Volume 26, Number 6 June 1st, 2019
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of

Computational

Analysis and

Applications

EUDOXUS PRESS,LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(fifteen times annually)

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Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

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Some identities involving generalized degenerate tangent polynomials arising from differential equations

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Abstract : In this paper, we study differential equations arising from the generating functions of generalized degenerate tangent polynomials. We give explicit identities for the generalized degenerate tangent polynomials arising from differential equations.

Key words : Differential equations, tangent numbers, higher-order tangent numbers, degenerate tangent polynomials, generalized degenerate tangent polynomials.

2000 Mathematics Subject Classification : 05A19, 11B83, 34A30, 65L99.

1. Introduction

Recently, many mathematicians have studied in the area of the degenerate Euler numbers, degenerate Bernoulli numbers, degenerate Genocchi numbers, and degenerate tangent numbers(see [1, 2, 3, 5, 6, 7, 8, 9, 10, 11]).

We first give the definitions of the tangent numbers and polynomials. It should be mentioned that the definition of tangent numbers T_n and polynomials $T_n(x)$ can be found in [5, 6]. The tangent numbers T_n and polynomials $T_n(x)$ are defined by means of the generating functions:

$$\frac{2}{e^{2t}+1} = \sum_{n=0}^{\infty} T_n \frac{t^n}{n!},$$

$$\left(\frac{2}{e^{2t}+1}\right) e^{xt} = \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!}. \tag{1.1}$$

Generalized tangent polynomials $T_n(x)(n \geq 0)$, were introduced by Ryoo. The generalized tangent polynomials $T_n(x)$ are defined by the generating function:

$$\left(\frac{2}{e^{2t}+1}\right)^x = \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!}. \tag{1.2}$$

Degenerate tangent numbers $T_{n,\lambda}$ and polynomials, $T_{n,\lambda}(x)(n \geq 0)$, were introduced by Ryoo(see [8]). The degenerate tangent numbers $T_{n,\lambda}$ are defined by the generating function:

$$\frac{2}{(1+\lambda t)^{2/\lambda}+1} = \sum_{n=0}^{\infty} \mathcal{T}_{n,\lambda} \frac{t^n}{n!}. \tag{1.3}$$

The generalized degenerate tangent polynomials $\mathcal{T}_{n,\lambda}(x)$ are defined by means of the following generating function

$$\left(\frac{2}{(1+\lambda t)^{2/\lambda}+1}\right)^x = \sum_{n=0}^{\infty} \mathcal{T}_{n,\lambda}(x) \frac{t^n}{n!}. \tag{1.4}$$

We recall that the classical Stirling numbers of the first kind $S_1(n,k)$ and $S_2(n,k)$ are defined by the relations(see [11])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k) x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,$$

respectively. Here $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. The symbol $< x >_n$ is used for the rising factorial: $< x >_n = x(x+1)\cdots(x+n-1)$. The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1}(x-\lambda k) \tag{1.5}$$

for positive integer n, with the convention $(x|\lambda)_0 = 1$. The generalized rising factorial $< x|\lambda >_n^{(N)}$ is defined by

$$< x|\lambda >_n^{(N)} = \prod_{k=0}^{n-1}(x+(N-k)\lambda) \tag{1.6}$$

for positive integer n, with the convention $< x|\lambda >_0^{(N)} = 1$. We also need the binomial theorem: for a variable x,

$$(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!}. \tag{1.7}$$

Many mathematicians have studied in the area of the linear and nonlinear differential equations arising from the generating functions of special polynomials in order to give explicit identities for special polynomials(see [3, 7, 9]). In this paper, we study differential equations arising from the generating functions of generalized degenerate tangent polynomials. We give explicit identities for the generalized degenerate tangent polynomials.

2. Differential equations associated with generalized degenerate tangent polynomials

In this section, we study differential equations arising from the generating functions of generalized degenerate tangent polynomials. Let

$$F = F(t,x,\lambda) = \left(\frac{2}{(1+\lambda t)^{2/\lambda}+1}\right)^x. \tag{2.1}$$

Then, by (2.1), we have

$$\begin{aligned} F^{(1)} &= \frac{\partial}{\partial t}F(t,x,\lambda) = \frac{\partial}{\partial t}\left(\frac{2}{(1+\lambda t)^{2/\lambda}+1}\right)^x \\ &= \frac{x}{1+\lambda t}\left(\frac{2}{(1+\lambda t)^{2/\lambda}+1}\right)^{x-1}\left(\frac{-4(1+\lambda t)^{2/\lambda}}{(1+\lambda t)^{2/\lambda}+1}\right) \\ &= \frac{xF(t,x+1,\lambda)-2xF(t,x,\lambda)}{1+\lambda t} \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} F^{(2)} = \frac{\partial}{\partial t}F^{(1)} &= \left(xF^{(1)}(t,x+1,\lambda)-2xF^{(1)}(t,x,\lambda)\right)(1+\lambda t)^{-1} \\ &\quad -\lambda\left(xF(t,x+1,\lambda)-2xF(t,x,\lambda)\right)(1+\lambda t)^{-2} \\ &= \frac{x(x+1)F(t,x+2,\lambda)}{(1+\lambda t)^2} - \frac{(4x^2+2x+\lambda x)F(t,x,\lambda)}{(1+\lambda t)^2} \\ &\quad + \frac{(4x^2+2x\lambda)F(t,x,i)}{(1+\lambda t)^2}, \end{aligned} \tag{2.3}$$

Continuing this process, we can guess that

$$\begin{aligned} F^{(N)} &= \left(\frac{\partial}{\partial t}\right)^N F(t,x,\lambda) \\ &= \sum_{i=0}^{N} a_i(N,x,\lambda)F(t,x+i,\lambda)(1+\lambda t)^{-N}, \quad (N=0,1,2,\ldots). \end{aligned} \tag{2.4}$$

Taking the derivative with respect to t in (2.4), we have

$$\begin{aligned} F^{(N+1)} &= \left(\frac{\partial}{\partial t}\right)^{N+1} F(t,x,\lambda) \\ &= \sum_{i=0}^{N} a_i(N,x,\lambda)(-N\lambda)F(t,x+i,\lambda)(1+\lambda t)^{-N-1} \\ &\quad + \sum_{i=0}^{N} a_i(N,x,\lambda)F^{(1)}(t,x+i,\lambda)(1+\lambda t)^{-N} \\ &= \sum_{i=0}^{N} a_i(N,x,\lambda)(-N\lambda)(t,x+i,\lambda)(1+\lambda t)^{-N-1} \\ &\quad + \sum_{i=0}^{N} a_i(N,x,\lambda)\left[(x+i)F(t,x+i+1,\lambda) - 2(x+i)F(t,x+i,\lambda)\right](1+\lambda t)^{-N} \\ &= \sum_{i=0}^{N}(-2x-2i-N\lambda)a_i(N,x,\lambda)F(t,x+i,\lambda)(1+\lambda t)^{-N-1} \\ &\quad + \sum_{i=1}^{N+1}(x+i-1)a_{i-1}(N,x,\lambda)F(t,x+i,\lambda)(1+\lambda t)^{-N-1}. \end{aligned} \tag{2.5}$$

On the other hand, by replacing N by $N+1$ in (2.4), we get

$$F^{(N+1)} = \sum_{i=0}^{N+1} a_i(N+1,x,\lambda)F(t,x+i,\lambda)(1+\lambda t)^{-N-1}. \tag{2.6}$$

By (2.5) and (2.6), we have

$$\begin{aligned} &\sum_{i=0}^{N}(-2x-2i-N\lambda)a_i(N,x,\lambda)F(t,x+i,\lambda)(1+\lambda t)^{-N-1} \\ &\quad + \sum_{i=1}^{N+1}(x+i-1)a_{i-1}(N,x,\lambda)F(t,x+i,\lambda)(1+\lambda t)^{-N-1} \\ &= \sum_{i=0}^{N+1} a_i(N+1,x,\lambda)F(t,x+i,\lambda)(1+\lambda t)^{-N-1}. \end{aligned} \tag{2.7}$$

Comparing the coefficients on both sides of (2.7), we obtain

$$\begin{aligned} a_0(N+1,x,\lambda) &= -(2x+N\lambda)a_0(N,x,\lambda), \\ a_{N+1}(N+1,x,\lambda) &= (x+N)a_N(N,x,\lambda), \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} a_i(N+1,x,\lambda) = (-1)&(2x+2i+N\lambda)a_i(N,x,\lambda) \\ &+ (x+i-1)a_{i-1}(N,x,\lambda), \quad (1 \le i \le N). \end{aligned} \tag{2.9}$$

In addition, by (2.2) and (2.4), we get

$$F = F^{(0)} = a_0(0,x,\lambda)F(t,x,\lambda) = F(t,x,\lambda). \tag{2.10}$$

Thus, by (2.10), we obtain

$$a_0(0,x,\lambda) = 1. \tag{2.11}$$

It is not difficult to show that

$$\begin{aligned} &xF(t,x+1,\lambda)(1+\lambda t)^{-1} - 2xF(t,x,\lambda)(1+\lambda t)^{-1} \\ &= \sum_{i=0}^{1} a_i(1,x,\lambda)F(t,x+i,\lambda)(1+\lambda t)^{-1} \\ &= a_0(1,x,\lambda)F(t,x,\lambda)(1+\lambda t)^{-1} + a_1(1,x,\lambda)F(t,x+1,\lambda)(1+\lambda t)^{-1}. \end{aligned} \tag{2.12}$$

Thus, by (2.12), we also get

$$a_0(1,x,\lambda) = -2x, \quad a_1(1,x,\lambda) = x. \tag{2.13}$$

From (2.8), we note that

$$a_0(N+1,x,\lambda) = -(2x+N\lambda)a_0(N,x,\lambda) = \cdots = (-1)^{N+1} < 2x|\lambda >_{N+1}^{(N)},$$

and

$$a_{N+1}(N+1,x,\lambda) = (x+N)a_N(N,x,\lambda) = \cdots = xa_0(0,x,\lambda) =< x >_{N+1} . \tag{2.14}$$

For $i = 1,2,3$ in (2.9), we get

$$\begin{aligned}
a_1(N+1,\alpha,x) &= x\sum_{k=0}^{N}(-1)^k < 2x+2|\lambda >_k^{(N)} a_0(N-k,x,\lambda), \\
a_2(N+1,x,\lambda) &= (x+1)\sum_{k=0}^{N}(-1)^k < 2x+4|\lambda >_k^{(N)} a_1(N-k,x,\lambda), \text{ and} \\
a_3(N+1,x,\lambda) &= (x+2)\sum_{k=0}^{N-2}(-1)^k < 2x+6|\lambda) >_k^{(N)} a_2(N-k,x,\lambda).
\end{aligned}$$

Continuing this process, we can deduce that, for $1 \leq i \leq N$,

$$a_i(N+1,x,\lambda) = (X+i-1)\sum_{k=0}^{N-i+1}(-1)^k < 2x+2i|\lambda) >_k^{(N)} a_{i-1}(N-k,x,\lambda). \tag{2.15}$$

Note that, here the matrix $a_i(j,x,\lambda)_{0\leq i,j\leq N+1}$ is given by

$$\begin{pmatrix}
1 & -2x & (2x)(2x+\lambda) & -(2x)(2x+\lambda)(2x+2\lambda) & \cdots & (-1)^{N+1} < 2x|\lambda >_{N+1}^{(N)} \\
0 & < x >_1 & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & < x >_2 & \cdot & \cdots & \cdot \\
0 & 0 & 0 & < x >_3 & \cdots & \cdot \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & < x >_{N+1}
\end{pmatrix}$$

Now, we give explicit expressions for $a_i(N+1,x,\lambda)$. By (2.14) and (2.15), we get

$$\begin{aligned}
a_1(N+1,x,\lambda) &= x\sum_{k_1=0}^{N}(-1)^{k_1} < 2x+2|\lambda >_{k_1}^{(N)} a_0(N-k_1,x,\lambda) \\
&= x\sum_{k_1=0}^{N}(-1)^{N} < 2x+2|\lambda >_{k_1}^{(N)}< 2x|\lambda >_{N-k_1}^{(N-k_1-1)} \\
&=< x >_1\sum_{k_1=0}^{N}(-1)^{N} < 2x+2|\lambda >_{k_1}^{(N)}< 2x|\lambda >_{N-k_1}^{(N-k_1-1)},
\end{aligned}$$

$$\begin{aligned}
a_2(N+1,x,\lambda) &= (x+1)\sum_{k_2=0}^{N-1}(-1)^{k_2} < 2x+4|\lambda >_{k_2}^{(N)} a_1(N-k_2,x,\lambda) \\
&=< x >_2\sum_{k_2=0}^{N-1}\sum_{k_1=0}^{N-k_2-1}(-1)^{N-1} < 2x+4|\lambda >_{k_2}^{(N)} \\
&\qquad \times < 2x+2|\lambda >_{k_1}^{(N-k_2-1)}< 2x|\lambda >_{N-k_2-k_1-1}^{(N-k_2-k_1-2)},
\end{aligned}$$

and

$$\begin{aligned}
&a_3(N+1,x,\lambda)\\
&=(x+2)\sum_{k_3=0}^{N-2}(-1)^{k_3}<2x+6|\lambda>_{k_3}^{(N)}a_2(N-k_3,x,\lambda)\\
&=<x>_3\sum_{k_3=0}^{N-2}\sum_{k_2=0}^{N-k_3-2}\sum_{k_1=0}^{N-k_3-k_2-2}(-1)^{N-2}<2x+6|\lambda>_{k_3}^{(N)}\\
&\qquad\times<2x+4|\lambda>_{k_2}^{(N-k_3-1)}<2x+2|\lambda>_{k_1}^{(N-k_3-k_2-2)}<2x|\lambda>_{N-k_3-k_2-k_1-2}^{(N-k_3-k_2-k_1-3)}.
\end{aligned}$$

Continuing this process, we obtain

$$\begin{aligned}
&a_i(N+1,x,\lambda)\\
&=<x>_i\sum_{k_i=0}^{N-i+1}\sum_{k_{i-1}=0}^{N-k_i-i+1}\cdots\sum_{k_1=0}^{N-k_i-\cdots-k_2-i+1}(-1)^{N-i+1}<2x+2i|\lambda>_{k_i}^{(N)}\\
&\qquad\times<2x+2(i-1)|\lambda>_{k_{i-1}}^{(N-k_i-1)}\cdots<2x|\lambda>_{N-k_i-k_{i-1}-\cdots-k_2-k_1-i+1}^{(N-k_i-k_{i-1}-\cdots-k_2-k_1-i)}.
\end{aligned}\tag{2.16}$$

Therefore, by (2.16), we obtain the following theorem.

Theorem 1. For $N=0,1,2,\ldots$, the functional equation

$$F^{(N)}=\sum_{i=0}^{N}a_i(N,x,\lambda)F(t,x+i,\lambda)(1+\lambda t)^{-N}$$

has a solution

$$F=F(t,x,\lambda)=\left(\frac{2}{(1+\lambda t)^{2/\lambda}+1}\right)^x,$$

where

$$\begin{aligned}
&a_0(N,x,\lambda)=(-1)^N<2x|\lambda>_N^{(N-1)},\\
&a_N(N,x,\lambda)=<x>_N,\\
&a_i(N,x,\lambda)\\
&=(-1)^i<\alpha>_i(\zeta q^h)^i\sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-k_i-i}\cdots\sum_{k_1=0}^{N-k_i-\cdots-k_2-i}(-1)^{N-i}<2x+2i|\lambda>_{k_i}^{(N-1)}\\
&\qquad\times<2x+2(i-1)|\lambda>_{k_{i-1}}^{(N-k_i-2)}\cdots<2x|\lambda>_{N-k_i-k_{i-1}-\cdots-k_2-k_1-i}^{(N-k_i-k_{i-1}-\cdots-k_2-k_1-i-1)},\\
&(1\le i\le N-1).
\end{aligned}$$

Here is a plot of the surface for this solution. We choose $\lambda=1/10$. The viewing windows is $\{(t,x):-4\le t\le 10, 0\le x\le 15\}$. In Figure 1(left), we plot of the surface for this solution. In Figure 1(right), we shows a higher-resolution density plot of the solution.

From (1.1), we note that

$$F^{(N)}=\left(\frac{\partial}{\partial t}\right)^N\sum_{n=0}^{\infty}\mathcal{T}_{n,\lambda}(x)\frac{t^n}{n!}=\sum_{k=0}^{\infty}\mathcal{T}_{n+N,\lambda}(x)\frac{t^k}{k!}.\tag{2.17}$$

From Theorem 1, (1.3), and (2.17), we can derive the following equation:

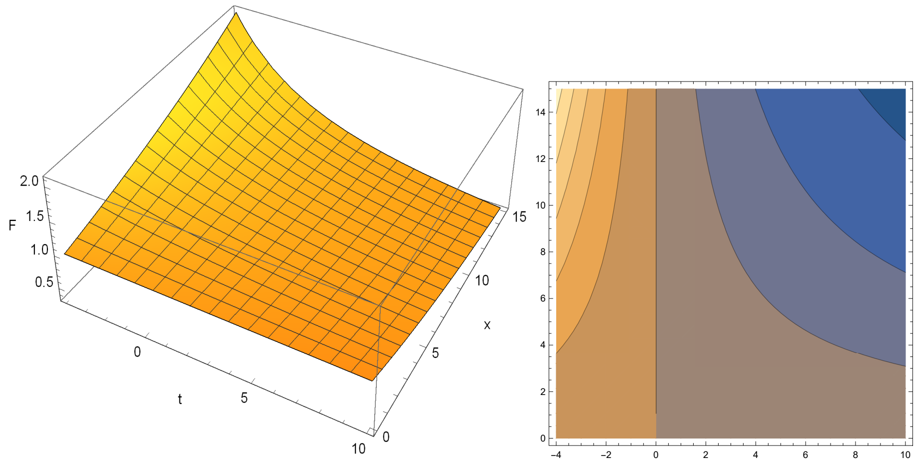

Figure 1: The surface for the solution $F(t, x, \lambda)$

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{T}_{n+N,\lambda}(x)\frac{t^n}{n!} &= F^{(N)} = \sum_{i=0}^{N} a_i(N,x,\lambda)F(t,x+i,\lambda)(1+\lambda t)^{-N} \\
&= \sum_{i=0}^{N} a_i(N,x,\lambda)(1+\lambda t)^{-N}\left(\frac{2}{(1+\lambda t)^{2/\lambda}+1}\right)^{x+i} \\
&= \sum_{i=0}^{N} a_i(N,x,\lambda)\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(-\lambda)^l\binom{N+l-1}{N-1}l!\mathcal{T}_{n-l,\lambda}(x+i)\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{N}\sum_{l=0}^{n}\binom{n}{l}\binom{N+l-1}{N-1}(-\lambda)^l l! a_i(N,x,\lambda)\mathcal{T}_{n-l,\lambda}(x+i)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.18}$$

By comparing the coefficients on both sides of (2.18), we obtain the following theorem.

Theorem 2. For $k = 0, 1, \dots,$ and $N = 0, 1, 2, \dots,$ we have

$$\mathcal{T}_{n+N,\lambda}(x) = \sum_{i=0}^{N}\sum_{l=0}^{n}\binom{n}{l}\binom{N+l-1}{N-1}(-\lambda)^l l! a_i(N,x,\lambda)\mathcal{T}_{n-l,\lambda}(x+i), \tag{2.19}$$

where

$$\begin{aligned}
&a_0(N,x,\lambda) = (-1)^N < 2x|\lambda >_N^{(N-1)}, \\
&a_N(N,x,\lambda) = < x >_N, \\
&a_i(N,x,\lambda) \\
&= (-1)^i < \alpha >_i (\zeta q^h)^i \sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-k_i-i}\cdots\sum_{k_1=0}^{N-k_i-\cdots-k_2-i}(-1)^{N-i} < 2x+2i|\lambda >_{k_i}^{(N-1)} \\
&\qquad \times < 2x+2(i-1)|\lambda >_{k_{i-1}}^{(N-k_i-2)} \cdots < 2x|\lambda >_{N-k_i-k_{i-1}-\cdots-k_2-k_1-i}^{(N-k_i-k_{i-1}-\cdots-k_2-k_1-i-1)}, \\
&(1 \le i \le N-1).
\end{aligned}$$

Let us take $n = 0$ in (2.19). Then, we have the following corollary.

Corollary 3. For $N = 0, 1, 2, \dots,$ we have

$$\mathcal{T}_{N,\lambda}(x) = \sum_{i=0}^{N} a_i(N,x,\lambda)\mathcal{T}_{0,\lambda}(x+i).$$

3. Zeros of the generalized degenerate tangent polynomial

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the generalized degenerate tangent polynomial $\mathcal{T}_{n,\lambda}(x)$. By using computer, the generalized degenerate tangent polynomials $\mathcal{T}_{n,\lambda}(x)$ can be determined explicitly. The first few of them are

$$\begin{aligned}
\mathcal{T}_{0,\lambda}(x) &= 1,\\
\mathcal{T}_{1,\lambda}(x) &= -x,\\
\mathcal{T}_{2,\lambda}(x) &= -x + \lambda x + x^2,\\
\mathcal{T}_{3,\lambda}(x) &= 3\lambda x - 2\lambda^2 x + 3x^2 - 3\lambda x^2 - x^3,\\
\mathcal{T}_{4,\lambda}(x) &= 2x - 11\lambda^2 x + 6\lambda^3 x + 3x^2 - 18\lambda x^2 + 11\lambda^2 x^2 - 6x^3 + 6\lambda x^3 + x^4,\\
\mathcal{T}_{5,\lambda}(x) &= -20\lambda x + 50\lambda^3 x - 24\lambda^4 x - 10x^2 - 30\lambda x^2 + 105\lambda^2 x^2 - 50\lambda^3 x^2\\
&\quad - 15x^3 + 60\lambda x^3 - 35\lambda^2 x^3 + 10x^4 - 10\lambda x^4 - x^5,\\
\mathcal{T}_{6,\lambda}(x) &= -16x + 170\lambda^2 x - 274\lambda^4 x + 120\lambda^5 x - 30x^2 + 150\lambda x^2 + 255\lambda^2 x^2\\
&\quad - 675\lambda^3 x^2 + 274\lambda^4 x^2 + 15x^3 + 225\lambda x^3 - 510\lambda^2 x^3 + 225\lambda^3 x^3\\
&\quad + 45x^4 - 150\lambda x^4 + 85\lambda^2 x^4 - 15x^5 + 15\lambda x^5 + x^6.
\end{aligned}$$

We investigate the beautiful zeros of the generalized degenerate tangent polynomials $\mathcal{T}_{n,\lambda}(x)$ by using a computer. We plot the zeros of the $\mathcal{T}_{n,\lambda}(x)$ for $n = 20, \lambda = 15/10, 10/10,\ 5/10, 1/10$, and $x \in \mathbb{C}$(Figure 2). In Figure 2(top-left), we choose $n = 20$ and $\lambda = 15/10$. In Figure 2(top-right), we choose $n = 20$ and $\lambda = 10/10$. In Figure 2(bottom-left), we choose $n = 20$ and $\lambda = 5/10$. In Figure 2(bottom-right), we choose $n = 20$and $\lambda = 1/10$. Prove that $\mathcal{T}_{n,\lambda}(x), x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry analytic complex functions(see Figure 2).

Stacks of zeros of the generalized degenerate tangent polynomials $\mathcal{T}_{n,\lambda}(x)$ for $1 \le n \le 20, \lambda = 1/10$ from a 3-D structure are presented(Figure 3).

Our numerical results for approximate solutions of real zeros of the generalized degenerate tangent polynomials $\mathcal{T}_{n,\lambda}(x) = 0, \lambda = 15/10$ are displayed(Tables 1, 2).

Table 1. Numbers of real and complex zeros of $\mathcal{T}_{n,\lambda}(x)$

degree n	real zeros	complex zeros
1	1	0
2	2	0
3	3	0
4	4	0
5	3	2
6	4	2
7	3	4
8	4	4
9	3	6
10	4	6
11	3	8
12	4	8
13	3	10
14	4	10

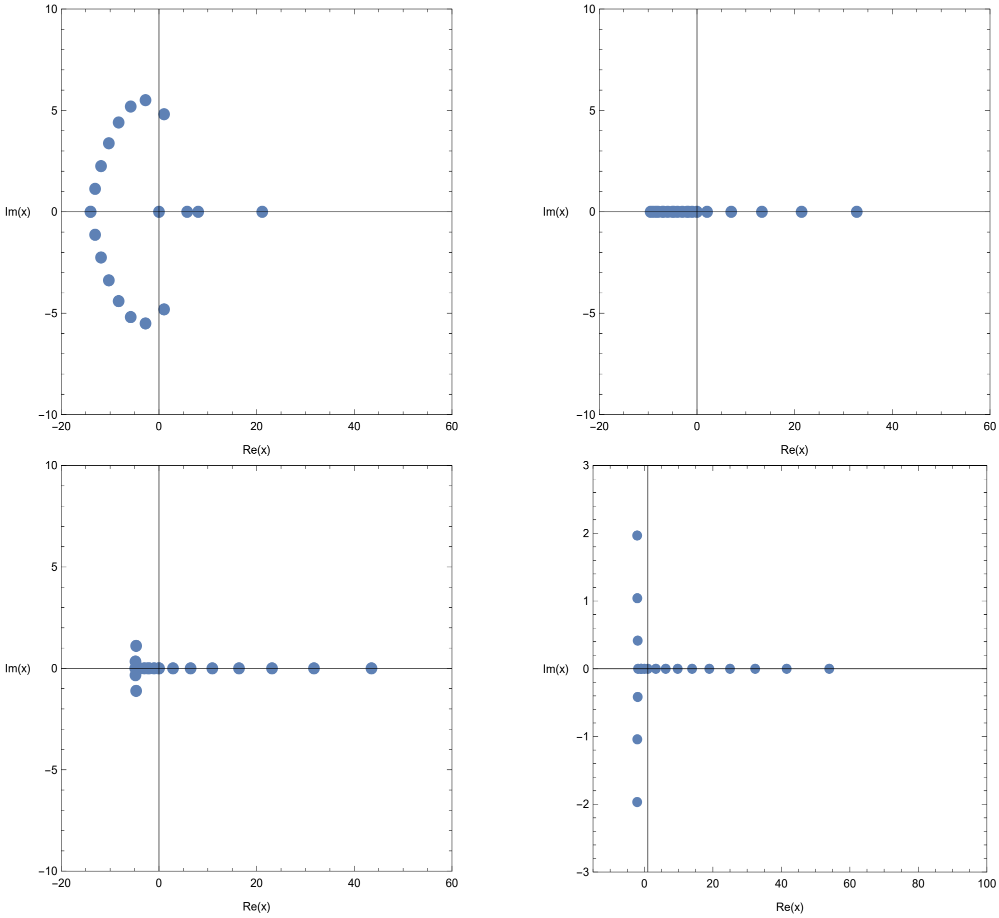

Figure 2: Zeros of $\mathcal{T}_{n,\lambda}(x)$

Plot of real zeros of $\mathcal{T}_{n,\lambda}(x)$ for $1 \leq n \leq 20$ structure are presented(Figure 4).

We observe a remarkably regular structure of the complex roots of the generalized degenerate tangent polynomials $\mathcal{T}_{n,\lambda}(x)$. We hope to verify a remarkably regular structure of the complex roots of the generalized degenerate tangent polynomials $\mathcal{T}_{n,\lambda}(x)$ (Table 1). Next, we calculated an

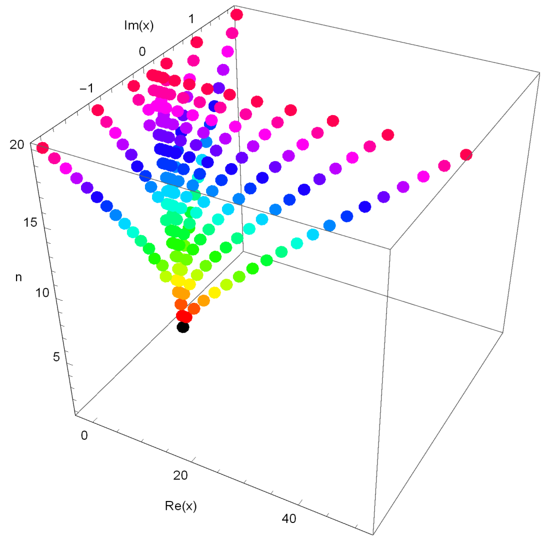

Figure 3: Stacks of zeros of $\mathcal{T}_{n,\lambda}(x), 1 \leq n \leq 20$

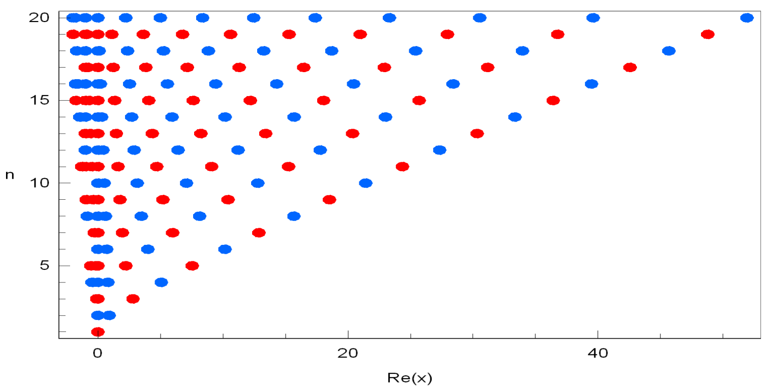

Figure 4: Real zeros of $\mathcal{T}_{n,\lambda}(x)$ for $1 \leq n \leq 20$

approximate solution satisfying $\mathcal{T}_{n,\lambda}(x) = 0, \lambda = 15/10, x \in \mathbb{R}$. The results are given in Table 2.

Table 2. Approximate solutions of $\mathcal{T}_{n,\lambda}(x) = 0, x \in \mathbb{R}$

degree n	x
1	0
2	$-0.50000, \quad 0$
3	$-1.5000, \quad 0, \quad 0$
4	$-2.0000, \quad -1.7247, \quad 0.7247, \quad 0$
5	$-3.0000, \quad 1.6113, \quad 0$
6	$-3.5000, \quad -3.3258, \quad 2.6128, \quad 0$
7	$-4.5000, \quad 3.6997, \quad 0$
8	$-5.0001, \quad -4.8845, \quad 4.8524, \quad 0$
9	$6.0575, \quad -6.0000, \quad 0$

Finally, we shall consider the more general problems. How many zeros does $\mathcal{T}_{n,\lambda}(x)$ have? $\mathcal{T}_{n,\lambda}(x)=0$ has not n distinct solutions(see Table 2). Find the numbers of complex zeros $C_{\mathcal{T}_{n,\lambda}(x)}$ of $\mathcal{T}_{n,\lambda}(x), Im(x)\neq 0$. Since n is the degree of the polynomial $\mathcal{T}_{n,\lambda}(x)$, the number of real zeros $R_{\mathcal{T}_{n,\lambda}(x)}$ lying on the real line $Im(x)=0$ is then $R_{\mathcal{T}_{n,\lambda}(x)}=n-C_{\mathcal{T}_{n,\lambda}(x)}$, where $C_{\mathcal{T}_{n,\lambda}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{\mathcal{T}_{n,\lambda}(x)}$ and $C_{\mathcal{T}_{n,\lambda}(x)}$. The author has no doubt that investigations along this line will lead to a new approach employing numerical method in the research field of the generalized degenerate tangent polynomials $\mathcal{T}_{n,\lambda}(x)$ to appear in mathematics and physics.

Acknowledgement: This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (No. 2017R1A2B4006092).

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Some New Inequalities of the Hermite–Hadamard Type for Extended s-Convex Functions

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Abstract

In the paper, the authors establish several new inequalities of the Hermite–Hadamard type for functions whose derivatives are extended s-convex in the absolute value and present some applications to special means of positive real numbers.

2010 Mathematics Subject Classification: Primary 26A51; Secondary 26D15, 26D20, 26E60, 41A55.

Key words and phrases: integral inequality of the Hermite–Hadamard type; extended s-convex function; application; mean.

1 Introduction

The following definitions are well known in the literature.

Definition 1.1. Let I be an interval in $\mathbb{R} = (-\infty, \infty)$. Then a function $f : I \to \mathbb{R}$ is said to be convex if $f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$ holds for all $x, y \in I$ and $t \in [0,1]$.

It is famous that, for any convex function f defined on $[a,b]$, the Hermite–Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \le \frac{f(a)+f(b)}{2}$$

holds true.

Definition 1.2 ([3, 6])**.** Let $s \in (0,1]$ be a real number. A function $f : \mathbb{R}_0 = [0,\infty) \to \mathbb{R}_0$ is said to be s-convex (in the second sense) if $f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y)$ holds for all $x, y \in I$ and $t \in [0,1]$.

Definition 1.3 ([12])**.** A function $f : \mathbb{R} \to \mathbb{R}$ is said to be extended s-convex if $f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y)$ holds for all $x, y \in I$ and $t \in (0,1)$ and for some fixed $s \in [-1,1]$.

In recent decades, a lot of integral inequalities of the Hermite–Hadamard type for various kinds of convex functions have been established. Some of them can be recited as follows.

Theorem 1.1 ([1, Theorem 6]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I^\circ$ with $a < b$ such that $f' \in L_1([a, b])$. If $|f'|^q$ is s-convex on $[a, b]$ for $s \in (0, 1]$, then*

$$\left|\frac{f(a)+rf(b)}{r+1}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{(r+1)(s+1)(s+2)} \times \left\{\left[s-r+1+\frac{2r^{s+2}}{(r+1)^{s+1}}\right]|f'(a)|+\left[r(s+1)-1+\frac{2}{(r+1)^{s+1}}\right]|f'(b)|\right\}. \quad (1.1)$$

Theorem 1.2 ([4, Theorem 3.1]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on I°, $a, b \in I$ with $a < b$, and $f' \in L_1([a, b])$. If $|f'|$ is s-convex on $[a, b]$ for some $s \in (0, 1]$, then*

$$\left|f(\lambda a+(1-\lambda)b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{(s+1)(s+2)}\bigl\{(1-\lambda)^2\bigl[|f'(a)|+(s+1)\bigl|f'(\lambda a+(1-\lambda)b)\bigr|\bigr] + \lambda^2\bigl[|f'(b)|+(s+1)\bigl|f'(\lambda a+(1-\lambda)b)\bigr|\bigr]\bigr\}. \quad (1.2)$$

Theorem 1.3 ([5, Theorem 2.2]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}.$$

Theorem 1.4 ([7, Theorems 4]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on I°, $a, b \in I$ with $a < b$, and $f' \in L_1([a, b])$. If $|f'|^q$ is s-convex on $[a, b]$ for some fixed $s \in (0, 1]$ and $q > 1$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{4}\left[\frac{1}{(s+1)(s+2)}\right]^{1/q}\left(\frac{1}{2}\right)^{1/p} \times\left\{\left[|f'(a)|^q+(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[|f'(b)|^q+(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.5 ([8, Theorems 1 and 3]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $|f'(x)|^q$ is s-convex on $[a, b]$ for some fixed $s \in (0, 1]$ and $q \ge 1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q}\left[\frac{2+1/2^s}{(s+1)(s+2)}\right]^{1/q}\bigl[|f'(a)|^q+|f'(b)|^q\bigr]^{1/q}.$$

Theorem 1.6 ([9, Theorems 1 and 2]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a, b]$ for $q \ge 1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}$$

and

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}.$$

Theorem 1.7 ([12, Theorems 3.1(2) and 3.2])**.** *Let $0 \le \lambda, \mu \le 1$ and $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, and $f' \in L_1[a, b]$ such that $|f'(x)|^q$ for $q \ge 1$ is extended s-convex on $[a, b]$ for some fixed $s \in [-1, 1]$.*

1. *If $-1 < s \le 1$, then*

$$\begin{aligned}\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2} f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{4}\left[\frac{1}{(s+1)(s+2)}\right]^{1/q}\\ \times\left\{\left(\frac{1}{2}-\lambda+\lambda^2\right)^{1-1/q}\left[\big(2(1-\lambda)^{s+2}+(s+2)\lambda-1\big)|f'(a)|^q+\big(2\lambda^{s+2}+s+1\right.\right.\\ \left.\left.-(s+2)\lambda\big)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left(\frac{1}{2}-\mu+\mu^2\right)^{1-1/q}\left[\big(2\mu^{s+2}+s+1\right.\\ \left.\left.-(s+2)\mu\big)\left|f'\left(\frac{a+b}{2}\right)\right|^q+\big(2(1-\mu)^{s+2}+(s+2)\mu-1\big)|f'(b)|^q\right]^{1/q}\right\}; \quad (1.3)\end{aligned}$$

2. *If $s = -1$, we have*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\ &\le \frac{b-a}{2^{3-2/q}}\left\{\big[(2\ln 2-1)|f'(a)|^q+|f'(b)|^q\big]^{1/q}+\big[|f'(a)|^q+(2\ln 2-1)|f'(b)|^q\big]^{1/q}\right\}. \quad (1.4)\end{aligned}$$

For recent generalizations of the Hermite–Hadamard type inequalities, please refer to [2, 10, 11, 13] and the references cited therein.

The main aim of this paper is to establish new inequalities of the Hermite–Hadamard type for the class of functions whose derivatives to certain powers are extended s-convex functions.

2 Lemmas

In order to prove our main results, we need the following lemmas.

Lemma 2.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $f' \in L_1([a, b])$, $\lambda, \mu \in \mathbb{R}$, and $\xi \in [0, 1]$, then*

$$\begin{aligned}\frac{\lambda f(a)+\mu f(b)}{2}&+\frac{2-\lambda-\mu}{2} f(\xi a+(1-\xi)b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\\ &=\frac{b-a}{2}\left[(1-\xi)\int_0^1 (2(1-\xi)t-\lambda) f'(t(\xi a+(1-\xi)b)+(1-t)a)\,\mathrm{d}\,t\right.\\ &\qquad\left.+\xi\int_0^1(\mu-2\xi t) f'(t(\xi a+(1-\xi)b)+(1-t)b)\,\mathrm{d}\,t\right].\end{aligned}$$

In particular, when $\xi = 0, 1$,

$$\lambda f(a)+(1-\lambda)f(b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x=(b-a)\int_0^1 (t-\lambda) f'((1-t)a+tb)\,\mathrm{d}\,t.$$

Proof. Integrating by part and changing variables of integration yield

$$
\begin{aligned}
&\frac{b-a}{2}\bigg[(1-\xi)\int_0^1(2(1-\xi)t-\lambda)f'(t(\xi a+(1-\xi)b)+(1-t)a)\,\mathrm{d}\,t \\
&+\xi\int_0^1(\mu-2\xi t)f'(t(\xi a+(1-\xi)b)+(1-t)b)\,\mathrm{d}\,t\bigg] \\
=&\frac{1}{2}\bigg[(2-2\xi-\lambda)f(\xi a+(1-\xi)b)+\lambda f(a)-\frac{2}{b-a}\int_a^{\xi a+(1-\xi)b}f(x)\,\mathrm{d}\,x \\
&+(2\xi-\mu)f(\xi a+(1-\xi)b)+\mu f(b)-\frac{2}{b-a}\int_{\xi a+(1-\xi)b}^b f(x)\,\mathrm{d}\,x\bigg] \\
=&\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f\Big(\xi a+(1-\xi)b\Big)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x.
\end{aligned}
$$

This completes the proof. ☐

Lemma 2.2. *Let $\lambda\in\mathbb{R}$ and $s>-1$. Then*

$$
\int_0^1|\lambda-t|t^s\,\mathrm{d}\,t=\begin{cases}\dfrac{(s+1)-(s+2)\lambda}{(s+1)(s+2)}, & \lambda\le0,\\ \dfrac{2\lambda^{s+2}-(s+2)\lambda+(s+1)}{(s+1)(s+2)}, & 0\le\lambda\le1,\\ \dfrac{(s+2)\lambda-(s+1)}{(s+1)(s+2)}, & \lambda\ge1\end{cases}
$$

and

$$
\int_0^1|\lambda-t|^s\,\mathrm{d}\,t=\frac{1}{s+1}\begin{cases}(1-\lambda)^{s+1}-(-\lambda)^{s+1}, & \lambda\le0,\\ \lambda^{s+1}+(1-\lambda)^{s+1}, & 0\le\lambda\le1,\\ \lambda^{s+1}-(\lambda-1)^{s+1}, & \lambda\ge1.\end{cases}
$$

Proof. These follow from straightforward computation of definite integrals. ☐

3 Main results

We are now in a position to establish some new integral inequalities of the Hermite–Hadamard type for differentiable and extended s-convex functions.

Theorem 3.1. *Let $0\le\lambda,\mu,\xi\le1$ and $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I$ with $a<b$, and $f'\in L_1([a,b])$ such that $|f'|^q$ for $q\ge1$ is extended s-convex on $[a,b]$ for some fixed $s\in[-1,1]$.*

1. *If $\xi\in(0,1)$ and $s\in(-1,1]$, then*

$$
\begin{aligned}
&\bigg|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(\xi a+(1-\xi)b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\bigg|\le\frac{b-a}{2}\Big\{(1-\xi) \\
&\times[E(1-\xi,\lambda,0)]^{1-1/q}\big[E(1-\xi,2-2\xi-\lambda,s)|f'(a)|^q+E(1-\xi,\lambda,s)|f'(\xi a+(1-\xi)b)|^q\big]^{1/q} \\
&+\xi[E(\xi,\mu,0)]^{1-1/q}\big[E(\xi,\mu,s)|f'(\xi a+(1-\xi)b)|^q+E(\xi,2\xi-\mu,s)|f'(b)|^q\big]^{1/q}\Big\}; \quad (3.1)
\end{aligned}
$$

2. If $\xi \in (0,1)$ and $s=-1$, we have

$$\left|f(\xi a+(1-\xi)b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{2^{1-1/q}}\big\{(1-\xi)^{2-2/q}\big[(\xi-1-\ln\xi)|f'(a)|^q + (1-\xi)|f'(b)|^q\big]^{1/q}+\xi^{2-2/q}\big[\xi|f'(a)|^q-(\xi+\ln(1-\xi))|f'(b)|^q\big]^{1/q}\big\}; \quad (3.2)$$

3. If $\xi=0,1$ and $s\ne -1$, we have

$$\left|\lambda f(a)+(1-\lambda)f(b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{2^{1-1/q}}\left[\frac{1}{(s+1)(s+2)}\right]^{1/q}(2\lambda^2-2\lambda+1)^{1-1/q} \times\big[\big(2(1-\lambda)^{s+2}+(s+2)\lambda-1\big)|f'(a)|^q+\big(2\lambda^{s+2}-(s+2)\lambda+s+1\big)|f'(b)|^q\big]^{1/q}, \quad (3.3)$$

where

$$E(\xi,\lambda,s)=\int_0^1 |2\xi t-\lambda|t^s\,\mathrm{d}\,t.$$

Proof. For $\xi \in (0,1)$ and $s\in(-1,1]$, from Lemma 2.1, using Hölder's integral inequality and extended s-convexity of $|f'|^q$, we have

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(\xi a+(1-\xi)b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\
&\le \frac{b-a}{2}\bigg[(1-\xi)\int_0^1 |2(1-\xi)t-\lambda||f'(t(\xi a+(1-\xi)b)+(1-t)a)|\,\mathrm{d}\,t \\
&\quad +\xi\int_0^1|\mu-2\xi t||f'(t(\xi a+(1-\xi)b)+(1-t)b)|\,\mathrm{d}\,t\bigg] \\
&\le \frac{b-a}{2}\bigg\{(1-\xi)\bigg(\int_0^1|2(1-\xi)t-\lambda|\,\mathrm{d}\,t\bigg)^{1-1/q} \\
&\quad \times\bigg[\int_0^1|2(1-\xi)t-\lambda||f'(t(\xi a+(1-\xi)b)+(1-t)a)|^q\,\mathrm{d}\,t\bigg]^{1/q} \\
&\quad +\xi\bigg(\int_0^1|\mu-2\xi t|\,\mathrm{d}\,t\bigg)^{1-1/q}\bigg[\int_0^1|\mu-2\xi t||f'(t(\xi a+(1-\xi)b)+(1-t)b)|^q\,\mathrm{d}\,t\bigg]^{1/q}\bigg\} \\
&\le \frac{b-a}{2}\bigg\{(1-\xi)\bigg(\int_0^1|2(1-\xi)t-\lambda|\,\mathrm{d}\,t\bigg)^{1-1/q}\bigg[\int_0^1|2(1-\xi)t-\lambda| \\
&\quad \times\big(t^s|f'(\xi a+(1-\xi)b)|^q+(1-t)^s|f'(a)|^q\big)\,\mathrm{d}\,t\bigg]^{1/q}+\xi\bigg(\int_0^1|\mu-2\xi t|\,\mathrm{d}\,t\bigg)^{1-1/q} \\
&\quad \times\bigg[\int_0^1|\mu-2\xi t|\big(t^s|f'(\xi a+(1-\xi)b)|^q+(1-t)^s|f'(b)|^q\big)\,\mathrm{d}\,t\bigg]^{1/q}\bigg\}.
\end{aligned} \quad (3.4)$$

From Lemma 2.2, we have

$$\int_0^1|2\xi t-\mu|\,\mathrm{d}\,t=E(\xi,\mu,0), \qquad \int_0^1|2\xi t-\mu|t^s\,\mathrm{d}\,t=E(\xi,\mu,s), \quad (3.5)$$

and

$$\int_0^1 |2\xi t-\mu|(1-t)^s \,\mathrm{d}\, t = E(\xi, 2\xi-\mu, s). \tag{3.6}$$

By virtue of (3.5) to (3.6) in (3.4), we obtain (3.1).

For $\xi \in (0,1)$ and $s=-1$, since $|f'|^q$ is extended s-convex, by Lemma 2.1 and Hölder's integral inequality, we have

$$\begin{aligned}
&\left|f(\xi a+(1-\xi)b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le (b-a)(1-\xi)^2 \\
&\times \int_0^1 t|f'((t\xi+1-t)a+(t-t\xi)b)|\,\mathrm{d}\,t+(b-a)\xi^2\int_0^1 t|f'(t\xi a+(1-t\xi)b)|\,\mathrm{d}\,t \\
&\le (b-a)(1-\xi)^2\left(\int_0^1 t\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1 t|f'((t\xi+1-t)a+(t-t\xi)b)|^q\,\mathrm{d}\,t\right]^{1/q} \\
&+(b-a)\xi^2\left(\int_0^1 t\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1 t|f'(t\xi a+(1-t\xi)b)|^q\,\mathrm{d}\,t\right]^{1/q} \\
&\le \frac{b-a}{2^{1-1/q}}\left\{(1-\xi)^2\left[\int_0^1 \left(t\big(t\xi+1-t\big)^{-1}|f'(a)|^q+t\big(t-t\xi\big)^{-1}|f'(b)|^q\right)\mathrm{d}\,t\right]^{1/q}\right. \\
&\left.+\xi^2\left[\int_0^1 \left(t\big(t\xi\big)^{-1}|f'(a)|^q+t(1-t\xi)^{-1}|f'(b)|^q\right)\mathrm{d}\,t\right]^{1/q}\right\}.
\end{aligned}$$

We thus deduce the inequality (3.2).

For $\xi=0,1$ and $s\ne -1$, by Lemma 2.1, Hölder's integral inequality, and extended s-convexity of $|f'|^q$, we have

$$\begin{aligned}
&\left|\lambda f(a)+(1-\lambda)f(b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le (b-a)\int_0^1 |t-\lambda||f'((1-t)a+tb)|\,\mathrm{d}\,t \\
&\le (b-a)\left(\int_0^1 |t-\lambda|\,\mathrm{d}\,t\right)^{1-1/q}\left(\int_0^1 |t-\lambda||f'((1-t)a+tb)|^q\,\mathrm{d}\,t\right)^{1/q} \\
&\le (b-a)\left(\int_0^1 |t-\lambda|\,\mathrm{d}\,t\right)^{1-1/q}\left(\int_0^1 |t-\lambda|\big((1-t)^s|f'(a)|^q+t^s|f'(b)|^q\big)\,\mathrm{d}\,t\right)^{1/q}.
\end{aligned}$$

We arrive at the inequality (3.3). Theorem 3.1 is proved. □

Corollary 3.1.1. *When $\xi\in(0,1)$ and $q=1$ in Theorem 3.1,*

1. *if $-1<s\le 1$, we have*

$$\begin{aligned}
&\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(\xi a+(1-\xi)b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\
&\le \frac{b-a}{2}\big\{(1-\xi)E(1-\xi, 2-2\xi-\lambda, s)|f'(a)| \\
&+\big[(1-\xi)E(1-\xi,\lambda,s)+\xi E(\xi,\mu,s)\big]|f'(\xi a+(1-\xi)b)|+\xi E(\xi,2\xi-\mu,s)|f'(b)|\big\};
\end{aligned}$$

2. *if $s=-1$, we have*

$$\left|f(\xi a+(1-\xi)b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\ \le (b-a)[(2\xi-1-\ln\xi)|f'(a)|+(1-2\xi-\ln(1-\xi))|f'(b)|].$$

Corollary 3.1.2. *Under conditions of Theorem 3.1,*

1. *if $-1<s\le 1$, then*

$$\left|\frac{1}{6}\left[f(a)+2f\left(\frac{2a+b}{3}\right)+2f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{b-a}{18(s+1)(s+2)} \\ \times\left[(s+5)|f'(a)|+(4s+5)\left|f'\left(\frac{2a+b}{3}\right)\right|+(4s+5)\left|f'\left(\frac{a+2b}{3}\right)\right|+(s+5)|f'(b)|\right];$$

2. *if $s=-1$, then*

$$\left|\frac{1}{2}\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{b-a}{2}(2\ln 3-\ln 2)(|f'(a)|+|f'(b)|).$$

Proof. Since

$$\left|\frac{1}{6}\left[f(a)+2f\left(\frac{2a+b}{3}\right)+2f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{1}{2}\left|\frac{1}{3}\left[f(a)\right.\right. \\ \left.\left.+2f\left(\frac{2a+b}{3}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|+\frac{1}{2}\left|\frac{1}{3}\left[2f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\ \le\frac{(b-a)[(s+5)|f'(a)|+(4s+5)|f'(\frac{2a+b}{3})|+(4s+5)|f'(\frac{a+2b}{3})|+(s+5)|f'(b)|]}{18(s+1)(s+2)}$$

and

$$\left|\frac{1}{2}\left[f\left(\frac{2a+b}{3}\right)+f\left(\frac{a+2b}{3}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{1}{2}\left|f\left(\frac{2a+b}{3}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\ +\frac{1}{2}\left|f\left(\frac{a+2b}{3}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{b-a}{2}(2\ln 3-\ln 2)(|f'(a)|+|f'(b)|).$$

Corollary 3.1.2 is thus proved. ☐

Remark 3.1. The inequality (1.2) can be deduced from (3.1) applied to $\lambda=\mu=0, q=1$, and $0<s\le 1$. The inequalities (1.3) and (1.4) can be deduced from (3.1) and (3.3) applied to $\xi=2^{-1}$. If we take $q=1$ and $\lambda=(r+1)^{-1}$ for $r\in[0,1]$ in (3.3), then the inequality (3.3) becomes (1.1). These show that Theorem 3.1 and its corollaries generalize some main results in [1, 4, 12].

Theorem 3.2. *Let $s\in(-1,1]$, $\lambda,\mu,\xi\in[0,1]$, $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I$ with $a<b$, and $f'\in L_1([a,b])$. When $|f'|^q$ for $q>1$ is extended s-convex on $[a,b]$,*

1. *if* $\xi \in (0,1)$, *then*

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(\xi a+(1-\xi)b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|$$
$$\le \frac{b-a}{2(s+1)^{1/q}}\left\{(1-\xi)\left[F\left(1-\xi,\lambda,\frac{q}{q-1}\right)\right]^{1-1/q}\left[|f'(a)|^q+|f'(\xi a+(1-\xi)b)|^q\right]^{1/q}\right.$$
$$\left.+\xi\left[F\left(\xi,\mu,\frac{q}{q-1}\right)\right]^{1-1/q}\left[|f'(b)|^q+|f'(\xi a+(1-\xi)b)|^q\right]^{1/q}\right\}; \quad (3.7)$$

2. *if* $\xi = 0,1$, *then*

$$\left|\lambda f(a)+(1-\lambda)f(b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{(s+1)^{1/q}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}$$
$$\times\left[\lambda^{(2q-1)/(q-1)}+(1-\lambda)^{(2q-1)/(q-1)}\right]^{1-1/q}[|f'(a)|^q+|f'(b)|^q]^{1/q}, \quad (3.8)$$

where

$$F(\xi,\lambda,s)=\int_0^1|2\xi t-\lambda|^s\,\mathrm{d}\,t.$$

Proof. For $\xi \in (0,1)$, by Lemma 2.1, Hölder's integral inequality, and the extended s-convexity of $|f'|^q$, we have

$$\left|\frac{\lambda f(a)+\mu f(b)}{2}+\frac{2-\lambda-\mu}{2}f(\xi a+(1-\xi)b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|$$
$$\le \frac{b-a}{2}\left[(1-\xi)\int_0^1|2(1-\xi)t-\lambda||f'(t(\xi a+(1-\xi)b)+(1-t)a)|\,\mathrm{d}\,t\right.$$
$$\left.+\xi\int_0^1|\mu-2\xi t||f'(t(\xi a+(1-\xi)b)+(1-t)b)|\,\mathrm{d}\,t\right]$$
$$\le \frac{b-a}{2}\left\{(1-\xi)\left(\int_0^1|2t(1-\xi)-\lambda|^{q/(q-1)}\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1|f'(t(\xi a+(1-\xi)b)+(1-t)a)|^q\,\mathrm{d}\,t\right]^{1/q}\right.$$
$$\left.+\xi\left(\int_0^1|\mu-2\xi t|^{q/(q-1)}\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1|f'(t(\xi a+(1-\xi)b)+(1-t)b)|^q\,\mathrm{d}\,t\right]^{1/q}\right\}$$
$$\le \frac{b-a}{2}\left\{(1-\xi)\left(\int_0^1|2t(1-\xi)-\lambda|^{q/(q-1)}\,\mathrm{d}\,t\right)^{1-1/q}\right.$$
$$\times\left[\int_0^1\left(t^s|f'(\xi a+(1-\xi)b)|^q+(1-t)^s|f'(a)|^q\right)\mathrm{d}\,t\right]^{1/q}+\xi\left(\int_0^1|\mu-2\xi t|^{q/(q-1)}\,\mathrm{d}\,t\right)^{1-1/q}$$
$$\left.\times\left[\int_0^1\left(t^s|f'(\xi a+(1-\xi)b)|^q+(1-t)^s|f'(b)|^q\right)\mathrm{d}\,t\right]^{1/q}\right\}.$$

From Lemma 2.2, we derive the inequality (3.7).

For $\xi = 0,1$, since $|f'|^q$ is extended s-convex, from Lemma 2.1 and by Hölder's integral inequality, we have

$$
\begin{aligned}
\left|\lambda f(a)+(1-\lambda)f(b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| &\le (b-a)\int_0^1 |t-\lambda||f'((1-t)a+tb)|\,\mathrm{d}\,t\\
&\le (b-a)\left(\int_0^1 |t-\lambda|^{q/(q-1)}\,\mathrm{d}\,t\right)^{1-1/q}\left(\int_0^1 |f'((1-t)a+tb)|^q\,\mathrm{d}\,t\right)^{1/q}\\
&\le (b-a)\left(\int_0^1 |t-\lambda|^{q/(q-1)}\,\mathrm{d}\,t\right)^{1-1/q}\left(\int_0^1 \big((1-t)^s|f'(a)|^q+t^s|f'(b)|^q\big)\,\mathrm{d}\,t\right)^{1/q}\\
&= \frac{b-a}{(s+1)^{1/q}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\big(\lambda^{(2q-1)/(q-1)}+(1-\lambda)^{(2q-1)/(q-1)}\big)^{1-1/q}(|f'(a)|^q+|f'(b)|^q)^{1/q}.
\end{aligned}
$$

Hence, we acquire the inequality (3.8). The proof of Theorem 3.2 is complete. □

Theorem 3.3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, and $f' \in L_1([a, b])$. Let $0 \le \xi \le 1$ and $0 \le \ell, r \le 1$. If $|f'|^q$ for $q > 1$ is extended s-convex on $[a, b]$ for $s \in (-1, 1]$, then*

$$
\begin{aligned}
\left|f(\xi a+(1-\xi)b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| &\le (b-a)\bigg\{(1-\xi)^2\left[\frac{q-1}{(2-l)q-1}\right]^{1-1/q}\\
\times\big[B(\ell q+1,s+1)|f'(a)|^q+(\ell q+s+1)^{-1}|f'(\xi a+(1-\xi)b)|^q\big]^{1/q}&+\xi^2\left[\frac{q-1}{(2-r)q-1}\right]^{1-1/q}\\
\times\big[(rq+s+1)^{-1}|f'(\xi a+(1-\xi)b)|^q&+B(rq+1,s+1)|f'(b)|^q\big]^{1/q}\bigg\},
\end{aligned}
$$

where

$$
B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}\,t,\quad \alpha,\beta>0
$$

is the noted beta function.

Proof. Since $|f'|^q$ is extended s-convex, from Lemma 2.1, using Hölder's integral inequality, we have

$$
\begin{aligned}
&\left|f(\xi a+(1-\xi)b)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\
&\le (b-a)(1-\xi)^2\left[\int_0^1 t^{(1-\ell)q/(q-1)}\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^1 t^{\ell q}|f'(t(\xi a+(1-\xi)b)+(1-t)a)|^q\,\mathrm{d}\,t\right]^{1/q}\\
&\quad+(b-a)\xi^2\left[\int_0^1 t^{(1-r)q/(q-1)}\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^1 t^{rq}|f'(t(\xi a+(1-\xi)b)+(1-t)b)|^q\,\mathrm{d}\,t\right]^{1/q}\\
&\le (b-a)(1-\xi)^2\left[\frac{q-1}{(2-\ell)q-1}\right]^{1-1/q}\left[\int_0^1 t^{\ell q}\big(t^s|f'(\xi a+(1-\xi)b)|^q+(1-t)^s|f'(a)|^q\big)\,\mathrm{d}\,t\right]^{1/q}\\
&\quad+(b-a)\xi^2\left[\frac{q-1}{(2-r)q-1}\right]^{1-1/q}\left[\int_0^1 t^{rq}\big(t^s|f'(\xi a+(1-\xi)b)|^q+(1-t)^s|f'(b)|^q\big)\,\mathrm{d}\,t\right]^{1/q}.
\end{aligned}
$$

Theorem 3.3 is thus proved. □

4 Applications to means

In this final section, we apply some inequalities of the Hermite–Hadamard type for extended s-convex functions to construct some inequalities for means.

For two positive numbers $a, b > 0$ and $s \in [-1, 1]$, define

$$A(a,b) = \frac{a+b}{2}, \quad A_\xi(a,b) = \xi a + (1-\xi)b, \quad \xi \in [0,1]$$

and

$$L_s(a,b) = \begin{cases} \left[\dfrac{b^{s+1} - a^{s+1}}{(s+1)(b-a)}\right]^{1/s}, & a \neq b, s \neq 0, -1; \\ \dfrac{b-a}{\ln b - \ln a}, & a \neq b, s = -1; \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}, & a \neq b, s = 0; \\ a, & a = b. \end{cases}$$

These means are respectively called the arithmetic, weighted arithmetic, and generalized logarithmic means of two positive number a and b.

Let $f(x) = \frac{x^{s+1}}{s+1}$ for $x > 0$, $-1 < s \le 1$, and $q \ge 1$. If $0 \le sq \le 1$, we have

$$|f'(\lambda x + (1-\lambda)y)|^q \le \lambda^{sq} x^{sq} + (1-\lambda)^{sq} y^{sq} \le \lambda^s |f'(x)|^q + (1-\lambda)^s |f'(y)|^q$$

for $x, y > 0$ and $\lambda \in (0,1)$. If $-1 < sq \le 0$, we have

$$|f'(\lambda x + (1-\lambda)y)|^q \le (x^{sq})^\lambda (y^{sq})^{1-\lambda} \le \lambda^s |f'(x)|^q + (1-\lambda)^s |f'(y)|^q$$

for $x, y > 0$ and $\lambda \in (0,1)$. These mean that, when $-1 < sq \le 1$, the function $|f'(x)|^q = x^{sq}$ is extended s-convex on $\mathbb{R}_+ = (0, \infty)$. Consequently, applying the inequality (3.3) to x^{sq} yields

Theorem 4.1. *Let $b > a > 0$, $q \ge 1$, $-1 < s \le 1$, $-1 < sq \le 1$, and $0 \le \xi \le 1$. Then*

$$\begin{aligned} \left|A_\xi\left(a^{s+1}, b^{s+1}\right) - L_{s+1}^{s+1}(a,b)\right| &\le \frac{b-a}{2^{1-1/q}}\left(\frac{1}{s+2}\right)^{1/q}\left[(s+1)(2\xi^2 - 2\xi + 1)\right]^{1-1/q} \\ &\quad \times \left[\left(2(1-\xi)^{s+2} + (s+2)\xi - 1\right)a^{sq} + \left(2\xi^{s+2} - (s+2)\xi + s + 1\right)b^{sq}\right]^{1/q}. \end{aligned}$$

In particular, if $\xi = \frac{1}{2}$, then

$$\left|A\left(a^{s+1}, b^{s+1}\right) - L_{s+1}^{s+1}(a,b)\right| \le \frac{b-a}{2^{2+(s-2)/q}}\left(\frac{1}{s+2}\right)^{1/q}(s+1)^{1-1/q}\left[(2^s s + 1)A(a^{sq}, b^{sq})\right]^{1/q}.$$

Taking $f(x) = \frac{x^{s+1}}{s+1}$ for $x > 0$, $-1 < s \le 1$ and $q \ge 1$ in Corollary 3.1.2 derives the following inequalities for means.

Theorem 4.2. *Let $b > a > 0$ and $-1 < s \le 1$. Then*

$$\begin{aligned} &\left|A\left(a^{s+1}, b^{s+1}\right) + 2A\left(\left(\frac{2a+b}{3}\right)^{s+1}, \left(\frac{a+2b}{3}\right)^{s+1}\right) - 3L_{s+1}^{s+1}(a,b)\right| \\ &\qquad \le \frac{b-a}{3(s+2)}\left[(s+5)A(a^s, b^s) + (4s+5)A\left(\left(\frac{2a+b}{3}\right)^s, \left(\frac{a+2b}{3}\right)^s\right)\right]. \end{aligned}$$

Applying the inequality (3.8) to x^{sq} yields

Theorem 4.3. *Let $b > a > 0$, $q > 1$, $-1 < s \le 1$, $-1 < sq \le 1$, and $0 \le \xi \le 1$. Then*

$$\left|A_\xi\left(a^{s+1}, b^{s+1}\right) - L_{s+1}^{s+1}(a,b)\right| \le 2^{1/q}(b-a)\left(\frac{(s+1)(q-1)}{2q-1}\right)^{1-1/q} \times \left[\xi^{(2q-1)/(q-1)} + (1-\xi)^{(2q-1)/(q-1)}\right]^{1-1/q}\left[A(a^{sq}, b^{sq})\right]^{1/q}.$$

Furthermore, if $\xi = \frac{1}{2}$, we have

$$\left|A\left(a^{s+1}, b^{s+1}\right) - L_{s+1}^{s+1}(a,b)\right| \le (b-a)\left(\frac{(s+1)(q-1)}{2(2q-1)}\right)^{1-1/q}[A(a^{sq}, b^{sq})]^{1/q}.$$

Acknowledgements

This work was partially supported by the National Natural Science Foundation of China (Grant No. 11361038) and by the Foundation of the Research Program of Science and Technology at Universities of Inner Mongolia Autonomous Region (Grant No. NMDSS1729).

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On non-convex hybrid algorithm for a family of countable quasi-Lipschitz mappings in Hilbert spaces

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Abstract

We can find many convex iterative algorithms for common fixed points for a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in the domains of Hilbert spaces and there are only few non-convex iterative algorithms. In this report, we present a new non-convex hybrid iteration algorithm concerning Suantai iterative scheme. We also establish strong convergence theorems of common fixed points for a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in the domains of Hilbert spaces.

2010 Mathematics Subject Classification: 47H05, 47H09, 47H10

Key words and phrases: hybrid algorithm, nonexpansive mapping, quasi-Lipschitz mapping, quasi-nonexpansive mapping

1 Introduction

Fixed point theory of special mappings like nonexpansive, asymptotically nonexpansive, contractive and other mappings is an active area of interest and finds applications in many related fields like image recovery, signal processing and geometry of objects. From time to time, some versions of theorems relating to fixed points of functions of special nature keep on appearing in almost in all branches of mathematics. Consequently, we apply them in industry, toy making, finance, aircrafts and manufacturing of new model cars. For example, a fixed-point iteration scheme has been applied in intensity modulated radiation

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therapy optimization to pre-compute dose-deposition coefficient matrix, see [21]. Because of its vast range of applications almost in all directions, the research in it is moving rapidly and an immense literature is present currently. The construction of fixed point theorems (e.g., Banach fixed point theorem) which not only claim the existence of a fixed point but yield an algorithm, too (in the Banach case fixed point iteration $x_{n+1} = f(x_n)$). Any equation that can be written as $x = f(x)$ for some map f that is contracting with respect to some (complete) metric on X will provide such a fixed point iteration. Mann's iteration method was the stepping stone in this regard and is invariably used in most of the occasions see [11]. But it only ensures weak convergence, see [3] but, we require strong convergence in many real world problems relating to Hilbert spaces, see [1]. So mathematician are in search for the modifications of the Mann's process to control and ensure the strong convergence, (see [2, 5, 7–9, 14–19], and references therein).

Most probably the first noticeable modification of Mann's Iteration process was proposed by Nakajo and Takahashi [13] in 2003. They introduced this modification for only one nonexpansive mapping in a Hilbert space where as Kim and Xu [6] introduced a modification for asymptotically nonexpansive mapping in the Hilbert space in 2006. In the same year Martinez-Yanes and Xu [12] introduced a modification of the Ishikawa Iteration process for a nonexpansive mapping for a Hilbert space. They also gave modification of Halpern iteration method in Hilbert space. Su and Qin. [20] gave a monotone hybrid iteration process for nonexpansive mapping in a Hilbert space. Liu et al. [10] gave a novel iteration method for finite family of quasi-asymptotically pseudo-contractive mapping in a Hilbert space. Hence, we can find many iterative methods for finding fixed point of different type of mappings in literature. If we talk about the iterative algorithms for common fixed points of a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in the domains of Hilbert spaces,

Let H be the fixed notation for Hilbert space and C be nonempty, closed and convex subset of it. First we recall some basic definitions that will accompany us throughout this paper. Let $P_c(\cdot)$ be the metric projection onto C.

A mapping $T : C \to C$ is said to be *non-expensive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. And $T : C \to C$ is said to be *quasi-Lipschitz* if $Fix(T) \neq \phi$ and For all $p \in Fix(T)$, $\|Tx - p\| \leq L\|x - p\|$, where L is a constant $1 \leq L < \infty$.

If $L = 1$, then T is known as quasi-nonexpansive. It is well-known that T is said to be *closed* if for $n \to \infty$, $x_n \to x$ and $\|Tx_n - x_n\| \to 0$ implies $Tx = x$. T is said to be *weak closed* if $x_n \rightharpoonup x$ and $\|Tx_n - x_n\| \to 0$ implies $Tx = x$. as $n \to \infty$. It is admitted fact that a mapping which is weak closed should be closed but converse is no longer true.

Let $\{T_n\}$ be a sequence of mappings having a non-empty fixed points set F. Then $\{T_n\}$ is defined to be *uniformly* closed if for all convergent sequences $\{z_n\} \subset C$ with conditions $\|T_n z_n - z_n\| \to 0$, $n \to \infty$ implies the limit of $\{z_n\}$ belongs to F.

In 1953 [11], Mann proposed an iterative scheme given as:

$$x_{n+1} = (1 - \alpha_n)x_n n + \alpha_n T(x_n), \quad n = 0, 1, 2, \ldots.$$

Guan et al. in [4] established the following non-convex hybrid iteration algorithm corresponding to Mann iterative scheme:

$$\begin{cases} x_0 \in C = Q_0, \quad \text{choosen arbitrarily,} \\ y_n = (1 - \alpha_n)x_n + \alpha_n T_n x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq (1 + (L_n - 1)\alpha_n)\|x_n - z\| \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0. \end{cases}$$

In [4] Guan et al. established non-convex hybrid iteration algorithm and proved some strong convergence results relating to common fixed points for a uniformly closed asymptotic family of countable quasi-Lipschitz mappings in H. They applied their results for the finite case to obtain fixed points. In this article, we establish a non-convex hybrid algorithms corresponding to Karakaya iteration scheme. Then we also establish strong convergence theorems with proofs about common fixed points related to a uniformly closed asymptotically family of countable quasi-Lipschitz mappings in the realm of Hilbert spaces. An application of this algorithm is also given. We fix $\overline{co}C_n$ for closed convex closure of C_n for all $n \geq 1$, $A = \{z \in H : \|z - P_F x_0\| \leq 1\}$, T_n for countable quasi-L_n-Lipschitz mappings from C into itself, and T be closed quasi-nonexpansive mapping from C into itself to avoid redundancy. We also present an application of our algorithm.

2 Main results

In this part we formulate our main results. We start with some basic definitions.

Definition 2.1. Let $\{T_n\}$ be a family of countable quasi-L_n-Lipschitz mappings from C into itself, where C is a closed convex subset of a Hilbert space H. Then $\{T_n\}$ is said to be *asymptotic* if $\lim_{n\to\infty} L_n = 1$.

Proposition 2.2. *Let C be a closed convex subset of a Hilbert space H. Then for $x \in H$ and $z \in C$, $z = P_C x$ if and only if we have $\langle x - z, z - y\rangle \geq 0$ for all $y \in C$.*

Proposition 2.3. *Let $\{T_n\}$ be a family of countable quasi-L_n-Lipschitz mappings from C into itself, where C is a closed convex subset of a Hilbert space H. Then the common fixed point set F is closed and convex.*

Proposition 2.4. *Let C be a closed convex subset of a Hilbert space H. Then for any given $x_0 \in H$, we have $p = P_C x_0$ if and only if $\langle p - z, x_0 - p\rangle \geq 0$, $\forall z \in C$.*

Theorem 2.5. *Let C be a closed convex subset of a Hilbert space H, and let $\{T_n\}$ be uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings from C into itself. Suppose that α_n, β_n, γ_n, a_n and $b_n \in [0,1]$, $\alpha_n + \beta_n \in [0,1]$ and $a_n + b_n \in [0,1]$ for all $n \in N$ and $\sum_{n=0}^{\infty}(\alpha_n + \beta_n) = \infty$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \textit{choosen arbitrarily}, \\ y_n = (1-\alpha_n-\beta_n)x_n + \alpha_n T_n z_n + \beta_n T_n t_n, \quad n \geq 0, \\ z_n = (1-a_n-b_n)x_n + a_n T_n t_n + b_n T_n x_n, \quad n \geq 0, \\ t_n = (1-\gamma_n)x_n + \gamma_n T_n x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1 + (L_n(1-a_n-b_n) \\ \qquad + L_n^2((1-\gamma_n)a_n + b_n)a_n\gamma_n L_n^3 - 1)\alpha_n \\ \qquad + (L_n(1-\gamma_n) - 1) + \gamma_n L_n^2)\beta_n]\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0 \end{cases}$$

converges strongly to $P_F x_0$.

Proof. We give our proof in following steps.

STEP 1. We know that $\overline{co}C_n$ and Q_n are closed and convex for all $n \geq 0$. Next, we show that $F \cap A \subset \overline{co}C_n$ for all $n \geq 0$. Indeed, for each $p \in F \cap A$, we have

$$
\begin{aligned}
&\|y_n - p\| \\
&= \|(1-\alpha_n-\beta_n)x_n + \alpha_n T_n z_n + \beta_n T_n t_n - p\| \\
&= \|(1-\alpha_n-\beta_n)x_n + \alpha_n T_n((1-a_n-b_n)x_n + a_n T_n t_n + b_n T_n x_n) + \beta_n T_n t_n - p\| \\
&= \|(1-\alpha_n-\beta_n)x_n + \alpha_n T_n[(1-a_n-b_n)x_n + a_n T_n((1-\gamma_n)x_n + \gamma_n T_n x_n) + b_n T_n x_n] \\
&\quad + \beta_n T_n[(1-\gamma_n)x_n + \gamma_n T_n x_n] - p\| \\
&= \|(1-\alpha_n-\beta_n)(x_n - p) + (\alpha_n - a_n\alpha_n - b_n\alpha_n + \beta_n - \beta_n\gamma_n)(T_n x_n - p) \\
&\quad + (a_n\alpha_n - a_n\alpha_n\gamma_n + b_n\alpha_n + \beta_n\gamma_n)(T_n^2 x_n - p) + a_n\alpha_n\gamma_n)(T_n^3 x_n - p)\| \\
&\leq (1-\alpha_n-\beta_n)\|x_n - p\| + (\alpha_n - a_n\alpha_n - b_n\alpha_n + \beta_n - \beta_n\gamma_n)L_n\|T_n x_n - p\| \\
&\quad + (a_n\alpha_n - a_n\alpha_n\gamma_n + b_n\alpha_n + \beta_n\gamma_n)L_n^2\|T_n^2 x_n - p\| + a_n\alpha_n\gamma_n)L_n^3\|T_n^3 x_n - p\| \\
&= [1 + (L_n(1-a_n-b_n) + L_n^2((1-\gamma_n)a_n + b_n)a_n\gamma_n L_n^3 - 1)\alpha_n \\
&+ (L_n(1-\gamma_n) - 1) + \gamma_n L_n^2)\beta_n]\|x_n - p\|,
\end{aligned}
$$

and $p \in A$, so $p \in C_n$ which implies that $F \cap A \subset C_n$ for all $n \geq 0$. therefore, $F \cap A \subset \overline{co}C_n$ for all $n \geq 0$.

STEP 2. We show that $F \cap A \subset \overline{co}C_n \cap Q_n$ for all $n \geq 0$. it suffices to show that $F \cap A \subset Q_n$, for all $n \geq 0$. We prove this by mathematical induction. For $n = 0$ we have $F \cap A \subset C = Q_0$. Assume that $F \cap A \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $\overline{co}C_n \cap Q_n$, from Proposition 2.2, we have

$$\langle x_{n+1} - z, x_{n+1} - x_0 \rangle \leq 0, \quad \forall z \in \overline{co}C_n \cap Q_n,$$

as

$$F \cap A \subset \overline{co}C_n \cap Q_n,$$

the last inequality holds, in particular, for all $z \in F \cap A$. This together with the definition of Q_{n+1} implies that $F \cap A \subset Q_{n+1}$. Hence the $F \cap A \subset \overline{co}C_n \cap Q_n$ holds for all $n \geq 0$.

STEP 3. We prove $\{x_n\}$ is bounded. Since F is a nonempty, closed, and convex subset of C, there exists a unique element $z_0 \in F$ such that $z_0 = P_F x_0$. From $x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0$, we have

$$\|x_{n+1} - x_0\| \leq \|z - x_0\|$$

for every $z \in \overline{co}C_n \cap Q_n$. As $z_0 \in F \cap A \subset \overline{co}C_n \cap Q_n$, we get

$$\|x_{n+1} - x_0\| \leq \|z_0 - x_0\|$$

for each $n \geq 0$. This implies that $\{x_n\}$ is bounded.

STEP 4. We show that $\{x_n\}$ converges strongly to a point of C (we show that $\{x_n\}$ is a cauchy sequence). As $x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0 \subset Q_n$ and $x_n = P_{Q_n} x_0$ (Proposition 2.4), we have

$$\|x_{n+1} - x_0\| \geq \|x_n - x_0\|$$

for every $n \geq 0$, which together with the boundedness of $\|x_n - x_0\|$ implies that there exsists the limit of $\|x_n - x_0\|$. On the other hand, from $x_{n+m} \in Q_n$, we have $\langle x_n - x_{n+m}, x_n - x_0 \rangle \leq 0$ and hence

$$
\begin{aligned}
\|x_{n+m} - x_n\|^2 &= \|(x_{n+m} - x_0) - (x_n - x_0)\|^2 \\
&\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+m} - x_n, x_n - x_0 \rangle \\
&\leq \|x_{n+m} - x_0\|^2 - \|x_n - x_0\|^2 \to 0, \quad n \to \infty
\end{aligned}
$$

for any $m \geq 1$. Therefore $\{x_n\}$ is a cauchy sequence in C, then there exists a point $q \in C$ such that $\lim_{n\to\infty} x_n = q$.

STEP 5. We show that $y_n \to q$, as $n \to \infty$. Let

$$D_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (L_n^3 + 2L_n^2 - L_n - 2)(L_n^3 + 2L_n^2 - L_n)\}.$$

From the definition of D_n, we have

$$\begin{aligned} D_n &= \{z \in C : \langle y_n - z, y_n - z\rangle \leq \langle x_n - z, x_n - z\rangle \\ &\quad + (L_n^3 + 2L_n^2 - L_n - 2)(L_n^3 + 2L_n^2 - L_n)\} \\ &= \{z \in C : \|y_n\|^2 - 2\langle y_n, z\rangle + \|z\|^2 \leq \|x_n\|^2 - 2\langle x_n, z\rangle + \|z\|^2 \\ &\quad + (L_n^3 + 2L_n^2 - L_n - 2)(L_n^3 + 2L_n^2 - L_n)\} \\ &= \{z \in C : 2\langle x_n - y_n, z\rangle \leq \|x_n\|^2 - \|y_n\|^2 \\ &\quad + (L_n^3 + 2L_n^2 - L_n - 2)(L_n^3 + 2L_n^2 - L_n)\}. \end{aligned}$$

This shows that D_n is convex and closed, $n \in \mathbb{Z}^+ \cup \{0\}$.

Next, we want to prove that $C_n \subset D_n$, $n \geq 0$. In fact, for any $z \in C_n$, we have

$$\begin{aligned} \|y_n - z\|^2 &\leq [1 + (L_n(1 - a_n - b_n) + L_n^2((1-\gamma_n)a_n + b_n)a_n\gamma_n L_n^3 - 1)\alpha_n \\ &\quad + (L_n(1-\gamma_n) - 1) + \gamma_n L_n^2)\beta_n]^2\|x_n - z\|^2 \\ &= \|x_n - z\|^2 + 2[(L_n(1 - a_n - b_n) + L_n^2((1-\gamma_n)a_n + b_n)a_n\gamma_n L_n^3 - 1)\alpha_n \\ &\quad + (L_n(1-\gamma_n) - 1) + \gamma_n L_n^2)\beta_n] + [(L_n(1 - a_n - b_n) \\ &\quad + L_n^2((1-\gamma_n)a_n + b_n)a_n\gamma_n L_n^3 - 1)\alpha_n \\ &\quad + (L_n(1-\gamma_n) - 1) + \gamma_n L_n^2)\beta_n]^2\|x_n - z\|^2 \\ &\leq \|x_n - z\|^2 + [2(L_n^3 + 2L_n^2 - L_n - 2) + (L_n^3 + 2L_n^2 - L_n - 2)^2]\|x_n - z\|^2 \\ &= \|x_n - z\|^2 + (L_n^3 + 2L_n^2 - L_n - 2)(L_n^3 + 2L_n^2 - L_n)\|x_n - z\|^2. \end{aligned}$$

From

$$\begin{aligned} C_n = \{z \in C : \|y_n - z\| &\leq [1 + (L_n(1 - a_n - b_n) + L_n^2((1-\gamma_n)a_n + b_n)a_n\gamma_n L_n^3 - 1)\alpha_n \\ &\quad + (L_n(1-\gamma_n) - 1) + \gamma_n L_n^2)\beta_n]\|x_n - z\|\} \cap A, \quad n \geq 0, \end{aligned}$$

we have $C_n \subset A$, $n \geq 0$. Since A is convex, we also have $\overline{co}C_n \subset A$, $n \geq 0$. Consider $x_n \in \overline{co}C_{n-1}$, we know that

$$\begin{aligned} \|y_n - z\| &\leq \|x_n - z\|^2 + (L_n^3 + 2L_n^2 - L_n - 2)(L_n^3 + 2L_n^2 - L_n)\|x_n - z\|^2 \\ &\leq \|x_n - z\|^2 + (L_n^3 + 2L_n^2 - L_n - 2)(L_n^3 + 2L_n^2 - L_n). \end{aligned}$$

This implies that $z \in D_n$ and hence $C_n \subset D_n$, $n \geq 0$. Sinnce D_n is convex, we have $\overline{co}(C_n) \subset D_n$, $n \geq 0$. Therefore

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + (L_n^3 + 2L_n^2 - L_n - 2)(L_n^3 + 2L_n^2 - L_n) \to 0$$

as $n \to \infty$. That is, $y_n \to q$ as $n \to \infty$.

STEP 6. We show that $q \in F$. From the definition of y_n, we have

$$\begin{aligned} &(\alpha_n + a_n\alpha_n T_n + b_n\alpha_n T_n + \beta_n + \beta_n\gamma_n T_n + a_n\alpha_n\gamma_n T_n^2)\|T_n x_n - x_n\| \\ &= \|y_n - x_n\| \to 0 \end{aligned}$$

as $n \to \infty$. Since $\alpha_n \in (a, 1] \subset [0, 1]$, from the above limit we have

$$\lim_{n\to\infty} \|T_n x_n - x_n\| = 0.$$

Since $\{T_n\}$ is uniformly closed and $x_n \to q$, we have $q \in F$.

STEP 7. We claim that $q = z_0 = P_F x_0$, if not, we have that $\|x_0 - p\| > \|x_0 - z_0\|$. There must exist a positive integer N, if $n > N$, then $\|x_0 - x_n\| > \|x_0 - z_0\|$, which leads to

$$\|z_0 - x_n\|^2 = \|z_0 - x_n + x_n - x_0\|^2 = \|z_0 - x_n\|^2 + \|x_n - x_0\|^2 + 2\langle z_0 - x_n, x_n - x_0\rangle.$$

It follows that $\langle z_0 - x_n, x_n - x_0\rangle < 0$, which implies that $z_0 \in \overline{Q_n}$, so that $z_0 \in F$, this is a contradiction. This completes the proof. □

Now, we present an example of C_n which does not involve a convex subset.

Example 2.6. Take $H = R^2$, and a sequence of mappings $T_n : R^2 \to R^2$ given by

$$T_n : (t_1, t_2) \mapsto \Big(\frac{1}{8}t_1, t_2\Big), \quad \forall (t_1, t_2) \in R^2,\ n \geq 0.$$

It is clear that $\{T_n\}$ satisfies the desired definition of with $F = \{(t_1, 0) : t_1 \in (-\infty, +\infty)\}$ common fixed point set. Take $x_0 = (4, 0)$, $a_0 = \frac{6}{7}$, we have

$$y_0 = \frac{1}{7}x_0 + \frac{6}{7}T_0 x_0 = \Big(4 \times \frac{1}{7} + \frac{4}{8} \times \frac{6}{7}, 0\Big) = (1, 0).$$

Take $1 + (L_0 - 1)a_0 = \sqrt{\frac{5}{2}}$, we have

$$C_0 = \Big\{z \in R^2 : \|y_0 - z\| \leq \sqrt{\frac{5}{2}}\|x_0 - z\|\Big\}.$$

It is easy to show that $z_1 = (1, 3)$, $z_2 = (-1, 3) \in C_0$. But

$$z' = \frac{1}{2}z_1 + \frac{1}{2}z_2 = (0, 3)\overline{\in} C_0,$$

since $\|y_0 - z\| = 2$, $\|x_0 - z\| = 1$. Therefore C_0 is not convex.

Corollary 2.7. *Let C be a closed convex subset of a Hilbert space H, and let T be a closed quasi-nonexpansive mapping from C into itself. Assume that α_n, β_n, γ_n, a_n and $b_n \in [0, 1]$, $\alpha_n + \beta_n \in [0, 1]$ and $a_n + b_n \in [0, 1]$ for all $n \in N$ and $\sum_{n=0}^{\infty}(\alpha_n + \beta_n) = \infty$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \textit{choosen arbitrarily}, \\ y_n = (1 - \alpha_n - \beta_n)x_n + \alpha_n T z_n + \beta_n T t_n, \quad n \geq 0, \\ z_n = (1 - a_n - b_n)x_n + a_n T t_n + b_n T x_n, \quad n \geq 0, \\ t_n = (1 - \gamma_n)x_n + \gamma_n T x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap Q_n} x_0 \end{cases}$$

converges strongly to $P_F x_0$.

Proof. Take $T_n = T$, $L_n = 1$ in Theorem 2.5, in this case, C_n is convex and closed and , for all $n \geq 0$, by using Theorem 2.5, we obtain Corollary 2.7. $\square$

Corollary 2.8. *Let C be a closed convex subset of a Hilbert space H, and let T be a nonexpansive mapping from C into itself. Assume that α_n, β_n, γ_n, a_n and $b_n \in [0,1]$, $\alpha_n + \beta_n \in [0,1]$ and $a_n + b_n \in [0,1]$ for all $n \in N$ and $\sum_{n=0}^{\infty}(\alpha_n + \beta_n) = \infty$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \textit{choosen arbitrarily,} \\ y_n = (1-\alpha_n-\beta_n)x_n + \alpha_n T z_n + \beta_n T t_n, \quad n \geq 0, \\ z_n = (1-a_n-b_n)x_n + a_n T t_n + b_n T x_n, \quad n \geq 0, \\ t_n = (1-\gamma_n)x_n + \gamma_n T x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap Q_n} x_0 \end{cases}$$

converges strongly to $P_{F(T)}x_0$.

3 Applications

Here, we give an application of our result for the following case of finite family of asymptotically quasi-nonexpansive mappings $\{T_n\}_{n=0}^{N-1}$. Let

$$\|T_i^j x - p\| \leq k_{i,j}\|x - p\|, \quad \forall x \in C,\ p \in F,$$

where F is common fixed point sets of $\{T_n\}_{n=0}^{N-1}$ and $\lim_{j\to\infty} k_{i,j} = 1$ for all $0 \leq i \leq N-1$. The finite family of asymptotically quasi-nonexpansive mappings $\{T_n\}_{n=0}^{N-1}$ is *uniformly L-Lipschitz* if

$$\|T_i^j x - T_i^j y\| \leq L_{i,j}\|x - y\|, \quad \forall x, y \in C,$$

for all $i \in \{0,1,2,...,N-1\}$, $j \geq 1$, where $L \geq 1$.

Theorem 3.1. *Let C be a closed convex subset of a Hilbert space H, and let $\{T_n\}_{n=0}^{N-1}$ be a finite uniformly L-Lipschitz family of asymptotically quasi-nonexpansive mappings with the nonempty common fixed point set F. Assume that α_n, β_n, γ_n, a_n and $b_n \in [0,1]$, $\alpha_n + \beta_n \in [0,1]$ and $a_n + b_n \in [0,1]$ for all $n \in N$ and $\sum_{n=0}^{\infty}(\alpha_n + \beta_n) = \infty$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C = Q_0, \quad \textit{arbitrarily,} \\ y_n = (1-\alpha_n-\beta_n)x_n + \alpha_n T_{i(n)}^{j(n)} z_n + \beta_n T_{i(n)}^{j(n)} t_n, \quad n \geq 0, \\ z_n = (1-a_n-b_n)x_n + a_n T_{i(n)}^{j(n)} t_n + b_n T_{i(n)}^{j(n)} x_n, \quad n \geq 0, \\ t_n = (1-\gamma_n)x_n + \gamma_n T_{i(n)}^{j(n)} x_n, \quad n \geq 0, \\ C_n = \{z \in C : \|y_n - z\| \leq [1 + (k_{i(n),j(n)}(1-a_n-b_n) \\ \qquad + k_{i(n),j(n)}^2((1-\gamma_n)a_n + b_n)a_n\gamma_n k_{i(n),j(n)}^3 - 1)\alpha_n \\ \qquad + (k_{i(n),j(n)}(1-\gamma_n) - 1) + \gamma_n k_{i(n),j(n)}^2)\beta_n]\|x_n - z\|\} \cap A, \quad n \geq 0, \\ Q_n = \{z \in Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{\overline{co}C_n \cap Q_n} x_0 \end{cases}$$

converges strongly to $P_F x_0$.

Proof. We can drive the prove from the following two conclusions.

Conclusion 1 $\{T_{n=0}^{N-1}\}_{n=0}^{\infty}$ is a uniformly closed asymptotically family of countable quasi-L_n-Lipschitz mappings from C into itself.

Conclusion 2
$F=\bigcap_{n=0}^{N}F(T_n)=\bigcap_{n=0}^{\infty}F(T_{i(n)}^{j(n)})$, where $F(T_n)$ denotes the fixed point set of the mappings T_n. □

Corollary 3.2. *Let C be a closed convex subset of a Hilbert space H, and let T be a L-Lipschitz asymptotically quasi-nonexpansive mapping with the nonempty common fixed point set F. Assume that α_n, β_n, γ_n, a_n and $b_n\in[0,1]$, $\alpha_n+\beta_n\in[0,1]$ and $a_n+b_n\in[0,1]$ for all $n\in N$ and $\sum_{n=0}^{\infty}(\alpha_n+\beta_n)=\infty$. Then $\{x_n\}$ generated by*

$$\begin{cases} x_0\in C=Q_0, \quad \textit{arbitrarily,}\\ y_n=(1-\alpha_n-\beta_n)x_n+\alpha_nT^nz_n+\beta_nT^nt_n, \quad n\geq 0,\\ z_n=(1-a_n-b_n)x_n+a_nT^nt_n+b_nT^nx_n, \quad n\geq 0,\\ t_n=(1-\gamma_n)x_n+\gamma_nT^nx_n, \quad n\geq 0,\\ C_n=\{z\in C:\|y_n-z\|\leq[1+(K_n(1-a_n-b_n)\\ \qquad +K_n^2((1-\gamma_n)a_n+b_n)a_n\gamma_nK_n^3-1)\alpha_n\\ \qquad +(K_n(1-\gamma_n)-1)+\gamma_nK_n^2)\beta_n]\|x_n-z\|\}\cap A, \quad n\geq 0,\\ Q_n=\{z\in Q_{n-1}:\langle x_n-z,x_0-x_n\rangle\geq 0\}, \quad n\geq 1,\\ x_{n+1}=P_{\overline{co}C_n\cap Q_n}x_0 \end{cases}$$

converges strongly to P_Fx_0.

Proof. Take $T_n=T$ in Theorem 3.1, we get the desired result. □

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Some Results of The Class of Functions with Bounded Radius Rotation

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Abstract

Let $\mathcal{A}$ be the family of functions $f(z) = z + a_2z^2 + ...$ which are analytic in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$, and denote by $\mathcal{P}$ of functions $p(z) = z + p_1z + p_2z^2 + ...$ analytic in $\mathbb{D}$ such that $p(z)$ is in $\mathcal{P}$ if and only if

$$p(z) \prec \frac{1+z}{1-z} \Leftrightarrow p(z) = \frac{1+\phi(z)}{1-\phi(z)},$$

for some Schwarz function $\phi(z)$ and every $z \in \mathbb{D}$.

Let $f(z)$ be an element of $\mathcal{A}$, and satisfies the condition

$$z\frac{f'(z)}{f(z)} = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z)$$

where $p_1(z), p_2(z) \in \mathcal{P}$ and $k \geq 2$, then $f(z)$ is called function with bounded radius rotation. The class of such functions is denoted by R_k. This class is generalization of starlike functions.

The main purpose is to give some properties of the class R_k.

1 Introduction

Let Ω be the family of functions $\phi(z)$ which are analytic in $\mathbb{D}$ and satisfy the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. If $f_1(z)$ and $f_2(z)$ are analytic functions in $\mathbb{D}$, then we say that $f_1(z)$ is subordinate to $f_2(z)$, written as $f_1(z) \prec f_2(z)$ if there exists a Schwarz function $\phi \in \Omega$ such that $f_1(z) = f_2(\phi(z)), z \in \mathbb{D}$. We also note that if f_2 univalent in $\mathbb{D}$, then $f_1(z) \prec f_2(z)$ if and only if $f_1(0) = f_2(0)$, $f_1(\mathbb{D}) \subset f_2(\mathbb{D})$ implies $f_1(\mathbb{D}_r) \subset f_2(\mathbb{D}_r)$, where $\mathbb{D}_r = \{z : |z| < r, 0 < r < 1\}$ (see [2]). Denote by $\mathcal{P}$ the family of functions $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots$ analytic in $\mathbb{D}$ such that p is in $\mathcal{P}$ if and only if

$$p(z) \prec \frac{1+z}{1-z} \Leftrightarrow p(z) = \frac{1+\phi(z)}{1-\phi(z)}, z \in \mathbb{D} \tag{1.1}$$

2010 *Mathematics Subject Classification:* 30C45
Key words and phrases: Bounded radius rotation, bounded boundary rotation, distortion theorem, growth theorem and coefficient inequality.
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Let $f(z)$ be an element of $\mathcal{A}$. Then $f(z)$ is called convex or starlike if it maps $\mathbb{D}$ onto a convex or starlike region, respectively. Corresponding classes are denoted by $\mathcal{C}$ and S^*. It is well known that $\mathcal{C} \subset S^*$, that both are subclasses of the univalent functions and have the following analytical representations.

$$f(z) \in \mathcal{C} \Longleftrightarrow Re\left(1 + z\frac{f''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D} \tag{1.2}$$

and

$$f(z) \in S^* \Longleftrightarrow Re\left(z\frac{f'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D} \tag{1.3}$$

More on these classes can be found in [2]. Let $f(z)$ be an element of $\mathcal{A}$. If there is a function $g(z)$ in $\mathcal{C}$ such that

$$Re\left(\frac{f'(z)}{g'(z)}\right) > 0, \quad z \in \mathbb{D} \tag{1.4}$$

then $f(z)$ is called close-to-convex function in $\mathbb{D}$ and the class of such functions are denoted by $\mathcal{CC}$.

A function analytic and locally univalent in a given simply connected domain is said to be of bounded boundary rotation if its range has bounded boundary rotation which is defined as the total variation of the direction angle of the tangent to the boundary curve under a complete circuit. Let V_k denote the class of functions $f(z) \in \mathcal{A}$ which maps $\mathbb{D}$ conformally onto an image domain of boundary rotation at most $k\pi$. The class of functions of bounded boundary rotation was introduced by Loewner [3] in 1917 and was developed by Paatero [5, 6] who systematically developed their properties and made an exhaustive study of the class V_k. Paatero has shown that $f(z) \in V_k$ if and only if

$$f'(z) = Exp\left[-\int_0^{2\pi} \log\left(1 - ze^{-it}\right) d\mu(t)\right], \tag{1.5}$$

where $\mu(t)$ is real-valued function of bounded variation for which

$$\int_0^{2\pi} d\mu(t) = 2 \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq k \tag{1.6}$$

for fixed $k \geq 2$ it can also be expressed as

$$\int_0^{2\pi} \left|Re\frac{(zf'(z))'}{f'(z)}\right| d\theta \leq 2k\pi, \quad z = re^{i\theta}. \tag{1.7}$$

Clearly, if $k_1 < k_2$ then $V_{k_1} \subset V_{k_2}$ that is the class V_k obviously expands on k increases. V_2 is the class of $\mathcal{C}$ of convex univalent functions. Paatero showed that $V_4 \subset \mathcal{S}$, where $\mathcal{S}$ is the class of normalized univalent functions. Later Pinchuk proved that V_k is close-to convex functions in $\mathbb{D}$ if $2 \leq k \leq 4$ [7].

Let R_k denote the class of analytic functions f of the form $f(z) = z + a_2z^2 + a_3z^3 + ...$ having the representation

$$f(z) = zExp\left[-\int_0^{2\pi} \log\left(1 - ze^{-it}\right) d\mu(t)\right], \tag{1.8}$$

where $\mu(t)$ is given in (1.6). We note that the class R_k was introduced by Pinchuk and Pinchuk showed that Alexander type relation between the classes V_k and R_k exist,

$$f \in V_k \Leftrightarrow zf'(z) \in R_k \tag{1.9}$$

R_k consists of those function $f(z)$ which satisfy

$$\int_0^{2\pi} \left| Re(re^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})}) \right| d\theta \leq k\pi, z = re^{i\theta}. \tag{1.10}$$

Geometrically, the condition is that the total variation of angle between radius vector $f(re^{i\theta})$ makes with positive real axis is bounded $k\pi$. Thus, R_k is the class of functions of bounded radius rotation bounded by $k\pi$, therefore R_k generalizes the starlike functions.

P_k denote the class of functions $p(0) = 1$ analytic in $\mathbb{D}$ and having representation

$$p(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t) \tag{1.11}$$

where $\mu(t)$ is given in (1.6). Clearly, $P_2 = P$ where P is the class of analytic functions with positive real part. For more details see [7]. From (1.11), one can easily find that $p(z) \in P_k$ can also written by

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \quad z \in \mathbb{D} \tag{1.12}$$

where $p_1(z), p_2(z) \in \mathcal{P}$. Pinchuk [7] has shown that the classes V_k and R_k can be defined by using the class P_k as gives below

$$f \in V_k \Leftrightarrow \frac{(zf'(z))'}{f'(z)} \in P_k \tag{1.13}$$

and

$$f \in R_k \Leftrightarrow \frac{zf'(z)}{f(z)} \in P_k \tag{1.14}$$

At the same time, we note that V_k generalizes of convex functions.

2 Main Results

Lemma 2.1. *Let $p(z)$ be an element of P_k, then*

$$\left| p(z) - \frac{1 + r^2}{1 - r^2} \right| \leq \frac{kr}{1 - r^2} \tag{2.1}$$

Proof. Let $f(z)$ be an element of V_k. Using (1.13), we can write

$$p(z) = 1 + \frac{f''(z)}{f'(z)}, p(z) \in \mathcal{P}_k \tag{2.2}$$

On the other hand M.S. Robertson [8] proved that if $f(z) \in V_k$, then

$$\left| z\frac{f''(z)}{f'(z)} - \frac{2r^2}{1 - r^2} \right| \leq \frac{kr}{1 - r^2} \tag{2.3}$$

Therefore the relation can be written in the following form,

$$\left| (1 + z\frac{f''(z)}{f'(z)}) - \frac{1 + r^2}{1 - r^2} \right| \leq \frac{kr}{1 - r^2} \tag{2.4}$$

Using the definition of the class V_k, we obtain (2.1). ☐

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Theorem 2.2. *Let $f(z)$ be an element of R_k, then*

$$\frac{r}{(1-r)^{\frac{2-k}{2}}(1+r)^{\frac{2+k}{2}}} \leq |f(z)| \leq \frac{r}{(1-r)^{\frac{2+k}{2}}(1+r)^{\frac{2-k}{2}}} \tag{2.5}$$

$$\frac{1-kr+r^2}{(1-r)^{2-\frac{k}{2}}(1+r)^{2+\frac{k}{2}}} \leq |f'(z)| \leq \frac{1+kr+r^2}{(1-r)^{2+\frac{k}{2}}(1+r)^{2-\frac{k}{2}}} \tag{2.6}$$

Proof. Using the definition of R_k, then we can write

$$\left|z\frac{f'(z)}{f(z)} - \frac{1+r^2}{1-r^2}\right| \leq \frac{kr}{1-r^2} \tag{2.7}$$

This inequality can be written in the following form,

$$\frac{1-kr+r^2}{1-r^2} \leq Rez\frac{f'(z)}{f(z)} \leq \frac{1+kr+r^2}{1-r^2} \tag{2.8}$$

On the other hand, we have

$$Rez\frac{f'(z)}{f(z)} = r.\frac{\partial}{\partial r}log|f(z)| \tag{2.9}$$

Thus we have

$$\frac{1-kr+r^2}{r(1-r^2)} \leq \frac{\partial}{\partial r}log|f(z)| \leq \frac{1+kr+r^2}{r(1-r^2)} \tag{2.10}$$

Integrating both sides (2.10), we get (2.5). The inequality (2.7) can be written in the form

$$\frac{1-kr+r^2}{1-r^2} \leq \left|z\frac{f'(z)}{f(z)}\right| \leq \frac{1+kr+r^2}{1-r^2} \tag{2.11}$$

In this step, if we use (2.5), we obtain (2.6). □

Corollary 2.3. *For $k = 2$ in (2.5), we obtain*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}$$

This is well known growth theorem for starlike functions [2].

Corollary 2.4. *For $k = 2$ in (2.6), we obtain*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}$$

This is well known distortion theorem for starlike functions [2].

Corollary 2.5. *The radius of starlikeness of R_k is*

$$R_{S^*} = \frac{k-\sqrt{k^2-4}}{2}, k \geq 2 \tag{2.12}$$

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Proof. Since

$$Re\left(z\frac{f'(z)}{f(z)}\right) > \frac{1-kr+r^2}{1-r^2}$$

□

Hence for $R < R_{S^*}$ the left hand side of the preceding inequality is positive which implies (2.12). We note that all results are sharp because of extremal function is

$$f_*(z) = \frac{z(1-z)^{\frac{k}{2}-1}}{(1+z)^{\frac{k}{2}+1}}$$

Indeed,

$$z\frac{f_*'(z)}{f_*(z)} = \frac{1-kz+z^2}{1-z^2} = \left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+z}{1-z} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-z}{1+z}$$

Thus, $f_*(z) \in R_k$ and $f_*(z)$ is extremal function.

Lemma 2.6. *Let $p(z) = 1 + p_1z + p_2z^2 + ...$ be an element of $\mathcal{P}_k$, then*

$$|p_n| \leq k$$

Proof. Method I. Since $p(z) \in \mathcal{P}_k$, then we have

$$\begin{aligned} p(z) &= \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z) \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)(1+a_1z+a_2z^2+...) - \left(\frac{k}{4}-\frac{1}{2}\right)(1+b_1z+b_2z^2+...) \end{aligned}$$

Then we have

$$p_n = \left(\frac{k}{4}+\frac{1}{2}\right)a_n - \left(\frac{k}{4}-\frac{1}{2}\right)b_n$$

Thus

$$\begin{aligned} |p_n| &= \left|\left(\frac{k}{4}+\frac{1}{2}\right)a_n - \left(\frac{k}{4}-\frac{1}{2}\right)b_n\right| \\ &\leq \left(\frac{k}{4}+\frac{1}{2}\right)|a_n| + \left(\frac{k}{4}-\frac{1}{2}\right)|b_n| \\ &\leq \left(\frac{k}{4}+\frac{1}{2}\right)2 + \left(\frac{k}{4}-\frac{1}{2}\right)2 \end{aligned}$$

This shows that,

$$|p_n| \leq k$$

Method II. Since $p(z) \in \mathcal{P}_k$, then $p(z)$ can be written in the form

$$p(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1+ze^{-it}}{1-ze^{-it}}d\mu(t)$$

5

and

$$\int_0^{2\pi} d\mu(t) = 2\pi \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq k\pi.$$

Then

$$\begin{aligned} p(z) = 1 + p_1 z + p_2 z^2 + ... &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1 + ze^{-it} - ze^{-it} + ze^{-it}}{1 - ze^{-it}} d\mu(t) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left(1 - \frac{2ze^{-it}}{1 - ze^{-it}}\right) d\mu(t) \\ |p_n| &\leq \frac{1}{\pi}\int_0^{2\pi} |d\mu(t)| \leq k \end{aligned}$$

is obtained.

We note that this lemma was proved first by K.I. Noor [4] (Method II). □

Theorem 2.7. *Let $f(z)$ be an element of R_k, then*

$$|a_n| \leq \frac{1}{(n-1)!}\prod_{\nu=0}^{n-2}(k+\nu) \tag{2.13}$$

Proof. Since $f(z) \in R_k$, then we have

$$z\frac{f'(z)}{f(z)} = p(z)$$

where $p(z) \in \mathcal{P}_k$. Thus

$$zf'(z) = f(z)p(z)$$

Comparing the coefficients in both sides of $zf'(z) = f(z)p(z)$, we obtain the recursion formula

$$a_n = \frac{1}{n-1}\sum_{\nu=1}^{n-1} p_{n-\nu}a_\nu, \quad n \geq 2$$

and therefore by Lemma 2.6,

$$|a_n| = \frac{k}{n-1}\sum_{\nu=1}^{n-1}|a_\nu|$$

Induction shows that

$$|a_n| \leq \frac{1}{(n-1)!}\prod_{\nu=0}^{n-2}(k+\nu).$$

□

Corollary 2.8. *For $k = 2$, we obtain $|a_n| \leq n$. This inequality is well known coefficient inequality for starlike functions.*

Indeed,

$$|a_n| \leq \frac{1}{(n-1)!}\prod_{\nu=0}^{n-2}(k+\nu) = \frac{k(k+1)(k+2)...(k+(n-2))}{(n-1)!}.$$

If we take $k=2$,

$$|a_n| \leq \frac{2.3.4...(n-2).(n-1).n}{(n-1)!} = n$$

Corollary 2.9. *Let* $f(z)$ *be an element of* V_k*, then*

$$|a_n| \leq \frac{1}{n!}\prod_{\nu=0}^{n-2}(k+\nu) \tag{2.14}$$

Proof. Using the theorem of Pinchuk

$$f(z) \in V_k \Leftrightarrow zf'(z) \in R_k$$

we get (2.14). □

Corollary 2.10. *For* $k=2$*, we obtain* $|a_n| \leq 1$*. This inequality is well known coefficient inequality for convex functions.*

We note that all these inequalities are sharp because extremal function is,

$$f_*(z) = \frac{z(1-z)^{\frac{k}{2}-1}}{(1+z)^{\frac{k}{2}+1}}.$$

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POLY-GENOCCHI POLYNOMIALS WITH UMBRAL CALCULUS VIEWPOINT

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Abstract. In this paper, we would like to exploit umbral calculus in order to derive explicit expressions, some properties, recurrence relations and identities for poly-Genocchi polynomials.

1. **Review on umbral calculus**

The purpose of this paper is to use umbral calculus in order to derive some new and interesting expressions, recurrence relations and identities for poly-Genocchi polynomials. To do that we first recall the umbral calculus very briefly. For more details, the reader may refer to [11, 12]. We denote the algebra of polynomials in a single variable x over $\mathbb{C}$ by $\mathbb{P}$ and the vector space of all linear functionals on $\mathbb{P}$ by $\mathbb{P}^*$. The action of a linear functional L on a polynomial $p(x)$ is denoted by $\langle L|p(x)\rangle$. We define the vector space structure on $\mathbb{P}^*$ by $\langle cL+c'L'|p(x)\rangle = c\langle L|p(x)\rangle + c'\langle L'|p(x)\rangle$, where $c, c' \in \mathbb{C}$. We define the algebra of formal power series in a single variable t to be

$$\mathcal{F} = \left\{ f(t) = \sum_{k\geq 0} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C} \right\}. \tag{1.1}$$

A power series $f(t) \in \mathcal{F}$ defines a linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n, \text{ for all } n \geq 0. \tag{1.2}$$

By (1.1) and (1.2), we have

$$\langle t^k|x^n\rangle = n!\delta_{n,k}, \text{ for all } n, k \geq 0, \tag{1.3}$$

where $\delta_{n,k}$ is the Kronecker's symbol. Let $f_L(t) = \sum_{n\geq 0}\langle L|x^n\rangle \frac{t^n}{n!}$. From (1.2), we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. So, the map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Thus, $\mathcal{F}$ is thought of as set of both formal power series and linear functionals. We call $\mathcal{F}$ the *umbral algebra*. The *umbral calculus* is the study of umbral algebra.

The *order* $O(f(t))$ of the non-zero power series $f(t) \in \mathcal{F}$ is the smallest integer k for which the coefficient of t^k does not vanish. Suppose that $f(t), g(t) \in \mathcal{F}$ such that $O(f(t)) = 1$ and $O(g(t)) = 0$, then there exists a unique sequence $s_n(x)$ of polynomials such that

$$\langle g(t)(f(t))^k|s_n(x)\rangle = n!\delta_{n,k}, \tag{1.4}$$

2010 *Mathematics Subject Classification.* 11B83, 42A16.

Key words and phrases. Poly-Genocchi polynomials, umbral calculus.

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where $n, k \geq 0$. The sequence $s_n(x)$ is called the *Sheffer* sequence for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$ (see [11, 12]). In particular, if $s_n(x) \sim (g(t), t)$, then $s_n(x)$ is called the Appell sequence for $g(t)$. For $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have $\langle e^{yt}|p(x)\rangle = p(y)$, $\langle f(t)g(t)|p(x)\rangle = \langle g(t)|f(t)p(x)\rangle$ and

$$f(t) = \sum_{n\geq 0} \langle f(t)|x^n\rangle \frac{t^n}{n!}, \quad p(x) = \sum_{n\geq 0} \langle t^n|p(x)\rangle \frac{x^n}{n!}. \tag{1.5}$$

From (1.5), we obtain $\langle t^k|p(x)\rangle = p^{(k)}(0)$ and $\langle 1|p^{(k)}(x)\rangle = p^{(k)}(0)$, where $p^{(k)}(0)$ denotes the k-th derivative of $p(x)$ with respect to x at $x = 0$. So, we get that $t^k p(x) = p^{(k)}(x) = \frac{d^k}{dx^k} p(x)$, for all $k \geq 0$. Let $s_n(x) \sim (g(t), f(t))$. Then we have

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{n\geq 0} s_n(y) \frac{t^n}{n!}, \tag{1.6}$$

for all $y \in \mathbb{C}$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$ satisfying $f(\bar{f}(t)) = \bar{f}(f(t)) = t$. Let $s_n(x) = \sum_{k=0}^{n} c_{n,k} r_k(x)$, for $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), \ell(t))$. Then we have

$$c_{n,k} = \frac{1}{k!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} (\ell(\bar{f}(t)))^k | x^n \right\rangle, \tag{1.7}$$

(see [11, 12]).

For $s_n(x) \sim (g(t), f(t))$, we have the recurrence relation

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)} \right) \frac{1}{f'(t)} s_n(x). \tag{1.8}$$

Finally, for any $h(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have the following.

$$\langle h(t)|xp(x)\rangle = \langle \partial_t h(t)|p(x)\rangle. \tag{1.9}$$

2. Introduction

Let r be any integer. We recall here that

$$Li_r(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^r}, \tag{2.1}$$

is the rth polylogarithm function for $r \geq 1$, and a rational function for $r \leq 0$. It is immediate to see that

$$\frac{d}{dx}(Li_{r+1}(x)) = \frac{1}{x} Li_r(x). \tag{2.2}$$

The Poly-Genocchi polynomials $G_n^{(r)}(x)$ of index r are given by

$$\frac{2Li_r(1-e^{-t})}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} G_n^{(r)}(x) \frac{t^n}{n!}. \tag{2.3}$$

For $x = 0$, $G_n^{(r)} = G_n^{(r)}(0)$ are called poly-Genocchi numbers of index r. In particular, if $r = 1$, $G_n^{(1)}(x) = G_n$ are the 'classical' Genocchi polynomials defined by

$$\frac{2t}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad \text{(see [8])}. \tag{2.4}$$

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The Poly-Genocchi polynomials $G_n^{(r)}(x)$ were first introduced in [3], even though they were called poly-Euler polynomials and denoted by $\mathbf{E}_n^{(r)}(x)$. For the obvious reason, it seems more appropriate to call them poly-Genocchi polynomials rather than poly-Euler polynomials. There are other definitions for poly-Euler numbers and poly-Euler polynomials. Indeed, in [10, 13] the poly-Euler numbers $E_m^{(r)}$ are defined by

$$\frac{Li_r(1-e^{-4t})}{4t\, cosht} = \sum_{m=0}^{\infty} E_m^{(r)} \frac{t^m}{m!}. \tag{2.5}$$

For poly-Euler polynomials, see [2]. The poly-Bernoulli polynomials $B_n^{(r)}(x)$ of index r are given by

$$\frac{Li_r(1-e^{-t})}{e^t-1} e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \quad \text{(see [1, 4, 6])}. \tag{2.6}$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$ are called poly-Bernoulli numbers of index r. In particular, if $r = 1$, $B_n^{(1)}(x) = B_n$ are the Bernoulli polynomials defined by

$$\frac{t}{e^t-1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{2.7}$$

The Euler polynomials $E_n(x)$ are given by

$$\frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{2.8}$$

As is well knwon,

$$E_n(x) = \frac{1}{n+1} G_{n+1}(x), \ (n \geq 0). \tag{2.9}$$

Writing $Li_r(1-e^{-t}) = \sum_{n=1}^{\infty} a_n \frac{t^n}{n!} = t + \sum_{n=2}^{\infty} a_n \frac{t^n}{n!}$, from (2.3) and (2.7) we see that

$$\sum_{n=0}^{\infty} G_n^{(r)}(x) \frac{t^n}{n!} = \sum_{n=1}^{\infty} \left(\sum_{l=0}^{n-1} \binom{n}{l} a_{n-l} E_l(x) \right) \frac{t^n}{n!}. \tag{2.10}$$

This implies that

$$G_0^{(r)}(x) = 0, \ G_1^{(r)}(x) = 1, \ deg\, G_n^{(r)}(x) = n-1, \ (n \geq 1). \tag{2.11}$$

In this paper, we would like that to exploit umbral calculus in order to derive explicit expressions, some properties, recurrence relations and identities for poly-Genocchi polynomials.

3. Explicit expressions

It is important to observe that sometimes we can not directly apply the umbral calculus techniques to the generating function (2.3) of poly-Genocchi polynomials, since $\frac{2Li_r(1-e^{-t})}{e^t+1}$ is a delta series, and hence is not invertible. Instead, we have to use the next generating function for $\frac{G_{n+1}(x)}{n+1}$, $(n \geq 1)$, which follows from (2.3) and (2.10).

$$\frac{2Li_r(1-e^{-t})}{t(e^t+1)}e^{xt} = \sum_{n=0}^{\infty} \frac{G_{n+1}^{(r)}(x)}{n+1}\frac{t^n}{n!}. \tag{3.1}$$

We see from (2.11) that $\frac{G_{n+1}^{(r)}(x)}{n+1}$ is the Appell sequence for $\frac{t(e^t+1)}{2Li_r(1-e^{-t})}$, namely

$$\frac{G_{n+1}^{(r)}(x)}{n+1} \sim \left(g(t) = \frac{t(e^t+1)}{2Li_r(1-e^{-t})}, f(t) = t\right). \tag{3.2}$$

We will compute $\left\langle \frac{Li_r(1-e^{-t})}{t} \mid x^{n+1} \right\rangle$ in four different ways in order to get interesting identities. Firstly, we have

$$\begin{aligned}
&\left\langle \frac{Li_r(1-e^{-t})}{t} | x^{n+1} \right\rangle \\
=& \left\langle \frac{1}{t}\sum_{m=1}^{\infty}(-1)^m \frac{(e^{-t}-1)^m}{m^r} | x^{n+1} \right\rangle \\
=& \sum_{m=1}^{n+2}(-1)^m \frac{m!}{m^r}\left\langle \frac{1}{t}\frac{1}{m!}(e^{-t}-1)^m | x^{n+1} \right\rangle \\
=& \sum_{m=1}^{n+2}(-1)^m \frac{m!}{m^r}\left\langle \sum_{j=m}^{\infty} S_2(j,m)\frac{(-1)^j}{j!}t^{j-1} | x^{n+1} \right\rangle \\
=& \sum_{m=1}^{n+2}(-1)^m \frac{m!}{m^r}\sum_{j=m}^{n+2} S_2(j,m)\frac{(-1)^j}{j!}(n+1)!\delta_{n+1,j-1} \\
=& \frac{1}{n+2}\sum_{m=1}^{n+2}(-1)^{m+n}\frac{m!}{m^r}S_2(n+2,m).
\end{aligned} \tag{3.3}$$

Secondly, we get

$$\begin{aligned}
&\left\langle \frac{Li_r(1-e^{-t})}{t} | x^{n+1} \right\rangle \\
=& \left\langle \frac{e^t-1}{t} \Big| \frac{Li_r(1-e^{-t})}{e^t-1} x^{n+1} \right\rangle \\
=& \left\langle \frac{e^t-1}{t} \Big| \sum_{m=0}^{\infty} B_m^{(r)}\frac{t^m}{m!} x^{n+1} \right\rangle \\
=& \sum_{m=0}^{n+1}\binom{n+1}{m}B_m^{(r)}\left\langle \frac{e^t-1}{t} | x^{n-m+1} \right\rangle \\
=& \sum_{m=0}^{n+1}\binom{n+1}{m}B_m^{(r)}\int_0^1 u^{n-m+1}du \\
=& \sum_{m=0}^{n+1}\binom{n+1}{m}B_m^{(r)}\frac{1}{n-m+2}.
\end{aligned} \tag{3.4}$$

Thirdly, we obtain

$$\begin{aligned}
&\left\langle \frac{Li_r(1-e^{-t})}{t} | x^{n+1} \right\rangle \\
=&\left\langle \frac{1}{t}\int_0^t (Li_r(1-e^{-s}))' ds | x^{n+1} \right\rangle \\
=&\left\langle \frac{1}{t}\int_0^t \frac{(Li_{r-1}(1-e^{-s}))}{e^s-1} ds | x^{n+1} \right\rangle \\
=&\left\langle \frac{1}{t}\int_0^t \sum_{m=0}^{\infty} B_m^{(r-1)} \frac{s^m}{m!} ds | x^{n+1} \right\rangle \\
=&\sum_{m=0}^{\infty} B_m^{(r-1)} \frac{1}{m!} \left\langle \frac{1}{t}\int_0^t s^m ds | x^{n+1} \right\rangle \\
=&\sum_{m=0}^{\infty} B_m^{(r-1)} \frac{1}{(m+1)!} \left\langle t^m | x^{n+1} \right\rangle \\
=&\sum_{m=0}^{\infty} B_m^{(r-1)} \frac{1}{(m+1)!} (n+1)! \delta_{n+1,m} \\
=&\frac{1}{n+2} B_{n+1}^{(r-1)}.
\end{aligned} \tag{3.5}$$

Lastly, in [7] we showed that

$$\begin{aligned}
Li_r(1-e^{-t}) = &\sum_{j_1=0}^{\infty} \cdots \sum_{j_{r-1}=0}^{\infty} t^{j_1+\cdots+j_{r-1}+1} \\
&\times \prod_{i=1}^{r-1} \frac{B_{j_i}}{j_i!(j_1+\cdots+j_i+1)}, \ (r \geq 2),
\end{aligned} \tag{3.6}$$

which follows from the well-known integral representation

$$Li_k(1-e^{-t}) = \int_0^t \underbrace{\frac{1}{e^y-1}\int_0^y \frac{1}{e^y-1}\int_0^y \cdots \frac{1}{e^y-1}\int_0^y}_{(k-2) \text{ times}} \frac{y}{e^y-1} dy \cdots dydydy, \tag{3.7}$$

Now,

$$\begin{aligned}
&\left\langle \frac{Li_r(1-e^{-t})}{t} | x^{n+1} \right\rangle \\
=&\sum_{j_1=0}^{\infty} \cdots \sum_{j_{r-1}=0}^{\infty} \prod_{i=1}^{r-1} \frac{B_{j_i}}{j_i!(j_1+\cdots+j_i+1)} \left\langle t^{j_1+\cdots+j_{r-1}} | x^{n+1} \right\rangle \\
=&\sum_{j_1=0}^{\infty} \cdots \sum_{j_{r-1}=0}^{\infty} \prod_{i=1}^{r-1} \frac{B_{j_i}}{j_i!(j_1+\cdots+j_i+1)} (n+1)! \delta_{n+1, j_1+\cdots+j_{r-1}} \\
=&(n+1)! \sum_{j_1+\cdots+j_{r-1}=n+1} \prod_{i=1}^{r-1} \frac{B_{j_i}}{j_i!(j_1+\cdots+j_i+1)}.
\end{aligned} \tag{3.8}$$

Theorem 3.1. *For all integers $r \geq 2$, and $n \geq -1$, we have the following.*

$$\begin{aligned}
&\left\langle \frac{Li_r(1-e^{-t})}{t} | x^{n+1} \right\rangle \\
=&\frac{1}{n+2}\sum_{m=1}^{n+2}(-1)^{m+n}\frac{m!}{m^r}S_2(n+2,m) \\
=&\sum_{m=0}^{n+1}\binom{n+1}{m}B_m^{(r)}\frac{1}{n-m+2} \\
=&\frac{1}{n+2}B_{n+1}^{(r-1)} \\
=&(n+1)!\sum_{j_1+\cdots+j_{r-1}=n+1}\prod_{i=1}^{r-1}\frac{B_{j_i}}{j_i!(j_1+\cdots+j_i+1)}.
\end{aligned}$$

Similarly, the following was derived in [7] except for the first one which is left as an exercise to the reader.

Theorem 3.2. *For all integers $r \geq 2$, and $n \geq -1$, we have the following.*

$$\begin{aligned}
&\left\langle Li_r(1-e^{-t}) | x^{n+1} \right\rangle \\
=&\sum_{m=1}^{n+1}(-1)^{m+n+1}\frac{m!}{m^r}S_2(n+1,m) \\
=&\sum_{m=0}^{n}\binom{n+1}{m}B_m^{(r)} \\
=&B_n^{(r-1)} \\
=&(n+1)!\sum_{j_1+\cdots+j_{r-1}=n}\prod_{i=1}^{r-1}\frac{B_{j_i}}{j_i!(j_1+\cdots+j_i+1)}.
\end{aligned}$$

The following is also immediate from (2.3). However, we derive it by using umbral calculus.

$$\begin{aligned}
G_n^{(r)}(y) &= \left\langle \sum_{m=0}^{\infty} G_m^{(r)}(y)\frac{t^m}{m!} | x^n \right\rangle \\
&= \left\langle \frac{2Li_r(1-e^{-t})}{e^t+1}e^{yt} | x^n \right\rangle \\
&= \left\langle \frac{2Li_r(1-e^{-t})}{e^t+1} | \sum_{l=0}^{\infty}\frac{y^l}{l!}t^l x^n \right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l}y^l\left\langle \frac{2Li_r(1-e^{-t})}{e^t+1} | x^{n-l} \right\rangle \\
&= \sum_{l=0}^{n}\binom{n}{l}y^l G_{n-l}^{(r)}.
\end{aligned} \tag{3.9}$$

Thus we have shown

$$G_n^{(r)}(x) = \sum_{l=0}^{n} \binom{n}{l} G_{n-l}^{(r)} x^l.$$

Next, in order to express poly-Genocchi polynomials in terms of Euler polynomials, we first observe the following.

$$\begin{aligned} G_n^{(r)}(y) &= \left\langle \frac{2Li_r(1-e^{-t})}{e^t+1} e^{yt} | x^n \right\rangle \\ &= \left\langle Li_r(1-e^{-t}) | \frac{2}{e^t+1} e^{yt} x^n \right\rangle \\ &= \left\langle Li_r(1-e^{-t}) | \sum_{l=0}^{\infty} E_l(y) \frac{t^l}{l!} x^n \right\rangle \\ &= \sum_{l=0}^{n} \binom{n}{l} E_l(y) \left\langle Li_r(1-e^{-t}) | x^{n-l} \right\rangle \end{aligned} \tag{3.10}$$

From this and Theorem 1.2, after simple manipulations, we obtain the following explicit expressions for $G_n^{(r)}(x)$, as linear combinations of Euler polynomials.

Theorem 3.3. *For any integer $n \geq 0$, we have*

$$\begin{aligned} G_n^{(r)}(x) &= \sum_{l=1}^{n} \sum_{m=1}^{l} \binom{n}{l} (-1)^{l+m} \frac{m!}{m^r} S_2(l,m) E_{n-l}(x) \\ &= \sum_{l=1}^{n} \sum_{m=0}^{l-1} \binom{n}{l} \binom{l}{m} B_m^{(r)} E_{n-l}(x) \\ &= \sum_{l=1}^{n} \binom{n}{l} B_{l-1}^{(r-1)} E_{n-l}(x) \\ &= \sum_{l=1}^{n} \sum_{j_1,\cdots,j_{r-1} \geq 0, j_1+\cdots+j_{r-1}=l-1} (n)_l \prod_{i=1}^{r-1} \frac{B_{j_i}}{j_i!(j_1+\cdots+j_i+1)} E_{n-l}(x). \end{aligned}$$

This time we want to express poly-Genocchi polynomials in terms of Genocchi polynomials. For this, we first observe the following.

$$\begin{aligned} G_n^{(r)}(y) &= \left\langle \frac{2Li_r(1-e^{-t})}{e^t+1} e^{yt} | x^n \right\rangle \\ &= \left\langle \frac{Li_r(1-e^{-t})}{t} | \frac{2t}{e^t+1} e^{yt} x^n \right\rangle \\ &= \left\langle \frac{Li_r(1-e^{-t})}{t} | \sum_{l=0}^{\infty} G_l(y) \frac{t^l}{l!} x^n \right\rangle \\ &= \sum_{l=0}^{n} \binom{n}{l} G_l(y) \left\langle \frac{Li_r(1-e^{-t})}{t} | x^{n-l} \right\rangle \end{aligned} \tag{3.11}$$

From this and Theorem 1.1, after simple manipulations, we get the following explicit expressions for $G_n^{(r)}(x)$, as linear combinations of Genocchi polynomials.

Theorem 3.4. *For any integer $n \geq 0$, we have*

$$
\begin{aligned}
G_n^{(r)}(x) &= \sum_{l=0}^{n-1} \sum_{m=1}^{l+1} \frac{1}{l+1} \binom{n}{l} (-1)^{l+m-1} \frac{m!}{m^r} S_2(l+1,m) G_{n-l}(x) \\
&= \sum_{l=0}^{n-1} \sum_{m=0}^{l} \frac{1}{l-m+1} \binom{n}{l} \binom{l}{m} B_m^{(r)} G_{n-l}(x) \\
&= \sum_{l=0}^{n-1} \frac{1}{l+1} \binom{n}{l} B_l^{(r-1)} G_{n-l}(x) \\
&= \sum_{l=0}^{n-1} \sum_{j_1,\cdots,j_{r-1} \geq 0, j_1+\cdots+j_{r-1}=l} (n)_l \prod_{i=1}^{r-1} \frac{B_{j_i}}{j_i!(j_1+\cdots+j_i+1)} G_{n-l}(x).
\end{aligned}
$$

As a final remark in this section, we mention the following Appell identity.

$$
B_n^{(r)}(x+y) = \sum_{j=0}^{n} \binom{n}{j} B_j^{(r)}(y) x^{n-j}. \tag{3.12}
$$

4. **Recurrence relations**

From (1.9), for $s_n(x) \sim (g(t), t)$ we have

$$
s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)} \right) s_n(x). \tag{4.1}
$$

Here we apply this recurrence relation to

$$
\frac{G_{n+1}^{(r)}(x)}{n+1} \sim \left(g(t) = \frac{t(e^t+1)}{2Li_r(1-e^{-t})}, t \right). \tag{4.2}
$$

Then

$$
\frac{G_{n+2}^{(r)}(x)}{n+2} = \frac{1}{n+1} x G_{n+1}^{(r)}(x) - \frac{g'(t)}{g(t)} \frac{1}{n+1} G_{n+1}^{(r)}(x). \tag{4.3}
$$

Observe first that

$$
\begin{aligned}
&\frac{g'(t)}{g(t)} = (\log g(t))' \\
=&\frac{1}{t} + \frac{e^t}{e^t+1} - \frac{(Li_r(1-e^{-t}))'}{Li_r(1-e^{-t})} \\
=&\frac{1}{t} + \frac{e^t}{e^t+1} - \frac{1}{Li_r(1-e^{-t})} \frac{Li_{r-1}(1-e^{-t})}{e^t-1} \\
=&\frac{1}{t} \left(1 + t - \frac{t}{e^t+1} - \frac{t}{Li_r(1-e^{-t})} \frac{Li_{r-1}(1-e^{-t})}{e^t-1} \right) \\
=&\frac{1}{t} \Big(\frac{2Li_r(1-e^{-t})}{t(e^t+1)} + \frac{2Li_r(1-e^{-t})}{e^t+1} - \frac{1}{2} \frac{2}{e^t+1} \frac{2Li_r(1-e^{-t})}{e^t+1} \\
&- \frac{2}{e^t+1} \frac{Li_{r-1}(1-e^{-t})}{e^t-1} \Big) \frac{t(e^t+1)}{2Li_r(1-e^{-t})}.
\end{aligned} \tag{4.4}
$$

Now,

$$\begin{aligned}
&\frac{g'(t)}{g(t)}\frac{1}{n+1}G_{n+1}^{(r)}(x)\\
=&\frac{1}{t}\Big(\frac{2Li_r(1-e^{-t})}{t(e^t+1)}+\frac{2Li_r(1-e^{-t})}{e^t+1}-\frac{1}{2}\frac{2}{e^t+1}\frac{2Li_r(1-e^{-t})}{e^t+1}\\
&-\frac{2}{e^t+1}\frac{Li_{r-1}(1-e^{-t})}{e^t-1}\Big)x^n. \qquad (4.5)\\
=&\frac{1}{n+1}\Big(\frac{2Li_r(1-e^{-t})}{t(e^t+1)}+\frac{2Li_r(1-e^{-t})}{e^t+1}-\frac{1}{2}\frac{2}{e^t+1}\frac{2Li_r(1-e^{-t})}{e^t+1}\\
&-\frac{2}{e^t+1}\frac{Li_{r-1}(1-e^{-t})}{e^t-1}\Big)x^{n+1}.
\end{aligned}$$

Note here that the expression in bracket of (4.5) has order ≥ 1, and

$$x^n=\frac{t(e^t+1)}{2Li_r(1-e^{-t})}\frac{G_{n+1}^{(r)}(x)}{n+1}. \qquad (4.6)$$

We now compute the four pieces in the expression of (**??**):

$$\begin{aligned}
&\frac{2Li_r(1-e^{-t})}{t(e^t+1)}x^{n+1}=\sum_{l=0}^{\infty}\frac{G_{l+1}^{(r)}}{l+1}\frac{t^l}{l!}x^{n+1}\\
=&\sum_{l=0}^{n+1}\frac{1}{l+1}\binom{n+1}{l}G_{l+1}^{(r)}x^{n+1-l},
\end{aligned} \qquad (4.7)$$

$$\begin{aligned}
&\frac{2Li_r(1-e^{-t})}{e^t+1}x^{n+1}=\sum_{l=0}^{\infty}G_l^{(r)}\frac{t^l}{l!}x^{n+1}\\
=&\sum_{l=0}^{n+1}\binom{n+1}{l}G_l^{(r)}x^{n+1-l},
\end{aligned} \qquad (4.8)$$

$$\begin{aligned}
\frac{2}{e^t+1}\frac{2Li_r(1-e^{-t})}{e^t+1}x^{n+1}=&\frac{2}{e^t+1}\sum_{l=0}^{n+1}\binom{n+1}{l}G_l^{(r)}x^{n+1-l}\\
=&\sum_{l=0}^{n+1}\binom{n+1}{l}G_l^{(r)}\frac{2}{e^t+1}x^{n+1-l}\\
=&\sum_{l=0}^{n+1}\binom{n+1}{l}G_l^{(r)}E_{n+1-l}(x),
\end{aligned} \qquad (4.9)$$

$$\begin{aligned}
\frac{2}{e^t+1}\frac{Li_{r-1}(1-e^{-t})}{e^t-1}x^{n+1}=&\frac{2}{e^t+1}\sum_{l=0}^{\infty}B_l^{(r-1)}\frac{t^l}{l!}x^{n+1}\\
=&\frac{2}{e^t+1}\sum_{l=0}^{n+1}\binom{n+1}{l}B_l^{(r-1)}x^{n+1-l}\\
=&\sum_{l=0}^{n+1}\binom{n+1}{l}B_l^{(r-1)}E_{n+1-l}(x).
\end{aligned} \qquad (4.10)$$

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Putting everything altogether, we arrive at the following theorem.

Theorem 4.1. *For any integer $n \geq 0$, we have*

$$\frac{G_{n+2}^{(r)}(x)}{n+2} = \frac{1}{n+1} x G_{n+1}^{(r)}(x) + \frac{1}{n+1}\Big(\sum_{l=0}^{n+1}\binom{n+1}{l}\big(\frac{1}{2}G_l^{(r)} + B_l^{(r-1)}\big)E_{n+1-l}(x) - \sum_{l=0}^{n+1}\binom{n+1}{l}\big(\frac{G_{l+1}^{(r)}}{l+1} + G_l^{(r)}\big)x^{n+1-l}\Big).$$

Assume that $n \geq 1$,

$$\begin{aligned} G_n^{(r)}(y) &= \left\langle \frac{2Li_r(1-e^{-t})}{e^t+1} e^{yt} | x^n \right\rangle \\ &= \left\langle \Big(\partial_t \frac{2Li_r(1-e^{-t})}{e^t+1}\Big) e^{yt} | x^{n-1} \right\rangle + \left\langle \frac{2Li_r(1-e^{-t})}{e^t+1} (\partial_t e^{yt}) | x^{n-1} \right\rangle \end{aligned} \tag{4.11}$$

It is easy to see that the second term in (4.11) is equal to $yG_n^{(r)}(y)$. For the first term, we observe that

$$\begin{aligned} &\partial_t\Big(\frac{2Li_r(1-e^{-t})}{e^t+1}\Big) \\ =& \frac{2\frac{Li_{r-1}(1-e^{-t})}{1-e^{-t}} e^{-t}(e^t+1) - 2Li_r(1-e^{-t})e^t}{(e^t+1)^2} \\ =& \frac{2}{e^t+1}\frac{Li_{r-1}(1-e^{-t})}{e^t-1} - \frac{2Li_r(1-e^{-t})}{e^t+1} + \frac{1}{2}\frac{2}{e^t+1}\frac{2Li_r(1-e^{-t})}{e^t+1} \end{aligned} \tag{4.12}$$

So the first term can be written as three sums:

$$\begin{aligned} &\left\langle \frac{2}{e^t+1}\frac{Li_{r-1}(1-e^{-t})}{e^t-1} e^{yt} | x^{n-1} \right\rangle - \left\langle \frac{2Li_r(1-e^{-t})}{e^t+1} e^{yt} \mid x^{n-1} \right\rangle \\ &+ \frac{1}{2}\left\langle \frac{2}{e^t+1}\frac{2Li_r(1-e^{-t})}{e^t+1} e^{yt} | x^{n-1} \right\rangle. \end{aligned} \tag{4.13}$$

We now compute the three terms in (4.13):

$$\begin{aligned} &\left\langle \frac{2}{e^t+1}\frac{Li_{r-1}(1-e^{-t})}{e^t-1} e^{yt} | x^{n-1} \right\rangle \\ =& \left\langle \frac{2}{e^t+1} | \frac{Li_{r-1}(1-e^{-t})}{e^t-1} e^{yt} x^{n-1} \right\rangle \\ =& \left\langle \frac{2}{e^t+1} | \sum_{l=0}^{\infty} B_l^{(r-1)}(y)\frac{t^l}{l!} x^{n-1} \right\rangle \\ =& \sum_{l=0}^{n-1}\binom{n-1}{l} B_l^{(r-1)}(y) \left\langle \frac{2}{e^t+1} | x^{n-1-l} \right\rangle \\ =& \sum_{l=0}^{n-1}\binom{n-1}{l} B_l^{(r-1)}(y) E_{n-1-l}, \end{aligned} \tag{4.14}$$

$$\left\langle \frac{2Li_r(1-e^{-t})}{e^t+1} e^{yt} | x^{n-1} \right\rangle = G_{n-1}^{(r)}(y), \tag{4.15}$$

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$$\begin{aligned}
&\left\langle \frac{2}{e^t+1}\frac{2Li_r(1-e^{-t})}{e^t+1}e^{yt}|x^{n-1}\right\rangle \\
=&\left\langle \frac{2}{e^t+1}|\frac{2Li_r(1-e^{-t})}{e^t+1}e^{yt}x^{n-1}\right\rangle \\
=&\left\langle \frac{2}{e^t+1}|\sum_{l=0}^{\infty}G_l^{(r)}(y)\frac{t^l}{l!}x^{n-1}\right\rangle \\
=&\sum_{l=0}^{n-1}\binom{n-1}{l}G_l^{(r)}(y)\left\langle \frac{2}{e^t+1}|x^{n-1-l}\right\rangle \\
=&\sum_{l=0}^{n-1}\binom{n-1}{l}G_l^{(r)}(y)E_{n-1-l}.
\end{aligned} \tag{4.16}$$

Putting everything altogether, we have the following theorem.

Theorem 4.2. *For any integer $n \geq 1$, we have the following recursive relation.*

$$\begin{aligned}
&(1-x)G_n^{(r)}(x)+G_{n-1}^{(r)}(x) \\
=&\sum_{l=0}^{n-1}\binom{n-1}{l}E_{n-1-l}(B_l^{(r-1)}(x)+\frac{1}{2}G_l^{(r)}(x)).
\end{aligned}$$

5. Connections with other families of polynomials

In this section, we will exploit (1.7) in order to express poly-Genocchi polynomials as linear combinations of well known families of polynomials. To express poly-Genocchi polynomials in terms of Bernoulli polynomials, with noting that

$$\frac{G_{n+1}^{(r)}(x)}{n+1} \sim \left(\frac{t(e^t+1)}{2Li_r(1-e^{-t})},t\right), B_n(x) \sim \left(\frac{e^t-1}{t},t\right), \tag{5.1}$$

we let $\frac{G_{n+1}^{(r)}(x)}{n+1} = \sum_{k=0}^{n} C_{n,k}B_k(x)$. Then

$$\begin{aligned}
C_{n,k} =&\frac{1}{k!}\left\langle \frac{e^t-1}{t}\frac{2Li_r(1-e^{-t})}{t(e^t+1)}t^k|x^n\right\rangle \\
=&\binom{n}{k}\left\langle \frac{e^t-1}{t}|\frac{2Li_r(1-e^{-t})}{t(e^t+1)}x^{n-k}\right\rangle \\
=&\binom{n}{k}\left\langle \frac{e^t-1}{t}|\sum_{l=0}^{\infty}\frac{G_{l+1}^{(r)}}{l+1}\frac{t^l}{l!}x^{n-k}\right\rangle \\
=&\binom{n}{k}\sum_{l=0}^{n-k}\frac{1}{l+1}\binom{n-k}{l}G_{l+1}^{(r)}\left\langle \frac{e^t-1}{t}|x^{n-k-l}\right\rangle
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
&=\binom{n}{k}\sum_{l=0}^{n-k}\frac{1}{l+1}\binom{n-k}{l}G_{l+1}^{(r)}\int_0^1 u^{n-k-l}du \\
&=\binom{n}{k}\sum_{l=0}^{n-k}\frac{1}{(l+1)(n-k-l+1)}\binom{n-k}{l}G_{l+1}^{(r)} \\
&=\frac{1}{(n+1)k}\sum_{l=0}^{n-k}\binom{n+1}{l+1}\binom{n-l}{k-1}G_{l+1}^{(r)}.
\end{aligned} \tag{5.3}$$

Thus we get the following result.

Theorem 5.1. *For any integer $n \geq 0$, we have the following.*

$$G_{n+1}^{(r)}(x)=\sum_{k=0}^{n}\sum_{l=0}^{n-k}\frac{1}{k}\binom{n+1}{l+1}\binom{n-l}{k-1}G_{l+1}^{(r)}B_k(x).$$

Write $\frac{G_{n+1}^{(r)}(x)}{n+1}=\sum_{k=0}^{n}C_{n,k}(x)_n$, with

$$\frac{G_{n+1}^{(r)}(x)}{n+1}\sim\left(\frac{t(e^t+1)}{2Li_r(1-e^{-t})},t\right),(x)_n\sim\left(1,e^t-1\right), \tag{5.4}$$

where $(x)_n$ are the lower factorial polynomials. Then

$$\begin{aligned}
C_{n,k}&=\frac{1}{k!}\left\langle\frac{2Li_r(1-e^{-t})}{t(e^t+1)}(e^t-1)^k|x^n\right\rangle \\
&=\left\langle\frac{2Li_r(1-e^{-t})}{t(e^t+1)}|\frac{1}{k!}(e^t-1)^k x^n\right\rangle \\
&=\left\langle\frac{2Li_r(1-e^{-t})}{t(e^t+1)}|\sum_{l=k}^{\infty}S_2(l,k)\frac{t^l}{l!}x^n\right\rangle \\
&=\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\left\langle\frac{2Li_r(1-e^{-t})}{t(e^t+1)}|x^{n-l}\right\rangle \\
&=\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\frac{G_{n-l+1}^{(r)}}{n-l+1} \\
&=\frac{1}{n+1}\sum_{l=k}^{n}\binom{n+1}{l}S_2(l,k)G_{n-l+1}^{(r)}.
\end{aligned} \tag{5.5}$$

Thus we obtain the following theorem.

Theorem 5.2. *For any integer $n \geq 0$, we have the following.*

$$G_{n+1}^{(r)}(x)=\sum_{k=0}^{n}\sum_{l=k}^{n}\binom{n+1}{l}S_2(l,k)G_{n-l+1}^{(r)}(x)_k.$$

Let $Ob_n(x)$ denote the ordered Bell polynomials given by

$$\frac{1}{2-e^t}e^{xt}=\sum_{n=0}^{\infty}Ob_n(x)\frac{t^n}{n!}. \tag{5.6}$$

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The ordered Bell polynomials have been of great use in number theory and enumerative combinatorics.

Here we would like to express the poly-Genocchi polynomials in terms of ordered Bell polynomials. With observing that

$$\frac{G_{n+1}^{(r)}(x)}{n+1} \sim \left(\frac{t(e^t+1)}{2Li_r(1-e^{-t})}, t\right), Ob_n(x) \sim \left(2-e^t, t\right), \tag{5.7}$$

we let $\frac{G_{n+1}^{(r)}(x)}{n+1} = \sum_{k=0}^{n} C_{n,k} Ob_k(x)$. Then

$$\begin{aligned} C_{n,k} =& \frac{1}{k!}\left\langle (2-e^t)\frac{2Li_r(1-e^{-t})}{t(e^t+1)}t^k | x^n \right\rangle \\ =& \binom{n}{k}\left\langle 2-e^t | \frac{2Li_r(1-e^{-t})}{t(e^t+1)} x^{n-k} \right\rangle \\ =& \binom{n}{k}\left\langle 2-e^t | \sum_{l=0}^{\infty} \frac{G_{l+1}^{(r)}}{l+1}\frac{t^l}{l!} x^{n-k} \right\rangle \\ =& \binom{n}{k}\sum_{l=0}^{n-k} \frac{G_{l+1}^{(r)}}{l+1}\binom{n-k}{l}\langle 2-e^t | x^{n-k-l}\rangle \\ =& \binom{n}{k}\sum_{l=0}^{n-k} \frac{G_{l+1}^{(r)}}{l+1}\binom{n-k}{l}(2\delta_{n-k}, l-1) \\ =& \frac{1}{n+1}\sum_{l=0}^{n-k} \binom{n+1}{l+1}\binom{n-l}{k} G_{l+1}^{(r)}(2\delta_{n-k,l}-1). \end{aligned} \tag{5.8}$$

Thus we get the following result.

Theorem 5.3. *For any integer $n \geq 0$, we have the following.*

$$G_{n+1}^{(r)}(x) = \sum_{k=0}^{n}\sum_{l=0}^{n-k} \binom{n+1}{l+1}\binom{n-l}{k} G_{l+1}^{(r)}(2\delta_{n-k,l}-1) Ob_k(x).$$

We recall here that the Bernoulli polynomials of the second kind $b_n(x)$ are given by

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}, \quad \text{(see [9])}. \tag{5.9}$$

With noting that

$$\frac{G_{n+1}^{(r)}(x)}{n+1} \sim \left(\frac{t(e^t+1)}{2Li_r(1-e^{-t})}, t\right), b_n(x) \sim \left(\frac{t}{e^t-1}, e^t-1\right), \tag{5.10}$$

we let $\frac{G^{(r)}_{n+1}(x)}{n+1} = \sum_{k=0}^{n} C_{n,k} b_k(x)$. Then

$$\begin{aligned} C_{n,k} =& \frac{1}{k!}\left\langle \frac{t}{e^t-1}\frac{2Li_r(1-e^{-t})}{t(e^t+1)}(e^t-1)^k | x^n \right\rangle \\ =& \left\langle \frac{t}{e^t-1}\frac{2Li_r(1-e^{-t})}{t(e^t+1)} | \frac{1}{k!}(e^t-1)^k x^n \right\rangle \\ =& \left\langle \frac{t}{e^t-1}\frac{2Li_r(1-e^{-t})}{t(e^t+1)} | \sum_{l=k}^{\infty} S_2(l,k)\frac{t^l}{l!} x^n \right\rangle \\ =& \sum_{l=k}^{n}\binom{n}{l} S_2(l,k)\left\langle \frac{t}{e^t-1} | \frac{2Li_r(1-e^{-t})}{t(e^t+1)} x^{n-l} \right\rangle \\ =& \sum_{l=k}^{n}\binom{n}{l} S_2(l,k)\left\langle \frac{t}{e^t-1} | \sum_{m=0}^{\infty}\frac{G^{(r)}_{m+1}}{m+1}\frac{t^m}{m!} x^{n-l} \right\rangle \\ =& \sum_{l=k}^{n}\binom{n}{l} S_2(l,k)\sum_{m=0}^{n-l}\frac{G^{(r)}_{m+1}}{m+1}\binom{n-l}{m}\left\langle \frac{t}{e^t-1} | x^{n-l-m} \right\rangle \\ =& \frac{1}{n+1}\sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n+1}{m+1}\binom{n-m}{l} S_2(l,k) G^{(r)}_{m+1} B_{n-l-m}. \end{aligned} \tag{5.11}$$

Thus we deduced the following theorem.

Theorem 5.4. *For any integer $n \geq 0$, we have the following.*

$$G^{(r)}_{n+1}(x) = \sum_{k=0}^{n}\sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n+1}{m+1}\binom{n-m}{l} S_2(l,k) G^{(r)}_{m+1} B_{n-l-m} b_k(x).$$

The exponential polynomials $\phi_n(x)$(also called Bell or Touchard polynomials) are given by

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty}\phi_n(x)\frac{t^n}{n!}. \tag{5.12}$$

With noting that

$$\frac{G^{(r)}_{n+1}(x)}{n+1} \sim \left(\frac{t(e^t+1)}{2Li_r(1-e^{-t})}, t\right), \phi_n(x) \sim (1, \log(1+t)), \tag{5.13}$$

we write $\frac{G^{(r)}_{n+1}(x)}{n+1} = \sum_{k=0}^{n} C_{n,k}\phi_k(x)$. Then

$$\begin{aligned} C_{n,k} =& \frac{1}{k!}\left\langle \frac{2Li_r(1-e^{-t})}{t(e^t+1)}(\log(1+t))^k | x^n \right\rangle \\ =& \left\langle \frac{2Li_r(1-e^{-t})}{t(e^t+1)} | \frac{1}{k!}(\log(1+t))^k x^n \right\rangle \\ =& \left\langle \frac{2Li_r(1-e^{-t})}{t(e^t+1)} | \sum_{l=k}^{\infty} S_1(l,k)\frac{t^l}{l!} x^n \right\rangle \end{aligned} \tag{5.14}$$

$$\begin{aligned}
&=\sum_{l=k}^{n}\binom{n}{l}S_1(l,k)\left\langle\frac{2Li_r(1-e^{-t})}{t(e^t+1)}|x^{n-l}\right\rangle\\
&=\sum_{l=k}^{n}\binom{n}{l}S_1(l,k)\frac{G_{n-l+1}^{(r)}}{n-l+1}\\
&=\frac{1}{n+1}\sum_{l=k}^{n}\binom{n+1}{l}S_1(l,k)G_{n-l+1}^{(r)}.
\end{aligned}\tag{5.15}$$

Thus we have the following result.

Theorem 5.5. *For any integer $n \geq 0$, we have the following.*

$$G_{n+1}^{(r)}(x)=\sum_{k=0}^{n}\sum_{l=k}^{n}\binom{n+1}{l}S_1(l,k)G_{n-l+1}^{(r)}\phi_k(x).$$

The Daehee polynomials $D_n(x)$ are given by

$$\frac{\log(1+t)}{t}(1+t)^x=\sum_{n=0}^{\infty}D_n(x)\frac{t^n}{n!}.\tag{5.16}$$

Let $\frac{G_{n+1}^{(r)}(x)}{n+1}=\sum_{k=0}^{n}C_{n,k}D_k(x)$, with noting that

$$\frac{G_{n+1}^{(r)}(x)}{n+1}\sim\left(\frac{t(e^t+1)}{2Li_r(1-e^{-t})},t\right),D_n(x)\sim\left(\frac{e^t-1}{t},e^t-1\right).\tag{5.17}$$

Then we have

$$\begin{aligned}
C_{n,k}&=\frac{1}{k!}\left\langle\frac{e^t-1}{t}\frac{2Li_r(1-e^{-t})}{t(e^t+1)}(e^t-1)^k|x^n\right\rangle\\
&=\left\langle\frac{e^t-1}{t}\frac{2Li_r(1-e^{-t})}{t(e^t+1)}|\frac{1}{k!}(e^t-1)^kx^n\right\rangle\\
&=\left\langle\frac{e^t-1}{t}\frac{2Li_r(1-e^{-t})}{t(e^t+1)}|\sum_{l=k}^{\infty}S_2(l,k)\frac{t^l}{l!}x^n\right\rangle\\
&=\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\left\langle\frac{e^t-1}{t}|\frac{2Li_r(1-e^{-t})}{t(e^t+1)}x^{n-l}\right\rangle\\
&=\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\left\langle\frac{e^t-1}{t}|\sum_{m=0}^{\infty}\frac{G_{m+1}^{(r)}}{m+1}\frac{t^m}{m!}x^{n-l}\right\rangle\\
&=\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\sum_{m=0}^{n-l}\frac{G_{m+1}^{(r)}}{m+1}\binom{n-l}{m}\left\langle\frac{e^t-1}{t}|x^{n-l-m}\right\rangle\\
&=\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\sum_{m=0}^{n-l}\frac{G_{m+1}^{(r)}}{m+1}\binom{n-l}{m}\int_0^1u^{n-l-m}du
\end{aligned}\tag{5.18}$$

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$$
\begin{aligned}
&=\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\sum_{m=0}^{n-l}\frac{G_{m+1}^{(r)}}{(m+1)(n-l-m+1)}\binom{n-l}{m}\\
&=\frac{1}{n+1}\sum_{l=k}^{n}\sum_{m=0}^{n-l}\frac{1}{m+1}\binom{n+1}{m}\binom{n-m+1}{l}S_2(l,k)G_{m+1}^{(r)}.
\end{aligned}
\tag{5.19}
$$

Thus we derived the following result.

Theorem 5.6. *For any integer $n \geq 0$, we have the following.*

$$
G_{n+1}^{(r)}(x)=\sum_{k=0}^{n}\sum_{l=k}^{n}\sum_{m=0}^{n-l}\frac{1}{m+1}\binom{n+1}{m}\binom{n-m+1}{l}S_2(l,k)G_{m+1}^{(r)}D_k(x).
$$

The Mittag-Leffler polynomials $M_n(x)$ are given by

$$
\left(\frac{1+t}{1-t}\right)^x=\sum_{n=0}^{\infty}M_n(x)\frac{t^n}{n!}. \tag{5.20}
$$

Write $\frac{G_{n+1}^{(r)}(x)}{n+1}=\sum_{k=0}^{n}C_{n,k}M_k(x)$, with observing that

$$
\frac{G_{n+1}^{(r)}(x)}{n+1}\sim\left(\frac{t(e^t+1)}{2Li_r(1-e^{-t})},t\right),M_n(x)\sim\left(1,\frac{e^t-1}{e^t+1}\right). \tag{5.21}
$$

Then we have

$$
\begin{aligned}
C_{n,k}=&\frac{1}{k!}\left\langle\frac{2Li_r(1-e^{-t})}{t(e^t+1)}\left(\frac{e^t-1}{e^t+1}\right)^k|x^n\right\rangle\\
=&2^{-k}\left\langle(\frac{2}{e^t+1})^k\frac{2Li_r(1-e^{-t})}{t(e^t+1)}|\frac{1}{k!}(e^t-1)^kx^n\right\rangle\\
=&2^{-k}\left\langle(\frac{2}{e^t+1})^k\frac{2Li_r(1-e^{-t})}{t(e^t+1)}|\sum_{l=k}^{\infty}S_2(l,k)\frac{t^l}{l!}x^n\right\rangle\\
=&2^{-k}\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\left\langle(\frac{2}{e^t+1})^k|\frac{2Li_r(1-e^{-t})}{t(e^t+1)}x^{n-l}\right\rangle\\
=&2^{-k}\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\left\langle(\frac{2}{e^t+1})^k|\sum_{m=0}^{\infty}\frac{G_{m+1}^{(r)}}{m+1}\frac{t^m}{m!}x^{n-l}\right\rangle\\
=&2^{-k}\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\sum_{m=0}^{n-l}\frac{G_{m+1}^{(r)}}{m+1}\binom{n-l}{m}\left\langle(\frac{2}{e^t+1})^k|x^{n-l-m}\right\rangle\\
=&2^{-k}\sum_{l=k}^{n}\binom{n}{l}S_2(l,k)\sum_{m=0}^{n-l}\frac{G_{m+1}^{(r)}}{m+1}\binom{n-l}{m}E_{n-l-m}^{(k)}\\
=&\frac{2^{-k}}{n+1}\sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n+1}{m+1}\binom{n-m}{l}S_2(l,k)G_{m+1}^{(r)}E_{n-l-m}^{(k)}.
\end{aligned}
\tag{5.22}
$$

Here $E_n^{(k)}$ are the Euler numbers of order k given by

$$\left(\frac{2}{e^t+1}\right)^k = \sum_{n=0}^{\infty} E_n^{(k)} \frac{t^n}{n!}. \tag{5.23}$$

Thus we deduced the following theorem.

Theorem 5.7. *For any integer $n \geq 0$, we have the following.*

$$G_{n+1}^{(r)}(x) = \sum_{k=0}^{n}\sum_{l=k}^{n}\sum_{m=0}^{n-l} 2^{-k}\binom{n+1}{m+1}\binom{n-m}{l} S_2(l,k) G_{m+1}^{(r)} E_{n-l-m}^{(k)} M_k(x).$$

The Boole polynomials $Bl_n(x)$ are given by

$$\frac{1}{1+(1+t)^{\lambda}}(1+t)^x = \sum_{n=0}^{\infty} Bl_n(x)\frac{t^n}{n!}. \tag{5.24}$$

To express the poly-Genocchi polynomials in terms of Boole polynomials, we let $\frac{G_{n+1}^{(r)}(x)}{n+1} = \sum_{k=0}^{n} C_{n,k} Bl_k(x)$, with noting that

$$\frac{G_{n+1}^{(r)}(x)}{n+1} \sim \left(\frac{t(e^t+1)}{2Li_r(1-e^{-t})}, t\right), Bl_n(x) \sim \left(1+e^{\lambda t}, e^t-1\right). \tag{5.25}$$

Then

$$\begin{aligned}
C_{n,k} =& \frac{1}{k!}\left\langle (1+e^{\lambda t})\frac{2Li_r(1-e^{-t})}{t(e^t+1)}(e^t-1)^k | x^n \right\rangle \\
=& \left\langle (1+e^{\lambda t})\frac{2Li_r(1-e^{-t})}{t(e^t+1)} | \frac{1}{k!}(e^t-1)^k x^n \right\rangle \\
=& \left\langle (1+e^{\lambda t})\frac{2Li_r(1-e^{-t})}{t(e^t+1)} | \sum_{l=k}^{\infty} S_2(l,k)\frac{t^l}{l!} x^n \right\rangle \\
=& \sum_{l=k}^{n}\binom{n}{l} S_2(l,k)\left\langle 1+e^{\lambda t} | \frac{2Li_r(1-e^{-t})}{t(e^t+1)} x^{n-l} \right\rangle \\
=& \sum_{l=k}^{n}\binom{n}{l} S_2(l,k)\left\langle 1+e^{\lambda t} | \sum_{m=0}^{\infty}\frac{G_{m+1}^{(r)}}{m+1}\frac{t^m}{m!} x^{n-l} \right\rangle \\
=& \sum_{l=k}^{n}\binom{n}{l} S_2(l,k)\sum_{m=0}^{n-l}\frac{G_{m+1}^{(r)}}{m+1}\binom{n-l}{m}\left\langle 1+e^{\lambda t} | x^{n-l-m}\right\rangle \\
=& \sum_{l=k}^{n}\binom{n}{l} S_2(l,k)\sum_{m=0}^{n-l}\frac{G_{m+1}^{(r)}}{m+1}\binom{n-l}{m}(\delta_{n-l,m}+\lambda^{n-l-m}) \\
=& \frac{1}{n+1}\sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n+1}{m+1}\binom{n-m}{l} S_2(l,k) G_{m+1}^{(r)}(\delta_{n-l,m}+\lambda^{n-l-m}).
\end{aligned} \tag{5.26}$$

So we obtained the following theorem.

Theorem 5.8. *For any integer $n \geq 0$, we have the following.*

$$G_{n+1}^{(r)}(x) = \sum_{k=0}^{n}\sum_{l=k}^{n}\sum_{m=0}^{n-l}\binom{n+1}{m+1}\binom{n-m}{l} S_2(l,k) G_{m+1}^{(r)}(\delta_{n-l,m}+\lambda^{n-l-m}) Bl_k(x).$$

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On a class of certain dynamic inequalities in three independent variables on time scales

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December 23, 2017

Abstract

The objective of this paper is to investigate and extend some Pachpatte type dynamic inequalities on time scales in three independent variables which provide explicit bounds on unknown functions and their derivatives. Some applications are also discussed here in order to illustrate the usefulness of our results.

Keywords and phrases: Time scales, integral inequality, dynamic inequality, explicit estimates .

2010 Mathematics Subject Classification: 26E70, 26D10, 34N05.

1 Introduction

The theory of time scales was created by Hilger [11] in order to unify the theories of differential equations and of difference equations and in order to extend those theories to other kinds of the so-called "dynamic equations". The two main features of the calculus on time scales are unification and extension of continuous and discrete analysis. Since then, many authors have studied different aspects of dynamic and integral inequalities on time scales by using various techniques(for example, see[1-22] and the references therein).

Our work is related to the explicit bounds of Pachpatte [15], [19] in the form of dynamic inequalities with three variables which can be used as handy tools to study the properties of certain differential and dynamic equations on time scales. We hope the results given here will assure greater importance in near future.

2 Notations and Preliminaries on Time Scales

Here, we begin by giving some necessary material for our study.

Throughout this paper, we assume that a time scale T is an arbitrary nonempty closed subset of R where R denotes the set of real numbers and $R_+ = [0, \infty)$. Also T_1 and T_2 be two time scales with atleast two points and $\Phi = T_1 \times T_2$ and $N = \Phi \times I$, where $J = [a, b]$. Furthermore $f : T \longrightarrow R$ is rd-continuous provided f is continuous right dense point T and has a finite left sided limit at each left dense point of T and will be denoted by C_{rd}. The partial delta derivative of $z(x, y)$ for $(x, y) \in N$ with respect to x is denoted by $z^{\triangle_1}(x, y)$.

Before giving our main results, we introduce the following lemma which is required in our theorems.

Lemma[8]: Let $u, a, f \in C_{rd}(T_1 \times T_2, R)$ and a is nondecreasing in each of the variables. If

$$u(x,y) \leq a(x,y) + \int_{x_0}^{x} \int_{y_0}^{y} f(s,t)u(s,t) \triangle t \triangle s, \tag{2.1}$$

for $(x, y) \in T_1 \times T_2$, then

$$u(x,y) \leq a(x,y) e_{C(x,y)}(x, x_0), \tag{2.2}$$

where

$$C(x,y) = \int_{y_0}^{y} f(x,t) \triangle t, \tag{2.3}$$

for $(x, y) \in T_1 \times T_2$.

3 Results and discussion

Our main results are based on the following theorems of integral inequalities with three independent variables which can be used in certain situations.

Theorem 3.1. *Let $u(x, y, z)$, $f(x, y, z)$ and $g(x, y, z)$ $\in C_{rd}(N, R_+)$ and c be a nonnegative constant. If*

$$u^2(x,y,z) \leq c^2 + 2 \int_{x_0}^{x} \int_{y_0}^{y} \int_{a}^{b} \Big[f(s,t,r)u^2(s,t,r) + g(s,t,r)u(s,t,r) \Big] \triangle r \triangle t \triangle s, \tag{3.1}$$

for (x,y,z) $\in N$, *then*

$$u(x,y,z) \leq p(x,y,z)e_{W(x,y)}(x,x_0), \tag{3.2}$$

where

$$p(x,y,z) = c + \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} g(s,t,r)\triangle r\triangle t\triangle s, \tag{3.3}$$

and

$$W(x,y) = \int_{y_0}^{y}\int_{a}^{b} f(x,t,r)\triangle r\triangle t, \tag{3.4}$$

for $(x,y,z) \in N$.

Proof. Let $c > 0$ and define a function $z(x,y)$ by the right hand side of (3.1), then

$$z(x_0,y) = c^2, u(x,y,z) \leq \sqrt{z}(x,y), \tag{3.5}$$

and

$$z(x,y) = c^2 + 2\int_{x_0}^{x}\int_{y_0}^{y} E(s,t)\triangle t\triangle s, \tag{3.6}$$

where

$$E(x,y) = \int_{a}^{b}\Big[f(x,y,r)u^2(x,y,r) + g(x,y,r)u(x,y,r)\Big]\triangle r. \tag{3.7}$$

From (3.5), (3.6) and (3.7), we notice that

$$z^{\triangle_1}(x,y) = 2\int_{y_0}^{y} E(x,t),$$

which implies

$$\frac{z^{\triangle_1}(x,y)}{\sqrt{z}(x,y)} \leq 2\int_{y_0}^{y}\int_{a}^{b}\Big[f(x,t,r)\sqrt{z}(x,t) + g(x,t,r)\Big]\triangle r\triangle t. \tag{3.8}$$

Now from (3.8) above we have by taking delta integral

$$\sqrt{z}(x,y) \leq p(x,y,z) + \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} f(s,t,r)\sqrt{z}(s,t)\triangle r\triangle t\triangle s, \tag{3.9}$$

where $p(x,y,z)$ be defined as in (3.3). Clearly $p(x,y,z)$ is nonnegative, continuous and nondecreasing $(x,y,z) \in N$. We assume that $p(x,y,z) > 0$ for $(x,y,z) \in N$. From (3.9), it is easy to observe that

$$\frac{\sqrt{z}(x,y)}{p(x,y,z)} \leq 1 + \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} f(s,t,r)\frac{\sqrt{z}(s,t)}{p(s,t,r)}\triangle r\triangle t\triangle s. \tag{3.10}$$

Define a function $v(x, y)$ by

$$v(x,y) = 1 + \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} f(s,t,r)\frac{\sqrt{z}(s,t)}{p(s,t,r)}\triangle r\triangle t\triangle s, \tag{3.11}$$

it follows from (3.10) and (3.11) that

$$v(x_0, y) = 1, \sqrt{z}(x, y) \leq p(x, y, z)v(x, y), \tag{3.12}$$

now from (3.11) and delta derivative with respect to x yields

$$\frac{v^{\triangle_1}(x,y)}{v(x,y)} \leq W(x,y), \tag{3.13}$$

where $W(x, y)$ be defined as in (3.4). Keeping y fixed and set $x = s$ and delta integrate the resulting inequality with respect to s from x_0 to x for $(x, y, z) \in N$ and using (3.12), we have

$$v(x, y) \leq e_{W(x,y)}(x, x_0). \tag{3.14}$$

The desired inequality in (3.2) follows by using (3.14) and (3.12) in (3.5). □

Remark1: If we take $f = 0$ and $T_1 = T_2 = R$, then Theorem 3.1 reduces to [18] Theorem 1(a_3).

Remark2: It is interesting to note that the inequalities established in Theorem 3.1 with three variables become the inequalities of Theorem 1 (a_1)and

Theorem 4 (b_1)with $T_1 = T_2 = R$ and $T_1 = T_2 = Z$ of one variable respectively given in [19].

Remark3: Theorem 3.1 reduces to [18] Theorem 2 (b_3) with $T_1 = T_2 = Z$ and $f = 0$.

Theorem 3.2. *Let $u(x, y, z)$, $f(x, y, z)$, $g(x, y, z)$, $h(x, y, z)$ and $m(x, y, z)$ $\in C_{rd}(N, R_+)$. If*

$$u(x,y,z) \leq g(x,y,z)+h(x,y,z)\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\Big[f(s,t,r)u(s,t,r)+m(s,t,r)\Big]\triangle r\triangle t\triangle s, \tag{3.15}$$

for $(x, y, z) \in N$, then

$$u(x, y, z) \leq g(x, y, z) + h(x, y, z)p_1(x, y, z)e_{W^\star(x,y)}(x, x_0),$$

where

$$W^\star(x,y) = \int_{y_0}^{y}\int_{a}^{b} f(x,t,r)h(x,t,r)\triangle r\triangle t, \tag{3.16}$$

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$$p_1(x,y,z)=\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\Big[f(s,t,r)g(s,t,r)+m(s,t,r)\Big]\triangle r\triangle t\triangle s, \tag{3.17}$$

for $(x,y,z)\in N$.

Proof. Define a function $z(x,y)$ by

$$z(x,y)=\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\Big[f(s,t,r)u(s,t,r)+m(s,t,r)\Big]\triangle r\triangle t\triangle s, \tag{3.18}$$

then

$$z(x_0,y)=0, u(x,y,z)\le g(x,y,z)+h(x,y,z)z(x,y), \tag{3.19}$$

and

$$z(x,y)=\int_{x_0}^{x}\int_{y_0}^{y}E(s,t)\triangle t\triangle s, \tag{3.20}$$

where

$$E(x,y)=\int_{a}^{b}\Big[f(x,y,r)u(x,y,r)+m(x,y,r)\Big]\triangle r. \tag{3.21}$$

From (3.19), (3.20) and (3.21), we notice that

$$z^{\triangle_1}(x,y)=2\int_{y_0}^{y}E(x,t),$$

$$z^{\triangle_1}(x,y)\le\int_{y_0}^{y}\int_{a}^{b}\Big[f(s,t,r)g(s,t,r)+m(s,t,r)\Big]\triangle r\triangle t$$
$$+\int_{y_0}^{y}\int_{a}^{b}\Big[f(x,t,r)h(x,t,r)z(x,t)\Big]\triangle r\triangle t,$$

which implies

$$z(x,y)\le p_1(x,y,z)+\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\Big[f(s,t,r)h(s,t,r)z(s,t)\Big]\triangle r\triangle t\triangle s,$$

where $p_1(x,y,z)$ be defined as in (3.17). The remaining proof can be completed by following a suitable modifications at the proof of Theorem 3.1 given above. Here we omit the details. □

Remark4: By taking m=0, it is easy to observe that the bound obtained in Theorem 3.2 reduces to the bound obtained in Theorem 2.1 given in [15].

Remark5: Theorem 3.2 with $T_1=T_2=R$ and m=0 reduces to Theorem 1(a_2) given in [18].

Remark6: If we take $T_1=T_2=Z$ and m=0, then Theorem 3.2 takes the form of Theorem 2(b_2) given in [18].

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Theorem 3.3. *Let $u(x,y,z)$, $f(x,y,z)$, $g(x,y,z)$ and c be defined as in Theorem 3.1. If*

$$u^2(x,y,z) \leq$$

$$c^2 + 2\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\Big[f(s,t,r)u(s,t,r)$$

$$\Big(u(s,t,r)+\int_{s_0}^{s}\int_{t_0}^{t}\int_{c}^{d} g(\sigma,\varsigma,\tau)u(\sigma,\varsigma,\tau)\triangle\tau\triangle\varsigma\triangle\sigma\Big)+h(s,t,r)u(s,t,r)\Big]\triangle r\triangle t\triangle s, \tag{3.22}$$

for $(x,y,z)\in N$, then

$$u(x,y,z) \leq p_2(x,y,z)e_{W_1(x,y)}(x,x_0), \tag{3.23}$$

where

$$p_2(x,y,z) = c + \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} h(s,t,r)\triangle r\triangle t\triangle s, \tag{3.24}$$

and

$$W_1(x,y) = \int_{y_0}^{y}\int_{a}^{b}\Big[f(x,t,r)+g(x,t,r)\Big]\triangle r\triangle t, \tag{3.25}$$

for $(x,y,z)\in N$.

Proof. Let $c>0$ and define a function $z(x,y)$ by the right hand side of (3.22), then

$$z(x_0,y) = c^2, u(x,y,z) \leq \sqrt{z}(x,y), \tag{3.26}$$

and

$$z(x,y) = c^2 + 2\int_{x_0}^{x}\int_{y_0}^{y} E(s,t)\triangle t\triangle s, \tag{3.27}$$

where

$$E(x,y) =$$

$$\int_{a}^{b}\Big[f(x,y,r)u(x,y,r)\Big(u(x,y,r)+\int_{x_0}^{x}\int_{y_0}^{y}\int_{c}^{d} g(s,t,\tau)u(,t,\tau)\triangle\tau\triangle t\triangle s\Big)$$

$$+h(x,y,r)u(x,y,r)\Big]\triangle r. \tag{3.28}$$

From (3.26), (3.27) and (3.28), we notice that

$$z^{\triangle_1}(x,y) = 2\int_{y_0}^{y} E(x,t),$$

which implies

$$\frac{z^{\triangle_1}(x,y)}{\sqrt{z}(x,y)} \leq 2\int_{y_0}^{y}\int_{a}^{b}\Big[f(x,t,r)\Big(\sqrt{z}(x,t)$$

$$+\int_{x_0}^{x}\int_{t_0}^{t}\int_{c}^{d} g(s,\varsigma,\tau)\sqrt{z}(s,\varsigma)\triangle\tau\triangle\varsigma\triangle s\Big) + h(x,t,r)\Big]\triangle r\triangle t, \tag{3.29}$$

now from (3.29) above we have by taking delta integral

$$\sqrt{z}(x,y) \leq p_2(x,y,z)$$

$$+\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\Big[f(s,t,r)\Big(\sqrt{z}(s,t)+\int_{s_0}^{s}\int_{t_0}^{t}\int_{c}^{d} g(\sigma,\varsigma,\tau)\sqrt{z}(\sigma,\varsigma)\triangle\tau\triangle\varsigma\triangle\sigma\Big)\Big]\triangle r\triangle t\triangle s, \tag{3.30}$$

where $p_2(x,y,z)$ be defined as in (3.24). Clearly $p_2(x,y,z)$ is nonnegative, continuous and nondecreasing $(x,y,z) \in N$. We assume that $p_2(x,y,z) > 0$ for $(x,y,z) \in N$. From (3.30), it is easy to observe that

$$\frac{\sqrt{z}(x,y)}{p_2(x,y,z)} \leq 1 + \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\Big[f(s,t,r)\Big(\frac{\sqrt{z}(s,t)}{p_2(s,t,r)}$$

$$+\int_{s_0}^{s}\int_{t_0}^{t}\int_{c}^{d} g(\sigma,\varsigma,\tau)\frac{\sqrt{z}(\sigma,\varsigma)}{p_2(\sigma,\varsigma,\tau)}\triangle\tau\triangle\varsigma\triangle\sigma\Big)\Big]\triangle r\triangle t\triangle s. \tag{3.31}$$

Define a function $v(x,y)$ by

$$v(x,y) = 1 + \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\Big[f(s,t,r)\Big(\frac{\sqrt{z}(s,t)}{p_2(s,t,r)}$$

$$+\int_{s_0}^{s}\int_{t_0}^{t}\int_{c}^{d} g(\sigma,\varsigma,\tau)\frac{\sqrt{z}(\sigma,\varsigma)}{p_2(\sigma,\varsigma,\tau)}\triangle\tau\triangle\varsigma\triangle\sigma\Big)\Big]\triangle r\triangle t\triangle s, \tag{3.32}$$

it follows from (3.31) and (3.32) that

$$v(x_0,y) = 1, \sqrt{z}(x,y) \leq p_2(x,y,z)v(x,y). \tag{3.33}$$

Now from (3.33) and delta derivative with respect to x yields

$$\frac{v^{\triangle_1}(x,y)}{v(x,y)} \leq W_1(x,y), \tag{3.34}$$

where $W_1(x,y)$ be defined as in (3.25). Keeping y fixed and set $x = s$ and delta integrate the resulting inequality with respect to s from x_0 to x for $(x,y,z) \in N$ and using (3.33), we have

$$v(x,y) \leq e_{W_1(x,y)}(x,x_0). \tag{3.35}$$

The desired inequality in (3.23) follows by using (3.33) and (3.35) in (3.26). □

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Remark7: We note that Theorem 3.3 is the further extension of Theorem 1(a_2) given in [19] with three variables.

Remark8: Theorem 3.3 with f=0 and $T_1 = T_2 = R$ converted into Theorem 1(a_3) given in [18].

Remark9: By taking g=0 and $T_1 = T_2 = R$ in Theorem 3.3, it reduces to Theorem 1(a_1) given in [19] with three variables.

Remark10: If we put g=0 and $T_1 = T_2 = Z$ in Theorem 3.3, then it reduces to Theorem 4(b_1) given in [19] with three variables.

Theorem 3.4. *Let $u(x, y, z)$, $f(x, y, z)$, $g(x, y, z)$ and c be defined as in Theorem 3.1. Let $L \in C_{rd}(N, R_+)$ which satisfies the condition*

$$0 \leq L(x, y, z, v) - L(x, y, z, w) \leq k(x, y, z, w)(v - w), \tag{3.36}$$

for $(x, y, z) \in N$ and $v \geq w \geq 0$ where $k \in C_{rd}(N, R_+)$. If

$$u^2(x, y, z) \leq c^2 + 2\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} \Big[f(s, t, r)u(s, t, r)L(s, t, r, u(s, t, r)) + g(s, t, r)u(s, t, r)\Big] \triangle r \triangle t \triangle s, \tag{3.37}$$

for $(x, y, z) \in N$, then

$$u(x, y, z) \leq p(x, y, z) + q(x, y, z)e_{W_2(x,y)}(x, x_0), \tag{3.38}$$

where p(x,y,z) be defined as in (3.3) and

$$q(x, y, z) = c + \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} f(s, t, r)L(s, t, r, p(s, t, r))\triangle r \triangle t \triangle s, \tag{3.39}$$

$$W_2(x, y) = \int_{y_0}^{y}\int_{a}^{b} f(x, t, r)k(x, t, r, p(x, t, r))\triangle r \triangle t, \tag{3.40}$$

for $(x, y, z) \in N$.

Proof. Let $c > 0$ and define a function $z(x, y)$ by the right hand side of (3.37), then

$$z(x_0, y) = c^2, u(x, y, z) \leq \sqrt{z}(x, y), \tag{3.41}$$

and

$$z(x, y) = c^2 + 2\int_{x_0}^{x}\int_{y_0}^{y} E(s, t)\triangle t \triangle s, \tag{3.42}$$

where

$$E(x,y)=\int_a^b \Big[f(x,y,r)u(x,y,r)L(x,y,r,u(x,y,r))+g(x,y,r)u(x,y,r)\Big]\triangle r. \tag{3.43}$$

From (3.41), (3.42) and (3.43), we notice that

$$z^{\triangle_1}(x,y)=2\int_{y_0}^{y} E(x,t),$$

which implies

$$\frac{z^{\triangle_1}(x,y)}{\sqrt{z}(x,y)}\le 2\int_{y_0}^{y}\int_a^b \Big[f(x,t,r)L(x,t,r,\sqrt{z}(x,t))+g(x,t,r)\Big]\triangle r\triangle t. \tag{3.44}$$

Now from (3.44) above we have by taking delta integral

$$\sqrt{z}(x,y)\le p(x,y,z)+\int_{x_0}^{x}\int_{y_0}^{y}\int_a^b f(s,t,r)L(s,t,r\sqrt{z}(s,t))\triangle r\triangle t\triangle s, \tag{3.45}$$

where $p(x,y,z)$ be defined as in (3.3). Let

$$v(x,y)=\int_{x_0}^{x}\int_{y_0}^{y}\int_a^b f(s,t,r)L(s,t,r,\sqrt{z}(s,t))\triangle r\triangle t\triangle s, \tag{3.46}$$

it follows from (3.45) and (3.46) that

$$v(x_0,y)=0, \sqrt{z}(x,y)\le p(x,y,z)+v(x,y). \tag{3.47}$$

Now from (3.46), (3.47) and (3.36), we observe that

$$v(x,y)\le q(x,y,z)+\int_{x_0}^{x}\int_{y_0}^{y}\int_a^b f(s,t,r)k(s,t,r,p(s,t,r))v(s,t)\triangle r\triangle t\triangle s, \tag{3.48}$$

where $q(x,y,z)$ be defined as in (3.39). Clearly $q(x,y,z)$ is nonnegative, continuous and nondecreasing $(x,y,z)\in N$. We assume that $q(x,y,z)>0$ for $(x,y,z)\in N$. From (3.48), it is easy to observe that

$$\frac{v(x,y)}{q(x,y,z)}\le R(x,y), \tag{3.49}$$

where

$$R(x,y)\le 1+\int_{x_0}^{x}\int_{y_0}^{y}\int_a^b f(s,t,r)k(s,t,r,p(s,t,r))\frac{v(s,t)}{q(s,t,r)}\triangle r\triangle t\triangle s, \tag{3.50}$$

and

$$R(x_0, y) = 1. \tag{3.51}$$

Now from (3.50) and delta derivative with respect to x yields

$$\frac{R^{\triangle_1}(x,y)}{R(x,y)} \leq W_2(x,y), \tag{3.52}$$

where $W_2(x,y)$ be defined as in (3.40). Keeping y fixed and set $x = s$ and delta integrate the resulting inequality with respect to s from x_0 to x for $(x,y,z) \in N$ and using (3.51), we have

$$R(x,y) \leq e_{W_2(x,y)}(x, x_0). \tag{3.53}$$

The desired inequality in (3.38) follows by using (3.47), (3.49) and (3.53) in (3.41). □

4 Some Applications

In this section, we present some applications of the Theorem 3.2. Consider the following dynamic integral equation of the form

$$u(x,y,z) = d(x,y,z) + \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} F(x,y,z,s,t,r,u(s,t,r))\triangle r\triangle t\triangle s, \tag{4.1}$$

where $(x,y,z) \in N$ and $d \in C_{rd}(N,R)$, $F \in C_{rd}(N^2 \times R, R)$.

First, we shall give the following theorem concerning the estimate on the solution of (4.1).

Theorem 4.1. *: Assume that the function F in (4.1) satisfies the condition*

$$| F(x,y,z,s,t,r,u(s,t,r)) | \leq q(x,y,z)\Big[f(s,t,r) \mid u \mid + h(s,t,r)\Big], \tag{4.2}$$

where $f, q, h \in C_{rd}(N,R)$. If $u(x,y,z)$ is a solution of (4.1), then

$$| u(x,y,z) | \leq d(x,y,z) + q(x,y,z)B(x,y,z)e_{M(x,y)}(x,x_0), \tag{4.3}$$

$$B(x,y,z) = \int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\Big[f(s,t,r) \mid d(s,t,r) + h(s,t,r) \mid \Big]\triangle r\triangle t\triangle s, \tag{4.4}$$

$$M(x,y) = \int_{y_0}^{y}\int_{a}^{b} f(x,t,r)q(x,t,r)\triangle r\triangle t, \tag{4.5}$$

for $(x,y,z) \in N$.

Proof. Let $u \in C_{rd}(N, R)$ be a solution of (4.1). Then from the hypotheses, we have

$$| u(x,y,z) | \leq | d(x,y,z) | + \int_{x_0}^{x} \int_{y_0}^{y} \int_{a}^{b} | F(x,y,z,s,t,r,u(s,t,r)) | \triangle r \triangle t \triangle s \tag{4.6}$$

$$\leq | d(x,y,z) | + q(x,y,z) \int_{x_0}^{x} \int_{y_0}^{y} \int_{a}^{b} \Big[f(s,t,r) | u(s,t,r) | + h(s,t,r) \Big] \triangle r \triangle t \triangle s, \tag{4.7}$$

for $(x,y,z) \in N$. Now an application of the inequality given in Theorem 3.2 to (4.7) yields the desired estimate in (4.3). $\square$

The next theorem gives the estimation on the solution of equation (4.1) assuming that the function F in equation (4.1) satisfies the Lipschitz type condition.

Theorem 4.2. *: Assume that the function F in (4.1) satisfies the condition*

$$| F(x,y,z,s,t,r,u) - F(x,y,z,s,t,r,v) | \leq q(x,y,z) \Big[f(s,t,r) | u - v | + h(s,t,r) \Big], \tag{4.8}$$

where $f, q, h \in C_{rd}(N, R)$. If $u(x,y,z)$ is a solution of (4.1), then

$$| u(x,y,z) - d(x,y,z) | \leq k(x,y,z) + q(x,y,z) B_1(x,y,z) e_{M(x,y)}(x, x_0), \tag{4.9}$$

where $M(x,y)$ be defined as in (4.5) and

$$k(x,y,z) = \int_{x_0}^{x} \int_{y_0}^{y} \int_{a}^{b} | F(x,y,z,s,t,r,d(s,t,r)) | \triangle r \triangle t \triangle s, \tag{4.10}$$

$$B_1(x,y,z) = \int_{x_0}^{x} \int_{y_0}^{y} \int_{a}^{b} f(s,t,r) \Big[| k(s,t,r) + h(s,t,r) | \Big] \triangle r \triangle t \triangle s, \tag{4.11}$$

for $(x,y,z) \in N$.

Proof. Let $u \in C_{rd}(N, R)$ be a solution of (4.1). Then from the hypotheses, we have

$$| u(x,y,z) - d(x,y,z) | \leq \int_{x_0}^{x} \int_{y_0}^{y} \int_{a}^{b} | F(x,y,z,s,t,r,u(s,t,r)) | \triangle r \triangle t \triangle s$$

$$\leq \int_{x_0}^{x} \int_{y_0}^{y} \int_{a}^{b} | F(x,y,z,s,t,r,u(s,t,r)) - F(x,y,z,s,t,r,d(s,t,r)) | \triangle r \triangle t \triangle s$$

$$+\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} \mid F(x,y,z,s,t,r,d(s,t,r)) \mid \triangle r\triangle t\triangle s$$

$$\leq k(x,y,z)+q(x,y,z)\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\Big[f(s,t,r)\mid u(s,t,r)-d(s,t,r)\mid +h(s,t,r)\Big]\triangle r\triangle t\triangle s, \tag{4.12}$$

for $(x,y,z)\in N$. Now an application of the inequality given in Theorem 3.2 to (4.12) yields the desired estimate in (4.9). □

We next consider the equation (4.1) and also the following integral equation

$$v(x,y,z)=g(x,y,z)+\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}L(x,y,z,s,t,r,v(s,t,r))\triangle r\triangle t\triangle s, \tag{4.13}$$

for $g\in C_{rd}(N,R)$, $L\in C_{rd}(N^2\times R,R)$.

Theorem 4.3. *: Suppose that the function F in (4.1) satisfies the condition (4.8). Then for every solution $v\in C_{rd}(N,R)$ of (4.13) and $u\in C_{rd}(N,R)$ a solution of equation (4.1), we have the estimates*

$$\mid u(x,y,z)-v(x,y,z)\mid\leq [d_1(x,y,z)+k_1(x,y,z)]+q(x,y,z)B_2(x,y,z)e_{M(x,y)}(x,x_0), \tag{4.14}$$

where $M(x,y)$ be defined as in (4.5) and

$$d_1(x,y,z)=\mid d(x,y,z)-g(x,y,z)\mid, \tag{4.15}$$

$$k_1(x,y,z)=\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\mid F(x,y,z,s,t,r,v(s,t,r))-L(x,y,z,s,t,r,v(s,t,r))\mid\triangle r\triangle t\triangle s, \tag{4.16}$$

$$B_2(x,y,z)=\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}f(s,t,r)\Big[d(s,t,r)+k(s,t,r)+h(s,t,r)\Big]\triangle r\triangle t\triangle s, \tag{4.17}$$

for $(x,y,z)\in N$.

Proof. Since $u(x,y,z)$ and $v(x,y,z)$ are respectively solutions of (4.1) and (4.13) we have

$$\mid u(x,y,z)-v(x,y,z)\mid\leq\mid d(x,y,z)-g(x,y,z)\mid$$

$$+\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\mid F(x,y,z,s,t,r,u(s,t,r))-F(x,y,z,s,t,r,v(s,t,r))\mid\triangle r\triangle t\triangle s$$

$$+\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b} \mid F(x,y,z,s,t,r,v(s,t,r)) - L(x,y,z,s,t,r,v(s,t,r)) \mid \triangle r \triangle t \triangle s, \tag{4.18}$$

$$\mid u(x,y,z) - v(x,y,z) \mid \leq d_1(x,y,z) + k_1(x,y,z)$$

$$+q(x,y,z)\int_{x_0}^{x}\int_{y_0}^{y}\int_{a}^{b}\Big[f(s,t,r) \mid u-v \mid + h(s,t,r)\Big]\triangle r \triangle t \triangle s, \tag{4.19}$$

for $(x,y,z) \in N$. Now an application of Theorem 3.2 to (4.19) yields (4.14). □

Conflict: There is no conflict of interest regarding the publication of this manuscript.

Acknowledgment: The author is very grateful to the editor and referee for their valuable evaluations.

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Divisibility of Generalized Catalan Numbers and Raney Numbers

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Abstract

The Raney numbers, also called Fuss-Catalan numbers, are defined by $R_k(n,r) = r\binom{kn+r}{n}/(kn+r)$. A generalized Lobb numbers is introduced. The relationship between Raney numbers and generalized Lobb numbers and the relationship between generalized Lobb numbers and generalized Catalan numbers are given. Based on the relationships among Raney numbers, generalized Lobb numbers, and generalized Catalan numbers, we present the divisibility of a certain class of those numbers.

AMS Subject Classification: 05A15, 65B10, 33C45, 39A70, 41A80.

Key Words and Phrases: Raney numbers, Fuss-Catalan numbers, Lobb numbers, generalized Lobb numbers, generalized Catalan numbers, Catalan numbers, divisibility.

1 Introduction

The Fuss-Catalan numbers or Raney numbers are numbers of the form

$$R_k(n,r) := \frac{r}{kn+r}\binom{kn+r}{n}, \tag{1}$$

which are named after N. I. Fuss and E. C. Catalan (see [5, 6, 13, 15, 17]) and initially studied by Raney in [17]. The Fuss-Catalan numbers have several combinatorial applications. They count for example (see, for instance, [8]):

(i) the number of ways of subdividing a convex polygon, with $n(k-1)+2$ vertices, into n disjoint $k+1$-gons by means of nonintersecting diagonals,

(ii) the number of sequences $(a_1, a_2, ..., a_{nk})$, where $a_i \in \{1, 1-k\}$, with all partial sums $a_1 + ... + a_k$ nonnegative and with $a_1 + ... + a_{nk} = 0$,

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(iii) the number of noncrossing partitions π of $1, 2, ..., n(k-1)$, such that $k-1$ divides the cardinality of every block of π,

(iv) the number of k-cacti formed of n polygons, etc. See [1, 3, 4, 6, 16, 18, 19] for more details and examples.

The generating function $R_k(t)$ for the Fuss-Catalan numbers, $\{R_k(n,1)\}_{n\geq 0}$ is called the generalized binomial series in [6], and it satisfies the function equation $R_k(t) = 1 + tR_k(t)^k$. Hence, from the Lambert's formula for the Taylor expansion of the powers of $R_k(t)$ (see [6]), we have

$$R_k^r \equiv R_k(t)^r = \sum_{n\geq 0} \frac{r}{mn+r}\binom{kn+r}{n}t^n \tag{2}$$

for all $r \in \mathbb{Z}$. Equation (2) implies the following formula of $R_k(t)$:

$$R_k(t) = 1 + tR_k^k(t). \tag{3}$$

Lobb [12] defines his Lobb numbers as

$$L_{n,m} := \frac{2n+1}{m+n+1}\binom{2n}{m+n}$$

for $n \geq m \geq 0$, which have the following combinatorial interpretation: Let $L_{n,m}$ be the number of sequences of length $2n$ with $n+m$ of the terms equal 1 and $n-m$ of the terms equal -1. It is natural to extend Lobb numbers to the number of sequences with $(k-1)n+m$ terms equal to 1 and $n-m$ terms equal to $1-k$. We denote the extended Lobb numbers by $L^k_{m,n}$ and define them as

$$L^k_{n,m} := \frac{km+1}{(k-1)n+m+1}\binom{kn}{n-m}. \tag{4}$$

Generalized Lobb numbers include many number sequences as their special cases. For instance, when $k=2$, $L^2_{n,m}$ are classical Lobb numbers; when $m=0$,

$$L^k_{n,0} = \frac{1}{(k-1)n+1}\binom{kn}{n} =: C_k(n) \tag{5}$$

are the generalized Catalan numbers; when $k=2$ and $m=0$, then

$$L^2_{n,0} = \frac{1}{n+1}\binom{2n}{n} =: C_2(n) \equiv C(n) \tag{6}$$

are the classical Catalan numbers; when $k=1$, then

$$L^1_{n,m} = \binom{n}{m}$$

are the binomial numbers. Other special cases can be seen in [7, 8]. The following relationship between generalized Lobb numbers and Raney numbers make us switch our results between the generalized Lobb numbers and the Raney numbers (see, for example, [9]):

$$L^k_{n,m} = R_k(n-m, km+1), \tag{7}$$

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which can be proved below. From (1) and using the transformation $n \to n-m$ and $r \to km+1$, we have

$$R_k(n-m, km+1) = \frac{km+1}{k(n-m)+km+1}\binom{k(n-m)+km+1}{n-m}$$
$$= \frac{km+1}{kn+1}\binom{kn+1}{n-m} = \frac{km+1}{kn+1}\frac{(kn+1)!}{((k-1)n+m+1)!(n-m)!}$$
$$= \frac{km+1}{(k-1)n+m+1}\binom{kn}{n-m} = L^k_{n,m},$$

or equivalently,

$$L^k_{n+\frac{r-1}{k},\frac{r-1}{k}} = R_k(n,r). \tag{8}$$

This paper is arranged as follows. In next section, we discuss the relationship between the generalized Lobb numbers and Raney numbers and the relationship between the generalized Lobb numbers and Ballot numbers. Some properties and identities of the generalized Lobb numbers are given. In Section 3, we discuss the divisibilities of the generalized Lobb numbers, Raney numbers, and generalized Catalan numbers.

2 Properties of the generalized Lobb numbers and Raney numbers

Proposition 2.1 *Let $L^k_{n,m}$ be defined by (4). Then*

$$L^k_{n,m} = \binom{kn}{n-m} - (k-1)\binom{kn}{n-m-1}. \tag{9}$$

Particularly,

$$L^2_{n,m} = \frac{2m+1}{n+m+1}\binom{2n}{n-m} = \binom{2n}{n-m} - \binom{2n}{n-m-1}. \tag{10}$$

For generalized Catalan numbers and Catalan numbers, there are

$$L^k_{n,0} = C_k(n) = \binom{kn}{n} - (k-1)\binom{kn}{n-1} \quad \textit{and}$$
$$L^2_{n,o} = C_2(n) = \binom{2n}{n} - \binom{2n}{n-1}. \tag{11}$$

Formula (9) also shows

$$L^1_{n,m} = \binom{n}{m}.$$

Proof. The right-hand side of (9) generates

$$\begin{aligned} RHS &= \binom{kn}{n-m} - \frac{(k-1)(n-m)}{kn-n+m+1}\binom{kn}{n-m} \\ &= \left[1 - \frac{(k-1)(n-m)}{kn-n+m+1}\right]\binom{kn}{n-m} \\ &= \frac{km+1}{(k-1)n+m+1}\binom{kn}{n-m} = L^k_{n,m}. \end{aligned}$$

The results for special cases are straightforward from (9).

Proposition 2.2 *Let $L^k_{n,m}$ be defined by (4). Then it can be written as*

$$L^k_{n,m} = \frac{km+1}{kn+1}\binom{kn+1}{n-m}. \tag{12}$$

Particularly,

$$L^2_{n,m} = \frac{2m+1}{2n+1}\binom{2n+1}{n-m} = \frac{2m+1}{n+m+1}\binom{2n}{n-m} = \frac{2m+1}{n+m+1}\binom{2n}{n+m}. \tag{13}$$

Proof. The right-hand side of (12) can be changed to

$$\begin{aligned} RHS &= \frac{km+1}{kn+1}\frac{(kn+1)!}{(n-m)!(kn-n+m+1)!} \\ &= \frac{km+1}{(k-1)n+m+1}\frac{(kn)!}{(n-m)!((k-1)n+m)!} = L^k_{n,m}. \end{aligned}$$

The special case (13) follows from (12).

Proposition 2.3 *Let $L^k_{n,m}$ be defined by (4). Then*

$$L^k_{n-m,\frac{r-1}{k}} = L^k_{n-m,\frac{r-2}{k}} + L^k_{n-m-1,\frac{r-2}{k}+1}. \tag{14}$$

Proof. From Corollary 3 of [14], we have

$$R_k(n,r) = R_k(n,r-1) + R_k(n-1,r+k-1), \tag{15}$$

which implies (14) by using (8).

Lobb numbers $L^2_{n,m}$ are also related to Ballot numbers (see, for example, [6])

$$B(a,b) = \frac{a-b}{a+b}\binom{a+b}{a} = \frac{a-b}{a+b}\binom{a+b}{b}. \tag{16}$$

Proposition 2.4 *Let $L^k_{n,m}$ and $B(a,b)$ be defined by (4) and (16), respectively. Then*

$$L^2_{n,m} = B(n+m+1, n-m), \tag{17}$$

or equivalently,

$$B(n,m) = L^2_{\frac{n+m-1}{2},\frac{n-m-1}{2}}. \tag{18}$$

Hence, $L^2_{n,m}$ is a special case of Ballot numbers.

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Proof. Substituting $a = n + m + 1$ and $b = n - m$ yields

$$B(n+m+1, n-m) = \frac{2m+1}{2n+1}\binom{2n+1}{n-m} = L^2_{n,m},$$

where the last equation is from (12).

Corollary 2.5 *Let $L^k_{n,m}$ be defined by (4). Then*

$$L^2_{n,m} = L^2_{\frac{2n-1}{2},\frac{2m-1}{2}} + L^2_{\frac{2n-1}{2},\frac{2m+1}{2}}. \tag{19}$$

Proof. From [6], we have

$$B(n,k) = B(n-1,k) + B(n,k-1).$$

Thus,

$$B(n+m+1, n-m) = B(n+m, n-m) + B(n+m+1, n-m-1),$$

which implies (19) by using (18).

3 Divisibility of generalized Catalan numbers, generalized Lobb numbers, and Raney numbers

We now consider the divisibility properties of generalized Lobb numbers, generalized Catalan numbers, and Raney numbers.

Theorem 3.1 *Let $C_k(n) := \binom{kn}{n}/((k-1)n+1)$ $(k \geq 2)$, and let $n = (k+1)t+1$ $(t = 0, 1, 2, \ldots)$. Then*

(a) If k is odd, then

$$((k-1)t+1) \,|\, C_k(n). \tag{20}$$

(b) If k is even and t is even, then

$$((k-1)t+1) \,|\, C_k(n). \tag{21}$$

(c) If k is even and t is odd, then

$$((k-1)t+1) \,|\, 2C_k(n). \tag{22}$$

Proof. First, we express Lobb numbers $L^k_{n,m}$ in terms of generalized Catalan numbers $C_k(n)$:

$$\begin{aligned} L^k_{n,m} &= \frac{km+1}{(k-1)n+m+1}\binom{kn}{n-m} \\ &= (km+1)\frac{(kn)!}{(n-m)!((k-1)n+m+1)!} \\ &= (km+1)\frac{(kn)!}{n!((k-1)n+1)!}\Pi^m_{j=1}\frac{n-j+1}{(k-1)n+j+1} \\ &= (km+1)C_k(n)\Pi^m_{j=1}\frac{n-j+1}{(k-1)n+j+1}. \end{aligned}$$

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Therefore, for non-negative integer t

$$L^k_{(k+1)t+1,1} = C_k((k+1)t+1)\frac{(k+1)((k+1)t+1)}{(k-1)((k+1)t+1)+2}$$
$$= C_k((k+1)t+1)\frac{(k+1)t+1}{(k-1)t+1}. \tag{23}$$

Secondly, we consider different cases for k. In case (a), let k be odd, i.e., $k = 2l+1$, $l = 0, 1, 2, \ldots$. Then,

$$(k-1)t+1 = 2lt+1$$

and

$$(k+1)t+1 = 2(l+1)t+1.$$

Noting $(k+1)t+1 = (k-1)t+1+2t$, we have

$$gcd\ [(k+1)t+1, (k-1)t+1] = gcd\ [2t, (k-1)t+1] = 1$$

because $(k-1)t+1$ is an odd integer. From (23), we have proved $((k-1)t+1)|\, C_k((k+1)t+1)$ when k is odd. In case (b), we assume $k = 2l$ ($l \in \mathbb{Z}$), an even number. Then

$$(k-1)t+1 = (2l-1)t+1,$$
$$(k+1)t+1 = (2l+1)t+1 = (2l-1)t+1+2t.$$

Thus,

$$gcd\ [(k+1)t+1, (k-1)t+1] = gcd\ [2t, (2l-1)t+1].$$

If t is even, then $gcd\ [2t, (2l-1)t+1] = 1$, which implies $((k-1)t+1)|\, C_k((k+1)t+1)$. Finally, considering the case (c), where k is an even number $2l$ and t is an odd number $2u+1$ ($l, u \in \mathbb{Z}$), we have

$$gcd\ [2t, (2l-1)t+1] = gcd\ [2(2u+1), (2l-1)(2u+1)+1]$$
$$= gcd\ [2(2u+1), -2u] = 2$$

So that $((k-1)t+1)|\, 2C_k((k+1)t+1)$, which completes the proof.

Example 3.1 For $k = 3$ and $t = 1$, we have $(k-1)t+1 = 3$ and $(k+1)t+1 = 5$. $C_3(5) = 273$ and $3|C_3(5)$. For $k = 3$ and $t = 2$, we have $(k-1)t+1 = 5$ and $(k+1)t+1 = 9$. Thus $5|C_3(9) = 246675$. For $k = 2$ and $t = 2$, we have $(k-1)t+1 = 3$ and $(k+1)t+1 = 7$, which implies $3|C_2(7) = 429$.

Example 3.2 For $k = 3$, from Theorem 3.1 there holds $2t+1|\, C_3(4t+1)$. Thus,

$$1|\, C_3(1),\ 3|\, C_3(5),\ 5|\, C_3(9),\ 7|\, C_3(13),\ 9|\, C_3(17), etc.$$

Here, $\{2t+1 : t = 0, 1, 2, \ldots\}$ and $\{4t+1 : t = 0, 1, 2, \ldots\}$ form arithmetical sequences.

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Remark From the expression of Lobb numbers $L^k_{n,m}$ in terms of generalized Catalan numbers $C_k(n)$, we have

$$L^k_{n,1} = \frac{k+1}{(k-1)n+2}\binom{kn}{n-1} = (k+1)C_k(n)\frac{n}{(k-1)n+2}. \tag{24}$$

Hence, provided

$$gcd\,((k+1)n, (k-1)n+2) = 1,$$

or equivalently,

$$gcd\,((k+1)n, -2(n-1)) = 1, \tag{25}$$

we have

$$((k-1)n+2)|C_k(n). \tag{26}$$

Note (25) implies that $(k+1)n$ must be odd, or equivalently, n is odd and k is even. In other words, if k is even and t is even, $C_k((k+1)t+1)$ has two divisors $(k-1)t+1$ and $(k^2-1)t+k+1$, which are given by (21) and (26) respectively.

Example 3.3 If $n=3$ and $k=2$, then $gcd\,((k+1)n, (k-1)n+2) = gcd\,(9.4) = 1$. From (26), $5|C_2(3)$. Actually, $C_2(3) = 5$. Similarly, if $n=3$ and $k=4$, then $gcd\,(11,4) = 1$, which implies $11|C_4(3)$. Actually, $C_4(3) = 22$. While $n=3$ and $k=6$ yield $17|C_6(3)$, where $C_6(3) = 51$, and $n=5$ and $k=2$ yield $7|C_2(5)$, where $C_2(5) = 42$. An non-example is given by $n=7$ and $k=2$, which yields $gcd\,(21,-12) = 3 \neq 1$. Since $C_2(7) = 429$, which does not have divisor 21.

Sury [20] proves if $n \neq (p^l-1)/(p-1)$ for any prime $p \geq 3$, then

$$p|\,C_p(n). \tag{27}$$

A natural question is what is a divisor of $C_p((p^l-1)/(p-1))$. We now apply Theorem 3.1 to answer this question.

Corollary 3.2 *Let $C_k(n)$ be the generalized Catalan numbers defined by* (5)*. Then for an odd integer l we have*

$$\left.\frac{p^l+1}{p+1}\right|C_p\left(\frac{p^l-1}{p-1}\right) \quad (p \geq 3). \tag{28}$$

Proof. If $k = p \geq 3$ and $n = (k+1)t+1 = (p^l-1)/(p-1)$, then

$$t = \frac{1}{k+1}\left(\frac{p^l-1}{p-1}-1\right) = \frac{p^l-p}{p^2-1},$$

where t is an integer because l is odd. Thus

$$(k-1)t+1 = (k+1)t+1-2t = \frac{p^l-1}{p-1} - 2\frac{p^l-p}{p^2-1} = \frac{p^l+1}{p+1}.$$

Substituting $k=p$, $n=(k+1)t+1 = (p^l-1)/(p-1)$, and $(k-1)t+1 = (p^l+1)/(p+1)$ into (20) of Theorem 3.1, we obtain (28).

In [11, 10], the following result is given

$$(2^{l+1}-3)\Big|\, C_2(M_l),$$

where M_l are the Mersenne numbers, 2^l-1 $(l=0,1,2,\ldots)$, and $C_2(n)=L^2_{n,0}$ are classical Catalan numbers. Thus, for $l=4$ and 5, there are $29|C_2(M_4)$ and $61|C_2(M_5)$, respectively.

We obtain the following corollary from Theorem 3.1, which extends the results shown in [11, 10].

Corollary 3.3 *Let $C_2(n)$ be the Catalan numbers. Then*

$$\frac{2^l+1}{3}\Bigg|\, C_2\left(2^l-1\right) \tag{29}$$

for $l=1,3,5,7,\ldots$. Combining [11], $C_2(M_k)$ has two different divisors, $2^{l+1}-3$ and $(2^l+1)/3$, when odd $l>1$. Furthermore, if l is odd and not a prime, then all of its divisors are divisors of $C_2(2^l-1)$.

Proof. Set $(k+1)t+1=2^l-1$. Then $t=(2^l-2)/(k+1)$. Let $k=2$, we have $t=(2^l-2)/3$. Here t is even because

$$3t=2^l-2$$

is even. Thus,

$$(k-1)t+1=(2-1)t+1=\frac{2^l+1}{3}.$$

From Theorem 3.1 (b), for $k=2$ and $t=(2^l-2)/3$ we obtain

$$\frac{2^\ell+1}{3}=((k-1)t+1)\Bigg|\, C_k((k+1)t+1)=C\left(2^l-1\right)$$

for $l=1,3,5,7,\ldots$. To prove that $C_2(M_l)$ has two different divisors, $2^{l+1}-3$ and $(2^l+1)/3$, when odd $l>1$, we only need to show

$$2^{l+1}-3\neq\frac{2^l+1}{3}$$

when $l>1$. This is clearly true, otherwise, there is a contradiction

$$3\cdot 2^l-2^{l-1}=5$$

for $l>1$. Finally, from [2], we know that $(2^l+1)/3$ is a prime only if l is a prime. Hence, if l is not a prime number, then $(2^l+1)/3$ is a composite number. Additionally, when l is odd and not a prime, then all of the divisors of such composite number are also divisors of $C_2(2^l-1)$ because of (29).

Example 3.4 For $l=1,3,5,$ and 7, Corollary 3.3 generates $\frac{2^l+1}{3}\Big|\, C_2\left(2^l-1\right)$ for $l=1,3,5,$ and 7. For examples,

$$1|C_2(1),\ 3|C_2(7),\ 11|C_2(31),\ \textit{and}\ 43|C_2(127).$$

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Among the above results, the second and fourth are new. Actually, we may give infinitely many new results from Corollary 3.3.

We now extend the result on Catalan numbers shown in Corollary 3.3 to generalized Catalan numbers.

Theorem 3.4 *Let $C_k(n) := L^k_{n,0}$ be the generalized Catalan numbers defined by (5). If k is even and $\ell \equiv 1$ (mod $\phi(k+1)$), where $\phi(n)$ is Euler's totient function, then*

$$\left((k-1)\frac{2^l-2}{k+1}+1\right)\Big| \, C_k(2^l-1). \tag{30}$$

Proof. Let

$$(k+1)t+1 = 2^l-1,$$

the Mersenne numbers. Then $t=(2^l-2)/(k+1)$, where t is even because k is even, and

$$(k-1)t+1 = (k-1)\frac{2l-2}{k+1}+1.$$

To prove (30), we need to show the right-hand side of the above equation is an integer, i.e.,

$$(k-1)(2^l-2) \equiv 0 \ (mod\ k+1).$$

The last equation is equivalent to

$$-4(2^{l-1}-1) \equiv 0 \ (mod\ k+1)$$

because

$$(k-1)(2^l-2) = (k+1)(2^l-2) - 4(2^{l-1}-1).$$

Therefore, if $gcd\ (4, k+1) = 1$, then we need

$$2^{l-1} \equiv 1 \ (mod\ k+1). \tag{31}$$

From Euler theorem, if $gcd\ (2, k+1) = 1$; i.e., k is even, then

$$2^{\phi(k+1)} \equiv 1 \ (mod\ k+1),$$

where $\phi(n)$ is Euler's totient function, i.e., the number of the positive integers less than or equal to n that are relatively prime to n. Comparing the above equation and equation (31), we should have

$$l-1 \equiv 0 \ (mod\ \phi(k+1)),$$

or equivalently,

$$\ell = u\phi(k+1)+1$$

for some integer u. Now, we assume that k is even and $\ell \equiv 1$ $(mod\ \phi(k+1))$, where ϕ is Euler's totient function. Under the conditions, $((k-1)t+1 = (k-1)(2^l-2)/(k+1)$ is an integer when $t=(2^l-2)/(k+1)$. Now k is even and t is even. Then by Theorem 3.1 (b)

$$\left((k-1)\frac{2^l-2}{k+1}+1\right)\Big| \, C_k(2^l-1).$$

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Example 3.4 Let $C_k(n) := L^k_{n,0}$ be the generalized Catalan numbers defined by (5). Since $k = 4$ is even, and $\phi(k+1) = \phi(5) = 4$, from Theorem 3.4, for $l \equiv 1\ (mod\ 4)$, i.e., $\ell = 1, 5, 9, \ldots$, we have

$$\left((4-1)\frac{2^l-2}{4+1}+1\right)\Big|\, C_4(2^l-1),$$

which implies

$$\frac{3\cdot 2^l-1}{5}\Big|\, C_4(2^l-1) \tag{32}$$

for $l = 1, 5, 9, \ldots$.

In Theorem 3.4, the condition $l \equiv 1\ (mod\ \phi(k+1))$ can be replaced by $l \equiv 1\ (mod\ k)$ when $k+1$ is a prime number greater than 3. In this case, the condition of that k is even is automatically satisfied. Hence, we have the following corollary of Theorem 3.4.

Corollary 3.5 *Let $C_k(n) := L^k_{n,0}$ be the generalized Catalan numbers defined by (5). If $k+1$ is a prime number greater than 3, and $\ell \equiv 1\ (mod\ k)$, then*

$$\left((k-1)\frac{2^l-2}{k+1}+1\right)\Big|\, C_k(2^l-1). \tag{33}$$

Proof. It is sufficient to note that if $k+1$ is a prime number greater than 3, then k is an even number and $\phi(k+1) = k$. Hence, Theorem 3.4 implies the corollary.

From the above discussion, the key to get divisibility of $C_k(n)$ by using (23) is

$$((k-1)t+1)|\,(k+1)t+1.$$

Hence, we may have a special case of Theorem 3.1, which is more easier to be applied.

Example 3.6 Let $C_k(n) := L^k_{n,0}$ be the generalized Catalan numbers defined by (5). If t is even, then

$$(t+1)|\,C_2(3t+1) \quad and \quad (3t+1)|\,C_4(5t+1). \tag{34}$$

Thus,

$$1|\,C_2(1),\ 3|\,C_2(7),\ 5|\,C_2(13),\ 7|\,C_2(19),\ 9|\,C_2(25), etc.$$

and

$$1|\,C_4(1),\ 7|\,C_4(11),\ 13|\,C_4(21),\ 19|\,C_4(31),\ 25|\,C_4(41), etc.$$

In general, if $t = 2m$, then we have

$$(2m+1)|\,C_2(6m+1),\ (6m+1)|\,C_4(10m+1),\ (10m+1)|\,C_6(14m+1), etc.$$

for $k = 2, 4, 6$, etc. More generally, for $k = 2u$ and $t = 2m$, we have

$$(2(2u-1)m+1)|\,C_{2u}(2(2u+1)m+1),$$

where the sequences of $\{2(2u-1)m+1 t = 0, 1, 2, \ldots\}$, $\{2(2u-1)m+1 m = 0, 1, 2, \ldots\}$, $\{2(2u+1)m+1 t = 0, 1, 2, \ldots\}$, and $\{2(2u+1)m+1\, m = 0, 1, 2, \ldots\}$ are arithmetical sequences.

We now transfer the divisibility from the generalized Lobb numbers to Raney numbers and Ballot numbers.

Theorem 3.6 *Let $R_k(n,m)$ be Raney numbers defined by* (1)*. If k is an odd integer, then we have*

$$((k-1)t+1)\,|\,R_k((k+1)t+1,1). \tag{35}$$

If k is an even integer and $n=(k+1)t+1$ is odd, then (35) *holds. If both k and $n=(k+1)t+1$ are even, then*

$$((k-1)t+1)\,|\,2R_k((k+1)t+1,1) \tag{36}$$

holds.

Proof. By using the relationship (7) between the generalized Lobb numbers and Raney numbers, we may establish Theorem 3.6 from Theorem 3.1.

Theorem 3.7 *Let $B(a,b)$ be Ballot numbers defined by* (16)*. If $n=3t+1$ is odd, then we have*

$$((k-1)t+1)\,|\,B((k+1)t+2,(k+1)t+1). \tag{37}$$

If $n=3t+1$ is even, then t is odd and

$$((k-1)t+1)\,|\,2B((k+1)t+2,(k+1)t+1) \tag{38}$$

holds.

Proof. From the relationship between $L^2_{n,m}$ and $B(a,b)$ shown in (17) and Theorem 3.1, we may obtain (37) and (38).

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COUPLED FIXED POINT THEOREMS FOR TWO MAPS IN CONE b-METRIC SPACES OVER BANACH ALGEBRAS

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ABSTRACT. In this paper, we obtain some coupled fixed point results for two mappings satisfying some contractive conditions in cone b-metric spaces over Banach algebras with a solid cone by virtue of the properties of spectral radius. Also we give an example as an applications of one of the main results.

1. Introduction

In 2007 the concept of cone metric space was introduced by Huang and Zhang in [4], where they generalized metric space by replacing the set of real numbers with an ordering Banach space, investigated the convergence in cone metric space and proved some fixed point theorems for contractive mappings on these spaces. Recently, in ([1],[3], [4], [6], [7], [8], [10], [11]) some common fixed point theorems have been proved for contractive maps on cone metric spaces. Gnana Bhaskar and Lakshmikantham([2]) introduced the concept of coupled fixed point of a mapping $F : X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered sets. Since then this new concept is extended and used in various directions([2], [5]).

In 2013, in order to generalize the Banach contraction principle to more general form, Liu and Xu([8]) introduced the concept of cone metric spaces over Banach algebras, by replacing Banach spaces with Banach algebras as the underlying spaces of cone metric spaces, and proved some fixed point theorems of generalized Lipschitz mappings with weaker and natural conditions on generalized Lipschitz constants by means of spectral radius. Furthermore, they gave an example to explain that the fixed point theorems in cone metric spaces over Banach algebras are not equivalent to those in metric spaces.

Motivated by the above works, in this paper, we obtain some coupled fixed point results for two mappings satisfying some contractive conditions in cone b-metric spaces over Banach algebras without the assumption of normal cones by virtue of the properties of spectral radius. Our main results generalize the corresponding main results

1991 *Mathematics Subject Classification.* 47H10, 54H25.

Key words and phrases. cone metric spaces over Banach algebras, coupled fixed point, spectral radius.

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in cone metric spaces obtained by H.K. Nashie, Y. Rohen and C. Thokchom([5]. Also we give an example as an applications of one of the main results.

Let A always be a real Banach algebra. That is, A is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for all $x, y, z \in A$, $\alpha \in \mathbb{R}$):

(1) $(xy)z = x(yz)$;
(2) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;
(3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
(4) $\|xy\| \le \|x\|\|y\|$.

In this paper, we shall assume that A is a real Banach algebra with a unit (i.e., a multiplicative identity) e. An element $x \in A$ is said to be invertible if there is an inverse element $y \in A$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}.

Let A be a real Banach algebra with a unit e and θ the zero element of A. A nonempty closed subset P of Banach algebra A is called a *cone* if

(i) $\{\theta, e\} \subset P$;
(ii) $\alpha P + \beta y P \subset P$ for all nonnegative real numbers α, β ;
(iii) $P^2 = PP \subset P$;
(iv) $P \cap (-P) = \{\theta\}$ i.e, $x \in P$ and $-x \in P$ imply $x = \theta$.

For any cone $P \subseteq A$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ stands for $x \preceq y$ but $x \neq y$. Also, we use $x \ll y$ to indicate that $y - x \in \operatorname{int} P$ where $\operatorname{int} P$ denotes the interior of P. If $\operatorname{int} P \neq \emptyset$ then P is called a *solid cone*.

Definition 1.1. Let X be a nonempty set, $s \ge 1$ be a constant and A be a real Banach algebra. Suppose the mapping $d : X \times X \to A$ satisfies the following conditions:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \preceq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then d is called a *cone b-metric* on X, and (X, d) is called a *cone b-metric space* over the Banach algebra A.

If $s = 1$,then every cone b-metric is a cone metric space.

Definition 1.2. Let (X, d) be a cone b-metric space over the Banach algebra A. Let $\{x_n\}$ be a sequence in X and $x \in X$.

(1) If for every $c \in A$ with $\theta \ll c$, there exists a natural number N such that $d(x_n, x) \ll c$ for all $n > N$, then $\{x_n\}$ is said to be *convergent* and $\{x_n\}$

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converges to x, and the point x is the *limit* of $\{x_n\}$. We denote this by

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x \quad (n \to \infty).$$

(2) If for all $c \in A$ with $\theta \ll c$, there exists a positive integer N such that $d(x_n, x_m) \ll c$ for all $m, n > N$, then $\{x_n\}$ is called a *Cauchy sequence* in X.

(3) A cone b-metric space (X, d) is said to be *complete* if every Cauchy sequence in X is convergent.

Definition 1.3. Let E be a real Banach space with a solid cone P. A sequens $\{x_n\} \subset P$ is called a $c-$*sequence* if for any $c \in A$ with $\theta \ll c$, there exists a positive integer N such that $x_n \ll c$ for all $n \geq N$.

Lemma 1.4. *([6], [8]) Let E be a real Banach space with a cone P. Then*

(p_1) *If $a \ll b$ and $b \ll c$, then $a \ll c$.*
(p_2) *If $a \preceq b$ and $b \ll c$, then $a \ll c$.*
(p_3) *If $a \preceq b + c$ for each $\theta \ll c$, then $a \preceq b$.*
(p_4) *If $\theta \preceq u \ll c$ for each $\theta \ll c$, then $u = \theta$.*
(p_5) *If $\{x_n\}, \{y_n\}$ are sequences in E such that $x_n \to x$, $y_n \to y$ and $x_n \preceq y_n$ for all $n \geq 1$, then $x \preceq y$.*

We define the spectral radius of $x \in A$ by

$$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\geq 1} \|x^n\|^{1/n}.$$

Lemma 1.5. *([8]) Let x, y be vectors in the Banach algebra A. If x and y commute, then the spectral radius ρ satisfies the following properties :*

(1) *$r(xy) \leq r(x)r(y)$;*
(2) *$r(x + y) \leq r(x) + r(y)$;*
(3) *$|r(x) - r(y)| \leq r(x - y)$.*

Lemma 1.6. *([8]) Let A ba a real Banach algebra with a unit e and $x \in A$. If $0 \leq r(x) < 1$, then*

(1) *$e - x$ is invertible, $(e - x)^{-1} = \sum_{i=0}^{\infty} x^i$ and*

$$r((e - x)^{-1}) \leq (1 - r(x))^{-1}.$$

(2) *$\|x^n\| \to 0$ as $n \to \infty$.*

Lemma 1.7. *([6]) Let P be a solid cone in the Banach algebra A and $\|x_n\| \to 0$ as $n \to \infty$, then $\{x_n\}$ is a $c-$sequence.*

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Lemma 1.8. *([8]) Let P ba a solid cone in a Banach space A and and $\{x_n\}$ be a sequence in P. If $k \in P$ is an arbitrarily given vector and $\{x_n\}$ is $c-$sequence in P, then $\{kx_n\}$ is a $c-$sequence.*

Lemma 1.9. *([8]) Let A be a Banach algebra with a unit e and let P be a solid cone in A. The following assertions hold true:*

(1) *For any $x, y \in A$, $a \in P$ with $x \preceq y$, we have $ax \preceq ay$.*
(2) *For any sequences $\{x_n\}, \{y_n\} \subset A$ with $x_n \to x$ $(n \to \infty)$ and $y_n \to y$ $(n \to \infty)$ where $x, y \in A$, we have $x_n y_n \to xy$ $(n \to \infty)$.*

Lemma 1.10. *([8]) Let (X, d) be a complete cone metric space over a Banach algebra A and let P be a solid cone in A. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to $x \in X$, then we have:*

(1) *$\{d(x_n, x)\}$ is a c-sequence.*
(2) *For any $p \in \mathbb{N}$, $\{d(x_n, x_{n+p})\}$ is a c-sequence.*

Lemma 1.11. *([8]) Let P be a solid cone in a real Banach algebra A and $k \in P$. If $r(k) < 1$,then the following assertions hold true:*

(1) *If $u \in P$ and $u \preceq ku$, then $u = \theta$.*
(2) *If $k \succeq \theta$,then $(e - k)^{-1} \succeq \theta$.*

Definition 1.12. Let (X, d) be a cone b-metric space over the Banach algebra A. An element $(x, y) \in X \times X$ is called a *coupled fixed point* of $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Note that if (x, y) is a coupled fixed point of F, then (y, x) is also a coupled fixed point of F.

2. Main results

In the following, we always assume that (X, d) is a cone b-metric space over the Banach algebra A. In this section, we establish a common coupled fixed point results for two mappings $S, T : X \times X \to X$ satisfying certain contractive condition on cone metric spaces over Banach algebras. The following results generalize the corresponding results in cone metric spaces obtained by H.K. Nashie, Y. Rohen and C. Thokchom([5]).

Theorem 2.1. *Let (X, d) be a complete cone b-metric space over the Banach algebra A with the coefficient $s \geq 1$ and let P be a solid cone in A. Suppose that S, T :*

$X \times X \to X$ are two mappings satisfying the condition

$$\begin{aligned} d(S(x,y),T(u,v)) &\preceq a_1 d(x,u) + a_2 d(S(x,y),x) + a_3 d(y,v) \\ &+ a_4 d(T(u,v),u) + a_5 d(S(x,y),u) + a_6 d(T(u,v),x) \end{aligned} \tag{2.2.1}$$

for all $x, y, u, v \in X$, where $a_i \in P$ and $a_i a_j = a_j a_i$ $(i, j = 1,2,3,4,5,6)$. If

$$s[r(a_1) + r(a_2) + r(a_3)] + r(a_4) + r(a_5) + (s^2 + s)r(a_6) < 1,$$

then S and T have a common coupled fixed point in X.

Proof. Let x_0 and y_0 be any points X. Let

$$x_{2k+1} = S(x_{2k}, y_{2k}), \quad y_{2k+1} = S(y_{2k}, x_{2k})$$

and

$$x_{2k+2} = T(x_{2k+1}, y_{2k+1}), \quad y_{2k+2} = T(y_{2k+1}, x_{2k+1})$$

for $k = 0, 1, 2, \cdots$. Then we have

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &= d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1})) \\ &\preceq a_1 d(x_{2k}, x_{2k+1}) + a_2 d(S(x_{2k}, y_{2k}), x_{2k}) + a_3 d(y_{2k}, y_{2k+1}) \\ &+ a_4 d(T(x_{2k+1}, y_{2k+1}), x_{2k+1}) + a_5 d(S(x_{2k}, y_{2k}), x_{2k+1}) \\ &+ a_6 d(T(x_{2k+1}, y_{2k+1}), x_{2k}) \\ &= a_1 d(x_{2k}, x_{2k+1}) + a_2 d(x_{2k+1}, x_{2k}) + a_3 d(y_{2k}, y_{2k+1}) \\ &+ a_4 d(x_{2k+2}, x_{2k+1}) + a_5 d(x_{2k+1}, x_{2k+1}) + a_6 d(x_{2k+2}, x_{2k}) \\ &\preceq a_1 d(x_{2k}, x_{2k+1}) + a_2 d(x_{2k+1}, x_{2k}) + a_3 d(y_{2k}, y_{2k+1}) \\ &+ a_4 d(x_{2k+2}, x_{2k+1}) + a_5 \cdot \theta \\ &+ s a_6 [d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2})]. \end{aligned}$$

which implies that

$$(e - a_4 - s a_6) d(x_{2k+1}, x_{2k+2}) \preceq (a_1 + a_2 + s a_6) d(x_{2k}, x_{2k+1}) + a_3 d(y_{2k}, y_{2k+1}).$$

By hypothesis and Lemma 1.8, $e - (a_4 + s a_6)$ is invertible. Putting $\alpha = (e - a_4 - s a_6)^{-1}(a_1 + a_2 + s a_6)$, $\beta = (e - a_4 - s a_6)^{-1} a_3$, we have

$$d(x_{2k+1}, x_{2k+2}) \preceq \alpha d(x_{2k}, x_{2k+1}) + \beta d(y_{2k}, y_{2k+1}). \tag{2.2.2}$$

Similarly,

$$\begin{aligned}
d(y_{2k+1}, y_{2k+2}) &= d(S(y_{2k}, x_{2k}), T(y_{2k+1}, x_{2k+1})) \\
&\preceq a_1 d(y_{2k}, y_{2k+1}) + a_2 d(S(y_{2k}, y_{2k}), y_{2k}) + a_3 d(x_{2k}, x_{2k+1}) \\
&+ a_4 d(T(y_{2k+1}, x_{2k+1}), y_{2k+1}) + a_5 d(S(y_{2k}, x_{2k}), y_{2k+1}) \\
&+ a_6 d(T(y_{2k+1}, x_{2k+1}), y_{2k}) \\
&= a_1 d(y_{2k}, y_{2k+1}) + a_2 d(y_{2k+1}, y_{2k}) + a_3 d(x_{2k}, x_{2k+1}) \\
&+ a_4 d(y_{2k+2}, y_{2k+1}) + a_5 d(y_{2k+1}, y_{2k+1}) + a_6 d(y_{2k+2}, y_{2k}) \\
&\preceq a_1 d(y_{2k}, y_{2k+1}) + a_2 d(y_{2k+1}, y_{2k}) + a_3 d(x_{2k}, x_{2k+1}) \\
&+ a_4 d(y_{2k+2}, y_{2k+1}) + a_5 \cdot \theta \\
&+ s a_6 [d(y_{2k}, y_{2k+1}) + d(y_{2k+1}, y_{2k+2})].
\end{aligned}$$

which implies that

$$d(y_{2k+1}, y_{2k+2}) \preceq \alpha d(y_{2k}, y_{2k+1}) + \beta d(x_{2k}, x_{2k+1}). \tag{2.2.3}$$

Adding both inequalities, we have

$$\begin{aligned}
d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2}) &\preceq (\alpha + \beta)[d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})] \\
&= h[d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})]
\end{aligned}$$

where $h = \alpha + \beta = (e - a_4 - sa_6)^{-1}(a_1 + a_2 + a_3 + sa_6)$. Also we have

$$d(x_{2k+2}, x_{2k+3}) + d(y_{2k+2}, y_{2k+3}) = h[d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2})].$$

Therefore

$$\begin{aligned}
d(x_n, x_{n+1}) + d(y_n, y_{n+1}) &\preceq h[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)] \\
&\preceq \cdots \preceq h^n [d(x_0, x_1) + d(y_0, y_1)]
\end{aligned}$$

By hypothesis, Lemma 1.7 and Lemma 1.8, we have

$$\begin{aligned}
r(h) &\leq r((e - a_4 - sa_6)^{-1}) r(a_1 + a_2 + a_3 + sa_6) \\
&\leq \frac{r(a_1) + r(a_2) + r(a_3) + sr(a_6)}{1 - r(a_4) - sr(a_6)} < \frac{1}{s}
\end{aligned}$$

which means that $e - h$ is invertible, $(e - h)^{-1} = \sum_{i=0}^{\infty} h^n$ and $\|h^n\| \to 0$ as $n \to \infty$.

Now if $\delta_n = d(x_n, x_{n+1}) + d(y_n, y_{n+1})$, then the above relation implies

$$\delta_n \preceq h\delta_{n-1} \preceq \cdots \preceq h^n \delta_0.$$

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For $m > n$, we have

$$\begin{aligned} d(x_n, x_m) + d(y_n, y_m) &\preceq \delta_{m-1} + \delta_{m-2} + \cdots + \delta_n \\ &\preceq (h^{m-1} + h^{m-2} + \cdots + h^n)\delta_0 \\ &= h^n(1 + h + \cdots + h^{m-n-1})\delta_0 \\ &\preceq h^n(\sum_{i=0}^{\infty} h^i)\delta_0 \\ &= (e-h)^{-1}h^n\delta_0 \end{aligned}$$

since $r(h) < 1$ and P is closed. Since $r(h) < 1$, $\|(e-h)^{-1}h^n\delta_0\| \to 0$ as $n \to \infty$, and so for any $c \in A$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that for any $n > m > N$, we have

$$d(x_n, x_m) + d(y_n, y_m) \preceq (e-h)^{-1}h^n\delta_0 \ll c.$$

Thus $\{d(x_n, x_m) + d(y_n, y_m)\}$ is a c-sequence in P. Since

$$\theta \preceq d(x_n, x_m), d(y_n, y_m) \preceq d(x_n, x_m) + d(y_n, y_m),$$

$\{d(x_n, x_m)\}$ and $\{d(y_n, y_m)\}$ are c-sequences and so Cauchy sequence in X. Since X is complete, there exists $x \in X$ and $y \in X$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Now we show that $x = S(x, y)$ and $y = S(y, x)$. On the contrary, let us assume that $x \neq S(x, y)$ or $y \neq S(y, x)$ so that $d(x, S(x, y)) = k \succ \theta$ and $d(y, S(y, x)) = l \succ \theta$. Then we have

$$\begin{aligned} k = d(x, S(x, y)) &\preceq d(x, x_{2k+2}) + d(x_{2k+2}, S(x, y)) \\ &= d(x, x_{2k+2}) + d(T(x_{2k+1}, y_{2k+1}), S(x, y)) \\ &\preceq d(x, x_{2k+2}) + a_1 d(x, x_{2k+1}) + a_2 d(S(x, y), x) + a_3 d(y, y_{2k+1}) \\ &+ a_4 d(T(x_{2k+1}, y_{2k+1}), x_{2k+1}) + a_5 d(S(x, y), x_{2k+1}) \\ &+ a_6 d(T(x_{2k+1}, y_{2k+1}), x) \\ &= d(x, x_{2k+2}) + a_1 d(x, x_{2k+1}) + a_2 d(S(x, y), x) + a_3 d(y, y_{2k+1}) \\ &+ a_4 d(x_{2k+2}, x_{2k+1}) + a_5 d(S(x, y), x_{2k+1}) + a_6 d(x_{2k+2}, x) \end{aligned}$$

which implies that

$$\begin{aligned} k = d(x, S(x, y)) &\preceq (e + a_6) d(x, x_{2k+2}) + a_1 d(x, x_{2k+1}) + a_2 d(x, S(x, y)) \\ &+ a_3 d(y, y_{2k+1}) + a_4 d(x_{2k+2}, x_{2k+1}) + a_5 d(S(x, y), x_{2k+1}). \end{aligned}$$

Taking $n \to \infty$, by Lemma 1.6 and Lemma 1.10, we have

$$\begin{aligned} k = d(x, S(x, y)) &\preceq (e + a_6)\theta + a_1 \cdot \theta + a_2 d(S(x, y), x) + a_3 \cdot \theta \\ &+ a_4 \cdot \theta + a_5 d(S(x, y), x) + a_6 \cdot \theta \end{aligned}$$

and so $d(x, S(x, y)) \preceq (a_2 + a_5) d(x, S(x, y))$. Since $r(a_2 + a_5) < 1$, by Lemma 1.11, $d(x, S(x, y)) = \theta$. Therefore $x = S(x, y)$. Similarly we can prove that $y = S(y, x)$. It follows similarly that

$$x = T(x, y) \quad \text{and} \quad y = T(y, x).$$

Therefore (x, y) is a common coupled fixed point of S and T.

In order to prove the uniqueness, let $(x', y') \in X \times X$ be another common coupled fixed point of S and T. Then

$$\begin{aligned} d(x, x') &= d(S(x, y), T(x', y')) \\ &\preceq a_1 d(x, x') + a_2 d(S(x, y), x) + a_3 d(y, y') \\ &+ a_4 d(T(x', y'), x') + a_5 d(S(x, y), x') + a_6 d(T(x', y'), x) \\ &= a_1 d(x, x') + a_2 d(x, x) + a_3 d(y, y') \\ &+ a_4 d(x', x') + a_5 d(x, x') + a_6 d(x', x) \\ &= (a_1 + a_5 + a_6) d(x', x) + a_3 d(y, y') \end{aligned}$$

which implies that

$$(e - a_1 - a_5 - a_6) d(x, x') \preceq a_3 d(y, y').$$

Since $r(a_1 + a_5 + a_6) < 1$, $e - (a_1 + a_5 + a_6)$ is invertible and

$$d(x, x') \preceq (e - a_1 - a_5 - a_6)^{-1} a_3 d(y, y').$$

Similarly we can prove that

$$d(y, y') \preceq (e - a_1 - a_5 - a_6)^{-1} a_3 d(x, x').$$

Adding both sides, we get

$$d(x, x') + d(y, y') \preceq (e - a_1 - a_5 - a_6)^{-1} a_3 [d(x, x') + d(y, y')],$$

Since $r((e - a_1 - a_5 - a_6)^{-1} a_3) < 1$, by Lemma 1.11, we have $d(x, x') + d(y, y') = \theta$. Therefore $x = x'$ and $y = y'$. □

The following results generalize the corresponding results in cone metric spaces obtained by H.K. Nashie, Y. Rohen and C. Thokchom([5]).

Corollary 2.2. *(Theorem 2.1 of [5]) Let (X, d) be a complete cone metric space with a solid cone P. Suppose that $S, T : X \times X \to X$ are two mappings satisfying the condition*

$$\begin{aligned} d(S(x, y), T(u, v)) &\preceq a_1 d(x, u) + a_2 d(S(x, y), x) + a_3 d(y, v) \\ &+ a_4 d(T(u, v), u) + a_5 d(S(x, y), u) + a_6 d(T(u, v), x) \end{aligned}$$

for all $x, y, u, v \in X$, where $a_i\,(i = 1, 2, 3, 4, 5, 6)$ are non-negative real numbers such that $\sum_{i=1}^{5} a_i + 2a_6 < 1$. Then S and T have a common coupled fixed point in X.

Proof. Taking $s = 1$ and letting A as a real Banach space in Theorem 2.1, we get the required result. □

Corollary 2.3. *Let (X,d) be a complete cone metric space over the Banach algebra A and let P be a solid cone in A. Suppose that $S,T : X \times X \to X$ are two mappings satisfying the condition*

$$\begin{aligned} d(S(x,y),T(u,v)) &\preceq a_1 d(x,u) + a_2 d(S(x,y),x) + a_3 d(y,v) \\ &+ a_4 d(T(u,v),u) + a_5 d(S(x,y),u) + a_6 d(T(u,v),x) \end{aligned} \tag{2.2.4}$$

for all $x,y,u,v \in X$, where $a_i \in P$ and $a_i a_j = a_j a_i$ $(i,j = 1,2,3,4,5,6)$. If

$$s(r(a_1) + r(a_2) + r(a_3)) + r(a_4) + r(a_5) + (s^2 + s)r(a_6) < 1,$$

then S and T have a common coupled fixed point in X.

Proof. Taking $s = 1$ in Theorem 2.1, we get the required result. □

Corollary 2.4. *Let (X,d) be a complete cone b-metric space over the Banach algebra A with the coefficient $s \geq 1$ and let P be a solid cone. Suppose that $T : X \times X \to X$ is a mapping satisfying the condition*

$$\begin{aligned} d(T(x,y),T(u,v)) &\preceq a_1 d(x,u) + a_2 d(T(x,y),x) + a_3 d(y,v) + a_4 d(T(u,v),u) \\ &+ a_5 d(T(x,y),u) + a_6 d(T(u,v),x) \end{aligned}$$

for all $x,y,u,v \in X$, where $a_i \in P$ and $a_i a_j = a_j a_i$ $(i,j = 1,2,3,4,5,6)$. If

$$s(r(a_1) + r(a_2) + r(a_3)) + r(a_4) + r(a_5) + (s^2 + s)r(a_6) < 1,$$

then T has a unique coupled fixed point in X.

Corollary 2.5. *Let (X,d) be a complete cone b-metric space over the Banach algebra A with the coefficient $s \geq 1$ and let P be a solid cone. Suppose that $S,T : X \times X \to X$ are two mappings satisfying the condition*

$$\begin{aligned} d(S(x,y),T(u,v)) &\preceq ad(x,u) + bd(y,v) + c[d(S(x,y),x) + d(T(u,v),u)] \\ &+ e[d(S(x,y),u) + d(T(u,v),x)] \end{aligned}$$

for all $x,y,u,v \in X$, where $a,b,c,e \in P$ are commuting. If

$$s(r(a) + r(b)) + (s+1)r(c)) + (s^2 + s + 1)r(e) < 1,$$

then S and T have a unique common coupled fixed point in X.

Corollary 2.6. *Let (X,d) be a complete cone b-metric space over the Banach algebra A with the coefficient $s \geq 1$ and let P be a solid cone. Suppose that $S,T : X \times X \to X$ are two mappings satisfying the condition*

$$\begin{aligned} d(T(x,y),T(u,v)) &\preceq ad(x,u) + bd(y,v) + c[d(T(x,y),x) + d(T(u,v),u)] \\ &+ e[d(T(x,y),u) + d(T(u,v),x)] \end{aligned}$$

for all $x, y, u, v \in X$, where $a, b, c, e \in P$ are commuting. If

$$s(r(a) + r(b)) + (s + 1)r(c)) + (s^2 + s + 1)r(e) < 1,$$

then T has a unique coupled fixed point in X.

Now we give an example showing that Theorem 2.1 is a proper extension of known results. In this example, the conditions of Theorem 2.1 are fulfilled.

Example 2.7. Let $A = C^1_{\mathbb{R}}[0, 1]$ and define a norm on A by $\|x\| = \|x\|_\infty + \|x'\|_\infty$ for $x \in A$. Define multiplication in A as just pointwise multiplication. Then A is a real Banach algebra with unit $e = 1(e(t) = 1$ for all $t \in [0, 1])$. The set $P = \{x \in A : x \geq 0\}$ is a cone in A. Moreover, P is not normal.

Let $X = \{1, 2, 3\}$. Define $d : X \times X \to A$ by $d(1, 2)(t) = d(2, 1)(t) = d(2, 3)(t) = d(3, 2)(t) = e^t, d(1, 3)(t) = d(3, 1)(t) = 3e^t, d(x, x)(t) = \theta$ for all $t \in [0, 1]$ and for each $x \in X$. Then (X, d) is a solid cone b-metric space over Banach algebra with the coefficient $s = \frac{3}{2}$. But it is not a cone metric space over Banach algebra since it does not satisfy the triangle inequality.

Define two mappings $S, T : X \times X \to X$ by $S(x, y) = 1$ for any $(x, y) \in X \times X$, and

$$T(x, y) = \begin{cases} 2, & (x, y) = (3, 1) \\ 1, & \text{otherwise} \end{cases}$$

Let $a_1, a_2, a_3, a_4, a_5, a_6 \in P$ defined with $a_1(t) = a_2(t) = a_3(t) = 0.2, a_4(t) = 0.1, a_5(t) = 0.4, a_6(t) = 0.05$ for all $t \in [0, 1]$. Then, by definition of spectral radius, $r(a_1) = r(a_2) = r(a_3) = 0.2, r(a_4) = 0.1, r(a_5) = 0.4, r(a_6) = 0.05$ and so

$$s[r(a_1) + r(a_2) + r(a_3)] + r(a_4) + r(a_5) + (s^2 + s)r(a_6) = 0.9875 < 1.$$

Since $d(S(x, y), T(3, 1))(t) = d(1, 2)(t)) = e^t$ for any $x, y \in X$, by careful calculations, we can get that for any $x, y, u, v \in X$, S and T satisfy the contractive condition (2.2.4) of Theorem 2.1. Hence the hypotheses are satisfied and so by Theorem 2.1, S and T have a common coupled fixed point in X. Since $S(1, 1) = 1 = T(1, 1)$, $(1, 1)$ is the unique coupled fixed point of S and T.

Acknowledgments.

The first author was supported by the 2018 scientific promotion program funded by Jeju National University.

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FOURIER SERIES OF SUMS OF PRODUCTS OF POLY-GENOCCHI FUNCTIONS

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Abstract. Recently, some authors introduced poly-Genocchi polynomials as an analogy to poly-Bernoulli polynomials. In this paper, we will consider three types of sums of products of poly-Genocchi functions and derive their Fourier expansions. In addition, we will express each of them in terms of Bernoulli functions.

1. Introduction

The *Bernoulli polynomials* $B_m(x)$ are given by the generating function

$$\frac{t}{e^t-1}e^{xt}=\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!}.$$

When $x=0$, $B_m=B_m(0)$ are called *Bernoulli numbers.*

The *Genocchi polynomials* $G_m(x)$ are defined by the generating function

$$\frac{2t}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}G_m(x)\frac{t^m}{m!}.$$

For $x=0$, $G_m=G_m(0)$ are called *Genocchi numbers.*

Let r be any integer. The *poly-Bernoulli polynomials* $\mathbb{B}_m^{(r)}(x)$ of index r are given by

$$\frac{Li_r(1-e^{-t})}{e^t-1}e^{xt}=\sum_{m=0}^{\infty}\mathbb{B}_m^{(r)}(x)\frac{t^m}{m!},$$

where $Li_r(x)=\sum_{m=1}^{\infty}\frac{x^m}{m^r}$ is the rth polylogarithm function for $r\geq 1$, and a rational function for $r\leq 0$. We note here that this definition of poly-Bernoulli polynomials are slightly different from the Kaneko's original definition [1, 2, 3, 5]. Indeed, if $\tilde{\mathbb{B}}_m^{(r)}(x)$ denotes the Kaneko's poly-Bernoulli polynomial of index r, then $\mathbb{B}_m^{(r)}(x)=\tilde{\mathbb{B}}_m^{(r)}(x-1)$. Also, for $x=0$, $\mathbb{B}_m^{(r)}=\mathbb{B}_m^{(r)}(0)$ are called *poly-Bernoulli numbers* of index r. Clearly,

$$\mathbb{B}_m^{(1)}(x)=B_m(x),\ \mathbb{B}_0^{(r)}(x)=1,\ \mathbb{B}_m^{(0)}(x)=x^m,$$
$$\mathbb{B}_m^{(0)}=\delta_{m,0},\ \frac{d}{dx}\mathbb{B}_m^{(r)}(x)=m\mathbb{B}_{m-1}^{(r)}(x),\ (m\geq 1).$$

2010 *Mathematics Subject Classification.* 11B68, 11B83, 42A16.

Key words and phrases. Fourier series, Bernoulli polynomial, Euler function, Genocchi polynomial, poly-Genocchi polynomial, poly-Bernoulli polynomial.

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As an analogy to this construction of poly-Bernoulli polynomials, the poly-Genocchi polynomials $\mathbb{G}_m^{(r)}(x)$ of index r are given by

$$\frac{2Li_r(1-e^{-t})}{e^t+1}e^{xt} = \sum_{m=0}^{\infty} \mathbb{G}_m^{(r)}(x)\frac{t^m}{m!}. \tag{1.1}$$

When $x=0$, $\mathbb{G}_m^{(r)} = \mathbb{G}_m^{(r)}(0)$ are called *poly-Genocchi numbers*. Unfortunately, the poly-Genocchi polynomials were named as poly-Euler polynomials. But, as we clearly have $\mathbb{G}_m^{(1)}(x) = G_m(x)$, it seems more appropriate to call them poly-Genocchi polynomials (see [6]). There are other definitions for poly-Euler numbers and polynomials. Indeed, in [7, 8], the poly-Euler numbers $E_m^{(r)}$ are defined by

$$\frac{Li_r(1-e^{-4t})}{4t\cosh t} = \sum_{m=0}^{\infty} E_m^{(r)}\frac{t^m}{m!}.$$

For poly-Euler polynomials, see [4].

As is known or one can see,

$$\frac{d}{dx}(Li_{r+1}(x)) = \frac{1}{x}Li_r(x).$$

In addition, since $\mathbb{G}_m^{(r)}(x)$ are Appell polynomials,

$$\frac{d}{dx}\mathbb{G}_m^{(r)}(x) = m\mathbb{G}_{m-1}^{(r)}(x),\ (m \geq 1).$$

Here we claim that

$$\mathbb{G}_m^{(r+1)}(1) + \mathbb{G}_m^{(r+1)}(0) = 2\mathbb{B}_{m-1}^{(r)},\ (m \geq 1). \tag{1.2}$$

From (1.1), we clearly have

$$\sum_{m=0}^{\infty}\left(\mathbb{G}_m^{(r+1)}(1) + \mathbb{G}_m^{(r+1)}(0)\right)\frac{t^m}{m!} = 2Li_{r+1}(1-e^{-t}). \tag{1.3}$$

Differentiation of LHS of (1.3) with respect to t gives

$$\sum_{m=0}^{\infty}\left(\mathbb{G}_{m+1}^{(r+1)}(1) + \mathbb{G}_{m+1}^{(r+1)}(0)\right)\frac{t^m}{m!}.$$

On the other hahd, differentiation of RHS of (1.3) with respect to t yields

$$\frac{2Li_r(1-e^{-t})}{1-e^{-t}}e^{-t} = 2\sum_{m=0}^{\infty}\mathbb{B}_m^{(r)}\frac{t^m}{m!}.$$

From these, we get the desired result. Writing $Li_r(1-e^{-t}) = \sum_{n=1}^{\infty} a_n \frac{t^n}{n!} = t + \sum_{n=2}^{\infty} a_n \frac{t^n}{n!}$, from (1.1) we obtain

$$\sum_{m=0}^{\infty}\mathbb{G}_m^{(r)}(x)\frac{t^m}{m!} = \sum_{m=1}^{\infty}\left(\sum_{l=0}^{m-1}\binom{m}{l}a_{m-l}E_l(x)\right)\frac{t^m}{m!},$$

where $E_m(x)$ are Euler polynomials given by

$$\frac{2}{e^t+1}e^{xt} = \sum_{m=0}^{\infty}E_m(x)\frac{t^m}{m!}.$$

In particular, this implies that

$$\mathbb{G}_0^{(r)}(x) = 0,\ \mathbb{G}_1^{(r)}(x) = 1,\ \deg\mathbb{G}_m^{(r)}(x) = m-1, \text{for } m \geq 1.$$

As a quick application of (1.2), we express $\mathbb{G}_m^{(r+1)}(x)$ as a linear combination of Euler polynomials. For this, we recall that, for a polynomial $p(x) \in \mathbb{Q}[x]$ with $\deg p(x) = m$,

$$p(x) = \sum_{j=0}^{m} b_j E_j(x),\ b_j \in \mathbb{Q},$$

where

$$b_j = \frac{1}{2j!}\left(p^{(j)}(1) + p^{(j)}(0)\right),\ j = 0, 1, \ldots, m.$$

We now apply this to the polynomial $p(x) = \mathbb{G}_m^{(r+1)}(x)$, and let

$$\mathbb{G}_m^{(r+1)}(x) = \sum_{j=0}^{m} b_j E_j(x).$$

Then

$$\begin{aligned} b_j &= \frac{(m)_j}{2j!}\left(\mathbb{G}_{m-j}^{(r+1)}(1) + \mathbb{G}_{m-j}^{(r+1)}(0)\right) \\ &= \begin{cases} \binom{m}{j}\mathbb{B}_{m-j-1}^{(r)}, & \text{for } 0 \leq j \leq m-1, \\ 0, & \text{for } j = m \end{cases} \end{aligned}$$

Thus

$$\mathbb{G}_m^{(r+1)}(x) = \sum_{j=0}^{m-1}\binom{m}{j}\mathbb{B}_{m-j-1}^{(r)} E_j(x),\ (m \geq 1).$$

Also, for $p(x) \in \mathbb{Q}[x]$, with $\deg p(x) = m$,

$$p(x) = \sum_{j=1}^{m+1} b_j G_j(x),\ b_j \in \mathbb{Q},$$

where $b_j = \frac{1}{2j!}\left(p^{(j-1)}(1) + p^{(j-1)}(0)\right)$, for $m = 1, \ldots, m+1$.

Applying this to $p(x) = G_m^{(r+1)}(x)$, we see that

$$b_j = \begin{cases} \frac{1}{m+1}\binom{m+1}{j}\mathbb{B}_{m-j}^{(r)}, & \text{for } 1 \leq j \leq m, \\ 0, & \text{for } j = m+1. \end{cases}$$

Thus we obtain

$$\mathbb{G}_m^{(r+1)}(x) = \frac{1}{m+1}\sum_{j=1}^{m}\binom{m+1}{j}\mathbb{B}_{m-j}^{(r)} G_j(x),\ (m \geq 1).$$

For any real number x, we let

$$\langle x \rangle = x - \lfloor x \rfloor \in [0,\ 1)$$

denote the fractional part of x.

Here we will consider the following three types of sums of products of poly-Genocchi functions $\alpha_m(\langle x \rangle)$, $\beta_m(\langle x \rangle)$, and $\gamma_m(\langle x \rangle)$ and derive their Fourier expansions. In addition, we will express each of them in terms of Bernoulli functions.

(a) $\alpha_m(\langle x \rangle) = \sum_{k=1}^{m-1} \mathbb{G}_k^{(r+1)}(\langle x \rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x \rangle),\ (m \geq 3);$

(b) $\beta_m(\langle x\rangle)=\sum_{k=1}^{m-1}\frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle)$ $(m\geq 3)$;

(c) $\gamma_m(\langle x\rangle)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle)$, $(m\geq 3)$.

2. The sums of products of poly-Genocchi functions, type I

For integers r,s,m, with $m\geq 3$, let

$$\alpha_m(x)=\sum_{k=1}^{m-1}\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k}^{(s+1)}(x).$$

$$\begin{aligned}\alpha_m'(x)=&\sum_{k=1}^{m-1}\left\{k\mathbb{G}_{k-1}^{(r+1)}(x)\mathbb{G}_{m-k}^{(s+1)}(x)+(m-k)\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k-1}^{(s+1)}(x)\right\}\\ =&\sum_{k=2}^{m-1}k\mathbb{G}_{k-1}^{(r+1)}(x)\mathbb{G}_{m-k}^{(s+1)}(x)+\sum_{k=1}^{m-2}(m-k)\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k-1}^{(s+1)}(x)\\ =&\sum_{k=1}^{m-2}(k+1)\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k-1}^{(s+1)}(x)+\sum_{k=1}^{m-2}(m-k)\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k-1}^{(s+1)}(x)\\ =&(m+1)\alpha_{m-1}(x).\end{aligned}$$

From this, we have

$$\left(\frac{\alpha_{m+1}(x)}{m+2}\right)'=\alpha_m(x),$$

$$\int_0^1\alpha_m(x)dx=\frac{1}{m+2}\left(\alpha_{m+1}(1)-\alpha_{m+1}(0)\right).$$

For $m\geq 3$, we put

$$\begin{aligned}\Delta_m=&\Delta_m(r,s)=\alpha_m(1)-\alpha_m(0)\\ =&\sum_{k=1}^{m-1}\left(\mathbb{G}_k^{(r+1)}(1)\mathbb{G}_{m-k}^{(s+1)}(1)-\mathbb{G}_k^{(r+1)}\mathbb{G}_{m-k}^{(s+1)}\right)\\ =&\sum_{k=1}^{m-1}\left(\left(-\mathbb{G}_k^{(r+1)}+2\mathbb{B}_{k-1}^{(r)}\right)\left(-\mathbb{G}_{m-k}^{(s+1)}+2\mathbb{B}_{m-k-1}^{(s)}\right)-\mathbb{G}_k^{(r+1)}\mathbb{G}_{m-k}^{(s+1)}\right)\\ =&-2\sum_{k=1}^{m-1}\left(\mathbb{G}_k^{(r+1)}\mathbb{B}_{m-k-1}^{(s)}+\mathbb{B}_{k-1}^{(r)}\mathbb{G}_{m-k}^{(s+1)}-2\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{m-k-1}^{(s)}\right).\end{aligned}$$

Thus

$$\begin{aligned}&\alpha_m(0)=\alpha_m(1)\\ \iff&\Delta_m=0\\ \iff&\sum_{k=1}^{m-1}\left(\mathbb{G}_k^{(r+1)}\mathbb{B}_{m-k-1}^{(s)}+\mathbb{B}_{k-1}^{(r)}\mathbb{G}_{m-k}^{(s+1)}-2\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{m-k-1}^{(s)}\right)=0,\end{aligned}$$

$$\int_0^1\alpha_m(x)dx=\frac{1}{m+2}\Delta_{m+1}.$$

We are now going to consider the function

$$\alpha_m(\langle x\rangle)=\sum_{k=1}^{m-1}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle),\ (m\geq 3),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\alpha_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty}A_n^{(m)}e^{2\pi inx},$$

where

$$A_n^{(m)}=\int_0^1\alpha_m(\langle x\rangle)e^{-2\pi inx}dx=\int_0^1\alpha_m(x)e^{-2\pi inx}dx.$$

Now, we would like to determine the Fourier coefficients $A_n^{(m)}$.

Case 1 : $n\neq 0$.

$$\begin{aligned}A_n^{(m)}=&\int_0^1\alpha_m(x)e^{-2\pi inx}dx\\
=&-\frac{1}{2\pi in}\left[\alpha_m(x)e^{-2\pi inx}\right]_0^1+\frac{1}{2\pi in}\int_0^1\alpha_m'(x)e^{-2\pi inx}dx\\
=&-\frac{1}{2\pi in}\left(\alpha_m(1)-\alpha_m(0)\right)+\frac{m+1}{2\pi in}\int_0^1\alpha_{m-1}(x)e^{-2\pi nx}dx\\
=&\frac{m+1}{2\pi in}A_n^{(m-1)}-\frac{1}{2\pi in}\Delta_m,\end{aligned}$$

from which we can easily deduce that

$$A_n^{(m)}=-\frac{1}{m+2}\sum_{j=1}^{m-2}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}.$$

Case 2 : $n=0$.

$$A_0^{(m)}=\int_0^1\alpha_m(x)dx=\frac{1}{m+2}\Delta_{m+1}.$$

We recall the following facts about Bernoulli functions $B_m(\langle x\rangle)$:

(a) for $m\geq 2$,

$$B_m(\langle x\rangle)=-m!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^m},$$

(b) for $m=1$,

$$-\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{2\pi in}=\begin{cases}B_1(\langle x\rangle), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}.\end{cases}$$

$\alpha_m(\langle x\rangle)$, $(m\geq 3)$ is piecewise C^∞. Moreover, $\alpha_m(\langle x\rangle)$ is continuous for those integers $m\geq 3$ with $\Delta_m=0$, and discontinuous with jump discontinuities at integers for those integers $m\geq 3$ with $\Delta_m\neq 0$.

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Assume first that m is an integer ≥ 3 with $\Delta_m = 0$. Then $\alpha_m(0) = \alpha_m(1)$. Hence $\alpha_m(\langle x \rangle)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\alpha_m(\langle x \rangle)$ converges uniformly to $\alpha_m(\langle x \rangle)$, and

$$\begin{aligned}
&\alpha_m(\langle x \rangle)\\
=&\frac{1}{m+2}\Delta_{m+1} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m-2}\frac{(m+2)_j}{(2\pi i n)^j}\Delta_{m-j+1}\right)e^{2\pi i n x}\\
=&\frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=1}^{m-2}\binom{m+2}{j}\Delta_{m-j+1}\left(-j!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^j}\right)\\
=&\frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m-2}\binom{m+2}{j}\Delta_{m-j+1}B_j(\langle x \rangle)\\
&+\Delta_m \times \begin{cases} B_1(\langle x \rangle), & \text{for } x \notin \mathbb{Z},\\ 0, & \text{for } x \in \mathbb{Z}.\end{cases}
\end{aligned}$$

Now, we are ready to state our first theorem.

Theorem 2.1. *For each integer $l \geq 3$, let*

$$\Delta_l = \Delta_l(r,s) = -2\sum_{k=1}^{l-1}\left(\mathbb{G}_k^{(r+1)}\mathbb{B}_{l-k-1}^{(s)} + \mathbb{B}_{k-1}^{(r)}\mathbb{G}_{l-k}^{(s+1)} - 2\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{l-k-1}^{(s)}\right).$$

Assume that $\Delta_m = 0$, for an integer $m \geq 3$. Then we have the following.

(a) $\sum_{k=1}^{m-1}\mathbb{G}_k^{(r+1)}(\langle x \rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x \rangle)$ *has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=1}^{m-1}\mathbb{G}_k^{(r+1)}(\langle x \rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x \rangle)\\
=&\frac{1}{m+2}\Delta_{m+1} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m-2}\frac{(m+2)_j}{(2\pi i n)^j}\Delta_{m-j+1}\right)e^{2\pi i n x},
\end{aligned}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

(b)

$$\sum_{k=1}^{m-1}\mathbb{G}_k^{(r+1)}(\langle x \rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x \rangle) = \frac{1}{m+2}\sum_{\substack{j=0\\ j\neq 1}}^{m-2}\binom{m+2}{j}\Delta_{m-j+1}B_j(\langle x \rangle),$$

for all $x \in \mathbb{R}$.

Assume next that m is an integer ≥ 3, with $\Delta_m \neq 0$. Then $\alpha_m(0) \neq \alpha_m(1)$. Hence $\alpha_m(\langle x \rangle)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Thus the Foureir series of $\alpha_m(\langle x \rangle)$ converges pointwise to $\alpha_m(\langle x \rangle)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\alpha_m(0) + \alpha_m(1)\right) = \alpha_m(0) + \frac{1}{2}\Delta_m,$$

for $x \in \mathbb{Z}$. We are now ready to state our second theorem.

Theorem 2.2. *For each integer $l \geq 3$, we let*

$$\Delta_l = \Delta_l(r,s) = -2\sum_{k=1}^{l-1}\left(\mathbb{G}_k^{(r+1)}\mathbb{B}_{l-k-1}^{(s)} + \mathbb{B}_{k-1}^{(r)}\mathbb{G}_{l-k}^{(s+1)} - 2\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{l-k-1}^{(s)}\right).$$

Assume that $\Delta_m \neq 0$, for an integer $m \geq 3$. Then we have the following

(a)

$$\frac{1}{m+2}\Delta_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m-2}\frac{(m+2)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx}$$
$$= \begin{cases} \sum_{k=1}^{m-1}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle), & \text{for } x \notin \mathbb{Z}, \\ \sum_{k=1}^{m-1}\mathbb{G}_k^{(r+1)}\mathbb{G}_{m-k}^{(s+1)} + \frac{1}{2}\Delta_m, & \text{for } x \in \mathbb{Z}. \end{cases}$$

(b)

$$\frac{1}{m+2}\sum_{j=0}^{m-2}\binom{m+2}{j}\Delta_{m-j+1}B_j(\langle x\rangle) = \sum_{k=1}^{m-1}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle), \ \text{for } x \notin \mathbb{Z};$$

$$\frac{1}{m+2}\sum_{\substack{j=0 \\ j\neq 1}}^{m-2}\binom{m+2}{j}\Delta_{m-j+1}B_j(\langle x\rangle) = \sum_{k=1}^{m-1}\mathbb{G}_k^{(r+1)}\mathbb{G}_{m-k}^{(s+1)} + \frac{1}{2}\Delta_m, \ \text{for } x \in \mathbb{Z}.$$

3. The sums of products of poly-Genocchi functions, type II

Let

$$\beta_m(x) = \sum_{k=1}^{m-1}\frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k}^{(s+1)}(x), \ (m \geq 3).$$

$$\beta_m'(x) = \sum_{k=1}^{m-1}\left\{\frac{k}{k!(m-k)!}\mathbb{G}_{k-1}^{(r+1)}(x)\mathbb{G}_{m-k}^{(s+1)}(x) + \frac{m-k}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k-1}^{(s+1)}(x)\right\}$$
$$= \sum_{k=2}^{m-1}\frac{1}{(k-1)!(m-k)!}\mathbb{G}_{k-1}^{(r+1)}(x)\mathbb{G}_{m-k}^{(s+1)}(x) + \sum_{k=1}^{m-2}\frac{1}{k!(m-k-1)!}\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k-1}^{(s+1)}(x)$$
$$= \sum_{k=1}^{m-2}\frac{1}{k!(m-k-1)!}\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k-1}^{(s+1)}(x) + \sum_{k=1}^{m-2}\frac{1}{k!(m-k-1)!}\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k-1}^{(s+1)}(x)$$
$$= 2\beta_{m-1}(x).$$

From this, we obtain that

$$\left(\frac{\beta_{m+1}(x)}{2}\right)' = \beta_m(x)$$
$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\left(\beta_{m+1}(1) - \beta_{m+1}(0)\right).$$

For $m \geq 3$, we let

$$\begin{aligned}
&\Omega_m = \Omega_m(r,s) = \beta_m(1) - \beta_m(0) \\
&= \sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \left(\mathbb{G}_k^{(r+1)}(1)\mathbb{G}_{m-k}^{(s+1)}(1) - \mathbb{G}_k^{(r+1)}\mathbb{G}_{m-k}^{(s+1)} \right) \\
&= \sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \left(\left(-\mathbb{G}_k^{(r+1)} + 2\mathbb{B}_{k-1}^{(r)} \right) \left(-\mathbb{G}_{m-k}^{(s+1)} + 2\mathbb{B}_{m-k-1}^{(s)} \right) - \mathbb{G}_k^{(r+1)}\mathbb{G}_{m-k}^{(s+1)} \right) \\
&= -2 \sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \left(\mathbb{G}_k^{(r+1)}\mathbb{B}_{m-k-1}^{(s)} + \mathbb{B}_{k-1}^{(r)}\mathbb{G}_{m-k}^{(s+1)} - 2\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{m-k-1}^{(s)} \right).
\end{aligned}$$

Then

$$\begin{aligned}
&\beta_m(0) = \beta_m(1) \iff \Omega_m = 0 \\
&\iff \sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \left(\mathbb{G}_k^{(r+1)}\mathbb{B}_{m-k-1}^{(s)} + \mathbb{B}_{k-1}^{(r)}\mathbb{G}_{m-k}^{(s+1)} - 2\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{m-k-1}^{(s)} \right) = 0,
\end{aligned}$$

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}.$$

We now would like to consider the function

$$\beta_m(\langle x \rangle) = \sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \mathbb{G}_k^{(r+1)}(\langle x \rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x \rangle), \ (m \geq 3),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\beta_m(\langle x \rangle)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi i n x},$$

where

$$B_n^{(m)} = \int_0^1 \beta_m(\langle x \rangle) e^{-2\pi i n x} dx = \int_0^1 \beta_m(x) e^{-2\pi i n x} dx.$$

Next, we want to determine the Fourier coefficients $B_n^{(m)}$.

Case $1: \ n \neq 0$.

$$\begin{aligned}
B_n^{(m)} &= \int_0^1 \beta_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} \left[\beta_m(x) e^{-2\pi i n x} \right]_0^1 + \frac{1}{2\pi i n} \int_0^1 \beta_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} \left(\beta_m(1) - \beta_m(0) \right) + \frac{2}{2\pi i n} \int_0^1 \beta_{m-1}(x) e^{-2\pi i n x} dx \\
&= \frac{2}{2\pi i n} B_n^{(m-1)} - \frac{1}{2\pi i n} \Omega_m,
\end{aligned}$$

from which by induction we can easily deduce that

$$B_n^{(m)} = -\frac{1}{2} \sum_{j=1}^{m-2} \frac{2^j}{(2\pi i n)^j} \Omega_{m-j+1}.$$

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Case $2:\ n=0$.

$$B_0^{(m)}=\int_0^1 \beta_m(x)dx=\frac{1}{2}\Omega_{m+1}.$$

$\beta_m(\langle x\rangle)$, $(m\geq 3)$ is piecewise C^∞. Moreover, it is continuous for those integers $m\geq 3$ with $\Omega_m=0$, and discontinuous with jump discontinuities at integers for those integers $m\geq 3$ with $\Omega_m\neq 0$.

Assume first that m is an integer ≥ 3 with $\Omega_m=0$. Then $\beta_m(0)=\beta_m(1)$. Hence $\beta_m(\langle x\rangle)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\beta_m(\langle x\rangle)$ converges uniformly to $\beta_m(\langle x\rangle)$, and

$$\begin{aligned}
&\beta_m(\langle x\rangle)\\
=&\frac{1}{2}\Omega_{m+1}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{2}\sum_{j=1}^{m-2}\frac{2^j}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx}\\
=&\frac{1}{2}\Omega_{m+1}+\frac{1}{2}\sum_{j=1}^{m-2}\frac{2^j}{j!}\Omega_{m-j+1}\left(-j!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right)\\
=&\frac{1}{2}\Omega_{m+1}+\frac{1}{2}\sum_{j=2}^{m-2}\frac{2^j}{j!}\Omega_{m-j+1}B_j(\langle x\rangle)+\Omega_m\times\begin{cases} B_1(\langle x\rangle), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}.\end{cases}
\end{aligned}$$

Now, we are ready to state our first theorem.

Theorem 3.1. *For each integer $l\geq 3$, let*

$$\Omega_l=\Omega_l(r,s)=-2\sum_{k=1}^{l-1}\frac{1}{k!(l-k)!}\left(\mathbb{G}_k^{(r+1)}\mathbb{B}_{l-k-1}^{(s)}+\mathbb{B}_{k-1}^{(r)}\mathbb{G}_{l-k}^{(s+1)}-2\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{l-k-1}^{(s)}\right).$$

Assume that $\Omega_m=0$, for an integer $m\geq 3$. Then we have the following.

(a) $\sum_{k=1}^{m-1}\frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle)$ *has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle)\\
=&\frac{1}{2}\Omega_{m+1}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{2}\sum_{j=1}^{m-2}\frac{2^j}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx},
\end{aligned}$$

for all $x\in\mathbb{R}$, where the convergence is uniform.

(b)

$$\sum_{k=1}^{m-1}\frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle)=\frac{1}{2}\sum_{\substack{j=0\\ j\neq 1}}^{m-2}\frac{2^j}{j!}\Omega_{m-j+1}B_j(\langle x\rangle),$$

for all $x\in\mathbb{R}$.

Assume next that m is an integer ≥ 3 with $\Omega_m\neq 0$. Then $\beta_m(0)\neq\beta_m(1)$, and hence $\beta_m(\langle x\rangle)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\beta_m(\langle x\rangle)$ converges pointwise to $\beta_m(\langle x\rangle)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\beta_m(0)+\beta_m(1)\right)=\beta_m(0)+\frac{1}{2}\Omega_m,$$

for $x \in \mathbb{Z}$.

We are now ready to state our second theorem.

Theorem 3.2. *For each integer $l \geq 3$, let*

$$\Omega_l=\Omega_l(r,s)=-2\sum_{k=1}^{l-1}\frac{1}{k!(l-k)!}\left(\mathbb{G}_k^{(r+1)}\mathbb{B}_{l-k-1}^{(s)}+\mathbb{B}_{k-1}^{(r)}\mathbb{G}_{l-k}^{(s+1)}-2\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{l-k-1}^{(s)}\right).$$

Assume $\Omega_m \neq 0$, for an integer $m \geq 3$. Then we have the following

(a)

$$\frac{1}{2}\Omega_{m+1}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{2}\sum_{j=1}^{m-2}\frac{2^j}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx}$$

$$=\begin{cases}\sum_{k=1}^{m-1}\frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle), & \textit{for } x\notin\mathbb{Z},\\ \sum_{k=1}^{m-1}\frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}\mathbb{G}_{m-k}^{(s+1)}+\frac{1}{2}\Omega_m, & \textit{for } x\in\mathbb{Z}.\end{cases}$$

(b)

$$\frac{1}{2}\sum_{j=0}^{m-2}\frac{2^j}{j!}\Omega_{m-j+1}B_j(\langle x\rangle)=\sum_{k=1}^{m-1}\frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle),\ \textit{for } x\notin\mathbb{Z};$$

$$\frac{1}{2}\sum_{\substack{j=0\\ j\neq 1}}^{m-2}\frac{2^j}{j!}\Omega_{m-j+1}B_j(\langle x\rangle)=\sum_{k=1}^{m-1}\frac{1}{k!(m-k)!}\mathbb{G}_k^{(r+1)}\mathbb{G}_{m-k}^{(s+1)}+\frac{1}{2}\Omega_m,\ \textit{for } x\in\mathbb{Z}.$$

4. The sums of products of poly-Genocchi functions, type III

Let

$$\gamma_m(x)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k}^{(s+1)}(x),\ (m\geq 3).$$

$$\gamma_m'(x)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\left\{k\mathbb{G}_{k-1}^{(r+1)}(x)\mathbb{G}_{m-k}^{(s+1)}(x)+(m-k)\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k-1}^{(s+1)}(x)\right\}$$

$$=\sum_{k=2}^{m-1}\frac{1}{m-k}\mathbb{G}_{k-1}^{(r+1)}(x)\mathbb{G}_{m-k}^{(s+1)}(x)+\sum_{k=1}^{m-2}\frac{1}{k}\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-1-k}^{(s+1)}(x)$$

$$=\sum_{k=1}^{m-2}\left(\frac{1}{m-k-1}+\frac{1}{k}\right)\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k-1}^{(s+1)}(x)$$

$$=(m-1)\sum_{k=1}^{m-2}\frac{1}{k(m-k-1)}\mathbb{G}_k^{(r+1)}(x)\mathbb{G}_{m-k-1}^{(s+1)}(x)$$

$$=(m-1)\gamma_{m-1}(x).$$

From this, we have

$$\left(\frac{\gamma_{m+1}(x)}{m}\right)' = \gamma_m(x),$$

and

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m}\left(\gamma_{m+1}(1) - \gamma_{m+1}(0)\right).$$

For $m \geq 3$, we let

$$\begin{aligned}
&\Lambda_m = \Lambda_m(r,s) = \gamma_m(1) - \gamma_m(0)\\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\left(\mathbb{G}_k^{(r+1)}(1)\mathbb{G}_{m-k}^{(s+1)}(1) - \mathbb{G}_k^{(r+1)}\mathbb{G}_{m-k}^{(s+1)}\right)\\
&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\left(\left(-\mathbb{G}_k^{(r+1)} + 2\mathbb{B}_{k-1}^{(r)}\right)\left(-\mathbb{G}_{m-k}^{(s+1)} + 2\mathbb{B}_{m-k-1}^{(s)}\right) - \mathbb{G}_k^{(r+1)}\mathbb{G}_{m-k}^{(s+1)}\right)\\
&= -2\sum_{k=1}^{m-1} \frac{1}{k(m-k)}\left(\mathbb{G}_k^{(r+1)}\mathbb{B}_{m-k-1}^{(s)} + \mathbb{B}_{k-1}^{(r)}\mathbb{G}_{m-k}^{(s+1)} - 2\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{m-k-1}^{(s)}\right).
\end{aligned}$$

Then $\gamma_m(0) = \gamma_m(1) \iff \Lambda_m = 0$, and

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m}\Lambda_{m+1}.$$

We are now going to consider

$$\gamma_m(\langle x\rangle) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle),\ (m \geq 3),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\gamma_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi i n x},$$

where

$$C_n^{(m)} = \int_0^1 \gamma_m(\langle x\rangle)e^{-2\pi i n x}dx = \int_0^1 \gamma_m(x)e^{-2\pi i n x}dx.$$

Now, we want to determine the Fourier coefficients $C_n^{(m)}$.

Case $1:\ n \neq 0$.

$$\begin{aligned}
C_n^{(m)} &= \int_0^1 \gamma_m(x)e^{-2\pi i n x}dx\\
&= -\frac{1}{2\pi i n}\left[\gamma_m(x)e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 \gamma_m'(x)e^{-2\pi i n x}dx\\
&= -\frac{1}{2\pi i n}\left(\gamma_m(1) - \gamma_m(0)\right) + \frac{m-1}{2\pi i n}\int_0^1 \gamma_{m-1}(x)e^{-2\pi i n x}dx\\
&= \frac{m-1}{2\pi i n}C_n^{(m-1)} - \frac{1}{2\pi i n}\Lambda_m,
\end{aligned}$$

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from which by induction on m we can deduce that

$$C_n^{(m)} = -\frac{1}{m}\sum_{j=1}^{m-2}\frac{(m)_j}{(2\pi in)^j}\wedge_{m-j+1}.$$

Case 2 : $n = 0$.

$$C_0^{(m)} = \int_0^1 \gamma_m(x)dx = \frac{1}{m}\Lambda_{m+1}.$$

$\gamma_m(\langle x\rangle)$, $(m \geq 3)$ is piecewise C^∞. Further, it is continuous for those integers $m \geq 3$ with $\Lambda_m = 0$, and discontinuous with jump discontinuities at integers for those integers $m \geq 3$ with $\Lambda_m \neq 0$.

Assume first that m is an integer ≥ 3 with $\Lambda_m = 0$. Then $\gamma_m(0) = \gamma_m(1)$. Hence $\gamma_m(\langle x\rangle)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\gamma_m(\langle x\rangle)$ converges uniformly to $\gamma_m(\langle x\rangle)$, and

$$\begin{aligned}
&\gamma_m(\langle x\rangle)\\
=&\frac{1}{m}\Lambda_{m+1} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m}\sum_{j=1}^{m-2}\frac{(m)_j}{(2\pi in)^j}\Lambda_{m-j+1}\right)e^{2\pi inx}\\
=&\frac{1}{m}\Lambda_{m+1} + \frac{1}{m}\sum_{j=1}^{m-2}\binom{m}{j}\Lambda_{m-j+1}\left(-j!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right)\\
=&\frac{1}{m}\Lambda_{m+1} + \frac{1}{m}\sum_{j=2}^{m-2}\binom{m}{j}\Lambda_{m-j+1}B_j(\langle x\rangle) + \Lambda_m \times \begin{cases} B_1(\langle x\rangle), & \text{for } x \notin \mathbb{Z},\\ 0, & \text{for } x \in \mathbb{Z}.\end{cases}
\end{aligned}$$

Now, we can state our first theorem.

Theorem 4.1. *For each integer $l \geq 3$, let*

$$\Lambda_l = \Lambda_l(r,s) = -2\sum_{k=1}^{l-1}\frac{1}{k(l-k)}\left(\mathbb{G}_k^{(r+1)}\mathbb{B}_{l-k-1}^{(s)} + \mathbb{B}_{k-1}^{(r)}\mathbb{G}_{l-k}^{(s+1)} - 2\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{l-k-1}^{(s)}\right).$$

Assume that $\Lambda_m = 0$, for an integer $m \geq 3$. Then we have the following

(a) $\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle)$ *has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle)\\
=&\frac{1}{m}\Lambda_{m+1} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m}\sum_{j=1}^{m-2}\frac{(m)_j}{(2\pi in)^j}\Lambda_{m-j+1}\right)e^{2\pi inx},
\end{aligned}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

(b)

$$\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(\langle x\rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x\rangle) = \frac{1}{m}\sum_{\substack{j=0\\ j\neq 1}}^{m-2}\binom{m}{j}\Lambda_{m-j+1}B_j(\langle x\rangle),$$

for all $x \in \mathbb{R}$.

Assume next that m is an integer ≥ 3 with $\wedge_m \neq 0$. Then $\gamma_m(0) \neq \gamma_m(1)$, and hence $\gamma_m(\langle x \rangle)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\gamma_m(\langle x \rangle)$ converges pointwise to $\gamma_m(\langle x \rangle)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\gamma_m(0) + \gamma_m(1)\right) = \gamma_m(0) + \frac{1}{2}\Lambda_m.$$

We can now state our second theorem.

Theorem 4.2. *For each integer $l \geq 3$, let*

$$\Lambda_l = \Lambda_l(r,s) = -2\sum_{k=1}^{l-1} \frac{1}{k(l-k)}\left(\mathbb{G}_k^{(r+1)}\mathbb{B}_{l-k-1}^{(s)} + \mathbb{B}_{k-1}^{(r)}\mathbb{G}_{l-k}^{(s+1)} - 2\mathbb{B}_{k-1}^{(r)}\mathbb{B}_{l-k-1}^{(s)}\right).$$

Assume that $\Lambda_m \neq 0$, for an integer $m \geq 3$. Then we have the following

(a)

$$\frac{1}{m}\Lambda_{m+1} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty}\left(-\frac{1}{m}\sum_{j=1}^{m-2}\frac{(m)_j}{(2\pi i n)^j}\Lambda_{m-j+1}\right)e^{2\pi i n x}$$

$$= \begin{cases} \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(\langle x \rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x \rangle), & \text{for } x \notin \mathbb{Z}, \\ \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}\mathbb{G}_{m-k}^{(s+1)} + \frac{1}{2}\Lambda_m, & \text{for } x \in \mathbb{Z}. \end{cases}$$

(b)

$$\frac{1}{m}\sum_{j=0}^{m-2}\binom{m}{j}\Lambda_{m-j+1}B_j(\langle x \rangle)$$

$$= \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}(\langle x \rangle)\mathbb{G}_{m-k}^{(s+1)}(\langle x \rangle), \text{ for } x \notin \mathbb{Z};$$

$$\frac{1}{m}\sum_{\substack{j=0 \\ j \neq 1}}^{m-2}\binom{m}{j}\Lambda_{m-j+1}B_j(\langle x \rangle)$$

$$= \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbb{G}_k^{(r+1)}\mathbb{G}_{m-k}^{(s+1)} + \frac{1}{2}\Lambda_m, \text{ for } x \in \mathbb{Z}.$$

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Hesitant fuzzy normal subalgebras in B-algebras

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Abstract. The notions of a hesitant fuzzy subalgebra and a hesitant fuzzy normal subalgebra of a B-algebra are introduced, and related properties are investigated. A quotient structure of a B-algebra using a hesitant fuzzy normal subalgebra is constructed. The fundamental homomorphism of a quotient B-algebra is established.

1. Introduction

The notions of Atanassov's intuitionistic fuzzy sets, type 2 fuzzy sets and fuzzy multisets etc. are a generalization of fuzzy sets. As another generalization of fuzzy sets, Torra [9] introduced the notion of hesitant fuzzy sets which are a very useful to express peoples hesitancy in daily life. The hesitant fuzzy set is a very useful tool to deal with uncertainty, which can be accurately and perfectly described in terms of the opinions of decision makers. Also, hesitant fuzzy set theory is used in decision making problem etc. [2, 3, 10, 11], and is applied to MTL-algebras [5]. On the while, J. Neggers and H. S. Kim [7] introduced the notion of B-algebra and investigated several properties. Y. B. Jun et al. [4] defined the notion of a fuzzy B-algebra and studied some related properties of it.

In this paper, we discuss applications of a hesitant fuzzy set in a (normal) subalgebra of a B-algebra. We introduce the notion of hesitant fuzzy (normal) subalgebra of a B-algebra, and investigate some properties of it. Also we consider a new construction of a quotient B-algebra induced by a hesitant fuzzy normal subalgebra. Finally, we establish the fundamental homomorphism of B-algebra.

2. Preliminaries

A *B-algebra* ([7]) is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying axioms:

(B1) $x * x = 0$,
(B2) $x * 0 = x$,
(B) $(x * y) * z = x * (z * (0 * y))$

[0]**2010 Mathematics Subject Classification**: 06F35, 03G25, 06D72.
[0]**Keywords**: γ-inclusive set; hesitant fuzzy (normal) subalgebra; B-algebra.
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[0]This study was supported by Gangneung-Wonju National University.

for any x, y, z in X. For brevity we call X a *B-algebra*. In X we can define a binary relation "$\leq$" by $x \leq y$ if and only if $x * y = 0$.

Proposition 2.1.([1, 7]) *Let $(X; *, 0)$ be a B-algebra. Then*

(i) *the left cancellation law holds in X, i.e., $x * y = x * z$ implies $y = z$,*
(ii) *if $x * y = 0$, then $x = y$ for any $x, y \in X$,*
(iii) *if $0 * x = 0 * y$, then $x = y$ for any $x, y \in X$,*
(iv) *$0 * (0 * x) = x$, for all $x \in X$,*
(v) *$x * (y * z) = (x * (0 * z)) * y$ for all $x, y, z \in X$.*

Let $(X; *_X, 0_X)$ and $(Y; *_Y, 0_Y)$ be B-algebras. A mapping $\varphi : X \to Y$ is called a *homomorphism* if $\varphi(x *_X y) = \varphi(x) *_Y \varphi(y)$ for any $x, y \in X$. A homomorphism $\varphi : X \to Y$ is called an *isomorphism* if φ is a bijection, and denote it by $X \cong Y$. Let $\varphi : X \to Y$ be a homomorphism. Then the subset $\{x \in X | \varphi(x) = 0_Y\}$ of X is called the *kernel* of the homomorphism φ, and denote it by $Ker\ \varphi$. A non-empty subset S of X is called a *subalgebra* of X if $x * y \in S$ for any $x, y \in X$.

A non-empty subset N of X is said to be *normal* if $(x * a) * (y * b) \in N$ for any $x * y, a * b \in N$. Then any normal subset N of a B-algebra X is a subalgebra of X, but the converse need not be true ([8]). A non-empty subset X of a B-algebra X is a called a *normal subalgebra* of X if it is both a subalgebra and normal.

Let X be a B-algebra and let N be a normal subalgebra of X. Define a relation $\sim_N$ on X by $x \sim_N y$ if and only if $x * y \in N$, where $x, y \in X$. Then it is a congruence relation on X ([13]). Denote the equivalence class containing x by $[x]_N$, i.e., $[x]_N := \{y \in X | x \sim_N y\}$ and let $X/N := \{[x]_N | x \in X\}$.

Theorem 2.2.([8]) *Let N be a normal subalgebra of a BG-algebra X. Then X/N is a B-algebra.*

The B-algebra X/N is discussed in Theorem 2.2 is called the *quotient B-algebra* of X by N.

Theorem 2.3.([8]) *Let N be a normal subalgebra of a B-algebra X. Then the mapping $\gamma : X \to X/N$ given by $\gamma(x) := [x]_N$ is a surjective homomorphism, and $Ker\gamma = N$.*

Theorem 2.4.([8]) *Let $\varphi : X \to Y$ be a homomorphism of B-algebras. Then $Ker\varphi$ is a normal subalgebra of X.*

Theorem 2.5.([8]) *Let $\varphi : X \to Y$ be a homomorphism of B-algebras. Then $X/Ker\varphi \cong Im\varphi$. In particular, if φ is surjective, then $X/Ker\varphi \cong Y$.*

Definition 2.6.([9]) Let E be a reference set. A *hesitant fuzzy set* on E is defined in terms of a function that when applied to E returns a subset of $[0, 1]$, which can be viewed as the following mathematical representation: $H_E := \{(e, h_E(e)) | e \in E\}$ where $h_E : E \to \mathscr{P}([0, 1])$.

Definition 2.7.([2]) Given a non-empty subset A of a set X, a hesitant fuzzy set $H_X := \{(x, h_X(x))|x \in X\}$ on satisfying the following condition: $h_X(x) = \emptyset$ for all $x \notin A$ (briefly, *A-hesitant fuzzy set*) on X, and is represented by $H_A := \{(x, h_A(x)) \mid x \in X\}$, where h_A is a mapping from X to $\mathscr{P}([0,1])$ with $h_A(x) = \emptyset$ for all $x \notin A$.

For a hesitant fuzzy set $H_X := \{(x, h_X(x)) \mid x \in X\}$ of a set X and a subset γ of $[0,1]$, the hesitant fuzzy γ-inclusive set of H_X, denoted by $H_X(\gamma)$, is defined to be the set $H_X(\gamma) := \{x \in X|\gamma \subseteq h_X(x)\}$. For any hesitant fuzzy set $H_X = \{(x, h_X(x)|x \in X\}$ and $G_X = \{(x, g_X(x))|x \in X\}$, we call H_X a *hesitant fuzzy subset* of G_X, denoted by $H_X \widetilde{\subseteq} G_X$, if $h_X(x) \subseteq g_X(x)$ for all $x \in X$. The *hesitant fuzzy union* of H_X and G_X, denoted by $H_X \widetilde{\cup} G_X$, is defined to be the hesitant fuzzy set $(h_X \widetilde{\cup} g_X)(x) = h_X(x) \cup g_X(x)$ for all $x \in X$. The *hesitant fuzzy intersection* of H_X and G_X, denoted by $H_X \widetilde{\cap} G_X$, is defined to be the hesitant fuzzy set $(h_X \widetilde{\cap} g_X)(x) = h_X(x) \cap g_X(x)$ for all $x \in X$.

3. Hesitant fuzzy normal subalgebra

In what follows let X denote a B-algebra X unless otherwise specified.

Definition 3.1. Let X be a B-algebra. Given a non-empty subset (subalgebra as much as possible) A of X, let $H_A := \{(x, h_A(x)) \mid x \in X\}$ be an A-hesitant fuzzy set on X. Then $H_A := \{(x, h_A(x)) \mid x \in X\}$ is called a hesitant fuzzy subalgebra of X related to A (briefly, A-hesitant fuzzy subalgebra of X) if it satisfies the following condition:

(3.1) $h_A(x) \cap h_A(y) \subseteq h_A(x * y)$ for all $x, y \in A$.

An A-hesitant fuzzy subalgebra of X with $A = X$ is called a hesitant fuzzy subalgebra of X.

Proposition 3.2. *Every hesitant fuzzy subalgebra $H_X := \{(x, h_X(x))|x \in X\}$ of a B-algebra X satisfies the following inclusion:*

(3.2) $h_X(x) \subseteq h_X(0)$ *for all* $x \in X$.

Proof. Using (3.1) and (B1), we have $h_X(x) = h_X(x) \cap h_X(x) \subseteq h_X(x * x) = h_X(0)$ for all $x \in X$. □

Example 3.3. Let $X = \{0, 1, 2, 3\}$ is a B-algebra ([6]) with the following Cayley table:

$*$	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Let $H_X := \{(x, h_X(x))|x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \left\{(0, [0,1]), (1, (\tfrac{3}{8}, \tfrac{5}{8})), (2, (\tfrac{3}{8}, \tfrac{5}{8}), (3, (\tfrac{1}{4}, \tfrac{3}{4})))\right\}.$$

It is easy to verify that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy subalgebra of X.

Theorem 3.4. *A hesitant fuzzy set $H_X := \{(x, h_X(x))|x \in X\}$ of a B-algebra is a hesitant fuzzy subalgebra of X if and only if $H_X(\gamma) := \{x \in X|\gamma \subseteq h_X(x)\}$ is a subalgebra of X for all $\gamma \in \mathscr{P}([0,1])$ whenever it is non-empty.*

Proof. Assume that $H_X := \{(x, h_X(x))|x \in X\}$ is a hesitant fuzzy subalgebra of X. Let $x, y \in X$ and $\gamma \in \mathscr{P}([0,1])$ be such that $x, y \in H_X(\gamma)$. Then $\gamma \subseteq h_X(x)$ and $\gamma \subseteq h_X(y)$. It follows from (3.1) that $\gamma \subseteq h_X(x) \cap h_X(y) \subseteq h_X(x * y)$ Hence $x * y \in h_X(\gamma)$. Thus $H_X(\gamma)$ is a subalgebra of X.

Conversely, suppose that $H_X(\gamma)$ is a subalgebra X for all $\gamma \in \mathscr{P}([0,1])$ with $H_X(\gamma) \neq \emptyset$. Let $x, y \in X$, be such that $h_X(x) = \gamma_x$ and $h_X(y) = \gamma_y$. Take $\gamma = \gamma_x \cap \gamma_y$. Then $x, y \in H_X(\gamma)$ and so $x * y \in H_X(\gamma)$ by assumption. Hence $h_X(x) \cap h_X(y) = \gamma_x \cap \gamma_y = \gamma \subseteq h_X(x * y)$. Thus $H_X := \{(x, h_X(x))|x \in X\}$ is a hesitant fuzzy subalgebra of X. □

Theorem 3.5. *Every subalgebra of a B-algebra can be represented as a γ-inclusive set of a hesitant fuzzy subalgebra.*

Proof. Let A be a subalgebra of a B-algebra X. For a subset γ of $[0,1]$, define a hesitant fuzzy set H_X on X by

$$h_X : X \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \gamma & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$$

Obviously, $A = H_X(\gamma)$. We now prove that H_X is a hesitant fuzzy subalgebra of X. Let $x, y \in X$. If $x, y \in A$, then $x*y \in A$ because A is a subalgebra of X. Hence $h_X(x) = h_X(y) = h_X(x*y) = \gamma$, and so $h_X(x) \cap h_X(y) \subseteq h_X(x * y)$. If $x \in A$ and $y \notin A$, then $h_X(x) = \gamma$ and $h_X(y) = \emptyset$ which imply that $h_X(x) \cap h_X(y) = \gamma \cap \emptyset = \emptyset \subseteq h_X(x * y)$. Similarly, if $x \notin A$ and $y \in A$, then $h_X(x) \cap h_X(y) \subseteq h_X(x * y)$. Obviously, if $x \notin A$ and $y \notin A$, then $h_X(x) \cap h_X(y) \subseteq h_X(x * y)$. Therefore H_X is a hesitant fuzzy subalgebra of X. □

Any subalgebra of a B-algebra X may not be represented as a γ-inclusive set of a hesitant fuzzy subalgebra of X in general (see Example 3.6).

Example 3.6. Let $X = \{0, 1, 2, 3\}$ be a B-algebra with the following Cayley table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Let $H_X := \{(x, h_X(x))|x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \left\{(0, [0,1]), (1, (\tfrac{3}{7}, \tfrac{5}{7})), (2, (\tfrac{3}{7}, \tfrac{5}{7}), (3, (\tfrac{3}{7}, \tfrac{5}{7})))\right\}.$$

It is easy to verify that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is a hesitant fuzzy subalgebra of X. The γ-inclusive set of H_X are described as follows:

$$H_X(\gamma) = \begin{cases} \{0\} & \text{if } \gamma \in \{[0,1]\} \\ X & \text{if } \gamma \in \{S|\emptyset \subseteq S \subseteq (\frac{3}{7}, \frac{5}{7})\} \\ \emptyset & \text{otherwise.} \end{cases}$$

The subalgebra $\{0, 1\}$ cannot be a γ-inclusive set $H_X(\gamma)$ since there is no $\gamma \subseteq [0, 1]$ such that $H_X(\gamma) = \{0, 1\}$.

Definition 3.7. A hesitant fuzzy set $H_X := \{(x, h_X)|x \in X\}$ on a B-algebra X is said to be *hesitant fuzzy normal* if it satisfies:

(3.3) $h_X(x * y) \cap h_X(a * b) \subseteq h_X((x * a) * (y * b))$ for all $x, y, a, b \in X$.

A hesitant fuzzy set H_X on a B-algebra X is called a *hesitant fuzzy normal subalgebra* of X if it satisfies (3.1) and (3.3).

Example 3.8. Let $X = \{0, 1, 2, 3\}$ be a B-algebra as in Example 3.3. Let $H_X := \{(x, h_X)|x \in X\}$ be a hesitant fuzzy set on X defined by

$$H_X = \left\{(0, [0, 1]), (1, (\tfrac{1}{4}, \tfrac{3}{4})), (2, (\tfrac{1}{4}, \tfrac{3}{4})), (3, [0, 1])\right\}.$$

It is easy to verify that $H_X := \{(x, h_X(x)) \mid x \in X\}$ is hesitant fuzzy normal.

Proposition 3.9. *Every hesitant fuzzy normal H_X of a B-algebra X is a hesitant fuzzy subalgebra of X.*

Proof. Put $y := 0, b := 0$ and $a := y$ in (3.3). Then $h_X(x*0) \cap h_X(y*0) \subseteq h_X((x*y)*(0*0))$ for any $x, y \in X$. Using (B2) and (B1), we have $h_X(x) \cap h_X(y) \subseteq h_X(x*y)$. Hence H_X is a hesitant fuzzy subalgebra of X. $\square$

The converse of Proposition 3.9 may not be true in general (see Example 3.10).

Example 3.10. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a B-algebra ([8]) with the following table:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Let H_X be a hesitant fuzzy set defined by

$$H_X = \{(0, \gamma_3), (1, \gamma_1), (2, \gamma_1), (3, \gamma_1), (4, \gamma_1), (5, \gamma_2)\}.$$

where γ_1, γ_2 and γ_3 are subsets of $[0,1]$ with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3$. It is easy to check that H_X is a hesitant fuzzy subalgebra of X. But it is not hesitant fuzzy normal since $h_X(1*4) \cap h_X(3*2) = h_X(5) \cap h_X(5) = \gamma_2 \nsubseteq \gamma_1 = h_X(1) = h_X((1*3)*(4*2))$.

Theorem 3.11. *A hesitant fuzzy set $H_X := \{(x, h_X(x)) | x \in X\}$ of a B-algebra is a hesitant fuzzy normal subalgebra of X if and only if $H_X(\gamma) := \{x \in X | \gamma \subseteq h_X(x)\}$ is a normal subalgebra of X for all $\gamma \in \mathscr{P}([0,1])$ whenever it is non-empty.*

Proof. Similar to Theorem 3.4. □

Proposition 3.12. *Let a hesitant fuzzy set H_X of a B-algebra X be hesitant fuzzy normal. Then $h_X(x*y) = h_X(y*x)$ for any $x, y \in X$.*

Proof. Let $x, y \in X$. By (B1) and (B2), we have $h_X(x*y) = h_X((x*y)*(x*x)) \supseteq h_X(x*x) \cap h_X(y*x) = h_X(0) \cap h_X(y*x) = h_X(y*x)$. Interchanging x with y, we obtain $h_X(y*x) \supseteq h_X(x*y)$, which proves the proposition. □

Theorem 3.13. *Let $H_X := \{(x, h_x(x)) | x \in X\}$ be a hesitant fuzzy normal subalgebra of a B-algebra X. Then the set $X_{h_X} = \{x \in X | h_X(x) = h_X(0)\}$ is a normal subalgebra of X.*

Proof. It is sufficient to show that X_{h_X} is normal. Let $a, b, x, y \in X$ be such that $x*y \in X_{h_X}$ and $a*b \in X_{h_X}$. Then $h_X(x*y) = h_X(0) = h_X(a*b)$. Since H_X is a hesitant fuzzy normal subalgebra of X, it follows that $h_X((x*a)*(y*b)) \supseteq h_X(x*y) \cap h_X(a*b) = h_X(0)$. Using (3.2), we conclude that $h_X((x*a)*(y*b)) = h_X(0)$. Hence $(x*a)*(y*b) \in X_{h_X}$. This completes the proof. □

Theorem 3.14. *The intersection of any set of a hesitant fuzzy normal subalgebra of a B-algebra X is also a hesitant fuzzy normal subalgebra.*

Proof. Let $\{(H_X)_\alpha | \alpha \in \Lambda\}$ be a family of hesitant fuzzy normal subalgebras of a B-algebra X and let $a, b, x, y \in X$. Then

$$\begin{aligned}
\cap_{\alpha\in\Lambda}(h_X)_\alpha((x*a)*(y*b)) &= \inf_{\alpha\in\Lambda}(h_X)_\alpha((x*a)*(y*b)) \\
&\geq \inf_{\alpha\in\Lambda}\{(h_X)_\alpha(x*y) \cap (h_X)_\alpha(a*b)\} \\
&=[\inf_{\alpha\in\Lambda}(h_X)_\alpha(x*y)] \cap [\inf_{\alpha\in\Lambda}(h_X)_\alpha(a*b)] \\
&=((\cap_{\alpha\in\Lambda}(h_X)_\alpha)(x*y)) \cap ((\cap_{\alpha\in\Lambda}(h_X)_\alpha)(a*b))
\end{aligned}$$

which shows that $\cap_{\alpha\in\Lambda}(H_X)_\alpha$ is hesitant fuzzy normal. By Proposition 3.9, $\cap_{\alpha\in\Lambda}(H_X)_\alpha$ is an int-soft normal subalgebra of X. □

The union of any set of hesitant fuzzy normal subalgebra of a B-algebra X need not be a hesitant fuzzy normal subalgebra of X.

Example 3.15. Let $X := \{0, 1, 2, 3, 4, 5\}$ be a B-algebra as in Example 3.10. Let $H_X := \{(x, h_X(x))|x \in X\}$ and $G_X := \{(x, g_X(x))|x \in X\}$ be hesitant fuzzy sets of X defined as follows:

$$h_X : X \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \gamma_3 & \text{if } x \in \{0,4\} \\ \gamma_1 & \text{if } x \in \{1,2,3,5\} \end{cases}$$

$$g_X : X \to \mathscr{P}([0,1]),\ x \mapsto \begin{cases} \gamma_3 & \text{if } x \in \{0,5\} \\ \gamma_2 & \text{if } x \in \{1,2,3,4\} \end{cases}$$

where $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3 \subseteq [0,1]$. It is easy to check that H_X and G_X are hesitant fuzzy subalgebras of X. But $H_X \cup G_X$ is not a hesitant fuzzy subalgebra of X because

$$\begin{aligned}(h_X \cup g_X)(4) \cap (h_X \cup g_X)(5) =& (h_X(4) \cup g_X(4)) \cap (h_X(5) \cup g_X(5)) \\ =& (\gamma_3 \cup \gamma_2) \cap (\gamma_1 \cup \gamma_3) = \gamma_3 \\ \not\subseteq& \gamma_2 = \gamma_1 \cup \gamma_2 = h_X(2) \cup g_X(2) \\ =& (h_X \cup g_X)(2) = (h_X \cup g_X)(4 * 5).\end{aligned}$$

Since every hesitant fuzzy normal subalgebra of a B-algebra X is a hesitant fuzzy subalgebra of X, the union of hesitant fuzzy normal subalgebra need not be a hesitant fuzzy normal subalgebra of a B-algebra.

4. Quotient B-algebras induced by a hesitant fuzzy normal subalgebra

Let $H_X := \{(x, h_X(x))|x \in X\}$ be a hesitant fuzzy normal subalgebra of a B-algebra X. For any $x, y \in X$, we define a binary operation " $\sim^{h_X}$ " on X as follows: $x \sim^{h_X} y \Leftrightarrow h_X(x*y) = h_X(0)$.

Lemma 4.1. *The operation $\sim^{h_X}$ is an equivalence relation on a B-algebra X.*

Proof. Obviously, it is reflexive. Let $x \sim^{h_X} y$. Then $h_X(x*y) = h_X(0)$. It follows from Proposition 3.12 that $h_X(0) = h_X(x * y) = h_X(y * x)$. Hence $\sim^{h_X}$ is symmetric. Let $x, y, z \in X$ be such that $x \sim^{h_X} y$ and $y \sim^{h_X} z$. Then $h_X(x * y) = h_X(0)$ and $h_X(y * z) = h_X(0)$. Using Proposition 3.12, (3.3), (B1), (B2) and (3.2), we have $h_X(0) = h_X(x * y) \cap h_X(y * z) = h_X(x * y) \cap h_X(z * y) \subseteq h_X((x * z) * (y * y)) = h_X((x * z) * 0) = h_X(x * z) \subseteq h_X(0)$. Hence $h_X(x * z) = h_X(0)$, i.e., $\sim^{h_X}$ is transitive. Therefore " $\sim^{h_X}$ " is an equivalence relation on X. □

Lemma 4.2. *For any $x, y, p, q \in X$, if $x \sim^{h_X} y$ and $p \sim^{h_X} q$, then $x * p \sim^{h_X} y * q$.*

Proof. Let $x, y, p, q \in X$ be such that $x \sim^{h_X} y$ and $p \sim^{h_X} q$. Then $h_X(x * y) = h_X(y * x) = h_X(0)$ and $h_X(p*q) = h_X(q*p) = h_X(0)$. Using (3.3) and (3.2), we have $h_X(0) = h_X(x*y) \cap h_X(p*q) \subseteq h_X((x * p) * (y * q)) \subseteq h_X(0)$. Hence $h_X((x * p) * (y * q)) = h_X(0)$. By similar way, we get $h_X((y * q) * (x * p)) = h_X(0)$. Therefore $x * p \sim^{h_X} y * q$. Thus " $\sim^{h_X}$ " is a congruence relation on X. □

Denote $(h_X)_x$ and X/h_X the equivalence class containing x and the set of all equivalence classes of X, respectively, i.e., $(h_X)_x := \{y \in X | y \sim^{h_X} x\}$ and $X/h_X := \{(h_X)_x | x \in X\}$. Define a binary relation $\bullet$ on X/h_X as follows: $(h_X)_x \bullet (h_X)_y = (h_X)_{x*y}$ for all $(h_X)_x, (h_X)_y \in X/h_X$. Then this operation is well-defined by Lemma 4.2.

Theorem 4.3. *If $H_X := \{(x, h_X(x)) | x \in X\}$ is a hesitant fuzzy normal subalgebra of a B-algebra X, then the quotient algebra $X/h_X := (X/h_X, \bullet, (h_X)_0)$ is a B-algebra.*

Proof. Straightforward. □

Proposition 4.4. *Let $\mu : X \to Y$ be a homomorphism of B-algebras. If $H_Y := \{(y, h_Y(y)) | y \in Y\}$ is a hesitant fuzzy normal subalgebra of Y, then $(h_Y \circ \mu, X)$ is a hesitant fuzzy normal subalgebra of X.*

Proof. For any $x, y, a, b \in X$, we have

$$\begin{aligned}(h_Y \circ \mu)((x *_X a) *_X (y *_X b)) =& h_Y(\mu((x *_X a) *_X (y *_X b)) \\ =& h_X((\mu(x) *_Y \mu(a)) *_Y (\mu(y) *_Y \mu(b))) \\ \supseteq& h_Y(\mu(x) *_Y \mu(y)) \cap h_Y(\mu(a) *_Y \mu(b)) \\ =& h_Y(\mu(x *_X y)) \cap h_Y(\mu(a *_X b)) \\ =& (h_Y \circ \mu)(x *_X y) \cap (h_Y \circ \mu)(a *_X b).\end{aligned}$$

Hence $h_Y \circ \mu$ is hesitant fuzzy normal. By Proposition 3.9, $(h_Y \circ \mu, X)$ is a hesitant fuzzy normal subalgebra of X. □

Proposition 4.5. *Let H_X be a hesitant fuzzy normal subalgebra of a B-algebra X. The mapping $\gamma : X \to X/h_X$, given by $\gamma(x) := (h_X)_x$, is a surjective homomorphism, and $Ker\gamma = \{x \in X | \gamma(x) = (h_X)_0\} = X_{h_X}$.*

Proof. Let $(h_X)_x \in X/h_X$. Then there exists an element $x \in X$ such that $\gamma(x) = (h_X)_x$. Hence γ is surjective. For any $x, y \in X$, we have $\gamma(x * y) = (h_X)_{x*y} = (h_X)_x \bullet (h_X)_y = \gamma(x) \bullet \gamma(y)$. Thus γ is a homomorphism. Moreover, $Ker\ \gamma = \{x \in X | \gamma(x) = (h_X)_0\} = \{x \in X | x \sim^{h_X} 0\} = \{x \in X | h_X(x) = h_X(0)\} = X_{h_X}$. □

Example 4.6. Let $X = \{0, 1, 2, 3\}$ be a B-algebra ([4]) with the following Cayley table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Let H_X be a hesitant fuzzy set defined by

$$h_X : X \to \mathscr{P}(U),\ x \mapsto \begin{cases} \gamma_2 & \text{if } x \in \{0,2\} \\ \gamma_1 & \text{if } x \in \{1,3\} \end{cases}$$

where $\gamma_1 \subsetneq \gamma_2 \subseteq [0,1]$. It is easy to check that H_X is a hesitant fuzzy normal subalgebras of X. Then $X_{h_X} = \{x \in X | h_X(x) = h_X(0)\} = \{0,2\}$. Define $x \sim^{h_X} y$ if and only if $h_X(x * y) = h_X(0)$. Then $(h_X)_0 = \{x \in X | x \sim^{h_X} 0\} = \{x \in X | h_X(x * 0) = h_X(0)\} = \{0,2\}$ and $(h_X)_1 = \{x \in X | x \sim^{h_X} 1\} = \{x \in X | h_X(x * 1) = h_X(0)\} = \{1,3\}$ Hence $X/h_X = \{(h_X)_0, (h_X)_1\}$. Let $\varphi : X \to X/h_X$ be a map defined by $\varphi(0) = \varphi(2) = (h_X)_0$ and $\varphi(1) = \varphi(3) = (h_X)_1$. It is easy to check that φ is a homomorphism and $Ker\varphi = \{x \in X | \varphi(x) = (h_X)_0\} = \{x \in X | x \sim^{h_X} 0\} = \{x \in X | h_X(x) = h_X(0)\} = X_{h_X}$.

Theorem 4.7. *Let $X := (X; *_X, 0_X)$ be a B-algebra and $Y := (Y; *_Y, 0_Y)$ be a B-algebra and let $\mu : X \to Y$ be an epimorphism. If $H_Y := \{(y, h_Y) | y \in Y\}$ is a hesitant fuzzy normal subalgebra of Y, then the quotient algebra $X/(h_Y \circ \mu) := (X/(h_Y \circ \mu), \bullet_X, (h_Y \circ \mu)_{0_X})$ is isomorphic to the quotient algebra $Y/h_X := (Y/h_Y, \bullet_Y, (h_Y)_{0_Y})$.*

Proof. By Theorem 4.3 and Proposition 4.4, $X/h_Y \circ \mu : (X/(h_Y \circ \mu), \bullet_X, (h_Y \circ \mu)_{0_X})$ is a B-algebra and $Y/h_Y := (Y/h_X, \bullet_Y, (h_Y)_{0_Y})$ is a B-algebra. Define a map

$$\eta : X/(h_Y \circ \mu) \to Y/h_Y,\ (h_Y \circ \mu)_x \mapsto (h_Y)_{\mu(x)}$$

for all $x \in X$. Then the function η is well-defined. In fact, assume that $(h_Y \circ \mu)_x = (h_Y \circ \mu)_y$ for all $x, y \in X$. Then we have $h_Y(\mu(x) *_Y \mu(y)) = h_Y(\mu(x *_X y)) = (h_Y \circ \mu)(x *_X y) = (h_Y \circ \mu)(0_X) = h_Y(\mu(0_X)) = h_Y(0_Y)$. Hence $(h_Y)_{\mu(x)} = (h_Y)_{\mu(y)}$.

For any $(h_Y \circ \mu)_x, (h_Y \circ \mu)_y \in X/(h_Y \circ \mu)$, we have $\eta((h_Y \circ \mu)_x \bullet_X (h_Y \circ \mu)_y) = \eta((h_Y \circ \mu)_{x*y}) = (h_Y)_{\mu(x *_X y)} = (h_Y)_{\mu(x) *_Y \mu(y)} = (h_Y)_{\mu(x)} \bullet (h_Y)_{\mu(y)} = \eta((h_Y \circ \mu)_x) \bullet_Y \eta((h_Y \circ \mu)_y)$. Therefore η is a homomorphism.

Let $(h_Y)_a \in Y/h_Y$. Then there exists $x \in X$ such that $\mu(x) = a$ since μ is surjective. Hence $\eta((h_X \circ \mu)_x) = (h_Y)_{\mu(x)} = (h_Y)_a$ and so η is surjective.

Let $x, y \in X$ be such that $(h_Y)_{\mu(x)} = (h_Y)_{\mu(y)}$. Then we have $(h_Y \circ \mu)(x *_X y) = h_Y(\mu(x *_X y)) = h_Y(\mu(x) *_Y \mu(y)) = h_Y(0_Y) = h_Y(\mu(0_X)) = (h_Y \circ \mu)(0_X)$. It follows that $(h_Y \circ \mu)_x = (h_Y \circ \mu)_y$. Thus η is injective. □

The homomorphism $\pi : X \to X/h_X,\ x \to (h_X)_x$, is called the *natural homomorphism* of X onto X/h_X. In Theorem 4.7, if we define natural homomorphisms $\pi_X : X \to X/h_Y \circ \mu$ and $\pi_Y : Y \to Y/h_Y$ then it is easy to show that $\eta \circ \pi_X = \pi_Y \circ \mu$, i.e., the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\mu} & Y \\ {\scriptstyle \pi_X}\Big\downarrow & & {\scriptstyle \pi_Y}\Big\downarrow \\ X/(h_Y \circ \mu) & \xrightarrow{\eta} & Y/h_Y. \end{array}$$

Proposition 4.8. *Let H_X be a hesitant fuzzy normal subalgebra of a B-algebras X. If J is a normal subalgebra of X, then J/h_X is a normal subalgebra of X/h_X.*

Proof. Let H_X be a hesitant fuzzy normal subalgebra of a B-algebras X and let J be a normal subalgebra of X. Then for any $x, y \in J$, $x*y \in J$. Let $(h_X)_x, (h_X)_y \in J/h_X$. Then $(h_X)_x \bullet (h_X)_y = (h_X)_{x*y} \in J/h_X$. Hence $J/h_X = \{(h_X)_x | x \in J\}$ is a subalgebra of X/h_X.

For any $x * y, a * b \in J$, $(x * a) * (y * b) \in J$, so for any $(h_X)_x \bullet (h_X)_y, (h_X)_a \bullet (h_X)_b \in J/h_X$, we have $((h_X)_x \bullet (h_X)_a) \bullet ((h_X)_y \bullet (h_X)_b) = (h_X)_{x*a} \bullet (h_X)_{y*b} = (h_X)_{(x*a)*(y*b)} \in J/h_X$. Thus J/h_X is a normal subalgebra of X/h_X. □

Theorem 4.9. *Let H_X be a hesitant fuzzy normal subalgebra of a B-algebras X. If J^* is a normal subalgebra of a B-algebra X/h_X, then there exists a normal subalgebra $J = \{x \in X | (h_X)_x \in J^*\}$ in X such that $J/h_X = J^*$.*

Proof. Since J^* is a normal subalgebra of X/h_X, so $(h_X)_x \bullet (h_X)_y = (h_X)_{x*y} \in J^*$ for any $(h_X)_x, (h_X)_y \in J^*$. Thus $x * y \in J$ for any $x, y \in J$. And $(h_X)_{x*a} \bullet (h_X)_{y*b} = (h_X)_{(x*a)*(y*b)} \in J^*$ for any $(h_X)_{x*y}, (h_X)_{a*b} \in J^*$. Thus $(x * a) * (y * b) \in J$ for any $x * y, a * b \in J$. Therefore J is a normal subalgebra of X. By Proposition 4.5, we have

$$\begin{aligned} J/h_X =& \{(h_X)_j | j \in J\} \\ =& \{(h_X)_j | \exists (h_X)_x \in J^* \text{ such that } j \sim^{h_X} x\} \\ =& \{(h_X)_j | \exists (h_X)_x \in J^* \text{ such that } (h_X)_x = (h_X)_j\} \\ =& \{(h_X)_j | (h_X)_j \in J^*\} = J^*. \end{aligned}$$

Theorem 4.10. *Let H_X be a hesitant fuzzy normal subalgebra of a B-algebra X. If J is a normal subalgebra of X, then $\dfrac{X/h_X}{J/h_X} \cong X/J$.*

Proof. Note that $\dfrac{X/h_X}{J/h_X} = \{[(h_X)_x]_{J/h_X} | h_X \in X/h_X\}$. If we define $\varphi : \dfrac{X/h_X}{J/h_X} \to X/J$ by $\varphi([(h_X)_x]_{J/h_X}) = [x]_J = \{y \in X | x \sim^J y\}$, then it is well defined. In fact, suppose that $[(h_X)_x]_{J/h_X} = [(h_X)_y]_{J/h_X}$. Then $(h_X)_x \sim^{J/h_X} (h_X)_y$ and so $(h_X)_{x*y} = (h_X)_x \bullet (h_X)_y \in J/h_X$. Hence $x * y \in J$. Therefore $x \sim^J y$, i.e., $[x]_J = [y]_J$. Given $[(h_X)_x]_{J/h_X}, [(h_X)_y]_{J/h_X} \in \dfrac{X/h_X}{J/h_X}$, we have $\varphi([(h_X)_x]_{J/h_X} \bullet [(h_X)_y]_{J/h_X}) = \varphi([(h_X)_x \bullet (h_X)_y]_{J/h_X}) = [x * y]_J = [x]_J * [y]_J$
$= \varphi([(h_X)_x]_{J/h_X}) * \varphi([(h_X)_y]_{J/h_X})$. Hence φ is a homomorphism.

Obviously, φ is onto. Finally, we show that φ is one-to-one. If $\varphi([(h_X)_x]_{J/h_X}) = \varphi([(h_X)_y]_{J/h_X})$, then $[x]_J = [y]_J$, i.e., $x \sim^J y$. If $(h_X)_a \in [(h_X)_x]_{J/h_X}$, then $(h_X)_a \sim^{J/h_X} (h_X)_x$ and hence $(h_X)_{a*x} \in J/h_X$. It follows that $a * x \in J$, i.e., $a \sim^J x$. Since $\sim^J$ is an equivalence relation, $a \sim^J y$ and so $J_a = J_y$. Hence $a * y \in J$ and so $(h_X)_{a*y} \in J/h_X$. Therefore $(h_X)_a \sim^{J/h_X} (h_X)_y$. Hence $(h_X)_a \in [(h_X)_y]_{J/h_X}$. Thus $[(h_X)_x]_{J/h_X} \subseteq [(h_X)_y]_{J/h_X}$. Similarly, we obtain $[(h_X)_y]_{J/h_X} \subseteq [(h_X)_x]_{J/h_X}$. Therefore $[(h_X)_x]_{J/h_X} = [(h_X)_y]_{J/h_X}$. This completes the proof. $\square$

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IMPULSIVE PERIODIC SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH SINGULARITY

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ABSTRACT. In this paper, we study the impulsive periodic solutions of second order singular ordinary differential equations. The proof of the main result relies on a nonlinear alternative principle of Leray-Schauder, together with a truncation technique and the result is applicable to the case of a strong singularity as well as the case of a weak singularity.

1. INTRODUCTION

Impulsive effects occur widely in many evolution processes in which their states are changed abruptly at certain moments of time, for example, in population biology, the radiation of electromagnetic waves, the spread of heat, the diffusion of chemicals, the maintenance of a species through instantaneous stocking, harvesting. The impulsive diffferential equation is also an adequate apparatus for the mathematical simulation of such processes and phenomena. For the general aspects of impulsive differential equations, we refer the reader to the classical monograph [9].

In this paper, we study the existence of positive solution for the periodic boundary value problem with impulse effects:

$$\begin{cases} x'' + a(t)x = f(t, x), & t \in \mathbb{J}', \\ x(0) - x(T) = x'(0) - x'(T) = 0, \end{cases} \tag{1.1}$$

under the impulse conditions

$$-\Delta x'|_{t=t_k} = I_k(x(t_k)), \quad k = 1, 2, \dots, p, \tag{1.2}$$

where $\mathbb{J} = [0, T], t_1, t_2, \dots, t_p \in \mathbb{J}$ with $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$, $\mathbb{J}' = \mathbb{J} \setminus \{t_1, t_2, \dots, t_p\}$; the nonlinearity $f(t, x)$ is continuous in $(t, x) \in \mathbb{J}' \times \mathbb{R}$, $f(t_k^+, x), f(t_k^-, x)$ exist, $f(t_k^-, x) = f(t_k, x)$ and $T-$periodic in t; $\Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-)$ with $x'(t_k^{\pm}) = \lim\limits_{t \to t_k^{\pm}} x'(t)$; $a(t)$ is continuous, $T-$periodic function; the impulsive $I_k : \mathbb{R} \to \mathbb{R} (k = 1, \dots, p)$ are continuous functions. We are mainly interested in the case that $f(t, x)$ presents a repulsive singularity at $x = 0$, which means that

$$\lim_{x \to 0^+} f(t, x) = +\infty, \text{ uniformly in } t.$$

By an impulsive periodic solution of (1.1), we mean that $x \in PC(\mathbb{J})$ satisfying (1.1). $PC(\mathbb{J})$ denotes the class of the maps $x : \mathbb{J} \to \mathbb{R}$ such that $x(t)$ is continuous at $t \neq t_k$, and left continuous at $t = t_k$, the right limit $x(t_k^+)$ exists for $k = 1, 2, \dots, p$. Note that $PC(\mathbb{J})$ is a Banach space with the norm $\|x\|_{PC} = \sup_{t \in \mathbb{J}} |x(t)|$.

2010 *Mathematics Subject Classification.* 34C25.

Key words and phrases. Periodic solution; Impulse; Singularity; Leray-Schauder alternative principle.

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Impulsive differential equations have been studied by many authors [3, 6, 16, 17, 20, 21, 22]. Some classical tools have been used to study such problems in the literature. These classical techniques include the obtention of a priori bounds for the possible solutions and then the applications of the coincidence degree theory of Mawhin [18], the method of upper and lower solutions with monotone technique [2] and some fixed point theorems [4] and variational methods [23, 24].

On the other hand, singular periodic problems without impulse effects have also been investigated extensively in the literature by variational methods [15], or topological methods [5, 8, 11, 12, 13], which were started with the pioneering paper of Lazer and Solimini [10], in this paper, they proved that a necessary and sufficient condition for the existence of a positive periodic solution for equation

$$x''(t) = \frac{1}{x^{\lambda}} + e(t)$$

is that the mean value of e is negative, $\bar{e} < 0$, here $\lambda \geq 1$, which is a strong force condition in a terminology first introduced by Gordon [7]. Moreover, if $0 < \lambda < 1$, which corresponds to a weak force condition, they found example of functions e with negative mean values and such that periodic solutions do not exist. Since then, the strong force condition became standard in the related works; see, for instance [26, 27]. The study of impulsive singular problems is more recent and the number of references is much smaller [14, 21]. In this paper, we will apply a nonlinear alternative principle of Leray-Schauder to study the impulsive periodic solutions of second-order singular differential equations (1.1) and (1.2). Our main aim is to obtain some new existence results for positive impulsive periodic solutions of the singular problem

$$x''(t) + a(t)x = \frac{1}{x^{\alpha}} + \mu x^{\beta}, \tag{1.3}$$

$$-\Delta x'|_{t=t_k} = c_k x, k = 1, \dots, p,$$

where $\alpha, \beta > 0$ and $\mu \in \mathbb{R}$ is a given parameter. Here we emphasize that new results are applicable to the case of a strong singularity as well as the case of a weak singularity.

The rest of this paper is organized as follows. In Section 2, some preliminary results will be given. In Section 3, we will state and prove the main results. To illustrate the new results, some applications are also given.

2. PRELIMINARIES

Let us consider the linear equation

$$x'' + a(t)x = 0. \tag{2.1}$$

When (2.1) is nonresonant, i.e., its unique T-periodic solution is the trivial one, as a consequence of Fredholm's alternative, the nonhomogeneous equation

$$x'' + a(t)x = h(t) \tag{2.2}$$

admits a unique T-periodic solution which can be written as

$$x(t) = \int_0^T G(t,s)h(s)ds,$$

where $G(t,s)$ is the Green's function of (2.1) associated with periodic boundary conditions

$$x(0) = x(T), \quad x'(0) = x'(T). \tag{2.3}$$

Throughout this paper, we always assume that the following standing hypothesis is satisfied:

(H) $a(t)$ is a continuous T-function and the Green's function of (2.1) is positive for all $(t,s) \in [0,T] \times [0,T]$.

In other words, the strict anti-maximum principle holds for (2.1)-(2.3). In order to guarantee the positivity of $G(t,s)$, it is prove in [25] that if $a(t)$ satisfies $a \succ 0$ then the positivity of $G(t,s)$ is equivalent to

$$\underline{\lambda}_1(a) > 0,$$

where the notation $a \succ 0$ means that $a(t) \geq 0$ for all $t \in [0,T]$ and $a(t) > 0$ for t in a subset of positive measure, $\underline{\lambda}_1(a)$ denotes the first anti-periodic eigenvalue of

$$x'' + (\lambda + a(t))x = 0$$

subject to the anti-periodic boundary conditions

$$x(0) = -x(T), \quad x'(0) = -x'(T).$$

Now we make condition (H) clear. When $a(t) \equiv k^2$, condition (H) is equivalent to saying that $0 < k^2 \leq \lambda_1 = (\pi/T)^2$, where λ_1 is the first eigenvalue of the homogeneous equation $x'' + k^2 x = 0$ with Dirichlet boundary conditions $x(0) = x(T) = 0$. For a non-constant function $a(t)$, there is an L^p-criterion proved in [25]. To describe these, we use $\|\cdot\|_q$ to denote the usual L^q-norm over $(0,T)$ for any given exponent $q \in [1,\infty]$. The conjugate exponent of q is denoted by $p : \frac{1}{p} + \frac{1}{q} = 1$. Let $\mathbf{M}(q)$ denote the best Sobolev constant in the following inequality

$$C\|u\|_q^2 \leq \|u'\|_2^2 \qquad \text{for all } u \in H_0^1(0,T).$$

The explicit formula for $\mathbf{M}(q)$ is

$$\mathbf{M}(q) = \begin{cases} \frac{2\pi}{qT^{1+2/q}} \left(\frac{2}{q+2}\right)^{1-2/q} \left(\frac{\Gamma(1/q)}{\Gamma(1/2+1/q)}\right)^2, & \text{for } 1 \leq q < \infty, \\ \frac{4}{T}, & \text{for } q = \infty, \end{cases}$$

where $\Gamma(\cdot)$ is the Gamma function of Euler.

Lemma 2.1 [25] Assume that $a \succ 0$ and $a \in L^p[0,T]$ for some $1 \leq p \leq +\infty$. If

$$\|a\|_p < \mathbf{M}(2q),$$

then (2.1) satisfies the standing hypothesis (H), i.e, $G(t,s) > 0$ for all $(t,s) \in [0,T] \times [0,T]$.

When $a(t) \equiv k^2$ and $0 < k \leq\leq \pi/T$, we have

$$G(t,s) = \begin{cases} \frac{\sin k(t-s) + \sin k(T-t+s)}{2k(1-\cos kT)}, & 0 \leq s \leq t \leq T, \\ \frac{\sin k(s-t) + \sin k(T-s+t)}{2k(1-\cos kT)}, & 0 \leq t \leq s \leq T. \end{cases}$$

Under hypothesis (H), we always denote

$$M = \max_{0 \leq s,t \leq T} G(t,s), \qquad m = \min_{0 \leq s,t \leq T} G(t,s), \qquad \sigma = \frac{m}{M}.$$

Thus $M > m > 0$ and $0 < \sigma < 1$.

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Now, we define the operator $T : PC(\mathbb{J}) \to PC(\mathbb{J})$ by

$$(Tx)(t) = \int_0^T G(t,s)f(s,x(s))ds + \sum_{k=1}^{p} G(t,t_k)I_k(x(t_k)).$$

Lemma 2.3 T is continuous and completely continuous. Moreover, $x(t)$ is an impulsive periodic solution of (1.1) and (1.2) if and only if $x(t)$ is a fixed point of T.

Proof. The proof is similar to that of [1], and therefore we omit the detail.

3. Main results

In this section, we state and prove the new existence results for (1.1). In order to prove our main results, the following nonlinear alternative of Leray-Schauder is need, which can be found in [19]. Let us define the function $\omega(x) = \int_0^T G(x,s)ds$ and use $\|\cdot\|_1$ denote the usual L^1- norm over $(0,T)$, by $\|\cdot\|$ the supremum norm of $\mathbb{C}[0,T]$.

Lemma 3.1 Assume Ω is a relatively compact subset of a convex set E in a normed space X. Let $T : \overline{\Omega} \to E$ be a compact map with $0 \in \Omega$. Then one of the following two conclusions holds:

(i) T has at least one fixed point in $\overline{\Omega}$.

(ii) There exist $u \in \partial\Omega$ and $0 < \lambda < 1$ such that $u = \lambda Tu$.

Now we present our main existence result of positive solution to problem (1.1).

Theorem 3.2 Suppose that (1.1) satisfies (H). Furthermore, assume that there exists a constant $r > 0$ such that

(H_1) There exists a continuous function $\phi_r \succ 0$ such that $f(t,x) \geq \phi_r(t)$ for all $(t,x) \in [0,T] \times (0,r]$.

(H_2) There exist continuous, non-negative functions $g(x)$, $h(x)$ and $\psi(x)$ on $(0,\infty)$ such that

$$f(t,x) \leq g(x) + h(x), \quad \text{for all } (t,x) \in [0,T] \times (0,\infty),$$

$$I_k(x) > 0, k = 1,\dots,p, \sum_{k=1}^{p} I_k(x) \leq \psi(x) \quad \text{for all } x \in (0,\infty),$$

where $g(x) > 0$ is non-increasing, $h(x)/g(x)$ and $\psi(x)$ is non-decreasing.

(H_3) The following inequality holds

$$\frac{r - M\psi(r)}{g(\sigma r)\left\{1 + \frac{h(r)}{g(r)}\right\}} > \|\omega\|,$$

.

Then (1.1) has at least one positive T-periodic solution x with $0 < \|x\| \leq r$.

Proof. Since (H_3) holds, let $N_0 = \{n_0, n_0+1, \cdots\}$, we can choose $n_0 \in \{1,2,\cdots\}$ such that $\dfrac{1}{n_0} < \sigma r$ and

$$\|\omega\| g(\sigma r)\left\{1 + \frac{h(r)}{g(r)}\right\} + M\psi(r) + \frac{1}{n_0} < r.$$

Consider the family of equations

$$x''(t) + a(t)x(t) = \lambda f_n(t,x(t)) + \frac{a(t)}{n}, \tag{3.1}$$

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associated with boundary conditions

$$x'(t_k^-) = x'(t_k^+) + I_{k,n}(x(t_k)), \quad k = 1, \dots, p, \tag{3.2}$$

where $\lambda \in [0,1]$, $n \in N_0$ and

$$f_n(t,x) = \begin{cases} f(t,x) & \text{if } x \geq 1/n, \\ f(t,1/n) & \text{if } x \leq 1/n. \end{cases}$$

and

$$I_{k,n}(x) = \begin{cases} I_k(x) & \text{if } x \geq 1/n, \\ I_k(1/n) & \text{if } x \leq 1/n. \end{cases}$$

Problem (3.1)-(3.2) is equivalent to the following fixed point of the operator equation

$$\begin{aligned} x(t) &= \lambda \int_0^T G(t,s) f_n(s,x(s))ds + \sum_{k=1}^p G(t,t_k) I_{k,n}(x(t_k)) + \frac{1}{n} \\ &= \lambda (T_n x)(t) + \frac{1}{n}. \end{aligned} \tag{3.3}$$

Now we show $\|x\| \neq r$ for any fixed point x of (3.3). If not, assume that x is a fixed point of (3.3) for some $\lambda \in [0,1]$ such that $\|x\| = r$. Note that

$$\begin{aligned} x(t) - \frac{1}{n} &= \lambda \int_0^T G(t,s) f_n(s,x(s))ds + \sum_{k=1}^p G(t,t_k) I_{k,n}(x(t_k)) \\ &\geq \lambda m \int_0^T f_n(s,x(s))ds + m \sum_{k=1}^p I_{k,n}(x(t_k)) \\ &= \sigma M \lambda \int_0^T f_n(s,x(s))ds + \sigma M \sum_{k=1}^p I_{k,n}(x(t_k)) \\ &\geq \sigma \max_{t \in [0,T]} \left\{ \lambda \int_0^T G(t,s) f_n(s,x(s))ds + \sum_{k=1}^p G(t,t_k) I_{k,n}(x(t_k)) \right\} \\ &= \sigma \|x - \frac{1}{n}\|. \end{aligned}$$

By the choice of n_0, $1/n \leq 1/n_0 < \sigma r$. Hence, we have

$$x(t) \geq \sigma \|x - \frac{1}{n}\| + \frac{1}{n} \geq \sigma \left(\|x\| - \frac{1}{n} \right) + \frac{1}{n} \geq \sigma r, \quad \text{for all} \quad 0 \leq x \leq T.$$

Thus, from condition (H_2) we have

$$
\begin{aligned}
x(t) &= \lambda \int_0^T G(t,s) f_n(s,x(s))ds + \sum_{k=1}^{p} G(t,t_k) I_{k,n}(x(t_k)) + \frac{1}{n} \\
&= \lambda \int_0^T G(t,s) f(s,x(s))ds + \sum_{k=1}^{p} G(t,t_k) I_k(x(t_k)) + \frac{1}{n} \\
&\le \int_0^T G(t,s) f(s,x(s))ds + \sum_{k=1}^{p} G(t,t_k) I_k(x(t_k)) + \frac{1}{n} \\
&\le \int_0^T G(x,s) g(x(s)) \left\{1 + \frac{h(x(s))}{g(x(s))}\right\} ds + M\psi(x(t_k)) + \frac{1}{n} \\
&\le g(\sigma r) \left\{1 + \frac{h(r)}{g(r)}\right\} \int_0^T G(t,s)ds + M\psi(r) + \frac{1}{n} \\
&\le g(\sigma r) \left\{1 + \frac{h(r)}{g(r)}\right\} \|\omega\| + M\psi(r) + \frac{1}{n_0}.
\end{aligned}
$$

Therefore,

$$
r = \|x\| \le g(\sigma r) \left\{1 + \frac{h(r)}{g(r)}\right\} \|\omega\| + \frac{1}{n_0}.
$$

This is a contradiction to the choice of n_0, so $\|x\| \neq r$.

Using Lemma 3.1, we know that

$$
x(t) = (T_n x)(t) + \frac{1}{n}
$$

has a fixed point, denoted by x_n, in $B_r = \{x \in PC(\mathbb{J}) : \|x\| < r\}$, that is, the equation

$$
x''(t) + a(t)x(t) = f_n(t,x(t)) + \frac{a(t)}{n}, \tag{3.4}
$$

has a periodic solution x_n with $\|x_n\| < r$. Since $x_n(t) \ge 1/n$ for all $t \in [0,T]$ and x_n is actually a positive solution of (3.4).

Next we claim that these solutions x_n have a uniform positive lower bound, i.e., there exists a constant $\delta > 0$, independent of $n \in N_0$, such that

$$
\min_{t \in [0,T]} x_n(t) \ge \delta
$$

for all $n \in N_0$. To see this, we know from (H_1) that there exists a function $\phi_r \succ 0$ such that $f(t,x) \ge \phi_r(t)$ for $(t,x) \in [0,T] \times (0,r]$. Now let $x_r(t)$ be the unique periodic solution to the problem (2.2) with $h = \phi_r(t)$. Then

$$
x_r(t) = \int_0^T G(t,s)\phi_r(s)ds \ge M\|\phi_r\|_1 > 0.
$$

Let

$$
\mathbb{E} = \left\{t \in [0,T] : x_n(t) \ge \frac{1}{n}\right\}, \quad \mathbb{E}' = [0,T] \backslash \mathbb{E}.
$$

So we have

$$\begin{aligned} x(t) &= \int_0^T G(t,s)f_n(s,x_n(s))ds + \sum_{k=1}^{p} G(t,t_k)I_{k,n}(x(t_k)) + \frac{1}{n} \\ &= \int_{\mathbb{E}} G(t,s)f(s,x_n(s))ds + \int_{\mathbb{E}'} G(t,s)f\left(s,\frac{1}{n}\right)ds + \sum_{k=1}^{p} G(t,t_k)I_k(x(t_k)) + \frac{1}{n} \\ &\geq \int_{\mathbb{E}} G(t,s)\phi_r ds + \int_{\mathbb{E}'} G(t,s)\phi_r ds \\ &= \int_0^T G(t,s)\phi_r(s)ds \geq M\|\phi_r\|_1 =: \delta. \end{aligned}$$

In order to pass the solutions of the truncation equation (3.1) (with $\lambda = 1$) to that of the original equation (1.1), we need the fact $\|x_n'\|$ is bounded. Now we show that

$$\|x_n'\| \leq H \tag{3.5}$$

for some constant $H > 0$ and for all $n \geq n_0$.

Integrating (3.1) from 0 to T (with $\lambda = 1$), we obtain

$$\int_0^T a(t)x_n(t)dt = \int_0^T \left[f_n(t,x_n(t)) + \frac{a(t)}{n}\right]dt.$$

Since $x(0) = x(T)$, there exists $t_0 \in [0,T]$ such that $x_n'(t_0) = 0$, therefore

$$\begin{aligned} \|x_n'\| &= \max_{0\leq t\leq T}|x_n'(t)| = \max_{0\leq t\leq T}\left|\int_{t_0}^t x_n''(s)ds\right| \\ &= \max_{0\leq t\leq T}\left|\int_{t_0}^t \left[f_n(s,x_n(s)) + \frac{a(s)}{n} - a(s)x_n(s)\right]ds\right| \\ &\leq \int_0^T \left[f_n(s,x_n(s)) + \frac{a(s)}{n}ds + \int_0^T a(s)x_n(s)\right]ds \\ &= 2\int_0^T a(s)x_n(s)ds = 2r\|a\|_1 =: H. \end{aligned}$$

The fact $\|x_n\| < r$ and $\|x_n'\| \leq H$ show that $\{x_n\}_{n\in N_0}$ is a bounded and equi-continuous family on $[0,T]$. Thus the Arzela-Ascoli Theorem guarantees that $\{x_n\}_{n\in N_0}$ has a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ converging uniformly on $[0,T]$ to a function $x \in \mathbb{C}[0,T]$. f is uniformly continuous since x_n satisfies $\delta \leq x_n(t) \leq r$ for all $t \in [0,T]$. Moreover, x_{n_i} satisfies the integral equation

$$x_{n_i}(t) = \int_0^T G(t,s)f(s,x_{n_i}(s))ds + \sum_{i=1}^{p} G(t,t_i)I_k(x_{n_i}(t)) + \frac{1}{n_i}.$$

Letting $i \to \infty$, we arrive at

$$x(t) = \int_0^T G(t,s)f(s,x(s))ds + \sum_{i=1}^{p} G(t,t_i)I_k(x(t)).$$

Therefore, x is a positive periodic solution of (1.1) and satisfies $0 < \|x\| \leq r$.

Corollary 3.3 Assume that $\alpha > 0, \beta \geq 0, c_k > 0, k = 1,2,\ldots,p, M\sum\limits_{k=1}^{p} c_k < 1$.

(i) if $\beta < 1$, then (1.3) has at least one positive periodic solution for each $\mu > 0$.
(ii) if $\beta \geq 1$, then (1.3) has at least one positive periodic solution for each $0 < \mu < \mu^*$, where μ^* is some positive constant.

Proof. We will apply Theorem 3.2. To this end, the assumption (H_1) is fulfilled with $\phi_r(t) = r^{-\alpha}$. If we take

$$g(x) = x^{-\alpha}, \quad h(x) = \mu x^{\beta}, \quad \psi(x) = \sum_{k=1}^{p} c_k x,$$

then conditions (H_2) is satisfied. Let $\omega(t) = \int_0^T G(t,s)ds$. Now the existence condition (H_3) becomes

$$\mu < \frac{r\left(1 - M\sum\limits_{k=1}^{p} c_k\right) - M\sum\limits_{k=1}^{p} c_k}{\|\omega\| r^{\alpha+\beta}(\sigma r)^{-\alpha}} - \frac{1}{r^{\alpha+\beta}}$$

for some $r > 0$. So (1.3) has at least one positive periodic solution for

$$0 < \mu < \mu^* := \sup_{r>0} \frac{r\left(1 - M\sum\limits_{k=1}^{p} c_k\right) - M\sum\limits_{k=1}^{p} c_k}{\|\omega\| r^{\alpha+\beta}(\sigma r)^{-\alpha}} - \frac{1}{r^{\alpha+\beta}}.$$

Note that $\mu^* = \infty$ if Since $M\sum\limits_{k=1}^{p} c_k < 1$, it is easy to see that $\mu_* = \infty$ if $\beta < 1$ and $\mu_* < \infty$ if $\beta \geq 1$. We have (i) and (ii).

Acknowledgment

This work is supported by the National Natural Science Foundation of China (Grant No.11461016), Hainan Natural Science Foundation(Grant No.117005), China Postdoctoral Science Foundation funded project (Grant No.2017M612577), Young Foundation of Hainan University (Grant No.hdkyxj201718).

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On Gauss diagrams of Knots: A modern approach

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Abstract

Gauss diagrams were introduced by Polyak and Viro as an appropriate device to describe finite-type invariants, which now appear as a very convenient way of coding knots in computer-recognizable form.

2010 Mathematics Subject Classification: 57M25, 57M27, 57M50

Key words and phrases: Gauss diagram, Gauss code, reidemeister move, braid knot

1 Introduction

Gauss diagrams were introduced by Polyak and Viro [14] as an appropriate device to describe finite type invariants in 1994.

Planar diagrams are convenient for presenting knots graphically, while Gauss diagrams are suited better for coding knots in a computer-recognizable form.

Goussarov [7] proved that any Vassiliev invariant can be calculated as a function of arrow polynomials on the knot diagram. Polyak used in [15] the notion of chord diagrams to define their representations in Gauss diagrams of plane curves. He also obtained invariants of generic plane and spherical curves in a systematic way via Gauss diagrams. Moreover, he proved that any Gauss diagram invariants are of finite degree. Fiedler showed in [6] that Gauss diagram invariants can be effectively used to show that a given knot is not isotopic to any closed braid. (Actually, it a well-known theorem of Alexander that each

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link in $\mathbb{R}^3$ is isotopic to a closed braid. But this is no longer the case for knots in the solid torus.) Mortier introduced in [10] decorated Gauss diagram as an efficient tool for recovering a knot diagram from it, and established characterization of the decorated Gauss diagrams of closed braids. Kauffman [9] gave a formula for Vassiliev invariants of a knot in terms of its chord diagram, which was related the Gauss diagram of the knot. Ochiai showed in [17] that the Gauss diagram formulas for the Kontsevich integral agree with the formulas for Vassiliev invariants which are introduced by Polyak and Viro [14]. Recently, Nizami [12] studied Kauffman bracket 2 and 3-strand braid links.

Our main contribution in this regard is the answer to the question "What happens to the Gauss diagram if a knot is mirrored and what happens to it if a knot is reversed ?" We prove that the Gauss diagram remains unchanged if a knot is mirrored, and is mirrored if the knot is reversed.

This paper is organized as follows: Section 2 includes basic, relevant material (including knots, braids, Gauss codes, Gauss diagrams, and Reidemeister moves) which is necessary to understand the results. We tried to make it interesting, particularly for a new reader. The results we got are presented in Section 3.

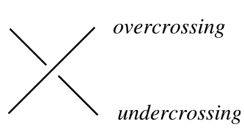
2 Preliminary Notions

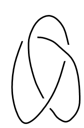
This section is devoted to basic notions, relevant to Gauss diagrams.

2.1 Knots

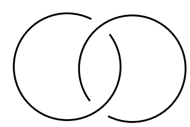
A *knot* is an embedding of the unit circle S^1 in $\mathbb{R}^3$. A *link* is an embedding of a disjoint union of such circles; each circle in a link is called a component. A 1-component link is actually a knot.

Knots are usually studied via projecting them on a plan; a projection with extra information of *overcrossing* and *undercrossing* is called the *knot diagram*.





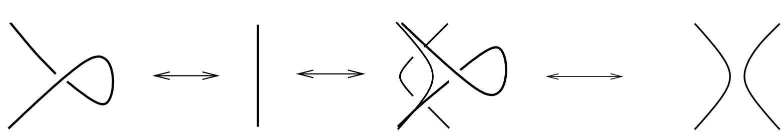
Trefoil knot



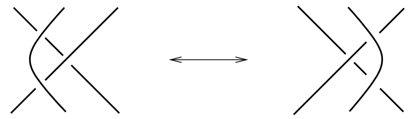
Hopf link

Two knots are called *isotopic* if one of them can be transformed to the other by a diffeomorphism of the ambient space onto itself. A fundamental result about the isotopic knot diagrams is:

Theorem 2.1. *[18] Two knots K_1 and K_2 are equivalent if and only if a diagram of K_1 can be transformed into a diagram of K_2 by a finite sequence of ambient isotopies of the plane and local (Reidemeister) moves:*



R_1 R_2

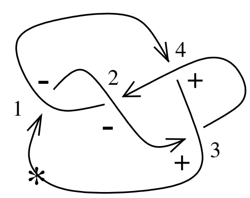


R_3

An *oriented knot* is an image of an embedding of S^1 into $\mathbb{R}^3$ together with the choice of one of the two possible directions on it. Each crossing of an oriented knot is either positive or negative:

positive crossing *negative crossing*

The *local writhe* of a crossing is defined as +1 or -1 for positive or negative crossing, respectively. The *writhe* (or total writhe) of a diagram is the sum of all the local writhes, or, equivalently, the difference between the number of positive and negative crossings.



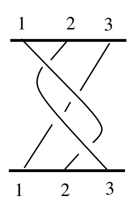
A knot with total writhe 0

The set of all knots that are equivalent to a knot K is called a *class* of K.

Remark 2.2. By a knot K we shall always mean a class of the knot K.

2.2 Braids

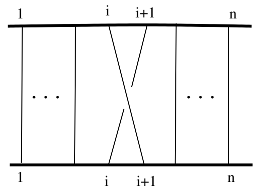
An n-strand *braid* is a set of n non intersecting smooth paths connecting n points on a horizontal plane to n points exactly below them on another horizontal plane in an arbitrary order. The smooth paths are called *strands* of the braid.



A 3-strand braid

The *product* ab of two n-strand braids is defined by putting the braid b below the braid a and then gluing their common end points.

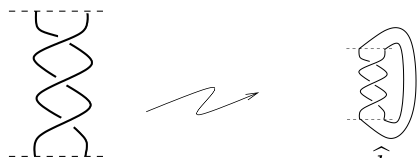
A braid with only one crossing is called the *elementary* braid; the ith elementary braid x_i with n strands is:



x_i

A useful property of elementary braids is that every braid can be written as a product of elementary braids. For instance, the above 3-strand braid is $x_1x_2x_1x_2$.

The *closure* of a braid b is the link $\widehat{b}$ obtained by connecting the lower ends of b with the corresponding upper ends.



b $\qquad$ $\widehat{b}$

Remark 2.3. 1. All braids are oriented from top to bottom.

2. By a braid b we shall mean the link $\widehat{b}$.
3. By a braid knot we shall mean a knot obtained as a closure of a braid.

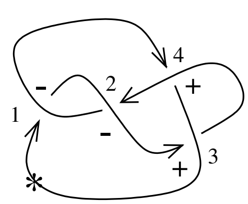
An important result connecting knots and braids is by Alexander:

Theorem 2.4. ([1]) *Each link can be represented as the closure of a braid.*

2.3 Gauss Diagram

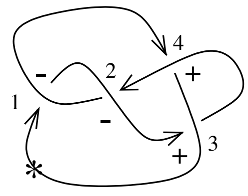
Planar diagrams are convenient for presenting knots graphically, while Gauss diagrams are suited better for coding knots in a computer-recognizable form.

A Gauss diagram is a diagrammatic representation of the classical Gauss code of the knot. The *Gauss code* is obtained from the oriented knot diagram by first labelling each crossing with a naming label (such as $1, 2, \ldots$) and also indicating the crossing type ($+1$ or -1). Then choose a basepoint on the knot diagram and begin walking along the diagram, recording the name of the crossings encountered, their sign and whether the walk takes you over or under that crossing. For example, if you go under crossing 1 whose sign is $+$ then you will record it as $U1+$. You may see the following knot along with its Gauss:

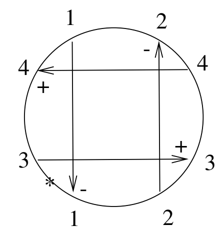


Gauss code: $U1 - O2 - U3 + O4 + U2 - O1 - U4 + O3+$

To form a Gauss diagram from a Gauss code, take an oriented circle with a basepoint chosen on the circle. Walk along the circle marking it with the labels for the crossings in the order of the Gauss code. Now draw chords between the points on the circle that have the same label. Orient each chord from overcrossing site to undercrossing site. Mark each chord with $+1$ or -1 according to the sign of the corresponding crossing in the Gauss code. The resulting labelled and basepointed graph is the (based) *Gauss diagram* for the knot. See, for instance, the knot and its Gauss diagram:



K



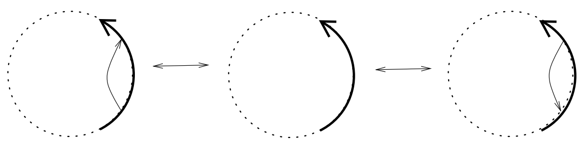
Gauss diagram of K

Remark 2.5. 1. A knot can be uniquely recovered from its Gauss diagrams and also from Gauss code.

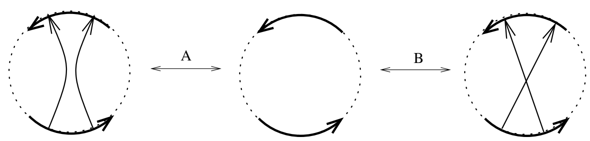
2. Gauss diagrams are considered up to orientation-preserving homeomorphisms of the circle.

2.4 Reidemeister moves for Gauss diagrams

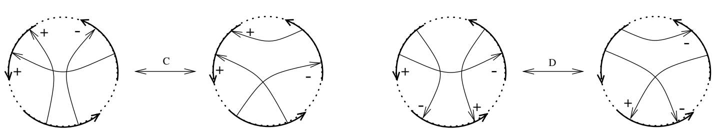
As we know, two oriented knot diagrams represent the same knot if and only if they are related by a sequence of oriented Reidemeister moves. The corresponding moves translated into the language of Gauss diagrams are:



$V\Omega_1$



$V\Omega_2$



$V\Omega_3(I)$ $V\Omega_3(II)$

3 The results

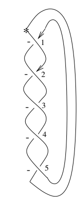
In this section we shall prove that the Gauss diagram remains unchanged if a knot is mirrored, and it is mirrored if the knot is reversed. Here we also show that the Reidemeister move $V\Omega_3$ for Gauss diagrams is a combination of the moves $V\Omega_2$ and $V\Omega_3'$, and that the move $V\Omega_3'$ is a combination of the moves $V\Omega_2$ and $V\Omega_3$.

Theorem 3.1. (*a*) *The Gauss diagram remains the same if a knot is mirrored. In this case all the crossings switch their signs.*

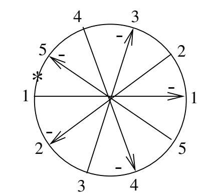
(*b*) *The Gauss diagram is mirrored if a knot is reversed.*

Proof. (*a*) In the mirror image $\overline{K}$ of a knot K the overcrossings remain overcrossings and undercrossings remain undercrossings. So, the sequence of over and under crossings in the Guass code of $\overline{K}$ remains the same as in the knot K. However, since the positive crossings change to negative and negative to positive in $\overline{K}$, the signs of chords in the Gauss diagram of $\overline{K}$ change accordingly. You may observe some examples:

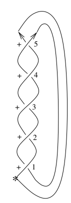
1.



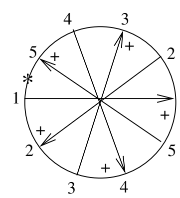
x_1^5



Gauss diagram of x_1^5

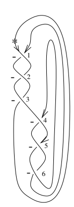


$\overline{x}_1^5$

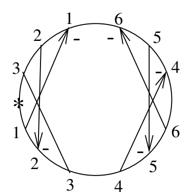


Gauss diagram $\overline{x}_1^5$

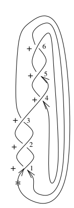
2.



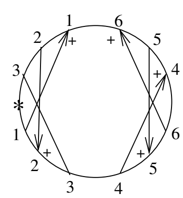
$x_1^3x_2^3$



Gauss diagram of $x_1^3x_2^3$

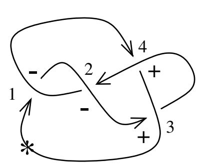


$\overline{x_1^3x_2^3}$

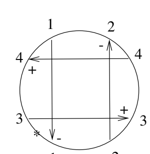


Gauss diagram of $\overline{x_1^3x_2^3}$

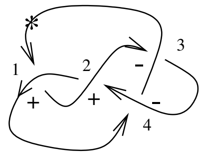
3.



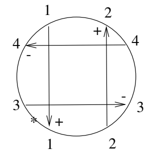
K: Figure-eight knot



Gauss diagram of K



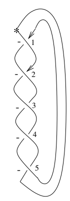
$\overline{K}$



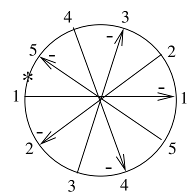
Gauss diagram of $\overline{K}$

(b) The proof will be finished with just two reasons: When a knot K is reversed, the sign of each crossing remains unchanged, a positive crossing remains positive and a negative crossing remains negative. However, the Gauss code of $-K$ reverses. Just have a look at the examples:

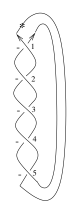
1.



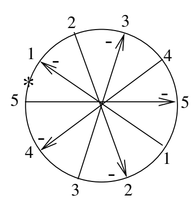
x_1^5



Gauss diagram of x_1^5

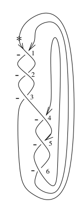


$-x_1^5$

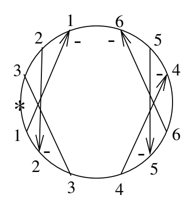


Gauss diagram of $-x_1^5$

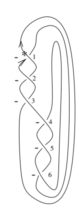
2.



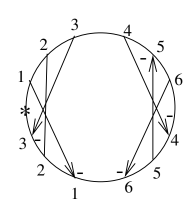
$x_1^3x_2^3$



Gauss diagram of $x_1^3x_2^3$

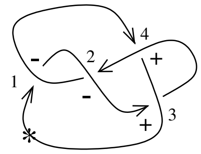


$-x_1^3x_2^3$

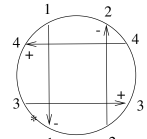


Gauss diagram of $-x_1^3x_2^3$

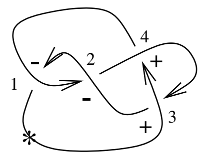
3.



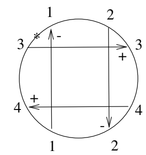
K: Figure-eight knot



Gauss diagram of K



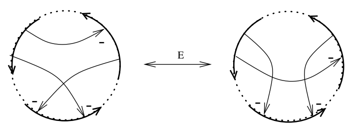
$-K$



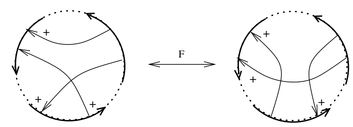
Gauss diagram of $-K$

□

We now show that in case of Gauss diagrams the second and third Reidemeister moves are related to two special moves, which we shall denote by $V\Omega_3'$:



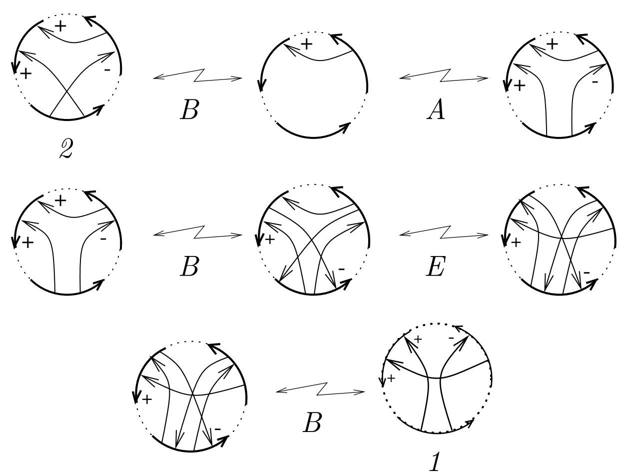
$V\Omega_3'(I)$



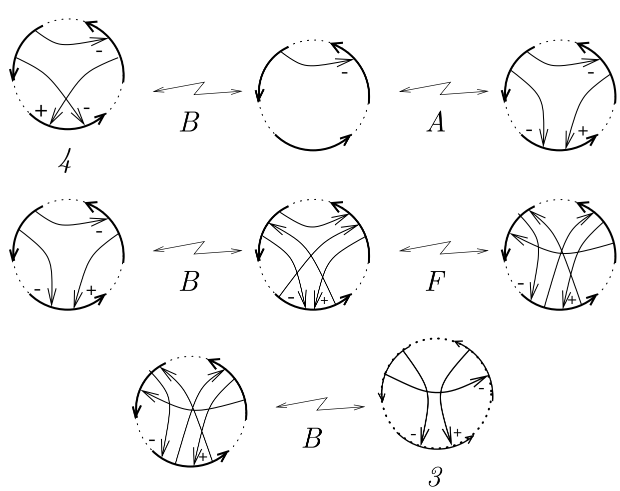
$V\Omega_3'(II)$

Theorem 3.2. (*a*) *Each of the moves* $V\Omega_3$ *is a combination of the moves* $V\Omega_2$ *and* $V\Omega_3'$.
(*b*) *Each of the moves* $V\Omega_3'$ *is a combination of the moves* $V\Omega_2$ *and* $V\Omega_3$.

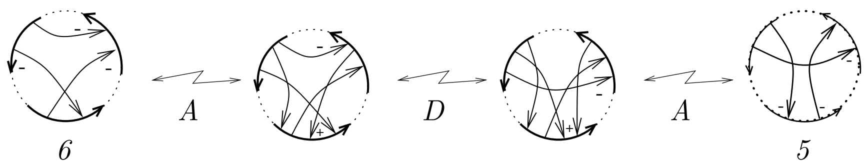
Proof. (*a*) Here is the proof of the first part (which is denoted by C) of the $V\Omega_3$:



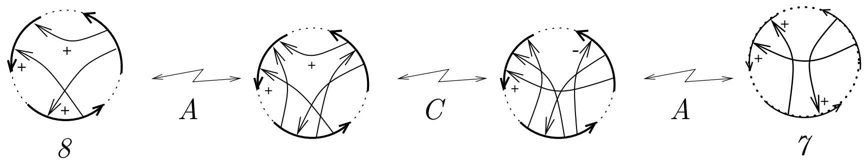
Now goes D:



(b) Just see the step-by-step application of the concerned moves:



F goes in a similar way:



□

Acknowledgement

This work was supported by the Dong-A University research fund.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 26, NO.6, 2019, COPYRIGHT 2019 EUDOXUS PRESS, LLC

The Jones polynomial of graph links via the Tutte polynomial

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Abstract

We give the Jones polynomial of the alternating links that correspond to a family of positive-signed connected planar graphs. We first find the general form of the Tutte polynomial of the family of graphs and then specializes it to the Jones polynomial. Then we recover the flow and chromatic polynomials from it as special cases. Finally, we give useful combinatorial information about the graph by evaluating the Tutte polynomial at some special points.

2010 Mathematics Subject Classification: 05C31, 57M27

Key words and phrases: Tutte polynomial, Jones polynomial, flow polynomial, chromatic polynomial

1 Introduction

The Tutte polynomial was introduced by Tutte [21] in 1954 as a generalization of chromatic polynomials studied by Birkhoff [1] and Whitney [24]. This graph invariant became popular because of its universal property that any multiplicative graph invariant with a deletion/contraction reduction must be an evaluation of it, and because of its applications in computer science, engineering, optimization, physics, biology, and knot theory.

In 1985, Jones [10] revolutionized knot theory by defining the Jones polynomial as a knot invariant via Von Neumann algebras. However, in 1987 Kauffman introduced in [13] a state-sum model construction of the Jones polynomial that was purely combinatorial and remarkably simple; we follow this construction.

Our primary motivation to study the Tutte polynomial came from the remarkable connection between the Tutte and the Jones polynomials that *up to a sign and multiplication*

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by a power of t the Jones polynomial $V_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_G(-t, -t^{-1})$. For detail study about knot theory, we refer [9,12,15,16,18,19].

This paper is organized as follows: In Section 2 we give some basic notions about graphs and knots along with definitions of the Tutte and the Jones polynomials. Moreover, in this section we give the relation between graphs and knots, and the relation between the Tutte and the Jones polynomials. Then the main result is given in Section 3. Finally, in Section 4 we specialize the Tutte polynomial to the Jones and the chromatic polynomials, and in Section 5 we give interpretations of some evaluations of the Tutte polynomial.

2 Preliminary notions

2.1 Basic concepts of graphs

A *graph* G is an ordered pair of disjoint sets (V, E) such that E is a subset of the set V^2 of unordered pairs of V. The set V is the set of vertices and E is the set of edges. If G is a graph, then $V = V(G)$ is the vertex set of G, and $E = E(G)$ is the edge set. An edge x, y is said to join the vertices x and y, and is denoted by xy; the vertices x and y are the end vertices of this edge. If $xy \in E(G)$, then x and y are *adjacent*, or neighboring, vertices of G, and the vertices x and y are *incident* with the edge xy. Two edges are adjacent if they have exactly one common end vertex.

We say that $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. In this case we write $G' \subset G$. If G' contains all edges of G that join two vertices in V' then G' is said to be the subgraph induced or spanned by V', and is denoted by $G[V']$. Thus, a subgraph G' of G is an *induced subgraph* if $G' = G[V(G')]$. If $V = V'$ then G' is said to be a *spanning subgraph* of G.

Two graphs are *isomorphic* if there is a correspondence between their vertex sets that preserves adjacency. Thus, $G = (V, E)$ is isomorphic to $G' = (V', E')$, denoted $G \simeq G'$, if there is a bijection $\varphi : V \to V'$ such that $xy \in E$ if and only if $\varphi(xy) \in E'$.

The dual notion of a cycle is that of cut or cocycle. If $\{V1, V2\}$ is a partition of the vertex set, and the set C, consisting of those edges with one end in V_1 and one end in V_2, is not empty, then C is called a *cut*. A cycle with one edge is called a *loop* and a cocycle with one edge is called a bridge. We refer to an edge that is neither a loop nor a bridge as *ordinary*.

A graph is *connected* if there is a path from one vertex to any other vertex of the graph. A connected subgraph of a graph G is called the *component* of G. We denote by $k(G)$ the number of connected components of a graph G, and by $c(G)$ the number of non-trivial connected components, that is the number of connected components not counting isolated vertices. A graph is k-connected if at least k vertices must be removed to disconnect the graph.

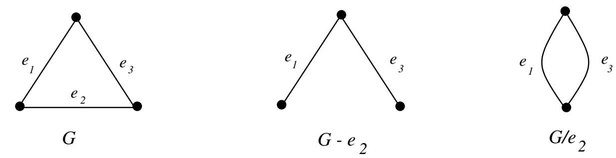
A *tree* is a connected graph without cycles. A *forest* is a graph whose connected components are all trees. (Spanning trees in connected graphs play a fundamental role in the theory of the Tutte polynomial.) Observe that a loop in a connected graph can be characterized as an edge that is in no spanning tree, while a bridge is an edge that is in every spanning tree.

A graph is *planar* if it can be drawn in the plane without edges crossings. A drawing of a graph in the plane separates the plane into regions called faces. Every plane graph G has a *dual graph*, G^*, formed by assigning a vertex of G^* to each face of G and joining two vertices of G^* by k edges if and only if the corresponding faces of G share k edges in their boundaries. Note that G^* is always connected. If G is connected, then $(G^*)^* = G$. If G is planar, it may have many dual graphs.

A *graph invariant* is a function f on the collection of all graphs such that $f(G_1) = f(G_2)$ whenever $G_1 \cong G_2$. A graph polynomial is a graph invariant where the image lies in some polynomial ring.

2.2 The Tutte polynomial

The following two operations are essential to understand the Tutte polynomial definition for a graph G. These are: *edge deletion* denoted by $G' = G - e$, and *edge contraction* $G'' = G/e$.



The deletion and contraction operations

Definition 2.1. ([21–23]) The *Tutte polynomial* of a graph G is a two-variable polynomial $T_G(x, y)$ defined as follows:

$$T_G(x,y) = \begin{cases} 1 & \text{if } E \text{ is empty,} \\ xT(G/e) & \text{if } e \text{ is a bridge,} \\ yT(G-e) & \text{if } e \text{ is a loop,} \\ T(G-e) + T(G/e) & \text{if } e \text{ is neither a bridge nor a loop.} \end{cases}$$

Example 2.2. Here is the Tutte polynomial of the graph $G =$.

$$\begin{aligned} T(\;) &= T(\;) + T(\;) \\ &= xT(\;) + T(\;) + T(\;) \\ &= x^2T(\bullet) + xT(\bullet) + y \\ &= x^2 + x + y. \end{aligned}$$

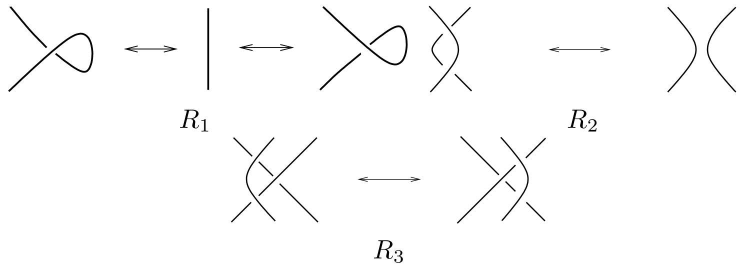
Remark 2.3. The definition of the Tutte polynomial outlines a simple recursive procedure to compute it, but the order of the rules applied is not fixed.

2.3 Basic concepts of Knots

A *knot* is a circle embedded in $\mathbb{R}^3$, and a *link* is an embedding of a union of such circles. Since knots are special cases of links, we shall often use the term link for both knots and links. Links are usually studied via projecting them on a plan; a projection with extra information of *overcrossing* and *undercrossing* is called the *link diagram*.

Trefoil knot *Hopf link*

Two links are called *isotopic* if one of them can be transformed to the other by a diffeomorphism of the ambient space onto itself. A fundamental result about the isotopic link diagrams is: *Two unoriented links L_1 and L_2 are equivalent if and only if a diagram of L_1 can be transformed into a diagram of L_2 by a finite sequence of ambient isotopies of the plane and local (Reidemeister) moves of the following three types:*



The set of all links that are equivalent to a link L is called a *class* of L. By a link L we shall always mean a class of the link L.

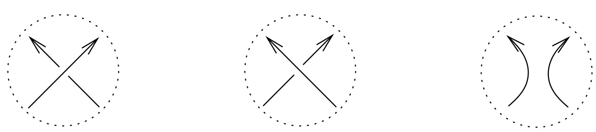
2.4 The Jones polynomial

The main question of knot theory is *Which two links are equivalent and which are not*? To address this question one needs a *knot invariant*, a function that gives one value on all links in a single class and gives different values (but not always) on links that belong to different classes. In 1985, Jones revolutionized knot theory by defining the Jones polynomial as a knot invariant via Von Neumann algebras [10]. However, in 1987 Kauffman introduced in [13] a state-sum model construction of the Jones polynomial that was purely combinatorial and remarkably simple.

Definition 2.4. [10,11,13] The *Jones polynomial* $V_K(t)$ of an oriented link L is a Laurent polynomial in the variable $\sqrt{t}$ satisfying the skein relation

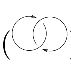
$$t^{-1}V_{L_+}(t) - tV_{L_-}(t) = (t^{1/2} - t^{-1/2})V_{L_0}(t),$$

and that the value of the unknot is 1. Here L_+, L_-, and L_0 are three oriented links having diagrams that are isotopic everywhere except at one crossing where they differ as in the figure below:



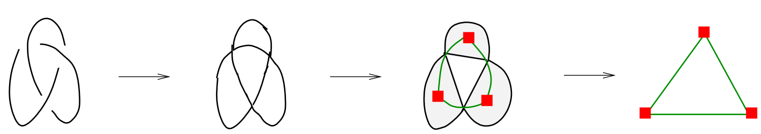
L_+ L_- L_0

Example 2.5. The Jones polynomials of the Hopf link and the trefoil knot are respectively

$V($$) = -t^{-5/2} - t^{-1/2}$ and $V($$) = -t^{-4} + t^{-3} + t.$

2.5 A connection between Knots and graphs

Corresponding to every connected link diagram we can find a connected signed planar graph and vice versa. The process is as follows: Suppose K is a knot and K' its projection. The projection K' divides the plane into several regions. Starting with the outermost region, we can color the regions either white or black. By our convention, we color the outermost region white. Now, we color the regions so that on either side of an edge the colors never agree.

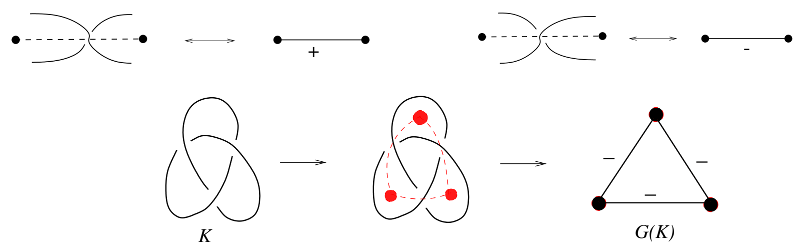


K K' G

The graph G corresponding to the knot projection K'

Next, choose a vertex in each black region. If two black regions R and R' have common crossing points $c_1, c_2, \ldots, c_n$, then we connect the selected vertices of R and R' by simple edges that pass through $c_1, c_2, \ldots, c_n$ and lie in these two black regions. In this way, we obtain from K' a plane graph G [17].

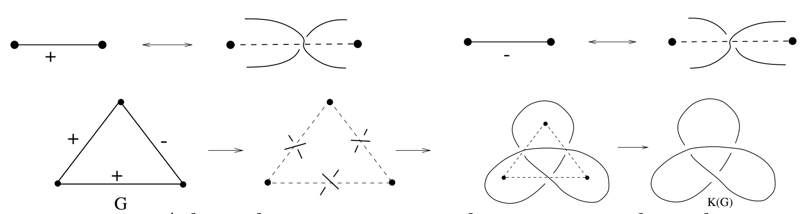
However, in order for the plane graph to embody some of the characteristics of the knot, we need to use the regular diagram rather than the projection. So, we need to consider the *under*- and *over*-crossings. To this end, we assign to each edge of G either the sign $+$ or $-$ as you can see in the following figure.



A signed graph corresponding to a knot diagram

A signed plane graph that has been formed by means of the above process is said to be the graph of the knot K [17].

Conversely, corresponding to a connected signed planar graph, we can find a connected planar link diagram. The construction is clear from the following figure.



A knot diagram corresponding to a signed graph

The fundamental combinatorial result connecting knots and graphs is:

Theorem 2.6. ([15]) *The collection of connected planar link diagrams is in one-to-one correspondence with the collection of connected signed planar graphs.*

2.6 Connection between the Tutte and the Jones polynomials

The primary motivation to study the Tutte polynomial came from the following remarkable connection between the Tutte and the Jones polynomials.

Theorem 2.7. ([9, 15, 19]) (Thistlethwaite) *Up to a sign and multiplication by a power of t the Jones polynomial $V_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_G(-t, -t^{-1})$.*

For positive-signed connected graphs, we have the precise connection:

Theorem 2.8. ([2]) *Let G be the positive-signed connected planar graph of an alternating oriented link diagram L. Then the Jones polynomial of the link L is*

$$V_L(t) = (-1)^{wr(L)} t^{\frac{b(L)-a(L)+3wr(L)}{4}} T_G(-t, -t^{-1}),$$

where $a(L)$ is the number of vertices in G, $b(L)$ is the number of vertices in the dual of G, and $wr(L)$ is the writhe of L.

Remark 2.9. In this paper, we shall compute Jones polynomials of links that correspond only to positive-signed graphs.

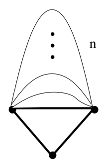
Example 2.10. Corresponding to the positive-signed graph G: , we receive the right-handed trefoil knot L: . It is easy to check, by definitions, that $V(\ ;t) = -t^4 + t^3 + t$ and $T(\ ;x,y) = x^2 + x + y$. Further note that the number of vertices in G is 3, number of vertices in the dual of G is 2, and withe of L is 3. Now notice that

$$V(\ ;t) = (-)^3 t^{\frac{2-3+3(3)}{4}} T(\ ;-t,-t^{-1}) = -t^2(t^2 - t - t^{-1}),$$

which agrees with the known value.

3 The main result

In this section we give the general form of the Tutte polynomial of the following graph:



$G_{3,n}$

For reference purposes, we denote this graph by $G_{3,n}$, where n is the number of edges parallel to one of the edges, as you can observe in the figure.

Theorem 3.1. *The Tutte polynomial of the graph $G_{3,n}$ is*

$$T_{G_{3,n}}(x,y) = (x + x^2) + (1+x)\sum_{i=1}^{n} y^i + y^{n+1}.$$

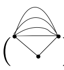
Proof. We prove it by induction on n. For $n = 1$, we have

$$\begin{aligned} vT(\ ) &= T(\ ) + T(\ ) \\ &= x^2 + x + y + T(\ ) + T(\ ) \\ &= x^2 + x + y + xy + y^2 \\ &= x + x^2 + (1+x)y + y^2 \\ &= (x + x^2) + (1+x)\sum_{i=1}^{1} y^i + y^{1+1}. \end{aligned}$$

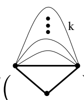
Just for authentication, we check for two more values of n. So, for $n = 2$ we get

$$\begin{aligned} T(\ ) &= T(\ ) + T(\ ) \\ &= x^2 + x + (x+1)y + y^2 + T(\ ) + T(\ ) \\ &= x^2 + x + (x+1)y + y^2 + xy^2 + y^3 \\ &= x^2 + x + (x+1)(y + y^2) + y^3 \\ &= (x + x^2) + (1+x)\sum_{i=1}^{2} y^i + y^{2+1}. \end{aligned}$$

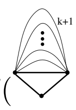
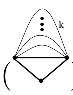
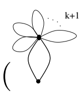
Similarly, if we take $n = 3$, then

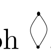
$T($  $)$ $$= x^2 + x + (x+1)(y + y^2 + y^3) + y^4 = (x + x^2) + (1 + x)\sum_{i=1}^{3} y^i + y^{3+1}.$$

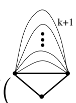
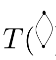
We now suppose the result holds for $n = k$, that is,

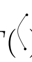
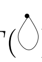
$T($  $)$ $$= (x + x^2) + (1 + x)\sum_{i=1}^{k} y^i + y^{k+1}. \tag{3.1}$$

Now for $n = k+1$ the Tutte polynomial becomes

$T($  $) = T($  $) + T($  $)$. (3.2)

Note that in the second term of equation (3.2) $k+1$ loops are attached to the graph . Now applying the inductive step on the first term and definition on the second term of equation (3.2), we get

$T($  $)$ $= [(x + x^2) + (1 + x)\sum_{i=1}^{k} y^i + y^{k+1}] + y^{k+1}T($  $)$

$= [(x + x^2) + (1 + x)\sum_{i=1}^{k} y^i + y^{k+1}] + y^{k+1}[T($  $) + T($  $)]$

$$
\begin{aligned}
&= [(x + x^2) + (1 + x)\sum_{i=1}^{k} y^i + y^{k+1}] + y^{k+1}[x + y] \\
&= (x + x^2) + (1 + x)\sum_{i=1}^{k} y^i + y^{k+1} + xy^{k+1} + y^{k+2} \\
&= (x + x^2) + (1 + x)\sum_{i=1}^{k} y^i + (1 + x)y^{k+1} + y^{k+2} \\
&= (x + x^2) + (1 + x)\sum_{i=1}^{k+1} y^i + y^{k+2},
\end{aligned}
$$

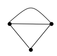
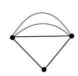
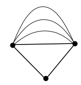
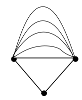
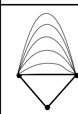
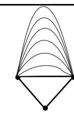
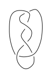
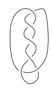
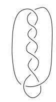
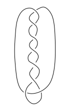
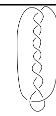
which is the desired result. □

4 Specializations

In this section we specialize the Tutte polynomial $T_{G_{3,n}}(x, y)$ to the chromatic and the Jones polynomials.

4.1 The Jones polynomial

The alternating links L that correspond to the graphs $G_{3,n}$ are given in the following table.

n	1	2	3	4	5	6	$\cdots$
G							$\cdots$
L							$\cdots$
$a(L)$	3	3	3	3	3	3	$\cdots$
$b(L)$	3	4	5	6	7	8	$\cdots$
$wr(L)$	**0**	5	**2**	7	**4**	9	$\cdots$

Lemma 4.1. *The number of vertices $b(L)$ in the dual of $G_{3,n}$ is $n+2$.*

Proof. Obvious from the table. □

Lemma 4.2. *The writhe of the link L corresponding to the graph $G_{3,n}$ is*

$$wr(L) = \begin{cases} n+3, & n \text{ is even,} \\ n-1, & n \text{ is odd.} \end{cases}$$

Proof. It is also obvious from the table. □

Proposition 4.3. *The Jones polynomial of the alternating link L that corresponds to the planar graph $G_{3,n}$, when n is a even, is*

$$V_L(t) = -t^{n+4} + t^{n+3} - t^{n+2} - 2\sum_{i=1}^{n-1}(-t)^{n+2-i} - t^2 + t.$$

Proof. We prove it by specializing the Tutte polynomial of the graph $G_{3,n}$ using Theorem 2.3, which says that

$$V_L(t) = (-1)^{wr(L)} t^{\frac{b(L)-a(L)+3wr(L)}{4}} T_{G_{3,n}}(-t, -t^{-1}).$$

Observe that, from Lemmas 4.1 and 4.2, the factor $(-1)^{wr(L)} t^{\frac{b(L)-a(L)+3wr(L)}{4}}$ reduces to $-t^{n+2}$. Now using this factor and substituting $x = -t$ and $y = -t^{-1}$ in Theorem 3.1, we have

$$\begin{aligned} V_L(t) &= (-t^{n+2})\big[-t + t^2 + (1-t)\sum_{i=1}^{n}(-t)^{-i} + (-t)^{-n-1}\big] \\ &= t^{n+3} - t^{n+4} + (t^{n+3} - t^{n+2})\sum_{i=1}^{n}(-t)^{-i} + t \\ &= -t^{n+4} + t^{n+3} + (t^{n+3} - t^{n+2})\big[-t^{-1} + t^{-2} - t^{-3} + \cdots + t^{-n+2} - t^{-n+1} + t^{-n}\big] + t \\ &= -t^{n+4} + t^{n+3} + \big[-t^{n+2} + t^{n+1} - t^{n} + \cdots + t^5 - t^4 + t^3\big] \\ &\quad \times \big[t^{n+1} - t^{n} + \cdots - t^4 + t^3 - t^2\big] + t \\ &= -t^{n+4} + t^{n+3} - t^{n+2} + 2\big[t^{n+1} - t^{n} + \cdots + t^5 - t^4 + t^3\big] - t^2 + t, \end{aligned}$$

which finally reduces to the desired result. □

Proposition 4.4. *The Jones polynomial of the alternating link L that corresponds to the planar graph $G_{3,n}$, when n is odd, is*

$$V_K(t) = t^{n+1} - t^n + t^{n-1} + 2\sum_{i=1}^{n-1}(-t)^{n-1-i} - t^{-1} + t^{-2}.$$

Proof. In this case, the factor $(-1)^{wr(L)}t^{\frac{b(L)-a(L)+3wr(L)}{4}}$ reduces to t^{n-1}. The proof is however similar to the proof of Proposition 4.3. □

With the understanding that *span* of $V_L(t)$ is the difference of the largest and smallest exponents of t, we have:

Proposition 4.5. *If L is the alternating link corresponding to the planar graph $G_{3,n}$, then* $span V_L(t) = n+3$ $(n \in \mathbb{N})$ *and* $\deg V_L(t) = \begin{cases} n+4, & n \text{ is even,} \\ n+1, & n \text{ is odd.} \end{cases}$

Proof. Obvious from Propositions 4.3 and 4.4. □

4.2 The flow polynomial

The flow polynomial was investigated by Tutte in 1947 in [20] as a function which could count the number of flows in a connected graph.

Definition 4.6. Let G be a graph with an arbitrary but fixed orientation, and let K be an Abelian group of order k and with 0 as its identity element. A K-flow is a mapping ϕ of the oriented edges $\overrightarrow{E}(G)$ into the elements of the group K such that:

$$\sum_{\overrightarrow{e}=u\to v} \phi(\overrightarrow{e}) + \sum_{\overrightarrow{e}=u\leftarrow v} \phi(\overrightarrow{e}) = 0 \tag{4.1}$$

for every vertex v, and where the first sum is taken over all arcs towards v and the second sum is over all arcs leaving v.

A K-flow is nowhere zero if ϕ never takes the value 0. The relation (4.1) is called the conservation law (that is, the Kirchhoff's law is satisfied at each vertex of G).

It is well known [2, 3, 5] that the number of proper K-flows does not depend on the structure of the group, but rather only on its order, and this number is a polynomial function of k that we refer to as the *flow polynomial*.

The following, due to Tutte [21], relates the Tutte polynomial of G with the number of nowhere zero flows of G over a finite Abelian group (which, in our case, is $\mathbb{Z}_k$).

Theorem 4.7. ([21]) *Let $G = (V, E)$ be a graph and K a finite Abelian group. If $F_G(k)$ denotes the number of nowhere zero K-flows then*

$$F_G(k) = (-1)^{|E|-|V|+k(G)}T(0, 1-k),$$

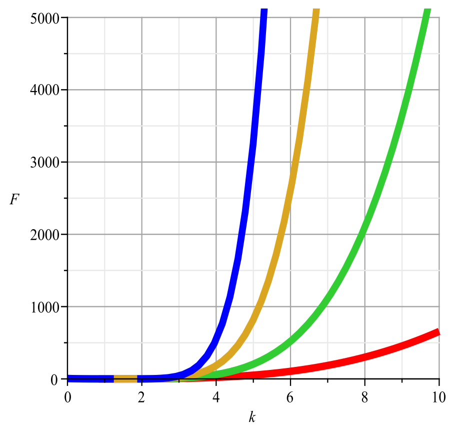
where $|E|$ is the number of edges, $|V|$ is the number of vertices, and $k(G)$ is the number of connected components of G.

Proposition 4.8. *The flow polynomial of the graph $G_{3,n}$ is*

$$F_{G_{3,n}}(k) = \frac{(-1)^n}{k}\big[(1-k)\big((1-k)^{n+1} - 1\big)\big].$$

Proof. We prove it by specializing the Tutte polynomial to the flow polynomial by the relation $F_{G_{3,n}}(k) = (-1)^{|E|-|V|+k(G)}T(0, 1-k)$.

Observe that in the graph $G_{3,n}$, $k(G) = 1$, $|E| = n+3$, and $|V| = 3$. Since the factors $(-1)^{|E|-|V|+k(G)}$ and $T(0, 1-k)$ reduces respectively to $(-1)^{n+1}$ and $\sum_{i=1}^{n+1}(1-k)^i$.



Flow polynomials F verses the order k of the group K
(The curves for n = 2, 3, 4, and 5 appear respecttively from right to left.)

The sum of the geometric series $\sum_{i=1}^{n+1}(1-k)^i$ (with first term $(1-k)$, common ratio $(1-k)$, and number of terms $n+1$) is $\frac{(1-k)}{-k}\big((1-k)^{n+1} - 1\big)$. Finally, applying Theorem 4.7, we receive the desired result. □

4.3 The chromatic polynomial

The chromatic polynomial, because of its theoretical and applied importance, has generated a large body of work. Chia [4] provides an extensive bibliography on the chromatic polynomial, and Dong, Koh, and Teo [6] give a comprehensive treatment.

For positive integer λ, a λ-coloring of a graph G is a mapping of $V(G)$ into the set $\{1, 2, 3, \cdots, \lambda\}$ of λ colors. Thus, there are exactly λ^n colorings for a graph on n vertices. If ϕ is a λ-coloring such that $\phi(u) \neq \phi(v)$ for all $uv \in E$, then ϕ is called a *proper* (or *admissible*) coloring.

Definition 4.9. The *chromatic polynomial* $P_G(\lambda)$ of a graph G is a one-variable graph invariant and is defined recursively by the following deletion-contraction relation:

$$P_G(\lambda) = P(G-e) - P(G/e)$$

We wish to find the number of admissible λ-colorings of a graph $G_{3,n}$. Since the chromatic polynomial counts the number of distinct ways to color a graph with λ colors, we recover it from the Tutte polynomial $T_{G_{3,n}}(x, y)$. The following theorem gives the precise relation between these polynomials.

Theorem 4.10. *[2] The chromatic polynomial of a graph $G = (V, E)$ is*

$$P_G(\lambda) = (-1)^{|V|-k(G)}\lambda^{k(G)}T_G(1-\lambda, 0),$$

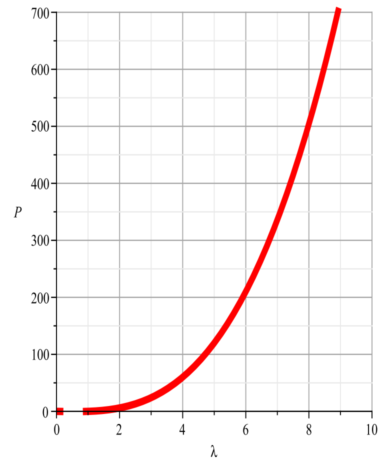
where $k(G)$ denote the number of connected components of G.

Proposition 4.11. *The chromatic polynomial of the graph $G_{3,n}$ is*

$$P_{G_{3,n}}(\lambda) = \lambda^3 - 3\lambda^2 + 2\lambda.$$

Proof. Although one can directly compute the chromatic polynomial of $G_{3,n}$ by definition, we recover it from the Tutte polynomial.

Since $|V| = 3$ and $k(G) = 1$, the factor $(-1)^{|V|-k(G)}\lambda^{k(G)}$ reduces to λ. Also, the factor $T_G(1-\lambda, 0)$ is $\lambda^2 - 3\lambda + 2$ for every $n \in \{0, 1, 2, \cdots\}$, and the result is thus established.



The chromatic polynomial **P** verses the number of colors **λ**

□

5 Evaluations

In this section, we evaluate $T_{G_{3,n}}(x, y)$ at some points, and give the corresponding useful combinatorial information about $G_{3,n}$.

Theorem 5.1. ([7]) *If $G = (V, E)$ is a connected graph, then*

1. $T_G(1, 1)$ *is the number of spanning trees of* G.
2. $T_G(2, 1)$ *equals the number of spanning forests of* G.
3. $T_G(1, 2)$ *is the number of spanning connected subgraphs of* G.
4. $T_G(2, 2)$ *equals* $2^{|E|}$, *and is the number of subgraphs of* G.

Proposition 5.2. *The following statements hold for the connected, planar graph* $G_{3,n}$.

1. $T_{G_{3,n}}(1, 1) = 2n + 3$.
2. $T_{G_{3,n}}(2, 1) = 3n + 7$.
3. $T_{G_{3,n}}(2, 2) = 2^{n+3}$.
4. $T_{G_{3,n}}(1, 2) = 3 \cdot 2^{n+1} - 2$.

Proof. We prove it step by step using directly Theorem 3.1:

1. For different values of n, we get the following different values of $T(1, 1)$.

n	1	2	3	4	$\cdots$
$T(1,1)$	5	7	9	11	$\cdots$

It is now clear that $T_{G_{3,n}}(1, 1) = 2n + 3$.

2. This result is similarly followed from the table:

n	1	2	3	4	$\cdots$
$T(2,1)$	10	13	16	19	$\cdots$

3. Since for the graph $G_{3,n}$ we have $|E| = n + 3$, the result follows from the Theorem 5.1.

4. Directly substituting $x = 1$ and $y = 2$ in Theorem 3.1 we receive

$$\begin{aligned} T_{G_{3,n}}(1,2) &= 2 + 2\sum_{i=1}^{n} 2^i + 2^{n+1} \\ &= (2 + 2^2 + 2^3 + \cdots + 2^{n+1}) + 2^{n+1} \\ &= 2\left(\frac{1-2^{n+1}}{1-2}\right) + 2^{n+1} \\ &= -2(1-2^{n+1}) + 2^{n+1}, \end{aligned}$$

which reduces to the desired result. □

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FOURIER SERIES OF SUMS OF PRODUCT OF POLY-BERNOULLI AND EULER FUNCTIONS AND THEIR APPLICATIONS

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Abstract. We consider three types of functions given by sums of products of poly-Bernoulli and Euler functions and derive their Fourier series expansions. In addition, we will express each of them in terms of Bernoulli and Euler functions.

1. Introduction and preliminaries

As is well known, the Euler polynomials $E_m(x)$ are given by the generating function

$$\frac{2}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}E_m(x)\frac{t^m}{m!},\quad \text{(see [6,10,11,13,14,16,19])}. \tag{1.1}$$

For any integer r, the poly-Bernoulli polynomials $\mathbf{B}_m^{(r)}(x)$ of index r are given by the generating function

$$\frac{Li_r(1-e^{-t})}{e^t-1}e^{xt}=\sum_{m=0}^{\infty}\mathbf{B}_m^{(r)}(x)\frac{t^m}{m!},\quad \text{(see [1-3,5,7,9,12,15])}, \tag{1.2}$$

w here $Li_r(x)=\sum_{m=0}^{\infty}\frac{x^m}{m^r}$ is the rth polylogarithmic function for $r\geq 1$ and a rational function for $r\leq 0$.

Observe here that

$$\frac{d}{dx}(Li_{r+1}(x))=\frac{1}{x}Li_r(x). \tag{1.3}$$

As to poly-Bernoulli polynomials, we note the following:

$$\frac{d}{dx}\mathbf{B}_m^{(r)}(x)=m\mathbf{B}_{m-1}^{(r)}(x),(m\geq 1). \tag{1.4}$$

$$\begin{aligned}&\mathbf{B}_m^{(1)}(x)=B_m(x),\mathbf{B}_0^{(r)}(x)=1,\mathbf{B}_m^{(0)}(x)=x^m,\\&\mathbf{B}_m^{(0)}=\delta_{m,0},\mathbf{B}_m^{(r+1)}(1)-\mathbf{B}_m^{(r+1)}(0)=\mathbf{B}_{m-1}^{(r)}(0),(m\geq 1).\end{aligned} \tag{1.5}$$

For any real number x, we let

$$<x>=x-[x]\in[0,1) \tag{1.6}$$

denote the fractional part of x.

2010 *Mathematics Subject Classification.* 42A16, 11B68, 11B83.
Key words and phrases. Fourier series, poly-Bernoulli function, Euler function.
* corresponding author.

Here we consider three types of functions given by sums of products of poly-Bernoulli and Euler functions and derive their Fourier series expansions. In addition, we will express each of them in terms of Bernoulli and Euler functions.

(1) $\alpha_m(x)=\sum_{k=0}^{m}\mathbf{B}_k^{(r+1)}(x)E_{m-k}(x),(m\geq 1)$,
(2) $\beta_m(<x>)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\mathbf{B}_k^{(r+1)}E_{m-k}(<x>),(m\geq 1)$,
(3) $\gamma_m(<x>)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbf{B}_k^{(r+1)}E_{m-k}(<x>),(m\geq 2)$.

For elementary facts about Fourier analysis, the reader may refer to any book (for example, see [4,18,20]). Some related works about Fourier series expansion for higher-order Bernoulli functions can be found in the recent papers in [8,17].

2. The function $\alpha_m(<x>)$

For integers r,m with $m\geq 1$, we let

$$\alpha_m(x)=\sum_{k=0}^{m}\mathbf{B}_k^{(r+1)}(x)E_{m-k}(x). \tag{2.1}$$

$$\begin{aligned}\alpha_m'(x)&=\sum_{k=0}^{m}\left(k\mathbf{B}_{k-1}^{(r+1)}(x)E_{m-k}(x)+(m-k)\mathbf{B}_k^{(r+1)}(x)E_{m-k-1}(x)\right)\\&=\sum_{k=1}^{m}k\mathbf{B}_{k-1}^{(r+1)}(x)E_{m-k}(x)+\sum_{k=0}^{m-1}(m-k)\mathbf{B}_k^{(r+1)}(x)E_{m-k-1}(x)\\&=\sum_{k=0}^{m-1}(k+1)\mathbf{B}_k^{(r+1)}(x)E_{m-1-k}(x)+\sum_{k=0}^{m-1}(m-k)\mathbf{B}_k^{(r+1)}(x)E_{m-1-k}(x)\\&=(m+1)\sum_{k=0}^{m-1}\mathbf{B}_k^{(r+1)}(x)E_{m-1-k}(x)\\&=(m+1)\alpha_{m-1}(x).\end{aligned}\tag{2.2}$$

$$\left(\frac{\alpha_{m+1}(x)}{m+2}\right)'=\alpha_m(x).$$

$$\int_0^1\alpha_m(x)dx=\frac{1}{m+2}\left(\alpha_{m+1}(1)-\alpha_{m+1}(0)\right). \tag{2.3}$$

$$\begin{aligned}\alpha_m(1)-\alpha_m(0)&=\sum_{k=0}^{m}\left(\mathbf{B}_k^{(r+1)}(1)E_{m-k}(1)-\mathbf{B}_k^{(r+1)}E_{m-k}\right)\\&=\sum_{k=1}^{m}\left(\left(\mathbf{B}_k^{(r+1)}+\mathbf{B}_{k-1}^{(r)}\right)(-E_{m-k}+2\delta_{m,k})-\mathbf{B}_k^{(r+1)}E_{m-k}\right)\\&+\mathbf{B}_0^{(r+1)}(1)E_m(1)-\mathbf{B}_0^{(r+1)}E_m\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^{m} \left(-\mathbf{B}_k^{(r+1)} E_{m-k} + 2\mathbf{B}_k^{(r+1)} \delta_{m,k} - \mathbf{B}_{k-1}^{(r)} E_{m-k} + 2\mathbf{B}_{k-1}^{(r)} \delta_{m,k} - \mathbf{B}_k^{(r+1)} E_{m-k} \right) \\
&- 2E_m + 2\delta_{m,0} \\
&= -2 \sum_{k=1}^{m} \mathbf{B}_k^{(r+1)} E_{m-k} - \sum_{k=1}^{m} \mathbf{B}_{k-1}^{(r)} E_{m-k} + 2\mathbf{B}_m^{(r+1)} + 2\mathbf{B}_{m-1}^{(r)} - 2E_m \\
&= -2 \sum_{k=0}^{m-1} \mathbf{B}_k^{(r+1)} E_{m-k} - \sum_{k=1}^{m-1} \mathbf{B}_{k-1}^{(r)} E_{m-k} + \mathbf{B}_{m-1}^{(r)}
\end{aligned} \tag{2.4}$$

For $m \geq 1$, we put,

$$\begin{aligned}
\Delta_m &= \alpha_m(1) - \alpha_m(0) \\
&= -2 \sum_{k=0}^{m-1} \mathbf{B}_k^{(r+1)} E_{m-k} - \sum_{k=1}^{m-1} \mathbf{B}_{k-1}^{(r)} E_{m-k} + \mathbf{B}_{m-1}^{(r)}.
\end{aligned} \tag{2.5}$$

Then $\alpha_m(1) = \alpha_m(0) \iff \Delta_m = 0$, and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2} \Delta_{m+1}. \tag{2.6}$$

Now, we will consider the function
$\alpha_m(< x >) = \sum_{k=0}^{m} \mathbf{B}_k^{(r+1)}(< x >) E_{m-k}(< x >), (m \geq 1)$
defined on $(-\infty, \infty)$, which is periodic with period 1.
The Fourier series of $\alpha_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi i n x},$$

where

$$\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(< x >) e^{-2\pi i n x} dx \\
&= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx.
\end{aligned} \tag{2.7}$$

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Now, we would like to determine the Fourier coefficients $A_n^{(m)}$.
$Case 1 : n \neq 0$.

$$\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\Big[\alpha_m(x) e^{-2\pi i n x}\Big]_0^1 + \frac{1}{2\pi i n}\int_0^1 \alpha_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\left(\alpha_m(1) - \alpha_m(0)\right) + \frac{m+1}{2\pi i n}\int_0^1 \alpha_{m-1}(x) e^{-2\pi i n x} dx \\
&= \frac{m+1}{2\pi i n} A_n^{(m-1)} - \frac{1}{2\pi i n}\Delta_m \\
&= \frac{m+1}{2\pi i n}\left(\frac{m}{2\pi i n} A_n^{(m-2)} - \frac{1}{2\pi i n}\Delta_{m-1}\right) - \frac{1}{2\pi i n}\Delta_m \\
&= \frac{(m+1)m}{(2\pi i n)^2} A_n^{(m-2)} - \frac{m+1}{(2\pi i n)^2}\Delta_{m-1} - \frac{1}{2\pi i n}\Delta_m \\
&= \frac{(m+1)m}{(2\pi i n)^2}\left(\frac{m-1}{2\pi i n} A_n^{(m-3)} - \frac{1}{2\pi i n}\Delta_{m-2}\right) - \frac{m+1}{(2\pi i n)^2}\Delta_{m-1} - \frac{1}{2\pi i n}\Delta_m \\
&= \frac{(m+1)_3}{(2\pi i n)^3} A_n^{(m-3)} - \sum_{j=1}^{3} \frac{(m+1)_{j-1}}{(2\pi i n)^j}\Delta_{m-j+1} \\
&= \cdots \\
&= \frac{(m+1)_m}{(2\pi i n)^m} A_n^{(0)} - \sum_{j=1}^{m} \frac{(m+1)_{j-1}}{(2\pi i n)^j}\Delta_{m-j+1} \\
&= -\frac{1}{m+2}\sum_{j=1}^{m} \frac{(m+2)_j}{(2\pi i n)^j}\Delta_{m-j+1}
\end{aligned} \tag{2.8}$$

where $A_n^{(0)} = \int_0^1 e^{-2\pi i n x} dx = 0$.

$Case 2 : n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x) dx = \frac{1}{m+2}\Delta_{m+1}. \tag{2.9}$$

We recall the following facts about Bernoulli functions $B_n(< x >)$:
(a) for $m \geq 2$,

$$B_m(< x >) = -m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^m}. \tag{2.10}$$

(b) for $m = 1$,

$$-\sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \tag{2.11}$$

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$\alpha_m(< x >), (m \geq 1)$ is piecewise C^∞. Moreover, $\alpha_m(< x >)$ is continuous for those positive integers m with $\Delta_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers m with $\Delta_m \neq 0$.

Assume first that m is a positive integer with $\Delta_m = 0$. Then $\alpha_m(1) = \alpha_m(0)$. $\alpha_m(< x >)$ is piecewise C^∞, and continuous. So the Fourier series of $\alpha_m(< x >)$ converges uniformly to $\alpha_m(< x >)$, and

$$\begin{aligned}
\alpha_m(< x >) &= \frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(-\frac{1}{m+2} \sum_{j=1}^{m} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x} \\
&= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=1}^{m} \binom{m+2}{j} \Delta_{m-j+1} \\
&\times \left(-j! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^j} \right) \\
&= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=1}^{m} \binom{m+2}{j} \Delta_{m-j+1} \times B_j(< x >) \\
&+ \Delta_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{2.12}$$

Now, we can state our first theorem.

Theorem 2.1. *For each positive integer l, let*

$$\Delta_l = -2 \sum_{k=0}^{l-1} \boldsymbol{B}_k^{(r+1)} E_{l-k} - \sum_{k=1}^{l-1} \boldsymbol{B}_{k-1}^{(r)} E_{l-k} + \boldsymbol{B}_{l-1}^{(r)}.$$

Assume that $\Delta_m = 0$, for a positive integer m. Then we have the following.

(a) $\sum_{k=0}^{m} \boldsymbol{B}_k^{(r+1)}(< x >) E_{m-k}(< x >)$ *has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=0}^{m} \boldsymbol{B}_k^{(r+1)}(< x >) E_{m-k}(< x >) \\
&= \frac{1}{m+2}\Delta_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(-\frac{1}{m+2} \sum_{j=1}^{m} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x},
\end{aligned}$$

for all $x \in (-\infty, \infty)$, where the convergence is uniform.

$$\begin{aligned}
\text{(b)} &\sum_{k=0}^{m} \boldsymbol{B}_k^{(r+1)}(< x >) E_{m-k}(< x >) \\
&= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2} \sum_{j=2}^{m} \binom{m+2}{j} \Delta_{m-j+1} B_j(< x >),
\end{aligned}$$

for all $x \in (-\infty, \infty)$, where $B_j(<x>)$ is the Bernoulli function.

Assume next that m is a positive integer with $\Delta_m \neq 0$. Then $\alpha_m(1) \neq \alpha_m(0)$. Thus $\alpha_m(<x>)$ is piecewise C^∞ , and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(<x>)$ converges pointwise to $\alpha_m(<x>)$, for $x \notin \mathbb{Z}$, and converges to

$$\begin{aligned}\frac{1}{2}\left(\alpha_m(0)+\alpha_m(1)\right) &= \alpha_m(0)+\frac{1}{2}\Delta_m \\ &= \sum_{k=0}^{m}\mathbf{B}_k^{(r+1)}E_{m-k}+\frac{1}{2}\Delta_m,\end{aligned} \tag{2.13}$$

for $x \in \mathbb{Z}$.

Next, we can state the second theorem.

Theorem 2.2. *For each positive integer l, let*

$$\Delta_l = -2\sum_{k=0}^{l-1}\boldsymbol{B}_k^{(r+1)}E_{l-k}-\sum_{k=1}^{l-1}\boldsymbol{B}_{k-1}^{(r)}E_{l-k}+\boldsymbol{B}_{l-1}^{(r)}.$$

Assume that $\Delta_m \neq 0$, for a positive integer m. Then we have the following.

$$\begin{aligned}&\text{(a)}\frac{1}{m+2}\Delta_{m+1} \\ &\quad+\sum_{n=-\infty,n\neq 0}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi i n)^j}\Delta_{m-j+1}\right)e^{2\pi i n x} \\ &\quad=\begin{cases}\sum_{k=0}^{m}\boldsymbol{B}_k^{(r+1)}(<x>)E_{m-k}(<x>), & \text{for } x\notin\mathbb{Z},\\ \sum_{k=0}^{m}\boldsymbol{B}_k^{(r+1)}E_{m-k}+\frac{1}{2}\Delta_m, & \text{for } x\in\mathbb{Z}.\end{cases}\end{aligned}$$

$$\begin{aligned}&\text{(b)}\frac{1}{m+2}\Delta_{m+1}+\frac{1}{m+2}\sum_{j=1}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>) \\ &=\sum_{k=0}^{m}\boldsymbol{B}_k^{(r+1)}(<x>)E_{m-k}(<x>),\ \text{for } x\notin\mathbb{Z},\end{aligned}$$

$$\begin{aligned}&\frac{1}{m+2}\Delta_{m+1}+\frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\Delta_{m-j+1}B_j(<x>) \\ &=\sum_{k=0}^{m}\boldsymbol{B}_k^{(r+1)}E_{m-k}+\frac{1}{2}\Delta_m,\ \text{for } x\in\mathbb{Z}.\end{aligned}$$

3. The function $\beta_m(< x >)$

Let $\beta_m(x) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!}\mathbf{B}_k^{(r+1)}(x)E_{m-k}(x),\ (m \geq 1)$.

$$\begin{aligned}
\beta_m'(x) &= \sum_{k=0}^{m}\left\{\frac{k}{k!(m-k)!}\mathbf{B}_{k-1}^{(r+1)}(x)E_{m-k}(x)\right. \\
&\quad \left.+\frac{m-k}{k!(m-k)!}\mathbf{B}_k^{(r+1)}(x)E_{m-k-1}(x)\right\} \\
&= \sum_{k=1}^{m}\frac{1}{(k-1)!(m-k)!}\mathbf{B}_{k-1}^{(r+1)}(x)E_{m-k}(x) \\
&\quad + \sum_{k=0}^{m-1}\frac{1}{k!(m-k-1)!}\mathbf{B}_k^{(r+1)}(x)E_{m-k-1}(x) \\
&= \sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}\mathbf{B}_k^{(r+1)}(x)E_{m-1-k}(x) \\
&\quad + \sum_{k=0}^{m-1}\frac{1}{k!(m-1-k)!}\mathbf{B}_k^{(r+1)}(x)E_{m-1-k}(x) \\
&= 2\beta_{m-1}(x).
\end{aligned} \tag{3.1}$$

So, $\beta_m'(x) = 2\beta_{m-1}(x)$, and from this we obtain $\left(\frac{\beta_{m+1}(x)}{2}\right)' = \beta_m(x)$.

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\Big(\beta_{m+1}(1) - \beta_{m+1}(0)\Big). \tag{3.2}$$

For $m \geq 1$, we have

$$\begin{aligned}
&\Omega_m = \Omega_m(r) = \beta_m(1) - \beta_m(0) \\
&= \sum_{k=0}^{m}\frac{1}{k!(m-k)!}\big(\mathbf{B}_k^{(r+1)}(1)E_{m-k}(1) - \mathbf{B}_k^{(r+1)}E_{m-k}\big) \\
&= \sum_{k=1}^{m}\frac{1}{k!(m-k)!}\left\{(\mathbf{B}_k^{(r+1)} + \mathbf{B}_{k-1}^{(r)})(-E_{m-k} + 2\delta_{m,k}) - \mathbf{B}_k^{(r+1)}E_{m-k}\right\} \\
&+ \frac{1}{m!}(-2E_m + 2\delta_{m,0}) \\
&= -2\sum_{k=1}^{m}\frac{\mathbf{B}_k^{(r+1)}E_{m-k}}{k!(m-k)!} - \sum_{k=1}^{m}\frac{\mathbf{B}_{k-1}^{(r)}E_{m-k}}{k!(m-k)!} + 2\frac{\mathbf{B}_m^{(r+1)}}{m!} + 2\frac{\mathbf{B}_{m-1}^{(r)}}{m!} - 2\frac{E_m}{m!} \\
&= -2\sum_{k=0}^{m-1}\frac{\mathbf{B}_k^{(r+1)}E_{m-k}}{k!(m-k)!} - \sum_{k=1}^{m-1}\frac{\mathbf{B}_{k-1}^{(r)}E_{m-k}}{k!(m-k)!} + \frac{1}{m!}\mathbf{B}_{m-1}^{(r)}.
\end{aligned} \tag{3.3}$$

Then $\beta_m(1) = \beta_m(0) \Longleftrightarrow \Omega_m = 0$.

Also,

$$\int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}.$$

Now, we are going to consider the function

$$\beta_m(< x >) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!}\mathbf{B}_k^{(r+1)}(< x >)E_{m-k}(< x >),\ (m \geq 1)$$

which is defined on $(-\infty, \infty)$, and periodic with period 1.

The Fourier series of $\beta_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi i n x},$$

where

$$B_n^{(m)} = \int_0^1 \beta_m(< x >)e^{-2\pi i n x}dx = \int_0^1 \beta_m(x)e^{-2\pi i n x}dx.$$

We are going to determine the Fourier coefficients $B_n^{(m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned}
B_n^{(m)} &= \int_0^1 \beta_m(x)e^{-2\pi i n x}dx \\
&= -\frac{1}{2\pi i n}\Big[\beta_m(x)e^{-2\pi i n x}\Big]_0^1 + \frac{1}{2\pi i n}\int_0^1 \beta_m'(x)e^{-2\pi i n x}dx \\
&= -\frac{1}{2\pi i n}\Big(\beta_m(1) - \beta_m(0)\Big) + \frac{2}{2\pi i n}\int_0^1 \beta_{m-1}(x)e^{-2\pi i n x}dx \\
&= \frac{2}{2\pi i n}B_n^{(m-1)} - \frac{1}{2\pi i n}\Omega_m \\
&= \frac{2}{2\pi i n}\Big(\frac{2}{2\pi i n}B_n^{(m-2)} - \frac{1}{2\pi i n}\Omega_{m-1}\Big) - \frac{1}{2\pi i n}\Omega_m \\
&= \Big(\frac{2}{2\pi i n}\Big)^2 B_n^{(m-2)} - \frac{2}{(2\pi i n)^2}\Omega_{m-1} - \frac{1}{2\pi i n}\Omega_m \\
&= \cdots \\
&= \Big(\frac{2}{2\pi i n}\Big)^m B_n^{(0)} - \sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1} \\
&= -\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1},
\end{aligned} \tag{3.4}$$

where $B_n^{(0)} = \int_0^1 e^{-2\pi i n x}dx = 0$.

Case 2: $n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x) = \frac{1}{2}\Omega_{m+1}. \tag{3.5}$$

$\beta_m(< x >)$, $(m \geq 1)$ is piecewise C^∞. Moreover, $\beta_m(< x >)$ is continuous for those positive integers m with $\Omega_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers m with $\Omega_m \neq 0$.

Assume first that m is a positive integer with $\Omega_m = 0$. Then $\beta_m(1) = \beta_m(0)$. $\beta_m(< x >)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned}
&\beta_m(< x >)\\
&= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \Big(-\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1} \Big) e^{2\pi i n x}\\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m} \frac{2^{j-1}}{j!}\Omega_{m-j+1} \times \Big(-j! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^j} \Big)\\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m} \frac{2^{j-1}}{j!}\Omega_{m-j+1} B_j(< x >)\\
&+ \Omega_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z},\\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{3.6}$$

Now, we are ready to state our first theorem.

Theorem 3.1. *For each positive integer l, let*

$$\Omega_l = -2\sum_{k=0}^{l-1} \frac{\boldsymbol{B}_k^{(r+1)} E_{l-k}}{k!(l-k)!} - \sum_{k=1}^{l-1} \frac{\boldsymbol{B}_{k-1}^{(r)} E_{l-k}}{k!(l-k)!} + \frac{1}{l!}\boldsymbol{B}_{l-1}^{(r)}.$$

Assume that $\Omega_m = 0$, for a positive integer m. Then we have the following.
(a) $\sum_{k=0}^{m} \frac{1}{k!(m-k)!}\boldsymbol{B}_k^{(r+1)}(< x >)E_{m-k}(< x >)$ *has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=0}^{m} \frac{1}{k!(m-k)!}\boldsymbol{B}_k^{(r+1)}(< x >)E_{m-k}(< x >)\\
&= \frac{1}{2}\Omega_{m+1} + \sum_{n=-\infty, n\neq 0}^{\infty} \Big(-\sum_{j=1}^{m} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1} \Big) e^{2\pi i n x},
\end{aligned}$$

for all $x \in (-\infty, \infty)$, where the convergence is uniform.

(b) $$\sum_{k=0}^{m} \frac{1}{k!(m-k)!}\boldsymbol{B}_k^{(r+1)}(< x >)E_{m-k}(< x >) = \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m} \frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(< x >),$$

for all $x \in (-\infty, \infty)$, where $B_j(< x >)$ is the Bernoulli function.

Assume next that m is a positive integer with $\Omega_m \neq 0$. Then, $\beta_m(1) \neq \beta_m(0)$. Thus $\beta_m(< x >)$ is piecewise C^∞ and discontinuous with jump discontinuities at

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integers. The Fourier series of $\beta_m(<x>)$ converges pointwise to $\beta_m(<x>)$, for $x \notin \mathbb{Z}$, and converges to

$$\begin{aligned}\frac{1}{2}(\beta_m(0)+\beta_m(1)) &= \beta_m(0)+\frac{1}{2}\Omega_m \\ &= \sum_{k=0}^{m}\frac{1}{k!(m-k)!}\mathbf{B}_k^{(r+1)}E_{m-k}+\frac{1}{2}\Omega_m,\end{aligned} \tag{3.7}$$

for $x \in \mathbb{Z}$.

Now, we can state our second theorem.

Theorem 3.2. *For each positive integer l, let*

$$\Omega_l = -2\sum_{k=0}^{l-1}\frac{\boldsymbol{B}_k^{(r+1)}E_{l-k}}{k!(l-k)!}-\sum_{k=1}^{l-1}\frac{\boldsymbol{B}_{k-1}^{(r)}E_{l-k}}{k!(l-k)!}+\frac{1}{l!}\boldsymbol{B}_{l-1}^{(r)}.$$

Assume that $\Omega_m \neq 0$, for a positive integer m. Then we have the following.

$$\begin{aligned}&(a)\frac{1}{2}\Omega_{m+1}+\sum_{n=-\infty,n\neq 0}^{\infty}\Big(-\sum_{j=1}^{m}\frac{2^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\Big)e^{2\pi inx} \\ &= \begin{cases}\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\boldsymbol{B}_k^{(r+1)}(<x>)E_{m-k}(<x>), & \text{for } x \notin \mathbb{Z}, \\ \sum_{k=0}^{m}\frac{1}{k!(m-k)!}\boldsymbol{B}_k^{(r+1)}E_{m-k}+\frac{1}{2}\Omega_m, & \text{for } x \in \mathbb{Z}.\end{cases}\end{aligned}$$

Here the convergence is pointwise.
(b)

$$\begin{aligned}&\frac{1}{2}\Omega_{m+1}+\sum_{j=1}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(<x>) \\ &= \sum_{k=0}^{m}\frac{1}{k!(m-k)!}\boldsymbol{B}_k^{(r+1)}(<x>)E_{m-k}(<x>), \quad \text{for } x \notin \mathbb{Z},\end{aligned}$$

$$\begin{aligned}&\frac{1}{2}\Omega_{m+1}+\sum_{j=2}^{m}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(<x>) \\ &= \sum_{k=0}^{m}\frac{1}{k!(m-k)!}\boldsymbol{B}_k^{(r+1)}E_{m-k}+\frac{1}{2}\Omega_m, \quad \text{for } x \in \mathbb{Z}.\end{aligned}$$

Here $B_k(<x>)$ is the Bernoulli function.

4. The fuction $\gamma_m(<x>)$

Let $\gamma_m(x) = \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\mathbf{B}_k^{(r+1)}(x)E_{m-k}(x)$, $(m \geq 2)$.

$$
\begin{aligned}
\gamma_m'(x) &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \Big(k\mathbf{B}_{k-1}^{(r+1)}(x) E_{m-k}(x) + (m-k)\mathbf{B}_k^{(r+1)}(x) E_{m-k-1}(x) \Big) \\
&= \sum_{k=0}^{m-2} \frac{1}{m-1-k} \mathbf{B}_k^{(r+1)}(x) E_{m-1-k}(x) + \sum_{k=1}^{m-1} \frac{1}{k} \mathbf{B}_k^{(r+1)}(x) E_{m-1-k}(x) \\
&= \frac{1}{m-1} E_{m-1}(x) + \sum_{k=1}^{m-2} \frac{1}{m-1-k} \mathbf{B}_k^{(r+1)}(x) E_{m-1-k}(x) \\
&+ \frac{1}{m-1} \mathbf{B}_{m-1}^{(r+1)}(x) + \sum_{k=1}^{m-2} \frac{1}{k} \mathbf{B}_k^{(r+1)}(x) E_{m-1-k}(x) \\
&= (m-1) \sum_{k=1}^{m-2} \frac{1}{k(m-1-k)} \mathbf{B}_k^{(r+1)}(x) E_{m-1-k}(x) + \frac{1}{m-1} E_{m-1}(x) + \frac{1}{m-1} \mathbf{B}_{m-1}^{(r+1)}(x) \\
&= \frac{1}{m-1} \Big(\mathbf{B}_{m-1}^{(r+1)}(x) + E_{m-1}(x) \Big) + (m-1)\gamma_{m-1}(x).
\end{aligned} \tag{4.1}
$$

So,

$$
\gamma_m'(x) = \frac{1}{m-1} \Big(\mathbf{B}_{m-1}^{(r+1)}(x) + E_{m-1}(x) \Big) + (m-1)\gamma_{m-1}(x).
$$

From this, we obtain

$$
\left(\frac{1}{m} \Big(\gamma_{m+1}(x) - \frac{1}{m(m+1)} \mathbf{B}_{m+1}^{(r+1)}(x) - \frac{1}{m(m+1)} E_{m+1}(x) \Big) \right)' = \gamma_m(x).
$$

$$
\begin{aligned}
&\int_0^1 \gamma_m(x) dx \\
&= \frac{1}{m} \Big[\gamma_{m+1}(x) - \frac{1}{m(m+1)} \mathbf{B}_{m+1}^{(r+1)}(x) - \frac{1}{m(m+1)} E_{m+1}(x) \Big]_0^1 \\
&= \frac{1}{m} \Big(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)} \Big(\mathbf{B}_{m+1}^{(r+1)}(1) - \mathbf{B}_{m+1}^{(r+1)}(0) \Big) \\
&- \frac{1}{m(m+1)} \Big(E_{m+1}(1) - E_{m+1}(0) \Big) \Big) \\
&= \frac{1}{m} \Big(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)} \mathbf{B}_m^{(r)} \\
&- \frac{1}{m(m+1)} \Big(-2E_{m+1} + 2\delta_{m+1,0} \Big) \Big) \\
&= \frac{1}{m} \Big(\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)} \mathbf{B}_m^{(r)} + \frac{2}{m(m+1)} E_{m+1} \Big).
\end{aligned} \tag{4.2}
$$

For $m \geq 2$, we let

$$\begin{aligned}\Lambda_m &= \Lambda_m(r) = \gamma_m(1) - \gamma_m(0)\\&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \left(\mathbf{B}_k^{(r+1)}(1) E_{m-k}(1) - \mathbf{B}_k^{(r+1)} E_{m-k}\right)\\&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \Big((\mathbf{B}_k^{(r+1)} + \mathbf{B}_{k-1}^{(r)})(-E_{m-k} + 2\delta_{m,k}) - \mathbf{B}_k^{(r+1)} E_{m-k}\Big)\\&= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \Big(-2\mathbf{B}_k^{(r+1)} E_{m-k} - \mathbf{B}_{k-1}^{(r)} E_{m-k}\Big)\\&= -\sum_{k=1}^{m-1} \frac{1}{k(m-k)} \Big(2\mathbf{B}_k^{(r+1)} + \mathbf{B}_{k-1}^{(r)}\Big) E_{m-k}.\end{aligned} \tag{4.3}$$

So,

$$\gamma_m(1) = \gamma_m(0) \;\Leftrightarrow\; \Lambda_m = 0. \tag{4.4}$$

Also,

$$\int_0^1 \gamma_m(x)dx = \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbf{B}_m^{(r)} + \frac{2}{m(m+1)}E_{m+1}\right). \tag{4.5}$$

We are now going to consider

$$\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \mathbf{B}_k^{(r+1)}(< x >) E_{m-k}(< x >), \tag{4.6}$$

which is defined on $(-\infty, \infty)$, and periodic with period 1.

The Fourier series of $\gamma_m(< x >)$ is

$$\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi i n x}, \tag{4.7}$$

where

$$C_n^{(m)} = \int_0^1 \gamma_m(< x >) e^{-2\pi i n x} dx = \int_0^1 \gamma_m(x) e^{-2\pi i n x} dx. \tag{4.8}$$

Now, we are ready to determine the Fourier coefficients $C_n^{(m)}$.
Case 1: $n \neq 0$.

$$\begin{aligned}
C_n^{(m)} &= \int_0^1 \gamma_m(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\Big[\gamma_m(x) e^{-2\pi i n x}\Big]_0^1 + \frac{1}{2\pi i n}\int_0^1 \gamma_m'(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\Big(\gamma_m(1) - \gamma_m(0)\Big) \\
&+ \frac{1}{2\pi i n}\int_0^1 \Big((m-1)\gamma_{m-1}(x) + \frac{1}{m-1}\mathbf{B}_{m-1}^{(r+1)}(x) + \frac{1}{m-1}E_{m-1}(x)\Big) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\Lambda_m + \frac{m-1}{2\pi i n} C_n^{(m-1)} \\
&+ \frac{1}{2\pi i n(m-1)}\int_0^1 \mathbf{B}_{m-1}^{(r+1)}(x) e^{-2\pi i n x} dx \\
&+ \frac{1}{2\pi i n(m-1)}\int_0^1 E_{m-1}(x) e^{-2\pi i n x} dx.
\end{aligned} \tag{4.9}$$

where, for $l \geq 1$ and $n \neq 0$,

$$\int_0^1 \mathbf{B}_l^{(r+1)}(x) e^{-2\pi i n x} dx = -\sum_{k=1}^{l} \frac{(l)_{k-1}}{(2\pi i n)^k} \mathbf{B}_{l-k}^{(r)},$$

$$\int_0^1 E_l(x) e^{-2\pi i n x} dx = 2\sum_{k=1}^{l} \frac{(l)_{k-1}}{(2\pi i n)^k} E_{l-k+1}.$$

Thus

$$\begin{aligned}
C_n^{(m)} &= \frac{m-1}{2\pi i n} C_n^{(m-1)} - \frac{1}{2\pi i n}\Lambda_m \\
&- \frac{1}{2\pi i n(m-1)}\Theta_m + \frac{2}{2\pi i n(m-1)}\Phi_m,
\end{aligned} \tag{4.10}$$

where, for $m \geq 2$,

$$\begin{aligned}
\Lambda_m &= \gamma_m(1) - \gamma_m(0) = -\sum_{k=1}^{m-1} \frac{1}{k(m-k)}\Big(2\mathbf{B}_k^{(r+1)} + \mathbf{B}_{k-1}^{(r)}\Big) E_{m-k}, \\
\Theta_m &= \sum_{k=1}^{m-1} \frac{(m-1)_{k-1}}{(2\pi i n)^k} \mathbf{B}_{m-k-1}^{(r)}, \\
\Phi_m &= \sum_{k=1}^{m-1} \frac{(m-1)_{k-1}}{(2\pi i n)^k} E_{m-k}.
\end{aligned} \tag{4.11}$$

$$
\begin{aligned}
C_n^{(m)} &= \frac{m-1}{2\pi in}C_n^{(m-1)} - \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Theta_m + \frac{2}{2\pi in(m-1)}\Phi_m \\
&= \frac{m-1}{2\pi in}\Big(\frac{m-2}{2\pi in}C_n^{(m-2)} - \frac{1}{2\pi in}\Lambda_{m-1} - \frac{1}{2\pi in(m-2)}\Theta_{m-1} + \frac{2}{2\pi in(m-1)}\Phi_{m-1}\Big) \\
&- \frac{1}{2\pi in}\Lambda_m - \frac{1}{2\pi in(m-1)}\Theta_m + \frac{2}{2\pi in(m-1)}\Phi_m \\
&= \frac{(m-1)(m-2)}{(2\pi in)^2}C_n^{(m-2)} - \frac{m-1}{(2\pi in)^2}\Lambda_{m-1} - \frac{1}{2\pi in}\Lambda_m - \frac{m-1}{(2\pi in)^2(m-2)}\Theta_{m-1} \\
&- \frac{1}{(2\pi in)(m-1)}\Theta_m + \frac{2(m-1)}{(2\pi in)^2(m-2)}\Phi_{m-1} + \frac{2}{2\pi in(m-1)}\Phi_m \\
&= \cdots \\
&= \frac{(m-1)!}{(2\pi in)^{m-2}}C_n^{(1)} - \sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} - \sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1} \\
&+ \sum_{j=1}^{m-1}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\
&= -\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j}\Lambda_{m-j+1} - \sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1} \\
&+ \sum_{j=1}^{m-1}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1},
\end{aligned}
\tag{4.12}
$$

where $C_n^{(1)} = 0$.

Before proceeding further, we note the following.

$$
\begin{aligned}
&\sum_{j=1}^{m-1}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Phi_{m-j+1} \\
&= \sum_{j=1}^{m-1}\frac{2(m-1)_{j-1}}{(2\pi in)^j(m-j)}\sum_{k=1}^{m-j}\frac{(m-j)_{k-1}}{(2\pi in)^k}E_{m-j-k+1} \\
&= \sum_{j=1}^{m-1}\sum_{k=1}^{m-j}\frac{2(m-1)_{j+k-2}}{(2\pi in)^{j+k}(m-j)}E_{m-j-k+1} \\
&= \sum_{j=1}^{m-1}\frac{1}{m-j}\sum_{s=j+1}^{m}\frac{2(m-1)_{s-2}}{(2\pi in)^s}E_{m-s+1}
\end{aligned}
\tag{4.13}
$$

$$=\sum_{s=2}^{m}\frac{2(m-1)_{s-2}}{(2\pi in)^s}E_{m-s+1}\sum_{j=1}^{s-1}\frac{1}{m-j}$$
$$=\frac{2}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\frac{H_{m-1}-H_{m-s}}{m-s+1}E_{m-s+1}.$$

$$\begin{aligned}
&\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\Theta_{m-j+1}\\
=&\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi in)^j(m-j)}\sum_{k=1}^{m-j}\frac{(m-j)_{k-1}}{(2\pi in)^k}\mathbf{B}_{m-j-k}^{(r)}\\
=&\sum_{j=1}^{m-1}\sum_{k=1}^{m-j}\frac{(m-1)_{j+k-2}}{(2\pi in)^{j+k}(m-j)}\mathbf{B}_{m-j-k}^{(r)}\\
=&\sum_{j=1}^{m-1}\frac{1}{m-j}\sum_{s=j+1}^{m}\frac{(m-1)_{s-2}}{(2\pi in)^s}\mathbf{B}_{m-s}^{(r)}\\
=&\sum_{s=2}^{m}\frac{(m-1)_{s-2}}{(2\pi in)^s}\mathbf{B}_{m-s}^{(r)}\sum_{j=1}^{s-1}\frac{1}{m-j}\\
=&\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\frac{H_{m-1}-H_{m-s}}{m-s+1}\mathbf{B}_{m-s}^{(r)}.
\end{aligned} \tag{4.14}$$

Putting everything together, we obtain

$$\begin{aligned}
&C_n^{(m)}=-\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\Lambda_{m-s+1}\\
&-\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\frac{H_{m-1}-H_{m-s}}{m-s+1}\mathbf{B}_{m-s}^{(r)}+\frac{2}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\frac{H_{m-1}-H_{m-s}}{m-s+1}E_{m-s+1}\\
&=-\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\left(\Lambda_{m-s+1}+\frac{H_{m-1}-H_{m-s}}{m-s+1}(\mathbf{B}_{m-s}^{(r)}-2E_{m-s+1})\right).
\end{aligned} \tag{4.15}$$

Case 2: $n=0$.

$$C_0^{(m)}=\int_0^1\gamma_m(x)dx=\frac{1}{m}\left(\Lambda_{m+1}-\frac{1}{m(m+1)}\mathbf{B}_m^{(r)}+\frac{2}{m(m+1)}E_{m+1}\right). \tag{4.16}$$

$\gamma_m(<x>)$, $(m\geq 2)$ is piecewise C^∞. Moreover, $\gamma_m(<x>)$ is continuous for those positive integers $m\geq 2$ with $\Lambda_m=0$ and discontinuous with jump discontinuities at integers for those positive integers $m\geq 2$ with $\Lambda_m\neq 0$.

Assume first that $\Lambda_m = 0$. Then $\gamma_m(1) = \gamma_m(0)$. $\gamma_m(< x >)$ is piecewise C^∞ and continuous. So the Fourier series of $\gamma_m(< x >)$ converges uniformly to $\gamma_m(< x >)$, and

$$
\begin{aligned}
&\gamma_m(< x >)\\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbf{B}_m^{(r)} + \frac{2}{m(m+1)}E_{m+1}\right)\\
&- \sum_{n=-\infty, n\neq 0}^{\infty}\left(\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(\mathbf{B}_{m-s}^{(r)} - 2E_{m-s+1})\right)\right)e^{2\pi inx}\\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbf{B}_m^{(r)} + \frac{2}{m(m+1)}E_{m+1}\right)\\
&+ \frac{1}{m}\sum_{s=1}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(\mathbf{B}_{m-s}^{(r)} - 2E_{m-s+1})\right)\\
&\times\left(-s!\sum_{n=-\infty, n\neq 0}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^s}\right)\\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\mathbf{B}_m^{(r)} + \frac{2}{m(m+1)}E_{m+1}\right)\\
&+ \frac{1}{m}\sum_{s=2}^{m}\binom{m}{s}\left(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(\mathbf{B}_{m-s}^{(r)} - 2E_{m-s+1})\right)B_s(< x >)\\
&+ \Lambda_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z},\\ 0, & \text{for } x \in \mathbb{Z}, \end{cases}
\end{aligned}
\tag{4.17}
$$

where $H_m = \sum_{k=1}^{m}\frac{1}{k}$.

Now, we are able to state our first theorem.

Theorem 4.1. *For each integer $l \geq 2$, let*

$$\Lambda_l = -\sum_{k=1}^{l-1}\frac{1}{k(l-k)}\left(2\boldsymbol{B}_k^{(r+1)} + \boldsymbol{B}_{k-1}^{(r)}\right)E_{l-k},$$

with $\Lambda_1 = 0$.

Assume that $\Lambda_m = 0$, for the an integer $m \geq 2$.
Then we have the following.

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(a) $\sum_{k=1}^{m-1} \frac{1}{k(m-k)} \boldsymbol{B}_k^{(r+1)}(< x >) E_{m-k}(< x >)$ *has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} \boldsymbol{B}_k^{(r+1)}(< x >) E_{m-k}(< x >) \\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\boldsymbol{B}_m^{(r)} + \frac{2}{m(m+1)}E_{m+1}\right) \\
&- \sum_{n=-\infty, n\neq 0}^{\infty} \Big(\frac{1}{m}\sum_{s=1}^{m} \frac{(m)_s}{(2\pi i n)^s}\Big(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(\boldsymbol{B}_{m-s}^{(r)} - 2E_{m-s+1})\Big)\Big) e^{2\pi i n x},
\end{aligned}$$

for all $x \in (-\infty, \infty)$. *Here the convergence is uniform.*

(b)

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} \boldsymbol{B}_k^{(r+1)}(< x >) E_{m-k}(< x >) \\
&= \frac{1}{m}\left(\Lambda_{m+1} - \frac{1}{m(m+1)}\boldsymbol{B}_m^{(r)} + \frac{2}{m(m+1)}E_{m+1}\right) \\
&+ \frac{1}{m}\sum_{s=2}^{m} \binom{m}{s}\Big(\Lambda_{m-s+1} + \frac{H_{m-1}-H_{m-s}}{m-s+1}(\boldsymbol{B}_{m-s}^{(r)} - 2E_{m-s+1})\Big) B_s(< x >),
\end{aligned}$$

for all $x \in (-\infty, \infty)$. *Here* $B_k(< x >)$ *is the Bernoulli function.*

Assume next that m is an integer ≥ 2 with $\Lambda_m \neq 0$. Then, $\gamma_m(1) \neq \gamma_m(0)$. Hence $\gamma_m(< x >)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\gamma_m(< x >)$ converges pointwise to $\gamma_m(< x >)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}(\gamma_m(0) + \gamma_m(1)) = \gamma_m(0) + \frac{1}{2}\Lambda_m = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \mathbf{B}_k^{(r+1)} E_{m-k} + \frac{1}{2}\Lambda_m, \tag{4.18}$$

for $x \in \mathbb{Z}$.

Next, we can state our second theorem.

Theorem 4.2. *For each integer* $l \geq 2$, *let*

$$\Lambda_l = -\sum_{k=1}^{l-1} \frac{1}{k(l-k)}\Big(2\boldsymbol{B}_k^{(r+1)} + \boldsymbol{B}_{k-1}^{(r)}\Big) E_{l-k},$$

with $\Lambda_1 = 0$.

Assume that $\Lambda_m \neq 0$, *for an integer* $m \geq 2$.
Then we have the following.

(a)

$$\frac{1}{m}\left(\Lambda_{m+1}-\frac{1}{m(m+1)}\boldsymbol{B}_m^{(r)}+\frac{2}{m(m+1)}E_{m+1}\right)$$
$$-\sum_{n=-\infty,n\neq 0}^{\infty}\Big(\frac{1}{m}\sum_{s=1}^{m}\frac{(m)_s}{(2\pi in)^s}\Big(\Lambda_{m-s+1}+\frac{H_{m-1}-H_{m-s}}{m-s+1}(\boldsymbol{B}_{m-s}^{(r)}-2E_{m-s+1})\Big)\Big)e^{2\pi inx}$$
$$=\begin{cases}\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\boldsymbol{B}_k^{(r+1)}(<x>)E_{m-k}(<x>), & \textit{for } x\notin\mathbb{Z},\\ \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\boldsymbol{B}_k^{(r+1)}E_{m-k}+\frac{1}{2}\Lambda_m, & \textit{for } x\in\mathbb{Z}.\end{cases}$$

Here the convergence is pointwise.

(b)

$$\frac{1}{m}\left(\Lambda_{m+1}-\frac{1}{m(m+1)}\boldsymbol{B}_m^{(r)}+\frac{2}{m(m+1)}E_{m+1}\right)$$
$$+\frac{1}{m}\sum_{s=2}^{m}\binom{m}{s}\Big(\Lambda_{m-s+1}+\frac{H_{m-1}-H_{m-s}}{m-s+1}(\boldsymbol{B}_{m-s}^{(r)}-2E_{m-s+1})\Big)B_s(<x>)$$
$$=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\boldsymbol{B}_k^{(r+1)}(<x>)E_{m-k}(<x>),\ \textit{for } x\notin\mathbb{Z},$$

$$\frac{1}{m}\left(\Lambda_{m+1}-\frac{1}{m(m+1)}\boldsymbol{B}_m^{(r)}+\frac{2}{m(m+1)}E_{m+1}\right)$$
$$+\frac{1}{m}\sum_{s=2}^{m}\binom{m}{s}\Big(\Lambda_{m-s+1}+\frac{H_{m-1}-H_{m-s}}{m-s+1}(\boldsymbol{B}_{m-s}^{(r)}-2E_{m-s+1})\Big)B_s(<x>)$$
$$=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\boldsymbol{B}_k^{(r+1)}E_{m-k}+\frac{1}{2}\Lambda_m,\ \textit{for } x\in\mathbb{Z}.$$

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Ellipticity of co-effective complex for locally conformally calibrated $\tilde{G}_2$-manifolds

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Abstract

First we characterize a differential subcopmlex of de de Rham complex for locally conformally calibrated $\tilde{G}_2$-manifolds. Then we give co-effective complex for $\tilde{G}_2$-manifolds and prove that in dimension different from 3 this complex is elliptic.

2010 Mathematics Subject Classification: 53C15, 53C10, 53C25, 53C30

Key words and phrases: locally conformally calibrated $\tilde{G}_2$-manifolds, co-effective complex, ellipticity of co-effective complex

1 Introduction

Recently, the theory of special G-structures on smooth manifolds has enjoyed a lot of success among mathematicians and physicist as they exhibit some nice properties. For example G_2-structure can be geometric models in the theory of super strings with torsion [16]. Also Donaldson and Segal [9] suggested recently that manifolds with non-vanishing torsion G_2-structure can be the right framework for guage theory in dimension 7. Main computable models for manifolds with G_2-structure are homogeneous spaces having cohomogeneity one [8, 22, 26].

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In 1884 Killing exposed a vigorous proof of the presence of smallest of the remarkable simple lie algebra g_2^C. In 1907, Reichel [25], a student of Engel [10], succeeded in achieving the uniform geometric explanation of the Lie groups G_2 and $\tilde{G}_2$, which are two real forms of G_2^C. In 1914, Cartan proved that G_2 and $\tilde{G}_2$ can be treated as the automorphism group of octonions and split-octonions respectively. Later these groups appeared in the Bereger's celebrated list of potential holonomy of pseudo-Riemannian mertic see [1]. In 1989 Bryant and Salamon [5] gave construction of first complete but non-compact Riemannian manifolds having holonomy G_2, while the first compact example was given by Joyce [17] in 1994. Fernández and Gray [13] classified all G_2-structures in 16 classes in 1982 by decomposing the covariant derivative in 4 irreducible components. A lot has already been said about these different classes. For example, in [24] Friedrich et all discussed special properties of nearly parallel G_2-structures and prove that they carry Einstein metric. Kath [18] initialized the study of psudo-Riemannian 7-manifolds with a $\tilde{G}_2$-structure. Munir and Nizami [24] gave classification of $\tilde{G}_2$-structures using intrinsic torsion with sixteen classes of algebraic types of $\tilde{G}_2$-structures and also proved some strict inclusion relations among the classes of these structures. Manifold with $\tilde{G}_2$ are relatively less explained as compared to those admitting G_2-structures. To our knowledge there are only a few papers discussing a few properties about them, see, for example, [4, 18–20, 22, 24].

We recall that a 7-dimensional smooth manifold M^7 is said to admit a $\tilde{G}_2$-structure if it has a section of the bundle $\mathcal{F}(M^7)/\tilde{G}_2$ on M^7, where $\mathcal{F}(M^7)$ is the frame bundle on M^7. It is noted that $\tilde{G}_2$ is the automorphism group of a 3-form $\tilde{\varphi}$ over $\mathbb{R}^7$ which is called a 3-form of $\tilde{G}_2$-type [21]. It is known that $GL(\mathbb{R}^7)$-orbit of $\tilde{\varphi}$ is an open orbit of the $GL(\mathbb{R}^7)$-action on $\Lambda^3(\mathbb{R}^7)$. A 3-form in that open orbit is known as indefinite 3-form. The presence of a $\tilde{G}_2$-structure on a manifold M^7 is equivalent to the presence of an indefinite differential 3-form $\tilde{\varphi}$ over M^7. A manifold with a $\tilde{G}_2$-structure is said to be parallel if $\nabla\tilde{\varphi} = 0$ or $d\tilde{\varphi} = d * \tilde{\varphi} = 0$ and almost parallel or calibrated if $d\tilde{\varphi} = 0$, locally conformal calibrated if $d\tilde{\varphi} = \theta \wedge \tilde{\varphi}$ where θ is the differential 1-form on M and $\theta = \frac{1}{4}(*(*d\tilde{\varphi} \wedge \tilde{\varphi})$ [3, 7, 11, 12].

In this paper, we study manifolds with a locally confromally calibrated $\tilde{G}_2$-structure which constitute the class $\mathcal{W}_2 \oplus \mathcal{W}_4$ of [24]. We first construct a differential sub-copmlex of de Rham complex for locally conformally calibrated $\tilde{G}_2$-manifolds, then we have a co-effective complex and determine its ellipticity. Bouche [2] constructed similar complex for symplectic manifolds where as Fernández and Ugrate [14] discussed the co-effective complex for G_2-manifolds. In Section 2 we describe some properties and representation of the group $\tilde{G}_2$ and construct the co-effective complex for locally conformal calibrated $\tilde{G}_2$-manifolds. We use this name as the complex is analogue to the complex developed by [2] for the case of symplectic manifolds. In Section 3 we discuss the ellipticity of this complex. However it is important to remark that we study these manifolds for two particular reasons. First, they having striking similarities with those admitting a G_2-structure and secondly, because of their interesting class in pseudo-Riemannian geometry, see [6, 27].

2 Co-effective complex for locally conformal calibrated $\tilde{G}_2$-manifolds

In this section first we introduce basic representations for $\tilde{G}_2$-manifolds. Then we give simple characterizations of locally conformal calibrated $\tilde{G}_2$-manifolds in the form of a complex.

Let $\Lambda^q(M)$ be the space of differential q-forms on M. Our main purpose is the study of those manifolds for which the sequence

$$\cdots \to \mathcal{B}^{q-1}(M) \xrightarrow{\hat{d}} \mathcal{B}^q(M) \xrightarrow{\hat{d}} \mathcal{B}^{q+1}(M) \to \cdots \tag{2.1}$$

is a differential complex. Here $\mathcal{B}^q(M)$ is the subspace of $\Lambda^q(M)$ defined by

$$\mathcal{B}^q(M) = \{\beta \in \Lambda^q(M) \mid \beta \wedge \tilde{\varphi} = 0\}$$

and $\hat{d}$ denotes the restriction to $\mathcal{B}^q(M)$ of the exterior differential d of M. A $\tilde{G}_2$-manifold is defined as a 7-dimensional Riemannian manifold M (in which a Riemannian metric $g_{\tilde{\varphi}} = (1,1,1,-1,-1,-1,-1)$ is defined) endowed with a 2-fold vector cross product P satisfying the following axioms

1. $\langle P(X_1, X_2), X_1\rangle = \langle P(X_1, X_2), X_2\rangle = 0$
2. $\|P(X_1, X_2)\|^2 = \|X_1\|^2\|X_2\|^2 - \langle X_1, X_2\rangle^2$

for $X_1, X_2 \in \mathfrak{X}(M)$. The fundamental 3-form on M is then defined as

$$\tilde{\varphi}(X_1, X_2, X_3) = \langle P(X_1, X_2), X_3\rangle$$

for $X_1, X_2, X_3 \in \mathfrak{X}(M)$ and inner product for $x, y \in \wedge^q(M)$ is defined as

$$\langle x, y\rangle V_M = x \wedge *y \tag{2.2}$$

where V_M is the volume form on M. It is proved that $\wedge^q(M)$ splits orthogonally into $\tilde{G}_2$-irreducible components $\wedge^q_l$ of dimension l [3]. An isometry known as Hodge star operator defined as $* : \wedge^q(M) \longrightarrow \wedge^{7-q}(M)$ make two irreducible component isomorphic. For example the representation of $\tilde{G}_2$ on $\wedge^1(M)$ and $\wedge^7(M)$ are isomorphic. So it is sufficient to describe the representation of $\tilde{G}_2$ on $\wedge^2(M)$ and $\wedge^3(M)$ as follows

$$\begin{cases} \wedge^2_7(M) = \{*(\alpha \wedge *\tilde{\varphi}) \mid \alpha \in \wedge^1(M)\} \\ \wedge^2_{14}(M) = \{\beta \in \wedge^2(M) \mid \beta \wedge *\tilde{\varphi} = 0\} \\ \wedge^3_1(M) = \{f\tilde{\varphi} \mid f \in \mathfrak{F}(M)\} \\ \wedge^3_7(M) = \{*(\alpha \wedge \tilde{\varphi}) \mid \alpha \in \wedge^1(M)\} \\ \wedge^3_{27}(M) = \{\gamma \in \wedge^3(M) \mid \gamma \wedge \tilde{\varphi} = \gamma \wedge *\tilde{\varphi} = 0. \end{cases} \tag{2.3}$$

From above, it is easy to compute

$$\wedge^3_1(M) \oplus \wedge^3_{27}(M) = \{\gamma \in \wedge^3(M) | \gamma \wedge \tilde{\varphi} = 0\}. \tag{2.4}$$

$$\wedge_7^4(M) \oplus \wedge_{27}^4(M) = \{\lambda \in \wedge^4(M) | \lambda \wedge \tilde{\varphi} = 0\}. \tag{2.5}$$

For a seven dimensional manifold M, a $\tilde{G}_2$-structure on M can be distinguished by a globally defined 3-form $\tilde{\varphi}$ which can be written at each point as

$$\tilde{\varphi} = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} + e^{347} + e^{356}$$

with respect to some local co frame $e^1, e^2, ..., e^7$ see [5]. It induces a Riemannian metric $g_{\tilde{\varphi}}$ and volume form $dV_{g\tilde{\varphi}}$ on M given by

$$g_{\tilde{\varphi}}(X, Y) = \frac{1}{6} i_X\tilde{\varphi} \wedge i_Y\tilde{\varphi} \wedge \tilde{\varphi}$$

for any pair of vector fields X, Y on M.

Now we have the following result [23].

Proposition 2.1. *Let M be a $\tilde{G}_2$-manifold with a fundamental 3-form $\tilde{\varphi}$. Then*

(1) *For any differential 1-form α on M, $*(*(\alpha \wedge \tilde{\varphi}) \wedge \tilde{\varphi}) = 4\alpha$.*

(2) *If there is a differential 1-form η on M such that $d\tilde{\varphi} = \eta \wedge \tilde{\varphi}$, then $\eta = \frac{1}{4}(*(*d\tilde{\varphi} \wedge \tilde{\varphi})$ and M is locally conformal calibrated.*

Definition 2.2. Let M be a $\tilde{G}_2$ manifold having 3-form $\tilde{\varphi}$. For each l, $0 \leq l \leq 7$, we denote the space $\mathcal{B}^l(M) = \{\lambda \in \Lambda^l(M) | \lambda \wedge \tilde{\varphi} = 0\}$. Also, the orthogonal compliment of $\mathcal{B}^l(M)$ in $\Lambda^q(M)$ is denoted by $\mathcal{A}^l(M)$.

Lemma 2.3. *Let M be a $\tilde{G}_2$-manifold. Then we have the following*

$$\begin{aligned} \mathcal{B}^l(M) &= \{0\} \quad \textit{for } 0 \leq l \leq 2, \\ \mathcal{B}^3(M) &= \Lambda_1^3(M) \oplus \Lambda_{27}^3(M), \\ \mathcal{B}^4(M) &= \Lambda_7^4(M) \oplus \Lambda_{27}^4(M), \\ \mathcal{B}^l(M) &= \Lambda^l(M) \quad \textit{for}\, 5 \leq l \leq 7. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{A}^l(M) &= \Lambda^l(M) \quad \textit{for } 0 \leq l \leq 2, \\ \mathcal{A}^3(M) &= \Lambda_7^3(M), \\ \mathcal{A}^4(M) &= \Lambda_1^4(M), \\ \mathcal{A}^q(M) &= \{0\} \quad \textit{for } 5 \leq l \leq 7. \end{aligned}$$

Proposition 2.4. *Let M be a $\tilde{G}_2$ manifold endowed with fundamental 3-form $\tilde{\varphi}$. Then M is locally conformal calibrated if and only if for any differential 3-form $\rho \in \Lambda_1^3(M) \oplus \Lambda_{27}^3(M)$, the exterior differential $d\rho \in \Lambda_7^4(M) \oplus \Lambda_{27}^4(M)$.*

In the following, we take $\mathcal{B}^3(M) = \Lambda_1^3(M) \oplus \Lambda_{27}^3(M)$ and $\mathcal{B}^4(M) = \Lambda_7^4(M) \oplus \Lambda_{27}^4(M)$.

Here we give the co-effective complex for locally conformal calibrated $\tilde{G}_2$-manifold.

Theorem 2.5. *Let M be a $\tilde{G}_2$ -manifold. Then M is locally conformal calibrated iff there exist the complex*

$$0 \to \Lambda_1^3(M) \oplus \Lambda_{27}^3(M) \xrightarrow{\hat{d}} \Lambda_7^4(M) \oplus \Lambda_{27}^4(M) \xrightarrow{\hat{d}} \Lambda^5(M) \xrightarrow{d} \Lambda^6(M) \xrightarrow{d} \Lambda^7(M) \to 0, \tag{2.6}$$

where $\hat{d}$ denotes the restriction to $\mathcal{B}^q(M)(q = 3, 4)$ of the exterior differential d of M

Proof. From Proposition 2.4 it is clear that (2.6) is a complex if M is locally conformal calibrated. To prove the converse, let us first show that for any $f \in \Im(M)$ and $y \in \mathcal{B}^3(M) = \Lambda^1_3(M) \oplus \Lambda^3_{27}(M)$ we have

$$\pi_4 od(fy) = f\pi_4 od(y), \tag{2.7}$$

that is, the operator $\pi_4 od : \mathcal{B}^3(M) \to \mathcal{A}^4(M)$ is tensorial, where π_4 denotes the orthogonal projection of $\Lambda^4(M)$ onto $\mathcal{A}^4(M) = \Lambda^4_1(M)$. In fact, since $y \in \Lambda^3_1(M) \oplus \Lambda^3_{27}(M)$, from equation (2.4) and equation (2.5) it follows that $df \wedge y \in \Lambda^4_7(M) \oplus \Lambda^4_{27}(M)$, that is, $\pi_4(df \wedge y) = 0$; thus

$$\pi_4 od(fy) = \pi_4(df \wedge y) + \pi_4(fdy) = f\pi_4(dy),$$

which shows equation (2.7) Now suppose that equation (2.6) is a complex, that is, $d(\hat{d}y) = 0$ for any $y \in \mathcal{B}^3(M)$. Since $dy = \pi_4 od(y) + \hat{d}y$, applying d to this equality we get

$$d(\pi_4 od(y)) = 0 \tag{2.8}$$

for any $y \in \mathcal{B}^3(M)$. Therefore, if f is any function on M, from equation (2.7) and equation (2.8) we get

$$0 = d(\pi_4 od(fy)) = d(f\pi_4 od(y)) = df \wedge \pi_4 od(y)$$

. Since $\pi_4 od(y) \in \Lambda^4_1(M)$, there is $h_y \in \Im(M)$ such that $\pi_4 od(y) = h_y * \tilde{\varphi}$ and thus $h_y(df \wedge *\tilde{\varphi}) = 0$, for any$f \in \Im(M)$. But $\alpha \wedge *\alpha = 0$ iff $\alpha = 0$, for $\alpha \in \Lambda^1(M)$, which implies that the function h_y must be zero. Therefore, $\pi_4 od(y) = 0$ for any $y \in \mathcal{B}^3(M)$, that is, $d(\mathcal{B}^3(M)) \subset \mathcal{B}^4(M)$, and Proposition 2.4 implies that M is locally calibrated. ☐

Definition 2.6. Let M be a $\tilde{G}_2$-manifold.For $0 \leq q \leq 3$, the map $\breve{d}_q : \mathcal{A}^q(M) \to \mathcal{A}^{q+1}(M)$ is defined by

$$\breve{d}_q = \pi_{q+1} od \tag{2.9}$$

where $\pi_{q+1} : \Lambda^{q+1}(M) \to \mathcal{A}^{q+1}(M)$ is the orthogonal projection of $\Lambda^{q+1}(M)$ onto $\mathcal{A}^{q+1}(M)$.

Theorem 2.7. *Let M be a $\tilde{G}_2$-manifold with fundamental 3-form φ. Then M is locally conformal calibrated if and if the sequence*

$$0 \to \Lambda^0(M) \xrightarrow{d} \Lambda^1(M) \xrightarrow{d} \Lambda^2(M) \xrightarrow{\breve{d}_2} \Lambda^3_7(M) \xrightarrow{\breve{d}_3} \Lambda^4_1(M) \to 0 \tag{2.10}$$

is a complex.

Proof. consider $\alpha \in \Lambda^1(M)$. From equation (2.9) we see that $\breve{d}_2(d\alpha) = \pi_3 od(d\alpha) = 0$. This proves that $\breve{d}_2 od = 0$. Now, let us suppose that M is locally conformal calibrated, and let $\beta \in \Lambda^2(M)$. Using the fact that

$$\Lambda^3(M) = \Lambda^3_1(M) \oplus \Lambda^3_7(M) \oplus \Lambda^3_{27}(M),$$

we have

$$d\beta = \breve{d}_2\beta + y, \tag{2.11}$$

where $\breve{d}_2\beta \in \mathcal{A}^3(M) = \Lambda^3_7(M)$ and $y \in \Lambda^3_1(M) \oplus \Lambda^3_{27}(M)$. Proposition 2.4 implies that $dy \in \Lambda^4_7(M) \oplus \Lambda^4_{27}(M)$. Then taking equation (2.11) the exterior differential d of M, we obtain

$$0 = d(\breve{d}_2\beta) + dy$$

which means that $d(\breve{d}_2\beta) \in \Lambda^4_7(M) \oplus \Lambda^4_{27}(M)$. Thus $\breve{d}_3(\breve{d}_2\beta) = 0$ because $\breve{d}_3(\breve{d}_2\beta)$ is the image of $d(\breve{d}_2\beta)$ by the orthogonal projection $\pi_4 : \Lambda^4(M) \to \mathcal{A}^4(M) = \Lambda^4_1(M)$. To prove the converse, let β be a 2-form on M. Therefore, the exterior differential $d\beta$ of β is

$$d\beta = \breve{d}_2\beta + y, \tag{2.12}$$

where $\breve{d}_2\beta \in \Lambda^3_7(M)$ and $y \in \Lambda^3_1(M) \oplus \Lambda^3_{27}(M)$. Appling exterior differential d of M on equation (2.12),we get

$$0 = d(\breve{d}_2\beta) + d\gamma. \tag{2.13}$$

Applying the projection π_4 to equation (2.13) and using equation (2.9) together with the hypothesis $\breve{d}_3 o \breve{d}_2 = 0$, we obtain

$$\begin{aligned} 0 &= \pi_4(d(\breve{d}_2\beta)) + \pi_4(d\gamma) \\ &= \breve{d}_3 o \breve{d}_2(\beta) + \pi_4(d\gamma) \\ &= \pi_4(d\gamma), \end{aligned}$$

which implies that $d\gamma \in \Lambda^4_7(M) \oplus \Lambda^4_{27}(M)$. Moreover, using equation (2.7) we conclude that

$$d(\Lambda^3_1(M) \oplus \Lambda^3_{27}(M)) \subset \Lambda^4_7(M) \oplus \Lambda^4_{27}(M).$$

From Proposition 2.4 it follows that M is locally conformal calibrated. □

3 Ellipticity of the coeffective complex

In this section we determine the ellipticity of the complex given in (2.6) and (2.10)

Theorem 3.1. *The complex given* $(\mathcal{A}^*(M), \breve{d})$ *in* (2.10) *is elliptic in degree* q *for any* $q \neq 2$.

Proof. It is obvious that the complex $(\mathcal{A}^*(M), \breve{d})$ is elliptic in degrees 0 and 1, because the de Rham complex $(\Lambda^*(M), d)$ of M is elliptic. The complex $(\mathcal{A}^*(M), \breve{d})$ is elliptic in degrees 3 and 4 if for any point $m \in M$ and for any 1-form μ non-zero at m, the complex

$$\Lambda^2(T^*_m M) \xrightarrow{\sigma_\mu(\breve{d}_2)} \Lambda^3_7(T^*_m M) \xrightarrow{\sigma_\mu(\breve{d}_3)} \Lambda^4_1(T^*_m M) \to 0$$

is exact in the steps 3 and 4, where $T^*_m M$ is the cotangent space of M at m,and

$$\sigma_\mu(\breve{d}_2)(\beta) = \pi_3(\mu \wedge \beta), \tag{3.1}$$

$$\sigma_\pi(\breve{d}_3(\gamma)) = \pi_4(\mu \wedge \gamma),$$

for $\beta \in \Lambda^2(T_m^*M)$ and $\gamma \in \Lambda_7^3(T_m^*M)$. Therefore, to prove that the complex $(\mathcal{A}^*(M), \breve{d})$ is elliptic in degree $q = 3$ it is sufficient to prove that

$$ker(\sigma_\pi(\breve{d}_3)) \subset Im(\sigma_\pi(\breve{d}_2)). \tag{3.2}$$

Let $\gamma \in \Lambda_7^3(T_m^*M)$ be such that $\gamma \in Ker(\sigma_\pi(\breve{d}_3))$, or equivalently $\pi_4(\mu \wedge \gamma) = 0$. This implies that $\mu \wedge \gamma \in \Lambda_7^4(T_m^*M) \oplus \Lambda_{27}^4(T_m^*M)$, and so $\mu \wedge \gamma \wedge \tilde{\varphi}_m = 0$. Since $\gamma \wedge \tilde{\varphi}_m \in \Lambda^6(T_m^*M)$,from the ellipticity of the de Rham complex it follows that there is $\eta \in \Lambda^5(T_m^*M)$ satisfying

$$\gamma \wedge \tilde{\varphi}_m = \mu \wedge \eta. \tag{3.3}$$

Now, we use the isomorphism $\Lambda\tilde{\varphi}_m : \Lambda^2(T_m^*M) \to \Lambda^5(T_m^*M)$ given by $\Lambda\tilde{\varphi}_m(\beta) = \beta \wedge \tilde{\varphi}_m$, for $\beta \in \Lambda^2(T_m^*M)$. This isomorphism implies that there is $\nu \in \Lambda^2(T_m^*M)$ such that $\eta = \nu \wedge \tilde{\varphi}_m$. Thus equation (3.2) becomes

$$\gamma \wedge \tilde{\varphi}_m = \mu \wedge \nu \wedge \tilde{\varphi}_m = \pi_3(\mu \wedge \nu) \wedge \tilde{\varphi}_m.$$

Therefore, we have

$$(\gamma - \pi_3(\mu \wedge \nu)) \wedge \tilde{\varphi}_m = 0. \tag{3.4}$$

But the wedge product by $\tilde{\varphi}_m$ is also an isomorphism $\Lambda\tilde{\varphi}_m : \Lambda_7^3(T_m^*M) \to \Lambda^6(T_m^*M)$ and so, from equation (3.4), it follows that $(\gamma - \pi_3(\mu \wedge \nu)) = 0$, using equation (3.1),

$$\gamma = \pi_3(\mu \wedge \nu) = \sigma_\pi(\breve{d}_2)(\nu),$$

which proves equation (3.2). To prove the ellipticity of the complex $(\mathcal{A}^*(M), \breve{d})$ in degree $q = 4$, we show

$$\Lambda_1^4(T_m^*M) \subset Im(\sigma_\pi(\breve{d}_3))$$

Let $\lambda \in \Lambda_1^4(T_m^*M)$. Then $\lambda \wedge \tilde{\varphi}_m \in \Lambda^7(T_m^*M)$. Now, from the ellipticity of the de Rham Complex of M, we conclude that

$$\mu \wedge \omega = \lambda \wedge \tilde{\varphi}_m, \tag{3.5}$$

for some $\omega \in \Lambda^6(T_m^*M)$. Using the isomorphism $\Lambda\tilde{\varphi}_m : \Lambda_7^3(T_m^*M) \to \Lambda^6(T_m^*M)$ again, we obtain $\omega = \gamma \wedge \tilde{\varphi}_m$ for some $\gamma \in \Lambda_7^3(T_m^*M)$. Then equation (3.5) becomes

$$\lambda \wedge \tilde{\varphi}_m = \mu \wedge \gamma \wedge \tilde{\varphi}_m = \pi_4(\mu \wedge \gamma) \wedge \tilde{\varphi}_m,$$

which implies that

$$(\lambda - \pi_4(\mu \wedge \gamma)) \wedge \tilde{\varphi}_m = 0. \tag{3.6}$$

But $\Lambda\tilde{\varphi}_m : \Lambda_1^4(T_m^*M) \to \Lambda^7(T_m^*M)$ is an isomorphism, and hence, from equation (3.6), we have

$$\lambda = \pi_4(\mu \wedge \gamma) = \sigma_\mu(\breve{d}_3)(\gamma).$$

Thus $\lambda \in Im(\sigma_\mu(\breve{d}_3))$. This completes the proof. $\square$

Remark 3.2. As

$$\sum_{q=0}(-1)^q dim(\mathcal{A}^q(T_m^*M)) = 1-7+21-7+1=9$$

so the complex $(\mathcal{A}^*(M), \check{d})$ is not elliptic in degree $q=2$.

Theorem 3.3. *The complex* $(\mathcal{B}^*(M), \hat{d})$ *given by* (2.6) *is elliptic in degree* q *for any* $q \neq 3$.

Proof. It is obvious that the complex $(\mathcal{B}^*(M), \hat{d})$ is elliptic in degrees 6 and 7, because it is the de Rham complex of M. Now we show that $(\mathcal{B}^*(M), \hat{d})$ is elliptic in degree $q=4$, we must prove that for $m \in M$ and for non-zero $\mu \in \mathrm{T}_m^*(M)$, the complex

$$\Lambda_1^3(T_m^*M) \oplus \Lambda_{27}^3(T_m^*M) \xrightarrow{\mu\wedge} \Lambda_7^4(T_m^*M) \oplus \Lambda_{27}^4(T_m^*M) \xrightarrow{\mu\wedge} \Lambda^5(T_m^*M) \tag{3.7}$$

is exact in degree 4. $\lambda \in \Lambda_7^4(T_m^*M) \oplus \Lambda_{27}^4(T_m^*M)$ satisfy $\mu \wedge \lambda = 0$. We must show that there is $\eta \in \Lambda_1^3(T_m^*M) \oplus \Lambda_{27}^3(T_m^*M)$ such that $\lambda = \mu \wedge \eta$. By the definition of ellipticity of the de Rham complex there exist $\eta_1 \in \Lambda^3(T_m^*M)$ such that

$$\lambda = \mu \wedge \eta_1, \tag{3.8}$$

where $\eta_1 = \eta_1{}' + \eta_1{}''$ with $\eta_1{}' \in \Lambda_7^3(T_m^*M)$ and $\eta_1{}'' \in \Lambda_1^3(T_m^*M) \oplus \Lambda_{27}^3(T_m^*M)$ Now equation (3.8) becomes

$$\lambda = \mu \wedge \eta_1 = \mu \wedge \eta_1{}' + \mu \wedge \eta_1{}''. \tag{3.9}$$

But λ and $\mu \wedge \eta_1{}'' \in \Lambda_7^4(T_m^*M) \oplus \Lambda_{27}^4(T_m^*M)$ hence $\pi_4(\mu \wedge \eta_1{}') = 0$, which implies that $\eta_1{}' \in Ker(\sigma_\mu(\check{d}_3))$. From Theorem 1.8 it follows that $\eta_1{}' \in Im(\sigma_\mu(\check{d}_2))$. This means that there exist $\omega \in \Lambda^2(T_m^*M)$ such that $\eta_1{}' \in \pi_3(\mu \wedge \omega)$. Let $\nu \in \Lambda_1^3(T_m^*M) \oplus \Lambda_{27}^3(T_m^*M)$ be the image of $\mu \wedge \omega$ by the orthogonal projection of $\Lambda^3(T_m^*M)$ onto $\Lambda_1^3(T_m^*M) \oplus \Lambda_{27}^3(T_m^*M)$. Then we get

$$0 = \mu \wedge (\mu \wedge \omega) = \mu \wedge \eta_1{}' + \mu \wedge \alpha$$

and we obtain $\lambda = \mu \wedge (-\alpha + \eta_1{}'')$. Now implies that the form $\eta = -\alpha + \eta_1{}''$ is such that $\eta \in \Lambda_1^3(T_m^*M) \oplus \Lambda_{27}^3(T_m^*M)$ and $\lambda \in= \mu \wedge \eta$. This proves that equation (3.7) is exact in degree 4.

Finally, we must prove that the complex

$$\Lambda_7^4(T_m^*M) \oplus \Lambda_{27}^4(T_m^*M) \xrightarrow{\mu\wedge} \Lambda^5(T_m^*M) \xrightarrow{\mu\wedge} \Lambda^6(T_m^*M)$$

is exact in degree 5. Let $\beta \in \Lambda^5(T_m^*M)$ satisfy $\mu \wedge \beta = 0$. We must find a 4-form $\xi \in \Lambda_7^4(T_m^*M) \oplus \Lambda_{27}^4(T_m^*M)$ such that

$$\beta = \mu \wedge \xi. \tag{3.10}$$

By the ellipticity of the de Rham complex of$M$ we see that there is $\alpha = \Lambda^4(T_m^*M)$ such that

$$\beta = \mu \wedge \alpha. \tag{3.11}$$

Because $\Lambda^4(T_m^*M) = \Lambda_1^4(T_m^*M) \oplus \Lambda_7^4(T_m^*M) \oplus \Lambda_{27}^4(T_m^*M)$ and $\alpha \in \Lambda^4(T_m^*M)$ we have

$$\alpha = \alpha' + \alpha'', \tag{3.12}$$

where $\alpha' \in \Lambda^4_1(T^*_mM)$ and $\alpha'' \in \Lambda^4_7(T^*_mM) \oplus \Lambda^4_{27}(T^*_mM)$. By Theorem 1.8 there exist $\eta \in \Lambda^3_7(T^*_mM)$ such that

$$\alpha' = \pi_4(\mu \wedge \eta) \tag{3.13}$$

from equation (3.13) it follows that

$$0 = \mu \wedge (\mu \wedge \eta) = \mu \wedge \alpha' + \mu \wedge \upsilon, \tag{3.14}$$

where υ is the image of $\nu \wedge \eta$ by the orthogonal projection of $\Lambda^4(T^*_mM)$ onto subspace $\Lambda^4_7(T^*_mM) \oplus \Lambda^4_{27}(T^*_mM)$. The identity equation (3.14) implies that $\mu \wedge \alpha' = -\mu \wedge \upsilon$. Thus from equation (3.11) and equation (3.12) we conclude that

$$\beta = \mu \wedge (-\upsilon + \alpha'')$$

Consider $\eta = -\upsilon + \alpha''$. Then $\xi \in \Lambda^4_7(T^*_mM) \oplus \Lambda^4_{27}(T^*_mM)$, and moreover $\beta = \mu \wedge \eta$. This proves equation (3.10) and completes the proof. $\square$

Remark 3.4.

$$\sum_{q=3}^{7} (-1)^q dim(\mathcal{B}^q(T^*_mM)) = -28 + 34 - 21 + 7 - 1$$

so complex $(\mathcal{B}^*(M), \hat{d})$ is not elliptic in degree $q = 3$.

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