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Editor in Chief: George Anastassiou
Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152-3240, U.S.A
ganastss@memphis.edu
http://www.msci.memphis.edu/~ganastss/jocaaa

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8000 Cummings Hall,
Hanover, NH 03755-8000
603-646-3843 (X 3546 Secr.)
e-mail:george.cybenko@dartmouth.edu
Approximation Theory and Neural Networks

Sever S. Dragomir
School of Computer Science and Mathematics, Victoria University,
PO Box 14428,
Melbourne City,
MC 8001, AUSTRALIA
Tel. +61 3 9688 4437
Fax +61 3 9688 4050
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Saber N. Elaydi
Department Of Mathematics
Trinity University
715 Stadium Dr.
San Antonio, TX 78212-7200
210-736-8246
e-mail: selaydi@trinity.edu
Ordinary Differential Equations, Difference Equations

J .A. Goldstein
Department of Mathematical Sciences
The University of Memphis
Memphis, TN 38152
901-678-3130
jgoldste@memphis.edu
Partial Differential Equations, Semigroups of Operators

H. H. Gonska
Department of Mathematics
University of Duisburg
Duisburg, D-47048
Germany
011-49-203-379-3542
e-mail: heiner.gonska@uni-due.de
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Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37304 USA
John-Graef@utc.edu
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Department of Mathematics
University of Iowa
Iowa City, IA 52242-1419
319-335-0770
e-mail: whan@math.uiowa.edu
Numerical analysis, Finite element method, Numerical PDE, Variational inequalities, Computational mechanics

Tian-Xiao He
Department of Mathematics and Computer Science
P.O. Box 2900, Illinois Wesleyan University
Bloomington, IL 61702-2900, USA
Tel (309)556-3089
Fax (309)556-3864
the@iwu.edu
Approximations, Wavelet, Integration Theory, Numerical Analysis, Analytic Combinatorics

Margareta Heilmann
Faculty of Mathematics and Natural Sciences, University of Wuppertal
Gaußstraße 20
D-42119 Wuppertal, Germany,
heilmann@math.uni-wuppertal.de
Approximation Theory (Positive Linear Operators)

Xing-Biao Hu
Institute of Computational Mathematics
AMSS, Chinese Academy of Sciences
Beijing, 100190, CHINA
hxb@lsec.cc.ac.cn

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Editor in Chief: George Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152-3240, U.S.A.

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Prof.George A. Anastassiou
Department of Mathematical Sciences
The University of Memphis
Memphis,TN 38152, USA.
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SOME COMPANIONS OF QUASI GRÜSS TYPE INEQUALITIES FOR COMPLEX FUNCTIONS DEFINED ON UNIT CIRCLE

JIAN ZHU AND QIAOLING XUE

Abstract. Several companions of quasi Grüss type inequalities for the Riemann-Stieltjes integral of continuous complex valued integrands defined on the complex unit circle $C(0,1)$ are given. Our results in special cases recapture some known results, and moreover, give a smaller estimator than that of these known results.

1. Introduction

Riemann-Stieltjes integral $\int_a^b f(t)du(t)$, where f is called the integrand and u the integrator, is an important concept in Mathematics. One can approximate the Riemann-Stieltjes integral $\int_a^b f(t)du(t)$ with the following simpler quantity (see [13, 14]):

$$\frac{u(b)-u(a)}{b-a}\int_a^b f(t)dt. \tag{1.1}$$

In order to provide a priory sharp bounds for the approximation error, Dragonir and Fedotov established the following functional in [13]:

$$D(f;u) := \int_a^b f(t)du(t) - \frac{u(b)-u(a)}{b-a}\int_a^b f(t)dt \tag{1.2}$$

and proved the following inequality of Grüss type for Riemann-Stieltjes integral

$$|D(f;u)| \le \frac{1}{2}K(b-a)\bigvee_a^b(u),$$

where u is of bounded variation on $[a,b]$ and f is Lipschitzian with the constant $K>0$, the constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity. In [1], the author studied a companion functional of (1.2). Introducing the functional

$$GS(f;u) := \int_a^{\frac{a+b}{2}} \frac{f(x)+f(a+b-x)}{2}du(x) - \frac{u\left(\frac{a+b}{2}\right)-u(a)}{b-a}\int_a^b f(t)dt, \tag{1.3}$$

provided that the Stieltjes integral $\int_a^{\frac{a+b}{2}} \frac{f(x)+f(a+b-x)}{2}du(x)$ and the Riemann integral $\int_a^b f(t)dt$ exist, the author proved several bounds for $GS(f;u)$. More specifically, the integrand f is assumed to be of $r-H$-Hölder's type and the integrator u is of bounded variation, Lipschitzian and monotonic, respectively.

For continuous functions $f: C(0,1)\to\mathbb{C}$, where $C(0,1)$ is the unit circle from $\mathbb{C}$ centered in O and $u:[a,b]\subseteq[0,2\pi]\to\mathbb{C}$ a function of bounded variation on $[a,b]$. In [15], Dragomir developed some quasi Grüss type inequalities for the Riemann-Stieltjes integral of continuous complex valued integrands defined on the complex unit circle $C(0,1)$.

2010 *Mathematics Subject Classification.* 26D15.

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Theorem 1.1. *Assume that $f : C(0,1) \to \mathbb{C}$ satisfies the following Hölder's type condition*

$$|f(a) - f(b)| \leq H |a-b|^r \tag{1.4}$$

for any $a, b \in C(0,1)$, where $H > 0$ and $r \in (0,1]$ are given. If $[a,b] \subseteq [0,2\pi]$ and the function $u : [a,b] \to \mathbb{C}$ is a function of bounded variation on $[a,b]$, then

$$\left| \int_a^b f\left(e^{it}\right) du(t) - \frac{u(b)-u(a)}{b-a} \int_a^b f\left(e^{it}\right) dt \right| \leq \frac{2^r H}{b-a} \max_{t\in[a,b]} B_r(a,b;t) \bigvee_a^b (u) \tag{1.5}$$

for any $t \in [a,b]$, where

$$B_r(a,b;t) := \int_a^t \left| \sin^r \left(\frac{t-s}{2} \right) \right| ds + \int_t^b \left| \sin^r \left(\frac{s-t}{2} \right) \right| ds.$$

For other inequalities for the Riemann-Stieltjes integral see [2]-[12], [16]-[26] and the references therein.

Motivated by the above facts, we consider in the present paper the problem of approximating the companions of Riemann-Stieltjes integral $\int_a^{\frac{a+b}{2}} \frac{f(e^{it}) + f(e^{i(a+b-t)})}{2} du(t)$. We denote the following functional of companions of quasi Grüss type:

$$D_c(f;u,a,b) := \int_a^{\frac{a+b}{2}} \frac{f(e^{it}) + f(e^{i(a+b-t)})}{2} du(t) - \frac{u\left(\frac{a+b}{2}\right) - u(a)}{b-a} \int_a^b f(e^{it}) dt. \tag{1.6}$$

In this paper we establish some bounds for the magnitude of $D_c(f;u,a,b)$ when the integrand $f : C(0,1) \to \mathbb{C}$ satisfies some Hölder's type conditions on the circle $C(0,1)$ while the integrator u is of bounded variation, Lipschitzian and monotonic, respectively.

2. The case of bounded variation integrators

Theorem 2.1. *Let $f : C(0,1) \to \mathbb{C}$ satisfy an H-r-Hölder's type condition on the circle $C(0,1)$, where $H > 0$ and $r \in (0,1]$ are given. If $u : [a,b] \subseteq [0,2\pi] \to \mathbb{C}$ is a function of bounded variation on $[a,b]$, then*

$$\begin{aligned} |D_c(f;u,a,b)| \leq & \frac{2^r H}{b-a} \max_{t\in[a,\frac{a+b}{2}]} B_r(a,b;t) \bigvee_a^{\frac{a+b}{2}} (u) \\ \leq & \frac{H}{r+1} (b-a)^r \bigvee_a^{\frac{a+b}{2}} (u), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} B_r(a,b;t) := & \int_a^t \sin^r \left(\frac{t-s}{2} \right) ds + \int_t^b \sin^r \left(\frac{s-t}{2} \right) ds \\ \leq & \frac{1}{2^r} \frac{(t-a)^{r+1} + (b-t)^{r+1}}{r+1} \end{aligned} \tag{2.2}$$

for any $t \in [a, \frac{a+b}{2}]$.

In particular, if f is Lipschitzian with the constant $L > 0$, and $[a,b] \subset [0,2\pi]$ with $b - a \neq 2\pi$, then we have the simpler inequality

$$|D_c(f;u,a,b)| \leq \frac{8L}{b-a} \sin^2 \left(\frac{b-a}{4} \right) \bigvee_a^{\frac{a+b}{2}} (u) \leq \frac{1}{2} L(b-a) \bigvee_a^{\frac{a+b}{2}} (u). \tag{2.3}$$

If $a = 0$ and $b = 2\pi$ and f is Lipschitzian with the constant $L > 0$, then

$$|D_c(f;u,0,2\pi)| \leq \frac{4L}{\pi} \bigvee_0^{\pi} (u). \tag{2.4}$$

Proof. We have

$$D_c(f;u,a,b)=\int_a^{\frac{a+b}{2}}\left[\frac{f(e^{it})+f(e^{i(a+b-t)})}{2}-\frac{1}{b-a}\int_a^b f(e^{is})ds\right]du(t) \tag{2.5}$$
$$=\frac{1}{b-a}\int_a^{\frac{a+b}{2}}\left(\int_a^b\left[\frac{f(e^{it})+f(e^{i(a+b-t)})}{2}-f(e^{is})\right]ds\right)du(t).$$

It is known that if $p:[c,d]\to\mathbb{C}$ is a continuous function and $v:[c,d]\to\mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integral $\int_c^d p(t)dv(t)$ exists and the following inequality holds

$$\left|\int_c^d p(t)dv(t)\right|\le\max_{t\in[c,d]}|p(t)|\bigvee_c^d(v). \tag{2.6}$$

Utilising this property and (2.5), we have

$$|D_c(f;u,a,b)|=\frac{1}{b-a}\left|\int_a^{\frac{a+b}{2}}\left(\int_a^b\left[\frac{f(e^{it})+f(e^{i(a+b-t)})}{2}-f(e^{is})\right]ds\right)du(t)\right| \tag{2.7}$$
$$\le\frac{1}{b-a}\max_{t\in[a,\frac{a+b}{2}]}\left|\int_a^b\left[\frac{f(e^{it})+f(e^{i(a+b-t)})}{2}-f(e^{is})\right]ds\right|\bigvee_a^{\frac{a+b}{2}}(u).$$

Utilising the properties of the Riemann integral and the fact that f is of H-r-Hölder's type on the circle $C(0,1)$ we have

$$\left|\int_a^b\left[\frac{f(e^{it})+f(e^{i(a+b-t)})}{2}-f(e^{is})\right]ds\right| \tag{2.8}$$
$$\le\int_a^b\left|\frac{f(e^{it})-f(e^{is})}{2}+\frac{f(e^{i(a+b-t)}-f(e^{is})}{2}\right|ds$$
$$\le\frac{1}{2}\int_a^b\left|f(e^{it})-f(e^{is})\right|ds+\frac{1}{2}\int_a^b\left|f(e^{i(a+b-t)}-f(e^{is})\right|ds$$
$$\le\frac{H}{2}\left(\int_a^b\left|e^{is}-e^{it}\right|^r ds+\int_a^b\left|e^{is}-e^{i(a+b-t)}\right|^r ds\right).$$

From [15], we have

$$\left|e^{is}-e^{it}\right|^r=2^r\left|\sin\left(\frac{s-t}{2}\right)\right|^r \tag{2.9}$$

for any $s,t\in\mathbb{R}$. Therefore

$$\int_a^b\left|e^{it}-e^{is}\right|^r ds+\int_a^b\left|e^{i(a+b-t)}-e^{is}\right|^r ds$$
$$=2^r\left(\int_a^b\left|\sin\left(\frac{s-t}{2}\right)\right|^r ds+\int_a^b\left|\sin(\frac{s+t-a-b}{2})\right|^r ds\right)$$
$$=2^r\left(\int_a^t\sin^r\left(\frac{t-s}{2}\right)ds+\int_t^b\sin^r\left(\frac{s-t}{2}\right)ds\right.$$
$$\left.+\int_a^{a+b-t}\sin^r\left(\frac{a+b-t-s}{2}\right)ds+\int_{a+b-t}^b\sin^r\left(\frac{s+t-a-b}{2}\right)ds\right).$$

Utilising the variable substitution $u = a + b - s$, we have

$$\int_a^{a+b-t} \sin^r\left(\frac{a+b-t-s}{2}\right) ds = \int_t^b \sin^r\left(\frac{s-t}{2}\right) ds$$

and

$$\int_{a+b-t}^b \sin^r\left(\frac{s+t-a-b}{2}\right) ds = \int_a^t \sin^r\left(\frac{t-s}{2}\right) ds.$$

So

$$\int_a^b \left|e^{it} - e^{is}\right|^r ds + \int_a^b \left|e^{i(a+b-t)} - e^{is}\right|^r ds = 2^{r+1}\left[\int_a^t \sin^r\left(\frac{t-s}{2}\right) ds + \int_t^b \sin^r\left(\frac{s-t}{2}\right) ds\right] \tag{2.10}$$

for any $t \in [a, \frac{a+b}{2}]$. Making use of (2.8) and (2.10), we have

$$\max_{t\in[a,\frac{a+b}{2}]}\left|\int_a^b \left[\frac{f(e^{it}) + f(e^{i(a+b-t)})}{2} - f(e^{is})\right] ds\right| \le 2^r H \max_{t\in[a,\frac{a+b}{2}]} B_r(a,b;t)$$

and the first inequality in (2.1) is proved.

Utilising the elementary inequality $|\sin(x)| \le |x|$, $x \in \mathbb{R}$, we have

$$B_r(a,b;t) \le \int_a^t \left(\frac{t-s}{2}\right)^r ds + \int_t^b \left(\frac{s-t}{2}\right)^r ds = \frac{1}{2^r}\frac{(t-a)^{r+1} + (b-t)^{r+1}}{r+1} \tag{2.11}$$

for any $t \in [a, \frac{a+b}{2}]$, and the inequality (2.2) is proved.

If we consider the auxiliary function $\varphi : [a, \frac{a+b}{2}] \to \mathbb{R}$,

$$\varphi(t) = (t-a)^{r+1} + (b-t)^{r+1},\ r \in (0,1],$$

then

$$\varphi'(t) = (r+1)[(t-a)^r - (b-t)^r]$$

and

$$\varphi''(t) = (r+1)r[(t-a)^{r-1} + (b-t)^{r-1}].$$

We have $\varphi'(t) = 0$ iff $t = \frac{a+b}{2}$ and $\varphi'(t) < 0$ for $t \in (a, \frac{a+b}{2})$. We also have $\varphi''(t) > 0$ for any $t \in (a, \frac{a+b}{2})$, which shows that φ is strictly decreasing on $(a, \frac{a+b}{2})$. In addition, we have

$$\min_{t\in[a,\frac{a+b}{2}]} \varphi(t) = \varphi\left(\frac{a+b}{2}\right) = \frac{(b-a)^{r+1}}{2^r}$$

and

$$\max_{t\in[a,\frac{a+b}{2}]} \varphi(t) = \varphi(a) = (b-a)^{r+1}.$$

Taking the maximum over $t \in [a, \frac{a+b}{2}]$ in (2.11) we deduce the second inequality in (2.1).

For $r = 1$ we have

$$B(a,b;t) := \int_a^t \sin\left(\frac{t-s}{2}\right) ds + \int_t^b \sin\left(\frac{s-t}{2}\right) ds = 4\left[\sin^2(\frac{t-a}{4}) + \sin^2(\frac{b-t}{4})\right]$$

for any $t \in [a, \frac{a+b}{2}]$.

Now, if we take the derivative in the first equality, we have

$$B'(a,b;t) = \sin\left(\frac{t-a}{2}\right) - \sin\left(\frac{b-t}{2}\right) = 2\sin\left(\frac{t-\frac{a+b}{2}}{2}\right)\cos\left(\frac{b-a}{4}\right),$$

for $[a,b] \subset [0,2\pi]$ and $b - a \ne 2\pi$.

We observe that $B'(a,b;t) = 0$ iff $t = \frac{a+b}{2}$, $B'(a,b;t) < 0$ for $t \in (a, \frac{a+b}{2})$. The second derivation of $B(a,b;t)$ is given by

$$B''(a,b;t) = \cos\left(\frac{t - \frac{a+b}{2}}{2}\right)\cos\left(\frac{b-a}{4}\right)$$

and we observe that $B''(a,b;t) > 0$ for $t \in (a, \frac{a+b}{2})$.

Therefore the function $B(a,b;t)$ is strictly decreasing on $(a, \frac{a+b}{2})$. It is also a strictly convex function on $(a, \frac{a+b}{2})$. We have

$$\min_{t\in[a,\frac{a+b}{2}]} B(a,b;t) = B\left(a, b; \frac{a+b}{2}\right) = 8\sin^2\left(\frac{b-a}{8}\right)$$

and

$$\max_{t\in[a,\frac{a+b}{2}]} B(a,b;t) = B(a,b;a) = 4\sin^2\left(\frac{b-a}{4}\right).$$

This proves the bound (2.3).

If $a = 0$ and $b = 2\pi$, then

$$B(0,2\pi;t) := 4\left[\sin^2\left(\frac{t}{4}\right) + \sin^2\left(\frac{2\pi - t}{4}\right)\right] = 4$$

and by (2.1) we get (2.4).

The proof is complete. □

3. The case of Lipschitzian integrators

The following result also holds.

Theorem 3.1. *Let $f : C(0,1) \to \mathbb{C}$ satisfy an H-r-Hölder's type condition on the circle $C(0,1)$, where $H > 0$ and $r \in (0,1]$ are given. If $u : [a,b] \subseteq [0,2\pi] \to \mathbb{C}$ is a function of Lipschitz type with the constant $K > 0$ on $[a,b]$, then*

$$|D_c(f;u,a,b)| \leq \frac{2^r HK}{b-a} C_r(a,b) \leq \frac{HK(b-a)^{r+1}}{(r+1)(r+2)}, \tag{3.1}$$

where

$$\begin{aligned} C_r(a,b) :=& \int_a^{\frac{a+b}{2}} \int_a^t \sin^r\left(\frac{t-s}{2}\right) dsdt + \int_a^{\frac{a+b}{2}} \int_t^b \sin^r\left(\frac{s-t}{2}\right) dsdt \\ \leq& \frac{(b-a)^{r+2}}{2^r(r+1)(r+2)}. \end{aligned} \tag{3.2}$$

In particular, if f is Lipschitzian with the constant $L > 0$, then we have the simpler inequality

$$\begin{aligned} |D_c(f;u,a,b)| \leq& \frac{8LK}{b-a}\left[\frac{b-a}{2} - \sin\left(\frac{b-a}{2}\right)\right] \\ \leq& \frac{LK(b-a)^2}{6}. \end{aligned} \tag{3.3}$$

Proof. It is known that if $p : [c,d] \to \mathbb{C}$ is a Riemann integrable function and $v : [c,d] \to \mathbb{C}$ is Lipschitzian with the constant $M > 0$, then the Riemann-Stieltjes integral $\int_c^d p(t)dv(t)$ exists and the following inequality holds

$$\left|\int_c^d p(t)dv(t)\right| \leq M \int_c^d |p(t)|\, d(t). \tag{3.4}$$

Utilising the equality (2.5) and this property, we have

$$
\begin{aligned}
|D_c(f;u,a,b)| =& \frac{1}{b-a}\left|\int_a^{\frac{a+b}{2}}\left(\int_a^b [\frac{f(e^{it})+f(e^{i(a+b-t)})}{2} - f(e^{is})]ds\right)du(t)\right| \\
\leq& \frac{K}{b-a}\int_a^{\frac{a+b}{2}}\left|\int_a^b\left[\frac{f(e^{it})+f(e^{i(a+b-t)})}{2} - f(e^{is})\right]ds\right|dt.
\end{aligned} \tag{3.5}
$$

From (2.8) and (2.10) we have

$$
\begin{aligned}
&\left|\int_a^b\left[\frac{f(e^{it})+f(e^{i(a+b-t)})}{2} - f(e^{is})\right]ds\right| \\
\leq& 2^r H\left[\int_a^t \sin^r\left(\frac{t-s}{2}\right)ds + \int_t^b \sin^r\left(\frac{s-t}{2}\right)ds\right],
\end{aligned} \tag{3.6}
$$

and by (3.5) we deduce the first part of (3.1).

By (2.11) we have

$$
\begin{aligned}
C_r(a,b) \leq& \int_a^{\frac{a+b}{2}}\left[\int_a^t\left(\frac{t-s}{2}\right)^r ds + \int_t^b\left(\frac{s-t}{2}\right)^r ds\right]dt \\
=& \frac{1}{2^r}\int_a^{\frac{a+b}{2}}\frac{(t-a)^{r+1}+(b-t)^{r+1}}{r+1}dt = \frac{(b-a)^{r+2}}{2^r(r+1)(r+2)},
\end{aligned}
$$

which proves the inequality (3.2).

For $r=1$ we have

$$
\begin{aligned}
C_1(a,b) :=& \int_a^{\frac{a+b}{2}}\left[\int_a^t \sin\left(\frac{t-s}{2}\right)ds + \int_t^b \sin\left(\frac{s-t}{2}\right)ds\right]dt \\
=& \int_a^{\frac{a+b}{2}}\left[4-2\cos\left(\frac{t-a}{a}\right)-2\cos\left(\frac{b-t}{a}\right)\right]dt = 4\left[\frac{b-a}{2}-\sin\left(\frac{b-a}{2}\right)\right],
\end{aligned}
$$

which by (3.1) produces the desired inequality (3.3). □

Remark 1. *For the case $a=0$ and $b=2\pi$ the inequality* (3.3) *is deduced to the simple inequality*

$$
|D_c(f;u,0,2\pi)| \leq 4Lk. \tag{3.7}
$$

4. The case of monotonic integrators

Theorem 4.1. *Let $f : C(0,1) \to \mathbb{C}$ satisfy an H-r-Hölder's type condition on the circle $C(0,1)$, where $H > 0$ and $r \in (0,1]$ are given. If $u : [a,b] \subseteq [0,2\pi] \to \mathbb{C}$ is a monotonically nondecreasing function on $[a,b]$, then*

$$
\begin{aligned}
|D_c(f;u,a,b)| \leq& \frac{2^r H}{b-a}D_r(a,b) \leq \frac{H}{(r+1)(b-a)}\int_a^{\frac{a+b}{2}}\left[(t-a)^{r+1}+(b-t)^{r+1}\right]du(t) \\
\leq& \frac{H}{r+1}(b-a)^r\left[u\left(\frac{a+b}{2}\right)-u(a)\right],
\end{aligned} \tag{4.1}
$$

where

$$
D_r(a,b) := \int_a^{\frac{a+b}{2}} B_r(a,b;t)du(t) \tag{4.2}
$$

and $B_r(a,b;t)$ is given by (2.2).

In particular, if f is Lipschitzian with the constant $L > 0$, then we have the simpler inequality

$$|D_c(f;u,a,b)| \leq \frac{8L}{b-a}\int_a^{\frac{a+b}{2}}\left[\sin^2\left(\frac{t-a}{4}\right)+\sin^2\left(\frac{b-t}{4}\right)\right]du(t) \tag{4.3}$$
$$\leq 2L(b-a)\left[u\left(\frac{a+b}{2}\right)-u(a)\right].$$

Proof. It is well known that if $p : [c,d] \to \mathbb{C}$ is a continuous function and $v : [c,d] \to \mathbb{R}$ is monotonically nondecreasing on $[c,d]$, then the Riemann-Stieltjes integral $\int_c^d p(t)dv(t)$ exists and the following inequality holds

$$\left|\int_c^d p(t)dv(t)\right| \leq \int_c^d |p(t)|\,dv(t). \tag{4.4}$$

Utilising this property and the identities (2.5) and (2.10) we have

$$|D_c(f;u,a,b)| = \frac{1}{b-a}\left|\int_a^{\frac{a+b}{2}}\left(\int_a^b\left[\frac{f(e^{it})+f(e^{i(a+b-t)})}{2}-f(e^{is})\right]ds\right)du(t)\right| \tag{4.5}$$
$$\leq \frac{1}{b-a}\int_a^{\frac{a+b}{2}}\left|\int_a^b\left[\frac{f(e^{it})+f(e^{i(a+b-t)})}{2}-f(e^{is})\right]ds\right|du(t)$$
$$\leq \frac{2^rH}{b-a}\int_a^{\frac{a+b}{2}}B_r(a,b;t)du(t) = \frac{2^rH}{b-a}D_r(a,b)$$
$$\leq \frac{H}{b-a}\int_a^{\frac{a+b}{2}}\frac{(t-a)^{r+1}+(b-t)^{r+1}}{r+1}du(t)$$

and the first part of the inequality (4.1) is proved.

Since $\max_{t\in[a,\frac{a+b}{2}]}[(t-a)^{r+1}+(b-t)^{r+1}] = (b-a)^{r+1}$, the last part of (4.1) is also proved.

For $r = 1$ we have

$$D_1(a,b) := \int_a^{\frac{a+b}{2}}B_1(a,b;t)du(t) = 4\int_a^{\frac{a+b}{2}}\left[\sin^2\left(\frac{t-a}{4}\right)+\sin^2\left(\frac{b-t}{4}\right)\right]du(t),$$

and the inequality (4.3) is obtained. □

Remark 2. *For the case $a = 0$, $b = 2\pi$ the inequality (4.3) can be stated as*

$$|D_c(f;u,0,2\pi)| \leq \frac{4L}{\pi}[u(\pi)-u(0)]. \tag{4.6}$$

Indeed, by (4.3) we have

$$|D_c(f;u,0,2\pi)| \leq \frac{8L}{2\pi}\int_0^{\pi}\left[\sin^2\left(\frac{t}{4}\right)+\sin^2\left(\frac{2\pi-t}{4}\right)\right]du(t)$$
$$= \frac{4L}{\pi}\int_0^{\pi}du(t) = \frac{4L}{\pi}[u(\pi)-u(0)].$$

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(J. Zhu) College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China

E-mail address: `zhujian1018@yeah.net`

(Q. Xue) College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China

A FIXED POINT APPROACH TO THE STABILITY OF QUADRATIC (ρ_1, ρ_2)-FUNCTIONAL INEQUALITIES

SUNGSIK YUN

Abstract. In this paper, we introduce and solve the following quadratic (ρ_1, ρ_2)-functional inequalities

$$\begin{aligned} &\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\ &\leq \left\|\rho_1\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\| \\ &+\left\|\rho_2\left(4f\left(\frac{x+y}{2}\right)+f\left(x-y\right)-2f(x)-2f(y)\right)\right\|, \end{aligned} \tag{0.1}$$

where ρ_1 and ρ_2 are fixed nonzero complex numbers with $\frac{|\rho_1|}{2}+|\rho_2|<1$, and

$$\begin{aligned} &\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\ &\leq \left\|\rho_1\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\| \\ &+\left\|\rho_2\left(2f\left(x+y\right)+2f\left(x-y\right)-f(2x)-f(2y)\right)\right\|, \end{aligned} \tag{0.2}$$

where ρ_1 and ρ_2 are fixed nonzero complex numbers with $\frac{|\rho_1|}{2}+2|\rho_2|<1$.

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic (ρ_1, ρ_2)-functional inequalities (0.1) and (0.2) in complex Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [29] concerning the stability of group homomorphisms.

The functional equation $f(x+y)=f(x)+f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [21] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [11] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The stability of quadratic functional equation was proved by Skof [28] for mappings $f: E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [8] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group.

Park [16, 17] defined additive ρ-functional inequalities and proved the Hyers-Ulam stability of the additive ρ-functional inequalities in Banach spaces and non-Archimedean Banach spaces. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 3, 7, 10, 15, 18, 19, 22, 23, 24, 25, 26, 27, 30]).

We recall a fundamental result in fixed point theory.

Theorem 1.1. [4, 9] *Let (X, d) be a complete generalized metric space and let $J: X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

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for all nonnegative integers n or there exists a positive integer n_0 such that
(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ for all $y \in Y$.*

In 1996, Isac and Rassias [13] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 20]).

In Section 2, we solve the quadratic (ρ_1, ρ_2)-functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic (ρ_1, ρ_2)-functional inequality (0.1) in Banach spaces by using the fixed point method.

In Section 3, we solve the quadratic (ρ_1, ρ_2)-functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic (ρ_1, ρ_2)-functional inequality (0.2) in Banach spaces by using the fixed point method.

Throughout this paper, let X be a real or complex normed space with norm $\|\cdot\|$ and Y a complex Banach space with norm $\|\cdot\|$.

2. Quadratic (ρ_1, ρ_2)-functional inequality (0.1)

Throughout this section, assume that ρ_1 and ρ_2 are fixed nonzero complex numbers with $\frac{|\rho_1|}{2} + |\rho_2| < 1$.

In this section, we solve and investigate the quadratic (ρ_1, ρ_2)-functional inequality (0.1) in complex Banach spaces.

Lemma 2.1. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\quad \leq \left\|\rho_1 \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \\ &\quad + \left\|\rho_2 \left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right)\right\| \end{aligned} \tag{2.1}$$

for all $x, y \in X$, then $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (2.1).

Letting $y = x$ in (2.1), we get $\|f(2x) - 4f(x)\| \leq 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4} f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\quad \leq \left\|\rho_1 \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \\ &\quad + \left\|\rho_2 \left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right)\right\| \\ &\quad = \left\|\frac{\rho_1}{2} \left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right)\right\| \\ &\quad + \|\rho_2 \left(f(x+y) + f(x-y) - 2f(x) - 2f(y)\right)\| \\ &\quad = \left(\frac{|\rho_1|}{2} + |\rho_2|\right) \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \end{aligned}$$

for all $x, y \in X$. Since $\frac{|\rho_1|}{2} + |\rho_2| < 1$, $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$. Thus f is quadratic. □

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic (ρ_1, ρ_2)-functional inequality (2.1) in complex Banach spaces.

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \le \frac{L}{4}\varphi(x, y) \tag{2.3}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\le \left\|\rho_1\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \\ &+ \left\|\rho_2\left(4f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - 2f(y)\right)\right\| + \varphi(x, y) \end{aligned} \tag{2.4}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{L}{4(1-L)}\varphi(x, x)$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.4), we get

$$\|f(2x) - 4f(x)\| \le \varphi(x, x) \tag{2.5}$$

for all $x \in X$.

Consider the set $S := \{h : X \to Y, \quad h(0) = 0\}$ and introduce the generalized metric on S:

$$d(g, h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \le \mu\varphi(x, x), \ \forall x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [14]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then $\|g(x) - h(x)\| \le \varepsilon\varphi(x, x)$ for all $x \in X$. Hence

$$\|Jg(x) - Jh(x)\| = \left\|4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right)\right\| \le 4\varepsilon\varphi\left(\frac{x}{2}, \frac{x}{2}\right) \le 4\varepsilon\frac{L}{4}\varphi(x, x) = L\varepsilon\varphi(x, x)$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that $d(Jg, Jh) \le Ld(g, h)$ for all $g, h \in S$.

It follows from (2.5) that

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \le \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \le \frac{L}{4}\varphi(x, x)$$

for all $x \in X$. So $d(f, Jf) \le \frac{L}{4}$.

By Theorem 1.1, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q(x) = 4Q\left(\frac{x}{2}\right) \tag{2.6}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0,\infty)$ satisfying $\|f(x)-Q(x)\| \le \mu\varphi(x,x)$ for all $x \in X$;

(2) $d(J^l f, Q) \to 0$ as $l \to \infty$. This implies the equality $\lim_{l\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$ for all $x \in X$;

(3) $d(f,Q) \le \frac{1}{1-L} d(f, Jf)$, which implies

$$\|f(x)-Q(x)\| \le \frac{L}{4(1-L)}\varphi(x,x)$$

for all $x \in X$.

It follows from (2.3) and (2.4) that

$$\begin{aligned}
&\|Q(x+y)+Q(x-y)-2Q(x)-2Q(y)\| \\
&= \lim_{n\to\infty} 4^n \left\|f\left(\frac{x+y}{2^n}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right\| \\
&\le \lim_{n\to\infty} 4^n|\rho_1| \left\|2f\left(\frac{x+y}{2^{n+1}}\right)+2f\left(\frac{x-y}{2^{n+1}}\right)-f\left(\frac{x}{2^n}\right)-f\left(\frac{y}{2^n}\right)\right\| \\
&+ \lim_{n\to\infty} 4^n|\rho_2| \left\|4f\left(\frac{x+y}{2^{n+1}}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)-2f\left(\frac{y}{2^n}\right)\right\| + \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n},\frac{y}{2^n}\right) \\
&= \left\|\rho_1\left(2Q\left(\frac{x+y}{2}\right)+2Q\left(\frac{x-y}{2}\right)-Q(x)-Q(y)\right)\right\| \\
&+ \left\|\rho_2\left(4Q\left(\frac{x+y}{2}\right)+Q(x-y)-2Q(x)-2Q(y)\right)\right\|
\end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned}
&\|Q(x+y)+Q(x-y)-2Q(x)-2Q(y)\| \\
&\le \left\|\rho_1\left(2Q\left(\frac{x+y}{2}\right)+2Q\left(\frac{x-y}{2}\right)-Q(x)-Q(y)\right)\right\| \\
&+ \left\|\rho_2\left(4Q\left(\frac{x+y}{2}\right)+Q(x-y)-2Q(x)-2Q(y)\right)\right\|
\end{aligned}$$

for all $x, y \in X$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic. □

Corollary 2.3. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned}
&\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\
&\le \left\|\rho_1\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\| \\
&+ \left\|\rho_2\left(4f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)-2f(y)\right)\right\| + \theta(\|x\|^r+\|y\|^r)
\end{aligned} \tag{2.7}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x)-Q(x)\| \le \frac{2\theta}{2^r-4}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x,y) = \theta(\|x\|^r+\|y\|^r)$ for all $x, y \in X$. Choosing $L = 2^{2-r}$, we obtain the desired result. □

Theorem 2.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le 4L\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.4). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{4(1-L)}\varphi(x, x)$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x)$$

for all $x \in X$.

It follows from (2.5) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{1}{4}\varphi(x, x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $r < 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.7). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{2\theta}{4 - 2^r}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Choosing $L = 2^{r-2}$, we obtain the desired result. □

Remark 2.6. If ρ is a real number such that $\frac{|\rho_1|}{2} + |\rho_2| < 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

3. Quadratic (ρ_1, ρ_2)-functional inequality (0.2)

Throughout this section, assume that ρ_1 and ρ_2 are fixed nonzero complex numbers with $\frac{|\rho_1|}{2} + 2|\rho_2| < 1$.

In this section, we solve and investigate the quadratic (ρ_1, ρ_2)-functional inequality (0.2) in complex Banach spaces.

Lemma 3.1. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\le \left\|\rho_1\left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\right)\right\| \\ &+ \|\rho_2\left(2f(x+y) + 2f(x-y) - f(2x) - f(2y)\right)\| \end{aligned} \tag{3.1}$$

for all $x, y \in X$, then $f : X \to Y$ is quadratic.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $y = x$ in (3.1), we get $\|f(2x) - 4f(x)\| \le 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}
&\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\
&\le \left\|\rho_1\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\| \\
&+\|\rho_2\left(2f\left(x+y\right)+2f\left(x-y\right)-f(2x)-f(2y)\right)\| \\
&= \left\|\frac{\rho_1}{2}\left(f(x+y)+f(x-y)-2f(x)-2f(y)\right)\right\| \\
&+\|2\rho_2\left(f(x+y)+f(x-y)-2f(x)-2f(y)\right)\| \\
&= \left(\frac{|\rho_1|}{2}+2|\rho_2|\right)\|f(x+y)+f(x-y)-2f(x)-2f(y)\|
\end{aligned}$$

for all $x,y\in X$. Since $\frac{|\rho_1|}{2}+2|\rho_2|<1$, $f(x+y)+f(x-y)=2f(x)+2f(y)$ for all $x,y\in X$. Thus f is quadratic. □

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic (ρ_1,ρ_2)-functional inequality (3.1) in complex Banach spaces.

Theorem 3.2. *Let $\varphi: X^2\to[0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi\left(\frac{x}{2},\frac{y}{2}\right)\le\frac{L}{4}\varphi\left(x,y\right)$$

for all $x,y\in X$. Let $f:X\to Y$ be a mapping satisfying $f(0)=0$ and

$$\begin{aligned}
&\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\
&\le \left\|\rho_1\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\| \\
&+\|\rho_2\left(2f\left(x+y\right)+2f\left(x-y\right)-f(2x)-f(2y)\right)\|+\varphi(x,y)
\end{aligned} \tag{3.3}$$

for all $x,y\in X$. Then there exists a unique quadratic mapping $Q:X\to Y$ such that

$$\|f(x)-Q(x)\|\le\frac{L}{4(1-L)}\varphi\left(x,x\right)$$

for all $x\in X$.

Proof. Letting $y=x$ in (3.3), we get

$$\|f(2x)-4f(x)\|\le\varphi(x,x) \tag{3.4}$$

for all $x\in X$.

Let (S,d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J:S\to S$ such that

$$Jg(x):=4g\left(\frac{x}{2}\right)$$

for all $x\in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.3. *Let $r>2$ and θ be nonnegative real numbers, and let $f:X\to Y$ be a mapping satisfying $f(0)=0$ and*

$$\begin{aligned}
&\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\
&\le \left\|\rho_1\left(2f\left(\frac{x+y}{2}\right)+2f\left(\frac{x-y}{2}\right)-f(x)-f(y)\right)\right\| \\
&+\|\rho_2\left(2f\left(x+y\right)+2f\left(x-y\right)-f(2x)-f(2y)\right)\|+\theta(\|x\|^r+\|y\|^r)
\end{aligned} \tag{3.5}$$

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for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{2\theta}{2^r - 4}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Choosing $L = 2^{2-r}$, we obtain the desired result. □

Theorem 3.4. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \le 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.3). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{4(1 - L)}\varphi(x, x)$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x)$$

for all $x \in X$.

It follows from (3.4) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{1}{4}\varphi(x, x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 3.5. *Let $r < 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.5). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{2\theta}{4 - 2^r}\|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Choosing $L = 2^{r-2}$, we obtain the desired result. □

Remark 3.6. If ρ is a real number such that $\frac{|\rho_1|}{2} + 2|\rho_2| < 1$ and Y is a real Banach space, then all the assertions in this section remain valid.

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Sungsik Yun
Department of Financial Mathematics, Hanshin University, Gyeonggi-do 18101, Korea
E-mail address: ssyun@hs.ac.kr

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A COMPARISON BETWEEN CAPUTO AND CANAVATI FRACTIONAL DERIVATIVES

AMMAR KHANFER

ABSTRACT. We investigate some properties of Caputo and Canavati fractional derivatives, and study some connections and comparisons between them. It turns out that the Canavati-type definition works more efficiently than the Caputo-type, and overcomes all the pitfalls of Caputo-type.

1. Introduction

The purpose of this paper is to make a comparison study between two of the important fractional derivatives, namely the Caputo derivative and the Canavati derivative. The Caputo-type has been proposed by Caputo and has been used in a wide spectrum of research for a long time and became popular among researchers due to some of its nice properties. The Canavati type has been proposed by Canavati [6], and has appeared in the work of Anastassiou [1,2,3] and in the work of M. Andric et al [4,5], and others.

2. Background

Definition 1. *Riemann - Liouville derivative*: For $n-1 \leq \alpha < n$, the α^{th} derivative of f is defined as

$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \cdot \frac{d^n}{dx^n} \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt,$$

where Γ is the gamma function.

For simplicity, throughout this paper we will consider $a = 0$. The major drawbacks of the R-L derivative are summarized into the following: 1. $D^{\alpha}(1) = \frac{x^{\alpha}}{\Gamma(1-\alpha)} \neq 0$, i.e. $D^{1/2}1 \neq 0$ and $D^{3/2}1 \neq 0$. 2. Taking the Laplace transform of the derivative gives $\mathcal{L}\{D^{\alpha} f\} = S^{\alpha} F(s) - \sum_{k=1}^{n} s^{n-k} [D^{\alpha-n+k-1} f(t)](0)$. So the initial conditions accompany the fractional differential equations of R-L type are usually expressed in terms of fractional derivatives, which have no obvious physical interpretation.

Definition 2. *Caputo derivative*: For $n - 1 \leq \alpha < n$, the α^{th} derivative of f is defined as

$$^{C}D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt.$$

Key words and phrases. fractional derivative, Canavati-type definition, Caputo-type definition, fractional differential equations.

One of the advantages of this derivative, is that taking the Laplace transform gives: $\mathcal{L}\{{}^C D^\alpha f\} = S^\alpha F(s) - \sum_{k=1}^{n} s^{\alpha-k} f^{(k-1)}(0)$, i.e., the initial conditions are expressed in terms of derivatives of integer order, which is fortunate to the physicists and engineers in their applications. However, the following are the major issues with the Caputo derivative:

(1) The Caputo definition finds the α^{th} derivative in terms of the n^{th} derivative for $\alpha < n$, i.e. we need to obtain the higher order derivatives in order to obtain the lower derivatives, which is the backward direction opposite to the natural process of differentiation. This also presumes the n- differentiability of f, so if $n-1 < \alpha < n$ then f needs to be n^{th} differentiable in order to be α^{th} differentiable.

(2) It's not always correct that ${}^C D^0 f(x) = f(x)$, unless $f(0) = 0$. For example, ${}^C D^0(x^2+1) = x^2$. This is due to the fact that the Caputo derivative obeys the formula $\lim_{\alpha\to n-1} {}^C D^\alpha f = f^{(n-1)}(t) - f^{(n-1)}(0)$ for any $n-1 < \alpha < n \in \mathbb{N}$. Nevertheless, subtracting from the function the value of the function at the lower terminal means that the function can be recovered with a difference by a constant term.

(3) ${}^C D^\alpha 1 = 0$ for all $\alpha \geq 0$. Although this may be fortunate when $\alpha \geq 1$, it's not the case for $\alpha < 1$.

Definition 3. *Canavati derivative.* Let $n \geq 1$ be an integer number, and $n-1 \leq \alpha < n$. Then the α^{th} derivative of $f(x)$ is given by

$$ {}^\star D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d}{dx} \int_0^x \frac{f^{(n-1)}(t)}{(x-t)^{\alpha-n+1}} dt, \tag{2.1}$$

where $f^{(n-1)}$ is the $(n-1)^{th}$ derivative of f. In the next section we will see that this definition overcomes all the aforementioned issues.

3. Properties

We state some results that discuss properties and relations between the three types of derivatives.

Proposition 4. *Assume that f has sufficient regularity on $[a,b]$. Then*

(1) ${}^\star D^\alpha 1 = 0$ for all $\alpha \geq 1$.
(2) ${}^\star D^\alpha (f'(x)) = \frac{d}{dx}({}^C D^\alpha f(x))$.

Proof. Immediate consequence from the definitions of derivatives. □

If both $D^\alpha f$ and ${}^C D^\alpha f$ exist, then is well known in the literature that

$$ {}^C D^\alpha f(t) = D^\alpha f(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(0), \tag{3.1}$$

for $n-1 \leq \alpha < n$ and $t > 0$. The proof can be found in [8] and [10]. Formula (4.1) shows that the R-L derivative and Caputo derivative are identical if the derivatives of the function up to $(n-1)^{th}$ derivative are vanished at zero or whatever the lower terminal of the definition is. The next result gives a simple sufficient condition for the existence of Canavati derivative and it's connection with the Caputo derivative.

Theorem 5. *Let $n-1 \leq \alpha < n$ and $f \in A^n[0,b]$ with $f^{(n-1)}(0)$ exists. Then ${}^{\star}D^{\alpha} f$ and ${}^{C}D^{\alpha} f$ exist a.e., and*

$$^{\star}D^{\alpha} f(x) = ^{C} D^{\alpha} f(x) + \frac{1}{\Gamma(n-\alpha)} \cdot \frac{f^{(n-1)}(0)}{x^{\alpha-n+1}}. \tag{3.2}$$

Proof. Let $D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d}{dx} \int_0^x \frac{f^{(n-1)}(t)}{(x-t)^{\alpha-n+1}} dt$. Since $f^{(n-1)}$ is absolutely continuous on $[0,b]$, then

$$^{\star}D^{\alpha} f = \frac{1}{\Gamma(n-\alpha)} \frac{d}{dx} \int_0^x [f^{(n-1)}(0) + \int_0^t f^{(n)}(u) du](x-t)^{n-\alpha-1} dt$$

But this is just equal to

$$^{\star}D^{\alpha} f = \frac{1}{\Gamma(n-\alpha)} \frac{f^{(n-1)}(0)}{x^{\alpha-n+1}} + \frac{1}{\Gamma(n-\alpha)} \frac{d}{dx} \int_0^x \int_0^t f^{(n)}(u)(x-t)^{n-\alpha-1} du dt.$$

Interchanging the order of integration using Fubini's theorem, this gives

$$^{\star}D^{\alpha} f = \frac{1}{\Gamma(n-\alpha)} \frac{f^{(n-1)}(0)}{x^{\alpha-n+1}} + \frac{1}{\Gamma(n-\alpha)} \frac{d}{dx} \int_0^x \int_u^x f^{(n)}(u)(x-t)^{n-\alpha-1} dt du.$$

$$= \frac{1}{\Gamma(n-\alpha)} \frac{f^{(n-1)}(0)}{x^{\alpha-n+1}} + \frac{1}{\Gamma(n-\alpha)} \frac{d}{dx} \int_0^x f^{(n)}(u) \frac{(x-u)^{n-\alpha}}{n-\alpha} dt du.$$

We then use Leibniz integral formula in the integral or results from classical measure theory, and this completes the proof. □

An immediate corollary which can be proved using (4.1) and (4.2) is the following.

Corollary 6. *Let $n-1 \leq \alpha < n$. Then*

$$^{\star}D^{\alpha} f(t) = D^{\alpha} f(t) - \sum_{k=0}^{n-2} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(0). \tag{3.3}$$

Example 7. Let $f(x) = x^2 + x + 1$. Then $f'(x) = 2x + 1$. Consider the following two cases: First, let $\alpha = 1/2$, then $n = 1$. We calculate ${}^{C}D^{1/2} f$ using each one of the definitions, we get ${}^{C}D^{1/2} f = \frac{1}{\Gamma(1/2)} \int_0^x \frac{2t+1}{(x-t)^{1/2}} dt$. Performing integration by parts, gives $\frac{1}{\sqrt{\pi}} \cdot (2\sqrt{x} + \frac{8}{3} x^{3/2})$. Similarly, ${}^{K}D^{1/2} f = \frac{1}{\Gamma(1/2)} \cdot \frac{d}{dx} \int_0^x \frac{t^2+t+1}{(x-t)^{1/2}} dt = D^{1/2} f$. Performing integration by parts gives: $\frac{1}{\sqrt{\pi}} . (2\sqrt{x} + \frac{8}{3} x^{3/2} + \frac{1}{\sqrt{x}}) = D_C^{1/2} f + \frac{1}{\sqrt{\pi x}}$. Note that the second term is in the form of $\frac{1}{\Gamma(n-\alpha)} \cdot \frac{f^{(n-1)}(0)}{x^{\alpha-n+1}}$ which is indeed the second term in (3.2). Let $\alpha = 3/2$. We calculate the α^{th} derivative of f using all three definitions, we obtain $D^\alpha f = \frac{4\sqrt{x}}{\Gamma(1/2)} + \frac{1}{\Gamma(1/2)\sqrt{x}} + \frac{x^{-3/2}}{\Gamma(-1/2)}$, ${}^K D^\alpha f = \frac{4\sqrt{x}}{\Gamma(1/2)} + \frac{1}{\Gamma(1/2)\sqrt{x}}$, and ${}^C D^\alpha f = \frac{4\sqrt{x}}{\Gamma(1/2)}$. It's clear that all three derivatives satisfy (3.1) and (3.2).

Another property that needs to be discussed is the compatibility condition. The condition reads: $D^\alpha f(x) \to f^{(n)}(x)$ as $\alpha \to n$ for any $\alpha \geq 0$. If $n = 0$ then the condition reduces to the identity condition: $D^0 f(x) \to f(x)$ as $\alpha \downarrow 0$. The property is essential in the theory as it demonstrates that the fractional derivative is the natural extension of the classical derivative.

Theorem 8. *Let f be such that $D^\alpha f(x)$ exists, and $n-1 \leq \alpha < n$. Then*

(1) $\lim\limits_{\alpha\to n} {}^C D^\alpha f(x) = f^{(n)}(x)$, and $\lim\limits_{\alpha\to n-1} {}^C D^\alpha f(x) = f^{(n-1)}(x) - f^{(n-1)}(0)$.
(2) $\lim\limits_{\alpha\to n} {}^\star D^\alpha f(x) = f^{(n)}(x)$, and $\lim\limits_{\alpha\to n-1} {}^\star D^\alpha f(x) = f^{(n-1)}(x)$.

Proof. For (1) see [7] or [8]. To prove (2), we perform integration by parts in the definition of ${}^\star D^\alpha$ to obtain

$$(3.4)\quad {}^\star D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha+1)}[(n-\alpha)\cdot f^{(n-1)}(0)\cdot x^{n-\alpha-1} + \frac{d}{dx}\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n}}dt.$$

Take $\alpha \to n$, we get

$$\lim_{\alpha\to n} {}^\star D^\alpha f(x) = \frac{d}{dx}[f^{(n)}(x) - f^{(n-1)}(0)] = f^{(n)}(x).$$

Also, take $\alpha \to n-1$ in (3.4) to get

$${}^\star D^\alpha f(x) = f^{(n-1)}(0) + \frac{d}{dx}\int_0^x (x-t) f^{(n)}(t)dt.$$

Perform integration by parts in the integral in the right hand side of the equation, then differentiate with respect to x gives the result. □

The theorem shows that the R-L and Canavati definitions works better than the Caputo type in terms of backward compatibility.

Definition 9. Let f be a function, and D be a derivative operator. If $D^r f = 0$ for some $r \in \mathbb{R}$, then we say that D^r is an "*$f-$annihilator*". If $D^{r_0} f = 0$ and $D^r f \neq 0$ for every $r < r_0$, then the number r_0 is called: "*the least order of $f-$ annihilator*", and $D^{r_0}$is called: "*$f-$annihilator of least order*".

The following theorem discusses the least order of an annihilator.

Theorem 10. *Let $f \in C^n$ and $n-1 \leq \alpha < n$. If $f^{(n-1)} \neq 0$, and $f^{(n)} = 0$, then ${}^\star D^\alpha f \neq 0$.*

Proof. Suppose on the contrary that ${}^\star D^\alpha f \equiv 0$. Then

$$(3.5)\quad \int_0^x \frac{f^{(n-1)}(t)}{(x-t)^{\alpha-n+1}}dt = c$$

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for some constant $c \in \mathbb{R}$. Performing integration by parts, and taking into account that $f^{(n)}(t) \equiv 0$, we obtain $\frac{f^{(n-1)}(0)x^{n-\alpha}}{n-\alpha} = c$, which is impossible unless $f^{(n-1)}(0) = c = 0$, but this implies from (3.5) that $f^{(n-1)} \equiv 0$, contradicting the fact that $f^{(n-1)} \neq 0$. □

Example 11. Let $f(x) = 1$. Then $D^{1/2}1 =^{\star} D^{1/2}1 = \frac{1}{\sqrt{\pi x}}$, but ${}^{C}D^{1/2}1 = 0$. Let $\alpha = 1-\delta$, for some $\delta > 0$. Then ${}^{\star}D^{1-\delta}1 = \frac{1}{\Gamma(\delta)}t^{\delta-1} = \frac{\delta}{\Gamma(\delta+1)}t^{\delta-1}$. Taking the limit as $\delta \downarrow 0$, we obtain ${}^{\star}D^{1}1 = 0$. So, the order 1 serves as the least order of $f-$annihilator. Recall that ${}^{C}D^{\alpha}1 = 0$, and ${}^{*}D^{3/2}1 =^{C} D^{3/2}1 = 0$, while $D^{3/2}1 = \frac{-1}{2\sqrt{\pi}x^{3/2}}$ for the R-L type.

Theorem 10 shows that in the ${}^{\star}D$ case, the least order of a function annihilator cannot be noninteger, so it must be of integer order. This is not the case in the Caputo type. Another result that supports this idea is the following:

Theorem 12. *Let $f \in C^n[a,b]$, $f^{(n)}$ be integrable, and ${}^{\star}D^{\alpha}f(x)$ exists on $[a,b]$ for $n-1 \leq \alpha < n$. Then ${}^{\star}D^{\alpha}f(x)(0) = 0$ if and only if $f^{(n-1)}(0) = 0$.*

Proof. Let ${}^{\star}D^{\alpha}f(x)(0) = 0$. Multiply both sides of (3.4) by $x^{\alpha-n+1}$ we obtain

$$0 = \frac{f^{(n-1)}(0)}{\Gamma(n-\alpha)} + \frac{x^{\alpha-n+1}}{\Gamma(n-\alpha+1)}\frac{d}{dx}\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n}}dt.$$

Letting $x \to 0$ gives $f^{(n-1)}(0) = 0$. For the other direction, let $f^{(n-1)}(0) = 0$. Then substituting in (3.4) and taking $x \to 0$ gives the result. □

The corresponding result for the R-L type is that $D^{\alpha}f(x)(0) = 0$ if and only if $f^{(k)}(0) = 0$ for $k = 0, 1, \cdots, n-1$ (See [11] for the details of the proof). This explains why the derivative of a nonzero constant function is not zero in the R-L type.

4. Applications to FDEs

The Mittag-Leffler function is defined to be $E_{\alpha,\beta}(t) = \sum\limits_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+\beta)}$. The following is well known in the literature (See for example [7], [8], [10], [11], or [12]) $\mathcal{L}^{-1}\{\frac{s^{\alpha-1}}{s^{\alpha}-\lambda}\} = E_{\alpha,1}(\lambda t^{\alpha})$, from which one can derive the following

$$\mathcal{L}^{-1}\{\frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}\} = t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha}). \tag{4.1}$$

Proposition 13. *Let $\mathcal{L}\{f(x); s\} = F(s)$. Then*

$$\mathcal{L}\{{}^{\star}D^{\alpha}f\} = S^{\alpha}F(s) - \sum_{k=1}^{n-1} s^{\alpha-k}f^{(k-1)}(0).$$

Proof. Let $g(x) = \int\limits_0^x f^{(n-1)}(t) \cdot (x-t)^{n-\alpha-1}dt$. Taking the Laplace transform of g gives: $\mathcal{L}\{g(x)\} = \mathcal{L}\{f^{(n-1)}(x)\} \cdot \mathcal{L}\{x^{n-\alpha-1}\}$. Applying the $n-$Laplacian transform for n^{th} derivative function, and the fact that $\mathcal{L}\{t^{n-\alpha-1}\} = \frac{\Gamma(n-\alpha)}{s^{n-\alpha}}$, we obtain

$$\mathcal{L}\{g'(x)\} = s\mathcal{L}\{g(x)\} - g(0) = s^{\alpha}F(s) - s^{\alpha-1}f(0) - \cdots s^{\alpha-n+1}f^{(n-2)}(0). \tag{4.2}$$

Performing the calculations, taking into account that $g(0) = 0$, we obtain the result. □

It is worth mentioning the following two observations.

(1) To find a solution of a fractional initial value problem of order α between $n-1$ and n using the Laplace transform, we need $n-1$ conditions to perform the Canavati derivative while n conditions is required to perform Caputo derivative. Let $1 < \alpha < 2$, then we obtain $\mathcal{L}\{{}^C D^\alpha f\} = s^\alpha F(s) - s^{\alpha-1} f(0) - s^{\alpha-2} f'(0)$, and $\mathcal{L}\{{}^\star D^\alpha f\} = s^\alpha F(s) - s^{\alpha-1} f(0)$. This shows that the Laplace transform for both definitions coincide when $f(0) = f'(0) = 0$. Otherwise, we need two conditions for the Caputo and one condition for Canavati definition. This gives two fundamental solutions for the Caputo type and only one solution for Canavati type for the case $1 < \alpha < 2$. This is due to the fact that ${}^\star D^\alpha 1 \neq 0$ and ${}^C D^\alpha 1 = 0$. This shows that we need less conditions to employ the Canavati definition. In fact, we need no conditions for the case $0 < \alpha < 1$.

(2) If $n-1 < \alpha < n$ then according to Theorem 8 we can study convergence of the Caputo solution in the case $\alpha \to n$, not $\alpha \to n-1$. In case of Canavati derivative, we can study convergence for $\alpha \to n-1$ so that $\mathcal{L}\{{}^\star D^\alpha f\} \to \mathcal{L}\{{}^\star D^{n-1} f\}$. The advantage of Canavati derivative comes from the fact that we cannot study convergence of the Caputo solution when $\alpha \to 0$.

For the sake of simplicity, we denote the solution to a fractional differential equation by y_f , the solution with respect to Caputo type by ${}^C y_f$, and the solution with respect to Canavati type by ${}^\star y_f$.

Example 14. Let $D^{4/3} y = 0$, $y(0) = 1$. Applying the Laplace transform for the Canavati definition, making use of (4.1), we obtain $Y(s) = \frac{1}{s}$ from which we get ${}^\star y_f(t) = 1$. To apply the Caputo derivative we need another initial condition, say $y'(0) = 1$. Then $y(t) = \frac{t^{-1/3}}{\Gamma(-1/3)} + 1$. In general, let $D^\alpha y = 0$ for $1 < \alpha < 2$. Then ${}^\star y_f(t) = 1$ and ${}^C y_f(t) = t + 1$. As shown above, Theorem 8 suggests that letting $\alpha \to 1$ won't lead to a convergence of ${}^C y_f(t)$ to the solution of the classical equation $y' = 0$. If ${}^\star D^\alpha y = 0$ for $0 < \alpha < 1$, then $s^\alpha Y = 0$, which implies that ${}^\star y_f(t) = 0$. For the Caputo type we need the condition $y(0) = 1$, then we have $s^\alpha Y - s^{\alpha-1} = 0$, which implies that ${}^C y_f(t) = 1$. The Canavati solution in the $0 < \alpha < 1$ case doesn't require any initial conditions.

Example 15. Let $D^\alpha y = \lambda y$ and $y(0) = a$ for $0 < \alpha < 1$. Applying Laplace transform for the Caputo type we have $Y(s) = c\frac{s^{\alpha-1}}{s^\alpha - \lambda}$. Thus we have ${}^C y_f(t) = a.E_{\alpha,1}(\lambda t^\alpha)$, where E is the Mittag-Leffler function. Taking $\alpha \to 1$ for the Caputo case, we get ${}^C y_f \to y$ where y is the solution to the classical equation $y' = \lambda y$. Theorem 8 won't allow the convergence $\alpha \to 0$. Now we apply the Laplace transform for Canavati type to get ${}^\star y_f(t) = 0$. This shows that no function can be the α^{th} derivative of itself for any $\alpha < 1$ in Canavati type. Let $\alpha \to 0$, we get the algebraic equation $y = \lambda y$ which has the solution $y = 0$ as well.

Example 16. Consider the fractional equation $D^\alpha y + y = xe^{-x}$ for $1 < \alpha < 2$ with the zeroth initial conditions. Applying Laplace transform for the Canavati type and then taking the Laplace inverse gives

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${}^{\star}y_f(x) = \int_0^x K(x-t)te^{-t}dt$, where $K(x) = x^{\alpha-1}E_{\alpha,\alpha}(-x^{\alpha})$, which is in full agreement with the result with respect to the R-L derivative shown by [13]. Now consider the nonhomogeneous problem, i.e. the same equation with $y(0) = a$ and $y'(0) = b$. We obtain ${}^{\star}y_f(x) = a.E_{\alpha,1}(-x^{\alpha}) + \int_0^x K(x-t)te^{-t}dt$, where $K(x) = x^{\alpha-1}E_{\alpha,\alpha}(-x^{\alpha})$. To study convergence, let $\alpha \to 1$, then $K(x) \to e^{-x}$, which implies that $y_f(x) \to \frac{x^2}{2}e^{-x}+ae^{-x}$ which is the solution to the classical differential equation $y' + y = xe^{-x}$. Let $\alpha \to 2$ then we use both initial conditions to get ${}^{\star}y_f(x) = a.E_{\alpha,1}(-x^{\alpha}) + bx.E_{\alpha,2}(-x^{\alpha}) + \int_0^x K(x-t)te^{-t}dt$,

and so $K(x) \to \sin x$, and $y_f(x) \to a\cos x + b\sin x + \int_0^x te^{-t}\sin(x-t)dt$, which is the solution to the corresponding classical equation of order 2.

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Al-Imam University, Department of mathematics and statistics, Al-Riyadh, Saudi Arabia.

E-mail address: ahkhanfer@imamu.edu.sa.

On the Asymptotic Behavior Of Some Nonlinear Difference Equations

A. M. Alotaibi[1], **M. S. M. Noorani**[1] **and M. A. El-Moneam**[2]
[1]School of Mathematical Sciences,
Faculty of Science and Technology,
Universiti Kebangsaan Malaysia, Malaysia.
[2]Mathematics Department, Faculty of Science,
Jazan University, Kingdom of Saudi Arabia.
E-mail: ab-alo@hotmail.com
E-mail: msn@ukm.edu.my
E-mail: maliahmedibrahim@jazanu.edu.sa

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Abstract

In this paper, some qualitative properties are discussed such as the boundedness, the periodicity and the global stability of the positive solutions of the nonlinear difference equation

$$y_{m+1} = Ay_m + \frac{\alpha_1 y_{m-1} + \alpha_2 y_{m-2} + \alpha_3 y_{m-3} + \alpha_4 y_{m-4} + \alpha_5 y_{m-5}}{\beta_1 y_{m-1} + \beta_2 y_{m-2} + \beta_3 y_{m-3} + \beta_4 y_{m-4} + \beta_5 y_{m-5}},$$

where the coefficients $A, \alpha_i, \beta_i \in (0, \infty),\ i = 1, ..., 5,$ while the initial conditions y_{-5},y_{-4},y_{-3},y_{-2}, y_{-1}, y_0 are arbitrary positive real numbers. Some numerical examples will be given to illustrate our results.

Keywords and Phrases: Difference equations, prime period two solution, boundedness character, locally asymptotically stable, global attractor, global stability, high orders.

AMS subject classifications:39A10,39A11,39A99,34C99.

1 Introduction

The study of difference equations is a diverse field that affects most aspects of mathematics including both applied and pure. Every dynamical system $a_{n+1} = f(a_n)$ determines a difference equation and vice versa.

Recently, there has been a significant increase in the study of difference equation. One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real life situations in population biology, economic, probability theory, genetics and psychology [2,3,21,24]. Note that most of these equation often show increasingly complex behavior such as the existence of a bounded.

In particular, there has been a huge development in studying of the boundedness character, the global attractivity and the periodicity nature of nonlinear difference equations. For example, in the articles [1, 6–9], closely related global convergence results were obtained which can be applied to nonlinear difference equations in proving that every solution of these equations converges to a period two solution. For other closely related results, (see [10–15]) and the references are cited therein. The study of these equations is challenging and rewarding and still actively investigated by researchers. Note that these results for nonlinear difference equations can be used to prove similar results for the case of non-linear rational difference equations.

The main focus of this article is to discuss some qualitative behavior of the solutions of the nonlinear difference equation

$$y_{m+1} = Ay_m + \frac{\alpha_1 y_{m-1} + \alpha_2 y_{m-2} + \alpha_3 y_{m-3} + \alpha_4 y_{m-4} + \alpha_5 y_{m-5}}{\beta_1 y_{m-1} + \beta_2 y_{m-2} + \beta_3 y_{m-3} + \beta_4 y_{m-4} + \beta_5 y_{m-5}}, \qquad m = 0, 1, 2, ..., \tag{1.1}$$

where the coefficients $A, \alpha_i, \beta_i \in (0, \infty),\ i = 1, ..., 5$, while the initial conditions y_{-5},y_{-4},y_{-3},y_{-2}, y_{-1}, y_0 are arbitrary positive real numbers. Note that the special case of Eq.(1.1) has been discussed in [4] when $\alpha_3 = \beta_3 = \alpha_4 = \beta_4 = \alpha_5 = \beta_5 = 0$ and Eq.(1.1) has been studied in [8] in the special case when $\alpha_4 = \beta_4 = \alpha_5 = \beta_5 = 0$ and Eq.(1.1) has been discussed in [5] in the special case when $\alpha_5 = \beta_5 = 0$.

Aboutaleb et al. [1] studied the global attractivity of the positive equilibrium of the rational recursive equation

$$y_{m+1} = \frac{A - \beta y_m}{P + y_{m-1}}, \qquad m = 0, 1, 2, ...,$$

where the coefficients A, β, P are non-negative real numbers.

E. M. Elabbasy et al. [2] investigated the periodic character and the global stability of all positive solutions of the equation

$$y_{m+1} = ay_m - \frac{by_m}{cy_m - dy_{m-1}}, \qquad m = 0, 1, 2, ...,$$

where the parameters a, b, c and d and the initial conditions y_{-1}, y_0 are positive real numbers.

E. M. Elabbasy et al. [3] investigated the periodic character and the global stability of all positive solutions of the equation

$$y_{m+1} = \frac{\alpha y_{m-l} + \beta y_{m-k}}{A y_{m-l} + B y_{m-k}}, \qquad m = 0, 1, 2, ...,$$

where the parameters α, β, A and B are positive real numbers and the initial conditions $y_{-r}, y_{-r+1}, ..., y_{-1}$ and $y_0 \in (0, \infty)$ where $r = \max\{l, k\}$.

Li and Sun [7] investigated the periodic character and the global stability of all positive solutions of the equation

$$y_{m+1} = \frac{p y_m + y_{m-k}}{q + y_{m-k}}, \qquad m = 0, 1, 2, ...,$$

where the parameters p and q and the initial conditions $y_{-k}, ..., y_{-1}, y_0$ are positive real numbers, $k = \{1, 2, 3, ...\}$.

M. Saleh et al. [9] investigated the periodic character and the global stability of all positive solutions of the equation

$$y_{m+1} = \frac{\beta y_m + \gamma y_{m-k}}{B y_m + C y_{m-k}}, \qquad m = 0, 1, 2, ...,$$

where the parameters β, γ and B, C and the initial conditions $y_{-k}, ..., y_{-1}, y_0$ are positive real numbers, $k = \{1, 2, 3, ...\}$.

Our main current objective is to examine the behavior of the solutions of Eq.(1.1) (for related work, (see [16-25])).

Definition 1 *Let*

$$H : V^{k+1} \to V,$$

where H is a continuously differentiable function. Then, for every set of initial conditions $y_{-k}, y_{-k+1}, \ldots, y_{-1}, y_0 \in V$, the difference equation of order $(k+1)$ is an equation of the form

$$y_{m+1} = H(y_m, y_{m-1}, ..., y_{m-k}), \qquad m = 0, 1, 2, ... \tag{1.2}$$

and it has a unique solution $\{y_m\}_{m=-k}^{\infty}$. An equilibrium point $\widetilde{y}$ of Eq.(1.2) is a point that satisfies the condition $\widetilde{y} = H(\widetilde{y}, \widetilde{y}, ..., \widetilde{y})$. That is, the constant sequence $\{y_m\}$ with $y_m = \widetilde{y}$ for all $m \geq 0$ is a solution of Eq.(1.2) or equivalently, $\widetilde{y}$ is a fixed point of H.

Definition 2 *Let $\widetilde{y} \in V$, be an equilibrium point of Eq.(1.2). Then, we have*

(i) An equilibrium point $\widetilde{y}$ of Eq.(1.2) is called locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $y_{-k}, y_{-k+1}, \ldots, y_{-1}, y_0 \in V$ with $|y_{-k} - \widetilde{y}| + |y_{-k+1} - \widetilde{y}| + \ldots + |y_{-1} - \widetilde{y}| + |y_0 - \widetilde{y}| < \delta$, then $|y_m - \widetilde{y}| < \varepsilon$ for all $m \geq -k$.

(ii) An equilibrium point $\widetilde{y}$ of Eq.(1.2) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that, if $y_{-k}, y_{-k+1}, \ldots, y_{-1}$, $y_0 \in V$ with $|y_{-k} - \widetilde{y}| + |y_{-k+1} - \widetilde{y}| + \ldots + |y_{-1} - \widetilde{y}| + |y_0 - \widetilde{y}| < \gamma$, then

$$\lim_{m \to \infty} y_m = \widetilde{y}.$$

(iii) An equilibrium point $\widetilde{y}$ of Eq.(1.2) is called a global attractor if for every $y_{-l}, \ldots, y_{-k}, \ldots, y_{-1}, y_0 \in (0, \infty)$ we have

$$\lim_{m \to \infty} y_m = \widetilde{y}.$$

(iv) An equilibrium point $\widetilde{y}$ of Eq.(1.2) is called globally asymptotically stable if it is locally stable and a global attractor.

(v) An equilibrium point $\widetilde{y}$ of Eq.(1.2) is called unstable if it is not locally stable.

Definition 3 *A sequence $\{y_m\}_{m=-k}^{\infty}$ is said to be periodic with period r if $y_{m+r} = y_m$ for all $m \geq -p$. A sequence $\{y_m\}_{m=-k}^{\infty}$ is said to be periodic with prime period r if r is the smallest positive integer having this property.*

Definition 4 *Eq.(1.2) is called permanent and bounded if there exists numbers n and N with $0 < n < N < \infty$ such that for any initial conditions $y_{-k}, y_{-k+1}, \ldots, y_{-1}, y_0 \in V$ there exists a positive integer M which depends on these initial conditions such that*

$$n \leq y_m \leq N \quad \text{for all} \quad m \geq M.$$

Definition 5 *The linearized equation of Eq.(1.2) about the equilibrium point $\widetilde{y}$ is defined by the equation*

$$z_{m+1} = \rho_0 z_m + \rho_1 z_{m-1} + \rho_2 z_{m-2} + \rho_3 z_{m-3} + \ldots = 0, \tag{1.3}$$

where

$$\rho_0 = \frac{\partial H(\widetilde{y}, \widetilde{y}, \ldots, \widetilde{y})}{\partial y_m},\ \rho_1 = \frac{\partial H(\widetilde{y}, \widetilde{y}, \ldots, \widetilde{y})}{\partial y_{m-1}},\ \rho_2 = \frac{\partial H(\widetilde{y}, \widetilde{y}, \ldots, \widetilde{y})}{\partial y_{m-2}},\ \rho_3 = \frac{\partial H(\widetilde{y}, \widetilde{y}, \ldots, \widetilde{y})}{\partial y_{m-3}}, \ldots$$

Theorem 1 *([6]). Assume that* $p_i \in R,\ i = 1, 2, ..., k$ *. Then,*

$$\sum_{i=1}^{k} |p_i| < 1, \tag{1.4}$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{m+k} + p_1 y_{m+k-1} + ... + p_k y_m = 0, \quad m = 0, 1, 2, ... \tag{1.5}$$

Theorem 2 *([6]). Let* $H : [a, b]^{k+1} \to [a, b]$ *be a continuous function, where* k *is a positive integer, and where* $[a, b]$ *is an interval of real numbers. Consider the difference equation (1.2). Suppose that* H *satisfies the following conditions:*

1. For each integer i *with* $1 \leq i \leq k + 1$*; the function* $H(z_1, z_2, ..., z_{k+1})$ *is weakly monotonic in* z_i *for fixed* $z_1, z_2, ..., z_{i-1}, z_{i+1}, ..., z_{k+1}$.

2. If (d, D) *is a solution of the system*

$$d = H(d_1, d_2, ..., d_{k+1}) \quad and \quad D = H(D_1, D_2, ..., D_{k+1}),$$

then $d = D$*, where for each* $i = 1, 2, ..., k + 1,$ *we set*

$$d_i = \begin{cases} d & if \ \ F \ \ is\ non-decreasing\ in\ z_i \\ D & if \ \ F \ \ is\ non-increasing\ in\ z_i \end{cases}$$

and

$$D_i = \begin{cases} D & if \ \ F \ \ is\ non-decreasing\ in\ z_i \\ d & if \ \ F \ \ is\ non-increasing\ in\ z_i. \end{cases}$$

Then there exists exactly one equilibrium $\widetilde{y}$ *of Eq.(1.2), and every solution of Eq.(1.2) converges to* $\widetilde{y}$.

2 The local stability of the solutions

In this section, the local stability of the solutions of Eq.(1.1) is investigated. The equilibrium point $\widetilde{y}$ of Eq.(1.1) is the positive solution of the equation

$$\widetilde{y} = A\widetilde{y} + \frac{\sum_{i=1}^{5} \alpha_i}{\sum_{i=1}^{5} \beta_i}. \tag{2.6}$$

Then, the only positive equilibrium point $\widetilde{y}$ of Eq.(1.1) is given by

$$\widetilde{y} = \frac{\sum_{i=1}^{5} \alpha_i}{(1-A)\left(\sum_{i=1}^{5} \beta_i \right)}, \tag{2.7}$$

provided that $A < 1$. Now, let us introduce a continuous function $H : (0,\infty)^6 \longrightarrow (0,\infty)$ which is defined by

$$H(u_0, ..., u_5) = Au_0 + \frac{\sum_{i=1}^{5}(\alpha_i u_i)}{\sum_{i=1}^{5}(\beta_i u_i)}. \tag{2.8}$$

Therefore, it follows that

$$\begin{cases} \frac{H(u_0,...,u_5)}{\partial u_0} = A, \\ \frac{H(u_0,...,u_5)}{\partial u_1} = \frac{\alpha_1 \left[\sum_{i=2}^{5}(\beta_i u_i)\right] - \beta_1 \left[\sum_{i=2}^{5}(\alpha_i u_i)\right]}{\left(\sum_{i=1}^{5}(\beta_i u_i)\right)^2}, \\ \frac{H(u_0,...,u_5)}{\partial u_2} = \frac{\alpha_2 \left[\beta_1 u_1 + \sum_{i=3}^{5}(\beta_i u_i)\right] - \beta_2 \left[\alpha_1 u_1 + \sum_{i=3}^{5}(\alpha_i u_i)\right]}{\left(\sum_{i=1}^{5}(\beta_i u_i)\right)^2}, \\ \frac{H(u_0,...,u_5)}{\partial u_3} = \frac{\alpha_3 \left[\sum_{i=1}^{2}(\beta_i u_i) + \sum_{i=4}^{5}(\beta_i u_i)\right] - \beta_3 \left[\sum_{i=1}^{2}(\alpha_i u_i) + \sum_{i=4}^{5}(\alpha_i u_i)\right]}{\left(\sum_{i=1}^{5}(\beta_i u_i)\right)^2}, \\ \frac{H(u_0,...,u_5)}{\partial u_4} = \frac{\alpha_4 \left[\sum_{i=1}^{3}(\beta_i u_i) + \beta_5 u_5\right] - \beta_4 \left[\sum_{i=1}^{3}(\alpha_i u_i) + \alpha_5 u_5\right]}{\left(\sum_{i=1}^{5}(\beta_i u_i)\right)^2}, \\ \frac{H(u_0,...,u_5)}{\partial u_5} = \frac{\alpha_5 \left[\sum_{i=1}^{4}(\beta_i u_i)\right] - \beta_5 \left[\sum_{i=1}^{4}(\alpha_i u_i)\right]}{\left(\sum_{i=1}^{5}(\beta_i u_i)\right)^2}, \end{cases}.$$

Consequently, we get

$$\begin{cases} \frac{\partial H(\widetilde{y},\ldots,\widetilde{y})}{\partial u_0} = A = -\ \rho_5, \\ \frac{\partial H(\widetilde{y},\ldots,\widetilde{y})}{\partial u_1} = \frac{(1-A)\left[\alpha_1\ \left(\ \sum_{i=2}^{5}\beta_i\right)\ -\ \beta_1\ \left(\ \sum_{i=2}^{5}\alpha_i\right)\right]}{\left(\ \sum_{i=1}^{5}\alpha_i\right)\left(\ \sum_{i=1}^{5}\beta_i\right)} = -\ \rho_4, \\ \frac{\partial H(\widetilde{y},\ldots,\widetilde{y})}{\partial u_2} = \frac{(1-A)\left[\alpha_2\ \left(\ \beta_1+\sum_{i=3}^{5}\beta_i\right)\ -\ \beta_2\ \left(\alpha_1+\ \sum_{i=3}^{5}\alpha_i\right)\right]}{\left(\ \sum_{i=1}^{5}\alpha_i\right)\left(\ \sum_{i=1}^{5}\beta_i\right)} = -\ \rho_3, \\ \frac{\partial H(\widetilde{y},\ldots,\widetilde{y})}{\partial u_3} = \frac{(1-A)\left[\alpha_3\ \left(\ \sum_{i=1}^{2}\beta_i+\sum_{i=4}^{5}\beta_i\right)\ -\ \beta_3\ \left(\sum_{i=1}^{2}\alpha_i+\ \sum_{i=4}^{5}\alpha_i\right)\right]}{\left(\ \sum_{i=1}^{5}\alpha_i\right)\left(\ \sum_{i=1}^{5}\beta_i\right)} = -\ \rho_2, \\ \frac{\partial H(\widetilde{y},\ldots,\widetilde{y})}{\partial u_4} = \frac{(1-A)\left[\alpha_4\ \left(\beta_5\ +\sum_{i=1}^{3}\beta_i\right)\ -\ \beta_4\ \left(\alpha_5+\ \sum_{i=3}^{5}\alpha_i\right)\right]}{\left(\ \sum_{i=1}^{5}\alpha_i\right)\left(\ \sum_{i=1}^{5}\beta_i\right)} = -\ \rho_1, \\ \frac{\partial H(\widetilde{y},\ldots,\widetilde{y})}{\partial u_4} = \frac{(1-A)\left[\alpha_5\ \left(\sum_{i=1}^{4}\beta_i\right)\ -\ \beta_5\ \left(\sum_{i=1}^{4}\alpha_i\right)\right]}{\left(\ \sum_{i=1}^{5}\alpha_i\right)\left(\ \sum_{i=1}^{5}\beta_i\right)} = -\ \rho_0. \end{cases} \tag{2.9}$$

Hence, the linearized equation of Eq.(1.1) about $\widetilde{y}$ takes the form

$$y_{m+1}+\rho_5 y_m+\rho_4 y_{m-1}+\rho_3 y_{m-2}+\rho_2 y_{m-3}+\rho_1 y_{m-4}+\rho_0 y_{m-5}=0, \tag{2.10}$$

where $\rho_0, \rho_1, \rho_2, \rho_3, \rho_4$ and ρ_5 are given by (2.9).

The characteristic equation associated with Eq.(2.10) is

$$\lambda^6+\rho_5\lambda^5+\rho_4\lambda^4+\rho_3\lambda^3+\rho_2\lambda^2+\rho_1\lambda+\rho_0=0, \tag{2.11}$$

Theorem 3 *Let $A<1$ and*

$$\begin{aligned} &\left|\alpha_1\ \left(\sum_{i=2}^{5}\beta_i\right)\ -\ \beta_1\ \left(\sum_{i=2}^{5}\alpha_i\right)\right|+\left|\alpha_2\ \left(\ \beta_1+\sum_{i=3}^{5}\beta_i\right)\ -\ \beta_2\ \left(\alpha_1+\ \sum_{i=3}^{5}\alpha_i\right)\right|+ \\ &\left|\alpha_3\ \left(\sum_{i=1}^{2}\beta_i+\sum_{i=4}^{5}\beta_i\right)\ -\ \beta_3\ \left(\sum_{i=1}^{2}\alpha_i+\ \sum_{i=4}^{5}\alpha_i\right)\right| \\ &+\left|\alpha_4\ \left(\beta_5\ +\sum_{i=1}^{3}\beta_i\right)\ -\ \beta_4\ \left(\alpha_5+\ \sum_{i=3}^{5}\alpha_i\right)\right| \\ &+\left|\alpha_5\ \left(\sum_{i=1}^{4}\beta_i\right)\ -\ \beta_5\ \left(\sum_{i=1}^{4}\alpha_i\right)\right|<\left(\ \sum_{i=1}^{5}\alpha_i\right)\left(\ \sum_{i=1}^{5}\beta_i\right), \end{aligned} \tag{2.12}$$

then the positive equilibrium point (2.7) of Eq.(1.1) is locally asymptotically stable.

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proof: It follows by Theorem 1 that Eq.(2.10) is asymptotically stable if all roots of Eq.(2.11) lie in the open disk is $|\lambda| < 1$ that is if $\sum_{i=0}^{5} |p_i| < 1$,

$$|A| + \left| \frac{(1-A)\left[\alpha_1 \left(\sum_{i=2}^{5} \beta_i \right) - \beta_1 \left(\sum_{i=2}^{5} \alpha_i \right)\right]}{\left(\sum_{i=1}^{5} \alpha_i \right)\left(\sum_{i=1}^{5} \beta_i \right)} \right|$$

$$+ \left| \frac{(1-A)\left[\alpha_2 \left(\beta_1 + \sum_{i=3}^{5} \beta_i \right) - \beta_2 \left(\alpha_1 + \sum_{i=3}^{5} \alpha_i \right)\right]}{\left(\sum_{i=1}^{5} \alpha_i \right)\left(\sum_{i=1}^{5} \beta_i \right)} \right|$$

$$+ \left| \frac{(1-A)\left[\alpha_3 \left(\sum_{i=1}^{2} \beta_i + \sum_{i=4}^{5} \beta_i \right) - \beta_3 \left(\sum_{i=1}^{2} \alpha_i + \sum_{i=4}^{5} \alpha_i \right)\right]}{\left(\sum_{i=1}^{5} \alpha_i \right)\left(\sum_{i=1}^{5} \beta_i \right)} \right|$$

$$+ \left| \frac{(1-A)\left[\alpha_4 \left(\beta_5 + \sum_{i=1}^{3} \beta_i \right) - \beta_4 \left(\alpha_5 + \sum_{i=3}^{5} \alpha_i \right)\right]}{\left(\sum_{i=1}^{5} \alpha_i \right)\left(\sum_{i=1}^{5} \beta_i \right)} \right|,$$

$$+ \left| \frac{(1-A)\left[\alpha_5 \left(\sum_{i=1}^{4} \beta_i \right) - \beta_5 \left(\sum_{i=1}^{4} \alpha_i \right)\right]}{\left(\sum_{i=1}^{5} \alpha_i \right)\left(\sum_{i=1}^{5} \beta_i \right)} \right| < 1.$$

and so

$$\left| \frac{(1-A)\left[\alpha_1 \left(\sum_{i=2}^{5} \beta_i \right) - \beta_1 \left(\sum_{i=2}^{5} \alpha_i \right)\right]}{\left(\sum_{i=1}^{5} \alpha_i \right)\left(\sum_{i=1}^{5} \beta_i \right)} \right|$$

$$+ \left| \frac{(1-A)\left[\alpha_2 \left(\beta_1 + \sum_{i=3}^{5} \beta_i \right) - \beta_2 \left(\alpha_1 + \sum_{i=3}^{5} \alpha_i \right)\right]}{\left(\sum_{i=1}^{5} \alpha_i \right)\left(\sum_{i=1}^{5} \beta_i \right)} \right|$$

$$+ \left| \frac{(1-A)\left[\alpha_3 \left(\sum_{i=1}^{2} \beta_i + \sum_{i=4}^{5} \beta_i \right) - \beta_3 \left(\sum_{i=1}^{2} \alpha_i + \sum_{i=4}^{5} \alpha_i \right)\right]}{\left(\sum_{i=1}^{5} \alpha_i \right)\left(\sum_{i=1}^{5} \beta_i \right)} \right|$$

$$+ \left| \frac{(1-A)\left[\alpha_4 \left(\beta_5 + \sum_{i=1}^{3} \beta_i \right) - \beta_4 \left(\alpha_5 + \sum_{i=3}^{5} \alpha_i \right)\right]}{\left(\sum_{i=1}^{5} \alpha_i \right)\left(\sum_{i=1}^{5} \beta_i \right)} \right|,$$

$$+ \left| \frac{(1-A)\left[\alpha_5 \left(\sum_{i=1}^{4} \beta_i \right) - \beta_5 \left(\sum_{i=1}^{4} \alpha_i \right)\right]}{\left(\sum_{i=1}^{5} \alpha_i \right)\left(\sum_{i=1}^{5} \beta_i \right)} \right| (1-A), \qquad A < 1,$$

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or

$$\left|\alpha_1 \left(\sum_{i=2}^{5}\beta_i\right) - \beta_1 \left(\sum_{i=2}^{5}\alpha_i\right)\right| + \left|\alpha_2 \left(\beta_1 + \sum_{i=3}^{5}\beta_i\right) - \beta_2 \left(\alpha_1 + \sum_{i=3}^{5}\alpha_i\right)\right| +$$

$$\left|\alpha_3 \left(\sum_{i=1}^{2}\beta_i + \sum_{i=4}^{5}\beta_i\right) - \beta_3 \left(\sum_{i=1}^{2}\alpha_i + \sum_{i=4}^{5}\alpha_i\right)\right|$$

$$+\left|\alpha_4 \left(\beta_5 + \sum_{i=1}^{3}\beta_i\right) - \beta_4 \left(\alpha_5 + \sum_{i=3}^{5}\alpha_i\right)\right|$$

$$+\left|\alpha_5 \left(\sum_{i=1}^{4}\beta_i\right) - \beta_5 \left(\sum_{i=1}^{4}\alpha_i\right)\right| < \left(\sum_{i=1}^{5}\alpha_i\right)\left(\sum_{i=1}^{5}\beta_i\right).$$

Thus, the proof is complete.

3 Boundedness of the solutions

In this section, the boundedness of the positive solutions of Eq.(1.1) is determined.

Theorem 4 *Every solution of Eq.(1.1) is bounded if* $A < 1$.

proof Let $\{y_m\}_{m=-5}^{\infty}$ be a solution of Eq.(1.1). It follows from Eq.(1.1) that

$$\begin{aligned}
y_{m+1} &= Ay_m + \frac{\alpha_1 y_{m-1} + \alpha_2 y_{m-2} + \alpha_3 y_{m-3} + \alpha_4 y_{m-4} + \alpha_5 y_{m-5}}{\beta_1 y_{m-1} + \beta_2 y_{m-2} + \beta_3 y_{m-3} + \beta_4 y_{m-4} + \beta_5 y_{m-5}} \\
&= Ay_m + \frac{\alpha_1 y_{m-1}}{\beta_1 y_{m-1} + \beta_2 y_{m-2} + \beta_3 y_{m-3} + \beta_4 y_{m-4} + \beta_5 y_{m-5}} \\
&\quad + \frac{\alpha_2 y_{m-2}}{\beta_1 y_{m-1} + \beta_2 y_{m-2} + \beta_3 y_{m-3} + \beta_4 y_{m-4} + \beta_5 y_{m-5}} \\
&\quad + \frac{\alpha_3 y_{m-3}}{\beta_1 y_{m-1} + \beta_2 y_{m-2} + \beta_3 y_{m-3} + \beta_4 y_{m-4} + \beta_5 y_{m-5}} \\
&\quad + \frac{\alpha_4 y_{m-4}}{\beta_1 y_{m-1} + \beta_2 y_{m-2} + \beta_3 y_{m-3} + \beta_4 y_{m-4} + \beta_5 y_{m-5}} \\
&\quad + \frac{\alpha_5 y_{m-5}}{\beta_1 y_{m-1} + \beta_2 y_{m-2} + \beta_3 y_{m-3} + \beta_4 y_{m-4} + \beta_5 y_{m-5}}.
\end{aligned}$$

Then

$$y_{m+1} \leq Ay_m + \frac{\alpha_1 y_{m-1}}{\beta_1 y_{m-1}} + \frac{\alpha_2 y_{m-2}}{\beta_2 y_{m-2}} + \frac{\alpha_3 y_{m-3}}{\beta_3 y_{m-3}} + \frac{\alpha_4 y_{m-4}}{\beta_4 y_{m-4}} + \frac{\alpha_5 y_{m-5}}{\beta_5 y_{m-5}} =$$

$$Ay_m + \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + \frac{\alpha_3}{\beta_3} + \frac{\alpha_4}{\beta_4} + \frac{\alpha_5}{\beta_5} \qquad for \quad all \quad m \geq 1.$$

By using a comparison, we can write the right hand side as follows

$$y_{m+1} = Ay_m + \frac{\alpha_1}{\beta_1} + \frac{\alpha_2}{\beta_2} + \frac{\alpha_3}{\beta_3} + \frac{\alpha_4}{\beta_4} + \frac{\alpha_5}{\beta_5}.$$

then

$$y_m = a^m y_0 + constant,$$

and this equation is locally asymptotically stable because $A < 1$, and converges to the equilibrium point

$$\widetilde{y} = \frac{\alpha_1\beta_2\beta_3\beta_4\beta_5 + \alpha_2\beta_1\beta_3\beta_4\beta_5 + \alpha_3\beta_1\beta_2\beta_4\beta_5 + \alpha_4\beta_1\beta_2\beta_3\beta_5 + \alpha_5\beta_1\beta_2\beta_3\beta_4}{\beta_1\beta_2\beta_3\beta_4\beta_5\,(1-A)}.$$

Therefore,

$$\lim_{m\to\infty} \sup y_m \leq \frac{\alpha_1\beta_2\beta_3\beta_4\beta_5 + \alpha_2\beta_1\beta_3\beta_4\beta_5 + \alpha_3\beta_1\beta_2\beta_4\beta_5 + \alpha_4\beta_1\beta_2\beta_3\beta_5 + \alpha_5\beta_1\beta_2\beta_3\beta_4}{\beta_1\beta_2\beta_3\beta_4\beta_5\,(1-A)}.$$

Thus, the solution of Eq.(1.1) is bounded and the proof is complete.

Theorem 5 *Every solution of Eq.(1.1) is unbounded if* $A > 1$.

proof: Let $\{y_n\}_{n=-5}^{\infty}$ be a solution of Eq.(1.1). Then from Eq.(1.1) we see that

$$y_{n+1} = Ay_n + \frac{\alpha_1 y_{n-1} + \alpha_2 y_{n-2} + \alpha_3 y_{n-3} + \alpha_4 y_{n-4} + \alpha_5 y_{n-5}}{\beta_1 y_{n-1} + \beta_2 y_{n-2} + \beta_3 y_{n-3} + \beta_4 y_{n-4} + \beta_5 y_{n-5}} > Ay_n \qquad for \quad all \quad n \geq 1.$$

We can see that the right hand side can be written as follows

$$x_{n+1} = ax_n \Rightarrow x_n = a^n x_0,$$

and this equation is unstable because $A > 1$, and

$$\lim_{n\to\infty} x_n = \infty.$$

Then, by using the ratio test $\{y_n\}_{n=-5}^{\infty}$ is unbounded from above. Thus, the proof is now obtained.

4 Periodic solutions

The following theorem states the necessary and sufficient conditions for the equation to have periodic solutions of prime period two.

Theorem 6 *If* $(\alpha_1+\alpha_3+\alpha_5)>(\alpha_2+\alpha_4)$ *and* $(\beta_1+\beta_3+\beta_5)>(\beta_2+\beta_4)$, *then the necessary and sufficient condition for Eq.(1.1) to have positive solutions of prime period two is that the inequality*

$$\begin{aligned}&[(A+1)\left((\beta_1+\beta_3+\beta_5)-(\beta_2+\beta_4)\right)]\left[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)\right]^2\\&+4\left[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)\right]\left[(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A(\beta_2+\beta_4)\ \ (\alpha_1+\alpha_3+\alpha_5)\right]>0.\end{aligned} \tag{4.13}$$

is valid.

proof: Suppose there exist positive distinctive solutions of prime period two

$$......., P, Q, P, Q, ...$$

of Eq.(1.1). From Eq.(1.1) we have

$$y_{m+1}=Ay_m+\frac{\alpha_1y_{m-1}+\alpha_2y_{m-2}+\alpha_3y_{m-3}+\alpha_4y_{m-4}+\alpha_5y_{m-5}}{\beta_1y_{m-1}+\beta_2y_{m-2}+\beta_3y_{m-3}+\beta_4y_{m-4}+\beta_5y_{m-5}}$$

$$P=AQ+\frac{(\alpha_1+\alpha_3+\alpha_5)P+(\alpha_2+\alpha_4)Q}{(\beta_1+\beta_3+\beta_5)P+(\beta_2+\beta_4)Q},\quad Q=AP+\frac{(\alpha_1+\alpha_3+\alpha_5)Q+(\alpha_2+\alpha_4)P}{(\beta_1+\beta_3+\beta_5)Q+(\beta_2+\beta_4)P}. \tag{4.14}$$

Consequently, we get

$$\begin{aligned}(\beta_1+\beta_3+\beta_5)P^2+(\beta_2+\beta_4)PQ &= A(\beta_1+\beta_3+\beta_5)PQ+A(\beta_2+\beta_4)Q^2\\&\quad+(\alpha_1+\alpha_3+\alpha_5)P+(\alpha_2+\alpha_4)Q,\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}(\beta_1+\beta_3+\beta_5)Q^2+(\beta_2+\beta_4)PQ &= A(\beta_1+\beta_3+\beta_5)PQ+A(\beta_2+\beta_4)P^2\\&\quad+(\alpha_1+\alpha_3+\alpha_5)Q+(\alpha_2+\alpha_4)P.\end{aligned} \tag{4.16}$$

By subtracting (4.15) from (4.16), we obtain

$$[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]\left(P^2-Q^2\right)=[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)](P-Q).$$

Since $P \neq Q$, it follows that

$$P+Q=\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]}{[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]}, \tag{4.17}$$

while, by adding (4.15) and (4.16) and by using the relation

$$P^2+Q^2=(P+Q)^2-2PQ \quad for\ all \quad P,Q\ \in R,$$

we have

$$PQ=\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]\,[(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A(\beta_2+\beta_4)\ (\alpha_1+\alpha_3+\alpha_5)]}{[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]^2\,[((\beta_2+\beta_4)-(\beta_1+\beta_3+\beta_5))(A+1)]} \tag{4.18}$$

Let P and Q are two distinct real roots of the quadratic equation

$$t^2-(\ P+Q)\,t+PQ=0.$$

$$\begin{aligned}&[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]\,t^2-[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]\,t\\&+\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]\,[(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A(\beta_2+\beta_4)\ (\alpha_1+\alpha_3+\alpha_5)]}{[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]\,[((\beta_2+\beta_4)-(\beta_1+\beta_3+\beta_5))(A+1)]}\\=\ &0,\end{aligned} \tag{4.19}$$

and so

$$\begin{aligned}&[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]^2\\&-\frac{4\,[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]\,[(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A(\beta_2+\beta_4)\ (\alpha_1+\alpha_3+\alpha_5)]}{[((\beta_2+\beta_4)-(\beta_1+\beta_3+\beta_5))(A+1)]}>0,\end{aligned}$$

or

$$\begin{aligned}&[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]^2\\&+\frac{4\,[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]\,[(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A(\beta_2+\beta_4)\ (\alpha_1+\alpha_3+\alpha_5)]}{[((\beta_1+\beta_3+\beta_5)-(\beta_2+\beta_4))(A+1)]}>0.\end{aligned} \tag{4.20}$$

From (4.20), we get

$$\begin{aligned}&[((\beta_1+\beta_3+\beta_5)-(\beta_2+\beta_4))(A+1)]\,[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]^2\\&+4\,[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]\,[(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A(\beta_2+\beta_4)\ (\alpha_1+\alpha_3+\alpha_5)]>0.\end{aligned}$$

Therefore, the condition (4.13) is valid. Alternatively, if we imagine that the condition (4.13) is valid where $(\alpha_1+\alpha_3+\alpha_5)>(\alpha_2+\alpha_4)$ and $(\beta_1+\beta_3+\beta_5)>(\beta_2+\beta_4)$. Then, we can immediately discover that the inequality stands.

There exist two positive distinctive real numbers P and Q representing two positive roots of Eq.(4.19) such that

$$P=\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]+\delta}{2\left[(\beta_1+\beta_3+\beta_5)+A\left(\beta_2+\beta_4\right)\right]} \tag{4.21}$$

and

$$Q=\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]-\delta}{2\left[(\beta_1+\beta_3+\beta_5)+A\left(\beta_2+\beta_4\right)\right]} \tag{4.22}$$

where

$$\delta=\sqrt{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]^2-\eta},$$

and

$$\eta=\frac{4\left[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)\right]\left[(\beta_1+\beta_3+\beta_5)\left(\alpha_2+\alpha_4\right)+A\left(\beta_2+\beta_4\right)\ \left(\alpha_1+\alpha_3+\alpha_5\right)\right]}{\left[\left((\beta_2+\beta_4)-(\beta_1+\beta_3+\beta_5)\right)(A+1)\right]}.$$

Now, let us prove that P and Q are positive solutions of prime period two of Eq.(1.1). To this end, we assume that $y_{-5}=P,\ \ y_{-4}=Q,\ \ y_{-3}=P,\ \ y_{-2}=Q,\ \ y_{-1}=P,\ \ y_0=Q$. Now, we are going to show that $y_1=P$ and $y_2=Q$.

From Eq.(1.1) we deduce that

$$\begin{aligned} y_1 &= Ay_0+\frac{\alpha_1y_{-1}+\alpha_2y_{-2}+\alpha_3y_{-3}+\alpha_4y_{-4}+\alpha_5y_{-5}}{\beta_1y_{-1}+\beta_2y_{-2}+\beta_3y_{-3}+\beta_4y_{-4}+\beta_5y_{-5}} \\ &= AQ+\frac{(\alpha_1+\alpha_3+\alpha_5)\,P+(\alpha_2+\alpha_4)\,Q}{(\beta_1+\beta_3+\beta_5)\,P+(\beta_2+\beta_4)\,Q}. \end{aligned} \tag{4.23}$$

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Substituting (4.21) and (4.22) into (4.23) we deduce that

$$\begin{aligned}
y_1 - P &= AQ + \frac{(\alpha_1+\alpha_3+\alpha_5)P + (\alpha_2+\alpha_4)Q}{(\beta_1+\beta_3+\beta_5)P + (\beta_2+\beta_4)Q} - P \\
&= \frac{[A(\beta_1+\beta_3+\beta_5) - (\beta_2+\beta_4)]PQ + A(\beta_2+\beta_4)Q^2 - (\beta_1+\beta_3+\beta_5)P^2}{(\beta_1+\beta_3+\beta_5)P + (\beta_2+\beta_4)Q} \\
&\quad + \frac{(\alpha_1+\alpha_3+\alpha_5)P + (\alpha_2+\alpha_4)Q}{(\beta_1+\beta_3+\beta_5)P + (\beta_2+\beta_4)Q} \\
&= \frac{\left(\frac{[A(\beta_1+\beta_3+\beta_5)-(\beta_2+\beta_4)][S_1][(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A(\beta_2+\beta_4)\ (\alpha_1+\alpha_3+\alpha_5)]}{[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]^2[((\beta_2+\beta_4)-(\beta_1+\beta_3+\beta_5))(A+1)]}\right)}{(\beta_1+\beta_3+\beta_5)\left(\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]+\delta}{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]}\right) + (\beta_2+\beta_4)\left(\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]-\delta}{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]}\right)} \\
&\quad + \frac{A(\beta_2+\beta_4)\left(\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]-\delta}{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]}\right)^2 - (\beta_1+\beta_3+\beta_5)\left(\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]+\delta}{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]}\right)^2}{(\beta_1+\beta_3+\beta_5)\left(\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]+\delta}{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]}\right) + (\beta_2+\beta_4)\left(\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]-\delta}{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]}\right)} \\
&\quad + \frac{(\alpha_1+\alpha_3+\alpha_5)\left(\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]+\delta}{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]}\right) + (\alpha_2+\alpha_4)\left(\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]-\delta}{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]}\right)}{(\beta_1+\beta_3+\beta_5)\left(\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]+\delta}{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]}\right) + (\beta_2+\beta_4)\left(\frac{[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]-\delta}{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]}\right)}
\end{aligned} \tag{4.24}$$

Multiplying the denominator and numerator of (4.24) by $4\left[(\beta_1+\beta_3+\beta_5) + A(\beta_2+\beta_4)\right]^2$ we get

$$\begin{aligned}
y_1 - P &= \frac{\frac{4[A(\beta_1+\beta_3+\beta_5)-(\beta_2+\beta_4)][S_1][(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A(\beta_2+\beta_4)\ (\alpha_1+\alpha_3+\alpha_5)]}{[((\beta_2+\beta_4)-(\beta_1+\beta_3+\beta_5))(A+1)]}}{S} \\
&\quad + \frac{A(\beta_2+\beta_4)(S_1-\delta)^2 - (\beta_1+\beta_3+\beta_5)(S_1+\delta)^2}{S} \\
&\quad + \frac{2[(\beta_1+\beta_3+\beta_5) + A(\beta_2+\beta_4)](\alpha_1+\alpha_3+\alpha_5)(S_1+\delta)}{S} \\
&\quad + \frac{2[(\beta_1+\beta_3+\beta_5) + A(\beta_2+\beta_4)](\alpha_2+\alpha_4)(S_1-\delta)}{S} \\
&= \frac{\frac{4[A(\beta_1+\beta_3+\beta_5)-(\beta_2+\beta_4)][S_1][(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A(\beta_2+\beta_4)\ (\alpha_1+\alpha_3+\alpha_5)]}{[((\beta_2+\beta_4)-(\beta_1+\beta_3+\beta_5))(A+1)]}}{S} \\
&\quad + \frac{A(\beta_2+\beta_4)[S_1]^2 - (\beta_1+\beta_3+\beta_5)[S_1]^2}{S} \\
&\quad + \frac{[A(\beta_2+\beta_4) - (\beta_1+\beta_3+\beta_5)]\delta^2}{S}
\end{aligned}$$

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$$
\begin{aligned}
&+\frac{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)](\alpha_2+\alpha_4)[S_1]}{S}\\
&+\frac{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)](\alpha_1+\alpha_3+\alpha_5)[S_1]}{S}\\
&-\frac{2A(\beta_2+\beta_4)[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)]\delta+2(\beta_1+\beta_3+\beta_5)[S_1]\delta}{S}\\
&+\frac{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)](\alpha_1+\alpha_3+\alpha_5)\delta-2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)](\alpha_2+\alpha_4)\delta}{S}
\end{aligned}
$$

$$
\begin{aligned}
= \;& \frac{\frac{4[A(\beta_1+\beta_3+\beta_5)-(\beta_2+\beta_4)][S_1][(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A(\beta_2+\beta_4)(\alpha_1+\alpha_3+\alpha_5)]}{[((\beta_2+\beta_4)-(\beta_1+\beta_3+\beta_5))(A+1)]}}{S}\\
&-\frac{\frac{4[S_1][(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A(\beta_2+\beta_4)(\alpha_1+\alpha_3+\alpha_5)][A(\beta_2+\beta_4)-(\beta_1+\beta_3+\beta_5)]}{[((\beta_2+\beta_4)-(\beta_1+\beta_3+\beta_5))(A+1)]}}{S}\\
&+\frac{A(\beta_2+\beta_4)[S_1]^2-(\beta_1+\beta_3+\beta_5)[S_1]^2}{S}\\
&+\frac{[A(\beta_2+\beta_4)-(\beta_1+\beta_3+\beta_5)][S_1]^2}{S}\\
&+\frac{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)](\alpha_2+\alpha_4)[S_1]}{S}\\
&+\frac{2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)](\alpha_1+\alpha_3+\alpha_5)[S_1]}{S}\\
&-\frac{2[S_1][(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]\delta}{S}\\
&+\frac{2[S_1][(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]\delta}{S}\\
= \;& \frac{4[S_1][(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A(\beta_2+\beta_4)(\alpha_1+\alpha_3+\alpha_5)]}{S}\\
&-\frac{4[S_1][(\beta_1+\beta_3+\beta_5)(\alpha_2+\alpha_4)+A(\beta_2+\beta_4)(\alpha_1+\alpha_3+\alpha_5)]}{S}\\
&-\frac{2[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)][(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]\delta}{S}\\
&+\frac{2[(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)][(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]\delta}{S}\\
= \;& 0.
\end{aligned}
$$

where

$$
\begin{aligned}
S \;=\;& 2[(\beta_1+\beta_3+\beta_5)+A(\beta_2+\beta_4)]\times\\
&[(\beta_1+\beta_3+\beta_5)(S_1+\delta)+(\beta_2+\beta_4)(S_1-\delta)]
\end{aligned}
$$

and

$$S_1 = [(\alpha_1+\alpha_3+\alpha_5)-(\alpha_2+\alpha_4)].$$

Similarly, we can show that

$$y_2 = Ay_1+\frac{\alpha_1 y_0+\alpha_2 y_{-1}+\alpha_3 y_{-2}+\alpha_4 y_{-3}+\alpha_5 y_{-4}}{\beta_1 y_0+\beta_2 y_{-1}+\beta_3 y_{-2}+\beta_4 y_{-3}+\beta_5 y_{-4}} = AP+\frac{(\alpha_1+\alpha_3+\alpha_5)\,Q+(\alpha_2+\alpha_4)\,P}{(\beta_1+\beta_3+\beta_5)\,Q+(\beta_2+\beta_4)\,P} = Q.$$

By using the mathematical induction, we have $y_m = P$ and $y_{m+1} = Q, \quad m \geq -5$.

5 Global stability

In this section, the global asymptotic stability of the positive solutions of Eq.(1.1) is discussed.

Theorem 7 *For any values of the quotient* $\sum_{i=1}^{5}\frac{\alpha_i}{\beta_i}$, *If* $A<1$, *then the positive equilibrium point* $\widetilde{y}$ *of Eq.(1.1) is a global attractor and the following conditions hold*

$$\begin{aligned}\alpha_1\beta_2 &\geq \alpha_2\beta_1,\ \alpha_1\beta_3\geq\alpha_3\beta_1,\ \alpha_1\beta_4\geq\alpha_4\beta_1,\ \alpha_1\beta_5\geq\alpha_5\beta_1,\ \alpha_2\beta_3\geq\alpha_3\beta_2,\ \alpha_2\beta_4\geq\alpha_4\beta_2,\\ \alpha_2\beta_5 &\geq \alpha_5\beta_2, \alpha_3\beta_4\geq\alpha_4\beta_3,\ \alpha_3\beta_5\geq\alpha_5\beta_3,\ \alpha_4\beta_5\geq\alpha_5\beta_4 \text{ and } \alpha_5\geq(\alpha_1+\alpha_2+\alpha_3+\alpha_4)\,.\end{aligned} \tag{5.25}$$

proof: Let $\{y_m\}_{m=-5}^{\infty}$ be a positive solution of Eq.(1.1). and let $H : (0,\infty)^6 \longrightarrow (0,\infty)$ be a continuous function which is defined by

$$H(u_0,...,u_5) = Au_0+\frac{\sum_{i=1}^{5}(\alpha_i u_i)}{\sum_{i=1}^{5}(\beta_i u_i)}.$$

By differentiating the function $H(u_0,...,u_5)$ with respect to u_i $(i=0,...,5)$, we obtain

$$H_{u_0} = A, \tag{5.26}$$

$$H_{u_1} = \frac{(\alpha_1\beta_2-\alpha_2\beta_1)\,u_2+(\alpha_1\beta_3-\alpha_3\beta_1)\,u_3+(\alpha_1\beta_4-\alpha_4\beta_1)\,u_4+(\alpha_1\beta_5-\alpha_5\beta_1)\,u_5}{\left(\sum_{i=1}^{5}(\beta_i u_i)\right)^2}, \tag{5.27}$$

$$H_{u_2} = \frac{-\ (\alpha_1\beta_2-\alpha_2\beta_1)\,u_1+(\alpha_2\beta_3-\alpha_3\beta_2)\,u_3+(\alpha_2\beta_4-\alpha_4\beta_2)\,u_4+(\alpha_2\beta_5-\alpha_5\beta_2)\,u_5}{\left(\sum_{i=1}^{5}(\beta_i u_i)\right)^2}, \tag{5.28}$$

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$$H_{u_3} = \frac{-\ (\alpha_1\beta_3 - \alpha_3\beta_1)\, u_1 - (\alpha_2\beta_3 - \alpha_3\beta_2)\, u_2 + (\alpha_3\beta_4 - \alpha_4\beta_3)\, u_4 + (\alpha_3\beta_5 - \alpha_5\beta_3)\, u_4}{\left(\sum_{i=1}^{5}(\beta_i u_i)\right)^2}, \tag{5.29}$$

$$H_{u_4} = \frac{-\ (\alpha_1\beta_4 - \alpha_4\beta_1)\, u_1 - (\alpha_2\beta_4 - \alpha_4\beta_2)\, u_2 - (\alpha_3\beta_4 - \alpha_4\beta_3)\, u_3 + (\alpha_4\beta_5 - \alpha_5\beta_4)\, u_5}{\left(\sum_{i=1}^{5}(\beta_i u_i)\right)^2}, \tag{5.30}$$

and

$$H_{u_5} = \frac{-\ (\alpha_1\beta_5 - \alpha_5\beta_1)\, u_1 - (\alpha_2\beta_5 - \alpha_5\beta_2)\, u_2 - (\alpha_3\beta_5 - \alpha_5\beta_3)\, u_3 - (\alpha_4\beta_5 - \alpha_5\beta_4)\, u_4}{\left(\sum_{i=1}^{5}(\beta_i u_i)\right)^2}. \tag{5.31}$$

It is observed that the function $H(u_0, ..., u_5)$ is non-decreasing in u_0,u_1 and non-increasing in u_5. Now, we consider four cases:

Case 1. Let the function $H(u_0, ..., u_5)$ is non-decreasing in u_0,u_1,u_2,u_3,u_4 and non-increasing in u_5. Suppose that (d, D) is a solution of the system

$$D = H(D, D, D, D, D, d) \quad \textit{and} \quad d = H(d, d, d, d, d, D).$$

Then we get

$$D = AD + \frac{\alpha_1 D + \alpha_2 D + \alpha_3 D + \alpha_4 D + \alpha_5 d}{\beta_1 D + \beta_2 D + \beta_3 D + \beta_4 D + \beta_5 d} \quad \textit{and} \quad d = Ad + \frac{\alpha_1 d + \alpha_2 d + \alpha_3 d + \alpha_4 d + \alpha_5 D}{\beta_1 d + \beta_2 d + \beta_3 d + \beta_4 d + \beta_5 D},$$

or

$$D\,(1-A) = \frac{(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\, D + \alpha_5 d}{(\beta_1 + \beta_2 + \beta_3 + \beta_4)\, D + \beta_5 d} \quad \textit{and} \quad d\,(1-A) = \frac{(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\, d + \alpha_5 D}{(\beta_1 + \beta_2 + \beta_3 + \beta_4)\, d + \beta_5 D}.$$

From which we have

$$(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\, D + \alpha_5 d - (1-A)\,(\beta_1 + \beta_2 + \beta_3 + \beta_4)\, D^2 = (1-A)\,\beta_5 D d \tag{5.32}$$

and

$$(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\, d + \alpha_5 D - (1-A)\,(\beta_1 + \beta_2 + \beta_3 + \beta_4)\, d^2 = (1-A)\,\beta_5 D d \tag{5.33}$$

From (5.32) and (5.33), we obtain

$$(d-D)\left\{[(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) - \alpha_5] - (1-A)\,(\beta_1 + \beta_2 + \beta_3 + \beta_4)\,(d+D)\right\} = 0. \tag{5.34}$$

Since $A < 1$ and $\alpha_5 \geq (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$, we deduce from (5.34) that $D = d$. It follows by Theorem 2, that $\widetilde{y}$ of Eq.(1.1) is a global attractor.

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Case 2. Let the function $H(u_0,...,u_5)$ is non-decreasing in u_0,u_1 and non-increasing in u_2,u_3,u_4,u_5.

Suppose that (d,D) is a solution of the system

$$D = H(D,D,d,d,d,d) \quad and \quad d = H(d,d,D,D,D,D).$$

Then we get

$$D = AD + \frac{\alpha_1 D + \alpha_2 d + \alpha_3 d + \alpha_4 d + \alpha_5 d}{\beta_1 D + \beta_2 d + \beta_3 d + \beta_4 d + \beta_5 d} \quad and \quad d = Ad + \frac{\alpha_1 d + \alpha_2 D + \alpha_3 D + \alpha_4 D + \alpha_5 D}{\beta_1 d + \beta_2 D + \beta_3 D + \beta_4 D + \beta_5 D},$$

or

$$D(1-A) = \frac{\alpha_1 D + (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)\, d}{\beta_1 D + (\beta_2 + \beta_3 + \beta_4 + \beta_5)\, d} \quad and \quad d(1-A) = \frac{\alpha_1 d + (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)\, D}{\beta_1 d + (\beta_2 + \beta_3 + \beta_4 + \beta_5)\, D}.$$

From which we have

$$\alpha_1 D + (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)\, d - \beta_1 (1-A) D^2 = (1-A)(\beta_2 + \beta_3 + \beta_4 + \beta_5)\, Dd \tag{5.35}$$

and

$$\alpha_1 d + (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)\, D - \beta_1 (1-A) d^2 = (1-A)(\beta_2 + \beta_3 + \beta_4 + \beta_5)\, Dd. \tag{5.36}$$

From (5.35) and (5.36), we obtain

$$(d-D)\{[\alpha_1 - (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)] - \beta_1 (1-A)(d+D)\} = 0. \tag{5.37}$$

Since $A < 1$ and $(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) \geq \alpha_1$, we deduce from (5.37) that $D = d$. It follows by Theorem 2, that $\widetilde{y}$ of Eq.(1.1) is a global attractor.

Case 3. Let the function $H(u_0,...,u_5)$ is non-decreasing in u_0,u_1,u_2 and non-increasing in u_3,u_4,u_5.

Suppose that (d,D) is a solution of the system

$$D = H(D,D,D,d,d,d) \quad and \quad d = H(d,d,d,D,D,D).$$

Then we get

$$D = AD + \frac{\alpha_1 D + \alpha_2 D + \alpha_3 d + \alpha_4 d + \alpha_5 d}{\beta_1 D + \beta_2 D + \beta_3 d + \beta_4 d + \beta_5 d} \quad and \quad d = Ad + \frac{\alpha_1 d + \alpha_2 d + \alpha_3 D + \alpha_4 D + \alpha_5 D}{\beta_1 d + \beta_2 d + \beta_3 D + \beta_4 D + \beta_5 D},$$

or

$$D(1-A) = \frac{(\alpha_1 + \alpha_2)\, D + (\alpha_3 + \alpha_4 + \alpha_5)\, d}{(\beta_1 + \beta_2)\, D + (\beta_3 + \beta_4 + \beta_5)\, d} \quad and \quad d(1-A) = \frac{(\alpha_1 + \alpha_2)\, d + (\alpha_3 + \alpha_4 + \alpha_5)\, D}{(\beta_1 + \beta_2)\, d + (\beta_3 + \beta_4 + \beta_5)\, D}$$

From which we have

$$(\alpha_1+\alpha_2)\,D+(\alpha_3+\alpha_4+\alpha_5)\,d-(1-A)\,(\beta_1+\beta_2)\,D^2=(1-A)\,(\beta_3+\beta_4+\beta_5)\,Dd \tag{5.38}$$

and

$$(\alpha_1+\alpha_2)\,d+(\alpha_3+\alpha_4+\alpha_5)\,D-(1-A)\,(\beta_1+\beta_2)\,d^2=(1-A)\,(\beta_3+\beta_4+\beta_5)\,Dd \tag{5.39}$$

From (5.38) and (5.39), we obtain

$$(d-D)\left\{[(\alpha_1+\alpha_2)-(\alpha_3+\alpha_4+\alpha_5)]-(1-A)\,(\beta_1+\beta_2)\,(d+D)\right\}=0. \tag{5.40}$$

Since $A<1$ and $(\alpha_3+\alpha_4+\alpha_5)\geq(\alpha_1+\alpha_2)$, we deduce from (5.40) that $D=d$. It follows by Theorem 2, that $\widetilde{y}$ of Eq.(1.1) is a global attractor.

Case 4. Let the function $H(u_0,...,u_5)$ is non-decreasing in u_0,u_1,u_3 and non-increasing in u_2,u_4,u_5.

Suppose that (d,D) is a solution of the system

$$D=H(D,D,d,D,d,d) \quad and \quad d=H(d,d,D,d,D,D).$$

Then we get

$$D=AD+\frac{\alpha_1D+\alpha_2d+\alpha_3D+\alpha_4d+\alpha_5d}{\beta_1D+\beta_2d+\beta_3D+\beta_4d+\beta_5d} \quad and \quad d=Ad+\frac{\alpha_1d+\alpha_2D+\alpha_3d+\alpha_4D+\alpha_5D}{\beta_1d+\beta_2D+\beta_3d+\beta_4D+\beta_5D},$$

or

$$D\,(1-A)=\frac{(\alpha_1+\alpha_3)\,D+(\alpha_2+\alpha_4+\alpha_5)\,d}{(\beta_1+\beta_3)\,D+(\beta_2+\beta_4+\beta_5)\,d} \quad and \quad d\,(1-A)=\frac{(\alpha_1+\alpha_3)\,d+(\alpha_2+\alpha_4+\alpha_5)\,D}{(\beta_1+\beta_3)\,d+(\beta_2+\beta_4+\beta_5)\,D}$$

From which we have

$$(\alpha_1+\alpha_3)\,D+(\alpha_2+\alpha_4+\alpha_5)\,d-(1-A)\,(\beta_1+\beta_3)\,D^2=(1-A)\,(\beta_2+\beta_4+\beta_5)\,Dd \tag{5.41}$$

and

$$(\alpha_1+\alpha_3)\,d+(\alpha_2+\alpha_4+\alpha_5)\,D-(1-A)\,(\beta_1+\beta_3)\,d^2=(1-A)\,(\beta_2+\beta_4+\beta_5)\,Dd \tag{5.42}$$

From (5.41) and (5.42), we obtain

$$(d-D)\left\{[(\alpha_1+\alpha_3)-(\alpha_2+\alpha_4+\alpha_5)]-(1-A)\,(\beta_1+\beta_3)\,(d+D)\right\}=0. \tag{5.43}$$

Since $A<1$ and $(\alpha_2+\alpha_4+\alpha_5)\geq(\alpha_1+\alpha_3)$, we deduce from (5.43) that $D=d$. It follows by Theorem 2, that $\widetilde{y}$ of Eq.(1.1) is a global attractor.

It follows by Theorem 2, that $\widetilde{y}$ of Eq.(1.1) is a global attractor and the proof is now completed.

6 Numerical examples

Some numerical examples are stated in this section in order to strengthen our theoretical results. These examples represent different types of qualitative behavior of solutions of Eq.(1.1).

Example 1. Figure 1, shows that the solution of Eq.(1.1) is unbounded if $y_{-5} = 1,\ y_{-4} = 2,\ y_{-3} = 3,\ y_{-2} = 4,\ y_{-1} = 5,\ y_0 = 6,\ \ A = 1.1,\ \alpha_1 = 10,\ \alpha_2 = 1,\ \alpha_3 = 12,\ \alpha_4 = 4,\ \alpha_5 = 6,\ \beta_1 = 2,\ \beta_2 = 3,\ \beta_3 = 40,\ \beta_4 = 50,\ \beta_5 = 60.$

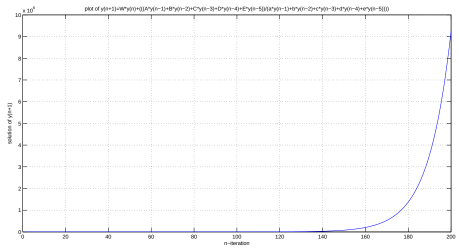

Figure 1: $(y_{m+1} = 1.1y_m + \frac{10y_{m-1}+y_{m-2}+12y_{m-3}+4y_{m-4}+6y_{m-5}}{2y_{m-1}+3y_{m-2}+40y_{m-3}+50y_{m-4}+60y_{m-5}})$

Example 2. Figure 2, shows that Eq.(1.1) has prime period two solutions if $y_{-5} = y_{-3} = y_{-1} \simeq 0.519,\ y_{-4} = y_{-2} = y_0 \simeq -0.0938,\ A = 1,\ \alpha_1 = 10,\ \alpha_2 = 3,\ \alpha_3 = 30,\ \alpha_4 = 8,\ \alpha_5 = 45,\ \beta_1 = 20,\ \beta_2 = 5,\ \beta_3 = 40,\ \beta_4 = 9,\ \beta_5 = 100.$

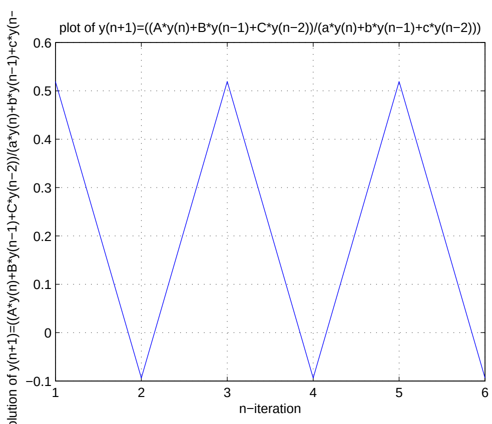

Figure 2: $(y_{m+1} = y_m + \frac{10y_{m-1}+3y_{m-2}+30y_{m-3}+8y_{m-4}+45y_{m-5}}{20y_{m-1}+5y_{m-2}+40y_{m-3}+9y_{m-4}+100y_{m-5}})$

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Example 3. Figure 3, shows that Eq.(1.1) is globally asymptotically stable if $y_{-5} = 1$, $y_{-4} = 2$, $y_{-3} = 3$, $y_{-2} = 4$, $y_{-1} = 5$, $y_0 = 6$, $A = 0.5$, $\alpha_1 = 10$, $\alpha_2 = 1$, $\alpha_3 = 12$, $\alpha_4 = 4$, $\alpha_5 = 30$, $\beta_1 = 2$, $\beta_2 = 3$, $\beta_3 = 40$, $\beta_4 = 50$, $\beta_5 = 400$.

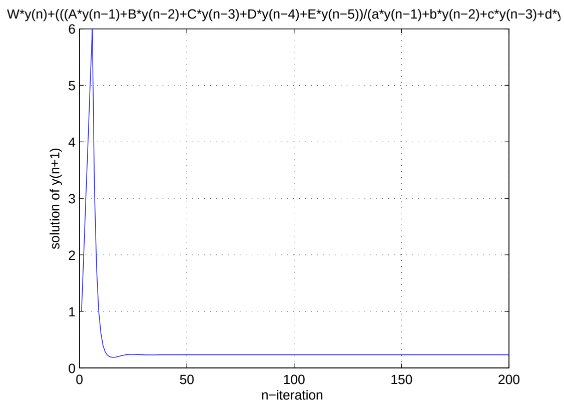

Figure 3: $(y_{m+1} = 0.5y_m + \frac{10y_{m-1}+y_{m-2}+12y_{m-3}+4y_{m-4}+30y_{m-5}}{2y_{m-1}+3y_{m-2}+40y_{m-3}+50y_{m-4}+400y_{m-5}})$

Example 4. Figure 4, shows that Eq.(1.1) is not globally asymptotically stable if $y_{-5} = 1$, $y_{-4} = 2$, $y_{-3} = 3$, $y_{-2} = 4$, $y_{-1} = 5$, $y_0 = 6$, $A = 100$, $\alpha_1 = 10$, $\alpha_2 = 1$, $\alpha_3 = 12$, $\alpha_4 = 4$, $\alpha_5 = 6$, $\beta_1 = 2$, $\beta_2 = 3$, $\beta_3 = 40$, $\beta_4 = 50, \beta_5 = 400$.

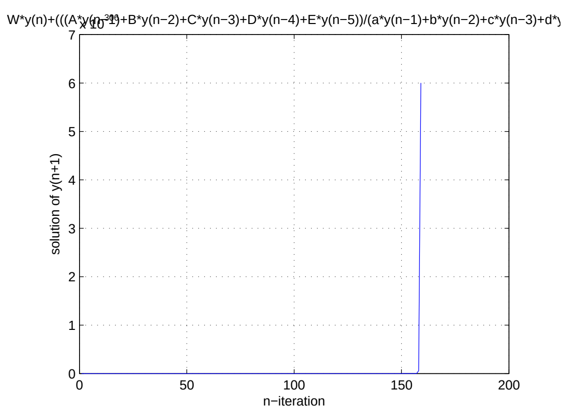

Figure 4: $(y_{m+1} = 100y_m + \frac{10y_{m-1}+y_{m-2}+12y_{m-3}+4y_{m-4}+6y_{m-5}}{2y_{m-1}+3y_{m-2}+40y_{m-3}+50y_{m-4}+400y_{m-5}})$

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7 Conclusion

We have discussed some properties of the nonlinear rational difference equation (1.1), such as the periodicity, the boundedness and the global stability of the positive solutions of this equation. We gave some figures to illustrate the behavior of these solutions, as generalization of the results obtained in Refs.[4,5,8]. Note that example 1 illustrates Theorem 5 which shows that the solution of Eq.(1.1) is unbounded and example 2 illustrates Theorem 6 which shows that Eq.(1.1) has prime period two solutions, while example 3 illustrates Theorems 3 and 7 which shows that Eq.(1.1) is globally asymptotically stable. But example 4 shows that Eq.(1.1) is not globally asymptotically stable if $A > 1$.

8 Acknowledgement

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On sequential fractional differential equations with nonlocal integral boundary conditions

N.I. Mahmudov and M. Awadalla
Department of Mathematics
Eastern Mediterranean University
Gazimagusa, Mersin 10, Turkey

Abstract

This article develops the existence theory for sequential fractional differential equations involving Caputo fractional derivative of order $1 < \alpha \leq 2$ with nonlocal integral boundary conditions. An example is given to demonstrate application of our results.

Keywords: fractional differential equations; mixed boundary value problem; fixed point theorem.
2010 AMS Subject Classification: 34A08; 34B

1 Introduction

The theory of fractional-order differential equations involving different kinds of boundary conditions has been a field of interest in pure and applied sciences. In addition to the classical two-point boundary conditions, great attention is paid to non-local multipoint and integral boundary conditions. Nonlocal conditions are used to describe certain features of physical, chemical or other processes occurring in the internal positions of the given region, while integral boundary conditions provide a plausible and practical approach to modeling the problems of blood flow. For more details and explanation, see, for instance [2], [1]. Some recent results on fractional-order boundary value problem can be found in a series of papers [3]-[20] and the references cited therein. Sequential fractional differential equations have also received considerable attention, for instance see [4]-[9]. To the best of our knowledge, the study of sequential fractional differential equations supplemented with nonlocal integral fractional boundary conditions has yet to be initiated.

We study the following nonlinear sequential fractional differential equation subject to nonseparated nonlocal integral fractional boundary conditions

$$\begin{cases} \left({}^{C}D^{\alpha} + \lambda\ {}^{C}D^{\alpha-1}\right) u(t) = f(t, u(t)), & 1 < \alpha \leq 2, \quad 0 \leq t \leq T, \\ \nu_1 u(\eta) + \mu_1 u(T) = \gamma_1 \int_0^{\xi} u(s)\, ds, & \\ \nu_2\ {}^{C}D^{\alpha-1} u(\eta) + \mu_2\ {}^{C}D^{\alpha-1} u(T) = \gamma_2 \int_{\zeta}^{T} u(s)\, ds, & \end{cases} \tag{1}$$

where $0 < \eta < T, 0 < \xi < \zeta < T$, $\lambda \in \mathbb{R}_+, \nu_1, \nu_2, \mu_1, \mu_2, \gamma_1, \gamma_2 \in \mathbb{R}$.

The rest of the paper is organized as follows. In Section 2, we recall some basic concepts of fractional calculus and obtain the integral solution for the linear variants of the given problems. Section 3 contains the existence results for problem (1) obtained by applying Leray-Schauder's nonlinear alternative, Banach's contraction mapping principle and Krasnoselskii's fixed point theorem. In Section 4, the main result is illustrated with the aid of an example.

2 Preliminaries

Definition 1 *The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $f : [0, +\infty) \to R$ is defined as*

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided that the right hand side of the integral is pointwise defined on $(0, +\infty)$ and Γ is the gamma function.

Definition 2 *The Caputo derivative of order $\alpha > 0$ for a function $f : [0, +\infty) \to R$ is written as*

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$

where $n = [\alpha] + 1, [\alpha]$ is integral part of α.

Lemma 3 *Let $\alpha > 0$. Then the differential equation $D_{0+}^{\alpha} f(t) = 0$ has solutions*

$$f(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1},$$

and

$$I_{0+}^{\alpha} D_{0+}^{\alpha} f(t) = f(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$ and $i = 1, 2, ..., n = [\alpha] + 1$.

In what follows we use the following notations:

$$\begin{aligned}
a_{11} &:= \nu_1 e^{-\lambda\eta} + \mu_1 e^{-\lambda T} - \frac{\gamma_1}{\lambda}\left(1 - e^{-\lambda\xi}\right), \quad a_{12} := \nu_1 + \mu_1 - \gamma_1 \xi, \\
a_{21} &:= \nu_2 \frac{\lambda}{\Gamma(2-\alpha)} \int_0^{\eta} (\eta - s)^{1-\alpha} e^{-\lambda s} ds + \mu_2 \frac{\lambda}{\Gamma(2-\alpha)} \int_0^{T} (T-s)^{1-\alpha} e^{-\lambda s} ds + \gamma_2 \int_{\zeta}^{T} e^{-\lambda t} dt, \\
a_{22} &:= \gamma_2 (T - \zeta), \quad \Delta := a_{11} a_{22} - a_{12} a_{21}, \quad \Delta \neq 0, \\
\varphi_1(t) &= \frac{a_{21} - a_{22} e^{-\lambda t}}{\Delta}, \quad \varphi_2(t) = \frac{a_{11} - a_{12} e^{-\lambda t}}{\Delta}, \\
K_1(t,s) &= \frac{1}{\Gamma(\alpha-1)} \int_s^t e^{-\lambda(t-r)} (r-s)^{\alpha-2} dr, \quad K_2(t,r) = \frac{1}{\Gamma(2-\alpha)} \int_r^t (t-s)^{1-\alpha} K_1(s,r) ds.
\end{aligned}$$

It is clear that

$$\begin{aligned}
|\varphi_1(t)| &\leq \max\left(\frac{|a_{21} - a_{22}|}{|\Delta|}, \frac{\left|a_{21} - a_{22} e^{-\lambda T}\right|}{|\Delta|}\right) := \phi_1, \\
|\varphi_2(t)| &\leq \max\left(\frac{|a_{11} - a_{12}|}{|\Delta|}, \frac{\left|a_{11} - a_{12} e^{-\lambda T}\right|}{|\Delta|}\right) := \phi_2,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^t e^{-\lambda(t-r)} I^{\alpha-1} h(r) dr &= \int_0^t K_1(t,s) h(s) ds, \\
\frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \int_0^s e^{-\lambda(s-r)} I^{\alpha-1} h(r) dr ds &= \int_0^t K_2(t,r) h(r) dr.
\end{aligned}$$

Lemma 4 *Let $h \in C([0,T];\mathbb{R})$. The the following boundary value problem*

$$\begin{cases} \left({}^{C}D^{\alpha} + \lambda\ {}^{C}D^{\alpha-1}\right) u(t) = h(t), & 1 < \alpha \leq 2, \quad 0 \leq t \leq T, \\ \nu_1 u(\eta) + \mu_1 u(T) = \gamma_1 \int_0^{\xi} u(s)\, ds, & \\ \nu_2\ {}^{C}D^{\alpha-1} u(\eta) + \mu_2\ {}^{C}D^{\alpha-1} u(T) = \gamma_2 \int_{\zeta}^{T} u(s)\, ds, & \end{cases} \tag{2}$$

is equivalent to the fractional integral equation

$$\begin{aligned} u(t) &= \int_0^t K_1(t,s) h(s)\, ds \\ &+ \nu_1 \varphi_1(t) \int_0^{\eta} K_1(\eta,s) h(s)\, ds + \mu_1 \varphi_1(t) \int_0^T K_1(T,s) h(s)\, ds \\ &- \gamma_1 \varphi_1(t) \int_0^{\xi} \int_0^r K_1(r,s) h(s)\, ds dr - \gamma_2 \varphi_2(t) \int_{\zeta}^T \int_0^t K_1(t,s) h(s)\, ds dt \\ &- \lambda \nu_2 \varphi_2(t) \int_0^{\eta} K_2(\eta,s) h(s)\, ds - \lambda \mu_2 \varphi_2(t) \int_0^T K_2(T,s) h(s)\, ds \\ &+ \nu_2 \varphi_2(t) \int_0^{\eta} h(s)\, ds + \mu_2 \varphi_2(t) \int_0^T h(s)\, ds. \end{aligned} \tag{3}$$

Proof. Applying $I^{\alpha-1}$ to both sides of (2) we get

$$\begin{aligned} I^{\alpha-1}\ {}^{C}D^{\alpha-1}(D+\lambda) u(t) &= I^{\alpha-1} h(t), \\ (D+\lambda) u(t) - c_0 &= I^{\alpha-1} h(t). \end{aligned}$$

We solve the above linear differential equation

$$\begin{aligned} u(t) &= (u(0) - c_0) e^{-\lambda t} + c_0 + \int_0^t e^{-\lambda(t-s)} I^{\alpha-1} h(s)\, ds, \\ u(t) &= c_1 e^{-\lambda t} + c_0 + \int_0^t e^{-\lambda(t-s)} I^{\alpha-1} h(s)\, ds. \end{aligned} \tag{4}$$

It is clear that

$$\begin{aligned} {}^{C}D^{\alpha-1} u(t) &= \frac{-\lambda c_1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} e^{-\lambda s} ds \\ &+ \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \left(I^{\alpha-1} h(s) - \lambda \int_0^s e^{-\lambda(s-r)} I^{\alpha-1} h(r)\, dr \right) ds. \end{aligned}$$

The first boundary condition implies that

$$\begin{aligned} &\nu_1 u(\eta) + \mu_1 u(T) \\ &= \nu_1 c_1 e^{-\lambda \eta} + \nu_1 c_0 + \nu_1 \int_0^{\eta} e^{-\lambda(\eta-s)} I^{\alpha-1} h(s)\, ds \\ &+ \mu_1 c_1 e^{-\lambda T} + \mu_1 c_0 + \mu_1 \int_0^T e^{-\lambda(T-s)} I^{\alpha-1} h(s)\, ds \\ &= \gamma_1 \int_0^{\xi} \left(c_1 e^{-\lambda r} + c_0 + \int_0^r e^{-\lambda(r-s)} I^{\alpha-1} h(s)\, ds \right) dr \\ &= \frac{\gamma_1 c_1}{\lambda} \left(1 - e^{-\lambda \xi}\right) + \gamma_1 c_0 \xi + \gamma_1 \int_0^{\xi} \int_0^r e^{-\lambda(r-s)} I^{\alpha-1} h(s)\, ds dr, \end{aligned}$$

$$\begin{aligned}
&\left(\nu_1 e^{-\lambda\eta} + \mu_1 e^{-\lambda T} - \frac{\gamma_1}{\lambda}\left(1-e^{-\lambda\xi}\right)\right)c_1 + \left(\nu_1+\mu_1-\gamma_1\xi\right)c_0 \\
&= \gamma_1 \int_0^{\xi}\int_0^{r} e^{-\lambda(r-s)} I^{\alpha-1}h(s)\,dsdr - \nu_1\int_0^{\eta} e^{-\lambda(\eta-s)} I^{\alpha-1}h(s)\,ds - \mu_1\int_0^{T} e^{-\lambda(T-s)} I^{\alpha-1}h(s)\,ds.
\end{aligned}$$

The second boundary condition implies that

$$\begin{aligned}
&\left(\nu_2\frac{\lambda}{\Gamma(2-\alpha)}\int_0^{\eta}(\eta-s)^{1-\alpha}e^{-\lambda s}ds + \mu_2\frac{\lambda}{\Gamma(2-\alpha)}\int_0^{T}(T-s)^{1-\alpha}e^{-\lambda s}ds + \gamma_2\int_{\zeta}^{T}e^{-\lambda t}dt\right)c_1 + \gamma_2(T-\zeta)c_0 \\
&= \nu_2\frac{1}{\Gamma(2-\alpha)}\int_0^{\eta}(\eta-s)^{1-\alpha}\left(I^{\alpha-1}h(s) - \lambda\int_0^{s}e^{-\lambda(s-r)}I^{\alpha-1}h(r)\,dr\right)ds \\
&+ \mu_2\frac{1}{\Gamma(2-\alpha)}\int_0^{T}(T-s)^{1-\alpha}\left(I^{\alpha-1}h(s) - \lambda\int_0^{s}e^{-\lambda(s-r)}I^{\alpha-1}h(r)\,dr\right)ds - \gamma_2\int_{\zeta}^{T}\int_0^{t}e^{-\lambda(t-s)}I^{\alpha-1}h(s)\,dsdt.
\end{aligned}$$

Thus

$$\begin{aligned}
a_{11}c_1 + a_{12}c_0 &= \gamma_1\int_0^{\xi}\int_0^{r}K_1(r,s)h(s)\,dsdr - \nu_1\int_0^{\eta}K_1(\eta,s)h(s)\,ds - \mu_1\int_0^{T}K_1(T,s)h(s)\,ds, \\
a_{21}c_1 + a_{22}c_0 &= \nu_2\int_0^{\eta}h(s)\,ds + \mu_2\int_0^{T}h(s)\,ds - \lambda\nu_2\int_0^{\eta}K_2(\eta,s)h(s)\,ds \\
&\quad - \lambda\mu_2\int_0^{T}K_2(T,s)h(s)\,ds - \gamma_2\int_{\zeta}^{T}\int_0^{t}K_1(t,s)h(s)\,dsdt.
\end{aligned}$$

Solving the above system of equations for c_0 and c_1, we get

$$\begin{aligned}
c_0 &= \frac{a_{11}}{\Delta}\nu_2\int_0^{\eta}h(s)\,ds + \frac{a_{11}}{\Delta}\mu_2\int_0^{T}h(s)\,ds - \frac{a_{11}}{\Delta}\lambda\nu_2\int_0^{\eta}K_2(\eta,s)h(s)\,ds \\
&- \frac{a_{11}}{\Delta}\lambda\mu_2\int_0^{T}K_2(T,s)h(s)\,ds - \frac{a_{11}}{\Delta}\gamma_2\int_{\zeta}^{T}\int_0^{t}K_1(t,s)h(s)\,dsdt \\
&- \frac{a_{21}}{\Delta}\gamma_1\int_0^{\xi}\int_0^{r}K_1(r,s)h(s)\,dsdr + \frac{a_{21}}{\Delta}\nu_1\int_0^{\eta}K_1(\eta,s)h(s)\,ds + \frac{a_{21}}{\Delta}\mu_1\int_0^{T}K_1(T,s)h(s)\,ds \\
c_1 &= \frac{a_{22}}{\Delta}\gamma_1\int_0^{\xi}\int_0^{r}K_1(r,s)h(s)\,dsdr - \frac{a_{22}}{\Delta}\nu_1\int_0^{\eta}K_1(\eta,s)h(s)\,ds - \frac{a_{22}}{\Delta}\mu_1\int_0^{T}K_1(T,s)h(s)\,ds \\
&- \frac{a_{12}}{\Delta}\nu_2\int_0^{\eta}h(s)\,ds - \frac{a_{12}}{\Delta}\mu_2\int_0^{T}h(s)\,ds + \frac{a_{12}}{\Delta}\lambda\nu_2\int_0^{\eta}K_2(\eta,s)h(s)\,ds \\
&+ \frac{a_{12}}{\Delta}\lambda\mu_2\int_0^{T}K_2(T,s)h(s)\,ds + \frac{a_{12}}{\Delta}\gamma_2\int_{\zeta}^{T}\int_0^{t}K_1(t,s)h(s)\,dsdt.
\end{aligned}$$

Inserting c_0 and c_1 in (4) we obtain the desired formula (3).

Conversely, assume that u satisfies (3). By a direct computation, it follows that the solution given by (3) satisfies (2). ■

Lemma 5 *For any $g, h \in C([0,T];\mathbb{R})$ we have*

$$\begin{aligned}
\left|\int_0^{t}K_1(t,s)g(s)\,ds - \int_0^{t}K_1(t,s)h(s)\,ds\right| &\le \frac{t^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda t}\right)\|g-h\|_C, \\
\left|\int_0^{t}K_2(t,s)g(s)\,ds - \int_0^{t}K_2(t,s)h(s)\,ds\right| &\le \frac{t}{\lambda}\left(1-e^{-\lambda t}\right)\|g-h\|_C.
\end{aligned}$$

Proof. Indeed,

$$
\begin{aligned}
& \left| \int_0^t K_1(t,s)\, g(s)\, ds - \int_0^t K_1(t,s)\, h(s)\, ds \right| \\
& \leq \int_0^t K_1(t,s) \left| g(s) - h(s) \right| ds \leq \int_0^t K_1(t,s)\, ds \left\| g-h \right\|_C \\
& \leq \frac{1}{\Gamma(\alpha-1)} \int_0^t \left(\int_s^t e^{-\lambda(t-r)} (r-s)^{\alpha-2}\, dr \right) ds \left\| g-h \right\|_C \\
& = \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^r e^{-\lambda(t-r)} (r-s)^{\alpha-2}\, ds dr \left\| g-h \right\|_C \\
& \leq \frac{t^{\alpha-1}}{\lambda(\alpha-1)\Gamma(\alpha-1)} \left(1-e^{-\lambda t}\right) \left\| g-h \right\|_C \\
& \leq \frac{T^{\alpha-1}}{\lambda\Gamma(\alpha)} \left(1-e^{-\lambda T}\right) \left\| g-h \right\|_C .
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
& \left| \int_0^t K_2(t,s)\, g(s)\, ds - \int_0^t K_2(t,s)\, h(s)\, ds \right| \\
& = \left| \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \int_0^s e^{-\lambda(s-r)} I^{\alpha-1} (g(r) - h(r))\, dr ds \right| \\
& = \frac{1}{\Gamma(2-\alpha)\Gamma(\alpha-1)} \left| \int_0^t (t-s)^{1-\alpha} \int_0^s e^{-\lambda(s-r)} \int_0^r (r-l)^{\alpha-2} (g(l) - h(l))\, dl dr ds \right| \\
& \leq \frac{1}{(\alpha-1)\Gamma(2-\alpha)\Gamma(\alpha-1)} \int_0^t (t-s)^{1-\alpha} \int_0^s e^{-\lambda(s-r)} r^{\alpha-1} dr ds \left\| g-h \right\|_C \\
& \leq \frac{1}{(\alpha-1)\Gamma(2-\alpha)\Gamma(\alpha-1)} \int_0^t (t-s)^{1-\alpha} s^{\alpha-1} \int_0^s e^{-\lambda(s-r)} dr ds \left\| g-h \right\|_C \\
& = \frac{\left(1-e^{-\lambda t}\right)}{\lambda(\alpha-1)\Gamma(2-\alpha)\Gamma(\alpha-1)} \int_0^t (t-s)^{1-\alpha} s^{\alpha-1} ds \left\| g-h \right\|_C \\
& = \frac{\left(1-e^{-\lambda t}\right) t}{\lambda(\alpha-1)\Gamma(2-\alpha)\Gamma(\alpha-1)} \int_0^1 (1-s)^{1-\alpha} s^{\alpha-1} ds \left\| g-h \right\|_C \\
& = \frac{\left(1-e^{-\lambda t}\right) t}{\lambda(\alpha-1)\Gamma(2-\alpha)\Gamma(\alpha-1)} B(\alpha, 2-\alpha) \left\| g-h \right\|_C \\
& = \frac{\left(1-e^{-\lambda t}\right) t}{\lambda(\alpha-1)\Gamma(2-\alpha)\Gamma(\alpha-1)} \frac{\Gamma(\alpha)\Gamma(2-\alpha)}{\Gamma(2)} \left\| g-h \right\|_C \\
& = \frac{t}{\lambda} \left(1-e^{-\lambda t}\right) \left\| g-h \right\|_C .
\end{aligned}
$$

■

3 Main results

We introduce a fixed point problem associated with the problem as follows:

$$\begin{aligned}
(\mathfrak{F}u)(t) &= \int_0^t K_1(t,s) f(s,u(s))\,ds \\
&+ \nu_1\varphi_1(t)\int_0^\eta K_1(\eta,s) f(s,u(s))\,ds + \mu_1\varphi_1(t)\int_0^T K_1(T,s) f(s,u(s))\,ds \\
&- \gamma_1\varphi_1(t)\int_0^\xi\int_0^r K_1(r,s) f(s,u(s))\,dsdr - \gamma_2\varphi_2(t)\int_\zeta^T\int_0^t K_1(t,s) f(s,u(s))\,dsdt \\
&- \lambda\nu_2\varphi_2(t)\int_0^\eta K_2(\eta,s) f(s,u(s))\,ds - \lambda\mu_2\varphi_2(t)\int_0^T K_2(T,s) f(s,u(s))\,ds \\
&+ \nu_2\varphi_2(t)\int_0^\eta f(s,u(s))\,ds + \mu_2\varphi_2(t)\int_0^T f(s,u(s))\,ds. \qquad (5)
\end{aligned}$$

Let

$$\begin{aligned}
R &:= \frac{T^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda T}\right) + |\nu_1|\,\phi_1\frac{\eta^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda\eta}\right) \\
&+ |\mu_1|\,\phi_1\frac{T^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda T}\right) + |\gamma_1|\,\phi_1\int_0^\xi\frac{r^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda r}\right)dr \\
&+ |\gamma_2|\,\phi_2\int_\zeta^T\frac{t^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda t}\right)dt + \lambda\,|\nu_2|\,\phi_2\frac{\eta}{\lambda}\left(1-e^{-\lambda\eta}\right) \\
&+ \lambda\,|\mu_2|\,\phi_2\frac{T}{\lambda}\left(1-e^{-\lambda T}\right) + |\nu_2|\,\phi_2\eta + |\mu_2|\,\phi_2 T, \\
R^* &:= R - \frac{T^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda T}\right).
\end{aligned}$$

Theorem 6 *Let $f.[0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function such that the following conditions hold:*

(A_1) there exists $L_f>0$ such that

$$|f(t,u)-f(t,v)| \le L_f\,|u-v|\,, \quad \forall (t,u),(t,v)\in[0,T]\times\mathbb{R};$$

(A_2) $L_f R<1$.
Then the problem (1) has a unique solution in $C([0,T],\mathbb{R})$.

Proof. Consider a ball

$$B_r := \{u\in C([0,T],\mathbb{R}) : \|u\|_C \le r\}$$

with $r\ge \frac{M_f R}{1-L_f R}$, where $M_f := \sup\{|f(t,0)| : 0\le t\le T\}$. It is clear that

$$|f(t,u)| \le L_f\,|u| + M_f, \quad u\in\mathbb{R}.$$

Using this inequality and Lemma 5 from (5) it follows that

$$\begin{aligned}
\left|(\mathfrak{F}u)(t)\right| &\leq \frac{t^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda t}\right)\|f(\cdot,u(\cdot))\|_C+|\nu_1||\varphi_1(t)|\frac{\eta^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda\eta}\right)\|f(\cdot,u(\cdot))\|_C \\
&+|\mu_1||\varphi_1(t)|\frac{T^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda T}\right)\|f(\cdot,u(\cdot))\|_C+|\gamma_1||\varphi_1(t)|\int_0^{\xi}\frac{r^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda r}\right)dr\,\|f(\cdot,u(\cdot))\|_C \\
&+|\gamma_2||\varphi_2(t)|\int_{\zeta}^{T}\frac{t^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda t}\right)dt\,\|f(\cdot,u(\cdot))\|_C+\lambda|\nu_2||\varphi_2(t)|\frac{\eta}{\lambda}\left(1-e^{-\lambda\eta}\right)\|f(\cdot,u(\cdot))\|_C \\
&+\lambda|\mu_2||\varphi_2(t)|\frac{T}{\lambda}\left(1-e^{-\lambda T}\right)\|f(\cdot,u(\cdot))\|_C+|\nu_2||\varphi_2(t)|\eta\|f(\cdot,u(\cdot))\|_C+|\mu_2||\varphi_2(t)|T\|f(\cdot,u(\cdot))\|_C \\
&\leq (L_f r+M_f)R\leq r.
\end{aligned}$$

This shows that $\mathfrak{F}B_r\subset B_r$. Next, using the condition (A_1), we obtain

$$\|\mathfrak{F}u-\mathfrak{F}v\|_C\leq L_f R\|u-v\|_C.$$

By (A_2) the operator $\mathfrak{F}$ is a contraction. Thus by the Banach fixed point theorem has $\mathfrak{F}$ a unique fixed point in $C([0,T],\mathbb{R})$. ■

Theorem 7 *Let $f.[0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function such that the following condition holds:*

(A_3) there exists $\gamma\in C([0,T],\mathbb{R}_+)$ and a nondecreasing function $\psi:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$|f(t,u)|\leq\gamma(t)\psi(|u|),\quad\forall(t,u)\in[0,T]\times\mathbb{R}.$$

(A_4) There exists $M>0$ such that

$$\frac{M}{\psi(M)\|\gamma\|_C R}>1.$$

Then the BVP (1) has at least one solution.

Proof. Step 1: Show that $\mathfrak{F}:C([0,T],\mathbb{R})\to C([0,T],\mathbb{R})$ maps bounded sets into bounded sets and is continuous.

Let B_r be a bounded set in $C([0,T],\mathbb{R})$.Then $|f(t,u(t))|\leq\|\gamma\|\psi(|u(t)|)\leq\|\gamma\|\psi(r)$ and by Lemma 5

$$\begin{aligned}
\left|(\mathfrak{F}u)(t)\right| &\leq \frac{t^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda t}\right)\|f(\cdot,u(\cdot))\|_C+|\nu_1||\varphi_1(t)|\frac{\eta^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda\eta}\right)\|f(\cdot,u(\cdot))\|_C \\
&+|\mu_1||\varphi_1(t)|\frac{T^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda T}\right)\|f(\cdot,u(\cdot))\|_C+|\gamma_1||\varphi_1(t)|\int_0^{\xi}\frac{t^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda t}\right)dt\,\|f(\cdot,u(\cdot))\|_C \\
&+|\gamma_2||\varphi_2(t)|\int_{\zeta}^{T}\frac{t^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1-e^{-\lambda t}\right)dt\,\|f(\cdot,u(\cdot))\|_C+\lambda|\nu_2||\varphi_2(t)|\frac{\eta}{\lambda}\left(1-e^{-\lambda\eta}\right)\|f(\cdot,u(\cdot))\|_C \\
&+\lambda|\mu_2||\varphi_2(t)|\frac{T}{\lambda}\left(1-e^{-\lambda T}\right)\|f(\cdot,u(\cdot))\|_C+|\nu_2||\varphi_2(t)|\eta\|f(\cdot,u(\cdot))\|_C+|\mu_2||\varphi_2(t)|T\|f(\cdot,u(\cdot))\|_C \\
&\leq \|\gamma\|_C\psi(r)R.
\end{aligned}$$

Step 2: Next we show that $\mathfrak{F}$ maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$.

Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and $u \in B_r$. Then we obtain

$$
\begin{aligned}
&\left|(\mathfrak{F}u)(t_1) - (\mathfrak{F}u)(t_2)\right| \\
&\le \left|\int_0^{t_1} \left(K_1(t_1,s) - K_2(t_1,s)\right) f(s,u(s))\,ds\right| + \left|\int_{t_1}^{t_2} K_2(t_1,s) f(s,u(s))\,ds\right| \\
&+ |\nu_1| \left(|\varphi_1(t_1) - \varphi_1(t_2)|\right) \frac{\eta^{\alpha-1}}{\lambda\Gamma(\alpha)} \left(1 - e^{-\lambda\eta}\right) \|\gamma\|_C \psi(r) \\
&+ |\mu_1| \left(|\varphi_1(t_1) - \varphi_1(t_2)|\right) \frac{T^{\alpha-1}}{\lambda\Gamma(\alpha)} \left(1 - e^{-\lambda T}\right) \|\gamma\|_C \psi(r) \\
&+ |\gamma_1| \left(|\varphi_1(t_1) - \varphi_1(t_2)|\right) \int_0^{\xi} \frac{r^{\alpha-1}}{\lambda\Gamma(\alpha)} \left(1 - e^{-\lambda r}\right) dr \, \|\gamma\|_C \psi(r) \\
&+ |\gamma_2| \, |\varphi_2(t_1) - \varphi_2(t_2)| \int_{\zeta}^{T} \frac{t^{\alpha-1}}{\lambda\Gamma(\alpha)} \left(1 - e^{-\lambda t}\right) dt \, \|\gamma\|_C \psi(r) \\
&+ \lambda |\nu_2| \, |\varphi_2(t_1) - \varphi_2(t_2)| \frac{\eta}{\lambda} \left(1 - e^{-\lambda\eta}\right) \|\gamma\|_C \psi(r) \\
&+ \lambda |\mu_2| \, |\varphi_2(t_1) - \varphi_2(t_2)| \frac{T}{\lambda} \left(1 - e^{-\lambda T}\right) \|\gamma\|_C \psi(r) \\
&+ \eta |\nu_2| \, |\varphi_2(t_1) - \varphi_2(t_2)| \, \|\gamma\|_C \psi(r) + T |\mu_2| \, |\varphi_2(t_1) - \varphi_2(t_2)| \, \|\gamma\|_C \psi(r).
\end{aligned}
$$

Obviously, the right-hand side of the above inequality tends to zero independently of $u \in B_r$ as $t_1 \to t_2$.As $\mathfrak{F}$ satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\mathfrak{F} : C([0,T], \mathbb{R}) \to C([0,T], \mathbb{R})$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative once we have proved the boundedness of the set of all solutions to equations $u = \theta\mathfrak{F}u$ for $0 \le \theta \le 1$.

Let u be a solution. Then using the computations employed in proving that $\mathfrak{F}$ is bounded, we have

$$|u(t)| = \theta \left|(\mathfrak{F}u)(t)\right| \le \|\gamma\|_C \psi(\|u\|_C) R.$$

Consequently, we have

$$\frac{\|u\|_C}{\|\gamma\|_C \psi(\|u\|_C) R} \le 1.$$

In view of (A_4), there exists M such that $\|u\|_C \ne M$. Let us set

$$\mathfrak{U} = \left\{u \in C([0,T], \mathbb{R}) : \|u\|_C < M \right\}.$$

Note that the operator $\mathfrak{F} : \overline{\mathfrak{U}} \to C([0,T], \mathbb{R})$ is continuous and completely continuous. From the choice of $\mathfrak{U}$, there is no $u \in \partial\mathfrak{U}$ such that $u = \theta\mathfrak{F}u$ for some $0 < \theta < 1$. Consequently, by the nonlinear alternative of Leray-Schauder type, we deduce that $\mathfrak{F}$ has a fixed point $u \in \overline{\mathfrak{U}}$ which is a solution of problem (1). This completes the proof. ■

Now, we result based on the Krasnoselskii theorem.

Theorem 8 *Let $f.[0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that the following conditions hold:*

(A_1) there exists $L_f > 0$ such that

$$|f(t,u) - f(t,v)| \le L_f |u - v|, \quad \forall (t,u), (t,v) \in [0,T] \times \mathbb{R};$$

(A_5) there exists $\gamma \in C([0,T], \mathbb{R}_+)$ such that

$$|f(t,u)| \le \gamma(t), \quad \forall (t,u) \in [0,T] \times \mathbb{R}.$$

(A_6) $L_f R^* < 1$.

Then the boundary value problem (1) has at least one solution in $C([0,T],\mathbb{R})$.

Proof. Consider the closed set $B_r := \{u \in C([0,T],\mathbb{R}) : \|u\|_C \leq r\}$ with $r \geq R\|\gamma\|_C$ and define the operators $\mathfrak{F}_1$ and $\mathfrak{F}_2$ on B_r as follows:

$$(\mathfrak{F}_1 u)(t) := \int_0^t K_1(t,s) f(s,u(s))\, ds,$$

$$\begin{aligned}(\mathfrak{F}_2 u)(t) &:= \nu_1\varphi_1(t)\int_0^\eta K_1(\eta,s) f(s,u(s))\,ds + \mu_1\varphi_1(t)\int_0^T K_1(T,s) f(s,u(s))\,ds \\ &- \gamma_1\varphi_1(t)\int_0^\xi\int_0^r K_1(r,s) f(s,u(s))\,dsdr - \gamma_2\varphi_2(t)\int_\zeta^T\int_0^t K_1(t,s) f(s,u(s))\,dsdt \\ &- \lambda\nu_2\varphi_2(t)\int_0^\eta K_2(\eta,s) f(s,u(s))\,ds - \lambda\mu_2\varphi_2(t)\int_0^T K_2(T,s) f(s,u(s))\,ds \\ &+ \nu_2\varphi_2(t)\int_0^\eta f(s,u(s))\,ds + \mu_2\varphi_2(t)\int_0^T f(s,u(s))\,ds.\end{aligned}$$

For $u, v \in B_r$, it is easy to verify that $\|\mathfrak{F}_1 u + \mathfrak{F}_2 v\|_C \leq R\|\gamma\|_C$. Thus, $\mathfrak{F}_1 u + \mathfrak{F}_2 v \in B_r$. One can easily show that

$$\|\mathfrak{F}_2 u - \mathfrak{F}_2 v\|_C \leq L_f R^* \|u - v\|_C.$$

By (A_6) $\mathfrak{F}_2$ is contraction. On the other hand, (i) continuity of f implies that the operator $\mathfrak{F}_1$ is continuous, (ii) $\mathfrak{F}_1$ is uniformly bounded on B_r :

$$\|\mathfrak{F}_1 u\|_C \leq \frac{T^{\alpha-1}}{\lambda\Gamma(\alpha)}\left(1 - e^{-\lambda T}\right)\|\gamma\|_C,$$

(iii) $\mathfrak{F}_1$ is equicontinuous on B_r. These imply that $\mathfrak{F}_1$ is compact on B_r. Thus all the assumptions of Krasnoselskii's theorem are satisfied. In consequence, It follows from the conclusion of Krasnoselskii's theorem that the problem (1) has at least one solution on $[0,T]$. ∎

4 Examples

Example 1. Consider the following problem

$$\begin{cases} \left({}^C D^{\frac{3}{2}} + 2\,{}^C D^{\frac{1}{2}}\right) u(t) = \frac{1}{\sqrt{t^2+49}}\left(\frac{t\sin u(t)}{49} + e^{-t}\cos t\right), \quad 0 \leq t \leq 4, \\ 2u(1) + 3u(4) = -\int_0^2 u(s)\,ds, \\ {}^C D^{\frac{1}{2}} u(1) + 5\,{}^C D^{\frac{1}{2}} u(4) = -\int_0^2 u(s)\,ds, \end{cases} \tag{6}$$

where $f(t,u) = \frac{1}{\sqrt{t^2+49}}\left(\frac{t\sin u}{49} + e^{-t}\cos t\right)$, $T = 4,\ \alpha = \frac{3}{2},\ \nu_1 = 2,\ \nu_2 = 1,\ \mu_1 = 3,\ \mu_2 = 5,\ \gamma_1 = -1,\ \gamma_2 = 5,\ \eta = 1,\ \xi = 2,\ \zeta = 3,\ \lambda = 2.$

A simple calculations show that

$$a_{11}=\nu_1 e^{-\lambda\eta}+\mu_1 e^{-\lambda T}-\frac{\gamma_1}{\lambda}\left(1-e^{-\lambda\xi}\right)\cong 0.761,$$

$$a_{21}=\nu_2\frac{\lambda}{\Gamma(2-\alpha)}\int_0^{\eta}(\eta-s)^{1-\alpha}e^{-\lambda s}ds+\mu_2\frac{\lambda}{\Gamma(2-\alpha)}\int_0^{T}(T-s)^{1-\alpha}e^{-\lambda s}ds$$

$$+\gamma_2\int_{\zeta}^{T}e^{-\lambda t}dt\nu_2\frac{\lambda}{\Gamma(2-\alpha)}\int_0^{\eta}(\eta-s)^{1-\alpha}e^{-\lambda s}ds+\mu_2\frac{\lambda}{\Gamma(2-\alpha)}\int_0^{T}(T-s)^{1-\alpha}e^{-\lambda s}ds+\gamma_2\int_{\zeta}^{T}e^{-\lambda t}dt\cong 24.8,$$

$$a_{22}=\gamma_2(T-\zeta)\cong 5,\quad a_{12}=\nu_1+\mu_1-\gamma_1\xi=6,\ \Delta=-145$$

$$\varphi_1=\max\left(\frac{24.8-5}{145},\frac{24.8-5e^{-8}}{145}\right)\cong 0.17,$$

$$\varphi_2=\max\left(\frac{0.76-6}{-145},\frac{0.76-6e^{-8}}{145}\right)\cong 0.036,$$

$$R<2.083.$$

To apply Theorem 6 we need to show conditions (A_1)and (A_2) are satisfied. Indeed,

(A_1) $|f(t,u)-f(t,v)|=\left|\frac{1}{\sqrt{t^2+49}}\frac{t}{49}(\sin u-\sin v)\right|\leq\frac{1}{49}|u-v|,$

(A_2) $L_f R<\frac{1}{49}2.083<0.043<1.$

Therefore, according to Theorem 6 the BVP (6) has a unique solution on $[0,4]$.

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Bieberbach-de Branges and Fekete-Szegö inequalities for certain families of $q-$ convex and $q-$ close-to-convex functions

Om P. Ahuja[1], Asena Çetinkaya[2,*], Yaşar Polatoğlu[3]

[1] Department of Mathematical Sciences, Kent State University, Ohio, 44021, U.S.A
e-mail: oahuja@kent.edu
[2,3] Department of Mathematics and Computer Sciences, İstanbul Kültür University, İstanbul, Turkey
Corresponding Author* e-mail: asnfigen@hotmail.com
e-mail: [3]y.polatoglu@iku.edu.tr

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Abstract

In this paper, we investigate $q-$ analogues of Bieberbach- de Branges theorems and Fekete-Szegö inequalities for certain families of $q-$ convex and $q-$close-to-convex functions.

Key words and phrases: $q-$ close-to-convex function, $q-$ convex function, coefficient inequality and Fekete-Szegö inequality.

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1 Introduction

Let $\mathcal{A}$ be the class of functions f, defined by $f(z) = z + a_2z^2 + a_3z^3 + \cdots$, that are analytic in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ and Ω be the family of functions w which are analytic in $\mathbb{D}$ and satisfy the conditions $w(0) = 0$, $|w(z)| < 1$ for all $z \in \mathbb{D}$. If f_1 and f_2 are analytic functions in $\mathbb{D}$, then we say that f_1 is subordinate to f_2, written as $f_1 \prec f_2$ if there exists a Schwarz function $w \in \Omega$ such that $f_1(z) = f_2(w(z)), z \in \mathbb{D}$. We also note that if f_2 univalent in $\mathbb{D}$, then $f_1 \prec f_2$ if and only if $f_1(0) = f_2(0)$, $f_1(\mathbb{D}) \subset f_2(\mathbb{D})$ implies $f_1(\mathbb{D}_r) \subset f_2(\mathbb{D}_r)$, where $\mathbb{D}_r = \{z : |z| < r, 0 < r < 1\}$ (see [7]). Let $f_1(z) = z + \sum_{n=2}^{\infty} a_nz^n$ and $f_2(z) = z + \sum_{n=2}^{\infty} b_nz^n$ be elements in $\mathcal{A}$. Then the convolution of these functions is defined by

$$f_1(z) * f_2(z) = z + \sum_{n=2}^{\infty} a_nb_nz^n. \tag{1.1}$$

Denote by $\mathcal{P}$ the family of functions p of the form $p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots$, analytic in $\mathbb{D}$ such that p is in $\mathcal{P}$ if and only if

$$p(z) \prec \frac{1+z}{1-z} \Leftrightarrow p(z) = \frac{1+w(z)}{1-w(z)}, \quad z \in \mathbb{D} \tag{1.2}$$

for some function $w \in \Omega$ and for all $z \in \mathbb{D}$. It is well known that a function f in $\mathcal{A}$ is called starlike ($f \in S^*$), convex ($f \in \mathcal{C}$) and close-to-convex ($f \in \mathcal{CC}$) if there exists a function p in $\mathcal{P}$ such that p may be expressed, respectively, by the following relations:

$$p(z) = z\frac{f'(z)}{f(z)}, p(z) = 1 + z\frac{f''(z)}{f'(z)}, p(z) = \frac{f'(z)}{g'(z)}$$

for all $z \in \mathbb{D}$. For definitions and properties of these classes, one may refer to [1] and [7].

The problem of maximizing the absolute value of $a_3 - \mu a_2^2$ is called Fekete-Szegö problem [3] when μ is a real number. Later, Pfluger [14] considered the problem when μ is complex. Many authors have considered the Fekete-Szegö problem for various subclasses of $\mathcal{A}$ (see [5, 12, 16]).

In 1909 and 1910, Jackson [8, 9, 10] initiated a study of $q-$ difference operator D_q defined by

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} \quad \text{for} \quad z \in B \setminus \{0\}, \tag{1.3}$$

where B is a subset of complex plane $\mathbb{C}$, called $q-$ geometric set if $qz \in B$, whenever $z \in B$. Note that if a subset B of $\mathbb{C}$ is $q-$ geometric, then it contains all geometric sequences $\{zq^n\}_0^\infty$, $zq \in B$. Obviously, $D_q f(z) \to f'(z)$ as $q \to 1^-$. The $q-$ difference operator (1.3) is also called Jackson $q-$ difference operator. Note that such an operator plays an important role in the theory of hypergeometric series and quantum physics (see for instance [2, 4, 6, 11]).

Also, note that $D_q f(0) \to f'(0)$ as $q \to 1^-$ and $D_q^2 f(z) = D_q(D_q f(z))$. In fact, $q-$ calculus is ordinary classical calculus without the notion of limits. Recent interest in $q-$ calculus is because of its applications in various branches of mathematics and physics. For definition and properties of $q-$ difference operator and $q-$ calculus, one may refer to [2, 4, 6, 11]. In particular, we recall the following definitions and properties:

Since

$$D_q z^n = \frac{1-q^n}{1-q} z^{n-1} = [n]_q z^{n-1},$$

where $[n]_q = \frac{1-q^n}{1-q}$, it follows that for any $f \in \mathcal{A}$ we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \tag{1.4}$$

$$D_q(zD_q f(z)) = 1 + \sum_{n=2}^{\infty} [n]_q^2 a_n z^{n-1}. \tag{1.5}$$

The $q-$ analogue of the factorial function is defined for positive integer n by

$$[n]_q! = \prod_{k=1}^{n} [k]_q,$$

where $q \in (0,1)$. Clearly, as $q \to 1^-$, $[n]_q \to n$ and $[n]_q! \to n!$. For notations, one may refer to [6].

We introduce a new generalized class of $q-$ convex functions as follows:

Definition 1.1. *A function $f \in \mathcal{A}$ is said to be in the $\mathcal{C}_q$ such that*

$$\mathcal{C}_q = \left\{ f \in \mathcal{A} : Re\left(\frac{D_q(zD_q f(z))}{D_q f(z)} \right) > 0, q \in (0,1), z \in \mathbb{D} \right\}.$$

When $q \to 1^-$ in the limiting sense, then the class $\mathcal{C}_q$ reduces to the traditional class $\mathcal{C}$.

We also introduce a new generalized class of $q-$ close-to-convex functions associated with $q-$ convex functions in $\mathbb{D}$.

Definition 1.2. *A function $f \in \mathcal{A}$ is said to be in the $\mathcal{CC}_q$ if there exists a function g in class $\mathcal{C}_q$ such that*

$$Re\left(\frac{D_q f(z)}{D_q g(z)}\right) > 0, \tag{1.6}$$

where $q \in (0,1), z \in \mathbb{D}$. As $D_q f(z) \to f'(z)$ and $D_q g(z) \to g'(z)$, when $q \to 1^-$ in the limiting sense, then the inequality (1.6) reduces to the traditional class $\mathcal{CC}$.

Definition 1.1 and Definition 1.2 are equivalent to the following classes

$$\mathcal{C}_q = \left\{ f \in \mathcal{A} : \left(\frac{D_q(zD_q f(z))}{D_q f(z)}\right) \prec \frac{1+z}{1-z}, q \in (0,1), z \in \mathbb{D} \right\},$$

$$\mathcal{CC}_q = \left\{ f \in \mathcal{A} : \left(\frac{D_q f(z)}{D_q g(z)}\right) \prec \frac{1+z}{1-z}, \quad g(z) \in \mathcal{C}_q \right\}.$$

In this paper, we investigate the Bieberbach-de Branges inequalities for the class $\mathcal{C}_q$ and $\mathcal{CC}_q$. We also obtain the Fekete-Szegö inequalities for both these classes.

2 The Bieberbach-De Branges Theorems

In order to find the Bieberbach-de Branges theorem for the class $\mathcal{C}_q$, we need the following result:

Lemma 2.1. [7]*(Caratheodory's lemma) If $p \in \mathcal{P}$ and $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, then $|c_n| \leq 2$ for $n \geq 1$. This inequality is sharp for each n.*

Theorem 2.2. *If $f \in \mathcal{C}_q$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then*

$$|a_n| \leq \frac{1}{[n]_q!} \prod_{k=0}^{n-2} \left([k]_q + \frac{2}{q}\right). \tag{2.1}$$

This result is sharp for all $n \geq 2$.

Proof. In view of Definition 1.1 and subordination principle, we can write

$$\frac{D_q(zD_q f(z))}{D_q f(z)} = p(z)$$

where $p \in \mathcal{P}$, $p(0) = 1$ and $Rep(z) > 0$.

In view of (1.4), (1.5) and $p(z) = 1 + c_1 z + c_2 z^2 + ...$, we get

$$\left(1 + \sum_{n=2}^{\infty} [n]_q^2 a_n z^{n-1}\right) = \left(1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}\right)\left(1 + \sum_{n=1}^{\infty} c_n z^n\right).$$

This equation yields,

$$1 + [2]_q^2 a_2 z + [3]_q^2 a_3 z^2 + ... = 1 + ([2]_q a_2 + c_1)z + ([3]_q a_3 + [2]_q a_2 c_1 + c_2)z^2 + ... \tag{2.2}$$

Comparing the coefficients of z^n on both sides, we obtain

$$[n+1]_q([n+1]_q - 1)a_{n+1} = [n]_q a_n c_1 + [n-1]_q a_{n-1} c_2 + ... + [2]_q a_2 c_{n-1} + c_n$$

or equivalently

$$q[n]_q[n+1]_q a_{n+1} = [n]_q a_n c_1 + [n-1]_q a_{n-1} c_2 + ... + [2]_q a_2 c_{n-1} + c_n.$$

In view of Lemma 2.1, we get

$$q[n]_q[n+1]_q |a_{n+1}| \leq 2\Bigg[[n]_q|a_n| + [n-1]_q|a_{n-1}| + ... + [2]_q|a_2| + 1\Bigg].$$

This shows that we have

$$q[n]_q[n+1]_q|a_{n+1}| \leq 2\Bigg(\sum_{k=1}^{n}[k]_q|a_k|\Bigg), |a_1| = 1.$$

This inequality is equivalent to

$$q[n-1]_q[n]_q|a_n| \leq 2\Bigg(\sum_{k=1}^{n-1}[k]_q|a_k|\Bigg), |a_1| = 1 \tag{2.3}$$

or

$$|a_n| \leq \frac{2}{q[n-1]_q[n]_q}\Bigg(\sum_{k=1}^{n-1}[k]_q|a_k|\Bigg), |a_1| = 1. \tag{2.4}$$

In order to prove (2.1), we will use the process of iteration. We first plug-in $n = 2$ and use our assumption $|a_1| = 1$ in (2.4). On simplification, we get

$$|a_2| \leq \frac{1}{[2]_q!}\frac{2}{q}. \tag{2.5}$$

This is equivalent to

$$|a_2| \leq \frac{1}{[2]_q!}\prod_{k=0}^{2-2}\Bigg([k]_q + \frac{2}{q}\Bigg).$$

Next by substituting $n = 3$ and using the output (2.5) in (2.4), we obtain

$$|a_3| \leq \frac{1}{[3]_q!}\frac{2}{q}(1 + \frac{2}{q}).$$

This is equivalent to

$$|a_3| \leq \frac{1}{[3]_q!}\prod_{k=0}^{3-2}\Bigg([k]_q + \frac{2}{q}\Bigg). \tag{2.6}$$

By repeating the above process by letting $n = 4$ and in view of (2.4), it is a routine process to prove

$$|a_4| \leq \frac{1}{[4]_q!}\frac{2}{q}(1 + \frac{2}{q})(1 + q + \frac{2}{q}),$$

that is,

$$|a_4| \leq \frac{1}{[4]_q!}\prod_{k=0}^{4-2}\left([k]_q + \frac{2}{q}\right). \tag{2.7}$$

By continuing the process of iterations, we get (2.1). The result in (2.1) is sharp for the functions $f(z) = \int (1-z)^{-\frac{2}{q}\frac{1-q}{logq^{-1}}} d_q z.$ □

Remark 2.3. *If we take limit for $q \to 1^-$ in inequality (2.1), we get*

$$|a_n| \leq 1$$

for all $n \geq 2$. This is the well known coefficient inequality for convex functions.

Theorem 2.2 helps us to establish the Bieberbach-de Branges theorem for the class $\mathcal{CC}_q$ in the next result.

Theorem 2.4. *If $f \in \mathcal{CC}_q$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then*

$$|a_n| \leq \frac{1}{[n]_q!}\prod_{k=0}^{n-2} A(k,q) + \frac{2}{[n]_q}\sum_{r=1}^{n-1}\left([n-r]_q \frac{1}{[n-r]_q!}\prod_{k=-1}^{n-r-2} A(k,q)\right), \tag{2.8}$$

where $A(k,q) = \left([k]_q + \frac{2}{q}\right)$. Extremal function is given by

$$f(z) = \int \frac{1+z}{1-z}(1-z)^{-\frac{2}{q}\frac{1-q}{logq^{-1}}} d_q z.$$

Proof. In view of Definition 1.2 and subordination principle, we can write

$$\frac{D_q f(z)}{D_q g(z)} = p(z) \tag{2.9}$$

for some $g \in \mathcal{C}_q$, where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $z \in \mathbb{D}$. Since $p(0) = 1$ and $Rep(z) > 0$, it shows that $p \in \mathcal{P}$, where $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. In view of (1.4), we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty}[n]_q a_n z^{n-1} \quad \text{and} \quad D_q g(z) = 1 + \sum_{n=2}^{\infty}[n]_q b_n z^{n-1}.$$

Therefore, (2.9) is equivalent to

$$\left(1 + \sum_{n=2}^{\infty}[n]_q a_n z^{n-1}\right) = \left(1 + \sum_{n=2}^{\infty}[n]_q b_n z^{n-1}\right)\left(1 + \sum_{n=1}^{\infty} c_n z^n\right).$$

This equation yields,

$$1 + [2]_q a_2 z + [3]_q a_3 z^2 + ... = 1 + ([2]_q b_2 + c_1)z + ([3]_q b_3 + [2]_q b_2 c_1 + c_2)z^2 + ... \tag{2.10}$$

Comparing the coefficients of z^{n-1} on both sides, we obtain

$$[n]_q a_n = [n]_q b_n + [n-1]_q b_{n-1} c_1 + [n-2]_q b_{n-2} c_2 + ... + [2]_q b_2 c_{n-2} + c_{n-1}.$$

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Using Lemma 2.1, we get

$$[n]_q|a_n| \leq [n]_q|b_n| + 2\bigg[[n-1]_q|b_{n-1}| + ... + [2]_q|b_2| + 1\bigg]$$

or equivalently,

$$[n]_q|a_n| \leq [n]_q|b_n| + 2\bigg(\sum_{r=1}^{n-1}[n-r]_q|b_{n-r}|\bigg), |b_1| = 1. \tag{2.11}$$

Using Theorem 2.2, (2.11) yields,

$$|a_n| \leq \frac{1}{[n]_q!}\prod_{k=0}^{n-2}\left([k]_q + \frac{2}{q}\right) + \frac{2}{[n]_q}\sum_{r=1}^{n-1}\left([n-r]_q\frac{1}{[n-r]_q!}\prod_{k=-1}^{n-r-2}\left([k]_q + \frac{2}{q}\right)\right).$$

This inequality gives (2.8), where $A(k,q) = \left([k]_q + \frac{2}{q}\right)$. Thus the proof is completed. □

Remark 2.5. *If we take limit for $q \to 1^-$ in inequality (2.8), we get*

$$|a_n| \leq n$$

for all $n \geq 2$. This is the well known coefficient inequality for close-to-convex functions.

3 Fekete-Szegö Inequalities

We now investigate Fekete-Szegö inequalities for the class $\mathcal{C}_q$ and $\mathcal{CC}_q$. For our main theorems we need the following result:

Lemma 3.1. ([*15*]) *If $p \in \mathcal{P}$ with $p(z) = 1 + c_1z + c_2z^2 + ...$, then*

$$\left|c_2 - \frac{c_1^2}{2}\right| \leq 2 - \frac{|c_1|^2}{2}.$$

Theorem 3.2. *If f belongs to the class $\mathcal{C}_q$, then*

$$|a_2| \leq \frac{2}{[2]_q q}, \tag{3.1}$$

$$|a_3| \leq \frac{2}{[3]_q[2]_q q}\left(1 + \frac{2}{q}\right), \tag{3.2}$$

$$\left|a_3 - \frac{[2]_q}{[3]_q}a_2^2\right| \leq \frac{2}{[3]_q[2]_q q}. \tag{3.3}$$

These results are sharp.

Proof. Using equation (2.2), we obtain

$$a_2 = \frac{c_1}{[2]_q([2]_q - 1)} = \frac{c_1}{[2]_q[1]_q q} \tag{3.4}$$

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and

$$a_3 = \frac{1}{[3]_q([3]_q - 1)}\left(c_2 + \frac{c_1^2}{[2]_q - 1}\right) = \frac{1}{[3]_q[2]_q q}\left(c_2 + \frac{c_1^2}{q}\right). \tag{3.5}$$

Taking into account Lemma 2.1 and Lemma 3.1, we obtain

$$|a_2| = \left|\frac{c_1}{[2]_q q}\right| \leq \frac{2}{[2]_q q}$$

and

$$|a_3| = \left|\frac{1}{[3]_q[2]_q q}\left(c_2 + \frac{c_1^2}{q}\right)\right| \leq \frac{2}{[3]_q[2]_q q}\left(1 + \frac{2}{q}\right).$$

Furthermore, using (3.4) and (3.5), we obtain

$$\left|a_3 - \frac{[2]_q}{[3]_q}a_2^2\right| = \left|\frac{c_2}{[3]_q[2]_q q}\right| \leq \frac{2}{[3]_q[2]_q q}.$$

These results are sharp for the functions

$$\frac{D_q(zD_qf(z))}{D_qf(z)} = \frac{1+z}{1-z} \Rightarrow f(z) = \int (1-z)^{-\frac{2}{q}\frac{1-q}{logq^{-1}}} d_qz, \tag{3.6}$$

$$\frac{D_q(zD_qf(z))}{D_qf(z)} = \frac{1+z^2}{1-z^2} \Rightarrow f(z) = \int (1-z^2)^{-\frac{2}{[2]_q q}\frac{1-q}{logq^{-1}}} d_qz. \tag{3.7}$$

In fact, Theorem 3.2 gives a special case of Fekete-Szegö problem for real $\mu = [2]_q/[3]_q$ which obtain the naturally and simple estimate. Thus the proof is completed. □

Motivated by the above-mentioned special case of Fekete-Szegö problem, we now find the next estimate of $|a_3 - \mu a_2^2|$ for complex μ.

Theorem 3.3. *Let μ be a nonzero complex number and let $f \in \mathcal{C}_q$, then*

$$|a_3 - \mu a_2^2| \leq \frac{2}{[3]_q[2]_q q} max\left\{1, \left|1 + \frac{2}{q}\left(1 - \frac{[3]_q}{[2]_q}\mu\right)\right|\right\}. \tag{3.8}$$

This result is sharp.

Proof. Applying (3.4) and (3.5), we have

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{[3]_q[2]_q q}\left[c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2}\left(1 + \frac{2}{q}\right)\right] - \mu\frac{c_1^2}{([2]_q)^2 q^2} \\ &= \frac{1}{[3]_q[2]_q q}\left[c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2}\left(1 + \frac{2}{q}\left(1 - \frac{[3]_q[2]_q q}{([2]_q)^2 q}\mu\right)\right)\right] \end{aligned}$$

In view of Lemma 2.1 and Lemma 3.1, we get

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{1}{[3]_q[2]_q q}\left[2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2}\left(\left|1 + \frac{2}{q}\left(1 - \frac{[3]_q}{[2]_q}\mu\right)\right|\right)\right] \\ &= \frac{1}{[3]_q[2]_q q}\left[2 + \frac{|c_1|^2}{2}\left(\left|1 + \frac{2}{q}\left(1 - \frac{[3]_q}{[2]_q}\mu\right)\right| - 1\right)\right] \\ &\leq \frac{2}{[3]_q[2]_q q}\max\left\{1, \left|1 + \frac{2}{q}\left(1 - \frac{[3]_q}{[2]_q}\mu\right)\right|\right\}. \end{aligned}$$

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This result is sharp for the functions $f(z)=\int(1-z)^{-\frac{2}{q}\frac{1-q}{logq^{-1}}}d_qz$ and $f(z)=\int(1-z^2)^{-\frac{2}{[2]_qq}\frac{1-q}{logq^{-1}}}d_qz$. □

We next consider the case, when μ is real. Then we have:

Theorem 3.4. *If f belongs to the class $\mathcal{C}_q$, then for $\mu\in\mathbb{R}$, we have*

$$|a_3-\mu a_2^2|\leq\begin{cases}\frac{2}{[3]_q[2]_qq}\left(1+\frac{2}{q}\left(1-\frac{[3]_q}{[2]_q}\mu\right)\right), & \mu\leq\frac{[2]_q}{[3]_q}\\ \frac{2}{[3]_q[2]_qq}, & \frac{[2]_q}{[3]_q}\leq\mu\leq\frac{q(2+\frac{2}{q})[2]_q}{2[3]_q}\\ \frac{2}{[3]_q[2]_qq}\left(\frac{2}{q}\frac{[3]_q}{[2]_q}\mu-1-\frac{2}{q}\right), & \mu\geq\frac{q(2+\frac{2}{q})[2]_q}{2[3]_q}\end{cases}$$

These results are sharp.

Proof. First, let $\mu\leq\frac{[2]_q}{[3]_q}$. In this case (3.4), (3.5), Lemma 2.1 and Lemma 3.1 give

$$\begin{aligned}|a_3-\mu a_2^2|&\leq\frac{1}{[3]_q[2]_qq}\left(2-\frac{|c_1|^2}{2}+\frac{|c_1|^2}{2}\left(1+\frac{2}{q}-\frac{2}{q}\frac{[3]_q}{[2]_q}\mu\right)\right)\\&\leq\frac{2}{[3]_q[2]_qq}\left(1+\frac{2}{q}\left(1-\frac{[3]_q}{[2]_q}\mu\right)\right).\end{aligned}$$

Let, now $\frac{[2]_q}{[3]_q}\leq\mu\leq\frac{q(2+\frac{2}{q})[2]_q}{2[3]_q}$. Then using the above calculations, we have

$$|a_3-\mu a_2^2|\leq\frac{2}{[3]_q[2]_qq}.$$

Finally, if $\mu\geq\frac{q(2+\frac{2}{q})[2]_q}{2[3]_q}$, then

$$\begin{aligned}|a_3-\mu a_2^2|&\leq\frac{1}{[3]_q[2]_qq}\left(2-\frac{|c_1|^2}{2}+\frac{|c_1|^2}{2}\left(\frac{2}{q}\frac{[3]_q}{[2]_q}\mu-1-\frac{2}{q}\right)\right)\\&\leq\frac{1}{[3]_q[2]_qq}\left(2+\frac{|c_1|^2}{2}\left(\frac{2}{q}\frac{[3]_q}{[2]_q}\mu-2-\frac{2}{q}\right)\right)\\&\leq\frac{2}{[3]_q[2]_qq}\left(\frac{2}{q}\frac{[3]_q}{[2]_q}\mu-1-\frac{2}{q}\right).\end{aligned}$$

Equality is attained for the second case on choosing $c_1=0,c_2=2$ in (3.6) and for the first and third case on choosing $c_1=2,c_2=2,c_1=2i,c_2=-2$ in (3.7), respectively. Thus the proof is completed. □

Remark 3.5. *Taking $q\to 1^-$ in Theorem 3.4, we get Fekete-Szegö inequality for convex functions which was found by Keogh and Merkes [13].*

Theorem 3.6. *If f belongs to the class $\mathcal{CC}_q$, then*

$$|a_2|\leq\frac{2}{[2]_q}(1+\frac{1}{q}),\tag{3.9}$$

$$|a_3| \leq \frac{2}{[2]_q q}\left(1+\frac{2}{q}\right), \tag{3.10}$$

$$\left|a_3 - \frac{1}{[2]_q}a_2^2\right| \leq \frac{2}{[2]_q q}. \tag{3.11}$$

These results are sharp.

Proof. Using equation (2.10), we obtain

$$a_2 = b_2 + \frac{c_1}{[2]_q} \tag{3.12}$$

and

$$a_3 = b_3 + \frac{[2]_q}{[3]_q}b_2 c_1 + \frac{c_2}{[3]_q}. \tag{3.13}$$

Since $b_2, b_3 \in \mathcal{C}_q$, applying equations (3.4) and (3.5) in (3.12) and (3.13), respectively, we get

$$a_2 = \frac{c_1}{[2]_q q} + \frac{c_1}{[2]_q} \tag{3.14}$$

and

$$a_3 = \frac{1}{[2]_q q}\left(c_2 + \frac{c_1^2}{q}\right). \tag{3.15}$$

Taking into account Lemma 2.1 and Lemma 3.1, we obtain

$$|a_2| = \left|\frac{c_1}{[2]_q q} + \frac{c_1}{[2]_q}\right| \leq \frac{2}{[2]_q}(1+\frac{1}{q})$$

and

$$|a_3| = \left|\frac{1}{[2]_q q}\left(c_2 + \frac{c_1^2}{q}\right)\right| \leq \frac{2}{[2]_q q}(1+\frac{2}{q}).$$

Furthermore, using (3.14) and (3.15), we obtain

$$\left|a_3 - \frac{1}{[2]_q}a_2^2\right| = \left|\frac{c_2}{[2]_q q}\right| \leq \frac{2}{[2]_q q}.$$

These results are sharp for the functions

$$\frac{D_q f(z)}{D_q g(z)} = \frac{1+z}{1-z} \Rightarrow f(z) = \int \frac{1+z}{1-z}(1-z)^{-\frac{2}{q}\frac{1-q}{logq^{-1}}} d_q z, \tag{3.16}$$

$$\frac{D_q f(z)}{D_q g(z)} = \frac{1+z^2}{1-z^2} \Rightarrow f(z) = \int \frac{1+z^2}{1-z^2}(1-z^2)^{-\frac{2}{[2]_q q}\frac{1-q}{logq^{-1}}} d_q z. \tag{3.17}$$

This completes the proof. □

Theorem 3.7. *Let μ be a nonzero complex number and let $f \in \mathcal{CC}_q$, then*

$$|a_3 - \mu a_2^2| \leq \frac{2}{[2]_q q} max\left\{1, \left|1 + \frac{2}{q}\left(1 - [2]_q \mu\right)\right|\right\}. \tag{3.18}$$

This result is sharp.

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Proof. Applying (3.14) and (3.15), we have

$$\begin{aligned} a_3-\mu a_2^2 &= \frac{1}{[2]_q q}\left[c_2-\frac{c_1^2}{2}+\frac{c_1^2}{2}\left(1+\frac{2}{q}\right)\right]-\mu\left(\frac{c_1}{[2]_q q}+\frac{c_1}{[2]_q}\right)^2 \\ &= \frac{1}{[2]_q q}\left[c_2-\frac{c_1^2}{2}+\frac{c_1^2}{2}\left(1+\frac{2}{q}\left(1-\frac{[2]_q q}{q}\mu\right)\right)\right] \end{aligned}$$

In view of Lemma 2.1 and Lemma 3.1, we obtain

$$\begin{aligned} |a_3-\mu a_2^2| &\le \frac{1}{[2]_q q}\left[2-\frac{|c_1|^2}{2}+\frac{|c_1|^2}{2}\left(\left|1+\frac{2}{q}\left(1-[2]_q\mu\right)\right|\right)\right] \\ &= \frac{1}{[2]_q q}\left[2+\frac{|c_1|^2}{2}\left(\left|1+\frac{2}{q}\left(1-[2]_q\mu\right)\right|-1\right)\right] \\ &\le \frac{2}{[2]_q q}\max\left\{1,\left|1+\frac{2}{q}\left(1-[2]_q\mu\right)\right|\right\}. \end{aligned}$$

This result is sharp for the functions given in (3.16) and (3.17). Thus the proof is completed. □

Remark 3.8. *Taking $q\to 1^-$ in Theorem 3.3 and Theorem 3.7, we obtain*

$$|a_3-\mu a_2^2| \le \frac{1}{3}max\{1,|1+2(1-\frac{3}{2}\mu)|\},$$

$$|a_3-\mu a_2^2| \le max\{1,|1+2(1-2\mu)|\}.$$

These results are sharp.

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Bilinear θ-Type Calderón-Zygmund Operators on Non-homogeneous Generalized Morrey Spaces

Guanghui Lu, Shuangping Tao*

(College of Mathematics and Statistics, Northwest Normal University, Lanzhou, Gansu, 730070, P.R. China)

Abstract: Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space which satisfies the geometrically doubling and the upper doubling conditions in the sense of Hytönen. In this paper, the authors prove that the bilinear θ-type Calderón-Zygmund operator and its corresponding commutator are bounded on the generalized Morrey space $\mathcal{L}^{p,\phi}(\mu)$ for $1 < p < \infty$. As an application, the authors also obtain that the bilinear θ-type Calderón-Zygmund operator and its commutator are bounded on the Morrey space $M_p^q(\mu)$.

Keywords: Non-homogeneous metric measure space, commutator, bilinear θ-type Calderón-Zygmund operator, RBMO(μ), generalized Morrey space.

2010 MR Subject Classification: 42B20, 42B35, 30L99.

1 Introduction

As we all know, in 2010, Hytönen [7] firstly introduced the non-homogeneous metric measure spaces including the upper doubling and the geometrically doubling conditions (see Definitions 1.1 and 1.2, respectively), to unify the homogeneous type spaces (see [1-3]) and the non-doubling measure spaces [9, 16, 18-22, 24, 27]. Since then, some properties for various of the singular integral operators and function spaces on non-homogeneous metric measure spaces have been obtained by researchers, for example, see [4-6, 8, 10-13, 17, 23, 25, 28-29] and their references.

In 1985, Yabuta [26] gave out the definition of the θ-type Calderón-Zygmund operator. Later, some researchers paid much attention to study the properties of the operator on different function spaces, for example, Ri and Zhang [16, 17] obtained the boundedness of the θ-type Calderón-Zygmund on Hardy spaces with non-doubling measures and non-homogeneous metric measure spaces, respectively. Besides, in 2009, Maldonado and Naibo [14] developed a theory of the bilinear Calderón-Zygmund operator of type $\omega(t)$ and generalized the consequences of Yabuta [26]. About the further development of the bilinear Calderón-Zygmund operator of type $\omega(t)$, we can see [28-29].

*Corresponding author and Email: taosp@nwnu.edu.cn (by S. Tao); lghwmm1989@126.com (by G. Lu).

In this paper, let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space in the sense of Hytönen [7]. The definition of the generalized Morrey space on $(\mathcal{X}, d, \mu)$ was given out by Lu and Tao in [11], furthermore, we also obtained the boundedness of some classical singular integral operators on generalized Morrey space. In [25], Xie et al. got the boundedness of the commutators generated by the bilinear θ-type Calderón-Zygmund operator and the spaces RBMO(μ). Inspired by this, we will study the boundedness of the bilinear θ-type Calderón-Zygmund operator and its commutator on generalized Morrey space. Moreover, as an application, we also study the boundedness of the bilinear θ-type Calderón-Zygmund operator and its commutator on Morrey space.

Before stating the main results of this article, we first recall some necessary notions. In [7], Hytönen originally introduced the following definition of the upper doubling metric measure space.

Definition 1.1. A metric measure space $(\mathcal{X}, d, \mu)$ is said to be *upper doubling* if μ is a Borel measure on $\mathcal{X}$ and there exist a dominating function $\lambda : \mathcal{X} \times (0, \infty) \to (0, \infty)$ and a positive constant C_λ such that, for each $x \in \mathcal{X}, r \to \lambda(x, r)$ is non-decreasing and, for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \leq \lambda(x, r) \leq C_\lambda \lambda(x, \frac{r}{2}). \tag{1.1}$$

Hytönen et al. [10] have showed that, there is another dominating function $\tilde{\lambda}$ such that $\tilde{\lambda} \leq \lambda$, $C_{\tilde{\lambda}} \leq C_\lambda$ and

$$\tilde{\lambda}(x, r) \leq C_{\tilde{\lambda}} \tilde{\lambda}(y, r), \tag{1.2}$$

where $x, y \in \mathcal{X}$ and $d(x, y) \leq r$. If there is no special instruction in this article, we always assume λ that in (1.1) satisfies (1.2).

Coifaman and Weiss in [2] firstly introduced the notion of the geometrically doubling as follows, which is well known in analysis on metric spaces.

Definition 1.2. A metric space $(\mathcal{X}, d)$ is said to be *geometrically doubling*, if there exists some $N_0 \in \mathbb{N}$ such that, for any ball $B(x, r) \subset \mathcal{X}$, there exists a finite ball covering $\{B(x_i, \frac{r}{2})\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most N_0.

Assume $(\mathcal{X}, d)$ is a metric space. In [7], Hytönen proved the following statements are mutually equivalent:

(1) $(\mathcal{X}, d)$ is geometrically doubling.

(2) For any $\epsilon \in (0, 1)$ and any ball $B(x, r) \subset \mathcal{X}$, there is a finite ball covering $\{B(x_i, \epsilon r)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most $N_0 \epsilon^{-n}$, where $n := \log_2 N_0$.

(3) For any $\epsilon \in (0, 1)$, any ball $B(x, r) \subset \mathcal{X}$ contains at most $N_0 \epsilon^{-n}$ centers of disjoint balls $\{B(x_i, \epsilon r)\}_i$.

(4) There is $M \in \mathbb{N}$ such that any ball $B(x, r) \subset \mathcal{X}$ contains at most M centers $\{x_i\}_i$ of disjoint balls $\{B(x_i, \frac{r}{4})\}_{i=1}^M$.

Now we recall the definition of the coefficient $K_{B,S}$ given in [7], which is analogous to the number $K_{Q,R}$ introduced by Tolsa in [20, 21], i.e., for any two balls $B \subset S$ in $\mathcal{X}$, set

$$K_{B,S} := 1 + \int_{2S \setminus B} \frac{1}{\lambda(c_B, d(x, c_B))} \mathrm{d}\mu(x), \tag{1.3}$$

where c_B is the center of the ball B.

Though the measure doubling condition is not assumed uniformly for all balls on $(\mathcal{X}, d, \mu)$, it was showed in [7] that there are many balls satisfying the property of the (α, η)-doubling, i.e., a ball $B \subset \mathcal{X}$ is said to belong to (α, η)-doubling if $\mu(\alpha B) \leq \eta\mu(B)$, for α, $\eta > 1$. In the latter of this paper, unless α and η_α are specified, otherwise, by an (α, η_α)-doubling ball we mean a $(6, \beta_6)$-doubling ball with a fixed number $\eta_6 > \max\{C_\lambda^{3\log_2 6}, 6^n\}$, where $n := \log_2 N_0$ is viewed as a geometric dimension of the space. In addition, the smallest $(6, \eta_6)$-doubling ball of the from $6^j B$ with $j \in \mathbb{N}$ is denoted by $\tilde{B}^6$, and $\tilde{B}^6$ is simply denoted by $\tilde{B}$.

Now we need to recall the following definition of $\mathrm{RBMO}(\mu)$ from [7].

Definition 1.3. Let $\rho \in (1, \infty)$. A function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to be in the *space* $\mathrm{RBMO}(\mu)$ if there exist a positive constant and, for any ball $B \subset \mathcal{X}$, a number f_B such that

$$\frac{1}{\mu(\rho B)} \int_B |f(x) - f_B| \mathrm{d}\mu(x) \leq C \tag{1.4}$$

and, for any two balls B and S such that $B \subset S$

$$|f_B - f_S| \leq CK_{B,S}. \tag{1.5}$$

The infimum of the positive constants C satisfying both (1.4) and (1.5) is defined to be the $\mathrm{RBMO}(\mu)$ norm of f and denoted by $\|f\|_{\mathrm{RBMO}(\mu)}$.

The following notion of the bilinear θ-type Calderón-Zygmund operator is given in [25].

Definition 1.4. Let θ be a non-negative and non-decreasing function on $(0, \infty)$ satisfying

$$\int_0^1 \frac{\theta(t)}{t} \mathrm{d}t < \infty.$$

A kernel $K(\cdot, \cdot, \cdot) \in L^1_{\mathrm{loc}}(\mathcal{X}^3 \setminus \{(x, y_1, y_2) : x = y_1 = y_2\})$ is called the bilinear θ-type Calderón-Zygmund kernel if it satisfies the following conditions:

(1) for all $(x, y_1, y_2) \in \mathcal{X}^3$ with $x \neq y_i$ for $i = 1, 2$,

$$|K(x, y_1, y_2)| \leq C \left[\sum_{i=1}^{2} \lambda(x, d(x, y_i)) \right]^{-2}; \tag{1.6}$$

(2) there exists a positive constant C such that, for all $x, x', y_1, y_2 \in \mathcal{X}$ with $Cd(x, x') \leq \max_{1 \leq i \leq 2} d(x, y_i)$,

$$|K(x, y_1, y_2) - K(x', y_1, y_2)| \leq \theta\left(\frac{d(x, x')}{\sum_{i=1}^{2} d(x, y_i)} \right) \left[\sum_{i=1}^{2} \lambda(x, d(x, y_i)) \right]^{-2}. \tag{1.7}$$

Let $L_b^\infty(\mu)$ be the space of all $L^\infty(\mu)$ functions with bounded support. A bilinear operator T_θ is called a bilinear θ-type Calderón-Zygmund operator with kernel K satisfying (1.6) and (1.7) if, for all $f_1, f_2 \in L_b^\infty(\mu)$ and $x \notin \bigcap_{i=1}^{2} \mathrm{supp} f_i$,

$$T_\theta(f_1, f_2)(x) := \int_{\mathcal{X}} \int_{\mathcal{X}} K(x, y_1, y_2) f_1(y_1) f_2(y_2) \mathrm{d}\mu(y_1) \mathrm{d}\mu(y_2). \tag{1.8}$$

The commutator closely related to the bilinear θ-type Calderón-Zygmund operator T_θ and $b_1, b_2 \in \mathrm{RBMO}(\mu)$ is defined by

$$[b_1, b_2, T_\theta](f_1, f_2)(x) := b_1(x)b_2(x)T_\theta(f_1, f_2)(x) - b_1(x)T_\theta(f_1, b_2f_2)(x)$$
$$-b_2(x)T_\theta(b_1f_1, f_2)(x) + T_\theta(b_1f_1, b_2f_2)(x). \tag{1.9}$$

Also, $[b_1, T_\theta]$ and $[b_2, T_\theta]$ are defined as follows, respectively:

$$[b_1, T_\theta](f_1, f_2)(x) = b_1(x)T_\theta(f_1, f_2)(x) - T_\theta(b_1f_1, f_2)(x), \tag{1.10}$$

$$[b_2, T_\theta](f_1, f_2)(x) = b_2(x)T_\theta(f_1, f_2)(x) - T_\theta(f_1, b_2f_2)(x). \tag{1.11}$$

Now we recall the definition of the generalized Morrey space $\mathcal{L}^{p,\phi}(\mu)$ from [11].

Definition 1.5. Let $\kappa > 1$ and $1 \le p < \infty$. Suppose that $\phi : (0, \infty) \to (0, \infty)$ is an increasing function. Then the generalized Morrey space $\mathcal{L}^{p,\phi}(\mu)$ is defined by

$$\mathcal{L}^{p,\phi}(\mu) := \{f \in L^p_{\mathrm{loc}}(\mu) : \|f\|_{L^{p,\phi}(\mu)} < \infty\},$$

where

$$\|f\|_{\mathcal{L}^{p,\phi}(\mu)} := \sup_B \left(\frac{1}{\phi(\mu(\kappa B))} \int_B |f(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}. \tag{1.12}$$

From [11, Remark 1.7], it follows that the generalized Morrey space $\mathcal{L}^{p,\phi}(\mu)$ is independent of the choice of $\kappa > 1$.

The following definition of the ϵ-weak reverse doubling condition is from [5].

Definition 1.6. Let $\epsilon \in (0, \infty)$. A dominating function λ is said to satisfy the ϵ-weak reverse doubling condition if, for all $s \in (0, 2\mathrm{diam}(\mathcal{X}))$ and $a \in (1, 2\mathrm{diam}(\mathcal{X})/s)$, there exists a number $C(a) \in [1, \infty)$, depending only a and $\mathcal{X}$, such that,

$$\lambda(x, as) \ge C(a)\lambda(x, s), \ x \in \mathcal{X}, \tag{1.13}$$

and, moreover,

$$\sum_{k=1}^{\infty} \frac{1}{[C(a^k)]^\epsilon} < \infty. \tag{1.14}$$

Now we can state the main theorems of this article as follows.

Theorem 1.7. *Let $1 < p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, K satisfy (1.6) and (1.7), λ satisfy the ϵ-weak reverse doubling condition, and let $\phi : (0, \infty) \to (0, \infty)$ be an increasing function. Suppose that T_θ is a bilinear Calderón-Zygmund operator and is bounded from $L^1(\mu) \times L^1(\mu)$ to $L^{\frac{1}{2},\infty}(\mu)$, the mapping $t \mapsto \frac{\phi(t)}{t}$ is almost decreasing and there is a constant $C > 0$ such that*

$$\frac{\phi(t)}{t} \le C\frac{\phi(s)}{s} \tag{1.15}$$

for $s \geq t$, in addition, ϕ also satisfies the following condition

$$\int_r^\infty \frac{\phi(t)}{t}\frac{\mathrm{d}t}{t} \leq C\frac{\phi(r)}{r}, \ \textit{for all } r > 0.$$

Then there exists a positive constant C, such that, for all $f_i \in \mathcal{L}^{p_i,\phi}(\mu)$ with $i = 1, 2$,

$$\|T_\theta(f_1, f_2)\|_{\mathcal{L}^{p,\phi}(\mu)} \leq C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}.$$

Theorem 1.8. *Under the same assumption of Theorem 1.7. Suppose that $b_1, b_2 \in \mathrm{RBMO}(\mu)$, and $[b_1, b_2, T_\theta](f_1, f_2)$ is as in (1.9). Then there is a positive constant C, such that, for all $f_i \in \mathcal{L}^{p_i,\phi}(\mu)$ with $i = 1, 2$,*

$$\|[b_1, b_2, T_\theta](f_1, f_2)\|_{\mathcal{L}^{p,\phi}(\mu)} \leq C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}.$$

In particular, if we take $\phi(t) = t^{1-\frac{p}{q}}$ with $1 \leq p \leq q < \infty$ and $t > 0$ in Definition 1.5, the generalized Morrey space is just Morrey space which was established by Cao and Zhou in [4], that is, for $k > 1$ and $1 \leq p \leq q < \infty$, the Morrey space $M_p^q(\mu)$ is defined as

$$M_p^q(\mu) := \{f \in L^p_{\mathrm{loc}}(\mu) : \|f\|_{M_p^q(\mu)} < \infty\}$$

with the norm

$$\|f\|_{M_p^q(\mu)} := \sup_B [\mu(kB)]^{\frac{1}{q}-\frac{1}{p}}\left(\int_B |f(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}. \tag{1.16}$$

Furthermore, based on the results of Theorems 1.7-1.8, it is not hard to find that the bilinear θ-type Calderón-Zygmund operator also holds on the Morrey space $M_p^q(\mu)$.

Theorem 1.9. *Assume that T_θ is a bilinear θ-type Calderón-Zygmund operator, and K satisfies (1.6) and (1.7). Suppose that T_θ is a bounded operator from $L^1(\mu) \times L^1(\mu)$ to $L^{\frac{1}{2},\infty}(\mu)$, then there exists a positive constant C, such that, for all $f_i \in M_{p_i}^{q_i}(\mu)$ with $i = 1, 2$,*

$$\|T_\theta(f_1, f_2)\|_{M_p^q(\mu)} \leq C\|f_1\|_{M_{p_1}^{q_1}(\mu)}\|f_2\|_{M_{p_2}^{q_2}(\mu)},$$

where $1 < p_i \leq q_i$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$.

Theorem 1.10. *Let $b_1, b_2 \in \mathrm{RBMO}(\mu)$, K satisfy (1.6) and (1.7). Assume λ satisfy the ϵ-weak reverse doubling condition, $f_1 \in M_{p_1}^{q_1}(\mu)$ and $f_2 \in M_{p_2}^{q_2}(\mu)$. If T_θ is a bounded operator from $L^1(\mu) \times L^1(\mu)$ to $L^{\frac{1}{2},\infty}(\mu)$, then there is a constant $C > 0$ such that*

$$\|[b_1, b_2, T_\theta](f_1, f_2)\|_{M_p^q(\mu)} \leq C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M_{p_1}^{q_1}(\mu)}\|f_2\|_{M_{p_2}^{q_2}(\mu)}.$$

where $1 < p_i \leq q_i < \infty$ for $i = 1, 2$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$.

Throughout the paper C will denote a positive constant whose value may change at each appearance. For a μ-measurable set E, χ_E denotes its characteristic function. For any $p \in [1, \infty]$, we denote by p' its conjugate index, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

2 Preliminaries

In this section, we need to recall some preliminary lemmas which will be used in the proofs of our main theorems. Firstly, we recall the following useful properties of $K_{B,S}$ from [7].

Lemma 2.1. (1) *For all balls $B \subset R \subset S$, it holds true that $K_{B,R} \leq K_{B,S}$.*

(2) *For any $\xi \in [1, \infty)$, there exists a positive constant C_ξ, depending on ξ, such that, for all balls $B \subset S$ with $r_S \leq \xi r_B$, $K_{B,S} \leq C_\xi$.*

(3) *For any $\varrho \in (1, \infty)$, there exists a positive constant C_ϱ, depending on ϱ, such that, for all balls B,$K_{B,\widetilde{B}^\varrho} \leq C_\varrho$.*

(4) *There is a positive constant c such that, for all balls $B \subset R \subset S$,$K_{B,S} \leq K_{B,R} + cK_{R,S}$. In particular, if B and R are concentric, then $c = 1$.*

(5) *There exists a positive constant $\widetilde{c}$ such that, for all balls $B \subset R \subset S$,$K_{B,R} \leq \widetilde{c}K_{B,S}$; moreover, if B and R are concentric, then $K_{R,S} \leq K_{B,S}$.*

Next, we need recall the boundedness of the bilinear θ-type Calderón-Zygmund T_θ and its commutator $[b_1, b_2, T_\theta](f_1, f_2)$ on Lebesgue space $L^p(\mu)$, see [28, 25], respectively.

Lemma 2.2. *Let K satisfy* (1.6) *and* (1.7), $1 < p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $f_1 \in L^{p_1}(\mu)$ *and $f_2 \in L^{p_2}(\mu)$. If T_θ is bounded from $L^1(\mu) \times L^1(\mu)$ to $L^{\frac{1}{2},\infty}(\mu)$, then there exists a positive constant C such that*

$$\|T_\theta(f_1, f_2)\|_{L^p(\mu)} \leq C\|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}.$$

Lemma 2.3. *Let $1 < p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $b_1, b_2 \in \mathrm{RBMO}(\mu)$. Assume that $f_1 \in L^{p_1}(\mu), f_2 \in L^{p_2}(\mu)$ with $\int_{\mathcal{X}} T_\theta(f_1, f_2)(x)\mathrm{d}\mu(x) = 0$, $\int_{\mathcal{X}}[b_1, T_\theta](f_1, f_2)(x)\mathrm{d}\mu(x) = 0$, $\int_{\mathcal{X}}[b_2, T_\theta](f_1, f_2)(x)\mathrm{d}\mu(x) = 0$, $\int_{\mathcal{X}}[b_1, b_2, T_\theta](f_1, f_2)(x)\mathrm{d}\mu(x) = 0$ if $\|\mu\| < \infty$. If T_θ is a bounded from $L^1(\mu) \times L^1(\mu)$ to $L^{\frac{1}{2},\infty}(\mu)$, then there exists a constant $C > 0$ such that*

$$\|[b_1, b_2, T_\theta](f_1, f_2)\|_{L^p(\mu)} \leq C\|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}.$$

Nakai [15] introduced the following lemma which ensures that the integrability of the functions can be boostered automatically.

Lemma 2.4. *Suppose that $\psi : (0, \infty) \to (0, \infty)$ be a function satisfying*

$$\int_r^\infty \psi(s)\frac{\mathrm{d}s}{s} \leq C\psi(r) \text{ for all } r > 0.$$

Then there exists $\varepsilon > 0$ such that $\int_r^\infty \psi(s)s^\varepsilon \frac{\mathrm{d}s}{s} \leq C\psi(r)r^\varepsilon$ for all $r > 0$. In particular, for every $\eta \leq 1$, there exists $c > 0$ such that $\int_r^\infty [\psi(s)]^\eta \frac{\mathrm{d}s}{s} \leq C[\psi(r)]^\eta$ for all $r > 0$.

Finally, we recall the following equivalent characterization of $\mathrm{RBMO}(\mu)$ in [6].

Lemma 2.5. *Suppose that $1 \le r < \infty$ and $1 < \rho < \infty$. Then $f \in \mathrm{RBMO}(\mu)$ if and only if for any ball $B \subset \mathcal{X}$,*

$$\left(\frac{1}{\mu(\rho B)}\int_B |f(x) - m_{\widetilde{B}}(f)|^r \mathrm{d}\mu(x)\right)^r \le C\|b\|_{\mathrm{RBMO}(\mu)}, \tag{2.1}$$

and for any doubling $B \subset S$,

$$|m_B(f) - m_S(f)| \le C\|f\|_{\mathrm{RBMO}(\mu)}, \tag{2.2}$$

where $m_B(f)$ is the mean value of f on B, namely,

$$m_B(f) := \frac{1}{\mu(B)}\int_B f(x)\mathrm{d}\mu(x).$$

Moreover, the infimum of the positive constants C satisfying both (2.1) *and* (2.2) *is an equivalent* $\mathrm{RBMO}(\mu)$ *norm of f.*

3 Proofs of the main results

Proof of Theorem 1.7. Without loss of generality, we may assume that $\kappa = 6$ in (1.12). Fix a doubling ball $B \in \mathcal{X}$, and decompose each f_i as $f_i = f_i^0 + f_i^\infty$ for $i = 1, 2$, where $f_i^0 := f_i\chi_{6B}$ and $f_i^\infty := f_i\chi_{\mathcal{X}\setminus 6B}$. Then, by Minkowski inequality, we have

$$\begin{aligned}
&\left(\frac{1}{\phi(\mu(6B))}\int_B |T_\theta(f_1, f_2)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}} \\
&\le \left(\frac{1}{\phi(\mu(6B))}\int_B |T_\theta(f_1^0, f_2^0)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}} + \left(\frac{1}{\phi(\mu(6B))}\int_B |T_\theta(f_1^0, f_2^\infty)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}} \\
&+\left(\frac{1}{\phi(\mu(6B))}\int_B |T_\theta(f_1^\infty, f_2^0)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}} + \left(\frac{1}{\phi(\mu(6B))}\int_B |T_\theta(f_1^\infty, f_2^\infty)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}} \\
&=: \mathrm{D}_1 + \mathrm{D}_2 + \mathrm{D}_3 + \mathrm{D}_4.
\end{aligned}$$

By applying Lemma 2.2 and Definition 1.5, one has

$$\begin{aligned}
\mathrm{D}_1 &= \left(\frac{1}{\phi(\mu(6B))}\int_B |T_\theta(f_1^0, f_2^0)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}} \\
&\le C\frac{1}{[\phi(\mu(6B))]^{\frac{1}{p_1}+\frac{1}{p_2}}}\|f_1^0\|_{L^{p_1}(\mu)}\|f_2^0\|_{L^{p_2}(\mu)} \\
&\le C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}.
\end{aligned}$$

Now let us turn to estimate D_2. For any $x \in B, y_1 \in 6B$ and $y_2 \in \mathcal{X} \setminus 6B$, we have $\lambda(x, d(x, y_1)) \le \lambda(x, d(x, y_2))$. By (1.6), (1.12), Hölder inequality and (1.13), one has

$$|T_\theta(f_1^0, f_2^\infty)(x)| \le \int_{6B} |f_1(y_1)| \int_{\mathcal{X}\setminus 6B} |K(x, y_1, y_2)||f_2(y)|\mathrm{d}\mu(y_2)\mathrm{d}\mu(y_1)$$

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$$
\begin{aligned}
&\leq C\int_{6B}|f_1(y_1)|\int_{\mathcal{X}\setminus 6B}\frac{|f_2(y)|}{[\lambda(x,d(x,y_1))+\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\mathrm{d}\mu(y_1)\\
&\leq C\int_{6B}|f_1(y_1)|\int_{\mathcal{X}\setminus 6B}\frac{|f_2(y)|}{[\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\mathrm{d}\mu(y_1)\\
&\leq C\int_{6B}|f_1(y_1)|\int_{\mathcal{X}\setminus 6B}\frac{|f_2(y)|}{[\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\mathrm{d}\mu(y_1)\\
&\leq C\int_{6B}|f_1(y_1)|\mathrm{d}\mu(y_1)\Bigg(\sum_{k=1}^{\infty}\int_{6^{k+1}B\setminus 6^kB}\frac{|f_2(y)|}{[\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\Bigg)\\
&\leq C\Bigg(\int_{6B}|f_1(y_1)|^{p_1}\mathrm{d}\mu(y_1)\Bigg)^{\frac{1}{p_1}}[\mu(6B)]^{1-\frac{1}{p_1}}\\
&\quad\times\Bigg\{\sum_{k=1}^{\infty}\frac{1}{[\lambda(x,6^kr)]^2}\Bigg(\int_{6^{k+1}B}|f_2(y)|^{p_2}\mathrm{d}\mu(y_2)\Bigg)^{\frac{1}{p_2}}[\mu(6^{k+1}B)]^{1-\frac{1}{p_2}}\Bigg\}\\
&\leq C\frac{1}{\lambda(x,r)}\Bigg(\int_{6B}|f_1(y_1)|^{p_1}\mathrm{d}\mu(y_1)\Bigg)^{\frac{1}{p_1}}[\mu(6B)]^{1-\frac{1}{p_1}}\\
&\quad\times\Bigg\{\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr)}\Bigg(\int_{6^{k+1}B}|f_2(y)|^{p_2}\mathrm{d}\mu(y_2)\Bigg)^{\frac{1}{p_2}}[\mu(6^{k+1}B)]^{1-\frac{1}{p_2}}\Bigg\}\\
&\leq C\|f_1\|_{L^{p_1,\phi}(\mu)}\|f_2\|_{L^{p_2,\phi}(\mu)}\left[\frac{\phi(\mu(6B))}{\mu(6B)}\right]^{\frac{1}{p_1}}\\
&\qquad\times\Bigg\{\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\left[\frac{\phi(\mu(6^{k+2}B))}{\mu(6^{k+2}B)}\right]^{\frac{1}{p_2}}\Bigg\},
\end{aligned}
$$

further, by condition (1.14), (1.15) and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, it follows that

$$
\begin{aligned}
\mathrm{D}_2&\leq C\|f_1\|_{L^{p_1,\phi}(\mu)}\|f_2\|_{L^{p_2,\phi}(\mu)}\left[\frac{\mu(6B)}{\phi(\mu(6B))}\right]^{\frac{1}{p}}\left[\frac{\phi(\mu(6B))}{\mu(6B)}\right]^{\frac{1}{p_1}}\\
&\qquad\times\Bigg\{\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\left[\frac{\phi(\mu(6^{k+2}B))}{\mu(6^{k+2}B)}\right]^{\frac{1}{p_2}}\Bigg\}\\
&\leq C\|f_1\|_{L^{p_1,\phi}(\mu)}\|f_2\|_{L^{p_2,\phi}(\mu)}.
\end{aligned}
$$

With an argument similar to that used in the proof of D_2, we can easily obtain

$$\mathrm{D}_3\leq C\|f_1\|_{L^{p_1,\phi}(\mu)}\|f_2\|_{L^{p_2,\phi}(\mu)}.$$

It remains to estimate D_4. Firstly, consider $|T_\theta(f_1^\infty,f_2^\infty)(x)|$, for any $x\in B$, by condition (1.6), we have

$$|T_\theta(f_1^\infty,f_2^\infty)(x)|\leq\int_{\mathcal{X}}\int_{\mathcal{X}}|K(x,y_1,y_2)||f_1^\infty(y_1)||f_2^\infty(y_2)|\mathrm{d}\mu(y_1)\mathrm{d}\mu(y_2)$$

$$\begin{aligned}
&\leq C\int_{\mathcal{X}\backslash 6B}\int_{\mathcal{X}\backslash 6B}\frac{|f_1(y_1)||f_2(y_2)|}{[\lambda(x,d(x,y_1))+\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_1)\mathrm{d}\mu(y_2)\\
&\leq C\sum_{k=1}^{\infty}\int_{6^{k+1}B\backslash 6^kB}\left(\sum_{j=1}^{\infty}\int_{6^{j+1}B\backslash 6^jB}\frac{|f_1(y_1)||f_2(y_2)|}{[\lambda(x,d(x,y_1))+\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_1)\right)\mathrm{d}\mu(y_2)\\
&\leq C\sum_{k=1}^{\infty}\int_{6^{k+1}B\backslash 6^kB}\frac{|f_2(y_2)|}{[\lambda(x,d(x,y_2))]^2}\left(\sum_{j=1}^{k-1}\int_{6^{j+1}B\backslash 6^jB}|f_1(y_1)|\mathrm{d}\mu(y_1)\right)\mathrm{d}\mu(y_2)\\
&\quad+C\sum_{k=1}^{\infty}\int_{6^{k+1}B\backslash 6^kB}|f_2(y_2)|\left(\sum_{j=k}^{\infty}\int_{6^{j+1}B\backslash 6^jB}\frac{|f_1(y_1)|}{[\lambda(x,d(x,y_1))]^2}\mathrm{d}\mu(y_1)\right)\mathrm{d}\mu(y_2)\\
&=: \mathrm{E}_1+\mathrm{E}_2.
\end{aligned}$$

For E_1. By applying the Hölder inequality and (1.12), we have

$$\begin{aligned}
\mathrm{E}_1&\leq C\sum_{k=1}^{\infty}\int_{6^{k+1}B\backslash 6^kB}\frac{|f_2(y_2)|}{\lambda(x,d(x,y_2))}\left(\sum_{j=1}^{k-1}\int_{6^{j+1}B\backslash 6^jB}\frac{|f_1(y_1)|}{\lambda(x,d(x,y_1))}\mathrm{d}\mu(y_1)\right)\mathrm{d}\mu(y_2)\\
&\leq C\sum_{k=1}^{\infty}\int_{6^{k+1}B\backslash 6^kB}\frac{|f_2(y_2)|}{\lambda(x,d(x,y_2))}\left(\sum_{j=1}^{k-1}\frac{1}{\lambda(x,6^jr)}\int_{6^{j+1}B\backslash 6^jB}|f_1(y_1)|\mathrm{d}\mu(y_1)\right)\mathrm{d}\mu(y_2)\\
&\leq C\sum_{k=1}^{\infty}\frac{1}{\lambda(x,6^kB)}\left(\int_{6^{k+1}B}|f_2(y_2)|^{p_2}\mathrm{d}\mu(y_2)\right)^{\frac{1}{p_2}}[\mu(6^{k+1}B)]^{1-\frac{1}{p_2}}\\
&\qquad\times\left\{\sum_{j=1}^{k-1}\frac{1}{\lambda(x,6^jr)}\left(\int_{6^{j+1}B}|f_1(y_1)|^{p_1}\mathrm{d}\mu(y_1)\right)[\mu(6^{j+1}B)]^{1-\frac{1}{p_1}}\right\}\\
&\leq C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left\{\sum_{k=1}^{\infty}\left[\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right]^{\frac{1}{p_2}}\right\}\left\{\sum_{j=1}^{k-1}\left[\frac{\phi(\mu(6^{j+1}B))}{\mu(6^{j+1}B)}\right]^{\frac{1}{p_1}}\right\}\\
&\leq C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left\{\sum_{k=1}^{\infty}\left[\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right]^{\frac{1}{p_2}}\left[\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right]^{\frac{1}{p_1}}\right\}\\
&\leq C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left\{\sum_{k=1}^{\infty}\left[\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right]^{\frac{1}{p}}\right\}
\end{aligned}$$

An argument similar to that used in the above proof, it is not difficult to obtain

$$\mathrm{E}_2\leq C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left\{\sum_{k=1}^{\infty}\left[\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right]^{\frac{1}{p}}\right\}.$$

Moreover, by applying the assumption $\int_r^{\infty}\frac{\phi(t)}{t}\frac{\mathrm{d}t}{t}\leq C\frac{\phi(r)}{r}$ and Lemma 2.4, lead to

$$\sum_{k=1}^{\infty}\left[\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right]^{\frac{1}{p}}\leq C\left[\frac{\phi(\mu(6^2B))}{\mu(6^2B)}\right]^{\frac{1}{p}},$$

combining the estimates for E_1 and E_2, and condition (1.15), it follows that

$$\begin{aligned}\mathrm{D}_4 &\le C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\mu(6B)}{\phi(\mu(6B))}\right]^{\frac{1}{p}}\left\{\sum_{k=1}^{\infty}\left[\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right]^{\frac{1}{p}}\right\}\\ &\le C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\mu(6B)}{\phi(\mu(6B))}\right]^{\frac{1}{p}}\left[\frac{\phi(\mu(6^2B))}{\mu(6^2B)}\right]^{\frac{1}{p}}\\ &\le C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)},\end{aligned}$$

which, summing up the estimates for D_1, D_2 and D_3, the proof of Theorem 1.7 is finished. □

Proof of Theorem 1.8. We decompose f_i as $f_i = f_i^0 + f_i^\infty$ in the proof of Theorem 1.7, where $f_i^0 := f_i\chi_{6B}, i = 1, 2$. Then

$$\begin{aligned}&\left(\frac{1}{\phi(\mu(6B))}\int_B |[b_1,b_2,T_\theta](f_1,f_2)(x)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\ &\le \left(\frac{1}{\phi(\mu(6B))}\int_B |[b_1,b_2,T_\theta](f_1^0,f_2^0)(x)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\ &\quad+\left(\frac{1}{\phi(\mu(6B))}\int_B |[b_1,b_2,T_\theta](f_1^0,f_2^\infty)(x)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\ &\quad+\left(\frac{1}{\phi(\mu(6B))}\int_B |[b_1,b_2,T_\theta](f_1^\infty,f_2^0)(x)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\ &\quad+\left(\frac{1}{\phi(\mu(6B))}\int_B |[b_1,b_2,T_\theta](f_1^\infty,f_2^\infty)(x)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\ &=: \mathrm{F}_1+\mathrm{F}_2+\mathrm{F}_3+\mathrm{F}_4.\end{aligned}$$

From Lemma 2.3, Definition 1.5 and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, it follows that,

$$\begin{aligned}\mathrm{F}_1 &\le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\frac{1}{[\phi(\mu(6B))]^{\frac{1}{p}}}\|f_1^0\|_{L^{p_1}(\mu)}\|f_2^0\|_{L^{p_2}(\mu)}\\ &\le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}.\end{aligned}$$

In order to estimate F_2, we firstly consider $[b_1,b_2,T_\theta](f_1^0,f_2^\infty)(x)$. For any $x\in B$, write

$$\begin{aligned}&|[b_1,b_2,T_\theta](f_1^0,f_2^\infty)(x)|\\ &\le \int_{6B}|b_1(x)-b_1(y_1)||f_1(y_1)|\int_{\mathcal{X}\setminus 6B}|K(x,y_1,y_2)||b_2(x)-b_2(y_2)||f_2(y_2)|\mathrm{d}\mu(y_2)\mathrm{d}\mu(y_1)\\ &\le C\int_{6B}|b_1(x)-b_1(y_1)||f_1(y_1)|\int_{\mathcal{X}\setminus 6B}\frac{|b_2(x)-b_2(y_2)||f_2(y_2)|}{[\lambda(x,d(x,y_1))+\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\mathrm{d}\mu(y_1)\end{aligned}$$

$$
\begin{aligned}
&\le C\int_{6B}|b_1(x)-b_1(y_1)||f_1(y_1)|\left(\sum_{k=1}^{\infty}\int_{6^{k+1}B\setminus 6^kB}\frac{|b_2(x)-b_2(y_2)||f_2(y_2)|}{[\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\right)\mathrm{d}\mu(y_1)\\
&\le C|b_1(x)-m_{\widetilde{6B}}(b_1)||b_2(x)-m_{\widetilde{6B}}(b_2)|\\
&\quad\times\int_{6B}|f_1(y_1)|\left(\sum_{k=1}^{\infty}\int_{6^{k+1}B\setminus 6^kB}\frac{|f_2(y_2)|}{[\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\right)\mathrm{d}\mu(y_1)\\
&\quad+C|b_1(x)-m_{\widetilde{6B}}(b_1)|\\
&\quad\times\int_{6B}|f_1(y_1)|\left(\sum_{k=1}^{\infty}\int_{6^{k+1}B\setminus 6^kB}\frac{|b_2(y_2)-m_{\widetilde{6B}}(b_2)||f_2(y_2)|}{[\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\right)\mathrm{d}\mu(y_1)\\
&\quad+C|b_2(x)-m_{\widetilde{6B}}(b_2)|\\
&\quad\times\int_{6B}|b_1(y_1)-m_{\widetilde{6B}}(b_1)||f_1(y_1)|\left(\sum_{k=1}^{\infty}\int_{6^{k+1}B\setminus 6^kB}\frac{|f_2(y_2)|}{[\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\right)\mathrm{d}\mu(y_1)\\
&\quad+C\int_{6B}|b_1(y_1)-m_{\widetilde{6B}}(b_1)||f_1(y_1)|\\
&\quad\times\left(\sum_{k=1}^{\infty}\int_{6^{k+1}B\setminus 6^kB}\frac{|b_2(y_2)-m_{\widetilde{6B}}(b_2)||f_2(y_2)|}{[\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\right)\mathrm{d}\mu(y_1)\\
&=:\mathrm{G}_1+\mathrm{G}_2+\mathrm{G}_3+\mathrm{G}_4.
\end{aligned}
$$

With an argument similar to that used in the proof of D_2 in Theorem 1.7, it follows that

$$
\begin{aligned}
\mathrm{G}_1\le\, & C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}|b_1(x)-m_{\widetilde{6B}}(b_1)||b_2(x)-m_{\widetilde{6B}}(b_2)|\\
&\times\left[\frac{\phi(\mu(6B))}{\mu(6B)}\right]^{\frac{1}{p_1}}\left\{\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\left[\frac{\phi(\mu(6^{k+2}B))}{\mu(6^{k+2}B)}\right]^{\frac{1}{p_2}}\right\}.
\end{aligned}
$$

By applying the Hölder inequality, (1.14), (2.1), we have

$$
\begin{aligned}
\mathrm{G}_2\le\, & C|b_1(x)-m_{\widetilde{6B}}(b_1)|\frac{1}{\lambda(x,r)}\int_{6B}|f_1(y_1)|\mathrm{d}\mu(y_1)\\
&\times\left(\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr))}\int_{6^{k+1}B}|b_2(y_2)-m_{\widetilde{6B}}(b_2)||f_2(y_2)|\mathrm{d}\mu(y_2)\right)\\
\le\, & C|b_1(x)-m_{\widetilde{6B}}(b_1)|[\mu(12B)]^{-\frac{1}{p_1}}\left(\int_{6B}|f_1(y_1)|^{p_1}\mathrm{d}\mu(y_1)\right)^{\frac{1}{p_1}}\\
&\times\left[\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr))}\int_{6^{k+1}B}\left(|b_2(y_2)-m_{\widetilde{6^{k+1}B}}(b_2)|\right.\right.\\
&\left.\left.\qquad+|m_{\widetilde{6^{k+1}B}}(b_2)-m_{\widetilde{6B}}(b_2)|\right)|f_2(y_2)|\mathrm{d}\mu(y_2)\right]
\end{aligned}
$$

$$\begin{aligned}
&\leq C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}|b_1(x)-m_{\widetilde{6B}}(b_1)|\left[\frac{\phi(\mu(12B))}{\mu(12B)}\right]^{\frac{1}{p_1}}\left[\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr))}\right.\\
&\quad\left.\times\int_{6^{k+1}B}\left(|b_2(y_2)-m_{\widetilde{6^{k+1}B}}(b_2)|+k\|b_2\|_{\mathrm{RBMO}(\mu)}\right)|f_2(y_2)|\mathrm{d}\mu(y_2)\right]\\
&\leq C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}|b_1(x)-m_{\widetilde{6B}}(b_1)|\left[\frac{\phi(\mu(12B))}{\mu(12B)}\right]^{\frac{1}{p_1}}\left\{\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr))}\right.\\
&\quad\times\left[\left(\int_{6^{k+1}B}|f_2(y_2)|^{p_2}\mathrm{d}\mu(y_2)\right)^{\frac{1}{p_2}}\left(\int_{6^{k+1}B}|b_2(y_2)-m_{\widetilde{6^{k+1}B}}(b_2)|^{p_2'}\mathrm{d}\mu(y_2)\right)^{\frac{1}{p_2'}}\right.\\
&\quad\left.\left.+k\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\mu(6^{k+1}B)\left(\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right)^{\frac{1}{p_2}}\right]\right\}\\
&\leq C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}|b_1(x)-m_{\widetilde{6B}}(b_1)|\left[\frac{\phi(\mu(12B))}{\mu(12B)}\right]^{\frac{1}{p_1}}\left[\sum_{k=1}^{\infty}\frac{k+1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr))}\right.\\
&\quad\left.\times\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\mu(6^{k+1}B)\left(\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right)^{\frac{1}{p_2}}\right]\\
&\leq C\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}|b_1(x)-m_{\widetilde{6B}}(b_1)|\left[\frac{\phi(\mu(12B))}{\mu(12B)}\right]^{\frac{1}{p_1}}\\
&\quad\times\left[\sum_{k=1}^{\infty}\frac{k+1}{[C(6^k)]^{\epsilon}}\left(\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right)^{\frac{1}{p_2}}\right],
\end{aligned}$$

where we have used the following fact that

$$|m_{\widetilde{6^{k+1}B}}(b_2)-m_{\widetilde{6B}}(b_2)|\leq C(k+1)\|b_2\|_{\mathrm{RBMO}(\mu)}. \tag{3.1}$$

By applying (1.12), the Hölder inequality, (1.14) and (2.1), one has

$$\begin{aligned}
\mathrm{G}_3&\leq C|b_2(x)-m_{\widetilde{6B}}(b_2)|\frac{1}{\lambda(x,r)}\int_{6B}|b_1(y_1)-m_{\widetilde{6B}}(b_1)||f_1(y_1)|\mathrm{d}\mu(y_1)\\
&\quad\times\left(\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr)}\int_{6^{k+1}B}|f_2(y_2)|\mathrm{d}\mu(y_2)\right)\\
&\leq C\|b_1\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}|b_2(x)-m_{\widetilde{6B}}(b_2)|\left[\frac{\phi(\mu(12B))}{\mu(12B)}\right]^{\frac{1}{p_1}}\\
&\quad\times\left[\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\left(\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right)^{\frac{1}{p_2}}\right].
\end{aligned}$$

It remains to estimate G_4. By (1.12), the Hölder inequality, (1.14), (2.1) and (3.1), we have

$$\begin{aligned}
\mathrm{G}_4 \leq& C\frac{1}{\lambda(x,r)}\int_{6B}|b_1(y_1)-m_{\widetilde{6B}}(b_1)||f_1(y_1)|\mathrm{d}\mu(y_1)\\
&\times\left(\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr)}\int_{6^{k+1}B}|b_2(y_2)-m_{\widetilde{6B}}(b_2)||f_2(y_2)|\mathrm{d}\mu(y_2)\right)\\
\leq& C\|b_1\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\left[\frac{\phi(\mu(12B))}{\mu(12B)}\right]^{\frac{1}{p_1}}\\
&\times\left[\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr)}\left(\int_{6^{k+1}B}|b_2(y_2)-m_{\widetilde{6^{k+1}B}}(b_2)||f_2(y_2)|\mathrm{d}\mu(y_2)\right.\right.\\
&\qquad\left.\left.+|m_{\widetilde{6^{k+1}B}}(b_2)-m_{\widetilde{6B}}(b_2)|\int_{6^{k+1}B}|f_2(y_2)|\mathrm{d}\mu(y_2)\right)\right]\\
\leq& C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\phi(\mu(12B))}{\mu(12B)}\right]^{\frac{1}{p_1}}\\
&\times\left[\sum_{k=1}^{\infty}\frac{k+1}{[C(6^k)]^{\epsilon}}\frac{\mu(6^{k+1}B)}{\lambda(x,6^kr)}\left(\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right)^{\frac{1}{p_2}}\right]\\
\leq& C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\phi(\mu(12B))}{\mu(12B)}\right]^{\frac{1}{p_1}}\\
&\times\left[\sum_{k=1}^{\infty}\frac{k+1}{[C(6^k)]^{\epsilon}}\left(\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right)^{\frac{1}{p_2}}\right].
\end{aligned}$$

Thus, by applying the estimates of G_1, G_2, G_3 and G_4, the Hölder inequality and the fact that $\frac{\phi(t)}{t}\leq C\frac{\phi(s)}{s}$ with $s\geq t$, it follows that

$$\begin{aligned}
\mathrm{F}_2 =& \left(\frac{1}{\phi(\mu(6B))}\int_B|[b_1,b_2,T_\theta](f_1^0,f_2^\infty)(x)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
\leq& C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left(\int_B|b_1(x)-m_{\widetilde{6B}}(b_1)|^p|b_2(x)-m_{\widetilde{6B}}(b_2)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\times[\phi(\mu(6B))]^{-\frac{1}{p}}\left[\frac{\phi(\mu(6B))}{\mu(6B)}\right]^{\frac{1}{p_1}}\left\{\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\left[\frac{\phi(\mu(6^{k+2}B))}{\mu(6^{k+2}B)}\right]^{\frac{1}{p_2}}\right\}\\
&+C\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left(\frac{1}{\phi(\mu(6B))}\int_B|b_1(x)-m_{\widetilde{6B}}(b_1)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&\times\left[\frac{\phi(\mu(12B))}{\mu(12B)}\right]^{\frac{1}{p_1}}\left[\sum_{k=1}^{\infty}\frac{k+1}{[C(6^k)]^{\epsilon}}\left(\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right)^{\frac{1}{p_2}}\right]\\
&+C\|b_1\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left(\frac{1}{\phi(\mu(6B))}\int_B|b_2(x)-m_{\widetilde{6B}}(b_2)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\times\left[\frac{\phi(\mu(12B))}{\mu(12B)}\right]^{\frac{1}{p_1}}\left[\sum_{k=1}^{\infty}\frac{k+1}{[C(6^k)]^{\epsilon}}\left(\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right)^{\frac{1}{p_2}}\right]\\
&+C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\mu(6B)}{\phi(\mu(6B))}\right]^{\frac{1}{p}}\\
&\times\left[\frac{\phi(\mu(12B))}{\mu(12B)}\right]^{\frac{1}{p_1}}\left[\sum_{k=1}^{\infty}\frac{k+1}{[C(6^k)]^{\epsilon}}\left(\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right)^{\frac{1}{p_2}}\right]\\
\le{}&C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\mu(6B)}{\phi(\mu(6B))}\right]^{\frac{1}{p}}\\
&\times\left[\frac{\phi(\mu(6B))}{\mu(6B)}\right]^{\frac{1}{p_1}}\left\{\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\left[\frac{\phi(\mu(6^{k+2}B))}{\mu(6^{k+2}B)}\right]^{\frac{1}{p_2}}\right\}\\
&+C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\mu(6B)}{\phi(\mu(6B))}\right]^{\frac{1}{p}}\\
&\times\left[\frac{\phi(\mu(12B))}{\mu(12B)}\right]^{\frac{1}{p_1}}\left[\sum_{k=1}^{\infty}\frac{k+1}{[C(6^k)]^{\epsilon}}\left(\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right)^{\frac{1}{p_2}}\right]\\
\le{}&C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)},
\end{aligned}
$$

where $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$.

Similarly, it is not difficult to obtain

$$\mathrm{F}_3\le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}.$$

Now let us turn to estimate F_4. For any $x\in B$, write

$$
\begin{aligned}
&|[b_1,b_2,T_\theta](f_1^\infty,f_2^\infty)(x)|\\
&\quad\le|b_1(x)-m_{\widetilde{B}}(b_1)||b_2(x)-m_{\widetilde{B}}(b_2)||T_\theta(f_1^\infty,f_2^\infty)(x)|\\
&\qquad+|b_1(x)-m_{\widetilde{B}}(b_1)||T_\theta(f_1^\infty,(b_2-m_{\widetilde{B}}(b_2)f_2^\infty)(x)|\\
&\qquad+|b_2(x)-m_{\widetilde{B}}(b_2)||T_\theta((b_1-m_{\widetilde{B}}(b_1)f_1^\infty,f_2^\infty)(x)|\\
&\qquad+|T_\theta((b_1-m_{\widetilde{B}}(b_1)f_1^\infty,(b_2-m_{\widetilde{B}}(b_2)f_2^\infty)(x)|\\
&\quad=:\mathrm{H}_1+\mathrm{H}_2+\mathrm{H}_3+\mathrm{H}_4.
\end{aligned}
$$

An argument similar to that used in the proof of D_4 in the Theorem 1.7, we have

$$\mathrm{H}_1 \leq C|b_1(x)-m_{\widetilde{B}}(b_1)||b_2(x)-m_{\widetilde{B}}(b_2)|\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left\{\sum_{k=1}^{\infty}\left[\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right]^{\frac{1}{p}}\right\}.$$

With a slight modified argument similar to that used in the proof of J_{21} in [25], it is not difficult to obtain that

$$\begin{aligned}\mathrm{H}_2+\mathrm{H}_3+\mathrm{H}_4 \leq & C|b_1(x)-m_{\widetilde{B}}(b_1)|\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\phi(\mu(6B))}{\mu(6B)}\right]^{\frac{1}{p}}\\ &+C|b_2(x)-m_{\widetilde{B}}(b_2)|\|b_1\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\phi(\mu(6B))}{\mu(6B)}\right]^{\frac{1}{p}}\\ &+C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\phi(\mu(6B))}{\mu(6B)}\right]^{\frac{1}{p}}.\end{aligned}$$

Further, by applying the Hölder inequality, Definition 1.5 and (2.1), we can deduce that

$$\begin{aligned}\mathrm{F}_4 \leq & C\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left\{\sum_{k=1}^{\infty}\left[\frac{\phi(\mu(6^{k+1}B))}{\mu(6^{k+1}B)}\right]^{\frac{1}{p}}\right\}\\ &\times\left(\frac{1}{\phi(\mu(6B))}\int_B |b_1(x)-m_{\widetilde{B}}(b_1)|^p|b_2(x)-m_{\widetilde{B}}(b_2)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\ &+C\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\phi(\mu(6B))}{\mu(6B)}\right]^{\frac{1}{p}}\\ &\times\left(\frac{1}{\phi(\mu(6B))}\int_B |b_1(x)-m_{\widetilde{B}}(b_1)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\ &+C\|b_1\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\phi(\mu(6B))}{\mu(6B)}\right]^{\frac{1}{p}}\\ &\times\left(\frac{1}{\phi(\mu(6B))}\int_B |b_1(x)-m_{\widetilde{B}}(b_1)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\ &+C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}\left[\frac{\phi(\mu(6B))}{\mu(6B)}\right]^{\frac{1}{p}}\left[\frac{\phi(\mu(6B))}{\mu(6B)}\right]^{-\frac{1}{p}}\\ \leq & C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{\mathcal{L}^{p_1,\phi}(\mu)}\|f_2\|_{\mathcal{L}^{p_2,\phi}(\mu)}.\end{aligned}$$

Which, combining the estimates of F_1, F_2 and F_3, the proof of Theorem 1.8 is finished. □

Remark 3.1. With an argument similar to those used in the proof of Theorem 1.6 in [28] and Remarks 6-7 in [25], it is not difficult to obtain Theorem 1.9. Thus, we omit the details in this article.

Proof of Theorem 1.10. Without loss of generality, we assume that $k = 6$ in (1.16), and decompose f_1 as $f_i = f_i^0 + f_i^\infty$ as in Theorem 1.7, where $f_i^0 := f_i\chi_{6B}$. Then

$$\begin{aligned}
&[\mu(6B)]^{\frac{1}{q}-\frac{1}{p}}\left(\int_B |[b_1,b_2,T_\theta](f_1,f_2)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\le [\mu(6B)]^{\frac{1}{q}-\frac{1}{p}}\left(\int_B |[b_1,b_2,T_\theta](f_1^0,f_2^0)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\quad+[\mu(6B)]^{\frac{1}{q}-\frac{1}{p}}\left(\int_B |[b_1,b_2,T_\theta](f_1^0,f_2^\infty)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\quad+[\mu(6B)]^{\frac{1}{q}-\frac{1}{p}}\left(\int_B |[b_1,b_2,T_\theta](f_1^\infty,f_2^0)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&\quad+[\mu(6B)]^{\frac{1}{q}-\frac{1}{p}}\left(\int_B |[b_1,b_2,T_\theta](f_1^\infty,f_2^\infty)(x)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}}\\
&=: \mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4.
\end{aligned}$$

By applying Lemma 2.3, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$ and $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$, we have

$$\begin{aligned}
\mathrm{I}_1 &\le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}[\mu(6B)]^{\frac{1}{q}-\frac{1}{p}}\|f_1^0\|_{L^{p_1}(\mu)}\|f_2^0\|_{L^{p_2}(\mu)}\\
&\le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}[\mu(6B)]^{\frac{1}{q}-\frac{1}{p}}[\mu(6B)]^{\frac{1}{p}-\frac{1}{q}}\\
&\le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}.
\end{aligned}$$

To estimate I_2. For any $x\in B$, we firstly consider $|[b_1,b_2,T_\theta](f_1^0,f_2^\infty)(x)|$. Write

$$\begin{aligned}
|[b_1,b_2,T_\theta](f_1^0,f_2^\infty)(x)| &\le |b_1(x)-m_{\widetilde{B}}(b_1)||b_2(x)-m_{\widetilde{B}}(b_2)||T_\theta(f_1^0,f_2^\infty)(x)|\\
&\quad+|b_1(x)-m_{\widetilde{B}}(b_1)||T_\theta(f_1^0,(b_2-m_{\widetilde{B}}(b_2)f_2^\infty)(x)|\\
&\quad+|b_2(x)-m_{\widetilde{B}}(b_2)||T_\theta((b_1-m_{\widetilde{B}}(b_1)f_1^0,f_2^\infty)(x)|\\
&\quad+|T_\theta((b_1-m_{\widetilde{B}}(b_1)f_1^0,(b_2-m_{\widetilde{B}}(b_2)f_2^\infty)(x)|\\
&=: \mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_3+\mathrm{J}_4.
\end{aligned}$$

With an argument similar to that used in the proof of H_2 in [28], it is not difficult to obtain that

$$\mathrm{J}_1 \le C|b_1(x)-m_{\widetilde{B}}(b_1)||b_2(x)-m_{\widetilde{B}}(b_2)|\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}[\mu(6B)]^{-\frac{1}{q}}.$$

By applying (1.6), (1.13), (1.14), the Hölder inequality, (2.1) and (3.1), we can deduce

$$\begin{aligned}
\mathrm{J}_2 &\le C|b_1(x)-m_{\widetilde{B}}(b_1)|\int_{6B}|f_1(y_1)|\int_{\mathcal{X}\setminus 6B}\frac{|b_2(y_1)-m_{\widetilde{B}}(b_2)||f_2(y_2)|}{[\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\mathrm{d}\mu(y_1)\\
&\le C|b_1(x)-m_{\widetilde{B}}(b_1)|\int_{6B}|f_1(y_1)|\mathrm{d}\mu(y_1)\\
&\quad\times\left(\sum_{k=1}^{\infty}\int_{6^{k+1}B\setminus 6^kB}\frac{|b_2(y_1)-m_{\widetilde{B}}(b_2)||f_2(y_2)|}{[\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\right)\\
&\le C|b_1(x)-m_{\widetilde{B}}(b_1)|\frac{1}{\lambda(x,r)}\int_{6B}|f_1(y_1)|\mathrm{d}\mu(y_1)\\
&\quad\times\left(\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr)}\int_{6^{k+1}B}|b_2(y_1)-m_{\widetilde{B}}(b_2)||f_2(y_2)|\mathrm{d}\mu(y_2)\right)\\
&\le C\|f_1\|_{M^{q_1}_{p_1}(\mu)}|b_1(x)-m_{\widetilde{B}}(b_1)|[\mu(6B)]^{-\frac{1}{q_1}}\left\{\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr)}\right.\\
&\quad\left.\times\int_{6^{k+1}B}[|b_2(y_1)-m_{\widetilde{6^{k+1}B}}(b_2)|+|m_{\widetilde{6^{k+1}B}}(b_2)-m_{\widetilde{B}}(b_2)|]|f_2(y_2)|\mathrm{d}\mu(y_2)\right\}\\
&\le C\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}|b_1(x)-m_{\widetilde{B}}(b_1)|[\mu(6B)]^{-\frac{1}{q_1}}\left\{\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr)}\right.\\
&\quad\left.\times(k+1)\|b_2\|_{\mathrm{RBMO}(\mu)}[\mu(6^{k+1}B)]^{1-\frac{1}{q_2}}\right\}\\
&\le C\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}|b_1(x)-m_{\widetilde{B}}(b_1)|[\mu(6B)]^{-\frac{1}{q}}
\end{aligned}$$

Similarly, we have

$$\mathrm{J}_3\le C\|b_1\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}|b_2(x)-m_{\widetilde{B}}(b_2)|[\mu(6B)]^{-\frac{1}{q}}.$$

Now let us turn to estimate J_4. With (1.6), (1.13), (1.14), the Hölder inequality, (2.1) and (3.1), lead to

$$\begin{aligned}
&|T_\theta((b_1-m_{\widetilde{B}}(b_1)f_1^0,(b_2-m_{\widetilde{B}}(b_2)f_2^{\infty})(x)|\\
&\le C\int_{6B}|b_1(y_1)-m_{\widetilde{B}}(b_1)||f_1(y_1)|\int_{\mathcal{X}\setminus 6B}\frac{|b_2(y_2)-m_{\widetilde{B}}(b_2)||f_2(y_2)|}{[\lambda(x,d(x,y_2))]^2}\mathrm{d}\mu(y_2)\mathrm{d}\mu(y_1)\\
&\le C\|b_1\|_{\mathrm{RBMO}(\mu)}[\mu(6B)]^{1-\frac{1}{p_1}}\left(\int_{6B}|f_1(y_1)|^{p_1}\mathrm{d}\mu(y_1)\right)^{\frac{1}{p_1}}\\
&\quad\times\left\{\sum_{k=1}^{\infty}\frac{1}{[\lambda(x,6^kr)]^2}\int_{6^{k+1}B}|b_2(y_2)-m_{\widetilde{B}}(b_2)||f_2(y_2)|\mathrm{d}\mu(y_2)\right\}
\end{aligned}$$

$$\leq C\|b_1\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}[\mu(6B)]^{-\frac{1}{q_1}}\left\{\sum_{k=1}^{\infty}\frac{1}{[C(6^k)]^{\epsilon}}\frac{1}{\lambda(x,6^kr)}\left[\int_{6^{k+1}B}|f_2(y_2)|\right.\right.$$
$$\left.\left.\times|b_2(y_2)-m_{\widetilde{6^{k+1}B}}(b_2)|\mathrm{d}\mu(y_2)+(k+1)\|b_2\|_{\mathrm{RBMO}(\mu)}\int_{6^{k+1}B}|f_2(y_2)|\mathrm{d}\mu(y_2)\right]\right\}$$
$$\leq C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}[\mu(6B)]^{-\frac{1}{q}}.$$

Combining the estimates of J_1, J_2, J_3 and J_4, we have

$$\mathrm{I}_2=[\mu(6B)]^{\frac{1}{q}-\frac{1}{p}}\left(\int_B|[b_1,b_2,T_\theta](f_1^0,f_2^\infty)(x)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}$$
$$\leq C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}[\mu(6B)]^{\frac{1}{p}-\frac{1}{q}}[\mu(6B)]^{\frac{1}{q}-\frac{1}{p}}$$
$$+C\|b_1\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}[\mu(6B)]^{-\frac{1}{p}}\left(\int_{6B}|b_2(x)-m_{\widetilde{B}}(b_2)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}$$
$$+C\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}[\mu(6B)]^{-\frac{1}{p}}\left(\int_{6B}|b_1(x)-m_{\widetilde{B}}(b_1)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}$$
$$+C\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}[\mu(6B)]^{-\frac{1}{p}}\left(\int_{6B}|b_2(x)-m_{\widetilde{B}}(b_2)|^p|b_1(x)-m_{\widetilde{B}}(b_1)|^p\mathrm{d}\mu(x)\right)^{\frac{1}{p}}$$
$$\leq C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}+C\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}[\mu(6B)]^{-\frac{1}{p}}$$
$$\times\left\{\left(\int_{6B}|b_1(x)-m_{\widetilde{B}}(b_1)|^{p_1}\mathrm{d}\mu(x)\right)^{\frac{p}{p_1}}\left(\int_{6B}|b_2(x)-m_{\widetilde{B}}(b_2)|^{p_2}\mathrm{d}\mu(x)\right)^{\frac{p}{p_2}}\right\}^{\frac{1}{p}}$$
$$\leq C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}.$$

By an argument similar to that used in the I_2, we have

$$\mathrm{I}_3\leq C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}.$$

It remains to estimate I_4. For any $x\in B$, write

$$\begin{aligned}|[b_1,b_2,T_\theta](f_1^\infty,f_2^\infty)(x)| &\leq |b_1(x)-m_{\widetilde{B}}(b_1)||b_2(x)-m_{\widetilde{B}}(b_2)||T_\theta(f_1^\infty,f_2^\infty)(x)|\\ &\quad+|b_1(x)-m_{\widetilde{B}}(b_1)||T_\theta(f_1^\infty,(b_2-m_{\widetilde{B}}(b_2)f_2^\infty)(x)|\\ &\quad+|b_2(x)-m_{\widetilde{B}}(b_2)||T_\theta((b_1-m_{\widetilde{B}}(b_1)f_1^\infty,f_2^\infty)(x)|\\ &\quad+|T_\theta((b_1-m_{\widetilde{B}}(b_1)f_1^\infty,(b_2-m_{\widetilde{B}}(b_2)f_2^\infty)(x)|\\ &=:\mathrm{U}_1+\mathrm{U}_2+\mathrm{U}_3+\mathrm{U}_4.\end{aligned}$$

For U_1, U_2, U_3 and U_4, by some arguments similar to those used in the proofs of H_4 in [28] and U_2' and U_2'' in [23], we can obtain

$$\mathrm{U}_1+\mathrm{U}_2+\mathrm{U}_3+\mathrm{U}_4\leq C|b_1(x)-m_{\widetilde{B}}(b_1)||b_2(x)-m_{\widetilde{B}}(b_2)|\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}$$

$$+C\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}|b_1(x)-m_{\widetilde{B}}(b_1)|[\mu(6B)]^{-\frac{1}{q}}$$
$$+C\|b_1\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}|b_2(x)-m_{\widetilde{B}}(b_2)|[\mu(6B)]^{-\frac{1}{q}}$$
$$+C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}[\mu(6B)]^{-\frac{1}{q}}.$$

Further, by a way similar to that used in the estimate of I_2, we can deduce

$$\mathrm{I}_4 \leq C\|b_1\|_{\mathrm{RBMO}(\mu)}\|b_2\|_{\mathrm{RBMO}(\mu)}\|f_1\|_{M^{q_1}_{p_1}(\mu)}\|f_2\|_{M^{q_2}_{p_2}(\mu)}.$$

Combining the estimates $\mathrm{I}_1 - \mathrm{I}_4$, we complete the proof of Theorem 1.10. □

Conflict of interest
The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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On Ulam-Hyers stability of decic functional equation in non-Archimedean spaces

Yali Ding[1] , Tian-Zhou Xu [1,*], John Michael Rassias[2]

(*1. School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, P. R. China*)

(*2. Pedagogical Department E.E., Section of Mathematics and Informatics, National and Capodistrian University of Athens, 4, Agamemnonos Str., Aghia Paraskevi, Athens 15342, Greece*)

Abstract In the current work, using the fixed point theorems due to Brzdęk and Ciepliński, we give some Ulam-Hyers stability results for the decic functional equation in non-Archimedean spaces.

Keywords Ulam-Hyers stability; decic functional equation; decic mapping; non-Archimedean space; fixed point method.

Mathematics Subject Classification Primary 39B82; Secondary 39B52, 47H10.

1 Introduction and preliminaries

Throughout this paper, $\mathbb{N}$ stands for the set of all positive integers, $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Let us recall (see, for instance, [9]) some basic definitions and facts concerning non-Archimedean normed spaces.

A non-Archimedean valuation on a field $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \to \mathbb{R}$ such that

(1) $|r| \geq 0$ and equality holds if and only if $r = 0$;

(2) $|rs| = |r||s|, \quad r, s \in \mathbb{K}$;

(3) $|r + s| \leq \max\{|r|, |s|\}, \quad r, s \in \mathbb{K}$.

Any field endowed with a non-Archimedean valuation is said to be a non-Archimedean field. In any non-Archimedean field we have $|1| = | - 1| = 1$ and $|n| \leq 1$ for $n \in \mathbb{N}_0$. The most important examples of non-Archimedean fields are p-adic numbers which have gained the interest of physicists for their research in some problems coming from quantum physics, p-adic strings and superstrings (see [9]).

Let X be a linear space over a field $\mathbb{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to \mathbb{R}_+$ is a non-Archimedean norm if it satisfies the following conditions:

(1) $\|x\| = 0$ if and only if $x = 0$;

(2) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$;

(3) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in X$.

Then $(X, \|\cdot\|)$ is called a non-Archimedean normed space.

Let X be a non-Archimedean normed space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent

*Corresponding author. E-mail: xutianzhou@bit.edu.cn

if there exists $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $\lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n\to\infty} \|x_{n+p} - x_n\| = 0$ for all $p = 1, 2, \ldots$. Due to the fact that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : \quad m \leq j \leq n-1\} \quad (n > m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space.

The first work on the Ulam-Hyers stability of functional equations in complete non-Archimedean normed spaces is [10]. After it, a lot of papers on the stability for various classes of functional equations in such spaces have been published, and there are many interesting results concerning this problem, see for instance [2–8, 12–15] and the references therein. The fixed point method is one of the most effective tools in studying these problems.

In this paper, we consider the decic functional equations which was introduced in [1, 11] as follows:

$$\begin{aligned} f(x+5y) - 10f(x+4y) + 45f(x+3y) - 120f(x+2y) + 210f(x+y) - 252f(x) \\ +210f(x-y) - 120f(x-2y) + 45f(x-3y) - 10f(x-4y) + f(x-5y) = 10!f(y). \end{aligned} \tag{1.1}$$

Since $f(x) = x^{10}$ is a solutions of (1.1), we say that it is a decic functional equation. Every solution of the decic functional equation is said to be a decic mapping. Indeed, general solution of the equation (1.1) was found in [11]. In this paper, we study some stability results concerning the functional equation (1.1) in the setting of non-Archimedean normed spaces.

2 Stability of the decic functional equation (1.1)

In this section, we show the generalized Ulam-Hyers stability of equation (1.1) in complete non-Archimedean normed spaces (its stability in quasi-β-Banach spaces was proved in [11]). The proof of our main resut is based on the following fixed point result obtained in [5, Theorem 1] (see also [2, Theorem 2.3] and [3, Theorem 2.2]).

Theorem 2.1 *Let the following three hypotheses be valid :*

(H1) *E is a nonempty set, Y is a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 2, $j \in \mathbb{N}, f_1, \ldots, f_j : E \to E$ and $L_1, \ldots, L_j : E \to \mathbb{R}_+$;*

(H2) *$\mathcal{T} : Y^E \to Y^E$ is an operator satisfying the inequality*

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \leq \max_{i\in\{1,\ldots,j\}} L_i(x)\|\xi(f_i(x)) - \mu(f_i(x))\|, \qquad \xi, \mu \in Y^E, x \in E; \tag{2.1}$$

(H3) *$\Lambda : \mathbb{R}_+^E \to \mathbb{R}_+^E$ is an operator defined by*

$$\Lambda\delta(x) := \max_{i\in\{1,\ldots,j\}} L_i(x)\delta(f_i(x)), \qquad \delta \in \mathbb{R}_+^E, x \in E. \tag{2.2}$$

Assume that the functions $\varepsilon : E \to \mathbb{R}_+$ and $\varphi : E \to Y$ fulfill the following two conditions :

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \leq \varepsilon(x), \quad x \in E, \tag{2.3}$$

and

$$\lim_{l\to\infty} \Lambda^l \varepsilon(x) = 0, \quad x \in E. \tag{2.4}$$

Then there exists a unique fixed point ψ of $\mathcal{T}$ with

$$\|\varphi(x) - \psi(x)\| \leq \sup_{l\in\mathbb{N}_0} \Lambda^l \varepsilon(x), \quad x \in E. \tag{2.5}$$

Moreover,

$$\psi(x) := \lim_{l\to\infty} \mathcal{T}^l \varphi(x), \quad x \in E. \tag{2.6}$$

Let $(X, +)$ is a commutative group and Y is a complete non-Archimedean normed space. Given $f : X \to Y, x, y \in X$, put

$$\begin{aligned} D_{10}(f)(x,y) := {} & f(x+5y) - 10f(x+4y) + 45f(x+3y) - 120f(x+2y) + 210f(x+y) - 252f(x) \\ & +210f(x-y) - 120f(x-2y) + 45f(x-3y) - 10f(x-4y) + f(x-5y) - 10!f(y). \end{aligned} \tag{2.7}$$

Theorem 2.2 *Assume that X be a commutative group uniquely divisible by 2 and let Y be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 210. Let $f : X \to Y$ and $\varphi : X^2 \to \mathbb{R}_+$ be mappings satisfying the inequality*

$$\|D_{10}(f)(x,y)\| \le \varphi(x,y), \qquad x, y \in X. \tag{2.8}$$

Assume also that there is an $s \in \{-1, 1\}$ such that

$$\lim_{l\to\infty} \left(\frac{1}{|2|^{10s}}\right)^l \varphi\left(2^{sl}x, 2^{sl}y\right) = 0, \qquad x, y \in X. \tag{2.9}$$

Then there exists a decic mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \le \sup_{l\in\mathbb{N}_0} \frac{1}{|2|^{5(s+1)}} \left(\frac{1}{|2|^{10s}}\right)^l \delta(2^{sl+\frac{s-1}{2}}x), \qquad x \in X, \tag{2.10}$$

where

$$\begin{aligned} \delta(x) &= \frac{1}{|10!|} \max\left\{|252|\varphi(0,x), |252|A(5x), |11340|A(3x), D(x)\right\}, \\ D(x) &= \max\{|90|\varphi(3x,x), |240|\varphi(2x,x), |420|\varphi(x,x), |420|A(4x), |240|A(3x), |4200|A(3x), \\ &\qquad |90|A(2x), |2400|A(2x), B(x)\}, \\ B(x) &= \max\{|2|\varphi(5x,x), |20|\varphi(4x,x), \varphi(0,2x), |2|C, A(10x), |10|A(8x), \\ &\qquad |45|A(6x), |120|A(4x), |210|A(2x), |20|A(x)\}, \\ A(x) &= \frac{1}{|10!|} \max\{\varphi(x,x), \varphi(x,-x)\}, \qquad C = \frac{1}{|10!|}\varphi(0,0). \end{aligned} \tag{2.11}$$

Proof. Replacing $x = y = 0$ in (2.8), we get

$$\|f(0)\| \le \frac{1}{|10!|}\varphi(0,0) := C. \tag{2.12}$$

Replacing x and y by x and x in (2.8), respectively, we get

$$\begin{aligned} &\|f(6x) - 10f(5x) + 45f(4x) - 120f(3x) + 210f(2x) - 252f(x) \\ &\quad +210f(0) - 120f(-x) + 45f(-2x) - 10f(-3x) + f(-4x) - 10!f(x)\| \le \varphi(x,x) \end{aligned} \tag{2.13}$$

for all $x \in X$. Replacing x and y by x and $-x$ in (2.8), respectively, we have

$$\begin{aligned} &\|f(-4x) - 10f(-3x) + 45f(-2x) - 120f(-x) + 210f(0) - 252f(x) + 210f(2x) \\ &\quad -120f(3x) + 45f(4x) - 10f(5x) + f(6x) - 10!f(-x)\| \le \varphi(x,-x) \end{aligned} \tag{2.14}$$

for all $x \in X$. By (2.13) and (2.14), we obtain

$$\|f(x) - f(-x)\| \le \frac{1}{|10!|} \max\{\varphi(x,x), \varphi(x,-x)\} := A(x) \tag{2.15}$$

for all $x \in X$. Replacing x and y by 0 and $2x$ in (2.8), respectively, and using (2.12) and (2.15), we find

$$\begin{aligned}&\|2f(10x) - 20f(8x) + 90f(6x) - 240f(4x) - (10! - 420)f(2x)\| \\ &\quad \leq \max\{\varphi(0, 2x), A(10x), |10|A(8x), |45|A(6x), |120|A(4x), |210|A(2x), |252|C\}\end{aligned} \tag{2.16}$$

for all $x \in X$. Replacing x and y by $5x$ and x in (2.8), respectively, we get

$$\begin{aligned}&\|f(10x) - 10f(9x) + 45f(8x) - 120f(7x) + 210f(6x) - 252f(5x) + 210f(4x) \\ &\quad -120f(3x) + 45f(2x) - (10! + 10)f(x)\| \leq \max\{\varphi(5x, x), C\}\end{aligned} \tag{2.17}$$

for all $x \in X$. By (2.16) and (2.17), we obtain

$$\begin{aligned}&\|20f(9x) - 110f(8x) + 240f(7x) - 330f(6x) + 504f(5x) - 660f(4x) \\ &\quad +240f(3x) - (10! - 330)f(2x) + (2 \cdot 10! + 20)f(x)\| \\ &\leq \max\{|2|\varphi(5x, x), \varphi(0, 2x), |2|C, A(10x), |10|A(8x), |45|A(6x), |120|A(4x), |210|A(2x)\}\end{aligned} \tag{2.18}$$

for all $x \in X$. Replacing x and y by $4x$ and x in (2.8), respectively, and using (2.12) we have

$$\begin{aligned}&\|f(9x) - 10f(8x) + 45f(7x) - 120f(6x) + 210f(5x) - 252f(4x) + 210f(3x) \\ &\quad -120f(2x) - (10! - 46)f(x)\| \leq \max\{\varphi(4x, x), |10|C, A(x)\}\end{aligned} \tag{2.19}$$

for all $x \in X$. By (2.18) and (2.19), we get

$$\begin{aligned}&\|90f(8x) - 660f(7x) + 2070f(6x) - 3696f(5x) + 4380f(4x) \\ &\quad -3960f(3x) - (10! - 2730)f(2x) + (22 \cdot 10! - 900)f(x)\| \\ &\leq \max\{|2|\varphi(5x, x), |20|\varphi(4x, x), \varphi(0, 2x), |2|C, A(10x), |10|A(8x), \\ &\quad |45|A(6x), |120|A(4x), |210|A(2x), |20|A(x)\} := B(x)\end{aligned} \tag{2.20}$$

for all $x \in X$. Replacing x and y by $3x$ and x in (2.8), respectively, then using (2.12) and (2.15), we have

$$\begin{aligned}&\|f(8x) - 10f(7x) + 45f(6x) - 120f(5x) + 210f(4x) - 252f(3x) + 211f(2x) - (10! + 130)f(x)\| \\ &\quad \leq \max\{\varphi(3x, x), |45|C, A(2x), |10|A(x)\}\end{aligned} \tag{2.21}$$

for all $x \in X$. By (2.20) and (2.21), we get

$$\begin{aligned}&\|240f(7x) - 1980f(6x) + 7104f(5x) - 14520f(4x) \\ &\quad +18720f(3x) - (10! + 16260)f(2x) + (112 \cdot 10! + 10800)f(x)\| \\ &\leq \max\{|90|\varphi(3x, x), |90|A(2x), B(x)\}\end{aligned} \tag{2.22}$$

for all $x \in X$. Replacing x and y by $2x$ and x in (2.8), respectively, then using (2.12) and (2.15), we have

$$\begin{aligned}&\|f(7x) - 10f(6x) + 45f(5x) - 120f(4x) + 211f(3x) - 262f(2x) - (10! - 255)f(x)\| \\ &\quad \leq \max\{\varphi(2x, x), A(3x), |10|A(2x), |45|A(x), |120|C\}\end{aligned} \tag{2.23}$$

for all $x \in X$. By (2.22) and (2.23), we get

$$\begin{aligned}&\|420f(6x) - 3696f(5x) + 14280f(4x) - 31920f(3x) - (10! - 46620)f(2x) \\ &\quad +(352 \cdot 10! - 50400)f(x)\| \\ &\leq \max\{|90|\varphi(3x, x), |240|\varphi(2x, x), |240|A(3x), |90|A(2x), |2400|A(2x), B(x)\}\end{aligned} \tag{2.24}$$

for all $x \in X$. Replacing x and y by x and x in (2.8), respectively, then using (2.12) and (2.15), we have

$$\begin{aligned} &\|f(6x) - 10f(5x) + 46f(4x) - 130f(3x) + 255f(2x) - (10! + 372)f(x)\| \\ &\leq \max\{\varphi(x,x), |210|C, |120|A(x), |45|A(2x), |10|A(3x), A(4x)\} \end{aligned} \tag{2.25}$$

for all $x \in X$. By (2.24) and (2.25), we get

$$\begin{aligned} &\|504f(5x) - 5040f(4x) + 22680f(3x) - (10! + 60480)f(2x) + (772 \cdot 10! + 105840)f(x)\| \\ &\leq \max\{|90|\varphi(3x,x), |240|\varphi(2x,x), |420|\varphi(x,x), |420|A(4x), |240|A(3x), |4200|A(3x), \\ &\quad |90|A(2x), |2400|A(2x), B(x)\} := D(x) \end{aligned} \tag{2.26}$$

for all $x \in X$. Replacing x and y by 0 and x in (2.8), respectively, then using (2.12) and (2.15), we have

$$\begin{aligned} &\|2f(5x) - 20f(4x) + 90f(3x) - 240f(2x) - (10! - 420)f(x)\| \\ &\leq \max\{\varphi(0,x), |252|C, A(5x), |10|A(4x), |45|A(3x), |120|A(2x), |210|A(x)\} \end{aligned} \tag{2.27}$$

for all $x \in X$. By (2.26) and (2.27), we get

$$\|f(2x) - 2^{10}f(x)\| \leq \frac{1}{|10!|}\max\{|252|\varphi(0,x), |252|A(5x), |11340|A(3x), D(x)\} := \delta(x) \tag{2.28}$$

for all $x \in X$. Thus

$$\left\|\frac{1}{2^{10}}f(2x) - f(x)\right\| \leq \frac{1}{|2|^{10}}\delta(x), \qquad x \in X. \tag{2.29}$$

Similarly,

$$\left\|2^{10}f\left(\frac{x}{2}\right) - f(x)\right\| \leq \delta\left(\frac{x}{2}\right), \qquad x \in X. \tag{2.30}$$

Fix an $x \in X$ and write

$$\mathcal{T}\xi(x) := \frac{1}{2^{10s}}\xi(2^s x), \qquad \xi \in Y^X, \tag{2.31}$$

$$\varepsilon(x) := \begin{cases} \frac{1}{|2|^{10}}\delta(x), & \text{if } s = 1, \\ \delta\left(\frac{x}{2}\right), & \text{if } s = -1. \end{cases} \tag{2.32}$$

Then, by (2.29) and (2.30), we obtain

$$\|\mathcal{T}f(x) - f(x)\| \leq \varepsilon(x), \qquad x \in X. \tag{2.33}$$

Next, put

$$\Lambda\eta(x) := \frac{1}{|2|^{10s}}\eta(2^s x), \qquad \eta \in \mathbb{R}_+^X, x \in X. \tag{2.34}$$

It is easily seen that Λ has the form described in (H3) with $E = X, j = 1$ and $f_1(x) = 2^s x, L_1(x) = \frac{1}{|2|^{10s}}$ for $x \in X$. Moreover, for any $\xi, \mu \in Y^X$ and $x \in X$ we have

$$\begin{aligned} \|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| &= \left\|\frac{1}{2^{10s}}\xi(2^s x) - \frac{1}{2^{10s}}\mu(2^s x)\right\| \\ &\leq L_1(x)\|\xi(f_1(x)) - \mu(f_1(x))\|, \end{aligned} \tag{2.35}$$

so hypothesis (H2) is also valid.

Finally, using induction, one can check that for any $l \in \mathbb{N}_0$ and $x \in X$ we have

$$\begin{aligned} \Lambda^l\varepsilon(x) &= \left(\frac{1}{|2|^{10s}}\right)^l \varepsilon(2^{sl}x) \\ &= \left(\frac{1}{|2|^{10\frac{s+1}{2}}}\right)\left(\frac{1}{|2|^{10s}}\right)^l \varepsilon(2^{sl+\frac{s-1}{2}}x), \end{aligned} \tag{2.36}$$

which, together with (2.9), shows that all assumptions of Theorem 2.1 are satisfied. Therefore, there exists a function $F : X \to Y$ such that

$$F(x) = \left(\frac{1}{|2|^{10s}}\right)^l F(2^{sl}x), \qquad x \in X, \tag{2.37}$$

and (2.10) holds. Moreover,

$$F(x) = \lim_{l\to\infty} \mathcal{T}^l f(x), \qquad x \in X. \tag{2.38}$$

One can now show, by induction, that

$$\|D_{10}(\mathcal{T}^l f)(x, y)\| \le \left(\frac{1}{|2|^{10s}}\right)^l \varphi(2^{sl}x, 2^{sl}y) \tag{2.39}$$

for $l \in \mathbb{N}_0, x, y \in X$. Letting $l \to \infty$ in (2.39) and using (2.9), we obtain

$$D_{10}(f)(x, y) = 0, \tag{2.40}$$

which means that the function F satisfies equation (1.1). Thus the mapping $\mathcal{T} : X \to Y$ is decic. □

Theorem 2.2 with $\varphi(x, y) = \epsilon > 0, \epsilon(\|x\|^p + \|y\|^p), \epsilon\|x\|^p \cdot \|y\|^q$, respectively, and $s = -1$ yields the following results.

Corollary 2.1 *Let ϵ be a positive real number, X be a commutative group uniquely divisible by 2 and Y be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 210 such that $|2| < 1$. If $f : X \to Y$ be a mapping satisfying*

$$\|D_{10}(f)(x, y)\| \le \epsilon \tag{2.41}$$

for $x, y \in X$, then there exists a decic mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \le \frac{\epsilon}{|10!|^2} \tag{2.42}$$

for all $x \in X$.

Corollary 2.2 *Let p, ϵ be positive real numbers with $p < 10$, X be a non-Archimedean normed space and Y be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 210 such that $|2| < 1$. If $f : X \to Y$ be a mapping satisfying*

$$\|D_{10}(f)(x, y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{2.43}$$

for $x, y \in X$, then there exists a decic mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \le \frac{2\epsilon\|x\|^p}{|10!|^2} \tag{2.44}$$

for all $x \in X$.

Corollary 2.3 *Let p, q, ϵ be positive real numbers with $p+q < 10$, X be a non-Archimedean normed space and Y be a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from 210 such that $|2| < 1$. If $f : X \to Y$ be a mapping satisfying*

$$\|D_{10}(f)(x, y)\| \le \epsilon\|x\|^p \cdot \|y\|^q \tag{2.45}$$

for $x, y \in X$, then there exists a decic mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \le \frac{\epsilon\|x\|^{p+q}}{|10!|^2} \tag{2.46}$$

for all $x \in X$.

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Existence of positive solution for fully third-order boundary value problems *

Yongxiang Li and Elyasa Ibrahim

Department of Mathematics, Northwest Normal University,
Lanzhou 730070, People's Republic of China
e-mail: liyxnwnu@163.com, elyasa15@yeah.net

Abstract

In this paper, we are concerned with the existence of positive solutions of the fully third-order boundary value problem

$$\begin{cases} -u'''(t) = f(t,\, u(t),\, u'(t),\, u''(t)), & t \in [0,\, 1], \\ u(0) = u'(0) = u'(1) = 0, \end{cases}$$

where $f : [0,\, 1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is continuous. Some inequality conditions on f to guarantee the existence of positive solution are presented. These inequality conditions allow that $f(t,\, x,\, y,\, z)$ may be superlinear or sublinear growth on x, y and z as $|(x,\, y,\, z)| \to 0$ and $|(x,\, y,\, z)| \to \infty$.

Key Words: fully third-order boundary value problem; Nagumo-type growth condition; positive solution; cone; fixed point index.

AMS Subject Classification: 34B18; 47H11; 47N20.

1 Introduction

In this paper we discuss existence of positive solution for third-order boundary value problem(BVP) with fully nonlinear term

$$\begin{cases} -u'''(t) = f(t,\, u(t),\, u'(t),\, u''(t)), & t \in [0,\, 1], \\ u(0) = u'(0) = u'(1) = 0, \end{cases} \tag{1.1}$$

where $f : [0,\, 1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ is continuous.

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The boundary value problems of third order ordinary differential equations arise in a variety of different areas of applied mathematics and physics, as the deflection of a curved beam having a constant or varying cross section, three layer beam, electromagnetic waves or gravity driven flows and so on [1,2]. These problems have attracted many authors' attention and concern, and some theorems and methods of nonlinear functional analysis have been applied to research the solvability of these problem, such as the topological transversality [3], the monotone iterative technique [4-6], the method of upper and lower solutions[7-9], Leray-Schauder degree [10-13], the fixed point theory of increasing operator[14,15]. Especially, in recent years the fixed-point theorem of Krasnoselskii's cone expansion or compression type have been availably applied to some special third-order boundary problems that nonlinearity f doesn't contain derivative terms u' and u'', and some results of existence and multiplicity of positive solutions have been obtained, see [16-18]. However, few people consider the existence of the positive solutions for the more general third-order boundary problems that nonlinearity explicitly contains first-order or second-order derivative term.

The purpose of this paper is to obtain existence result of positive solution for B-VP (1.1) with full nonlinearity. We will use the fixed point index theory in cones to discuss this problem. We present some inequality conditions on f to guarantee the existence of positive solution. These inequality conditions allow that $f(t,\, x,\, y,\, z)$ may be superlinear or sublinear growth on x, y and z as $|(x,\, y,\, z)| \to 0$ and $|(x,\, y,\, z)| \to \infty$, where $|(x,\, y,\, z)| = \sqrt{x^2+y^2+z^2}$. For the superlinear growth case as $|(x,\, y,\, z)| \to \infty$, a Nagumo-type condition is presented to restrict the growth of f on z. We choose a proper cone K in the work space $C^2[0,\, 1]$ and convert the BVP(1.1) to a fixed point problem of a completely continuous cone mapping $A : K \to K$, then apply the fixed point index theory in cones and a-priori estimates in $C^2[0,\, 1]$ to prove our existence results.

Let $I = [0,\, 1]$, $G = I \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$. Our main results as follows:

Theorem 1.1 *Let $f : I \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ be continuous and satisfy the following conditions*

(F1) *There exist constants $a,\, b,\, c \geq 0$ and $\delta > 0$, $0 < \frac{a}{\sqrt{2}\pi^2} + \frac{b}{\pi^2} + \frac{c}{\pi} < 1$, such that*

$$f(t,\, x,\, y,\, z) \leq a\,x + b\,y + c\,|z|, \quad \text{for } (t,x,y,z) \in G \text{ such that } (x,y,z)| < \delta;$$

(F2) *there exists constants $a_1,\, b_1 \geq 0$ and $H > \delta$, $\frac{a_1}{12\pi^2} + \frac{2b_1}{\pi^4} > 1$, such that*

$$f(t,\, x,\, y,\, z) \geq a_1\,x + b_1\,y, \quad \text{for } (t,x,y,z) \in G \text{ such that } |(x,y,z)| > H;$$

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(F3) *Given any $M > 0$, there is a positive continuous function $g_M(\rho)$ on $\mathbb{R}^+$ satisfying*

$$\int_0^{+\infty} \frac{\rho\, d\rho}{g_M(\rho)+1} = +\infty, \tag{1.2}$$

such that

$$f(t,\, x,\, y,\, z) \le g_M(|z|), \quad (t, x, y, z) \in [0,\, 1] \times [0,\, M] \times [0,\, M] \times \mathbb{R}. \tag{1.3}$$

Then BVP(1.1) has at least one positive solution.

Theorem 1.2 *Let $f : I \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ be continuous and satisfy the following conditions*

(F4) *there exists constants $a,\, b \ge 0$ and $\delta > 0$, $\frac{a}{12\pi^2} + \frac{2b}{\pi^4} > 1$, such that*

$$f(t,\, x,\, y,\, z) \ge a\, x + b\, y, \quad \text{for } (t, x, y, z) \in G \text{ such that } |(x, y, z)| < \delta;$$

(F5) *There exist constants $a_1,\, b_1,\, c_1 \ge 0$ and $H > \delta$, $0 < \frac{a_1}{\sqrt{2}\pi^2} + \frac{b_1}{\pi^2} + \frac{c_1}{\pi} < 1$, such that*

$$f(t,\, x,\, y,\, z) \le a_1 x + b_1 y + c_1 |z|, \quad \text{for } (t, x, y, z) \in G \text{ such that } |(x, y, z)| > H;$$

Then BVP(1.1) has at least one positive solution.

In Theorem 1.1, the condition (F1) and (F2) allow that $f(t,\, x,\, y,\, z)$ is superlinear growth on x, y and z as $|(x,\, y,\, z)| \to 0$ and $|(x,\, y,\, z)| \to \infty$, respectively. The condition (F3) is a Nagumo type growth condition on z which restricts the growth of f on z is quadric. For example, the power function

$$f(t,\, x,\, y,\, z) = |x|^\alpha + |y|^\beta + |z|^\gamma \tag{1.4}$$

satisfies Condition (F1) and (F2) when $\alpha,\, \beta,\, \gamma > 1$. But only when $\gamma \le 2$, Condition (F3) holds. In Theorem 2.2, the condition (F4) and (F5) allow that $f(t,\, x,\, y,\, z)$ is sublinear growth on on x, y and z as $|(x,\, y,\, z)| \to 0$ and $|(x,\, y,\, z)| \to \infty$, respectively. For example, the power function defined by (1.4) satisfies Condition (F4) and (F5) when $0 < \alpha,\, \beta,\, \gamma < 1$.

The conditions (F1)-(F2) and (F4)-(F5) also allow that f may be asymptotically linear on x, y and z as $|(x,\, y,\, z)| \to 0$ and $|(x,\, y,\, z)| \to \infty$. Indeed, we have the following results:

Corollary 1.3 *Let $f : I \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ be continuous and satisfy the following conditions*

(H1) There exist constants $a,\, b,\, c \geq 0$, $\frac{a}{\sqrt{2}\pi^2} + \frac{b}{\pi^2} + \frac{c}{\pi} < 1$, such that

$$f(t,\, x,\, y,\, z) = a\,x + b\,y + c\,|z| + o(|(x, y, z)|), \quad \text{as } \ |(x, y, z)| \to 0;$$

(H2) there exists constants $a_1,\, b_1,\, c_1 > 0$, $\frac{a_1}{12\pi^2} + \frac{2b_1}{\pi^4} > 1$, such that

$$f(t,\, x,\, y,\, z) = a_1 x + b_1 y + c_1 |z| + o(|(x, y, z)|), \quad \text{as } \ |(x, y, z)| \to \infty.$$

Then BVP(1.1) has at least one positive solution.

Corollary 1.4 *Let $f : I \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ be continuous and satisfy the following conditions*

(H4) There exist constants $a,\, b,\, c > 0$, $\frac{a}{12\pi^2} + \frac{2b}{\pi^4} > 1$, such that

$$f(t,\, x,\, y,\, z) = a\,x + b\,y + c\,|z| + o(|(x, y, z)|), \quad \text{as } \ |(x, y, z)| \to 0;$$

(H5) There exist constants $a_1,\, b_1,\, c_1 \geq 0$, $\frac{a_1}{\sqrt{2}\pi^2} + \frac{b_1}{\pi^2} + \frac{c_2}{\pi} < 1$, such that

$$f(t,\, x,\, y,\, z) = a_1 x + b_1 y + c_1 |z| + o(|(x, y, z)|), \quad \text{as } \ |(x, y, z)| \to \infty.$$

Then BVP(1.1) has at least one positive solution.

In (H2) and (H5), $o(|(x, y, z)|)$ denote a term of f which is less than $|(x, y, z)|$ as $|(x, y, z)| \to \infty$, that is, $\lim_{|(x,y,z)| \to \infty} \frac{o(|(x,y,z)|)}{|(x,y,z)|} = 0$. We can easily obtain the following facts:

$$\text{(H1)} \Longrightarrow \text{(F1) holds}, \qquad \text{(H2)} \Longrightarrow \text{(F2) and (F3) hold};$$

$$\text{(H4)} \Longrightarrow \text{(F4) holds}, \qquad \text{(H5)} \Longrightarrow \text{(F5) holds}.$$

Hence, by Theorem 1.1 and Theorem 1.2, the conclusions of Corollary 1.3 and 1.4 hold.

The proofs of Theorem 1.1 and 1.2 will be given in Section 3. Some preliminaries to discuss BVP(1.1) are presented in Section 2. In section 4, we use Theorem 1.1 and 1.2 to induce two new existence results.

2 Preliminaries

Let $C(I)$ denote the Banach space of all continuous function $u(t)$ on I with the norm $\|u\|_C = \max_{t \in I} |u(t)|$. Generally, for $n \in \mathbb{N}$, we use $C^n(I)$ to denote the Banach space of all nth-order continuous differentiable function on I with the norm $\|u\|_{C^n} = \max\{\, \|u\|_C,\, \|u'\|_C,\, \cdots,\, \|u^{(n)}\|_C\}$. Let $C^+(I)$ be the cone of nonnegative functions in $C(I)$. Let $H = L^2(I)$ be the usual Hilbert space with the inner product $(u,\, v) =$

$\int_0^1 u(t)v(t)dt$ and the norm $\|u\|_2 = \left(\int_0^1 |u(t)|^2 dt\right)^{1/2}$. Let $H^n(I)$ be the usual Sobolev space. $u \in H^n(I)$ means that $u \in C^{n-1}(I)$, $u^{(n-1)}(t)$ is absolutely continuous on I and $u^{(n)} \in L^2(I)$. In $H^n(I)$, we use the norm $\|u\|_{H^n} = \max\{\, \|u\|_2,\, \|u'\|_2,\, \cdots,\, \|u^{(n)}\|_2\}$.

To discuss BVP(1.1), we firstly consider the corresponding linear boundary value problem (LBVP)

$$\begin{cases} -u'''(t) = h(t)\,, & t \in I, \\ u(0) = u'(0) = u'(1) = 0\,, \end{cases} \tag{2.1}$$

where $h \in L^2(I)$.

Lemma 2.1 *For every $h \in L^2(I)$, LBVP(2.1) has a unique solution $u := Sh \in H^3(I)$, which satisfies*

$$\|u\|_2 \le \frac{1}{\sqrt{2}}\,\|u'\|_2, \quad \|u'\|_2 \le \frac{1}{\pi}\,\|u''\|_2, \quad \|u''\|_2 \le \frac{1}{\pi}\,\|u'''\|_2. \tag{2.2}$$

Moreover, the solution operator $S : L^2(I) \to H^3(I)$ is a bounded linear operator. When $h \in C(I)$, the solution $u = Sh \in C^3(I)$, and the solution operator $S : C(I) \to C^2(I)$ is completely continuous.

Proof. Let $h \in H^2(I)$. It is well-known the linear second-order boundary value problem

$$\begin{cases} -v''(t) = h(t)\,, & t \in [0,\, 1], \\ v(0) = v(1) = 0\,, \end{cases} \tag{2.3}$$

has a unique solution $v \in H^2(I)$ given by

$$v(t) = \int_0^1 G(t,\, s)\, h(s)\, ds, \tag{2.4}$$

where $G(t,\, s)$ is the corresponding Green function

$$G(t,\, s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1\,, \\ s(1-t), & 0 \le s \le t \le 1\,. \end{cases} \tag{2.5}$$

Hence,

$$u(t) = \int_0^t v(\tau) d\tau = \int_0^t \int_0^1 G(\tau,\, s) h(s) ds d\tau := Sh(t) \tag{2.6}$$

belongs to $H^3(I)$ and is a unique solution of LBVP(2.1).

Since sine system $\{\sin k\pi t \mid k \in \mathbb{N}\}$ is a complete orthogonal system of $L^2(I)$, every $h \in L^2(I)$ can be expressed by the Fourier series expansion

$$h(t) = \sum_{k=1}^{\infty} b_k \, \sin k\pi t\,, \tag{2.7}$$

where $b_k = 2\int_0^1 h(s)\sin k\pi s\, ds,\ \ k = 1,\, 2,\, \cdots$, and the Paserval equality

$$\|h\|_2{}^2 = \frac{1}{2}\sum_{k=1}^{\infty} |b_k|^2 \tag{2.8}$$

holds. Since $u = Sh \in H^3(I)$, u' and u''' belong to $L^2(I)$ and they can also be expressed by the Fourier series expansion of the sine system. Since $u''' = -h$, by the integral formula of Fourier coefficient, we have

$$u'(t) = \sum_{k=1}^{\infty} \frac{b_k}{k^2\pi^2}\ \sin k\pi t\,. \tag{2.9}$$

On the other hand, since cosine system $\{\cos k\pi t \mid k = 0,\, 1,\, 2, \cdots\}$ is another complete orthogonal system of $L^2(I)$, every $w \in L^2(I)$ can be expressed by the cosine series expansion

$$w(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\, \cos k\pi t\,,$$

where $a_k = 2\int_0^1 w(s)\cos k\pi s\, ds,\ \ k = 0,\, 1,\, 2,\, \cdots$. For the $u'' \in L^2(I)$, by the integral formula of the coefficient of cosine series, we obtain its cosine series expansion:

$$u''(t) = \sum_{k=1}^{\infty} \frac{b_k}{k\pi}\ \cos k\pi t\,. \tag{2.10}$$

By (2.7), (2.9), (2.10) and Paserval equality, we obtain that

$$\|u'\|_2{}^2 = \frac{1}{2}\sum_{k=1}^{\infty}\left|\frac{b_k}{k^2\pi^2}\right|^2 \le \frac{1}{2\pi^2}\sum_{k=1}^{\infty}\left|\frac{b_k}{k\pi}\right|^2 = \frac{1}{\pi^2}\,\|u''\|_2{}^2\,,$$

$$\|u''\|_2{}^2 = \frac{1}{2}\sum_{k=1}^{\infty}\left|\frac{b_k}{k\pi}\right|^2 \le \frac{1}{2\pi^2}\sum_{k=1}^{\infty}|b_k|^2 = \frac{1}{\pi^2}\,\|h\|_2{}^2 = \frac{1}{\pi^2}\,\|u'''\|_2{}^2\,.$$

In addition, since $u(t) = \int_0^t u'(s)ds$, by Hölder inequality,

$$\|u\|_2{}^2 = \int_0^1 \Big|\int_0^t u'(s)ds\Big|^2 dt \le \int_0^1 t\int_0^t |u(s)|^2 ds dt \le \frac{1}{2}\|u'\|_2{}^2.$$

Hence (2.2) holds.

By the expression (2.6) of the solution $u = Sh$, $S : L^2(I) \to H^3(I)$ is a bounded linear operator. When $h \in C(I)$, by (2.4) and (2.6), $u \in C^3(I)$ and the solution operator $S : C(I) \to C^3(I)$ is bounded. By the compactness of the embedding $C^3(I) \hookrightarrow C^2(I)$, $S : C(I) \to C^2(I)$ is completely continuous. □

Lemma 2.2 *Let $h \in C^+(I)$. Then the solution u of LBVP(2.1) belongs to $C^3(I)$ and has the following properties:*

(1) $u \geq 0,\ u' \geq 0,\ u''' \leq 0$ and $\|u\|_C \leq \|u'\|_C \leq \|u''\|$;

(2) $u'(t) \geq t(1-t)\,\|u'\|_C\,,\ \forall\, t \in I;\quad \|u'\|_C \leq \frac{\pi^3}{4}\int_0^1 u'(t)\,\sin\pi t\,dt;$

(3) $u(t) \geq \frac{1}{6}t^2(3-2t)\,\|u'\|_C\,,\ \forall\, t \in I;\quad \|u'\|_C \leq 6\pi\int_0^1 u(t)\,\sin\pi t\,dt;$

(4) there exists $\xi \in (0,\,1)$ such that $u''(\xi) = 0$, $u''(t) \geq 0$ for $t \in [0,\,\xi]$ and $u''(t) \leq 0$ for $t \in [\xi,\,1]$. Moreover, $\|u''\|_C = \max\{u''(0),\,-u''(1)\}$.

Proof. Let $h \in C^+(I)$ and $u = Sh$ be the unique solution of BVP(2.1). By Lemma 2.1, $u \in C^3(I)$ and $u''' = -h \leq 0$. Set $v = u'$, then $v \in C^2(I)$ is a unique solution of LBVP(2.3) and given by (2.4). Hence, $v \geq 0$. For every $t \in I$, we have $u(t) = \int_0^1 v(s)ds \geq 0$, and

$$|u(t)| = \int_0^t v(s)\,ds \leq t\,\|v\|_C \leq \|u'\|_C.$$

Hence, $\|u\|_C \leq \|u'\|_C$. By the boundary conditions of LBVP(2.1), there exists $\xi \in (0,\,1)$ such that $u''(\xi) = 0$, and for every $t \in I$, $u'(t) = \int_\xi^t u''(s)\,ds$. Hence,

$$|u'(t)| = \Big|\int_\xi^t u''(s)\,ds\Big| \leq |t-\xi|\,\|u''\|_C \leq \|u''\|_C,$$

so we have $\|u'\|_C \leq \|u''\|_C$. Hence, the conclusion of Lemma 2.2(1) holds.

From the expression (2.5) we easily see that the Green function $G(t,s)$ has the following properties

(i) $0 \leq G(t,\,s) \leq G(s,\,s) \quad \forall\, t,\ s \in I$;

(ii) $G(t,\,s) \geq G(t,\,t)\,G(s,\,s), \quad \forall\, t,\ s \in I$.

For every $t \in I$, by (2.4) and the property (i) of G we have

$$v(t) = \int_0^1 G(t,\,s)\,h(s)\,ds \leq \int_0^1 G(s,\,s)\,h(s)\,ds.$$

Hence

$$\|v\|_C \leq \int_0^1 G(s,\,s)\,h(s)\,ds.$$

By the property (ii) of G and this inequality, we have

$$\begin{aligned} v(t) = \int_0^1 G(t,\,s)\,h(s)\,ds &\geq G(t,\,t)\int_0^1 G(s,\,s)\,h(s)\,ds \\ &\geq G(t,\,t)\,\|v\|_C = t(1-t)\,\|v\|_C, \quad t \in I. \end{aligned} \tag{2.11}$$

Multiplying this inequality by $\sin \pi t$ and integrating on I, we have

$$\int_0^1 v(t)\,\sin \pi t\,dt \geq \|v\|_C \int_0^1 t(1-t)\,\sin \pi t\,dt = \frac{4}{\pi^3}\,\|v\|_C\,.$$

Thus, the conclusion (2) holds.

By (2.11), we have

$$u(t) = \int_0^t v(s)\,ds \geq \int_0^t s(1-s)\|v\|_C ds = \frac{1}{6}t^2(3-2t)\|u'\|_C, \quad t \in I.$$

Multiplying this inequality by $\sin \pi t$ and integrating on I, we obtain that

$$\int_0^1 u(t)\,\sin \pi t\,dt \geq \frac{\|u'\|_C}{6} \int_0^1 t^2(3-2t)\,\sin \pi t\,dt = \frac{\|u'\|_C}{6\pi}.$$

Hence, the conclusion (3) holds.

Since $u' \geq 0$, from the boundary conditions of LBVP(2.1) we see that $u''(0) \geq 0$ and $u''(1) \geq 0$. Since $u'''(t) = -h(t) \geq 0$ for every $t \in I$, it follows that $u''(t)$ is a monotone increasing function on I. From these we conclude that, there exists $\xi \in (0,\,1)$ such that $u''(\xi) = 0$, $u''(t) \geq 0$ for $t \in [0,\,\xi]$ and $u''(t) \geq 0$ for $t \in [\xi,\,1]$. Moreover $\|u''\|_C = \max_{t\in I}|u''(t)| = \max\{u''(0),\,-u''(1)\}$. Hence, the conclusion of Lemma2.2(4) holds. □

Now, we define a closed convex cone K in $C^2(I)$ by

$$K = \big\{\, u \in C^2(I)\;:\; u(t) \geq 0,\;\; u'(t) \geq 0,\; \forall\, t \in I \,\big\}. \tag{2.12}$$

By Lemma 2.2(1), we have that $S(C^+(I)) \subset K$. Let $f : I \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ be continuous. For every $u \in K$, set

$$F(u)(t) := f(t,\,u(t),\,u'(t),\,u''(t)), \qquad t \in I. \tag{2.13}$$

Then $F : K \to C^+(I)$ is continuous and it maps every bounded in K into a bounded set in $C^+(I)$. Define a mapping $A : K \to K$ by

$$A = S \circ F. \tag{2.14}$$

By Lemma 2.1, $A : K \to K$ is a completely continuous mapping. By the definitions of S and K, the positive solution of BVP(1.1) is equivalent to the nonzero fixed point of A. We will find the nonzero fixed point of A by using the fixed point index theory in cones.

Let E be a Banach space and $K \subset E$ be a closed convex cone in E. Assume Ω is a bounded open subset of E with boundary $\partial\Omega$, and $K \cap \Omega \neq \emptyset$. Let $A : K \cap \overline{\Omega} \to K$ be a completely continuous mapping. If $Au \neq u$ for any $u \in K \cap \partial\Omega$, then the fixed point index $i\,(A, K \cap \Omega, K)$ is well defined. The following lemmas in [19, 20] are needed in our discussion.

Lemma 2.3 *Let Ω be a bounded open subset of E with $\theta \in \Omega$, and $A : K \cap \overline{\Omega} \to K$ a completely continuous mapping. If $\mu\, Au \neq u$ for every $u \in K \cap \partial\Omega$ and $0 < \mu \leq 1$, then $i\,(A,\, K \cap \Omega,\, K) = 1$.*

Lemma 2.4 *Let Ω be a bounded open subset of E and $A : K \cap \overline{\Omega} \to K$ a completely continuous mapping. If there exists $v_0 \in K \setminus \{\theta\}$ such that $u - Au \neq \tau\, v_0$ for every $u \in K \cap \partial\Omega$ and $\tau \geq 0$, then $i\,(A,\, K \cap \Omega,\, K) = 0$.*

Lemma 2.5 *Let Ω be a bounded open subset of E, and $A,\, A_1 : K \cap \overline{\Omega} \to K$ be two completely continuous mappings. If $(1-s)Au + sA_1u \neq u$ for every $u \in K \cap \partial\Omega$ and $0 \leq s \leq 1$, then $i\,(A,\, K \cap \Omega,\, K) = i\,(A_1,\, K \cap \Omega,\, K)$.*

3 Proof of the Main Results

In this section, we use the fixed point index theory in cones to prove Theorem 1.1 and 1.2. Let $E = C^2(I)$, $K \subset C^2(I)$ be the closed convex cone defined by (2.12) and $A : K \to K$ be the completely continuous mapping defined by (2.14). Then the positive solution of BVP(1.1) is equivalent to the nontrivial fixed point of A. We use Lemma 2.3-2.5 to find the nontrivial fixed point of A.

Proof of Theorem 1.1. Let $0 < r < R < +\infty$ and set

$$\Omega_1 = \{u \in C^2(I) \mid \|u\|_{C^2} < r\}, \qquad \Omega_2 = \{u \in C^2(I) \mid \|u\|_{C^2} < R\}. \tag{3.1}$$

We show that A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$ when r is small enough and R large enough.

Choose $r \in (0,\, \delta/\sqrt{3})$, where δ is the positive constant in Condition (F1). We prove that A satisfies the condition of Lemma 2.5 in $K \cap \partial\Omega_1$, namely

$$\mu\, Au \neq u, \qquad \forall\, u \in K \cap \partial\Omega_1,\;\; 0 < \mu \leq 1. \tag{3.2}$$

In fact, if (3.2) doesn't hold, there exist $u_0 \in K \cap \partial\Omega_1$ and $0 < \mu_0 \leq 1$ such that $\mu_0\, Au_0 = u_0$. Since $u_0 = S(\mu_0 F(u_0))$, by the definition of S, $u_0 \in C^3(I)$ is the unique solution of LBVP(2.1) for $h = \mu_0 F(u_0) \in C^+(I)$. Hence, $u_0 \in C^2(I)$ satisfies the differential equation

$$\begin{cases} -u_0'''(t) = \mu_0\, f(t,\, u_0(t),\, u_0'(t),\, u_0''(t)), & t \in [0,\, 1], \\ u_0(0) = u_0'(0) = u_0'(1) = 0\,. \end{cases} \tag{3.3}$$

Since $u_0 \in K \cap \partial\Omega_1$, by the definitions of K and Ω_1, we have

$$(t,\, u_0(t),\, u_0'(t),\, u_0''(t)) \in G, \quad |(u_0(t),\, u_0'(t),\, u_0''(t))| < \delta, \quad t \in I.$$

Hence by Condition (F1), we have

$$0 \le f(t,\, u_0(t),\, {u_0}'(t),\, {u_0}''(t)) \le a\, u_0(t) + b\, {u_0}'(t) + c\, |{u_0}''(t)|, \quad t \in I,$$

Combining this inequality with Equation (3.3), we obtain that

$$\begin{aligned} |{u_0}'''(t)| &= \mu_0\, f(t,\, u_0(t),\, {u_0}'(t),\, {u_0}''(t)) \\ &\le f(t,\, u_0(t),\, {u_0}'(t),\, {u_0}''(t)) \\ &\le a\, |u_0(t)| + b\, |{u_0}'(t)| + c\, |{u_0}''(t)|, \qquad t \in I. \end{aligned}$$

From this inequality and Lemma 2.1 it follows that

$$\|{u_0}'''\|_2 \le a\|u_0\|_2 + b\|{u_0}'\|_2 + c\|{u_0}''\|_2 \le \Big(\frac{a}{\sqrt{2}\pi^2} + \frac{b}{\pi^2} + \frac{c}{\pi}\Big)\|{u_0}'''\|_2. \tag{3.4}$$

Since $\|u_0\|_{C^2} > 0$, from boundary condition in Equation (3.3) we easily see that $\|{u_0}'''\|_2 > 0$. Hence by (3.5) we obtain that $\frac{a}{\sqrt{2}\pi^2} + \frac{b}{\pi^2} + \frac{c}{\pi} \ge 1$, which contradicts the assumption in Condition (F1). Hence (3.2) holds, namely A satisfies the condition of Lemma 2.3 in $K \cap \partial\Omega_1$. By Lemma 2.3, we have

$$i\,(A,\, K \cap \Omega_1,\, K) = 1. \tag{3.5}$$

Set $C_0 = \max\{\, |f(t,x,y,z) - (a_1\, x + b_1\, y)| : (t,x,y,z) \in G,\ |(x,\, y,\, z)| \le H\} + 1$, where H is the constant in Condition (F2). By Condition (F2), we have

$$f(t,\, x,\, y,\, z) \ge a_1\, x + b_1\, y - C_0, \qquad \forall\, (t,x,y,z) \in G. \tag{3.6}$$

Define a mapping $F_1 : K \to C^+(I)$ by

$$F_1(u)(t) := f(t,\, u(t),\, u'(t),\, u''(t)) + C_0 = F(u)(t) + C_0, \qquad t \in I, \tag{3.7}$$

and set

$$A_1 = S \circ F_1. \tag{3.8}$$

Then $A_1 : K \to K$ is a completely continuous mapping.

Let $R > \delta$. We show that A_1 satisfies that

$$i\,(A_1,\, K \cap \Omega_2,\, K) = 0. \tag{3.9}$$

Choose $v_0 = 1 - \cos \pi t$ and $w_0 = \pi^3 \sin \pi t$. since $-{v_0}'''(t) = \pi^3 \sin \pi t = w_0$, by the definition of S and Lemma 2.2(1), $v_0 = S(w_0) \in K \setminus \{\theta\}$. We show that A_1 satisfies the condition of Lemma 2.4 in $K \cap \partial\Omega_2$, namely

$$u - A_1 u \ne \tau\, v_0, \qquad \forall\, u \in K \cap \partial\Omega_2, \quad \tau \ge 0. \tag{3.10}$$

In fact, if (3.10) doesn't hold, there exist $u_1 \in K \cap \partial\Omega_2$ and $\tau_1 \geq 0$ such that $u_1 - A_1 u_1 = \tau_1 v_0$. Since $u_1 = A_1 u_1 + \tau_1 v_0 = S(F(u_1) + C_0 + \tau_1 w_0)$, by the definition of S, u_1 is the unique solution of LBVP(2.1) for $h = F(u_1) + C_0 + \tau_1 w_0 \in C^+(I)$. Hence, $u_1 \in C^3(I)$ satisfies the differential equation

$$\begin{cases} -u_1'''(t) = f(t,\, u_1(t),\, u_1'(t),\, u_1''(t)) + C_0 + \tau_1 w_0(t), \quad t \in I, \\ u_1(0) = u_1'(0) = u_1'(1) = 0\,. \end{cases} \tag{3.11}$$

Since $u_1 \in K \cap \partial\Omega_2$, by the definition of K, $(t,\, u_1(t),\, u_1'(t),\, u_1''(t)) \in G$, $t \in I$. Hence by (3.6), we have

$$f(t,\, u_1(t),\, u_1'(t),\, u_1''(t)) \geq a_1\, u_1(t) + b_1\, u_1'(t) - C_0, \qquad t \in I.$$

From this inequality and Equation (3.11), we conclude that

$$\begin{aligned} -u_1'''(t) \;&=\; f(t,\, u_1(t),\, u_1'(t),\, u_1''(t)) + C_0 + \tau_1 w_0(t) \\ &\geq\; a_1\, u_1(t) + b_1\, u_1'(t) + \tau_1 w_0(t) \\ &\geq\; a_1\, u_1(t) + b_1\, u_1'(t), \qquad t \in I. \end{aligned}$$

Multiplying this inequality by $\sin \pi t$ and integrating on I, then using integration by parts for the left side, we have

$$\pi^2 \int_0^1 u_1'(t)\, \sin \pi t\, dt \geq a_1 \int_0^1 u_1(t)\, \sin \pi t\, dt + b_1 \int_0^1 u_1'(t)\, \sin \pi t\, dt. \tag{3.12}$$

By Lemma 2.2 (2) and (3),

$$\int_0^1 u_1(t)\, \sin \pi t\, dt \geq \frac{1}{6\pi} \|u_1'\|_C, \quad \int_0^1 u_1'(t)\, \sin \pi t\, dt \geq \frac{4}{\pi^3} \|u_1'\|_C. \tag{3.13}$$

Since $\pi^2 \int_0^1 u_1'(t)\, \sin \pi t\, dt \leq 2\pi \|u_1'\|_C$, from (3.12) and (3.13) it follows that

$$\begin{aligned} 2\pi \|u_1'\|_C \geq \pi^2 &\int_0^1 u_1'(t) \sin \pi t\, dt \\ &\geq a_1 \int_0^1 u_1(t) \sin \pi t\, dt + b_1 \int_0^1 u_1'(t) \sin \pi t\, dt \\ &\geq \Big(\frac{a_1}{6\pi} + \frac{4b_1}{\pi^3} \Big) \|u_1'\|_C. \end{aligned}$$

Since $\|u_1'\|_C > 0$, by this inequality we obtain that $\frac{a_1}{12\pi^2} + \frac{2b_1}{\pi^4} \leq 1$, which contradicts the assumption in (F2). Hence (3.10) holds, namely A_1 satisfies the condition of Lemma 2.4 in $K \cap \partial\Omega_2$. By Lemma 2.4, (3.9) holds.

Next, we show that A and A_1 satisfy the condition of Lemma 2.5 in $K\cap\partial\Omega_2$ when R is large enough, namely

$$(1-s)Au+sA_1u\neq u,\qquad \forall\, u\in K\cap\partial\Omega_2,\quad 0\leq s\leq 1. \tag{3.14}$$

If (3.14) is not valid, there exist $u_2\in K\cap\partial\Omega_2$ and $s_0\in[0,\,1]$, such that $(1-s_0)Au_2+s_0A_1u_2=u_2$. Since $u_2=S((1-s_0)\,F(u_2)+s_0F_1(u_2))$, by the definition of S, u_2 is the unique solution of LBVP(2.1) for $h=(1-s_0)\,F(u_2)+s_0F_1(u_2)=F(u_2)+s_0C_0\in C^+(I)$. Hence, $u_2\in C^3(I)$ satisfies the differential equation

$$\begin{cases} -u_2{}'''(t)=f(t,\,u_2(t),\,u_2{}'(t),\,u_2{}''(t))+s_0\,C_0, & t\in I,\\ u_2(0)=u_2{}'(0)=u_2{}'(1)=0\,. \end{cases} \tag{3.15}$$

Since $u_2\in K\cap\partial\Omega_2$, by the definition of K, $(t,\,u_2(t),\,u_2{}'(t),\,u_2{}''(t))\in G$, $t\in I$. Hence by (3.6), we have

$$f(t,\,u_2(t),\,u_2{}'(t),\,u_2{}''(t))\geq a_1\,u_2(t)+b_1\,u_2{}'(t)-C_0,\qquad t\in I.$$

From this inequality and Equation (3.15), we obtain that

$$\begin{aligned} -u_2{}'''(t)&=f(t,\,u_2(t),\,u_2{}'(t),\,u_2{}''(t))+s_0\,C_0\\ &\geq a_1\,u_2(t)+b_1\,u_2{}'(t)-(1-s_0)\,C_0,\\ &\geq a_1\,u_2(t)+b_1\,u_2{}'(t)-C_0,\qquad t\in I. \end{aligned}$$

Multiplying this inequality by $\sin\pi t$ and integrating on I, then using integration by parts for the left side, we have

$$\pi^2\int_0^1 u_2{}'(t)\sin\pi t\,dt\geq a_1\int_0^1 u_2(t)\sin\pi t\,dt+b_1\int_0^1 u_2{}'(t)\sin\pi t\,dt-\frac{2C_0}{\pi}.$$

Using this inequality and Lemma 2.2 (2) and (3), we obtain that

$$\begin{aligned} 2\pi\|u_2{}'\|_C&\geq\pi^2\int_0^1 u_2{}'(t)\sin\pi t\,dt\\ &\geq a_1\int_0^1 u_2(t)\sin\pi t\,dt+b_1\int_0^1 u_2{}'(t)\sin\pi t\,dt-\frac{2C_0}{\pi}\\ &\geq\Big(\frac{a_1}{6\pi}+\frac{4b_1}{\pi^3}\Big)\|u_2{}'\|_C-\frac{2C_0}{\pi}. \end{aligned}$$

From this inequality it follows that

$$\|u_2{}'\|_C\leq\frac{C_0}{(\frac{a_1}{12\pi^2}+\frac{2b_1}{\pi^4}-1)\pi^2}:=M\,.$$

Hence, by Lemma 2.2(1) we obtain that

$$\|u_2\|_C \leq \|u_2'\|_C \leq M. \tag{3.16}$$

For this $M > 0$, by Assumption (F3), there is a positive continuous function $g_M(\rho)$ on $\mathbb{R}^+$ satisfying (1.2) such that (1.3) holds. By (3.16) and definition of K, $0 \leq u_2(t) \leq M$, $0 \leq u_2'(t) \leq M$, $t \in I$. Hence from (1.3) it follows that

$$f(t,\, u_2(t),\, u_2'(t),\, u_2''(t)) \leq g_M(|u_2''(t)|), \qquad t \in I.$$

Combining this inequality with Equation (3.15), we obtain that

$$-u_2'''(t) \leq g_M(|u_2''(t)|) + C_0, \qquad t \in I. \tag{3.17}$$

From (1.3) we easily obtain that

$$\int_0^{+\infty} \frac{\rho\, d\rho}{g_M(\rho) + C_0} = +\infty.$$

Hence there exists a positive constant $M_1 \geq M$ such that

$$\int_0^{M_1} \frac{\rho\, d\rho}{g_M(\rho) + C_0} > M. \tag{3.18}$$

By Lemma 2.2(4), there exists $\xi \in (0,\, 1)$ such that $u_2''(\xi) = 0$, $u_2''(t) \geq 0$ for $t \in [0,\, \xi]$, $u_2''(t) \leq 0$ for $t \in [\xi,\, 1]$, and $\|u_2''\|_C = \max\{u_2''(0),\, -u_2''(1)\}$. Hence $\|u_2''\|_C = u_2''(0)$ or $\|u_2''\|_C = -u_2''(1)$. We only consider the case of that $\|u_2''\|_C = u_2''(0)$, and the other case can be treated with a same way.

Since $u_2''(t) \geq 0$ for $t \in [0,\, \xi]$, multiplying both sides of the inequality (3.17) by $u_2''(t)$, we obtain that

$$\frac{-u_2'''(t)\, u_2''(t)}{g_M(u_2''(t)) + C_0} \ \leq\ u_2''(t), \qquad t \in [0,\, \xi].$$

Integrating both sides of this inequality on $[0,\, \xi]$ and making the variable transformation $\rho = u_2''(t)$ for the left side, we have

$$\int_0^{u_2''(0)} \frac{\rho\, d\rho}{g_M(\rho) + C_0} \leq u_2'(\xi) - u_2'(0) \leq \|u_2'\|_C.$$

Since $\|u_2''\|_C = u_2''(0)$, from this inequality and (3.16) it follows that

$$\int_0^{\|u_2''\|_C} \frac{\rho\, d\rho}{g_M(\rho) + C_0} \leq M.$$

Using this inequality and (3.18), we conclude that

$$\|u_2''\|_C \leq M_1. \tag{3.19}$$

Hence, from this inequality and (3.16) it follows that

$$\|u_2\|_{C^2} \leq M_1. \tag{3.20}$$

Let $R > \max\{M_1,\, \delta\}$. Since $u_2 \in K \cap \partial\Omega_2$, by the definition of Ω_2, $\|u_2\|_{C^2} = R > M_1$. This contradicts (3.20). Hence, (3.14) holds, namely A and A_1 satisfy the condition of Lemma 2.5 in $K \cap \partial\Omega_2$. By Lemma 2.5, we have

$$i\,(A,\, K \cap \Omega_2,\, K) = i\,(A_1,\, K \cap \Omega_2,\, K). \tag{3.21}$$

Hence, from (3.21) and (3.9) it follows that

$$i\,(A,\, K \cap \Omega_2,\, K) = 0. \tag{3.22}$$

Now using the additivity of the fixed point index, from (3.5) and (3.22), we conclude that

$$i\,(A,\, K \cap (\Omega_2 \setminus \overline{\Omega}_1),\, K) = i\,(A,\, K \cap \Omega_2,\, K) - i\,(A,\, K \cap \Omega_1,\, K) = -1.$$

Hence A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$, which is a positive solution of BVP(1.1). The proof of Theorem 1.1 is completed. □

Proof of Theorem 1.2. Let $\Omega_1,\, \Omega_2 \subset C^2(I)$ be defined by (3.1). We prove that the completely continuous mapping $A : K \to K$ defined by (2.14) has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$ when r is small enough and R large enough.

Let $r \in (0,\, \delta/\sqrt{3})$, where δ is the positive constant in Condition (F4). Choose $v_0 = 1 - cos\pi t$ and $w_0 = \pi^3 \sin \pi t$. Then $S(w_0) = v_0$, and $v_0 \in K \setminus \{\theta\}$. We show that A satisfies the condition of Lemma 2.4 in $K \cap \partial\Omega_1$, namely

$$u - Au \neq \tau\, v_0, \qquad \forall\, u \in K \cap \partial\Omega_1, \quad \tau \geq 0. \tag{3.23}$$

In fact, if (3.23) is not valid, there exist $u_0 \in K \cap \partial\Omega_1$ and $\tau_0 \geq 0$ such that $u_0 - Au_0 = \tau_0 v_0$. Since $u_0 = Au_0 + \tau_0 v_0 = S(F(u_0) + \tau_0 w_0)$, by definition of S, u_0 is the unique solution of LBVP(2.1) for $h = F(u_0) + \tau_0 w_0 \in C^+(I)$. Hence $u_0 \in C^3(I)$ satisfies the differential equation

$$\begin{cases} -u_0'''(t) = f(t,\, u_0(t),\, u_0'(t),\, u_0''(t)) + \tau_0 w_0(t), \quad t \in I, \\ u_0(0) = u_0'(0) = u_0'(1) = 0\,. \end{cases} \tag{3.24}$$

Since $u_0 \in K \cap \partial\Omega_1$, by the definitions of K and Ω_1, we have

$$(t,\, u_0(t),\, u_0'(t),\, u_0''(t)) \in G, \quad |(u_0(t),\, u_0'(t),\, u_0''(t))| < \delta, \quad t \in I.$$

Hence by Condition (F5) we have

$$f(t,\, u_0(t), {u_0}'(t), {u_0}''(t)) \ge a\, u_0(t) + b\, {u_0}'(t), \qquad t \in I.$$

From this inequality and Equation (3.24) it follows that

$$-{u_0}'''(t) \ge a\, u_0(t) + b\, {u_0}'(t), \qquad t \in I.$$

Multiplying this inequality by $\sin \pi t$ and integrating on I, then using integration by parts for the left side, we have

$$\pi^2 \int_0^1 {u_0}'(t) \sin \pi t\, dt \;\ge\; a \int_0^1 u_0(t) \sin \pi t\, dt + b \int_0^1 {u_0}'(t) \sin \pi t\, dt.$$

Using this inequality and Lemma 2.2 (2) and (3), we obtain that

$$\begin{aligned} 2\pi \|{u_0}'\|_C &\ge \pi^2 \int_0^1 {u_0}'(t)\, \sin \pi t\, dt \\ &\ge a_1 \int_0^1 u_0(t)\, \sin \pi t\, dt + b_1 \int_0^1 {u_0}'(t)\, \sin \pi t\, dt \\ &\ge \Big(\frac{a_1}{6\pi} + \frac{4b_1}{\pi^3}\Big) \|{u_0}'\|_C. \end{aligned}$$

Since $\|u_0\|_C > 0$, from this inequality it follows that $\frac{a_1}{12\pi^2} + \frac{2b_1}{\pi^4} \le 1$, which contradicts the assumption in (F4). Hence (3.23) holds, namely A satisfies the condition of Lemma 2.4 in $K \cap \partial\Omega_1$. By Lemma 2.4, we have

$$i\,(A,\, K \cap \Omega_1,\, K) = 0. \tag{3.25}$$

Let $R > \delta$ be large enough. We show that A satisfies the condition of Lemma 2.3 in $K \cap \partial\Omega_2$, namely

$$\mu\, Au \ne u, \qquad \forall\, u \in K \cap \partial\Omega_2, \quad 0 < \mu \le 1. \tag{3.26}$$

In fact, if (3.26) is not valid, there exist $u_1 \in K \cap \partial\Omega_2$ and $0 < \mu_1 \le 1$ such that $\mu_1\, Au_1 = u_1$. Since $u_1 = S(\mu_1 F(u_1))$, by the definition of S, $u_1 \in C^3(I)$ is the unique solution of LBVP(2.1) for $h = \mu_1 F(u_1) \in C^+(I)$. Hence $u_1 \in C^3(I)$ satisfies the differential equation

$$\begin{cases} -{u_1}'''(t) = \mu_1 f(t,\, u_1(t),\, {u_1}'(t),\, {u_1}''(t)), \quad t \in I, \\ u_1(0) = {u_1}'(0) = {u_1}'(1) = 0\,. \end{cases} \tag{3.27}$$

Set $C_1 = \max\{|f(t,x,y,z) - (a_1\, x + b_1\, y + c_1|z|)| : (t,x,y,z) \in G,\ |(x,\, y,\, z)| \le H\} + 1$, where H is the constant in Condition (F5). Then by Condition (F5), we have

$$f(t,\, x,\, y,\, z) \le a_1\, x + b_1\, y + c_1|z| + C_1, \qquad \forall\, (t,x,y,z) \in G. \tag{3.28}$$

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Since $u_1 \in K \cap \partial\Omega_2$, by the definition of K, $(t,\, u_1(t),\, {u_1}'(t),\, {u_1}''(t)) \in G$, $t \in I$. Hence by (3.28), we have

$$f(t,\, u_1(t),\, {u_1}'(t),\, {u_1}''(t)) \leq a_1\, u_1(t) + b_1\, {u_1}'(t) + c_1|{u_1}''(t)| + C_1, \qquad t \in I.$$

From this inequality with Equation (3.3), we obtain that

$$\begin{aligned}|{u_1}'''(t)| &= \mu_1\, f(t,\, u_1(t),\, {u_1}'(t),\, {u_1}''(t))\\ &\leq f(t,\, u_1(t),\, {u_1}'(t),\, {u_1}''(t))\\ &\leq a_1\, u_1(t) + b_1\, {u_1}'(t) + c_1|{u_1}''(t)| + C_1, \qquad t \in I.\end{aligned}$$

Using this inequality and Lemma 2.1, we have

$$\begin{aligned}\|{u_1}'''\|_2 &\leq a_1\|u_1\|_2 + b_1\|{u_1}'\|_2 + c_1\|{u_1}''\|_2 + C_1\\ &\leq \Big(\frac{a_1}{\sqrt{2}\pi^2} + \frac{b_1}{\pi^2} + \frac{c_1}{\pi}\Big)\|{u_1}'''\|_2 + C_1.\end{aligned}$$

Consequently,

$$\|{u_1}'''\|_2 \leq \frac{C_1}{1 - \big(\frac{a_1}{\sqrt{2}\pi^2} + \frac{b_1}{\pi^2} + \frac{c_1}{\pi}\big)} := R_1.$$

Hence by (2.2), we have

$$\|u_1\|_{H^3} = \max\{\|u_1\|_2,\, \|{u_1}'\|_2,\, \|{u_1}''\|_2,\, \|{u_1}'''\|_2\} = \|{u_1}'''\|_2 \leq R_1.$$

By this estimate and the boundedness of Sobolev embedding $H^3(I) \hookrightarrow C^2(I)$, we have

$$\|u_1\|_{C^2} \leq C\,\|u_1\|_{H^3} \leq CR_1 := R_2, \tag{3.29}$$

where C is the Sobolev embedding constant.

Choose $R > \max\{R_2,\, \delta\}$. Since $u_1 \in K \cap \partial\Omega_2$, by the definition of Ω_2, we see that $\|u_1\|_{C^2} = R > R_2$, which contradicts (3.29). Hence, (3.26) holds, namely A satisfies the condition of Lemma 2.3 in $K \cap \partial\Omega_2$. By Lemma 2.3, we have

$$i\,(A,\, K \cap \Omega_2,\, K) = 1. \tag{3.30}$$

Now, from (3.25) and (3.30) it follows that

$$i\,(A,\, K \cap (\Omega_2 \setminus \overline{\Omega}_1),\, K) = i\,(A,\, K \cap \Omega_2,\, K) - i\,(A,\, K \cap \Omega_1,\, K) = 1.$$

Hence A has a fixed-point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$, which is a positive solution of BVP(1.1). The proof of Theorem 1.2 is completed. □

4 Applications

In Theorem 1.1 and Theorem 1.2, we use the inequality conditions to describe the growth of the nonlinearity f as $|(x,y,z)| \to 0$ and $|(x,y,z)| \to \infty$. These inequality conditions can be replaced by the following upper and lower limits:

$$\begin{aligned} f_0 &= \liminf_{|(x,y,z)|\to 0} \min_{t\in I} \frac{f(t,\,x,\,y,\,z)}{|(x,y,z)|}, \quad f^0 = \limsup_{|(x,y,z)|\to 0} \max_{t\in I} \frac{f(t,\,x,\,y,\,z)}{|(x,y,z)|}, \\ f_\infty &= \liminf_{|(x,y,z)|\to \infty} \min_{t\in I} \frac{f(t,\,x,\,y,\,z)}{|(x,y,z)|}, \quad f^\infty = \limsup_{|(x,y,z)|\to \infty} \max_{t\in I} \frac{f(t,\,x,\,y,\,z)}{|(x,y,z)|}. \end{aligned} \tag{4.1}$$

Set

$$A = \frac{\sqrt{2}\pi^2}{1+\sqrt{2}(1+\pi)}, \qquad B = \frac{12\sqrt{3}\pi^4}{\pi^2+6}. \tag{4.2}$$

By the definition (4.1), we can verify that

$$\begin{aligned} f^0 &< A \implies \text{(F1) holds;} \\ f_\infty &> B \implies \text{(F2) holds;} \\ f_0 &> B \implies \text{(F4) holds;} \\ f^\infty &< A \implies \text{(F5) holds.} \end{aligned}$$

We only show the third assertion, and the other assertions can be showed with a similar way. Since $f_0 > B$, we may choose positive constant $\sigma > 0$ such that $f_0 > B+\sigma$. By definition f_0, there exists $\delta > 0$ such that

$$\frac{f(t,\,x,\,y,\,z)}{|(x,y,z)|} > B+\sigma, \qquad t\in I, \quad 0<|(x,y,z)|<\delta.$$

This implies that

$$f(t,\,x,\,y,\,z) > \frac{B+\sigma}{\sqrt{3}}(|x|+|y|+|z|), \qquad t\in I, \quad 0<|(x,y,z)|<\delta.$$

Choose $a = b = \frac{B+\sigma}{\sqrt{3}}$. Then $\frac{a}{12\pi^2} + \frac{2b}{\pi^4} = \frac{\pi^2+6}{12\sqrt{3}\pi^4}(B+\sigma) > 1$. The above inequality means that (F4) holds for these a, b and δ.

Hence, by Theorem 1.1 and Theorem 1.2, we obtain that

Theorem 4.1 *Let $f : I\times\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}^+$ be continuous. If f satisfies Assumption* (F3) *and the following condition*

(C1) $f^0 < A$, $f_\infty > B$,

then BVP(1.1) has at least one positive solution.

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Theorem 4.2 *Let $f : I \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ be continuous and satisfy the following condition*

(C2) $f_0 > B, \quad f^\infty < A.$

Then BVP(1.1) has at least one positive solution.

Example 4.1 Consider the superlinear third-order boundary value problem

$$\begin{cases} -u'''(t) = u^4(t) + (u'(t))^4 + (u'''(t))^2, & t \in [0,\, 1], \\ u(0) = u'(0) = u'(1) = 0\,. \end{cases} \tag{4.3}$$

We easily verify that the corresponding nonlinearity

$$f(t,\, x,\, y,\, z) = z^4 + y^4 + z^2$$

satisfies the conditions (F3) and (C1). By Theorem 4.1, the equation (4.3) has at least one positive solution.

Example 4.2 Consider the sublinear third-order boundary value problem

$$\begin{cases} -u'''(t) = \sqrt[3]{|u(t)|^2 + |u'(t)|^2 + |u''(t)|^2}, & t \in [0,\, 1], \\ u(0) = u'(0) = u'(1) = 0\,. \end{cases} \tag{4.4}$$

It is easy to see that the corresponding nonlinearity

$$f(t,\, x,\, y,\, z) = \sqrt[3]{|x|^2 + |y|^2 + |z|^2}$$

satisfies the condition (C2). By Theorem 4.2, the equation (4.4) has at least one positive solution.

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Alghamdi et al. Iteration Scheme for Hemicontractive Operators in Arbitrary Banach Spaces

Shin Min Kang[1,2], Arif Rafiq[3], Young Chel Kwun[4,*] and Faisal Ali[5]

[1]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

[2]Center for General Education, China Medical University, Taichung 40402, Taiwan

[3]Department of Mathematics and Statistics, Virtual University of Pakistan, Lahore 54000, Pakistan
e-mail: aarafiq@gmail.com

[4]Department of Mathematics, Dong-A University, Busan 49315, Korea
e-mail: yckwun@dau.ac.kr

[5]Center for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan 60800, Pakistan
e-mail: faisalali@bzu.edu.pk

Abstract

The purpose of this paper is to characterize the conditions for the convergence of the iterative scheme in the sense of Alghamdi et al. [The implicit midpoint rule for nonexpansive mappings, *Fixed Point Theory Appl.*, **2014** (2014), Article ID 96, 9 pages] associated with ϕ-hemicontractive mappings in a nonempty convex subset of an arbitrary Banach space.

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1 Introduction and Preliminaries

Let K be a nonempty subset of an arbitrary Banach space X and X^* be its dual space. The symbols $D(T)$ and $F(T)$ stand for the domain and the set of fixed points of T (for a single-valued map $T : X \to X$, $x \in X$ is called a fixed point of T iff $Tx = x$). We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}.$$

*Corresponding author

Let $T : D(T) \subseteq X \to X$ be an operator.

Definition 1.1. T is called *Lipshitzian* if there exists $L > 1$ such that

$$\|Tx - Ty\| \leqslant L \|x - y\|,$$

for all $x, y \in K$. If $L = 1$, then T is called *non-expansive* and if $0 \leqslant L < 1$, T is called *contraction.*

Definition 1.2. ([2,4,6]) (1) T is said to be *strongly pseudocontractive* if there exists a $t > 1$ such that for each $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ satisfying

$$\operatorname{Re} \langle Tx - Ty, j(x - y)\rangle \leq \frac{1}{t} \|x - y\|^2.$$

(2) T is said to be *strictly hemicontractive* if $F(T) \neq \emptyset$ and if there exists a $t > 1$ such that for each $x \in D(T)$ and $q \in F(T)$, there exists $j(x - y) \in J(x - y)$ satisfying

$$\operatorname{Re} \langle Tx - q, j(x - q)\rangle \leq \frac{1}{t} \|x - q\|^2.$$

(3) T is said to be *ϕ-strongly pseudocontractive* if there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that for each $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ satisfying

$$\operatorname{Re} \langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \phi(\|x - y\|) \|x - y\|.$$

(4) T is said to be *ϕ-hemicontractive* if $F(T) \neq \emptyset$ and if there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that for each $x \in D(T)$ and $q \in F(T)$, there exists $j(x - y) \in J(x - y)$ satisfying

$$\operatorname{Re} \langle Tx - q, j(x - q)\rangle \leq \|x - q\|^2 - \phi(\|x - q\|) \|x - q\|.$$

Clearly, each strictly hemicontractive operator is ϕ-hemicontractive.

For a nonempty convex subset K of a normed space X, $T : K \to K$ is an operator

(a) the Mann iteration scheme [9] is defined by the following sequence $\{x_n\}$:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1 - b_n) x_n + b_n T x_n, \quad n \geq 1, \end{cases}$$

where $\{b_n\}$ is a sequence in $[0, 1]$.

(b) the sequence $\{x_n\}$ defined by

$$\begin{cases} x_1 \in K, \\ y_n = (1 - b'_n) x_n + b'_n T x_n, \\ x_{n+1} = (1 - b_n) x_n + b_n T y_n, \quad n \geq 1, \end{cases}$$

where $\{b_n\}$ and $\{b'_n\}$ are sequences in $[0, 1]$ is known as the Ishikawa iteration scheme [4].

Chidume [2] established that the Mann iteration sequence converges strongly to the unique fixed point of T in case T is a Lipschitz strongly pseudo-contractive mapping from a bounded closed convex subset of L_p (or l_p) into itself. Afterwards, several authors generalized this result of Chidume in various directions [3, 5–8, 11, 12, 15, 16].

For a finite family of nonexpansive mappings $\{T_i : i \in \{1, 2, \ldots, N\}\}$ with a real sequence $\{t_n\} \in (0, 1)$, and $\varrho_0 \in X$, where X is an arbitrary Banach space, the following implicit iteration process is due to Xu and Ori [14]:

$$\begin{aligned} x_1 &= (1-t_1)x_0 + t_1 T_1 x_1, \\ x_2 &= (1-t_2)x_1 + t_2 T_2 x_2, \\ &\vdots \\ x_N &= (1-t_N)x_{N-1} + t_N T_N x_N, \\ x_{N+1} &= (1-t_{N+1})x_N + t_{N+1} T_{N+1} x_{N+1}, \\ &\vdots \end{aligned}$$

which can be written in the following compact form:

$$x_n = (1-t_n)x_{n-1} + t_n T_n x_n, \text{ for all } n \geq 1, \tag{XO}$$

where $T_n = T_{n(mod\, N \in \{1,2,\ldots,N\})}$. For the common fixed point of the finite family of nonexpansive mappings defined in a Hilbert space, Xu and Ori [14] proved the weak convergence of the implicit iteration process.

Lately Alghamdi et al. [1] introduced the following iteration process in a Hilbert space H:

Algorithm 1.3. *Initialize $x_0 \in H$ arbitrarily and define*

$$x_{n+1} = (1-t_n)x_n + t_n T\left(\frac{x_n + x_{n+1}}{2}\right),$$

where $t_n \in (0, 1)$ for all $n \in \mathbb{N} \cup \{0\}$

For the approximation of fixed points of nonexpansive mappings under the setting of Hilbert spaces, they proved the following results:

Lemma 1.4. *Let $\{x_n\}$ be the sequence generated by Algorithm 1.3. Then*

(i) $\|x_{n+1} - p\| \leq \|x_n - p\|$ *for all $n \geq 0$ and $p \in F(T)$,*

(ii) $\sum_{n=1}^{\infty} t_n \|x_n - x_{n+1}\|^2 < \infty$,

(iii) $\sum_{n=1}^{\infty} t_n(1-t_n)\|x_n - T(\frac{x_n + x_{n+1}}{2})\|^2 < \infty$.

Lemma 1.5. *Let $\{x_n\}$ be the sequence generated by Algorithm 1.3. Suppose that $t_{n+1}^2 \leq a t_n$ for all $n \geq 0$ and $a > 0$. Then*

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Lemma 1.6. *Assume that*

(i) $t_{n+1}^2 \leq at_n$ *for all* $n \geq 0$ *and* $a > 0$,

(ii) $\liminf_{n\to\infty} t_n > 0$.

Then the sequence $\{x_n\}$ *generated by Algorithm 1.3 satisfies the property*

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Theorem 1.7. *Let* H *be a Hilbert space and* $T : H \to H$ *be a nonexpansive mapping with* $F(T) \neq \emptyset$*. Assume that* $\{x_n\}$ *is generated by Algorithm 1.3, where the sequence* $\{t_n\}$ *of parameters satisfies the conditions* (i) *and* (ii) *of Lemma 1.6.*

Then $\{x_n\}$ *converges weakly to a fixed point of* T*.*

The purpose of this paper is to characterize conditions for the convergence of the iterative scheme in the sense of Alghamdi et al. [1] associated with ϕ-hemicontractive mappings in a nonempty convex subset of an arbitrary Banach space. Our results improve and generalize most results in recent literature [1–3, 6–8, 15, 16].

2 Main results

The following result is now well known.

Lemma 2.1. [13] *For all* $x, y \in X$ *and* $j(x+y) \in J(x+y)$,

$$\|x+y\|^2 \leq \|x\|^2 + 2Re\,\langle y, j(x+y)\rangle\,.$$

Now we prove our main results.

Theorem 2.2. *Let* K *be a nonempty closed and convex subset of an arbitrary Banach space* X *and* $T : K \to K$ *be continuous* ϕ*-hemicontractive mappings. For any* $x_1 \in K$*, define the sequence* $\{x_n\}_{n=1}^{\infty}$ *inductively as follows:*

$$x_n = (1-t_n)x_{n-1} + t_n T\left(\frac{x_{n-1}+x_n}{2}\right), \quad n \geq 1, \tag{2.1}$$

where $\{t_n\}_{n=1}^{\infty}$ *is a sequence in* $[0,1]$ *satisfying the following conditions*

(i) $\lim_{n\to\infty} t_n = 0$ *and*

(ii) $\sum_{n=1}^{\infty} t_n = \infty$.

Then the following conditions are equivalent:

(a) $\{x_n\}_{n=1}^{\infty}$ *converges strongly to the fixed point* q *of* T*.*

(b) $\left\{T\left(\frac{x_{n-1}+x_n}{2}\right)\right\}_{n=1}^{\infty}$ *is bounded.*

Proof. First we prove that (a) implies (b).

Since T is ϕ-hemicontractive, it follows that $F(T)$ is a singleton. Let $F(T) = \{q\}$ for some $q \in K$.

Suppose that $\lim_{n\to\infty} x_n = q$. Then the continuity of T yield that

$$\lim_{n\to\infty} T\left(\frac{x_{n-1}+x_n}{2}\right) = q.$$

Therefore $\{T(\frac{x_{n-1}+x_n}{2})\}_{n=1}^{\infty}$ is bounded.

Second we need to show that (b) implies (a).

Put

$$M_1 = \|x_0 - q\| + \sup_{n\geq 1}\left\|T\left(\frac{x_{n-1}+x_n}{2}\right) - q\right\|.$$

Obviously $M_1 < \infty$. It is clear that $\|x_0 - q\| \leq M_1$. Let $\|x_{n-1} - q\| \leq M_1$. Next we will prove that $\|x_n - q\| \leq M_1$.

Consider

$$\begin{aligned}
\|x_n - q\| &= \left\|(1-t_n)x_{n-1} + t_n T\left(\frac{x_{n-1}+x_n}{2}\right) - q\right\| \\
&= \left\|(1-t_n)(x_{n-1}-q) + t_n(T\left(\frac{x_{n-1}+x_n}{2}\right) - q)\right\| \\
&\leq (1-t_n)\|x_{n-1}-q\| + t_n\left\|T\left(\frac{x_{n-1}+x_n}{2}\right) - q\right\| \\
&\leq (t_n + (1-t_n))M_1 \\
&= M_1.
\end{aligned}$$

So, from the above discussion, we can conclude that the sequence $\{x_n - p\}_{n\geq 0}$ is bounded. Thus there is a constant $M_2 > 0$ satisfying

$$M_2 = \sup_{n\geq 1}\|x_n - q\| + \sup_{n\geq 1}\left\|T\left(\frac{x_{n-1}+x_n}{2}\right) - q\right\|. \tag{2.2}$$

Denote $M = M_1 + M_2$. Obviously $M < \infty$.

Let $w_n = \left\|Tx_n - T(\frac{x_{n-1}+x_n}{2})\right\|$ for each $n \geq 1$. The continuity of T ensures that

$$\lim_{n\to\infty} w_n = 0, \tag{2.3}$$

because

$$\begin{aligned}
\left\|x_n - \frac{x_{n-1}+x_n}{2}\right\| &= \frac{1}{2}\|x_{n-1} - x_n\| \\
&= \frac{1}{2}t_n\left\|x_{n-1} - T\left(\frac{x_{n-1}+x_n}{2}\right)\right\| \\
&\leq Mt_n \\
&\to 0
\end{aligned}$$

as $n \to \infty$.

By virtue of Lemma 3 and (2.1), we infer that

$$\begin{aligned}
\|x_n - q\|^2 &= \left\|(1-t_n)x_{n-1} + t_n T\left(\frac{x_{n-1}+x_n}{2}\right) - q\right\| \\
&= \left\|(1-t_n)(x_{n-1}-q) + t_n(T\left(\frac{x_{n-1}+x_n}{2}\right) - q)\right\|^2 \\
&\le (1-t_n)^2\|x_{n-1}-q\|^2 + 2t_n\mathrm{Re}\left\langle T\left(\frac{x_{n-1}+x_n}{2}\right) - q, j(x_n-q)\right\rangle \\
&\le (1-t_n)^2\|x_{n-1}-q\|^2 + 2t_n\mathrm{Re}\left\langle Tx_n - T\left(\frac{x_{n-1}+x_n}{2}\right), j(x_n-q)\right\rangle \\
&\quad + 2t_n\mathrm{Re}\left\langle Tx_n - q, j(x_n-q)\right\rangle \\
&\le (1-t_n)^2\|x_{n-1}-q\|^2 + 2t_n\left\|Tx_n - T\left(\frac{x_{n-1}+x_n}{2}\right)\right\|\|x_n-q\| \\
&\quad + 2t_n\|x_n-q\|^2 - 2t_n\phi(\|x_n-q\|)\|x_n-q\| \\
&\le (1-t_n)^2\|x_{n-1}-q\|^2 + 2Mt_nw_n + 2t_n\|x_n-q\|^2 \\
&\quad - 2t_n\phi(\|x_n-q\|)\|x_n-q\|.
\end{aligned} \tag{2.4}$$

Also

$$\begin{aligned}
\|x_n - q\|^2 &= \left\|(1-t_n)x_{n-1} + t_n T\left(\frac{x_{n-1}+x_n}{2}\right) - q\right\|^2 \\
&= \left\|(1-t_n)(x_{n-1}-q) + t_n(T\left(\frac{x_{n-1}+x_n}{2}\right) - q)\right\|^2 \\
&\le \left((1-t_n)\|x_{n-1}-p\| + t_n\left\|T\left(\frac{x_{n-1}+x_n}{2}\right) - p\right\|\right)^2 \\
&\le (1-t_n)\|x_{n-1}-p\|^2 + t_n\left\|T\left(\frac{x_{n-1}+x_n}{2}\right) - p\right\|^2 \\
&\le (1-t_n)\|x_{n-1}-p\|^2 + M^2t_n,
\end{aligned} \tag{2.5}$$

where the second inequality holds by the convexity of $\|\cdot\|^2$.

By substituting (2.5) in (2.4), we get

$$\begin{aligned}
\|x_n - q\|^2 &\le \left((1-t_n)^2 + 2t_n(1-t_n)\right)\|x_{n-1}-p\|^2 + 2Mt_n(w_n + Mt_n) \\
&\quad - 2t_n\phi(\|x_n-q\|)\|x_n-q\| \\
&= \left(1-t_n^2\right)\|x_{n-1}-p\|^2 + 2Mt_n(w_n + Mt_n) \\
&\quad - 2t_n\phi(\|x_n-q\|)\|x_n-q\| \\
&\le \|x_{n-1}-p\|^2 + 2Mt_n(w_n + Mt_n) \\
&\quad - 2t_n\phi(\|x_n-q\|)\|x_n-q\| \\
&= \|x_{n-1}-q\|^2 + t_nl_n - 2t_n\phi(\|x_n-q\|)\|x_n-q\|,
\end{aligned} \tag{2.6}$$

where

$$l_n = 2M(w_n + Mt_n) \to 0 \tag{2.7}$$

as $n \to \infty$.

Let $\delta = \inf\{\|x_n - q\| : n \geq 0\}$.

We claim that $\delta = 0$. Otherwise $\delta > 0$. Thus (2.7) implies that there exists a positive integer $N_1 > N_0$ such that $l_n < \phi(\delta)\delta$ for each $n \geq N_1$. In view of (2.6), we conclude that

$$\|x_n - q\|^2 \leq \|x_{n-1} - q\|^2 - \phi(\delta)\delta t_n, \quad n \geq N_1,$$

which implies that

$$\phi(\delta)\delta \sum_{n=N_1}^{\infty} t_n \leq \|x_{N_1} - q\|^2, \tag{2.8}$$

which contradicts (ii). Therefore $\delta = 0$. Thus there exists a subsequence $\{x_{n_i}\}_{n=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$such that

$$\lim_{i\to\infty} x_{n_i} = q. \tag{2.9}$$

Let $\epsilon > 0$ be a fixed number. By virtue of (2.7)and (2.9), we can select a positive integer $i_0 > N_1$ such that

$$\left\|x_{n_{i_0}} - q\right\| < \epsilon, \quad l_n < \phi(\epsilon)\epsilon, \, n \geq n_{i_0}. \tag{2.10}$$

Let $p = n_{i_0}$. By induction, we show that

$$\|x_{p+m} - q\| < \epsilon, \quad m \geq 1. \tag{2.11}$$

Observe that (2.6) means that (2.11) is true for $m = 1$. Suppose that (2.11) is true for some $m \geq 1$. If $\|x_{p+m} - q\| \geq \epsilon$, by (2.6) and (2.10) we know that

$$\begin{aligned}
\epsilon^2 &\leq \|x_{p+m} - q\|^2 \\
&\leq \|x_{p+m-1} - q\|^2 + t_{p+m} l_{p+m} - 2t_{p+m}\phi(\|x_{p+m} - q\|)\, \|x_{p+m} - q\| \\
&< \epsilon^2 + t_{p+m}\phi(\epsilon)\epsilon - 2t_{p+m}\phi(\epsilon)\epsilon \\
&= \epsilon^2 - t_{p+m}\phi(\epsilon)\epsilon \\
&< \epsilon^2,
\end{aligned}$$

which is impossible. Hence $\|x_{p+m} - q\| < \epsilon$. That is, (2.11) holds for all $m \geq 1$. Thus (2.11) ensures that $\lim_{n\to\infty} x_n = q$. This completes the proof. □

Taking $x_{n-1} \simeq x_n$ in Theorem 2.2, we get

Theorem 2.3. *Let K be a nonempty closed and convex subset of an arbitrary Banach space X, $T : K \to K$ be continuous ϕ-hemicontractive mapping. For any $x_1 \in K$, define the sequence $\{x_n\}_{n=1}^{\infty}$ inductively as follows:*

$$x_n = (1 - t_n)x_{n-1} + t_n T x_n, \quad n \geq 1,$$

where $\{t_n\}_{n=1}^{\infty}$ is a sequence in $[0, 1]$ satisfying the conditions (i) *and* (ii) *of Theorem 2.2.*

Then the following conditions are equivalent:

(a) $\{x_n\}_{n=1}^{\infty}$ converges strongly to the fixed point q of T.

(b) $\{Tx_n\}_{n=1}^{\infty}$ is bounded.

Remark 2.4. 1. All the results can also be proved for the same iterative scheme with error terms.

2. The known results for strongly pseudocontractive mappings are weakened by the ϕ-hemicontractive mappings.

3. Our results hold in arbitrary Banach spaces, where as other known results are restricted for L_p (or l_p) spaces and q-uniformly smooth Banach spaces.

4. Theorem 2.2 is more general in comparison to the results of Alghamdi et al. [1] and Sahu et al. [10] in the context of the class of ϕ-hemicontractive mappings.

3 Applications

Theorem 3.1. *Let X be an arbitrary real Banach space and let $T : X \to X$ be continuous ϕ-strongly accretive operators. For any $x_1 \in X$, define the sequence $\{x_n\}_{n=1}^{\infty}$ inductively as follows:*

$$x_n = (1 - t_n)x_{n-1} + t_n(f + (I - T)x_n), \quad n \geq 1,$$

where $\{t_n\}_{n=1}^{\infty}$ be the sequence in $[0, 1]$ satisfying the conditions (i) *and* (ii) *of Theorem 2.2.*

Then the following conditions are equivalent:

(a) $\{x_n\}_{n=1}^{\infty}$ converges strongly to the solution of the system $f = Tx$.

(b) $\{(I - T)x_n\}_{n=1}^{\infty}$ is bounded.

Proof. Suppose that x^* is the solution of the system $f = Tx$. Define $G : X \to X$ by $Gx = f + (I - S)x$. Then x^* is the fixed point of G. Thus Theorem 3.1 follows from Theorem 2.2. □

Example 3.2. Let $X = \mathbb{R}$ be the reals with the usual norm and $K = [0, 1]$. Define $T : K \to K$ by

$$Tx = x - \tan x \quad \text{for all } x \in K.$$

By mean value theorem, we have

$$|Tx - Ty| \leq \sup_{c \in (0,1)} |T'c||x - y| \quad \text{for all } x, y \in K.$$

Noticing that $T'c = c - \sec^2 c$ and $1 < \sup_{c \in (0,1)} |T'c| = 2.4255$. Hence

$$|Tx - Ty| \leq L|x - y| \quad \text{for all } x, y \in K,$$

where $L = 2.4255$. It is easy to verify that T is ϕ-hemicontractive mapping with $\phi : [0, \infty) \to [0, \infty)$ defined by $\phi(t) = \tan t$ for all $t \in [0, \infty)$. Moreover, 0 is the fixed point of T.

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Lyapunov-type inequalities for fractional differential equations under multi-point boundary conditions

Youyu Wang ✉ and **Qichao Wang**

Department of Mathematics, Tianjin University of Finance and Economics, Tianjin 300222, P. R. China

Abstract. In this work, we establish new Lyapunov-type inequalities for fractional differential equations with multi-point boundary conditions. Our conclusions cover many results in the literature.

Keywords: Lyapunov inequality, fractional differential equation, multi-point boundary value problem, Green's function.

2010 Mathematics Subject Classification: 34A40, 26A33, 34B05.

1 Introduction

The well-known result of Lyapunov [9] states that if $u(t)$ is a nontrivial solution of the differential system

$$\begin{aligned} &u''(t) + r(t)u(t) = 0, \qquad t \in (a,b), \\ &u(a) = 0 = u(b), \end{aligned} \tag{1.1}$$

where $r(t)$ is a continuous function defined in $[a,b]$, then

$$\int_a^b |r(t)|dt > \frac{4}{b-a}, \tag{1.2}$$

and the constant 4 cannot be replaced by a larger number.

Lyapunov inequality (1.2) is a useful tool in various branches of mathematics including disconjugacy, oscillation theory, and eigenvalue problems. Many improvements and generalizations of the inequality (1.2) have appeared in the literature. A thorough literature review of continuous and discrete Lyapunov-type inequalities and their applications can be found in the survey articles by Cheng [3], Brown and Hinton [1] and Tiryaki [12].

The study of Lyapunov-type inequalities for the differential equation depends on a fractional differential operator was initiated by Rui A. C. Ferreira [4]. He first obtained a Lyapunov-type inequality when the differential equation depends on the Riemann-Liouville fractional derivative, the main result is as follows.

Theorem 1.1. *If the following fractional boundary value problem*

$$(D_{a^+}^{\alpha} u)(t) + q(t)u(t) = 0, \quad a < t < b, \ 1 < \alpha \leq 2, \tag{1.3}$$

$$u(a) = 0 = u(b), \tag{1.4}$$

has a nontrivial solution, where q is a real and continuous function, then

$$\int_a^b |q(s)|ds > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}. \tag{1.5}$$

✉ Corresponding author. Email: wang_youyu@163.com

Recently, some Lyapunov-type inequalities were obtained for different fractional boundary value problems. In this direction, we refer to Ferreira [5], Jleli and Samet [6, 7], O'Regan and Samet [10], Rong and Bai [11] and Cabrera, Sadarangani, and Samet [2].

For example, Cabrera, Sadarangani, and Samet [2] obtain some Lyapunov-type inequalities for a higher-order nonlocal fractional boundary value problem, they give the following Lyapunov inequalities.

Theorem 1.2. *If the fractional boundary value problem*

$$(D_{a^+}^{\alpha}u)(t)+q(t)u(t)=0, \quad a<t<b,\ 2<\alpha\leq 3, \tag{1.6}$$

$$u(a)=u'(a)=0, \quad u'(b)=\beta u(\xi), \tag{1.7}$$

has a nontrivial solution, where q is a real and continuous function, $a<\xi<b, 0\leq\beta(\xi-a)^{\alpha-1}<(\alpha-1)(b-a)^{\alpha-2}$, then

$$\int_a^b (b-s)^{\alpha-2}(s-a)|q(s)|ds \geq \left(1+\frac{\beta(b-a)^{\alpha-1}}{(\alpha-1)(b-a)^{\alpha-2}-\beta(\xi-a)^{\alpha-1}}\right)^{-1}\Gamma(\alpha). \tag{1.8}$$

Theorem 1.3. *If the fractional boundary value problem*

$$(D_{a^+}^{\alpha}u)(t)+q(t)u(t)=0, \quad a<t<b,\ 2<\alpha\leq 3, \tag{1.9}$$

$$u(a)=u'(a)=0, \quad u'(b)=\beta u(\xi), \tag{1.10}$$

has a nontrivial solution, where q is a real and continuous function, $a<\xi<b, 0\leq\beta(\xi-a)^{\alpha-1}<(\alpha-1)(b-a)^{\alpha-2}$, then

$$\int_a^b |q(s)|ds \geq \frac{\Gamma(\alpha)(\alpha-1)^{\alpha-1}}{(b-a)^{\alpha-1}(\alpha-2)^{\alpha-2}}\left(1+\frac{\beta(b-a)^{\alpha-1}}{(\alpha-1)(b-a)^{\alpha-2}-\beta(\xi-a)^{\alpha-1}}\right)^{-1}. \tag{1.11}$$

Motivated by [2], in this paper, we study the problem of finding some Lyapunov-type inequalities for the fractional differential equations with multi-point boundary conditions.

$$(D_{a^+}^{\alpha}u)(t)+q(t)u(t)=0, \quad a<t<b,\ 2<\alpha\leq 3, \tag{1.12}$$

$$u(a)=u'(a)=0, \quad (D_{a^+}^{\beta+1}u)(b)=\sum_{i=1}^{m-2} b_i(D_{a^+}^{\beta}u)(\xi_i), \tag{1.13}$$

where $D_{a^+}^{\alpha}$ denotes the standard Riemann-Liouville fractional derivative of order $\alpha, \alpha>\beta+2, 0\leq\beta<1$, $a<\xi_1<\xi_2<\cdots<\xi_{m-2}<b$, $b_i\geq 0 (i=1,2,\cdots,m-2), 0\leq\sum_{i=1}^{m-2}b_i(\xi_i-a)^{\alpha-\beta-1}<(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}$ and $q:[a,b]\to\mathbb{R}$ is a continuous function.

2 Preliminaries

In this section, we recall the concepts of the Riemann-Liouville fractional integral, the Riemann-Liouville fractional derivative of order $\alpha\geq 0$.

Definition 2.1. [8] Let $\alpha\geq 0$ and f be a real function defined on $[a,b]$. The Riemann-Liouville fractional integral of order α is defined by $(I_{a^+}^0 f)\equiv f$ and

$$(I_{a^+}^{\alpha}f)(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds, \quad \alpha>0,\ t\in[a,b].$$

Definition 2.2. [8] The Riemann-Liouville fractional derivative of order $\alpha \geq 0$ is defined by $(D_{a^+}^0 f) \equiv f$ and

$$(D_{a^+}^\alpha f)(t) = (D^m I_{a^+}^{m-\alpha} f)(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^m \int_a^t (t-s)^{m-\alpha-1} f(s)ds,$$

for $\alpha > 0$, where m is the smallest integer greater or equal to α.

Lemma 2.3. *[8] Assume that $u \in C(a,b) \cap L(a,b)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(a,b) \cap L(a,b)$. Then*

$$I_{a^+}^\alpha (D_{a^+}^\alpha u)(t) = u(t) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \cdots + c_n(t-a)^{\alpha-n},$$

where $c_i \in \mathbb{R}, i = 1, 2, \cdots, n$, and $n = [\alpha] + 1$.

Lemma 2.4. *For $2 < \alpha \leq 3, 0 \leq \beta < 1$, we have*

$$(D_{a^+}^\beta (s-a)^{\alpha-1})(t) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(t-a)^{\alpha-\beta-1},$$
$$(D_{a^+}^{\beta+1} (s-a)^{\alpha-1})(t) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta-1)}(t-a)^{\alpha-\beta-2}.$$

3 Main Results

We begin by writing problems (1.12)-(1.13) in its equivalent integral form.

Lemma 3.1. *We have that $u \in C[a,b]$ is a solution to the boundary value problem (1.12)-(1.13) if and only if u satisfies the integral equation*

$$u(t) = \int_a^b G(t,s)q(s)u(s)ds + T(t) \int_a^b \left(\sum_{i=1}^{m-2} b_i H(\xi, s)q(s)u(s) \right) ds, \tag{3.1}$$

where $G(t,s)$, $H(t,s)$ and $T(t)$ defined by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}} - (t-s)^{\alpha-1}, & a \leq s \leq t \leq b, \\ \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}}, & a \leq t \leq s \leq b. \end{cases}$$

$$H(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{(t-a)^{\alpha-\beta-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}} - (t-s)^{\alpha-\beta-1}, & a \leq s \leq t \leq b, \\ \frac{(t-a)^{\alpha-\beta-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}}, & a \leq t \leq s \leq b, \end{cases}$$

$$T(t) = \frac{(t-a)^{\alpha-1}}{(\alpha-\beta-1)(b-a)^{\alpha-\beta-2} - \sum_{i=1}^{m-2} b_i(\xi_i - a)^{\alpha-\beta-1}}, \ t \geq a.$$

Proof. From Lemma 2.3, $u \in C[a,b]$ is a solution to the boundary value problem (1.12)-(1.13) if and only if

$$u(t) = c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + c_3(t-a)^{\alpha-3} - (I_{a^+}^\alpha qu)(t),$$

for some real constants c_1, c_2, c_3. Using the boundary condition $u(a) = u'(a) = 0$, we obtain $c_2 = c_3 = 0$. Thus

$$u(t) = c_1(t-a)^{\alpha-1} - (I_{a^+}^\alpha qu)(t).$$

Applying Lemma 2.4, we obtain

$$(D^{\beta}_{a^+}u)(t) = c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(t-a)^{\alpha-\beta-1} - (I^{\alpha-\beta}_{a^+}qu)(t),$$
$$(D^{\beta+1}_{a^+}u)(t) = c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta-1)}(t-a)^{\alpha-\beta-2} - (I^{\alpha-\beta-1}_{a^+}qu)(t),$$

the boundary condition $(D^{\beta+1}_{a^+}u)(b) = \sum_{i=1}^{m-2} b_i(D^{\beta}_{a^+}u)(\xi_i)$ imply that

$$\begin{aligned}
&c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta-1)}(b-a)^{\alpha-\beta-2} - \frac{1}{\Gamma(\alpha-\beta-1)}\int_a^b (b-s)^{\alpha-\beta-2}q(s)u(s)ds \\
&= \sum_{i=1}^{m-2} b_i\left[c_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(\xi_i-a)^{\alpha-\beta-1} - \frac{1}{\Gamma(\alpha-\beta)}\int_a^{\xi_i}(\xi_i-s)^{\alpha-\beta-1}q(s)u(s)ds\right],
\end{aligned}$$

thus

$$\begin{aligned}
c_1 =&\frac{\alpha-\beta-1}{[(\alpha-\beta-1)(b-a)^{\alpha-\beta-2} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}]\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-\beta-2}q(s)u(s)ds \\
&- \frac{1}{[(\alpha-\beta-1)(b-a)^{\alpha-\beta-2} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}]\Gamma(\alpha)}\sum_{i=1}^{m-2} b_i\int_a^{\xi_i}(\xi_i-s)^{\alpha-\beta-1}q(s)u(s)ds.
\end{aligned}$$

By the relation

$$\begin{aligned}
&\frac{1}{(\alpha-\beta-1)(b-a)^{\alpha-\beta-2} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}} = \frac{1}{(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}} \\
&+ \frac{\sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}}{(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}[(\alpha-\beta-1)(b-a)^{\alpha-\beta-2} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}]},
\end{aligned}$$

we obtain

$$\begin{aligned}
c_1 =&\frac{1}{\Gamma(\alpha)}\int_a^b \frac{(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}}q(s)u(s)ds \\
&+ \frac{\sum_{i=1}^{m-2} b_i\int_a^b \frac{(\xi_i-a)^{\alpha-\beta-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}}q(s)u(s)ds}{[(\alpha-\beta-1)(b-a)^{\alpha-\beta-2} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}]\Gamma(\alpha)} \\
&- \frac{\sum_{i=1}^{m-2} b_i\int_a^{\xi_i}(\xi_i-s)^{\alpha-\beta-1}q(s)u(s)ds}{[(\alpha-\beta-1)(b-a)^{\alpha-\beta-2} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}]\Gamma(\alpha)},
\end{aligned}$$

therefore

$$\begin{aligned}
u(t) =&c_1(t-a)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}q(s)u(s)ds \\
=&\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}\int_a^b \frac{(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}}q(s)u(s)ds - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}q(s)u(s)ds \\
&+ \frac{(t-a)^{\alpha-1}\sum_{i=1}^{m-2} b_i\int_a^b \frac{(\xi_i-a)^{\alpha-\beta-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}}q(s)u(s)ds}{[(\alpha-\beta-1)(b-a)^{\alpha-\beta-2} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}]\Gamma(\alpha)} \\
&- \frac{(t-a)^{\alpha-1}\sum_{i=1}^{m-2} b_i\int_a^{\xi_i}(\xi_i-s)^{\alpha-\beta-1}q(s)u(s)ds}{[(\alpha-\beta-1)(b-a)^{\alpha-\beta-2} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}]\Gamma(\alpha)} \\
=&\int_a^b G(t,s)q(s)u(s)ds + T(t)\int_a^b\left(\sum_{i=1}^{m-2} b_iH(\xi,s)q(s)u(s)\right)ds,
\end{aligned}$$

which concludes the proof. □

Lemma 3.2. *The function G defined in Lemma 3.1 satisfy the following properties:*

(**i**) $G(t,s) \geq 0$, *for all* $(t,s) \in [a,b] \times [a,b]$;

(**ii**) $G(t,s)$ *is non-decreasing with respect to the first variable;*

(**iii**) $0 \leq G(a,s) \leq G(t,s) \leq G(b,s) = \frac{1}{\Gamma(\alpha)}(b-s)^{\alpha-\beta-2}[(b-a)^{\beta+1} - (b-s)^{\beta+1}]$, $(t,s) \in [a,b] \times [a,b]$.

(**iv**) *For any* $s \in [a,b]$,

$$\max_{s\in[a,b]} G(b,s) = \frac{\beta+1}{\alpha-1} \cdot \left(\frac{\alpha-\beta-2}{\alpha-1}\right)^{\frac{\alpha-\beta-2}{\beta+1}} \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}.$$

Proof. Let us define two functions

$$g_1(t,s) = \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}} - (t-s)^{\alpha-1}, \quad a \leq s \leq t \leq b,$$
$$g_2(t,s) = \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}}, \qquad a \leq t \leq s \leq b.$$

(i) It is clear that for $a \leq t \leq s \leq b$, $G(t,s) = \frac{1}{\Gamma(\alpha)} g_2(t,s) \geq 0$. On the other hand, for $a \leq s \leq t \leq b$, by the relation $\frac{b-s}{b-a} \geq \frac{t-s}{t-a}, \beta \geq 0, \alpha > 2$, we obtain

$$\begin{aligned}
\Gamma(\alpha)G(t,s) &= g_1(t,s) \\
&= \frac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}} - (t-s)^{\alpha-1} \\
&= (t-a)^{\alpha-1}\left[\left(\frac{b-s}{b-a}\right)^{\alpha-\beta-2} - \left(\frac{t-s}{t-a}\right)^{\alpha-1}\right] \\
&= (t-a)^{\alpha-1}\left[\left(\frac{b-a}{b-s}\right)^{\beta+1}\left(\frac{b-s}{b-a}\right)^{\alpha-1} - \left(\frac{t-s}{t-a}\right)^{\alpha-1}\right] \\
&\geq (t-a)^{\alpha-1}\left[\left(\frac{b-s}{b-a}\right)^{\alpha-1} - \left(\frac{t-s}{t-a}\right)^{\alpha-1}\right] \\
&\geq 0.
\end{aligned}$$

Then (i) is proved.

(ii) For $a \leq t \leq s \leq b$, we have

$$\Gamma(\alpha)\frac{\partial G(t,s)}{\partial t} = \frac{\partial g_2(t,s)}{\partial t} = \frac{(\alpha-1)(t-a)^{\alpha-2}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}} \geq 0.$$

For $a \leq s \leq t \leq b$, by the relation $\frac{b-s}{b-a} \geq \frac{t-s}{t-a}, \beta \geq 0, \alpha - 2 > 0$, we have $\left(\frac{b-s}{b-a}\right)^{\alpha-\beta-2} - \left(\frac{t-s}{t-a}\right)^{\alpha-2} = \left(\frac{b-a}{b-s}\right)^{\beta}\left(\frac{b-s}{b-a}\right)^{\alpha-2} - \left(\frac{t-s}{t-a}\right)^{\alpha-2} \geq \left(\frac{b-s}{b-a}\right)^{\alpha-2} - \left(\frac{t-s}{t-a}\right)^{\alpha-2} \geq 0$, so we obtain

$$\begin{aligned}
\Gamma(\alpha)\frac{\partial G(t,s)}{\partial t} = \frac{\partial g_1(t,s)}{\partial t} &= (\alpha-1)\left[\left(\frac{b-s}{b-a}\right)^{\alpha-\beta-2}(t-a)^{\alpha-2} - (t-s)^{\alpha-2}\right] \\
&= (\alpha-1)(t-a)^{\alpha-2}\left[\left(\frac{b-s}{b-a}\right)^{\alpha-\beta-2} - \left(\frac{t-s}{t-a}\right)^{\alpha-2}\right] \\
&\geq 0.
\end{aligned}$$

Then we proved that $G(t,s)$ is non-decreasing with respect to the first variable t.

(iii) The result follows immediately from **(ii)**.

(iv) Let

$$\varphi(s) = \Gamma(\alpha)G(b,s) = (b-a)^{\beta+1}(b-s)^{\alpha-\beta-2} - (b-s)^{\alpha-1},\ s \in [a,b].$$

We have

$$\begin{aligned}\varphi'(s) &= -(\alpha-\beta-2)(b-a)^{\beta+1}(b-s)^{\alpha-\beta-3} + (\alpha-1)(b-s)^{\alpha-2} \\ &= (b-s)^{\alpha-\beta-3}[(\alpha-1)(b-s)^{\beta+1} - (\alpha-\beta-2)(b-a)^{\beta+1}].\end{aligned}$$

Moreover,

$$\varphi'(s) = 0,\ s \in (a,b) \Leftrightarrow (b-s^*)^{\beta+1} = \frac{\alpha-\beta-2}{\alpha-1}(b-a)^{\beta+1}.$$

It is not difficult to observe that $\varphi'(s) \geq 0$, if $s \leq s^*$ and $\varphi'(s) < 0$, if $s > s^*$. Therefore,

$$\max_{s\in[a,b]} \varphi(s) = \varphi(s^*) = \frac{\beta+1}{\alpha-1} \cdot \left(\frac{\alpha-\beta-2}{\alpha-1}\right)^{\frac{\alpha-\beta-2}{\beta+1}} (b-a)^{\alpha-1}.$$

□

Lemma 3.3. *The function H defined in Lemma 3.1 satisfy the following properties:*

(**i**) $H(t,s) \geq 0$, *for all* $(t,s) \in [a,b] \times [a,b]$;

(**ii**) $H(t,s)$ *is non-decreasing with respect to the first variable;*

(**iii**) $0 \leq H(a,s) \leq H(t,s) \leq H(b,s) = \frac{1}{\Gamma(\alpha)}(b-s)^{\alpha-\beta-2}(s-a),\ (t,s) \in [a,b] \times [a,b]$.

(**iV**)

$$\max_{s\in[a,b]} H(b,s) = H(b,s^*) = \frac{(\alpha-\beta-2)^{\alpha-\beta-2}}{\Gamma(\alpha)} \left(\frac{b-a}{\alpha-\beta-1}\right)^{\alpha-\beta-1}.$$

where $s^* = \frac{\alpha-\beta-2}{\alpha-\beta-1}a + \frac{1}{\alpha-\beta-1}b$.

Proof. Let us define two functions

$$\begin{aligned} h_1(t,s) &= \frac{(t-a)^{\alpha-\beta-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}} - (t-s)^{\alpha-\beta-1}, && a \leq s \leq t \leq b, \\ h_2(t,s) &= \frac{(t-a)^{\alpha-\beta-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}}, && a \leq t \leq s \leq b. \end{aligned}$$

(i) It is clear that for $a \leq t \leq s \leq b$, $H(t,s) = \frac{1}{\Gamma(\alpha)}h_2(t,s) \geq 0$. On the other hand, for $a \leq s \leq t \leq b$, by the relation $\frac{b-s}{b-a} \geq \frac{t-s}{t-a}, \beta \geq 0, \alpha > \beta+2$, we obtain

$$\begin{aligned}\Gamma(\alpha)H(t,s) &= h_1(t,s) \\ &= \frac{(t-a)^{\alpha-\beta-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}} - (t-s)^{\alpha-\beta-1} \\ &= (t-a)^{\alpha-\beta-1}\left[\left(\frac{b-s}{b-a}\right)^{\alpha-\beta-2} - \left(\frac{t-s}{t-a}\right)^{\alpha-\beta-1}\right] \\ &= (t-a)^{\alpha-\beta-1}\left[\left(\frac{b-a}{b-s}\right)\left(\frac{b-s}{b-a}\right)^{\alpha-\beta-1} - \left(\frac{t-s}{t-a}\right)^{\alpha-\beta-1}\right] \\ &\geq (t-a)^{\alpha-\beta-1}\left[\left(\frac{b-s}{b-a}\right)^{\alpha-\beta-1} - \left(\frac{t-s}{t-a}\right)^{\alpha-\beta-1}\right] \\ &\geq 0.\end{aligned}$$

Then (i) is proved.

(ii) For $a \le t \le s \le b$, we have

$$\Gamma(\alpha)\frac{\partial H(t,s)}{\partial t} = \frac{\partial h_2(t,s)}{\partial t} = \frac{(\alpha-\beta-1)(t-a)^{\alpha-\beta-2}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}} \ge 0.$$

For $a \le s \le t \le b$, by the relation $\frac{b-s}{b-a} \ge \frac{t-s}{t-a}, \beta \ge 0, \alpha-\beta-2 > 0$, we obtain

$$\begin{aligned}\Gamma(\alpha)\frac{\partial H(t,s)}{\partial t} = \frac{\partial h_1(t,s)}{\partial t} &= (\alpha-\beta-1)\left[\left(\frac{b-s}{b-a}\right)^{\alpha-\beta-2}(t-a)^{\alpha-\beta-2} - (t-s)^{\alpha-\beta-2}\right] \\ &= (\alpha-\beta-1)(t-a)^{\alpha-\beta-2}\left[\left(\frac{b-s}{b-a}\right)^{\alpha-\beta-2} - \left(\frac{t-s}{t-a}\right)^{\alpha-\beta-2}\right] \\ &\ge 0.\end{aligned}$$

Then we proved that $H(t,s)$ is non-decreasing with respect to the first variable t.

(iii) The result follows immediately from **(ii)**.

(iv) Let

$$\psi(s) = \Gamma(\alpha)H(b,s) = (b-s)^{\alpha-\beta-2}(s-a),\ s \in [a,b].$$

We have

$$\begin{aligned}\psi'(s) &= -(\alpha-\beta-2)(b-s)^{\alpha-\beta-3}(s-a) + (b-s)^{\alpha-\beta-2} \\ &= (b-s)^{\alpha-\beta-3}[(b-s) - (\alpha-\beta-2)(s-a)].\end{aligned}$$

Moreover,

$$\psi'(s) = 0,\ s \in (a,b) \Leftrightarrow s = s^* = \frac{\alpha-\beta-2}{\alpha-\beta-1}a + \frac{1}{\alpha-\beta-1}b.$$

It is not difficult to observe that $\psi'(s) \ge 0$, if $s \le s^*$ and $\psi'(s) < 0$, if $s > s^*$. Therefore,

$$\max_{s\in[a,b]} \psi(s) = \psi(s^*) = (\alpha-\beta-2)^{\alpha-\beta-2}\left(\frac{b-a}{\alpha-\beta-1}\right)^{\alpha-\beta-1}.$$

□

Now, we are ready to prove our first Lyapunov-type inequality.

Theorem 3.4. *If a nontrivial continuous solution of the fractional boundary value problem*

$$\begin{aligned}&(D^{\alpha}_{a^+}u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 2 < \alpha \le 3, \\ &u(a) = u'(a) = 0, \quad (D^{\beta+1}_{a^+}u)(b) = \sum_{i=1}^{m-2} b_i(D^{\beta}_{a^+}u)(\xi_i),\end{aligned}$$

exists, then

$$\int_a^b (b-s)^{\alpha-\beta-2}\left[(b-a)^{\beta+1} - (b-s)^{\beta+1} + \sum_{i=1}^{m-2} b_i T(b)(s-a)\right]|q(s)|ds \ge \Gamma(\alpha),$$

where

$$T(b) = \frac{(b-a)^{\alpha-1}}{(\alpha-\beta-1)(b-a)^{\alpha-\beta-2} - \sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}}.$$

Proof. Let $B = C[a,b]$ be the Banach space endowed with norm $\|u\| = \sup_{t\in[a,b]}|u(t)|$. It follows from Lemma 3.1 that a solution u to the boundary value problem satisfies the integral equation

$$u(t) = \int_a^b G(t,s)q(s)u(s)ds + T(t)\int_a^b \left(\sum_{i=1}^{m-2} b_i H(\xi,s)q(s)u(s)\right) ds.$$

Now, using Lemma 3.2, we obtain

$$\|u\| \le \|u\| \int_a^b G(b,s)|q(s)|ds + \|u\| \sum_{i=1}^{m-2} b_i T(b)\int_a^b H(b,s)|q(s)|ds,$$

which yields

$$\|u\| \le \|u\| \int_a^b \left(G(b,s) + \sum_{i=1}^{m-2} b_i T(b) H(b,s)\right)|q(s)|ds,$$

as

$$\begin{aligned}
&\Gamma(\alpha)\left[G(b,s) + \sum_{i=1}^{m-2} b_i T(b)H(b,s)\right] \\
&= (b-a)^{\beta+1}(b-s)^{\alpha-\beta-2} - (b-s)^{\alpha-1} + \sum_{i=1}^{m-2} b_i T(b)(b-s)^{\alpha-\beta-2}(s-a) \\
&= (b-s)^{\alpha-\beta-2}\left[(b-a)^{\beta+1} - (b-s)^{\beta+1} + \sum_{i=1}^{m-2} b_i T(b)(s-a)\right],
\end{aligned}$$

therefore, if u is a nontrivial continuous solution to (1.12)-(1.13), we have

$$\int_a^b (b-s)^{\alpha-\beta-2}\left[(b-a)^{\beta+1} - (b-s)^{\beta+1} + \sum_{i=1}^{m-2} b_i T(b)(s-a)\right]|q(s)|ds \ge \Gamma(\alpha).$$

□

Now, from Theorem 3.4 and Lemma 3.2 (iv), Lemma 3.3 (iv), we have

$$\begin{aligned}
&\Gamma(\alpha)\left[G(b,s) + \sum_{i=1}^{m-2} b_i T(b)H(b,s)\right] \\
&\le \Gamma(\alpha)\left[\max_{s\in[a,b]} G(b,s) + \sum_{i=1}^{m-2} b_i T(b) \max_{s\in[a,b]} H(b,s)\right] \\
&= \frac{\beta+1}{\alpha-1}\cdot\left(\frac{\alpha-\beta-2}{\alpha-1}\right)^{\frac{\alpha-\beta-2}{\beta+1}}(b-a)^{\alpha-1} + \sum_{i=1}^{m-2} b_i T(b)(\alpha-\beta-2)^{\alpha-\beta-2}\left(\frac{b-a}{\alpha-\beta-1}\right)^{\alpha-\beta-1}.
\end{aligned}$$

So, if problem (1.12)-(1.13) has a nontrivial continuous solution, then we have the following result.

Corollary 3.5. *If a nontrivial continuous solution of the fractional boundary value problem*

$$\begin{aligned}
&(D_{a^+}^{\alpha}u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 2 < \alpha \le 3, \\
&u(a) = u'(a) = 0, \quad (D_{a^+}^{\beta+1}u)(b) = \sum_{i=1}^{m-2} b_i (D_{a^+}^{\beta}u)(\xi_i),
\end{aligned}$$

exists, then

$$\int_a^b |q(s)|ds \geq \frac{\Gamma(\alpha)}{\frac{\beta+1}{\alpha-1}\cdot\left(\frac{\alpha-\beta-2}{\alpha-1}\right)^{\frac{\alpha-\beta-2}{\beta+1}}(b-a)^{\alpha-1}+\sum_{i=1}^{m-2} b_i T(b)(\alpha-\beta-2)^{\alpha-\beta-2}\left(\frac{b-a}{\alpha-\beta-1}\right)^{\alpha-\beta-1}}.$$

Let $\beta = 0$ in Theorem 3.4, we obtain

Corollary 3.6. *If a nontrivial continuous solution of the fractional boundary value problem*

$$\begin{aligned} &(D_{a^+}^{\alpha} u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 2 < \alpha \leq 3, \\ &u(a) = u'(a) = 0, \quad u'(b) = \sum_{i=1}^{m-2} b_i u(\xi_i), \end{aligned}$$

exists, then

$$\begin{aligned} &\int_a^b (b-s)^{\alpha-2}(s-a)|q(s)|ds \\ &\geq \frac{\Gamma(\alpha)}{1+\sum_{i=1}^{m-2} b_i T(b)} \\ &= \frac{(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}-\sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}}{(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}-\sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}+\sum_{i=1}^{m-2} b_i(b-a)^{\alpha-1}}\Gamma(\alpha). \end{aligned}$$

Let $\beta = 0$ in Corollary 3.5, we have the following result.

Corollary 3.7. *If a nontrivial continuous solution of the fractional boundary value problem*

$$\begin{aligned} &(D_{a^+}^{\alpha} u)(t) + q(t)u(t) = 0, \quad a < t < b,\ 2 < \alpha \leq 3, \\ &u(a) = u'(a) = 0, \quad u'(b) = \sum_{i=1}^{m-2} b_i u(\xi_i), \end{aligned}$$

exists, then

$$\begin{aligned} &\int_a^b |q(s)|ds \geq \frac{\Gamma(\alpha)}{1+\sum_{i=1}^{m-2} b_i T(b)}\cdot\frac{(\alpha-1)^{\alpha-1}}{(b-a)^{\alpha-1}(\alpha-2)^{\alpha-2}} \\ &= \frac{(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}-\sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}}{(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}-\sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}+\sum_{i=1}^{m-2} b_i(b-a)^{\alpha-1}}\cdot\frac{\Gamma(\alpha)(\alpha-1)^{\alpha-1}}{(b-a)^{\alpha-1}(\alpha-2)^{\alpha-2}}. \end{aligned}$$

Remark 3.8. Let $b_1 = \delta, b_2 = b_3 = \cdots = b_{m-2} = 0, \xi_1 = \xi$ in Corollary 3.6, we obtain (1.8), let $b_1 = \delta, b_2 = b_3 = \cdots = b_{m-2} = 0, \xi_1 = \xi$ in Corollary 3.7, we obtain (1.11).

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On a new generalized integral-type operator from mixed-norm spaces to Bloch-type spaces

Fang Zhang [1] Yongmin Liu[2*]

1. *Department of Applied Mathematics, Changzhou University, Changzhou 213164, China*

2. *School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou, 221116, China*

Abstract Let φ be an analytic self-map of unit disk $\mathbb{D}$, $H(\mathbb{D})$ the space of analytic functions on $\mathbb{D}$, and $g \in H(\mathbb{D})$. For an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\mathbb{D}$, the generalized integral-type operator $C_{\varphi,g}^{[\beta]}$ is defined by

$$\left(C_{\varphi,g}^{[\beta]} f\right)(z) = \int_0^z f^{[\beta]}(\varphi(w))g(w)dw, \ z \in \mathbb{D},$$

where $\beta \geq 0$, $f^{[\beta]}(z) = \sum\limits_{n=0}^{\infty} \frac{\Gamma(n+1+\beta)}{\Gamma(n+1)} a_n z^n$ and $f^{[0]}(z) = f(z)$. The boundedness and compactness of $C_{\varphi,g}^{[\beta]}$ from mixed-norm spaces $H(p,q,\mu)$ to Bloch-type spaces $\mathbb{B}^{\omega}$ are discussed in this paper.

Keywords. Generalized integral-type operator; Mixed-norm space; Bloch-type space

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1 Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, and $H(\mathbb{D})$ the set of all analytic functions on $\mathbb{D}$. The Pochhammer's symbol/shifted factorial is defined by

$$(a)_n := a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n \in \mathbb{N},$$

and $(a)_0 = 1$ for $a \neq 0$. Here a is a complex number such that $a \neq -m, m = 0, 1, 2, \ldots$. The classical/Gaussian hypergeometric series is defined by the power series expansion

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \quad |z| < 1.$$

*Corresponding author. Email: 6019820099@jsnu.edu.cn

For two analytic functions $f(z)=\sum_{n=0}^{\infty}a_nz^n$, $g(z)=\sum_{n=0}^{\infty}b_nz^n$ in $|z|<R$, the Hadamard product (or convolution) of f and g denoted by $f*g$ and is defined as follows

$$(f*g)(z)=\sum_{n=0}^{\infty}a_nb_nz^n,\quad |z|<R^2.$$

Furthermore,

$$(f*g)(z)=\frac{1}{2\pi i}\int_{|w|=\rho}f(w)g\left(\frac{z}{w}\right)\frac{dw}{w},\quad |z|<\rho R<R^2.$$

In particular, if $f,g\in H(\mathbb{D})$, we have

$$(f*g)(\rho z)=\frac{1}{2\pi}\int_0^{2\pi}f(\rho e^{it})g(ze^{-it})dt,\quad 0<\rho<1,$$

(see, e.g. [1]).

If $f(z)=\sum_{n=0}^{\infty}a_nz^n\in H(\mathbb{D})$ and $\beta>0$, then the fractional derivative $f^{[\beta]}$ of order β which introduced by Hardy and Littlewood [4], is defined as follows

$$f^{[\beta]}(z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+1+\beta)}{\Gamma(n+1)}a_nz^n.$$

It is easy to check that

$$f^{[\beta]}(z)=\Gamma(1+\beta)\left(f(z)*F(1,1+\beta;1;z)\right).$$

For $\beta=0$, we defined $f^{[0]}(z)=f(z)$. It is obvious to find that the fractional derivative and the ordinary derivative satisfy

$$f^{[k]}(z)=\frac{d^k}{dz^k}\left(z^kf(z)\right),\quad k=0,1,2,\ldots.$$

A positive continuous function μ on the interval [0,1) is called normal (see, e.g. [22]) if there exist positive numbers s,t $(0<s<t)$ and $\delta\in[0,1)$, such that

$$\frac{\mu(r)}{(1-r)^s}\text{ is decreasing for }\ \delta\le r<1\text{ and }\lim_{r\to1}\frac{\mu(r)}{(1-r)^s}=0;$$

$$\frac{\mu(r)}{(1-r)^t}\text{ is increasing for }\delta\le r<1\text{ and }\lim_{r\to1}\frac{\mu(r)}{(1-r)^t}=\infty.$$

From now on we always assume that μ is a normal function on $[0,1)$.

Let $0\le r<1, f\in H(\mathbb{D})$, we set

$$M_q(f,r)=\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^qd\theta\right)^{1/q},\quad 0<q<\infty,$$

$$M_\infty(f,r)=\sup_{0\le\theta\le2\pi}|f(re^{i\theta})|.$$

For $0 < p, q \leq \infty$, a function $f \in H(\mathbb{D})$ is said to belong to the mixed-norm space $H(p, q, \mu)$ if

$$\|f\|_{H(p,q,\mu)} = \left(\int_0^1 M_q^p(f,r)\frac{\mu^p(r)}{1-r}dr\right)^{1/p} < \infty.$$

The Bloch-type space (or ω-Bloch space), denoted by $\mathbb{B}^\omega = \mathbb{B}^\omega(\mathbb{D})$, consists of those functions $f \in H(\mathbb{D})$ such that

$$B_\omega(f) = \sup_{z\in\mathbb{D}} \omega(z)|f'(z)| < \infty,$$

where $\omega(z)$ is a continuous nonincreasing function such that

$$\omega(z) = \omega(|z|), \quad z \in \mathbb{D} \; and \lim_{|z|\to 1} \omega(z) = 0. \tag{1.1}$$

Functions ω that satisfy condition (1.1) are called almost classic weights.

With the norm $\|f\|_{\mathbb{B}^\omega} = |f(0)| + B_\omega(f)$, the ω-Bloch space becomes a Banach space. The little ω-Bloch space $\mathbb{B}_0^\omega$ is the subspace of $\mathbb{B}^\omega$ consisting of those $f \in \mathbb{B}^\omega$ such that

$$\lim_{|z|\to 1} \omega(z)|f'(z)| = 0.$$

For $\omega(z) = (1-|z|^2)^\alpha, \quad \alpha > 0$, ω-Bloch space becomes the α-Bloch space (see, e.g. [6, 19, 23, 29]).

Let $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. For $\beta \geq 0$, we introduce a new generalized integral-type operator $C_{\varphi,g}^{[\beta]}$ as follows:

$$\left(C_{\varphi,g}^{[\beta]} f\right)(z) = \int_0^z f^{[\beta]}(\varphi(w))g(w)dw, \; z \in \mathbb{D}, \; f \in H(\mathbb{D}).$$

The operator $C_{\varphi,g}^{[\beta]}$ is a generalization of the operator $C_{\varphi,g}^n$, which is defined as

$$\left(C_{\varphi,g}^n f\right)(z) = \int_0^z f^{(n)}(\varphi(w))g(w)dw, \quad f \in H(\mathbb{D}).$$

The operator $C_{\varphi,g}^n$ was introduced in [32] and studied in [3, 5, 14, 20, 21, 28]. When $n = 1$, then

$$\left(C_{\varphi,g}^1 f\right)(z) = \left(C_\varphi^g f\right)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi,$$

which is the generalized composition operator defined by Li and Stević in [11, 13], and studied in [9, 10, 12, 13, 24, 25, 26, 27, 30, 31, 33]. When $n = 0$, then $C_{\varphi,g}^{[\beta]} = C_{\varphi,g}^0$ is the Volterra composition operator defined by Li in [7], and studied in [8, 12, 15, 16]. In [17], Long and Wu characterized the boundedness and compactness of the integral-type operator $C_{\varphi,g}^n$ from mixed-norm spaces to the ω-Bloch spaces. Besides, Borgohain and Naik [2] initiated a generalized integral type operator as follows:

$$\left(C^{\beta}_{\varphi,g}f\right)(z)=\int_0^z f^{\beta}(\varphi(\xi))g(\xi)d\xi,$$

where f^{β} is the fractional derivative of order β ($\beta>0$) defined as

$$f^{\beta}(z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1-\beta)}a_n z^{n-\beta}.$$

They discussed the boundedness and compactness of the operator $C^{\beta}_{\varphi,g}$ from Zygmund spaces to Bloch type spaces in [2].

In [1], Borgohain and Naik defined an operator $D^{\beta}_{\varphi,u}$, called a weighted fractional differentiation composition operator, by

$$\left(D^{\beta}_{\varphi,u}f\right)(z)=u(z)f^{[\beta]}(\varphi(z)).$$

They discussed the boundedness and compactness of $D^{\beta}_{\varphi,u}$ from mixed-norm space $H(p,q,\phi)$ to weighted-type space H^{∞}_{μ}.

Motivated by [1, 2, 17, 32], we consider the boundedness and compactness of the operator $C^{[\beta]}_{\varphi,g}$ from mixed-norm spaces to the ω-Bloch spaces in this paper. Our results can be viewed as generalizations of the results in [17].

Throughout this paper, we will use the symbol C to denote a finite positive number, and it may differ from one occurrence to another.

2 Auxiliary results

In order to formulate our main results, we need some auxiliary results which are incorporated in the following lemmas.

The fisrt lemma is important. It gave an estimate which involves fractional derivative $f^{[\beta]}$ of $f\in H(p,q,\mu)$.

Lemma 2.1 *([1]) Assume $0<p\le\infty$, $1\le q\le\infty$, μ is normal, and $f\in H(p,q,\mu)$. Then for every $\beta\ge 0$, there is a positive constant C independent of f such that*

$$\left|f^{[\beta]}(z)\right|\le C\frac{\|f\|_{H(p,q,\mu)}}{(1-|z|^2)^{\beta+1/q}\mu(|z|)},\quad \forall z\in\mathbb{D}.$$

The following lemma, can be proved in a standard way (see, e.g. [18]).

Lemma 2.2 *Assume $\beta\ge 0$, $0<p\le\infty$, $1\le q\le\infty$, $g\in H(\mathbb{D})$, μ is normal, ω is a almost classic weight, and φ is an analytic self-map of $\mathbb{D}$. Then $C^{[\beta]}_{\varphi,g}:H(p,q,\mu)\to\mathbb{B}^{\omega}$ is compact if and only if $C^{[\beta]}_{\varphi,g}:H(p,q,\mu)\to\mathbb{B}^{\omega}$ is bounded and for any bounded sequence f_k in $H(p,q,\mu)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$, we have $\|C^{[\beta]}_{\varphi,g}f_k\|_{\mathbb{B}^{\omega}}\to 0$ as $k\to\infty$.*

Lemma 2.3 *([24]) Assume that ω is an almost classic weight. A closed set K in $\mathbb{B}_0^\omega$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in K}\omega(z)|f'(z)|=0.$$

3 Main results and proofs

In this section we consider the boundedness and the compactness of the operator $C_{\varphi,g}^{[\beta]}: H(p,q,\mu)\to \mathbb{B}^\omega$ (or $\mathbb{B}_0^\omega$).

Theorem 3.1 *Assume $\beta\ge 0$, $0<p\le\infty$, $1\le q\le\infty$, $g\in H(\mathbb{D})$, μ is normal, ω is an almost classic weight, and φ is an analytic self-map of $\mathbb{D}$. Then $C_{\varphi,g}^{[\beta]}: H(p,q,\mu)\to \mathbb{B}^\omega$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}}\frac{\omega(z)|g(z)|}{\mu(|\varphi(z)|)(1-|\varphi(z)|^2)^{\beta+1/q}}<\infty. \tag{3.1}$$

Proof Suppose that (3.1) holds. For any $z\in\mathbb{D}$ and $f\in H(p,q,\mu)$, by Lemma 2.1 we have

$$\begin{aligned}\omega(z)\left|\left(C_{\varphi,g}^{[\beta]}f\right)'(z)\right| &= \omega(z)|g(z)|\left|f^{[\beta]}(\varphi(z))\right|\\ &\le C\|f\|_{H(p,q,\mu)}\frac{\omega(z)|g(z)|}{\mu(|\varphi(z)|)(1-|\varphi(z)|^2)^{\beta+1/q}},\end{aligned}$$

and $(C_{\varphi,g}^{[\beta]}f)(0)=0$. This shows that $C_{\varphi,g}^{[\beta]}$ is bounded.

Conversely, assume that $C_{\varphi,g}^{[\beta]}: H(p,q,\mu)\to\mathbb{B}^\omega$ is bounded. Fix $a\in\mathbb{D}$, we take the test functions

$$f_a(z)=\frac{(1-|a|^2)^{t+1}F(\frac{1}{q}+\beta+t+1,1;1+\beta;\overline{a}z)}{\mu(|a|)}, \tag{3.2}$$

where the constant t is from the definition of the function μ. By elementary calculations similar to those outlined in Theorem 5 of [1], we see that $f_a\in H(p,q,\mu)$. In addition,

$$f_a^{[\beta]}(z)=\frac{\Gamma(1+\beta)(1-|a|^2)^{t+1}}{\mu(|a|)(1-\overline{a}z)^{\beta+t+1+1/q}}. \tag{3.3}$$

By the boundedness of $C_{\varphi,g}^{[\beta]}$, for every $\lambda\in\mathbb{D}$, we get

$$\begin{aligned}\infty &> C\|C_{\varphi,g}^{[\beta]}\|_{H(p,q,\mu)\to\mathbb{B}^\omega}\\ &\ge \|C_{\varphi,g}^{[\beta]}f_{\varphi(\lambda)}\|_{\mathbb{B}^\omega}\\ &\ge \sup_{z\in\mathbb{D}}\omega(z)\left|\left(C_{\varphi,g}^{[\beta]}f_{\varphi(\lambda)}\right)'(z)\right|\\ &\ge \frac{\Gamma(1+\beta)\omega(\lambda)|g(\lambda)|(1-|\varphi(\lambda)|^2)^{t+1}}{\mu(|\varphi(\lambda)|)(1-|\varphi(\lambda)|^2)^{\beta+t+1+1/q}}\\ &= \frac{\Gamma(1+\beta)\omega(\lambda)|g(\lambda)|}{\mu(|\varphi(\lambda)|)(1-|\varphi(\lambda)|^2)^{\beta+1/q}}.\end{aligned}$$

Therefore

$$\sup_{z\in\mathbb{D}} \frac{\omega(z)|g(z)|}{\mu(|\varphi(z)|)(1-|\varphi(z)|^2)^{\beta+1/q}} < \infty.$$

Theorem 3.2 *Assume $\beta \geq 0$, $0 < p \leq \infty$, $1 \leq q \leq \infty$, $g \in H(\mathbb{D})$, μ is normal, ω is an almost classic weight, and φ is an analytic self-map of $\mathbb{D}$. Then $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}_0^{\omega}$ is bounded if and only if $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}^{\omega}$ is bounded and*

$$\lim_{|z|\to 1} \omega(z)|g(z)| = 0. \tag{3.4}$$

Proof Suppose that $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}^{\omega}$ is bounded and (3.4) holds. For each polynomial $p(z)$, we get

$$\omega(z)\left|\left(C_{\varphi,g}^{[\beta]}p\right)'(z)\right| = \omega(z)|g(z)|\left|p^{[\beta]}(\varphi(z))\right|.$$

Let $p(z) = \sum_{n=0}^{k} a_n z^n$, $k \in \mathbb{N}$. From the proof of Theorem 7 in [1], we see that

$$p^{[\beta]}(z) = \Gamma(1+\beta)\left(\sum_{n=0}^{k} \frac{(1+\beta)_n}{(1)_n} a_n z^n\right).$$

Then we have $p^{[\beta]}(z)$ is bounded in $|z| < 1$. From (3.4), we see that $C_{\varphi,g}^{[\beta]}p \in \mathbb{B}_0^{\omega}$. Since the set of all polynomials is dense in $H(p,q,\mu)$, we have that for every $f \in H(p,q,\mu)$, there is a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$ such that $\|f - p_k\|_{H(p,q,\mu)} \to 0$ as $k \to \infty$. Hence by the boundedness of the operator $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}^{\omega}$, we have

$$\|C_{\varphi,g}^{[\beta]}f - C_{\varphi,g}^{[\beta]}p_k\|_{\mathbb{B}^{\omega}} \leq \|C_{\varphi,g}^{[\beta]}\|_{H(p,q,\mu)\to\mathbb{B}^{\omega}}\|f - p_k\|_{H(p,q,\mu)} \to 0,$$

as $k \to \infty$. Since $\mathbb{B}_0^{\omega}$ is the closed subset of $\mathbb{B}^{\omega}$, we see that $C_{\varphi,g}^{[\beta]}f \in \mathbb{B}_0^{\omega}$, and consequently $C_{\varphi,g}^{[\beta]}(H(p,q,\mu)) \subset \mathbb{B}_0^{\omega}$, so $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}_0^{\omega}$ is bounded.

For the converse, suppose that $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}_0^{\omega}$ is bounded. It is clear that $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}^{\omega}$ is bounded. We take the test functions $f(z) = \frac{1}{\Gamma(1+\beta)} \in H(p,q,\mu)$ for $z \in \mathbb{D}$, it follows that

$$\begin{aligned} f^{[\beta]}(z) \quad &= \quad \Gamma(1+\beta)\left(f(z) * F(1,1+\beta;1,z)\right) \\ &= \Gamma(1+\beta)\left(\frac{1}{\Gamma(1+\beta)} * \sum_{n=0}^{\infty} \frac{(1+\beta)_n}{(1)_n} z^n\right) \\ &= \Gamma(1+\beta)\left(\frac{1}{\Gamma(1+\beta)} \cdot \frac{(1+\beta)_0}{(1)_0}\right) \\ &= 1. \end{aligned}$$

By the assumption, we have

$$\begin{aligned}
&\lim_{|z|\to 1}\omega(z)\left|\left(C_{\varphi,g}^{[\beta]}f\right)'(z)\right| \\
&= \lim_{|z|\to 1}\omega(z)|g(z)|\left|f^{[\beta]}(\varphi(z))\right| \\
&= \lim_{|z|\to 1}\omega(z)|g(z)| \\
&= 0.
\end{aligned}$$

Theorem 3.3 *Assume $\beta \geq 0$, $0 < p \leq \infty$, $1 \leq q \leq \infty$, $g \in H(\mathbb{D})$, μ is normal, ω is an almost classic weight, and φ is an analytic self-map of $\mathbb{D}$. Then $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}^{\omega}$ is compact if and only if $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}^{\omega}$ is bounded and*

$$\lim_{|\varphi(z)|\to 1}\frac{\omega(z)|g(z)|}{\mu(|\varphi(z)|)(1-|\varphi(z)|^2)^{\beta+1/q}} = 0. \tag{3.5}$$

Proof. Assume that $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}^{\omega}$ is bounded and (3.5) holds. Let $\{f_n\}$ be a bounded sequence in $H(p,q,\mu)$ with $\|f_n\|_{H(p,q,\mu)} \leq 1$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. In light of Lemma 2.2, we only need to show that

$$\|C_{\varphi,g}^{[\beta]}f_n\|_{\mathbb{B}^{\omega}} \to 0, \qquad (n\to\infty).$$

From (3.5), we have that for every $\varepsilon > 0$, there exists a constant δ, $0<\delta<1$, such that $\delta < |\varphi(z)| < 1$ implies

$$\frac{\omega(z)|g(z)|}{\mu(|\varphi(z)|)(1-|\varphi(z)|^2)^{\beta+1/q}} < \varepsilon.$$

Since $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}^{\omega}$ is bounded, taking $f(z) = \frac{1}{\Gamma(1+\beta)}$, we see that $M_1 = \sup_{z\in\mathbb{D}}\omega(z)|g(z)| < \infty$. Since

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}\omega(z)\left|\left(C_{\varphi,g}^{[\beta]}f_n\right)'(z)\right| \\
\leq\ & \sup_{\{|\varphi(z)|\leq\delta\}} w(z_n)|g(z_n)|\left|f_n^{[\beta]}(\varphi(z_n))\right| + \sup_{\{|\varphi(z)|>\delta\}} w(z_n)|g(z_n)|\left|f_n^{[\beta]}(\varphi(z_n))\right| \\
\leq\ & M_1\sup_{\{|\varphi(z)|\leq\delta\}}\left|f_n^{[\beta]}(\varphi(z_n))\right| + C\|f_n\|_{H(p,q,\mu)}\sup_{\{|\varphi(z)|>\delta\}}\frac{\omega(z)|g(z)|}{\mu(|\varphi(z)|)(1-|\varphi(z)|^2)^{\beta+1/q}} \\
<\ & M_1\sup_{\{|\varphi(z)|\leq\delta\}}\left|f_n^{[\beta]}(\varphi(z_n))\right| + C\varepsilon.
\end{aligned}$$

From the proof of Theorem 10 in [1], $\{f_n^{[\beta]}\}$ converges uniformly to 0 on compact subsets of $\mathbb{D}$. Then

$$\|C_{\varphi,g}^{[\beta]}f_n\|_{\mathbb{B}^{\omega}} \to 0 \quad \text{as } n\to\infty.$$

Conversely, suppose that $C^{[\beta]}_{\varphi,g}$ is compact from $H(p,q,\mu)$ to $\mathbb{B}^\omega$. From which we can easily obtain the boundedness of $C^{[\beta]}_{\varphi,g} : H(p,q,\mu) \to \mathbb{B}^\omega$. Next we only need to show that (3.5) holds. Let $\{z_n\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. We now consider the function

$$h_n(z) = \frac{(1-|\varphi(z_n)|^2)^{t+1} F\left(\beta+t+1+1/q, 1; 1+\beta; \overline{\varphi(z_n)}z\right)}{\mu(|\varphi(z_n)|)}. \tag{3.6}$$

It is easy to check that $h_n \in H(p,q,\mu)$. Moreover, from (3.3)

$$h_n^{[\beta]}(\varphi(z_n)) = f^{[\beta]}_{\varphi(z_n)}(\varphi(z_n)) = \frac{\Gamma(1+\beta)(1-|\varphi(z_n)|^2)^{t+1}}{\mu(|\varphi(z_n)|)(1-\overline{\varphi(z_n)}z)^{t+1+1/q}}. \tag{3.7}$$

It is easy to show that $\{h_n\}$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. Therefore, using Lemma 2.2, we have $\lim_{n\to\infty} \|C^{[\beta]}_{\varphi,g} h_n\|_{\mathbb{B}^\omega} = 0$. From this and since

$$\begin{aligned} \|C^{[\beta]}_{\varphi,g} h_n\|_{\mathbb{B}^\omega} &\geq \sup_{z\in\mathbb{D}} \omega(z) \left| \left(C^{[\beta]}_{\varphi,g} h_n\right)'(z)\right| \\ &\geq w(z_n)|g(z_n)| \left|h_n^{[\beta]}(\varphi(z_n))\right| \\ &= \frac{\Gamma(1+\beta)\omega(z_n)|g(z_n)|}{\mu(|\varphi(z_n)|)(1-|\varphi(z_n)|^2)^{\beta+1/q}}, \end{aligned}$$

it follows that

$$\lim_{|\varphi(z)|\to 1} \frac{\omega(z)|g(z)|}{\mu(|\varphi(z)|)(1-|\varphi(z)|^2)^{\beta+1/q}} = 0.$$

Theorem 3.4 *Assume $\beta \geq 0$, $0 < p \leq \infty$, $1 \leq q \leq \infty$, $g \in H(\mathbb{D})$, ω is an almost classic weight, and φ is an analytic self-map of $\mathbb{D}$. Then $C^{[\beta]}_{\varphi,g} : H(p,q,\mu) \to \mathbb{B}^\omega_0$ is compact if and only if*

$$\lim_{|z|\to 1} \frac{\omega(z)|g(z)|}{\mu(|\varphi(z)|)(1-|\varphi(z)|^2)^{\beta+1/q}} = 0. \tag{3.8}$$

Proof Suppose that (3.8) holds. Then, from Lemma 2.3, $C^{[\beta]}_{\varphi,g} : H(p,q,\mu) \to \mathbb{B}^\omega_0$ is compact if and only if

$$\lim_{|z|\to 1} \sup_{\|f\|_{H(p,q,\mu)}\leq 1} \omega(z) \left| \left(C^{[\beta]}_{\varphi,g} f\right)'(z)\right| = 0. \tag{3.9}$$

For any $z \in \mathbb{D}$ and $f \in H(p,q,\mu)$, by Lemma 2.1 we have

$$\omega(z) \left| \left(C^{[\beta]}_{\varphi,g} f\right)'(z)\right| \leq C\|f\|_{H(p,q,\mu)} \frac{\omega(z)|g(z)|}{\mu(|\varphi(z)|)(1-|\varphi(z)|^2)^{\beta+1/q}}. \tag{3.10}$$

From (3.9) and (3.10), the implication follows.

Conversely, assume that $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}_0^\omega$ is compact. Then $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}^\omega$ is compact, and $C_{\varphi,g}^{[\beta]} : H(p,q,\mu) \to \mathbb{B}_0^\omega$ is bounded. Hence, by Theorems 3.2 and 3.3, we see that (3.4) and (3.5) hold. By (3.5), for every $\varepsilon > 0$ there exists an $r \in (0,1)$ such that

$$\frac{\omega(z)|g(z)|}{\mu(|\varphi(z)|)(1-|\varphi(z)|^2)^{\beta+1/q}} < \varepsilon,$$

when $r < |\varphi(z)| < 1$. By (3.4), there exists a $\delta \in (0,1)$ such that

$$\omega(z)|g(z)| < \varepsilon \inf_{t\in[0,\delta]} \mu(t)(1-t^2)^{\beta+1/q},$$

when $\sigma < |z| < 1$. Therefore, when $\sigma < |z| < 1$ and $r < |\varphi(z)| < 1$, we have that

$$\frac{\omega(z)|g(z)|}{\mu(|\varphi(z)|)(1-|\varphi(z)|^2)^{\beta+1/q}} < \varepsilon. \tag{3.11}$$

If $\sigma < |z| < 1$ and $|\varphi(z)| \leq r$, then we obtain

$$\frac{\omega(z)|g(z)|}{\mu(|\varphi(z)|)(1-|\varphi(z)|^2)^{\beta+1/q}} < \frac{\omega(z)|g(z)|}{\inf_{t\in[0,\delta]} \mu(t)(1-t^2)^{\beta+1/q}} < \varepsilon. \tag{3.12}$$

Combining (3.11) with (3.12), we obtain (3.9).

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Conformal automorphisms for exact locally conformally callibrated $\tilde{G}_2$-structures

Mobeen Munir[1], Waqas Nazeer[2], Asma Ashraf[3] and Shin Min Kang[4,5,*]

[1]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: mmunir@ue.edu.pk

[2]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: nazeer.waqas@ue.edu.pk

[3]Division of Science and Technology, University of Education, Lahore 54000, Pakistan
e-mail: asmaa.azaam@yahoo.com

[4]Department of Mathematics and RINS, Gyeongsang National University, Jinju 52828, Korea
e-mail: smkang@gnu.ac.kr

[5]Center for General Education, China Medical University, Taichung 40402, Taiwan

Abstract

First we characterize a conformal automorphism for exact locally conformally calibrated $\tilde{G}_2$-structures and give Lie derivative of the fundamental 3-form defining $\tilde{G}_2$-structures for this class of manifolds. In the end we prove some nice properties for this class.

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1 Introduction

Recently, the theory of special G-structures on smooth manifolds has been an astonishing success story among mathematicians and physicist as they exhibit some nice properties. For example G_2-structure can be geometric models in the theory of super strings with torsion [19]. Also Donaldson and Segal [10] suggested recently that manifolds with non-vanishing torsion G_2-structure can be the right framework for guage theory in dimension 7. Main computable models for manifolds with G_2-structure are homogeneous spaces having co-homogeneity one [9, 25, 29].

* Corresponding author

Historically the first sign of g_2^C(remarkable exceptional simple Lie algebra) appeared in 1884, when Killing gave a proof of its existence. In 1907, Reichel [28], a student of Engel [11], proved that Lie groups G_2 and $\tilde{G}_2$ are two real forms of G_2^C. In 1914, Cartan proved that G_2 and $\tilde{G}_2$ can be regarded as the automorphism group of octonions and split-octonions respectively in 1914. Later these groups appeared in the Bereger's celebrated list of potential holonomy of pseudo-Riemannian mertic (see [2]). Quest for examples of metrics having holonomy G_2 and $\tilde{G}_2$ remained unsuccessful until 1989 when Bryant and Salamon [6] constructed first complete but non-compact Riemannian manifolds having holonomy G_2. The construction of first compact example by Joyce [20] in 1994 was a huge breakthrough.

We recall that a smooth manifold M^7 is said to have a $\tilde{G}_2$-structure if it has a section of the bundle $\mathcal{F}(M^7)/\tilde{G}_2$ on M^7, where $\mathcal{F}(M^7)$ is the frame bundle on M^7. It is noted that the automorphism group of a 3-form $\tilde{\varphi}$ over $\mathbb{R}^7$ is $\tilde{G}_2$ which is called a 3-form of $\tilde{G}_2$-type [15]. It is known that $GL(\mathbb{R}^7)$-orbit of $\tilde{\varphi}$ is an open orbit of the $GL(\mathbb{R}^7)$-action on $\Lambda^3(\mathbb{R}^7)$. A 3-form in that open orbit is known as indefinite 3-form. The presence of a $\tilde{G}_2$-structure on a manifold M^7 is equivalent to the presence of an indefinite differential 3-form $\tilde{\varphi}$ over M^7. A $\tilde{G}_2$-structure $\tilde{\varphi}$ on a manifold is called parallel if $\nabla\tilde{\varphi} = 0$ or $d\tilde{\varphi} = d * \tilde{\varphi} = 0$ and almost parallel or calibrated if $d\tilde{\varphi} = 0$, locally conformal calibrated if $d\tilde{\varphi} = \theta \wedge \tilde{\varphi}$ with a differential 1-form θ on M and $\theta = \frac{1}{4}(*(*d\tilde{\varphi} \wedge \tilde{\varphi})$ [4, 8, 12, 13].

We say that a locally conformal calibrated G_2-structure is d_θ-exact with $\tilde{\varphi} = d_\theta\omega = d\omega - \theta \wedge \omega$, where θ is 1-form and ω is a 2-form on M. Manifold carrying these special structure have been extensively studies for some nice properties. In [1] Bangaya described locally conformal symplectic manifolds. In [14] authors discussed locally conformal calibrated G_2-manifolds.

Fernández and Gray [15] classified all G_2-structures in 16 classes in 1982 by decomposing the covariant derivative of the 3-form defining the G_2-structures in 4 irreducible components. A lot has already been said about these different classes. For example, in [18] Friedrich et al. discussed special properties of nearly parallel G_2-structures and proved that they carry Einstein metrics. In [16] Fernández and Ugrate gave a differential sub-copmlex of de Rham complex for locally conformal calibrated $\tilde{G}_2$-manifolds and determined its ellipticity. A deep insight about these classes were described by Cabrera et al. [8]. In [7] Cabrrera discussed the inclusion relations of these classes and discovered strict inclusion in particular two classes. Kath [21] started the study of psudo-Riemannian 7-manifolds with a $\tilde{G}_2$-structure. Munir and Nizami in [27] gave classification of $\tilde{G}_2$-structures based on intrinsic torsion with sixteen classes of algebraic types of $\tilde{G}_2$-structures and also proved some strict inclusion relations among the classes of these structures. Generally speaking, manifold with $\tilde{G}_2$-structures are relatively less understood as compared to those admitting G_2. To our knowledge there are only a few papers discussing about them, (see for example [5, 21, 22, 23, 25, 27]).

In this paper, we study manifolds endowed with a locally conformal calibrated $\tilde{G}_2$-structure which constitute the class $\mathcal{W}_2 \oplus \mathcal{W}_4$ of [27]. We focus on its subspace where we have exact locally conformal calibrated $\tilde{G}_2$-structure. However it is worth mentioning that we study these manifolds for two particular reasons. First, they have striking similarities with those admitting a G_2-structure and secondly, because of their interesting class in pseudo-Riemannian geometry, see [7, 30].

2 Locally conformal calibrated $\tilde{G}_2$-structure

Here we first introduce the basic representations for $\tilde{G}_2$-manifolds. Then we give simple characterizations of locally conformal calibrated $\tilde{G}_2$-manifolds. These results are known facts see for example [25, 27]. These fact will help a lot to prove our main results in next part. Let $\Lambda^q(M)$ be the space of differential q-forms on M and $\mathcal{B}^q(M)$ is the subspace of $\Lambda^q(M)$ defined by

$$\mathcal{B}^q(M) = \{\beta \epsilon \Lambda^q(M) \mid \beta \wedge \tilde{\varphi} = 0\}.$$

A $\tilde{G}_2$-manifold is defined as a 7-dimensional Riemannian manifold M (in which a Riemannian metric $g_{\tilde{\varphi}} = (1, 1, 1, -1, -1, -1, -1)$ is defined) endowed with a 2-fold vector cross product P satisfying the following axioms

1. $\langle P(X_1, X_2), X_1\rangle = \langle P(X_1, X_2), X_2\rangle = 0,$
2. $\|P(X_1, X_2)\|^2 = \|X_1\|^2\|X_2\|^2 - \langle X_1, X_2\rangle^2$

for $X_1, X_2 \in \mathfrak{X}(M)$. The fundamental 3-form on M is then defined as

$$\tilde{\varphi}(X_1, X_2, X_3) = \langle P(X_1, X_2), X_3\rangle$$

for $X_1, X_2, X_3 \in \mathfrak{X}(M)$ and inner product for $x, y \in \wedge^q(M)$ is defined as

$$\langle x, y\rangle V_M = x \wedge *y, \tag{2.1}$$

where V_M is the volume form on M. It is proved that $\wedge^q(M)$ splits orthogonally into $\tilde{G}_2$-irreducible components $\wedge^q_l$ of dimension l [4]. An isometry known as Hodge star operator defined as $* : \wedge^q(M) \longrightarrow \wedge^{7-q}(M)$ make two irreducible component isomorphic. For example the representation of $\tilde{G}_2$ on $\wedge^1(M)$ and $\wedge^7(M)$ are isomorphic. So it is sufficient to describe the representation of $\tilde{G}_2$ on $\wedge^2(M)$ and $\wedge^3(M)$ as follows

$$\begin{cases} \wedge^2_7(M) = \{*(\alpha \wedge *\tilde{\varphi}) \mid \alpha \in \wedge^1(M)\} \\ \wedge^2_{14}(M) = \{\beta \in \wedge^2(M) \mid \beta \wedge *\tilde{\varphi} = 0\} \\ \wedge^3_1(M) = \{f\tilde{\varphi} \mid f \in \mathfrak{F}(M)\} \\ \wedge^3_7(M) = \{*(\alpha \wedge \tilde{\varphi}) \mid \alpha \in \wedge^1(M)\} \\ \wedge^3_{27}(M) = \{\gamma \in \wedge^3(M) \mid \gamma \wedge \tilde{\varphi} = \gamma \wedge *\tilde{\varphi} = 0. \end{cases} \tag{2.2}$$

From above, it is easy to compute

$$\wedge^3_1(M) \oplus \wedge^3_{27}(M) = \{\gamma \in \wedge^3(M) \mid \gamma \wedge \tilde{\varphi} = 0\}. \tag{2.3}$$

$$\wedge_7^4(M) \oplus \wedge_{27}^4(M) = \{\lambda \in \wedge^4(M) \mid \lambda \wedge \tilde{\varphi} = 0\}. \tag{2.4}$$

For M^7, most general $\tilde{G}_2$-structure can be distinguished by a globally defined 3-form $\tilde{\varphi}$, which has local representation

$$\tilde{\varphi} = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} + e^{347} + e^{356} \tag{2.5}$$

with respect to some local co frame $e^1, e^2, ..., e^7$ see [3]. It induces $g_{\tilde{\varphi}}$ and $dV_{g\tilde{\varphi}}$ on M given by

$$g_{\tilde{\varphi}}(X, Y) = \frac{1}{6} i_X\tilde{\varphi} \wedge i_Y\tilde{\varphi} \wedge \tilde{\varphi}$$

for all vector fields X, Y on M, where $g_{\tilde{\varphi}}$ is a Riemannian metric and $dV_{g\tilde{\varphi}}$ is a volume form.

Now we have the following result.

Proposition 2.1. *Let M be a manifold endowed a $\tilde{G}_2$-structure $\tilde{\varphi}$. Then*

(1) *For any differential 1-form α on M, $*(*(\alpha \wedge \tilde{\varphi}) \wedge \tilde{\varphi}) = 4\alpha$*

(2) *If there is a differential 1-form η on M such that $d\tilde{\varphi} = \eta \wedge \tilde{\varphi}$, then $\eta = \frac{1}{4}(*(*d\tilde{\varphi} \wedge \tilde{\varphi})$ and M is locally conformal calibrated.*

Proof. (1) Let $\tilde{\varphi}$ be 3-form given as in (2.5), and $\alpha = \sum_{i=1}^{7} e^i$ be a 1-form on M then from simple computation it can be easily verified that

$$*(*(\alpha \wedge \tilde{\varphi}) \wedge \tilde{\varphi}) = 4\alpha.$$

(2) Let η be a differential 1-form on M and $d\tilde{\varphi} = \eta \wedge \tilde{\varphi}$ then $*d\tilde{\varphi} = *(\eta \wedge \tilde{\varphi})$.

By taking wedge product by $\tilde{\varphi}$, we get

$$*d\tilde{\varphi} \wedge \tilde{\varphi} = *(\eta \wedge \tilde{\varphi}) \wedge \tilde{\varphi}.$$

Applying $*$ on both sides

$$*(*d\tilde{\varphi} \wedge \tilde{\varphi}) = *(*(\eta \wedge \tilde{\varphi}) \wedge \tilde{\varphi}) = 4\eta.$$

From above $\eta = \frac{1}{4} * (*d\tilde{\varphi} \wedge \tilde{\varphi})$, which implies M is locally conformal calibrated. ☐

Definition 2.2. Let M be a $\tilde{G}_2$ manifold having 3-form $\tilde{\varphi}$. For each l, $0 \leq l \leq 7$, we denote the space $\mathcal{B}^l(M) = \{\lambda \in \Lambda^l(M) | \lambda \wedge \tilde{\varphi} = 0\}$. Also, the orthogonal compliment of $\mathcal{B}^l(M)$ in $\Lambda^q(M)$ is denoted by $\mathcal{A}^l(M)$.

Lemma 2.3. *Let M be a $\tilde{G}_2$-manifold. Then we have the following*

$$\begin{aligned} \mathcal{B}^l(M) &= \{0\} \quad \textit{for } 0 \leq l \leq 2, \\ \mathcal{B}^3(M) &= \Lambda_1^3(M) \oplus \Lambda_{27}^3(M), \\ \mathcal{B}^4(M) &= \Lambda_7^4(M) \oplus \Lambda_{27}^4(M), \\ \mathcal{B}^l(M) &= \Lambda^l(M) \quad \textit{for } 5 \leq l \leq 7. \end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{A}^l(M) &= \Lambda^l(M) \quad \textit{for } 0 \le l \le 2,\\
\mathcal{A}^3(M) &= \Lambda^3_7(M),\\
\mathcal{A}^4(M) &= \Lambda^4_1(M),\\
\mathcal{A}^q(M) &= \{0\} \quad \textit{for } 5 \le l \le 7.
\end{aligned}$$

Proposition 2.4. *Let M be a $\tilde{G}_2$ manifold endowed with fundamental 3-form $\tilde{\varphi}$. Then M is locally conformal calibrated if and only if for any differential 3-form $\rho \in \Lambda^3_1(M) \oplus \Lambda^3_{27}(M)$, the exterior differential $d\rho \in \Lambda^4_7(M) \oplus \Lambda^4_{27}(M)$.*

Proof. Let M be a locally conformal calibrated G_2 and $d\tilde{\varphi} = \theta \wedge \tilde{\varphi}$. Also let $\rho \in \Lambda^3_1(M) \oplus \Lambda^3_{27}(M)$. From equation (2.4) follows that

$$d\rho \wedge \tilde{\varphi} = d(\rho \wedge \tilde{\varphi}) - \rho \wedge d\tilde{\varphi} = -\rho \wedge \theta \wedge \tilde{\varphi} = 0$$

using equation (2.4) $d\rho \in \Lambda^4_7(M) \oplus \Lambda^4_{27}(M)$.
Conversely, let $d\tilde{\varphi} \in \Lambda^4_7(M) \oplus \Lambda^4_{27}(M)$ because $\tilde{\varphi} \in \Lambda^3_1(M)$. Also we have

$$\tilde{\varphi} = \theta \wedge \tilde{\varphi} \wedge *\rho, \tag{2.6}$$

where $\theta \wedge \tilde{\varphi} \in \Lambda^4_7(M)$ and $\rho \in \Lambda^3_{27}(M)$. Thus $d\rho \wedge \tilde{\varphi} = 0$, and we deduce that

$$\rho \wedge d\tilde{\varphi} = d\rho \wedge \tilde{\varphi} - d(\rho \wedge \tilde{\varphi}) = 0 \tag{2.7}$$

Taking wedge product by y in equation (2.6), and using equation (2.7), we get

$$\begin{aligned}
0 &= y \wedge d\tilde{\varphi}\\
&= y \wedge \theta \wedge \tilde{\varphi} + y \wedge *y\\
&= y \wedge *y,
\end{aligned}$$

which implies that $y = 0$. Then equation (2.6) becomes

$$d\tilde{\varphi} = \theta \wedge \tilde{\varphi},$$

which, by Proposition 2.1, proves that M is locally conformal calibrated. □

3 Exact locally conformal calibrated $\tilde{G}_2$-structure

In this part we mainly use the concept developed in previous section. In [1] on locally conformally symplectic manifolds, authors found some characterizations, so on following similar track we find for d_θ-exact locally conformal calibrated $\tilde{G}_2$-structures $\tilde{\varphi}$ having 1-form θ, called Lee form. Then we give some characterization of conformal automorphisms for exact locally conformal calibrated $\tilde{G}_2$-structures and derive some new useful properties for these manifolds.

We already know that $Y \in \mathfrak{X}(M)$, smooth vector fields on M is a conformal infinitesimal automorphism of $\tilde{\varphi}$ iff there exists a function ρ_Y which is smooth on M satisfying $\mathfrak{L}\tilde{\varphi} = \rho_Y \tilde{\varphi}$ and vector field Y is said to be conformal automorphism of $\tilde{\varphi}$ if $\rho_Y \equiv 0$.

First we have the following proof.

Proposition 3.1. *Let $\tilde{\varphi}$ be a $\tilde{G}_2$-structure on M^7. Let $Y \in \mathfrak{X}(M)$ be a vector field and ω (a 2-form) satisfying $\omega = i_Y\tilde{\varphi}$. Then we have*

$$|\omega|^2 = 3|Y|^2$$

Proof. The identity implies that $\tilde{\varphi} \wedge (i_Y\tilde{\varphi}) = 2 * (i_Y\tilde{\varphi})$,our case becomes $\tilde{\varphi} \wedge \omega = 2 * \omega$ and

$$\begin{aligned} |\omega|^2 * 1 &= \omega \wedge *\omega \\ &= \frac{1}{2}\omega \wedge \tilde{\varphi} \wedge \omega \\ &= \frac{1}{2}(i_Y\tilde{\varphi}) \wedge (i_Y\tilde{\varphi}) \wedge \tilde{\varphi} \\ &= 3|Y|^2 * 1. \end{aligned}$$

□

Which leads to the desired conclusion.

Proposition 3.2. *Let $(M, \tilde{\varphi})$ be a locally conformal calibrated $\tilde{G}_2$-structure having Lee form θ.*

(1) *A vector field $Y \in \mathfrak{X}(M)$ is a conformal infinitesimal automorphism of $\tilde{\varphi}$ if and only if there exists a function which is smooth $f_Y \in C^\infty(M)$ satisfying $d_\theta\omega = f_Y\tilde{\varphi}$, where $\omega = i_Y\tilde{\varphi}$.*

(2) *For M to be connected, f_Y is constant.*

Proof. (1) Here we have by the following expression

$$\begin{aligned} \mathfrak{L}_{Y\varphi} &= d(i_Y\tilde{\varphi}) + i_Y(d\tilde{\varphi}) \\ &= d\omega + i_Y(\theta \wedge \tilde{\varphi}) \\ &= d\omega + \theta(Y)\tilde{\varphi} - \theta \wedge (i_Y\tilde{\varphi}) \\ &= d\omega - \theta \wedge \omega + \theta(Y)\tilde{\varphi} \\ &= d_{\theta\omega} + \theta(Y)\tilde{\varphi}, \end{aligned}$$

where $\omega = i_Y\tilde{\varphi}$. Hence, Y is a conformal infinitesimal automorphism of $\tilde{\varphi}$ with $\mathfrak{L}_Y\tilde{\varphi} = \rho_Y\tilde{\varphi}$ iff $d_\theta\omega = f_Y\tilde{\varphi}$, where $f_Y =$ a function which is smooth on M and $f_Y = \rho_Y + \theta(Y)$.

(2) If we take M be a connected and Y a conformal infinitesimal automorphism of $\tilde{\varphi}$. As $d_\theta\omega = f_Y\tilde{\varphi}$ for some $f_Y \in C^\infty(M)$. We have

$$\begin{aligned} 0 &= d_\theta(d_{\theta\omega}) \\ &= d_\theta(f_Y\tilde{\varphi}) \\ &= d(f_Y\tilde{\varphi})_\theta \wedge (f_Y\tilde{\varphi}) \\ &= df_Y \wedge \tilde{\varphi} + f_Y d\tilde{\varphi} - f_Y(\theta \wedge \tilde{\varphi}) \\ &= df_Y \wedge \tilde{\varphi} + f_Y d\tilde{\varphi} - f_Y d\tilde{\varphi} \\ &= df_Y \wedge \tilde{\varphi}. \end{aligned}$$

As we know that the mapping $\wedge\tilde{\varphi} : \Lambda^1(M) \to \Lambda^4(M)$ is a linear injective mapping and we obtain $df_Y = 0$ consequently as M is connected so f_Y is constant. □

Proposition 3.3. *If Y be a conformal infinitesimal automorphism of $\tilde{\varphi}$ with $f_Y \neq 0$, then $\tilde{\varphi}$ is d_θ-exact.*

Proof.

$$\tilde{\varphi} = \frac{1}{f_Y} d_\theta \omega = d_\theta\bigg(\frac{\omega}{f_Y}\bigg).$$

So $\tilde{\varphi}$ is d_θ-exact. □

Now we give an important result that can evaluate some integrals of a conformal infinitesimal automorphism of $\tilde{\varphi}$. We have

$$\int_M \mathfrak{L}_Y \tilde{\varphi} \wedge *f\tilde{\varphi} = -3\int_M df \wedge *Y^b$$

for a compact M^7,$f \in C^\infty(M)$, Y as a conformal infinitesimal automorphism of $\tilde{\varphi}$ with $\mathfrak{L}_Y \tilde{\varphi} = \rho_Y \tilde{\varphi}$.

Proposition 3.4. *Let Y be a conformal infinitesimal automorphism of $\tilde{\varphi}$ with $\mathfrak{L}_Y \tilde{\varphi} = \rho_Y \tilde{\varphi}$ we have $\int_M f_Y dV_{g\tilde{\varphi}} = 0$*

Proof. For the case of $\tilde{G}_2$-structure we modify the result of [26], that says, for a compact manifold $(M^7, \tilde{\phi})$ where $\tilde{\phi}$ is any general $\tilde{G}_2$-structure with

$$\int_M \mathfrak{L}_Y \tilde{\varphi} \wedge *f\tilde{\varphi} = -3\int_M df \wedge *Y^b$$

where $f \in C^\infty(M)$, Y as a conformal infinitesimal automorphism of $\tilde{\varphi}$ with $\mathfrak{L}_Y \tilde{\varphi} = \rho_Y \tilde{\varphi}$.

Take $f \equiv 1$, we arrive at

$$\int_M \rho_Y dV_{g\tilde{\varphi}} = 0.$$

Using Proposition 3.3, we get

$$\int_M \theta(Y) dV_{g\tilde{\varphi}} = \int_M f_Y dV_{g\tilde{\varphi}} = f_Y Vol(M)$$

this confirms the constancy of Riemann integeral of $\theta(Y)$ over M. □

As the consequences of above results, now we are able to give important characterizations of exact locally conformal calibrated $\tilde{G}_2$-structures.

Proposition 3.5. *Let $(M^7, \tilde{\varphi})$ be a connected locally conformal calibrated $\tilde{G}_2$-manifold and θ be associated Lee form. Let $g_{\tilde{\varphi}}$ be a dual vector field of θ denoted by Y satisfying $\theta(\cdot) = g_{\tilde{\varphi}}(Y, \cdot)$, and $\omega := i_Y \tilde{\varphi}$, where ω is a 2-form. Then we have the following results*

(1) *$\mathfrak{L}_Y \tilde{\varphi} = 0$ if and only if $\theta(Y)\tilde{\varphi} = d_\theta \omega$.*

(2) *If $\mathfrak{L}_Y \tilde{\varphi} = 0$, then $\theta(Y) = |Y|^2 \neq 0$ (a constant).*

Proof. (1) Here it is

$$\begin{aligned}\mathfrak{L}_Y\tilde{\varphi} &= d(i_Y\tilde{\varphi}) + i_Y d\tilde{\varphi}\\ &= d\omega + i_Y(\theta\wedge\tilde{\varphi})\\ &= d\omega + \theta(Y)\tilde{\varphi} - \theta\wedge\omega.\end{aligned}$$

Hence, $\mathfrak{L}_Y\tilde{\varphi}$ vanishes if and ony if $\theta(Y)\tilde{\varphi} = -d_\theta\omega$.

(2) From Proposition 3.2, If $\mathfrak{L}_Y\tilde{\varphi} = 0$ then $\theta(Y) = |Y|^2 \neq 0$ (a constant). Since $\theta(Y)\tilde{\varphi} = d_\theta\omega$ and $Y = \theta^t$, where the map $t : \Lambda^1(M) \to \mathfrak{Y}(M)$ is an isomorphism. $\square$

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FOURIER SERIES OF FINITE PRODUCTS OF BERNOULLI AND EULER FUNCTIONS

TAEKYUN KIM[1,2], DAE SAN KIM[3], DMITRY V. DOLGY[4], AND JIN-WOO PARK[5,*]

Abstract. In this paper, we will consider three types of sums of finite products of Bernoulli and Euler functions, and derive the Fourier series expansions of them. In addition, we will express each of them in terms of Bernoulli functions.

1. Introduction

Let $B_m(x)$ be the *Bernoulli polynomials* given by the generating function

$$\frac{t}{e^t-1}e^{xt}=\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!},\ \ (\text{see }\ [7,13,23]).$$

For $x=0$, $B_m=B_m(0)$ are called *Bernoulli numbers.*

Also, let $E_m(x)$ be the *Euler polynomials* defined by he generating function

$$\frac{2}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}E_m(x)\frac{t^m}{m!},\ \ (\text{see }\ [4,19,23]).$$

For $x=0$, $E_m=E_m(0)$ are called *Euler numbers.*

It is well known that the Bernoulli and Euler polynomials have the following properties

$$\frac{d}{dx}B_m(x)=mB_{m-1}(x),\ \frac{d}{dx}E_m(x)=mE_{m-1}(x),\ (m\geq 1),$$
$$B_m(1)=B_m+\delta_{1,m},\ E_m(1)=-E_m+2\delta_{0,m},\ (m\geq 0).$$

For any real number x, we let

$$\langle x\rangle = x-\lfloor x\rfloor\in[0,\ 1)$$

denote the fractional part of x.

We will need the following facts about Bernoulli functions $B_m(\langle x\rangle)$:

(i) for $m\geq 2$,

$$B_m(\langle x\rangle)=-m!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^m},$$

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* Corresponding author.

(ii) for $m = 1$,

$$-\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi inx}}{2\pi in} = \begin{cases} B_1(\langle x\rangle), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}$$

Throughout this paper, we will assume that r, s are nonnegative integers with $r + s \geq 1$. Here we will consider three types of sums of finite products of Bernoulli and Euler functions $\alpha_m(\langle x\rangle)$, $\beta_m(\langle x\rangle)$, and $\gamma_m(\langle x\rangle)$ and derive the Fourier series expansions of them. In addition, we will express each of them in terms of Bernoulli functions.

(1)

$$\alpha_m(\langle x\rangle) = \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle) \times E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle),\ (m \geq 1);$$

(2)

$$\beta_m(\langle x\rangle) = \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1!\cdots i_r! j_1!\cdots j_s!} B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle) \times E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle),\ (m \geq 1);$$

(3)

$$\beta_m(\langle x\rangle) = \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1\cdots i_r j_1\cdots j_s} B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle) \times E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle),\ (m \geq r+s).$$

Here the sums for (1) and (2) are over all nonnegative integers $i_1, \ldots, i_r, j_1, \ldots, j_s$ with $i_1 + \cdots + i_r + j_1 + \cdots + j_s = m$, and the sums for (3) are over all positive integers $i_1, \ldots, i_r, j_1, \ldots, j_s$ with $i_1 + \cdots + i_r + j_1 + \cdots + j_s = m$.

For elementary facts about Fourier analysis, the reader may refer to any book (for example, see [1,20,24]). As to $\alpha_m(\langle x\rangle)$, we note that the polynomial identity (1.1) follows immediately from Theorems 2.1 and 2.2, which is in turn derived from the Fourier series expansion of $\alpha_m(\langle x\rangle)$:

$$\begin{aligned} &\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x) \\ &= \frac{1}{m+r+s}\sum_{j=0}^{m}\binom{m+r+s}{j}\Delta_{m-j+1}B_j(x), \end{aligned} \tag{1.1}$$

where, for each positive integer l,

$$\Delta_l = \sum_{\substack{0\leq a\leq r \\ 0\leq c\leq s \\ r-l\leq a\leq r}} \binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c} \sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=a+l-r} B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c} - \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}.$$

The obvious polynomial identities can be derived also for $\beta_m(\langle x\rangle)$ from Theorems 3.1 and 3.2. It is noteworthy that from the Fourier series expansion of the function $\sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(\langle x \rangle) B_{m-k}(\langle x \rangle)$ we can derive the Faber-Pandharipande-Zagier identity (see [5,6,9-12,21,22]) and the Miki's identity (see [3,9-12]). The reader may refer to the recent papers [8,14-16,18] for the related results.

2. Sums of finite products of Bernoulli and Euler functions of the first type

Let

$$\alpha_m(x) = \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} B_{i_1}(x)\cdots B_{i_r}(x) E_{j_1}(x)\cdots E_{j_s}(x), \ (m \geq 1),$$

where the sum runs over all nonnegative integers $i_1, \dots, i_r, j_1, \dots, j_s$ satisfying $i_1 + \cdots + i_r + j_1 + \cdots + j_s = m$. Then we will consider the function

$$\alpha_m(\langle x \rangle) = \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} B_{i_1}(\langle x \rangle)\cdots B_{i_r}(\langle x \rangle) E_{j_1}(\langle x \rangle)\cdots E_{j_s}(\langle x \rangle),$$

defined on $\mathbb{R}$ which is periodic with period 1.

The Fourier series of $\alpha_m(\langle x \rangle)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi i n x},$$

where

$$A_n^{(m)} = \int_0^1 \alpha_m(\langle x \rangle) e^{-2\pi i n x} dx = \int_0^1 \alpha_m(x) e^{-2\pi i n x} dx.$$

To continue our discussion, we need to observe the following.

$$\begin{aligned}
&\alpha_m'(x) \\
=& \sum_{\substack{i_1+\cdots+i_r+j_1+\cdots+j_s=m \\ i_1 \geq 1}} i_1 B_{i_1-1}(x) B_{i_2}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_s}(x) \\
&+ \cdots + \sum_{\substack{i_1+\cdots+i_r+j_1+\cdots+j_s=m \\ i_r \geq 1}} B_{i_1}(x) \cdots B_{i_{r-1}}(x) i_r B_{i_r-1}(x) E_{j_1}(x) \cdots E_{j_s}(x) \\
&+ \sum_{\substack{i_1+\cdots+i_r+j_1+\cdots+j_s=m \\ j_1 \geq 1}} B_{i_1}(x) B_{i_2}(x) \cdots B_{i_r}(x) j_1 E_{j_1-1}(x) E_{j_2}(x) \cdots E_{j_s}(x) \\
&+ \cdots + \sum_{\substack{i_1+\cdots+i_r+j_1+\cdots+j_s=m \\ j_s \geq 1}} B_{i_1}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_{s-1}}(x) j_s E_{j_s-1}(x) \\
=& \sum_{i_1+\cdots+j_r+j_1+\cdots+j_s=m-1} (i_1+1) B_{i_1}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_s}(x) \\
&+ \cdots + \sum_{i_1+\cdots+j_r+j_1+\cdots+j_s=m-1} (i_r+1) B_{i_1}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_s}(x) \\
&+ \sum_{i_1+\cdots+j_r+j_1+\cdots+j_s=m-1} (j_1+1) B_{i_1}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_s}(x) \\
&+ \cdots + \sum_{i_1+\cdots+j_r+j_1+\cdots+j_s=m-1} (j_s+1) B_{i_1}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_s}(x) \\
=&(m+r+s-1)\alpha_{m-1}(x).
\end{aligned}$$

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From this, we obtain

$$\left(\frac{\alpha_{m+1}(x)}{m+r+s}\right)' = \alpha_m(x),$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+r+s}\left(\alpha_{m+1}(1) - \alpha_{m+1}(0)\right).$$

For $m \geq 1$, we set

$$\begin{aligned}
&\Delta_m = \alpha_m(1) - \alpha_m(0)\\
&= \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \left(B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1) - B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right)\\
&= \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} (B_{i_1}+\delta_{1,i_1})\cdots(B_{ir}+\delta_{1,i_r})(-E_{j_1}+2\delta_{0,j_1})\cdots(-E_{j_s}+2\delta_{0,j_s})\\
&\quad - \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\\
&= \sum_{\substack{0\leq a\leq r\\ 0\leq c\leq s\\ r-m\leq a\leq r}} \binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c} \sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=a+m-r} B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}\\
&\quad - \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}.
\end{aligned}$$

Note here that the sum over all $i_1+\cdots+i_r+j_1+\cdots+j_s=m$ of any term with a of B_{i_e}, $r-a$ of δ_{1,i_f} $(1\leq e,f\leq r)$, c of $-E_{j_u}$, and $s-c$ of $2\delta_{0,j_v}$ $(1\leq u,v\leq s)$ all give the same sum

$$\begin{aligned}
&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} B_{i_1}\cdots B_{i_a}\delta_{1,i_{a+1}}\cdots\delta_{1,i_r}(-E_{j_1})\cdots(-E_{j_c})(2\delta_{0,j_{c+1}})\cdots(2\delta_{0,j_s})\\
&= \sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=m+a-r} (-1)^c 2^{s-c} B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c},
\end{aligned}$$

which is not an empty sum as long as $m+a-r\geq 0$.

We now see that

$$\alpha_m(0) = \alpha_m(1) \Longleftrightarrow \Delta_m = 0,$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+r+s}\Delta_{m+1}.$$

We are now going to determine the Fourier coefficients $A_n^{(m)}$.

Case $1:\ n\neq 0$.

$$\begin{aligned}
A_n^{(m)} &= \int_0^1 \alpha_m(x)e^{-2\pi inx}dx\\
&= -\frac{1}{2\pi in}\left[\alpha_m(x)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 \alpha_m'(x)e^{-2\pi inx}dx\\
&= -\frac{1}{2\pi in}\left(\alpha_m(1)-\alpha_m(0)\right) + \frac{m+r+s-1}{2\pi in}\int_0^1 \alpha_{m-1}(x)e^{-2\pi inx}dx\\
&= \frac{m+r+s-1}{2\pi in}A_n^{(m-1)} - \frac{1}{2\pi in}\Delta_m,
\end{aligned}$$

from which by induction we can easily deduce

$$A_n^{(m)} = -\sum_{j=1}^{m} \frac{(m+r+s-1)_{j-1}}{(2\pi in)^j}\Delta_{m-j+1}$$
$$= -\frac{1}{m+r+s}\sum_{j=1}^{m}\frac{(m+r+s)_j}{(2\pi in)^j}\Delta_{m-j+1}.$$

Case 2 : $n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x)dx = \frac{1}{m+r+s}\Delta_{m+1}.$$

$\alpha_m(\langle x\rangle)$, $(m \geq 1)$ is piecewise C^∞. In addition, $\alpha_m(\langle x\rangle)$ is continuous for those positive integers with $\Delta_m = 0$ and discontinuous with jump discontinuities at integers for those positive integers with $\Delta_m \neq 0$.

Assume first that m is a positive integer with $\Delta_m = 0$. Then $\alpha_m(0) = \alpha_m(1)$. Hence $\alpha_m(\langle x\rangle)$ is piecewise C^∞ and continuous. Thus the Fourier series of $\alpha_m(\langle x\rangle)$ converges uniformly to $\alpha_m(\langle x\rangle)$, and

$$\begin{aligned}
&\alpha_m(\langle x\rangle)\\
=&\frac{1}{m+r+s}\Delta_{m+1} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+r+s}\sum_{j=1}^{m}\frac{(m+r+s)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx}\\
=&\frac{1}{m+r+s}\Delta_{m+1} + \frac{1}{m+r+s}\sum_{j=1}^{m}\binom{m+r+s}{j}\Delta_{m-j+1}\left(-j!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right)\\
=&\frac{1}{m+r+s}\Delta_{m+1} + \frac{1}{m+r+s}\sum_{j=2}^{m}\binom{m+r+s}{j}\Delta_{m-j+1}B_j(\langle x\rangle)\\
&+\Delta_m \times \begin{cases} B_1(\langle x\rangle), & \text{for } x \notin \mathbb{Z},\\ 0, & \text{for } x \in \mathbb{Z}.\end{cases}
\end{aligned}$$

Now, we are ready to state our first result.

Theorem 2.1. *For each positive integer l, we let*

$$\Delta_l = \sum_{\substack{0\leq a\leq r\\ 0\leq c\leq s\\ r-l\leq a\leq r}}\binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c}\sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=a+l-r}B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}$$
$$-\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=l}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}.$$

Assume that m is a positive integer with $\Delta_m = 0$. Then we have the following.

(a)

$$\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle)E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle)$$

has the Fourier expansion

$$\begin{aligned}&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle)E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle)\\ =&\frac{1}{m+r+s}\Delta_{m+1}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+r+s}\sum_{j=1}^{m}\frac{(m+r+s)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx},\end{aligned}$$

for all $x\in\mathbb{R}$, where the convergence is uniform.

(b)

$$\begin{aligned}&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle)E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle)\\ =&\frac{1}{m+r+s}\Delta_{m+1}+\frac{1}{m+r+s}\sum_{j=2}^{m}\binom{m+r+s}{j}\Delta_{m-j+1}B_j(\langle x\rangle),\end{aligned}$$

for all $x\in\mathbb{R}$, where $B_j(\langle x\rangle)$ is the Bernoulli function.

Assume next that $\Delta_m\neq 0$, for a positive integer m. Then $\alpha_m(0)\neq\alpha_m(1)$. Hence $\alpha_m(\langle x\rangle)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(\langle x\rangle)$ converges pointwise to $\alpha_m(\langle x\rangle)$, for $x\notin\mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\alpha_m(0)+\alpha_m(1)\right)=\alpha_m(0)+\frac{1}{2}\Delta_m,$$

for $x\in\mathbb{Z}$.

We are now ready to state our second result.

Theorem 2.2. *For each positive integer l, we let*

$$\begin{aligned}\Delta_l=&\sum_{\substack{0\le a\le r\\ 0\le c\le s\\ r-l\le a\le r}}\binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c}\sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=a+l-r}B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}\\ &-\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=l}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}.\end{aligned}$$

Assume that m is a positive integer with $\Delta_m\neq 0$. Then we have the following.

(a)

$$\begin{aligned}&\frac{1}{m+r+s}\Delta_{m+1}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+r+s}\sum_{j=1}^{m}\frac{(m+r+s)_j}{(2\pi in)^j}\Delta_{m-j+1}\right)e^{2\pi inx}\\ =&\begin{cases}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle)E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle), & \text{for } x\notin\mathbb{Z},\\ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}+\frac{1}{2}\Delta_m, & \text{for } x\in\mathbb{Z}.\end{cases}\end{aligned}$$

(b)

$$\begin{aligned}&\frac{1}{m+r+s}\sum_{j=0}^{m}\binom{m+r+s}{j}\Delta_{m-j+1}B_j(\langle x\rangle)\\ =&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle)E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle),\ \text{for } x\notin\mathbb{Z};\end{aligned}$$

$$\frac{1}{m+r+s}\sum_{\substack{j=0\\ j\neq 1}}^{m}\binom{m+r+s}{j}\Delta_{m-j+1}B_j(\langle x\rangle)$$
$$=\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}+\frac{1}{2}\Delta_m,\ \textit{for } x\in\mathbb{Z}.$$

3. Sums of finite products of Bernoulli and Euler functions of the second type

Let

$$\beta_m(x)=\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}\frac{1}{i_1!\cdots i_r!j_1!\cdots j_s!}B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x),$$
$$(m\geq 1),$$

where the sum runs over all nonnegative integers $i_1,\ldots,i_r,j_1,\ldots,j_s$ satisfying $i_1+\cdots+i_r+j_1+\cdots+j_s=m$.

Then we consider function

$$\beta_m(\langle x\rangle)=\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}\frac{1}{i_1!\cdots i_r!j_1!\cdots j_s!}B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle)$$
$$\times E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\beta_m(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty}B_n^{(m)}e^{2\pi inx},$$

where

$$B_n^{(m)}=\int_0^1\beta_m(\langle x\rangle)e^{-2\pi inx}dx=\int_0^1\beta_m(x)e^{-2\pi inx}dx.$$

To proceed further, we need to observe the following.

$$\beta_m'(x)=\sum_{\substack{i_1+\cdots+i_r+j_1+\cdots+j_s=m\\ i_1\geq 1}}\frac{1}{(i_1-1)!i_2!\cdots i_r!j_1!\cdots j_s!}B_{i_1-1}(x)B_{i_2}(x)\cdots B_{i_r}(x)$$
$$\times E_{j_1}(x)\cdots E_{j_s}(x)$$
$$+\cdots+\sum_{\substack{i_1+\cdots+i_r+j_1+\cdots+j_s=m\\ i_r\geq 1}}\frac{1}{i_1!\cdots i_{r-1}!(i_r-1)!j_1!\cdots j_s!}B_{i_1}(x)\cdots B_{i_{r-1}}(x)B_{i_r-1}(x)$$
$$\times E_{j_1}(x)\cdots E_{j_s}(x)$$
$$+\sum_{\substack{i_1+\cdots+i_r+j_1+\cdots+j_s=m\\ j_1\geq 1}}\frac{1}{i_1!\cdots i_r!(j_1-1)!j_2!\cdots j_s!}B_{i_1}(x)\cdots B_{i_r}(x)$$
$$\times E_{j_1-1}(x)E_{j_2}(x)\cdots E_{j_s}(x)$$
$$+\cdots+\sum_{\substack{i_1+\cdots+i_r+j_1+\cdots+j_s=m\\ j_s\geq 1}}\frac{1}{i_1!\cdots i_r!j_1!\cdots j_{s-1}!(j_s-1)!}B_{i_1}(x)\cdots B_{i_r}(x)$$
$$\times E_{j_1}(x)\cdots E_{j_{s-1}}(x)E_{j_s-1}(x)$$

$$
\begin{aligned}
&= \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m-1} \frac{1}{i_1!\cdots i_r! j_1!\cdots j_s!} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x) \\
&\quad + \cdots + \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m-1} \frac{1}{i_1!\cdots i_r! j_1!\cdots j_s!} B_{i_1}(x)\cdots B_{i_r}(x) \\
&\quad \times E_{j_1}(x)\cdots E_{j_s}(x) \\
&\quad + \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m-1} \frac{1}{i_1!\cdots i_r! j_1!\cdots j_s!} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x) \\
&\quad + \cdots + \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m-1} \frac{1}{i_1!\cdots i_r! j_1!\cdots j_s!} B_{i_1}(x)\cdots B_{i_r}(x) \\
&\quad \times E_{j_1}(x)\cdots E_{j_s}(x) \\
&=(r+s)\beta_{m-1}(x).
\end{aligned}
$$

From this, we have

$$\left(\frac{\beta_{m+1}(x)}{r+s}\right)' = \beta_m(x),$$

and

$$\int_0^1 \beta_m(x)dx = \frac{1}{r+s}\left(\beta_{m+1}(1)-\beta_{m+1}(0)\right).$$

For $m \geq 1$, we put

$$
\begin{aligned}
&\Omega_m = \beta_m(1) - \beta_m(0) \\
&= \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1!\cdots i_r! j_1!\cdots j_s!} \\
&\quad \times \left(B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1) - B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right) \\
&= \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1!\cdots i_r! j_1!\cdots j_s!} \\
&\quad \times (B_{i_1}+\delta_{1,i_1})\cdots(B_{i_r}+\delta_{1,i_r})(-E_{j_1}+2\delta_{0,j_1})\cdots(-E_{j_s}+2\delta_{0,j_s}) \\
&\quad - \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!\cdots i_r! j_1!\cdots j_s!} \\
&= \sum_{\substack{0\leq a\leq r\\ 0\leq c\leq s\\ r-m\leq a\leq r}} \binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c} \sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=m+a-r} \frac{B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}}{i_1!\cdots i_a! j_1!\cdots j_c!} \\
&- \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!\cdots i_r! j_1!\cdots js!}.
\end{aligned}
$$

We now see that

$$\beta_m(0)=\beta_m(1) \Longleftrightarrow \Omega_m = 0,$$

and

$$\int_0^1 \beta_m(x)dx = \frac{1}{r+s}\Omega_{m+1}.$$

Now, we would like to determine the Fourier coefficients $B_n^{(m)}$.

Case $1 : n \neq 0$.

$$\begin{aligned}B_n^{(m)} &= \int_0^1 \beta_m(x) e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n}\left[\beta_m(x) e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 \beta_m^{'}(x) e^{-2\pi i n x} dx \\ &= -\frac{1}{2\pi i n}\left(\beta_m(1) - \beta_m(0)\right) + \frac{r+s}{2\pi i n}\int_0^1 \beta_{m-1}(x) e^{-2\pi i n x} dx \\ &= \frac{r+s}{2\pi i n} B_n^{(m-1)} - \frac{1}{2\pi i n}\Omega_m,\end{aligned}$$

from which we can deduce that

$$B_n^{(m)} = \sum_{j=1}^{m} \frac{(r+s)^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}.$$

Case $2 : n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x) dx = \frac{1}{r+s}\Omega_{m+1}.$$

$\beta_m(\langle x\rangle)$, $(m \geq 1)$ is piecewise C^∞. Furthermore, $\beta_m(\langle x\rangle)$ is continuous for those positive integers m with $\Omega_m = 0$, and discontinuous with jump discontinuities at integers for those positive integers m with $\Omega_m \neq 0$.

Assume first that $\Omega_m = 0$, for a positive integer m. Then $\beta_m(0) = \beta_m(1)$. Hence $\beta_m(\langle x\rangle)$ is piecewise C^∞, and continuous. Thus the Fourier series of $\beta_m(\langle x\rangle)$ converges uniformly to $\beta_m(\langle x\rangle)$, and

$$\begin{aligned}B_n^{(m)} &= \frac{1}{r+s}\Omega_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(-\sum_{j=1}^{m}\frac{(r+s)^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}\right) e^{2\pi i n x} \\ &= \frac{1}{r+s}\Omega_{m+1} + \sum_{j=1}^{m}\frac{(r+s)^{j-1}}{j!}\Omega_{m-j+1}\left(-j! \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^j}\right) \\ &= \frac{1}{r+s}\Omega_{m+1} + \sum_{j=2}^{m}\frac{(r+s)^{j-1}}{j!}\Omega_{m-j+1} B_j(\langle x\rangle) \\ &\quad + \Omega_m \times \begin{cases} B_1(\langle x\rangle), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}.\end{cases}\end{aligned}$$

Now, we are ready to state our first result.

Theorem 3.1. *For each positive integer l, we let*

$$\begin{aligned}\Omega_l = &\sum_{\substack{0\leq a\leq r \\ 0\leq c\leq s \\ r-l\leq a\leq r}}\binom{r}{u}\binom{s}{c}(-1)^c 2^{s-c}\sum_{i_1+\cdots+i_a+i_1+\cdots+j_c=l+a-r}\frac{B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}}{i_1!\cdots i_a! j_1!\cdots j_c!} \\ &-\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=l}\frac{B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!\cdots i_r! j_1!\cdots j_s!}.\end{aligned}$$

Assume that m is a positive integer with $\Omega_m = 0$. Then we have the following.

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(a)

$$\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1!\cdots i_r! j_1!\cdots j_s!} B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle) E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle)$$

has the Fourier series expansion

$$\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1!\cdots i_r! j_1!\cdots j_s!} B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle) E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle)$$

$$=\frac{1}{r+s}\Omega_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(-\sum_{j=1}^{m} \frac{(r+s)^{j-1}}{(2\pi i n)^j} \Omega_{m-j+1} \right) e^{2\pi i n x},$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

(b)

$$\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1!\cdots i_r! j_1!\cdots j_s!} B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle) E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle)$$

$$=\sum_{\substack{j=0 \\ j\neq 1}}^{m} \frac{(r+s)^{j-1}}{j!} \Omega_{m-j+1} B_j(\langle x\rangle),$$

for all $x \in \mathbb{R}$, where $B_j(\langle x\rangle)$ is the Bernoulli function.

Assume next that m is a positive integer with $\Omega_m \neq 0$. Then $\beta_m(0) \neq \beta_m(1)$. Hence $\beta_m(\langle x\rangle)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\beta_m(\langle x\rangle)$ converges pointwise to $\beta_m(\langle x\rangle)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\beta_m(0) + \beta_m(1)\right) = \beta_m(0) + \frac{1}{2}\Omega_m,$$

for $x \in \mathbb{Z}$.

Now, we are ready to state our second result.

Theorem 3.2. *For each positive integer l, we let*

$$\Omega_l = \sum_{\substack{0\le a\le r \\ 0\le c\le s \\ r-l\le a\le r}} \binom{r}{u}\binom{s}{c}(-1)^c 2^{s-c} \sum_{i_1+\cdots+i_a+i_1+\cdots+j_c=l+a-r} \frac{B_{i_1}\cdots B_{i_a} E_{j_1}\cdots E_{j_c}}{i_1!\cdots i_a! j_1!\cdots j_c!}$$

$$- \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=l} \frac{B_{i_1}\cdots B_{i_r} E_{j_1}\cdots E_{j_s}}{i_1!\cdots i_r! j_1!\cdots j_s!}.$$

Assume that m is a positive integer with $\Omega_m \neq 0$, for a positive integer m. Then we have the following.

(a)

$$\sum_{j=0}^{m} \frac{(r+s)^{j-1}}{j!} \Omega_{m-j+1} B_j(\langle x\rangle)$$

$$= \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1!\cdots i_r! j_1!\cdots j_s!} B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle) E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle),$$

for $x \notin \mathbb{Z}$;

$$\sum_{\substack{j=0\\j\neq 1}}^{m}\frac{(r+s)^{j-1}}{j!}\Omega_{m-j+1}B_j(\langle x\rangle)$$
$$=\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}\frac{B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!\cdots i_r!j_1!\cdots j_s!}+\frac{1}{2}\Omega_m,\ for\ x\in\mathbb{Z}.$$

(b)

$$\frac{1}{r+s}\Omega_{m+1}+\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\left(-\sum_{j=1}^{m}\frac{(r+s)^{j-1}}{(2\pi in)^j}\Omega_{m-j+1}\right)e^{2\pi inx}$$
$$=\begin{cases}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}\frac{1}{i_1!\cdots i_r!j_1!\cdots j_s!}B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle)E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle),\\ \quad for\ x\notin\mathbb{Z},\\ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}\frac{B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!\cdots i_r!j_1!\cdots j_s!}+\frac{1}{2}\Omega_m,\\ \quad for\ x\in\mathbb{Z}.\end{cases}$$

4. Sums of finite products of Bernoulli and Euler functions of the Third type

Let

$$\gamma_{r,s,m}(x)=\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}\frac{1}{i_1\cdots i_rj_1\cdots j_s}B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x),$$
$$(m\geq r+s),$$

where the sum runs over all positive integers $i_1,\cdots i_r,j_1,\cdots j_s$ satisfying $i_1+\cdots i_r+j_1\cdots+j_s=m$.

Then we consider function

$$\gamma_{r,s,m}(\langle x\rangle)$$
$$=\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}\frac{1}{i_1\cdots i_rj_1\cdots j_s}B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle)E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\gamma_{r,s,m}(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty}C_n^{(r,s,m)}(x)e^{2\pi inx},$$

where

$$C_n^{(r,s,m)}=\int_0^1\gamma_m(\langle x\rangle)e^{-2\pi inx}dx=\int_0^1\gamma_m(x)e^{-2\pi inx}dx.$$

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To continue our discussion, we need to observe the following.

$$
\begin{aligned}
&\gamma'_{r,s,m}(x) = \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_2\cdots i_r j_1\cdots j_s} B_{i_1-1}(x)B_{i_2}(x)\cdots B_{i_r}(x)\\
&\times E_{j_1}(x)\cdots E_{j_s}(x)\\
&+\cdots+ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1\cdots i_{r-1} j_1\cdots j_s} B_{i_1}(x)\cdots B_{i_{r-1}}(x)B_{i_r-1}(x)\\
&\times E_{j_1}(x)\cdots E_{j_s}(x)\\
&+ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1\cdots i_r j_2\cdots j_s} B_{i_1}(x)\cdots B_{i_r}(x)\\
&\times E_{j_1-1}(x)E_{j_2}(x)\cdots E_{j_s}(x)\\
&+\cdots+ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1\cdots i_r j_1\cdots j_{s-1}} B_{i_1}(x)\cdots B_{i_r}(x)\\
&\times E_{j_1}(x)\cdots E_{j_{s-1}}(x)E_{j_s-1}(x)\\
=& \sum_{i_2+\cdots+i_r+j_1+\cdots+j_s=m-1} \frac{1}{i_2\cdots i_r j_1\cdots j_s} B_{i_2}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)\\
&+ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m-1} \frac{1}{i_2\cdots i_r j_1\cdots j_s} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)\\
&+\cdots+ \sum_{i_1+\cdots+i_{r-1}+j_1+\cdots+j_s=m-1} \frac{1}{i_1\cdots i_{r-1} j_1\cdots j_s} B_{i_1}(x)\cdots B_{i_{r-1}}(x)\\
&\times E_{j_1}(x)\cdots E_{j_s}(x)\\
&+ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m-1} \frac{1}{i_1\cdots i_{r-1} j_1\cdots j_s} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)\\
+& \sum_{i_1+\cdots+i_r+j_2+\cdots+j_s=m-1} \frac{1}{i_1\cdots i_r j_2\cdots j_s} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_2}(x)\cdots E_{j_s}(x)\\
&+ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m-1} \frac{1}{i_1\cdots i_r j_2\cdots j_s} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)\\
&+\cdots+ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_{s-1}=m-1} \frac{1}{i_1\cdots i_r j_1\cdots j_{s-1}} B_{i_1}(x)\cdots B_{i_r}(x)\\
&\times E_{j_1}(x)\cdots E_{j_{s-1}}(x)\\
&+ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m-1} \frac{1}{i_1\cdots i_r j_1\cdots j_{s-1}} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)\\
=& r\gamma_{r-1,s,m-1}(x) + s\gamma_{r,s-1,m-1}(x)\\
&+ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m-1} \left\{ \frac{1}{i_2\cdots i_r j_1\cdots j_s} +\cdots+ \frac{1}{i_1\cdots i_{r-1} j_1\cdots j_{s-1}} \right.\\
&\left. + \frac{1}{i_1\cdots i_r j_2\cdots j_s} +\cdots+ \frac{1}{i_1\cdots i_r j_1\cdots j_{s-1}} \right\} B_{i_1}(x)\cdots B_{i_r}(x)\\
&\times E_{j_1}(x)\cdots E_{j_s}(x)\\
=& r\gamma_{r-1,s,m-1}(x) + s\gamma_{r,s-1,m-1}(x) + (m-1)\gamma_{r,s,m-1}(x).
\end{aligned}
$$

So we obtained that

$$\gamma'_{r,s,m}(x) = r\gamma_{r-1,s,m-1}(x) + s\gamma_{r,s-1,m-1}(x) + (m-1)\gamma_{r,s,m-1}(x), \tag{4.1}$$

with $\gamma_{r,s,r+s-1}(x) = 0$.

For $m \geq r+s$, let us put

$$\begin{aligned}
&\Lambda_{r,s,m} = \gamma_{r,s,m}(1) - \gamma_{r,s,m}(0)\\
=&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1\cdots i_r j_1\cdots j_s}\\
&\times (B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1) - B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s})\\
=&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1\cdots i_r j_1\cdots j_s}((B_{i_1}+\delta_{1,i_1})\cdots(B_{i_r}+\delta_{1,i_r})\\
&\times(-E_{j_1}+2\delta_{0,j_1})\cdots(-E_{j_s}+2\delta_{0,j_s}) - B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s})\\
=&\sum_{a=0}^{r}\binom{r}{a}\sum_{i_1+\cdots+i_a+j_1+\cdots+j_s=m+a-r}\frac{(-1)^s}{i_1\cdots i_a j_1\cdots j_s}B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_s}\\
&-\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}\frac{1}{i_1\cdots i_r j_1\cdots j_s}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}.
\end{aligned}$$

Note here that $m+a-r \geq a+s$ and hence that none of the inner sum for each a $(0 \leq a \leq r)$ are empty.

Let us denote $\int_0^1 \gamma_{r,s,m}(x)dx$ by $a_{r,s,m}$. Then. from (4.1) we have

$$\gamma_{r,s,m}(x) = -\frac{r}{m}\gamma_{r-1,s,m}(x) - \frac{s}{m}\gamma_{r,s-1,m}(x) + \frac{1}{m}\gamma'_{r,s,m+1}(x),$$

and hence obtain

$$a_{r,s,m} = -\frac{r}{m}a_{r-1,s,m} - \frac{s}{m}a_{r,s-1,m} + \frac{1}{m}\Lambda_{r,s,m+1}. \tag{4.2}$$

In [2], we showed that

$$a_{r,0,m} = \int_0^1 \gamma_{r,0,m}(x)dx = \sum_{j=1}^{r}\frac{(-1)^{j-1}(r)_{j-1}}{m^j}\Lambda_{r-j+1,0,m+1},\ (r \geq 1). \tag{4.3}$$

Also, in [17], we derived that

$$a_{0,s,m} = \int_0^1 \gamma_{0,s,m}(x)dx = \sum_{j=1}^{s}\frac{(-1)^{j-1}(s)_{j-1}}{m^j}\Lambda_{0,s-j+1,m+1},\ (s \geq 1). \tag{4.4}$$

We now observe that (4.2) together with (4.3) and (4.4) determines $a_{r,s,m}$ recursively for all r, s, m, with $m \geq r+s \geq 1$.

Also, we note that

$$\gamma_{r,s,m}(0) = \gamma_{r,s,m}(1) \Longleftrightarrow \Lambda_{r,s,m} = 0.$$

Now, we would like to determine the Fourier coefficients $C_n^{(r,s,m)}$.

Case $1 : n \neq 0$. Note that

$$\begin{aligned}
&C_n^{(r,s,r+s)} = \int_0^1 \gamma_{r,s,r+s}(x) e^{-2\pi i n x} dx \\
&= \int_0^1 B_1(x)^r E_1(x)^s e^{-2\pi i n x} dx \\
&= \int_0^1 \left(x - \frac{1}{2}\right)^{r+s} e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} \left[\left(x - \frac{1}{2}\right)^{r+s} e^{-2\pi i n x}\right]_0^1 + \frac{r+s}{2\pi i n} \int_0^1 \left(x - \frac{1}{2}\right)^{r+s-1} e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} \left(\left(\frac{1}{2}\right)^{r+s} - \left(-\frac{1}{2}\right)^{r+s}\right) + \frac{r+s}{2\pi i n} \int_0^1 \left(x - \frac{1}{2}\right)^{r+s-1} e^{-2\pi i n x} dx,
\end{aligned} \tag{4.5}$$

$$C_n^{(r-1,s,r+s-1)} = C_n^{(r,s-1,r+s-1)} = \int_0^1 \left(x - \frac{1}{2}\right)^{r+s-1} e^{-2\pi i n x} dx, \tag{4.6}$$

and

$$\Lambda_{r,s,r+s} = B_1(x)^r E_1(x)^s - B_1^r E_1^s = \left(\frac{1}{2}\right)^{r+s} - \left(-\frac{1}{2}\right)^{r+s}. \tag{4.7}$$

By (4.5), (4.6) and (4.7),

$$\begin{aligned}
&C_n^{(r,s,m)} = \int_0^1 \gamma_{r,s,m}(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} \left[\gamma_{r,s,m}(x) e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n} \int_0^1 \gamma'_{r,s,m}(x) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} \left(\gamma_{r,s,m}(1) - \gamma_{r,s,m}(0)\right) \\
&\quad + \frac{1}{2\pi i n} \int_0^1 \left\{r\gamma_{r-1,s,m-1}(x) + s\gamma_{r,s-1,m-1}(x) + (m-1)\gamma_{r,s,m-1}(x)\right\} e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n} \Lambda_{r,s,m} + \frac{1}{2\pi i n} \left(rC_n^{(r-1,s,m-1)} + sC_n^{(r,s-1,m-1)} + (m-1)C_n^{(r,s,m-1)}\right) \\
&= \frac{m-1}{2\pi i n} C_n^{(r,s,m-1)} + \frac{r}{2\pi i n} C_n^{(r-1,s,m-1)} + \frac{s}{2\pi i n} C_n^{(r,s-1,m-1)} - \frac{1}{2\pi i n} \Lambda_{r,s,m} \\
&= \frac{m-1}{2\pi i n} \Big(\frac{m-1}{2\pi i n} C_n^{(r,s,m-2)} + \frac{r}{2\pi i n} C_n^{(r-1,s,m-2)} + \frac{s}{2\pi i n} C_n^{(r,s-1,m-2)} \\
&\quad - \frac{1}{2\pi i n} \Lambda_{r,s,m-1}\Big) + \frac{r}{2\pi i n} C_n^{(r-1,s,m-1)} + \frac{s}{2\pi i n} C_n^{(r,s-1,m-1)} - \frac{1}{2\pi i n} \Lambda_{r,s,m} \\
&= \frac{(m-1)_2}{(2\pi i n)^2} C_n^{(r,s,m-2)} + \sum_{j=1}^{2} \frac{r(m-1)_j - 1}{(2\pi i n)^j} C_n^{(r-1,s,m-j)} \\
&\quad + \sum_{j=1}^{2} \frac{s(m-1)_{j-1}}{(2\pi i n)^j} C_n^{(r,s-1,m-j)} - \sum_{j=1}^{2} \frac{(m-1)_{j-1}}{(2\pi i n)^j} \Lambda_{r,s,m-j+1}
\end{aligned}$$

$$
\begin{aligned}
&= \cdots \\
&= \frac{(m-1)_{m-(r+s)}}{(2\pi i n)^{m-(r+s)}} C_n^{(r,s,r+s)} + \sum_{j=1}^{m-(r+s)} \frac{r(m-1)_{j-1}}{(2\pi i n)^j} C_n^{(r-1,s,m-j)} \\
&\quad + \sum_{j=1}^{m-(r+s)} \frac{s(m-1)_{j-1}}{(2\pi i n)^j} C_n^{(r,s-1,m-j)} - \sum_{j=1}^{m-(r+s)} \frac{(m-1)_{j-1}}{(2\pi i n)^j} \Lambda_{r,s,m-j+1} \\
&= \sum_{j=1}^{m-(r+s)+1} \frac{r(m-1)_{j-1}}{(2\pi i n)^j} C_n^{(r-1,s,m-j)} + \sum_{j=1}^{m-(r+s)+1} \frac{s(m-1)_{j-1}}{(2\pi i n)^j} C_n^{(r,s-1,m-j)} \\
&\quad - \sum_{j=1}^{m-(r+s)+1} \frac{(m-1)_{j-1}}{(2\pi i n)^j} \Lambda_{r,s,m-j+1}.
\end{aligned}
$$

So we have shown that

$$
\begin{aligned}
C_n^{(r,s,m)} &= \sum_{j=1}^{m-(r+s)+1} \frac{r(m-1)_{j-1}}{(2\pi i n)^j} C_n^{(r-1,s,m-j)} \\
&+ \sum_{j=1}^{m-(r+s)+1} \frac{s(m-1)_{j-1}}{(2\pi i n)^j} C_n^{(r,s-1,m-j)} - \sum_{j=1}^{m-(r+s)+1} \frac{(m-1)_{j-1}}{(2\pi i n)^j} \Lambda_{r,s,m-j+1}.
\end{aligned} \tag{4.8}
$$

Also, we recall from [2] and [17] that

$$
C_n^{(r,0,m)} = \sum_{j=1}^{m-r+1} \frac{r(m-1)_{j-1}}{(2\pi i n)^j} C_n^{(r-1,0,m-j)} - \sum_{j=1}^{m-r+1} \frac{(m-1)_{j-1}}{(2\pi i n)^j} \Lambda_{r,0,m-j+1}, \ (r \geq 2), \tag{4.9}
$$

$$
C_n^{(1,0,m)} = -\frac{(m-1)!}{(2\pi i n)^m}, \tag{4.10}
$$

$$
C_n^{(0,s,m)} = \sum_{j=1}^{m-s+1} \frac{s(m-1)_{j-1}}{(2\pi i n)^j} C_n^{(0,s-1,m-j)} - \sum_{j=1}^{m-s+1} \frac{(m-1)_{j-1}}{(2\pi i n)^j} \Lambda_{0,s,m-j+1}, \ (s \geq 2), \tag{4.11}
$$

$$
C_n^{(0,1,m)} = \frac{2}{m} \sum_{j=1}^{m} \frac{(m)_{j-1}}{(2\pi i n)^j} E_{m-j+1}. \tag{4.12}
$$

Now, we see that $C_n^{(r,s,m)}$ $(n \neq 0)$ can be determined for all $m \geq r+s \geq 1$ from (4.8)-(4.12).

Case $2: \ n = 0$.

$$
C_0^{(r,s,m)} = \int_0^1 \gamma_{r,s,m}(x) dx
$$

can be determined for all $m \geq r+s \geq 1$ from (4.2)-(4.4).

$\gamma_{r,s,m}(\langle x \rangle)$, $(m \geq r+s \geq 1)$ is piecewise C^{∞}. In addition, $\gamma_{r,s,m}(\langle x \rangle)$ is continuous for those r, s, m with $\Lambda_{r,s,m} = 0$ and discontinuous with jump discontinuities at integers for those r, s, m with $\Lambda_{r,s,m} \neq 0$.

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Assume first that $\Lambda_{r,s,m}=0$, for some integers r,s,m with $m\geq r+s\geq 1$. Then $\gamma_{r,s,m}(0)=\gamma_{r,s,m}(1)$. $\gamma_{r,s,m}(\langle x\rangle)$ is piecewise C^{∞}, and continuous. So the Fourier series of $\gamma_{r,s,m}(\langle x\rangle)$ converges uniformly to $\gamma_{r,s,m}(\langle x\rangle)$, and

$$\gamma_{r,s,m}(\langle x\rangle)=C_0^{(r,s,m)}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}C_n^{(r,s,m)}e^{2\pi inx},$$

where $C_0^{(r,s,m)}$ are determined by (4.2)-(4.4) and $C_n^{(r,s,m)}$ $(n\neq 0)$ by (4.8)-(4.12).

Now, we are ready to state our first result.

Theorem 4.1. *For all integers r,s,l with $l\geq r+s\geq 1$, we let*

$$\begin{aligned}\Lambda_{r,s,l}=&\sum_{a=0}^{r}\binom{r}{a}\sum_{i_1+\cdots+i_a+j_1+\cdots+j_s=l+a-r}\frac{(-1)^s}{i_1\cdots i_a j_1\cdots j_s}B_1\cdots B_{i_a}E_{j_1}\cdots E_{j_s}\\ -&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=l}\frac{1}{i_1\cdots i_r j_1\cdots j_s}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}.\end{aligned}$$

Assume that $\Lambda_{r,s,m}=0$, for some integers r,s,m with $m\geq r+s\geq 1$. Then we have the following.

$$\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}\frac{1}{i_1\cdots i_r j_1\cdots j_s}B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle)E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle)$$

has the Fourier series expansion

$$\begin{aligned}&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m}\frac{1}{i_1\cdots i_r j_1\cdots j_s}B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle)E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle)\\ =&C_0^{(r,s,m)}+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}C_n^{(r,s,m)}e^{2\pi inx},\end{aligned}$$

where $C_0^{(r,s,m)}$ are determined by (4.2)-(4.4) and $C_n^{(r,s,m)}$ $(n\neq 0)$ by (4.8)-(4.12).

Here the convergence is uniform.

Next, assume that $\Lambda_{r,sm}\neq 0$, for some integers r,s,m with $m\geq r+s\geq 1$. Then $\gamma_{r,1,m}(0)\neq\gamma_{r,s,m}(1)$. Hence $\gamma_{r,s,m}(\langle x\rangle)$ is piecewise C^{∞} and discontinuous with jump discontinuities at integers. Then the Fourier series of $\gamma_{r,s,m}(\langle x\rangle)$ converges pointwise to $\gamma_{r,s,m}(\langle x\rangle)$, for $x\notin\mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\gamma_{r,s,m}(0)+\gamma_{r,s,m}(1)\right)=\gamma_{r,s,m}(0)+\frac{1}{2}\Lambda_{r,s,m},$$

for $x\in\mathbb{Z}$.

Now, we can state our second result.

Theorem 4.2. *For all integers r,s,l with $l\geq r+s\geq 1$, we let*

$$\begin{aligned}\Lambda_{r,s,l}=&\sum_{a=0}^{r}\binom{r}{a}\sum_{i_1+\cdots+i_a+j_1+\cdots+j_s=l+a-r}\frac{(-1)^s}{i_1\cdots i_a j_1\cdots j_s}B_1\cdots B_{i_a}E_{j_1}\cdots E_{j_s}\\ -&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=l}\frac{1}{i_1\cdots i_r j_1\cdots j_s}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}.\end{aligned}$$

Assume that $\Lambda_{r,s,m} \neq 0$, for some integers r, s, m with $m \geq r+s \geq 1$. Then we have the following.

$$C_0^{(r,s,m)} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} C_n^{(r,s,m)} e^{2\pi i n x}$$

$$= \begin{cases} \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1\cdots i_r j_1\cdots j_s} B_{i_1}(\langle x\rangle)\cdots B_{i_r}(\langle x\rangle) E_{j_1}(\langle x\rangle)\cdots E_{j_s}(\langle x\rangle), \\ \quad \text{for } x \notin \mathbb{Z}, \\ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=m} \frac{1}{i_1\cdots i_r j_1\cdots j_s} B_{i_1}\cdots B_{i_r} E_{j_1}\cdots E_{j_s} + \frac{1}{2}\Lambda_{r,s,m}, \\ \quad \text{for } x \in \mathbb{Z}, \end{cases}$$

where $C_0^{(r,s,m)}$ are determined by (4.2)-(4.4) *and $C_n^{(r,s,m)}$ $(n \neq 0)$ by* (4.8)-(4.12).

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[1] Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin, 300160, China

[2] Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea.
E-mail address: tkkim@kw.ac.kr

[3] Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea.
E-mail address: dskim@sogang.ac.kr

[4] Hanrimwon, Kwangwoon University, Seoul 139-701, Republic of Korea.
E-mail address: dvdolgy@gmail.com

[5] Department of Mathematics Education, Daegu University, Gyeongsan-si, Gyeongsangbuk-do, 712-714, Republic of Korea.
E-mail address: a0417001@knu.ac.kr

A NOTE ON APPELL-TYPE DEGENERATE q-BERNOULLI POLYNOMIALS AND NUMBERS

JONGKYUM KWON AND JIN-WOO PARK[2,*]

ABSTRACT. Recently, several researchers have studied for Appell-type of various polynomials (see [18-20,22]). In this paper, we consider some families of Appell-type q-Bernoulli polynomials and numbers. In particular, we derive some interesting identities for the Appell-type degenerate q-Bernoulli polynomials by using the some properties of those polynomials.

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Let q be an indeterminate in $\mathbb{C}_p$ such that $|q-1|_p < p^{-\frac{1}{p-1}}$. The *$q$-analogue of number x* is defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$.

As is well known, the Bernoulli polynomials are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad \text{(see [1-10,12-17,21,23,24])}. \tag{1.1}$$

When $x = 0$, $B_n = B_n(0)$ are called Bernoulli numbers.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1} f(x)q^x, \ \text{(see [4,7-13])}, \tag{1.2}$$

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* corresponding author.

where $[x]_q = \frac{1-q^x}{1-q}$.

From (1.2), we note that

$$q^n I_{-q}(f_n) - I_{-q}(f) = (q-1)\sum_{l=0}^{n-1} q^l f(l) + \frac{q-1}{\log q}\sum_{l=0}^{n-1} f^{'}(l) q^l, \tag{1.3}$$

L. Carlitz considered the degenerate Bernoulli polynomials which are defined by the generating function to be

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(x \mid \lambda)\frac{t^n}{n!}, \text{ (see [2-4])} \tag{1.4}$$

when $x = 0$, $\beta_n(0|\lambda) = \beta_n(\lambda)$ are called Carlitz's q-Bernoulli numbers.

In [15], T. Kim introduced the degenerate Carlitz q-Bernoulli polynomials which are defined by the generating function to be

$$\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda}[x+y]_q} d\mu_q(y) = \sum_{n=0}^{\infty} \beta_{n,q}(x \mid \lambda)\frac{t^n}{n!}, \tag{1.5}$$

when $x = 0$, $\beta_{n,q}(0|\lambda) = \beta_{n,q}(\lambda)$ are called the degenerate Carlitz's q-Bernoulli numbers.

It is well known that the Bell polynomials are defined by the generating function to be

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} Bel_n(x)\frac{t^n}{n!}, \quad \text{(see [22])}. \tag{1.6}$$

As is well known, the Apostol-Bernoulli polynomials are defined by the generating function to be

$$\left(\frac{t}{qe^t-1}\right)e^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n(x \mid q)\frac{t^n}{n!}, \quad \text{(see [5])}. \tag{1.7}$$

When $x = 0$, $\mathfrak{B}_n = \mathfrak{B}_n(0 \mid q)$ are called Apostol-Bernoulli numbers.

The Stirling numbers of the second kind are defined by

$$(e^t-1)^n = n!\sum_{l=n}^{\infty} S_2(n,l)\frac{t^l}{l!}, \quad \text{(see [20])}. \tag{1.8}$$

The gamma and beta function are defined by the following definite integrals: for $(\alpha > 0, \beta > 0)$,

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt \tag{1.9}$$

and

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \int_0^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}dt, \quad \text{(see [22]).} \tag{1.10}$$

Thus by (1.9) and (1.10), we get

$$\Gamma(\alpha+1)=\alpha\Gamma(\alpha), \qquad B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{1.11}$$

Recently, several researchers have studied for Appell-type of various polynomials (see [18-20,22]). In this paper, we consider the Appell-type degenerate q-Bernoulli polynomials and derive some properties of those polynomials.

2. The Appell-type degenerate q-Bernoullli polynomials

In this section, we define the Appell-type degenerate q-Bernoulli polynomials which are given by

$$\frac{t}{q(1+\lambda t)^{\frac{1}{\lambda}}-1}e^{xt}=\sum_{n=0}^{\infty}\widetilde{B}_{n,\lambda,q}(x)\frac{t^n}{n!}, \tag{2.1}$$

when $x=0$, the Appell-type degenerate degenerate Bernoulli numbers $\widetilde{B}_{n,\lambda}=\widetilde{B}_{n,\lambda}(0)$ are equal to the degenerate q-Bernoulli numbers.

From (2.1), we have

$$\widetilde{B}_{m,\lambda,q}(x)=\sum_{m=0}^{n}\binom{n}{m}\widetilde{B}_{m,\lambda,q}x^{n-m}. \tag{2.2}$$

By (2.2), we obtain

$$\frac{d}{dx}\widetilde{B}_{n,\lambda,q}(x)=n\widetilde{B}_{n-1,\lambda,q}(x), n\geq 1. \tag{2.3}$$

From (2.3), we show that

$$\int_0^1 \widetilde{B}_{n,\lambda,q}(x)dx = \frac{1}{n+1}\int_0^1 \frac{d}{dx}\widetilde{B}_{n+1,\lambda,q}(x)dx = \frac{1}{n+1}\left(\widetilde{B}_{n+1,\lambda,q}(1)-\widetilde{B}_{n+1,\lambda,q}\right). \tag{2.4}$$

We observe that

$$\begin{aligned}\int_0^1 y^n \widetilde{B}_{n,\lambda,q}(x+y)dy &= \sum_{m=0}^{n}\binom{n}{m}\widetilde{B}_{n-m,\lambda,q}(x)\int_0^1 y^{n+m}dy \\ &= \sum_{m=0}^{n}\binom{n}{m}\frac{\widetilde{B}_{n-m,\lambda,q}(x)}{n+m+1}.\end{aligned} \tag{2.5}$$

On the other hand, we derive

$$\begin{aligned}\int_0^1 y^n \widetilde{B}_{n,\lambda,q}(x+y)dy &= \sum_{m=0}^{n}\binom{n}{m}\widetilde{B}_{n-m,\lambda,q}(x+1)(-1)^m\int_0^1 y^n(1-y)^m dy \\ &= \sum_{m=0}^{n}\binom{n}{m}\widetilde{B}_{n-m,\lambda,q}(x+1)(-1)^m\frac{\Gamma(n+1)\Gamma(m+1)}{\Gamma(n+m+2)} \\ &= \sum_{m=0}^{n}\binom{n}{m}\widetilde{B}_{n-m,\lambda,q}(x+1)(-1)^m\frac{n!\ m!}{(n+m+1)!}.\end{aligned} \tag{2.6}$$

Therefore, by (2.5) and (2.6), we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{N}$, we have*

$$\sum_{m=0}^{n}\binom{n}{m}\frac{\widetilde{B}_{n-m,\lambda,q}(x)}{n+m+1} = \sum_{m=0}^{n}\binom{n}{m}\widetilde{B}_{n-m,\lambda,q}(x+1)(-1)^m\frac{n!\ m!}{(n+m+1)!},$$

when, $x=0$, $\sum_{m=0}^{n}\binom{n}{m}\frac{\widetilde{B}_{n-m,\lambda,q}}{n+m+1} = \sum_{m=0}^{n}\binom{n}{m}\widetilde{B}_{n-m,\lambda,q}(1)(-1)^m\frac{n!\ m!}{(n+m+1)!}$.

We also observe that

$$\begin{aligned}&\int_0^1 y^n \widetilde{B}_{n,\lambda,q}(x+y)dy \\ =&\frac{\widetilde{B}_{n,\lambda,q}(x+1)}{n+1} - \frac{n}{n+1}\int_0^1 y^{n+1}\widetilde{B}_{n-1,\lambda,q}(x+y)dy \\ =&\frac{\widetilde{B}_{n,\lambda,q}(x+1)}{n+1} - \frac{n}{n+1}\frac{\widetilde{B}_{n-1,\lambda,q}(x+1)}{n+2} \\ &+(-1)^2\frac{n(n-1)}{(n+1)(n+2)}\int_0^1 y^{n+2}\widetilde{B}_{n-2,\lambda,q}(x+y)dy \\ =&\frac{\widetilde{B}_{n,\lambda,q}(x+1)}{n+1} - \frac{n\widetilde{B}_{n-1,\lambda,q}(x+1)}{(n+1)(n+2)} + (-1)^2\frac{n(n-1)\widetilde{B}_{n-2,\lambda,q}(x+1)}{(n+1)(n+2)(n+3)} \\ &+(-1)^3\frac{n(n-1)(n-2)}{(n+1)(n+2)(n+3)}\int_0^1 y^{n+3}\widetilde{B}_{n-3,\lambda,q}(x+y)dy.\end{aligned} \tag{2.7}$$

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Continuing this process, we get

$$\begin{aligned}
&\int_0^1 y^{2n-1}\widetilde{B}_{1,\lambda,q}(x+y)dy \\
=&\frac{\widetilde{B}_{1,\lambda,q}(x+1)}{2n}-\frac{1}{2n}\int_0^1 y^{2n}\widetilde{B}_{0,\lambda}(x+y)dy \\
=&\frac{\widetilde{B}_{1,\lambda,q}(x+1)}{2n}-\frac{1}{2n}\frac{1}{2n+1}.
\end{aligned} \tag{2.8}$$

Therefore, by (2.7) and (2.8), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{N}$, we have*

$$\sum_{m=0}^{n}\binom{n}{m}\frac{\widetilde{B}_{n-m,\lambda,q}(x)}{n+m+1}=\sum_{m=0}^{n}\frac{n(n-1)\cdots(n-m+1)}{(n+1)(n+2)\cdots(n+m)}(-1)^m\widetilde{B}_{n-m,\lambda,q}(x+1).$$

For $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\int_0^1 y^{n}\widetilde{B}_{n,\lambda,q}(x+y)dy \\
=&\frac{\widetilde{B}_{n+1,\lambda,q}(x+1)}{n+1}-\frac{n}{n+1}\int_0^1 y^{n-1}\widetilde{B}_{n+1,\lambda,q}(x+y)dy \\
=&\frac{\widetilde{B}_{n+1,\lambda,q}(x+1)}{n+1}-\frac{n}{n+1}\frac{\widetilde{B}_{n+2,\lambda,q}(x+1)}{n+2}+(-1)^2\frac{n}{n+1}\frac{n-1}{n+2}\int_0^1 y^{n-2}\widetilde{B}_{n+2,\lambda,q}(x+y)dy \\
=&\frac{\widetilde{B}_{n+1,\lambda,q}(x+1)}{n+1}-\frac{n}{n+1}\sum_{m=0}^{n+1}\binom{n+1}{m}\widetilde{B}_{n+1-m,\lambda,q}(x+1)(-1)^m\int_0^1 y^{n-l}(1-y)^m dy \\
=&\frac{\widetilde{B}_{n+1,\lambda,q}(x+1)}{n+1}-\frac{n}{n+1}\sum_{m=0}^{n+1}\binom{n+1}{m}\widetilde{B}_{n+1-m,\lambda,q}(x+1)(-1)^m B(n,m+1),
\end{aligned} \tag{2.9}$$

where $B(n, m+1)$ is a beta function.

Therefore, by (2.5) and (2.9), we obtain the following theorem.

Theorem 2.3. *For $n \in \mathbb{N}$, we have*

$$\begin{aligned}
&\sum_{m=0}^{n}\binom{n}{m}\frac{\widetilde{B}_{n-m,\lambda,q}(x)}{n+m+1} \\
=&\frac{\widetilde{B}_{n+1,\lambda,q}(x+1)}{n+1}-\frac{n}{n+1}\sum_{m=0}^{n+1}\binom{n+1}{m}\widetilde{B}_{n+1-m,\lambda,q}(x+1)(-1)^m B(n,m+1).
\end{aligned}$$

Now, we observe that

$$
\begin{aligned}
&\int_0^1 \widetilde{B}_{m,\lambda,q}(x)\widetilde{B}_{n,\lambda,q}(x)dx \\
=&\sum_{l=0}^{n}\binom{n}{l}\widetilde{B}_{l,\lambda,q}\sum_{k=0}^{m}\binom{m}{k}\widetilde{B}_{k,\lambda,q}(1)(-1)^{m-k}\int_0^1 x^{n-l}(1-x)^{m-k}dx \\
=&\sum_{l=0}^{n}\sum_{k=0}^{m}\binom{n}{l}\binom{m}{k}(-1)^{m-k}\widetilde{B}_{k,\lambda,q}(1)\widetilde{B}_{l,\lambda,q}B(n-l+1,m-k+1) \\
=&\sum_{l=0}^{n}\sum_{k=0}^{m}\binom{n}{l}\binom{m}{k}(-1)^{m-k}\widetilde{B}_{k,\lambda,q}(1)\widetilde{B}_{l,\lambda,q}\frac{\Gamma(n-l+1)\Gamma(m-k+1)}{\Gamma(n+m-l-k+2)}.
\end{aligned}
\tag{2.10}
$$

On the other hand,

$$
\int_0^1 \widetilde{B}_{m,\lambda,q}(x)\widetilde{B}_{n,\lambda,q}(x)dx=\sum_{l=0}^{n}\sum_{k=0}^{m}\binom{n}{l}\binom{m}{k}\frac{\widetilde{B}_{m-k,\lambda,q}\widetilde{B}_{n-l,\lambda,q}}{k+l+1}. \tag{2.11}
$$

Therefore, by (2.10) and (2.11), we obtain the following theorem.

Theorem 2.4. *For $n\in\mathbb{N}$, we have*

$$
\begin{aligned}
&\sum_{l=0}^{n}\sum_{k=0}^{m}\binom{n}{l}\binom{m}{k}(-1)^{m-k}\widetilde{B}_{k,\lambda,q}(1)\widetilde{B}_{l,\lambda,q}\frac{\Gamma(n-l+1)\Gamma(m-k+1)}{\Gamma(n+m-l-k+2)} \\
=&\sum_{l=0}^{n}\sum_{k=0}^{m}\binom{n}{l}\binom{m}{k}\frac{\widetilde{B}_{m-k,\lambda,q}\widetilde{B}_{n-l,\lambda,q}}{k+l+1}.
\end{aligned}
$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (2.1), we get

$$\begin{aligned}
&\frac{\frac{1}{\lambda}(e^{\lambda t}-1)}{q(1+\lambda\frac{1}{\lambda}(e^{\lambda t}-1))^{\frac{1}{\lambda}}-1}e^{x\frac{1}{\lambda}(e^{\lambda t}-1)}\\
=&\frac{\frac{1}{\lambda}(e^{\lambda t}-1)}{qe^{t}-1}e^{\frac{x}{\lambda}(e^{\lambda t}-1)}\\
=&\left(\frac{t}{qe^{t}-1}\right)\left(\frac{e^{\lambda t}-1}{\lambda t}\right)\left(e^{\frac{1}{\lambda}x(e^{\lambda t}-1)}\right)\\
=&\left(\sum_{n=0}^{\infty}\mathfrak{B}_n(x\mid q)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}\lambda^j\frac{t^j}{j!}\right)\left(\sum_{m=0}^{\infty}Bel_m(\frac{x}{\lambda})\frac{\lambda^m t^m}{m!}\right)\\
=&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{m}{l}\binom{n}{m}\lambda^{n-l}\mathfrak{B}_l(x\mid q)Bel_{n-m}(\frac{x}{\lambda})\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.12}$$

On the other hand,

$$\begin{aligned}
\sum_{m=0}^{\infty}\widetilde{B}_{m,\lambda,q}(x)\frac{1}{m!}\frac{1}{\lambda^m}(e^{\lambda t}-1)^m&=\sum_{m=0}^{\infty}\widetilde{B}_{m,\lambda,q}(x)\frac{1}{\lambda^m}\sum_{n=m}^{\infty}S_2(n,m)\frac{\lambda^n t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\widetilde{B}_{m,\lambda,q}(x)\lambda^{n-m}S_2(n,m)\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.13}$$

where $S_2(n,m)$ is the Stirling numbers of the second kind.

Therefore, by (2.12) and (2.13), we obtain the following theorem.

Theorem 2.5. *For $n\in\mathbb{N}$, we have*

$$\sum_{m=0}^{n}\widetilde{B}_{m,\lambda,q}(x)\lambda^{n-m}S_2(n,m)=\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{m}{l}\binom{n}{m}\lambda^{n-l}\mathfrak{B}_l(x\mid q)Bel_{n-m}(\frac{x}{\lambda}).$$

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Department of Mathematics Education and RINS, Gyeongsang National University, Jinju, Gyeongsangnamdo, 52828, Republic of Korea
E-mail address: mathkjk26@gnu.ac.kr

[2,*] Department of Mathematics Education, Daegu University, Gyeongsan-si, Kyungsangbukdo, 38453, Republic of Korea
E-mail address: a0417001@knu.ac.kr

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