15] for linear mappings by considering an unbounded Cauchy difference, the latter of which has influenced many developments in the stability theory. This area is then referred to as the generalized Hyers-Ulam stability. In 1994, P. Găvruta [2] generalized these theorems for approximate additive mappings controlled by the unbounded Cauchy difference with regular conditions.

Recently, Sadeghi [16] presented a fixed point method to prove the generalized Hyers-Ulam stability of functional equations in modular spaces with the  $\Delta_2$ -condition and using the fixed point theorem Lemma 1.1, Wongkum, Chaipunya, and Kumam [19] proved the generalized Hyers-Ulam stability for quadratic mappings in a modular space whose modular is convex, lower semi-continuous but do not satisfy the  $\Delta_2$ -condition.

Lee and Jung [8] proved a general uniqueness theorem that can be easily applied to the (generalized) Hyers-Ulam stability of the Cauchy additive functional equation, the quadratic functional equation, and the quadratic-additive type functional equations in Banach spaces.

In this paper, we investigate a fixed point theorem in modular spaces, whose induced modular is lower semi-continuous, for a mapping with some conditions in place of the condition of a bounded orbit. Using this fixed point theorem, we will prove a general uniqueness theorem that can be applied to the generalized Hyers-Ulam stability of additive-quadratic functional equations in modular spaces without  $\Delta_2$ -conditions.

## 2. FIXED POINT THEOREMS IN MODULAR SPACES

In this section, we will prove a fixed point theorem in modular spaces, whose induced modular is (convex) lower semi-continuous, for a mapping with some conditions in place of the condition of a bounded orbit.

**Definition 2.1.** Let  $X$  be a vector space over a field  $\mathbb{K}(=\mathbb{R}$  or  $\mathbb{C})$ .

- (1) A generalized functional  $\rho : X \rightarrow [0, \infty]$  is called a *modular* if
  - (M1)  $\rho(x) = 0$  if and only if  $x = 0$ ,
  - (M2)  $\rho(\alpha x) = \rho(x)$  for every scalar  $\alpha$  with  $|\alpha| = 1$ , and
  - (M3)  $\rho(z) \leq \rho(x) + \rho(y)$  whenever  $z$  is a convex combination of  $x$  and  $y$ .
- (2) If (M3) is replaced by
  - (M4)  $\rho(\alpha x + \beta y) \leq \alpha \rho(x) + \beta \rho(y)$

for all  $x, y \in V$  and for all nonnegative real numbers  $\alpha, \beta$  with  $\alpha + \beta = 1$ , then we say that  $\rho$  is *convex*.

**Remark 2.2.** Let  $\rho$  be a modular on a vector space  $X$ . Then by (M1) and (M3), we can easily show that for any positive real number  $\delta$  with  $\delta < 1$ ,

$$\rho(\delta x) \leq \rho(x)$$

for all  $x \in X$  and hence we have

$$\rho(x) \leq \rho(2x)$$

for all  $x \in X$ .

For any modular  $\rho$  on  $X$ , the modular space  $X_\rho$  is defined by

$$X_\rho := \{x \in X \mid \rho(\lambda x) \rightarrow 0 \text{ as } \lambda \rightarrow 0\}.$$

Let  $X_\rho$  be a modular space and let  $\{x_n\}$  be a sequence in  $X_\rho$ . Then (i)  $\{x_n\}$  is called  $\rho$ -convergent to a point  $x \in X_\rho$ , denoted by

$$\lim_{n \rightarrow \infty} x_n =_\rho x \text{ or } x =_\rho \lim_{n \rightarrow \infty} x_n,$$

if  $\rho(x_n - x) \rightarrow 0$  as  $n \rightarrow \infty$ , (ii)  $\{x_n\}$  is called  $\rho$ -Cauchy if for any  $\epsilon > 0$ , one has  $\rho(x_n - x_m) < \epsilon$  for sufficiently large  $m, n \in \mathbb{N}$ , and (iii) a subset  $K$  of  $X_\rho$  is called  $\rho$ -complete if each  $\rho$ -Cauchy sequence in  $K$  is  $\rho$ -convergent to a point in  $K$ .

**Proposition 2.3.** *Let  $\rho$  be a modular on a vector space  $X$  and  $S : X \rightarrow X$  an one-to-one linear map. Define a map  $\rho_S : X \rightarrow [0, \infty]$  by*

$$\rho_S(x) = \rho(S(x)), \forall x \in X.$$

*Then we have*

- (1)  $\rho_S$  is a modular on  $X$ ,
- (2) if  $\rho$  is a convex modular on  $X$ , then  $\rho_S$  is a convex modular on  $X$ , and
- (3) if  $\rho$  is lower semi-continuous, then  $\rho_S$  is lower semi-continuous.

*Suppose that  $S$  is an isomorphism. Then we have*

- (4)  $S(X_{\rho_S}) = X_\rho$  and
- (5) if  $C$  is a  $\rho$ -complete subset of  $X_\rho$  and  $S(C) = C$ , then  $C$  is a  $\rho_S$ -complete subset of  $X_{\rho_S}$ .

*Proof.* (1) Suppose that  $\rho_S(x) = 0$ . Then by (M1),  $S(x) = 0$  and since  $S$  is one-to-one,  $x = 0$ . If  $x = 0$ , then  $\rho_S(0) = \rho(S(0)) = \rho(0) = 0$ . Hence  $\rho_S$  satisfies (M1). Since  $S$  is a linear map,  $\rho_S$  satisfies (M2). Let  $z = \alpha x + \beta y$  for  $x, y \in X$  and non-negative real numbers  $\alpha, \beta$  with  $\alpha + \beta = 1$ . Since  $S$  is a linear map and  $\rho$  is a modular, by (M3), we have

$$\rho_S(\alpha x + \beta y) = \rho(\alpha S(x) + \beta S(y)) \leq \rho(S(x)) + \rho(S(y)) \leq \rho_S(x) + \rho_S(y)$$

and thus  $\rho_S$  satisfies (M3).

(2) is trivial.

(3) Suppose that  $\{x_n\}$  is a sequence in  $X_{\rho_S}$  such that  $\{x_n\}$  is  $\rho_S$ -convergent to  $x$  in  $X_{\rho_S}$ . Then  $\{S(x_n)\}$  is  $\rho$ -convergent to  $S(x)$ . Since  $\rho$  is lower semi-continuous and  $\{S(x_n)\}$  is  $\rho$ -convergent to  $S(x)$ ,

$$(2.1) \quad \rho_S(x) = \rho(S(x)) \leq \liminf_{n \rightarrow \infty} \rho(S(x_n)) \leq \liminf_{n \rightarrow \infty} \rho_S(x_n).$$

and hence  $\rho_S$  is lower semi-continuous.

(4) Let  $x \in X_{\rho_S}$ . Then

$$\lim_{\lambda \rightarrow 0} \rho(\lambda x) = \lim_{\lambda \rightarrow 0} \rho_S(\lambda S^{-1}(x)) = 0$$

and so  $S^{-1}(x) \in X_\rho$ . Hence  $X_\rho \subseteq S(X_{\rho_S})$ . For the converse, let  $x \in X_{\rho_S}$ . Then

$$\lim_{\lambda \rightarrow 0} \rho_S(\lambda x) = \lim_{\lambda \rightarrow 0} \rho(\lambda S(x)) = 0$$



and so  $S(x) \in X_\rho$ . Hence  $S(X_{\rho_S}) \subseteq X_\rho$ .

(5) Suppose that  $C$  is a  $\rho$ -complete subset of  $X_\rho$  with  $S(C) = C$ . By (4),  $C \subseteq X_{\rho_S}$ . Let  $\{x_n\}$  be a  $\rho_S$ -Cauchy sequence in  $C$ . For any  $n, m \in \mathbb{N}$ ,

$$\rho(S(x_n) - S(x_m)) = \rho_S(x_n - x_m),$$

$\{S(x_n)\}$  is a  $\rho$ -Cauchy sequence in  $C$ . Since  $C$  is a  $\rho$ -complete subset of  $X_\rho$ , there is an  $y \in C$  such that  $\{S(x_n)\}$  is  $\rho$ -convergent to  $y$ . Then clearly,  $\{x_n\}$  is  $\rho_S$ -convergent to  $S^{-1}(y) \in C$  and so  $C$  is a  $\rho_S$ -complete subset of  $X_{\rho_S}$ .  $\square$

A modular space  $X_\rho$  is said to *satisfy the  $\Delta_2$ -condition* if there exists  $k \geq 2$  such that  $\rho(2x) \leq k\rho(x)$  for all  $x \in X$ .

Now, we will prove a fixed point theorem in modular spaces where the map  $T$  do not assume to be the boundedness of an orbit. Our results exploit one unifying hypothesis in which some conditions are assumed.

**Lemma 2.4.** *Let  $X_\rho$  be a modular space whose induced modular is lower semi-continuous and let  $C \subseteq X_\rho$  be a  $\rho$ -complete subset. Let  $S : X \rightarrow X$  be an isomorphism and  $T : C \rightarrow C$  a mapping such that  $S(C) = C$  and  $STx = TSx$  for all  $x \in C$ . Suppose that there is a constant  $L \in [0, 1)$  and  $x_0 \in C$  such that  $\rho(Tx_0 - x_0) < \infty$  and*

$$(2.2) \quad \rho(x + y) \leq \rho_S(x) + \rho_S(y), \quad \rho_S(Tx - Ty) \leq L\rho(x - y), \quad \forall x, y \in C.$$

*Then there is a unique fixed point  $w \in C$  of  $T$  such that*

$$(2.3) \quad \rho(S^{-2}x_0 - w) \leq \frac{2}{1-L}\rho(Tx_0 - x_0).$$

*Further, we have*

$$\lim_{n \rightarrow \infty} \rho(T^n S^{-2}x_0) =_\rho w.$$

*Proof.* By Proposition 2.3,  $\rho_S$  is a modular,  $C$  is a  $\rho_S$ -complete subset of  $X_{\rho_S}$ , and  $S(X_{\rho_S}) = X_\rho$ . By (M1) and (2.2), we have

$$\rho(x) \leq \rho_S(x), \quad \rho_S(Tx - Ty) \leq L\rho(x - y) \leq L\rho_S(x - y)$$

for all  $x, y \in C$  and so  $T$  is a  $\rho_S$ -contraction. By (M1) and (2.2), we have

$$\begin{aligned} \rho_S(S^{-2}T^2x_0 - S^{-2}x_0) &= \rho(S^{-1}T^2x_0 - S^{-1}x_0) \\ &\leq \rho_S(S^{-1}T^2x_0 - S^{-1}Tx_0) + \rho_S(S^{-1}Tx_0 - S^{-1}x_0) \\ &\leq L\rho(S^{-1}Tx_0 - S^{-1}x_0) + \rho(Tx_0 - x_0) \\ &\leq (L + 1)\rho(Tx_0 - x_0). \end{aligned}$$

and

$$\begin{aligned} \rho_S(S^{-2}T^n x_0 - S^{-2}x_0) &\leq \rho_S(S^{-1}T^n x_0 - S^{-1}Tx_0) + \rho_S(S^{-1}Tx_0 - S^{-1}x_0) \\ &\leq L\rho(S^{-1}T^{n-1}x_0 - S^{-1}x_0) + \rho(Tx_0 - x_0) \\ &= L\rho_S(S^{-2}T^{n-1}x_0 - S^{-2}x_0) + \rho(Tx_0 - x_0) \end{aligned}$$

for all  $n \in \mathbb{N}$ . By induction, we have

$$\rho_S(S^{-2}T^n x_0 - S^{-2}x_0) \leq \sum_{k=0}^{n-1} L^k \rho(Tx_0 - x_0) \leq \frac{1}{1-L} \rho(Tx_0 - x_0)$$

for all  $n \in \mathbb{N}$ . For any non-negative integers  $m, n$  with  $m > n$ ,

$$\begin{aligned}
 (2.4) \quad \rho_S(S^{-3}T^n x_0 - S^{-3}T^m x_0) &= \rho(S^{-2}T^n x_0 - S^{-2}T^m x_0) \\
 &\leq \rho_S(S^{-2}T^n x_0 - S^{-2}x_0) + \rho_S(S^{-2}T^m x_0 - S^{-2}x_0) \\
 &\leq \frac{2}{1-L} \rho(Tx_0 - x_0).
 \end{aligned}$$

Since  $STx = TSx$  for all  $x \in C$ ,  $T$  has a bounded orbit at a point  $S^{-3}x_0$  in  $C \subseteq X_{\rho_S}$  and thus by Lemma 1.1,  $\{T^n S^{-3}x_0\}$  is  $\rho_S$ -convergent to a point  $\omega_0 \in C$ . Let  $w = S\omega_0$ . Then

$$\lim_{n \rightarrow \infty} \rho(T^n S^{-2}x_0) =_{\rho} w$$

and since  $\rho_S$  is lower semi-continuous, by (2.4), we have (2.3).

Now, we claim that  $w$  is a unique fixed point of  $T$ . Since  $\rho_S$  is a lower semi-continuous, we have

$$\begin{aligned}
 \rho(w - Tw) = \rho_S(w_0 - Tw_0) &\leq \liminf_{n \rightarrow \infty} \rho_S(T^{n+1}S^{-3}x_0 - Tw_0) \\
 &\leq \liminf_{n \rightarrow \infty} L\rho(T^n S^{-3}x_0 - w_0) = 0
 \end{aligned}$$

and hence  $w$  is a fixed point of  $T$ . Suppose that  $v$  is another fixed point of  $T$ . Since  $STx = TSx$  for all  $x \in C$ , by (2.2) and (2.3), we have

$$\begin{aligned}
 \rho(S^{-1}w - S^{-1}v) &\leq \rho_S(T^n w - T^n S^{-2}x_0) + \rho_S(T^n S^{-2}x_0 - T^n v) \\
 &\leq L^n \frac{4}{1-L} \rho(Tx_0 - x_0)
 \end{aligned}$$

for all  $n \in \mathbb{N}$  and since  $0 \leq L < 1$ ,  $w = v$ . □

Replacing (2.2) by (2.5), we have the following lemma. The proof is similar to the proof of Lemma 2.4.

**Lemma 2.5.** *Let  $X_{\rho}$  be a modular space whose induced modular is lower semi-continuous and let  $C \subseteq X_{\rho}$  be a  $\rho$ -complete subset. Let  $S : X \rightarrow X$  be an isomorphism and  $T : C \rightarrow C$  a mapping such that  $S(C) = C$  and  $STx = TSx$  for all  $x \in C$ . Suppose that there are real numbers  $r, L$  and  $x_0 \in C$  such that  $0 < r < 1$ ,  $L \in [0, \frac{1}{r})$ ,  $\rho(Tx_0 - x_0) < \infty$ , and*

$$(2.5) \quad \rho(x + y) \leq r\rho_S(x) + r\rho_S(y), \quad \rho_S(Tx - Ty) \leq L\rho(x - y), \quad \forall x, y \in C.$$

*Then there is a unique fixed point  $w \in C$  of  $T$  such that*

$$\lim_{n \rightarrow \infty} \rho(T^n S^{-2}x_0) =_{\rho} w$$

and

$$(2.6) \quad \rho(S^{-2}x_0 - w) \leq \frac{2r^2}{1-rL} \rho(Tx_0 - x_0).$$

*Proof.* By Proposition 2.3,  $\rho_S$  is a modular,  $C$  is a  $\rho_S$ -complete subset of  $X_{\rho_S}$ , and  $S(X_{\rho_S}) = X_{\rho}$ . By (M1) and (2.5), we have

$$\rho(x) \leq \rho_S(x), \quad \rho_S(Tx - Ty) \leq L\rho(x - y) \leq rL\rho_S(x - y)$$

for all  $x, y \in C$  and so  $T$  is a  $\rho_S$ -contraction. By (M1) and (2.5), we have

$$\begin{aligned} \rho_S(S^{-2}T^2x_0 - S^{-2}x_0) &= \rho_S(S^{-1}T^2x_0 - S^{-1}x_0) \\ &\leq r\rho_S(S^{-1}T^2x_0 - S^{-1}Tx_0) + r\rho_S(S^{-1}Tx_0 - S^{-1}x_0) \\ &\leq rL\rho(S^{-1}Tx_0 - S^{-1}x_0) + r\rho(Tx_0 - x_0) \\ &\leq r(rL + 1)\rho(Tx_0 - x_0). \end{aligned}$$

and

$$\begin{aligned} \rho_S(S^{-2}T^nx_0 - S^{-2}x_0) &\leq r\rho_S(S^{-1}T^nx_0 - S^{-1}Tx_0) + r\rho_S(S^{-1}Tx_0 - S^{-1}x_0) \\ &\leq rL\rho(S^{-1}T^{n-1}x_0 - S^{-1}x_0) + r\rho(Tx_0 - x_0) \\ &= rL\rho_S(S^{-2}T^{n-1}x_0 - S^{-2}x_0) + r\rho(Tx_0 - x_0) \end{aligned}$$

for all  $n \in \mathbb{N}$ . By induction, we have

$$\rho_S(S^{-2}T^nx_0 - S^{-2}x_0) \leq \sum_{k=0}^{n-1} r^{k+1} L^k \rho(Tx_0 - x_0) \leq \frac{r}{1 - rL} \rho(Tx_0 - x_0)$$

for all  $n \in \mathbb{N}$ . For any non-negative integers  $m, n$  with  $m > n$ ,

$$\begin{aligned} \rho_S(S^{-3}T^nx_0 - S^{-3}T^mx_0) &= \rho(S^{-2}T^nx_0 - S^{-2}T^mx_0) \\ &\leq r\rho_S(S^{-2}T^nx_0 - S^{-2}x_0) + r\rho_S(S^{-2}T^mx_0 - S^{-2}x_0) \\ &\leq \frac{2r^2}{1 - rL} \rho(Tx_0 - x_0). \end{aligned}$$

The rest of the proof is similar to the proof of Lemma 2.4 . □

### 3. UNIQUENESS THEOREM FOR THE STABILITY OF FUNCTIONAL EQUATIONS AND ITS APPLICATIONS

Throughout this section, we assume that  $V$  is a linear space and  $X_\rho$  is a  $\rho$ -complete modular space whose induced modular is lower semi-continuous. In this section, we prove that, if for given map  $f : V \rightarrow X_\rho$ , there is a mapping  $F : V \rightarrow X_\rho$ , which is near  $f$  in  $X_\rho$ , with some properties possessed by additive-quadratic mappings, then  $F$  is uniquely determined.

Define a set  $\mathbb{M}$  by

$$\mathbb{M} := \{g : V \rightarrow X_\rho \mid g(0) = 0\}$$

and a generalized function  $\tilde{\rho}$  on  $\mathbb{M}$  by for each  $g \in \mathbb{M}$ ,

$$\tilde{\rho}(g) := \inf \{c > 0 \mid \rho(g(x)) \leq c\Psi(x), \forall x \in V\},$$

where  $\Psi : V \rightarrow [0, \infty)$  is a mapping.

Similar to Lemma 10 in [19], we have the following lemma :

**Lemma 3.1.** *We have the following :*

- (1)  $\mathbb{M}$  is a linear space,
- (2)  $\tilde{\rho}$  is a modular on  $\mathbb{M}$ , and
- (3) if  $\rho$  is convex, then  $\tilde{\rho}$  is convex,
- (4)  $\mathbb{M}_{\tilde{\rho}} = \mathbb{M}$  and  $\mathbb{M}_{\tilde{\rho}}$  is  $\tilde{\rho}$ -complete, and
- (5)  $\tilde{\rho}$  is lower semi-continuous.

*Proof.* (1), (2), and (3) are trivial.

(4) By the definition of  $\mathbb{M}_{\tilde{\rho}}$ ,  $\mathbb{M}_{\tilde{\rho}} = \mathbb{M}$ . Take any  $\tilde{\rho}$ -Cauchy sequence  $\{g_n\}$  in  $\mathbb{M}_{\tilde{\rho}}$ . Then  $\{g_n(x)\}$  is a  $\rho$ -Cauchy sequence in  $X_\rho$  for all  $x \in X$ . Since  $X_\rho$  is  $\rho$ -complete, there is a mapping  $g : V \rightarrow X_\rho$  such that  $\rho(g_n(x) - g(x)) \rightarrow 0$  as  $n \rightarrow \infty$  for all  $x \in X$ . For any  $\epsilon > 0$ , there is an  $m \in \mathbb{N}$  such that

$$\rho(g_m(0) - g(0)) = \rho(g(0)) \leq \epsilon$$

and hence  $g \in \mathbb{M}_{\tilde{\rho}} = \mathbb{M}$ . Let  $\delta > 0$  be given. Since  $\{g_n\}$  is a  $\tilde{\rho}$ -Cauchy sequence in  $\mathbb{M}_{\tilde{\rho}}$ , there is a  $k \in \mathbb{N}$  such that for any  $n, m \in \mathbb{N}$  with  $n, m \geq k$ ,

$$\rho(g_n(x) - g_m(x)) \leq \delta \Psi(x), \forall x \in V$$

and since  $\rho$  is a lower semi-continuous, we have

$$\rho(g_n(x) - g(x)) \leq \liminf_{m \rightarrow \infty} \rho(g_n(x) - g_m(x)) \leq \delta \Psi(x)$$

for all  $x \in X$ . Hence  $\{g_n\}$  is  $\tilde{\rho}$ -convergent and thus  $\mathbb{M}_{\tilde{\rho}}$  is  $\tilde{\rho}$ -complete.

(5) Suppose that  $\{g_n\}$  is a sequence in  $\mathbb{M}_{\tilde{\rho}}$  which is  $\tilde{\rho}$ -convergent to  $g \in \mathbb{M}_{\tilde{\rho}}$ . Then  $\{g_n(x)\}$  is  $\rho$ -convergent to  $g(x)$  for all  $x \in V$ . Let  $\epsilon > 0$  be given. Then for any  $n \in \mathbb{N}$ , there is a positive real number  $c_n$  such that

$$\tilde{\rho}(g_n) \leq c_n \leq \tilde{\rho}(g_n) + \epsilon$$

and so

$$\rho(g(x)) \leq \liminf_{n \rightarrow \infty} \rho(g_n(x)) \leq \liminf_{n \rightarrow \infty} c_n \Psi(x) \leq \left( \liminf_{n \rightarrow \infty} \tilde{\rho}(g_n) + \epsilon \right) \Psi(x)$$

for all  $x \in X$ . Hence  $\tilde{\rho}$  is lower semi-continuous. □

Now, with Lemma 2.4 and Lemma 3.1, we will show the following uniqueness concerning the stability of additive-quadratic mappings and using these, we prove the generalized Hyers-Ulam stability for additive-quadratic mappings

**Theorem 3.2.** *Let  $\Phi : V \rightarrow [0, \infty)$  be a mapping and  $L$  a positive real number such that  $0 \leq L < \frac{1}{2}$  and*

$$(3.1) \quad \Phi(2x) \leq L\Phi(x)$$

for all  $x \in V$ . Let  $f, F : V \rightarrow X_\rho$  be mappings such that

$$(3.2) \quad \rho(f(x) - F(x)) \leq M[\Phi(x) + \Phi(-x)]$$

for all  $x \in V$  and some non-negative real number  $M$  and

$$(3.3) \quad F(2x) = 3F(x) + F(-x)$$

for all  $x \in X$ . Then  $F$  is determined by

$$(3.4) \quad \frac{1}{8}F(x) =_\rho \lim_{n \rightarrow \infty} \left[ \left( \frac{1}{2^{2n+4}} + \frac{1}{2^{n+4}} \right) f(2^n x) + \left( \frac{1}{2^{2n+4}} - \frac{1}{2^{n+4}} \right) f(-2^n x) \right]$$

for all  $x \in V$ .

*Proof.* Let  $\Psi(x) = \Phi(x) + \Phi(-x)$  for all  $x \in V$ . By Lemma 3.1,  $\mathbb{M}_{\tilde{\rho}} = \mathbb{M}$  is  $\tilde{\rho}$ -complete and  $\tilde{\rho}$  is lower semi-continuous. Define  $T : \mathbb{M}_{\tilde{\rho}} \rightarrow \mathbb{M}_{\tilde{\rho}}$  by  $Tg(x) = \frac{3}{8}g(2x) - \frac{1}{8}g(-2x)$  for all  $g \in \mathbb{M}_{\tilde{\rho}}$  and all  $x \in V$  and  $S : \mathbb{M}_{\tilde{\rho}} \rightarrow \mathbb{M}_{\tilde{\rho}}$  by  $Sg = 2g$  for

all  $g \in \mathbb{M}_{\tilde{\rho}}$ . Then  $S$  is an isomorphism. Suppose that  $g, h \in \mathbb{M}_{\tilde{\rho}}$  and  $\tilde{\rho}(g - h) \leq c$  for some positive real number  $c$ . By (M3) and (3.1), we have

$$\begin{aligned} \rho_S(Tg(x) - Th(x)) &= \rho\left(\frac{3}{4}g(2x) - \frac{1}{4}g(-2x) - \frac{3}{4}h(2x) + \frac{1}{4}h(-2x)\right) \\ &\leq \rho(g(2x) - h(2x)) + \rho(g(-2x) - h(-2x)) \\ &\leq c(\Psi(2x) + \Psi(-2x)) \\ &\leq 2cL\Psi(x) \end{aligned}$$

for all  $x \in V$  and so

$$\tilde{\rho}_S(Tg - Th) \leq 2L\tilde{\rho}(g - h).$$

By (M3), we have

$$\tilde{\rho}(g - h) \leq \tilde{\rho}_S(g) + \tilde{\rho}_S(h)$$

for all  $g, h \in \mathbb{M}_{\tilde{\rho}}$ . By (3.3),  $F$  is a fixed point of  $T$  and since  $\tilde{\rho}_S = \tilde{\rho}_S$ , by (3.2), we get

$$\begin{aligned} \rho(S^{-1}f(x) - TS^{-1}f(x)) &\leq \rho(f(x) - F(x)) + \rho(TF(x) - Tf(x)) \\ &\leq M\Psi(x) + \rho_S(TF(x) - Tf(x)) \\ &\leq M\Psi(x) + 2L\rho(F(x) - f(x)) \end{aligned}$$

for all  $x \in V$  and thus

$$\tilde{\rho}(S^{-1}f - TS^{-1}f) \leq (1 + 2L)M < \infty.$$

By Lemma 2.4, there is a unique fixed point  $G \in \mathbb{M}_{\tilde{\rho}}$  of  $T$  such that

$$\lim_{n \rightarrow \infty} \tilde{\rho}(T^n S^{-3}f - G) = 0$$

and

$$\tilde{\rho}(S^{-3}f - G) \leq \frac{2}{1 - 2L}\tilde{\rho}(S^{-1}f - TS^{-1}f).$$

Since  $F$  is a fixed point of  $T$ ,  $S^{-3}F$  is a fixed point of  $T$  and by (3.2), we have

$$\tilde{\rho}(S^{-3}f - S^{-3}F) \leq \tilde{\rho}(S^{-1}f - S^{-1}F) \leq \frac{2}{1 - 2L}\tilde{\rho}(S^{-1}f - TS^{-1}f)$$

because  $1 < \frac{2}{1 - 2L}$ . Hence by the uniqueness of  $G$  in Lemma 2.4,  $S^{-3}F = G$  and

$$(3.5) \quad \lim_{n \rightarrow \infty} \tilde{\rho}(T^n S^{-3}f - S^{-3}F) = 0.$$

By induction, there are sequences  $\{a_n\}$  and  $\{b_n\}$  such that

$$T^n f(x) = a_n f(2^n x) + b_n f(-2^n x)$$

for all  $x \in V$  and all  $n \in \mathbb{N}$ . By the definition of  $T$ ,

$$T^{n+1}f(x) = \left(\frac{3}{8}a_n - \frac{1}{8}b_n\right)f(2^{n+1}x) + \left(\frac{3}{8}b_n - \frac{1}{8}a_n\right)f(-2^{n+1}x)$$

and so

$$\begin{cases} a_{n+1} = \frac{3}{8}a_n - \frac{1}{8}b_n \\ b_{n+1} = \frac{3}{8}b_n - \frac{1}{8}a_n \end{cases}$$

for all  $n \in \mathbb{N}$ . Hence

$$\begin{cases} a_{n+1} + b_{n+1} = \frac{1}{4}(a_n + b_n) \\ a_{n+1} - b_{n+1} = \frac{1}{2}(a_n - b_n) \end{cases}$$

for all  $n \in \mathbb{N}$  and so we get

$$\begin{cases} a_n = \frac{1}{2^{2n+1}} + \frac{1}{2^{n+1}} \\ b_n = \frac{1}{2^{2n+1}} - \frac{1}{2^{n+1}} \end{cases}$$

for all  $n \in \mathbb{N}$ . Thus

$$S^{-3}F(x) = \rho \lim_{n \rightarrow \infty} \left[ \left( \frac{1}{2^{2n+4}} + \frac{1}{2^{n+4}} \right) f(2^n x) + \left( \frac{1}{2^{2n+4}} - \frac{1}{2^{n+4}} \right) f(-2^n x) \right]$$

for all  $x \in V$  and hence we have (3.4).  $\square$

For any mapping  $f : V \rightarrow X_\rho$ , let

$$f_o(x) = \frac{f(x) - f(-x)}{2}, \quad f_e(x) = \frac{f(x) + f(-x)}{2}.$$

Then  $f_o$  is odd and  $f_e$  is even. By the fact that  $f(x) = f_o(x) + f_e(x)$  for all  $x \in V$ , we can easily show the following corollary :

**Corollary 3.3.** *All conditions in Theorem 3.2 are assumed. Then  $F$  is determined by*

$$(3.6) \quad \frac{1}{8}F_o(x) = \rho \lim_{n \rightarrow \infty} \frac{1}{2^{n+3}} f_o(2^n x),$$

$$(3.7) \quad \frac{1}{8}F_e(x) = \rho \lim_{n \rightarrow \infty} \frac{1}{2^{2n+3}} f_e(2^n x),$$

and

$$(3.8) \quad \frac{1}{16}F(x) = \rho \lim_{n \rightarrow \infty} \left( \frac{1}{2^{2n+4}} f_e(2^n x) + \frac{1}{2^{n+4}} f_o(2^n x) \right)$$

for all  $x \in V$ .

*Proof.* Note that

$$\begin{aligned} & \rho \left[ \frac{1}{8}F_o(x) - \frac{1}{2^{n+3}} f_o(2^n x) \right] \\ &= \rho \left[ \frac{1}{16}F(x) - \frac{1}{16}F(-x) - \frac{1}{2^{n+4}} f(2^n x) + \frac{1}{2^{n+4}} f(-2^n x) \right] \\ &\leq \rho \left[ \frac{1}{8}F(x) - \left( \frac{1}{2^{2n+4}} + \frac{1}{2^{n+4}} \right) f(2^n x) - \left( \frac{1}{2^{2n+4}} - \frac{1}{2^{n+4}} \right) f(-2^n x) \right] \\ &+ \rho \left[ \frac{1}{8}F(-x) - \left( \frac{1}{2^{2n+4}} + \frac{1}{2^{n+4}} \right) f(-2^n x) - \left( \frac{1}{2^{2n+4}} - \frac{1}{2^{n+4}} \right) f(2^n x) \right] \end{aligned}$$

for all  $x \in V$  and for all  $n \in \mathbb{N}$ . By (3.4), we have (3.6) and similarly, we have (3.7). Thus we get (3.8).  $\square$

If  $F$  is additive or quadratic or additive-quadratic, then  $F$  satisfies (3.3) and hence we have the following corollary :

**Corollary 3.4.** *All conditions in Theorem 3.2 are assumed. If  $F$  is additive (quadratic, additive-quadratic, resp.) then  $F$  is determined by*

$$\begin{aligned} \frac{1}{8}F(x) &= \rho \lim_{n \rightarrow \infty} \frac{1}{2^{n+3}} f_o(2^n x) \quad \left( \frac{1}{8}F(x) = \rho \lim_{n \rightarrow \infty} \frac{1}{2^{2n+3}} f_e(2^n x), \right. \\ \frac{1}{16}F(x) &= \rho \lim_{n \rightarrow \infty} \left( \frac{1}{2^{2n+4}} f_e(2^n x) + \frac{1}{2^{n+4}} f_o(2^n x) \right), \quad \left. \text{resp.} \right) \end{aligned}$$

for all  $x \in V$ .

Similar to the proof of Theorem 3.2, we can show the following theorem for modular spaces with convex modulars.

**Theorem 3.5.** *All conditions in Theorem 3.2 are assumed. Suppose that  $\rho$  is convex and  $L$  is a positive real number such that  $0 \leq L < 2$ . Then  $F$  is determined by*

$$(3.9) \quad \frac{1}{8}F(x) = \rho \lim_{n \rightarrow \infty} \left[ \left( \frac{1}{2^{2n+4}} + \frac{1}{2^{n+4}} \right) f(2^n x) + \left( \frac{1}{2^{2n+4}} - \frac{1}{2^{n+4}} \right) f(-2^n x) \right]$$

for all  $x \in V$ .

*Proof.* Let  $\Psi(x) = \Phi(x) + \Phi(-x)$  for all  $x \in V$ . By Lemma 3.1,  $\mathbb{M}_{\tilde{\rho}} = \mathbb{M}$  is  $\tilde{\rho}$ -complete and  $\tilde{\rho}$  is lower semi-continuous. Define  $T : \mathbb{M}_{\tilde{\rho}} \rightarrow \mathbb{M}_{\tilde{\rho}}$  by  $Tg(x) = \frac{3}{8}g(2x) - \frac{1}{8}g(-2x)$  for all  $g \in \mathbb{M}_{\tilde{\rho}}$  and all  $x \in V$  and  $S : \mathbb{M}_{\tilde{\rho}} \rightarrow \mathbb{M}_{\tilde{\rho}}$  by  $Sg = 2g$  for all  $g \in \mathbb{M}_{\tilde{\rho}}$ . Then  $S$  is an isomorphism. Suppose that  $g, h \in \mathbb{M}_{\tilde{\rho}}$  and  $\tilde{\rho}(g - h) \leq c$  for some positive real number  $c$ . By (M3) and (3.1), we have

$$\begin{aligned} \rho_S(Tg(x) - Th(x)) &= \rho\left(\frac{3}{4}g(2x) - \frac{1}{4}g(-2x) - \frac{3}{4}h(2x) + \frac{1}{4}h(-2x)\right) \\ &\leq \frac{3}{4}\rho(g(2x) - h(2x)) + \frac{1}{4}\rho(g(-2x) - h(-2x)) \\ &\leq cL\Psi(x) \end{aligned}$$

for all  $x \in V$  and so

$$\tilde{\rho}_S(Tg - Th) \leq L\tilde{\rho}(g - h).$$

Further, clearly we have

$$\tilde{\rho}(g - h) \leq \frac{1}{2}\tilde{\rho}_S(g) + \frac{1}{2}\tilde{\rho}_S(h)$$

for all  $g, h \in \mathbb{M}_{\tilde{\rho}}$ . By (3.3),  $F$  is a fixed point of  $T$  and by (3.2), we get

$$\begin{aligned} \rho(S^{-1}f(x) - TS^{-1}f(x)) &\leq \frac{1}{2}\rho(f(x) - F(x)) + \frac{1}{2}\rho(TF(x) - Tf(x)) \\ &\leq \frac{1}{2}M\Psi(x) + \frac{1}{2}\rho_S(TF(x) - Tf(x)) \\ &\leq \frac{1}{2}(1 + L)M\Psi(x) \end{aligned}$$

for all  $x \in V$  and thus

$$\tilde{\rho}(S^{-1}f - TS^{-1}f) \leq \frac{1}{2}(1 + L)M < \infty.$$

By Lemma 2.5, there is a unique fixed point  $G \in \mathbb{M}_{\tilde{\rho}}$  of  $T$  such that

$$\tilde{\rho}(S^{-3}f - G) \leq \frac{1}{2-L}\tilde{\rho}(S^{-1}f - TS^{-1}f).$$

and further, we have

$$\lim_{n \rightarrow \infty} \tilde{\rho}(T^n S^{-3}f - G) = 0.$$

Since  $F$  is a fixed point of  $T$ ,  $S^{-3}F$  is a fixed point of  $T$  and

$$\tilde{\rho}(S^{-3}f - S^{-3}F) \leq \frac{1}{4} \tilde{\rho}(S^{-1}f - S^{-1}F) \leq \frac{1}{2-L} \tilde{\rho}(S^{-1}f - TS^{-1}f)$$

because  $\frac{1}{2} \leq \frac{1}{2-L}$ . Hence by the uniqueness of  $G$  in Lemma 2.4,  $S^{-3}F = G$ . The rest proof is similar to the proof of Theorem 3.2.  $\square$

Using Lemma 2.5 and Theorem 3.5, we can show the generalized Hyers-Ulam stability for additive-quadratic mappings.

**Corollary 3.6.** *Let  $V$  be a linear space and  $X_\rho$  a  $\rho$ -complete modular space whose induced modular is convex lower semi-continuous. Suppose that  $f : V \rightarrow X_\rho$  is a mapping such that*

$$(3.10) \quad \rho(f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)) \leq \phi(x, y)$$

for all  $x, y \in V$  and let  $\phi : V^2 \rightarrow [0, \infty)$  be a mapping satisfying

$$(3.11) \quad \phi(2x, 2y) \leq L\phi(x, y), \quad \forall x, y \in V$$

for some real number  $L$  with  $0 \leq L < 2$ . Then there is a unique additive-quadratic mapping  $G : V \rightarrow X_\rho$  such that

$$(3.12) \quad \rho\left(\frac{1}{4}f(x) - G(x)\right) \leq \frac{3}{8(2-L)}[\phi(x, x) + \phi(-x, -x)]$$

for all  $x \in V$ .

*Proof.* Let  $\Phi(x) = \phi(x, x)$  and  $\Psi(x) = \Phi(x) + \Phi(-x)$  for all  $x \in V$ . By Lemma 3.1,  $\mathbb{M}_{\tilde{\rho}} = \mathbb{M}$  is  $\tilde{\rho}$ -complete and  $\tilde{\rho}$  is lower semi-continuous. Define  $T : \mathbb{M}_{\tilde{\rho}} \rightarrow \mathbb{M}_{\tilde{\rho}}$  by  $Tg(x) = \frac{3}{8}g(2x) - \frac{1}{8}g(-2x)$  for all  $g \in \mathbb{M}_{\tilde{\rho}}$  and all  $x \in V$  and  $S : \mathbb{M}_{\tilde{\rho}} \rightarrow \mathbb{M}_{\tilde{\rho}}$  by  $Sg = 2g$  for all  $g \in \mathbb{M}_{\tilde{\rho}}$ . Then  $S$  is an isomorphism and (2.5) in Lemma 2.5 holds for  $r = \frac{1}{2}$ . Letting  $y = x$  in (3.10), we get

$$(3.13) \quad \rho(f(2x) - 3f(x) - f(-x)) \leq \phi(x, x)$$

for all  $x \in V$  and by (3.13), we have

$$\rho(Tf(x) - f(x)) \leq \frac{3}{8}\phi(x, x) + \frac{1}{8}\phi(-x, -x) \leq \frac{3}{8}\Psi(x)$$

for all  $x \in V$ . Hence we get

$$(3.14) \quad \tilde{\rho}(Tf - f) \leq \frac{3}{8}$$

and by Lemma 2.5, there is a unique fixed point  $G \in \mathbb{M}_{\tilde{\rho}}$  of  $T$  such that

$$\tilde{\rho}(S^{-2}f - G) \leq \frac{3}{8(2-L)}.$$

For any  $n \in \mathbb{N}$ , let

$$a_n = \frac{1}{2^{2n+4}} + \frac{1}{2^{n+4}}, \quad b_n = \frac{1}{2^{2n+4}} - \frac{1}{2^{n+4}}.$$



Since  $G$  is a fixed point of  $T$ ,  $G$  satisfies (3.3) and by Theorem 3.5, we have

$$\frac{1}{8}G(x) =_{\rho} \lim_{n \rightarrow \infty} [a_n f(2^n x) + b_n f(-2^n x)]$$

for all  $x \in V$ . By (M3), we get

$$\begin{aligned} (3.15) \quad & \rho \left( \frac{1}{2^9}G(x+y) + \frac{1}{2^9}G(x-y) - \frac{1}{2^8}G(x) - \frac{1}{2^9}G(y) - \frac{1}{2^9}G(-y) \right) \\ & \leq \frac{1}{2^6} \rho \left( \frac{1}{8}G(x+y) - a_n f(2^n(x+y)) - b_n f(-2^n(x+y)) \right) \\ & + \frac{1}{2^6} \rho \left( \frac{1}{8}G(x-y) - a_n f(2^n(x-y)) - b_n f(-2^n(x-y)) \right) \\ & + \frac{1}{2^6} \rho \left( \frac{1}{4}G(x) - 2a_n f(2^n x) - 2b_n f(-2^n x) \right) \\ & + \frac{1}{2^6} \rho \left( \frac{1}{8}G(y) - a_n f(2^n y) - b_n f(-2^n y) \right) \\ & + \frac{1}{2^6} \rho \left( \frac{1}{8}G(-y) - a_n f(2^n(-y)) - b_n f(-2^n(-y)) \right) \\ & + \frac{a_n}{2^6} \rho \left( f(2^n(x+y)) + f(2^n(x-y)) - 2f(2^n x) - f(2^n y) - f(2^n(-y)) \right) \\ & + \frac{|b_n|}{2^6} \rho \left( f(-2^n(x+y)) + f(-2^n(x-y)) - 2f(-2^n x) - f(-2^n y) - f(-2^n(-y)) \right) \end{aligned}$$

and by (3.11), we have

$$\begin{aligned} (3.16) \quad & \frac{a_n}{2^6} \rho \left( f(2^n(x+y)) + f(2^n(x-y)) - 2f(2^n x) - f(2^n y) - f(2^n(-y)) \right) \\ & + \frac{|b_n|}{2^6} \rho \left( f(-2^n(x+y)) + f(-2^n(x-y)) - 2f(-2^n x) - f(-2^n y) - f(-2^n(-y)) \right) \\ & \leq \frac{a_n}{2^6} \phi(2^n x, 2^n y) + \frac{|b_n|}{2^6} \phi(-2^n x, -2^n y) \\ & \leq L^n \left[ \frac{a_n}{2^6} \phi(x, y) + \frac{|b_n|}{2^6} \phi(-x, -y) \right] \end{aligned}$$

for all  $x, y \in V$  and for all  $n \in \mathbb{N}$ . Since  $0 < a_n, |b_n| < 2^{-n}$ , by (3.15) and (3.16), we can show that  $G$  is an additive-quadratic mapping. Since every additive-quadratic mapping satisfies (3.3),  $G$  is a unique additive-quadratic mapping with (3.12).  $\square$

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## Rough fuzzy ideals in $BCK/BCI$ -algebras

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**Abstract.** The notions of rough ideals and rough fuzzy ideals in  $BCK/BCI$ -algebras are introduced and some properties of such ideals are investigated. The relations between the upper(lower) rough ideals and the upper (lower) approximations of their homomorphic images are discussed.

### 1. Introduction

The notion of rough sets was introduced by Pawlak ([11]). The theory of rough sets has emerged as another major mathematical approach for managing uncertainty that arises from inexact, noisy, or incomplete information. It is turning out to be methodologically significant to the domains of artificial intelligence and cognitive sciences, especially in the representation of reasoning with vague and/or imprecise knowledge, data analysis, machine learning, and knowledge discovery ([11,12]). The algebraic approach to rough sets was studied in [8]. Biswas and Nanda ([1]) introduced the notion of rough subgroups, and Kuroki and Morderson ([6]) discussed the structure of rough sets and rough groups. Kuroki and Wang ([7]) gave some properties of lower and upper approximations with respect to the normal subgroups and the fuzzy normal subgroups, and Kuroki ([5]) introduced the notion of rough ideals in semigroup, which is an extended notion of ideals in semigroups, and gave some properties of such ideals. Xiao and Zhang ([13]) established the notion of rough prime ideals and rough fuzzy prime ideals in a semigroup. Imai and Iséki ([2]) introduced two classes of abstract algebras :  $BCK$ -algebras and  $BCI$ -algebras. It is known that the class of  $BCK$ -algebras is a proper subclass of the class of  $BCI$ -algebras. C. R. Lim and H. S. Kim ([8]) introduced the notion of a rough set in  $BCK/BCI$ -algebras. By introducing the notion of a quick ideal in  $BCK/BCI$ -algebras, they obtained some relations between quick ideals and upper(lower) rough quick ideals in  $BCK/BCI$ -algebras.

In this paper, we introduce the notion of rough ideals and rough fuzzy ideals in  $BCK/BCI$ -algebras, and we give some properties of such ideals. Also, we discuss the relations between the upper(lower) rough ideals and the upper (lower) approximations of their homomorphic images.

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## 2. Preliminaries

A *BCI-algebra* ([9]) is a non-empty set  $X$  with a constant  $0$  and a binary operation “ $*$ ” satisfying the axioms, for all  $x, y, z \in X$ :

- (i)  $((x * y) * (x * z)) * (z * y) = 0$ ,
- (ii)  $(x * (x * y)) * y = 0$ ,
- (iii)  $x * x = 0$ ,
- (iv)  $x * y = 0$  and  $y * x = 0$  imply  $x = y$ .

A *BCK-algebra* is a *BCI-algebra* satisfying the axiom:

- (v)  $0 * x = 0$  for all  $x \in X$ .

We can define a partial ordering  $\leq$  on  $X$  by  $x \leq y$  if and only  $x * y = 0$ . In any *BCI-algebra*  $X$ , the following hold: for any  $x, y, z \in X$ ,

- (1)  $x * 0 = x$ ,
- (2)  $(x * y) * z = (x * z) * y$ ,
- (3)  $x \leq y$  implies  $x * z \leq y * z$  and  $z * y \leq z * x$ ,
- (4)  $(x * z) * (y * z) \leq (x * z)$ .

Let  $X$  be a *BCK/BCI-algebra* and let  $0 \in I \subseteq X$ . A set  $I$  is called an *ideal* of  $X$  if for all  $x, y \in X$ ,  $x * y \in I$  and  $y \in I$  imply  $x \in I$ . An ideal  $I$  is said to be *closed* if  $0 * x \in I$  whenever  $x \in I$ . Let  $S$  be a non-empty subset of  $X$ . Then  $S$  is called a *subalgebra* of  $X$  if, for any  $x, y \in S$ ,  $x * y \in S$ . A closed ideal of a *BCK/BCI-algebra*  $X$  is a subalgebra of  $X$ . An equivalence relation  $\rho$  on  $X$  is called a *congruence relation* on  $X$  if  $(x * u, y * v) \in \rho$  for any  $(x, y), (u, v) \in \rho$ . We denote by  $[a]_\rho$  the  $\rho$ -congruence class containing the element  $a \in X$ . Let  $X/\rho$  be the set of all  $\rho$ -equivalence classes on  $X$ , i.e.,  $X/\rho := \{[a]_\rho | a \in X\}$ . For any  $[x]_\rho, [y]_\rho \in X/\rho$ , if we define

$$[x]_\rho * [y]_\rho := [x * y]_\rho = \{z \in X | (z, x * y) \in \rho\},$$

then it is well defined, since  $\rho$  is a congruence relation. A congruence relation  $\rho$  on a *BCK/BCI-algebra*  $X$  is said to be *regular* if  $[x]_\rho * [y]_\rho = [0]_\rho = [y]_\rho * [x]_\rho$  implies  $[x]_\rho = [y]_\rho$  for any  $[x]_\rho, [y]_\rho \in X/\rho$ .

**Theorem 2.1.** ([9]) *Let  $X$  be a BCK-algebra and let  $\rho$  be a congruence relation on  $X$ . Then  $\rho$  is regular if and only if  $X/\rho$  is a BCK-algebra.*

Let  $I$  be an ideal of  $X$ . We define a relation  $\rho_I$  on  $X$  as follows:

$$\rho_I := \{(x, y) | x * y, y * x \in I\}. \tag{*}$$

Then  $\rho_I$  is a regular congruence relation ([4]). Let  $Con(X)$  be the set of all congruences on  $X$ . We define a subset  $I_\rho$  of  $X$  from  $\rho \in Con(X)$  by  $I_\rho := \{x * y | (x, y) \in \rho\}$ .

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**Proposition 2.2.** ([4]) *Let  $A$  be an ideal. If  $A$  is closed, then  $A = I_{\rho_A}$ .*

Let  $X$  be a  $BCK/BCI$ -algebra and let  $\rho$  be a congruence relation on  $X$ . Let  $\mathcal{P}(X)$  denote the power set of  $X$  and  $\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\emptyset\}$ . Define the functions  $\rho, \rho^- : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$  as follows: for any  $\emptyset \neq A \in \mathcal{P}(X)$ ,  $\rho_-(A) := \{x \in X \mid [x]_\rho \subseteq A\}$  and  $\rho^-(A) := \{x \in X \mid [x]_\rho \cap A \neq \emptyset\}$ . The set  $\rho_-(A)$  is called the  $\rho$ -lower approximation of  $A$ , while  $\rho^-(A)$  is called the  $\rho$ -upper approximation of  $A$ . For a non-empty subset  $A$  of  $X$ ,  $\rho(A) = (\rho_-(A), \rho^-(A))$  is called a *rough set* with respect to  $\rho$  of  $\mathcal{P}(X) \times \mathcal{P}(X)$  if  $\rho_-(A) \neq \rho^-(A)$ . A subset  $A$  of  $X$  is said to be *definable* if  $\rho_-(A) = \rho^-(A)$ . The pair  $(X, \rho)$  is called an *approximation space*. A congruence relation  $\rho$  on a set  $X$  is called *complete* if  $[x]_\rho * [y]_\rho = [x * y]_\rho$  for any  $x, y \in X$ .

3. Rough ideals in  $BCK/BCI$ -algebras

Let  $X$  be a  $BCK/BCI$ -algebra and let  $\emptyset \neq A \subseteq X$ . Let  $\rho$  be a congruence relation on  $X$ . Then  $A$  is called an *upper* (a *lower*, respectively) *rough ideal* of  $X$  if  $\rho^-(A)$  ( $\rho_-(A)$ , respectively) is an *ideal* of  $X$ .

**Theorem 3.1.** *Let  $\rho_I$  be a congruences relation on a  $BCK/BCI$ -algebra  $X$  as in (\*). If  $A$  is a closed ideal of  $X$ , then it is an upper rough ideal of  $X$ .*

*Proof.* Since  $A$  is an ideal of  $X$ ,  $0 \in A$ . Hence  $A \cap [0]_{\rho_I} \neq \emptyset$ . Therefore  $0 \in \rho_I^-(A)$ .

Let  $x, y \in X$  with  $x * y, y \in \rho_I^-(A)$ . Then  $([x]_{\rho_I} * [y]_{\rho_I}) \cap A = ([x * y]_{\rho_I}) \cap A \neq \emptyset$  and  $[y]_{\rho_I} \cap A \neq \emptyset$ . Hence there exist  $\alpha, \beta \in A$  such that  $\alpha \in [x]_{\rho_I} * [y]_{\rho_I} = [x * y]_{\rho_I}$  and  $\beta \in [y]_{\rho_I}$ . Therefore  $\alpha = p * q$  for some  $p \in [x]_{\rho_I}, q \in [y]_{\rho_I}$ . Since  $\beta, q \in [y]_{\rho_I}$ , we have  $(\beta, y), (y, q) \in \rho_I$  and so  $(\beta, q) \in \rho_I$ . Hence  $[\beta]_{\rho_I} = [q]_{\rho_I}$ . Since  $(q * \beta, q * q) = (q * \beta, 0) \in \rho_I$ , we have  $(q * \beta) * 0 = q * \beta \in A$  by Proposition 2.2. Using  $\beta \in A$ , we have  $q \in A$ . Since  $p * q, q \in A$  and  $A$  is an ideal of  $X$ , we obtain  $p \in A$ . Therefore  $p \in [x]_{\rho_I} \cap A \neq \emptyset$ . Thus  $x \in \rho_I^-(A)$ , completing the proof.  $\square$

Theorem 3.1 shows that the notion of an upper rough ideal is an extended notion of a closed ideal in  $BCK/BCI$ -algebras.

**Example 3.2.** Let  $X := \{0, 1, 2, 3\}$  be a  $BCK$ -algebra with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	1
3	3	3	3	0

If we take  $A := \{0, 2\}$ , then it is not an ideal of  $X$ , since  $1 * 2 = 0 \in A$ , but  $1 \notin A$ . On the while, let  $\rho$  be a congruence relation on  $X$  such that  $\{0, 1, 2\}, \{3\}$  are all  $\rho$ -congruence classes of  $X$ . Then  $\rho^-(A) = \{0, 1, 2\}$  is an ideal of  $X$ .

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**Theorem 3.3.** *Let  $X$  be a BCK/BCI-algebra and let  $A$  be a closed ideal of  $X$ . Then  $\rho_-(A)$ , if it is non-empty, is an ideal of  $X$ .*

*Proof.* Since  $A$  is a closed ideal of  $X$ , it is a subalgebra of  $X$ . Since  $\rho_-(A) \neq \emptyset$ ,  $\rho_-(A)$  is a subalgebra of  $X$ . Hence  $0 \in \rho_-(A)$ . Let  $x, y \in X$  with  $x * y, y \in \rho_-(A)$ . Then  $[x]_\rho * [y]_\rho = [x * y]_\rho \subseteq A$ ,  $[y]_\rho \subseteq A$ . If  $\alpha \in [x]_\rho$ , then  $(\alpha, x) \in \rho$ . Since  $\rho$  is a congruence relation on  $X$ , we have  $(\alpha * y, x * y) \in \rho$  and so  $\alpha * y \in [x * y]_\rho \subseteq A$ . Since  $A$  is an ideal of  $X$  and  $y \in A$ , we get  $\alpha \in A$ , i.e.,  $[x]_\rho \subseteq A$ , proving that  $x \in \rho_-(A)$ .  $\square$

Let  $\rho$  be a regular congruence relation on a BCK-algebra  $X$  and let  $\emptyset \neq A \subseteq X$ . The lower and upper approximations can be presented in an equivalent form as shown below:

$$\begin{aligned} \rho_-(A)/\rho &= \{[x]_\rho \in X/\rho \mid [x]_\rho \subseteq A\} \\ \rho^-(A)/\rho &= \{[x]_\rho \in X/\rho \mid [x]_\rho \cap A \neq \emptyset\}. \end{aligned}$$

**Proposition 3.4.** *Let  $\rho$  be a regular congruence relation on a BCK-algebra  $X$ . If  $A$  is a subalgebra of  $X$ , then  $\rho^-(A)/\rho$  is a subalgebra of the quotient BCK-algebra  $X/\rho$ .*

*Proof.* Since  $A$  is a subalgebra of  $X$ , there exists an element  $x \in A$  such that  $[x]_\rho \cap A \neq \emptyset$ , i.e.,  $\rho^-(A)/\rho \neq \emptyset$ . Let  $[x]_\rho$  and  $[y]_\rho$  be any elements of  $\rho^-(A)/\rho$ . Then  $[x]_\rho \cap A \neq \emptyset$  and  $[y]_\rho \cap A \neq \emptyset$ . This means that there exist  $a, b \in X$  such that  $a \in [x]_\rho \cap A$  and  $b \in [y]_\rho \cap A$ . Then  $a * b \in [x]_\rho * [y]_\rho$ . Since  $A$  is a subalgebra of  $X$ ,  $a * b \in A$ . This means that  $[x]_\rho * [y]_\rho \in \rho^-(A)/\rho$ , completing the proof.  $\square$

**Proposition 3.5.** *Let  $\rho$  be a regular congruence relation on a BCK-algebra  $X$ . If  $A$  is a subalgebra of  $X$ , then  $\rho_-(A)/\rho$  is, if it is non-empty, a subalgebra of the quotient BCK-algebra  $X/\rho$ .*

*Proof.* Straightforward.  $\square$

**Theorem 3.6.** *Let  $\rho_I$  be a regular congruence relation on a BCK-algebra  $X$  as in (\*). If  $A$  is an ideal of  $X$ , then  $\rho_I^-(A)/\rho_I$  is an ideal of the quotient BCK-algebra  $X/\rho_I$ .*

*Proof.* Since  $0 \in \rho_I^-(A)$ , we have  $[0]_{\rho_I} \cap A \neq \emptyset$  and hence  $[0]_{\rho_I} \in \rho_I^-(A)/\rho_I$ . Let  $[x]_{\rho_I} * [y]_{\rho_I}, [y]_{\rho_I} \in \rho_I^-(A)/\rho_I$ . Then  $([x]_{\rho_I} * [y]_{\rho_I}) \cap A = [x * y]_{\rho_I} \cap A \neq \emptyset$  and  $[y]_{\rho_I} \cap A \neq \emptyset$ . Hence there exist  $\alpha \in A$  with  $\alpha \in [x]_{\rho_I} * [y]_{\rho_I} = [x * y]_{\rho_I}$  and  $\beta \in A$  for some  $\beta \in [y]_{\rho_I}$ . Therefore  $\alpha = p * q$  for some  $p \in [x]_{\rho_I}, q \in [y]_{\rho_I}$ . Since  $\beta, q \in [y]_{\rho_I}$ , we have  $(\beta, y), (y, q) \in \rho_I$  and so  $(\beta, q) \in \rho_I$ . Hence  $[\beta]_{\rho_I} = [q]_{\rho_I}$ . Since  $(q * \beta, q * q) = (q * \beta, 0) \in \rho_I$ , we have  $(q * \beta) * 0 = q * \beta \in A$  by Proposition 2.2. Using  $\beta \in A$ , we have  $q \in A$ . Since  $A$  is an ideal of  $X$  and  $q \in A$ , we have  $p \in A$ . Thus  $p \in [x]_{\rho_I} \cap A$ , proving  $[x]_{\rho_I} \in \rho_I^-(A)/\rho_I$ .  $\square$

**Theorem 3.7.** *Let  $\rho$  be a regular congruence relation on a BCK-algebra  $X$ . If  $A$  is an ideal of  $X$ , then  $\rho_-(A)/\rho$ , if it is non-empty, an ideal of the quotient BCK-algebra  $X/\rho$ .*

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*Proof.* Since  $\rho_-(A)/\rho \neq \emptyset$ ,  $\rho_-(A)/\rho$  is a subalgebra of  $X/\rho$ . Hence  $[0]_\rho \in \rho_-(A)/\rho$ . Let  $[x]_\rho * [y]_\rho, [y]_\rho \in \rho_-(A)/\rho$  for some  $[x]_\rho \in X/\rho$ . Hence  $[x * y]_\rho \subseteq A$  and  $[y]_\rho \subseteq A$ . Therefore  $x * y \in \rho_-(A), y \in \rho_-(A)$ . If  $\alpha \in [x]_\rho$ , then  $(\alpha, x) \in \rho$ . Since  $\rho$  is a congruence relation on  $X$ , we have  $(\alpha * y, x * y) \in \rho$ . Hence  $\alpha * y \in [x * y]_\rho \subseteq A$ . Hence  $\alpha \in A$ , because  $A$  is an ideal of  $X$  and  $y \in A$ . Therefore  $[x]_\rho \subseteq A$ , proving that  $[x]_\rho \in \rho_-(A)/\rho$ .  $\square$

**Theorem 3.8.** *Let  $\rho$  be a regular congruence relation on a *BCK*-algebra  $X$ . If  $A$  is an upper rough ideal of  $X$ , then  $\rho^-(A)/\rho$  is an ideal of the quotient algebra  $X/\rho$ .*

*Proof.* Since  $0 \in \rho^-(A)$ , we have  $[0]_\rho \cap A \neq \emptyset$  and hence  $[0]_\rho \in \rho^-(A)/\rho$ . Let  $[x]_\rho * [y]_\rho = [x * y]_\rho, [y]_\rho \in \rho^-(A)/\rho$  for some  $[x]_\rho \in X/\rho$ . Then  $([x]_\rho * [y]_\rho) \cap A = [x * y]_\rho \cap A \neq \emptyset$  and  $[y]_\rho \cap A \neq \emptyset$ . Hence  $x * y, y \in \rho^-(A)$ . Since  $\rho^-(A)$  is an ideal of  $X$ , we have  $x \in A$ . Thus  $x \in [x]_\rho \cap A \neq \emptyset$ , proving  $[x]_\rho \in \rho^-(A)/\rho$ .  $\square$

**Theorem 3.9.** *Let  $\rho$  be a regular congruence relation on a *BCK*-algebra  $X$ . If  $A$  is a lower rough ideal of  $X$ , then  $\rho_-(A)/\rho$  is, if it is non-empty, an ideal of the quotient *BCK*-algebra  $X/\rho$ .*

*Proof.* Since  $\rho_-(A)/\rho \neq \emptyset$ ,  $\rho_-(A)/\rho$  is a subalgebra of  $X/\rho$  and hence  $[0]_\rho \in \rho_-(A)/\rho$ . Let  $[x]_\rho * [y]_\rho, [y]_\rho \in \rho_-(A)/\rho$  for some  $[x]_\rho \in X/\rho$ . Hence  $[x * y]_\rho \subseteq A$  and  $[y]_\rho \subseteq A$ . Therefore  $x * y \in \rho_-(A), y \in \rho_-(A)$ . Since  $\rho_-(A)$  is an ideal of  $X$ , we have  $x \in \rho_-(A)$ . Thus  $[x]_\rho \subseteq A$ .  $\square$

4. Approximations of fuzzy sets

Let  $\mu$  and  $\lambda$  be two fuzzy subsets of  $X$ . The inclusion  $\lambda \subseteq \mu$  is denoted by  $\lambda(x) \leq \mu(x)$  for all  $x \in X$ , and  $\mu \cap \lambda$  is defined by  $(\mu \cap \lambda)(x) = \mu(x) \wedge \lambda(x)$  for all  $x \in X$ .

**Definition 4.1.** Let  $\rho$  be a congruence relation on a *BCK/BCI*-algebra  $X$  and  $\mu$  a fuzzy subset of  $X$ . Then we define the fuzzy sets  $\rho_-(\mu)$  and  $\rho^-(\mu)$  as follows:

$$\rho_-(\mu)(x) := \bigwedge_{a \in [x]_\rho} \mu(a) \text{ and } \rho^-(\mu)(x) := \bigvee_{a \in [x]_\rho} \mu(a).$$

The fuzzy sets  $\rho_-(\mu)$  and  $\rho^-(\mu)$  are called the  $\rho$ -lower approximations and  $\rho$ -upper approximations of the fuzzy set  $\mu$ , respectively. A set  $\rho(\mu) = (\rho_-(\mu), \rho^-(\mu))$  is called a rough fuzzy set with respect to  $\rho$  if  $\rho_-(\mu) \neq \rho^-(\mu)$ .

**Definition 4.2.** ([3]) A fuzzy set  $\mu$  of a *BCK/BCI*-algebra  $X$  is called a fuzzy ideal of  $X$  if

- (F<sub>1</sub>)  $\mu(0) \geq \mu(x)$  for all  $x \in X$ ,
- (F<sub>2</sub>)  $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$  for all  $x, y \in X$ .

Let  $\mu$  and  $\nu$  be fuzzy ideals of a *BCK/BCI*-algebra  $X$ . Then  $\mu \cap \nu$  is also a fuzzy ideal of  $X$ .

A fuzzy subset  $\mu$  of a *BCK/BCI*-algebra  $X$  is called an upper (a lower, respectively) rough fuzzy ideal of  $X$  if  $\rho^-(\mu)$  ( $\rho_-(\mu)$ , respectively) is a fuzzy ideal of  $X$ .

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**Theorem 4.3.** *Let  $\rho$  be a congruence relation on a BCK/BCI-algebra  $X$ . If  $\mu$  is a fuzzy ideal of  $X$ , then  $\rho^-(\mu)$  is a fuzzy ideal of  $X$ .*

*Proof.* Since  $\mu$  is a fuzzy ideal of  $X$ ,  $\mu(0) \geq \mu(x)$  for all  $x \in X$ . Hence we obtain

$$\rho^-(\mu)(0) = \bigvee_{z \in [0]_\rho} \mu(z) \geq \bigvee_{x' \in [x]_\rho} \mu(x') = \rho^-(\mu)(x).$$

For any  $x, y \in X$ , we have

$$\begin{aligned} \rho^-(\mu)(x) &= \bigvee_{x' \in [x]_\rho} \mu(x') \geq \bigvee_{x' * y' \in [x]_\rho * [y]_\rho, y' \in [y]_\rho} \{\min\{\mu(x' * y'), \mu(y')\}\} \\ &= \bigvee_{x' * y' \in [x * y]_\rho, y' \in [y]_\rho} \{\min\{\mu(x' * y'), \mu(y')\}\} \\ &\geq \min\{\bigvee_{x' * y' \in [x * y]_\rho} \mu(x' * y'), \bigvee_{y' \in [y]_\rho} \mu(y')\} \\ &= \min\{\rho^-(\mu)(x * y), \rho^-(\mu)(y)\}. \end{aligned}$$

Thus  $\rho^-(\mu)$  is a fuzzy ideal of  $X$ . □

**Theorem 4.4.** *Let  $\rho$  be a congruence relation on a BCK/BCI-algebra  $X$ . If  $\mu$  is a fuzzy ideal of  $X$ , then  $\rho_-(\mu)$  is, if it is non-empty, a fuzzy ideal of  $X$ .*

*Proof.* Since  $\mu$  is a fuzzy ideal of  $X$ ,  $\mu(0) \geq \mu(x)$  for all  $x \in X$ . Hence for all  $x \in X$ , we have

$$\rho_-(\mu)(0) = \bigwedge_{z \in [0]_\rho} \mu(z) \geq \bigwedge_{z' \in [x]_\rho} \mu(z') = \rho_-(\mu)(x).$$

For any  $x, y \in X$ , we obtain

$$\begin{aligned} \rho_-(\mu)(x) &= \bigwedge_{x' \in [x]_\rho} \mu(x') \geq \bigwedge_{x' * y' \in [x]_\rho * [y]_\rho, y' \in [y]_\rho} \{\min\{\mu(x' * y'), \mu(y')\}\} \\ &= \bigwedge_{x' * y' \in [x * y]_\rho, y' \in [y]_\rho} \{\min\{\mu(x' * y'), \mu(y')\}\} \\ &= \min\{\bigwedge_{x' * y' \in [x * y]_\rho} \mu(x' * y'), \bigwedge_{y' \in [y]_\rho} \mu(y')\} \\ &= \min\{\rho_-(\mu)(x * y), \rho_-(\mu)(y)\}. \end{aligned}$$

Thus  $\rho_-(\mu)$  is a fuzzy ideal of  $X$ . □

Let  $\mu$  be a fuzzy subset of a BCK/BCI-algebra  $X$  and let  $(\rho_-(\mu), \rho^-(\mu))$  be a rough fuzzy set. If  $\rho_-(\mu)$  and  $\rho^-(\mu)$  are fuzzy ideals of a BCK/BCI-algebra  $X$ , then we call  $(\rho_-(\mu), \rho^-(\mu))$  a *rough fuzzy ideal* of  $X$ . Therefore we have:

**Corollary 4.5.** *If  $\mu$  is a fuzzy ideal of a BCK/BCI-algebra  $X$ , then  $(\rho_-(\mu), \rho^-(\mu))$  is a rough fuzzy ideal of  $X$ . If  $\mu, \lambda$  are fuzzy ideals of a BCK/BCI-algebra  $X$ , then  $(\rho_-(\mu \cap \lambda), \rho^-(\mu \cap \lambda))$  is a rough fuzzy ideal of  $X$ .*

Let  $\mu$  be a fuzzy subset of a BCK/BCI-algebra  $X$ . Then the sets

$$\mu_t := \{x \in X \mid \mu(x) \geq t\}, \quad \mu_t^X := \{x \in X \mid \mu(x) > t\},$$

where  $t \in [0, 1]$ , are called respectively, a *t-level subset* and a *t-strong level subset* of  $\mu$ .

**Theorem 4.6.** ([3]) *Let  $\mu$  be a fuzzy subset of a BCK/BCI-algebra  $X$ . Then  $\mu$  is a fuzzy ideal of  $X$  if and only if  $\mu_t$  and  $\mu_t^X$  are, if they are non-empty, ideals of  $X$  for every  $t \in [0, 1]$ .*



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**Lemma 4.7.** *Let  $\rho$  be a congruence relation on a *BCK/BCI*-algebra  $X$ . If  $\mu$  is a fuzzy subset of  $X$  and  $t \in [0, 1]$ , then*

- (1)  $(\rho_-(\mu))_t = \rho_-(\mu_t)$ ,
- (2)  $(\rho^-(\mu))_t^X = \rho^-(\mu_t^X)$ .

*Proof.* (1) We have

$$\begin{aligned} x \in (\rho_-(\mu))_t &\Leftrightarrow \rho_-(\mu)(x) \geq t \Leftrightarrow \bigwedge_{a \in [x]_\rho} \mu(a) \geq t \\ &\Leftrightarrow \forall a \in [x]_\rho, \mu(a) \geq t \Leftrightarrow [x]_\rho \subseteq \mu_t \Leftrightarrow x \in \rho_-(\mu_t). \end{aligned}$$

(2) Also we have

$$\begin{aligned} x \in (\rho^-(\mu))_t^X &\Leftrightarrow \rho^-(\mu)(x) > t \Leftrightarrow \bigvee_{a \in [x]_\rho} \mu(a) > t \\ &\Leftrightarrow \exists a \in [x]_\rho, \mu(a) > t \Leftrightarrow [x]_\rho \cap \mu_t^X \neq \emptyset \Leftrightarrow x \in \rho^-(\mu_t^X). \end{aligned}$$

□

**Theorem 4.8.** *Let  $\rho$  be a congruence relation on a *BCK/BCI*-algebra  $X$ . Then  $\mu$  is a lower (an upper) rough fuzzy ideal of  $X$  if and only if  $\mu_t, \mu_t^X$  are, if they are non-empty, lower (upper) rough ideals of  $X$  for every  $t \in [0, 1]$ .*

*Proof.* By Theorem 4.6 and Lemma 4.7, we can obtain the conclusion easily. □

### 5. Problems of Homomorphism

**Lemma 5.1.** *Let  $f$  be a surjective homomorphism of a *BCK/BCI*-algebra  $X$  to a *BCK/BCI*-algebra  $Y$  and let  $A$  be any subset of  $X$ . Let  $\rho_2$  be a congruence relation on  $Y$ , and  $\rho_1 := \{(x_1, x_2) \in X \times X \mid (f(x_1), f(x_2)) \in \rho_2\}$ . Then*

- (1)  $\rho_1$  is a congruence relation on  $X$ ,
- (2) If  $\rho_2$  is complete and  $f$  is single-valued, then  $\rho_1$  is complete,
- (3)  $f(\rho_1^-(A)) = \rho_2^-(f(A))$ ,
- (4)  $f(\rho_{1-}(A)) \subseteq \rho_{2-}(f(A))$ . If  $f$  is single-valued, then  $f(\rho_{1-}(A)) = \rho_{2-}(f(A))$ .

*Proof.* (1) It is clear that  $\rho_1$  is a congruence relation on  $X$ .

(2) Let  $x'$  be any element of  $[x_1 * x_2]_{\rho_1}$ . Since  $\rho_2$  is complete, by the definition of  $\rho_1$ , we know that  $f(x') \in [f(x_1 * x_2)]_{\rho_2} = [f(x_1)]_{\rho_2} * [f(x_2)]_{\rho_2}$ . Since  $f$  is surjective, there exist  $x'_1, x'_2 \in X$  such that  $f(x'_1) \in [f(x_1)]_{\rho_2}, f(x'_2) \in [f(x_2)]_{\rho_2}$ , and  $f(x') = f(x'_1) * f(x'_2) = f(x'_1 * x'_2)$ . Since  $f$  is single-valued, by the definition of  $\rho_1$ , we have  $x'_1 \in [x_1]_{\rho_1}, x'_2 \in [x_2]_{\rho_1}$ , such that  $x' = x'_1 * x'_2$ . Thus  $x' \in [x_1]_{\rho_1} * [x_2]_{\rho_1}$ . This means that  $[x_1 * x_2]_{\rho_1} \subseteq [x_1]_{\rho_1} * [x_2]_{\rho_1}$ . On the other hand, we have  $[x_1]_{\rho_1} * [x_2]_{\rho_1} \subseteq [x_1 * x_2]_{\rho_1}$ . Therefore  $\rho_1$  is complete.

(3) Let  $y$  be any element of  $f(\rho_1^-(A))$ . Then there exists  $x \in \rho_1^-(A)$  such that  $f(x) = y$ . Hence  $[x]_{\rho_1} \cap A \neq \emptyset$ . Then there exists  $x' \in [x]_{\rho_1} \cap A$ . Then  $f(x') \in f(A)$  and by the definition of  $\rho_1$ ,

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we have  $f(x') \in [f(x)]_{\rho_2}$ . So  $[f(x)]_{\rho_2} \cap f(A) \neq \emptyset$ , which implies  $y = f(x) \in \rho_2^-(f(A))$ . Thus  $f(\rho_1^-(A)) \subseteq \rho_2^-(f(A))$ .

Conversely, let  $y \in \rho_2^-(f(A))$ . Then there exists  $x \in X$  such that  $f(x) = y$ . Hence  $[f(x)]_{\rho_2} \cap f(A) \neq \emptyset$ . So there exists  $x' \in A$  such that  $f(x') \in f(A)$  and  $f(x') \in [f(x)]_{\rho_2}$ . Then by the definition of  $\rho_1$ , we have  $x' \in [x]_{\rho_1}$ . Thus  $[x]_{\rho_1} \cap A \neq \emptyset$  which implies  $x \in \rho_1^-(A)$ . So  $y = f(x) \in f(\rho_1^-(A))$ . It means that  $\rho_2^-(f(A)) \subseteq f(\rho_1^-(A))$ . From the above, we have  $f(\rho_1^-(A)) = \rho_2^-(f(A))$ .

(4) Let  $y$  be any element of  $f(\rho_{1-}(A))$ . Then there exists  $x \in \rho_{1-}(A)$  such that  $f(x) = y$ , so we have  $[x]_{\rho_1} \subseteq A$ . Let  $y' \in [y]_{\rho_2}$ . Then there exists  $x' \in X$  such that  $f(x') = y'$  and  $f(x') \in [f(x)]_{\rho_2}$ . Hence  $x' \in [x]_{\rho_1} \subseteq A$ , and so  $y' = f(x') \in f(A)$ . Thus  $[y]_{\rho_2} \subseteq f(A)$  which yields that  $y \in \rho_{2-}(f(A))$ . So we have  $f(\rho_{1-}(A)) \subseteq \rho_{2-}(f(A))$ .

Assume that  $f$  is single-valued and suppose  $y \in \rho_{2-}(f(A))$ . Then there exist  $x \in X$  such that  $f(x) = y$  and  $[f(x)]_{\rho_2} \subseteq f(A)$ . Let  $x' \in [x]_{\rho_1}$ . Then  $f(x') \in [f(x)]_{\rho_2} \subseteq f(A)$ , and so  $x' \in A$ . Thus  $[x]_{\rho_1} \subseteq A$  which yields  $x \in \rho_{1-}(A)$ . Then  $y = f(x) \in f(\rho_{1-}(A))$ , and so  $\rho_{2-}(f(A)) \subseteq f(\rho_{1-}(A))$ . From the above, we have  $f(\rho_{1-}(A)) = \rho_{2-}(f(A))$ .  $\square$

**Theorem 5.2.** *Let  $f$  be a surjective homomorphism of a BCK/BCI-algebra  $X$  to a BCK/BCI-algebra  $Y$ . Let  $\rho_2$  be a congruence relation on  $Y$  and  $A$  be a subset of  $X$ . If  $\rho_1 := \{(x_1, x_2) \in X \times X \mid (f(x_1), f(x_2)) \in \rho_2\}$ , then  $\rho_1^-(A)$  is an ideal of  $X$  if and only if  $\rho_2^-(f(A))$  is an ideal of  $Y$ .*

*Proof.* Assume that  $\rho_1^-(A)$  is an ideal of  $X$ . Since  $0 \in \rho_1^-(A)$ ,  $[0]_{\rho_1} \cap A \neq \emptyset$ . Hence there exists  $x' \in [0]_{\rho_1} \cap A$ . Then  $f(x') \in f(A)$ , and by the definition of  $\rho_1$ , we have  $f(x') \in [f(0)]_{\rho_2}$ . So  $[f(0)]_{\rho_2} \cap f(A) \neq \emptyset$  which means  $f(0) \in \rho_2^-(f(A))$ . Let  $x', y' \in Y$  with  $x', y' * x' \in \rho_2^-(f(A))$ . Then there exist  $x, z \in A$  such that  $f(x) = x'$  and  $f(z) = y' * x'$ . Hence  $[f(x)]_{\rho_2} \cap f(A) \neq \emptyset$  and  $[f(z)]_{\rho_2} \cap f(A) \neq \emptyset$ . Therefore there exists  $b \in A$  such that  $f(b) \in [f(x)]_{\rho_2}$ . By the definition of  $\rho_1$ ,  $b \in [x]_{\rho_1}$  and so  $b \in [x]_{\rho_1} \cap A$ . Hence  $[x]_{\rho_1} \cap A \neq \emptyset$ . Thus  $x \in \rho_1^-(A)$ . Since  $f$  is surjective, there exists  $y \in X$  such that  $f(y) = y'$ . Put  $u := y * ((y * x) * z)$ . Then  $u \in X$ . Since

$$\begin{aligned} f((y * x) * z) &= f(y * x) * f(z) \\ &= f(y * x) * y' * x' \quad (\because f(z) = y' * x') \\ &= (f(y) * f(x)) * (y' * x') \\ &= (y' * x') * (y' * x') = 0', \end{aligned}$$

we have  $f(u) = f(y * ((y * x) * z)) = f(y) * f((y * x) * z) = f(y) * 0' = f(y) = y'$ . Since  $[f(z)]_{\rho_2} \cap f(A) \neq \emptyset$ , we obtain

$$\begin{aligned} [y' * x']_{\rho_2} \cap f(A) &= ([y']_{\rho_2} * [x']_{\rho_2}) \cap f(A) \\ &= ([f(u)]_{\rho_2} * [f(x)]_{\rho_2}) \cap f(A) \\ &= [f(u * x)]_{\rho_2} \cap f(A) \neq \emptyset. \end{aligned}$$

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Then there exists  $a \in A$  such that  $f(a) \in f(A)$  and  $f(a) \in [f(u * x)]_{\rho_2}$ . By the definition of  $\rho_1$ , we have  $a \in [u * x]_{\rho_1}$ . Hence  $[u * x]_{\rho_1} \cap A \neq \emptyset$  and so  $u * x \in \rho_1^-(A)$ . Since  $\rho_1^-(A)$  is an ideal of  $X$  and  $x \in \rho_1^-(A)$ , we get  $u \in \rho_1^-(A)$ . Therefore  $f(u) = y' \in f(\rho_1^-(A)) = \rho_2^-(f(A))$ . Thus  $\rho_2^-(f(A))$  is an ideal of  $Y$ .

Conversely, suppose that  $\rho_2^-(f(A))$  is an ideal of  $Y$ . Since  $f(0) = 0' \in \rho_2^-(f(A))$ ,  $[f(0)]_{\rho_2} \cap f(A) \neq \emptyset$ . Hence there exists  $y' \in [f(0)]_{\rho_2} \cap f(A)$ . Since  $f$  is surjective, there exists  $x' \in X$  such that  $f(x') = y'$ . Hence  $f(x') \in [f(0)]_{\rho_2} \cap f(A)$ . Therefore  $f(x') \in f(A)$ . By the definition of  $\rho_1$ ,  $x' \in [0]_{\rho_1}$  and  $x' \in A$ . Hence  $[0]_{\rho_1} \cap A \neq \emptyset$ , which means  $0 \in \rho_1^-(A)$ .

Let  $x_1, x_2 \in X$  with  $x_1 * x_2, x_2 \in \rho_1^-(A)$ . By Lemma 5.1, we obtain that  $f(x_1 * x_2) = f(x_1) * f(x_2), f(x_2) \in f(\rho_1^-(A)) = \rho_2^-(f(A))$ . Since  $\rho_2^-(f(A))$  is an ideal of  $Y$ , we have  $f(x_1) \in \rho_2^-(f(A))$ . Hence  $[f(x_1)]_{\rho_2} \cap f(A) \neq \emptyset$ . Therefore  $y' \in [f(x_1)]_{\rho_2} \cap f(A)$ . Since  $f$  is surjective, there exists  $x' \in X$  such that  $f(x') = y'$ . Hence  $f(x') = y' \in [f(x_1)]_{\rho_2} \cap f(A)$ . Therefore  $f(x') \in f(A)$ . By the definition of  $\rho_1$ , there exists  $x' \in [x_1]_{\rho_1}$  and  $x' \in A$ . Therefore  $[x_1]_{\rho_1} \cap A \neq \emptyset$ , which means  $x_1 \in \rho_1^-(A)$ . Thus  $\rho_1^-(A)$  is an ideal of  $X$ .  $\square$

**Theorem 5.3.** *Let  $f$  be an isomorphism of a *BCK/BCI*-algebra  $X$  to a *BCK/BCI*-algebra  $Y$ . Let  $\rho_2$  be a complete congruence relation on  $Y$  and let  $A$  be a subset of  $X$ . If  $\rho_1 := \{(x_1, x_2) \in X \times X | (f(x_1), f(x_2)) \in \rho_2\}$ , then  $\rho_{1-}(A)$  is an ideal of  $X$  if and only if  $\rho_{2-}(f(A))$  is an ideal of  $Y$ .*

*Proof.* By Lemma 5.1, we have  $f(\rho_{1-}(A)) = \rho_{2-}(f(A))$ . The proof is similar to the proof of Theorem 5.2.  $\square$

By Theorem 5.2 and Theorem 5.3, we can obtain the following conclusion easily in quotient *BCK/BCI*-algebras.

**Corollary 5.4.** *Let  $f$  be an isomorphism of a *BCK/BCI*-algebra  $X$  to a *BCK/BCI*-algebra  $Y$ . Let  $\rho_2$  be a complete congruence relation on  $Y$  and let  $A$  be a subset of  $X$ . If  $\rho_1 := \{(x_1, x_2) \in X \times X | (f(x_1), f(x_2)) \in \rho_2\}$ , then  $\rho_{1-}(A)/\rho_1$  (resp.  $\rho_1^-(A)/\rho_1$ ) is an ideal of  $X/\rho_1$  if and only if  $\rho_{2-}(f(A))/\rho_2$  (resp.  $\rho_2^-(f(A))/\rho_2$ ) is an ideal of  $Y/\rho_2$ .*

**Theorem 5.5.** *Let  $f$  be a surjective homomorphism of a *BCK/BCI*-algebra  $X$  to a *BCK/BCI*-algebra  $Y$ . Let  $\rho_2$  be a complete congruence relation on  $Y$  and let  $A$  be a fuzzy subset of  $X$ . If  $\rho_1 := \{(x_1, x_2) \in X \times X | (f(x_1), f(x_2)) \in \rho_2\}$ , then*

- (1)  $\rho_1^-(A)$  is a fuzzy ideal of  $X$  if and only if  $\rho_2^-(f(A))$  is a fuzzy ideal of  $Y$ .
- (2) If  $f$  is single-valued, then  $\rho_{1-}(A)$  is a fuzzy ideal of  $X$  if and only if  $\rho_{2-}(f(A))$  is a fuzzy ideal of  $Y$ .

*Proof.* (1) By Theorem 4.6, we obtain that  $\rho_1^-(A)$  is a fuzzy ideal of  $X$  if and only if  $(\rho_1^-(A))_t^X$  is, if it is non-empty, an ideal of  $X$  for every  $t \in [0, 1]$ . By Lemma 4.7, we have  $(\rho_1^-(A))_t^X = \rho_1^-(A_t^X)$ .

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By Theorem 5.2, we obtain that  $\rho_1^-(A_t^X)$  is an ideal of  $X$  if and only if  $\rho_2^-(f(A_t^X))$  is an ideal of  $Y$ . It is clear that  $f(A_t^X) = (f(A))_t^X$ . From this and Lemma 4.7, we have

$$\rho_2^-(f(A_t^X)) = \rho_2^-(f(A))_t^X = (\rho_2^-(f(A)))_t^X.$$

By Theorem 4.6, we obtain that  $(\rho_2^-(f(A)))_t^X$  is an ideal of  $Y$  for every  $t \in [0, 1]$  if and only if  $\rho_2^-(f(A))$  is a fuzzy ideal of  $Y$ . Thus the conclusion holds.

(2) Since  $f$  is single valued, by Lemma 5.1, we have  $f(\rho_{1-}(A)) = \rho_{2-}(f(A))$ . The proof is similar to that of (1).  $\square$

**Corollary 5.6.** *Let  $f$  be an isomorphism of a BCK/BCI-algebra  $X$  to a BCK/BCI-algebra  $Y$ . Let  $\rho_2$  be a complete congruence relation on  $Y$  and  $A$  a fuzzy subset of  $X$ . If  $\rho_1 := \{(x_1, x_2) \in X \times X | (f(x_1), f(x_2)) \in \rho_2\}$ , then  $\rho_{1-}(A_t)/\rho_1$  (resp.  $\rho_1^-(A_t^X)/\rho_1$ ) is an ideal of  $X/\rho_1$  if and only if  $\rho_{2-}(f(A_t))/\rho_2$  (resp.  $\rho_2^-(f(A_t^X))/\rho_2$ ) is an ideal of  $Y/\rho_2$ .*

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# A Lebesgue integrable space of Boehmians for a class of $\mathcal{D}_\kappa$ transformations

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**Abstract** Boehmians are objects obtained by an abstract algebraic construction similar to that of field of quotients and it in some cases just gives the field of quotients. As Boehmian spaces are represented by convolution quotients, integral transforms have a natural extension onto appropriately defined spaces of Boehmians. In this paper, we have defined convolution products and a class of delta sequences and have examined the axioms necessary for generating the  $\mathcal{D}_\kappa$  spaces of Boehmians. The extended  $\mathcal{D}_\kappa$  transformation has therefore been defined as a one-to-one onto mapping continuous with respect to  $\Delta$  and  $\delta$  convergences. Over and above, it has been asserted that the necessary and sufficient conditions for an integrable sequence to be in the range of the  $\mathcal{D}_\kappa$  transformation is that the class of quotients belongs to the range of the representative. Further results related to the inverse problem are also discussed.

**keywords:** Integral transform; analogue system; generalized integral; discrete system; Boehmian.

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## 1 Introduction

As some physical situations were determined by initial value problems which are not smoothly enough but are generalized functions, numerous integral transforms were defined in a context of distributions, ultradistributions, tempered distributions, tempered ultradistributions and Boehmian spaces. The Laplace transform method of right-side distributions was treated in [17] and [18] to solve various types of ordinary differential equations. In [19] Loonker and Banerji have given a solution of Volterra-Abel integral equations by aid of a distributional wavelet transform. Indeed, if the differential equation  $\acute{u} = w$ ,  $w$  being the heaviside step function, is considered then no classical conclusion can be drawn at this point. But, on generalized sense, if  $S$  denotes a space of rapid descents (rapidly decreasing functions) and  $S'$  be its dual of slow growth, then for every

$v \in S$  we write

$$\begin{aligned}
 \dot{u}(v) &= -u(\dot{v}) \\
 &= -\int_{-\infty}^{\infty} u(x) \dot{v}(x) dx \\
 &= -\int_{-\infty}^0 \alpha \dot{v}(x) dx - \int_0^{\infty} (x + \alpha) \dot{v}(x) dx \\
 &= -\alpha v(0) + \alpha v(0) - \int_0^{\infty} x \dot{v}(x) dx \\
 &= \int_0^{\infty} v(x) dx \\
 &= \int_{-\infty}^{\infty} w(x) v(x) dx \\
 &= w(v)
 \end{aligned}$$

where  $\alpha$  is some suitable constant.

Let  $\kappa$  be a sampling period and  $v_\alpha$  be an analogue function. In some engineering applications, the classical  $\mathcal{D}$  transform was presented as an equivalence between discrete and analogue systems as [8]

$$\mathcal{D}v_\alpha(r, \kappa) := \mathcal{D}v_\alpha(r) := \frac{1}{r!} \int_{R_+} v_\alpha(t) e^{-t\kappa^{-1}} (t\kappa^{-1})^r dt \tag{1}$$

where  $\mathcal{D}(v_\alpha * y_\alpha)(r, \kappa) = \sum_0^r \mathcal{D}y_\alpha(r - k, \kappa) \mathcal{D}v_\alpha(r, \kappa)$ ,  $*$  being the Fourier convolution product defined by [7]

$$(v_\alpha * y_\alpha)(t) := \int_{R_+} v_\alpha(\tau) y_\alpha(t - \tau) dt. \tag{2}$$

Let  $x_\alpha$  be an analogue function and  $\kappa$  be a sampling period. Then, treating  $r$  as a positive real number, say  $\xi$ , then the existed integral, denoted by  $\mathcal{D}_\kappa$ , is given as

$$\mathcal{D}_\kappa v_\alpha(\xi, \kappa) = \frac{1}{\xi!} \int_{R_+} v_\alpha(t) e^{-t\kappa^{-1}} (t\kappa^{-1})^\xi dt, \tag{3}$$

where  $\xi \in R_+$ ;  $R_+ := (0, \infty)$ .

In this paper, without reading the efficiency of this integral in discrete and analogue systems, we attempt to investigate the extension of this integral to a class of Boehmians, being recent in the space of generalized functions. We derive virtuous products, give definitions and derive some properties of the existence of the given integral in the class of generalized functions.

Boehmian spaces were inaugurated by the idea of regular operators which is a

subalgebra of Mikusiński operators. According to literature, we briefly recall the general construction of Boehmian spaces. Let  $\mathcal{G}$  be a group and  $\mathcal{S}$  be a subgroup of  $\mathcal{G}$ . We assume to each pair of elements  $f \in \mathcal{G}$  and  $\omega \in \mathcal{S}$ , is assigned the product  $f * g$  such that :

- (1)  $\omega, \psi \in \mathcal{S}$  implies  $\omega * \psi \in \mathcal{S}$  and  $\omega * \psi = \psi * \omega$ .
- (2)  $f \in \mathcal{G}$  and  $\omega, \psi \in \mathcal{S}$  implies  $(f * \omega) * \psi = f * (\omega * \psi)$ .
- (3)  $f, g \in \mathcal{G}, \omega \in \mathcal{S}$  and  $\lambda \in R$ , implies  $(f + g) * \omega = f * \omega + g * \omega, \lambda (f * \omega) = (\lambda f) * \omega$ . Let  $\Delta$  be a family of sequences from  $\mathcal{S}$  such that:
  - (1)  $f, g \in \mathcal{G}, (\delta_n) \in \Delta$  and  $f * \delta_n = g * \delta_n (n = 1, 2, \dots)$  implies  $f = g, n \in N$ .
  - (2)  $(\omega_n), (\delta_n) \in \Delta$  implies  $(\omega_n * \psi_n) \in \Delta$ .

Members of  $\Delta$  are called delta sequences.

Let  $\mathcal{A}$  be a pair of sequences defined by  $\mathcal{A} = \{((f_n), (\omega_n)) : (f_n) \subseteq \mathcal{G}^N, (\omega_n) \in \Delta\}$ , where  $n \in N$ , then members of  $((f_n), (\omega_n)) \in \mathcal{A}$  are called quotient of sequences, denoted by  $[f_n/\omega_n]$ , if  $f_n * \omega_m = f_m * \omega_n, \forall n, m \in N$ . Two quotients of sequences  $f_n/\omega_n$  and  $g_n/\psi_n$  are equivalent,  $f_n/\omega_n \sim g_n/\psi_n$ , if  $f_n * \psi_m = g_m * \omega_n, \forall n, m \in N$ .

The relation  $\sim$  is an equivalent relation on  $\mathcal{A}$ . The equivalence class containing  $f_n/\omega_n$  is denoted by  $[f_n/\omega_n]$ . These equivalence classes are called Boehmians. The space of all Boehmians is denoted by  $\beta_1$ . The sum of two Boehmians and multiplication by a scalar can be defined in a natural way  $[f_n/\omega_n] + [g_n/\psi_n] = [(f_n * \psi_n + g_n * \omega_n) / (\omega_n * \psi_n)], \alpha [f_n/\omega_n] = [\alpha f_n/\omega_n], \alpha \in C$ , space of complex numbers.

The operations  $*$  and  $\mathcal{D}^\alpha$  are given by  $[f_n/\omega_n] * [g_n/\psi_n] = [(f_n * g_n) / (\omega_n * \psi_n)]$  and  $\mathcal{D}^\alpha [f_n/\omega_n] = [\mathcal{D}^\alpha f_n/\omega_n]$  whereas,  $*$  can be extended to  $\beta \times \mathcal{S}$  in the form that if  $[f_n/\omega_n] \in \beta_1$  and  $\omega \in \mathcal{S}$ , then  $[f_n/\omega_n] * \omega = [f_n * \omega/\omega_n]$ .

However, soon after the topic has been initiated, numerous integral transforms were extended to Boehmian spaces by many authors in [1], [2], [6], [9 – 16], [20 – 23] and many others.

**Definition 1** The Mellin type convolution product  $\odot$  between two signals  $x_\alpha$  and  $y_\alpha$  is defined by the integral equation ( see [4])

$$(v_\alpha \odot y_\alpha)(x) = \int_{R_+} v_\alpha(y^{-1}x) y_\alpha(x) y^{-1} dy \tag{4}$$

when the integral exists.

The Lebesgue space of integrable functions defined on  $R_+^2$  is denoted by  $L^1(R_+^2)$  and the set of smooth functions of bounded supports over  $R_+$  is denoted by  $\vartheta(R_+)$  (see [3] for definition, properties and convergence in  $\vartheta(R_+)$ ).

## 2 Convolution products and Boehmians

In this section, we establish the prerequisite axioms of the Boehmian space  $\mathcal{B}(L^1(R_+^2), \vartheta, \odot, \bullet)$  with the operations  $\odot$  and  $\bullet$  where  $\bullet$  is a convolution product defined as follows.

**Definition 2** Let the casual analogue signals  $v_\alpha, y_\alpha \in \mathbf{I}^1(R_+^2)$  be given. Then, between  $v_\alpha$  and  $y_\alpha$ , we define a product  $\bullet$  given as

$$(v_\alpha \bullet y_\alpha)(\xi, \kappa) = \int_{R_+} v_\alpha(\xi, y^{-1}\kappa) y_\alpha(y) dy \tag{5}$$

provided the above integral exists.

Proving axioms of the space  $\mathcal{B}(\mathbf{I}^1(R_+^2), \vartheta, \odot, \bullet)$  begins with the following theorem.

**Theorem 3** Given  $v_\alpha \in \mathbf{I}^1(R_+^2)$  and  $y_\alpha \in \vartheta(R_+)$ . Then we get  $v_\alpha \bullet y_\alpha \in \mathbf{I}^1(R_+^2)$ .

**Proof** The hypothesis that  $v_\alpha \in \mathbf{I}^1(R_+^2)$  implies  $\int_{R_+^2} |v_\alpha(\xi, y^{-1}\kappa)| d\xi d\kappa < M_1 (y > 0)$ . Hence, with the aid of the Fubini's theorem together with the hypothesis that  $y_\alpha \in \vartheta(R_+)$  we confirm

$$\begin{aligned} \int_{R_+^2} |(v_\alpha \bullet y_\alpha)(\xi, \kappa)| d\xi d\kappa &= \int_{R_+^2} \left| \int_{R_+} v_\alpha(\xi, y^{-1}\kappa) y_\alpha(y) dy \right| d\xi d\kappa \\ &\leq \int_{R_+^2} \int_P |v_\alpha(\xi, y^{-1}\kappa)| |y_\alpha(y)| dy d\xi d\kappa \\ &\leq M_1 \int_P |y_\alpha(y)| dy < \infty \end{aligned}$$

where  $P$  is an interval in  $R_+$  including the support of  $y_\alpha$ .

Hence the theorem is finished.

**Theorem 4** Let  $v_\alpha \in \mathbf{I}^1(R_+^2)$  and that  $y_\alpha, z_\alpha \in \vartheta(R_+)$  be analogue signals. Then

$$v_\alpha \bullet (y_\alpha \odot z_\alpha) = (v_\alpha \bullet y_\alpha) \bullet z_\alpha.$$

**Proof** On account of (4) and (5) we are permitted to write

$$(v_\alpha \bullet (y_\alpha \odot z_\alpha))(\xi, \kappa) = \int_{R_+^2} v_\alpha(\xi, y^{-1}\kappa) y_\alpha(t^{-1}y) dy z_\alpha(t) t^{-1} dt. \tag{6}$$

The substitution  $u = yt^{-1}$  implies  $dy = tdu$ . Therefore, (6) can be expressed as

$$\begin{aligned} (v_\alpha \bullet (y_\alpha \odot z_\alpha))(\xi, \kappa) &= \int_{R_+^2} v_\alpha(\xi, u^{-1}(t^{-1}\kappa)) y_\alpha(u) z_\alpha(t) dudt \\ &= \int_{R_+} (v_\alpha \bullet y_\alpha)(\xi, t^{-1}\kappa) z_\alpha(t) dt. \end{aligned}$$



The proof is therefore finished.

**Theorem 5** Given  $v_\alpha \in \mathbf{I}^1 (R_+^2)$ . For every  $y_\alpha \in \vartheta (R_+)$ , we get

$$\mathcal{D}_\kappa (v_\alpha \odot y_\alpha) (\xi, \kappa) = ((\mathcal{D}_\kappa v_\alpha) \bullet y_\alpha) (\xi, \kappa)$$

**Proof** Applying (3) to (4) gives

$$\begin{aligned} \mathcal{D}_\kappa (v_\alpha \bullet y_\alpha) (\xi, \kappa) &= \frac{1}{\xi!} \int_{R_+^2} v_\alpha (y^{-1}x) y_\alpha (y) y^{-1} dy \left( e^{-x\kappa^{-1}} \right) (x\kappa^{-1})^\xi dx \\ &= \frac{1}{\xi!} \int_{R_+^2} v_\alpha (y^{-1}x) \left( e^{-x\kappa^{-1}} \right) (x\kappa^{-1})^\xi y_\alpha (y) y^{-1} dx dy \end{aligned} \tag{7}$$

Let  $zy = x$ , then  $dx = ydz$ . Therefore, on account of (7) we obtain that

$$\mathcal{D}_\kappa (v_\alpha \odot y_\alpha) (\xi, \kappa) = \frac{1}{\xi!} \int_{R_+^2} v_\alpha (x) \left( e^{-yz\kappa^{-1}} \right) (x\kappa^{-1})^\xi y_\alpha (y) dz dy. \tag{8}$$

Hence, by the Fubini's theorem, we finish the proof of the theorem.

**Theorem 6** Given  $\check{r} \in C$ ,  $v_\alpha \in \mathbf{I}^1 (R_+^2)$  and  $y_\alpha \in \vartheta (R_+)$ . We get

$$(\check{r}v_\alpha) \bullet y_\alpha = \check{r} (v_\alpha \bullet y_\alpha).$$

Proof of this theorem is straightforward follows from definitions. Hence it is omitted.

**Theorem 7** Given  $v_\alpha, z_\alpha \in \mathbf{I}^1 (R_+^2)$ . For every  $y_\alpha \in \vartheta (R_+)$ , we get

$$(v_\alpha + z_\alpha) \bullet y_\alpha = v_\alpha \bullet y_\alpha + z_\alpha \bullet y_\alpha.$$

Proof of above theorem follows from simple integration. Details are therefore omitted.

**Theorem 8** Given  $v_{\alpha,n} \rightarrow v_\alpha$  as  $n \rightarrow \infty$  in  $\mathbf{I}^1 (R_+^2)$ . For every  $y_\alpha \in \vartheta (R_+)$ , we get  $v_{\alpha,n} \bullet y_\alpha \rightarrow v_\alpha \bullet y_\alpha$  as  $n \rightarrow \infty$ .

Proof of above theorem is a direct conclusion of Theorem 4. Hence it is avoided.

By  $\Delta$  we mean the subset of  $\vartheta (R_+)$  such that for every sequence  $(\mu_{\alpha,n})_0^\infty \in \vartheta (R_+)$ ,  $n \in N$ , we have.

$$i' : \int_{R_+} \mu_{\alpha,n} dy = 1; i'' : \int_{R_+} |\mu_{\alpha,n}| dy < \infty; i''' : \text{supp} \mu_{\alpha,n} \subseteq (0, a_n), \lim_{n \rightarrow \infty} a_n = 0.$$

Elements of  $\Delta$  are said to be delta sequences or approximating identities.

**Theorem 9** Given  $v_\alpha \in \mathbf{I}^1 (R_+^2)$ . For every  $(\mu_{\alpha,n}) \in \vartheta (R_+)$ , we get  $\lim_{n \rightarrow \infty} v_\alpha \bullet \mu_{\alpha,n} = v_\alpha$ .

**Proof** Let  $v_\alpha \in \mathbf{I}^1(R_+^2)$  and  $\vartheta(R_+^2)$  be the set of smooth functions of bounded supports over  $R^2$ , then  $\vartheta(R_+^2)$  is dense in  $\mathbf{I}^1(R_+^2)$ . Hence, for a given  $\epsilon > 0$ , we can find  $\psi_\alpha \in \vartheta(R_+^2)$  such that

$$\|v_\alpha - \psi_\alpha\| < \epsilon. \tag{9}$$

Define  $g_\alpha(y) = \psi_\alpha(\xi, y^{-1}\kappa)$ , then  $g_\alpha(y)$  is uniformly continuous mapping in  $\vartheta(R_+)$  for every  $\xi, \kappa > 0$ . Therefore, for each  $\epsilon > 0$  we find  $\delta > 0$  so that  $-\epsilon < g_\alpha(y_1) - g_\alpha(y_2) < \epsilon$  wherever  $-\delta < y - y^{-1} < \delta$ . Since  $y$  and  $y^{-1}$  belong to  $R_+$  and that  $g_\alpha \in \vartheta(R_+)$ , we get

$$-\epsilon < g_\alpha(y) - g_\alpha(y^{-1}) < \epsilon \tag{10}$$

when  $-\delta < y - y^{-1} < \delta$ .

Also, since  $\psi_\alpha$  is of bounded support in  $\vartheta(R_+^2)$  it follows  $\text{supp}\psi_\alpha(\xi, y^{-1}\kappa) \subseteq [a_1, a_2] \times P$  for some compact subset  $P$  of  $R_+$ . Hence

$$\psi_\alpha(\xi, y^{-1}\kappa) = 0, (\xi, y^{-1}\kappa) \notin [a_1 - \delta, a_2 + \delta] \times P. \tag{11}$$

The hypothesis that  $\text{supp}\mu_{\alpha,n} \rightarrow 0, n \rightarrow \infty$  asserts that we can find  $N \in \mathbb{N}$  such that

$$\text{supp}\mu_{\alpha,n}(y) \subset [0, \delta] \tag{12}$$

for every  $n \geq N$ . By the property  $\int_{R_+} \mu_{\alpha,n} dy = 1$  we write

$$\begin{aligned} \|(\psi_\alpha \bullet \mu_{\alpha,n}) - \psi_\alpha\| &= \int_{R_+^2} \left( \left| \int_{R_+} (\psi_\alpha(\xi, y^{-1}\kappa) - \psi_\alpha(\xi, \kappa) \mu_{\alpha,n}(y)) dy \right| \right) d\xi d\kappa \\ &\leq \int_{R_+^3} (|g_\alpha(y) - g_\alpha(y^{-1})| |\mu_{\alpha,n}(y)|) dy d\xi d\kappa \\ &= \int_P \int_{a_1}^{a_2+\delta} \int_0^\delta (|g_\alpha(y) - g_\alpha(y^{-1})| |\mu_{\alpha,n}(y)|) dy d\xi d\kappa. \end{aligned}$$

By virtue of (10) and (12) and the assumption that  $\int_{R_+} |\mu_{\alpha,n}| dy < M$  we have

$$\begin{aligned} \|\psi_\alpha \bullet \mu_{\alpha,n} - \psi_\alpha\| &< \epsilon (a_2 + \delta) \int_{a_1}^{a_2+\delta} \left( \int_0^\delta |\mu_{\alpha,n}(y)| dy \right) d\xi \\ &= \epsilon (a_2 + \delta) M (a_2 - a_1 + \delta) \end{aligned} \tag{13}$$

Finally, we write

$$\|(v_\alpha \bullet \mu_{\alpha,n}) - v_\alpha\| \leq \|(v_\alpha - \psi_\alpha) \bullet \mu_{\alpha,n}\| + \|(\psi_\alpha \bullet \mu_{\alpha,n}) - \psi_\alpha\| + \|\psi_\alpha - v_\alpha\|. \tag{14}$$

Hence, on account of (9), (10) and (13), (14) yields

$$\|(v_\alpha \bullet \mu_n) - v_\alpha\| \leq \epsilon (M + M (a_2 - a_1 + 2\delta) + 1).$$

This finishes the proof of our result.

The Boehmian space  $\mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet)$  is therefore entirely performed. Similarly, one can proceed to generate the space  $\mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \odot)$ .

We introduce addition in  $\mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet)$  as  $[(u_n)/(\epsilon_n)] + [(v_n)/(\epsilon_n)] = [((u_n \bullet \epsilon_n) + (v_n \bullet \epsilon_n))/(\epsilon_n \odot \epsilon_n)]$ . In  $\mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet)$  we define scalar multiplication as  $\Omega [(u_n)/(\epsilon_n)] = [\Omega (u_n)/(\epsilon_n)] = [(\Omega u_n)/(\epsilon_n)]$ ,  $\Omega \in C$ . We define the convolution  $\bullet$  in  $\mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet)$  as  $[(\varphi_n)/(\epsilon_n)] \bullet [(v_n)/(\epsilon_n)] = [(\varphi_n \bullet v_n)/(\epsilon_n \odot \epsilon_n)]$ . Also, we define differentiation in  $\mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet)$  as  $D^\alpha [(u_n)/(\epsilon_n)] = [(D^\alpha u_n)/(\epsilon_n)]$ ,  $\alpha$  is a real number. For  $\mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet) \bullet \mathbf{1}^1(R_+^2)$  the product is given as  $[(u_n)/(\epsilon_n)] \bullet \varphi = [(u_n \bullet \varphi)/(\epsilon_n)]$  where  $[(u_n)(\delta_n)] \in \mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet)$  and  $\varphi \in \mathbf{1}^1(R_+^2)$ .

$(\beta_n)$  of  $\mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet)$  is  $\delta$  convergent to  $\beta$  in  $\mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet)$ ; i.e.  $(\beta_n \rightarrow \beta)$ , if we can find  $(\epsilon_n)$  in  $\Delta$  such that  $\beta_n \bullet \epsilon_k; \beta \bullet \epsilon_k \in \mathbf{1}^1(R_+^2) (\forall k, n \in N)$ , where  $\lim_{n \rightarrow \infty} \beta_n \bullet \epsilon_k = \beta \bullet \epsilon_k$  in  $\mathbf{1}^1(R_+^2) (\forall k \in N)$ .

$(\beta_n)$  of  $\mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet)$  is  $\Delta$ -convergent to  $\beta$  in  $\mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet)$ ; i.e.  $(\beta_n \rightarrow \beta)$ , if  $(\epsilon_n) \in \Delta$ , with  $(\beta_n - \beta) \bullet \epsilon_n \in \mathbf{1}^1(R_+^2) (\forall n \in N)$ , and  $\lim_{n \rightarrow \infty} (\beta_n - \beta) \bullet \epsilon_n = 0$  in  $\mathbf{1}^1(R_+^2)$ .

### 3 $\mathcal{D}_\kappa$ transform of Boehmians

Let  $\beta = [(v_{\alpha,n})/(\mu_{\alpha,n})] \in \mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \odot)$ . Then we present the generalized transform  $\widehat{\mathcal{D}}_\kappa$  of  $\beta$  as

$$\widehat{\mathcal{D}}_\kappa \beta = [(D_\kappa v_{\alpha,n})/(\mu_{\alpha,n})] \tag{15}$$

in the space  $\mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet)$ .

**Theorem 10** The mapping  $\widehat{\mathcal{D}}_\kappa \beta : \mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \odot) \rightarrow \mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet)$  is well-defined and linear.

**Proof** Given  $[(v_{\alpha,n})/(\mu_{\alpha,n})] = [(y_{\alpha,n})/(\psi_{\alpha,n})] \in \mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \odot)$ . Then it follows that  $v_{\alpha,n} \odot y_{\alpha,m} = y_{\alpha,n} \odot r_{\alpha,m}$ . Employing  $\mathcal{D}_\kappa$  for the previous equation directly gives  $\mathcal{D}_\kappa v_{\alpha,n} \bullet y_{\alpha,m} = \mathcal{D}_\kappa y_{\alpha,n} \bullet r_{\alpha,m}, (\forall n, m \in N)$ . That is,

$$\mathcal{D}_\kappa v_{\alpha,n}/r_{\alpha,n} \sim \mathcal{D}_\kappa y_{\alpha,n}/\psi_{\alpha,n}. \tag{16}$$

Finally, let the Boehmians  $[(v_{\alpha,n})/(\mu_{\alpha,n})]$  and  $[(y_{\alpha,n})/(\psi_{\alpha,n})]$  be equivalent in the space  $\mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \odot)$ . Then by definitions and Theorem 5 the proof of our result follows.

**Theorem 11** The extended  $\widehat{\mathcal{D}}_\kappa$  integral is consistent with  $\mathcal{D}_\kappa : \mathbf{1}^1(R_+^2) \rightarrow \mathbf{1}^1(R_+^2)$ .

**Proof** For every  $v_\alpha \in \mathbf{1}^1(R_+^2)$ , let  $\beta$  be its representative in  $\mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \odot)$ . Then indeed  $\beta = [(v_\alpha \odot (\varphi_{\alpha,n})) / (\varphi_{\alpha,n})]$  where  $(\varphi_{\alpha,n}) \in \Delta (\forall n \in N)$ . On the other hand, its clear that  $(\varphi_{\alpha,n})$  and its representative are independent for  $n \in N$ . Therefore by Theorem 5 we deduce  $\widehat{\mathcal{D}}_\kappa(\beta) = [(\mathcal{D}_\kappa(v_\alpha \odot (\varphi_{\alpha,n}))) / (\varphi_{\alpha,n})] = [(\mathcal{D}_\kappa(v_\alpha \bullet (\varphi_{\alpha,n}))) / (\varphi_{\alpha,n})]$ ; that is the representative of  $\mathcal{D}_\kappa f$  in  $\mathbf{1}^1(R_+^2)$ . Hence the proof is finished.

**Theorem 12** Given  $[(g_{\alpha,n}) / (\psi_{\alpha,n})] \in \mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet)$ . Then  $[(g_{\alpha,n}) / (\psi_{\alpha,n})]$  is in the range of  $\widehat{\mathcal{D}}_\kappa$  iff  $g_{\alpha,n}$  is in the range of  $\mathcal{D}_\kappa (\forall n \in N)$ .

**Proof** Indeed, when  $[(g_{\alpha,n}) / (\psi_{\alpha,n})]$  is in the range of  $\widehat{\mathcal{D}}_\kappa$ ; then it is obvious that  $g_{\alpha,n}$  is in the range of  $\mathcal{D}_\kappa (\forall n \in N)$ .

On the other hand, if  $g_{\alpha,n}$  is in the range of  $\mathcal{D}_\kappa (\forall n \in N)$ , then it can be obtained an element  $v_{\alpha,n} \in \mathbf{1}^1(R_+^2)$  satisfying  $\mathcal{D}_\kappa v_{\alpha,n} = g_{\alpha,n} (\forall n \in N)$ . Due to the assumption that  $[(g_{\alpha,n}) / (\psi_{\alpha,n})] \in \mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet)$ , we have  $g_{\alpha,n} \bullet \psi_{\alpha,m} = g_{\alpha,m} \bullet \psi_{\alpha,n} (\forall m, n \in N)$ . That is,  $\mathcal{D}_\kappa(v_{\alpha,n} \odot \varphi_{\alpha,n}) = \mathcal{D}_\kappa(v_{\alpha,m} \odot \varphi_{\alpha,n}) (m, n \in N)$ , where  $v_{\alpha,n} \in \mathbf{1}^1(R_+^2)$  and  $\varphi_{\alpha,n} \in \Delta, \forall n \in N$ .

Now, injectivity of  $\mathcal{D}_\kappa$  leads to  $v_{\alpha,n} \odot \varphi_{\alpha,m} = v_{\alpha,m} \odot \varphi_{\alpha,n} (m, n \in N)$ .

Hence, the quotient  $v_{\alpha,n} / \varphi_{\alpha,n}$  defines a quotient in  $\mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \odot)$ . Therefore, we write  $[(v_{\alpha,n}) / (\varphi_{\alpha,n})] \in \mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \odot)$  and  $\widehat{\mathcal{D}}_\kappa([(v_{\alpha,n}) / (\varphi_{\alpha,n})]) = [(g_{\alpha,n}) / (\psi_{\alpha,n})]$ .

Hence the proof is finished.

**Theorem 13**  $\widehat{\mathcal{D}}_\kappa : \mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \odot) \rightarrow \mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet)$  is an isomorphism.

**Proof** Let  $\widehat{\mathcal{D}}_\kappa([(v_{\alpha,n}) / (\varphi_{\alpha,n})]) \cong \widehat{\mathcal{D}}_\kappa([(g_{\alpha,n}) / (\psi_{\alpha,n})])$  in  $\mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet)$ . By aid of Theorem 5 and the idea involving quotients of  $\mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet)$  we deduce that  $\mathcal{D}_\kappa v_{\alpha,n} \bullet \psi_{\alpha,m} = \mathcal{D}_\kappa v_{\alpha,m} \bullet \psi_{\alpha,n}$ . Theorem 5 reveals  $\mathcal{D}_\kappa(v_{\alpha,n} \odot \psi_{\alpha,m}) = \mathcal{D}_\kappa(g_{\alpha,m} \odot \psi_{\alpha,n})$ . Properties of  $\mathcal{D}_\kappa$  implies  $v_{\alpha,n} \odot \psi_{\alpha,m} = g_{\alpha,m} \odot \psi_{\alpha,n}$ . Therefore, the idea of quotients of  $\mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \odot)$  leads to  $[(v_{\alpha,n}) / (\varphi_{\alpha,n})] = [(g_{\alpha,n}) / (\psi_{\alpha,n})]$ . Surjectivity of  $\widehat{\mathcal{D}}_\kappa$  can be derived as in the following manner. Let  $[(\mathcal{D}_\kappa v_{\alpha,n}) / (\phi_{\alpha,n})] \in \mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet)$  be given arbitrary. Then  $\mathcal{D}_\kappa v_{\alpha,n} \bullet \phi_{\alpha,m} = \mathcal{D}_\kappa v_{\alpha,m} \bullet \phi_{\alpha,n}$  for every  $m, n \in N$ .

Once again, Theorem 5 leads to  $\mathcal{D}_\kappa(v_{\alpha,n} \odot \varphi_{\alpha,m}) = \mathcal{D}_\kappa(v_{\alpha,m} \odot \varphi_{\alpha,n})$ . Hence the proof is finished.

**Theorem 14** Given  $\delta_1, \delta_2 \in \mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \odot)$ . We get  $\widehat{\mathcal{D}}_\kappa(\delta_1 \odot \delta_2) = \widehat{\mathcal{D}}_\kappa \delta_1 \bullet \delta_2$ .

**Proof** Let  $\delta_1 = [(v_{\alpha,n}) / (\varphi_{\alpha,n})] \in \mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \odot)$ . By (15) and calculations, we get  $\widehat{\mathcal{D}}_\kappa([(v_{\alpha,n}) / (\varphi_{\alpha,n})] \odot [(\kappa_{\alpha,n}) / (\phi_{\alpha,n})]) = [(\widehat{\mathcal{D}}_\kappa(v_{\alpha,n})) / (\varphi_{\alpha,n})] \bullet [(\kappa_{\alpha,n}) / (\phi_{\alpha,n})]$ .

This finishes the proof of the theorem.

**Definition 15** Given  $[(\widehat{\mathcal{D}}_\kappa(v_{\alpha,n})) / (\varphi_{\alpha,n})] \in \mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet)$ . We define  $\widehat{\mathcal{D}}_\kappa^{-1}$  as the inverse integral of  $\widehat{\mathcal{D}}_\kappa$  as

$$\widehat{\mathcal{D}}_\kappa^{-1} \left[ \left[ \widehat{\mathcal{D}}_\kappa(v_{\alpha,n}) \right] / (\varphi_{\alpha,n}) \right] = [(v_{\alpha,n}) / (\varphi_{\alpha,n})], \quad (17)$$

$(\phi_{\alpha,n}) \in \Delta$ .

**Theorem 16** Given  $[(v_{\alpha,n}) / (\varphi_{\alpha,n})] \in \mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet)$ . For every  $\phi_\alpha \in \vartheta(R_+)$  we get  $\widehat{\mathcal{D}}_\kappa([(v_{\alpha,n}) / (\varphi_{\alpha,n})] \odot \phi_\alpha) = \left[ \left( \widehat{\mathcal{D}}_\kappa(v_{\alpha,n}) \right) / (\varphi_{\alpha,n}) \right] \bullet \phi_\alpha$ .

**Proof** Let  $[(v_{\alpha,n}) / (\varphi_{\alpha,n})] \in \mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet)$  and  $\phi_\alpha \in \vartheta(R_+)$ . Applying (15) and Theorem 5 give  $\widehat{\mathcal{D}}_\kappa([(v_{\alpha,n}) / (\varphi_{\alpha,n})] \odot \phi_\alpha) = \widehat{\mathcal{D}}_\kappa \left( \left[ \left( \widehat{\mathcal{D}}_\kappa(v_{\alpha,n}) \right) / (\varphi_{\alpha,n}) \right] \bullet \phi_\alpha \right)$ . Hence the proof is finished.

**Theorem 17**  $\widehat{\mathcal{D}}_\kappa$  and  $\widehat{\mathcal{D}}_\kappa^{-1}$  are continuous in terms of convergence of  $\delta$  and  $\Delta$  types.

**Proof** We now confirm that  $\widehat{\mathcal{D}}_\kappa$  and  $\widehat{\mathcal{D}}_\kappa^{-1}$  are continuous in terms of  $\delta$ . Let  $\beta_n \rightarrow \beta \in \mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \odot)$  as  $n \rightarrow \infty$ . By virtue of Theorem 1 we can find  $v_{\alpha,n,k}$  and  $v_{\alpha,k} \in \mathbf{1}^1(R_+^2)$  such that  $\beta_n = [(v_{\alpha,n,k}) / (\phi_{\alpha,k})]$  and  $\beta = [(v_{\alpha,k}) / (\phi_{\alpha,k})]$  with  $\lim_{n \rightarrow \infty} v_{\alpha,n,k} = v_{\alpha,k} \ (\forall k \in N)$ . Continuity of  $\mathcal{D}_\kappa$  transform yields  $\lim_{n \rightarrow \infty} \mathcal{D}_\kappa v_{\alpha,n,k} = \mathcal{D}_\kappa v_{\alpha,k}$  in  $\mathbf{1}^1(R_+^2)$ . Thus  $[(\mathcal{D}_\kappa v_{\alpha,n,k}) / (\phi_{\alpha,k})] \rightarrow [(\mathcal{D}_\kappa v_{\alpha,k}) / (\phi_{\alpha,k})] \in \mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet)$  as  $n \rightarrow \infty$ . in the sense of  $\delta$ .

On the other hand, we show continuity of the inverse integral with respect to  $\delta$  convergence. Let  $g_{\alpha,n} \rightarrow g_\alpha$  in  $\mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet)$  as  $n \rightarrow \infty$ . Then, a parity of Theorem 1 implies that we can write  $g_{\alpha,n} = [(\mathcal{D}_\kappa v_{\alpha,n,k}) / (\phi_{\alpha,k})]$  and that  $g_\alpha = [(\mathcal{D}_\kappa v_{\alpha,k}) / (\phi_{\alpha,k})]$  with the property that  $\mathcal{D}_\kappa v_{\alpha,n,k} \rightarrow \mathcal{D}_\kappa v_{\alpha,k}$  as  $n \rightarrow \infty$ . Hence  $v_{\alpha,n,k} \rightarrow v_{\alpha,k}$  as  $n \rightarrow \infty$ . Therefore,  $[(v_{\alpha,n,k}) / (\phi_{\alpha,k})] \rightarrow [(v_{\alpha,k}) / (\phi_{\alpha,k})]$  as  $n \rightarrow \infty$ . By using (17) we get  $\widehat{\mathcal{D}}_\kappa^{-1} [(\mathcal{D}_\kappa v_{\alpha,n,k}) / (\phi_{\alpha,k})] \rightarrow \widehat{\mathcal{D}}_\kappa^{-1} [(\mathcal{D}_\kappa v_{\alpha,k}) / (\phi_{\alpha,k})]$  as  $n \rightarrow \infty$ .

Now, we establish continuity of  $\widehat{\mathcal{D}}_\kappa$  and  $\widehat{\mathcal{D}}_\kappa^{-1}$  with respect to  $\Delta$  convergence. Let  $\beta_n \rightarrow \beta \in \mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \odot)$  as  $n \rightarrow \infty$ . Then there exist  $v_{\alpha,n} \in \mathbf{1}^1(R_+^2)$  and  $(\phi_{\alpha,n}) \in \Delta$  such that  $(\beta_n - \beta) \odot \phi_{\alpha,n} = [(v_{\alpha,n}) \odot \phi_{\alpha,k}] / (\phi_{\alpha,k})$  and  $v_{\alpha,n} \rightarrow 0$  as  $n \rightarrow \infty$ . Employing (16) reveals  $\widehat{\mathcal{D}}_\kappa((\beta_n - \beta) \odot \phi_{\alpha,n}) = [(\mathcal{D}_\kappa((v_{\alpha,n}) \odot \phi_{\alpha,k})) / (\phi_{\alpha,k})]$ . Hence, it follows

$$\widehat{\mathcal{D}}_\kappa((\beta_n - \beta) \odot \phi_{\alpha,n}) = [(\mathcal{D}_\kappa(v_{\alpha,n,k} \bullet \phi_{\alpha,k})) / (\phi_{\alpha,k})] = \mathcal{D}_\kappa(v_{\alpha,n}) \rightarrow 0$$

as  $n \rightarrow \infty$  in  $\mathbf{1}^1(R_+^2)$ .

Therefore,  $\widehat{\mathcal{D}}_\kappa((\beta_n - \beta) \odot \phi_{\alpha,n}) = (\widehat{\mathcal{D}}_\kappa \beta_n - \widehat{\mathcal{D}}_\kappa \beta) \bullet \phi_{\alpha,n}$  as  $n \rightarrow \infty$ . Hence,  $\widehat{\mathcal{D}}_\kappa \beta_n \rightarrow \widehat{\mathcal{D}}_\kappa \beta$  as  $n \rightarrow \infty$  in  $\Delta$  convergence.

Finally, let  $g_{\alpha,n} \rightarrow g_\alpha$  in  $\mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \bullet)$  as  $n \rightarrow \infty$  then by Theorem 1 we find  $\mathcal{D}_\kappa v_{\alpha,k} \in \mathbf{1}^1(R_+^2)$  such that  $(g_{\alpha,n} - g_\alpha) \bullet \phi_{\alpha,n} = [(\mathcal{D}_\kappa v_{\alpha,k} \bullet \phi_{\alpha,k}) / (\phi_{\alpha,k})]$  where  $\mathcal{D}_\kappa v_{\alpha,k} \rightarrow 0$  as  $n \rightarrow \infty$  for some  $(\phi_{\alpha,n}) \in \Delta$ .

Using (17), we obtain

$$\widehat{\mathcal{D}}_\kappa^{-1}((g_{\alpha,n} - g_\alpha) \bullet \phi_{\alpha,n}) = \left[ \left( \widehat{\mathcal{D}}_\kappa^{-1}(\mathcal{D}_\kappa v_{\alpha,k} \bullet \phi_{\alpha,k}) \right) / (\phi_{\alpha,k}) \right].$$

Theorem 5 implies  $\widehat{\mathcal{D}}_\kappa^{-1}((g_{\alpha,n} - g_\alpha) \bullet \phi_{\alpha,n}) = [((v_{\alpha,k}) \odot \phi_{\alpha,k}) / (\phi_{\alpha,k})] \sim v_{\alpha,n} \rightarrow 0$  as  $n \rightarrow \infty$ . Thus,  $\widehat{\mathcal{D}}_\kappa^{-1}((g_{\alpha,n} - g_\alpha) \bullet \phi_{\alpha,n}) = (\widehat{\mathcal{D}}_\kappa^{-1} g_{\alpha,n} - \widehat{\mathcal{D}}_\kappa^{-1} g_\alpha) \odot \phi_{\alpha,n}$

as  $n \rightarrow \infty$ . Hence, we get

$$\widehat{\mathcal{D}}_{\kappa}^{-1} g_{\alpha, n} \rightarrow g_{\alpha} \in \mathcal{B}(\mathbf{1}^1(R_+^2), \vartheta, \odot, \odot)$$

as  $n \rightarrow \infty$ . (18)

This finishes the proof of the theorem.

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## STABILITY OF THE SINE-COSINE FUNCTIONAL EQUATION IN HYPERFUNCTIONS

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ABSTRACT. Let  $\mathcal{D}'_{L^\infty}(\mathbb{R}^{2n})$  and  $\mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})$  be the spaces of bounded distributions and bounded hyperfunctions respectively. In this paper we consider the Ulam's stability of the sine-cosine functional equation

$$u \circ T - u \otimes v + v \otimes u \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n}) \text{ [resp. } \mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})],$$

where  $u, v$  are Gelfand hyperfunctions,  $T : \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$  such that  $T(x, y) = x - y$  for all  $x, y \in \mathbb{R}^n$ , and  $\circ, \otimes$  denote pullback and tensor product of generalized functions respectively.

### 1. INTRODUCTION

A certain formula or equation is applicable to model a physical process if a small change in the formula or equation gives rise to a small change in the corresponding result. When this happens we say the formula or equation is stable. In an application, a functional equation like the additive Cauchy functional equation  $f(x + y) - f(x) - f(y) = 0$  may not be true for all  $x, y \in \mathbb{R}$  but it may be true approximately, that is

$$f(x + y) - f(x) - f(y) \approx 0$$

for all  $x, y \in \mathbb{R}$ . This can be stated mathematically as

$$(1.1) \quad |f(x + y) - f(x) - f(y)| \leq \varepsilon$$

for some small positive  $\varepsilon$  and for all  $x, y \in \mathbb{R}$ . We would like to know when small changes in a particular equation like the additive Cauchy functional equation have only small effects on its solutions. This is the essence of stability theory. In 1940, S.M. Ulam asked the following question:

*Let  $f$  be a mapping from a group  $G_1$  to a metric group  $G_2$  with metric  $d(\cdot, \cdot)$  such that*

$$d(f(xy), f(x)f(y)) \leq \varepsilon.$$

*Then does there exist a group homomorphism  $h$  and  $\delta_\varepsilon > 0$  such that*

$$d(f(x), h(x)) \leq \delta_\varepsilon$$

*for all  $x \in G_1$ ?*

This problem was solved affirmatively by D. H. Hyers under the assumption that  $G_2$  is a Banach space (see Hyers [19], Hyers-Isac-Rassias [20]). Since then Ulam problems of many other functional

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equations have been investigated [13, 14, 15, 21, 23, 24, 25, 26, 27, 28]. Among the results, Székelyhidi has developed his idea of using invariant subspaces of functions defined on a group or semigroup he prove the Ulam-Hyers stability problem for functional equation

$$(1.2) \quad f(x + y) = f(x)g(y) + g(x)f(y), \quad x, y \in \mathbb{R}^n,$$

which arises from the sine addition formula [30, 31]. Using his elegant idea, Chung and Chang [7] prove the parallel Ulam-Hyers stability problem for functional equation

$$(1.3) \quad f(x - y) = f(x)g(y) - g(x)f(y), \quad x, y \in \mathbb{R}^n,$$

which arises from the sine subtraction formula. As a result it was proved that if  $f, g : \mathbb{R}^n \rightarrow \mathbb{C}$  satisfy

$$(1.4) \quad |f(x - y) - f(x)g(y) + g(x)f(y)| \leq M, \quad x, y \in \mathbb{R}^n$$

for some  $M > 0$ , then either there exist  $\lambda_1, \lambda_2 \in \mathbb{C}$ , not both zero, and  $L > 0$  such that

$$(1.5) \quad |\lambda_1 f(x) - \lambda_2 g(x)| \leq L$$

for all  $x \in \mathbb{R}^n$ , or else

$$(1.6) \quad f(x - y) = f(x)g(y) - g(x)f(y)$$

for all  $x, y \in \mathbb{R}^n$ . Also in the sequel, the functions  $f$  and  $g$  satisfying both (1.4) and (1.5) were investigated.

Schwartz introduced the theory of distributions in his monograph *Théorie des distributions* [29] in which Schwartz systematizes the theory of generalized functions, basing it on the theory of linear topological spaces, relates all the earlier approaches, and obtains many important results. After his elegant theory appeared, many important concepts and results on the classical spaces of functions have been generalized to the space of distributions. For example, the space  $L^\infty(\mathbb{R}^n)$  of bounded measurable functions on  $\mathbb{R}^n$  has been generalized to the space  $\mathcal{D}'_{L^\infty}(\mathbb{R}^n)$  of *bounded distributions* as a subspace of distributions and later the space  $\mathcal{D}'_{L^\infty}(\mathbb{R}^n)$  is further generalized to the space  $\mathcal{A}'_{L^\infty}(\mathbb{R}^n)$  of *bounded hyperfunctions*. It is very natural to consider the following stability problem for the functional equation in distributions and hyperfunctions  $u, v$  with respect to bounded distributions and bounded hyperfunctions

$$(1.7) \quad u \circ T - u \otimes v + v \otimes u \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n}) \text{ [resp. } \mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})],$$

where  $\mathcal{D}'_{L^\infty}(\mathbb{R}^{2n})$  and  $\mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})$  are the spaces of bounded distributions and bounded hyperfunctions,  $T : \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$  such that  $T(x, y) = x - y$  for all  $x, y \in \mathbb{R}^n$ , and  $\circ, \otimes$  denote pullback and tensor product of generalized functions respectively. In [10] the distributional version of the stability of (1.2) was proved. In this paper, as a parallel result we prove the stability of (1.7). As in [10] the main tool is *the heat kernel method* initiated by T. Matsuzawa [22] which represents the generalized functions in some class as the initial values of solutions of the heat equation with appropriate growth

conditions [12, 22]. Making use of the heat kernel method we can convert (1.7) to the classical Ulam-Hyers stability problem of the functional inequality; there exist  $C > 0$  and  $N \geq 0$  [resp. for every  $\epsilon > 0$  there exists  $C_\epsilon > 0$ ] such that

$$(1.8) \quad |\tilde{u}(x - y, t + s) - \tilde{u}(x, t)\tilde{v}(y, s) + \tilde{v}(x, t)\tilde{u}(y, s)| \leq C \left(\frac{1}{t} + \frac{1}{s}\right)^N \quad [\text{resp. } C_\epsilon e^{\epsilon(1/t+1/s)}]$$

for all  $x, y \in \mathbb{R}^n$ ,  $t, s > 0$ , where  $\tilde{u}, \tilde{v}, \tilde{w}, \tilde{k} : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{C}$  are solutions of the heat equation whose initial values are  $u, v, w, k$  respectively. In Section 3, we consider the stability problem (1.8) with a more general setting, which will be used, combined with the heat kernel method [12, 22], to prove the stability problem of (1.7).

## 2. DISTRIBUTIONS AND HYPERFUNCTIONS

We first introduce the spaces  $\mathcal{S}'$  of Schwartz tempered distributions and  $\mathcal{G}'$  of Gelfand hyperfunctions(see [16, 17, 18, 22, 29] for more details of these spaces). We use the notations:  $|\alpha| = \alpha_1 + \dots + \alpha_n$ ,  $\alpha! = \alpha_1! \dots \alpha_n!$ ,  $|x| = \sqrt{x_1^2 + \dots + x_n^2}$ ,  $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$  and  $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$ , for  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ ,  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ , where  $\mathbb{N}_0$  is the set of non-negative integers and  $\partial_j = \frac{\partial}{\partial x_j}$ .

**Definition 2.1.** [29] We denote by  $\mathcal{S}$  or  $\mathcal{S}(\mathbb{R}^n)$  the Schwartz space of all infinitely differentiable functions  $\varphi$  in  $\mathbb{R}^n$  such that

$$(2.1) \quad \|\varphi\|_{\alpha, \beta} = \sup_x |x^\alpha \partial^\beta \varphi(x)| < \infty$$

for all  $\alpha, \beta \in \mathbb{N}_0^n$ , equipped with the topology defined by the seminorms  $\|\cdot\|_{\alpha, \beta}$ . The elements of  $\mathcal{S}$  are called rapidly decreasing functions and the elements of the dual space  $\mathcal{S}'$  are called tempered distributions.

**Definition 2.2.** [16, 17] We denote by  $\mathcal{G}$  or  $\mathcal{G}(\mathbb{R}^n)$  the Gelfand space of all infinitely differentiable functions  $\varphi$  in  $\mathbb{R}^n$  such that

$$\|\varphi\|_{h, k} = \sup_{x \in \mathbb{R}^n, \alpha, \beta \in \mathbb{N}_0^n} \frac{|x^\alpha \partial^\beta \varphi(x)|}{h^{|\alpha|} k^{|\beta|} \alpha!^{1/2} \beta!^{1/2}} < \infty$$

for some  $h, k > 0$ . We say that  $\varphi_j \rightarrow 0$  as  $j \rightarrow \infty$  if  $\|\varphi_j\|_{h, k} \rightarrow 0$  as  $j \rightarrow \infty$  for some  $h, k$ , and denote by  $\mathcal{G}'$  the strong dual space of  $\mathcal{G}$  and call its elements Gelfand hyperfunctions.

As a generalization of the space  $L^\infty$  of bounded measurable functions, L. Schwartz introduced the space  $\mathcal{D}'_{L^\infty}$  of bounded distributions as a subspace of tempered distributions.

**Definition 2.3.** [29] We denote by  $\mathcal{D}_{L^1}(\mathbb{R}^n)$  the space of smooth functions on  $\mathbb{R}^n$  such that  $\partial^\alpha \varphi \in L^1(\mathbb{R}^n)$  for all  $\alpha \in \mathbb{N}_0^n$  equipped with the topology defined by the countable family of seminorms

$$\|\varphi\|_m = \sum_{|\alpha| \leq m} \|\partial^\alpha \varphi\|_{L^1}, \quad m \in \mathbb{N}_0.$$

We denote by  $\mathcal{D}'_{L^\infty}$  the strong dual space of  $\mathcal{D}_{L^1}$  and call its elements bounded distributions.

As a generalization of bounded distributions, the space  $\mathcal{A}'_{L^\infty}$  of bounded hyperfunctions has been introduced as a subspace of  $\mathcal{G}'$ .

**Definition 2.4.** [12] We denote by  $\mathcal{A}_{L^1}$  the space of smooth functions on  $\mathbb{R}^n$  satisfying

$$\|\varphi\|_h = \sup_{\alpha} \frac{\|\partial^\alpha \varphi\|_{L^1}}{h^{|\alpha|} \alpha!} < \infty$$

for some constant  $h > 0$ . We say that  $\varphi_j \rightarrow 0$  in  $\mathcal{A}_{L^1}$  as  $j \rightarrow \infty$  if there is a positive constant  $h$  such that

$$\sup_{\alpha} \frac{\|\partial^\alpha \varphi_j\|_{L^1}}{h^{|\alpha|} \alpha!} \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

We denote by  $\mathcal{A}'_{L^\infty}$  the strong dual space of  $\mathcal{A}_{L^1}$ .

It is well known that the following topological inclusions hold:

$$\begin{aligned} \mathcal{G} &\hookrightarrow \mathcal{S} \hookrightarrow \mathcal{D}_{L^1}, & \mathcal{D}'_{L^\infty} &\hookrightarrow \mathcal{S}' \hookrightarrow \mathcal{G}', \\ \mathcal{G} &\hookrightarrow \mathcal{A}_{L^1} \hookrightarrow \mathcal{D}_{L^1}, & \mathcal{D}'_{L^\infty} &\hookrightarrow \mathcal{A}'_{L^\infty} \hookrightarrow \mathcal{G}'. \end{aligned}$$

It is known that the space  $\mathcal{G}(\mathbb{R}^n)$  consists of all infinitely differentiable functions  $\varphi(x)$  on  $\mathbb{R}^n$  which can be extended to an entire function on  $\mathbb{C}^n$  satisfying

$$(2.2) \quad |\varphi(x + iy)| \leq C \exp(-a|x|^2 + b|y|^2), \quad x, y \in \mathbb{R}^n$$

for some  $a, b, C > 0$  (see [16]).

**Definition 2.5.** Let  $u_j \in \mathcal{G}'(\mathbb{R}^{n_j})$  for  $j = 1, 2$ . Then the tensor product  $u_1 \otimes u_2$  of  $u_1$  and  $u_2$ , defined by

$$\langle u_1 \otimes u_2, \varphi(x_1, x_2) \rangle = \langle u_1, \langle u_2, \varphi(x_1, x_2) \rangle \rangle$$

for  $\varphi(x_1, x_2) \in \mathcal{G}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ , belongs to  $\mathcal{G}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ .

### 3. STABILITY OF (1.8)

Throughout this paper  $\langle G, + \rangle$  is a 2-divisible commutative group,  $f, g : G \times (0, \infty) \rightarrow \mathbb{C}$  and  $N$  denotes a fixed nonnegative real number. We consider the stability problems of each of the following functional inequalities;

there exist  $C > 0$  and  $d > 0$  such that

$$(3.1) \quad |f(x - y, t + s) - f(x, t)g(y, s) + g(x, t)f(y, s)| \leq C \left( \frac{1}{t} + \frac{1}{s} \right)^N + d, \quad \forall x, y \in G, t, s > 0;$$

for every  $\epsilon > 0$ , there exists  $C_\epsilon > 0$  which depends on  $\epsilon$  such that

$$(3.2) \quad |f(x - y, t + s) - f(x, t)g(y, s) + g(x, t)f(y, s)| \leq C_\epsilon e^{\epsilon(1/t+1/s)}, \quad \forall x, y \in G, t, s > 0.$$

From now on, a function  $a$  from a semigroup  $\langle S, + \rangle$  to the field  $\mathbb{C}$  of complex numbers is said to be an additive function provided  $a(x + y) = a(x) + a(y)$  for all  $x, y \in S$  and  $m : S \rightarrow \mathbb{C}$  is said to be an exponential function provided  $m(x + y) = m(x)m(y)$  for all  $x, y \in S$ .

We introduce the following conditions (3.3) and (3.4) on  $f : G \times (0, \infty) \rightarrow \mathbb{C}$  and  $N$ ;

$$(3.3) \quad \begin{aligned} & \text{there exist } C > 0 \text{ and } d > 0 \text{ such that} \\ & |f(x, t)| \leq Ct^{-N} + d, \quad \forall x \in G, t > 0; \end{aligned}$$

$$(3.4) \quad \begin{aligned} & \text{for every } \epsilon > 0, \text{ there exists } C_\epsilon > 0 \text{ which depends on } \epsilon \text{ such that} \\ & |f(x, t)| \leq C_\epsilon e^{\epsilon/t}, \quad \forall x \in G, t > 0. \end{aligned}$$

Using the idea in [20, p. 104] we obtain the following (See [10] for the proofs).

**Lemma 3.1.** *Let  $f, g : G \times (0, \infty) \rightarrow \mathbb{C}$  satisfy the inequality; for each  $y \in G$  and  $s > 0$  there exist positive constants  $C = C(y, s)$  and  $d = d(y, s)$  [resp. for each  $y \in G, s > 0$  and  $\epsilon > 0$  there exists a positive constant  $C_\epsilon = C_\epsilon(y, s)$ ] such that*

$$(3.5) \quad |f(x - y, t + s) - f(x, t)g(y, s)| \leq Ct^{-N} + d \text{ [resp. } C_\epsilon e^{\epsilon/t}]$$

for all  $x \in G, t > 0$ . Then either  $f$  satisfies (3.3) [resp.(3.4)] or  $g$  is an exponential function.

**Lemma 3.2.** *Let  $m : G \times (0, \infty) \rightarrow \mathbb{C}$  be a nonzero exponential function satisfying (3.3) [resp.(3.4)]. Then  $m$  can be written in the form*

$$m(x, t) = m_1(x)m_2(t),$$

where  $m_1 : G \rightarrow \mathbb{C}, m_2 : (0, \infty) \rightarrow \mathbb{C}$  is exponential functions satisfying  $|m_1(x)| = 1$  for all  $x \in G$ .

**Lemma 3.3.** *Let  $m$  be a nonzero exponential function satisfying (3.3) [resp.(3.4)]. Suppose that  $f : G \times (0, \infty) \rightarrow \mathbb{C}$  satisfies the inequality; there exist positive constants  $C$  and  $d$  [resp. for each  $\epsilon > 0$ , there exists a positive constant  $C_\epsilon$ ] such that*

$$(3.6) \quad |f(x + y, t + s) - f(x, t)m(y, s) - f(y, s)m(x, t)| \leq C \left( \frac{1}{t} + \frac{1}{s} \right)^N + d \text{ [resp. } C_\epsilon e^{\epsilon/t}]$$

for all  $x, y \in G, t, s > 0$ . Then we have

$$f(x, t) = a(x)m_1(x)m_2(t) + 2f \left( 0, \frac{t}{2} \right) m_1(x)m_2 \left( \frac{t}{2} \right) + R(x, t),$$

where  $a : G \rightarrow \mathbb{C}$  is an additive function,  $m : (0, \infty) \rightarrow \mathbb{C}$  is an exponential function,  $\lambda \in \mathbb{C}$  and  $R : G \times (0, \infty) \rightarrow \mathbb{C}$  satisfies

$$|R(x, t)| \leq Ct^{-N} + d \text{ [resp.(3.4)]}$$

for all  $x \in G, t > 0$ .

**Theorem 3.4.** *Suppose that  $f, g : G \times (0, \infty) \rightarrow \mathbb{C}$  satisfy the inequality (3.1) [resp.(3.2)]. Then either*

$$(3.7) \quad f(x - y, t + s) - f(x, t)g(y, s) + g(x, t)f(y, s) = 0$$

for all  $x, y \in G, t, s > 0$ , or else there exist  $\lambda_1, \lambda_2 \in \mathbb{C}$ , not both zero, such that  $\lambda_1 f(x, t) - \lambda_2 g(x, t)$  satisfies (3.1) [resp.(3.2)].

*Proof.* It suffices to prove that  $f, g$  satisfies (3.7) when  $\lambda_1 f(x, t) - \lambda_2 g(x, t)$  satisfies (3.3) [resp.(3.4)] only for  $\lambda_1 = \lambda_2 = 0$ . Let

$$(3.8) \quad F(x, y, t, s) = f(x - y, t + s) - f(x, t)g(y, s) + g(x, t)f(y, s).$$

Choosing  $y_1$  and  $s_1$  with  $f(y_1, s_1) \neq 0$  we have

$$(3.9) \quad g(x, t) = k_1 f(x, t) + k_2 f(x - y_1, t + s_1) - k_2 F(x, y_1, t, s_1),$$

where  $k_1 = \frac{g(y_1, s_1)}{f(y_1, s_1)}$  and  $k_2 = \frac{-1}{f(y_1, s_1)}$ . From (3.8) and (3.9) we have

$$(3.10) \quad \begin{aligned} & f((x - y) - z, (t + s) + r) \\ &= f(x - y, t + s)g(z, r) - g(x - y, t + s)f(z, r) + F(x - y, z, t + s, r) \\ &= f(x - y, t + s)g(z, r) \\ &\quad - \left( k_1 f(x - y, t + s) + k_2 f(x - y - y_1, t + s + s_1) - k_2 F(x - y, y_1, t + s, s_1) \right) f(z, r) \\ &\quad + F(x - y, z, t + s, r) \\ &= \left( f(x, t)g(y, s) - g(x, t)f(y, s) + F(x, y, t, s) \right) g(z, r) \\ &\quad - k_1 \left( f(x, t)g(y, s) - g(x, t)f(y, s) + F(x, y, t, s) \right) f(z, r) \\ &\quad + k_2 \left( f(x, t)g(y + y_1, s + s_1) - g(x, t)f(y + y_1, s + s_1) \right. \\ &\quad \left. + F(x, y + y_1, t, s + s_1) - F(x - y, y_1, t + s, s_1) \right) f(z, r) \\ &\quad + F(x - y, z, t + s, r), \end{aligned}$$

and also we have

$$(3.11) \quad f(x - (y + z), t + (s + r)) = f(x, t)g(y + z, s + r) - g(x, t)f(y + z, s + r) + F(x, y + z, t, s + r).$$

From (3.10) and (3.11) we have

$$(3.12) \quad \begin{aligned} & f(x, t) \left( g(y, s)g(z, r) - k_1 g(y, s)f(z, r) + k_2 g(y + y_1, s + s_1)f(z, r) - g(y + z, s + r) \right) \\ &\quad + g(x, t) \left( -f(y, s)g(z, r) + k_1 f(y, s)f(z, r) - k_2 f(y + y_1, s + s_1)f(z, r) + f(y + z, s + r) \right) \\ &= F(x, y + z, t, s + r) - F(x - y, z, t + s, r) - F(x, y, t, s)g(z, r) + k_1 F(x, y, t, s)f(z, r) \\ &\quad - k_2 \left( F(x, y + y_1, t, s + s_1) - F(x - y, y_1, t + s, s_1) \right) f(z, r). \end{aligned}$$

Fixing  $y, z, s, r$  in (3.12), using (3.1) and (3.8) we have

$$\begin{aligned} & |F(x, y + z, t, s + r) - F(x - y, z, t + s, r) - F(x, y, t, s)g(z, r) + k_1 F(x, y, t, s)f(z, r) \\ &\quad - k_2 \left( F(x, y + y_1, t, s + s_1) - F(x - y, y_1, t + s, s_1) \right) f(z, r)| \\ &\leq 2C \left( \frac{1}{t} + \frac{1}{r} \right)^N + 2d + C_1 \left( \frac{1}{t} + \frac{1}{s} \right)^N + d_1 + C_2 \left( \frac{1}{t} + \frac{1}{s_1} \right)^N + d_2 \\ &\leq C't^{-N} + d', \end{aligned}$$

where  $C' = 2^N(2C + C_1 + C_2)$ ,  $d' = 2^N(2Cr^{-N} + C_1s^{-N} + C_2s_1^{-N}) + 2d + d_1 + d_2$ .

Similarly, using (3.2) we obtain that for every  $\epsilon > 0$  there exists  $C_\epsilon > 0$  such that

$$\begin{aligned} & |F(x, y + z, t, s + r) - F(x - y, z, t + s, r) - F(x, y, t, s)g(z, r) + k_1F(x, y, t, s)f(z, r) \\ & - k_2(F(x, y + y_1, t, s + s_1) - F(x - y, y_1, t + s, s_1))f(z, r)| \\ & \leq 2C_\epsilon e^{\epsilon(1/t+1/r)} + C_1C_\epsilon e^{\epsilon(1/t+1/s)} + C_2C_\epsilon e^{\epsilon(1/t+1/s_1)} \\ & \leq C'_\epsilon e^{\epsilon/t}, \end{aligned}$$

where  $C'_\epsilon = C_\epsilon(2e^{\epsilon/r} + C_1e^{\epsilon/s} + C_2e^{\epsilon/s_1})$ .

Thus, by the assumption that  $\lambda_1f(x, t) - \lambda_2g(x, t)$  satisfies (3.3) [resp.(3.4)] only for  $\lambda_1 = \lambda_2 = 0$  we have

$$\begin{aligned} & g(y, s)g(z, r) - k_1g(y, s)f(z, r) + k_2g(y + y_1, s + s_1)f(z, r) - g(y + z, s + r) \\ & = f(y, s)g(z, r) - k_1f(y, s)f(z, r) + k_2f(y + y_1, s + s_1)f(z, r) - f(y + z, s + r) = 0. \end{aligned}$$

Thus, it follows that

$$\begin{aligned} (3.13) \quad & F(x, y + z, t, s + r) - F(x - y, z, t + s, r) \\ & = \left( -k_1F(x, y, t, s) + k_2F(x, y + y_1, t, s + s_1) - k_2F(x - y, y_1, t + s, s_1) \right) f(z, r) \\ & \quad + F(x, y, t, s)g(z, r). \end{aligned}$$

Now, if we fix  $x, y, t, s$ , the left hand side of (3.13) satisfies (3.3) [resp. (3.4)] as a function of  $z$  and  $r$ . From the right hand side of (3.13), using the assumption that  $\lambda_1f(x, t) - \lambda_2g(x, t)$  satisfies (3.3) [resp.(3.4)] only for  $\lambda_1 = \lambda_2 = 0$  it follows that  $F \equiv 0$ . This completes the proof.  $\square$

**Theorem 3.5.** *Let  $f, g : G \times (0, \infty) \rightarrow \mathbb{C}$  satisfy (3.1) [resp. (3.2)]. Then  $(f, g)$  satisfies one of the following :*

- (i) both  $f$  and  $g$  satisfy (3.3) [resp.(3.4)],
- (ii)  $f(x, t) = a(x)m(t) + R(x, t)$ ,  $g(x, t) = \lambda f(x, t) + m(t)$  for all  $x \in G, t > 0$ , where  $a : G \rightarrow \mathbb{C}$  is an additive function,  $m : (0, \infty) \rightarrow \mathbb{C}$  is an exponential function,  $\lambda \in \mathbb{C}$  and  $R : G \times (0, \infty) \rightarrow \mathbb{C}$  satisfies  $|R(x, t)| \leq Ct^{-2N} + d$  [resp.(3.4)] for all  $x \in G, t > 0$  and for some  $C, d > 0$ ,
- (iii)  $f(x - y, t + s) - f(x, t)g(y, s) + g(x, t)f(y, s) = 0$  for all  $x, y \in G, t, s > 0$ .

*Proof.* Assume that  $(f, g)$  does not satisfy (iii). Then by Lemma 3.4 there exist  $\lambda_1, \lambda_2 \in \mathbb{C}$ , not both zero, such that  $\lambda_1f(x, t) - \lambda_2g(x, t)$  satisfies (3.3) [resp.(3.4)].

(Case 1)  $f(\neq 0)$  satisfies (3.3) [resp.(3.4)].

Assume that  $f(\neq 0)$  satisfies (3.3). Choosing  $y_0 \in G, s_0 > 0$  such that  $f(y_0, s_0) \neq 0$ , dividing  $|f(y_0, s_0)|$  in both sides of (3.1) and using the triangle inequality we have

$$\begin{aligned} |g(x, t)| & \leq \frac{1}{|f(y_0, s_0)|} \left( |f(x - y_0, t + s_0)| + |f(x, t)g(y_0, s_0)| + C \left( \frac{1}{t} + \frac{1}{s_0} \right)^N + d \right) \\ & \leq C_1(t + s_0)^{-N} + d_1 + C_2t^{-N} + d_2 + C_3 \left( \frac{1}{t} + \frac{1}{s_0} \right)^N + d_3 \\ & \leq C't^{-N} + d' \end{aligned}$$

for all  $x \in G, t > 0$  and for some positive constants  $C_1, C_2, C_3, d_1, d_2, d_3, C'$  and  $d'$ . Similarly, if  $f$  satisfies (3.4) we can show that for every  $\epsilon > 0$  there exists  $C'_\epsilon > 0$  such that

$$|g(x, t)| \leq C'_\epsilon e^{\epsilon/t}$$

for all  $x \in G, t > 0$ . Thus, we obtain the case (ii).

(Case 2)  $f$  does not satisfy (3.3) [resp. (3.4)].

Assume that  $f$  does not satisfy (3.3). In this case we must have  $\lambda_2 \neq 0$  and we can write

$$(3.14) \quad g(x, t) = -\frac{\lambda_1}{\lambda_2} f(x, t) + B(x, t) := \lambda f(x, t) + B(x, t)$$

for all  $x \in G, t > 0$ , where  $R$  satisfies (3.3) [resp. (3.4)]. Putting (3.14) in (3.1) we have

$$(3.15) \quad |f(x - y, t + s) - f(x, t)B(y, s) + B(x, t)f(y, s)| \leq C \left( \frac{1}{t} + \frac{1}{s} \right)^N + d$$

for all  $x, y \in G, t, s > 0$ . Using the triangle inequality and fixing  $y$  and  $s$  in (3.15) we have

$$|f(x - y, t + s) - f(x, t)B(y, s)| \leq |B(x, t)f(y, s)| + C \left( \frac{1}{t} + \frac{1}{s} \right)^N + d \leq C't^{-N} + d'$$

for all  $x, y \in G, t, s > 0$  and for some positive constants  $C'$  and  $d'$ . Applying Lemma 3.1 we have

$$(3.16) \quad B(x, t) = m(x, t)$$

for all  $x \in G, t > 0$ , where  $m$  is an exponential function on  $G \times (0, \infty)$ . Now, applying Lemma 3.2 we have

$$(3.17) \quad R(x, t) = m(x, t) = m_1(x)m_2(t)$$

for all  $x \in G, t > 0$ , where  $m_1 : G \rightarrow \mathbb{C}, m_2 : (0, \infty) \rightarrow \mathbb{C}$  are exponential functions. Replacing  $(x, t)$  by  $(y, s)$  in (3.15) we have

$$(3.18) \quad |f(-x + y, t + s) - f(y, s)B(x, t) + B(y, s)f(x, t)| \leq C \left( \frac{1}{t} + \frac{1}{s} \right)^N + d$$

for all  $x, y \in G, t, s > 0$ . From (3.15) and (3.18), using the triangle inequality, putting  $y = 0$  and replacing  $t, s$  by  $\frac{t}{2}$  we have

$$(3.19) \quad |f(x, t) + f(-x, t)| \leq C2^{2N+1}t^{-N} + 2d$$

for all  $x \in G, t > 0$ . Replacing  $x$  by  $-x, y$  by  $-y$  in (3.15), we have

$$(3.20) \quad |f(-x + y, t + s) - f(-x, t)B(-y, s) + B(-x, t)f(-y, s)| \leq C \left( \frac{1}{t} + \frac{1}{s} \right)^N + d$$

for all  $x, y \in G, t, s > 0$ . From (3.20) and using (3.19) with fixing  $y$  and  $s$  we have

$$(3.21) \quad |f(-x + y, t + s) - f(x, t)B(-y, s) + B(-x, t)f(y, s)| \leq C_1t^{-N} + d_1$$

for all  $x \in G, t > 0$ . From (3.20) and (3.21) with fixing  $y$  and  $s$  we have

$$(3.22) \quad |f(x, t)(B(y, s) - B(-y, s)) - f(y, s)(B(x, t) - B(-x, t))| \leq C_2t^{-N} + d_2$$

for all  $x, y \in G, t, s > 0$ . Since  $f$  does not satisfy (3.3), it follows from (3.22) that

$$B(y, s) = B(-y, s)$$

for all  $y \in G, s > 0$  and hence  $m_1(y) = 1$  for all  $y \in G$ . Thus, we have

$$(3.23) \quad g(x, t) = \lambda f(x, t) + m_2(t)$$

for all  $x \in G, t > 0$ . From (3.15), (3.17) and (3.19) we have

$$(3.24) \quad \begin{aligned} |f(x + y, t + s) - f(x, t)m_2(s) - f(y, s)m_2(t)| &\leq |f(y, s) + f(-y, s)||m_2(t)| + C \left(\frac{1}{t} + \frac{1}{s}\right)^N + d \\ &\leq (C2^{2N+1}t^{-N} + 2d)Ct^{-N} + C \left(\frac{1}{t} + \frac{1}{s}\right)^N + d \\ &\leq C' \left(\frac{1}{t} + \frac{1}{s}\right)^{2N} + d' \end{aligned}$$

for all  $x, y \in G, t, s > 0$  and for some  $C' > 0, d' > 0$ .

Similarly, if  $f$  satisfies (3.4) we can show that for every  $\epsilon > 0$  there exists  $C'_\epsilon > 0$  such that

$$(3.25) \quad |f(x + y, t + s) - f(x, t)m_2(s) - f(y, s)m_2(t)| \leq C'_\epsilon e^{\epsilon/t}$$

for all  $x \in G, t > 0$ . Applying Lemma 3.3 with (3.23) and (3.24) we have

$$(3.26) \quad f(x, t) = a(x)m_2(t) + 2f\left(0, \frac{t}{2}\right)m_2\left(\frac{t}{2}\right) + R(x, t)$$

for all  $x \in G, t > 0$ , where  $a$  is an additive mapping and  $R$  satisfies (3.3)[resp. (3.4)]. Replacing  $(y, s)$  by  $(x, t)$  in (3.1) we see that  $f(0, t)$  satisfies (3.3). Thus,  $2f\left(0, \frac{t}{2}\right)m_2\left(\frac{t}{2}\right) + R(x, t)$  satisfies (3.3)[resp. (3.4)]. Replacing  $2f\left(0, \frac{t}{2}\right)m_2\left(\frac{t}{2}\right) + R(x, t)$  by  $R(x, t)$  and  $m_2$  by  $m$  we get the case (iii). This completes the proof. □

#### 4. MAIN RESULTS

In this section as a main result of the paper we consider the stability of (1.6). The main tools of our proof are based on *structure theorems* for generalized functions and *the heat kernel method* initiated by T. Matsuzawa [22] which represents the generalized functions as initial values of solutions of the heat equation with appropriate growth conditions [8, 9, 11, 12, 22]. For the proof of our theorem we employ the  $n$ -dimensional heat kernel  $E_t(x)$  given by

$$E_t(x) = (4\pi t)^{-n/2} \exp(-|x|^2/4t), \quad t > 0.$$

In view of (2.2), we can see that the heat kernel  $E_t$  belongs to the Gelfand space  $\mathcal{G}(\mathbb{R}^n)$  for each  $t > 0$ . Thus, for each  $u \in \mathcal{G}'(\mathbb{R}^n)$ , the convolution  $(u * E_t)(x) := \langle u_y, E_t(x - y) \rangle$  is well defined. We call  $(u * E_t)(x)$  the *Gauss transform* of  $u$ . From now on we denote by  $\tilde{u}(x, t)$  the Gauss transform of



$u$ . It is well known that the Gauss transform  $\tilde{u}(x, t)$  is a smooth solution of the heat equation such that  $\tilde{u}(x, t) \rightarrow u$  in weak star topology as  $t \rightarrow 0^+$ , i.e.,

$$\langle u, \varphi \rangle = \lim_{t \rightarrow 0^+} \int \tilde{u}(x, t) \varphi(x) dx$$

for all  $\varphi \in \mathcal{G}$ .

We first discuss the solutions of the corresponding trigonometric functional equations in the space  $\mathcal{G}$  of Gelfand generalized functions.

**Lemma 4.1.** *The solutions  $u, v \in \mathcal{G}'$  of the equation*

$$(4.1) \quad u \circ T - u \otimes v + v \otimes u = 0$$

are either

$$(4.2) \quad u = \lambda(e^{c \cdot x} - e^{-c \cdot x}), \quad v = \gamma e^{c \cdot x} + (1 - \gamma)e^{-c \cdot x}$$

or else

$$(4.3) \quad u = c \cdot x, \quad v = 1 + \lambda c \cdot x.$$

*Proof.* As a consequence of the results in [4, 15] the solutions  $(u, v)$  of (4.1) are equal to the smooth solutions  $(f, g)$  of the equation

$$(4.4) \quad f(x - y) - f(x)g(y) + f(y)g(x) = 0$$

for all  $x, y \in \mathbb{R}^n$ . By [2, Theorem 11] all solutions of (4.4) are given by

$$(4.5) \quad f(x) = \lambda(m(x) - m(-x)), \quad g(x) = \gamma m(x) + (1 - \gamma)m(-x)$$

or else

$$(4.6) \quad f(x) = a(x), \quad g(x) = 1 + \lambda a(x),$$

where  $m$  is an exponential function and  $a$  is an additive function. From (4.5) and (4.6)  $m$  and  $a$  are smooth functions and hence  $m(x) = e^{c \cdot x}$  and  $a(x) = c \cdot x$  for some  $c \in \mathbb{C}^n$ . Thus, we get (4.2) and (4.3). This completes the proof.  $\square$

The proof of Theorem 2.3 of [11] works even when  $p = \infty$ , i.e., we obtain the following.

**Lemma 4.2.** [11] *The Gauss transform  $\tilde{u}(x, t) := (u * E)(x, t)$  of  $u \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$  is a smooth solution of the heat equation  $(\Delta - \partial/\partial t)\tilde{u} = 0$  satisfying:*

(i) *There exist constants  $C > 0, N \geq 0$  such that*

$$(4.7) \quad |\tilde{u}(x, t)| \leq Ct^{-N} \quad \text{for all } x \in \mathbb{R}^n, t > 0.$$

(ii)  *$\tilde{u}(x, t) \rightarrow u$  as  $t \rightarrow 0^+$  in the sense that for every  $\varphi \in \mathcal{D}_{L^1}$ ,*

$$\langle u, \varphi \rangle = \lim_{t \rightarrow 0^+} \int \tilde{u}(x, t) \varphi(x) dx.$$

*Conversely, every smooth solution  $\tilde{u}(x, t)$  of the heat equation satisfying the estimate (4.7) can be uniquely expressed as  $\tilde{u}(x, t) = (u * E)(x, t)$  for some  $u \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ .*

The following lemma is a special case of Theorem 3.5 of [12] when  $p = \infty$  where the space  $\mathcal{A}'_{L^\infty}(\mathbb{R}^n)$  is denoted by  $\mathcal{B}_{L^\infty}(\mathbb{R}^n)$ .

**Lemma 4.3.** [12] *The Gauss transform  $\tilde{u}(x, t) := (u * E)(x, t)$  of  $u \in \mathcal{A}'_{L^\infty}(\mathbb{R}^n)$  is a smooth solution of the heat equation  $(\Delta - \partial/\partial t)\tilde{u} = 0$  satisfying:*

(i) *For every  $\epsilon > 0$  there exists a constant  $C_\epsilon > 0$  such that*

$$(4.8) \quad |\tilde{u}(x, t)| \leq C_\epsilon e^{\epsilon/t} \quad \text{for all } x \in \mathbb{R}^n, t > 0.$$

(ii)  *$\tilde{u}(x, t) \rightarrow u$  as  $t \rightarrow 0^+$  in the sense that for every  $\varphi \in \mathcal{A}_{L^1}$ ,*

$$\langle u, \varphi \rangle = \lim_{t \rightarrow 0^+} \int \tilde{u}(x, t) \varphi(x) dx.$$

*Conversely, every smooth solution  $\tilde{u}(x, t)$  of the heat equation satisfying the estimate (4.8) can be uniquely expressed as  $\tilde{u}(x, t) = (u * E)(x, t)$  for some  $u \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$ .*

The following structure theorem for bounded distributions is well known.

**Lemma 4.4.** [29] *Every  $u \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$  can be expressed as*

$$(4.9) \quad u = \sum_{|\alpha| \leq p} \partial^\alpha f_\alpha$$

*for some  $p \in \mathbb{N}_0$  where  $f_\alpha$  are bounded continuous functions on  $\mathbb{R}^n$ . The equality (4.9) implies that*

$$\langle u, \varphi \rangle = \sum_{|\alpha| \leq p} (-1)^{|\alpha|} \int f_\alpha(x) \partial^\alpha \varphi(x) dx$$

*for all  $\varphi \in \mathcal{D}_{L^1}$ .*

As a special case of Theorem 3.4 of [12] when  $p = \infty$  where the space  $\mathcal{A}'_{L^\infty}(\mathbb{R}^n)$  is denoted by  $\mathcal{B}_{L^\infty}(\mathbb{R}^n)$  we obtain the following.

**Lemma 4.5.** [12] *Every  $u \in \mathcal{A}'_{L^\infty}(\mathbb{R}^n)$  can be expressed by*

$$(4.10) \quad u = \left( \sum_{k=0}^{\infty} a_k \Delta^k \right) g + h$$

*where  $\Delta$  denotes the Laplacian,  $g, h$  are bounded continuous functions on  $\mathbb{R}^n$  and  $a_k, k = 0, 1, 2, \dots$  satisfy the following estimates; for every  $L > 0$  there exists  $C > 0$  such that*

$$|a_k| \leq CL^k / k!^2$$

*for all  $k = 0, 1, 2, \dots$*

The following properties of the heat kernel will be useful, which can be found in [22].

**Proposition 4.6.** [22] *For each  $t > 0$ ,  $E_t(\cdot)$  is an entire function and the following estimate holds; there exists  $C > 0$  such that*

$$(4.11) \quad |\partial_x^\alpha E_t(x)| \leq C^{|\alpha|} t^{-(n+|\alpha|)/2} \alpha!^{1/2} \exp(-|x|^2/8t).$$

*Also for each  $t, s > 0$  we have*

$$(4.12) \quad (E_t * E_s)(x) := \int E_t(x - y) E_s(y) dy = E_{t+s}(x).$$

Now, we state and prove the main theorem.

**Theorem 4.7.** *Let  $u, v \in \mathcal{G}'(\mathbb{R}^n)$ . Then  $(u, v)$  satisfies (4.1) if and only if  $(u, v)$  satisfies one of the followings:*

- (i)  $u, v \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$  [resp.  $\mathcal{A}'_{L^\infty}(\mathbb{R}^n)$ ],
- (ii)  $u = c \cdot x + r, v = 1 + \lambda c \cdot x$  for some  $c \in \mathbb{C}^n, \lambda \in \mathbb{C}$  and  $r \in \mathcal{D}'_{L^\infty}(\mathbb{R}^n)$  [resp.  $\mathcal{A}'_{L^\infty}(\mathbb{R}^n)$ ],
- (iii)  $u = \lambda(e^{c \cdot x} - e^{-c \cdot x}), v = \gamma e^{c \cdot x} + (1 - \gamma)e^{-c \cdot x}$  for some  $c \in \mathbb{C}^n, \lambda, \gamma \in \mathbb{C}$ .

*Proof.* We use the same method as in the proof of [10, Theorem 4.6]. Here we give the proof for the reader. Convolving the tensor product  $E_t(x)E_s(y)$  of n-dimensional heat kernels in the left hand side of (4.1), in view of the semigroup property  $(E_t * E_s)(x) = E_{t+s}(x)$  of the heat kernel we have

$$\begin{aligned}
 (4.13) \quad [(u \circ T) * (E_t(\xi)E_s(\eta))](x, y) &= \langle u_\xi, \int E_t(x - \xi - \eta)E_s(y - \eta) d\eta \rangle \\
 &= \langle u_\xi, (E_t * E_s)(x - y - \xi) \rangle \\
 &= \langle u_\xi, E_{t+s}(x - y - \xi) \rangle \\
 &= \tilde{u}(x - y, t + s).
 \end{aligned}$$

Similarly we have

$$\begin{aligned}
 (4.14) \quad [(u \otimes v) * (E_t(\xi)E_s(\eta))](x, y) &= \tilde{u}(x, t)\tilde{v}(y, s), \\
 [(v \otimes u) * (E_t(\xi)E_s(\eta))](x, y) &= \tilde{v}(x, t)\tilde{u}(y, s),
 \end{aligned}$$

where  $\tilde{u}(x, t), \tilde{v}(x, t)$  are the Gauss transforms of  $u, v$ , respectively. Let  $w := u \circ T - u \otimes v + v \otimes u$ . Then  $w \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n})$  [resp.  $\mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})$ ]. First, we suppose that  $w \in \mathcal{D}'_{L^\infty}(\mathbb{R}^{2n})$ . Using (4.9) and (4.11) we have

$$\begin{aligned}
 |[w * (E_t(\xi)E_s(\eta))](x, y)| &\leq \sum_{|\alpha| \leq p} |[\partial^\alpha f_\alpha * (E_t(\xi)E_s(\eta))](x, y)| \\
 &\leq \sum_{|\alpha| \leq p} |[f_\alpha * \partial_{\xi, \eta}^\alpha (E_t(\xi)E_s(\eta))](x, y)| \\
 &\leq \sum_{|\alpha| \leq p} \|f_\alpha\|_{L^\infty} \|\partial_{\xi, \eta}^\alpha (E_t(\xi)E_s(\eta))\|_{L^1} \\
 &\leq C_1 \sum_{|\beta| + |\gamma| \leq p} \|\partial_\xi^\beta E_t(\xi)\|_{L^1} \|\partial_\eta^\gamma E_s(\eta)\|_{L^1} \\
 &\leq C_2 \sum_{|\beta| + |\gamma| \leq p} t^{-(n+|\beta|)/2} s^{-(n+|\gamma|)/2} \\
 &\leq C \left( \frac{1}{t} + \frac{1}{s} \right)^N + d,
 \end{aligned}$$

where  $N = n + p/2$  and the constants  $C$  and  $d$  depend only on  $p$ . Secondly we suppose that  $w \in \mathcal{A}'_{L^\infty}(\mathbb{R}^{2n})$ . Then, using (4.11) we have

$$\begin{aligned} \|\Delta^k(E_t(\xi)E_s(\eta))\|_{L^1} &\leq \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|\partial^{2\alpha}(E_t(\xi)E_s(\eta))\|_{L^1} \\ &\leq \sum_{|\beta|+|\gamma|=k} \frac{k!}{\beta!\gamma!} \|\partial_\xi^{2\beta} E_t(\xi)\|_{L^1} \|\partial_\eta^{2\gamma} E_s(\eta)\|_{L^1} \\ &\leq \sum_{|\beta|+|\gamma|=k} \frac{k!(2\beta)!^{1/2}(2\gamma)!^{1/2}M^{2k}}{\beta!\gamma!} t^{-n/2-|\beta|} s^{-n/2-|\gamma|} \\ &\leq \sum_{|\beta|+|\gamma|=k} k!(2M)^{2k} t^{-n/2-|\beta|} s^{-n/2-|\gamma|} \\ &\leq k!(2\sqrt{n}M)^{2k} (1/t + 1/s)^{n+k}. \end{aligned}$$

Now, by the structure (4.10) of bounded hyperfunctions together with the growth condition of  $a_k$ ,  $k = 0, 1, 2, \dots$  we have

$$\begin{aligned} |[w * (E_t(\xi)E_s(\eta))](x, y)| &\leq \sum_{k=0}^\infty \|a_k(\Delta^k g) * (E_t(\xi)E_s(\eta))\|_{L^\infty} + \|h * (E_t(\xi)E_s(\eta))\|_{L^\infty} \\ &\leq \|g\|_{L^\infty} \sum_{k=0}^\infty \|a_k \Delta^k (E_t(\xi)E_s(\eta))\|_{L^1} + \|h\|_{L^\infty} \|E_t(\xi)E_s(\eta)\|_{L^1} \\ &\leq C_1 \sum_{k=0}^\infty \frac{1}{k!} (4nM^2L)^k (1/t + 1/s)^{n+k} + \|h\|_{L^\infty} \\ &\leq C_2 \sum_{k=0}^\infty \frac{1}{k!} \epsilon^k (1/t + 1/s)^{n+k} + \|h\|_{L^\infty} \\ &\leq C_\epsilon e^{\epsilon(1/t+1/s)}, \end{aligned}$$

where  $L$  is taken so that  $4nM^2L < \epsilon$  and the constant  $C_\epsilon$  depends only on  $w$  and  $\epsilon$ . Thus, we have the inequality; there exist  $C > 0$  and  $d > 0$  [resp. for every  $\epsilon > 0$  there exists  $C_\epsilon > 0$ ] such that

$$(4.15) \quad |\tilde{u}(x - y, t + s) - \tilde{u}(x, t)\tilde{v}(y, s) + \tilde{v}(x, t)\tilde{u}(y, s)| \leq C \left(\frac{1}{t} + \frac{1}{s}\right)^N + d \text{ [resp. } C_\epsilon e^{\epsilon(1/t+1/s)} \text{]}$$

where  $\tilde{u}, \tilde{v}$  are the Gauss transforms of  $u, v$ , respectively, given in Lemma 4.2. Replacing  $f$  by  $\tilde{u}, g$  by  $\tilde{v}$  in Theorem 3.5 and using the continuity of  $\tilde{u}$  and  $\tilde{v}$  we obtain one of the followings (I) ~ (III):

- (I) both  $\tilde{u}$  and  $\tilde{v}$  satisfy (3.3) [resp.(3.4)],
  - (II)  $\tilde{u}(x, t) = c \cdot x e^{bt} + R(x, t)$ ,  $\tilde{v}(x, t) = \lambda \tilde{u}(x, t) + e^{bt}$ ,
- where  $c \in \mathbb{C}^n$ ,  $b, \lambda \in \mathbb{C}$  and  $R : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{C}$  satisfies

$$|R(x, t)| \leq Ct^{-2N} + d \text{ [resp.(3.4)]}$$

for all  $x \in \mathbb{R}^n, t > 0$  and for some  $C, d > 0$ ,

- (III)  $\tilde{u}(x - y, t + s) - \tilde{u}(x, t)\tilde{v}(y, s) + \tilde{v}(x, t)\tilde{u}(y, s) = 0$  for all  $x, y \in \mathbb{R}^n, t, s > 0$ .

By Lemma 4.2, case (I) implies (i). For the case (II), since  $\tilde{u}, \tilde{v}$  are solutions of the heat equation we must have  $b = 0$  and so is  $R(x, t) = \tilde{u}(x, t) - c \cdot x$ . Letting  $t \rightarrow 0^+$  in (II) we obtain case (ii).

Finally, letting  $t \rightarrow 0^+$  in (III) we have

$$(4.16) \quad u \circ T - u \otimes v + v \otimes u = 0.$$

The nontrivial solutions of the equation (4.16) are given by (iii) or  $u = c \cdot x$ ,  $v = 1 + \lambda c \cdot x$  which is included in the case (ii). This completes the proof.  $\square$

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# Effect of cytotoxic T lymphocytes on HIV-1 dynamics

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## Abstract

The purpose of this paper is to study the effect of the cytotoxic T lymphocytes (CTLs) on an HIV-1 dynamics. The model considers that the virus infects the macrophages in addition to the CD4<sup>+</sup> T cells. The role of the CTLs is to kill the infected macrophages and CD4<sup>+</sup> T cells. The time delay which accounts the time of infection and the time of producing new active HIV-1 is modeled. The HIV-1 dynamics is modeled as a 6-dimensional nonlinear delay differential equations. The incidence rate of infection and killer rate of infected cells are given by general nonlinear functions. We study the qualitative behavior of the system. The global stability analysis has been established using Lyapunov method and LaSalle invariance principle. We present an example and perform numerical simulations to emphasize our theoretical results.

*Keywords:* Global stability; HIV infection; time delay; Immune response; Direct Lyapunov method.

## 1 Introduction

Recently, the study of Human Immunodeficiency Virus type-1 (HIV-1) and Acquired Immunodeficiency Syndrome (AIDS) has become a topic of interest in the mathematical literature. In the pursuit of understanding the interaction between the HIV-1 and immune system, several mathematical models have been proposed.

The following is the basic model of HIV-1 infection dynamics that has been described and studied in [1]:

$$\begin{aligned}\dot{x} &= \lambda - dx - \beta xv, \\ \dot{y} &= \beta xv - \delta y, \\ \dot{v} &= ky - rv.\end{aligned}$$

Here, the concentrations of uninfected CD4<sup>+</sup> T cells, infected CD4<sup>+</sup> T cells and virus are represented by  $x$ ,  $y$  and  $v$ , respectively. The production rate of CD4<sup>+</sup> T cells is represented by  $\lambda$ , while the infection rate, and thus the infected CD4<sup>+</sup> T cell production rate, is represented by  $\beta xv$ , where  $\beta$  is the infection rate constant. The uninfected cells and infected cells are die with rate  $dx$  and  $\delta y$ , respectively.  $k$  represents the rate constant of virion generation by CD4<sup>+</sup> T cells while  $r$  represents the rate constant of viral particle emptying from the plasma.

Replication models assume cytotoxic T lymphocyte cells (CTLs) to be the main host defence restricting viral replication in vivo and thus the main determinant of viral load. Nowak and Bangham [2] constructed the first model of HIV taking into account CTLs as:

$$\begin{aligned}\dot{x} &= \lambda - dx - \beta xv, \\ \dot{y} &= \beta xv - \delta y - pyz, \\ \dot{v} &= ky - rv, \\ \dot{z} &= cyz - bz.\end{aligned}$$

where  $z$  represent the concentration of CTLs which multiply at a rate  $cyz$  when stimulated by infected cells, while  $bz$  represents the death rate of this population of CTLs.

Delay differential equations are used to introduce delays into the infection equations and/or equations for virus production to account for the intracellular phase of the viral life cycle. This delay is defined as period between infection of a  $CD4^+$  T-cell and the point at which the infected cell begins to produce viral particles (see, e.g. [3], [5], [6], [7], [8], [9], [10]). Complications have been shown to occur ([11], [12], [13]) when time delays are introduced into infection models with immune responses. Such complications include stable periodic solutions and chaos. The use of general kernel function to represent distributed intracellular delays has been motivated by the argument that constant delays may not biologically realistic ([22],[23],[24]). In contrast to Nakata's [17] investigation of the stability of an immunity mediated HIV-1 model with two finite distributed intracellular delays, Wang et al. [16] and Li and Shu [14] examined the stability of an infection model with infinite distributed intracellular delays by constructing Lyapunov functionals. Yuan and Zou [15] proposed and developed an appropriate mathematical model for HIV-1 infection by incorporating distributed delay into the cell infection equation and another virus production equation and nonlinear incidence rate and a nonlinear removal rate for the infected cells. However, the presence of the macrophages has been neglected.

Our aim in this paper is to study the effect of the CTL immune response of the global dynamics of a distributed delayed HIV-1 model which describe the interaction between the virus and two target cells,  $CD4^+$  T cells and macrophages. The motivation for considering the two target cell model is the observation that the rate of viral load decline was considerably lower after the rapid first phase of decay during the 1-2 weeks after antiretroviral treatment ([3],[4],[18]). The model is a 6-dimensional nonlinear ODES that takes into account cytotoxic T lymphocyte cells (CTLs) with nonlinear incidence rate and distributed delays using distributed kernels reflecting the variance in time required for viral entry into cells and the variability in time required for intracellular virion reproduction. The positive invariance properties and the boundedness of the solutions for the model are studied. By constructing explicit Lyapunov functionals and using the LaSalle invariance principle, which are extensions and modified forms of the Lyapunov functionals given in [15], we prove that the steady states of the model are globally asymptotically stable (GAS) and the dynamics of the system is fully determined by the basic reproduction number  $R_0$ .

## 2 Mathematical model

We shall examine a deterministic model of HIV infection, which represents the interaction of HIV with two co-circulation populations of target cells, representing  $CD4^+$  T and macrophages cells. The system takes into consideration the distributed invasion and production delays and (i) We assume that the incidence rate is given by a nonlinear form. (ii) The model takes into consideration cytotoxic T lymphocyte cells (CTLs) immune response:

$$\dot{x}_1(t) = \mu_1 - k_1x_1(t) - \alpha_1x_1(t)f_1(v(t)), \tag{1}$$

$$\dot{y}_1(t) = \alpha_1 \int_0^\infty e^{-m_1\tau} G_1(\tau)x_1(t-\tau)f_1(v(t-\tau))d\tau - ry_1(t) - \beta y_1(t)h_1(z(t)), \tag{2}$$

$$\dot{x}_2(t) = \mu_2 - k_2x_2(t) - \alpha_2x_2(t)f_2(v(t)), \tag{3}$$

$$\dot{y}_2(t) = \alpha_2 \int_0^\infty e^{-m_2\tau} G_2(\tau)x_2(t-\tau)f_2(v(t-\tau))d\tau - ry_2(t) - \beta y_2(t)h_2(z(t)), \tag{4}$$

$$\dot{v}(t) = Nr \left( \int_0^\infty e^{-n_1\tau} \Psi_1(\tau)y_1(t-\tau)d\tau + \int_0^\infty e^{-n_2\tau} \Psi_2(\tau)y_2(t-\tau)d\tau \right) - dv(t), \tag{5}$$

$$\dot{z}(t) = \lambda (y_1(t) + y_2(t)) - qz(t). \tag{6}$$



The state variables describes the plasma concentrations of:  $x_1, y_1$ , represent the uninfected and infected CD4<sup>+</sup> T cells;  $x_2, y_2$ , represent the uninfected and infected macrophages. Eq. (1) and (3) describe the populations of target cells, where  $\mu_1$  and  $\mu_2$  perform the rates of new generations of CD4<sup>+</sup> T cell and macrophages from sources within the body,  $k_1, k_2$  are the death rate constants, and  $\alpha_1, \alpha_2$  are the infection rate constants. Equation (2) and (4) represent the population dynamics of the infected target cells, where  $r$  represent the clearance rate and it killed at rate  $\beta y_1(t)h_1(z(t))$  and  $\beta y_2(t)h_2(z(t))$ , respectively. The CTL cells are produced at a rate  $\lambda(y_1 + y_2)z$  and are decayed at a rate  $qz$ . Assume the kernel functions  $G_i$  and  $\Psi_i, i = 1, 2$  satisfy  $G_i(\tau_i) > 0, \Psi_i(\tau_i) > 0$ . Let us denote  $a_i = \int_0^\infty e^{-m_i\tau} G_i(\tau) d\tau, b_i = \int_0^\infty e_i^{-n_i\tau} \Psi_i(\tau) d\tau, i = 1, 2$ . Thus  $0 < a_i \leq 1, 0 < b_i \leq 1$

All parameters are assumed to be positive. The function  $f_i(v)$  and  $h_i(z)$  are continuously differentiable and guarantee this conditions are met:

- (C1):  $f_i(0) = 0, f'_i(\xi_i)$  exists and satisfies  $f'_i(\xi_i) \geq 0$  and  $\left(\frac{f_i(\xi_i)}{\xi_i}\right)' \leq 0$  in  $(0, \infty)$ ,
- (C2):  $h_i(0) = 0, h_i(\zeta_i)$  is strictly increasing in  $(0, \infty)$ ,

### 2.1 Positively and Boundedness

To prove the positively and the boundedness of the solutions, it is biologically reasonable to consider the following non-negative initial conditions for the system (1-6), define the Banach space of fading memory type

$$C_\alpha = \{\varphi \in C((-\infty, 0], R) : \varphi(\theta)e^{\alpha\theta} \text{ is uniformly continuous for } \theta \in (-\infty, 0] \text{ and } \|\varphi\| < \infty\}$$

where  $\alpha$  is a positive constant and  $\|\varphi\| = \sup_{\theta \leq 0} |\varphi(\theta)| e^{\alpha\theta}$ . Let  $C_\alpha^+ = \{\varphi \in C_\alpha : \varphi(\theta) \geq 0 \text{ for } \theta \in (-\infty, 0]\}$ . The initial conditions for system (1-6) are given as:

$$\begin{aligned} x_1(\theta) &= \varphi_1(\theta), y_1(\theta) = \varphi_2(\theta), x_2(\theta) = \varphi_3(\theta), y_2(\theta) = \varphi_4(\theta), v(\theta) = \varphi_5(\theta), \\ z(\theta) &= \varphi_6(\theta) \text{ for } \theta \in [-\infty, 0], \varphi_i \in C_\alpha^+, i = 1, 2, \dots, 6. \end{aligned} \tag{7}$$

By the fundamental theory of functional differential equations (see [20] and [21]), model (1-6) with initial conditions (7) has a unique solution and the following lemma establishes the positivity and boundedness of the solutions.

**Lemma 1.** Let  $(x_1(t), y_1(t), x_2(t), y_2(t), v(t), z(t))$  be the solution of system (1-6) with the initial conditions (7), then  $x_1(t), y_1(t), x_2(t), y_2(t), v(t)$  and  $z(t)$  are all positive and bounded for all  $t > 0$ .

**Proof.** First, we will prove that  $x_i(t) > 0, i = 1, 2$ , for all  $t \geq 0$ . Assume that  $x_i(t)$  loses its nonnegativity on some local existence interval  $[0, v]$  for some constant  $v$  and let  $t^* \in [0, v]$  be such that  $x_i(t^*) = 0$ . From (1) and (3) we have  $x_i(t^*) = \mu_i > 0$ . Hence  $x_i(t^*) > 0$  for some  $t \in (t^*, t^* + \epsilon)$ , where  $\epsilon > 0$  is sufficiently small. This leads to a contradiction and hence  $x_i(t) > 0$ , for all  $t \geq 0$ . Further by using the variation of parameters method and Eq. (2), (4) and (5) we have

$$\begin{aligned} y_i(t) &= y_i(0)e^{-\int_0^t (r+\beta h(z(s))) ds} \\ &+ \alpha_i \int_0^t e^{-\int_s^t (r+h(z(\eta))) d\eta} \int_0^\infty e^{-m_i\eta} G_i(\eta) x(s-\eta) f_i(v(s-\eta)) d\eta ds; \quad i = 1, 2. \\ v(t) &= v(0)e^{-dt} + Nr \int_0^t e^{-d(t-s)} \int_0^\infty \sum_{i=1}^2 e^{-n_i\eta} \Psi_i(\eta) y_i(s-\eta) d\eta ds, \end{aligned}$$

confirming that  $y_i(t) \geq 0, i = 1, 2$ , and  $v(t) \geq 0$  for all  $t \geq 0$ . Now from (6) we get

$$z(t) = z(0)e^{-qt} + \lambda \int_0^t e^{-q(t-s)} \sum_{i=1}^2 y_i(s) ds.$$

Then  $z(t) \geq 0$ , for all  $t \geq 0$ , and this prove the positively of the solution. Now we shall prove that the solution are bounded, from Eq.(1) and (3), we have  $\dot{x}_i(t) \leq \mu_i - k_i x_i(t)$ , this implies  $\limsup_{t \rightarrow \infty} x_i(t) \leq \frac{\mu_i}{k_i}, i = 1, 2$ , let

$U_i(t) = \int_0^\infty e^{-m_i\tau} G_i(\tau) x_i(t - \tau) d\tau + y_i(t)$ , then

$$\begin{aligned} \dot{U}_i(t) &= \int_0^\infty e^{-m_i\tau} G_i(\tau) \dot{x}_i(t - \tau) d\tau + \dot{y}_i(t), \\ &= \int_0^\infty e^{-m_i\tau} G_i(\tau) (\mu_i - k_i x_i(t - \tau) - \alpha_i x_i(t - \tau) f_i(v(t - \tau))) d\tau \\ &\quad + \alpha_i \int_0^\infty e^{-m_i\tau} G_i(\tau) x_i(t - \tau) f_i(v(t - \tau)) d\tau - r y_i(t) - \beta y_i(t) h_i(z(t)), \\ &= \mu_i a_i - \alpha_i \int_0^\infty e^{-m_i\tau} G_i(\tau) x_i(t - \tau) d\tau - r y_i(t) - \beta y_i(t) h_i(z(t)), \\ &\leq \mu_i a_i - p_i U_i(t). \end{aligned}$$

It follows that  $\limsup_{t \rightarrow \infty} U_i(t) \leq \frac{\mu_i a_i}{p_i}$ , where  $p_i = \min\{\alpha_1, \alpha_2, r\}$ . Since  $x_i(t) > 0$ ,  $y_i(t) \geq 0$  and  $z(t) \geq 0$  then  $\limsup \sum_{i=1}^2 y_i(t)_{t \rightarrow \infty} \leq L_i$ ,  $\limsup y_i(t)_{t \rightarrow \infty} \leq L_i$ ,  $i = 1, 2$ . On the other hand,

$$\dot{v}(t) \leq Nr(b_1 L_1 + b_2 L_2) - dv(t) \leq NrbL - dv(t).$$

where  $b_i = \int_0^\infty e^{-n_i\tau} \Psi_i(\tau) d\tau$ . From Eq.(6) we get  $\dot{z}(t) \leq \lambda L - qz(t)$ . Then  $\limsup_{t \rightarrow \infty} v(t) \leq \frac{NrbL}{d}$  and  $\limsup_{t \rightarrow \infty} z(t) \leq \frac{\lambda L}{q}$ . Therefore,  $x_1(t)$ ,  $y_1(t)$ ,  $x_2(t)$ ,  $y_2(t)$ ,  $v(t)$ , and  $z(t)$  are ultimately bounded and this complete the proof of boundedness of solutions.

## 2.2 Basic reproduction number and steady state

To obtain the equilibrium points of model (1)-(6), we consider the following equations:

$$\begin{aligned} \mu_1 - k_1 x_1 - \alpha_1 x_1 f_1(v) &= 0, \\ \alpha_1 a_1 x_1 f_1(v) - r y_1 - \beta y_1 h_1(z) &= 0, \\ \mu_2 - k_2 x_2 - \alpha_2 x_2 f_2(v) &= 0, \\ \alpha_2 a_2 x_2 f_2(v) - r y_2 - \beta y_2 h_2(z) &= 0, \\ Nr(b_1 y_1 + b_2 y_2) - dv &= 0, \\ \lambda(y_1 + y_2) - qz &= 0. \end{aligned}$$

We find that if  $z \neq 0$ , there is two steady states  $E_0 = (x_1^0, 0, x_2^0, 0, 0, 0)$  where  $x_1^0 = \frac{\mu_1}{k_1}$ ,  $x_2^0 = \frac{\mu_2}{k_2}$ , and  $E^* = (x_1^*, y_1^*, x_2^*, y_2^*, v^*, z^*)$  satisfies the equations:

$$\begin{aligned} x_i^* &= \frac{\mu_i}{k_i + \alpha_i f_i(v^*)}, & \sum_{i=1}^2 y_i^* &= \frac{d}{Nrb} v^*, \\ z^* &= \frac{\lambda}{q} \sum_{i=1}^2 y_i^* = \frac{\lambda d}{Nrbq} v^*, & y_i^* &= \frac{\alpha_i a_i f_i(v^*)}{r + \beta h_i(z^*)} x_i^*. \end{aligned}$$

The basic reproduction number,  $R_0$ , for system (1)-(6) is given by:

$$R_0 = \frac{\mu_1 \alpha_1 a_1 N b_1 f_1'(0)}{k_1 d} + \frac{\mu_2 \alpha_2 a_2 N b_2 f_2'(0)}{k_2 d} = R_1 + R_2,$$

where,  $R_1$  and  $R_2$  are the basic reproduction numbers for CD4<sup>+</sup> T cells and macrophages cells, severally. Now, we shall prove that  $R_0 > 1$  is a sufficient condition to ensure the existence of an infected steady state  $E^* = (x_1^*, y_1^*, x_2^*, y_2^*, v^*, z^*)$ . Using the above calculations the existence of an infected equilibrium is equivalent to the existence of a positive root of the equation  $L(v) = 0$ , where

$$L(v^*) = \alpha_1 a_1 f_1(v^*) \frac{\mu_1}{k_1 + \alpha_1 f_1(v^*)} + \alpha_2 a_2 f_2(v^*) \frac{\mu_2}{k_2 + \alpha_2 f_2(v^*)} - \frac{rd}{Nrb} v^* - \frac{\beta d}{Nrb} v^* h \left( \frac{\lambda d v^*}{Nrbq} \right),$$

and it satisfies  $L(0) = 0$ ,  $L(+\infty) = -\infty$ , and

$$\begin{aligned} L'(0) &= \alpha_1 a_1 f_1'(0) \frac{\mu_1}{k_1} + \alpha_2 a_2 f_2'(0) \frac{\mu_2}{k_2} - \frac{rd}{Nrb} \\ &= \frac{d}{Nb} \frac{\alpha_1 a_1 \mu_1 N b f_1'(0)}{k_1 d} + \frac{\alpha_2 a_2 \mu_2 N b f_2'(0)}{k_2 d} - 1 = \frac{d}{Nb} (R_0 - 1) > 0. \end{aligned}$$

It follows from the continuity of the function  $L(v)$  in  $[0, \infty)$  that  $L(v) = 0$  has at least one positive root. Hence, we see that the condition at least has one infected equilibrium  $E^*$  when  $R_0 > 1$ . We can rewrite the model as:

$$\dot{x}_1(t) = \mu_1 - k_1 x_1(t) - \alpha_1 x_1(t) f_1(v(t)), \tag{8}$$

$$\dot{y}_1(t) = \vartheta_1 \int_0^\infty e^{-m_1 \zeta} g_1(\zeta) x_1(t - \zeta) f_1(v(t - \zeta)) d\zeta - r y_1(t) - \beta y_1(t) h(z(t)), \tag{9}$$

$$\dot{x}_2(t) = \mu_2 - k_2 x_2(t) - \alpha_2 x_2(t) f_2(v(t)), \tag{10}$$

$$\dot{y}_2(t) = \vartheta_2 \int_0^\infty e^{-m_2 \zeta} g_2(\zeta) x_2(t - \zeta) f_2(v(t - \zeta)) d\zeta - r y_2(t) - \beta y_2(t) h(z(t)), \tag{11}$$

$$\dot{v}(t) = \gamma_1 \int_0^\infty e^{-n_1 \zeta} \psi_1(\zeta) y_1(t - \zeta) d\zeta + \gamma_2 \int_0^\infty e^{-n_2 \zeta} \psi_2(\zeta) y_2(t - \zeta) d\zeta - d v(t), \tag{12}$$

$$\dot{z}(t) = \lambda (y_1(t) + y_2(t)) - q z(t). \tag{13}$$

For simplify, we taked  $h_1 = h_2 = h$ ,  $\vartheta_i = \alpha_i a_i$ ,  $\gamma_i = N r b_i$ ,  $g_i(\zeta) = \frac{G_i(\zeta)}{a_i}$ ,  $\psi_i(\zeta) = \frac{\Psi_i(\zeta)}{b_i}$ .

### 3 Global stability

In this section, we going to show that the steady states satisfy the global stability condition:

**Theorem 1.** Let Conditions C1 and C2 hold true and  $R_0 \leq 1$ , then the infection-free equilibrium  $E_0$  is globally asymptotically stable.

**Proof.** Define  $H_i(t) = \int_t^\infty g_i(\zeta) d\zeta$ ,  $P_i(t) = \int_t^\infty \psi_i(\zeta) d\zeta$ , and consider laypunov function  $W(t) = \sum_{i=1}^3 W_i(t)$ , where,

$$\begin{aligned} W_1(t) &= \sum_{i=1}^2 \frac{1}{2} \left( x_i(t) - \frac{\mu_i}{k_i} \right)^2 + \frac{\alpha_i \mu_i}{\vartheta_i k_i} y_i(t) + \frac{\alpha_i \mu_i r}{\vartheta_1 \vartheta_2 k_i} v(t) + \frac{\alpha_i \mu_i \beta}{\vartheta_i k_i \lambda} \int_0^{z(t)} h(\zeta) d\zeta, \\ W_2(t) &= \sum_{i=1}^2 \frac{\alpha_i \mu_i}{k_i} \int_0^\infty H_i(\zeta_i) x_i(t - \zeta_i) f_i(v(t - \zeta_i)) d\zeta_i, \\ W_3(t) &= \sum_{i=1}^2 \frac{\alpha_i \mu_i r}{k_i \vartheta_i} \int_0^\infty P_i(\zeta_i) y_i(t - \zeta_i) d\zeta_i, \end{aligned}$$

It clear that,  $W(t) \geq 0$  and  $W(t) = 0$  if and only if  $x_i(t) = \frac{\mu_i}{k_i}$  and  $y_i(t) = v(t) = z(t) = 0$ . The derivative  $\dot{W}_i(t)$  of along the solution is:

$$\begin{aligned} \dot{W}_1(t) &= \sum_{i=1}^2 \left[ \left( x_i(t) - \frac{\mu_i}{k_i} \right) (\mu_i - k_i x_i(t) - \alpha_i x_i(t) f(v(t))) + \frac{\alpha_i \mu_i}{k_i} \int_0^\infty g_i(\zeta_i) x_i(t - \zeta_i) f_i(v(t - \zeta_i)) d\zeta_i \right. \\ &\quad - \frac{\alpha_i \mu_i r}{k_i \vartheta_i} y_i(t) - \frac{\alpha_i \mu_i \beta}{k_i \vartheta_i} y_i(t) h(z(t)) + \frac{\alpha_i \mu_i r}{k_i \vartheta_i} \int_0^\infty \psi_i(\zeta_i) y_i(t - \zeta_i) d\zeta_i - \frac{\alpha_i \mu_i r d}{k_i \vartheta_i \gamma_i} v(t) \\ &\quad \left. + \frac{\alpha_i \mu_i \beta}{k_i \lambda \vartheta_i} h(z(t)) [\lambda y_i(t) - q z(t)] \right], \end{aligned}$$

Note that  $H_i(0) = 1, H_i(\infty) = 0$  and  $dH_i(t) = -g_i(t)dt$ . Using integration by parts, we calculate the derivative of  $W_2$ :

$$\begin{aligned} \dot{W}_2(t) &= \sum_{i=1}^2 \frac{\alpha_i \mu_i}{k_i} \int_0^\infty H_i(\zeta_i) \frac{d(x_i(t - \zeta_i) f_i(v(t - \zeta_i)))}{dt} d\zeta_i = - \sum_{i=1}^2 \frac{\alpha_i \mu_i}{k_i} \int_0^\infty H_i(\zeta_i) \frac{d(x_i(t - \zeta_i) f_i(v(t - \zeta_i)))}{d\zeta_i} d\zeta_i, \\ &= - \sum_{i=1}^2 \left( \frac{\alpha_i \mu_i}{k_i} H_i(\zeta_i) x_i(t - \zeta_i) f_i(v(t - \zeta_i)) \Big|_{\zeta_i=0}^\infty + \frac{\alpha_i \mu_i}{k_i} \int_0^\infty x_i(t - \zeta_i) f_i(v(t - \zeta_i)) dH_i(\zeta_i) \right), \\ &= \sum_{i=1}^2 \left( \frac{\alpha_i \mu_i}{k_i} x_i(t) f_i(v(t)) - \frac{\alpha_i \mu_i}{k_i} \int_0^\infty g_i(\zeta_i) x_i(t - \zeta_i) f_i(v(t - \zeta_i)) d\zeta_i \right). \end{aligned}$$

Similarly

$$\dot{W}_3(t) = \sum_{i=1}^2 \left( \frac{\alpha_i \mu_i r}{k_i \vartheta_i} y_i(t) - \frac{\alpha_i \mu_i r}{k_i \vartheta_i} \int_0^\infty \psi_i(\zeta_i) y_i(t - \zeta_i) d\zeta_i \right),$$

Therefore

$$\begin{aligned} \dot{W}(t) &= \sum_{i=1}^2 \left[ -k_i \left( x_i(t) - \frac{\mu_i}{k_i} \right)^2 - \alpha_i x_i^2(t) f_i(v(t)) + \frac{\alpha_i \mu_i}{k_i} x_i(t) f_i(v(t)) \right. \\ &\quad + \frac{\alpha_i \mu_i}{k_i} \int_0^\infty g_i(\zeta_i) x_i(t - \zeta_i) f_i(v(t - \zeta_i)) d\zeta_i - \frac{\alpha_i \mu_i r}{k_i \vartheta_i} y_i(t) - \frac{\alpha_i \mu_i \beta}{k_i \eta_i} y_i(t) h(z(t)) \\ &\quad + \frac{\alpha_i \mu_i r}{k_i \vartheta_i} \int_0^\infty \psi_i(\zeta_i) y_i(t - \zeta_i) d\zeta_i - \frac{\alpha_i \mu_i r d}{k_i \vartheta_i \gamma_i} v(t) + \frac{\alpha_i \mu_i \beta}{k_i \vartheta_i} y_i(t) h(z(t)) \\ &\quad - \frac{\alpha_i \mu_i \beta q}{k_i \lambda \vartheta_i} z(t) h(z(t)) + \frac{\alpha_i \mu_i}{k_i} x_i(t) f_i(v(t)) - \frac{\alpha_i \mu_i}{k_i} \int_0^\infty g_i(\zeta_i) x_i(t - \zeta_i) f_i(v(t - \zeta_i)) d\zeta_i \\ &\quad \left. + \frac{\alpha_i \mu_i r}{k_i \vartheta_i} y_i(t) - \frac{\alpha_i \mu_i r}{k_i \vartheta_i} \int_0^\infty \psi_i(\zeta_i) y_i(t - \zeta_i) d\zeta_i \right]. \\ \dot{W}(t) &= \sum_{i=1}^2 \left[ -k_i \left( x_i(t) - \frac{\mu_i}{k_i} \right)^2 - \alpha_i f_i(v(t)) \left( x_i^2(t) - \frac{2\mu_i}{k_i} x_i(t) + \frac{\mu_i^2}{k_i^2} \right) \right. \\ &\quad \left. + \frac{\alpha_i \mu_i^2}{k_i^2} f_i(v(t)) - \frac{\alpha_i \mu_i r d}{k_i \eta_i \gamma_i} v(t) - \frac{\alpha_i \mu_i \beta q}{k_i \lambda \eta_i} z(t) h(z(t)) \right]. \end{aligned}$$

Hence

$$\begin{aligned} \dot{W}(t) &= \sum_{i=1}^2 \left[ -k_i \left( x_i(t) - \frac{\mu_i}{k_i} \right)^2 - \alpha_i f_i(v(t)) \left( x_i(t) - \frac{\mu_i}{k_i} \right)^2 - \frac{\alpha_i \mu_i \beta q}{k_i \lambda \vartheta_i} z(t) h(z(t)) \right. \\ &\quad \left. + \frac{\alpha_i \mu_i r d}{k_i \vartheta_i \gamma_i} v(t) \left( \frac{\mu_i \vartheta_i \gamma_i}{k_i r d} \frac{f_i(v(t))}{v(t)} - 1 \right) \right]. \end{aligned}$$

But from Condition (C1), we have  $\frac{f_i(v(t))}{v(t)} \leq f_i'(0)$ . Hence

$$\dot{W}(t) \leq \sum_{i=1}^2 \left[ -k_i \left( x_i(t) - \frac{\mu_i}{k_i} \right)^2 - \alpha_i f_i(v(t)) \left( x_i(t) - \frac{\mu_i}{k_i} \right)^2 + \frac{\alpha_i \mu_i r d}{k_i \vartheta_i \gamma_i} (R_0 - 1) v(t) - \frac{\alpha_i \mu_i \beta q}{k_i \lambda \vartheta_i} z(t) h(z(t)) \right].$$

If  $R_0 \leq 1$ , then,  $\dot{W}_1 \leq 0$ . To prove the global stability of the infected equilibrium, we need to use this lemma:

**Lemma 2.** If satisfies Condition (C1), then  $g(F(\sigma)) \leq g(\sigma)$ ,  $\sigma > 0$  with the equality holding only at  $\sigma = 1$ , where  $F(\sigma) = \frac{f(v^*\sigma)}{f(v^*)}$ , and  $g(u) = u - 1 - \ln u$  and with  $g : (0, \infty) \rightarrow [0, \infty)$  has the global minimum  $g(1) = 0$  and positive elsewhere for  $\zeta_i \in (0, \infty)$ .

**Proof.** Since  $F(1) = 1$  and the derivative of  $g(\sigma)$  has the same sign as  $\sigma - 1$  for  $\sigma > 0$ , we need only to prove that  $\sigma \leq F(\sigma) \leq 1$  for  $\sigma \in (0, 1)$  and  $1 \leq F(\sigma) \leq \sigma$  for  $\sigma \in [1, \infty)$ . The proof of case  $\sigma \in [1, \infty)$  is similar to that case of  $\sigma \in (0, 1)$ , so we will only consider the case when  $\sigma \in (0, 1)$ . Note that  $\sigma \leq F(\sigma) \leq 1$  is equivalent to  $\frac{f(v^*)}{v^*} \leq \frac{f(v^*\sigma)}{v^*\sigma} \leq \frac{f(v^*)}{v^*\sigma}$  for  $\sigma \in (0, 1)$ , from Condition (C1) we completed the proof.

**Theorem 2.** Let conditions C1 and C2 hold true and  $R_0 > 1$ , then the chronic infection equilibrium  $E^*$  is globally asymptotically stable for all positive solution.

**Proof.** Define

$$\begin{aligned} V_1 &= \sum_{i=1}^2 g_i \left( \frac{x_i(t)}{x_i^*} \right), & V_2 &= \sum_{i=1}^2 \int_0^\infty H_i(\zeta_i) g_i \left( \frac{x_i(t - \zeta_i) f_i(v(t - \zeta_i))}{x_i^* f_i(v^*)} \right) d\zeta_i, \\ V_3 &= \sum_{i=1}^2 g_i \left( \frac{y_i(t)}{y_i^*} \right), & V_4 &= \sum_{i=1}^2 g_i \left( \frac{v(t)}{v^*} \right), \\ V_5 &= \int_{z^*}^{z(t)} [h(\zeta_i) - h(z^*)] d\zeta_i, & V_6 &= \sum_{i=1}^2 \int_0^\infty \Psi_i(\zeta_i) g_i \left( \frac{y_i(s)}{y_i^*} \right) ds. \end{aligned}$$

with the infected steady state conditions:

$$\begin{aligned} \mu_i &= k_i x_i^* + \alpha_i x_i^* f_i(v^*), & \gamma_i y_i^* &= dv^*, \\ \vartheta_i x_i^* f_i(v^*) &= r y_i^* + \beta y_i^* h(z^*), & \lambda y_i^* &= q z^*. \end{aligned} \tag{14}$$

we will let the function  $V(t)$  and study the derivative of the Lyapunov functional as:

$$\begin{aligned} V(t) &= x_i^* V_{1i}(t) + \alpha_i x_i^* f_i(v^*) V_{2i}(t) + \frac{\alpha_i y_i^*}{\vartheta_i} V_{3i}(t) + \frac{\alpha_i r v^*}{\vartheta_i \gamma_i} V_{4i}(t) + \frac{\alpha_i \beta}{\lambda \vartheta_i} V_{5i}(t) + \frac{\alpha_i \beta y_i^* h(z^*)}{d \vartheta_i} V_{4i}(t) \\ &+ \alpha_i x_i^* f_i(v^*) V_{6i}(t). \end{aligned}$$

satisfies  $V(t) \geq 0$  with the equality holding if and only  $x_i(t) = x_i^*$ ,  $y_i(t) = y_i^*$ ,  $v(t) = v^*$ ,  $z(t) = z^*$  and  $x_i(t - \zeta_i) f_i(v(t - \zeta_i)) = x_i^* f_i(v^*)$ ,  $y_i(t - \zeta_i) = y_i^*$ . We get

$$\begin{aligned} \dot{V}_{1i}(t) &= \frac{1}{x_i^*} \left( 1 - \frac{x_i^*}{x_i(t)} \right) (k_i x_i^* + \alpha_i x_i^* f_i(v^*) - k_i x_i(t) - \alpha_i x_i(t) f_i(v(t))), \\ &= \frac{-k_i (x_i(t) - x_i^*)^2}{x_i^* x_i(t)} + \alpha_i f_i(v^*) \left[ 1 - \frac{x_i^*}{x_i(t)} - \frac{x_i(t) f_i(v(t))}{x_i^* f_i(v^*)} + \frac{f_i(v(t))}{f_i(v^*)} \right]. \end{aligned} \tag{15}$$

$$\begin{aligned} \dot{V}_{2i}(t) &= \frac{x_i(t) f_i(v(t))}{x_i^* f_i(v^*)} - \ln \left( \frac{x_i(t) f_i(v(t))}{x_i^* f_i(v^*)} \right) - \int_0^\infty g_i(\zeta_i) \frac{x_i(t - \zeta_i) f_i(v(t - \zeta_i))}{x_i^* f_i(v^*)} d\zeta_i \\ &+ \int_0^\infty g_i(\zeta_i) \ln \left( \frac{x_i(t - \zeta_i) f_i(v(t - \zeta_i))}{x_i^* f_i(v^*)} \right) d\zeta_i, \end{aligned}$$

where

$$H_i(0) = 1, \quad H_i(\infty) = 0, \quad dH_i(t) = -g_i(t) dt.$$

$$\dot{V}_{3i}(t) = \frac{1}{y_i^*} \left( 1 - \frac{y_i^*}{y_i(t)} \right) \left( \eta_i \int_0^\infty g_i(\zeta_i) x_i(t - \zeta_i) f_i(v(t - \zeta_i)) d\zeta_i - r y_i(t) - \beta y_i(t) h(z(t)) \right).$$

Using Eq. (14)

$$\begin{aligned} \dot{V}_{3i}(t) &= \frac{\vartheta_i x_i^* f_i(v^*)}{y_i^*} \int_0^\infty g_i(\zeta_i) \frac{x_i(t-\zeta_i) f_i(v(t-\zeta_i))}{x_i^* f_i(v^*)} d\zeta_i - \frac{\vartheta_i x_i^* f_i(v^*)}{y_i^*} \int_0^\infty g_i(\zeta_i) \frac{x_i(t-\zeta_i) y_i^* f_i(v(t-\zeta_i))}{x_i^* y_i(t) f_i(v^*)} d\zeta_i, \\ &- \frac{\vartheta_i x_i^* f_i(v^*)}{y_i^*} \left[ \frac{y_i(t) h(z(t))}{y_i^* h(z^*)} - \frac{h(z(t))}{h(z^*)} \right] + r \left[ 1 - \frac{y_i(t)}{y_i^*} + \frac{y_i(t) h(z(t))}{y_i^* h(z^*)} - \frac{h(z(t))}{h(z^*)} \right]. \quad (16) \\ \dot{V}_{4i}(t) &= \frac{1}{y_i^*} \left( 1 - \frac{v^*}{v(t)} \right) \left( \gamma_1 \int_0^\infty \psi_1(\zeta_1) y_1(t-\zeta_1) d\zeta_1 + \gamma_2 \int_0^\infty \psi_2(\zeta_2) y_2(t-\zeta_2) d\zeta_2 - dv(t) \right), \end{aligned}$$

Using Eq. (14)

$$\begin{aligned} \dot{V}_{4i}(t) &= d \left[ 1 - \frac{v(t)}{v^*} + \int_0^\infty \psi_i(\zeta_i) \frac{y_i(t-\zeta_i)}{y_i^*} d\zeta_i - \int_0^\infty \psi_i(\zeta_i) \frac{v^* y_i(t-\zeta_i)}{v(t) y_i^*} d\zeta_i \right], \\ \dot{V}_{5i}(t) &= [h(z(t)) - h(z^*)] [\lambda y_i(t) - qz(t)]. \end{aligned}$$

Using Eq. (14)

$$\dot{V}_{5i}(t) = -q [h(z(t)) - h(z^*)] [z(t) - z^*] + \lambda y_i^* h(z^*) \left[ 1 - \frac{y_i(t)}{y_i^*} + \frac{y_i(t) h(z(t))}{y_i^* h(z^*)} - \frac{h(z(t))}{h(z^*)} \right].$$

Similar to  $\dot{V}_{2i}(t)$  the derivative of  $\dot{V}_{6i}(t)$ , differentiating gives

$$\dot{V}_{6i}(t) = \frac{y_i(t)}{y_i^*} - \int_0^\infty \psi_i(\zeta_i) \frac{y_i(t-\zeta_i)}{y_i^*} d\zeta_i + \int_0^\infty \psi_i(\zeta_i) \ln \left( \frac{y_i(t-\zeta_i)}{y_i(t)} \right) d\zeta_i,$$

It follows that

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^2 \left[ \frac{-k_i (x_i(t) - x_i^*)^2}{x_i(t)} + \alpha_i x_i^* f_i(v^*) \left[ 1 - \frac{x_i^*}{x_i(t)} + \frac{f_i(v(t))}{f_i(v^*)} \right] \right. \\ &+ \alpha_i x_i^* f_i(v^*) \int_0^\infty g_i(\zeta_i) \ln \left( \frac{x_i(t-\zeta_i) f_i(v(t-\zeta_i))}{x_i^* f_i(v^*)} \right) d\zeta_i - \alpha_i x_i^* f_i(v^*) \int_0^\infty g_i(\zeta_i) \frac{x_i(t-\zeta_i) y_i^* f_i(v(t-\zeta_i))}{x_i^* y_i(t) f_i(v^*)} d\zeta_i \\ &+ \frac{\alpha_i r y_i^*}{\vartheta_i} \left[ 1 - \frac{y_i(t)}{y_i^*} + \frac{y_i(t) h(z(t))}{y_i^* h(z^*)} - \frac{h(z(t))}{h(z^*)} \right] - \alpha_i x_i^* f_i(v^*) \left[ \frac{y_i(t) h(z(t))}{y_i^* h(z^*)} - \frac{h(z(t))}{h(z^*)} \right] \\ &+ \frac{\alpha_i r v^* d}{\vartheta_i \gamma_i} \left[ 1 - \frac{v(t)}{v^*} + \int_0^\infty \psi_i(\zeta_i) \frac{y_i(t-\zeta_i)}{y_i^*} d\zeta_i - \int_0^\infty \psi_i(\zeta_i) \frac{v^* y_i(t-\zeta_i)}{v(t) y_i^*} d\zeta_i \right] \\ &+ \frac{\alpha_i \beta y_i^* h(z^*)}{\vartheta_i} \left[ 1 - \frac{v(t)}{v^*} + \int_0^\infty \psi_i(\zeta_i) \frac{y_i(t-\zeta_i)}{y_i^*} d\zeta_i - \int_0^\infty \psi_i(\zeta_i) \frac{v^* y_i(t-\zeta_i)}{v(t) y_i^*} d\zeta_i \right] \\ &- \frac{\alpha_i \beta q}{\lambda \vartheta_i} [h(z(t)) - h(z^*)] [z(t) - z^*] + \frac{\alpha_i \beta y_i^* h(z^*)}{\vartheta_i} \left[ 1 - \frac{y_i(t)}{y_i^*} + \frac{y_i(t) h(z(t))}{y_i^* h(z^*)} - \frac{h(z(t))}{h(z^*)} \right] \\ &\left. + \alpha_i x_i^* f_i(v^*) \frac{y_i(t)}{y_i^*} - \alpha_i x_i^* f_i(v^*) \int_0^\infty \psi_i(\zeta_i) \frac{y_i(t-\zeta_i)}{y_i^*} d\zeta_i + \alpha_i x_i^* f_i(v^*) \int_0^\infty \psi_i(\zeta_i) \ln \left( \frac{y_i(t-\zeta_i)}{y_i(t)} \right) d\zeta_i \right], \end{aligned}$$

We have  $\frac{rdv^*}{\gamma_i} = ry_i^* = \vartheta_i x_i^* f_i(v^*) - \beta y_i^* h(z^*)$ , then

$$\begin{aligned} \dot{V}(t) = & \sum_{i=1}^2 \frac{-k_i(x_i(t) - x_i^*)^2}{x_i(t)} + \alpha_i x_i^* f_i(v^*) \left[ 3 - \frac{x_i^*}{x_i(t)} + \frac{f_i(v(t))}{f_i(v^*)} \right] - \alpha_i x_i^* f_i(v^*) \frac{v(t)}{v^*} \\ & - \alpha_i x_i^* f_i(v^*) \int_0^\infty g_i(\zeta_i) \frac{x_i(t - \zeta_i) y_i^* f_i(v(t - \zeta_i))}{x_i^* y_i(t) f_i(v^*)} d\zeta_i + \alpha_i x_i^* f_i(v^*) \int_0^\infty g_i(\zeta_i) \ln \left( \frac{x_i(t - \zeta_i) f_i(v(t - \zeta_i))}{x_i^* f_i(v^*)} \right) d\zeta_i \\ & - \alpha_i x_i^* f_i(v^*) \left[ \frac{y_i(t) h(z(t))}{y_i^* h(z^*)} - \frac{h(z(t))}{h(z^*)} \right] - \alpha_i x_i^* f_i(v^*) \int_0^\infty \psi_i(\zeta_i) \frac{y_i(t - \zeta_i)}{y_i^*} d\zeta_i \\ & - \frac{\alpha_i \beta q}{\lambda \vartheta_i} [h(z(t)) - h(z^*)] [z(t) - z^*] + \alpha_i x_i^* f_i(v^*) \int_0^\infty \psi_i(\zeta_i) \ln \left( \frac{y_i(t - \zeta_i)}{y_i(t)} \right) d\zeta_i, \end{aligned}$$

We can write

$$3 = 2 \int_0^\infty g_i(\zeta_i) d\zeta_i + \int_0^\infty \psi_i(\zeta_i) d\zeta_i.$$

Hence

$$\begin{aligned} \dot{V}(t) = & \sum_{i=1}^2 \left[ \frac{-k_i(x_i(t) - x_i^*)^2}{x_i(t)} - \frac{\alpha_i \beta q}{\lambda \vartheta_i} [h(z(t)) - h(z^*)] [z(t) - z^*] \right. \\ & + \alpha_i x_i^* f_i(v^*) \left[ \int_0^\infty g_i(\zeta_i) \left( -g \left( \frac{x_i^*}{x_i(t)} \right) - g \left( \frac{x_i(t - \zeta_i) y_i^* f_i(v(t - \zeta_i))}{x_i^* y_i(t) f_i(v^*)} \right) - \ln \frac{x^*}{x(t)} \right. \right. \\ & \left. \left. - \ln \frac{x(t - \zeta_i) y^* f_i(v(t - \zeta_i))}{x_i^* y_i(t) f_i(v^*)} + \ln \frac{x_i(t - \zeta_i) f(v(t - \zeta_i))}{x_i(t) f_i(v(t))} \right) d\zeta_i + \int_0^\infty \psi_i(\zeta_i) \left( -g \left( \frac{v^* y_i(t - \zeta_i)}{v(t) y_i^*} \right) \right. \right. \\ & \left. \left. - \ln \frac{v^* y_i(t - \zeta_i)}{v(t) y_i^*} + \ln \frac{y_i(t - \zeta_i)}{y_i(t)} \right) d\zeta_i - \frac{f_i(v(t))}{f_i(v^*)} - \frac{v(t)}{v^*} \right], \\ & \leq \sum_{i=1}^2 \left[ \alpha_i x_i^* f_i(v^*) \left[ \frac{f_i(v(t))}{f_i(v^*)} - \ln \left( \frac{f_i(v(t))}{f_i(v^*)} \right) - \frac{v(t)}{v^*} + \ln \left( \frac{v(t)}{v^*} \right) \right] \right], \\ & = \sum_{i=1}^2 [\alpha_i x_i^* f_i(v^*) (g(F(\sigma)) - g(\sigma))], \end{aligned}$$

where  $\sigma = \frac{v(t)}{v^*}$  and using Lemma 2, we get  $\dot{V}(t) \leq 0$  and  $\dot{V}(t) = 0$  if and only if  $x_i(t) = x_i^*, z(t) = z^*, y_i^* f_i(v(t - \zeta_i)) = y_i(t) f_i(v^*), v^* y_i(t - \zeta_i) = v(t) y_i^*$  and  $v(t) = v^*$  for  $\zeta_i \in [0, \infty)$ . Then the solutions converge to  $\Gamma$ , which is the largest invariant subset of  $\{\dot{V}(t) = 0\}$  and by conforming LaSalle's invariance principle, we get that  $E^*$  is GAS in  $\Gamma$ .

### 4 Numerical simulations

In this section, we present an instance to explain the main results given in Theorem 1 and 2 by using the Lyapunov direct method. We have determined a set of conditions which guarantee that the steady states of model (1)-(6) are GAS. Table 1 have the estimate values of model (1)-(6) parameters. The effects of two main factors on the qualitative behavior of the system which include therapy efficacy  $\varepsilon$  and time delay  $\tau$  will be studied below in details. Using MATLAB we have implemented all computations. This example is obtained from the model (1)-(6) by choosing particular template of the functions  $f_i(v(t))$  and  $h_i(z(t))$  as follow:

$$f_1(v(t)) = \frac{v}{1 + \omega_1 v}, f_2(v(t)) = \frac{v}{1 + \omega_2 v}, h_1(z(t)) = z(t), h_2(z(t)) = z(t),$$

where  $\omega_1, \omega_2 \geq 0$  are constants. Further more, we are going to choose a particular form of the probability distribution functions  $G_i(\tau)$  and  $\Psi_i(\tau)$  as  $G_i(\tau) = \delta(\tau - \tau_i), \Psi_i(\tau) = \delta(\tau - \tau_i), i = 1, 2$ , where  $\delta(\cdot)$  is the

Dirac delta function,  $\tau_1$  and  $\tau_2$  are constants where  $\tau_i \in [0, \infty]$ ,  $i = 1, 2$ . are constants where  $\int_0^\infty G_i(\tau)d\tau = \int_0^\infty \Psi_i(\tau)d\tau = 1$ . Using Dirac delta function properties we get:

$$g_i = \int_0^\infty e^{-m_i\tau} \delta(\tau - \tau_i)d\tau = e^{-m_i\tau_i} \int_0^\infty \delta(\tau - \tau_i)d\tau = e^{-m_i\tau_i}$$

$$\psi_i = \int_0^\infty e^{-n_i\tau} \delta(\tau - \tau_i)d\tau = e^{-n_i\tau_i} \int_0^\infty \delta(\tau - \tau_i)d\tau = e^{-n_i\tau_i},$$

$$\int_0^\infty \delta(\tau - \tau_i)e^{-m_i\tau} x_i(t - \tau)f_i(v(t - \tau))d\tau = e^{-m_i\tau_i} x_i(t - \tau_i)f_i(t - \tau_i),$$

$$\int_0^\infty \delta(\tau - \tau_i)y_i(t - \tau_i)d\tau = e^{-n_i\tau_i} y_i(t - \tau_i).$$

Referring to the previous relations, we can rewrite model (1)-(6) as follows

$$\dot{x}_1(t) = \mu_1 - k_1x_1(t) - \alpha_1x_1(t)\frac{v(t)}{1 + \omega_1v(t)}, \tag{17}$$

$$\dot{y}_1(t) = \alpha_1e^{-m_1\tau_1}x_1(t - \tau_1)\frac{v(t - \tau_1)}{1 + \omega_1v(t - \tau_1)} - ry_1(t) - \beta y_1(t)z(t), \tag{18}$$

$$\dot{x}_2(t) = \mu_2 - k_2x_2(t) - \alpha_2x_2(t)\frac{v(t)}{1 + \omega_2v(t)} \tag{19}$$

$$\dot{y}_1(t) = \alpha_2e^{-m_2\tau_2}F_2x_2(t - \tau_2)\frac{v(t - \tau_2)}{1 + \omega_2v(t - \tau_2)} - ry_2(t) - \beta y_2(t)z(t), \tag{20}$$

$$\dot{v}(t) = Nr(e^{-n_1\tau_1}y_1(t - \tau_1) + e^{-n_2\tau_2}y_2(t - \tau_2)) - dv(t), \tag{21}$$

$$\dot{z}(t) = \lambda(y_1(t) + y_2(t)) - qz(t). \tag{22}$$

To study the effect of drug efficacy, we choose  $\alpha_1 = (1 - \varepsilon)a_0$  and  $\alpha_2 = (1 - \varepsilon)b_0$ . We have chosen the initial conditions:

IC:  $\varphi_1(u) = 600$ ,  $\varphi_2(u) = 1$ ,  $\varphi_3(u) = 500$ ,  $\varphi_4(u) = 1$ ,  $\varphi_5(u) = 10$  and  $\varphi_6(u) = 40$ ,  $u \in [-\infty, 0]$ .

**Table 1:** We define the parameter values of model (17-22) as follow:

Parameter	Value	Parameter	Value
$\mu_1$	10 cells mm <sup>-3</sup> day <sup>-1</sup>	$\mu_2$	6 cells mm <sup>-3</sup> day <sup>-1</sup>
$k_1$	0.01 day <sup>-1</sup>	$k_2$	0.01 day <sup>-1</sup>
$a_0$	0.004 day <sup>-1</sup>	$b_0$	0.001 day <sup>-1</sup>
$\omega_1$	0.05 virus <sup>-1</sup> mm <sup>3</sup>	$\omega_2$	0.05 cells <sup>-1</sup> mm <sup>3</sup>
$r$	0.3 day <sup>-1</sup>	$m_1$	1 day <sup>-1</sup>
$\beta$	0.001	$m_2$	1 day <sup>-1</sup>
$N$	5 virus cells <sup>-1</sup>	$n_1$	1 day <sup>-1</sup>
$d$	3 day <sup>-1</sup>	$n_2$	1 day <sup>-1</sup>
$\lambda$	3 day <sup>-1</sup>	$\tau_1 = \tau_2$	varied
$q$	0.1 day <sup>-1</sup>	$\varepsilon$	varied

**Case I: Effect of drug efficacy on the dynamical behavior of the system:**

In this case, we fix the delay parameter  $\tau_1 = \tau_2 = \tau = 0.5$ . Figures 1-6 show the effect of drug efficacy on the stability of the steady states and the evolution of the uninfected and infected for each CD4+ T cells and macrophages, free virus particles and immune response. We observe that, as the drug efficacy is increased from



$\varepsilon = 0$  to  $\varepsilon = 0.8$ ,  $E_1$  still exists and is a globally asymptotically stable. Moreover the concentrations of uninfected  $CD4^+T$  cells and macrophages are increasing and converging to their normal values  $\frac{\mu_1}{k_1} = 1000$  cells  $mm^{-3}$ ,  $\frac{\mu_2}{k_2} = 600$  cells  $mm^{-3}$ , respectively. While the concentrations of  $CD4^+T$ , macrophages infected cells and free viruses are decaying and tend to zero when  $\varepsilon = 0.8$ . The concentration of cytotoxic T lymphocytes (CTLs) immune response is increasing for the values of equal to 0,0.2,0.5 and tend to zero when  $\varepsilon$  equal to 0.8. It means that the numerical results are consistent with the theoretical results that are given in theorem 1,2. We can see from the simulation results that the treatment with such drug efficacy succeeded to eliminate the HIV virus from the blood.

**Case II: Effect of time delay on the dynamical behavior of the system:**

In this case, we confirm the effect of delay parameter in pre-treatment case where  $\varepsilon = 0.0$ . Figures 7-12 show the effect of time delay on the stability of the steady states and the evolution of the uninfected and infected for each  $CD4^+$  T cells and macrophages, free virus particles and immune response. We observe that, as time delay is increased from  $\tau = 0.1$  to 0.9,  $E_1$  still exists and is a globally asymptotically stable. Moreover the concentrations of uninfected  $CD4^+T$  cells and macrophages are increasing for the values of  $\tau$  except  $\tau = 0.1$ . The concentrations of  $CD4^+T$ , macrophages infected cells and free viruses are decaying with the increasing of time delay values and tend to zero when  $\tau = 0.9$ . While the concentration of cytotoxic T lymphocytes (CTLs) immune response is increasing for the values of equal to 0.1,0.3,0.5 and it tend to zero when  $\tau$  equal to 0.9. It means that the numerical results are consistent with the theoretical results that are given in theorem 1,2. Moreover from a biological point of view, the intracellular delay plays a similar role as an antiviral treatment in eliminating the virus. Where, sufficiently large delay repress viral replication and works on virus clearance. This awaken us to the significance of medications running on the prolong of intracellular delay period.

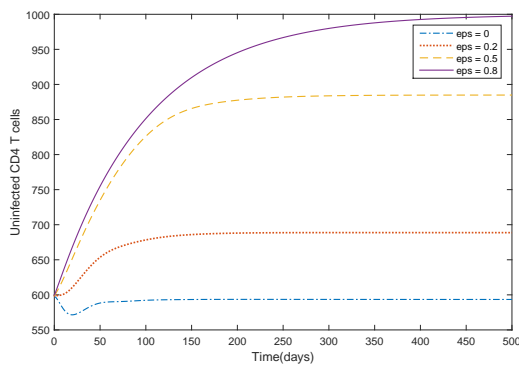


Figure 1: The evolution of uninfected  $CD4^+T$  cells against time with constant time delay  $\tau = 0.5$ .

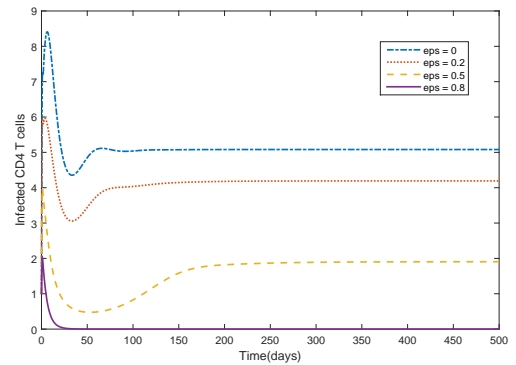


Figure 2: The evolution of infected  $CD4^+T$  cells against time with constant time delay  $\tau = 0.5$ .

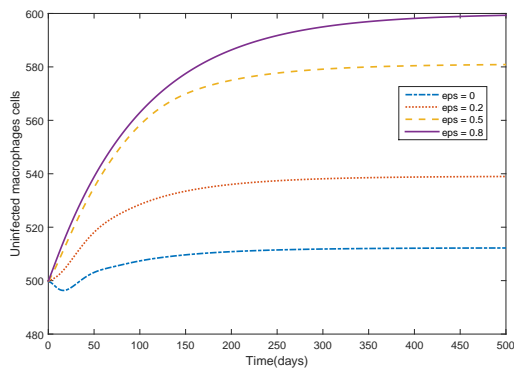


Figure 3: The evolution of uninfected macrophages cells against time with constant time delay  $\tau = 0.5$ .

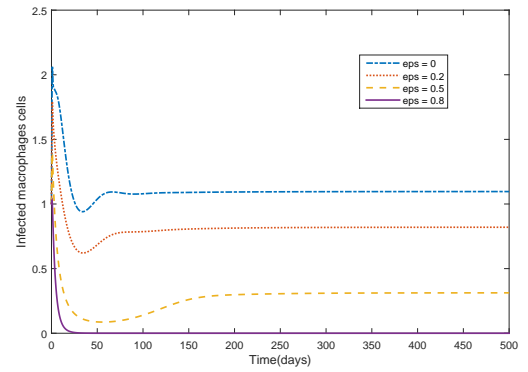


Figure 4: The evolution of infected macrophages cells against time with constant time delay  $\tau = 0.5$ .

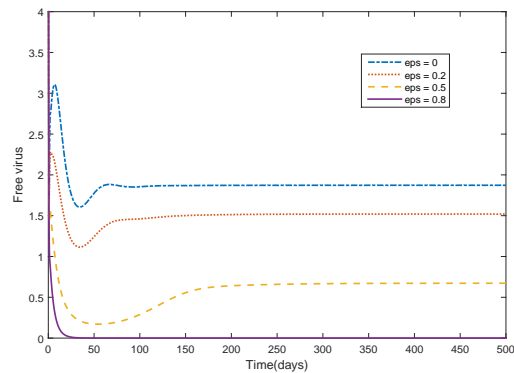


Figure 5: The evolution of free viruses against time with constant time delay  $\tau = 0.5$ .

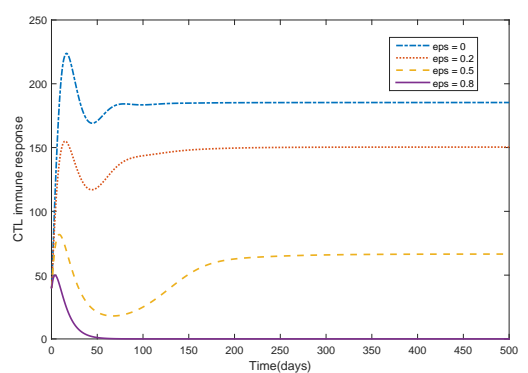


Figure 6: The evolution of immune response against time with constant time delay  $\tau = 0.5$ .

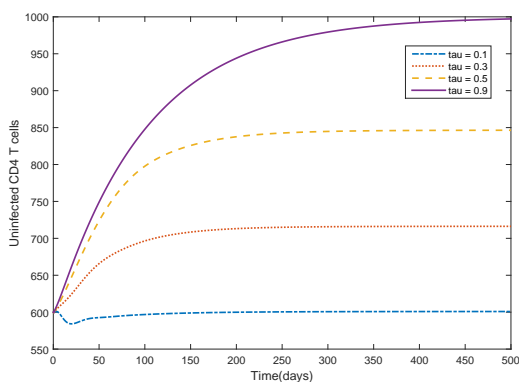


Figure 7: The pre-treatment evolution of uninfected CD4+T cells against time.

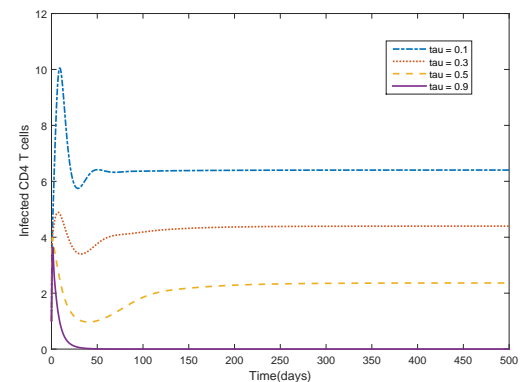


Figure 8: The pre-treatment evolution of infected CD4+T cells against time.

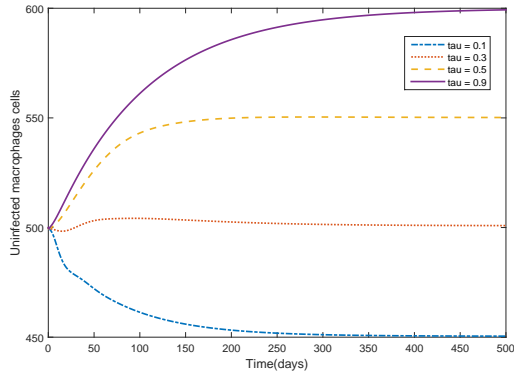


Figure 9: The pre-treatment evolution of uninfected macrophages cells against time.

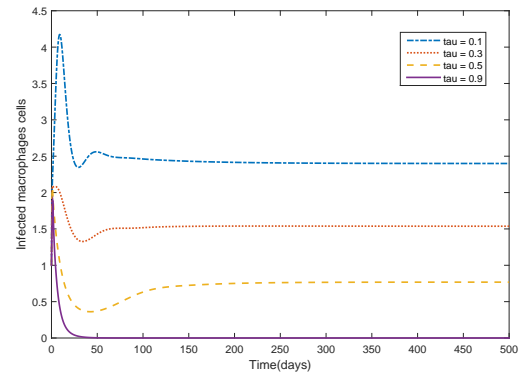


Figure 10: The pre-treatment evolution of infected macrophages cells against time.

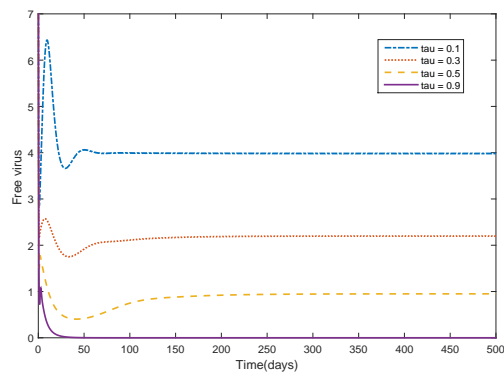


Figure 11: The pre-treatment evolution of free viruses against time.

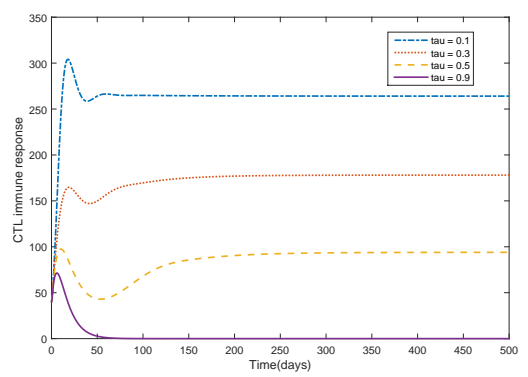


Figure 12: The pre-treatment evolution of immune response against time.

## 5 Conclusion

In this paper, we suggested a distributed delayed human immunodeficiency virus (HIV) models with CTL and two target cells as a system of nonlinear ODES. We demonstrated the positivity and boundedness of the solutions and calculate the steady states of the model. Besides we have used suitable Lyapunov functions to set the global asymptotic stability of the steady states. We have derived the basic reproduction number  $R_0$  and established that the global dynamics are completely established by the value of the related reproduction number.

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# The pseudo- $T$ -direction and pseudo-Nevalinna direction of $K$ -quasi-meromorphic mapping \*

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## Abstract

By applying Ahlfors' theory of covering surfaces, we prove that for quasi-meromorphic mapping  $f$  satisfying  $\limsup_{r \rightarrow \infty} \frac{T(r, f)}{(\log r)^2} = +\infty$ , there exists at least one pseudo- $T$ -direction of  $f$ . We also prove that there exists at least one pseudo-Nevalinna direction of  $f$  which is also pseudo- $T$ -direction of  $f$  under the same condition.

**Key words:**  $K$ -quasi-meromorphic mapping; pseudo- $T$ -direction; pseudo-Nevalinna direction  
**2000 Mathematics Subject Classification :** 30D 60.

## 1 Introduction, definitions and results

It is very interesting topic on singular directions of meromorphic functions in the fields of complex analysis([3, 6, 8, 13, 10, 14]), such as Julia direction, Borel direction,  $T$ -direction, Hayman direction, and so on. In 1997, Sun and Yang [7] extended the value distribution theory of meromorphic functions (see [3, 13] for standard references) to the corresponding theory of quasi-meromorphic mappings [1, 7]. In fact, for value distribution of quasi-meromorphic mappings  $f$ , the singular direction for  $f$  is also one of the main research objects. In [7], Sun and Yang obtained an existence theorem of the Borel direction by using the filling disc theorem of quasi-meromorphic mappings. Later, there were some important results about singular directions for quasi-meromorphic mappings. In 1999, Chen and Sun [1] gave the definition of Nevalinna directions of quasi-meromorphic mappings on the complex plane and proved that there exists at least one Nevalinna direction for quasi-meromorphic mappings of infinite order by using type function, and they also obtained that the Nevalinna direction for quasi-meromorphic mappings of infinite order is also one Borel direction with respect to the type function. In 2004, Liu and Yang [4] studied the relationship between the Julia direction and the Nevalinna direction of quasi-meromorphic mappings by applying a fundamental inequality of quasi-meromorphic mappings on an angular domain.

For a meromorphic function  $f$ , Zheng [14] introduced a new singular direction called a  $T$ -direction conjectured that a transcendental meromorphic function  $f$  must have at least one  $T$ -direction and proved that  $\limsup_{r \rightarrow \infty} \frac{T(r, f)}{(\log r)^2} = +\infty$ . Later, H. Guo, J. H. Zheng and T. W. Ng [2] proved that the conjecture is true by using Ahlfors-Shimizu character  $T(r, \Omega)$  of a meromorphic function in an angular domain  $\Omega$ . Xuan [12] studied the existence of  $T$ -direction of algebroid function dealing with multiple values. In 2006, Li and Gu [5] proved that there exists at least one Nevalinna direction for a  $K$ -quasi-meromorphic mapping  $f$  under the condition  $\limsup_{r \rightarrow \infty} \frac{T(r, f)}{(\log r)^2} =$

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$+\infty$ . In this paper we will further investigate some new singular direction of  $K$ -quasi-meromorphic mapping  $f$ . Before stating our main results, we will introduce some definitions and notations, which can be found in [7, 11].

**Definition 1.1** (see [7]). Let  $f$  be a complex and continuous functions in a region  $D$ . If for any rectangle  $R = \{x + iy; a < x < b, c < y < d\}$  in  $D$ ,  $f(x + iy)$  is an absolutely continuous function of  $y$  for almost every  $x \in (a, b)$ , and  $f(x + iy)$  is an absolutely continuous function of  $x$  for almost every  $y \in (c, d)$ , then  $f$  is said to be absolutely continuous on lines in the region  $D$ . We also call that  $f$  is ACL in  $D$ .

**Definition 1.2** (see [7, Definition 1.1]). Let  $f$  be a homomorphism from  $D$  to  $D'$ . If

(i)  $f$  is ACL in  $D$ ,

(ii) there exists  $K \geq 1$  such that  $f(z) = u(x, y) + iv(x, y)$  satisfies  $|f_z| + |f_{\bar{z}}| \leq K(|f_z| - |f_{\bar{z}}|)$  a. e. in  $D$ , then  $f$  is called an univalent  $K$ -quasiconformal mapping in  $D$ . If  $D'$  is a region on Riemann sphere  $V$ , then  $f$  is named an univalent  $K$ -quasi-meromorphic mapping in  $D$ .

**Definition 1.3** (see [7, Definition 1.2]) Let  $f$  be a complex and continuous function in the region  $D$ . For every point  $z_0$  in  $D$ , if there is a neighborhood  $U(\subset D)$  and a positive integer  $n$  depending on  $z_0$ , such that

$$F(z) = \begin{cases} (f(z))^{\frac{1}{n}}, & f(z_0) = \infty, \\ (f(z) - f(z_0))^{\frac{1}{n}} + f(z_0), & f(z_0) \neq \infty. \end{cases}$$

is an univalent  $K$ -quasi-meromorphic mapping, then  $f$  is named  $n$ -valent  $K$ -quasi-meromorphic mapping at point  $z_0$ . If  $f$  is  $n$ -valent  $K$ -quasi-meromorphic at every point of  $D$ , then  $f$  is called a  $K$ -quasi-meromorphic mapping in  $D$ .

Let  $V$  be the Riemann sphere whose diameter is 1. For any complex number  $a$ , let  $n(r, a)$  be the number of zero points of  $f(z) - a$  in disc  $|z| < r$ , counted according to their multiplicities,  $n^{(l)}(r, a)$  be the number of zeros of  $f(z) - a$  with multiplicity  $\leq l$  in disc  $|z| < r$ , counted according to their multiplicities. Let  $F_r$  be the covering surface  $f(z) = u(x, y) + iv(x, y)$  on sphere  $V$  and  $S(r, f)$  be the average covering times of  $F_r$  to  $V$ ,

$$S(r, f) = \frac{|F_r|}{|V|} = \frac{1}{\pi} \int_0^r \int_0^{2\pi} \frac{|f_z|^2 - |f_{\bar{z}}|^2}{(1 + |f|^2)^2} r d\varphi dr,$$

where  $|F_r|$  and  $|V|$  are the areas of  $F_r$  and  $V$  respectively,

$$T(r, f) = \int_0^r \frac{S(r, f)}{r} dr,$$

$$N(r, a) = \int_0^r \frac{n(t, a) - n(0, a)}{t} dt + n(0, a) \log r,$$

$$N^{(l)}(r, a) = \int_0^r \frac{n^{(l)}(t, a) - n^{(l)}(0, a)}{t} dt + n^{(l)}(0, a) \log r.$$

Let  $\Omega(\varphi_1, \varphi_2) = \{z \in \mathbb{C} : \varphi_1 < \arg z < \varphi_2\} (0 \leq \varphi_1 < \varphi_2 \leq 2\pi)$ , we denote

$$S(r, \varphi_1, \varphi_2; f) = \frac{|F_r|}{|V|} = \frac{1}{\pi} \int_0^r \int_{\varphi_1}^{\varphi_2} \frac{|f_z|^2 - |f_{\bar{z}}|^2}{(1 + |f|^2)^2} r d\varphi dr,$$

$$T(r, \varphi_1, \varphi_2; f) = \int_0^r \frac{S(r, \varphi_1, \varphi_2; f)}{r} dr,$$

when  $\varphi_1 = 0, \varphi_2 = 2\pi$ , we note  $S(r, 0, 2\pi; f) = S(r, f), T(r, 0, 2\pi; f) = T(r, f)$ .

For any complex number  $a$ , let  $n(r, \varphi_1, \varphi_2; a)$  be the number of zero points of  $f(z) - a$  in sector  $\Omega(\varphi_1, \varphi_2) \cap \{z : |z| < r\}$ , counted according to their multiplicities,  $n^l(r, \varphi_1, \varphi_2; a)$  be the number of zeros of  $f(z) - a$  with multiplicity  $\leq l$  in sector  $\Omega(\varphi_1, \varphi_2) \cap \{z : |z| < r\}$ , counted according to their multiplicities. We define

$$N(r, \varphi_1, \varphi_2; a) = \int_0^r \frac{n(t, \varphi_1, \varphi_2; a) - n(0, \varphi_1, \varphi_2; a)}{t} dt + n(0, \varphi_1, \varphi_2; a) \log r,$$

$$N^l(r, \varphi_1, \varphi_2; a) = \int_0^r \frac{n^l(t, \varphi_1, \varphi_2; a) - n^l(0, \varphi_1, \varphi_2; a)}{t} dt + n^l(0, \varphi_1, \varphi_2; a) \log r.$$

Next we give the definitions concerning the Nevanlinna direction of  $K$ -quasi-meromorphic mappings dealing with multiple values .

**Definition 1.4** Let  $f$  be a  $K$ -quasi-meromorphic mapping and  $l$  be a positive integer. Then we call  $\delta^l(a, \varphi_0)$  the deficiency of the value  $a$  in the direction  $\Delta(\varphi_0) : \arg z = \varphi_0, 0 \leq \varphi_0 < 2\pi$ . We call  $a$  the deficiency value of  $f$  in the direction  $\Delta(\varphi_0)$  if  $\delta^l(a, \varphi_0) > 0$ , where

$$\delta^l(a, \varphi_0) = 1 - \limsup_{\varepsilon \rightarrow +0} \limsup_{r \rightarrow \infty} \frac{N^l(r, \varphi_0 - \varepsilon, \varphi_0 + \varepsilon; a)}{T(r, \varphi_0 - \varepsilon, \varphi_0 + \varepsilon; f)}.$$

**Definition 1.5** We call  $\Delta(\varphi_0) : \arg z = \varphi_0$  the pseudo-Nevanlinna direction of  $f$  if, for any system  $a_j \in \mathbb{C} \cup \{\infty\} (j = 1, 2, \dots, q)$  of distinct values and any system  $k_j (j = 1, 2, \dots, q)$  such that  $k_j$  is a positive integer or  $+\infty$  such that

$$\sum_{j=1}^q \left(1 - \frac{1}{k_j + 1}\right) > 2, \tag{1}$$

and

$$\sum_{j=1}^q \frac{k_j}{k_j + 1} \delta^{k_j}(a_j, \varphi_0) \leq 2.$$

Similarly, we give the pseudo- $T$ -direction of  $K$ -quasi-meromorphic mapping as follows.

**Definition 1.6** Let  $f$  be the  $K$ -quasi-meromorphic mapping. A direction  $B : \arg z = \varphi_0 (0 \leq \varphi_0 \leq 2\pi)$  is called a  $T$ -direction of  $f$  if, for any  $\varepsilon (0 < \varepsilon < \frac{\pi}{2})$ , and any system  $a_j \in \mathbb{C} \cup \{\infty\} (j = 1, 2, \dots, q)$  of distinct values and any system  $k_j (j = 1, 2, \dots, q)$  such that  $k_j$  is a positive integer or  $+\infty$  satisfying (2), there exists at least one integer  $j (1 \leq j \leq q)$  such that

$$\limsup_{r \rightarrow \infty} \frac{N^{k_j}(r, \varphi_0 - \varepsilon, \varphi_0 + \varepsilon, a_j)}{T(r, f)} > 0.$$

Now, we will give an existence theorem of pseudo- $T$ -direction of  $K$ -quasi-meromorphic mapping  $f$  as follows.

**Theorem 1.1** Let  $f$  be the  $K$ -quasi-meromorphic mapping satisfying

$$\limsup_{r \rightarrow \infty} \frac{T(r, f)}{(\log r)^2} = +\infty, \tag{2}$$

then there exists at least one pseudo- $T$ -direction of  $f$ .

We also investigate the problem on the relationship between pseudo-Nevanlinna direction and pseudo- $T$ -direction of  $f$  under the condition  $\limsup_{r \rightarrow \infty} \frac{T(r, f)}{(\log r)^2} = +\infty$ , and obtain the following result:

**Theorem 1.2** Let  $f$  be the  $K$ -quasi-meromorphic mapping satisfying (2). Then there exists at least one direction which is both one pseudo-Nevanlinna direction of  $f$  and one pseudo- $T$ -direction of  $f$ .



## 2 Some Lemmas

Let  $F$  be a finite covering surface of  $F_1$ ,  $F$  is bounded by a finite number of analytic closed Jordan curves, its boundary is denoted by  $\partial F$ . We call the part of  $\partial F$ , which lies the interior of  $F_1$ , the relative boundary of  $F$ , and denote its length by  $L$ . Let  $D$  be a domain of  $F_1$ , its boundary consists of finite number of points or analytic closed Jordan curves, and  $F(D)$  be the part of  $F$ , which lies above  $D$ . We denote the area of  $F, F_1, F(D)$  and  $D$  by  $|F|, |F_1|, |F(D)|$  and  $|D|$ , respectively. We call

$$S = \frac{|F|}{|F_1|}, \quad S(D) = \frac{|F(D)|}{|D|}$$

the mean covering numbering of  $F$  relative to  $F_1, D$ , respectively.

**Lemma 2.1** (see [9, Theorem 3]) *Let  $F$  be a simply connected finite covering surface on the unit sphere  $V$ , and let  $k_j(j = 1, 2, \dots, q)$  be  $q$  positive integers. Let  $D_j(j = 1, 2, \dots, q)$  be  $q(\geq 2)$  disjoint spherical disks with radius  $\delta/3(> 0)$  on  $V$  and without a pair of  $D_j$  such that their spherical distance is less than  $\delta$  and let  $n_j^{k_j}$  be the number of simply connected islands in  $F(D_j)$ , which consist of not more than  $k_j$  sheets, then*

$$\sum_{j=1}^q \frac{k_j}{k_j + 1} n_j^{k_j} \geq \left( \sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) - 2 \right) S - \frac{C + 9\pi h}{\delta^3} L,$$

where  $L$  is the length of the relative boundary of  $F$ .

By applying Lemma 2.1, we can get an important inequality of  $K$ -quasi-meromorphic mapping in an angular domain as follows.

**Lemma 2.2** *Suppose that  $f(z)$  is a  $K$ -quasi-meromorphic mapping, and let  $k_j(j = 1, 2, \dots, q)$  be  $q$  positive integers, and  $\{a_j\}$  are  $q(\geq 3)$  distinct points on  $V$  and without a pair of  $\{a_j\}$  such that their spherical distance is less than  $\delta + 2\delta/3$ ,  $n_j^{k_j}$  be the number of zeros of  $f(z) - a_j$ , which are consisted of not more than  $k_j$  multiplicities, then*

$$\sum_{j=1}^q \frac{k_j}{k_j + 1} n_j^{k_j} \geq \left( \sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) - 2 \right) S - \frac{C + 9\pi h}{\delta^3} L.$$

**Lemma 2.3** (see [5, Lemma 2.2]). *Let  $f(z)$  be a  $K$ -quasi-meromorphic mapping on the angular domain  $\Omega(\varphi_0 - \delta, \varphi_0 + \delta)$ ,  $a_1, \dots, a_q(q \geq 3)$  are distinct points on the unit sphere  $V$  and the spherical distance of any two points is no smaller than  $\gamma \in (0, \frac{1}{2})$ . Let  $F_0 = V \setminus \{a_1, a_2, \dots, a_q\}$ ,  $D = \Omega(r, \varphi_0 - \varphi, \varphi_0 + \varphi) \cap \{z : |z| > 1\} \setminus \{f^{-1}(a_1), f^{-1}(a_2), \dots, f^{-1}(a_q)\}$  and  $D_r = D \cap \{z : |z| < r\}(r > 1)$ ,  $F_r = f(D_r) \subset V$ , then for any positive number  $\varphi$  satisfying  $0 < \varphi < \delta$ , we have*

$$L(\partial f(D_r)) \leq \sqrt{2K}\pi \left[ \frac{d(S(r, \varphi_0 - \varphi, \varphi_0 + \varphi; f) - S(1, \varphi_0 - \varphi, \varphi_0 + \varphi; f))}{d\varphi} \right]^{\frac{1}{2}} (\log r)^{\frac{1}{2}} \quad (3)$$

$$+ \sqrt{2K\delta r} \mu^{\frac{1}{2}}(r, \varphi_0 - \delta, \varphi_0 + \delta) + \sqrt{2K\delta} \mu^{\frac{1}{2}}(1, \varphi_0 - \delta, \varphi_0 + \delta).$$

where  $F_r$  is the covering surface of  $F_0$  and  $L(\partial f(D_r))$  is the length of the relative boundary of  $F_r$  relative to  $F_0$ , and

$$\mu(r, \varphi_0 - \delta, \varphi_0 + \delta) = \int_{\varphi_0 - \delta}^{\varphi_0 + \delta} \frac{|f_z|^2 - |f_{\bar{z}}|^2}{(1 + |f(re^{i\varphi})|^2)^2} r d\varphi.$$

**Lemma 2.4** *Let  $f(z)$  be a  $K$ -quasi-meromorphic mapping on the angular domain  $\Omega(\varphi_0 - \delta, \varphi_0 + \delta)$ , and  $k_j (j = 1, 2, \dots, q)$   $q$  positive integers. If  $a_1, \dots, a_q (q \geq 3)$  are distinct points on the unit sphere  $V$  and the spherical distance of any two points is no small than  $\gamma \in (0, \frac{1}{2})$ . Then*

$$\begin{aligned} & \left( \sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) - 2 \right) S(r, \varphi_0 - \varphi, \varphi_0 + \varphi; f) \\ & \leq \sum_{j=1}^q \frac{k_j}{k_j + 1} n^{k_j}(r, \varphi_0 - \delta, \varphi_0 + \delta; a_j) + \frac{2C^2 \gamma^{-6} \pi^2 K}{\left( \sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) - 2 \right) (\delta - \varphi)} \log r \\ & \quad + \left( \sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) - 2 \right) S(1, \varphi_0 - \varphi, \varphi_0 + \varphi; f) + 2C\gamma^{-3} \delta^{\frac{1}{2}} K^{\frac{1}{2}} r^{\frac{1}{2}} \mu^{\frac{1}{2}}(r, \varphi_0 - \delta, \varphi_0 + \delta) \\ & \quad + 2C\gamma^{-3} \delta^{\frac{1}{2}} K^{\frac{1}{2}} \mu^{\frac{1}{2}}(1, \varphi_0 - \delta, \varphi_0 + \delta) \end{aligned} \tag{4}$$

and

$$\begin{aligned} & \left( \sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) - 2 \right) T(r, \varphi_0 - \varphi, \varphi_0 + \varphi; f) \\ & \leq \sum_{j=1}^q \frac{k_j}{k_j + 1} N^{k_j}(r, \varphi_0 - \delta, \varphi_0 + \delta; a_j) + \frac{2C^2 \gamma^{-6} \pi^2 K}{\left( \sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) - 2 \right) (\delta - \varphi)} (\log r)^2 \\ & \quad + \left( \sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) - 2 \right) T(1, \varphi_0 - \varphi, \varphi_0 + \varphi; f) \\ & \quad + \left( \sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) - 2 \right) S(1, \varphi_0 - \varphi, \varphi_0 + \varphi; f) \log r \\ & \quad + 2C\gamma^{-3} \delta^{\frac{1}{2}} K^{\frac{1}{2}} \mu^{\frac{1}{2}}(1, \varphi_0 - \delta, \varphi_0 + \delta) \log r + \lambda(r, \varphi_0 - \delta, \varphi_0 + \delta) \end{aligned} \tag{5}$$

for any  $\varphi, 0 < \varphi < \delta$ , where  $C$  is a constant depending only on  $\{a_1, a_2, \dots, a_q\}$ .  $\lambda(r, \varphi_0 - \delta, \varphi_0 + \delta) = 2C\gamma^{-3} \delta^{\frac{1}{2}} K^{\frac{1}{2}} \int_1^r \left( \frac{\mu(r, \varphi_0 - \delta, \varphi_0 + \delta)}{r} \right)^{\frac{1}{2}} dr$ ,  $(\mu(r, \varphi_0 - \delta, \varphi_0 + \delta) = \int_{\varphi_0 - \delta}^{\varphi_0 + \delta} \frac{|f_z|^2 - |f_{\bar{z}}|^2}{(1 + |f(re^{i\varphi})|^2)^2} r d\varphi)$

$$\lambda(r, \varphi_0 - \delta, \varphi_0 + \delta) \leq 2C\gamma^{-3} \delta^{\frac{1}{2}} \pi^{\frac{1}{2}} K^{\frac{1}{2}} (T(r, \varphi_0 - \delta, \varphi_0 + \delta; f))^{\frac{1}{2}} \log T(r, \varphi_0 - \delta, \varphi_0 + \delta; f) \tag{6}$$

outside a set  $E_\delta$  of  $r$  at most, where  $E_\delta$  consists of a series of intervals and satisfies  $\int_{E_\delta} (r \log r)^{-1} dr < +\infty$ .

*Proof:* Under the condition of Lemma 2.3 and Lemma 2.2, we have

$$S(D_r) = S(r, \varphi_0 - \varphi, \varphi_0 + \varphi; f) - S(1, \varphi_0 - \varphi, \varphi_0 + \varphi; f). \tag{7}$$

Using Lemma 2.1, we easily obtain

$$\begin{aligned} & \left( \sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) - 2 \right) [S(r, \varphi_0 - \varphi, \varphi_0 + \varphi; f) - S(1, \varphi_0 - \varphi, \varphi_0 + \varphi; f)] \\ & \leq \sum_{j=1}^q \frac{k_j}{k_j + 1} n^{k_j}(r, \varphi_0 - \delta, \varphi_0 + \delta; a_j) + C\gamma^{-3} L(\partial(D_r)). \end{aligned} \tag{8}$$

where  $C$  is a constant depending only on  $\{a_1, a_2, \dots, a_q\}$ .

Taking (3) into (8), we have

$$\begin{aligned} & \left( \sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) - 2 \right) [S(r, \varphi_0 - \varphi, \varphi_0 + \varphi; f) - S(1, \varphi_0 - \varphi, \varphi_0 + \varphi; f)] \\ & - \sum_{j=1}^q \frac{k_j}{k_j + 1} n^{k_j}(r, \varphi_0 - \delta, \varphi_0 + \delta; a_j) - C\gamma^{-3} \sqrt{2K\delta r} \mu^{\frac{1}{2}}(r, \varphi_0 - \delta, \varphi_0 + \delta) \\ & - C\gamma^{-3} \sqrt{2K\delta} \mu^{\frac{1}{2}}(1, \varphi_0 - \delta, \varphi_0 + \delta) \\ & \leq C\gamma^{-3} \sqrt{2K} \pi \left[ \frac{d(S(r, \varphi_0 - \varphi, \varphi_0 + \varphi; f) - S(1, \varphi_0 - \varphi, \varphi_0 + \varphi; f))}{d\varphi} \right]^{\frac{1}{2}} (\log r)^{\frac{1}{2}}. \end{aligned} \quad (9)$$

We denote

$$\begin{aligned} A(r, \varphi) &= \left( \sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) - 2 \right) [S(r, \varphi_0 - \varphi, \varphi_0 + \varphi; f) - S(1, \varphi_0 - \varphi, \varphi_0 + \varphi; f)] \\ & - \sum_{j=1}^q \frac{k_j}{k_j + 1} n^{k_j}(r, \varphi_0 - \delta, \varphi_0 + \delta; a_j) - C\gamma^{-3} \sqrt{2K\delta r} \mu^{\frac{1}{2}}(r, \varphi_0 - \delta, \varphi_0 + \delta) \\ & - C\gamma^{-3} \sqrt{2K\delta} \mu^{\frac{1}{2}}(1, \varphi_0 - \delta, \varphi_0 + \delta). \end{aligned} \quad (10)$$

By (9) and (10), we have

$$A(r, \varphi) \leq C\gamma^{-3} \sqrt{2K} \pi \left[ \frac{d(S(r, \varphi_0 - \varphi, \varphi_0 + \varphi; f) - S(1, \varphi_0 - \varphi, \varphi_0 + \varphi; f))}{d\varphi} \right]^{\frac{1}{2}} (\log r)^{\frac{1}{2}}. \quad (11)$$

And from (10), it follows that  $A(r, \varphi)$  is an increasing function of  $\varphi$ . Thus, there exists  $\delta_0 > 0$ , such that  $A(r, \varphi) \leq 0$  for  $0 < \varphi \leq \delta_0$  and  $A(r, \varphi) > 0$  for  $\varphi > \delta_0$ .

Now, two following cases will be considered:

**Case 1.** For  $\varphi > \delta_0$ , by (11) we have

$$[A(r, \varphi)]^2 \leq 2C^2\gamma^{-6} K \pi^2 \frac{d(S(r, \varphi_0 - \varphi, \varphi_0 + \varphi; f) - S(1, \varphi_0 - \varphi, \varphi_0 + \varphi; f))}{d\varphi} \log r. \quad (12)$$

By (10) we have

$$\frac{dA(r, \varphi)}{d\varphi} = \left( \sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) - 2 \right) \frac{d(S(r, \varphi_0 - \varphi, \varphi_0 + \varphi; f) - S(1, \varphi_0 - \varphi, \varphi_0 + \varphi; f))}{d\varphi}. \quad (13)$$

From (12) and (13) we have

$$[A(r, \varphi)]^2 \leq \frac{2C^2\gamma^{-6} K \pi^2 \log r}{\sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) - 2} \cdot \frac{dA(r, \varphi)}{d\varphi},$$

*i.e.*,

$$d\varphi \leq \frac{2C^2\gamma^{-6} K \pi^2 \log r}{\sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) - 2} \cdot \frac{dA(r, \varphi)}{[A(r, \varphi)]^2}.$$

For the above inequality, by integrating its two sides, we have

$$\delta - \varphi = \int_{\varphi}^{\delta} d\varphi \leq \frac{2C^2\gamma^{-6} K \pi^2 \log r}{\sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) - 2} \int_{\varphi}^{\delta} \frac{dA(r, \varphi)}{[A(r, \varphi)]^2} \leq \frac{2C^2\gamma^{-6} K \pi^2 \log r}{\sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) - 2} \cdot \frac{1}{A(r, \varphi)}.$$

Thus

$$A(r, \varphi) \leq \frac{2C^2\gamma^{-6}K\pi^2 \log r}{\left(\sum_{j=1}^q \left(1 - \frac{1}{k_j+1}\right) - 2\right) (\delta - \varphi)}. \tag{14}$$

**Case 2.** Because  $A(r, \varphi) \leq 0$  when  $0 < \varphi \leq \delta_0$ , the above inequality also holds. From Case 1 and Case 2, we can easily get

$$A(r, \varphi) \leq \frac{2C^2\gamma^{-6}K\pi^2 \log r}{\left(\sum_{j=1}^q \left(1 - \frac{1}{k_j+1}\right) - 2\right) (\delta - \varphi)},$$

for any  $\varphi, 0 < \varphi < \delta$ . Thus, from (10) we can get (4) easily.

Then, by dividing  $r$  and integrating from 1 to  $r$  on each sides of (4) we get

$$\begin{aligned} & \left(\sum_{j=1}^q \left(1 - \frac{1}{k_j+1}\right) - 2\right) T(r, \varphi_0 - \varphi, \varphi_0 + \varphi; f) \\ \leq & \sum_{j=1}^q \frac{k_j}{k_j+1} N^{k_j}(r, \varphi_0 - \delta, \varphi_0 + \delta; a_j) + \frac{2C^2\gamma^{-6}\pi^2 K}{\left(\sum_{j=1}^q \left(1 - \frac{1}{k_j+1}\right) - 2\right) (\delta - \varphi)} (\log r)^2 \\ & + \left(\sum_{j=1}^q \left(1 - \frac{1}{k_j+1}\right) - 2\right) T(1, \varphi_0 - \varphi, \varphi_0 + \varphi; f) \\ & + \left(\sum_{j=1}^q \left(1 - \frac{1}{k_j+1}\right) - 2\right) S(1, \varphi_0 - \varphi, \varphi_0 + \varphi; f) \log r \\ & + 2C\gamma^{-3}\delta^{\frac{1}{2}}K^{\frac{1}{2}}\mu^{\frac{1}{2}}(1, \varphi_0 - \delta, \varphi_0 + \delta) \log r + 2C\gamma^{-3}\delta^{\frac{1}{2}}K^{\frac{1}{2}} \int_1^r \left[\frac{\mu(r, \varphi_0 - \delta, \varphi_0 + \delta)}{r}\right]^{\frac{1}{2}} dr. \end{aligned}$$

From the definitions of  $S(r, \varphi_1, \varphi_2; f)$ ,  $\mu(r, \varphi_0 - \delta, \varphi_0 + \delta)$  and  $\lambda(r, \varphi_0 - \delta, \varphi_0 + \delta)$ , and Schwarz's inequality we get

$$\begin{aligned} (\lambda(r, \varphi_0 - \delta, \varphi_0 + \delta))^2 &= 4C^2\gamma^{-6}\delta K \left[ \int_1^r \left(\frac{\mu(r, \varphi_0 - \delta, \varphi_0 + \delta)}{r}\right)^{\frac{1}{2}} dr \right]^2 \\ &\leq 4C^2\gamma^{-6}\delta K \int_1^r \mu(r, \varphi_0 - \delta, \varphi_0 + \delta) dr \int_1^r r^{-1} dr \\ &\leq 4C^2\gamma^{-6}\pi\delta K \log r \int_1^r dS(r, \varphi_0 - \delta, \varphi_0 + \delta; f) \\ &\leq 4C^2\gamma^{-6}\pi\delta K S(r, \varphi_0 - \delta, \varphi_0 + \delta; f) \log r \\ &= 4C^2\gamma^{-6}\pi\delta K \frac{dT(r, \varphi_0 - \delta, \varphi_0 + \delta; f)}{dr} r \log r. \end{aligned} \tag{15}$$

Choosing  $r_0, r_0 > 0$  such that  $T(r_0, \varphi_0 - \delta, \varphi_0 + \delta; f) > 1$ , and setting  $E_\delta = \{r_0 < r < \infty : (\lambda(r, \varphi_0 - \delta, \varphi_0 + \delta))^2 > 4C^2\gamma^{-6}\pi\delta K T(r, \varphi_0 - \delta, \varphi_0 + \delta; f)(\log T(r, \varphi_0 - \delta, \varphi_0 + \delta; f))^2\}$ , thus we have

$$\begin{aligned} \int_{E_\delta} \frac{dr}{r \log r} &\leq \int_{E_\delta} \frac{dT(r, \varphi_0 - \delta, \varphi_0 + \delta; f)}{T(r, \varphi_0 - \delta, \varphi_0 + \delta; f)[\log T(r, \varphi_0 - \delta, \varphi_0 + \delta; f)]^2} \\ &\leq [\log T(r_0, \varphi_0 - \delta, \varphi_0 + \delta; f)]^{-1} < +\infty. \end{aligned} \tag{16}$$

Then for  $r > r_0$  and  $r \notin E_\delta$ , we have (5).

Thus, the proof of Lemma 2.4 is completed. □

**Lemma 2.5** (see [11, Lemma 2.4] or [12]). Let  $F(r)$  be a positive nondecreasing function defined for  $1 < r < +\infty$  and satisfy

$$\limsup_{r \rightarrow \infty} \frac{F(r)}{(\log r)^2} = +\infty. \tag{17}$$

Then, for any subset  $E \subset (1, +\infty)$  satisfying  $\int_E \frac{dr}{r \log r} < \frac{1}{\varphi} (\varphi \geq 2)$ ,

$$\limsup_{r \rightarrow \infty, r \in (1, +\infty) \setminus E} \frac{F(r)}{(\log r)^2} = +\infty.$$

**Lemma 2.6** Let  $f(z)$  be the  $K$ -quasi-meromorphic mapping and  $m(m > 1)$  be a positive integer. Put  $\varphi_0 = 0, \varphi_1 = \frac{2\pi}{m}, \dots, \varphi_{m-1} = (m-1)\frac{2\pi}{m}$ . Let

$$\Delta(\varphi_i) = \left\{ z \mid |\arg z - \varphi_i| < \frac{2\pi}{m} \right\} \quad (0 \leq i \leq m-1).$$

Then among these  $m$  angular domains  $\{\Delta(\varphi_i)\}$ , there exists at least an angular domain  $\Delta(\varphi_i)$  such that for any system  $a_j (j = 1, 2, \dots, q)$  of distinct values and any system  $k_j (j = 1, 2, \dots, q)$  such that  $k_j$  is a positive integer or  $+\infty$  and that

$$\sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) > 2,$$

there exists at least one integer  $j (1 \leq j \leq q)$  such that

$$\limsup_{r \rightarrow \infty} \frac{N^{k_j}(r, \Delta(\varphi_i), a)}{T(r, f)} > 0.$$

*Proof:* Suppose that the conclusion is false. Then for any  $\Delta(\varphi_i) (i = 0, 1, \dots, m-1)$ , there is a system  $a_j^i (j = 1, 2, \dots, q)$  of distinct values and a system  $k_j^i (j = 1, 2, \dots, q)$  such that  $k_j^i$  is a positive integer or  $+\infty$  and that

$$\sum_{j=1}^q \left( 1 - \frac{1}{k_j^i + 1} \right) > 2,$$

for any  $j (1 \leq j \leq q)$ , we have

$$\limsup_{r \rightarrow \infty} \frac{N^{k_j^i}(r, \Delta(\varphi_i), a_j^i)}{T(r, f)} = 0. \tag{18}$$

Let  $\beta$  be any positive integer. Put  $\varphi_{i,k} = \frac{2\pi}{m}i + \frac{2k\pi}{\beta m}, 0 \leq i \leq m-1, 0 \leq k \leq \beta-1$ . For any given number  $r > 1$ , writing

$$\Delta_{i,k}(r) = \{z \mid |z| < r, \varphi_{i,k} < \arg z < \varphi_{i,k+1}\}.$$

Then

$$\{|z| < r\} = \sum_{k=0}^{\beta-1} \sum_{i=0}^{m-1} \Delta_{i,k}(r).$$

Put

$$\begin{aligned} \bar{\Delta}_i &= \left\{ z \mid \frac{\varphi_{i,0} + \varphi_{i,1}}{2} \leq \arg z \leq \frac{\varphi_{i+1,0} + \varphi_{i+1,1}}{2} \right\}, \\ \Delta_i^0 &= \{z \mid \varphi_{i,0} < \arg z < \varphi_{i+1,1}\}, \quad 0 \leq i \leq m-1, \\ \sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) &= \min_{1 \leq i \leq m} \left\{ \sum_{j=1}^q \left( 1 - \frac{1}{k_j^i + 1} \right) \right\} > 2. \end{aligned}$$

From Lemma 2.4 we have

$$\left( \sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) - 2 \right) S(r, \bar{\Delta}_i, f) \leq \sum_{j=1}^q \frac{k_j^i}{k_j^i + 1} n^{k_j^i}(r, \Delta_i^0, a_j^i) + O(\log r) + h_i r^{\frac{1}{2}} \mu^{\frac{1}{2}}(r, \varphi_{i,0}, \varphi_{i+1,1}).$$

Add from  $i = 0$  to  $m - 1$  and divide both sides of this inequality by  $r$  and integrate both sides from 1 to  $r$ , and since  $T(r, f) = \sum_{i=0}^{m-1} T(r, \bar{\Delta}_i, f)$ , then the following inequality can be obtained

$$\begin{aligned} & \left( \sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) - 2 \right) T(r, f) \\ & \leq \sum_{i=0}^{m-1} \sum_{j=1}^q \frac{k_j^i}{k_j^i + 1} N^{k_j^i}(r, \Delta_i^0, a_j^i) + O((\log r)^2) + \sum_{i=0}^{m-1} \lambda(r, \varphi_{i,0}, \varphi_{i+1,1}), \end{aligned} \tag{19}$$

where

$$\lambda(r, \varphi_{i,0}, \varphi_{i+1,1}) \leq h_i \left[ \frac{2\pi}{m} \left( 1 + \frac{1}{\beta} \right) \right]^{\frac{1}{2}} (T(r, \varphi_{i,0}, \varphi_{i+1,1}; f))^{\frac{1}{2}} \log T(r, \varphi_{i,0}, \varphi_{i+1,1}; f)$$

at most outside a set  $E_i$  of  $r$ , where  $E_i$  satisfies  $\int_{E_i} (r \log r)^{-1} dr < +\infty (i = 0, 1, \dots, m - 1)$ .

For any  $i \in \{0, 1, \dots, m - 1\}$  and  $\wp \geq 2$ , there exists  $r_i > 0$  such that  $T(r_i, \varphi_{i,0}, \varphi_{i+1,1}; f) > e^{\wp m}$  for  $r > r_i$ . Then it follows from (16) that

$$\int_{E_i} \frac{1}{r \log r} dr \leq \frac{1}{\log T(r, \varphi_{i,0}, \varphi_{i+1,1}; f)} < \frac{1}{\wp m} < \frac{1}{\wp}.$$

Put  $E = \cup_{i=0}^{m-1} E_i$ , then

$$\int_E \frac{1}{r \log r} dr \leq \sum_{i=0}^{m-1} \int_{E_i} \frac{1}{r \log r} dr \leq m \max_{0 \leq i \leq m-1} \int_{E_i} \frac{1}{r \log r} dr < m \cdot \frac{1}{\wp m} < \frac{1}{\wp}.$$

By applying Lemma 2.5 to this set  $E$  and  $T(r, f)$ , we obtain that

$$\limsup_{r \rightarrow \infty, r \in (1, \infty) \setminus E} \frac{T(r, f)}{(\log r)^2} = +\infty.$$

There exists  $\{r_n\} \in (r, +\infty) \setminus E$ ,

$$\lim_{n \rightarrow \infty} \frac{T(r_n, f)}{(\log r_n)^2} = +\infty.$$

For this sequence  $\{r_n\}$ , by (19) we have

$$\begin{aligned} & \left( \sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) - 2 \right) T(r_n, f) \\ & \leq \sum_{i=0}^{m-1} \sum_{j=1}^q \frac{k_j^i}{k_j^i + 1} N^{k_j^i}(r_n, \Delta_i^0, a_j^i) + O((\log r_n)^2) + \sum_{i=0}^{m-1} \lambda(r_n, \varphi_{i,0}, \varphi_{i+1,1}). \end{aligned}$$

From (18), by dividing both sides of the above inequality by  $T(r_n, f)$  and letting  $n \rightarrow \infty$ , we obtain  $\sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) - 2 \leq 0$  that is,  $\sum_{j=1}^q \left( 1 - \frac{1}{k_j + 1} \right) \leq 2$ , a contradict.

Thus, this completes the proof of Lemma 2.6. □

### 3 The Proof of Theorem 1.1

*Proof:* By Lemma 2.6, we can choose subsequence of  $\{\theta_m\}$ , assume that  $\theta_m \rightarrow \theta_0$  when  $m \rightarrow \infty$ . Then  $B : \arg z = \theta_0$  is a pseudo-T-direction of  $f$ .

In fact, for any  $\varepsilon(0 < \varepsilon < \frac{\pi}{2})$ , when  $m$  is sufficiently large, we have  $\Delta(\theta_m) \subset \Omega(\theta_0, \varepsilon)$ . By Lemma 2.6, we have

$$\limsup_{r \rightarrow \infty} \frac{N^{k_j}(r, \theta_0, \varepsilon, a_j)}{T(r, f)} \geq \limsup_{r \rightarrow \infty} \frac{N^{k_j}(r, \Delta(\theta_m), a_j)}{T(r, f)} > 0.$$

Thus, we complete the proof of Theorem 1.1. □

### 4 The Proof of Theorem 1.2

*Proof:* Suppose  $\delta \in (0, 2\pi)$ , we can choose  $r_0 > 0$  such that  $T(r_0, \varphi_0 - \delta, \varphi_0 + \delta; f) > e^\varphi$ . Then it follows from (16) that

$$\int_{E_\delta} \frac{1}{r \log r} dr \leq \frac{1}{\log T(r_0, \varphi_0 - \delta, \varphi_0 + \delta; f)} < \frac{1}{\varphi}.$$

By applying Lemma 2.4 for the set  $E_\delta$  and  $T(r, f)$ , it follows that

$$\limsup_{r \rightarrow \infty, r \in (1, \infty) \setminus E_\delta} \frac{T(r, f)}{(\log r)^2} = +\infty.$$

So, there exists a sequence  $\{r_n\} \in (r, +\infty) \setminus E_\delta$ ,

$$\lim_{n \rightarrow \infty} \frac{T(r_n, f)}{(\log r_n)^2} = +\infty. \tag{20}$$

By applying the finite covering theorem at  $[0, 2\pi]$ , there exists some  $\varphi_0$  such that  $\varphi_0 \in [0, 2\pi]$  and

$$\limsup_{n \rightarrow \infty} \frac{T(r_n, \varphi_0 - \varphi, \varphi_0 + \varphi; f)}{T(r_n, f)} > 0 \tag{21}$$

for an arbitrary  $\varphi, 0 < \varphi < \varphi_0$ . Thus, we will prove that the direction  $\Delta(\varphi_0) : \arg z = \varphi_0$  is one pseudo-Nevalinna direction of  $f(z)$  which is also the pseudo-T-direction of  $f(z)$ .

**Step one.** We firstly prove that the direction  $\Delta(\varphi_0) : \arg z = \varphi_0$  is one pseudo-Nevalinna direction of  $f(z)$ .

Otherwise, for an arbitrary positive number  $\varepsilon_0 > 0$ , there exists a system  $a_j \in \mathbb{C} \cup \{\infty\} (j = 1, 2, \dots, q)$  of distinct values and a system  $k_j (j = 1, 2, \dots, q)$  such that  $k_j$  is a positive integer or  $+\infty$  and that

$$\sum_{j=1}^q \left(1 - \frac{1}{k_j + 1}\right) > 2, \tag{22}$$

the following inequality holds

$$\sum_{j=1}^q \frac{k_j}{k_j + 1} \delta^{k_j}(a_j, \varphi_0) > 2 + \varepsilon_0.$$

From the definition of  $\delta^{k_j}(a_j, \varphi_0)$ , we get

$$\limsup_{\varphi \rightarrow +0} \limsup_{r \rightarrow +\infty} \sum_{j=1}^q \frac{k_j}{k_j + 1} \frac{N^{k_j}(r, \varphi_0 - \varphi, \varphi_0 + \varphi; a_j)}{T(r, \varphi_0 - \varphi, \varphi_0 + \varphi; f)} < \sum_{j=1}^q \left(1 - \frac{1}{k_j + 1}\right) - 2 - \varepsilon_0.$$

Thus, there exists some  $\varphi' (> 0)$ , and for any  $\varphi, 0 < \varphi < \varphi'$ , we have

$$\limsup_{r \rightarrow +\infty} \sum_{j=1}^q \frac{k_j}{k_j + 1} \frac{N^{k_j}(r, \varphi_0 - \varphi, \varphi_0 + \varphi; a_j)}{T(r, \varphi_0 - \varphi, \varphi_0 + \varphi; f)} < \sum_{j=1}^q \left(1 - \frac{1}{k_j + 1}\right) - 2 - \varepsilon_0. \tag{23}$$

Then for any  $\varphi, 0 < \varphi < \varphi'$ , set

$$T(\varphi) = \limsup_{n \rightarrow +\infty} \frac{T(r_n, \varphi_0 - \varphi, \varphi_0 + \varphi; f)}{T(r_n, f)}. \tag{24}$$

Obviously,  $T(\varphi)$  is an increasing function in interval  $[0, \varphi']$ . From (21) we have  $T(\varphi) > 0$ . So,  $0 < T(\varphi) \leq 1$ . Since the increasing of  $T(\varphi)$  in interval  $[0, \varphi']$  and the continuous theorem for monotonous functions, we can see that all discontinuous points of  $T(\varphi)$  constitute a countable set at most. Then, by Lemma 2.4, we can get

$$\begin{aligned} & \left( \sum_{j=1}^q \left(1 - \frac{1}{k_j + 1}\right) - 2 \right) T(r_n, \varphi_0 - \varphi, \varphi_0 + \varphi; f) \\ & \leq \sum_{j=1}^q \frac{k_j}{k_j + 1} N^{k_j}(r_n, \varphi_0 - \delta, \varphi_0 + \delta; a_j) + O(\log r_n)^2 \\ & \quad + O((T(r_n, \varphi_0 - \delta, \varphi_0 + \delta; f))^{\frac{1}{2}} \log T(r_n, \varphi_0 - \delta, \varphi_0 + \delta; f)) \end{aligned} \tag{25}$$

for  $0 < \varphi < \delta < \varphi'$  and  $r_n \notin E_\delta$ .

Thus, it follows from (23)-(25) that

$$\left( \sum_{j=1}^q \left(1 - \frac{1}{k_j + 1}\right) - 2 \right) T(\varphi) < \left( \sum_{j=1}^q \left(1 - \frac{1}{k_j + 1}\right) - 2 - \varepsilon_0 \right) T(\delta). \tag{26}$$

Then, we get from (26)

$$T(\varphi) \rightarrow T(\delta), \quad \varphi \rightarrow \delta. \tag{27}$$

By combining (26) with (27), we can obtain  $T(\delta) = 0$ , which is a contradiction to  $T(\delta) > 0$ . Then  $\Delta(\varphi_0) : argz = \varphi_0$  is the pseudo-Nevalinna direction of  $f(z)$ .

**Step two.** We will prove that  $\Delta(\varphi_0) : argz = \varphi_0$  is the pseudo- $T$ -direction of  $f(z)$ .

Otherwise, there exists  $\varepsilon_0 > 0$  and there is a system  $a_j (j = 1, 2, \dots, q)$  of distinct values and a system  $k_j (j = 1, 2, \dots, q)$  such that  $k_j$  is a positive integer or  $+\infty$  and that

$$\sum_{j=1}^q \left(1 - \frac{1}{k_j + 1}\right) > 2,$$

for any  $j (1 \leq j \leq q)$ , we have

$$\limsup_{r \rightarrow \infty} \frac{N^{k_j}(r, \varphi_0 - \varepsilon_0, \varphi_0 + \varepsilon_0, a_j)}{T(r, f)} = 0.$$

Then there exists a sequence  $\{r_n\}$  such that

$$\lim_{r \rightarrow \infty} \frac{N^{k_j}(r_n, \varphi_0 - \varepsilon_0, \varphi_0 + \varepsilon_0, a_j)}{T(r_n, f)} = 0. \tag{28}$$

For  $\varphi \in (0, \varepsilon_0)$ , similar to (24), we define  $T(\varphi)$ , then  $0 < T(\varphi) \leq 1$ . By Lemma 2.4, for the above sequence  $\{r_n\} \subset (1, +\infty) \setminus E_\delta$  and  $0 < \varphi < \varphi' < \delta$ , we have

$$\left( \sum_{j=1}^q \left(1 - \frac{1}{k_j + 1}\right) - 2 \right) T(r_n, \varphi_0 - \varphi, \varphi_0 + \varphi; f) \leq \sum_{j=1}^q \frac{k_j}{k_j + 1} N^{k_j}(r_n, \varphi_0 - \varepsilon_0, \varphi_0 + \varepsilon_0; a_j) + O((\log r_n)^2)$$



$$+O((T(r_n, \varphi_0 - \varepsilon_0, \varphi_0 + \varepsilon_0; f))^{\frac{1}{2}} \log T(r_n, \varphi_0 - \varepsilon_0, \varphi_0 + \varepsilon_0; f)). \quad (29)$$

By (21), (28), (29) and  $\sum_{j=1}^q (1 - \frac{1}{k_j+1}) > 2$ , we can obtain  $T(\varphi) \leq 0$  which is a contradiction with  $T(\varphi) > 0$ . Therefore  $\Delta(\varphi_0) : argz = \varphi_0$  is the pseudo- $T$ -direction of  $f(z)$ .

Thus, this completes the proof of Theorem 1.2. □

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# Some inequalities on small functions and derivatives of meromorphic functions on annuli \*

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## Abstract

In this paper, we firstly establish the second main theorem about meromorphic functions on annuli concerning small functions. Then, by using this theorem, we deal with the uniqueness of meromorphic functions sharing some small functions on annuli and obtain the results of meromorphic functions sharing five small functions on annuli, which is an answer to the question of Cao and Yi. In addition, we investigate the properties of meromorphic functions on annuli, and obtain a form of Yang's inequality on annuli by reducing the coefficients of Hayman's inequality. Moreover, we also study Hayman's inequality in different forms, and obtain accurate estimates of sums of deficiencies.

**Key words:** Small function, Nevanlinna theory, the annulus.

**Mathematical Subject Classification (2010):** 30D30, 30D35.

## 1 Introduction

Firstly, we always assumed that the reader is familiar with the notations of the Nevanlinna theory such as  $T(r, f)$ ,  $m(r, f)$ ,  $N(r, f)$  and so on (see [6, 22, 23]).

In 1920s, R. Nevanlinna (see [17]) first established the famous Nevanlinna characteristic of meromorphic functions. It is well known that the Nevanlinna characteristic is powerful, and Nevanlinna theory of value distribution play an important role in the research of complex analysis, which has been used to deal with various complex problems, such as: complex differential equation, complex difference equation, uniqueness of meromorphic functions, complex dynamic systems, etc. Among many basic theorems in Nevanlinna theory, the second main theorem is very important to study the value distribution, uniqueness, singular direction, which is listed as follows.

**Theorem 1.1** (see [6, 23]). *Let  $f(z)$  be a non-identically-constant meromorphic function, let  $a_1, \dots, a_q$  be distinct complex numbers, one of which can be equal to  $\infty$ . Then*

$$\sum_{j=1}^q m\left(r, \frac{1}{f - a_j}\right) < 2T(r, f) - N_1(r, f) + S(r, f),$$

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$$(q - 2)T(r, f) < \sum_{j=1}^q \bar{N}(r, \frac{1}{f - a_j}) + S(r, f),$$

where

$$N_1(r, f) = N(r, \frac{1}{f'}) + 2N(r, f) - N(r, f'),$$

and  $S(r, f) = O(\log r)$  as  $r \rightarrow \infty$ , if  $f$  is of finite order,  $S(r, f) = O(\log(rT(r, f)))$  as  $r \rightarrow \infty$ , excluding, possibly, some set of segments in  $[0, \infty)$  with finite total length, if  $f$  is of infinite order.

As a corollary we get the following result about deficiencies

$$\sum_{a \in \bar{\mathbb{C}}} \delta(a, f) \leq 2,$$

where

$$\delta(a, f) = \liminf_{r \rightarrow \infty} \frac{m(r, \frac{1}{f-a})}{T(r, f)} = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, \frac{1}{f-a})}{T(r, f)}.$$

Nevanlinna asked whether Theorem 1.1 is still true when we replace constants  $a_i$  to arbitrary collection of distinct small functions  $a_i(z)$  with respect to  $f$ . This question is very interesting and attracted many investigations (for references, see [5, 18]). In 2004, Yamanoi [21] extended the second main theorem to the case of small functions and obtained the following result

**Theorem 1.2** (see [21]). *Let  $f(z)$  be a non-constant meromorphic function and  $a_1(z), a_2(z), \dots, a_q(z)$  be  $q$  distinct small functions of  $f(z)$ . Then, for all  $\varepsilon > 0$*

$$(q - 2 - \varepsilon)T(r, f) < \sum_{j=1}^q \bar{N}(r, \frac{1}{f - a_j}),$$

as  $r \rightarrow \infty, r \notin E$ , where  $E$  is a subset of  $[0, \infty)$  such that  $E$  is of finite linear measure, and the defect relation:

$$\sum \delta(a(z), f) \leq 2.$$

After their theorems, there are vast references on the value distribution of meromorphic functions in the whole complex plane (see [6, 13, 14, 19, 23]). Moreover, it is an interesting topic how to extend some important results of Nevanlinna value distribution in the complex plane to some subset of the whole complex plane, such as, the unit disc, the angular domain, the annuli. In 2003, J. H. Zheng firstly took into account the value distribution of meromorphic functions in an angular domain and extended the second main theorem in the complex plane to an angular domain (see [24, 25]).

**Theorem 1.3** (see [26, pp.59 and pp.85]). *Let  $f$  be a nonconstant meromorphic functions in an angular domain  $\Omega(\alpha, \beta) = \{z : \alpha < \arg z < \beta\} (0 < \beta - \alpha < 2\pi)$  let  $a_1, \dots, a_q$  be distinct complex numbers, one of which can be equal to  $\infty$ . Then*

$$(q - 2)\mathfrak{T}_{\alpha, \beta}(r, f) < \sum_{j=1}^q \bar{\mathfrak{N}}_{\alpha, \beta}(r, \frac{1}{f - a_j}) + Q_{\alpha, \beta}(r, f),$$

where  $Q_{\alpha, \beta}(r, f) = O(\log r + \log^+ \mathfrak{T}_{\alpha, \beta}(r, f))$  as  $r \rightarrow \infty$ , possibly except some set of  $r$  with finite linear measure.

**Theorem 1.4** (see [26, Theorem 2.3.2]). *Let  $f$  be a nonconstant meromorphic functions in an angular domain  $\Omega(\alpha, \beta) = \{z : \alpha < \arg z < \beta\} (0 < \beta - \alpha < 2\pi)$  let  $a_1, \dots, a_q (q \geq 3)$  be distinct small functions with respect to  $f(z)$ . Then, for any positive number  $\varepsilon$ , we have*

$$(q - 2 - \varepsilon)\mathfrak{T}_{\alpha, \beta}(r, f) < \sum_{j=1}^q \bar{\mathfrak{N}}_{\alpha, \beta}(r, \frac{1}{f - a_j}) + Q_{\alpha, \beta}(r, f).$$

In recent, there have some results on the Nevanlinna Theory of meromorphic functions on annulus (see [8, 9, 11, 15, 16, 20]). In 2005, Khrystyanyyn and Kondratyuk [8, 9] proposed the Nevanlinna theory for meromorphic functions on annuli (see also [10]). Later, the other forms of the second fundamental theorem on annuli were given by Cao, Yi and Xu [3].

**Theorem 1.5** (see [3, Theorem 2.3]). (The second fundamental theorem) Let  $f$  be a nonconstant meromorphic function on the annulus  $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$ , where  $1 < R_0 \leq +\infty$ . Let  $a_1, a_2, \dots, a_q$  be  $q$  distinct complex numbers in the extended complex plane  $\overline{\mathbb{C}}$ . Let  $k_1, k_2, \dots, k_q$  be  $q$  positive integers, and let  $\lambda \geq 0$ . Then

$$(q - 2)T_0(r, f) < \sum_{j=1}^q N_0(r, \frac{1}{f - a_j}) - N_0^{(1)}(r, f) + S_1(r, f),$$

$$(q - 2)T_0(r, f) < \sum_{j=1}^q \overline{N}_0(r, \frac{1}{f - a_j}) + S_1(r, f),$$

where

$$N_0^{(1)}(r, f) = N_0(r, \frac{1}{f'}) + 2N_0(r, f) - N_0(r, f'),$$

and  $S_1(r, f)$  is stated as in Lemma 2.1.

The basic notions of the Nevanlinna theory on annuli will be showed in the next section. Lund and Ye [15] in 2009 studied functions meromorphic on the annuli with the form  $\{z : R_1 < |z| < R_2\}$ , where  $R_1 \geq 0$  and  $R_2 \leq \infty$ . In 2009 and 2011, Cao [2, 3] investigated the uniqueness of meromorphic functions on annuli sharing some values and some sets, and obtained an analog of Nevanlinna's famous five-value theorem. From Theorems 1.1-1.4, we can pose the following question

**Question 1.1** Whether Theorem 1.5 is still true when we replace constants  $a_i$  to arbitrary collection of distinct small functions  $a_i(z)$  with respect to  $f$ ?

In [6], W. K. Hayman obtained the following well-known theorem by investigating the characteristic functions of meromorphic function and its derivative in the complex plane.

**Theorem 1.6** (see [6, Hayman inequality]). Let  $f$  be a transcendental meromorphic function on complex plane, then for any positive integer  $k$ , we have

$$T(r, f) < \left(2 + \frac{1}{k}\right) N\left(r, \frac{1}{f}\right) + \left(2 + \frac{2}{k}\right) N\left(r, \frac{1}{f^{(k)} - 1}\right) + S(r, f).$$

W. K. Hayman [6] gave a question: whether the coefficients of two counting functions  $N(r, \frac{1}{f})$  and  $N(r, \frac{1}{f^{(k)} - 1})$  are best in Theorem 1.1? Hayman's question attracted many investigations (for references, see [23, 27, 4]). In [23], Yang Lo studied the above question and established the well-known Yang Lo's inequality, in which the coefficients of the counting functions is more precise than the ones of Hayman inequality.

**Theorem 1.7** (see [23]). Let  $f$  be a transcendental meromorphic function on the complex plane, then for any  $\varepsilon > 0$  and positive integer  $k$ , we have

$$T(r, f) < \left(1 + \frac{1}{k}\right) N\left(r, \frac{1}{f}\right) + \left(1 + \frac{1}{k}\right) N\left(r, \frac{1}{f^{(k)} - 1}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + \varepsilon T(r, f) + S(r, f).$$

## 2 Conclusions

The main purpose of this paper is to extend Theorem 1.5 to the case of small functions and obtained the following result.

**Theorem 2.1** *Let  $f$  be a nonconstant meromorphic function on the annulus  $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$ , where  $1 < R_0 \leq +\infty$ . let  $a_1, \dots, a_q (q \geq 3)$  be distinct small functions with respect to  $f(z)$ . Then, for any positive number  $\varepsilon$ , we have*

$$(q - 2 - \varepsilon)T_0(r, f) < \sum_{j=1}^q N_0(r, \frac{1}{f - a_j}) + S_1(r, f).$$

From Theorem 2.1, we can get the following result immediately.

**Theorem 2.2** *Let  $f_1$  and  $f_2$  be two transcendental or admissible meromorphic functions on the annulus  $\mathbb{A} = \{z : 0 < |z| < \infty\}$ . Let  $a_j(z) (j = 1, 2, 3, 4, 5)$  be five distinct small functions with respect to  $f_1$  and  $f_2$ . If  $f_1, f_2$  share  $a_j(z)$  CM for  $j = 1, 2, 3, 4, 5$ , then  $f_1(z) \equiv f_2(z)$ .*

The other purpose of this paper is to study the Hayman inequality of meromorphic function on annuli. We obtain:

**Theorem 2.3** *Let  $f$  be a transcendental or admissible meromorphic function on the annulus  $\mathbb{A} = \{z : 0 < |z| < \infty\}$ , then for any  $\varepsilon > 0$  and positive integer  $k$ , we have*

$$T_0(r, f) < \left(1 + \frac{1}{k}\right) N_0\left(r, \frac{1}{f}\right) + \left(1 + \frac{1}{k}\right) N_0\left(r, \frac{1}{f^{(k)} - 1}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + \varepsilon T_0(r, f) + S_1(r, f).$$

Furthermore, when  $a, b$  are two finite complex number,  $a \neq b$  and  $b \neq 0$ . Then we have

$$\delta_0(a, f) + \delta_0^k(b, f^{(k)}) \leq \frac{k + 2}{k + 1}.$$

**Remark 2.1** *For  $a \in \widehat{\mathbb{C}}$ , we define*

$$\delta_0(a, f) = 1 - \limsup_{r \rightarrow \infty} \frac{N_0\left(r, \frac{1}{f-a}\right)}{T_0(r, f)}, \quad \delta_0^k(a, f^{(k)}) = 1 - \limsup_{r \rightarrow \infty} \frac{N_0\left(r, \frac{1}{f^{(k)}-a}\right)}{T_0(r, f)}.$$

**Definition 2.1** *Let  $f(z)$  be a non-constant meromorphic function on the annulus  $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$ , where  $1 < R_0 \leq +\infty$ . The function  $f$  is called a transcendental or admissible meromorphic function on the annulus  $\mathbb{A}$  provided that*

$$\limsup_{r \rightarrow \infty} \frac{T_0(r, f)}{\log r} = \infty, \quad 1 < r < R_0 = +\infty$$

or

$$\limsup_{r \rightarrow R_0} \frac{T_0(r, f)}{-\log(R_0 - r)} = \infty, \quad 1 < r < R_0 < +\infty,$$

respectively.

Moreover, we also investigated other kind of precise inequalities, and obtained accurate estimation of the sum of deficiencies as follows.

**Theorem 2.4** *Let  $f$  be a transcendental or admissible meromorphic function on the annulus  $\mathbb{A} = \{z : 0 < |z| < \infty\}$ , then for any finite complex numbers  $a, b(a \neq b), \varepsilon > 0$  and positive integer  $k$ , we have*

$$T_0(r, f^{(k)}) < \left(1 + \frac{1}{2k}\right) N_0\left(r, \frac{1}{f^{(k)} - a}\right) + \left(1 + \frac{1}{2k}\right) N_0\left(r, \frac{1}{f^{(k)} - b}\right) - N_0\left(r, \frac{1}{f^{(k)}}\right) + \varepsilon T_0(r, f) + S_1(r, f^{(k)}),$$

Furthermore, we have

$$\delta_0(a, f^{(k)}) + \delta_0(b, f^{(k)}) \leq 1 + \frac{1}{2k + 1}.$$

### 3 Preliminaries and some lemmas

Now, we will introduce the basic notations and conclusion about meromorphic functions on annuli (see [8, 9, 10]). From the Doubly Connected Mapping Theorem [1], we can get that each doubly connected domain is conformally equivalent to the annulus  $\{z : r < |z| < R\}$ ,  $0 \leq r < R \leq +\infty$ . For two cases:  $r = 0, R = +\infty$  simultaneously and  $0 < r < R < +\infty$ , the latter case the homothety  $z \mapsto \frac{z}{\sqrt{rR}}$  reduces the given domain to the annulus  $\{z : \frac{1}{R_0} < |z| < R_0\}$ , where  $R_0 = \sqrt{\frac{R}{r}}$ . Thus, every annulus is invariant with respect to the inversion  $z \mapsto \frac{1}{z}$  in two cases.

Let  $f$  be a meromorphic function on the annulus  $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$ , where  $1 < r < R_0 \leq +\infty$ , the Nevanlinna characteristic of  $f$  on the annulus  $\mathbb{A}$  is defined by

$$T_0(r, f) = m_0(r, f) + N_0(r, f).$$

Some basic conclusions and properties of  $T_0(r, f), N_0(r, f), m_0(r, f)$  had been introduced in (see [3, 8, 9, 10]).

In 2005, the lemma on the logarithmic derivative on the the annulus  $\mathbb{A}$  was obtained by Khrystianyn and Kondratyuk [9].

**Lemma 3.1** (see [9, Lemma on the logarithmic derivative]). *Let  $f$  be a nonconstant meromorphic function on the annulus  $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$ , where  $R_0 \leq +\infty$ , and let  $\lambda > 0$ . Then*

$$m_0\left(r, \frac{f'}{f}\right) = S_1(r, f),$$

where (i) in the case  $R_0 = +\infty$ ,

$$S_1(r, *) = O(\log(rT_0(r, *)))$$

for  $r \in (1, +\infty)$  except for the set  $\Delta_r$  such that  $\int_{\Delta_r} r^{\lambda-1} dr < +\infty$ ;

(ii) if  $R_0 < +\infty$ , then

$$S_1(r, *) = O\left(\log\left(\frac{T_0(r, *)}{R_0 - r}\right)\right)$$

for  $r \in (1, R_0)$  except for the set  $\Delta'_r$  such that  $\int_{\Delta'_r} \frac{dr}{(R_0 - r)^{\lambda-1}} < +\infty$ .

**Remark 3.1** *If  $f$  is a transcendental or admissible meromorphic function on the annulus  $\mathbb{A}$ , then  $S_1(r, f) = o(T_0(r, f))$  holds for all  $1 < r < R_0$  except for the set  $\Delta_r$  or the set  $\Delta'_r$  mentioned in Theorem 3.1, respectively.*

By using the same argument as in (Valiron-Mohon'ko) ([12]), we can get the following lemma.

**Lemma 3.2** *Let  $f$  be a nonconstant meromorphic function on the annulus  $\mathbb{A}$ . Then for all irreducible rational functions in  $f$ ,*

$$R(z, f(z)) = \frac{\sum_{i=0}^m a_i(z)f(z)^i}{\sum_{j=0}^n b_j(z)f(z)^j},$$

where meromorphic coefficients  $a_i(z), b_j(z)$  are small functions with respect of  $f$ , then the characteristic function of  $R(z, f(z))$  satisfies that

$$T_0(r, R(z, f(z))) = dT_0(r, f) + S_1(r, f),$$

where  $d = \max\{m, n\}$ .

In order to prove the following lemma, we firstly give the definition of Wronskian determinant of  $a_1(z), \dots, a_p(z)$

$$W(a_1(z), \dots, a_p(z)) = \begin{vmatrix} a_1(z) & a_2(z) & \cdots & a_p(z) \\ a'_1(z) & a'_2(z) & \cdots & a'_p(z) \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{(p-1)}(z) & a_2^{(p-1)}(z) & \cdots & a_p^{(p-1)}(z) \end{vmatrix},$$

where  $a_1(z), \dots, a_p(z)$  are meromorphic in annuli.

**Lemma 3.3** *Let  $f$  be a transcendental or admissible meromorphic function on the annulus  $\mathbb{A} = \{z : 0 < |z| < \infty\}$  and  $\varepsilon > 0$ . Then for  $p = 1, 2, \dots$ , we have*

$$(p-1)\overline{N}_0(r, f) \leq (1+\varepsilon)N_0\left(r, \frac{1}{f^{(p)}}\right) + (1+\varepsilon)N_0^1(r, f) + S_1(r, f), \tag{1}$$

where  $N_0^1(r, f) = N_0(r, f) - \overline{N}_0(r, f)$ .

*Proof:* For any given  $\varepsilon > 0$  and positive integer  $n$ , we choose a positive integer  $n (> \frac{p}{\varepsilon})$ , and consider for all  $z \in \mathbb{A}$ . Let  $W(z) = W(1, z, z^2, \dots, z^{p+n-1}, f, zf, \dots, z^n f)$  as the Wronskian determinant of  $1, z, z^2, \dots, z^{p+n-1}, f, zf, \dots, z^n f$ . Since  $f$  is a transcendental meromorphic function, we can suppose that  $W(z) \neq 0$ . It is easy to see that  $W(z)$  is a homogeneous differential polynomial of degree  $p+1$  in  $f$  with polynomial coefficients of  $z$  and without  $f^{(j)}(z) (j < p)$  in each term of  $W(z)$ .

Let  $B(z) = W(z) \cdot (f^{(p)}(z))^{-n-1}$ , from Lemma 3.1, it follows

$$m_0(r, B) = S_1(r, f).$$

From the first fundamental theorem for meromorphic function on annuli, we have

$$\begin{aligned} N_0\left(r, \frac{1}{B}\right) &\leq T_0(r, B) + O(1) = N_0(r, B) + m_0(r, B) + O(1) \\ &\leq N_0(r, B) + S_1(r, f). \end{aligned} \tag{2}$$

Next, we will estimate the number of zeros and poles of  $B$  on  $\mathbb{A}$ . From the definition of  $W(z)$ , we have

$$W(z) = f^{p+2n+1}W(f^{-1}, zf^{-1}, \dots, z^{p+n-1}f^{-1}, 1, z, \dots, z^n).$$

If  $z_0$  is a pole of  $f$  of order  $t$ , then

$$W(z) = O((z - z_0)^{-t(p+2n+1)}), \quad z \rightarrow z_0.$$

Hence

$$\begin{aligned} B(z) &= O\left((z - z_0)^{(n+1)(p+t) - t(p+2n+1)}\right) \\ &= O\left((z - z_0)^{n(p-1) - (p+n)(t-1)}\right), \end{aligned} \tag{3}$$

as  $z \rightarrow z_0$ .

Let  $\bar{N}_t^0(r)$ ,  $\bar{N}_t^\infty(r)$  and  $\bar{N}_t^*(r)$  be the counting functions for those poles of  $f$  of order  $t$  on  $\mathbb{A}$ , where  $B(z)$  has a zero, pole or finite nonzero value, respectively, each pole being counted only once. From (2) and (3), we get

$$\begin{aligned} \sum_{t=1}^{\infty} (n(p-1) - (p+n)(t-1)) \bar{N}_p^0(r) &\leq N_0(r, \frac{1}{B}) \leq N_0(r, B) + S_1(r, f) \\ &\leq \sum_{t=1}^{\infty} ((p+n)(t-1) - n(p-1)) \bar{N}_t^\infty(r) \\ &\quad + (n+1)N_0\left(r, \frac{1}{f^{(p)}}\right) + S_1(r, f). \end{aligned} \tag{4}$$

If a pole of  $f$  contributes to  $\bar{N}_t^*(r)$ , then from (3) it follows  $n(p-1) - (p+n)(t-1) \leq 0$  and

$$n(p-1) \bar{N}_t^*(r) \leq (p+n)(t-1) \bar{N}_t^*(r).$$

Summing for  $t = 1, 2, \dots$  in above and substituting to (4), we obtain

$$n(p-1) \sum_{t=1}^{\infty} \bar{N}_t(r) \leq (p+n) \sum_{t=1}^{\infty} (t-1) \bar{N}_t(r) + (n+1)N_0\left(r, \frac{1}{f^{(p)}}\right) + S_1(r, f), \tag{5}$$

where  $\bar{N}_t(r) = \bar{N}_t^0(r) + \bar{N}_t^\infty(r) + \bar{N}_t^*(r)$ .

Noting

$$\begin{aligned} \sum_{t=1}^{\infty} (t-1) \bar{N}_t(r) &= \sum_{t=1}^{\infty} [t \bar{N}_t(r) - \bar{N}_t(r)] \\ &= \sum_{t=1}^{\infty} [N_t(r) - \bar{N}_t(r)] = N_0(r, f) - \bar{N}_0(r, f), \end{aligned}$$

since  $n > \frac{p}{\varepsilon}$  and (5), we have proved Lemma 3.3. □

By using the same argument as in the proof of Lemma 3.3, we can get the following lemma.

**Lemma 3.4** *Let  $f(z)$  and  $a_j(z) (j = 1, 2, \dots, p; p \geq 3)$  be stated as in Theorem 2.1. Set  $W(f) = W(a_1(z), a_2(z), \dots, a_p(z), f(z))$ . If  $a_j(z) (j = 1, 2, \dots, p; p \geq 3)$  are linearly independent, then for  $\varepsilon > 0$ , we have*

$$p \bar{N}_0(r, f) \leq N_0\left(r, \frac{1}{W(f)}\right) + (1 + \varepsilon)N_0(r, f) + S_1(r, f).$$

**Lemma 3.5** *Let  $f$  be a transcendental or admissible meromorphic function on the annulus  $\mathbb{A} = \{z : 0 < |z| < \infty\}$ . Then for any  $\varepsilon > 0$  and positive integer  $k$ , we have*

$$\bar{N}_0(r, f) < \frac{1}{k} N_0\left(r, \frac{1}{f^{(k)}}\right) + \frac{1}{k} N_0(r, f) + \varepsilon T_0(r, f) + S_1(r, f). \tag{6}$$

*Proof:* Replacing  $\varepsilon$  with  $\frac{\varepsilon}{3}$  in Lemma 3.3, it follows

$$\begin{aligned} \bar{N}_0(r, f) &< \frac{1}{k} N_0\left(r, \frac{1}{f^{(k)}}\right) + \frac{1}{k} N_0(r, f) + \frac{\varepsilon}{3k} N_0\left(r, \frac{1}{f^{(k)}}\right) \\ &\quad + \frac{\varepsilon}{3k} N_0(r, f) + S_1(r, f). \end{aligned} \tag{7}$$



Since

$$\begin{aligned} N_0\left(r, \frac{1}{f^{(k)}}\right) &\leq T_0(r, f^{(k)}) + O(1) \\ &\leq m_0\left(r, \frac{f^{(k)}}{f}\right) + m_0(r, f) + N_0(r, f^{(k)}) + O(1) \\ &\leq m_0(r, f) + N_0(r, f) + k\bar{N}_0(r, f) + S_1(r, f) \\ &\leq (k + 1)T_0(r, f) + S_1(r, f), \end{aligned}$$

from (7), we have

$$\begin{aligned} \frac{\varepsilon}{3k}N_0\left(r, \frac{1}{f^{(k)}}\right) + \frac{\varepsilon}{3k}N_0(r, f) &\leq \frac{k+2}{3k}\varepsilon T_0(r, f) + S_1(r, f) \\ &\leq \varepsilon T_0(r, f) + S_1(r, f). \end{aligned} \tag{8}$$

From (7) and (8), we get (6) easily. □

**Lemma 3.6** (see [7, Theorem 2]). *Let  $f$  be a transcendental or admissible meromorphic function on the annulus  $\mathbb{A} = \{z : 0 < |z| < \infty\}$ , and  $k$  be a positive integers. Then*

$$T_0(r, f) < \bar{N}_0(r, f) + N_0\left(r, \frac{1}{f}\right) + N_0\left(r, \frac{1}{f^{(k)} - 1}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + S_1(r, f).$$

**Lemma 3.7** *Let  $f$  be a transcendental or admissible meromorphic function on the annulus  $\mathbb{A} = \{z : 0 < |z| < \infty\}$ . Then for any  $\varepsilon > 0$  and positive integer  $k$ , we have*

$$\bar{N}_0(r, f) < \frac{1}{k}N_0\left(r, \frac{1}{f}\right) + \frac{1}{k}N_0\left(r, \frac{1}{f^{(k)} - 1}\right) + \varepsilon T_0(r, f) + S_1(r, f). \tag{9}$$

*Proof:* From Lemma 3.5 we have

$$N_0\left(r, \frac{1}{f^{(k+1)}}\right) > (k + 1)\bar{N}_0(r, f) - N_0(r, f) - \frac{k+1}{2}\varepsilon T_0(r, f) - S_1(r, f).$$

Substituting the above inequality back into Lemma 3.6, we obtain

$$\begin{aligned} k\bar{N}_0(r, f) &< N_0\left(r, \frac{1}{f}\right) + N_0\left(r, \frac{1}{f^{(k)} - 1}\right) + (N_0(r, f) - T_0(r, f)) \\ &\quad + \frac{k+1}{2}\varepsilon T_0(r, f) + S_1(r, f). \end{aligned}$$

Therefore

$$\begin{aligned} \bar{N}_0(r, f) &< \frac{1}{k}N_0\left(r, \frac{1}{f}\right) + \frac{1}{k}N_0\left(r, \frac{1}{f^{(k)} - 1}\right) + \frac{k+1}{2k}\varepsilon T_0(r, f) + S_1(r, f) \\ &< \frac{1}{k}N_0\left(r, \frac{1}{f}\right) + \frac{1}{k}N_0\left(r, \frac{1}{f^{(k)} - 1}\right) + \varepsilon' T_0(r, f) + S_1(r, f). \end{aligned}$$

Thus, this completes the proof of Lemma 3.7. □

From Theorem 1.5, we get the following conclusion easily.

**Lemma 3.8** *Let  $f$  be a transcendental or admissible meromorphic function on the annulus  $\mathbb{A} = \{z : 0 < |z| < \infty\}$ . Then for any finite complex number  $a, b(a \neq b)$ , we have*

$$T_0(r, f) \leq N_0(r, f) + N_0\left(r, \frac{1}{f-a}\right) + N_0\left(r, \frac{1}{f-b}\right) - N_0^0(r) + S_1(r, f),$$

where  $N_0^0(r) = 2N_0(r, f) - N_0(r, f') + N_0\left(r, \frac{1}{f'}\right)$ .

**Lemma 3.9** *Let  $f$  be a transcendental or admissible meromorphic functions on the annulus  $\mathbb{A} = \{z : 0 < |z| < \infty\}$ . Then for any finite complex numbers  $a, b(a \neq b)$ , positive number  $\varepsilon > 0$  and positive integer  $k$ , we have*

$$\bar{N}_0(r, f) < \frac{1}{2k}N_0\left(r, \frac{1}{f^{(k)} - a}\right) + \frac{1}{2k}N_0\left(r, \frac{1}{f^{(k)} - b}\right) + \varepsilon T_0(r, f) + S_1(r, f).$$

*Proof:* By using Lemma 3.8 for  $f^{(k)}$  and three distinct complex numbers  $a, b, \infty$ , we have

$$\begin{aligned} T_0\left(r, f^{(k)}\right) &\leq N_0\left(r, f^{(k)}\right) + N_0\left(r, \frac{1}{f^{(k)} - a}\right) + N_0\left(r, \frac{1}{f^{(k)} - b}\right) \\ &\quad - N_0^0(r) + S_1\left(r, f^{(k)}\right), \end{aligned}$$

where  $N_0^0(r) = 2N_0\left(r, f^{(k)}\right) - N_0\left(r, f^{(k+1)}\right) + N_0\left(r, \frac{1}{f^{(k+1)}}\right)$ .

Thus, we get

$$\begin{aligned} T_0\left(r, f^{(k)}\right) &\leq \bar{N}_0(r, f) + N_0\left(r, \frac{1}{f^{(k)} - a}\right) + N_0\left(r, \frac{1}{f^{(k)} - b}\right) \\ &\quad - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + S_1\left(r, f^{(k)}\right). \end{aligned} \tag{10}$$

Since  $T_0(r, f^{(k)}) = m_0(r, f^{(k)}) + N_0(r, f) + k\bar{N}_0(r, f)$ , then by applying Lemma 3.7 for  $f^{(k+1)}$ , it follows

$$N_0\left(r, \frac{1}{f^{(k+1)}}\right) > (k+1)\bar{N}_0(r, f) - N_0(r, f) - (k+1)\varepsilon T_0(r, f) - (k+1)S_1(r, f).$$

Substituting the two above inequalities back into (10), we get

$$\begin{aligned} \bar{N}_0(r, f) &< \frac{1}{2k}N_0\left(r, \frac{1}{f^{(k)} - a}\right) + \frac{1}{2k}N_0\left(r, \frac{1}{f^{(k)} - b}\right) + \frac{k+1}{2k}\varepsilon T_0(r, f) \\ &\quad + \frac{k+2}{2k}S_1(r, f^{(k)}) \\ &< \frac{1}{2k}N_0\left(r, \frac{1}{f^{(k)} - a}\right) + \frac{1}{2k}N_0\left(r, \frac{1}{f^{(k)} - b}\right) + \varepsilon' T_0(r, f) \\ &\quad + S_1(r, f^{(k)}). \end{aligned}$$

From the definition of  $S_1(r, *)$  and  $T_0(r, f) \leq T_0(r, f^{(k)}) \leq (k+1)T_0(r, f) + S_1(r, f)$ , where  $S_1(r, f)$  is as stated in Lemma 3.1, we can get the conclusion of Lemma 3.9.

Thus, we can complete the proof of Lemma 3.9. □

## 4 Proofs of Theorem 2.1 and Theorem 2.2

### 4.1 The proof of Theorem 2.1

Without any loss of generalities, suppose that  $\{a_1(z), a_2(z), \dots, a_p(z)\}$  is a maximum linearly independent subset of  $a_j(z) (j = 1, 2, \dots, q)$ , then  $p \leq q$  and each  $a_j(z) (j = 1, 2, \dots, q)$  can be linearly expressed in terms of  $a_j(z) (j = 1, 2, \dots, p)$ . Set  $W(f) = W(a_1, a_2, \dots, a_p(z), f)$ , then

$$W(f) = b_p f^{(p)} + b_{p-1} f^{(p-1)} + \dots + b_1 f' + b_0 f, \tag{11}$$

where  $b_j (j = 1, 2, \dots, p)$  are small functions with respect to  $f$ . It follows from (11) that

$$N_0(r, W(f)) = p\bar{N}_0(r, f) + N_0(r, f) + S_1(r, f) \tag{12}$$

and

$$m_0(r, W(f)) \leq m_0(r, f) + m_0(r, \frac{W(f)}{f}) = m_0(r, f) + S_1(r, f). \tag{13}$$

Thus from (12), (13) it follows that

$$T_0(r, W(f)) \leq p\bar{N}_0(r, f) + T_0(r, f) + S_1(r, f). \tag{14}$$

From the definition of  $W(f)$ , we have  $W(f - a_j) = W(f)$  for  $j = 1, 2, \dots, q$ . Thus, it follows by Lemma 3.1 that

$$m_0(r, \frac{W(f)}{f - a_j}) = m_0(r, \frac{W(f - a_j)}{f - a_j}) = S_1(r, f). \tag{15}$$

Set

$$F(z) = \sum_{j=1}^q \frac{1}{f(z) - a_j(z)}.$$

Then it follows from (14),(15) and by Lemma 3.4 that

$$\begin{aligned} m_0(r, F) &\leq m_0(r, \frac{1}{W(f)}) + m_0(r, FW(f)) \\ &\leq T_0(r, W(f)) - N_0(r, \frac{1}{W(f)}) + S_1(r, f) \\ &\leq p\bar{N}_0(r, f) + T_0(r, f) - N_0(r, \frac{1}{W(f)}) + S_1(r, f) \\ &\leq T_0(r, f) + (1 + \varepsilon)N_0(r, f) + S_1(r, f). \end{aligned}$$

Then it follows by Lemma 3.2 that

$$\begin{aligned} qT_0(r, f) &= T_0(r, F) + S_1(r, f) \\ &\leq \sum_{j=1}^q N_0(r, \frac{1}{f - a_j}) + T_0(r, f) + (1 + \varepsilon)N_0(r, f) + S_1(r, f) \\ &\leq \sum_{j=1}^q N_0(r, \frac{1}{f - a_j}) + (2 + \varepsilon)T_0(r, f) + S_1(r, f). \end{aligned}$$

Hence, this completes the proof of Theorem 2.1.

### 4.2 The proof of Theorem 2.2

Suppose  $f(z) \not\equiv g(z)$ . By applying Theorem 2.1, since  $f$  and  $g$  share  $a_1, \dots, a_5$  CM, we have

$$\begin{aligned} (3 - \varepsilon)T_0(r, f) &\leq \sum_{j=1}^5 N_0(r, \frac{1}{f - a_j}) + S_1(r, f) \leq N_0(r, \frac{1}{f - g}) + S_1(r, f) \\ &\leq T_0(r, f) + T_0(r, g) + S_1(r, f), \end{aligned}$$

that is,

$$(2 - \varepsilon)T_0(r, f) \leq T_0(r, g) + S_1(r, f). \tag{16}$$

Similarly, we have

$$(2 - \varepsilon)T_0(r, g) \leq T_0(r, f) + S_1(r, g). \tag{17}$$

Thus for any small number  $\varepsilon(> 0)$ , it follows from (16) and (17) that

$$(1 - \varepsilon)[T_0(r, f) + T_0(r, g)] \leq S_1(r, f) + S_1(r, g),$$

which is a contradiction with the assumption that  $f, g$  are transcendental or admission.

Therefore, we have  $f \equiv g$ .

## 5 Proofs of Theorem 2.3 and Theorem 3.4

### 5.1 The proof of Theorem 2.3

From Lemma 3.4 and Lemma 3.7, we get

$$T_0(r, f) < \left(1 + \frac{1}{k}\right) N_0\left(r, \frac{1}{f}\right) + \left(1 + \frac{1}{k}\right) N_0\left(r, \frac{1}{f^{(k)} - 1}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + \varepsilon T_0(r, f) + S_1(r, f).$$

Now, we will prove the inequality of the sum of deficiencies as follows. First, by using the above inequality for the function  $\frac{f-a}{b}$ , then it follows

$$\begin{aligned} T_0(r, f) &< \left(1 + \frac{1}{k}\right) N_0\left(r, \frac{1}{f-a}\right) + \left(1 + \frac{1}{k}\right) N_0\left(r, \frac{1}{f^{(k)} - b}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) \\ &\quad + \varepsilon T_0(r, f) + S_1(r, f) \\ &< \left(1 + \frac{1}{k}\right) N_0\left(r, \frac{1}{f-a}\right) + \left(1 + \frac{1}{k}\right) N_0\left(r, \frac{1}{f^{(k)} - b}\right) + \varepsilon T_0(r, f) + S_1(r, f). \end{aligned}$$

Dividing the both sides of the above inequality by  $T_0(r, f)$ , we have

$$\left(1 + \frac{1}{k}\right) \left(1 - \frac{N_0\left(r, \frac{1}{f-a}\right)}{T_0(r, f)} + 1 - \frac{N_0\left(r, \frac{1}{f^{(k)} - b}\right)}{T_0(r, f)}\right) < 1 + \frac{2}{k} + \varepsilon + \frac{S_1(r, f)}{T_0(r, f)}. \tag{18}$$

From the definitions of  $\delta_0(a, f), \delta_0^k(b, f^{(k)})$ , then it follows from (18) that

$$\begin{aligned} &\left(1 + \frac{1}{k}\right) \left(\delta_0(a, f) + \delta_0^k(b, f^{(k)})\right) \\ &\leq \left(1 + \frac{1}{k}\right) \liminf_{r \rightarrow \infty} \left(1 - \frac{N_0\left(r, \frac{1}{f-a}\right)}{T_0(r, f)} + 1 - \frac{N_0\left(r, \frac{1}{f^{(k)} - b}\right)}{T_0(r, f)}\right) \\ &\leq \limsup_{r \rightarrow \infty} \left(1 + \frac{2}{k} + \varepsilon\right) + \liminf_{r \rightarrow \infty} \frac{S_1(r, f)}{T_0(r, f)}. \end{aligned}$$

Since  $f$  is a transcendental or admission on annuli, we have

$$\lim_{r \rightarrow \infty} \frac{S_1(r, f)}{T_0(r, f)} = 0. \tag{19}$$

Hence,

$$\delta_0(a, f) + \delta_0^k(b, f^{(k)}) \leq \frac{k+2}{k+1}.$$

Thus, this completes the proof of Theorem 2.3.

### 5.2 The proof of Theorem 2.4

By Lemma 3.9, it follows from (10) that

$$\begin{aligned} T_0\left(r, f^{(k)}\right) &\leq \left(1 + \frac{1}{2k}\right) N_0\left(r, \frac{1}{f^{(k)} - a}\right) + \left(1 + \frac{1}{2k}\right) N_0\left(r, \frac{1}{f^{(k)} - b}\right) \\ &\quad - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + \varepsilon T_0(r, f) + S_1\left(r, f^{(k)}\right). \end{aligned}$$

The above inequality implies

$$\left(1 + \frac{1}{2k}\right) \left(2 - \frac{N_0\left(r, \frac{1}{f^{(k)}-a}\right)}{T_0(r, f^{(k)})} - \frac{N_0\left(r, \frac{1}{f^{(k)}-b}\right)}{T_0(r, f^{(k)})}\right) < 1 + \frac{1}{k} + \varepsilon + \frac{S_1(r, f^{(k)})}{T_0(r, f^{(k)})}. \tag{20}$$

Thus, it follows from (20) and the definition of  $\delta_{\alpha,\beta}(a, f)$  that

$$\begin{aligned} & \left(1 + \frac{1}{2k}\right) \left(\delta_0(a, f^{(k)}) + \delta_0(b, f^{(k)})\right) \\ & \leq \left(1 + \frac{1}{2k}\right) \liminf_{r \rightarrow \infty} \left(1 - \frac{N_0\left(r, \frac{1}{f^{(k)}-a}\right)}{T_0(r, f^{(k)})} + 1 - \frac{N_0\left(r, \frac{1}{f^{(k)}-b}\right)}{T_0(r, f^{(k)})}\right) \\ & \leq \limsup_{r \rightarrow \infty} \left(1 + \frac{1}{k} + \varepsilon\right) + \liminf_{r \rightarrow \infty} \frac{S_1(r, f^{(k)})}{T_0(r, f^{(k)})}. \end{aligned}$$

Since  $f$  is a transcendental or admission on annuli, we have the following equalities easily

$$\liminf_{r \rightarrow \infty} \frac{S_1(r, f^{(k)})}{T_0(r, f^{(k)})} = 0. \tag{21}$$

Since  $\varepsilon$  is arbitrary, it follows from (21) that

$$\delta_0(a, f^{(k)}) + \delta_0(b, f^{(k)}) \leq 1 + \frac{1}{2k+1}.$$

Therefore, we complete the proof of Theorem 2.4.

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**WEIGHTED COMPOSITION OPERATORS BETWEEN  
WEIGHTED HILBERTIAN BERGMAN SPACES IN THE  
UNIT POLYDISK**

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ABSTRACT. In this paper, we prove that the topology spaces of nonzero weighted composition operators acting on some Hilbert spaces of holomorphic functions in the unit polydisk are path connected, which generalized Hosokawa, Izuchi and Ohno's results in single complex variables' case [9].

KEYWORDS: Weighted Hilbertian Bergman spaces, weighted composition operator, polydisk, norm topology, Hilbert-Schmidt topology.

1. INTRODUCTION

Let  $H(\mathbb{D}^N)$  be the space of analytic functions on the open unit polydisk

$$\mathbb{D}^N := \{z = (z_1, \dots, z_N) \in \mathbb{C}^N : |z_i| < 1, i = 1, 2, \dots, N\}$$

and  $H^\infty$  the space of bounded analytic functions on  $\mathbb{D}^N$  with the supremum norm  $\|\cdot\|_\infty$ . When  $N = 1$ , the unit polydisk reduces to the unit open disc  $\mathbb{D}$  in the complex plane  $\mathbb{C}$ . Let  $\mathcal{S}(\mathbb{D}^N)$  be the set of analytic self-maps of  $\mathbb{D}^N$ . Every  $\varphi = (\varphi_1, \dots, \varphi_N) \in \mathcal{S}(\mathbb{D}^N)$  induces the composition operator  $C_\varphi$  defined by  $C_\varphi f = f \circ \varphi$  for  $f \in H(\mathbb{D}^N)$ . If  $u \in H(\mathbb{D}^N)$ , the multiplication,  $M_u : H(\mathbb{D}^N) \rightarrow H(\mathbb{D}^N)$ , is defined by

$$M_u(f)(z) = u(z) \cdot f(z)$$

for any  $f \in H(\mathbb{D}^N)$  and  $z \in \mathbb{D}^N$ . If  $u \in H(\mathbb{D}^N)$  and  $\varphi \in \mathcal{S}(\mathbb{D}^N)$ , we call the operator  $M_u C_\varphi$  to be the weighted composition operator.

Much effort has been expended on characterizing those analytic maps which induce bounded or compact composition operators between those classic spaces of analytic functions. Readers interested in this topic can refer to the books [16] by Shapiro, [7] by Cowen and MacCluer, and [23, 24] by Zhu.

An active topic is the topological structure of the space of composition operators acting on function spaces. If  $X$  is a Banach space of analytic functions,

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we employ the symbol  $\mathcal{C}(X)$  to represent the space of composition operators on  $X$  equipped with the operator norm topology. In 1981, Berkson [3] firstly studied the topological structure of  $\mathcal{C}(H^2(\mathbb{D}))$ . Central problem focuses on both the structure of  $\mathcal{C}(H^2(\mathbb{D}))$  and the compact differences of its members.

In 1989, MacCluer [12] showed that, on the weighted Bergman space  $A_s^2(\mathbb{D})$  for  $s \geq -1$ , all the compact composition operators can be connected by paths, and she gave necessary conditions for two composition operators to have compact difference. At about the same time, Shapiro and Sundberg [17] gave further results on compact difference and isolation and they believe that the compact composition operators should form a connected component of the set  $\mathcal{C}(H^2(\mathbb{D}))$ .

In 2008, Gallardo-Gutiérrez and co-workers [8] showed that there exists noncompact composition operators in the component generated by all compact composition operators. In 2005, Moorhouse [13] characterized compact difference for composition operators acting on  $A_\lambda^2(\mathbb{D})$ ,  $\lambda > -1$ , and gave a partial answer to the component structure of  $\mathcal{C}(A_\lambda^2(\mathbb{D}))$ . Later, Kriete and Moorhouse [11] extended that results to linear combinations. In 2012, Saukko [14, 15] obtained a complete characterization of bounded and compact differences between standard weighted Bergman spaces. Recently, Choe, Koo and Park [5, 6] extend Moorhouses characterization to the Bergman spaces in unit polydisk and unit ball. In 2015, Hosokawa, Izuchi and Ohno [9] investigate the topology space of weighted composition operators acting between some Hilbert spaces on  $\mathbb{D}$  in general, and they also consider the Hilbert-Schmidt norm topology. Readers interested in those related topic can refer recent papers [19, 20, 21, 22] and the references therein.

Generally speaking, theory of composition operators on the spaces of holomorphic functions in the unit polydisk are far from complete. To completely characterize the boundedness and compactness of composition operators on Hardy spaces and weighted Bergman spaces is still open. In [2], Bayart showed that the study of boundedness of composition operators on the polydisk is a difficult problem, and many obstacles are caused by differences between the torus of  $\mathbb{D}^N$  (distinguishing boundary of  $\mathbb{D}^N$ ) and the whole boundary. Stessin and Zhu [18] characterized the boundedness of composition operators between different weighted Bergman spaces in the polydisk. Inspired by [2, 9, 18], we continue to investigate the topology spaces of weighted composition operators between different weighted Bergman spaces in the unit polydisk. On those spaces, we will also consider the Hilbert Schmidt topology spaces of weighed composition operators.

## 2. PRELIMINARIES

Let  $dA(z) = dx dy / \pi$  denote the normalized area measure of  $\mathbb{D}$ . For  $s > -1$  the weighted Hilbertian Bergman space  $A_s^2(\mathbb{D}^N)$  consists of all functions



$f \in H(\mathbb{D}^N)$  such that

$$\|f\|_s^2 = \int_{\mathbb{D}^N} |f(z)|^2 dv_s(z) < \infty,$$

where  $dv_s(z) = dA_s(z_1) \cdots dA_s(z_N)$  and  $dA_s(z_j) = (s+1)(1-|z_j|^2)^s dA(z_j)$ .  
The inner product of  $A_s^2(\mathbb{D}^N)$  is given by

$$\langle f, g \rangle_s = \int_{\mathbb{D}^N} f(z) \overline{g(z)} dv_s(z),$$

where  $f, g \in A_s^2$ . And the reproducing kernel of  $A_s^2(\mathbb{D}^N)$  is given by

$$k_{A_s^2, w}(z) = \prod_{j=1}^N \frac{1}{(1-\bar{w}_j z_j)^{2+s}}.$$

When  $s = 0$ ,  $A_0^2(\mathbb{D}^N)$  is the classical Bergman space. It is well known that

$$\|z^\alpha\|_s^2 = \int_{\mathbb{D}^N} |z^\alpha|^2 dv_s(z) = \prod_{j=1}^N \int_{\mathbb{D}} |z_j^{\alpha_j}|^2 dA_s(z_j) = \Gamma^N(s+1) \prod_{j=1}^N \frac{\Gamma(\alpha_j+1)}{\Gamma(\alpha_j+s+2)}$$

where  $\alpha = (\alpha_1, \dots, \alpha_N)$  and  $z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N}$ .

We believe the following pointwise estimate is well known, and we list it with a simple proof for the completeness.

**Lemma 2.1.** *Let  $p > 0$  and  $s > -1$ . If  $f \in A_s^p(\mathbb{D}^N)$ , then*

$$|f(w)| \leq \frac{\|f\|_s}{\prod_{i=1}^N (1-|w_i|^2)^{\frac{2+\alpha}{p}}}$$

for each  $w = (w_1, \dots, w_N) \in \mathbb{D}^N$ .

*Proof.* Fixing  $w_2, \dots, w_N$ , function  $f(\zeta_1, w_2, \dots, w_N)$  is analytic with respect to  $\zeta_1 \in \mathbb{D}$ . It is well known that

$$\begin{aligned} |f(w_1, w_2, \dots, w_N)|^p &\leq \frac{\|f(\zeta_1, w_2, \dots, w_N)\|_s^p}{(1-|w_1|^2)^{2+s}} \\ &= \frac{\int_{\mathbb{D}} |f(\zeta_1, w_2, \dots, w_N)|^p dA_s(\zeta_1)}{(1-|w_1|^2)^{2+s}}. \end{aligned}$$

We can also estimate  $|f(\zeta_1, w_2, \dots, w_N)|^p$  by the similar inequality with respect to  $w_2$ , and then up to  $w_N$ , hence we have

$$\begin{aligned} & |f(w_1, w_2, \dots, w_N)|^p \\ \leq & \frac{\int_{\mathbb{D}} |f(\zeta_1, w_2, \dots, w_N)|^p dA_s(\zeta_1)}{(1 - |w_1|^2)^{2+s}} \\ \leq & \frac{\int_{\mathbb{D}} \int_{\mathbb{D}} |f(\zeta_1, \zeta_2, \dots, w_N)|^p dA_s(\zeta_2) dA_s(\zeta_1)}{(1 - |w_1|^2)^{2+s} (1 - |w_2|^2)^{2+s}} \\ \dots & \\ \leq & \frac{\int_{\mathbb{D}} \dots \int_{\mathbb{D}} |f(\zeta_1, \zeta_2, \dots, \zeta_N)|^p dA_s(\zeta_N) \dots dA_s(\zeta_2) dA_s(\zeta_1)}{(1 - |w_1|^2)^{2+s} \dots (1 - |w_N|^2)^{2+s}} \end{aligned}$$

That is

$$|f(w)| \leq \frac{\|f\|_s}{\prod_{i=1}^N (1 - |w_i|^2)^{\frac{2+\alpha}{p}}}.$$

□

It is an obvious consequence that the point evaluation  $\epsilon_w : f \mapsto f(w) = f(w_1, \dots, w_N)$  is a bounded linear functional on  $A_s^p(\mathbb{D}^N)$ , and

$$\max_{i=1, \dots, N} \sup_{|w_i| \leq r} \|\epsilon_w\|_{A_s^p} < \infty$$

for every  $0 < r < 1$ .

If  $-1 < s' < s$ , we can immediately have that  $A_{s'}^2 \subset A_s^2$  by a direct computation  $\|f\|_s \leq C\|f\|_{s'}$  where the constant  $C$  depends only on  $s$  and  $s'$ . Let  $\mathcal{C}_w(A_{s'}^2, A_s^2)$  be the space of nonzero bounded weighted composition operators from a weighted Hilbertian Bergman space  $A_{s'}^2$  to another  $A_s^2$  with the operator norm topology, that is,

$$\mathcal{C}_w(A_{s'}^2, A_s^2) = \{M_u C_\varphi : M_u C_\varphi : A_{s'}^2 \rightarrow A_s^2 \text{ is bounded, } u \neq 0\}.$$

And  $\mathcal{C}_w(A_s^2) = \mathcal{C}_w(A_s^2, A_s^2)$ . For a bounded linear operator  $T : X' \rightarrow X$ , we write  $\|T\|_{X', X}$  its operator norm. If  $-1 < s' < s$ , for  $M_u C_\varphi \in \mathcal{C}_w(A_{s'}^2)$ , we have

$$\|M_u C_\varphi f\|_{A_s^2} \leq \|M_u C_\varphi\|_{A_s^2} \|f\|_s \leq C \cdot \|M_u C_\varphi\|_{A_s^2} \|f\|_{s'}$$

for every  $f \in A_{s'}^2$ . Hence  $M_u C_\varphi : A_{s'}^2 \rightarrow A_s^2$  is bounded and

$$(2.1) \quad \|M_u C_\varphi\|_{A_{s'}^2, A_s^2} \leq \|M_u C_\varphi\|_{A_s^2} \quad \text{for every } M_u C_\varphi \in \mathcal{C}_w(A_{s'}^2).$$

Restricting  $M_u C_\varphi \in \mathcal{C}_w(A_s^2)$  on  $A_{s'}^2$ , we may consider that  $M_u C_\varphi$  is also a bounded linear operator from  $A_{s'}^2$  to  $A_s^2$ . We note that

$$\mathcal{C}_w(A_{s'}^2, A_s^2) = \mathcal{C}_w(A_s^2)$$

as sets, and if  $M_u C_\varphi \in \mathcal{C}_w(A_{s'}^2, A_s^2)$ , then  $u \in A_s^2$ .

3. OPERATOR NORM TOPOLOGY SPACES

Now we discuss the operator norm topology spaces  $C_w(A_s^2, A_s^2)$ .

**Lemma 3.1.** *If  $\varphi = (\varphi_1, \dots, \varphi_N) \in \mathcal{S}(\mathbb{D}^N)$  and  $\|\varphi\|_\infty := \max_{i=1, \dots, N} \|\varphi_i\|_\infty < 1$ , then  $C_\varphi f \in H^\infty$  for every  $f \in A_s^2$  and*

$$\|C_\varphi f\|_\infty \leq \|f\|_s \max_{i=1, \dots, N} \sup_{|w_i| \leq \|\varphi\|_\infty} \|\epsilon_w\|_{A_s^2}.$$

*Proof.* For  $f \in \mathcal{H}$  and  $z \in \mathbb{D}^N$ , we have

$$|(C_\varphi f)(z)| = |f(\varphi(z))| \leq \|f\|_s \|\epsilon_{\varphi(z)}\|_{A_s^2} \leq \|f\|_s \max_{i=1, \dots, N} \sup_{|w_i| \leq \|\varphi\|_\infty} \|\epsilon_w\|_{A_s^2},$$

so we get the assertion. □

The main result is following.

**Theorem 3.2.** *If  $-1 < s' < s$ , then the space  $C_w(A_{s'}^2, A_s^2)$  is path connected.*

*Proof.* Let  $M_u C_\varphi \in C_w(A_{s'}^2, A_s^2)$ . Since  $C_w(A_{s'}^2, A_s^2) = C_w(A_s^2)$  as sets, we have  $u \in A_s^2$  and  $\|M_u C_\varphi\|_{A_{s'}^2, A_s^2} < \infty$ . Let  $0 \leq r < 1$ . For  $f \in A_s^2$ , by Lemma 3.1 we have  $f \circ r\varphi = f(r\varphi_1, \dots, r\varphi_N) \in H^\infty$  and by Lemma 2.1,

$$\begin{aligned} \|M_u C_{r\varphi} f\|_s &= \|u(f \circ r\varphi)\|_s \leq \|f \circ r\varphi\|_\infty \|u\|_s \\ &\leq \|u\|_s \|f\|_s \max_{i=1, \dots, N} \sup_{|w_i| \leq r} \|\epsilon_w\|_{A_s^2}. \end{aligned}$$

Hence  $M_u C_{r\varphi} \in C_w(A_s^2)$ , so  $M_u C_{r\varphi} \in C_w(A_{s'}^2, A_s^2)$ .

Fixing  $0 \leq t_0 \leq 1$ , we apply the similar method in [9] to show that  $\|M_u C_{t_0\varphi} - M_u C_{t\varphi}\|_{A_{s'}^2, A_s^2} \rightarrow 0$  as  $t \rightarrow t_0$ . Let  $g(z) = \sum_\alpha c_\alpha z^\alpha \in A_{s'}^2$ . For each  $0 \leq t \leq 1$ , let

$$g_t(z) = \sum_\alpha c_\alpha (t_0^{|\alpha|} - t^{|\alpha|}) z^\alpha.$$

Since  $A_{s'}^2 \subset A_s^2$ , we have  $g \in A_s^2$ . Note that

$$\begin{aligned} \|g_t\|_s^2 &= \langle g_t, g_t \rangle_s = \int_{\mathbb{D}^N} \sum_\alpha c_\alpha (t_0^{|\alpha|} - t^{|\alpha|}) z^\alpha \overline{\sum_\alpha c_\alpha (t_0^{|\alpha|} - t^{|\alpha|}) z^\alpha} dv_s(z) \\ &= \sum_\alpha (t_0^{|\alpha|} - t^{|\alpha|})^2 \int_{\mathbb{D}^N} |c_\alpha|^2 |z|^\alpha dv_s(z) \\ &\leq 4 \sum_\alpha \int_{\mathbb{D}^N} |c_\alpha|^2 |z|^\alpha dv_s(z) = 4 \|g\|_s^2, \end{aligned}$$

we have  $g_t \in A_s^2$ . Hence

$$\begin{aligned} & \| (M_u C_{t_0 \varphi} - M_u C_{t \varphi}) g \|_s^2 \\ &= \left\| u \sum_{\alpha} c_{\alpha} (t_0^{|\alpha|} - t^{|\alpha|}) \varphi^{\alpha} \right\|_s^2 \\ &= \| M_u C_{\varphi} g_t \|_s^2 \leq \| M_u C_{\varphi} \|_{A_s^2}^2 \| g_t \|_s^2 \\ &= \| M_u C_{\varphi} \|_{A_s^2}^2 \sum_{\alpha} |c_{\alpha}|^2 |t_0^{|\alpha|} - t^{|\alpha|}|^2 \| z^{\alpha} \|_s^2 \\ &\leq \| M_u C_{\varphi} \|_{A_s^2}^2 \sup_{|\alpha| > 0} \left( |t_0^{|\alpha|} - t^{|\alpha|}|^2 \frac{\| z^{\alpha} \|_s^2}{\| z^{\alpha} \|_{s'}^2} \right) \sum_{\alpha} |c_{\alpha}|^2 \| z^{\alpha} \|_{s'}^2 \\ &\leq \| M_u C_{\varphi} \|_{A_s^2}^2 \sup_{|\alpha| > 0} \left( |t_0^{|\alpha|} - t^{|\alpha|}| \frac{\| z^{\alpha} \|_s}{\| z^{\alpha} \|_{s'}} \right)^2 \| g \|_{s'}^2. \end{aligned}$$

Then

$$\| M_u C_{t_0 \varphi} - M_u C_{t \varphi} \|_{A_{s'}, A_s^2} \leq \| M_u C_{\varphi} \|_{A_s^2} \sup_{|\alpha| > 0} \left( |t_0^{|\alpha|} - t^{|\alpha|}| \frac{\| z^{\alpha} \|_s}{\| z^{\alpha} \|_{s'}} \right).$$

For any positive integer  $n$ , we have

$$\sup_{|\alpha| > 0} \left( |t_0^{|\alpha|} - t^{|\alpha|}| \frac{\| z^{\alpha} \|_s}{\| z^{\alpha} \|_{s'}} \right) \leq \sum_{|\alpha| < n^2} \left( |t_0^{|\alpha|} - t^{|\alpha|}| \frac{\| z^{\alpha} \|_s}{\| z^{\alpha} \|_{s'}} \right) + 2 \sup_{|\alpha| \geq n^2} \frac{\| z^{\alpha} \|_s}{\| z^{\alpha} \|_{s'}}.$$

Hence

$$(3.1) \quad \limsup_{t \rightarrow t_0} \| M_u C_{t_0 \varphi} - M_u C_{t \varphi} \|_{A_{s'}, A_s^2} \leq 2 \| M_u C_{\varphi} \|_{A_s^2} \sup_{|\alpha| \geq n^2} \frac{\| z^{\alpha} \|_s}{\| z^{\alpha} \|_{s'}}.$$

According to the Stirling's approximation, we have

$$\begin{aligned} \sup_{|\alpha| \geq n^2} \frac{\| z^{\alpha} \|_s}{\| z^{\alpha} \|_{s'}} &= \sup_{|\alpha| \geq n^2} \frac{\Gamma^N(s+1) \prod_{j=1}^N \frac{\Gamma(\alpha_j+1)}{\Gamma(\alpha_j+s+2)}}{\Gamma^N(s'+1) \prod_{j=1}^N \frac{\Gamma(\alpha_j+1)}{\Gamma(\alpha_j+s'+2)}} \\ &\leq C \cdot \sup_{|\alpha| \geq n^2} \prod_{j=1}^N \left( \frac{\alpha_j + s' + 2}{\alpha_j + s + 2} \right)^{\alpha_j + s' + 2} \cdot \left( \frac{1}{\alpha_j + s + 2} \right)^{s-s'} \end{aligned}$$

Since  $|\alpha| \geq n$ , there is at least one  $\alpha_j \geq n$ . Considering that

$$\left( \frac{\alpha_j + s' + 2}{\alpha_j + s + 2} \right)^{\alpha_j + s' + 2}$$

is bounded for  $j = 1, \dots, N$ , we get  $M_u C_{t \varphi} \rightarrow M_u C_{t_0 \varphi}$  as  $t \rightarrow t_0$  in  $\mathcal{C}_w(A_{s'}, A_s^2)$  by letting  $n \rightarrow \infty$  in (3.1). □

4. HILBERT-SCHMIDT NORM TOPOLOGY SPACES

We will consider the topology spaces of Hilbert-Schmidt weighted composition operators. If  $X$  is a separable Hilbert spaces with orthonormal bases  $\{e_m\}$  and  $X'$  is another Hilbert space, recall that a linear operator  $T : X \rightarrow X'$  is said to be Hilbert-Schmidt if

$$\|T\|_{X, X', HS}^2 := \sum_{m=0}^{\infty} \|Te_m\|_{X'}^2 \leq \infty$$

Since  $\{z^\alpha / \|z^\alpha\|_s\}$  is an orthonormal bases of  $A_s^2$ ,  $M_u C_\varphi \in \mathcal{C}_w(A_s^2)$  is Hilbert-Schmidt if and only if

$$\|M_u C_\varphi\|_{A_s^2, HS}^2 := \sum_{|\alpha| \geq 0} \frac{\|M_u C_\varphi(z^\alpha)\|_s^2}{\|z^\alpha\|_s^2} = \sum_{|\alpha| \geq 0} \frac{\|u\varphi_1^{\alpha_1} \cdots \varphi_N^{\alpha_N}\|_s^2}{\|z_1^{\alpha_1} \cdots z_N^{\alpha_N}\|_s^2} < \infty.$$

We denote by  $\mathcal{C}_{w, HS}(A_s^2)$  the space of Hilbert-Schmidt operators  $M_u C_\varphi$  in  $\mathcal{C}_w(A_s^2)$  with the Hilbert-Schmidt norm topology.

We have that  $M_u C_\varphi \in \mathcal{C}_w(A_{s'}^2, A_s^2)$  is Hilbert-Schmidt if and only if

$$\|M_u C_\varphi\|_{A_{s'}^2, A_s^2, HS}^2 := \sum_{\alpha} \frac{\|u\varphi^\alpha\|_s^2}{\|z^\alpha\|_{s'}^2} < \infty.$$

We denote by  $\mathcal{C}_{w, HS}(A_{s'}^2, A_s^2)$  the space of  $M_u C_\varphi \in \mathcal{C}_w(A_{s'}^2, A_s^2)$  which are Hilbert-Schmidt operators. We consider the Hilbert-Schmidt norm topology on  $\mathcal{C}_{w, HS}(A_{s'}^2, A_s^2)$ . The topology on  $\mathcal{C}_{w, HS}(A_{s'}^2, A_s^2)$  is stronger than the operator norm one. So a path connected set in  $\mathcal{C}_{w, HS}(A_{s'}^2, A_s^2)$  is so in  $\mathcal{C}_w(A_{s'}^2, A_s^2)$ . Since

$$\|M_u C_\varphi\|_{A_{s'}^2, A_s^2, HS}^2 \leq C \sum_{\alpha} \frac{\|u\varphi^\alpha\|_s^2}{\|z^\alpha\|_s^2} = C \|M_u C_\varphi\|_{A_s^2, HS}^2,$$

we have  $\mathcal{C}_{w, HS}(\mathcal{H}) \subset \mathcal{C}_{w, HS}(A_{s'}^2, A_s^2)$ . Next lemma will be employed in the main result of this section.

**Lemma 4.1.** *Let  $s', s > -1$ . For each  $0 \leq r \leq 1$ , if the operator  $M_u C_\varphi : A_{s'}^2 \rightarrow A_s^2$  is a Hilbert-Schmidt operator for some nonzero  $u \in A_s^2$  and  $\varphi \in \mathcal{S}(\mathbb{D}^N)$ , then  $M_u C_{r\varphi}$  is also a Hilbert-Schmidt operator from  $A_{s'}^2$  to  $A_s^2$ .*

*Proof.* The Lemma follows immediately from the computations that

$$\begin{aligned} \|M_u C_{r\varphi}\|_{A_{s'}^2, A_s^2, HS}^2 &= \sum_{\alpha \geq 0} \frac{\|M_u C_{r\varphi}(z^\alpha)\|_s^2}{\|z^\alpha\|_{s'}^2} \\ &= \sum_{\alpha \geq 0} r^{2|\alpha|} \frac{\|u\varphi^\alpha\|_s^2}{\|z^\alpha\|_{s'}^2} \leq \|M_u C_\varphi\|_{A_{s'}^2, A_s^2, HS}^2 < \infty. \end{aligned}$$

□

**Theorem 4.2.** *If  $-1 < s' < s$ , then the topology space  $\mathcal{C}_{w, HS}(A_{s'}^2, A_s^2)$  is path connected.*

*Proof.* If  $M_u C_\varphi \in \mathcal{C}_{w,HS}(A_{s'}^2, A_s^2)$ , recall that  $\{z^\alpha / \|z^\alpha\| : \alpha\}$  is an orthonormal basis, and we have

$$(4.1) \quad \|M_u C_\varphi\|_{A_{s'}^2, A_s^2, HS}^2 = \sum_{|\alpha| \geq 0} \frac{\|u\varphi^\alpha\|_s^2}{\|z^\alpha\|_{s'}^2} < \infty.$$

Besides,  $M_u C_{t\varphi} \in \mathcal{C}_{w,HS}(A_{s'}^2, A_s^2)$  for every  $0 \leq t \leq 1$  by Lemma 4.1.

If we fix  $0 \leq t_0 \leq 1$ . we can also prove  $\|M_u C_{t_0\varphi} - M_u C_{t\varphi}\|_{A_{s'}^2, A_s^2, HS} \rightarrow 0$  as  $t \rightarrow t_0$  as following statements. For any positive integer  $N$ , we have

$$\begin{aligned} \|M_u C_{t_0\varphi} - M_u C_{t\varphi}\|_{A_{s'}^2, A_s^2, HS}^2 &= \sum_{|\alpha| \geq 0} \frac{\|u(t_0^{|\alpha|} - t^{|\alpha|})\varphi^\alpha\|_s^2}{\|z^\alpha\|_{s'}^2} \\ &\leq \sum_{|\alpha| \leq N} |t_0^{|\alpha|} - t^{|\alpha|}|^2 \frac{\|u\varphi^\alpha\|_s^2}{\|z^\alpha\|_{s'}^2} + \sum_{|\alpha| > N} \frac{\|u\varphi^\alpha\|_s^2}{\|z^\alpha\|_{s'}^2}. \end{aligned}$$

Take  $\varepsilon > 0$  arbitrary. Then by (4.1), we may take  $N$  large enough so that

$$\sum_{|\alpha| > N} \frac{\|u\varphi^\alpha\|_s^2}{\|z^\alpha\|_{s'}^2} < \varepsilon.$$

Hence

$$\|M_u C_{t_0\varphi} - M_u C_{t\varphi}\|_{A_{s'}^2, A_s^2, HS}^2 < \varepsilon + \sum_{|\alpha| \leq N} |t_0^{|\alpha|} - t^{|\alpha|}|^2 \frac{\|u\varphi^\alpha\|_s^2}{\|z^\alpha\|_{s'}^2}$$

By letting  $t \rightarrow t_0$ , we have

$$\limsup_{t \rightarrow t_0} \|M_u C_{t_0\varphi} - M_u C_{t\varphi}\|_{A_{s'}^2, A_s^2, HS}^2 < \varepsilon.$$

Let  $\varepsilon \rightarrow 0$  then, we have the topology space  $\mathcal{C}_{w,HS}(A_{s'}^2, A_s^2)$  is path connected. □

### 5. FINAL REMARKS

**Lemma 5.1.** *Let  $s > -1$ . If  $\|\varphi\|_\infty < 1$  and  $u \in A_s^2$ , then  $M_u C_\varphi \in \mathcal{C}_w(A_s^2)$  and is compact.*

*Proof.* By the first paragraph of the proof in Theorem 3.2, we have  $M_u C_\varphi \in \mathcal{C}_w(A_s^2)$ .

To show that  $M_u C_\varphi$  is compact, let  $\{f_n\}$  be a sequence in  $A_s^2$  such that there is a positive constant  $K$  satisfying  $\|f_n\|_s < K$  for every  $n$ . By the pointwise estimate (Lemma 2.1), we may assume that  $f_n$  converges to some  $f \in H(\mathbb{D}^N)$  uniformly on compact subsets of  $\mathbb{D}^N$ . By the assumption,  $f_n \circ \varphi \rightarrow f \circ \varphi$  in  $H^\infty$ . Hence both  $u(f_n \circ \varphi)$  and  $u(f \circ \varphi)$  are in  $A_s^2$ , and

$$\|M_u C_\varphi f_n - u(f \circ \varphi)\|_s \leq \|u\|_s \|f_n \circ \varphi - f \circ \varphi\|_\infty \rightarrow 0, \quad n \rightarrow \infty.$$

Thus  $M_u C_\varphi \in \mathcal{C}_w(A_s^2)$  is compact. □

**Proposition 5.2.** *If  $-1 < s' < s$ , then  $\mathcal{C}_w(A_{s'}^2, A_s^2)$  consists of all compact weighted composition operators.*

*Proof.* For  $0 < r < 1$ , by Lemma 5.1  $M_u C_{r\varphi} \in \mathcal{C}_w(A_s^2)$  is compact. Hence  $M_u C_{r\varphi} : A_s^2 \rightarrow A_s^2$ , which can be regarded as the composition of  $\text{id} : A_{s'}^2 \rightarrow A_s^2$  and  $M_u C_{r\varphi} : A_s^2 \rightarrow A_s^2$ , is compact. Since the algebra of compact operators is closed in norm topology, we get  $M_u C_\varphi$  is compact since it can be approximated by compact operators  $M_u C_{r\varphi}$  by Theorem 3.2.  $\square$

We note that on many spaces, the compact (weighted) composition operators form a path connected subset in the topology space of bounded (weighted) composition operators. The compactness has played an important role in the proof of the main result.

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# On the $L_\infty$ convergence of a nonlinear difference scheme for Schrödinger equations

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## Abstract

In this article, a nonlinear difference scheme for Schrödinger equations is studied. The existence of the difference solution is proved by Brouwer fixed point theorem. With the aid of the fact that the difference solution satisfies two conservation laws, the difference solution is proved to be bounded in the  $L_\infty$  norm. Then, the difference solution is shown to be unique and second order convergent in the  $L_\infty$  norm. Finally, a convergent iterative algorithm is presented.

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**Keywords:** Schrödinger equations, Nonlinear difference scheme, Solvability, Convergence

## 1 Introduction

The Schrödinger equation is one of the most important equations in quantum mechanics. This model equation also arises in many other branches of science and technology, e.g. optics, seismology and plasma physics. Recently, a growing interest is on the numerical solution to the nonlinear Schrödinger equations. Many authors investigated the finite difference methods for solving this kind of equations, including the conservation, solvability, stability, convergence and the symplectic geometry (see [1] – [8]).

Consider nonlinear Schrödinger equations

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + q|u|^2u = 0, \quad 0 < x < 1, 0 < t \leq T, \quad (1.1)$$

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq 1, \quad (1.2)$$

$$u(0, t) = 0, \quad u(1, t) = 0, \quad 0 < t \leq T, \quad (1.3)$$

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where  $q$  is a real constant,  $\varphi(x)$  is a known function and  $\varphi(0) = \varphi(1) = 0$ ,  $u(x, t)$  is an unknown function. Take two positive integers  $m$  and  $n$ . Denote  $h = 1/m, \tau = T/n$ , so we have

$$x_j = jh, 0 \leq j \leq m, t_k = k\tau, 0 \leq k \leq n.$$

Denote  $\Omega_{h\tau} = \{(x_j, t_k) \mid 0 \leq j \leq m, 0 \leq k \leq n\}$ . Suppose  $u = \{u_j^k \mid 0 \leq j \leq m, 0 \leq k \leq n\}$  be a discrete grid function on  $\Omega_{h\tau}$ . Introduce the following notations:

$$u_j^{k+\frac{1}{2}} = \frac{1}{2} (u_j^{k+1} + u_j^k), \delta_t u_j^{k+\frac{1}{2}} = \frac{1}{\tau} (u_j^{k+1} - u_j^k), D_t u_j^k = \frac{1}{2\tau} (u_j^{k+1} - u_j^{k-1}),$$

$$\delta_x u_{j+\frac{1}{2}}^k = \frac{1}{h} (u_{j+1}^k - u_j^k), \delta_x^2 u_j^k = \frac{1}{h} (\delta_x u_{j+\frac{1}{2}}^k - \delta_x u_{j-\frac{1}{2}}^k).$$

The author of [9] developed the following nonlinear difference scheme for (1.1)-(1.3)

$$i\delta_t u_j^{k+\frac{1}{2}} + \delta_x^2 u_j^{k+\frac{1}{2}} + \frac{q}{2} (|u_j^k|^2 + |u_j^{k+1}|^2) u_j^{k+\frac{1}{2}} = 0, \quad (1.4)$$

$$1 \leq j \leq m-1, 0 < k \leq n-1,$$

$$u_j^0 = \varphi(x_j), \quad 1 \leq j \leq m-1, \quad (1.5)$$

$$u_0^k = 0, u_m^k = 0, \quad 0 < k \leq n. \quad (1.6)$$

The contents in [9] pointed out that the difference scheme preserves the densities and the energy of the solution, and the author also proved that the difference scheme is uniquely solvable and convergent with the convergence order of  $(\tau^2 + h^2)$  in  $L_2$  norm under some constraints on the stepsizes. On this basis, we proof further that this difference scheme is convergent with the convergence order of  $(\tau^2 + h^2)$  in  $L_\infty$  norm.

In this paper, we will analyze the difference scheme (1.4)-(1.6). The remainder of the paper is arranged as follows. In Section 2, the existence of the difference solution is shown by the Brouwer fixed point theorem. Then with the aid of the conversations of the difference solution, the boundedness and uniqueness of difference solution are proved. In Section 3, the convergence of the difference scheme is discussed. The difference scheme is proved to be convergent with the convergence order of  $\mathbf{O}(\tau^2 + h^2)$  in  $L_\infty$  norm. In Section 4, an iterative algorithm for the difference scheme with the proof of the convergence is given. A short conclusion section ends the paper.

## 2 The existence of the difference solution

In this section, we will prove that the finite difference scheme (1.4)-(1.6) exists a solution.

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Let  $H = \{v \mid v = (v_0, v_1, \dots, v_m), v_j \in C, 0 \leq j \leq m, v_0 = v_m = 0\}$  be the space of complex grid functions on  $\Omega_h$ . Given any complex grid functions  $v, w \in H$ , denote the inner product

$$(v, w) = h \sum_{j=1}^{m-1} v_j \bar{w}_j.$$

The discrete  $L_p$  norm  $\|\cdot\|_p$ , maximum-norm  $\|\cdot\|_\infty$  and  $H_0^1$  norm  $|\cdot|_1$  are defined respectively as follows

$$\|v\|_p = \sqrt[p]{h \sum_{j=1}^{m-1} |v_j|^p}, p \geq 1, \quad \|v\|_\infty = \max_{0 \leq i \leq m} |v_i|,$$

$$|v|_1 = \sqrt{h \sum_{j=0}^{m-1} \left| \frac{v_{j+1} - v_j}{h} \right|^2}.$$

For abbreviation, we write  $\|\cdot\|_2$  as  $\|\cdot\|$ .

In order to illustrate the existence of the difference solution, we need the following lemma.

**Lemma 2.1.** (Brouwer Fixed Point Theorem [10]) Let  $(H, (\cdot, \cdot))$  be a finite dimensional inner product space,  $\|\cdot\|$  the associated norm, and  $\Pi : H \rightarrow H$  be continuous. Assume moreover that

$$\exists \alpha > 0, \forall z \in H, \|z\| = \alpha, \operatorname{Re}(\Pi(z), z) \geq 0.$$

Then, there exists an element  $z^* \in H$  such that  $\Pi(z^*) = 0$  and  $\|z^*\| \leq \alpha$ .

**Theorem 2.2.** The solution of difference scheme (1.4) – (1.6) exists.

*Proof.* Suppose  $\{u_j^k \mid 0 \leq j \leq m\}$  be the numerical solution. Using the notation introduced before, we rewrite the difference scheme (1.4)-(1.6) in the following form

$$i \frac{2}{\tau} \left( u_j^{k+\frac{1}{2}} - u_j^k \right) + \delta_x^2 u_j^{k+\frac{1}{2}} + \frac{q}{2} \left( |u_j^k|^2 + |2u_j^{k+\frac{1}{2}} - u_j^k|^2 \right) u_j^{k+\frac{1}{2}} = 0, \quad (2.1)$$

$$1 \leq j \leq m - 1,$$

$$u_0^{k+\frac{1}{2}} = 0, \quad u_m^{k+\frac{1}{2}} = 0. \quad (2.2)$$

Let

$$v_j = u_j^{k+\frac{1}{2}}, \quad 0 \leq j \leq m,$$

then (2.1)-(2.2) can be written as

$$i \frac{2}{\tau} (v_j - u_j^k) + \delta_x^2 v_j + \frac{q}{2} (|u_j^k|^2 + |2v_j - u_j^k|^2) v_j = 0, \quad 1 \leq j \leq m - 1, \quad (2.3)$$

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$$v_0 = 0, v_m = 0. \tag{2.4}$$

Define the mapping  $\Pi : H \rightarrow H$ ,

$$(\Pi(v))_j = \begin{cases} 0, & j = 0, m, \\ v_j - u_j^k - i\frac{2}{\tau} [\delta_x^2 v_j + \frac{q}{2}(|u_j^k|^2 + |2v_j - u_j^k|^2)v_j], & 1 \leq j \leq m - 1. \end{cases}$$

Computing the inner product of  $\Pi(v)$  and  $v$ , we obtain

$$\begin{aligned} (\Pi(v), v) &= h \sum_{j=1}^{m-1} [v_j - u_j^k - i\frac{\tau}{2}\delta_x^2 v_j - i\frac{\tau q}{4}(|u_j^k|^2 + |2v_j - u_j^k|^2)v_j] \bar{v}_j \\ &= \|v\|^2 - (u^k, v) - i\frac{\tau}{2}h \sum_{j=1}^{m-1} (\delta_x^2 v_j) \bar{v}_j - i\frac{\tau q}{4}(|u_j^k|^2 + |2v_j - u_j^k|^2) |v_j|^2 \\ &= \|v\|^2 - (u^k, v) - i\frac{\tau}{2}h \sum_{j=1}^m \left| \delta_x v_{j-\frac{1}{2}} \right|^2 - i\frac{\tau q}{4} (|u_j^k|^2 + |2v_j - u_j^k|^2) |v_j|^2. \end{aligned}$$

So taking the real part of the inner product

$$\begin{aligned} Re(\Pi(v), v) &= \|v\|^2 - Re(u^k, v) \\ &= \|v\|^2 - Re \left( h \sum_{j=1}^{m-1} u_j^k \bar{v}_j \right) \\ &\geq \|v\|^2 - Re \left[ \frac{h}{2} \left( \sum_{j=1}^{m-1} u_j^k \right)^2 + \left( \sum_{j=1}^{m-1} \bar{v}_j \right)^2 \right] \\ &= \|v\|^2 - \frac{1}{2} (\|u^k\|^2 + \|v\|^2) \\ &= \frac{1}{2} (\|v\|^2 - \|u^k\|^2) \end{aligned}$$

When  $\|v\| = \|u^k\|$ ,  $Re(\Pi(v), v) \geq 0$ . Using Lemma 2.1, we have  $\forall v \in H, \|v\| = \|u^{k+\frac{1}{2}}\| = \|u^k\| > 0, Re(\Pi(v), v) \geq 0$ . Then there exists an element  $v^* \in H$  such that  $\Pi(v^*) = 0$  and  $\|v^*\| \leq \|u^k\|$ . Hence, it is easily seen that the solution  $\{v_j \mid 0 \leq j \leq m\}$  satisfies the difference scheme (2.3)-(2.4). The proof is complete.  $\square$

### 3 The uniqueness of the difference solution

**Theorem 3.1.** ([9]) *The solution of difference scheme (1.4) – (1.6) is conservative. In more precisely, let  $\{u_j^k \mid 0 \leq j \leq m, 0 \leq k \leq n\}$  be the solution of (1.4) – (1.6), we have*

$$\|u^k\|^2 = \|u^0\|^2, E^k = E^0, 1 \leq k \leq n,$$

where

$$E^k = h \sum_{j=0}^{m-1} \left| \delta_x u_{j+\frac{1}{2}}^k \right|^2 - \frac{q}{2} h \sum_{j=1}^{m-1} |u_j^k|^4.$$

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**Theorem 3.2.** ([9]) *The difference solution of (1.4) – (1.6) is bounded in the discrete  $L_\infty$  norm. In more precisely, let  $\{u_j^k \mid 0 \leq j \leq m, 0 \leq k \leq n\}$  be the solution of (1.4) – (1.6), we have*

$$\|u^k\|_\infty^2 \leq \frac{1}{2} \left( \frac{1}{8} |q|^2 \|u^0\|^6 + |u^0|_1^2 - \frac{q}{2} \|u^0\|_4^4 \right), \quad 1 \leq k \leq n,$$

Using Theorems 3.1 and 3.2, we can obtain

**Theorem 3.3.** *The difference solution of (1.4) – (1.6) is unique.*

*Proof.* To proof this theorem, we can prove the solution of difference scheme (2.3)-(2.4) is unique.

Let  $\{v_j \mid 0 \leq j \leq m\}$  and  $\{w_j \mid 0 \leq j \leq m\}$  be the solutions of (2.3)-(2.4). Then we have

$$i \frac{2}{\tau} (w_j - u_j^k) + \delta_x^2 w_j + \frac{q}{2} (|u_j^k|^2 + |2w_j - u_j^k|^2) w_j = 0, \quad 1 \leq j \leq m - 1, \quad (3.1)$$

$$w_0 = 0, \quad w_m = 0. \quad (3.2)$$

Denote

$$\theta_j = v_j - w_j, \quad 0 \leq j \leq m.$$

Subtracting (3.1)-(3.2) from (2.3)-(2.4) respectively, we obtain the following equations

$$i \frac{2}{\tau} \theta_j + \delta_x^2 \theta_j + \frac{q}{2} |u_j^k|^2 \theta_j + \frac{q}{2} (|2v_j - u_j^k|^2 v_j - |2w_j - u_j^k|^2 w_j) = 0, \quad 1 \leq j \leq m - 1, \quad (3.3)$$

$$\theta_0 = 0, \quad \theta_m = 0. \quad (3.4)$$

Multiplying (3.3) by  $-ih\bar{\theta}_j$ , summing up for  $j$  from 1 to  $m - 1$ , we can obtain

$$\begin{aligned} & \frac{2}{\tau} h \sum_{j=1}^{m-1} |\theta_j|^2 - ih \sum_{j=1}^{m-1} (\delta_x^2 \theta_j) \bar{\theta}_j - i \frac{q}{2} h \sum_{j=1}^{m-1} |u_j^k|^2 |\theta_j|^2 \\ & - i \frac{q}{2} h \sum_{j=1}^{m-1} (|2v_j - u_j^k|^2 v_j - |2w_j - u_j^k|^2 w_j) \bar{\theta}_j = 0. \end{aligned} \quad (3.5)$$

Adding the term  $-|2v_j - u_j^k|^2 w_j$  to the part of the forth term in (3.5). Meanwhile, noticing the equality  $|a|^2 - |b|^2 = (a - b)\bar{a} + b\overline{(a - b)}$  where both  $a$  and  $b$  are complex functions, we have

$$\begin{aligned} & |2v_j - u_j^k|^2 v_j - |2w_j - u_j^k|^2 w_j \\ = & |2v_j - u_j^k|^2 (v_j - w_j) + (|2v_j - u_j^k|^2 - |2w_j - u_j^k|^2) w_j \\ = & |2v_j - u_j^k|^2 \theta_j + \left[ 2(v_j - w_j) \overline{2v_j - u_j^k} + 2(v_j - w_j) \overline{(2w_j - u_j^k)} \right] w_j \\ = & |2v_j - u_j^k|^2 \theta_j + 2 \left[ \theta_j \overline{2v_j - u_j^k} + \bar{\theta}_j \overline{(2w_j - u_j^k)} \right] w_j. \end{aligned}$$

Thus, taking the real part of (3.5) with the rewritten forth term, we obtain

$$\frac{2}{\tau} \|\theta\|^2 - \text{Im} \left\{ \frac{q}{2} h \sum_{j=1}^{m-1} 2 \left[ |\theta_j|^2 \overline{(2v_j - u_j^k)} + (2w_j - u_j^k) w_j \right] \right\} = 0.$$

According to Theorem 3.2 which illustrates that the solution  $v, w$  are boundedness, we easily get that  $|v|, |w| \leq \|u^k\|_\infty$ . So using Theorem 3.2 and Cauchy-Schwarz inequality, we have

$$\begin{aligned} \frac{2}{\tau} \|\theta\|^2 &\leq |q|h \sum_{j=1}^{m-1} (|2v_j - u_j^k| + |2w_j - u_j^k|) |\theta_j|^2 |w_j| \\ &\leq |q|h \sum_{j=1}^{m-1} (2|v_j| + |u_j^k| + 2|w_j| + |u_j^k|) \|\theta\|^2 |w_j| \\ &\leq 6|q| \cdot \|u^k\|_\infty^2 \|\theta\|^2 \end{aligned}$$

Denote the right term of the inequality in Theorem 3.2 be a constant  $c_1$ , we have

$$\frac{2}{\tau} \|\theta\|^2 \leq 6c_1^2 |q| \cdot \|\theta\|^2.$$

When  $\tau < \frac{1}{3c_1^2 |q|}$ , we get  $\|\theta\|^2 = 0$ . Hence,  $v_j = w_j, 0 \leq j \leq m$ . The proof is complete.  $\square$

## 4 The convergence of the finite difference scheme

Suppose that the continuous problem (1.1)-(1.3) has a smooth solution  $u$ , and  $U_j^k = \{u(x_j, t_k) \mid 0 \leq j \leq m, 0 \leq k \leq n\}$  is the solution  $u$  under the mapping  $\Omega_{h\tau}$ . In this section, we will illustrate that the solution  $u_j^k$  of the difference scheme (1.4)-(1.6) is convergent to the solution  $U_j^k$  with the convergence order of  $\mathbf{O}(\tau^2 + h^2)$  in the  $L_\infty$  norm.

Denote

$$c_0 = \max_{0 \leq t \leq T} \|u(\cdot, t)\|_\infty, \tag{4.1}$$

$$e_j^k = U_j^k - u_j^k, \quad 0 \leq j \leq m, \quad 0 \leq k \leq n.$$

**Lemma 4.1.** (Gronwall Inequality [9]) Assume  $\{G^n \mid n \geq 0\}$  is a nonnegative sequence, and satisfies

$$G^{n+1} \leq (1 + c\tau)G^n + \tau g, \quad n = 0, 1, 2, \dots,$$

where  $c$  and  $g$  are nonnegative constants. Then  $G$  satisfies

$$G^n \leq e^{cn\tau} \left( G^0 + \frac{g}{c} \right), \quad n = 0, 1, 2, \dots$$

**Lemma 4.2.** ([11]) For any complex functions  $U, V, u, v$ , one has

$$||U|^2 V - |u|^2 v| \leq (\max\{|U|, |V|, |u|, |v|\})^2 \cdot (2|U - u| + |V - v|).$$

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**Lemma 4.3.** ([9]) Denote

$$\begin{aligned} V_h &= \{v \mid v = \{v_i \mid 0 \leq i \leq m\} \text{ is the grid function on } \Omega_h\}, \\ \dot{V}_h &= \{v \mid v = \{v_i \mid 0 \leq i \leq m\} \in V_h, v_0 = v_m = 0\}. \end{aligned}$$

(1) Suppose  $v \in \dot{V}_h$ , so there is

$$\|v\|_\infty \leq \frac{1}{2}|v|_1.$$

(2) Suppose  $v \in \dot{V}_h$ . For any  $\varepsilon > 0$ , there is

$$\|v\|_\infty^2 \leq \varepsilon|v|_1^2 + \frac{1}{4\varepsilon}\|v\|^2.$$

**Theorem 4.4.** Suppose that the difference scheme (1.4) – (1.6) has the solution  $u_j^k$  and the equations (1.1) – (1.3) has the solution  $U_j^k$ . When  $\tau$  is small enough, there exists a constant  $C$  independent of  $h, \tau$  such that

$$\|e^k\|_\infty \leq C(\tau^2 + h^2), \quad 0 \leq k \leq n. \quad (4.2)$$

*Proof.* Subtracting(1.4)-(1.6) from (1.1)-(1.3) respectively, we obtain the error equations

$$\begin{aligned} i\delta_t e_i^{k+\frac{1}{2}} + \delta_x^2 e_i^{k+\frac{1}{2}} + \frac{g}{2} \left[ (|U_j^k|^2 + |U_j^{k+1}|^2)U_j^{k+\frac{1}{2}} - (|u_j^k|^2 + |u_j^{k+1}|^2)u_j^{k+\frac{1}{2}} \right] \\ = R_j^k, \quad 1 \leq j \leq m-1, \quad 0 \leq k \leq n-1, \end{aligned} \quad (4.3)$$

$$e_j^0 = 0, \quad 1 \leq j \leq m-1, \quad (4.4)$$

$$e_0^k = 0, e_m^k = 0, \quad 0 \leq k \leq n. \quad (4.5)$$

In using the Taylor expansion with Lagrange remainder, we can get

$$R_j^k = \frac{\tau^2}{24} \frac{\partial^3 u}{\partial t^3}(x_j, \eta_{jk}) + \frac{h^2}{24} \left[ \frac{\partial^4 u}{\partial x^4}(\xi_{jk}, t_k) + \frac{\partial^4 u}{\partial x^4}(\xi'_{j,k+1}, t_{k+1}) \right] + \frac{\tau^2}{8} \frac{\partial^4 u}{\partial x^2 \partial t^2}(x_j, \zeta_{jk}),$$

where

$$\eta_{jk}, \zeta_{jk} \in (t_k, t_{k+1}), \quad \xi_{jk}, \xi'_{j,k+1} \in (x_{j-1}, x_{j+1}).$$

Therefore there exists a constant  $c_2$  such that

$$|R_j^k| \leq c_2(\tau^2 + h^2), \quad 1 \leq j \leq m-1, \quad 0 < k \leq n-1. \quad (4.6)$$

Let  $G_j^{k+\frac{1}{2}} = (|U_j^k|^2 + |U_j^{k+1}|^2)U_j^{k+\frac{1}{2}} - (|u_j^k|^2 + |u_j^{k+1}|^2)u_j^{k+\frac{1}{2}}$ , and add the term  $-(|U_j^k|^2 + |U_j^{k+1}|^2)u_j^{k+\frac{1}{2}}$  to the  $G_j^{k+\frac{1}{2}}$ , we obtain

$$\begin{aligned} G_j^{k+\frac{1}{2}} &= (|U_j^k|^2 + |U_j^{k+1}|^2)e_j^{k+\frac{1}{2}} + (|U_j^k|^2 + |U_j^{k+1}|^2 - |u_j^k|^2 - |u_j^{k+1}|^2)u_j^{k+\frac{1}{2}} \\ &= (|U_j^k|^2 + |U_j^{k+1}|^2)e_j^{k+\frac{1}{2}} + [(U_j^k - u_j^k)\bar{U}_j^k + u_j^k(U_j^k - u_j^k)] \\ &\quad + (U_j^{k+1} - u_j^{k+1})\bar{U}_j^{k+1} + u_j^{k+1}(U_j^{k+1} - u_j^{k+1})u_j^{k+\frac{1}{2}} \\ &= (|U_j^k|^2 + |U_j^{k+1}|^2)e_j^{k+\frac{1}{2}} + (e_j^k \bar{U}_j^k + u_j^k \bar{e}_j^k + e_j^{k+1} \bar{U}_j^{k+1} + u_j^{k+1} \bar{e}_j^{k+1})u_j^{k+\frac{1}{2}}. \end{aligned}$$

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Noticing the initial-boundary value conditions (1.2)-(1.3) and (1.5)-(1.6), we have

$$\begin{aligned} G_0^{k+\frac{1}{2}} &= (|U_0^k|^2 + |U_0^{k+1}|^2) U_0^{k+\frac{1}{2}} - (|u_0^k|^2 + |u_0^{k+1}|^2) u_0^{k+\frac{1}{2}} = 0, \\ G_m^{k+\frac{1}{2}} &= (|U_m^k|^2 + |U_m^{k+1}|^2) U_m^{k+\frac{1}{2}} - (|u_m^k|^2 + |u_m^{k+1}|^2) u_m^{k+\frac{1}{2}} = 0. \end{aligned}$$

According to Lemma 4.2, we get

$$|G_j^{k+\frac{1}{2}}| \leq \left( \max \left\{ |U_j^k|, |U_j^{k+\frac{1}{2}}|, |u_j^k|, |u_j^{k+\frac{1}{2}}| \right\} \right)^2 \cdot \left( 2|e_j^k| + |e_j^{k+\frac{1}{2}}| \right),$$

or we can say there exists a positive constant  $c_3$  such that

$$\|G^{k+\frac{1}{2}}\|^2 \leq c_3 (\|e^k\|^2 + \|e^{k+1}\|^2), \quad 0 < k \leq n-1, \quad (4.7)$$

$$|G^{k+\frac{1}{2}}|_1^2 \leq c_3 (\|e^k\|^2 + \|e^{k+1}\|^2 + |e^k|_1^2 + |e^{k+1}|_1^2), \quad 0 < k \leq n-1. \quad (4.8)$$

Multiplying the (4.3) by  $h\bar{e}_j^{k+\frac{1}{2}}$ , summing  $j$  from 1 to  $m-1$ , we have

$$\begin{aligned} ih \sum_{j=1}^{m-1} (\delta_t e_j^{k+\frac{1}{2}}) \bar{e}_j^{k+\frac{1}{2}} + h \sum_{j=1}^{m-1} (\delta_x^2 e_j^{k+\frac{1}{2}}) \bar{e}_j^{k+\frac{1}{2}} + \frac{q}{2} h \sum_{j=1}^{m-1} G_j^{k+\frac{1}{2}} \bar{e}_j^{k+\frac{1}{2}} &= h \sum_{j=1}^{m-1} R_j^k \bar{e}_j^{k+\frac{1}{2}}, \\ i \frac{1}{2\tau} (\|e^{k+1}\|^2 - \|e^k\|^2) - h \sum_{j=1}^m |\delta_x e_{j-\frac{1}{2}}^{k+\frac{1}{2}}|^2 + \frac{q}{2} h \sum_{j=1}^{m-1} (|U_j^k|^2 + |U_j^{k+1}|^2) |e_j^{k+\frac{1}{2}}|^2 \\ + \frac{q}{2} h \sum_{j=1}^{m-1} (e_j^k \bar{U}_j^k + u_j^k \bar{e}_j^k + e_j^{k+1} \bar{U}_j^{k+1} + u_j^{k+1} \bar{e}_j^{k+1}) u_j^{k+\frac{1}{2}} \bar{e}_j^{k+\frac{1}{2}} &= h \sum_{j=1}^{m-1} R_j^k \bar{e}_j^{k+\frac{1}{2}}. \end{aligned}$$

Taking the imaginary part and then using (4.1), (4.6) and Theorem 3.2, we can get

$$\begin{aligned} \frac{1}{2\tau} (\|e^{k+1}\|^2 - \|e^k\|^2) &\leq \frac{|q|}{2} h \sum_{j=1}^{m-1} | (e_j^k \bar{U}_j^k + u_j^k \bar{e}_j^k + e_j^{k+1} \bar{U}_j^{k+1} + u_j^{k+1} \bar{e}_j^{k+1}) \\ &\quad \cdot u_j^{k+\frac{1}{2}} \bar{e}_j^{k+\frac{1}{2}} | + h \sum_{j=1}^{m-1} |R_j^k \bar{e}_j^{k+\frac{1}{2}}| \\ &\leq \frac{|q|}{2} h \sum_{j=1}^{m-1} (|e_j^k| c_0 + c_1 |e_j^k| + |e_j^{k+1}| c_0 + c_1 |e_j^{k+1}|) \\ &\quad \cdot c_1 |e_j^{k+\frac{1}{2}}| + h \sum_{j=1}^{m-1} |R_j^k| |\bar{e}_j^{k+\frac{1}{2}}| \\ &\leq \frac{|q|}{2} h (c_0 + c_1) c_1 \sum_{j=1}^{m-1} (|e_j^k| + |e_j^{k+1}|) \cdot \frac{1}{2} (|e_j^k| + |e_j^{k+1}|) \\ &\quad + \frac{1}{2} c_2^2 (\tau^2 + h^2)^2 + \frac{h}{2} \sum_{j=1}^{m-1} [\frac{1}{2} (|e_j^k|^2 + |e_j^{k+1}|^2)] \\ &\leq \left[ \frac{|q|}{2} (c_0 + c_1) c_1 + \frac{1}{4} \right] (\|e^k\|^2 + \|e^{k+1}\|^2) \\ &\quad + \frac{1}{2} c_2^2 (\tau^2 + h^2)^2, \quad 0 \leq k \leq n-1. \end{aligned}$$

Thus,

$$\begin{aligned} &(1 - \tau (|q|(c_0 + c_1)c_1 + \frac{1}{2})) \|e^{k+1}\|^2 \\ &\leq \left( 1 + \tau \left( |q|(c_0 + c_1)c_1 + \frac{|q|}{2} \right) \right) \|e^k\|^2 + \tau c_2^2 (\tau^2 + h^2)^2, \quad 0 \leq k \leq n-1. \end{aligned}$$



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Let  $\beta = \tau (|q|(c_0 + c_1)c_1 + \frac{1}{2})$ . When  $\beta \leq \frac{1}{3}$ , we have

$$\|e^{k+1}\|^2 \leq (1 + 3\beta)\|e^k\|^2 + \frac{1}{1 - \beta}\tau c_2^2(\tau^2 + h^2)^2,$$

or we can say

$$\|e^{k+1}\|^2 \leq \left[1 + 3\tau \left(|q|(c_0 + c_1)c_1 + \frac{1}{2}\right)\right] \|e^k\|^2 + \frac{3}{2}\tau c_2^2(\tau^2 + h^2)^2, \quad 0 \leq k \leq n-1.$$

According to Gronwall Inequality in Lemma 4.1, we obtain

$$\|e^k\|^2 \leq \exp \left[3k\tau \left(|q|(c_0 + c_1)c_1 + \frac{1}{2}\right)\right] \cdot \left[\|e^0\|^2 + \frac{c_2^2(\tau^2 + h^2)^2}{2|q|(c_0 + c_1)c_1 + 1}\right], \quad 1 \leq k \leq n.$$

By the initial-boundary value conditions, we could easily know  $\|e^0\| = 0$ , so

$$\begin{aligned} \|e^k\| &\leq \exp \left[\frac{3}{2}T(|q|(c_0 + c_1)c_1 + \frac{1}{2})\right] \frac{c_2}{\sqrt{2|q|(c_0 + c_1)c_1 + 1}}(\tau^2 + h^2) \\ &= c_4(\tau^2 + h^2), \quad 0 \leq k \leq n. \end{aligned} \quad (4.9)$$

Multiplying the (4.3) by  $-h\delta_t \bar{e}_j^{k+\frac{1}{2}}$ , summing  $j$  from 1 to  $m-1$  and taking the real part, we have

$$\begin{aligned} &-Re \left\{ h \sum_{j=1}^{m-1} \left( \delta_x^2 e_j^{k+\frac{1}{2}} \right) \left( \delta_t \bar{e}_j^{k+\frac{1}{2}} \right) \right\} \\ &= \frac{q}{2} Re \left\{ h \sum_{j=1}^{m-1} G_j^{k+\frac{1}{2}} \left( \delta_t \bar{e}_j^{k+\frac{1}{2}} \right) \right\} - Re \left\{ h \sum_{j=1}^{m-1} R_j^k \left( \delta_t \bar{e}_j^{k+\frac{1}{2}} \right) \right\}. \end{aligned} \quad (4.10)$$

Now, we estimate each term of (4.10).

Firstly, simplifying the left of (4.10), we obtain

$$\begin{aligned} &-Re \left\{ h \sum_{j=1}^{m-1} \left( \delta_x^2 e_j^{k+\frac{1}{2}} \right) \left( \delta_t \bar{e}_j^{k+\frac{1}{2}} \right) \right\} \\ &= h \sum_{j=1}^m \left( \delta_x e_{j-\frac{1}{2}}^{k+\frac{1}{2}} \right) \cdot \delta_x \left( \delta_t \bar{e}_{j-\frac{1}{2}}^{k+\frac{1}{2}} \right) \\ &= h \sum_{j=1}^m \left( \delta_x e_{j-\frac{1}{2}}^{k+\frac{1}{2}} \right) \cdot \delta_t \left( \delta_x \bar{e}_{j-\frac{1}{2}}^{k+\frac{1}{2}} \right) \\ &= h \sum_{j=1}^m \frac{1}{2} \left( \delta_x e_{j-\frac{1}{2}}^{k+1} + \delta_x e_{j-\frac{1}{2}}^k \right) \cdot \frac{1}{\tau} \left( \delta_x \bar{e}_{j-\frac{1}{2}}^{k+1} - \delta_x \bar{e}_{j-\frac{1}{2}}^k \right) \\ &= \frac{1}{2\tau} h \sum_{j=1}^m \left( |\delta_x e_{j-\frac{1}{2}}^{k+1}|^2 - |\delta_x e_{j-\frac{1}{2}}^k|^2 \right), \end{aligned}$$

that is

$$-Re \left\{ h \sum_{j=1}^{m-1} \left( \delta_x^2 e_j^{k+\frac{1}{2}} \right) \left( \delta_t \bar{e}_j^{k+\frac{1}{2}} \right) \right\} = \frac{1}{2\tau} (|e^{k+1}|_1^2 - |e^k|_1^2). \quad (4.11)$$

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Let the right term of (4.10) be  $A_1, A_2$  separately. By the error equation (4.3), we have

$$\delta_t \bar{e}_j^{k+\frac{1}{2}} = -i\delta_x^2 \bar{e}_j^{k+\frac{1}{2}} - \frac{q}{2} i \bar{G}_j^{k+\frac{1}{2}} + i \bar{R}_j^k. \quad (4.12)$$

Substituting (4.12) into  $A_1$ , we obtain

$$\begin{aligned} & \operatorname{Re} \left\{ h \sum_{j=1}^{m-1} G_j^{k+\frac{1}{2}} \left( \delta_t \bar{e}_j^{k+\frac{1}{2}} \right) \right\} \\ = & \operatorname{Re} \left\{ h \sum_{j=1}^{m-1} G_j^{k+\frac{1}{2}} \left( -i\delta_x^2 \bar{e}_j^{k+\frac{1}{2}} - \frac{q}{2} i \bar{G}_j^{k+\frac{1}{2}} + i \bar{R}_j^k \right) \right\} \\ = & \operatorname{Re} \left\{ -ih \sum_{j=1}^{m-1} \delta_x^2 \bar{e}_j^{k+\frac{1}{2}} G_j^{k+\frac{1}{2}} - i\frac{q}{2} \|G^{k+\frac{1}{2}}\|^2 + ih \sum_{j=1}^{m-1} \bar{R}_j^k G_j^{k+\frac{1}{2}} \right\} \\ = & \operatorname{Im} \left\{ -h \sum_{j=1}^{m-1} \delta_x^2 \bar{e}_j^{k+\frac{1}{2}} G_j^{k+\frac{1}{2}} + h \sum_{j=1}^{m-1} \bar{R}_j^k G_j^{k+\frac{1}{2}} \right\} \\ = & B_1 + B_2, \end{aligned}$$

where

$$\begin{aligned} B_1 & \leq \left| h \sum_{j=1}^{m-1} \delta_x^2 \bar{e}_j^{k+\frac{1}{2}} G_j^{k+\frac{1}{2}} \right| \leq \left| -h \sum_{j=1}^m \delta_x \bar{e}_{j-\frac{1}{2}}^{k+\frac{1}{2}} \delta_x G_j^{k+\frac{1}{2}} \right| \leq \frac{1}{2} \left( |e^{k+\frac{1}{2}}|_1^2 + |G^{k+\frac{1}{2}}|_1^2 \right), \\ B_2 & \leq \left| h \sum_{j=1}^{m-1} \bar{R}_j^k G_j^{k+\frac{1}{2}} \right| \leq \frac{h}{2} \left( \sum_{j=1}^{m-1} |\bar{R}_j^k|^2 + \sum_{j=1}^{m-1} |G_j^{k+\frac{1}{2}}|^2 \right) \leq \frac{1}{2} \left( \|\bar{R}^k\|^2 + \|G^{k+\frac{1}{2}}\|^2 \right). \end{aligned}$$

Then according to (4.6)-(4.8), we can estimate the first right term  $A_1$  as follow

$$\begin{aligned} A_1 & \leq \frac{q}{4} \left( |e^{k+\frac{1}{2}}|_1^2 + |G^{k+\frac{1}{2}}|_1^2 + \|\bar{R}^k\|^2 + \|G^{k+\frac{1}{2}}\|^2 \right) \\ & \leq \frac{q}{4} |e^{k+\frac{1}{2}}|_1^2 + \frac{q}{4} c_3 \left( \|e^k\|^2 + \|e^{k+1}\|^2 + |e^k|_1^2 + |e^{k+1}|_1^2 \right) \\ & \quad + \frac{q}{4} c_2^2 (\tau^2 + h^2)^2 + \frac{q}{4} c_3 \left( \|e^k\|^2 + \|e^{k+1}\|^2 \right) \\ & = \frac{q}{4} \left[ |e^{k+\frac{1}{2}}|_1^2 + c_2^2 (\tau^2 + h^2)^2 \right] + \frac{q}{2} c_3 \left( \|e^k\|^2 + \|e^{k+1}\|^2 \right) \\ & \quad + \frac{q}{4} c_3 \left( |e^k|_1^2 + |e^{k+1}|_1^2 \right) \\ & \leq \frac{q}{4} \left[ |e^{k+\frac{1}{2}}|_1^2 + c_2^2 (\tau^2 + h^2)^2 \right] + qc_3 c_4^2 (\tau^2 + h^2)^2 + \frac{q}{4} c_3 \left( |e^k|_1^2 + |e^{k+1}|_1^2 \right) \\ & = \frac{q}{4} c_3 \left( |e^k|_1^2 + |e^{k+1}|_1^2 \right) + \frac{q}{4} |e^{k+\frac{1}{2}}|_1^2 + \left( qc_3 c_4^2 + \frac{q}{4} c_2^2 \right) (\tau^2 + h^2)^2. \end{aligned}$$

According to (4.6), we can also estimate the second right term  $A_2$  as follow

$$A_2 \leq \frac{1}{2} \left( \|\bar{R}^k\|^2 + |e^{k+\frac{1}{2}}|_1^2 \right) \leq \frac{1}{2} \left( c_1^2 (\tau^2 + h^2)^2 + |e^{k+\frac{1}{2}}|_1^2 \right).$$

Now, substituting the three estimations just represented before into (4.10),

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we obtain

$$\begin{aligned} & \frac{1}{2\tau} (|e^{k+1}|_1^2 - |e^k|_1^2) \\ & \leq \frac{q}{4}c_3 (|e^k|_1^2 + |e^{k+1}|_1^2) + \frac{q}{4}|e^{k+\frac{1}{2}}|_1^2 + (qc_3c_4^2 + \frac{q}{4}c_2^2) (\tau^2 + h^2)^2 \\ & \quad + \frac{1}{2}c_2^2(\tau^2 + h^2)^2 + \frac{1}{2}|e^{k+\frac{1}{2}}|_1^2 \\ & \leq \frac{q}{4}c_3 (|e^k|_1^2 + |e^{k+1}|_1^2) + \frac{q+2}{4}|e^{k+\frac{1}{2}}|_1^2 + (qc_3c_4^2 + \frac{q+2}{4}c_2^2) (\tau^2 + h^2)^2 \\ & \leq \frac{q}{4}c_3 (|e^k|_1^2 + |e^{k+1}|_1^2) + \frac{q+2}{4}\frac{1}{2}(|e^k|_1^2 + |e^{k+1}|_1^2) + (qc_3c_4^2 + \frac{q+2}{4}c_2^2) (\tau^2 + h^2)^2 \\ & = \frac{2qc_3+q+2}{8} (|e^k|_1^2 + |e^{k+1}|_1^2) + (qc_3c_4^2 + \frac{q+2}{4}c_2^2) (\tau^2 + h^2)^2, \end{aligned}$$

that is

$$(1 - \tau \frac{2qc_3+q+2}{4}) |e^{k+1}|_1^2 \leq (1 + \tau \frac{2qc_3+q+2}{4}) |e^k|_1^2 + (2qc_3c_4^2 + \frac{q+2}{2}c_2^2) (\tau^2 + h^2)^2, \quad 0 \leq k \leq n-1.$$

Let  $\beta = \tau \frac{2qc_3+q+2}{4}$ . When  $\beta \leq \frac{1}{3}$ , we have

$$|e^{k+1}|_1^2 \leq \left(1 + 3\tau \frac{2qc_3+q+2}{4}\right) |e^k|_1^2 + \frac{3}{2}\tau \left(2qc_3c_4^2 + \frac{q+2}{2}c_2^2\right) (\tau^2 + h^2)^2, \quad 0 \leq k \leq n-1.$$

Denote

$$c_5 = \frac{2qc_3+q+2}{4}, \quad c_6 = 2qc_3c_4^2 + \frac{q+2}{2}c_2^2,$$

then we rewrite the inequality as follow

$$|e^{k+1}|_1^2 \leq (1 + 3\tau c_5)|e^k|_1^2 + \frac{3}{2}\tau c_6(\tau^2 + h^2)^2, \quad 0 \leq k \leq n-1.$$

Using Gronwall inequality, we get

$$|e^k|_1^2 \leq \exp(3k\tau c_5) \cdot \left(|e^0|_1^2 + \frac{c_6(\tau^2 + h^2)^2}{2c_5}\right), \quad 1 \leq k \leq n.$$

By the initial-boundary value conditions, we also know  $|e^0|_1^2 = 0$ , so

$$\begin{aligned} |e^k|_1^2 & \leq \exp(3k\tau c_5) \cdot \frac{c_6(\tau^2+h^2)^2}{2c_5} \\ & \leq \exp(3c_5T) \frac{c_6}{2c_5} (\tau^2 + h^2)^2, \quad 0 \leq k \leq n. \end{aligned} \tag{4.13}$$

According to (1) in Lemma 4.3, we have

$$\begin{aligned} \|e^k\|_\infty^2 & \leq \frac{1}{4}|e^k|_1^2 \\ & \leq \frac{c_6}{8c_5} \exp(3c_5T) (\tau^2 + h^2)^2, \quad 0 \leq k \leq n. \end{aligned}$$

Denote

$$C = \sqrt{\frac{c_6}{8c_5} \exp(3c_5T)}.$$

Therefore, when  $\tau$  is small enough, there exists a constant  $C$  independent of  $h, \tau$  such that

$$\|e^k\|_\infty \leq C(\tau^2 + h^2), \quad 0 \leq k \leq n. \tag{4.14}$$

This completes the proof. □

### 5 Iterative algorithm

There are some discrete methods about the nonlinear Schrödinger equations [12]-[13]. In this section, we use an iterative method [9] to compute the solution of the difference scheme (2.3)-(2.4).

Define the following iterative method

$$i \frac{2}{\tau} \left( v_j^{(l)} - u_j^k \right) + \delta_x^2 v_j^{(l)} + \frac{q}{2} \left( |u_j^k|^2 + |2v_j^{(l-1)} - u_j^k|^2 \right) v_j^{(l)} = 0, \tag{5.1}$$

$$1 \leq j \leq m-1, 0 \leq k \leq n-1,$$

$$v_0^{(l)} = 0, v_m^{(l)} = 0, \tag{5.2}$$

where  $v_j^{(0)} = u_j^k, 0 \leq j \leq m, l = 1, 2, \dots$

Multiplying the (5.1) by  $h\bar{v}_j^{(l)}$ , summing  $j$  from 1 to  $m-1$  and taking the imaginary part, we have

$$\frac{2}{\tau} \|v^{(l)}\|^2 - \text{Re}h \sum_{j=1}^{m-1} u_j^k \bar{v}_j^{(l)} = 0,$$

that is

$$\begin{aligned} \frac{2}{\tau} \|v^{(l)}\|^2 &= \text{Re}h \sum_{j=1}^{m-1} u_j^k \bar{v}_j^{(l)} \\ &\leq h \left( \sum_{j=1}^{m-1} (u_j^k)^2 \cdot \sum_{j=1}^{m-1} (\bar{v}_j^{(l)})^2 \right)^{\frac{1}{2}} \\ &= \left( h \sum_{j=1}^{m-1} (u_j^k)^2 \right)^{\frac{1}{2}} \cdot \left( h \sum_{j=1}^{m-1} (\bar{v}_j^{(l)})^2 \right)^{\frac{1}{2}} \\ &= \|u^k\| \cdot \|v^{(l)}\|. \end{aligned}$$

Thus,

$$\|v^{(l)}\| \leq \|u^k\|, l = 1, 2, \dots \tag{5.3}$$

Denote

$$\varepsilon_j^{(l)} = v_j - v_j^{(l)}, 0 \leq j \leq m.$$

**Theorem 5.1.** *Suppose that the solution is  $\{u_j^k \mid 0 \leq j \leq m\}$ ,  $\tau$  is sufficiently small enough, then the iterative method (5.1) – (5.2) is convergent.*

*Proof.* Subtracting (5.1)-(5.2) from (2.3)-(2.4), we obtain

$$i \frac{2}{\tau} \varepsilon_j^{(l)} + \delta_x^2 \varepsilon_j^{(l)} + \frac{q}{2} \left[ (|u_j^k|^2 + |2v_j - u_j^k|^2) v_j - (|u_j^k|^2 + |2v_j^{(l-1)} - u_j^k|^2) v_j^{(l)} \right] = 0, 1 \leq j \leq m-1, \tag{5.4}$$

$$\varepsilon_0^{(l)} = \varepsilon_m^{(l)} = 0. \tag{5.5}$$

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Multiplying the (5.4) by  $h\bar{\varepsilon}_j^{(l)}$ , summing  $j$  from 1 to  $m - 1$ , we have

$$\begin{aligned}
 & i\frac{2}{\tau}h \sum_{j=1}^{m-1} |\varepsilon_j^{(l)}|^2 + h \sum_{j=1}^{m-1} (\delta_x^2 \varepsilon_j^{(l)}) \bar{\varepsilon}_j^{(l)} + \frac{q}{2}h \sum_{j=1}^{m-1} |u_j^k|^2 |\varepsilon_j^{(l)}|^2 \\
 & + \frac{q}{2}h \sum_{j=1}^{m-1} \left[ |2v_j - u_j^k|^2 v_j - |2v_j^{(l-1)} - u_j^k|^2 v_j^{(l)} \right] \bar{\varepsilon}_j^{(l)} = 0.
 \end{aligned} \tag{5.6}$$

Noticing the term in brackets, we add  $|2v_j^{(l-1)} - u_j^k|^2 v_j$  to this term as follow

$$\begin{aligned}
 & |2v_j - u_j^k|^2 v_j - |2v_j^{(l-1)} - u_j^k|^2 v_j^{(l)} \\
 = & |2v_j^{(l-1)} - u_j^k|^2 (v_j - v_j^{(l)}) + (|2v_j - u_j^k|^2 - |2v_j^{(l-1)} - u_j^k|^2) v_j \\
 = & |2v_j^{(l-1)} - u_j^k|^2 \varepsilon_j^{(l)} + \left( 2\varepsilon_j^{(l-1)} \overline{(2v_j - u_j^k)} + (2v_j^{(l-1)} - u_j^k) 2\bar{\varepsilon}_j^{(l-1)} \right) v_j.
 \end{aligned}$$

Then, substituting this rewritten term into (5.6) and taking the imaginary part, we have

$$\begin{aligned}
 & \frac{2}{\tau} \|\varepsilon^{(l)}\|^2 \\
 \leq & \frac{|q|}{2} h \sum_{j=1}^{m-1} \left| \left( 2\varepsilon_j^{(l-1)} \overline{(2v_j - u_j^k)} + (2v_j^{(l-1)} - u_j^k) 2\bar{\varepsilon}_j^{(l-1)} \right) v_j \bar{\varepsilon}_j^{(l)} \right| \\
 \leq & |q| \cdot \|\varepsilon^{(l-1)}\|_\infty \cdot \|\varepsilon^{(l)}\|_\infty \cdot h \sum_{j=1}^{m-1} \left( |2v_j - u_j^k| + |2v_j^{(l-1)} - u_j^k| \right) |v_j| \\
 \leq & |q| \cdot \|\varepsilon^{(l-1)}\|_\infty \cdot \|\varepsilon^{(l)}\|_\infty \cdot (\|2v\| + \|u^k\| + \|2v^{(l-1)}\| + \|u^k\|) \|v\| \\
 \leq & |q| \cdot \|\varepsilon^{(l-1)}\|_\infty \cdot \|\varepsilon^{(l)}\|_\infty \cdot 6\|v\|^2.
 \end{aligned}$$

According to (5.3) and Theorem 3.1, we obtain

$$\begin{aligned}
 \|\varepsilon^{(l)}\|^2 & \leq 3\tau|q| \cdot \|\varepsilon^{(l-1)}\|_\infty \cdot \|\varepsilon^{(l)}\|_\infty \cdot \|u^k\|^2 \\
 & = 3\tau|q| \cdot \|\varepsilon^{(l-1)}\|_\infty \cdot \|\varepsilon^{(l)}\|_\infty \cdot \|u^0\|^2.
 \end{aligned} \tag{5.7}$$

Similarly, taking the real part of (5.6), we have

$$\begin{aligned}
 & |\varepsilon^{(l)}|_1^2 \\
 \leq & \frac{|q|}{2} \|u^k\|^2 \cdot \|\varepsilon^{(l)}\|_\infty^2 + \frac{|q|}{2} \|\varepsilon^{(l)}\|_\infty^2 \cdot h \sum_{j=1}^{m-1} \left[ 4(v_j^{(l-1)})^2 + (u_j^k)^2 - 4v_j^{(l-1)} u_j^k \right] \\
 & + |q| \cdot \|\varepsilon^{(l-1)}\|_\infty \cdot \|\varepsilon^{(l)}\|_\infty \cdot 6\|v\|^2 \\
 \leq & \frac{|q|}{2} \|u^k\|^2 \|\varepsilon^{(l)}\|_\infty^2 + \frac{|q|}{2} \|\varepsilon^{(l)}\|_\infty^2 \cdot (4\|u^k\|^2 + \|u^k\|^2 + 4\|u^k\|^2) \\
 & + 6|q| \cdot \|u^k\|^2 \|\varepsilon^{(l-1)}\|_\infty \|\varepsilon^{(l)}\|_\infty \\
 \leq & 5|q| \cdot \|u^k\|^2 \|\varepsilon^{(l)}\|_\infty^2 + 6|q| \cdot \|u^k\|^2 \|\varepsilon^{(l-1)}\|_\infty \|\varepsilon^{(l)}\|_\infty \\
 \leq & 5|q| \cdot \|u^0\|^2 \|\varepsilon^{(l)}\|_\infty^2 + 6|q| \cdot \|u^0\|^2 \|\varepsilon^{(l-1)}\|_\infty \|\varepsilon^{(l)}\|_\infty.
 \end{aligned}$$

According to (2) in Lemma 4.3, for any  $\alpha > 0$ , there is

$$\begin{aligned}
 \|\varepsilon^{(l)}\|_\infty^2 & \leq \alpha |\varepsilon^{(l)}|_1^2 + \frac{1}{4\alpha} \|\varepsilon^{(l)}\|^2 \\
 & \leq \alpha \left( 5|q| \cdot \|u^0\|^2 \|\varepsilon^{(l)}\|_\infty^2 + 6|q| \cdot \|u^0\|^2 \|\varepsilon^{(l-1)}\|_\infty \|\varepsilon^{(l)}\|_\infty \right) \\
 & \quad + \frac{1}{4\alpha} \cdot 3\tau|q| \cdot \|u^0\|^2 \|\varepsilon^{(l-1)}\|_\infty \|\varepsilon^{(l)}\|_\infty.
 \end{aligned}$$

That is

$$\|\varepsilon^{(l)}\|_\infty \leq \alpha \left( 5|q| \cdot \|u^0\|^2 \|\varepsilon^{(l)}\|_\infty + 6|q| \cdot \|u^0\|^2 \|\varepsilon^{(l-1)}\|_\infty \right) + \frac{1}{4\alpha} \cdot 3\tau|q| \cdot \|u_0\|^2 \|\varepsilon^{(l-1)}\|_\infty.$$

Taking  $\alpha = 1/(12|q| \cdot \|u_0\|^2)$ , we get

$$\|\varepsilon^{(l)}\|_\infty \leq \frac{5}{12} \|\varepsilon^{(l)}\|_\infty + \frac{1}{2} \|\varepsilon^{(l-1)}\|_\infty + 9\tau q^2 \|u_0\|^4 \|\varepsilon^{(l-1)}\|_\infty,$$

that is

$$\|\varepsilon^{(l)}\|_\infty \leq \frac{12}{7} \left( \frac{1}{2} + 9\tau q^2 \|u_0\|^4 \right) \|\varepsilon^{(l-1)}\|_\infty.$$

When  $\tau q^2 \|u_0\|^4 \leq \frac{1}{144}$ , we have

$$\|\varepsilon^{(l)}\|_\infty \leq \frac{12}{7} \left( \frac{1}{2} + \frac{9}{144} \right) \|\varepsilon^{(l-1)}\|_\infty = \frac{27}{28} \|\varepsilon^{(l-1)}\|_\infty.$$

This completes the proof. □

## 6 Conclusion

In this paper, we consider a nonlinear finite difference scheme for the Schrödinger equations. We prove that the difference scheme has a unique and bounded solution and the finite difference solution is convergent with the convergence order of  $\mathbf{O}(\tau^2 + h^2)$  in  $L_\infty$  norm. Finally we give a convergent iterative method to compute the solution of the difference scheme.

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**SINGLE POINT V.S. TOTAL BLOW-UP FOR A REACTION  
DIFFUSION EQUATION WITH NONLOCAL SOURCE**

DENGMING LIU

ABSTRACT. In this paper, we consider the following initial-boundary value problem of a semilinear parabolic equation with local and nonlocal sources

$$u_t = \Delta u + u^p + \int_B u^q(x, t) dx, \quad (x, t) \in B \times (0, T),$$

where  $p, q > 0$ ,  $B = \{x \in \mathbb{R}^N : |x| < R\}$ . We completely classify blow-up solutions into total blow-up case and single point blow-up case according to the different values of the nonlinear parameters, and give the blow-up rates of solutions near the blow-up time.

1. INTRODUCTION

In this paper, we deal with the property of the blow-up solution of the following reaction-diffusion equation with local and nonlocal sources

$$\begin{cases} u_t = \Delta u + u^p + \int_B u^q(x, t) dx, & (x, t) \in B \times (0, T), \\ u(x, t) = 0, & (x, t) \in \partial B \times (0, T), \\ u(x, 0) = u_0(x), & x \in \bar{B}, \end{cases} \quad (1.1)$$

where  $p, q > 0$ ,  $B = \{x \in \mathbb{R}^N : |x| < R\}$ . Throughout this paper, we assume that the initial data  $u_0 \in C^2(B) \cap C(\bar{B})$ ,  $u_0(x) = u(r) \geq 0$  with  $r = |x|$ , and  $u'_0(r) < 0$  for  $r \in (0, R]$ . Moreover, we assume that there exists a positive constant  $\delta$  such that  $\Delta u_0 + u_0^p + \int_B u_0^q dx \geq \delta$ . When  $\min\{p, q\} \geq 1$ , we can easily show the local existence and uniqueness of classical solution of problem (1.1). If  $\min\{p, q\} < 1$ , the existence of maximal solution can be proved. Moreover, if  $\max\{p, q\} > 1$ , we can prove that the solution of (1.1) blows up in finite time for large initial data. In this paper, we consider the blow-up set of problem (1.1) and denote the blow-up time by  $T$ . We now begin with the definition of the blow-up point for a blow-up solution.

**Definition 1.1.** A point  $x \in \bar{B}$  is called a blow-up point if there exists a sequence  $(x_n, t_n)$  such that  $x_n \rightarrow x$ ,  $t_n \nearrow T$  and  $u(x_n, t_n) \rightarrow \infty$  as  $n \rightarrow \infty$ .

The set of all blow-up points is called the blow-up set. For simplification, we denote the blow-up set by  $S$ . When  $S = \bar{B}$ , we call this phenomenon “total blow-up” and when the blow-up set include only one point, we call this “single point blow-up”.

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In 1984, Weissler(see [1]) considered the property of the blow-up solution for the following one-dimensional initial-boundary value problem

$$\begin{cases} u_t = \Delta u + u^p, & (x, t) \in (-R, R) \times (0, T), \\ u(x, t) = 0, & (x, t) \in \{-R, R\} \times (0, T), \\ u(x, 0) = u_0(x), & x \in [-R, R], \end{cases} \quad (1.2)$$

where  $p > 1$ , and obtained the single point blow-up phenomenon under some suitable conditions. In [2], Friedman and McLeod generalized Weissler’s results to  $N$ -dimensional case, and showed that the blow-up point is only the origin, namely,  $S = \{0\}$ .

Chadam et al. in [3] studied the following problem with localized reaction term

$$\begin{cases} u_t = \Delta u + u^q(x^*, t), & (x, t) \in B \times (0, T), \\ u(x, t) = 0, & (x, t) \in \partial B \times (0, T), \\ u(x, 0) = u_0(x), & x \in \bar{B}, \end{cases} \quad (1.3)$$

and proved that total blow-up occurs whenever a solution blows up, that is,  $S = \bar{B}$ . Souplet [4,5] extended the results in [3] to the case for the moving source  $x^*(t)$  and obtained the precise blow-up profiles of the total blow-up solution.

Recently, Okada and Fukuda in [7] dealt with the single point and total blow-up for the following problem

$$\begin{cases} u_t = \Delta u + u^p + u^q(x^*, t), & (x, t) \in B \times (0, T), \\ u(x, t) = 0, & (x, t) \in \partial B \times (0, T), \\ u(x, 0) = u_0(x), & x \in \bar{B}. \end{cases} \quad (1.4)$$

They showed that  $p = q + 1$  is a cut off between the single point blow-up and the total blow-up for  $x^* = 0$ , and  $p = q$  is the critical exponent of the single point blow-up and the total blow-up for  $x^* \neq 0$ .

Motivated by above works, we investigate problem (1.1). Similar to [7], the main purpose of this article is to evaluate the effect of the competition between  $u^p$  and  $\int_B u^q dx$  on the single blow-up and total blow-up. Motivated by the idea of Souplet in [6], through modifying the construction of auxiliary functions used in [7], we completely classify blow-up solutions into total blow-up case and single point blow-up case according to the different values of  $p$  and  $q$ , and give the blow-up rates of solutions near the blow-up time.

In order to state our results, we first let  $\varphi$  be a solution of

$$\begin{cases} \varphi_t = \Delta \varphi, & (x, t) \in B \times (0, T), \\ \varphi(x, t) = 0, & (x, t) \in \partial B \times (0, T), \\ \varphi(x, 0) = \varphi_0(x) \geq 0, & x \in \bar{B}, \end{cases} \quad (1.5)$$

where  $\varphi_0 \in C^2(B) \cap C(\bar{B})$ ,  $\varphi_0(x) = \varphi(r)$  with  $r = |x|$ , and  $\varphi'_0(r) < 0$  for  $r \in (0, R]$ . The main results of this article are stated as follows.

**Theorem 1.2.** *Suppose  $q > p$  and  $q > 1$ , and let  $u$  be a solution of (1.1) with  $u_0 = \lambda\varphi_0$  ( $\lambda > 0$ ), then there exists a positive constant  $\lambda_0(\varphi_0)$  such that if  $\lambda > \lambda_0(\varphi_0)$ , then  $u$  blows up on the whole domain, that is,  $S = \overline{B}$ ; Moreover, the following estimate*

$$C_1(T-t)^{-\frac{1}{q-1}} \leq u(x,t) \leq C_2(T-t)^{-\frac{1}{q-1}}, \quad t \rightarrow T \tag{1.6}$$

*holds for any compact subset of  $B$ , here  $C_1, C_2$  are positive constants.*

**Theorem 1.3.** *Suppose  $p \geq q$  and  $p > 1$ , then all blow-up solutions of problem (1.1) blow up only at the origin, namely,  $S = \{0\}$ ; Moreover, there exist positive constants  $C_3$  and  $C_4$  such that*

$$C_3(T-t)^{-\frac{1}{p-1}} \leq u(0,t) \leq C_4(T-t)^{-\frac{1}{p-1}}, \quad t \rightarrow T. \tag{1.7}$$

**Remark 1.4.** *From Theorems 1.1 and 1.2, we know that  $p = q$  is the critical exponent for single point blow-up and total blow-up.*

This paper is organized as follows. In the next section, we will give some lemmas. In section 3, we concern with the single point blow-up and the total blow-up, and give the proofs of Theorems 1.1 and 1.2, respectively.

## 2. PRELIMINARY

In this section, we will state two important lemmas, which will be used in the sequel.

**Lemma 2.1.** *Suppose  $q > 1$  and  $q > p$ , let  $u(x,t)$  be a solution of (1.1) with  $u_0(x) = \lambda\varphi_0(x)$ , then there exists a positive constant  $\lambda_0(\varphi_0)$  such that*

$$u(x,t) \geq \frac{\varphi(x,t)}{2\varphi_0(0)}u(0,t) \equiv \psi(x,t)u(0,t), \quad (x,t) \in B \times [0,T], \tag{2.1}$$

*holds if  $\lambda > \lambda_0(\varphi_0)$ .*

*Proof.* Using maximum principle (see [4]), we have

$$0 \leq \varphi(x,t) \leq \varphi_0(x) \leq \varphi_0(0).$$

Because of  $q > p$  and  $u_0(0) = \lambda\varphi_0(0)$ , we can choose  $\lambda$  large enough such that

$$2\psi(x,t)u_0^{p-q}(0) \leq \{\lambda\varphi_0(0)\}^{p-q} \leq \int_B \psi^q(x,t) dx. \tag{2.2}$$

Now, letting

$$U = u(x,t) - \psi(x,t)u(0,t),$$

after a series of simple computation, we have

$$\begin{aligned} U_t - \Delta U &= u^p + \int_B u^q dx - \psi(x,t) \left( \Delta u(0,t) + u^p(0,t) + \int_B u^q dx \right) \\ &\geq \frac{1}{2} \int_B u^q dx - \psi(x,t)u^p(0,t). \end{aligned} \tag{2.3}$$

On the other hand,  $\Delta u_0 + u_0^p + \int_B u_0^q dx \geq \delta$  means  $u_t \geq 0$ , thus, for any  $t \in [0, T)$ , we see

$$u(0,t) \geq u_0(0). \tag{2.4}$$

Combining (2.2), (2.3) with (2.4), we know

$$\begin{aligned} U_t - \Delta U &\geq \frac{1}{2} \left( \int_B u^q dx - 2\psi(x, t) u_0^{p-q}(0) u^q(0, t) \right) \\ &\geq \frac{1}{2} \left( \int_B u^q dx - \int_B \psi^q u^q(0, t) dx \right) \\ &= \frac{q}{2} \int_B U \Phi dx, \end{aligned} \tag{2.5}$$

where

$$\Phi = \int_0^1 [\theta u + (1 - \theta) \psi u(0, t)]^{q-1} d\theta.$$

In addition, for any  $(x, t) \in \partial B \times (0, T)$ , we have

$$U(x, t) = 0, \tag{2.6}$$

and

$$U(x, 0) = u_0(x) - \psi(x, 0) u_0(0) = \lambda \varphi_0(x) - \frac{\varphi_0(x)}{2\varphi_0(0)} \lambda \varphi_0(0) = \frac{\lambda \varphi_0(x)}{2} \geq 0. \tag{2.7}$$

From (2.5), (2.6), (2.7) and maximum principle, it follows that

$$U(x, t) \geq 0, \quad (x, t) \in B \times [0, T),$$

which leads to (2.1). The proof of Lemma 2.1 is complete.  $\square$

**Lemma 2.2.** *Suppose  $p > 1$  and  $q \geq 0$ , let  $u(x, t)$  be a solution of (1.1), then there exists a positive constant  $\eta$  such that*

$$u_t \geq \eta \varphi(x, t) \left( u^p + \int_B u^q dx \right), \quad (x, t) \in B \times [0, T). \tag{2.8}$$

*Proof.* Putting

$$J(x, t) = u_t(x, t) - \eta \varphi(x, t) \left( u^p + \int_B u^q dx \right),$$

where  $\eta$  will be chosen later. Computing directly, we obtain

$$\begin{aligned} J_t - \Delta J - pu^{p-1}J &\geq \eta p \varphi u^{p-1} \left( u^p + \int_B u^q dx - u_t + \Delta u \right) \\ &\quad - \eta (\varphi_t - \Delta \varphi) \left( u^p + \int_B u^q dx \right) + q(1 - \eta \varphi) \int_B u^{q-1} u_t dx \\ &\quad + 2\eta p u^{p-1} \nabla u \cdot \nabla \varphi + \eta p(p-1) \varphi u^{p-2} |\nabla u|^2 \\ &\geq 2\eta p u^{p-1} \nabla u \cdot \nabla \varphi + q(1 - \eta \varphi) \int_B u^{q-1} u_t dx. \end{aligned} \tag{2.9}$$

Since  $u$  and  $\varphi$  are radially symmetric and monotone decreasing with respect to  $r$ , we have

$$\nabla u \cdot \nabla \varphi = u_r \left( \frac{x_1}{r}, \frac{x_2}{r}, \dots, \frac{x_n}{r} \right) \cdot \varphi_r \left( \frac{x_1}{r}, \frac{x_2}{r}, \dots, \frac{x_n}{r} \right) = u_r \varphi_r \geq 0.$$

On the other hand, by maximum principle, the assumption  $\Delta u_0 + u_0^p + \int_B u_0^q dx \geq \delta$  implies that  $u_t \geq 0$ . Choosing  $\eta$  small enough such that

$$1 - \eta \varphi(x, t) \geq 0, \tag{2.10}$$

we then have

$$J_t - \Delta J - pu^{p-1}J \geq 0. \tag{2.11}$$

Moreover, we can verify that

$$\begin{aligned} J(x, 0) &= u_t(x, 0) - \eta\varphi_0(x) \left( u_0^p(x) + \int_B u_0^q(x) dx \right) \\ &= \Delta u_0(x) + u_0^p(x) + \int_B u_0^q(x) dx - \eta\varphi_0(x) \left( u_0^p(x) + \int_B u_0^q(x) dx \right) \\ &\geq \mu - \eta\varphi_0(0) \left( u_0^p(0) + \int_B u_0^q(0) dx \right) \\ &\geq 0, \end{aligned} \tag{2.12}$$

holds for sufficiently small  $\eta$ . In addition, for any  $(x, t) \in \partial B \times (0, T)$ , we have

$$J(x, t) = 0. \tag{2.13}$$

From (2.11), (2.12), (2.13) and maximum principle, it follows that

$$J(x, t) \geq 0, \quad (x, t) \in B \times [0, T),$$

which yields (2.8). The proof of Lemma 2.2 is complete.  $\square$

### 3. PROOF OF THEOREMS 1.1 AND 1.2

In this section, we will discuss the single point and total blow-up phenomena according to the different values of  $p$  and  $q$ . Firstly, we give the proof of Theorem 1.1.

*Proof of Theorem 1.1.* Since

$$\lim_{t \rightarrow T} u(0, t) = +\infty,$$

we can easily show the total blow-up result under the condition  $q > p$  from Lemma 2.1.

Moreover, noticing the fact that

$$\max_{x \in \bar{B}} u(x, t) = u(0, t),$$

we have

$$\Delta u(0, t) \leq 0.$$

On the other hand, thanks to  $q > p$  and  $q > 1$ , there exists  $t_1 \in (0, T)$  such that

$$\begin{aligned} u_t(0, t) &= \Delta u(0, t) + u^p(0, t) + \int_B u^q(0, t) dx \\ &\leq u^p(0, t) + \int_B u^q(0, t) dx \\ &\leq (|B| + 1) u^q(0, t), \quad t \in (t_1, T). \end{aligned} \tag{3.1}$$

Combining (3.1) with Lemma 2.1, we obtain

$$C_1 (T - t)^{-\frac{1}{q-1}} \leq C u(0, t) \leq u(x, t), \quad (x, t) \in K \times (t_1, T), \tag{3.2}$$

where  $K$  is any compact subset of  $B$ .

Furthermore, it follows that, from (2.1) and (2.8),

$$u_t(0, t) \geq \eta \varphi(0, t) \int_B u^q dx \geq \eta |B| \varphi(0, t) \psi^q(0, t) u^q(0, t). \quad (3.3)$$

Integrating (3.3) from 0 to  $t$ , we conclude

$$u(x, t) \leq u(0, t) \leq C_2 (T - t)^{-\frac{1}{q-1}}, \quad (x, t) \in B \times (0, T). \quad (3.4)$$

Combining (3.2) with (3.3), we get (1.6). The proof of Theorem 1.1 is complete.  $\square$

Now, by the method of contradiction, we give the proof of Theorem 1.2.

*Proof of Theorem 1.2.* Using mean value theorem, and noticing the fact that  $u(x, t)$  is radially symmetric and monotone decreasing with respect to  $r$ , we know that there exists a unique point  $x^*$  such that

$$\int_B u^q dx = |B| u^q(x^*, t), \quad x^* \neq 0 \text{ and } x^* \notin \partial B.$$

We suppose that, on the contrary, there is a blow-up point  $x_0 \neq 0$  ( $|x_0| \leq |x^*| = r_0$ ). Since  $u(x, t)$  is radially symmetric and monotone decreasing on  $r$ , then for any  $r_1 \in [0, r_0]$ , we see

$$\lim_{t \rightarrow T} u(r_1, t) = \infty.$$

Letting  $0 < \mu_1 < \mu < \mu_2 < r_0$ , and

$$S_0(\mu, \gamma) = \{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : \mu < x_1 < \mu + \gamma, 0 < x_j < \gamma (j = 2, \dots, N)\},$$

here  $\gamma$  is a sufficiently small constant such that

$$\overline{S_0(\mu, \gamma)} \in B(\mu_2) \setminus \overline{B}(\mu_1).$$

Defining an auxiliary function as the form

$$F(x, t) = u_{x_1}(x, t) + \epsilon b(x) u^m(x, t), \quad (x, t) \in S_0 \times [0, T], \quad (3.5)$$

where  $\epsilon, m$  will be determined later, and

$$b(x) = \sin \frac{\pi(x_1 - \mu)}{\gamma} \prod_{j=2}^N \sin \frac{\pi x_j}{\gamma}.$$

Calculating directly, we obtain

$$\begin{aligned} & F_t - \Delta F - \left( pu^{p-1} - \frac{2\epsilon m \nabla b \cdot \nabla u}{u_{x_1}} u^{m-1} \right) F \\ &= \epsilon(m-p) bu^{p+m-1} + \epsilon mbu^{m-1} \int_B u^q dx + \frac{\epsilon \pi^2 N}{\gamma^2} bu^m \\ &= \epsilon(m-p) bu^{p+m-1} + \epsilon mb |B| u^{m-1} u^q(x^*, t) + \frac{\epsilon \pi^2 N}{\gamma^2} bu^m \\ &\quad - \epsilon m(m-1) bu^{m-2} |\nabla u|^2 + \frac{2m\epsilon^2 \nabla b \cdot \nabla u}{u_{x_1}} bu^{2m-1} \\ &\leq \epsilon(m-p) bu^{p+m-1} + \epsilon mb |B| u^{m+q-1} + \frac{\epsilon \pi^2 N}{\gamma^2} bu^m + \frac{2m\epsilon^2 \nabla b \cdot \nabla u}{u_{x_1}} bu^{2m-1}. \end{aligned}$$

On the other hand, it is easy to verify that

$$0 < \frac{2m\epsilon \nabla b \cdot \nabla u}{u_{x_1}} < \frac{2m\pi\epsilon N \mu_2}{\gamma \mu_1}.$$

For the case  $p = q$ , we can choose  $m < \frac{p}{1+|B|}$  and  $\tau_1$  large enough such that

$$\begin{aligned} & F_t - \Delta F - \left( pu^{p-1} - \frac{2\epsilon m \nabla b \cdot \nabla u}{u_{x_1}} u^{m-1} \right) F \\ & \leq -\epsilon b u^{m-1} \left\{ [p - m(1 + |B|)] u^p - \frac{\pi^2 N}{\gamma^2} u - \frac{2m\pi\epsilon N \mu_2}{\gamma \mu_1} u^m \right\} \\ & \leq 0, \end{aligned} \tag{3.6}$$

holds for every  $(x, t) \in S_0 \times [\tau_1, T)$ .

For the case  $p > q$ , we can take  $m < p$  and  $\tau_2$  large enough such that, for any  $(x, t) \in S_0 \times [\tau_2, T)$ , the following inequality holds

$$\begin{aligned} & F_t - \Delta F - \left( pu^{p-1} - \frac{2\epsilon m \nabla b \cdot \nabla u}{u_{x_1}} u^{m-1} \right) F \\ & \leq -\epsilon b u^{m-1} \left\{ (p - m) u^p - m |B| u^q - \frac{\pi^2 N}{\gamma^2} u - \frac{2m\pi\epsilon N \mu_2}{\gamma \mu_1} u^m \right\} \\ & \leq 0. \end{aligned} \tag{3.7}$$

Next, putting  $\tau = \max \{ \tau_1, \tau_2 \}$ , and taking  $\epsilon$  small enough, such that

$$\begin{aligned} F(x, \tau) &= u_{x_1}(x, \tau) + \epsilon b(x) u^m(x, \tau) \\ &\leq \max_{x \in S_0} u_{x_1}(x, \tau) + \epsilon \max_{x \in S_0} u^m(x, \tau) < 0. \end{aligned} \tag{3.8}$$

In addition, we can easily check that,

$$F(x, t) = u_{x_1}(x, t) < 0, \quad (x, t) \in \partial S_0 \times [\tau, T). \tag{3.9}$$

Combining (3.6), (3.7) and (3.8) with (3.9), we conclude immediately that

$$F(x, t) \leq 0, \quad \text{for any } (x, t) \in S_0 \times [\tau, T),$$

which implies

$$-u^{-m} u_{x_1} \geq \epsilon b(x). \tag{3.10}$$

Fixing

$$a' = (a_2, \dots, a_N) \in \mathbb{R}^{N-1},$$

and denoting

$$a_1 = (\mu + \gamma, a_2, \dots, a_N).$$

Integrating (3.10) by  $x_1$  from  $\mu$  to  $\mu + \gamma$ , we have

$$0 < \int_{\mu}^{\mu+\gamma} \epsilon b(x) dx_1 = \frac{2\epsilon\gamma}{\pi} \prod_{j=2}^N \sin \frac{\pi x_j}{\gamma} \leq \frac{1}{(m-1) u^{m-1}(a_1, t)}. \tag{3.11}$$

Since

$$\lim_{t \rightarrow T} u(a_1, t) = +\infty,$$

from (3.11), we have a contradiction. Hence,  $u(x, t)$  blows up only at the origin.

In light of

$$\max_{x \in B} u(x, t) = u(0, t),$$

then similar to the process of the derivation of (3.1) and (3.2), we find that

$$C_3 (T - t)^{-\frac{1}{p-1}} \leq u(0, t), \quad t \rightarrow T. \tag{3.12}$$

Now, using Lemma 2.2, we get

$$u_t(0, t) \geq \eta \varphi(0, t) u^p(0, t). \quad (3.13)$$

From (3.13), it follows immediately that

$$u(0, t) \leq C_4 (T - t)^{-\frac{1}{p-1}}, \quad t \rightarrow T. \quad (3.14)$$

Combining (3.12) with (3.14), we arrive at (1.7). The proof of Theorem 1.2 is complete.  $\square$

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**COMMON FIXED POINT RESULTS FOR WEAKLY COMPATIBLE MAPPINGS USING C-CLASS FUNCTIONS**

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ABSTRACT. In this paper, using the concept of C-class function, we prove the existence of common fixed point for generalized Zamfirescu-type mappings and generalized weakly Zamfirescu-type mappings. Our results generalize so many results in the literature.

1. INTRODUCTION

In 1922, Banach proved the existence of fixed point on complete metric space  $(X, d)$ . A mapping  $f$  has been considered to be a contraction and a self-mapping.

**Definition 1.1.** Let  $(X, d)$  be a metric space. A mapping  $f : X \rightarrow X$  is said to be a contraction mapping if there exists  $k \in [0, 1)$  such that

$$d(f(x), f(y)) \leq kd(x, y).$$

Later many authours have proved fixed point existence on several type of generalized contractions. Kannan type and Chatterjea type mappings were significant type of mappings since they provided existence of fixed point for non-continuous mappings in literature (see [4, 6]).

In 1972, Zamfirescu [7] generalized functions of Banach, Kannan and Chatterjea by introducing a new kind of mapping and proved the existence of fixed points for mappings.

**Definition 1.2.** Let  $(X, d)$  be a metric space. A mapping  $f : D \rightarrow X$  is said to be a Zamfirescu mapping if for all  $x, y \in X$  it satisfies the condition

$$d(f(x), f(y)) \leq kM_f(x, y)$$

for some  $k \in [0, 1)$ , where

$$M_f(x, y) := \max \left\{ d(x, y), \frac{1}{2}[d(x, f(x)) + d(y, f(y))], \frac{1}{2}[d(x, f(y)) + d(y, f(x))] \right\}.$$

Apart all these generalizations, Dugundji and Granas [5] in 1978 generalized the contraction mapping as follows.

**Definition 1.3.** [5] Let  $(X, d)$  be a metric space and  $D \subset X$ . A mapping  $f : D \rightarrow X$  is said to be a weakly contraction mapping if there exists  $\alpha : D \times D \rightarrow [0, 1]$  such that  $\Theta(a, b) := \sup\{\alpha(x, y) : a \leq d(x, y) \leq b\} < 1$  for every  $0 < a \leq b$  and for all  $x, y \in D$ ,

$$d(f(x), f(y)) \leq \alpha(x, y)d(x, y).$$

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In 2014, the concept of  $C$ -class functions was introduced by Ansari [2]. By using this concept, we can generalize many fixed point theorems in the literature.

**Definition 1.4.** [2] A function  $F : [0, \infty)^2 \rightarrow \mathbb{R}$  is called a  $C$ -class function (also denoted as  $\mathcal{C}$ ) if it is continuous and satisfies the following:

- (1)  $F(s, t) \leq s$ ;
- (2)  $F(s, t) = s$  implies that either  $s = 0$  or  $t = 0$  for all  $s, t \in [0, \infty)$ .

**Example 1.5.** [2] The following functions  $f : [0, \infty)^2 \rightarrow \mathbb{R}$  are elements of  $\mathcal{C}$ , for all  $s, t \in [0, \infty)$ :

- (1)  $f(s, t) = s - t, f(s, t) = s \Rightarrow t = 0$ ;
- (2)  $f(s, t) = ms, 0 < m < 1, f(s, t) = s \Rightarrow s = 0$ ;
- (3)  $f(s, t) = \frac{s}{(1+t)^r}; r \in (0, \infty), f(s, t) = s \Rightarrow s = 0$  or  $t = 0$ ;
- (4)  $f(s, t) = \log(t + a^s)/(1 + t), a > 1, f(s, t) = s \Rightarrow s = 0$  or  $t = 0$ ;
- (5)  $f(s, t) = \ln(1 + a^s)/2, a > e, f(s, 1) = s \Rightarrow s = 0$ .

**Definition 1.6.** Let  $\Psi$  denote all the functions  $\psi : [0, \infty) \rightarrow [0, \infty)$  which satisfy

- (i)  $\psi(t) = 0$  if and only if  $t = 0$ ,
- (ii)  $\psi$  is continuous,
- (iii)  $\psi(s) \leq s, \forall s > 0$ .

**Definition 1.7.** Let  $(X, d)$  be a metric space. Then  $f, g : X \rightarrow X$  are said to be weakly compatible if  $fg(x) = gf(x)$  for  $x \in X$  whenever  $f(x) = g(x)$ .

**Lemma 1.8.** ([3]) Suppose that  $(X, d)$  is a metric space. Let  $\{x_n\}$  be a sequence in  $X$  such that  $d(x_n, x_{n+1}) \rightarrow 0$  as  $n \rightarrow \infty$ . If  $\{x_n\}$  is not a Cauchy sequence, then there exist an  $\varepsilon > 0$  and sequences of positive integers  $\{m(k)\}$  and  $\{n(k)\}$  with  $m(k) > n(k) > k$  such that  $d(x_{m(k)}, x_{n(k)}) \geq \varepsilon, d(x_{m(k)-1}, x_{n(k)}) < \varepsilon$  and

- (i)  $\lim_{k \rightarrow \infty} d(x_{m(k)-1}, x_{n(k)+1}) = \varepsilon$ ;
- (ii)  $\lim_{k \rightarrow \infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon$ ;
- (iii)  $\lim_{k \rightarrow \infty} d(x_{m(k)-1}, x_{n(k)}) = \varepsilon$ .

In this paper, we prove the existence of common fixed point for two weakly compatible mappings on a complete metric space.

## 2. MAIN RESULTS

**Definition 2.1.** Let  $(X, d)$  be a metric space. Consider two self-mappings  $f$  and  $g$  on  $X$  and let  $\alpha : X \times X \rightarrow [0, 1]$  be a function. Then  $g$  is an  $f$ -weakly generalized Zamfirescu type mapping if, for all  $F \in \mathcal{C}, \psi \in \Psi$  and for all  $x, y \in X$ ,

$$d(g(x), g(y)) \leq F\left(\alpha(f(x), f(y)) \max\left\{d(f(x), f(y)), \frac{1}{2}[d(f(x), g(x)) + d(f(y), g(y))], \frac{1}{2}[d(f(x), g(y)) + d(f(y), g(x))]\right\}, \psi\left(\alpha(f(x), f(y)) \max\left\{d(f(x), f(y)), \frac{1}{2}[d(f(x), g(x)) + d(f(y), g(y))], \frac{1}{2}[d(f(x), g(y)) + d(f(y), g(x))]\right\}\right)\right).$$

**Theorem 2.2.** *Let  $(X, d)$  be a complete metric space and  $f, g : X \rightarrow X$  mappings such that  $g$  is an  $f$ -weakly generalized Zamfirescu type mapping. Then  $f$  and  $g$  have a unique common fixed point on  $X$  if the following conditions are satisfied:*

- (1)  $g(X) \subset f(X)$ ;
- (2)  $f(X)$  is complete;
- (3)  $f, g$  are weakly compatible.

*Proof.* Choose  $x_0 \in Y$  arbitrarily and  $x_n \in X$  such that  $f(x_n) = g(x_{n-1})$ . Then

$$\begin{aligned}
 & d(f(x_n), f(x_{n+1})) = d(g(x_{n-1}), g(x_n)) \tag{2.1} \\
 & \leq F\left(\alpha(f(x_{n-1}), f(x_n)) \max\left\{d(f(x_{n-1}), f(x_n)), \frac{1}{2}[d(f(x_{n-1}), g(x_{n-1})) + d(f(x_n), g(x_n))], \right. \right. \\
 & \qquad \qquad \qquad \left. \left. \frac{1}{2}[d(f(x_n), g(x_{n-1})) + d(f(x_{n-1}), g(x_n))]\right\}, \right. \\
 & \qquad \left. \psi\left(\alpha(f(x_{n-1}), f(x_n)) \max\left\{d(f(x_{n-1}), f(x_n)), \frac{1}{2}[d(f(x_{n-1}), g(x_{n-1})) + d(f(x_n), g(x_n))], \right. \right. \right. \\
 & \qquad \qquad \qquad \left. \left. \frac{1}{2}[d(f(x_n), g(x_{n-1})) + d(f(x_{n-1}), g(x_n))]\right\}\right)\right) \\
 & \leq \alpha(f(x_{n-1}), f(x_n)) \max\left\{d(f(x_{n-1}), f(x_n)), \frac{1}{2}[d(f(x_{n-1}), g(x_{n-1})) + d(f(x_n), g(x_n))], \right. \\
 & \qquad \qquad \qquad \left. \frac{1}{2}[d(f(x_n), g(x_{n-1})) + d(f(x_{n-1}), g(x_n))]\right\} \\
 & \leq \alpha(f(x_{n-1}), f(x_n)) \max\left\{d(f(x_{n-1}), f(x_n)), \frac{1}{2}[d(f(x_{n-1}), f(x_n)) + d(f(x_n), f(x_{n+1}))], \right. \\
 & \qquad \qquad \qquad \left. \frac{1}{2}[d(f(x_n), f(x_n)) + d(f(x_{n-1}), f(x_{n+1}))]\right\} \\
 & \leq \alpha(f(x_{n-1}), f(x_n)) \max\left\{d(f(x_{n-1}), f(x_n)), \frac{1}{2}[d(f(x_{n-1}), f(x_n)) + d(f(x_n), f(x_{n+1}))], \right. \\
 & \qquad \qquad \qquad \left. \frac{1}{2}[d(f(x_{n-1}), f(x_n)) + d(f(x_n), f(x_{n+1}))]\right\} \\
 & \leq \alpha(f(x_{n-1}), f(x_n)) \max\left\{d(f(x_{n-1}), f(x_n)), \frac{1}{2}[d(f(x_{n-1}), f(x_n)) + d(f(x_n), f(x_{n+1}))]\right\}.
 \end{aligned}$$

**Claim:**  $d(f(x_n), f(x_{n+1})) \leq \alpha(f(x_{n-1}), f(x_n))d(f(x_{n-1}), f(x_n))$ .

Suppose that

$$\begin{aligned}
 d(f(x_n), f(x_{n+1})) & \leq \frac{\alpha(f(x_{n-1}), f(x_n))}{2} [d(f(x_{n-1}), f(x_n)) + d(f(x_n), f(x_{n+1}))] \\
 \implies d(f(x_n), f(x_{n+1})) & \leq \frac{\alpha(f(x_{n-1}), f(x_n))}{2 - \alpha(f(x_{n-1}), f(x_n))} d(f(x_{n-1}), f(x_n)) \\
 & \leq \alpha(f(x_{n-1}), f(x_n)) d(f(x_{n-1}), f(x_n)).
 \end{aligned}$$

Then  $\{d(f(x_n), f(x_{n+1}))\}$  is positive, decreasing and converges to some  $d \in [0, \infty)$ .

Now, letting  $n \rightarrow \infty$  in (2.1), we get

$$\begin{aligned}
 d & \leq \lim_{n \rightarrow \infty} \alpha(f(x_{n-1}), f(x_n)) \max\left\{d(f(x_{n-1}), f(x_n)), \frac{1}{2}[d(f(x_{n-1}), g(x_{n-1})) + d(f(x_n), g(x_n))], \right. \\
 & \qquad \qquad \qquad \left. \frac{1}{2}[d(f(x_n), g(x_{n-1})) + d(f(x_{n-1}), g(x_n))]\right\} \\
 & \leq d,
 \end{aligned}$$

which implies that

$$d = \lim_{n \rightarrow \infty} \alpha(f(x_{n-1}), f(x_n)) \max \left\{ d(f(x_{n-1}), f(x_n)), \right. \\ \left. \frac{1}{2} [d(f(x_{n-1}), g(x_{n-1})) + d(f(x_n), g(x_n))], \frac{1}{2} [d(f(x_n), g(x_{n-1})) + d(f(x_{n-1}), g(x_n))] \right\}. \quad (2.2)$$

Again, letting  $n \rightarrow \infty$  in (2.1) and using (2.2), we get

$$d \leq F(d, \psi(d)) \leq d.$$

Thus  $F(d, \psi(d)) = d$ , which implies that  $d = 0$ .

Next, we prove that  $\{f(x_n)\}$  is Cauchy. Suppose not. Then by Lemma 1.8, there exist sequences of positive integers  $\{m(k)\}$  and  $\{n(k)\}$  with  $m_k > n_k \geq k$  such that  $d(f(x_{m_k-1}), f(x_{n_k}))$  and  $d(f(x_{m_k}), f(x_{n_k}))$  converge to some  $\delta > 0$ . So

$$d(f(x_{m_k}), f(x_{n_k})) = d(g(x_{m_k-1}), g(x_{n_k-1})) \quad (2.3) \\ \leq F \left( \alpha(f(x_{m_k-1}), f(x_{n_k-1})) \max \left\{ d(f(x_{m_k-1}), f(x_{n_k-1})), \right. \right. \\ \left. \left. \frac{1}{2} [d(f(x_{m_k-1}), g(x_{m_k-1})) + d(f(x_{n_k-1}), g(x_{n_k-1}))], \right. \right. \\ \left. \left. \frac{1}{2} [d(f(x_{m_k-1}), g(x_{n_k-1})) + d(f(x_{n_k-1}), g(x_{m_k-1}))] \right\}, \right. \\ \left. \psi \left( \alpha(f(x_{m_k-1}), f(x_{n_k-1})) \max \left\{ d(f(x_{m_k-1}), f(x_{n_k-1})), \right. \right. \right. \\ \left. \left. \frac{1}{2} [d(f(x_{m_k-1}), g(x_{m_k-1})) + d(f(x_{n_k-1}), g(x_{n_k-1}))], \right. \right. \\ \left. \left. \frac{1}{2} [d(f(x_{m_k-1}), g(x_{n_k-1})) + d(f(x_{n_k-1}), g(x_{m_k-1}))] \right\} \right) \right) \\ \leq \alpha(f(x_{m_k-1}), f(x_{n_k-1})) \max \left\{ d(f(x_{m_k-1}), f(x_{n_k-1})), \right. \\ \left. \frac{1}{2} [d(f(x_{m_k-1}), g(x_{m_k-1})) + d(f(x_{n_k-1}), g(x_{n_k-1}))], \right. \\ \left. \frac{1}{2} [d(f(x_{m_k-1}), g(x_{n_k-1})) + d(f(x_{n_k-1}), g(x_{m_k-1}))] \right\} \\ \leq \max \left\{ d(f(x_{m_k-1}), f(x_{n_k-1})), \frac{1}{2} [d(f(x_{m_k-1}), g(x_{m_k-1})) + d(f(x_{n_k-1}), g(x_{n_k-1}))], \right. \\ \left. \frac{1}{2} [d(f(x_{m_k-1}), g(x_{n_k-1})) + d(f(x_{n_k-1}), g(x_{m_k-1}))] \right\} \\ \leq \max \left\{ [d(f(x_{m_k-1}), f(x_{n_k})) + d(f(x_{n_k}), f(x_{n_k-1}))], d(f(x_{n_k-1}), f(x_{n_k})), \right. \\ \left. \frac{1}{2} [d(f(x_{m_k-1}), f(x_{n_k})) + d(f(x_{n_k-1}), f(x_{n_k})) + d(f(x_{n_k}), f(x_{m_k}))] \right\}.$$

Letting  $n \rightarrow \infty$  in (2.3), we get

$$\delta = \lim_{k \rightarrow \infty} \alpha(f(x_{m_k-1}), f(x_{n_k-1})) \max \left\{ d(f(x_{m_k-1}), f(x_{n_k})), \right. \\ \left. \frac{1}{2} [d(f(x_{m_k-1}), g(x_{m_k-1})) + d(f(x_{n_k}), g(x_{n_k}))], \right. \\ \left. \frac{1}{2} [d(f(x_{m_k-1}), g(x_{n_k})) + d(f(x_{n_k}), g(x_{m_k-1}))] \right\}. \quad (2.4)$$

Again, letting  $n \rightarrow \infty$  in (2.3) and using (2.4),

$$\delta \leq F(\delta, \psi(\delta)) \leq \delta.$$

Thus  $F(\delta, \psi(\delta)) = \delta$ , which implies that  $\delta = 0$ , which is a contradiction.

Therefore,  $\{f(x_m)\}$  is Cauchy and converges to  $x = f(u)$  for some  $x \in X$ .

Next, we prove that  $d(f(u), g(u)) = 0$ .

$$\begin{aligned} d(g(u), x) &= \lim_{n \rightarrow \infty} d(g(u), g(x_n)) \\ &\leq \lim_{n \rightarrow \infty} F\left(\alpha(f(u), f(x_n)) \max\left\{d(f(u), f(x_n)), \frac{1}{2}[d(f(u), g(u)) + d(f(x_n), g(x_n))], \right. \right. \\ &\quad \left. \left. \frac{1}{2}[d(f(x_n), g(u)) + d(f(u), g(x_n))]\right\}, \right. \\ &\quad \left. \psi\left(\alpha(f(u), f(x_n)) \max\left\{d(f(u), f(x_n)), \frac{1}{2}[d(f(u), g(u)) + d(f(x_n), g(x_n))], \right. \right. \right. \\ &\quad \left. \left. \frac{1}{2}[d(f(x_n), g(u)) + d(f(u), g(x_n))]\right\}\right)\right) \\ &\leq \lim_{n \rightarrow \infty} \alpha(f(u), f(x_n)) \max\left\{d(f(u), f(x_n)), \frac{1}{2}[d(f(u), g(u)) + d(f(x_n), g(x_n))], \right. \\ &\quad \left. \frac{1}{2}[d(f(x_n), g(u)) + d(f(u), g(x_n))]\right\} \\ &= \lim_{n \rightarrow \infty} \alpha(f(u), f(x_n)) \max\left\{d(f(u), f(x_n)), \frac{1}{2}[d(f(u), g(u)) + d(f(x_n), f(x_{n+1}))], \right. \\ &\quad \left. \frac{1}{2}[d(f(x_n), g(u)) + d(f(u), f(x_{n+1}))]\right\} \\ &\leq \frac{1}{2}d(f(u), g(u)) \leq \frac{1}{2}d(x, g(u)). \end{aligned}$$

So  $x = g(u) = f(u)$  on  $X$ . Therefore, by the weak compatibility of  $f$  and  $g$ , we have  $f(x) = fg(u) = gf(u) = g(x)$ .

**Claim:**  $x$  is a common fixed point of  $f$  and  $g$ .

$$\begin{aligned} d(x, g(x)) &= d(g(u), g(x)) \\ &\leq F\left(\alpha(f(u), f(x)) \max\left\{d(f(u), f(x)), \frac{1}{2}[d(f(u), g(u)) + d(f(x), g(x))], \right. \right. \\ &\quad \left. \left. \frac{1}{2}[d(f(u), g(x)) + d(f(x), g(u))]\right\}, \right. \\ &\quad \left. \psi\left(\alpha(f(u), f(x)) \max\left\{d(f(u), f(x)), \frac{1}{2}[d(f(u), g(u)) + d(f(x), g(x))], \right. \right. \right. \\ &\quad \left. \left. \frac{1}{2}[d(f(u), g(x)) + d(f(x), g(u))]\right\}\right)\right) \\ &\leq \alpha(f(u), f(x)) \max\left\{d(f(u), f(x)), \frac{1}{2}[d(f(u), g(u)) + d(f(x), g(x))], \right. \\ &\quad \left. \frac{1}{2}[d(f(u), g(x)) + d(f(x), g(u))]\right\} \\ &\leq d(x, g(x)). \end{aligned}$$

Thus  $F(d(x, g(x)), \psi(d(x, g(x)))) = d(x, g(x))$ , which implies  $d(x, g(x)) = 0$ . So  $x$  is a common fixed point of  $f$  and  $g$  on  $X$ .

**Uniqueness of common fixed point:**

Suppose that  $x$  and  $x'$  are common fixed points of  $f$  and  $g$ . Then

$$\begin{aligned} d(x, x') &= (f(x), f(x')) = d(g(x), g(x')) \\ &\leq F\left(\alpha(f(x), f(x')) \max\left\{d(f(x), f(x')), \frac{1}{2}[d(f(x), g(x)) + d(f(x'), g(x'))], \right. \right. \\ &\quad \left. \left. \frac{1}{2}[d(f(x), g(x')) + d(f(x'), g(x))]\right\}, \right. \\ &\quad \left. \psi\left(\alpha(f(x), f(x')) \max\left\{d(f(x), f(x')), \frac{1}{2}[d(f(x), g(x)) + d(f(x'), g(x'))], \right. \right. \right. \\ &\quad \left. \left. \frac{1}{2}[d(f(x), g(x')) + d(f(x'), g(x))]\right\}\right)\right) \\ &\leq \alpha(f(x), f(x')) \max\left\{d(f(x), f(x')), \frac{1}{2}[d(f(x), g(x)) + d(f(x'), g(x'))], \right. \\ &\quad \left. \frac{1}{2}[d(f(x), g(x')) + d(f(x'), g(x))]\right\} \\ &\leq \alpha(f(x), f(x')) \max\left\{d(x, x'), \frac{1}{2}[d(x, x') + d(x', x)]\right\} \\ &\leq d(x, x'). \end{aligned}$$

Therefore,  $F(d(x, x'), \psi(d(x, x'))) = d(x, x')$ , which implies  $d(x, x') = 0$ . So  $x$  is the unique common fixed point of  $f$  and  $g$  on  $X$ .  $\square$

**Definition 2.3.** Let  $(X, d)$  be a metric space. Consider two self-mappings  $f$  and  $g$  on  $X$  and let  $\alpha : X \times X \rightarrow [0, 1]$  be a function. Then  $g$  is said to satisfy condition (A) on  $f$  if, for all  $F \in \mathcal{C}$ ,  $\psi \in \Psi$ ,  $k \in [0, 1]$  and for all  $x, y \in X$ ,

$$\begin{aligned} d(g(x), g(y)) &\leq k \alpha(f(x), f(y)) \max\left\{(d(f(x), f(y)), \frac{1}{2}[d(f(x), g(x)) + d(f(y), g(y))], \right. \\ &\quad \left. \frac{1}{2}[d(f(x), g(y)) + d(f(y), g(x))]\right\}. \end{aligned}$$

**Corollary 2.4.** Let  $(X, d)$  be a complete metric space and  $f, g : X \rightarrow X$  mappings such that  $g$  satisfies the condition (A) on  $f$ . Then  $f$  and  $g$  have a unique common fixed point on  $X$  if the following conditions are satisfied:

- (1)  $g(X) \subset f(X)$ ;
- (2)  $f(X)$  is complete;
- (3)  $f, g$  are weakly compatible.

*Proof.* Choose  $x_0 \in Y$  arbitrarily. Let  $x_n \in X$  be the element such that  $f(x_n) = g(x_{n-1})$  and define a function  $F_1 : [0, \infty)^2 \rightarrow \mathbb{R}$  as  $F_1(s, t) = ks$  for all  $k \in [0, 1]$  which is a  $C$ -class function. Since  $f$  and  $g$  satisfy the condition (A),

$$\begin{aligned} d(g(x), g(y)) &\leq k \alpha(f(x), f(y)) \max\left\{(d(f(x), f(y)), \frac{1}{2}[d(f(x), g(x)) + d(f(y), g(y))], \right. \\ &\quad \left. \frac{1}{2}[d(f(x), g(y)) + d(f(y), g(x))]\right\} \end{aligned}$$

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$$\begin{aligned}
 &= F_1\left(\alpha(f(x), f(y))\max\left\{(d(f(x), f(y)), \frac{1}{2}[d(f(x), g(x)) + d(f(y), g(y))], \right. \right. \\
 &\qquad\qquad\qquad \left. \left. \frac{1}{2}[d(f(x), g(y)) + d(f(y), g(x))]\right\}\right. \\
 &\quad \left. \psi\left(\alpha(f(x), f(y))\max\left\{(d(f(x), f(y)), \frac{1}{2}[d(f(x), g(x)) + d(f(y), g(y))], \right. \right. \right. \\
 &\qquad\qquad\qquad \left. \left. \frac{1}{2}[d(f(x), g(y)) + d(f(y), g(x))]\right\}\right)\right).
 \end{aligned}$$

Hence by Theorem 2.2,  $f$  and  $g$  have a unique common fixed point in  $X$ . □

**Definition 2.5.** Let  $(X, d)$  be a metric space. Let  $f$  and  $g$  be two self-mappings on  $X$ . Then  $g$  is said to satisfy condition (B) on  $f$  if, for all  $F \in \mathcal{C}$ ,  $\psi \in \Psi$  and for all  $x, y \in X$ ,

$$\begin{aligned}
 &d(g(x), g(y)) \\
 &\leq F\left(\max\left\{(d(f(x), f(y)), \frac{1}{2}[d(f(x), g(x)) + d(f(y), g(y))], \frac{1}{2}[d(f(x), g(y)) + d(f(y), g(x))]\right\}, \right. \\
 &\quad \left. \psi\left(\max\left\{(d(f(x), f(y)), \frac{1}{2}[d(f(x), g(x)) + d(f(y), g(y))], \frac{1}{2}[d(f(x), g(y)) + d(f(y), g(x))]\right\}\right)\right).
 \end{aligned}$$

**Corollary 2.6.** Let  $(X, d)$  be a complete metric space and  $f, g : X \rightarrow X$  mappings such that  $g$  satisfies the condition (B) on  $f$ . Then  $f$  and  $g$  have a unique common fixed point on  $X$  if the following conditions are satisfied:

- (1)  $g(X) \subset f(X)$ ;
- (2)  $f(X)$  is complete;
- (3)  $f, g$  are weakly compatible.

*Proof.* By Theorem 2.2, if  $\alpha(x, y) = 1$  for all  $x, y \in X$ , then the mappings  $f$  and  $g$  have a unique common fixed point on  $X$ . □

**Definition 2.7.** Let  $(X, d)$  be a metric space and  $f, g$  be two self-mappings on  $X$ . Then  $g$  is said to satisfy condition (C) on  $f$  if, for all  $x, y \in X$  and  $a, b, c \in [0, 1)$ ,

$$\begin{aligned}
 d(g(x), g(y)) \leq \max\left\{a(d(f(x), f(y)), \frac{b}{2}[d(f(x), g(x)) + d(f(y), g(y))], \right. \\
 \left. \frac{c}{2}[d(f(x), g(y)) + d(f(y), g(x))]\right\}.
 \end{aligned}$$

*Remark 2.8.* If we choose  $f = I^X$  ( $I^X$  is the identity mapping in the condition (C), then we obtain the definition of Zamfierscu mapping [7].

**Corollary 2.9.** Let  $(X, d)$  be a complete metric space and  $f, g : X \rightarrow X$  mappings such that  $g$  satisfies the condition (C) on  $f$ . Then  $f$  and  $g$  have a unique common fixed point on  $X$  if the following conditions are satisfied:

- (1)  $g(X) \subset f(X)$ ;
- (2)  $f(X)$  is complete;
- (3)  $f, g$  are weakly compatible.

*Proof.* Choose  $x_0 \in Y$  arbitrarily. Let  $x_n \in X$  be elements such that  $f(x_n) = g(x_{n-1})$  and define a function  $F_1 : [0, \infty)^2 \rightarrow \mathbb{R}$  as  $F_1(s, t) = ks$  for all  $k \in [0, 1)$  which is a *C-class* function. Since  $f$  and  $g$  satisfy the condition (C),

$$\begin{aligned} d(g(x), g(y)) &\leq \max \left\{ a(d(f(x), f(y)), \frac{b}{2}d(f(x), g(x)) + d(f(y), g(y))), \right. \\ &\quad \left. \frac{c}{2}[d(f(x), g(y)) + d(f(y), g(x))] \right\} \\ &\leq k \max \left\{ (d(f(x), f(y)), \frac{1}{2}[d(f(x), g(x)) + d(f(y), g(y))]), \right. \\ &\quad \left. \frac{1}{2}[d(f(x), g(y)) + d(f(y), g(x))] \right\} \\ &= F_1 \left( \max \left\{ (d(f(x), f(y)), \frac{1}{2}[d(f(x), g(x)) + d(f(y), g(y))]), \right. \right. \\ &\quad \left. \left. \frac{1}{2}[d(f(x), g(y)) + d(f(y), g(x))] \right\}, \right. \\ &\quad \left. \psi \left( \max \left\{ (d(f(x), f(y)), \frac{1}{2}[d(f(x), g(x)) + d(f(y), g(y))]), \right. \right. \right. \\ &\quad \left. \left. \frac{1}{2}[d(f(x), g(y)) + d(f(y), g(x))] \right\} \right) \right). \end{aligned}$$

Hence by Corollary 2.6,  $f$  and  $g$  have a unique common fixed point in  $X$ . □

**Definition 2.10.** Let  $(X, d)$  be a metric space and  $f$  and  $g$  two self-mappings on  $X$ . Then  $g$  is said to satisfy condition (D) on  $f$  if, for all  $x, y \in X$ ,

$$\begin{aligned} d(g(x), g(y)) &\leq \max \left\{ (d(f(x), f(y)), \frac{1}{2}[d(f(x), g(x)) + d(f(y), g(y))]), \right. \\ &\quad \left. \frac{1}{2}[d(f(x), g(y)) + d(f(y), g(x))] \right\} \\ &\quad - \Psi \left( \max \left\{ (d(f(x), f(y)), \frac{1}{2}[d(f(x), g(x)) + d(f(y), g(y))]), \right. \right. \\ &\quad \left. \left. \frac{1}{2}[d(f(x), g(y)) + d(f(y), g(x))] \right\} \right) \end{aligned}$$

**Corollary 2.11.** Let  $(X, d)$  be a complete metric space and  $f, g : X \rightarrow X$  mappings such that  $g$  satisfies the condition (D) on  $f$ . Then  $f$  and  $g$  have a unique common fixed point on  $X$  if the following conditions are satisfied:

- (1)  $g(X) \subset f(X)$ ;
- (2)  $f(X)$  is complete;
- (3)  $f, g$  are weakly compatible.

*Proof.* Choose  $x_0 \in Y$  arbitrarily. Let  $x_n \in X$  be elements such that  $f(x_n) = g(x_{n-1})$  and define a function  $F_2 : [0, \infty)^2 \rightarrow \mathbb{R}$  as  $F_2(s, t) = s - t$  which is a *C-class* function. Since  $f$  and  $g$  satisfies the condition (D),

$$\begin{aligned} d(g(x), g(y)) &\leq \max \left\{ (d(f(x), f(y)), \frac{1}{2}[d(f(x), g(x)) + d(f(y), g(y))]), \right. \\ &\quad \left. \frac{1}{2}[d(f(x), g(y)) + d(f(y), g(x))] \right\} \end{aligned}$$

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$$\begin{aligned}
 & - \Psi \left( \max \left\{ (d(f(x), f(y)), \frac{1}{2} [d(f(x), g(x)) + d(f(y), g(y))], \right. \right. \\
 & \qquad \qquad \qquad \left. \left. \frac{1}{2} [d(f(x), g(y)) + d(f(y), g(x))] \right\} \right) \\
 & = F_2 \left( \max \left\{ (d(f(x), f(y)), \frac{1}{2} [d(f(x), g(x)) + d(f(y), g(y))], \right. \right. \\
 & \qquad \qquad \qquad \left. \left. \frac{1}{2} [d(f(x), g(y)) + d(f(y), g(x))] \right\}, \right. \\
 & \qquad \left. \psi \left( \max \left\{ (d(f(x), f(y)), \frac{1}{2} [d(f(x), g(x)) + d(f(y), g(y))], \right. \right. \right. \\
 & \qquad \qquad \qquad \left. \left. \frac{1}{2} [d(f(x), g(y)) + d(f(y), g(x))] \right\} \right) \right).
 \end{aligned}$$

Hence by Corollary 2.6,  $f$  and  $g$  have a unique common fixed point in  $X$ . □

**Definition 2.12.** [1] Let  $X$  be a normed linear space. Then a set  $Y \in X$  is called  $q$ -starshaped with  $q \in Y$  if the segment  $[q, x] = \{(1 - k)q + kx : 0 \leq k \leq 1\}$  joining  $q$  to  $x$  is contained in  $Y$  for all  $x \in Y$ .

**Definition 2.13.** Let  $(X, d)$  be a metric space,  $f, T$  two self-mappings on  $X$  and let  $\alpha : X \times X \rightarrow [0, 1]$  be a function. Then  $T$  is said to be a  $f$ -weakly generalized almost Zamfirescu mapping if, for all  $x, y \in X$  and  $a, b, c \in (0, 1)$ ,

$$\begin{aligned}
 & \|T(x) - T(y)\| \\
 & \leq F \left( \alpha(f(x) - f(y)) \max \left\{ a \|f(x) - f(y)\|, \frac{b}{2} [dist(f(x), [q, T(x)]) + dist(f(y), [q, T(y)])], \right. \right. \\
 & \qquad \qquad \qquad \left. \left. \frac{c}{2} [dist(f(x), [q, T(y)]) + dist(f(y), [q, T(x)])] \right\}, \right. \\
 & \qquad \left. \Psi \left( \alpha(f(x) - f(y)) \max \left\{ a \|f(x) - f(y)\|, \frac{b}{2} [dist(f(x), [q, T(x)]) + dist(f(y), [q, T(y)])], \right. \right. \right. \\
 & \qquad \qquad \qquad \left. \left. \frac{c}{2} [dist(f(x), [q, T(y)]) + dist(f(y), [q, T(x)])] \right\} \right) \right).
 \end{aligned}$$

**Theorem 2.14.** Let  $f$  and  $T$  be self-mappings on a nonempty  $q$ -starshaped subset  $Y$  of a Banach space  $X$ , where  $T$  is a  $f$ -weakly generalized almost Zamfirescu mapping and satisfies the following conditions:

- (1)  $f$  is linear and  $q = f(q)$ ;
- (2)  $T(X) \subset f(X)$ ;
- (3)  $f(X)$  is complete;
- (4)  $f, T$  are weakly compatible.

Define a mapping  $T_n$  on  $Y$  by

$$T_n(x) = (1 - \beta_n)q + \beta_n T(x),$$

where  $\{\beta_n\}$  is a sequence of numbers in  $(0, 1)$ . Then for each  $n$ ,  $T_n$  and  $f$  have exactly one common fixed point  $x_n$  in  $Y$  such that  $f(x_n) = x_n = (1 - \beta_n)q + \beta_n T(x_n)$ . Also  $T$  and  $f$  have a common fixed point  $x \in Y$ . Moreover, if  $\{x_n\}$  is Cauchy and  $\lim_{n \rightarrow \infty} \beta_n = 1$ , then  $x_n \rightarrow x$ .

*Proof.* By definition,

$$\begin{aligned}
 & \|T(x) - T(y)\| \\
 & \leq F \left( \alpha(f(x) - f(y)) \max \left\{ a \|f(x) - f(y)\|, \frac{b}{2} [dist(f(x), [q, T(x)]) + dist(f(y), [q, T(y)])], \right. \right.
 \end{aligned}$$



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$$\begin{aligned}
 & \left. \frac{c}{2} [dist(f(x), [q, T(y)]) + dist(f(y), [q, T(x)])] \right\}, \\
 \Psi & \left( \alpha(f(x) - f(y)) \max \left\{ a\|f(x) - f(y)\|, \frac{b}{2} [dist(f(x), [q, T(x)]) + dist(f(y), [q, T(y)])], \right. \right. \\
 & \left. \left. \frac{c}{2} [dist(f(x), [q, T(y)]) + dist(f(y), [q, T(x)])] \right\} \right) \\
 \leq & \alpha(f(x) - f(y)) \max \left\{ a\|f(x) - f(y)\|, \frac{b}{2} [dist(f(x), [q, T(x)]) + dist(f(y), [q, T(y)])], \right. \\
 & \left. \frac{c}{2} [dist(f(x), [q, T(y)]) + dist(f(y), [q, T(x)])] \right\}, \\
 \leq & \max \left\{ a\|f(x) - f(y)\|, \frac{b}{2} [dist(f(x), [q, T(x)]) + dist(f(y), [q, T(y)])], \right. \\
 & \left. \frac{c}{2} [dist(f(x), [q, T(y)]) + dist(f(y), [q, T(x)])] \right\}, \\
 \leq & \max \left\{ a\|f(x) - f(y)\|, \frac{b}{2} [\|f(x) - T(x)\| + \|f(y) - T(y)\|], \right. \\
 & \left. \frac{c}{2} [\|f(x) - T(y)\| + \|f(y) - T(x)\|] \right\}.
 \end{aligned}$$

Therefore, by Corollary 2.9,  $T$  and  $f$  have a common fixed point  $x \in Y$ .

By definition,

$$\begin{aligned}
 \|T_n(x) - T_n(y)\| &= \beta_n \|T(x) - T(y)\| \\
 \leq & \beta_n F \left( \alpha(f(x) - f(y)) \max \left\{ a\|f(x) - f(y)\|, \frac{b}{2} [dist(f(x), [q, T(x)]) + dist(f(y), [q, T(y)])], \right. \right. \\
 & \left. \left. \frac{c}{2} [dist(f(x), [q, T(y)]) + dist(f(y), [q, T(x)])] \right\}, \right. \\
 \Psi & \left( \alpha(f(x) - f(y)) \max \left\{ a\|f(x) - f(y)\|, \frac{b}{2} [dist(f(x), [q, T(x)]) + dist(f(y), [q, T(y)])], \right. \right. \\
 & \left. \left. \frac{c}{2} [dist(f(x), [q, T(y)]) + dist(f(y), [q, T(x)])] \right\} \right) \\
 \leq & \beta_n \alpha(f(x) - f(y)) \max \left\{ a\|f(x) - f(y)\|, \frac{b}{2} [dist(f(x), [q, T(x)]) + dist(f(y), [q, T(y)])], \right. \\
 & \left. \frac{c}{2} [dist(f(x), [q, T(y)]) + dist(f(y), [q, T(x)])] \right\}, \\
 \leq & \beta_n \alpha(f(x) - f(y)) \max \left\{ \|f(x) - f(y)\|, \frac{1}{2} [\|f(x) - T(x)\| + \|f(y) - T(y)\|], \right. \\
 & \left. \frac{1}{2} [\|f(x) - T(y)\| + \|f(y) - T(x)\|] \right\}.
 \end{aligned}$$

Therefore, by Corollary 2.4,  $T_n$  and  $f$  have a common fixed point  $x \in Y$ .

By the assumption that  $\{x_n\}$  is Cauchy, let us consider  $\{x_n\} \rightarrow y$ . If  $\lim_{n \rightarrow \infty} \beta_n = 1$ , then

$$\begin{aligned}
 \|x_n - x\| &= \|T_n(x_n) - T(x)\| \\
 &= \|(1 - \beta_n)p + \beta_n T(x_n) - T(x)\| \\
 &\leq \|(1 - \beta_n)p\| + \beta_n \|T(x_n) - T(x)\|
 \end{aligned}$$

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$$\begin{aligned} &\leq \|(1 - \beta_n)p\| + \beta_n F \left( \alpha(f(x) - f(y)) \max \left\{ a\|f(x_n) - f(x)\|, \right. \right. \\ &\quad \left. \left. \frac{b}{2} [dist(f(x_n), [q, T(x_n)]) + dist(f(x), [q, T(x)])], \right. \right. \\ &\quad \left. \left. \frac{c}{2} [dist(f(x_n), [q, T(x)]) + dist(f(x), [q, T(x_n)])] \right\}, \right. \\ &\quad \Psi \left( \alpha(f(x) - f(y)) \max \left\{ a\|f(x_n) - f(x)\|, \right. \right. \\ &\quad \left. \left. \frac{b}{2} [dist(f(x_n), [q, T(x_n)]) + dist(f(x), [q, T(x)])], \right. \right. \\ &\quad \left. \left. \frac{c}{2} [dist(f(x_n), [q, T(x)]) + dist(f(x), [q, T(x_n)])] \right\} \right) \\ &\leq \|(1 - \beta_n)p\| + \beta_n \max \left\{ a\|f(x_n) - f(x)\|, \right. \\ &\quad \left. \frac{b}{2} [\|f(x_n) - T(x_n)\| + \|f(x) - T(x)\|], \right. \\ &\quad \left. \frac{c}{2} [\|f(x_n) - T(x)\| + \|f(x) - T(x_n)\|] \right\} \\ &\leq \|(1 - \beta_n)p\| + \beta_n \max \left\{ a\|x_n - x\|, \frac{b}{2} \left( \frac{1 - \beta_n}{\beta_n} \right) \|x_n - p\|, \right. \\ &\quad \left. \frac{c}{2} [\|x_n - x\| + \|x - \frac{1}{\beta_n}x_n + \left( \frac{1 - \beta_n}{\beta_n} \right)p\|] \right\}. \end{aligned}$$

Letting  $n \rightarrow \infty$ , we obtain

$$\|y - x\| \leq \max \left\{ a\|y - x\|, 0, \frac{c}{2} [\|y - x\| + \|x - y\|] \right\} \leq k\|y - x\|,$$

where  $k = \max\{a, c\}$ . Therefore,  $x = y$  and so  $x_n \rightarrow x$ . □

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